Post-Newtonian constraints on $f(R)$ cosmologies in metric formalism

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We compute the complete post-Newtonian limit of the metric form of $f(R)$ gravities using a scalar-tensor representation. By comparing the predictions of these theories with laboratory and solar system experiments, we find a set of inequalities that any lagrangian $f(R)$ must satisfy. The constraints imposed by those inequalities allow us to find explicit bounds to the possible nonlinear terms of the lagrangian. We conclude that the lagrangian $f(R)$ must be almost linear in $R$ and that corrections that grow at low curvatures are incompatible with observations. This result shows that modifications of gravity at very low cosmic densities cannot be responsible for the observed cosmic speed-up.

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I. INTRODUCTION

It is now generally accepted that the universe is undergoing a period of accelerated expansion [1, 2], which cannot be justified by the description provided by the equations of motion of General Relativity (GR) and a universe filled with standard sources of matter and energy. It has been suggested that this effect could have its origin in, among other possibilities, corrections to the equations of motion of GR generated by nonlinear contributions of the scalar curvature in the gravity lagrangian $f(R)$ [3, 4, 5] (see also [6]). Reasons for considering nonlinear curvature terms in the gravity lagrangian can be found in quantum effects in curved space [7] or in certain low-energy limits of string/M-theories [8]. The nonlinearity of the lagrangian can also be related to the existence of scalar degrees of freedom in the gravitational interaction [8]. In any case, the fact that certain $f(R)$ lagrangians naturally lead to early-time inflationary behaviors is the main motivation to study possible new gravitational effects in the late-time cosmic expansion.

Once a nonlinear lagrangian $f(R)$ has been proposed, the equations of motion for the metric can be derived in two inequivalent ways. On the one hand, one can follow the standard metric formalism, in which variation of the action with respect to the metric leads to a system of fourth-order equations. On the other hand, one may assume that metric and connection are independent fields and then take variations of the action with respect to the metric and with respect to the connection. In this case, the resulting equations of motion for the metric are second-order. Only when the function $f(R)$ is linear in $R$, GR and GR plus cosmological constant, metric and Palatini formalisms lead to the same equations of motion. In this work we will be exclusively concerned with the metric formalism. The Palatini formalism will be considered elsewhere [10].

Though much work has been carried out in the last few years with regard to $f(R)$ gravities in the cosmological regime, very little is known about the form that the gravity lagrangian should have in order to be compatible with the most recent cosmological observations [11]. The main reason for this seems to be the fact that the precision of the supernovae luminosity distance data and other currently available tests supporting the late-time cosmic speed-up is not enough to discriminate with confidence between one model or another. It would be thus desirable to have a new arena where to test these theories with higher precision. In our opinion, the solar system represents a scenario more suitable than the cosmological one to study the possible constraints on the lagrangian $f(R)$. In fact, if in addition to modified gravitational dynamics, sources of dark energy were acting in the cosmic expansion, it would be very difficult to distinguish their effect from a purely gravitational one. In the solar system, however, it is ordinary matter which dominates the gravitational dynamics, being the contribution of dark sources negligible. Therefore, we should see the solar system as a suitable laboratory to impose the first useful constraints on $f(R)$ cosmologies.

In order to confront the predictions of a given gravity theory with experiment in the solar system, it is necessary to compute its weak-field, slow-motion (or post-Newtonian) limit. This limit has been computed for many metric theories of gravity and put in a standardized form [12], which depends on a set of parameters that change from theory to theory (Parametrized Post-Newtonian [PPN] formalism). However, for $f(R)$ gravities this limit has not yet been computed in detail. The Newtonian limit of these theories was recently studied in [13]. On the other hand, since the metric form of $f(R)$ gravities can be represented as the case $\omega = 0$.
of Brans-Dicke-like scalar-tensor theories, it is tempting to use the post-Newtonian limit of those theories given in the literature 12 14 (see also 13) to check the viability of particular models. This was proposed in 16, where it was concluded that the Carroll et al. model 4 (in metric formalism) was ruled out according to the observational constraints on the parameter $\gamma$ corresponding to that model. That result was based on the fact that the scalar field had a small effective mass, which was computed in terms of the second derivative of the potential. However, that prescription is usually derived under the assumption that the potential and its first derivative vanish (see for instance 12, 17), conditions that, in general, cannot be imposed on $f(R)$ theories (see section III).

In this work we will use the scalar-tensor representation introduced in 16 to compute the post-Newtonian limit of $f(R)$ gravities taking into account all the terms associated to the potential of the scalar field. In other words, we will not make any assumption or simplification about the function $f(R)$ that defines the lagrangian. We will actually compute the post-Newtonian limit corresponding to Brans-Dicke-like scalar-tensor theories with arbitrary potential and a generic constant value of $\omega$ and will then particularize to the case $\omega = 0$, which corresponds to the metric form of $f(R)$ gravities. In this manner we generalize the results of the literature so as to include all the terms that are relevant for our discussion. The Palatini form of $f(R)$ gravities, which can be represented as the case $\omega = -3/2$ of Brans-Dicke-like theories, represents an exception of the general case $\omega = constant$ and will be studied elsewhere 10.

The resulting post-Newtonian metric will allow us to confront the predictions of these theories with the observational data. In this way, we will find a series of constraints for the lagrangian, which is a priori completely unknown. Those constraints turn out to be so strong that the lagrangians compatible with observations are bounded by a well defined function that prohibits the growing of the nonlinear terms at low curvatures. This result will be enough to invalidate the arguments supporting the cosmic speed-up as due to new gravitational effects at low curvatures.

