Dynamics of inhomogeneities of metric
in the vicinity of a singularity
in multidimensional cosmology

A.A.Kirillov, V.N.Melnikov†

†Center for Surface and Vacuum Research,
8 Kravchenko str., Moscow 117331, Russia
Institute for Applied Mathematics and Cybernetics,
10 ulyanova str., Nizhny Novgorod 603005, Russia
e-mail: mel@cvsi.uucp.free.msk.su
e-mail: kirillov@focus.nnov.su

ABSTRACT

The problem of construction of a general inhomogeneous solution of $D$-dimensional Einstein equations with any matter sources satisfying the inequality $\epsilon \geq p$ in the vicinity of a cosmological singularity is considered. It is shown that near the singularity a local behavior of metric functions is described by a billiard on a space of a constant negative curvature. The billiard is shown to have a finite volume and consequently to be a mixing one. Dynamics of inhomogeneities of metric is studied and it is shown that its statistical properties admit a complete description. An invariant measure describing statistics of inhomogeneities is obtained and a role of a minimally-coupled scalar field in dynamics of the inhomogeneities is also considered.

Moscow 1994
1 Introduction

As is well known a number of unified theories predict that dimension of the Universe exceeds that of we normally experience at macroscopic level [1]. It is assumed that presently additional dimensions are hidden, for they are compactified to the Planckian size, and they do not display themselves in macroscopic and even in microscopic processes. However, the situation must be changed as we come back with time to the very beginning of the evolution of our Universe. Standard cosmological models predict the existence of a singular point at the very beginning and, therefore, the universe size could approach to the Planckian scale. Thus, in the early universe the additional dimensions, if exist, must not be different from ordinary dimensions and should be taken into account. Moreover, one could expect that the existence of additional dimensions may drastically change properties of the singularity and even remove it. The main aim of this paper is to construct a general solution of multidimensional Einstein equations near a singularity and to investigate properties of inhomogeneities.

The way to construct a general solution with singularity was indicated first by Belinsky et al. in Ref. [2] for $D = 4$, where $D$ is the dimension of a spacetime. Dynamics of metric at a particular point of space was shown to resemble the behaviour of the well studied "mixmaster" (or of the type-IX) homogeneous model and the last one has a complex stochastic nature [3,4]. Subsequent utilizing of that construction has been done in Ref. [5] where the so-called scalar-vector-tensor theory (or the case $D = 5$) was considered and the main feature of the mixmaster model, i.e. the complex oscillatory regime was shown to be also present in the 5-dimensional case.

An investigation of inhomogeneities of metric based on the general solutions has been considered first in Ref. [6]. The case of the scalar-tensor theory (or $D = 4$ + scalar fields) was considered and it turned out that the oscillatory regime leads to the fractioning of the coordinate scale $\lambda$ of the inhomogeneities of Kasner exponents ($\lambda \approx \lambda_0 2^{-N}$, where $N$ is the number of elapsed Kasner epochs and $\lambda_0$ is the initial scale of inhomogeneities). However, the methods by means of which the properties and statistics of the inhomogeneities were investigated turned out to be unapplicable for general case (i.e. for the absence of scalar fields as well as for the expanding universe). This problem has been solved recently in Ref. [7]. In this paper we generalize the results obtained in Ref. [7] to the case of arbitrary number of dimensions $D$.

As it was mentioned above the main features of the dynamics of an inhomogeneous gravitational field nearby the singularity in 4-dimensional case may be summarized as follows:

1. Locally dynamics of metric functions resembles the behaviour of the most general homogeneous "mixmaster" model [2], which has stochastic behaviour [3,4]. Just the stochastic behaviour leads to a monotonic decrease of the coordinate scale of the metric inhomogeneities [6,7].

