The impact of model risk on dynamic portfolio selection under multi-period mean-standard-deviation criterion

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Abstract

We quantify model risk of a financial portfolio whereby a multi-period mean-standard-deviation criterion is used as a selection criterion. In this work, model risk is defined as the loss due to uncertainty of the underlying distribution of the returns of the assets in the portfolio. The uncertainty is measured by the Kullback-Leibler divergence, i.e., the relative entropy. In the worst case scenario, the optimal robust strategy can be obtained in a semi-analytical form as a solution of a system of nonlinear equations. Several numerical results are presented which allow us to compare the performance of this robust strategy with the optimal non-robust strategy. For illustration, we also quantify the model risk associated with an empirical dataset.

Keywords: Multivariate statistics, Uncertainty modelling, Robust portfolio allocation, Pseudo dynamic programming, Mean-standard-deviation, Kullback-Leibler divergence

1. Introduction.

Portfolio selection has been studied extensively over the last few decades (see e.g., Markowitz (1952); Li and Ng (2000)). Investors face the problem of choosing the best possible investment strategy among thousands of assets. One significant difficulty in choosing optimal strategies is magnified by the fact that the essential information needed to make an optimal decision, namely, the distribution of assets, is typically unknown or only...
vaguely known. Another difficulty is that the distribution changes over time and a dynamic approach is needed to model it. A typical case is when the cross-sectional distribution of the assets in the portfolio is assumed to be “slightly deviating” from some nominal multivariate distribution. The deviation can be measured by a divergence measure such as the Kullback-Leibler (KL) divergence, i.e., the relative entropy, or more generally by the $\alpha$-divergence. Intuitively, the bigger the divergence, the more significant the impact on the optimal investment decision that is calculated under the nominal distribution assumption. However, the magnitude of the divergence that significantly impacts the investment decision depends on the nominal distribution and on the type of deviation. When the distributional assumptions are violated but only “slightly”, it may be prudent to use the optimal investment strategy under the nominal distribution. This may in fact deliver better results (for example, in the sense of a higher expectation of the terminal wealth) since the robust approach—focusing on safeguarding against the worst possible outcome—may deliver too pessimistic a strategy that may be disadvantageous when the nominal distribution is violated only slightly. Ideally, a ball of certain radius $\eta_0$ around the nominal distribution is given such that for distributions within the radius the nominal distribution is recommended, whereas when the model uncertainty is bigger than $\eta_0$ then the robust approach is recommended. It is worth noting that although the optimal investment strategy under the nominal distribution may perform better in the sense of a higher expectation of the terminal wealth, the model risk as defined from the standard risk management perspective may also be large. Thus, it is essential to quantify such model risk.

Unlike in robust optimization, where uncertainty is often measured by an uncertainty set (see e.g., Kapsos et al. (2014); Kim et al. (2014)), the deviation between distributions from a statistical point of view has been most commonly measured by a divergence measure. Glasserman and Xu (2014) interpreted the KL divergence as a measure of the amount of extra information required to adopt an alternative distribution, and disregard the nominal distribution. However, it has also been pointed out in Glasserman and Xu (2014) that the KL divergence is not suitable for heavy tailed distributions since it relies on the assumption that the moment generating function of the underlying random variables must exist in some open set containing the origin. Thus, the so-called $\alpha$-divergence is used as a substitute. In contrast, Schneider and Schweizer (2015) argued that the use of $\alpha$-divergence implicitly assumes that the tail of the deviating model is not heavier than that of the nominal distribution. In fact, the popularity
of the KL divergence and of the \( \alpha \)-divergence is due to the existence of a closed form solution when one considers a worst case scenario approach or, in other words, the robust optimization approach to quantify model risk in risk management (see Glasserman and Xu (2013), Schneider and Schweizer (2015)). If one only considers the measuring of model risk, alternative divergence measures are also possible. This has been discussed in the recent work of Breuer and Csiszár (2016) and in the references therein. Another interesting result worth mentioning in this area is the recent work by Lam (2016), deriving an asymptotic expansion of the worst risk measure in the case of KL divergence.

The main focus in this paper is to investigate the impact of uncertainty of the distribution of returns of assets on the optimal portfolio allocation model in Bannister et al (2016). As in Bannister et al (2016), the selection criterion is the multi-period mean-standard-deviation and portfolio selection is performed in a dynamic way. To measure uncertainty we use the KL divergence, which is reasonable if the underlying random variable (a function of asset returns) is not heavy tailed. If we consider short term re-balancing (daily or weekly) as we do in our numerical examples, this is a reasonable and acceptable assumption. Inspired by Kang and Filar (2006), we find what we call a time consistent optimal robust strategy (see Definition 1). This reduces to solving a sequence of single period portfolio selection problems. For each single period, we apply a robust optimization approach. Thus, we have to solve an inner and an outer optimization problem. The inner problem is an infinite dimensional optimization where we try to find a worst case distribution from a set of alternative distributions (which have positive distances to the nominal distribution). A closed form solution to the inner optimization problem is available from past literature, see for example Lam (2016). The outer optimization problem is a standard convex optimization problem. By solving this, we derive a system of equations which an optimal robust strategy should satisfy. This is our first contribution. To be more precise, we have derived an optimal strategy in a semi-analytical form for the portfolio selection model in Bannister et al (2016) but with added uncertainty of the distribution of the returns, where the uncertainty is measured by the KL divergence. This optimal robust strategy can easily be calculated numerically in combination with a simple Monte Carlo approach from Glasserman and Xu (2014). Our second contribution is to examine the impact of the uncertainty on portfolio selection by using the constructed model. Additionally, we compare the performance of the optimal robust strategy and of the non-robust strategy (the optimal strategy
when there is no distributional uncertainty) under various scenarios. Moreover, we define model risk from the standard risk management perspective and quantify model risk using an empirical dataset. This provides a way to examine model risk for practical purposes and is yet another contribution of our work.

The study of the impact of uncertainty of the underlying distribution of asset returns on the optimal strategy is an important one. In fact, the impact on the optimal strategy “under the worst case” was also raised as one of the five questions in the implementation of a robust risk management process by Schneider and Schweizer (2015). Their work focused on the remaining four questions. Although there are several works devoted to the topic of this paper, there are some essential differences to our work. Calafiore (2007) designed algorithms to solve mean-variance and mean-absolute-deviation static portfolio allocation under uncertainty. In contrast, Glasserman and Xu (2014) derived an analytical (or semi-analytical) solution for a static portfolio allocation problem under model uncertainty in which the mean-variance selection criterion is used. Glasserman and Xu (2013) also explored a dynamic setting using a factor model. However, their paper explicitly exploits an assumption of multivariate normality for the return vector. In our paper, no such assumption is required.

The paper is organized as follows. In Section 2, we define and formulate the problem of interest. In Section 3, we obtain the optimal robust strategy in a semi-analytical form. Some discussions regarding the model and its computation are presented in Section 4. Section 5 is devoted to numerical examples and discussions about quantifying model risk. We conclude the paper in Section 6.

2. Problem Formulation.

We consider a market of \( d > 1 \) risky assets in which a risk free asset is not available. Suppose that an investor wants to invest all of their money over a fixed investment horizon \([0, N]\) into these \( d \) risky assets. The return of each asset over the \( n \)th period \([n, n+1]\), \( n = 0, ..., N-1 \), is denoted as \( r_{n+1} = (r_{n+1}^1, ..., r_{n+1}^d)^T \), where \( r_{n+1}^i, i = 1, ..., d \) represents the return of the \( i \)th asset over the \( n \)th period. We assume that all random quantities are defined on a filtered complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})\) with the sample space \( \Omega \), the sigma-algebra \( \mathcal{F} \), filtration \((\mathcal{F}_n)\), and the probability measure \( \mathbb{P} \), where the sigma-algebra \( \mathcal{F}_n = \sigma(r_m, 1 \leq m \leq n) \) and \( \mathcal{F}_0 \) is
trivial. Moreover, the return vector $r_{n+1}$ has finite second moments in the $L^2$ sense, i.e., $\mathbb{E}(|r_{n+1}|^2) < \infty$.

At each time $n = 0, ..., N - 1$, the investor re-balances the portfolio using a re-balancing strategy $u = (u_0, ..., u_{N-1})^T$, where $u_i^n \in \mathbb{R}$, $i = 1, ..., d$ denotes the proportional allocation of the wealth of the investor into the $i$th asset. We denote by $\mathcal{U}^0$ the set of admissible strategies at time 0 such that all $u_n$, where $n = 0, ..., N - 1$, take values in the set

$$U = \{ u \in \mathbb{R}^d : 1^T u = 1 \}.$$ 

For $m > 0$, we use $\mathcal{U}^m$ to denote the set of admissible sub-strategies $u^m = (u_n)_{n \geq m}$ such that all $u_n, n = m, ..., N - 1$, take values in the set $U$.

