WEIGHTED ESTIMATES FOR MULTILINEAR COMMUTATORS OF MARCINKIEWICZ INTEGRALS WITH BOUNDED KERNEL

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Let \( \mu_\Omega, b \) be a multilinear commutator generalized by the \( n \)-dimensional Marcinkiewicz integral with bounded kernel \( \mu_\Omega \) and let \( b_j \in \text{Osc}_{\exp L^r} \), \( 1 \leq j \leq m \). We prove the following weighted inequalities for \( \omega \in A_\infty \) and \( 0 < p < \infty \):

\[
\| \mu_\Omega(f) \|_{L^p(\omega)} \leq C \| M(f) \|_{L^p(\omega)}, \quad \| \mu_{\Omega,b}(f) \|_{L^p(\omega)} \leq C \| M_L(\log L)^{1/r}(f) \|_{L^p(\omega)}.
\]

The weighted weak \( L(\log L)^{1/r} \)-type estimate is also established for \( p = 1 \) and \( \omega \in A_1 \).

1. Introduction and Main Results

Suppose that \( S^{n-1} \) is a unit sphere in \( \mathbb{R}^n \), \( n \geq 2 \), equipped with the normalized Lebesgue measure \( d\sigma \). Let \( \Omega \in L^1(S^{n-1}) \) be a homogeneous function of degree zero satisfying the cancellation condition

\[
\int_{S^{n-1}} \Omega(x') \, dx' = 0, \quad (1.1)
\]

where \( x' = x/|x| \) (\( \forall x \neq 0 \)).

The \( n \)-dimensional Marcinkiewicz integral corresponding to the Littlewood–Paley \( g \)-function introduced by Stein [1] is defined by the formula

\[
\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy.
\]

As usual, we denote the Muckenhoupt’s weights class by \( A_p, 1 \leq p < \infty \), and take \([\omega]_{A_p}\) as the constant \( A_p \) (see [2], Chapter V, or [3], Chapter 9 for details). The operators mapping \( L^p \) into \( L^q \) are called the operators of strong \((p,q)\) type and the operators mapping \( L^p \) into \( L^{q,\infty} \) are called the operators of weak \((p,q)\) type (see [3, p. 32]). Let

\[
\log^+ t = \max(\log t, 0) = \begin{cases} \log t, & \text{for } t > 1, \\ 0, & \text{for } 0 \leq t \leq 1, \end{cases}
\]

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where \( \log t = \ln t \). By \( L(\log L) \), we denote the set of all \( f \) with
\[
\int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)| \, dx < \infty
\]
(see [2, p.128], [3], § 7.5.a). Here and in what follows, \( \|b\|_{\ast} \) denotes the BMO-norm of \( b \) (see [3], Chapter 7 for details).

In 1958, Stein [1] proved that \( \mu_{\Omega} \) is of strong \((p, p)\) type for \( 1 < p \leq 2 \) and of weak \((1, 1)\) type for \( \Omega \in \text{Lip}_\alpha \), \( 0 < \alpha \leq 1 \), i.e., there is a constant \( C > 0 \) such that
\[
|\Omega(x') - \Omega(y')| \leq C|x' - y'|^{\alpha} \quad \forall x', y' \in S^{n-1}.
\] (1.2)

In 1990, Torchinsky and Wang [4] studied the weighted \( L^p \)-boundedness of \( \mu_{\Omega} \) with \( \Omega \) satisfying (1.1) and (1.2). They also considered the weighted \( L^p \)-norm inequality for the commutator of the Marcinkiewicz integral defined by the equality
\[
\mu_{\Omega, b}^m(f)(x) = \left( \int_0^\infty \left| \int_{|x-y| \leq t} \frac{(b(x) - b(y)) \Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy \right|^2 \, \frac{dt}{t^3} \right)^{1/2}, \quad m \in \mathbb{N}.
\]

In 2004, Ding, Lu, and Zhang [5] studied the weighted weak \( L(\log L) \)-type estimates for \( \mu_{\Omega, b}^m \). More precisely, if \( \omega \in A_1 \), \( b \in \text{BMO} \), and \( \Omega \) satisfies (1.1) and (1.2), then, for all \( \lambda > 0 \), there exists a constant \( C > 0 \), such that
\[
\omega\left( \{ x \in \mathbb{R}^n : \mu_{\Omega, b}^m(f)(x) > \lambda \} \right) \leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left( 1 + \log^+ \frac{|f(x)|}{\lambda} \right)^m \omega(x) \, dx.
\]

In 2008, Zhang [6] studied the weighted boundedness for the multilinear commutator of the Marcinkiewicz integral \( \mu_{\Omega, b}^m \) with \( \Omega \in \text{Lip}_\alpha \), \( 0 < \alpha \leq 1 \), \( 0 < p < \infty \), and \( \omega \in A_\infty \) (see [3], § 9.3) and established the weighted weak \( L(\log L)^{1/p} \)-type estimate for \( p = 1 \) and \( \omega \in A_1 \), where
\[
\mu_{\Omega, b}^m(f)(x) = \left( \int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left( \prod_{j=1}^m (b_j(x) - b_j(y)) \right) f(y) \, dy \right|^2 \, \frac{dt}{t^3} \right)^{1/2}, \quad m \in \mathbb{N}.
\]

Moreover, in 2012, Zhang, Wu, and Liu [7] established the weighted weak \( L(\log L)^m \)-type estimate for \( \mu_{\Omega, b}^m \) with \( \Omega \) satisfying a kind of the Dini conditions.

