4D spacetimes embedded in 5D light-like Kasner universes

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Abstract

We consider spatially homogeneous, anisotropic cosmological models in 5D whose line element can be written as

\[ dS^2 = A(u, v)du dv - B_{ij}(u, v)dx^idx^j, \]

\( (i, j = 1, 2, 3) \), where \( u \) and \( v \) are light-like coordinates. In the case where \( B_{ij} \) is diagonal, we construct three families of analytic solutions to the 5D vacuum field equations \( R_{AB} = 0 \) \( (A, B = 0, 1, 2, 3, 4) \). Among them, there is a family of self-similar homothetic solutions that contains, as a particular case, the so-called light-like Kasner universes. In this work we provide a detailed study of the different types of 4D scenarios that can be embedded in such universes. For the sake of generality of the discussion, and applicability of the results, in our analysis we consider the two versions of non-compactified 5D relativity in vogue, viz., braneworld theory and induced matter theory. We find a great variety of cosmological models in 4D which are anisotropic versions of the FRW ones. We obtain models on the brane with a non-vanishing cosmological term \( \Lambda(4) \), which inflate \textit{à la} de Sitter without satisfying the classical false-vacuum equation of state. Using the symmetry of the solutions, we construct a class of non-static vacuum solutions on the brane. We also develop static pancake-like distributions where the matter is concentrated in a thin surface (near \( z = 0 \)), similar to those proposed by Zel’dovich for the shape of the first collapsed objects in an expanding anisotropic universe. The solutions discussed here can be applied in a variety of physical situations.

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1 Introduction

In recent years there has been an increased interest in theories that envision our spacetime as embedded in a universe with more than four large dimensions. There are several reasons that justify this interest, among them that extensions of four-dimensional general relativity to five and more dimensions seem to provide the best route to unification of gravity with the interactions of particle physics [1]-[4]. In 5D there are two versions of relativity where the extra dimension is not assumed to be compactified. These are membrane theory [5] and space-time-matter (or induced matter) theory [6]. They lead to a great variety of models both in the cosmological context and in the description of local self-gravitating objects (see, e.g., [7], [8]). Most of these models have been obtained in coordinates where the metric in 5D can be written as

\[ ds^2 = g_{\mu\nu}(x^\rho, \psi)dx^\mu dx^\nu + c\Phi^2(x^\rho, \psi)d\psi^2, \]  

(1)

in such a way that our 4D spacetime can be recovered by going onto a hypersurface \( \Sigma_\psi \): \( \psi = \psi_0 = \) constant, which is orthogonal to the 5D unit vector

\[ \hat{n}^A = \frac{\delta^A_1}{\sqrt{g_{44}}}, \quad n_A\hat{n}^A = \epsilon, \]  

(2)

along the extra dimension, and \( g_{\mu\nu} \) can be interpreted as the metric of the spacetime.

In this framework, the effective equations for gravity in 4D are obtained from dimensional reduction of the Einstein field equations in 5D. The reduction is based on Campbell’s theorem [10], [11] and consists in isolating the 4D part of the relevant 5D geometric quantities and use them to construct the 4D Einstein tensor \((^{(4)}G_{\alpha\beta})\). The crucial result is that, even in the case where the energy-momentum tensor (EMT) in 5D is zero, to an observer confined to making physical measurements in our ordinary spacetime, and not aware of the extra dimension, the spacetime is not empty but contains (effective) matter whose EMT, \((^{(4)}T_{\alpha\beta})\), is determined by the Einstein equations in 4D, namely

\[ ^{(4)}G_{\alpha\beta} = 8\pi ^{(4)}T_{\alpha\beta} = -\epsilon (K_{\alpha\lambda}K^\lambda_{\beta} - K^\lambda_{\lambda}K_{\alpha\beta}) + \frac{\epsilon}{2}g_{\alpha\beta} (K_{\lambda\rho}K^{\lambda\rho} - (K^\lambda_{\lambda})^2) - \epsilon E_{\alpha\beta}, \]  

(3)

where \( K_{\mu\nu} \) is the extrinsic curvature

\[ K_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta}g_{\alpha\beta} = \frac{1}{2\Phi} \frac{\partial g_{\alpha\beta}}{\partial \psi}; \]  

(4)

\( E_{\mu\nu} \) is the projection of the bulk Weyl tensor \((^{(5)}C_{ABCD})\) orthogonal to \( \hat{n}^A \), i.e., “parallel” to spacetime, viz.,

\[ E_{\alpha\beta} = ^{(5)}C_{ABCD}\hat{n}^A \hat{n}^B = -\frac{1}{\Phi} \frac{\partial K_{\alpha\beta}}{\partial \psi} + K_{\alpha\mu}K^{\mu}_{\beta} - \epsilon \Phi_{;\alpha\beta} \]  

(5)

and \( \Phi_{\alpha} \equiv \partial \Phi/\partial x^\alpha \). Before going on, it is worthwhile to emphasize that the above dimensional reduction of the field equations in 5D is a standard technique that leads to the same effective matter content in 4D, i.e., \((^{(4)}T_{\alpha\beta})\), regardless of whether the line element [11] is interpreted within the context of brane theory with \( \mathbb{Z}_2 \) symmetry [12] or space-time-matter (STM) theory [13]. In this sense these two approaches to 5D relativity are mathematically equivalent. However, they are different as regards physical interpretation and motivation [14]. In brane theory there is a singular hypersurface that defines spacetime, and the properties of matter in that hypersurface are, in general, not identical to the ones of induced matter calculated in STM from the effective EMT defined by \((^{(4)}T_{\alpha\beta})\).

In the cosmological realm, nearly all models assume spatial homogeneity and isotropy, which means that the line element in 4D is taken to be an extended version of the conventional Friedmann-Roberson-Walker (FRW) metric in 4D, namely

\[ ds^2 = n^2(t, \psi)dt^2 - a^2(t, \psi)\gamma_{ij}dx^i dx^j + c\Phi^2(t, \psi)d\psi^2, \quad i, j = 1, 2, 3, \]  

(6)

\footnotetext{Notation: \( x^\mu = (x^0, x^1, x^2, x^3) \) are the coordinates in 4D and \( \psi \) is the coordinate along the extra dimension. We use spacetime signature \((+, -, -, -)\), while \( \epsilon = \pm 1 \) allows for spacelike or timelike extra dimension, both of which are physically admissible for a detailed discussion see, e.g., [9].}
where $\gamma_{ij}$ is a maximally symmetric 3-dimensional metric, with curvature index $k = 0, \pm 1$. In these coordinates the full integration of the vacuum Einstein field equations in 5D requires the specification of two additional assumptions. One of them is usually an assumption of geometric nature, e.g., that $\Phi = 1$, or $n = 1$. The second one is usually an equation of state for the matter quantities in 4D [15], [19].

Observations indicate that on large scales ($\gg 100$ Mpc) the universe is homogeneous and isotropic and well described by spatially-flat FRW cosmologies. However, there is no reason to expect that such features should hold at the early stages of the evolution of the universe. Rather, it is generally accepted that anisotropy could have played a significant role in the early universe and that it has been fading away in the course of cosmic evolution. In the framework of 4-dimensional spacetime, a prototype for anisotropic vacuum cosmologies is provided by the Kasner metric [17], which mimics the behavior of more general solutions near the singularity during some finite periods of time. Various higher dimensional extensions of the vacuum Kasner model have been discussed in the literature [18], [19].

In this paper we consider spatially homogeneous but anisotropic cosmological models whose metric in 5D has the form

$$dS^2 = A(u, v)dudv - B_{ij}(u, v)dx^idx^j,$$

where $A$ and $B$ are some functions of the “light-like” coordinates $u$ and $v$. These metrics are different from (6) in various aspects: (i) they do not contain the time or extra dimension in an explicit way; (ii) the hypersurfaces of constant $u$ or $v$ are three-dimensional instead of 4D; (ii) a priori it is not clear how to define the 5D unit vector $\hat{n}^A$ along the extra dimension, which in turn is needed for defining the spacetime sections and for constructing the appropriate projected quantities in 4D.

Here, for the case where $B_{ij}$ is diagonal we construct three families of analytic solutions to the 5D vacuum field equations $R_{AB} = 0$ ($A, B = 0, 1, 2, 3, 4$). The simplest one is a family of self-similar solutions (3) which contains as a particular case the so-called light-like Kasner universes. From a physical point of view, self-similar homothetic models are interesting because they may serve as asymptotic regimes, i.e., near the initial cosmological singularity and at late times, for many homogeneous and inhomogeneous cosmological models [24]. The other two families of solutions are obtained under the assumption that some of the metric coefficients are separable functions of their arguments.

In view of their potential relevance to the "similarity hypothesis" [24], in this work we focus our attention to the family of self-similar 5D spacetimes mentioned above. The main question under study is what kind of 4D scenarios can be embedded in such spacetimes. For the sake of generality of the discussion, and applicability of our results, in our analysis we consider both versions of non-compactified 5D relativity, viz., induced-matter and brane theory. Unfortunately, the expressions obtained in 4D as projections of the 5D solutions are quite complex and cumbersome. Therefore, to obtain manageable mathematical expressions in 4D, in sections 3, 4 and 5 we simplify the algebra (but not the physics) by restricting our discussion to the subset of light-like Kasner solutions.

