A REMARK ON DESCENT FOR COXETER GROUPS

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Abstract. Let \( \Gamma \) be a finite Coxeter group with reflection representation \( R \). We show that a \( \Gamma \)-equivariant quasicoherent sheaf on \( R \) descends to the quotient space \( R/\Gamma \) if it descends to the quotient space \( R/(s_i) \) for every simple reflection \( s_i \in \Gamma \).

Contents

1. Introduction 1
2. Two Algebras 2
3. Proof 5
References 7

1. Introduction

1.1. Let \( W \) be a finite Weyl group with reflection representation \( t \) over \( \mathbb{C} \), and let \( \widetilde{W} \) denote the extended affine Weyl group which also acts on \( t \) in the natural manner. In \cite[Thm 1.2]{R}, the author has demonstrated that there is an equivalence of categories between modules for a certain algebra, denoted \( H \), and the full subcategory of the category of \( \widetilde{W} \)-equivariant quasicoherent sheaves on \( t \) which descend to \( t/\Gamma \) for every finite parabolic subgroup \( \Gamma \) of the affine Weyl group \( W^\text{aff} \). On the other hand, it follows directly from the result of V. Ginzburg, \cite[Prop 6.2.5]{G}, that the category of \( H \)-modules is equivalent to the full subcategory of \( \widetilde{W} \)-equivariant quasicoherent sheaves on \( t \) which descend to \( t/W \). We would like to explain this apparent discrepancy.

1.2. In \cite[Lem 2.1.1]{W}, it is shown that every reflection in \( W^\text{aff} \) is conjugate in \( \widetilde{W} \) to a simple reflection of \( W^\text{aff} \). One sees therefore that a \( \widetilde{W} \)-equivariant quasicoherent sheaf on \( t \) which descends to \( t/W \) descends also to \( t/(s) \) for every reflection \( s \) in \( W^\text{aff} \). Now fix a finite parabolic subgroup \( \Gamma \) of \( W^\text{aff} \). Then \( \Gamma \) is generated by its reflections, and is the stabilizer in \( W^\text{aff} \) of some point of \( t \). Translating to 0, one sees that \( \Gamma \) is the Weyl group of some root subsystem of the root system of \( W \). It follows that \( t \) is the direct sum of the reflection representation \( t_\Gamma \) of \( \Gamma \) and the invariant subspace \( (t)^\Gamma \). Thus Ginzburg’s claim is seen to follow from the author’s in light of the following:

Theorem 1.1. Let \( \Gamma \) be a finite Coxeter group with reflection representation \( R \) over \( \mathbb{C} \). Let \( X \) be a scheme over \( \mathbb{C} \) and let \( \mathcal{Y} \) be a \( \Gamma \)-equivariant \( R \)-bundle over \( X \).

\footnote{This fact may be standard but this is the only reference the author is aware of.}
Let $M$ be a $\Gamma$-equivariant quasicoherent sheaf on $Y$. Then $M$ descends to $Y//\Gamma$ if it descends to $Y//\langle s_i \rangle$ for every simple reflection $s_i$ in $\Gamma$.

We emphasize that the content of this theorem is essentially all contained in the case $Y = R$.

1.3. Bad grammar. Let $\Gamma$ be a finite group acting on a scheme $Y$ with GIT quotient $q : Y \to Y//\Gamma$. Suppose $M$ is a quasicoherent sheaf on $Y$. We say $M$ descends to $Y//\Gamma$ to mean that $M$ is equipped with an isomorphism $M \cong q^*M'$ for some quasicoherent sheaf $M'$ on $Y//\Gamma$. This is not a property of $M$, but rather additional data. Note that in that case $M$ receives a $\Gamma$-equivariant structure. Now suppose instead that $M$ is a $\Gamma$-equivariant quasicoherent sheaf on $Y$. We employ the same phrase: $M$ descends to $Y//\Gamma$ to mean that the underlying quasicoherent sheaf descends to $Y//\Gamma$ (in the previous sense), and moreover that the induced $W$-equivariant structure coincides with the original one. More generally, for a subgroup $\Gamma' \subset \Gamma$, we will say that $M$ descends to $Y//\Gamma'$ to mean merely that the underlying $\Gamma'$-equivariant quasicoherent sheaf descends to $Y//\Gamma'$.

