Character expansion of matrix integrals

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Abstract

We consider expansions of certain multiple integrals and BKP tau functions in characters of orthogonal and symplectic groups. In particular we consider character expansions of integrals over orthogonal and over symplectic matrices.

Keywords: random matrices, characters of the symplectic and orthogonal groups, TL and BKP tau functions, multiple integrals

1. Introduction

The character expansion of the matrix models used in physics was first presented in the works [16, 34] and was used in [25–27, 38, 57, 59] for various problems, in particular in the context of relationships between matrix models and integrable systems. In all the works mentioned above, expansions in terms of Schur functions were used. The importance of such representation was shown in a set of papers, for instance, in the study of matrix models [34, 47, 74], for random processes and random partitions [3, 28], in communications [7, 9, 71], for counting problems [6, 20–23, 49, 52, 58], for relations of quantum and classical models [4, 5, 10, 32, 73] and some others. Here we write expansions for the characters of orthogonal and symplectic groups. We hope that this will also be useful.

Technically, our work is based on the Cauchy identities for the symplectic and orthogonal groups (31)–(33) found by Littlewood, see [44].

1.1. Some notations

Let us recall that the characters of the unitary group $U(n)$ are labeled by partitions and coincide with the so-called Schur functions [45]. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a set of nonnegative
integers $\lambda_i$ which are called parts of $\lambda$ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of $\lambda$ is called the length of the partition $\lambda$, and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum \lambda_i$ is called the weight of $\lambda$. The set of all partitions will be denoted by $\mathcal{P}$.

The Schur function corresponding to $\lambda$ is defined as the following symmetric function in $\lambda$ variables $x = (x_1, \ldots, x_n)$:

$$s_\lambda(x) = \frac{\det [x_\lambda^{-i-j+n}]}{\det [x_i^{-i-j+n}]}_{i,j}$$

(1)

in case $\ell(\lambda) \leq n$ and vanishes otherwise. One can see that $s_\lambda(x)$ is a symmetric homogeneous polynomial of degree $|\lambda|$ in the variables $x_1, \ldots, x_n$.

**Remark 1.** In case the set $x$ is the set of eigenvalues of a matrix $X$, we also write $s_\lambda(X)$ instead of $s_\lambda(x)$.

There is a different definition of the Schur function as a quasi-homogeneous non-symmetric polynomial of degree $|\lambda|$ in other variables, $\mathbf{p} = (p_1, p_2, \ldots)$:

$$s_\lambda(\mathbf{p}) = \det [s_{\lambda_i-j}(\mathbf{p})]_{i,j}$$

(2)

and the Schur functions $s_{(i)}$ are defined by $\sum_{m \geq 0} x^m = \sum_{m \geq 0} s_{(i)}(\mathbf{p}) x^i$. The Schur functions defined by (1) and by (2) are equal, $s_\lambda(\mathbf{p}) = s_\lambda(x)$, provided that the variables $\mathbf{p}$ and $x$ are related by

$$p_m = \sum_i x_i^m.$$  

(3)

From now on, we will use, where the argument of $s_\lambda$ is written in bold, the definition (2), and we imply the definition (1) otherwise.

**Remark 2.** For functions $f(\mathbf{p}) = f(\mathbf{p}(A))$, where $p_m(A) := \text{Tr } A^m$, $m = 1, 2, \ldots$ and $A$ is a given matrix we may equally write either $f(\mathbf{p}(A))$ or $f(A)$ where the capital letter implies a matrix. In particular, under this convention we may write $s_\lambda(A)$ and $\tau(A)$ instead of $s_\lambda(\mathbf{p}(A))$ and $\tau(\mathbf{p}(A))$.

12. Integrals over the unitary group. Unitary matrix model

The unitary matrix model was introduced in [12] (see also [48]). It is the following integral over the unitary group which depends on two semi-infinite sets of variables $\mathbf{p} = (p_1, p_2, \ldots)$ and $\mathbf{p}^* = (p_1^*, p_2^*, \ldots)$, which are free parameters

$$I_{U(n)}(\mathbf{p}, \mathbf{p}^*) := \int_{U(n)} e^{i\mathbf{p}^* \mathbf{U}\mathbf{p}} dU$$

(4)

$$= \frac{1}{(2\pi)^n} \int_{0 \leq \theta_1 \leq \ldots \leq \theta_n \leq 2\pi} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{-i\theta_k}|^2 \prod_{j=1}^n e^{\sum_{m>0} \frac{1}{m} (p_m e^{i\theta_j} + p_m^* e^{-i\theta_j})} d\theta_j$$

(5)

$$V(\mathbf{p}, x) := \sum_{n>0} \frac{1}{n} p_n x^n.$$  

(6)
Here, \( d_U \) is the Haar measure of the group \( \mathbb{U}(n) \), see (A.35) in the appendix, and \( e^{i\theta_1}, \ldots, e^{i\theta_n} \) are eigenvalues of \( U \in \mathbb{U}(n) \). The exponential factors inside the integral may be treated as a perturbation of the Haar measure and parameters \( p, p^* \) are called coupling constants. Being rewritten as the integral over the eigenvalues of \( U \in \mathbb{U}(n) \), the integral (4) may also be treated as an integrable deformation of the known \( \beta = 2 \) circular ensemble, where the deformation parameters \( p_i = g^{-2}\delta_i \) may be identified with the so-called higher times of the relativistic Toda lattice [72]. Where \( p_i = g^{-2}\delta_i \), the integral (4) coincides with Wilson’s lattice version of 2D quantum chromodynamics, \( g \) being the coupling constant [13, 14, 22].

Using the Cauchy–Littlewood identity
\[
\tau(p|p^*) := e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m p_m^*} = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(p^*) s_{\lambda}(p)
\]
and the orthogonality of the irreducible characters of the unitary group
\[
\int s_{\lambda}(U)s_{\mu}(U^{-1})d_U = \delta_{\lambda,\mu},
\]
we obtain that
\[
I_{U(n)}(p, p^*) = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(p)s_{\lambda}(p^*),
\]
which expresses the integral over unitary matrices as the perturbation series in coupling constants.

The formula (9) first appeared in [47] in the context of the study of the Brezin–Gross–Witten model [14, 22]. It was shown there that the integral \( I_{U(n)}(p, p^*) \) is equal to a certain Toda lattice tau function of [31] and [72]. Then, the series in Schur functions (9) can also be related to the double Schur functions series, found in [68] and [69].

Note that the series on the right-hand side of (9) may be viewed as the partition function related to the so-called Schur measure introduced in [50] and studied intensively (see, [11, 51, 53]). Let us also mention that the idea of the Schur measure has been widely used and developed in many directions. We apologize for not giving references to the numerous interesting papers in the topic (we can only mention the names of some of the researchers who crucially contributed the topic: Okounkov, Borodin, Tracy, Widom, Adler, van Moerbeke, Forrester and others). We focus on the connections of various series over partitions similar to (9), topics of random matrices and multiple integrals to various integrable hierarchies. In this context let us note the links between the KP and Toda lattice hierarchies and matrix integrals [19, 35], and the matrix integrals together with the sums over partition viewed as perturbation series in coupling constants, see [6, 25, 27, 38, 54, 57], and many others. Next, we mention the CKP hierarchy of [31] which gives rise to the series of partitions described in [41]. The BKP hierarchy of the Kyoto school [31] results in a series of strict partitions of projective Schur functions [29, 42, 55], which may be viewed as modifications of (9) (see [70]). The ‘large’ BKP hierarchy may be related to a number of matrix integrals (see [2, 40, 61, 62]) and to different modifications of (9) described in [60] and used in [49, 58]. Here, we continue the study of both the KP and of the ‘large’ BKP hierarchy of [33]. Our goal is to replace the Schur functions by the characters of symplectic and orthogonal groups. In particular, we express integrals over the symplectic and over the orthogonal groups, \( I_{\mathbb{Sp}(N)}(p) \) and \( I_{\mathbb{U}(N)}(p) \), respectively, as sums of the product of characters of orthogonal and symplectic groups, i.e. to obtain the analogues

\[ \text{Note that the DKP hierarchy has other names. It was rediscovered in [2] using the approach different of [31] and called the Pfaff lattice. It was also called the coupled KP equation in [30].} \]
of the relation (9) and relate these integrals and sums to integrable systems. We shall relate $I_{sp(p)}(p)$ and $I_{O(2n)}(p)$ to the DKP, and we shall relate $I_{O(2n+1)}(p)$ to the BKP tau functions introduced respectively in [31] and [33] and obtain Pfaffian representation for these integrals. On the other hand, one can relate these integrals to the Toda lattice (TL) tau function [31, 72] which yields the determinant representation.

2. Polynomials $o_\lambda(p)$ and $sp_\lambda(p)$ and TL tau functions $\tau_{\pm}(p|p^*)$

The orthogonal and symplectic characters are also labeled by partitions. They are given respectively by the following expressions

$$o_\lambda = \frac{\det \left[ x_{ij}^{\lambda_i+n-i+1} - x_{ij}^{-\lambda_i-n+i} \right]}{\det \left[ x_{ij}^{\lambda_i+n-i+1} - x_{ij}^{-\lambda_i-n+i} \right]}_{1 \leq i, j \leq n}$$

(10)

and

$$sp_\lambda = \frac{\det \left[ x_{ij}^{\lambda_i+n-i+1} - x_{ij}^{-\lambda_i-n+i-1} \right]}{\det \left[ x_{ij}^{\lambda_i+n-i+1} - x_{ij}^{-\lambda_i-n+i-1} \right]}_{1 \leq i, j \leq n}$$

(11)

if $\ell(\lambda) \leq n$ and vanish otherwise. See [18] or appendix A.4 for more information. Baker [8] realized that these characters can be obtained from the corresponding Schur functions $s_\lambda$ by the action of some operator. For this, it will be convenient to use the Schur functions in terms of the power sums $p_m$. As usual, we write $\tilde{\partial} = (\partial_{p_1}, 2\partial_{p_2}, 3\partial_{p_3}, \ldots)$. Let

$$\Omega_\pm(p) = \sum_{m \geq 0} \left( -\frac{1}{2m} p_m^2 \mp \frac{1}{2m} p_{2m} \right), \quad \Omega_\pm(\tilde{\partial}) := \sum_{m \geq 0} \left( -\frac{m}{2} (\partial_m)^2 \mp \partial_{2m} \right).$$

(12)

then

$$o_\lambda(p) = e^{\Omega_-(\tilde{\partial})} \cdot s_\lambda(p), \quad sp_\lambda(p) = e^{\Omega_+(\tilde{\partial})} \cdot s_\lambda(p).$$

(13)

Hence, if we let the operator $\Omega_\pm(\tilde{\partial}^*)$ act on the Cauchy–Littlewood identity (7), we obtain

$$\tau_-(p|p^*) = \sum_\lambda o_\lambda(p^*) s_\lambda(p)$$

(14)

and

$$\tau_+(p|p^*) = \sum_\lambda sp_\lambda(p^*) s_\lambda(p),$$

(15)

where

$$\tau_+(p|p^*) = e^{\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m^2} p_m^2 \mp \sum_{m=1}^{\infty} \frac{1}{m^2} p_{2m} \mp \sum_{m=1}^{\infty} \frac{1}{m^2} p_{2m}^2}.$$  

(16)

Remark 3. Note that

$$\tau_+(p|p^*) = e^{\Omega_+(\tilde{\partial})} \tau_0(p|p^*),$$

(17)

where $\tau_0(p|p^*) = e^{\sum_{m=1}^{\infty} \frac{1}{m} p_m^2}$ is known to be the simplest tau function of the TL hierarchy (this simplest tau function does not depend on the discrete TL time $p_0$).
It is well known that the function \( \tau_0(p[p^*]) \), for the variables \( p = (p_1, p_2, \ldots) \), is a solution of the Hirota bilinear equations for the KP hierarchy:

\[
\int \frac{dz}{2\pi i} e^{V(p' - p - z)} \tau_0(p' - [z^{-1}][p^*]) \tau_0(p + [z^{-1}][p^*]) = 0. \tag{18}
\]

Here, \( V \) is given by (6) and the variables \( p^* = (p_1^*, p_2^*, \ldots) \) play the role of auxiliary parameters. Here and below the notation \( [a] \) serves to denote the following set of power sums: \( (a, a^2, a^3, \ldots) \). The action of \( e^{\Omega_z(\bar{\sigma})} e^{\Omega_{\bar{z}}(\bar{\sigma}^*)} \) on (18) gives

\[
\int \frac{dz}{2\pi i} e^{V(p' - p - z)} \tau_0(p' - [z^{-1}][p^*]) \tau_0(p + [z^{-1}][p^*]) = 0, \tag{19}
\]

hence \( \tau_{\pm}(p[p^*]) \) is also a tau function of the KP hierarchy. Then it follows from remark 3 that both \( \tau_{\pm}(p[p^*]) \) are TL tau functions where \( p \) and \( p^* \) are two sets of higher times. These tau functions do not depend on the discrete TL variable \( \rho_0 \) because \( \tau_0(p[p^*]) \) does not depend on it.

According to Sato [64] a KP tau function may be related to any element of an infinite dimensional Grassmannian as a series in the Schur functions

\[
\tau_{\text{KP}}(p) = \sum_{\lambda} \pi_{\lambda} s_{\lambda}(p),
\]

where \( \pi_{\lambda} \) are the Plücker coordinates of the element. Hence, according to (7), (14) and (15), the functions \( s_{\lambda}(p^*) \), \( o_{\lambda}(p^*) \) and \( sp_{\lambda}(p^*) \) are the Plücker coordinates of \( \tau_0(p[p^*]) \), \( \tau_-(p[p^*]) \) and \( \tau_+(p[p^*]) \), respectively. The fermionic construction of the Plücker coordinates is written in appendix A.8.

