A DETERMINANTAL EXPRESSION AND A RECURSIVE RELATION OF THE DELANNOY NUMBERS

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Dedicated to people facing and fighting COVID-19

Abstract. In the paper, by a general and fundamental, but non-extensivel y circulated, formula for derivatives of a ratio of two differentiable functions and by a recursive relation of the Hessenberg determinant, the author finds a new determinantal expression and a new recursive relation of the Delannoy numbers. Consequently, the author derives a recursive relation for computing central Delannoy numbers in terms of related Delannoy numbers.

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1. Motivations

The Delannoy numbers, denoted by $D(p, q)$ for $p, q \geq 0$, form an array of positive integers which are related to lattice paths enumeration and other problems in combinatorics. For more information on their history and status in combinatorics, please refer to [2] and closely related references therein.

In [2, Section 2] and [64], the explicit formulas

$$D(p, q) = \sum_{i=0}^{p} \binom{p}{i} \binom{q}{i} 2^i$$

and

$$D(p, q) = \sum_{i=0}^{q} \binom{q}{i} \binom{p + q - i}{q}$$

were given. The first few values of the Delannoy numbers $D(p, q)$ for $0 \leq p, q \leq 8$ are listed in Table 1. It is well known [64] that the Delannoy numbers $D(p, q)$ satisfy a simple recurrence

$$D(p, q) = D(p - 1, q) + D(p - 1, q - 1) + D(p, q - 1) \quad (1.1)$$
Table 1. Values of the Delannoy numbers $D(p, q)$ for $0 \leq p, q \leq 8$

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
| 1 | 5 | 13 | 25 | 41 | 61 | 85 | 113 |
| 1 | 7 | 25 | 63 | 129 | 231 | 377 | 575 |
| 1 | 9 | 41 | 129 | 321 | 681 | 1289 | 2241 |
| 1 | 11 | 61 | 231 | 681 | 1683 | 3653 | 7183 |
| 1 | 13 | 85 | 377 | 1289 | 3653 | 8989 | 19825 |
| 1 | 15 | 113 | 575 | 2241 | 7183 | 19825 | 48639 |

and can be generated by

$$\frac{1}{1 - x - y - xy} = \sum_{p,q=0}^{\infty} D(p, q) x^p y^q.$$  

When taking $n = p = q$, the numbers $D(n) = D(n, n)$ are known [64] as central Delannoy numbers which have the generating function

$$\frac{1}{\sqrt{1 - 6x + x^2}} = \sum_{n=0}^{\infty} D(n) x^n = 1 + 3x + 13x^2 + 63x^3 + \cdots.$$  

(1.2)

The first nine central Delannoy numbers $D(n)$ for $0 \leq n \leq 8$ are listed as blacked numbers in Table 1.

We found on the MathSciNet on 27 March 2020 that the phrase “Delannoy number” appeared in the titles of the references [1, 6, 7, 9, 10, 12, 13, 14, 15, 23, 24, 25, 26, 27, 55, 57, 58, 59, 60, 61, 63, 64]. The Delannoy numbers $D(p, q)$ were also investigated in [11, 19, 20, 21]. The Delannoy numbers $D(p, q)$ have some connections with the Schröder numbers. For some recent results on the Schröder numbers, please refer to [36, 39, 46, 47, 48, 56] and closely related references.

In [50, Theorems 1.1 and 1.3], considering the generating function (1.2), among other things, the authors expressed central Delannoy numbers $D(n)$ by an integral

$$D(n) = \frac{1}{\pi} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{\sqrt{(t - 3 + 2\sqrt{2})(3 + 2\sqrt{2} - t)}} \frac{1}{t^{n+1}} dt.$$  

(1.3)

and by a determinant

$$D(n) = (-1)^n \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_1 & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}.$$  

(1.4)

for $n \in \mathbb{N}$, where

$$a_n = \frac{(-1)^{n+1}}{6^n} \sum_{\ell=1}^{n} (-1)^{\ell} 6^{2\ell} \frac{(2\ell - 3)!!}{(2\ell)!!} \binom{\ell}{n-\ell}.$$
Making use of the integral expression 1.3, the authors derived in [50] some new analytic properties, including some product inequalities and determinant inequalities, of central Delannoy numbers $D(n)$.

