FUNCTIONAL LÖwner ELLIPSOIDS

GRIGORY IVANOV AND IGOR TSIUTSIURUPA

Abstract. We extend the notion of the smallest volume ellipsoid containing a convex body in $\mathbb{R}^d$ to the setting of logarithmically concave functions. We consider a vast class of logarithmically concave functions whose superlevel sets are concentric ellipsoids. For a fixed function from this class, we consider the set of all its 'affine' positions. For any log-concave function $f$ on $\mathbb{R}^d$, we consider functions belonging to this set of 'affine' positions, and find the one with the smallest integral under the condition that it is pointwise greater than or equal to $f$. We study the properties of existence and uniqueness of the solution of this problem. For any $s \in [0, \infty)$, we consider the construction dual to the recently defined John $s$-function [IN20]. We prove that such a construction determines a unique function and call it the Löwner $s$-function of $f$. We study the Löwner $s$-functions as $s$ tends to zero and to infinity. Finally, extending the notion the outer volume ratio, we define the outer integral ratio of a log-concave function and give an asymptotically tight bound on it.

1. Introduction

We recall that the John ellipsoid of a convex body $K$ is the ellipsoid of maximal volume contained within $K$, and the Löwner ellipsoid of a convex body $K$ is the ellipsoid of minimal volume containing $K$. In [Joh14], Fritz John proves that each convex body in $\mathbb{R}^d$ contains a unique ellipsoid of maximal volume and characterizes all convex bodies $K$ such that the ellipsoid of maximal volume in $K$ is the Euclidean unit ball. The John and Löwner ellipsoids are cornerstones of the modern convex analysis. These objects appear in different areas of mathematics, computational mathematics and physics. Enormous problems has been solved using properties of these two deeply connected objects.

On the other hand, the idea of extending the results of convex analysis to the more general setting of log-concave functions has been investigated for the last few decades (we refer to [BGVV14], [AS17]). To the best of our knowledge, the authors of [AGMJV18] were the first who extended the notion of the John ellipsoid to the setting of integrable log-concave functions. The John function in the sense of [AGMJV18] is defined as follows.

The $L_\infty$ norm of a function $f$ is denoted by $\| f \|$. We will say that a function $f_1 : \mathbb{R}^d \to \mathbb{R}$ is below a function $f_2 : \mathbb{R}^d \to \mathbb{R}$, if $f_1$ is pointwise less than or equal to $f_2$, that is, $f_1(x) \leq f_2(x)$ for all $x \in \mathbb{R}^d$. We use $f_1 \leq f_2$ to denote the fact that $f_1 : \mathbb{R}^d \to \mathbb{R}$ is below $f_2 : \mathbb{R}^d \to \mathbb{R}$. We say that a log-concave function $f$ on $\mathbb{R}^d$ is a proper log-concave function, if $f$ is upper semi-continuous and has a finite positive integral. Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Consider the following class of functions

$$\mathcal{J}^d = \{ \alpha \chi_E : E \subset \mathbb{R}^d \text{ is an ellipsoid, } \alpha > 0 \},$$

and the following problem:

$$\max_{h \in \mathcal{J}^d} \int_{\mathbb{R}^d} h \quad \text{subject to } \quad h \leq f.$$ 

As shown in [AGMJV18], there is a unique solution of this problem, which was called by the authors of [AGMJV18] the John function of $f$.

In some sense the dual problem is studied in [LSW19], where, to the best of our knowledge, the first extension of the notion of Löwner ellipsoid to the setting of log-concave functions is

\begin{itemize}
  \item [1991 Mathematics Subject Classification.] Primary 52A23; Secondary 52A40, 46T12.
  \item [Key words and phrases.] Löwner ellipsoid, John ellipsoid, logarithmically concave function.
\end{itemize}
introduced. The authors of [LSW19] consider the following class of functions
\[ \mathcal{L}^d[\psi_0] = \{ \alpha e^{-|A(x-a)|} : A \in \text{GL}(d), \: \alpha > 0, \: a \in \mathbb{R}^d \}. \]

Since the log-conjugate function (see definition in Subsection 2.3) of the characteristic function of the unit ball is \( e^{-|x|} \), the classes \( \mathcal{J}^d \) and \( \mathcal{L}^d[\psi_0] \) consist of translates of the polar to each other functions. As shown in [LSW19], there is unique solution of problem
\[
\min_{\ell \in \mathcal{L}^d[\psi_0]} \int_{\mathbb{R}^d} \ell \text{ subject to } f \leq \ell,
\]
for a proper log-concave function \( f : \mathbb{R}^d \to [0, \infty) \), which we call the Löwner function in the sense of [LSW19].

Recently, a more general approach to the definition of John function has been considered in [IN20], where the John s-functions of an integrable log-concave function are constructed for all \( s \in (0, \infty) \). We define and discuss these functions in Subsection 6.1. It was shown [IN20, Theorem 7.1] that the John function in the sense of [AGMJV18] is the limit (in a rather strong sense) of the John s-function as \( s \to 0 \). Apart of this limit result, a characterization of the John s-function similar to the one given by John in his fundamental theorem is given in [IN20, Theorem 5.1].

Combining ideas of [LSW19] and [IN20], we consider the ‘dual’ problem and define the Löwner s-function below. In hindsight, we understood that it is easier to see the main ideas and geometric construction considering a more abstract approach than in [IN20].

1.1. The main results. For a function \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \), we consider the following class of functions
\[ \mathcal{L}^d[\psi] = \{ \alpha e^{-\psi(A(x-a))} : A \in \text{GL}(d), \: \alpha > 0, \: a \in \mathbb{R}^d \}. \]

For a given proper log-concave function \( f \), we consider the following optimization problem:
\[
(1.1) \quad \min_{\ell \in \mathcal{L}^d[\psi]} \int_{\mathbb{R}^d} \ell \text{ subject to } f \leq \ell.
\]

We say that \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) is an admissible function, if the function \( t \mapsto e^{-\psi(|t|)} \), \( t \in \mathbb{R} \), is a proper log-concave function. We describe the class of admissible convex functions in Subsection 2.5.

First, we study the uniqueness of the solution of (1.1).

**Theorem 1.1.** Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be an admissible function. Further, let \( \psi \) satisfy one of the following conditions:

1. \( \psi \) is strictly increasing and takes only finite values;
2. the effective domain of \( \psi \) is bounded.

Then, if there exists \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \) for a proper log-concave function \( f \), then there exists a unique solution of problem (1.1). Moreover, for the solution \( L_f[\psi] \), we have \( \|L_f[\psi]\| \leq e^d \|f\| \).

In Subsection 4.2, we show that there is no uniqueness of the solution in general if \( \psi \) does not satisfy both conditions in Theorem 1.1. We discuss the properties of uniqueness of the solution in Section 3. We note here that we use different geometric ideas when we consider admissible functions satisfying different conditions in Theorem 1.1.

The question of existence of \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \) is simple. As shown in Lemma 5.2, it suffices \( \psi \) to be of linear growth at infinity.

As it will be shown in Subsection 6.1, the following admissible functions
\[
(1.2) \quad \psi_s(t) = \begin{cases} 
{t}, & s = 0 \\
\frac{t}{2} \left[ \sqrt{1 + 4 \left( \frac{t}{s} \right)^2} - \ln \left( \frac{1 + \sqrt{1+4\left( \frac{t}{s} \right)^2}}{2} \right) - 1 \right], & s \in (0, \infty) \\
\frac{t^2}{2}, & s = \infty
\end{cases}
\]
define classes of functions polar to the classes considered in [IN20] for the construction of John $s$-functions.

As a simple consequence of Theorem 1.1, we obtain the following result.

**Theorem 1.2.** Fix $s \in [0, \infty)$ and let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Then, there exists a unique solution of problem

$$
\min_{\ell \in \mathcal{L}^d[\psi_s]} \int_{\mathbb{R}^d} \ell \quad \text{subject to} \quad f \leq \ell.
$$

We will refer to the solution of problem (1.3) for a fixed $s \in [0, \infty)$ as the *Löwner s-function* of $f$, and denote it by $(s)L_f$. Note that the Löwner function in the sense of [LSW19] is precisely $(0)L_f$.

As in the case of John $s$-functions, we get the following limit result as $s$ tends to zero.

**Theorem 1.3.** Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Then, the Löwner $s$-functions of $f$ converge uniformly to the Löwner 0-function of $f$ as $s \to +0$.

That is, the Löwner $s$-functions of $f$ can be considered as a reasonable extension of the Löwner function in the sense of [LSW19].

Probably, the most striking difference between the John $s$-functions and the Löwner $s$-functions appears in case $s = \infty$.

We prove the following statement.

**Theorem 1.4.** Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Then the following assertions are equivalent

1. $$
\liminf_{s \to \infty} \int_{\mathbb{R}^d} (s)L_f < \infty.
$$
2. There exists a Gaussian density $G$ such that $f \leq G$.
3. There exists a unique solution of problem

$$
\min_{\ell \in \mathcal{L}^d[\psi_{\infty}]} \int_{\mathbb{R}^d} \ell \quad \text{subject to} \quad f \leq \ell.
$$

Also, the Löwner $s$-functions of $f$ converge uniformly to the solution of problem (1.4) as $s \to \infty$.

Surprisingly enough, it was shown in [IN20] that the Gaussian density of maximal integral below a given proper log-concave function $f$ is not necessarily unique.

Let $K$ be a convex body in $\mathbb{R}^d$ and $L_K$ be its Löwner ellipsoid. The ratio $$
\left( \frac{\text{vol}_d L_K}{\text{vol}_d K} \right)^{1/d}
$$
is called the outer volume ratio of $K$. F. Barthe [Bar98, Proposition 11] showed that any simplex has the maximum outer volume ratio among all convex bodies in $\mathbb{R}^d$. That is, for any positive integer $d$ and any convex body $K \in \mathbb{R}^d$, the outer volume ratio of $K$ is at most $\Theta \sqrt{d}$ for some positive constant $\Theta$. We extend this result to the setting of log-concave functions.

**Theorem 1.5.** For any $s \in [0, \infty)$, there exists $\Theta_s$ such that for any positive integer $d$ and a proper log-concave function $f : \mathbb{R}^d \to [0, \infty)$ the following inequality holds

$$
\left( \frac{\int_{\mathbb{R}^d} (s)L_f}{\int_{\mathbb{R}^d} f} \right)^{1/d} \leq \Theta_s \sqrt{d}.
$$
2. Notation, Basic Terminology

2.1. Matrices. We will use \(<\) to denote the standard partial order on the cone of positive semi-definite matrices, that is, we will write \(A < B\) if \(B - A\) is positive definite. We recall the additive and the multiplicative form of Minkowski’s determinant inequality. Let \(A\) and \(B\) be positive definite matrices of order \(d\). Then, for any \(\lambda \in (0, 1)\),

\[
(\det (\lambda A + (1 - \lambda)B))^{1/d} \geq \lambda (\det A)^{1/d} + (1 - \lambda) (\det B)^{1/d},
\]

with equality if, and only if, \(A = cB\) for some \(c > 0\); and

\[
\det (\lambda A + (1 - \lambda)B) \geq (\det A)^{\lambda} \cdot (\det B)^{1 - \lambda},
\]

with equality if, and only if, \(A = B\).

2.2. Functions. Log-concave functions. A function \(\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}\) is called convex if \(\psi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\psi(x) + \lambda\psi(y)\) for every \(x, y \in \mathbb{R}^d\) and \(\lambda \in [0, 1]\). A function \(f\) on \(\mathbb{R}^d\) is logarithmically concave (or, log-concave for short) if \(f = e^{-\psi}\) for a convex function \(\psi\) on \(\mathbb{R}^d\).

Clearly, if \(f\) and \(g\) are log-concave functions, then

\[
f \leq g \iff -\log g \leq -\log f.
\]

For a function \(f : \mathbb{R}^d \to \mathbb{R}\) and a scalar \(\alpha \in \mathbb{R}\), we denote the superlevel set of \(f\) by

\[\{f \geq \alpha\} = \{x \in \mathbb{R}^d : f(x) \geq \alpha\}.
\]

The \(L_\infty\) norm of a function \(f\) is denoted as \(\|f\|\). Recall that the effective domain of a convex function \(\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty, \infty\}\) is the set \(\varphi = \{x \in \mathbb{R}^d : \varphi(x) < +\infty\}\).