The Plücker coordinates \( o_{\lambda}(p^*) \) and \( sp_{\lambda}(p^*) \) may be evaluated respectively as follows

\[
o_{\lambda}(p^*) = \left( s_{\lambda}(\tilde{\sigma}) \cdot e^{-\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} p_0^m + \frac{1}{m} p_0 \sum_{n=1}^{\infty} \frac{1}{k} p_n p_n^*} \right) |_{p=0} \tag{20}
\]

and

\[
sp_{\lambda}(p^*) = \left( s_{\lambda}(\tilde{\sigma}) \cdot e^{-\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} p_0^m + \frac{1}{m} p_0 \sum_{n=1}^{\infty} \frac{1}{k} p_n p_n^*} \right) |_{p=0}, \tag{21}
\]

which may be compared with the identity for the Schur functions

\[
s_{\lambda}(p^*) = \left( s_{\lambda}(\tilde{\sigma}) \cdot e^{\sum_{n=1}^{\infty} \frac{1}{k} p_n p_n^*} \right) |_{p=0}. \tag{22}
\]

As in the previous section, let us assign weight \( k \) to \( p_k \). We recall that the polynomials \( s_{\lambda} \) are quasi-homogeneous in the variables \( p_m \) of weight \( |\lambda| \). As we see from (20) and (21), polynomials \( o_{\lambda} \) and \( sp_{\lambda} \) are not quasi-homogeneous: they may both be presented as \( s_{\lambda} \) plus polynomials of less weight.

For instance

\[
o_{(1)}(p) = sp_{(1)}(p) = s_{(1)}(p) = p_1
\]

\[
o_{(2)}(p) = s_{(2)}(p) - 1 = \frac{1}{2} p_2 + \frac{1}{2} p_1^2 - 1, \quad sp_{(2)}(p) = s_{(2)}(p) = \frac{1}{2} p_2 + \frac{1}{2} p_1^2
\]

\[
o_{(2)}(p) = s_{(2)}(p) - 1 = -\frac{1}{2} p_2 + \frac{1}{2} p_1^2, \quad sp_{(2)}(p) = s_{(2)}(p) - 1 = -\frac{1}{2} p_2 + \frac{1}{2} p_1^2;
\]

\[
sp_{(2)}(p) = sp_{(2)}(p) = s_{(2)}(p) - 1 = -\frac{1}{2} p_2 + \frac{1}{2} p_1^2 - 1.
\]
Next, from $\tau_-(\mathbf{p} - \mathbf{p}^*) = \tau_+(\mathbf{p}|\mathbf{p}^*)$ and from $s_{\lambda}(\mathbf{p}) = (-)^{|\lambda|} s_{\lambda^\vee}(-\mathbf{p})$, we get
\begin{equation}
sp_{\lambda}(\mathbf{p}) = (-)^{|\lambda|} o_{\lambda^\vee}(-\mathbf{p}),
\end{equation}
where $-\mathbf{p} = (-p_1, -p_2, -p_3, \ldots)$.

From the following well-known formulas (see [45], pp 76 and 77 or [44] p 238)
\begin{equation}
e^{-\sum_{n>0} \frac{1}{n} \rho_n^2 + \sum_{n>0, \text{odd}} \rho_n} = \sum_{\mu \in \mathcal{P}} s_{\mu}(\mathbf{p}), \quad e^{-\Omega_+ (\mathbf{p})} = \sum_{\mu \in \mathcal{P}_{\text{even}}} s_{\mu}(\mathbf{p}), \quad \text{and} \quad e^{-\Omega_- (\mathbf{p})} = \sum_{\mu \in \mathcal{P}_{\text{odd}}} s_{\mu}(\mathbf{p})
\end{equation}
where $\mathcal{P}_{\text{even}}$ is the set of all partitions with even parts (including (0)), one deduces

**Lemma 1.**
\begin{equation}
\sum_{\mu \in \mathcal{P}} e^{\lambda_1|n|} s_{\mu}(\tilde{\partial}) = e^{\lambda_1\sum_{n=1}^\infty m \rho_n^2 + \sum_{m>0, \text{odd}} \rho_n} e^{\tilde{\lambda_2} \partial_n}
\end{equation}
\begin{equation}
\sum_{\mu \in \mathcal{P}_{\text{even}}} e^{\lambda_1|n|} s_{\mu}(\tilde{\partial}) = e^{\lambda_1\sum_{n=1}^\infty m \rho_n^2} e^{-\sum_{m>0, \text{even}} \rho_n} e^{\tilde{\lambda_2} \partial_n} = e^{-\Omega_+ (\mathbf{e}^n \partial_n)}
\end{equation}
\begin{equation}
\sum_{\mu \in \mathcal{P}_{\text{odd}}} e^{\lambda_1|n|} s_{\mu}(\tilde{\partial}) = e^{\lambda_1\sum_{n=1}^\infty m \rho_n^2 + \sum_{m>0, \text{even}} \rho_n} e^{\tilde{\lambda_2} \partial_n} = e^{-\Omega_- (\mathbf{e}^n \partial_n)},
\end{equation}
where $s_{\lambda}(\tilde{\partial})$ is defined as in (20) and (21).

(See also [60] where in (25) and (26) there is the opposite sign for the linear term in the exponents, which is a misprint.)

**Remark 4.** From lemma 1 a number of relations may be obtained. We present two examples:
\begin{equation}
e^{-\sum_{n>0} \frac{1}{n} \rho_n^2 - \sum_{m>0, \text{odd}} \frac{1}{2} \rho_n} \sum_{\mu \in \mathcal{P}} s_{\mu/\lambda}(\mathbf{p}) = \sum_{\mu \in \mathcal{P}} s_{\mu/\lambda}(\mathbf{p}) = e^{\sum_{n=1}^\infty \rho_n \rho_n} o_{\lambda}(\mathbf{p}),
\end{equation}
where the first equality is obtained from example 27(a) in I.5 of [45]. The second example follows from (13) and (24) and from $s_{\mu}(\tilde{\partial}) \cdot s_{\lambda}(\mathbf{p}) = s_{\lambda/\mu}(\mathbf{p})$:
\begin{equation}
\sum_{\mu \in \mathcal{P}} s_{\lambda/\mu}(\mathbf{p}) = e^{\sum_{n=1}^\infty \rho_n \rho_n} o_{\lambda}(\mathbf{p}).
\end{equation}
Note, that the constant term $\left[e^{\sum_{n=1}^\infty \rho_n \rho_n} o_{\lambda}(\mathbf{p})\right]_{\mathbf{p}=0}$ of (27) is equal to 1 for any $\lambda$ of the form $\mu \cup \mu$, and vanishes otherwise.

2.1. Relation to irreducible characters of the orthogonal and symplectic groups

We shall use the notations explained in remark 2 with $p_m(U) = \text{Tr} \ U^m$.

In this notation we write
\begin{equation}
\tau_+(U|\mathbf{p}^*) = \prod_{i<j} (1 - x_{ij}) \prod_{k=1}^n e^{\sum_{n=1}^\infty \frac{1}{n} \rho_n \rho_n} \sum_{\ell(\lambda) \leq s} sp_{\lambda}(\mathbf{p}^*) s_{\lambda}(U)
\end{equation}
\[ \tau_-(U|p^*) = \prod_{i \leq j < n} (1 - x_i x_j) \prod_{k=1}^n e^{\sum_{i=1}^n \lambda_i z^k} = \sum_{\ell(\lambda) \leq e} o_\lambda(p^*) s_\lambda(U). \] (29)

Here \( x_1, \ldots, x_n \) are the eigenvalues of \( U \in \mathbb{U}(n) \). Let us note that

\[ \tau_-(U|p^*) = \tau_+(U|p^*)\det (1 - U^2). \] (30)

Now take \( Z \in Sp(2n) \) and let \( z_1, z_1^{-1}, \ldots, z_n, z_n^{-1} \) be the eigenvalues of \( Z \). Then formula (28) gives

\[ \prod_{i \leq j < n} (1 - x_i x_j) \prod_{i,j=1}^n (1 - x_i z_j)^{-1} = \sum_{\ell(\lambda) \leq e} sp_\lambda(Z) s_\lambda(U), \quad Z \in Sp(2n), \quad U \in \mathbb{U}(n). \] (31)

This relation is known as the Cauchy identity for the irreducible characters of \( Sp(2n) \), see [44]. Thus, from the completeness of the Schur functions \( s_\lambda \) in the space of the symmetric functions in \( x_1, \ldots, x_n \), it follows that the polynomials \( sp_\lambda(p(Z)) = sp_\lambda(Z) \) coincide with the characters of \( Sp(2n) \).

Similarly, for \( Z \in O(2n) \) with the eigenvalues \( z_1, z_1^{-1}, \ldots, z_n, z_n^{-1} \), and for \( U \in \mathbb{U}(n) \) with eigenvalues \( x_1, \ldots, x_n \), we obtain

\[ \prod_{i \leq j < n} (1 - x_i x_j) \prod_{i,j=1}^n (1 - x_i z_j)^{-1} = \sum_{\ell(\lambda) \leq e} o_\lambda(Z) s_\lambda(U), \quad Z \in O(2n), \quad U \in \mathbb{U}(n). \] (32)

While for \( Z \in O(2n + 1) \) with the eigenvalues \( z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}, 1 \), we obtain

\[ \prod_{i \leq j < n} (1 - x_i x_j) \prod_{i,j=1}^n (1 - x_i z_j)^{-1} = \sum_{\ell(\lambda) \leq e} o_\lambda(Z) s_\lambda(U), \quad Z \in O(2n + 1), \quad U \in \mathbb{U}(n). \] (33)

Relations (32) and (33) are known as the Cauchy identities for the orthogonal group, the polynomials \( o_\lambda(Z) \) are irreducible characters of the orthogonal group.

We have

\[ \tau_+(U|Z) = \sum_{\ell(\lambda) \leq e(m(n,a))} sp_\lambda(Z) s_\lambda(U), \quad Z \in Sp(2n), \quad U \in \mathbb{U}(m) \]

and

\[ \tau_-(U|Z) = \sum_{\ell(\lambda) \leq e(m(n,a))} o_\lambda(Z) s_\lambda(U), \quad Z \in O(N), \quad U \in \mathbb{U}(m), \quad n = \left\lceil \frac{N}{2} \right\rceil, \]

where \( sp_\lambda \) and \( o_\lambda \) are characters, respectively, of symplectic and orthogonal groups.

The content of this section may be compared with [8, 37] where universal characters of classical groups were considered.
3. Characters and fermions

Since all Schur functions $s_\lambda$ are in the $GL_\infty$ group orbit, they are KP tau functions, i.e. they satisfy the bilinear identity:

$$\int \frac{dz}{2\pi i} \psi(z) \tau \otimes \psi^\dagger(z) \tau = 0,$$

where $\psi(z) = \sum_{i \in \mathbb{Z}} \psi_i \xi^i$ and $\psi^\dagger(z) = \sum_{i \in \mathbb{Z}} \psi_i^\dagger \xi^{-i}$ are free fermionic fields (see [31]), whose Fourier components anti-commute as follows $\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0$ and $\psi_i \psi_j^\dagger + \psi_j \psi_i^\dagger = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker symbol. We put

$$\psi_i(0) = \psi_i^\dagger(-1-i)|0\rangle = \langle 0|\psi_{-1-i} = \langle 0|\psi_i^\dagger = 0, \quad \text{for } i < 0,$$

where $\langle 0|$ and $|0\rangle$ are left and right vacuum vectors of the fermionic Fock space, $\langle 0| \cdot 1 \cdot |0\rangle = 1$. Let

$$\langle n \rangle = \begin{cases} \langle 0| \psi_0^\dagger \cdots \psi_{n-1}^\dagger \rangle & \text{if } n > 0, \\
\langle 0| \psi_{-1}^\dagger \cdots \psi_{-n}^\dagger \rangle & \text{if } n < 0,
\end{cases} \quad |n\rangle = \begin{cases} \psi_{n-1} \cdots \psi_0 |0\rangle & \text{if } n > 0, \\
\psi_{-n}^\dagger \cdots \psi_{-1}^\dagger |0\rangle & \text{if } n < 0,
\end{cases}
$$

then $\langle n \rangle \cdot 1 \cdot |m\rangle = \delta_{nm}$. The integer $n$ is called the charge of the vacuum. The Fock space is the direct sum of the components labelled by $n \in \mathbb{Z}$ (in the context of integrable systems see [31] or [33]).

