Abstract. Let $S$ and $R$ be the rings of regular functions on affine algebraic varieties over a field of characteristic 0, $R$ be embedded as a subring in $S$, and $F : S \to S$ be an endomorphism such that $F(R) \subset R$. Suppose that every ideal of height 1 in $R$ generates a proper ideal in $S$, and the spectrum of $R$ has no selfintersection points. We show that if $F$ is an automorphism so is $F|_R : R \to R$. When $R$ and $S$ have the same transcendence degree then the fact that $F|_R$ is an automorphisms implies that $F$ is an automorphism.

1. Introduction. In [CZ] E. Connell and J. Zweibel proved the following fact. Let $k$ be a field of characteristic 0, $S$ and $R$ be isomorphic to $k[x_1, \ldots, x_n]$, $R$ be a subring of $S$, and $F : S \to S$ be an endomorphism for which $F(R) \subset R$. Then $F$ is an automorphism iff $F|_R : R \to R$ is an automorphism.

Though the result is very natural the proof is not simple and it is based to a great extend on the Zariski Main theorem. We shall study the question when an analogue of this theorem holds for a wider class of rings. If one suppose that $S$ and $R \subset S$ are the rings of regular functions on affine algebraic varieties (over $k$) then a similar theorem is not valid without an extra assumption. Put $S = k[x, x^{-1}, y]$ and $R = k[x, y]$. Consider the automorphism of $S$ that sends $x, x^{-1}, y$ to $x, x^{-1}, xy$ respectively. Then its restriction to $R$ is not an automorphism though the image of $R$ is contained in $R$. This counterexample is based on the fact that $x$ is a unit in $S$ but not in $R$. Meanwhile it is easy to check that under the assumption of the Connell-Zweibel theorem every element of the subring which is invertible in the ambient ring must be automatically invertible in the subring. It turns out that this property is crucial in the case when $R$ is a UFD. In a more general setting we prove

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Theorem A. Let $S$ and $R$ be affine domains over a field $k$ of characteristic 0, $R$ be embedded as a subring in $S$, and $F : S \to S$ be an endomorphism for which $F(R) \subset R$.

(i) Suppose that $R$ is the ring of regular functions on an affine algebraic variety without selfintersection points (for instance, $R$ is integrally closed) and every ideal of height 1 in $R$ generates a proper ideal in $S$. Then if $F$ is an automorphism so is $F|_R : R \to R$.

(ii) Let $S$ and $R$ have the same transcendence degree. Then if $F|_R$ is an automorphism so is $F$.

Using the “Lefschetz principle” (e.g., see [BCW]) one can reduce the problem to the case when $k = \mathbb{C}$. Furthermore, we prefer to work with a geometrical reformulation of this theorem. More precisely, Theorem A is a consequence of

Theorem B. Let $X$ and $Y$ be irreducible complex affine algebraic varieties. Suppose that $\rho : X \to Y$, $f : X \to X$, and $g : Y \to Y$ are morphisms such that $\rho$ is dominant and the following diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow \rho & & \downarrow \rho \\
Y & \xrightarrow{g} & Y
\end{array}
$$

(i) Suppose that $Y$ has no selfintersection points, $f$ is an automorphism, and $g$ is not. Then there exists a closed hypersurface $D \subset Y$ such that $\text{codim}_Y g(D) \geq 2$ and $\rho^{-1}(D)$ is empty.

(ii) Let $\dim Y = \dim X$. Then if $g$ is an automorphism so is $f$.

Besides the Zariski Main Theorem [H, Ch. 5, Th. 5.2] our other main tool follows from a remarkable theorem of Ax [A] (later rediscovered by Kawamata [I])

Theorem. Let $Z$ be a complex algebraic variety and let $h : Z \to Z$ be an injective morphism. Then $h$ is an automorphism.

\footnote{The absence of selfintersection points is essential. Indeed, in the example above we can replace $R$ by its subring which consists of polynomials in $k[x, y]$ taking the same values at points $(0,0)$ and $(1,1)$. This gives a counterexample to Theorem A in the presence of selfintersection points.}
The idea of the proof is the following. Using the Zariski and Ax theorems we prove that if \( g \) (resp. \( f \)) is not an automorphism under the assumption of Theorem B (i) (resp. B(ii)) then there exists a divisor \( D \subset Y \) (resp. \( E \subset X \)) such that \( \text{codim}_Y g(D) \geq 2 \) and \( g(D) \subset D \) (resp. \( \text{codim}_X f(E) \geq 2 \) and \( f(E) \subset E \)). The next argument is especially simple in the smooth equidimensional case: we show that the zero multiplicity of the Jacobians of \( g^s \circ \rho \) and \( \rho \circ f^s \) are different at \( x \in \rho^{-1}(D) \) (resp. \( x \in E \)) for some \( s > 0 \). In the non-smooth case we show that the dimensions of the images of a \( k \)-jet space at \( x \) under \( g^s \circ \rho \) and \( \rho \circ f^s \) are different.

It is our pleasure to thank M. Miyanishi for drawing our attention to the paper of Ax.

2. The existence of the exceptional divisor.

2.1. Replacing \( X \) and \( Y \) in diagram (1) with their normalizations \( X^0 \) and \( Y^0 \) (which are also affine) we get a commutative diagram

\[
\begin{array}{ccc}
X^0 & \xrightarrow{f^0} & X^0 \\
\downarrow \rho^0 & & \downarrow \rho^0 \\
Y^0 & \xrightarrow{g^0} & Y^0
\end{array}
\]

As \( Y \) has no selfintersection points the normalization \( Y^0 \to Y \) is a homeomorphism. Hence for any divisor \( D \subset Y \) and its proper transform \( D^0 \subset Y^0 \) we have \( (\rho^0)^{-1}(D^0) \neq \emptyset \) iff \( \rho^{-1}(D) \neq \emptyset \). Hence it is not difficult to prove the following.

**Lemma.** Theorem B is true if it is true under the additional assumption that \( X \) and \( Y \) are normal.

2.2. **Lemma.** Let \( X \) and \( Y \) be as in diagram (1). Then

(a) if \( f \) is birational so is \( g \),

(b) if \( \dim X = \dim Y \) and \( g \) is birational then \( f \) is birational.

**Proof.** Consider (a). It follows from the semi-continuity theorem [H, Ch. 3, Th. 12.8] that the number of connected components in \( \rho^{-1}(y) \) is an upper semi-continuous function on \( Y \). In particular, this number is the same for general points \( y \in Y \). Denote it by \( n \). Note that \( g \) is dominant since otherwise \( f \) is not dominant. Let \( k \) be the number of components in the preimage of a general point of \( y \in Y \) under \( g \). There are
n components in \((\rho \circ f)^{-1}(y)\) and \(kn\) components in \((g \circ \rho)^{-1}(y)\). By commutativity of diagram (1) we have \(k = 1\). That is, the degree of \(g\) is 1 and \(g\) is birational. The proof of (b) is similar. □

**Corollary.** Under the assumption of Theorem B \(f\) is birational iff \(g\) is birational.

2.3. By the semi-continuity theorem \(X_1 = \{x \in X | \text{dim} f^{-1}(f(x)) > \text{dim} X - \text{dim} Y\}\) is a closed algebraic subvariety of \(X\). Let \(X_0 = X \setminus X_1\) and \(Y^0\) be the largest Zariski open subset of \(\rho(X_0)\). In Theorem B (ii) we need also the Zariski open subset \(X^0\) of \(X\) that is the largest subset such that \(\rho|_{X^0}\) is quasi-finite.

