On quasi-local charges and Newman–Penrose type quantities in Yang–Mills theories

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Abstract
We generalize the notion of quasi-local charges, introduced by P Tod for Yang–Mills fields with unitary gauge groups, to non-Abelian gauge theories with arbitrary gauge groups, and calculate its small sphere and large sphere limits both at spatial and null infinity. We show that for semisimple gauge groups no reasonable definition yield conserved total charges and Newman–Penrose (NP) type quantities at null infinity in generic, radiative configurations. The conditions of their conservation, both in terms of the field configurations and the structure of the gauge group, are clarified. We also calculate the NP quantities for stationary, asymptotic solutions of the field equations with vanishing magnetic charges, and illustrate these by explicit solutions with various gauge groups.

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1. Introduction

Just a couple of years after the introduction of the concept of (non-Abelian) gauge fields, in his classical paper Utiyama [1] already intended to interpret general relativity as a gauge theory, in which the Lorentz group emerges as the gauge group of transformations taking locally defined orthonormal bases to orthonormal ones. Then Møller [2] formulated general relativity as a dynamical theory of frame fields, and in this new form Einstein’s theory showed indeed remarkable resemblance to Yang–Mills theories. This similarity of the two theories made it possible to import techniques developed in one field to the other (see e.g. [3]), yielded a deeper understanding of both (see e.g. [4, 5]), or motivated the search for a proper gauge theoretic foundation of general relativity (see e.g. [6, 7]). In particular, the concept of charge density of non-Abelian gauge fields appeared to be similar to the (ill-defined) concept of the energy-momentum density of the gravitational field: the vanishing of the gauge-covariant divergence
of the (gauge-covariant) non-Abelian charge 4-current (e.g. of the fermionic matter) in itself does not yield any gauge-covariant notion of conserved charges.

To overcome this difficulty, Tod [8] introduced the notion of quasi-local charges in non-Abelian Yang–Mills theories, following the analogous twistorial definition of quasi-local energy momentum and angular momentum of Penrose in general relativity. His construction is based on the use of the holomorphic and anti-holomorphic cross sections of a Hermitian vector bundle. Hence the original construction could give a well-defined notion of charge in theories with (pseudo-)unitary gauge groups (or subgroups of them). However, the Lorentz group is not such a group; thus, to be able to see the closest analogy between general relativity and proper non-Abelian gauge theories as it could be possible, it would have to be considered the latter with the Lorentz (or its covering group, the \( SL(2, \mathbb{C}) \)) gauge group. Therefore, we need to generalize the notion of charge of Tod to arbitrary gauge groups, or at least to the groups above.

In general relativity, the energy momentum and (relativistic) angular momentum take the form of 2-surface integrals, like the charge in Maxwell theory. However, in Einstein’s theory there are ten additional quantities (and in the coupled Einstein–Maxwell system 16 quantities), taking the form of 2-surface integrals on spherical cuts of future null infinity, which are absolutely conserved. These are the Newman–Penrose (or NP) quantities [9, 10]. Though their meaning is still not quite well understood, in the linear approximation they appear to characterize the radiation profile of the incoming radiation, while in stationary configurations they are connected with the multipole structure of the source. Thus the analogy between Einstein’s theory and the non-Abelian gauge theories raises the question if the NP quantities can be introduced in Yang–Mills theories, and whether or not they are conserved. Though there is an argumentation that the analogous NP quantities cannot be conserved [11, 12], as far as we are aware no such mathematical analysis of the question has been given.

Although the charges and the NP type quantities are conceptually different physical quantities, technically they are similar. Thus it seems natural to consider both. Therefore, the aim of the present paper is twofold: first, to generalize Tod’s definition of quasi-local charges to arbitrary gauge group and investigate the resulting expression in the standard small sphere and large sphere limits. The second is to form the analog of the NP quantities and clarify the conditions of their conservation.

Mostly to fix the notation, in section 2, we review the theoretical background and recall some basic facts that we use. Then, in section 3, we introduce the general notion of the self-dual/anti-self-dual charge integrals. We will see that the notion of electric and magnetic charges can be formulated and studied in terms of these more naturally and easier than the original charge integrals of the field strength. In subsection 3.2 we summarize the key points of Tod’s construction for Hermitian vector bundles, and its generalization for arbitrary vector bundles and connections is suggested in subsection 3.3, using either anti-holomorphic or holomorphic cross sections of the vector bundle. Here it is also shown that for globally trivializable bundles, the modified construction also gives vanishing magnetic charges.

In subsection 3.4 the limit of the quasi-local charge integrals are calculated in three limits. First, the charges are associated with small spheres near a point. We show that in the first two non-trivial orders the electric charge integral is independent of the actual construction, i.e. has some universal nature. The solution of the field equations near a point with the necessary accuracy is given in appendix A.1. Then we calculate the spatial infinity limit. We show that under the standard boundary conditions for the gauge fields this limit is finite and constant in time, i.e. these define conserved total charges. In the case of the vanishing total magnetic charges, the total electric charges in the holomorphic and in the anti-holomorphic constructions coincide. We also calculate the null infinity limit of the general charge integrals.
We show that there is no reasonable definition for the total charges at null infinity that, for semisimple gauge groups, could be conserved in general radiative configurations. There can be charge conservation only in special (e.g. not radiative) field configurations or for gauge groups whose Lie algebra contains a non-trivial solvable ideal, provided the charges are introduced in the anti-holomorphic construction. The holomorphic construction appears naturally at past null infinity.

Section 4 is devoted to the NP type quantities. First we show that for semisimple gauge groups no reasonable definition yields conserved NP type quantities in the generic, radiative configurations, and their conservation can be expected only in special configurations or for special gauge groups. We also calculate them in special stationary field configurations and find that they are built from the expansion coefficients of the field strength with dipole and higher moments. To be able to do these calculations we had to increase the accuracy of the asymptotic solutions of the Yang–Mills equations of Newman [3] by two orders. These solutions are given in appendix A.2.

The signature of the spacetime metric is $-2$, and in connection with the spin weighted spherical harmonics we use the conventions of [13].

2. The theoretical background

Let $E(M)$ be a real or complex vector bundle over the spacetime manifold $M$ with an $n$ dimensional real or complex vector space $\mathbb{R}^n$ as the typical fiber, let $\{E^A_p\}$, $A = 1, \ldots, n$, be a locally defined frame field on some open $U \subseteq M$, i.e. at each point $p \in U$ the vectors $E^1_p$, $\ldots, E^n_p$ span the fiber $\mathbb{R}^n_p$ at $p$. (Thus the capital Latin indices $A, B, \ldots$ are abstract indices referring to the internal fiber space, while the capital boldface Latin indices are concrete name indices taking numerical values. Small Latin indices are abstract spacetime indices.) The dual frame field will be denoted by $\{E^A_p\}$, and hence $E^A_p E^B_p = \delta^A_B$ and $E^A_p E^A_p = \delta^B_p$ hold. Clearly, such a frame field is not unique, and we allow the vector basis to be transformed as $E^A_p \mapsto E^A_p A^B_p$, where $A^B_p$ is a function on $U$ taking its values in some subgroup $G$ of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. This yields the transformation $E^A_p \mapsto E^A_p A^B_p$ of the dual frame field on $U$, where $A^B_p A^C_p = \delta^C_p$. Thus we think of the set of the frames above as a principal fiber bundle $P(M,G)$ with structure group $G$, and the matrices $A^B_p$ provide the free action of $G$ on $P$.

Let a connection $D_p$ be given on $E(M)$. This can be specified by its action on the vectors $E^A_p$ of the frame field, yielding the connection 1-form $A^A_{B_p} := E_A^A(D_p E^A_p)$ on $U$. The connection is a $G$-connection if $A^A_{B_p}$, as a locally defined matrix valued 1-form, takes its values in the Lie algebra $\mathfrak{g}$ of the Lie group $G$. The curvature (or field strength), also in the local frame field on $U$, is the $\mathfrak{g}$ valued 2-form $F^A_{Bcd} = \nabla^A_{B} A^C_{cd} - \nabla^A_{C} A^B_{cd} - \nabla^A_{d} A^B_{cd}$. It satisfies the Bianchi identity $(\nabla^A_{Bef} + A^A_{B} A^C_{ef} - A^A_{C} A^B_{ef} A^C_{B}) \delta_{abc} = 0$. $F^A_{Bcd} := E^A_{B} E^A_{C} A^C_{B} F^A_{Bcd}$ is already gauge covariant, and globally well defined.

The Yang–Mills equations with gauge group $G$ and gauge-covariant source current $j^A_{B_e}$ are

$$g^{cb} (\nabla^c F^A_{Bcb} + A^A_{C} F^C_{Bcb} - F^A_{Cba} A^C_{B}) = -4\pi j^A_{B_e}.$$  \hspace{1cm} (2.1)

These equations imply, in particular, that the spacetime and gauge-covariant divergence of $j^A_{B_e}$ is vanishing, and hence its spacetime divergence in itself is not $\nabla^a j^A_{B_a} = g^{cb} (j^A_{C_b} A^C_{B} - A^A_{B_c} j^C_{B_e})$. Therefore, by taking the flux integral on some spacelike hypersurface $\Sigma$, it is the $\nabla^c$-divergence free and not gauge-covariant current.
\[ J^A_{\alpha B} := J^A_{\alpha B} + \frac{1}{3\pi} (F^{A}_{\alpha B} C_{\beta \mu} - F^{C}_{\alpha C} F^{C}_{\beta B}) \] that should be used to define charge. Thus it is more convenient to use the non-gauge-covariant form:

\[ \nabla^\alpha F^A_{\beta \gamma} = -4\pi J^A_{\beta \gamma} \] (2.2)

of the field equations. In the rest of the paper we frequently use the matrix notation instead of writing out the boldface (matrix Lie algebra) or abstract fiber indices explicitly. For example, equation (2.1) takes the form \( g^{ij} (\nabla_i F_{j\alpha} + [A_i, F_{j\alpha}]) = -4\pi j_{\alpha} \).

If we fix a basis \( \{ e^A_{\alpha B} \} \) in the Lie algebra \( \mathfrak{g} \), \( \alpha = 1, \ldots, \dim \mathfrak{g} \), then we can write \( A^A_{\alpha B} = A^\mu_{\alpha B} e^A_{\mu B} \) and \( F^A_{\alpha \beta} = F^{\mu \nu}_{\alpha \beta} e^A_{\mu \nu} \). We define the structure constant \( c^\mu_{\alpha \beta} \) of \( \mathfrak{g} \) in this basis in the usual way by

\[ e^A_{\alpha B} e^B_{\beta C} - e^A_{\beta B} e^B_{\alpha C} =: c^\mu_{\alpha \beta} e^A_{\mu C}. \]

Then the definitions yield

\[ F^{\mu \nu}_{\alpha \beta} = \nabla_\mu A^\nu_{\beta} - \nabla_\nu A^\mu_{\beta} + c^\rho_{\mu \nu} A^\rho_{\alpha \beta}, \]

and that the gauge-covariant derivative of any field \( e \) e.g. with components \( X^A_{\alpha B} \) can also be written in the form \( D_e X^A_{\alpha B} = (\nabla_e X^\nu + c^\rho_{\nu \mu} A^\rho_{\alpha \beta}) e^A_{\mu B} \).

