Dynamical zeta functions for Artin’s billiard and the Venkov–Zograf factorization formula

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Abstract

Dynamical zeta functions are expected to relate the Schrödinger operator’s spectrum to the periodic orbits of the corresponding fully chaotic Hamiltonian system. The relationship is exact in the case of surfaces of constant negative curvature. The recently found factorisation of the Selberg zeta function for the modular surface is known to correspond to a decomposition of the Schrödinger operator’s eigenfunctions into two sets obeying different boundary condition on Artin’s billiard. Here we express zeta functions for Artin’s billiard in terms of generalized transfer operators, providing thereby a new dynamical proof of the above interpretation of the factorization formula. This dynamical proof is then extended to the Artin–Venkov–Zograf formula for finite coverings of the modular surface.

1 Introduction

Dynamical zeta functions for flows $\phi_t : M \to M$ were introduced in the sixties by Smale after he learned from Sinai about such an interpretation for Selberg’s zeta function in terms of the geodesic flow on a surface of constant negative curvature defined by a discrete group acting on Poincaré’s hyperbolic upper half plane, the so called Fuchsian group. It turned out that Ruelle’s and Bowen’s thermodynamic formalism provides a new approach for understanding the properties of these functions completely different from Selberg’s original one, which was based mainly on his famous trace formula. In the new approach the dynamical zeta function gets expressed in terms of Fredholm determinants of so called transfer operators. These operators have their origin in the transfer matrix method of statistical mechanics and were used in the early days of the thermodynamic formalism to characterize the ergodic properties of dynamical systems.
Since in the case of the geodesic flow on surfaces of constant negative curvature there exists a close connection between the nontrivial zeros of the corresponding dynamical zeta function and the spectrum of the free Schrödinger operator on the surface, the transfer operator, which is a purely classical object, when analytically continued to complex temperatures, hence determines also the quantum properties of these systems [8], [9]. One could then expect that something like this could be true also for general chaotic Hamiltonian systems at least in the semiclassical limit of the systems’ quantization. Indeed a heuristic approach to this question was recently provided in [9] where the quantization condition is again expressed in terms of a Fredholm determinant of a so-called quantum operator, which coincides with the aforementioned transfer operator at least in the case of the geodesic flow on a compact surface of constant negative curvature [10].

Not so much is known for such surfaces in case they are not compact, but have a finite hyperbolic area [11]. Among them the modular surface, defined through the action of the modular group $SL(2, \mathbb{Z})$ on the Poincaré half plane $H = \{z = x + iy : y > 0\}$ plays a special role: for its geodesic flow the above theory works perfectly and the dynamical zeta function can be expressed again in terms of the Fredholm determinant of a transfer operator [11], providing at the same time a much easier and shorter approach to Selberg’s results for such non-compact surfaces of constant negative curvature. Furthermore, an interesting factorization formula of the dynamical zeta function follows from the transfer operator approach, closely related to the description of the geodesic flow on the modular surface in terms of the symbolic dynamics of Series [12] resp. Adler and Flatto [13]. The conjecture in [8], namely, that this factorization corresponds to a decomposition of the spectrum of the free Schrödinger–operator on the modular surface into even respectively odd eigenfunctions under reflection of its fundamental domain on the imaginary axis, was proved recently in [14] by using number theoretic methods. A formal argument was given also in [10].

In the present paper we give a rigorous proof of this conjecture based completely on dynamical properties of the flow and its relation to the Artin billiard. It turns out that our proof shows at the same time that the factorization of the Selberg function for the geodesic flow on the modular surface can be interpreted as an extension of the Artin–Venkov–Zograf factorization [14] for the Selberg function for normal subgroups $\Gamma'$ of Fuchsian groups $\Gamma$ in terms of Selberg functions with unitary representation of this group $\Gamma$. Indeed the modular group $SL(2, \mathbb{Z})$ is a normal subgroup of $GL(2, \mathbb{Z})$ and the Selberg functions of the Artin billiard with Neumann resp. Dirichlet boundary conditions are just generalized zeta functions of the latter group corresponding to the two one dimensional unitary representations of $GL(2, \mathbb{Z})$ with kernel $SL(2, \mathbb{Z})$. We then show how our approach to the factorization formula for the modular surface can be extended immediately to any finite covering of the modular surface, defined by a normal subgroup $\Gamma'$ of the modular group $SL(2, \mathbb{Z})$, and results in a simple dynamical proof of the Artin–Venkov–Zograf formula. An interesting case is the principal
congruence subgroup \( \Gamma(2) = \{ g \in SL(2, \mathbb{Z}) : g \mod 2 = 1 \} \) which defines a sixfold covering of the modular surface. Its geodesic flow is closely related to certain approximate solutions of the so-called mixmaster universe [16].

In a first section we determine the symbolic dynamics of the billiard flow on Artin’s billiard from the one determined by Series [12] resp. Adler and Flutto [13] some time ago for the geodesic flow on the modular surface. This allows us to express the dynamical zeta functions for the billiard flow in terms of Fredholm determinants of the generalized transfer operator for the Gauß map. We show that the product of these two functions gives exactly the dynamical zeta function for the geodesic flow on the modular surface. Using well-known results of Venkov this then proves our conjecture relating the factorization to the even and odd spectrum of the Schrödinger operator. Next we show that our factorization can be regarded also as an extension of the Artin–Venkov–Zograf formula for subgroups of Fuchsian groups to the case of the subgroup \( SL(2, \mathbb{Z}) \) of the group \( GL(2, \mathbb{Z}) \).

