The $L^2$ decay for the 2D co-rotation FENE dumbbell model of polymeric flows

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Abstract

In this paper we mainly study the long time behaviour of solutions to the finite extensible nonlinear elastic (FENE) dumbbell model with dimension two in the co-rotation case. Firstly, we obtain the $L^2$ decay rate of the velocity of the 2D co-rotation FENE model is $(1 + t)^{-\frac{1}{2}}$ with small data. Then, by virtue of the Littlewood-Paley theory, we can remove the small condition. Our obtained sharp result improves considerably the recent results in [8, 14].

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1 Introduction

In this paper we consider the finite extensible nonlinear elastic (FENE) dumbbell model \[2, 4\]:

\[
\begin{aligned}
&u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla P = \text{div } \tau, \quad \text{div } u = 0, \\
&\psi_t + (u \cdot \nabla)\psi = \text{div}_R[-\sigma(u) \cdot R\psi + \beta \nabla_R \psi + \nabla_R \mathcal{U}_R], \\
&\tau_{ij} = \int_B (R_i \nabla_j \mathcal{U}) \psi dR, \\
&u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0, \\
&\left(\beta \nabla_R \psi + \nabla_R \mathcal{U}_R\right) \cdot n = 0 \quad \text{on} \quad \partial B(0, R_0).
\end{aligned}
\]

(1.1)

In (1.1) \((t, x, R)\) denotes the distribution function for the internal configuration and \(u(t, x)\) stands for the velocity of the polymeric liquid, where \(x \in \mathbb{R}^d\) and \(d \geq 2\) means the dimension. Here the polymer elongation \(R\) is bounded in ball \(B = B(0, R_0)\) of \(\mathbb{R}^d\) which means that the extensibility of the polymers is finite. \(\beta = \frac{2k_B T_a}{\lambda}\), where \(k_B\) is the Boltzmann constant, \(T_a\) is the absolute temperature and \(\lambda\) is the friction coefficient. \(\nu > 0\) is the viscosity of the fluid, \(\tau\) is an additional stress tensor and \(P\) is the pressure. The Reynolds number \(Re = \frac{2}{\nu}\) with \(\gamma \in (0, 1)\) and the density \(\rho = \int_B \psi dR\).

Moreover the potential \(\mathcal{U}(R) = -k \log(1 - (\frac{|R|}{|R_0|})^2)\) for some constant \(k > 0\). \(\sigma(u)\) is the drag term. In general, \(\sigma(u) = \nabla u\). For the co-rotation case, \(\sigma(u) = \frac{\nabla u - (\nabla u)^T}{2}\).

This model describes the system coupling fluids and polymers. The system is of great interest in many branches of physics, chemistry, and biology, see \[2, 4\]. In this model, a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring that can be modeled by a vector \(R\). At the level of liquid, the system couples the Navier-Stokes equation for the fluid velocity with a Fokker-Planck equation describing the evolution of the polymer density. This is a micro-macro model (For more details, one can refer to \[2, 4, 9\] and \[10\]).

In the paper we will take \(\beta = 1\), \(\nu = 1\) and \(R_0 = 1\). Notice that \((u, \psi)\) with \(u = 0\) and

\[
\psi_{\infty}(R) = \frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R)} dR} = \frac{(1 - |R|^2)^k}{\int_B (1 - |R|^2)^k dR}.
\]

is a trivial solution of (1.1). By a simple calculation, we can rewrite (1.1) for the following system:

\[
\begin{aligned}
&u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla P = \text{div } \tau, \quad \text{div } u = 0, \\
&\psi_t + (u \cdot \nabla)\psi = \text{div}_R[-\sigma(u) \cdot R\psi + \psi_{\infty} \nabla_R \frac{\psi}{\psi_{\infty}}], \\
&\tau_{ij} = \int_B (R_i \nabla_j \mathcal{U}) \psi dR, \\
&u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0, \\
&\psi_{\infty} \nabla_R \frac{\psi}{\psi_{\infty}} \cdot n = 0 \quad \text{on} \quad \partial B(0, 1).
\end{aligned}
\]

(1.2)

Remark. As in the reference \[10\], one can deduce that \(\psi = 0\) on the boundary.

There are a lot of mathematical results about the FENE dumbbell model. M. Renardy \[11\] established the local well-posedness in Sobolev spaces with potential \(\mathcal{U}(R) = (1 - |R|^2)^{1-\sigma}\) for \(\sigma > 1\). Later, B. Jourdain, T. Lelièvre, and C. Le Bris \[3\] proved local existence of a stochastic differential equation with potential \(\mathcal{U}(R) = -k \log(1 - |R|^2)\) in the case \(k > 3\) for a Couette flow. H. Zhang and P. Zhang \[15\] proved local well-posedness of (1.4) with \(d = 3\) in weighted Sobolev spaces. For the co-rotation
case, F. Lin, P. Zhang, and Z. Zhang [6] obtain a global existence results with \( d = 2 \) and \( k > 6 \). If the initial data is perturbation around equilibrium, N. Masmoudi [9] proved global well-posedness of (1.4) for \( k > 0 \). In the co-rotation case with \( d = 2 \), he [9] obtained a global result for \( k > 0 \) without any small conditions. In the co-rotation case, A. V. Busuioc, I. S. Ciuperca, D. Iftimie and L. I. Palade [3] obtain a global existence result with only the small condition on \( \psi_0 \). The global existence of weak solutions in \( L^2 \) was proved recently by N. Masmoudi [10] under some entropy conditions.

