Matrix factorizations and families of curves of genus 15

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Introduction

The moduli spaces $\mathcal{M}_g$ of curves of genus $g$ is

- unirational for $g \leq 14$, [Severi, Sernesi, Chang-Ran, Verra],
- of general type for $g = 22$ and $g \geq 24$, [Harris-Mumford, Eisenbud-Harris, Farkas].

The cases in between are not fully understood:

- $\mathcal{M}_{23}$ has positive Kodaira dimension [Farkas],
- $\mathcal{M}_{15}$ is rationally connected [Bruno-Verra],
- $\mathcal{M}_{16}$ is uniruled [Chang-Ran, Farkas].

In this talk I report on an attempt to prove the unirationality of $\mathcal{M}_{15}$. 
By Brill-Noether theory,

$$W^r_d(C) = \{ L \in \text{Pic}^d C \mid h^0(L) \geq r + 1 \}$$

has dimension at least

$$\rho = g - (r + 1)(g - d + r),$$
By Brill-Noether theory,
a general curve of genus $15 = 5 \cdot 3$ has a finite number of
smooth models of degree 16 in $\mathbb{P}^4$. 
By Brill-Noether theory, a general curve of genus $15 = 5 \cdot 3$ has a finite number of smooth models of degree 16 in $\mathbb{P}^4$. Let

$$\mathcal{H} \subset \text{Hilb}_{16t+1-15}(\mathbb{P}^4)$$

be the corresponding component of the Hilbert scheme, and let

$$\tilde{\mathcal{M}}_{15} \subset \{(C, L) \mid C \in \mathcal{M}_{15}, L \in W^4_{16}(C)\} \to \mathcal{M}_{15}$$

be the component of the Hurwitz scheme, which dominates generically finite to one. So $\mathcal{H}//\text{PGL}(5)$ is birational to $\tilde{\mathcal{M}}_{15}$.
By Brill-Noether theory, a general curve of genus $15 = 5 \cdot 3$ has a finite number of smooth models of degree 16 in $\mathbb{P}^4$. Let

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be the corresponding component of the Hilbert scheme, and let

$$\tilde{M}_{15} \subset \{(C, L) \mid C \in M_{15}, L \in W_{16}^4(C)\} \to M_{15}$$

be the component of the Hurwitz scheme, which dominates generically finite to one. So $\mathcal{H} // \text{PGL}(5)$ is birational to $\tilde{M}_{15}$.

Our main result connects the moduli space $\tilde{M}_{15}$ to a moduli space of certain matrix factorizations of cubic threefolds.
Main Results

Theorem

The moduli space $\widetilde{M}_{15}$ of curves of genus 15 together with a $g^4_{16}$ is birational to a component of the moduli space of matrix factorizations of type

$$\mathcal{O}^{18}(-3) \xrightarrow{\psi} \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2) \xrightarrow{\varphi} \mathcal{O}^{18}$$

of cubic forms on $\mathbb{P}^4$.

Theorem

$\widetilde{M}_{15}$ is uniruled.
Overview

1. Introduction
2. Review of matrix factorizations
3. The structure theorem
4. Constructions
5. Tangent space computations
6. Conclusion
Matrix factorizations [Eisenbud, 1980]

$R$ a regular local ring, $f \in \mathfrak{m}^2$. A matrix factorization of $f$ is a pair $(\varphi, \psi)$ of matrices satisfying

$$\psi \circ \varphi = f \ id_G \quad \text{and} \quad \varphi \circ \psi = f \ id_F.$$ 

$M = \text{coker} \ \varphi$ is a maximal Cohen-Macaulay $R/f$-module.
Matrix factorizations [Eisenbud, 1980]

Let $R$ be a regular local ring, $f \in m^2$. A matrix factorization of $f$ is a pair $(\varphi, \psi)$ of matrices satisfying

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$M = \text{coker } \varphi$ is a maximal Cohen-Macaulay $R/f$-module.