Our solutions generalize a number of isotropic cosmological models and give back previous ones in the literature (see e.g., [25], [27] and references therein). Although we are not discussing particular applications here, they could be useful in the study of generalizations of Mixmaster or Belinskii-Khalatnikov-Lifshitz oscillations in theories with a single extra dimension [21], [22], [28], [30]. They could also be applied to studying conjectures about isotropic Big Bang singularities in braneworlds [31], [33]. Certain cosmological models, such as the cyclic universe model, also require an understanding of the behavior of Kasner-like solutions and are based on brane-type models [34].

The paper is organized as follows. In section 2 we derive our self-similar solution (the other two families of solutions are presented in the Appendix) and introduce a timelike coordinate $t$ and a spacelike coordinate $\psi$ along the extra dimension. This is equivalent to introducing two additional degrees of freedom, which are expressed in terms of two functions of $t$ and $\psi$. We will see that, as in the familiar FRW picture (6), these two functions can be related to the specific choice for embedding $\Sigma_\psi$ in 5D and to the physics in 4D. In section 3, within the context of STM we show that the light-like Kasner solutions generate a great variety of cosmological models in 4D, including

\[Without\ entering\ into\ technical\ details:\ extrapolating\ backwards\ in\ time\ towards\ the\ singularity,\ one\ finds\ an\ infinite\ number\ of\ alternating\ quasi-periodic\ Kasner-like\ epochs\ with\ different\ expansion\ rates\ [20]-[22].\]

\[In\ the\ traditional\ interpretation\ of\ Sedov,\ Taub\ and\ Ze’ldovich\ [23],\ self-similarity\ means\ that\ all\ dimensionless\ quantities\ in\ the\ theory\ can\ be\ expressed\ as\ functions\ only\ of\ a\ single\ similarity\ variable,\ which\ is\ some\ combination\ of\ the\ independent\ coordinates.\ In\ this\ way\ the\ field\ equations\ become\ a\ system\ of\ ordinary,\ instead\ of\ partial,\ differential\ equations.\]
the de Sitter, Milne and power-law FRW models. In section 4, within the context of the braneworld paradigm we find that they embed 4D cosmological models with a non-vanishing cosmological term $\Lambda_{(4)}$, which in principle can be either constant or time-dependent. In the case of constant $\Lambda_{(4)}$ the 3D space exponentially inflates, regardless of the specific embedding. We also show that by virtue of the symmetry $(x, y, z) \leftrightarrow \psi$, they generate a class of non-static vacuum solutions on the brane. In section 5, once again using symmetry properties, we demonstrate that the Kasner-like metric \cite{22} can be used to generate static pancake-like distributions of matter in 4D, similar to those proposed by Zel’ dovich for the shape of the first collapsed objects in an expanding anisotropic universe \cite{35}. In section 6 we present a summary of our results.

2 Cosmological models in 5D. Light-like coordinates

2.1 Solving the field equations. Part I

In this work we obtain three families of solutions to the field equations $R_{AB} = 0$. However, to facilitate the discussion, in this subsection we present only one of them. Specifically, we present the family of solutions that we will use throughout the paper, which (as we will see in sections 3-5) may be interpreted or used as 5D embeddings for a number of 4D universes. The derivation of the other two families of solutions, whose 4D interpretation is not discussed here, is deferred to the Appendix (Solving the field equations. Part II).

To simplify the shape of the field equations let us momentarily denote $B_{11} = e^{\lambda(u,v)}$, $B_{22} = e^{\mu(u,v)}$ and $B_{33} = e^{\sigma(u,v)}$. From $R_{xx} = 0$, $R_{yy} = 0$ $R_{zz} = 0$ we obtain the equations
\[
\begin{align*}
4\lambda_{uv} + \lambda_{u}(\sigma_{v} + 2\lambda_{v} + \mu_{v}) + \lambda_{v}(\mu_{u} + \sigma_{u}) &= 0, \\
4\mu_{uv} + \mu_{u}(\lambda_{v} + 2\mu_{v} + \sigma_{v}) + \mu_{v}(\sigma_{u} + \lambda_{u}) &= 0, \\
4\sigma_{uv} + \sigma_{u}(\mu_{v} + 2\sigma_{v} + \lambda_{v}) + \sigma_{v}(\lambda_{u} + \mu_{u}) &= 0. 
\end{align*}
\]

(8)

Here the subscripts $u, v$ indicate partial derivatives with respect to those arguments. The above equations show cyclic permutation symmetry, i.e., starting from any of them by means of the transformation $\lambda \rightarrow \mu \rightarrow \sigma \rightarrow \lambda$ we obtain the other two.

- Self-similar solutions: First we solve the field equations under the assumption that the metric (7) possesses self-similar symmetry. This assumption is motivated by a number of studies suggesting that self-similar models play a significant role at asymptotic regimes \cite{24}. From a mathematical point of view, it means that by a suitable transformation of coordinates all the dimensionless quantities can be put in a form where they are functions only of a single variable (say $\zeta$) \cite{23}. In our particular case, this implies that $\lambda = \lambda(\zeta)$, $\mu = \mu(\zeta)$, $\sigma = \sigma(\zeta)$, where $\zeta$ is some function of $u$ and $v$, viz.,
\[
\zeta = \zeta(u,v). 
\]

(9)

With this assumption the first equation in (8) reduces to
\[
2 \frac{\lambda_{\zeta\zeta}}{\lambda_{\zeta}} + (\lambda_{\zeta} + \mu_{\zeta} + \sigma_{\zeta}) + 2\frac{\zeta_{uv}}{\zeta_{u}\zeta_{v}} = 0. 
\]

(10)

The assumed symmetry requires $(\zeta_{uv}/\zeta_{u}\zeta_{v})$ to be some function of $\zeta$, say $Z(\zeta) = (\zeta_{uv}/\zeta_{u}\zeta_{v})$. Integrating we get
\[
\lambda_{\zeta} = 2\alpha e^{-(\lambda+\mu+\sigma)/2}e^{-\int Z(\zeta)d\zeta} \equiv 2\alpha \left( \frac{f_{\zeta}}{f} \right), 
\]

(11)

where $\alpha$ is an arbitrary constant of integration. Similar equations, with new constants, e.g., $\beta$ and $\gamma$, are obtained for $\mu$ and $\sigma$ by means of a cyclic transformation. What this means is that
\[
\frac{\lambda_{\zeta}}{2\alpha} = \frac{\mu_{\zeta}}{2\beta}, \quad \frac{\sigma_{\zeta}}{2\gamma} = \frac{f_{\zeta}}{f},
\]

(12)

which upon integration yields
\[
e^{\lambda} = C_{1}f^{2\alpha}, \quad e^{\mu} = C_{2}f^{2\beta}, \quad e^{\sigma} = C_{3}f^{2\gamma},
\]

(13)

4
where \( C_1, C_2, C_3 \) are constants of integration. A single differential equation for \( f(\zeta) = f(u, v) \) can be easily obtained by substituting (13) into any (11), namely
\[
ff_{uv} + (a - 1) f_u f_v = 0.
\]

**Notation:** Here and henceforth we denote
\[
a \equiv \alpha + \beta + \gamma, \quad b \equiv \alpha^2 + \beta^2 + \gamma^2 - \alpha - \beta - \gamma, \quad c \equiv \alpha\beta + \alpha\gamma + \beta\gamma,
\]
where \( \alpha, \beta, \gamma \) are arbitrary parameters.

A simple integration gives
\[
f = [h(u) + g(v)]^{1/a},
\]
where \( h(u) \) and \( g(v) \) are arbitrary functions of their arguments. Clearly, in the present case \( B_{ij} \) are power-law type solutions of the similarity variable \( \zeta = [h(u) + g(v)] \).

To simplify the discussion, and eliminate spurious degrees of freedom, we now make the coordinate transformation
\[
h(u) = c_1 \bar{u}, \quad g(v) = c_2 \bar{v},
\]
where \( c_1 \) and \( c_2 \) are constants. In these new coordinates
\[
\mathcal{A}(u, v) du dv \rightarrow \bar{\mathcal{A}}(\bar{u}, \bar{v}) d\bar{u} d\bar{v},
\]
where \( \bar{\mathcal{A}}(\bar{u}, \bar{v}) = [\mathcal{A}(u, v)/h_u g_v] \) with \( u \) and \( v \) expressed in terms of \( \bar{u}, \bar{v} \). Then relabeling the coordinates (dropping the overbars) the metric becomes
\[
dS^2 = \mathcal{A}(u, v) du dv - C_1 [c_1 u + c_2 v]^{2a / a} dx^2 - C_2 [c_1 u + c_2 v]^{2\beta / a} dy^2 - C_3 [c_1 u + c_2 v]^{2\gamma / a} dz^2.
\]
For this metric the field equations \( R_{uu} = 0 \) and \( R_{vv} = 0 \) yield the equations
\[
a^2 [c_1 u + c_2 v] A_u + 2c c_1 A = 0,
\]
\[
a^2 [c_1 u + c_2 v] A_v + 2c c_2 A = 0,
\]
which have a unique solution given by \( \mathcal{A} = C_0 (c_1 u + c_2 v)^{-2c / a^2} \), where \( C_0 \) is a constant of integration. Now, it is easy to verify that \( R_{uv} = 0 \) is identically satisfied. In summary, the final form of the self-similar solution is given by
\[
\mathcal{A} = (c_1 u + c_2 v)^{-2c / a^2}, \quad B_{11} = \mathcal{A}^{-\alpha c / c}, \quad B_{22} = \mathcal{A}^{-\beta c / c}, \quad B_{33} = \mathcal{A}^{-\gamma c / c}, \quad B_{ij} = 0, \quad i \neq j.
\]

