Strength of the $\Sigma$ Single-Particle Potential in Nuclei
From Semiclassical Distorted Wave Model Analysis of $(\pi^-, K^+)$ Inclusive Spectrum

M. Kohno$^1$, Y. Fujiwara$^2$, Y. Watanabe$^3$, K. Ogata$^4$ and M. Kawai$^4$

$^1$Physics Division, Kyushu Dental College, Kitakyushu 803-8580, Japan
$^2$Department of Physics, Kyoto University, Kyoto 606-8502, Japan
$^3$Department of Advanced Energy Engineering Science, Kyushu University, Kasuga, Fukuoka 816-8580, Japan
$^4$Department of Physics, Kyushu University, Fukuoka 812-8581, Japan

Semiclassical distorted wave model is developed to describe $(\pi^-, K^+)$ inclusive spectra related to $\Sigma^-$-formation measured at KEK with $p_\pi = 1.2$ GeV/c. The shape and magnitude of the spectrum on $^{28}\text{Si}$ target are satisfactorily reproduced by a repulsive $\Sigma$-nucleus potential, the strength of which is of the order of 30$\sim$50 MeV. This strength is not so strong as more than 100 MeV suggested by the estimation presented in the report of the experiment.

The study of the $\Sigma$-N interaction is essential to understanding the physics of octet baryons. In contrast to the other strange particle $\Lambda$, however, the $\Sigma$ single-particle (s.p.) potential in nuclear medium, which reflects basic properties of the $\Sigma$-N interaction, has not been established, since the experimental information has been scarce. Even the sign of it has been controversial.

$\Sigma$ formation spectra in $(\pi, K)$ or $(K, \pi)$ reactions on nuclei are not expected to have narrow peaks because of the strong $\Sigma N \rightarrow \Lambda N$ coupling. In spite of these circumstances, the early $(\pi, K)$ experimental spectra$^{1,2}$ were interpreted to indicate an attractive $\Sigma$ s.p. particle potential of around 10 MeV.$^{2,3}$ The experimental finding of $^4\text{He}$ $^{4,5}$ has shown that the $\Sigma$-N interaction in $T = 1/2$ channel is attractive to support the bound state, as was discussed by Harada.$^6$ It has been recognized, however, that due to the strong isospin dependence $\Sigma$ bound states are unlikely to be observed in heavier nuclei, which has been experimentally supported.$^7$ In the different context, Batty, Friedman and Gal$^8$ reexamined the $\Sigma^-$ atomic data to conclude that the $\Sigma$ potential might be repulsive in a nucleus. The analysis of the $(\pi, K)$ spectra from BNL$^9$ by Dąbrowski$^{10}$ has also suggested that the $\Sigma$ potential is repulsive of the order of 20 MeV.

Theoretical studies have also been elusive for the $\Sigma$-N interaction. In a standard OBEP model for the hyperon-nucleon interactions, there are uncertainties in the coupling constants, although $SU_3$ relations are basically imposed. The Nijmegen group constructed, in 1970’s, hard-core hyperon-nucleon potentials, models D and F.$^{11}$ Yamamoto and Bando’s calculation$^{12}$ showed that the model D gave $-16.3$ MeV for the $\Sigma$ potential in nuclear matter ($k_F = 1.35$ fm$^{-1}$) and the model F repulsive 5.3 MeV. The later soft-core version$^{13}$ was shown$^{14}$ to predict smaller attraction than the model D.

In recent years, a non-relativistic $SU_6$ quark model has been developed by Kyoto-
Niigata group\textsuperscript{15)--17) for the unifying description of octet baryon-baryon interactions. G matrix calculations in the lowest order Brueckner theory\textsuperscript{18) with this potential showed that the Σ s.p. potential in symmetric nuclear matter is repulsive of the order of 20 MeV due to the strong repulsion in the $T = \frac{3}{2}$ channel, which originates from the quark Pauli effects.

Recently, $(\pi^-, K^+)$ spectra corresponding to Σ formation were measured on various nuclei at KEK\textsuperscript{19) with better precision, using 1.2 GeV/c $\pi^-$. The striking report\textsuperscript{19) in the experimental presentation of results on $^{28}$Si was that the Σ potential deduced from their DWIA analysis was strongly repulsive as large as 100 MeV.