The expression between the parentheses is just the gauge-covariant derivative on the vector bundle defined via the adjoint representation of \( G \) as an abstract Lie group, i.e. in which the typical fiber is \( \mathfrak{g} \) as its abstract Lie algebra. Then it is straightforward to rewrite the Yang–Mills equations in terms of these notions. Also for later use, we introduce the symmetric metric \( g_{\alpha \beta} := e^A_{\alpha B} e^B_{\beta A} \) on the Lie algebra \( \mathfrak{g} \). It is a direct calculation to check that \( e^A_{\alpha B} \) which are constant functions on the domain \( U \) of the frame field \( \{ E^A_{\alpha} \} \) as a field on \( U \) is constant with respect to the gauge-covariant derivative \( D_e \), too. Therefore, the connection is compatible with this metric: \( D_e g_{\alpha \beta} = 0 \). In the adjoint representation, in which the basis \( \{ e^A_{\alpha B} \} \) is chosen to be the structure constants \( c^\mu_{\alpha \beta} \), this metric reduces to the Killing–Cartan metric \( g_{\alpha \beta} \) on \( \mathfrak{g} \).

\[ g_{\alpha \beta} \] is well known to be non-singular precisely for semisimple \( \mathfrak{g} \), and this is negative definite precisely for Lie algebras of compact semisimple (and hence real) Lie groups.

### 3. Quasi-local charges

#### 3.1. The concept of quasi-local charge

Let \( S \) be any smooth, orientable, closed spacelike 2-surface. Let us fix a NP complex null tetrad \( \{ l^\mu, n^\mu, \bar{m}^\mu, \bar{m}^\mu \} \) on \( S \) (i.e. \( l^\mu \) and \( n^\mu \) are future pointing real null vectors orthogonal to \( S \), \( m^\mu \) and \( \bar{m}^\mu \) are complex null tangents of \( S \), and they satisfy the normalization conditions \( l^\mu l_\mu = 1 \) and \( m^\mu \bar{m}_\mu = -1 \)), and recall that the area 2-form on \( S \) can also be given as

\[ \varepsilon_{\mu \nu} = -2i m_\mu \bar{m}_\nu, \]

and the area element is \( dS = \frac{1}{2} \varepsilon_{\mu \nu} \) (see e.g. [14]). Note that while \( l^\mu \) and \( n^\mu \) are globally defined on \( S \), \( m^\mu \) and \( \bar{m}^\mu \) are defined only on the local trivialization domains of the tangent bundle of \( S \). Then we search for the electric and magnetic charges in the form

\[ \begin{align*}
\mathcal{E}[\varepsilon] &:= \frac{i}{4\pi} \oint_S e^A_{\alpha B} F^B_{\alpha \beta} \bar{m}^\beta m^\alpha dS = \frac{1}{4\pi} \oint_S \text{Tr}(\varepsilon(\chi_1 + \bar{\chi}_1)) dS, \\
\mathcal{M}[\mu] &:= \frac{i}{4\pi} \oint_S \mu^A_{\alpha B} F^B_{\alpha \beta} \bar{m}^\beta m^\alpha dS = \frac{1}{4\pi} \oint_S \text{Tr}(\mu(\chi_1 - \bar{\chi}_1)) dS,
\end{align*} \] (3.1)

where \( e^A_{\alpha B} \) and \( \mu^A_{\alpha B} \) are two appropriately chosen globally defined cross sections of \( E(S) \otimes E^*(S) \), where \( E^*(S) \) is the dual bundle, \( \ast F_{\alpha \beta} := \frac{1}{2} \varepsilon_{\alpha \beta \gamma \delta} F_{\gamma \delta} \) is the dual of the field strength, \( \varepsilon_{\alpha \beta \gamma \delta} \) is the spacetime volume 4-form. In the expressions on the right we adopted the matrix notation, in which \( \chi_1 := \frac{1}{2} F_{ab}(l^a n^b + \bar{m}^a \bar{m}^b) \) is just the \( \bar{m}^a \bar{m}^b \) component of the anti-self-dual part of the curvature, and \( \bar{\chi}_1 \) is the \( m^a m^b \) component of the self-dual part of the curvature (see e.g. [3, 14, 15]). (N.B.: \( \ast F_{ab} \bar{m}^a \bar{m}^b = -i F_{ab} n^a n^b \). Moreover, though \( \chi_1 \) is defined as the \( \bar{m}^a \bar{m}^b \) component of \( F_{ab} \) and the vectors \( m^a \) and \( \bar{m}^a \) are only locally defined, \( \chi_1 \) is globally well defined. In fact, it can be rewritten into the form \( \chi_1 = -\frac{1}{2} F_{ab}(l^a e^{ab} + i e^{ab}) \),
where $\varepsilon_{ab} := n_{a}l_{b} - l_{a}n_{b}$ is the area 2-form on the timelike 2-planes orthogonal to $S$. In terms of the electric and magnetic field strengths defined with respect to some timelike unit vector field $r^{\nu}$ orthogonal to $S$, $E_{a} := F_{ab}r^{b}$ and $B_{a} := *F_{ab}r^{b}$, respectively, $\chi_{1} = \frac{1}{2}(E_{a} + iB_{a})v^{a}$ holds, where $v^{a}$ is the outward pointing unit spacelike normal of $S$ such that $v^{2}v_{a} = 0$. For real bundle and connection $\chi_{1}$ and $\tilde{\chi}_{1}$ are complex conjugate of each other, but in the complex case these are independent. Note that for $\varepsilon^{A}B = \mu^{A}B = \delta^{A}_{B}$ the first integral can be rewritten into the flux integral of the trace (in the Lie algebra indices) of the current $J_{a}$ on any spacelike hypersurface $\Sigma$ whose boundary is $S$, while the second is precisely the integral of the first Chern class $\varepsilon_{1}$ of $E(S)$, the pull back of $E(M)$ to $S$. Thus $\mathcal{E}[\varepsilon]$ and $\mathcal{M}[\mu]$ are interpreted as the electric and magnetic charges, respectively, surrounded by $S$ and defined with respect to the fields $\varepsilon^{A}B$ and $\mu^{A}B$.

If we define $q^{A}B := \varepsilon^{A}B + i\mu^{A}B$, then

$$Q[q] := \mathcal{E}[q] - i\mathcal{M}[q] = \frac{1}{2\pi} \oint_{S} Tr(q\chi_{1}) \, dS,$$

in which only the anti-self-dual curvature component $\chi_{1}$ appears. $\tilde{\chi}_{1}$ appears in $\tilde{Q}[q] := \mathcal{E}[\tilde{q}] + i\mathcal{M}[\tilde{q}]$ with $q^{\dagger}B := \varepsilon^{A}B + i\mu^{A}B$. If the bundle $E(M)$ and the connection $D_{a}$ are real and $\varepsilon^{A}B$ and $\mu^{A}B$ are chosen to be real, then $\tilde{Q}[q]$ is simply the complex conjugate of $Q[q]$. For complex bundles and connections $Q[q]$ and $\tilde{Q}[q]$ are independent. As we will see, it is more convenient to work with these anti-self-dual and self-dual charges than the original $\mathcal{E}[\varepsilon]$ and $\mathcal{M}[\mu]$.

Since $\varepsilon^{A}B$ and $\mu^{A}B$ are still arbitrary, the number of the ‘charges’ $\mathcal{E}[\varepsilon]$ and $\mathcal{M}[\mu]$ is infinite. However, we expect that the number of the independent charges be the number of the gauge potentials, i.e. $\dim \mathfrak{g}$. Thus $\varepsilon^{A}B$ and $\mu^{A}B$ should be restricted so that the number of independent charges be the correct ones, and the resulting integrals define reasonable electric and magnetic charge multiplets.

### 3.2. Tod’s suggestion

Tod’s suggestion [8] for the quasi-local charges has the form (3.1)–(3.2) with $\varepsilon^{A}B = \mu^{A}B = \phi^{A}\psi^{B}$ for appropriately chosen cross sections $\phi^{A}$ of $E(S)$ and $\psi^{B}$ of $E^{*}(S)$: suppose that $E(M)$ is a complex vector bundle endowed with a non-degenerate Hermitian fiber metric $H_{AB}$, and assume that the connection $D_{a}$ on $E(M)$ is compatible with this fiber metric, $D_{a}H_{AB} = 0$. Thus the gauge group is assumed to be some (pseudo) unitary group (or a subgroup of such). Then by means of $H_{AB}$ the complex conjugate bundle $E(M)$ can be identified with $E^{*}(M)$ via the fiber map $\bar{\phi}^{A} \mapsto \bar{\psi}^{B} := H_{BA}\bar{\phi}^{A}$. Finally, the section $\phi^{A}$ was considered to be anti-holomorphic and $\bar{\phi}^{A}$ to be holomorphic, i.e. on the local trivialization domains to be satisfying

$$0 = m\tilde{\nu}D_{a}\phi^{A} = E^{A}_{\bar{\alpha}}(\delta\bar{\phi}^{A} + m\tilde{\nu}A^{A}_{B}\phi^{B}),$$

$$0 = \bar{m}\nu D_{a}\bar{\phi}^{A} = E^{A}_{\alpha}(\bar{\phi}\psi^{A} + \bar{m}\nu A^{A}_{B}\bar{\psi}^{B}).$$

Here $\delta$ and $\bar{\delta}$ are the standard edth operators on $S$ and $\phi^{A}$ and $\bar{\phi}^{A}$ are of zero-spin-weight functions on the domain of the frame field (see e.g. [9, 14]). If $\mathcal{V}$ is the space of solutions $\phi^{A}$ of (3.4) and $\mathcal{U}$ is the space of solutions $\bar{\phi}^{A}$ of (3.5), then Tod defines the quasi-local charges to be the eigenvalues of the bilinear mappings $\mathcal{E} : \mathcal{V} \times \mathcal{U} \mapsto \mathcal{C} : (\phi^{A}, \bar{\phi}^{B}) \mapsto \mathcal{E}[\phi^{A}\bar{\phi}^{B}]$ and

3 Since we adopt the convention for the wedge product in which the wedge product of the two 1-forms $\alpha_{a}$ and $\beta_{b}$ is $(\alpha \wedge \beta)_{ab} = \frac{i}{2}(\alpha_{a}\beta_{b} - \beta_{a}\alpha_{b})$ (rather than only $\alpha_{a}\beta_{b} - \beta_{a}\alpha_{b}$), the proper normalization factor of the $k$th Chern class is $\frac{1}{k!}((iv)^{k})$ (instead of $\frac{1}{k!(iv)^{k}}$).
In the generic case on $S \approx S^2$ this construction gives $n$ (not necessarily real) charges, independently of the dimension of the gauge group. The magnetic charges are identically vanishing for globally trivializable bundle $E(S)$. (For the details see [8].)

3.3. The modified construction

Our aim is to modify Tod’s construction in such a way that (1) the gauge group can be arbitrary, (2) the number of the electric and the magnetic charges (at least in the generic case) be just $\dim \mathfrak{g}$, and (3) the charges be real for real bundles and connections; but at the same time keeping the advantageous properties of the original construction (especially the vanishing of the magnetic charges for globally trivializable bundles).