In 2004, Lee and Rim [8] proved the \( L^p \) boundedness for \( \mu_{\Omega} \) in the case where there exist constants \( C > 0 \) and \( \rho > 1 \) such that
\[
|\Omega(x') - \Omega(y')| \leq \frac{C}{\left( \log \frac{1}{|x' - y'|} \right)^{\rho}}
\] (1.3)
holds uniformly in \( x', y' \in S^{n-1} \) and \( \Omega \in L^\infty(S^{n-1}) \) is a homogeneous function of degree zero with the can-
cellation property (1.1). In 2005, Ding [9] studied the weak (1, 1)-type estimate in the case where \( \rho > 2 \) and \( \Omega \) satisfies (1.1) and (1.3).

In what follows, we always assume that \( \Omega \in L^\infty(S^{n-1}) \) and satisfies (1.1) and (1.3) with \( \rho > 2 \). Let \( m \) be a positive integer. For \( \vec{b} = (b_1, b_2, \ldots, b_m) \), \( b_j \in \text{Osc}_{\exp L^r_j} \), \( r_j \geq 1, 1 \leq j \leq m \), we denote

\[
\frac{1}{r} = \frac{1}{r_1} + \ldots + \frac{1}{r_m}, \quad \|\vec{b}\| = \prod_{j=1}^m \|b_j\|_{\text{Osc}_{\exp L^r_j}}. \tag{1.4}
\]

For the definitions of \( \text{Osc}_{\exp L^r_j} \), \( \|\cdot\|_{\text{Osc}_{\exp L^r}} \) and \( M_{L(\log L)^{1/r}} \), see Section 2.

Our results can be formulated as follows:

**Theorem 1.1.** Let \( 0 < p < \infty \) and let \( \omega \in A_\infty \). For \( \rho > 2 \), \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies (1.1) and (1.3). Then there is a positive constant \( C \), such that

\[
\int_{\mathbb{R}^n} |\mu_\omega(f)(x)|^p \omega(x) \, dx \leq C[\omega]_{A_\infty}^p \int_{\mathbb{R}^n} [M(f)(x)]^p \omega(x) \, dx
\]

for all bounded functions \( f \) with compact support.

**Theorem 1.2.** Let \( 0 < p < \infty \), \( \omega \in A_\infty \), and \( b_j \in \text{Osc}_{\exp L^r_j} \), \( r_j \geq 1, 1 \leq j \leq m \), and let \( r \) and \( \|\vec{b}\| \) be as in (1.4). For \( \rho > 2 \), \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies (1.1) and (1.3). Then there is a positive constant \( C \) such that

\[
\int_{\mathbb{R}^n} |\mu_{\Omega, \vec{b}}(f)(x)|^p \omega(x) \, dx \leq C\|\vec{b}\|^p \int_{\mathbb{R}^n} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x) \, dx \tag{1.5}
\]

for all bounded functions \( f \) with compact support.

Since \( r_j \geq 1, j = 1, 2, \ldots, m \), we conclude that \( M_{L(\log L)^{1/r}} \) is pointwise smaller than \( M_{L(\log L)^m} \). In view of the fact that \( M_{L(\log L)^m} \) is equivalent to \( M^{m+1} \), by using the \( m+1 \) iterations of the Hardy–Littlewood maximal operator \( M \) (see (21) in [10]), the weighted \( L^p \)-boundedness of \( M \), and Theorem 1.2, we get the following result:

**Corollary 1.1.** Let \( 1 < p < \infty \), \( \omega \in A_p \), \( b_j \in \text{Osc}_{\exp L^r_j} \), \( r_j \geq 1, 1 \leq j \leq m \), and let \( r \) and \( \|\vec{b}\| \) be as in (1.4). For \( \rho > 2 \), \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies (1.1) and (1.3). Then there is a positive constant \( C \), such that

\[
\int_{\mathbb{R}^n} |\mu_{\Omega, \vec{b}}(f)(x)|^p \omega(x) \, dx \leq C\|\vec{b}\|^p \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx
\]

for all bounded functions \( f \) with compact support.

**Theorem 1.3.** Let \( \omega \in A_1 \), \( b_j \in \text{Osc}_{\exp L^r_j} \), \( r_j \geq 1, 1 \leq j \leq m \), and let \( r \) and \( \|\vec{b}\| \) be as in (1.4). For \( \rho > 2 \), \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies (1.1) and (1.3). Also let

\[
\Phi(t) = t \log^{1/r}(e + t).
\]
Then there is a positive constant \( C \) (for all bounded functions \( f \) with compact support and all \( \lambda > 0 \)) such that

\[
\omega(\{ x \in \mathbb{R}^n : \mu_{\Omega,b}(f)(x) > \lambda \}) \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{\| \tilde{b} \| f(y) \|}{\lambda} \right) \omega(y) \, dy.
\]

The remaining part of the paper is organized as follows. In Section 2, we recall some notation and known results necessary for our subsequent presentation and establish the basic estimates for sharp functions. In Section 3, we prove Theorems 1.1 and 1.2. In the last section, we prove Theorem 1.3.

Throughout the paper, \( C \) denotes a constant independent of the main parameters involved. However, its values may differ from line to line. For any index \( p \in [1, \infty) \), we denote by \( p' \) its conjugate index, namely, \( 1/p + 1/p' = 1 \).

The relationship \( A \subset B \) means that there exists a constant \( C > 0 \) such that \( C^{-1}B \subset A \subset CB \).

2. Preliminaries and Estimates for Sharp Functions

As usual, \( M \) stands for the Hardy–Littlewood maximal operator. For a ball \( B \) in \( \mathbb{R}^n \), denote

\[
f_B = |B|^{-1} \int_B f(y) \, dy.
\]

We need the following versions of \( M \) and the Fefferman–Stein’s sharp function: For \( \delta > 0 \), we define

\[
M_\delta(f)(x) = \left[ M(|f|^\delta)(x) \right]^{1/\delta}, \quad M_\delta^\sharp(f)(x) = \left[ M^2(|f|^\delta)(x) \right]^{1/\delta},
\]

where

\[
M^\sharp(f)(x) = \sup_{B \ni x} \inf_{c} \frac{1}{|B|} \int_B |f(y) - c| \, dy \approx \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy.
\]

The relationships between \( M_\delta^\sharp \) and \( M_\delta \) used in what follows are versions of the classical relationships proposed by Fefferman and Stein (see [2, p. 153]).

