ALMOST H-SEMISLANT RIEMANNIAN MAPS

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ABSTRACT. As a generalization of slant Riemannian maps [17], semi-slant Riemannian maps [14], almost h-slant submersions [12], and almost h-semi-slant submersions [13], we introduce the notion of almost h-semi-slant Riemannian maps from almost quaternionic Hermitian manifolds to Riemannian manifolds. We investigate the integrability of distributions, the harmonicity of such maps, the geometry of fibers, etc. We also deal with the condition for such maps to be totally geodesic and study some decomposition theorems. Moreover, we give some examples.

1. INTRODUCTION

Let \( F \) be a \( C^\infty \)-map from a Riemannian manifold \((M, g_M)\) to a Riemannian manifold \((N, g_N)\), according to the conditions on the map \( F \), the map \( F \) is said to be a harmonic map [1], a totally geodesic map [1], an isometric immersion [4], a Riemannian submersion ([11], [18], [8]), a Riemannian map [7], etc. As we know, if we consider the notions of an isometric immersion and a Riemannian submersion as the Riemannian generalization of the notions of an immersion and a submersion, then the notion of a Riemannian map may be the Riemannian generalization of the notion of a subimmersion [7].

Given an isometric immersion \( F \), its research is originated from Gauss' work, which studied surfaces in the Euclidean space \( \mathbb{R}^3 \) and there are lots of papers and books on this topic. In particular, B. Y. Chen introduced and studied some notions: generic submanifolds [2] and slant submanifolds [3]. The notion of generic submanifolds contains the notions of real hypersurfaces, complex submanifolds, totally real submanifolds, anti-holomorphic submanifolds, purely real submanifolds, and CR-submanifolds. And the notion of slant submanifolds has some similarities with the notions of slant submersions [16], semi-slant submersions [15], almost h-slant submersions [12], almost h-semi-slant submersions [13], slant Riemannian maps [17], and semi-slant Riemannian maps [14]. For the Riemannian submersion \( F \), B. O'Neill [11] and A. Gray [9] firstly studied the map \( F \). Since then, there are several kinds of Riemannian submersions ([13], references therein). A. Fischer [7] defined a Riemannian map \( F \), which generalizes and unifies the notions of an isometric immersion, a Riemannian submersion, and an isometry. After that, there are a lot of papers on this topic. Moreover, B. Sahin introduced a slant Riemannian map [17] and the author defined a semi-slant Riemannian map [14]. As a generalization of slant Riemannian maps [17], semi-slant Riemannian maps [14], almost h-slant submersions [12], and almost h-semi-slant submersions [13], we will define an almost h-semi-slant Riemannian map and a h-semi-slant Riemannian map. And
as we know, the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear σ-models with supersymmetry [5].

The paper is organized as follows. In section 2 we recall some notions, which are needed for later use. In section 3 we define the notions of an almost h-semi-slant Riemannian map and a h-semi-slant Riemannian map and obtain some properties on them. In section 4 using both an almost h-semi-slant Riemannian map and a h-semi-slant Riemannian map, we get some decomposition theorems. In section 5 we obtain some examples.

2. Preliminaries

Let \((M, E, g)\) be an almost quaternionic Hermitian manifold, where \(M\) is a \(4m\)-dimensional differentiable manifold, \(g\) is a Riemannian metric on \(M\), and \(E\) is a rank 3 subbundle of \(\text{End}(TM)\) such that for any point \(p \in M\) with its some neighborhood \(U\), there exists a local basis \(\{J_1, J_2, J_3\}\) of sections of \(E\) on \(U\) satisfying for all \(\alpha \in \{1, 2, 3\}\)

\[
J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2},
\]

\[
g(J_\alpha X, J_\alpha Y) = g(X, Y)
\]

for all vector fields \(X, Y \in \Gamma(TM)\), where the indices are taken from \(\{1, 2, 3\}\) modulo 3. The above basis \(\{J_1, J_2, J_3\}\) is said to be a quaternionic Hermitian basis. We call \((M, E, g)\) a quaternionic Kähler manifold if there exist locally defined 1-forms \(\omega_1, \omega_2, \omega_3\) such that for \(\alpha \in \{1, 2, 3\}\)

\[
\nabla_X J_\alpha = \omega_{\alpha+2}(X)J_{\alpha+1} - \omega_{\alpha+1}(X)J_{\alpha+2}
\]

for any vector field \(X \in \Gamma(TM)\), where the indices are taken from \(\{1, 2, 3\}\) modulo 3. If there exists a global parallel quaternionic Hermitian basis \(\{J_1, J_2, J_3\}\) of sections of \(E\) on \(M\), then \((M, E, g)\) is said to be hyperkähler. Furthermore, we call \((J_1, J_2, J_3, g)\) a hyperkähler structure on \(M\) and \(g\) a hyperkähler metric.

Let \((M, g_M)\) and \((N, g_N)\) be Riemannian manifolds, where \(M, N\) are \(C^\infty\)-manifolds and \(g_M, g_N\) are Riemannian metrics on \(M, N\), respectively. Let \(F : (M, g_M) \rightarrow (N, g_N)\) be a \(C^\infty\)-map. We call the map \(F\) a \(C^\infty\)-submersion if \(F\) is surjective and the differential \((F_*)_p\) has a maximal rank for any \(p \in M\). The map \(F\) is said to be a Riemannian submersion \([14]\) if \(F\) is a \(C^\infty\)-submersion and \((F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \rightarrow ((T_{F(p)}N, (g_N)_{F(p)})\) is a linear isometry for each \(p \in M\), where \((\ker(F_*)_p)^\perp\) is the orthogonal complement of the space \(\ker(F_*)_p\) in the tangent space \(T_pM\) of \(M\) at \(p\). We call the map \(F\) a Riemannian map \([17]\) if \((F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \rightarrow ((\text{range}F_*)_F(p), (g_N)_{F(p)})\) is a linear isometry for each \(p \in M\), where \((\text{range}F_*)_F(p) := (F_*)_p((\ker(F_*)_p)^\perp)\) for \(p \in M\).

Let \((M, g_M, J)\) be an almost Hermitian manifold and \((N, g_N)\) a Riemannian manifold, where \(J\) is an almost complex structure on \(M\). Let \(F : (M, g_M, J) \rightarrow (N, g_N)\) be a \(C^\infty\)-map. We call the map \(F\) a slant submersion \([16]\) if \(F\) is a Riemannian submersion and the angle \(\theta = \theta(X)\) between \(JX\) and the space \(\ker(F_*)_p\) is constant for nonzero \(X \in \ker(F_*)_p\) and \(p \in M\).

We call the angle \(\theta\) a slant angle.

The map \(F\) is said to be a semi-slant submersion \([15]\) if \(F\) is a Riemannian submersion and there is a distribution \(D_1 \subset \ker F_*\) such that

\[
\ker F_* = D_1 \oplus D_2, \quad J(D_1) = D_1,
\]
and the angle $\theta = \theta(X)$ between $JX$ and the space $(D_2)_q$ is constant for nonzero $X \in (D_2)_q$ and $q \in M$, where $D_2$ is the orthogonal complement of $D_1$ in $\ker F_*$. We call the angle $\theta$ a semi-slant angle.

