Stochastic Implicit Natural Gradient for Black-box Optimization

Yueming Lyu
University of Technology Sydney

Ivor Tsang
University of Technology Sydney

Abstract

Black-box optimization is primarily important for many compute-intensive applications, including reinforcement learning (RL), robot control, etc. This paper presents a novel theoretical framework for black-box optimization, in which our method performs stochastic update within a trust region defined with KL-divergence. We show that this update is equivalent to a natural gradient step w.r.t. natural parameters of an exponential-family distribution. Theoretically, we prove the convergence rate of our framework for convex functions. Our theoretical results also hold for non-differentiable black-box functions. Empirically, our method achieves superior performance compared with the state-of-the-art method CMA-ES on separable benchmark test problems.

1 Introduction

Given a proper function \( f(\mathbf{x}) : \mathbb{R}^d \rightarrow \mathbb{R} \) such that \( f(\mathbf{x}) > -\infty \), we aim at minimizing \( f(\mathbf{x}) \) by using function queries only, which is known as black-box optimization. It has a wide range of applications, such as automatic hyper-parameters tuning in machine learning and computer vision problems [28], adjusting parameters for robot control and reinforcement learning [9] [15], black-box architecture search in engineering design [30] and drug discovery [22].

Several kinds of approaches have been widely studied for black-box optimization, including Bayesian optimization (BO) methods [29] [8] [21], evolution strategies (ES) [15] and genetic algorithms [13], all these are instances of derivative free optimization (DFO). Among them, Bayesian optimization methods are good at dealing with low-dimensional expensive black-box optimization, while ES methods are better for relatively high-dimensional problems with cheaper evaluations compared with BO methods. ES-type algorithms can well support parallel evaluation, and have drawn more and more attention because of its success in reinforcement learning problems [11] [27] [14], recently.

CMA-ES [15] [1] is one of state-of-the-art ES methods with many successful applications. It uses second-order information to search candidate solutions by updating the mean and covariance matrix of the likelihood of candidate distributions. Despite of its successful performance, the update rule combines several sophisticated components, which is not well understood. Natural evolution strategies (NES) [31] estimate the natural gradient for black-box optimization. However, they use the Monte Carlo sampling to approximate the Fisher information matrix (FIM), which incurs additional error and computation cost unavoidably. Along this line, [2] show the connection between the rank-\( \mu \) update of CMA-ES and NES [31]. [24] further show that several ES methods can be included in an unified framework. Despite of these theoretical attempts, the practical performance of these methods are still inferior to CMA-ES. Moreover, these works do not provide any convergence rate analysis, which is the key insight to expedite black-box optimizations.

Another line of research for ES-type algorithms is to reduce the variance of gradient estimators. Choromanski et al. [11] proposed to employ Quasi Monte Carlo (QMC) sampling to achieve more accurate gradients estimate. Recently, they further proposed to construct gradient estimators based on active subspace techniques [10]. Although these works can reduce sample complexity, how does the variance of these estimators influence the convergence rate remains unclear.

To take advantage of second-order information for the acceleration of black-box optimizations, we propose a novel theoretical framework: stochastic Implicit Natural Gradient Optimization (INGO) algorithms, from the perspective of information geometry. Our methods derive an implicit natural gradient update that can
avoid computing the inverse of the Fisher information matrix. This is done by taking a stochastic update within a trust region defined with KL-divergence from the previous step. Theoretically, this update is equivalent to a stochastic natural gradient step w.r.t. natural parameters of an exponential-family distribution.

Our contributions are summarized as follows:

- We propose a novel stochastic implicit natural gradient descent framework for black-box optimization (INGO). Our methods construct stochastic implicit natural gradient update without computing the FIM. Our INGO can adaptively control the stochastic update by taking advantage of the second-order information, which is able to accelerate convergence and is primarily important for ill-conditioned problems; yet it is very simple to implement. Moreover, our INGO has fewer hyperparameters.

- In terms of evaluations, our INGO belongs to ES regime, thus can well support batch evaluation, which can take advantage of parallel evaluations, e.g., parallel evaluations of policy candidates in RL tasks.

- Theoretically, we prove the convergence rate of our framework for convex functions. Our theoretical results also hold for non-differentiable convex black-box functions. This is distinct from most literature works that need Lipschitz continuous gradients (\(L\)-smooth) assumption. Our theoretical results can include many interesting problems with non-smooth structures. Our theoretical results also shows that reducing variance of gradient estimators can lead to a small regret bound.

- Empirically, our method achieves a superior performance compared with the state-of-the-art method CMA-ES on separable benchmark test problems. We further show the effecness of our methods on RL control problems.

2 Notation and Symbols

Denote \(\| \cdot \|_2\) and \(\| \cdot \|_F\) as the spectral norm and Frobenius norm for matrices, respectively. Define \(\| Y \|_{tr} := \sum_i |\lambda_i|\), where \(\lambda_i\) denotes the \(i^{th}\) eigenvalue of matrix \(Y\). Notation \(\| \cdot \|_2\) will also denote \(L_2\)-norm for vectors. Symbol \(\langle \cdot, \cdot \rangle\) denotes inner product under \(L_2\)-norm for vectors and inner product under Frobenius norm for matrices. Define \(\| x \|_C := \langle x, Cx \rangle\). Denote \(S^+\) and \(S^{++}\) as the set of positive semi-definite matrices and the set of positive definite matrices, respectively. Denote \(\Sigma^\dagger\) as the symmetric matrix such that \(\Sigma = \Sigma^\dagger \Sigma^\dagger\) for \(\Sigma \in S^+\).

3 Methodology

3.1 Optimization for Exponential-family

We aim at minimizing a proper function \(f(x), x \in X\) with only function queries, which is known as black-box optimization.

Due to the difficulty of the lack of gradient information for black-box optimization, we relax the problem to an augmented problem. Specifically, we relax the objective as an expectation of \(f(x)\) under a parametric distribution \(p(x; \eta)\) with parameter \(\eta\), i.e., \(J(\eta) := \mathbb{E}_{p(x; \eta)}[f(x)]\). Thus, we optimize the parameter \(\eta\) to minimize the augmented problem \(J(\eta)\) as

\[
\min_{\eta} \{ \mathbb{E}_{p(x; \eta)}[f(x)] \}. \tag{1}
\]

The relaxed problem is minimized when the probability mass all assigned on minimum of \(f(x)\).

In this work, we assume that the distribution \(p(x; \eta)\) is an exponential-family distribution:

\[
p(x; \eta) = h(x) \exp \{\langle \phi(x), \eta \rangle - A(\eta)\}, \tag{2}
\]

where \(\eta\) and \(\phi(x)\) are the natural parameter and sufficient statistic, respectively. And \(A(\eta)\) is the log partition function defined as:

\[
A(\eta) = \log \int \exp \{\langle \phi(x), \eta \rangle h(x)dx \} \tag{3}
\]

We call an exponential-family distribution minimal when there is a one-to-one mapping between the mean parameter \(m := \mathbb{E}_p[\phi(x)]\) and natural parameter \(\eta\). This one-to-one mapping ensures that we can reparameterize \(J(\eta)\) as \(J(m) = J(\eta)\) \[4] \[16]. \(\hat{J}\) is w.r.t parameter \(m\), while \(J\) is w.r.t parameter \(\eta\).

To minimize the objective \(\hat{J}(m)\), we desire the updated distribution lying in a trust region of the previous distribution at each step. Formally, we update the mean parameters by solving the following optimization problem.

\[
m_{t+1} = \arg \min_m \left\{ m, \nabla_m \hat{J}(m_t) \right\} + \frac{1}{\beta_t} \text{KL}(p_m || p_{m_t}) \tag{4}
\]

where \(\nabla_m \hat{J}(m_t)\) denotes the gradient at \(m = m_t\).

The KL-divergence term measures how close the updated distribution and the previous distribution. For an exponential-family distribution, the KL-divergence term in [4] is equal to Bregman divergence between \(m\) and \(m_t\) [5].

\[
\text{KL}(p_m || p_{m_t}) = A^*(m) - A^*(m_t) - \langle m - m_t, \nabla_mA^*(m_t) \rangle \tag{5}
\]

where \(A^*(m)\) is the convex conjugate of \(A(\eta)\). Thus, the problem [4] is a convex optimization problem, and it has a closed-form solution.
3.2 Implicit Natural Gradient

Natural gradient \[ B \] can capture information geometry structure during optimization, which enables us to take advantage of the second-order information to accelerate convergence. Direct computation of natural gradient needs the inverse of Fisher information matrix (FIM), which needs to estimate the FIM. The method in \[ 23 \] provides an alternative way to compute natural gradient without computation of FIM. However, it relies on the exact gradient, which is impossible for black-box optimization. We thus propose a novel stochastic implicit natural gradient methods for black-box optimization.

We first show how to compute the implicit natural gradient. In problem Eq.(4), take the derivative w.r.t \( \eta \), and set it to zero, also note that \( \nabla A(m) = \eta \) \[ 23 \], we can obtain that

\[
\eta_{t+1} = \eta_t - \beta_t \nabla m \tilde{J}(m_t)
\]  

(6)

Natural parameters \( \eta \) of the distribution lies on a Riemannian manifold with metric tensor specified by the Fisher Information Matrix:

\[
F(\eta) := \mathbb{E}_p \left[ \nabla_\eta \log p(x; \eta) \nabla_\eta \log p(x; \eta)^\top \right]
\]  

(7)

For exponential-family with the minimal representation, the natural gradient has a simple form for computation.

**Theorem 1.** \[ 17, 23 \] For an exponential-family in the minimal representation, the natural gradient w.r.t \( \eta \) is equal to the gradient w.r.t. \( m \), i.e.,

\[
F(\eta)^{-1} \nabla_\eta J(\eta) = \nabla m \tilde{J}(m)
\]  

(8)

Theorem 1 can be easily obtained by the chain rule and the fact \( F(\eta) = \frac{\partial A(\eta)}{\partial \eta} \). It enables us to compute the natural gradient implicitly without computing the inverse of the Fisher information matrix. As shown in Theorem 1, the update rule in (9) is equivalent to the natural gradient update w.r.t. \( \eta \) in (9):

\[
\eta_{t+1} = \eta_t - \beta_t F(\eta_t)^{-1} \nabla_\eta J(\eta_t)
\]  

(9)

Thus, update rule in (9) selects the steepest descent direction along the Riemannian manifold induced by the Fisher information matrix as natural gradient descent. It can take the second-order information to accelerate convergence.

3.3 Update Rule for Gaussian

We first present an update method for the Gaussian case. For other distributions, we can derive the update rule in a similar manner.

For a Gaussian distribution \( p := N(\mu, \Sigma) \) with mean \( \mu \) and covariance matrix \( \Sigma \), the natural parameters \( \eta = \{\eta_1, \eta_2\} \) are given as follows:

\[
\eta_1 := \Sigma^{-1} \mu
\]  

(10)

\[
\eta_2 := -\frac{1}{2} \Sigma^{-1}
\]  

(11)

The related mean parameters \( \eta = \{m_1, m_2\} \) are given as:

\[
m_1 := \mathbb{E}_p[x] = \mu
\]  

(12)

\[
m_2 := \mathbb{E}_p[x x^\top] = \mu \mu^\top + \Sigma
\]  

(13)

Using the chain rule, the gradient with respect to mean parameters can be expressed in terms of the gradients w.r.t \( \mu \) and \( \Sigma \) \[ 16, 18 \]:

\[
\nabla m_1 \tilde{J}(m) = \nabla_\mu \tilde{J}(m) - 2[\nabla_\Sigma \tilde{J}(m)] \mu
\]  

(14)

\[
\nabla m_2 \tilde{J}(m) = \nabla_\Sigma \tilde{J}(m)
\]  

(15)

Now, we can give the natural gradient update. Plug Eq.(10) and (11) into update formula Eq.(6), we know that

\[
-\frac{1}{2} \Sigma_{t+1}^{-1} = -\frac{1}{2} \Sigma_t^{-1} - \beta_t \nabla m_2 \tilde{J}(m_t)
\]  

(16)

\[
\Sigma_t^{-1} \mu_{t+1} = \Sigma_t^{-1} \mu_t - \beta_t \nabla m_1 \tilde{J}(m_t)
\]  

(17)

Then, plug (14) and (15) into (16) and (17), we obtain that

\[
\Sigma_{t+1}^{-1} = \Sigma_t^{-1} + 2 \beta_t \nabla_\Sigma \tilde{J}(m_t)
\]  

(18)

\[
\mu_{t+1} = \mu_t - \beta_t \Sigma_t \nabla_\Sigma \tilde{J}(m_t)
\]  

(19)

Note that \( \tilde{J}(m) = \mathbb{E}_p[f(x)] \), we give the gradients of \( \tilde{J}(m) \) w.r.t \( \mu \) and \( \Sigma \) as in Theorem 2.