2. Two Algebras

2.1. The nil Hecke algebra. Let $\Gamma$ be a finite Coxeter group with simple reflections $\{s_i\}_{i \in \Sigma}$. Denote by $\alpha_i$ the corresponding simple roots; then $\mathcal{O}(R)$ is naturally identified with the symmetric algebra on the various $\alpha_i$. For $i, j \in \Sigma$ let $m_{i,j}$ denote the order of $s_is_j$. Following [4], the nil Hecke algebra is the algebra generated by the symbols $\{D_i\}_{i \in \Sigma}$ subject to the relations

$$D_i^2 = 0$$

and

$$D_iD_jD_i\ldots = D_jD_iD_j\ldots \quad \text{mod} \quad m_{i,j}$$

for all $i, j \in \Sigma$. The latter relations imply that for any $w \in \Gamma$ the element

$$D_w := D_{i_1}D_{i_2}\ldots D_{i_l}$$

is independent of choice of reduced expression $w = s_{i_1}s_{i_2}\ldots s_{i_l}$. Together with the former relations, one sees that $\{D_w\}_{w \in \Gamma}$ form a basis for the nil Hecke algebra, and also that the nil Hecke algebra is graded with each $D_i$ in degree $-1$ by convention.

2.2. This algebra acts on $\mathcal{O}(R)$ by setting

$$D_i(f) = \alpha_i^{-1}.(1 - s_i)(f),$$

for any $f \in \mathcal{O}(R)$; this is the so-called Demazure operator. $D_i$ acts by $s_i$-derivations, that is:

$$D_i(f.g) = D_i(f).g + s_i(f).D_i(g)$$

for any $f, g \in \mathcal{O}(R)$. One may accordingly take the smash product of the nil Hecke algebra with $\mathcal{O}(R)$. The resulting algebra, denoted $\mathcal{H}$, is free over its subalgebra $\mathcal{O}(R)$ with respect to both left and right multiplication, with basis $\{D_w\}_{w \in \Gamma}$ and is generated by the nil Hecke algebra and $\mathcal{O}(R)$ subject to the relations:

$$D_i.f = D_i(f) + s_i(f).D_i.$$
In fact it suffices to impose these relations when \( f = \theta \) is a linear function, in which case the relations become:

\[
D_i.\theta = \langle \alpha_i^\vee, \theta \rangle + s_i(\theta).D_i
\]

where \( \alpha_i^\vee \) is the coroot dual to \( \alpha_i \). We extend the grading on the nil Hecke algebra to one on \( \mathcal{H} \) by putting the linear functions in degree 1.

2.3. The algebra \( \mathcal{H} \) may also be understood as the largest subalgebra of the smash product \( \Gamma \# \text{Frac}(\mathcal{O}(R)) \) which stabilizes \( \mathcal{O}(R) \) in its action on \( \text{Frac}(\mathcal{O}(R)) \); \( D_i \) corresponds to the element \( \alpha_i^{-1}.(1 - s_i) \). Also \( \mathcal{H} \) receives an \( \mathcal{O}(R) \)-algebra map from \( \Gamma \# \mathcal{O}(R) \) determined by sending \( s_i \) to \( 1 - \alpha_i.D_i \) in a diagram

\[
\Gamma \# \mathcal{O}(R) \to \mathcal{H} \to \Gamma \# \text{Frac}(\mathcal{O}(R)).
\]

2.4. The main point about \( \mathcal{H} \) is that it is in an appropriate sense dual to the Hopf algebroid \( \mathcal{O}(R \times_{R/\Gamma} R) \), whose comodules are equivalent to quasicoherent sheaves on \( R/\Gamma \) since \( R/\Gamma \) is faithfully flat. We therefore see that:

**Lemma 2.1.** The \( \Gamma \)-equivariant quasicoherent sheaf \( M \) on \( R \) descends to \( R//\Gamma \) if and only if the \( \Gamma \# \mathcal{O}(R) \)-module structure on \( M \) may be extended to an \( \mathcal{H} \)-module structure.