We call the map $F$ a \textit{slant Riemannian map [17]} if $F$ is a Riemannian map and the angle $\theta = \theta(X)$ between $JX$ and the space $\ker(F_*)_p$ is constant for nonzero $X \in \ker(F_*)_p$ and $p \in M$. We call the angle $\theta$ a \textit{slant angle}.

The map $F$ is said to be a \textit{semi-slant Riemannian map [14]} if $F$ is a Riemannian map and there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus D_2, \quad J(D_1) = D_1,$$

and the angle $\theta = \theta(X)$ between $JX$ and the space $(D_2)_q$ is constant for nonzero $X \in (D_2)_q$ and $q \in M$, where $D_2$ is the orthogonal complement of $D_1$ in $\ker F_*$. We call the angle $\theta$ a \textit{semi-slant angle}.

Let $(\mathcal{M}, E, g_M)$ be an almost quaternionic Hermitian manifold and $(\mathcal{N}, g_N)$ a Riemannian manifold. A Riemannian submersion $F : (\mathcal{M}, E, g_M) \rightarrow (\mathcal{N}, g_N)$ is said to be an \textit{almost h-slant submersion [12]} if given a point $p \in M$ with its some neighborhood $U$, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of $E$ on $U$ such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between $RX$ and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$. We call such a basis $\{I, J, K\}$ an \textit{almost h-slant basis} and the angles $\{\theta_I, \theta_J, \theta_K\}$ \textit{almost h-slant angles}.

A Riemannian submersion $F : (\mathcal{M}, E, g_M) \rightarrow (\mathcal{N}, g_N)$ is called a \textit{h-slant submersion [12]} if given a point $p \in M$ with its some neighborhood $U$, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of $E$ on $U$ such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between $RX$ and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, and $\theta = \theta_I = \theta_J = \theta_K$.

We call such a basis $\{I, J, K\}$ a \textit{h-slant basis} and the angle $\theta$ a \textit{h-slant angle}.

A Riemannian submersion $F : (\mathcal{M}, E, g_M) \rightarrow (\mathcal{N}, g_N)$ is called a \textit{h-semi-slant submersion [13]} if given a point $p \in M$ with its some neighborhood $U$, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of $E$ on $U$ such that for any $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1 \subset \ker F_*$ on $U$ such that

$$\ker F_* = \mathcal{D}_1 \oplus D_2, \quad R(D_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between $RX$ and the space $(D_2)_q$ is constant for nonzero $X \in (D_2)_q$ and $q \in U$, where $D_2$ is the orthogonal complement of $D_1$ in $\ker F_*$. We call such a basis $\{I, J, K\}$ a \textit{h-semi-slant basis} and the angles $\{\theta_I, \theta_J, \theta_K\}$ \textit{h-semi-slant angles}.

Furthermore, if we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call the map $F : (\mathcal{M}, E, g_M) \rightarrow (\mathcal{N}, g_N)$ a \textit{strictly h-semi-slant submersion}, $\{I, J, K\}$ a \textit{strictly h-semi-slant basis}, and the angle $\theta$ a \textit{strictly h-semi-slant angle}.

A Riemannian submersion $F : (\mathcal{M}, E, g_M) \rightarrow (\mathcal{N}, g_N)$ is called an \textit{almost h-semi-slant submersion [13]} if given a point $p \in M$ with its some neighborhood $U$, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of $E$ on $U$ such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on $U$ such that

$$\ker F_* = \mathcal{D}_1^R \oplus D_2^R, \quad R(D_1^R) = D_1^R,$$
and the angle $\theta_R = \theta_R(X)$ between $RX$ and the space $(\mathcal{D}_q^R)_q$ is constant for nonzero $X \in (\mathcal{D}_q^R)_q$ and $q \in U$, where $\mathcal{D}_q^R$ is the orthogonal complement of $\mathcal{D}_q^R$ in ker $F_s$.

We call such a basis $\{I, J, K\}$ an almost h-semi-slant basis and the angles $\{\theta_I, \theta_J, \theta_K\}$ almost h-semi-slant angles.

Let $(M, E_M, g_M)$ and $(N, E_N, g_N)$ be almost quaternionic Hermitian manifolds. A map $F: M \rightarrow N$ is called a $(E_M, E_N)$-holomorphic map if given a point $x \in M$, for any $J \in (E_M)_x$ there exists $J' \in (E_N)_{F(x)}$ such that

(2.4) $F^* \circ J = J' \circ F^*$.

A Riemannian submersion $F: M \rightarrow N$ which is a $(E_M, E_N)$-holomorphic map is called a quaternionic submersion. Moreover, if $(M, E_M, g_M)$ is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that $F$ is a quaternionic Kähler submersion (or a hyperkähler submersion) \[10\].

Let $F: (M, g_M) \rightarrow (N, g_N)$ be a $C^\infty$-map. The second fundamental form of $F$ is given by

(2.5) $(\nabla F_e)(X, Y) := \nabla^N_X F_e Y - F_e(\nabla_X Y)$ for $X, Y \in \Gamma(TM)$,

where $\nabla^F$ is the pullback connection and we denote conveniently by $\nabla$ the Levi-Civita connections of the metrics $g_M$ and $g_N$ \[1\]. Recall that $F$ is said to be harmonic if we have the tension field $\tau(F) := \text{trace}(\nabla F_e) = 0$ and we call the map $F$ a totally geodesic map if $(\nabla F_e)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ \[1\]. Denote the range of $F_e$ by range$F_e$ as a subset of the pullback bundle $F^{-1}TN$. With its orthogonal complement $(\text{range} F_e)^\perp$ we obtain the following decomposition

(2.6) $F^{-1}TN = \text{range} F_e \oplus (\text{range} F_e)^\perp$.

Moreover, we have

(2.7) $TM = \ker F_e \oplus (\ker F_e)^\perp$.

Then we easily get

**Lemma 2.1.** Let $F$ be a Riemannian map from a Riemannian manifold $(M, g_M)$ to a Riemannian manifold $(N, g_N)$. Then

(2.8) $(\nabla F_e)(X, Y) \in \Gamma((\text{range} F_e)^\perp)$ for $X, Y \in \Gamma((\ker F_e)^\perp)$.

**Lemma 2.2.** Let $F$ be a Riemannian map from a Riemannian manifold $(M, g_M)$ to a Riemannian manifold $(N, g_N)$. Then the map $F$ satisfies a generalized eikonal equation \[7\]

(2.9) $2e(F) = ||F_e||^2 = \text{rank} F$.

