Assessing the Stability of Noisy Quantum Computation

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ABSTRACT

Quantum computation has made considerable progress in the last decade with multiple emerging technologies providing proof-of-principle experimental demonstrations of such calculations. However, these experimental demonstrations of quantum computation face technical challenges due to the noise and errors that arise from imperfect implementation of the technology. Here, we frame the concepts of computational accuracy, result reproducibility, device reliability and program stability in the context of quantum computation. We provide intuitive definitions for these concepts in the context of quantum computation that lead to operationally meaningful bounds on program output. Our assessment highlights the continuing need for statistical analyses of quantum computing program to increase our confidence in the burgeoning field of quantum information science.

Keywords: quantum computing, stability characterization, computational accuracy, result reproducibility, device reliability

1. INTRODUCTION

The stability of a computing system is an important concern for assessing computational performance. System stability plays an essential role in qualifying the reproducibility of scientific results as well the validation, verification, and quantification of uncertainty. For the burgeoning field of quantum computing, the underlying quantum device stability also represents an essential concern for both reliable and reproducible results. Unique sources of quantum device instability include the varying quality of the quantum register implementation, which suffers from non-uniform spontaneous decay, energy loss to the environment, cross-talk, the sensitivity of gate operations for initialization, measurement, and unitary operations to imprecise control pulses as well as fluctuations in the thermodynamic controls. In the presence of inhomogeneous and non-stationary noise processes, an unstable quantum device presents a unique challenge for useful computation as it becomes difficult to attribute errors and quantify uncertainty. While there have been many successful demonstrations of accurate quantum computing, these are generally point solutions using high calibrated and tuned devices. Thus, little attention has been paid to whether such results are reproducible across varying technologies or instances of a given technology. The resulting notion of stability tests the preciseness of repeating a quantum computation, for example, within a defined tolerance, as well as the reproducibility of the results as measured by the similarity between independent outcome distributions.

In this study, we investigate how to assess the stability of a quantum device and link this metric to result reproducibility and device reliability. The notions of accuracy, reproducibility, reliability and stability are related yet distinct. We use the term accuracy to quantify how close the experimentally realized output of a quantum program matches the known, correct output.\textsuperscript{1} Reproducibility is used to quantify how close an instance of a program is to a previously executed instance on the same or potentially different device.\textsuperscript{2,3} Reliability studies how select parameters characterizing the device components, such as gate fidelities, deviate from expectation, while stability quantifies whether the program behavior stays bounded in the presence of fluctuations due to noise.\textsuperscript{4} In the following sections, we define and quantify these notions and, with the help of depolarizing channel as a running example, explore their behaviors for noisy quantum circuits.

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2. THEORY

Consider an idealized quantum computer composed of an n-qubit register that encodes a $2^n$-dimensional Hilbert space $\mathbb{C}^{2^\otimes n}$ and a set of operations that transform the register. Projective operations initialize the state of the register, e.g., in the computational basis state $|\psi\rangle = |0\rangle^{\otimes n}$, while a unitary operation $U$, aka a gate, transforms the state as $|\psi\rangle \to U|\psi\rangle$. Additional projective operations implement measurement of the register labeled by the n-bit string $s$, where $s \in \{0, 1\}^{\otimes n}$, and the probability to observe this outcome is $\Pr(s) = |\langle s|\psi\rangle|^2$. Practical efforts to realize a quantum computer introduce many additional physical processes, identified here as noise, that complicate the operational description above.\textsuperscript{5–8} In all cases, an important question is to understand whether the resulting computational output is accurate, reproducible and stable. We next consider these notions in the presence of quantum channels that model noisy operations.

2.1 Noisy circuit model

Consider the initial density matrix representing the state of a quantum register as $\rho_0$ and let the noiseless application of a circuit unitary $U_C$ prepare the state $\rho = U_C\rho_0 U_C^\dagger$. By comparison, let the action of a noisy circuit on this state be modeled as $\rho' = \mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$, where $\mathcal{E}$ is a channel operator expressed in terms of Kraus operators $(M_k)$. In practice, such a simple model may be replaced by more complex, constructive models for the individual gate operations.\textsuperscript{1} In the orthonormal computational basis $\{|i\rangle\}_{i=0}^{N-1}$ with $N = 2^n$, a corresponding projective measurement is modeled as $\Pi_i = |i\rangle \langle i|$ for $i = 0$ to $N - 1$. The noiseless circuits yield the probability for the $i$-th outcome as $p_i = \text{Tr}[\Pi_i \rho]$, while the noisy circuit model yields the probability

$$p_i = \text{Tr}[\Pi_i \rho'] = \text{Tr}[\Pi_i^\dagger \sum_k M_k \rho M_k^\dagger]\Pi_i]$$