Let $W_n$ denote the wealth of the investor at time $n$, where $n = 0, ..., N$. We assume that $W_n$ and $r_{n+1}$ are independent. This can be achieved, for example, by assuming that $r_n$ are independent, identically distributed. During the period $[n, n+1]$, the investor’s wealth changes to

$$W_{n+1} = W_n (1 + r_{n+1})^T u_n = W_n R_{n+1}^T u_n,$$

where $R_{n+1} = 1 + r_{n+1}$. For $x \in \mathbb{R}$, at any time $m$, the aim of the investor is to optimize

$$J_{m,x}(u^m) = \mathbb{E}\left( \sum_{n=m}^{N-1} J_{n,W_n}(W_{n+1}) | W_m = x \right),$$

(1)

where

$$J_{n,W_n}(W_{n+1}) = W_{n+1} - \kappa_n \sqrt{\text{Var}_{n,W_n}(W_{n+1})} = W_n \left( R_{n+1}^T u_n - \kappa_n \sqrt{u_n^T \Sigma_n u_n} \right),$$

$$\text{Var}_{n,W_n}(W_{n+1}) = \text{Var}(W_{n+1} | W_n), \quad \text{and} \quad \Sigma_n = \text{Var}(r_{n+1}),$$

where $\text{Var}(r_{n+1})$ is the variance of $r_{n+1}$, and the parameter $\kappa_n$ characterizes the risk aversion of the investor. The above criterion is a multi-period selection criterion of mean-standard-deviation (MSD) type. We note that because of the scaling property of the single period mean-standard-deviation criterion, the optimization of the intermediate wealth contributes directly to the optimization of the terminal wealth. For more properties and discussions of this objective we refer to Bannister et al. (2016).

**Remark 1.** One may note that we do not include any discounting here to reflect the time value of money. This is because we assume a market of risky assets only. Since no risk free asset is available, disregarding discounting would be appropriate.
The value function of this control problem takes the form
\[ V(m, x) = \sup_{u^m \in U^m} J_{m,x}(u^m). \tag{2} \]

We will use the Kullback-Leibler (KL) divergence
\[ R(\mathcal{E}) = \mathbb{E}\left( \mathcal{E}\log \mathcal{E} \right), \]
where \( \mathcal{E} \) is the ratio of the density of an alternative distribution to the density of the nominal distribution, as a deviation measure between different distributions. For the reader’s convenience we recall the concept of the KL divergence in Appendix A. Now, for a given \( \eta > 0 \), a KL divergence ball is defined as
\[ B_\eta = \{ \mathcal{E} : R(\mathcal{E}) \leq \eta \}. \tag{3} \]

Next, to quantify the model risk for the investor, we formulate a robust version of the problem in (2). We first define a sequence of KL divergence balls
\[ B_{\eta_n} = \{ \mathcal{E} : R(\mathcal{E}) \leq \eta_n \}, \quad \text{where } n = 0, ..., N - 1. \]

One may note that all moments through this paper are defined with respect to the nominal distribution.

Given any starting time \( m = 0, ..., N - 1 \), we denote the set of \( \mathcal{E}^m = (\mathcal{E}_m, ..., \mathcal{E}_{N-1}) \) such that each \( \mathcal{E}_n \in B_{\eta_n} \), where \( n = m, ..., N - 1 \), by \( \mathcal{B}^m \). Here, \( \mathcal{E}_n, n = m, ..., N - 1 \), is the ratio of the density of an alternative distribution to the density of the nominal distribution over \([n, n + 1]\). The robust version of (2) is defined then by
\[ V(m, x) = \sup_{u^m \in U^m} \inf_{\mathcal{E}^m \in \mathcal{B}^m} J_{m,x}(\mathcal{E}^m, u^m), \tag{4} \]
where
\[
J_{m,x}(\mathcal{E}^m, u^m) = \mathbb{E}\left( \mathcal{E}_m W_m \left( R_{m+1}^T u_m - \kappa_m \sqrt{u_m^T \Sigma_m u_m} \right) \right.
\]
\[
+ \sum_{n=m+1}^{N-1} e^{-\eta_n c_n \kappa_n} \mathcal{E}_n J_{n,x}(W_n(W_{n+1}) | W_m = x),
\]
where \( c_n \) are scaling parameters. Note that if \( m \geq N - 1 \), the summation term is set to zero.
Remark 2. Here we take the infimum over the set of all possible distributions within a KL ball. This corresponds to the worst case scenario. We then find the best strategy under the worst case scenario. The robustification process has an interpretation from game theory. For readers interested in finding more about this interpretation, we refer to Glasserman and Xu (2014).

Remark 3. At time $n$, we scale the future payoff by a factor $e^{-\eta_n c_n \kappa_n}$. The term is added for mathematical convenience and guarantees the existence of an optimal solution. Indeed, we reduce the impact of the future uncertainty on the current stage according to the investor’s risk aversion and the radius of the divergence ball in the following period. It is worth noting that as the radius of the divergence ball $\eta_n \to 0$, we return to the non-robust case.

3. Semi-Analytical Optimal Solution under KL Divergence.

To solve the robust control problem in (4), we apply a strategy that we call a strongly time consistent optimal robust strategy. It represents a robustified version of a strong time consistent optimal strategy inspired by Kang and Filar (2006) (see also (Bannister et al., 2016, definition 2)). The exact definition is given below.

Definition 1. Given any starting time $m = 0, ..., N - 1$, a strategy $u_{m,*} = (u_m^*, ..., u_{N-1}^*)$ is said to be a strongly time consistent optimal robust strategy with respect to $J_{m,x}(E^m, u^m)$ if it satisfies the following two conditions.

- **Condition 1**: Let $A^m \subset U^m$ be a set of strategies of the form $u^m = (v, u_{m+1}^*, ..., u_{N-1}^*)$, where $v \in \mathbb{R}^d$ is arbitrary. Then there exists $E^{m,*} \in B^m$ such that

  $$\inf_{E^m \in B^m} J_{m,x}(E^m, \cdot) = J_{m,x}(E^{m,*}(\cdot), \cdot).$$  

- **Condition 2**: For $n = m + 1, ..., N - 1$, $u^n = (u_n, u_{n+1}, ..., u_{N-1}) \in U^n$, there exists $E^{n,*} \in B^n$ such that

  $$\sup_{u^n \in U^n} J_{n,x}(E^{n,*}(u^n), \cdot) = J_{n,x}(E^{n,*}(u^n), u^n).$$

If only **Condition 1** is satisfied then we say that the strategy is a weakly time consistent optimal robust strategy with respect to $J_{m,x}(\cdot)$. 

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Remark 4. The time consistency that we defined here refers to the time consistency of a strategy with respect to the particular criterion that we choose. There are other definitions of time consistency such as the time consistency of a selection criterion itself (see Chen et al., [2013, definition 2]).

Since the value function of the robust control problem given by (4) is separable (in the sense that it can be written as a sum of expectations), by a similar argument as in the proof of (Chen et al., 2013, theorem 3) we know that a weakly time consistent optimal strategy, which can be found by period-wise optimization, is also a strongly time consistent optimal strategy.

The rest of this section is devoted to the following theorem and its proof. This theorem summarizes one of our main findings, i.e., a system of nonlinear equations that an optimal strategy should satisfy.

Theorem 1. Suppose that \((u^*_m, m = 0, 1, \ldots, N-1)\) is a strategy where there exists a sequence \((\theta^*_m, m = 0, 1, \ldots, N-1)\) such that

\[
\mathbb{E}\left( \exp \left(-R_{m+1}^T u_m^* \frac{2}{\theta_m^*} \right) \right) < \infty,
\]

and

\[
u^*_m = S^*_m \left( \Sigma^{-1}_m X^*_m - \frac{b^*_m \Sigma^{-1}_m}{a^*_m} \right) + \frac{\Sigma^{-1}_m}{a^*_m}, \quad (9)
\]

\[
S^*_m = \sqrt{1 - \frac{h^*_m}{\kappa_m} + \frac{(b^*_m)^2}{\kappa_m a^*_m}} \sqrt{1 - \frac{1}{\kappa_m} S^*}, \quad (10)
\]

\[
X^*_m = \frac{E\left( \exp \left(-R_{m+1}^T u_m^* \frac{1}{\theta_m^*} \right) R_{m+1} \right)}{E\left( \exp \left(-R_{m+1}^T u_m^* \frac{1}{\theta_m^*} \right) \right) + e^{-\eta_m + \kappa_m + 1 \kappa_m \log(\mathcal{E}_m)}} G_{m+1}(u_{m+1}^*, \theta_{m+1}^*) \mathbb{E}(R_{m+1})}, \quad (11)
\]

\[
E(\mathcal{E}_m \log(\mathcal{E}_m)) = \eta_m, \quad (12)
\]

where

\[
g^*_m = h^*_m - \frac{(b^*_m)^2}{a^*_m}, \quad h^*_m = (X^*_m)^T \Sigma^{-1}_m X^*_m, \quad a^*_m = 1^T \Sigma^{-1}_m 1,
\]

\[
b^*_m = 1^T \Sigma^{-1}_m X^*_m, \quad \epsilon^*_m = \frac{\exp \left(-R_{m+1}^T u_m^* \frac{1}{\theta_m^*} \right)}{E\left( \exp \left(-R_{m+1}^T u_m^* \frac{1}{\theta_m^*} \right) \right)} \text{ P.-a.s.},
\]

\[
G_m(u^*_m, \theta^*_m) = -\theta^*_m \log \mathbb{E}\left( \exp \left(-R_{m+1}^T u_m^* \frac{1}{\theta_m^*} \right) \right) + e^{-\eta_m + \kappa_m + 1 \kappa_m \log(\mathcal{E}_m)} \times
\]

\[
G_{m+1}(u_{m+1}^*, \theta_{m+1}^*) \mathbb{E}(R_{m+1}^T u_m^* \frac{1}{\theta_m^*}) - \kappa_m S^*_m - \eta_m \theta^*_m.
\]
Then, \((u_m^*, \theta_m^*)\) is optimal, and the value function is given by
\[ V(m, x) = xG_m(u_m^*, \theta_m^*), \]
where \(x \in (0, \infty)\).