We note that this solution admits a homothetic Killing vector in 5D for any values of \( \alpha, \beta \) and \( \gamma \), namely,
\[
\mathcal{L}_\zeta g_{AB} = 2g_{AB}, \quad \text{with} \quad \zeta^C = [\eta_0 u, \eta_0 v, (1 - \alpha \eta_0 / \alpha)x, (1 - \beta \eta_0 / \alpha)y, (1 - \gamma \eta_0 / \alpha)z],
\]
where \( g_{AB} \) is the metric (20), \( \mathcal{L}_\zeta \) denotes the Lie derivative along the 5D vector \( \zeta^C \) and \( \eta_0 \equiv a^2/(a^2 - c) \). In addition, by setting one of the constants equal to zero, say \( c_2 = 0 \), and making the coordinate transformation \( u^{-2c/a} du \rightarrow d\bar{u} \), it reduces to
\[
dS^2 = d\bar{u}^2 - A\bar{u} d\bar{x}^2 - B\bar{u}^2 d\bar{y}^2 - C\bar{u}^3 d\bar{z}^2,
\]
where \( A, B, C \) are constants with the appropriate units, and \( p_1, p_2, p_3 \) denote
\[
p_1 = \frac{2\alpha}{\alpha^2 + \beta^2 + \gamma}, \quad p_2 = \frac{2\beta}{\alpha^2 + \beta^2 + \gamma^2}, \quad p_3 = \frac{2\gamma}{\alpha^2 + \beta^2 + \gamma^2},
\]
which satisfy the relation \( \sum_{i=1}^{3} (p_i - 1)^2 = 3 \) for any values of \( \alpha, \beta \) and \( \gamma \). The metric (22) is usually called light-like Kasner solution. In this case the 5D homothetic vector is given by \( \zeta^C = [\bar{u}, \bar{v}, \bar{u} d\bar{x}, \bar{v} d\bar{y}, \bar{u} d\bar{z}] \).

\[\text{The proportionality coefficients} \quad C_0, C_1, C_2, C_3 \quad \text{can be set equal to unity without any loss of generality.}\]
2.2 Introducing the timelike and “extra” coordinates

In order to be able to apply the standard dimensional reduction (3) to metrics (7) one has to introduce coordinates that are adapted to the spacetime sections \( \Sigma_\psi \). With this aim we make the coordinate transformation

\[ u = F(t, \psi), \quad v = V(t, \psi), \quad \text{(24)} \]

where \( t \) is assumed to be the timelike coordinate; \( \psi \) the “extra” coordinate; \( F \) and \( V \) are, in principle, arbitrary differentiable functions of their arguments, except from the condition that the Jacobian of the transformation must be nonzero.

With this transformation we obtain

\[ du dv = \left( \dot{F} V dt^2 + F' V' d\psi^2 \right) + \left( \dot{F}' V + F' \dot{V} \right) dt d\psi, \quad \text{(25)} \]

where dots and primes denote derivatives with respect to \( t \) and \( \psi \), respectively. We can choose the coordinates \( t, \psi \) in such a way that the 5D metric be diagonal. This requires

\[ V' = -\frac{F' \dot{V}}{F}. \quad \text{(26)} \]

As a consequence, the line element (7) becomes

\[ dS^2 = \tilde{A}(t, \psi) \dot{F} \dot{V} dt^2 - \tilde{B}_{ij}(t, \psi) dx^i dx^j - \tilde{A}(t, \psi) \frac{F'^2 V}{F} d\psi^2, \quad \text{(27)} \]

where \( \tilde{A}(t, \psi) \equiv A(F, V) \) and \( \tilde{B}_{ij}(t, \psi) \equiv B_{ij}(F, V) \). A couple of points should be noticed here. Firstly, that the physical requirement \( g_{00} > 0 \) demands \( \psi \) to be spacelike. Secondly, that the line element (27) contains two arbitrary functions, which are not present in the original solution (20). The question is, why? Is this a mathematical, or gauge, artifact?

The answer to this question is that the arbitrary functions in (27) are not gauge artifacts. They reflect the physical reality that there are many ways of embedding a 4D spacetime in 5D while satisfying the field equations. If we choose some particular embedding we obtain a differential constraint connecting \( V \) and \( F \), which allows us to obtain one of them in terms of the other, e.g., \( V \) in terms of \( F \). Then, the remaining unknown function, e.g., \( F \), can be determined from the physics in 4D. As an illustration of the former assertion, let us consider two common embeddings that arise from the choice of the coordinate/reference system.

**Gaussian normal coordinate system:** A popular choice in the literature is to use the five degrees of coordinate freedom to set \( g_{44} = 0 \) and \( g_{44} = -1 \). This is the so-called ‘Gaussian normal coordinate system’ based on \( \Sigma_\psi \). Consequently, in such coordinates \( \dot{V} = (\dot{F}/\dot{F}^2) \) and (20) becomes \( V' = -(1/\dot{A}) \). Now the condition \( \partial \dot{V} / \partial \psi \) yields

\[ a^2 (c_1 F + c_2 V) F'' - 2c_1 F'^2 = 0. \quad \text{(28)} \]

If \( c_2 \neq 0 \), this equation gives \( V(t, \psi) \) for any smooth function \( F(t, \psi) \), and the metric (27) becomes

\[ dS^2 = \left( \frac{\dot{F}}{F'} \right)^2 dt^2 - \tilde{B}_{ij}(t, \psi) dx^i dx^j - d\psi^2. \quad \text{(29)} \]

If \( c_2 = 0 \), then (28) is an equation for \( F \). Integrating it we find

\[ F = [l(t) + \psi h(t)]^{\alpha^2/(\alpha+b)}, \quad \text{(30)} \]

where \( l(t) \) and \( h(t) \) are arbitrary differentiable functions.
**Synchronous reference system:** The choice $g_{00} = 1$ is usual in cosmology: it corresponds to the so-called synchronous reference system where the coordinate $t$ is the proper time at each point. Thus, setting $\dot{V} = (1/\dot{A}\dot{F})$, the line element (27) becomes

$$dS^2 = dt^2 - \dot{B}_{ij}(t, \psi)dx^i dx^j - \left(\frac{F'}{F}\right)^2 d\psi^2.$$  

(31)

In these coordinates (20) reduces to $V' = -(F'/\dot{A}\dot{F}^2)$ and $(\partial \dot{V} / \partial \psi) = (\partial V' / \partial t)$ yields

$$a^2 (c_1 F + c_2 V) \dot{F} - 2cc_1 \dot{F}^2 = 0.$$  

(32)

Thus, for $c_2 = 0$ we get

$$F = [M(\psi) + tN(\psi)]^{a^2/(a+b)},$$  

(33)

where $M$ and $N$ are arbitrary differentiable functions of $\psi$. For any other $c_2 \neq 0$, we obtain $V$ from (32) after choosing some smooth function $F(t, \psi)$.

Thus, in principle the function $V$ can be determined if we know $F$. At this point the question arises of whether we can single out the function $F$ from “physical” considerations in 4D. Further analysis of the field equations shows that if we assume an equation of state for the matter in 4D, then we obtain an extra differential equation connecting $V$ and $F$, which in addition to (28) or (32), allows to express the solution (27) in terms of $t$ and $\psi$. This is what is required for the 4 + 1 dimensional reduction of the 5D solutions. The general calculations are straightforward, but the equations are notational cumbersome in both STM and braneworld theory. On the other hand, (30) and (33) indicate that a great algebraic simplification is attained if $c_2 = 0$. In fact, in this case we can re-scale the function $F$, as $F \rightarrow \dot{F} a^2/(a+b)$, after which the solution (27) with $c_2 = 0$ reduces to

$$dS^2 = \dot{F}V dt^2 - AF^{p_1} dx^2 - BF^{p_2} dy^2 - CF^{p_3} dz^2 - \frac{F'^2 \dot{V}}{F} d\psi^2,$$  

(34)

where $p_1, p_2, p_3$ are the parameters introduced in (33). In sections 3, 4 and 5 we use this line element, which we call Kasner-like, for illustrating the fact that physics in 4D determines $F$.

### 3 Cosmological models in 4D. The STM approach

The aim of this section is to determine $F$ within the context of induced matter theory. To this end we assume an equation of state for the effective matter quantities. Our results show that the light-like Kasner metrics (22) can be used, or interpreted, as 5-dimensional embeddings for a number of cosmological models in 4D that are spatially anisotropic extensions of the FRW ones.

