\section*{$\mathcal{N}$-fold Parasupersymmetry}

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\begin{abstract}
We find a new type of non-linear supersymmetries, called $\mathcal{N}$-fold parasupersymmetry, which is a generalization of both $\mathcal{N}$-fold supersymmetry and parasupersymmetry. We provide a general formulation of this new symmetry and then construct a second-order $\mathcal{N}$-fold parasupersymmetric quantum system where all the components of $\mathcal{N}$-fold parasupercharges are given by type A $\mathcal{N}$-fold supercharges. We show that this system exactly reduces to the Rubakov–Spiridonov model when $\mathcal{N} = 1$ and admits a generalized type C $2\mathcal{N}$-fold superalgebra. We conjecture the existence of other ‘$\mathcal{N}$-fold generalizations’ such as $\mathcal{N}$-fold fractional supersymmetry, $\mathcal{N}$-fold orthosupersymmetry, and so on.

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\end{abstract}
I. INTRODUCTION

It goes without saying that the concept of symmetry has played a central role in the development of modern theoretical physics and mathematical science. This means in particular that a new discovery of a new symmetry enlarges our ability and possibility to describe new phenomena in any field of science, regardless of whether or not they have already been observed in nature. In this letter, we report on a new type of non-linear supersymmetries, called \( \mathcal{N} \)-fold parasupersymmetry, which is a generalization of both \( \mathcal{N} \)-fold supersymmetry \([1, 2, 3]\) and parasupersymmetry \([4, 5, 6, 7, 8]\). In our previous paper \([9]\), we generically formulated parasupersymmetry and showed that parasupersymmetric quantum systems as well as \( \mathcal{N} \)-fold supersymmetric ones are weakly quasi-solvable. The latter fact implies that corresponding higher-dimensional quantum field theories, if exist, would satisfy some kinds of perturbative non-renormalization theorems, as in the case of ordinary supersymmetric quantum field theory, since quasi-solvability is a one-dimensional analog and, in a sense, a generalization of the theorems \([2]\). Hence, \( \mathcal{N} \)-fold parasupersymmetric quantum systems presented in this letter would provide up to now the most general framework for describing a physical system which has characteristic features like some non-renormalization theorems. For the formulation, we fully employ the general formalism of parafermionic algebra and parasupersymmetry previously proposed by us in Ref. \([9]\) and omit technical details of them in this letter. Hence, for the details see Ref. \([9]\) and the references cited therein.

II. DEFINITION

First of all, let us introduce parafermionic algebra of order \( p (\in \mathbb{N}) \). It is an associative algebra composed of the identity operator \( I \) and two parafermionic operators \( \psi^- \) and \( \psi^+ \) of order \( p \) which satisfy the nilpotency:

\[
(\psi^-)^p \neq 0, \quad (\psi^+)^p \neq 0, \quad (\psi^-)^{p+1} = (\psi^+)^{p+1} = 0.
\]  

(1)

Hence, we immediately have \( 2p + 1 \) non-zero elements \( \{ I, \psi^-, \cdots, (\psi^-)^p, \psi^+, \cdots, (\psi^+)^p \} \). We call them the fundamental elements of parafermionic algebra of order \( p \). Parafermionic algebra is characterized by anti-commutation relation \( \{ A, B \} = AB + BA \) and commutation relation \( [A, B] = AB - BA \) among the fundamental elements.

We shall next define parafermionic Fock spaces \( V_p \) of order \( p \) on which the parafermionic operators act. The latter space is \( (p + 1) \) dimensional and its \( p + 1 \) bases \( | k \rangle \) \((k = 0, \ldots, p)\) are defined by

\[
\psi^-|0\rangle = 0, \quad |k\rangle = (\psi^+)^k|0\rangle, \quad \psi^-|k\rangle = |k-1\rangle \quad (k = 1, \ldots, p).
\]  

(2)

