On graphs with a large chromatic number containing no small odd cycles

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Abstract

In this paper, we present the lower bounds for the number of vertices in a graph with a large chromatic number containing no small odd cycles.

1 Introduction

P. Erdős [2] showed that for every integer \( n > 1 \) and \( p > 2 \), there exists a graph of girth \( g \) and chromatic number greater than \( n \) which contains not more than \( n^{2g+1} \) vertices. Later, he conjectured [3] that for every positive integer \( s \) there exists a constant \( c_s \) such that for every graph \( G \) having \( N \) vertices and containing no odd cycles of length less than \( c_s N^{1/s} \), its chromatic number does not exceed \( s + 1 \).

This conjecture was proved by Kierstead, Szemerédi, and Trotter [4]; in fact, they have proved a more general result. In our case, their result states that the chromatic number of any graph on \( N \) vertices containing no odd cycles of length at most \( 4sN^{1/s} + 1 \) does not exceed \( s + 1 \).

Basing on these results, we introduce the following notation.

Definition 1. Assume that \( n, k > 1 \) are two integers. Denote by \( f(n, k) \) the maximal integer \( f \) satisfying the following property: If a graph \( G = (V, E) \) contains no odd cycles of length at most \( 2k - 1 \), and \( |V| \leq f \), then there exists a proper coloring of its vertices in \( n \) colors.

Notice that a graph contains no odd cycles of length at most \( 2k - 1 \) if and only if it contains no simple odd cycles of the same lengths.

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The results mentioned above imply that \( f(n, k) < n^{4k+1} \) and \( f(s + 1, [2sN^{1/s}] + 1) \geq N \). One can obtain that the latter inequality is equivalent to the bound

\[
f(n, k) \geq \left( \frac{k}{2(n-1)} \right)^{n-1} - 1. \tag{1}
\]

A different upper bound for \( f(n, k) \) can be obtained from the following graph constructed by Schrijver \([6]\). Let \( m, d \) be some positive integers. Set \( X = \{1, 2, \ldots, 2m + d\} \), \( V = \{x \subset X : |x| = m, 1 < |i - j| < 2m + d - 1 \text{ for all pairs of distinct } i, j \in x\} \), \( E = \{(x, y) \in V^2 : x \cap y = \emptyset\} \). The Schrijver graph \((V, E)\) is \((d+2)\)-chromatic, whilst it does not contain odd cycles of length less than \( \frac{2m+d}{d} \). Next, we have \(|V| = \frac{(2m+d)}{m+d} \binom{m+d}{d} \); now it is easy to obtain that

\[
f(n, k) < \frac{(n-1)(2k-1)+2}{(n-1)k+1} \binom{(n-1)k+1}{n-1}. \tag{2}
\]

When we fix the value of \( n \), the bounds (1) and (2) become the polynomials in \( k \) of the same degree; hence, in some sense they are close to each other. On the contrary, when we fix the value of \( k \) and consider the values \( n > k/(2e)+1 \), we see that the right-hand part of (1) decreases (as a function in \( n \)). Hence for larger values of \( n \) this estimate does not provide any additional information.

On the other hand, for \( k = 2 \) the asymptotics of \( f(n, 2) \) is tightly connected with the asymptotics of Ramsey numbers \( R_n, 3 \). In the papers of Ajtai, Komlós, and E. Szemerédi \([1]\) and Kim \([5]\) it is shown that \( c_1 \frac{n^2}{\log n} \leq R(n, 3) \leq c_2 \frac{n^2}{\log n} \) for some absolute constants \( c_1, c_2 \). One can check that these results imply the bounds

\[
c_3 n^2 \log n \leq f(n, 2) \leq c_4 n^2 \log n
\]

for some absolute constants \( c_3, c_4 \).