As we can see, $||F_e||^2$ is continuous on $M$ and rank $F$ is an integer-valued function on $M$ so that rank $F$ is locally constant. Hence, if $M$ is connected, then rank $F$ is a constant function \[7\]. In \[7\], A. Fischer suggested that using (2.9), we may build a quantum model of nature. And if we can do it, then there will be an interesting relationship between the mathematical side from Riemannian maps, harmonic maps, and Lagrangian field theory and the physical side from Maxwell’s equation, Schrödinger’s equation, and their proposed generalization.
ALMOST H-SEMISLANT RIEMANNIAN MAPS

3. ALMOST H-SEMISLANT RIEMANNIAN MAPS

Definition 3.1. Let \((M, E, g_M)\) be an almost quaternionic Hermitian manifold and \((N, g_N)\) a Riemannian manifold. A Riemannian map \(F : (M, E, g_M) \rightarrow (N, g_N)\) is called a h-semi-slant Riemannian map if given a point \(p \in M\) with its some neighborhood \(U\), there exists a quaternionic Hermitian basis \(\{I, J, K\}\) of sections of \(E\) on \(U\) such that for any \(R \in \{I, J, K\}\), there is a distribution \(D_1 \subset \ker F_*\) on \(U\) such that

\[
\ker F_* = D_1 \oplus D_2, \quad R(D_1) = D_1,
\]

and the angle \(\theta_R = \theta_R(X)\) between \(RX\) and the space \((D_2)_q\) is constant for nonzero \(X \in (D_2)_q\) and \(q \in U\), where \(D_2\) is the orthogonal complement of \(D_1\) in \(\ker F_*\).

We call such a basis \(\{I, J, K\}\) a h-semi-slant basis and the angles \(\{\theta_I, \theta_J, \theta_K\}\) h-semi-slant angles.

Furthermore, if we have

\[
\theta = \theta_I = \theta_J = \theta_K,
\]

then we call the map \(F : (M, E, g_M) \rightarrow (N, g_N)\) a strictly h-semi-slant Riemannian map, \(\{I, J, K\}\) a strictly h-semi-slant basis, and the angle \(\theta\) a strictly h-semi-slant angle.

Definition 3.2. Let \((M, E, g_M)\) be an almost quaternionic Hermitian manifold and \((N, g_N)\) a Riemannian manifold. A Riemannian map \(F : (M, E, g_M) \rightarrow (N, g_N)\) is called an almost h-semi-slant Riemannian map if given a point \(p \in M\) with its some neighborhood \(U\), there exists a quaternionic Hermitian basis \(\{I, J, K\}\) of sections of \(E\) on \(U\) such that for each \(R \in \{I, J, K\}\), there is a distribution \(D_1^R \subset \ker F_*\) on \(U\) such that

\[
\ker F_* = D_1^R \oplus D_2^R, \quad R(D_1^R) = D_1^R,
\]

and the angle \(\theta_R = \theta_R(X)\) between \(RX\) and the space \((D_2^R)_q\) is constant for nonzero \(X \in (D_2^R)_q\) and \(q \in U\), where \(D_2^R\) is the orthogonal complement of \(D_1^R\) in \(\ker F_*\).

We call such a basis \(\{I, J, K\}\) an almost h-semi-slant basis and the angles \(\{\theta_I, \theta_J, \theta_K\}\) almost h-semi-slant angles.

Let \(F : (M, E, g_M) \rightarrow (N, g_N)\) be an almost h-semi-slant Riemannian map. Then given a point \(p \in M\) with its some neighborhood \(U\), there exists a quaternionic Hermitian basis \(\{I, J, K\}\) of sections of \(E\) on \(U\) such that for each \(R \in \{I, J, K\}\), there is a distribution \(D_1^R \subset \ker F_*\) on \(U\) such that

\[
\ker F_* = D_1^R \oplus D_2^R, \quad R(D_1^R) = D_1^R,
\]

and the angle \(\theta_R = \theta_R(X)\) between \(RX\) and the space \((D_2^R)_q\) is constant for nonzero \(X \in (D_2^R)_q\) and \(q \in U\), where \(D_2^R\) is the orthogonal complement of \(D_1^R\) in \(\ker F_*\).

Then for \(X \in \Gamma(\ker F_*),\) we write

\[
X = P_RX + Q_RX,
\]

where \(P_RX \in \Gamma(D_1^R)\) and \(Q_RX \in \Gamma(D_2^R)\).

For \(X \in \Gamma(\ker F_*),\) we obtain

\[
RX = \phi_RX + \omega_RX,
\]

where \(\phi_RX \in \Gamma(\ker F_*)\) and \(\omega_RX \in \Gamma((\ker F_*)^\perp)\).

For \(Z \in \Gamma((\ker F_*)^\perp),\) we have

\[
RZ = B_RZ + C_RZ,
\]
where $B_{R}Z \in \Gamma(\ker F_*)$ and $C_{R}Z \in \Gamma((\ker F_*)^\perp)$.

For $U \in \Gamma(TM)$, we get

\[ U = U' + U'' \]

where $U' \in \Gamma(\ker F_*)$ and $U'' \in \Gamma((\ker F_*)^\perp)$.

For $W \in \Gamma(F^{-1}TN)$, we write

\[ W = PW + QW, \]

where $PW \in \Gamma(\text{range}F^*)$ and $QW \in \Gamma((\text{range}F_*)^\perp)$.

Then

\[ (\ker F_*)^\perp = \omega_{R}D_{2}R^\perp \oplus \mu_{R}, \]

where $\mu_{R}$ is the orthogonal complement of $\omega_{R}D_{2}R^\perp$ in $(\ker F_*)^\perp$ and is invariant under $R$.

Furthermore,

\[ \phi_{R}D_{2}1^R = D_{1}^R, \quad \omega_{R}D_{2}1^R = 0, \quad \phi_{R}D_{2}2^R \subset D_{2}2^R, \quad B_{R}(\ker F_*)^\perp = D_{2}R^R \]

\[ \phi_{R}^2 + B_{R}\omega_{R} = -id, \quad C_{R} + \omega_{R}B_{R} = -id, \quad \omega_{R}\phi_{R} + C_{R}\omega_{R} = 0, \quad B_{R}C_{R} + \phi_{R}B_{R} = 0. \]

Define the tensors $T$ and $A$ by

\[ A_EF = H\nabla_{HE}VF + V\nabla_{HE}HF \]

\[ T_EF = H\nabla_{VE}VF + V\nabla_{VE}HF \]

for $E, F \in \Gamma(TM)$, where $\nabla$ is the Levi-Civita connection of $g_M$.

For $X, Y \in \Gamma(\ker F_*)$, define

\[ \widehat{\nabla}_X Y := \nabla_X Y \]

\[ (\nabla_X \phi)Y := \widehat{\nabla}_X \phi Y - \phi \widehat{\nabla}_X Y \]

\[ (\nabla_X \omega)Y := H\nabla_X \omega Y - \omega \widehat{\nabla}_X Y. \]

Then we easily obtain

**Lemma 3.3.** Let $F$ be an almost h-semi-slant Riemannian map from a hyperkähler manifold $(M, I, J, K, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-semi-slant basis. Then we get

1. \[ \widehat{\nabla}_X \phi Y + T_X\omega Y = \phi R \widehat{\nabla}_X Y + B_R T_X Y \]
   \[ T_X\phi Y + H\nabla_X \omega Y = \omega R \widehat{\nabla}_X Y + C_R T_X Y \]
   for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.
2. \[ \nabla_Z B_{R}W + A_ZC_{R}W = \phi_{R}A_ZW + B_{R}H\nabla_Z W \]
   \[ A_Z B_{R}W + H\nabla_Z C_{R}W = \omega_{R}A_ZW + C_{R}H\nabla_Z W \]
   for $Z, W \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$.  