Conversely, if $M$ is a MCM on $R/f$, then as $R$-module $M$ has a short resolution

$$0 \leftarrow M \leftarrow F \leftarrow G \leftarrow 0.$$ 

and multiplication with $f$ on this complex is null homotopic

$$0 \leftarrow M \leftarrow F \leftarrow G \leftarrow 0$$

which yields a matrix factorization $(\varphi, \psi)$. 

\[ \begin{array}{c}
\downarrow 0 \\
\downarrow \varphi \\
\downarrow f \\
\downarrow \psi \\
\downarrow f
\end{array} \]

\[ \begin{array}{c}
\downarrow 0 \\
\downarrow \varphi \\
\downarrow f \\
\downarrow \psi \\
\downarrow f
\end{array} \]
2-periodic resolutions

As an $R/f$-module, $M$ has the infinite 2-periodic resolution

$$0 \leftarrow M \leftarrow \overline{F} \leftarrow \overline{G} \leftarrow \overline{F} \leftarrow \overline{G} \leftarrow \ldots$$

where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$. In particular, this sequence is exact, and the dual sequence corresponding to the matrix factorization $(\psi^t, \varphi^t)$ is exact as well.
2-periodic resolutions

As an $R/f$-module, $M$ has the infinite 2-periodic resolution

$$0 \leftarrow M \leftarrow \overline{F} \leftarrow \overline{G} \leftarrow \overline{F} \leftarrow \overline{G} \leftarrow \ldots$$

where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$.

The resolution of an arbitrary $R/f$-module $N$ is eventually 2-periodic. If

$$0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \leftarrow F_c \leftarrow 0$$

is the finite resolution of $N$ as $R$-module then

$$0 \leftarrow N \leftarrow \overline{F}_0 \leftarrow \overline{F}_1 \leftarrow \overline{F}_2 \oplus \overline{F}_0 \leftarrow \overline{F}_3 \oplus \overline{F}_1 \leftarrow \ldots \leftarrow \overline{F}_{ev} \leftarrow \overline{F}_{odd} \leftarrow \ldots$$

is a $R/f$-resolution, where

$$F_{ev} = \bigoplus_{i \equiv 0 \mod 2} F_i \quad \text{and} \quad F_{odd} = \bigoplus_{i \equiv 1 \mod 2} F_i.$$
MCM-approximation

The high syzygy modules over a Cohen-Macaulay ring are MCM.

In case of an hypersurface, \( M = \text{coker}(\overline{F}_{\text{odd}} \to \overline{F}_{\text{ev}}) \) is a MCM module. There is a natural surjection from \( M \) to \( N \) with kernel \( P \),

\[
0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0
\]

where \( P \) is a module of finite projective dimension

\[
\text{pd}_{R/f} P < \infty.
\]
The graded case: replace $R$ by $S = k[x_0, \ldots, x_n]$

If $f \in S$ is a homogeneous form of degree $d$ then we have to take the grading into account:

- A matrix factorization is now given by a pair

\[ \begin{array}{ccc}
G & \xrightarrow{\varphi} & F & \xrightarrow{\psi} & G(d)
\end{array} \]

of maps between graded free $S$-modules.

- The $i$-th term in the (not necessarily minimal) eventually 2-periodic $S/f$-resolution obtained from an $S$-resolution $F_\bullet$ is

\[ \overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \ldots \oplus \overline{F}_0(-id/2) \]

or

\[ \overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \ldots \oplus \overline{F}_1(-(i-1)d/2) \]

in case $i$ is even or odd, respectively.
Vector bundles on hypersurfaces

If \( X = V(f) \subset \mathbb{P}^n \) is a smooth hypersurface then an MCM module

\[
M = \text{coker } \varphi
\]

sheafifies to a vector bundle

\[
\mathcal{F} = \tilde{M}
\]
on \( X \) with no intermediate cohomology,

\[
H^p(X, \mathcal{F}(t)) = 0 \text{ for all } p \text{ with } 0 < p < \dim X.
\]

If \( \det \varphi = \lambda f' \) with \( \lambda \in K \) a scalar, then

\[
\text{rank } \mathcal{F} = r.
\]
We begin now with the proof of the main theorem.

