Doubles of Quasi–Quantum Groups

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Abstract

In [Dr1] Drinfeld showed that any finite dimensional Hopf algebra $G$ extends to a quasitriangular Hopf algebra $D(G)$, the quantum double of $G$. Based on the construction of a so–called diagonal crossed product developed by the authors in [HN], we generalize this result to the case of quasi–Hopf algebras $G$. As for ordinary Hopf algebras, as a vector space the “quasi–quantum double” $D(G)$ is isomorphic to $\hat{G} \otimes G$, where $\hat{G}$ denotes the dual of $G$. We give explicit formulas for the product, the coproduct, the $R$–matrix and the antipode on $D(G)$ and prove that they fulfill Drinfeld’s axioms of a quasitriangular quasi–Hopf algebra. In particular $D(G)$ becomes an associative algebra containing $G \equiv 1 \otimes \hat{G}$ as a quasi–Hopf subalgebra. On the other hand, $\hat{G} \equiv \hat{G} \otimes 1$ is not a subalgebra of $D(G)$ unless the coproduct on $G$ is strictly coassociative. It is shown that the category $\text{Rep}D(G)$ of finite dimensional representations of $D(G)$ coincides with what has been called the double category of $G$–modules by S. Majid [M2]. Thus our construction gives a concrete realization of Majid’s abstract definition of quasi–quantum doubles in terms of a Tannaka–Krein–like reconstruction procedure. The whole construction is shown to generalize to weak quasi–Hopf algebras with $D(G)$ now being linearly isomorphic to a subspace of $\hat{G} \otimes G$.

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1 Introduction

Given a finite dimensional Hopf algebra $G$ and its dual $\hat{G}$ Drinfeld [Dr1] has introduced the quantum double $D(G) \supset G$ as the universal Hopf algebra extension of $G$ satisfying

1. There exists a unital algebra embedding $D: \hat{G} \rightarrow D(G)$ such that $D(G)$ is algebraically generated by $G$ and $D(\hat{G})$.

2. Let $e_\mu \in G$ be a basis with dual basis $e^\mu \in \hat{G}$. Then $R_D := e_\mu \otimes D(e^\mu) \in D(G) \otimes D(G)$ is quasitriangular.

It follows that as a coalgebra $D(G) = G^{\text{cop}} \otimes G$, where “cop” refers to the opposite coproduct. However, when realized on $\hat{G} \otimes G$, the algebraic structure of $D(G)$ becomes more involved. It has been analyzed in detail by S. Majid as a particular example of his notion of double crossed products, see [M3,M4] and references therein. The dual version of the quantum double has been introduced for infinite dimensional compact quantum groups in [PW] as the mathematical structure underlying the quantum Lorentz group.

During the 90’s the quantum double has become of increasing importance as a quantum symmetry in two–dimensional lattice and continuum QFT. In continuum theories the quantum double $D(G)$ of a finite group $G$ (i.e. $G = \mathbb{C}G$) has first been applied (mostly in a twisted version, which we will come back to below) to describe the symmetry underlying the sector structure of orbifold models in [DPR]. Quite interestingly, the same structure appears as a residual generalized “dyon–symmetry” in spontaneously broken (2+1)-dimensional Higgs models with a finite unbroken subgroup $G$ [BaWi]. For the role of quantum doubles in integrable field theories see, e.g. [BL].

More recently, in the framework of algebraic QFT, M. Müger [Mü] has also found the double of a finite group $G$ acting as a global symmetry on a “disorder–field extension” $\hat{F}$ of a massive 2–dimensional field algebra $F$ with global gauge symmetry $G$. As opposed to the above cases, in this type of models the “disorder–part” $\hat{G}$ of the double is also spontaneously broken, corresponding to a violation of Haag duality (for double cones) for the $D(G)$–invariant observable algebra $A \subset \hat{F}$. The Haag dual extension $\hat{A} \supset A$ is then recovered as the invariant subalgebra of $\hat{F}$ under the unbroken symmetry $G$.

On the lattice, related but prior to Müger’s work, the double of a finite group $G$ has been realized by K. Szlachányi and P. Vecsernyés as a symmetry realized on the order×disorder field algebra of a $G$–spin quantum chain [SzV]. Since for $G = \mathbb{Z}_N$ the double coincides with (the group algebra of) $\mathbb{Z}_N \times \mathbb{Z}_N$, this generalizes the well known order×disorder symmetry of abelian $G$–spin models. This investigation has been substantially extended to arbitrary finite dimensional $C^*$–Hopf algebras $G$ in [NSz], where the authors show that such “Hopf spin models” always have $D(G)$ as a universal localized cosymmetry. This means that under the assumption of a Haag dual vacuum representation (i.e. absence of spontaneous symmetry breaking) the full superselection structure of these models is precisely created by the irreducible representations of $D(G)$. The formulation of [NSz] also allowed for a generalization of duality transformations to the non–commutative and non–cocommutative setting.

As it has turned out meanwhile, very much related results have been obtained independently for lattice current algebras on finite periodic lattice chains by A. Alekseev et al. [AFFS]. For these models the authors have completely determined the representation category, showing that it is in one-to-one correspondence with $\text{Rep} K_1$, where $K_1$ is the algebra living on a minimal loop consisting of one site and one link biting into its own tail. Using the braided–group theory of [M5] (see also [M4]), it has been realized by one of us [N1], that $K_1$ is in fact again isomorphic to a quantum double $D(G)$. Also, requiring $G$ to be a modular Hopf algebra as in [AFFS], the Hopf spin model of [NSz] has been shown in [N1] to be iso-
morphic to the lattice current algebra of [AFFS] by a local transformation of the generators.

As a common feature of all these models we emphasize that under the quantum physical requirement of positivity they only give rise to quantum symmetries with integer \( q \)-dimensions \([N2]\). Thus, to construct “rational” models with a finite sector theory and non–integer dimensions one is inevitably forced to depart from ordinary Hopf algebras \( G \). Here, the most fashionable candidates are the truncated semisimple versions of the \( q \)-deformations \( U_q(\mathfrak{g}) \), \( \mathfrak{g} \) a simple Lie algebra, at roots of unity, \( q^N = 1 \). Also, since lattice current algebras have been invented as regularized versions of WZNW–models [AFFS, AFSV, AFS, ByS, Fa, FG], they should eventually be studied at roots of unity.

Following G. Mack and V. Schomerus [MS], truncated quantum groups at \( q^N = 1 \) have to be described as weak quasi–Hopf algebras in the sense of Drinfeld [Dr2], with the additional feature \( \Delta(1) \neq 1 \otimes 1 \), where \( \Delta : G \to G \otimes G \) denotes the coproduct.

To formulate lattice current algebras at roots of unity one may now combine the methods of [AFFS] with those developed by [AGS,AS] for lattice Chern–Simons theories. However, it remains unclear whether and in what sense in such models universal localized cosymmetries \( \rho : A \to A \otimes \mathcal{G} \) still provide coactions and whether \( \mathcal{G} \) would still be (an analogue of) a quantum double of a quasi-Hopf algebra.

In fact, a definition of a quantum double \( D(\mathcal{G}) \) for quasi–Hopf algebras \( \mathcal{G} \) has recently been proposed by S. Majid [M2]. Unfortunately this has only been done in form of an implicit Tannaka–Krein reconstruction procedure, which makes it hard to identify this algebra in terms of generators and relations in concrete models.

In [HN] we have started a program where we generalize standard notions of Hopf algebra theory (like coactions and crossed products) to (weak) quasi–Hopf algebras and apply them to quantum chains based on weak quasi–quantum groups in the spirit of [NSz, AFFS]. As a central mathematical structure underlying these constructions we have developed the concept of a diagonal crossed product by the dual \( \hat{\mathcal{G}} \) of a (weak) quasi–quantum group \( \mathcal{G} \). In this way we have obtained as one of our main nontrivial examples an explicit algebraic definition of the double \( D(\mathcal{G}) \). We have shown that, as for ordinary Hopf algebras, \( D(\mathcal{G}) \) may be realized as a new quasi–bialgebra structure on \( \hat{\mathcal{G}} \otimes \mathcal{G} \) (or, in the weak case, a certain subspace thereof) containing \( \mathcal{G} \equiv 1_{\mathcal{G}} \otimes \mathcal{G} \) as a sub-bialgebra. Generalizing the results of [NSz, AFFS] we have also constructed the above lattice models for weak quasi–Hopf algebras \( \mathcal{G} \) and established that they always admit localized coactions of \( D(\mathcal{G}) \) in the sense of [NSz].

In this work we extend our analysis of \( D(\mathcal{G}) \) by proving that it is always a quasitriangular quasi–Hopf algebra, which is weak if and only if \( \Delta(1_{\mathcal{G}}) \neq 1_{\mathcal{G}} \otimes 1_{\mathcal{G}} \). Our main results are summarized by the following

**Theorem A** Let \((\mathcal{G}, \Delta, \phi, S)\) be a finite dimensional quasi–Hopf algebra with coproduct \( \Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G} \), reassociator \( \phi \in \mathcal{G}^{\otimes 3} \) and invertible antipode \( S \). Assume \( D(\mathcal{G}) \) to be a quasi–Hopf algebra extension \( D(\mathcal{G}) \supset \mathcal{G} \) satisfying

---

1 More generally, even without these assumptions on \( \mathcal{G} \), periodic Hopf spin chains are meanwhile known to be isomorphic to \( D(\mathcal{G}) \otimes \text{Mat}(N_L) \), where \( N_L \in \mathbb{N} \) depends on the length of the loop \( L \) [Sz], thus explaining the representation theory of [AFFS].

2 This result is frequently ignored in the literature and relates to the finite dimensionality of \( \mathcal{G} \). By a Perron–Frobenius argument it also applies to the twisted double of [DPR].

3 The twisted double of [DPR] is also a quasi–Hopf algebra but it still satisfies \( \Delta(1) = 1 \otimes 1 \).

4 This has also been announced in [M2].
(i) There exists a linear map \( D : \hat{G} \rightarrow \mathcal{D}(G) \) such that \( \mathcal{D}(G) \) is algebraically generated by \( \hat{G} \) and \( D(\hat{G}) \)

(ii) \( R_D := \sum \mu e_\mu \otimes D(e^\mu) \in \mathcal{D}(G) \otimes \mathcal{D}(G) \) is quasitriangular.

(iii) If \( \hat{G} \supset G \) and \( \hat{D} : \hat{G} \rightarrow \hat{D} \) have the same properties, then there exists a bialgebra homomorphism \( f : \mathcal{D}(\hat{G}) \rightarrow \hat{D} \) restricting to the identity on \( G \) and satisfying \( f \circ D = \hat{D} \).

Then \( \mathcal{D}(G) \) exists uniquely up to equivalence and the map \( \mu : \hat{G} \otimes G \rightarrow \mathcal{D}(G) \) given by

\[
\mu(\varphi \otimes a) = (\text{id} \otimes \varphi(1))(q_\rho) D(\varphi(2)), \quad \text{where} \quad q_\rho = \phi^1 \otimes S^{-1}(\alpha \phi^3) \phi^2 \in G \otimes G
\]  

provides a linear bijection.

Theorem A will be proven at the end of Section 3.4. We will also have a generalization to weak quasi–Hopf algebras, which is stated as Theorem B in Section 4.

The major achievement of Theorem A in comparison with [HN] consists in the construction of the antipode on \( \mathcal{D}(G) \). To this end, as a central technical result we establish a formula for \((S \otimes S)(R)\) and the relations between \( R^{-1} \), \((S \otimes \text{id})(R)\) and \((\text{id} \otimes S^{-1})(R)\) for a quasitriangular \( R \in G \otimes G \) in any quasi–Hopf algebra \( G \). Recall, that in ordinary Hopf algebras the last three quantities coincide and therefore \((S \otimes S)(R) = R\).

To prove these results we combine the methods of [HN] with the very efficient graphical calculus developed by [RT,T,AC]. This will also allow to give nice intuitive interpretations of many of our almost untraceable identities derived in [HN]. In fact, without this graphical machinery we would have been lost in proving or even only trying to guess these formulas. In particular, a purely algebraic proof of the formulas for \( R^{-1} \) and \((S \otimes S)(R)\) in Theorem 2.1 would most likely be unreadable and therefore also untrustworthy. This is why we think it worthwhile to put more emphasis on this graphical technique in the present paper.

We start in Section 2.1 with shortly reviewing Drinfeld’s theory of quasi–Hopf algebras and introduce our graphical conventions in Section 2.2. In Section 2.3 we derive our main formulas for \( R^{-1} \) and \((S \otimes S)(R)\) for any quasi–triangular \( R \in G \otimes G \). In Section 3.1 we review our construction [HN] of the double \( \mathcal{D}(G) \) as an associative algebra on the vector space \( \hat{G} \otimes G \). In Section 3.2 we reformulate this construction in the spirit of [N1] in terms of the universal \( \Delta \)-flip operator \( D \in \hat{G} \otimes \mathcal{D}(G) \). Section 3.3 roughly sketches, how the double may also be realized on the vector space \( G \otimes \hat{G} \). In Section 3.4 we establish the quasitriangular quasi–Hopf structure of \( \mathcal{D}(G) \) and prove Theorem A. Finally, in Section 3.5 we identify the category Rep\( \mathcal{D}(G) \) as the double of the category Rep\( G \) in the sense of [M2], thus proving that our construction of \( \mathcal{D}(G) \) provides a concrete realization of Majid’s Tannaka–Krein like reconstruction procedure. In Section 4 we generalize our results to weak quasi–Hopf algebras \( G \). As an application we discuss the twisted double \( \mathcal{D}^\omega(G) \) of [DPR] in Appendix A and generalize the results of [N1] on the relation with the monodromy algebras of [AGS,AS] in Appendix B.

Throughout, all linear spaces are assumed finite dimensional over the field \( \mathbb{C} \). We will use standard Hopf algebra notations, see e.g. [A,Sw,K,M3]. By an extension \( B \supset A \) of algebras we always mean a unital injective algebra morphism \( A \rightarrow B \). Two extensions \( B_1 \supset A \) and \( B_2 \supset A \) are called equivalent, if there exists an isomorphism of algebras \( B_1 \cong B_2 \) restricting to the identity on \( A \).

\[\text{Footnote 5: Here } \alpha \in G \text{ is one of the two structural elements appearing in Drinfeld’s antipode axioms, see Sect. 2.1.}\]
2 Quasitriangular quasi–Hopf algebras

2.1 Basic definitions and properties

In this subsection we review the basic definitions and properties of quasitriangular quasi–Hopf algebras as introduced by Drinfeld [Dr2], where the interested reader will find a more detailed discussion.

A quasi-bialgebra \((G, \Delta, \epsilon, \phi)\) is an associative algebra \(G\) with unity, algebra morphisms \(\Delta: G \to G \otimes G\) and \(\epsilon: G \to \mathbb{C}\), and an invertible element \(\phi \in G \otimes G \otimes G\), such that

\[
(id \otimes \Delta)(\Delta(a))\phi = \phi(\Delta \otimes id)(\Delta(a)), \quad a \in G \tag{2.1}
\]

\[
(id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1), \quad \tag{2.2}
\]

\[
(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta, \tag{2.3}
\]

\[
(id \otimes \epsilon \otimes id)(\phi) = 1 \otimes 1 \tag{2.4}
\]

The map \(\Delta\) is called the coproduct and \(\epsilon\) the counit. A coproduct with the above properties is called quasi-coassociative and the element \(\phi\) will be called the reassociator. The identities \((2.2)\) and \((2.4)\) together imply

\[
(\epsilon \otimes id \otimes id)(\phi) = (id \otimes id \otimes \epsilon)(\phi) = 1 \otimes 1. \tag{2.5}
\]

Let us briefly recall some of the main consequences of these definitions for the representation theory of \(G\). Let \(\text{Rep} \ G\) be the category of finite dimensional representations of \(G\), i.e. of pairs \((\pi_V, V)\), where \(V\) is a finite dimensional vector space and \(\pi_V: G \to \text{End}_\mathbb{C}(V)\) is a unital algebra morphism. We will also use the equivalent notion of a \(G\)-module \(V\) with multiplication \(g \cdot v \equiv \pi_V(g)v\). Given two pairs \((\pi_V, V), (\pi_U, U)\), the coproduct allows for the definition of a tensor product \((\pi_{V \otimes U}, V \otimes U)\) by setting \(\pi_{V \otimes U} = (\pi_V \otimes \pi_U) \circ \Delta\). The counit defines a one dimensional representation. Equation \((2.3)\) says, that this representation is a left and right unit with respect to the tensor product, and \((2.4)\) says that given three representations \((\pi_U, \pi_V, \pi_W)\), then \(\pi_{(U \otimes V) \otimes W} \cong \pi_{V \otimes (W \otimes U)}\) with intertwiner \(\phi_{UVW} = (\pi_U \otimes \pi_V \otimes \pi_W)(\phi)\).