**Lemma.** (1) Under the assumption of Theorem B (i) the restriction of \(g\) to \(Y^0\) is an automorphism provided that \(Y\) is normal.

(2) Under the assumption of Theorem B (ii) the restriction of \(f\) to \(X^0\) is an automorphism provided that \(X\) is normal.

**Proof.** The commutativity of diagram (1) implies that \(f(X_1) \subset X_1\) in the first statement. By the Ax theorem the restriction of \(f\) to \(X_1\) is an automorphism of \(X_1\) whence we have the similar fact for \(X_0\). The commutativity of diagram (1) implies that the restriction of \(g\) to \(\rho(X_0)\) is a homeomorphism of \(\rho(X_0)\) whence (1) follows from the Zariski Main theorem. In (2) let \(E \subset X\) be the set of points where \(f\) is not étale. By the Zariski Main Theorem any \(x \in E\) is not a connected component of \(f^{-1}(f(x))\), and by the commutativity of diagram (1) \(f^{-1}(f(x))\) is contained in \(\rho^{-1}(\rho(x))\). Thus \(X^0 \subset X \setminus E\). As \(f = g \circ \rho \circ f^{-1}\) for \(x \in X^0\) we have \(f(x) \in X^0\), i.e. \(X^0 \subset X^0\) whence by the Ax theorem \(f|_{X^0}: X^0 \to X^0\) is an automorphism. □

2.4. **Proposition.** Let \(g : Y \to Y\) be a birational endomorphism of a normal affine algebraic variety which is not an automorphism, but for a Zariski open subset \(Y^0\) of \(Y\) the restriction of \(g\) to \(Y^0\) is an automorphism. Then there exists an exceptional divisor \(D\) with respect to \(g\) (i.e. \(\text{codim}_Y g(D) \geq 2\)). Furthermore, replacing \(g\) with \(g^m\) for some \(m > 0\) one can suppose that \(g(D) \subset D\).

**Proof.** Let \(D' = Y \setminus Y^0\). Denote by \(D'_0\) the Zariski open subset of \(D'\) that consists of points such that the restriction of \(g\) to a neighborhood of any of these points is a quasi-finite morphism. For every \(y \in D'\) its image \(g(y)\) cannot belong to \(Y^0\) (that is, \(g(D') \subset D'\)) since otherwise \(g(y_1) = g(y)\) for some \(y_1 \in Y^0\) whence the preimage of \(g(y)\) is not connected contrary to the Zariski Main theorem. The same theorem

\[\begin{align*}
\text{Corollary.} & \quad \text{Under the assumption of Theorem B } f \text{ is birational iff } g \text{ is birational.} \\
2.3. \quad & \text{By the semi-continuity theorem } X_1 = \{x \in X | \text{dim} f^{-1}(f(x)) > \text{dim} X - \text{dim} Y\} \text{ is a closed algebraic subvariety of } X. \text{ Let } X_0 = X \setminus X_1 \text{ and } Y^0 \text{ be the largest Zariski open subset of } \rho(X_0). \text{ In Theorem B (ii) we need also the Zariski open subset } X^0 \text{ of } X \text{ that is the largest subset such that } \rho|_{X^0} \text{ is quasi-finite.} \\
\text{Lemma.} & \quad (1) \text{ Under the assumption of Theorem B (i) the restriction of } g \text{ to } Y^0 \text{ is an automorphism provided that } Y \text{ is normal.} \\
& \quad (2) \text{ Under the assumption of Theorem B (ii) the restriction of } f \text{ to } X^0 \text{ is an automorphism provided that } X \text{ is normal.} \\
\text{Proof.} \quad \text{The commutativity of diagram (1) implies that } f(X_1) \subset X_1 \text{ in the first statement. By the Ax theorem the restriction of } f \text{ to } X_1 \text{ is an automorphism of } X_1 \text{ whence we have the similar fact for } X_0. \text{ The commutativity of diagram (1) implies that the restriction of } g \text{ to } \rho(X_0) \text{ is a homeomorphism of } \rho(X_0) \text{ whence (1) follows from the Zariski Main theorem. In (2) let } E \subset X \text{ be the set of points where } f \text{ is not étale. By the Zariski Main Theorem any } x \in E \text{ is not a connected component of } f^{-1}(f(x)), \text{ and by the commutativity of diagram (1) } f^{-1}(f(x)) \text{ is contained in } \rho^{-1}(\rho(x)). \text{ Thus } X^0 \subset X \setminus E. \text{ As } f = g \circ \rho \circ f^{-1} \text{ for } x \in X^0 \text{ we have } f(x) \in X^0, \text{ i.e. } X^0 \subset X^0 \text{ whence by the Ax theorem } f|_{X^0}: X^0 \to X^0 \text{ is an automorphism. □} \\
2.4. \quad \text{Proposition.} \quad \text{Let } g : Y \to Y \text{ be a birational endomorphism of a normal affine algebraic variety which is not an automorphism, but for a Zariski open subset } Y^0 \text{ of } Y \text{ the restriction of } g \text{ to } Y^0 \text{ is an automorphism. Then there exists an exceptional divisor } D \text{ with respect to } g \text{ (i.e. } \text{codim}_Y g(D) \geq 2\). \text{ Furthermore, replacing } g \text{ with } g^m \text{ for some } m > 0 \text{ one can suppose that } g(D) \subset D. \\
\text{Proof.} \quad \text{Let } D' = Y \setminus Y^0. \text{ Denote by } D'_0 \text{ the Zariski open subset of } D' \text{ that consists of points such that the restriction of } g \text{ to a neighborhood of any of these points is a quasi-finite morphism. For every } y \in D' \text{ its image } g(y) \text{ cannot belong to } Y^0 \text{ (that is, } g(D') \subset D') \text{ since otherwise } g(y_1) = g(y) \text{ for some } y_1 \in Y^0 \text{ whence the preimage of } g(y) \text{ is not connected contrary to the Zariski Main theorem. The same theorem}
\end{align*}\]
implies that the restriction of \( g \) to \( Y^0 \cup D'_0 \) is an embedding. Suppose that \( C \) is an irreducible component of \( D' \) which is a hypersurface and which meets \( D'_0 \) (i.e. \( D'_0 \cap C \) is dense in \( C \)), and let \( D_1 \) be the union of such hypersurfaces. Then the closure of \( g(C) \) is also a hypersurface which is an irreducible component of \( D' \). Assume that this component is not contained in \( D_1 \). Denote by \( D''_0 \) the subset of \( D' \) that consists of points such that the restriction of \( g^2 \) to a neighborhood of any of these points is a quasi-finite morphism. Note that under this assumption \( C \) does not meet \( D''_0 \). Thus replacing, if necessary, \( g \) with \( g^m \) for some natural \( m \) we can suppose that \( g(D_1) \subset D_1 \).

In particular, \( D_1 \setminus g(D_1 \cap D'_0) \) is of codimension at least 2 in \( Y \). Assume that \( D' \) does not contain an exceptional divisor with respect to \( g \). Then the codimension of the complement to \( g(Y^0 \cup D'_0) \) in \( Y \) is at least 2. Since \( g^{-1} \) is well-defined on \( g(Y^0 \cup D'_0) \) it can be extended to \( Y \) by the theorem about deleting singularities for normal algebraic varieties in codimension 2 [D, Ch. 7.1]. This contradicts the assumption that \( g \) is not an automorphism whence there exists an exceptional divisor \( D \) with respect to \( g \) which is, of course, contained in \( D' \).