**Theorem 2.** The gradient of the expectation of an integrable function \( f(x) \) under a Gaussian distribution \( p := N(\mu, \Sigma) \) with respect to the mean \( \mu \) and the covariance \( \Sigma \) can be expressed as Eq. (20) and Eq. (21), respectively.

\[
\nabla_\mu \mathbb{E}_p[f(x)] = \mathbb{E}_p \left[ \Sigma^{-1} (x - \mu) f(x) \right]
\]  

(20)

\[
\nabla_\Sigma \mathbb{E}_p[f(x)] = \frac{1}{2} \mathbb{E}_p \left[ (\Sigma^{-1} (x - \mu) (x - \mu)^\top) \Sigma^{-1} - \Sigma^{-1} \right] f(x)
\]  

(21)

If \( f(x) \) is twice differentiable, we can express the gradients as in Theorem 3.

**Theorem 3.** Suppose \( f(x) \) be an integrable and twice differentiable function under a Gaussian distribution \( p := N(\mu, \Sigma) \) such that \( \mathbb{E}_p[\nabla_x f(x)] \) and \( \mathbb{E}_p \left[ \frac{\partial^2 f(x)}{\partial x^2} \right] \)
Given as Eq. (26) and Eq. (27): Update rule employing Monte Carlo sampling are property is very important for black-box optimization. Eq. (24), (25) enables us to estimate the gradient of a black-box function. However, this expectation exists. Then, the expectation of the gradient and Hessian of \( f(x) \) can be expressed as Eq. (22) and Eq. (23), respectively.

\[
\mathbb{E}_p \left[ \nabla_x f(x) \right] = \mathbb{E}_p \left[ \Sigma^{-1}(x - \mu) f(x) \right] \tag{22}
\]

\[
\mathbb{E}_p \left[ \frac{\partial^2 f(x)}{\partial x^2} \right] = \mathbb{E}_p \left[ (\Sigma^{-1}(x - \mu)(x - \mu)^\top \Sigma^{-1} - \Sigma^{-1}) f(x) \right] \tag{23}
\]

Together Theorem 2 with Eq. (18) and (19), we present the update with only function queries as:

\[
\Sigma_{i+1}^{-1} = \Sigma_i^{-1} + \beta_i \mathbb{E}_p \left[ (\Sigma_i^{-1}(x - \mu_i)(x - \mu_i)^\top \Sigma_i^{-1} - \Sigma_i^{-1}) f(x) \right] \tag{24}
\]

\[
\mu_{i+1} = \mu_i - \beta_i \Sigma_{i+1} \mathbb{E}_p \left[ \Sigma_i^{-1}(x - \mu_i) f(x) \right] \tag{25}
\]

3.4 Stochastic Update

The true gradient update above needs the expectation of a black-box function. However, this expectation does not have a closed form solution. Thus, we estimate the gradient w.r.t. \( \mu \) and \( \Sigma \) by Monte Carlo sampling. Eq. (24), (25) enables us to estimate the gradient by the function queries of \( f(x) \) instead of \( \nabla f(x) \). This property is very important for black-box optimization because gradient information (\( \nabla f(x) \)) is not available.

Update rule employing Monte Carlo sampling are given as Eq. (26) and Eq. (27):

\[
\Sigma_{i+1}^{-1} = \Sigma_i^{-1} + \beta_i \sum_{i=1}^N \left( (\Sigma_i^{-1}(x_i - \mu_i)(x_i - \mu_i)^\top \Sigma_i^{-1} - \Sigma_i^{-1}) f(x_i) \right) \tag{26}
\]

\[
\mu_{i+1} = \mu_i - \beta_i \frac{1}{N} \sum_{i=1}^N \Sigma_{i+1} \Sigma_i^{-1}(x_i - \mu_i) f(x_i) \tag{27}
\]

Algorithm 1: Implicit Natural Gradient Optimization

**Input:** Number of Samples \( N \), step-size \( \beta \).

**while** Termination condition not satisfied **do**

Take i.i.d samples \( z_i \sim \mathcal{N}(0, I) \) for \( i \in \{1, \ldots, N/2\} \).

Set \( z_{i+N/2} = -z_i \) for \( i \in \{1, \ldots, N/2\} \).

Set \( x_i = \mu_i + \Sigma_i^{1/2} z_i \) for \( i \in \{1, \ldots, N\} \).

Query the batch observations \( \{f(x_1), \ldots, f(x_N)\} \) (or ranking scores \( \{h(x_1), \ldots, h(x_N)\} \)).

Compute weights \( w_i = \log \frac{1}{\sum_{j=1}^N \log j} \) for \( i \in \{1, \ldots, N\} \) according to Eq. (28).

Set \( \Sigma_{i+1}^{-1} = (1 - \beta_i) \Sigma_i^{-1} + \beta_i \sum_{i=1}^N w_i \Sigma_i^{-1/2} z_i z_i^\top \Sigma_i^{-1/2} \).

Set \( \mu_{i+1} = \mu_i - \beta_i \Sigma_{i+1} \sum_{i=1}^N \frac{1}{N} \sum_{i=1}^N (x_i - \mu_i) f(x_i) \).

end while

Algorithm 2: INGO-u

**Input:** Number of Samples \( N \), step-size \( \beta \).

**while** Termination condition not satisfied **do**

Take i.i.d samples \( z_i \sim \mathcal{N}(0, I) \) for \( i \in \{1, \ldots, N/2\} \).

Set \( z_{i+N/2} = -z_i \) for \( i \in \{1, \ldots, N/2\} \).

Set \( x_i = \mu_i + \Sigma_i^{1/2} z_i \) for \( i \in \{1, \ldots, N\} \).

Query the batch observations \( \{f(x_1), \ldots, f(x_N)\} \)

Compute weights \( w_i = \log \frac{1}{\sum_{j=1}^N \log j} \) for \( i \in \{1, \ldots, N\} \) according to Eq. (28).

Set \( \Sigma_{i+1}^{-1} = (1 - \beta_i) \Sigma_i^{-1} + \beta_i \sum_{i=1}^N w_i \Sigma_i^{-1/2} z_i z_i^\top \Sigma_i^{-1/2} \).

Compute \( \tilde{\sigma} = \text{std}(f_1, \ldots, f_N) \).

Set \( \mu_{i+1} = \mu_i - \beta_i \frac{1}{N} \sum_{i=1}^N \frac{1}{N} \sum_{i=1}^N (x_i - \mu_i) \).

end while

To avoid scaling problem of \( f(x) \), we use a monotonic score function \( h(\cdot) \) to transform \( f(x) \) as:

\[
h(f(x_i)) = \frac{\log \hat{i}}{\sum_{j=1}^N \log j} \tag{28}
\]

where \( \hat{i} \) denotes the ranking index of \( f(x_i) \) (index after non-decreasingly sort) among \( N \) samples \( f(x_1), \ldots, f(x_N) \). (Break ties by some determinate rules).

Plug Eq. (28) into Eq. (26) and (27), we obtain that

\[
\Sigma_{i+1}^{-1} = (1 - \beta_i) \Sigma_i^{-1} + \beta_i \sum_{i=1}^N w_i (x_i - \mu_i)(x_i - \mu_i)^\top \Sigma_i^{-1} \tag{29}
\]

\[
\mu_{i+1} = \mu_i - \beta_i \frac{1}{N} \sum_{i=1}^N w_i \Sigma_{i+1} \Sigma_i^{-1}(x_i - \mu_i) \tag{30}
\]

where \( w_i = \log \frac{1}{\sum_{j=1}^N \log j} \).

We present our black-box optimization algorithm in Algorithm 1. In Algorithm 1 we select a set of random samples \( [z, -z] \), this trick still leads to an unbiased estimator for the Gaussian distribution [20]. We can use the structured samples in [20] to reduce the variance of estimators. Algorithm 1 employs the ranking scores \( w_i \) for update.

It is worth to noting that Algorithm 1 does not require the exact function value. It needs only the ranking of the observations for a set of samples. This is useful when the function value is not observable, e.g. only preference list of customers is given.

The downside of update based on ranking is that the gradient estimator is biased. In addition, the information of function is lost. To take advantage of the
Algorithm 3 Fast INGO

**Input:** Number of Samples $N$, step-size $\beta$.

**while** Termination condition not satisfied **do**

Take i.i.d samples $z_i \sim \mathcal{N}(0, I)$ for $i \in \{1, \cdots \cdot N/2\}$.

Set $z_{i+N/2} = -z_i$ for $i \in \{1, \cdots \cdot N/2\}$.

Set $x_i = \mu_i + \sigma_i \odot z_i$ for $i \in \{1, \cdots \cdot N\}$.

Query the batch observations $\{f(x_1), \ldots, f(x_N)\}$ (or ranking scores $\{h(x_1), \ldots, h(x_N)\}$)

Compute weights $w_i = \log i / \sum_{j=1}^{N} \log j$ for $i \in \{1, \cdots \cdot N\}$ according to Eq. (28).

Set $\sigma_t^2 = (1 - \beta)\sigma_t^{-2} + \beta \sigma_t^{-2} \odot \left(\sum_{i=1}^{N} w_i z_i^2\right)$.

Set $\mu_{t+1} = \mu_t - \beta \sigma_t \odot \sigma_t^{-1} \odot \left(\sum_{i=1}^{N} w_i z_i\right)$.

**end while**

Algorithm 4 Fast INGO-u

**Input:** Number of Samples $N$, step-size $\beta$.

**while** Termination condition not satisfied **do**

Take i.i.d samples $z_i \sim \mathcal{N}(0, I)$ for $i \in \{1, \cdots \cdot N/2\}$.

Set $z_{i+N/2} = -z_i$ for $i \in \{1, \cdots \cdot N/2\}$.

Set $x_i = \mu_i + \sigma_i \odot z_i$ for $i \in \{1, \cdots \cdot N\}$.

Query the batch observations $\{f(x_1), \ldots, f(x_N)\}$

Compute weights $w_i = \log i / \sum_{j=1}^{N} \log j$ for $i \in \{1, \cdots \cdot N\}$ according to Eq. (25).

Set $\sigma_t^2 = (1 - \beta)\sigma_t^{-2} + \beta \sigma_t^{-2} \odot \left(\sum_{i=1}^{N} w_i z_i^2\right)$.

Compute $\bar{\sigma} = \text{std}(f_1, \ldots, f_N)$.

Set $\mu_{t+1} = \mu_t - \beta \sigma_t \odot \sigma_t^{-1} \odot \left(\sum_{i=1}^{N} \bar{\sigma} z_i\right)$.

**end while**

smoothness of function while avoid scale problem, we present an unbiased estimator for gradient update as follows:

$$\mu_{t+1} = \mu_t - \beta \sum_{i=1}^{N} \frac{f_i}{N\bar{\sigma}} \sigma_t^{-1} (x_i - \mu_t)$$  \hspace{0.5cm} (31)

where $\bar{\sigma}$ denotes the stand deviation of function values in a batch of samples, i.e., $\bar{\sigma} = \text{std}(f_1, \ldots, f_N)$.