The meaning of \((2.2)\) is the commutativity of the pentagon

\[
((U \otimes V) \otimes W) \otimes X \quad (U \otimes V) \otimes (W \otimes X) \quad U \otimes (V \otimes (W \otimes X))
\]

\[
(U \otimes (V \otimes W)) \otimes X \quad U \otimes ((V \otimes W) \otimes X),
\]

(2.6)

where the arrows stand for the corresponding rebracketing intertwiners. For example the first one is given by \((\pi_{U \otimes V} \otimes \pi_W \otimes \pi_X)(\phi) = (\pi_U \otimes \pi_V \otimes \pi_W \otimes \pi_X)((\Delta \otimes id \otimes id)(\phi))\). The diagram \((2.6)\) explains the name pentagon identity for equation \((2.2)\). The importance of axiom \((2.2)\) lies in the fact, that in any tensor product representation the intertwiner connecting two different bracket conventions is given by a suitable product of \(\phi\)'s, as in \((2.4)\).

The pentagon identity then guarantees, that this intertwiner is independent of the chosen sequence of intermediate rebracketings. This is known as Mac Lanes coherence theorem [ML].

A quasi–bialgebra \(G\) is called quasi-Hopf algebra, if there is a linear antimorphism \(S: G \to G\) and elements \(\alpha, \beta \in G\) satisfying (for all \(a \in G\))

\[
\sum_i S(a_{(1)}^i)aa_{(2)}^i = \alpha \epsilon(a), \quad \sum_i a_{(1)}^i \beta S(a_{(2)}^i) = \beta \epsilon(a) \tag{2.7}
\]

\[
\sum_j X^j \beta S(Y^j)\alpha Z^j = 1 = \sum_j S(P^j)\alpha Q^j \beta S(R^j). \tag{2.8}
\]
Here and throughout we use the notation \( \sum_i a_i^{(1)} \otimes a_i^{(2)} = \Delta(a) \) and
\[
\phi = \sum_j X^j \otimes Y^j \otimes Z^j; \quad \phi^{-1} = \sum_j P^j \otimes Q^j \otimes R^j. \tag{2.9}
\]

To simplify the notation, we will in the following also frequently suppress the summation symbol and write \( \phi = X^i \otimes Y^i \otimes Z^i, \Delta(a) = a^{(1)} \otimes a^{(2)}, \) etc. The map \( S \) is called an antipode. We will also always suppose that \( S \) is invertible. Note that as opposed to ordinary Hopf algebras, an antipode is not uniquely determined, provided it exists. The antipode allows to define the (left) dual representation \((\ast, \ast V)\) of \((\pi, V)\), where \( \ast V \) is the dual space of \( V \), by \( \ast \pi(a) = \pi(S(a))^t \), the superscript \( t \) denoting the transposed map. Analogously one defines a right dual representation \((\pi^*, V^*)\), where \( V^* \equiv \ast V \) and \( \pi^*(a) = \pi(S^{-1}(a))^t \).

A quasi-Hopf algebra \( G \) is called quasitriangular, if there exists an invertible element \( R \in \mathcal{G} \otimes \mathcal{G} \), such that
\[
\Delta^{op}(a)R = R\Delta(a), \quad a \in \mathcal{G} \tag{2.10}
\]
\[
(\Delta \otimes \text{id})(R) = \phi^{12} R^{13} (\phi^{-1})^{132} R^{23} \phi \tag{2.11}
\]
\[
(\text{id} \otimes \Delta)(R) = (\phi^{-1})^{231} R^{13} \phi^{213} R^{12} \phi^{-1}, \tag{2.12}
\]
where we use the following notation: If \( \psi = \sum \psi^1_i \otimes \ldots \otimes \psi^n_i \in \mathcal{G}^{\otimes n} \), then, for \( m \leq n \), \( \psi^{n_1n_2..n_m} \in \mathcal{G}^{\otimes n} \) denotes the element of \( \mathcal{G}^{\otimes n} \) having \( \psi^k_i \) in the \( n_k \)th slot and 1 in the remaining ones. The element \( R \) is called the R-matrix. The above relations imply the quasi-Yang-Baxter equation
\[
R^{12} \phi^{312} R^{13} (\phi^{-1})^{132} R^{23} \phi = \phi^{321} R^{23} (\phi^{-1})^{231} R^{13} \phi^{213} R^{12} \tag{2.13}
\]
and the property
\[
(\epsilon \otimes \text{id})(R) = (\text{id} \otimes \epsilon)(R) = 1. \tag{2.14}
\]

Eq. \((2.10)\) implies, that for any pair \( \pi_U, \pi_V \) the two representations \((\pi_U \otimes V, U \otimes V)\) and \((\pi_V \otimes U, V \otimes U)\) are equivalent with intertwiner \( B^{UV} := \tau^{12} \circ (\pi_U \otimes \pi_V)(R) \), where \( \tau \) denotes the permutation of tensor factors in \( U \otimes V \). Eqs. \((2.11), (2.12)\) imply the commutativity of two hexagon diagrams obtained by taking \( \pi_U \otimes \pi_V \otimes \pi_W \) on both sides.

\( \mathcal{G} \) being a quasitriangular quasi-Hopf algebra implies that \( \text{Rep} \mathcal{G} \) is a rigid monoidal category with braiding, where the associativity and commutativity constraints for the tensor product functor \( \otimes : \text{Rep} \mathcal{G} \times \text{Rep} \mathcal{G} \rightarrow \text{Rep} \mathcal{G} \) are given by the natural families \( \phi_{UVW} \) and \( \tau^{12} \circ R_{UV} \) and the (left) duality is defined with the help of the antipode \( S \) and the elements \( \alpha, \beta, \) see \((2.29), (2.31)\) below.

Together with a quasi–Hopf algebra \( G \equiv (G, \Delta, \epsilon, \phi, S, \alpha, \beta) \) we also have \( G_{\text{op}}, G_{\text{cop}} \) and \( G_{\text{cop}}^{\text{op}} \) as quasi–Hopf algebras, where “op” means opposite multiplication and “cop” means opposite comultiplication. The quasi–Hopf structures are obtained by putting \( \phi_{\text{op}} := \phi^{-1}, \phi_{\text{cop}} := (\phi^{-1})^{321}, \phi_{\text{cop}} := \phi^{321}, S_{\text{op}} = S^{\text{cop}} = (S_{\text{cop}}^{-1})^{-1} := S^{-1}, \alpha_{\text{op}} := S^{-1}(\beta), \beta_{\text{op}} := S^{-1}(\alpha), \alpha_{\text{cop}} := S^{-1}(\alpha), \beta_{\text{cop}} := S^{-1}(\beta), \alpha_{\text{cop}} := \beta \) and \( \beta_{\text{cop}} := \alpha \). Also if \( R \in G \otimes G \) is quasitriangular in \( G \), then \( R^{12} \) is quasitriangular in \( G_{\text{op}}, R_{21} \) is quasitriangular in \( G_{\text{cop}} \) and \( (R^{-1})^{21} \) is quasitriangular in \( G_{\text{cop}}^{\text{op}} \).

Next we recall that the definition of a quasitriangular quasi-Hopf algebra is ‘twist covariant’ in the following sense: An element \( F \in \mathcal{G} \otimes \mathcal{G} \) which is invertible and satisfies \((\epsilon \otimes \text{id})(F) = (\text{id} \otimes \epsilon)(F) = 1\), induces a so–called twist transformation
\[
\Delta_F(a) := F\Delta(a)F, \tag{2.15}
\]
\[
\phi_F := (1 \otimes F)(\text{id} \otimes \Delta)(F)\phi(\Delta \otimes \text{id})(F^{-1})(F^{-1} \otimes 1) \tag{2.16}
\]
It has been noticed by Drinfel’d [Dr2] that \((\mathcal{G}, \Delta_F, \epsilon, \phi_F)\) is again a quasi–bialgebra. Setting
\[
\alpha_F := S(h^i)ak^i, \quad \beta_F := f^i\beta S(g^i),
\]
where \(h^i \otimes k^i = F^{-1}\) and \(f^i \otimes g^i = F\), \((\mathcal{G}, \Delta_F, \epsilon, \phi_F, S, \alpha_F, \beta_F)\) is also a quasi-Hopf algebra. Moreover, if \(R\) is quasitriangular with respect to \((\Delta, \phi)\), then
\[
R_F := F^{21}RF^{-1}
\]
is quasitriangular w.r.t. \((\Delta_F, \phi_F)\). This means that a twist preserves the class of quasitriangular quasi-Hopf algebras [Dr2].

For Hopf algebras, one knows, that the antipode is an anti coalgebra morphism, i.e. \(\Delta(a) = (S \otimes S)(\Delta^{op}(S^{-1}(a)))\). For quasi-Hopf algebras this is true only up to a twist: Following Drinfeld we define the elements \(\gamma, \delta \in \mathcal{G} \otimes \mathcal{G}\) by setting
\[
\gamma := (S(U^i) \otimes S(T^i)) \cdot (\alpha \otimes \alpha) \cdot (V^i \otimes W^i)
\]
\[
\delta := (K^j \otimes L^j) \cdot (\beta \otimes \beta) \cdot (S(N^j) \otimes S(M^j))
\]
(2.18) \hspace{1cm} (2.19)
where
\[
T^i \otimes U^i \otimes V^i \otimes W^i = (1 \otimes \phi^{-1}) \cdot (id \otimes id \otimes \Delta)(\phi),
\]
\[
K^j \otimes L^j \otimes M^j \otimes N^j = (\Delta \otimes id \otimes id)(\phi) \cdot (\phi^{-1} \otimes 1).
\]
(2.20) \hspace{1cm} (2.21)

With these definitions Drinfel’d has shown in [Dr2], that \(f \in \mathcal{G} \otimes \mathcal{G}\) given by
\[
f := (S \otimes S)(\Delta^{op}(P^i)) \cdot \gamma \cdot \Delta(Q^i\beta R^i).
\]
(2.22)
defines a twist with inverse given by
\[
f^{-1} = \Delta(S(P^j)\alpha Q^j) \cdot \delta \cdot (S \otimes S)(\Delta^{op}(R^i)),
\]
(2.23)
such that for all \(a \in \mathcal{G}\)
\[
f\Delta(a)f^{-1} = (S \otimes S)(\Delta^{op}(S^{-1}(a))).
\]
(2.24)

The elements \(\gamma, \delta\) and the twist \(f\) fulfill the relation
\[
f \Delta(\alpha) = \gamma, \quad \Delta(\beta) f^{-1} = \delta
\]
(2.25)

Furthermore, the corresponding twisted reassociator \((2.10)\) is given by
\[
\phi_f = (S \otimes S \otimes S)(\phi^{321}).
\]
(2.26)

Setting \(h := (S^{-1} \otimes S^{-1})(f^{21})\), the above relations imply
\[
h\Delta(a)h^{-1} = (S^{-1} \otimes S^{-1})(\Delta^{op}(S(a)))
\]
(2.27)
\[
\phi_h = (S^{-1} \otimes S^{-1} \otimes S^{-1})(\phi^{321}).
\]
(2.28)
The importance of the twist \(f\) for the representation theory of \(\mathcal{G}\) is the existence of an intertwiner \(U \otimes V \longrightarrow (V \otimes U)^*\) given by \(\tau^{12} \circ (\pi_U \otimes \pi_V)(f)\).

Finally we introduce \(\hat{\mathcal{G}}\) as the dual space of \(\mathcal{G}\) with its natural coassociative coalgebra structure \((\hat{\Delta}, \hat{\epsilon})\) given by \(\langle \hat{\Delta}(\varphi) \mid a \otimes b \rangle := \langle \varphi \mid ab \rangle\) and \(\hat{\epsilon}(\varphi) := \langle \varphi \mid 1_\mathcal{G} \rangle\), where \(\varphi \in \hat{\mathcal{G}}, a, b \in \mathcal{G}\)^{6}

\(^{6}\) suppressing summation symbols
and where $\langle \cdot \mid \cdot \rangle : \hat{G} \otimes \hat{G} \to \mathbb{C}$ denotes the dual pairing. On $\hat{G}$ we have the natural left and right $G$–actions

$$a \to \varphi := \varphi(1) \langle \varphi(2) \mid a \rangle, \quad \varphi \leftarrow a := \varphi(2) \langle \varphi(1) \mid a \rangle,$$

where $a \in G$, $\varphi \in \hat{G}$. By transposing the coproduct on $G$ we also get a multiplication $\hat{G} \otimes \hat{G} \to \hat{G}$, which however is no longer associative

$$\langle \varphi \psi \mid a \rangle := \langle \varphi \otimes \psi \mid \Delta(a) \rangle, \quad \langle 1_{\hat{G}} \mid a \rangle := \epsilon(a).$$

Yet, we have the identities $1_{\hat{G}} \varphi = \varphi 1_{\hat{G}} = \varphi$, $\hat{\Delta}(\varphi \psi) = \hat{\Delta}(\varphi) \hat{\Delta}(\psi)$, $a \to (\varphi \psi) = (a(1) \to \varphi) (a(2) \to \psi)$ and $(\varphi \psi) \leftarrow a = (\varphi \leftarrow a(1))(\psi \leftarrow a(2))$ for all $\varphi, \psi \in \hat{G}$ and $a \in G$. We also introduce $\hat{S} : \hat{G} \to \hat{G}$ as the coalgebra anti-morphism dual to $S$, i.e. $\langle \hat{S}(\varphi) \mid a \rangle := \langle \varphi \mid S(a) \rangle$.