For the second statement note that for every \( y \in D \) its image \( y_1 = g(y) \) must belong to an irreducible component of \( D' \) which is a hypersurface since otherwise \( g^{-1} \) can be extended to \( y_1 \) by the theorem about deleting singularities in codimension 2. Suppose that \( C \) and \( D_1 \) are as above. In particular, the closure of \( g(D_1) \) is \( D_1 \), and \( g(C \cap D'_0) \) is dense in \( g(C) \). Let \( C_0 \) be the complement in the closure of \( g(C) \) to the union of the other components of \( D' \) that are hypersurfaces. Note that \( g^{-1}(C_0) \) is contained in \( D'_0 \) by the theorem about deleting singularities. Furthermore, applying this theorem again we see that \( g^{-1}(D_1 \setminus D) \) is also contained in \( D'_0 \), i.e. \( y_1 \) cannot belong to \( D_1 \setminus D \). Thus \( y_1 \in D \) and \( g(D) \subset D \). □

**Corollary.** If \( g \) (resp. \( f \)) is not an automorphism under the assumption of Theorem B (i) (resp. B(ii)) then there exists an exceptional divisor \( D \) with respect to \( g \) (resp. \( E \) with respect to \( f \)). Furthermore, one can suppose that \( g(D) \subset D \) (resp. \( f(E) \subset E \)).

**2.5.** We can already prove Theorem B in the case of smooth varieties \( X \) and \( Y \) (for simplicity we shall consider the case when \( X \) and \( Y \) are of the same dimension). Consider a holomorphic mapping \( h : V \to U \) of equidimensional complex manifolds \( V \) and \( U \) and the Jacobian of this mapping in local coordinate systems at \( v \in V \) and
\( u = h(v) \), i.e. the determinant of the Jacobi matrix. The Jacobian itself depends on the choice of these local coordinate systems but the order of its zeros at \( v \) does not. We denote this order by \( Jd_h(v) \). The following the two properties of \( Jd_h \) are simple.

(a) \( Jd_h(v) > 0 \) iff \( h \) is not a local embedding in a neighborhood of \( v \);

(\( \beta \)) if \( e : U \to W \) is another holomorphic mapping of equidimensional complex manifolds then \( Jd_{e\circ h} \geq Jd_h(v) + Jd_e(u) \), and the equality holds in the case when either \( h \) is a local embedding at \( v \) or \( e \) is a local embedding at \( u \).

Let the assumption of Theorem B (i) hold and \( D \) be as in Corollary 2.4. Assume that \( \rho^{-1}(D) \neq \emptyset \), \( x \in \rho^{-1}(D), x' = f(x), \) and \( y = \rho(x) \). Since \( \rho \circ f = g \circ \rho \) and \( f \) is an automorphism we have by (\( \beta \)) \( \rho \circ f = Jd_{\rho \circ f}(x) = Jd_\rho(x') = Jd_\rho(x) + Jd_\rho(y) \). Since \( g \) is not a local embedding at \( y \) we see that \( Jd_\rho(y) > 0 \). Furthermore, since \( g(D) \subset D \), replacing \( g \) (resp. \( f \)) by \( g^m \) (resp. \( f^m \)) we can make \( Jd_\rho(y) >> 0 \). One the other hand \( Jd_\rho(x') \) is bounded as \( Jd_\rho \) is bounded on \( X \). This contradiction concludes the proof of Theorem B (i) in the smooth case. The proof of Theorem B (ii) in the smooth case is similar.

3. Jets on manifolds.

3.1. In order to deal with the general case we need to consider the variety of \( k \)-jets \( J^k(M) \) from the germ \((C, 0)\) of the complex line at the origin into a complex manifold \( M \). The following notation and simple facts will be used. For \( k \geq l \) we denote by \( \tau_M^{k,l} : J^k(M) \to J^l(M) \) the natural projection. The map \( \tau_M^{k,0} : J^k(M) \to J^0(M) \cong M \) is a \( C^* \)-fibration where \( s = k \dim M \). This fibration admits a natural \( C^* \)-action generated by the \( C^* \)-action on \((C, 0)\). The restriction of this action to any fiber generates an embedding of this fiber into a weighted projective space. Hence we can extend \( \tau_M^{k,0} \) to a proper holomorphic fibration \( \tilde{\tau}_M^{k,0} : J^k(M) \to M \) whose fibers are isomorphic to this weighted projective space. For every subset \( Z \) of \( J^l(M) \) we denote by \( J_Z^k(M) \) the set \( \{ j \in J^k(M) | \tau_M^{k,l}(j) \in Z \} \). Note that if \( Z \) is a variety then

\[
\dim J_Z^k(M) = \dim Z + (k - l) \dim M.
\] (2)

Any holomorphic map of complex manifolds \( \varphi : M \to N \) generates a holomorphic map \( \varphi^{(k)} : J^k(M) \to J^k(N) \) such that \( \tau_N^{k,l} \circ \varphi^{(k)} = \varphi^{(l)} \circ \tau_M^{k,l} \). In particular, if \( Z \subset J^l(M), Z^k \subset J_Z^k(M), W = \varphi^{(l)}(Z), \) and \( W^k = \varphi^{(k)}(Z^k) \) then \( W^k \subset J_W^k(N) \). Another useful observation is that \( \varphi^{(k)} \) commutes naturally with the \( C^* \)-actions on \( J^k(M) \) and \( J^k(N) \) whence it can be extended to a holomorphic map \( \tilde{\varphi}^{(k)} : J^k(M) \to J^k(N) \).
3.2. Proposition. Let \( \varphi : M \to N \) be a non-degenerate holomorphic map of complex manifolds. Let \( l \geq 0 \) and \( Z_0 \) be an algebraic subvariety of \( J^l(M) \). Then there exists \( r \geq l \) such that for every \( k \geq r \), \( Z = J^r_{Z_0}(M), Z^k = J^k_Z(M), W = \varphi^{(r)}(Z), \) and \( W^k = \varphi^{(k)}(Z^k) \) we have \( \dim W^k = \dim W + (k - r) \dim N \) (i.e., by (2), \( W^k \) is dense in \( J^k_{W}(N) \)).

Proof. First note that we can suppose that \( Z_0 \) is irreducible (then in the non-irreducible case for every irreducible component of \( Z_0 \) one can find its own number \( r \) and take the maximum of these numbers in the statement of Lemma).

Step 1. Let us show that for every \( l \geq 0 \) it suffices to prove the statement under the additional assumption that \( \tau^{l,0}_M(Z_0) \) is a point \( x_0 \in M \).

Let \( B = \tau^{r,0}_M(Z) = \tau^{l,0}_M(Z_0) \) and \( x \in B \). Put \( \theta^k = \tau^{k,r}_N|_{W^k} : W^k \to W, Z^k = \{ j \in Z^k | \tau^r_M(j) = x \} \), and \( W^k = \varphi^{(r)}(Z^k) \). Since \( \theta^k(W^k) = W^r \) it suffices to show that for general \( x \in B \) we have \( \dim W^k_x - \dim W^r_x = (k - r) \dim N \). Since we assume that Lemma is correct under the additional assumption, for every \( x \in B \) the number \( r \) can chosen so that we have this equality. Furthermore, by Baire's category theorem we can suppose that there exists \( r \) for which the equality holds for every \( x \) in a subset \( L \subset B \) which is not contained in any analytic subset of \( B \). Consider \( \tilde{W}^k \) equal to the image of \( Z^k \) in \( B \times W^k \) under the holomorphic map \( (\tau^{k,r}_M, \varphi^{(k)}) \). Note that \( W^k_x \) can be viewed as a fiber of the natural projection \( \tilde{W}^k \to \tau^{k,0}_M(Z^k) = B \). As \( \varphi^{(k)} \) and \( \tau^{k,0}_M \) can be extended to \( \tilde{\varphi}^{(k)} \) and \( \tilde{\tau}^{k,0}_M \) from 3.1 this projection can be extended to a proper holomorphic map into \( B \) whence, by semi-continuity theorem (e.g., see [BN, Th. 2.3]) the dimension of \( W^k_x \) is constant on a complement \( U \) to a proper analytic subset of \( B \). Consider the natural projection \( \tilde{\theta}^k : \tilde{W}^k \to \tilde{W}^r \) generated by \( \tau^{k,r}_N \) and its restriction to \( W^k_x \) which may be viewed as \( \theta^k \). The dimension of a general fiber of \( \tilde{\theta}^k \) is \( \dim \tilde{W}^k - \dim \tilde{W}^r \) where the last number coincides with \( \dim W^k_x - \dim W^r_x \) for general \( x \), i.e. for \( x \in U \). Since \( L \) meets \( U \) we see that \( \dim W^k_x - \dim W^r_x = (k - r) \dim N \) for \( x \in U \) which concludes the first step.

