Representations of Munn algebras and related semigroups

Yuriy A. Drozd and Andriana I. Plakosh
National Academy of Sciences of Ukraine, Institute of Mathematics, Kyiv, Ukraine

ABSTRACT
We establish representation types (finite, tame or wild) of finite dimensional Munn algebras with semisimple bases. As an application, we establish representation types of finite Rees matrix semigroups, in particular, 0-simple semigroups, and their mutually annihilating unions.

ARTICLE HISTORY
Received 21 March 2022
Communicated by Steffen Oppermann

KEYWORDS
Munn algebras; Rees matrix semigroups; representations of valued graphs; representation types

2020 MATHEMATICS SUBJECT CLASSIFICATION
20M30; 16G60; 20M25; 16G20

1. Introduction
Munn algebras appeared in the theory of semigroups as semigroup algebras of completely 0-simple semigroups [1, 6]. They were immediately used for the study of representations of such semigroups. An important input was made by Ponizovskii in the paper [7], where he established the cases when a finite 0-simple semigroup is representation finite, i.e., only has finitely many indecomposable representations, over an algebraically closed field \( k \) whose characteristic does not divide the order of the underlying group of its Rees matrix presentation [1, Th. 3.5]. He also considered the case of semigroups that are unions of mutually annihilating 0-simple semigroups with common zero.

The questions remained what happens if the field is not algebraically closed and when the representation type of such a semigroup is tame, i.e., indecomposable representations of each dimension form a finite number of 1-parameter families. In this article we give a complete answer to these questions (also for the fields of characteristics that does non divide the orders of the underlying groups). Of course, in the case of an algebraically closed field our criterion of finiteness coincides with that of Ponizovskii. Actually, we obtain criteria of finiteness and tameness for all Munn algebras with semisimple base, even in a bit more wide context than they are considered in [1]. To prove these results, we establish a relation of modules over Munn algebras with representations of valued graphs in the sense of [2] (in the algebraically closed case they are just representations of quivers in the sense of [4]). Then we apply the criteria from this paper.

It follows from [3] (and can be easily checked directly) that in all other cases the Munn algebra \( \mathcal{M} \) (or the corresponding semigroup) is representation wild over the field \( k \), i.e., for every finitely generated \( k \)-algebra \( A \) there is an exact functor \( A \text{-Mod} \rightarrow \mathcal{M} \text{-Mod} \) mapping non-isomorphic modules to non-isomorphic and indecomposable to indecomposable.
2. Munn algebras

In this paper algebra means an associative algebra over a commutative ring \( \mathbb{k} \). We do not suppose that such an algebra is unital, but always suppose that modules over such algebra are also \( \mathbb{k} \)-modules and the multiplication by elements of the algebra is \( \mathbb{k} \)-bilinear. We denote by \( A\text{-Mod} \) and \( \text{Mod}-A \), respectively, the categories of left and right \( A \)-modules. By \( A \text{-}1 \) we denote the algebra obtained from an algebra \( A \) by the formal attachment of unit. Then the categories of \( A \)-modules and unital \( A \text{-}1 \)-modules are equivalent. So \( A \) and \( B \) are Morita equivalent if and only if so are \( A \text{-}1 \) and \( B \text{-}1 \). We consider the elements from \( A \text{-}1 \) as formal sums \( \lambda + a \), where \( a \in A \), \( \lambda \in \mathbb{k} \).

Definition 2.1.

(1) Let \( R \) be a \( \mathbb{k} \)-algebra and \( \mu : N \rightarrow M \) be a homomorphism of \( R \)-modules. Define a multiplication on \( \text{Hom}_R(M,N) \) setting \( a \cdot b = \mu b a \). The resulting ring is called a Munn algebra and denoted by \( \mathbb{M}(R,M,N,\mu) \). We say that this Munn algebra is based on the algebra \( R \). We denote by \( \mathbb{M}^1(R,M,N,\mu) \) the algebra obtained from \( \mathbb{M}(R,M,N,\mu) \) by the formal attachment of unit.

(2) A Munn algebra \( \mathbb{M}(R,M,N,\mu) \) is said to be regular if the homomorphism \( \mu \) is von Neumann regular, i.e., there is a homomorphism \( \theta : M \rightarrow N \) such that \( \mu \theta \mu = \mu \). For instance, this is the case if \( R \) is von Neumann regular, while \( M \) and \( N \) are finitely generated and projective and \( \mu \neq 0 \) (it follows from [5, Th. 1.7]).

Remark 2.2. One can see that \( \mathbb{M}(R,M,N,\mu) \) has a unit if and only if there are decompositions \( M \simeq M_1 \oplus M_2 \) and \( N \simeq N_1 \oplus N_2 \) such that \( \text{Hom}_R(M_2,N) = \text{Hom}_R(M,N_2) = 0 \) and the map \( \bar{\mu} = \text{pr}_1 \circ \mu|_{N_1} \) is an isomorphism \( N_1 \xrightarrow{\sim} M_1 \). Then the unit \( u : M \rightarrow N \) coincides with \( \bar{\mu}^{-1} \). Actually, in this case \( \mathbb{M}(R,M,N,\mu) \simeq \mathbb{M}(R,M_1,N_1,\bar{\mu}) \simeq \text{End}_R M_1 \).