**Lemma 2.1** [10–12].

(a) Let \( \omega \in A_\infty \) and let \( \phi : (0, \infty) \to (0, \infty) \) be doubling. Then there exists a positive constant \( C \) depending on the doubling condition of \( \phi \) such that, for all \( \lambda, \delta > 0 \)

\[
\sup_{\lambda > 0} \phi(\lambda) \omega(\{ y \in \mathbb{R}^n : M_\delta(f)(y) > \lambda \}) \leq C[\omega]_{A_\infty} \sup_{\lambda > 0} \phi(\lambda) \omega(\{ y \in \mathbb{R}^n : M_\delta^\sharp(f)(y) > \lambda \}),
\]

for every function \( f \) such that the left-hand side is finite.

(b) Let \( \omega \in A_\infty \) and \( 0 < p, \delta < \infty \). Then there exists a positive constant \( C \) depending on \( p \) such that

\[
\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p \omega(x) \, dx \leq C[\omega]_{A_\infty}^{p} \int_{\mathbb{R}^n} [M_\delta^\sharp(f)(x)]^p \omega(x) \, dx,
\]

for every function \( f \) such that the left-hand side is finite.
A function $\Phi$ defined on $[0, \infty)$ is called a Young function if $\Phi$ is a continuous nonnegative strictly increasing convex function with $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$. We define the $\Phi$-average of a function $f$ over a ball $B$ as follows:

$$
\|f\|_{\Phi,B} = \inf \left\{ \lambda > 0 : \frac{1}{|B|} \int_B \Phi\left(\frac{|f(y)|}{\lambda}\right) \, dy \leq 1 \right\}.
$$

The maximal operator $M_{\Phi}$ associated with the $\Phi$-average $\| \cdot \|_{\Phi,B}$, is defined as

$$
M_{\Phi}(f)(x) = \sup_{B \ni x} \|f\|_{\Phi,B},
$$

where the supremum is taken over all balls $B$ containing $x$.

For $\Phi(t) = t \log^r(e + t)$, we denote $\| \cdot \|_{\Phi,B}$ and $M_{\Phi}$ by $\| \cdot \|_{L(\log L)^r,B}$ and $M_{L(\log L)^r}$, respectively. For $\Phi(t) = e^t - 1$, we denote $\| \cdot \|_{\Phi,B}$ and $M_{\Phi}$ by $\| \cdot \|_{\exp L^r,B}$ and $M_{\exp L^r}$, respectively. If $k \in \mathbb{N}$, then

$$
M_{L(\log L)^m} \sim M^{m+1}
$$

(see (21) in [10]).

We get the generalized Hölder’s inequality as follows (for details and more general cases, see Lemma 2.3 in [11]):

**Lemma 2.2** [11]. Let $r_1, \ldots, r_m \geq 1$ with $1/r = 1/r_1 + \ldots + 1/r_m$ and let $B$ be a ball in $\mathbb{R}^n$. Then the generalized Hölder inequality is true:

$$
\frac{1}{|B|} \int_B |f_1(x) \ldots f_m(x) g(x)| \, dx \leq C \left( \prod_{i=1}^m \|f_i\|_{\exp L^{r_i},B} \right) \|g\|_{L(\log L)^{1/r},B}.
$$

For $r \geq 1$, we say that $f \in \text{Osc}_{\exp L^r}$ if $f \in L_{1,\text{loc}}^1(\mathbb{R}^n)$ and $\|f\|_{\text{Osc}_{\exp L^r}} < \infty$, where

$$
\|f\|_{\text{Osc}_{\exp L^r}} = \sup_B \|f - f_B\|_{\exp L^r,B},
$$

and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

By the John–Nirenberg theorem (see [2] or [13]), it is not difficult to show that $\text{Osc}_{\exp L^1} = \text{BMO}(\mathbb{R}^n)$ and $\text{Osc}_{\exp L^r}$ is properly contained in $\text{BMO}(\mathbb{R}^n)$ for $r > 1$ (see [14]). Furthermore, $\|b\|_\ast \leq C \|b\|_{\text{Osc}_{\exp L^r}}$ for $b \in \text{Osc}_{\exp L^r}$ and $r \geq 1$ (see [6]). For more information on Orlicz spaces, see [15].

We use the viewpoint of vector-valued singular integrals proposed by Benedek, Calderón, and Panzone [16]. Let $\mathcal{H}$ be a Hilbert space defined by

$$
\mathcal{H} = \left\{ h : \|h\|_{\mathcal{H}} = \left( \int_0^\infty \frac{|h(t)|^2}{t^\frac{3}{2}} \, dt \right)^{1/2} < \infty \right\}.
$$

For all $x \in \mathbb{R}^n$ and $t > 0$, let

$$
F_{\Omega,b,t}(f)(x) = \int_{|x-y| \leq t} \Omega(x-y) \frac{1}{|x-y|^{n-1}} \left( \prod_{j=1}^m (b_j(x) - b_j(y)) \right) f(y) \, dy, \quad m \in \mathbb{N}.
$$
Then, for any fixed \( x \in \mathbb{R}^n \), \( F_{\Omega,t}(f)(x) \) and \( F_{\Omega,t}(f)(x) \) can be regarded as mappings from \( [0, \infty) \) to \( \mathcal{H} \) and
\[
\mu_{\Omega}(f)(x) = \| F_{\Omega,t}(f)(x) \|_{\mathcal{H}}, \quad \mu_{\Omega,t}(f)(x) = \| F_{\Omega,t}(f)(x) \|_{\mathcal{H}}.
\]

The following pointwise estimates for the sharp function of \( \mu \) are obtained from [17]:

**Lemma 2.3** [17]. Let \( 0 < \delta < 1 \) and let both \( f \) and \( \mu_{\Omega}(f) \) be locally integrable functions. For \( \rho > 2 \), \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies (1.1) and (1.3). Then there is a positive constant \( C \) independent of \( f \) and \( x \) such that
\[
M_{\delta}^2(\mu_{\Omega}(f))(x) \leq CM(f)(x), \quad \text{a.e.} \ x \in \mathbb{R}^n.
\]

Some ideas used in the proof of Lemma 2.3 come from [5]. For details and additional information, see Lemma 3.2.4 in [17].

