PHANTOM COVERS IN EXACT CATEGORIES

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Abstract. We define the notion of \( P \)-phantom map with respect to a class of conflations in a locally \( \lambda \)-presentable exact additive category \((C; P)\) and we give sufficient conditions to ensure that the ideal \( \Phi(P) \) of \( P \)-phantom maps is a (special) covering ideal. As a byproduct of this result, we infer the existence of various covering ideals in categories of sheaves which have a meaningful geometrical motivation. In particular we deal with a Zariski-local notion of phantom maps in categories of sheaves. We would like to point up that our approach is necessarily different from [10], as the categories involved in most of the examples we are interested in do not have enough projective morphisms.

1. Introduction

Let \( C \) be an abelian category. Approximation Theory by Objects is typically used in Relative Homological Algebra to compute (co)resolutions from either an additive class of objects of \( C \) or an additive subfunctor of \( \text{Ext} \). In the first case, the necessary and sufficient condition for a class \( \mathcal{F} \) to provide such (co)resolutions is to be either (pre)covering or (pre)enveloping. We recall that a morphism \( \phi: F \to X \) with \( F \in \mathcal{F} \) is said to be an \( \mathcal{F} \)-precover of \( X \) if \( \text{Hom}(F', F) \to \text{Hom}(F', X) \to 0 \) is exact, for every \( F' \in \mathcal{F} \); equivalently, the natural transformation of functors \( \text{Hom}(\cdot, F)|_\mathcal{F} \to \text{Hom}(\cdot, A)|_\mathcal{F} \to 0 \) is exact. And the \( \mathcal{F} \)-precover \( \phi \) is called an \( \mathcal{F} \)-cover if any endomorphism \( f: F \to F \) is an isomorphism whenever \( \phi \circ f = \phi \). The class \( \mathcal{F} \) is said to be a (pre)covering class when every object has an \( \mathcal{F} \)-(pre)cover. (Pre)envelopes and (pre)enveloping classes are defined in a dual way.

When \( \mathcal{F} \) is a precovering class, we get for each object \( A \in \mathcal{C} \) a \( \text{Hom}(\mathcal{F}, \cdot) \)-exact sequence

\[
\ldots \to F_1 \to F_0 \to A \to 0
\]

with \( F_i \in \mathcal{F}, i \geq 0 \), which is unique up to homotopy.

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In the second situation of subfunctors of Ext, these (co)resolutions are constructed as follows. Let us recall from Mac Lane [26, Section XII.4] that a class \( \mathcal{P} \) of short exact sequences in \( \mathcal{C} \) is called \textit{proper} if it is closed under certain operations (isomorphisms and pullbacks and pushouts of \( \mathcal{P} \)-epimorphisms and \( \mathcal{P} \)-monomorphisms by any other morphism, respectively) and contains all the split short exact sequences. These axioms ensure that this class \( \mathcal{P} \) gives rise to an additive subfunctor \( \text{Ext}_{\mathcal{P}} \) of \( \text{Ext} \). Conversely, any additive subfunctor of \( \text{Ext} \) provides a proper class. In particular, for any morphism \( f \), the image of the restriction of \( \text{Ext}(f, -) \) or \( \text{Ext}(-, f) \) to \( \mathcal{P} \) is contained in \( \mathcal{P} \). So we may speak of injective objects relative to \( \mathcal{P} \) (i.e., \( \mathcal{P} \)-injective objects) and projective objects relative to \( \mathcal{P} \) (\( \mathcal{P} \)-projective objects). They are objects having the extension and the lifting property respect to short exact sequences in \( \mathcal{P} \), respectively. Therefore, resolutions of objects are computed by using \( \mathcal{P} \)-projectives and \( \mathcal{P} \)-epimorphisms; and coresolutions, by using \( \mathcal{P} \)-injectives and \( \mathcal{P} \)-monomorphisms. For later use in this introduction, we will denote by \( \mathcal{P}_{\text{sp}} \) the proper class associated to the zero subfunctor of \( \text{Ext} \) (i.e., the smallest proper class whose elements are all split short exact sequences).

In [16] Fu, Guil Asensio, Herzog and Torrecillas introduced the new concept of Approximations by Ideals on an additive exact category \( (\mathcal{A}; \mathcal{E}) \). This notion not only encompasses the usual Approximation Theory by Objects with respect to an additive subcategory of an abelian category \( \mathcal{C} \), but it also provides an additive subfunctor of \( \text{Ext}^1 \) (and hence, a proper class of \( \mathcal{C} \)). Recall that an \textit{ideal} \( \mathcal{I} \) of an additive category \( \mathcal{C} \) is just an additive subfunctor of \( \text{Hom} \). Thus, it associates subgroups \( \mathcal{I}(A, A') \subseteq \text{Hom}(A, A') \) to each couple of objects \( A, A' \in \mathcal{C} \) which are closed under left and right composition by other morphisms. This means that one can define functors

\[
\mathcal{I}(-, f): \mathcal{I}(-, M) \to \mathcal{I}(-, N)
\]

for any morphism \( f: M \to N \), as in the case of the absolute \( \text{Ext}^1(\mathcal{A}, \mathcal{A'}) \)-bifunctor. Let us also note that a morphism \( f: M \to N \) belongs to \( \mathcal{I} \) if and only if \( \text{Im} \text{Hom}(-, f) \subseteq \mathcal{I}(-, N) \); i.e., the natural transformation

\[
\text{Hom}(-, f): \text{Hom}(-, M) \to \text{Hom}(-, N)
\]

factors through \( \mathcal{I}(-, N) \).

An \textit{\( \mathcal{I} \)-precover} of an object \( A \in \mathcal{C} \) is defined as a morphism \( f: M \to A \) in \( \mathcal{I} \) such that any other morphism \( f': M' \to A \in \mathcal{I} \) factors through \( f \). In other words, a morphism \( f: M \to A \in \mathcal{I} \) is an \( \mathcal{I} \)-precover of \( A \) if and only if \( \text{Im} \text{Hom}(-, f) = \mathcal{I}(-, A) \). This \( \mathcal{I} \)-precover \( f: M \to A \) of \( A \) is said to be \textit{special} if \( \text{Ext}(g, \text{Ker} f) = 0 \) for each \( g \in \mathcal{I} \). And an ideal \( \mathcal{I} \) is called \textit{(special) precovering} if every object in \( \mathcal{C} \) has a (special) \( \mathcal{I} \)-precover. So if \( \mathcal{I} \) is a precovering ideal then, for each \( A \in \mathcal{C} \), \( \mathcal{I}(-, A): \mathcal{C}^\text{op} \to \text{Ab} \) has a
Given an additive subcategory $\mathcal{F}$ of an abelian category $\mathcal{C}$, one may define the ideal $\mathcal{I}(\mathcal{F})$ generated by $\mathcal{F}$ as the smallest ideal containing the identity morphisms $\text{id}_F$ for each $F \in \mathcal{F}$. That is, the ideal consisting of all morphisms which factor through objects in $\mathcal{F}$. This ideal is called in [16, Section 2] the object ideal generated by $\mathcal{F}$. Conversely, given an ideal $\mathcal{I}$, the subcategory $\text{Ob}(\mathcal{I})$ of all objects $A \in \mathcal{C}$ with $\text{id}_A \in \mathcal{I}$ is additive. Thus, there exists a bijection between the class of all additive subcategories of $\mathcal{C}$ and the class of all object ideals of $\mathcal{C}$ (see Fu and Herzog [17, Proposition 1]). With this notation, an additive subcategory $\mathcal{F}$ of $\mathcal{C}$ is precovering if and only if the object ideal $\mathcal{I}(\mathcal{F})$ is a precovering ideal. Therefore, Approximation Theory by Ideals encompasses the usual Approximation Theory by Objects.

