Abstract

The uniformly hyperbolic Anosov C-systems defined on a torus have exponential instability of their trajectories, and as such C-systems have mixing of all orders and nonzero Kolmogorov entropy. The mixing property of all orders means that all its correlation functions tend to zero and the question of a fundamental interest is a speed at which they tend to zero. It was proven that the speed of decay in the C-systems is exponential, that is, the observables on the phase space become independent and uncorrelated exponentially fast. It is important to specify the properties of the C-system which quantify the exponential decay of correlations. We have found that the upper bound on the exponential decay of the correlation functions universally depends on the value of a system entropy. A quintessence of the analyses is that local and homogeneous instability of the C-system phase space trajectories translated into the exponential decay of the correlation functions at the rate which is proportional to the Kolmogorov entropy, one of the fundamental characteristics of the Anosov automorphisms. This result allows to define the decorrelation and relaxation times of a C-system in terms of its entropy and characterise the statistical properties of a broad class of dynamical systems, including pseudorandom number generators and gravitational systems.
1 Introduction

A uniformly hyperbolic Anosov C-systems defined on a torus have exponential instability of all trajectories \[1\] and as such have mixing of all orders and nonzero Kolmogorov entropy \[1, 2, 3, 4, 5, 6, 7, 8\]. The statistical properties of a deterministic dynamical system essentially depend on behaviour of the correlation functions defined on a corresponding phase space. The question of a fundamental interest is a speed at which correlation functions of C-systems tend to zero \[9, 10, 12, 13, 17, 18\]. It was proven that for the hyperbolic Anosov C-systems the speed of decay is exponential, that is, the observables on the phase space become independent and uncorrelated exponentially fast \[21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\]. It is important to specify the properties of the C-system which quantify the exponential decay of the correlation functions.

In this paper we shall study statistical properties of observables \(\{f(x)\}\) defined on an \(N\)-dimensional torus phase space \(M\) ( \(x \in M\) ) of the Anosov C-system diffeomorphisms \(T\) and specify the rate at which the exponential decay takes place. The statistical properties of the deterministic dynamical system defined by the map \(\forall x \in M : x \rightarrow x_n = T^n x\) are characterised by the behaviour of the corresponding correlation functions \(D_n(f, g)\). The very fact that the C-systems have mixing of all orders means that the correlation function of any two observables \(f(x)\) and \(g(x)\) tends to zero \[\ast\] when iteration/interaction time \(t = n\) tends to infinity \(n \rightarrow \infty\):

\[
D_n(f, g) = \langle f(x)g(T^n x) \rangle - \langle f(x)\rangle \langle g(x) \rangle \rightarrow 0.
\]

This function measures the dependence between the values of \(f(x)\) at zero time and values of \(g(x)\) at the time \(n\) and tells that the overlapping integral between observables \(f(x)\) and \(g(T^n x)\) tends to zero, so that they become independent and uncorrelated. A fundamental question which was raised in this respect is a question of a speed at which a correlation function \(D_n(f, g)\) tends to zero in (1.1). It was proven in the mathematical literature \[21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\] that for the hyperbolic Anosov diffeomorphisms and continuous flows the speed of decay is exponential:

\[
|D_n(f, g)| = |\langle f(x)g(T^n x) \rangle - \langle f(x)\rangle \langle g(x) \rangle| \leq C(f, g) \Lambda^n = C(f, g) \exp (-n \log 1/\Lambda),
\]

where the exponential upper bound on the correlation functions \(\Lambda_{T,f,g} < 1\) depends on properties of the dynamical system \(T\) and of the observables \(f(x)\) and \(g(x)\). A constant \(C(f, g)\) depends only on the observables \(f(x)\) and \(g(x)\). This outstanding result tells that the observables on the phase space of a C-system become uncorrelated exponentially fast and represent independent random variables.

In statistical physics the autocorrelation functions \(D_n(f, f)\) define the important physical properties of a dynamical system \(T\), such as its relaxation time \(\tau\), as well as temperature, diffusion, viscosity and other macroscopic characteristics \[11, 13, 18, 19\].

\[\ast\] In fact, correlation functions of any number of observables \(\langle f_1(T^{n_1} x)f_2(T^{n_2} x)\ldots f_r(T^{n_r} x) \rangle\) of a C-system tend to zero as \(n_i \rightarrow \infty\) \[11, 17\]. We denote by \(\langle f \rangle\) the average value of a function \(f\) with respect to invariant measure on \(M\).
It is important to express Λ and the relaxation time τ in terms of C-system quantitative characteristics. For that one should specify the properties of a C-system which quantify the exponential decay of the correlation functions in (1.1). We have found that the upper bound on the exponential decay of the correlation functions is universal and is defined by the value of the system entropy $h(T)$:

$$|D_n(f,g)| \leq C e^{-nh(T)\nu},$$  \hspace{1cm} (1.1)

where $C(f,g)$ and $\nu(f,g)$ depend only on observables and are positive numbers. This result allows to define the decorrelation time $\tau_0$ for a physical observable $f(x)$ as

$$\tau_0 = \frac{1}{h(T)\nu_f}. \hspace{1cm} (1.2)$$

A local and homogeneous instability of the C-system phase trajectories is translated into the exponential decay of the correlation functions at rate which is expressed in terms of the system entropy $h(T)$. The expression of the decorrelation time in terms of system entropy allows to characterise statistical properties of a broad class of dynamical systems, including gravitational systems and pseudorandom number generators [13, 18, 14, 15, 16].