For the multilinear commutators \( \mu_{\Omega,\vec{b}} \), we have a similar pointwise estimate. To deduce it, we first introduce some notation. For all \( 1 \leq j \leq m \), we denote by \( \mathcal{C}_j^m \) the family of all finite subsets \( \sigma = \{ \sigma(1), \ldots, \sigma(j) \} \) of \( \{1, 2, \ldots, m\} \) with \( j \) different elements. For any \( \sigma \in \mathcal{C}_j^m \) and \( \vec{b} = (b_1, \ldots, b_m) \), we define
\[
\sigma' = \{1, 2, \ldots, m\} \setminus \sigma, \quad \vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)}), \quad \text{and} \quad b_\sigma = b_{\sigma(1)} \ldots b_{\sigma(j)}.
\]

For any vector \( (r_{\sigma(1)}, \ldots, r_{\sigma(j)}) \) of \( j \) positive numbers and \( 1/r_\sigma = 1/r_{\sigma(1)} + \ldots + 1/r_{\sigma(j)} \), we write
\[
\|\vec{b}_\sigma\| = \|\vec{b}_\sigma\|_{\text{Osc}_{\exp L^{r_\sigma}}} = \|b_{\sigma(1)}\|_{\text{Osc}_{\exp L^{r_{\sigma(1)}}}} \ldots \|b_{\sigma(j)}\|_{\text{Osc}_{\exp L^{r_{\sigma(j)}}}}.
\]

Moreover, for any \( \sigma = \{\sigma(1), \ldots, \sigma(j)\} \in \mathcal{C}_j^m \) and \( \vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)}) \), we have
\[
F_{\Omega,\vec{b}_\sigma,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left( \prod_{i=1}^j (b_{\sigma(i)}(x) - b_{\sigma(i)}(y)) \right) f(y) \, dy
\]
and
\[
\mu_{\Omega,\vec{b}_\sigma}(f)(x) = \left\| F_{\Omega,\vec{b}_\sigma,t}(f)(x) \right\|_{\mathcal{H}}.
\]

If \( \sigma = \{1, \ldots, m\} \), then \( \sigma' = \emptyset \). We also understand that
\[
\mu_{\Omega,\vec{b}_\sigma} = \mu_{\Omega,\vec{b}} \quad \text{and} \quad \mu_{\Omega,\vec{b}_{\sigma'}} = \mu_{\Omega}.
\]

**Lemma 2.4** [17]. Let \( r_j \geq 1 \), \( b_j \in \text{Osc}_{\exp L^{r_j}} \), \( 1 \leq j \leq m \), and let \( r \) and \( \|\vec{b}\| \) be as in (1.4). For \( \rho > 2 \), \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies (1.1) and (1.3). Then, for any \( \delta \) and \( \varepsilon \) with \( 0 < \delta < \varepsilon < 1 \), there is a constant \( C > 0 \) depending only on \( \delta \) and \( \varepsilon \) and such that, for any bounded function \( f \) with compact support,
\[
M_{\delta}^2(\mu_{\Omega,\vec{b}}(f))(x) \leq C \left( \|\vec{b}\| M_\rho(f)(x) + \sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^m} \|\vec{b}_\sigma\|_{\text{Osc}_{\exp L^{r_\sigma}}} M_\varepsilon(\mu_{\Omega,\vec{b}_\sigma}(f))(x) \right).
\]
Some ideas used in the proof of Lemma 2.4 come from [5, 6, 10, 11]. For details and additional information, see Lemma 3.2.5 in [17].

Remark 2.1. In view of the fact that (1.3) is weaker than the condition $\text{Lip}_\alpha$, $0 < \alpha \leq 1$, the main results of the present paper improve the main results in [6]. Moreover, Theorem 1.3 is equivalent to Theorem 4.1.1 in [18] for $b_1 = b_2 = \ldots = b_m$.

3. Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 is similar to the proof of Theorem 1.1 in [6]. Hence, we omit the details and present only the proof of Theorem 1.2. For the sake of brevity, we write

$$\| h \|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} |h(x)|^p \omega(x) \, dx \right)^{1/p} \quad \text{for} \quad 0 < p < \infty.$$  

Proof of Theorem 1.2. Without loss of generality, we assume

$$\int_{\mathbb{R}^n} \left[ \mathcal{M}_{L(\log L)^{1/r}}(f)(x) \right]^p \omega(x) \, dx < \infty,$$  

(3.1)

since otherwise there is nothing to be proven. We divide the proof into two cases.

Case I. Suppose that $\omega$ and $b_j$, $1 \leq j \leq m$, are all bounded. Firstly, we take it for granted that, for all bounded functions $f$ with compact supports, the inequality

$$\int_{\mathbb{R}^n} \left[ \mathcal{M}_\delta (\mu_{\Omega, \bar{b}}(f))(x) \right]^p \omega(x) \, dx < \infty$$  

holds for $0 < p < \infty$ and appropriate $\delta$ with $0 < \delta < 1$.