That is, \( \psi^- \) and \( \psi^+ \) act as annihilation and creation operators of parafermions, respectively. The state \( |0\rangle \) is called the parafermionic vacuum. The subspace spanned by each state \( |k\rangle \) \((k = 0, \ldots, p)\) is called the \( k \)-parafermionic subspace and is denoted by \( V_p^{(k)} \). We can now define a set of projection operators \( \Pi_k : V_p \rightarrow V_p^{(k)} \) \((k = 0, \ldots, p)\) which satisfy

\[
\Pi_k|l\rangle = \delta_{k,l}|k\rangle, \quad \Pi_k\Pi_l = \delta_{k,l}\Pi_k, \quad \sum_{k=0}^{p} \Pi_k = I.
\]  

(3)
From the definitions (2) and (3), we obtain

\[ \Pi_{k+1} \psi^+ = \psi^+ \Pi_k, \quad \psi^- \Pi_{k+1} = \Pi_k \psi^-, \] (4)

where and hereafter we put \( \Pi_k \equiv 0 \) for all \( k < 0 \) and \( k > p \).

Parasupersymmetry of order 2 in quantum mechanics was first introduced by Rubakov and Spiridonov [4] and was later generalized to arbitrary order independently by Tomiya [6] and by Khare [7]. A different formulation for order 2 was proposed by Beckers and Debergh [5] and a generalization of the latter to arbitrary order was attempted by Chenaghlou and Fakhri [8]. Thus, we call them RSTK and BDCF formalism, respectively. We shall generalize them such that they reduce to \( N \)-fold supersymmetry in Ref. [2] when parafermionic order is 1. For this purpose, we first introduce a pair of \( N \)-fold parasupercharges \( Q^+ \) of order \( p \) which satisfy

\[ (Q^+)^p \neq 0, \quad (Q^-)^p \neq 0, \quad (Q^+)^{p+1} = (Q^-)^{p+1} = 0. \] (5)

A system \( \mathbf{H} \) is said to have \( \mathcal{N} \)-fold parasupersymmetry of order \( p \) if it commutes with the \( \mathcal{N} \)-fold parasupercharges of order \( p \)

\[ [Q^-, \mathbf{H}] = [Q^+, \mathbf{H}] = 0, \] (6)

and satisfies the non-linear relations in a generalized RSTK formalism

\[ \sum_{k=0}^{p} (Q^-)^{p-k} Q^+ (Q^-)^k = C_p (Q^-)^{p-1} P_N(\mathbf{H}), \] (7a)

\[ \sum_{k=0}^{p} (Q^+)^{p-k} Q^- (Q^+)^k = C_p P_N(\mathbf{H}) (Q^+)^{p-1}, \] (7b)

where \( P_N(x) \) is a monic polynomial of degree \( \mathcal{N} \) in \( x \) and \( C_p \) is a constant, or in a generalized BDCF formalism

\[ \underbrace{[Q^-_N, \cdots, [Q^-_N, \cdots, [Q^-_N, [Q^-_N, \cdots]}_{(p-1) \text{ times}} = (-1)^p C_p (Q^-)^{p-1} P_N(\mathbf{H}), \] (8a)

\[ \underbrace{[Q^+_N, \cdots, [Q^+_N, \cdots, [Q^+_N, [Q^+_N, \cdots]}_{(p-1) \text{ times}} = C_p P_N(\mathbf{H}) (Q^+)^{p-1}, \] (8b)

It is evident that in both generalizations (7) and (8), which are completely new algebraic relations and have never appeared in the past literature, they reduce to ordinary parasupersymmetry of the corresponding formulations when \( \mathcal{N} = 1 \). As we pointed out in Ref. [9], an apparent drawback of the (generalized) BDCF formalism is that the relations (8) do not reduce to the ordinary \( \mathcal{N} \)-fold supersymmetric anti-commutation relation \( \{Q^-, Q^+\} = C_1 P_N(\mathbf{H}) \) when \( p = 1 \), in contrast to the RSTK relation (7). For this reason, we only consider the (generalized) RSTK formalism in this paper.

