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Bifurcation of solutions of the second order boundary value problems in the Hilbert spaces.

Conditions of the existence of solutions of linear and perturbed linear boundary value problems in the Hilbert spaces for the second order evolution equation are obtained.

Consider the following boundary value problem (BVP) in the Hilbert spaces

\[ y''(t, \varepsilon) + A(t)y(t, \varepsilon) = \varepsilon A_1(t)y(t, \varepsilon) + f(t), \]

\[ l(y(\cdot, \varepsilon), y'(\cdot, \varepsilon))^T = \alpha, \]

where \( y : J \to \mathcal{H} \) is a vector-function \( y \in C^2(J, \mathcal{H}) \), \( J \subset \mathbb{R} \), the closed operator-valued function \( A(t) \) acts from \( J \) into the dense domain \( D = D(A(t)) \subset \mathcal{H} \) which is independent from \( t \), \( l \) is a linear and bounded operator which translates solutions of (1) into the Hilbert space \( \mathcal{H}_1 \), \( A_1(t) \) is a linear and bounded operator valued function \( |||A_1||| = \sup_{t \in J} |||A_1(t)||| < \infty, \alpha \in \mathcal{H}_1 \).

**Linear case.** At first we find the necessary and sufficient conditions of the existence of solutions of linear nonhomogeneous boundary value problem

\[ y''_0(t) + A(t)y_0(t) = f(t), \quad l(y_0(\cdot), y'_0(\cdot))^T = \alpha. \]

Let \( x_1(t) = y_0(t), \ x_2(t) = y'_0(t), \ x(t) = (x_1(t), x_2(t))^T \), then we can rewrite boundary value problem (3) in the form of the operator system

\[ x'_0(t) = B(t)x_0(t) + g(t), \quad lx_0(\cdot) = \alpha, \]

where

\[ B(t) = \begin{pmatrix} 0 & I \\ -A(t) & 0 \end{pmatrix}, \quad g(t) = (0, f(t))^T. \]

Denote by \( U(t) \) the evolution operator of homogeneous equation \( U'(t) = B(t)U(t), U(0) = I \). Then the set of solutions of (4) has the form

\[ x_0(t, c) = U(t)c + \int_0^t U(t)U^{-1}(\tau)g(\tau)d\tau. \]

Substituting in \( lx_0(\cdot) = \alpha \) we obtain the following operator equation

\[ Qc = \alpha - l \int_0^\infty U(\cdot)U^{-1}(\tau)g(\tau)d\tau, \quad Q = lU(\cdot) : \mathcal{H} \to \mathcal{H}_1. \]

Using the theory of strong generalized solutions [2, 3] we obtain the following result.
Theorem 1. 1. a) Boundary value problem (3) has strongly generalized solutions if and only if the following condition holds

$$\mathcal{P}_{N(\overline{Q})}\{\alpha - l \int_{0}^{\tau} U(\cdot)U^{-1}(\cdot)g(\cdot)d\cdot\} = 0;$$  \hspace{1cm} (7)

if $$\alpha - l \int_{0}^{\tau} U(\cdot)U^{-1}(\cdot)f(\cdot)d\cdot \in R(Q)$$ then generalized solutions will be classical;

b) under condition (7) the set of solutions has the form

$$x_0(t, c) = U(t)\mathcal{P}_{N(\overline{Q})}c + (G[g, \alpha])(t),$$

where $$\mathcal{P}_{N(\overline{Q})}, \mathcal{P}_{N(\overline{Q}^\ast)}$$ are the orthoprojectors onto the kernel and cokernel of the operator $$\overline{Q}$$ respectively,

$$\overline{(G[g, \alpha])}(t) = \int_{0}^{t} U(t)U^{-1}(\cdot)g(\cdot)d\cdot + \overline{Q}^\ast\{\alpha - l \int_{0}^{\tau} U(\cdot)U^{-1}(\cdot)g(\cdot)d\cdot\}$$

is a generalized Green’s operator;