The paper is organized as follows. We first derive the equations of motion in the original $f(R)$ form and show how to obtain the scalar-tensor representation out of them. Then we comment on the choice of coordinates and boundary conditions. The post-Newtonian limit is computed in the Appendix and commented in section IV where we obtain constraints on the lagrangian from the experimental data. Section V is devoted to the discussion of particular models. In section VI we find the form of the lagrangian that satisfies the constraints. We conclude the paper with a brief summary and conclusions.

II. EQUATIONS OF MOTION

The action that defines $f(R)$ gravities in the metric formalism is the following

$$S[f; g, \psi_m] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m[g_{\mu\nu}, \psi_m]$$

where $S_m[g, \psi_m]$ represents the matter action, which depends on the metric $g_{\mu\nu}$ and the matter fields $\psi_m$. For notational purposes, we remark that $T_{\mu\nu} = \nabla_\mu \psi_m$, and that the scalar curvature $R$ is defined as the contraction $R = g^{\mu\nu} R_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci tensor

$$R_{\mu\nu} = -\partial_\mu \Gamma^\lambda_{\nu\lambda} + \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\rho}$$

and $\Gamma^\alpha_{\beta\gamma}$ is the Levi-Civita connection

$$\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\lambda}}{2} (\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\beta\lambda} - \partial_\lambda g_{\beta\gamma})$$

From eq. (1) we obtain the following equations of motion:

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) = \kappa^2 T_{\mu\nu}$$

where $f'(R) \equiv df/dR$. According to eq. (4), we see that, in general, the metric satisfies a system of fourth-order partial differential equations. The higher order derivatives come from the terms $\nabla_\mu \nabla_\nu f'$ and $\Box f'$. Only when $f(R)$ is a linear function of the scalar curvature, $f(R) = a + b R$, the equations of motion are second-order. The trace of eq. (4) is given by

$$3 \Box f' + f'R - 2f = \kappa^2 T$$

This equation can be interpreted as the equation of motion of a self-interacting scalar field, where the self-interaction terms are represented by $f'R - 2f$. This can be seen by algebraically inverting the function $f'(R)$ and writing $R$ as $R = R(f')$. In this way, defining

$$\phi \equiv f'$$

$$V(\phi) \equiv R(\phi) f' - f(\phi)$$

we can write eqs. (4) and (5) as

$$G_{\mu\nu} \equiv \frac{\kappa^2}{\phi} T_{\mu\nu} - \frac{V(\phi)}{2\phi} g_{\mu\nu} + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi)$$

$$3 \Box \phi + 2V(\phi) - \phi \frac{dV}{d\phi} \equiv \kappa^2 T$$

The above equations of motion can also be obtained from the following action

$$S[g_{\mu\nu}, \phi, \psi_m] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [\phi R(g) - \frac{w(\phi)}{\phi} (\partial_\mu \phi \partial^\mu \phi) - V(\phi)] + S_m[g_{\mu\nu}, \psi_m]$$
which represents a Brans-Dicke-like scalar-tensor theory, in the particular case $\omega = 0$. For more details on the scalar-tensor representation and a different derivation of this result see [18].

III. COORDINATES AND BOUNDARY CONDITIONS

In order to obtain the metric in the solar system we will follow the basic guidelines outlined in chapter 4 of Will’s book [12]. First we solve eqs. (8) and (9) for the metric and the scalar field in the cosmic regime, where the high degree of homogeneity and isotropy leads to a Friedman-Robertson-Walker metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 dx_i dx^i$$

(11)

and to a space-independent value of the scalar field, $\phi = \hat{\phi}(t)$. At smaller scales there appear local deviations from the cosmic values of the fields. In the solar system, for instance, the local deviations will be small, thereby allowing us to treat them as a perturbation with respect to the background cosmic boundary values $g_{\mu\nu}$ and $\phi^R(t)$. In our computations we will use coordinates $(\bar{t}, \bar{x}^i)$ in which the outer regions of the local system are in free fall with respect to the surrounding cosmological model. Neglecting the second-order corrections, the local and background coordinates are simply related by $\bar{t}(t_0, x_0; t, x) \approx (t - t_0)$ and $\bar{x}^i(t_0, x_0; t, x) \approx a_0(x - x_0)^i$. From now on we will omit the bar on the local coordinates and will denote $\phi_0$, $\dot{\phi}_0$ the asymptotic boundary values of the scalar field at the cosmic time $t_0$, i.e., $\phi_0 = \hat{\phi}(t_0)$ and $\dot{\phi}_0 = \dot{\hat{\phi}}(t_0)$.

For approximately static solutions, corresponding to gravitating masses such as the Sun or Earth, to lowest-order, we can drop the terms involving time derivatives from the equations of motion. In our local coordinate system, the metric can be expanded about its Minkowskian value as $g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}$. The solution for the scalar field can be expressed in the form $\phi = \phi_0 + \varphi(t, x)$, where $\phi_0 \equiv \phi(t_0)$ is the asymptotic cosmic value, which is a slowly-varying function of the cosmic time $t_0$, and $\varphi(t, x)$ represents the local deviation from $\phi_0$, which vanishes far from the local system. We want to remark that since $\phi_0$ and $\dot{\phi}_0$ depend on $R_0$ and $\dot{R}_0$, the metric of the local post-Newtonian system will also depend on the background cosmic values $R_0$ and $\dot{R}_0$. The dependence on these background quantities will make the metric change adiabatically in a cosmic timescale. This adiabatic evolution could make a theory be compatible with the current experimental tests during some cosmic era but fail in other periods. We will give below some examples to illustrate this effect. In particular, we will comment on the Carroll et al. 1/R model [4].