2. In the vicinity of a singularity a scalar field is the only kind of matter effecting the dynamics of metric [5].

These facts may be simply understood under the following qualitative estimates (that is confirmed by subsequent consideration). As is well known in cosmology the horizon size $l_h$ is a natural scale measuring a distance from the singularity. Therefore, inhomogeneities may be divided into the large-scale ($l_i \gg l_h$) and small-scale ($l_i \ll l_h$) ones. The horison
size varies with time as \( l_h \sim t \) (where \( t \) is the time in synchronous reference system) whereas the characteristic spatial dimension of the inhomogeneity may be estimated as \( l_i \sim t^\alpha \) (as \( t \to 0 \)). In a linear theory for an isotropic background the exponent \( \alpha \) may be expressed via the state equation of matter as \( \alpha = \frac{2}{3(p+\epsilon)} \) and what is important \( \alpha < 1 \). Thus, it is clear that an arbitrary inhomogeneous field becomes large-scale in the sufficient closeness to the singularity. Since the inhomogeneities are large-scale there are no effects connected with propagating of gravitational waves etc, and this would mean that inhomogeneities become passive. Consequently, dynamics of the field may be approximately described by the most general homogeneous model depending parametrically upon the spatial coordinates. Note, however, that the homogeneous model would appear to be in a general non-diagonal form.

The second fact may be understood in the same way. As it was shown in Ref. [8] the gravitational part of the Einstein equations at the singular point varies with time, in the leading order, as \( R_{\alpha\beta} \sim t^{-2} \) whereas the matter has the order \( T_{\alpha\beta} \sim t^{-2k} \), where \( k \) depends upon the state equation as \( k = \frac{2\epsilon + p}{2} \). Thus, one can see that for the equation of state satisfying the inequality \( p < \epsilon \) we have \( k < 1 \) and only for the limited case \( p = \epsilon \) the both sides turn out to be of the same order. We note that in the vicinity of a singularity scalar fields give just this equation of state.

As it is well known (see for example Ref.[5,9]) additional dimensions may be treated in ordinary gravity as a set of nonminimally coupled scalar and vector fields. Therefore, one could expect that the main contribution in dynamics in the vicinity of a singularity would be given by those dynamical functions which are connected with scalar fields whereas other functions would play a passive role.

Thus, one could expect that in multidimensional cosmology local behaviour of the metric functions (at a particular point of space) will be described by a most general homogeneous model. Here, it is necessary to recall the important property of the mixmaster universe that is the stochastic behaviour. The problem of stochasticity of homogeneous multidimensional cosmological models has been investigated in a number of papers [10]. In particular, the conclusion was made that chaos is absent in spaces whose dimension is large enough. This negative result is, apparently, connected with the fact that in the vicinity of a singularity the homogeneous model would appear to be in a general nondiagonal form, instead of simple diagonal models considered in Refs.[10]. Besides, recently in Ref.[11] it was shown that a wide class of multidimensional models has stochastic behaviour. In this paper we shall show that stochastic behaviour is the general property of multidimensional cosmology.

2 Generalized Kasner Solution, Generalized Kasner Variables

We consider the theory in canonical formulation. Basic variables are the Riemann metric components \( g_{\alpha\beta} \) with signature \((+,-,\ldots,-)\) and a scalar field \( \phi \) specified on the n-manifold \( S \), and its conjugate momentum \( \Pi^\alpha = \sqrt{g}(K^\alpha - g^\alpha\beta K) \) and \( \Pi_\phi \), where \( \alpha = 1,\ldots,n \) and \( K^\alpha \) is the extrinsic curvature of \( S \). For the sake of simplicity we shall consider \( S \) to be
compact i.e. $\partial S = 0$. The action has in Planck units the following form

$$I = \int_S (\Pi^i_0 \partial g_{ij} \partial t + \Pi_0 \partial \phi \partial t - NH^0 - N_a H^\alpha) d^nx dt,$$

(2.1)

where

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \Pi^\alpha_\beta \Pi^\beta_\alpha - \frac{1}{n-1} (\Pi^\alpha_\alpha)^2 + \frac{1}{2} \Pi^2_\phi + g(W(\phi) - R) \right\},$$

(2.2)

$$H^\alpha = -2 \Pi^\alpha_\beta + g^\alpha_\beta \partial_\phi \Pi_\phi,$$

(2.3)

and

$$W(\phi) = \frac{1}{2} \left\{ g^\alpha_\beta \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right\}.$$

(2.4)