In combinatorial number theory, it is a significant and meaningful work to express concrete sequences or arrays of integer numbers or polynomials in terms of tridiagonal determinants or the Hessenberg determinants. In this respect, the Bernoulli numbers and polynomials [3, 8, 30, 33, 40, 62], the Euler numbers and polynomials [35, 37, 65], (central) Delannoy numbers and polynomials [32, 36, 39, 47, 48, 49, 50], the Horadam polynomials [43], (generalized) Fibonacci numbers and polynomials [17, 18, 22, 31, 38, 49, 52], the Lucas polynomials [43], and the like, have been represented via tridiagonal determinants or the Hessenberg determinants, and consequently many remarkable relations have been obtained. For more information in this area and direction, please refer to [16, 28, 29, 41, 42, 44, 45, 51, 53, 54] and closely related references therein.

In this paper, by a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions in [4, p. 40] and by a recursive relation of the Hessenberg determinant in [5, p. 222, Theorem], we find a new determinantal expression and a new recursive relation of the Delannoy numbers $D(p, q)$. Consequently, we derive a recursive relation for computing central Delannoy numbers $D(n)$ in terms of related Delannoy numbers $D(p, q)$.

2. A determinantal expression of the Delannoy numbers

In this section, by virtue of a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions in [4, p. 40], we find a new determinantal expression of the Delannoy numbers $D(p, q)$.

Theorem 2.1. For $p, q \geq 0$, the Delannoy numbers $D(p, q)$ can be determinantly expressed by

$$D(p, q) = \frac{(-1)^q}{q!} |L_{(q+1)\times 1}(p) M_{(q+1)\times q}(p)|_{(q+1)\times (q+1)},$$

(2.1)

where

$$L_{(q+1)\times 1}(p) = (\langle p \rangle_0, \langle p \rangle_1, \ldots, \langle p \rangle_q)^T,$$

$$M_{(q+1)\times q}(p) = \begin{pmatrix} (-1)^{i-j} \frac{(-p+1)^{j-i}}{(i-j)} \langle p+1 \rangle_{i-j} \\ 1 \leq i \leq q+1 \\ 1 \leq j \leq q \end{pmatrix},$$

$$\langle z \rangle_n = \begin{cases} z(z-1) \cdots (z-n+1), & n \geq 1; \\ 1, & n = 0 \end{cases}$$

is known as the $n$-th falling factorial of the number $z \in \mathbb{C}$, and $T$ denotes the transpose of a matrix. Consequently, central Delannoy numbers $D(n)$ for $n \geq 0$ can be determinantly expressed as

$$D(n) = \frac{(-1)^n}{n!} |L_{(n+1)\times 1}(n) M_{(n+1)\times n}(n)|_{(n+1)\times (n+1)}.$$  

(2.2)

Proof. We recall a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions. Let $u(t)$ and $v(t) \neq 0$ be two $n$-th differentiable functions for $n \in \mathbb{N}$. Exercise 5) in [4, p. 40] reads that the
\( n \)-th derivative of the ratio \( \frac{u(t)}{v(t)} \) can be computed by
\[
\frac{d^n}{dt^n} \left[ \frac{u(t)}{v(t)} \right] = (-1)^n \frac{|W_{(n+1)\times(n+1)}(t)|}{v^{n+1}(t)},
\] (2.3)

where \( U_{(n+1)\times1}(t) \) is an \((n+1)\times1\) matrix whose elements satisfy \( u_{k,1}(t) = u^{(k-1)}(t) \) for \( 1 \leq k \leq n+1 \), \( V_{(n+1)\times n}(t) \) is an \((n+1)\times n\) matrix whose elements meet \( v_{i,j}(t) = \left( \frac{1}{n-1} \right) v^{(i-j)}(t) \) for \( 1 \leq i \leq n+1 \) and \( 1 \leq j \leq n \), and \(|W_{(n+1)\times(n+1)}(t)|\) is the determinant of the \((n+1)\times(n+1)\) matrix
\[
W_{(n+1)\times(n+1)}(t) = (U_{(n+1)\times1}(t) \ V_{(n+1)\times n}(t))_{(n+1)\times(n+1)}.
\]

