Evolution of mesoscopic interactions and scattering solutions of the Boltzmann equation

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1 Introduction

The purpose of this writing is to study the evolution of interacting moving bodies in the mesoscopic scale. Consider a differential equation of the form:

\[ \partial_t f(x, \xi, t) + \xi \cdot \nabla_x f(x, \xi, t) = I(f, x, \xi, t) \]  \hspace{1cm} \text{(1.1)}

\[ f(x, \xi, 0) = f_0(x, \xi) \]

In the equation defined above \( x, \xi \in \mathbb{R}^n \) and \( t \in [0, \infty) \). We expect \( f(x, \xi, t) \) to be non-negative and interpret it as density or amplitude of bodies or particles with velocity \( \xi \) located at \( x \). The left hand side of this equation resembles the principle of inertia, that implies particles will get transported linearly along the trajectory of their velocities, while the right hand side is representative of change via possibly non linear interactions of particles. The interaction \( I(f, x, \xi, t) \) is expected to be a mesoscopic interaction in the sense defined below.
Definition 1.1. \( I(f, x, \xi, t) \) is a mesoscopic interaction if:

\[
\int_{\mathbb{R}^n} I(f, x, \xi, t)d\xi = \int_{\mathbb{R}^n} I(f, x, \xi, t)|\xi|^2d\xi = \int_{\mathbb{R}^n} I(f, x, \xi, t)\xi_i d\xi = 0
\]

1 ≤ \( i \) ≤ \( n \)

One immediate consequence of the previous definition is that for solutions of equation (1.1) the total amount of mass, momentum and energy remain invariant. For example, conservation of mass can be proven by the argument below using the definition of a mesoscopic interaction and the other two are similar.

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)dxd\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d}{dt} f(x + t\xi, \xi, t)dxd\xi = \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(f, x + t\xi, \xi, t)dxd\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(f, x, \xi, t)dxd\xi = 0
\]

We will use the notation \( M, E \) and \( V \) respectively for the macroscopic quantities mass, energy and momentum defined below.

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)dxd\xi = M \quad (1.2)
\]

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)|\xi|^2dxd\xi = E
\]

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)\xi dxd\xi = V
\]

The narrative of this writing is grounded in concrete examples and the generality of the theory is built upon that. Some of the most important examples of mesoscopic interactions are the linear transport equation for \( I = 0 \) and different variants of the Boltzmann equation where \( I \) is the Boltzmann collision operator. Lanford \[7\] writes:

*The conceptual foundations of the Boltzmann equation seem to me to merit careful study not so much for their own sake as because the Boltzmann equation is a prototype of a mathematical construct central to the theory of time-dependent phenomena in large systems.*

When we think of the particles as discrete elastic bodies, that is the microscopic scale. In that setting, classical mechanics governs the evolution of interacting particles, namely the principle of inertia and conservation laws of momentum and energy. On the other hand, we have differential equations that represent the evolution of macroscopic descriptions, where particles are not discrete objects anymore. Crucial examples of such equations are Navier–Stokes and Euler equations in the context of fluid dynamics. The role of the mesoscopic scale, with the Boltzmann equation as its central example, is to create a
connection between microscopic and macroscopic descriptions.

A rigorous derivation of the Boltzmann equation from the dynamics of the microscopic scale was created first by Lanford in an asymptotic regime that is often called the Boltzmann-Grad limit, subject to a short time interval of validity. His ideas were later improved upon in the landmark works of Raymond[2] and others. Another successful and unique approach for the justification of the Boltzmann equation, without limitation on time but under stochastic assumptions for the microscopic dynamics, appears in the works of Rezakanlou[8].

Similar to the attempts for justification of the Boltzmann equation from the microscopic scale, there has been a vast amount of research dedicated to the relationship between the Boltzmann equation and the aforementioned higher scale macroscopic equations. Assuming the validity of laws of classical mechanics in the microscopic scale is self-evident, the hierarchy described before demonstrates that the Boltzmann equation can be a scaffold, to illuminate the validity of the most important equations appearing in fluid mechanics, which were originally derived independent of the Boltzmann equation. More than often, mathematical arguments related to the transition from the mesoscopic scale to the macroscopic one remain valid if we replace the Boltzmann collision operator with an arbitrary mesoscopic interaction. For example in [1], we see that it is possible, at least formally, to derive the compressible Euler equations from the Boltzmann equation, using some constitutive relations in fluid dynamics, yet an identical argument is also valid for any solution of equation (1.1) with a mesoscopic interaction.

Intrinsic properties of the Boltzmann equation, independent of their relationship to the microscopic or macroscopic scales, lead to a series of heuristic arguments that establish the relevance of the equation to the actual physical problem of interacting bodies. For example, conservation of mass, momentum and energy remain true for any mesoscopic interaction and are not solely a property of the Boltzmann collision operator. Another feature of the Boltzmann equation is the monotonicity of the famous Boltzmann’s entropy formula, which can be interpreted as a mathematical formulation of the second law of thermodynamics. The previous statement makes the idea of a thermodynamic equilibrium relevant of which Maxwell–Boltzmann distributions—often called Maxwellsians—are the principal examples. However we will show there exists other monotone quantities for any general mesoscopic interaction, that one can harness to study the evolution of solutions.

The objective here is to study the mesoscopic scale in more generality independent of the specific structure of interactions, we will follow this goal in sections 2 and 3, and we will return afterwards to the Boltzmann equation to show the existence of a class of solutions that can serve as examples for the results of the previous sections. In section 2, under the assumption of having a solution to equation (1.1), we introduce the concept of uncertainty associ-
ated to the solution. Under a minimal set of priori assumptions we will show that the evolution of moving interacting particles is subject to dispersion and furthermore uncertainty goes to infinity with time. Section 3 is dedicated to understanding the role of a norm in measuring dispersion of particles and developing the concept of scattering for mesoscopic interactions. The contents of sections 2 and 3 include the construction of monotone quantities and the results on convergence of solutions to equation (1.1), which are subject to an arbitrary mesoscopic interaction. For this reason, hereinafter we purposely avoid any reference to the term entropy. The mathematical analysis of the next two sections illuminates a competition between linear and possibly non-linear parts of equation (1.1). These ideas have analogies to the theory of linear and non-linear Schrodinger’s equations as Tao [4] describes very well. Finally in section 4, we will use the insights from section 3 to create a class of solutions to the Boltzmann equation, namely the scattering solutions.

We intend this writing to be clear and accessible for a reader with background knowledge in the field and hope that it reflects the sage advice of Halmos [9]:

\begin{quote}
A good attitude to the preparation of written mathematical exposition is to pretend that it is spoken. Pretend that you are explaining the subject to a friend on a long walk in the woods.
\end{quote}
2 Dispersive evolution of interacting particles

Throughout this section we assume $f$ is a non negative solution of the equation (1.1) for some mesoscopic interaction $I$. We expect that at time zero the mass, momentum and energy as represented in (1.2) are well defined quantities. Furthermore we assume that at time zero the integral below is convergent.