In the present paper, we find nontrivial lower bounds for \( f(n, k) \) for all values of \( n \geq 2 \) and \( k \geq 2 \). In Section 2, we make some combinatorial considerations leading to the recurrent bounds for \( f(n, k) \). In Section 3, we obtain explicit bounds following from those results. In particular, we show (see Theorem \([3]\)) that

\[
f(n, k) \geq \frac{(n+k)(n+k+1) \cdots (n+2k-1)}{2^{k-1}k^k}
\]

for all \( n \geq 2 \) and \( k \geq 2 \).
2 Recurrent bounds

Firstly, we introduce some notation.

Let \( G = (V, E) \) be an (unoriented) graph. We denote the distance between the vertices \( u, v \in V \) by \( \text{dist}_G(u, v) \).

Consider a vertex \( v \in V \), and let \( r \) be a nonnegative integer. We denote by \( U_r(v, G) = \{ u \in V \mid \text{dist}_G(u, v) \leq r \} \) the ball of radius \( r \) with the center at \( v \), and by \( S_r(v, G) = \{ u \in V \mid \text{dist}_G(u, v) = r \} \) the sphere with the same radius and center. In particular, \( S_0(v, G) = U_0(v, G) = \{ v \} \). Denote also by \( \partial_G^\text{int}V_i = \{ u \in V \setminus V_i \mid \exists v \in V_i : (u, v) \in E \} \) the outer boundary of a set \( V_i \subseteq V \). In particular, \( S_r(v, G) = \partial_G^\text{int}U_{r-1}(v, G) \).

For a set \( V_1 \subseteq V \), we denote by \( G(V_1) \) the induced subgraph on the set of vertices \( V_1 \).

Let us fix some integers \( n \) and \( k \) which are greater than 1. We need the following easy proposition.

**Proposition 1.** Graph \( G \) does not contain odd cycles of length not exceeding \( 2k - 1 \) if and only if for each vertex \( v \in V \) and each positive integer \( r < k \), the subgraph \( G(S_r(v, G)) \) contains no edges.

**Proof.** Assume that the subgraph \( G(S_r(v, G)) \) contains an edge \((u_1, u_2)\). Supplementing this edge by shortest paths from \( v \) to \( u_1 \) and \( u_2 \), we obtain a cycle of length \( 2r + 1 \leq 2k - 1 \).

Conversely, assume that \( G \) contains a cycle of length \( \leq 2k - 1 \). Consider such a cycle \( C \) of the minimal length \( 2r + 1 \) (then \( r < k \)). Choose any its vertex \( v \), and let \( u_1, u_2 \) be two vertices of \( C \) such that \( \text{dist}_C(v, u_1) = \text{dist}_C(v, u_2) = r \). In fact, we have \( \text{dist}_G(u_1, v) = \text{dist}_G(u_2, v) = r \). Actually, assume that \( \text{dist}_G(v, u_1) < r \), and choose a path \( P \) of the minimal length connecting \( v \) and \( u_1 \). Then one can supplement it by one of the two subpaths of \( C \) connecting \( u_1 \) and \( v \) to obtain an odd cycle \( C' \). The length of \( C' \) is smaller than \( r + (r + 1) = 2r + 1 \), that contradicts the choice of \( C \).

Thus, \( u_1, u_2 \in S_r(v, G) \), and the graph \( G(S_r(v, G)) \) contains an edge. \( \square \)

Now let us fix an arbitrary graph \( G = (V, E) \) with a minimal number of vertices such that it contains no odd cycles of length not exceeding \( 2k - 1 \), and \( \chi(G) > n \) (hence \( |V| = f(n, k)+1 \)). By the minimality condition, the graph \( G \) is connected. Moreover, for every \( v \in V \) and \( 0 \leq r \leq k \), the sphere \( S_r(v, G) \) is nonempty. Otherwise we would have \( G = \bigcup_{i=0}^{r-1} S_i(v, G) \), where all the graphs \( G(S_i(v, G)) \) contain no edges by Proposition 1. Therefore, it is possible to color this graph properly in two colors: the vertices of the sets \( S_i(v, G) \) with even \( i \) in color 1, while those for odd \( i \) — in color 2 (the vertex \( v \) should be colored in color 1).