The determination of the Σ-$N$ interaction should have fundamental influence on such a problem of the neutron star matter and the heavy ion collision, since the baryonic component of these hadronic matter, especially the hyperon admixture, is governed by the basic baryon-baryon interactions. In view of the importance of the understanding of the Σ-$N$ interaction for our description of the whole octet baryon-baryon interactions, it is desirable to carry out an independent analysis of the KEK experiments. In this Letter, we develop a semiclassical method for DWIA approach and apply it to $(\pi^\pm, K^\mp)$ inclusive spectra. The semiclassical distorted wave (SCDW) model was originally considered for describing intermediate-energy nucleon inelastic reactions on nuclei.\textsuperscript{20) Applications to various $(p, p')$ and $(p, n)$ inclusive spectra\textsuperscript{21), 22) have proved that the method works well.

The double differential cross section for the $(\pi, K)$ hyperon ($Y$) production inclusive reaction is expressed as

$$
\frac{d^2\sigma}{dWd\Omega} = \frac{\omega_i\omega_f p_f}{(2\pi)^2 p_i} \int \int dr dr' \sum_{p, h} \frac{1}{4\omega_i\omega_f} \chi_f^{(-)}(r) v_{f, p, i, h} \chi_i^{(+)}(r) \chi_f^{(-)}(r') \times \chi_i^{(+)}(r') \phi_p(r) \phi_h(r') \phi_p^*(r') \delta(W - \epsilon_p + \epsilon_h) \theta(\epsilon_F - \epsilon_h).
$$

where $\chi_i^{(+)}$ and $\chi_f^{(-)}$ describe the incident pion and final kaon wave functions with energies $\omega_i$ and $\omega_f$, respectively, and $W = \omega_i - \omega_f$ is the energy transfer. The $p$ and $h$ denote the unobserved outgoing hyperon ($\Lambda$ or Σ) and nucleon hole states, and the Fermi energy of the target nucleus is specified by $\epsilon_F$. The transition strength of the elementary process $\pi + N \rightarrow K + Y$ is represented by $v_{f, p, i, h}$, which depends on the energy and angle of the scattering particles, though not explicitly written. Denoting the c.m. and relative coordinates of $r$ and $r'$ by $R = \frac{r + r'}{2}$ and $s = r' - r$, respectively, we introduce a semiclassical approximation:

$$
\chi_f^{(-)}(R \pm \frac{1}{2}s) \simeq e^{\pm i\frac{1}{2}s} k_f(R) \chi_f^{(-)}(R),
$$

$$
\chi_i^{(+)}(R \pm \frac{1}{2}s) \simeq e^{\pm i\frac{1}{2}s} k_i(R) \chi_i^{(+)}(R).
$$

The hyperon wave functions $\phi_p(r)$ and $\phi_p(r')$ are also treated in the same way. In these expressions, $k_i(R)$ is local classical momentum at the position $R$, which is defined as follows. First, the quantum mechanical expectation value of the momentum
Letters

is calculated by

\[ k_\theta(R) = \frac{\Re \{ \chi^{(\pm)}(R)(-i)\nabla \chi^{(\pm)}(R) \} }{|\chi^{(\pm)}(R)|^2}, \tag{4} \]

where \( \Re \) stands for taking a real part, and then the magnitude is renormalized by the energy-momentum relation \( \hbar^2/2m k^2(R) + U_R(R) = E \). \( U_R(R) \) is the real part of an optical potential which describes the distorted wave function \( \chi \) of each particle with the energy \( E \). Along with the above approximation, we employ the Thomas-Fermi approximation for the summation of hole states. Namely, the Bloch density

\[ C(r, r'; \beta) \equiv \sum_i \phi_i(r) \phi_i^*(r') e^{-\beta \epsilon_i}, \tag{5} \]

where the summation with respect to the single particle state \( \phi_i \) of the potential \( U \) with the energy \( \epsilon_i \) goes over the complete spectrum, is replaced by that in the Thomas-Fermi approximation:\textsuperscript{23)}

\[ C_{TF}(r, r'; \beta) = \frac{2}{(2\pi)^3} \int dK e^{-\beta(U_F(R) + \hbar^2 K^2/2m)} e^{iK \cdot s}. \tag{6} \]

The \( \theta(\epsilon_F - \epsilon_h) \) function in eq. (1) actually restricts the integration over \( K \) below the local Fermi momentum \( k_F(R) \) defined by the nucleon density \( \rho_F(R) \), with \( \tau \) specifying the proton in the present case, as \( k_F(R) = [3\pi^2 \rho_F(R)]^{1/3} \).