(3)
\[
\hat{\nabla} X B_R Z + T_X C_R Z = \phi_R T_X Z + B_R \nabla_X Z \\
T_X B_R Z + \nabla_X C_R Z = \omega_R T_X Z + C_R \nabla_X Z \\
\nabla_Y \omega_R X + A_Z \omega_R X = \phi_R \nabla_Z X + B_R A_Z X \\
A_Z \phi_R X + \nabla_Z \omega_R X = \omega_R \nabla_Z X + C_R A_Z X
\]

for \( X \in \Gamma(\ker F_\ast) \), \( Z \in \Gamma((\ker F_\ast)^\perp) \), and \( R \in \{I, J, K\} \).

Using the h-semi-slant Riemannian map \( F \), we have

**Theorem 3.4.** Let \( F \) be a h-semi-slant Riemannian map from a hyperkähler manifold \((M, I, J, K, g_M)\) to a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is a h-semi-slant basis. Then the following conditions are equivalent:

a) the complex distribution \( D_1 \) is integrable.

b) \( Q_f(\hat{\nabla}_X \phi_I Y - \hat{\nabla}_Y \phi_I X) = 0 \) and \( T_X \phi_I Y = T_Y \phi_I X \) for \( X, Y \in \Gamma(D_1) \).

c) \( Q_f(\hat{\nabla}_X \phi_J Y - \hat{\nabla}_Y \phi_J X) = 0 \) and \( T_X \phi_J Y = T_Y \phi_J X \) for \( X, Y \in \Gamma(D_1) \).

d) \( Q_f(\hat{\nabla}_X \phi_K Y - \hat{\nabla}_Y \phi_K X) = 0 \) and \( T_X \phi_K Y = T_Y \phi_K X \) for \( X, Y \in \Gamma(D_1) \).

**Proof.** Given \( X, Y \in \Gamma(D_1) \) and \( R \in \{I, J, K\} \), we obtain

\[
\begin{align*}
R[X, Y] &= R(\nabla_X Y - \nabla_Y X) = \nabla_X RY - \nabla_Y RX \\
&= \hat{\nabla}_X \phi_R Y - \hat{\nabla}_Y \phi_R X + T_X \phi_R Y - T_Y \phi_R X.
\end{align*}
\]

Hence, we get

\[ a) \iff b), \quad a) \iff c), \quad a) \iff d). \]

Therefore, the result follows. \qed

**Theorem 3.5.** Let \( F \) be a h-semi-slant Riemannian map from a hyperkähler manifold \((M, I, J, K, g_M)\) to a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is a h-semi-slant basis. Then the following conditions are equivalent:

a) the slant distribution \( D_2 \) is integrable.

b) \( P_f(\hat{\nabla}_X \phi_I Y - \hat{\nabla}_Y \phi_I X + T_X \omega_I Y - T_Y \omega_I X) = 0 \) for \( X, Y \in \Gamma(D_2) \).

c) \( P_f(\hat{\nabla}_X \phi_J Y - \hat{\nabla}_Y \phi_J X + T_X \omega_J Y - T_Y \omega_J X) = 0 \) for \( X, Y \in \Gamma(D_2) \).

d) \( P_f(\hat{\nabla}_X \phi_K Y - \hat{\nabla}_Y \phi_K X + T_X \omega_K Y - T_Y \omega_K X) = 0 \) for \( X, Y \in \Gamma(D_2) \).

**Proof.** Given \( X, Y \in \Gamma(D_2) \), \( Z \in \Gamma(D_1) \), and \( R \in \{I, J, K\} \), we obtain

\[
g_M(R[X, Y], Z) = g_M(\nabla_X RY - \nabla_Y RX, Z)
\]

\[
= g_M(\hat{\nabla}_X \phi_R Y + T_X \phi_I Y + T_X \omega_R Y + \nabla_X \omega_R Y - \hat{\nabla}_Y \phi_R X - T_Y \phi_I X - T_Y \omega_R X - \nabla_Y \omega_R X, Z)
\]

\[
= g_M(\hat{\nabla}_X \phi_R Y + T_X \omega_R Y - \hat{\nabla}_Y \phi_R X - T_Y \omega_R X, Z).
\]

Since \([X, Y] \in \Gamma(\ker F_\ast)\), we get

\[ a) \iff b), \quad a) \iff c), \quad a) \iff d). \]

Therefore, we have the result. \qed

In the same way with the proof of Proposition 2.6 in [13], we can show
Proposition 3.6. Let $F$ be an almost h-semi-slant Riemannian map from an almost quaternionic Hermitian manifold $(M, E, g_M)$ to a Riemannian manifold $(N, g_N)$. Then we have

$$\phi_R^2 X = -\cos^2 \theta_R X \quad \text{for } X \in \Gamma(D_2^R) \quad \text{and } R \in \{I, J, K\},$$

where \(\{I, J, K\}\) is an almost h-semi-slant basis with the almost h-semi-slant angles \(\{\theta_I, \theta_J, \theta_K\}\).

Remark 3.7. In particular, it is easy to obtain that the converse of Proposition 3.6 is also true.

Since $g_M(\phi_R X, \phi_R Y) = \cos^2 \theta_R g_M(X, Y)$ and $g_M(\omega_R X, \omega_R Y) = \sin^2 \theta_R g_M(X, Y)$ for $X, Y \in \Gamma(D_2^R)$, if $\theta_R \in (0, \frac{\pi}{2})$, then we can locally choose an orthonormal frame \(\{f_1, \sec \theta_R \phi_R f_1, \ldots, f_s, \sec \theta_R \phi_R f_s\}\) of $D_2^R$.

Using (3.12), in a similar way with Lemma 3.5 in [14], we can obtain Lemma 3.8.

Let $F$ be an almost h-semi-slant Riemannian map from a hyperkähler manifold $(M, I, J, K, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-semi-slant basis with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$. If the tensor $\omega_R$ is parallel, then we have

$$T_{\phi_R X} \phi_R X = -\cos^2 \theta_R \cdot T_X X \quad \text{for } X \in \Gamma(D_2^R),$$

where $R \in \{I, J, K\}$.

Given an almost h-semi-slant Riemannian map $F$ from an almost quaternionic Hermitian manifold $(M, E, g_M)$ to a Riemannian manifold $(N, g_N)$, for some $R \in \{I, J, K\}$ with $\theta_R \in (0, \frac{\pi}{2})$, we can define an endomorphism $\widehat{R}$ of ker $F_*$ by

$$\widehat{R} := RP_R + \sec \theta_R \phi_R Q_R.$$

Then

$$\widehat{R}^2 = -id \text{ on } \ker F_*.$$

Note that the distribution ker $F_*$ is integrable. But its dimension may be odd. With the endomorphism $\widehat{R}$ we get

Theorem 3.9. Let $F$ be an almost h-semi-slant Riemannian map from an almost quaternionic Hermitian manifold $(M, E, g_M)$ to a Riemannian manifold $(N, g_N)$ with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$ not all $\frac{\pi}{2}$. Then the fibers $F^{-1}(x)$ are even dimensional submanifolds of $M$ for $x \in M$.