Let us finally observe that we can also associate an ideal to any proper class of $\mathcal{C}$. Let us outline the process. Given a proper class $\mathcal{P}$ in $\mathcal{C}$, a morphism $f: A \to B$ is called $\mathcal{P}$-phantom if $\text{Ext}(f,-)$ transforms short exact sequences in $\mathcal{C}$ into distinguished short exact sequences in $\mathcal{P}$; that is, $\text{Ext}(f,-): \text{Ext}(B,-) \to \text{Ext}_\mathcal{P}(A,-)$. The class of $\mathcal{P}$-phantoms morphisms constitutes an ideal of $\mathcal{C}$ which is usually denoted by $\Phi(\mathcal{P})$. Conversely, we may associate to any ideal $\mathcal{I}$, the proper class $\text{PB}(\mathcal{I})$ consisting of all short exact sequences arising as pullbacks along morphisms in $\mathcal{I}$. For more details on this relation between ideals and proper classes, see [16].

This definition of $\mathcal{P}$-phantom morphisms generalizes that of phantom morphisms, which have their origin in triangulated categories. Let us briefly explain it. Let $\lambda$ be an infinite regular cardinal. Following the terminology of Adámek and Rosický in [1], an additive category $\mathcal{C}$ is called locally $\lambda$-presentable if it is cocomplete and there is a set of $\lambda$-presentable objects in $\mathcal{C}$ whose $\lambda$-directed colimit completion is $\mathcal{C}$ itself, that is, every object of $\mathcal{C}$ can be written as a $\lambda$-directed colimit of objects in this set. This means that objects in locally presentable categories are controlled by a set and hence, they are ‘small’ in terms of presentability for sufficiently large cardinals. This means that the Yoneda functor

$$Y: \mathcal{C} \to \text{Add}(\mathcal{C}_\lambda^{\text{op}}, \text{Ab}),$$

$$A \mapsto \text{Hom}(-, A)|_{\mathcal{C}_\lambda}$$

is fully faithful, where $\text{Add}(\mathcal{C}_\lambda^{\text{op}}, \text{Ab})$ denotes the category of all abelian valued contravariant additive functors in the subcategory $\mathcal{C}_\lambda$ of all $\lambda$-presentable objects of $\mathcal{C}$. Therefore, this embedding induces an equivalence between $\mathcal{C}$
and the full subcategory of all $\lambda$-cocontinuous functors in $\text{Add}(\mathcal{C}_\lambda^\text{op}, \text{Ab})$; i.e., $\mathcal{C}$ is the $\lambda$-free cocompletion of $\mathcal{C}_\lambda$ (see [11, Theorem 1.46]).

The corresponding notion for triangulated categories is obtained by replacing $\lambda$-presentable objects by $\lambda$-compact (or just compact when $\lambda = \aleph_0$) objects in the above definition, see Neeman [28]. However, the notion of generation by a class of objects in triangulated categories is weaker than in the case of (co)complete categories. As a consequence, one may not have the analogous representation type theorem for triangulated categories, as the corresponding version of the Yoneda functor (1.1) for a compactly generated additive triangulated category $\mathcal{T}$ may fail to be either faithful or full. I.e., if $\mathcal{T}_c$ is the class of all compact objects in $\mathcal{T}$, then the canonical functor

$$Y : \mathcal{T} \rightarrow H(\mathcal{T}_c),$$

does not need to be faithful nor full, where $H(\mathcal{T}_c)$ is the category of cohomological functors on $\mathcal{T}_c$. Morphisms in $\mathcal{T}$ sent to zero by $Y$ are called phantom, see Neeman [27, Definition 2.4].

A concrete example of a compactly generated triangulated category is the stable module category $kG\text{-Mod}$ of modules over $kG$, where $k$ is a field and $G$ is a finite group. Its objects are the $kG$-modules and the morphisms are the usual ones modulo those which factor through projective objects. A $kG$-module $F$ is compact in $kG\text{-Mod}$ if and only if it is finitely generated. So a morphism $f : M \rightarrow N$ in $kG\text{-Mod}$ is phantom if and only if $f \circ g$ factors through a projective $kG$-module, for any morphism $g : F \rightarrow M \in kG\text{-Mod}$ with $F$ finitely generated $kG$-module (see also [19]).

This fact motivated Herzog [21] to define phantom morphisms for categories of modules over arbitrary rings $R$ by replacing the stable module category $kG\text{-Mod}$ by the stable $R\text{-Mod}$ whose objects are the left $R$-modules and its morphisms are the classes of morphisms in $R\text{-Mod}$ modulo those which factor through flat modules. In this situation, the subcategory $R\text{-mod}$ of all finitely presented modules is just the quotient category of $R\text{-mod}$ modulo finitely generated projective modules. Therefore, he defines that a morphism $f : M \rightarrow N$ of $R$-modules is phantom if $f \in R\text{-mod}$ is in the kernel of the Yoneda functor

$$Y : R\text{-mod} \rightarrow \text{Add}(R\text{-mod}^\text{op}, \text{Ab});$$

that is, for every morphism $g : F \rightarrow M$ with $F$ a finitely presented $R$-module, the map $f \circ g$ factors through a projective module.

Note that a morphism $f$ is phantom in $R\text{-Mod}$ with this definition if and only if $f$ is $\mathcal{P}_{\aleph_0}$-phantom (i.e. $f \in \Phi(\mathcal{P}_{\aleph_0})$) according to our definition, where $\mathcal{P}_{\aleph_0}$ is the proper class of all pure short exact sequences in $R\text{-Mod}$. This easy observation provides an interesting connection between phantom morphisms and purity. This was first noticed by Christensen ([11]) and Christensen and Strickland ([5]), who showed the existence of phantom precovers in the category of homotopy spectra. Coming back to the category $R\text{-Mod}$ of left $R$-modules over an arbitrary ring $R$, Herzog ([21 Theorem 7]) proved that
this ideal $\Phi(P_{\aleph_0})$ is special covering in $R\text{-Mod}$ (see also Estrada, Guil Asensi and Ozbek [12] for another proof using quiver representations). More generally, in [16, Theorem 17], the authors give sufficient conditions that ensure that the ideal $\Phi(P)$ is special precovering in an exact category $(A; E)$ with enough projective morphisms. We recall that a morphism $f$ in an arbitrary exact category $(A; E)$ is projective if $f \in \Phi(E_{sp})$.

The above observation that a morphism $f$ in $R\text{-Mod}$ is phantom if and only if $f \in \Phi(P_{\aleph_0})$, suggests that the right setup for defining a concept of phantom morphisms is the framework of locally finitely presented additive categories. The reason is that it is possible to define a Theory of Purity for these categories in the sense of Crawley-Boevey [6]. Namely, a complex $E = 0 \to A \to B \to C \to 0$ is pure if $\text{Hom}(F,E)$ is a short exact sequence, for every finitely presented object $F$. Now, in its recent work [25, Theorem 1.1] Krause, has proven the existence of right minimal morphisms determined by a set $C$ of finitely presented objects in a locally finitely presented additive category $A$. It is easy to realize that, as a by-product of his result [25, Proposition 1.13], one can show that the ideal $\Phi(P_{\aleph_0})$ of phantom maps in $A$ is special covering. To prove it, one just needs to apply [25, Theorem 1.1] to an object $Y$ in $A$, setting $C$ as the skeleton of the finitely presented objects in $A$ and $H$, the set of all projective morphisms in $\text{Hom}(C, Y)$.

From this perspective, our notion of $P$-phantom morphisms naturally extends phantom morphisms. And we use the Purity Theory developed in [11] to set locally $\lambda$-presentable (rather than locally finitely presented) additive category as our ambient category. Thus, the present paper is devoted to showing sufficient conditions that guarantee that the ideal $\Phi(P)$ associated to a locally $\lambda$-presentable abelian category is special covering. Namely, one of the main results in this paper is the following theorem (see Theorem 3.2 and Proposition 3.4).

**Theorem 1.** Let $C$ be a locally $\lambda$-presentable abelian category and $P$, a proper class which is closed under direct limits. Then $\Phi(P)$ is a covering ideal. If moreover if $P$ has enough injectives then $\Phi(P)$ is special covering.

As a consequence of this theorem, we recover in Corollary 3.5 (1) the aforementioned result of Krause for the existence of special phantom covers in locally finitely presented categories.