In view of the inequality (3.2), we will proceed the proof by induction on $m$. For $m = 1$, we have $\bar{b} = b_1$, $\mu_{\Omega, \bar{b}} = \mu_{\Omega, b_1}$. By Lemma 2.1(b) and Lemma 2.4, for $0 < \delta < \varepsilon < 1$, we have

$$\| \mu_{\Omega, b_1}(f) \|_{L^p(\omega)} \leq \| \mathcal{M}_\delta(\mu_{\Omega, b_1}(f)) \|_{L^p(\omega)} \leq C \| \mathcal{M}_{\delta/2}(\mu_{\Omega, b_1}(f)) \|_{L^p(\omega)} \leq C \| b_1 \|_{(\text{osc})_{L^{p/\delta}}} \left( \| \mathcal{M}_{L(\log L)^{1/r_1}}(f) \|_{L^p(\omega)} + \| \mathcal{M}_\varepsilon(\mu(f)) \|_{L^p(\omega)} \right).$$  

(3.3)

Since $\omega \in A_\infty$, there is a $p_0 > 1$, such that $\omega \in A_{p_0}$. We can choose $\delta > 0$ small enough, so that $p/\delta > p_0$. Thus, $\omega \in A_{p/\delta}$. Then by the definition of $\mathcal{M}_\delta$ and the weighted $L^{p/\delta}$-boundedness of the Hardy–Littlewood maximal operator $\mathcal{M}$, we get

$$\int_{\mathbb{R}^n} \left[ \mathcal{M}_\delta(\mu(f))(x) \right]^p \omega(x) \, dx = \int_{\mathbb{R}^n} \left[ \mathcal{M}(\mu(f))^{\delta/\delta}(x) \right]^{p/\delta} \omega(x) \, dx \leq \int_{\mathbb{R}^n} |\mu(f)(x)|^p \omega(x) \, dx.$$  

(3.4)
This, together with (3.3), Theorem 1.1, and the fact that \( M(f) \leq C M_{L(\log L)^{1/s}}(f) \) for any \( s > 0 \), gives
\[
\|\mu_{\Omega, \delta}(f)\|_{L^p(\omega)} \leq C \|b_1\|_{\text{Osc}_{\exp L^r \sigma}} \left( \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} + \|\mu_{\Omega}(f)\|_{L^p(\omega)} \right)
\]
\[
\leq C \|b_1\|_{\text{Osc}_{\exp L^r \sigma}} \left( \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} + \|M(f)\|_{L^p(\omega)} \right)
\]
\[
\leq C \|b_1\|_{\text{Osc}_{\exp L^r \sigma}} \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)}.
\]

We now suppose that the theorem is true for \( 1, 2, \ldots, m - 1 \) and prove it for \( m \). Recall that, if
\[
\sigma = \{\sigma(1), \ldots, \sigma(j)\}, \quad 1 \leq j \leq m,
\]
and the corresponding \( r_{\sigma} \) satisfies the equality
\[
1/r_{\sigma} = 1/r_{\sigma(1)} + \ldots + 1/r_{\sigma(j)},
\]
then \( \sigma' = \{1, \ldots, m\} \setminus \sigma \) and the corresponding \( r_{\sigma'} \) satisfies the equality
\[
1/r_{\sigma'} = 1/r - 1/r_{\sigma}.
\]
Reasoning as in (3.4), for sufficiently small \( \theta > 0 \), we obtain
\[
\int_{\mathbb{R}^n} \left[ M_{\theta}(\mu_{\Omega, \delta}(f))(x) \right]^p \omega(x) \, dx \leq C \int_{\mathbb{R}^n} \left| \mu_{\Omega, \delta}(f)(x) \right|^p \omega(x) \, dx. \tag{3.5}
\]

The same argument as used above and the induction hypothesis enable us to conclude that
\[
\|\mu_{\Omega, \delta}(f)\|_{L^p(\omega)} \leq \|M_\delta(\mu_{\Omega, \delta}(f))\|_{L^p(\omega)} \leq C \|M_\delta^2(\mu_{\Omega, \delta}(f))\|_{L^p(\omega)}
\]
\[
\leq C \|\tilde{b}\| \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} + C \sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^n} \|\tilde{b}_\sigma\|_{\text{Osc}_{\exp L^r \sigma}} \|M_{\sigma}(\mu_{\Omega, \delta}(f))\|_{L^p(\omega)}
\]
\[
\leq C \|\tilde{b}\| \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)} + C \sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^n} \|\tilde{b}_\sigma\|_{\text{Osc}_{\exp L^r \sigma}} \|\mu_{\Omega, \delta}(f)\|_{L^p(\omega)}
\]
\[
\leq C \|\tilde{b}\| \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)}
\]
\[
+ C \sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j^n} \|\tilde{b}_\sigma\|_{\text{Osc}_{\exp L^r \sigma}} \|\tilde{b}_\sigma\|_{\text{Osc}_{\exp L^r \sigma'}} \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)}
\]
\[
\leq C \|\tilde{b}\| \|M_{L(\log L)^{1/r}}(f)\|_{L^p(\omega)}.
\]
where the fourth inequality follows from (3.5) and the last inequality follows from the fact that

\[ M_{\log L}^{1/r}(f) \leq M_{\log L}^{1/r}(f). \]

To complete the proof of this special case of Theorem 1.2, we need to check (3.2). From (3.5), it suffices to prove

\[ \int_{\mathbb{R}^n} \left| \mu_{\Omega, b}(f)(x) \right|^p \omega(x) \, dx < \infty, \quad 0 < p < \infty, \quad (3.6) \]

whenever the weight \( \omega \) and the functions \( b_j, 1 \leq j \leq m \), are all bounded.