For the Kasner-like metric (34), the components of the effective EMT induced on $\Sigma_\psi$: $\psi = \psi_0 = \text{constant}$ are given by (in what follows we simplify the notation by omitting the bar over $F$ in (34) and the index $^{(4)}$ in $^{(4)}T_{\mu\nu}$)

$$8\pi GT_0^0 = \frac{a^2c\dot{F}}{(a + b)^2VF^2},$$

$$8\pi GT_1^1 = \frac{a\left\{(\gamma + \beta)(a + b)F\left[\frac{\dot{F}}{F} - \frac{\dot{V}}{V}\right] - 2c(\alpha - \beta - \gamma)\dot{F}\right\}}{2(a + b)^2VF^2},$$

$$8\pi GT_2^2 = \frac{a\left\{(\gamma + \alpha)(a + b)F\left[\frac{\dot{F}}{F} - \frac{\dot{V}}{V}\right] - 2c(\beta - \alpha - \gamma)\dot{F}\right\}}{2(a + b)^2VF^2},$$

$$8\pi GT_3^3 = \frac{a\left\{(\alpha + \beta)(a + b)F\left[\frac{\dot{F}}{F} - \frac{\dot{V}}{V}\right] - 2c(\gamma - \alpha - \beta)\dot{F}\right\}}{2(a + b)^2VF^2}. $$

(35)
Certainly the specific shape of the EMT depends on the embedding. However, there are a number of relationships, between the components of the EMT, which are “embedding-independent”. These are

\[(\gamma - \beta)T_1^1 + (\alpha - \gamma)T_2^2 + (\beta - \alpha)T_3^3 = 0,\]  

(36)

and

\[(\alpha + \gamma)T_1^1 - (\beta + \gamma)T_2^2 = (\beta - \alpha)T_0^0,\]
\[(\alpha + \beta)T_1^1 - (\beta + \gamma)T_3^3 = (\gamma - \alpha)T_0^0,\]
\[(\alpha + \beta)T_2^2 - (\alpha + \gamma)T_3^3 = (\gamma - \beta)T_0^0.\]  

(37)

Let us notice some particular cases: (i) If two of the parameters are equal to each other (axial symmetry), say \(\alpha = \beta\), then \(T_1^1 = T_2^2\); (ii) If \(T_1^1 = T_2^2\) but \(\alpha \neq \beta\), then \(T_1^1 = T_2^2 = T_3^3 = -T_0^0\); (iii) If \(\alpha = -\beta\), then \(T_3^3 = -T_0^0\); (iv) In the case of isotropic expansion \((\alpha = \beta = \gamma)\) then \(T_1^1 = T_2^2 = T_3^3\) (but not necessarily \(T_1^1 = T_2^2 = T_3^3 = -T_0^0\)).

### 3.1 Perfect Fluid

Let us consider the case where the effective EMT behaves like a perfect fluid. From (35) we find that \(T_1^1 = T_2^2 = T_3^3\) requires

\[(a + b) \left( \frac{F'}{F} - \frac{\dot{V}}{V} \right) + 4c \left( \frac{\dot{F}}{F} \right) = 0,\]  

(38)

which implies \(\dot{V} \propto F^{4c/(a+b)}\). Substituting this into (36) and using \((\partial \dot{V}/\partial \psi) = (\partial V'/\partial t)\) we find that \(F\) must satisfy the equation\(^5\)

\[(a + b)F F' + 4c\dot{F}F'' = 0,\]  

(39)

from which we get

\[F = [f(t) + g(\psi)]^{(a+b)/(4c+a+b)},\]  

(40)

where \(f(t)\) and \(g(\psi)\) are arbitrary functions of their arguments. The effective energy density \(\rho(\text{eff}) \equiv T_0^0\) and pressure \(p(\text{eff}) \equiv -T_1^1 = -T_2^2 = -T_3^3\) are given by

\[\rho(\text{eff}) = p(\text{eff}), \quad 8\pi G\rho(\text{eff}) = \frac{ca^2}{(a+b)^2 F^{2a^2/(a+b)}}.\]  

(41)

### 3.2 Ultra-relativistic matter and radiation

It is well-known that in the case of radiation as well as for ultra-relativistic matter (i.e., particles with finite rest mass moving close to the speed of light) the trace of the EMT vanishes identically. From (35) we find that \(T = T_0^0 + T_1^1 + T_2^2 + T_3^3 = 0\) requires

\[(a + b) \left( \frac{\dot{F}}{F} - \frac{\dot{V}}{V} \right) + 2c \left( \frac{\dot{F}}{F} \right) = 0.\]  

(42)

This equation is the analogue of (35). Following the same procedure as above we find \(F = [f(t) + g(\psi)]^{(a+b)/(2c+a+b)}\).

Therefore, the solution for radiation-like matter resembles that of perfect fluid in the sense that the effective stresses \(p_x(\text{eff}) \equiv -T_1^1, p_y(\text{eff}) \equiv -T_2^2, p_z(\text{eff}) \equiv -T_3^3\) are proportional to the energy density, viz.,

\[p_x(\text{eff}) = n_x\rho(\text{eff}), \quad p_y(\text{eff}) = n_y\rho(\text{eff}), \quad p_z(\text{eff}) = n_z\rho(\text{eff}),\]  

(43)

where \(n_x, n_y\) and \(n_z\) are constants satisfying \(n_x + n_y + n_z = 1\). If we average over the three spatial directions, this is equivalent to saying that the equation of state is \(\bar{p}(\text{eff}) = \rho(\text{eff})/3\), where \(\bar{p}(\text{eff}) \equiv -T_1^1/3\).

\(^5\)We note that (38) remains invariant under the re-scaling \(F \to F^{a^2/(a+b)}\).
3.3 Barotropic linear equation of state

For the sake of generality, and in order to keep contact with isotropic FRW cosmologies, let us study the scenario where the effective matter is barotropic, that is the ratio $\bar{p}/\bar{\rho}$ is constant. Thus we set

$$\bar{p} = n \bar{\rho}, \quad n = \text{constant},$$  \hspace{1cm} (44)

which for $n = 1$ and $n = 1/3$ gets back the above-discussed perfect fluid and radiation-like scenarios, respectively. Substituting into (33) we obtain an equation similar to (38) and (42), but with the coefficient $(1 + 3n)$ in front of the term $c\dot{F}/F$. Consequently, $\dot{V} \propto \dot{F}^{(c(1+3n))/(a+b)}$. The condition $(\partial \dot{V}/\partial \psi) = (\partial V'/\partial t)$ then requires

$$F = [f(t) + g(\psi)]_{(a+b)}^{(3n+1)c}. \hspace{1cm} (45)$$

Substituting this expression into (34) and making the coordinate transformation $(df/dt)dt \to \tilde{t}$, $(dg/d\psi)d\psi \to d\tilde{\psi}$, the line element in 5$D$ can be written as

$$dS^2 = \frac{Ddt^2}{\tilde{H}^{(3n+1)c}} - A\tilde{H}^{2\alpha_{x}} dx^2 - B\tilde{H}^{2\alpha_{y}} dy^2 - C\tilde{H}^{2\alpha_{z}} dz^2 - \frac{Dd\tilde{\psi}^2}{\tilde{H}^{(3n+1)c}}, \hspace{1cm} (46)$$

where $\tilde{H} = (i + E\tilde{\psi})^{\alpha_{x}+\alpha_{y}+\alpha_{z}}$. $E$ is an arbitrary constant for $n = 1/3$, but $E = \pm 1$ for any other $n \neq 1/3$; $D$ is a positive constant introduced for dimensional considerations.

3.3.1 Kasner universe in 5$D$

We immediately note that the case where $E = 0$, which requires $n = 1/3$, gives back the well-known Kasner universe in 5$D$. In fact, setting $i \propto \tau^{(a+b+2c)/(a+b+c)}$ the line element (46) reduces to

$$dS^2 = d\tau^2 - A\tau^{q_1} dx^2 - B\tau^{q_2} dy^2 - C\tau^{q_3} dz^2 \pm D\tau^{q_4} d\psi^2, \hspace{1cm} (47)$$

where

$$q_1 = \frac{\alpha a}{(a+b+c)}, \quad q_2 = \frac{\beta a}{(a+b+c)}, \quad q_3 = \frac{\gamma a}{(a+b+c)}, \quad q_4 = -\frac{c}{a+b+c}, \hspace{1cm} (48)$$

satisfy $\Sigma_{i=1}^{4} q_i = \Sigma_{i=1}^{4} q_i = 1$, typical of the Kasner universe in 5$D$. In order to avoid misunderstandings, it may be useful to reiterate our terminology: (i) the light-like Kasner metric is (22), which depends on the light-like variable $u$; (ii) by Kasner-like metric we refer to (44), which depends on one arbitrary function of $t$ and $\psi$, and (iii) Kasner metric is the usual name given to (47), which depends only on $\tau$.

In general, for the 4-dimensional interpretation of (46) we should notice that on every hypersurface $\tilde{\psi} = \tilde{\psi}_0 = \text{constant}$ ($\psi = \psi_0 = \text{constant}$) the proper time $\tau$ is given by

$$d\tau = \pm \frac{\sqrt{Ddt}}{(i + E\tilde{\psi}_0)^m}, \quad m = \frac{(3n+1)c}{2[a+b+(3n+1)c]} \hspace{1cm} (49)$$

Bellow we consider several cases.

3.3.2 Anisotropic Milne universe

If $m = 0$, then$^{6}$

$$n = -\frac{1}{3}. \hspace{1cm} (50)$$

$^{6}$We exclude $c = 0$ because it corresponds to empty space, i.e., $T_{\mu\nu} = 0$. 

9
In terms of the proper time \( \tau = \sqrt{D} (t + E \psi_0) \), the metric induced on 4-dimensional hypersurfaces \( \Sigma_\psi \) can be written as
\[
ds^2 = dS^2_{\Sigma_\psi} = d\tau^2 - \tilde{A} \tau^{p_1} dx^2 - \tilde{B} \tau^{p_2} dy^2 - \tilde{C} \tau^{p_3} dz^2,
\]
where \( p_i \) are the parameters defined in \( \text{(23)} \) and \( \tilde{A}, \tilde{B} \) and \( \tilde{C} \) are some new constants. In addition, \( n_i \), the ratios of the anisotropic stresses to the energy density \( \text{(43)} \) are given by
\[
n_x = \frac{(\alpha - \beta - \gamma)}{a}, \quad n_y = \frac{(\beta - \alpha - \gamma)}{a}, \quad n_z = \frac{(\gamma - \alpha - \beta)}{a}, \quad 8\pi G \rho^{(\text{eff})} = \frac{a^2 c}{(a + b)^2 \tau^2}.
\]
For \( \alpha = \beta = \gamma \) we find \( n_x = n_y = n_z = -1/3 \) and consequently we recover Milne’s universe, as expected.