In a last section we argue how our approach can be used to provide a dynamical proof of the Artin–Venkov–Zograf formula for finite coverings of the modular surface. These authors used in their work the special group theoretic structure of the corresponding Fuchsian groups whereas ours is mainly of dynamical nature.

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2 Dynamical zeta function for Artin’s billiard

2.1 Geodesic flow on the modular surface

The Poincaré half plane \( \mathbb{H} = \{ z = x + iy : y > 0 \} \) is shown in figure [3]. Due to its metric \( (dx^2 + dy^2)/y^2 \) the geodesics are circles (or straight lines) orthogonal to the \( x \)-axis. The modular surface is defined by identifying points \( z, Tz = z + 1 \) and \( Qz = -1/z \). \( T \) and \( Q \) are the generators of the group \( PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{+1, -1\} \)

acting on \( \mathbb{H} \) as

\[
\{ f : \mathbb{H} \rightarrow \mathbb{H} | f(z) = \frac{\alpha z + \beta}{\gamma z + \delta} ; \alpha \delta - \beta \gamma = 1; \alpha, \beta, \gamma, \delta \in \mathbb{Z} \} \quad (2)
\]

The fundamental domain \( \{ z \in \mathbb{H} : |z| \geq 1; |\Re z| \leq 1/2 \} \) tiles the plane \( \mathbb{H} \) under the action of this group. Artin’s billiard is the part of the fundamental domain with \( x > 0 \). At intersections with the boundary of this domain the geodesic
flow of the billiard is reflected while the geodesic flow of the modular surface is mapped under $Q$ or $T$ or $T^{-1}$ to an equivalent point of the fundamental domain of the modular group $SL(2, \mathbb{Z})$.

The symbolic dynamics of the geodesic flow on the modular surface was investigated by Adler and Flatto [13] resp. Series [12]. They introduced a Poincaré section, which can be finally induced on $I_2 \times \mathbb{Z}_2$. The coordinates $(\chi, \psi, \epsilon) \in I_2 \times \mathbb{Z}_2$ of this induced Poincaré section are determined from the orbit’s $\gamma$ forward and backward intersections $x_+\infty$ and $x_-\infty$ with the $x$-axis. Immediately before the intersection with the unit circle we have $x_+\infty \in (-1,1)$ and:

\begin{align*}
\chi &:= |x_+\infty| \in (0,1) \\
\psi &:= 1/|x_-\infty| \in (0,1) \\
\epsilon &:= \text{sgn}(x_+\infty - x_-\infty) \in \{-1,1\}.
\end{align*}

In these coordinates the Poincaré Map $\tilde{T}$ takes the simple form [20]:

$$\tilde{T}(\chi, \psi, \epsilon) = (T_G(\chi), \frac{1}{\psi + \lfloor 1/\chi \rfloor}, -\epsilon)$$

where $\lfloor x \rfloor$ denotes the integer part of $x$ and $T_G$ is the Gauss map on the unit interval:

$$T_G(\chi) := \frac{1}{\chi} \mod 1, \chi \neq 0.$$  \hspace{1cm} (7)

The $x$-component of the direction of motion changes its sign $\epsilon$ at every intersection with the unit circle. Thus the number of these intersections along a periodic orbit is equal to the number of Poincaré mappings. Enumerating the monotonic branches of the Gauss map with the natural numbers one gets a complete and exact symbolic dynamics of the Gauss map. This generalizes to a symbolic dynamics of the Poincaré map $\tilde{T}$. Thus all the periodic orbits $\gamma$ of the modular surface are classified. Their length $l(\gamma)$ was determined by Pollicott [20] as a sum over all their intersections with the Poincaré section:

$$l(\gamma) = \sum_i r(\chi_i, \psi_i, \epsilon_i)$$

where $r(\chi, \psi, \epsilon) := \ln |T_G(\chi)|$.  \hspace{1cm} (9)

2.2 Periodic orbits of the modular surface and the Artin billiard

Let us denote the periodic orbits of the geodesic flow on the Artin billiard by $\gamma_b$ and on the modular surface by $\gamma_m$. We will derive a relationship between their lengths and numbers.

This relationship depends on the symmetry properties of the orbits under the reflection at the $y$-axis $J(x, y) = (-x, y)$. According to definitions [3] to [3]
this reflection acts on the Poincaré section as:

\[ J(\chi, \psi, \epsilon) = (\chi, \psi, -\epsilon) \quad . \]  

(10)

Applying this reflection \( J \) we can transform periodic orbits \( \gamma_m \) of the modular surface into periodic orbits \( \gamma_b \) of the Artin billiard and vice versa. Any prime periodic orbit \( \gamma_m \) of the modular surface is cut into segments by its intersections with the \( y \)-axis and the boundary of the fundamental domain. On these segments it is a simple geodesic flow. Applying \( J \) to every second segment we get an orbit, that fulfills the boundary conditions of the orbits for the Artin billiard. Starting from a point \( P \) with positive \( x \)-coordinate in the direction \( \underline{u} \) of \( \gamma_m \) and transversing the orbit \( \gamma_m \), we thus construct an orbit \( \tilde{\gamma}_b \) of the Artin billiard, which is periodic. The prime periodic part of \( \tilde{\gamma}_b \) is a unique function \( G(\gamma_m) \) of the prime periodic orbit \( \gamma_m \). Only \( \gamma_m \) and its mirror image \( J \gamma_m \), but no other prime periodic orbit of the modular surface, yield the same orbit \( G(\gamma_m) = G(J \gamma_m) \).