2.3. Concept of duality. Recall the definition of the classical convex conjugate (or Legendre transform) transform \(\mathcal{L}\) defined for functions \(\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty, \infty\}\) by

\[\mathcal{L}\varphi(y) = \sup_{x \in \mathbb{R}^d} \{\langle x, y \rangle - \varphi(x)\}.
\]

The reasonable extension (justified in [AAM07]) of this notion to the setting of log-concave functions is the following. Let \(f = e^{-\psi} : \mathbb{R}^d \to [0, \infty]\), then its log-conjugate (or polar) function is defined by

\[f^\circ(y) = e^{-\mathcal{L}\psi(y)} = \inf_{x \in \mathbb{R}^d} \frac{e^{-\langle x, y \rangle}}{f(x)}.
\]

Clearly, the log-conjugate function of any function is log-concave. It is known [AAKM04] that the log-conjugate function of a proper log-concave function is a proper log-concave function. Also, if \(f\) and \(g\) are log-concave functions, then

\[
f \leq g \iff f^\circ \geq g^\circ.
\]

The following result is proven in [AAKM04, Lemma 3.2].

Lemma 2.1. Let \(f, \{f_i\}_1^\infty : \mathbb{R}^d \to [0, \infty)\) be log-concave functions such that \(f_n \to f\) on a dense set \(A \subset \mathbb{R}^d\). Then,

(1) \(\int_{\mathbb{R}^d} f_i \to \int_{\mathbb{R}^d} f\)

(2) \((f_n)^\circ \to f^\circ\) locally uniformly on the interior of the support of \(f^\circ\).

2.4. Ellipsoids. We denote the Euclidean unit ball in \(\mathbb{R}^n\) by \(B^n\), where \(n\) will mostly be \(d\) or \(d + 1\).

For a matrix \(A \in \mathbb{R}^{d \times d}\) and a number \(\alpha\), \(A \oplus \alpha\) denotes \((d + 1) \times (d + 1)\) matrix

\[A \oplus \alpha = \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}.
\]

We introduce the convex cone

\[\mathcal{E} = \{ (A \oplus \alpha, a) : A \in \mathbb{R}^{d \times d} \text{ is positive definite, } \alpha > 0 \}.
\]
There is a one-to-one correspondence between $E$ and the class of $(d+1)$-dimensional ellipsoids in $\mathbb{R}^{d+1}$ symmetric with respect to $\mathbb{R}^d$ (for example, $(A \oplus \alpha, a) \mapsto (A \oplus \alpha)B^{d+1} + a$). We will refer to the elements of $E$ as to $d$-ellipsoids.

2.5. Admissible convex functions. Clearly, $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ is an admissible function if, and only if, $\psi$ has the following properties:

- $\psi$ is convex with minimum at 0 (otherwise, $e^{-\psi(|t|)}$ is not log-concave);
- $\psi$ is lower semi-continuous (otherwise, $e^{-\psi(|t|)}$ is not upper semi-continuous);
- $\lim_{t \to +\infty} \psi(t) = +\infty$ (otherwise, the integral of $e^{-\psi(|t|)}$ equals $+\infty$);
- $\text{dom } \psi$ has positive measure (otherwise, the integral of $e^{-\psi(|t|)}$ equals zero).

Recall that a convex function is continuous (even locally Lipschitz) on the interior of the effective functions.

2.6. Classes of ellipsoidal functions. We say that the functions of $L^d[\psi]$ are $\psi$-ellipsoidal functions. If $\psi$ is an admissible function then all functions of $L^d[\psi]$ are proper log-concave functions.

For any $(A \oplus \alpha, a) \in E$, we say that the $\psi$-ellipsoidal function on $\mathbb{R}^d$ defined as $x \mapsto \alpha e^{-\psi(|A(x-a)|)}$ is represented by $(A \oplus \alpha, a)$. We use $\ell_{\psi, \overline{E}}$ to denote the $\psi$-ellipsoidal function represented by $\overline{E} = (A \oplus \alpha, a) \in E$. By definition, we have
\[
\ell_{\psi, \overline{E}}(x) = \alpha \cdot \ell_{\psi, B^{d+1}}(A(x - a)).
\]

This simple identity plays a crucial role in our proofs.

By the polar decomposition theorem, any $\psi$-ellipsoidal function is represented by a unique element of $E$. Thus,
\[
L^d[\psi] = \{ \alpha e^{-\psi(|A(x-a)|)} : (A \oplus \alpha, a) \in E \} = \{ \ell_{\psi, \overline{E}} : (A \oplus \alpha, a) \in E \}.
\]
The number $\alpha$ is called the height of the $\psi$-ellipsoidal function $\ell_{\psi, (A \oplus \alpha, a)}$.

We use $\lambda_d[\psi]$ to denote the integral of $\psi$-ellipsoidal function represented by the unit Euclidean ball $B^{d+1}$, that is,
\[
\lambda_d[\psi] = \int_{\mathbb{R}^d} \ell_{\psi, \overline{E}^{d+1}}.
\]
Clearly, $\lambda_d[\psi]$ is a positive number. Also, for an arbitrary $\overline{E} = (A \oplus \alpha, a) \in E$, we have that
\[
\int_{\mathbb{R}^d} \ell_{\psi, \overline{E}} = \frac{\alpha}{\det A} \cdot \lambda_d[\psi].
\]

2.7. Classes of $\psi_s$-ellipsoidal functions. To stress the fact that we restrict ourself to the admissible functions $\psi_s$ in the last four Sections. We will use the following notations related to $\psi_s$:

- $s$-ellipsoidal function instead of $\psi_s$-ellipsoidal function;
- $(s)\ell_{\overline{E}}$ instead of $\ell_{\psi_s, \overline{E}}$;
- $(s)\lambda_d$ instead of $\lambda_d[\psi]$.  

3. Classes of $\psi$-ellipsoidal functions

3.1. Boundedness. In this subsection, we show that for any admissible function $\psi$, we can restrict ourself to the bounded in $\mathcal{E}$ set of parameters in problem (1.1).

Define

$$S(\psi, f, \delta) = \left\{ E \in \mathcal{E} : f \leq \ell_{\psi,E} \text{ and } \int_{\mathbb{R}^d} \ell_{\psi,E} \leq \delta \right\}$$

The main result of this subsection is the following.

**Lemma 3.1.** Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function, and $\delta > 0$. Then there exist $\nu, \rho, \rho_1, \rho_2 > 0$ such that for any admissible function $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ and $(A \oplus \alpha, a) \in S(\psi, f, \delta)$, the following inequalities

$$\|f\| \leq \alpha e^{-\psi(0)} \leq \nu, \quad \text{and} \quad |a| \leq \rho$$

holds, and

$$\rho_1 I < \frac{A}{(\lambda_d[\psi])^{1/d}} < \rho_2 I.$$

**Proof.** Denote $\bar{\alpha} = \alpha e^{-\psi(0)}$. We assume that the origin is in the interior of support of $f$. Then, by continuity, there exist $\theta, \Theta > 0$ such that $\Theta \cdot \chi_{\partial B^d} \leq f$. Let $E = (A \oplus \alpha, a) \in S(\psi, f, \delta)$.

The maximum of $\ell_{\psi,E}$ is attained at $a$. Thus, by the log-concavity of a $\psi$-ellipsoidal function, we have that $\ell_{\psi,E}$ is attained at $\Theta$ on the set $\{\nu B^d, a\}$. Hence,

$$\delta \geq \int_{\mathbb{R}^d} \ell_{\psi,E} \geq \Theta \int_{\mathbb{R}^d} \chi_{co(\nu B^d, a)} = \Theta \det co\{\nu B^d, a\} \geq \frac{\Theta \rho^{d-1} \vol_{d-1} B^{d-1}}{d} |a|.$$

Thus, there exists $\rho > 0$ such that $a \in \rho B^d$.

Obviously, $\bar{\alpha} \geq \|f\|$. We proceed with an upper bound on $\bar{\alpha}$. Assume $\bar{\alpha} > 1$. Using again the log-concavity of $\ell_{\psi,E}$, we get that

$$\ell_{\psi,E}(0) \geq \Theta \frac{\rho^d}{d} \bar{\alpha}^{\frac{d}{d-1}} \geq \min \left\{ 1, \Theta \frac{\rho^d}{d} \right\} \bar{\alpha}^{\frac{d}{d-1}}.$$

Denote the right-hand side of this inequality by $C(\bar{\alpha})$. For any $x \in \nu B^d$, using the log-concavity of $\ell_{\psi,E}$, we have that

$$\ell_{\psi,E}(x) \geq C(\bar{\alpha}) \left( \frac{C(\bar{\alpha})}{\Theta} \right)^{-\frac{|x|}{d}}.$$

Therefore,

$$\delta \geq \int_{\mathbb{R}^d} \ell_{\psi,E} \geq \int_{\nu B^d} \ell_{\psi,E} \geq C(\bar{\alpha}) \int_{\nu B^d} \left( \frac{C(\bar{\alpha})}{\Theta} \right)^{-\frac{|x|}{d}} \, dx.$$

By routine computation, the right-hand side tends to $+\infty$ as $\bar{\alpha} \to \infty$. Thus, one sees that there exists $\nu$ such that $\bar{\alpha} \leq \nu$. This completes the proof of inequality (3.1).

Next, by the assumption of the lemma and by (2.6), we have

$$\int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} \ell_{\psi,E} = \frac{\bar{\alpha} \lambda_d[\psi]}{\det A} \leq \delta.$$

Let $\beta$ be the smallest eigenvalue of $A$. Since $\bar{\alpha} \in [\|f\|, \nu]$, the previous inequality implies that

$$\frac{\|f\|}{\delta} \cdot \lambda_d[\psi] \leq \det A \leq \beta \|A\|^{d-1} \leq \|A\|^d \quad \text{and} \quad \beta^d \leq \beta^{d-1} \|A\| \leq \det A \leq \frac{\nu}{\delta} \cdot \lambda_d[\psi].$$

By routine, we get inequality (3.2).

As an immediate consequence of Lemma 3.1, we have
Corollary 3.1. Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be an admissible function and let \( f : \mathbb{R}^d \to [0, \infty) \) be a proper log-concave function. If there exists \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \), then there exists a solution of problem (1.1).

Proof. If there is \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \), then, by Lemma 3.1, \( S(\psi, f, \int_{\mathbb{R}^d} \ell) \) is bounded in \( \mathcal{E} \). Hence, by (3.1) and (3.2), there is a minimizing sequence \( \{(A_i \oplus \alpha_i, a_i)\}_{i=1}^{\infty} \) of elements of \( S(\psi, f, \int_{\mathbb{R}^d} \ell) \) such that

\[
\lim_{i \to \infty} (A_i \oplus \alpha_i, a_i) = (A \oplus \alpha, a) \in \mathcal{E}
\]

and

\[
\lim_{i \to \infty} \int_{\mathbb{R}^d} \ell_{\psi,(A_i \oplus \alpha_i, a_i)} = \inf_{\ell \in \mathcal{L}^d[\psi]} \left\{ \int_{\mathbb{R}^d} \ell : f \leq \ell \right\}.
\]

Next, by continuity of a convex function \( \psi \) on the interior of its effective domain, (3.3) implies that \( \ell_{\psi,(A_i \oplus \alpha_i, a_i)} \to \ell_{\psi,(A \oplus \alpha, a)} \) on the interior of the support of \( \ell_{\psi,(A \oplus \alpha, a)} \). Thus, by assertion (1) of Lemma 2.1, we have that \( \ell_{\psi,(A \oplus \alpha, a)} \) is a solution of problem (1.1). \( \square \)

3.2. Interpolation between \( \psi \)-ellipsoidal functions.

Lemma 3.2 (Interpolated \( \psi \)-ellipsoidal function). Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be an admissible function. Let \( f_1 \) and \( f_2 \) be proper log-concave functions on \( \mathbb{R}^d \). Let \( \mathcal{E}_1 = (A_1 \oplus \alpha_1, a_1) \) and \( \mathcal{E}_2 = (A_2 \oplus \alpha_2, a_2) \) be \( d \)-ellipsoids in \( \mathcal{E} \) such that \( f_1 \leq \ell_{\psi,\mathcal{E}_1} \) and \( f_2 \leq \ell_{\psi,\mathcal{E}_2} \). Let \( \beta_1, \beta_2 > 0 \) be such that \( \beta_1 + \beta_2 = 1 \). Put

\[
A = \beta_1 A_1 + \beta_2 A_2, \quad \alpha = \alpha_1^\beta_1 \alpha_2^\beta_2, \quad a = A^{-1} (\beta_1 A_1 a_1 + \beta_2 A_2 a_2) \quad \text{and} \quad \mathcal{E} = (A \oplus \alpha, a).
\]

Then \( \ell_{\psi,\mathcal{E}} \) satisfies the following inequalities:

\[
f_1^\beta_1 f_2^\beta_2 \leq \ell_{\psi,\mathcal{E}}
\]

and

\[
\int \ell_{\psi,\mathcal{E}} \leq \left( \int \ell_{\psi,\mathcal{E}_1} \right)^{\beta_1} \left( \int \ell_{\psi,\mathcal{E}_2} \right)^{\beta_2},
\]

with equality in (3.7) if, and only if, \( A_1 = A_2 \).