Now, we consider the harmonicity of such maps. Let $F$ be a $C^\infty$-map from a Riemannian manifold $(M, g_M)$ to a Riemannian manifold $(N, g_N)$. We can canonically define a function $e(F) : M \mapsto [0, \infty)$ given by

$$e(F)(x) := \frac{1}{2}|(F_*)_x|^2, \quad x \in M,$$

where $|(F_*)_x|$ denotes the Hilbert-Schmidt norm of $(F_*)_x$. Then the function $e(F)$ is said to be the energy density of $F$. Let $D$ be a compact domain of $M$, i.e.,
$D$ is the compact closure $\bar{U}$ of a non-empty connected open subset $U$ of $M$. The
energy integral of $F$ over $D$ is the integral of its energy density:

\begin{equation}
E(F; D) := \int_D e(F)v_{g_M} = \frac{1}{2} \int_D |F_v|^2 v_{g_M},
\end{equation}

where $v_{g_M}$ is the volume form on $(M, g_M)$. Let $C^\infty(M, N)$ denote the space of all $C^\infty$-maps from $M$ to $N$. A $C^\infty$-map $F : M \to N$ is said to be harmonic if it is a critical point of the energy functional $E(\cdot, D) : C^\infty(M, N) \to \mathbb{R}$ for any compact domain $D \subset M$. By the result of J. Eells and J. Sampson [6], we see that the map $F$ is harmonic if and only if the tension field $\tau(F) := \text{trace} \nabla F_* = 0$.

**Theorem 3.10.** Let $F$ be an almost h-semi-slant Riemannian map from a hyperkähler manifold $(M, I, J, K, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-semi-slant basis. Assume that $\bar{H} = 0$, where $\bar{H}$ denotes the mean curvature vector field of range$F_*$. Then each of the following conditions implies that $F$ is harmonic.

1. $D^I_1$ is integrable and $\text{trace} \nabla F_* = 0$ on $D^I_1$.
2. $D^J_1$ is integrable and $\text{trace} \nabla F_* = 0$ on $D^J_1$.
3. $D^K_1$ is integrable and $\text{trace} \nabla F_* = 0$ on $D^K_1$.

**Proof.** Using Lemma 2.1, we get $\text{trace} \nabla F_*|_{\ker F_*} \in \Gamma(\text{range} F_*)$ and $\text{trace} \nabla F_*|_{\ker F_*} \perp \in \Gamma(\text{range} F_*) \perp$ so that from (2.7), we have

\[ \text{trace} \nabla F_* = 0 \iff \text{trace} \nabla F_*|_{\ker F_*} = 0 \text{ and } \text{trace} \nabla F_*|_{\ker F_*} \perp = 0. \]

Moreover, we easily obtain

\[ \text{trace} \nabla F_*|_{\ker F_*} \perp = l\bar{H} \quad \text{for } l := \dim(\ker F_*)^\perp \]

so that

\[ \text{trace} \nabla F_*|_{\ker F_*} \perp = 0 \iff \bar{H} = 0. \]

Given $R \in \{I, J, K\}$, since $D^K_1 = R(D^I_1)$, we can choose locally an orthonormal frame $\{e_1, Re_1, \cdots, e_k, Re_k\}$ of $D^K_1$ so that

\[
(\nabla F_*)(Re_i, Re_i) = -F_*\nabla Re_i Re_i = -F_*(\nabla e_i, Re_i + [Re_i, e_i]) = F_*\nabla e_i e_i - F_* R[Re_i, e_i] = -F_* R e_i, e_i - F_* R[Re_i, e_i]
\]

for $1 \leq i \leq k$.

Thus,

$D^I_1$ is integrable $\implies$ $\text{trace} \nabla F_*|_{D^I_1} = 0.$

Since $D^I_2$ is the orthogonal complement of $D^I_1$ in $\ker F_*$, we have the result. \hfill $\square$

Using Lemma 3.8, we obtain

**Corollary 3.11.** Let $F$ be an almost h-semi-slant Riemannian map from a hyperkähler manifold $(M, I, J, K, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-semi-slant basis with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$. Assume that $\bar{H} = 0$. Then each of the following conditions implies that $F$ is harmonic.

1. $D^I_1$ is integrable, the tensor $\omega_I$ is parallel, and $\theta_I \in \left[0, \frac{\pi}{2}\right)$.
2. $D^J_1$ is integrable, the tensor $\omega_J$ is parallel, and $\theta_J \in \left[0, \frac{\pi}{2}\right)$.
3. $D^K_1$ is integrable, the tensor $\omega_K$ is parallel, and $\theta_K \in \left[0, \frac{\pi}{2}\right)$.

We now investigate the condition for such a map $F$ to be totally geodesic.
Theorem 3.12. Let $F$ be an almost h-semi-slant Riemannian map from a hyperkähler manifold $(M, I, J, K, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-semi-slant basis. Assume that $\bar{Q}(\nabla^F_{Z_1} F, Z_2) = 0$ for $Z_1, Z_2 \in \Gamma((\ker F)\perp)$. Then the following conditions are equivalent:

a) $F$ is a totally geodesic map.

\[ \omega_I(\nabla_X \phi Y + T_X \omega Y) + C_I(T_X \phi Y + \mathcal{H}_X \omega Y) = 0 \]
\[ \omega_I(\nabla_X B_I Z + T_X C_I Z) + C_I(T_X B_I Z + \mathcal{H}_X C_I Z) = 0 \]

for $X, Y \in \Gamma(\ker F_\ast)$ and $Z \in \Gamma((\ker F_\ast)\perp)$.

b) $\omega_J(\nabla_X \phi J Y + T_X \omega J Y) + C_J(T_X \phi J Y + \mathcal{H}_X \omega J Y) = 0$
\[ \omega_J(\nabla_X B_J Z + T_X C_J Z) + C_J(T_X B_J Z + \mathcal{H}_X C_J Z) = 0 \]

for $X, Y \in \Gamma(\ker F_\ast)$ and $Z \in \Gamma((\ker F_\ast)\perp)$.

c) $\omega_K(\nabla_X \phi K Y + T_X \omega K Y) + C_K(T_X \phi K Y + \mathcal{H}_X \omega K Y) = 0$
\[ \omega_K(\nabla_X B_K Z + T_X C_K Z) + C_K(T_X B_K Z + \mathcal{H}_X C_K Z) = 0 \]

for $X, Y \in \Gamma(\ker F_\ast)$ and $Z \in \Gamma((\ker F_\ast)\perp)$.