Step 2. If \( Z \subset J_{x_0}(M) \) the fact becomes local analytic, and one can suppose that \( M \) (resp. \( N \)) coincides with the germ \( (\mathbb{C}^m, o_m) \) (resp. \( (\mathbb{C}^n, o_n) \)) of a Euclidean space at the origin (of course, we put \( x_0 = o_m \)). Let \( (\varphi_1, \ldots, \varphi_n) \) be the coordinate form of a holomorphic map \( \varphi \) and let \( \varphi_{i,0} \) the the minor homogeneous form in the Taylor decomposition of \( \varphi_i \). We need

Claim. For \( Z_0 \subset J_{x_0}(M) \) it suffices to prove the local version of Lemma in the
case of homogeneous \( \varphi \), i.e. \( \varphi_i = \varphi_{i,0} \) for every \( i \) and the degrees of these coordinate functions are the same number \( s \).

First note that if \( \theta : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map Lemma holds for morphism \( \varphi \) provided it holds for morphism \( \phi = \theta \circ \varphi \). The coordinate functions \( \phi_1, \ldots, \phi_n \) of \( \phi \) are elements of the algebra generated by \( \varphi_1, \ldots, \varphi_n \). These elements can be chosen so that there minor homogeneous forms are algebraically independent [M-L]. Thus we can suppose from the beginning that \( \varphi_{1,0}, \ldots, \varphi_{n,0} \) are algebraically independent (i.e. morphism \( \psi_0 = (\varphi_{1,0}, \ldots, \varphi_{n,0}) \) is dominant). Furthermore, replacing \( \varphi_1, \ldots, \varphi_n \) by their powers, we suppose that each \( \varphi_{i,0} \) has the same degree \( s \). Let \( \xi = (\xi_1, \ldots, \xi_m) \) be a coordinate system on \( \mathbb{C}^m \). Put \( \psi_{i,c} = c^{-s} \varphi_{i,0}(c \xi) \) where \( c \in \mathbb{C}^* \), and put \( \psi_c = (\psi_{1,c}, \ldots, \psi_{n,c}) \). Clearly, \( \varphi^{(k)}(J^k_z(M)) \) and \( \psi_c^{(k)}(J^k_z(M)) \) are isomorphic for \( c \neq 0 \), and \( \psi_c \to \psi_0 \) as \( c \to 0 \). This yields a surjective morphism from \( \varphi^{(k)}(J^k_z(M)) \) to \( \psi_0^{(k)}(J^k_z(M)) \) which implies the statement of the Claim and concludes Step 2.

Step 3. We shall use induction by \( l \). Let \( l = 0 \). By Step 1 we can suppose that \( Z_0 \) consists of one element \( j_0 \) which is presented by a constant map from \( (\mathbb{C}, 0) \) into a point \( x_0 \in M \). That is, \( j_0(t) = x_0 \) where \( t \) is a coordinate on \( (\mathbb{C}, 0) \). By Step 2 we can suppose that \( M = \mathbb{C}^m, x_0 = o_m, N = \mathbb{C}^n \), and \( \varphi : \mathbb{C}^m \to \mathbb{C}^n \) is homogeneous of degree \( s \). If \( n = m \) then, since \( \varphi \) is dominant, it is a local analytic isomorphism at a general point \( x \) of \( \mathbb{C}^n \). Hence for \( y = \varphi(x) \) the restriction of \( \varphi^k \) to \( J^k_x(M) \) is an isomorphism between \( J^k_x(M) \) and \( J^k_y(N) \). In the case when \( m > n \) applying the above argument to the restrictions of \( \varphi \) to general \( n \)-dimensional submanifolds of \( M \) we can see that the restriction of \( \varphi^{(k)} \) to \( J^k_x(M) \) is an epimorphism onto \( J^k_y(N) \) for general \( x \in M \). Every \( j \in J^k_{\text{om}}(M) \) is of form

\[
  j(t) = tj_1(t)
\]

where \( j_1 \in J^{k-1}_x(M) \) and \( x \in \mathbb{C}^n \). Put \( r = s \) and consider the Zariski open subset of \( J^r_{\text{om}}(M) \) which consists of \( j^0 \) such that \( j^0(t) = tj_1^0(t) \) where \( j_1^0 \) is an element of \( J^{r-1}_x(M) \) for which \( x = j_1^0(0) \) is a general point of \( \mathbb{C}^n \). In particular, \( j^0 \) is a general element of \( Z = J^r_{\text{om}}(M) \), and the restriction of \( \varphi^{(k)} \) to \( J^k_x(M) \) is an epimorphism onto \( J^k_y(N) \) where \( y = \varphi(x) \). Let \( j \in J^r_{\text{om}}(M) \) and \( j_1 \in J^{k-1}_{\text{om}}(M) \) be as in (3). Note that

\[
  \varphi^{(k)}(j) = t^s \varphi^{(k-s)}(j_2)
\]

where \( j_2 = \tau^{k-1}_{M}(j_1) \). Hence \( \varphi^{(r)}(j^0) = t^sy \). Since the restriction of \( \varphi^{(k-s)} \) to \( J^{k-s}_x(M) \) is an epimorphism onto \( J^{k-s}_y(N) \) we see that the restriction of \( \varphi^{(k)} \) to \( J^k_{\text{om}}(M) \)
is an epimorphism onto $J^{k}_{\varphi(o)}(N)$ which proves the statement for $l = 0$ and concludes Step 3.

Step 4. Assume that Lemma is proven for $l - 1$. That is, for every $Z'_0 \subset J^{l-1}_x(M)$ there exists $r_0 \geq l - 1$ such that for $Z' = J^{r_0}_x(M)$, $W' = \varphi^{(r_0)}(Z')$, and every $k \geq r_0$ the image $\varphi^{(k)}(J^{r_0}_x(M))$ is dense in $J^{k}_W(N)$. By Step 1 we can suppose that $\tau^{l,0}_M(Z_0) = x_0$ whence by Step 2 $M = C^n, x_0 = o_m, N = C^n,$ and $\varphi : C^n \to C^n$ is homogeneous of degree $s$. This means that $Z_0$ is of form $Z_0 = tZ'_0$ and $Z = tZ'$.