Thus we have constructed a map \( G \) from the set \( M_m \) of prime periodic orbits on the modular surface to the set \( M_b \) of prime periodic orbits in the Artin billiard. \( M_m \) is divided into symmetric \( M_m^s = \{ \gamma_m \in M_m | J \gamma_m = \gamma_m \} \) and asymmetric \( M_m^a = \{ \gamma_m \in M_m | J \gamma_m \neq \gamma_m \} \) orbits. The map \( G \) is injective on \( M_m^a \) and two-to-one on \( M_m^s \).

In a similar geometric way we can construct the inverse of the map \( G \). Any prime periodic orbit \( \gamma_b \) of the Artin billiard is cut into segments by its reflections from the wall. Applying \( J \) to every second segment we get an orbit \( \tilde{\gamma}_m \), which fulfills the boundary conditions of the modular surface. Starting at \( P \) in the direction \( \underline{u} \) of \( \gamma_b \) and transversing \( \gamma_b \) once, we thus construct an orbit, which ends either at \( ( P, \underline{u} ) \) or at \( ( J P, J \underline{u} ) \), depending on the number \( n(\gamma_b) \) of intersections with the unit circle:

1. \( n(\gamma_b) \) even \( \Rightarrow \tilde{\gamma}_m \) ends at \( ( P, \underline{u} ) \). Then \( \gamma_m := \tilde{\gamma}_m \) is a prime periodic orbit with the same length \( l \) and the same number \( n \) of intersections with the unit circle as \( \gamma_b \). It is different from its mirror image \( J \gamma_m \).

2. \( n(\gamma_b) \) odd \( \Rightarrow \tilde{\gamma}_m \) ends at \( ( J P, J \underline{u} ) \). Then we create an orbit \( \gamma_m \) by concatenating \( J \tilde{\gamma}_m \) and \( \tilde{\gamma}_m \). This doubles the length: \( l(\gamma_m) = 2 \cdot l(\gamma_b) \) and the number \( n \) of intersections with the unit circle: \( n(\gamma_m) = 2 \cdot n(\gamma_b) \). The orbit \( \gamma_m \) ends at \( ( J^2 P, J^2 \underline{u} ) = ( P, \underline{u} ) \), thus it is prime periodic. It is also symmetric: \( \gamma_m = J \gamma_m \).

As there is no third possibility, all orbits \( \gamma_b \in G(M_m^a) \) belong to the first case, all orbits \( \gamma_b \in G(M_m^s) \) to the second. It follows that \( G(M_m^a) \) and \( G(M_m^s) \) are disjoint and that \( G \) is surjective: \( M_b = G(M_m^a) \cup G(M_m^s) \).

1. \( n(\gamma_b) \) even \( \iff \gamma_b \in G(M_m^a) \)

\[ \Rightarrow G^{-1}(\{ \gamma_b \}) = \{ \gamma_m, J \gamma_m \} \quad . \]  

(11)
\[ l(\gamma_b) = l(\gamma_m) \]  
\[ n(\gamma_b) = n(\gamma_m) \]  

2. \( n(\gamma_b) \) odd ⇔ \( \gamma_b \in G(M^n_m) \)

\[ \Rightarrow G^{-1}(\{\gamma_b\}) = \{\gamma_m\} \]  
\[ l(\gamma_b) = l(\gamma_m)/2 \]  
\[ n(\gamma_b) = n(\gamma_m)/2 \]

2.3 Periodic orbits of the Poincaré map and the Gauß map

Let us denote the periodic orbits of the Poincaré map \( \tilde{T} \) by \( \gamma_P \) and of the Gauß map \( T_G \) by \( \gamma_G \). If the length of an periodic orbit \( \gamma \) is \( n \), then the geometric length of the corresponding orbit of the geodesic flow is:

\[ l(\gamma) = \sum_{i=1}^{n} r(\chi_i) \]  

where \( r(\chi) \) was defined in equation (8). In analogy to the last section we divide the set \( M_P \) of prime periodic orbits of the Poincaré map \( \tilde{T} \) into symmetric \( M^s_P = \{ \gamma_P \in M_P | J \gamma_P = \gamma_P \} \) and asymmetric \( M^a_P = \{ \gamma_P \in M_P | J \gamma_P \neq \gamma_P \} \) orbits.

As the first component \( T_G(\chi) \) of the image \( \tilde{T}(\chi, \psi, \epsilon) \) does not depend on \( \psi \) and \( \epsilon \), there is a trivial projection \( (\chi, \psi, \epsilon) \mapsto \chi \), which maps any prime periodic orbit \( \gamma_P \) of the Poincaré map \( \tilde{T} \) onto a periodic orbit \( \gamma_G \) of the Gauß map \( T_G \). The prime periodic part of \( \gamma_G \) is a function \( H(\gamma_P) \) of the prime periodic orbit \( \gamma_P \).