**Theorem**

The moduli space $\tilde{M}_{15}$ of curves of genus 15 together with a $g^4_{16}$ is birational to a component of the moduli space of matrix factorizations of type

$$\mathcal{O}^{18}(-3) \xrightarrow{\psi} \mathcal{O}^{15}(-1) \oplus \mathcal{O}^{3}(-2) \xrightarrow{\varphi} \mathcal{O}^{18}$$

of cubic forms on $\mathbb{P}^4$. 
Postulation

For $C \subseteq \mathbb{P}^4$ be a smooth curve of degree $d = 16$ and genus $g = 15$. We have

1. $S_C = S/I_C$, the homogeneous coordinate ring, and
2. $H^0_*(\mathcal{O}_C) = \bigoplus_n H^0(\mathcal{O}_C(n))$, the ring of sections.

Proposition

As $S$-modules these rings have free resolution with Betti tables

$$
\begin{array}{cccc}
0 & 1 & \ldots & \\
1 & . & . & . \\
2 & . & 1 & . \\
3 & 15 & 30 & 18 & 3
\end{array}
$$

and

$$
\begin{array}{cccc}
0 & 1 & \ldots & \\
1 & . & . & . \\
2 & 3 & 16 & 15 & 0 \\
3 & . & 3 & 
\end{array}
$$

iff $C$ has maximal rank and $(C, L)$ is not a ramification point of $\tilde{M}_{15} \to M_{15}$. In particular a general curve $C$ lies on a unique cubic $X$. 
Postulation

For $C \subset \mathbb{P}^4$ be a smooth curve of degree $d = 16$ and genus $g = 15$. We have

- $S_C = S/I_C$, the homogeneous coordinate ring, and
- $H^0_*(\mathcal{O}_C) = \bigoplus_n H^0(\mathcal{O}_C(n))$, the ring of sections.

Proposition

As $S$-modules these rings have free resolution with Betti tables

$$
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
0 & . & . & . & . \\
1 & . & . & . & . \\
2 & 1 & . & . & . \\
3 & 15 & 30 & 18 & 3
\end{array}
\quad\quad
\begin{array}{cccccc}
0 & 1 & 2 & 3 \\
0 & . & . & . & . \\
1 & . & . & . & . \\
2 & 3 & 16 & 15 & 0 \\
3 & . & . & 0 & 3
\end{array}
$$

iff $C$ has maximal rank and $(C, L)$ is not a ramification point of $\widetilde{\mathcal{M}}_{15} \to \mathcal{M}_{15}$. In particular a general curve $C$ lies on a unique cubic $X$. 

Syzygies $H^0_*(\mathcal{O}_C)$ of as $S_X$-module

From now on, $C \subset \mathbb{P}^4$ will always denote a general curve of degree 16 and genus 15.

The eventual 2-periodic resolution of $H^0_*(\mathcal{O}_C)$ as an $S_X = S/f$ has the shape

|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  |  ... |
|---|----|----|----|----|----|----|----|-------|
| 0 | 1  | .  | .  | .  | .  | .  | .  |  ...  |
| 1 | .  | .  | 1  | .  | .  | .  | .  |  ...  |
| 2 | 3  | 16 | 15 | .  | 1  | .  | .  |  ...  |
| 3 | .  | .  | 3  | 3+16| 15 | .  | 1  |  ...  |
| 4 | .  | .  | .  | 3  | 19 | 15 | .  |  ...  |
| ...|   | .  | .  | .  | .  | .  | 3  |  ...  |

This is not a minimal resolution.
Syzygies $H^0_*(\mathcal{O}_C)$ of as $S_X$-module

From now on, $C \subset \mathbb{P}^4$ will always denote a general curve of degree 16 and genus 15.

**Proposition**

The minimal resolution of $H^0_*(\mathcal{O}_C)$ as an $S_X = S/f$ has the shape

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
0 & 1 & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . \\
2 & 3 & 15 & 15 & . & . & . & . \\
3 & . & . & 3 & 18 & 15 & . & . \\
4 & . & . & . & 3 & 18 & 15 & \cdots \\
\vdots & . & . & . & . & . & . & 3 & \cdots \\
\end{array}
\]
From $C$ to a matrix factorization

Corollary

A general curve $C$ determines a matrix factorization of shape

$$
\begin{array}{c|ccc}
  & 0 & 1 & 2 \\
\hline
1 & 15 & . & . \\
2 & 3 & 18 & 15 \\
3 & . & . & 3
\end{array}
$$
From $C$ to a matrix factorization