A generalized Kasner solution is realized under the following assumption

$$\sqrt{gT} \sim (\Pi^\alpha_\beta, \Pi^\phi) \gg V = g(W - R),$$

(2.5)

where $\sqrt{gT}$ denotes the first three terms in (2.2). Then, using (2.1) one can find the following solution of the multidimensional Einstein equations

$$ds^2 = dt^2 - \sum_{a=0}^{n-1} t^a l^a_\alpha l^a_\beta dx^\alpha dx^\beta$$

(2.6)

where $l^a_\alpha$, $s^a$ are functions of space coordinates. Kasner exponents $s^a$ satisfy the identities

$$\sum s^a = \sum s^2^a + q^2 = 1,$$

and run the domain $-\frac{n-2}{n} \leq s^a \leq 1$ (here $q^2 = \frac{(n-1)^2}{2} \frac{\Pi^2_\phi}{(\Pi^2_\phi)^2}$). Since, as it was shown in Ref.[2,5] the generalized Kasner solution takes a substantial portion of the evolution of metric it is convenient to introduce a Kasner-like parametrization of the dynamical variables [7]. We consider the following representation for metric components and their conjugate momenta

$$g_{\alpha\beta} = \sum_a \exp \left\{ q^a \right\} l^a_\alpha l^a_\beta,$$

(2.7)

$$\Pi^\alpha_\beta = \sum a p_a L^a_\alpha l^a_\beta,$$

(2.8)

here $L^a_\alpha l^b_\beta = \delta^b_a (a, b = 0, \ldots, (n - 1))$, and the vectors $l^a_\alpha$ contain only $n(n - 1)$ arbitrary functions of spatial coordinates. Further parametrization may be taken in the following form

$$l^a_\alpha = U^a_\alpha S^a_b, U^a_b \in SO(n), S^a_\alpha = \delta^a_\alpha + R^a_\alpha$$

(2.9)

where $R^a_\alpha$ denotes a triangle matrix ($R^a_\alpha = 0$ as $a \leq \alpha$). Substituting (2.7) - (2.9) into (2.1) one gets the following expression for the action functional

$$I = \int_S (p_a \partial q^a \partial t + T^a_\alpha \partial R^a_\alpha \partial t + \Pi^\alpha_\phi \partial \phi \partial t - NH^0 - N_a H^\alpha) d^nx dt,$$

(2.10)

here $T^a_\alpha = 2 \sum_b p_b L^a_\beta U^b_\alpha$ and the Hamiltonian constraint takes the form

$$H^0 = \frac{1}{\sqrt{g}} \left\{ \sum p^2_a - \frac{1}{n-1} (\sum p_a)^2 + \frac{1}{2} \Pi^2_\phi + V \right\}.$$

(2.11)
In the case of \( n = 3 \) the functions \( R^a_\alpha \) are connected purely with transformations of a coordinate system and may be removed by solving momentum constraints \( H^\alpha = 0 \). In the multidimensional case the functions \( R^a_\alpha \) contain \( \frac{n(n-3)}{2} \) dynamical functions as well. Now it is easy to see that the choice of Kasner-like parametrization simplifies the procedure of the constructing of the generalized Kasner solution. Indeed, if we now neglect the potential term in (2.10) and put \( N^\alpha = 0 \) we find that Hamiltonian does not depend on the scale functions and other dynamical variables contained in Kasner vectors introduced by expressions (2.7) (2.8).

3 The asymptotic model in the vicinity of a cosmological singularity

As it is well known, [2], [5], the Kasner regime (2.6) turns out to be unstable in a general case. This happens due to the violation of the condition (2.5) because the potential \( V \) contains increasing terms which lead to replacement of Kasner regimes. To find out the law of replacement it is more convenient to use an asymptotic expression for the potential [7], [11]. For this aim we put the potential in the following form

\[
V = \sum_{A=1}^{k} \lambda_A g^{u_A},
\]

(3.1)

here \( \lambda_A \) is a set of functions of all dynamical variables and of their derivatives and \( u_a \) are linear functions of the anisotropy parameters \( Q_a = \frac{g_a^\alpha}{\sum g^\beta} (u_A = u_A(Q)) \). Assuming the finiteness of the functions \( \lambda \) and considering the limit \( g \to 0 \) we find that the potential \( V \) may be modeled by potential walls

\[
g^{u_A} \to \theta_\infty[\theta(u_A(Q))] = \begin{cases} +\infty, & u_A < 0, \\ 0, & u_a > 0 \end{cases}
\]

(3.2)