**Proof.** We proceed by using an induction argument.

**Step 1:** Firstly, let \(m = N - 1\). The optimization problem becomes
\[
\sup_{u_{N-1} \in U} \inf_{E_{N-1} \in \mathcal{B}_{\theta_{N-1}}} \mathbb{E}\left( E_{N-1}W_{N-1}\left( u_{N-1}^T R_N - \kappa_{N-1} \sqrt{u_{N-1}^T \Sigma_{N-1} u_{N-1}} \right) \bigg| W_{N-1} = x \right).
\]
which reduces to
\[
\sup_{u_{N-1} \in U} \inf_{E_{N-1} \in \mathcal{B}_{\theta_{N-1}}} \left( \mathbb{E}(E_{N-1}u_{N-1}^T R_N) - \kappa_{N-1} \sqrt{u_{N-1}^T \Sigma_{N-1} u_{N-1}} \right) \right). \tag{13}
\]
Let us look at the inner optimization problem, i.e.,
\[
\inf_{E_{N-1} \in \mathcal{B}_{\theta_{N-1}}} \left( \mathbb{E}(E_{N-1}u_{N-1}^T R_N) - \kappa_{N-1} \sqrt{u_{N-1}^T \Sigma_{N-1} u_{N-1}} \right). \tag{14}
\]
We can write down the Lagrangian as
\[
L_{N-1}(E_{N-1}, \theta_{N-1}) = \mathbb{E}(E_{N-1}u_{N-1}^T R_N) - \kappa_{N-1} \sqrt{u_{N-1}^T \Sigma_{N-1} u_{N-1}} + \theta_{N-1} \left( \mathbb{E}(E_{N-1} \log(E_{N-1})) - \eta_{N-1} \right).
\]
By setting the derivative (with respect to \(E_{N-1}\)) of the expression under the expectation of the Lagrangian to be equal to zero, we obtain
\[
u_{N-1} R_N + \theta_{N-1} \log(E_{N-1}) + \theta_{N-1} = 0. \quad \mathbb{P}\text{-a.s.,}
\]
Solving the above equation together with the fact that all alternative distributions have a proper density, i.e.,
\[
\mathbb{E}(E_{N-1}) = 1,
\]
we obtain
\[
E_{N-1}^* = \frac{\exp \left( -R_{N-1}^T u_{N-1} \frac{1}{\theta_{N-1}} \right)}{\mathbb{E} \left( \exp \left( -R_{N-1}^T u_{N-1} \frac{1}{\theta_{N-1}} \right) \right)} \quad \mathbb{P}\text{-a.s.,} \tag{15}
\]
for some $\theta_{N-1}$ such that
\[ E\left( \exp \left( - R_N^T u_{N-1}^* \frac{1}{\theta_{N-1}} \right) \right) < \infty. \]

We can verify that (15) is indeed the optimal solution by using a convexity argument. We refer to the proof of Lam, 2016, proposition 3.1) for more details.

Now, since the set
\[ \{ \mathcal{E} : \mathcal{R}(\mathcal{E}) < \eta_{N-1} \} \]
is not empty, by Ben-Tal et al., 1988, theorem 2.1, strong duality holds. This implies (see Boyd and Vandenberghe, 2004, pp. 242–243) that the optimal solution $\mathcal{E}_{N-1}^*$ and its corresponding $\theta_{N-1}^*$ satisfies the following system:
\[
\begin{align*}
\theta_{N-1} \left( E(\mathcal{E}_{N-1}^* \log(\mathcal{E}_{N-1}^*)) - \eta_{N-1} \right) &= 0, \\
E(\mathcal{E}_{N-1}^* \log(\mathcal{E}_{N-1}^*)) &\leq \eta_{N-1}, \\
\theta_{N-1} &> 0.
\end{align*}
\]

We denote the solution $\theta_{N-1}$ of this system as $\theta_{N-1}^*$.

Next, with (15), the optimization problem (13) becomes
\[
\sup_{u_{N-1} \in U} \left( - \theta_{N-1}^* \log E\left( \exp \left( - R_N^T u_{N-1} \frac{1}{\theta_{N-1}} \right) \right) \right)
- \kappa_{N-1} \sqrt{u_{N-1}^T \Sigma_{N-1} u_{N-1} - \eta_{N-1} \theta_{N-1}^*}. \tag{16}
\]

One may note that the expression under the supremum is actually the optimal dual of (13). This can be confirmed by applying Ben-Tal et al., 1988, lemma 2.1).

By Lemma B.1 (see Appendix B), we obtain the unique optimum of (16) which satisfies the following system of nonlinear equations:
\[
\begin{align*}
u_{N-1}^* &= \frac{S_{N-1}^*}{\kappa_{N-1}} \left( \Sigma_{N-1}^{-1} X_{N-1}^* - \frac{b_{N-1}^* \Sigma_{N-1}^{-1} 1}{a_{N-1}} \right) + \frac{\Sigma_{N-1}^{-1} 1}{a_{N-1}}, \\
S_{N-1}^* &= \sqrt{\frac{1}{\kappa_{N-1}^2 + \frac{(\theta_{N-1}^*)^2}{\kappa_{N-1} a_{N-1}}} \left( 1 - \frac{1}{\kappa_{N-1} \theta_{N-1}^*} \right) \frac{1}{\kappa_{N-1} a_{N-1}}}. \tag{16}
\end{align*}
\]

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\[ X_{N-1}^* = \frac{\mathbb{E}(\exp(-R_{N-1}^T u_{N-1}^* \frac{1}{\theta_{N-1}^*})) R_N}{\mathbb{E}(\exp(-R_{N-1}^T u_{N-1}^* \frac{1}{\theta_{N-1}^*}))} . \]

This follows from the proof of \cite{Bannister et al., 2016, theorem 4}. Thus, the corresponding value function is given by

\[ V(N-1, x) = x G_{N-1}(u_{N-1}^*, \theta_{N-1}^*), \]

where

\[ G_{N-1}(u_{N-1}^*, \theta_{N-1}^*) = -\theta_{N-1}^* \log \mathbb{E}(\exp(-R_{N-1}^T u_{N-1}^* \frac{1}{\theta_{N-1}^*})) - \kappa_{N-1} S_{N-1}^* - \eta_{N-1} \theta_{N-1}^*. \]

**Step 2:** Now, when \( m = N - 2 \), the optimization problem becomes

\[
\sup_{u_{N-2} \in U} \inf_{\mathcal{E}_{N-2} \in \mathcal{B}_{\eta_{N-2}}} \left( \mathbb{E}(\mathcal{E}_{N-2} R_{N-1}^T u_{N-2}) + e^{-\eta_{N-1} c_{N-1}} \kappa_{N-1} \times G_{N-1}(u_{N-1}^*, \theta_{N-1}^*) \mathbb{E}(R_{N-1}^T u_{N-2}) \right.
\]

\[
\left. -\kappa_{N-2} \sqrt{u_{N-2}^T \Sigma_{N-2} u_{N-2}} \right). \tag{17}
\]

Again, let us write down the Lagrangian

\[ L_{N-2}(\mathcal{E}_{N-2}, \theta_{N-2}) = \mathbb{E}(\mathcal{E}_{N-2} R_{N-1}^T u_{N-2}) + e^{-\eta_{N-1} c_{N-1}} \kappa_{N-1} \times G_{N-1}(u_{N-1}^*, \theta_{N-1}^*) \mathbb{E}(R_{N-1}^T u_{N-2}) \]

\[ -\kappa_{N-2} \sqrt{u_{N-2}^T \Sigma_{N-2} u_{N-2}} + \theta_{N-2} \left( \mathbb{E}(\mathcal{E}_{N-2} \log(\mathcal{E}_{N-2})) - \eta_{N-2} \right). \]

As in **Step 1**, we obtain the optimal \( \mathcal{E}_{N-2}^* \):

\[
\mathcal{E}_{N-2}^* = \frac{\exp\left(-R_{N-1}^T u_{N-2}^* \frac{1}{\theta_{N-2}^*}\right)}{\mathbb{E}(\exp(-R_{N-1}^T u_{N-2}^* \frac{1}{\theta_{N-2}^*}))} \quad \text{P-a.s.,} \tag{18}
\]

by solving

\[ u_{N-2}^T R_{N-1} + \theta_{N-2} \log(\mathcal{E}_{N-2}) + \theta_{N-2} = 0, \quad \text{P-a.s.,} \]

together with the fact that all alternative distributions have a proper density, i.e.,

\[ \mathbb{E}(\mathcal{E}_{N-2}) = 1. \]
Moreover, the optimal $E_{N-2}^*$ and its associated optimal $\theta_{N-2}$ satisfy the following system:

\[
\theta_{N-2} \left( \mathbb{E}(E_{N-2} \log(E_{N-2})) - \eta_{N-2} \right) = 0, \\
\mathbb{E}(E_{N-2} \log(E_{N-2})) \leq \eta_{N-2}, \\
\theta_{N-2} > 0.
\]

We denote such $\theta_{N-2}$ as $\theta_{N-2}^*$. 