Put $r = r_0 + s, Z'' = \tau^{r-1,s-s}_M(Z')$ and $W'' = \varphi^{(r-s)}(Z'')$. Then $Z'' = J^{r_0}_x(M)$ since $\tau^{r-1,l-1}_M = \tau^{r-s,l-1}_M \circ \tau^{r-1,s-s}_M$. By (4), $W = tsW''$ and the statement of Lemma is equivalent to the fact that $\varphi^{(k-s)}(J^{k-s}_x(M))$ is dense in $J^{k-s}_W(N)$. But this is true by the induction assumption for $l - 1$. □

4. Jets on algebraic varieties.

4.1. We need an analogue of $J^{k}(M)$ in the case of non-smooth algebraic varieties. In the rest of the paper for every algebraic variety (resp. analytic set) $Y$ and $y \in Y$ we denote by $(Y, y)$ the germ of $Y$ at $y$ in the Zariski (resp. Euclidean) topology. Let $(Y, y) \hookrightarrow (C^n, o_n)$ be a closed embedding where $o_n$ is the origin in $C^n$. Let $t$ be a coordinate on $(C, 0)$. We denote by $\hat{J}C^n$ the set of formal jets $\hat{j}$ which are $n$-tuples $\hat{j} = (\hat{j}_1, \ldots, \hat{j}_n)$ of formal power series in $t$. Its subset $\hat{J}_oC^n$ consists of $\hat{j}$ such that $\hat{j}_i(0) = 0$ for every $i$. We define the set of formal jets $\hat{J}_yY$ of $Y$ at $y$ as a subset of $\hat{J}_oC^n$ such that $\hat{j} \in \hat{J}_yY$ iff for every regular function $h$ from the defining ideal of $(Y, y)$ in $(C^n, o_n)$ the formal series $h \circ \hat{j}$ is zero.

**Definition.** Let $\tau^k : \hat{J}C^n \to J^kC^n$ be the forgetting projection. The set of $k$-jets of $Y$ at $y$ is $J^k_yY := \tau^k(\hat{J}_yY)$.

**Remark.** We call $\hat{j} \in (\tau^k)^{-1}(j)$ a formal extension of $j \in J^kC^n$. By Artin’s theorem [P, Th. 4.4] for $j \in J^k_yY$ its formal extension $\hat{j} \in \hat{J}_yY$ can be chosen convergent. That is, we can treat $\hat{j}(t)$ as a germ of a curve in $Y$.

4.2. Lemma. The closure of $J^k_yY$ in $J^k_{o_n}C^n$ is an algebraic variety, it is independent (up to an isomorphism) from the choice of a coordinate $t$ on $(C, 0)$ and from the choice of the closed embedding $(Y, y) \hookrightarrow (C^n, o_n)$, and $\tau^k_{Y^*}(J^k_yY) = J^k_Y$ where $\tau^k_Y = \tau^{k,l}_C|_{J^k_Y}$ and $l \leq k$. Furthermore, any morphism $\varphi : (Y, y) \to (Z, z)$ generates a
morphism \( \varphi^{(k)} : J^k_y Y \to J^k_z Z \).

Proof. For the first statement note that \( \hat{J}_y Y \) is given in \( \hat{J}_{o_n} C^n \) by a countable number of polynomial equations on the coefficients of the coordinates of formal jets \( \hat{j} = (\hat{j}_1, \ldots, \hat{j}_n) \). This implies that \( \tau^k_{o_n}(\hat{J}_y Y) \) is the intersection of at most countable number of constructive sets whence the closure of \( J^k_y Y \) is an algebraic variety. The other statements are immediate consequence of the definition. \( \square \)

4.3. Consider a coordinate form \((j_1(t), \ldots, j_n(t))\) of a \( j \in J^k_{o_n} C^n \) where \( t \in (C, 0) \) and each \( j_i \) is a polynomial in \( t \) of degree at most \( k \). We say that the multiplicity of \( j \) is \( m = \min\{s : \exists t : \frac{d^s}{dt^s} j(t) \neq 0\} \). The subset of jets of multiplicity \( m \) in \( J^k_{o_n}(C^n) \) will be denoted by \( J^k_{o_n,m}(C^n) \). Notation \( \hat{J}^m_{o_n} C^n, J^m_y Y, J^m_{y,m} Y \) have the similar meaning. For any \( h \) from the defining ideal of \((Y, y)\) in \((C^n, o_n)\) consider its homogeneous decomposition \( h = h_0 + h_1 + \ldots \) where \( h_0 \) is the minor homogeneous form. One can treat the tangent space of \( C^n \) at \( o_n \) as 1-jets. Then the reduced tangent cone \( C_y Y \) consists of all 1-jets \( j(t) \) such that \( h_0 \circ j(t) = 0 \) for any \( h_0 \) as above. This implies.

Lemma. Every \( j \in J^k_y Y \) is of form \( j(t) = j^1(t^k) \) where \( j^1 \in C_y Y \).

4.4. Let \( \sigma : \tilde{C}^n \to (C^n, o_n) \) be the blowing-up of \((C^n, o_n)\) at \( o_n \) and \( E \) be its exceptional divisor. Let \((\xi_1, \ldots, \xi_n)\) be a coordinate system on \((C^n, o_n)\). Then \( \tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n) = (\xi_1, \xi_2/\xi_1, \ldots, \xi_n/\xi_1) \) is a local coordinate system on \( \tilde{C}^n \). Without loss of generality we can suppose that \( \frac{d^m}{dt^m} J_1(0) \neq 0 \). Then \((j_1, j_2/j_1, \ldots, j_n/j_1)\) can be viewed as an \( n \)-tuple of power series so that the first \((k - m)\) terms of every entry are well-defined. This enables us to define for \( l = 0, \ldots, k - m \) morphism \( \theta_{n,m,l}^k : J^k_{o_n} C^n \to J^l_E \tilde{C}^n \) such that in this local coordinate system \( \tilde{\xi} \) we have \( \theta_{n,m,l}^k(j) = ([j_1], [j_2/j_1], \ldots, [j_n/j_1]) \) where for every power series \( a \) we denote by \([a]_l\) the sum of its first \( l \) terms. Let us discuss the dimension of fibers of \( \theta_{n,m,k-m}^k \). Among the last \( m \) coefficients of \( j_1 \) (which is a polynomial of degree at most \( k \)) there are at most \( \min(m, k - m + 1) \) nonzero ones. Knowing these coefficients and \( \theta_{n,m,k-m}^k(j) \) one can recover \( j \). Thus fibers of \( \theta_{n,m,k-m}^k \) are of dimension \( \min(m, k - m + 1) \).

For formal jets we define the similar morphism \( \hat{\theta}_{n}^m : \hat{J}^m_{o_n} C^n \to \hat{J}^m_E \tilde{C}^n := \bigcup_{\hat{y} \in E} \hat{J}_y \hat{C}^n \) given locally by \( (\hat{j}_1, \ldots, \hat{j}_n) \to (\hat{j}_1, \hat{j}_2/\hat{j}_1, \ldots, \hat{j}_n/\hat{j}_1) \).

4.5. As usual we consider a closed embedding \((Y, y) \hookrightarrow (C^n, o_n) \). Let \( \hat{Y} \) as a proper transform of \( Y \), \( E_Y = E \cap \hat{Y} \), and \( \sigma_Y = \sigma|_{\hat{y}} \), i.e. \( \sigma_Y : \hat{Y} \to Y \) is the blowing-
up of \((Y, y)\) at \(y\). Put \(\theta^{k,m,l}_Y = \theta^{k,m,l}_n|_{j^k_m Y}\) and \(\hat{\theta}^m_Y = \hat{\theta}^m_n|_{j^m Y}\). It is easy to see that \(\theta^{k,m,0}_Y(j^{k,m} Y) \subset E_Y\).