With regard to the potential defined for the scalar field, see eq. (4), it is easy to see that $dV/d\phi = R$ [18]. Since, the curvature can be expressed as $R = R_0 + \dot{R}(t, x)$, where $\dot{R}(t, x)$ denotes the local deviation from the background cosmic value $R_0$, it is easy to see that the scalar field will not, in general, satisfy the extremum condition $dV/d\phi = 0$. This is to be contrasted with the results of the literature regarding the post-Newtonian limit of Brans-Dicke-like theories, where it is generally assumed that the field is near an extremum [14]. We thus see that for $f(R)$ gravities (or $\omega = 0$ Brans-Dicke-like theories) it is necessary to consider all the terms associated to the potential. This slight complication, on the other hand, has its own advantages, since at the end of the calculations we may ask Nature about the admissible forms that the lagrangian $f(R)$ may have.

IV. POST-NEWTONIAN METRIC

As we advanced above, we will expand the equations of motion around the background values of the metric and the scalar field. In particular, we will take $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, $\phi_{\mu\nu} \approx \eta_{\mu\nu} - h_{\mu\nu}$, $\phi = \phi_0 + \varphi(t, x)$ and $V(\phi) \approx V_0 + \varphi V'_0 + \varphi^2 V''_0/2 + \ldots$. The complete post-Newtonian limit needs the different components of the metric and the scalar field evaluated to the following orders $\phi_0 \sim O(2) + O(4)$, $\varphi_{ij} \sim O(3)$, $\varphi_t \sim O(2)$ and $\phi \sim O(2) + O(4)$ (see [12]). The details of the calculations and the complete post-Newtonian limit for the theories defined in eq. (10) can be found in the Appendix. For convenience, we will discuss here only the lowest order corrections, $g_{00} \sim O(2)$, $g_{ij} \sim O(2)$ and $\phi \sim O(2)$, of the case we are interested in, say, $\omega = 0$. The order of approximation will be denoted by a superindex. This approximation will be enough to place tight constraints on the gravity lagrangian. To this order, the metric satisfies the following equations

$$-\frac{1}{2} \nabla^2 \left[ h_{00}^{(2)} - \frac{\varphi^{(2)}}{\phi_0} \right] = \frac{\kappa^2 \rho}{2\phi_0} + \left( \frac{3}{2} \frac{\phi_0}{2\phi_0} - \frac{V_0}{2\phi_0} \right)$$

(12)

$$-\frac{1}{2} \nabla^2 \left[ \delta_{ij}^{(2)} + \frac{\varphi^{(2)}}{\phi_0} \right] = \delta_{ij} \left[ \frac{\kappa^2 \rho}{2\phi_0} - \frac{\dot{\phi}_0}{2\phi_0} + \frac{V_0}{2\phi_0} \right]$$

(13)

where the gauge condition $h_{k;\mu} - \frac{1}{2} h^{\mu}_{\mu,k} = \partial_k \varphi^{(2)}/\phi_0$ has been used. In eliminating the zeroth-order terms in the field equation for $\varphi$, corresponding to the cosmological solution for $\phi_0$, the equation for the scalar field to this order boils down to

$$\nabla^2 m^2 + m^2 \varphi^{(2)}(t, x) = -\frac{\kappa^2 \rho}{3}$$

(14)

where $m^2 x$ is a slowly-varying function of the cosmological times which given by

$$m^2 \equiv \frac{\phi_0 V''_0 - V'_0}{3}$$

(15)
Note that, despite our notation, there is no a priori restriction on the sign of $m^2\gamma$. The equations of above can be easily integrated to give

$$\varphi^{(2)}(t, x) = \frac{\kappa^2}{3} \frac{1}{4\pi} \int d^3 x' \frac{\rho(t, x')}{|x - x'|} F(|x - x'|)$$

(16)

$$h^{(2)}_{00}(t, x) = \frac{\kappa^2}{\phi_0} \frac{1}{4\pi} \int d^3 x' \frac{\rho(t, x')}{|x - x'|} \left[ 1 + \frac{F(|x - x'|)}{3} \right] -$$

(17)

$$h^{(2)}_{ij}(t, x) = \left( \frac{\kappa^2}{\phi_0} \frac{1}{4\pi} \int d^3 x' \frac{\rho(t, x')}{|x - x'|} \right) \left[ 1 - \frac{F(|x - x'|)}{3} \right] +$$

(18)



where $x_c$ is an arbitrary constant vector\(^1\) and the function $F(|x - x'|)$ is given by

$$F(|x - x'|) = \begin{cases} 
  e^{-m_\gamma|x-x'|} & \text{if } m^2\gamma > 0 \\
  \cos(m_\gamma|x-x'|) & \text{if } m^2\gamma < 0 
\end{cases}$$

(19)

Note that the term $\dot{\phi}_0$ does not appear in eqs. (16), (17) and (18) and, therefore, the fact that $R_0$ may not be strictly zero does not affect the Newtonian limit \(^{13}\). In the post-Newtonian limit it contributes to $h^{(4)}_{00}$ (see the Appendix). In any case, since to all effects $\phi_0$ is almost constant, we can neglect the contributions due to $\phi_0$ and $\dot{\phi}_0$.