An immediate consequence of the commutativity (6) is that each \( n \)-th power of the \( \mathcal{N} \)-fold parasupercharges \( (2 \leq n \leq p) \) also commutes with the system \( \mathbf{H} \)

\[ ([Q^-_N]^n, \mathbf{H}) = ([Q^+_N]^n, \mathbf{H}) = 0 \quad (2 \leq n \leq p). \] (9)
Hence, every $\mathcal{N}$-fold parasupersymmetric system $\mathbf{H}$ satisfying (10) always has $2p$ conserved charges.

To realize $\mathcal{N}$-fold parasupersymmetry in quantum mechanical systems, we usually consider a vector space $\mathfrak{F} \times \mathbb{V}_p$ where $\mathfrak{F}$ is a linear space of complex functions such as the Hilbert space $L^2$ in Hermitian quantum theory and the Krein space $L^2_P$ in $\mathcal{PT}$-symmetric quantum theory [10, 11]. A parafermionic quantum system $\mathbf{H}$ is introduced by

$$H = \sum_{k=0}^{p} H_k \Pi_k,$$

where $H_k$ ($k = 0, \ldots, p$) are scalar Hamiltonians acting on $\mathfrak{F}$:

$$H_k = -\frac{1}{2} \frac{d^2}{d q^2} + V_k(q) \quad (k = 0, \ldots, p).$$  

(11)

Two $\mathcal{N}$-fold parasupercharges $\mathbf{Q}^\pm_{\mathcal{N}}$ are defined by

$$Q^+_\mathcal{N} = \sum_{k=0}^{p} Q^+_{\mathcal{N},k} \psi^- \Pi_k, \quad Q^-_\mathcal{N} = \sum_{k=0}^{p} Q^-_{\mathcal{N},k} \Pi_k \psi^+,$$  

(12)

where $Q^+_{\mathcal{N},k}$ ($k = 0, \ldots, p$) are $\mathcal{N}$-th-order linear differential operators acting on $\mathfrak{F}$

$$Q^+_{\mathcal{N},k} = \sum_{l=0}^{\mathcal{N}} w_{k,l}(q) \frac{d^l}{d q^l} \quad (k = 0, \ldots, p),$$  

(13)

and for each $k$ $Q^-_{\mathcal{N},k}$ is given by a certain ‘adjoint’ of $Q^+_{\mathcal{N},k}$, e.g., the (ordinary) adjoint $Q^-_{\mathcal{N},k} = (Q^+_{\mathcal{N},k})^\dagger$ in the Hilbert space $L^2$, the $\mathcal{P}$-adjoint $Q^-_{\mathcal{N},k} = \mathcal{P}(Q^+_{\mathcal{N},k})^\dagger \mathcal{P}$ in the Krein space $L^2_P$, and so on. For all $k \leq 0$ we put $Q^+_{\mathcal{N},k} \equiv 0$. As we will show shortly, the novel realization of parafermionic supercharges (12) in terms of $\mathcal{N}$-th-order linear differential operators (13) indeed enables us to realize an $\mathcal{N}$-fold parasupersymmetry in quantum mechanical systems, we usually consider $\mathcal{N}$-fold parasupersymmetric quantum systems defined by Eqs. (5)–(13) provide a natural generalization of ordinary $\mathcal{N}$-fold supersymmetric quantum mechanics. It is easy to check that the $\mathcal{N}$-fold parasupercharges $\mathbf{Q}^\pm$ defined by Eq. (12) already satisfy the nilpotency (5) and that the commutativity (6) is satisfied if and only if

$$H_{k-1} Q^-_{\mathcal{N},k} = Q^-_{\mathcal{N},k} H_{k-1} = H_k Q^+_{\mathcal{N},k}, \quad \forall k = 1, \ldots, p.$$  

(16)