The latter models may also be extended to consider noisy readout channels.\textsuperscript{9}

Differences in the probabilities influence the observed measurement results as well as the derived expectation values. For example, consider the case of a register with $n = 1$ qubits in the presence of depolarizing noise. The latter channel operator is characterized by a noise parameter $\epsilon$ for which the Kraus operators $M_k \in \{\sqrt{1 - \epsilon}I, \sqrt{\epsilon}X, \sqrt{\epsilon}Y, \sqrt{\epsilon}Z\}$ yield

$$\rho' = \mathcal{E}_e(\rho) = (1 - \epsilon)\rho + \frac{\epsilon}{3} X \rho X + \frac{\epsilon}{3} Y \rho Y + \frac{\epsilon}{3} Z \rho Z$$

Assuming $\rho = |\psi\rangle \langle \psi|$ with $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, then $q_0 = |\alpha|^2$ and $q_1 = |\beta|^2$, while $p_0 = q_0 - 2\epsilon(1 - 2|\beta|^2)/3$ and $p_1 = q_1 + 2\epsilon(1 - 2|\beta|^2)/3$ and the noisy expectation value of the diagonal observable $Z$ is $\langle Z_{\text{noisy}} \rangle = (-1)p_0 + (+1)p_1 = (2|\beta|^2 - 1)(1 - \frac{4}{3}\epsilon)$.

2.2 Computational accuracy

It is apparent above that noise induces discrepancies in the probabilities for measurement outcomes and that these errors will make the resulting observables deviate from their expected behavior. We quantify the difference in the noisy and noiseless probability distributions, $p$ and $q$, in terms of the well-known Hellinger distance. The Hellinger distance between $p$ and the reference distribution $q$ is defined as $d_H(p, q) = \sqrt{1 - BC(p, q)}$ where the Bhattacharyya coefficient is given by $BC(p, q) = \sum_i \sqrt{p_i q_i}$ with $p_i, q_i$ the $i$-th discrete output of these distributions. It is notable that the Hellinger distance vanishes when the distributions are identical and grows to unity as the distributions become completely disjoint. Using our expression for noisy circuit execution per the model discussed in the previous section, the Hellinger distance may be expressed as

$$d_H = \sqrt{1 - \sum_i \sum_k \text{Tr}[\Pi_i^\dagger M_k \rho M_k^\dagger] \text{Tr}[\Pi_i \rho]}$$

Given this measure of similarity, we now say the distribution $p$ is $\epsilon$-accurate if $d_H \leq \epsilon$. Checking this condition for a given error tolerance $\epsilon$ requires a simple testing after computing the Hellinger distance.
However, the above definition of accuracy requires a priori knowledge of the noiseless reference distribution $q$, and this may be an impractical requirement when testing the accuracy of noisy quantum circuits whose noiseless distribution of outcomes is unknown, e.g., for large problem sizes in the realm of quantum advantage. In such cases, accuracy of the measurement probabilities may be verified indirectly. In particular, we can define the error in a computed observable as

$$d_O = |\langle O_{\text{noisy}} \rangle - \langle O_{\text{noiseless}} \rangle|$$  \hspace{1cm} (4)

and the $\epsilon-$accuracy condition can be stated as:

$$d_O \leq \epsilon$$  \hspace{1cm} (5)

$$\Rightarrow |\sum_m m(p(m) - q(m))| \leq \epsilon$$  \hspace{1cm} (6)

$$\Rightarrow |\sum_m m\text{Tr}[\Pi_m \mathcal{E}_e(\rho) - \Pi_m \rho]| \leq \epsilon$$  \hspace{1cm} (7)

where $\mathcal{E}_e(\cdot)$ denotes the noise channel for the parameter instance $e$. Here, we define $q(m) = \text{Tr}[\Pi_m \rho]$ as the probability of observing eigenvalue $m$ of the observable $O$ when projecting onto the corresponding $m$-th eigenstate by the measurement $\Pi_m$, and $p(m) = \text{Tr}[\Pi_m \mathcal{E}_e(\rho)]$ as the same for the noisy channel $\mathcal{E}_e(\cdot)$.