**Corollary 1.** Let $C$ be a locally finitely presented category. The ideal $\Phi(P_{\aleph_0})$ is special covering.

However, we are also interested in very different frameworks. One of the main reasons why we want to introduce $P$-phantom morphisms comes from a quite different source of examples. Let us fix a closed symmetric monoidal Grothendieck category $(C, \otimes, [\cdot, \cdot])$. According to Fox [15], a monomorphism $f : A \to B$ is called $\otimes$-pure if for all $Z \in C$, the morphism $f \otimes Z$ is monic. Following the terminology introduced in Estrada, Gillespie and Odabaşı [12], we will call this kind of purity geometrical and we will denote...
by \( \mathcal{P}_\otimes \) the proper class of all geometrical pure short exact sequences in \( \mathcal{C} \). We prove in Corollary 3.5(2) that the ideal \( \Phi(\mathcal{P}_\otimes) \) of \( \mathcal{P}_\otimes \)-phantom maps is special covering in \( \mathcal{C} \).

**Corollary 2.** Let \( \mathcal{C} \) be a closed symmetric monoidal Grothendieck category. The ideal \( \Phi(\mathcal{P}_\otimes) \) is special covering.

It is well-known that both notions of purity coincide for the closed symmetric monoidal Grothendieck category \( R\text{-Mod} \) (where \( R \) is a commutative ring), so the \( \mathcal{P}_\otimes \)-phantom maps are just the usual phantom maps. But this is no longer true for arbitrary closed symmetric monoidal Grothendieck categories. Maybe one of the most interesting situations in which both definitions do not coincide appears when one considers the category \( \mathcal{Qco}(X) \) of all quasi-coherent sheaves over a (non-affine) scheme \( X \). This is a closed symmetric monoidal Grothendieck category. The closed structure is given by applying the coherator functor \( Q : \mathcal{O}_X\text{-Mod} \to \mathcal{Qco}(X) \), to the usual sheaf hom functor. We recall that the coherator is defined as the right adjoint functor of the inclusion functor \( \mathcal{Qco}(X) \to \mathcal{O}_X\text{-Mod} \). In most practical cases (\( X \) quasi-compact and quasi-separated) the category \( \mathcal{Qco}(X) \) is also locally finitely presented (see Grothendieck and Dieudonné [20, I.6.9.12] or Garkusha [18] for a precise statement). So it is possible to define phantom maps in terms of the proper class \( \mathcal{P}_{\aleph_0} \) of all categorical pure short exact sequences. However, we show in Corollary 4.4 that, unless the scheme \( X \) is affine, it is unlikely to find non-trivial phantom maps in \( \mathcal{Qco}(X) \). Indeed, the categorical purity is not Zariski-local, see Estrada and Saorín [14] (again, unless \( X \) is affine) whereas the geometrical purity is a local concept for these schemes. We refer to [7], [8, §27.4], [13] and [33, 13.5] for a recent update on Zariski-local properties of modules.

Motivated by these arguments, we devote the second part of this paper to define a good Zariski-local notion of phantom maps in \( \mathcal{Qco}(X) \) (see Definition 4.8 and Theorem 4.10).

**Definition 1.** A morphism \( f : \mathcal{G} \to \mathcal{F} \) in \( \mathcal{Qco}(X) \) is said to be locally phantom if there exists an open affine covering \( \mathcal{U} \) of \( X \) such that \( f_U \) is phantom in \( \mathcal{O}_X(U)\text{-Mod} \), for every open affine subset \( U \in \mathcal{U} \). We will denote by \( \Phi' \) the ideal of all locally phantom maps in \( \mathcal{Qco}(X) \).

A different notion of purity is also introduced in [10] for \( \mathcal{Qco}(X) \): the stalk-wise purity \( \mathcal{P}_{st} \). That is, a short exact sequence \( \mathcal{E} \) in \( \mathcal{Qco}(X) \) is stalk-wise pure if the induced short exact sequence on the stalks \( \mathcal{E}_x \) is pure in \( \mathcal{O}_{X,x}\text{-Mod} \) for every \( x \in X \). We then show in Corollary 4.14 that all of these notions of \( \mathcal{P} \)-phantom maps in \( \mathcal{Qco}(X) \) give rise to covering ideals.

**Theorem 2.** The ideals \( \Phi(\mathcal{P}_{st}) \) and \( \Phi(\mathcal{P}_\otimes) \) are special covering in \( \mathcal{Qco}(X) \) and the ideal \( \Phi' \) is covering in \( \mathcal{Qco}(X) \). If moreover, \( \mathcal{Qco}(X) \) has a flat generator (for instance, if \( X \) is quasi-compact and semi-separated), then all these covers are epimorphisms.
We finish this paper by showing that, for semi-separated schemes, these three new definitions of $\mathcal{P}$-phantoms ideals in $\mathcal{O}(X)$ (which are different of the ideal $\Phi(\mathcal{P})$ of phantom maps) coincide. Thus we have the following result (see Proposition 4.12).

**Proposition 1.** Let $X$ be a semi-separated scheme. Then $\Phi(\mathcal{P}) = \Phi(\mathcal{P}_{st}) = \Phi'$.

2. **Purity in locally presentable categories**

Along this paper, the symbol $\mathcal{C}$ will stand for an abelian category. Recall that, given $A, A' \in \mathcal{C}$, the class of Yoneda extensions $\text{Ext}(A, A')$ is the class of all representatives of isomorphism classes of short exact sequences in

\[ 0 \to A' \to X \to A \to 0. \]

We also point out that defining an exact structure $(\mathcal{C}; \mathcal{P})$ in the sense of Quillen [30] in $\mathcal{C}$ is the same as defining a proper class $\mathcal{P}$ in the sense of Mac Lane [26, Section XII.4]. Given two objects $A, A' \in \mathcal{C}$ and a proper class $\mathcal{P}$, we shall denote by $\text{Ext}_{\mathcal{P}}(A, A')$ the class of all representatives of isomorphism classes of short exact sequences in $\mathcal{P}$. The class $\mathcal{P}$ is called *injectively generated* (resp., *projectively generated*) by a class $\mathcal{M}$ if a short exact sequence $E$ belongs to $\mathcal{P}$ if and only if $\text{Hom}(E, M)$ is exact (resp. $\text{Hom}(M, E)$ is exact), for all $M \in \mathcal{M}$. Each proper class $\mathcal{P}$ with enough injectives (resp., projectives) is injectively generated (resp. projectively generated) by the class of $\mathcal{P}$-injective objects (resp. $\mathcal{P}$-projective objects). I.e., the class of objects in $\mathcal{C}$ which are injective (resp., projective) respect to any short exact sequence in $\mathcal{P}$.

We can now state our definition of phantom morphism.

**Definition 2.1.** A map $\phi: M \to N$ in $\mathcal{C}$ is called *$\mathcal{P}$-phantom* if

\[ \text{Im} (\text{Ext}(\phi, -)) \subseteq \text{Ext}_{\mathcal{P}}(N, -). \]

We will denote by $\Phi(\mathcal{P})$ the ideal of $\mathcal{P}$-phantom maps in $\mathcal{C}$.

Let $\mathcal{A}$ be a category and let us fix a cardinal, which we will always assume that is infinite and regular. An object $A$ in $\mathcal{A}$ is called *$\lambda$-presentable* if the functor $\text{Hom}_{\mathcal{A}}(A, -)$ preserves $\lambda$-directed colimits. The category $\mathcal{A}$ is called *locally $\lambda$-presentable* if it is cocomplete and there is a set $\mathcal{S}$ of $\lambda$-presentable objects in $\mathcal{A}$ such that any other object in $\mathcal{A}$ is a $\lambda$-directed colimit of objects in $\mathcal{S}$. For short, $\aleph_0$-directed colimits will be just called *direct limits*; locally $\aleph_0$-presentable categories, *locally finitely presented*; and $\aleph_0$-presentable objects, *finitely presented*. 