That is, each pair of $H_{k-1}$ and $H_k$ must satisfy the intertwining relations with respect to the $\mathcal{N}$th-order linear differential operators $Q^-_{\mathcal{N},k}$ and $Q^+_{\mathcal{N},k}$. Similarly, the commutativity (6)
between \((Q^\pm_N)^n\) and \(H\) \((2 \leq n \leq p)\) means that any pair of \(H_{k-n}\) and \(H_k\) \((1 \leq n \leq k \leq p)\) satisfies

\[
H_{k-n}Q^{-}_{N,k-n+1} \cdots Q^{-}_{N,k-1}Q^{-}_{N,k} = Q^{-}_{N,k-n+1} \cdots Q^{-}_{N,k-1}Q^{-}_{N,k}H_k, \tag{17a}
\]

\[
Q^+_{N,k}Q^+_{N,k-1} \cdots Q^+_{N,k-n+1}H_{k-n} = H_k Q^+_{N,k}Q^+_{N,k-1} \cdots Q^+_{N,k-n+1}, \tag{17b}
\]

which means that \(H_{k-n}\) and \(H_k\) constitute a pair of \(n\mathcal{N}\)-fold supersymmetry. The relations (17) can also be derived by repeated applications of Eq. (16). Since \(\mathcal{N}\)-fold supersymmetry is essentially equivalent to weak quasi-solvability \([2, 12]\), \(\mathcal{N}\)-fold parasupersymmetric quantum systems also possess weak quasi-solvability. To see the structure of weak quasi-solvability in the \(\mathcal{N}\)-fold parasupersymmetric system \(H\) more precisely, let us first define

\[
\mathcal{V}^-_{n,k} = \ker(Q^-_{N,k-n+1} \cdots Q^-_{N,k}), \quad \mathcal{V}^+_{n,k} = \ker(Q^+_{N,k} \cdots Q^+_{N,k-n+1}) \quad (1 \leq n \leq k \leq p). \tag{18}
\]

By the definition of (18), the vector spaces \(\mathcal{V}^\pm_{n,k}\) for each fixed \(k\) are related as

\[
\mathcal{V}^-_{1,k} \subset \mathcal{V}^-_{2,k} \subset \cdots \subset \mathcal{V}^-_{k,k}, \quad \mathcal{V}^+_{1,k} \subset \mathcal{V}^+_{2,k} \subset \cdots \subset \mathcal{V}^+_{k,k}. \tag{19}
\]

On the other hand, it is evident from the intertwining relations (17) that each Hamiltonian \(H_k\) \((0 \leq k \leq p)\) preserves vector spaces as follows:

\[
H_k \mathcal{V}^-_{n,k} \subset \mathcal{V}^-_{n,k} \quad (1 \leq n \leq k), \tag{20a}
\]

\[
H_k \mathcal{V}^+_{n,k+n} \subset \mathcal{V}^+_{n,k+n} \quad (1 \leq n \leq p - k). \tag{20b}
\]

From Eqs. (19) and (20), the largest space preserved by each \(H_k\) \((0 \leq k \leq p)\) is given by

\[
\mathcal{V}^-_{k,k} + \mathcal{V}^+_{p-k,p} \quad (0 \leq k \leq p). \tag{21}
\]

Needless to say, each Hamiltonian \(H_k\) preserves the two spaces in Eq. (21) separately. The intertwining relations (16) and (17) ensure that all the component Hamiltonians \(H_k\) \((k = 0, \ldots, p)\) of the system \(H\) are isospectral outside the sectors \(\mathcal{V}^\pm_{n,k}\) \((1 \leq n \leq k \leq p)\). The spectral degeneracy of \(H\) in these sectors depends on the form of each component of the \(\mathcal{N}\)-fold parasupersymmetric quantum system \(Q^\pm_{N,k}\) \((k = 1, \ldots, p)\).

In addition to these ‘power-type’ symmetries, every \(\mathcal{N}\)-fold parasupersymmetric quantum system \(H\) defined in Eq. (10) can have ‘discrete-type’ ones. The conserved charges of this type are given by

\[
Q^\pm_{N,\{n\}} = \{[(\psi^-)^n, (\psi^+)^n], Q^\pm_N\}, \quad Q^\pm_{N,\{n\}} = \{[(\psi^-)^n, (\psi^+)^n], Q^\pm_N\} \quad (n = 1, \ldots, p). \tag{22}
\]

It follows from Jacobi identity that they indeed commute with \(H\):

\[
[Q^\pm_{N,\{n\}}, H] = [Q^\pm_{N,\{n\}}, H] = 0 \quad (n = 1, \ldots, p). \tag{23}
\]