It is easy to see that
\[
\frac{\partial^p}{\partial x^p} \left( \frac{1}{1-x-y-xy} \right) = \frac{p!(1+y)^p}{[1-x-(1+x)y]^{p+1}}.
\]

Making use of the formula (2.3) gives
\[
\frac{\partial^{q+p}}{\partial y^q \partial x^p} \left( \frac{1}{1-x-y-xy} \right) = \frac{p!}{[1-x-(1+x)y]^{p+1}} \frac{(1+y)^p}{[1-x-(1+x)y]^{(p+1)(q+1)}}
\]
\[
\times \begin{vmatrix}
(1+y)^p & [1-x-(1+x)y]^{p+1} \\
\langle p \rangle 1(1+y)^{p-1} & (-1)^{1} \langle p+1 \rangle 1(1+x)^{1} [1-x-(1+x)y]^{p} \\
\langle p \rangle 2(1+y)^{p-2} & (-1)^{2} \langle p+1 \rangle 2(1+x)^{2} [1-x-(1+x)y]^{p-1} \\
\langle p \rangle 3(1+y)^{p-3} & (-1)^{3} \langle p+1 \rangle 3(1+x)^{3} [1-x-(1+x)y]^{p-2} \\
\vdots & \vdots \\
\langle p \rangle q-2(1+y)^{p-q+2} & (-1)^{q-2} \langle p+1 \rangle q-2(1+x)^{q-2} [1-x-(1+x)y]^{p-q+3} \\
\langle p \rangle q-1(1+y)^{p-q+1} & (-1)^{q-1} \langle p+1 \rangle q-1(1+x)^{q-1} [1-x-(1+x)y]^{p-q+2} \\
\langle p \rangle q(1+y)^{p-q} & (-1)^{q} \langle p+1 \rangle q(1+x)^{q} [1-x-(1+x)y]^{p-q+1} \\
0 & 0 \\
\end{vmatrix}
\]
\[
\begin{pmatrix}
\binom{p}{q} \langle 1 \rangle [1-x-(1+x)y]^{p+1} \\
\binom{p+1}{q} \langle 1 \rangle [1-x-(1+x)y]^{p} \\
\binom{p+1}{q-1} \langle 1 \rangle [1-x-(1+x)y]^{p-1} \\
\vdots & \vdots \\
\binom{p+1}{q-p} \langle 1 \rangle [1-x-(1+x)y]^{p-q+4} \\
\binom{p+1}{q-p+3} \langle 1 \rangle [1-x-(1+x)y]^{p-q+3} \\
\binom{p+1}{q-p+2} \langle 1 \rangle [1-x-(1+x)y]^{p-q+2} \\
0 & 0 \\
\binom{p+1}{q-p} [1-x-(1+x)y]^{p+1} \\
\binom{p+1}{q-p+1} [1-x-(1+x)y]^{p} \\
\vdots & \vdots \\
\binom{p+1}{q-p+5} [1-x-(1+x)y]^{p-q+5} \\
\binom{p+1}{q-p+4} [1-x-(1+x)y]^{p-q+4} \\
\binom{p+1}{q-p+3} [1-x-(1+x)y]^{p-q+3} \\
\binom{p+1}{q-p+2} [1-x-(1+x)y]^{p-q+2} \\
\end{pmatrix}
\]

The determinantal expression (2.1) is thus proved.