On the other hand, if \( \gamma_G \) is a prime periodic orbit \( \{\chi_i\}_{i \in \mathbb{Z}} \): \( \chi_{i+n} = \chi_i; \chi_{i+1} = T_G(\chi_i) \) of length \( n(\gamma_G) \) of the Gauß map, we can find corresponding orbits of the Poincaré map. The coordinate \( \psi_i \) is fixed uniquely by the periodic continued fraction

\[ \psi_i = \frac{1}{1/\chi_i + \frac{1}{1/\chi_{i-1} + \ldots}} \]  

Again we have to distinguish two cases:

1. \( n(\gamma_G) \) even ⇒ \( (\chi_0, \psi_0, 1) \) and \( (\chi_0, \psi_0, -1) \) are starting points of two different prime periodic orbits

\[ \{\gamma_P, J \gamma_P\} = H^{-1}(\{\gamma_G\}) \]  

of length

\[ n(\gamma_P) = n(\gamma_G) \]  

The corresponding geometric lengths are also equal:

\[ l(\gamma_P) = l(\gamma_G) \]
2. $n(\gamma_G)$ odd $\Rightarrow (\chi_0, \psi_0, 1)$ and $(\chi_0, \psi_0, -1)$ belong to the same symmetric prime periodic orbit
\[ \{\gamma_P\} = H^{-1}(\{\gamma_G\}) \] (22)
of length
\[ n(\gamma_P) = 2 \cdot n(\gamma_G) . \] (23)
The corresponding geometric length, too, has doubled:
\[ l(\gamma_P) = 2 \cdot l(\gamma_G) . \] (24)

Thus the set $M_G$ of prime periodic orbits of the Gauß map $T_G$ is split into two disjoint parts $H(M_P^a)$ and $H(M_P^s)$.

2.4 Partition sums

Now we have related the prime periodic orbits of the Artin billiard to those of the modular surface, those to the ones of the Poincaré map and finally the latter ones to those of the Gauß map. In order to determine dynamical zeta functions we have to calculate sums over all prime periodic orbits of a quantity $\phi$, that depends on the geometric length $l(\gamma)$ of the orbits. The sum for the Artin billiard is:
\[
\sum_{\gamma_b \in M_b} \phi(l(\gamma_b)) = \sum_{\gamma_b \in G(M_m^a)} \phi(l(\gamma_b)) + \sum_{\gamma_b \in G(M_m^s)} \phi(l(\gamma_b))
\]
\[
= \frac{1}{2} \sum_{\gamma_m \in M_m^a} \phi(l(\gamma_m)) + \sum_{\gamma_m \in M_m^s} \phi(\frac{1}{2} \cdot l(\gamma_m))
\]
\[
= \frac{1}{2} \sum_{\gamma_P \in M_P^a} \phi(l(\gamma_P)) + \sum_{\gamma_P \in M_P^s} \phi(\frac{1}{2} \cdot l(\gamma_P))
\]
\[
= \frac{1}{2} \cdot 2 \cdot \sum_{\gamma_G \in H(M_P^a)} \phi(l(\gamma_G)) + \sum_{\gamma_G \in H(M_P^s)} \phi(\frac{1}{2} \cdot 2 \cdot l(\gamma_G))
\]
\[
= \sum_{\gamma_G \in M_G} \phi(l(\gamma_G)) \] (25)
equal to the sum for the Gauß map.

In the same way one shows, that the number $n(\gamma_b)$ of intersections of an Artin billiard’s orbit with the unit circle is equal to the length $n(\gamma_G)$ of the Gauß map’s orbit:

1. $n(\gamma_G)$ even $\Rightarrow n(\gamma_G) = n(\gamma_P) = n(\gamma_m) = n(\gamma_b)$
2. $n(\gamma_G)$ odd $\Rightarrow n(\gamma_G) = n(\gamma_P)/2 = n(\gamma_m)/2 = n(\gamma_b)$

Thus we can calculate the dynamical zeta functions for the Artin billiard as easily as those for the Gauß map.
2.5 Dynamical zeta functions

When calculating the Ruelle zeta function \( Z^D_R \) of a billiard with Dirichlet boundary condition we have to include the phase factor \( \exp(\pi i) = (-1) \) at every reflection from a wall \[22\] and thus the phase factor \( \exp(2\pi i) = 1 \) at every reflection from a corner in between two walls. In the Artin billiard the \( x \)-component of the orbit’s direction changes its sign at every reflection from the lines \( x = 0 \) or \( x = 1/2 \), but it keeps it sign when reflected from the unit circle. As the total number of sign changes along a periodic orbit must be even, it is sufficient to include the phase factor \((-1)\) at every reflection from the unit circle, that is \( n(\gamma_b) = n(\gamma_G) \) times.

The Ruelle zeta function \( Z^D_R \) of the Artin billiard with Dirichlet boundary condition is defined for large enough real part \( \Re \beta \) as a product over all prime periodic orbits \( \gamma \):

\[
\frac{1}{Z^D_R}(\beta) = \prod_{\gamma} (1 - \exp(-\beta \cdot l(\gamma)) \cdot (-1)^{n(\gamma)})
\]  

(26)

This Euler product can be rewritten as a Dirichlet sum-like formula \[6\]:

\[
\frac{1}{Z^D_R}(\beta) = \exp(- \sum_{\gamma_b \in M_b} \sum_{m=1}^{\infty} \frac{1}{m} \cdot (e^{-\beta l(\gamma_b)} \cdot (-1)^{n(\gamma_b)})^m) 
\]  

(27)

\[
= \exp(- \sum_{\gamma_G \in M_G} \sum_{m=1}^{\infty} \frac{1}{m} \cdot (e^{-\beta l(\gamma_G)} \cdot (-1)^{n(\gamma_G)})^m) 
\]  

(28)

\[
= \exp(- \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sum_{x \in \text{Fix}T^G_n} e^{-\beta l_n(x)} \cdot (-1)^n), 
\]  

(29)

where \( \text{Fix}T^G_n \) is the set of fixpoints of \( T^G_n \) and

\[
l_n(x) := \sum_{i=0}^{n-1} r(T^G_i(x)) .
\]  

(30)

The sum of exponentials can be calculated by the transfer operator method \[6\].