Proposition 2.3. Let \( \mathbb{M}(R,M,N,\mu) \) be a regular Munn algebra. There are isomorphisms \( M \simeq L \oplus M' \) and \( N \simeq L \oplus N' \) such that with respect to these decompositions \( \mu = \begin{pmatrix} 1_L & 0 \\ 0 & 0 \end{pmatrix} \).

Proof. Let \( \theta : M \rightarrow N \) be such that \( \mu \theta \mu = \mu \). Then \( \mu \theta : M \rightarrow M \) and \( \theta \mu : N \rightarrow N \) are idempotents. Therefore, \( M = M_1 \oplus M_2 \), where \( M_1 = \text{Im} \mu \theta \), \( M_2 = \text{Ker} \mu \theta \) and \( N = N_1 \oplus N_2 \), where \( N_1 = \text{Im} \theta \mu \), \( N_2 = \text{Ker} \theta \mu \). One easily sees that \( \text{Ker} \mu = \text{Ker} \theta \mu \) and \( \text{Im} \mu = \text{Im} \mu \theta \), so \( \bar{\mu} = \text{pr}_1 \circ \mu|_{N_1} \) is an isomorphism and \( \bar{\mu}^{-1} = \text{pr}_1 \circ \theta|_{M_1} \), while \( \mu|_{N_1} = 0 \) and \( \text{pr}_2 \circ \mu = 0 \), hence \( \mu = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) with respect to these decompositions. Obviously, it implies the claim.

Definition 2.4. We write \( \mathbb{M}(R,L,M,N) \) instead of \( \mathbb{M}(R,L \oplus M, L \oplus N, \mu) \), where \( \mu = \begin{pmatrix} 1_L & 0 \\ 0 & 0 \end{pmatrix} \), and call such a Munn algebra normal. Thus every regular Munn algebra is isomorphic to a normal one. As above, we denote by \( \mathbb{M}^1(R,L,M,N) \) the algebra obtained from \( \mathbb{M}(R,L,M,N) \) by the formal attachment of unit.

Lemma 2.5. Let \( A \) and \( C \) be two rings, \( P \) be a right \( C \)-module, \( M \) be a right \( A \)-module and \( N \) be a right \( A \)-left \( C \)-bimodule. Define the natural map \( \phi : P \otimes_C \text{Hom}_A(M,N) \rightarrow \text{Hom}_A(M,P \otimes_C N) \) mapping \( P \otimes f \) to the homomorphism \( x \mapsto P \otimes f(x) \). If \( P \) is projective and either \( P \) or \( M \) is finitely generated, \( \phi \) is an isomorphism.

The proof is obvious.

\footnote{This definition is a bit more general than that from [1, 6], where only the case of free modules is considered.}
Lemma 2.6. Let $A$ be a unital ring, $1 = e_1 + e_2$, where $e_1, e_2$ are orthogonal idempotents. We denote $A_i = e_i A$, $A_{ij} = e_i A e_j \simeq \text{Hom}_A(A_j, A_i)$ and identify $A$ with the ring of matrices

$$
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
$$

(2.1)

Let $P$ be a progenerator of the category $\text{Mod-}A_{11}$. Then $P^e = (P \otimes_{A_{11}} A_1) \oplus A_2$ is a progenerator of the category $\text{Mod-}A$, hence $A-\text{Mod} \simeq B-\text{Mod}$, where $B = \text{End}_A P^e$. The ring $B$ can be identified with the ring of matrices

$$
B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix},
$$

(2.2)

where $B_{11} = \text{End}_{A_{11}} P$, $B_{12} = P \otimes_{A_{11}} A_{12}$, $B_{21} = \text{Hom}_{A_{11}}(P, A_{21})$, $B_{22} = A_{22}$.

Proof. For some $m$ there is an epimorphism of $A_{11}$-modules $P^m \twoheadrightarrow A_{11}$, which induces an epimorphism $(P \otimes_{A_{11}} A_1)^m \twoheadrightarrow A_1$. Hence, $A$ is a direct summand of $(P \otimes_{A_{11}} A_1)^m \oplus A_2$ and $P^e$ is a progenerator of $A-\text{Mod}$. Using Lemma 2.5, we obtain:

$$
\text{Hom}_A(P \otimes_{A_{11}} A_1, P \otimes_{A_{11}} A_1) \simeq
\simeq \text{Hom}_{A_{11}}(P, \text{Hom}_A(A_1, P \otimes_{A_{11}} A_1)) \simeq
\simeq \text{Hom}_{A_{11}}(P, P \otimes_{A_{11}} A_{11}) \simeq \text{End}_{A_{11}} P;
$$

$$
\text{Hom}_A(A_2, P \otimes_{A_{11}} A_1) \simeq P \otimes_{A_{11}} A_{12};
$$

$$
\text{Hom}_A(P \otimes_{A_{11}} A_1, A_2) \simeq \text{Hom}_{A_{11}}(P, A_{21}).
$$

It gives the presentation (2.2) for $\text{End}_A P^e$. \hfill $\square$

Theorem 2.7. Let $\mathcal{M} = \mathcal{M}(R, L, M, N)$ be a normal Munn algebra, $C = \text{End}_R L$ and $P$ be a progenerator of the category $\text{Mod-}C$. Then $\mathcal{M}$ is Morita equivalent to the normal Munn algebra $\mathcal{M}(R, P \otimes_C L, M, N)$.