2. a) Boundary value problem (3) has strongly quasisolutions if and only if the following condition holds

$$\mathcal{P}_{N(\overline{Q}^\ast)}\{\alpha - l \int_{0}^{\tau} U(\cdot)U^{-1}(\cdot)f(\cdot)d\cdot\} \neq 0;$$  \hspace{1cm} (8)

b) under condition (8) the set of strongly quasisolutions has the form

$$x_0(t, c) = U(t)\mathcal{P}_{N(\overline{Q}^\ast)}c + (G[g, \alpha])(t).$$

Bifurcation conditions. a) Suppose that condition (8) is hold. We obtain the condition on $$A_1(t)$$ such that the perturbed boundary value problem

$$x'(t, \varepsilon) = B(t)x(t, \varepsilon) + g(t) + \varepsilon B_1(t)x(t, \varepsilon),$$  \hspace{1cm} (9)

$$lx(\cdot, \varepsilon) = \alpha,$$  \hspace{1cm} (10)

have the generalized solutions. Here is the operator-valued function $$B_1(t)$$ has the following form:

$$B_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & A_1(t) \end{pmatrix}, \quad g(t) = (0, f(t))^T,$$  \hspace{1cm} (11)

$$x(t, \varepsilon) = (x_1(t, \varepsilon), x_2(t, \varepsilon))^T, \quad x_1(t, \varepsilon) = y(t, \varepsilon), x_2(t, \varepsilon) = y'(t, \varepsilon).$$

We will use the modification of the well-known Vishik-Lyusternik method. A solution of problem (9), (10) is sought in the form of a segment of the Laurent series in powers of the small parameter $$\varepsilon$$:

$$x(t, \varepsilon) = \sum_{i=-1}^{+\infty} \varepsilon^i x_i(t) = \frac{x_{-1}(t)}{\varepsilon} + x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ....$$  \hspace{1cm} (12)
Substituting series (12) into problem (9), (10) and equating the coefficients of $\varepsilon^{-1}$, we obtain the following boundary value problem for finding the coefficient $x_{-1}(t)$ of series (12):

$$x'_{-1}(t) = B(t)x_{-1}(t), \quad lx_{-1}(\cdot) = 0. \quad (13)$$

Problem (13), (14) has a family of solutions:

$$x_{-1}(t, c_{-1}) = U(t)P_{N(Q)}c_{-1}, \quad c_{-1} \in \mathcal{H}. \quad (14)$$

An arbitrary element $c_{-1}$ is determined by the condition for the solvability of the following linear inhomogeneous boundary value problem for finding the coefficient $x_{0}(t)$ in series (12):

$$x'_{0}(t) = B(t)x_{0}(t) + B_{1}(t)x_{-1}(t) + g(t), \quad lx_{0}(\cdot) = \alpha. \quad (15)$$

A necessary and sufficient condition for the solvability of problem (15), (16) is given by

$$\mathcal{P}_{N(Q)}\{\alpha - l \int_{0}^{\cdot} U(\cdot)U^{-1}(\tau)(B_{1}(\tau)x_{-1}(\tau, c_{-1}) + g(\tau))d\tau\} = 0. \quad (18)$$

From this, in view of the form of $x_{-1}(t, c_{-1})$, we obtain an operator equation for $c_{-1} \in \mathcal{H}$:

$$B_{0}c_{-1} = \mathcal{P}_{N(Q)}\{\alpha - l \int_{0}^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau\}, \quad (17)$$

where

$$B_{0} = \mathcal{P}_{N(Q)}l \int_{0}^{\cdot} U(\cdot)U^{-1}(\tau)B_{1}(\tau)U(\tau)d\tau \mathcal{P}_{N(Q)}. \quad$$

A necessary and sufficient condition for the generalized solvability of this operator equation is

$$\mathcal{P}_{N(Q)}\mathcal{P}_{N(Q)}\{\alpha - l \int_{0}^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau\} = 0. \quad (18)$$

Suppose that $\mathcal{P}_{N(Q)}\mathcal{P}_{N(Q)} = 0$. Then condition (18) is hold. The solution set of operator equation for $c_{-1} \in \mathcal{H}$ has the form

$$c_{-1} = \bar{c}_{-1} + \mathcal{P}_{N(Q)}c_{p}, \quad \forall c_{p} \in \mathcal{H},$$

where

$$\bar{c}_{-1} = \mathcal{P}_{N(Q)}\{\alpha - l \int_{0}^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau\}. \quad (18)$$
In view of the expression for $c_{-1}$, the homogeneous boundary value problem (15), (16) has a $\rho$ - parameter family of solutions

$$x_{-1}(t, c_\rho) = \overline{x}_{-1}(t, \overline{c}_{-1}) + U(t)\mathcal{P}_{N(\overline{Q})}\mathcal{P}_{N(\overline{B}_0)}c_\rho,$$

(19)

where

$$\overline{x}_{-1}(t, \overline{c}_{-1}) = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c}_{-1}.$$ 

The general solution of problem (15), (16) has the form

$$x_0(t, c_0) = U(t)\mathcal{P}_{N(\overline{Q})}c_0 + F_{-1}(t) + K_{-1}(t)\mathcal{P}_{N(\overline{B}_0)}c_\rho,$$

where

$$F_{-1}(t) = (G[g + B_1\overline{x}_{-1}, \alpha])(t), \quad K_{-1}(t) = (G[U, 0])(t)\mathcal{P}_{N(\overline{Q})},$$

where $c_0$ is an element of the space $\mathcal{H}$, which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient $x_1(t)$ in series (12).