We note, however, that they are in general not linearly independent and we cannot determine the number of linearly independent conserved charges without the knowledge of parafermionic algebra of each order.
The non-linear constraints (7) can also be calculated in a similar way. The first non-linear relation in Eq. (7) is satisfied if and only if the following two identities hold:

\[ Q_{N,1} \cdots Q_{N,p} Q_{N,p}^+ + \sum_{k=1}^{p-1} Q_{N,1}^{-} \cdots Q_{N,p-k}^{-} Q_{N,p-k}^+ Q_{N,p-k}^{-} \cdots Q_{N,p-1}^{-} = C_p Q_{N,1}^{-} \cdots Q_{N,p-1}^{-} P_N(H_{p-1}) \]  
\[ \sum_{k=1}^{p-1} Q_{N,2}^{-} \cdots Q_{N,p-k+1}^{-} Q_{N,p-k+1}^+ Q_{N,p-k+1}^{-} \cdots Q_{N,p}^{-} Q_{N,1}^+ Q_{N,1}^{-} \cdots Q_{N,p}^{-} = C_p Q_{N,2}^{-} \cdots Q_{N,p}^{-} P_N(H_p) \]  

(24a)  
(24b)

The conditions for the second non-linear relation in Eq. (7) are apparently given by the ‘adjoint’ of Eqs. (24).

An \( \mathcal{N} \)-fold generalization of quasi-parasupersymmetry proposed in Ref. [9] is also straightforward.

III. AN EXAMPLE

Let us now construct a second-order \( \mathcal{N} \)-fold parasupersymmetric quantum system. In the case of \( p = 2 \), the triple \((H, Q_N, Q_N^+)\) defined in Eqs. (10) and (12) is given by

\[ H = H_0(\psi^-)^2(\psi^+)^2 + H_1(\psi^+\psi^+ - (\psi^+)^2(\psi^-)^2) + H_2(\psi^+)(\psi^-)^2, \]  
\[ Q_N^- = Q_{N,1}^- \psi^- + Q_{N,2}^- \psi^+ (\psi^-)^2, \]  
\[ Q_N^+ = Q_{N,1}^+ \psi^- (\psi^+)^2 + Q_{N,2}^+ \psi^+ (\psi^+)^2. \]  

(25)  
(26)  
(27)

We recall the fact that the above second-order parasupercharges (26) and (27) already satisfy the nilpotent condition (5) for \( p = 2 \), \( (Q_N^-)^3 = (Q_N^+)^3 = 0 \). From Eqs. (10) and (24), the commutativity (3) and the non-linear constraints (7) for \( p = 2 \) hold if and only if the following conditions

\[ H_0 Q_{N,1}^- = Q_{N,1}^- H_1, \quad H_1 Q_{N,2}^- = Q_{N,2}^- H_2, \]  
\[ Q_{N,1}^+ Q_{N,2}^- Q_{N,2}^+ + Q_{N,1}^+ Q_{N,1}^- Q_{N,1}^- = C_2 Q_{N,1}^- P_N(H_1), \]  
\[ Q_{N,2}^+ Q_{N,2}^- Q_{N,2}^+ + Q_{N,1}^+ Q_{N,1}^- Q_{N,1}^- = C_2 Q_{N,2}^- P_N(H_2), \]  

(28)  
(29)  
(30)

and their ‘adjoint’ relations

\[ Q_{N,1}^+ H_0 = H_1 Q_{N,1}^+, \quad Q_{N,2}^+ H_1 = H_2 Q_{N,2}^+, \]  
\[ Q_{N,1}^+ Q_{N,1}^- Q_{N,1}^- + Q_{N,2}^+ Q_{N,2}^- Q_{N,2}^- = C_2 P_N(H_1) Q_{N,1}^+, \]  
\[ Q_{N,2}^+ Q_{N,1}^- Q_{N,1}^- + Q_{N,2}^+ Q_{N,2}^- Q_{N,2}^- = C_2 P_N(H_2) Q_{N,2}^+. \]  

(31)  
(32)  
(33)

are satisfied. In general, we do not need to solve the ‘adjoint’ conditions.