When the dimension $N$ of the C-system (2.10) on a torus is increasing, its index $\nu_f$ is increasing linearly with dimension $\nu_f = 2pN$, where $p$ is the order of smoothness of the observable/function $f(x)$. The entropy $h(T)$ of the C-system (2.11) increases linearly as well $h(T) = \frac{2}{\pi}N$, therefore

$$\tau_0 = \frac{\pi}{4pN^2}. \hspace{1cm} (1.3)$$

Considering a set of initial trajectories occupying a small volume $\delta v_0$ in the phase space of a C-system, one can ask how fast this small phase volume will be uniformly distributed over the whole phase space. This characteristic time interval $\tau$ defines the relaxation time at which the system reaches a stationary distribution. Because the entropy defines the expansion rate of the phase space volume one can derive that [13]

$$\tau = \frac{1}{h(T)} \ln \frac{1}{\delta v_0}. \hspace{1cm} (1.4)$$

Thus there are three characteristic time scales associated with the C-system [13]:

$$\begin{pmatrix} \text{Decorrelation time} \\ \tau_0 = \frac{\pi}{4pN^2} \end{pmatrix} < \begin{pmatrix} \text{Interaction time} \\ t_{int} = n = 1 \end{pmatrix} < \begin{pmatrix} \text{Stationary distribution time} \\ \tau = \frac{1}{h(T)} \ln \frac{1}{\delta v_0} \end{pmatrix}. \hspace{1cm} (1.5)$$

This result defines important physical characteristics of the C-systems and measures the "level of chaos" developed in the system and justifies the statistical/probabilistic description of the system [11]. Indeed, the appearance of well developed statistical properties has important consequences in the form of the central limit theorem for Anosov diffeomorphisms. The time average of the observable $f(x)$ on $\mathcal{M}$

$$\bar{f}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$
behaves as a superposition of quantities which are statistically independent [36]. It has been proven that the fluctuations of the time averages (1.6) from the phase space average

$$\langle f \rangle = \int_{\mathcal{M}} f(x) d\mu(x)$$

multiplied by $\sqrt{n}$ have at large $n \to \infty$ the Gaussian distribution [35, 36, 37, 38, 39]:

$$\lim_{n \to \infty} \mu \left\{ x : \sqrt{n} \left( \bar{f}_n(x) - \langle f \rangle \right) < z \right\} = \frac{1}{\sqrt{2\pi\sigma_f^2}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy ,$$

where the value of the standard deviation $\sigma_f$ is a sum

$$\sigma_f^2 = \langle f^2(x) \rangle - \langle f(x) \rangle^2 + 2 \sum_{n=1}^{\infty} \left[ \langle f(T^n x) f(x) \rangle - \langle f(x) \rangle^2 \right].$$

Using our result (1.1) one can explicitly estimate the standard deviation in terms of entropy:

$$\sigma_f^2 \leq C \frac{1 + e^{-h(T)\nu}}{1 - e^{-h(T)\nu}} .$$

The earlier publications concerning the application of the modern results of the ergodic theory to concrete physical systems can be found in [11, 12, 13, 17, 18, 19]. These articles contain review material as well. The present paper is organised as follows. In section two we shall overview the basic properties of a C-system defined on a high dimensional torus, its spectral properties and its entropy. In section three we shall calculate the correlation functions and shall express the upper bound on the correlation functions in terms of entropy for the system on two-dimensional torus. In the fourth section we shall extend these results to the high dimensional C-systems and shall define three characteristic time scales associated with the C-systems. In section five the time scales associated with the MIXMAX pseudorandom number generators [13, 14, 15, 16] will be estimated. In conclusion we summaries the results.

2 C-systems on a Torus

A particular system chosen for investigation is the one realising linear automorphisms of the unit hypercube $\mathcal{M}^N$ in Euclidean space $E^N$ with coordinates $(x_1, \ldots, x_N)$ [1, 13, 14, 15]:

$$x_i^{(k+1)} = \sum_{j=1}^{N} T_{ij} x_j^{(k)} \mod 1, \quad k = 0, 1, 2, \ldots \quad (2.6)$$

where the components of the vector $x^{(k)}$ are defined as $x^{(k)} = (x_1^{(k)}, \ldots, x_N^{(k)})$. The phase space $\mathcal{M}^N$ of the systems (2.6) can also be considered a N-dimensional torus [11, 13, 14, 15], appearing at factorisation of the Euclidean space $E^N$ with coordinates $x = (x_1, \ldots, x_N)$ over an integer lattice $\mathbb{Z}^N$. The operator $T$ acts on the initial vector $x^{(0)}$ and produces a phase space trajectory $x^{(n)} = T^n x^{(0)}$ on a torus.
The dynamical system defined by the integer matrix $T$ has a determinant equal to one $\text{Det } T = 1$ and has no eigenvalues on the unit circle \[1\]. The spectrum $\{\lambda_1, ..., \lambda_N\}$ of the matrix $T$ fulfils therefore the following two conditions:

\begin{align}
1) \quad \text{Det } T = \lambda_1 \lambda_2 ... \lambda_N = 1, & \quad 2) \quad |\lambda_i| \neq 1, \quad \forall \ i. \tag{2.7}
\end{align}

The Liouville’s measure $d\mu = dx_1 ... dx_N$ is invariant under the action of $T$, and $T$ is an automorphism of the unit hypercube onto itself. The conditions (2.7) on the eigenvalues of the matrix $T$ are sufficient to prove that the system represents an Anosov C-system [1] and therefore as such it also represents a Kolmogorov K-system [2, 3, 4, 5, 6] with mixing of all orders and of nonzero entropy. The eigenvalues of the matrix $T$ can be divided into two sets $\{\lambda_\alpha\}$ and $\{\lambda_\beta\}$ with modulus smaller and larger than one:

\begin{align}
0 < |\lambda_\alpha| < 1 & \quad \text{for } \alpha = 1 ... d \\
1 < |\lambda_\beta| < \infty & \quad \text{for } \beta = d+1 ... N. \tag{2.8}
\end{align}