From (2.1), we readily see that, when \( p = q, \) central Delannoy numbers \( D(p, q) \) for \( n \geq 0 \) can be expressed as (2.2). The proof of Theorem 2.1 is complete. □
Remark 2.1. Since the symmetric property $D(p, q) = D(q, p)$, from the determinantal expression (2.1) in Theorem 2.1, it follows that

$$
\left| L_{(q+1) \times 1}(p) \ M_{(q+1) \times q}(p) \right|_{(q+1) \times (q+1)} = (-1)^{p-q} \frac{q!}{p!} \left| L_{(p+1) \times 1}(q) \ M_{(p+1) \times p}(q) \right|_{(p+1) \times (p+1)}.
$$

For example, when $p = 8$ and $q = 3$, we have

$$
\left| L_{4 \times 1}(8) \ M_{4 \times 3}(8) \right|_{4 \times 4} = -\frac{31}{8!} \left| L_{9 \times 1}(3) \ M_{9 \times 8}(3) \right|_{9 \times 9}.
$$

This means that, when computing $D(p, q)$, if $p \geq q$, it would be sufficient to compute the determinant

$$
\left| L_{(q+1) \times 1}(p) \ M_{(q+1) \times q}(p) \right|_{(q+1) \times (q+1)}.
$$

Remark 2.2. The determinantal expression (2.2) is different from and simpler than (1.4) established in [50, Theorem 1.1].

3. A recursive relation of the Delannoy numbers

In this section, by virtue of a recursive relation of the Hessenberg determinant in [5, p. 222, Theorem], we will find a new recursive relation of the Delannoy numbers $D(p, q)$.

**Theorem 3.1.** For $p, q \geq 0$, the Delannoy numbers $D(p, q)$ satisfy the recursive relation

$$
D(p, q) = \left( \frac{p}{q} \right) + (-1)^{q-1} \sum_{r=0}^{q-1} (-1)^r \left( \frac{p+1}{q-r} \right) D(p, r). \tag{3.1}
$$

Consequently, central Delannoy numbers $D(n)$ for $n \geq 0$ satisfy

$$
D(n) = 1 + (-1)^{n+1} \sum_{r=0}^{n-1} (-1)^r \left( \frac{n+1}{r+1} \right) D(n, r). \tag{3.2}
$$

**Proof.** Let $Q_0 = 1$ and

$$
Q_n = \begin{bmatrix}
\varepsilon_{1,1} & \varepsilon_{1,2} & 0 & \ldots & 0 & 0 \\
\varepsilon_{2,1} & \varepsilon_{2,2} & \varepsilon_{2,3} & \ldots & 0 & 0 \\
\varepsilon_{3,1} & \varepsilon_{3,2} & \varepsilon_{3,3} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\varepsilon_{n-2,1} & \varepsilon_{n-2,2} & \varepsilon_{n-2,3} & \ldots & \varepsilon_{n-2,n-1} & 0 \\
\varepsilon_{n-1,1} & \varepsilon_{n-1,2} & \varepsilon_{n-1,3} & \ldots & \varepsilon_{n-1,n-1} & \varepsilon_{n-1,n} \\
\varepsilon_{n,1} & \varepsilon_{n,2} & \varepsilon_{n,3} & \ldots & \varepsilon_{n,n-1} & \varepsilon_{n,n}
\end{bmatrix}
$$

for $n \in \mathbb{N}$. In [5, p. 222, Theorem], it was proved that the sequence $Q_n$ for $n \geq 0$ satisfies $Q_1 = \varepsilon_{1,1}$ and

$$
Q_n = \sum_{r=1}^{n} (-1)^{n-r} e_{n,r} \left( \prod_{j=r}^{n-1} e_{j,j+1} \right) Q_{r-1} \tag{3.3}
$$

for $n \geq 2$, where the empty product is understood to be 1. Replacing the determinant $Q_r$ by $(-1)^{r-1}(r-1)!D(p, r-1)$ in (2.1) for $1 \leq r \leq n$ in the recursive
relation (3.3) and simplifying give
\[ D(p, n - 1) = \frac{(p)_{n-1}}{(n-1)!} + (-1)^n \sum_{r=2}^{n} (-1)^r \frac{(p+1)_{n-r+1}}{(n-r+1)!} D(p, r-2) \]
which is equivalent to the recursive relation (3.1).