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)|x|^2 dx d\xi < \infty$$ (2.1)

Results that appear here hold true under the assumptions described above and are independent of the specific structure of the interaction $I$, consequently they are true for the solutions of the Boltzmann equation as well. Discussion about the existence of such solutions is omitted until the next section.

Definition 2.1. Let $A(t)$ be the angular momentum associated to $f$ at time $t$:

$$A(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)x.\xi dx d\xi$$

Theorem 2.1. For solutions of equation (1.1), the angular momentum is a linear function of time increasing proportional to total energy defined in (1.2):

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)x.\xi dx d\xi = A(0) + tE$$

Proof.

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)x.\xi dx d\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + t\xi, \xi, t)(x + t\xi).\xi dx d\xi =$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + t\xi, \xi, t)x.\xi dx d\xi + t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + t\xi, \xi, t)|\xi|^2 dx d\xi =$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dx d\xi + \int_{0}^{t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(x + s\xi, \xi, s)x.\xi dx d\xi ds +$$

$$t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)|\xi|^2 dx d\xi =$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dx d\xi + \int_{0}^{t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(x, \xi, s)(x - s\xi).\xi dx d\xi ds +$$

$$t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)|\xi|^2 dx d\xi = A(0) + tE$$

The last equality is a consequence of definition 1.1, which completes the proof.

$\square$
Definition 2.2. Let \( U(t) \) be the uncertainty associated to \( f \) at time \( t \):

\[
U(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)|x||\xi|dx\xi
\]

Remark. It is possible to interpret uncertainty defined above akin to its quantum mechanics counterpart, that is by highlighting a fundamental limit on the precision of the physical measurements. As in figure 1, consider an observer located at the origin and assume speed of light is \( C \). For each particle located at \( x \) there is a minimum delay of \( T = C^{-1}|x| \) between the actual time of measurement and observation. The quantity \( T \times f(x, \xi, t)||\xi|| = C^{-1}f(x, \xi, t)|x||\xi| \) represents uncertainty of measurement due to the interval of delay for a particle at position \( x \) with velocity \( \xi \). Re-scaling this quantity with \( C = 1 \) and integrating over the space of the positions and velocities one can get the definition above.

Definition 2.3. Let \( \|f\|_G \) be the relative angular norm defined below in terms of uncertainty and angular momentum:

\[
\|f\|_G = \sup_t \left( U(t) - A(t) \right)
\]

Theorem 2.2. Relative angular norm for the solutions of the equation (1.1) is bounded:

\[
\|f\|_G \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)(|x|^2 + |\xi|^2)dx\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dx\xi
\]

Furthermore, uncertainty goes to infinity with time:

\[
\lim_{t \to \infty} U(t) = \infty
\]
Proof. Start by the following observation using the theorem 2.1:

\[
U(t) - A(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)|x||\xi|dxd\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)x.\xi dxd\xi = \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + t\xi, \xi, t)|x + t\xi||\xi|dxd\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi - \\
t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)|\xi|^2dxd\xi
\]

Continue with considering the non negativity of \( f \), the triangle inequality and definition of a mesoscopic interaction. We get:

\[
U(t) - A(t) \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi - t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)|\xi|^2dxd\xi = \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + t\xi, \xi, t)|x||\xi|dxd\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi \leq \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + t\xi, \xi, t)(|x|^2 + |\xi|^2)dxd\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi = \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)(|x|^2 + |\xi|^2)dxd\xi + \\
\int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(f, x + s\xi, \xi, s)(|x|^2 + |\xi|^2)dxd\xi ds - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi = \\
\int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(f, x, \xi, s)(|x - s\xi|^2 + |\xi|^2)dxd\xi ds - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi = \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)(|x|^2 + |\xi|^2)dxd\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi
\]

Recall assumption (2.1), now since the bound found above is independent of time, it follows:

\[
\|f\|_G = \sup_t (U(t) - A(t)) \leq \\
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)(|x|^2 + |\xi|^2)dxd\xi - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, 0)x.\xi dxd\xi
\]

Finally theorem 2.1 implies that \( \lim_{t \to \infty} A(t) = \infty \). On the other hand we just proved \( \|f\|_G < \infty \), therefore:

\[
\lim_{t \to \infty} U(t) = \infty
\]

\( \square \)

Corollary 2.2.1. The previous theorem implies that uncertainty on average increases linearly.
Theorem 2.3. For the solutions of the equation (1.1), energy contained inside any bounded subset of the spatial variable is integrable over time. Equivalently, if $X \subseteq \mathbb{R}^n$ and is bounded, then:

$$\int_0^\infty \int_{\mathbb{R}^n} f(x, \xi, t)|\xi|^2 dx d\xi dt < \infty$$

Proof. Let $\frac{(x - x_0) \cdot \xi}{|x - x_0|}$ be the localized angular momentum at point $x_0$. For any $x_0 \in \mathbb{R}^n$, after multiplying the both sides of equation (1.1) with this quantity, we have:

$$\partial_t f(x, \xi, t) \frac{(x - x_0) \cdot \xi}{|x - x_0|} + \sum_{i=1}^{n} \frac{(x - x_0) \cdot \xi_i}{|x - x_0|} \partial_{x_i} f(x, \xi, t) = I(f, x, \xi, t) \frac{(x - x_0) \cdot \xi}{|x - x_0|}$$

Integrating the left and right sides of this equation over the space of the positions and velocities, following by the change of variables $x \to x + x_0$ we get:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_t f(x + x_0, \xi, t) \frac{(x \cdot \xi)}{|x|} dx d\xi +$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{(x \cdot \xi_i)}{|x|} \xi_i \partial_{x_i} f(x + x_0, \xi, t) dx d\xi =$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(f, x + x_0, \xi, t) \frac{(x \cdot \xi)}{|x|} dx d\xi = 0$$

The last line is true since $I$ is a mesoscopic interaction. Let $\theta(x, \xi)$ be the angle between $x$ and $\xi$, then continue by integration by parts with respect to $x$:

$$\partial_t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + x_0, \xi, t) \frac{(x \cdot \xi)}{|x|} dx d\xi +$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{1}{|x|^3} (x \cdot \xi)^2 f(x + x_0, \xi, t) - \frac{1}{|x|} |\xi|^2 f(x + x_0, \xi, t) \right) dx d\xi = 0 \rightarrow$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x|} |\xi|^2 \sin^2(\theta(x, \xi)) f(x + x_0, \xi, t) dx d\xi =$$

$$\partial_t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + x_0, \xi, t) \frac{(x \cdot \xi)}{|x|} dx d\xi$$

The derivative appearing in last line of the computation above, is the time derivative of a bounded quantity:

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + x_0, \xi, t) \frac{(x \cdot \xi)}{|x|} dx d\xi \right| \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + x_0, \xi, t) |\xi| dx d\xi \leq$$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t) (1 + |\xi|^2) dx d\xi \leq M + E < \infty$$

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Positivity of the derivative of this bounded quantity implies that for an arbitrary \( x_0 \in \mathbb{R}^n \) following limit exists:

\[
\lim_{t \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + x_0, \xi, t) \frac{(x, \xi)}{|x|} \, dx \, d\xi < \infty
\]