2.5. Suppose \( X \) is a scheme over \( \mathbb{C} \). Since \( R \) is an irreducible representation of \( W \), the category of \( W \)-equivariant \( R \)-bundles over \( X \) is equivalent to the category of \( \mathbb{G}_m \)-torsors over \( X \). Recall that \( \mathcal{H} \) is graded, extending the grading of \( \mathcal{O}(R) \) which comes from the dilation of \( R \). Therefore given a \( W \)-equivariant \( R \)-torsor \( \pi : Y \to X \) one may take the underlying \( \mathbb{G}_m \)-torsor \( \pi : \mathcal{L} \to X \) and form

\[
\mathcal{H}(Y) := \pi_* \mathcal{O}(\mathcal{L}) \otimes^{\mathbb{G}_m} \mathcal{H}.
\]

This is a quasicoherent sheaf of Hopf algebroids over \( \pi_* \mathcal{O}(Y) \) on \( X \) whose fibers are copies of \( \mathcal{H} \) and which receives a natural map from \( \Gamma \# \mathcal{O}(Y) := \pi_* \mathcal{O}(\mathcal{L}) \otimes^{\mathbb{G}_m} (\Gamma \# \mathcal{O}(R)) \). It is dual over \( \pi_* \mathcal{O}(Y) \) to the Hopf algebroid

\[
\pi_* \mathcal{O}(Y \times_{Y//\Gamma} Y)
\]

whose comodules over \( \pi_* \mathcal{O}(Y) \) are equivalent to quasicoherent sheaves on \( Y//\Gamma \), since \( Y \to Y//\Gamma \) is faithfully flat. Thus one obtains the generalization of Lemma 2.1.

**Lemma 2.2.** The \( \Gamma \)-equivariant quasicoherent sheaf \( M \) on \( Y \) descends to \( Y//\Gamma \) if and only if the \( \Gamma \# \mathcal{O}(Y) \)-module structure on \( \pi_* M \) may be extended to an \( \mathcal{H}(Y) \)-module structure.

2.6. The Demazure descent algebra. Now fix a simple reflection \( s_i \) and let \( M \) be an \( (s_i) \)-equivariant quasicoherent sheaf on \( R \). Then \( M \) descends to \( R//\langle s_i \rangle \) if and only if for every \( m \in M \) there is a unique element \( m' \in M \) such that \( \alpha_i.m' = (1 - s_i)(m) \). In that case one may define the operator

\[
G_i : M \to M
\]

\[
m \mapsto m'.
\]

By the uniqueness of \( m' \), one sees that \( G_i \) is linear and satisfies the relations

\[
G_i^2 = 0
\]

and

\[
G_i(f.m) = D_i(f).m + s_i(f).G_i(m)
\]
for any \( f \in \mathcal{O}(R) \), \( m \in M \). This leads us to define the \textit{Demazure descent algebra}, \( D \), to be the algebra generated by \( \mathcal{O}(R) \) and the symbols \( \{G_i\}_{i \in \Sigma} \) subject to the relations
\[
G_i^2 = 0
\]
and
\[
G_i.f = D_i(f) + s_i(f).G_i
\]
for any \( f \in \mathcal{O}(R) \). As for \( \mathcal{H} \) the second relation follows from the relation
\[
G_i.\theta = \langle \alpha_i^\vee, \theta \rangle + s_i(\theta).G_i
\]
for any linear function \( \theta \in \mathcal{O}(R) \). It is easy to see that \( D \) is free over its subalgebra \( \mathcal{O}(R) \) with respect to both left and right multiplication, with basis consisting of all words in \( \{G_i\}_{i \in \Sigma} \) without double letters.

