Exact Controllability for Stochastic Transport Equations*

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Abstract

This paper is addressed to studying the exact controllability for stochastic transport equations by two controls: one is a boundary control imposed on the drift term and the other is an internal control imposed on the diffusion term. By means of the duality argument, this controllability problem can be reduced to an observability problem for backward stochastic transport equations, and the desired observability estimate is obtained by a new global Carleman estimate. Also, we present some results about the lack of exact controllability, which show that the action of two controls is necessary. To some extent, this indicates that the controllability problems for stochastic PDEs differ from their deterministic counterpart.

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1 Introduction

Let $T > 0$ and $G \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a strictly convex bounded domain with a $C^1$ boundary $\Gamma$. Denote by $\nu(x) = (\nu^1(x), \cdots, \nu^d(x))$ the unit outward normal vector of $G$ at $x \in \Gamma$. Let $\bar{x}_1, \bar{x}_2 \in \Gamma$ satisfy that

$$|\bar{x}_1 - \bar{x}_2|_{\mathbb{R}^d} = \max_{x_1, x_2 \in G} |x_1 - x_2|_{\mathbb{R}^d}.$$

Without loss of generality, we assume that $0 \in G$ and $0 = \bar{x}_1 + \bar{x}_2$. Put $R = \max_{x \in \Gamma} |x|_{\mathbb{R}^d}$. Let

$$S^{d-1} \triangleq \{x \in \mathbb{R}^d : |x|_{\mathbb{R}^d} = 1\}.$$

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Denote by
\[ \Gamma^-_S = \{(x, U) \in \Gamma \times S^{d-1} : U \cdot \nu(x) \leq 0\}, \quad \Gamma^+_S = (\Gamma \times S^{d-1}) \setminus \Gamma^-_S. \]

Let us define a Hilbert space \( L^2_w(\Gamma^-_S) \) as the completion of all \( h \in C^0(\Gamma^-_S \times S^{d-1}) \) with the norm
\[ |h|_{L^2_w(\Gamma^-_S)} \equiv \left( -\int_{\Gamma^-_S} U \cdot \nu |h|^2 d\Gamma^-_S \right)^{1/2}, \]
where \( d\Gamma^-_S \) denotes the Lebesgue measure on \( \Gamma^-_S \). Clearly, \( L^2(\Gamma^-_S) \) is dense in \( L^2_w(\Gamma^-_S) \).

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \) be a complete filtered probability space on which a one-dimensional standard Brownian motion \( \{B(t)\}_{t \geq 0} \) is defined such that \( \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration generated by \( \{B(t)\}_{t \geq 0} \), augmented by all the \( P \)-null sets in \( \mathcal{F} \). Let \( H \) be a Banach space. We denote by \( L^2_{\mathcal{F}}(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes \( X(\cdot) \) such that \( \mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty \); by \( L^\infty_{\mathcal{F}}(0, T; H) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted bounded processes; by \( L^2_{\mathcal{F}}(\Omega; C([0, T]; H)) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted continuous processes \( X(\cdot) \) such that \( \mathbb{E}(|X(\cdot)|^2_{C([0, T]; H)}) < \infty \); and by \( C_{\mathcal{F}}([0, T]; L^2(\Omega; H)) \) the Banach space consisting of all \( H \)-valued \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted continuous processes \( X(\cdot) \) such that \( \mathbb{E}(|X(\cdot)|^2) \) is continuous (similarly, one can define \( L^2_{\mathcal{F}}(\Omega; C^k([0, T]; H)) \) and \( C^k_{\mathcal{F}}([0, T]; L^2(\Omega; H)) \) for any positive integer \( k \)). All of the above spaces are endowed with the canonical norm.

The main purpose of this paper is to study the exact controllability of the following controlled linear forward stochastic transport equation:

\[
\begin{cases}
  dy + U \cdot \nabla y dt = \left[ a_1 y + \int_{S^{d-1}} a_2(t, x, U, V) y(t, x, V) dS^{d-1}(V) + f \right] dt \\
  \quad + (a_3 y + v) dB(t) \\
  y = u & \text{in } (0, T) \times G \times S^{d-1}, \\
  y(0) = y_0 & \text{on } (0, T) \times \Gamma^-_S, \\
  & \text{in } G \times S^{d-1}.
\end{cases}
\] (1.1)

Here and in what follows, \( \nabla \) denotes the gradient operator with respect to \( x \),

\[
\begin{cases}
  y_0 \in L^2(G \times S^{d-1}), \\
  a_1 \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G \times S^{d-1})), \\
  a_2 \in L^\infty_{\mathcal{F}}(\Omega; C([0, T]; C(G \times S^{d-1} \times S^{d-1}))), \\
  a_3 \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G \times S^{d-1})), \\
  f \in L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1})).
\end{cases}
\]

The boundary control function \( u \in L^2_{\mathcal{F}}(0, T; L^2_w(\Gamma^-_S)) \) and the internal control function \( v \in L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1})) \).

We begin with the definition of solution to the system (1.1).
Definition 1.1 A solution to (1.1) is a process \( y \in L^2_F(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \) such that for every \( t \in [0,T] \) and \( \phi \in C^1(G \times S^{d-1}) \) with \( \phi = 0 \) on \( \Gamma^+_S \), it holds that

\[
\int_G \int_{S^{d-1}} y(t,x,U) \phi(x,U) dS^{d-1} dx - \int_G \int_{S^{d-1}} y_0(x,U) \phi(x,U) dS^{d-1} dx \\
- \int_0^t \int_G \int_{S^{d-1}} y(s,x,U) \nabla \phi(x,U) dS^{d-1} dx ds + \int_0^t \int_{\Gamma^-_S} u(s,x,U) \phi(x,U) U \cdot v d\Gamma^-_S ds \\
= \int_0^t \int_G \int_{S^{d-1}} [a_1(s,x,U)y(s,x,U) + f(s,x,U)] \phi(x,U) dS^{d-1} dx ds \\
+ \int_0^t \int_G \int_{S^{d-1}} [\int_{S^{d-1}} a_2(s,x,V)y(s,x,V) dS^{d-1}(V)] \phi(x,U) dS^{d-1}(U) dx ds \\
+ \int_0^t \int_G \int_{S^{d-1}} [a_3(s,x,U)y(s,x,U) + v(s,x,U)] \phi(x,U) dS^{d-1} dx dB(s), \quad P-a.s.
\]  

(1.2)

In Section 2, we will prove the following well-posedness result for (1.1).

Proposition 1.1 For each \( y_0 \in L^2(G \times S^{d-1}) \), the system (1.1) admits a unique solution \( y \) such that

\[
|y|_{L^2(\Omega; C([0,T]; L^2(G \times S^{d-1})))} \\
\leq e^{Cr_1} \left( |y_0|_{L^2(G \times S^{d-1})} + \int_0^T |y_s|_{L^2(G \times S^{d-1})} \right) + |u|_{L^2_F(0,T; L^2_{\mathbb{F}}(\Omega \times S^{d-1})))} + |v|_{L^2_F(0,T; L^2_{\mathbb{F}}(\Gamma^- \times S^{d-1})))}.
\]  

(1.3)

Here \( C > 0 \) is a constant which is independent of \( y_0 \) and \( r_1 = |a_1|_{L^2_{\mathbb{F}}}[0,T; L^\infty(G \times S^{d-1})] + |a_2|_{L^2_{\mathbb{F}}(\Omega; C([0,T]; C(\overline{G} \times S^{d-1} \times S^{d-1})))} + |a_3|_{L^2_{\mathbb{F}}}[0,T; L^\infty(G \times S^{d-1})] + 1.

Now we introduce the notion of exact controllability for the system (1.1).

Definition 1.2 System (1.1) is said to be exactly controllable at time \( T \) if for every \( y_0 \in L^2(G \times S^{d-1}) \) and \( y_1 \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})) \), one can find a pair of controls \( (u,v) \in L^2_{\mathbb{F}}(0,T; L^2(\Omega \times S^{d-1}))) \times L^2_{\mathbb{F}}(0,T; L^2(G \times S^{d-1})) \) such that the solution \( y \) with \( y(0) = y_0 \) of the system (1.1) satisfies that \( y(T) = y_1 \).

Remark 1.1 Since the control \( v \) in the diffusion term is effective in the whole domain, one may expect to eliminate the randomness of the system (1.1) by taking \( v = -a_3 y \) and reduce this system to a controlled random transport equation. However, the randomness in (1.1) comes from not only the stochastic noise \( dB \), but also its coefficients. Although one can take a feedback control to get rid of the noise term, we still need to deal with the random coefficients, which cannot be handled by the classical controllability theory of deterministic transport equations.

We have the following result for the exact controllability of the system (1.1).
**Theorem 1.1**  If \( T > 2R \), then the system (1.1) is exactly controllable at time \( T \).