Proof. If $Z_1, Z_2 \in \Gamma((\ker F_\ast)\perp)$, then by Lemma 2.1 we get
\[ (\nabla F_\ast)(Z_1, Z_2) = 0 \iff \bar{Q}(\nabla F_\ast)(Z_1, Z_2) = \bar{Q}(\nabla^F_{Z_1} F, Z_2) = 0. \]

For $X, Y \in \Gamma(\ker F_\ast)$, we obtain
\[ (\nabla F_\ast)(X, Y) = -F_\ast(\nabla_X Y) = F_\ast(I \nabla_X (\phi I Y + \omega I Y)) \]
\[ = F_\ast(\phi_I \nabla_X \phi_I Y + \omega_I \nabla_X \phi_I Y + B_I T_X \phi_I Y + C_I T_X \phi_I Y + \phi_I T_X \omega_I Y + \omega_I T_X \omega_I Y + B_I \mathcal{H}_X \omega_I Y + C_I \mathcal{H}_X \omega_I Y). \]

Thus,
\[ (\nabla F_\ast)(X, Y) = 0 \iff \omega_I(\nabla_X \phi I Y + T_X \omega I Y) + C_I(T_X \phi I Y + \mathcal{H}_X \omega I Y) = 0. \]

Given $X \in \Gamma(\ker F_\ast)$ and $Z \in \Gamma((\ker F_\ast)\perp)$, since $(\nabla F_\ast)(X, Z) = (\nabla F_\ast)(Z, X)$, it is sufficient to consider the following case:
\[ (\nabla F_\ast)(X, Z) = -F_\ast(\nabla_X Z) = F_\ast(I \nabla_X (B_I Z + C_I Z)) \]
\[ = F_\ast(\phi_I \nabla_X B_I Z + \omega_I \nabla_X B_I Z + B_I T_X B_I Z + C_I T_X B_I Z + \phi_I T_X C_I Z + \omega_I T_X C_I Z + C_I \mathcal{H}_X C_I Z + B_I \mathcal{H}_X C_I Z) \]
\[ \text{so that} \]
\[ (\nabla F_\ast)(X, Z) = 0 \iff \omega_I(\nabla_X B_I Z + T_X C_I Z) + C_I(T_X B_I Z + \mathcal{H}_X C_I Z) = 0. \]

Hence,
\[ \text{a) } \iff \text{b).} \]

Similarly,
\[ \text{a) } \iff \text{c) } \quad \text{and} \quad \text{a) } \iff \text{d).} \]

Therefore, we get the result. \qed
Let \( F : (M, g_M) \mapsto (N, g_N) \) be a Riemannian map. The map \( F \) is called a Riemannian map with \textit{totally umbilical fibers} if
\[
T_XY = g_M(X,Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*),
\]
where \( H \) is the mean curvature vector field of the fiber.

In a similar way with Lemma 2.17 in [13], we have

**Lemma 3.13.** Let \( F \) be an almost h-semi-slant Riemannian map with totally umbilical fibers from a hyperkähler manifold \( (M, I, J, K, g_M) \) to a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost h-semi-slant basis. Then we get
\[
(3.18) \quad H \in \Gamma(\omega_R D^R_2) \quad \text{for } R \in \{I, J, K\}.
\]

Using Lemma 3.13, we obtain

**Corollary 3.14.** Let \( F \) be an almost h-semi-slant Riemannian map with totally umbilical fibers from a hyperkähler manifold \( (M, I, J, K, g_M) \) to a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost h-semi-slant basis with the almost h-semi-slant angles \( \{\theta_I, \theta_J, \theta_K\} \). Assume that \( \theta_R = 0 \) for some \( R \in \{I, J, K\} \). Then the fibers of \( F \) are minimal submanifolds of \( M \).

### 4. Decomposition theorems

Let \( (M, g_M) \) be a Riemannian manifold and \( D \) a \((C^\infty)\)-distribution on \( M \). The distribution \( D \) is said to be \textit{autoparallel} (or a \textit{totally geodesic foliation}) if \( \nabla_X Y \in \Gamma(D) \) for \( X, Y \in \Gamma(D) \). Given an autoparallel distribution \( D \) on \( M \), it is easy to see that \( D \) is integrable and its leaves are totally geodesic in \( M \). Moreover, we call the distribution \( D \) \textit{parallel} if \( \nabla_Z Y \in \Gamma(D) \) for \( Y \in \Gamma(D) \) and \( Z \in \Gamma(TM) \). Given a parallel distribution \( D \) on \( M \), we easily obtain that its orthogonal complementary distribution \( D^\perp \) is also parallel. In this case, \( M \) is locally a Riemannian product manifold of the leaves of \( D \) and \( D^\perp \). We can also obtain that if the distributions \( D \) and \( D^\perp \) are simultaneously autoparallel, then they are also parallel. Using this fact, we have

**Theorem 4.1.** Let \( F \) be an almost h-semi-slant Riemannian map from a hyperkähler manifold \( (M, I, J, K, g_M) \) to a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost h-semi-slant basis. Then the following conditions are equivalent:

a) \((M, g_M)\) is locally a Riemannian product manifold of the leaves of \( \ker F_* \) and \((\ker F_*)^\perp\)

b) \[
\omega_1(\hat{\nabla}_X \phi_I Y + T_X \omega_I Y) + C_I(\hat{\nabla}_X \phi_I Y + \mathcal{H} \nabla_X \omega_I Y) = 0 \quad \text{for } X, Y \in \Gamma(\ker F_*),
\]
\[
\phi_1(\hat{\nabla}_X B_I W + A_Z C_I W) + B_I(A_Z B_I W + \mathcal{H} \nabla_Z C_I W) = 0 \quad \text{for } Z, W \in \Gamma((\ker F_*)^\perp).
\]

c) \[
\omega_J(\hat{\nabla}_X \phi_J Y + T_X \omega_J Y) + C_J(\hat{\nabla}_X \phi_J Y + \mathcal{H} \nabla_X \omega_J Y) = 0 \quad \text{for } X, Y \in \Gamma(\ker F_*),
\]
\[
\phi_J(\hat{\nabla}_X B_J W + A_Z C_J W) + B_J(A_Z B_J W + \mathcal{H} \nabla_Z C_J W) = 0 \quad \text{for } Z, W \in \Gamma((\ker F_*)^\perp).
\]

d) \[
\omega_K(\hat{\nabla}_X \phi_K Y + T_X \omega_K Y) + C_K(\hat{\nabla}_X \phi_K Y + \mathcal{H} \nabla_X \omega_K Y) = 0 \quad \text{for } X, Y \in \Gamma(\ker F_*),
\]
\[
\phi_K(\hat{\nabla}_X B_K W + A_Z C_K W) + B_K(A_Z B_K W + \mathcal{H} \nabla_Z C_K W) = 0 \quad \text{for } Z, W \in \Gamma((\ker F_*)^\perp).
\]
Proof. Given $R \in \{I, J, K\}$, for $X, Y \in \Gamma(\ker F_*)$, we get

$$\nabla_X Y = -R\nabla_X R Y = -R(\tilde{\nabla}_X \phi_R Y + \nabla_X \omega_R Y + H\nabla_X \omega_R Y)$$

$$= -(\phi_R \nabla_X \phi_R Y + \omega_R \nabla_X \phi_R Y + B_R \nabla_X \phi_R Y + C_R \nabla_X \phi_R Y + \phi_R \nabla_X \omega_R Y + \omega_R \nabla_X \omega_R Y + B_R H\nabla_X \omega_R Y + C_R H\nabla_X \omega_R Y).$$