Recently, M. Schonbek [14] studied the \( L^2 \) decay of the velocity for the co-rotation FENE dumbbell model, and obtained the decay rate \((1 + t)^{-\frac{d}{4} + \frac{1}{2}}\), \( d \geq 2 \) with \( u_0 \in L^1 \). Moreover, she conjectured that the sharp decay rate should be \((1 + t)^{-\frac{d}{4}}\), \( d \geq 2 \). However, she failed to get it because she could not use the bootstrap argument as in [12] due to the additional stress tensor. More recently, W. Luo and Z. Yin [8] improved Schonbek’s result and showed that the decay rate is \((1 + t)^{-\frac{d}{4}}\) with \( d \geq 3 \) and \( \ln^{-l}(1 + t) \) with \( d = 2 \) for any \( l \in \mathbb{N}^+ \). This result shows that M. Schonbek’s conjecture is true when \( d \geq 3 \). However, there is no any result to show that M. Schonbek’s conjecture is true when \( d = 2 \).

In this paper, we are going on to prove that M. Schonbek’s conjecture holds true when \( d = 2 \). Firstly, we show that the \( L^2 \) decay rate is \((1 + t)^{-\frac{d}{4}}\), \( d \geq 2 \) for the velocity with the small initial data. The main idea is that we can obtain a estimate for the \( L^3_\infty(L^2) \) norm of the initial data being small in \( L^1 \). Since we are interested in the large time behaviour of the weak solutions, we can consider the evolution system after a large time \( T_0 \). Note that the \( L^2 \) norm of the solutions will decay to zero. Thus we can remove the small \( L^2 \) norm condition of the initial data. And then the main difficult is to get the \( L^1 \)-estimate for the velocity. Since the Leray project operator is not bounded in \( L^1 \), it follows that one can not obtain the \( L^1 \)-estimate directly by the heat kernel estimate. Instead, we will use the Littlewood-Paley theory to estimate the \( B^s_{1,1} \) norm in place of the \( L^1 \) norm. Finally, we can prove that the sharp \( L^2 \) decay rate is \((1 + t)^{-\frac{d}{4}}\), \( d \geq 2 \) for the velocity with the large initial data.

The paper is organized as follows. In Section 2 we introduce some notations and give some preliminaries which will be used in the sequel. In Section 3 we study the \( L^2 \) decay of solutions to the 2D co-rotation FENE model by using the Fourier splitting method and the Littlewood-Paley theory.

## 2 Preliminaries

In this section we will introduce some notations and useful lemmas which will be used in the sequel.

If the function spaces are over \( \mathbb{R}^d \) and \( B \) with respect to the variable \( x \) and \( R \), for simplicity, we drop \( \mathbb{R}^d \) and \( B \) in the notation of function spaces if there is no ambiguity.

For \( p \geq 1 \), we denote by \( \mathcal{L}^p \) the space

\[
\mathcal{L}^p = \left\{ \psi \mid \| \psi \|_{\mathcal{L}^p}^p = \int_{\mathbb{R}^d} \left| \frac{\psi}{\psi \infty} \right|^p dR < \infty \right\}.
\]

We will use the notation \( L^p_x(\mathcal{L}^q) \) to denote \( L^p[\mathbb{R}^d; \mathcal{L}^q] \):

\[
L^p_x(\mathcal{L}^q) = \left\{ \psi \mid \| \psi \|_{L^p_x(\mathcal{L}^q)} = \left( \int_{\mathbb{R}^d} \left( \int_B \left| \frac{\psi}{\psi \infty} \right|^q dR \right)^\frac{p}{q} dx \right)^\frac{1}{p} < \infty \right\}.
\]

When \( p = q \), we also use the short notation \( \mathcal{L}^p \) for \( L^p_x(\mathcal{L}^p) \) if there is no ambiguity.

The symbol \( \hat{f} = \mathcal{F}(f) \) denotes the Fourier transform of \( f \).
Moreover, we denote by $\dot{H}^1$ the space
\[
\dot{H}^1 = \{ g \| g \|_{\dot{H}^1} = (\int_B |\nabla_R g|^2 \psi_\infty dR)^{\frac{1}{2}} \}.
\]
Sometimes we write $f \lesssim g$ instead of $f \leq Cg$, where $C$ is a constant. We agree that $\nabla$ stands for $\nabla_x$ and $\text{div}$ stands for $\text{div}_x$.

The following lemma allows us to estimate the extra stress tensor $\tau$.