Proof. Let $A = \mathcal{M}^1(R, L, M, N)$. Consider the idempotents $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = 1 - e_1$. The presentation (2.1) of the algebra $A$ is of the form

$$
\begin{pmatrix}
C & \text{Hom}_R(M, L) \\
\text{Hom}_R(L, N) & k + \text{Hom}_R(M, N)
\end{pmatrix}
$$

(2.3)

By Lemma 2.6, $A$ is Morita equivalent to the algebra $B$ of the matrices of the form (2.2), where, due to Lemma 2.5,

$$
B_{11} = \text{Hom}_C(P, P) \simeq \text{Hom}_C(P, P \otimes_C \text{Hom}_R(L, L)) \simeq \text{Hom}_C(P, P \otimes_C \text{Hom}_R(L, L));
$$

$$
B_{12} = P \otimes_C \text{Hom}_R(M, L) \simeq \text{Hom}_R(M, P \otimes_C L);
$$

$$
B_{21} = \text{Hom}_C(P, \text{Hom}_R(L, N)) \simeq \text{Hom}_R(P \otimes_C L, N);
$$

$$
B_{22} = k + \text{Hom}_R(M, N).
$$

But it is just the matrix presentation of $\mathcal{M}^1(R, P \otimes_C L, M, N)$. \hfill $\square$

The following fact is evident.

Proposition 2.8. $\prod_{k=1}^i \mathcal{M}(R_k, M_k, N_k, \mu_k) \simeq \mathcal{M}(R, M, N, \mu)$, where $R = \prod_{k=1}^i R_k$, $M = \bigoplus_{k=1}^i M_k$, $N = \bigoplus_{k=1}^i N_k$ and $\mu|_{N_k} = \mu_k$.

Remark 2.9. Note that $\prod_{k=1}^i \mathcal{M}^1(R_k, M_k, N_k, \mu_k) \ncong \mathcal{M}^1(R, M, N, \mu)$. 

Let now $R$ be a semisimple ring. Then $R = \prod_{k=1}^{e} R_k$, where $R_k = \text{Mat}(d_k, F_k)$ for some integers $d_k$ and some skewfields $F_k$. So any Munn algebra based on $R$ is a product of Munn algebras based on the simple algebras $R_k$. All of them are regular, so can be supposed normal.

**Proposition 2.10.** Let $R = \text{Mat}(d, F)$, where $F$ is a skewfield, $U$ be the simple $R$-module, $L = U^\tau$, $M = U^m$, $N = U^n$. The algebra $\mathcal{M}(R, L, M, N)$, up to isomorphism, only depends on $r, m, n$ and does not depend on $d$. In particular, it is isomorphic to $\mathcal{M}(F, F^r, F^m, F^n)$.

We denote the algebra $\mathcal{M}(F, F^r, F^m, F^n)$ by $\mathcal{M}(F, r, m, n)$.

**Proof.** Indeed, $\text{Hom}(U^k, U^j) \simeq \text{Mat}(l \times k, F)$ does not depend on $d$ and with respect to such isomorphisms $\mathcal{M}(R, L, M, N) = \mathcal{M}(r (r + n) \times (r + m), F)$ with the multiplication $a \cdot b = a \mu b$, where $\mu = (0 \ 1)$ (of size $(r + m) \times (r + n)$) and $I$ is the $r \times r$ unit matrix.

**Theorem 2.11.** Let $\mathcal{M} = \prod_{k=1}^{e} \mathcal{M}(F_k, r_k, m_k, n_k)$, where $F_k$ are skewfields. Then $\mathcal{M}$ is Morita equivalent to $\prod_{k=1}^{e} \mathcal{M}(F_k, 1, m_k, n_k)$.

**Proof.** Let $R = \prod_{k=1}^{e} F_k, L_k = F_k^{1k}$ and $L = \prod_{k=1}^{e} L_k$. Then $C_k = \text{End}_R L_k \simeq \text{Mat}(r_k \times r_k, F)$. Let $P_k$ be the simple right $C_k$-module. It is a progenitor of the category $\text{Mod-}C_k$ and $P_k \otimes C_k L_k \simeq F_k$. Now apply Theorem 2.7.