In principle, there are several ways to ensure the correct number of charges. The first is to consider those cross sections whose components $(\varepsilon^A B)$ and $\mu^{AB}$ in a local frame field $\{E^A\}$ (as real or complex $n \times n$ matrix valued functions on the domain of the frame field) take their values in the matrix Lie algebra $\mathfrak{g}$. In this case, we could always write them as $\varepsilon^a B^A$ and $\mu^{aA}$ for some, still arbitrary (real or complex) valued functions $\varepsilon^a$ and $\mu^a$. Then it would be the (all together $2 \dim \mathfrak{g}$) functions $\varepsilon^a$ and $\mu^a$ that would have to be restricted appropriately to depend only on $2 \dim \mathfrak{g}$ parameters in the generic case. Though conceptually this appears to be the most natural approach, it turns out that this framework is less flexible as it is more difficult to prove statements (e.g. the vanishing of the magnetic charges for globally trivializable bundles) than e.g. in the third one below.

Another strategy is to build the charge integrals from the curvature and the fields $\varepsilon^A B$ and $\mu^{AB}$ in the adjoint representation, i.e. when the typical fiber $\mathfrak{g}^A$ of the bundle $E(M)$ is $\mathfrak{g}$ as an abstract Lie algebra, and it is the structure constants that provide the matrix representation of a basis of $\mathfrak{g}$. However, the kernel of the action of the Lie algebra on itself via the adjoint map is not trivial, it is the center of the Lie algebra. Thus this construction may work e.g. for semisimple Lie algebras, but it would not yield any charge e.g. for Abelian Lie algebras.

The third possibility is to construct the charge integrals in a vector bundle based on a representation in which the dimension $n$ of the fiber space is just $\dim \mathfrak{g}$ (though the representation is not a priori the adjoint one). Since typically the dimension of the classical Lie algebras grows with the square of the dimension of their defining representation, the representation space with the dimension of the Lie algebra appears to provide enough room for a faithful representation. Restricting the original construction of Tod in this way we get the correct number of charges. For the sake of simplicity we follow this strategy, but, instead of $\varepsilon^A B$ and $\mu^{AB}$, we formulate our conditions in terms of $q^A B$ and $\tilde{q}^A B$.

We search for $q^A B$ and $\tilde{q}^A B$ in the form $\phi^A \psi_B$ and $\tilde{\phi}^A \tilde{\psi}_B$, respectively, where $\phi^A$ and $\psi_B$ are anti-holomorphic while $\tilde{\phi}^A$ and $\tilde{\psi}_B$ are holomorphic. If the bundle and the connection are real, then $\phi^A$ and $\psi_B$ are anti-holomorphic cross sections of the complexified vector bundles $E(S) \otimes \mathbb{C}$ and $E^*(S) \otimes \mathbb{C}$, respectively, and the tilde denotes complex conjugation. In terms of their locally defined components these conditions are

$$\delta \phi^A + m^\beta A^A_B \phi^B = 0, \quad \delta \psi^A_B = m^\alpha A^A_B \psi^B = 0,$$

(3.6)

$$\delta \tilde{\phi}^A + \tilde{m}^\alpha A^\alpha_B \tilde{\phi}^B = 0, \quad \delta \tilde{\psi}^A_B = \tilde{m}^\beta A^\beta_B \tilde{\psi}^B = 0.$$

(3.7)

Then clearly $q^A B = \phi^A \psi_B$ is anti-holomorphic and $\tilde{q}^A B = \tilde{\phi}^A \tilde{\psi}_B$ is holomorphic.

Next let us calculate the integrand of $\mathcal{M}[\mu]$. Since $2i \mu^A B = \tilde{q}^A B - q^A B$, by forming total derivatives and using (3.6) and (3.7), the integrand of $\mathcal{M}[\mu]$ is

$$2i \mu^A B F^B_{AC} m^\gamma m^\delta (\delta (\delta (\tilde{\phi} - q) (\delta \tilde{q})) - \delta (\delta (\tilde{q} - q) (\delta \tilde{q})) + \delta (\delta \tilde{q}) (\delta \tilde{q}) + (\delta q) (\delta q)).$$
where \( q_{AB} = \delta_{AB} \) is not an element of \( \mathfrak{g} \), it can be considered as the generator of the central extension of the matrix Lie algebra \( \mathfrak{g} \). Since it is both holomorphic and anti-holomorphic, it is worth considering this extension. \( \mathcal{E}[\delta] \) and \( \mathcal{M}[\delta] \) can be thought of as the ‘mean’ electric and magnetic charge, respectively, and, as we already noted, the latter is just the first Chern class of \( E(S) \).

### 3.4. Three limits

Though the quasi-local charge integrals \( Q[q] \) are well defined even in curved spacetime, for the sake of simplicity in the rest of the paper we assume that the spacetime is the Minkowski spacetime. In particular, we calculate their small and large sphere limits in the flat spacetime.

#### 3.4.1. Small spheres

Let \( p \in M \) be an arbitrary point, \( t^a \) a future pointing timelike unit vector at \( p \), and let us consider the future pointing null geodesics starting from \( p \) with affine parameter \( r \) and tangents \( t^a \) satisfying \( t^a t_a = 1 \) at \( p \). Then for sufficiently small \( r \) the set \( S \) of the points on these null geodesics whose affine distance from \( p \) is \( r \) is a smooth spacelike 2-surface, which is homeomorphic to \( S^2 \). Such a surface is called a small sphere about \( p \). Our aim is to calculate the charge integral \( Q[q] \) for small spheres of radius \( r \), and to determine the coefficients \( Q^{(k)} \) in its expansion \( Q[q] = Q^{(0)} + \cdots + r^k Q^{(k)} + O(r^5) \). (Since \( S \) is contractible to \( p \), any bundle over \( S \) is globally trivializable, and hence \( \mathcal{M}[\mu] \) is identically vanishing.) The strategy is to expand the curvature component \( \chi_1 \) and the solutions \( \phi \) and \( \psi \) of (3.6) (or of (3.7)) by powers of \( r \). Since \( \text{d}S = r^2 \text{d}S_1 \), clearly \( Q^{(0)} = 0 \) and \( Q^{(1)} = 0 \), and to calculate \( Q[q] \) with \( r^3 \) accuracy we need to know \( \chi_1 \), \( \phi \) and \( \psi \) with \( r^2 \) accuracy.
Similarly, expanding the component $\gamma_{0\nu} := A_\nu m^\nu$ of the connection 1-form as a series of $r$, substituting this to (3.6) and taking into account that $\delta = \frac{1}{r} \partial \delta$, where $\partial \delta$ is the eddith operator on the unit sphere (and is given explicitly in the appendix), we see that we need to know the expansion coefficients $\gamma_{0\nu}^{(l)}$ for $l \leq 2$. However, as the analysis of appendix A.1 shows, to determine the solution of the Yang–Mills equations on the light cone (more precisely, on the null boundary of the chronological future) of $p$ for $\chi_1$ with $r^2$ accuracy, formally $\gamma_{0\nu}^{(1)}$, $\gamma_{1\mu}^{(1)}$ and $\gamma_{1\nu}^{(0)}$ must be expanded with accuracy $r^3$. The solution of the hypersurface equations (A.1)–(A.3) is given in appendix A.1, by means of which the equations (3.6) themselves are

$$\partial \delta \phi^{(0)} = 0, \quad \partial \delta \phi^{(1)} = 0, \quad \partial \delta \phi^{(2)} = -\frac{1}{2} \chi_0^{(0)} \phi^{(0)};$$

$$\partial \delta \psi^{(0)} = 0, \quad \partial \delta \psi^{(1)} = 0, \quad \partial \delta \psi^{(2)} = \frac{1}{2} \chi_0^{(0)} \phi^{(0)}.$$

(3.11)

(3.12)

Since $\phi$ and $\psi$ are multiplets of type (0, 0) scalars, $\phi^{(0)}, \phi^{(1)}, \psi^{(0)}$ and $\psi^{(1)}$ are constant on $\mathcal{S}$.

To solve the third of (3.11) and (3.12) first let us recall that in terms of the spinor form of the field strength, $F_{ab} = \Phi_{ABEAB} + \Phi_{ABEAB}$, and the GHP spinor dyad $\{\sigma^A, \iota^A\}$ the curvature component $\chi_0^{(0)}$ is given by $\Phi_{AB} \sigma^A \iota^B$. (In the present subsection we use two-component spinors, and hence, in particular, the capital Latin indices $A, B, \ldots$ in this subsection are abstract spacetime spinor indices. These should not be confused with the indices referring to the internal fiber space or the concrete name indices $A, B, \ldots$ etc.) However, in terms of the Cartesian spinor dyad $\{O^A, I^A\}$ at $p$ the GHP dyad is

$$\sigma^A = \frac{i \sqrt{2}}{\sqrt{1 + \xi \zeta}} (\xi O^A + I^A), \quad \iota^A = \frac{i}{\sqrt{2} \sqrt{1 + \xi \zeta}} (O^A - \xi I^A),$$

and hence it is straightforward to integrate (3.11) and (3.12) in the coordinates $(\xi, \zeta)$:

$$\phi^{(2)} = -\frac{1}{1 + \xi \zeta} (\zeta \Phi_0 + 2 \Phi_1 - \xi \Phi_2) \phi^{(0)} + \phi^{(2)}_0,$$

$$\psi^{(2)} = \frac{1}{1 + \xi \zeta} \psi^{(0)} (\zeta \Phi_0 + 2 \Phi_1 - \xi \Phi_2) + \psi^{(2)}_0.$$

(3.13)

(3.14)

Here $\Phi_0 := \Phi_{AB} O^A O^B$, $\Phi_1 := \Phi_{AB} O^A I^B$, $\Phi_2 := \Phi_{AB} I^A I^B$; and $\phi^{(2)}_0$ and $\psi^{(2)}_0$ are arbitrary constant sections, being the general solution of the homogeneous equation. Then, using the pattern of (3.13)–(3.14), it is easy to write down the holomorphic solutions, too.

Though for small spheres the solution spaces $\mathcal{H}$ and $\mathcal{H}^*$ are dim $g$ dimensional, the space of the $O(r^k)$ accurate solutions is $(k + 1) \dim g$ dimensional. The presence of the ‘extra’, spurious solutions is a consequence of the lack of a natural isomorphism between the solution spaces on different two-surfaces, and hence, in particular, with different radii. These yield some ambiguity in higher accurate approximations. (For a more detailed discussion of this issue in general relativity, see e.g. [16]. Also, in the calculation of certain integrals, we will use equation (2.5) of [16].)