**Lemma 2.1.** [9] If $\int_B \psi dR = 0$ and $\int_B \left| \nabla_R \left( \frac{\psi}{\psi_\infty} \right) \right|^2 \psi_\infty dR < \infty$ with $p \geq 2$, then there exists a constant $C$ such that
\[
\int_B \frac{|\psi|^2}{\psi_\infty} dR \leq C \int_B \left| \nabla_R \left( \frac{\psi}{\psi_\infty} \right) \right|^2 \psi_\infty dR.
\]

**Lemma 2.2.** [9] For all $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that
\[
|\tau|^2 \leq \varepsilon \int_B \psi_\infty |\nabla_R \left( \frac{\psi}{\psi_\infty} \right)|^2 dR + C_\varepsilon \int_B \frac{|\psi|^2}{\psi_\infty} dR,
\]
or
\[
|\tau|^2 \leq C \left( \int_B \frac{|\psi|^2}{\psi_\infty} dR \right)^{\frac{1}{2}} \left( \int_B \psi_\infty |\nabla_R \left( \frac{\psi}{\psi_\infty} \right)|^2 dR \right)^{\frac{1}{2}}.
\]

The following lemma is the well-known $(L^p, L^q)$-estimates which can be easily deduced from the properties of the heat kernel.

**Lemma 2.3.** [1] Let $1 \leq p \leq q \leq \infty$. For all $f \in L^p$, there exists a constant $C$ such that
\[
\| e^{-t \Delta} f \|_{L^q} \leq C t^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_{L^p}, \quad \| e^{-t \Delta} \nabla f \|_{L^q} \leq C t^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right) - \frac{1}{4}} \| f \|_{L^p}.
\]

**Theorem 2.4.** [8] Let $(u, \psi)$ be a weak solution of (1.2) with the initial data $u_0 \in L^2 \cap L^1$ and $\psi_0$ satisfies $\psi_0 - \psi_\infty \in L^2_x(L^2)$ and $\int_B \psi_0 = 1$ a.e. in $x$. Then there exists a constant $C$ such that
\[
(2.1) \quad \int_{\mathbb{R}^d \times B} \frac{|\psi - \psi_\infty|^2}{\psi_\infty} dxdR \leq C \exp(-Ct),
\]

\[
(2.2) \quad \|u\|_{L^2} \leq C(1 + t)^{-\frac{4}{d}}, \quad \text{if} \quad d \geq 3, \quad \|u\|_{L^2} \leq C l \ln^{-l}(e + t), \quad \text{if} \quad d = 2,
\]

where $l > 0$ is arbitrarily integer and $C_l$ is a constant dependent on $l$.

## 3 Main results

This section is devoted to investigating the long time behaviour for the velocity of the co-rotation FENE dumbbell model with dimension $d = 2$. More precisely, we prove the $L^2$ decay for the weak solutions of the 2D co-rotation FENE dumbbell model and obtain the $L^2$ decay rate. The existence of the solutions in $L^2$ was established in [7, 14]. Then our main result can be stated as follows.
3.1. The $L^2$ decay with small data

**Theorem 3.1.** Let $(u, \psi)$ be a weak solution of (1.2) with the initial data $u_0 \in L^2 \cap L^1$ and $\psi_0$ satisfies $\psi_0 - \psi_\infty \in L^2_2(C^2)$ and $\int_B \psi_0 = 1$ a.e. in $x$. A constant $\varepsilon$ exists such that if

$$\|u_0\|_{L^2} + \|\psi_0 - \psi_\infty\|_{C^2} \leq \varepsilon,$$

then we have

$$\int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi_\infty|^2}{\psi_\infty} \, dx \, dR \leq C \exp(-Ct),$$

(3.1)

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C(1 + t)^{-\frac{1}{2}},$$

(3.2)

where $C$ is a constant dependent on the initial data.

**Proof.** By the standard density argument, we only need to prove that the conclusion holds for the smooth solution. Since $\psi_\infty = \frac{(1 - |R|^2)^k}{\int_B (1 - |R|^2)^k \, dR} = \frac{(1 - |R|^2)^k}{C_0}$, it follows that

$$\text{div}_R([\nabla u - (\nabla u)^T] \psi_\infty) = \sum_{i,j} \partial_{R_i} \left[ \partial_j u - \partial_j u^i R_j \psi_\infty \right]
= \sum_{i,j} (\partial_i u^j - \partial_j u^i) \delta_{ij} \psi_\infty + \sum_{i,j} 2k (\partial_i u^j - \partial_j u^i) R_i R_j (1 - |R|^2)^{k-1} = 0.$$

(3.3)

By virtue of the second equation of (1.2), we have

$$\frac{\psi - \psi_\infty}{\psi_\infty} \frac{\psi - \psi_\infty}{\psi_\infty} = \text{div}_R \left[ -\sigma(u) \cdot R (\psi - \psi_\infty) + \psi_\infty \nabla_R \frac{\psi - \psi_\infty}{\psi_\infty} \right].$$

(3.4)

Multiplying $\frac{\psi - \psi_\infty}{\psi_\infty}$ by both sides of the above equation and integrating over $B$ with $R$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \int_B \frac{|\psi - \psi_\infty|^2}{\psi_\infty} + \frac{1}{2} u \cdot \nabla_x \int_B \frac{|\psi - \psi_\infty|^2}{\psi_\infty} + \int_B \psi_\infty |\nabla_R \frac{\psi - \psi_\infty}{\psi_\infty}|^2 \right) = \int_B \sigma(u) R (\psi - \psi_\infty) \nabla_R \frac{\psi - \psi_\infty}{\psi_\infty}.$$

(3.5)