For the second-order case, we have one new \( \mathcal{N} \)-fold quasi-parasupersymmetry, namely, that of order (2, 2). The conditions are given by Eqs. (28)–(33) but the first-order intertwining relations (28) and (31) are replaced by the second-order intertwining relations

\[ H_0 Q_{N,1}^- Q_{N,2}^- = Q_{N,1}^- Q_{N,2}^- H_2, \quad Q_{N,2}^+ Q_{N,1}^- H_0 = H_2 Q_{N,2}^+ Q_{N,1}^- . \]  

(34)
Let us next put $C_2 = 2^{N+1}$ and

$$H_k = -\frac{1}{2} \frac{d^2}{dq^2} + V_k(q), \quad Q_{X,k}^+ = \prod_{i=0}^{N-1} \left( \frac{d}{dq} + W_k(q) + \frac{N - 1 - 2i}{2} E_k(q) \right), \quad (35)$$

where the product of operators is ordered according to $\prod_{i=0}^{N-1} A_i = A_i A_{i-1} \cdots A_0$. Each component $Q_{X,k}^+$ of $N$-fold parasupercharges given in Eq. (35) is so-called type A $N$-fold supercharge, and the necessary and sufficient condition for two Hamiltonians to be intertwined by it is already well known [12]: the conditions (28) and (31) are satisfied if and only if

$$H_0 = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} W_1(q)^2 + \frac{N^2 - 1}{24} (E_1(q)^2 - 2E_1'(q)) - \frac{N}{2} W_1'(q) - R_1, \quad (36)$$

$$H_1 = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} W_2(q)^2 + \frac{N^2 - 1}{24} (E_1(q)^2 - 2E_1'(q)) + \frac{N}{2} W_2'(q) - R_1$$

$$H_2 = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} W_2(q)^2 + \frac{N^2 - 1}{24} (E_2(q)^2 - 2E_2'(q)) + \frac{N}{2} W_2'(q) - R_2, \quad (37)$$

where $R_k$ ($k = 1, 2$) are constants and the functions $E_k$ and $W_k$ ($k = 1, 2$) must satisfy the following non-linear differential equations:

$$\left( \frac{d}{dq} - E_k(q) \right) \frac{d}{dq} \left( \frac{d}{dq} + E_k(q) \right) W_k(q) = 0 \text{ for } N \geq 2, \quad (39)$$

$$\left( \frac{d}{dq} - 2E_k(q) \right) \frac{d}{dq} \left( \frac{d}{dq} + E_k(q) \right) E_k(q) = 0 \text{ for } N \geq 3. \quad (40)$$

We note that the formula (37) for $H_1$ implies the following condition among $E_k$ and $W_k$:

$$W_1^2 + \frac{N^2 - 1}{12} (E_1^2 - 2E_1') + NW_1 - 2R_1 = W_2^2 + \frac{N^2 - 1}{12} (E_2^2 - 2E_2') - NW_1 - 2R_2. \quad (41)$$

It is worth pointing out that it is similar to but less restrictive than the condition for simultaneous type A $N$-fold supersymmetry with two different values of $N$, cf. Eqs. (15) and (16) in Ref. [13]. When the conditions (36)–(40) are all satisfied, it was shown [12] that the following relations hold

$$Q_{N,1}^- Q_{N,1}^+ = 2^N \pi_{1,N}^{[N]}(H_0), \quad Q_{N,1}^+ Q_{N,1}^- = 2^N \pi_{1,N}^{[N]}(H_1), \quad (42a)$$

$$Q_{N,2}^- Q_{N,2}^+ = 2^N \pi_{2,N}^{[N]}(H_1), \quad Q_{N,2}^+ Q_{N,2}^- = 2^N \pi_{2,N}^{[N]}(H_2), \quad (42b)$$

where $\pi_{k,N}^{[N]}$ are the $N$th critical generalized Bender–Dunne polynomials associated with each system labeled by the indices $k = 1, 2$. Substituting Eqs. (42) into the second condition (29), we have

$$2^N Q_{N,1}^+ \pi_{2,N}^{[N]}(H_1) + 2^N Q_{N,1}^- \pi_{1,N}^{[N]}(H_1) = C_2 Q_{N,1}^- P_N(H_1), \quad (43)$$