There exist two hyperplanes $X = \{X_\alpha\}$ and $Y = \{Y_\beta\}$ which are spanned by the corresponding eigenvectors $\{e_\alpha\}$ and $\{e_\beta\}$. These invariant planes define invariant spaces on which the phase trajectories are expanding and contracting under the transformation $T$ at an exponential rate. The C-system (2.6) has a nonzero Kolmogorov entropy $h(T)$ [1, 4, 6, 7, 8, 15]:

\begin{align}
h(A) = \sum_\beta \ln |\lambda_\beta| \tag{2.9}
\end{align}

which is expressed in terms of the eigenvalues $\lambda_\beta$ of the operator $T$. The entropy quantitatively characterises the instability of a C-system trajectories and its value depends on the spectral properties of the evolution operator $T$. We shall consider a family of operators of dimension $N$ introduced in [14]:

\begin{equation}
T = \begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 2 & 1 & 1 & \ldots & 1 & 1 \\
1 & 2 & 2 & 1 & \ldots & 1 & 1 \\
1 & 4 & 3 & 2 & \ldots & 1 & 1 \\
\vdots \\
1 & N & N-1 & N-2 & \ldots & 3 & 2
\end{pmatrix} \tag{2.10}
\end{equation}

The operator $T$ fulfils the C-condition (2.7) and represents a C-system [13]. The spectrum of the operator $T$ and of its inverse $T^{-1}$ are presented in Fig.1 [14, 16]. The entropy of the C-system $T$ can be calculated for large values of $N$ [14, 16]:

\begin{align}
h(A) = \sum_\beta \ln |\lambda_\beta| \approx \frac{2}{\pi} N \tag{2.11}
\end{align}
and increases linearly with the dimension $N$. Our aim is to study the behaviour of the observables \( \{f(x)\} \) defined on the torus phase space $\mathcal{M}^N$ of the dynamical system $T$ (2.10) and, in particular, a speed at which the correlation functions decay.

3 \textbf{Correlation Functions}

The general form of the correlations we are intending to consider are:

$$D_n(f,g) = \langle f(x)g(T^n x) \rangle - \langle f(x) \rangle \langle g(x) \rangle,$$

where the $f(x)$ and $g(x)$ are the observables/functions defined on the torus phase space $\mathcal{M}^N$. We shall consider the observables belonging to a general class of functions which are $p$-times differentiable $f, g \in C^p$, where $p$ is an integer, or functions which are in the $\alpha$-Hölder class.

In this section the operator $T$ is a two-dimensional matrix, $N = 2$ in (2.10):

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{mod} \ 1, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} \{x_1 + x_2\} \\ \{x_1 + 2x_2\} \end{pmatrix},$$

where $\{x\} \equiv x \mod 1$. To have an idea about how the correlation $D_n$ behaves as a function of the time step $n$ we shall calculate a ”one-step” correlation $D_1(f,g)$ when the observables are separated by one unit of time $n = 1$ and $f$ and $g$ are some polynomial functions. A simple example will be of the form

$$D_1(f,g) = \langle f(x) \rangle \langle x \rangle - \langle f(x) \rangle \langle x \rangle,$$

where $f = x_1x_2^r$ or $f = (x_1x_2)^r$, $r = 0, 1, ...$ and $g = x$. The result of calculations is presented on Fig. 2 and demonstrates that a one-step correlation decreases, $D_1(r) \rightarrow 0$, as the order of the polynomials $f$ increases, $r \rightarrow \infty$. In order to confirm this behaviour analytically let us consider the
Figure 2: The correlation function $D_1(r) = \langle x_1 x_2^1 \{x_1 + x_2\} \rangle - \langle x_1 x_2^1 \rangle \langle x_1 + x_2 \rangle$ and the correlator $K_1(r) = \langle x_1 x_2^1 \{x_1 + 2x_2\}^r \rangle - \langle x_1 x_2^1 \rangle \langle x_1 + 2x_2 \rangle^r$, where $\{x\} \equiv x \mod 1$.

Observables of the form

$$f(x) = \sum_{i_1,i_2=1}^{\infty} a_{i_1i_2} \sin(2\pi i_1 x_1) \cos(2\pi i_2 x_2), \quad <f(x)> = 0, \quad g(x) = x, \quad <g(x)> = \frac{1}{2},$$

where the numbers $(i_1,i_2)$ define the oscillation frequencies of the observable $f$. The correlation function (3.14) will take the form

$$D_1(f,g) = \sum_{i_1,i_2=1}^{\infty} a_{i_1i_2} \int_0^1 dx_1 dx_2 \sin(2\pi i_1 x_1) \cos(2\pi i_2 x_2) \{x_1 + x_2\} = -\sum_{r=1}^{\infty} \frac{a_{rr}}{4\pi r}, \quad (3.14)$$

where we used a trigonometric representation of the $\mod 1$ operation:

$$(x_1 + x_2) \mod 1 \equiv \{x_1 + x_2\} = \frac{1}{2} - \frac{1}{\pi} \sum_{r=1}^{\infty} \frac{1}{r} \sin(2\pi r(x_1 + x_2)).$$

In order to estimate the behaviour of the Fourier coefficients in (3.14) we shall consider the functions $f(x)$ which have $(2p_1,2p_2)$ continuous partial derivatives $f(x_1,x_2) \in C^{2p_1,2p_2}(\mathcal{M})$. The behaviour of the Fourier coefficients can be found performing a partial integrations and using the periodicity of the functions $f(x+1) = f(x)$. Thus the Fourier coefficients can be represented in the form

$$a_{i_1i_2} = \frac{4(-1)^{p_1+p_2}}{(2\pi i_1)^{2p_1}(2\pi i_2)^{2p_2}} \int_0^1 dx_1 dx_2 \sin(2\pi i_1 x_1) \cos(2\pi i_2 x_2) \partial_{x_1}^{(2p_1)} \partial_{x_2}^{(2p_2)} f(x_1,x_2). \quad (3.15)$$