Furthermore previous argument indicates:

\[
egin{align*}
&\int_{0}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x|} |\xi|^2 \sin^2(\theta(x, \xi)) f(x + x_0, \xi, t) \, dx \, d\xi \, dt = \\
&\int_{0}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x - x_0|} |\xi|^2 \sin^2(\theta(x - x_0, \xi)) f(x, \xi, t) \, dx \, d\xi \, dt < \infty
\end{align*}
\] (2.2)

Now define the blind cone at \( x \) with respect to observer \( x_0 \) for \( c > 0 \) as:

\[
C_{x_0}(x, c) = \{ \xi \in \mathbb{R}^n | \theta(x - x_0, \xi) \notin [c, \pi - c] \}
\] (2.3)

Let \( B(x_0, R) \) be the ball of radius \( R \) centered at \( x_0 \). Continue by integrating inside this ball in the space of positions and outside of the blind cones in space of velocities. Using (2.2), one can get:

\[
\sin^2(c) \int_{0}^{\infty} \int_{B(x_0, R)} \int_{\mathbb{R}^n - C_{x_0}(x, c)} |\xi|^2 f(x, \xi, t) \, d\xi \, dx \, dt < \\
\int_{0}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x - x_0|} |\xi|^2 \sin^2(\theta(x - x_0, \xi)) f(x, \xi, t) \, dx \, d\xi \, dt < \infty
\]

This implies that for any \( x_0 \in \mathbb{R}^n, c > 0 \) and \( R > 0 \) we have:

\[
\int_{0}^{\infty} \int_{B(x_0, R)} \int_{\mathbb{R}^n - C_{x_0}(x, c)} |\xi|^2 f(x, \xi, t) \, d\xi \, dx \, dt < \infty
\] (2.4)

Choose any three distinct observers: \( O_1, O_2, O_3 \in \partial B(0, R) \) and for an arbitrary \( x \in B(0, R) \) define \( P = \partial B(0, R) \cap C_{O_1}(x, 2c) \). If there exists an \( O_i \) such that \( O_i \notin P \) then the blind cone \( C_{O_i}(x) \) and \( C_{O_1}(x) \) have an empty intersection. Set \( P \) defined on \( \partial B(0, R) \) is made of two path connected components, consider the longest short path on each component and set \( K \) to be the maximum length of the two, for a fixed \( R \) it is possible to choose \( c \) small enough such that \( K \) becomes as small as desired. Now set \( c \) small enough such that \( K \) becomes smaller than the shortest path on the sphere between any two of the there observers. The pigeon hole principle implies that since each path connected component of \( S \) can only contain maximum one of the observers, there exists an \( O_i \) such that \( O_i \notin P \).
Figure 2: One possible configuration of three observers. This drawing includes blind cones \(C_{O_1}(x,c), C_{O_1}(x,2c), C_{O_3}(x,c)\) and ball \(B(0,R)\), as well as a fragment of \(B(O_1,2R)\). Here \(O_3 \notin P\).

The previous argument implies that for any \(x \in B(0,R)\) we have:

\[C_{O_1}(x) \cap C_{O_1}(x) \cap C_{O_1}(x) = \emptyset\]

Now let \(A, B\) and \(C\) be the integrals defined below. Convergence of these integrals is consequence of (2.4).

\[
A = \int_0^\infty \int_{B(O_1,2R)} \int_{\mathbb{R}^n-C_{O_1}(x)} |\xi|^2 f(x, \xi, t) d\xi dx dt < \infty
\]

\[
B = \int_0^\infty \int_{B(O_2,2R)} \int_{\mathbb{R}^n-C_{O_2}(x)} |\xi|^2 f(x, \xi, t) d\xi dx dt < \infty
\]

\[
C = \int_0^\infty \int_{B(O_3,2R)} \int_{\mathbb{R}^n-C_{O_3}(x)} |\xi|^2 f(x, \xi, t) d\xi dx dt < \infty
\]

Consider that \(B(0,R) \subset B(O_1,2R) \cap B(O_2,2R) \cap B(O_3,2R)\), furthermore for any \(x \in B(0,R)\) we have \(C_{O_1}(x) \cap C_{O_1}(x) \cap C_{O_3}(x) = \emptyset\), this shows that any subset of \(B(0,R) \times \mathbb{R}^n\) is at least covered once in the domains of integration.
for $A, B$ and $C$. Therefore since integrands are positive we have:

$$\int_0^\infty \int_{B(0,R)} \int_{\mathbb{R}^n} f(x, \xi, t)|\xi|^2d\xi dx dt < A + B + C < \infty$$

Finally, this argument can be replicated for a ball centered at an arbitrary point. Accordingly because any bounded set is contained within a ball of finite radius and that $f$ is positive, proof is complete.

\[ \square \]

**Corollary 2.3.1.** As a result of the previous theorem we have that inside any bounded subset of the space of positions, like $X$, total mass of the particles with magnitude of velocity greater or equal to $v$, for any fixed $0 < v$, is integrable over time. Equivalently:

$$\int_0^\infty \int_{\mathbb{R}^n - B(0, v)} \int_X f(x, \xi, t)dx d\xi dt < \infty$$

**Remark.** Consider the quantity defined below and recall (2.1).

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t)|x|^2 dxd\xi$$

Using integration by parts one can show that it is convex over time and:

$$\partial_t^2 \int \int f(x, \xi, t)|x|^2 = 2 \int \int |\xi|^2 f(x, \xi, t) d\xi dx$$

However it is not possible to use this convex quantity or monotonicity of the angular momentum to get similar results as the previous theorem, because these quantities are not priori bounded. Also the previous argument is not useful for unbounded sets, since any two different observers will have overlapping blind cones when $x$ is distant enough from both of them. This implies that it is not possible to cover an unbounded set with finitely many such observers as we did before.

The previous theorem and corollary are proved independent of the specific structure of the interactions. Including more priori assumptions about the solutions of equation (1.1), paired with theorem 2.3 or corollary 2.3.1 can illuminate the use of the term dispersion. For example, consider a class of solutions to equation (1.1) which are bounded and have bounded derivatives with respect to time. Using positivity of $f$ and corollary 2.3.1, one can conclude that if $X \subset R^n$ is an arbitrary bounded subset of the spatial variable and $v$ is any fixed positive number then for almost every $(x, \xi) \in X \times (R^n - B(0, v))$ we have that:

$$\lim_{t \to \infty} f(x, \xi, t) = 0$$

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This indicates that inside any bounded set of the spatial variable like $X$, as time goes to infinity, almost every particle with a magnitude of velocity greater than $v$ will inevitably leave the bounded set. The previous assertion is equivalent to the statement that only the particles with an arbitrarily small velocity can remain inside $X$. Thus, as time goes to infinity, an idle observer will almost only identify particles with velocity $\xi = 0$ inside any bounded set. One can continue using the Galilean invariance of the setting. Assume that the same idle observer starts to move with a non zero arbitrary constant velocity like $\xi$. As a consequence, when time goes to infinity, inside any bounded moving region with the same constant velocity, like $X + t\xi \subset \mathbb{R}^n$, the particles with velocity $\xi$ are being observed as motionless with respect to the moving observer, while almost every particle with a different velocity will vanish from the moving region. However, this decay may or may not be uniform. This shows that, in a way, the different velocities or frequencies separate from each other along the evolution of the solutions to equation (1.1). Each particle has a tendency to travel with other particles whose velocities are indistinguishable from each other and as a result avoid interaction with particles that move at different velocities.