### 3.3.3 Anisotropic de Sitter universe

If \( m = 1 \), then
\[
n = -\frac{1}{3} - \frac{2(\alpha^2 + \beta^2 + \gamma^2)}{3(\alpha \beta + \alpha \gamma + \beta \gamma)}.
\]
From \( \text{(49)} \) we get \( (t + E \psi_0) \propto e^{+\tau/\sqrt{D}} \). Taking the negative sign, the induced metric in 4D can be expressed as
\[
ds^2 = dS^2_{\Sigma_\psi} = d\tau^2 - \tilde{A} e^{\rho_1 \tau/\sqrt{D}} dx^2 - \tilde{B} e^{\rho_2 \tau/\sqrt{D}} dy^2 - \tilde{C} e^{\rho_3 \tau/\sqrt{D}} dz^2.
\]
For this metric we find
\[
n_x = -\frac{(\beta^2 + \gamma^2 + \beta \gamma)}{c}, \quad n_y = -\frac{(\alpha^2 + \gamma^2 + \alpha \gamma)}{c}, \quad n_z = -\frac{(\alpha^2 + \beta^2 + \alpha \beta)}{c}, \quad 8\pi G \rho^{(\text{eff})} = \frac{a^2 c}{(a + b)^2 D}.
\]

In the case of isotropic expansion \( (\alpha = \beta = \gamma) \) these equations yield \( n_x = n_y = n_z = n = -1 \) and reduces to the familiar de Sitter metric with cosmological constant \( \Lambda_{(4)} = 3/D \). An interesting conclusion here is that an anisotropic universe can enter a phase of exponential expansion (inflation), without satisfying the classical “false-vacuum” equation \( p = -\rho \) (see \( \text{(53)} \)).

### 3.3.4 Anisotropic power-law FRW universe

For \( m \neq 1 \), from \( \text{(49)} \) we obtain
\[
(t + E \psi_0) = \left[ \frac{(1 - m)}{\sqrt{D}} (\tau - \tau_0) \right]^{1/(1-m)},
\]
where \( \tau_0 \) is a constant of integration. Thus, the induced metric in 4D becomes
\[
ds^2 = dS^2_{\Sigma_\psi} = d\tau^2 - A \tau^{\kappa} dx^2 - B \tau^{\beta \kappa} dy^2 - C \tau^{\gamma \kappa} dz^2,
\]
where we have set \( \tau_0 = 0 \); \( A, B, C \) are constants with the appropriate units, and \( \kappa \) is given by
\[
\kappa = \frac{4a}{2(a + b) + (3n + 1)c} = \frac{4(\alpha + \beta + \gamma)}{2(\alpha^2 + \beta^2 + \gamma^2) + (3n + 1)(\alpha \beta + \alpha \gamma + \beta \gamma)}.
\]
We note that the denominator of \( \kappa \) is non-zero because by assumption here \( m \neq 1 \), see \( \text{(53)} \). The effective matter quantities are
\[
\rho^{(\text{eff})} = n \rho^{(\text{eff})}, \quad 8\pi G \rho^{(\text{eff})} = \frac{\kappa^2 c}{4\tau^2}.
\]
We also find
\[
n_x = \frac{2\alpha + (3n - 1)(\beta + \gamma)}{2a}, \quad n_y = \frac{2\beta + (3n - 1)(\alpha + \gamma)}{2a}, \quad n_z = \frac{2\gamma + (3n - 1)(\alpha + \beta)}{2a}.
\]
We note that for \( c = 0 \), the space is empty \( (\rho^{(\text{eff})} = 0) \), and the line element \( \text{(57)} \) yields the well-known Kasner solution in 4D. Besides, for \( n = 1 \) and \( n = 1/3 \) the above expressions reduce to those obtained for perfect fluid and radiation-like matter discussed in sections 3.1 and 3.2, respectively.
3.3.5 Isotropic expansion: spatially flat FRW universe

The above expressions evidence the fact that for anisotropic expansion the effective EMT behaves like a perfect fluid only for \( n = 1 \). In contrast, isotropic expansion allows perfect fluid for any value of \( n \). In this case the 5D metric \([46]\) can be written as (we omit the tilde over \( t \) and \( \psi \))

\[
dS^2 = \frac{Ddt^2}{(t + E\psi)^{(3n+1)/(3n+2)}} - C(t + \psi)^{2/(2+3n)} \left[ dx^2 + dy^2 + dz^2 \right] - \frac{Dd\psi^2}{(t + E\psi)^{(3n+1)/(3n+2)}},
\]

For \( n \neq -1 \), on every hypersurface \( \Sigma_\psi \) it reduces to

\[
ds^2 = d\tau^2 - C\tau^{4/(n+1)} \left[ dx^2 + dy^2 + dz^2 \right],
\]

which is the familiar flat FRW model with perfect fluid

\[
p = n\rho, \quad 8\pi G \rho = \frac{4}{3(n + 1)^2\tau^2}.
\]

For \( n = -1 \) we recover the de Sitter spacetime as shown in \([51]\).

To finish this section we would like to emphasize that although the metrics with \( a = 0 \) and \( c = 0 \) correspond to empty space (Ricci-flat in 4D), they are different in nature. For \( a = 0 \) the spacetime is Minkowski (Riemann-flat) in 5D and 4D, while for \( c = 0 \) the components of the Riemann tensor are nonzero in 5D and in the 4D subspace \( \Sigma_\psi \).

4 Cosmological models in 4D. The braneworld approach

The preceding discussion shows that, in the framework of STM the Kasner-like metric \([34]\) embeds a large family of 4D cosmological models that are anisotropic versions of the FRW ones. However, one could argue that the effective matter quantities \([35]\) do not have to satisfy the regular energy conditions \([36]\), or any physically motivated equation of state, because they involve terms of geometric origin \([1\])

In this section we will see that the 5D metric \([54]\) can be completely determined if one imposes an equation of state on the matter in the brane. Although the concept is the same as in section 3, the physics here is different. Namely, in this approach the spacetime is a singular hypersurface and, for the 5D Kasner metrics under consideration, there is an effective non-vanishing cosmological term in 4D (the brane). As a consequence, the time evolution as well as the interpretation of the solutions in 4D is distinct from the one obtained, under similar conditions, in the framework of STM.

4.1 The braneworld paradigm

In order to make the paper self-consistent, and set the notation, we give a brief sketch of the technical details that we need in our discussion. In the simplest RS2 braneworld scenario our universe is identified with a fixed singular hypersurface \( \Sigma_{\psi_b} \) (called brane) embedded in a 5-dimensional bulk with \( \mathbb{Z}_2 \) symmetry with respect to the brane. The discontinuity of the extrinsic curvature across \( \Sigma_{\psi_b} \) is related to the presence of matter on the brane, which is described by an EMT that we denote as \( \tau_{\mu\nu} \). Thus, now the Einstein field equations in 5D are \( G_{AB} = k_{(5)}^2 T_{AB}^{(brane)} \), where \( k_{(5)}^2 \) is a constant with the appropriate units and \( T_{AB}^{(brane)} = \delta_A^A \delta_B^B \tau_{\mu\nu} \delta(\psi)/\Phi \).

Israel’s boundary conditions \([37]\) relate the jump of \( K_{\mu\nu} \) to \( \tau_{\mu\nu} \), namely,

\[
(K_{\mu\nu}|_{\Sigma^+_{\psi_b}} - K_{\mu\nu}|_{\Sigma^-_{\psi_b}}) = -\frac{k_{(5)}^2}{2} (\tau_{\mu\nu} - \frac{1}{3} \tau g_{\mu\nu}).
\]

Now, the assumed \( \mathbb{Z}_2 \) symmetry implies \( K_{\mu\nu}|_{\Sigma^+_{\psi_b}} = -K_{\mu\nu}|_{\Sigma^-_{\psi_b}} \). Consequently,

In fact, the effective EMT defined by \([38]\) contains a contribution, given by \( E_{\alpha\beta} \), which is the spacetime projection of the 5D Weyl tensor and connects the physics in 4D with the geometry in 5D.
\[ \tau_{\mu
u} = -\frac{2\epsilon}{k^{(5)}} (K_{\mu
u} - g_{\mu
u} K), \]

where the extrinsic curvature \( K_{\mu
u} \) has to be evaluated on \( \Sigma^+_{\psi_b} \). From \( G_{\nu\lambda} = 0 \) it follows that \( \tau^{\mu}_{\nu} = 0 \). Thus \( \tau_{\mu
u} \) represents the total, vacuum plus matter, conserved energy-momentum tensor on the brane. It is usually separated in two parts \[38]\],

\[ \tau_{\mu
u} = \sigma g_{\mu
u} + T_{\mu
u}, \]

where \( \sigma \) is the tension of the brane, which is interpreted as the vacuum energy density, and \( T_{\mu
u} \) represents the energy-momentum tensor of ordinary matter in 4D.