The transfer operator \( L_\beta \) is the generalised Frobenius Perron operator of the Gauß map:

\[
L_\beta f(x) := \sum_{w : T_G(w) = x} \frac{f(w)}{|T_G(w)|^\beta}
\]  

(31)

\[
= \sum_{w : T_G(w) = x} f(w) \cdot e^{-\beta r(x)} .
\]  

(32)
The result is:

\[
\sum_{x \in \text{Fix}T_n} e^{-\beta L_n(x)} = \sum_{x \in \text{Fix}T_n} \prod_{i=0}^{n-1} \exp(-\beta r(T_n^i(x)) \quad (33)
\]

\[
= \text{Tr}(L^R_n) - \text{Tr}((-L_{\beta+1})^n) \quad . (34)
\]

Inserting this into equation (29) we get:

\[
\frac{1}{Z_R^D(\beta)} = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \cdot (-1)^n \cdot (\text{Tr}(L^R_n) - \text{Tr}((-L_{\beta+1})^n))\right) \quad (35)
\]

\[
= \frac{\det(1 + L_{\beta})}{\det(1 - L_{\beta+1})} \quad . (36)
\]

The Selberg zeta function hence is given by:

\[
Z^D_S(\beta) = \prod_{k=0}^{\infty} \frac{1}{Z_R^D(\beta + k)} \quad (37)
\]

\[
= \det(1 + L_{\beta}) \cdot \prod_{k=1}^{\infty} \frac{\det(1 + L_{\beta+k})}{\det(1 - L_{\beta+k})} \quad . (38)
\]

In the same way, only replacing the phase factor \((-1)\) by 1, one gets the Ruelle zeta function \(Z_R^N\) of the Artin billiard with Neumann boundary conditions:

\[
\frac{1}{Z_R^N(\beta)} = \frac{\det(1 - L_{\beta})}{\det(1 + L_{\beta+1})} \quad . (39)
\]

The Selberg zeta function for Neumann boundary conditions then is:

\[
Z^N_S(\beta) = \det(1 - L_{\beta}) \cdot \prod_{k=1}^{\infty} \frac{\det(1 - L_{\beta+k})}{\det(1 + L_{\beta+k})} \quad . (40)
\]

In [10] Bogomolny and Carioli conjectured, that the Selberg zeta functions for the two cases of Dirichlet and Neumann boundary conditions should be equal to \(\det(1 + L_{\beta})\) and \(\det(1 - L_{\beta})\). This conjecture is violated by the additional factors in equations (38) and (40).

The poles and zeros on the line \(\Re \beta = 1/2\), which determine the spectrum of the Laplace-Beltrami operator \(L\), are, however, not affected by these additional factors: It follows from a result by Pollicott in [21], that for \(\Re \beta > 1\) all the eigenvalues of \(L_{\beta}\) have a modulus smaller than 1. Thus \(\det(1 + L_{\beta+k})\) and \(\det(1 - L_{\beta+k})\) have no zeros for \(\Re \beta = 1/2\) and \(k \geq 1\). In [17] it was shown, that \(\beta = (1 - j)/2; j \in \mathbb{N}_0\) are the only poles of \(L_{\beta}\). Thus \(\det(1 + L_{\beta+k})\) and \(\det(1 - L_{\beta+k})\) have no poles for \(\Re \beta = 1/2\) and \(k \geq 1\).
Finally, multiplying the Selberg zeta functions for Dirichlet and Neumann boundary conditions, we get the Selberg zeta function $Z^m_S$ for the modular surface $\Gamma$ as:

$$Z^m_S(\beta) = Z^N_S(\beta) \cdot Z^D_S(\beta)$$  \hspace{0.5cm} (41)

$$= \det(1 - L_\beta) \cdot \det(1 + L_\beta) \quad .$$ \hspace{0.5cm} (42)

3 The Artin–Venkov–Zograf factorization formula

In [15] Venkov and Zograf proved a remarkable factorization formula for the Selberg zeta function for normal subgroups $\Gamma'$ of a Fuchsian group $\Gamma$ with finite factor group $\Gamma/\Gamma'$. This formula is well known in algebraic number theory and was found for number theoretic zeta functions by Artin and Tagaki [18].

Starting from the relation of the spectra of the free Schrödinger–operator on the corresponding surfaces for the groups $\Gamma$ and $\Gamma'$ they showed that the Selberg zeta function for the subgroup $\Gamma'$ can be simply expressed as a product of zeta functions for $\Gamma$ with all finite dimensional unitary representations of this group which have $\Gamma'$ in their kernel. To be more precise, denote by $\chi^*(\Gamma/\Gamma')$ all inequivalent unitary irreducible representations of the factor group $\Gamma/\Gamma' = \{g \cdot \Gamma' : g \in \Gamma\}$, whose elements we denote by $\{g\}$. If $\tilde{\chi} \in \chi^*(\Gamma/\Gamma')$ then $\tilde{\chi}$ obviously defines also a unitary representation $\chi$ of the group $\Gamma$ by

$$\chi(g) = \tilde{\chi}(g \cdot \Gamma')$$ \hspace{0.5cm} (43)

For $g \in \Gamma'$ this gives

$$\chi(g) = \tilde{\chi}(g \cdot \Gamma') = \tilde{\chi}(1 \cdot \Gamma') = 1$$ \hspace{0.5cm} (44)

and hence $\Gamma' \in \text{kernel} \chi$.