### 2.2 Graphical calculus

In the following it will be useful to have a graphical notation for the identities and definitions given so far. The graphical calculus introduced below has been developed and used in many papers, e.g. [RT,AC,T], mainly in the setting of ribbon–Hopf algebras. Formally speaking, it consists of a functor from the braided monoidal category $\text{Rep} \ G$ into a category of colored graphs. For an introduction into the category terminology see [K], [T]. We will use the graphical notation to have a pictorial way to understand - and deduce - certain relations and identities between morphisms (intertwiners) in $\text{Rep} \ G$, which - written out algebraically - would look very complicated. By morphisms in $\text{Rep} \ G$ we mean elements $t \in \text{Hom}_G(U,V)$, i.e. linear maps $t : U \to V$ satisfying $t \pi_U(a) = \pi_V(t) a$, $\forall a \in G$. As discussed in Section 2.1, the $n$–fold tensor product of $G$–modules is again a $G$–module (where one has to take care of the bracketing of the tensor factors). A morphism $t$ from an $n$-fold to an $m$-fold tensor product of $G$–modules is represented by a graph consisting of a “coupon” with $n$ lower legs and $m$ upper legs “coloured” with the source and target modules respectively. The upper and lower legs are always equipped with a definite bracketing corresponding to the bracketing defining the associated tensor module. For example the picture

![Graphical representation of morphism](https://example.com/graph.png)

corresponds to the morphism

$$t : X \otimes [(Y \otimes Z) \otimes U] \to (U \otimes V) \otimes X$$

The tensor product of two morphisms corresponds to the juxtaposition of diagrams and the composition of morphisms is depicted by gluing the corresponding graphs together. Here one has to take care that the gluing $t \circ k$ is only admissible if source($t$) = target($k$), which in particular implies that the bracketing conventions of the associated tensor factors have to coincide. We also use the convention that the lower legs always represent the source, i.e. the graph $t \circ k$ is obtained by gluing $t$ on top of $k$. 
Following the conventions of [AC] we now give a list of some special morphisms depicted by the following graphs:

\[ V := \text{id}_V, \]

\[ V^* \Rightarrow V := b_V, \]

\[ C \]

\[ V^* \Rightarrow V \Rightarrow b_V \]

\[ V^* \Rightarrow V := a_V. \]

\[ V^* \Rightarrow V \Rightarrow a_V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

\[ V \]

\[ W \]

where \( C \) stands for the one dimensional representation given by the counit and where

\[ b_V : C \to V \otimes V, \quad 1 \mapsto \sum_i \beta_i v_i \otimes v_i \]

\[ a_V : V^* \otimes V \to C, \quad \hat{v} \otimes w \mapsto \langle \hat{v} | \alpha_V w \rangle \]

\[ B_{VW} = \tau_{VW} \circ R_{VW}, \quad B_{VW}^{-1} = R_{VW}^{-1} \circ \tau_{VW}. \]

Here \( \{ v_i \} \) is a basis of \( V \) with dual basis \( \{ v_i^* \} \) and \( \tau_{VW} \) denotes the permutation of tensor factors in \( V \otimes W \). We also use the shortcut notation \( \alpha_V \equiv \pi_V(\alpha), \quad R_{VW} \equiv (\pi_V \otimes \pi_W)(R), \) etc. The properties of \( \mathcal{G} \) being a quasitriangular quasi-Hopf algebra ensure, that the above defined maps are in fact intertwiners (morphisms of \( \mathcal{G} \)-modules). Note that within higher tensor products the graphs (2.30) and (2.31) are only admissible if their legs are “bracketed together”. In order to change the bracket convention one has to use rebracketing morphisms. These are given as products of the basic elements

\[ \phi_{U VW}, \quad \phi_{V W}^{-1} \]

where each of the three individual legs in (2.32) may again represent a tensor product of \( \mathcal{G} \)-modules. In this way we adopt the convention that any empty coupon with the same number of upper and lower legs - where the colouring only differs by the bracket convention - always represents the associated unique rebracketing morphism in \( \text{Rep} \mathcal{G} \) given in terms of suitable products of \( \phi \)'s. We have already remarked that the uniqueness of this rebracketing morphism (i.e. the independence of the chosen sequence of intermediate rebracketings) is guaranteed by McLane’s coherence theorem and the “pentagon axiom” (2.2). This is why it
is often not even necessary to spell out one of the possible formulas for such an intertwiner. Explicitly, the pentagon identity \(2.2\) may be expressed as

\[
\begin{align*}
\text{(id \otimes \Delta \otimes \text{id})(\phi)} & \cdot \text{(\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1})} \\
= (1 \otimes \phi^{-1})(\text{id} \otimes \text{id} \otimes \Delta)(\phi)
\end{align*}
\]

which is the graphical notation for

\[
\begin{align*}
\text{(id \otimes \Delta \otimes \text{id})(\phi)} & \cdot \text{(\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1})} \\
= (1 \otimes \phi^{-1})(\text{id} \otimes \text{id} \otimes \Delta)(\phi)
\end{align*}
\]

In the same philosophy one may rewrite a simple rebracketing as a product of more complicated ones, as long as the overall source and target brackets coincide, for example

\[
\begin{align*}
\text{(id \otimes \Delta \otimes \text{id})(\phi)} & \cdot \text{(\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1})} \\
= (1 \otimes \phi^{-1})(\text{id} \otimes \text{id} \otimes \Delta)(\phi)
\end{align*}
\]

As done in the above pictures we will frequently not specify the modules sitting at the source and target legs. Also note that by Eqs. \(2.4\) and \(2.5\) the rebracketing of the invisible “white” leg corresponding to the trivial \(\mathcal{G}\)-module \(C\) is always given by the trivial identification.

If \(\mathcal{G}\) is finite dimensional, it may itself be viewed as a \(\mathcal{G}\)-module under left multiplication and algebraic identities may directly be translated into identities of the corresponding graphs and vice versa. So e.g. Eq. \(2.8\) is equivalent to

\[
\begin{align*}
\text{(id \otimes \Delta \otimes \text{id})(\phi)} & \cdot \text{(\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1})} \\
= (1 \otimes \phi^{-1})(\text{id} \otimes \text{id} \otimes \Delta)(\phi)
\end{align*}
\]

and Eqs. \(2.4\) and \(2.5\) together with \(2.7\) imply

\[
\begin{align*}
\text{(id \otimes \Delta \otimes \text{id})(\phi)} & \cdot \text{(\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1})} \\
= (1 \otimes \phi^{-1})(\text{id} \otimes \text{id} \otimes \Delta)(\phi)
\end{align*}
\]

(2.35)
and

\[
\begin{array}{c}
( \\
| \\
( \\
| )
\end{array}
= \begin{array}{c}
( \\
| \\
( \\
| )
\end{array}
\]

(2.37)

as well as the upside-down and left-right mirror images of (2.36) and (2.37) and the graphs obtained by rotating by 180° in the drawing plane. In general, with every graphical rule, where the graph is build from elementary graphs of the above list, the rotated as well as the upside-down and left-right mirror images are also valid and are proven analogously. This induces a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) - symmetry action on all graphical identities given below, which in fact is already apparent in the axioms of a quasi–triangular quasi–Hopf algebra given in Section 2.1 by taking \( G^{op}, G^{cop} \) or \( G^{op}G^{cop} \) instead of \( G \).

Finally we point out the important “pull through” rule saying that morphisms built from representation matrices of special elements in \( G \) (like the braiding (2.31) or the reassociator (2.32)) always “commute” with all other intertwiners in the appropriate sense i.e. by changing colours and orderings accordingly. For example one has

\[
\begin{array}{c}
U \\
| \\
V
\end{array}
\begin{array}{c}
h
\end{array}
\begin{array}{c}
X \\
U
\end{array}
= \begin{array}{c}
U \\
| \\
V
\end{array}
\begin{array}{c}
h \hspace{1cm} h'
\end{array}
\begin{array}{c}
X \\
U
\end{array}

(2.38)

In the language of categories this means that the braidings and the reassociators provide natural transformations [ML].

### 2.3 The antipode image of the R–matrix

In this subsection we exploit the full power of our graphical machinery by proving various important identities involving the action of the antipode on a quasitriangular R–matrix. We recall that for ordinary Hopf algebras (i.e. where \( \phi, \alpha \) and \( \beta \) are trivial) one has \( (S \otimes \text{id})(R) = R^{-1} = (\text{id} \otimes S^{-1})(R) \) and \( (S \otimes S)(R) = R \). To generalize these identities to the quasi–Hopf case we introduce the following four elements in \( G \otimes G \) (using the notation (2.9))

\[
\begin{align*}
p_\lambda &:= Y^i S^{-1}(X^i \beta) \otimes Z^i \\
q_\lambda &:= S(P^i) \alpha Q^i \otimes R^i
\end{align*}
\]

\[
\begin{align*}
p_\rho &:= P^i \otimes Q^i \beta S(R^i) \\
q_\rho &:= X^i \otimes S^{-1}(\alpha Z^i) Y^i
\end{align*}
\]

These elements have already been considered by [Dr2,S], see also Eqs. (9.20)-(9.23) of [HN]. They obey the commutation relations (for all \( a \in G \))

\[
\begin{align*}
\Delta(a(2)) p_\lambda [S^{-1}(a(1)) \otimes 1] & = p_\lambda [1 \otimes a], \\
\Delta(a(1)) p_\rho [1 \otimes S(a(2))] & = p_\rho [a \otimes 1] \\
[S(a(1)) \otimes 1] q_\lambda \Delta(a(2)) & = [1 \otimes a] q_\lambda, \\
[1 \otimes S^{-1}(a(2))] q_\rho \Delta(a(1)) & = [a \otimes 1] q_\rho
\end{align*}
\]

(2.40)

see e.g. [HN,Lem. 9.1]. A graphical interpretation of these identities will be given in Eqs. (2.45) below. With these definitions we now have
Theorem 2.1. Let \((\mathcal{G}, \Delta, \phi, S, \alpha, \beta)\) be a finite dimensional quasi–Hopf algebra, let \(\gamma \in \mathcal{G} \otimes \mathcal{G}\) be as in (2.18) and let \(R \in \mathcal{G} \otimes \mathcal{G}\) be quasitriangular. Then

\[
R^{-1} = [\Delta(R^j) (Y^i \otimes Z^i)] \cdot [(S \otimes \text{id})(q^\text{op} R)] \cdot [(R^i \otimes Q^j) \Delta^\text{op}(Z^j)] \\
= [\Delta(R^j) (Y^i \otimes Z^i)] \cdot [(S \otimes S^{-1})(R p_R)] \cdot [1 \otimes S^{-1}(\alpha Q^j X^i) P^j] \\
(S \otimes S)(R) \gamma = \gamma^{21} R
\]

(2.41)

A direct implication of equation (2.42) is the following formula, which has already been stated without proof in [AC].

Corollary 2.2. Under the conditions of Theorem 2.1 let \(f \in \mathcal{G} \otimes \mathcal{G}\) be the twist defined in (2.22), then

\[
f^\text{op} R f^{-1} = (S \otimes S)(R)
\]

(2.43)

Proof of Corollary 2.2. Using the formula (2.22) for \(f\) and (2.42) one computes

\[
f^\text{op} R = (S \otimes S)(\Delta(P^i)) \gamma^\text{op} \Delta^\text{op}(Q^j \beta R^i) R \\
= (S \otimes S)(\Delta(P^i)) (S \otimes S)(R) \gamma \Delta(Q^j \beta R^i) \\
= (S \otimes S)(R) \gamma \Delta(Q^j \beta R^i) = (S \otimes S)(R) f.
\]

To prepare the proof of Theorem 2.1 we need the following 3 Lemmata. First we have

Lemma 2.3.

\[
\begin{align*}
\begin{array}{c}
\text{Q}\end{array}
\end{align*}
\]

and three mirror images.

Proof. This is straightforward and left to the reader. (Use first \((2.33)\), then \((2.36)\) and \((2.37)\) (or the suitable mirror image) and finally \((2.33)\).

To give an algebraic formulation of the four identities of Lemma 2.3 let us introduce the notation

\[
P_{VW}^\lambda := \\
Q_{VW}^\lambda := \\
P_{WV}^\rho := \\
Q_{WV}^\rho :=
\]

(2.45)
In a more general scenario these morphisms have already been introduced in [HN]. Algebraically, they are given by (using the module notation)

\[
\begin{align*}
P^\lambda_{VW} : w &\mapsto v^i \otimes p_\lambda \cdot (v_i \otimes w), \\
P^\rho_{WV} &\cdot (w \otimes v_i) \otimes v^i, \\
Q^\lambda_{VW} : v \otimes v \otimes w &\mapsto (v \otimes \id)(q_\lambda \cdot (v \otimes w)), \\
Q^\rho_{WV} : w \otimes v \otimes \hat{v} &\mapsto (\id \otimes \hat{v})(q_\rho \cdot (w \otimes v))
\end{align*}
\]

see Eqs. (9.36),(9.37),(9.42) and (9.43) of [HN]. Note that the identities (2.40) precisely reflect the fact that these maps are morphisms in \(\text{Rep} \, G\). In this way Lemma 2.3 is also contained in [HN, Lem 9.1], since it is equivalent to the four identities, respectively

\[
\begin{align*}
[S(p^1_\lambda) \otimes 1] q_\lambda \Delta(p^2_\lambda) &= 1 \otimes 1, & [1 \otimes S^{-1}(p^2_\rho)] q_\rho \Delta(p^1_\rho) &= 1 \otimes 1, \\
\Delta(q^1_\lambda) p_\lambda [S^{-1}(q^1_\lambda) \otimes 1] &= 1 \otimes 1, & \Delta(q^1_\rho) p_\rho [1 \otimes S(q^2_\rho)] &= 1 \otimes 1.
\end{align*}
\]

Next we define intertwiners \(g_{VW} : (*W \otimes *V) \otimes (V \otimes W) \rightarrow \mathbb{C}\) and \(d_{VW} : \mathbb{C} \rightarrow (V \otimes W) \otimes (*W \otimes *V)\) by

\[
\begin{align*}
g_{VW} := &\begin{array}{|c|c|c|}
\hline
& *W & *V \\
\hline
V & W & \\
\hline
\end{array}
\quad & d_{VW} := &\begin{array}{|c|c|c|}
\hline
& V & *W \\
\hline
*V & \\
\hline
W & \\
\hline
\end{array}
\end{align*}
\]

One directly verifies that

\[
\begin{align*}
g_{VW}(\hat{w} \otimes \hat{v} \otimes v \otimes w) &= (\hat{v} \otimes \hat{w}) |_{\gamma_{VW}(v \otimes w)} \\
d_{VW}(1) &= \sum_i (\delta_{VW}(v_i \otimes w_i)) \otimes (w^i \otimes v^i),
\end{align*}
\]

where \(\hat{v} \in *V, \hat{w} \in *W, v \in V, w \in W\) and where \(\{v_i \otimes w_i\}\) is a basis of \(V \otimes W\) with dual basis \(\{v^i \otimes w^i\}\) in \(*V \otimes *W\). Here \(\gamma, \delta \in G \otimes G\) are given in (2.18),(2.19) and \(\gamma_{VW} = (\pi_V \otimes \pi_W)(\gamma), \delta_{VW} = (\pi_V \otimes \pi_W)(\delta)\). We remark that in terms of these intertwiners the identities (2.27) may now be depicted as

\[
\begin{align*}
\begin{array}{|c|c|c|}
\hline
*W & V \otimes W & \\
\hline
(V \otimes V) & & \\
\hline
\end{array}
\quad = &\begin{array}{|c|c|c|}
\hline
& g_{VW} & \\
\hline
f^{-1} & & \\
\hline
id_{V \otimes W} & & \\
\hline
\end{array}
\quad (V \otimes W)^* &\begin{array}{|c|c|c|}
\hline
& (V \otimes W) & \\
\hline
id_{V \otimes W} & & \\
\hline
\end{array}
\quad = &\begin{array}{|c|c|c|}
\hline
& f & \\
\hline
& & \\
\hline
d_{V \cdot W*} & & \\
\hline
\end{array}
\end{align*}
\]

Moreover, we have the following

**Lemma 2.4.**

\[
\begin{align*}
g_{VW} &= \begin{array}{|c|c|c|}
\hline
& *W & *V \\
\hline
V & W & \\
\hline
\end{array} \\
\quad d_{VW} &= \begin{array}{|c|c|c|}
\hline
& *V & *W \\
\hline
V & W & \\
\hline
\end{array}
\end{align*}
\]
Proof. We prove the first identity:

where in the second equality we have plugged in the pentagon identity (2.33) and in the third we have used (2.36). The second identity in (2.51) is the upside–down mirror image of the first one and is proved analogously.

We invite the reader to check that algebraically Lemma 2.4 implies the following identities for $\gamma$ and $\delta$ defined in (2.18) and (2.19):

$$
\gamma = (S(\tilde{U}^i) \otimes S(\tilde{T}^i)) \cdot (\alpha \otimes \alpha) \cdot (\tilde{V}^i \otimes \tilde{W}^i)
$$

$$
\delta = (\tilde{K}^j \otimes \tilde{L}^j) \cdot (\beta \otimes \beta) \cdot (S(\tilde{N}^j) \otimes S(\tilde{M}^j)), \quad \text{where}
$$

$$
\tilde{T}^i \otimes \tilde{U}^i \otimes \tilde{V}^i \otimes \tilde{W}^i = (\phi \otimes 1) \cdot (\Delta \otimes \text{id} \otimes \text{id}) (\phi^{-1}),
$$

$$
\tilde{K}^j \otimes \tilde{L}^j \otimes \tilde{M}^j \otimes \tilde{N}^j = (\text{id} \otimes \text{id} \otimes \Delta)(\phi^{-1}) \cdot (1 \otimes \phi).
$$

These identities have already been obtained by Drinfel’d [Dr2].

Finally we note the following linear isomorphisms of intertwiner spaces holding in fact in any rigid monoidal category.