Thus, putting \( N^\alpha = 0 \) we can remove the passive dynamical function \( T^a_\alpha, R^a_\alpha \) from the action (2.10) and get the reduced dynamical system

\[
I = \int_S \left\{ p_a \frac{\partial p^a}{\partial t} + \Pi_\phi \frac{\partial \phi}{\partial t} - \lambda \left\{ \sum p^2 - \frac{1}{n-1}(\sum p)^2 + \frac{1}{2} \Pi_\phi^2 + U(Q) \right\} \right\} d^n x dt,
\]

(3.3)

here \( \lambda \) is expressed via the lapse function as \( \lambda = \frac{N}{\sqrt{g}} \). In harmonic variables the action (3.3) takes the form formally coincided with the action for a relativistic particle

\[
I = \int_S \left\{ P_i \frac{\partial P^i}{\partial t} - \lambda'(P^2 + U - P^2_0) \right\} d^n x dt,
\]

(3.4)

here \( r = 0, ..., n, i = 1, ..., n, \frac{q^a}{z^i} = A^a_j z^j + z^0 (j = 1, ..., n - 1), \phi = \sqrt{n(n-1)} \phi \) and the constant matrix \( A^a_j \) obeys the following conditions

\[
\sum_a A^a_j = 0, \sum_a A^a_j A^a_k = n(n-1)\delta_{jk}
\]

(3.5)
and can be expressed in the following form

\[ A^a_j = \sqrt{\frac{n(n-1)}{j(j-1)}} (\theta^a_j - j \delta^a_j), \]

where \( \theta^a_j = \begin{cases} 1, & j > a \\ 0, & j \leq a \end{cases} \).

Since the timelike variable \( z^0 \) varies during the evolution as \( z^0 \sim \ln g \) the positions of potential walls turn out to be moving. It is more convenient to fix the positions of walls. This may be done by using the so-called Misner-Chitre like variables [11] \((\vec{y} = y^j)\)

\[ z^0 = -e^{-\tau} \frac{1 + y^2}{1 - y^2}, \quad z = -2e^{-\tau} \frac{\vec{y}}{1 - y^2}, \quad y = |\vec{y}| < 1. \quad (3.6) \]

Using these variables one can find the following expressions for the anisotropy parameters

\[ Q_a(y) = \frac{1}{n} \left( 1 + \frac{2A^a_j y^j}{1 + y^2} \right), \quad (3.7) \]

which are now independent of timelike variable \( \tau \). From (3.7) one can find the range of the anisotropy functions \(-\frac{n}{n-2} \leq Q_a \leq 1\).

Choosing as a time variable the quantity \( \tau \) (i.e. in the gauge \( N = \frac{n(n-1)}{2} \sqrt{g} \exp(-2\tau)/P^0 \)) we put the action (3.4) into the ADM form

\[ I = \int_S \left\{ \vec{P} \frac{\partial}{\partial \tau} \vec{y} + P^n \frac{\partial}{\partial \tau} z^n - P^0(P, y) \right\} d^nxd\tau, \quad (3.8) \]

where the quantity

\[ P^0(P, y) = (\epsilon^2(\vec{y}, \vec{P}) + V[y] + (P^n)^2 e^{-2\tau})^{1/2}, \quad (3.9) \]

plays the role of the ADM Hamiltonian density and

\[ \epsilon^2 = \frac{1}{4} (1 - y^2)^2 \vec{F}^2. \quad (3.10) \]

The part of the configuration space connected with the variables \( \vec{y} \) is a realization of the \((n-1)\) -dimensional Lobachevsky space [12] and the potential \( V \) cuts a part of it. Thus, locally (at a particular point of \( S \)) the action (3.9) describes a billiard on the Lobachevsky space. The positions of walls which form the boundary of the billiard are determined, due to (3.1) by the inequalities

\[ \sigma_{abc} = 1 + Q_a - Q_b - Q_c \geq 0, \quad a \neq b \neq c \quad (3.11) \]

and the total number of walls is \( \frac{n(n-1)(n-2)}{2} \). Using the matrix (3.5) one can find that the walls are formed by spheres determined by the equations

\[ \sigma_{abc} = \frac{n-1}{n(1 + y^2)} \left\{ (\vec{y} + \vec{B}_{abc})^2 + 1 - B^2_{abc} \right\}, \quad \vec{B}_{abc} = \frac{1}{n-1} (\vec{A}^a - \vec{A}^b - \vec{A}^c), \quad (3.12) \]
here for arbitrary \( a, b, c \) we have \( B^2 = 1 + \frac{2n}{n-1} \). In a general case \( n \) points of the billiard having the coordinates \( \vec{P}_a = \frac{1}{n-1} \vec{A}^a \) lie on the absolute (at infinity of the Lobachevsky space). Nevertheless, one can show that the volume of the billiard is finite. We give two simplest examples for illustration of the billiards on fig.1. The case \( n = 3 \) on fig.1a coincides with the well-known "mixmaster" model and on fig.1b we illustrate the case of \( n = 4 \) considered in Ref.[5].