We use the vertex operator expression for $\psi(z)$ and $\psi^\dagger(z)$

$$\psi(z) = e^{\alpha_0} z^{\alpha_0} e^{-\sum_{i \in \mathbb{Z}} \frac{\alpha_0}{2} e^{-\frac{\alpha_0}{2}}}$$

$$\psi^\dagger(z) = e^{-\alpha_0} z^{-\alpha_0} e^{\sum_{i \in \mathbb{Z}} \frac{\alpha_0}{2} e^{\frac{\alpha_0}{2}}}$$

where

$$\alpha_m = \sum_{i \in \mathbb{Z}} :\psi_i \psi_i^\dagger:\ + m.$$  

Note that $[\alpha_i, \alpha_j] = i\delta_{i-j}$, hence they form a Heisenberg algebra. Now use the standard realization of the Heisenberg algebra

$$\alpha_k = \frac{\partial_k}{k}, \quad \alpha_{-k} = p_k, \quad \alpha_0 = q\partial_q, \quad e^{\alpha_0} = q.$$  

Then (37) and (38), respectively, turn into

$$\psi(z) = q e^{q\partial_q} e^{\sum_{i=1}^{\infty} \frac{\partial_i}{i} e^{-\sum_{i=1}^{\infty} \partial_i z^{-i}}}$$

$$\psi^\dagger(z) = q^{-1} z^{-q\partial_q} e^{-\sum_{i=1}^{\infty} \frac{\partial_i}{i} e^{\sum_{i=1}^{\infty} \partial_i z^{-i}}}.$$  

For future use we also introduce

$$\gamma(p) := e^{\sum_{i=1}^{\infty} \frac{p_i}{i} \alpha_0}, \quad \gamma^\dagger(p) := e^{\sum_{i=1}^{\infty} \frac{p_i}{i} \alpha_{-i}}.$$  

Note that $e^{\Omega z}$ is an automorphism of the Fock space. It maps any charge sector into itself and maps Schur functions into orthogonal and symplectic characters, see (13). Thus $\alpha_\lambda$ (for $e^{\Omega z}$) and $\alpha_{\lambda'}$ (for $e^{\Omega z}$) satisfies

$$\int \frac{dz}{2\pi i} e^{\Omega z} (\tilde{\alpha}) \psi(z) e^{-\Omega z} (\tilde{\alpha}) \sigma \otimes e^{\Omega z} (\tilde{\alpha}) \psi^\dagger(z) e^{-\Omega z} (\tilde{\alpha}) \sigma = 0.$$
where \( \sigma = e^{\Omega \tau} \). We now want to calculate \( \Psi_\mp(z) = e^{\Omega \tau} \langle \partial^z \rangle \psi(z) e^{-\Omega \tau} \) and \( \Psi_\mp^\dagger(z) = e^{\Omega \tau} \langle \partial^z \rangle \psi^\dagger(z) e^{-\Omega \tau} \). Using the following formulas, which can easily be deduced from the Cambell–Baker–Hausdorff formula:

\[
e^{a\partial_x} e^{b\partial_x} = e^{b \partial_x} e^{a \partial_x} e^{(b - a)\partial_x^2}, \quad e^{a\partial_x} e^{b\partial_x} = e^{b \partial_x} e^{a \partial_x} e^{(b - a)^2\partial_x^2},
\]

one thus obtains:

\[
\Psi_\mp(z) = (1 - z^2)^{i\frac{1}{2}} e^{\Omega \epsilon \sum_{j=1}^\infty \frac{1}{z^j}} e^{-\sum_{j=1}^\infty \Omega \epsilon (z^{-j} + z^j)}
\]

(44)

\[
\Psi_\mp^\dagger(z) = (1 - z^2)^{i\frac{1}{2}} e^{-\sum_{j=1}^\infty \Omega \epsilon (z^{-j} + z^j)}
\]

(45)

Hence the orthogonal and symplectic characters satisfy the bilinear equation:

\[
\oint \frac{dz}{2\pi i} \Psi_\mp(z) \sigma_\mp \otimes \Psi_\mp^\dagger(z) \sigma_\mp = 0,
\]

(46)

or equivalently, let \([z] = (z, z^2, z^3, \ldots)\), then

\[
\oint \frac{dz}{2\pi i} (1 - z^2) e^{\sum_{j=1}^\infty \frac{1}{z^j}} \sigma_\mp (p - [z] - [z^{-1}]) \sigma_\mp (q + [z] + [z^{-1}]) = 0,
\]

(47)

which is equation (5.4) of [8].

Now note that if we write \( \Psi_\mp(z) = \sum_{i\in\mathbb{Z}} \Psi_\mp \chi^i \) and \( \Psi_\mp^\dagger(z) = \sum_{i\in\mathbb{Z}} \Psi_\mp^\dagger \chi^{-i} \), then the modes still satisfy the usual relations.

\[
\Psi_\mp_i \Psi_\mp_j + \Psi_\mp_j \Psi_\mp_i = 0 = \Psi_\mp_i \Psi_\mp_j^\dagger + \Psi_\mp_j \Psi_\mp_i^\dagger,
\]

\[
\Psi_\mp_i \Psi_\mp_j^\dagger + \Psi_\mp_j \Psi_\mp_i^\dagger = \delta_{ij}.
\]

Using the above vertex operators on the vacuum \([0] = q^0\), one still has

\[
\Psi_\mp_i [0] = 0 = \Psi_\mp_i^\dagger [0], \quad i < 0.
\]

Another approach is as follows, equation (46) still generates the \( GL_\infty \) group orbit of the vacuum; however, one has to take a different realization of \( gl_\infty \), viz. \( \Psi_\mp \Psi_\mp^\dagger \) (\( i, j \in \mathbb{Z} \)) still forms a basis of \( g_\infty \), it is the coefficient of \( z^i y^{j-1} \) in the expansion

\[
X_\mp(y, z) = e^{\Omega \tau} \langle \partial^z \rangle \psi(y) e^{-\Omega \tau} = \Psi_\mp(z) \Psi_\mp^\dagger(y).
\]

Using the above vertex operators, we find

\[
X_\mp(y, z) = \frac{(1 - z^2)^{i\frac{1}{2}} (1 - y^2)^{i\frac{1}{2}}}{(z - y)(1 - zy)} e^{\sum_{j=1}^\infty \frac{1}{z^j}} e^{-\sum_{j=1}^\infty \Omega \epsilon (z^{-j} + z^j)}.
\]

Clearly the standard Heisenberg algebra also changes. Now define \( \beta_\mp = e^{\Omega \epsilon} \alpha_k e^{\Omega \tau} \), then the \( \beta_k \) still has the standard commutation relations, \([\beta_\mp^\dagger, \beta_\mp^\dagger] = i \delta_{\mp j} \), however, these elements are realized in a different way. Using

\[
e^{a\partial_x} x = (x + a) e^{a\partial_x} \quad e^{a\partial_x} x = (x + 2a \partial_x) e^{a\partial_x},
\]

Hence,

\[
\beta_k^\mp = \frac{\delta_k}{k} \quad \beta^\dagger_\mp = p_k - \frac{\delta_k}{k} \mp \delta_{k \text{even}}, \quad \beta^\dagger_0 = q \partial_q.
\]

And clearly
\[ \Psi_{\mp}(z) = e^{\frac{\partial}{\partial x^\mp}} z^\mp e^{-\sum_{i<0} \frac{\partial}{\partial x^i}} e^{-\sum_{i>0} \frac{\partial}{\partial x^i}} \]

or equivalently
\[ \Psi_{\mp}(z) = q e^{\partial_\mp} e^{\sum_{i=1}^{\infty} \left( \frac{\partial z}{\partial y^i} - \partial \right)} e^{\sum_{i=1}^{\infty} \partial z^{-i}} \]

or equivalently
\[ \Psi_{\mp}(z) = q^{-1} e^{-\partial_\mp} e^{\sum_{i=1}^{\infty} \left( \frac{\partial z}{\partial y^i} - \partial \right)} e^{\sum_{i=1}^{\infty} \partial z^{-i}}. \] (48)

(49)

Note that one obtains (44) and (45) from (48) and (49), respectively, if one moves the differential operator part to the right. One can use
\[ e^{a \partial_b + b \partial_a} = e^{a^\dagger b} e^{b a \partial_b}. \]

Note that
\[ X_\mp(y, z) = \left( \frac{z}{y} \right) e^{\sum_{i=1}^{\infty} \left( \frac{\partial z}{\partial y^i} - \partial \right)(z^\mp - y^\mp)} e^{-\sum_{i=1}^{\infty} \partial (z^{-1} - y^{-1})}. \]

The above suggests that we can take the normal Clifford algebra in \( \psi_{\mp} \) and \( \psi_{\mp}^\dagger \), but choose another realization of the Heisenberg algebra, viz., the ones given by the \( \beta_i^\dagger \), such that the fields \( \psi(z) \) and \( \psi^\dagger(z) \) are given by (48) and (49), respectively. Then the tau function, which is in the KP hierarchy given by
\[ \tau(p) = \langle 0 | e^{\sum_{\gamma>0} \frac{\partial}{\partial x^\gamma}} g | 0 \rangle \quad \text{for } g \in \text{Gl}_\infty, \]
changes into \( \sigma_\mp(p) = e^{i \Omega \mp} \tau(p) \), which is equal to
\[ \sigma_\mp(p) = \langle 0 | e^{\sum_{\gamma>0} \frac{\partial}{\partial x^\gamma}} - \frac{1}{2} \gamma \alpha - \frac{1}{2} \gamma \alpha^\dagger + \frac{1}{2} \gamma \alpha g | 0 \rangle \quad \text{for } g \in \text{Gl}_\infty, \]
which corresponds to the modified Hamiltonian of [8], section 3, approach I. Next, we calculate
\[ \langle 0 | e^{\sum_{\gamma>0} \frac{\partial}{\partial x^\gamma}} - \frac{1}{2} \gamma \alpha - \frac{1}{2} \gamma \alpha^\dagger + \frac{1}{2} \gamma \alpha g e^{\sum_{\gamma>0} \frac{\partial}{\partial x^\gamma}} - \frac{1}{2} \gamma \alpha | 0 \rangle = \tau_\mp(p^* | p). \]

Hence, it makes sense to look at
\[ \langle 0 | e^{\sum_{\gamma>0} \frac{\partial}{\partial x^\gamma}} - \frac{1}{2} \gamma \alpha - \frac{1}{2} \gamma \alpha^\dagger + \frac{1}{2} \gamma \alpha g e^{\sum_{\gamma>0} \frac{\partial}{\partial x^\gamma}} - \frac{1}{2} \gamma \alpha | 0 \rangle \quad \text{for } g \in \text{Gl}_\infty. \]

**Remark 5.** Actually, we have
\[ \tau(p, p^*) \rightarrow \tau_{\pm}(p, p^*) = e^{i \Omega \pm} \cdot \tau(p, p^*). \] (50)

If
\[ \tau(p, p^*) = \sum_{\lambda, \mu \in \mathcal{P}} s_\lambda(p) \pi_{\lambda, \mu} s_\mu(p^*), \]
where
\[ \pi_{\lambda, \mu} = \langle 0 | s_\lambda(\tilde{\alpha}) g s_\mu(\tilde{\alpha}^*) | 0 \rangle, \]
then
\[ \tau^\pm(p, p^*) = \sum_{\lambda, \mu \in \mathbb{P}} \lambda_{\lambda, \mu} \pi^\pm_{\lambda, \mu}(p^*), \]

where

\[ \pi^+_{\lambda, \mu} = \langle 0 | s_{\lambda}(\tilde{\alpha}) g s_{\mu}(\tilde{\alpha}^*) | 0 \rangle, \quad \pi^-_{\lambda, \mu} = \langle 0 | o_{\lambda}(\tilde{\alpha}) g s_{\mu}(\tilde{\alpha}^*) | 0 \rangle. \]

Similarly, one can consider \( \tau^{ab} \) with \( a, b = \pm \).

4. Integrals over the symplectic group and over orthogonal groups

4.1. Haar measures and generating functions for characters

Lemma A.6 in appendix A.1 and certain formulas of appendix A.7 result in the following lemma, which we shall need.

**Lemma 2.** The Haar measures of the symplectic group \( \text{Sp}(2n) \) and of the unitary group \( \text{U}(n) \) are related as follows:

\[ \varepsilon^o \text{V}^{\text{Sp}} d_s S = 2^{-n} \tau_-(U|p) \tau_-(U^{-1}|p) d_s U \]

\[ = 2^{-n} \tau_+(U|p) \tau_+(U^{-1}|p) \det(1 - U^2) \det(1 - U^{-2}) d_s U \]

\[ = 2^{-n} \tau_+(U|p) \tau_+(U^{-1}|p) \det(1 - U^2) d_s U, \]

where \( e^{i\theta}, e^{-i\theta}, \ldots, e^{i\theta_0}, e^{-i\theta_0} \) are eigenvalues of \( S \in \text{Sp}(2n) \) while \( e^{i\theta}, \ldots, e^{i\theta_0} \) are eigenvalues of \( U \in \text{U}(n) \).

**Lemma 3.** The Haar measures of the orthogonal group \( \text{O}(2n) \) and of the unitary group \( \text{U}(n) \) are related as follows:

\[ \varepsilon^o \text{V}^{\text{O}} d_s O = 2^{-n} \tau_+(U|p) \tau_+(U^{-1}|p) d_s U \]

\[ = 2^{-n} \tau_-(U|p) \tau_-(U^{-1}|p) \det(1 - U^2)^{-1} \det(1 - U^{-2})^{-1} d_s U \]

\[ = 2^{-n} \tau_-(U|p) \tau_+(U^{-1}|p) \det(1 - U^2)^{-1} d_s U. \]

**Lemma 4.** The Haar measures of the orthogonal group \( \text{O}(2n + 1) \) and of the unitary group \( \text{U}(n) \) are related as follows:

\[ \varepsilon^o \text{V}^{\text{O}} d_s O = 2^{-n} \tau_+(U|p) \tau_+(U^{-1}|p) \det(1 - U) \det(1 - U^{-1}) d_s U \]

\[ = 2^{-n} \tau_-(U|p) \tau_-(U^{-1}|p) \det(1 + U) \det(1 + U^{-1})^{-1} d_s U \]

\[ = 2^{-n} \tau_-(U|p) \tau_+(U^{-1}|p) \det \frac{1 - U^{-1}}{1 + U} d_s U, \]

where \( e^{i\theta}, e^{-i\theta}, \ldots, e^{i\theta_0}, e^{-i\theta_0}, 1 \) are eigenvalues of \( O \in \text{O}(2n + 1) \), while \( e^{i\theta}, \ldots, e^{i\theta_0} \) are eigenvalues of \( U \in \text{U}(n) \).
4.2. Integrals over the symplectic group

Consider the following integral over the symplectic group:

\[ I_{Sp(2n)}(p) = \int_{S \in Sp(2n)} e^{-\sum_{i=1}^{2n} \text{tr} S^i} d_S, \]  

(60)

where \( d_S \) is the corresponding Haar measure. Explicitly,

\[ I_{Sp(2n)}(p) = \frac{2^{n^2}}{n!} \int_{0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \pi} \prod_{i < j}^{n} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^{n} e^{2 \sum_{m=1}^{\infty} t_m \cos m \theta_i \sin^2 \theta_i} d \theta, \]

(61)

where \( e^{\pm i \theta_1}, \ldots, e^{\pm i \theta_n} \) are the eigenvalues of \( S \).

By analogy with matrix models studied in physics we call the parameters \( p = (p_1, p_2, \ldots) \) coupling constants, and (asymptotic) series in these parameters are called perturbation series.

In this subsection we will calculate the integrals in various ways.

4.2.1. Perturbation series for integrals over symplectic group as series in characters \( o_\lambda(p), sp_\lambda(p), s_\lambda(p) \).