Consider our single-qubit working example in the presence of depolarizing noise discussed in the previous section. Then, the Hellinger distance is given as

$$d_H = \left(1 - |\alpha|^2 \right)^{1/2} \sqrt{1 - \frac{2e}{3} \left(1 - \frac{|\beta|^2}{|\alpha|^2}\right)} - |\beta|^2 \sqrt{1 - \frac{2e}{3} \left(1 - \frac{|\alpha|^2}{|\beta|^2}\right)}$$  \hspace{1cm} (8)

while the error in our diagonal observable becomes

$$d_O = |\langle Z_{\text{noisy}} \rangle - \langle Z_{\text{noiseless}} \rangle| = \left|\frac{4e(1 - 2|\beta|^2)}{3}\right|$$  \hspace{1cm} (9)

Assuming a desired computational accuracy that is bounded by error $\epsilon$, this places an upper bound on the depolarizing channel parameter as $e \leq \frac{3\epsilon}{4}$

### 2.3 Result reproducibility

In the previous section, we elaborated on the meaning of accuracy in the context of quantum computing, for which the output from a program instance can be tested to satisfy as desired accuracy conditions. However, such
tests are instance specific and do not address the question of whether the program is reproducible, in the sense that repeated executions will generate similar results. The latter statistical concept requires consideration of program output across multiple instances, which may include execution at different times or on different devices [e.g., see Fig. 1]. In particular, the channel parameters, denoted by $e$, may be different across such instances. We therefore consider the Hellinger distance of the program output with respect to multiple execution instances and pose the confidence interval denoted by $\Pr(d_H^e \leq \epsilon) \geq 1 - \delta$, where $\delta$ denotes the desired confidence level (e.g., $0.05$) and $e$ signifies the instance of channel noise. With respect to error in an observable, we may similarly pose the reproducibility condition as

$$\Pr(d_O^e \leq \epsilon) \geq 1 - \delta$$

where $d_O^e = |\langle O^{\text{ideal}} \rangle - \langle O^{\text{noisy}} \rangle_e|$ and $\langle O^{\text{noisy}} \rangle_e$ is a random variable due to the presence of both shot noise as well as the randomness of $e$. This reproducibility condition may be used to derive a bound on the error model. For example, in our working demonstration, we find that

$$\Pr \left( e \leq \frac{3\epsilon}{4(1 - 2\beta^2)} \right) > 1 - \delta$$

If the parameter $e$ is drawn from a probability distribution $p(e)$ that follows an exponential distribution with mean $1/\lambda$, i.e.,

$$p_\lambda(e) = \lambda \exp^{-\lambda e},$$

then the reproducibility condition requires $\lambda \geq 4|\log \delta|/3\epsilon$.

### 2.4 Device reliability

We call a device reliable if the characteristic parameters describing its quantum noise channels (such as $e$ and $\lambda$ in our working example) remain nearly stationary across multiple instances of program execution. These parameters are dependent upon the model selected to characterize the device and we have previously proposed DiVincenzo’s criteria to generate a minimal set. Those characteristic metrics include the register capacity, initialization fidelity, gate fidelity, duty cycle, which we define as the ratio of register coherence time to gate duration, and register addressability. The latter is quantified in terms of the mutual information between register elements. These metrics are often used as characteristic parameters that inform more explicit noise models for quantum devices.

For example, variations in the distribution of the initialization fidelity for the ibmq_washington device are shown in Fig. 2. Here, the fidelity metric observed during two different periods of time has been fit to the beta distribution, indicating stark differences of the behavior during these two execution windows. The noise parameter characterizing the readout error on May 31, 2022 fluctuates significantly across the device as well as time. Thus, the assumption that a NISQ noise channel is stationary does not hold up to experimental scrutiny. However, noise characterizations are assumed to be constant in current research when applying error mitigation. Thus, a natural question arises as to how sensitive is the output to non-stationary noise processes.