We denote the algebra $M(F, 1, m, n)$ by $\mathcal{M}(F, m, n)$. It is the algebra of $(n + 1) \times (m + 1)$ matrices over $F$ with the multiplication $a \cdot b = a \mu b$, where $\mu$ is the $(m + 1) \times (n + 1)$ matrix with 1 at the $(1, 1)$-place and 0 elsewhere.

### 3. Representations

In this section we consider representations of finite dimensional regular Munn algebras over a field $k$ with a semisimple base. According to Theorem 2.11, such an algebra is Morita equivalent to a direct product $\mathcal{M} = \prod_{k=1}^{e} \mathcal{M}_k$, where $\mathcal{M}_k = \mathcal{M}(F_k, m_k, n_k)$ and $F_k$ are skewfields. If $m_k = n_k = 0$, $\mathcal{M}(F_k, m_k, n_k) = F_k$ and is a direct factor of $\mathcal{M}^1$. So we can and will suppose that there are no such components in $\mathcal{M}$. The algebra $\mathcal{M}_k$ contains an idempotent $e_k$ which is the $(n_k + 1) \times (m_k + 1)$ matrix with 1 at the $(1, 1)$-place and 0 elsewhere. Let $e_0 = 1 - \sum_{k=1}^{e} e_k$. Then, if $k \neq 0$, $e_0 \mathcal{M}^1 e_k = F_k, e_0 \mathcal{M}^1 e_k = F_k^{1k}, e_k \mathcal{M}^1 e_0 = F_k^{mk}, e_0 \mathcal{M}^1 e_0 = k + \bigoplus_{k=1}^{e} M_k$, where $M_k \simeq \text{Mat}(n_k \times m_k, F_k)$, and $e_k \mathcal{M}^1 e_0 = 0$ if $0 \neq k \neq 1 \neq 0$. Choose an $F_k$-basis $\{a_{k1}, a_{k2}, \ldots, a_{km_k}\}$ in each space $e_k \mathcal{M}^1 e_0$ and an $F_k$-basis $\{b_{k1}, b_{k2}, \ldots, b_{kn_k}\}$ in each space $e_0 \mathcal{M}^1 e_k$. Then $a_k b_{kj} = 0$ for all $k, i, j, b_{ki}a_{ij} = 0$ if $k \neq l$ and $\{b_{ki}a_{ij}\}$ is a basis of $M_k$. For every $\mathcal{M}^1$-module $V$ set $V_k = e_k V (0 \leq k \leq s)$. It is a vector space over $F_k$. The multiplication by $a_{ki}$ gives rise to a $k$-linear map $\alpha_{ki} : V_0 \to V_k$ and the multiplication by $b_{kj}$ gives rise to a $k$-linear map $\beta_{kj} : V_k \to V_0$. Since $\text{Hom}_k(V_0, V_k) \simeq \text{Hom}_F(F_k, V_0 \otimes_k V_k)$ and $\text{Hom}_k(V_k, V_0) \simeq \text{Hom}_k(V_k, \text{Hom}_k(F_k, V_0))$, both $\alpha$ and $\beta$ can be considered as matrices over $F_k$ of appropriate sizes. So $V$ is defined by the set of maps (or of matrices) $\{\alpha_{ki}, \beta_{kj}\}$ such that $\alpha_{ki} \beta_{kj} = 0$ for all $k, l, i, j$. We present it by the diagram

$$
\begin{array}{c}
V : \{V_k\} \xrightarrow{\{\alpha_{ki}\} \ | \ {\beta_{kj}\}} V_0,
\end{array}
$$

2 Ponizovskii [7] denotes this algebra by $\mathcal{M}(E_{m+r,p+r+1}, F)$. 
A homomorphism \( \phi : V \to V' \) is given by a set of \( F_k \)-linear maps \( \phi_k : V_k \to V_k' \) \((0 \leq k \leq s)\), where \( F_0 = k \), such that \( \phi_k \alpha_{ki} = \alpha'_{ki} \phi_0 \) and \( \phi_0 \beta_{kj} = \beta'_{kj} \phi_k \), i.e. the following diagram is commutative:

\[
\begin{array}{c}
\{V_k\} \xrightarrow{\phi_k} V_0 \\
\{\phi_k\} \\
\{V_k'\} \xrightarrow{\alpha'_{ki}} V'_0 \\
\{\beta_{kj}\} \\
\end{array}
\]

(3.1)

\( \phi \) is an isomorphism if and only if so are all \( \phi_k \).