Since $\phi^{(0)}$ and $\psi^{(0)}$ are constant, by (A.11) the $O(r^2)$ order term $Q^{(2)}$ is vanishing. For the $O(r^3)$ order term we obtain

$$Q^{(3)} = \frac{1}{2 \pi} \oint (\psi^{(0)} \chi_1^{(0)} \phi^{(1)} + \psi^{(0)} \chi_1^{(1)} \phi^{(0)} + \psi^{(1)} \chi_1^{(0)} \phi^{(0)}) dS_1$$

$$= \frac{4 \pi}{3} I^A \iota^A (\psi^{(0)} \iota_A \phi^{(0)}).$$

(3.15)

where the integration is taken on the unit sphere. Thus $Q^{(3)}$ is the volume of the 3-ball of radius $r$ times the timelike component of the gauge-covariant current at $p$. This result coincides
with that of Tod, too. In fact, let us observe that in $Q^{(3)}$ only the zeroth order terms of the cross sections $\phi$ and $\psi$ matter (the others integrate to zero), which are constant both for the holomorphic and anti-holomorphic sections.

In the fourth order we obtain

$$Q^{(4)} = \frac{1}{3} \int_{\Omega} t^{\alpha} t^{\beta} \epsilon^{\mu} \Phi_{\alpha\beta} \Phi^{\mu} (\psi^{(0)} e_\mu \phi^{(0)}) + \frac{2\pi}{3} (t^{\alpha} t^{\beta} a^{\mu} \Phi_{\alpha\beta} \epsilon^\mu (\psi^{(0)} e_\mu \phi^{(0)}))$$

$$+ \frac{4\pi}{3} t^\alpha j^\beta (\psi^{(1)} e_\mu \phi^{(0)} + \psi^{(0)} e_\mu \phi^{(1)}).$$

(3.16)

Though formally $Q^{(4)}$ is built from the $O(r^2)$ accurate cross sections, their $O(r^2)$ order correction terms are integrate to zero. Hence we obtain the same result in the holomorphic and the anti-holomorphic constructions. On the other hand, in general $Q^{(4)}$ does depend on the $O(r)$ order ‘gauge solutions’ $\phi^{(1)}$ and $\psi^{(1)}$: $Q^{(4)}$ is well defined only if $Q^{(3)} = 0$, i.e. when $j^\beta t^\alpha = 0$ at $p$.

The calculation of $Q^{(5)}$ is much longer and technically more difficult, so we summarize only the main message of that. First, ambiguities similar to that in $Q^{(4)}$, parameterized by the ‘extra’ solutions $\phi^{(3)}$, $\psi^{(1)}$, . . . etc, appear, unless the third and fourth order terms, $Q^{(3)}$ and $Q^{(4)}$, are vanishing. The difference of the holomorphic and anti-holomorphic constructions can appear in this order. Thus $Q^{(3)}$ and $Q^{(4)}$ have some ‘universal’ nature, not being sensitive to the details of the defining equations for the cross sections $\phi$ and $\psi$, provided $\phi$ and $\psi$ are constant in the zeroth and first orders.

3.4.2. Large spheres at spatial infinity. Let $(t, r, \zeta, \bar{\zeta})$ be the standard spherical, complex stereographic coordinate system (at least in a neighbourhood of spatial infinity) in Minkowski spacetime, and consider the charge integral $Q[q]$ on the metric 2-spheres $S_r$ of radius $r$ in some $t = \text{const}$ spacelike hyperplane. Let $t^\alpha$ and $\nu^\alpha$ be their future pointing unit timelike and outward pointing unit spacelike normals, respectively, such that $t^\alpha$ is orthogonal to the $t = \text{const}$ hyperplanes and $t^\alpha \nu_\alpha = 0$. Let $m^\alpha$ and $\bar{m}^\alpha$ be the complex null tangents of the 2-spheres. Then we impose the following a priori fall-off conditions for the scalar potential and the radial and tangential parts of the spatial vector potential: $\Phi := A_\nu t^\nu = O(r^{-a})$, $A_a t^a = O(r^{-b})$, $\gamma := A_\nu m^\nu = O(r^{-c})$, $\bar{\gamma} := A_{\bar{\nu}} \bar{m}^\nu = O(r^{-c})$, respectively, for some $a, b, c > 0$. In addition, we assume that the time derivative of the last three satisfy $(A_\nu v^\nu) = O(r^{-C})$, $\dot{\gamma} = O(r^{-\gamma})$ and $\dot{\bar{\gamma}} = O(r^{-\gamma})$ also for some $\beta, \gamma > 0$. By technical reasons we also assume that the differentiation with respect to $r$ reduces these orders by one, but the differentiation with respect to $\zeta$ and $\bar{\zeta}$ does not change these orders. Then, expressing $\chi_1 = \frac{1}{2}(E_a + iB_a) v^a$ by the potentials, we find that $\chi_1 = O(r^{-2})$ if $a = b = c = 1$ and $\beta = 2$. Therefore, these boundary conditions ensure the existence of the $r \to \infty$ limit of the charge integrals $Q[q]$ on 2-spheres $S_r$ of radius $r$ provided $q = O(1)$. (If $q$ were not involved in the 2-surface integral of the curvature, i.e. if they were only the 2-surface integral of the components of the field strength in some globally defined frame $E_A^a$, then one could rewrite that as the 3-volume integral of its divergence on some spacelike hypersurface $\Sigma$ with boundary $S$. Then, using the constraint equation, one could show that even weaker boundary conditions could ensure the finiteness of the original 2-surface integral [17]. Since, however, $q$ is defined only on $S$, here this strategy cannot be followed, and hence we have the above ‘conservative’ fall-off conditions.)

Next let us determine the conditions that ensure the vanishing of the $r \to \infty$ limit of the time derivative of $Q[q]$:}

$$\frac{d}{dt} Q[q] = \frac{1}{2\pi} \oint \text{Tr} \left( q \chi_1 + q \frac{1}{2} (E_a + iB_a) v^a \right) dS.$$

(3.17)
To determine the order of the second term of the integrand we use the evolution equations for the field strengths in the present geometrical situation (i.e. flat spacetime foliated by spacelike 3-planes with vanishing shift vector). These are

$$
E_a = (\mathbb{D}_a B_d) \epsilon^{cd} a^b t^b - [\Phi, E_a], \quad B_a = -(\mathbb{D}_a E_d) \epsilon^{cd} a^b t^b - [\Phi, B_a],
$$

where $\mathbb{D}_a$ denotes the space- and spatial gauge-covariant derivative operator (i.e. for any action on any cross section $\phi^A = E_A^B \phi^B$ of $E(M)$ near the spatial infinity is $\mathbb{D}_a \phi^A = E_A^B (\delta_a \phi^B + A^B_{ab} \phi^b) p_a^b$ and $p_a^b := \delta_a^b - t^b t_a$ is the orthogonal projection to the $t = \text{const hyperplanes}$). By these equations $\dot{\chi}_1 = O(r^{-2-\epsilon})$ if $\gamma = 1 + \epsilon$. Thus if $\epsilon > 0$ and $\dot{q} = O(r^{-\epsilon})$, then the $r \to \infty$ limit of the time derivative of $Q[q]$ is zero. Note that by $\gamma = 1 + \epsilon$ the asymptotic form of $\gamma$ (and of $\dot{\gamma}$, too) is

$$
\gamma = \frac{1}{r} \gamma^{(0)}(\xi, \bar{\xi}) + \frac{1}{r^{1+\epsilon}} \gamma^{(1)}(t, \xi, \bar{\xi}) + o(r^{-1-\epsilon}),
$$

i.e. its leading order term does not depend on $t$.

Finally, let us write $q = q^{(0)} + r^{-\epsilon} q^{(1)} + o(r^{-\epsilon})$ and substitute into the equation of the anti-holomorphicity of the cross section $q$. We find

$$
0 = \delta q + [\gamma, q] = \frac{1}{r} (\delta q^{(0)} + [\gamma^{(0)}, q^{(0)})] + \frac{1}{r^{1+\epsilon}} (\delta q^{(1)} + [\gamma^{(0)}, q^{(1)}] + [\gamma^{(1)}, q^{(0)})] + o(r^{-1-\epsilon}).
$$

Thus its solution is such that the leading term $q^{(0)}$ does not depend on time, and hence $\dot{q} = O(r^{-\epsilon})$. Therefore, under the fall-off conditions for the vector potential above, the spatial infinity limit of $Q[q]$ exists and is conserved in time. It might be interesting to note that here we used only the anti-holomorphicity of $q^A$, but did not assume that it has the dyadic product structure $\phi^A \psi_B$.

Note that in general $\gamma^{(0)}$ cannot be transformed by gauge transformations to zero globally unless the leading term of the $r^\mu$ component of the magnetic field strength is vanishing. Thus $q^{(0)}$ is not constant on the unit sphere, and hence the total charge depends on whether $q$ is anti-holomorphic or holomorphic. If the connection coefficients $\gamma^{(0)}$ and $\tilde{\gamma}^{(0)}$ can be made vanishing globally, then the total magnetic charges are vanishing and the total electric charges introduced in the holomorphic and anti-holomorphic constructions coincide.

3.4.3. Large spheres at null infinity. Let us consider the charge integral $Q[q]$ on any $u = \text{const cut of } \mathcal{S}^*$ for still arbitrary globally defined $q^A$. (For a summary of the relevant results on the asymptotic structure of the Yang–Mills fields near $\mathcal{S}^*$, see appendix A.2.) Because of the peel-off this integral can be written in terms of the expansion coefficient $\chi_1^0$ as $\frac{1}{2\pi} \oint \text{Tr}(q \chi_1^0) \text{d}S_1$, where the integration is taken on the unit sphere. Hence this integral is still a function of the retarded time coordinate. In this subsection we show that for semisimple gauge groups no reasonable definition yields conserved total charge in general, radiative configurations. In particular, neither Tod’s original definition (as pointed out in [8]) nor the modified one yield conserved total charges.

By integration by parts and using (A.15), for the $u$-derivative of $Q[q]$ we obtain

$$
\dot{Q}[q] = \frac{1}{2\pi} \oint (\dot{q} \chi_1^0 - (\delta q + [\gamma, q]) \chi_2^0) \text{d}S_1. \quad (3.19)
$$

Since in any given retarded time instant $u = u_0$ in a generic (radiative) configuration $\chi_1^0$ and $\chi_2^0$ are independent data (because $\chi_1^0 + \tilde{\chi}_1^0$ and $\tilde{\gamma}^0$ can be specified independently, see appendix A.2.1), the integrand can be zero precisely when the coefficient of $\chi_1^0$ and of $\chi_2^0$ are vanishing. In particular, $q^A$ must be anti-holomorphic, but in addition it would have to be
constant in time even though the anti-holomorphic structure might be changing from cut to cut on $\mathcal{I}^+$:

$$\dot{q} = 0, \quad \partial q + [\gamma^0, q] = 0. \quad (3.20)$$

The condition of the integrability of the system of these equations is $[q, \dot{\gamma}^0] = 0$. Thus the charge $\mathcal{Q}[q]$ with anti-holomorphic $q^A B$ is conserved in the absence of outgoing radiation, $\dot{\gamma}^0 = 0$, or when $q$ is anti-holomorphic and belongs to the center of the Lie algebra $\mathfrak{g}$, too. Therefore, for semisimple gauge groups in the generic radiative configurations there is no total conserved charge. Charge conservation requires the non-triviality of the radical (i.e. the maximal solvable subalgebra) of $\mathfrak{g}$ in its Levi–Malcev decomposition.