Using integration by parts and (3.3), we see that

$$\int_B \sigma(u) R (\psi - \psi_\infty) \nabla_R \frac{\psi - \psi_\infty}{\psi_\infty} = \int_B \sigma(u) R \psi_\infty \left[ \frac{1}{2} \nabla_R \left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)^2 \right] = -\frac{1}{2} \int_B \text{div}_R \left( [\nabla u - (\nabla u)^T] \psi_\infty \right) \left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)^2 = 0.$$

(3.6)

Plugging (3.6) into (3.5) and using the fact that $\text{div} u = 0$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi_\infty|^2}{\psi_\infty} + \int_{\mathbb{R}^2 \times B} \psi_\infty |\nabla_R \left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 \right) = 0.$$

(3.7)

By virtue of the equation (1.2), we have $\int_B \psi_0 dR = \int_B \psi_0 dR = 1$, which leads to $\int_B (\psi - \psi_\infty) dR = 0$. Taking advantage of Lemma 2.1, we infer that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi_\infty|^2}{\psi_\infty} + C \int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi_\infty|^2}{\psi_\infty} \leq 0,$$

(3.8)
which leads to

\[
(3.9) \quad \frac{d}{dt} \left[ \exp(Ct) \int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi\infty|^2}{\psi\infty} \right] \leq 0 \Rightarrow \int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi\infty|^2}{\psi\infty} \leq \exp(-Ct) \int_{\mathbb{R}^2 \times B} \frac{|\psi_0 - \psi\infty|^2}{\psi_\infty}.
\]

Since \( \partial_t \psi\infty = 0 \), it follows that \( \text{div}\tau = \text{div} \int_B (R \otimes \nabla R\psi) dR = \text{div} \int_B (R \otimes \nabla R\psi)(\psi - \psi\infty) dR \). Then, we may assume that \( \tau = \int_B (R \otimes \nabla R\psi)(\psi - \psi\infty) dR \). By the standard energy estimate for the Navier-Stokes equations, we get

\[
(3.10) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = - \int_{\mathbb{R}^2} \tau : \nabla u \leq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\tau\|_{L^2}^2.
\]

Using Lemmas 2.1, 2.2 we verify that

\[
(3.11) \quad \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq \|\tau\|_{L^2}^2 \leq K \int_{\mathbb{R}^2 \times B} \psi\infty |\nabla R(\frac{\psi - \psi\infty}{\psi\infty})|^2.
\]

Let \( \lambda \geq 2K \) be a sufficiently large constant. From the above inequality and (3.7), we deduce that

\[
(3.12) \quad \frac{d}{dt} (\lambda \|\psi - \psi\infty\|_{L^2}^2 + \|u\|_{L^2}^2) + \lambda \int_{\mathbb{R}^2 \times B} \psi\infty |\nabla R(\frac{\psi - \psi\infty}{\psi\infty})|^2 \leq 0.
\]

Taking \( \lambda = 2K \), we have

\[
(3.13) \quad \|u\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + 2K \|\psi_0 - \psi\infty\|_{L^2}^2 < \infty.
\]

From (3.12), we have

\[
(3.14) \quad \frac{d}{dt} ((1 + t)^2 \lambda \|\psi - \psi\infty\|_{L^2}^2 + (1 + t)^2 \|\tilde{u}\|_{L^2}^2) + \lambda (1 + t)^2 \int_{\mathbb{R}^2 \times B} \psi\infty |\nabla R(\frac{\psi - \psi\infty}{\psi\infty})|^2 + (1 + t)^2 \int_{\mathbb{R}^2} |\xi|^2 |\tilde{u}|^2 d\xi
\]
\[
\leq 2(1 + t) \lambda \|\psi - \psi\infty\|_{L^2}^2 + 2(1 + t) \|\tilde{u}\|_{L^2}^2.
\]

Setting \( S(t) = \{ \xi : |\xi|^2 \leq \frac{2}{1 + t} \} \), then we obtain

\[
(3.15) \quad \frac{d}{dt} ((1 + t)^2 \lambda \|\psi - \psi\infty\|_{L^2}^2 + (1 + t)^2 \|\tilde{u}\|_{L^2}^2) + \lambda (1 + t)^2 \int_{\mathbb{R}^2 \times B} \psi\infty |\nabla R(\frac{\psi - \psi\infty}{\psi\infty})|^2 + (1 + t)^2 \int_{\mathbb{R}^2} |\xi|^2 |\tilde{u}|^2 d\xi
\]
\[
\leq 2(1 + t) \lambda \|\psi - \psi\infty\|_{L^2}^2 + 2(1 + t) \int_{S(t)} |\tilde{u}|^2 d\xi.
\]