2.7. A \( D \)-module is precisely the same thing as a quasicoherent sheaf on \( R \) which descends to \( R/\langle s_i \rangle \) for each \( i \in \Sigma \). By Lemma 2.1, a quasicoherent sheaf on \( R \) which descends to \( R/\Gamma \) is the same thing as a module for the quotient
\[
\mathcal{H} = D/(D B D)
\]
where \( B_D \) is the set of Demazure braid relations,
\[
B_D = \{G_i G_j \tilde{G}_i \ldots - G_j G_i \tilde{G}_j \ldots \}_{i,j} \in \Gamma.
\]
On the other hand, a \( \Gamma \)-equivariant quasicoherent sheaf on \( R \) which descends to \( R/\langle s_i \rangle \) for every simple reflection \( s_i \) is the same thing as a module for the quotient
\[
\mathcal{I} := D/(D B D)
\]
where \( B \) is the set of Coxeter braid relations,
\[
B = \{s_i s_j s_i \ldots - s_j s_i s_j \ldots \}_{i,j} \in \Gamma
\]
where we have set \( s_i = 1 - \alpha_i.G_i \in D \) for each \( i \in \Sigma \).

2.8. Note that \( D \) is graded, with each \( G_i \) in degree \(-1\) and each linear operator \( \theta \) on \( R \) in degree \( 1 \), so that the quotient map
\[
D \to \mathcal{H} = D/(D B D)
\]
respects the grading. Since the generators of \( B \) are also homogeneous, \( D/(D B D) \) is also graded and the quotient map
\[
D \to \mathcal{I} = D/(D B D)
\]
respects the grading. As in [24], for a \( W \)-equivariant \( R \)-bundle \( \pi : Y \to X \), we obtain the sheaves of algebras \( D(Y) \), \( \mathcal{I}(Y) \) on \( X \). They both receive algebra maps from \( \pi_* \mathcal{O}(Y) \), and are locally free on both sides over \( \pi_* \mathcal{O}(Y) \), and we have the sequence of surjective algebra homomorphisms:
\[
D(Y) \to \mathcal{I}(Y) \to \mathcal{H}(Y)
\]
which affine locally is identified up to \( \mathbb{G}_m \)-action with \( D \to I \to H \). A \( D(Y) \)-module over \( \pi_* \mathcal{O}(Y) \) is the same thing as a quasicoherent sheaf on \( Y \) which descends to \( Y/\langle s_i \rangle \) for each simple reflection \( s_i \). An \( \mathcal{I}(Y) \)-module over \( \pi_* \mathcal{O}(Y) \) is the same thing as a \( \Gamma \)-equivariant quasicoherent sheaf which descends to \( Y/\langle s_i \rangle \) for each simple reflection \( s_i \). Therefore to prove Theorem 1.1, it suffices to prove that the
natural map $\mathcal{I}(Y) \to \mathcal{H}(Y)$ is an isomorphism. For this it is enough to prove the case $Y = R$.

3. Proof

3.1. Recall we have the projection map $\mathcal{I} \to \mathcal{H}$, so that $\mathcal{D}B\mathcal{D} \subset \mathcal{D}B\mathcal{D}$. Let

$$B_{k,l} = s_{k_1}s_{k_2}\cdots s_{l_1}s_{l_2}\cdots$$

be one of the elements of $B$. Recall that $s_i = 1 - \alpha_i G_i$ for each $i$. Expanding $B_{k,l}$ with respect to the basis as a left $O(R)$-module consisting of monomials in the symbols $G_i$, we get

$$B_{k,l} = (-1)^{m_{k,l}}(\alpha_k s_k(\alpha_l), s_k s_l(\alpha_k) \cdots G_k G_l G_k \cdots - \alpha_l s_l(\alpha_k) s_l s_k(\alpha_l) \cdots G_l G_k G_l \cdots)$$

$$+l.o.t.$$.