**Corollary**

* A general curve $C$ determines a matrix factorization of shape

|   | 0 | 1 | 2 |
|---|---|---|---|
| 1 | 15|   |   |
| 2 | 3 | 18| 15|
| 3 |   |   | 3 |

Define $\mathcal{F}$ via

$$0 \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3) \leftarrow \mathcal{O}_X^{15}(-4) \oplus \mathcal{O}_X^3(-5)).$$

The composition

$$\mathcal{O}_X^3(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3)$$

is surjective with a summand $\mathcal{O}_X^3(-3)$ in the kernel, since there are only 5 linear forms on $\mathbb{P}^4$. 
From the matrix factorization back to \( C \)

**Theorem (Structure Theorem)**

*Given the matrix factorization associated to \( C \) then the complex*

\[
0 \leftarrow \mathcal{O}_X^3(-2) \xleftarrow{\alpha} \mathcal{F} \xleftarrow{\beta} \mathcal{O}_X^3(-3) \leftarrow 0
\]

*is a monad for the ideal sheaf \( \mathcal{I}_{C/X} \) of \( C \subset X \), i.e. \( \alpha \) is surjective, \( \beta \) injective and*

\[
\mathcal{I}_{C/X} \cong \ker \alpha / \text{im} \beta.
\]

*\( \mathcal{F} \) is a rank 7 vector bundle on the cubic \( X \), because*

\[
\deg \det \begin{pmatrix}
18 & 15 \\
3 & 3
\end{pmatrix} = 15 + 3 \cdot 2 = 7 \cdot 3.
\]
Proof of the main theorem

Since it is an open condition on matrix factorizations of shape

\[
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
1 & 15 & . & . \\
2 & 3 & 18 & 15 \\
3 & . & . & 3 \\
\end{array}
\]

to lead to a monad of a smooth curve of genus 15 and degree 16, this completes the proof of the main theorem.

We now could study the moduli space \( \mathcal{M}_X(7; c_1 \mathcal{F}, c_2 \mathcal{F}, c_3 \mathcal{F}) \) of vector bundles on the cubic threefold \( X \).
Section 4. Constructions

Different approach: construct auxiliary modules $N$, whose syzygies would lead to a desired matrix factorization.

Possible shape of Betti tables $\beta(N)$ are

|   | 0   | 1   | 2   | 3   | 4   |
|---|-----|-----|-----|-----|-----|
| 0 | $a$  | .   | .   | .   | .   |
| 1 | $b$  | $c$ | $d$ | .   | .   |
| 2 | .    | $e$ | $f$ | $h$ |     |
| 3 | .    | .   | .   | .   | $i$ |

or

|   | 0   | 1   | 2   | 3   | 4   |
|---|-----|-----|-----|-----|-----|
| 0 | $a$  | $b$ | .   | .   | .   |
| 1 | .    | $c$ | $d$ | $e$ | .   |
| 2 | .    | .   | $f$ | $h$ |     |
| 3 | .    | .   | .   | .   | .   |

A computation shows: There are 39 of the tables in the Boij-Söderberg cone with $\text{codim} \beta(N) \geq 3$, in all case we have equality.
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|-----|---|---|---|---|---|
| 0   |   | a | . | . | . |
| 1   | b | c | d | . | . |
| 2   | . | . | e | f | h |
| 3   | . | . | . | i |   |

or

|     | 0  | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|----|
| 0   | a  | b  | .  | .  | .  |
| 1   | .  | c  | d  | e  | .  |
| 2   | .  | .  | .  | f  | h  |

with $(a + d + h, b + e + i, c + f) = (3, 15, 18)$ or $(15, 3, 18)$ for the first case.
Section 4. Constructions

Different approach: construct auxiliary modules $N$, whose syzygies would lead to a desired matrix factorization.