Since in the solar system the sun represents the main contribution to the metric, we can approximate the expressions of above far from the sources by

$$h^{(2)}_{00} \approx 2G \frac{M_\odot}{r} + \frac{V_0}{6\phi_0} r^2$$

(20)

$$h^{(2)}_{ij} \approx \delta_{ij} \left[ 2\gamma G \frac{M_\odot}{r} - \frac{V_0}{6\phi_0} r^2 \right]$$

(21)

where $M_\odot = \int d^3 x' \rho_{\text{sun}}(t, x')$ is the Newtonian mass of the sun and the $\phi_0$ contribution has been neglected for simplicity. We have defined the effective Newton’s constant $G$ as

$$G = \frac{\kappa^2}{8\pi\phi_0} \left[ 1 + \frac{F(r)}{3} \right]$$

(22)

and the effective PPN parameter $\gamma$ as

$$\gamma = \frac{3 - F(r)}{3 + F(r)}$$

(23)

We shall show now that the oscillatory solutions, $m^2\gamma < 0 \rightarrow F(r) = \cos(m_\gamma r)$, are always unphysical. For this case, the inverse-square law gets modified as follows

$$\frac{M_\odot}{r^2} \rightarrow \left( 1 + \frac{\cos(m_\gamma r) + (m_\gamma r) \sin(m_\gamma r)}{2} \right) \frac{M_\odot}{r^2}$$

(24)

For very light fields, which represent long-range interactions, the argument of the sinus and the cosine is very small in solar system scales ($m_\gamma r \ll 1$). We can thus approximate $\cos(m_\gamma r) \approx 1$ and $\sin(m_\gamma r) \approx 0$ and recover the usual Newtonian limit up to an irrelevant redefinition of Newton’s constant. However, these approximations also lead to $\gamma \approx 1/2$, which is observationally unacceptable since $\gamma_{\text{obs}} \approx 1$ \(^{12}\). If the scalar interaction were short- or mid-range, the Newtonian limit would get dramatically modified. In fact, the leading order term is then oscillating, $\sin(m_\gamma r)M_\odot/r$, and is clearly incompatible with observations. We are thus led to consider only the damped solutions $F(r) = e^{-m_\gamma r}$.

The Yukawa-type correction in the Newtonian potential has not been observed over distances that range from meters to planetary scales. In addition, since the post-Newtonian parameter $\gamma$ is observationally very close to unity, we see that the effective mass in eqs. (23) and (22) must satisfy the constraint $m^2_\gamma L^2 \gg 1$, where $L$ represents a typical experimental length scale. Note that when $\omega$ is not fixed (see the Appendix), there is also the possibility of having a very light (long-range) field that yields almost space-independent values of $G$ and $\gamma$. In that case, the theory behaves as a Brans-Dicke theory with $\gamma$ given by

$$\gamma = \frac{1 + \omega}{2 + \omega}$$

(25)

and it takes $\omega > 40000$ to satisfy the observational constraints \(^{20}\).

The cosmological constant term $(V_0/6\phi_0)^2$ appearing in eqs. (20) and (21) also imposes constraints on the particular model, since this contribution must be very small in order not to modify the gravitational dynamics of local systems ranging from the solar system to clusters of galaxies. In the terminology of $f(R)$ gravities, the constraint from the cosmological constant term is

$$\left| \frac{f_0 - f_0'}{f_0'} \right| L_L^2 \ll 1$$

(26)

where $L_L$ may represent a (relatively large) length scale the same order or greater than the solar system. The constraint $m^2_\gamma L_S^2 \gg 1$ associated to the effective mass can be reexpressed as

$$\left| \frac{f_0 - f_0'}{f_0'} \right| L_S^2 \gg 1$$

(27)

where $L_S$ represents a (relatively short) length scale that can range from meters to planetary scales, depending on

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\(^1\) This vector could be taken, for instance, as representing the center of mass of the system.
the particular test used to verify the theory. It is worth noting that a generic lagrangian of the form

\[ f(R) = R + \lambda h(R) \]  \hspace{1cm} (28)

with \( \lambda \) a suitable small parameter, satisfies the two constraints of above if \( h(R), h'(R) \) or \( h''(R) \) are finite or vanish as the universe expands. General Relativity, which can be seen as the limit \( \lambda \to 0 \), saturates those constraints. We will consider in the next section some examples of theories with the form proposed in eq. (28). In section VI we will analyze in detail the implications of the constraint of eq. (27).

