Finite-time self-similar rupture in a generalized elastohydrodynamic lubrication model

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Abstract

Thin film rupture is a type of nonlinear instability that causes the solution to touch down to zero at finite time. We investigate the finite-time rupture behavior of a generalized elastohydrodynamic lubrication model. This model features the interplay between destabilizing disjoining pressure and stabilizing elastic bending pressure and surface tension. The governing equation is a sixth-order nonlinear degenerate parabolic partial differential equation parameterized by exponents in the mobility function and the disjoining pressure, respectively. Asymptotic self-similar finite-time rupture solutions governed by a sixth-order leading-order equation are analyzed. In the weak elasticity limit, transient self-similar dynamics governed by a fourth-order similarity equation are also identified.

Keywords: high-order nonlinear PDEs, degenerate PDEs, singularities, thin films

1. Introduction

This paper presents a study of the development of finite-time singularities in a one-dimensional sixth-order partial differential equation for \( h(x,t) \) on a finite domain, \( 0 \leq x \leq L \),

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left[ h^n \frac{\partial}{\partial x} \left( B \frac{\partial^4 h}{\partial x^4} - \frac{\partial^2 h}{\partial x^2} + \frac{1}{m h^m} \right) \right],
\]

where the parameters \( B, m, n > 0 \). This model is motivated by the work by Carlson and Mahadevan \cite{1} on adhesive elastohydrodynamic touchdown that occurs as an elastic sheet begins to adhere to a wall. The PDE \cite{1} fits into the framework of classical lubrication theory which has been widely studied for the dynamics of thin layers of slow viscous fluids spreading over solid surfaces \cite{2,3}. Under the long-wave approximation, the lubrication equation for the evolution of the thickness (or the height \( h \) of the free-surface) of the fluid layer can be derived from Navier-Stokes equations in the low Reynolds number limit,

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( \mathcal{M}(h) \frac{\partial p}{\partial x} \right),
\]

where the mobility function \( \mathcal{M}(h) = h^n \) with \( n > 0 \). Here, \( n = 3 \) corresponds to the no-slip boundary condition at the liquid-solid interface, and more general Navier slip condition can be incorporated via \( \mathcal{M}(h) = h^3 + \lambda h^2 \). Following the work of Young and Stone \cite{4}, we define the dynamic pressure \( p \) to incorporate the elastohydrodynamic effects,

\[
p = B \frac{\partial^4 h}{\partial x^4} - \frac{\partial^2 h}{\partial x^2} + \Pi(h), \quad B > 0,
\]
where $\partial^4 h/\partial x^4$ represents the elastic bending pressure due to long-wavelength sheet deformations, $B > 0$ is a scaling parameter for the bending pressure, $\partial^2 h/\partial x^2$ represents the surface tension between the elastic sheet and liquid, and the disjoining pressure

$$\Pi(h) = \frac{A}{h^2}, \quad m > 0, \quad A = \frac{1}{m} > 0$$

characterizes the wetting property of the solid substrate, where $A > 0$ is the Hamaker constant. For $m = 3$, $\Pi(h) = A/h^3$ corresponds to the van der Waals model [5] for the destabilizing intermolecular adhesion pressure [1]. Other elastohydrodynamic lubrication models [4, 6] have also used the disjoining pressure to prevent thin film rupture from happening [7–9].

Starting from positive and finite-mass initial data $h_0(x) > 0$ at time $t = 0$, the dynamics of the model (1) are governed by the interaction between the higher-order elastic bending pressure, the surface tension, and the disjoining pressure. Following the work of Young and Stone [4], we consider the no-flux boundary conditions at $x = 0$ and $x = L$,

$$h_x = h_{xxx} = h_{xxxxx} = 0, \quad \text{at } x = 0, L.$$  

The dynamics of (1) can also be described by a monotone decreasing energy functional

$$\mathcal{E} = \int_0^L B \left( \frac{\partial^2 h}{\partial x^2} \right)^2 + \frac{1}{2} \left( \frac{\partial h}{\partial x} \right)^2 + U(h) \, dx, \quad \text{with} \quad \frac{d\mathcal{E}}{dt} = - \int_0^L h^n \left( \frac{\partial h}{\partial x} \right)^2 \, dx \leq 0,$$

where $U(h)$ is the interaction potential that satisfies $U'(h) = \Pi(h)$.