**Lemma 2.5.** Let $X, V, W$ be finite dimensional $G$-modules. Then there exist linear bijections

$$
\Psi^V_{X,W} : \text{Hom}_G(X \otimes W, V) \rightarrow \text{Hom}_G(X, V \otimes ^*W) \quad (2.52)
$$

$$
\Phi^V_{X,W} : \text{Hom}_G(X, V \otimes W) \rightarrow \text{Hom}_G(^*V \otimes X, W) \quad (2.53)
$$
given by

\[ \Psi^V_{X,W} : \quad h' \quad \mapsto \quad h' \quad \quad ; \]

\[ (\Psi^V_{X,W})^{-1} : \quad h \quad \mapsto \quad h \quad \quad ; \]

\[ \Phi^V_W : \quad h \quad \mapsto \quad h \quad \quad ; \]

\[ (\Phi^V_W)^{-1} : \quad h' \quad \mapsto \quad h' \quad \quad ; \]

Proof. We prove \( \Psi^V_{X,W} \circ (\Psi^V_{X,W})^{-1} = \text{id} \) by determine its action on \( h \in \text{Hom}_G(X, V \otimes *W) \)
as follows:

where in the first equality we have used a “pull through” rule for $h$, and in the second
equality a left-right mirror image of (2.44). Analogously one shows that $\Psi^{-1} \circ \Psi = \text{id}$ and
$\Phi \circ \Phi^{-1} = \text{id} = \Phi^{-1} \circ \Phi$. \hfill\Box

We are now in the position to prove Eqs. (2.41) and (2.42) of Theorem 2.1 by rewriting
them as graphical identities as follows:

Lemma 2.6. For all finite dimensional $\mathcal{G}$-modules the inverse braiding $B^{-1}_{UV}$ obeys

and the (left) conjugate braiding $B_{*U,V}$ obeys

Taking $\mathcal{G}$ itself as a $\mathcal{G}$-module yields (2.54) $\Leftrightarrow$ (2.41) and (2.55) $\Leftrightarrow$ (2.42) and therefore the
above identities prove Theorem 2.1.
Proof. To prove the first equation in (2.54) note, that (2.11) and (2.14) imply the identity

\[
U U V \equiv U V \quad \text{(2.56)}
\]

Now we apply the isomorphism \((\Phi_{U,V})^{-1}\) of Lemma 2.5 to both sides of (2.56) to obtain

\[
\begin{align*}
U V & \equiv U V \\
\equiv & \quad \text{(2.57)}
\end{align*}
\]

where in the left identity we have used a “pull through” rule for the braiding. By the identity (2.37) the top of the graph in the middle of (2.57) may be replaced by

\[
\begin{align*}
\begin{array}{c}
\vdots \\
\end{array} & \equiv \begin{array}{c}
\vdots \\
\end{array} \\
\text{(2.58)}
\end{align*}
\]
and using Lemma 2.3 for the r.h.s. of (2.57) we end up with

\[ U \otimes V = U \otimes V \]

Hence we get the first equality in (2.54). Analogously one starts with (2.12),(2.14) to get

\[ U \otimes V = U \otimes V \]

where we have used that the trivial identification \( * (V^*) = V \). Taking the mirror image of the above proof yields the second equality in (2.54).

Eq. (2.55) follows from Lemma 2.7 below by putting \( h' = B_{U V} \) and \( h = B_{U \otimes V} \).

The identifications (2.54) \( \iff \) (2.41) and (2.55) \( \iff \) (2.42) are straightforward and are left to the reader. This concludes the proof of Lemma 2.6 and therefore also of Theorem 2.1. \( \square \)

We end this section with a Lemma used in the above proof, which will also be used in the next section.

**Lemma 2.7.** Let \( V, W, X, Y \) be finite dimensional \( G \)-modules with intertwiners

\[ h : X \otimes Y \longrightarrow W \otimes \nabla, \quad h' : V \otimes W \longrightarrow Y \otimes X \]

\(^7\)By finite dimensionality it suffices to prove the left inverse property.
then the following two identities are equivalent

(2.61)

Proof. Using (2.48), (2.51), a “pull through” rule for \( h \) and \( h' \) and rebracketing the source legs, Eq. (2.61) is equivalent to

(2.62)

Note that the l.h.s. of (2.63) is of the form \( \Psi^{-1}(\cdots) \) (with a “white” target leg). Now we apply the isomorphism \( \Psi^{\circ 2}_{X \otimes (Y \otimes V),W} \) of Lemma 2.5 to both sides of (2.63). Then using for
the bottom part of the r.h.s. the identity

\[
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
= \\
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
\]

(2.64)

and a pull through rule to raise the upper box of the r.h.s. of (2.64) to the top we conclude that (2.63) is equivalent to

\[
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
= \\
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
\begin{array}{c}
\{[ ] [ ] [ ] [ ] \} \\
\{[ ] [ ] [ ] \} \\
\{[ ] [ ] \} \\
\{[ ] \}
\end{array}
\]

(2.65)

The proof is finished by applying the isomorphism \((\Phi^{-1})_{X,Y,V}\). \qed

3 Doubles of quasi-Hopf algebras

3.1 \(\mathcal{D}(\mathcal{G})\) as an associative algebra

In this section we review the definition of the double \(\mathcal{D}(\mathcal{G})\) of a quasi–Hopf algebra \(\mathcal{G}\) as a diagonal crossed product as introduced in [HN]. We also give a graphical description of this construction. Consider

\[
\delta := (\Delta \otimes \text{id}) \circ \Delta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}
\]

(3.1)

and let \(\Phi \in \mathcal{G}^{\otimes 3}\) be given by

\[
\Phi := [(\text{id} \otimes \Delta \otimes \text{id})(\phi) \otimes 1] [\phi \otimes 1 \otimes 1] [((\delta \otimes \text{id} \otimes \text{id})(\phi^{-1})].
\]

(3.2)

Then the pair \((\delta, \Phi)\) provides a two–sided coaction of \(\mathcal{G}\) on itself as defined in [HN], i.e the following axioms are satisfied:

(i) The map \(\delta\) is a unital algebra morphism satisfying \((\varepsilon \otimes \text{id} \otimes \varepsilon) \circ \delta = \text{id}\).
(ii) The element $\Phi \in G^{\otimes 5}$ is invertible and fulfills
\begin{equation}
(id \otimes \delta \otimes id)((\delta(a)) \Phi = \Phi (\Delta \otimes id \otimes \Delta)(\delta(a)), \quad \forall a \in G \tag{3.3}
\end{equation}
\begin{equation}
(1 \otimes \Phi \otimes 1)(id \otimes \Delta \otimes id \otimes \Delta \otimes id)(\Phi)(\phi \otimes 1 \otimes \phi^{-1})
\end{equation}
\begin{equation}
= (id^{\otimes 2} \otimes \delta \otimes id^{\otimes 2})(\Phi)(\Delta \otimes id^{\otimes 3} \otimes \Delta)(\Phi) \tag{3.4}
\end{equation}
\begin{equation}
(id \otimes \epsilon \otimes id \otimes \epsilon \otimes id)(\Phi) = (\epsilon \otimes id^{\otimes 3} \otimes \epsilon)(\Phi) = 1 \otimes 1 \otimes 1 \tag{3.5}
\end{equation}

Next we define $\Omega \equiv \Omega^1 \otimes \Omega^2 \otimes \Omega^3 \otimes \Omega^4 \otimes \Omega^5 \in G^{\otimes 5}$ by
\begin{equation}
\Omega := (id^{\otimes 3} \otimes S^{-1} \otimes S^{-1})(f^{45} \cdot \Phi^{-1}) = (id^{\otimes 3} \otimes S^{-1} \otimes S^{-1})(\Phi^{-1}) \cdot h^{54} \tag{3.6}
\end{equation}
where $f, h \in G \otimes G$ are the twists defined in (2.22) and (2.27).

Let now $\hat{G}$ be the coalgebra dual to $G$ with its natural left and right $G$–action and the nonassociative multiplication given by $\langle \varphi \psi \mid a \rangle := \langle \varphi \otimes \psi \mid \Delta(a) \rangle$.

With $\delta : G \rightarrow G^{\otimes 3}$ being a two–sided coaction we then write $\varphi \triangleright a \triangleleft \psi := \langle \psi \otimes id \otimes \varphi \rangle(\delta(a))$. Note that for the two–sided coaction $\delta$ in Eq. (3.1) we have the identity $\varphi \triangleright a \triangleleft \psi = (\varphi \rightarrow a) \leftarrow \psi$.

Considered as an element of $\hat{G}$ we also write $1 \equiv 1_G \equiv \epsilon$. The following proposition has been proven in [HN]:

**Proposition 3.1.** Let $(\delta, \Phi)$ be a two–sided coaction of $G$ on $G$ and define the diagonal crossed product $\hat{G} \bowtie G$ to be the vector space $\hat{G} \otimes G$ with multiplication rule
\begin{equation}
(\varphi \bowtie a)(\psi \bowtie b) := \left[\Omega^1 \rightarrow \varphi \leftarrow \Omega^5\right](\Omega^2 \rightarrow \psi_2 \leftarrow \Omega^4) \bowtie \left[\Omega^3(\hat{S}^{-1}(\psi_{(1)}) \triangleright a \triangleleft \psi_{(3)}) b\right], \tag{3.7}
\end{equation}
where we write $(\varphi \triangleright a)$ in place of $(\varphi \bowtie a)$ to distinguish the new algebraic structure.

Then $\hat{G} \bowtie G$ is an associative algebra with unit $\hat{1} \bowtie 1$, containing $G \equiv 1 \bowtie G$ as a unital subalgebra.

The associativity of the above product follows from the axioms for two–sided coactions as has been proven in Theorem 10.2 of [HN]. Note that in general the subspace $\hat{G} \bowtie 1$ is not a subalgebra of $\hat{G} \bowtie G$. On the other hand if $G$ is an ordinary Hopf algebra with $\phi \equiv 1 \otimes 1 \otimes 1$ then Eq. (3.7) becomes
\begin{equation}
(\varphi \bowtie a)(\psi \bowtie b) = (\varphi \psi_{(2)} \bowtie (\hat{S}^{-1}(\psi_{(1)}) \triangleright a \triangleleft \psi_{(3)}) b) \tag{3.8}
\end{equation}
which is the standard multiplication rule in the quantum double $D(G)$ [Dr1,M3]. This motivates the

**Definition 3.2.** [HN] The diagonal crossed product $\hat{G} \bowtie G$ defined in Proposition 3.1 with $(\delta, \Phi)$ given by (3.1), (3.2) is called the **quantum double** of $G$, denoted by $D(G) \equiv \hat{G} \bowtie G$.

We will now rewrite the multiplication (3.7) given in Proposition 3.1 using the “generating matrix” formalism of the St. Petersburg school. In this way we will be able to give a graphical (i.e. a categorical) description of the algebraic relations in $\hat{G} \bowtie G$ which should convince the reader, that the multiplication given in (3.7) is indeed associative. The following Corollary is a generalization of [N, Lemma 5.2] and coincides with [HN,Cor. 10.4] applied to the present scenario.
Corollary 3.3. [HN] Let $\mathcal{A}$ be some unital algebra and $\gamma : \mathcal{G} \rightarrow \mathcal{A}$ a unital algebra map. Then the relation

$$\gamma_L(\varphi \otimes a) = (\varphi \otimes \text{id})(L) \cdot \gamma(a) \quad (3.9)$$

provides a one to one correspondence between unital algebra morphisms $\gamma_L : \hat{\mathcal{G}} \rightarrow \mathcal{A}$ extending $\gamma$ and elements $L \in \mathcal{G} \otimes \mathcal{A}$ satisfying $(\epsilon \otimes \text{id})(L) = 1_\mathcal{A}$ and

$$[1_{\mathcal{G}} \otimes \gamma(a)]L = [S^{-1}(a_{(1)}) \otimes 1_{\mathcal{A}}]L [a_{(-1)} \otimes \gamma(a_{(0)})], \quad \forall a \in \mathcal{G} \quad (3.10)$$

$$L^{13}L^{23} = [\Omega^5 \otimes \Omega^4 \otimes 1_{\mathcal{A}}][(\Delta \otimes \text{id})(L)][\Omega^1 \otimes \Omega^2 \otimes \gamma(\Omega^3)], \quad (3.11)$$

where $\Omega$ has been defined in (3.6) and where $\delta(a) = a_{(-1)} \otimes a_{(0)} \otimes a_{(1)}$.

An element $L \in \mathcal{G} \otimes \mathcal{A}$ satisfying $(\epsilon \otimes \text{id})(L) = 1_\mathcal{A}$ and (3.10)/(3.11) is called a normal coherent (left diagonal) $\delta$–implementer (with respect to $\gamma$), see [HN].

Note that by choosing $\mathcal{A} = \hat{\mathcal{G}} \triangleright \triangleright \mathcal{G}$ and $\gamma = \text{id}$, Cor. 3.3 implies that the multiplication on $\hat{\mathcal{G}} \triangleright \triangleright \mathcal{G}$ may uniquely be described by the relations of one “generating matrix” $L = \sum e_\mu \otimes (e_\mu \otimes 1)$, where $\{e_\mu\}$ is a basis in $\mathcal{G}$ with dual basis $\{e^\mu\}$. In fact this formulation is used in [HN] to prove the associativity of the multiplication (3.7).

We now give a graphical interpretation of the identities (3.10),(3.11) by using that any unital algebra map $\gamma : \mathcal{G} \rightarrow \mathcal{A}$ defines a $\mathcal{G}$–module structure on $\mathcal{A}$ via $b \cdot A := \pi_A(b)A := \gamma(b)A$. Moreover, given two $\mathcal{G}$–modules $V,W$ then $(V \otimes \mathcal{A}) \otimes W$ becomes a $\mathcal{G}$–module by setting $\pi_V \otimes A \otimes W(a) = (\pi_V \otimes \pi_A \otimes \pi_W)(\delta(a))$. Considering $\mathcal{G}$ as a $\mathcal{G}$–module by left multiplication we now define the map

$$L^A_{(\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^*} : (\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^* \rightarrow \mathcal{A}, \quad (b \otimes A \otimes \varphi) \mapsto (\varphi \otimes \text{id})(L \cdot (b \otimes A))$$

Then Eq. (3.10) is equivalent to $L^A_{(\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^*}$ being an intertwiner of $\mathcal{G}$–modules, i.e. to $L^A_{(\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^*} \in \text{Hom}_\mathcal{G}((\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^*, \mathcal{G})$. We depict this intertwiner as

$$L^A_{(\mathcal{G} \otimes \mathcal{A}) \otimes \mathcal{G}^*} := \begin{array}{c}
A \\
G \\
G \quad A \\
\{ \\
G \quad G \quad A \quad G^* \}
\end{array},$$

and call this a $d$–fork (≡ down fork) graph. The “coherence condition” (3.11) is now equivalent to the graphical identity

$$\begin{array}{c}
\begin{array}{c}
A \\
G \\
G \quad A \\
\{ \\
G \quad G \quad A \quad G^* \}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
A \\
G \\
\{ \\
G \quad G \quad A \quad G^* \}
\end{array}
\end{array}$$

(3.12)

Note that the lowest box on the r.h.s. represents the rebracketing morphism $\Phi^{-1}$ defined in (3.2). This explains why one has to chose the complicated multiplication rule (3.7) instead of (3.8) if $\phi$ and therefore $f$ and $\Phi$ are non–trivial.
3.2 Coherent $\Delta$–flip operators

We are now going to provide another set of generators in $\mathcal{D}(\mathcal{G})$ which later will be more appropriate for defining the (quasitriangular) quasi–Hopf structure. Associated with any coherent (left diagonal) $\delta$–implementer $L$ we define the element $T \in \mathcal{G} \otimes \mathcal{A}$ by

$$T := [S^{-1}(p^2_\rho \otimes 1)] \cdot L \cdot (\text{id} \otimes \gamma)(\Delta(p^1_\rho)), \quad (3.13)$$

where $p_\rho \equiv p^1_\rho \otimes p^2_\rho$ has been given in (2.33).

**Proposition 3.4.** [HN] The relation (3.13) defines a one-to-one correspondence between elements $L \in \mathcal{G} \otimes \mathcal{A}$ satisfying (3.10) and (3.11) and elements $T \in \mathcal{G} \otimes \mathcal{A}$ satisfying

$$(\text{id} \otimes \gamma)(\Delta^{\text{op}}(a)) T = T (\text{id} \otimes \gamma)(\Delta(a)), \quad \forall a \in \mathcal{G} \quad (3.14)$$

$$\phi^{312}_A T^{13}(\phi^{-1})^{132}_A T^{23}\phi_A = (\Delta \otimes \text{id})(T), \quad (3.15)$$

where $\phi_A = (\text{id} \otimes \text{id} \otimes \gamma)(\phi)$. $L$ is recovered from $T$ by

$$L = (\text{id} \otimes \gamma)(q^{op}_\rho) T \quad (3.16)$$

Moreover $(e \otimes \text{id})(T) = 1_A$ if and only if $(e \otimes \text{id})(L) = 1_A$.