Now, with (18), the optimization problem (17) becomes

\[
\sup_{u_{N-2} \in U} \left( -\theta_{N-2} \log \mathbb{E} \left( \exp \left( -R_{N-1}^T u_{N-2} \frac{1}{\theta_{N-2}} \right) \right) + e^{-\eta_{N-1} \epsilon_{N-1} N^{-1} \log N} \times \\
G_{N-1}(u_{N-1}^*, \theta_{N-1}^*) \mathbb{E}(R_{N-1}^T u_{N-2}) \\
- \kappa_{N-2} \sqrt{u_{N-2}^T \Sigma_{N-2} u_{N-2} - \eta_{N-2} \theta_{N-2}^*}. \right)
\]

By Lemma B.1, we obtain the unique optimum of (19) which satisfies the following system of nonlinear equations:

\[
u_{N-2}^* = \frac{S_{N-2}^*}{\kappa_{N-2}} \left( \frac{1}{a_{N-2}} \Sigma_{N-2}^{-1} X_{N-2}^* - \frac{b_{N-2}^*}{a_{N-2}} \Sigma_{N-2}^{-1} \right), \\
S_{N-2}^* = \sqrt{\frac{1}{1 - \frac{b_{N-2}^*}{\kappa_{N-2} a_{N-2}}}} = \sqrt{\frac{1}{1 - \frac{\kappa_{N-2}}{\kappa_{N-2} - \theta_{N-2}^*}}}, \\
X_{N-2}^* = \frac{\mathbb{E}(\exp(-R_{N-1}^T u_{N-2}^* \frac{1}{\theta_{N-2}^*}) R_{N-1}^T)}{\mathbb{E}(\exp(-R_{N-1}^T u_{N-2}^* \frac{1}{\theta_{N-2}^*}))} \times \\
+ e^{-\eta_{N-1} \epsilon_{N-1} N^{-1} \log N} G_{N-1}(u_{N-1}^*, \theta_{N-1}^*) \mathbb{E}(R_{N-1}).
\]

This again follows from the proof of \cite{Bannister et al., 2016}, theorem 4). Thus, the corresponding value function is given by

\[
V(N-2, x) = x \mathbb{E} G_{N-2}(u_{N-2}^*, \theta_{N-2}^*),
\]

where

\[
G_{N-2}(u_{N-2}^*, \theta_{N-2}^*) = -\theta_{N-2}^* \log \mathbb{E} \left( \exp \left( -R_{N-1}^T u_{N-2}^* \frac{1}{\theta_{N-2}^*} \right) \right) \\
+ e^{-\eta_{N-1} \epsilon_{N-1} N^{-1} \log N} G_{N-1}(u_{N-1}^*, \theta_{N-1}^*) \times \\
\mathbb{E}(R_{N-1}^T u_{N-2}^*) - \kappa_{N-2} S_{N-2}^* - \eta_{N-2} \theta_{N-2}^*.
\]
**Step 3**: Next, for \( m = N - 2, N - 1, \ldots, 1, 0 \), we use a backward induction step. Assume that the claim holds for \( m = n + 1 \). We need to show that it holds for \( m = n \). When \( m = n \), the optimization problem becomes

\[
\sup_{u_n \in U} \inf_{\mathcal{E}_n \in \mathcal{B}_n} \left( \mathbb{E}(\mathcal{E}_n R_{n+1}^T u_n) + e^{-\eta_{n+1}c_{n+1} + \kappa_n + 1} G_{n+1}(u_{n+1}^*, \theta_{n+1}^*) \mathbb{E}(R_{n+1}^T u_n) \right)
\]

\[
-\kappa_n \sqrt{u_n^T \Sigma_n u_n} \right).
\]

The Lagrangian can be written as

\[
L_n(\mathcal{E}_n, \theta_n) = \mathbb{E}(\mathcal{E}_n R_{n+1}^T u_n) + e^{-\eta_{n+1}c_{n+1} + \kappa_n + 1} G_{n+1}(u_{n+1}^*, \theta_{n+1}^*) \times 
\mathbb{E}(R_{n+1}^T u_n) - \kappa_n \sqrt{u_n^T \Sigma_n u_n} + \theta_n \left( \mathbb{E}(\mathcal{E}_n \log(\mathcal{E}_n)) - \eta_n \right).
\]

As in **Step 1** or **Step 2**, we obtain the optimal \( \mathcal{E}_n^* \):

\[
\mathcal{E}_n^* = \frac{\exp \left( -\frac{R_{n+1}^T u_n}{\theta_n} \right)}{\mathbb{E} \left( \exp \left( -\frac{R_{n+1}^T u_n}{\theta_n} \right) \right)} \quad \mathbb{P}\text{-a.s.}, \tag{21}
\]

by solving

\[
u_n^T R_{n+1} + \theta_n \log(\mathcal{E}_n) + \theta_n = 0, \quad \mathbb{P}\text{-a.s.},
\]

together with the fact that the alternative distributions have a proper density, i.e.,

\[
\mathbb{E}(\mathcal{E}_n) = 1.
\]

Moreover, the optimal \( \mathcal{E}_n^* \) and its associated optimal \( \theta_n \) satisfy the following system:

\[
\theta_n \left( \mathbb{E}(\mathcal{E}_n \log(\mathcal{E}_n)) - \eta_n \right) = 0,
\]

\[
\mathbb{E}(\mathcal{E}_n \log(\mathcal{E}_n)) \leq \eta_n,
\]

\[
\theta_n > 0.
\]

We denote such solution as \( \theta_n^* \). Now, with (21), the optimization problem (17) becomes

\[
\sup_{u_n \in U} \left( -\theta_n^* \log \mathbb{E} \left( \exp \left( -\frac{R_{n+1}^T u_n}{\theta_n^*} \right) \right) + e^{-\eta_{n+1}c_{n+1} + \kappa_n + 1} \times \right.
\]

\[
G_{n+1}(u_{n+1}^*, \theta_{n+1}^*) \mathbb{E}(R_{n+1}^T u_n) - \kappa_n \sqrt{u_n^T \Sigma_n u_n - \eta_n \theta_n^*} \right).
\]

(22)
By Lemma B.1, we obtain the unique optimum of (22) which satisfies the following system of nonlinear equations:

\[
\begin{align*}
    u^*_n &= \frac{S^*_n}{\kappa_n} \left( \Sigma^{-1}_n X^*_n - \frac{b^*_n \Sigma^{-1}_n}{a_n} \right) + \frac{\Sigma^{-1}_n}{a_n}, \\
    S^*_n &= \sqrt{1 - \frac{\alpha_n}{\kappa_n} + \left( \frac{b^*_n}{\kappa_n a_n} \right)^2} = \sqrt{1 - \frac{1}{\kappa_n^2 g^*_n}}, \\
    X^*_n &= \frac{E \left( \exp(-R^T_{n+1} u^*_n \frac{1}{\theta^*_n}) R_{n+1} \right)}{E \left( \exp(-R^T_{n+1} u^*_n \frac{1}{\theta^*_n}) \right)} + e^{-\eta_{n+1} \epsilon_{n+1} \kappa_{n+1}} G_{n+1}(u^*_{n+1}, \theta^*_{n+1}) E(R_{n+1}).
\end{align*}
\]

The corresponding value function is given by

\[ V(n, x) = x G_n(u^*_n, \theta^*_n), \]

where

\[
G_n(u^*_n, \theta^*_n) = -\theta^*_n \log E \left( \exp \left( - R^T_{n+1} u^*_n \frac{1}{\theta^*_n} \right) \right) + e^{-\eta_{n+1} \epsilon_{n+1} \kappa_{n+1}} G_{n+1}(u^*_{n+1}, \theta^*_{n+1}) E(R_{n+1}^T u^*_n) - \kappa_n S^*_n - \eta_n \theta^*_n.
\]

This completes the proof. \( \square \)

4. Some Discussions of the Model.

In this section we discuss some modelling, theoretical and computational issues that may arise when we implement our approach. Also, we briefly discuss how to handle short selling constraints and a generalization to the case of \( \alpha \)-divergence.