Here the lower order terms are left $O(R)$-linear combinations of double-letter-free words in $G_k$, $G_l$ of length strictly less than $m_{k,l}$. Such words correspond to elements of $\Gamma$ with unique reduced expressions. Therefore in $\mathcal{H}$, the images of such words, together with the common image of $G_k G_l G_k \cdots$ and $G_l G_k G_l \cdots$, are $O(R)$-linearly independent. It follows that the lower order terms are zero, and also that the two coefficients are equal:

$$\alpha_k s_k(\alpha_l), s_k s_l(\alpha_k) \cdots = \alpha_l s_l(\alpha_k), s_l s_k(\alpha_l) \cdots.$$

We are able to compute them exactly. Let us embed the rank two root system $X_{k,l}$ consisting of those roots in the span of $\alpha_k, \alpha_l$ inside two-dimensional real Euclidean space in the usual manner, so that the angle measured clockwise from $\alpha_k$ to $\alpha_l$ is less than $\pi$. Then the positive roots in $X_{k,l}$ are precisely those which lie in the $\mathbf{R}_+^+$-cone swept out by rotating the half line $\mathbf{R}_+^+ \alpha_k$ clockwise as far as $\mathbf{R}_+^+ \alpha_l$. Then $s_k \alpha_l$ is the root in $X_{k,l}$ closest in angle clockwise from $\alpha_k$, and the rotation $s_k s_l$ sends any root $\alpha$ in $X_{k,l}$ to the root second closest in angle clockwise from $\alpha$. It follows that the roots

$$\alpha_k, s_k(\alpha_l), s_k s_l(\alpha_k) \cdots$$

are precisely the positive roots of $X_{k,l}$, ordered clockwise from $\alpha_k$ to $\alpha_l$. Likewise the roots

$$\alpha_l, s_l(\alpha_k), s_l s_k(\alpha_l) \cdots$$

are precisely the positive roots of $X_{k,l}$ ordered in the other direction. We may therefore write

$$B_{k,l} = (-1)^{m_{k,l}}\Delta_{k,l} B_{k,l}^D$$

where $\Delta_{k,l}$ is the product of positive roots in $X_{k,l}$ and $B_{k,l}^D = G_k G_l G_k \cdots - G_l G_k G_l \cdots$. 

3.2. We are required to prove that $B_{k,l}^D \in \mathcal{D} B D$.

**Lemma 3.1.** For every integer $n > 0$ and every word $\Xi$ in the symbols $D_k, D_l$, the elements $\Xi(\Delta_{k,l}), G_kG_lG_k \ldots$ and $\Xi(\Delta_{k,l}), G_lG_kG_l \ldots$ are both contained in $\mathcal{D} B D$.

**Proof.** For brevity we set
\[
A_k^n := G_kG_lG_k \ldots \tag{m_k,j+n}
\]
and
\[
A_l^n := G_lG_kG_l \ldots \tag{m_k,j+n}
\]
We proceed by induction on the length of the word $\Xi$. The case of length 0 follows by multiplying $\pm B_{k,l}$ on the right by words in $G_k, G_l$. Suppose next that the claim is known for all $\Xi$ of length at most $p$. Let $\Xi$ be a word of length $p + 1$, say $\Xi = D_\ell \Xi'$ for some word $\Xi'$ of length $p$ (the proof is the same if $\Xi$ starts with $D_l$). By hypothesis,
\[
\Xi'(\Delta_{k,l}).A_k^n
\]
and
\[
\Xi'(\Delta_{k,l}).A_l^n
\]
are both contained in $\mathcal{D} B D$. Multiplying the first element on the left by $G_k$, we get that
\[
G_k.\Xi'(\Delta_{k,l}).A_k^n = \Xi(\Delta_{k,l}).A_k^n + s_k(\Xi'(\Delta_{k,l})).G_k \cdot A_k^n
\]
and
\[
G_k.\Xi'(\Delta_{k,l}).A_l^n = \Xi(\Delta_{k,l}).A_l^n + \Xi'(\Delta_{k,l}).A_k^{n+1} - \alpha_k.\Xi(\Delta_{k,l}).A_k^{n+1}
\]
is contained in $\mathcal{D} B D$. The second term of the RHS is contained in $\mathcal{D} B D$ by hypothesis, while the third term of the RHS is contained in $\mathcal{D} B D$, as one sees from the previous equation (substituting $n + 1$ for $n$ and multiplying $\alpha_k$). Thus $\Xi(\Delta_{k,l}).A_k^n$ is contained in $\mathcal{D} B D$, as required. $\square$