Proof. Since \( f_1 \leq \ell_{\psi,\mathcal{E}_1} \) and \( f_2 \leq \ell_{\psi,\mathcal{E}_2} \), we have that

\[
f_1^\beta_1 f_2^\beta_2 \leq \left( \ell_{\psi,\mathcal{E}_1} \right)^{\beta_1} \left( \ell_{\psi,\mathcal{E}_2} \right)^{\beta_2}.
\]

Since \( \ell_{\psi,\mathcal{B}^{d+1}} \) is log-concave and since \( \beta_1 A_1(x - a_1) + \beta_2 A_2(x - a_2) = A(x - a) \), inequality

\[
\ell_{\psi,\mathcal{B}^{d+1}}(A(x - a)) \geq \left( \ell_{\psi,\mathcal{B}^{d+1}}(A_1(x - a_1)) \right)^{\beta_1} \left( \ell_{\psi,\mathcal{B}^{d+1}}(A_2(x - a_2)) \right)^{\beta_2}
\]

holds for all \( x \in \mathbb{R}^d \). Using here identities (2.5) and \( \alpha = \alpha_1^\beta_1 \alpha_2^\beta_2 \), we get \( \left( \ell_{\psi,\mathcal{E}_1} \right)^{\beta_1} \left( \ell_{\psi,\mathcal{E}_2} \right)^{\beta_2} \leq \ell_{\psi,\mathcal{E}} \).

This and inequality (3.8) imply inequality (3.6).

Using (2.6), we see that inequality (3.7) is equivalent to inequality

\[
\frac{\lambda_d[\psi]}{\det(\beta_1 A_1 + \beta_2 A_2)} \leq \frac{\lambda_d[\psi]}{(\det A_1)^{\beta_1} (\det A_2)^{\beta_2}},
\]

which is equivalent to

\( (\det A_1)^{\beta_1} (\det A_2)^{\beta_2} \leq \det(\beta_1 A_1 + \beta_2 A_2) \).

That is, inequality (3.7) and the equality condition in it follow from Minkowski's determinant inequality (2.2) and the equality condition therein. \( \square \)

Lemma 3.2 and Corollary 3.1 imply
Theorem 3.1. Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be an admissible function and \( f \) be a proper log-concave function. If there exists \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \), then there exists a solution of problem (1.1). Moreover, all solutions of this problem are translates of each other.

Proof. By Corollary 3.1, if the set \( \{ \ell \in \mathcal{L}^d[\psi] : f \leq \ell \} \) is nonempty then the minimum in (1.1) is attained on some \( \psi \)-ellipsoidal function. Let \( \overline{E}_1 = (A_1 \oplus \alpha_1, a_1) \) and \( \overline{E}_2 = (A_2 \oplus \alpha_2, a_2) \) be \( d \)-ellipsoids in \( \mathcal{E} \) such that \( \ell_{\psi, \overline{E}_1} \) and \( \ell_{\psi, \overline{E}_2} \) are the solutions of problem (1.1).

Set \( \beta_1 = \beta_2 = 1/2 \) and let \( \overline{E} \) be given by (3.5). By Lemma 3.2, \( \ell_{\psi, \overline{E}} \) is a \( \psi \)-ellipsoidal function such that \( f \leq \ell_{\psi, \overline{E}} \) and

\[
\int \ell_{\psi, \overline{E}_1} \leq \int \ell_{\psi, \overline{E}} \leq \sqrt{\int \ell_{\psi, \overline{E}_1} \cdot \int \ell_{\psi, \overline{E}_2}} = \int \ell_{\psi, \overline{E}}.
\]

Hence, by the equality condition in (3.7), we have \( A_1 = A_2 \). Since the integrals of \( \ell_{\psi, \overline{E}_1} \) and \( \ell_{\psi, \overline{E}_2} \) are equal, and by (2.6), we obtain \( \alpha_1 = \alpha_2 \). This completes the proof. \( \square \)

4. When the solution is unique?

Let \( \ell_1 \) and \( \ell_2 \) be two \( \psi \)-ellipsoidal function that are translates of each other. Summarizing the already proved results, to show the uniqueness of a solution of problem (1.1), we need to be able to `squeeze' a \( \psi \)-ellipsoidal function \( \ell \) between \( \ell_1 \) and \( \ell_2 \) in such a way that \( \int_{\mathbb{R}^d} \ell < \int_{\mathbb{R}^d} \ell_1 = \int_{\mathbb{R}^d} \ell_2 \) and \( \min\{\ell_1, \ell_2\} \leq \ell \). We are able to find such an interpolation for two classes (actually intersected) of admissible functions. They are the class of strictly increasing admissible functions and the class of admissible functions with bounded effective domain. We will prove the following two lemmas about interpolation between two translated \( \psi \)-ellipsoidal functions:

Lemma 4.1. Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be a strictly increasing admissible function. Let \( \alpha_1 \neq \alpha_2 \) and let \( \overline{E}_1 = (A \oplus \alpha, a_1) \) and \( \overline{E}_2 = (A \oplus \alpha, a_2) \) be elements of \( \mathcal{E} \). Then, there exists \( \overline{E} \in \mathcal{E} \) such that

\[
\int_{\mathbb{R}^d} \ell_{\psi, \overline{E}_1} < \int_{\mathbb{R}^d} \ell_{\psi, \overline{E}_2} = \int_{\mathbb{R}^d} \ell_{\psi, \overline{E}} \quad \text{and} \quad \min\{\ell_{\psi, \overline{E}_1}, \ell_{\psi, \overline{E}_2}\} \leq \ell_{\psi, \overline{E}}.
\]

Lemma 4.2. Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be an admissible function with bounded effective domain. Let \( \alpha_1 \neq \alpha_2 \) and let \( \overline{E}_1 = (A \oplus \alpha, a_1) \) and \( \overline{E}_2 = (A \oplus \alpha, a_2) \) be elements of \( \mathcal{E} \). Then, there exists \( \overline{E} \in \mathcal{E} \) such that

\[
\int_{\mathbb{R}^d} \ell_{\psi, \overline{E}_1} < \int_{\mathbb{R}^d} \ell_{\psi, \overline{E}_2} = \int_{\mathbb{R}^d} \ell_{\psi, \overline{E}} \quad \text{and} \quad \min\{\ell_{\psi, \overline{E}_1}, \ell_{\psi, \overline{E}_2}\} \leq \ell_{\psi, \overline{E}}.
\]

We use two simple, but different ideas of how to construct a \( \psi \)-ellipsoidal function \( \ell_{\psi, \overline{E}} \) in these lemmas.

Before we proceed with the proofs of Lemmas 4.1 and 4.2, let us show how they imply the uniqueness result.

Theorem 4.1. Let \( \psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) be an admissible function. Additionally, let \( \psi \) be a strictly increasing function or the effective domain of \( \psi \) be bounded. Let \( f : \mathbb{R}^d \to [0, \infty) \) be a proper log-concave function. If there exists \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \), then there exists a unique solution of problem (1.1).

Proof. Assume there exists \( \ell \in \mathcal{L}^d[\psi] \) such that \( f \leq \ell \). By Corollary 3.1, the solution of problem (1.1) exists. Assume there are at least two different solutions \( \ell_1 \) and \( \ell_2 \). By construction, we have that \( f \leq \min\{\ell_1, \ell_2\} \). This means that a \( \psi \)-ellipsoidal function \( \ell \) such that \( \min\{\ell_1, \ell_2\} \leq \ell \) also satisfies inequality \( f \leq \ell \). By Theorem 3.1, the \( \psi \)-ellipsoidal functions \( \ell_1 \) and \( \ell_2 \) are translates of each other. Hence, by Lemma 4.1 (in case \( \psi \) is strictly increasing) and by Lemma 4.2 (in case \( \psi \) has bounded effective domain), \( \ell_1 \) and \( \ell_2 \) are not solutions of problem (1.1). This contradicts the choice of \( \ell_1 \) and \( \ell_2 \). \( \square \)
4.1. Proofs of Lemma 4.1 and Lemma 4.2. In the sake of simplicity, we assume that $\psi(0) = 0$. Without loss of generality, we assume that $a_1$ and $a_2$ are the opposite vectors, that is, $a_2 = -a_1 \neq 0$.

Set $a = -Aa_1$. Consider the half-space $H^+ = \{x \in \mathbb{R}^d : \langle Ax, a \rangle \geq 0\}$. Clearly, we have

$$\ell_{\psi,E}(x) \leq \ell_{\psi,E}(x) \quad \forall x \in H^+. \quad (4.1)$$

Proof of Lemma 4.1. The idea of our construction is as follows. We consider $\psi$-ellipsoidal functions $\ell_{\psi,E}$, where $E$ has the form $(\rho_1A \oplus \rho_2\alpha, 0)$. Next, we find suitable $\rho_1, \rho_2 \in (0, 1)$. That is, we decrease the height and the determinant of operator.

First, we construct $\psi$-ellipsoidal functions that satisfy inequality $\min\{\ell_{\psi,E_1}, \ell_{\psi,E_2}\} \leq \ell_{\psi,E}$. We denote $\varepsilon_1 = \frac{1}{2d} \inf \{\psi(|Ax + a|) - \psi(|Ax|) : \langle Ax, a \rangle \geq 0 \text{ and } \psi(|Ax|) \leq 2d\}$. Then, $\varepsilon_1 > 0$ and for any $\varepsilon \in [0, \min\{\varepsilon_1, 1\}]$, the $\psi$-ellipsoidal function $\ell_{\psi,E}$ with

$$E = \left((1 - \varepsilon)A \oplus \alpha e^{-2d\varepsilon}, 0\right), \quad (4.3)$$

satisfies inequality $\min\{\ell_{\psi,E_1}, \ell_{\psi,E_2}\} \leq \ell_{\psi,E}$.

Proof. Let us show that $\varepsilon_1 > 0$. Since $\psi$ is a convex function, for any $\tau, t > 0$, we have that

$$\psi(\tau) = \psi(\tau) - \psi(0) \leq \psi(\tau + t) - \psi(t).$$

Hence, $\psi(|Ax + a|) - \psi(|Ax|) \geq \psi(|Ax + a| - |Ax|)$. Since $\psi$ is a convex function and since $\lim_{t \to \infty} \psi(t) = \infty$, the set $\{x \in \mathbb{R}^d : \psi(|Ax|) \leq 2d\}$ is bounded. Clearly, $|Ax + a| - |Ax|$ is bounded from below by some positive constant on the bounded set $H^+ \cap \{x : \psi(|Ax|) \leq 2d\}$. Since $\psi$ is strictly increasing, we conclude that $\varepsilon_1 > 0$.

Let us check inequality $\min\{\ell_{\psi,E_1}, \ell_{\psi,E_2}\} \leq \ell_{\psi,E}$. Using (2.3) and (4.1), we see that it suffices to prove

$$\psi\left((1 - \varepsilon)Ax\right) + 2d \cdot \varepsilon \leq \psi(|Ax + a|) \quad \text{for all } x \in H^+.$$ 

Since $\psi$ is a convex function, we have

$$\psi\left((1 - \varepsilon)Ax\right) \leq (1 - \varepsilon)\psi(|Ax|) + \varepsilon\psi(0) = (1 - \varepsilon)\psi(|Ax|)$$

for any $\varepsilon \in [0, 1)$. Therefore, it suffices to show that

$$(1 - \varepsilon)\psi(|Ax|) + 2d \cdot \varepsilon \leq \psi(|Ax + a|) \quad \text{for all } x \in H^+. \quad (4.4)$$

Since $x \in H^+$, we have $|Ax + a| > |Ax|$, if $\psi(|Ax|) = +\infty$ then $\psi(|Ax + a|) = +\infty$; therefore, inequality (4.4) trivially holds. Again, since $|Ax + a| > |Ax|$, the right-hand side in (4.4) is positive. Thus, inequality (4.4) trivially holds for any nonnegative $\varepsilon$ and any $x$ in $\{y \in \mathbb{R}^d : \psi(|Ay|) \geq 2d\}$.