4 Dynamics of inhomogeneities

The system (3.8) has the form of the direct product of "homogeneous" local systems. Each local system in (3.8) has two variables \( \epsilon \) and \( P^n \) as integrals of motion. The solution of this local system for remaining functions represents a geodesic flow on a manifold with negative curvature. As it is well known the geodesic flow on a manifold with negative curvature is characterized by exponential instability [12]. This means that during the motion along a geodesic the normal deviations grow no slower than the exponential of the traversed path \( \xi \approx \xi_0 e^s \), where the traversed path is determined by the expression

\[
s = \int_{\tau_0}^{\tau} dl = \int_{\tau_0}^{\tau} \frac{2 | \frac{\partial y}{\partial x} |}{(1 - y^2)} d\tau = \frac{1}{2} \ln \left| \frac{P^0 - \epsilon}{P^0 + \epsilon} \right| \tau_0 .
\]

(4.1)

This instability leads to the stochastic nature of the corresponding geodesic flow. The system possesses the mixing property [13] and an invariant measure induced by the Liouvillean one

\[
d\mu(y, P) = const \delta(E - \epsilon) d^{n-1}y d^{n-1}P ,
\]

(4.2)

where \( E \) is a constant. Integrating this expression over \( \epsilon \) we find

\[
d\mu(y, s) = const \frac{d^{n-1}y d^{n-2}s}{(1 - y^2)^n} ,
\]

(4.3)

where \( s = \frac{\vec{P}}{\epsilon} \), \( |s| = 1 \).

Since the inhomogeneous system (3.8) is the direct product of "homogeneous" systems one can simply describe its behaviour as in ref [7]. In particular, the scale of the inhomogeneity decreases as

\[
\lambda_i \sim (\frac{\partial y}{\partial x})^{-1} \sim \lambda_0^i \exp(-s)
\]

(4.4)

and after sufficiently large time (\( s(\tau) \to \infty \)) the dynamical functions \( \vec{y}(x), \vec{P}(x) \) become a random functions of the spatial coordinates. Their statistics is described by the invariant distribution (4.3) and asymptotic expressions for averages and correlating functions have the form

\[
< \vec{y}(x) > = < \vec{P}(x) > = 0, < y_k(x), y_l(x') > = < y_k, y_l > \delta(x, x') ,
\]

(4.5)

for \( |x - x'| \gg \lambda_0^i \exp(-s) \).

Here it is necessary to point out a role of the scalar field in dynamics and statistical properties of inhomogeneities. As may be easily seen from (4.1) in the absence of a scalar
field (i.e. $P^n = 0$) the transversed path coincides with the duration of motion (we have $s = \Delta \tau = \tau - \tau_0$ instead of (4.1)). Thus, the effect of scalar fields is displayed in the replacement of the dependence for transversed path of time variable and, therefore, in the replacement of the rate of increasing of the inhomogeneities. This replacement does not change qualitatively the evolution of the universe in the case of cosmological expansion. But in the case of the contracting universe the situation changes drastically. Indeed, in the limit $\tau \to -\infty$ from (4.1) we find that the transversed path $s$ takes a limited value $s_0$ and therefore the increasing of inhomogeneities turns out to be finite. One of consequences of such behaviour is the fact that at the singularity the functions $\vec{y}$ and $\vec{P}$ take constant values. In other words in the presence of scalar fields a cosmological collapse ends with a stable Kasner-like regime (2.6). This fact may be seen in the other way. Indeed, in the limit $\tau \to -\infty$ the scalar field gives the leading contribution in ADM Hamiltonian (3.9) and $P^0$ does not depend on gravitational variables at all.