Remarkably enough, though in the presence of outgoing radiation there is no charge conservation, the structure of the Yang–Mills fields at future null infinity singles out just the anti-holomorphic cross sections $q^A B$ in a natural way. Thus our suggestion for the quasi-local charges is justified by the conservation properties of the total charges at future null infinity. At past null infinity the holomorphic cross sections would appear.

Even for semisimple Lie algebras and non-stationary, but special configurations, in which $\chi^0_0$ and $\tilde{\chi}^0_1 + \chi^1_0$ are not quite independent, the total charge can be conserved. Such are the Liénard–Wiechert type solutions. In [18] Tod defined them geometrically as the algebraically special solutions of the Yang–Mills equations for which the field strength components $\chi^0_0$ and $\chi^0_0$ are vanishing, and, in contrast to our present framework, the origin of the coordinate system $(u, r, \zeta, \bar{\zeta})$ (i.e. the ‘source’s world line’) is allowed to be an arbitrary timelike curve. He showed that there is an anti-holomorphic frame in which $\chi^0_0$ is constant both in time and on the cut of $\mathcal{I}^+$, and hence that the total charge is, in fact, conserved. (For another approach to the Liénard–Wiechert type configurations, based on certain assumptions on the explicit form of the 4-potential in terms of a pointlike source, the so-called colour, see [19, 20], and for a recent review of them see e.g. [21]).

In the special stationary case discussed in appendix A.2.2 $\gamma^0$ and $\tilde{\gamma}^0$ are vanishing and $\chi^0_0$ are not quite independent, the total charge can be conserved. Such are the Liénard–Wiechert type solutions. In [18] Tod defined them geometrically as the algebraically special solutions of the Yang–Mills equations for which the field strength components $\chi^0_0$ and $\chi^0_0$ are vanishing, and, in contrast to our present framework, the origin of the coordinate system $(u, r, \zeta, \bar{\zeta})$ (i.e. the ‘source’s world line’) is allowed to be an arbitrary timelike curve. He showed that there is an anti-holomorphic frame in which $\chi^0_0$ is constant both in time and on the cut of $\mathcal{I}^+$, and hence that the total charge is, in fact, conserved. (For another approach to the Liénard–Wiechert type configurations, based on certain assumptions on the explicit form of the 4-potential in terms of a pointlike source, the so-called colour, see [19, 20], and for a recent review of them see e.g. [21]).

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4. Newman–Penrose type quantities

The NP conserved quantities in general relativity and in Maxwell theory (and, in fact, in any theory based on a linear zero-rest-mass field equation with any spin or on the conformally invariant wave equation) are built from the second nontrivial expansion coefficient of the tetrad component of the curvature with the highest spin-weight, weighted with appropriate spin-weighted spherical harmonics [9, 10]. Thus it is natural to define

$$\Omega[\omega] := \oint \text{Tr}(\omega \chi^0_0) \, dS_1, \quad (4.1)$$

where $\omega$ is a cross section of (in the case of real bundles, the complexified) $E(M) \otimes E^*(M)$ over the $u = u_0$ cut of $\mathcal{I}^+$ with $-1$ spin weight so that the integrand of (4.1) is globally well defined.
4.1. The (non-)conservation of the Yang–Mills NP quantities

Here we show that for semisimple gauge groups and any non-trivial choice for \( \omega \) the quantity \( \Omega[\omega] \) is not conserved in generic, radiative configurations. Using (A.17) a straightforward calculation (by integration by parts) yields

\[
\Omega[\omega] := \int \text{Tr} \left\{ \omega \chi_0^0 - (\omega \delta(\omega^0, \omega) + [\hat{\gamma}^0, \omega]) + [\gamma^0, (\omega \delta^0\omega + [\hat{\gamma}^0, \omega])] - 3[\omega, \chi_0^0] \chi_0^0 \right\} dS_1.
\]

(4.2)

Since \( \chi_0^0 = \chi_0^0(\zeta, \bar{\zeta}) \) and \( \chi_0^1 = \chi_0^1(\zeta, \bar{\zeta}) \) are freely and independently specifiable functions at \( u = u_0 \) (see appendix A.2.1), \( \Omega[\omega] \) is constant in time precisely when their coefficients in (4.2) are vanishing:

\[
\hat{\omega} = 0,
\]

\[
\delta(\omega^0, \omega) + [\hat{\gamma}^0, \omega] = 0.
\]

(4.3)

Using (A.14)–(A.15) of the appendix, the integrability condition of this system of equations is

\[
[\chi_0^0, (\omega \delta^0\omega + [\hat{\gamma}^0, \omega])] + [\chi_0^0, (\omega \delta^0\omega + [\hat{\gamma}^0, \omega])] = 2[\omega, \chi_0^0] \omega.
\]

Thus, in particular, \( \Omega[\omega] \) is conserved if \( \omega \) satisfies (4.3) and there is no outgoing radiation, i.e. \( \chi_0^0 = 0 \); or if \( \omega \) belongs to the center of the Lie algebra \( g \) and solves \( \omega \delta^0\omega = 3[\omega, \chi_0^0] = 0 \). Therefore, the NP type quantities could be conserved in special situations (e.g. for a collection of independent Maxwell fields), but not conserved in general. However, this non-conservation of the NP quantities in non-Abelian Yang–Mills theories appears to be known for a while, as Bazanski pointed out to Newman in an unpublished private communication [11, 12].

Since for the Liénard–Wiechert type solutions of the Yang–Mills equations, considered in [18], the curvature components \( \chi_0 \) and \( \bar{\chi}_0 \) are identically vanishing, the NP quantities for these solutions are vanishing for any gauge group.

4.2. Yang–Mills NP quantities in stationary configurations

It is known that in stationary configurations the NP quantities in the theories based on the linear zero-rest-mass field equations (e.g. in particular in Maxwell theory or the linearized Einstein theory) are vanishing. On the other hand, in stationary spacetime the gravitational NP quantities of the full nonlinear theory reduce to a nontrivial combination of the mass and Einstein theory) are vanishing. On the other hand, in stationary spacetime the gravitational NP quantities of the full nonlinear theory reduce to a nontrivial combination of the mass and dipole and quadrupole moments of the source [9]. Thus their non-vanishing is due to the nonlinearity of the theory. This result raises the question whether the analogous quantities \( \Omega[\omega] \) in non-Abelian Yang–Mills theories could be non-vanishing in the stationary configurations, and if they could, then what would be their meaning?

In the special stationary case discussed in appendix A.2.2 equation (A.18) reduces to

\[
2\chi_0^1 = -\omega \delta^0\omega \chi_0^1 - 5[\chi_0^0, \chi_0^0] - [\omega \delta^0\omega, \chi_0^0] - 2[\chi_0^0, \omega \delta^0\omega] - 3[\chi_0^0, \omega \delta^0\omega],
\]

while the defining equation (4.3) of \( \omega \) to

\[
\omega \delta^0\omega = 3[\omega, \chi_0^0].
\]

(4.4)

Writing \( \omega = \omega^\mu e_{iB}^\mu \) for some spin weight \(-1\) functions \( \omega^\mu \) and expanding the latter as \( \omega^\mu = \sum_{j-1}^\infty \sum_{k=-j}^j \omega^\mu_{jm} \equiv Y_{jm} \), equation (4.4) yields

\[
C^\mu \omega^\mu_{jm} := (C^\mu \omega^\mu_{jm} = \frac{1}{2}(j + 1)(j + 2)\omega^\mu_{jm}.
\]

(4.5)

Thus \( \omega^\mu_{jm} \) can always be non-zero, and if \( C^\mu \), the ‘charge matrix’ in the adjoint representation, has only one eigenvector with zero eigenvalue then \( \omega^\mu_{jm} = C^\mu \omega_{jm} \) for arbitrary constants \( \omega_{jm} \).
If, in addition to $C^\mu$, there are other eigenvectors with zero eigenvalue, say $\hat{C}^\mu$, etc, then $\omega^\mu_1$ will be a linear combination of all them with arbitrary coefficients $\omega^\mu_1, \omega^\mu_2, \ldots$ etc. However, $\omega^\mu_j$ for $j \geq 2$ can be nonzero only in the exceptional case when $\frac{1}{2}(j-1)(j+2)$ is an eigenvalue of $C^\mu$. (For a more detailed discussion of $C^\mu$ see appendix A.2.2.) Substituting these into the definition of $\Omega[\omega]$, by integration by parts we obtain

$$\Omega[\omega] = -\frac{1}{2} \int \omega^\mu G_{\mu\nu} \left( \sum_{m=-1}^1 \left( C^\nu a k^\mu_1 + H^\nu_1 \right) \right) dS_1,$$

where the functions $h^\mu, \hat{h}^\mu$ and $k^\mu$ have been introduced via $\chi^0 = h^\mu e^A_\mu, \hat{\chi}^0 = \hat{h}^\mu e^A_\mu$ and $\chi^1 = k^\mu e^A_\mu$, respectively. Thus, in particular, if the Lie algebra $g$ is Abelian, then by (4.6) $\Omega[\omega] = 0$, in accordance with the known fact that the NP quantities are vanishing for an Abelian multiplet of stationary Maxwell fields. Thus in the rest of the paper we assume that $g$ is not Abelian.

First suppose that $C^\mu$ is generic (see appendix A.2.3). Then the only non-zero expansion coefficients of $\omega^\mu$ in terms of spin weighted spherical harmonics is $\omega^\mu_1$, while the expansion coefficients of $h^\mu$ and $k^\mu$ (and in the complex case of $\hat{h}^\mu$, too) are determined in appendix A.2.3. Substituting those solutions into equation (4.6), using the notation of the appendix and elementary properties of the $\delta$th operators and the spin weighted spherical harmonics (e.g. the orthogonality of these harmonics with different indices), we obtain

$$\Omega[\omega] = -\frac{1}{2} \int \omega^\mu G_{\mu\nu} \left( \sum_{m=-1}^1 \left( C^\nu a k^\mu_1 + H^\nu_1 \right) \right) dS_1.$$

In the special case when the only eigenvector of $C^\mu$, with zero eigenvalue is $C^\mu$ and $-\frac{4}{3}$ is not an eigenvalue of $C^\mu$ this reduces to zero. Thus the non-Abelian nature of the Lie algebra $g$ in itself is not enough to have non-trivial NP quantities. If, however, $-\frac{4}{3}$ is an eigenvalue and $k^\mu_1$ are the corresponding eigenvectors, then $\Omega[\omega]$ can be non-zero. In the other extreme case when $C^\mu = 0$ (e.g. when the total charge is vanishing) it gives $\frac{1}{2}G_{\mu\nu} \sum_{m=-1}^1 (-)^m H^\nu_{1m} \omega^\mu_{-m}$, which is a homogeneous quadratic expression of the coefficients $h^\mu_{m}$ and $\hat{h}^\mu_{m}$ characterizing the dipole structure of the Yang–Mills field.

We saw in appendix A.2.3 that for the Lie algebras $so(3)$, $sl(2, \mathbb{R})$, and $sl(2, \mathbb{C})$ the charge matrix has only one eigenvector with zero eigenvalue while for $so(1, 3)$ there are two, moreover there could be exceptional configurations for the last three Lie algebras, but not for $so(3)$. Therefore, for the Lie algebra $so(3)$ in general and for $sl(2, \mathbb{R})$, $sl(2, \mathbb{C})$ and $so(1, 3)$ in the generic case the NP quantities can be non-vanishing e.g. when the total charge is zero.