Therefore, inequality (4.4) holds if, and only if, the following inequality holds

$$\varepsilon(2d - \psi(|Ax|)) \leq \psi(|Ax + a|) - \psi(|Ax|) \quad \text{for all } x \in H^+ : \psi(|Ax|) \leq 2d. \quad (4.5)$$

Set

$$\varepsilon_0 = \inf_{x \in H^+, \psi(|Ax|) \leq 2d} \left\{ \psi(|Ax + a|) - \psi(|Ax|) \right\} \cdot \frac{1}{2d - \psi(|Ax|)}. \quad \varepsilon_0 \in [0, 1].$$

We have that inequality (4.5) holds for any $\varepsilon \in [0, \varepsilon_0]$. Clearly, $\varepsilon_0 \geq \varepsilon_1$. By symmetry, $\varepsilon_1$ given by (4.2) satisfies the required property.}

Using (2.6) for $E$ given by (4.3), we obtain that

$$\int \ell_{\psi,E} = \frac{e^{-2d\varepsilon}}{(1 - \varepsilon)^d} \cdot \int \ell_{\psi,E_1}.$$ 

However, since $\frac{e^{-2d\varepsilon}}{(1 - \varepsilon)^d} < 1$ for any $\varepsilon \in (0, 1/2]$, we conclude that $\int \ell_{\psi,E} < \int \ell_{\psi,E_1}$ for a sufficiently small positive $\varepsilon$. This completes the proof of Lemma 4.1.

\[\square\]
**Proof of Lemma 4.2.** Here we use the same idea as in the setting of convex sets, that is, if the set \((B_d + a_1) \cap (B_d + a_2)\) with \(a_1 \neq a_2\) is nonempty, then it is a subset of \(\rho B_d + \frac{a_1 + a_2}{2}\) for some \(\rho \in (0, 1)\). We apply this observation for the superlevel sets of \(\psi\)-ellipsoidal functions and consider \(\psi\)-ellipsoidal functions \(\ell_{\psi, E}\), where \(E\) has the form \((A_1 \oplus \alpha, 0)\) with \(A < A_1\). The boundedness of the effective domain allows us to show that there exists a suitable \(A_1\).

Denote \(\delta = |a|\). Since \(\text{dom } \psi = [0, \tau]\), we have that supp \(\ell_{\psi, E} = \{x \in \mathbb{R}^d : |Ax + a| \leq \tau\}\). Hence, the interior of supports of \(\ell_{\psi, E_1}\) and \(\ell_{\psi, E_2}\) do not intersect if \(\delta \geq \tau\). Thus, any \(E\) of the form \((\rho A \oplus \alpha, 0)\) with any \(\rho > 1\) suits us in this case.

Assume that \(\delta < \tau\). We claim that the function \(\ell_{\psi, E}\) with
\[
E = (MA \oplus \alpha, 0), \quad \text{where } M = \text{diag} \left\{ \frac{\tau}{\tau - \delta}, 1, \ldots, 1 \right\},
\]
satisfies the condition of the lemma.

The bound on the integral of \(\ell_{\psi, E}\) trivially holds.

Let us check inequality \(\min\{\ell_{\psi, E_1}, \ell_{\psi, E_2}\} \leq \ell_{\psi, E}\). Therefore, using (2.3), we see that it suffices to prove
\[
(4.6) \quad \psi(|MAx|) \leq \psi(|Ax + a|) \quad \text{for all } x \in H^+ \cap \text{supp } \ell_{\psi, E}.
\]
Since an admissible function is increasing on its domain, (4.6) follows from the following
\[
(4.7) \quad |MAx| \leq |Ax + a| \quad \text{for all } x \in H^+ \cap \{v \in \mathbb{R}^d : |Av + a| \leq \tau\}.
\]
Let \(y = Ax\) and let \(y = \lambda a + z\), where \(z\) is orthogonal to \(a\). We have
\[
|M y|^2 = |M(\lambda a + z)|^2 = \left| \frac{\tau}{\tau - \delta} \lambda a + z \right|^2 = \left( \frac{\tau}{\tau - \delta} \right)^2 \lambda^2 \delta^2 + |z|^2
\]
and \(|y + a|^2 = (1 + \lambda)^2 \delta^2 + |z|^2\). Since \(\lambda \geq 0\) for \(x \in H^+\), inequality (4.7) is equivalent to
\[
\frac{\tau}{\tau - \delta} \lambda \leq 1 + \lambda,
\]
which trivially holds since \((1 + \lambda) \delta \leq |y + a| \leq \tau\). Thus, inequality (4.6) holds. We conclude that
\[\min\{\ell_{\psi, E_1}, \ell_{\psi, E_2}\} \leq \ell_{\psi, E},\]
completing the proof of Lemma 4.2. \(\square\)

### 4.2. Chimeras

Let \(\psi : [0, \infty) \to \mathbb{R}\) be an admissible function such that it is constant on some interval \([0, \tau]\), \(\tau > 0\), and it takes only finite values. We claim that, given such an admissible function and a log-concave function \(f\), a solution of problem (1.1) is not necessarily unique, which we show in the following example. For simplicity, we assume that \(\psi(0) = 0\).

**Example 4.1.** Let \(f : \mathbb{R}^d \to [0, \infty)\) be given by \(f = \min\{\ell_{\psi, (\text{Id} \oplus 1, -c)}, \ell_{\psi, (\text{Id} \oplus 1, c)}\}\), where \(|c| = \tau\). Then the functions of the form \(\ell_{\psi, (\text{Id} \oplus 1, pc)}\) with \(|\rho| \leq 1\) are solutions of problem (1.1).

**Proof.** By monotonicity of the Euclidean norm, we see that any function of the form \(\ell_{\psi, (\text{Id} \oplus 1, pc)}\) with \(|\rho| \leq 1\) satisfies inequality \(f \leq \ell_{\psi, (\text{Id} \oplus 1, pc)}\). Thus, we need to show that these functions are of minimal integral.

Let \((A \oplus \alpha, a) \in \mathcal{E}\) be such that \(f \leq \ell_{\psi, (A \oplus \alpha, a)}\). Since \(f(0) = 1\), we conclude that \(\alpha \geq 1\). By this and by (2.6), it suffices to show that \(\|A\| \leq 1\). Assume the contrary: There is an eigenvalue \(\xi\) of \(A\) such that \(\xi > 1\). Denote by \(u\) a unit eigenvector corresponding to \(\xi\) such that \(<u, c> \geq 0\). Then, for all \(\lambda > 0\), the following inequality holds:
\[
\alpha e^{-\psi(|\xi \lambda u - Aa|)} \geq e^{-\psi(|\lambda u + c|)}.
\]

Or, equivalently,
\[
(4.8) \quad \psi(|\lambda u + c|) - \psi(|\xi \lambda u - Aa|) \geq - \ln \alpha.
\]
By convexity of \(\psi\), we have that the left-hand side in (4.8) is at most \(-\psi(|\xi \lambda u - Aa| - |\lambda u + c|)\).

Since \(\xi > 1\), the argument of \(\psi\) tends to infinity as \(\lambda\) tends to infinity. Thus, the left-hand side in (4.8) is strictly less than \(-\ln \alpha\) for a sufficiently large \(\lambda\). We obtain a contradiction. \(\square\)
5. When the solution exists?

In this Section, we address the question of existence of a solution of problem (1.1). By Corollary 3.1, it suffices to find one $\psi$-ellipsoidal function $\ell$ such that $f \leq \ell$. This is a simple technical question.

**Example 5.1.** Let $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ be an admissible function which is not of linear growth. Then for any $d \in \mathbb{N}$, the set

$$\{ \ell \in \mathcal{L}^d[\psi] : e^{-|x|} \leq \ell \}$$

is empty.

We show below that if $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ is an admissible function of linear growth then for any proper log-concave function $f : \mathbb{R}^d \to [0, \infty)$, there is a $\psi$-ellipsoidal function $\ell$ such that $f \leq \ell$. Next, we bound the norm of the maximizers of (1.1) in Lemma 5.3, and complete the proof of Theorem 1.1.

5.1. Existence of a 'covering'.

**Lemma 5.1.** Let $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ be an admissible function and $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. If there exists $\overline{E} \in \mathcal{E}$ such that $f \leq \ell_{\psi, \overline{E}}$, then for an arbitrary $\alpha > \|f\|$, there exists $(A \oplus \alpha, a) \in \mathcal{E}$ such that $f \leq \ell_{\psi,(A \oplus \alpha,a)}$.

**Proof.** Without loss of generality, we assume that $\overline{E} = \mathcal{B}^{d+1}$ and that $\psi(0) = 0$. Case $\alpha > 1$ is trivial. Consider $\|f\| < \alpha < 1$. Set $S = \{ x \in \mathbb{R}^d : \ell_{\psi,(1/2, \text{Id} \oplus \alpha,0)} \leq \ell_{\psi, \mathcal{B}^{d+1}} \}$. Since $\psi$ is convex and admissible, we have that $S$ is bounded. Clearly, for a sufficiently small $\rho \in (0,1/2)$, we have that $\|f\| \leq \ell_{\psi, (\rho, \text{Id} \oplus \alpha,0)}(x)$ for all $x \in S$. By monotonicity, $\ell_{\psi, (0.5, \text{Id} \oplus \alpha,0)} \leq \ell_{\psi, (\rho, \text{Id} \oplus \alpha,0)}$. Thus, by construction, $f \leq \ell_{\psi,(\rho,\text{Id} \oplus \alpha,0)}$. \hfill $\Box$

**Lemma 5.2.** Let $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ be an admissible function of linear growth with $\psi(0) = 0$ and $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Then for any $(A \oplus \alpha, a) \in \mathcal{E}$ with $\alpha > \|f\|$, there exists $\gamma > 0$ such that $f \leq \ell_{\psi,(\gamma A \oplus \alpha,a)}$.

**Proof.** By monotonicity of $\psi$, it suffices to prove the lemma for $A = \text{Id}$. Without loss of generality, we assume that $a = 0$. It is known that (see [BGVV14, Lemma 2.2.1]) for any proper log-concave function $f$ on $\mathbb{R}^d$, there are $\Theta, \vartheta > 0$ such that

$$f(x) \leq \Theta e^{-\vartheta |x|}, \text{ for all } x \in \mathbb{R}^d.$$ 

On the other hand, since $\psi$ is convex and of linear growth, there are $\gamma_1 > 0$ and $C > 0$ such that

$$\Theta e^{-\vartheta |x|} \leq \alpha e^{-\psi(\gamma_1 |x|)} \text{ for all } x \text{ such that } |x| > C.$$ 

By monotonicity of $\psi$, this inequality holds for all $\gamma \in (0, \gamma_1)$.

However, by continuity, there exists $\gamma_2 > 0$ such that inequality

$$\|f\| \leq \alpha e^{-\psi(\gamma_2 |x|)} \text{ holds for all } x \text{ such that } |x| \leq C.$$ 

That is, $\gamma = \min\{\gamma_1, \gamma_2\}$ satisfies the required property. \hfill $\Box$

As an immediate consequence of Corollary 3.1 and Lemma 5.2, we get the following.

**Corollary 5.1.** Let $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ be an admissible function of linear growth. Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Then, there exists a solution of problem (1.1).
5.2. Bound on the height. The following result is an extension of the analogous result from [IN20] about the John $s$-ellipsoid with a similar proof. The idea of the proof can be traced back to [AGMJV18].