We introduce two controls into the system (1.1). Moreover, the control \( v \) acts on the whole domain and \( T \) needs to be larger than \( 2R \). Compared with the deterministic transport equations, it seems that our choice of controls is too restrictive. One may consider the following four weaker cases for designing the control:

1. Only one control is acted on the system, that is, \( u = 0 \) or \( v = 0 \) in (1.1).
2. Neither \( u \) nor \( v \) is zero. But \( v = 0 \) in \((0, T) \times G_0 \times S^{d-1}\), where \( G_0 \) is a nonempty open subset of \( G \).
3. Two controls are imposed on the system. But both of them are in the drift term.
4. The time \( T < 2R \).

It is easy to see that the exact controllability of (1.1) does not hold for the fourth case. Indeed, if the system (1.1) would be exactly controllable at some time \( T < 2R \), then one could deduce the exact controllability of a deterministic transport equation on \( G \) at time \( T \) with a boundary control acted on \( \Gamma^- \), but this is obviously impossible. For the other three cases, according to the controllability result for deterministic transport equations (see [18]), it seems that the corresponding system should be exactly controllable. However, as we shall see later, it is not the truth, either.

**Theorem 1.2**  If \( u \equiv 0 \) or \( v \equiv 0 \) in the system (1.1), then this system is not exactly controllable at any time \( T \).

Theorem 1.2 indicates that it is necessary to use two controls to obtain the desired exact controllability property for the system (1.1). Nevertheless, one may expect the exact controllability of (1.1) with the control \( v \) (in the diffusion term) acted only in a proper subdomain of \( G \) rather than the whole domain \( G \). But this is impossible, either. Indeed, we have the following negative result.

**Theorem 1.3** Let \( G_0 \) be a nonempty open subset of \( G \). If \( v \equiv 0 \) in \((0, T) \times G_0 \times S^{d-1}\), then the system (1.1) is not exactly controllable at any time \( T \).

For the third case, we consider the following controlled equation:

\[
\begin{cases}
  dy + U \cdot \nabla y dt = \left[ a_1 y + \int_{S^{d-1}} a_2(t, x, U, V) y(t, x, V) dS^{d-1} + f + \ell \right] dt \\
  + a_3 y dB(t) \\
  y = u \\
  y(0) = y_0
\end{cases}
\quad \text{in } (0, T) \times G \times S^{d-1},
\]

\[
\text{on } (0, T) \times \Gamma^-.
\]

\[
(1.4)
\]

Here \( \ell \in \mathcal{L}^2(0, T; \mathcal{L}^2(G \times S^{d-1})) \) is another control. Similar to Definition 1.2, one can define the exact controllability of (1.4). We have the following negative result.

**Theorem 1.4** System (1.4) is not exactly controllable for any \( T > 0 \).
In order to prove Theorem 1.1, we make use of the duality argument. We obtain the exact controllability of the system (1.1) by establishing an observability estimate for the following backward stochastic transport equation:

\[
\begin{aligned}
    dz + U \cdot \nabla z dt &= \left[ b_1 z + \int_{S_{d-1}} b_2(t, x, V) z(t, x, V) dS^{d-1}(V) + b_3 z \right] dt \\
    &+ (b_4 z + Z) dB(t)
\end{aligned}
\]

\[z = 0\]

\[z(T) = z_T\]

Here

\[
\begin{aligned}
    z_T &\in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})), \\
    b_1 &\in L^\infty_F(0, T; L^\infty(G \times S^{d-1})), \\
    b_2 &\in L^\infty_F(\Omega; C([0, T]; C(\overline{G} \times S^{d-1} \times S^{d-1}))), \\
    b_3 &\in L^\infty_F(0, T; L^\infty(G \times S^{d-1})), \\
    b_4 &\in L^\infty_F(0, T; L^\infty(G \times S^{d-1})).
\end{aligned}
\]

The definition of solution to (1.5) is given as follows.

**Definition 1.3** A solution to the equation (1.5) is a pair of stochastic processes

\[(z, Z) \in L^2_F(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \times L^2_F(0, T; L^2(G \times S^{d-1}))\]

such that for every \(\psi \in C^1(\overline{G} \times S^{d-1})\) with \(\psi = 0\) on \(\Gamma^-_S\) and \(t \in [0, T]\), it holds that

\[
\begin{aligned}
    &\int_G \int_{S^{d-1}} z_T(x, U) \psi(x, U) dS^{d-1} dx - \int_G \int_{S^{d-1}} z(t, x, U) \psi(x, U) dS^{d-1} dx \\
    &- \int_t^T \int_G \int_{S^{d-1}} z(s, x, U) U \cdot \nabla \psi(x, U) dS^{d-1} dx ds \\
    &= \int_t^T \int_G \int_{S^{d-1}} \left[ b_1(s, x, U) z(s, x, U) + b_3(s, x, U) Z(s, x, U) \right] \psi(x, U) dS^{d-1} dx ds \\
    &+ \int_t^T \int_G \int_{S^{d-1}} \left[ \int_{S^{d-1}} b_2(s, x, V) z(s, x, V) dS^{d-1}(V) \right] \psi(x, U) dS^{d-1}(U) dx ds \\
    &+ \int_t^T \int_G \int_{S^{d-1}} \left[ b_4(s, x, U) z(s, x, U) + Z(s, x, U) \right] \psi(x, U) dS^{d-1} dx dB(s), \text{ P-a.s.}
\end{aligned}
\]

In Section 2, we will establish the following well-posedness result for (1.5).

**Proposition 1.2** For any \(z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))\), the equation (1.5) admits a unique solution \((z, Z)\) such that

\[
\|z\|_{L^2(\Omega; C([0, T]; L^2(G \times S^{d-1})))} + \|Z\|_{L^2_F(0, T; L^2(G \times S^{d-1}))} \leq e^{C_T} \|z_T\|_{L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1}))},
\]

(1.7)
where \( C \) is a constant which is independent of \( z_T \) and

\[
 r_2 \triangleq \sum_{i=1,i\neq 2}^{4} |b_i|_{L^\infty_T(L^\infty_2(G \times S^{d-1}))}^4 + |b_2|_{L^\infty_T(\Omega; C([0,T]; C(\mathbb{T} \times S^{d-1} \times S^{d-1})))}^2 + 1.
\]

Now we give the definition of the continuous observability for the equation (1.5).

**Definition 1.4** Equation (1.5) is said to be continuously observable in \([0,T]\) if there is a constant \( C(b_1,b_2,b_3,b_4) > 0 \) such that all solutions of the equation (1.5) satisfy that

\[
 |z_T|_{L^2(\Omega; \mathcal{F}_T,P; L^2(G \times S^{d-1})))} \leq C(b_1,b_2,b_3,b_4)(|z|_{L^2_T(0,T; L^2_\mathcal{G}^\perp)} + |Z|_{L^2_T(0,T; L^2(G \times S^{d-1})))}.
\]

The solution \( z \in L^2_T(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \), hence, it is not obvious that \( z|_{\mathcal{G}^\perp} \) belongs to \( L^2_T(0,T; L^2_\mathcal{G}^\perp) \). This is indeed guaranteed by the following regularity result for (1.5).

**Proposition 1.3** Let \((z,Z)\) solve the equation (1.5) with the terminal state \( z_T \). Then

\[
 |z|^2_{L^2_T(0,T; L^2_\mathcal{G}^\perp)} \leq C_{r_2}\mathbb{E}|z_T|^2_{L^2(G \times S^{d-1})).
\]

**Remark 1.2** The fact that \( z|_{\mathcal{G}^\perp} \in L^2_T(0,T; L^2_\mathcal{G}^\perp) \) is sometimes called a hidden regularity property. It does not follow directly from the classical trace theorem of Sobolev space.

It follows from Proposition 1.3 that \( |z|^2_{L^2_T(0,T; L^2_\mathcal{G}^\perp)} \) makes sense. Now we give the observability result for the equation (1.5).

**Theorem 1.5** If \( T > 2R \), then the equation (1.5) is continuously observable in \([0,T]\).

In spite of its simple linear form, the transport equation governs many diffusion processes (see [10] for example). Moreover, it is a linearized Boltzmann equation, and it is related to the equations of fluid dynamics such as the Euler and the Navier-Stokes equations. It is desired to study the stochastic transport equation since it is a model when the system governed by the transport equation is perturbed by some stochastic influence. The stochastic transport equation is extensively studied now (see [1, 3, 6, 8, 23] and the rich references cited therein).

The controllability problems for linear and nonlinear deterministic transport equations are well studied in the literature (see [4, 9, 11, 18, 24] and the rich references cited therein).

On the contrast, to the author’s best knowledge, there is no published paper addressed to the controllability of stochastic transport equations.