Let us introduce the number \( d = \max_{v \in V} |U_{k-1}(v, G)|. \)
Lemma 1. For every vertex \( v \in V \), we have \( |U_{k-1}(v, G)| \geq n(k-1) + 1 \). In particular, \( d \geq n(k-1) + 1 \).

Proof. Notice that \( U_{k-1}(v, G) = \bigcup_{r=0}^{k-1} S_r(v, G) \). Assume that \( |S_r(v, G)| \geq n \) for every \( r = 1, \ldots, k-1 \); then

\[
|U_{k-1}(v, G)| = \sum_{r=0}^{k-1} |S_r(v, G)| \geq 1 + (k-1) \cdot n,
\]

as desired.

Assume now that \( |S_r(v, G)| < n \) for some \( 1 \leq r \leq k-1 \). Consider a subgraph \( G' = G(V \setminus U_{r-1}(v, G)) \). From the minimality condition, it can be colored properly in \( n \) colors. Consider an arbitrary such proper coloring; then the vertices of \( S_r(v, G) \) are colored in at most \( n-1 \) colors, so there exists a color (say, color 1) different from them. Let us now color the vertices of \( S_{r-1}(v, G) \) in color 1, and then color all the remaining vertices of the sets \( S_i(v, G) \) \( (i < r-1) \) alternately: we use colors 1 and 2 (here 2 is any remaining color) for even and odd values of \( i-(r-1) \), respectively. It follows from Proposition II that this coloring is proper. This contradicts the choice of \( G \).

Lemma 2. \( |V| \geq f(n-1, k) + d + 1 \).

Proof. Choose a vertex \( v \) such that \( d = |U_{k-1}(v, G)| \). Assume that \( |V \setminus U_{k-1}(v, G)| \leq f(n-1, k) \); then one can color properly vertices of the set \( V \setminus U_{k-1}(v, G) \) in \( n-1 \) colors. Now we can color the vertices of the set \( U_{k-1}(v, G) \) in colors 1 and \( n \) (where \( n \) is a new color, and 1 is any of the old colors) in the following way: we color all the vertices of \( S_r(v, G) \) in color 1 or \( n \), if \( r-(k-1) \) is odd or even, respectively. It follows from Proposition II we obtain a proper coloring of \( G \) in \( n \) colors which is impossible.

Thus, our assumption is wrong, so \( |V \setminus U_{k-1}(v, G)| \geq f(n-1, k) + 1 \), and

\[
|V| \geq f(n-1, k) + 1 + |U_{k-1}(v, G)| = f(n-1, k) + d + 1.
\]

Lemma 3. \( |V| \geq \frac{d^{1/(k-1)}}{d^{1/(k-1)}-1} (f(n-2, k) + 1) \).

Proof. We will construct inductively a sequence of partitions of \( V \) into non-intersecting parts,

\[
V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_s \sqcup N_s \sqcup V_s,
\]

such that the following conditions are satisfied:

(i) for all \( i = 1, \ldots, s \) we have \( \partial_G^{\text{in}} U_i \subseteq N_s \); moreover, \( \partial_G^{\text{out}} V_s \subseteq N_s \);
(ii) for every $i = 1, 2, \ldots, s$ the graph $G(U_i)$ is bipartite (in fact, $U_i$ is a ball with radius not exceeding $k - 1$ in a certain subgraph of $G$);

(iii) $(d^{1/(k-1)} - 1)(|U_1| + \cdots + |U_s|) \geq |N_s|.$

For the base case $s = 0$, we may set $V_0 = V$, $N_0 = \emptyset$ (there are no sets $U_i$ in this case).