Before concluding this section, we shall briefly discuss some simplifications that we may carry out from the above considerations in the complete post-Newtonian metric given in the Appendix. First of all, it is worth noting that with a tiny \( V_0 \) we can eliminate part of the cosmological constant terms. This fact together with our definition for \( \lambda \) with \( \alpha = 0 \), coincides with the one corresponding to GR. On the other hand, a massive field would allow us to neglect the exponential terms and the \( \phi^2 \) contributions. Furthermore, this family of models yields an acceptable weak-field limit. In fact, it seems reasonable to think that characterizes the post-Newtonian metric. It is given by

\[ m_p^2 = \frac{R_0}{3(n-1)} \left[ \frac{1}{n} \left( \frac{M^2}{R_0} \right)^{-n-1} - (n-2) \right] = \frac{M^2}{3n} \left( \frac{n}{\phi_0 - 1} \right)^{n-1} \left[ 1 - \frac{(n-2)}{(n-1)} \phi_0 \right] \]  \hspace{1cm} (31)

where \( \phi_0 \equiv f'/(R_0) \) and \( R_0 \) represent the cosmological values of \( \phi \) and \( R \) at the moment \( t_0 \). The time-time component of eq. (4) can be used to extract some information about the cosmological evolution of \( R \). This will help us to understand the adiabatic change in the post-Newtonian metric. The expansion factor satisfies the following equation

\[ 3 \left( \frac{\dot{a}}{a} \right)^2 = \kappa^2 \rho - \left( \frac{R}{M^2} \right)^{n-1} \left[ 3n \left( \frac{\dot{a}}{a} \right)^2 - \frac{(n-1)}{2} \frac{R}{a} \right] \]  \hspace{1cm} (32)

Inserting \( a(t) = a_0 e^{\gamma t} \) in eq. (32) and taking \( \rho = 0 \) for simplicity, it follows that at early-times the evolution is dominated by the \( (R/M^2)^n \) contribution with \( \gamma = (M^2/12)^{-1/2} \). After the early-time inflation predicted by these expansion factors, as the curvature decays below the scale defined by \( M^2 \), the \( (R/M^2)^n \) effect is suppressed and the subsequent evolution is governed by GR, with \( a(t) = a_0 t^s \) and \( s = 1/2 \) during the radiation dominated era, and \( s = 2/3 \) during the matter dominated era. Thus, at all times after the inflationary period, we have \( M^2/R \gg 1 \), or equivalently \( (\phi - 1) \to 0 \). This leads to a very large effective mass for the scalar field and a tiny cosmological constant term \( V_0/(\phi_0) \to 0 \). In consequence, this family of models yields an acceptable weak-field limit. In fact, it seems reasonable to think that these theories are compatible with GR in all astrophysical applications, since the curvature is expected to be much smaller than \( M^2 \) in all situations but at the very early universe.

V. EXAMPLES

We shall now illustrate with some simple examples how the parameters that define the post-Newtonian metric are subject to a slow adiabatic evolution due to the cosmic expansion. The aim of this section is to point out the relevance of the cosmic boundary values of the fields in the description of isolated systems. We want to make special emphasis on the fact that the gravitational dynamical properties of a local system at a given time may not be completely determined by its own internal characteristics, but can be affected by the state of the universe as a whole at that moment. Only if the \( f(R) \) lagrangian is linear in \( R \) or if the scalar field is non-dynamical (Palatini formalism), the post-Newtonian metric is completely determined by the properties of the local system.
B. Negative powers of \( R \)

A well-known example of this type is the Carroll et al. model \([4]\), defined by \( f(R) = R - \mu^4/R \), where \( \mu \) represents a tiny mass scale of order \( 10^{-33} \text{ eV} \). The reason for the minus sign in front of \( \mu^4 \) is intriguing, since this definition leads to a negative effective mass

\[
m^2_\varphi = -\frac{R}{6\mu^4}(R^2 + 3\mu^4) \tag{33}
\]

which we have shown to be in conflict with the post-Newtonian limit (see eq. (24)). An improved formulation of the theory could be obtained by changing the sign in front of \( \mu^4 \) in the definition of \( f(R) \). In this way, we can easily extend the results of the examples of above to the models \( f(R) = R + \mu^{2n+2}/R^n \). A direct consequence of the positive sign in front of \( \mu^{2n+2} \) is the loss of exponential solutions for \( a(t) \) at late-times, since the relation between \( \gamma \) and \( \mu \) turns into \( \gamma^{2(n+1)} = -(n+2)(\mu^2/12)^{n+1} \). These models are characterized by

\[
\phi \equiv f' = 1 - n \left( \frac{\mu^2}{R} \right)^{n+1}, \tag{34}
\]

\[
V(\phi) = -\mu^2(n-1) \left( \frac{\mu^2}{R} \right)^n = \mu^2(n+1) \left( \frac{n}{n-\phi} \right)^{n+1}. \tag{35}
\]

The effective mass of the scalar field takes the form

\[
m^2_\varphi = \frac{R_0}{3(n+1)} \left[ \frac{1}{n} \left( \frac{R_0}{\mu^2} \right)^{n+1} - (n+2) \right] = \mu^2 \frac{(n+1)}{(n-\phi_0)} \left( \frac{(n+2)}{(n+1)} \phi_0 - 1 \right) \tag{36}
\]

We will restrict our discussion to the cases with \( n \geq 1 \). The cosmological evolution of these models during the radiation dominated era requires a complete solution of the model, since a simple power law expansion is ill-defined. We will just concentrate on the matter dominated era, \( a(t) = a_0 t^{\gamma/3} \), and beyond, \( a(t) = a_0 t^n \) with \( s_n = (2n+1)(n+1)/(n+2) \). These solutions imply that the curvature decays with the cosmic time as \( R = 6s(2s-1)/t^2 \). One can numerically check that the transition from the matter dominated era, \( s = 2/3 \), to its final value \( s_n \) is smooth (we took \( \kappa^2 \rho_{m_0}/\mu^2 = 3/7 \)). During the matter dominated era, \( \mu^2/R \to 0 \) and \( \phi = 1 \), eqs. (36) and (35) indicate that \( m^2_\varphi \) is very large and \( V_0/\phi_0 \) very small. In consequence, these models yield a valid post-Newtonian limit. However, as the universe expands and the curvature approaches the critical value \( (R_0/\mu^2)^{n+1} = n(n+2) \), in which \( m^2_\varphi = 0 \), the effective mass is small and the post-Newtonian limit tends to that of a Brans-Dicke theory with \( \omega = 0 \). At later times, \( m^2_\varphi \) becomes negative and the weak-field approximation is ill-defined, as we discussed above. We can, thus, conclude that these theories do not seem to have a good alternative to explain the late-time cosmic speed-up, since they have an unacceptable weak-field limit at the present time.