When \( n = p = q \) in (3.1), we can see that central Delannoy numbers \( D(n) \) satisfy the recursive relation (3.2). The proof of Theorem 3.1 is complete. \( \square \)

**Remark 3.1.** The recursive relation (3.1) is different from (1.1).

**Remark 3.2.** The recursive relation (3.2) demonstrates that we can compute central Delannoy numbers \( D(n) \) in terms of the Delannoy numbers \( D(p, q) \). However, the Delannoy numbers \( D(p, q) \) cannot be expressed in terms of central Delannoy numbers \( D(n) \).

**Remark 3.3.** The recursive relation (3.2) can be rearranged as
\[ D(n) = 1 - \sum_{r=0}^{n-1} (-1)^{n-r} \binom{n+1}{r+1} D(n, r) = 1 - \sum_{r=1}^{n} (-1)^r \binom{n+1}{r} D(n, n-r). \]

**Remark 3.4.** The recursive relation (3.2) can also be written as
\[ 1 - D(1) = \frac{-1}{1!} D(1, 0), \]
\[ 1 - D(2) = \frac{-1}{2!} (3) D(2, 1) + \frac{1}{2!} (3) D(2, 0), \]
\[ 1 - D(3) = \frac{-1}{3!} (4) D(3, 2) + \frac{1}{2!} (4) D(3, 1) + \frac{-1}{3!} (4) D(3, 0), \]
\[ 1 - D(4) = \frac{-1}{4!} (5) D(4, 3) + \frac{1}{2!} (5) D(4, 2) + \frac{-1}{3!} (5) D(4, 1) + \frac{1}{4!} (5) D(4, 0), \]
and so on. Consequently, making use of the symmetry relation \( D(p, q) = D(q, p) \), we can reformulate the recursive relation (3.2) as
\[
\begin{pmatrix}
D(1) \\
D(2) \\
D(3) \\
D(4) \\
\vdots \\
D(n-1) \\
D(n)
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
\vdots \\
1 \\
1
\end{pmatrix}
- M_{n \times n}
\begin{pmatrix}
\frac{-1}{1!} \\
\frac{-2}{2!} \\
\frac{-3}{3!} \\
\frac{-4}{4!} \\
\vdots \\
\frac{(-1)^{n-1}}{(n-1)!}
\end{pmatrix},
\]
where
\[
M_{n \times n} = \begin{pmatrix}
\langle 2 \rangle D(0, 1) & 0 & 0 \\
\langle 3 \rangle D(1, 2) & \langle 3 \rangle D(0, 2) & 0 \\
\langle 4 \rangle D(2, 3) & \langle 4 \rangle D(1, 3) & \langle 4 \rangle D(0, 3) \\
\langle 5 \rangle D(3, 4) & D(5) D(2, 4) & D(5) D(1, 4) \\
\vdots & \vdots & \vdots \\
\langle n \rangle D(n-2, n-1) & \langle n \rangle D(n-3, n-1) & \langle n \rangle D(n-4, n-1) \\
\langle n+1 \rangle D(n-1, n) & \langle n+1 \rangle D(n-2, n) & \langle n+1 \rangle D(n-3, n)
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
\langle 5 \rangle_4 D(0, 4) & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\langle n \rangle_4 D(n-5, n-1) & \cdots & \langle n \rangle_{n-1} D(0, n-1) & 0 \\
\langle n + 1 \rangle_4 D(n-4, n) & \cdots & \langle n + 1 \rangle_{n-1} D(1, n) & \langle n + 1 \rangle_n D(0, n)
\end{pmatrix}
\]

**Remark 3.5.** The recursive relation (3.2) should be new. Therefore, the recursive relation (3.1) should be new. Furthermore, the determinantal expression (2.1) should also be new. Interesting!

**Remark 3.6.** This paper is a revised version of the electronic preprint [32].

**Acknowledgements.** The author thanks Christer Oscar Kiselman (Department of Information Technology, Uppsala University, Sweden) for his careful corrections, helpful suggestions, and valuable comments in April 2020 on the original version of this paper.

The author thanks anonymous referees for their careful corrections and valuable comments on the original version of this paper.

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