For the induction step, suppose that the partition $V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_{s-1} \sqcup N_{s-1} \sqcup V_{s-1}$ has been constructed, and assume that the set $V_{s-1}$ is nonempty. Consider the graph $G_{s-1} = G(V_{s-1})$ and choose an arbitrary vertex $v \in V_{s-1}$. Now consider the sets $U_0(v, G_{s-1}) = \{v\}, \ U_1(v, G_{s-1}), \ldots, \ U_{k-1}(v, G_{s-1}).$

One of the ratios
\[
\frac{|U_1(v, G_{s-1})|}{|U_0(v, G_{s-1})|}, \frac{|U_2(v, G_{s-1})|}{|U_1(v, G_{s-1})|}, \ldots, \frac{|U_{k-1}(v, G_{s-1})|}{|U_{k-2}(v, G_{s-1})|}
\]
does not exceed $d^{1/(k-1)}$, since the product of these ratios is
\[
|U_{k-1}(v, G_{s-1})| \leq |U_{k-1}(v, G)| \leq d.
\]
So, let us choose $1 \leq m \leq k - 1$ such that
\[
\frac{|U_m(v, G_{s-1})|}{|U_{m-1}(v, G_{s-1})|} \leq d^{1/(k-1)}.
\]

Now we set
\[
U_s = U_{m-1}(v, G_{s-1}), \quad N_s = N_{s-1} \cup S_m(v, G_{s-1}), \quad V_s = V_{s-1} \setminus U_m(v, G_{s-1}).
\]
Since the condition (i) was satisfied on the previous step, we have
\[
\partial_G^\text{out} V_s \subseteq \partial_G^\text{out} V_{s-1} \cup S_m(v, G_{s-1}) \subseteq N_s
\]
and
\[
\partial_G^\text{out} U_s \subseteq \partial_G^\text{out} V_{s-1} \cup S_m(v, G_{s-1}) \subseteq N_s,
\]
so this condition also holds now. The condition (ii) is satisfied by Proposition 1. Finally, the choice of $m$ and the condition (iii) for the previous step imply that
\[
d^{1/(k-1)}|U_s| = d^{1/(k-1)}|U_{m-1}(v, G_{s-1})| \geq |U_m(v, G_{s-1})|,
\]
\[
(d^{1/(k-1)} - 1)(|U_1| + \cdots + |U_{s-1}|) \geq |N_{s-1}|
\]
and hence
\[(d^{1/(k-1)}-1)(|U_1|+\cdots+|U_s|) \geq |N_{s-1}|+|U_m(v, G_{s-1})|-|U_{m-1}(v, G_{s-1})| = |N_s|.
\]
Thus, the condition (iii) also holds on this step.

Continuing the construction in this manner, we will eventually come to the partition with \(V_s = \emptyset\) since the value of \(|V_s|\) strictly decreases. As the result, we obtain the partition \(V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_s \sqcup N_s\) such that \(|N_s| \leq (d^{1/(k-1)}-1)(|U_1|+\cdots+|U_s|)\). So,
\[
d^{1/(k-1)}|N_s| \leq (d^{1/(k-1)}-1)(|U_1|+\cdots+|U_s|)+(d^{1/(k-1)}-1)|N_s| = |V|(d^{1/(k-1)}-1),
\]
or \(|N_s| \leq |V|d^{1/(k-1)}-1\).

Assume now that \(|N_s| \leq f(n-2, k)\); then one may color the vertices of \(G(N_s)\) in \(n-2\) colors, and then color the vertices of each bipartite graph \(G(U_i)\) in two remaining colors. This coloring might be not proper only if some vertices of two subgraphs \(G(U_i)\) and \(G(U_j)\) (\(i \neq j\)) are adjacent, which is impossible by the condition (i). So, \(G\) is \(n\)-colorable which is wrong. Therefore, \(|N_s| \geq f(n-2, k)+1\) and hence \(|V| \geq \frac{d^{1/(k-1)}}{d^{1/(k-1)}-1}|N_s| \geq \frac{d^{1/(k-1)}-1}{d^{1/(k-1)}-1}(f(n-2, k)+1)\), as desired. \(\square\)

1. In the statement of the Lemma above, one may use the number \(d' = \max_{\emptyset \neq V' \subseteq V} \min_{u \in V'} |U_{k-1}(u, G(V'))|\) instead of \(d\). For reaching that, on each step it is sufficient to choose the vertex \(v \in V_{s-1}\) such that
\[
|U_{k-1}(v, G_{s-1})| = \min_{u \in V_{s-1}} |U_{k-1}(u, G_{s-1})|.
\]
Clearly, we have \(d' \leq d\).