Introducing these approximations, we obtain the following expression for the double differential cross section, the detailed derivation of which is discussed in a separate paper:

\[
\frac{d^2\sigma}{dWd\Omega} = \frac{\omega_f \omega_f}{(2\pi)^2} \frac{p_f}{p_i} \int dR \int_{K<k_F(R)} dK \sum_p \frac{1}{4\omega_f \omega_f} |\chi_f^{(-)}(R)|^2 |\chi_i^{(+)}(R)|^2 |\phi_p(R)|^2 \\
\times |v_{f,p,i,h}(K, k_i)|^2 \delta(K + k_i(R) - k_f(R) - k_p(R)) \\
\times \delta(W - \epsilon_p + \hbar^2 K^2/2m + U_F(R)). \tag{7} \]

This expression has simple interpretation that the reaction of \( \pi + N \) into \( K + Y \) takes place at the position \( R \) with satisfying the conservation of local semiclassical momenta. These momenta \( k_i(R), k_f(R) \) and \( k_p(R) \) are calculated by eq. (4), using \( \pi, K \) and \( Y \) distorted wave functions in an optical model description. It should be stressed that we have avoided the naive introduction of an averaged differential cross section of the elementary process over the proton momentum distribution \( \rho(k) \),

\[
\frac{d\sigma(\pi^{-} p \rightarrow K^{+} \Sigma^{-})}{d\Omega} = \int \rho(k) \frac{d\sigma}{d\Omega}(\Omega_k) \delta(k - P)dk \\
\int \rho(k) \delta(k - P)dk \tag{8} \]

with \( P = k_K + k_Y - k_\pi \), which was used in the analysis of ref.\textsuperscript{19)}

The transition strength \( v_{f,p,i,h} \) is related to the elementary cross section by

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4\pi} \frac{E_N E_Y}{k_\pi} |v|^2, \tag{9} \]

Letters

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Fig. 1. \((\pi^+, K^+) \Lambda\) formation inclusive spectra on \(^{28}\text{Si}\) at \(\theta_K = 6^\circ \mp 2^\circ\) for the pion of \(p_\pi = 1.2\ \text{GeV/c}\), obtained by various choices of \(U_0^\Lambda\) in a Woods-Saxon potential form, compared with the KEK data.\(^{27}\)

Fig. 2. \((\pi^-, K^+) \Sigma\) formation inclusive spectra on \(^{28}\text{Si}\) at \(\theta_K = 6^\circ \mp 2^\circ\) for the pion of \(p_\pi = 1.2\ \text{GeV/c}\), obtained by various choices of \(U_0^\Sigma\) in a Woods-Saxon potential form, compared with the KEK data.\(^{19}\)

where \(s\) is the invariant mass squared. We are able to take into account the angular dependence of the \(\pi + N \rightarrow K + Y\) elementary process.

The local Fermi momentum \(k_F(R)\) for \(^{28}\text{Si}\) is prepared with the nucleon density distribution obtained by the density-dependent Hartree-Fock method of Campi-Sprung.\(^{25}\) The distorted waves for the incident pion and outgoing kaon are described by a simple absorptive potential \(U(r) = -i\frac{k^2}{2\pi}b_0\rho(r)\) with \(\rho(r)\) being the nucleon density distribution. The parameter \(b_0\) is related to the spin-isospin averaged total cross section of the elementary process by \(b_0 \sim \frac{1}{k_F}\langle \sigma_{\text{tot}} \rangle\). Referring to the PDG data,\(^{26}\) we use \(b_0 = 0.58\ \text{fm}^3\) for the incident 1.2 GeV/c pion and \(b_0 = \frac{1}{2p_K}(2.1(\log p_K - 2) + 0.84)\ \text{fm}^3\ (p_K \text{ in fm}^{-1})\) for the outgoing kaon, respectively.