Generally speaking, there are three methods to establish the exact controllability of deterministic transport equations. The first and most straightforward one is utilizing the explicit formula of the solution. By this method, for some simple transport equations, one can explicitly give a control steering the system from every given initial state to any given final state, provided that the time is large enough. It seems that this method cannot be used to solve our problem since generally we do not have the explicit formula for solutions to the system (1.1). Nevertheless, we shall borrow this idea to prove one of our negative
results (i.e., Theorem 1.2). The second one is the extension method. This method was first introduced in [25] to prove the exact controllability of wave equations. It is effective to solve the exact controllability problem for many hyperbolic-type equations. However, it seems that it is only valid for time reversible systems. The third and most popular method is based on the duality between controllability and observability, via which the exact controllability problem is reduced to suitable observability estimate for the dual system, and the desired observability estimate is obtained by some global Carleman estimate (see [18] for example).

Similar to the deterministic setting, we shall use a stochastic version of the global Carleman estimate to derive the inequality (1.8). For this, we borrow some idea from the proof of the observability estimate for the deterministic transport equations (see [18] for example). However, the stochastic setting will produce some extra difficulties. We cannot simply mimic the method in [18] to solve our problem.

Generally speaking, the nonlocal term, say the term \( \int_{S^{d-1}} b_2(t, x, V, U) z(t, x, V) dS^{d-1}(V) \) for our problem, will lead some trouble for obtaining the observability estimate from the Carleman estimate, because one cannot simply interchange the integral operator and the weight function. However, this will not happen in our case, for the reason that we choose a weight function \( \theta \) which independent of the variable \( U \).

Compared with the extensive results for Carleman estimate of partial differential equations, there are a very few works addressed to its stochastic counterpart. In [2] and [26], the authors established some Carleman type inequalities for forward and backward stochastic parabolic equations, and via which the controllability problems for these equations were addressed. On the other hand, the authors in [12], [13] and [28] obtained some different Carleman type inequalities for studying unique continuation problems for stochastic parabolic equations. In [29], a Carleman type inequality for stochastic wave equations was first obtained. The result in [29] was improved in [15] and [17] to solve some inverse problems for stochastic wave equations. In [14], the author got a Carleman type inequality for stochastic Schrödinger equations and used it to study a state observation problem for these equations. A Carleman type inequality for backward stochastic Schrödinger equations was established in [16] to prove the exact controllability of (forward) stochastic Schrödinger equations.

In the literature, in order to obtain the observability estimate, people usually combine a Carleman estimate and an Energy estimate (see [18] and [27] for example). In this paper, we deduce the inequality (1.4) by a new global Carleman estimate directly (without using the energy estimate). Indeed, our method even provide a proof which is simpler than that in [18] for the observability estimate for deterministic transport equations.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results, including the proofs of Propositions 1.1-1.3 and a weighted identity which is used to prove Theorem 1.5. In Section 3, we prove Theorem 1.5 and in Section 4, we prove Theorem 1.1. Finally, Section 5 is addressed to the proofs of Theorems 1.2-1.4.

### 2 Some preliminaries

This section is addressed to present some preliminary results. We divided it into four subsections. Proofs of Propositions 1.1-1.3 are given in the first three subsections. Next, we present a weighted identity for the stochastic transport operator \( d + U \cdot \nabla dt \), which plays an
2.1 Well-posedness of (1.1)

In this subsection, we prove Proposition 1.1. Equation (1.1) is a nonhomogeneous boundary value problem. Usually, the well-posedness of such kind of equations is established in the sense of transposition solutions (see [19] and [20] for example). However, fortunately, for our problem, we can obtain the well-posedness of (1.1) in the context of weak solution. The key point for doing this is to establish some suitable a priori estimate (see the inequality (2.8) below).

Proof of Proposition 1.1: Let us first deal with the case in which

\[
\begin{align*}
\begin{cases}
y_0 &\in L^2(\Omega; \mathcal{F}_0, P; H^1(\Omega \times S^{d-1})) \\
f, v &\in L^2_T(0, T; H^1_0(\Omega \times S^{d-1})), \ u \in Y.
\end{cases}
\end{align*}
\]

Here

\[
Y \triangleq \{ u : u = \tilde{u}\rvert_{[0,T] \times \Gamma^-_S} \text{ for some } \tilde{u} \in L^2_T(\Omega; C^1([0,T]; H^1(\Omega \times S^{d-1}))), \ \tilde{u}(0, \cdot, \cdot) = 0 \text{ on } \Gamma^-_S, \ P\text{-a.s.} \}.
\]

It is clear that \( Y \) is dense in \( L^2_T(0, T; L^2(\Gamma^-_S)) \).

Let us consider the following equation:

\[
\begin{align*}
\begin{cases}
dw + U \cdot \nabla wd\tau = &\left(a_1w + \int_{S^{d-1}} a_2(t, x, U, V)w(t, x, V)dS^{d-1} + \tilde{f}\right)dt \\
+ (a_3w + v)dB(t) + a_3\tilde{u}dB(t) &\text{in } (0, T) \times \Omega, \quad \text{in } (0, T) \times \Omega, \\
w(t, 0) = &0, \quad \text{in } \Omega, \\
w(0) = &y_0, \quad \text{in } \Omega.
\end{cases}
\end{align*}
\]

Here

\[
\tilde{f} = -\tilde{u}_t - U \cdot \nabla \tilde{u} + a_1\tilde{u} + \int_{S^{d-1}} a_2(t, x, U, V)\tilde{u}(t, x, V)dS^{d-1} + f.
\]

Clearly, \( \tilde{f} \in L^2_T(0, T; H^1(\Omega \times S^{d-1})) \). Define an unbounded operator \( A \) on \( L^2(\Omega \times S^{d-1}) \) as follows:

\[
\begin{align*}
\begin{cases}
D(A) = &\{ h \in H^1(\Omega \times S^{d-1}) : h = 0 \text{ on } \Gamma^-_S \}, \\
Ah = &-U \cdot \nabla h, \quad \forall h \in D(A).
\end{cases}
\end{align*}
\]

It is an easy matter to see that \( D(A) \) is dense in \( L^2(\Omega \times S^{d-1}) \) and \( A \) is closed. Furthermore, for every \( h \in D(A) \),

\[
(Ah, h)_{L^2(\Omega \times S^{d-1})} = -\int_{\Omega} \int_{S^{d-1}} hU \cdot \nabla h ds^{d-1} dx = -\int_{\Gamma^+_S} U \cdot \nu |h|^2 d\Gamma^+_S \leq 0.
\]

One can easily check that the adjoint operator of \( A \) is

\[
\begin{align*}
\begin{cases}
D(A^*) = &\{ h \in H^1(\Omega \times S^{d-1}) : h = 0 \text{ on } \Gamma^+_S \}, \\
A^*h = &U \cdot \nabla h, \quad \forall h \in D(A^*).
\end{cases}
\end{align*}
\]
For every $h \in D(A^*)$, it holds that
\[
(A^*h, h)_{L^2 \times S^{d-1}} = \int_G \int_{S^{d-1}} hU \cdot \nabla h dS^{d-1} dx = \int_{\Gamma^-} U \cdot \nu |h|^2 d\Gamma^- \leq 0.
\]
Hence, both $A$ and $A^*$ are dissipative operators. Recalling that $D(A)$ is dense in $L^2(G \times S^{d-1})$ and $A$ is closed. From the standard operator semigroup theory (see [7, Page 84] for example), we conclude that $A$ generates a $C_0$-semigroup $\{S(t)\}_{t \geq 0}$ on $L^2(G \times S^{d-1})$ and $A^*$ generates its dual semigroup $\{S^*(t)\}_{t \geq 0}$ on $L^2(G \times S^{d-1})$. Therefore, by the classical theory for stochastic partial differential equations (see [5, Chapter 6]), the system (2.2) admits a unique solution
\[
w \in L^2_{\mathcal{F}}(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \cap L^2_F(0, T; D(A))
\]
such that
\[
\int_G \int_{S^{d-1}} w(t, x)\phi(x) dS^{d-1} dx - \int_G \int_{S^{d-1}} y_0(x)\phi(x) dS^{d-1} dx
\]
\[
- \int_0^t \int_G \int_{S^{d-1}} w(s, x)U \cdot \nabla \phi(x) dS^{d-1} dx ds
\]
\[
= \int_0^t \int_G \int_{S^{d-1}} \left[ a_1(s, x, U)w(s, x, U) + \tilde{f}(s, x, U) \right] \phi(x, U) dS^{d-1} dx ds
\]
\[
+ \int_0^t \int_G \int_{S^{d-1}} a_2(s, x, V)w(s, x, V) dS^{d-1}(V) \phi(x, U) dS^{d-1}(U) dx ds
\]
\[
+ \int_0^t \int_G \int_{S^{d-1}} \left\{ a_3(s, x, U)\left[ w(s, x, U) + \tilde{u}(s, x, U) \right] + v(s, x, U) \right\} \phi(x, U) dS^{d-1} dx dB(s),
\]
P-a.s., for any $\phi \in C^1(G \times S^{d-1})$ with $\phi = 0$ on $\Gamma^+_S$ and $t \in [0, T]$.