Possible shape of Betti tables $\beta(N)$ are

\[
\begin{array}{c|cccc}
0 & a & . & . & . \\
1 & b & c & d & . \\
2 & . & . & e & f & h \\
3 & . & . & . & . & i \\
\end{array}
\quad\text{or}\quad
\begin{array}{c|cccc}
0 & a & b & . & . \\
1 & . & c & d & e \\
2 & . & . & . & f & h \\
\end{array}
\]

with $(a + d + h, b + e + i, c + f) = (3, 15, 18)$ or $(15, 3, 18)$ for the first case, and $(a + d + h, b + e, c + f) = (18, 15, 3)$ or $(18, 3, 15)$ in the second case.
Different approach: construct auxiliary modules \( N \), whose syzygies would lead to a desired matrix factorization.

Possible shape of Betti tables \( \beta(N) \) are

\[
\begin{array}{c|cccc}
0 & 1 & 2 & 3 & 4 \\
\hline
0 & a & . & . & . \\
1 & b & c & d & . \\
2 & . & . & e & f & h \\
3 & . & . & . & . & i \\
\end{array}
\]

or

\[
\begin{array}{c|cccc}
0 & 1 & 2 & 3 & 4 \\
\hline
0 & a & b & . & . \\
1 & . & c & d & e \\
2 & . & . & f & h \\
3 & . & . & . & . \\
\end{array}
\]

A computation shows: There are 39 of the tables in the Boij-Söderberg cone with \( \text{codim} \, \beta(N) \geq 3 \), in all case we have equality.
Four candidate tables

| deg $\beta(N) = 11$ | deg $\beta(N) = 14$ |
|---------------------|---------------------|
| $0$ | $0$ | $0$ | $0$ |
| $1$ | $2$ | $2$ | $6$ |
| $2$ | $3$ | $13$ | $9$ |
| $3$ | $9$ | $13$ | $6$ |

| deg $\beta(N) = 13$ | deg $\beta(N) = 14$ |
|---------------------|---------------------|
| $0$ | $0$ | $0$ | $0$ |
| $1$ | $2$ | $2$ | $14$ |
| $2$ | $3$ | $13$ | $9$ |
| $3$ | $9$ | $13$ | $6$ |

| deg $\beta(N) = 14$ | deg $\beta(N) = 14$ |
|---------------------|---------------------|
| $0$ | $0$ | $0$ | $0$ |
| $1$ | $2$ | $2$ | $12$ |
| $2$ | $3$ | $13$ | $9$ |
| $3$ | $9$ | $13$ | $6$ |
How to think about $N$?
How to think about $\mathcal{N}$?

- In all cases we will assume that

$$\mathcal{L} = \tilde{\mathcal{N}}$$

is a line bundle on an auxiliary curve $E$ of degree

$$d_E = \deg \beta(\mathcal{N}).$$
How to think about $\mathcal{N}$?

- In all cases we will assume that
  \[ \mathcal{L} = \tilde{\mathcal{N}} \]
  is a line bundle on a auxiliary curve $E$ of degree $d_E = \deg \beta(\mathcal{N})$.

- Since $\text{pd}_S(\mathcal{N}) \leq 4$, $\mathcal{N} \subset H^0_*(\mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}(n))$ and local cohomology measures the difference
  \[ 0 \to \mathcal{N} \to H^0_*(\mathcal{L}) \to H^1_{m}(\mathcal{N}) \to 0. \]
How to think about $N$?

- In all cases we will assume that

$$\mathcal{L} = \tilde{N}$$

is a line bundle on a auxiliary curve $E$ of degree

$$d_E = \deg \beta(N).$$

- Since $\text{pd}_S(N) \leq 4$, $N \subset H^0_*(\mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}(n))$ and local cohomology measures the difference

$$0 \to N \to H^0_*(\mathcal{L}) \to H^1_m(N) \to 0.$$

- Since $H^1_m(N)$ is dual to $\text{Ext}^4_S(N, S(-5))$ the 4-th map in the resolution gives us an idea about $N$. 
How to think about $N$?

- In all cases we will assume that
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  is a line bundle on a auxiliary curve $E$ of degree $d_E = \deg \beta(N)$.

- Since $pd_S(N) \leq 4$, $N \subset H^0_*(\mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{L}(n))$ and local cohomology measures the difference
  \[
  0 \rightarrow N \rightarrow H^0_*(\mathcal{L}) \rightarrow H^1_m(N) \rightarrow 0.
  \]

- Since $H^1_m(N)$ is dual to $\text{Ext}^4_S(N, S(-5))$ the 4-th map in the resolution gives us an idea about $N$.