Thus,

$$\nabla_X Y \in \Gamma(\ker F_*) \iff \omega_R(\tilde{\nabla}_X \phi_R Y + \nabla_X \omega_R Y) + C_R(\nabla_X \phi_R Y + H\nabla_X \omega_R Y) = 0.$$ 

For $Z, W \in \Gamma((\ker F_*)^\perp)$, we have

$$\nabla_Z W = -R\nabla_Z R W = -R(V\nabla_Z B_R W + A_Z B_R W + A_Z C_R W + H\nabla_Z C_R W)$$

$$= -((\phi_R V\nabla_Z B_R W + \omega_R V\nabla_Z B_R W + B_R A_Z B_R W + C_R A_Z B_R W + \phi_R A_Z C_R W + \omega_R A_Z C_R W + B_R H\nabla_Z C_R W + C_R H\nabla_Z C_R W).$$

Thus,

$$\nabla_Z W \in \Gamma((\ker F_*)^\perp) \iff \phi_R(V\nabla_Z B_R W + A_Z C_R W) + B_R(A_Z B_R W + H\nabla_Z C_R W) = 0.$$ 

Hence, we obtain

$$a) \iff b), \quad a) \iff c), \quad a) \iff d).$$

Therefore, the result follows. \qed

**Theorem 4.2.** Let $F$ be a h-semi-slant Riemannian map from a hyperkähler manifold $(M, I, J, K, g_M)$ to a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is a h-semi-slant basis. Then the following conditions are equivalent:

a) the fibers of $F$ are locally Riemannian product manifolds of the leaves of $D_1$ and $D_2$

b) $Q_I(\phi_I \tilde{\nabla}_U \phi_I V + B_I \nabla_U \phi_I V) = 0$ and $\omega_I \tilde{\nabla}_U \phi_I V + C_I \nabla_U \phi_I V = 0$

for $U, V \in \Gamma(D_1)$,

c) $P_I(\phi_I(\tilde{\nabla}_X \phi_I Y + \nabla_X \omega_I Y) + B_I(\nabla_X \phi_I Y + H\nabla_X \omega_I Y)) = 0$ and $\omega_I(\tilde{\nabla}_X \phi_I Y + \nabla_X \omega_I Y) + C_I(\nabla_X \phi_I Y + H\nabla_X \omega_I Y) = 0$

for $X, Y \in \Gamma(D_2)$.

d) $Q_J(\phi_J \tilde{\nabla}_U \phi_J V + B_J \nabla_U \phi_J V) = 0$ and $\omega_J \tilde{\nabla}_U \phi_J V + C_J \nabla_U \phi_J V = 0$

for $U, V \in \Gamma(D_1)$,

e) $P_J(\phi_J(\tilde{\nabla}_X \phi_J Y + \nabla_X \omega_J Y) + B_J(\nabla_X \phi_J Y + H\nabla_X \omega_J Y)) = 0$ and $\omega_J(\tilde{\nabla}_X \phi_J Y + \nabla_X \omega_J Y) + C_J(\nabla_X \phi_J Y + H\nabla_X \omega_J Y) = 0$

for $X, Y \in \Gamma(D_2)$.

f) $Q_K(\phi_K \tilde{\nabla}_U \phi_K V + B_K \nabla_U \phi_K V) = 0$ and $\omega_K \tilde{\nabla}_U \phi_K V + C_K \nabla_U \phi_K V = 0$

for $U, V \in \Gamma(D_1)$,
\[ P_K(\phi_K(\nabla_X \phi_R Y + T_X \omega_R Y) + B_K(T_X \phi_K Y + H \nabla_X \omega_R Y)) = 0 \]
\[ \omega_K(\nabla_X \phi_K Y + T_X \omega_R Y) + C_K(T_X \phi_K Y + H \nabla_X \omega_K Y) = 0 \]
for \( X, Y \in \Gamma(D_2) \).

**Proof.** Given \( R \in \{I, J, K\} \), for \( U, V \in \Gamma(D_1) \), we get
\[ \nabla_U V = -J \nabla_U JV = -J(\nabla_U \phi V + T_U \phi V) \]
\[ = -\phi \nabla_U \phi V + \omega \nabla_U \phi V + B_T U \phi V + C_T U \phi V). \]
Thus,
\[ \nabla_U V \in \Gamma(D_1) \iff Q(\phi \nabla_U \phi V + B_T U \phi V) = 0 \]
and \( \omega \nabla_U \phi V + C_T U \phi V = 0 \).
For \( X, Y \in \Gamma(D_2) \), we have
\[ \nabla_X Y = -R \nabla_X RY = -R(\nabla_X \phi_R Y + T_X \phi_R Y + T_X \omega_R Y + H \nabla_X \omega_R Y) \]
\[ = -\phi_R \nabla_X \phi_R Y + \omega_R \nabla_X \phi_R Y + B_R T_X \phi_R Y + C_R T_X \phi_R Y + \phi_R T_X \omega_R Y \]
\[ + \omega_R T_X \omega_R Y + B_R H \nabla_X \omega_R Y + C_R H \nabla_X \omega_R Y) \]
Thus,
\[ \nabla_X Y \in \Gamma(D_2) \iff \]
\[ \begin{cases} P_R(\phi_R(\nabla_X \phi_R Y + T_X \omega_R Y) + B_R(T_X \phi_R Y + H \nabla_X \omega_R Y)) = 0, \\ \omega_R(\nabla_X \phi_R Y + T_X \omega_R Y) + C_R(T_X \phi_R Y + H \nabla_X \omega_R Y) = 0. \end{cases} \]
Hence, we have
\[ a \iff b), \quad a \iff c), \quad a \iff d). \]
Therefore, we obtain the result. \( \square \)

5. **Examples**

Note that given an Euclidean space \( \mathbb{R}^{4m} \) with coordinates \((x_1, x_2, \ldots, x_{4m})\), we can canonically choose complex structures \( I, J, K \) on \( \mathbb{R}^{4m} \) as follows:
\[ I(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+2}}, \quad I(\frac{\partial}{\partial x_{4k+2}}) = -\frac{\partial}{\partial x_{4k+1}}, \quad I(\frac{\partial}{\partial x_{4k+3}}) = \frac{\partial}{\partial x_{4k+4}}, \quad I(\frac{\partial}{\partial x_{4k+4}}) = -\frac{\partial}{\partial x_{4k+3}}, \]
\[ J(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+3}}, \quad J(\frac{\partial}{\partial x_{4k+3}}) = -\frac{\partial}{\partial x_{4k+1}}, \quad J(\frac{\partial}{\partial x_{4k+4}}) = \frac{\partial}{\partial x_{4k+2}}, \quad J(\frac{\partial}{\partial x_{4k+2}}) = -\frac{\partial}{\partial x_{4k+4}}, \]
\[ K(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+4}}, \quad K(\frac{\partial}{\partial x_{4k+4}}) = \frac{\partial}{\partial x_{4k+3}}, \quad K(\frac{\partial}{\partial x_{4k+3}}) = -\frac{\partial}{\partial x_{4k+2}}, \quad K(\frac{\partial}{\partial x_{4k+2}}) = -\frac{\partial}{\partial x_{4k+1}} \]
for \( k \in \{0, 1, \ldots, m - 1\} \). Then it is easy to check that \( (I, J, K, <, >) \) is a hyperkähler structure on \( \mathbb{R}^{4m} \), where \(<, > \) denotes the Euclidean metric on \( \mathbb{R}^{4m} \).
Throughout this section, we will use these notations.