7
and thus obtain a solution to the condition \((29)\) as

\[
\pi_{1,N}^{[N]}(x) + \pi_{2,N}^{[N]}(x) = 2P_N(x).
\] (44)

Finally, substituting Eqs. \((42)\) and \((44)\) into the third condition \((30)\), we have

\[
\pi_{2,N}^{[N]}(H_1)Q_{N,2}^- + \pi_{1,N}^{[N]}(H_1)Q_{N,2}^- = 2P_N(H_1)Q_{N,2}^- = 2Q_{N,2}^-P_N(H_2).
\] (45)

It is evident that this condition is already satisfied since we have constructed the system so that \(H_1\) and \(H_2\) satisfy the second intertwining relation in Eq. \((28)\). Therefore, the system \((35)\)–\((38)\) constitutes a type of \(\mathcal{N}\)-fold superalgebra. Indeed, if we restrict the linear space \((41)\) are all satisfied. We note that this system exactly reduces to the parasupersymmetric quantum system of Rubakov–Spiridonov (RS) type \([4]\) when \(\mathcal{N} = 1\) and \(R_1 + R_2 = 0\).

In our previous paper \([9]\), we found that the RS model admits a generalized 2-fold superalgebra. In the following, we show that the above \(\mathcal{N}\)-fold parasupersymmetric system also satisfies a novel non-linear algebra. Using the relation \((12)\) and applying the intertwining relation \((31)\), we obtain for the system \((35)\)–\((38)\) the following formulas:

\[
Q_{N}^-Q_{N}^- = 2^N \pi_{1,N}^{[N]}(H_0)\Pi_0 + 2^N \pi_{1,N}^{[N]}(H_1)\Pi_1,
\] (46)

\[
Q_{N}^-Q_{N}^- = 2^N \pi_{1,N}^{[N]}(H_1)\Pi_1 + 2^N \pi_{2,N}^{[N]}(H_2)\Pi_2,
\] (47)

\[
(Q_{N}^-)^2(Q_{N}^+)^2 = 2^N Q_{N,1}^{[N]}\pi_{2,N,1}^{[N]}(H_1)Q_{N,1}^{[N]}\Pi_0 = 2^N Q_{N,1}^{[N]}Q_{N,1}^{[N]}\Pi_0
\]

\[
- 2^N \pi_{1,N}^{[N]}(H_0)\pi_{2,N}^{[N]}(H_0)\Pi_0
\] (48)

\[
(Q_{N}^+)^2(Q_{N}^-)^2 = 2^N Q_{N,2}^{[N]}\pi_{1,N,2}^{[N]}(H_1)Q_{N,2}^{[N]}\Pi_2 = 2^N \pi_{1,N}^{[N]}(H_2)Q_{N,2}^{[N]}\Pi_2
\]

\[
- 2^N \pi_{1,N}^{[N]}(H_2)\pi_{2,N}^{[N]}(H_2)\Pi_2.
\] (49)

Hence, we can easily find the following non-linear relation:

\[
(Q_{N}^-)^2(Q_{N}^+)^2 + Q_{N}^+(Q_{N}^+)^2Q_{N}^- + (Q_{N}^+)^2(Q_{N}^-)^2 = 2^{2N} \pi_{1,N}^{[N]}(H)\pi_{2,N}^{[N]}(H).
\] (50)

It is interesting to note that this non-linear relation can be regarded as a generalization of \(2\mathcal{N}\)-fold superalgebra. Indeed, if we restrict the linear space \(\mathfrak{g} \times V_2\) on which the system \(H\) acts to \(\mathfrak{g} \times (V_2^{(0)} + V_2^{(2)})\) (cf. the definition between Eqs. \((2)\) and \((3)\)), we have

\[
\{(Q_{N}^-)^2, (Q_{N}^+)^2\} = \{(Q_{N}^+)^2, (Q_{N}^-)^2\} = \{(Q_{N}^+)^2, H\} = 0.
\] (51)