The genus $g_E$ and the degree $d_\mathcal{L} = \deg \mathcal{L}$ are however not yet determined. Their choice is motivated by a dimension count.
Example 1.

The easiest case is perhaps $d_E = 11$ with Betti table

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 5 & 9 & . & . \\
1 & . & 3 & 13 & 6 \\
2 & . & . & . & 0 \\
\end{array}
\]

It is natural to assume that $h^0 \mathcal{O}_E(1) = 5$. Riemann-Roch $\Rightarrow h^1 \mathcal{O}_E(1) = g_E - d_E + 4 = g_E - 7$.

Parameter count:

\[
\dim \{(E, \mathcal{O}_E(1))\} = 4g_E - 3 - 5 \cdot h^1 \mathcal{O}_E(1) = 32 - g_E
\]

\[
\dim \{X | X \supset E\} = 34 - (3d_E + 1 - g_E) = g_E
\]

Finally $h^1 (L) = 0$ can be read of the Betti table, so $L$ is non-special and we obtain further $g_E$ parameters.

Altogether we get $g_E + 32$ parameters, and to obtain (at least) 42 motivates the choice $g_E = 10$. 
Example 1.

The easiest case is perhaps $d_E = 11$ with Betti table

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 5 | 9 | . | . |
| 1 | . | 3 | 13| 6 |
| 2 | . | . | . | 0 |

It is natural to assume that $h^0 \mathcal{O}_E(1) = 5$. Riemann-Roch \(\Rightarrow h^1 \mathcal{O}_E(1) = g_E - d_E + 4 = g_E - 7.\)

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Altogether
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|     | 0  | 1  | 2  | 3  |
|-----|----|----|----|----|
| 0   | 5  | 9  | .  | .  |
| 1   | .  | 3  | 13 | 6  |
| 2   | .  | .  | .  | 0  |

It is natural to assume that $h^0\mathcal{O}_E(1) = 5$. Riemann-Roch $\Rightarrow h^1\mathcal{O}_E(1) = g_E - d_E + 4 = g_E - 7$. Parameter count:

$$\dim\{(E, \mathcal{O}_E(1))\} = 4g_E - 3 - 5 \cdot h^1\mathcal{O}_E(1) = 32 - g_E$$

$$\dim\{X \mid X \supset E\} = 34 - (3d_E + 1 - g_E) = g_E$$

Finally $h^1(\mathcal{L}) = 0$ can be read of the Betti table, so $\mathcal{L}$ is non-special and we obtain further $g_E$ parameters. Altogether we get $g_E + 32$ parameters, and to obtain (at least) 42 motivates the choice $g_E = 10$. 
Example 1.

\[ g_E = 10 \Rightarrow h^1(\mathcal{O}_E(1)) = 3, \text{ so } E \text{ has a plane model } E' \text{ of degree } 18 - 11 = 7 \text{ with } \delta = \binom{6}{2} - 10 = 5 \text{ double points. So we can choose } 5 + 10 \text{ points in } \mathbb{P}^2, \]

\[ E' \in |7h - \sum_{i=1}^{5} 2p_i - \sum_{j=1}^{10} q_j|, \]

and take \( \mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \ldots + q_{10})). \)
Example 1.

\[ g_E = 10 \Rightarrow h^1(\mathcal{O}_E(1)) = 3, \] so \( E \) has a plane model \( E' \) of degree \( 18 - 11 = 7 \) with \( \delta = \binom{6}{2} - 10 = 5 \) double points. So we can choose \( 5 + 10 \) points in \( \mathbb{P}^2 \),

\[ E' \in |7h - \sum_{i=1}^{5} 2p_i - \sum_{j=1}^{10} q_j|, \]

and take \( \mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \ldots + q_{10})) \). By checking an example with Macaulay2 over a finite field we conclude:

**Theorem (Family 1)**

There exists a 42-dimensional unirational family of tuples

\[ (E, \mathcal{O}_E(1), X, \mathcal{L}) \] with \( (d_E, g_E, d_\mathcal{L}) = (11, 10, 14) \)

such that \( N = H^0_*(\mathcal{L}) \) leads to a matrix factorization of desired shape. The general one gives a smooth curve \( C \subset \mathbb{P}^4 \) of degree 16 and genus 15.
Example 2.