VI. CONSTRAINED LAGRANGIAN

A qualitative analysis of the constraint given in eq. (27) can be used to argue that, in general, \( f(R) \) gravities with terms that become dominant at low cosmic curvatures are not viable theories in solar system scales and, therefore, cannot represent an acceptable mechanism for the cosmic expansion. Roughly speaking, eq. (27) says that the smaller the term \( f''_0 \) with \( f''_0 > 0 \) to guarantee \( m^2_\varphi > 0 \), the heavier the scalar field \( \phi \). In other words, the smaller \( f''_0 \), the shorter the interaction range of the field. In the limit \( f''_0 \to 0 \), corresponding to GR, the scalar interaction is completely suppressed. Thus, if the nonlinearity of the gravity lagrangian had become dominant in the last few billions of years (at low cosmic curvatures), the scalar field interaction range would have increased accordingly. In consequence, gravitating systems such as the solar system, globular clusters, galaxies… would have experienced (or will experience) observable changes in their gravitational dynamics. Since there is no experimental evidence supporting such a change and all currently available solar system gravitational experiments are compatible with GR, it seems unlikely that the nonlinear corrections may be dominant at the current epoch.

Let us now analyze in detail the constraint given in eq. (27). That equation can be rewritten as follows

\[
R_0 \left[ \frac{f'(R_0)}{R_0 f''(R_0)} - 1 \right] L_S^2 \gg 1 \tag{37}
\]

We are interested in the form of the lagrangian at intermediate and low cosmic curvatures, i.e., from the matter dominated to the vacuum dominated eras. We shall now demand that the interaction range of the scalar field remains as short as today or decreases with time so as to avoid dramatic modifications of the gravitational dynamics in post-Newtonian systems with the cosmic expansion. This can be implemented imposing

\[
\left[ \frac{f'(R)}{R f''(R)} - 1 \right] \geq \frac{1}{L^2 R} \tag{38}
\]

as \( R \to 0 \), where \( L^2 \ll L_S^2 \) represents a bound to the current interaction range of the scalar field. Thus, eq. (38) means that the interaction range of the field must decrease or remain short, \( \sim L^2 \), with the expansion of the

\[2\] Note that \( \phi \equiv f' \) must be positive in order to have a well-posed theory.
bounded by a lagrangian at intermediate and low scalar curvatures is negative, the lagrangian is also bounded from below, i.e., $f(R) \geq A \equiv −2\Lambda$ must be of order a cosmological constant $2\Lambda \sim 10^{-53}$ m$^2$. We thus conclude that the gravity lagrangian at intermediate and low scalar curvatures is bounded by

$$-2\Lambda \leq f(R) \leq R - 2\Lambda + \frac{\ell^2 R^2}{2}$$

which can be integrated twice to give the following inequality

$$f(R) \leq A + B \left( R + \frac{\ell^2 R^2}{2} \right)$$

where $B$ is a positive constant, which can be set to unity without loss of generality. Since $f'$ and $f''$ are positive, the lagrangian is also bounded from below, i.e., $f(R) \geq A$. In addition, according to the cosmological data, $A \equiv -2\Lambda$ must be of order a cosmological constant $2\Lambda \sim 10^{-53}$ m$^2$. We thus conclude that the gravity lagrangian at intermediate and low scalar curvatures is bounded by

$$\frac{d\log[f'(R)]}{dR} \leq \frac{\ell^2}{1 + \ell^2 R}$$

VII. SUMMARY AND CONCLUSIONS

In this work we have computed the post-Newtonian limit of $f(R)$ gravities in the metric approach using a scalar-tensor representation. This representation allows to encode the higher-order derivatives of the metric in a self-interacting scalar field defined by $\phi \equiv df/dR$. In this manner, the equations of motion turn into a system of second-order equations for the metric plus a second-order equation for the scalar field. The post-Newtonian metric is thus characterized by several quantities related to the scalar field, say, the intensity $V_0$ of the potential, the length scale $\sim m_\phi^{-1}$ of its interaction, and the boundary values $\phi_0$ and $\phi_0$. Since those magnitudes are given in terms of $f$ and its derivatives, we found the constraints given in eq. (26) and eq. (27) necessary to have agreement between the predictions of these theories and the observational data in the solar system. Those constraints show that the gravity lagrangian $f(R)$ at relatively low curvatures is bounded from above an from below according to eq. (11). Therefore, $f(R)$ gravities with nonlinear terms that grow with the expansion of the universe are incompatible with observations and cannot represent a valid mechanism to justify the observed cosmic speed-up.