From (65) and (66) we get

\[ K_{\mu
u} = -\frac{\epsilon k^{(5)}_2}{2} \left( T_{\mu
u} - \frac{1}{3} g_{\mu\nu} (T + \sigma) \right). \]

Substituting this expression into (3) we obtain \[12\]

\[ (4) G_{\mu
u} = \Lambda (4) g_{\mu\nu} + 8\pi G T_{\mu
u} - \epsilon k^{(5)}_4 \Pi_{\mu\nu} - \epsilon E_{\mu\nu}, \]

where

\[ \Lambda (4) = -\frac{k^{(5)}_4 \sigma^2}{12}, \]

\[ 8\pi G = -\frac{k^{(5)}_4 \sigma}{6}, \]

and

\[ \Pi_{\mu\nu} = \frac{1}{4} T_{\mu\alpha} T_{\nu}^{\alpha} - \frac{1}{12} T T_{\mu\nu} - \frac{1}{8} g_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} + \frac{1}{24} g_{\mu\nu} T^2. \]

All these four-dimensional quantities have to be evaluated on \( \Sigma^+_{\psi_b} \). They contain two important features; they give a working definition of the fundamental quantities \( \Lambda (4) \) and \( G \) and contain higher-dimensional modifications to general relativity. Namely, local quadratic energy-momentum corrections via the tensor \( \Pi_{\mu\nu} \), and the nonlocal effects from the free gravitational field in the bulk, transmitted by \( E_{\mu\nu} \).

### 4.2 Matter in the brane. Gaussian coordinates

In the braneworld literature the use of Gaussian coordinates in quite common. In these coordinates the function \( F \) is given by (30), which under the re-scaling \( F \to \bar{F}_{a^2/(a+b)} \) becomes \( F = l(t) + \psi h(t) \).

If we locate the brane at \( \psi = 0 \), then the metric of the bulk is given by:

1. For \( \psi > 0 \)

\[ ds^2_{(+)} = \left[ \frac{i + \psi h}{h} \right]^2 dt^2 - A [l(t) + \psi h(t)] dx^2 - B [l(t) + \psi h(t)] dy^2 - C [l(t) + \psi h(t)] dz^2 - d\psi^2. \]

2. For \( \psi < 0 \)

\[ ds^2_{(-)} = \left[ \frac{i - \psi h}{h} \right]^2 dt^2 - A [l(t) - \psi h(t)] dx^2 - B [l(t) - \psi h(t)] dy^2 - C [l(t) - \psi h(t)] dz^2 - d\psi^2. \]

Using \[47\] we calculate the non-vanishing components of \( K_{\mu\nu} = K_{\mu\nu}|_{\Sigma^+_{\psi_b}} \). These are

\[ K_{00} = \frac{i\bar{h}}{h^2}, \quad K_{11} = -\frac{A\alpha l^{(p_1-1)}h}{(a+b)}, \quad K_{22} = -\frac{B\beta l^{(p_2-1)}h}{(a+b)}, \quad K_{33} = -\frac{C\gamma l^{(p_3-1)}h}{(a+b)}. \]
We assume that the matter in the brane satisfies the equation of state
\[ p = n \rho, \tag{75} \]
where \( \rho = T^{0}_{0}, \ p = (p_{x} + p_{y} + p_{z})/3 \) and \( p_{x} = -T^{1}_{1}, \ p_{y} = -T^{2}_{2}, \ p_{z} = -T^{3}_{3} \). Using these expressions, from (65), with \( \epsilon = -1 \), and (66) we obtain
\[
\begin{align*}
\kappa_{(5)}^{2} \sigma &= -\frac{2}{(1 + n)} \left[ \frac{\dot{h}}{\dot{t}} + \frac{(2 + 3n)a^{2}}{3(a + b)} \left( \frac{h}{t} \right) \right], \\
\kappa_{(5)}^{2} \rho &= \frac{2}{(1 + n)} \left[ \frac{\dot{h}}{\dot{t}} - \frac{a^{2}}{3(a + b)} \left( \frac{h}{t} \right) \right], \quad n \neq -1, \ \dot{t} \neq 0. \tag{76}
\end{align*}
\]
We notice that in cosmological applications the metric function \( g_{00} \) is subjected to the condition \( 15, 39 \)
\[ g_{00}|_{brane} = 1. \tag{77} \]
Thus
\[ h(t) = s \dot{t}(t), \quad s = \pm 1. \tag{78} \]
Therefore, we have two equations for the three unknown \( \sigma, \rho \) and \( \dot{t}(t) \). Taking the covariant divergence of (66), it follows that to conserve both the total brane energy-momentum tensor \( T_{\mu \nu} \) and the matter energy-momentum tensor \( T_{\mu \nu} \), we must have \( \sigma = \sigma_{0} = \) constant. Then, using (78) we integrate the first equation in (76) and obtain the scale factor as\(^8\)
\[ l(t) = \left[ C_{1} e^{-s(n+1)\kappa_{(5)}^{2} \sigma_{0} t^{2}/2} + C_{2} \right]^{\eta}, \quad \eta = \frac{3(a + b)}{(2 + 3n)a^{2} + 3(a + b)}, \tag{79} \]
where \( C_{1} \) and \( C_{2} \) are constants of integration. We note that \( \eta \) is positive for arbitrary values of \( \alpha, \beta, \gamma \) and \( n > -1 \). Therefore, if we choose \( s = -1 \) and set \( C_{2} = 0 \), then the “origin” \( l = 0 \) is located at \( t = -\infty \). Thus, from (78) we find \( \rho \propto \sigma = \sigma_{0} \) for all \( t \), regardless of the value of \( n \) (but \( n \neq -1 \)). The resulting metric on the brane is de Sitter-like, with different rates of exponential expansion in every direction, similar to the models discussed in (54).

### 4.3 Non-Gaussian embeddings

The question may arise of whether the simplicity of the above scenario is not a consequence of the simplifying assumption of Gaussian coordinates. In order to investigate this question, we consider here the embedding that arises from the choice\(^9\)
\[ \dot{V} \propto \frac{F^{q_{c}/(a+b)}}{F^{2q}}, \tag{80} \]
where \( q \) is some constant. With this choice the metric in 5D can be written as
\[
\begin{align*}
dS^{2} &= \frac{F^{q_{c}/(a+b)}}{F^{2}} \frac{\dot{F}^{2}}{F^{q}} dt^{2} - A F^{p_{1}} dx^{2} - B F^{p_{2}} dy^{2} - C F^{p_{3}} dz^{2} - F^{q_{c}/(a+b)} d\psi^{2}. \tag{81}
\end{align*}
\]
Now, using \( \partial \dot{V}/\partial \psi = (\partial V'/\partial t) \) we find
\[ F = \left[ l(t) + \psi h(t) \right]^{(a+b)/(a+b-q_{c})}, \quad \text{for} \quad q \neq \frac{a+b}{c}. \tag{82} \]

\(^8\)From (66), with \( \epsilon = -1 \), it follows that \( \sigma \) must be positive in order to ensure \( G > 0 \).

\(^9\)This is a simple combination between \( \dot{V} \propto F^{(c(1+3n)/(a+b))} \) for the anisotropic FRW models considered in Section 3.3, and \( \dot{V} \propto \dot{F}/F^{2} \) for the Gaussian embedding discussed in Section 4.2.
we find

\[ H = l(t)e^{\psi(t)}, \quad \text{for} \quad q = \frac{a + b}{c}. \tag{83} \]

where \( l(t) \) and \( h(t) \) are arbitrary functions of integration. Again, if we locate the brane at \( \psi = 0 \), then the metric in the \( Z_2 \)-symmetric bulk is obtained by replacing \( \psi \rightarrow |\psi| \) in \( (82) \) and \( (83) \). Following the steps used in Section (4.2) we find

\[ k^2_{(5)}\sigma = -\frac{2s}{n + 1} \left[ \frac{\ell}{\ell} + \frac{(2 + 3n)a^2 + 3qc}{3(a + b - qc)} \right], \quad q \neq \frac{a + b}{c}. \tag{84} \]

It is interesting to note that in the case where \( q = (a + b)/c \), the equation for \( \sigma \) can formally be obtained from \( (84) \) by setting \( q = 0 \). Consequently, \( (83) \) yields models on the brane that are identical to those in Gaussian coordinates \( (70) \), although the metric in the 5D bulk is completely different in both cases.

The conclusion emanating from \( (84) \) is that, within the context of the 5-dimensional Kasner spacetimes under consideration, Gaussian and non-Gaussian embeddings generate the same physics on the brane. In particular, the assumption of constant \( \sigma \), which is equivalent to a constant cosmological term \( \Lambda_{(4)} \), obliges the universe to expand in a de Sitter anisotropic form regardless of (the choice of) the embedding. This is quite analogous to the cosmological “no-hair” theorem/conjecture of general relativity.

### 4.4 Vacuum solutions on the brane

Since the extra dimension is spacelike, the solutions to the field equations are invariant under the transformation \((x, y, z) \leftrightarrow \psi\). However, the physics in 4D crucially depends on how we choose our ordinary 3D space.