Assume that \( \text{supp} f \subset B = B(0, R) \) for some \( R > 0 \) and write

\[ \int_{\mathbb{R}^n} |\mu_{\Omega, b}(f)(x)|^p \omega(x) \, dx = \int_{2B} |\mu_{\Omega, b}(f)(x)|^p \omega(x) \, dx + \int_{(2B)^c} |\mu_{\Omega, b}(f)(x)|^p \omega(x) \, dx = I + II. \]

By using the fact that \( \omega \) and \( b_j \) are all bounded, the Hölder inequality, the induction hypothesis, the fact that \( M_{\log L}^{m} \sim M^{m+1} \), and the \( L^{p/\delta} \)-boundedness of \( M \), we get

\[ \int_{2B} |b_\sigma(x)|^p |\mu_{\Omega, b_\sigma}(f)(x)|^p \omega(x) \, dx \leq C_\omega \|b_\sigma\|_{L^\infty(\mathbb{R}^n)} |B|^{1-\delta} \|b_\sigma f\|_{L^\delta/(\mathbb{R}^n)}^p < \infty. \]

This and the definition of \( \mu_{\Omega, b}(f) \) give us the following inequality:

\[ I \leq C \sum_{j=1}^m \sum_{\sigma \in \mathcal{C}_j} \int_{2B} |b_\sigma(x)|^p |\mu_{\Omega, b_\sigma}(f)(x)|^p \omega(x) \, dx < \infty. \quad (3.7) \]

For II, we first estimate \( \mu_{\Omega, b}(f)(x) \) for \( x \in (2B)^c \). We have

\[ |x|/2 \leq |x - y| \leq 3|x|/2 \]

when \( x \in (2B)^c \) and \( y \in B \). Noting that \( \Omega \in L^\infty(S^{n-1}) \), \( \omega \) and \( b_j \) are bounded functions and \( |x| \sim |x - y| \) when \( x \in (2B)^c \) and \( y \in B \), we get that there exists a constant \( C_{\Omega, b_\omega} \), depending on the \( L^\infty \)-norm of \( \Omega \), \( b_j \) and \( \omega \), such that

\[ \mu_{\Omega, b}(f)(x) \leq C \|\Omega\|_{L^\infty(S^{n-1})} \|b\|_{L^\infty(\mathbb{R}^n)} \left( \int_0^\infty \left( \int_{|x-y| \leq t} \frac{|f(y)|}{|x-y|^{n-1}} \, dy \right)^2 \frac{dt}{t^3} \right)^{1/2} \]

\[ \leq C_{\Omega, b_\omega} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \left( \int_{|x-y| \leq t} \frac{dt}{t^3} \right)^{1/2} \, dy \]

\[ \leq C_{\Omega, b_\omega} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n}} \, dy \leq C_{\Omega, b_\omega} \frac{1}{2B} \int_{\mathbb{R}^n} |f(y)| \, dy \leq C_{\Omega, b_\omega} M(f)(x). \quad (3.8) \]
By (3.8) and the fact that $M(f)(x) \leq CM_{L(\log L)^{1/r}}(f)(x)$, we obtain from (3.1) that
\[
II \leq C_{\Omega, \tilde{b}, \omega} \int_{(2B)^c} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x) \, dx < \infty.
\]
This, together with (3.7), shows that (3.6) is true when $\omega$ and $b_j$ are bounded functions, hence, (3.2) is also true. Thus, Theorem 1.2 is proved for this special case.

**Case II.** For unbounded $\omega$ and $b_j$, we truncate the weight $\omega$ and the functions $b_j$, $j = 1, \ldots, m$, as follows: Let $N$ be a positive integer, and denote $\omega_N = \inf \{\omega, N\}$ and $\tilde{b}^N = (b_1^N, \ldots, b_m^N)$, where $b_j^N$ is defined by
\[
b_j^N(x) = \begin{cases} 
N, & \text{for } b_j(x) > N, \\
b_j(x), & \text{for } |b_j(x)| \leq N, \\
-N, & \text{for } b_j(x) < -N.
\end{cases}
\]

By Lemma 2.4 in [11], there is a positive constant $C$ independent of $N$ such that
\[
\|b_j^N\|_{\text{Osc}_{\exp L^p}} \leq \|b_j\|_{\text{Osc}_{\exp L^p}}. \tag{3.9}
\]
Applying (1.5) for $\tilde{b}^N$ and $\omega_N$, and using (3.9), we have
\[
\int_{\mathbb{R}^n} |\mu_{\Omega, \tilde{b}^N}(f)(x)|^p \omega_N(x) \, dx \leq C \|\tilde{b}\|^p \int_{\mathbb{R}^n} [M_{L(\log L)^{1/r}}(f)(x)]^p \omega(x) \, dx. \tag{3.10}
\]
Further, in view of the fact that $f$ has a compact support, we conclude that $b_j^N$ converges to $b_j$ and
\[
b_{\sigma(1)}^N \cdots b_{\sigma(j)}^N f \text{ converges to } b_{\sigma(1)} \cdots b_{\sigma(j)} f
\]
in any space $L^p$ with $p > 1$ as $N \to \infty$. Recalling that $\mu_{\Omega}$ is $L^p$-bounded, we claim that, at least for a subsequence, \(\{\|\mu_{\Omega, \tilde{b}^N}(f)(x)\|^p \omega_N(x)\}_{N=1}^\infty\) converges pointwise almost everywhere to \(\|\mu_{\Omega, \tilde{b}}(f)(x)\|^p \omega(x)\) as $N \to \infty$.

This fact, together with (3.10) and the Fatou lemma, completes the proof of Theorem 1.2.