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APPENDIX A: DETAILED CALCULATIONS

We will take $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, $g^\mu{}^\nu \approx \eta^\mu{}^\nu - h^{\mu\nu}$ and $\phi = \phi_0 + \varphi(t,x)$. For convenience, we will rewrite the equations of motion corresponding to the action of eq. (11) in the following form

$$R_{\mu\nu} = \frac{\kappa^2}{\phi} \left[ T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right] + \frac{\omega}{\phi^2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{\phi} \nabla_\mu \nabla_\nu \phi + \frac{1}{2\phi} g_{\mu\nu} \left[ \Box \phi + V(\phi) \right]$$

We keep the term with $\omega$ because at the same price we can compute the post-Newtonian limit of any Brans-Dicke-like theory. At the end of the calculations we can parameterize to the case $\omega = 0$ to obtain the desired result. It is also useful to keep the term with $\omega$ to check that, when the potential terms are neglected, we recover the expected limit of Brans-Dicke theories.

The expansion of the Ricci tensor around the Minkowski metric can be written as follows

$$R_{ij} = -\frac{1}{2} \nabla^2 h_{ij}^{(2)} + \frac{1}{2} \partial_i \left[ h_{j,\mu}^{\mu} - \frac{1}{2} h_{\mu,j}^{\mu} \right] + \frac{1}{2} \partial_j \left[ h_{i,\mu}^{\mu} - \frac{1}{2} h_{\mu,i}^{\mu} \right]$$

$$R_{0j} = -\frac{1}{2} \nabla^2 h_{0j}^{(3)} + \frac{1}{2} \partial_j \left[ h_{0,\mu}^{\mu} - \frac{1}{2} h_{\mu,0}^{\mu} \right] + \frac{1}{2} \partial_0 \left[ h_{j,\mu}^{\mu} - \frac{1}{2} h_{\mu,j}^{\mu} \right]$$
\[ R_{00} = -\frac{1}{2} \nabla^2 \left[ h_{00}^{(4)} + \frac{(h_{00}^{(2)})^2}{2} \right] + \]
\[ + \partial_0 \left[ h_{0,\mu}^{(2)} - \frac{1}{2} h_{\mu,0}^{(2)} + \frac{1}{2} h_{00,0} \right] + \]
\[ + \frac{1}{2} h_{j,\mu}^{(2)} - \frac{1}{2} h_{\mu,j}^{(2)} \partial^j h_{00}^{(2)} + \]
\[ + \frac{1}{2} h_{00}^{(2)} \nabla^2 h_{00}^{(2)} + \frac{1}{2} h^{(2)ij} \partial_i \partial_j h_{00}^{(2)} \quad (A4) \]

where all the indices are raised and lowered with the Minkowski metric. Assuming a perfect fluid, the elements on the right-hand side of eq. (A14) are given, up to the necessary order, by

\[ \tau_{ij} = \frac{\kappa^2 \rho}{2 \phi_0} \phi_{ij} + \rho O(\nu^2) \quad (A5) \]
\[ \tau_{0j} = -\frac{\kappa^2 \rho}{\phi_0} \phi_{0j} + \rho O(\nu^3) \quad (A6) \]
\[ \tau_{00} = \frac{\kappa^2 \rho}{2 \phi_0} \left[ 1 + \Pi + 2 \nu^2 - \left( h_{00}^{(2)} + \frac{\phi_{00}^{(2)}}{\phi_0} \right) + \frac{3P}{\rho} \right] + \rho O(\nu^4) \quad (A7) \]

where

\[ \tau_{\mu\nu} = \frac{\kappa^2}{\phi} \left[ T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right] \quad (A8) \]

We can also define the contribution due to the scalar field as

\[ \tau_{\mu\nu} = \frac{\omega}{\phi^2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{\phi} \nabla_\mu \nabla_\nu \phi + \frac{1}{2 \phi} g_{\mu\nu} \left[ \Box \phi + V(\phi) \right] \quad (A9) \]

Its components are

\[ \tau_{\phi j}^\phi = \partial_\phi \partial_j \left( \frac{\phi_{00}^{(2)}}{\phi_0} \right) + \frac{\delta_{ij}}{2 \phi_0} \left[ V_0 - \phi_0 + \nabla^2 \phi_0 \right] \quad (A10) \]
\[ \tau_{\phi j} = \frac{1}{2} \partial_j \left[ 2 \omega \frac{\phi_0}{\phi_0} \frac{\phi_{00}^{(2)}}{\phi_0} + \frac{\phi_0}{\phi_0} h_{00}^{(2)} + \frac{\phi_{00}^{(2)}}{\phi_0} \right] + \]
\[ + \frac{1}{2} \partial_\phi \partial_j \left( \frac{\phi_{00}^{(2)}}{\phi_0} \right) + \frac{1}{2 \phi_0} \partial_j \left( \frac{\phi_{00}^{(2)}}{\phi_0} \right) \quad (A11) \]