Thin film rupture is a type of nonlinear instability that leads to finite-time singularities as the film thickness approaches zero at a point. That is, $h \to 0$ at an isolated point, $x = x_c$, at a finite critical time $t = t_c$. It was shown in [10] that thin film equations can yield self-similar rupture singularities driven by van der Waals forces. Different types of finite-time rupture dynamics have been investigated in a family of generalized lubrication equations parametrized by exponents in conservative and non-conservative loss terms, respectively [7][11]. In this work, we focus on the impact of the sixth-order bending pressure and the fourth-order surface tension terms on the rupture dynamics of the generalized elastohydrodynamic lubrication equation (1).

Finite-time singularities in thin film equations can result from growth in spatial perturbations due to strong instabilities. To perform a stability analysis of flat film solutions in (1), we perturb the spatially-uniform base state $h = \bar{h}$ by an infinitesimal Fourier mode $h(x,t) = \bar{h} + \delta e^{ik\pi x/L + \lambda t} + O(\delta^2)$, where $k$ is the wave number, $\lambda$ is the growth rate of disturbances, and the initial amplitude $\delta \ll 1$. Substituting the expansion into model (1) and linearizing about $h = \bar{h}$ yields the dispersion relation

$$\lambda = -\bar{h}^n \left( \frac{k\pi}{L} \right)^2 \left[ B \left( \frac{k\pi}{L} \right)^4 + \left( \frac{k\pi}{L} \right)^2 - \frac{1}{m+1} \right].$$

This relation indicates that the uniform film $\bar{h} < h_c$ is long-wave unstable with respect to perturbations associated with any wave number $k \in \mathbb{Z}^+$, where the critical film thickness $\bar{h}_c = \left[ B(k\pi/L)^4 + (k\pi/L)^2 \right]^{-1/(m+1)}$. Moreover, the relation (5) also shows that the disjoining pressure $\Pi(h) = 1/mh^n$ is destabilizing, and both the elastic bending pressure $B\partial^4 h/\partial x^4$ and the surface tension $-\partial^2 h/\partial x^2$ are stabilizing in the PDE (1).

The structure of the paper is as follows. In Section 2 we analyze the asymptotic self-similar rupture solutions in (1), with a focus on the role of the bending pressure term. Numerical studies for the singularity solutions are presented in Section 3 followed by concluding remarks in Section 4.
2. Self-similar rupture solutions

The solutions of (1) leading to rupture at a critical location \( x = x_c \) for \( t \to t_c \) can take the form of self-similar solutions. Various self-similar rupture solutions of thin-film type equations have been previously analyzed \([1, 7, 12]\). Specifically, the work of Carlson and Mahadevan \([1]\) investigated the self-similar rupture solutions to a model that is equivalent to (1) for \( m = n = 3 \) without the fourth-order surface tension term.

We express the solutions of model (1) using the following self-similar ansatz,

\[
h(x,t) \sim \tau^\alpha H(\eta), \quad \tau = t_c - t, \quad \eta = \frac{x-x_c}{\tau^{\beta}}, \quad \alpha, \beta > 0,
\]

where the scaling parameter \( \alpha > 0 \) corresponds to finite-time touchdown, \( h \to 0 \), at \( t = t_c \), and the scaling parameter \( \beta \) describes the spatial focusing at \( x_c \) as \( \tau \to 0 \). Moreover, the far-field solution \( h \) away from the critical location \( x_c \) should evolve slowly in time as the finite-time singularity is approached. That is, for any fixed point away from the critical location \( x_c \), the time derivative term \( h_t \) is bounded. This leads to the far-field boundary condition on the similarity solution \( H(\eta) \),

\[
\alpha H - \beta \eta H_{\eta} = 0 \quad \text{as } |\eta| \to \infty.
\]

Substituting the ansatz (6) into the PDE (1) leads to the ordinary differential equation

\[
\tau^{\alpha-1} \left( -\alpha H + \beta H_{\eta} \frac{dH}{d\eta} \right) = \tau^{\alpha-2\beta} \frac{d}{d\eta} \left[ H^n \frac{d}{d\eta} \left( \tau^{\alpha-4\beta} B \frac{d^4H}{d\eta^4} \right) + \tau^{m\alpha} \right].
\]

For PDE models with exact similarity solutions, the values of the scaling parameters \( \alpha \) and \( \beta \) can be identified by separating out \( \tau \) and reducing the PDE to an ODE for the similarity solution \( H(\eta) \). However, it is impossible to find an exact similarity solution for (8) due to the number of terms in the equation. Instead, we seek an asymptotically self-similar solution of the PDE determined by the leading-order dominant balance of terms for the limit \( \tau \to 0 \).