**Corollary 2.3.2.** As a consequence of the Galilean invariance for the solutions of equation (1.1), theorem 2.1 and corollary 2.3.1 can be replicated for a bounded region that moves with constant velocity $\xi$.

The previous discussion is only one route to demonstrate the universality of dispersion along evolution of particles. One can support the idea of dispersion, even without assumptions on boundedness of the solutions or their derivatives. The rest of this section is faithful to this idea.

Recall (2.3) as the definition of the blind cone $C_{x_0}(x, c)$ for some $c > 0$. By allowing velocities that their magnitude is smaller than some fixed $0 < v$, define $K_{x_0}(x, v, c)$ as the punctured blind cone at $x \in \mathbb{R}^n$ with respect to observer $x_0 \in \mathbb{R}^n$:

$$K_{x_0}(x, v, c) = C_{x_0}(x, c) \cup B(0, v)$$

**Definition 2.4.** Let $\Gamma_{x_0}(c, v)$ be the collection of the punctured blind cones over the spatial variable, for $0 < c, v$ and $x_0 \in \mathbb{R}^n$ defined below.

$$\Gamma_{x_0}(c, v) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n | \xi \in K_{x_0}(x)\}$$

**Theorem 2.4.** Under assumption (2.1), along the evolution of the solutions to equation (1.1), total mass is contained within any collection of the punctured blind cones, on the average sense defined below. For any positive $c$ and $v$ and observer $x_0 \in \mathbb{R}^n$ we have:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\Gamma_{x_0}(c, v)} f(x, \xi, t) dx d\xi dt = M$$
Figure 3: An illustration for collection of punctured blind cones $\Gamma_{x_0}(c, v) \subset \mathbb{R}^n \times \mathbb{R}^n$. In this drawing balls are of radius $v$ and cones with angle $c$ are facing the observer $x_0$ at the center.

Proof. Define the following two sets:

$$N_R = \{(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n - K_{x_0}(x)) | |x| > R\}$$
$$M_R = \{(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n - K_{x_0}(x)) | |x| \leq R\}$$

Using boundedness of the relative angular norm from theorem 2.2, it is possible to control total amount of mass inside $N_R$:

$$\iint_{N_R} f(x, \xi, t)(|x||\xi| - x, \xi)dx d\xi =$$
$$\iint_{N_R} f(x, \xi, t)|x||\xi|(1 - \cos(\theta))dx d\xi \leq ||f||_G \to$$
$$Re(1 - \cos(c)) \iint_{N_R} f(x, \xi, t)dx d\xi <$$
$$\iint_{N_R} f(x, \xi, t)(|x||\xi| - x, \xi)dx d\xi \leq ||f||_G \to$$
$$\iint_{N_R} f(x, \xi, t)dx d\xi \leq \frac{||f||_G}{Re(1 - \cos(c))}$$
Now starting with the conservation of mass over time:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t) dx d\xi dt = M \to \\
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \int_{\mathbb{R}^n} f(x, \xi, t) dx d\xi + \int_{\Gamma_{x_0}(c,v)} f(x, \xi, t) dx d\xi \right) dt = M \to \\
M - \frac{\|f\|_G}{Rv(1 - \cos(c))} \leq \lim \inf_{T \to \infty} \frac{1}{T} \int_0^T \left( \int_{\Gamma_{x_0}(c,v)} f(x, \xi, t) dx d\xi \right) dt \leq M
\]

Because \( R \) can be arbitrary large and that \( \|f\|_G \) bounded we have:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \int_{\Gamma_{x_0}(c,v)} f(x, \xi, t) dx d\xi \right) dt = M
\]

To finish this section we will demonstrate that combining corollary 2.3.1 with theorem 2.4 can improve the previous results. To do so we replace the punctured blind cones in the definition of \( \Gamma_{x_0}(c,v) \) with the balls of radius \( v \), inside any fixed bounded region \( X \) and keep \( \Gamma_{x_0}(c,v) \) intact outside of the bounded set \( X \).
3 Scattering phenomenon and existence theory

A possible and beneficial narrative that one can attribute to the results of the previous section, that are proven independent from a specific structure of interactions, is that they are evidence for an inevitable weak dispersion of particles. By weak we mean averaged over time or within a bounded set of the spatial variable. We will use the term scattering as a specific type of dispersion, that is a strong dispersion in a norm. This subject will naturally bring a discussion of existence theory with itself. Ideas discussed here will be used in section 4 for the purpose of creating a specific category of solutions to the Boltzmann equation.

One could argue to start with the bare minimum, by searching for a norm independent of time, which if for any specific slice of time it is bounded, then the energy, mass, momentum and interactions are well defined at that time.

Definition 3.1. We say $\| \|_O$ is an observer’s norm if it is a norm defined on position and velocity variables, omitting time. We expect that if $\| f(x, \xi, t) \|_O < \infty$ for a slice of time like $t$ then $I(f, x, \xi, t)$ be well defined at that time. Furthermore energy, momentum and mass are convergent integrals:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t) dx d\xi, \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t) \xi dx d\xi, \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, \xi, t) |\xi|^2 dx d\xi < \infty$$

Definition 3.2. Let $\| \|_S$ be the scattering norm associated to the observers norm $\| \|_O$ defined as:

$$\| f \|_S = \sup_t \| f(x + t\xi, \xi, t) \|_O$$

Definition 3.3. For $C > 0$ let $O(C)$ be observer’s space and $S(C)$ be scattering space defined as:

$$f(x, \xi, t) \in O(C) \iff \| f(x, \xi, t) \|_O \leq C$$
$$f(x, \xi, t) \in S(C) \iff \| f(x, \xi, t) \|_S \leq C$$

Definition 3.4. $f$ is a scattering solution of equation (1.1), if there exists an observer’s norm such that:

$$\| f(x, \xi, 0) \|_O < \infty$$
$$\int_0^\infty \| I(f, x + t\xi, \xi, t) \|_O dt < \infty$$
Figure 4: $f_z$ and $f_\infty$ are solutions to the linear transport equation. $f$ scatters to $f_\infty$ as time goes to infinity.