The treatment of the unobserved hyperon is in order. Actually the hyperon optical potential should be complex, because inelastic processes are present. Effects of these inelastic channels may be treated by the Green function method. Here we adopt simplified prescription to use a real local potential of the standard Woods-Saxon form

\[
U_Y = \frac{U_0^Y}{1 + \exp((r - r_0)/a)},
\]

and convolute the result of the calculated spectrum by a Lorentz type distribution function. The half width is taken to be 5 MeV for the \(\Lambda\) and 20 MeV for the \(\Sigma\), based on the imaginary part of the \(\Lambda\) and \(\Sigma\) s.p. potentials\(^{18}\) in symmetric nuclear matter calculated with the quark-model potential FSS. As the first application, we take the standard geometry parameters: \(r_0 = 1.2 \times (A - 1)^{1/3}\ \text{fm}\) and \(a = 0.6\ \text{fm}\). The Coulomb interaction is incorporated.

We first apply our model to the \((\pi^+, K^+) \Lambda\) formation inclusive spectrum on \(^{28}\text{Si}\) measured at KEK\(^{27}\) with the incident \(\pi^+\) momentum \(p_\pi = 1.2\ \text{GeV/c}\). The strength and the angular dependence of the elementary process are parameterized according to the available experimental data.\(^{28}, 29\) Figure 1 shows calculated spectra
with various strengths of the $\Lambda$ s.p. potential of $V_\Lambda^0 = -50, -30, -10$ and $10$ MeV, respectively, to see the potential dependence of the calculated spectra, though the $\Lambda$ s.p. potential has been established as $V_\Lambda^0 \sim -30$ MeV from various $\Lambda$ hypernuclear data. Bearing in mind various ambiguities in the elementary amplitudes which would be modified in nuclear medium and additional two-step contributions, our model is seen to be capable to describe the inclusive spectra.

Figure 2 shows calculated $\Sigma^-$ formation ($\pi^-, K^+$) inclusive spectra, compared with the KEK experimental data. In the present calculation, we assume an isotropic angular dependence. The energy dependence of $|v|^2$ is taken from the parameterization by Tsushima et al.\cite{24} Their overall strength was normalized by factor 0.82 to match the experimental data taken at KEK.\cite{19} Several curves correspond to the supposed $\Sigma$ potential of $U_\Sigma^0 = -10, 10, 30, 50$ and $90$ MeV, respectively. No overall renormalization factor is introduced. It is seen that the shape and absolute value are satisfactorily reproduced by the repulsive strength of $30$ MeV. Expecting the contributions from multi-step processes, the repulsive strength may be larger to be $50$ MeV, though it is premature to draw the final conclusion before taking into account various effects discussed later. This order of the repulsive magnitude of the $\Sigma$ s.p. potential is in line with the estimation by Dąbrowski\cite{10} for BNL data that the $\Sigma^-$-nucleus potential for $^9$Be is about $20$ MeV. It was noted in ref.\cite{19} that the peak position at as high as $150$ MeV was hard to reproduce if the repulsion of the $\Sigma$-nucleus potential is not so strong. Present calculations suggest, however, that the $\Sigma$ s.p. potential is not necessary to be strongly repulsive. The attractive potential fails by predicting much larger cross sections. On the other hand stronger repulsive $\Sigma$ s.p. potential tends to underestimate the cross section. The cause of the different result from ref.\cite{19} might be due to avoiding the use of the factorization approximation by the average cross section, eq. (18). This point deserves further investigation.

As for the $SU_6$ quark model, it is interesting to observe that this order of magnitude of a few ten MeV is consistent with the prediction of the calculation in nuclear matter.\cite{18} The quark model description of the $\Sigma N$ interaction predicts definite strong repulsive nature in the isospin $T = \frac{3}{2}$ channel due to the quark antisymmetrization effect. Thus the $\Sigma^-$-nucleus potential is expected to become more repulsive in the circumstance of the neutron excess. In this respect, the analysis of the ($\pi^-, K^+$) data on heavier nuclei would be interesting to investigate whether such quantitative isospin dependence actually exists.

There are various simplified treatments in the present calculations. The smearing by the Lorentz type convolution should be replaced by the Green’s function method. Quantitative estimation of the contribution from multistep processes is needed. More sophisticated description of the elementary process has to be employed. Contributions from more than two-step processes tend to increase the cross section. On the other hand, the possible modification of the elementary process in nuclear medium would reduce the cross section. These problems are future subjects to be investigated. The present SCDW framework serves as a quantitatively reliable model to discuss possible change of properties of intermediate baryonic states.