This representation allows to estimate the one-step correlation function $D_1(f,g)$:

$$|D_1(f,g)| = |\langle f(x) T x \rangle - \langle f(x) \rangle \langle x \rangle| = \left| \sum_{r=1}^{\infty} \frac{a_{rr}}{4\pi r} \right| \leq \sum_{r=1}^{\infty} \frac{2M_p}{(2\pi r)^{2p_1+2p_2+1}}, \quad (3.16)$$

where

$$|\partial_{x_1}^{(2p_1)} \partial_{x_2}^{(2p_2)} f(x_1,x_2)| \leq M_p.$$

\(^1\)If a function is finite and continuous together with its $p-2$ derivatives and if its $p-1$ derivative is bound and has finite number of discontinuities, then there exists a finite constant $M_p$ such that $|a_i| < M_p/i^p$.\(^1\)
This result confirms our numerical observation demonstrated on Fig.2 that the one-step correlation \( D_1(f, g) \) decreases when the oscillation frequency \( r \) of the observable \( f \) increases:

\[
D_1(r) \sim \frac{1}{r^{2p_1+2p_2+1}} \to 0. \tag{3.17}
\]

The oscillations frequencies of the observable \( f(x) \) are defined by \((i_1, i_2)\) in (3.14) and we are considering the limit when \((i_1, i_2) \sim r \to \infty\). Performing a similar calculation one can get convinced that the result holds for more general functions \( f(x) \) in (3.14):

\[
f(x) = \sum_{i_1, i_2=0}^{\infty} a_{i_1i_2} \cos(2\pi i_1 x_1) \cos(2\pi i_2 x_2) + \ldots
\]

The other independent correlation function of \( Tx \) is of the form

\[
< f(x) T x > = \sum_{i_1, i_2=1}^{\infty} a_{i_1i_2} \int_0^1 dx_1 dx_2 \sin(2\pi i_1 x_1) \cos(2\pi i_2 x_2) \{x_1 + 2x_2\} = -\sum_{r=1}^{\infty} \frac{a_{r,r+2}}{4\pi r}.
\]

As one can see, the second index of the Fourier coefficient was shifted by two units \( a_{r,r+2} \). Examining its behaviour by using (3.15) we conclude that the coefficients decay faster:

\[
|D_1(f, g)| = \left| \sum_{r=1}^{\infty} \frac{a_{r,r+2}}{4\pi r} \right| \leq \sum_{r=1}^{\infty} \frac{2M_p}{(2\pi r)^{2p_1+1}(2\pi (r + 2))^{2p_2}}. \tag{3.18}
\]

Of our primary interest is to find out the behaviour of the correlations \( D_n(f, g) \) for observables \( f(x) \) and \( g(T^n x) \) separated by \( n \) time steps (3.12). For that one should generalise our previous calculation in two directions, considering \( n \geq 1 \) and the general functions \( g(x) \) in (3.12). Thus we shall consider a Fourier representation of the function \( g(x) \) on a torus of the form

\[
g(x_1, x_2) = \sum_{j_1, j_2=1}^{\infty} b_{j_1j_2} \cos(2\pi j_1 x_1) \cos(2\pi j_2 x_2). \tag{3.19}
\]

In order to calculate the observable \( g(x) \) after \( n \)-steps \( g(T^n x) = g(\{x_1 + 2x_2\}, \{x_1 + 2x_2\}) \) we have to define a \textit{mod} 1 operation acting on a nonlinear function \( g(x) \). The \textit{mod} 1 operation can be easily realised on trigonometric functions, since \( \sin(2\pi j x) = \sin(2\pi j (\{x\} + \text{integer})) = \sin(2\pi j \{x\}) \), \( \cos(2\pi j x) = \cos(2\pi j (\{x\} + \text{integer})) = \cos(2\pi j \{x\}) \), therefore

\[
g(\{x_1\}, \{x_2\}) = \sum_{j_1, j_2=1}^{\infty} b_{j_1j_2} \cos(2\pi j_1 x_1) \cos(2\pi j_2 x_2). \tag{3.20}
\]

This is an important observation for a successful calculation of \( D_n(f, g) \), because if one considers a polynomial expansion of \( f \) and \( g \), then the \textit{mod} 1 operation on polynomials is much more difficult to execute. Thus we shall consider a Fourier representation of the observables:

\[
f(x) = \sum_{i_1, i_2=1}^{\infty} a_{i_1i_2} \cos(2\pi i_1 x_1) \cos(2\pi i_2 x_2), \quad \{T^n x\} = \begin{cases} a_n x_1 + b_n x_2 \\ c_n x_1 + d_n x_2 \end{cases}, \tag{3.21}
\]

\[
g(T^n x) = \sum_{j_1, j_2=1}^{\infty} b_{j_1j_2} \cos(2\pi j_1 (a_n x_1 + b_n x_2)) \cos(2\pi j_2 (c_n x_1 + d_n x_2)),
\]

\]
where the coefficients \((a_n, b_n, c_n, d_n)\) define the \(n\)th power of the operator \(T\) and can be expressed in terms of Fibonacci numbers \(F_{2n}\):

\[
T^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} F_{2n} - F_{2n-2} & F_{2n} \\ F_{2n} & 2F_{2n} - F_{2n-2} \end{pmatrix}, \quad F_{2n} = \frac{\lambda^n - \lambda^{-n}}{\sqrt{5}}.
\] (3.22)

The \(\lambda = \frac{3 + \sqrt{5}}{2} > 1\) is the eigenvalue of the matrix \(T\) and

\[
det T^n = a_nb_n - b_nc_n = 1, \quad d_n > b_n = c_n > a_n.
\] (3.23)

Using the above formulas we can express the general correlation functions in the form

\[
\langle f(x)g(T^n x) \rangle = \sum_{i_1, i_2, j_1, j_2 = 1}^{\infty} a_{i_1 i_2} b_{j_1 j_2} \frac{1}{8} \left( \delta_{i_1 j_1 a_n + j_2 c_n} \delta_{i_2 j_1 b_n + j_2 d_n} + \delta_{j_1 a_n, i_1 + j_2 c_n} \delta_{j_1 b_n, i_2 + j_2 d_n} + \right.
\]

\[
\left. + \delta_{j_2 c_n, i_1 + j_1 a_n} \delta_{j_2 d_n, i_2 + j_1 b_n} + \delta_{j_2 c_n, i_1 + j_1 a_n} \delta_{j_2 d_n, i_2 + j_1 b_n} \right)\]

In order to execute the last four delta functions in (3.24) one should solve the linear equations

\[
a_n j_1 - c_n j_2 = i_1, \quad -a_n j_1 + c_n j_2 = i_1
\]

\[
\pm b_n j_1 \mp d_n j_2 = i_2, \quad \mp b_n j_1 \pm d_n j_2 = i_2
\] (3.25)

with respect to the \(j_1, j_2\). All these equations have unique solutions because the corresponding determinants are equal to \(\pm 1\) (3.23) and we have to select the solutions with positive \(j_1, j_2\). The last delta function in (3.24) has no positive solutions and therefore does not contribute to the correlation.