This, together with the trivial (anti-)commutation relations

\[
\{(Q_{N}^-)^2, (Q_{N}^-)^2\} = \{(Q_{N}^+)^2, (Q_{N}^+)^2\} = \{(Q_{N}^+)^2, H\} = 0,
\] (52)

constitutes a type of \(2\mathcal{N}\)-fold superalgebra in the sector \(\mathfrak{g} \times (V_2^{(0)} + V_2^{(2)})\). We also note that the anti-commutation relation \((51)\) is reminiscent of the one appeared in type \(\mathcal{C}\) \(\mathcal{N}\)-fold supersymmetry, cf. Eq. (5.11b) in Ref. \([14]\). It is not accidental. Indeed, on one hand it follows from Eqs. \((28)\) and \((31)\) that \(H_0\) and \(H_2\) are intertwined by \(Q_{N,2}^-Q_{N,1}^+\), which is the component of \((Q_{N}^+)^2\), and on the other hand if we put

\[
E_1 = E_2 = E, \quad W_1 - \frac{N}{2}E_1 = W, \quad W_2 + \frac{N}{2}E_2 = W + (\mathcal{N} - \lambda)F,
\] (53)
where $\lambda$ is a parameter, the operator $Q_{N,2}^+ Q_{N,1}^+$ is expressed as

$$Q_{N,2}^+ Q_{N,1}^+ = \prod_{i=N}^{2N-1} \left( \frac{d}{dq} + W + (N - \lambda)F + \frac{2N - 1 - 2i E}{2} \right) \times \prod_{i=0}^{N-1} \left( \frac{d}{dq} + W + \frac{2N - 1 - 2i E}{2} \right), \quad (54)$$

which is nothing but a type C $2N$-fold supercharge with $N_1 = N_2 = N$ (cf. Eq. (3.22) in Ref. [14]). Hence, $H_0$ and $H_2$ can be regarded as a type C $2N$-fold supersymmetric pair and thus the formula (51) can be naturally understood.

We also note that as in the case of (ordinary) parasupersymmetry, quasi-parasupersymmetry of order $(2,2)$ may not produce any new result over parasupersymmetry of order 2. The reason is that in most cases the general solution to the conditions [29] and [32] would be given by

$$Q_{N,1}^+ Q_{N,1}^- + Q_{N,2}^+ Q_{N,2}^- = C_2 P_N(H_1). \quad (55)$$

In this case, the conditions [30] and [33] are equivalent to

$$P_N(H_1) Q_{N,2}^- = Q_{N,2}^- P_N(H_2), \quad Q_{N,2}^+ P_N(H_1) = P_N(H_2) Q_{N,2}^+, \quad (56)$$

which is close to the second relation in Eqs. (28) and (31). Hence, in most of the second-order cases, $N$-fold quasi-parasupersymmetry may be identical to $N$-fold parasupersymmetry.

Finally, we would like to refer to the fact that some different types of generalized supersymmetries have been shown to have intimate relation with each other. For instance, according to Ref. [15] every orthosupersymmetric system [16] admits both parasupersymmetry and fractional supersymmetry [17]. Combining this fact with the present results shown in this letter, we conjecture the existence of ‘$N$-fold generalization’ of other supersymmetric variants such as $N$-fold fractional supersymmetry characterized by the following non-linear relation

$$(Q_{N,i})^p = C_p P_N(H), \quad (57)$$

$N$-fold orthosupersymmetry characterized by the following non-linear relation

$$Q_{N,\alpha}^+ Q_{N,\beta}^- = 0, \quad Q_{N,\alpha}^- Q_{N,\beta}^+ + \delta_{\alpha,\beta} \sum_{\gamma=1}^{p} Q_{N,\gamma}^+ Q_{N,\gamma}^- = C_p \delta_{\alpha,\beta} P_N(H), \quad (58)$$

and so on.

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