Let
\[
y(t, x, U) = w(t, x, U) + \tilde{u}(t, x, U), \quad \text{for } (t, x, U) \in [0, T] \times G \times S^{d-1}.
\]
Clearly,
\[
y \in L^2_{\mathcal{F}}(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \cap L^2_F(0, T; H^1(G \times S^{d-1})).
\]
From (2.5), we know that $y$ satisfies
\[
\int_G \int_{S^{d-1}} y(t, x, U)\phi(x, U) dS^{d-1} dx - \int_G \int_{S^{d-1}} y_0(x, U)\phi(x, U) dS^{d-1} dx
\]
\[
- \int_0^t \int_G \int_{S^{d-1}} y(s, x, U)U \cdot \nabla \phi(x, U) dS^{d-1} dx ds + \int_0^t \int_G \int_{S^{d-1}} \tilde{u}(s, x, U)U \cdot \nabla \phi(x, U) dS^{d-1} dx ds
\]
\[
= \int_0^t \int_G \int_{S^{d-1}} \left[ a_1(s, x, U)y(s, x, U) + f(s, x, U) - U \cdot \nabla \tilde{u}(s, x, U) \right] \phi(x) dS^{d-1} dx ds
\]
\[
+ \int_0^t \int_G \int_{S^{d-1}} a_2(s, x, V)y(s, x, V) dS^{d-1}(V) \phi(x, U) dS^{d-1}(U) dx ds
\]
\[
+ \int_0^t \int_G \int_{S^{d-1}} a_3(s, x, U)y(s, x, U) + v(s, x, U) \phi(x, U) dS^{d-1} dx dB(s),
\]
P-a.s., for all $\phi \in C^1(G \times S^{d-1})$ with $\phi = 0$ on $\Gamma^+_S$ and $t \in [0, T]$. 

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Utilizing integration by parts again, we see that the equality (1.2) holds. Therefore, \( y \) is a solution to the system (1.1) under the assumption (2.1). Furthermore, by means of Itô's formula,

\[
|y(t)|^2_{L^2(G \times S^{d-1})} = |y_0|^2_{L^2(G \times S^{d-1})} - 2 \int_0^t \int_G \int_{S^{d-1}} yU \cdot \nabla ydS^{d-1} \, dxds \\
+ 2 \int_0^t \int_G \int_{S^{d-1}} \left( \int_{S^{d-1}} a_2 ydS^{d-1}(V) \right) ydS^{d-1}(U) \, dxds \\
+ \int_0^t \int_G \int_{S^{d-1}} \left[ 2a_1 y^2 + 2fy + (a_3y + v)^2 \right] dS^{d-1} \, dxds \\
+ 2 \int_0^t \int_G \int_{S^{d-1}} y(a_3y + v) dS^{d-1} \, dx dB(s).
\]

This, together with the Burkholder-Davis-Gundy inequality, implies that

\[
\mathbb{E} \sup_{s \in [0,t]} |y(s)|^2_{L^2(G \times S^{d-1})} \\
\leq |y_0|^2_{L^2(G \times S^{d-1})} - 2 \mathbb{E} \int_0^t \int_G U \cdot \nu u^2 d\Gamma^{-}_S \, ds + 2\mathbb{E} \int_0^t \int_G |a_2|_{C(\mathcal{C} \times S^{d-1} \times S^{d-1})} \int_{S^{d-1}} y^2 dS^{d-1} \, dxds \\
+ 4\mathbb{E} \int_0^t \int_G \int_{S^{d-1}} \left[ a_1 y^2 + y^2 + f^2 + a_3^2 y^2 + v^2 \right] dS^{d-1} \, dxds \\
\leq |y_0|^2_{L^2(G \times S^{d-1})} + 4r_1 \mathbb{E} \int_0^t \left[ \sup_{\sigma \in [0,s]} |y(\sigma)|^2_{L^2(G \times S^{d-1})} \right] ds \mathbb{E} \int_0^t \int_G U \cdot \nu u^2 d\Gamma^{-}_S \, ds \\
+ 4\mathbb{E} \int_0^t \int_G \int_{S^{d-1}} (f^2 + v^2) dS^{d-1} \, dxds.
\]

Hence, by Gronwall’s inequality, we obtain that

\[
|y|_{L^2_t(\Omega; C([0,T]; L^2(G \times S^{d-1})))} \\
\leq e^{Cr^1} \left( |y_0|_{L^2_t(G \times S^{d-1})} + |u|_{L^2_t(0,T; L^2(\Gamma^{-}_S))} + |f|_{L^2_t(0,T; L^2(G \times S^{d-1}))} + |v|_{L^2_t(0,T; L^2(G \times S^{d-1}))} \right).
\]

By a similar argument, we can show that if

\[
(\hat{y}_0, \hat{u}, \hat{f}, \hat{v}) \in D(A) \times Y \times L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1})) \times L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1}))
\]

and

\[
(\check{y}_0, \check{u}, \check{f}, \check{v}) \in D(A) \times Y \times L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1})) \times L^2_{\mathcal{F}}(0,T; H^1_0(G \times S^{d-1}))
\]

then we can find corresponding solutions

\[
\hat{y}, \check{y} \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \cap L^2_{\mathcal{F}}(0,T; H^1(G \times S^{d-1}))
\]

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such that
\[ |\hat{y} - \hat{y}|_2^2(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \]
\[ \leq e^{C_1^\prime} (|\hat{y}_0 - \hat{y}_0|_2^2(\Omega; C([0,T]; L^2(G \times S^{d-1}))) + |\hat{u} - \hat{u}|_2^2(\Omega; C([0,T]; L^2(\Gamma_S^-))) + |\hat{f} - \hat{f}|_2^2(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \]
\[ + |\hat{v} - \hat{v}|_2^2(\Omega; C([0,T]; L^2(G \times S^{d-1}))). \]

Now for \( y_0 \in L^2(G \times S^{d-1}), u \in L^2_\tau(0,T; L^2_w(\Gamma_S^-)), f, v \in L^2_\tau(0,T; L^2(G \times S^{d-1})), \) let us choose
\[ \{y^n\}_{n=1}^{+\infty} \subset D(A), \quad \{u^n\}_{n=1}^{+\infty} \subset Y, \quad \{f^n\}_{n=1}^{+\infty} \subset L^2_\tau(0,T; H^1_0(G \times S^{d-1})), \]
\[ \{v^n\}_{n=1}^{+\infty} \subset L^2_\tau(0,T; H^1_0(G \times S^{d-1})), \]
such that
\[ \begin{aligned}
\lim_{n \to \infty} y^n &= y_0 \text{ in } L^2(G \times S^{d-1}); \\
\lim_{n \to \infty} u^n &= u \text{ in } L^2_\tau(0,T; L^2(\Gamma_S^-)); \\
\lim_{n \to \infty} f^n &= f \text{ in } L^2_\tau(0,T; L^2(G \times S^{d-1})); \\
\lim_{n \to \infty} v^n &= v \text{ in } L^2_\tau(0,T; L^2(G \times S^{d-1})).
\end{aligned} \quad (2.9) \]

For every given \( (y^n_0, u^n, f^n, v^n) \), by the argument above, we know that there is a unique solution \( y_n(\cdot, \cdot) \) to the system (1.1), which satisfies
\[ \begin{aligned}
\int_G \int_{S^{d-1}} y_n(t,x,U)\phi(x)dS^{d-1}dx &= \int_G \int_{S^{d-1}} y^n_0(x,U)\phi(x,U)dS^{d-1}dx \\
- \int_0^T \int_G \int_{S^{d-1}} y_n(s,x,U)U \cdot \nabla \phi(x,U)dS^{d-1}dsdx &= \int_0^T \int_{\Gamma_S^-} U \cdot \nu u^n(s,x,U)\phi(x,U)d\Gamma^-_Sds \\
= & \int_0^T \int_G \int_{S^{d-1}} \left[ a_1(s,x,U)y_n(s,x,U) + f^n(s,x,U) \right] \phi(x)dS^{d-1}ds \\
+ & \int_0^T \int_G \int_{S^{d-1}} \left[ \int_{S^{d-1}} a_2(s,x,U)V y_n(s,x,V) dS^{d-1}(V) \right] \phi(x,U)dS^{d-1}(U)ds \\
+ & \int_0^T \int_G \int_{S^{d-1}} \left[ a_3(s,x,U)y_n(s,x,U) + v^n(s,x,U) \right] \phi(x,U)dS^{d-1}d\nu B(s), \\
P-a.s., & \text{ for any } \phi \in C^1(\overline{G} \times S^{d-1}) \text{ with } \phi = 0 \text{ on } \Gamma_S^+ \text{ and } \tau \in [0,T], \\
\text{and} \quad y_n \in L^2_\tau(\Omega; C([0,T]; L^2(G \times S^{d-1}))) \quad (2.10) \quad \text{and} \quad \begin{aligned}
|y_n|_{L^2_\tau(\Omega; C([0,T]; L^2(G \times S^{d-1})))} \leq & e^{C_1^\prime} (|y^n_0|_{L^2(G \times S^{d-1})} + |u^n|_{L^2_\tau(0,T; L^2(\Gamma_S^-))} + |f^n|_{L^2_\tau(0,T; L^2(G \times S^{d-1}))} + |v^n|_{L^2_\tau(0,T; L^2(G \times S^{d-1}))}) \\
& \quad (2.11) \end{aligned} \]