Definition 2.2. [1, Definition 2.27] A morphism $f : A \to B$ in $\mathcal{A}$ is said to be $\lambda$-pure if for any commutative diagram

\[
\begin{array}{c}
\begin{array}{cccc}
A' & \xrightarrow{f'} & B' \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{f} & B
\end{array}
\end{array}
\]

with $A', B'$ $\lambda$-presentable, there is a morphism $g : B' \to A$ such that $u = g \circ f'$.

If the category $\mathcal{A}$ is locally $\lambda$-presentable, we infer from [1, Proposition 2.30] that a morphism in $\mathcal{A}$ is $\lambda$-pure if and only if it is a $\lambda$-directed colimit of sections.

When the considered category $\mathcal{C}$ is abelian, every $\lambda$-pure morphism gives rise to a short exact sequence. Hence a short exact sequence is $\lambda$-pure if and only if it is a $\lambda$-directed colimit of split short exact sequences. In the sequel, we will refer to $\lambda$-pure short exact sequences as categorical pure short exact sequences and we will denote by $\mathcal{P}_{\lambda}$ the proper class of all categorical pure short exact sequences. For a detailed treatment on the theory, see [1].

Recall that a morphism $f$ is called projective if $f \in \Phi(\mathcal{P}_{sp})$, where, $\mathcal{P}_{sp}$ is the smallest proper class whose elements are all split short exact sequences. Note that if $\mathcal{C}$ has enough projective objects, then a morphism $f$ is projective if and only if it factors through a projective object.

Proposition 2.3. Let $\mathcal{C}$ be a locally finitely presented category. Then $\phi \in \Phi(\mathcal{P}_{R_0})$ if and only if it is a direct limit of projective morphisms.

Proof. Let $\phi \in \Phi(\mathcal{P}_{R_0})$, $\phi : M \to N$. We may write $M = \lim \rightarrow M_i$ as a direct limit of finitely presented objects. Let $\{\tau_i : M_i \to M\}$ be the structural morphisms. It is easy to check that the family $\{\phi \circ \tau_i\}$ is a directed system of projective morphisms and that $\phi = \lim \rightarrow (\phi \circ \tau_i)$. Conversely assume that $\{\phi : M_i \to N_i\}$ is a morphism of directed systems with each $\phi_i$, a projective morphism and call $\phi = \lim \rightarrow \phi_i$. We need to check that the upper row $\mathcal{E}\phi$ in the following pullback diagram

\[
\begin{array}{ccc}
\mathcal{E}\phi : 0 & \rightarrow K & \xrightarrow{P} \lim \rightarrow M_i \rightarrow 0 \\
\mathcal{E} : 0 & \rightarrow K & \xrightarrow{X} \lim \rightarrow N_i \rightarrow 0
\end{array}
\]

is categorical pure (i.e. it belongs to $\Phi(\mathcal{P}_{R_0})$). To see this, let $T \to \lim \rightarrow M_i$ be a morphism with $T$, finitely presented. Then $T \to \lim \rightarrow M_i$ factors through
$T \to M_i \to \varinjlim M_i$ for some index $i$. Let us compute the pullback

$$
\begin{array}{ccccccc}
0 & \rightarrow & K & \rightarrow & P' & \rightarrow & N_i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \kappa_i \\
0 & \rightarrow & K & \rightarrow & X & \rightarrow & \varinjlim N_i & \rightarrow & 0
\end{array}
$$

where $\kappa_i: N_i \to \varinjlim N_i$ is the structural morphism. As $\phi_i: M_i \to N_i$ is projective, we can find a morphism $M_i \to P'$ making the obvious diagram commutative. Finally, since the $\mathbb{E}\phi$ is a pullback, there is a $T \to P$ such that $T \to P \to \varinjlim M_i$ equals to $T \to \varinjlim M_i$. Hence, the short exact sequence $\mathbb{E}\phi$ is categorical pure.

For later use, we will denote by $\text{Mor}(\mathcal{C})$ the category whose objects are all morphisms among objects in $\mathcal{C}$. It is easy to check that $\text{Mor}(\mathcal{C})$ is also abelian, and if $\mathcal{C}$ is, in addition, locally $\lambda$-presentable, so is $\text{Mor}(\mathcal{C})$. In fact, the $\lambda$-presentable objects in $\text{Mor}(\mathcal{C})$ are just the morphisms $A \to B$ in $\mathcal{C}$ with $A$ and $B$ $\lambda$-presentable objects in $\mathcal{C}$. Given a proper class $\mathcal{P}$ in $\mathcal{C}$, the ideal $\Phi(\mathcal{P})$ may be regarded as an additive subcategory in $\text{Mor}(\mathcal{C})$. Indeed, it is easy to observe that if $\Phi(\mathcal{P})$ is a (pre)covering class in $\text{Mor}(\mathcal{C})$ then $\Phi(\mathcal{P})$ is a (pre)covering ideal in $\mathcal{C}$ (see Estrada, Guil Asensio and Ozbek [12, Proof of Theorem 3.2] for a detailed explanation).

### 3. Phantom morphisms in Grothendieck categories

Let $\mathcal{P}$ be a proper class in $R$-$\text{Mod}$. For any left $R$-module $B$ there is always a short exact sequence that serves as a test sequence to check whether $f: A \to B$ is $\mathcal{P}$-phantom. Indeed, just take a short exact sequence

$$
\mathbb{E}: \quad 0 \to K \to P \to B \to 0
$$

with a projective module $P$. Then any short exact sequence $\mathbb{E}'$ ending in $B$

$$
\begin{array}{ccccccc}
\mathbb{E}: & 0 & \rightarrow & K' & \rightarrow & P & \rightarrow & B & \rightarrow & 0 \\
\mathbb{E}': & 0 & \rightarrow & K & \rightarrow & X & \rightarrow & B & \rightarrow & 0
\end{array}
$$

is in fact a pushout of $\mathbb{E}$, i.e. $\mathbb{E}' \cong g\mathbb{E}$ for some morphism $g$. Now, to check that a morphism $f: A \to B$ is $\mathcal{P}$-phantom, it suffices to show that the pullback of $\mathbb{E}$ along $f$, $\mathbb{E}f$, belongs to $\mathcal{P}$, since any pullback $\mathbb{E}'f \cong (g\mathbb{E})f \cong g(\mathbb{E}f)$ and $\mathcal{P}$ is closed under pushouts. This observation plays an important role in proving the existence of special phantom precovers, see [22, page 67].

The above arguments cannot be applied to more general categories unless they have enough projective morphisms. However, for Grothendieck categories $\mathcal{C}$, we can still get, for each object $B \in \mathcal{C}$, a set of short exact sequences

$$
\mathbb{E}_i: \quad 0 \to K_i \to T_i \to B \to 0,
$$
indexed by $i \in I_B$, such that every short exact sequence $E$ ending in $B$ is a pushout of some of them. That is, $E \cong gE_i$, for some $i \in I_B$ and some morphism $g$, see Šťovíček [32, Proposition 5.3]. By the same argument as above, a morphism $f: A \to B$ in $C$ is $P$-phantom if and only if $E_i f$ belongs to $P$, for each $i \in I_B$.

As explained in the introduction, the ideal of (classical) phantom morphisms has a direct relation with the usual notion of purity in $R$-Mod. More generally, with the categorical purity $P_{\aleph_0}$ in a locally finitely presented Grothendieck category. However, there are interesting situations in which different kinds of purity are also meaningful. For example, the so-called stalkwise-purity in $\Omega_{\text{co}}(X)$, which will be detailed in the next section, plays an important role in several problems in the category of quasi-coherent sheaves $\Omega_{\text{co}}(X)$.

Unfortunately, when dealing with a notion of purity which is different from the categorical one, one can not use the standard arguments introduced in [21, 22, 12] to deduce the existence of phantom precovers. For example, the category $\Omega_{\text{co}}(X)$ has rarely projective objects and the proper class of stalkwise purity is not known to be projectively generated. Therefore the goal of this section will be to give a different approach to the existence of (pre)covering ideals in more general categories which extends the original one in the classical situations. To pursue this aim, we first recall the following result by Krause [24, Theorem 2.1].