Consider next the generalized dynamical zeta function $Z^\Gamma_S(\beta, \chi)$ of the geodesic flow on the surface determined by the group $\Gamma$ [2]:

$$Z^\Gamma_S(\beta, \chi) := \prod_{\gamma} \prod_{k=0}^{\infty} \det \left(1 - \chi(P_\gamma)e^{-(\beta+k)l(\gamma)}\right) \quad , \hspace{0.5cm} (45)$$

where $P_\gamma$ denotes an element of $\Gamma$ which fixes the closed orbit $\gamma$, that means $P_\gamma x_\infty = x_\infty$, $P_\gamma x_{-\infty} = x_{-\infty}$, if $\gamma = (x_{-\infty}, x_{+\infty})$ is this orbit in the Poincaré half–plane $H$ and $\chi$ is a representation of $\Gamma$ as discussed before.

Obviously for the trivial one dimensional representation $\tilde{\chi}_0$ with $\tilde{\chi}_0(g \cdot \Gamma') \equiv 1$ we also have $\chi_0(g) = 1$ for all $g \in \Gamma$ and hence

$$Z^\Gamma_S(\beta, \chi_0) = Z^\Gamma_S(\beta) = \prod_{\gamma} \prod_{k=0}^{\infty} \left(1 - e^{-(\beta+k)l(\gamma)}\right)$$ \hspace{0.5cm} (46)
is the ordinary dynamical zeta function for the geodesic flow on the surface \( \mathbb{H}/\Gamma \).\(^1\)\(^2\)

The Artin–Venkov–Zograf formula then states\(^3\):

\[
Z^\Gamma_S(\beta) = \prod_{\tilde{\chi} \in \chi^*(\Gamma/\Gamma')} Z^\Gamma_S(\beta, \chi)^{\dim \chi} .
\] (47)

The proof by the above authors uses the specific nature of the algebraic structure of Fuchsian groups. However, it turns out that this formula has an almost trivial interpretation when the dynamics of the involved geodesic flows is taken into account. This approach works at least for compact surfaces of constant negative curvature and their finite sheeted coverings. More interestingly, it works also for the modular surfaces, that means the surface defined by the modular group \( \text{SL}(2, \mathbb{Z}) \) and its finite coverings. It is expected that the same arguments can be applied to general constant negative curvature surfaces with finite volume, as soon as the thermodynamic formalism approach to their dynamical zeta functions has been worked out\(^4\). Interestingly enough, this approach even extends to a case strictly speaking not covered by the Venkov–Zograf paper, namely the group \( \text{GL}(2, \mathbb{Z}) \) of all \( 2 \times 2 \) integer matrices with determinant \( \pm 1 \) — which is not a Fuchsian group.

We will show now, that the factorization of the dynamical zeta function for the modular surface as discussed in the first sections of this paper can indeed be interpreted in this way. To see this, remember the transfer operator \( \tilde{L}_\beta \) of the geodesic flow on the modular surface\(^5\):

\[
\tilde{L}_\beta f(z, \epsilon) = \sum_{n=1}^{\infty} \left( \frac{1}{z + n} \right)^{2\beta} f \left( \frac{1}{z + n}, -\epsilon \right) .
\] (48)

The group \( \text{SL}(2, \mathbb{Z}) \) is a normal subgroup of \( \text{GL}(2, \mathbb{Z}) \) and \( \text{GL}(2, \mathbb{Z})/\text{SL}(2, \mathbb{Z}) \) has just two elements which we denote by \( \{g\} = \pm 1 \) corresponding to the two classes \( g \cdot \text{SL}(2, \mathbb{Z}) \) with \( g \in \text{GL}(2, \mathbb{Z}) \) and \( \det g = \pm 1 \). The group \( \text{GL}(2, \mathbb{Z})/\text{SL}(2, \mathbb{Z}) \) has just two finite dimensional irreducible unitary representations \( \tilde{\chi}_1, \tilde{\chi}_2 \), both one dimensional with

\[
\tilde{\chi}_1(g \cdot \text{SL}(2, \mathbb{Z})) = 1
\] (49)

and

\[
\tilde{\chi}_2(g \cdot \text{SL}(2, \mathbb{Z})) = \det g
\] (50)

for all \( g \in \text{GL}(2, \mathbb{Z}) \).\(^6\) The corresponding representations \( \chi_1, \chi_2 \) of \( \text{GL}(2, \mathbb{Z}) \) with \( \text{SL}(2, \mathbb{Z}) \) in their kernel are

\[
\chi_1(g) = 1
\] (51)

and

\[
\chi_2(g) = \det g .
\] (52)
The above transfer operator $\tilde{L}_\beta$ can be rewritten also as follows:

\[
\tilde{L}_\beta f(z, \{g\}) = \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} f\left( \frac{1}{z+n}, \{JQT^{-n}\} \{g\} \right) \]

(53)

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} \tilde{\chi}_L(\{T^nQJ\}) f\left( \frac{1}{z+n}, \{g\} \right)
\]

(54)

where

\[
\tilde{\chi}_L : GL(2, \mathbb{Z})/SL(2, \mathbb{Z}) \to \text{Aut}(C(GL(2, \mathbb{Z})/SL(2, \mathbb{Z})))
\]

(55)

denotes the so called left–regular representation of the group $GL(2, \mathbb{Z})/SL(2, \mathbb{Z})$ on the two dimensional space of complex valued functions on the group $GL(2, \mathbb{Z})/SL(2, \mathbb{Z})$ defined quite generally for any finite group $G$ as [19]:

\[
\tilde{\chi}_L(g')f(g) = f(g'^{-1}g) \text{ for } g, g' \in G
\]

(56)

and $f \in C(G)$, the space of complex functions on $G$ with $\dim C(G) = \text{order of } G$.