Thus we have

\[
\langle f(x)g(T^n x) \rangle = \frac{1}{8} \sum_{j_1, j_2 = 1}^{\infty} a_{j_1 a_n + j_2 c_n, j_1 b_n + j_2 d_n} b_{j_1 j_2} + \frac{1}{8} \sum_{i_1, i_2 = 1}^{\infty} a_{i_1, i_2} b_{i_1 d_n + i_2 c_n, i_1 b_n + i_2 a_n} +
\]

\[
+ \frac{1}{8} \sum_{i_1 d_n > i_2 c_n, i_1 b_n > i_2 a_n} a_{i_1, i_2} b_{i_1 d_n - i_2 c_n, i_1 b_n - i_2 a_n} + \frac{1}{8} \sum_{i_2 c_n > i_1 d_n, i_2 a_n > i_1 b_n} a_{i_1, i_2} b_{i_1 d_n + i_2 c_n, -i_1 b_n + i_2 a_n}.
\] (3.26)

The subtraction terms in (3.12) are:

\[
< f(x) > = \sum_{i_1, i_2 = 1}^{\infty} a_{i_1 i_2} \int_0^1 dx_1 dx_2 \cos(2\pi i_1 x_1) \cos(2\pi i_2 x_2) = 0,
\]

\[
< g(x) > = \sum_{j_1, j_2 = 1}^{\infty} b_{j_1 j_2} \int_0^1 dx_1 dx_2 \cos(2\pi j_1 x_1) \cos(2\pi j_2 x_1) = 0,
\]

and the total expression for the correlation will take the following form:

\[
D_n(f, g) = < f(x)g(T^n x) > - < f(x) > < g(x) > =
\]

\[
= \frac{1}{8} \sum_{j_1, j_2 = 1}^{\infty} a_{j_1 a_n + j_2 c_n, j_1 b_n + j_2 d_n} b_{j_1 j_2} + \frac{1}{8} \sum_{i_1, i_2 = 1}^{\infty} a_{i_1, i_2} b_{i_1 d_n + i_2 c_n, i_1 b_n + i_2 a_n} +
\]

\[
+ \frac{1}{8} \sum_{i_1 d_n > i_2 c_n, i_1 b_n > i_2 a_n} a_{i_1, i_2} b_{i_1 d_n - i_2 c_n, i_1 b_n - i_2 a_n} + \frac{1}{8} \sum_{i_2 c_n > i_1 d_n, i_2 a_n > i_1 b_n} a_{i_1, i_2} b_{i_1 d_n + i_2 c_n, -i_1 b_n + i_2 a_n}.
\] (3.27)
Thus we have to estimate each term in the above expression. Using the representation (3.15) we shall evaluate the first term in the correlator $D_n(f,g)$:

\[
\frac{1}{8} \sum_{r_1, r_2 = 1}^{\infty} a_{r_1 a_n + r_2 c_n, r_1 b_n + r_2 d_n, b_{r_1} r_2} \leq \\
2 \sum_{r_1, r_2 = 1}^{\infty} \frac{M_p}{(2\pi(r_1 a_n + r_2 c_n)^{2p_1} (2\pi(r_1 b_n + r_2 d_n))^{2p_2} (2\pi r_1)^{2q_1} (2\pi r_2)^{2q_2}} = \\
\frac{2}{(b_n)^{2q_2} (c_n)^{2q_1}} \sum_{r_1, r_2 = 1}^{\infty} \frac{M_p}{(2\pi(r_2 + r_1 a_n c_n)^{2p_1} (2\pi(r_1 + r_2 d_n a_n))^{2p_2} (2\pi r_1)^{2q_1} (2\pi r_2)^{2q_2}} \leq \\
e^{-2q_1 \ln b_n - 2q_2 \ln c_n} \sum_{r_1, r_2 = 1}^{\infty} \frac{M_p}{(2\pi r_2)^{2p_2} (2\pi r_1)^{2p_2} (2\pi r_1)^{2q_1} (2\pi r_2)^{2q_2}} ,
\]

where

\[
|\partial_{x_1}^{(2p_1)} \partial_{x_2}^{(2p_2)} f(x_1, x_2)| \leq M_p, \quad |\partial_{x_1}^{(2q_1)} \partial_{x_2}^{(2q_2)} g(x_1, x_2)| \leq M_q . \tag{3.28}
\]

The logarithm in the exponent can be expressed as

\[
\ln b_n = \ln c_n = \ln F_{2n} = \ln \frac{\lambda^n - \lambda^{-n}}{\sqrt{5}} = n \ln \lambda + \ln(1 - \lambda^{-2n}) - \frac{1}{2} \ln 5 > n \ln \lambda.
\]