**Lemma 3.1.** Let $C$ be a locally presentable category and let $F$ be an additive subcategory of $C$ which is closed under direct limits. If $F$ is closed under $\lambda$-pure subobjects or $\lambda$-quotients for some regular cardinal $\lambda$, then it is a precovering class in $C$.

We can now prove.

**Theorem 3.2.** Let $C$ be a locally $\lambda$-presentable additive category and $P$, a proper class which is closed under direct limits. Then $\Phi(P)$ is a covering ideal.

**Proof.** Note that any proper class contains all the split short exact sequences. Since $P$ is closed under direct limits, it also contains all direct limits of split short exact sequences, which implies that $P_\lambda \subseteq P$. The category of morphisms in $C$, $\text{Mor}(C)$, is also locally $\lambda$-presentable. As pointed out in Section 2, the ideal $\Phi(P)$ may be regarded as an additive subcategory of $\text{Mor}(C)$. It is also closed under direct limits because a pullback diagram is a finite limit diagram. Moreover, finite limits and direct limits commute and $P$ is closed under direct limits. To apply Lemma 3.1 for $\Phi(P)$ as a subcategory of $\text{Mor}(C)$, we claim that $\Phi(P)$ is also closed under $\lambda$-pure quotients.

Let $f: A \to B$ be a morphism in $\Phi(P)$ and $a: f \to f'$, a $\lambda$-pure epimorphism. Then it is of the form
The fact that $a$ is a $\lambda$-pure epimorphism in $\text{Mor}(C)$ implies that $a_0$ and $a_1$ are $\lambda$-pure epimorphisms in $C$ as well, since the $\lambda$-presentable objects in $\text{Mor}(C)$ are exactly the morphisms with $\lambda$-presentable domain and codomain. By assumption,

$$\text{Ext}(a_1 \circ f, -) = \text{Ext}(f' \circ a_0, -): \text{Ext}(B', -) \to \text{Ext}(P, A, -).$$

Then, for any short exact sequence of the form $\mathbb{E}f'$, we have that $(\mathbb{E}f')a_0 = \mathbb{E}(f' \circ a_0)$ belongs to $P$.

As $a_0$ is a $P_\lambda$-epimorphism, it is also a $P$-epimorphism. Then $a_0 \circ g$ is a $P$-epimorphism, which implies $\mathbb{E}f' \in P$. Therefore, we infer from Lemma 3.1 that $\Phi(P)$ is a precovering class in $\text{Mor}(C)$. Since $\Phi(P)$ is closed under direct limits, it is in fact a covering ideal (note that the argument of Xu [34, Theorem 2.2.8] for modules carries over to our setting). Hence, the ideal $\Phi(P)$ is covering by the comments at the end of Section 2. □

**Proposition 3.3.** Let $P$ be a proper class with enough injective or projective objects. Consider the following pushout diagram

$$\begin{array}{ccc}
0 & \to & K \\
\downarrow & & \downarrow \\
0 & \to & K''
\end{array}
\quad
\begin{array}{ccc}
0 & \to & B \\
\downarrow & & \downarrow \\
0 & \to & B'
\end{array}
\quad
\begin{array}{ccc}
0 & \to & A \\
\downarrow & & \downarrow \\
0 & \to & A'
\end{array}
$$

where $v$ is $P$-monic and $\varphi$ is $P$-phantom. Then $\varphi'$ is $P$-phantom, as well.
Proof. First, suppose that \( \mathcal{P} \) has enough projectives. Let \( f : T \to B' \) be a morphism with \( T \) a \( \mathcal{P} \)-projective object. Since \( a \) is a \( \mathcal{P} \)-epimorphism, there is a morphism \( f' : T \to K' \) such that \( a \circ f' = a' \circ f \). So \( a' \circ (f - u' \circ f') = 0 \). Then there exists a unique morphism \( t : T \to K \) such that \( \varphi \circ t = \varphi' \circ (f - u' \circ f') \), which means that the following diagram is commutative

\[
\begin{array}{c}
T \xrightarrow{t} B \\
\downarrow f \quad \downarrow \varphi \\
B' \xrightarrow{\varphi'} A
\end{array}
\]

Finally, the assertion follows from the fact that any pullback of an exact sequence over \( \varphi' \circ f = \varphi \circ t \) splits because \( \varphi \) is \( \mathcal{P} \)-phantom and \( T \) is \( \mathcal{P} \)-projective.

Suppose now that \( \mathcal{P} \) has enough injectives. Let

\[
E : 0 \to Y \to X \to A \to 0
\]

be an exact sequence. Then there is a commutative diagram

\[
\begin{array}{c}
0 \to Y \xrightarrow{a''} X'' \xrightarrow{\varphi} B \to 0 \\
\downarrow t' \quad \downarrow \varphi' \quad \downarrow \varphi \\
0 \to Y' \xrightarrow{a'} X' \xrightarrow{\varphi'} B' \to 0 \\
\downarrow v' \quad \downarrow v \quad \downarrow \varphi \\
0 \to Y \xrightarrow{a} X \xrightarrow{\varphi} A \to 0
\end{array}
\]

We have to show that \( \text{Hom}(E\varphi', H) \) is exact for every \( \mathcal{P} \)-injective object \( H \). Let \( h : Y \to H \) be a morphism. Since \( a'' \) is \( \mathcal{P} \)-monic by assumption, there exists an \( h'' : X'' \to H \) such that \( h'' \circ a'' = h \). But \( v \) is also \( \mathcal{P} \)-monic, so there is a morphism \( h' : K' \to H \) such that \( h' \circ v = h'' \circ t \). But the left-face of the upper cube is a pushout diagram because \( \text{Coker} v = \text{Coker} v'' \). So there exists an \( h' : X' \to H \) such that \( h' \circ v' = h'' \circ t' \). Then \( h' \circ a' = h' \circ v'' \circ a'' = h' \circ a'' = h \). Therefore, \( E\varphi' \) belongs to \( \mathcal{P} \).

\( \square \)
Proposition 3.4. Let $\mathcal{P}$ be a proper class with enough injectives. Then the kernel of any $\mathcal{P}$-phantom cover is $\mathcal{P}$-injective.

Proof. Let $\varphi: B \to A$ be a phantom cover and call $K := \text{Ker } \varphi$. Consider the commutative diagram

$$
\begin{array}{ccc}
0 & \to & K & \to & B & \to & A \\
\downarrow & & \downarrow & & \varphi & & \downarrow \\
0 & \to & E & \to & B' & \to & A
\end{array}
$$

where $K \hookrightarrow E$ is the $\mathcal{P}$-injective envelope of $K$. By Proposition 3.3, $\varphi'$ is $\mathcal{P}$-phantom. Therefore, $K$ is a direct summand of $E$ since $\varphi$ is a $\mathcal{P}$-phantom cover. \qed

We close this section by introducing new significant examples of $\mathcal{P}$-phantom morphisms in several categories. Recall that a monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $- \otimes -: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, subject to certain coherence conditions which ensure that all relevant diagrams commute. A monoidal category is called symmetric if, for every pair of objects $A, B$ in $\mathcal{C}$, there is an isomorphism $A \otimes B \cong B \otimes A$, which is natural in both $A$ and $B$. A monoidal structure $- \otimes -: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ on $\mathcal{C}$ is said to be closed if for each $A \in \mathcal{C}$, the functor $- \otimes A: \mathcal{C} \to \mathcal{C}$ has a right adjoint $[A, -]: \mathcal{C} \to \mathcal{C}$. For the whole axioms and examples of closed symmetric monoidal categories, see Kelly [23].

A monomorphism $f: X \to Y$ in a closed symmetric monoidal category $\mathcal{C}$ is called $\otimes$-pure if for all $Z \in \mathcal{C}$, $f \otimes Z$ remains monic, see [15]. In our context, we will call them geometrical pure, as in [11].