Lemma 5.3. Let $\psi : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ be an admissible function and $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function such that $f \leq \ell_{\psi,E}$ for a $d$-ellipsoid $E \in \mathcal{E}$. Further, let $L_f[\psi]$ be a solution of (1.1). Then

$$
\|L_f[\psi]\| \leq e^d \|f\|. \tag{5.1}
$$

Proof. Without loss of generality, we assume that $\psi(0) = 0$. There is nothing to prove if $\|f\| = \|L_f[\psi]\|$. Assume $\|f\| < \|L_f[\psi]\|$. We define a function $\Psi : (\|f\|, +\infty) \to [0, +\infty)$ as follows. Let $\alpha > \|f\|$. By Lemma 5.1, there exists $E = (A + \alpha, a) \in \mathcal{E}$ such that $f \leq \ell_{\psi,E}$. By Lemma 3.1 and a standard compactness argument, among $\psi$-ellipsoidal functions $\ell$ with height $\alpha$ and such that $f \leq \ell$, there is a function of minimal integral. We denote by $A$ the operator corresponding to any of such a function and set $\Psi(\alpha) = \det A$. The function $\Psi$ is well-defined. Indeed, by (2.6), we have that the determinants of operators that correspond to two different $\psi$-ellipsoidal functions with the same heights and the same integrals, are equal.

Claim 5.1. For any $\alpha_1, \alpha_2 \in (\|f\|, \infty)$ and $\lambda \in [0, 1]$, we have

$$
\Psi(\alpha_1^{1-\lambda} \alpha_2^{\lambda}) \geq \lambda \Psi(\alpha_1)^{1/d} + (1 - \lambda) \Psi(\alpha_2)^{1/d}. \tag{5.2}
$$

Proof. Let $(A_1 + \alpha_1, a_1), (A_2 + \alpha_2, a_2) \in \mathcal{E}$ be such that

$$
f \leq \ell_{\psi,(A_1+\alpha_1,a_1)} \quad \text{and} \quad f \leq \ell_{\psi,(A_2+\alpha_2,a_2)},
$$

and

$$
\Psi(\alpha_1) = \det A_1 \quad \text{and} \quad \Psi(\alpha_2) = \det A_2.
$$

By Lemma 3.2 and (2.6), we have that

$$
\Psi(\alpha_1^{1-\lambda} \alpha_2^{\lambda}) \geq \det (\lambda A_1 + (1 - \lambda) A_2).
$$

Now, (5.2) follows immediately from Minkowski’s determinant inequality (2.1). \hfill \Box

Set $\Phi(t) = \Psi(e^t)^{1/d}$ for all $t \in (\log \|f\|, +\infty)$. Inequality (5.2) implies that $\Phi$ is a concave function on its domain.

Let $\alpha_0$ be the height of $L_f[\psi]$. Then, by (2.6), for any $\alpha$ in the domain of $\Psi$, we have that

$$
\frac{\alpha_0}{\Psi(\alpha_0)} \leq \frac{\alpha}{\Psi(\alpha)}.
$$

Setting $t_0 = \log \alpha_0$ and taking root of order $d$, we obtain

$$
\Phi(t) \leq \Phi(t_0)e^{\frac{1}{d}(t-t_0)}
$$

for any $t$ in the domain of $\Phi$. The expression on the right-hand side is a convex function of $t$, while $\Phi$ is a concave function. Since these functions take the same value at $t = t_0$, we conclude that the graph of $\Phi$ lies below the tangent line to graph of $\Phi(t_0)e^{\frac{1}{d}(t-t_0)}$ at point $t_0$. That is,

$$
\Phi(t) \leq \Phi(t_0) \left(1 + \frac{1}{d}(t-t_0)\right).
$$

Passing to the limit as $t \to \log \|f\|$ and since the values of $\Phi$ are positive, we get

$$
0 \leq 1 + \frac{\log \|f\| - t_0}{d}.
$$

Or, equivalently, $t_0 \leq d + \log \|f\|$. Therefore, $\alpha_0 = \|L_f[\psi]\| \leq e^d \|f\|$. This completes the proof of Lemma 5.3. \hfill \Box

5.3. Proof of Theorem 1.1. Since $\mathcal{L}^d[\psi] = \mathcal{L}^d[\psi + c]$ for any constant $c$, Theorem 1.1 is an immediate consequence of Theorem 4.1 and Lemma 5.3.
6. Classes of \( s \)-ellipsoidal functions

In this Section, we discuss \( s \)-ellipsoidal functions and their properties. We recall the definition of the John \( s \)-function and explain basic properties of duality between two optimization problems. Next, we prove Theorem 1.2 and 1.3.

6.1. Basic properties and the John \( s \)-functions. Recall that for any \( E = (A \oplus \alpha, a) \in \mathcal{E} \) and \( s \in (0, \infty) \), the \( s \)-ellipsoidal function \((s)\ell_E \) is given by

\[
(s)\ell_E(x) = \alpha e^{-\psi_s(|A(x-a)|)} = \alpha \left[ 1 + \frac{1}{2} \frac{|A(x-a)|^2}{s^2} \right]^{s/2} \cdot \exp \left( 1 - \sqrt{1 + \frac{4}{s^2} |A(x-a)|^2} \right). \tag{6.1}
\]

However, we have not shown that \( \psi_s \) is an admissible function. Avoiding some simple boring computations, we show the basic properties of \( s \)-ellipsoidal functions using their log-conjugate functions. At the same time, we reveal some duality between our definition of Löwner \( s \)-function and the definition of John \( s \)-function introduced in [IN20].

For any \( d \)-ellipsoid \( E = (A \oplus \alpha, a) \in \mathcal{E} \) and \( s > 0 \), define

\[
(s)h_E(x) = \begin{cases} \frac{1}{\alpha} \left[ 1 - |A^{-1}(x-a)|^2 \right]^{s/2}, & \text{if } x \in AB^d + a, \\ 0, & \text{otherwise}. \end{cases} \tag{6.2}
\]

The geometric sense of this functions is as follows. The graph of \((1)H_{(A\oplus\alpha,a)}\) restricted to its support coincides with the upper hemisphere of the \( d \)-ellipsoid \((A \oplus \alpha^{-1/s})B^{d+1} + a\). It follows that \((s)h_E \leq f\) if, and only if, the \( d \)-ellipsoid \((A \oplus \frac{1}{s})B^{d+1} + a\) is contained within the subgraph of \(f^{1/s}\). Another consequence of observation that \(1)H_{E}\) is a 'height' function of the \( d \)-ellipsoid is that \((1)h_E\) is concave on its support. Hence, \((s)h_E\) is log-concave. Clearly, \((s)h_E\) is a proper log-concave function.

For a \( d \)-ellipsoid \( E = (A \oplus \alpha, a) \in \mathcal{E} \), we put

\[
(0)h_E = \frac{1}{\alpha} \chi_{AB^d + a} \quad \text{and} \quad (s)h_E = \frac{1}{\alpha} e^{-|A^{-1}(x-a)|^2}. \tag{6.3}
\]

The authors of [IN20] consider the problem

\[
\max_{E \in \mathcal{E}} \int (s)h_E \quad \text{subject to} \quad (s)h_E \leq f;
\]

and show that for any fixed \( s \in [0, \infty) \), the solution of this problem exists and is unique ([IN20, Theorems 4.1 and 7.1]) for any proper log-concave function \( f : \mathbb{R}^d \to [0, \infty) \). The case \( s = 0 \) was studied earlier in [AGMJV18]. For a proper log-concave function \( f : \mathbb{R}^d \to [0, \infty) \), we call the unique solution of problem (6.2) a John \( s \)-function of \( f \), and denote it by \((s)f_E\).

We prove in the next lemma that for any \( s \in [0, \infty] \), the affine classes \( \{(s)h_E : E \in \mathcal{E}\} \) and \( L^d[\psi] = \{(s)\ell_E : E \in \mathcal{E}\} \) are the affine classes of two polar to each other functions. That is, problems (6.2) and (1.3) are in some sense dual. This gives us a reason for considering classes of \( s \)-ellipsoidal functions.

Lemma 6.1. Let \( E = (A \oplus \alpha, 0) \in \mathcal{E} \) and \( s \in [0, \infty] \). Then

\[
(s)h_E^\circ = (s)\ell_E^\circ \quad \text{and} \quad (s)\ell_E^\circ = (s)H_E. \tag{6.3}
\]

Before we prove this lemma, let us mention several corollaries. First, it follows that for any \( s \in [0, \infty] \), \( s \)-ellipsoidal functions are proper log-concave functions and \( \psi_s \) are indeed strictly increasing admissible functions. Moreover, as the support of \((s)h_{B^{d+1}}\) is bounded for any \( s \in [0, \infty) \),
we conclude that $\psi_s$ is of linear growth for any $s \in [0, \infty)$. Also, we will need the following simple observations:

For any $E \in \mathcal{E}$, we have

$$\int_{\mathbb{R}^d} (s) h_{E^\circ} \cdot \int_{\mathbb{R}^d} (s) f_{E^\circ} = \int_{\mathbb{R}^d} (s) h_{B^{d+1}} \cdot \int_{\mathbb{R}^d} (s) f_{B^{d+1}}.$$  

Clearly, $(s) h_{B^{d+1}}(x)$ is a strictly decreasing function of $s$ on $[0, +\infty)$ for all $x \in B^d \setminus \{0\}$. By this and (2.4), we have

$$(6.5) \quad (s) f_{B^{d+1}}(x) \text{ is a strictly increasing function of } s \text{ on } [0, +\infty) \text{ for all } x \in \mathbb{R}^d \setminus \{0\}.$$  

**Proof of Lemma 6.1.** Identities (6.3) are trivial in two cases $s = 0$ and $s = \infty$. Assume $s \in (0, \infty)$. By definition, for any log-concave function $f$ and invertible operator $A$ on $\mathbb{R}^d$, we have $(f \circ A)^\circ = f^\circ \circ (A^{-1})^s$. Thus, it suffices to consider $E = B^{d+1}$.

Consider $f = \left( (s) h_{B^{d+1}} \right)^{2/s} = 1 - x^2$, simple computations yield

$$f_\circ(y) = \inf_{|x| < 1} \frac{e^{-(x,y)}}{1 - |x|^2} = \inf_{t \in [0,1)} \frac{e^{-t|y|}}{1 - t^2} = \frac{1 + \sqrt{1 + |y|^2}}{2} \exp \left( 1 - \sqrt{1 + |y|^2} \right).$$  

Again, by the definition of the log-conjugate function, we have $(f^\circ)^\circ(qx) = (f^\circ(x))^q$ for all $x \in \mathbb{R}^d$ and $q > 0$. Thus, we prove the leftmost identity in (6.3), which implies the rightmost identity. \hfill \square

6.2. **Löwner $s$-functions.** We already proved all required results to understand the properties of existence and uniqueness of a solution of problem (1.3).

**Proof of Theorem 1.2.** The result follows from Corollary 5.1, Theorem 4.1 and the fact that $\psi_s$ is a strictly increasing admissible function of linear growth for a fixed $s \in [0, \infty)$. \hfill \square

The following lemma says that for fixed finite $s_1$ and $s_2$, the integrals of the Löwner functions $(s_1) L_f$ and $(s_2) L_f$ are bounded by each other.

Note that by (6.5), we have that $(s) \lambda_d$ is an increasing function of $s \in [0, \infty)$. By simple computation, we have $(0) \lambda_d = d! \operatorname{vol}_d B^d$. Actually, $(s) \lambda_d$ might be computed using hypergeometric functions. We claim without proof that

$$(s) \lambda_d = \pi^{d/2} s^d \left( d \cdot U \left( \frac{d}{2} + 1; d + s/2 + 1; s \right) + U \left( \frac{d}{2}, d + s/2 + 1; s \right) \right)$$

where $U(a; b; z)$ is the **hypergeometric Tricomi function** defined by

$$U(a; b; z) = \frac{1}{\Gamma(a)} \int_0^{+\infty} v^{a-1} (v + 1)^{b-a-1} e^{-zv} \, dv.$$  

**Lemma 6.2.** Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function and $0 < s_1 < s_2$. Then

$$\frac{(s_1) \lambda_d}{(s_2) \lambda_d} \leq \frac{\int_{\mathbb{R}^d} (s_1) L_f}{\int_{\mathbb{R}^d} (s_2) L_f} \leq \sqrt{\left( \frac{d + s_2}{s_2} \right)^{s_2} \left( \frac{d + s_2}{d} \right)^d} \cdot \frac{(s_2) \lambda_d}{(s_1) \lambda_d}.$$  

**Proof.** We prove the leftmost inequality in (6.6) first. Without loss of generality, assume that $(s_1) f_{B^{d+1}}$ is the Löwner $s_1$-function of $f$. By (6.5), we have that $f \leq (s_2) f_{B^{d+1}}$. Hence, $\int_{\mathbb{R}^d} (s_2) L_f \leq (s_2) \lambda_d$. The leftmost inequality in (6.6) follows.