Firstly, it is worth noting that the uncertainty of the underlying distribution only enters into the expectation part, and the standard deviation part is added as a further penalization. Mathematically, it is difficult to include the uncertainty in the standard deviation part as trying to do so leads to losing the time consistency property. From a modelling and risk management perspective, since the error of estimation in the expectation part is far more serious than the standard deviation (see, e.g., Chopra and Ziemba (1993)), handling uncertainty in the expectation part is more important.
Secondly, as discussed in Bannister et al. (2016), the strategy calculated in Theorem 1 is optimal provided that the wealth stays positive. Of course, there is no guarantee that this will always be the case. However, depending on risk tolerance, the investor may as well be happy to adopt such a strategy if the probability that the wealth stays positive exceeds a certain threshold. For more detailed discussions, we refer to Bannister et al. (2016). To obtain such an optimal strategy, and to determine whether the investor should adopt such a strategy, we modify (Bannister et al., 2016, algorithm 1). This yields Algorithm A. It is worth noting that unlike algorithm 1, there is no explicitly given lower bound on the risk aversion parameter $\kappa_n$. Instead, we constrain $\kappa_n$, so that the $\kappa_n$ chosen by the investor is a valid risk aversion parameter in the sense that the system of nonlinear equations in Theorem 1 is well defined.

**Algorithm A** Multi-period MSD Robust Portfolio Selection

1: set abandon = false;
2: for $n = N - 1, \ldots, 0$ do
3:   set $W_n = 1$ and select a $\kappa_n > 0$;
4:   solve (9) and (12) simultaneously subject to $1 - \frac{2\kappa_n}{\kappa_n^2} > 0$;
5:   calculate $p_n(u_n) = P(W_{n+1} > 0)$;
6:   if $p_n(u_n) > 1 - \exp(-\kappa_n)$ then
7:     keep the strategy $u_n$;
8:   else
9:     abandon = true;
10: end if
11: end for
12: if abandon == false then
13:   take the investment;
14: else
15:   abandon the investment;
16: end if

In terms of computation, we see that to compute the robust strategy, we have to solve the system (9) - (12) simultaneously which requires evaluation of the expectations in (11) - (12). It is almost impossible to evaluate such expectations directly in this system because of the complicated interdependence of these equations. This difficulty can be resolved by applying a Monte Carlo type approach (see Glasserman and Xu (2014, section 3)). The idea is to replace the theoretical expectations by sample means via simulations. In this way, we end up with a system of nonlinear equations which can then be solved numerically.
It is worth noting that we have assumed that short selling is allowed. In portfolio selection, it is often required to impose short selling constraints. In our case, this corresponds to replacing the set $U$ by

$$U^{\text{short}} = \left\{ u \in \mathbb{R}^d : 1^T u = 1, \ell_i < u^i < b_i, \text{ for } i = 1, ..., d, \text{ and } \ell_i, b_i \in \mathbb{R} \right\}.$$  

Indeed, by a straightforward Kuhn-Tucker argument (see, e.g., Boyd and Vandenberghe (2004)), it is easy to see that we can still find the optimal strategy without losing the semi-analytical form under a short selling restriction. It is worth noting that if we change the strict inequality constraints in $U^{\text{short}}$ to inequality constraints, this adds a further difficulty and we may lose the semi-analytical form of the optimal strategy.

Also note the possibility of a risk free asset in the portfolio, which often occurs in practice and is of interest. Mathematically, it causes difficulty in that the matrix $\Sigma_n$ becomes singular and non-invertible. The work of Landsman and Makov (2012) deals with the presence of a risk free asset in a non-robust single-period scenario. Specifically, their Corollary 1 points out that when a single constraint in the form $1^T u = 1$ is imposed and the risk aversion parameter $\kappa$ is large enough then only a trivial solution exists. The trivial solution implies that one should be fully invested in the risk free asset. Since in our case we are working under the condition of this single constraint, the inclusion of a risk free asset would not provide any further insights.

If further linear equality constraints are imposed then Theorem 1 in Landsman and Makov (2012) tells us that for a specific form of these constraints, under specific assumptions on the distribution of returns and for a large enough $\kappa$ (that is, for a risk-averse investor), that a non-trivial solution can be obtained. This solution corresponds to putting non-zero weights to both the risky and the risk free components. However, at this stage it is difficult to generalize Theorem 1 of Landsman and Makov (2012) for our multi-period robust portfolio selection scenario and this generalization has been left as a future research agenda.

Finally, it is well known that the KL divergence can be generalized to the so-called $\alpha$-divergence (see Appendix A). Our approach can be applied to such a case. However, several issues arise when using $\alpha$-divergence. To have a properly defined worst case distribution, the underlying random variable, i.e., the return of the assets, must be bounded (see Glasserman and Xu, 2016).
2014, proposition 2.3)). Also, it is not clear to us whether the optimal strategy exists for all \( \alpha \). To properly handle the uncertainty in the case where the underlying distribution is heavy tailed, another divergence measure may be needed. We are very interested in this case but it will be dealt with in another paper.

5. Numerical Examples.

In this section, we demonstrate the use of our model to select optimal strategy and to quantify model risk. Suppose our interest is to find the best allocation of a portfolio of three stocks from the customer service industry–Navitas, Domino and Tabcorp–over an investment horizon of 5 days, i.e., \( N = 5 \). The historical daily prices of these stocks traded on the Australian Securities Exchange\(^1\) have been collected over the period 1 Jan 2015 - 31 Dec 2015. The corresponding daily returns form a set of 261 data points. In this section, without loss of generality, a few assumptions will be made. The risk aversion parameter of the investor \( \kappa_n \) is assumed to be constant and equals to three (i.e., \( \kappa_n = 3 \) for \( n = 0, 1, 2, 3, 4 \)). The initial wealth is assumed to be one dollar, i.e., \( W_0 = 1 \). The random daily returns are assumed to be independent and identically distributed over the investment horizon and have mean \( \mu \) and covariance matrix \( \Sigma \) under the nominal distribution.

5.1. Comparison of Optimal Robust and Non-Robust Portfolio.

Let us first consider a special case of uncertainty in distribution (i.e., the uncertainty in parameters) and compare the performance of the optimal robust and non-robust strategies. The convenience of this simple scenario is the fact that the closed form formula for the KL divergence is sometimes available. For example, if the nominal distribution is a \( d \)-dimensional multivariate normal distribution with mean \( \mu \) and covariance matrix \( \Sigma \), and the worst case distribution is a \( d \)-dimensional multivariate normal distribution with mean \( \bar{\mu} \) and covariance matrix \( \bar{\Sigma} \), then the KL divergence can be calculated as (see, e.g., (Nielsen et al., 2017, p. 296)):

\[
\mathcal{R}(\mathcal{E}) = \frac{1}{2} \left( \text{trace}(\Sigma^{-1} \bar{\Sigma}) + (\mu - \bar{\mu})^T \Sigma^{-1}(\mu - \bar{\mu}) - d + \log \left( \frac{|\Sigma|}{|\bar{\Sigma}|} \right) \right),
\]

(23)

where \(|\cdot|\) denotes the determinant of a matrix. For the purpose of illustration, we will consider the case where \( \bar{\mu} = \gamma \times \mu \) for some \( \gamma \in \mathbb{R} \) and \( \bar{\Sigma} = \Sigma \). Based

\(^1\)Data obtained from Yahoo Finance https://au.finance.yahoo.com/
on the collected data, we calculate the expected returns and the covariance matrix of the returns (under the nominal distribution) as listed below:

\[
\mu = \begin{pmatrix} 0.0007 \\ 0.0022 \\ 0.0016 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.0003 & 0.0001 & 0.0001 \\ 0.0001 & 0.0004 & 0.0001 \\ 0.0001 & 0.0001 & 0.0003 \end{pmatrix}.
\]

Formally, the \( \mu \) and \( \Sigma \) are estimates, however, for simplicity of notation, we do not use \( \hat{\mu} \) and \( \hat{\Sigma} \).

Since in the worst case scenario for the model disturbance, the alternative distribution is on the boundary of the KL divergence ball, then the divergence between the two models is equal to \( \eta_n \) (see Theorem 1). Without loss of generality, let us assume the radius of divergence ball \( \eta_n \) to be constant over the entire investment horizon. This allows us to find a time homogeneous \( \gamma \). For simplicity, we drop the time dependence of \( \eta_n \) and simply write \( \eta \). In addition, from now on, we will always choose \((c_2, c_3, c_4, c_5)\) such that \((c_2 \eta \kappa, c_3 \eta \kappa, c_4 \eta \kappa, c_5 \eta \kappa) = (7.5, 8.0, 8.5, 9.0)\). The choice of such numbers seems to be arbitrary but one may consider these values as the investor’s risk tolerance for uncertainty of distribution (in contrast to \( \kappa \) which is the risk aversion of the investor’s preference for a fixed distribution). Thus, it solely depends on the investor’s choice. As a consequence, the investor will have their own freedom to choose the amount of penalization (i.e. the effect of \((-c_n \eta \kappa\)) that they would like to take when selecting the portfolio.