Set \( V_+ = \sum_{i,j} \text{Im} \beta_{ij} \subseteq V_0 \), \( V_- = V_0/V_+ \). Then \( \alpha_{ki}(V_+) = 0 \). Hence \( \alpha_{ki} \) can be considered as a map \( V_- \to V_k \) and we obtain a diagram

\[
\tilde{V} : \{V_k\} \xrightarrow{\alpha_{ki}} V_- \\
\{\beta_{kj}\} \\
\{V_k\} \xrightarrow{V_+}
\]

with the condition \( \sum_{k,j} \text{Im} \beta_{kj} = V_+ \). Such diagram can be considered as a representation of the realization \((\mathfrak{M}, \Omega)\) of the valued graph \((\Gamma, d)\) in the sense of [2]. Namely the vertices of the graph \( \Gamma \) are \{+, -, 1, 2, \ldots, s\}, \( d_k = \dim_k F_k \), \( d_{k,+} = (m_k, m_k d_k) \), \( d_{k,-} = (n_k d_k, n_k) \) and \( d_{ij} = 0 \) otherwise. The orientation \( \Omega \) of the edge \( \{k, +\} \) is \( k \to + \) and that of the edge \( \{-, k\} \) is \( - \to k \). The modulation \( \mathfrak{M} \) of \( \Gamma \) is given by the algebras \( F_k \) and \( F_{\pm} = k \), \( F_{k,k-} \)-bimodules \( k M_- = m_k F_k \) and \( F_{+,k} \)-bimodules \( +M_k = n_k F_k \). Thus a representation of this realization is indeed given by a set of \( F_k \)-vector spaces \( V_k \), \( F_0 \)-vector spaces \( V_\pm \) and a set of linear maps \( \tilde{\alpha}_k : n_k V_- \to V_k \) and \( \tilde{\beta}_1 : m_1 V_1 \to V_+ \). There components are just \( \alpha_{ki} \) and \( \beta_{kj} \).

**Theorem 3.1.** Let \( \text{Rep}^+(\mathfrak{M}, \Omega) \) be the full subcategory of the category of representations of \((\mathfrak{M}, \Omega)\) such that \( \sum_{i=1}^{s} \text{Im} \beta_i = V_+ \) and \( \bigcap_{k,i} \text{Ker} \tilde{\alpha}_k = 0 \). Let also \( \mathcal{M} \text{-Mod}^+ \) be the full subcategory of \( \mathcal{M} \text{-Mod} \) consisting of such modules \( V \) that \( \sum_{i,j} \text{Im} \beta_{ij} = \bigcap_i \text{Ker} \tilde{\alpha}_{ki} \). Denote by \( \mathcal{I} \) the ideal of the category \( \mathcal{M} \text{-Mod}^+ \) consisting of all morphisms \( \phi : V \to V' \) such that \( \phi_k = 0 \) for \( k \neq 0 \), \( \phi_0(V_+) = 0 \) and \( \text{Im} \phi_0 \subseteq V'_+ \). Then \( \mathcal{M} \text{-Mod}^+ / \mathcal{I} \simeq \text{Rep}^+(\mathfrak{M}, \Omega) \) and \( \mathcal{I}^2 = 0 \).

**Proof.** We have already constructed, for any \( \mathcal{M} \)-module \( V \), the representation \( \tilde{V} \). By definition, \( \tilde{V} \in \text{Rep}^+(\mathfrak{M}, \Omega) \). Given a homomorphism \( \phi = (\phi_k) : V \to V' \) as in (3.1), we obtain linear maps \( \phi_+ : V_+ \to V'_+ \) and \( \phi_- : V_- \to V'_- \) such that together with the maps \( \phi_k \) they give a morphism \( \tilde{\phi} : \tilde{V} \to \tilde{V}' \). Obviously, \( \tilde{\phi} = 0 \) if and only if \( \phi \in \mathcal{I} \). Thus we obtain a functor \( \Phi : \mathcal{M} \text{-Mod}^+ / \mathcal{I} \to \text{Rep}^+(\mathfrak{M}, \Omega) \). Obviously \( \mathcal{I}^2 = 0 \).

Let \( W = (W_k, W_+, W_-, \alpha_{ki}, \beta_{kj} \mid 1 \leq k \leq s) \) be a representation from \( \text{Rep}^+(\mathfrak{M}, \Omega) \). Set \( \tilde{W}_0 = W_+ \oplus W_-, \alpha_{ki} : W_0 \to W_k \) the maps that are 0 on \( W_+ \) and coincide with the components of \( \alpha_k \) on \( W_- \), and take for \( \tilde{\beta}_{kj} : W_i \to W_0 \) the components of \( \beta_i : W_i \to W_+ \). It defines an \( \mathcal{M} \)-module \( \tilde{W} \in \mathcal{M} \text{-Mod}^+ \). If \( \psi : W \to W' \) is a morphism of representations, set \( \tilde{\psi}_0(w) = \psi_+(w_+) + \psi_-(w_-) \) if \( w = w_+ + w_- \), where \( w_\pm \in W_\pm \). It gives a homomorphism \( \tilde{\psi} : \tilde{W} \to \tilde{W}' \). Taking its class modulo \( \mathcal{I} \), we obtain a functor \( \Psi : \text{Rep}^+(\mathfrak{M}, \Omega) \to \mathcal{M} \text{-Mod}^+ / \mathcal{I} \). One easily verifies that this functor is quasi-inverse to \( \Phi \).