Following [HN,Sect. 11] we call the elements $T \in \mathcal{G} \otimes \mathcal{A}$ satisfying (3.14) and (3.15) coherent $\Delta$–flip operators. They are special versions of coherent $\lambda \rho$–intertwiners [HN, Def. 10.8] associated with quasi–commuting pairs $(\lambda, \rho)$ of left $\mathcal{G}$–coactions $\lambda$ and right $\mathcal{G}$–coactions $\rho$ on an algebra $\mathcal{M}$. Proposition 3.4 has been proven algebraically in [HN,Prop. 10.10]. Before giving an alternative proof below, using the graphical calculus developed in Section 2.2 and 2.3, let us state the following central consequence

**Theorem 3.5.** Define the element $D \in \mathcal{G} \otimes (\hat{\mathcal{G}} \otimes \mathcal{G})$ by

$$D := \sum_{\mu} S^{-1}(p^2_\rho \otimes 1) \varepsilon_\mu p^1_{\rho(1)} \otimes (e^\mu \otimes p^1_{\rho(2)}) \quad (3.17)$$

and denote $i_D : \mathcal{G} \hookrightarrow \hat{\mathcal{G}} \otimes \mathcal{G}$ the canonical embedding $i_D(a) := 1_{\hat{\mathcal{G}}} \otimes a$. Then there is a unique algebra structure on the vector space $\hat{\mathcal{G}} \otimes \mathcal{G}$ satisfying

$$i_D(a) i_D(b) = i_D(ab) \quad \forall a, b \in \mathcal{G} \quad (3.18)$$

$$D \cdot (\text{id} \otimes i_D)(\Delta(a)) = (\text{id} \otimes i_D)(\Delta^{\text{op}}(a)) \cdot D \quad \forall a \in \mathcal{G} \quad (3.19)$$

$$\phi^{312} D^{13}(\phi^{-1})^{132} D^{23} \phi = (\Delta \otimes \text{id})(D). \quad (3.20)$$

where we have identified $\phi \equiv (\text{id} \otimes \text{id} \otimes i_D)(\phi) \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{D}(\mathcal{G})$. This algebra is precisely the quantum double $\mathcal{D}(\mathcal{G})$ defined in Prop. 3.7 and we have

$$\varphi \otimes a = (i_D \otimes \varphi(1))(q_\rho)(\varphi(2) \otimes \text{id})(D) i_D(a) \quad (3.21)$$

**Proof.** Follows from Proposition 3.4 and Cor. 3.3 by choosing $\mathcal{A} = \mathcal{D}(\mathcal{G})$ which means that $T = D$. \hfill \Box

We call $D$ the universal $\Delta$–flip operator in $\mathcal{D}(\mathcal{G})$. The description of the quantum double $\mathcal{D}(\mathcal{G})$ as given in Theorem 3.5 will be used in the next section to derive the quasi–Hopf structure of $\mathcal{D}(\mathcal{G})$. We now prove Proposition 3.4.

---

9 as before $\{e_\mu\}$ denotes a basis of $\mathcal{G}$ with dual basis $\{e^\mu\}$
Proof of Proposition 3.4. The equivalence \((\epsilon \otimes \text{id})(T) = 1_A \iff (\epsilon \otimes \text{id})(L) = 1_A\) follows from property (2.5) of \(\phi\). To show that the relations (3.10) and (3.11) for \(L\) are equivalent to the relations (3.14) and (3.15) for \(T\), respectively, we use the graphical calculus. First we use the isomorphism \(\Psi^A_{(G \otimes A), G^*}\) of Lemma 2.5 to define the intertwiner \(T_{GA} \in \text{Hom}_G(G \otimes A, A \otimes G)\) by

\[
T_{GA} \equiv \begin{array}{ccc}
\begin{array}{c}
G
\end{array} & \begin{array}{c}
A
\end{array} & \begin{array}{c}
G
\end{array} \\
\begin{array}{c}
G
\end{array} & \begin{array}{c}
A
\end{array} & \begin{array}{c}
G
\end{array}
\end{array} := \begin{array}{c}
L
\end{array} \begin{array}{c}
G
\end{array} (3.22)
\]

Algebraically Definition (3.22) translate into \(T_{GA}(b \otimes A) := T^{2^1} \cdot (A \otimes b)\), where \(T \in G \otimes A\) is expressed in terms of \(L\) by (3.13). Now note that the property of \(T_{GA}\) being an intertwiner of \(G\)-modules is equivalent to \(T\) satisfying (3.14). Thus we have proven the equivalence (3.10) \(\iff\) (3.14) and since the map \(\Psi^A_{(G \otimes A), G^*}\) is invertible also the invertibility of the transformation (3.13). In fact, (3.16) is equivalent to

\[
\begin{array}{ccc}
\begin{array}{c}
A
\end{array} & \begin{array}{c}
L
\end{array} & \begin{array}{c}
G
\end{array} \\
\begin{array}{c}
G
\end{array} & \begin{array}{c}
A
\end{array} & \begin{array}{c}
G^*
\end{array}
\end{array} = \begin{array}{c}
\text{id}_{G \otimes G}
\end{array} \begin{array}{c}
A
\end{array} (3.23)
\]

We are left to show that (3.11) \(\iff\) (3.15) (under the conditions (3.10) and (3.14), respectively). To this end we use that Eq. (3.13) is graphically expressed as

\[
\begin{array}{ccc}
\begin{array}{c}
A
\end{array} & \begin{array}{c}
G
\end{array} & \begin{array}{c}
G
\end{array} \\
\begin{array}{c}
G
\end{array} & \begin{array}{c}
G
\end{array} & \begin{array}{c}
A
\end{array}
\end{array} = \begin{array}{c}
\text{id}_{G \otimes G}
\end{array} \begin{array}{c}
G
\end{array} (3.23)
\]

Thus the following Lemma proves the equivalence (3.11) \(\iff\) (3.15) and therefore completes the proof of Proposition 3.4.

\(\square\)
Lemma 3.6. The graphical identities (3.12) for $L^A_{(A\otimes G)\otimes G}$ and (3.23) for $T_{G,A}$ are equivalent.

Proof. Let us prove (3.12) $\Rightarrow$ (3.23): Using the definition (3.22) we get for the l.h.s. of (3.23)

$$\text{l.h.s. of (3.23)} = \begin{array}{c}
\text{L} \\
\{(\quad)\quad\}
\end{array}$$

Here we have used a pull through rule to push the lower $d$–fork up and then we have combined all rebracketing morphisms in one box. Now plugging in Eq. (3.12), splitting the rebracketing morphism at the bottom into four factors and pushing two of them up, one obtains

$$\text{l.h.s. of (3.23)} = \begin{array}{c}
\text{L} \\
\{(\quad)\quad\}
\end{array} \begin{array}{c}
\text{id} \\
\{(\quad)\quad\}
\end{array}$$

Using the identity (2.50) the last picture equals the r.h.s. of (3.23). Hence we have shown (3.12) $\Rightarrow$ (3.23). The implication (3.23) $\Rightarrow$ (3.12) is shown similarly by bending the two upper $G$–legs in (3.23) down again. Thus we have proved Lemma 3.6 and therefore also Proposition 3.4. \qed
3.3 Left and right diagonal crossed products

In this subsection we sketch how the quantum double $D(G)$ may equivalently be modeled on $G \otimes \hat{G}$ instead of $\hat{G} \otimes G$. (In fact this is true for any diagonal crossed product as has been shown in [HN,Thm. 10.2].) With the notation as in Proposition 3.1 and with $\Omega_R \in G \otimes \hat{G}$ given by

$$\Omega_R := (h - 1)^{21} \cdot (S^{-1} \otimes S^{-1} \otimes \text{id}_G^3)(\Phi)$$

the right diagonal crossed product $G \triangleright \triangleleft \hat{G}$ is defined to be the vector space $G \otimes \hat{G}$ with multiplication rule

$$(a \triangleright \triangleleft \varphi)(b \triangleright \triangleleft \psi) := \left[ a (\varphi(1) \triangleright b \triangleleft S^{-1}(\varphi(3))) \right] \triangleright \left[ (\Omega_R^2 \rightarrow \varphi(2) \leftarrow \Omega_R^4)(\Omega_R^1 \rightarrow \psi \leftarrow \Omega_R^5) \right].$$

(3.24)

This makes $G \triangleright \triangleleft \hat{G}$ an associative algebra with unit $1 \otimes \hat{1}$, containing $G \equiv G \triangleright \triangleleft \hat{1}$ as a unital subalgebra. To see that the two algebras $\hat{G} \triangleright \triangleleft G$ and $G \triangleright \triangleleft \hat{G}$ are isomorphic let us begin with stating the analogue of Lemma 3.3: Let $\gamma : G \rightarrow \mathcal{A}$ be a unital algebra map into some target algebra $\mathcal{A}$. Then the relation

$$\gamma_R(a \triangleright \varphi) = \gamma(a) \cdot (\varphi \otimes \text{id})(R)$$

(3.25)

provides a one–to–one correspondence between unital algebra morphisms $\gamma_R : G \triangleright \triangleleft \hat{G} \rightarrow \mathcal{A}$ extending $\gamma$ and elements $R \in G \otimes \mathcal{A}$ satisfying $(\epsilon \otimes \text{id})(R) = 1_\mathcal{A}$ and

$$R [1_G \otimes \gamma(a)] = [a_{(1)} \otimes \gamma(a_{(0)})] R [S^{-1}(a_{(-1)}) \otimes 1_\mathcal{A}], \quad \forall a \in G$$

(3.26)

and

$$R^{13} R^{23} = [\Omega_R^2 \otimes \Omega_R^5 \otimes \gamma(\Omega_R^3)] (\Delta \otimes \text{id})(R) [\Omega_R^2 \otimes \Omega_R^1 \otimes 1_\mathcal{A}]$$

(3.27)

We call such elements normal coherent right diagonal $\delta$–implementers [HN]. With this definition one gets

**Lemma 3.7.** Let $\gamma : G \rightarrow \mathcal{A}$ be some unital algebra map. Then the relation

$$R := [\Phi^5 S^{-1}(\Phi^4 \beta) \otimes 1_\mathcal{A}] L [\Phi^2 S^{-1}(\Phi^1 \beta) \otimes \gamma(\Phi^3)]$$

(3.28)

defines a one–to–one correspondence between unital algebra maps $\gamma_L : \hat{G} \triangleright \triangleright G \rightarrow \mathcal{A}$ and unital algebra maps $\gamma_R : G \triangleright \triangleright \hat{G} \rightarrow \mathcal{A}$ extending $\gamma$, as defined in (3.3) and (3.25), respectively.

**Proof.** We will sketch the proof, using graphical methods. For more details see [HN, Prop. 10.5]. Defining the map $R^{(G \otimes A) \otimes G}_A : \mathcal{A} \rightarrow (G^* \otimes A) \otimes G$ by

$$R^{(G^* \otimes A) \otimes G}_A(A) := \sum e^\mu \otimes [R^{21} \cdot (A \otimes e_\mu)], \quad A \in \mathcal{A},$$

property (3.26) of $R$ is equivalent to $R^{(G^* \otimes A) \otimes G}_A$ being an intertwiner of $G$–modules. Depicting this intertwiner as a $u$–fork ($\equiv$ up fork) graph

$\begin{array}{c}
R \\
A
\end{array}$

is equivalent to

$\begin{array}{c}
\hat{G}^* \\
A
\end{array}$

$\begin{array}{c}
G \\
A
\end{array}$

$\begin{array}{c}
A
\end{array}$

$\begin{array}{c}
G
\end{array}$

$\begin{array}{c}
A
\end{array}$

$\begin{array}{c}
R
\end{array}$

$\begin{array}{c}
\hat{G}^*
\end{array}$

$\begin{array}{c}
\hat{G}
\end{array}$

$\begin{array}{c}
A
\end{array}$

$\begin{array}{c}
\hat{1}
\end{array}$

$\begin{array}{c}
\hat{G}
\end{array}$
the relation (3.28) may graphically be expressed as

\[
\begin{array}{c}
\mathcal{G}^* \\
\downarrow \\
\mathcal{A} \\
\downarrow \\
\mathcal{G}
\end{array}
\xrightarrow{R} 
\begin{array}{c}
\mathcal{A} \\
\downarrow \\
\mathcal{G}^* \\
\downarrow \\
\mathcal{G}
\end{array}
\]

(3.29)

Since the r.h.s. of (3.29) defines a \( \mathcal{G} \)-module intertwiner if and only if \( L \) satisfies (3.10), the element \( R \) defined by (3.28), (3.29) satisfies (3.26) if and only if \( L \) satisfies (3.10). The equivalence of the coherence conditions (3.11) and (3.27) is shown by first expressing (3.27) as the graphical identity (3.12), then plugging in the definition (3.29) and using a pull through rule to collect all rebracketing morphisms at the bottom of the graph and finally using the identities (2.50). We leave it to the reader to draw the corresponding pictures.

We are left to show that relation (3.28) may be inverted. This follows by a “two–sided version” of Lemma 2.5. The reader is invited to check that the inverse is given by

\[
L := [S^{-1}(\alpha \tilde{\Phi}^5) \Phi^4 \otimes \gamma(\Phi^3)] R [S^{-1}(\alpha \Phi^2) \Phi^1 \otimes 1_A],
\]

with \( \Phi^{-1} =: \Phi^1 \otimes \tilde{\Phi}^2 \otimes \tilde{\Phi}^3 \otimes \Phi^4 \otimes \Phi^5 \).

Eq. (3.30) may be expressed graphically as the upside–down mirror image of picture (3.29) with \( R \) and \( L \) as well as \( \mathcal{G} \) and \( \mathcal{G}^* \) exchanged.

Putting \( A = \hat{\mathcal{G}} \bowtie \mathcal{G} \) and \( \gamma_L = \text{id} \) or \( A = \mathcal{G} \bowtie \hat{\mathcal{G}} \) and \( \gamma_R = \text{id} \), respectively, Lemma 3.7 implies

**Corollary 3.8.** Let \((\delta, \Phi)\) be the two–sided coaction of \( \mathcal{G} \) on \( \mathcal{G} \) given in (3.2) and define \( V \equiv V^1 \otimes V^2 \otimes V^3 \), \( W \equiv W^1 \otimes W^2 \otimes W^3 \in \mathcal{G} \otimes^3 \) by

\[
V := S(\Phi^1) \alpha \tilde{\Phi}^2 \otimes \tilde{\Phi}^3 \otimes S^{-1}(\alpha \Phi^2) \Phi^4, \quad W := \Phi^2 S^{-1}(\Phi^1 \beta \otimes \Phi^5 \otimes \Phi^4 \beta S(\Phi^5))
\]

Then the map

\[
\hat{\mathcal{G}} \bowtie \mathcal{G} \ni (\varphi \bowtie a) \mapsto \left((V^2 \bowtie (S^{-1}(V^1) \rightarrow \varphi \leftarrow V^3)) \cdot (a \bowtie 1)\right) \in \hat{\mathcal{G}} \bowtie \mathcal{G}
\]

provides an algebra isomorphism with inverse given by

\[
\mathcal{G} \bowtie \hat{\mathcal{G}} \ni (a \bowtie \varphi) \mapsto (1 \bowtie a) \cdot \left(((W^1 \leftarrow \varphi \rightarrow S^{-1}(W^3)) \bowtie W^2)\right) \in \hat{\mathcal{G}} \bowtie \mathcal{G}.
\]

Corollary 3.8 has been proven for general diagonal crossed products in [HN, Thm. 10.2.iii] using the notation \( V \equiv q_8 \) and \( W \equiv p_8 \). We also remark that one may equally well use the two–sided \( \mathcal{G} \)-coaction \( \delta' := (\text{id} \otimes \Delta) \circ \Delta \) with reassociator \( \Phi' := [1 \otimes (\text{id} \otimes \Delta \otimes \text{id})(\phi^{-1})][1 \otimes 1 \otimes \phi^{-1}][(\text{id} \otimes \text{id} \otimes \delta')(\phi)] \) to construct another to versions of quantum doubles \( \hat{\mathcal{G}} \bowtie_{\Psi} \mathcal{G} \) and \( \mathcal{G} \bowtie_{\delta'} \hat{\mathcal{G}} \). Since the two–sided coactions \((\delta, \Phi)\) and \((\delta', \Phi')\) are twist equivalent [HN, Prop.8.4], these constructions are also isomorphic to the previous ones, i.e. all four diagonal crossed products define equivalent extensions of \( \mathcal{G} \) [HN, Prop. 10.6].