2.1. Sixth-order similarity solution for \( B = O(1) \) and \( 0 < n < (3m+3)/2 \)

For \( \tau \to 0 \) with \( B = O(1) \), there are four possible leading-order terms in (8), the time derivative term \( \tau^{\alpha-1}(-\alpha H + \beta H_{\eta}) \), the elastic bending pressure term \( B\tau^{(n+1)\alpha-6\beta}(H^n H(5)_{\eta})_\eta \), the surface tension term \( \tau^{(n+1)\alpha-4\beta}(H^n H_{\eta\eta\eta})_{\eta} \), and the disjoining pressure term \( \tau^{(n-m)\alpha-2\beta}(H^n \frac{1}{m} H^{-m} \eta)_{\eta} \). In the limit \( \tau \to 0 \), we have \( \tau^{(n+1)\alpha-6\beta} \gg \tau^{(n+1)\alpha-4\beta} \). Therefore, the dominant balance for dynamic solutions is given by the system of equations \( \alpha - 1 = (n+1)\alpha - 6\beta = (n-m)\alpha - 2\beta \), yielding the scalings

\[
\alpha = \frac{2}{3m - 2n + 3}, \quad \beta = \frac{m+1}{6m - 4n + 6},
\]

and \( H(\eta) \) satisfies the sixth-order similarity ODE

\[
-\alpha H + \beta \eta \frac{dH}{d\eta} = \frac{d}{d\eta} \left[ H^n \frac{d}{d\eta} \left( B \frac{d^4H}{d\eta^4} + \frac{1}{mH^m} \right) \right].
\]

With the scalings (9), the far-field boundary condition (7) reduces to \( H - \frac{m+1}{4} \eta \frac{dH}{d\eta} = 0 \) as \( |\eta| \to \infty \), which indicates the asymptotic far-field behavior \( H(\eta) \sim C \eta^{4/(m+1)} \) as \( |\eta| \to \infty \). The similarity equation (10) corresponds to the sixth-order leading-order PDE

\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left[ h^n \frac{\partial}{\partial x} \left( B \frac{\partial^4 h}{\partial x^4} + \frac{1}{mH^m} \right) \right], \quad B, m, n > 0,
\]

which characterizes the balance between the sixth-order stabilizing elastic bending pressure term and the second-order destabilizing disjoining pressure term. Since the scaling parameters \( \alpha, \beta > 0 \) in (10), we need \( 3m - 2n + 3 > 0 \), or \( n < (3m + 3)/2 \) for the self-similar rupture solution ansatz (6) to hold.
2.2. Transient self-similar dynamics for $B \ll 1$ and $0 < n \leq m$

In the weak elasticity limit, $B \ll 1$, for the regime when $B \ll \tau^{2\beta}$, the surface tension term $d^2H/\eta^2$ dominates over the bending pressure term $Bd^4H/d\eta^4$ in (8). Therefore, we have the equations of dominant balance between the time derivative term, the fourth-order stabilizing term, and the second-order destabilizing term, $\alpha - 1 = (n + 1)\alpha - 4\beta = (n - m)\alpha - 2\beta$, which leads to the scaling

$$\alpha = \frac{1}{2m - n + 2}, \quad \beta = \frac{m + 1}{4m - 2n + 4},$$

and the similarity solution $H(\eta)$ satisfies the fourth-order nonlinear ODE

$$-\alpha H + \beta H \frac{dH}{d\eta} = \frac{d}{d\eta} \left[ H^n \frac{d}{d\eta} \left( \frac{d^2H}{d\eta^2} + \frac{1}{mH^m} \right) \right].$$

The leading order terms involve represent the time derivative, the surface tension, and the disjoining pressure. In this case, the far-field boundary condition (7) becomes $H - \frac{m+1}{2} \eta \frac{dH}{d\eta} = 0$ as $|\eta| \to \infty$, which indicates the asymptotic far-field behavior $H(\eta) \sim C\eta^{2/(m+1)}$ as $|\eta| \to \infty$.