By virtue of (1.2), we get

\[
(3.16) \quad \tilde{u} = e^{-t|\xi|^2} \tilde{u}_0 + \int_0^t e^{-(t-s)|\xi|^2} \xi \mathcal{F}(\mathbb{P}(u \otimes u) + \mathbb{P}\tau) ds,
\]

where \( \mathbb{P} \) stands for Leray’s project operator. Using the fact that \( |\hat{f}| \leq \|f\|_{L^1} \), we have

\[
|\tilde{u}| \leq e^{-t|\xi|^2} |\tilde{u}_0| + |\xi| \int_0^t \|u\|_{L^2}^2 ds + |\xi| t^\frac{1}{2} (\int_0^t |\tilde{\tau}|^2 ds)^{\frac{1}{2}}
\]
\[
\leq C + |\xi| (t \int_0^t \|u\|_{L^2}^2 ds)^{\frac{1}{2}} + |\xi| t^\frac{1}{2} (\int_0^t |\tilde{\tau}|^2 ds)^{\frac{1}{2}}.
\]
Using the system (1.2), we have

\[ u = e^{-t\Delta}u_0 + \int_0^t e^{-(t-s)\Delta}(\mathbb{P}(u\nabla u) + \mathbb{P} \text{div} \tau)\,ds. \]

Taking advantage of Lemma 2.3, we obtain

\[ \|u\|_{L^2} \leq t^{-\frac{1}{6}}\|u_0\|_{L^\frac{6}{5}} + C \int_0^t (t-s)^{-\frac{1}{7}}\|u\nabla u\|_{L^1} + (t-s)^{-\frac{1}{6}}\|\nabla u\|_{L^2} \,ds \]

\[ \leq t^{-\frac{1}{6}}\|u_0\|_{L^\frac{6}{5}} + C \int_0^t (t-s)^{-\frac{1}{7}}\|u\|_{L^2}\|\nabla u\|_{L^2} + (t-s)^{-\frac{1}{6}}\|\tau\|_{L^2} \,ds. \]

Note that \(1 + \frac{1}{3} = \frac{5}{6} + \frac{1}{2}\). Using the generalized Young inequality, we deduce that

\[ \left( \int_0^t \|u(s)\|_{L^2}^3 \,ds \right)^{\frac{1}{3}} \leq t^{\frac{1}{6}}\|u_0\|_{L^\frac{6}{5}} + C\|s^{-\frac{1}{7}}1_{[0,t]}\|_{L^\frac{6}{5}}\left( \int_0^t \left(\|u(s)\|_{L^2}\|\nabla u(s)\|_{L^2} + \|\tau(s)\|_{L^2}\right) \,ds \right)^{\frac{1}{3}} \]

\[ \leq t^{\frac{1}{6}}\|u_0\|_{L^\frac{6}{5}} + C\left( \int_0^t \|u(s)\|_{L^2}^3 \,ds \right)^{\frac{1}{3}} \left( \int_0^t \|\nabla u(s)\|_{L^2}^2 \,ds \right)^{\frac{1}{3}} + \left( \int_0^t \|\tau(s)\|_{L^2}^2 \,ds \right)^{\frac{1}{3}} \]

\[ \leq t^{\frac{1}{6}}\|u_0\|_{L^\frac{6}{5}} + aC\left( \int_0^t \|u(s)\|_{L^2}^3 \,ds \right)^{\frac{1}{3}} + \left( \int_0^t \|\tau(s)\|_{L^2}^2 \,ds \right)^{\frac{1}{3}}, \]

where \(a = (\|u_0\|_{L^2}^2 + 2K\|\psi_0 - \psi_\infty\|_{L^2}^2)^{\frac{1}{3}}\). Applying Lemma 2.2, we verify that

\[ \left( \int_0^t \|\tau(s)\|_{L^2}^\frac{9}{2} \,ds \right)^{\frac{2}{9}} \leq \left( \int_0^t \|\psi - \psi_\infty\|_{L^2}^\frac{4}{3} \left( \int_{\mathbb{R}^2 \times B} |\nabla R(\frac{\psi - \psi_\infty}{\psi_\infty})^2 \psi_\infty \,dx \,dR \right)^{\frac{1}{2}} \,ds \right)^{\frac{2}{9}} \]

\[ \leq \left( \int_0^t \|\psi - \psi_\infty\|_{L^2}^\frac{4}{3} \right)^{\frac{2}{9}} \left( \int_{\mathbb{R}^2 \times B} |\nabla R(\frac{\psi - \psi_\infty}{\psi_\infty})^2 \psi_\infty \,dx \,dR \right)^{\frac{1}{2}} \]

\[ \leq \|\psi_0 - \psi_\infty\|_{L^2} \left( \int_0^t \exp(-Ct) \,ds \right)^{\frac{1}{2}} \leq C\|\psi_0 - \psi_\infty\|_{L^2}. \]

Plugging (3.21) into (3.20) yields that

\[ \left( \int_0^t \|u(s)\|_{L^2}^3 \,ds \right)^{\frac{1}{3}} \leq C(1+t)^{\frac{1}{6}} + aC\left( \int_0^t \|u(s)\|_{L^2}^3 \,ds \right)^{\frac{1}{3}}. \]

If \(aC \leq \frac{1}{2}\), we then have

\[ \left( \int_0^t \|u(s)\|_{L^2}^3 \,ds \right)^{\frac{1}{3}} \leq C(1+t)^{\frac{1}{6}}. \]

