Are known maximal extensions of the Kerr and Kerr-Newman spacetimes physically meaningful and analytic?

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Abstract

We argue that the well-known maximal extensions of the Kerr and Kerr-Newman spacetimes characterized by two asymptotically flat regions with ADM masses of opposite signs are physically inconsistent and actually non-analytic. We also demonstrate that the problem of the construction of genuine maximal analytic extensions of these spacetimes may be successfully solved by using models with only one asymptotically flat region, in which case the analyticity inside a ring singularity must be ensured by a correct coordinate choice.

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Unlike the maximal analytic extensions (MAEs) of the Schwarzschild \[1, 2\] and Reissner-Nordström \[3–6\] metrics in which the range of the radial coordinate \(r\) is restricted to non-negative values only, the MAEs of the Kerr \[7, 8\] and Kerr-Newman (KN) \[9, 10\] spacetimes involve both positive and negative values of \(r\). This seems needed because the intrinsic curvature singularity located at \(r = 0\) in the equatorial plane of each of the latter two spacetimes has topology of a ring and the extension within the same spacetime, through the disk enclosed by the ring singularity, is known to be non-analytic in the Boyer-Lindquist coordinates. The procedure of the continuation of \(r\) into negative values is well described in the classical books on general relativity \[11, 12\] and leads to appearance of the second asymptotically flat region of negative ADM \[13\] mass, provided the mass of the first asymptotically flat region is positive definite. The very fact that the same singularity may look as having positive or negative mass depending on a particular asymptotically flat region in which an observer is situated seems rather unphysical, but quite surprisingly it has never been questioned or objected in the literature. An additional undesirable feature of the known MAEs of the Kerr and KN solutions is that in the static limit they do not reduce straightforwardly to the MAEs of static spacetimes with one asymptotically flat region. Although it is commonly believed that the specific gluing of the two asymptotically flat regions employed in those MAEs is smooth, a recent study of the Kerr and KN solutions endowed with negative mass \[14, 15\] has revealed, however, that the curvature singularities in the negative-mass case are massless and in addition are located outside both the symmetry axis and the mass distributions. This makes them completely different from the singularities in the positive-mass case, thus raising serious doubts about the analyticity of the corresponding MAEs on the disks joining the regions of positive and negative ADM masses. In the present letter we will discuss the physical and mathematical inconsistencies of the MAEs from Refs. \[8, 10\] and give arguments favoring the construction of the maximally extended Kerr and KN solutions within the framework of the models with only one asymptotically flat region. Our main result is a demonstration that, in an appropriate coordinate system, a KN spacetime can be analytically continued from one hemisphere to another through the part of the equatorial plane encircled by the ring singularity, thus making the introduction of the second asymptotically flat region unnecessary.
The KN metric in the Boyer-Lindquist coordinates \((r, \theta, \varphi, t)\) reads as

\[
ds^2 = \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} [adt - (r^2 + a^2)d\varphi]^2 - \frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\varphi)^2,
\]

(1)

with

\[
\Delta = r^2 - 2Mr + a^2 + Q^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta,
\]

(2)

where \(M, a\) and \(Q\) are the mass, rotational and charge parameters, respectively, and the coordinate range is: \(0 \leq r < \infty, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi, -\infty < t < \infty\). When \(Q = 0\), Eqs. (1) and (2) define the Kerr metric. The curvature singularity is determined by \(\Sigma = 0\), i.e., it is located at \(r = 0\) of the equatorial \((\theta = \frac{\pi}{2})\) plane, the singularity for \(M \neq 0\), \(a \neq 0\) being ring-like rather than point-like, which allows one to extend the KN metric into the negative values of the radial coordinate \(r\). Then a “standard” extension of the region I \((0 \leq r < \infty)\), with \(M > 0\), consists in letting \(r\) take negative values while \(M\) remains unchanged, thus giving rise to the region II \((-\infty < r \leq 0)\) which is equivalent, in view of the invariance of (1) under the discrete transformation \(r \rightarrow -r, M \rightarrow -M\), with the KN spacetime corresponding to \(M < 0, r \geq 0\) and characterized by a negative ADM mass. In the MAEs from [8, 10] it is assumed implicitly that the ring singularity present in the Kerr and KN solutions with positive mass is the same as in the negative-mass case, so that in order to formally pass from the region I to region II and vice versa one uses the sole singularity, the latter being seen as having positive mass in the region I, and having negative mass in the region II. This situation, when there exists the “other side” where the singularity’s mass changes its sign, really looks surpassing any logic and imagination, especially if one recalls that already in 1935 Einstein and Rosen [16] gave a nice example of a spacetime with two asymptotically flat regions and the same ADM mass (the famous Einstein-Rosen bridge). It is not then surprising that the question of how the singularity’s mass may have simultaneously two opposite values has never been clarified in the literature.