The similarity equation (13) corresponds to the fourth-order PDE

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left[ h^n \frac{\partial}{\partial x} \left( \frac{\partial^2 h}{\partial x^2} + \frac{1}{mH^m} \right) \right], \quad m, n > 0,$$

which falls into a class of thin film-type equations studied by Bertozzi and Pugh [13] and Chou and Kwong [14], $h_t + (h^n h_{xxx})_x + (h^r h_x)_x = 0$, where $n, r \in \mathbb{R}$ and $n > 0$. This equation is identical to (14) with $r = n - m - 1$, and the conditions for the global existence of its solutions and finite-time singularities have been established in terms of the exponents $n$ and $r$ in the competing second- and fourth-order terms. Specifically, it was shown that ruptures in finite time can occur for the ranges $n > 0$ and $r \leq -1$. Therefore, the established rupture criterion for the fourth-order PDE (14) is that the exponents $m, n$ satisfy $0 < n \leq m$. This range also guarantees that the scaling coefficients $\alpha, \beta > 0$ in (12).

3. Numerical studies

Next, we numerically solve the nonlinear PDE (1) using a fully implicit second-order finite difference method with adaptive time stepping. The sixth-order PDE is expressed as a discretized, cell-centered system of six first-order differential equations for $h, k \equiv h_x, p \equiv k_x, q \equiv p_t, s \equiv Bq_x - p + \frac{3}{n} h^{-m}$ and $w \equiv h^n s_x$. To identify the dynamic transition from the transient fourth-order self-similar solution to the later stage sixth-order self-similar rupture profiles, it is useful to track the relationship between the local feature of the PDE solution at the critical location $x = x_c$. The form of the self-similar ansatz (6) indicates that at $x = x_c$, we have $h(x_c, t) = \tau^\alpha H(0), h_{xx}(x_c, t) = \tau^{\alpha-2\beta} H'(0)$, and $h_{xxxx}(x_c, t) = \tau^{\alpha-4\beta} H^{(4)}(0)$. Therefore, we obtain the relation between the linearized curvature $h_{xx}(x_c, t)$ and the solution $h(x_c, t)$ at $x = x_c$,

$$h_{xx}(x_c, t) = C h(x_c, t)^{\nu}, \quad \text{where } \nu = 1 - \frac{2\beta}{\alpha},$$

where the coefficient $C$ is uniquely determined by the local property of the similarity function $H(\eta)$. Based on the scaling coefficients (12) for the fourth-order self-similar dynamics and the coefficients (9) for the sixth-order self-similar rupture solutions, we define the critical fourth-order and sixth-order curvature-magnitude exponent, $\nu = \nu_4(m)$ and $\nu = \nu_6(m)$, respectively,

$$\nu_4(m) = -m, \quad \nu_6(m) = (1 - m)/2.$$
Transient behavior satisfies the exponents $\nu = 1$, starting from the initial condition $h_0(x) = 0.5 + 0.01 \cos(\pi x)$ leading to finite-time rupture. The PDE solution for $t < 0.33174$ scaled as $H(\eta/H_0^2)/H_0^* = 0.732$. (Right) Later stage dynamics for $0.33178 < t < t_e = 0.33179$ showing that the PDE solution scaled as $H(\eta/H_0^2)/H_0^*$ converges to the similarity solution $H(\eta)$ of the sixth-order ODE (10), where $H_0^* = H(0) = 2.111$.

Figure 2: Plots of (left) $h(x_c,t)$ vs. $h(x_c,t)$ and (right) $h(x_c,t)$ vs. $-h_{xxxx}(x_c,t)$ for the simulation in Fig. 1. The transition between the early stage and the later stage self-similar behaviors agree with the analytical predictions (15) and (17) with the scaling parameters given by (12) for the early stage transient behavior and (9) for the later stage rupture behavior.

We note that these critical exponents only depend on the disjoining pressure exponent $m$. Similarly, we have

$$h_{xxxx}(x_c,t) = C_2 h(x_c,t)^\mu, \quad \text{where} \quad \mu = 1 - \frac{4\beta}{\alpha},$$

which represents the relation between the elastic bending pressure and the film thickness at $x = x_c$.