Let $\mathcal{C}$ be a Grothendieck category with a closed monoidal structure. Then $\mathcal{C}$ is locally $\lambda$-presentable for some regular cardinal $\lambda$. And two different nontrivial proper classes naturally arise. On the one hand, the class $\mathcal{P}_\lambda$ of all $\lambda$-pure (categorical) short exact sequences. And, on the other, the class $\mathcal{P}_\otimes$ of all geometrical pure short exact sequences. It was proved in [11] that $\mathcal{P}_\otimes$ has enough injectives and, in fact, it is injectively generated by a set.

Indeed, it is easy to check that each $[A, \mathcal{E}]$ is geometric pure-injective for any injective cogenerator $\mathcal{E}$ of $\mathcal{C}$ and any object $A \in \mathcal{C}$. Let us show that an exact sequence $E$ is geometric pure if and only if $\text{Hom}(E, [A, \mathcal{E}])$ is exact for every $\lambda$-presentable object $A$ of $\mathcal{C}$. The necessity is clear since $[A, \mathcal{E}]$ is geometric pure-injective. For the sufficiency, it is enough to show that $E \otimes A$ is exact in $\mathcal{C}$ for any $\lambda$-presentable object $A$, as $\otimes$ preserves any colimit. The assertion now follows since $\text{Hom}(E, [A, \mathcal{E}]) \cong \text{Hom}(E \otimes A, \mathcal{E})$ is exact and $\mathcal{E}$ is a cogenerator.

On the other hand, one may consider, for any set $\mathcal{S}$ of objects, the proper class flatly generated by $\mathcal{S}$, $\tau^{-1}(\mathcal{S})$. This class consists of all short exact sequences $E$ which remain exact under $- \otimes S$, for every $S \in \mathcal{S}$. Note that it
is clearly closed under direct limits and has enough injectives. In particular, when $C = R$-Mod and $S$ is a set of finitely presented $R$-modules, the proper class of all $\text{Hom}(S, -)$-exact sequences coincides with $\tau^{-1}(Tr(S))$, where $Tr$ is the Auslander-Bridger transpose of the finitely presented $R$-modules, see [31, Theorem 8.3].

**Corollary 3.5.** The following ideals are special covering in $C$.

(i) $\Phi(P_{\aleph_0})$, for any locally finitely presented category $C$,

(ii) $\Phi(P_{\otimes})$, for any Grothendieck closed symmetric monoidal category $C$.

The main advantage of geometrical purity is that it provides a wide class of categories whose proper purity concept may be obtained through a monoidal structure and recovers many of the known exact structures. For instance, the usual purity in $R$-Mod; the componentwise purity, as well as, the categorical purity in the category $C(R)$ of complexes of $R$-modules; the usual purity on stalks in the category $O_X$-Mod of $O_X$-modules; or the stalkwise purity in the category $Qco(X)$ of quasi-coherent sheaves over a quasi-separated scheme $X$.

4. **Phantom maps in $Qco(X)$**

The goal of this section will be to introduce a notion of phantom maps in the category $Qco(X)$ of quasi-coherent sheaves over a scheme $X$. It was proved in Enochs and Estrada [22] that $Qco(X)$ is always a Grothendieck category for any scheme. Moreover, it is locally finitely presented when $X$ is quasi-compact and quasi-separated (see [20, I.6.9.12] or [18, Proposition 7] for a precise formulation).

Let us start by discussing a simple example which will show that the usual notion of (categorical) phantom maps is not suitable in this framework. We are going to check that, in case the scheme $X = P^1(R)$ is the projective line over any commutative ring $R$, there are no non-zero classical phantom maps in $Qco(X)$. Let us cover $X$ by the usual affine open subsets $U = U \leftarrow U \cap V \rightarrow V$. The structure sheaf of $X$ is given by the following representation of $U$,

$$O = R[x] \hookrightarrow R[x,x^{-1}] \hookrightarrow R[x^{-1}].$$

The Serre’s twisting sheaves $O(n)$ are given by

$$O(n) = R[x] \hookrightarrow R[x,x^{-1}] \xleftarrow{\cdot x^n} R[x^{-1}],$$

with $n \in \mathbb{Z}$. It is known that the family of twisting sheaves $\{O(n)\}_{n \in \mathbb{Z}}$ generates the category $Qco(P^1(R))$. Indeed it suffices to take the family $\{O(-n)\}_{n \in \mathbb{N}}$ to generate $Qco(P^1(R))$.

On the other hand, any quasi-coherent sheaf $\mathcal{M} \in Qco(P^1(R))$ is determined by a representation of $U$

$$\mathcal{M} = M \xrightarrow{f} P \xleftarrow{g} N,$$
where $M \in R[x]\text{-Mod}$, $N \in R[x^{-1}]\text{-Mod}$, $P \in R[x^{-1},x]\text{-Mod}$, $f$ is an $R[x]$-linear map and $g$, a $R[x^{-1}]$-linear, satisfying that $S^{-1}f : S^{-1}M \to P$ and $T^{-1}g : T^{-1}N \to P$ are isomorphisms, where $S = \{1,x,x^2,\ldots\}$ and $T = \{1,x^{-1},x^{-2},\ldots\}$.

**Lemma 4.1.** Let $0 \neq T = (M \xrightarrow{f} P \xrightarrow{g} N) \in \mathcal{Qco}(P^1(R))$. Given $0 \neq m \in M$ (resp., $0 \neq p \in P$, $0 \neq y \in N$), there exists a natural number $k_m$ (resp., $k_p$, $k_y$) such that for every $l \geq k_m$ (resp., $l \geq k_p$, $l \geq k_y$) and every subset $\Delta \subseteq \mathbb{Z}$, any morphism

$$(\gamma_1,\gamma_2) : T \to \oplus_{n \in \Delta} \mathcal{O}(-n - l)$$

maps $m$ (resp., $p$, $y$) to zero.

**Proof.** Since $T^{-1}g : T^{-1}N \to P$ is an isomorphism, we get that $f(m) = g(a)/x^{-l}$, for some $a \in N$ and $l \in \mathbb{N}$. That is, $g(a) = x^{-l}f(m)$. Set $k = l + 1$.

Let $\Delta \subseteq \mathbb{Z}$ and consider a morphism $(\gamma_1,\gamma_2) : T \to \oplus_{n \in \Delta} \mathcal{O}(-n - k)$. Let us write

$$\gamma_1(m) = (\ldots,p_1(x),\ldots,p_k(x),\ldots)$$

and

$$\gamma_2(a) = (\ldots,q_1(x^{-1}),\ldots,q_t(x^{-1}),\ldots).$$

Then,

$$\gamma \circ g(a) = \gamma(x^{-l}f(m)) = x^{-l}\gamma(f(m)) = \gamma_1(m) = x^{-l}(p_1(x),\ldots,p_k(x)).$$

Thus, $ord(x^{-l}p_i(x)) \geq -l$, for every $1 \leq i \leq k$. But, by the commutativity of the diagram, we also get that

$$\gamma \circ g(a) = \oplus_{n \in \Delta} x^{-n+k}(\gamma_2(a)) = \oplus_{n \in \Delta} x^{-n+k}(q_1(x^{-1}),\ldots,q_t(x^{-1}))$$

$$= x^{-k}r_1(x^{-1}),\ldots,r_t(x^{-1})),$$

with $ord(x^{-k}r_i(x^{-1})) \leq -k = -l - 1$, for all $1 \leq i \leq t$. This shows that $\gamma_1(m) = 0$. \qed

**Corollary 4.2.** Assume that $T \in \mathcal{Qco}(P^1(R))$ is finitely presented. Then, there exists a natural number $k = k(T)$ such that there are no nonzero morphisms from $T$ into an arbitrary direct sum of the elements of the family

$\{\mathcal{O}(-n - k) : n \in \mathbb{N}\}.$

**Proof.** Without lost of generality, we may assume that $M$ is generated by $m_1,\ldots,m_s$; $P$ is generated by $p_1,\ldots,p_s$; and $N$ is generated by $y_1,\ldots,y_t$. Then, in view of Lemma 4.1, we just have to take $k \geq \max\{k_{m_i},k_{p_i},k_{y_i} : i = 1,\ldots,s\}$. \qed

**Proposition 4.3.** The only projective morphism in $\mathcal{Qco}(P^1(R))$ is the zero map.
Proof. Suppose that $\mathcal{M} \rightarrow \mathcal{L}$ is a projective morphism in $\mathcal{Qco}(\mathbb{P}^1(R))$. Since $\mathcal{Qco}(\mathbb{P}^1(R))$ is locally finitely presented, we can assume that $\mathcal{M}$ is finitely presented. There exists an epimorphism $\bigoplus_{n \in \mathbb{N}} \mathcal{O}(-n)^{(X_n)} \rightarrow \mathcal{L}$. Let $(r,s,v)_n$ be the composition map $\mathcal{O}(-n) \hookrightarrow \bigoplus_{n \in \mathbb{N}} \mathcal{O}(-n)^{(X_n)} \rightarrow \mathcal{L}$.