**Lemma.** For every \(\tilde{y} \in E_Y\) the fiber \(E^{\tilde{y}} = (\theta^{k,m,0}_Y)^{-1}(\tilde{y})\) is of dimension at most \(\dim j^{k-m}\tilde{Y} + \min(m, k - m + 1)\) and for any \(l = 0, \ldots, k - m\) the image \(\theta^{k,m,l}_Y(E^{\tilde{y}})\) is contained in \(j^{l}\tilde{Y}\).

**Proof.** The first statement follows from the second one for \(l = k - m\), the fact that \(\theta^{k,m,l}_Y = \tau_{Y,l}^Y \circ \theta^{k,m,r}_Y\) for \(r > l\), and the remark about the dimension of \(\theta^{n,m,k-m}\)-fibers in 4.4. For the second statement put \(\tilde{j} = \theta^{k,m,l}_Y(j)\) where \(j \in J^{k,m}_Y\). Consider a formal extension \(\tilde{j} \in \tilde{J}_Y\) of \(j\), i.e. for every regular function \(h\) from the defining ideal of \((Y, y)\) in \((C^n, o_n)\) we have \(h \circ \tilde{j} = 0\). Note that \(\theta^m_Y(\tilde{j})\) is a formal extension of \(\tilde{j}(t)\).

Suppose that \(j_i\) and \(\tilde{\xi}\) are as in 4.3 and 4.4, and \(\frac{d^{m}}{dt} j_1(0) \neq 0\). Then for every regular function \(h\) from the defining ideal of \((\tilde{Y}, \tilde{y})\) there exist \(h\) as above and \(s > 0\) so that \(h(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_l) = \xi^s_i h(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)\). Hence \(h \circ \hat{\theta}^m_Y(\tilde{j}) = 0\) and \(\tilde{j} \in j^{l}\tilde{Y}\). \(\Box\)

Induction on \(k\) and Lemma 4.5 imply.

**Corollary.** The dimension of \(J^{k,m}_Y(Y)\) is at most \((k - m + 1) \dim Y + \min(m - 1, k - m)\). In particular, \(\dim J^k_Y(Y) \leq k \dim Y\).

**4.6. Lemma.** Let \(\varphi : (Y, y) \to (Z, z)\) be a morphism, \(j \in J^{k,m}_Y(Y)\) be such that the multiplicity of \(\varphi^{(k)}(j)\) is \(m\) (i.e. \(\varphi^{(k)}(j) \in J^{k,m}_Z(Z)\)). Let \(\sigma : \tilde{Y} \to Y\) (resp. \(\delta : \tilde{Z} \to Z\)) be the blowing-up of \(Y\) at \(y\) (resp. \(Z\) at \(z\)) and \(\psi : \tilde{Y} \to \tilde{Z}\) be the rational map generated by \(\varphi\). Then \(\psi\) is regular at \(\tilde{y} = \theta^{k,m,0}_Y(j)\) and sends it to \(\tilde{z} = \theta^{k,m,0}_Z(\varphi^{(k)}(j))\).

Furthermore, for \(E^{\tilde{y}} = (\theta^{k,m,0}_Z)^{-1}(\tilde{y})\) we have \(\theta^{k,m,l}_Z \circ \varphi^{(k)}|_{E^{\tilde{y}}} = \psi(0) \circ \theta^{k,m,l}_Y|_{E^{\tilde{y}}}\).

**Proof.** Let \((Y, y) \hookrightarrow (C^n, o_n)\) and \(\tilde{C}^n\) as in 4.5. In particular, \(\tilde{Y}\) can be viewed as a subvariety of \(\tilde{C}^n\). Let \((C^*, o_s)\) and \(\tilde{C}^s\) play the similar role for \((Z, z)\). Then \(\varphi\) is a restriction of a morphism \(\Phi : (C^n, o_n) \to (C^*, o_s)\) which generates a rational map \(\Psi : \tilde{C}^n \to \tilde{C}^s\) such that \(\psi\) is the restriction of \(\Psi\). Thus we can suppose that \((Y, y) = (C^n, o_n)\) and \((Z, z) = (C^*, o_s)\). Let \(\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)\) (resp. \(\tilde{\zeta} = (\tilde{\zeta}_1, \ldots, \tilde{\zeta}_s)\) be a local coordinate system on \(\tilde{C}^n\) (resp. \(\tilde{C}^s\)). Making linear coordinate changes we can suppose that \(\tilde{y} = (1, 0, \ldots, 0)\) in this local coordinate system \(\tilde{\xi}\) (resp. \(\tilde{z} = (1, 0, \ldots, 0)\) in \(\tilde{\zeta}\)) and \(\sigma(\tilde{\xi}) = (\tilde{\xi}_1, \tilde{\xi}_2 \tilde{\xi}_3, \ldots, \tilde{\xi}_l)\) (resp. \(\delta(\tilde{\zeta}) = (\tilde{\zeta}_1, \tilde{\zeta}_2 \tilde{\zeta}_3, \ldots, \tilde{\zeta}_s)\)). This implies that locally the coordinate form of \(\psi\) is \((\varphi_1 \circ \sigma, \varphi_2 \circ \sigma/\varphi_1 \circ \sigma, \ldots, \varphi_s \circ \sigma/\varphi_1 \circ \sigma)\) where \(\varphi = (\varphi_1, \ldots, \varphi_s)\), and for every \(j = (j_1, \ldots, j_n) \in E^{\tilde{y}}\) we have \(\frac{d^{m}}{dt} j_1(0) \neq 0\) (resp.
Thus changing the coordinate $t$ on $(C, 0)$ we can suppose that $j_1(t) = t^n$. Recall that for every power series $a(t)$ the sum of its first $l$ terms is denoted by $[a]_l$. Treating each $j_i$ as a polynomial of degree at most $k$ we have $\theta^{k, m, l}_s(\varphi(k)(j)) = ([\varphi \circ j]_1, [\varphi_2 \circ j / \varphi_1 \circ j]_1, \ldots, [\varphi_s \circ j / \varphi_1 \circ j]_1)$. Since $j_1(t) = t^n$ we have on the other hand $\sigma \circ \theta^{k, m, l}_n(j) = ([j_1], [j_2/j_1]_1, \ldots, [j_\alpha/j_1]_1[\bar{j}_1]_1) = ([j'_1], [j'_2/j_1]_1, \ldots, [\bar{j}_\alpha/j_1]_1[\bar{j}_1]_1)$, where $j'_1 = [j_1]_{l+m}$ for $l \geq m$, and $j'_1$ is zero for $l < m$. Hence $\psi^{(t)} \circ \theta^{k, m, l}_n(j) = ([\varphi \circ j]_1, [\varphi_2 \circ j'/\varphi_1 \circ j']_1, \ldots, [\varphi_s \circ j'/\varphi_1 \circ j']_1)$. As $\frac{d^m}{dt^m}(\varphi \circ j)(0) \neq 0$ one can see that the last expression coincides with those for $\theta^{k, m, l}_s(\varphi(k)(j))$.

4.7. Let $h : Y_1 \to Y_2$ be a morphism of algebraic varieties, $y_1 \in Y$, and $y_2 = h(y_1)$. Then $h$ generates a morphism $h_* : C_{y_1} Y_1 \to C_{y_2} Y_2$ of the reduced tangent cones at $y_1$ and $y_2$ respectively where $h_*$ is just the restriction of the induced linear map of the tangent spaces $T_{y_1} Y_1 \to T_{y_2} Y_2$. It is known [D, Ch. 2.5.2] that if $h$ is not unramified at $y_1$ (in particular, when it $y_1$ is not a connected component of $h^{-1}(y_2)$) then the induced map of (non-reduced) tangent cones is not an embedding. We need a similar claim for reduced tangent cones.