Further, for any \( m, n \in \mathbb{N} \), we have
\[ \begin{aligned}
|y_n - y_m|_{L^2_\tau(\Omega; C([0,T]; L^2(G \times S^{d-1})))} \leq & e^{C_1^\prime} (|y^n_0 - y^m_0|_{L^2(G \times S^{d-1})} + |u^n - u^m|_{L^2_\tau(0,T; L^2(\Gamma_S^-))} + |f^n - f^m|_{L^2_\tau(0,T; L^2(G \times S^{d-1}))} + |v^n - v^m|_{L^2_\tau(0,T; L^2(G \times S^{d-1}))}) \\
& \quad (2.12) \end{aligned} \]
From (2.9) and (2.12), we obtain that \( \{y_n\}_{n=1}^{+\infty} \) is a Cauchy sequence in \( L_F^2(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \). Hence, there exists a unique \( y \in L_F^2(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \) such that

\[
y_n \to y \text{ in } L_F^2(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \quad \text{as } n \to +\infty. \tag{2.13}
\]

Combining (2.10) and (2.13), we find that \( y \) satisfies (1.2). Hence, \( y \) is a solution to the system (1.1).

Further, from (2.11) and (2.13), we obtain that \( y \) satisfies the inequality (1.3).

The uniqueness of the solution to (1.1) follows from (1.3) immediately. This completes the proof of Proposition 1.1.

\[\square\]

### 2.2 Well-posedness of (1.5)

This subsection is devoted to a proof of Proposition 1.2.

We first recall the definition of the mild solution to backward stochastic evolution equations.

Let \( X \) be a Hilbert space and \( A : D(A) \subset X \to X \) be a linear operator which generates a \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \) on \( X \). Let \( F_1 : [0, T] \times X \times X \to X \) satisfy that

- there exists an \( L_1 > 0 \) such that

\[
|F_1(t, \eta_1, \eta_2) - F_1(t, \hat{\eta}_1, \hat{\eta}_2)|_X \leq L_1(|\eta_1 - \hat{\eta}_1|_X + |\eta_2 - \hat{\eta}_2|_X) \quad \text{for all } t \in [0, T], \eta_1, \hat{\eta}_1, \eta_2, \hat{\eta}_2 \in X;
\]

- \( F_1(\cdot, 0, 0) \in L^2(0, T; X) \).

Let \( F_2(\cdot, \cdot) : [0, T] \times X \to X \) satisfy that

- there exists an \( L_2 > 0 \) such that

\[
|F_2(t, \eta_1) - F_2(t, \hat{\eta}_1)|_X \leq L_2 |\phi - \hat{\phi}|_X \quad \text{for all } t \in [0, T], \eta_1, \hat{\eta}_1 \in X;
\]

- \( F_2(\cdot, \cdot) \in L^2(0, T; X) \).

Consider the following backward stochastic evolution equation

\[
\begin{cases}
\quad d\phi = -[A\phi(t) + F_1(t, \phi(t), \Phi(t))] dt - [F_2(t, \phi(t)) + \Phi(t)] dB(t) & \text{in } [0, T], \\
\quad \phi(T) = \phi_T,
\end{cases} \tag{2.14}
\]

where \( \phi_T \in L^2(\Omega, F_T, P; X) \).

A pair of processes \((\phi, \Phi) \in L_F^2(\Omega; C([0, T]; X)) \times L_F^2(0, T; X)\) is a mild solution of (2.14) if for all \( t \in [0, T] \), they satisfy that

\[
\phi(t) = S(T - t)\phi_T + \int_t^T S(s - t)F_1(s, \phi(s), \Phi(s)) ds + \int_t^T S(s - t)[F_2(s, \phi(s)) + \Phi(s)] dB(s), \quad P\text{-a.s.} \tag{2.15}
\]
Lemma 2.1 [27, Theorem 9] The equation (2.14) admits a unique mild solution $(\phi, \Phi)$.

We are now in a position to prove Proposition 1.2.

Proof of Proposition 1.2: Let $X = L^2(G \times S^{d-1})$, $A = A^*$,

$$
\left\{
\begin{array}{l}
F_1(t, \phi, \Phi) = - \left[ b_1 \phi + \int_{S^{d-1}} b_2(t, x, V, U) \phi(t, x, V) dS^{d-1}(V) + b_3 \Phi \right], \\
F_2(t, \phi) = -b_4 \phi.
\end{array}
\right.
$$

We have $S(t) = S^*(t)$. By Lemma 2.1, we conclude that (1.5) admits a unique mild solution $(z, Z)$ such that

$$
z(t) = S^*(T - t)z_T - \int_t^T S^*(s - t) \left[ b_1 z + \int_{S^{d-1}} b_2(s, x, V, U) z(s, x, V) dS^{d-1}(V) + b_3 Z \right] ds
$$

$$
- \int_t^T S^*(s - t) (b_4 z + Z) dB(s), \quad P\text{-a.s.}
$$

(2.16)

From (2.16), for any $\psi \in C^1(G \times S^{d-1})$ with $\psi = 0$ on $\Gamma_0$, we have that

$$
\langle z(t), A\psi \rangle_{L^2(G \times S^{d-1})}
$$

$$
= \langle S^*(T - t)z_T, A\psi \rangle_{L^2(G \times S^{d-1})} + \int_t^T \left\langle S^*(s - t) F_1(s, z(s), Z(s)), A\psi \right\rangle_{L^2(G \times S^{d-1})} ds
$$

$$
- \int_t^T \left\langle S^*(s - t) (b_4 z + Z), A\psi \right\rangle_{L^2(G \times S^{d-1})} dB(s)
$$

$$
= \langle z_T, S(T - t) A\psi \rangle_{L^2(G \times S^{d-1})} + \int_t^T \left\langle F_1(s, z(s), Z(s)), S(s - t) A\psi \right\rangle_{L^2(G \times S^{d-1})} ds
$$

$$
- \int_t^T \left\langle b_4 z + Z, S(s - t) A\psi \right\rangle_{L^2(G \times S^{d-1})} dB(s)
$$

$$
\triangleq I_1 + I_2 - I_3.
$$

(2.17)

Integrating (2.17) from $t$ to $T$, we obtain that

$$
\int_t^T \langle z(t), A\psi \rangle_{L^2(G \times S^{d-1})} ds = \int_t^T (I_1 + I_2 - I_3) ds.
$$

(2.18)

Clearly,

$$
\int_t^T I_1 ds = \int_t^T \langle z_T, S(T - s) A\psi \rangle_{L^2(G \times S^{d-1})} ds
$$

$$
= \langle z_T, S(T - t) \psi \rangle_{L^2(G \times S^{d-1})} - \langle z_T, \psi \rangle_{L^2(G \times S^{d-1})}.
$$

(2.19)
By the Fubini’s theorem, we have that
\[ \int_t^T I_2 ds = \int_t^T \int_s^T \left\langle F_1(r, z(r), Z(r)), S(r-s)A\psi \right\rangle_{L^2(G \times S^{d-1})} dr ds \]
\[ = \int_t^T \left\langle F_1(r, z(r), Z(r)), \int_t^r S(r-s)A\psi ds \right\rangle_{L^2(G \times S^{d-1})} dr \]
\[ = \int_t^T \left\langle F_1(r, z(r), Z(r)), S(r-t)\psi - \psi \right\rangle_{L^2(G \times S^{d-1})} dr \]
\[ = \int_t^T \left\langle S^* (r-t)F_1(r, z(r), Z(r)), \psi \right\rangle_{L^2(G \times S^{d-1})} dr \]
\[ - \int_t^T \left\langle F_1(r, z(r), Z(r)), \psi \right\rangle_{L^2(G \times S^{d-1})} dr, \tag{2.20} \]
and by the stochastic Fubini’s theorem (see [5, page 109] for example), we find that
\[ \int_t^T I_3 ds = \int_t^T \int_s^T \left\langle b_4(r)z(r) + Z(r), S(r-s)A\psi \right\rangle_{L^2(G \times S^{d-1})} dr dB(s) \]
\[ = \int_t^T \left\langle b_4(r)z(r) + Z(r), \int_t^r S(r-s)A\psi ds \right\rangle_{L^2(G \times S^{d-1})} dB(r) \]
\[ = \int_t^T \left\langle b_4(r)z(r) + Z(r), S(r-t)\psi - \psi \right\rangle_{L^2(G \times S^{d-1})} dB(r) \]
\[ = \int_t^T \left\langle S^* (r-t)[b_4(r)z(r) + Z(r)], \psi \right\rangle_{L^2(G \times S^{d-1})} dB(r) \]
\[ - \int_t^T \left\langle b_4(r)z(r) + Z(r), \psi \right\rangle_{L^2(G \times S^{d-1})} dB(r). \tag{2.21} \]

From (2.17)–(2.21), we obtain that \((z, Z)\) satisfies (1.6).