In order to illustrate this, let us permute \( \psi \leftrightarrow z \) in the solution given \( (81) \) and \( (82) \). Also, to avoid misunderstanding we change \( F(t, \psi) \rightarrow H(t, z) \). Using this notation, we find that the metric

\[ dS^2 = \frac{H^{q/(a+b)}H^2}{H^2_z}dt^2 - AD^{p_1}dx^2 - BH^2_zdy^2 - H^{q/(a+b)}dz^2 \pm CH^3dz^2, \tag{85} \]

where \( H_z = \partial H/\partial z \) and \( H = [l(t) + z h(t)]^{(a+b)/(a+b-qc)} \), is also a solution of the field equations \( R_{AB} = 0 \). Although \( (81) \) and \( (85) \) are diffeomorphic in 5D, their interpretation in 4D is quite different. Specifically, unlike \( (81) \) in \( (85) \): (i) the extra dimension can be either spacelike or timelike, (ii) the spacetime slices \( \Sigma_\psi \) are non-flat, and (iii) the metric of the spacetime is independent of \( \psi \). As a consequence of the latter, the extrinsic curvature \( K_{\mu \nu} \), defined by \( (4) \), vanishes identically. Which in turn, by virtue of \( (65) \), implies \( T_{\mu \nu} = 0 \), i.e., the spacetime (the brane) is devoid of matter \( (T_{\mu \nu} = 0) \) and \( \Lambda_{(4)} = 0 \).

Clearly, other 5D metrics with properties similar to \( (85) \) can be constructed from the solutions \( (20)-(33) \) of section 2 as well as from \( (10) \) and \( (15) \) of section 3. The conclusion here is that the spacetime part of the 5D Kasner-like metric \( (31) \), after the transformation \( \psi \leftrightarrow (x, y, z) \), can be interpreted as vacuum solutions in a braneworld \( Z_2 \)-symmetric scenario.

### 5 Static embeddings

As we noted above, when a 5D metric is independent of \( \psi \), the extra dimension can be either spacelike or timelike. This is a general feature of the 5D field equations \( (40) \). Therefore, after the transformation \( \psi \rightarrow t \) the 5D metric still satisfies the field equations \( R_{AB} = 0 \). The interesting feature here is that after such a transformation the line element induced on 4D hypersurfaces \( \Sigma_\psi \) is static, instead of dynamic as in sections 3 and 4.

A simple 5D line element that illustrates this feature, in a quite general way, can be obtained from the Kasner-like metric \( (44) \) in the synchronous reference system. In fact, making the transformations \( \psi \leftrightarrow z \), \( F(t, \psi) \rightarrow H(t, z) \); and \( \psi \leftrightarrow t \), \( H(t, z) \rightarrow W(\psi, z) \), from \( (81) \) and \( (83) \) we obtain

\[ dS^2 = CW^{p_1}dt^2 - AW^{p_1}dx^2 - BW^{p_2}dy^2 - \left( \frac{W_z}{W^7} \right)^2 dz^2 + d\psi^2, \tag{86} \]
where (We recall the re-scaling of F introduced at the end of section 2.)

\[ W = M(z) + \psi N(z). \]  

This is a solution of the 5D equations \( R_{AB} = 0 \) for any arbitrary functions \( M(z) \) and \( N(z) \). It explicitly depends on the extra dimension \( \psi \), which now is timelike. At this point it is worthwhile to emphasize that in modern noncompactified 5D theories both, spacelike and timelike extra dimensions are physically admissible \[^9\].

Once again the choice of the functions \( M(z) \) and \( N(z) \) depends on the version of 5D relativity we use to evaluate the properties of the matter content in 4D. Bellow we illustrate this by considering the induced matter approach, used in section 3, and the braneworld paradigm used in section 4.

### 5.1 Static solutions with planar symmetry in conventional 4D general relativity

It is not difficult to show that the components of the effective EMT, induced on spacetime hypersurfaces \( \Sigma_\psi : \psi = \psi_0 = \text{constant} \), for the metric (86) satisfy algebraic relations similar, but not identical, \[^10\] to those in (36) and (37), which are independent on the specific choice of \( M \) and \( N \). We omit them here and present the case where the effective matter quantities satisfy the barotropic linear equation of state (44). In such a case we find

\[ M(z) = \bar{C}N(z)^k - \psi_0N(z), \quad k = -\frac{(\alpha^2 + \beta^2 + \gamma^2)(3n + 1)(\alpha + \beta) + 2\gamma}{(\alpha + \beta + \alpha\gamma + \beta\gamma)(3n + 1)(\alpha + \beta - \gamma) + 4\gamma}, \]

where \( \bar{C} \) is a constant of integration. Thus, in the 5D solution (86): \( W = \bar{C}N^k + (\psi - \psi_0)N \), which implies that the metric induced in 4D is independent of the choice of the hypersurface \( \Sigma_\psi \). The effective energy density in 4D is given by

\[ 8\pi G\rho^{(\text{eff})} = \frac{2a^2cN^{2(1-k)}}{C^2[(3n + 1)(\alpha + \beta) + 2\gamma][(\alpha + b)^2]}, \]

and the stresses are

\[ \frac{p_x^{(\text{eff})}}{\rho^{(\text{eff})}} = \frac{(3n - 1)\gamma - \alpha(3n + 1)}{2\gamma}, \quad \frac{p_y^{(\text{eff})}}{\rho^{(\text{eff})}} = \frac{(3n + 1)\gamma - \beta(3n + 1)}{2\gamma}, \quad \frac{p_z^{(\text{eff})}}{\rho^{(\text{eff})}} = \frac{(3n + 1)(\alpha + \beta) + 2\gamma}{2\gamma}. \]

We note that for \( k = 1 \) these quantities are constants and \( \rho^{(\text{eff})} < 0 \) for all values of \( \alpha, \beta \) and \( \gamma \). In what follows we assume \( k \neq 1 \).

Since \( g_{33|_\psi} = -\bar{C}^2a^2N^{2(k-2)}(dN/dz)^2 \) we can make the coordinate transformation \( N^{(k-2)}dN \to d\bar{z} \), i.e., \( N \sim \bar{z}^{1/(k-1)} \). In terms of this new coordinate the static metric in 5D is generated by (henceforth we omit the bar over \( z \))

\[ W(t, \psi) = \bar{C} \left[ z^{k/(k-1)} + (\psi - \psi_0)z^{1/(k-1)} \right] \]

The matter quantities induced in 4D decrease as \( 1/z^2 \). Therefore, the above equations represent static “pancake-like” distributions where the matter is concentrated near the plane \( z = 0 \), while far from it \( \rho \to 0 \).

Except for the singularity at \( z = 0 \), the matter distribution presents “reasonable” physical properties. Indeed, for every value of \( n \), the “physical” conditions \( \rho^{(\text{eff})} > 0 \) and \( \rho \geq |p_{x,y,z}| \) are satisfied in a wide range of parameters \( \alpha, \beta \) and \( \gamma \). As an illustration, in the case of axial symmetry with respect to \( z \), for \( n = 0 \) these conditions hold in the range \(-2\beta/3 < \gamma < -\beta/2 \) (\( \alpha = \beta > 0 \)) or \(-\beta/2 < \gamma < -2\beta/3 \) (\( \alpha = \beta < 0 \)). For \( n = 1/3 \), they hold if \(-2\beta < \gamma \leq -\beta \) (\( \alpha = \beta > 0 \)) or \(-\beta \leq \gamma < -2\beta \) (\( \alpha = \beta < 0 \)). A similar analysis can be extended for other values of \( n \).

A simpler solution can be obtained from the above expressions in the limiting case where \( k = \infty \), which occurs for \((3n + 1)(\alpha + \beta - \gamma) + 4\gamma = 0 \). In this case (91) simplifies to \( W = \bar{C}[z + (\psi - \psi_0)] \) and the matter quantities are obtained from (83), (50) by replacing \( \gamma \to [(3n + 1)(\alpha + \beta)/3(n - 1)] \). In the case of axial symmetry, the line element becomes independent of the parameters \( \alpha, \beta, \gamma \) and depends only on \( n \). The effective matter quantities satisfy \( \rho^{(\text{eff})} > 0 \) and \( \rho \geq |p_{x,y,z}| \) for any \( n \) in the range \(-1/3 \leq n < 1/3 \).

\[^9\]For example (36) is now replaced by \(- (\alpha + \gamma)T_1^1 + (\beta + \gamma)T_2^2 + (\beta - \alpha)T_3^3 = 0 \).
5.2 Static solutions on the brane

We now proceed to use the braneworld technique for evaluating the matter quantities. If we locate the brane at $\psi = 0$, then the metric in the $\mathbb{Z}_2$-symmetric bulk is generated by $W = M(z) + |\psi| N(z)$. From (65), with $\epsilon = 1$, and (66) we obtain the components of $T_{\mu\nu}$. Now the barotropic equation of state (75) yields a differential equation linking $M(z)$, $N(z)$ and $\sigma$, which can be easily integrated for constant vacuum energy, $\sigma = \sigma_0$. Namely, we obtain

$$N(z) = \frac{3k_2(\gamma)\sigma_0(a + b)(1 + n)M(z)}{2(5 + 3n)[\gamma^2 + \gamma(\alpha + \beta)] + 4(2 + 3n)[\alpha^2 + \beta(\alpha + \beta)]} - E M(z)^{-\frac{\alpha(2 + 3n)(\alpha + \beta) + \gamma n}{(4 + 3n)k(\alpha + \beta)}},$$

(92)

where $E$ is a constant of integration. Using this expression we obtain

$$k_2(\gamma)\rho = \frac{6E\alpha\gamma}{(a + b)(2 + 3n)M(z)^k} + \frac{2k_2(\gamma)\sigma_0[\alpha^2 + \beta^2 - \gamma^2 + \alpha\beta - \gamma(\alpha + \beta)]}{k(\alpha + b)(2 + 3n)}.$$

(93)

where

$$\tilde{k} = \frac{(5 + 3n)[\gamma^2 + \gamma(\alpha + \beta)] + 2(2 + 3n)[\alpha^2 + \beta(\alpha + \beta)]}{(\alpha^2 + \beta^2 + \gamma^2)(2 + 3n)}.$$

(94)

We note that $\rho = \text{constant}$ for $\gamma = 0$. Therefore, in what follows we assume $\gamma > 0$. Since $M(z)$ is an arbitrary function, without loss of generality we can choose it as

$$M(z) \sim z^{2/k},$$

(95)

which is suggested by the decrease of the effective density discussed in section 5.1. It is not difficult to see that $\rho$ is positive for a large number of values of $\alpha, \beta, \gamma$. The positivity of the first term is guaranteed by the constant of integration $E$. To illustrate the positivity of the second term, we once again consider the case with axial symmetry with respect to $z$. In this case we find

$$\lim_{z \to \pm \infty} \rho = \frac{2\sigma_0(3\beta + \gamma)(\beta - \gamma)}{\gamma(5 + 3n)(2\beta + \gamma) + 6\beta^2(3n + 2)},$$

(96)

which is positive for any $\beta > \gamma$ and $n \geq -2/3$.