Lemma. Let $h : (Y_1, y_1) \to (Y_2, y_2)$ be a morphism such that $y_1 \neq h^{-1}(y_2) \cap Z_1$ for some irreducible analytic branch $(Z_1, y_1)$ of $(Y_1, y_1)$. Let $(Z_2, y_2)$ be the proper transform of $(Z_1, y_1)$ under $h$.

1. Then $h_*$ is not an embedding.

2. Let $V_i$ be the subspace of $T_{y_i} Y_i$ generated by $C_{y_i} Z_i$ and dim $V_1 \leq$ dim $V_2$. Then the closure of $h_*(C_{y_1} Z_1)$ is a proper subvariety of $C_{y_2} Z_2$, i.e. dim $h_*(C_{y_1} Z_1) <$ dim $Y_2$.

Proof. Let $Y_i$ be a closed subvariety of $C^n$ with coordinates $x_1, \ldots, x_n$ so that $y_i$ is the origin. Consider the homotety $(x_1, \ldots, x_n) \to (tx_1, \ldots, tx_n)$ where $t \in C^*$ and the image of $Y_i$ in $C^n \simeq C^n \times t$ under it. The closure of the union of these images is a subvariety $\tilde{Y}_i$ of $C^{n+1} \simeq C^n \times C_t$ such that for the natural projection $\tau_1 : \tilde{Y}_i \to C$ to the $t$-axis, $\tau_1^{-1}(0)$ is isomorphic to $C_{y_i} Y_i$ and there is an isomorphism $\varphi_i : Y_i \times C^* \to \tau_1^{-1}(C^*)$ over $C^*$ (e.g., see [D, Ch. 3.6.2]). Moreover, $h$ generates a morphism $\tilde{h} : \tilde{Y}_1 \to Y_2$ such that $\tau_1 = \tau_2 \circ \tilde{h}, \varphi_2^{-1} \circ \tilde{h} \circ \varphi_1|_{Y_1 \times t} = h$ for nonzero $t$, and $\tilde{h}|_{\tau_1^{-1}(0)} = h_*$. The closure $\tilde{Z}_i$ of $\varphi_i(Z_i \times C^*)$ is an irreducible analytic subvariety of $\tilde{Y}_i$ and $\tilde{Z}_i \cap \tau_1^{-1}(0)$ is isomorphic to $C_{y_i} Z_i$. By the assumption for any fixed $t \in C^*$ the variety $\tilde{Z}^t_1 := \tilde{Z}_1 \cap \tilde{h}^{-1}(\varphi_2(y_2 \times t)) \neq \varphi_1(y_1 \times t)$, i.e. $\tilde{Z}^t_1$ is at least of dimension 1 and the closure $\tilde{Z}_1$ of $\bigcup_{t \in C^*} \tilde{Z}^t_1$ is at least of dimension 2. Hence $\tilde{Z}^0_1 = \tilde{Z}_1 \cap \tau_1^{-1}(0)$
contains a curve. Let \( v_i \) be the vertex of \( C_y Y_i \). Note that \( \varphi_i(y_i \times t) \) approaches \( v_i \) as \( t \to 0 \). Hence by continuity \( \hat{h}(\hat{Z}_i^t) = v_2 \), and, therefore, \( h_* \) is not an embedding.

As \((Z_i, y_i)\) is an irreducible analytic branch of \((Y_i, y_i)\) it follows easily from [M, Ch. 5A] that \( C_y Z_i \) is irreducible. Hence if the closure of \( h_*(C_y Z_i) \) is not a proper subvariety, it coincides with \( C_y Z_2 \) and, therefore, \( h_*(C_y Z_1) \) generates \( V_2 \). This implies that \( h_*(V_1) = V_2 \) whence \( h_*|_{V_1} : V_1 \to V_2 \) is an isomorphism as \( \dim V_1 \leq \dim V_2 \). Thus the restriction of \( h_* \) to \( C_y Z_1 \subset V_1 \) is an embedding contrary to (1). \( \Box \)

4.8. Lemma. Let \((Y_1, y_1) \to (Y_2, y_2) \to \ldots \to (Y_s, y_s)\) be a sequence of birational morphisms of germs of algebraic varieties and \( g_{i_1, i_2} : (Y_{i_1}, y_{i_1}) \to (Y_{i_2}, y_{i_2})\) be the composite morphisms for \( i_1 < i_2 \). Suppose that \( \dim Y_i = n \geq 2 \) and \( g_{i_1, i_2}^{-1}(y_{i_1}) \) contains a curve. Let \( v_i \) be the vertex of \( C_y Y_i \). Note that \( \varphi_i(y_i \times t) \) approaches \( v_i \) as \( t \to 0 \). Hence by continuity \( \hat{h}(\hat{Z}_i^t) = v_2 \), and, therefore, \( h_* \) is not an embedding.

4.9. In order to generalize the above Lemma to the case of \( k \)-jet cones we need

Lemma. Let \((Y, y)\) be an irreducible germ of an analytic set, \( \sigma : \hat{Y} \to (Y, y) \) be its blowing-up at \( y \), and \( E_Y \) be the exceptional divisor. Then \( E_Y \cap (\hat{Z}, \hat{y}) \) is a divisor in \( \hat{Y} \) where \((\hat{Z}, \hat{y})\) is any irreducible analytic branch of \( \hat{Y} \) at any point \( \hat{y} \in E_Y \).

Proof. Consider the union \( U \) of all irreducible germs \((\hat{Z}, \hat{y}), \hat{y} \in E_Y \) of \( \hat{Y} \) that do not contain an open subset of \( E_Y \). If \( U \neq \emptyset \) then \( U \) is a proper analytic subset of \( \hat{Y} \) of the same dimension. Hence if \( \nu : \hat{Y}_\nu \to \hat{Y} \) is normalization then \( \hat{Y}_\nu \) contains at least two connected components: the proper transform \( U' \) of \( U \) and another component \( U'' \) such that \( \nu(U'') \supset E_Y \). There is a natural proper morphism from \( \hat{Y}_\nu \) into the normalization \( Y_\nu \) of \((Y, y)\). As \((Y, y)\) is irreducible the preimage \( y_\nu \) of \( y \) in \( Y_\nu \) is a point. But the preimage of \( y_\nu \) in \( \hat{Y}_\nu \) is not connected (it has points in both \( U' \) and \( U'' \) in contradiction with the Zariski Main Theorem. \( \Box \)

4.10. Lemma. Let the assumption of Lemma 4.8 hold and \( k > 0 \). Suppose that \( s = (2l)^{k-1}l \) and \( l \geq \max Z \dim T Z \) where \( Z \) is the result of any sequence of \( r \) blowing-ups of any \((Y_i, y_i)\) at \( y_i \) and infinitely near points with \( 1 \leq i \leq s \) and \( 0 \leq r \leq k - 1 \).
Then \( \dim g^{(k)}_{1,s}(J^{k,m}_{y_i}Y_1) \leq (k-m+1)(n-1) + \min(m-1,k-m) \) for every \( 1 \leq m \leq k \). In particular, \( \dim g^{(k)}_{1,s}(J^k_{y_i}Y_1) \leq k(n-1) \).