The modified algorithm with unbiased gradient update is presented in Algorithm 2. Algorithm 2 employs Eq. (31) for $\mu$ update. This update needs function value instead of ranking scores.

3.5 Mean field approximation for acceleration

Algorithm 1 and Algorithm 2 need to compute the inverse of covariance matrix, which has $O(d^3)$ complexity. In this section, we present a fast algorithm for separate problems.

Suppose the covariance matrix is diagonal with diagonal elements denoted as $\sigma^2$. From (29) and (30), we can obtain update rule as

$$\sigma_{t+1}^2 = (1 - \beta_t)\sigma_t^{-2} + \beta_t \sigma_t^{-2} \odot \left(\sum_{i=1}^{N} w_i z_i^2\right)$$  \hspace{0.5cm} (32)

$$\mu_{t+1} = \mu_t - \beta_t \sigma_{t+1}^2 \odot \sigma_t^{-1} \odot \left(\sum_{i=1}^{N} w_i z_i\right)$$  \hspace{0.5cm} (33)

where $\odot$ denotes element-wise product. And the power operation denotes element-wise operation. And $x_i = \mu_t + \sigma_t \odot z_i$.

Our fast algorithm is given in Algorithm 3. It only involves element-wise operation in vectors, which is very simple to implement. The complexity in per iteration is $O(Nd)$, which scales well for high-dimensional optimization.

Algorithm 3 employs biased gradient estimator, we present an algorithm using unbiased gradient estimator in Algorithm 4. Both Algorithm 3 and Algorithm 4 support parallel evaluation; yet they are very simple to implement.

3.6 Direct Update for $\mu$ and $\Sigma$

We provide an alternative updating equation with simple concept and derivation. The implicit natural gradient algorithms are working on the natural parameter space. Alternatively, we can also directly work on the $\mu$ and $\Sigma$ parameter space. Formally, we derive the update rule by solving the following trust region optimization problem.

$$\theta_{t+1} = \arg \min_\theta \langle \theta, \nabla_\theta J(\theta) \rangle + \frac{1}{\beta_t} \text{KL} (p_{\theta} || p_{\theta_t})$$  \hspace{0.5cm} (34)

where $\theta := \{\mu, \Sigma\}$ and $J(\theta) := \mathbb{E}_{p(x; \theta)}[f(x)] = J(\eta)$.

For Gaussian distribution, the optimization problem in (34) is a convex optimization problem. We can achieve a closed-form update given in Theorem 4.

**Theorem 4.** For Gaussian distribution with parameter $\theta := \{\mu, \Sigma\}$, problem (34) is convex w.r.t $\theta$. The optimum of problem (34) leads to closed-form update (35) and (36):

$$\Sigma_{t+1} = \Sigma_t^{-1} + 2\beta_t \nabla_\Sigma J(\theta_t)$$  \hspace{0.5cm} (35)

$$\mu_{t+1} = \mu_t - \beta_t \Sigma_t \nabla_\mu J(\theta_t)$$  \hspace{0.5cm} (36)

Comparing the update rule in Theorem 4 with Eq. (18) and (19), we can observe that the only difference is in the update of $\mu$. In Eq. (36), the update employs $\Sigma_t$, while the update in Eq. (19) employs $\Sigma_{t+1}$. The update
Algorithm 5 General Framework

Input: Number of Samples $N$, step-size $\beta$.

while Termination condition not satisfied do

Construct unbiased estimator $\hat{g}_t$.

Construct unbiased/biased estimator $\hat{G}_t \in S^{++}$ such that $bI \preceq \hat{G}_t \preceq \frac{2}{\gamma}I$

Set $\Sigma^{t}_{-1} = \Sigma^{t-1} + 2\beta\hat{G}_t$.

Set $\mu^{t+1} = \mu^{t} - \beta\Sigma^{t}_{-1}\hat{g}_t$.

end while

in Eq. (19) takes one step look ahead information, it helps to improve sample efficiency.

We can obtain the black-box update for $\mu$ and $\Sigma$ by Theorem 4 and Theorem 2. The update rule is given as follows:

$$
\Sigma^{t+1} = \Sigma^{t-1} + \beta E_p \left[ (\frac{1}{\rho} f(x) - \hat{f}_t) \right] \Sigma^{t}_{-1}
$$

(37)

$$
\mu^{t+1} = \mu^{t} - \beta \Sigma^{t}_{-1} \hat{g}_t
$$

(38)

Using the score function $h()$, we can obtain Monte Carlo approximation update as

$$
\Sigma^{t+1} = (1 - \beta) \Sigma^{t-1} + \beta \sum_{i=1}^{N} w_i \left( \frac{1}{\rho} f(x_i) - \hat{f}_t \right) \Sigma^{t-1}
$$

(39)

$$
\mu^{t+1} = \mu^{t} - \beta \sum_{i=1}^{N} w_i (x_i - \mu^{t})
$$

(40)

where $w_i = \log \frac{i}{\sum_{j=1}^{N} \log j}$.

From Eq. (40), we can see that the update rule for $\mu$ is similar to that of CMA-ES. In contrast, the update rule for covariance matrix $\Sigma$ is the same as the implicit natural gradient update in Eq. (29).

4 Convergence Rate

We first show a general framework in Algorithm 5. Algorithm 3 employs an unbiased estimator $(\hat{g}_t)$ for gradient $\nabla_{\mu} J(\theta)$). In contrast, it can employ both the unbiased and biased estimators $\hat{G}_t$ for update. It is worth noting that $\hat{g}_t$ can be both the first-order estimate (stochastic gradient) and the zeroth-order estimate (function value based estimator).

The update step of $\mu$ and $\Sigma$ is achieved by solving the following convex minimization problem as shown in Theorem 5.

$$
\mu^{t+1} = \arg \min_{\mu \in \mathcal{M}} \beta_t \langle \mu, \hat{v}_t \rangle + KL (p_m || p_{m^*})
$$

(41)

where $\mathcal{M} := \{m_1, m_2\} = \{\mu, \Sigma + \beta \mu \Sigma^{-1} \}$ and $\hat{v}_t = \{\hat{g}_t - 2\hat{G}_t \mu, \hat{G}_t\}$.

Theorem 5. For Gaussian distribution with parameter $\mathcal{M} := \{m_1, m_2\} = \{\mu, \Sigma + \beta \mu \Sigma^{-1} \}$. The optimum of problem (41) leads to closed-form update (42) and (43):

$$
\Sigma^{t+1} = \Sigma^{t-1} + 2\beta \hat{G}_t
$$

(42)

$$
\mu^{t+1} = \mu^{t} - \beta \Sigma^{t}_{-1} \hat{g}_t
$$

(43)

The trust-region optimization problem in Eq. (41) is defined w. r. t. the stochastic estimator instead of the true gradient. It enables us to analyze the convergence rate of our stochastic framework. Note that $\hat{G}_t$ can be a biased estimator, it is different from [18] which is derived for the true gradient.

General Stochastic Case: The convergence rate of Algorithm 5 is shown in Theorem 6.

Theorem 6. Given a convex function $f(x)$, define $J(\theta) := E_{p(x; \theta)} [f(x)]$ for Gaussian distribution with parameter $\Theta := \{\mu, \Sigma\} \in \Theta$ and $\Theta := \{\mu, \Sigma\} \in \Theta$. Suppose $J(\theta)$ is $\gamma$-strongly convex. Let $\hat{G}_t$ be positive semi-definite matrix such that $bI \preceq \hat{G}_t \preceq \frac{2}{\gamma}I$. Suppose $\Sigma_t \in S^{++}$ and $\|\Sigma_t\| \leq \rho$, $\hat{G}_t = \nabla_{p=\mu} J$. Assume furthermore $\|\nabla_{p=\mu} J\|_{tr} \leq B_1$ and $\|\mu^* - \mu\|_{tr} \leq R$, $\hat{G}_t \|\| \leq B$. Set $\beta_t = \beta$, then Algorithm 5 can achieve

$$
\frac{1}{T} \sum_{t=1}^{T} f(\theta_t) - f(\theta^*) \leq \frac{2bR + 2b\beta \rho (4B_1 + \beta B)}{4\beta B} + 4B_1 (1 + \log T) + (1 + \log T) \beta B
$$

(44)

$$
= O \left( \frac{\log T}{T} \right)
$$

(45)

Remark: Theorem 6 does not require the function $f(x)$ to be differentiable. It holds for non-smooth function $f(x)$. Theorem 6 holds for convex function $f(x)$, as long as $J(\theta) := E_{p(x; \theta)} [f(x)]$ is $\gamma$-strongly convex. Particularly, when $f(x)$ is $\gamma$-strongly convex, we know $J(\theta)$ is $\gamma$-strongly convex [12]. Thus, the assumption here is weaker than strongly convex assumption of $f(x)$. Moreover, Theorem 6 does not require the boundedness of the domain. It only requires the boundedness of the distance between the initialization point and an optimal point. Theorem 6 shows that the bound depends on the bound of $\|\hat{G}_t\|^2_2$, which means that reducing variance of the gradient estimators can lead to a small regret bound.

Black-box Case: For black-box optimization, we can only access the function value instead of the gradient. We give an unbiased estimator of $\nabla_{\mu} J(\theta_t)$ using function values as below

$$
\hat{g}_t = \frac{1}{\rho} \left( f(\mu_t + \Sigma^{-1}_t z) - f(\mu_t) \right)
$$

(47)
where $z \sim \mathcal{N}(0, I)$.

The estimator $\hat{g}_t$ is unbiased, i.e., $\mathbb{E}[\hat{g}_t] = \nabla_{\mu} \bar{J}(\theta_t)$. The proof of unbiasedness of the estimator $\hat{g}_t$ is given in Lemma 7 in Appendix. With this estimator, we give the convergence rate of Algorithm 3 for convex black-box optimization as in Theorem 7.

**Theorem 7.** For a $L$-Lipschitz continuous convex black box function $f(x)$, define $\bar{J}(\theta) := \mathbb{E}_{x \sim \mathcal{G}(\theta)}[f(x)]$ for Gaussian distribution with parameter $\theta := (\mu, \Sigma^2) \in \Theta$ and $\Theta := \{\mu, \Sigma^2 \mid \mu \in \mathbb{R}^d, \Sigma \in S^+\}$. Suppose $\bar{J}(\theta)$ be $\gamma$-strongly convex. Let $\hat{G}_t$ be positive semi-definite matrix such that $bI \preceq \hat{G}_t \preceq \frac{2}{\gamma}I$. Suppose $\Sigma_1 \in S^{++}$ and $\|\Sigma_1\|_2 \leq \rho$. Assume furthermore $\|\nabla_{\Sigma_1} \bar{J}\|_{tr} \leq B_1$ and $\|\mu^* - \mu_1\|_{\Sigma_1^{-1}} \leq R$. Set $\beta_t = \beta$ and employ estimator $\hat{g}_t$ in Eq. (47), then Algorithm 4 can achieve

\[
\frac{1}{T} \sum_{i=1}^{T} f(\theta_i) - f(\theta^*) \leq \frac{bR + 4b\beta(4B_1 + 2\beta L^2(d + 4)^2)}{2bT} + \frac{4B_1(1 + \log T) + (1 + \log T)\beta L^2(d + 4)^2}{4\beta T}
\]

\[
= \mathcal{O}\left(\frac{d^2 \log T}{T}\right)
\]

Remark: Theorem 7 holds for non-differentiable function $f(x)$. Thus, Theorem 7 can cover more interesting cases e.g. sparse black box optimization. In contrast, Krishnakumar et al. [6] require function $f(x)$ has Lipschitz continuous gradients.