Proposition 1.

\[ I_{Sp(2n)}(p) = \sum_{\lambda, \ell(\lambda) \leq n} s_{\lambda \cup \lambda}(p), \]

(62)

\[ = 2^{-n} \sum_{\ell(\lambda) \leq n} (o_\lambda(p))^2 = 2^{-n} \sum_{\lambda, \ell(\lambda) \leq n} (sp_\lambda(-p))^2. \]

(63)

Formula (62) may be derived using of Cauchy-Littlewood formula

\[ e^{\sum_{m=1}^{\infty} \frac{1}{m} \text{tr} S^m} = \sum_{\ell(\lambda) \leq 2n} s_\lambda(S)s_\lambda(p) \]

and the known relation (see for instance (6.13)–(6.15) in sections 6 and 7 in [45])

\[ \int_{S \in Sp(2n)} s_\lambda(S) d_S = \begin{cases} 1 & \lambda^t \text{is even} \\ 0 & \text{otherwise}, \end{cases} \]

(64)

where \( \lambda^t \) is the partition conjugated to \( \lambda \), see [45]. (This relation may be easily obtained by the evaluation of the Schur function \( s_\lambda(z) \) where \( z_i = x_i + x_i^{-1} \) inside the integral over symplectic group.)

The left-hand side of (63) is obtained from (51), (29) and (8), and the right-hand side of (63) is the result of (23).

Proposition 2.

\[ I_{Sp(2n)}(p + p^*) e^{-\frac{1}{2} \sum_{i=1}^{n} (p_i + p_i^*)^2 + \sum_{i=1}^{n} \frac{1}{m} (p_{2i-1} + p_{2i})^2} = \frac{I_{U(n)}(p, p^*)}{I_{U(\infty)}(p, p^*)}. \]

(65)

Proof. Let us use

\[ e^{-\frac{1}{2} \sum_{i=1}^{n} \frac{1}{m} p_i^2 + \sum_{i=1}^{n} \frac{1}{m} p_{2i-1}^2 + \sum_{i=1}^{n} \frac{1}{m} p_{2i}^2} = \sum_{\ell(\lambda) \leq n} s_\lambda(p) s_\lambda(S), \]

(66)
which follows from (16) and (95) and the orthogonality of relation for characters [44]
\[ \int_{S(2n)} s_\lambda(S) s_\mu(S) d_s S = \delta_{\lambda, \mu}. \] (67)

We get
\[ e^{-\frac{1}{2} \sum_{\ell=1}^\infty \frac{1}{\ell} (p_\ell^2 + (p_\ell^*)^2)} \prod_{\ell=1}^\infty \frac{1}{\ell} (p_{2\ell} + p_{2\ell}^*) \int_{SP(2n)} e^{\sum_{m>0} (p_m + p_m^*) m} S^m d_s S = \delta_{\lambda, \mu}. \]

Let us multiply the left-hand side on \( I_{U(\infty)}(p, p^*) e^{-\sum_{m>0} \frac{1}{m} p_m p_m^*} = 1 \) (see (9) where \( N \to +\infty \) and the second equality in (7)), then by (9) this formula may be rewritten as (65).

\[ \text{□} \]

4.2.2. Integrals over the symplectic group as DKP tau functions. Here we need the fermionic language of section 1, see also appendix A.3.

Proposition 3.
\[ I_{SP(2n)}(p) = \frac{1}{n!} \langle 2n | \gamma(p) e^{\frac{i}{4\pi} \int \psi(x-1) \psi(x)(x-x^{-1}) \frac{dx}{x}} | 0 \rangle \] (68)
\[ = \frac{1}{n!} \langle 2n | \gamma(p) e^{\sum_{i \in \mathbb{Z}} \psi_i \psi_i^{-1}} | 0 \rangle. \] (69)

Proof. The second equality follows from
\[ \frac{1}{4\pi i} \int \psi(x^{-1}) \psi(x)(x-x^{-1}) \frac{dx}{x} = \sum_{i \in \mathbb{Z}} \psi_i \psi_i^{-1}. \] (70)

Let us consider the Taylor series of the exponential on the right-hand side. The first equality follows from
\[ \langle N | \psi(x_1) \cdots \psi(x_N) | 0 \rangle = \prod_{i<j} (x_i - x_j) =: \Delta_N(x) \] (71)

and
\[ \Delta_{2n}(e^{-i\theta_1}, e^{i\theta_1}, \ldots, e^{-i\theta_n}, e^{i\theta_n}) = (-i)^n 2^n \prod_{k<l} (\cos \theta_k - \cos \theta_l)^2 \prod_{k=1}^n \sin \theta_k. \] (72)

Let us note that formula (62) follows from (70) and from results of [60] (see the formulas for \( S_4^{(1)} \) in [60]). There also exists a

\[ \text{□} \]

4.2.3. Pfaffian representation. Here we need appendix A.5. Let us note that, thanks to Wick’s rule, we can directly obtain the Pfaffian representation of the integral (60) as follows.
Proposition 4.

\[ I_{Sp(2n)}(\mathbf{p}) = \text{Pf} [M_{ij}(\mathbf{p})]_{k,j=1,\ldots,2n}, \tag{73} \]

where \( M \) is the following Toeplitz matrix

\[ M_{ij}(\mathbf{p}) = -M_{ji}(\mathbf{p}) = \frac{1}{4\pi i} \oint (x^{i-j} - x^{j-i}) \left( x - x^{-1} \right) e^{\sum_{m=1}^{\infty} \frac{1}{m} \mu_m (x^m + x^{-m})} \frac{dx}{x}. \tag{74} \]

4.2.4. Relation of the integral over \( Sp(2n) \) to an integral over \( U(2n) \). We refer to results of [43] about the interplay between BKP and two-component KP tau functions, or to the following. Consider

\[ I_{U(2n)}(\mathbf{p}) := \int_{U \in U(2n)} \det (U - U^\dagger) e^{\sum_{m=1}^{\infty} \frac{1}{m} \mu_m (U^m + U^{-m})} d_U. \tag{75} \]

Written as an integral over eigenvalues it is

\[ I_{U(2n)}(\mathbf{p}) = \frac{1}{(2n)! (2\pi i)^{2n}} \int \Delta_{2n}(x) \Delta_{2n}(x^{-1}) \prod_{i=1}^{2n} (x_i - x_i^{-1}) e^{\sum_{m=1}^{\infty} \frac{1}{m} \mu_m (x_i^m + x_i^{-m})} \frac{dx_i}{x_i}. \tag{76} \]

We have

Lemma 5.

\[ I_{U(2n)}(\mathbf{p}) = \langle 2n, -2n | e^{\sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \mu_\alpha (a^{(\alpha)}_\alpha - a^{(\alpha)}_\alpha^\dagger)} e^{\frac{1}{2} \oint (\psi^{(1)}_\alpha (x^{-1}) \psi^{(2)}_\alpha (x) - \psi^{(2)}_\alpha (x) \psi^{(1)}_\alpha (x^{-1}))} \rangle | 0, 0 \rangle. \tag{77} \]

For the two-component fermions see the appendix, \( \alpha^{(a)}_n := \sum_{j \in \mathbb{Z}} \psi_j^{(a)} \psi_{i+n}^{(\dagger)(a)} \), \( n > 0 \).

The lemma is the direct result of [27] (see proposition 4 there). We also have

Proposition 5.

\[ \left( \int_{S \in Sp(2n)} e^{\sum_{\alpha=1}^{\infty} \frac{1}{\alpha} \mu_\alpha tr S^\alpha S} d_S \right)^2 = \int_{U \in U(2n)} \det (U - U^\dagger) e^{\sum_{m=1}^{\infty} \frac{1}{m} \mu_m (U^m + U^{-m})} d_U \tag{78} \]

\[ = \sum_{\lambda, \mu} J_{\lambda, \mu} s_\lambda(\mathbf{p}) s_\mu(\mathbf{p}) \tag{79} \]

where

\[ J_{\lambda, \mu} = \int_{U \in U(2n)} s_\lambda(U) s_\mu(U^{-1}) \det (U - U^\dagger) d_U. \]

In terms of \( h_i = \lambda_i - i + 2n \) and \( h'_i = \mu_i - i + 2n \), the last formula may be rewritten in form

\[ J_{\lambda, \mu} = \prod_{i=1}^{2n} \delta_{h_i, h'_i - 1} - \prod_{i=1}^{2n} \delta_{h_i, h'_i + 1}, \]

where \( \delta_{ij} \) is the Kronecker symbol.
Proof. One way to prove it is to present \( I_{U(2n)} \) as a determinant of \( M \) of (74). It is easy using Wick’s rule for the vacuum expectation value in lemma 5. Via Wick’s rule we directly obtain

\[
I_{U(2n)}(p) = \det \left[ 2M_{ij} \right]_{i,j=1,...,2n}
\]

with the same matrix \( M \) as in Pfaffian representation (73). This proves (5).

The other way is to apply the results of [43] (see proposition 4 there). Then we have

\[
(2^n I_{Sp(2n)}(p))^2 = (2n_{\mu} - 2n |e^{\sum_{i=1}^\infty \frac{1}{2} \rho_m (\alpha_m^{(i)} - \alpha_m^{(j)})}| e^{i \pi \sum_{i=1}^\infty \psi^{(i)}(\xi - 1) \psi^{(i)}(\xi - 1) \xi} | 0, 0),
\]

which coincides with the right-hand side of (77). This ends the proof.

4.2.5. Different fermionic representation and Toda chain-AKNS tau function. Denote \( z = x + x^{-1} \). Introduce variables \( \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots) \) with the help of

\[
\sum_{n=1}^\infty \frac{1}{n} p_n (x^n + x^{-n}) = \sum_{n=1}^\infty \frac{1}{n} p_n e^n - c(\tilde{p}), \quad c(\tilde{p}) = \sum_n \frac{(2n)!}{|n|!} \tilde{p}_{2n},
\]

where the sets of \( \{p_{2n}, n > 0\} \) and \( \{\tilde{p}_{2n+1}, n \geq 0\} \) may be expressed via the sets \( \{p_{2n}, n > 0\} \) and \( \{p_{2n+1}, n \geq 0\} \), respectively, by triangle transformations with binomial entries.

Then, thanks to (A.5) and (A.9) in appendix A.1, we have

\[
I_{Sp(2n)}(p(\tilde{p})) = \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \prod_{i<j}^n (z_i - z_j) \prod_{i=1}^n e^{\sum_{n=1}^\infty \frac{1}{2} \tilde{p}_n e^n (4 - z_i^2)} dz_i.
\]

The last integral is an example of well-studied \( \beta = 2 \) ensemble and may be presented in the form of a determinant:

\[
I_{Sp(2n)}(p(\tilde{p})) = \det [N_{ij}(\tilde{p})]_{i,j=1,...,n},
\]

where \( N_{ij} \) are the so-called moments. In our case

\[
N_{ij}(\tilde{p}) = \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \prod_{n=1}^\infty \frac{1}{2} \tilde{p}_n e^n (4 - z_i^2) dz_i.
\]

The fermionic representation for \( \beta = 2 \) ensembles is known, see [35, 36]. In our case, it may be written as

\[
I_{Sp(2n)}(p(\tilde{p})) = \langle n, -n | e^{\sum_{i,j=1}^\infty \frac{1}{2} \tilde{p}_n (\alpha_\mu^{(i)} - \alpha_\mu^{(j)})} e^{\int_{-\pi}^{\pi} \psi^{(1)}(z) \psi^{(2)}(z) (4 - z^2) dz} | 0, 0 \rangle
\]

(see lemma 5 for the notations).

4.3. Integrals over the orthogonal group

Consider the following integral over the orthogonal group \( O(N) \)

\[
I_{O(N)}(p) = \int_{O(2n)} e^{\sum_{i,j=1}^\infty \frac{1}{2} \rho_m (\alpha_m^{(i)} - \alpha_m^{(j)})} d_{i,j} O,
\]

where \( d_{i,j} O \) is the corresponding Haar measure. Explicitly,

\[
I_{O(2n)}(p) = \frac{2^{(n-1)^2}}{\pi^n n!} \int_{0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \pi} \prod_{i<j} (\cos \theta_i - \cos \theta_j)^2 \prod_{i=1}^n e^{\sum_{n=1}^\infty \frac{1}{2} \rho_m (\alpha_m^{(i)} - \alpha_m^{(j)})} \cos \theta_i d\theta_i.
\]
where $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_n}$ are the eigenvalues of $O(2n)$. And

$$I_{O(2n+1)}(p) = \frac{2\pi^n}{\pi^{n+1}} \int_{0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \pi} \left( \prod_{i<j} (\cos \theta_i - \cos \theta_j) \right) \left( \sum_{i=1}^{\infty} \frac{1}{2} p_{\nu_i} (1 + 2 \cos \nu_i \theta_i) \right) \sin^2 \frac{\theta_i}{2} d\theta_i,$$

where $e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}$, 1 are eigenvalues of $O(2n+1)$. As in the previous subsection, we calculate this integral in various ways.

**Remark 6.** Note that

$$I_{O(2n)}(p) = 2I_{O(2n)}(p + [1] + [-1])$$

According to (54) and (29) and (28) we obtain

$$I_{O(2n)}(p) = 2^{1-n} \int_{U(n)} \tau_+(U|p) \tau_+(U^{-1}|p) d_U U$$

$$I_{O(2n+1)}(p) = 2^{1-n} \int_{U(n)} \tau_+(U|p) \tau_+(U^{-1}|p) (1 - U) \det (1 - U^{-1}) d_U U.$$

**Proposition 6.**

$$I_{O(n)}(p + p^*) e^{-\frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{2} (p_{\nu_i} + p_{\nu_i}^*)^2} e^{-\frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{2} (p_{2\nu_i} + p_{2\nu_i}^*)^2} = \frac{I_{U(n)}(p, p^*)}{I_{U(\infty)}(p, p^*)}.$$  

The proof repeats the proof of proposition 2.