Finally we consider the exceptional configurations of appendix A.2.3. In this case the solution of (4.5) is $\omega^\mu = \sum_{m=-1}^1 \omega^\mu_{1m} Y_{1m} + \sum_{m=-j}^j \omega^\mu_{jm} Y_{jm}$. Substituting this and the solution of (A.33) into (4.6), and using the orthonormality of the spin weighted spherical harmonics on the unit sphere, we obtain that

$$\Omega[\omega] = \frac{1}{4} G_{\mu\nu} \left( \sum_{m=-1}^1 (-)^m \omega^\mu_{-m} \left( 2C^\nu a k^\mu_1 + K^\nu_{1m} \right) + \sum_{m=-j}^j (-)^m \omega^\mu_{jm} \left( 2C^\nu a k^\mu_{jm} + K^\nu_{jm} \right) \right),$$

which can also be non-zero in general, even if the charge matrix has a single eigenvector with zero eigenvalue. In this case, however, $\Omega[\omega]$ is built from higher, actually the $2^j$th order multipole moments $h^\mu_{jm}$ and $\hat{h}^\mu_{jm}$ of the curvature.

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Appendix: The solution of the Yang–Mills equations

In this appendix we summarize the structure of the Yang–Mills fields near a point (subsection A.1) and near the future null infinity in Minkowski spacetime (subsection A.2).

For the generalization to the coupled Einstein–Yang–Mills equations in the NP formalism, see [15].

Following [3] we use the coordinates \((u, r, \zeta, \bar{\zeta})\), where \(u := t - r\) is the standard retarded null coordinate and \((\zeta, \bar{\zeta})\) are the complex stereographic coordinates on the unit sphere. The complex null NP tetrad \([h^\mu, n^a, m^a, \bar{m}^a]\) is adapted to the \(u = \text{const}, r = \text{const}\) 2-spheres, the edth operators on the unit metric sphere will be denoted by \(\partial_\delta\) and \(\partial_\delta^\prime\) and their action on a scalar \(\eta\) of GHP type \((p, q)\) is \(\partial_\delta \eta := \frac{1}{\sqrt{\zeta}}(1 + \zeta \bar{\zeta}) \frac{\partial}{\partial \zeta} + \frac{1}{\sqrt{\zeta}}(p - q) \eta\) and \(\partial_\delta^\prime \eta := \frac{1}{\sqrt{\zeta}}(1 + \zeta \bar{\zeta}) \frac{\partial}{\partial \zeta} - \frac{1}{\sqrt{\zeta}}(p - q) \eta\), respectively. The components of the connection 1-form \(\chi\) in the NP basis are denoted by \(\gamma_{00}, \gamma_{11}, \gamma_{01}\) and \(\gamma_{10}\), respectively. The components of the curvature are \(\chi_0 := F_{a0}^b m^a, 2 \chi_1 := F_{a0}^b (\bar{m}^a + m^a) b, \chi_2 := F_{a0}^b \bar{m}^a m^a b, \chi_3 := F_{a0}^b \bar{m}^a \bar{m}^a b\). Note that for real connections \(\gamma_{00}\) and \(\gamma_{11}\) are real and \(\gamma_{10}\) is the complex conjugate of \(\gamma_{01}\). Similarly, also for real connections, \(\tilde{\chi}_1, \tilde{\chi}_2\) and \(\tilde{\chi}_3\) are complex conjugate of \(\chi_0, \chi_1\) and \(\chi_2\), respectively, but for general complex connections they are independent.

In the gauge \(\gamma_{00} = 0\) of Newman [3] the expression of the components of the field strength by the components of the connection 1-forms and the field equations (2.1) together with the Bianchi identities take the form

\[
\chi_0 = \frac{\partial \gamma_{01}}{\partial r} + \frac{1}{r} \gamma_{01}, \quad (A.1)
\]

\[
2 \chi_1 = \frac{\partial \gamma_{11}}{\partial r} - \frac{1}{r} \partial_\delta \gamma_{01} + \frac{1}{r} \partial_\delta^\prime \gamma_{01} + [\gamma_{10}, \gamma_{01}], \quad (A.2)
\]

\[
\partial \chi_1 + \frac{1}{2} \chi_1 = - \frac{1}{r} \partial_\delta^\prime \chi_0 = - [\chi_0, \gamma_{10}] + 2 \pi f_{00}, \quad (A.3)
\]

\[
\partial \chi_2 + \frac{1}{2} \chi_2 = - \frac{1}{r} \partial_\delta \chi_1 = - [\chi_1, \gamma_{01}] + 2 \pi f_{10}, \quad (A.4)
\]

\[
\partial \gamma_{10} \quad (A.5)
\]

\[
\frac{\partial \gamma_{00}}{\partial u} = - \frac{1}{2} \chi_1 + \frac{1}{2r} \frac{\partial \gamma_{00}}{\partial r} + \frac{1}{2r} \gamma_{00} + \frac{1}{r} \partial_\delta^\prime \gamma_{11} + [\gamma_{00}, \gamma_{11}], \quad (A.5)
\]

\[
\frac{\partial \chi_0}{\partial u} = \frac{1}{2} \frac{\partial \chi_0}{\partial r} + \frac{1}{2r} \chi_0 + \frac{1}{r} \partial_\delta \chi_1 + [\chi_0, \gamma_{11}] - [\chi_1, \gamma_{01}] - 2 \pi f_{01}, \quad (A.6)
\]

\[
\frac{\partial \chi_1}{\partial u} = \frac{1}{2} \frac{\partial \chi_1}{\partial r} + \frac{1}{2} \chi_1 + \frac{1}{r} \partial_\delta \chi_2 + [\chi_1, \gamma_{11}] - [\chi_2, \gamma_{01}] - 2 \pi f_{11}. \quad (A.7)
\]

The first four are only the constraint (or ‘hypersurface’) equations, while the remaining three are the evolution equations. In the complex case there are analogous equations for the independent curvature components \(\tilde{\chi}_0, \tilde{\chi}_1\) and \(\tilde{\chi}_2\), too.
A.1. The solution of the hypersurface equations near a point

In the small sphere limit calculations of subsection 3.4.1 we need to know only \( \chi_1 \) and \( \gamma_{00} \) on the light cone of the point \( p \in M \). Thus we need to solve only (A.1)–(A.3) with the accuracy required by the small sphere calculations. Hence we expand the components of the curvature and the connection 1-form as

\[
\gamma = \gamma^{(0)} + r \gamma^{(1)} + r^2 \gamma^{(2)} + O(r^3) \quad \text{and} \quad \chi = \chi^{(0)} + \cdots + r^3 \chi^{(3)} + O(r^4),
\]

respectively, as well as \( j = j^{(0)} + rf^{(1)} + O(r^2) \).

Then the solution of (A.1) and its tilded version is

\[
\begin{align*}
\gamma_{00}^{(0)} &= 0, & \gamma_{00}^{(1)} &= \frac{1}{2} \chi_0^{(0)}, & \gamma_{00}^{(2)} &= \frac{1}{2} \chi_0^{(1)}, & \gamma_{00}^{(3)} &= \frac{1}{4} \chi_0^{(2)}, \\
\gamma_{00}^{(0)} &= 0, & \gamma_{00}^{(1)} &= \frac{1}{2} \tilde{\chi}_0^{(0)}, & \gamma_{00}^{(2)} &= \frac{1}{2} \tilde{\chi}_0^{(1)}, & \gamma_{00}^{(3)} &= \frac{1}{4} \tilde{\chi}_0^{(2)}.
\end{align*}
\]  

(A.8)

(A.9)

The sum of (A.2) and its tilded version yield

\[
\begin{align*}
\gamma_{11}^{(0)} &= \chi_1^{(0)} + \tilde{\chi}_1^{(0)}, & \gamma_{11}^{(2)} &= \frac{1}{2} (\chi_1^{(1)} + \tilde{\chi}_1^{(1)}), & \gamma_{11}^{(3)} &= \frac{1}{4} (\chi_1^{(2)} + \tilde{\chi}_1^{(2)}).
\end{align*}
\]  

(A.10)

Equations (A.8)–(A.10) give all the components of the connection 1-form in terms of the curvature components \( \chi_1 \), which remains undetermined. This, however, can be made zero by an appropriate gauge transformation. The various components of the curvature are not independent either, as they are restricted by (A.3) and (A.4).

A.2. The asymptotics of Yang–Mills fields near null infinity

In [3] Newman solved the source free Yang–Mills equations near the future null infinity, and determined the freely specifiable functions of such solutions. In subsection A.2.1 we present the results of the analogous analysis with the higher accuracy of the asymptotic expansion required by the analysis of section 4. In subsection A.2.2 the asymptotic field equations are solved in certain special stationary configurations. In subsection A.2.3 we discuss some examples with special gauge groups.

A.2.1. The asymptotic solution of the Yang–Mills equations. As Newman showed [3], the hypersurface equations can be integrated in the gauge \( \gamma_{00} = 0 \), and if we assume that the curvature components \( \chi_0 \) and \( \tilde{\chi}_0 \) have the asymptotic form

\[
\begin{align*}
\chi_0 &= \frac{1}{r^3} \chi_0^{(0)} + \frac{1}{r^2} \chi_0^{(1)} + \frac{1}{r} \chi_0^{(2)} + O(r^{-3}), & \tilde{\chi}_0 &= \frac{1}{r^3} \tilde{\chi}_0^{(0)} + \frac{1}{r^2} \tilde{\chi}_0^{(1)} + \frac{1}{r} \tilde{\chi}_0^{(2)} + O(r^{-3}),
\end{align*}
\]

then the evolution equations imply, in particular, the peeling

\[
\begin{align*}
\chi_1 &= \frac{1}{r^2} \chi_1^{(0)} + \frac{1}{r^3} \chi_1^{(1)} + \frac{1}{r} \chi_1^{(2)} + O(r^{-3}), & \chi_2 &= \frac{1}{r^2} \chi_2^{(0)} + \frac{1}{r^3} \chi_2^{(1)} + \frac{1}{r} \chi_2^{(2)} + O(r^{-3}),
\end{align*}
\]
and similar peeling for \( \tilde{\chi}_1 \) and \( \tilde{\chi}_2 \), too. The functions in these expansions satisfy

\[
\chi^0_2 = -\tilde{\varphi}^0. 
\] (A.14)

\[
\dot{\chi}_1^0 = -\left(\partial_\alpha \tilde{\varphi}^0 + [\varphi^0, \tilde{\varphi}^0]\right). 
\] (A.15)

\[
\dot{\chi}^0_0 = \partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]. 
\] (A.16)

\[
\dot{\chi}_1^0 = -\partial_\alpha \left(\partial^\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right) - \left[\tilde{\varphi}^0, \partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right] + 3\left[\chi^0_1, \chi^0_0\right]. 
\] (A.17)

\[
2\dot{\chi}^0_0 = -\partial_\alpha \left(\partial^\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right) + \left[\chi^0_0, \chi^0_0\right] - \left[\tilde{\varphi}^0, 3\partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right] + \left[\tilde{\varphi}^0, \chi^0_0\right]. 
\] (A.18)

and there are the analogous ones for \( \tilde{\chi}_1^0 \), \( \tilde{\chi}_2^0 \), etc obtained from (A.14)–(A.18) by formally taking their complex conjugate. (The dot denotes derivative with respect to the retarded time coordinate \( u \).) Here \( \varphi^0 = \tilde{\varphi}^0(u, \xi, \tilde{\xi}) \) is an arbitrary function (and, for complex bundle and connection, \( \tilde{\varphi}^0 = \tilde{\varphi}^0(u, \xi, \tilde{\xi}) \) also), but the functions \( \chi^0_0 = \chi^0_0(u, \xi, \tilde{\xi}) \), \( i = 0, 1, 2 \), and, in the complex case, \( \tilde{\chi}^0_0 \), too can be specified freely only at a given retarded time instant \( u = u_0 \). Their time evolution is governed by (A.16)–(A.18) (and the analogous tilded equations). Similarly, the sum \( \chi^0_1 + \tilde{\chi}^0_1 \) can be freely specified at \( u = u_0 \), but the difference \( \chi^0_1 - \tilde{\chi}^0_1 \) is already determined by \( \varphi^0 \) and \( \tilde{\varphi}^0 \) via

\[
\chi^0_1 - \tilde{\chi}^0_1 = \partial_\alpha \varphi^0 - \partial_\alpha \tilde{\varphi}^0 + [\varphi^0, \tilde{\varphi}^0]. 
\] (A.19)