Plugging (3.23) into (3.17) yields that

\[ |\tilde{u}| \leq C + |\xi|(1+t)^{\frac{1}{6}} + |\xi|t^{\frac{1}{6}} \left( \int_0^t |\tilde{\tau}|^2 \,ds \right)^{\frac{1}{2}}, \]

which leads to

\[ \int_{S(t)} |\tilde{u}|^2 \,d\xi \leq \int_{S(t)} d\xi + (1+t)^{\frac{1}{6}} \int_{S(t)} |\xi|^2 \,d\xi + t \int_{S(t)} |\xi|^2 \left( \int_0^t |\tilde{\tau}|^2 \,ds \right) \,d\xi \]

\[ \leq \int_0^t \sqrt{\frac{2t}{1+t}} \,d\tau + (1+t)^{\frac{1}{6}} \int_0^t \sqrt{\frac{2t}{1+t}} \,d\tau \leq \int_0^t \|\|L^2\|_{L^2}^2 \,ds \]

\[ \leq \int_0^t \sqrt{\frac{2t}{1+t}} \,d\tau + (1+t)^{\frac{1}{6}} \int_0^t \sqrt{\frac{2t}{1+t}} \,d\tau \leq \int_0^t \|\|L^2\|_{L^2}^2 \,ds \]

\[ \leq \int_0^t \sqrt{\frac{2t}{1+t}} \,d\tau + (1+t)^{\frac{1}{6}} \int_0^t \sqrt{\frac{2t}{1+t}} \,d\tau \leq \int_0^t \|\|L^2\|_{L^2}^2 \,ds \]
\[ \lesssim (1+t)^{-\frac{1}{2}} + \int_0^t \int_{R^d \times B} \psi_\infty |\nabla R\left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 ds. \]

Plugging (3.25) into (3.15) and using the fact that \( \|\psi - \psi_\infty\|_{L^2} \lesssim \exp(-Ct) \) yield that
\[ (3.26) \quad \frac{d}{dt}((1+t)^2\lambda\|\psi - \psi_\infty\|_{L^2}^2 + (1+t)^2\|\hat{u}\|_{L^2}^2) + \lambda(1+t)^2 \int_{R^d \times B} \psi_\infty |\nabla R\left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 \leq C(1+t)^{\frac{1}{2}} + C(1+t) \int_0^t \int_{R^d \times B} \psi_\infty |\nabla R\left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 ds. \]

By taking \( \lambda \) sufficiently large, we deduce that
\[ (3.27) \quad (1+t)^2\lambda\|\psi - \psi_\infty\|_{L^2}^2 + (1+t)^2\|u\|_{L^2}^2 \lesssim 1 + \int_0^t (1+t')^{\frac{1}{2}} dt' \lesssim (1+t)^{\frac{1}{2}}, \]
which implies that
\[ (3.28) \quad \|u\|_{L^2}^2 \lesssim (1+t)^{-\frac{1}{2}}. \]

From (3.16) we have
\[ (3.29) \quad \|\hat{u}\| \leq e^{-t|\xi|^2}|\hat{u}_0| + |\xi| \int_0^t \|u\|_{L^2}^2 ds + |\xi|t^{\frac{1}{2}} \left( \int_0^t |\hat{\tau}|^2 ds \right)^{\frac{1}{2}} \leq \|u_0\|_{L^1} + C|\xi| \int_0^t (1+s)^{-\frac{1}{2}} ds + |\xi|t^{\frac{1}{2}} \left( \int_0^t |\hat{\tau}|^2 ds \right)^{\frac{1}{2}} \leq \|u_0\|_{L^1} + C|\xi| \int_0^t (1+s)^{-\frac{1}{2}} ds + |\xi|t^{\frac{1}{2}} \left( \int_0^t |\hat{\tau}|^2 ds \right)^{\frac{1}{2}} = \|u_0\|_{L^1} + C|\xi| \sqrt{1+t} + |\xi|t^{\frac{1}{2}} \left( \int_0^t |\hat{\tau}|^2 ds \right)^{\frac{1}{2}}, \]
which leads to
\[ (3.30) \quad \int_{S(t)} |\hat{u}|^2 \leq \int_{S(t)} d\xi + (1+t) \int_{S(t)} |\xi|^2 d\xi + t \int_{S(t)} |\xi|^2 \left( \int_0^t |\hat{\tau}|^2 ds \right) d\xi \leq \int_0^t \frac{t}{1+t} r dr + (1+t) \int_0^t \frac{\sqrt{t}}{1+t} r^3 dr + \frac{2t}{1+t} \int_0^t \|u\|_{L^2}^2 ds \lesssim (1+t)^{-1} + \int_0^t \int_{R^d \times B} \psi_\infty |\nabla R\left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 ds. \]

Plugging (3.30) into (3.15) yields that
\[ (3.31) \quad \frac{d}{dt}((1+t)^2\lambda\|\psi - \psi_\infty\|_{L^2}^2 + (1+t)^2\|\hat{u}\|_{L^2}^2) + \lambda(1+t)^2 \int_{R^d \times B} \psi_\infty |\nabla R\left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 \leq C + C(1+t) \int_0^t \int_{R^d \times B} \psi_\infty |\nabla R\left( \frac{\psi - \psi_\infty}{\psi_\infty} \right)|^2 ds. \]

By taking \( \lambda \) sufficiently large, we get
\[ (3.32) \quad (1+t)^2\lambda\|\psi - \psi_\infty\|_{L^2}^2 + (1+t)^2\|u\|_{L^2}^2 \lesssim 1 + \int_0^t dt' \lesssim (1+t), \]
which implies that
\[ (3.33) \quad \|u\|_{L^2}^2 \lesssim (1+t)^{-1}. \]
3.2. *The $L^2$ decay with large data*