### 2.5 Output stability

We next introduce a stability condition to bound the acceptable fluctuations in the parameters of a quantum channel. Assuming a distribution $p(e)$ for the depolarizing channel parameter $e$, then the stability of the Hellinger distance $d_H$ takes the form

$$E_e [d_H^e] < \epsilon, \quad \forall C, t$$

where

$$E_e (\cdot) = \int \cdot p(e) de$$
Figure 2. Device reliability measures a device’s consistency with respect to characteristic parameters. (a) Wide variance seen in the distribution of Initialization Fidelity, indicating an unstable readout channel. Data shown for qubit #26. (b) Wide spatial variation seen across the 127 qubits of ibmq_washington. Data shown for 31-May-2022.

is the expectation operator with respect to the distribution of error parameter $e$. It is important to note that the stability condition is defined with respect to all circuits $C$ that may run on the device and at any instance, or time $t$, for such executions. For an observable $O$, the stability condition can be similarly stated as

$$E_e[(O_{noisy}^e) - E_e[(O_{noisy}^e)]^2] < e, \quad \forall C, t$$

(15)

where

$$E_e[(O_{noisy}^e)] = \int e \sum m Tr[\mathcal{E}_e(C \rho C^\dagger) \Pi_m] p(e) de$$

(16)

Our model for the channel $\mathcal{E}_e(\cdot)$ is now dependent on a model for the distribution $p(e)$, which characterizes the device reliability. Moreover, $p(e) = p(e; t)$ due to non-stationary noise processes, such that the stability conditions above must be assessed at different points in time. For example, consider a non-stationary depolarizing noise channel with error parameter $e$ drawn from an exponential distribution whose parameter $\lambda$ is time dependent. Let the random variable $f_e = \langle O_{noisy}^e \rangle - E_e[\langle O_{noisy}^e \rangle]$ denote the fluctuation attributable only to channel non-stationarity in $e$ but not to the intrinsic quantum fluctuation in the observable $O$. Hence, it is a measure of stability with respect to the channel, and we want $f_e$ to be small in the mean square sense. As

$$E_e[f_e^2] = \frac{16}{9\lambda^2} (1 - 2|\beta|^2)^2$$

(17)

then applying the stability condition yields

$$\lambda_{stable} \geq \frac{4}{3\sqrt{\epsilon}}$$

(18)

Our analysis has identified bounds on channel parameters assuming a given instance of the distribution $p(e)$. In practice, device calibrations may tune these parameters and help mitigate such fluctuations. Our framework enables the question of time-scale for which such calibrations are necessary to be addressed. Suppose the rate of non-stationary dynamics increases with time. This process can then be modeled by assuming that the spread of the error parameter $e$ characterizing the channel also increases with time. Since the spread (or variance) of an exponential distribution $\propto \lambda^{-2}$, we can then model this situation assuming channel-drift $\lambda(t) = \lambda_0 - \eta t$ i.e. $\lambda$ decreases at a rate $\eta$. The time-scale at which the device transitions from stable to unstable (in the absence of re-calibration) can be estimated as

$$\lambda_0 - \eta \delta t \geq \lambda_{stable}$$

(19)

$$\Rightarrow \delta t_{stable} \leq \frac{\lambda_0}{\eta} \left( 1 - \frac{4}{3\lambda_0 \sqrt{\epsilon}} \right)$$

(20)
Thus, in presence of this channel-drift model, we expect a stable device to remain stable until $t = \delta t_{\text{stable}}$. This gives an estimate of the frequency of device calibration required to meet the stability condition.

3. CONCLUSIONS

Unstable fluctuations in device parameters represent a significant concern for the reproducibility of NISQ computing demonstrations. Many current experimental demonstrations rely on quantum circuits calibrated immediately prior to program execution and tuned during run-time. While this approach is successful for singular demonstrations, the resulting circuits and calibrations are implicitly dependent on the device parameters, which fluctuate significantly over time, across the chip and technology. Unstable devices are not suitable for error attribution or uncertainty quantification (let alone producing reliable results). Most of the research till date has focused on accuracy but little has been done on the reproducibility and stability of quantum computers. In this study, we define computational accuracy, result reproducibility and device reliability and discuss how they are linked using a framework for stability of quantum information. Without additional efforts to make current experimental results reproducible, the knowledge and insights gained from today’s burgeoning field of quantum computer research may be undercut by low confidence in the reported results.

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