Let us fix a natural number $n_0$. For any $n \in \mathbb{N}$, we can consider the morphisms $\mathcal{O}(-n-n_0) \rightarrow \mathcal{L}$ given by $(r,s,x^{-n_0}v)_n$ and $(x^{n_0}r,x^{n_0}s,v)_{n+n_0}$, respectively. These two morphisms induce a morphism $\mathcal{O}(-n-n_0) \oplus \mathcal{O}(-n-n_0) \rightarrow \mathcal{L}$.

In turn, these morphisms induce an epimorphism $\bigoplus_{n \in \mathbb{N}} \left( \mathcal{O}(-n-n_0)^{(X_n)} \oplus \mathcal{O}(-n-n_0)^{(X_n)} \right) \rightarrow \psi \mathcal{L}$.

Now, as $\mathcal{M} \in \mathcal{Qco}(\mathbb{P}^1(R))$ is finitely presented, Corollary 4.2 states that there exists an $n_0 \in \mathbb{N}$ such that it is not possible to factorize any nonzero morphism $\mathcal{M} \rightarrow \mathcal{L}$ through $\psi$. $\square$

Corollary 4.4. There are no non-trivial phantom maps in $\mathcal{Qco}(\mathbb{P}^1(R))$.

Proof. The category $\mathcal{Qco}(\mathbb{P}^1(R))$ is locally finitely presented, so the result follows from propositions 4.3 and 2.3. $\square$

In view of this example, we will devote the rest of this section to introduce a new (Zariski-local) definition of phantom morphisms in $\mathcal{Qco}(X)$. Recall that $\mathcal{Qco}(X)$ is a coreflective subcategory of the category $\mathcal{O}_X$-Mod of $\mathcal{O}_X$-modules; that is, the inclusion $i: \mathcal{Qco}(X) \hookrightarrow \mathcal{O}_X$-Mod has a right adjoint, called coherator. Note that if the scheme $X$ is quasi-separated then, for each open affine $U$ and each inclusion $\iota: U \rightarrow X$, the restriction functor $\text{res}_U: \mathcal{Qco}(X) \rightarrow \mathcal{Qco}(U)$ is a left adjoint functor of the direct image functor $\iota_*: \mathcal{Qco}(U) \rightarrow \mathcal{Qco}(X)$ which implies that $\text{res}_U$ preserves all colimits. Therefore a pushout diagram in $\mathcal{Qco}(X)$ gives rise to a pushout diagram on each module of sections over any affine open subset. Conversely, for any family $\{\mathcal{F}_i\}_{i \in I}$ of quasi-coherent sheaves, $(\text{colim}_I \mathcal{F})(U) \cong \text{colim}_I (\mathcal{F}_i(U))$, where $U$ is an affine open subset of $X$.

Lemma 4.5. Let

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
\end{array}
$$

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
\end{array}
$$

be a commutative diagram in $\mathcal{Qco}(X)$ with exact rows. Then the diagram induced on the modules of sections over any affine open subset $U$ is also a pullback.
Proof. Let $U$ be an affine open subset. Then we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{F}(U) & \rightarrow & \mathcal{G}(U) & \rightarrow & \mathcal{H}(U) & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{F}(U) & \rightarrow & \mathcal{G}''(U) & \rightarrow & \mathcal{H}''(U) & \rightarrow & 0
\end{array}
$$

in $\mathcal{O}_X(U)$-Mod. The right square of such a diagram is always a pullback. □

Our next lemma is straightforward to prove.

Lemma 4.6. Let

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{G} & \rightarrow & \mathcal{H} & \rightarrow & 0
\end{array}
$$

be a diagram in $\mathfrak{Qco}(X)$ with the row exact. Let us construct the family of pullback diagrams

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{F}(U) & \rightarrow & M_U & \rightarrow & \mathcal{G}(U) & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{F}(U) & \rightarrow & \mathcal{G}'(U) & \rightarrow & \mathcal{G}''(U) & \rightarrow & 0
\end{array}
$$

for each affine open subset $U$ and the pullback diagram in $\mathfrak{Qco}(X)$

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{G}' & \rightarrow & \mathcal{G} & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{G}' & \rightarrow & \mathcal{G}'' & \rightarrow & 0
\end{array}
$$

Then $\mathcal{G}'(U) \simeq M_U$ for every affine open subset $U$.

We can now introduce a Zariski-local notion of purity in $\mathfrak{Qco}(X)$. The following result was proved in [10, Proposition 3.4].

Proposition 4.7. Let $X$ be a scheme and $\mathcal{F}, \mathcal{G} \in \mathfrak{Qco}(X)$. The following statements are equivalent:

(i) $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}$ is geometrical pure exact in $\mathcal{O}_X$-Mod;

(ii) There exists an open covering of $X$ by affine open sets, $\mathcal{U} = \{U_i\}$, such that $0 \rightarrow \mathcal{F}(U_i) \rightarrow \mathcal{G}(U_i)$ is pure in $\mathcal{O}_X(U_i)$-Mod;

(iii) $0 \rightarrow \mathcal{F}_x \rightarrow \mathcal{G}_x$ is pure in $\mathcal{O}_{X,x}$-Mod, for each $x \in X$;

We will call a short exact sequence in $\mathfrak{Qco}(X)$ stalkwise pure if it satisfies the above equivalent conditions. We will denote by $\mathcal{P}_{st}$ the corresponding proper class. Since stalks, colimits and tensor products in $\mathfrak{Qco}(X)$ commute, we have the following ordering of proper classes in $\mathfrak{Qco}(X)$

$$
\mathcal{P}_{st} \subseteq \mathcal{P}_{st} \subseteq \mathcal{P}_{\otimes},
$$
where $\lambda$ is a regular cardinal for which $\mathcal{Q}co(X)$ is locally $\lambda$-presentable. Therefore, $\Phi(P_\lambda) \subseteq \Phi(P_{st}) \subseteq \Phi(P_\emptyset)$. We want to study how $P_{st}$-phantom morphisms carry phantom-property on sections.

**Definition 4.8.** A morphism $f : \mathcal{G} \to \mathcal{F}$ in $\mathcal{Q}co(X)$ is said to be **locally phantom** if $f_U$ is phantom in $\mathcal{O}_X(U)$-Mod, for each affine open subset $U$. The ideal of locally phantom morphisms in $\mathcal{Q}co(X)$ will be denoted by $\Phi'$. Let us begin by characterizing phantom maps among modules over a commutative ring in terms of prime ideals.

**Lemma 4.9.** Let $R$ be a commutative ring and $f : M' \to M$, a homomorphism of $R$-modules. Then $f$ is phantom if and only if $f_P$ is phantom in $R_P$-Mod for every $P \in \text{Spec}(R)$.

**Proof.** Suppose that $f$ is a phantom morphism in $R$-Mod and consider the pullback diagram in $R_P$-Mod for $P \in \text{Spec}(R)$

$$
\begin{array}{ccc}
0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & M'_P & \longrightarrow & 0 \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & M_P & \longrightarrow & 0.
\end{array}
$$

Using the canonical morphisms $M \to M_P$ and $M' \to M'_P$, we get a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & A & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & 0 \\
0 & \longrightarrow & A & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\
0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & M'_P & \longrightarrow & 0 \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & M_P & \longrightarrow & 0
\end{array}
$$

in which each face is a diagram in $R$-Mod except to the bottom face, which is in $R_P$-Mod. The upper rectangle is a pullback diagram, so by our assumption, the upper row is pure-exact. But purity is preserved under localization and $A_P \simeq A$ when we think of $A$ as $R$-module, see Pinzon[29] Remark 3.8]. Therefore, $f_P$ is phantom in $R_P$-Mod.