Remark. if $f$ is a scattering solution for initial value $f_0$, integrating over characteristics of equation (1.1) gives us:

$$f(x, \xi, t) = f_0(x - t\xi, \xi) + \int_0^t I(f, x - t\xi + s\xi, \xi, s)ds$$

This implies that:

$$\|f(x, \xi, t)\|_{S} = \sup_t \|f(x + t\xi, \xi, t)\|_{O} \leq \|f(x, \xi, 0)\|_{O}$$

$$+ \int_0^\infty |I(x + t\xi, \xi, t)|dt \|_{O} \leq \|f(x, \xi, 0)\|_{O} + \int_0^\infty \|I(x + t\xi, \xi, t)\|_{O}dt$$

So we have:

$$f \in S(\|f(x, \xi, 0)\|_{O} + \int_0^\infty \|I(x + t\xi, \xi, t)\|_{O}dt)$$

**Theorem 3.1.** If $f$ is a scattering solution of equation 1.1 such that:

$$N = \|f(x, \xi, 0)\|_{O} + \int_0^\infty \|I(x + t\xi, \xi, t)\|_{O}dt$$

Then there exists a solution of linear transport equation like $f_\infty \in S(N)$ that $f$ scatters to as time goes to infinity in the sense defined bellow:

$$\lim_{t \to \infty} \|f(x, \xi, t) - f_\infty(x, \xi, t)\|_{O} = 0$$
Recalling definition of scattering norm again and using the fact that $f^\infty$ converges to another solution of linear transport equation like $s$ are solutions of linear transport equation: $\|f^\infty - f_s\|_S = \|f(x - (t - z_1)) - f(x - (t - z_2))\|_S = 0$.

Proof. Consider figure 4. For $0 \leq z$, let $f_z$ be the solutions to linear transport equation defined below.

$$f_z(x, \xi, t) = f(x - (t - z_1)\xi, \xi, z_1)$$

We have $f, f_z \in S(N)$. Using the scattering norm we will show that sequence $f_z$ is a Cauchy sequence:

$$\|f_{s_1} - f_{s_2}\|_S = \|f(x - (t - z_1)) - f(x - (t - z_2))\|_S = \|\int_{z_1}^{z_2} f(x - (t - z)\xi, \xi, z)dz\|_S = \sup_t \|\int_{z_1}^{z_2} f(x + z\xi, \xi, z)dz\|_O \leq \|\int_{z_1}^{z_2} |f(x + z\xi, \xi, z)|dz\|_O$$

Now since $f$ is a scattering solution:

$$\forall \epsilon > 0 \exists T > 0 \ s.t \ T < z_1 \leq z_2 \rightarrow \|\int_{z_1}^{z_2} |f(x + z\xi, \xi, z)|dz\|_O < \epsilon$$

This implies that sequence of functions $f_z$ is a Cauchy sequence and they converge to another solution of linear transport equation like $f_\infty \in S(N)$. Also consider that for for any fixed $0 < z_0$ we have:

$$\|f(x, \xi, t + z_0) - f_{z_0}(x, \xi, t + z_0)\|_S = \sup_t \|f(x + t\xi, \xi, t + z_0) - f_{z_0}(x + t\xi, \xi, t + z_0)\|_O \leq \|\int_{t_0}^{t_1} f(x + z\xi, \xi, z)dz - f_z(x, \xi, z_0)\|_O \leq \|\int_{z_0}^{\infty} |f(x + z\xi, \xi, z)|dz\|_O \leq \|\int_{z_0}^{\infty} |f(x + z\xi, \xi, z)|dz\|_O$$

While definition of scattering norm implies that:

$$\|f(x, \xi, z_0) - f_{z_0}(x, \xi, z_0)\|_O \leq \|f(x, \xi, t + z_0) - f_{z_0}(x, \xi, t + z_0)\|_S$$

Combining the two above inequalities we get:

$$\|f(x, \xi, z_0) - f_{z_0}(x, \xi, z_0)\|_O \leq \int_{z_0}^{\infty} \|f(x + z\xi, \xi, z)|dz\|_O dt \rightarrow \lim_{z_0 \rightarrow \infty} \|f(x, \xi, z_0) - f_{z_0}(x, \xi, z_0)\|_O = 0 \quad (3.1)$$

Recalling definition of scattering norm again and using the fact that $f_\infty$ and $f_z$s are solutions of linear transport equation:

$$\|f_\infty(x, \xi, z_0 + t) - f_{z_0}(x, \xi, z_0 + t)\|_S = \|f_\infty(x, \xi, z_0) - f_{z_0}(x, \xi, z_0)\|_O \rightarrow \lim_{z_0 \rightarrow \infty} \|f_\infty(x, \xi, z_0) - f_{z_0}(x, \xi, z_0)\|_O = 0 \quad (3.2)$$
Finally using the triangle inequality and combining (3.1) and (3.2) proof is complete:

$$\lim_{z_0 \to \infty} \| f(x, \xi, z_0) - f_{\infty}(x, \xi, z_0) \|_0 \leq \lim_{z_0 \to \infty} \| f(x, \xi, z_0) - f_{z_0}(x, \xi, z_0) \|_0 + \lim_{z_0 \to \infty} \| f_{z_0}(x, \xi, z_0) - f_{\infty}(x, \xi, z_0) \|_0 = 0$$

Previous theorem suggests that if $\lambda(x, \xi, t)$ is a solution to the linear transport equation, such that for every $x$ and $\xi$ we have: $f_{\infty}(x, \xi, 0) \leq \lambda(x, \xi, 0)$ then the following norms of solution $f$ are well defined:

$$\sup_{x, \xi} \frac{1}{\lambda(x + t\xi, \xi, t)} f(x, \xi, t) < \infty$$

$$\sup_{x, \xi, t} \frac{1}{\lambda(x + t\xi, \xi, t)} f(x + t\xi, \xi, t) < \infty$$

Norms defined above can be thought of respectively as the observer’s and scattering norms. Consider a the solution map for equation (1.1) as written below, one can think of solutions as fixed points of this map.

$$\Phi(f)(x, \xi, t) = f_0(x - t\xi, \xi) + \int_0^t I(f, x - t\xi + s\xi, \xi, s)ds$$

Using a bootstrap like argument, it is possible to have a scattering space which iterations of the solution map remain within the same space, therefore variety of fixed point methods become applicable for the existence theory. Question of whether this map have a fixed point or not needs more information about the specific structure of interactions. We will implement this next chapter for the Boltzmann equation.
4 The Boltzmann equation

In this section, we briefly introduce the Boltzmann equation in the case of hard spheres. A comprehensive treatment of the mathematical theory behind the classical solutions to the Boltzmann equation can be found in the works of Bressan[3], Cercignani[5], and Ukai & Yang [1]. Furthermore, a concept of weak solutions exists as introduced by DiPerna and Lions[6] that shows a very general existence theory. It is ambiguous whether these weak solutions preserve mass and there exists no statement about their uniqueness. These missing links make it infeasible to confidently relate the weak solutions to the underlying physical phenomena which the Boltzmann equation is intended to describe.

Here, we pursue classical solutions to demonstrate the applicability of the previous section’s results. After discussing the preliminaries we will show the existence of scattering solutions to the Boltzmann equation in the case of hard spheres, for a class of initial values with a special property. Furthermore, we will show for any solution to the linear transport equation in an appropriate space, there exists a unique solution to the Boltzmann equation that scatters to the aforementioned solution of the linear transport equation.