In case of

|   | 0   | 1   | 2   | 3   | 4   |
|---|-----|-----|-----|-----|-----|
| 0 | 2   | .   | .   | .   | .   |
| 1 | 1   | 9   | .   | .   | .   |
| 2 | .   | .   | 14  | 9   | 1   |

we have \( N \subset H^0_*(\mathcal{L}) \) with cokern \( K(-1) \). The resolution of \( N \) and \( H^0_*(\mathcal{L}) \) differ by a Koszul complex on 5 linear forms.
Example 2.

In case of

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 2 |   |   |   |   |
| 1 | 1 | 9 |   |   |   |
| 2 |   | 14| 9 | 1 |   |

we have \( N \subset H^0_*(\mathcal{L}) \) with cokern \( K(-1) \). The resolution of \( N \) and \( H^0_*(\mathcal{L}) \) differ by a Koszul complex on 5 linear forms. Thus the Betti table is \( H^0_*(\mathcal{L}) \) is

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 2 |   |   |   |
| 1 | 2 | 14| 10|   |
| 2 |   | 4 | 4 |   |
Example 2.

In case of

\[
\begin{array}{cccccc}
& 0 & 1 & 2 & 3 & 4 \\
0 & 2 & . & . & . & . \\
1 & 1 & 9 & . & . & . \\
2 & . & . & 14 & 9 & 1 \\
\end{array}
\]

we have \( N \subset H^0_*(\mathcal{L}) \) with cokern \( K(-1) \). The resolution of \( N \) and \( H^0_*(\mathcal{L}) \) differ by a Koszul complex on 5 linear forms. Thus the Betti table is \( H^0_*(\mathcal{L}) \) is

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& 0 & 1 & 2 & 3 \\
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1 & 2 & 14 & 10 & . \\
2 & . & . & 4 & 4 \\
\end{array}
\]

So \( E \) has a model in \( \mathbb{P}^3 \) and to pass from \( H^0_*(\mathcal{L}) \) to \( N \) amounts to the choice a point in a \( \mathbb{P}^1 \).
Example 2.

The dimension count suggest to take $g_E = 11$. Riemann-Roch
\[ h^1(O_E(1)) = 1, \]
hence
\[ O_E(1) \cong \omega_E(-(p_1 + \ldots + p_6)). \]

Theorem (Family 2)

There exists a 42-dimensional uniruled family of tuples
\[(E, O_E(1), X, \mathcal{L}, N) \text{ with } (d_E, g_E, d_\mathcal{L}) = (14, 11, 8)\]
such the general tuple gives a smooth curve $C \subset \mathbb{P}^4$ of degree 16 and genus 15.
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Take the line bundle \( \mathcal{L} = \omega_E(-h) \), where \( h \) denotes the hyperplane class of the model \( E \subset \mathbb{P}^3 \) of degree 12, that is a Chang-Ran curve of genus 11.

---

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Take the line bundle $\mathcal{L} = \omega_E(-h)$, where $h$ denotes the
hyperplane class of the model $E \subset \mathbb{P}^3$ of degree 12, that is a
Chang-Ran curve of genus 11. I do not know how to choose a
Chang-Ran curve together with 6 points unirationally.

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But over a finite field \( \mathbb{F}_q \) there are plenty of points in \( E(\mathbb{F}_q) \) which are easy to pick with a probabilistic method.

**Theorem (Family 2)**

There exists a 42-dimensional uniruled family of tuples

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such the general tuple gives a smooth curve \( C \subset \mathbb{P}^4 \) of degree 16 and genus 15.
All what is needed to conclude from family 1 that $\widetilde{M}_{15}$ is unirational, is to prove that the map gives an isomorphism on tangent spaces in a random example. Since the association

$$(N, X) \mapsto (M, X)$$

might not be surjective, this is a nontrivial assertion. So we want to study the natural map

$$\text{Ext}^1_{S_x}(N, N)_0 \to \text{Ext}^1_{S_x}(M, M)_0.$$
5. Tangent space diagram