To determine the coefficient $x_1(t)$ of $\varepsilon^1$ in series (12), we obtain the following boundary value problem

$$x_1'(t) = B(t)x_1(t) + B_1(t)x_0(t, c_0),$$

(20)

$$lx_1(\cdot) = 0.$$ 

(21)

Under condition of solvability

$$\mathcal{P}_{N(\overline{Q})}l\int_0^t U(\cdot)U^{-1}(\tau)B_1(\tau)x_0(\tau, c_0)d\tau = 0,$$

BVP (20), (21) has the set of solutions in the form

$$x_1(t, c_1) = U(t)\mathcal{P}_{N(\overline{Q})}c_1 + (G[B_1U\mathcal{P}_{N(\overline{Q})}c_0 + F_{-1} + K_{-1}, 0])(t).$$

The condition for the solvability of the boundary condition for the element $c_0$ is

$$B_0c_0 = -\mathcal{P}_{N(\overline{Q})}l\int_0^t U(\cdot)U^{-1}(\tau)B_1(\tau)F_{-1}(\tau)d\tau -$$

(22)

$$-\mathcal{P}_{N(\overline{Q})}l\int_0^t U(\cdot)U^{-1}(\tau)B_1(\tau)K_{-1}(\tau)d\tau\mathcal{P}_{N(\overline{Q})}\mathcal{P}_{N(\overline{B}_0)}c_\rho.$$ 

From the condition $\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(\overline{Q})} = 0$ follows solvability of equation (22) with the set of solutions in the following form

$$c_0 = -B_0^+\mathcal{P}_{N(\overline{Q})}l\int_0^t U(\cdot)U^{-1}(\tau)B_1(\tau)F_{-1}(\tau)d\tau -$$

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\[-B_0^+\mathcal{P}_{N,Q}l \int_0^l U(\cdot) U^{-1}(\tau) B_1(\tau) K_{-1}(\tau) d\tau \mathcal{P}_{N,Q} \mathcal{P}_{N,B_0} c_{\rho} + \mathcal{P}_{N,B_0} c_{\rho},\]

\[c_0 = \overline{c}_0 + D_0 \mathcal{P}_{N,B_0} c_{\rho}; \quad \forall c_{\rho} \in \mathcal{H},\]

where

\[\overline{c}_0 = -B_0^+ \mathcal{P}_{N,Q} l \int_0^l U(\cdot) U^{-1}(\tau) B_1(\tau) F_{-1}(\tau) d\tau,\]

\[D_0 = I - B_0^+ \mathcal{P}_{N,Q} l \int_0^l U(\cdot) U^{-1}(\tau) B_1(\tau) K_{-1}(\tau) d\tau \mathcal{P}_{N,Q}.$

Thus, problem (15), (16) has a \(\rho\)-parameter family of solutions:

\[x_0(t, c_0) = \overline{x}_0(t, \overline{c}_0) + X_0(t) \mathcal{P}_{N,B_0} c_{\rho}, \quad \forall c_{\rho} \in \mathcal{H},\]

where

\[\overline{x}_0(t, \overline{c}_0) = U(t) \mathcal{P}_{N,Q} \overline{c}_0 + F_{-1}(t),\]

\[X_0(t) = U(t) \mathcal{P}_{N,Q} D_0 + K_{-1}(t).\]

Then problem (20), (21) has a \(\rho\)-parameter family of solutions

\[x_0(t, c_{\rho}) = \overline{x}_0(t, \overline{c}_0) + X_0(t) \mathcal{P}_{N,B_0} c_{\rho},\]

where

\[\overline{x}_0(t, \overline{c}_0) = U(t) \mathcal{P}_{N,Q} \overline{c}_0 + F_{-1}(t),\]

\[X_0(t) = U(t) \mathcal{P}_{N,Q} D_0 + K_{-1}(t).\]