Both Algorithm 2 and Algorithm 4 employ an unbiased gradient estimator $\bar{g}$ for $\mu$ update and biased estimator $\hat{G}$ for variance $\Sigma (\sigma^2)$ update. When further ensure $bI \preceq \hat{G}_t \preceq \frac{2}{\gamma}I$, Theorem 7 holds for Alg. 2 and Alg. 4. Theorem 4 is derived for single sample per iteration. We can reduce the variance of estimators by constructing a set of structured samples that are conjugate of inverse covariance matrix in a batch, i.e., $z_i \Sigma_i^{-1} z_j = 0, i \neq j$. Particularly, when we use $\bar{\Sigma}_t = \Sigma_t I$, sampling $N = d$ orthogonal samples per iteration can lead to a convergence rate $\mathcal{O}\left(\frac{d \log T}{T}\right)$. For $N > d$ samples, we can use the method in [20] with a random rotation to reduce variance.

Our methods can capture information geometry structure during optimization, which enables us to take advantage of the second-order information to accelerate convergence.

5 Empirical Study

### Evaluation on synthetic test benchmarks

We evaluate the proposed Fast-INGO (Algorithm 3) and Fast-INGO-u (Algorithm 4) by comparing with one of the state-of-the-art method CMA-ES [15] according to [1], and vanilla ES with antithetic gradient estimators [27] on several synthetic benchmark test problems. All the test problems are listed in Table 1.

**Parameter Settings:** For both Fast-INGO and Fast-INGO-u, we set step size $\beta = 1/\sqrt{d}$, where $d$ is the dimension of the test problems. The number of samples per iteration is set to $N = 2[3 + \lfloor 3 \times \ln d \rfloor / 2]$ for all the methods, where $\lfloor \cdot \rfloor$ denotes the floor function. This setting ensures $N$ to be an even number. We set $\sigma_1 = 0.5 \times 1$ and sample $\mu_1 \sim \text{Uni}(0, 1)$ as the same initialization for all the methods, where $\text{Uni}(0, 1)$ denotes the uniform distribution in $[0, 1]$. For ES [27], we use the default step-size hyper-parameters.

The mean value of $f(x)$ over 20 independent runs for 100-dimensional problems are show in Figure 2. From Figure 2, we can see that both Fast INGO and Fast INGO-u converge linearly in log scale. They can arrive $10^{-10}$ precision. on five cases except the highly non-convex Rastrigrin10 problem. Moreover, CMA-ES con-
Figure 2: Mean value of $f(x)$ in log scale over 20 independent runs for 100-dimensional problems

Evaluation on RL test problems

We further evaluate the proposed Fast-INGO-u (Algorithm 4) by comparing ES with antithetic gradient estimators [27] on Box2D control problems: BipedalWalker, LunarLanderContinuous and MuJoCo control problems: Swimmer, HalfCheetah, HumanoidStandup, InvertedDoublePendulum, in Open-AI Gym environments. CMA-ES is too slow due to the computation of eigendecomposition for high-dimensional problems.

We use three layer feed-forward neural network with tanh activation function as policy architecture. The number of hidden unit is set to $h = 16$ for all problems. The goal is to find the parameters of this policy network to achieve large reward. The same policy architecture is used for all the methods on all test problems. The number of samples per iteration is set to $N = 20 + 4 \left\lfloor \frac{3 + \ln d}{2} \right\rfloor$ for all the methods. For Fast-INGO-u, we set $\beta_1 = 0.1$ for $\sigma$ update and $\beta_2 = 1$ for $\mu$ update. We set $\sigma_1 = 0.1 \times 1$ and $\mu_1 = 0$ as the initialization for all methods. For ES [27], we use the default step-size hyper-parameters. Five independent runs are performed.

The experimental results are shown in Figure 1. We can observe that Fast INGO-u increase AverageReward faster than ES on all six cases. This shows that the update using second-order information in Fast INGO-u can help accelerate convergence.

6 Conclusions

We proposed a novel stochastic implicit natural gradient frameworks for black-box optimization. Theoretically, we proved the $O(\log T/T)$ convergence rate for general stochastic update for non-differentiable convex function under expectation $\gamma$-strongly convex assumption. We proved $O(d^2 \log T/T)$ converge rate for black-box function under same assumptions above. For isometric Gaussian case, we proved the $O(d \log T/T)$ converge rate when using $d$ orthogonal samples per iteration, which well supports parallel
evaluation. Empirically, our methods converge faster than CMAES on separate test problems, which shows the efficiency of our methods. On RL control problems, our methods increase AverageReward faster than ES, which shows employing second order information can help accelerate convergence.

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A Proof of Theorem 2

Proof. For Gaussian distribution $p := \mathcal{N}(\mu, \Sigma)$, the gradient of $\mathbb{E}_p[f(x)]$ w.r.t $\mu$ can be derived as follows:

$$\nabla_{\mu} \mathbb{E}_p[f(x)] = \mathbb{E}_p[f(x) \nabla_{\mu} \log(p(x; \mu, \Sigma))]= \mathbb{E}_p \left[ f(x) \nabla_{\mu} \left[ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right] \right]$$

(51)

$$= \mathbb{E}_p \left[ \Sigma^{-1} (x - \mu) f(x) \right]$$

(52)

The gradient of $\mathbb{E}_p[f(x)]$ w.r.t $\Sigma$ can be derived as follows:

$$\nabla_{\Sigma} \mathbb{E}_p[f(x)] = \mathbb{E}_p[f(x) \nabla_{\Sigma} \log(p(x; \mu, \Sigma))]$$

(54)

$$= \mathbb{E}_p \left[ f(x) \nabla_{\Sigma} \left[ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) - \frac{1}{2} \log \det(\Sigma) \right] \right]$$

(55)

$$= \frac{1}{2} \mathbb{E}_p \left[ (\Sigma^{-1} (x - \mu)(x - \mu)^T \Sigma^{-1} - \Sigma) f(x) \right]$$

(56)

B Proof of Theorem 3

Proof. For Gaussian distribution, from Bonnet’s theorem [26], we know that

$$\nabla_{\mu} \mathbb{E}_p[f(x)] = \mathbb{E}_p [\nabla_x f(x)].$$

(57)

From Theorem 2, we know that

$$\nabla_{\mu} \mathbb{E}_p[f(x)] = \mathbb{E}_p \left[ \Sigma^{-1} (x - \mu) f(x) \right].$$

(58)

Thus, we can obtain that

$$\mathbb{E}_p [\nabla_x f(x)] = \mathbb{E}_p \left[ \Sigma^{-1} (x - \mu) f(x) \right].$$

(59)

From Price’s Theorem [26], we know that

$$\nabla_{\Sigma} \mathbb{E}_p[f(x)] = \mathbb{E}_p \left[ \frac{\partial^2 f(x)}{\partial x \partial x^T} \right].$$

(60)

From Theorem 2, we know that

$$\nabla_{\Sigma} \mathbb{E}_p[f(x)] = \frac{1}{2} \mathbb{E}_p \left[ (\Sigma^{-1} (x - \mu)(x - \mu)^T \Sigma^{-1} - \Sigma) f(x) \right]$$

(61)

It follows that

$$\mathbb{E}_p \left[ \frac{\partial^2 f(x)}{\partial x \partial x^T} \right] = \mathbb{E}_p \left[ (\Sigma^{-1} (x - \mu)(x - \mu)^T \Sigma^{-1} - \Sigma) f(x) \right].$$

(62)

C Proof of Theorem 4

Proof. For Gaussian distribution with parameter $\theta := \{\mu, \Sigma\}$, problem (34) can be rewritten as

$$\langle \theta, \nabla_\theta \mathcal{J}(\theta) \rangle + \frac{1}{\beta_t} \text{KL} (p_0 || p_\theta) = \mu^T \nabla_{\mu} \mathcal{J}(\theta) + \text{tr}(\Sigma \nabla_\Sigma \mathcal{J}(\theta)) + \frac{1}{2 \beta_t} \left[ \text{tr}(\Sigma^{-1} \Sigma) + (\mu - \mu_\Sigma)^T \Sigma^{-1} (\mu - \mu_\Sigma) + \log \left| \frac{\Sigma}{\Sigma} \right| - d \right]$$

(63)
where $\nabla_\mu \tilde{J}(\theta_t)$ denotes the derivative w.r.t $\mu$ taking at $\mu = \mu_t$, $\Sigma = \Sigma_t$. $\nabla_\Sigma \tilde{J}(\theta_t)$ denotes the derivative w.r.t $\Sigma$ taking at $\mu = \mu_t$, $\Sigma = \Sigma_t$. Note that $\nabla_\mu \tilde{J}(\theta_t)$ and $\nabla_\Sigma \tilde{J}(\theta_t)$ are not functions now.

From Eq.\,(63), we can see that the problem is convex with respect to $\mu$ and $\Sigma$. Taking the derivative of Eq.\,(63) w.r.t $\mu$ and $\Sigma$, and setting them to zero, we can obtain that

$$\nabla_\mu \tilde{J}(\theta_t) + \frac{1}{\beta_t} \Sigma_t^{-1}(\mu - \mu_t) = 0$$

$$\nabla_\Sigma \tilde{J}(\theta_t) + \frac{1}{2\beta_t} [\Sigma_t^{-1} - \Sigma^{-1}] = 0$$

It follows that

$$\mu = \mu_t - \beta_t \Sigma_t \nabla_\mu \tilde{J}(\theta_t)$$

$$\Sigma^{-1} = \Sigma_t^{-1} + 2\beta_t \nabla_\Sigma \tilde{J}(\theta_t)$$

By definition, $\mu_{t+1}$ and $\Sigma_{t+1}$ are the optimum of this convex optimization problem. Thus, we achieve that

$$\mu_{t+1} = \mu_t - \beta_t \Sigma_t \nabla_\mu \tilde{J}(\theta_t)$$

$$\Sigma_{t+1}^{-1} = \Sigma_t^{-1} + 2\beta_t \nabla_\Sigma \tilde{J}(\theta_t)$$

### D Proof of Theorem 5

**Proof.** For Gaussian distribution with mean parameter $m := \{m_1, m_2\} = \{\mu, \Sigma + \mu \mu^\top\}$, also note that $\hat{v}_t := \{\hat{g}_t - 2\hat{G}_t \mu_t, \hat{G}_t\}$, the problem \[11\] can be rewrited as

$$\beta_t \langle m, \hat{v}_t \rangle + \text{KL}(p_m \| p_{m^*}) = \beta_t \langle m_1, \hat{g}_t - 2\hat{G}_t \mu_t \rangle + \beta_t \langle m_2, \hat{G}_t \rangle + \text{KL}(p_m \| p_{m^*})$$

Taking derivative and set to zero, also note that $\nabla_m \text{KL}(p_m \| p_{m^*}) = \eta - \eta^t$, $\eta_1 := \Sigma^{-1}\mu$ and $\eta_2 := -\frac{1}{2} \Sigma^{-1}$, we can obtain that

$$- \frac{1}{2} \Sigma_{t+1}^{-1} = - \frac{1}{2} \Sigma_t^{-1} - \beta_t \hat{G}_t$$

$$\Sigma_{t+1}^{-1} \mu_{t+1} = \Sigma_t^{-1} \mu_t - \beta_t \left( \hat{g}_t - 2\hat{G}_t \mu_t \right)$$

Rearrange terms, we can obtain that

$$\Sigma_{t+1}^{-1} = \Sigma_t^{-1} + 2\beta_t \hat{G}_t$$

$$\mu_{t+1} = \Sigma_{t+1}^{-1} \mu_t - \beta_t \Sigma_{t+1} \left( \hat{g}_t - 2\hat{G}_t \mu_t \right)$$