Using the gauge conditions

\[ h_{\mu,\mu}^{(2)} - \frac{1}{2} h_{\mu,0}^{(2)} = \frac{\partial_k \phi_{00}^{(2)}}{\phi_0} \quad (A13) \]

the equations of motion boil down to

\[ -\frac{1}{2} \nabla^2 \left[ h_{ij}^{(2)} + \frac{\phi_{00}^{(2)}}{\phi_0} \right] = \frac{\delta_{ij}}{2 \phi_0} \left[ \kappa^2 \rho + V_0 - \phi_0 \right] \quad (A15) \]
\[ -\frac{1}{2} \nabla^2 h_{ij}^{(2)} - \frac{1}{4} h_{ij,00}^{(2)} = -\kappa^2 \rho \nu_{ij} \quad (A16) \]
The equation for the scalar field is given by

$$-\frac{1}{2} \nabla^2 \left[ h_{00}^{(4)} - \frac{\varphi'^4}{\phi_0} + \frac{(h_{00}^{(2)})^2}{2} + \frac{1}{2} \left( \varphi'' \right)^2 \right] = \frac{\kappa^2 \rho}{2 \phi_0} \left[ 1 + \Pi + 2 \omega^2 + \frac{h_{ij}^{(2)}}{\phi_0} \varphi'' \right] + \frac{\omega^2}{\phi_0} \left[ 1 + h_{00}^{(2)} + h_{ij}^{(2)} \right] - \frac{1}{2 \phi_0} \left[ V_0 \left( 1 + h_{ij}^{(2)} \right) - \frac{h_{00}^{(2)}}{2} - 2 \omega \varphi'' \right] + \frac{\dot{\varphi}'^2}{\phi_0} + \omega \left( \frac{\dot{\varphi}}{\phi_0} \right)^2 \left[ 1 + h_{00}^{(2)} + h_{ij}^{(2)} \right] - \frac{1}{2 \phi_0} \left[ V_0 \left( 1 + h_{ij}^{(2)} \right) - \frac{h_{00}^{(2)}}{2} - 2 \omega \varphi'' \right] + \frac{\dot{\varphi}_0}{\phi_0} \left[ 3 \left( 1 + h_{ij}^{(2)} \right) - \frac{h_{00}^{(2)}}{2} - 2 \omega \varphi'' \right] + \frac{\dot{\varphi}_0}{\phi_0} \left[ 2 \omega \frac{\dot{\varphi}_0}{\phi_0} \varphi'' + \frac{\dot{\varphi}_0}{\phi_0} h_{00}^{(2)} \right] + \frac{\dot{\varphi}_0}{\phi_0} \left[ \frac{(\varphi''^2)}{2 \phi_0} \left[ \frac{\phi_0 V''_0}{2} - \frac{m_\varphi^2}{2} \right] \right] (A17)

where \( h_{ij} \) simply states the relation \( h_{ij} = \delta_{ij} h_{[ij]} \). The equation for the scalar field is given by

$$\left( \nabla^2 - m_\varphi^2 \right) \left[ \frac{\varphi'^4}{\phi_0} - \frac{1}{2} \left( \varphi'' \right)^2 \right] = -\frac{\kappa^2 \rho}{3 + 2 \omega} \left[ 1 + \Pi - \frac{3P}{\rho} + h_{ij}^{(2)} - \frac{\varphi''}{\phi_0} \right] + \varphi'' + m_\varphi^2 \varphi (h_{ij}^{(2)}) - \frac{\dot{\varphi}_0}{\phi_0} \left[ 2 \omega \frac{\dot{\varphi}_0}{\phi_0} \varphi'' + \frac{\dot{\varphi}_0}{\phi_0} h_{00}^{(2)} \right] + \frac{(\varphi''^2)}{2 \phi_0} \left[ \frac{\phi_0 V''_0}{3 + 2 \omega} - \frac{m_\varphi^2}{2} \right] (A18)

where we have defined

$$m_\varphi^2 = \frac{\phi_0 V''_0 - V'_0}{3 + 2 \omega} \tag{A19}$$

The solutions are formally given by

$$\frac{\varphi'(t,x)}{\phi_0} = \frac{\kappa^2}{4 \pi \phi_0} \int d^3x' \rho(t,x') F(|x - x'|) \left( 3 + 2 \omega \right) \tag{A20}$$

$$h_{00}^{(2)}(t,x) = \frac{\kappa^2}{4 \pi \phi_0} \int d^3x' \rho(t,x') \left[ 1 + F(|x - x'|) \left( 3 + 2 \omega \right) \right] + \left[ V_0 - 3 \frac{\dot{\varphi}_0}{\phi_0} V'_0 - 2 \omega \left( \frac{\dot{\varphi}_0}{\phi_0} \right)^2 \right] \frac{|x - x'|^2}{6} \tag{A21}$$

$$h_{ij}^{(2)}(t,x) = \frac{\kappa^2}{4 \pi \phi_0} \int d^3x' \rho(t,x') \left[ 1 - F(|x - x'|) \left( 3 + 2 \omega \right) \right] - \left( \frac{V_0 - \dot{\varphi}_0}{\phi_0} \right) \frac{|x - x'|^2}{6} \delta_{ij} \tag{A22}$$

$$h_{0j}^{(3)}(t,x) = -\frac{\kappa^2}{4 \pi \phi_0} \int d^3x' 2 \rho(t,x') \phi_0 \left[ 1 + \Pi + 2 \omega \varphi'' \right] + \frac{1}{4 \pi} \int d^3x' \frac{h_{00,0i}^{(2)}}{2|x - x'|} \tag{A23}$$

where the function \( F(|x - x'|) \) denotes

$$F(|x - x'|) = \begin{cases} 
  e^{-m_\varphi|x - x'|} & \text{if } m_\varphi^2 > 0 \\
  \cos(m_\varphi|x - x'|) & \text{if } m_\varphi^2 < 0 
\end{cases} \tag{A26}$$
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