2. On the other hand, the number \(d^{1/(k-1)}\) in the same statement can be replaced by \((f(n, k)+1)^{1/k}\). Now, in the proof one may deal with \(k+1\) sets
\[
U_0(v, G_{s-1}) = \{v\}, \quad U_1(v, G_{s-1}), \quad \ldots, \quad U_k(v, G_{s-1})
\]
and use the condition \(|U_k(v, G_{s-1})| \leq |V| = f(n, k)+1\).

The next theorem follows immediately from the Lemmas 2 and 3.

**Theorem 1.** For all integer \(n, k \geq 2\), we have
\[
f(n, k) \geq \min_{t \geq n(k-1)+1} \max \left\{ f(n-1, k) + t, \frac{t^{1/(k-1)}}{t^{1/(k-1)}-1}(f(n-2, k)+1) - 1 \right\}.
\]
Proof. From the choice of \( G \) we have \( f(n,k) = |G| - 1 \). From Lemmas 2 and 3 it follows that

\[
|G| \geq \max \left\{ f(n-1,k) + d, \frac{d^{1/(k-1)}}{d^{1/(k-1)} - 1}(f(n-2,k) + 1) - 1 \right\} + 1.
\]

Since \( d \geq n(k - 1) + 1 \) by Lemma 1, the statement holds.

Corollary 1. For every real \( g > 1 \), we have

\[
f(n,k) \geq \min \left\{ f(n-1,k) + g, \frac{g^{1/(k-1)}}{g^{1/(k-1)} - 1}(f(n-2,k) + 1) - 1 \right\}.
\]

(4)

Proof. Let \( t_0 \) be the integer for which the minimum in (3) is achieved. As \( t > 1 \) increases, the value of \( f(n-1,k) + t \) also increases, while the value of \( \frac{t^{1/(k-1)}}{t^{1/(k-1)} - 1}(f(n-2,k) + 1) - 1 \) decreases. Thus, if \( g \leq t_0 \), then we have

\[
f(n-1,k) + g \leq f(n-1,k) + t_0 \leq f(n,k).
\]

Otherwise, we have \( g > t_0 \) and

\[
\frac{g^{1/(k-1)}}{g^{1/(k-1)} - 1}(f(n-2,k) + 1) - 1 \leq \frac{t_0^{1/(k-1)}}{t_0^{1/(k-1)} - 1}(f(n-2,k) + 1) - 1 \leq f(n,k).
\]

3 Explicit bounds

Now we present the explicit lower bounds for \( f(n,k) \) following from the results of the previous section.

Notice that for every \( k \) we have \( f(1,k) = 1 \) and \( f(2,k) = 2k \). Lemma 2 implies now the following statement.

Theorem 2. For all integer \( n \geq 1 \) and \( k \geq 2 \) the inequality \( f(n,k) \geq n + \frac{(k-1)(n-1)(n+2)}{2} \) holds.

Proof. Induction on \( n \). In the base cases \( n = 1 \) or \( n = 2 \) the statement holds. Assume now that \( n > 2 \). By Lemmas 1 and 2 we have \( f(n,k) \geq f(n-1,k) + n(k-1) + 1 \). Next, the hypothesis of the induction implies that

\[
f(n-1,k) \geq (n-1) + \frac{(k-1)(n-2)(n+1)}{2}.
\]
Therefore,
\[ f(n, k) \geq f(n - 1, k) + n(k - 1) + 1 \geq n + \frac{(k - 1)(n - 1)(n + 2)}{2}, \]
as desired.

The next estimate uses the whole statement of the Theorem \[\square\] For the convenience, we use the notation \(n^k = n(n + 1) \ldots (n + k - 1)\).

**Lemma 4.** Suppose that for some integer \(n_0 \geq 1\), integer \(k \geq 2\), and real \(a\), the inequality
\[ f(m, k) \geq \frac{(m + a)^k}{2^{k-1}k^k} \]
holds for two values \(m = n_0\) and \(m = n_0 + 1\). Then the same estimate holds for all integer \(m \geq n_0\).