**Example 5.1.** Let \( F \) be an almost \( h \)-slant submersion from an almost quaternionic Hermitian manifold \((M, E, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then the map \( F : (M, E, g_M) \to (N, g_N) \) is a \( h \)-semi-slant Riemannian map with \( D_2 = \ker F_\ast \).

**Example 5.2.** Let \( F \) be an almost \( h \)-semi-slant submersion from an almost quaternionic Hermitian manifold \((M, E, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then the map \( F : (M, E, g_M) \to (N, g_N) \) is an almost \( h \)-semi-slant Riemannian map \([13]\).
Example 5.3. Let $(M, E, g)$ be an almost quaternionic Hermitian manifold. Let $\pi : TM \rightarrow M$ be the natural projection. Then the map $\pi$ is a strictly $h$-semi-slant Riemannian map such that $D_1 = \ker \pi_*$ and the strictly $h$-semi-slant angle $\theta = 0$ [10].

Example 5.4. Let $(M, E_M, g_M)$ and $(N, E_N, g_N)$ be almost quaternionic Hermitian manifolds. Let $F : M \rightarrow N$ be a quaternionic submersion. Then the map $F$ is a strictly $h$-semi-slant Riemannian map such that $D_1 = \ker F_*$ and the strictly $h$-semi-slant angle $\theta = 0$ [10].

Example 5.5. Define a map $F : \mathbb{R}^8 \rightarrow \mathbb{R}^4$ by
$$F(x_1, \cdots, x_8) = (x_2, x_1 \sin \alpha - x_3 \cos \alpha, 2012, x_4),$$
where $\alpha$ is constant. Then the map $F$ is a strictly $h$-semi-slant Riemannian map such that
$$D_1 = \left< \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right> \quad \text{and} \quad D_2 = \left< \cos \alpha \frac{\partial}{\partial x_1} + \sin \alpha \frac{\partial}{\partial x_3} \right>$$
with the strictly $h$-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.6. Let $(M, E, g_M)$ be a $4m$-dimensional almost quaternionic Hermitian manifold and $(N, g_N)$ a $(4m - 1)$-dimensional Riemannian manifold. Let $F : (M, E, g_M) \rightarrow (N, g_N)$ be a Riemannian map with rank $F = 4m - 1$. Then the map $F$ is a strictly $h$-semi-slant Riemannian map such that $D_2 = \ker F_*$ and the strictly $h$-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.7. Define a map $F : \mathbb{R}^{12} \rightarrow \mathbb{R}^5$ by
$$F(x_1, \cdots, x_{12}) = (x_6, \frac{x_1 - x_3}{\sqrt{2}}, c, x_4, x_5 - x_7),$$
where $c$ is constant. Then the map $F$ is a $h$-semi-slant Riemannian map such that
$$D_1 = \left< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right> \quad \text{and} \quad D_2 = \left< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \right>$$
with the $h$-semi-slant angles $\{\theta_I = \frac{\pi}{4}, \theta_J = \frac{\pi}{4}, \theta_K = \frac{\pi}{4}\}$.

Example 5.8. Define a map $F : \mathbb{R}^{12} \rightarrow \mathbb{R}^7$ by
$$F(x_1, \cdots, x_{12}) = (x_5 \cos \alpha - x_7 \sin \alpha, \gamma, x_6 \sin \beta - x_8 \cos \beta, x_9, x_{11}, x_{12}, x_{10}),$$
where $\alpha, \beta,$ and $\gamma$ are constant. Then the map $F$ is a $h$-semi-slant Riemannian map such that
$$D_1 = \left< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right>$$
and
$$D_2 = \left< \sin \alpha \frac{\partial}{\partial x_5} + \cos \alpha \frac{\partial}{\partial x_7}, \cos \beta \frac{\partial}{\partial x_6} + \sin \beta \frac{\partial}{\partial x_8} \right>$$
with the $h$-semi-slant angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K\}$ such that $\cos \theta_I = |\sin(\alpha + \beta)|$ and $\cos \theta_K = |\cos(\alpha + \beta)|$.

Example 5.9. Define a map $F : \mathbb{R}^{12} \rightarrow \mathbb{R}^7$ by
$$F(x_1, \cdots, x_{12}) = (x_3, x_4, 0, x_7, x_5, x_6, x_8).$$
Then the map $F$ is an almost h-semi-slant Riemannian map such that
\[
D_I^1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \rangle,
\]
\[
D_I^2 = D_I^K = \langle \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \rangle,
\]
\[
D_J^2 = 0, \quad D_J^K = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle .
\]
with the almost h-semi-slant angles \{\theta_I = 0, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}.

**Example 5.10.** Define a map $F : \mathbb{R}^{12} \rightarrow \mathbb{R}^6$ by
\[
F(x_1, \cdots, x_{12}) = (x_2, x_5, \alpha, x_1, \beta, x_7),
\]
where $\alpha$ and $\beta$ are constant. Then the map $F$ is an almost h-semi-slant Riemannian map such that
\[
D_I^1 = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \rangle,
\]
\[
D_I^2 = \langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \rangle,
\]
\[
D_I^K = \langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \rangle,
\]
\[
D_J^2 = \langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} \rangle, \quad D_J^K = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} \rangle,
\]
with the almost h-semi-slant angles \{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}.

**Example 5.11.** Let $\tilde{F}$ be a h-semi-slant Riemannian map from an almost quaternionic Hermitian manifold $(M_1, E_1, g_{M_1})$ to a Riemannian manifold $(N, g_N)$ with $D_2 = \ker \tilde{F}_\ast$. Let $(M_2, E_2, g_{M_2})$ be an almost quaternionic Hermitian manifold. Denote by $(M, E, g_M)$ the warped product of $(M_1, E_1, g_{M_1})$ and $(M_2, E_2, g_{M_2})$ by a positive function $g$ on $M_1$ \cite{3}, where $E = E_1 \times E_2$.

Define a map $F : (M, E, g_M) \rightarrow (N, g_N)$ by
\[
F(x, y) = \tilde{F}(x) \quad \text{for } x \in M_1 \text{ and } y \in M_2.
\]
Then the map $F$ is a h-semi-slant Riemannian map such that
\[
D_1 = T M_2 \quad \text{and} \quad D_2 = \ker \tilde{F}_\ast
\]
with the h-semi-slant angles \{\theta_I, \theta_J, \theta_K\}, where \{I,J,K\} is a h-slant basis for the map $\tilde{F}$ with the h-semi-slant angles \{\theta_I, \theta_J, \theta_K\}.

Note that as a generalization of an almost h-slant submersion \cite{12}, we call the map $F$ an almost h-slant Riemannian map.

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