In equation (53) $J$ denotes the reflection $Jz = -z^*$ and $T$ and $Q$ are the generators $Tz = z + 1$ resp. $Qz = -1/z$ of the group $PSL(2, \mathbb{Z})$. They are given by the corresponding matrices in $GL(2, \mathbb{Z})$. We also made use of the fact that $\{JQT^{-n}\} \{g\} = -\{g\}$ for all $n$ and all $\{g\} = \pm 1$ since $\det(JQT^{-n}) = -1$ for all $n$.

To understand expression (54) better remember that the Poincaré map for the Artin-billiard was simply

\[
P(x, y) = \left( T_Gx, \frac{1}{y + |1/x|} \right)
\]

(57)

Consider then a geodesic on the Artin billiard starting in the point $x, y$ of the Poincaré section. On the modular surface this point is covered by the two points $(x, y, \{g\})$ with $\{g\} = \pm 1 \in GL(2, \mathbb{Z})/SL(2, \mathbb{Z})$. When starting in one of them, say $(x, y, 1)$, the point will come back to a lift of the points $P(x, y)$ in the Poincaré section of the Artin billiard. From the symbolic dynamics of Series et al. [12], [13] it follows that the Poincaré map $P$ just corresponds to the map $JGT^{-n}$ of the endpoints $x_-$ and $x_+$ of the corresponding half circle $\gamma$ in the upper half plane and hence the geodesic arrives in the point $P(x, y), \{JQT^{-n}\} = P(x, y), -1$, which then defines the Poincaré map for the geodesic flow on the modular surface. The left regular representation $\tilde{\chi}_L$ can be decomposed completely into its irreducible components [19] which are just all the finite–dimensional irreducible unitary representations $\chi^\ast(GL(2, \mathbb{Z})/SL(2, \mathbb{Z}))$ and therefore $\tilde{\chi}_1$ and $\tilde{\chi}_2$. It is known [19] that each of these representations occurs in $\tilde{\chi}_L$ just as many times as given by its dimension,
hence exactly once. The transfer operator $\tilde{L}_\beta$ hence can be written as

$$\tilde{L}_\beta = L_{\beta,\chi_1} \oplus L_{\beta,\chi_2}$$

with

$$L_{\beta,\chi_1} f(z) = \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} f\left( \frac{1}{z+n} \right)$$

and

$$L_{\beta,\chi_2} f(z) = -\sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} f\left( \frac{1}{z+n} \right) = -L_{\beta,\chi_1} f(z)$$

since $\chi_2(T^nQJ) = -1$ for all $n$.

(61)

It is then straightforward to show, that the zeta functions for the Artin billiard with Neumann and Dirichlet boundary conditions coincide with the zeta functions for $GL(2,\mathbb{Z})$ with representations $\chi_1$ and $\chi_2$:

$$Z^N_S(\beta) = Z^{GL(2,\mathbb{Z})}_S(\beta,\chi_1)$$

and

$$Z^D_S(\beta) = Z^{GL(2,\mathbb{Z})}_S(\beta,\chi_2)$$

and our factorization

$$Z^{SL(2,\mathbb{Z})}_S(\beta) = Z^D_S(\beta) \cdot Z^N_S(\beta)$$

coincides indeed with the Venkov–Zograf formula for $SL(2,\mathbb{Z})$ when considered a subgroup of $GL(2,\mathbb{Z})$.

The above sequence of arguments extends immediately to any finite sheeted covering of the modular surface. If $\Gamma'$ denotes the corresponding subgroup of the group $\Gamma = SL(2,\mathbb{Z})$, then the covering group of $H/\Gamma'$ with respect to $H/\Gamma$ is just $\Gamma/\Gamma'$. The Poincaré map for the geodesic flow on the covering surface can be constructed from the Poincaré map $\tilde{T} : I_2 \times \mathbb{Z} \to I_2 \times \mathbb{Z}$ of the modular surface in complete analogy to our procedure in going from the Artin billiard to the modular surface: in the present case every point in the Poincaré section of the geodesic flow on the modular surface is covered by $d = |\Gamma : \Gamma'| = \#\{ \{ g \} \in \Gamma/\Gamma' \}$ points given by $(x, y, \epsilon, \{ g \})$, $\{ g \} \in \Gamma/\Gamma'$. A geodesic starting at the point $(x, y, \epsilon)$ in the Poincaré section of the modular surface and arriving at $\tilde{T}(x, y, \epsilon)$ gets therefore lifted to a geodesic on the covering surface starting in $(x, y, \epsilon, \{ g \})$ and arriving at the point $(\tilde{T}(x, y, \epsilon), \{ QT^{-n}\} \{ g \})$ in the lifted Poincaré section, where $n = [1/x]$. This gives the following Poincaré map:

$$\tilde{T}^\nu(x, y, \epsilon, \{ g \}) = (\tilde{T}(x, y, \epsilon), \{ QT^{-n}\} \{ g \}), \quad n = [1/x] \quad .$$