**4. Proof of Theorem 1.3**

The idea of the proof of Theorem 1.3 follows the idea of the proof of Theorem 1.5 in [11]. We first prove the following lemma:

**Lemma 4.1.** Let $\omega \in A_\infty$, let $\Phi(t) = t \log^{1/r}(e + t)$, and let $\tilde{b}$, $r$, and $r_j$ be the same as in Theorem 1.3. Then, for $\rho > 2$, $\Omega \in L^\infty(S^{n-1})$ is homogeneous of degree zero and satisfies (1.1) and (1.3). Moreover, there exists a positive constant $C$ such that
\[
\sup_{t > 0} \frac{\omega(\{y \in \mathbb{R}^n : M_\Omega^r(\mu_{\Omega, \tilde{b}})(f)(y) > t\})}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega(\{y \in \mathbb{R}^n : M_\Phi(\|\tilde{b}\|f)(y) > t\})}{\Phi(t)} \tag{4.1}
\]
for all bounded functions $f$ with compact support and all $0 < \delta < 1$. 
Proof. To use Lemma 2.1(a), we first check that

$$\sup_{t > 0} \frac{1}{\Phi(1/t)} \omega(\{ x \in \mathbb{R}^n : M_\varepsilon(\mu_{\Omega,\check{b}})(f)(x) > t \} ) < \infty$$

(4.2)

for all bounded functions $f$ with compact support and all $0 < \delta < 1$.

We only prove (4.2) for the special case where $\omega$ and $b_j$ are bounded functions. In the general case, we consider the truncations of $\omega$ and $\check{b}$ as in the proof of Theorem 1.2. For the sake of brevity, this time, we take into account the weak $(1,1)$ boundedness of $\mu_{\Omega}$ that gives the convergence in measure. Then we can obtain (4.2) for all $\omega$ and $\check{b}$ with the help of Lemma 4.1. Thus, we omit the details.

Assume that $\text{supp } f \subset B = B(0, R)$. Then, for any $0 < \varepsilon < 1$, we have

$$\sup_{t > 0} \frac{\omega(\{ x \in \mathbb{R}^n : M_\varepsilon(\mu_{\Omega,\check{b}})(f)(x) > t \} )}{\Phi(1/t)} \leq C_\varepsilon \sup_{t > 0} \frac{\omega(\{ x \in \mathbb{R}^n : M_\varepsilon(\chi_{2B}\mu_{\Omega,\check{b}})(f)(x) > t/2 \} )}{\Phi(1/t)}$$

$$+ C_\varepsilon \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega(\{ x \in \mathbb{R}^n : M_\varepsilon(\chi_{(2B)^c}\mu_{\Omega,\check{b}})(f)(x) > t/2 \} )$$

$$= C_\varepsilon (I + II),$$

(4.3)

where $C_\varepsilon$ is a positive constant depending on $\varepsilon$.

For $I$, making use of the weak $(1,1)$ boundedness of $M$ and $[\Phi(1/t)]^{-1} \leq Ct$, we note that $\omega$ and $b_j$ are all bounded. Then there exists a positive constant $C_\omega$, depending on $\omega$, such that

$$I \leq C_\omega \sup_{t > 0} t |\{ x \in \mathbb{R}^n : M_\varepsilon(\chi_{2B}\mu_{\Omega,\check{b}})(f)(x) > t/2 \} |$$

$$\leq C_\omega \int_{2B} |\mu_{\Omega,\check{b}}(f)(x)| dx \leq C_\omega |B|^{1/2} \left( \int_{2B} |\mu_{\Omega,\check{b}}(f)(x)|^2 dx \right)^{1/2} < \infty,$$

where the last step follows from (3.7).

Using the fact that $(M(f))^{\varepsilon} \in A_1$ for $0 < \varepsilon < 1$ and $f$ is locally integrable, we get

$$M_\varepsilon(M(f))(x) = [M(|M(f)|^\varepsilon)(x)]^{1/\varepsilon} \leq CM(f)(x).$$

Noting that $\omega$ is bounded, it follows from (3.8) and the weak $(1,1)$ boundedness of $M$ that

$$II \leq C_\omega \sup_{t > 0} t \cdot \omega(\{ x \in \mathbb{R}^n : M_\varepsilon(M(f))(x) > Ct \} )$$

$$\leq C_\omega \sup_{t > 0} t \cdot \omega(\{ x \in \mathbb{R}^n : M(f)(x) > Ct \} )$$

$$\leq C_\omega \int_{\mathbb{R}^n} |f(x)| dx < \infty.$$
Combining (4.3) and the estimates for I and II, we get (4.2).

We now prove (4.1) by induction. For $\tilde{b} \in \text{Osc}_{\exp L^r}$, we write $\tilde{b} = \frac{b}{\|b\|}$. Then $\|\tilde{b}\| = 1$ and

$$\frac{\mu_{\Omega, \tilde{b}}(f)}{\|\tilde{b}\|} = \mu_{\Omega, b/\|b\|}(f) = \mu_{\Omega, b}(f).$$

Hence, we can assume that $\|\tilde{b}\| = 1$. For $m = 1$, we have $\tilde{b} = b$, $\|\tilde{b}\| = \|b\|_{\text{Osc}_{\exp L^r}} = 1$, $\mu_{\Omega, \tilde{b}}(f) = \mu_{\Omega, b}(f)$. Therefore, to prove (4.1), it suffices to show that

$$\sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M^2_\delta(\mu_{\Omega, b}(f))(y) > t\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M_{L(\log L)^{1/r}}(f)(y) > t\}}{\Phi(1/t)}$$

for all bounded functions $f$ with compact support.