Figure 1 presents the dynamic solution of the PDE (1) approaching a finite-time singularity at $x_c = 1$, starting from the initial condition $h_0(x) = 0.5 + 0.01 \cos(\pi x)$ on a domain $0 \leq x \leq 2$. This simulation corresponds to $m = n = 3$ in the weak elasticity case $B = 10^{-5}$. For the transient self-similar stage with $B \ll \tau^{2\beta}$, the scaling parameters in the self-similar ansatz are $\alpha = 1/5$ and $\beta = 2/5$ based on (12), leading to the exponents $\nu = \nu_4 = -3$ and $\mu = -7$ in the analytical predictions (15) and (17). That is, the early-stage transient behavior satisfies $h_{xx}(x_c,t) = O(h(x_c,t)^{-3})$ and $-h_{xxxxx}(x_c,t) = O(h(x_c,t)^{-7})$ at the critical location $x = x_c$. Following (15), we use finite difference methods to numerically solve the fourth-order similarity ODE (13) associated with the far field boundary condition (7) as $|\eta| \rightarrow 0$ and identify a discrete family of similarity solutions. Fig. 1 (center) shows that the transient solution for $0 < t < 0.33174$, rescaled by $h_{\text{min}} = \min h(x,t)$, converges to the primary similarity solution $H(\eta)$ of equation (13) as $h_{\text{min}}$ decreases.

As the solution approaches the finite-time singularity with $\tau = t_e - t \rightarrow 0$, the condition $B \ll \tau^{2\beta}$ is no longer valid. Therefore, the PDE solution evolves following the similarity scalings (9) with $\alpha = \beta = 1/3$, and the exponents in the analytical predictions (15) and (17) become $\nu = \nu_4 = -1$ and $\mu = -3$, indicating that the solution satisfies $h_{xx}(x_c,t) = O(h(x_c,t)^{-1})$ and $-h_{xxxxx}(x_c,t) = O(h(x_c,t)^{-3})$ as the critical time $t_e$ is approached. We plot the later stage solutions rescaled by $h_{\text{min}}$ in Fig. 1 (right) against the primary similarity solution.
solution of the sixth-order ODE \[10\]. This transition in scaling is visible in Fig. 2, which depicts the relation between \(h(x_c, t)\), \(h_{xx}(x_c, t)\), and \(h_{xxxxxxx}(x_c, t)\) for the PDE simulation shown in Fig. 1. As \(t \to t_c\), the numerically observed relations of \(h(x_c, t)\) vs. \(h_{xx}(x_c, t)\) and \(h(x_c, t)\) vs. \(-h_{xxxxxxx}(x_c, t)\) agree well with analytical predictions.

To further investigate the rupture solution behavior and the transient dynamics in \[1\], we conduct a sequence of PDE simulations with fixed \(n = 3\) and over a range of \(m = 2, 3, 4\), for both the weak (\(B = 10^{-5}\)) and strong (\(B = 1\)) elasticity cases. Numerical simulations starting from the initial condition \(h_0(x) = \bar{h} + 0.01 \cos(\pi x / L)\) all lead to finite-time singularities. To identify the rupture behaviors, we track the relation between the critical curvature \(h_{xx}(x_c, t)\) and \(h(x_c, t)\) in time and compare them against the predictions \[15\] – \[16\]. Fig. 3 (left) shows that in the strong elasticity case (\(B = 1\)), the sixth-order bending pressure dominates the rupture dynamics, leading to self-similar rupture solutions following the prediction \[15\] with \(\nu = \nu_6(m)\).

Based on the discussion in Sec. 2.2 in the weak elasticity case \(B \ll 1\) with \(0 < n \leq m\), the solution is expected to follow the transient fourth-order self-similar dynamics for \(B \ll \tau^2 B\) with the critical exponent \(\nu_4(m)\). For the later stage dynamics towards the final rupture, the self-similar singularity occurs following the sixth-order similarity ODE \[17\] and the prediction \[15\] with \(\nu = \nu_6(m)\). Figure 3 (right) shows that in the weak elasticity case \(B = 10^{-5}\), the \(h(x_c, t)\) vs. \(h_{xx}(x_c, t)\) curves present a clear transition from \(\nu = \nu_4\) to \(\nu = \nu_6\) for the cases \(n \leq m = 3, 4\). Such transition is not observed for the case \(n > m = 2\), which does not satisfy the rupture criteria \[14\] for the fourth-order PDE \[14\]. This observation confirms our analysis in Sec. 2.2 for the transient fourth-order rupture behavior in the weak elasticity limit.

### 4. Conclusions

This paper presents a study of the finite-time self-similar rupture dynamics in the generalized elasto-hydrodynamic lubrication model \(1\) parameterized by exponents \((m, n)\) in disjoining pressure and mobility function, respectively. Asymptotically self-similar rupture solutions governed by a sixth-order nonlinear ODE are identified and numerically studied for this model. In the weak elasticity limit with \(B \ll 1\) and \(0 < n \leq m\), an interesting transition from fourth-order self-similar dynamics to the final stage sixth-order rupture solution is numerically investigated.

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