**Proof.** We prove by induction on \(n \geq n_0\) that the estimate (5) holds for \(m = n\). The base cases \(m = n_0\) and \(m = n_0 + 1\) follow from the theorem assumptions.

For the induction step, suppose that \(n \geq n_0 + 2\). Let \(c = 2^{1-k}k^{-k}\), \(g = ck(n + a)^{k-1}\). By the induction hypothesis, we have
\[ f(n-1, k) + g \geq c(n+a-1)^k + ck(n+a)^{k-1} = c(n+a)^{k-1}(n+a-1+k) = c(n+a)^k. \]
Notice that Lemmas 1 and 2 imply that \(f(n, k) \geq f(n-1, k) + n(k - 1) + 1\). Hence, if \(g \leq n(k - 1) + 1\), then \(f(n, k) \geq f(n-1, k) + g \geq c(n+a)^k\), as desired.

Thus we may deal only with the case \(g > n(k - 1) + 1\); in particular, \(g > 1\). We intend to use Corollary 1 for this, let us estimate the second term in the right-hand part of (4).

From the AM–GM inequality we have
\[ g^{1/(k-1)} = (ck)^{1/(k-1)} \left( (n + a)^{k-1} \right)^{1/(k-1)} \leq \frac{1}{2k} \left( n + a + \frac{k}{2} - 1 \right). \]
Let \(s = n + a + \frac{k}{2} - 1\); then \(s \geq 2kg^{1/(k-1)} > 2k\). Therefore,
\[
\frac{g^{1/(k-1)}}{g^{1/(k-1)} - 1} \geq \frac{s}{s - 2k} \geq \frac{s + k - 1}{s - (k + 1)} \geq \frac{s^2 + s(k - 1) + \frac{k(k-2)}{4}}{s^2 - s(k + 1) + \frac{k(k+2)}{4}} = \frac{(n + a + k - 2)(n + a + k - 1)}{(n + a - 2)(n + a - 1)}.
\]
Finally, from the induction hypothesis we get
\[
\frac{g^{1/(k-1)}}{g^{1/(k-1)} - 1} (f(n - 2, k) + 1) - 1 \geq \frac{g^{1/(k-1)}}{g^{1/(k-1)} - 1} f(n - 2, k) \geq \\
\geq \frac{(n + a + k - 2)(n + a + k - 1)}{(n + a - 2)(n + a - 1)} \cdot c(n + a - 2)^k = c(n + a)^k. \quad (7)
\]

Thus, for the value of \( g \) chosen above, Corollary 1 and the estimates (6) and (7) provide that
\[
f(n, k) \geq \min \left\{ f(n - 1, k) + g, \frac{g^{1/(k-1)}}{g^{1/(k-1)} - 1} (f(n - 2, k) + 1) - 1 \right\} \geq c(n+a)^k,
\]
as desired.

Finally, let us show that the constant \( a \) in the previous Lemma can be chosen relatively large.

**Theorem 3.** For all \( k \geq 2 \) and \( n \geq 2 \), we have \( f(n, k) \geq \frac{(n+k)^k}{2^{k-1}k^k} \).

**Proof.** Set \( a = k \). Let us check the inequality (5) for \( n = 2 \) and \( n = 3 \). Recall that \( f(2, k) = 2k \). Now for \( m = 2 \) we get
\[
2^{k-1}k^k f(2, k) = 2^k k^{k+1} = (2k)^{k-1} \cdot 2k \geq (k+2)(k+3) \ldots 2k \cdot (2k+1) = (k+2)^k.
\]

For \( m = 3 \), Theorem 2 yields \( f(3, k) \geq 5k - 2 \), and the previous estimate now implies that
\[
2^{k-1}k^k f(3, k) \geq 2^{k-1}k^k (5k - 2) \geq 2^k k^k f(2, k) \geq 2(k+2)^k > (k+3)^k.
\]

Thus, the inequality (5) holds for \( m = 2 \) and \( n = 3 \), and hence for all \( n \geq 2 \) by Lemma 4.

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