**Remark 3.2.** Since \( \mathcal{I}^2 = 0 \), the isomorphism classes of objects in \( \mathcal{M} \text{-Mod}^+ \) are the same as in \( \mathcal{M} \text{-Mod}^+ / \mathcal{I} \). The only indecomposable representations not belonging to \( \text{Rep}^+(\mathfrak{M}, \Omega) \) are two trivial representations such that \( V_+ = k \) (or \( V_- = k \)) and \( V_k = 0 \) for \( k \neq \pm 1 \) (respectively, for \( k \neq -1 \)). The only indecomposable \( \mathcal{M} \)-module not belonging to \( \mathcal{M} \text{-Mod}^+ \) is the 1-dimensional vector space with zero
multiplication by the elements of \(M\). Therefore, the representation type of the algebra \(M\) (finite, tame or wild) is the same as that of the realization \((\mathfrak{M}, \Omega)\) of the valued graph \(\Gamma\).

It is proved in [2] that the representation type of \((\mathfrak{M}, \Omega)\) actually only depends on the valued graph itself. Namely, it is representation finite if and only if all its connected components are Dynkin graphs and representation tame if and only if all of them are Dynkin or Euclidean (extended Dynkin) graphs and at least one Euclidean graph occurs. For the list of these graphs see [2, p. 3]. In all other cases it is representation wild.

Taking into account the construction of the valued graph \(\Gamma\) from the algebra \(M\), we can establish the representation type of any finite dimensional Munn algebra with a semisimple base. Actually it only depends on the set of triples \(\{(d_k, m_k, n_k)\}\), where \(d_k = \text{dim}_{k} F_k\). We use the following notations:

\[
\Sigma(d_1, \ldots, d_r | d_{r+1}, \ldots, d_s) = \{(d_1, 1, 0), \ldots, (d_r, 1, 0), (d_{r+1}, 0, 1), \ldots, (d_s, 0, 1)\},
\]

and, for \(\Sigma = \Sigma(d_1, \ldots, d_r | d_{r+1}, \ldots, d_s)\),

\[
S^- (\Sigma) = \sum_{k=1}^{r} d_k, \\
S^+ (\Sigma) = \sum_{k=r+1}^{s} d_k, \\
S(\Sigma) = S^- (\Sigma) + S^+ (\Sigma).
\]

Certainly, maybe \(r = 0\) or \(s = 0\).

**Theorem 3.3.** Let \(M = \prod_{k=1}^{\infty} M_k(F_k, m_k, n_k)\), \(\Sigma = \{(d_k, m_k, n_k) | (m_k, n_k) \neq (0, 0)\}\), where \(d_k = \text{dim}_{k} F_k\).

(1)\(^3\) \(M\) is representation finite if and only if \(\Sigma = \Sigma_0 \cup \Sigma_1\), where \(\Sigma_0 = \Sigma(d_1, \ldots, d_r | d_{r+1}, \ldots, d_s)\) for some \(d_k\) and

(a) either \(\Sigma_1 = \emptyset\) and \(\max\{S^- (\Sigma_0), S^+ (\Sigma_0)\} \leq 3\)

(b) or \(\Sigma_1 = \{(1, 1, 1)\}\), \(S(\Sigma_0) \leq 3\) and \(\max\{S^- (\Sigma_0), S^+ (\Sigma_0)\} \leq 2\).

(2) \(M\) is representation tame if and only if \(\Sigma = \Sigma_0 \cup \Sigma_1\), where \(\Sigma_0 = \Sigma(d_1, \ldots, d_r | d_{r+1}, \ldots, d_s)\) for some \(d_k\) and

(a) either \(\Sigma_0 = \emptyset\) and \(\Sigma_1\) is one of the sets

\(\{(1, 1, 1), (1, 1, 1), (2, 1, 1), ((1, 2, 0), (1, 0, 2),\}\)

(b) or \(\Sigma_1 = \emptyset\) and \(\max\{S^- (\Sigma_0), S^+ (\Sigma_0)\} = 4\)

(c) or \(\Sigma_1 = \{(1, 1, 1)\}\) and \(S^- (\Sigma_0) = S^+ (\Sigma_0) = 2\).

(3) In all other cases \(M\) is representation wild.

**Proof.**

(1a) In this case the graph \(\Gamma\) is a disjoint union of 2 graphs of the types \(A_2, A_3, D_4, B_2, \) or \(B_3\).

(1b) In this case \(\Gamma\) is of one of the types \(A_3, A_4, A_5, D_5, D_6, B_4, \) or \(B_5\).

In other cases \(\Gamma\) is not a disjoint union of Dynkin graphs.

From now on we only list the cases when \(M\) is not representation finite.

(2a) In these cases \(\Gamma\) is, respectively, of type \(\tilde{A}_3, \) or \(\tilde{B}_2, \) or \(\tilde{A}_{12}\).

(2b) In this case \(\Gamma\) is a disjoint union of two graphs, where either both are of types \(\tilde{D}_4, \tilde{B}D_3, \tilde{B}_2, \tilde{A}_{11}, \) or \(\tilde{G}_2\) or one is of one of these types while the other is of a type cited in case (1a).