It is a simple calculation to check that the evolution equation (A.15) and its tilded version preserve (A.19). Geometrically \( \varphi^0 \) and \( \tilde{\varphi}^0 \) are the leading \( (r^{-1}) \) order term in the asymptotic expansion of \( \gamma^0_0 \) and \( \gamma^0_0 \), respectively, and they are analogous to the asymptotic shear in general relativity, while \( \chi^0_0 \) and \( \tilde{\chi}^0_0 \) represent the outgoing radiation. In terms of these the expansion coefficients \( \chi^0_1, \chi^0_2, \chi^0_1 \) and \( \chi^0_2 \) are given by

\[
\chi^0_1 = -\left(\partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right). 
\] (A.20)

\[
\chi^0_2 = -\frac{1}{2} \left(\partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right) + \left[\chi^0_0, \chi^0_0\right]. 
\] (A.21)

\[
\chi^0_1 = -\left(\partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_1]\right). 
\] (A.22)

\[
\chi^0_2 = \frac{1}{2} \left(\partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right) + \left[\varphi^0, \partial_\alpha \chi^0_0 + [\varphi^0, \chi^0_0]\right] - \left[\chi^0_1, \chi^0_0\right], 
\] (A.23)

and there are analogous expressions for \( \tilde{\chi}^0_1 \), \( \tilde{\chi}^0_2 \), \( \tilde{\chi}^0_1 \) and \( \tilde{\chi}^0_2 \), too. The gauge is not completely fixed: gauge transformations with arbitrary \( \Lambda^A_B \equiv \Lambda^A_B(\xi, \tilde{\xi}) \) can still be carried out.

A.2.2. Special stationary configurations. Let the Yang–Mills field be stationary with respect to \( u \), i.e. all the dot derivatives are vanishing. Newman and Penrose showed [9] that in the analogous gravitational case the asymptotic shear can be made zero by an appropriate supertranslation of the origin cut. However, now the integrability condition of the local existence of a gauge transformation \( \Lambda(\xi, \tilde{\xi}) \) taking \( \varphi^0 \) and \( \tilde{\varphi}^0 \) to zero is the vanishing of the corresponding curvature, which is proportional to the right-hand side of (A.19). Hence, for example, in the presence of magnetic charges (see equation (3.2)) there is no such a gauge transformation.

Thus in the rest of this subsection we consider those special stationary solutions of the asymptotic field equations in which \( \varphi^0 = 0 \) and \( \tilde{\varphi}^0 = 0 \) can be ensured even globally, and hence \( \chi^0_1 = \tilde{\chi}^0_1 \) holds. Thus the total magnetic charges are vanishing. (If the connection is real, then \( \chi^0_1 \) is also real.) Then equation (A.16) immediately gives that \( \chi^0_1 \) is constant on \( S \),
and hence has the form $x_i^0 = \frac{1}{2} C_i^\mu e_A^\mu e_B^\nu$ for some constants $C_i^\mu$. These constants are real for real bundles and connections, while in the complex case they are complex but $C_i^\mu = \bar{C}_i^\mu$. We saw in subsection 3.4.3 that the constants $C_i^\mu$ represent the total charges.

To solve equation (A.17), let us write $\chi_0 = h^a e_A^a k^\mu$ for some functions $h^a$ with spin weight one. Then (A.17) takes the simple form

$$2i\delta_0 m h^\mu = 3(C^a e_A^a)^{\mu\nu} h^\nu = 3C^\mu, h^\nu.$$ 

Thus $C^\nu$ is a constant $\dim \mathfrak{g} \times \dim \mathfrak{g}$ real or complex matrix. Expanding $h^\mu$ in terms of the $s = 1$ spin weighted spherical harmonics with the constant coefficients $h^a_{jm}$ as

$$h^\mu = \sum_{j=1}^\infty \sum_{m=-j}^j h^a_{jm} Y_{jm}$$

and recalling that the action of the edth operators on spin-$s$ spherical harmonics is

$$0\partial_s Y_{jm} = -\frac{1}{\sqrt{j}} \sqrt{(j-s)(j+s+1)} Y_{jm}, \quad 0\partial_s Y_{jm} = \frac{1}{\sqrt{j}} \sqrt{(j+s)(j-s+1)} Y_{jm},$$

equation (A.24) leads to

$$C^\nu h^a_{jm} = -\frac{1}{2} (j-1)(j+2) h^a_{jm}. \quad (A.26)$$

Thus the expansion coefficients $h^a_{jm}$ must be eigenvectors of $C^\nu$ with the real non-positive eigenvalues $-\frac{1}{2} (j-1)(j+2)$. By the definition of the charge matrix $C^\nu C^\nu = 0$ holds, i.e. $\lambda = 0$ is always an eigenvalue with eigenvector $C^\mu$. Thus, in particular, $h^a_{jm}$ can always be non-zero, and if $C^\nu$ is the only eigenvector of $C^\nu$ with zero eigenvalue then $h^a_{jm} = C^a h^a_{jm}$ for some constants $h^a_{jm}$. If the dimension of the kernel space of $C^\nu$ is $k > 1$, then $h^a_{jm}$ will be the linear combination of the vectors of a basis of this kernel with arbitrary constants $h^a_{jm}$, $\ldots$ etc. However, since $C^\nu$ is a finite matrix, it may have only finitely many eigenvalues, and hence only finitely many coefficients $h^a_{jm}$ can be non-zero. Nevertheless, since the total (electric) charges $C^\mu$ can be specified freely, in general no eigenvalues of $C^\nu$ has the form $-\frac{1}{2} (j-1)(j+2)$ for some $j \geq 2$. Therefore, apart from the exceptional cases in which the charges have this structure, the expansion coefficient functions $\chi_0^0$ and $\tilde{\chi}_0^0$ are combinations only of the spherical harmonics $0 Y_{jm}$ and $\tilde{1} Y_{jm}$, respectively.

Similarly, if we write $\chi_0^0 = k^a e_A^a$, then equation (A.18) reduces to the linear, but inhomogeneous equation

$$0\partial_0 k^a + 2k^a + \frac{5}{2} C^\nu k^\nu = -e^\mu_{a\beta} ((0\partial h^a)^\beta + 2h^a (0\partial h^a)^\beta + 3h^a (0\partial_h a_{0}\beta))) \quad (A.27)$$

Following the strategy of the previous paragraph and expanding $k^a$ as

$$k^a = \sum_{j=1}^\infty \sum_{m=-j}^j k^a_{jm} Y_{jm}, \quad (A.28)$$

for the expansion coefficients $k^a_{jm}$ of the solution of the homogeneous equation we obtain the condition

$$C^\nu k^a_{jm} = \frac{1}{2} ((j-1)(j+2) - 4) k^a_{jm}. \quad (A.29)$$

Thus the general solution of the homogeneous equation contains non-trivial $k^a_{jm}$ for $j \neq 2$ only if the matrix $C^\nu$ is chosen to be very special. This solution depends only on finitely many parameters. A particular solution of the inhomogeneous equation (A.27) can also be found by using the spin weighted spherical harmonics and the expansion of their product, $Y_{jm} Y_{jm'}$, by the appropriate Clebsch–Gordan coefficients and the spherical harmonics $s s Y_{jm+jm'}$, max{$s + s', |j - j'|$} $\leq J \leq j + j'$.
Thus, to summarize, in the present special stationary case the asymptotic field equations can be integrated completely and these solutions (with the second order accuracy, i.e. keeping only $\chi^0$, $\chi^1$ and $\chi^j$) can be indexed by the freely specifiable total (electric) charges $C^\mu$ and the expansion coefficients $h^\mu_1$ and $k^\mu_2$ (and, in the complex case, by $\tilde{h}^\mu_1$ and $\tilde{k}^\mu_2$, too). Other parameters may occur in the solutions (namely in the expansion coefficients $\chi^0$, $\chi^1$, etc) only when the charges are chosen in very specific ways.

A.2.3. Special stationary solutions: examples. First consider the generic case, i.e. when no non-vanishing eigenvalue of $C^\mu$, has the form $\pm \frac{1}{2}(j-1)(j+2)$ for some $j \in \mathbb{N}$. Then by (A.26) we have that $h^\mu = \sum_{m=-1}^1 h^\mu_{1m}Y_{1m}$ and $\tilde{h}^\mu = \sum_{m=-1}^1 \tilde{h}^\mu_{1m}Y_{1m}$, and then, using the expressions how the edth operators act on the spin weighted spherical harmonics, (A.27) reduces to

$$0 = \frac{1}{2} \sum_{j=1}^\infty \sum_{m=-j}^j \left( 5C^\mu k^\mu_{jm} + (4 - (j - 1)(j + 2))k^\mu_{jm} \right)Y_{jm}$$

$$+ c_{a\beta} \sum_{m,m'=1}^1 \left( 3h^\mu_{1m}h^\beta_{1m'} - 2h^\mu_{1m}h^\beta_{1m'} \right)Y_{1m}0Y_{1m'}$$

Its second line can be written as $\sum_{m=-1}^1 h^\mu_{1m}Y_{1m} + \sum_{m=-2}^2 h^\mu_{2m}Y_{2m}$, where $H^\mu_{jm}$, $j = 1, 2$, are explicitly given, homogeneous quadratic expressions of $h^\mu_1$ and $h^\mu_2$, and are built also from the structure constants and the Clebsch–Gordan coefficients. With this substitution we obtain that

$$(C^\mu + \frac{4}{3}h^\mu)k^\mu_{1m} = -\frac{2}{3}H^\mu_{1m}, \quad C^\mu k^\mu_{2m} = -\frac{2}{3}H^\mu_{2m}.$$  \hspace{1cm} (A.31)

Moreover, if $C^\mu$ has an eigenvalue $\frac{1}{2}(J - 1)(J + 2)$ for some $J \geq 3$ then $k^\mu_{jm}$ is an eigenvector while $h^\mu_{jm} = 0$ for all $j = 3, \ldots, J - 1, J + 1, \ldots$; otherwise, the coefficients $h^\mu_{jm}$ are vanishing for all $j \geq 3$. The first of (A.31) can be inverted to find the unique solution $k^\mu_{1m}$ precisely when $\pm \frac{1}{2}$ is not an eigenvalue of $C^\mu$, otherwise the corresponding eigenvector can be freely added to the particular solution. Similarly, $k^\mu_{2m}$ is determined only up to the general solution of the homogeneous equation $C^\mu k^\mu_{2m} = 0$.