Merge terms in Eq.\,[74], we get that

$$\mu_{t+1} = \Sigma_{t+1}^{-1} \mu_t - \beta_t \Sigma_{t+1} \left( \hat{g}_t - 2\hat{G}_t \mu_t \right)$$

$$\mu_{t+1} = \Sigma_{t+1} \left( \Sigma_t^{-1} + 2\beta_t \hat{G}_t \right) \mu_t - \beta_t \Sigma_{t+1} \hat{g}_t$$

$$\mu_{t+1} = \mu_t - \beta_t \Sigma_{t+1} \hat{g}_t$$

\[\square\]
E Proof of Theorem 6

Lemma 1. For Gaussian distribution with parameter $\theta := \{\mu, \Sigma\} \in \Theta$. Let $F_t(m) = \beta_t \langle m, \tilde{v}_t \rangle$ for all $t \geq 1$, where $m = \{m_1, m_2\} = \{\mu, \Sigma + \mu \mu^T\} \in \mathcal{M}$, $\mathcal{M}$ denotes a convex set. Let $m^{t+1}$ as the solution of

$$m^{t+1} = \arg \min_{m \in \mathcal{M}} F_t(m) + KL(p_m||p_{m^t})$$ \hspace{1cm} (79)

Then, for $\forall m \in \mathcal{M}$, we have

$$F(m) + KL(p_m||p_{m^t}) \geq F(m^{t+1}) + KL(p_m||p_{m^t}) + KL(p_m||p_{m^{t+1}})$$ \hspace{1cm} (80)

Proof. Since KL-divergence of Gaussian is a Bregman divergence associated with base function $A^*(m)$ w.r.t mean parameter $m$, we know problem in Eq.(79) is convex. Since $m^{t+1}$ is the optimum of the convex optimization problem in Eq.(79), we have that

$$\langle \beta_t \tilde{v}_t + \nabla_{m^t} KL(p_{m^t}), m - m^{t+1} \rangle \geq 0, \forall m \in \mathcal{M}$$ \hspace{1cm} (81)

Note that $\nabla_{m^t} KL(p_{m^t}) = -\nabla A^*(m^{t+1}) + \nabla A^*(m^t)$. For $\forall m \in \mathcal{M}$ we have that

$$F(m) = \beta_t \langle \tilde{v}_t, m^{t+1} \rangle + \langle \beta_t \tilde{v}_t, m - m^{t+1} \rangle$$ \hspace{1cm} (82)

Rewritten the term $-\langle \nabla A^*(m^{t+1}) - \nabla A^*(m^t), m - m^{t+1} \rangle$, we have that

$$-\langle \nabla A^*(m^{t+1}) - \nabla A^*(m^t), m - m^{t+1} \rangle = A^*(m^{t+1}) - A^*(m^t) - \langle \nabla A^*(m^t), m - m^{t+1} \rangle$$ \hspace{1cm} (83)

$$= A^*(m) + A^*(m^t) - \langle \nabla A^*(m^t), m - m^{t+1} \rangle$$ \hspace{1cm} (84)

$$= KL(p_m||p_{m^t}) - KL(p_m||p_{m^t}) + KL(p_m||p_{m^{t+1}})$$ \hspace{1cm} (85)

Plug Eq.(87) into (83), we obtain that

$$F(m) + KL(p_m||p_{m^t}) \geq F(m^{t+1}) + KL(p_m||p_{m^t}) + KL(p_m||p_{m^{t+1}})$$ \hspace{1cm} (86)

$$\square$$

Lemma 2. Let $\tilde{v}_t = \{\tilde{g}_t - 2\tilde{G}_t \mu_t, \tilde{G}_t\}$, updating parameter as (79), then we have

$$\frac{1}{2}\|\mu^* - \mu_{t+1}\|^2_{\Sigma_{t+1}} \leq \frac{1}{2}\|\mu^* - \mu_t\|^2_{\Sigma_t} + \beta_t \langle \tilde{g}_t, \mu^* - \mu_{t+1} \rangle - \frac{1}{2}\|\mu_{t+1} - \mu_t\|^2_{\Sigma_{t+1}} + \beta_t \|\mu^* - \mu_t\|^2_{\tilde{G}_t}$$ \hspace{1cm} (87)

Proof. First, recall that the KL-divergence is defined as

$$KL(p_m||p_{m^t}) = \frac{1}{2} \left\{ \|\mu - \mu_t\|^2_{\Sigma_t} + \text{tr} \left( \Sigma_{t+1}^{-1} \right) \log \frac{\Sigma_t}{\Sigma} - d \right\}$$ \hspace{1cm} (88)

From Lemma 1 we know that

$$KL(p_m||p_{m^t}) \leq KL(p_m||p_{m^t}) - KL(p_m||p_{m^t}) + F(m^*) - F(m^{t+1})$$ \hspace{1cm} (89)

It follows that

$$\frac{1}{2} \left\{ \|\mu^* - \mu_{t+1}\|^2_{\Sigma_{t+1}} + \text{tr} \left( \Sigma^* \Sigma_{t+1}^{-1} \right) \log \frac{\Sigma_{t+1}}{\Sigma^*} \right\} \leq \frac{1}{2} \left\{ \|\mu^* - \mu_t\|^2_{\Sigma_t} + \text{tr} \left( \Sigma^* \Sigma_t^{-1} \right) \log \frac{\Sigma_t}{\Sigma^*} \right\}$$ \hspace{1cm} (90)

$$- \frac{1}{2} \left\{ \|\mu_{t+1} - \mu_t\|^2_{\Sigma_{t+1}} + \text{tr} \left( \Sigma_{t+1} \Sigma_t^{-1} \right) \log \frac{\Sigma_t}{\Sigma_{t+1}} \right\}$$ \hspace{1cm} (91)

$$\beta_t \langle \tilde{v}_t, \mu^* - m^{t+1} \rangle$$
Then, we obtain that
\[
\frac{1}{2} \left\{ \| \mu^* - \mu_{t+1} \|_{\Sigma_t^{-1}}^2 + \text{tr} \left( \Sigma^* \Sigma_t^{-1} \right) \right\} \leq \frac{1}{2} \left\{ \| \mu^* - \mu_t \|_{\Sigma_t^{-1}}^2 + \text{tr} \left( \Sigma^* \Sigma_t^{-1} \right) \right\} - \frac{1}{2} \left\{ \| \mu_{t+1} - \mu_t \|_{\Sigma_t^{-1}}^2 + \text{tr} \left( \Sigma_{t+1} \Sigma_t^{-1} \right) - d \right\} + \beta_t \langle \tilde{v}_t, m^* - m^{t+1} \rangle
\] (93)

In addition, we have that
\[
\text{tr} \left( \Sigma^* \Sigma_t^{-1} \right) - \text{tr} \left( \Sigma^* \Sigma_{t+1}^{-1} \right) - \text{tr} \left( \Sigma_{t+1} \Sigma_t^{-1} \right) + d = \text{tr} \left( \Sigma^* (\Sigma_t^{-1} - \Sigma_{t+1}^{-1}) \right) - \text{tr} \left( \Sigma_{t+1} (\Sigma_t^{-1} - \Sigma_{t+1}^{-1}) \right) = \text{tr} \left( (\Sigma^* - \Sigma_{t+1}) (\Sigma_t^{-1} - \Sigma_{t+1}^{-1}) \right) = \text{tr} \left( m_2^* - \mu^* \mu^T - m_2^{t+1} + \mu_{t+1} \mu_{t+1}^T (\Sigma_t^{-1} - \Sigma_{t+1}^{-1}) \right)
\] (94)

Note that \( \Sigma_t^{-1} - \Sigma_{t+1}^{-1} = -2\beta_t \hat{G}_t \) by updating rule, it follows that
\[
\text{tr} \left( \Sigma^* \Sigma_t^{-1} \right) - \text{tr} \left( \Sigma^* \Sigma_{t+1}^{-1} \right) - \text{tr} \left( \Sigma_{t+1} \Sigma_t^{-1} \right) + d = -2\beta_t \text{tr} \left( m_2^* - \mu^* \mu^T - m_2^{t+1} + \mu_{t+1} \mu_{t+1}^T \right) \hat{G}_t
\] (98)

Then, recall that
\[
\langle \tilde{v}_t, m^* - m^{t+1} \rangle = \langle \tilde{g}_t - 2 \hat{G}_t \mu_t, \mu^* - \mu_{t+1} \rangle + \text{tr} \left( m_2^* - m_2^{t+1} \right) \hat{G}_t
\] (99)

Plug (99) and (98) into (93), we get that
\[
\frac{1}{2} \| \mu^* - \mu_{t+1} \|_{\Sigma_t^{-1}}^2 \leq \frac{1}{2} \| \mu^* - \mu_t \|_{\Sigma_t^{-1}}^2 - \frac{1}{2} \| \mu_{t+1} - \mu_t \|_{\Sigma_t^{-1}}^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_{t+1} \rangle - 2\beta_t \left( \langle \tilde{G}_t \mu_t, \mu^* - \mu_{t+1} \rangle + \text{tr} \left( \mu^* \mu^T - \mu_{t+1} \mu_{t+1}^T \right) \hat{G}_t \right)
\] (100)

Note that
\[
-2 \left( \langle \tilde{G}_t \mu_t, \mu^* - \mu_{t+1} \rangle + \text{tr} \left( \mu^* \mu^T - \mu_{t+1} \mu_{t+1}^T \right) \hat{G}_t \right) = \langle \tilde{G}_t \mu^*, \mu^* \rangle - 2 \langle \tilde{G}_t \mu_t, \mu^* \rangle + \langle \tilde{G}_t \mu_t, \mu_t \rangle - \langle \tilde{G}_t \mu_t, \mu_{t+1} \rangle + 2 \langle \tilde{G}_t \mu_t, \mu_{t+1} \rangle - \langle \tilde{G}_t \mu_{t+1}, \mu_{t+1} \rangle
\] (101)

Plug into (100), we can get that
\[
\frac{1}{2} \| \mu^* - \mu_{t+1} \|_{\Sigma_t^{-1}}^2 \leq \frac{1}{2} \| \mu^* - \mu_t \|_{\Sigma_t^{-1}}^2 - \frac{1}{2} \| \mu_t - \mu_t \|_{\Sigma_t^{-1}}^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_{t+1} \rangle + \beta_t \| \mu^* - \mu_{t+1} \|_{\tilde{G}_t}^2
\] (103)

Also note that \( \frac{1}{2} \| \mu_{t+1} - \mu^* \|_{\Sigma_t^{-1}}^2 = \frac{1}{2} \| \mu_{t+1} - \mu_t \|_{\tilde{G}_t}^2 + \frac{1}{2} \| \mu_t - \mu^* \|_{\Sigma_t^{-1}}^2 \), we can get that
\[
\frac{1}{2} \| \mu^* - \mu_{t+1} \|_{\Sigma_t^{-1}}^2 \leq \frac{1}{2} \| \mu^* - \mu_t \|_{\Sigma_t^{-1}}^2 - \frac{1}{2} \| \mu_{t+1} - \mu_t \|_{\Sigma_t^{-1}}^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_{t+1} \rangle + \beta_t \| \mu^* - \mu_{t+1} \|_{\tilde{G}_t}^2
\] (104)

**Lemma 3.** Given a convex function \( f(x) \), for Gaussian distribution with parameters \( \theta := \{\mu, \Sigma^2\} \), let \( \tilde{J}(\theta) := \mathbb{E}_{p(x|\theta)}[f(x)] \). Then \( \tilde{J}(\theta) \) is a convex function with respect to \( \theta \).