Now, as it follows from the above (see also [17]), the known MAEs of the Kerr and KN metrics can be envisaged, roughly speaking, as a unification of two physically different manifolds: (i) the usual black-hole or hyperextreme spacetime \((M > 0, r \geq 0)\) and (ii) a portion of spacetime created by negative mass \((M < 0, r \geq 0, \text{which is equivalent to region II: } M > 0, r \leq 0)\). However, a key point to observe here is that the ring singularity \(r = 0, \theta = \frac{\pi}{2}\) is not the same object in the \(M > 0\) and \(M < 0\) cases. This follows directly from the recent study [14, 15] of the Kerr and KN solutions with negative mass: if for
instance $M^2 > a^2 + Q^2$, the singularity in the positive-mass case is massive and lies inside the horizon [18], while in the $M < 0$ case it is massless and lies off the symmetry axis outside the stationary limit surface [25] (of course, one supposes that $r \geq 0$ in both the cases); the only common feature shared by the two singularities is their $S^1$ topology. In other words, for $M \neq 0$, the limit $r \to 0$ leads to two different singular rings corresponding to $M > 0$ and $M < 0$, thus spoiling the analyticity property of the standard extensions of the Kerr and KN solutions which are not smooth on the disks joining the two asymptotically flat regions. Mention also that in the absence of rotation ($a = 0$), when Eqs. (1), (2) define the Reissner-Nordström metric, the curvature singularity in the $M > 0$ case is point-like while in the $M < 0$ case it has topology of a sphere, which means that the “standard” MAE of the KN solution does not admit a mathematically correct electrostatic limit.

Apparently, the main physical inconsistency of the known MAEs of the Kerr and KN solutions is the non-correspondence of the asymptotics of region II to the positive-mass singularity as a source for that region artificially introduced instead of the genuine source of negative mass. In this respect one should bear in mind that region II will be always characterized asymptotically by the same ADM mass independently of which source one puts at its $r = 0$; however, the source’s mass must be the same as measured asymptotically if one wants to get a physically consistent model. On the other hand, the main mathematical problem of the known MAEs is their non-analyticity on the disks encircled by ring singularities because of the intrinsic differences between the regions with positive and negative mass, and also due to a restrictive character of the Boyer-Lindquist coordinates spoiling the differentiability properties of the Kerr and KN spacetimes at $r = 0$ [20].

At first glance it seems that the physical inconsistency inherent in the aforementioned MAEs can be remedied by only a slight modification of the extension scheme: while the region I corresponds, as in Refs. [8, 10], to positive values of $r$ and $M$, the region II should involve negative values of $r$ and $M$. Consequently, the newly defined region II, with $r \leq 0$, $M < 0$, will be identical with the region I (again by virtue of invariance of the metric (1), (2) under the discrete transformation $r \to -r$, $M \to -M$) and of course will have the same asymptotics as region I. The MAE performed in this way looks physically consistent because the resulting extended manifold does not create any conflict of singularities, which are this time the same curvature ring singularity endowed with positive mass, and both regions I and II are described by the same ADM mass coinciding with the singularity’s mass. A certain
support in favor of this way of continuing $r$ into negative values in the Kerr and KN solutions comes from Carter’s MAE of KN spacetime itself: in the case $M = 0$, this MAE becomes physically reasonable and coincides with the above construction, being composed of two identical asymptotically flat regions. The KN solution in this case represents the exterior field of a massless charged magnetic dipole, and the metric is still stationary due to a specific frame-dragging effect [21, 22], with $g_{t\phi} = aQ^2 \sin^2 \theta/(r^2 + a^2 \cos^2 \theta)$ and a zero-mass ring singularity. However, the use of the same procedure of attaching the region II to region I as described in detail by Hawking and Ellis for the Kerr MAE [11], which consists in identifying the top side of the first disk with the bottom side of the second one (and vice versa), in the case of two identical KN spacetimes is similar to the problem of analytically extending a single KN metric through the disk encircled by the ring singularity, and the latter extension is commonly thought to be not $C^1$ (see, e.g., [17]). Thus one arrives at the following delicate situation: the known MAEs of the Kerr and KN solutions are not satisfactory both physically and mathematically, but a rectified extension procedure within the framework of the same philosophy of two asymptotically flat regions, being physically consistent, leads to the mathematical problem that long ago had already forced the researchers to abandon the philosophy of a sole-spacetime MAE. Once the possible physically reasonable extensions of the Kerr and KN solutions, independently of the number of asymptotically flat regions they may involve, require the same mathematical difficulty to be clarified, it would be plausible to analyze that difficulty (supposing that the correct MAEs do exist) within the simplest model of a single KN spacetime.