Now, we prove the rightmost inequality in (6.6). Assume that $(s_2) f_{B^{d+1}}$ is the Löwner $s_2$-function of $f$. For a fixed $\rho \in (0, 1)$, consider

$$E_{\rho} = \left( \rho \cdot \operatorname{Id} \oplus \left( \sqrt{1 - \rho^2} \right)^{-s_2/s_1}, 0 \right).$$
We claim that $\ell_{\mathcal{B}^{d+1}} \leq \ell_{\mathcal{E}}$ for all $\rho \in (0, 1)$. By (2.4) and by (6.3), it is equivalent to the inequality $(s_1) h_{\mathcal{B}^{d+1}} \leq (s_2) h_{\mathcal{B}^{d+1}}$ holds. Since for any $\rho \in (0, 1)$, the cylinder $\rho \mathcal{B}^d \times [0, \sqrt{1 - \rho^2}]$ contained in $\mathcal{B}^{d+1}$, we conclude that

$$
(s_1) h_{\mathcal{E}} \leq \left(1 - \rho^2\right)^{s_2} \cdot \chi_{\rho \mathcal{B}^d} \leq \left(1 - \rho^2\right)^{s_2} \cdot \chi_{\mathcal{B}^{d+1}} = (s_2) h_{\mathcal{B}^{d+1}}.
$$

Thus, $(s_1) \ell_{\mathcal{B}^{d+1}} \leq (s_2) \ell_{\mathcal{E}}$ for all $\rho \in (0, 1)$. Choosing $\rho = \sqrt{\frac{d}{d+s_2}}$, we obtain

$$
\frac{\int_{\mathbb{R}^d}^d}{} \leq \int_{\mathbb{R}^d}^d \leq \int_{\mathbb{R}^d}^d = \sqrt{\left(1 + s_2\right)^{s_2}} \cdot \left(1 + s_2\right)^{d} \cdot \left(s_1\right)^{\lambda_d} / \left(s_2\right)^{\lambda_d}.
$$

6.3. The limit as $s \to 0$. In this subsection, we prove Theorem 1.2.

**Proof of Theorem 1.3.** Let the Löwner $s$-function $(s) L_f$ be represented by $(A_s + \alpha_s, a_s) \in \mathcal{E}$ for every $s \in [0, 1]$. Clearly, it suffices to show that

$$
\lim_{s \to 0^+} (A_s + \alpha_s, a_s) = (A_0 + \alpha_0, a_0).
$$

Assume the contrary. By Lemma 6.2, we see that the integrals of $(s) L_f$ for $s \in [0, 1]$ are bounded from above by some finite constant. By monotonicity, $(s) \lambda_d \in \left[0 \lambda_d, 1 \lambda_d\right]$, $s \in [0, 1]$. Thus, applying Lemma 3.1 for $\psi = \psi_s$, $s \in [0, 1]$, we get that the set $\{(A_s + \alpha_s, a_s) : s \in [0, 1]\}$ is bounded in $\mathcal{E}$. Now, we see that there exists a sequence of positive numbers $\{s_i\}_{i=1}^\infty$ with $\lim s_i = 0$ such that

$$
\lim_{i \to \infty} (A_{s_i} + \alpha_{s_i}, a_{s_i}) = (A + \alpha, a) \in \mathcal{E} \quad \text{and} \quad (A + \alpha, a) \neq (A_0 + \alpha_0, a_0).
$$

Clearly, we have that

$$
\partial \leq (0) \ell_{(A\oplus \alpha, a)} \quad \text{and} \quad \int_{\mathbb{R}^d}^d = \lim_{i \to \infty} \int_{\mathbb{R}^d} (s) \ell_f.
$$

By (6.5), we have $f \leq (s) \ell_{(A_0\oplus \alpha_0, a_0)}$ for any positive $s$. This and (6.8) imply that

$$
\int_{\mathbb{R}^d}^d = \lim_{i \to \infty} \int_{\mathbb{R}^d} (s) \ell_{(A_0\oplus \alpha_0, a_0)} = \int_{\mathbb{R}^d}^d
$$

This contradicts the choice of $(0) L_f$.

7. Gaussian densities. The limit as $s \to \infty$

In this Section, we prove Theorem 1.4. That is, we show that Gaussian densities, or in our notation $\psi_\infty$-ellipsoidal functions, are the natural extension of the classes of s-ellipsoidal functions.

7.1. Basic identities. For any positive-definite operator $A$, we have

$$
\left(e^{-|A^{-1}x|^2}\right)^0 = e^{-\frac{d}{2}|Ax|^2}.
$$

By direct computations, we find that

$$
\lim_{s \to \infty} \psi_s(\sqrt{s} \cdot t) = \frac{t^2}{2} \quad \text{for any } t \in [0, +\infty).
$$

This implies that the functions $(s) \ell_{(c(s)\mathbb{B}, 1, 0)}$, where $c(s) = \sqrt{s}$, converge to $e^{-|x|^2/2}$ as $s \to \infty$. By this and using assertion (1) of Lemma 2.1, we have

$$
\lim_{s \to \infty} (s) \lambda_d \cdot s^{-d/2} = (2\pi)^{d/2}.
$$

As an immediate consequence of Lemma 3.1 and identity (7.3), we obtain
Corollary 7.1. Let \( f : \mathbb{R}^d \to [0, \infty) \) be a proper log-concave function such that there is a sequence \( \{s_i\}_1^\infty \) of positive numbers with \( \lim_{i \to \infty} s_i = \infty \) and

\[
\lim_{i \to \infty} \int_{\mathbb{R}^d} (s_i) L_f = \lambda < \infty.
\]

Further, let \((A_s \oplus \alpha_s, a_s)\) represent the Löwner s-function of \( f, s \in [0, \infty) \). Then, inequality

\[
\rho_1 \cdot \text{Id} < \frac{A_{s_i}}{\sqrt{s_i}} < \rho_2 \cdot \text{Id}.
\]

holds for some positive \( \rho_1, \rho_2 \).

Using (6.3) in assertion (2) of Lemma 2.1, we conclude that \((s)i h(\sqrt{\pi} \text{Id} \oplus 1, 0)(x) \to e^{-|x|^2/2} \). Again, by assertion (1) of Lemma 2.1, we get

\[
\lim_{s \to \infty} s^{d/2} \int_{\mathbb{R}^d} (s) h_{B^{d+1}} = (2\pi)^{d/2}.
\]

Thus, we conclude that

(7.4)

\[
\lim_{s \to \infty} (s) \lambda \cdot \int_{\mathbb{R}^d} (s) h_{B^{d+1}} = (2\pi)^d.
\]

For any \((A \oplus \alpha, a) \in E\), identity (2.6) gives

(7.5)

\[
\int_{\mathbb{R}^d} (s) \ell(A \oplus \alpha, a) = \frac{\alpha^d/2}{\det A}.
\]

7.2. Limits of the centered sequences.

Lemma 7.1. Let \( \{s_i\}_1^\infty \) be a sequence of positive scalars such that \( \lim_{i \to \infty} s_i = \infty \), let \( \{A_i\}_1^\infty \) be a sequence of positive definite operators such that \( \lim_{i \to \infty} \frac{A_i}{\|A_i\|} = A \), where \( A \) is positive definite, and the ellipsoids \( E_i \), represented by \((A_i \oplus 1, 0)\), satisfy \( \lim_{i \to \infty} \int_{\mathbb{R}^d} (s_i) \ell(E_i) = \lambda < \infty \). Then, the functions \( \{s_i\} \ell(E_i) \) converge uniformly to the Gaussian density \( G[[A_L \oplus 1, 0]] \), where

(7.6)

\[
A_L = \frac{\sqrt{\pi}}{(\lambda \cdot \det A)^{1/d}} A.
\]

Proof. Consider functions \( \{s_i h_{(A_i \oplus 1, 0)}\}_1^\infty \). By (7.4), we have that

\[
\lim_{i \to \infty} \int_{\mathbb{R}^d} (s_i) \ell_{E_i} = \frac{(2\pi)^d}{\lambda}.
\]

As shown in [IN20, Claim 8.1], functions \( s_i h_{E_i} \) converge uniformly to the Gaussian density \( e^{-\langle A_j^{-1} x, A_j^{-1} x \rangle} \) with

\[
A_j = \frac{2\sqrt{\pi}}{(\lambda \cdot \det A)^{1/d}} A.
\]

Thus, by (7.1) and by assertion (2) of Lemma 2.1, we get that the functions \( (s_i) \ell_{E_i} \) converge locally uniformly to the Gaussian density \( G[[A_L \oplus 1, 0]] \) with \( A_L \) given by (7.6). Since

\[
\lim_{i \to \infty} \int_{\mathbb{R}^d} (s_i) \ell_{E_i} = \int_{\mathbb{R}^d} G[[A_L \oplus 1, 0]],
\]

we conclude that the functions \( \{s_i\} \ell_{E_i} \) are uniformly convergent.

Lemma 7.2. Let \( G \) be a Gaussian density. Then, the functions \( (s) L_G \) converge uniformly to \( G \) as \( s \to \infty \).
Proof. We assume that $G(x) = e^{-|x|^2/2}$. First, we relax the condition and prove that it suffices to approximate $G$ by any suitable $s$-ellipsoidal functions.

Claim 7.1. If there is a function $c : [1, \infty) \to [1, \infty)$ such that

$$\lim_{s \to \infty} \int_{\mathbb{R}^d} \ell(c(s) I_{\|s\|}) = \int_{\mathbb{R}^d} G = (2\pi)^{d/2},$$

and $G \leq \ell(c(s) I_{\|s\|})$ for all $s \geq 1$, then the functions $\ell G$ converge uniformly to $G$ as $s \to \infty$.

Proof. By Theorem 1.2, the Löwner function $(^s) G$ exists and is unique for any positive $s$. By symmetry, we see that $(^s) G$ is of the form $(^s) \ell(\beta(s) I_{\|s\|}, 0)$, where $\beta : [1, \infty) \to (0, \infty)$ and $\alpha : [1, \infty) \to (0, 1)$. By (7.7), we obtain that

$$\lim_{s \to \infty} \int_{\mathbb{R}^d} G = \int_{\mathbb{R}^d} G.$$ 

This implies that $\alpha(s) \to 1$ as $s \to \infty$. Hence, the functions $(^s) G = \ell(\beta(s) I_{\|s\|}, 0)$ converge uniformly to the same function as the functions $(^s) \ell(\beta(s) I_{\|s\|}, 0)$ as $s$ tends to $\infty$ (if the latter converges). However, by Lemma 7.1, the functions $(^s) \ell(\beta(s) I_{\|s\|}, 0)$ converge uniformly to $G$ as $s \to \infty$.

It is not hard to find a suitable function $c(s)$.

Claim 7.2. Let $c(s) = \sqrt{s}$. Then, $G \leq \ell(c(s) I_{\|s\|})$ for all $s \geq 1$, and identity (7.7) holds.

Proof. Identity (7.7) is an immediate consequence of (2.6) and (7.3).

Inequality $G \leq \ell(c(s) I_{\|s\|})$ is purely technical. By (7.2), for any $x \in \mathbb{R}^d$, we have

$$\lim_{s \to \infty} \ell(c(s) I_{\|s\|}) = G(x).$$

We claim that $(c(s) I_{\|s\|})$ is a decreasing function of $s \in [1, \infty)$ for any fixed $x \in \mathbb{R}^d$. Or, equivalently, $\psi_s(\sqrt{s} \cdot t)$ is an increasing function of $s \in [1, \infty)$ for any fixed $t \in (0, \infty)$. The derivative of $\psi_s(\sqrt{s} \cdot t)$ as a function of $s$ is

$$\frac{1}{2\sqrt{1 + t^2/s + 1}} - \ln \left( \sqrt{1 + \frac{t^2}{s}} + 1 \right) + \ln 2.$$

It is a function of $t^2/s$, and we use $\Phi(t^2/s)$ to denote this function. Making the substitution $z = t^2/s$ and computing the derivative of $\Phi(z)$, we get that

$$\Phi'(z) = \frac{z}{4\sqrt{z + 1} (1 + \sqrt{z + 1})^2} > 0.$$

By this and since $\Phi(z) = 0$, we conclude that the derivative of $\psi_s(\sqrt{s} t)$ as a function of $s$ is positive for any $s \in [1, \infty)$. This completes the proof.