**Proof.** For \( \lambda \in [0, 1] \), we have
\[
\lambda \tilde{J}(\theta_1) + (1 - \lambda) \tilde{J}(\theta_2) = \lambda \mathbb{E}[f(\mu_1 + \Sigma_1^2 z)] + (1 - \lambda) \mathbb{E}[f(\mu_2 + \Sigma_2^2 z)]
\] (105)
\[
= \mathbb{E}[\lambda f(\mu_1 + \Sigma_1^2 z) + (1 - \lambda) f(\mu_2 + \Sigma_2^2 z)]
\] (106)
\[
\geq \mathbb{E}[f(\lambda \mu_1 + (1 - \lambda)\mu_2 + (\lambda \Sigma_1^2 + (1 - \lambda)\Sigma_2^2) z)]
\] (107)
\[
= \tilde{J}(\lambda \theta_1 + (1 - \lambda) \theta_2)
\] (108)
Lemma 4. Given a $\gamma$-strongly convex function $f(x)$, define $\tilde{J}(\theta) := E_{p(x|\theta)}[f(x)]$ for Gaussian distribution with parameter $\theta := (\mu, \Sigma)^2 \in \Theta$ and $\Theta := \{(\mu, \Sigma^2) \mid \|\mu\|_2^2 \leq R, 0 < \Sigma \preceq \rho I, \Sigma \in S^+\}$. Suppose $bI \preceq \tilde{G}_t \preceq \frac{\gamma}{2} I$ be positive semi-definite matrix and $\Sigma_1 \in \Theta$, then we have

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 + \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 + \beta_t(\tilde{J}(\theta^*) - \tilde{J}(\theta_t)) + \beta_t \langle G_t, 2\Sigma_t \rangle + \frac{\beta^2_t}{2}E\|\Sigma_{t+1}\|_2\|\tilde{g}_t\|_2^2
$$

(109)

Proof. From Lemma [2], we know that

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 \leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 - \frac{1}{2}\|\mu_{t+1} - \mu_t\|_{\Sigma_t}^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2
$$

(110)

It follows that

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 \leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 - \frac{1}{2}\|\mu_{t+1} - \mu_t\|_{\Sigma_t}^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \langle \tilde{g}_t, \mu_t - \mu_{t+1} \rangle + \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2
$$

(111)

Note that

$$
-\frac{1}{2}\|\mu_{t+1} - \mu_t\|_{\Sigma_t}^2 + \beta_t \langle \tilde{g}_t, \mu_t - \mu_{t+1} \rangle = -\frac{1}{2}\|\mu_{t+1} - \mu_t\|_{\Sigma_t}^2 + \beta_t \langle \tilde{g}_t, \mu_t - \mu_{t+1} \rangle + \beta_t \langle \tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2
$$

(112)

$$
= -\frac{1}{2}\|\mu_{t+1} - \mu_t\|_{\Sigma_t}^2 + \beta_t \Sigma_{t+1} \tilde{g}_t \leq \frac{\beta^2_t}{2}\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(113)

$$
\leq \frac{\beta^2_t}{2} \|\tilde{g}_t\|_2^2 \leq \frac{\beta^2_t}{2} \|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(114)

Note that $\Sigma_t = \Sigma_{t-1} + 2\beta_t \tilde{G}_t$ and $\tilde{G}_t \succeq bI$, we have smallest eigenvalues $\lambda_{\min}(\Sigma_t) \geq \lambda_{\min}(\Sigma_{t-1}) \geq \cdots \geq \lambda_{\min}(\Sigma_1)$. Then, we know $\|\Sigma_{t+1}\|_2 \leq \|\Sigma_1\|_2 \leq \rho$. In addition, $\Sigma_{t+1} \in \Theta$ for $t \in \{1, 2, 3, \cdots\}$.

Plug (114) into (111), we can achieve that

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 \leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2 + \frac{\beta^2_t}{2} \|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(115)

Since $bI \preceq \tilde{G}_t \preceq \frac{\gamma}{2} I$, we get that

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 \leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 + \beta_t \langle \tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2 + \frac{\beta^2_t}{2} \|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(116)

Taking conditional expectation on both sides, we obtain that

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 \leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 + \beta_t \langle E\tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2 + \frac{\beta^2_t}{2}E\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(117)

$$
\leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 + \beta_t \langle E\tilde{g}_t, \mu^* - \mu_t \rangle + \beta_t \langle G_t, 2\Sigma_t \rangle + \beta_t \langle G_t, 2\Sigma_t \rangle
$$

$$
+ \beta_t \|\mu^* - \mu_t\|_{\tilde{G}_t}^2 + \frac{\beta^2_t}{2} E\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(118)

Note that $g_t = E\tilde{g}_t = \nabla_{\mu^*} \tilde{J}$ and $G_t = \Sigma_{\Sigma^2} \tilde{J}$ and $\nabla_{\Sigma^2} \tilde{J} = \Sigma_{\Sigma^2} \tilde{J}$ and $\nabla_{\Sigma^2} \tilde{J} = \Sigma_{\Sigma^2} \tilde{J}$ are symmetric matrix. Since $\tilde{J}(\theta)$ is a $\gamma$-strongly convex function with optimum at $\theta^* = \{\mu^*, 0\}$, we have that

$$
\langle \nabla_{\mu^*} \tilde{J}, \mu^* - \mu_t \rangle + \langle \nabla_{\Sigma^2} \tilde{J} \frac{\Sigma^2}{2}, 0 - \Sigma_t \rangle = \langle g_t, \mu^* - \mu_t \rangle + \langle \Sigma_t^2 G_t + G_t^2 \Sigma_t^2, 0 - \Sigma_t \rangle
$$

(119)

$$
= \langle g_t, \mu^* - \mu_t \rangle + \langle G_t, 2\Sigma_t \rangle
$$

(120)

$$
\leq (\tilde{J}(\theta^*) - \tilde{J}(\theta_t)) - \frac{\gamma}{2} \|\mu^* - \mu_t\|_2^2
$$

(121)

Plug it into (118), we can obtain that

$$
\frac{1}{2}E\|\mu^* - \mu_{t+1}\|_2^2 \leq \frac{1}{2}E\|\mu^* - \mu_t\|_2^2 + \beta_t (\tilde{J}(\theta^*) - \tilde{J}(\theta_t)) + \beta_t \langle G_t, 2\Sigma_t \rangle + \frac{\beta^2_t}{2}E\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(122)
Lemma 5. Given a symmetric matrix $X$ and a symmetric positive semi-definite matrix $Y$, then we have $\text{tr}(XY) \leq \|Y\|_2 \|X\|_{tr}$, where $\|X\|_{tr} := \sum_{i=1}^d |\lambda_i|$ with $\lambda_i$ denotes the eigenvalues.

Proof. Since $X$ is symmetric, it can be orthogonal diagonalized as $X = U\Lambda U^\top$, where $\Lambda$ is a diagonal matrix contains eigenvalues $\lambda_i, i \in \{1, \cdots, d\}$. Since $Y$ is a symmetric positive semi-definite matrix, it can be written as $Y = Y^\frac{1}{2}Y^\frac{1}{2}$. It follows that

$$\text{tr}(XY) = \text{tr}\left(U\Lambda U^\top Y^\frac{1}{2}Y^\frac{1}{2}\right) = \text{tr}\left(Y^\frac{1}{2}U\Lambda U^\top Y^\frac{1}{2}\right) = \sum_{i=1}^d \lambda_i a_i^\top a_i$$

(123)

where $a_i$ denotes the $i^{th}$ column of the matrix $A = U^\top Y^{\frac{1}{2}}$. Then, we have

$$\text{tr}(XY) \leq \sum_{i=1}^d |\lambda_i| a_i^\top a_i = \text{tr}\left(Y^\frac{1}{2}U|\Lambda|U^\top Y^\frac{1}{2}\right) = \text{tr}\left(Y^\frac{1}{2}X^\frac{1}{2}X^\frac{1}{2}Y^\frac{1}{2}\right) = \|Y^\frac{1}{2}X^\frac{1}{2}\|_F^2$$

(124)

where $X^\frac{1}{2} = U|\Lambda|^\frac{1}{2}U^\top$.

Using the fact $\|Y^\frac{1}{2}X^\frac{1}{2}\|_F^2 \leq \|Y^\frac{1}{2}\|_2^2\|X^\frac{1}{2}\|_F^2$, we can obtain that

$$\text{tr}(XY) = \|Y^\frac{1}{2}X^\frac{1}{2}\|_F^2 \leq \|Y^\frac{1}{2}\|_2^2\|X^\frac{1}{2}\|_F^2 = \|Y\|_2\|X\|_{tr}$$

(125)

Lemma 6. Suppose gradients $\|G_t\|_{tr} \leq B_1$ and $\hat{G}_t \succeq bI$ with $b > 0$, by setting $\beta = \beta$ as a constant step size, we have

$$\sum_{t=1}^T \beta_t \langle G_t, 2\Sigma_t \rangle \leq 2B_1 \left(\beta\|\Sigma_1\|_2 + \frac{1 + \log T}{2b}\right)$$

(126)

Proof. Note that $\Sigma_{t+1}^{-1} - \Sigma_t^{-1} = 2\beta_t \hat{G}_t$ and $\hat{G}_t \succeq bI$ with $b > 0$, we know the smallest eigenvalue of $\Sigma_{t+1}^{-1}$, i.e. $\lambda_{\min}(\Sigma_{t+1}^{-1})$ satisfies that

$$\lambda_{\min}(\Sigma_{t+1}^{-1}) \geq \lambda_{\min}(\Sigma_t^{-1}) + 2\beta_t b \geq 2 \sum_{i=1}^t \beta_i b \geq 2 \sum_{i=1}^t \beta_i b$$

(127)

Thus, we know that

$$\|\Sigma_{t+1}\|_2 = \frac{1}{\lambda_{\min}(\Sigma_{t+1}^{-1})} \leq \frac{1}{2 \sum_{i=1}^t \beta_i b} = \frac{1}{2t\beta b}$$

(128)

Note that $\Sigma_t$ is symmetric positive semi-definite and $G_t$ is symmetric. From Lemma 5, we know that $\text{tr}(G_t\Sigma_t) \leq \|\Sigma_t\|_2\|G_t\|_{tr}$. It follows that

$$\sum_{t=1}^T \beta_t \langle G_t, 2\Sigma_t \rangle \leq 2\beta \sum_{t=1}^T \|G_t\|_{tr}\|\Sigma_t\|_2 \leq 2\beta B_1 \sum_{t=1}^T \|\Sigma_t\|_2$$

(129)

$$\leq 2\beta B_1 \|\Sigma_1\|_2 + 2B_1 \left(\sum_{t=1}^{T-1} \frac{1}{2bt}\right)$$

(130)

Since $\sum_{t=1}^T \frac{1}{t} \leq 1 + \log T$, we know that

$$\sum_{t=1}^T \beta_t \langle G_t, 2\Sigma_t \rangle \leq 2\beta B_1 \|\Sigma_1\|_2 + 2B_1 \left(\frac{1 + \log T}{2b}\right) = 2B_1 \left(\beta\|\Sigma_1\|_2 + \frac{1 + \log T}{2b}\right)$$

(131)
**Theorem.** Given a convex function $f(x)$, define $\tilde{J}(\theta) := \mathbb{E}_{p(x, \theta)}[f(x)]$ for Gaussian distribution with parameter $\theta := \{\mu, \Sigma\}$ in $\Theta$ and $\Phi := \{\mu, \Sigma\}$ $\mu \in \mathbb{R}^d, \Sigma \in \mathcal{S}^+$. Suppose $\tilde{J}(\theta)$ be $\gamma$-strongly convex. Let $\tilde{G}_i$ be positive semi-definite matrix such that $bI \preceq \tilde{G}_i \preceq \frac{\gamma}{2}I$. Suppose $\Sigma_1 \in \mathcal{S}^{++}$ and $\|\Sigma_1\| \leq \rho$, $\mathbb{E}\tilde{g}_i = \nabla_{\mu=\mu_i} \tilde{J}$. Assume furthermore $\|\nabla_{\theta=\Sigma_1} \tilde{J}\|_F \leq B_1$ and $\|\mu^* - \mu_1\|_{\Sigma_1^{-1}} \leq R, \mathbb{E}\|\tilde{g}_i\|^2 \leq B$. Set $\beta_t = \beta$, then Algorithm 5 can achieve

$$
\frac{1}{T} \left[ \sum_{t=1}^{T} f(\theta_t) \right] - f(\theta^*) \leq \frac{2bR + 2b\beta \rho (4B_1 + \beta B) + 4B_1 (1 + \log T) + (1 + \log T) \beta B}{4\beta b T} = O \left( \frac{\log T}{T} \right)
$$

(132)