The proof of the inequality (1.7) is very similar to the one of (1.3). Indeed, by Itô’s formula, we can easily obtain that
\[ |z_T|^2_{L^2(G \times S^{d-1})} - |z(t)|^2_{L^2(G \times S^{d-1})} \]
\[ \geq 2 \int_t^T \int_G \int_{S^{d-1}} z \left[ b_1 z + \int_{S^{d-1}} b_2(r, x, V)z(t, x, V) dS^{d-1}(V) + b_3 Z \right] dS^{d-1} dx ds \]
\[ + 2 \int_t^T \int_G \int_{S^{d-1}} (b_4 z + Z) dS^{d-1} dx ds + \int_t^T \int_G \int_{S^{d-1}} (b_4 z + Z)^2 dS^{d-1} dx ds. \tag{2.22} \]

By Burkholder-Davis-Gundy inequality, we find that
\[ \mathbb{E} \sup_{s \in [t,T]} |z(t)|^2_{L^2(G \times S^{d-1})} + |Z|^2_{L^2(t,T;L^2(G \times S^{d-1}))} \]
\[ \leq |z_T|^2_{L^2(G \times S^{d-1})} + Cr_2 \mathbb{E} \int_t^T |z(s)|^2_{L^2(G \times S^{d-1})}. \tag{2.23} \]

Then, by the Gronwall’s inequality, we get (1.7) immediately. The uniqueness of the solution follows from the inequality (1.7). This completes the proof of Proposition 1.2. \(\square\)
2.3 Hidden regularity for solutions to backward stochastic transport equations

In this subsection, we give a proof of Proposition 1.3.

Proof of Proposition 1.3: The proof is almost standard. Here we give it for the sake of completeness. Let

\[ \mathcal{X} \triangleq \{ h \in H^1(G \times S^{d-1}) : h = 0 \text{ on } \Gamma^+_S \}. \]

Following the proof of Proposition 1.2 (for this, one needs numerous but small changes), one can show that if \( z_T \in L^2(\Omega, \mathcal{F}_T, P; \mathcal{X}) \), then the solution

\[ (z, Z) \in \left( L^2(\Omega; C([0, T]; L^2(G \times S^{d-1}))) \cap L^2_T(0, T; \mathcal{X}) \right) \times L^2_T(0, T; L^2(G \times S^{d-1})). \]

Then, by Itô’s formula, we see that

\[
\begin{align*}
E[z_T^2_{L^2(G \times S^{d-1})} - |z(0)|^2_{L^2(G \times S^{d-1})}] & = -E \int_0^T \int_G \int_{S^{d-1}} zU \cdot \nabla z dS^{d-1} dx dt + E \int_0^T \int_G \int_{S^{d-1}} \left[ 2z(b_1 z + b_3 Z) + (b_4 z + Z)^2 \right] dS^{d-1} dx dt \\
& \quad + E \int_0^T \int_G \int_{S^{d-1}} z(t, x, U) \left[ \int_{S^{d-1}} b_2(t, x, U, V) z(t, x, V) dS^{d-1}(V) \right] dS^{d-1}(U) dx dt.
\end{align*}
\]

Therefore, we find that

\[
-\mathbb{E} \int_0^T \int_{\Gamma^+_S} U \cdot \nu z^2 d\Gamma^-_S dt
= \mathbb{E}|z_T^2_{L^2(G \times S^{d-1})} - |z(0)|^2_{L^2(G \times S^{d-1})} | + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \left[ 2z(b_1 z + b_3 Z) + (b_4 z + Z)^2 \right] dS^{d-1} dx dt
\]

\[
\quad + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} z(t, x, U) \left[ \int_{S^{d-1}} b_2(t, x, U, V) z(t, x, V) dS^{d-1}(V) \right] dS^{d-1}(U) dx dt
\]

\[
\leq e^{C_{r_2}} \mathbb{E}|z_T^2_{L^2(G \times S^{d-1})} | \quad (2.24)
\]

For any \( z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})) \), we can find a sequence \( \{ z_T^{(n)} \}_{n=1}^{\infty} \subset L^2(\Omega, \mathcal{F}_T, P; \mathcal{X}) \) such that

\[
\lim_{n \to \infty} z_T^{(n)} = z_T \text{ in } L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})).
\]

Hence, we know that the inequality (2.25) also holds for \( z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G \times S^{d-1})) \).

2.4 Identity for a stochastic transport operator

In this subsection, we introduce a weighted identity for the stochastic transport operator \( d + U \cdot \nabla dt \), which will play a key role in the proof of Theorem 1.5. Let \( \lambda > 0 \), and let \( 0 < c < 1 \) such that \( cT > 2R \). Put

\[
l = \lambda \left[ |x|^2 - c \left( t - \frac{T}{2} \right)^2 \right] \quad \text{and} \quad \theta = e^l. \quad (2.26)
\]
We have the following weighted identity involving $\theta$ and $l$.

**Proposition 2.1** Assume that $q$ is an $H^1(\mathbb{R}^n) \times L^2(S^{d-1})$-valued continuous semi-martingale. Put $p = \theta q$. We have the following equality

\[
-\theta(l_t + U \cdot \nabla l)p\left[ dq + U \cdot \nabla q \, dt \right] \\
= -\frac{1}{2} d\left[ (l_t + U \cdot \nabla l)p^2 \right] - \frac{1}{2} U \cdot \nabla \left[ (l_t + U \cdot \nabla l)p^2 \right] + \frac{1}{2} l_{tt} + U \cdot \nabla (U \cdot \nabla l) \\
+ 2U \cdot \nabla l_t p^2 + \frac{1}{2} (l_t + U \cdot \nabla l)(dp)^2 + (l_t + U \cdot \nabla l)^2 p^2.
\]

(2.27)

**Proof of Proposition 2.1**: By the definition of $p$, we have

\[
\theta(dq + U \cdot \nabla q) = \theta d(\theta^{-1} p) + \theta U \cdot \nabla (\theta^{-1} p) = dp + U \cdot \nabla p - (l_t + U \cdot \nabla l)p.
\]

Thus,

\[
-\theta(l_t + U \cdot \nabla l)p(dq + U \cdot \nabla q) \\
= -(l_t + U \cdot \nabla l)p\left[ dp + U \cdot \nabla p - (l_t + U \cdot \nabla l)p \right] \\
= -(l_t + U \cdot \nabla l)p(dp + U \cdot \nabla p) + (l_t + U \cdot \nabla l)^2 p^2.
\]

(2.28)

It is easy to see that

\[
\begin{align*}
-l_t dp &= -\frac{1}{2} d(l_t p^2) + \frac{1}{2} l_t p^2 + \frac{1}{2} l_t dp^2, \\
-U \cdot \nabla l dp &= -\frac{1}{2} d(U \cdot \nabla l p^2) + \frac{1}{2} (U \cdot \nabla l) dp^2 + \frac{1}{2} U \cdot \nabla l(dp)^2, \\
-l_t U \cdot \nabla p &= -\frac{1}{2} U \cdot \nabla (l_t p^2) + \frac{1}{2} U \cdot \nabla l_t p^2, \\
-U \cdot \nabla l U \cdot \nabla p &= -\frac{1}{2} U \cdot \nabla (U \cdot \nabla l p^2) + \frac{1}{2} U \cdot \nabla (U \cdot \nabla l) p^2.
\end{align*}
\]

(2.29)

From (2.28) and (2.29), we obtain the equality (2.27). \(\square\)

### 3 Proof of Theorem 1.5

This section is devoted to proving Theorem 1.5 by means of a suitable global Carleman estimate for the equation (1.5).