Lemma 7.2 follows from Claims 7.1 and 7.2.

7.3. Proof of Theorem 1.4. Implication (3) $\Rightarrow$ (2) is trivial. Implication (2) $\Rightarrow$ (1) is a direct consequence of Lemma 7.2.

We proceed with implication (1) $\Rightarrow$ (3). Let the Löwner $s$-function $(^s) L_f$ be represented by the ellipsoid $(A_s \oplus \alpha_s, a_s) \in E$ for all $s \in [1, \infty)$. Applying Lemma 3.1 for $\psi = \psi_s$, one sees that there exists a sequence $\{s_i\}$ with $\lim_{i \to \infty} s_i = \infty$ such that

$$\lim_{i \to \infty} \int_{\mathbb{R}^d} (s_i) L_f \to \lim_{s \to \infty} \inf_{s \in \mathbb{R}^d} (s) L_f, \frac{A_s}{\|A_s\|} \to A, \alpha \to \alpha \text{ and } a_{s_i} \to a$$

for some positive semidefinite matrix $A \in \mathbb{R}^{d \times d}$, a number $\alpha > 0$ and $a \in \mathbb{R}^d$, as $i$ tends to $\infty$. By Corollary 7.1, we conclude that $A$ is positive-definite. Thus, by Lemma 7.1, we have
that the functions \( (s) L_f \) converge uniformly to some Gaussian density \( G \). Clearly, \( f \leq G \) and \( \int_{\mathbb{R}^d} G = \lim_{s \to \infty} \int_{\mathbb{R}^d} (s) L_f \). Hence, by Theorem 1.1, there exists a unique solution of problem (1.4). We use \( G_L \) to denote this solution. By Lemma 7.2, we have

\[
\int_{\mathbb{R}^d} G_L = \lim_{s \to \infty} \int_{\mathbb{R}^d} (s) L_{G_L} \geq \lim_{s \to \infty} \int_{\mathbb{R}^d} (s) L_f = \int_{\mathbb{R}^d} G.
\]

Thus, by the choice of \( G_L \), we conclude that \( G_L = G \). Again, by Lemma 7.2, we have that

\[
\lim_{s \to \infty} \int_{\mathbb{R}^d} (s) L_f = \lim_{s \to \infty} \int_{\mathbb{R}^d} (s) L_f = \int_{\mathbb{R}^d} G_L.
\]

Hence, \( A_s \to A \), \( \alpha_s \to \alpha \) and \( a_s \to a \) as \( s \to \infty \). Indeed, otherwise using Lemma 7.1, we see that there is another Gaussian density \( G_2 \) such that \( f \leq G_2 \) and \( \int_{\mathbb{R}^d} G_2 = \int_{\mathbb{R}^d} G_L \), which contradicts the choice of \( G_L \). Thus, we conclude that the functions \( (s) L_f \) converge uniformly as \( s \to \infty \) to \( G_L \) and complete the proof.

**Proposition 7.1.** Let \( K \subset \mathbb{R}^d \) be a compact convex set with non-empty interior, and let \( \|f\|_K \) denote the gauge function of \( K \), that is, \( \|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\} \). Let \( A^{-1}(B^d) \) be the smallest volume origin centered ellipsoid containing \( K \), where \( \lambda \) is a positive definite matrix. Then the \( \alpha \)-ellipsoidal function \( \langle A \rangle \) represented by \( (A \oplus 1, 0) \) is the unique solution of (14) with \( f = e^{-\|x\|_K^2} \).

**Proof.** Let \( (A' \oplus \alpha', a') \in \mathcal{E} \) be such that \( f \leq G[(A' \oplus \alpha', a')] \). First, we show that \( K \subset (A')^{-1}B^d \). Indeed, we have

\[
\langle A'(x - a'), A'(x - a') \rangle - \ln(\alpha') \leq \|x\|_K^2
\]

for every \( x \in \mathbb{R}^d \). Suppose for a contradiction that there is \( y \in K \setminus (A')^{-1}B^d \). Consider \( x = \vartheta y \), and substitute into the previous inequality. We obtain

\[
\vartheta^2 \|A'y\|^2 - 2\vartheta \langle A'a', A'y \rangle + \|A'a'\|^2 - \ln(\alpha') \leq \vartheta^2 \|y\|_K^2 < \vartheta^2.
\]

As \( |A'y| > 1 \), letting \( \vartheta \) tend to infinity, we obtain a contradiction. Thus, \( K \subset (A')^{-1}B^d \).

Hence, \( \det(A') \leq \det(A) \). On the other hand, \( \alpha' \geq \|f\| = 1 \). The proposition now easily follows from (7.5). \( \square \)

8. Outer integral ratio

The notion of the volume ratio was extended to the setting of log-concave functions in [AGMJV18], and then in [IN20] for John s-functions. For any \( s \in [0, \infty) \), the \( s \)-integral ratio of \( f \) is defined by

\[
(s) \text{I.rat}(f) = \left( \frac{\int_{\mathbb{R}^d} f}{\int_{\mathbb{R}^d} (s) J_f} \right)^{1/d}.
\]

Corollary 1.3 of [AGMJV18] states that there exists \( \Theta > 0 \) such that

\[
(0) \text{I.rat}(f) \leq \Theta \sqrt{d}
\]

for any proper log-concave function \( f : \mathbb{R}^d \to [0, \infty) \) and any positive integer \( d \). As a simple consequence of this result, the authors of [IN20] generalized this asymptotically tight bound to the John s-functions with fixed \( s \in [0, \infty) \).

However, the similar property of the Löwner ellipsoid is not obtained in [LSW19]. For any \( s \in [0, \infty) \), it is reasonable to define the outer \( s \)-integral ratio of \( f \) by

\[
(s) \text{I.or}(f) = \left( \frac{\int_{\mathbb{R}^d} (s) L_f}{\int_{\mathbb{R}^d} f} \right)^{1/d}.
\]

Theorem 1.5 is an immediate corollary of the following result and Lemma 6.2.
Theorem 8.1. There exists $\Theta_0$ such that for any positive integer $d$ and a proper log-concave function $f : \mathbb{R}^d \to [0, \infty)$ the following inequality holds

$$(0) \text{l.or}(f) \leq \Theta_0 d.$$ 

Proof. Since the outer integral ratio is the same for all functions of the form $\alpha f(x-a)$ with $\alpha > 0$ and $a \in \mathbb{R}^d$, we assume that $f(0) = \|f\| = e^{-d}$. By Lemma 5.2, there is a $d$-ellipsoid $(A \oplus 1, 0)$ such that $f \leq (0) \ell_{(A \oplus 1, 0)}$. By Lemma 3.1 and Corollary 3.1, there exists a 0-ellipsoidal function of minimal integral among those even 0-ellipsoidal functions with height 1 that are pointwise greater than or equal to $f$. Applying a suitable linear transform, we assume that the minimal integral is attained at $(0) \ell_{B_{d+1}} = e^{-|x|}$. That is, we assume that $f \leq (0) \ell_{B_{d+1}}$ and $(0) \ell_{B_{d+1}}$ is the solution of problem

$$\min_{(A \oplus 1, 0) \in \mathcal{E}} \int_{\mathbb{R}^d} (0) \ell_{(A \oplus 1, 0)} \quad \text{subject to} \quad f \leq (0) \ell_{(A \oplus 1, 0)}.$$ 

Let us show the geometric necessary condition for $(0) \ell_{(A \oplus 1, 0)}$ to be the solution of this problem. Define the convex set

$$K_f = \text{cl} \left( \bigcup_{r \geq d} \frac{1}{r} [f \geq e^{-r}] \right).$$ 

Claim 8.1. The unit ball $B^d$ is the smallest volume origin centered ellipsoid containing $K_f$.

Proof. By identity $f(0) = \|f\| = e^{-d}$ and inequality $f \leq (0) \ell_{B_{d+1}}$, we see that each of the convex sets $\frac{1}{r} [f \geq e^{-r}]$ with $r \geq d$ is nonempty and is a subset of the unit ball $B^d$. Thus, $K_f \subset B^d$. By construction, the origin belongs to $K_f$.

It is easy to see that the smallest volume origin centered ellipsoid containing $K_f$ is unique. Assume for a contrary that the ellipsoid $A^{-1}(B^d)$ with det $A > 1$ contains $K_f$. Consider the function $(0) \ell_{(A \oplus 1, 0)}$. Fix $r \geq d$. By definition, we have

$$\left[ (0) \ell_{(A \oplus 1, 0)} \geq e^{-r} \right] = \left\{ x \in \mathbb{R}^d : e^{-|Ax|} \geq e^{-r} \right\} = \left\{ x \in \mathbb{R}^d : |Ax| \leq r \right\} = r \cdot A^{-1}(B^d).$$

By this and since $\frac{1}{r} [f \geq e^{-r}] \subset A^{-1}(B^d)$, we conclude that

$$[f \geq e^{-r}] \subset \left[ (0) \ell_{(A \oplus 1, 0)} \geq e^{-r} \right].$$

Thus, $f \leq (0) \ell_{(A \oplus 1, 0)}$ and $\int_{\mathbb{R}^d} (0) \ell_{(A \oplus 1, 0)} < \int_{\mathbb{R}^d} (0) \ell_{B_{d+1}}$. We obtain a contradiction. \qed

Adjusting the celebrated John’s theorem (for example, following the proof given in [Bar97]) to our case, we get the following statement, the proof of which is given in the Appendix A.

Claim 8.2. Let $K$ be a convex body in $\mathbb{R}^d$. If $B^d$ is the smallest volume origin symmetric ellipsoid containing $K$, then there exist unit vectors $(u_i)_{i=1}^m$ on the boundary of $K$ and positive weights $(\alpha_i)_{i=1}^m$ satisfying John’s condition

$$\sum_{i=1}^m \alpha_i u_i \otimes u_i = \text{Id} \quad \text{and} \quad \sum_{i=1}^m \alpha_i = 1.$$

Another our key tool related to John’s condition is the Brascamp–Lieb inequality. We recall the reverse Brascamp–Lieb inequality of F. Barthe [Bar98, Theorem 5]. It states that if unit vectors $(u_i)_{i=1}^m$ and positive weights $(\alpha_i)_{i=1}^m$ satisfy John’s condition (8.1), then for measurable non-negative functions $q_i$ on $\mathbb{R}$, $i \in [m]$, one has

$$\int_{\mathbb{R}^d} \sup \left\{ \prod_{i=1}^m q_i(\Theta_i) \alpha_i : x = \sum_{i=1}^m \alpha_i \Theta_i u_i \right\} \, dx \geq \prod_{i=1}^m \left( \int_{\mathbb{R}} q_i \right)^{\alpha_i}.$$ 

Let unit vectors $(u_i)_{i=1}^m$ on the boundary of $K_f$ and positive weights $(\alpha_i)_{i=1}^m$ satisfy John’s condition (8.1). Fix $i \in [m]$. Let $f_i$ be the restriction of $f$ on the line with directional vector $u_i$, that is,
$f_i(t) = f(tu_i)$, $t \in \mathbb{R}$. What does it mean that $u_i$ belongs to the boundary of $K_j$? The answer is simple: Either there exists a $t \geq d$ such that $f_i(t) = e^{-t}$ or, by convexity, $u_i$ is an accumulation point of segments $\{ \frac{1}{r} [f_i \geq e^{-r}] \}_{r \geq d}$ and does not belong to any of these segments. In the former case, we set $t_i = \min \{ t \geq 0 : f(tu_i) = e^{-t} \}$. Otherwise, put $t_i = +\infty$. Clearly, $t_i \geq d$. Define

$$g_i(t) = \begin{cases} e^{-d}e^{-t(1-\frac{d}{r})}, & \text{if } t \in [0, t_i), \\ 0, & \text{otherwise} \end{cases}$$

for a finite $t_i$, and

$$g_i(t) = \begin{cases} e^{-d}e^{-t}, & \text{if } t \in [0, \infty), \\ 0, & \text{otherwise} \end{cases}$$

for $t_i = +\infty$.