4.1 Preliminaries

The Boltzmann equation has a similar form to equation (1.1). The notation $Q(f, f)(x, \xi, t)$ represents the interaction of particles and is called the Boltzmann collision operator.

$$\partial_t f(x, \xi, t) + \xi \cdot \nabla_x f(x, \xi, t) = Q(f, f)(x, \xi, t)$$

$$f(0, x, \xi) = f_0(x, \xi)$$

In the case which this operator represents collisions of hard elastic spheres, $Q$ obtains the form below.

$$Q(f, f)(x, \xi, t) = \int_{\mathbb{R}^n} \int_{S^{n-1}} (f' f' \ast - ff \ast) |n.(\xi - \xi \ast)|d\xi \ast d\mathbf{n}$$

In the expression above, $\mathbf{n}$ is the normal unit vector to the $n-1$ dimensional unit sphere $S^{n-1}$ and as is customary in the field we use the standard notation:

$$\xi' = \xi - \mathbf{n}(\xi - \xi \ast)\mathbf{n}$$

$$\xi \ast = \xi + \mathbf{n}(\xi - \xi \ast)\mathbf{n}$$

$$f(x, \xi, t) = f, f(x, \xi \ast, t) = f \ast$$

$$f(x, \xi', t) = f', f(x, \xi' \ast, t) = f' \ast$$
Assume that $\xi$ and $\xi^*$ are velocities of two colliding elastic balls with unit mass and let $n$ or equivalently $-n$, represent the unit normal vector to the plane which uniquely describes the relative position of these two spheres upon collision. Then $\xi'$ and $\xi'^*$ are the velocities of spheres after the collision. For a fixed $n$, these velocities are unique solutions to the conservation laws of momentum and energy written below.

$$|\xi|^2 + |\xi^*|^2 = |\xi'|^2 + |\xi'^*|^2$$

$$\xi + \xi^* = \xi' + \xi'^*.$$ 

As shown by Boltzmann, the collision operator satisfies the conditions of definition 1.1 and is a mesoscopic interaction. We will replicate his argument here. Consider the following well-known change of variables:

$$(\xi \rightarrow \xi', \xi^* \rightarrow \xi'^*) \quad (\xi \rightarrow \xi^*, \xi^* \rightarrow \xi') \quad (\xi \rightarrow \xi^*, \xi^* \rightarrow \xi)$$

The transformations above are measure preserving, by implementing them for the collision operator and an arbitrary $\phi(\xi)$ we get:

$$\int_{\mathbb{R}^n} Q(f, f)(x, \xi, t) \phi(\xi) d\xi =$$

$$\frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} (f' f_s' - f f_s)(\phi(\xi) + \phi(\xi^*) - \phi(\xi') - \phi(\xi'^*)) n.(\xi - \xi^*) |d n| d\xi d\xi$$

By setting $\phi(\xi)$ equal to either 1, $|\xi|^2$ or $\xi_i$ for $1 \leq i \leq n$ and using the relation above one can conclude that $Q$ has the properties of definition 1.1. The notation $Q(f, f)$ is to emphasize that the collision operator is quadratic. Obtaining a similar notation for $g$ as was used for $f$:

$$Q(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \frac{1}{2}(f' g_s' + g' f_s' - f g_s - f g_s^*) n.(\xi - \xi^*) |d\xi| d\xi$$

Note that the conservation laws imply $|\xi - \xi^*| = |\xi' - \xi'^*|$. Finally it is common to break this operator into gain and loss terms, which are respectively defined:

$$Q^+(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \frac{1}{2}(f' g_s' + g' f_s' - f g_s - f g_s^*) n.(\xi - \xi^*) |d\xi| d\xi$$

$$Q^-(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} \frac{1}{2}(g f_s + f g_s^*) n.(\xi - \xi^*) |d\xi| d\xi$$

Where we have:

$$Q(f, g) = Q^+(f, g) - Q^-(f, g)$$ 

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4.2 Scattering solutions

Starting with an initial value \( f_0(x, \xi) \) for the Boltzmann equation, the purpose is to create scattering solutions associated to this initial value. It is expected that the initial value has finite mass, energy and momentum as defined in (1.2). To accomplish this goal, we propose an extra condition for the initial value that makes it suitable for scattering. We call this condition the scattering property of the initial value.

**Definition 4.1.** We say initial value \( f_0(x, \xi) \) has the scattering property if for \( \lambda(x, \xi, t) = f_0(x - t\xi, \xi) \) we have:

\[
\alpha = \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} Q^+(\lambda, \lambda)(x + t\xi, \xi, t) dt + \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} Q^-(\lambda, \lambda)(x + t\xi, \xi, t) dt < \infty
\]

**Remark.** In order to interpret the property defined above, consider the evolution of particles associated to the initial value \( f_0(x, \xi) \) that are not interacting with one another, therefore particles move along their own trajectories without change. This evolution is described by \( \lambda(x, \xi, t) \). In this case, any two particles can meet at most once, only under the conditions that their trajectories intersect and timing is ideal. Although these particles are moving with inertia and effects of the collisions are not being implemented in their motion, still one can measure the total amount of possible interaction and that is represented in \( \alpha \). One could argue that in the non linear case, although particles are subject to dispersion, it is still possible for any two particles to meet more than once. This leads to the expectation that, starting from an identical initial value, the non linear case is subject to more collisions compared to the nonexistent collisions of the linear equation. Recall definition 3.4 in which one expects the observer’s norm of interactions to be absolutely integrable along the characteristics of equation (1.1), then prior to that one should expect a similar condition for the linear case \( \lambda \) associated to the initial value. Therefore the linear equation can be used as a frame of reference for the non linear one.

**Theorem 4.1.** If the initial value \( f_0(x, \xi) \) has the scattering property, then there is a \( 0 < k_0 \leq 1 \) such that for any \( k \leq k_0 \) we have a unique scattering solution to the Boltzmann equation in the sense defined below, where the term scattering here refers to definition 3.4.

\[
f(x, \xi, 0) = kf_0(x, \xi) \\
\partial_t f(x, \xi, t) + \xi \cdot \nabla_x f(x, \xi, t) = Q(f, f)(x, \xi, t)
\]

Conversely for any solution to the linear transport equation like \( f_\infty \) which at time zero \( f_\infty(x, \xi, 0) \leq k_0 f_0(x, \xi) \) there exists a unique solution to Boltzmann equation that scatters to \( f_\infty \) in the sense described at the theorem 3.1.
Proof. Let the observer’s norm be defined as below:

$$
\|f(x, \xi, t)\|_O = \sup_{x, \xi} \frac{1}{f_0(x, \xi)} g(x, \xi, t)
$$

Set $$\lambda(x, \xi, t) = f_0(x - t \xi, \xi)$$ and define the scattering norm as:

$$
\|f(x, \xi, t)\|_S = \sup_{x, \xi, t} \frac{1}{f_0(x, \xi)} g(x + t \xi, \xi, t) = \sup_{x, \xi, t} \frac{1}{\lambda(x, \xi, t)} g(x, \xi, t)
$$