**Proof of Theorem 1.5**: To begin with, applying Proposition 2.1 to the equation (1.5) with $v = z$, integrating (2.27) on $(0, T) \times G \times S^{d-1}$ and using integration by parts, and taking
expectation, we get that

\[-2\mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l) z(dz + U \cdot \nabla z dt) dS^{d-1} dx dt\]

\[= \lambda \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (cT - 2U \cdot x) \theta^2(T) z^2(T) dS^{d-1} dx + \lambda \int_0^T \int_G \int_{S^{d-1}} (cT + 2U \cdot x) \theta^2(0) z^2(0) dS^{d-1} dx\]

\[+ \lambda \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} U \cdot \nu [c(T - 2t) - 2U \cdot x] \theta^2 z^2 d\Gamma^- S dt + 2(1 - c) \lambda \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dS^{d-1} dx dt\]

\[+ \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l) (b_4 z + Z)^2 dS^{d-1} dx dt + 2 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l) z^2 dS^{d-1} dx dt.\]

By virtue of that \(z\) solves the equation (1.5), we see that

\[-2\mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l) z(dz + U \cdot \nabla z dt) dS^{d-1} dx dt\]

\[= 2 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l) z \left(b_1 z + \int_{S^{d-1}} b_2(t, x, U, V) z(t, x, V) dS^{d-1}(V) + b_3 Z\right) dS^{d-1}(U) dx dt\]

\[\leq \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)^2 z^2 dS^{d-1} dx dt + 3 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(b_1^2 z^2 + b_3^2 Z^2) dS^{d-1} dx dt\]

\[+ 3 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 \left| \int_{S^{d-1}} b_2(t, x, U, V) z(t, x, V) dS^{d-1}(V) \right|^2 dS^{d-1}(U) dx dt.\]

This, together with the equality (3.1), implies that

\[\lambda \mathbb{E} \int_G \int_{S^{d-1}} (cT - 2U \cdot x) \theta^2(T) z^2(T) dS^{d-1} dx + \lambda \int_G \int_{S^{d-1}} (cT + 2U \cdot x) \theta^2(0) z^2(0) dS^{d-1} dx\]

\[+ 2(1 - c) \lambda \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dS^{d-1} dx dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l) (b_4 z + Z)^2 dS^{d-1} dx dt\]

\[+ \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)^2 z^2 dS^{d-1} dx dt\]

\[\leq 3 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(b_1^2 z^2 + b_3^2 Z^2) dS^{d-1} dx dt - \lambda \mathbb{E} \int_0^T \int_{\Gamma^- S} U \cdot \nu [c(T - 2t) - 2U \cdot x] \theta^2 z^2 d\Gamma^- S dt\]

\[+ 3 |b_2|^2 \mathbb{E}_{L^\infty((0, T); C(\overline{S^{d-1}} \times \overline{S^{d-1}}))} \int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dS^{d-1} dx dt.\]
Since
\[
\mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)(b_4 z + Z)^2 dS^{d-1} dx dt \\
\leq \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(l_t + U \cdot \nabla l)^2 z^2 dS^{d-1} dx dt + \frac{1}{2} \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(b_4^2 + 2b_4^2) z^2 dS^{d-1} dx dt \\
+ \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2|l_t + U \cdot \nabla l + 2|Z^2 dS^{d-1} dx dt,
\]
by means of the inequality (3.3), we find
\[
\lambda \mathbb{E} \int_G \int_{S^{d-1}} (cT - 2U \cdot x) \theta^2(T) z^2(T) dS^{d-1} dx + \lambda \mathbb{E} \int_G \int_{S^{d-1}} (cT + 2U \cdot x) \theta^2(0) z^2(0) dS^{d-1} dx \\
+ 2(1 - c) \lambda \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dS^{d-1} dx dt - 3 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(b_4^2 + b_4^4 + b_4^2) z^2 dS^{d-1} dx dt \\
- 3 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 b_4^2 |2 + \lambda x - c\lambda t| Z^2 dS^{d-1} dx dt \\
- \lambda \mathbb{E} \int_0^T \int_{\Gamma_S} U \cdot \nu [c(T - 2t) - 2U \cdot x] \theta^2 z^2 d\Gamma_S^- dt.
\]
Noting that \(|x| < 2R\), we know that
\[
\begin{align*}
&\left\{ \begin{array}{ll}
(cT - 2R) \mathbb{E} \int_G \int_{S^{d-1}} \theta^2(T) z^2(T) dS^{d-1} dx &\leq \mathbb{E} \int_G \int_{S^{d-1}} \theta^2(T)(cT - U \cdot x) z^2(T) dS^{d-1} dx, \\
(cT - 2R) \int_G \int_{S^{d-1}} \theta^2(0) z^2(0) dS^{d-1} dx &\leq \int_G \int_{S^{d-1}} \theta^2(0)(cT + U \cdot x) z^2(0) dS^{d-1} dx.
\end{array} \right.
\end{align*}
\]
Taking
\[
\lambda_1 = \frac{3}{2(1 - c)} \left( |b_4|^2_{L^\infty(0,T;L^\infty(G \times S^{d-1}))} + |b_4|^4_{L^\infty(0,T;L^\infty(G \times S^{d-1} \times S^{d-1}))} \right),
\]
for any \(\lambda \geq \lambda_1\), we conclude that
\[
\begin{align*}
&3 \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2(b_4^2 + b_4^4 + b_4^2) z^2 dS^{d-1} dx dt \\
&+ 3 |b_4|^2_{L^\infty(0,T;L^\infty(G \times S^{d-1} \times S^{d-1}))} \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dS^{d-1} dx dt \\
&\leq 2(1 - c) \lambda \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \theta^2 z^2 dS^{d-1} dx dt.
\end{align*}
\]
From (3.4)–(3.6), and noting that $cT > 2R$, we find that

$$E \int_G \int_{S^{d-1}} \theta^2(T, x)z^2(T, x) dS^{d-1} dx$$

$$\leq C E \int_0^T \int_G \int_{S^{d-1}} \theta^2(b^2 + 2 + |\lambda x - c\lambda t|) z^2 dS^{d-1} dt$$

$$- C E \int_0^T \int_{\Gamma_S} U \cdot \nu [c(T - 2t) - 2U \cdot x] \theta^2 z^2 d\Gamma_S dt.$$  

(3.7)

By the definition of $\theta$, we have

$$e^{-c\lambda T^2} \leq \theta \leq e^{4\lambda R^2}.$$  

This, together with the inequality (3.7), indicates that

$$e^{-2c\lambda T^2} E \int_G \int_{S^{d-1}} z^2 dS^{d-1} dx$$

$$\leq Ce^{8\lambda R^2}\left\{ E \int_0^T \int_G \int_{S^{d-1}} Z^2 dS^{d-1} dt - E \int_0^T \int_{\Gamma_S} U \cdot \nu z^2 d\Gamma_S dt \right\},$$

which implies that

$$E \int_G \int_{S^{d-1}} z^2 dS^{d-1} dx$$

$$\leq Ce^{2\lambda T^2 + 8\lambda R^2}\left\{ E \int_0^T \int_G \int_{S^{d-1}} Z^2 dS^{d-1} dt - E \int_0^T \int_{\Gamma_S} U \cdot \nu z^2 d\Gamma_S dt \right\}$$

$$\leq e^{4\lambda^2 T} \left\{ E \int_0^T \int_G \int_{S^{d-1}} Z^2 dS^{d-1} dt - E \int_0^T \int_{\Gamma_S} U \cdot \nu z^2 d\Gamma_S dt \right\}.$$  

(3.9)

This completes the proof.

\[\square\]

4 Proof of Theorem 1.1

This section is addressed to a proof of Theorem 1.1.

Proof of Theorem 1.1: Since the system (1.1) is linear, we only need to show that the attainable set $A_T$ at time $T$ with initial datum $y(0) = 0$ is $L^2(\Omega, F_T, P; L^2(G \times S^{d-1}))$, that is, for any $y_1 \in L^2(\Omega, F_T, P; L^2(G \times S^{d-1}))$, we can find a pair of control

$$(u, v) \in L^2_{\mathcal{F}}(0, T; L^2_w(\Gamma_S^c)) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$$

such that the solution to the system (1.1) satisfies that $y(T) = y_1$ in $L^2(G \times S^{d-1})$, $P$-a.s. We achieve this goal by the duality argument.