**Claim 8.3.** For any $i \in [m]$, we have that $g_i \leq f_i$.

**Proof.** If $t_i$ is finite, the result follows from the log-concavity of $f_i$. Consider the case $t_i = +\infty$. Then for any $\varepsilon \in (0, 1)$, there exists $r > 1/\varepsilon$ such that $f_i((1-\varepsilon)r) \geq e^{-r}$. The log-concavity of $f_i$ yields

$$f_i(t) \geq e^{-d}e^{-t[\frac{1}{r} - (1-\frac{d}{r})r]}$$

for all $t \in [0, 1/\varepsilon]$. Taking the limit as $\varepsilon$ tends to 0, we see that $f_i(t) \geq e^{-d}e^{-t}$ for all $t \in [0, \infty)$. This completes the proof. □

By the log-concavity of $f$ and the rightmost identity in (8.1), we get

$$f(x) \geq \sup \left\{ \prod_{i=1}^{m} f_i(d \cdot \Theta_i)^{c_i/d} : x = \sum_{i=1}^{m} c_i \Theta_i u_i \right\}. \quad (8.3)$$

Using (8.3) in the reverse Brascamp–Lieb inequality (8.2) and by Claim 8.3, we obtain

$$\int_{\mathbb{R}^d} f \geq \prod_{i=1}^{m} \left( \int_{\mathbb{R}} f_i(td)^{1/d} dt \right)^{c_i} \geq \prod_{i=1}^{m} \left( \int_{\mathbb{R}} g_i(td)^{1/d} dt \right)^{c_i} = \prod_{i=1}^{m} \left( \int_{0}^{t_i/d} g_i(td)^{1/d} dt \right)^{c_i}$$

For any $\tau \geq d$, define $p(\tau) = \int_{0}^{\tau/d} e^{-t(1-\frac{d}{r})} dt$. Additionally, put $p(+\infty) = e^{-d} \int_{0}^{+\infty} e^{-t} dt$. Therefore, by the previous inequality and the rightmost identity in (8.1), we get

$$\int_{\mathbb{R}^d} f \geq \prod_{i=1}^{m} \left( \int_{0}^{t_i/d} g_i(td)^{1/d} dt \right)^{c_i} = \prod_{i=1}^{m} \left( \frac{1}{e} \int_{0}^{t_i/d} e^{-t(1-\frac{d}{r})} dt \right)^{c_i} \geq e^{-d} \cdot \left( \inf_{\tau \geq d} p(\tau) \right)^{d}.$$  

We claim that $\inf_{\tau \geq d} p(\tau) = 1$. Indeed, $p(d) = 1$ and $\lim_{\tau \to +\infty} p(\tau) = p(+\infty) = 1$; on the other hand, if $+\infty > \tau > d$, we have

$$p(\tau) = \frac{1 - e^{1-\tau/d}}{1 - \frac{d}{\tau}} > 1 \iff e^{\tau} > e^{d \cdot \frac{\tau}{d}}.$$  

The last inequality is simple and holds for any $+\infty > \tau > d$. Thus, $\int_{\mathbb{R}^d} f \geq e^{-d}$, and we conclude that

$$(0)_{1. \text{ or } (f) = \left( \frac{\int_{\mathbb{R}^d} f}{\int_{\mathbb{R}^d} (0)^{1/d} L_f} \right)^{1/d} \leq \left( \frac{\int_{\mathbb{R}^d} e^{-|x|} dx}{\int_{\mathbb{R}^d} f} \right)^{1/d} \leq \left( \frac{d! \text{vol}_d B^d}{e^{-d}} \right)^{1/d} = e \sqrt{\pi} \left( \frac{d!}{\Gamma(1 + \frac{d}{2})} \right)^{1/d},$$

where $\Gamma(\cdot)$ is Euler’s Gamma function. The existence of $\Theta_0$ follows. □
9. Duality

Recall that the polar set $K^\circ$ of a body $K \subset \mathbb{R}^d$ is defined by

$K^\circ = \{y : \langle y, x \rangle \leq 1 \quad \forall x \in K\}.$

It is known that for a convex body $K \subset \mathbb{R}^d$ with John ellipsoid $J_K$ centered at the origin, we have

$$(J_K)^\circ = L_{K^\circ},$$

where $L_{K^\circ}$ is the Löwner ellipsoid of $K^\circ$.

One would expect the same properties for John and Löwner functions. However, as was shown in [LSW19], there is no duality between the John and Löwner 0-functions. That is, assuming the John 0-function of $f$ is represented by an ellipsoid centered at the origin, we might have $\left((0)J_f\right)^\circ \neq \left((0)L_f\right)^\circ$. By continuity and Theorem 1.3, it follows that there exist functions $f$ such that $\left((s)J_f\right)^\circ \neq \left((s)L_f\right)^\circ$ for a sufficiently small positive $s$. We conjecture that for any positive $s$ there is such an example.

The problem is that the centers of $d$-ellipsoids representing $(s)J_f$ and $(s)L_f^\circ$ might be different. It is not the case for the setting of convex sets (key observation: the polar of an ellipsoid containing the origin in the interior is an ellipsoid). However, for any $s \in [0, \infty]$, if $d$-ellipsoids representing $(s)J_f$ and $(s)L_f^\circ$ are origin symmetric, we have duality, that is, the following identity

$$(9.1) \quad \left((s)J_f\right)^\circ = (s)L_f^\circ \quad \text{and} \quad \left((s)L_f\right)^\circ = (s)J_f^\circ$$

holds. Indeed, it follows from Lemma 6.1, identity (6.4) and (2.4).

Here are some examples when there is duality. By symmetry, we have the following.

**Example 9.1.** Let $s \in [0, \infty]$ and $f : \mathbb{R}^d \to [0, \infty)$ be a proper even log-concave function on $\mathbb{R}^d$. Then identity (9.1) holds.

In Theorem 5.1 of [IN20], the authors give a necessary and sufficient condition for a proper log-concave functions $f$ to satisfy $(s)h_{\mathbb{B}^{d+1}} = (s)J_f$ ($s \in (0, \infty)$). This and Example 9.1 yield the same type result for proper even log-concave functions.

In the following example, the centers were computed directly. By Proposition 7.1 and Proposition 8.1 of [IN20], we obtain

**Example 9.2.** Let $K \subset \mathbb{R}^d$ be a compact convex set with non-empty interior, and let $\|\cdot\|_K$ denote the gauge function of $K$, that is, $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$. Set $f = e^{-\|x\|_K}$. Then

$$\left((s)J_f\right)^\circ = (s)L_f^\circ \quad \text{and} \quad \left((s)L_f\right)^\circ = (s)J_f^\circ.$$

Finally, we note that duality helps us understand the properties of the constructed Löwner functions. The key tool for the interpolation between ellipsoidal functions $(s)h_{\ell_1}$ is the Asplund sum (or, sup-convolution), which is defined for two functions $f_1$ and $f_2$ on $\mathbb{R}^d$ by

$$(f_1 \star f_2)(x) = \sup_{x_1 + x_2 = x} f_1(x_1)f_2(x_2).$$

In our case everything is easier; we use the product of two $(s)\ell_1$ functions. It is easy to check that

$$(f_1 \star f_2)^\circ = f_1^\circ \cdot f_2^\circ.$$

That is, despite that there is no duality between the John and Löwner $s$-functions in general, there is duality between the methods.
Again, by the standard compactness argument, it follows that the re exists $\delta \in \delta$ such that there exists $K$ the smallest volume origin ellipsoid containing $H$. Then, there is a symmetric matrix $\hat{H}$ satisfying the equivalence of $(\text{8.1})$ and complete the proof of Claim $\text{A.1}$. Indeed, taking the trace of both sides in the leftmost identity in $(\text{8.1})$, we conclude that there exists $\delta > 0$ such that for any $\delta \in (0, \delta_1)$, we obtain

$$\det (\Id + \delta H) = 1 + \delta (\Id, H) + o(\delta) > 1.$$ 

Hence, the ellipsoid $(\Id + \delta H)^{-1} (\mathcal{B}^d)$ has the volume strictly less than the volume of $\mathcal{B}^d$ for any $\delta \in (0, \delta_1)$. Thus, it is suffice to show that that there exists $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0)$, the ellipsoid $(\Id + \delta H)^{-1} (\mathcal{B}^d)$ contains $K$. Let us prove this assertion. Fix any $u \in C$ and denote $x = x(\delta, u) = (\Id + \delta H) u$.

$$|x|^2 = \langle (\Id + \delta H) u, (\Id + \delta H) u \rangle = 1 + 2\delta \langle u, H u \rangle + \delta^2 |Hu|^2.$$ 

By $(\text{A.1})$, we conclude that there exists $\delta_u > 0$ such that $|x| < 1$ for all $\delta \in (0, \delta_u)$. Hence, $u = (\Id + \delta H)^{-1} x$ belongs to the interior of $(\Id + \delta H)^{-1} (\mathcal{B}^d)$ for all $\delta \in (0, \delta_u)$. By compactness, there exists $\delta_2 > 0$ such that $C$ belongs to the interior of $(\Id + \delta H)^{-1} (\mathcal{B}^d)$ for all $\delta \in (0, \delta_2)$. Again, by the standard compactness argument, it follows that there exists $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0)$, the ellipsoid $(\Id + \delta H)^{-1} (\mathcal{B}^d)$ contains $K$. That is, the original symmetric ellipsoid $(\Id + \delta H)^{-1} (\mathcal{B}^d)$ contains $K$ and has the volume strictly less than the volume of $\mathcal{B}^d$. We obtain a contradiction with the choice of the smallest volume origin symmetric ellipsoid containing $K$, and complete the proof of Claim 8.2. \hfill $\Box$

Remark A.1. It is not hard to proof that under the assertion of Claim 8.2, the unit ball $\mathcal{B}^d$ is the smallest volume origin ellipsoid containing $K$ if, and only if, John’s condition $(\text{8.1})$ holds.

Acknowledgement. The authors acknowledge the financial support from the Ministry of Educational and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926.


FUNCTIONAL LÖWNER ELLIPSOIDS

REFERENCES

[AAKM04] Shiri Artstein-Avidan, Bo’az Klartag, and Vitali Milman. The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika*, 51(1-2):33–48, 2004.

[AAM07] Shiri Artstein-Avidan and Vitali Milman. A characterization of the concept of duality. *Electronic Research Announcements in Mathematical Sciences*, 14:42–59, 2007.

[AGMJV18] David Alonso-Gutiérrez, Bernardo González Merino, C. Hugo Jiménez, and Rafael Villa. John’s ellipsoid and the integral ratio of a log-concave function. *The Journal of Geometric Analysis*, 28(2):1182–1201, 2018.

[AS17] Guillaume Aubrun and Stanislaw J. Szarek. *Alice and Bob Meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory*, volume 223. American Mathematical Soc., 2017.

[Bal97] Keith Ball. An elementary introduction to modern convex geometry. *Flavors of geometry*, 31:1–58, 1997.

[Bar98] Franck Barthe. On a reverse form of the Brascamp–Lieb inequality. *Inventiones mathematicae*, 134(2):335–361, 1998.

[BGVV14] Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou. *Geometry of isotropic convex bodies*, volume 196. American Mathematical Society Providence, 2014.

[Cla90] Frank H. Clarke. *Optimization and nonsmooth analysis*. SIAM, 1990.

[Gru07] Peter M. Gruber. *Convex and Discrete Geometry*. Springer Berlin Heidelberg, 2007.

[IN20] Grigory Ivanov and Márton Naszódi. Functional John Ellipsoids. *arXiv preprint arXiv:2006.09934*, 2020.

[Joh14] Fritz John. Extremum problems with inequalities as subsidiary conditions. In *Traces and emergence of nonlinear programming*, pages 197–215. Springer, 2014.

[LSW19] Ben Li, Carsten Schütt, and Elisabeth M. Werner. The Löwner function of a log-concave function. *The Journal of Geometric Analysis*, pages 1–34, 2019.

GRIGORY IVANOV: INSTITUTE OF SCIENCE AND TECHNOLOGY AUSTRIA (IST AUSTRIA), KLEUSTENEUBURG, AUSTRIA. MOSCOW INST. OF PHYSICS AND TECHNOLOGY, MOSCOW, RUSSIA

E-mail address: GRIMIVANOV@GMAIL.COM

IGOR TSIUTSIURUPA: MOSCOW INST. OF PHYSICS AND TECHNOLOGY, MOSCOW, RUSSIA

E-mail address: IGOR.TSUTSIURUPA97@GMAIL.COM