Estimating the gain term:

$$
\frac{1}{2} \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} Q^+(f, g)(x + t \xi, \xi, t) dt = 
$$

$$
\frac{1}{2} \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} \int_{S^{n-1}} \int_{\mathbb{R}^n} (f(x + t \xi, \xi', t) g(x + t \xi, \xi', t) + g(x + t \xi, \xi', t) f(x + t \xi, \xi', t)) \frac{d\xi_s}{d\xi} d\xi d\tau d\tau d\tau \leq 
$$

$$
\|f\|_S \|g\|_S \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} \int_{S^{n-1}} \int_{\mathbb{R}^n} \lambda(x + t \xi, \xi', t) \lambda(x + s \xi, \xi, t) \frac{d\xi}{d\xi_s} d\tau d\tau d\tau
$$

Estimating the loss term:

$$
\frac{1}{2} \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} Q^-(f, g)(x + t \xi, \xi, t) dt \leq 
$$

$$
\|f\|_S \|g\|_S \int_0^\infty \sup_{x, \xi} \frac{1}{f_0(x, \xi)} \int_{S^{n-1}} \int_{\mathbb{R}^n} \lambda(x + t \xi, \xi, t) \lambda(x + t \xi, \xi, t) \frac{d\xi}{d\xi_s} d\tau d\tau d\tau
$$

Combining the two estimates above we get:

$$
\int_0^\infty \|Q(f, g)(x + t \xi, \xi, t)\|_O dt \leq \alpha \|f\|_S \|g\|_S
$$

(4.1)

Above, $$\alpha$$ refers to definition 4.1. Now consider the solution map defined below for some $$k > 0$$:

$$
\Phi(f)(x, \xi, t) = k \times f_0(x - t \xi, \xi) + \int_0^t Q(f, f)(x - t \xi + s \xi, \xi, s) ds
$$
Following bound is true for scattering norm of $\Phi$:

$$
\|\Phi(f)(x, \xi, t)\|_S = \sup_t \|\Phi(f)(x + t\xi, \xi, t)\|_O = 
\sup_t \|k f_0(x, \xi) + \int_0^t Q(f, f)(x + \xi s, \xi, s)ds\|_O \leq 
\|f_0(x, \xi)\|_O + \int_0^\infty \|Q(f, f)(x + \xi s, \xi, s)\|_O ds \leq k \|f_0\|_O + \alpha \|f\|_S^2
$$

This implies:

$$
\|f\|_S < \infty \rightarrow \Phi(f) \in S(k \|f_0\|_O + \alpha \|f\|_S^2) \quad (4.2)
$$

Furthermore we have:

$$
\|\Phi(u_1) - \Phi(u_2)\|_S = \left\| \int_0^t Q(u_1, u_1)(x, \xi, z) - Q(u_2, u_2)(x, \xi, z)dz \right\|_S = 
\left\| \int_0^t Q(u_1 - u_2, u_1 + u_2)(x, \xi, z)dz \right\|_S \leq 
\int_0^\infty \|Q(u_1 - u_2, u_1 + u_2)(x + z\xi, \xi, z)\|_O dz \leq 
\alpha \|u_1 - u_2\|_S \|u_1 + u_2\|_S \leq 2\alpha N \|u_1 - u_2\|_S
$$

Where we are using estimate (4.1). This leads to:

$$
\|\Phi(u_1) - \Phi(u_2)\|_S \leq 2\alpha N \|u_1 - u_2\|_S \quad (4.3)
$$

Now choose $N$ such that $\alpha N^2 < N$ and $2\alpha N < 1$. There exists $0 < k_0 \leq 1$ such that $k_0 \|f_0\|_O = N - \alpha N^2$. Now since any for $k \leq k_0$ we have $k \|f_0\|_O + \alpha \|f\|_S^2 \leq N$, (4.2) implies:

$$
f \in S(N) \rightarrow \Phi(f) \in S(N)
$$

Using (4.3) we conclude $\Phi$ is a contraction on $S(N)$. The Banach fixed point theorem implies there exists a unique fixed point for the solution map $\Phi$ and this completes the proof of the first part of the theorem.

For the converse, consider an arbitrary solution to the linear transport equation $f_\infty \in S(k_0)$. Define a family of solution maps for $0 \leq z$ as:

$$
\Phi_z(g)(x, \xi, t) = f_\infty(x - t\xi + z\xi, \xi, z) + \int_z^t Q(g, g)(x - t\xi + s\xi, \xi, s)ds
$$

Since $f_\infty \in S(K_0)$, using the previous part we conclude that iterations of maps defined above converge to solutions of the Boltzmann equation like $u_z$, or equivalently:

$$
\lim_{n \to \infty} \Phi^n_z = u_z(x, \xi, t)
$$
Figure 5: $u_{z_1}$ and $u_{z_2}$ are two solutions of the Boltzmann equation that coincide with $f_\infty$ respectively at times $z_1$ and $z_2$. $u_\infty$ is a unique solution to the Boltzmann equation in space $S(k_0)$ that scatters to $f_\infty$.

Continue with the following approximation:

$$\|u_{z_1}(x - t\xi + z_2\xi, \xi, z_2) - f_\infty(x - t\xi + z_2\xi, \xi, z_2)\|_S =$$

$$\|u_{z_1}(x + z_2\xi, \xi, z_2) - f_\infty(x + z_2\xi, \xi, z_2)\|_O =$$

$$\|u_{z_1}(x + z_1\xi, \xi, z_1) + \int_{z_1}^{z_2} Q(u_{z_1}, u_{z_1})(x + z\xi, \xi, z)dz - f_\infty(x + z_1\xi, \xi, z_1)\|_O =$$

$$\|\int_{z_1}^{z_2} Q(u_{z_1}, u_{z_1})(x + z\xi, \xi, z)dz\|_O \leq \int_{z_1}^{z_2} \|Q(u_{z_1}, u_{z_1})(x + z\xi, \xi, z)\|_O dz$$

(4.4)

Define $\Psi_1$ and $\Psi_2$ as solution maps:

$$\Psi_1(g) = u_{z_1}(x - t\xi + z_2\xi, \xi, z_2) + \int_{z_2}^{t} Q(g, g)(x - t\xi + s\xi, \xi, s)ds$$

$$\Psi_2(g) = f_\infty(x - t\xi + z_2\xi, \xi, z_2) + \int_{z_2}^{t} Q(g, g)(x - t\xi + s\xi, \xi, s)ds$$

Due to the uniqueness of solutions to the Boltzmann equation in the scattering space $S(k_0)$, iterations of the maps $\Psi_1$ and $\Psi_2$ will converge respectively to $u_{z_1}$.
and \( u_{z_2} \), furthermore following bound is true for a single iteration:

\[
\| \Psi_1(g_1) - \Psi_2(g_2) \|_S = \\
\sup_t \| u_{z_1}(x + z_2 \xi, \xi, z_2) + \int_{z_2}^{t} Q(g, g)(x + z \xi, \xi, z) \\
- f_\infty(x + z_2 \xi, \xi, z_2) - \int_{z_2}^{t} Q(g, g)(x + z \xi, \xi, z)\rho \leq \\
\| u_{z_1}(x + z_2 \xi, \xi, z_2) - f_\infty(x + z_2 \xi, \xi, z_2) \|_O + \\
\int_{0}^{\infty} \| Q(g_1 - g_2, g_1 + g_2)(x + z \xi, \xi, z) \|_O dz \leq \\
\| u_{z_1}(x + z_2 \xi, \xi, z_2) - f_\infty(x + z_2 \xi, \xi, z_2) \|_O + 2\alpha k_0 g_1 - g_2 \|_O (4.5)
\]