Then problem (20), (21) has a \(\rho\)-parameter family of solutions

\[x_1(t, c_1) = U(t) \mathcal{P}_{N,Q} c_1 + F_0(t) + K_0(t) \mathcal{P}_{N,B_0} c_{\rho},\]

where

\[F_0(t) = (G[B_1 U \mathcal{P}_{N,Q} \overline{c}_0 + F_{-1} + K_{-1}, 0])(t),\]

\[K_0(t) = (G[B_1 U \mathcal{P}_{N,Q} D_0, 0])(t),\]

where \(c_1\) is an element of the Hilbert space \(\mathcal{H}\), which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient \(x_3(t)\) in series (12). By induction, we can prove that the coefficients \(x_i(t)\) in series (12) are determined by solving the boundary value problem

\[x'_i(t) = B(t)x_i(t) + B_1(t)x_{i-1}(t, c_{i-1}), \quad (23)\]

\[lx_i(\cdot) = 0 \quad (24)\]
which under condition of solvability has a \( \rho \)-parameter family of solutions

\[
x_i(t, c_i) = \Xi_i(t, c_i) + X_i(t)P_{N(B_0)}c_\rho, \quad \forall c_\rho \in \mathcal{H}
\]  

(25)

where all the terms are determined by the iterative procedure

\[
\Xi_i(t, c_i) = U(t)P_{N(\overline{Q})}c_i + F_{i-1}(t), \quad (26)
\]

\[
X_i(t) = U(t)P_{N(\overline{Q})}D_i + K_{i-1}(t), \quad (27)
\]

\[
D_i = I - \overline{P}_{0}P_{N(Q)}l \int_{0}^{t} U(\cdot)U^{-1}(\tau)B_1(\tau)K_{i-1}(\tau) d\tau P_{N(\overline{Q})}, \quad (28)
\]

\[
F_{i-1}(t) = (G[B_1 U P_{N(\overline{Q})} c_{i-1} + F_{i-2} + K_{i-2}, 0])(t), \quad (29)
\]

\[
K_{i-1}(t) = (G[B_1 U P_{N(\overline{Q})} D_{i-1}, 0])(t). \quad (30)
\]

The convergence of series (12) is proved in the same manner as in [11]. Thus, the following result holds.

**Theorem 1.** The boundary value problem (9), (10) with the condition \( P_{N(B_0)}P_{N(Q)} = 0 \) has a \( \rho \)-parameter family of solutions in the form of the Laurent series segment

\[
x(t, c_\rho) = \sum_{i=-1}^{+\infty} \varepsilon^i[\Xi_i(t, c_i) + X_i(t)P_{N(B_0)}c_\rho], \quad \forall c_\rho \in \mathcal{H},
\]

whose coefficients are given by formulas (26)-(30).

b) Suppose that condition (7) is hold. We obtain the condition on \( A_1(t) \) such that the perturbed boundary value problem

\[
x'(t, \varepsilon) = B(t)x(t, \varepsilon) + g(t) + \varepsilon B_1(t)x(t, \varepsilon), \quad (31)
\]

\[
lx(\cdot, \varepsilon) = \alpha, \quad (32)
\]

have the generalized solutions. A solution of problem (31), (32) is sought in the form of a segment of the Taylor series in powers of the small parameter \( \varepsilon \):

\[
x(t, \varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^ix_i(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + .... \quad (33)
\]

Substituting series (33) into problem (31), (32) and equating the coefficients of \( \varepsilon^0 \), we obtain the following boundary value problem for finding the coefficient \( x_0(t) \) of series (33):

\[
x'_0(t) = B(t)x_0(t) + g(t), \quad (34)
\]
\[ l x_0(\cdot) = \alpha. \] (35)

Problem (34), (35) has a family of solutions:
\[ x_0(t, c) = U(t) P_{N(Q)} c_0 + \overline{(G[g, \alpha])}(t). \]

An arbitrary element \( c_0 \) is determined by the condition for the solvability of the following linear inhomogeneous boundary value problem for finding the coefficient \( x_1(t) \) in series (33):
\[
\begin{align*}
&x_1'(t) = B(t) x_1(t) + B_1(t) x_0(t, c_0), \\
&l x_1(\cdot) = 0.
\end{align*}
\] (36) (37)

A necessary and sufficient condition for the solvability of problem (36), (37) is given by
\[
P_{N(Q^*)} \{ l \int_0^\infty U(\cdot) U^{-1}(\tau) B_1(\tau) x_0(\tau, c_0) d\tau \} = 0.
\]