(65)

The generalized transfer operator hence has the form:

$$\tilde{L}_\beta^\nu f(z, \epsilon, \{ g \}) = \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} f\left( \frac{1}{z+n}, -\epsilon, \{ T^{n}Q \} \{ g \} \right) \quad .$$

(66)
Introducing again the left regular representation \( \tilde{\chi}_L \) of the group \( \Gamma / \Gamma' \), on the space of complex functions on \( \Gamma / \Gamma' \) we can write \( \tilde{\chi}_L \) also as

\[
\tilde{\chi}_L \beta f(z, \epsilon, \{g\}) = \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} \tilde{\chi}_L(QT^{-ne}) f \left( \frac{1}{z+n}, -\epsilon, \{g\} \right) . \tag{67}
\]

The left regular representation \( \tilde{\chi}_L \) decomposes again as [19]:

\[
\tilde{\chi}_L = \bigoplus_{\chi_i \in \chi^*(\Gamma / \Gamma')} n_i \tilde{\chi}_i \tag{68}
\]

with \( n_i \) being the dimension of the irreducible representation \( \tilde{\chi}_i \) of \( \Gamma / \Gamma' \). This shows that the transfer operator \( \tilde{L}_\beta^{\Gamma'} \) for the geodesic flow on the covering surface \( H / \Gamma' \) of the modular surface can be rewritten in the appropriate basis of the space of complex functions on \( \Gamma / \Gamma' \) as

\[
\tilde{L}_\beta^{\Gamma'} f(z, \epsilon) = \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} \chi_i(QT^{-ne}) f \left( \frac{1}{z+n}, -\epsilon \right) \tag{69}
\]

But this shows that

\[
\det(1 - \tilde{L}_\beta^{\Gamma'}) = \prod_{i=1}^{k} \det \left( 1 - \tilde{L}_{\beta, \chi_i}^{\Gamma} \right)^{n_i} \tag{70}
\]

where

\[
\tilde{L}_{\beta, \chi_i}^{\Gamma} f(z, \epsilon) = \sum_{n=1}^{\infty} \left( \frac{1}{z+n} \right)^{2\beta} \chi_i(QT^{-ne}) f \left( \frac{1}{z+n}, -\epsilon \right) \tag{71}
\]

is the transfer operator for the geodesic flow on the modular surface with unitary representation \( \chi_i \) of the group \( \Gamma = SL(2, \mathbb{Z}) \) with \( \Gamma' \) in its kernel. From this the Venkov–Zograf formula [17] follows immediately, since

\[
Z_S^\Gamma(\beta, \chi_i) = \det(1 - \tilde{L}_{\beta, \chi_i}^{\Gamma}) \tag{72}
\]

as can be verified by standard arguments.
Of special interest is the case of the principal congruence subgroup $\Gamma(2)$ of the modular group with

$$
\Gamma(2) := \{ g \in SL(2, \mathbb{Z}) : g \text{ mod } 2 = 1 \}.
$$

Since

$$
SL(2, \mathbb{Z})/\Gamma(2) \cong SL(2, \mathbb{Z}_2) \cong S_3,
$$

where $S_3$ is the group of permutations of three elements, $H/\Gamma(2)$ is a sixfold covering of the modular surface. Indeed, its fundamental domain can be chosen as

$$
F' = \{ z \in H : 0 \leq |\Re z| \leq 1, |z \pm 1/2| \geq 1/2 \}.
$$

Its zeta function $Z_{S}^{\Gamma(2)}(\beta)$ can be written as

$$
Z_{S}^{\Gamma(2)}(\beta) = Z_{S}^{SL(2, \mathbb{Z})}(\beta, \chi_1) \cdot Z_{S}^{SL(2, \mathbb{Z})}(\beta, \chi_2) \cdot \left(Z_{S}^{SL(2, \mathbb{Z})}(\beta, \chi_3)\right)^2
$$

where $\chi_1$ denotes the trivial representation of $S_3$, $\chi_2$ is the one dimensional representation with $\chi_2(\tau) = \pm 1$ if $\tau$ is an even resp. odd permutation. The two dimensional representation $\chi_3$ is given by the rotation of the plane by the angle $\pm 2\pi/3$ respectively the reflection on the $y$-axis: $x \to -x$ and $y \to y$.

15
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Figure 1: The shaded area is the domain of the Artin billiard. It is half as big as the fundamental domain of the modular surface. The geodesic flow takes place along vertical lines or halfcircles centered on the $x$-axis. Examples are the halfcircles $\gamma_1$, $\gamma_2$ and its reflection image $J\gamma_2$.

Comment for users of the automated preprint bulletin board: This figure has not been added to the file, as it is purely introductory. Pictures of the symbolic plane can be found for example in the book by Gutzwiller or Terras.
Contents

1 Introduction 1

2 Dynamical zeta function for Artin’s billiard 3
   2.1 Geodesic flow on the modular surface 3
   2.2 Periodic orbits of the modular surface and the Artin billiard 4
   2.3 Periodic orbits of the Poincaré map and the Gauss map 6
   2.4 Partition sums 7
   2.5 Dynamical zeta functions 8

3 The Artin–Venkov–Zograf factorization formula 10