**Lemma 3.2.** There exists an element $Z$ of the left $\mathbb{C}[\alpha_k, \alpha_l]$-submodule of $\mathcal{H}$ spanned by the words in $D_k, D_l$ such that $Z(\Delta_{k,l}) = 1$.

**Proof.** We note that the submodule in question is the subalgebra generated by $\alpha_k, \alpha_l, D_k$ and $D_l$, which is the algebra associated to the rank 2 root system generated by $\alpha_k, \alpha_l$ in the same way that $\mathcal{H}$ is associated to the root system of $\Gamma$. We will call this algebra $\mathcal{H}_{k,l}$ and write $\Gamma_{k,l}$ for its associated Coxeter group. As we have already remarked, $\mathcal{H}_{k,l}$ is equal to the subalgebra of $\Gamma_{k,l} \# \mathbb{C}[\alpha_k, \alpha_l]$ consisting of all those elements which send $\mathbb{C}[\alpha_k, \alpha_l]$ to itself in the natural action on $\mathbb{C}(\alpha_k, \alpha_l)$. We note that
\[
\Delta_{k,l}^{-1} \sum_{g \in \Gamma_{k,l}} sgn(g) g
\]
is such an element, and it sends $\Delta_{k,l}$ to the non-zero scalar $|\Gamma_{k,l}|$. $\square$

In combination, we obtain:
Lemma 3.3. The elements $G_kG_1G_k\ldots$ and $G_1G_kG_1\ldots$ are both contained in $\mathcal{D}B\mathcal{D}$.

Finally, we have:

Theorem 3.4. $B^D_{k,l} \in \mathcal{D}B\mathcal{D}$.

Proof. By Lemma 3.2 it suffices to show that for every word $\Xi$ in the symbols $D_k, D_l$, the element $\Xi(\Delta_{k,l}).B^D_{k,l}$ is contained in $\mathcal{D}B\mathcal{D}$. Again we proceed by induction on the length of $\Xi$. The length 0 case is the fact that $B_{k,l} \in \mathcal{D}B\mathcal{D}$. Suppose next that the claim is known for all $\Xi$ of length at most $p$. Let $\Xi$ be a word of length $p + 1$, say $\Xi = D_k\Xi'$ for some word $\Xi'$ of length $p$ (the proof is the same if $\Xi$ starts with $D_l$). By hypothesis, $\Xi'(\Delta_{k,l}).B^D_{k,l}$ is contained in $\mathcal{D}B\mathcal{D}$. Therefore

$$G_k.\Xi'(\Delta_{k,l}).B^D_{k,l} = \Xi(\Delta_{k,l}).B^D_{k,l} + s_k(\Xi'(\Delta_{k,l})).G_k.B^D_{k,l}$$

$$= \Xi(\Delta_{k,l}).B^D_{k,l} - s_k(\Xi'(\Delta_{k,l})).G_kG_1G_k\ldots$$

is contained in $\mathcal{D}B\mathcal{D}$. By Lemma 3.3 we are done. 

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