Applying Lemma 2.4 for $m = 1$ and any $\varepsilon$ with $0 < \delta < \varepsilon < 1$, one can easily show that the left-hand side of (4.4) is dominated by

$$\sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M^2_\delta(\mu_{\Omega, b}(f))(y) > t\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M_{L(\log L)^{1/r}}(f)(y) > t/2\}}{\Phi(1/t)} + C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M_{\varepsilon}(\mu_{\Omega}(f))(y) > t/2\}}{\Phi(1/t)}.$$

Recall that (4.2) is true and, since $[\Phi(1/t)]^{-1}$ is doubling, by using Lemma 2.1(a), Lemma 2.3, and the inequality $M(f) \leq M_{L(\log L)^{1/r}}(f)$, we conclude that

$$\sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M^2_\delta(\mu_{\Omega, b}(f))(y) > t\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M_{L(\log L)^{1/r}}(f)(y) > t\}}{\Phi(1/t)} + C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M^2_\delta(\mu_{\Omega}(f))(y) > t\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M_{L(\log L)^{1/r}}(f)(y) > t\}}{\Phi(1/t)} + C \sup_{t > 0} \frac{\omega\{y \in \mathbb{R}^n : M(f)(y) > t\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{y \in \mathbb{R}^n : M_{L(\log L)^{1/r}}(f)(y) > t\}.$$
We now check (4.1) in the general case $m \geq 2$. Suppose that (4.1) holds for $m - 1$. It is necessary to prove this inequality for $m$. We note that (4.2) is true and recall the fact that $[\Phi(1/t)]^{-1}$ is doubling. Thus, by Lemmas 2.3 and 2.4 for $\epsilon$ with $0 < \delta < \epsilon$, Lemma 2.1(a), and the induction hypothesis for (4.1), we obtain

$$\sup_{t > 0} \frac{\omega\{(y \in \mathbb{R}^n : M^t_{\delta}(\mu_{\Omega_{\delta}}(f))(y) > t)\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega\{(y \in \mathbb{R}^n : M_{\Phi}(f)(y) > t/C_m)\}}{\Phi(1/t)}$$

$$+ C \sum_{j=1}^{m} \sum_{\sigma \in e_j} \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M_{\Phi}(\mu_{\Omega_{\delta}}(\|\tilde{b}_{\sigma}\|f))(y) > t/C_m)\}$$

$$\leq C_m \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M_{\Phi}(f)(y) > t)\}$$

$$+ C_m \sum_{j=1}^{m} \sum_{\sigma \in e_j} \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M^2_{\epsilon}(\mu_{\Omega_{\delta}}(\|\tilde{b}_{\sigma}\|f))(y) > t)\}$$

$$\leq C_m \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M_{\Phi}(f)(y) > t)\}$$

$$+ C_m \sum_{j=1}^{m} \sum_{\sigma \in e_j} \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M_{\Phi}(\|\tilde{b}_{\sigma}\|f)(y) > t)\}$$

$$\leq C_m \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M_{\Phi}(f)(y) > t)\}$$

$$+ C_m \sum_{j=1}^{m} \sum_{\sigma \in e_j} \sup_{t > 0} \frac{1}{\Phi(1/t)} \omega\{(y \in \mathbb{R}^n : M_{\Phi}(f)(y) > t)\},$$

where $\|\tilde{b}_{\sigma}\|$ and $\|\tilde{b}_{\sigma'}\|$ are as in (2.1). In the last step, we use the fact that $\|\tilde{b}_{\sigma'}\|\|\tilde{b}_{\sigma}\| = \|\tilde{b}\| = 1$.

This completes the proof (4.1) for all $m$. Hence, the proof of Lemma 4.1 is also completed.

**Lemma 4.2.** Let $\omega \in A_\infty$, let $\Phi(t) = t \log^{1/\epsilon}(e + t)$, and let $\tilde{b}$, $r$, and $r_j$ be the same as in Theorem 1.3. For $\rho > 2$, $\Omega \in L^\infty(S^{m-1})$ is homogeneous of degree zero and satisfies (1.1) and (1.3). Moreover, there exists a positive constant $C$ such that

$$\sup_{t > 0} \frac{\omega\{(y \in \mathbb{R}^n : \mu_{\Omega_{\delta}}(f)(y) > t)\}}{\Phi(1/t)} \leq C \sup_{t > 0} \frac{\omega\{(y \in \mathbb{R}^n : M_{\Phi}(\|\tilde{b}\|f)(y) > t)\}}{\Phi(1/t)}$$

for all bounded functions $f$ with compact support.

The proof is similar to the proof of Lemma 4.2 in [6]. Thus, we omit the details.
To prove Theorem 1.3, we need the following weighted weak-type inequality due to Pérez and Trujillo–González [11]:

**Lemma 4.3** [11]. Let $\omega \in A_1$ and let $\Phi(t) = t \log^{1/r}(e + t)$. Then for any $\lambda > 0$ and any locally integrable function $f$, there exists a positive constant $C$ such that

$$\omega(\{ y \in \mathbb{R}^n : M_\Phi(f)(y) > \lambda \}) \leq C \int_{\mathbb{R}^n} \Phi\left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy.$$ 

**Proof of Theorem 1.3.** By the homogeneity of $\tilde{b}$, we can assume that $\lambda = ||\tilde{b}|| = 1$. Thus, it remains to prove that

$$\omega(\{ y \in \mathbb{R}^n : \mu_{\Omega, \tilde{b}}(f)(y) > 1 \}) \leq C \int_{\mathbb{R}^n} \Phi(|f(y)|) \omega(y) \, dy.$$ 

In view of the inequality $\Phi(ab) \leq 2\Phi(a)\Phi(b)$, $a, b \geq 0$, and Lemmas 4.2 and 4.3, we find

$$\omega(\{ y \in \mathbb{R}^n : \mu_{\Omega, \tilde{b}}(f)(y) > 1 \}) \leq \sup_{\lambda > 0} \frac{1}{\Phi(1/\lambda)} \omega(\{ y \in \mathbb{R}^n : M_\Phi(f)(y) > \lambda \})$$

$$\leq \sup_{\lambda > 0} \frac{1}{\Phi(1/\lambda)} \int_{\mathbb{R}^n} \Phi\left( \frac{|f(y)|}{\lambda} \right) \omega(y) \, dy$$

$$\leq \sup_{\lambda > 0} \frac{1}{\Phi(1/\lambda)} \int_{\mathbb{R}^n} \Phi(|f(y)|) \Phi(1/\lambda) \omega(y) \, dy$$

$$\leq C \int_{\mathbb{R}^n} \Phi(|f(y)|) \omega(y) \, dy.$$ 

Theorem 1.3 is proved.

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