### 4.3.1. Perturbation series for integrals over orthogonal group as series in characters $sp_{\lambda}(p)$, $o_{\lambda}(p)$, $s_{\lambda}(p)$

**Proposition 7.**

$$I_{O(n)}(p) = \sum_{\lambda \in s \otimes \lambda \subset N} s_{\lambda}(p).$$

Equality (94) follows from the Cauchy–Littlewood formula

$$e^{\sum_{i=1}^{\infty} \frac{1}{2} p_{\nu_i} O^*} = \sum_{\lambda \in s \otimes \lambda \subset N} s_{\lambda}(O) s_{\lambda}(p)$$

and (see for instance (3.19)–(3.21) in sections 3 and 7 in [45])

$$\int_{O \in O(n)} s_{\lambda}(O) d_O O = \begin{cases} 1 & \text{if } \lambda \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 8.**

$$I_{O(2n)}(p) = 2^{1-n} \sum_{\lambda \in s \otimes \lambda \subset N} (sp_{\lambda}(p))^2 = 2^{1-n} \sum_{\lambda \in s \otimes \lambda \subset s} (o_{\lambda}(-p))^2.$$
The first equality in (96) follows from (54), (28) and (8), then the second equality is the result of (23).

**Proposition 9.**

\[ I_{(2n+1)}(\mathbf{p}) = 2^{1-n} \sum_{\lambda, \mu \in \mathcal{P}} s_{\lambda}(U) a_{\lambda, \mu}(\mathbf{p}) = 2^{1-n} \sum_{\lambda, \mu \in \mathcal{P}} a_{\lambda, \mu}(\mathbf{p}), \]  

where

\[ a_{\lambda, \mu} = \int_{U(n)} s_{\lambda}(U) s_{\mu}(U^{-1}) \det (2 - U - U^{-1}) dU. \]

The last formula may be written in terms of \( h_i := \lambda_i - i + n \) and \( h_i' := \mu_i - i + n \) as follows:

\[ a_{\lambda, \mu} = 2^n \prod_{i=1}^n h_i - n \prod_{i=1}^n h_i' - 1 - \prod_{i=1}^n h_i + 1. \]

The first equality of (97) follows from (57), (92), (28) and (8), then the second one follows from (23).

4.3.2. Integrals over the orthogonal group as BKP \( \tau \) functions.

**Proposition 10.**

\[ n! I_{(N)}(\mathbf{p}) = \langle N | \gamma(\mathbf{p}) e^{\frac{i}{\pi n} \int \psi^{(x^{-1})} \psi(x)(x^{-1} - x)^{-1} \frac{dx}{x}} \psi(\mathbf{1}) \rangle \]  

where \( N \) may be even or odd.

**Proof.** It is convenient to rewrite the integral in the fermionic exponent in (99):

\[ \frac{1}{4\pi i} \int \frac{\psi^{(x^{-1})} \psi(x)(x^{-1} - x)^{-1} \frac{dx}{x}}{\sin \theta} = \frac{i}{\sqrt{2}} \int_0^\pi \frac{\psi^{(1)} \psi(\mathbf{1})}{\sin \theta} d\theta \]

(one may note that \( \frac{\psi^{(1)}}{\sin \theta} - \psi^{(1)} \psi(\mathbf{1}) \) is not singular at \( \theta = 0 \)). The proof basically repeats the proof of proposition 3. For \( N = 2n \), the vacuum expectation value in the right-hand side of (99), we obtain

\[ \frac{1}{n!} \left( \frac{i^n}{\pi^n n!} \int_0^\pi \cdots \int_0^\pi \Delta_{2n}(e^{-i\theta_1}, e^{i\theta_1}, \ldots, e^{-i\theta_1}, e^{i\theta_1}) \prod_{i=1}^n e^{2\pi i \theta_i} \frac{d\theta_i}{\sin \theta_i} \right). \]

For \( N = 2n + 1 \), the vacuum expectation value is

\[ \frac{1}{2n!} \left( \frac{i^n}{\pi^n n!} \int_0^\pi \cdots \int_0^\pi \Delta_{2n}(e^{-i\theta_1}, e^{i\theta_1}, \ldots, e^{-i\theta_1}, e^{i\theta_1}, 1) \prod_{i=1}^n e^{2\pi i \theta_i} \frac{d\theta_i}{\sin \theta_i} \right). \]

Then formula (72) proves (99). \( \square \)

4.3.3. Pfaffian representation. As before, applying Wick’s rule, we obtain the Pfaffian representation of the integral (87).
Proposition 11. Where the rank of the orthogonal group is even, we have

\[
I_{\mathbb{O}(2n)}(\mathbf{p}) = \text{Pr} \left[ M_{ij}(\mathbf{p}) \right]_{k,j=1,\ldots,2n},
\]

where

\[
M_{ij}(\mathbf{p}) = \frac{1}{4\pi i} \int \left( x^{i-k} - x^{k-i} \right) \left( x - x^{-1} \right)^{-1} e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (x^n + x^{-n})} \, dx.
\]

In the case of an odd rank we get

\[
I_{\mathbb{O}(2n+1)}(\mathbf{p}) = \epsilon^{\sum_{m>0} \frac{1}{\beta} p_m} \text{Pr} \left[ M_{ij}(\mathbf{p}) \right]_{k,j=1,\ldots,2n+2},
\]

where, for \( k,j \leq 2n+1 \), the entries \( M_{ij} \) are given by the same formula (101), and

\[
M_{k,2n+2} = -M_{2n+2,k} = \frac{1}{2\pi i} \int x^{k-2} e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (x^n + x^{-n})} \, dx.
\]

For the proof we notice that from Wick’s rule we get

\[
M_{ij}(\mathbf{p}) = \frac{1}{2\pi i} \langle 0 | \psi_i^\dagger \psi_j^\dagger \int \frac{\psi(x^{-1}) \psi(x)}{x^{-1} - x} e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (x^n + x^{-n})} \, dx | 0 \rangle.
\]

And for the last column/row of the matrix \( M \) we use

\[
e^{\frac{i}{\sqrt{2}} \phi(1)} | 0 \rangle = \left( 1 + \frac{1}{\sqrt{2}} \phi(1) \right) | 0 \rangle
\]

(see appendix A.3) and apply Wick’s rule.

4.3.4. Relation of the integral over \( \mathbb{O}(N) \) to an integral over \( U(2n) \). Next turn to its two-component KP counterpart. According to proposition 4 in [43] we have

\[
\left( I_{\mathbb{O}(N)}(\mathbf{p}) \right)^2 = \langle N, -N | e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (n^{(1)}_{1,1} - n^{(2)}_{1,1})} e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (n^{(2)}_{1,1} (x^{-1})^{(1)} (x^{-1})^{(2)} \langle 0, 0 \rangle, \right.
\]

where we use the additional fermions \( \psi, \psi^\dagger \) which anticommute with \( \psi^{(1)}(x) \) and \( \psi^{(2)}(x) \), also \( \psi \psi^\dagger + \psi^\dagger \psi = 1 \) and \( \psi^\dagger |*, \ast \rangle = 0 \), see section 2 in [43]. Other notations are the same as in lemma 5.

4.3.5. As a 1DTL-NLS \( \tau \)-function. Denote \( z = x + x^{-1} \) and introduce variables \( \tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_2, \ldots) \) with the help of (82). Then, similar to (83), we get

\[
I_{\mathbb{O}(2n)}(\tilde{\mathbf{p}}) = \int_{-2}^{2} \cdots \int_{-2}^{2} \prod_{i<j} (z_i - z_j)^2 \prod_{j=1}^{n} e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (\tilde{p} - \tilde{c}(\tilde{\mathbf{p}})) \, dz_i
\]

\[
I_{\mathbb{O}(2n+1)}(\tilde{\mathbf{p}}) = \int_{-2}^{2} \cdots \int_{-2}^{2} \prod_{i<j} (z_i - z_j)^2 \prod_{j=1}^{n} e^{\sum_{m=1}^{\infty} \frac{2}{\beta} p_m (\tilde{p} - \tilde{c}(\tilde{\mathbf{p}})) \, (2 - z_i) \, dz_i
\]
4.4. On some integrals over unitary matrices

(1) Here we consider a Cauchy-like integral:

\[
(-1)^{\frac{1}{2}(N^2-N)} \int_{U(N)} e^{\sum_{i>j} \frac{1}{z_i z_j}} t^{i-j} (1 - U^{-1})^{-1} \det U^{2N-2} dU = (-1)^{\frac{1}{2}(N^2-N)} \exp \sum_{n \geq 0} \frac{pn}{n} \tag{107}
\]

\[
= \sum_{\lambda \in \Lambda : \lambda \leq N} s_{\lambda}(\mathbf{p}) = \sum_{\lambda \in \Lambda : \lambda \leq N} (-1)^{|\lambda|} o_{\lambda}(-\mathbf{p}), \tag{108}
\]

where we imply \( \det (1 - U^{-1})^{-1} = \sum_\lambda s_\lambda(U^\dagger) s_\lambda(p^*) \), \( p^* = (1, 1, 1, \ldots) \) (Cauchy–Littlewood identity).

To prove (107) we need

\[
(2\pi)^{-N} \int \det \left[ \frac{e^{\delta_{i\lambda}}}{z_i} \right] \prod_{i<j} \frac{dz_i}{z_i} \prod_{i>j} \frac{dz_j}{z_j} = \prod_{i} \prod_{j} (1 - \frac{1}{z_i z_j})^{-1} \to \int_{U(N)} s_{\lambda}(U) s_{\lambda}(U^\dagger) \det U^d dU = \delta_{\lambda + \lambda', \lambda'}, \tag{109}
\]

which generalizes (8). Let us substitute the Cauchy–Littlewood series for the exponential and for the determinant \( (1 - U^{-1})^{-1} \) inside the integral (107). These series are equal, respectively, to \( \sum_\lambda s_{\lambda}(\mathbf{p}) s_{\lambda}(U) \) and to \( \sum_\lambda s_{\lambda}(U^\dagger) s_{\lambda}(\mathbf{p}^*) \). Using (109) where \( L = N - 1 \), up to the sign factor we obtain \( \sum_\lambda s_{\lambda}(\mathbf{p}) s_{\lambda}(\mathbf{p}^*) \), where \( \lambda \to \lambda + N - 1 \). Notice that each \( s_{\lambda}(\mathbf{p}^*) = 1 \) for \( \lambda = (n) \), \( n = 0, 1, 2 \ldots \) and vanishes otherwise. Then, the first equality is true, because \( \sum_{n \geq 0} s_{(n)}(\mathbf{p}) = \exp \sum_{n \geq 0} \frac{pn}{n} \).

The second equality may be obtained with the help of the Schur–Littlewood relation (see sections 1 and 5, example 4 in [45], p 76)

\[
\sum_\lambda s_{\lambda}(U^\dagger) = \prod_{i<j} (1 - z_i z_j)^{-1} \prod_k (1 - z_k)^{-1}, \]

where \( z_i, i = 1, \ldots, N \) are eigenvalues of \( U \) written in the form

\[
\prod_i (1 - z_i^{-1}) \prod_j (1 - z_j^{-1}) \sum_{\lambda} s_{\lambda}(U^\dagger) = 1
\]

and further rewritten in the form

\[
(-1)^{\frac{1}{2}(N^2-N)} (\det U)^{1-N} \det (1 - U^\dagger) \left( \prod_{i<j} (1 - z_i z_j) \sum_{\lambda} s_{\lambda}(U^\dagger) \right) = 1,
\]

where \( z_i, i = 1, \ldots, N \) are eigenvalues of \( U \). Then, inserting the left-hand side of the last equality inside the integral and using first (28) and then (8) we obtain (108). The third equality follows from (23). Relation (107) proves that sums of characters (108) are the elementary KP tau function with \( \mathbf{p} \) being the KP higher times.

(2) Similarly, for \( N > 1 \)

\[
0 = (-1)^{\frac{1}{2}(N^2-N)} \int_{U(N)} e^{\sum_{i>j} \frac{1}{z_i z_j}} t^{i-j} (1 - U^2)^{-1} \det U^{2N-2} dU = \sum_{\lambda \in \Lambda : \lambda \leq N} o_{\lambda, \lambda}^{\lambda}(\mathbf{p}) = \sum_{\lambda \in \Lambda : \lambda \leq N} s_{\lambda}(\mathbf{p}). \tag{110}
\]
Proof. To get the second equality, we use
\[ \sum_{\lambda^e} s_{\lambda}(U) = \prod_{i < j} (1 - z_i z_j)^{-1} \]
(see sections 1 and 5, example 5(b) in [45] p 77) written in the form
\[
(-1)^{(N^2 - N)} \det U^{1-N} \det (1 - U^2)^{-1} \left( \prod_{i < j} (1 - z_i z_j) \sum_{\lambda} s_{\lambda}(U) \right) = 1
\]
and relations (29) and (8). The third equality results from (23). The first equality in (110) is obtained using \( \det (1 - U^2) = \sum_{\lambda} s_{\lambda}(U)s_{\lambda}(p_o) \), \( p_o = (0, 1, 0, 1, \ldots) \) and (109), which gives 0 for each \( N \).

\[ \square \]

5. The character expansion for \( \beta = 2 \) ensembles

5.1. The character expansions

In this section we want to obtain various character expansions of certain matrix integrals.

Lemma A.6 in appendix A.1 and series (28) and (29) induce the character expansion for a number of matrix integrals. Relations (A.2)–(A.4) may be used in the study of the so-called \( \beta = 2 \) ensembles. Then (A.9) yields a link of these ensembles with the BKP tau function [33].

Let
\[
V(z, T) = \sum_{m > 1} \frac{1}{m!} z^m T_m
\]
and variables \( T = (T_1, T_2, \ldots) \) and \( p = (p_1, p_2, \ldots) \) are linearly dependent and related as follows
\[
V(z, T) - c(T) = V(x, p) + V(x^{-1}, p) \quad \quad c(T) = \sum_{n=1}^{\infty} \frac{(2n)!}{n! n!} T_{2n} \frac{1}{2n}.
\]

For instance, \( T_1 = t_1 - \sum_{n=1}^{\infty} (2n + 1) t_{2n+1} \).