The finiteness of the transversed path $s(\tau)$ leads, generally speaking, to the destruction of the mixing properties [13], since for establishment of the invariant measure it is necessary to satisfy the condition $s_0 \to \infty$. Evidently, this condition requires the smallness of the energy density for scalar field as compared with the ADM energy of gravitational field (the last term in (3.9) in comparison with the first ones). Indeed, in this case $s_0$ is determined by the expression $s_0 = -\ln \frac{P_0}{2e^{\tau_0}}$, which follows from (4.1), and as $P^n \to 0$ one get $s_0 \to \infty$ (i.e. $s$ can have arbitrary large values).

Thus, in the case of cosmological contraction one may speak of the mixing and, therefore, of establishment of the invariant statistical distribution just only for those spatial domains which have sufficiently small energy density of the scalar field.

5 Estimates and concluding remarks

In this manner the large-scale structure of the space in the vicinity of singularity acquires a quasi-isotropic nature. A distribution of inhomogeneities is determined by the set of functions of spatial coordinates $\epsilon(x)$, $\Pi_{\phi}(x)$ and $R_\alpha^a$ which conserve during the evolution a primordial degree of inhomogeneity of the space. The scale of inhomogeneity of other functions grows as $\lambda \approx \lambda_0 e^{-s(\tau)}$. In this section we give some estimates clarifying the behaviour of the inhomogeneities. For simplicity we consider the case when the scalar field is absent.

To find the estimate for growth of the inhomogeneity in a synchronous time $t$ ($dt = Nd\tau$) we put $y = 0$. Then for variation of the variable $\tau$ one may find the following estimate $\sqrt{g} \sim \exp(-\frac{\tau}{2}e^{-\tau}) \sim P^0 t$, (here the point $t = 0$ corresponds to the singularity). According to (4.4) the dependence of the coordinate scale of inhomogeneity upon the time $t$ takes the form

$$\lambda \approx \lambda_0 \ln(1/g_0)/\ln(1/g)$$

in the case of contracting ($g \to 0$) and

$$\lambda \approx \lambda_0 \ln(1/g)/\ln(1/g_0)$$

in the case of the expanding universe.
A rapid generation of the more and more small scales leads to the formation of spatial chaos in metric functions and so the large-scale structure acquires a quasi-isotropic nature. Speeds of the scale growing (Hubble constants) for different directions turn out to be equal after averaging over a spatial domains having the size $\approx \lambda_0$. Indeed, using (3.7) one may find the expressions for averages $\langle Q_a \rangle = 1/n$.

Besides, it is necessary to mention one more characteristic feature of the oscillatory regime in the inhomogeneous case. This is the formation of a cellular structure in the scale functions $Q_a$ during the evolution which demonstrate explicitly the stochastic process of development of inhomogeneities. Indeed, let us consider some region of coordinate space $\Delta V$. Two functions $\vec{y}(\vec{x})$ define the map of that region on some square $\Sigma \in K$ (see fig.1c). During the evolution the size of the square $\Sigma$ grows $\approx e^{s(\tau)}$ and $\Sigma$ covers the domain of the billiard $K$ many times. Each covering determines its own preimage in $\Delta V$. In this manner the initial coordinate volume is splitted up in ”cells” $\Delta V = \bigcup_i \Delta V_i$. In the every cell the vector $\vec{y}(\vec{x})$ takes almost all admissible values $\vec{y} \in K$ and that of the functions $Q_a$ (for $n = 3$ $Q_a \in [0, 1]$). Such a structure turns out to be depending of time and the number of cells increases as $N \approx N_0 e^{s(\tau)}$. However, the situation will be changed if we consider a contracting space filled with a scalar field. Then the evolution of this structure in the limit $g \to 0$ ends, because the functions $Q_a$ become independent of time, and on the final stage of the collapse one would have a real cellular structure [6].

In spite of the isotropic nature of the spatial distribution of the field the large local anisotropy displays itself in the anomalous dependence of spatial lengths upon time variable for vectors and curves. Indeed, in the case of $D = 3$ a moment of scale function $\langle g^{M Q_a} \rangle$ (where $M > 0$) is decreased in the asymptotic $g \to o$ as the Laplace integral $\int_0^1 g^{M Q_a} \rho(Q_a) dQ_a$, where $\rho(Q_a)$ is a distribution which follows from (4.3) and has the form

$$\rho(H) = \frac{2}{\pi} (Q(1 - Q))^{-1/2} (1 + 3Q)^{-1}. \quad (5.1)$$

As $Q \ll 1$ one has $\rho(Q_a) \approx \frac{2}{\pi} Q_a^{-1/2}$ and, thus, in the limit $g \to 0$ we get the estimate

$$\langle g^{M Q_a} \rangle \approx (M \ln(1/g))^{-1/2}. \quad (5.2)$$

When $D > 3$ the similar analysis may be done.