Let $b_1 = -a_1$, $b_2 = -a_2$, $b_3 = -a_3$ and $b_4 = 0$ in the equation (1.5). We introduce the following linear subspace of $L^2_{\mathcal{F}}(0, T; L^2_w(\Gamma_S^c)) \times L^2_{\mathcal{F}}(0, T; L^2(G \times S^{d-1}))$:

$$\mathcal{Y} \triangleq \left\{( - z|_{\Gamma_S^c}, Z) \mid (z, Z) \text{ solves the equation (1.5) with some } z_T \in L^2(\Omega, F_T, P; L^2(G \times S^{d-1})) \right\}$$
and define a linear functional $\mathcal{L}$ on $\mathcal{Y}$ as follows:

$$\mathcal{L}(z|_{\Gamma^{-}_S}, Z) = \mathbb{E} \int_G \int_{S^{d-1}} y_1 z_T dS^{d-1} dx - \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} z f dS^{d-1} dx dt.$$ 

From Theorem 1.5, we see that $\mathcal{L}$ is a bounded linear functional on $\mathcal{Y}$. By means of the Hahn-Banach theorem, $\mathcal{L}$ can be extended to be a bounded linear functional on the space $L^2_\mathcal{F}(0, T; L^2_w(\Gamma^{-}_S)) \times L^2_\mathcal{F}(0, T; L^2(\mathcal{G} \times S^{d-1}))$. For simplicity, we still use $\mathcal{L}$ to denote this extension. Now, by the Riesz representation theorem, there is a pair of random fields $(u, v) \in L^2_\mathcal{F}(0, T; L^2_w(\Gamma^{-}_S)) \times L^2_\mathcal{F}(0, T; L^2(\mathcal{G} \times S^{d-1}))$

so that

$$\mathbb{E} \int_G \int_{S^{d-1}} y_1 z_T dS^{d-1} dx - \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} z f dS^{d-1} dx dt$$

$$= -\mathbb{E} \int_0^T \int_{\Gamma^{-}_S} U \cdot \nu z ud\Gamma^{-}_S dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} v Z dS^{d-1} dx dt.$$ 

(4.1)

We claim that this pair of random fields $(u, v)$ is the desired controls. Indeed, by Itô’s formula, we have

$$\mathbb{E} \int_G \int_{S^{d-1}} y(T, \cdot) z_T dS^{d-1} dx$$

$$= \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (-U \cdot \nabla y + a_1 yz + f z) dS^{d-1} dx dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (a_3 yZ + v Z) dS^{d-1} dx dt$$

$$+ \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \left( \int_{S^{d-1}} a_2 y dS^{d-1}(V) \right) z dS^{d-1}(U) dx dt$$

$$+ \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} (-U \cdot \nabla y z - a_1 yz - a_3 yZ) dS^{d-1} dx dt$$

$$- \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} \left( \int_{S^{d-1}} a_2 y dS^{d-1}(V) \right) z dS^{d-1}(U) dx dt.$$ 

(4.2)

Hence,

$$\mathbb{E} \int_G \int_{S^{d-1}} y(T, \cdot) z_T dS^{d-1} dx - \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} z f dS^{d-1} dx dt$$

$$= -\mathbb{E} \int_0^T \int_{\Gamma^{-}_S} U \cdot \nu z ud\Gamma^{-}_S dt + \mathbb{E} \int_0^T \int_G \int_{S^{d-1}} v Z dS^{d-1} dx dt.$$ 

(4.3)

From (4.1) and (4.3), we see that

$$\mathbb{E} \int_G \int_{S^{d-1}} y_1 z_T dS^{d-1} dx = \mathbb{E} \int_G \int_{S^{d-1}} y(T, \cdot) z_T dS^{d-1} dx.$$ 

(4.4)

Since $z_T$ can be an arbitrary element in $L^2(\Omega, \mathcal{F}_T, P; L^2(\mathcal{G} \times S^{d-1}))$, from the equality (4.4), we conclude that $y(T) = y_1$ in $L^2(\mathcal{G} \times S^{d-1}), P$-a.s. This completes the proof of Theorem 1.1.

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5 Proof of the lack of exact controllability

The purpose of this section is to give proofs of Theorems 1.2–1.4. In order to present the
key idea in the simplest way, we only consider a very special case of the system (1.1), that
is, \( G = (0, 1), a_1 = 0, a_2 = 0, a_3 = 1 \) and \( f = 0 \). The argument for the general case is very
similar.

Proof of Theorem 1.2: The case that \( v = 0 \) in \( L^2_F(0, T; L^2(0, 1)) \) is covered in Theorem
(1.3). Hence, we only prove Theorem 1.2 for \( u \equiv 0 \). In this case, the system (1.1) reads as

\[
\begin{cases}
  dy + y_x dt = (y + v) dB(t) & \text{in } (0, T) \times (0, 1), \\
  y(t, 0) = 0 & \text{on } (0, T) \times \{0\}, \\
  y(0) = y_0 & \text{in } (0, 1).
\end{cases}
\]  

(5.1)

Since the system (5.1) is linear, we only need to show that the attainable set \( A_T \) of this
system at time \( T \) for the initial datum \( y_0 = 0 \) is not \( L^2(\Omega, \mathcal{F}_T, P; L^2(0, 1)) \). For
\( y_0 = 0 \), the solution of this system is

\[
y(T) = \int_0^T S(T - s)[y(s) + v(s)] dB(s).
\]  

(5.2)

Here \( \{S(t)\}_{t \geq 0} \) is the semigroup introduced in Section 2. We refer to [3, Chapter 6] for the
details of establishing (5.2). From (5.2), we find that \( \mathbb{E}(y(T)) = 0 \). Thus, if we choose a
\( y_1 \in L^2(\Omega, \mathcal{F}_T, P; L^2(0, 1)) \) such that \( \mathbb{E}(y_1) \neq 0 \), then \( y_1 \) is not in \( A_T \), which completes the
proof.

To prove Theorems 1.3–1.4, we first recall the following known result.

Set

\[
\eta(t) = \begin{cases} 
  1, & \text{if } t \in [(1 - 2^{-2i})T, (1 - 2^{-2i-1})T], \quad i = 0, 1, \ldots, \\
  -1, & \text{otherwise in } [0, T]
\end{cases}
\]

and

\[
\xi = \int_0^T \eta(t) dB(t).
\]  

(5.3)

We have the following result.

Lemma 5.1 [22, Lemma 2.1] It is impossible to find

\[
(\varphi_1, \varphi_2) \in L^2_F(0, T; \mathbb{R}) \times C_F([0, T]; L^2(\Omega; \mathbb{R}))
\]

and \( x \in \mathbb{R} \) such that

\[
\xi = x + \int_0^T \varphi_1(t) dt + \int_0^T \varphi_2(t) dB(t).
\]  

(5.4)

Proof of Theorem 1.3: Put

\[
\mathcal{V} \triangleq \{v \in L^2_F(0, T; L^2(0, 1)) : v = 0 \text{ in } (0, T) \times G_0\}.
\]
Let $\xi$ be given in (5.3). Choose a $\psi \in C_0^\infty(G)$ such that $|\psi|_{L^2(G)} = 1$ and set $y_T = \xi \psi$. We will show that $y_T$ cannot be attained for any $y_0 \in \mathbb{R}$, $u \in L^2_T(0,T;\mathbb{R})$ and $\ell \in \mathcal{V}$. This goal is achieved by the contradiction argument. If there exist a $y_0 \in \mathbb{R}$, a $u \in L^2_T(0,T;\mathbb{R})$ and a $\ell \in \mathcal{V}$ such that the corresponding solution $y(\cdot)$ satisfies $y(T) = y_T$, then by the definition of the solution to (1.1), we obtain that

$$
\xi = \int_G y_T \psi dx = \int_G y_0 \psi dx + \int_0^T \left( \int_G \psi_x y dx \right) dt + \int_0^T \left( \int_G \psi y dx \right) dB(t). \quad (5.5)
$$

It is clear that both $\int_G \psi_x y dx$ and $\int_G \psi y dx$ belong to $C_F([0,T];L^2(\Omega;\mathbb{R}))$. This, together with (5.5), contradicts Lemma 5.1.

**Proof of Theorem 1.4:** The proof is similar to the one for Theorem 1.3.

Let $\xi$ be given by (5.3). Choose a $\psi \in C_0^\infty(G)$ such that $|\psi|_{L^2(G)} = 1$ and set $y_T = \xi \psi$. We will show that $y_T$ cannot be attained for any $y_0 \in \mathbb{R}$, $u \in L^2_T(0,T;\mathbb{R})$ and $\ell \in L^2_T(0,T;L^2(0,1))$. It is done by the contradiction argument too. If there exist a $u \in L^2_T(0,T;\mathbb{R})$ and an $\ell \in L^2_T(0,T;L^2(0,1))$ such that the corresponding solution $y(\cdot)$ satisfies $y(T) = y_T$, then, from the definition of the solution to (1.1), we obtain

$$
\xi = \int_G y_T \psi dx = \int_G y_0 \psi dx + \int_0^T \left( \int_G \psi_x y dx + \int_G \psi \ell dx \right) dt + \int_0^T \left( \int_G \psi y dx \right) dB(t). \quad (5.6)
$$

It is clear that $\int_G \psi_x y dx + \int_G \psi \ell dx \in L^2_T(0,T;\mathbb{R})$ and $\int_G \psi y \in C_F([0,T];L^2(\Omega;\mathbb{R}))$. These, together with (5.6), contradict Lemma 5.1.

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