The main conclusion from this section is that regardless of whether we use the braneworld paradigm or the induced matter approach, the basic picture in $4D$ is essentially the same. Namely that the $4D$ part of (80) represents static pancake-like distributions of matter.

6 Summary

The vacuum Einstein field equations for the $5D$ FRW line element [10] allow complete integration in a number of cases. In particular for $\Phi = 1$, or $n = 1$, the $(t, \psi)$-component of the field equations provides a relation that leads to a set of first integrals [15], [16]. However, for the simplest anisotropic extension of (11), namely, the diagonal Bianchi type-I metric

$$dS^2 = n^2(t, \psi)dt^2 - \sum_{i=1}^3 b_i(t, \psi) (dx^i)^2 + c\Phi^2(t, \psi) d\psi^2,$$

(97)

this procedure does not work. (For a discussion, and a new point of view in the context of braneworld, see [11].)

Here we have pointed out that making the coordinate transformation $du \propto (ndt - \Phi d\psi)$, $dv \propto (ndt + \Phi d\psi)$, in [97] with $\epsilon = -1$, the field equations allow complete integration in several physical situations, viz., [20], [A-7], [A-8]. The $4D$ interpretation of the $5D$ solutions requires the introduction of coordinates adapted to spacetime sections $\Sigma_\psi$.

11 We note that in the case under consideration ($\gamma \neq 0$), $\tilde{k}$ never vanishes. In fact, for real parameters $\alpha$ and $\beta$ the quantity $[\alpha^2 + \beta(\alpha + \beta)]$ is always positive. On the other hand, $[\gamma^2 + \gamma(\alpha + \beta)] = 0$ requires $\alpha + \beta + \gamma = 0$, i.e., $a = 0$, which corresponds to Minkowski space in $5D$ and $4D$. 

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We introduced the \((t, \psi)\) coordinates by setting \(u = F(t, \psi)\) and \(v = V(t, \psi)\), and used a foliation of the 5D manifold such that \(\Sigma_\psi\) is a hypersurface of the foliation that is orthogonal to the extra dimension with tangent \(\hat{n}^A = \delta^A_4/\Phi\). From a mathematical point of view the functions \(F\) and \(V\) can be arbitrary, except for the fact that they have to satisfy (26). However, from a physical point of view, they are related to two important aspects of the construction of the spacetime, namely: (i) the choice of coordinates in 5D, e.g., Gaussian normal coordinates adapted to \(\Sigma_\psi\), and (ii) the formulation of physical conditions on the matter fields in 4D, e.g., some an equation of state.

Our study shows that there is a great freedom for embedding a 4D spacetime in an anisotropic 5D cosmological model. Similar results but in a distinct context have been found in [41]. To simplify the algebraic expressions, but not the physics, in sections 3, 4 and 5 we have devoted our attention to the study of 4D spacetimes embedded in the light-like Kasner cosmological metric, which is a simplified version of (27).

We have seen that the simple one-variable line element (22) can accommodate a great variety of models in 4D. Indeed, within the context of STM and braneworld theories, we have shown here that the Kasner-like metrics (34) may be used or interpreted as embeddings for a large number of cosmological and static spacetimes in 4D. Thus, apparently “different” astrophysical and cosmological scenarios in 4D might just be distinct versions of the same physics in 5D [40].

This investigation can be extended, or generalized, in different ways. In particular, we have not fully examined the possible 4D interpretation of the self-similar homothetic solution (20). Neither, have we investigated the solutions (A-7) and (A-8). An important future development here is the question of how these solutions can be applied in the generalizations of Mixmaster or Belinskii-Khalatnikov-Lifschits oscillations, as well as other issues mentioned in section 1, which appear in theories with one extra dimension.

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Appendix: Solving the field equations. Part II

Here we show two more families of analytic solutions to the field equations (8). With this aim we notice that these equations are greatly simplified if we introduce the function 

\[
\mathcal{V} = e^{\lambda + \mu + \sigma},
\]

in terms of which (8) become

\[
\begin{align*}
4\lambda_{uv} + \frac{\lambda_u V_v}{\mathcal{V}} + \frac{\lambda_v V_u}{\mathcal{V}} &= 0, \\
4\mu_{uv} + \frac{\mu_u V_v}{\mathcal{V}} + \frac{\mu_v V_u}{\mathcal{V}} &= 0, \\
4\sigma_{uv} + \frac{\sigma_u V_v}{\mathcal{V}} + \frac{\sigma_v V_u}{\mathcal{V}} &= 0.
\end{align*}
\]

Adding these equations and using (A-1) we obtain an equation for \(\mathcal{V}\), namely,

\[
2\mathcal{V} \mathcal{V}_{uv} - \mathcal{V}_u \mathcal{V}_v = 0,
\]

whose general solution can be written as

\[
\mathcal{V} = \left[\mathcal{h}(u) + \mathcal{g}(v)\right]^2,
\]

where \(\mathcal{h}\) and \(\mathcal{g}\) are arbitrary functions of their arguments. Clearly, the self-similar solution discussed in section 2.1 corresponds to the particular choice

\[
e^\lambda \propto \mathcal{V}^{\alpha/a}, \quad e^\mu \propto \mathcal{V}^{\beta/a}, \quad e^\sigma \propto \mathcal{V}^{\gamma/a},
\]
which satisfies (A-2) and (A-3) identically.

In what follows, as in section 2.1, we introduce a new set of null coordinates \( \tilde{u} \) and \( \tilde{v} \) by the relations \( \tilde{h}(u) = \tilde{c}_1 \tilde{u} \) and \( \tilde{g}(v) = \tilde{c}_2 \tilde{v} \), where \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are constants. In terms of these new coordinates \( \mathcal{V} = (\tilde{c}_1 \tilde{u} + \tilde{c}_2 \tilde{v})^2 \). Substituting this expression into the first of the equations (A-2), and dropping the tilde characters, we obtain

\[
2(c_1 u + c_2 v) \lambda_{uv} + c_1 \lambda_v + c_2 \lambda_u = 0.
\]

A similar expression holds for \( \mu \). The solutions below are obtained under the assumption that \( e^\lambda \) and \( e^\mu \) are separable functions of their arguments. In which case, the above equation implies that they are proportional to \( e^{\pm (c_1 u - c_2 v)} \). The metric function \( e^\nu = \mathcal{V} e^{-(\lambda + \mu)} \) automatically satisfies the third equation in (A-2) and is non-separable. Consequently, there are two different families of solutions corresponding to whether \( e^\lambda \propto e^{-\mu} \) or \( e^\lambda \propto e^\mu \).

- In the case where \( e^\lambda \propto e^{-\mu} \), the field equations \( R_{uv} = 0 \) and \( R_{uv} = 0 \) reduce to

\[
2A_u - c_1 (c_1 u + c_2 v) A = 0, \quad \text{and} \quad 2A_v - c_2 (c_1 u + c_2 v) A = 0.
\]

These equations completely determine the function \( A \) and assure the fulfillment of \( R_{uv} = 0 \). The final form of the solution is given by

\[
dS^2 = C_0 e^{[(c_1 u + c_2 v)^2/4]} du dv - C_1 e^{(c_1 u - c_2 v)} dx^2 - C_2 e^{-(c_1 u - c_2 v)} dy^2 - (C_1 C_2)^{-1} (c_1 u + c_2 v)^2 dz^2.
\]

- Following the same steps as above we find that when \( e^\lambda \propto e^\mu \) the solution is

\[
dS^2 = C_0 e^{[3(c_1 u + c_2 v)^2/4] - 2(c_1 u - c_2 v)} du dv - C e^{(c_1 u - c_2 v)} (dx^2 + dy^2) - C^{-2} (c_1 u + c_2 v)^2 e^{-2(c_1 u - c_2 v)} dz^2.
\]

In the above line elements \( (C, C_0, C_1, C_2) \) are constants of integration. We note that the resulting solutions are quite complicated even in the case where either \( c_1 \) or \( c_2 \) are set equal to zero. Although this is a great obstacle for the analytical interpretation of these metrics in \( 4D \), it allows us to appreciate the simplicity of the self-similar solutions discussed in the main text.

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