Thus we have

\[
\frac{1}{8} \sum_{r_1, r_2 = 1}^{\infty} a_{r_1 a_n + r_2 c_n, r_1 b_n + r_2 d_n, b_{r_1} r_2} \leq e^{-n(2p_1+2p_2)} \ln \lambda \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p M_q}{(2\pi r_1)^{2q_1+2q_2} (2\pi r_2)^{2p_1+2q_2}}
\]

and remembering that the entropy of the system is $h(T) = \ln \lambda (2.9)$, we have for the first term

\[
\frac{1}{8} \sum_{r_1, r_2 = 1}^{\infty} a_{r_1 a_n + r_2 c_n, r_1 b_n + r_2 d_n, b_{r_1} r_2} \leq C_1(f, g) e^{-nh(T)\nu_1(f,g)} , \tag{3.29}
\]

where the numerical factors

\[
C_1(f, g) = \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p M_q}{(2\pi r_1)^{2q_1+2q_2} (2\pi r_2)^{2p_1+2q_2}}, \quad \nu_1(f,g) = 2p_1 + 2p_2
\]

depend only on the observables and are independent of the system dynamics $T$. For the second term we have a similar estimate:

\[
\frac{1}{8} \sum_{r_1, r_2 = 1}^{\infty} a_{r_1 r_2 b_{r_1 d_n + r_2 c_n, r_1 b_n + r_2 d_n}} \leq \\
2M_p \sum_{r_1, r_2 = 1}^{\infty} \frac{M_q}{(2\pi r_1)^{2p_1} (2\pi r_2)^{2p_2} (2\pi (r_1 d_n + r_2 c_n))^{2q_1} (2\pi (r_1 b_n + r_2 d_n))^{2q_2}} = \\
\frac{2}{(b_n)^{2q_2} (c_n)^{2q_1}} \sum_{r_1, r_2 = 1}^{\infty} \frac{M_p}{(2\pi r_2)^{2p_2} (2\pi r_1)^{2p_2} (2\pi r_1)^{2q_1} (2\pi r_2)^{2q_2}} \leq \\
e^{-2q_2 \ln b_n - 2q_1 \ln c_n} \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p}{(2\pi r_2)^{2p_2} (2\pi r_1)^{2p_2} (2\pi r_1)^{2q_1} (2\pi r_2)^{2q_2}} \leq \\
e^{-n(2q_1+2q_2)} \ln \lambda \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p M_q}{(2\pi r_1)^{2p_1+2q_2} (2\pi r_2)^{2p_1+2q_2}} ,
\]

\[
1
\]

9
Thus
\[
\frac{1}{8} \left| \sum_{r_1, r_2 = 1}^{\infty} a_{r_1, r_2} b_{r_1 d_n + r_2 c_n, r_1 b_n + r_2 a_n} \right| \leq C_2(f, g) e^{-nh(T)\nu_2(f, g)}, \quad (3.30)
\]
where the numerical factors are:
\[
C_2(f, g) = \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p M_q}{(2\pi r_1)^{2p_1 + 2q_2} (2\pi r_2)^{2q_1 + 2p_2}}, \quad \nu_2(f, g) = 2q_1 + 2q_2,
\]
and they are independent of the system dynamics $T$. The third and the fourth terms of $D_n(f, g)$ in (3.27) are
\[
+ \frac{1}{8} \sum_{i_1 d_n > i_2 c_n; i_1 b_n > i_2 a_n} a_{i_1, i_2} b_{i_1 d_n - i_2 c_n, i_1 b_n - i_2 a_n} + \frac{1}{8} \sum_{i_2 c_n > i_1 d_n; i_2 a_n > i_1 b_n} a_{i_1, i_2} b_{i_1 d_n + i_2 c_n, -i_1 b_n + i_2 a_n}
\]
and using the representation (3.15) for the Fourier coefficients one can find that
\[
\left| \sum_{r_1 d_n > r_2 c_n; r_1 b_n > r_2 a_n}^{\infty} a_{r_1, r_2} b_{r_1 d_n - r_2 c_n, r_1 b_n - r_2 a_n} \right| \leq
\]
\[
\leq \sum_{r_1 d_n > r_2 c_n; r_1 b_n > r_2 a_n}^{\infty} \frac{2M_p}{(2\pi r_1)^{2p_1} (2\pi r_2)^{2p_2}} \frac{M_q}{(2\pi (r_1 d_n - r_2 c_n))^{2q_1} (2\pi (r_1 b_n - r_2 a_n))^{2q_2}} \leq
\]
\[
= \frac{2}{(b_n)^{2q_1} (d_n)^{2q_1}} \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p}{(2\pi r_1)^{2p_1} (2\pi r_2)^{2p_2}} \frac{9M_q}{(2\pi (r_1 - r_2 \frac{c_n}{d_n}))^{2q_1} (2\pi (r_1 - r_2 \frac{a_n}{b_n}))^{2q_2}} \leq
\]
\[
\leq e^{-n(2q_1 + 2q_2) \ln \lambda} \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p}{(2\pi r_1)^{2p_1} (2\pi r_2)^{2p_2}} \frac{9M_q}{(2\pi (r_1 - r_2 \frac{c_n}{d_n}))^{2q_1} (2\pi (r_1 - r_2 \frac{a_n}{b_n}))^{2q_2}}.
\]
Thus for the third term we have
\[
\left| \sum_{r_1 d_n > r_2 c_n; r_1 b_n > r_2 a_n}^{\infty} a_{r_1, r_2} b_{r_1 d_n - r_2 c_n, r_1 b_n - r_2 a_n} \right| \leq C(f, g) e^{-nh(T)\nu(f, g)}, \quad (3.31)
\]
where the numerical factors are
\[
C(f, g) = \sum_{r_1, r_2 = 1}^{\infty} \frac{2M_p}{(2\pi r_1)^{2p_1} (2\pi r_2)^{2p_2}} \frac{9M_q}{(2\pi)^{2q_1} (2\pi)^{2q_2}}, \quad \nu(f, g) = 2q_1 + 2q_2,
\]
and they are $T$ independent. The fourth term has an identical upper bound:
\[
\left| \sum_{i_2 c_n > i_1 d_n; i_2 a_n > i_1 b_n}^{\infty} a_{i_1, i_2} b_{i_1 d_n + i_2 c_n, -i_1 b_n + i_2 a_n} \right| \leq C(f, g) e^{-nh(T)\nu(f, g)}.
\]
We arrive to the following upper bound on the correlation function:
\[
|D_n(f, g)| \leq C_1 e^{-nh(T)\nu_1} + C_2 e^{-nh(T)\nu_2} + 2C e^{-nh(T)\nu}. \quad (3.32)
\]
It follows that the dependence on the system dynamics $T$ appears in the exponential factor $e^{-nh(T)\nu}$ through its fundamental characteristic, the entropy $h(T)$. The coefficients $C_i(f,g), C(f,g)$ and $\nu_i(f,g), \nu(f,g)$ depend only on observables through their smoothness indices $p_i$ and $q_i$ and the upper bounds on derivatives $M_p$ and $M_q$.