Then we have

Proposition 12.

\[
I_n(T) = \int \Delta(z_1, \ldots, z_n)^2 \prod_{i=1}^{n} z_i^{N} e^{V(z, T)} d\mu(z_i)
\]
\[
= \sum_{h_1 > \ldots > h_n \geq 0} s_{\{h\}}(T) \pi_{\{h\}}(N)
\]
\[
= \sum_{h_1 > \ldots > h_n \geq 0} s_{\{h\}}(p^{(1)}) \pi_{\{h, \delta\}}^{00}(N) s_{\{\delta\}}(p^{(2)})
\]
\[
= \sum_{h_1 > \ldots > h_n \geq 0} e^{c(p)} \sigma_{\{h\}}(p) \pi_{\{h, \delta\}}(N) o_{\{\delta\}}(p) e^{c(p)}
\]
\begin{equation}
\pi_{ij}^{0+}(N) = \int z^{i+j+N} d\mu(z) 
\end{equation}

\begin{equation}
\pi_{ij}^{-+}(N) = \int x^{i-j}(x+x^{-1})^{N} d\mu(z(x)) 
\end{equation}

\begin{equation}
\pi_{ij}^{--}(N) = \int x^{i-j}(x-x^{-1})^{N} \frac{d\mu(z(x))}{1-x^{-2}} 
\end{equation}

\begin{equation}
\pi_{ij}^{+-}(N) = \int x^{i-j}(x-x^{-1})^{N} \frac{d\mu(z(x))}{1-x^{-2}} 
\end{equation}

**Proof.** We use lemma A.6 of appendix A.1 and relations (114), (28) and (29). The Schur functions, which are involved in (28) and (29), we present as ratios of the determinants (1). Finally, we use the simple identity (sometimes called the Andreif identity)

\[ \int \det [\tau_i(x)]_{i=1, \ldots, n} \det [\tau_j(x)]_{i=1, \ldots, n} = n! \det \left[ \int \tau_i(x) \tau_j(x) d\mu(x) \right]_{i, j=1, \ldots, n} \]

\[ \int \det [\tau_i(x)]_{i=1, \ldots, n} \det [\tau_j(x)]_{i=1, \ldots, n} \prod_{i=1}^{n} \mu(x_i) = n! \det \left[ \int \tau_i(x) \tau_j(x) d\mu(x) \right]_{i, j=1, \ldots, n} \]

to get moment matrices (123)–(126).

**Remark 7.** Series (116) is a special case of (117) where \( p^{(2)} = 0 \).

### 5.2. \( \beta = 2 \) ensembles as the BKP \( \tau \) function

Using (A.9) of the appendix, one may verify that

\[ I_{q}(T(p)) = n! e^{\mathbf{c}^{T}(p)} \sum_{\alpha>0} \sum_{\beta} \frac{1}{P_{\pi}(\alpha, \beta)} e^{\int \frac{1}{2}(\mathbf{c}(\alpha) + \mathbf{d}(\beta)) d\mu(z(x)) + \sqrt{2} \int c(x) d\mu(x) \mathbf{d}(\alpha)} \phi(0). \]

Applying Wick’s rule, we get the Pfaffian representation
Now we consider the two-matrix models. Let

\[ I_\nu(T(p)) = n! Pf [A_\nu(p),] \tag{128} \]

for \( n \) even.

**Remark 8.** The integrals (115) may be related both to the BKP hierarchy where \( p \) are BKP higher times and to the 1D Toda chain with higher times \( \tilde{p} \), see (82).

6. The character expansion of two-matrix models

Now we consider the two-matrix models. Let

\[ z = x + x^{-1}, \quad \tilde{z} = \tilde{x} + \tilde{x}^{-1}. \tag{129} \]

We express the variables \( p \) as a linear combination of the variables \( p^{(1)} \) and the variables \( \tilde{p} \) as a linear combination of the variables \( p^{(2)} \):

\[ p_m = \sum_{n=1}^\infty D_{mn}p_n^{(1)}, \quad \tilde{p}_m = \sum_{n=1}^\infty \tilde{D}_{mn}p_n^{(2)} \]

in such a way that

\[ V \left( \frac{az + b}{cz + d} \cdot p^{(1)} \right) = V(x, p), \quad V \left( \frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}} \cdot \tilde{p}^{(1)} \right) = V(\tilde{x}, \tilde{p}). \tag{130} \]

A similar method as in (A.2) and (A.3) of appendix A.1 gives a character expansion of

\[ I_\nu(p^{(1)}, p^{(2)}) = \int \Delta(z_1, \ldots, z_n) \Delta(\tilde{z}_1, \ldots, \tilde{z}_n) \prod_{i=1}^n \eta_i e^{V(z, p^{(1)})} e^{V(\tilde{z}, p^{(2)})} d\mu(z, \tilde{z}). \tag{131} \]

For the sake of simplicity let us take \( d\mu(z, \tilde{z}) = 0 \) if either \( z \geq 1 \) or \( \tilde{z} \geq 1 \), in (131) and in addition

\[ V(z^{-1}, p^{(1)}) = V(x, p), \quad V(\tilde{z}^{-1}, p^{(1)}) = V(\tilde{x}, \tilde{p}) \]

such that \( p_1 = p_1^{(1)}, p_2 = p_2^{(1)}, p_3 = p_3^{(1)} - t(1)_1, p_4 = p_4^{(1)} - 2t(1)_2 \) and so on. Then

\[ I_\nu(p^{(1)}, p^{(2)}) = \sum_{k_1 > \cdots > k_2 \geq 0} s_{i(h)}(p^{(1)}) G_{h,i}^{(0)} s_{i(h)}(p^{(2)}) \tag{132} \]

\[ = \sum_{k_1 > \cdots > k_2 \geq 0} e^{c(h)} \delta_{i(h)}(p) G_{h,i}^{(0)} \delta_{i(h)}(\tilde{p}) e^{c(\tilde{p})} \tag{133} \]

\[ = \sum_{k_1 > \cdots > k_2 \geq 0} e^{c(h)} \delta_{i(h)}(p) G_{h,i}^{(0)} \delta_{i(h)}(\tilde{p}) e^{c(\tilde{p})} \tag{134} \]

\[ = \sum_{k_1 > \cdots > k_2 \geq 0} e^{c(h)} \delta_{i(h)}(p) G_{h,i}^{(0)} \delta_{i(h)}(\tilde{p}) e^{c(\tilde{p})} \tag{135} \]

\[ = \sum_{k_1 > \cdots > k_2 \geq 0} e^{c(h)} \delta_{i(h)}(p) G_{h,i}^{(0)} \delta_{i(h)}(\tilde{p}) e^{c(\tilde{p})} \tag{136} \]
\[ G_{(k,\tilde{k})}^{\alpha\beta} = \det \left[ C_{\alpha \beta, \tilde{\alpha} \tilde{\beta}} \right] \quad \alpha, \beta = \pm, 0 \]  

where

\[ G_{ij}^{00} = \int \overline{z}^{j+N} \overline{z}^{j+N} d\mu(z, \overline{z}) \]  

\[ G_{ij}^{+} = \int x^{-\frac{m(1-1)}{2}} \overline{x}^{-\frac{m(1-1)}{2}} (x + x^{-1})^N (\overline{x} + \overline{x}^{-1})^N d\mu(z(x), \overline{z}(\overline{x})) \]  

\[ G_{ij}^{-} = \int x^{-\frac{m(1-1)}{2}} \overline{x}^{-\frac{m(1-1)}{2}} (x + x^{-1})^N (\overline{x} + \overline{x}^{-1})^N \frac{d\mu(z(x), \overline{z}(\overline{x}))}{(1-x^2)(1-x^4)} \]  

\[ G_{ij}^{++} = \int x^{-\frac{m(1-1)}{2}} \overline{x}^{-\frac{m(1-1)}{2}} (x + x^{-1})^N (\overline{x} + \overline{x}^{-1})^N \frac{d\mu(z(x), \overline{z}(\overline{x}))}{1-x^2} \]  

\[ G_{ij}^{0+} = \int \overline{z}^{j+N} \overline{z}^{j+N} d\mu(z, \overline{z}) \]  

\[ G_{ij}^{0-} = \int \overline{z}^{j+N} \overline{z}^{j+N} d\mu(z, \overline{z}) \]  

**Example.** Take \( \tilde{z} = \tilde{z}, \quad x = \bar{x}, \quad t^{(1)} = \bar{t}^{(2)}, \quad t_m = \bar{t}_m, \quad \text{and} \quad N = 0. \) Take also \( d\mu(z, \overline{z}) = f(|z|) \frac{d\bar{t}}{|1-x^2|}, \) where \( z \) and \( x \) are related by (129). Then

\[ I_{m}(p^{(1)}, \bar{p}^{(2)}) = \sum_{\lambda} r_{\lambda} s_{\lambda} (p) s_{\lambda} (\bar{p}). \]

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**Appendix**

A.1. **Rewriting Vandermonde determinants**

We have useful elementary

**Lemma A.6.** Let

\[ z = x + x^{-1}, \quad x = \frac{z}{2} \pm \frac{1}{2} \sqrt{z^2 - 4}. \]  

\[ (A.1) \]
(Joukovsky transform.)

Then
\[
\prod_{1 \leq k < j \leq n} (z_k - z_j) = \prod_{k < j \leq n} (1 - x_k x_j) \prod_{1 \leq k < j \leq n} (x_k^{-1} - x_j^{-1}) \tag{A.2}
\]

\[
= \tau_+ (X|0) \prod_{1 \leq k < j \leq n} (x_k^{-1} - x_j^{-1}) \tag{A.3}
\]

\[
= \tau_- (X|0) \prod_{1 \leq k < j \leq n} (x_k^{-1} - x_j^{-1}) \frac{n}{\prod_{j=1}^{n} 1 - x_j^2}. \tag{A.4}
\]

And, as a result,
\[
\prod_{1 \leq k < j \leq n} (z_k - z_j)^2 = \prod_{1 \leq k < j \leq n} (1 - x_k x_j)(1 - x_k^{-1} x_j^{-1}) \prod_{1 \leq k < j \leq n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \tag{A.5}
\]

\[
= \tau_+ (X|0) \tau_+ (X^{-1}|0) \prod_{1 \leq k < j \leq n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \tag{A.6}
\]

\[
= \tau_- (X|0) \tau_- (X^{-1}|0) \prod_{1 \leq k < j \leq n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \frac{1}{\prod_{j=1}^{n} (1 - x_j^2)(1 - x_j^{-2})}. \tag{A.7}
\]

\[
= \tau_- (X|0) \tau_+ (X^{-1}|0) \prod_{1 \leq k < j \leq n} (x_k - x_j)(x_k^{-1} - x_j^{-1}) \frac{n}{\prod_{j=1}^{n} 1 - x_j^2}. \tag{A.8}
\]

\[
= \Delta_{2n}(x_1^{-1}, x_1, \ldots, x_n^{-1}, x_n) \prod_{j=1}^{n} \frac{1}{x_j^{-1} - x_j}. \tag{A.9}
\]

Indeed, we have
\[
(1 - x_j^{-1} x_k^{-1})(x_j - x_k) = (1 - x_j x_k)(x_j^{-1} - x_k^{-1}) = z_j - z_k
\]
and obtain (A.5) and (A.6).

A.2. Vertex operators

Vertex operators we need are as follows
\[
\hat{X}(L, p, \lambda) := e^{\sum_{n=1}^{\infty} \lambda^n p e^{-\sum_{n=1}^{\infty} \frac{\lambda^n}{n}}} \quad \hat{X}^\dagger(L, p, \lambda) := e^{-\sum_{n=1}^{\infty} \lambda^n p e^{\sum_{n=1}^{\infty} \frac{\lambda^n}{n}}} \tag{A.10}
\]

\[
\hat{Y}(L, s, \lambda) := e^{-\sum_{n=1}^{\infty} \lambda^n s e^{\sum_{n=1}^{\infty} \frac{\lambda^n}{n}}} \quad \hat{Y}^\dagger(L, s, \lambda) := e^{\sum_{n=1}^{\infty} \lambda^n s e^{-\sum_{n=1}^{\infty} \frac{\lambda^n}{n}}} \tag{A.11}
\]

An interesting historical fact is that the formula which relates fermions to bosons was first found in [63].

The following bosonization relation is useful
\[
(L + N|\Gamma(p + \sum_{i=1}^{N} [p_i]) = \frac{\langle L|\psi^{\dagger}(p_1^{-1})\cdots\psi^{\dagger}(p_N^{-1})\gamma_{+}(p)\rangle}{\prod_{i=1}^{N} p_i^{(L+1)(N-1)} \prod_{j>i}(p_i - p_j)}.
\]

Introduce
\[
\hat{\Omega}_n(L, \mathbf{p}) = \text{res}_\lambda \left( \frac{\partial^n}\partial\lambda^n \vec{\partial}\hat{X}(L, \mathbf{p}, \lambda) \right), \quad \hat{\Omega}_n^{s}(L, \mathbf{s}) = \text{res}_\lambda \left( \hat{\hat{Y}}(L, \mathbf{s}, \lambda) \frac{\partial^n}\partial\lambda^n \hat{\hat{Y}}(L, \mathbf{s}, \lambda) \right).
\]

A.3. Fermions

Recall the fermionic fields \(\psi_i(z) = \sum_i \psi_i z^i\) and \(\psi_i^\dagger(z) = \sum_i \psi_i^\dagger z^{-i-1}\) from Section 1, their modes satisfy:
\[
\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0 \quad \text{and} \quad \psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = \delta_{ij}.
\]

We have
\[
\hat{X}(L, \mathbf{p}, \lambda) X^{\dagger}(L, \mu) \langle N + L|\Gamma(p)g\Gamma^{\dagger}(s)|L \rangle = \langle N + L|\Gamma(p)\psi(\lambda)\psi^{\dagger}(\mu)g\Gamma^{\dagger}(s)|L \rangle.
\]
\[
\hat{Y}(L, \mathbf{s}, \mu) Y(\lambda) \langle N + L|\Gamma(p)g\Gamma^{\dagger}(s)|L \rangle = \langle N + L|\Gamma(p)\psi(\lambda)\psi^{\dagger}(\mu)\Gamma^{\dagger}(s)|L \rangle.
\]
Then it follows that
\[
\hat{\Omega}_n(L, \mathbf{p}) \langle N + L|\Gamma(p)g\Gamma^{\dagger}(s)|L \rangle = \langle N + L|\Gamma(p)\hat{\Omega}_n g\Gamma^{\dagger}(s)|L \rangle
\]
\[
\hat{\Omega}_n(L, \mathbf{p}) \langle N + L|\Gamma(p)g\Gamma^{\dagger}(s)|L \rangle = \langle N + L|\Gamma(p)\hat{\Omega}_n g\Gamma^{\dagger}(s)|L \rangle,
\]
where
\[
\hat{\Omega}_n = \text{res}_\lambda \left( \frac{\partial^n\psi(\lambda)}{\partial\lambda^n} \psi^{\dagger}(\lambda) \right).
\]

Using the fermionic representation, one may verify that tau functions related to the considered ensembles (dd-OE, dd-GinOE, dd-SE, dd-GinSE) obey the constraints
\[
\left( \hat{\Omega}_n(L, \mathbf{p}) - \hat{\Omega}_n^{s}(L, \mathbf{s}) \right) \tau(L, \mathbf{p}', \mathbf{s}') = 0, \quad n \geq 1, \text{ odd},
\]
where \(t'_k = t_k - \frac{1}{2}\delta_{2,k} s'_k = s_k - \frac{1}{2}\delta_{2,k} (\text{this shift appears due to the Gauss measure in undeformed ensembles}).