In conclusion we briefly repeat the main results. The general inhomogeneous solution of $D$-dimensional Einstein equations with any matter sources satisfying the inequality $\epsilon \geq p$ near the cosmological singularity is constructed. It is shown that near the singularity a local behavior of metric functions ( at a particular point of the coordinate space) is described by a billiard on the $(D - 1)$-dimensional Lobachevsky space. In contrast to diagonal homogeneous models [10] in the inhomogeneous case the billiard is shown to have always a finite volume and consequently to be a mixing one. The rate of growth of inhomogeneities of metric is obtained. Statistical properties of inhomogeneities are described by the invariant measure. It is shown that a minimally-coupled scalar field leads, in general, to the distraction of stochastic properties of the inhomogeneous model.

Acknowledgments

This work was supported in part by the Russian Ministry of Science.
References

[1] M.B.Green, J.H.Schwarz, and E.Witten, Superstring Theory, Cambridge Univ. Press, 1988.

[2] V.A.Belinskii, E.M.Lifshitz and I.M.Khalatnikov, Adv. Phys. 31, 639 (1982); Zh. Eksp. Teor. Fiz. 62, 1606 (1972) [Sov. Phys. JETP 35, 838 (1972)].

[3] V.A.Belinskii, E.M.Lifshitz and I.M.Khalatnikov, Usp. Fiz. Nauk 102, 463 (1970) [Sov. Phys. Usp. 13, 745 (1971)].

[4] I.M.Khalatnikov, E.M.Lifshitz, K.M.Khanin, L.M.Shchur, and Ya.G.Sinai, Pis’ma Zh. Eksp. Teor. Fiz. 38, 79 (1983) [JETP Lett. 38, 91 (1983)].

[5] V.A.Belinskii, I.M.Khalatnikov, Zh. Eksp. Teor. Fiz. 63, 1121 (1972) [Sov. Phys. JETP 36, 591 (1973)].

[6] A.A.Kirillov, A.A.Kochnev, Pis’ma Zh. Eksp. Teor. Fiz. 46, 345 (1987). [JETP Lett. 46, 435 (1987)].

[7] A.A.Kirillov, Zh. Eksp. Teor. Fiz. 103, 721 (1993). [Sov. Phys. JETP 76, 355 (1993)].

[8] E.M.Lifshitz and I.M.Khalatnikov, Adv. Phys. 12, 185 (1963).

[9] V.N.Melnikov, In Results of Science and Technology. Ser. Classical Field Theory and Gravitation. Gravitation and Cosmology. Ed. V.N.Melnikov. VINITI Publ., Moscow, Vol.1, 1991, 49 [in Russian].

K.A.Bronnikov and V.N.Melnikov, ibid., Vol. 4, 1992, 67.

V.N.Melnikov. Preprint CBPF-NF-051/93, Rio de Janeiro, Brazil, 1993. Proc.VII Brazilian School on Cosmology and Gravitation, 1994.

V.D.Ivashchuk and V.N.Melnikov, Int. J. Mod. Phys. D, December 1994.

[10] M.Szydlowski, J.Szczesny, M.Biesiada, GRG, 19, 1118 (1987); M.Szydlowski, G.Pajdosz, Class. Quant. Grav., 6, 1391 (1989); J.Demaret, Y.de Rop, M.Henneaux, Int. J. Theor. Phys., 28, 250 (1989).

[11] V.D.Ivashchuk, A.A.Kirillov, V.N.Melnikov, JETP Letters,60, N4 (1994). Izv. Vuz. Fizika (1994).

[12] D.V.Anosov, Geodesic Flow on Manifolds with Negative Curvature, Trudy Steklov Math. Inst., Moscow, 1967 [in Russian].

[13] I.P.Kornfel’d, Ya.G.Sinai, and S.V.Fomin, Ergodic Theory, Nauka, Moscow, 1980. [in Russian].
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9408004v1