**Theorem 3.2.** Suppose that $p \in [1, \infty]$ and $pk > 1$. Let $(u, \psi)$ be a weak solution of (1.2) with the initial data $u_0 \in L^2 \cap \dot{B}^{0}_{1,1}$ and $\psi_0$ satisfies $\psi_0 - \psi_\infty \in L^2_x(C^2) \cap L^1_x(L^p)$ and $\int_B \psi_0 = 1$ a.e. in $x$. Then there exists a constants such that

$$\int_{\mathbb{R}^2 \times B} \frac{|\psi - \psi_\infty|^2}{\psi_\infty} dx dR \leq C \exp(-Ct),$$

(3.34)

$$\|u\|_{L^2(\mathbb{R}^2)} \leq C(1 + t)^{-\frac{1}{2}}.$$  

(3.35)

In order to prove the above theorem, we need to use the Littlewood-Paley decomposition and some basic lemma for the homogeneous Besov space. (see [1] for more details)

Let $C$ be the annulus \{ $\xi \in \mathbb{R}^d | \frac{3}{4} \leq |\xi| \leq \frac{5}{4}$ \}. There exists radial function $\varphi$, valued in the interval $[0, 1]$, such that

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,$$

(3.36)

$$|j - j'| \geq 2 \Rightarrow \text{Supp} \ \varphi(2^{-j} \xi) \cap \text{Supp} \ \varphi(2^{-j'} \xi) = \emptyset.$$  

(3.37)

The homogeneous dyadic blocks $\tilde{\Delta}_j$ are defined by

$$\tilde{\Delta}_j u = \varphi(2^{-j} D) u = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) u(x - y) dy,$$

(3.38)

$$\tilde{\mathcal{S}}_j u = \chi(2^{-j} D) u = \int_{\mathbb{R}^d} \tilde{h}(2^j y) u(x - y) dy.$$  

(3.39)

The homogeneous Besov space is denoted by $\dot{B}^s_{p,r}$, that is

$$\dot{B}^s_{p,r} = \{ u \in S_{\delta}' \| \| u \|_{B^s_{p,r}} \equiv \| 2^{js} \| D^j u \|_{L^p_x} \| r < \infty \}.$$  

The following lemmas will be useful to obtain the estimates for the solutions of the Navier-Stokes equations.

**Lemma 3.3.** Let $C$ be an annulus and $B$ a ball. A constant $C$ exists such that for any nonnegative integer $k$, any couple $(p, q)$ in $[1, \infty]^2$ with $q \geq p \geq 1$, and any function $u$ of $L^p$, we have

$$\text{Supp} \ \hat{u} \subseteq \lambda B \Rightarrow \| D^k u \|_{L^q} \triangleq \sup_{|\alpha| \leq k} \| \partial^\alpha u \|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p} - \frac{1}{2})} \| u \|_{L^p},$$

$$\text{Supp} \ \hat{u} \subseteq \lambda C \Rightarrow C^{-k-1} \lambda^k \| u \|_{L^p} \leq \| D^k u \|_{L^p} \leq C^{k+1} \lambda^k \| u \|_{L^p},$$

$$\text{Supp} \ \hat{u} \subseteq \lambda C \Rightarrow \| e^{t \Delta} u \|_{L^p} \leq C e^{-ct \lambda^2} \| u \|_{L^p}.$$  

**Lemma 3.4.** For any positive $s$, we have

$$\sup_{t > 0} \sum_j t^{s} 2^{2js} e^{-ct2^j} < \infty.$$  

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Lemma 3.5. Suppose that \( u_0 \in L^2 \cap \dot{B}^{0}_{1,1} \) and \( \tau \in L^2_T(L^2) \cap L^{\infty}_T(L^1) \). If \( u \in L^{\infty}_T(L^2) \cap L^{2}_T(\dot{H}^1) \) is the solution of

\[
\begin{align*}
\left\{ \begin{array}{ll}
 u_t + (u \cdot \nabla)u - \Delta u + \nabla P = \div \tau, & \div u = 0, \\
 u|_{t=0} = u_0,
\end{array} \right.
\end{align*}
\]

then we have

\[
\sup_{t \in [0, T]} \|u\|_{\dot{B}^{0}_{1,1}} \leq \|u_0\|_{\dot{B}^{0}_{1,1}} + C \sqrt{T}(\|u_0\|_{L^2}^2 + \int_0^T \|\tau\|_{L^2}^2 ds + \|\tau\|_{L^{\infty}_T(L^1)}).
\]

Proof. By the standard energy estimate, we have

\[
\|u\|_{L^2}^2 + \frac{1}{2} \int_0^T \|\nabla u\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + \int_0^T \|\tau\|_{L^2}^2 ds.
\]

By virtue of (3.40), we can write that

\[
u = e^{-t\Delta} u_0 + \int_0^t e^{-(t-s)\Delta}(\overline{\mathcal{P}} \div (u \otimes u + \tau)) ds.
\]