Now, the two-component Fermi fields used in section 6 are defined as
\[
\psi^{(i)}(z) = \sum_{n \in \mathbb{Z}} z^n \psi_{2n+i}, \quad \psi^{(i)^\dagger}(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} \psi_{2n+i}^\dagger
\]
where \(i = 1, 2, 3\). More details about multi-component fermions may be found in [31, 33], and, in relation to matrix models, in [27].

A.4. Characters of classical Lie groups/algebras

We recall some information about the characters of classical Lie algebras as presented in [18], section 24. The character of a simple Lie algebra is given by the Weyl character formula. For a dominant integral weight \(\lambda\) the character \(c_{\lambda}\) is equal to
\[ ch_\lambda = \frac{A_{\lambda + \rho}}{A_\rho}, \quad \text{where} \quad A_\mu = \sum_{w \in W} \sgn(w) e(w(\mu)). \tag{A.21} \]

Here \( W \) is the Weyl group of the simple Lie algebra, and \( \rho \) the sum of all the fundamental weights, or equivalently half the sum of all positive roots \( R^+ \). One has \( A_\rho = \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) = e(\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha)) = \prod_{\alpha \in R^+} e(-\rho)(e(\alpha) - 1) \).

**A.4.1. First case: \( sl_n \) or rather \( gl_n \).** We identify the standard Cartan subalgebra, the diagonal matrices of \( gl_n \), with its dual via the trace form \( (a, b) = \text{trace}(ab) \). Let \( \epsilon_i = E_{ii} \), and assume \( \lambda = \sum_{i=1}^n \lambda_i \epsilon_i \). In this case \( R^+ \) consists of all elements \( \epsilon_i - \epsilon_j \), with \( i < j \) and \( \rho = \frac{1}{2} \sum_{i<j} (\epsilon_i - \epsilon_j) = \frac{n(n+1)}{2} - n \epsilon_i \).

Fulton and Harris [18] have a different formula for \( \rho \), they claim \( \rho = \sum_i (n-i) \epsilon_i \), which is wrong. The Weyl group is the group \( S_n \), the permutation group that permutes the elements \( \epsilon_i \). Denote by \( x_i = e(\epsilon_i) \), then

\[ A_{\lambda + \rho} = \sum_{w \in S_n} \sgn(w) x_{\lambda_1 + \frac{n+1}{2}} x_{\lambda_2 + \frac{n+1}{2} - 1} \cdots x_{\lambda_n + \frac{n+1}{2} - n} = \det [x_j^{\lambda_i + \frac{n+1}{2} - i}]_{1 \leq i < j \leq n}. \tag{A.22} \]

Taking \( \lambda = 0 \) in (A.22), we obtain \( A_\rho = \det [x_j^{\frac{n+1}{2} - i}] \). Thus

\[ ch_\lambda = s_\lambda = \frac{\det [x_j^{\lambda_i + \frac{n+1}{2} - i}]_{1 \leq i < j \leq n}}{\det [x_j^{\frac{n+1}{2} - i}]_{1 \leq i < j \leq n}} = \frac{\det [x_j^{\lambda_i + n - i}]_{1 \leq i < j \leq n}}{\det [x_j^{\frac{n}{2} - i}]_{1 \leq i < j \leq n}}. \]

Another presentation of the character is the so-called Giambelli or determinantal formula. One expresses the character in the elementary Schur functions \( s_{(k)}(x) \):

\[ s_\lambda(x) = \det [s_{(\lambda_i + j-i)}(x)]_{1 \leq i < j \leq n}. \]

**A.4.2. Second case: \( sp_{2n} \).** The positive roots are now \( \epsilon_i - \epsilon_j \), with \( 1 \leq i < j \leq n \) and \( \epsilon_i + \epsilon_j \) with \( i \leq j \). The element \( \rho = \sum_{i=1}^n (n - i) \epsilon_i \) and the Weyl group is the group that permutes all \( \epsilon_i \) allowing also all possible sign changes, i.e. \( \epsilon_i \mapsto \pm \epsilon_i \), hence it is the semi-direct product of \( S_n \) and \( \mathbb{Z}_2^n \). Then

\[ A_\mu = \sum_{w \in S_n} \sum_{\sigma \in \mathbb{Z}_2^n} \sgn(w) \sgn(\sigma) e(\sum_{i} (-)^{\sigma_i} \mu_i w(\epsilon_i)). \]

Here \( \sigma = (\sigma_1, \ldots, \sigma_n) \) and \( \sgn(\sigma) = (-)^{\sum_\sigma \sigma} \). Using again \( x_i = e(\epsilon_i) \), it is straightforward to check

\[ A_\mu = \sum_{w \in S_n} \sgn(w) (x_\sigma^\mu - x_{-\sigma}^\mu). \tag{A.23} \]

Hence
\[ ch_\lambda = sp_\lambda = \frac{\det \left[ x_j^{\lambda_i + n-i+1} - x_j^{-\lambda_i - n+i-1} \right]_{1 \leq i,j \leq n}}{\det \left[ x_j^{n-i+1} - x_j^{-n+i-1} \right]_{1 \leq i,j \leq n}}. \]

The determinantal formula in this case is due to Koike en Terada [39]

\[ sp_\lambda(x) = \frac{1}{2} \det \left[ s_{(\lambda_i - i + j)}(x) + s_{(\lambda_i - i - j)}(x) \right]_{ij = 1, \ldots, n}, \]

where we substitute \( x_{n+i} = x_i^{-1} \).

**A.4.3. Third case: \( SO_{2n+1} \)** The positive roots are now \( \epsilon_i - \epsilon_j \), with \( 1 \leq i < j \leq n \) and \( \epsilon_i + \epsilon_j \) with \( i < j \) and all \( \epsilon_i \). The element \( \rho = \sum_{i=1}^n (n + \frac{1}{2} - i) \epsilon_i \). The Weyl group is the same as in the \( sp_{2n} \) case. Hence, also \( A_\mu \) is the same, viz. (A.23), as for \( sp_{2n} \). Hence

\[ ch_\lambda = o_\lambda = \frac{\det \left[ x_j^{\lambda_i + n-i+\frac{1}{2}} - x_j^{-\lambda_i - n+i-\frac{1}{2}} \right]_{1 \leq i,j \leq n}}{\det \left[ x_j^{n-i+\frac{1}{2}} - x_j^{-n+i-\frac{1}{2}} \right]_{1 \leq i,j \leq n}} = \frac{\det \left[ x_j^{\lambda_i + n-i+1} - x_j^{-\lambda_i - n+i-1} \right]_{1 \leq i,j \leq n}}{\det \left[ x_j^{n-i+1} - x_j^{-n+i} \right]_{1 \leq i,j \leq n}}. \]

The determinantal formula is equal to

\[ o_\lambda(x) = \frac{1}{2} \det \left[ s_{(\lambda_i - i + j)}(x) + s_{(\lambda_i - i - j)}(x) \right]_{ij = 1, \ldots, n}, \]

here \( x_{n+i} = x_i^{-1} \) and \( x_{2n+1} = 1 \). In term of irreducible characters \( s^\circ(k)(x) = s(k)(x) - s(k-2)(x) \), one can rewrite (A.24) to

\[ o_\lambda(x) = \frac{1}{2} \det \left[ s_{(\lambda_i - i + j)}(x) + s_{(\lambda_i - i - j)}(x) \right]_{ij = 1, \ldots, n}. \]

**A.4.4. Fourth case: \( SO_{2n} \)** The positive roots are now \( \epsilon_i - \epsilon_j \), with \( 1 \leq i < j \leq n \) and \( \epsilon_i + \epsilon_j \) with \( i < j \) and \( \rho \) is the same as for \( gl_n \), viz. \( \rho = \sum_{i=1}^n (n - i) \epsilon_i \). However, the Weyl group is a subgroup of the Weyl group of \( sp_{2n+1} \), one only allows an even number of sign changes. This leads to

\[ A_\mu = \frac{1}{2} \left( \det \left[ x_j^{\mu_i} - x_j^{-\mu_i} \right]_{1 \leq i,j \leq n} + \det \left[ x_j^{\mu_i} + x_j^{-\mu_i} \right]_{1 \leq i,j \leq n} \right) \]

and hence

\[ ch_\lambda = \frac{\det \left[ x_j^{\lambda_i + n-i} - x_j^{-\lambda_i - n+i} \right]_{1 \leq i,j \leq n}}{\det \left[ x_j^{n-i} + x_j^{-n+i} \right]_{1 \leq i,j \leq n}}. \]

The determinantal formulas (A.24) and (A.25) also hold in this case with the restriction that in this case \( x_{n+i} = x_i^{-1} \), the element \( x_{2n+1} \) does not exist.

**A.5. Pfaffians**

If \( A \) an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say \( k \), the following multilinear form in \( A_{ij}, i < j \leq k \)

\[ \text{Pf}[A] := \sum_{\sigma} \text{sgn} \,(\sigma) \, A_{\sigma(1),\sigma(2)}A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(k-1),\sigma(k)}, \]

(A.26)
where sum runs over all permutation restricted by
\[ \sigma : \sigma (2i - 1) < \sigma (2i), \quad \sigma (1) < \sigma (3) < \cdots < \sigma (k-1), \] (A.27)
coincide with the square root of det $A$ and is called the Pfaffian of $A$, see, for instance [46]. As one can see, the Pfaffian contains $1 \cdot 3 \cdot 5 \cdots (k-1) =: (k-1)!!$ terms.

A.5.1. Wick’s relations. Let each of $w_i$ be a linear combination of Fermi operators:
\[ \hat{w}_i = \sum_{m \in \mathbb{Z}} v_{im} \psi_m^\dagger + \sum_{m \in \mathbb{Z}} u_{im} \psi_m^\dagger, \quad i = 1, \ldots, n. \]
Then the Wick’s formula is
\[ \langle l|\hat{w}_1 \cdots \hat{w}_n|l \rangle = \begin{cases} \text{Pf} [A]_{i,j=1 \ldots n} & \text{if } n \text{ is even} \\ 0 & \text{otherwise}, \end{cases} \] (A.28)
where $A$ is an $n \times n$ antisymmetric matrix with entries $A_{ij} = \langle l|\hat{w}_i \hat{w}_j|l \rangle$, $i < j$.

A.6. Hirota equations

KP Hirota equation:
\[ \oint \frac{dz}{2\pi i} e^{(p^\prime - p - z)z} \tau_\pm (p - \gamma^{-1}[p^\prime]) \tau_\pm (p + \gamma^{-1}[p^\prime]) = 0. \] (A.29)
For $\tau_\pm$ defined by (16) we have for the left-hand side
\[ \oint \frac{dz}{2\pi i} (1 - z^{-2}) e^{\sum_{k \geq 1} \frac{1}{2k} z^{2k}} e^{-\sum_{k \geq 1} \frac{1}{2k} \gamma^{-1}[z^{2k}]}, \]
which is equal to zero. Thus, $\tau_\pm (p|p^\prime)$ is the KP tau function with respect to the $p$ variables.

Now in view of Giambelli relations for characters [8]
\[ o\lambda (p) = \det (o_{(\alpha_{i,\beta})}(p)), \quad sp\lambda (p) = \det (sp_{(\alpha_{i,\beta})}(p)), \quad \lambda = (\alpha_{i,\beta}) \] (A.30)
we can write
\[ \tau_- (p|p^\prime) = \langle 0|e^{\sum_{i > j} \frac{1}{2} \gamma_{ij} e^{\sum_{k \geq 0} (-1)^k i_{(\alpha_{ij})}(p^\prime) \psi_{i-1}^\dagger}} e^{\sum_{k \geq 0} (-1)^k i_{(\alpha_{ij})}(p^\prime) \psi_{i-1}^\dagger}|0 \rangle \] (A.31)
\[ \tau_+ (p|p^\prime) = \langle 0|e^{\sum_{i > j} \frac{1}{2} \gamma_{ij} e^{\sum_{k \geq 0} (-1)^k sp_{(\alpha_{ij})}(p^\prime) \psi_{i-1}^\dagger}} e^{\sum_{k \geq 0} (-1)^k sp_{(\alpha_{ij})}(p^\prime) \psi_{i-1}^\dagger}|0 \rangle. \] (A.32)
Hirota equations for the large BKP hierarchy were written in [33]. For 2-BKP hierarchy the Hirota equations are as follows [60]
\[ \oint \frac{dz}{2\pi i} e^{(l^\prime, p^\prime - p - z)z} \tau_{N-1}^N (l, p - \gamma^{-1}[z]) \tau_{N+1}^N (l, p + \gamma^{-1}[z]) \]
\[ + \oint \frac{dz}{2\pi i} e^{(l^\prime, p^\prime - p - z)z} \tau_{N+1}^N (l, p - \gamma^{-1}[z]) \tau_{N-1}^N (l, p + \gamma^{-1}[z]) \]
\[ = \oint \frac{dz}{2\pi i} e^{(l^\prime - l, p^\prime - p - z)s} \tau_{N-1}^N (l, p - \gamma^{-1}[z]) \tau_{N+1}^N (l - 1, p, s - \gamma^{-1}[z]) \]
\[ + \oint \frac{dz}{2\pi i} e^{(l^\prime - l, p^\prime - p - z)s} \tau_{N+1}^N (l, p - \gamma^{-1}[z]) \tau_{N-1}^N (l + 1, p, s + \gamma^{-1}[z]) \]
\[ + \frac{(-1)^{l+l^\prime}}{2} (1 - (1)^{N+1}) \tau_{N^2}^N (l^\prime, p^\prime, s^\prime) \tau_{N}^N (l, p, s). \] (A.33)
The difference Hirota equation may be obtained from the previous one [60]

\[-\frac{\beta}{\alpha - \beta} \tau_N(l, p + [\beta^{-1}]) \tau_{N+1}(l, p + [\alpha^{-1}]) - \frac{\alpha}{\beta - \alpha} \tau_N(l, p + [\alpha^{-1}]) \tau_{N+1}(l, p + [\beta^{-1}])