The relevant diagram is

\[
\begin{array}{c}
\Ext^1_{S_X} (M, P) \rightarrow \Ext^1_{S_X} (M, M) \rightarrow \Ext^1_{S_X} (M, N) \rightarrow \Ext^2_{S_X} (M, P) \\
\uparrow \\
\Ext^1_{S_X} (N, N) \\
\uparrow \\
\Hom_{S_X} (P, N)
\end{array}
\]

deduced from the MCM approximation

\[
0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0.
\]
5. Tangent space diagram

The relevant diagram is

\[
\begin{array}{ccc}
\Ext^1_{S_X}(M, P) & \rightarrow & \Ext^1_{S_X}(M, M) \\
0 & \| & 0 \\
\Ext^1_{S_X}(N, N) & \rightarrow & \Ext^1_{S_X}(M, N) \\
\Ext^1_{S_X}(N, N) & \| & \Ext^2_{S_X}(M, P) \\
\Hom_{S_X}(P, N) & \| & \\
\end{array}
\]

deduced from the MCM approximation

\[0 \longleftrightarrow N \longleftrightarrow M \longleftrightarrow P \longleftrightarrow 0.\]
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The relevant diagram is

\[
\begin{array}{cccccc}
\Ext^1_{S_X}(M, P) & \to & \Ext^1_{S_X}(M, M) & \xrightarrow{\text{IR}} & \Ext^1_{S_X}(M, N) & \to \Ext^2_{S_X}(M, P) \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & \Ext^1_{S_X}(N, N) & \uparrow & 0 \\
\uparrow & & \uparrow & & \uparrow & \\
\Hom_{S_X}(P, N) & & & & & \\
\end{array}
\]

deduced from the MCM approximation

\[
0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0.
\]

\[
\dim \Ext^1_{S_X}(M, M)_0 = \dim \Ext^1_{S_X}(N, N)_0 = 32 \text{ as expected,}
\]

\[
\Hom_{S_X}(P, N)_0 \leftrightarrow \Ext^1_{S_X}(N, N)_0, \text{ but}
\]
Dimensions of the families

Proposition

For a randomly chosen example,

\[
\dim \text{Hom}_{S^X}(P, N)_0 = \begin{cases} 
3 & \text{in case of family 1} \\
0 & \text{in case of family 2}
\end{cases}
\]

Hence family 1 leads to a 39-dimensional subvariety of \( \widetilde{M}_{15} \) and family 2 dominates. In particular \( \widetilde{M}_{15} \) is uniruled.
Altogether I managed to construct 20 families of pairs \((X, N)\) all of dimension at least 42, of which

- 17 families are unirational,
- 3 are (possibly) not, since they required the choice of additional points on the auxiliary curve \(E\).
- The three non-unirational families dominate.
- Most of the unirational families lead to 39-dimensional subvarieties of \(\widetilde{\mathcal{M}}_{15}\). One has dimension 40, another one dimension 41.

Could this be just bad luck?
A conjecture
I think no.
A conjecture

I think no. A good explanation could be

Conjecture

The maximal rationally connected fibration of $\widetilde{\mathcal{M}}_{15}$ has a three dimensional base.

Theorem

The probabilistic algorithm, which for a finite field $\mathbb{F}_q$ selects randomly curves of genus $15$, has running time $O((\log q)^3)$.

▶ I expect that the algorithm picks points from a subset of $\mathcal{M}_{15}(\mathbb{F}_q)$ of density about 47%.

The image of $\widetilde{\mathcal{M}}_{15}(\mathbb{F}_q)$ should have density about 63%.

The same should hold for the image of the $\mathbb{F}_q$-rational points of the parameter space in $\widetilde{\mathcal{M}}_{15}(\mathbb{F}_q)$.

▶ A unirational description of $\mathcal{M}_{15}$ would lead to an algorithm with running time $O((\log q)^2)$.

▶ For any fixed genus $g$ there exists an algorithm which selects points from a subset of $\mathcal{M}_g(\mathbb{F}_q)$ of positive density in running time $O((\log q)^3)$.

Thank you!
A conjecture and a complexity result

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- A unirational description of $M_{15}$ would lead to an algorithm with running time $O((\log q)^2)$. 

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