Applying \( \hat{\Delta}_j \) to the above equation and taking the \( L^1 \)-norm, we deduce from Lemma 3.3 that

\[
\|\hat{\Delta}_j u\|_{L^1} \leq \|\hat{\Delta}_j u_0\|_{L^1} + \int_0^t e^{-(t-s)2^j} 2^j (\|\hat{\Delta}_j (u \otimes u)\|_{L^1} + \|\hat{\Delta}_j \tau\|_{L^1}) ds
\]

which leads to

\[
\sum_j \|\hat{\Delta}_j u\|_{L^1} \leq \|u_0\|_{\dot{B}^{0}_{1,1}} + \int_0^t (t-s)^{-\frac{j}{2}} \sum_j (t-s)^{\frac{j}{2}} e^{-(t-s)2^j} 2^j (\|u \otimes u\|_{L^1} + \|\tau\|_{L^1}) ds
\]

\[
\leq \|u_0\|_{\dot{B}^{0}_{1,1}} + \int_0^t (t-s)^{-\frac{j}{2}} ds \sup_{t-s > 0} \sum_j (t-s)^{\frac{j}{2}} e^{-(t-s)2^j} 2^j (\|u\|_{L^{\infty}_T(L^2)} + \|\tau\|_{L^{\infty}_T(L^1)})
\]

\[
\leq \|u_0\|_{\dot{B}^{0}_{1,1}} + C \frac{2^j}{2^j} \int_0^T \|\tau\|_{L^2}^2 ds + \|\tau\|_{L^{\infty}_T(L^1)}.
\]

Proof of Theorem 3.2: Multiplying \( p |\psi - \psi|^{p-2} \psi - \psi \) by both sides of (3.3) and integrating over \( B \) with \( R \), we obtain

\[
\frac{d}{dt} \int_B |\psi - \psi|^{p} \psi \psi + u \cdot \nabla x \int_B |\psi - \psi|^{p} \psi + \frac{4(p-1)}{p} \int_B |\psi| \nabla R(\frac{\psi - \psi}{\psi})^2 = \int_B \sigma(u) R(\psi - \psi) \nabla R(\psi - \psi) p.
\]

Using integration by parts and (3.3), we see that

\[
\frac{d}{dt} \int_B |\psi - \psi|^{p} \psi + u \cdot \nabla x \int_B |\psi - \psi|^{p} \psi + \frac{4(p-1)}{p} \int_B |\psi| \nabla R(\frac{\psi - \psi}{\psi})^2 = 0,
\]
which leads to
\[
\frac{d}{dt} \int_B \frac{\psi - \psi_\infty}{\psi_\infty} |p\psi_\infty + u \cdot \nabla \phi| \leq 0.
\]

Since \( \text{div} u = 0 \), it follows that
\[
\|\psi - \psi_\infty\|_{L^1} \leq \|\psi_0 - \psi_\infty\|_{L^1}.
\]

Taking advantage of Hölder’s inequality and using the fact that \( pk > 1 \), we have
\[
|\tau| \leq \int_B \frac{|\psi - \psi_\infty|}{1 + |R|} dR \leq \int_B \frac{(\psi_\infty)^{\frac{p}{p+1}}}{(\psi_\infty)^{\frac{p}{p+1}}} dR \leq C(\int_B |\frac{\psi - \psi_\infty}{\psi_\infty}|^p \psi_\infty dR)^{\frac{1}{p}},
\]
which leads to
\[
\|\tau\|_{L^1} \leq \|\psi - \psi_\infty\|_{L^p} \leq \|\psi_0 - \psi_\infty\|_{L^p}.
\]

By virtue of Lemma 3.5, we deduce that
\[
\sup_{t \in [0,T]} \|u\|_{\dot{B}^0_{1,1}} \leq C_T,
\]
for any \( T < \infty \). By virtue of Theorem 2.4 we see that
\[
\|u\|_{L^2} \leq C \ln^{-1}(e + t), \quad \|\psi - \psi_\infty\|_{L^2} \leq \exp(1) \|u\|_{L^2} T,
\]
which implies that for any \( \varepsilon > 0 \) there exists \( T_0 \) such that
\[
\|u(T_0)\|_{L^1} + \|\psi(T_0) - \psi_\infty\|_{L^2} < \varepsilon.
\]

Since \( \dot{B}^0_{1,1} \hookrightarrow L^1 \), it follows that \( \|u(T_0)\|_{L^1} \leq C T_0 \). Let \( (u(T_0), \psi(T_0)) \) be the initial data. Applying Theorem 3.1 we complete the proof.

Remark 3.6. By Theorem 3.2 together with Theorem 2.4, we see that the conjecture proposed by M. Schonbek in [14] holds true for all \( d \geq 2 \). In [13], M. Schonbek showed that \( (1 + t)^{-\frac{d}{4}} \), \( d \geq 2 \) is the optimal \( L^2 \) decay rate for the Navier-Stokes equations with \( u_0 \in L^1 \). Note that if \( \psi \) is independent on \( x \), then \( \text{div} \tau = 0 \). Then, the co-rotation FENE model is reduced to the Navier-Stokes equations. Thus, the \( L^2 \) decay rate for the co-rotation FENE model which we obtained in Theorem 2.4 and Theorem 3.1 is sharp for all \( d \geq 2 \).

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