The above calculation provides a qualitative understanding of how the exponential decay of the correlation functions appears and its rate. Under repeated action of the dynamic system $T$ on the observable $g(T^n x)$ its oscillating frequencies are stretching apart toward the high frequency modes and the overlapping integral with the fixed observable $f(x)$ falls exponentially.

In order to estimate the upper bound (3.32) let us consider the observables of the same order of smoothness $p_1 = p_2 = q_1 = q_2 = p$. In that case the formula simplifies:

$$|D_n(f,g)| \leq 4C(f,g) e^{-nh(T)\nu},$$

(3.33)

where

$$\nu = 4p, \quad 4C(f,g) = \frac{72M_p^2}{(16\pi^4)^{2p}} \left(\sum_{r=1}^{\infty} \frac{1}{r^{2p}}\right)^2.$$

This result allows to define the decorrelation time for the physical observable $f(x)$ as in [11, 13, 15]:

$$\tau_0 = \frac{1}{h(T)\nu_f}.$$

(3.34)

The value of the standard deviation $\sigma_f$ is a sum

$$\sigma_f^2 = \sum_{n=-\infty}^{+\infty} \left[\langle f(T^n x) f(x) \rangle - \langle f(x) \rangle^2\right]$$

and using the (4.40) one can explicitly estimate the standard deviation $\sigma_f$

$$\sigma_f^2 \leq 4C_f \frac{1 + e^{-h(T) \nu_f}}{1 - e^{-h(T) \nu_f}}.$$

(3.35)

4 High Dimensional C-systems

Let us now consider the operators $T$ of high dimension, $N > 2$ in [2,10]. In that case the characteristic polynomial is of high order and it is difficult to find out the general analytical expression for the coefficients of the operator $T^n$ similar to the formula (3.22). We shall use a computer to calculate the coefficients of the matrix $T^n$. What we need is the rate at which the coefficients grow as a function of $n$. The numerical experiments demonstrate that the fastest growing coefficients in each row are the ones which are the next to the last column and for the columns they are on the last row. Let us consider $N = 3$:

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}, \quad T^n = \begin{pmatrix} a_{11}^n & a_{12}^n & a_{13}^n \\ a_{21}^n & a_{22}^n & a_{23}^n \\ a_{31}^n & a_{32}^n & a_{33}^n \end{pmatrix}.$$

(4.36)
There is only one eigenvalue which is outside of the unit circle $\lambda = 4.0796...$. The largest coefficients in each row are $a_n^{12}, a_n^{22}, a_n^{32}$ correspondingly and the largest coefficients in each column are $a_n^{31}, a_n^{32}, a_n^{33}$ correspondingly, and they all grow at the rate $\propto \lambda^n$. The observables are

$$
f(x) = \sum_{i_1,i_2,i_3=1}^{\infty} a_{i_1i_2i_3} \cos(2\pi i_1 x_1) \cos(2\pi i_2 x_2) \cos(2\pi i_3 x_3), \quad (4.37)
$$

$$
g(T^n x) = \sum_{j_1,j_2,j_3=1}^{\infty} b_{j_1j_2j_3} \cos(2\pi j_1(a_n^{11} x_1 + a_n^{12} x_2 + a_n^{13} x_3)) \cos(2\pi j_2(a_n^{21} x_1 + a_n^{22} x_2 + a_n^{23} x_3)) \cos(2\pi j_3(a_n^{31} x_1 + a_n^{32} x_2 + a_n^{33} x_3)).
$$

A typical term in the correlation function $D_n(f,g)$ will have a form similar to the (3.26):

$$
\sum_{j_1,j_2,j_3=1}^{\infty} a_{j_1a_n^{11}+j_2a_n^{21}+j_3a_n^{31}}, a_{j_1a_n^{12}+j_2a_n^{22}+j_3a_n^{32}}, a_{j_1a_n^{13}+j_2a_n^{23}+j_3a_n^{33}} b_{j_1j_2j_3}, \quad (4.38)
$$

and it can be bound from above:

$$
| \sum_{j_1,j_2,j_3=1}^{\infty} a_{j_1a_n^{11}+j_2a_n^{21}+j_3a_n^{31}}, a_{j_1a_n^{12}+j_2a_n^{22}+j_3a_n^{32}}, a_{j_1a_n^{13}+j_2a_n^{23}+j_3a_n^{33}} b_{j_1j_2j_3}| \leq e^{-(2p_1+2p_2+2p_3)n\ln \lambda} \sum_{r_1,r_2,r_3=1}^{\infty} \frac{M_p}{(2\pi r_3)^{2p_1}(2\pi r_3)^{2p_2}(2\pi r_3)^{2p_3}} \frac{M_q}{(2\pi r_1)^{2p_1}(2\pi r_2)^{2p_2}(2\pi r_3)^{2p_3}} \leq 2M_p 2M_q \sum_{r_1,r_2,r_3=1}^{\infty} \frac{M_p}{(2\pi r_3)^{2p_1+2p_2+2p_3}} \frac{M_q}{(2\pi r_1)^{2p_1}(2\pi r_2)^{2p_2}(2\pi r_3)^{2p_3}},
$$

where $\ln \lambda = h(T)$ and

$$
|\partial_{x_1}^{(2p_1)} \partial_{x_2}^{(2p_2)} \partial_{x_3}^{(2p_3)} f(x_1, x_2, x_3)| \leq M_p, \quad |\partial_{x_1}^{(2q_1)} \partial_{x_2}^{(2q_2)} \partial_{x_3}^{(2q_3)} g(x_1, x_2, x_3)| \leq M_q. \quad (4.39)
$$