Proof. Let \( k = m \). Then every \( j \in J^{k,k}_{y_i}Y_1 \) is of form \( j = j_0 \circ h \) where \( j_0 \in C_{y_i}Y_i \) and \( h : C \to C, t \to t^k \). Hence \( g^{(k)}_{i_1,i_2}(j) = g^{(1)}_{i_1,i_2}(j_0) \circ h \). In this case the statement follows from Lemma 4.8. In particular, Lemma is true for \( k = 1 \). We use now induction on \( k \) and inside it induction on \( k-m \). Let \( s_0 = s/2+1 \) and for \( i < s_0 \) let \( S^0_i \) be the subvariety of \( J^{k,m}_{y_i}Y_i \) such that \( g^{(k)}_{i,s_0}(S^0_i) \subset J^{k,m+1}_{y_{s_0}}Y_{s_0} \). By induction \( g^{(k)}_{1,s}(S^0_1) \subset g^{(k)}_{1,s}(J^{k,m+1}_{y_{s_0}}Y_{s_0}) \) is of dimension at most \( (k-m)(n-1) + \min(m,k-m-1) \). Thus (since \( n > 1 \)) it suffices to consider jets from \( S_1 \) where \( S_1 = J^{k,m}_{y_i}Y_i \setminus S^0_1 \).

Let \( \tilde{Y}_i \) be the blowing-up of \( Y_i \) at \( y_i \). Its exceptional divisor \( E_i \) is naturally isomorphic to the base of the cone \( C_{y_i}Y_i \) and \( g_{i_1,i_2} \) generates a birational map \( h_{i_1,i_2} : \tilde{Y}_{i_1} \to \tilde{Y}_{i_2} \). Deleting the indeterminacy points (i.e. replacing \( E_i \) by its Zariski open subset \( E_i^* \)) we can suppose that \( h_{i_1,i_2} \) is regular on \( \tilde{Y}_{i_1}^* = (\tilde{Y}_i \setminus E_i) \cup E_i^* \) for \( i_2 \leq s_0 \). Note that for every \( j \in S_{i_1} \) the multiplicity of \( g_{i_1,i_2}(j) \) is \( m \) whence by Lemma 4.6 \( \theta_{y_{i_1}}^{k,m,0}(S_{i_1}) \subset E_{i_2}^* \). By Lemma 4.8 for every \( (2l)^{k-1} > q \geq 0 \) there exist \( lq < i_1 < i_2 \leq l(q+1) \) such that the dimension of \( g^{(1)}_{i_1,i_2}(C_{y_i}Y_i) \) is at most \( n-1 \) whence \( \dim h_{i_1,i_2}(E^*_{i_2}) \leq n-2 \), i.e. \( E_{i_2}^* \) is the exceptional divisor of \( h_{i_1,i_2} \). Put \( e_{i_1,i_2} = h_{i_1,i_2,i_2} \) and \( Z_i = \tilde{Y}_{i_2}^* \). We get a sequence of birational morphisms \( Z_1 \to Z_2 \to \ldots \to Z_{s_1} \) where \( s_1 = (2l)^{k-2}l \). Note that \( E_{i_1} \) is an exceptional divisor of \( e_{i_1,i_2} \) and, by Lemma 4.9 it meets every irreducible analytic branch \((Z^1_i, z_i)\) of \((Z_i, z_i)\) where \( z_i \) is any point of \( E_{i_1}^* \). Thus this new sequence of birational morphisms satisfies the assumption of this Lemma. By induction \( \dim e^{(k-m)}_{1,s_1}(J^{k-m}_{z_1}Z_1) \leq (k-m)(n-1) \), and Lemma 4.6 implies that \( \theta_{y_{i_2}}^{k,m,k-m} \circ g^{(k)}_{1,s_1}(S_{i_1}) \subset \bigcup_{y_{i_2} \in E_{i_2}} h^{(k-m)}_{1,s_1}(J^{k-m}_{y_{i_2}}Y_{i_2}) \subset \bigcup_{z_{i_2} \in h_{1,i}(E_{i_2})} e^{(k-m)}_{1,s_1}(J^{k-m}_{z_1}Z_1) \). As \( \dim h_{1,i}(E_{i_2}^*) \leq n-2 \), we have \( \dim \theta_{y_{i_2}}^{k,m,k-m} \circ g^{(k)}_{1,s_1}(S_{i_1}) \leq (k-m+1)(n-1) - 1 \). Taking into consideration the remark about the dimension of \( \theta_{i}^{n,m,k-m} \)-fibers in 4.4 we get the desired conclusion. □

5. The proof of Theorems B and A.

5.1 By Lemma 2.1 we can suppose that \( X \) and \( Y \) are normal in Theorem B. In the case when \( n = \dim Y = 1 \) the result follows from the fact that a bijective morphism of smooth curves is an isomorphism. Consider \( n > 1 \). Suppose that under the assumption of Theorem B (i) \( f \) is an automorphism and \( g \) is not. By Corollary 2.4 there exists an exceptional divisor \( D \) for \( g \). Assume \( x \) be a general point in \( \rho^{-1}(D) \neq \emptyset \).
In particular, \( x_s = f^s(x) \) is also a general point in \( \rho^{-1}(D) \). Let \( \psi : Y \to \mathbb{C}^n \) be a dominant morphism. As \( f^s \) is an automorphism \( \dim((\psi \circ \rho \circ f^s)^{(k)}(J^k_X)) = (\psi \circ \rho)^{(k)}(J^k_X) \) as both \( x \) and \( x_s \) are general points. By Proposition 3.2 there exists \( n_0 \) such that for any \( k \) we have \( \dim((\psi \circ \rho)(J^k_X)) \geq kn - n_0 \). On the other hand \( Y \) is locally analytically irreducible since it is normal. By Lemma 4.10 \( \dim((\psi \circ g^s \circ \rho)(J^k_X)) \leq k(n-1) \) for sufficiently large \( s \). Since \( \psi \circ \rho \circ f^s = \psi \circ g^s \circ \rho \) we get a contradiction which proves that \( g \) is an automorphism. The proof of Theorem B (ii) is similar. \( \square \)

5.2. Theorem B yields Theorem A in the case of \( k = \mathbb{C} \). We need to reduce the general case to this one. Let \( \bar{k} \) be the algebraic closure of the field \( k \). Recall that \( \bar{k} \) is a faithfully flat \( k \)-module. This means that an endomorphism \( \varphi : T \to T \) of a \( k \)-algebra \( T \) is an automorphism iff the endomorphism \( \varphi \otimes_k Id_{\bar{k}} : T \otimes_k \bar{k} \to T \otimes_k \bar{k} \) is an automorphism. Thus we can replace the rings \( S \) and \( R \) in Theorem A by \( S \otimes_k \bar{k} \) and \( R \otimes_k \bar{k} \) respectively. That is, we can suppose from the beginning that \( k \) is algebraically closed. We consider case (i) only since the other case is similar. It is equivalent to the analogue of Theorem B (i) in which \( X \) and \( Y \) are already affine algebraic varieties over \( k \). Note that Lemmas 2.1 and 2.2 hold for every algebraically closed field whence we can suppose that \( Y \) is normal and \( g \) is birational. Hence if we assume that \( g \) is not an automorphism then a coordinate function of \( g^{-1} \) has a pole at a point \( y_0 \in Y \). Let \( k' \) be the subfield of \( k \) generated by a finite number of elements which include the coordinates of \( y_0 \) in the ambient Euclidean space, the coefficients of coordinate functions of \( \rho, g, f \) and \( f^{-1} \) (as polynomials over \( k \)), and the coefficients of generators of the defining ideals of \( X \) and \( Y \). Consider our varieties and morphisms over \( k' \) instead of \( k \) and denote the corresponding objects by \( X', Y', f', g', \) and \( \rho' \). Note that \( g' \) is not an automorphism as \( y_0 \in Y' \). But \( k' \) can be embedded as a subfield in \( \mathbb{C} \) by the “Lefschetz principle” [BCW]. Hence theorem B (i) implies that the coordinate functions of \((g')^{-1}\) cannot have a pole at \( y_0 \). Contradiction. \( \square \)

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