**Proof.** From Lemma 1 to Lemma 4, we know that

$$
\frac{1}{2} \mathbb{E}\|\mu^* - \mu_{t+1}\|_{\Sigma^{-1}_{t+1}} \leq \frac{1}{2} \mathbb{E}\|\mu^* - \mu_t\|_{\Sigma^{-1}_t} + \beta_t (\tilde{J}(\theta^*) - \tilde{J}(\theta_t)) + \beta_t \langle G_t, 2\Sigma_t \rangle + \frac{\beta^2_t}{2} \mathbb{E}\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(133)

Sum up both sides from $t = 1$ to $t = T$ and rearrange terms, we get

$$
\sum_{t=1}^{T} \beta_t [\tilde{J}(\theta_t) - \tilde{J}(\theta^*)] \leq \frac{1}{2} \mathbb{E}\|\mu^* - \mu_t\|_{\Sigma^{-1}_t} - \frac{1}{2} \mathbb{E}\|\mu^* - \mu_{T+1}\|_{\Sigma^{-1}_{T+1}} + \frac{\beta^2}{2} \sum_{t=1}^{T} \mathbb{E}\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(134)

$$
+ \sum_{t=1}^{T} \beta_t \langle G_t, 2\Sigma_t \rangle + \sum_{t=1}^{T} \frac{\beta^2_t}{2} \mathbb{E}\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2
$$

(135)

Since $\beta_t = \beta$, we can obtain that

$$
\frac{1}{T} \left[ \sum_{t=1}^{T} \tilde{J}(\theta_t) \right] - \tilde{J}(\theta^*) \leq \frac{\frac{1}{2} \mathbb{E}\|\mu^* - \mu_1\|_{\Sigma^{-1}_1} + \sum_{t=1}^{T} \beta_t \langle G_t, 2\Sigma_t \rangle + \frac{\beta^2}{2} \sum_{t=1}^{T} \mathbb{E}\|\Sigma_{t+1}\|_2 \|\tilde{g}_t\|_2^2}{\beta}
$$

(136)

$$
\leq \frac{\frac{1}{2} R + \sum_{t=1}^{T} \beta_t \langle G_t, 2\Sigma_t \rangle + \frac{\beta^2}{2} B \sum_{t=1}^{T} \mathbb{E}\|\Sigma_{t+1}\|_2}{\beta}
$$

(137)

From Eq. (128), we know that

$$
\|\Sigma_{t+1}\|_2 \leq \frac{1}{2b\beta}
$$

(138)

Since $\sum_{t=1}^{T} \frac{1}{2} \leq 1 + \log T$, we know that $\sum_{t=1}^{T} \mathbb{E}\|\Sigma_{t+1}\|_2 \leq \|\Sigma_1\|_2 + \frac{1+\log T}{2b\beta} \leq \rho + \frac{1+\log T}{2b\beta}$

In addition, from Lemma 5, we know that

$$
\sum_{t=1}^{T} \beta_t \langle G_t, 2\Sigma_t \rangle \leq 2B_1 \left( \beta \|\Sigma_1\|_2 + \frac{1+\log T}{2b} \right) \leq 2B_1 \left( \beta \rho + \frac{1+\log T}{2b} \right)
$$

(139)

Plug all them into (137), we can get

$$
\frac{1}{T} \left[ \sum_{t=1}^{T} \tilde{J}(\theta_t) \right] - \tilde{J}(\theta^*) \leq \frac{\frac{1}{2} R + 2B_1 \left( \beta \rho + \frac{1+\log T}{2b} \right) + \frac{\beta^2}{2} B \left( \frac{1+\log T}{2b} + \frac{1+\log T}{4\beta} + \frac{1+\log T}{4\beta b T} \right)}{\beta}
$$

(140)

$$
= \frac{2bR + 4B_1 (1 + \log T) + 2b\beta \rho (4B_1 + \beta B) + (1 + \log T) \beta B}{4\beta b T}
$$

(141)

$$
= \frac{2bR + 2b\beta \rho (4B_1 + \beta B) + 4B_1 (1 + \log T) + (1 + \log T) \beta B}{4\beta b T}
$$

(142)

$$
= O \left( \frac{\log T}{T} \right)
$$

(143)
Since \( f(x) \) is a convex function, we know \( f(\mu) \leq \bar{J}(\mu, \Sigma) = \mathbb{E}[f(x)] \). Note that for an optimum point \( \mu^* \) of \( f(x), \theta^* = (\mu^*, 0) \) is an optimum of \( \bar{J}(\theta) \), i.e., \( f(\mu^*) = \bar{J}(\theta^*) \). Thus, we can obtain that

\[
\frac{1}{T} \left[ \sum_{t=1}^{T} f(\theta_t) \right] - f(\theta^*) \leq \frac{1}{T} \left[ \sum_{t=1}^{T} \bar{J}(\theta_t) \right] - \bar{J}(\theta^*) \leq \frac{2bR + 2b\beta\rho(4B_1 \beta + 4B_1 (1 + \log T) + (1 + \log T)\beta E)}{4\beta b T}
\]

\[
\leq \mathcal{O}\left( \frac{\log T}{T} \right)
\]

(F) Proof of Theorem 7

**Lemma 7.** For a \( L \)-Lipschitz continuous black box function \( f(x) \). Let \( \bar{G} \) be positive semi-definite matrix such that \( bI \leq \bar{G} \) with \( b > 0 \). Suppose the gradient estimator \( \hat{g}_i \) is defined as

\[
\hat{g}_i = \Sigma_i^{-\frac{1}{2}} z \left( f(\mu_i + \Sigma_i^\frac{1}{2} z) - f(\mu_i) \right)
\]

where \( z \sim \mathcal{N}(0, I) \). Then \( \hat{g}_i \) is an unbiased estimator of \( \nabla_{\mu} \mathbb{E}_p[f(x)] \) and \( \mathbb{E}||\Sigma_{t+1}||_2 \| \hat{g}_t \|_2^2 \leq L^2 ||\Sigma_t||_2 (d+4)^2 \)

**Proof.** We first show the unbiased estimator.

\[
\mathbb{E}[\hat{g}_i] = \mathbb{E} \left[ \Sigma_i^{-\frac{1}{2}} z f(\mu_i + \Sigma_i^\frac{1}{2} z) \right] - \mathbb{E} \left[ \Sigma_i^{-\frac{1}{2}} z f(\mu_i) \right]
\]

\[
= \mathbb{E} \left[ \Sigma_i^{-\frac{1}{2}} z f(\mu_i + \Sigma_i^\frac{1}{2} z) \right]
\]

\[
= \mathbb{E}_{p(\mu, \Sigma_i)} \left[ \Sigma_i^{-1} (x - \mu_i) f(x) \right]
\]

\[
= \nabla_{\mu} \mathbb{E}_p[f(x)]
\]

The last equality holds by Theorem 2.

Now, we prove the bound of \( \mathbb{E}_{p} ||\Sigma_{t+1}||_2 \| \hat{g}_t \|_2^2 \).

\[
||\Sigma_{t+1}||_2 \| \hat{g}_t \|_2^2 = ||\Sigma_{t+1}||_2 \| \Sigma_i^{-\frac{1}{2}} z \|_2^2 \left( f(\mu_i + \Sigma_i^\frac{1}{2} z) - f(\mu_i) \right)^2
\]

\[\leq ||\Sigma_{t+1}||_2 \| \Sigma_i^{-\frac{1}{2}} z \|_2^2 L^2 ||\Sigma_i^\frac{1}{2} z \|_2^2
\]

\[\leq ||\Sigma_{t+1}||_2 \| \Sigma_i^{-\frac{1}{2}} z \|_2^2 \| z \|_2^2 L^2 ||\Sigma_i^\frac{1}{2} z \|_2^2
\]

\[= ||\Sigma_{t+1}||_2 \| \Sigma_i^{-1} z \|_2^2 L^2 ||\Sigma_i^\frac{1}{2} z \|_2^2
\]

Since \( ||\Sigma_{t+1}||_2 \leq ||\Sigma_i||_2 \) proved in Lemma 4 (below Eq. (114)), we get that

\[||\Sigma_{t+1}||_2 \| \hat{g}_t \|_2^2 \leq || z \|_2^2 L^2 ||\Sigma_i^\frac{1}{2} z \|_2^2 \leq L^2 ||\Sigma_{t+1}||_2 \| z \|_2^2
\]

Since \( \mathbb{E} || z \|_2^2 \leq (d+4)^2 \) shown in [24], we can obtain that

\[\mathbb{E}_{p} ||\Sigma_{t+1}||_2 \| \hat{g}_t \|_2^2 \leq L^2 ||\Sigma_{t+1}||_2 (d+4)^2
\]

**Theorem.** For a \( L \)-Lipschitz continuous convex black box function \( f(x) \), define \( \bar{J}(\theta) := \mathbb{E}_{p(x, \theta)}[f(x)] \) for Gaussian distribution with parameter \( \theta := \{\mu, \Sigma \} \in \Theta \) and \( \Theta := \{\mu, \Sigma \mid \mu \in \mathbb{R}^d, \Sigma \in \mathcal{S}^+ \} \). Suppose \( \bar{J}(\theta) \) be \( \gamma \)-strongly convex. Let \( \bar{G} \) be positive semi-definite matrix such that \( bI \preceq \bar{G} \preceq \frac{\gamma}{4} I \). Suppose \( \Sigma_1 \in \mathcal{S}^{++} \) and...
From Lemma 7, we know $E$.

Proof. We are now ready to prove Theorem 6. From Lemma 7, we know $E\|\Sigma_{t+1}\|_2^2 \leq L^2\|\Sigma_t\|_2(d+4)^2$. Note that $\|\Sigma_{t+1}\|_2 \leq \frac{1}{2T}\beta b$ from Eq. (128), we can obtain that

$$E\|\Sigma_{t+1}\|_2^2 \leq L^2\|\Sigma_t\|_2(d+4)^2 \leq \frac{L^2(d+4)^2}{2(t-1)\beta b}$$

Plug it into Eq. (136), also note that $\|\Sigma_2\|_2 \leq \|\Sigma_1\|_2$, we get that

$$\frac{1}{T} \left[ \sum_{t=1}^T \tilde{J}^{\beta}(\theta_t) \right] - \tilde{J}^{\beta}(\theta^*) \leq \frac{\frac{1}{2}E\|\mu^* - \mu_1\|_{\Sigma_1}^2 + \sum_{t=1}^T \beta_t (G_t, 2\Sigma_t) + \beta^2\|\Sigma_1\|_2L^2(d+4)^2 + \frac{\beta L^2(d+4)^2}{4b} \sum_{t=1}^T \frac{1}{T} \right]$$

$$\leq \frac{1}{2}R + \sum_{t=1}^T \beta_t (G_t, 2\Sigma_t) + \beta^2\|\Sigma_1\|_2L^2(d+4)^2 + \frac{\beta L^2(d+4)^2}{4b} (1 + \log T)$$

In addition, from Lemma 6 we know that

$$\sum_{t=1}^T \beta_t (G_t, 2\Sigma_t) \leq 2B_1 \left( \beta\|\Sigma_1\|_2 + \frac{1 + \log T}{2b} \right) \leq 2B_1 \left( \beta \rho + \frac{1 + \log T}{2b} \right)$$

Then, we can get that

$$\frac{1}{T} \left[ \sum_{t=1}^T \tilde{J}^{\beta}(\theta_t) \right] - \tilde{J}^{\beta}(\theta^*) \leq \frac{\frac{1}{2}R + 2B_1 \left( \beta \rho + \frac{1 + \log T}{2b} \right) + \beta^2 \rho L^2(d+4)^2 + \frac{\beta L^2(d+4)^2}{4b} (1 + \log T)}{T\beta}$$

$$= \frac{2bR + 2b\beta \rho (4B_1 + 2\beta L^2(d+4)^2) + 4B_1 (1 + \log T) + (1 + \log T)\beta L^2(d+4)^2}{4\beta b T}$$

$$= \mathcal{O} \left( \frac{d^2 \log T}{T} \right)$$

Since $f(x)$ is a convex function, we know that

$$\frac{1}{T} \left[ \sum_{t=1}^T f(\theta_t) \right] - f(\theta^*) \leq \frac{1}{T} \left[ \sum_{t=1}^T \tilde{J}(\theta_t) \right] - \tilde{J}(\theta^*)$$

$$\leq \frac{2bR + 2b\beta \rho (4B_1 + 2\beta L^2(d+4)^2) + 4B_1 (1 + \log T) + (1 + \log T)\beta L^2(d+4)^2}{4\beta b T}$$

$$= \mathcal{O} \left( \frac{d^2 \log T}{T} \right)$$
G Variance Reduction

Lemma 8. For a $L$-Lipschitz continuous black box function $f(x)$. Suppose $\Sigma_t = \sigma_t^2 I$ with $\sigma_t > 0$ for $t \in \{1, \ldots, T\}$. Suppose the gradient estimator $\widehat{g}_t$ is defined as

$$\widehat{g}_t = \frac{1}{N} \sum_{i=1}^{N} \Sigma_t^{-\frac{1}{2}} z_i \left( f(\mu_t + \Sigma_t^\frac{1}{2} z_i) - f(\mu_t) \right) \quad (169)$$

where $Z = [z_1, \ldots, z_N]$ has marginal distribution $\mathcal{N}(0, I)$ and $Z^\top Z = I$. Then $\widehat{g}_t$ is an unbiased estimator of $\nabla \mathbb{E}_q[f(x)]$ and $\mathbb{E}_Z\|\Sigma_{t+1}\|_2\|\widehat{g}_t\|_2^2 \leq \frac{\sigma_{t+1}^2 L^2(d+4)^2}{N}$ for $N \leq d$.