Actually, the question of whether or not the KN spacetime can be smoothly continued through the ring singularity from one hemisphere to another is of paramount importance, being a key point for the construction of a correct MAE, and until now it had a negative answer [17, 23]. The usual “non-traversability” argument is the following: the disk inside the ring singularity is assumed to be described by the hypersurface $r = 0$, and across the disk there are discontinuities in the first derivatives of the metric tensor components that can be associated with some matter sources. Then one “naturally” concludes that the analytic continuation through the disk is impossible. Nonetheless, as was already observed in [20], the Boyer-Lindquist coordinates are not quite appropriate for the analysis of the Kerr geometry in the vicinity of $r = 0$. As will be seen later on, the true problems caused by these coordinates are really enormous, and below we shall demonstrate that (i) the hypersurface
$r = 0$ does not actually describe a disk, but a spheroid, and (ii) that in an appropriately chosen coordinate system a smooth extension across the ring singularity is possible.

To make our demonstration more visual, we will first consider the KN solution with negative mass. Resorting to this case does not affect the generality of our future conclusions but will permit us to use an already known coordinate transformation for making various important statements. According to [15], the KN spacetime with negative mass can be conveniently tackled in the usual Weyl-Papapetrou cylindrical coordinates $\rho$ and $z$ ($\rho \geq 0$, $-\infty < z < \infty$) which are related to the Boyer-Lindquist coordinates $(r, \theta)$ via the formulas

$$r = M + \frac{1}{2}(r_+ + r_-), \quad \cos \theta = \frac{1}{2\kappa}(r_+ - r_-),$$
$$r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2}, \quad \kappa = \sqrt{M^2 - a^2 - Q^2}.$$  \hspace{1cm} (3)

With the aid of (3), in Fig. 1 we have plotted the hypersurface $r = 0$ in cylindrical coordinates for the particular parameter choice $M = -2$, $a = 1$, $Q = 0.25$, and one can see that it represents a spheroid with the poles $z = \pm 2$, $\rho = 0$ and the ring singularity $z = 0$, $\rho \approx 1.031$ as its equator. In the general case the spheroid has the poles $z = \mp M$, $\rho = 0$, and the ring singularity $z = 0$, $\rho = \sqrt{a^2 + Q^2}$ is its equator. In the Boyer-Lindquist coordinates the spheroid is removed from the general manifold, which explains the existence of the aforementioned discontinuities in the first derivatives of the metric tensor inside the singular ring, as well as interpretation of the surface $r = 0$ as a specific cut used in the paper [23]. The spheroid degenerates to an infinitesimally thin disk in the case $M = 0$ when the discontinuities disappear. Therefore, when $M \neq 0$, the Boyer-Lindquist coordinates are unable to provide one with a correct description of the part of the equatorial plane encircled by the ring singularity. This does not occur after rewriting the KN metric in cylindrical coordinates when one becomes capable to reach analytically the part $0 < \rho < \sqrt{a^2 + Q^2}$ of the equatorial plane and then smoothly cross the latter in the absence of discontinuities in the derivatives of the metric tensor. Moreover, in the equatorial plane the metric coefficient $g_{tt}$ of the KN metric with $M < 0$, $M^2 > a^2 + Q^2$ takes the form

$$g_{tt} = \frac{\rho^2 - a^2}{M + \sqrt{\rho^2 + m^2 - a^2 - Q^2}},$$ \hspace{1cm} (4)

whence it follows that at the part of the (genuine) equatorial plane encircled by the ring singularity the KN metric is sensitive to the sign of mass, thus excluding the possibility of gluing there spacetimes characterized by masses of opposite signs.
By introducing instead of \( r \) a new radial coordinate \( \tilde{r} \) via the formula

\[
\tilde{r} = \alpha M + \frac{1}{2}(r_+ + r_-), \quad \alpha = \text{const},
\]