The converse follows from the fact that the localization functor preserves pullback diagrams of epimorphisms and that a monomorphism $\iota$ is pure if and only if each localization $\iota_P$, $P \in \text{Spec}(R)$, is pure. \qed

**Theorem 4.10.** Let $f : \mathcal{G} \to \mathcal{F}$ be a morphism in $\mathcal{Q}co(X)$. The following are equivalent:

(i) $f$ is locally phantom;
(ii) There is a cover $\mathcal{U}$ of $X$ consisting of affine open subsets such that $f_U$ is phantom for every $U \in \mathcal{U}$;
(iii) $f_x$ is phantom in $\mathcal{O}_{X,x}$-Mod, for all $x \in X$;
Proof. Follows from Lemma 4.9.

Proposition 4.11. Let \( f : \mathcal{G} \to \mathcal{F} \) be a morphism in \( \Omega \mathfrak{co}(X) \). If \( f \) is locally phantom then it is also \( \mathcal{P}_{st} \)-phantom. That is, \( \Phi' \subseteq \Phi(\mathcal{P}_{st}) \).

Proof. Let

\[
\begin{array}{ccc}
\mathbb{E}f : 0 & \longrightarrow & \mathcal{F}' \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow 0 \\
\downarrow & & \downarrow f \\
\mathbb{E} : 0 & \longrightarrow & \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow \mathcal{F} \longrightarrow 0
\end{array}
\]

be a pullback diagram in \( \Omega \mathfrak{co}(X) \). Note that \( \mathbb{E}(U) \) is exact for each affine open subset \( U \subseteq X \) and \( (\mathbb{E}f)(U) = \mathbb{E}(U)f_U \) by Lemma 4.5. By assumption, \( (\mathbb{E}f)(U) \) is pure-exact for each affine open subset \( U \subseteq X \). Hence, \( \mathbb{E} \in \mathcal{P}_{st} \).

Proposition 4.12. If \( X \) is semi-separated, then \( f : \mathcal{G} \to \mathcal{F} \) is \( \mathcal{P}_{st} \)-phantom in \( \Omega \mathfrak{co}(X) \) if and only if it is locally phantom. Moreover, in this case, \( \Phi(\mathcal{P}_\otimes) = \Phi(\mathcal{P}_{st}) = \Phi' \).

Proof. For an affine open subset \( U \), let

\[
\begin{array}{ccc}
0 & \longrightarrow & N \longrightarrow M' \longrightarrow \mathcal{G}(U) \longrightarrow 0 \\
| & & | \\
0 & \longrightarrow & N \longrightarrow M \longrightarrow \mathcal{F}(U) \longrightarrow 0 \\
| & & | \\
0 & \longrightarrow & i_*\tilde{N} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{G} \longrightarrow 0 \\
| & & | \\
0 & \longrightarrow & i_*\tilde{N} \longrightarrow \mathcal{F} \longrightarrow 0 \\
| & & | \\
0 & \longrightarrow & i_*\tilde{N} \longrightarrow t_*\tilde{M} \longrightarrow \mathcal{F} |_U \longrightarrow 0
\end{array}
\]

be a pullback diagram in \( \mathcal{O}_X(U) \)-Mod. Note that \( i_* \) is an exact functor from \( \Omega \mathfrak{co}(U) \) to \( \Omega \mathfrak{co}(X) \) since \( X \) is semi-separated. Consider the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & i_*\tilde{N} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{G} \longrightarrow 0 \\
| & & | \\
0 & \longrightarrow & i_*\tilde{N} \longrightarrow \mathcal{F} \longrightarrow 0 \\
| & & | \\
0 & \longrightarrow & i_*\tilde{N} \longrightarrow t_*\tilde{M} \longrightarrow \mathcal{F} |_U \longrightarrow 0
\end{array}
\]

where each face is a pullback diagram in \( \Omega \mathfrak{co}(X) \). By assumption, the upper exact sequence is stalkwise pure. Then the short exact sequence of modules of sections over \( U \), \( 0 \to N \to \mathcal{F}'(U) \to \mathcal{G}(U) \to 0 \), which is isomorphic to \( 0 \to N \to M' \to \mathcal{G}(U) \), is pure-exact. So \( f_U \) is a phantom morphism. This shows that \( \Phi' = \Phi(\mathcal{P}_{st}) \). Finally, by [12, Proposition 2.10], we have that \( \mathcal{P}_\otimes = \mathcal{P}_{st} \) for any quasi-separated scheme (so, in particular, for any semi-separated scheme). Therefore, \( \Phi' = \Phi(\mathcal{P}_{st}) = \Phi(\mathcal{P}_\otimes) \).

Lemma 4.13. The ideals \( \Phi(\mathcal{P}_{st}) \), \( \Phi(\mathcal{P}_\otimes) \) and \( \Phi' \) in \( \Omega \mathfrak{co}(X) \) are closed under direct limits. The ideal \( \Phi(\mathcal{P}_\lambda) \) is closed under direct limits when \( \lambda = \aleph_0 \), that is, when \( \Omega \mathfrak{co}(X) \) is locally finitely presented.
Proof. It follows from the fact that $\mathcal{P}_{st}$ and $\mathcal{P}_\otimes$ are closed under direct limits and finite coproducts. Finally, note that the result holds for $\mathcal{P}_\lambda$ when $\lambda = \aleph_0$. □

Let $\mathcal{P}_{O_X}$ and $\mathcal{P}_{Qco(X)}$ denote the proper classes of geometrical pure short exact sequences in $O_X$-Mod and $\mathfrak{Q}_{co}(X)$, respectively. As noted in the comments before Corollary 3.5, both proper classes are injectively generated by a set. Then, the proper class $\mathcal{P}_{st}$ in $\mathfrak{Q}_{co}(X)$ is also injectively generated by a set. Indeed, from [10, Lemma 4.7], we know that the cohera tor functor $Q$ transforms geometrical pure injectives in $O_X$-Mod into stalkwise pure injectives in $\mathfrak{Q}_{co}(X)$. By Lemma 4.7, a short exact sequence $E$ of quasi-coherent sheaves is stalkwise pure if and only if $i(E)$ is geometrical pure in $O_X$-Mod (where $i: : \mathfrak{Q}_{co}(X) \to O_X$-Mod is the inclusion functor). Since $\mathcal{P}_{O_X}$ is injectively generated by a set, say $S$, it follows that $\mathcal{P}_{st}$ is injectively generated by the set $Q(S)$ because $(i, Q)$ is an adjoint pair. Thus, we have proved the following

Corollary 4.14. The following holds for $\mathcal{C} := \mathfrak{Q}_{co}(X)$:

(i) The ideals $\Phi(\mathcal{P}_{st})$ and $\Phi(\mathcal{P}_\otimes)$, are special covering in $\mathfrak{Q}_{co}(X)$.

(ii) The ideal $\Phi'$ is covering in $\mathfrak{Q}_{co}(X)$.

Let us close the paper by making the following observation. It is easy to check that any morphism $f: \mathcal{F} \to \mathcal{G}$ in $\mathfrak{Q}_{co}(X)$, with $\mathcal{F}$ a flat quasi-coherent sheaf, belongs to $\Phi'$ (and thus, to $\Phi(\mathcal{P}_{st}) \subseteq \Phi(\mathcal{P}_\otimes)$). This means that, when $\mathfrak{Q}_{co}(X)$ has a flat generator, covers with respect to any of the ideals $\Phi' \subseteq \Phi(\mathcal{P}_{st}) \subseteq \Phi(\mathcal{P}_\otimes)$ are epimorphisms. For instance, this is the case when $X$ is a quasi-compact and semi-separated scheme (see Alonso Tarrío, Jeremías López and Lipman [2]).

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