In summary, we have developed a semiclassical distorted wave model for ($\pi, K$) inclusive spectra corresponding to the $\Lambda$ or $\Sigma$ formation. The expression of the
double differential cross section consists of the incoming pion distorted wave function, outgoing kaon distorted wave function and undetected hyperon distorted wave function at each collision point, where the conservation of the classical local momentum is respected. The bound nucleon in the target nucleus is described by a local Fermi gas model. The present framework is easily applied to describe other inclusive spectra, such as \((K, \pi)\), \((\pi, \eta)\), \((\gamma, K)\) and so on. It is also straightforward to consider multistep contributions, as was carried out for \((p, p')\) and \((p, n)\) inclusive spectra.\(^{21,22}\) The application of this model to the \((\pi^-, K^+)\) inclusive spectrum on \(^{28}\)Si taken at KEK\(^{19}\) has shown that the spectrum is satisfactorily reproduced by a repulsive \(\Sigma\)-nucleus potential of the order of 30~50 MeV. This magnitude, in turn, constrains the \(\Sigma\)-\(N\) potential model to improve our understanding of the interactions between complete octet baryons.

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1) R. Bertini et al., Phys. Lett. 90B (1980), 375.
2) M. Kohno, R. Hausmann, P. Siegel and W. Weise, Nucl. Phys. A470 (1987), 609.
3) C.B. Dover, D.J. Millener and A. Gal, Phys. Rep. 184 (1989), 1.
4) R.S. Hayano et al., Phys. Lett. B231 (1989), 355.
5) T. Nagae et al., Phys. Rev. Lett. 80 (1998), 1605.
6) T. Harada, Phys. Rev. Lett. 81 (1998), 5287.
7) S. Bart et al., Phys. Rev. Lett. 83 (1999), 5238.
8) C.J. Batty, E. Friedman and A. Gal, Prog. Theor. Phys. Suppl. 117 (1994), 227.
9) R. Sawata, Nucl. Phys. A585 (1995), 103c; A639 (1998), 103c.
10) J. Dąbrowski, Phys. Rev. C60 (1999), 025205.
11) M.M. Nagels, T.A. Rijken and J.J. de Swart, Phys. Rev. D12 (1975), 744; D15 (1977), 2547; D20 (1979), 1633.
12) Y. Yamamoto and H. Bando, Prog. Theor. Phys. Suppl. 81 (1985), 9.
13) P.M.M. Maassen, T.A. Rijken and J.J. de Swart, Phys. Rev. C40 (1989), 2226.
14) Y. Yamamoto, T. Motoba, H. Himeno, K. Ikeda and S. Nagata, Prog. Theor. Phys. Suppl. 117 (1994), 361.
15) Y. Fujiiwara, C. Nakamoto and Y. Suzuki, Phys. Rev. Lett. 76 (1996), 2242.
16) Y. Fujiiwara, C. Nakamoto and Y. Suzuki, Phys. Rev. C54 (1996), 2180.
17) Y. Fujiiwara, M. Kohno, C. Nakamoto and Y. Suzuki, Phys. Rev. C64 (2001), 054001.
18) M. Kohno et al., Nucl. Phys. A674 (2000), 229.
19) H. Noumi et al., Phys. Rev. Lett. 89 (2002), 072301; 90 (2003), 049902(E).
20) Y.L. Luo and M. Kawai, Phys. Lett. B235 (1990), 211; Phys. Rev. C43 (1991), 2367.
21) Y. Watanabe et al., Phys. Rev. C59 (1999), 2136.
22) K. Ogata et al., Phys. Rev. C60 (1999), 054605.
23) M. Brack, C. Guet and H.-B. Häkansson, Phys. Rep. 123 (1985), 275.
24) K. Tsushima, S.W. Huang and A. Faessler, Phys. Lett. B337 (1994), 245.
25) X. Campi and D.W. Sprung, Nucl. Phys. A194 (1972), 401.
26) S. Eidelman et al., Phys. Lett. B592 (2004), 1.
27) P.K. Saha, private communication.
28) A. Baldini et al., Numerical Data and Functional Relationships in Science and Technology, Landolt-Börnstein Vol. 12a (Springer, Berlin, 1988).
29) D.H. Saxon et al., Nucl. Phys. B162 (1980), 522.