\(^{3}\)If the field \(k\) is algebraically closed, hence all \(d_k = 1\), this result coincides with that of Ponizovskii [7, \(n^3\) 5].
(3) In all other cases the graph $\Gamma$ is not a disjoint union of Dynkin and Euclidean graphs.

4. Semigroups

We apply the obtained result to representations of finite Rees matrix semigroups. Recall [1, Section 3.1] that such semigroup $\mathcal{M}(G, p, q, \mu)$ is given by a finite group $G$ and a matrix $\mu$ of size $p \times q$ with coefficients from the group $G$. The elements of $\mathcal{M}(G, p, q, \mu)$ are $q \times p$ matrices with coefficients from $G^0 = G \cup \{0\}$ containing at most one non-zero element and the multiplication is defined by the rule $a \cdot b = a\mu b$. If the sandwich matrix $\mu$ is regular, i.e., every column and every row of $\mu$ contains a non-zero element, the semigroup $\mathcal{M}(G, p, q, \mu)$ is 0-simple (hence completely 0-simple) and every finite 0-simple semigroup is isomorphic to a Rees matrix semigroup with a regular sandwich matrix [1, Th. 3.5]. We always suppose that the matrix $\mu$ is non-zero; otherwise $\mathcal{M}(G, p, q, \mu)$ is just a semigroup with zero multiplication.

Let $k$ be a field, $R = kG$ and $\mathcal{M} = \mathcal{M}(G, p, q, \mu)$. Obviously, $k\mathcal{M} = \mathcal{M}(R, R^p, R^q, \mu)$, where $\mu$ is considered as an element of $\text{Mat}(p \times q, R)$ and is identified with an $R$-homomorphism $R^q \to R^p$. We suppose that $\text{char } k \nmid \#(G)$. Then $R$ is semisimple. Namely, let $U_1, U_2, \ldots, U_s$ be all irreducible representations of $G$ over $k$, $F_k = \text{End}_k U_k$, $d_k = \dim_k F_k$ and $u_k = \dim_k U_k$. Set $c_k = \frac{u_k}{d_k}$. Then $R \cong \prod_{k=1}^s R_k$, where $R_k = \text{Mat}(c_k \times c_k, F_k)$, and $\text{Mat}(p \times q, R_k) = \text{Mat}(p_{c_k} \times q_{c_k}, F_k)$. Denote by $\mu_k$ the projection of $\mu$ onto $\text{Mat}(p_{c_k} \times q_{c_k}, F_k)$ and set $r_k = rk \mu_k$. As $k \neq 0$, also all $\mu_k \neq 0$ and the Munn algebra $k\mathcal{M}$ is regular. Then $k\mathcal{M} \cong \prod_{k=1}^s \mathcal{M}(F_k, r_k, m_k, n_k)$, where $m_k = p_{c_k} - r_k$ and $n_k = q_{c_k} - r_k$.

**Theorem 4.1**. $k\mathcal{M}$ is Morita equivalent to $\prod_{k=1}^s \mathcal{M}(F_k, m_k, n_k)$.

**Remark 4.2**. Note that $c_k \mid m_k - n_k$ and $\frac{m_k - n_k}{c_k} = p - q$ does not depend on $k$. In particular, if $m_k = n_k$, or $m_k > n_k$, or $m_k < n_k$ for some $k$, the same holds for all $k$.

From Corollary 4.1 and Theorem 3.3, taking into account Remark 4.2, we obtain a classification of representation types of Rees matrix semigroups, in particular, of 0-simple semigroups. In the next theorem we use the just introduced notations.

**Theorem 4.3**. Let $\mathcal{M} = \mathcal{M}(G, p, q, \mu)$ be a finite Rees matrix semigroup, $k$ be a field such that $\text{char } k \nmid \#(G)$. Set $\Sigma(\mathcal{M}) = \{(d_k, m_k, n_k) \mid (m_k, n_k) \neq (0, 0)\}$.

(1) $\mathcal{M}$ is representation finite over the field $k$ if and only if

(a) either $\Sigma = \{(1, 1, 1)\}$
(b) or $\#(G) \leq 3$ and $\Sigma$ contains either only triples $(d_k, 1, 0)$ or only triples $(d_k, 0, 1)$.

(2) $\mathcal{M}$ is representation tame over the field $k$ if and only if

(a) either $\Sigma(\mathcal{M}) = \{(1, 1, 1), (1, 1, 1)\}$, or $\Sigma(\mathcal{M}) = \{(2, 1, 1)\}$
(b) or $\#(G) = 4$ and $\Sigma(\mathcal{M})$ contains either only triples $(d_k, 1, 0)$ or only triples $(d_k, 0, 1)$,
(c) $G = \{1\}$ and $\Sigma(\mathcal{M}) = \{(1, 2, 0)\}$ or $\Sigma(\mathcal{M}) = \{(1, 0, 2)\}$.

(3) In all other cases $\mathcal{M}$ is representation wild over the field $k$.