From this, in view of the form of \( x_0(t, c_0) \), we obtain an operator equation for \( c_0 \in \mathcal{H} \):
\[
B_0 c_0 = -P_{N(Q^*)} l \int_0^\infty U(\cdot) U^{-1}(\tau) B_1(\tau) \overline{(G[g, \alpha])}(\tau) d\tau.
\] (38)

Under condition \( P_{N(B_0^*)} P_{N(Q^*)} = 0 \) the equation (38) is solvable. The solution set of operator equation for \( c_{-1} \in \mathcal{H} \) has the form
\[
c_0 = \overline{c}_0 + P_{N(Q^*)} c_\rho, \forall c_\rho \in \mathcal{H},
\]
where
\[
\overline{c}_0 = -B_0^* P_{N(Q^*)} l \int_0^\infty U(\cdot) U^{-1}(\tau) B_1(\tau) \overline{(G[g, \alpha])}(\tau) d\tau.
\]

In view of the expression for \( c_0 \), the homogeneous boundary value problem (36), (37) has a \( \rho \) - parameter family of solutions
\[
x_0(t, c_\rho) = \overline{x}_0(t, \overline{c}_0) + U(t) P_{N(Q^*)} P_{N(B_0)} c_\rho, \] (39)
where
\[
\overline{x}_0(t, \overline{c}_0) = U(t) P_{N(Q^*)} \overline{c}_0 + \overline{(G[g, \alpha])}(t).
\]

The general solution of problem (36), (37) has the form
\[
x_0(t, c_0) = U(t) P_{N(Q^*)} c_0 + F_{-1}(t) + K_{-1}(t) P_{N(B_0)} c_\rho,
\]
where
\[
F_{-1}(t) = \overline{(G[g + B_1 \overline{c}_{-1}, \alpha])}(t), K_{-1}(t) = \overline{(G[U, 0])}(t) P_{N(Q^*)},
\]

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where $c_0$ is an element of the space $\mathcal{H}$, which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient $x_1(t)$ in series (33). To determine the coefficient $x_1(t)$ of $\varepsilon^1$ in series (33), we obtain the following boundary value problem

$$x_1'(t) = B(t)x_1(t) + B_1(t)x_0(t, c_0), \quad l x_1(\cdot) = 0.$$  

(40)

(41)

Under condition of solvability

$$\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)x_0(\tau, c_0)d\tau = 0,$$

BVP (40), (41) has the set of solutions in the form

$$x_1(t, c_1) = U(t)\mathcal{P}_{N(\overline{\mathcal{Q}})} c_1 + \left( G[B_1U\mathcal{P}_{N(\overline{\mathcal{Q}})} c_0 + F_{-1} + K_{-1}, 0]\right)(t).$$

The condition for the solvability of the boundary condition for the element $c_0$ is

$$B_0 c_0 = -\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)F_{-1}(\tau)d\tau -$$

$$\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)K_{-1}(\tau)d\tau \mathcal{P}_{N(\overline{\mathcal{Q}})} \mathcal{P}_{N(\overline{\mathcal{Q}})} c_0.$$  

(42)

From the condition $\mathcal{P}_{N(B_0)}\mathcal{P}_{N(\overline{\mathcal{Q}})} = 0$ follows solvability of equation (22) with the set of solutions in the following form

$$c_0 = -B_0^+\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)F_{-1}(\tau)d\tau -$$

$$-B_0^+\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)K_{-1}(\tau)d\tau \mathcal{P}_{N(\overline{\mathcal{Q}})} \mathcal{P}_{N(\overline{\mathcal{Q}})} c_\rho + \mathcal{P}_{N(\overline{\mathcal{Q}})} c_\rho,$$

$$c_0 = \overline{c}_0 + D_0 \mathcal{P}_{N(\overline{\mathcal{Q}})} c_\rho, \quad \forall c_\rho \in \mathcal{H},$$

where

$$\overline{c}_0 = -B_0^+\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)F_{-1}(\tau)d\tau,$$

$$D_0 = I - B_0^+\mathcal{P}_{N(\overline{\mathcal{Q}})} l \int_0^1 U(\cdot)U^{-1}(\tau)B_1(\tau)K_{-1}(\tau)d\tau \mathcal{P}_{N(\overline{\mathcal{Q}})}.$$  

Thus, problem (36), (37) has a $\rho$-parameter family of solutions:

$$x_0(t, c_0) = \overline{x}_0(t, \overline{c}_0) + \mathcal{X}_0(t)\mathcal{P}_{N(\overline{\mathcal{Q}})} c_\rho, \quad \forall c_\rho \in \mathcal{H},$$