If the only eigenvector of $C^\mu$ with zero eigenvalue is the charge vector $C^\mu$, then $h^\mu_{1m} = C^\mu h_m$ and $\tilde{h}^\mu_{1m} = C^\mu \tilde{h}_m$, which imply that $H^\mu_{1m} = 0$. (For the sake of simplicity we assume that the structure constants are real.) Hence $k^\mu_{1m}$ is either an arbitrary combination of the eigenvectors of $C^\mu$ with eigenvalue $\pm \frac{1}{2}$ or zero, while $k^\mu_{2m}$ is a solution of $C^\mu k^\mu_{2m} = 0$. If, in addition to $C^\mu$, $\tilde{C}^\mu$ is another eigenvector of $C^\mu$, with zero eigenvalue, then $h^\mu = \sum_{m=-1}^1 (C^\mu h_m + \tilde{C}^\mu \tilde{h}_m)Y_{1m}$, implying that $H^\mu_{1m}$ are proportional to $c_{a\beta}C^\alpha \tilde{C}^\beta = \frac{1}{2}C^\alpha \tilde{C}^\beta = 0$, i.e. $H^\mu_{1m} = 0$ again. In the other extreme special case when the charge matrix is vanishing (e.g. if $C^\mu = 0$) $k^\mu_{1m}$ holds, $k^\mu_{2m}$ are arbitrary, but now $H^\mu_{2m} = 0$ must hold. This yields restrictions on the coefficients $h^\mu_1$ and $\tilde{h}^\mu_1$. The evaluation of the explicit formulae shows that then $H^\mu_{11} = 0$, but $H^\mu_{1\pm 1}$ are still non-zero.

To illustrate the above possibilities, let us specify the Lie algebra $\mathfrak{g}$. First it is chosen to be the abstract Lie algebra $sl(2, \mathbb{R})$, $so(3)$ (i.e. $su(2)$) or $sl(2, \mathbb{C})$; and let us choose the basis $\{e_1, e_2, e_3\}$ such that $\left\{e_1, e_2, e_3\right\} = \mp e_1, \left\{e_3, e_1\right\} = e_2$ and $\left\{e_1, e_2\right\} = e_3$, where the upper sign corresponds to $sl(2, \mathbb{R})$ and the lower to $so(3)$ and $sl(2, \mathbb{C})$. In this basis the components of the Killing–Cartan metric are $g_{a\beta} = 2 \text{diag}(-1, \pm 1, \pm 1)$. The eigenvalues of $C^\mu$, are $\lambda = 0$, in which case the corresponding eigenvector is proportional to the total charge vector $C^\mu$, and $\lambda = \pm \frac{1}{\sqrt{2}} \sqrt{g_{a\beta}C^a C^\beta}$. The non-zero eigenvalues can be real for $\mathfrak{g} = sl(2, \mathbb{R})$ if
$g_{\alpha\beta} C^\alpha C^\beta > 0$ and for $g = sl(2, \mathbb{C})$ also in special cases (e.g. for purely imaginary $C^\mu$), but never for $g = so(3)$. Thus the configurations for $g = so(3)$ are always generic.

From the point of view of general relativity the most interesting choice for $g$ is the abstract Lorentz Lie algebra $so(1, 3)$. We choose its basis $\{e_1, \cdots, e_6\}$ to be the standard one, i.e. in which $\{e_1, e_2, e_3\}$ is the basis above for $so(3)$, while $e_4$, $e_5$ and $e_6$ are the ‘boost generators’ such that $[e_1, e_4] = 0$, $[e_1, e_5] = e_6$, . . . etc. In this basis the Killing–Cartan metric is $g_{\alpha\beta} = 4\text{diag}(-1, -1, -1, 1, 1, 1)$. One eigenvalue of the charge matrix is zero with eigenvectors $C^\mu$ and $\hat{C}^\mu := (-C^4, -C^5, -C^6, C^1, C^2, C^3)$ (as a column vector). Clearly, $g_{\alpha\beta} C^\alpha C^\beta = -g_{\alpha\beta} \hat{C}^\alpha \hat{C}^\beta$. The (square of the) non-zero eigenvalues are $\lambda^2 = \frac{1}{4}(g_{\alpha\beta} C^\alpha C^\beta \pm i|g_{\alpha\beta} C^\alpha C^\beta|)$. Thus $\lambda$ is real precisely when $g_{\alpha\beta} C^\alpha C^\beta = 0$ and $g_{\alpha\beta} C^\alpha C^\beta > 0$, and, in contrast to $so(3)$, for special values of $C^\mu$ the non-zero eigenvalues could have the form $\pm \frac{1}{2}(j - 1)(j + 2)$ or $\frac{1}{2}(j - 1)(j + 2) - 4$ for some $j \in \mathbb{N}$.

Finally we summarize the results in an exceptional case, i.e. when $\pm \frac{1}{2}(J - 1)(J + 2)$ are eigenvalues of $C^\mu_\nu$ for some integer $J \geq 2$. Solving (A.26), we have that

$$h^\mu = \sum_{m=-1}^{1} h^\mu_{lm} Y_{lm} + \sum_{m=-j}^{j} h^\mu_{jm} Y_{jm},$$

(A.32)

where $h^\mu_{lm}$ and $h^\mu_{jm}$ are arbitrary combinations of the eigenvectors of $C^\mu_\nu$ with zero and $-\frac{1}{2}(J - 1)(J + 2)$ eigenvalue, respectively. If we substitute this and the analogous expression for $h^\mu$ into (A.27), use elementary properties of the edth operators and the spin weighted spherical harmonics and write the products of such harmonics as combinations of spherical harmonics with Clebsch–Gordan coefficients, we obtain

$$0 = \frac{1}{2} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} (5C^\mu_\nu k^\nu_{jm} + (4 - (j - 1)(j + 2))k^\mu_{jm}) Y_{jm} + \sum_{m'=1}^{1} \sum_{m=-1}^{1} (3h^\alpha_{lm} h^\beta_{lm'} - 2h^\alpha_{lm} \bar{h}^\beta_{lm'}) Y_{lm} Y_{lm'} + \frac{1}{2} \sum_{j=1}^{2} \sum_{m=-1}^{j} \sum_{m=-j}^{j} K^\mu_{jm} Y_{jm}.$$  

(A.33)

Here $K^\mu_{jm}$ can be given explicitly in terms of $h^\nu_{lm}$, $\bar{h}^\nu_{lm}$, $h^\mu_{jm}$, $\bar{h}^\mu_{jm}$, the structure constants and Clebsch–Gordan coefficients. Since we assumed that $C^\mu_\nu$ has non-zero eigenvalues, it cannot be identically zero, and, for the sake of simplicity, let us assume that $C^\mu_\nu$ does not have more than two eigenvectors with zero eigenvalue. Then $C^\mu_\nu (3h^\alpha_{lm} h^\beta_{lm'} - 2h^\alpha_{lm} \bar{h}^\beta_{lm'}) = 0$ (see the paragraph following equation (A.31)). Then, evaluating (A.33), we find that $k^\mu_{jm} = 0$ for $j \geq 2J + 1$, $k^\mu_{jm}$ are uniquely determined for $j = 1, 3, 4, \ldots, 2J$, and $k^\mu_{2m}$ are determined up to the addition of the general solution of the homogeneous equation $C^\mu_\nu k^\nu_{2m} = 0$.

References

[1] Utiyama R 1956 Invariant theoretical interpretation of interaction Phys. Rev. 101 1597–1607
[2] Møller C 1961 Conservation laws and absolute parallelism in general relativity Mat.-Fys. Skr. K. Danske Vid. Selsk. 1 1–50
[3] Newman E T 1978 Source free Yang–Mills theories Phys. Rev. D 18 2901–2908
[4] Trautman A 1980 Fiber bundles, gauge fields and gravitation General Relativity and Gravitation vol 1 ed A Held (New York: Plenum)
[5] Dubois-Violette M 1982 Einstein equations, Yang–Mills equations and classical field theory as compatibility conditions of linear partial differential operators Phys. Lett. B 119 157–61
[6] Dubois-Violette M 1983 Remarks on the local structure of Yang–Mills and Einstein equations Phys. Lett. B 131 323–6
[7] Carmeli M, Lebowitz E and Nissan N 1990 Gravitation: SL(2, C) Gauge Theory and Conservation Laws (Singapore: World Scientific)
[7] Hehl F W, McCrean J D, Mielke E W and Ne’eman Y 1995 Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance Phys. Rep. 285 1–171
[8] Tod K P 1983 Quasi-local charges in Yang–Mills theory Proc. R. Soc. A 389 369–77
[9] Newman E T and Penrose R 1968 New conservation laws for zero rest-mass fields in asymptotically flat space-time Proc. Roy. Soc. A 305 175–204
[10] Exton A R, Newman E T and Penrose R 1069 Conserved quantities in the Einstein–Maxwell theory J. Math. Phys. 10 1566–70
[11] Newman E T 2010 Private communication
[12] Bazanski S 2010 Private communication
[13] Penrose R and Rindler W 1984 Spinors and Spacetime Vol 1 (Cambridge: Cambridge University Press)
[14] Hugget S A and Tod K P 1985 An Introduction to Twistor Theory (Cambridge: Cambridge University Press)
[15] Newman E T and Tod K P 1980 Asymptotically flat space-times General Relativity and Gravitation vol 2 ed A Held (New York: Plenum)
[16] Szabados L B 1999 On certain quasi-local spin-angular momentum expressions for small spheres Class. Quantum Grav. 16 2889–904 arXiv:gr-qc/9901068
[17] Chrusciel P T and Kondracki W 1987 Some global charges in classical Yang–Mills theory Phys. Rev. D 36 1874–1881
[18] Tod K P 1993 Liénard–Wiechert Yang–Mills fields Spinors, Twistors, Clifford Algebras and Quantum Deformations ed Z Oziewicz, B Jancewicz and A Borowiec (Dordrecht: Kluwer)
[19] Trautman A 1981 Radiation of energy and charge in color of a point source of the Yang–Mills field Phys. Rev. Lett. 46 875–7
[20] Tafel J and Trautman A 1983 Can poles change color? J. Math. Phys. 24 1087–92
[21] Sarıoğlu Ö 2002 Liénard–Wiechert potentials of a non-Abelian Yang–Mills charge Phys. Rev. D 66 085005