Table 1: Performance of robust and non-robust optimal solution: Comparison 1

| \( \gamma \) | \( \eta \) | \( \eta \) | number of times robust outperforms non-robust | \% |
|---------------|---------------|---------------|---------------------------------|----|
| 0.2139        | 0.0050        | 244429        | 48.89%                          |
| -1.4859       | 0.0500        | 285828        | 57.17%                          |
| -2.5156       | 0.1000        | 309814        | 61.96%                          |
| -3.9718       | 0.2000        | 336583        | 67.32%                          |
| -5.0892       | 0.3000        | 362909        | 72.58%                          |
| -6.0312       | 0.4000        | 378952        | 75.79%                          |
| -6.8611       | 0.5000        | 391459        | 78.29%                          |

Table 2: Performance of robust and non-robust optimal solution: Comparison 2

| \( \gamma \) | \( \eta \) | \( \mathbb{E}(W_N) \) | \( \mathbb{E}(W_N) - W_0 \) | \( \mathbb{E}(W_N) - W_0 \) | \% |
|---------------|---------------|------------------|------------------|------------------|----|
| -1.4859       | 0.0500        | 0.9891           | -0.0012          | -0.3253          | 0.0428 |
| -2.5156       | 0.1000        | 0.9816           | 0.0026           | -0.5267          | 0.1010 |
| -3.9718       | 0.2000        | 0.9711           | 0.0050           | -0.7831          | 0.2129 |
| -5.0892       | 0.3000        | 0.9631           | 0.0084           | -0.9073          | 0.3735 |
| -6.0312       | 0.4000        | 0.9564           | 0.0114           | -0.9965          | 0.5259 |
| -6.8611       | 0.5000        | 0.9505           | 0.0145           | -1.0526          | 0.6837 |
Now, suppose that we calculate under the worst case distribution. By generating data from this distribution, we compare the performance under the optimal robust and non-robust strategies for different values of $\eta$. The optimal robust strategies are calculated by using 500,000 Monte Carlo simulations. Then, we simulate 500,000 daily return paths (over 5 days), and calculate the number of times, as well as the corresponding percentage, when the simulated terminal wealth under the robust case out-performs the non-robust case (see Table 1). Figure 1 shows how the out-performance varies for different values of $\eta$. Other comparison metrics that we have also calculated include the expected terminal wealth under both the robust and the non-robust case $^2$ and the ratio of the difference between the expected terminal wealth and the initial wealth to the standard deviation of the terminal wealth (see Table 2). These are plotted in Figure 2 and Figure 3 respectively.

From Table 1 we see that when $\eta$ is small, there is more than 50% chance for the non-robust strategy to outperform the robust one. The corresponding expected terminal wealth under the robust case is also higher than under the non-robust case. This suggests that if the worst case distribution is close to the nominal one (in the sense of a small enough KL divergence), it may be hard to separate the two distributions and thus it may be appropriate to use the nominal distribution. However, when $\eta$ is large, it is clear that the robust strategy starts outperforming the non-robust one (with respect to each of the criteria that make sense in our discussion). This suggests that

\[^2\text{the optimal non-robust strategy can be calculated by following Bannister et al. (2016).}\]
when the worst case distribution is far from the nominal one, it is worth switching to the robust strategy. Furthermore, we notice that a profit is made when the radius is small and a loss is made when the radius is large. Thus, another suggestion could be that the optimal robust strategy is protecting against a loss of a portfolio due to the distribution uncertainty and its impact is more apparent if a portfolio made a loss.

Another case where we have a closed form formula for the KL divergence is when both the nominal and alternative distributions are multivariate skew-normal. We will see in what follows that in the extreme case where the nominal distribution degenerates to normal, we obtain similar comparison results as in the first example.

Let a nominal distribution be a multivariate skew-normal with location parameter $\mu$, scale parameter $\Sigma$ and skewness parameter $\xi$. Next, we take the worst case distribution to be a multivariate skew-normal with location parameter $\bar{\mu}$, scale parameter $\bar{\Sigma}$ and skewness parameter $\bar{\xi}$. The $d$-dimensional versions of these models are denoted by $Y \sim SN_d(\mu, \Sigma, \xi)$ and $\bar{Y} \sim SN_d(\bar{\mu}, \bar{\Sigma}, \bar{\xi})$ respectively. The closed form KL divergence is summarized in the following result, for which the proof is in Appendix C.

**Proposition 1.** Given a nominal distribution $Y \sim SN_d(\mu, \Sigma, \xi)$ and an alternative distribution $\bar{Y} \sim SN_d(\bar{\mu}, \bar{\Sigma}, \bar{\xi})$, then the KL divergence between
Figure 3: robust vs non-robust: ratio of the difference between the expected terminal wealth and the initial wealth to the standard deviation of the terminal wealth.

The nominal and the alternative distributions is given by:

\[
\mathcal{R}_{\text{skew}}(E) = \mathcal{R}(E) + 2\sqrt{\frac{2}{\pi}} (\bar{\mu} - \mu)^T \Sigma^{-1} \Sigma^{\frac{1}{2}} \xi - \mathbb{E}\left( \log\left( 2\Phi(\Xi_1 | 1 - \xi^T \xi) \right) \right) + \mathbb{E}\left( \log\left( 2\Phi(\Xi_2 | 1 - \xi^T \xi) \right) \right),
\]

where \( \mathcal{R}(E) \) is given in (23), \( \Phi(\cdot | \sigma^2) \) is the cumulative distribution function of a normal random variable with mean 0 and variance \( \sigma^2 \), and

\[
\Xi_1 \sim SN_1(0, \xi^T \xi, \sqrt{\xi^T \xi}),
\]

\[
\Xi_2 \sim SN_1\left( \xi^T \Sigma^{-\frac{1}{2}} (\mu - \bar{\mu}), \xi^T \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}} \xi, \frac{\xi^T \Sigma^{-\frac{1}{2}} \bar{\Sigma} \Sigma^{-\frac{1}{2}} \xi}{\sqrt{\xi^T \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}} \xi}} \right).
\]

It is worth noting that if both skewness parameters are equal to zero, we retain (23). For more detailed discussions of (24) and of the multivariate skew-normal distribution, we refer to Contreras-Reyes and Arellano-Valle (2012); Arellano-Valle and Genton (2005).

For illustration, we take a \( d \)-dimensional multivariate normal distribution with mean \( \mu \) and covariance matrix \( \Sigma \) as the nominal distribution. The worst case distribution is assumed to be a \( d \)-dimensional multivariate skew-normal distribution with a location parameter \( \bar{\mu} \), a scale parameter \( \bar{\Sigma} \) and a skewness parameter \( \bar{\xi} \), such that \( \bar{\mu} = \mu \), and \( \bar{\Sigma} = \Sigma \). We will choose \( \bar{\xi} \).
such that the mean of the worst case distribution is changed to \( \beta \)\% of the mean of the nominal distribution. We note that the location parameter \( \mu \), and the scale parameter \( \Sigma \) are not the mean and the covariance matrix of the multivariate skew-normal. The reason that we choose this scenario is to show that the robust case is indeed a strategy to safeguard against losses due to a shift of the mean. This can be easily seen from the optimization procedure. Since we assume that \( \eta \) is constant over time, we also have the same worst case distributions across time.

| \( \beta \)\% | \( \xi \) | \( \eta \) | number of times robust outperforms non-robust | % |
|---|---|---|---|---|
| -78.30\% | -0.0135 | 0.0050 | 244577 | 48.92\% |
| -242.54\% | -0.3042 | 0.0500 | 284998 | 57.00\% |
| -334.61\% | -0.4196 | 0.1000 | 307050 | 61.41\% |
| -449.99\% | -0.5643 | 0.2000 | 329841 | 65.97\% |
| -523.30\% | -0.6563 | 0.3000 | 351005 | 70.20\% |
| -572.42\% | -0.7178 | 0.4000 | 361064 | 72.21\% |
| -604.20\% | -0.7576 | 0.5000 | 366798 | 73.36\% |

Figure 4: the number of times robust outperforms non-robust

We again run 500,000 simulations and calculate the number of times, as
Table 4: Performance of robust and non-robust optimal solution (skew-normal): Comparison 2

| β%     | ξ         | η         | $E(W_N)$ | $E(W_N)$ - $W_0$ | $\sqrt{Var(W_N)}$ | robust | non-robust | difference | robust | non-robust | difference |
|--------|-----------|-----------|----------|------------------|-------------------|--------|------------|------------|--------|------------|------------|
| -78.30%| -0.0135   | -0.0092   | -0.0759  | 0.0050           | 1.0016            | 0.0000 | 0.0528     | 0.0557     | 0.0029 |
| -242.54%| -0.0417  | -0.3942   | -0.2351  | 0.0500           | 0.9907            | 0.0011 | -0.3232    | -0.3682    | 0.0450 |
| -334.61%| -0.0574  | -0.4196   | -0.3244  | 0.1000           | 0.9853            | 0.0024 | -0.5187    | -0.6316    | 0.1129 |
| -449.99%| -0.0773  | -0.5641   | -0.4362  | 0.2000           | 0.9789            | 0.0044 | -0.7541    | -1.0143    | 0.2602 |
| -523.30%| -0.0893  | -0.6563   | -0.5073  | 0.3000           | 0.9762            | 0.0070 | -0.8334    | -1.3091    | 0.4757 |
| -572.42%| -0.0983  | -0.7178   | -0.5549  | 0.4000           | 0.9747            | 0.0090 | -0.8661    | -1.5428    | 0.6767 |
| -604.20%| -0.1038  | -0.7576   | -0.5857  | 0.5000           | 0.9742            | 0.0108 | -0.8565    | -1.7159    | 0.8594 |

well as the corresponding percentage when the simulated terminal wealth under the robust case outperforms the non-robust case for a range of divergences (see Table 3). The divergences are calculated by (24) using Monte Carlo simulation with 500,000 simulations. Figure 4 shows how the outperformance varies for different divergences. We also calculate the expected terminal wealth under both the robust and the non-robust case, and the ratio of the difference between the expected terminal wealth and the initial wealth to the standard deviation of the terminal wealth (see Table 4). These have been plotted in Figure 5 and Figure 6, respectively.