(5)

with \( \tilde{r} \geq 0 \), it is easy to see that an appropriate choice of the parameter \( \alpha \) permits one to get a better description of the KN solution than give us the usual Boyer-Lindquist coordinates. In Fig. 2 we have plotted the hypersurface \( \tilde{r} = 0 \) for two particular values of \( \alpha \) (the values of \( M, a \) and \( Q \) are the same as in Fig. 1), and the case \( \alpha = 12/13 \) gives an extension of the Boyer-Lindquist coordinates, while the choice \( \alpha = 13/12 \) leads to a larger restriction of the KN manifold than in Fig. 1 (remember that \( M < 0 \)). In general, for all \( \alpha M > -\kappa \), the new coordinates \( (\tilde{r}, \theta) \) may be considered as extensions of the cylindrical coordinates \( (\rho, z) \).

Now we are ready to briefly discuss the case \( M > 0 \) which turns out analogous to the previous case of negative mass. Since the non-negative values of the radial Boyer-Lindquist coordinate \( r \) do not permit one to go beyond the ring singularity, an extension of \( r \) is needed to describe geometry inside the singular ring. Like in the \( M < 0 \) case, the hypersurface \( r = 0 \) for \( M > 0 \) does not represent a disk, being a spheroid, and this situation should be considered as exclusively the product of the use of bad coordinates. It is clear that direct extension of \( r \) into negative values with the aid of the scheme worked out in the papers [8, 10] does not cure the artificial, purely coordinate problem of the surface \( r = 0 \), and besides introduces new pathological regions into the extended manifold. To make the interior of the ring singularity analytic and traversable for geodesics, one has to introduce a new radial coordinate \( \tilde{r} \) which in a natural way would extend the spacetime inside the ring. A possible transformation might have the form

\[
r = \sqrt{M^2 + a^2 + \frac{1}{2}(r_+ + r_-)}, \quad \cos \theta = \frac{1}{2\kappa}(r_+ - r_-),
\]

(6)

with \( r_\pm \) and \( \kappa \) earlier defined in (3). In the coordinates (6) the KN metric takes the form

\[
\begin{align*}
   ds^2 &= \Sigma' \left( \frac{d\tilde{r}^2}{\Delta'} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma'} \left( adt - \left[ (\tilde{r} + M - \mu)^2 + a^2 \right] d\varphi \right)^2 - \frac{\Delta'}{\Sigma'} \left( dt - a \sin^2 \theta d\varphi \right)^2,
\end{align*}
\]

(7)

where

\[
   \Delta' = (\tilde{r} - \mu)^2 - \kappa^2, \quad \Sigma' = (\tilde{r} + M - \mu)^2 + a^2 \cos^2 \theta, \quad \mu = \sqrt{M^2 + a^2},
\]

(8)

and it may be used as a starting point for elaborating, via the usual procedures, a MAE involving non-negative \( \tilde{r} \). The construction of such MAE in explicit form, however, goes beyond the scope of the present letter.
It is worth noting that various extended KN identical spacetimes can be used for obtaining MAEs of more complicated topologies by first making cuts in the equatorial plane inside the ring singularities and then gluing appropriately different sides of the cuts of different spacetimes. In this respect a model with two asymptotically flat regions discussed by us earlier will be analytic in some good coordinate system, giving a possible example of such an exotic MAE. Mention in conclusion that the extension of our results to the case of the Kerr-Newman-(anti-)de Sitter solution [24] is straightforward.

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Location of the singularity of the KN solution with negative mass off the symmetry axis (at $\rho > 0$) has been also noticed by Meinel [19] (see a footnote on p. 10 of his paper).

FIG. 1: The form of the surface $r = 0$ in cylindrical coordinates.
FIG. 2: The surface $\tilde{r} = 0$ for two particular values of the parameter $\alpha$. 