Consider figure 5 and define \( \epsilon \) as below.

\[
\epsilon = \| u_{z_1}(x + z_2 \xi, \xi, z_2) - f_\infty(x + z_2 \xi, \xi, z_2) \|_O
\]

We Continue from (4.5):

\[
\| \Psi_j^i(g_1) - \Psi_j^i(g_2) \|_S \leq \epsilon \sum_{i=0}^{j} (2\alpha k_0)^i \\
\lim_{j\to\infty} \| \Psi_j(g_1) - \Psi_j(g_2) \|_S = \| u_{z_1} - u_{z_2} \|_S \leq \epsilon \sum_{i=0}^{\infty} (2\alpha k_0)^i
\]

Recalling that \( u_z \)s are scattering solutions, (4.4) implies that as \( z_1 \) and \( z_2 \) go to infinity, \( \epsilon \) goes to zero. On the other hand summation at the right hand side of the inequality above is a convergent geometric series with ratio \( 2\alpha k_0 < 1 \). In consequence we have that \( u_z \) is a Cauchy sequence in scattering norm, so there exists an \( u_\infty \) which is a solution of the Boltzmann equation such that:

\[
\lim_{z\to\infty} \| u_z(x, \xi, t) - u_\infty(x, \xi, t) \|_S = 0
\]

Which scatters to \( f_\infty \):

\[
\lim_{t\to\infty} \| u_\infty(x, \xi, t) - f_\infty(x, \xi, t) \|_O = 0
\]

Completing proof of the theorem.

We conclude by examining a set of prominent initial values, often called Maxwellians, and show that they have the scattering property. Computations below are the paraphrasing of what appear in notes of Calogero[10].

**Theorem 4.2.** Any initial value \( f_0(x, \xi) \) such that for some \( C > 0 \) we have \( f_0 \leq C e^{-|x|^2 - |\xi|^2} \) has the scattering property.
**Proof.** Observe that the quantity below is bounded.

\[ \Omega = \sup_{x, \xi} \int_0^\infty \int_{\mathbb{R}^n} e^{-|x+s\xi-s\xi_*|^2-|\xi_*|^2} |\xi - \xi_*| d\xi_* ds < \infty \]

Starting with change of variables \( s \to \frac{s}{|\xi - \xi_*|} \) and let \( n = |\xi - \xi_*| \in S^{n-1} \)

\[
\sup_{x, \xi} \int_0^\infty \int_{\mathbb{R}^n} e^{-|x+s\xi-s\xi_*|^2-|\xi_*|^2} |\xi - \xi_*| d\xi_* ds = \\
\sup_{x, \xi} \int_0^\infty \int_{\mathbb{R}^n} e^{-|x+n|^2-|\xi_*|^2} d\xi_* ds \leq \\
\sup_{x, \xi} \int_0^\infty \int_{\mathbb{R}^n} e^{-|x|^2+2|x-s| - |\xi_*|^2} d\xi_* ds = \\
\sup_{x, \xi} \int_{-|x|}^{|x|} \int_{\mathbb{R}^n} e^{-s^2-|\xi_*|^2} d\xi_* ds \leq \\
\sup_{x, \xi} \int_{-\infty}^\infty \int_{\mathbb{R}^n} e^{-s^2-|\xi_*|^2} d\xi_* ds < \infty
\]

Now we will show function \( f_0 \) has the scattering property. Let \( V_{n-1} \) be the volume of the \( n-1 \) dimensional unit sphere and start by estimating the loss term:

\[
V_{n-1} C^2 \int_0^\infty \int_{\mathbb{R}^n} e^{-|x+s\xi-s\xi_*|^2-|\xi_*|^2} |\xi - \xi_*| d\xi_* ds \leq \\
V_{n-1} C^2 \Omega
\]

Using the following identities gain term can be estimated similarly:

\[
|x+s\xi-s\xi_*|^2 + |x+s\xi-s\xi_*|^2 = |x|^2 + |x+s\xi-s\xi_*|^2 \\
|\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi_*|^2
\]

\[
C \int_0^\infty \int_{S^{n-1}} \int_{\mathbb{R}^n} e^{||x|^2+|\xi|^2 - |x+s\xi-s\xi_*|^2-|\xi_*|^2} \\
\times e^{-|x+s\xi-s\xi_*|^2-|\xi_*|^2} |\xi - \xi_*| d\xi_* d\sigma_{n-1} ds \leq \\
V_{n-1} C^2 \int_0^\infty \int_{\mathbb{R}^n} e^{-|x+s\xi-s\xi_*|^2-|\xi_*|^2} |\xi - \xi_*| d\xi_* \leq \\
V_{n-1} C^2 \Omega
\]
Previous computations lead to:

\[
\alpha = \int_0^\infty \|Q^+(f, f)(x + \xi s, \xi, s)\| ods + \int_0^\infty \|Q^-(f, f)(x + \xi s, \xi, s)\| ods \leq 2V_{n-1} \Omega C^2
\]

This shows that \(f_0\) has the scattering property.

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References

[1] Seiji Ukai and Tong Yang. *Mathematical Theory of Boltzmann Equation*, Department of Mathematics and Liu Bie Ju Center for Mathematical Sciences, City University of Hong Kong.

[2] Isabelle Gallagher, Laure Saint-Raymond and Benjamin Texier. *From Newton to Boltzmann: hard spheres and short-range potentials*. arXiv:1208.5753v2 [math.AP] 15 Jan 2013.

[3] Alberto Bressan. *Notes on the Boltzmann Equation*. [Lecture notes for a summer course given at S.I.S.S.A., Trieste, 2005]

[4] Terence Tao. *CBMS Regional Conference Series in Mathematics, Nonlinear Dispersive Equations: Local and Global Analysis*.

[5] Carlo Cercignani. *The Boltzmann equation and its applications*. ISBN: 978-1-4612-6995-3.

[6] R. J. DiPerna and P. L. Lions. *On the Cauchy Problem for Boltzmann Equations: Global Existence and Weak Stability*. Annals of Mathematics Second Series, Vol. 130, No. 2 (Sep., 1989), pp. 321-366 (46 pages).

[7] Oscar E. Lanford, III. *Time evolution of large classical systems*. Lect. Notes in Physics 38, J. Moser ed., 1–111, Springer Verlag.

[8] Fraydoun Rezakhanlou, *Boltzmann-Grad Limits for Stochastic Hard Sphere Models*. Communications in Mathematical Physics volume 248, pages 553–637 (2004)

[9] Paul R. Halmos. *How to Write Mathematics*.

[10] Simone Calogero. *Lectures notes on Boltzmann’s equation*. 