### 3.4 The quasitriangular quasi–Hopf structure

In [HN] we have shown that \( \mathcal{D}(\mathcal{G}) \) is a quasi–bialgebra. As one might expect, \( \mathcal{D}(\mathcal{G}) \) is even a quasitriangular quasi-Hopf algebra. This is the content of the next theorem.
Theorem 3.9. Let $\mathcal{D}(\mathcal{G})$ be the associative algebra defined in Theorem 2.3. Then $(\mathcal{D}(\mathcal{G}), \Delta_D, \epsilon_D, \phi_D, S_D, \alpha_D, \beta_D, R_D)$ is a quasitriangular quasi-Hopf algebra, where

$$\phi_D := (i_D \otimes i_D \otimes i_D)(\phi), \quad (3.31)$$

$$R_D := (i_D \otimes \text{id})(\text{D}) = \sum_{\mu} i_D(\epsilon_\mu) \otimes \text{D}(\epsilon^\mu) \quad (3.32)$$

and where the structural maps are given by

$$\Delta_D(i_D(a)) := (i_D \otimes i_D)(\Delta(a)), \quad \forall a \in \mathcal{G} \quad (3.33)$$

$$\epsilon_D(i_D(a)) := \epsilon(a), \quad \forall a \in \mathcal{G}, \quad (3.35)$$

where $\epsilon$ is a counit for $\Delta_D$.

Furthermore the antipode $S_D$ is defined by

$$S_D(i_D(a)) := i_D(S(a)), \quad \forall a \in \mathcal{G} \quad (3.37)$$

$$(S \otimes S_D)(\text{D}) := (\text{id} \otimes i_D)(f^{op})(\text{id} \otimes i_D)(f^{-1}) \quad (3.38)$$

where $f \in \mathcal{G} \otimes \mathcal{G}$ is the twist defined in (2.22). The elements $\alpha_D, \beta_D$ are given by

$$\alpha_D := i_D(\alpha), \quad \beta_D := i_D(\beta). \quad (3.39)$$

Proof. To simplify the notation we will frequently suppress the embedding $i_D$, if no confusion is possible, i.e. we write $\alpha \equiv i_D(\alpha) = \alpha_D, \quad (\text{id} \otimes \text{id} \otimes i_D)(\phi) \equiv \phi$ etc. To show that (3.33) and (3.34) define an algebra morphism $\Delta_D : \mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G})$, it is sufficient to check the consistency with the defining relations (3.18) - (3.20). Consistency with (3.18) is obvious because of (3.33). Let us go on with (3.18). For the r.h.s. we get

$$(\text{id} \otimes \Delta_D)(\phi_D(a) \text{D}) = [(\Delta \otimes \text{id})(\Delta(a))]^{231} \cdot (\phi^{-1})^{231} \varnothing_{213} \varnothing_{12} \varnothing^{-1}$$

The l.h.s. yields

$$(\text{id} \otimes \Delta_D)(\text{D}(\Delta(a))) = (\phi^{-1})^{231} \varnothing_{213} \varnothing_{12} \varnothing^{-1}[(\text{id} \otimes \Delta)(\Delta(a))],$$

which, using (3.19) and the property $\phi(\Delta \otimes \text{id})(\Delta) = (\text{id} \otimes \Delta)(\Delta)\phi$ to shift the factor $(\text{id} \otimes \Delta)(\Delta(a))$ to the left, equals the r.h.s.

Consistency with the relation (3.20) may be checked in a longer but analogous calculation, where one also has to use the pentagon equation for $\phi$ several times, as in the proof of [HN, Lem. 11.2]. Hence $\Delta_D$ is an algebra map. To show that $\Delta_D$ is quasi-coassociative we compute by a similar calculation

$$(1 \otimes \phi_D) \circ (\text{id} \otimes \Delta_D \otimes \text{id})(\text{D}) = (\text{id} \otimes \text{id} \otimes \Delta_D)(\text{D}) \cdot (1 \otimes \phi_D),$$

by using again the pentagon equation for $\phi$ and the covariance property (3.19).

The property of $\epsilon_D$ being a counit for $\Delta_D$ follows directly from the fact that $(\text{id} \otimes \epsilon \otimes \text{id})(\phi) = 1 \otimes 1$. Hence $(\mathcal{D}(\mathcal{G}), \Delta_D, \epsilon_D, \phi_D)$ is a quasi-bialgebra, see [HN, Thm. 11.3] for more details.

To show quasitriangularity we first note that the element $R_D = (i_D \otimes \text{id})(\text{D})$ fulfills (2.11) and (2.12) so as to say by definition because of (3.20) and (3.34). The invertibility of $R_D$
is equivalent to the invertibility of $D$ which will be proved in Lemma 3.10 (i) below. We are left to show that $R_D$ intertwines $\Delta_D$ and $\Delta_D^{op}$, i.e.
\[
\Delta_D^{op}(id_D(a)) \cdot R_D = R_D \cdot \Delta_D(id(D(a)), \quad \forall a \in \mathcal{G}
\] (3.40)
\[
(id \otimes \Delta_D^{op}(D)) \cdot R_D^{23} = R_D^{23} \cdot (id \otimes \Delta_D)(D).
\] (3.41)

Now Eq. (3.41) follows from (3.19). Hence we also get in $\mathcal{D}(\mathcal{G})^{\otimes^3}$
\[
R_D^{12} \cdot (\Delta_D \otimes id)(R_D) = (\Delta_D^{op} \otimes id)(R_D) \cdot R_D^{12},
\] (3.42)
which together with (2.11) implies the quasi-Yang Baxter equation
\[
(\phi_D^{-1})^{321} R_D^{12} \phi_D^{312} R_D^{13} (\phi_D^{-1})^{132} R_D^{23} = R_D^{23} (\phi_D^{-1})^{231} R_D^{13} \phi_D^{213} R_D^{12} \phi_D^{-1}.
\] (3.43)

Using the Definition (3.34), Eq. (3.43) is further equivalent to
\[
(i_D \otimes \Delta_D^{op}(D)) \cdot R_D^{23} = R_D^{12} \cdot (i_D \otimes \Delta_D)(D)
\]
which also proves (3.41). Hence $R_D$ is quasi-triangular.

In order to prove that the definition of $S_D$ in (3.37) may be extended anti-multiplicatively to the entire algebra $\mathcal{D}(\mathcal{G})$, we have to show that this continuation is consistent with the defining relations (3.19), (3.20). This amounts to showing
\[
(S \otimes S_D)(D) \cdot (S \otimes S_D)(\Delta^{op}(a)) = (S \otimes S_D)(\Delta(a)) \cdot (S \otimes S_D)(D), \quad \text{and} \quad (S \otimes S_D)(D) \cdot ((\Delta \otimes id)(D)) = (S \otimes S_D)(\Delta)(D) \cdot (S \otimes S_D)(D),
\] (3.44)
\[
(S \otimes S \otimes S_D)((\Delta \otimes id)(D)) = (S \otimes S \otimes S_D)(\phi) \cdot (S \otimes S \otimes S_D)(D^{23}) \cdot (S \otimes S \otimes S_D)(\phi^{312}).
\] (3.45)
Since by definition $(S \otimes S_D)(D) = f^{op}Df^{-1}$ equation (3.44) follows directly from (3.19) and the fact, that by (2.24) $f$ has the property $f \cdot \Delta(S(a)) = (S \otimes S)(\Delta^{op}(a)) \cdot f$. For the proof of (3.45) let us recall, that $\Delta^f := f \Delta(\cdot)f^{-1}$ defines a twist equivalent quasi-coassociative coproduct on $\mathcal{G}$ with twisted reassociator $\phi_f$ defined in (2.16) satisfying $\phi_f = (S \otimes S \otimes S)(\phi^{321})$ (see (2.26)). Thus we get for the l.h.s. of (3.45) (with $D_f := f^{op}Df^{-1}$)
\[
(S \otimes S \otimes S_D)((\Delta \otimes id)(D)) = (\Delta^{op}_f \otimes id)(S \otimes S_D)(D)
\]
\[
= (\Delta^{op}_f \otimes id)(D_f)
\]
\[
= \phi_f^{321} D_f^{23} (\phi_f^{-1})^{231} D_f^{13} \phi_f^{213},
\]
where the last equality is exactly the transformation property of a quasi-triangular R-matrix under a twist [Dr2] and may be proven analogously using (3.19). By (2.24) this equals the r.h.s. of (3.45). Hence $S_D$ defines an anti-algebra morphism on $\mathcal{D}(\mathcal{G})$.

We are left to show that the map $S_D$ fulfills the antipode axioms given in (2.7) and (2.8). Axiom (2.8) is clearly fulfilled since we have $S_D \circ i_D = i_D \circ S$ and $\alpha_D = i_D(\alpha), \beta_D = i_D(\beta), \phi_D = (i_D \otimes i_D \otimes i_D)(\phi)$. Noting that $\Delta_D(id_D(a)) = (id_D \otimes i_D)(\Delta(a)), a \in \mathcal{G}$, the validity of axiom (2.7) follows from its validity in $\mathcal{G}$ and Lemma 3.10 (ii) below. □
Lemma 3.10.

(i) The universal flip operator \( D \in \mathcal{G} \otimes \mathcal{D}(\mathcal{G}) \) is invertible where the inverse is given by

\[
D^{-1} = [X^j \beta S(P^i Y^j) \otimes 1] \cdot [(S \otimes \text{id})(q^D \otimes D)] \cdot [(R^i \otimes Q^j) \Delta_{op}(Z^j)],
\]  

where \( q^D \in \mathcal{G} \otimes \mathcal{G} \) has been defined in (2.39).

(ii) Let \( \mu_D \) denote the multiplication map \( \mu_D : \mathcal{D}(\mathcal{G}) \otimes \mathcal{D}(\mathcal{G}) \longrightarrow \mathcal{D}(\mathcal{G}) \), then

\[
(id \otimes \mu_D) \circ (id \otimes S_D \otimes \text{id})( (id \otimes \Delta_D)(D) \cdot (1_\mathcal{G} \otimes 1_\mathcal{G} \otimes \alpha_D) ) = 1_\mathcal{G} \otimes \alpha_D \quad (3.47)
\]

\[
(id \otimes \mu_D) \circ (id \otimes \text{id} \otimes S_D)( (id \otimes \Delta_D)(D) \cdot (1_\mathcal{G} \otimes \beta_D \otimes 1_G) ) = 1_\mathcal{G} \otimes \beta_D \quad (3.48)
\]

Proof. We will use the graphical methods adopted in Sections 2.2/2.3. To this end let us view \( \mathcal{G} \) and \( \mathcal{D} \equiv \mathcal{D}(\mathcal{G}) \) as left \( \mathcal{G} \)–modules. Then, due to (3.19), \( \tilde{B}_{GD} := \tau_{GD} \circ D \) defines an intertwiner \( \tilde{B}_{GD} : \mathcal{G} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{G} \) which will be depicted as

\[
\tilde{B}_{GD} =:
\]

(In fact this is the intertwiner \( T_{GA} \) defined in (3.22) for the special case \( A = \mathcal{D} \). For the left modules \( \ast \mathcal{G} \) and \( \ast \mathcal{D} \) the corresponding intertwiners \( \tilde{B}_{G\ast D}, \tilde{B}_{G\ast D}, \tilde{B}_{\ast G\ast D} \) are defined with the help of the map \( S \) and/or \( S_D \). Graphically they are represented by the same picture, except that the colours of the legs are replaced by \( \ast \mathcal{G} \) and(or) \( \ast \mathcal{D} \), respectively. The reason for distinguishing the \( \mathcal{D} \)–line from the \( \mathcal{G} \)–line lies in the fact that unlike in (2.31) \( \tilde{B}_{GD} \) is not given in terms of a quasitriangular R–matrix in \( \mathcal{G} \otimes \mathcal{G} \), which is why we write \( \tilde{B}_{GD} \) in place of \( B_{GD} \). Correspondingly, the identities derived in Section 2.3 are not automatically valid for \( \tilde{B}_{GD} \). We now show, which of them still hold. First, since \( S \) is an antipode for \( \Delta \), Eq. (3.20) together with \( (\epsilon \otimes \text{id})(D) = 1_D \) implies the equality (compare with (2.54))

\[
\text{(3.49)}
\]
and a step by step repetition of the prove of \( \eqref{2.59} \) yields

\[
(\tilde{\beta}_{GD})^{-1} =: \begin{array}{c}
\text{Diagram}
\end{array} = \begin{array}{c}
\text{Diagram}
\end{array}
\]

This means that algebraically we get the analogue of the first identity in Eq. \( \eqref{2.41} \) which yields \( \eqref{3.46} \). Thus we have proven part (i)

To prove (ii) let us translate the two claims \( \eqref{3.47} \) and \( \eqref{3.48} \) into the graphical language as

\[
\eqref{3.47} \Leftrightarrow \begin{array}{c}
\text{Diagram}
\end{array} = \begin{array}{c}
\text{Diagram}
\end{array}
\]

and

\[
\eqref{3.48} \Leftrightarrow \begin{array}{c}
\text{Diagram}
\end{array} = \begin{array}{c}
\text{Diagram}
\end{array}
\]

Note that as opposed to \( \eqref{3.49} \) the identities \( \eqref{3.51} \) and \( \eqref{3.52} \) are not automatically satisfied, since \( S_D \) is not yet proved to be an antipode for \( \Delta_D \). To prove \( \eqref{3.51} \) and \( \eqref{3.52} \) we now
proceed backwards along the proof of Lemma 2.6, i.e. we use Lemma 2.5 to show that either of these two identities is equivalent to

\[
\begin{array}{c}
\text{G} \quad \text{*D} \\
\text{G} \quad \text{G} \\
\text{G} \quad \text{G}
\end{array}
\]

\[
= \quad \text{.*D} \quad \text{G}
\]

\[
\text{D} \quad \text{G}
\]

<snip>

More precisely (3.51) is equivalent to (3.53) just as (2.60) is equivalent to the second equation in (2.54), and “rotating” this proof by 180° in the drawing plane we also get (3.52) ⇔ (3.53). Thus we are left with proving (3.53). To this end we remark, that (3.49) equally holds if we replace \( \text{D} \) by \( \text{*D} \), and therefore (3.50) also holds with \( \text{D} \) replaced by \( \text{*D} \). Hence (3.53) follows from (3.50) provided we can show

\[
\begin{array}{c}
\text{G} \quad \text{*D} \\
\text{G} \quad \text{G} \\
\text{G} \quad \text{G}
\end{array}
\]

\[
= \quad \text{.*D} \quad \text{G}
\]

\[
\text{D} \quad \text{G}
\]

<snip>

By Lemma 2.7 this further equivalent to

\[
\begin{array}{c}
\text{G} \quad \text{*D} \\
\text{G} \quad \text{G} \\
\text{G} \quad \text{G}
\end{array}
\]

\[
= \quad \text{.*D} \quad \text{G}
\]

\[
\text{D} \quad \text{G}
\]

<snip>

Using (2.48) and (2.49), Eq. (3.55) is algebraically equivalent to

\[\gamma^{op} \text{D} = (S \otimes S_D)(\text{D}) \gamma,\]

which finally holds by (3.38), (3.19) and (2.23). This concludes the proof of Lemma 3.10 (ii) and therefore of Theorem 3.9. \(\square\)
Clearly, if $G$ is a Hopf algebra and $\phi = 1 \otimes 1 \otimes 1$, one recovers the well-known definitions of $\Delta_D, S_D$ and $R_D$ in Drinfeld’s quantum double

\[
\begin{align*}
\Delta_D(i_D(g)) &= (i_D \otimes i_D)(\Delta(g)) \\
\Delta_D(D(\varphi)) &= (D \otimes D)(\hat{\Delta}^{op}(\varphi)) \\
S_D(i_D(g)) &= i_D(S(g)) \\
S_D(D(\varphi)) &= D(S^{-1}(\varphi)) \\
R_D &= (\hat{1} \otimes e_\mu) \otimes (e_\mu \otimes 1),
\end{align*}
\]

where $D(\varphi) := (\varphi \otimes \text{id})(\mathcal{D})$, $\varphi \in \hat{G}$. As in the Hopf algebra case, one may take the construction of the quasitriangular R-Matrix in $\mathcal{D}(\mathcal{G})$ as the starting point and formulate Theorem 3.5 together with Theorem 3.9 differently:

**Corollary 3.11.** Let $G$ be a finite dimensional quasi-Hopf algebra with invertible antipode. Then there exists a unique quasi-Hopf algebra $D(G)$ such that

(i) $D(G) = \hat{G} \otimes G$ as a vector space,

(ii) the canonical embedding $i_D : G \rightarrow 1 \hat{G} \otimes G \subset D(G)$ is a unital injective homomorphism of quasi-Hopf algebras,

(iii) Let $D \in G \otimes D(G)$ be given by Eq. (3.17), then the element $R_D := (i_D \otimes \text{id})(D) \in D(G) \otimes D(G)$ is quasitriangular.