One can estimate the upper bound on the correlation function considering for simplicity the observables of the same order of smoothness $p_i = q_i = p$. In that case the formula takes the following form:

$$
|D_n(f,g)| \leq C(f,g) e^{-nh(T)\nu}, \quad (4.40)
$$

where

$$
\nu = 6p, \quad C(f,g) = \frac{M_p^2}{(2\pi)^{2p}} \left( \sum_{r=1}^{\infty} \frac{1}{r^{2p}} \right)^3.
$$

Generalising this calculation to the operators $T$ on dimension $N$ we shall get

$$
\nu_f = 2pN, \quad C(f,g) = \frac{M_p^2}{(2\pi)^{4pN}} \left( \sum_{r=1}^{\infty} \frac{1}{r^{2p}} \right)^N,
$$

and the formula for the decorrelation time takes the form

$$
\tau_0 = \frac{1}{\nu_f h(T)} = \frac{1}{2pN h(T)}. \quad (4.41)
$$
The exponential decay of the correlation functions is getting faster as the dimension $N$ of the operators $T$ is increasing. Taking into consideration that the entropy $h(T)$ of our system is linearly increasing with $N$, $h(T) \approx \frac{2}{\pi} N$ in (2.11), we shall get the following expression for the decorrelation time:

$$\tau_0 = \frac{\pi}{4pN^2}. \quad (4.42)$$

Considering a set of initial trajectories occupying a small volume $\delta v_0$ in the phase space of the C-system it is important to know how fast the volume $\delta v_0$ will be spread/distributed over the whole phase space during the evolution of the system. This characteristic time interval $\tau$ defines the time at which the system reaches a stationary distribution. The entropy of the system defines the expansion rate of the phase space volume elements and one can derive therefore that [13]

$$\tau = \frac{1}{h(T)} \ln \frac{1}{\delta v_0}. \quad (4.43)$$

Thus there are three characteristic time scales associated with the C-system [13]:

$$\left( \text{Decorrelation time} \right) \left( \tau_0 = \frac{\pi}{4pN^2} \right) < \left( \text{Interaction time} \right) \left( t_{int} = 1 \right) < \left( \text{Stationary distribution time} \right) \left( \tau = \frac{1}{h(T)} \ln \frac{1}{\delta v_0} \right). \quad (4.44)$$

This result defines important physical characteristics of the C-systems and measures the "level of chaos" developed in the system and justifies the physical conditions at which the statistical/probabilistic description of the system is available.

5 MIXMAX Random Number Generator

One of the interesting applications of the Anosov C-systems (2.10) is associated with the so called MIXMAX generator of pseudorandom numbers [13, 14, 15, 16]. It was demonstrated in [14] that the MIXMAX pseudorandom number generators are passing strong statistical U01-tests [41] when the entropy of the generators is larger than fifty, $h(T) > 50$. Using the formulas (4.42), (4.43) and (4.44) of the last section one can estimate characteristic time scales associated with the MIXMAX generators. The generator $N = 256$ in Table 1 of the article [16] has the entropy $h(T) = 194$ and the smallest phase volume is of order $\delta v_0 = 2^{-61.256}$, therefore the characteristic time scales for this generator are

$$\left( \text{Decorrelation time} \right) \left( \tau_0 = 0.000012 \right) < \left( \text{Interaction time} \right) \left( t_{int} = 1 \right) < \left( \text{Stationary distribution time} \right) \left( \tau = 95 \right). \quad (5.45)$$

The MIXMAX generator which has much higher entropy was presented in Table 3 of the article [16]. It has the entropy $h(T) = 8679$ and the smallest volume $\delta v_0 = 2^{-61.240}$, therefore the characteristic
time scales for this generator are
\[
\begin{pmatrix}
\text{Decorrelation time} \\
\tau_0 = 0.000004
\end{pmatrix}
<
\begin{pmatrix}
\text{Interaction time} \\
t_{\text{int}} = 1
\end{pmatrix}
<
\begin{pmatrix}
\text{Stationary distribution time} \\
\tau = 1.17
\end{pmatrix}.
\]
(5.46)

Both generators have very short decorrelation time. The second generator \( N = 240 \) has much bigger entropy and therefore its relaxation time \( \tau \) is much smaller, of order 1.17, and is close to the interaction time. In that sense it has very strong stochastic/chaotic properties, it much faster spreads trajectories over the whole phase space and reaches the equilibrium. Therefore it should not be surprising that these generators are passing all the tests in the BigCrush U01-suite [41]. These generators have the best combination of speed, reasonable size of the state and are currently available generators in the ROOT and CLHEP software packages at CERN for Monte-Carlo simulations and scientific calculation [42, 43, 44].

6 Conclusion

Our analyses of the \( N \) dimensional hyperbolic Anosov C-system (2.10) indicates that its basic statistical characteristics are expressible in terms of entropy. The decorrelation time \( \tau_0 \) and the relaxation time \( \tau \) are inversely proportional to the entropy of the system and indicate that these time scales become shorter as entropy increases. This is an intuitively appealing result because the entropy measures the uncertainty in the description of the physical systems and here it is translated into the important time scales characteristics. As a result a perfectly deterministic dynamical system shows up a fast thermalisation and well developed statistical properties. When measuring different observables of the hyperbolic Anosov C-system it will be difficult to recognise that in reality the data are coming out from a perfectly deterministic dynamical system.

The exponential decay of the correlation functions has been found earlier in classical dynamics of the N-body gravitating systems and can be used to justify a statistical description of globular clusters and elliptic galaxies [18, 19, 20].

It was suggested in the literature that the outgoing Hawking radiation [45] may be not exactly thermal, but had subtle correlations [46, 47, 48, 49]. In that respect one can suppose that the effective description of the black hole radiation can be understood in analogy with the behaviour of the hyperbolic systems of the type considered above [50].

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