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where
\[
\begin{align*}
\overline{x}_0(t, c_0) &= U(t)\mathcal{P}_{N(\mathcal{Q})}c_0 + F_{-1}(t), \\
\overline{X}_0(t) &= U(t)\mathcal{P}_{N(\mathcal{Q})}D_0 + K_{-1}(t).
\end{align*}
\]

Then problem (40), (41) has a \(\rho\)-parameter family of solutions
\[
x_0(t, c_\rho) = \overline{x}_0(t, c_0) + \overline{X}_0(t)\mathcal{P}_{N(B_0)}c_\rho,
\]

where
\[
\begin{align*}
\overline{x}_0(t, c_0) &= U(t)\mathcal{P}_{N(\mathcal{Q})}c_0 + F_{-1}(t), \\
\overline{X}_0(t) &= U(t)\mathcal{P}_{N(\mathcal{Q})}D_0 + K_{-1}(t).
\end{align*}
\]

Then problem (40), (41) has a \(\rho\)-parameter family of solutions
\[
x_1(t, c_1) = U(t)\mathcal{P}_{N(\mathcal{Q})}c_1 + F_0(t) + K_0(t)\mathcal{P}_{N(B_0)}c_\rho,
\]

where
\[
\begin{align*}
F_0(t) &= (G[B_1U\mathcal{P}_{N(\mathcal{Q})}c_0 + F_{-1} + K_{-1}, 0])(t), \\
K_0(t) &= (G[B_1U\mathcal{P}_{N(\mathcal{Q})}D_0, 0])(t),
\end{align*}
\]

where \(c_1\) is an element of the Hilbert space \(\mathcal{H}\), which is determined at the next step from the condition for the solvability of the boundary value problem for finding the coefficient \(x_2(t)\) in series (33). By induction, we can prove that the coefficients \(x_i(t)\) in series (33) are determined by solving the boundary value problem
\[
x'_i(t) = B(t)x_i(t) + B_1(t)x_{i-1}(t, c_{i-1}),
\]

\[lx_i(\cdot) = 0\] (44)

which under condition of solvability has a \(\rho\)-parameter family of solutions
\[
x_i(t, c_i) = \overline{x}_i(t, \overline{c}_i) + \overline{X}_i(t)\mathcal{P}_{N(B_0)}c_\rho, \quad \forall c_\rho \in \mathcal{H}
\]

where all the terms are determined by the iterative procedure
\[
\begin{align*}
\overline{x}_i(t, \overline{c}_i) &= U(t)\mathcal{P}_{N(\mathcal{Q})}\overline{c}_i + F_{i-1}(t), \\
\overline{X}_i(t) &= U(t)\mathcal{P}_{N(\mathcal{Q})}D_i + K_{i-1}(t),
\end{align*}
\]

\[
D_i = I - B_0^+\mathcal{P}_{N(\mathcal{Q})}l \int_0^t U(\cdot)U^{-1}(\tau)B_1(\tau)K_{i-1}(\tau)d\tau\mathcal{P}_{N(\mathcal{Q})},
\]

\[
F_{i-1}(t) = (G[B_1U\mathcal{P}_{N(\mathcal{Q})}\overline{c}_{i-1} + F_{i-2} + K_{i-2}, 0])(t),
\]

\[
F_{i-1}(t) = (G[B_1U\mathcal{P}_{N(\mathcal{Q})}\overline{c}_{i-1} + F_{i-2} + K_{i-2}, 0])(t),
\]

\[
F_{i-1}(t) = (G[B_1U\mathcal{P}_{N(\mathcal{Q})}\overline{c}_{i-1} + F_{i-2} + K_{i-2}, 0])(t),
\]

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\[ K_{i-1}(t) = \overline{(G[B_1 U \mathcal{P}_{N([\overline{Q}])} D_{i-1}, 0])}(t). \] (50)

The convergence of series (33) is proved in the same manner as in [11]. Thus, the following result holds.

**Theorem 2.** The boundary value problem (31), (32) with the condition \( \mathcal{P}_{N(B_0)} \mathcal{P}_{N(Q^*)} = 0 \) has a \( \rho \)-parameter family of solutions in the form of the Laurent series segment

\[
x(t, c_\rho) = \sum_{i=-\infty}^{+\infty} [x_i(t, c_\rho) + X_i(t) \mathcal{P}_{N(B_0)} c_\rho], \quad \forall c_\rho \in \mathcal{H},
\]

whose coefficients are given by formulas (46)-(50).

**References**

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