This quasi-Hopf algebra structure is given by the definitions in Theorem 3.5 and 3.9.

**Proof.** The property (ii) implies (3.18), (3.33), (3.31), (3.35) and (3.39), yielding also $f_D = (i_D \otimes i_D)(f)$. The quasitriangularity of $R_D$ implies (3.19), (3.20), (3.34) and (3.36) and according to (2.43) $(S_D \otimes S_D)(R_D) = f_D^{op} R_D f_D^{-1}$. Hence the antipode is uniquely fixed to be the one defined in Theorem 3.9. \qed

We are now in the position to prove Theorem A.

**Proof of Theorem A.** First note that Corollary 3.11 already proves the existence parts (i),(ii) of Theorem A by putting $D(\varphi) := (\varphi \otimes \text{id})(\mathcal{D})$. The fact that $\mu : \hat{G} \otimes G \rightarrow D(G)$ provides a linear isomorphism follows from the last statement in Theorem 2.5. Moreover, if $\bar{D} \supset G$ is another Hopf algebra extension and if $\bar{D} : \hat{G} \rightarrow \bar{D}$ is a linear map such that $\bar{D}$ is algebraically generated by $G$ and $\bar{D}(\mathcal{G})$ and $R_{\bar{D}} := e_\mu \otimes \bar{D}(e_\mu) \in \bar{D} \otimes \bar{D}$ is quasitriangular, then $\nu : D(\mathcal{G}) \rightarrow \bar{D}$

\[
\nu(\varphi \otimes a) := (\text{id} \otimes \varphi(1))(q_\mu) \bar{D}(\varphi(2)) a \quad (3.56)
\]

is a uniquely and well defined algebra map satisfying $\nu \circ D = \bar{D}$ by Prop. 3.4. In fact, the quasitriangularity of $R_{\bar{D}}$ implies that $\nu$ is even a quasi–bialgebra homomorphism. Thus $\mathcal{D}(\mathcal{G})$ also solves the universality property (iii) of Theorem A. In particular the extension $\mathcal{D}(\mathcal{G}) \supset G$ is unique up to equivalence. \qed

### 3.5 The category $\text{Rep} \mathcal{D}(\mathcal{G})$

We will now give a representation theoretical interpretation of the quantum double $\mathcal{D}(\mathcal{G})$ by describing its representation category in terms of the representation category of the underlying quasi-Hopf algebra $\mathcal{G}$. In this way we will show that $\mathcal{D}(\mathcal{G})$ is a concrete realization of the quantum double as defined by Majid in [M2] with the help of a Tannaka-Krein-like
reconstruction theorem. We denote the monoidal category of finite dimensional unital representations of \( \mathcal{D}(\mathcal{G}) \) and of \( \mathcal{G} \) by \( \text{Rep} \mathcal{D}(\mathcal{G}) \) and \( \text{Rep} \mathcal{G} \), respectively. The next proposition states a necessary and sufficient condition, under which a representation of \( \mathcal{G} \) extends to a representation of \( \mathcal{D}(\mathcal{G}) \):

**Proposition 3.12.**

1.) The objects of \( \text{Rep} \mathcal{D}(\mathcal{G}) \) are in one to one correspondence with pairs \( \{ (\pi_V, V), D_V \} \), where \( (\pi_V, V) \) is a finite dimensional representation of \( \mathcal{G} \) and where \( D_V \in \mathcal{G} \otimes \text{End}_C(V) \) is a normal coherent \( \Delta \)-flip, i.e.

\[
(i) \quad (\epsilon \otimes \text{id})(D_V) = \text{id}_V
\]

\[
(ii) \quad D_V \cdot (\text{id} \otimes \pi_V)(\Delta(a)) = (\text{id} \otimes \pi_V)(\Delta^{\text{op}}(a)) \cdot D_V, \quad \forall a \in \mathcal{G}
\]

\[
(iii) \quad \phi_V^{312} D_V^{13} (\phi_V^{-1})_{132} D_V^{23} \phi_V = (\Delta \otimes \text{id})(D_V), \text{ where } \phi_V := (\text{id} \otimes \text{id} \otimes \pi_V)(\phi).
\]

2.) Let \( \{ (\pi_V, V), D_V \} \) and \( \{ (\pi_W, W), D_W \} \) be as above, then

\[
\text{Hom}_{\mathcal{D}(\mathcal{G})} = \{ t \in \text{Hom}_C(V, W) \mid (\text{id} \otimes t)(D_V) = D_W \}
\]

**Proof.** We define the extended representation \( \pi_V^D \) on the generators of \( \mathcal{D}(\mathcal{G}) \) by

\[
\pi_V^D(i_D(g)) := \pi_V(g), \quad g \in \mathcal{G}
\]

\[
\pi_V^D(D(\varphi)) := (\varphi \otimes \text{id}_{\text{End}_V})(D_V), \quad \varphi \in \hat{\mathcal{G}}
\]

Condition (i) implies that \( \pi_V^D \) is unital whereas conditions (ii),(iii) just reflect the defining relations (3.19) and (3.20) of \( \mathcal{D}(\mathcal{G}) \), which ensures, that \( \pi_V^D \) is a well defined algebra morphism. On the other hand, given a representation \( \{ \pi_V^D, V \} \) of \( \mathcal{D}(\mathcal{G}) \), we define

\[
D_V := (\text{id}_\mathcal{G} \otimes \pi_V^D)(D)
\]

which clearly satisfies conditions (i) - (iii). This proves part 1. Part 2. follows trivially. \( \square \)

To get the relation with Majid’s formalism [M2] we now write \( a \cdot v := \pi_V(a) v, a \in \mathcal{G}, v \in V \) and define \( \beta_V : V \rightarrow \mathcal{G} \otimes V; \quad v \mapsto v(1) \otimes v(2) := D_V(1_\mathcal{G} \otimes v) \). With this notation we get the following Corollary:

**Corollary 3.13.** The conditions (i)-(iii) of Proposition 3.12 are equivalent to the following three conditions for \( \beta_V \) (as before denoting \( P^i \otimes Q^i \otimes R^i = \phi^{-1} \)):

\[
(i') \quad (\epsilon \otimes \text{id}_V) \circ \beta_V = \text{id}_V
\]

\[
(ii') \quad (a(2) \cdot v)^{(1)} a(1) \otimes (a(2) \cdot v)^{(2)} = a(2) v^{(1)} \otimes a(1) \cdot v^{(2)}, \quad \forall v \in V
\]

\[
(iii') \quad R^i v^{(1)} \otimes (Q^i \cdot v^{(2)})^{(1)} P^i \otimes ((Q^i \cdot v^{(2)})^{(2)} = (\phi^{-1})^{321} \cdot \left[ (R^i v^{(1)})^{(2)} Q^i \otimes (R^i \cdot v^{(2)})^{(1)} P^i \otimes (R^i \cdot v^{(2)})^{(2)} \right], \quad \forall v \in V
\]

**Proof.** The equivalences (i) \( \Leftrightarrow \) (i') and (ii) \( \Leftrightarrow \) (ii') are obvious. The equivalence (iii) \( \Leftrightarrow \) (iii') follows by multiplying (iii) with \( (\phi_V^{-1})^{312} \) from the left and with \( \phi_V^{-1} \) from the right and permuting the first two tensor factors. \( \square \)

The conditions stated in the above Corollary agree with those formulated in [M2, Prop.2.2] by taking \( \mathcal{G}^{\text{op}} \equiv (\mathcal{G}, \Delta^{\text{op}}, (\phi^{-1})^{321}) \) instead of \( (\mathcal{G}, \Delta, \phi) \) as the underlying quasi-bialgebra. This means that we have identified the category \( \text{Rep} \mathcal{D}(\mathcal{G}) \) with what is called the double category of modules over \( \mathcal{G} \) in [M2].
4 Doubles of weak quasi–Hopf algebras

Allowing the coproduct $\Delta$ to be non–unital (i.e. $\Delta(1) \neq 1 \otimes 1$) leads to the definition of weak quasi–Hopf algebras as introduced by G. Mack and V. Schomerus in [MS]. In this Section we sketch how the construction of the quantum double $D(G)$ generalizes to this case. As it will turn out, there are only minor adjustments to be made. The reason for this lies in the fact, that we have used mostly graphical identities (i.e. identities in $\text{Rep} G$) to derive and describe our results. But since $\text{Rep} G$ is a rigid monoidal category also in the case of weak quasi–Hopf algebras $G$, all graphical identities in Section 2.2 and 2.3 stay valid. Thus the only adjustments required refer to those points, where we have translated graphical identities into algebraic ones.

Following [MS] we define a weak quasi–Hopf algebra $(G, \Delta, \epsilon, \phi)$ to be an associative algebra $G$ with unit $1$, a non–unital algebra map $\Delta : G \rightarrow G \otimes G$, an algebra map $\epsilon : G \rightarrow \mathbb{C}$ and an element $\phi \in G \otimes G \otimes G$ satisfying (2.1)-(2.3), whereas (2.4) is replaced by

$$\quad (\text{id} \otimes \epsilon \otimes \text{id})(\phi) = \Delta(1) \quad (4.1)$$

and where in place of invertibility $\phi$ is supposed to have a quasi–inverse $\tilde{\phi} \equiv \phi^{-1}$ with respect to the intertwining property (2.1). By this we mean that $\tilde{\phi}$ satisfies $\phi \tilde{\phi} \phi = \phi$, $\tilde{\phi} \phi \phi = \tilde{\phi}$ as well as

$$\quad \phi = (\text{id} \otimes \Delta)(\Delta(1)), \quad \tilde{\phi} = (\Delta \otimes \text{id})(\Delta(1))$$

which implies the further identities

$$\quad (\text{id} \otimes \Delta)(\Delta(a)) = \phi (\text{id} \otimes \text{id})(\Delta(a)) \tilde{\phi}, \quad \forall a \in G \quad (4.3)$$

$$\quad \phi = \phi (\text{id} \otimes \Delta)(\Delta(1)), \quad \tilde{\phi} = \tilde{\phi} (\text{id} \otimes \Delta)(\Delta(1)) \quad (4.4)$$

$$\quad (\text{id} \otimes \epsilon \otimes \text{id})(\tilde{\phi}) = \Delta(1) \quad (4.5)$$

More generally we call an element $t \in A$ an intertwiner between two (possibly non–unital) algebra maps $\alpha, \beta : G \rightarrow A$, if

$$\quad t \alpha(a) = \beta(a) t, \forall a \in G \quad \text{and} \quad t \alpha(1) \equiv \beta(1) t = t \quad (4.6)$$

In this case by a quasi–inverse of $t$ (with respect to this intertwiner property) we mean the unique (if existing) element $\tilde{t} \equiv t^{-1} \in A$ satisfying $\tilde{t} t = \alpha(1)$, $\tilde{t} \beta = \beta(1)$ and $\tilde{t} \alpha(1) \equiv \tilde{t} t = \tilde{t}$. Note that this implies

$$\quad \tilde{t} \beta(a) = \alpha(a) \tilde{t}, \quad \tilde{t} \beta(1) \equiv \alpha(1) \tilde{t} = \tilde{t} \quad (4.7)$$

and therefore $t$ is also the quasi–inverse of $t^{-1}$.

A weak quasi–bialgebra is called weak quasi–Hopf algebra, if there exists a unital algebra antimorphism $S : G \rightarrow G$ and elements $\alpha, \beta \in G$ satisfying (2.7) and (2.8). We will also always suppose that $S$ is invertible.

Furthermore, $G$ is said to be quasitriangular if there exists an element $R \in G \otimes G$ satisfying (2.10)-(2.12) and possessing a quasi–inverse $\tilde{R} \equiv R^{-1}$ with respect to the intertwining property (2.11).

With these substitutions Theorem A generalizes as follows

**Theorem B** Let $(G, \Delta, \phi)$ be a finite dimensional weak quasi–Hopf algebra with invertible antipode $S$. Assume $D(G) \supset G$ to be a weak quasi–Hopf algebra extension satisfying (i)-(iii)
of Theorem A. Then \( \mathcal{D}(\mathcal{G}) \) exists uniquely up to equivalence and the linear map \( \mu : \hat{\mathcal{G}} \otimes \mathcal{G} \to \mathcal{D}(\mathcal{G}) \)

\[
\mu(\varphi \otimes a) := (\text{id} \otimes \varphi(1))(q_{\rho}) D(\varphi(2))
\]

is surjective with \( \text{Ker} \mu = \text{Ker} P \), where \( P : \hat{\mathcal{G}} \otimes \mathcal{G} \to \hat{\mathcal{G}} \otimes \mathcal{G} \) is the linear projection

\[
P(\varphi \otimes a) := \varphi(2) \otimes (\hat{S}^{-1}(\varphi(1)) \cdot 1_{\mathcal{G}}) \leftarrow \varphi(3) \equiv : \varphi \triangleright a
\]

To adapt our previous strategy to weak quasi–Hopf algebras we first recall that due to the coproduct being non–unital the definition of the tensor product functor in \( \text{Rep} \mathcal{G} \) has to be slightly modified. First note that the element \( \Delta(1) \) (as well as higher coproducts of \( 1 \)) is idempotent and commutes with all elements in \( \Delta(\mathcal{G}) \). Thus, given two representations \((V, \pi_V), (W, \pi_W)\), the operator \((\pi_V \otimes \pi_W)(\Delta(1))\) is a projector, whose image is precisely the \( \mathcal{G} \)–invariant subspace of \( V \otimes W \) on which the tensor product representation operates non trivial. Thus one is led to define the tensor product \( \boxtimes \) of two representations of \( \mathcal{G} \) by setting

\[
V \boxtimes W := (\pi_V \otimes \pi_W)(\Delta(1)) (V \otimes W), \quad \pi_V \boxtimes \pi_W := (\pi_V \otimes \pi_W) \circ \Delta_{|V \boxtimes W}
\]

One readily verifies that with these definitions \( \phi_{UVW} \) - restricted to the subspace \( (U \boxtimes V) \boxtimes W \) - furnish a natural family of isomorphisms defining an associativity constraint for the tensor product functor \( \boxtimes \), where the tensor product of morphisms is defined by restricting the “usual” tensor product map to the truncated subspace.

With these adjustments, the graphical calculus described in Section 2.2 and 2.3 carries over to the present case. The collection of colored upper (or lower) legs represent the (truncated) tensor product of \( \mathcal{G} \)–modules associated with the individual legs. One just has to take care when translating the pictures into algebraic identities. For example the graph

\[
\begin{array}{c}
\hat{\mathcal{G}} \\
\equiv \\
\mathcal{G} \boxtimes \mathcal{G}
\end{array}
\]

is a pictorial representation of \( \Delta(1) \) and not of \( 1 \otimes 1 \)! Thus the graph \( (2.44) \) is equivalent to the algebraic identity \([S(p_1^{1}) \otimes 1_{\mathcal{G}} \Delta(p_2^{1}) = \Delta(1)] \) in place of the first equation of \( (2.47) \), etc. In this way all graphical identities of Section 2 stay valid as well as Theorem 2.1, where now \( R^{-1} \) is meant to be the quasi–inverse of \( R \).