We notice that the performance comparison in Figure 5 and Figure 6 exhibits a similar pattern like in the previous example. The number of times that the optimal robust strategy outperforms the non-robust strategy increases as the radius of the divergence increases. The values in the difference columns in Table 4 also tend to increase as the radius of the KL ball increases. Hence, a larger uncertainty (i.e., a larger radius) makes the advantage of the optimal robust strategy more apparent. As in the first example, we will choose the non-robust optimal strategy when the divergence is “small” and choose the robust optimal strategy when the divergence is “large”.

Up to this point, we only discussed when to choose an optimal non-robust strategy and when to adopt the robust one. It is worth noting that,
although in some cases it may be worth choosing the non-robust optimal strategy, we still need to quantify the amount of model risk involved in this action. In the next section, we define model risk through the standard definition of risk in risk management, that is, as the quantile of a ‘profit-loss distribution’ (see, e.g., [Connor et al., 2010, p. 12]). We also provide a procedure to estimate the model risk by using empirical data.

5.2. Quantification of Model Risk with Empirical Data.

In this section, we discuss how to quantify model risk regarding our optimal portfolio when only the empirical data is available and the true distribution is not known.

Before we go into details, let us define model risk in terms of the quantile of a ‘profit-loss distribution’. Let $Q$ denote the probability measure of a worst case distribution, i.e., the empirical measure. The optimal portfolio is said to have a model risk of $\theta$ with a confidence level $q$ if

$$Q \left( W_N^{non-robust} - W_N^{robust} \leq -\theta \right) = 1 - q.$$  

In other words, we define model risk as the $(1 - q)$th-quantile of the distribution of the difference between the terminal wealth under the non-robust strategy and the robust strategy.

Now, to quantify the model risk of our optimal portfolio, we divide the dataset into two subsets. The first subset, which we call dataset 1 (and it
contains 201 data points), is used to estimate the expected value and the covariance matrix of the nominal distribution, which yields:

\[ \tilde{\mu} = \begin{pmatrix} 0.0009 \\ 0.0019 \\ 0.0014 \end{pmatrix}, \quad \tilde{\Sigma} = \begin{pmatrix} 0.0003 & 0.0001 & 0.0001 \\ 0.0001 & 0.0003 & 0.0001 \\ 0.0001 & 0.0001 & 0.0002 \end{pmatrix}. \] (25)

The return distribution under the nominal distribution is again assumed to be a \(d\)-dimensional multivariate normal with mean and covariance matrix as given in (25). The second subset, which contains 60 data points, is labelled as dataset 2. The distribution formed by taking equal probability for each data point in dataset 2 is assumed to be a forecasted distribution of the incoming daily returns. We take this as the alternative distribution, and in the following five days, assume that it is time-homogeneous, i.e., it is the same for each time period.

Now, the next task is to estimate the divergence between the nominal distribution and the alternative distribution. We adapt an estimation procedure which is based on the \(k\)th-nearest-neighbor approach (see, e.g., Pérez-Cruz (2008); Schneider and Schweizer (2015); Wang et al. (2009) for this method). Each time, a sample of 60 data is generated from the nominal distribution and the divergence between the nominal distribution and the alternative distribution is estimated by using this sample, the dataset 2, and Equation (A.1). To reduce variance, we repeat this procedure by taking the average over 100,000 repetitions (see, e.g., Wang et al. (2009)). Then, the
estimated divergence is obtained as:
\[ \hat{R}(\mathcal{E}) \approx 0.6455. \]

By knowing the KL divergence, we use a bootstrapping type of approach to sample 100,000 data points from dataset 2. This allows us to construct the distribution of \( W^\text{non-robust}_N - W^\text{robust}_N \) (see Figure 7) from which the model risk can be estimated. The estimated model risk at \( q = 95\% \) confidence level is 0.0128 (see the red vertical line in Figure 7). The interpretation is that if the optimal non-robust strategy is applied but the optimal robust strategy turns out to be more appropriate, then 95\% of the time we would lose no more than 1.28 cents for every one dollar. The complete procedure of quantifying the model risk is summarized in Algorithm B.

6. Conclusion.

In this work, we have derived a semi-analytical form of an optimal robust strategy for an investment portfolio in which an uncertainty of distribution of return is involved. The uncertainty is measured by the Kullback-Leibler divergence. We have applied our approach to several numerical examples and have suggested whether to adopt the optimal robust or non-robust strategy. In addition, we define model risk from the standard risk management perspective and present an algorithm for quantifying the model risk by using empirical data. This delivers a convenient way of quantifying model risk in
Algorithm B  Quantification of model risk by using dataset 1 and dataset 2

1: estimate $\hat{\mu}$ and $\hat{\Sigma}$ by using dataset 1;
2: for $i = 1, 2, \ldots, 100,000$ do
3: generate 60 independent sample from $N(\hat{\mu}, \hat{\Sigma})$ (the normal sample);
4: calculate ith divergence by using (A.1), the normal sample and dataset 2;
5: if ith divergence > 0 then
6: keep the ith divergence;
7: else
8: abandon the ith divergence;
9: end if
10: end for
11: estimate the divergence by taking the average of all the positive divergence that have been calculated above;
12: calculate the optimal robust strategy by using Theorem 1 with 500,000 simulations;
13: calculate the optimal non-robust strategy;
14: for $j = 1, \ldots, 100,000$ do
15: calculate $W_N^{\text{non-robust}} - W_N^{\text{robust}}$;
16: end for
17: Find the $(1 - q)$th-quantile of the distribution of $W_N^{\text{non-robust}} - W_N^{\text{robust}}$.

There are some possible variations and extensions of our work that deserve further investigation. One research question is about designing a fair way to perform an out-of-sample comparison of our method with the non-robust method. The purpose would be to compare the performance of the robust and the non-robust strategy directly rather than via assessing the risk of applying the non-robust strategy that we have presented here. Another research question is to scrutinize the cases where the size of the portfolio is very large (for example, $d > 100$). We have performed some initial testing of the performance of the solution for portfolios up to size 50 but do not currently have theoretical criteria to guarantee the existence of a solution of the system in Theorem 1. In addition, it is worth investigating the computational cost as the problem scales to very large portfolio sizes.

Appendix A. Kullback-Leibler Divergence: Concept, Extension, and Estimation.

The Kullback-Leibler (KL) divergence is a well known deviation measure between distributions, and has been discussed in many papers. Here, we present a brief summary of its concept, extension and estimation by mainly
following Breuer and Csiszár (2016); Glasserman and Xu (2014); Schneider and Schweizer (2015).

Fix a probability space \((\Omega, \mathcal{F}, P)\), and let us assume that the nominal distribution is described by the probability measure \(P\), and denote its density by \(f\). An alternative distribution described by a probability measure \(Q\) is assumed to be absolutely continuous with respect to \(P\), i.e., for \(A \in \mathcal{F}\) and \(P(A) = 0\), we have \(Q(A) = 0\). The density of an alternative distribution is denoted by \(g\). Let \(E = g/f\). The KL divergence between the nominal and the alternative distribution is defined as

\[
R(E) = \mathbb{E}(E \log E) := \mathbb{E}(E(\zeta) \log E(\zeta)) = \int \left( \frac{g(\zeta)}{f(\zeta)} \right) \log \left( \frac{g(\zeta)}{f(\zeta)} \right) f(\zeta) d\zeta.
\]

The KL divergence can be generalized to the so-called \(\alpha\)-divergence. For \(\alpha > 1\), the \(\alpha\)-divergence is defined as

\[
R(E) = \mathbb{E}(E^\alpha - \alpha(E - 1) - 1) = \mathbb{E}(E(\zeta)^\alpha - \alpha(E(\zeta) - 1) - 1) = \int \left( \frac{g(\zeta)}{f(\zeta)} \right)^\alpha - \alpha \left( \frac{g(\zeta)}{f(\zeta)} - 1 \right) f(\zeta) d\zeta.
\]

By applying the L'Hôpital's rule, one can show that the \(\alpha\)-divergence converges to the KL divergence as \(\alpha \to 1\). In other words, the KL divergence is the limiting case of the \(\alpha\)-divergence.

When we know the nominal and the alternative distributions precisely, sometimes we may be able to calculate the KL divergence analytically. In practice, however, we often only have samples of the distributions. This requires estimation of the KL divergence. One way to estimate the KL divergence is by using the \(k\)th-nearest-neighbor estimation approach. Suppose we have an independent identically distributed (i.i.d) sample \((Y_i)\) from the nominal distribution and another i.i.d sample \((\tilde{Y}_i)\) from the alternative distribution. The estimated KL divergence between the two models by using the \(k\)th-nearest-neighbor approach is given by

\[
\hat{R}(E) = \frac{1}{K} \sum_{i=1}^{K} \log \left( \frac{K(y_i(i))_d}{(K-1)(\tilde{y}_i(i))_d} \right).
\]
Appendix B. A Useful Lemma.