*Note that in cases (1a) and (2a) $p = q$, while in cases (1b) and (2b) the group $G$ is commutative.*

\[^4\]If the field $k$ is algebraically closed, hence all $d_k = 1$, this result was proved by Ponizovskii [7].
Remark 4.4. According to Proposition 2.10, the algebra $\mathbb{k}M(G, p, q, \mu)$ only depends on the ranks $r_k$. Elementary transformations of the matrix $\mu$ do not change these ranks. Obviously, using them one can obtain a matrix $\mu'$ such that there is a non-zero element in every row and in every column. Therefore, $\mathbb{k}M(G, p, q, \mu) \simeq \mathbb{k}M(G, p, q, \mu')$ and $M(G, p, q, \mu')$ is a 0-simple semigroup [1, Thm.3.3]. Thus, for every Rees matrix semigroup with a non-zero sandwich matrix there is a 0-simple semigroup with the same representation theory.

If a finite semigroup $S = \bigvee_{i=1}^t M_i$ is a union of pairwise annihilating Rees matrix semigroups $M_i$ with common 0, its semigroup algebra $\mathbb{k}S$ is a direct product of semigroup algebras $\mathbb{k}M_i$ and all of them are Munn algebras. So we obtain the following result.

Theorem 4.5. Let $S = \bigvee_{i=1}^t M_i$, where $M_i = M(G_i, m_i, n_i, \mu_i)$ are finite Rees matrix semigroups, $\mathbb{k}$ be a field such that $\text{char} \mathbb{k} \nmid \#(G_i)$ for all $i$. Denote

$$T_\succ = \sum_{n_i > n_j} \#(G_i),$$
$$T_\prec = \sum_{n_i < n_j} \#(G_i),$$
$$\mathcal{T}_0 = \bigcup_{m_i \neq n_j} \mathcal{T}(M_i),$$
$$\mathcal{T}_1 = \bigcup_{m_i = n_j} \mathcal{T}(M_i).$$

(1) $S$ is representation finite over the field $\mathbb{k}$ if and only if

(a) either $\mathcal{T}_1 = \emptyset$, $\max\{T_\succ, T_\prec\} \leq 3$ and all triples from $\mathcal{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$,
(b) or $\mathcal{T}_1 = \{(1, 1, 1)\}$, $T_\succ + T_\prec \leq 3$, $\max\{T_\succ, T_\prec\} \leq 2$ and all triples from $\mathcal{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$.

(2) $S$ is representation tame over the field $\mathbb{k}$ if and only if

(a) $\mathcal{T}_0 = \emptyset$, $\max\{T_\succ, T_\prec\} = 4$ and all triples from $\mathcal{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$,
(b) or $\mathcal{T}_1 = \{(1, 1, 1)\}$, $T_\succ = T_\prec = 2$ and all triples from $\mathcal{T}_0$ are either $(d_k, 1, 0)$ or $(d_k, 0, 1)$,
(c) or $\mathcal{T}_0 = \emptyset$ and either $\mathcal{T}_1 = \{(1, 1, 1), (1, 1, 1)\}$ or $\mathcal{T}_1 = \{(2, 1, 1)\}$,
(d) or $\mathcal{T}_1 = \emptyset$ and $\mathcal{T}_0 = \{(1, 2, 0)\}$ or $\mathcal{T}_0 = \{(1, 0, 2)\}$.

In the last case there is a unique index $i$ such that $m_i \neq n_i$ and the corresponding group $G_i = \{1\}$.

(3) In all other cases $S$ is representation wild over the field $\mathbb{k}$.

Acknowledgments

The final version of the paper was prepared during the stay of the first author in the Max-Plank-Institute for Mathematics (Bonn).

Funding

This work was supported within the framework of the program of support of priority for the state scientific researches and scientific and technical (experimental) developments of the Department of Mathematics NAS of Ukraine for 2022-2023 (Project “Innovative methods in the theory of differential equations, computational mathematics and mathematical modeling”, No. 7/1/241).

If the field $\mathbb{k}$ is algebraically closed, this result easily follows from that of Ponizovskii [7, n°5] and Remark 4.2.
References

[1] Clifford, A. H., Preston, G. B. (1961). The Algebraic Theory of Semigroups. Vol. I. Providence, RI: American Mathematical Society.

[2] Dlab, V., Ringel, C. M. (1976). Indecomposable representations of graphs and algebras. Mem. Am. Math. Soc. 173:1–57.

[3] Drozd, Y. A. (1986). Tame and wild matrix problems. Transl. Ser. 2, Am. Math. Soc. 128:31–55.

[4] Gabriel, P. (1972). Unzerlegbare Darstellungen. I. Manuscr. Math. 6:71–103.

[5] Goodearl, K. R. (1991). Von Neumann Regular Rings. Malabar, FL: Krieger Publishing Company.

[6] Okniński, J. (1990). Semigroup Algebras. New York: Marcel Dekker, Inc.

[7] Ponizovskii, I. S. (1975). On the finiteness of type of a semigroup algebra of a finite fully prime semigroup. J. Sov. Math. 3:700–709.