+ \frac{1}{\alpha \beta} \tau_{N+2}(l, p + [\alpha^{-1}] + [\beta^{-1}]) \tau_{N-1}(l, p) = \tau_{N+1}(l, p + [\alpha^{-1}] + [\beta^{-1}]) \tau_{N}(l, p).\]

(A.34)

A.7 Integrals over the unitary, orthogonal and symplectic groups

For $U(n)$ the Haar measure is

\[dU = \frac{1}{(2\pi)^n} \prod_{1 \leq i < k \leq n} |\epsilon^{\theta_i} - \epsilon^{\theta_k}|^2 \prod_{i=1}^{n} d\theta_i, \quad -\pi \leq \theta_1 < \ldots < \theta_n \leq \pi,\]

(A.35)

where $\epsilon^{\theta_1}, \ldots, \epsilon^{\theta_n}$ are eigenvalues of $U \in U(n)$. For $O(2n)$, the Haar measure is

\[dO = \frac{2^{(n-1)^2}}{\pi^n} \prod_{i<j} \left(\cos \theta_i - \cos \theta_j\right)^2 \prod_{i=1}^{n} d\theta_i, \quad 0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \pi,\]

(A.36)

where $\epsilon^{\theta_1}, \epsilon^{-\theta_1}, \ldots, \epsilon^{\theta_n}, \epsilon^{-\theta_n}$ are eigenvalues of $O \in O(2n)$. For $O(2n+1)$ the Haar measure is

\[dO = \frac{2^n}{\pi^n} \prod_{i<j} \left(\cos \theta_i - \cos \theta_j\right)^2 \prod_{i=1}^{n} \sin^2 \theta_i \frac{1}{2} d\theta_i, \quad 0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \pi,\]

(A.37)

where $\epsilon^{\theta_1}, \epsilon^{-\theta_1}, \ldots, \epsilon^{\theta_n}, \epsilon^{-\theta_n}, 1$ are eigenvalues of $O \in O(2n+1)$. For $Sp(2n)$ the Haar measure is

\[dS = \frac{2^n}{\pi^n} \prod_{i<j} \left(\cos \theta_i - \cos \theta_j\right)^2 \prod_{i=1}^{n} \sin^2 \theta_i d\theta_i, \quad 0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \pi,\]

(A.38)

where $\epsilon^{\theta_1}, \epsilon^{-\theta_1}, \ldots, \epsilon^{\theta_n}, \epsilon^{-\theta_n}$ are eigenvalues of $S \in Sp(2n)$.

A.8 Characters as vacuum expectation values

We can write the determinantal expression (2) for $s_{\lambda}(p)$ as a (vacuum) expectation value [31]

\[s_{\lambda}(p) = \langle k | \Gamma(p) \psi_{\lambda_1} \psi_{\lambda_2-1} \psi_{\lambda_3-2} \cdots \psi_{\lambda_n-0} | k - n \rangle.\]

(A.39)

Now using the commutation relations of $\Gamma(p)$ with the fermions $\psi_j$, we define

\[\psi_j(p) := \Gamma(p) \psi_j \Gamma(p)^{-1} = \sum_{i=0}^{\infty} s_{(i)}(p) \psi_{j-i}, \quad \psi_j^\dagger(p) := \Gamma(p) \psi_j^\dagger \Gamma(p)^{-1} = \sum_{i=0}^{\infty} s_{(i)}(-p) \psi_{j+i}^\dagger.\]

(A.40)

Hence,

\[s_{\lambda}(p) = \langle k | \psi_{\lambda_1-1}(p) \psi_{\lambda_2-2}(p) \cdots \psi_{\lambda_r-n}(p) | k - n \rangle,\]

(A.41)

which is exactly the determinant of (2). On the other hand, if we write $\lambda$ in the Frobenius notation $\lambda = (a_1, a_2, \ldots, a_r | b_1, b_2, \ldots, b_s)$, where $a_i = \lambda_i - i \geq 0$ for $1 \leq i \leq r$ and $\lambda_{r+1} - r - 1 < 0$, with the fermions $\psi_j$, we define

\[\psi_j(p) := \Gamma(p) \psi_j \Gamma(p)^{-1} = \sum_{i=0}^{\infty} s_{(i)}(p) \psi_{j-i}, \quad \psi_j^\dagger(p) := \Gamma(p) \psi_j^\dagger \Gamma(p)^{-1} = \sum_{i=0}^{\infty} s_{(i)}(-p) \psi_{j+i}^\dagger.\]

(A.40)

Hence,

\[s_{\lambda}(p) = \langle k | \psi_{\lambda_1-1}(p) \psi_{\lambda_2-2}(p) \cdots \psi_{\lambda_r-n}(p) | k - n \rangle,\]

(A.41)
and \( b_t = \lambda'_t - i \geq 0 \) for \( 1 \leq i \leq r \) and \( \lambda'_{r+1} - r - 1 < 0 \) for \( \lambda' \) the conjugate partition of \( \lambda \). If we drop the condition that \( a_i = \lambda_i - i \geq 0 \) and \( b_t = \lambda'_t - i \geq 0 \), then we can see

\[
\lambda = (a_1, a_2, \ldots, b_1, b_2, \ldots)
\]

with \( a_i = \lambda_i - i \), and \( b_t = \lambda'_t - i \)

as two infinite sequences. It is then well known (see e.g. [65]) that the sets

\[
A = \{a_1, a_2, a_3, \ldots\} \quad \text{and} \quad B = \{-b_1 - 1, -b_2 - 1, -b_3 - 1, \ldots\}
\]

form a partition of \( \mathbb{Z} \), thus

\[
\psi_{k+\lambda_1-1,1} \psi_{k+\lambda_2-2,1} \ldots \psi_{k+\lambda_n-1,1} |k - n\rangle
\]

\[
= (-)^{b_1 + \cdots + b_t} \psi_{k+a_1,1}^\dagger \psi_{k+a_2,1}^\dagger \ldots \psi_{k+b_{b_1 - 1},1}^\dagger \psi_{k+b_{b_2 - 1},1}^\dagger \ldots \psi_{k-b_{b_1 - 1},1}^\dagger |k\rangle
\]

\( \text{(A.42)} \)

and

\[
s_\lambda(p) = (-)^{b_1 + \cdots + b_t + \sum_{i=1}^n (k | \psi_{k+a_i}(p) \psi_{k+a_i}(p) \ldots \psi_{k+b_{b_1 - 1}}(p) \psi_{k+b_{b_2 - 1}}(p) \ldots \psi_{k-b_{b_1 - 1}}(p) | k\rangle).
\]

\( \text{(A.43)} \)

Now using Wick’s relation, see appendix A.5, this is equal to the Pfaffian expression

\[
(-)^{b_1 + \cdots + b_t + \sum_{i=1}^n} \text{Pf}[A_{ij}]_{i,j=1,\ldots,r} \quad \text{where} \quad A_{ij} = \begin{pmatrix} 0 & B \\ -B^{\dagger} & 0 \end{pmatrix}, \quad \text{for} \quad B_{ij} = \langle k | \psi_{k+a_i}(p) \psi_{k+b_{b_1 - 1}}(p) | k\rangle.
\]

\( \text{(A.44)} \)

Now

\[
\text{Pf}[A_{ij}]_{i,j=1,\ldots,r} = (-)^{\frac{n(n-1)}{2}} \det |B_{ij}|_{i,j=1,\ldots,r}
\]

and

\[
B_{ij} = \langle k | \Gamma(p) \psi_{k+a_i} \psi_{k+b_{b_1 - 1}}^\dagger | k\rangle = (-)^b s_{(a_i,b_i)}(p).
\]

Thus

\[
s_\lambda(p) = \det [s_{(a_i,b_i)}(p)]_{i,j=1,\ldots,r}
\]

\( \text{(A.45)} \)

which is a well-known result of Littlewood [44] (see also [65]).

The orthogonal and symplectic characters can also be expressed as determinants (see e.g. [39], theorems 1.3.2 and 1.3.3)

\[
o_{\lambda}(p) = \det [s_{(\lambda_i+i+j)}(p) - s_{(\lambda_i-i-j)}(p)]_{i,j=1,\ldots,n},
\]

\[
s_{\lambda}(p) = \frac{1}{2} \det [s_{(\lambda_i+i+j)}(p) + s_{(\lambda_i-i-j+2)}(p)]_{i,j=1,\ldots,n}
\]

\( \text{(A.46)} \)

and similarly one has a formula (A.45), see [65]

\[
o_{\lambda}(p) = \det [o_{(a_i,b_i)}(p)]_{i,j=1,\ldots,r} \quad \text{and} \quad s_{\lambda}(p) = \det [s_{(a_i,b_i)}(p)]_{i,j=1,\ldots,r}.
\]

\( \text{(A.47)} \)

Using (A.47), we calculate \( o_{(a_i,b_i)}(p) = o_{(a_i,b_i)}(p) \) explicitly

\[
o_{(a_i,b_i)} = \det \begin{bmatrix}
-s_{(a_1-1)} & s_{(a_1-2)} & s_{(a_1-2)} & \cdots & s_{(a_1-1)} \\
0 & s_{(1)} & s_{(2)} & \cdots & s_{(b)} \\
0 & 0 & s_{(1)} & \cdots & s_{(b-1)} \\
0 & 0 & 0 & \cdots & s_{(1)}
\end{bmatrix}
\]

\( \text{(A.48)} \)
Now let
\[ s_{(a)b} - s_{(a-2)b} + \det \begin{bmatrix} 0 & s_{(a)} - s_{(a-2)} & s_{(a+1)} - s_{(a-3)} & \cdots & s_{(a+b-1)} - s_{(a-b-1)} \\ s_{(0)} & s_{(1)} & s_{(2)} & \cdots & s_{(b)} \\ 0 & s_{(0)} & s_{(1)} & \cdots & s_{(b-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_{(1)} \end{bmatrix} \]

\[ = s_{(a)b} - s_{(a-2)b} - \alpha_{(a-1)b-1} = \sum_{j=0}^{b} (-1)^j s_{(a-j)b-j} - s_{(a-j-2)b-j}. \]

A similar calculation, using again (A.47), shows that \( s_{p(a+1,1')} + s_{p(a)j} = sp_{(a)j}(p) \) is equal to

\[ s_{p(a)b} = s_{(a)b} - s_{(a-1)b-1} - \alpha_{(a)b-2} = \sum_{j=0}^{b} (-1)^j s_{(a-j)b-j} - s_{(a-j)b-j-2}, \]

which is also equal to

\[ s_{p(a)b} = \sum_{j=0}^{b} (-1)^j s_{(a-j)b-j} - s_{(a-j)b-j-2}. \]

We can write both (A.48) and (A.50) as vacuum expectation values, viz.

\[ \alpha_{(a)b}(p) = (-)^b \sum_{j=0}^{\infty} \langle k| \Gamma(p)(\psi_{k+a-j} - \psi_{k+a-j-2})\psi_{k-b+j-1}^\dagger |k \rangle \]

\[ = (-)^b \sum_{j=0}^{\infty} \mathcal{R}_{z=0} \mathcal{R}_{w=0} \left( \frac{z^{a-1} - z^{a+1}}{1-zw} \right)^k \left( \frac{w}{z} \right)^k \langle k| \Gamma(p)\psi(z)\psi^\dagger(w)|k \rangle, \]

where

\[ \psi(z) = \sum_{j \in \mathbb{Z}} \psi_j z^j, \quad \psi^\dagger(z) = \sum_{j \in \mathbb{Z}} \psi_j^\dagger z^{-j-1}. \]

And

\[ s_{p(a)b}(p) = (-)^b \sum_{j=0}^{\infty} \langle k| \Gamma(p)\psi_{k+a-j}^\dagger (\psi_{k-b+j-1}^\dagger - \psi_{k-b+j+1}) |k \rangle \]

\[ = (-)^b \sum_{j=0}^{\infty} \mathcal{R}_{z=0} \mathcal{R}_{w=0} \left( \frac{z^{a-1} - w^{-b-1} - w^{-b+1}}{1-zw} \right)^k \left( \frac{w}{z} \right)^k \langle k| \Gamma(p)\psi(z)\psi^\dagger(w)|k \rangle. \]

Now let \( \lambda = (a_1, a_2, \cdots, a_r, b_1, b_2, \cdots, b_r) \), then

\[ \alpha_{\lambda}(p) = (-)^r \prod_{i=1}^{r} \mathcal{R}_{z_i=0} \mathcal{R}_{w_i=0} \left( z_i^{a_i-1} - z_i^{a_i+1} \right) \]

\[ w_i^{-b_i-1} \det \left[ \left( \frac{w_j}{z_j} \right)^k \langle k| \Gamma(p)\psi(z_j)\psi^\dagger(w_j)|k \rangle \right]_{j=1, \ldots, r}. \]

\[ 31 \]
And, similarly,

\[
sp_{\lambda}(p) = (-)^{\sum b_{i}} \prod_{i=1}^{r} \text{Res}_{w_{j}=0} \text{Res}_{w_{j}=0} z_{i}^{-a_{i}-1} \left( w_{j}^{-b_{j}-1} - w_{j}^{-b_{j}+1} \right) \det \left[ \left( \frac{w_{i}}{z_{j}} \right)^{k} \langle k | \Gamma(p) | w_{j} \rangle \right]_{i,j=1,\ldots,r}. \tag{A.55}
\]

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