**Theorem 2** (Lemma B.1). The function

\[
h : u \in U \rightarrow \left( -\theta \log \mathbb{E} \left( \exp \left( -u^\top R_{\theta}^{-1} \right) \right) - \kappa \sqrt{u^\top \Sigma u} \right)
\]

is strictly concave, where \( R \in \mathbb{R}^d \) is a random vector, \( \kappa, \theta > 0 \), and \( \Sigma \) is positive definite.

**Proof.** The proof that \( -\kappa \sqrt{u^\top \Sigma u} \) is strictly concave follows from (Owadally, 2012, p. 4430). Thus, it is sufficient to prove that the other part is concave. The latter follows from Hölder’s inequality. Indeed, for \( u, v \in \mathbb{R}^d \), and \( t \in (0, 1) \), we see that

\[
\mathbb{E} \left( \exp \left( -\left( tu + (1-t)v\right)^\top R_{\theta}^{-1} \right) \right) \leq \left( \mathbb{E} \left( \exp \left( -u^\top R_{\theta}^{-1} \right) \right) \right)^t \left( \mathbb{E} \left( \exp \left( -v^\top R_{\theta}^{-1} \right) \right) \right)^{1-t}.
\]

By taking logarithms of both sides and by multiplying by \( -\theta \), we obtain the desired result. \( \square \)

Appendix C. Proof of Proposition 1.

In order to prove Proposition 1, we start with some basics from Arellano-Valle and Genton (2005); Contreras-Reyes and Arellano-Valle (2012).

Given \( Y \sim SN_d(\mu, \Sigma, \xi) \), the density of \( Y \) is given by

\[
f_Y(y) = 2|\Sigma^{\frac{1}{2}}| \phi_d \left( \Sigma^{\frac{1}{2}}(y - \mu) \right) \Phi \left( \xi^\top \Sigma^{\frac{1}{2}}(y - \mu) \right) \left| 1 - \xi^\top \xi \right|, \tag{C.1}
\]

where \( \phi_d \) is the density of a \( d \)-dimensional standard normal, and \( \Phi(\cdot | \sigma^2) \) is the cumulative distribution function of a standard normal with mean 0 and variance \( \sigma^2 \).

Moreover, we have

\[
\mathbb{E}(Y) = \mu + \sqrt{\frac{2}{\pi}} \Sigma^{\frac{1}{2}} \xi \quad \text{and} \quad \text{Var}(Y) = \Sigma - \frac{2}{\pi} \Sigma^{\frac{1}{2}} \xi \xi^\top \Sigma^{\frac{1}{2}}. \tag{C.2}
\]

and

\[
Y = \mu + \Sigma^{\frac{1}{2}} Y^* \overset{d}{=} \mu + \Sigma^{\frac{1}{2}} \left( \xi | Z_0 \rangle + (I_d - \xi \xi^\top)^{\frac{1}{2}} Z \right),
\]

29
where \( Z_0 \sim N(0, 1) \) and \( Z \sim N_d(0, I_d) \) are independent one-dimensional Normal and \( d \)-dimensional Multivariate Normal distributions.

If the nominal model is \( Y \sim SN_d(\mu, \Sigma, \xi) \) and the alternative model is \( \bar{Y} \sim SN_d(\bar{\mu}, \bar{\Sigma}, \bar{\xi}) \), the KL divergence between the two models is given by

\[
\mathcal{R}_{skew}(\mathcal{E}) = C(Y, \bar{Y}) - C(\bar{Y}, \bar{Y}),
\]

where

\[
C(Y, \bar{Y}) = -\mathbb{E}\left( \log (f_Y(\bar{Y})) \right)
\]

is the cross-entropy (see [Contreras-Reyes and Arellano-Vallé, 2012, p. 14]).

Next, we proceed to the proof of Proposition 1.

**Proof.** Since \( Y \sim SN_d(\mu, \Sigma, \xi) \), by (C.1), it is easy to see that

\[
\log(f_Y(y)) = \log \left( \left| \Sigma \right|^{-\frac{1}{2}} \phi_d (\Sigma^{-\frac{1}{2}}(y - \mu)) \right) + \log \left( 2\Phi (\xi^T \Sigma^{-\frac{1}{2}}(y - \mu) | 1 - \xi^T \xi) \right)
\]

\[
= -\frac{1}{2} \left( \log(\left| \Sigma \right|^{-1}) + d \log(2\pi) + (y - \mu)^T \Sigma^{-1}(y - \mu) \right) + \log \left( 2\Phi (\xi^T \Sigma^{-\frac{1}{2}}(y - \mu) | 1 - \xi^T \xi) \right)
\]

The cross-entropy between \( Y \sim SN_d(\mu, \Sigma, \xi) \) and \( \bar{Y} \sim SN_d(\bar{\mu}, \bar{\Sigma}, \bar{\xi}) \) is then given by

\[
C(Y, \bar{Y}) = -\mathbb{E}\left( \log (f_Y(\bar{Y})) \right)
\]

\[
= \frac{1}{2} \left( \log(\left| \Sigma \right|^{-1}) + d \log(2\pi) + \mathbb{E}\left( (\bar{Y} - \mu)^T \Sigma^{-1}(\bar{Y} - \mu) \right) \right)
\]

\[
- \mathbb{E}\left( \log \left( 2\Phi (\xi^T \Sigma^{-\frac{1}{2}}(\bar{Y} - \mu) | 1 - \xi^T \xi) \right) \right).
\]

As a consequence of [Contreras-Reyes and Arellano-Vallé, 2012, part (iii) of Lemma 1] and (C.2), we obtain

\[
\mathbb{E}\left( (\bar{Y} - \mu)^T \Sigma^{-1}(\bar{Y} - \mu) \right) = tr \left( \Sigma^{-1} \Sigma \right) + (\bar{\mu} - \mu)^T \Sigma^{-1}(\bar{\mu} - \mu)
\]

\[
+ 2\sqrt{\frac{2}{\pi}} (\bar{\mu} - \mu)^T \Sigma^{-1} \Sigma^{\frac{1}{2}} \bar{\xi}.
\]
which implies
\[
C(Y, \bar{Y}) = \frac{1}{2} \left( \log(|\Sigma|^{-1}) + d \log(2\pi) + tr \left( \Sigma^{-1} \Sigma^{-1} \right) + (\mu - \bar{\mu})^T \Sigma^{-1} (\mu - \bar{\mu}) \right) \\
+ 2 \sqrt{\frac{d}{\pi}} (\mu - \bar{\mu})^T \Sigma^{-1} \bar{2} \xi - \mathbb{E} \left( \log \left( 2\Phi \left( \xi^T \Sigma^{-1/2} (Y - \mu) | 1 - \xi^T \xi \right) \right) \right).
\]

Since \( \bar{Y} \sim SN(\bar{\mu}, \bar{\Sigma}, \bar{\xi}) \), this yields
\[
\Xi_2 = \xi^T \Sigma^{-1/2} (Y - \mu) \\
= \xi^T \Sigma^{-1/2} (\mu - \mu) + \xi^T \Sigma^{-1/2} \bar{\Sigma}^{1/2} \bar{Y}^* \\
= d \xi^T \Sigma^{-1/2} (\mu - \mu) + \xi^T \Sigma^{-1/2} \bar{\Sigma}^{1/2} \xi |Z_0| \\
+ \sqrt{\xi^T \Sigma^{-1/2} \bar{\Sigma} \Sigma^{-1/2} \xi - (\xi^T \Sigma^{-1/2} \bar{\Sigma}^{1/2} \xi)^2} Z_1 \\
\sim SN \left( \xi^T \Sigma^{-1/2} (\mu - \mu), \xi^T \Sigma^{-1/2} \Sigma \Sigma^{-1/2} \xi, \xi^T \Sigma^{-1/2} \Sigma \Sigma^{-1/2} \xi \right),
\]
if \( \xi \) is not \( 0 \), and \( \Xi_2 = 0 \) otherwise, where \( Z_1 \sim N(0, 1) \) is independent of \( Z_0 \). This then implies
\[
C(Y, \bar{Y}) = C(Y_0, \bar{Y}_0) + 2 \sqrt{\frac{d}{\pi}} (\mu - \mu)^T \Sigma^{-1} \bar{2} \xi \\
- \mathbb{E} \left( \log \left( 2\Phi \left( \Xi_2 | 1 - \xi^T \xi \right) \right) \right).
\]

where \( Y_0 \sim SN_d(\mu, \Sigma, 0) \) and \( \bar{Y}_0 \sim SN_d(\bar{\mu}, \bar{\Sigma}, 0) \).

Since
\[
\mathcal{R}_{skew}(\mathcal{E}) = C(Y, \bar{Y}) - C(\bar{Y}, \bar{Y}),
\]
after some simple algebra we obtain the desired result. \( \square \)

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