Proof. We first show the unbiased estimator.

$$\mathbb{E}_Z[\widehat{g}_t] = \mathbb{E}_Z \left[ \frac{1}{N} \sum_{i=1}^{N} \Sigma_t^{-\frac{1}{2}} z_i \left( f(\mu_t + \Sigma_t^\frac{1}{2} z_i) - f(\mu_t) \right) \right] \quad (170)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_Z \left[ \Sigma_t^{-\frac{1}{2}} z_i \left( f(\mu_t + \Sigma_t^\frac{1}{2} z_i) - f(\mu_t) \right) \right] \quad (171)$$

$$= \mathbb{E}_Z \left[ \Sigma_t^{-\frac{1}{2}} z_i f(\mu_t + \Sigma_t^\frac{1}{2} z_i) \right] - \mathbb{E}_Z \left[ \Sigma_t^{-\frac{1}{2}} z_i f(\mu_t) \right] \quad (172)$$

$$= \mathbb{E}_Z \left[ \Sigma_t^{-\frac{1}{2}} z f(\mu_t + \Sigma_t^\frac{1}{2} z) \right] \quad (173)$$

$$= \mathbb{E}_Z \left[ \Sigma_t^{-\frac{1}{2}} z f(\mu_t + \Sigma_t^\frac{1}{2} z) \right] \quad (174)$$

$$= \mathbb{E}_p(\mu, \Sigma_\cdot) \left[ \Sigma_t^{-1}(x - \mu) f(x) \right] \quad (175)$$

$$= \nabla \mathbb{E}_p[f(x)] \quad (176)$$

The last equality holds by Theorem 2.

Now, we prove the bound of $\mathbb{E}_p\|\Sigma_{t+1}\|_2\|\widehat{g}_t\|_2^2$.

$$\|\Sigma_{t+1}\|_2\|\widehat{g}_t\|_2^2 \quad (177)$$

$$= \sigma_{t+1}^2 \left\| \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{-1} z_i (f(\mu_t + \sigma_t z_i) - f(\mu_t)) \right\|_2^2 \quad (178)$$

$$= \sigma_{t+1}^2 \sum_{i=1}^{N} \|\sigma_i^{-1} z_i (f(\mu_t + \sigma_t z_i) - f(\mu_t))\|_2^2 + \frac{\sigma_{t+1}^2 \sigma_t^{-2}}{N^2} \sum_{i=1}^{N} \sum_{i \neq j} z_i^\top z_j (f(\mu_t + \sigma_t z_i) - f(\mu_i))(f(\mu_t + \sigma_t z_j) - f(\mu_i)) \quad (179)$$

$$= \sigma_{t+1}^2 \sum_{i=1}^{N} \|\sigma_i^{-1} z_i (f(\mu_t + \sigma_t z_i) - f(\mu_t))\|_2^2 \quad (180)$$

$$\leq \frac{\sigma_{t+1}^2 \sigma_t^{-2} \sigma_t^2 L^2}{N^2} \sum_{i=1}^{N} \|z_i\|_2^4 = \frac{\sigma_{t+1}^2 L^2}{N^2} \sum_{i=1}^{N} \|z_i\|_2^4 \quad (181)$$

Thus, we know that

$$\mathbb{E}_Z \left[ \|\Sigma_{t+1}\|_2\|\widehat{g}_t\|_2^2 \right] \leq \frac{\sigma_{t+1}^2 L^2}{N^2} \mathbb{E}_Z \sum_{i=1}^{N} \|z_i\|_2^4 = \frac{\sigma_{t+1}^2 L^2}{N} \mathbb{E}_Z[\|z\|_2^4] \quad (182)$$

Since $\mathbb{E}_Z[\|z\|_2^4] \leq (d + 4)^2$ shown in [23], we can obtain that

$$\mathbb{E}_Z\|\Sigma_{t+1}\|_2\|\widehat{g}_t\|_2^2 \leq \frac{\sigma_{t+1}^2 L^2(d+4)^2}{N} \quad (183)$$
Theorem. For a $L$-Lipschitz continuous convex black box function $f(x)$, define $\bar{J}(\theta) := \mathbb{E}_{\rho(x, \theta)}[f(x)]$ for Guassian distribution with parameter $\theta := \{\mu, \sigma \in \mathbb{R}^d \mid \mu \in \mathbb{R}^d, \Sigma \in S^d\}$. Suppose $\bar{J}(\theta)$ be $\gamma$-strongly convex. Let $G_t = bI$ with $b \leq \frac{\gamma}{2}$. Suppose $\|\Sigma\|_2 \leq \rho = \frac{1}{L}$. Assume furthermore $\|\nabla_{\Sigma = \Sigma_t} \bar{J}\|_{tr} \leq B_1$ and $\|\mu^* - \mu_t\|_{2-t} \leq R_t$. Set $\beta_t = T$ and employ orthogonal estimator $\hat{g}_t$ in Eq. (169) with $N = d$, then Algorithm 3 can achieve

\[
\frac{1}{T} \left[ \sum_{t=1}^{T} f(\theta_t) \right] - f(\theta^*) \leq \frac{2bR + 2\beta(4B_1/d + 2\beta L^2(d + 4)^2/d) + 4B_1(1 + \log T) + (1 + \log T)\beta L^2(d + 4)^2/d}{4\beta T} \tag{184}
\]

\[
= \mathcal{O} \left( \frac{d \log T}{T} \right) \tag{185}
\]

**Proof.** The proof is similar to the proof of Theorem 7.

From Lemma 8 and $N = d$, we know $\mathbb{E}\|\Sigma_{t+1}\|_2 \|\hat{g}_t\|_2^2 \leq \sigma_{t+1}^2 \Sigma_{t+1}^2$. Note that $\sigma_{t+1}^2 = ||\Sigma_{t+1}||_2^2 \leq \frac{1}{2\beta b}$ from Eq. (128), we can obtain that

\[
\mathbb{E}\|\Sigma_{t+1}\|_2 \|\hat{g}_t\|_2^2 \leq \sigma_{t+1}^2 \Sigma_{t+1}^2 \leq \frac{L^2(d + 4)^2}{2\beta b} \tag{186}
\]

Plug it into Eq. (136), we get that

\[
\frac{1}{T} \left[ \sum_{t=1}^{T} J(\theta_t) \right] - J(\theta^*) \leq \frac{\frac{1}{T} \mathbb{E}\|\mu^* - \mu_t\|_2^2 + \sum_{t=1}^{T} \beta_t \{G_t, 2\Sigma_t\} + 2\beta\|\Sigma_t\|_2 \Sigma_{t+1}^2 + \frac{\beta L^2(d + 4)^2}{4\beta d} \sum_{t=1}^{T} \frac{1}{T}}{T \beta} \tag{187}
\]

\[
\leq \frac{\frac{1}{T} R + \sum_{t=1}^{T} \beta_t \{G_t, 2\Sigma_t\} + 2\beta\|\Sigma_t\|_2 \Sigma_{t+1}^2 + \frac{\beta L^2(d + 4)^2}{4\beta d} (1 + \log T)}{T \beta} \tag{188}
\]

In addition, from Lemma 6, we know that

\[
\sum_{t=1}^{T} \beta_t \{G_t, 2\Sigma_t\} \leq 2B_1 \left( \beta\|\Sigma_t\|_2 \frac{1 + \log T}{2b} \right) \leq 2B_1 \left( \beta \rho \frac{1 + \log T}{2b} \right) \tag{189}
\]

Then, we can get that

\[
\frac{1}{T} \left[ \sum_{t=1}^{T} J(\theta_t) \right] - J(\theta^*) \leq \frac{\frac{1}{T} R + 2B_1 \left( \beta \rho \frac{1 + \log T}{2b} \right) + 2\beta \rho L^2(d + 4)^2 + \frac{\beta L^2(d + 4)^2}{4\beta d} (1 + \log T)}{T \beta} \tag{190}
\]

\[
= \frac{2bR + 2\beta \rho (4B_1 + 2\beta L^2(d + 4)^2) + 4B_1(1 + \log T) + (1 + \log T)\beta L^2(d + 4)^2/d}{4\beta T} \tag{191}
\]

\[
= \mathcal{O} \left( \frac{d \log T}{T} \right) \tag{193}
\]

Since $f(x)$ is a convex function, we know that

\[
\frac{1}{T} \left[ \sum_{t=1}^{T} f(\theta_t) \right] - f(\theta^*) \leq \frac{1}{T} \left[ \sum_{t=1}^{T} J(\theta_t) \right] - J(\theta^*) \leq \mathcal{O} \left( \frac{d \log T}{T} \right) \tag{194}
\]

\[
\square
\]
H Test Problems

Table 1: Test functions

| name         | function                                                                 |
|--------------|---------------------------------------------------------------------------|
| Ellipsoid    | \( f(x) := \sum_{i=1}^{d} 10^{\frac{i-1}{d-1}} x_i^2 \)                  |
| Discus       | \( f(x) := 10^6 x_1 + \sum_{i=2}^{d} x_i^2 \)                            |
| \( \ell_1 \)-Ellipsoid | \( f(x) := \sum_{i=1}^{d} 10^{\frac{i-1}{d-1}} |x_i| \)                   |
| \( \ell_{\frac{1}{2}} \)-Ellipsoid | \( f(x) := \sum_{i=1}^{d} 10^{\frac{i-1}{d-1}} |x_i|^{\frac{1}{2}} \)   |
| Levy         | \( f(x) := \sin^2(\pi w_1) + \sum_{i=1}^{d-1} (w_i - 1)(1 + 10\sin^2(\pi w_i + 1)) + (w_d - 1)^2(1 + \sin^2(2\pi w_d)) \) |
|              | where \( w_i = 1 + (x_i - 1)/4, \ i \in \{1, ..., d\} \)                |
| Rastrigin10  | \( f(x) := 10d + \sum_{i=1}^{d} (10 \pi^{-\frac{i-1}{d}} x_i)^2 - 10 \cos(2\pi 10 \pi^{-\frac{i-1}{d}} x_i) \) |

I Experiments on 1000-dimensional test problems

Figure 3: Mean value of \( f(x) \) in \( \log_{10} \) scale over 20 independent runs for 1000-dimensional problems