The definition of the diagonal crossed product in Proposition \( 3.1 \) yields an associative algebra which in general is not unital, but \((1_{\hat{\mathcal{G}}} \otimes 1_{\mathcal{G}})\) is still a right unit and in particular idempotent [HN, Thm. 14.2]. This may be cured by taking the right ideal generated by \((1_{\hat{\mathcal{G}}} \otimes 1_{\mathcal{G}})\). Thus, let \( P : \hat{\mathcal{G}} \otimes \mathcal{G} \to \hat{\mathcal{G}} \otimes \mathcal{G} \) be the linear projection given by left multiplication with \( 1_{\hat{\mathcal{G}}} \otimes 1_{\mathcal{G}} \) with respect to the algebra structure \((3.7)\), i.e.

\[
P(\varphi \otimes a) := \varphi(2) \otimes \hat{S}^{-1}(\varphi(1)) \cdot 1_{\mathcal{G}} \varphi(3) \equiv : \varphi \triangleright a
\]

As in [HN, Sect. 14] we introduce the notation

\[
\varphi \triangleright a := P(\varphi \otimes a) \in \hat{\mathcal{G}} \otimes \mathcal{G}
\]

and define the quantum double \( \mathcal{D} (\mathcal{G}) \) as the subalgebra

\[
\mathcal{D}(\mathcal{G}) := \mathcal{G} \triangleright \mathcal{G} \equiv P(\hat{\mathcal{G}} \otimes \mathcal{G})
\]
Then $1_G \triangleright \triangleleft 1_G \equiv 1_G \otimes 1_G$ is the unit of $D(G)$ and in terms of the notation (4.8) the multiplication in $D(G)$ is still given by (3.7). In particular

$$i_D : G \ni a \mapsto 1_G \triangleright \triangleleft a \equiv 1_G \otimes a \in D(G)$$

still provides a unital algebra inclusion. Interpreting Eq. (3.9) also via (4.8), Corollary 3.3 likewise extends to the present scenario. However note that now the definition (3.10) for left diagonal $\delta$–implementers $L \in G \otimes A$ also implies the nontrivial relation

$$L \equiv [1_G \otimes 1_A] L = [S^{-1}((1_1) \otimes 1_A)] L [1_{(-1)} \otimes \tau(1_{(0)})],$$

(4.10)

This leads to a slight modification of Proposition 3.4 where one has to add the requirement that $\Delta$–flip operators $T$ fulfill also

$$T (id \otimes \tau)(\Delta(1)) \equiv (id \otimes \tau)(\Delta^{op}(1)) T = T$$

which follows directly from (4.10) or by multiplicating both sides of (3.13) from the right with $(id \otimes \tau)(\Delta(1))$.

Taking this additional identity into account, Theorem 3.5 now reads

**Theorem 4.1.** Using the notation (4.8) we define the element $D \in G \otimes D(G)$ by

$$D := \sum_{\mu} S^{-1}(p_\mu^2) e_\mu p^1_{\rho(1)} \otimes (e_\mu \triangleright \triangleleft p^1_{\rho(2)})$$

Then the multiplication (3.7) is the unique algebra structure on $D(G)$ satisfying (3.18)–(3.20) together with

$$D (id \otimes i_D)(\Delta(1)) \equiv (id \otimes i_D)(\Delta^{op}(1)) D = D$$

(4.11)

Moreover the identity (3.21) also stays valid.

The quasitriangular quasi–Hopf structure is now defined precisely as in Theorem 3.8 and is proven analogously, where in (3.38) $f^{-1}$ is the quasi–inverse of $f$. Correspondingly $D^{-1}$ given by (3.41) becomes the quasi–inverse of $D$ with respect to the $\Delta$–flip property (3.19). Thus we arrive at a

**Proof of Theorem B.** The existence parts (i),(ii) of Theorem B follow by putting as before $D(\varphi) = (\varphi \otimes id)(D)$. The universality (and therefore uniqueness) property (iii) follows analogously as in the proof of Theorem A, Eq. (3.56). Here one just has to note that by Prop. 3.4

$$\tilde{L} := q^{op}_\mu \tilde{D} \in G \otimes \tilde{D}$$

is a normal coherent $\delta$–implementer, where $\tilde{D} := \sum e_\mu \otimes \tilde{D}(e_\mu)$. Hence $\tilde{L}$ satisfies (4.10) and therefore the map $\nu : \tilde{G} \otimes G \rightarrow \tilde{D}$

$$\nu(\varphi \otimes a) := (\varphi \otimes id)(\tilde{L}) a$$

satisfies $\nu \circ P = \nu$, where $P$ is the projection (4.7). Since as in Corollary 3.3 the relations (3.10), (3.11) guarantee that $\nu$ is an algebra map with respect to the multiplication (7.7) on $G \otimes G$, it passes down to a well defined algebra map $f : D(G) \rightarrow \tilde{D}$, $f(\varphi \triangleright \triangleleft a) := \nu(\varphi \triangleright \triangleleft a)$, thus proving (iii). Since $\tilde{D}$ (as an element in $D \otimes \tilde{D}$) is also required to be quasitriangular, $f$ is even a quasi–bialgebra homomorphism. In the case $\tilde{D} = D(G)$ we have $\nu = \mu$ and $Ker P = Ker \mu$ by definition, proving also the second part of Theorem B. □
A  The twisted double of a finite group

As an application we now use Theorem 3.3 and Theorem 3.9 to recover the “twisted” quantum double $D^\omega(G)$ of [DPR] where $G$ is a finite group and $\omega : G \times G \times G \to U(1)$ is a normalized 3-cocycle. By definition this means $\omega(g, h, k) = 1$ whenever at least one of the three arguments is equal to the unit $e$ of $G$ and

$$\omega(g, x, y)\omega(gx, y, z)^{-1}\omega(g, xy, z)\omega(g, x, yz)^{-1}\omega(x, y, z) = 1, \quad \forall g, x, y, z \in G.$$  

The Hopf algebra $G := Fun(G)$ of functions on $G$ may then also be viewed as a quasi-Hopf algebra with its standard coproduct, counit and antipode but with reassociator given by

$$\phi := \sum_{g, h, k \in G} \omega(g, h, k) \cdot (\delta_g \otimes \delta_h \otimes \delta_k), \quad \text{(A.1)}$$

where $\delta_g(x) := \delta_{g,x}$. The identities (2.2) and (2.4) for $\phi$ are equivalent to $\omega$ being a normalized 3-cocycle. Also note that choosing $\alpha = 1_G$ the antipode axioms now require $\beta = \sum_g \omega(g^{-1}, g, g^{-1})\delta_g$. In this special example our quantum double $D(G) \equiv \hat{G} \triangleright \triangleright G$ allows for another identification with the linear space $\hat{G} \otimes G$.

Lemma A.1. Let $G$ be as above and define $\sigma : \hat{G} \otimes G \to D(G)$ by $\sigma(\varphi \otimes a) := D(\varphi) a, \varphi \in \hat{G}, a \in G$. Then $\sigma$ is a linear bijection.

Proof. Since $(\hat{G}, \Delta, \epsilon, S)$ is also an ordinary Hopf algebra, the relation (3.19) is equivalent to (suppressing the symbol $i_D$)

$$a \Delta(\varphi) = D(a_{(1)} \to \varphi \leftarrow S^{-1}(a_{(3)})) a_{(2)}, \quad \forall a \in G, \varphi \in \hat{G}. \quad \text{(A.2)}$$

Using (3.21) this implies

$$\varphi \triangleright a \equiv (\id \otimes \varphi_{(1)})(q_{\rho}) D(\varphi_{(2)}) a = D(q_{\rho(1)} \to \varphi \leftarrow (q_{\rho} S^{-1}(q_{\rho(3)}))) q_{\rho(2)} a,$$

which lies in the image of $\sigma$. Hence, $\sigma$ is surjective and therefore also injective. \qed

We note that in general the map $\sigma$ need not be surjective (nor injective). Due to Lemma A.1 we may now identify $D(G)$ with the new algebraic structure on $\hat{G} \otimes \hat{G}$ induced by $\sigma^{-1}$. We call this algebra $\hat{G} \otimes_D G$. Putting $a \equiv \hat{1} \otimes_D a, a \in G$ and $D := e_\mu \otimes (e^\mu \otimes_D 1) \in \hat{G} \otimes (\hat{G} \otimes_D G)$ it is described by the relations (A.2), (3.20) and the requirement of $G \equiv \hat{1} \otimes_D G$ being a unital subalgebra. To compute these multiplication rules we now use that the group elements $g \in G$ provide a basis in $\hat{G}$ with dual basis $\delta_g \in \hat{G}$. Hence a basis of $\hat{G} \otimes_D \hat{G}$ is given by $\{h \otimes_D \delta_g\}_{h, g \in G}$. In this basis the generating matrix $D$ is given by

$$D = \sum_{k \in G} \delta_k \otimes (k \otimes_D 1_G), \quad 1_G = \sum_{h \in G} \delta_h. \quad \text{(A.3)}$$

Let us know compute the multiplication laws according to the definitions in Theorem 3.5. To begin with, we have

$$(h \otimes 1_G)(e \otimes \delta_g) = (h \otimes \delta_g) \quad \text{and} \quad (g \otimes 1_G)(h \otimes 1_G) = (gh \otimes 1_G).$$

Taking $(x \otimes \id)$ of both sides of (3.19), where $x \in G$, and using $\Delta(h) = \sum_{k \in G} \delta_k \otimes \delta_{k^{-1}g}$ we get

$$(x \otimes 1_G)(e \otimes \delta_{x^{-1}g}) = (e \otimes \delta_{gx^{-1}})(x \otimes 1_G),$$
or equivalently
\[(e \otimes \delta_g)(x \otimes 1_G) = (x \otimes \delta_{x^{-1}gx}). \tag{A.4}\]

Finally, pairing equation (3.20) with \(x \otimes y \in \hat{G} \otimes \hat{G}\) in the two auxiliary spaces, the l.h.s. yields

\[
\sum_{s,r,t \in G} \omega(s, x, y)(1 \otimes \delta_s)(x \otimes 1_G) \cdot \omega(x, r, y)^{-1}(e \otimes \delta_r)(y \otimes 1_G) \cdot \omega(x, y, t)(1 \otimes \delta_t)
\]

\[
= (x \otimes 1_G)(y \otimes 1_G) \cdot \sum_{s,r,t \in G} \frac{\omega(s, x, y)\omega(x, y, t)}{\omega(x, r, y)}(e \otimes \delta_{(xy)^{-1}sxy}^{-1}g_{y^{-1}ry} \delta_t)
\]

\[
= \sum_{t \in G} (x \otimes 1_G)(y \otimes 1_G)(1 \otimes \delta_t) \frac{\omega(xyty^{-1}, x, y)\omega(x, y, t)}{\omega(x, yty^{-1}, y)},
\]

where we have used (A.4) in the first equality. The right hand side of (3.20) gives \((xy \otimes 1_G)\) so that we end up with

\[
(x \otimes 1_G)(y \otimes 1_G) = \sum_{t \in G} \frac{\omega(x, yty^{-1}, y)}{\omega(xyty^{-1}, x, y)\omega(x, y, t)}(xy \otimes \delta_t). \tag{A.5}\]

Similarly the coproduct is computed as \(\Delta_D(e \otimes \delta_g) = \sum_{k \in G} (e \otimes \delta_k) \otimes (e \otimes \delta_{k^{-1}g})\) and

\[
\Delta_D(x \otimes 1_G) = \sum_{r,s \in G} \frac{\omega(xrx^{-1}, x, s)}{\omega(x, r, s)\omega(xrx^{-1}, xsx^{-1}, x)} ((x \otimes \delta_r) \otimes (x \otimes \delta_s)). \tag{A.6}\]

The above construction agrees with the definition of \(D^\omega(G)\) given in [DPR] up to the convention, that they have build \(D(G)\) on \(\hat{G} \otimes \hat{G}\) instead of \(\hat{G} \otimes \hat{G}\).

**B The Monodromy Algebra**

The definition of monodromy algebras (see e.g. [AFLS]) associated with quasitriangular Hopf algebras may now easily be generalized to the case of quasi-Hopf algebras. This has already been done in [AGS]. We will give an explicit proof that the defining relations of [AGS] indeed define an associative algebra structure on \(\hat{G} \otimes \hat{G}\), which in fact is isomorphic to our quantum double \(D(G)\). For ordinary Hopf algebras this has recently been shown in [N1].

Let \(\hat{G}\) be a finite dimensional quasi-Hopf algebra with quasitriangular R-matrix \(R \in \hat{G} \otimes \hat{G}\). Following [N1] we define the monodromy matrix \(M \in \hat{G} \otimes D(\hat{G})\) to be

\[
M := (id \otimes i_D)(R^{op}) D.
\]

Defining also \(\hat{R} \in \hat{G} \otimes \hat{G} \otimes D(\hat{G})\) by

\[
\hat{R} := \phi^{213} R^{12} \phi^{-1},
\]

we get the following Lemma:

**Lemma B.1.** The monodromy matrix \(M\) obeys the following three conditions (dropping the symbol \(i_D\)):

\[
(e \otimes \text{id})(M) = 1_{D(\hat{G})}, \tag{B.1}
\]

\[
\Delta(a) M = M \Delta(a), \quad a \in \hat{G} \tag{B.2}
\]

\[
M^{13} \hat{R} M^{23} = \hat{R} \phi (\Delta \otimes \text{id})(M) \phi^{-1} \tag{B.3}
\]
Proof. We will freely suppress the embedding $i_D$. Since the R-Matrix has the property $(\text{id} \otimes \epsilon)(R) = 1$, equation (B.1) follows from $(\epsilon \otimes \text{id})(D) = 1_{D(G)}$. The identity (B.2) is implied by (B.19) and the intertwiner property of the R–Matrix. Let us now compute the l.h.s. of (B.3):

$$M^{13} \hat{R} M^{23} = R^{31} D^{13} \phi^{213} R^{12} \phi^{-1} R^{32} D^{23}$$

$$= R^{31} D^{13} [(\Delta \otimes \text{id})(R)\phi^{-1}]_{132} D^{23}$$

$$= [(R \otimes 1) \cdot (\Delta \otimes \text{id})(R)]^{312} D^{13} (\phi^{-1})_{132} D^{23}$$

where we have used the quasitriangularity of $R$ in the second line and property (B.19) of $D$ in the third line. The r.h.s. of (B.3) yields

$$\hat{R} \phi (\Delta \otimes \text{id})(M) \phi^{-1} = \phi^{213} \phi^{312} D^{13} (\phi^{-1})^{132} D^{23}$$

$$= [\phi^{321} (1 \otimes R) (\text{id} \otimes \Delta)(R) \phi]^{312} D^{13} (\phi^{-1})^{132} D^{23},$$

where we have used the definitions of $M$ and $\hat{R}$ and (3.20). Now, the quasitriangularity of $R$ implies

$$(R \otimes 1) (\Delta \otimes \text{id})(R) = \phi^{321} (1 \otimes R) (\text{id} \otimes \Delta)(R) \phi$$

which finally proves (B.3). \qed

Note that the relations (B.1) - (B.3) are precisely the defining relations postulated by [AGS] to describe the algebra generated by the entries of a monodromy matrix around a closed loop together with the quantum group of gauge transformations sitting at the initial ($\equiv$ end) point of the loop. Thus we define similarly as in [N1]

**Definition B.2.** The gauged monodromy algebra $M_R(G) \supset G$ is the algebra extension generated by $G$ and elements $M(\varphi), \varphi \in \hat{G}$ with defining relations given by (B.1) - (B.3), where $M(\varphi) = (\varphi \otimes \text{id})(M)$.

Lemma B.1 then implies the immediate

**Corollary B.3.** Let $(G, R)$ be a finite dimensional quasitriangular quasi–Hopf algebra. Then the monodromy algebra $M_R(G)$ and the quantum double $D(G)$ are equivalent extensions of $G$, where the isomorphism is given on the generators by

$$(M(\varphi) \leftrightarrow (\varphi \otimes \text{id})(R^{op} D))$$

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