Entanglement control in coupled two-mode boson systems

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A two-mode boson model, widely used for the physics of fast rotating nuclei and Bose-Einstein condensates, is studied in the context of entanglement control. We derive an analytical expression for the entanglement between the fields in this model as a function of time. We found that depending on the interaction strengths between boson modes and the nature of the initial boson states the dynamical evolution of the entanglement and the squeezing can occur independently.

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There is currently an enormous effort underway to understand the rich dynamics of interacting Bose systems. Among various problems are the properties of Bose-Einstein condensate (BEC) in external fields \cite{1}, the Bose coherent effects of excitons and polaritons in semiconductor microcavities \cite{2}, and the production of scattered radiation due to interaction of the incident laser beam with vibrational modes of a medium (the Raman effect). The generic feature of interacting Bose systems is the formation of Bose-field collective states with nonclassical (squeezed, sub-Poissonian, etc) statistical and fluctuation properties. Another prominent feature is an entanglement produced by the interaction between different system constituents, which is one of the most subtle and intriguing phenomena in nature (see for a recent review \cite{3, 4}). Nowadays, there is explosive activity in the study of the entanglement due to its potential usefulness in quantum teleportation, quantum cryptography, and, in general, in quantum information theory.

The dynamical interplay between quantum entanglement and nonclassical properties of various Bose systems can be traced within a model of two-coupled harmonic oscillators. A particular example of interest is the Hamiltonian

\begin{equation}
\hat{H} = \hbar \omega_1 \hat{c}_1^{\dagger} \hat{c}_1 + \hbar \omega_2 \hat{c}_2^{\dagger} \hat{c}_2 + i \hbar g_1 \left( \hat{c}_1^{\dagger} \hat{c}_2 - \hat{c}_2^{\dagger} \hat{c}_1 \right) - i \hbar g_2 \left( \hat{c}_1^{\dagger} \hat{c}_2^{\dagger} - \hat{c}_2 \hat{c}_1 \right). \quad (1)
\end{equation}

Here $\hat{c}_{1, 2}$ ($\hat{c}_{1, 2}^{\dagger}$) are the annihilation (creation) boson operators of the fields with the energies $\hbar \omega_{1, 2}$. The coupling between these fields is governed by dint of the coupling constants $g_{1, 2}$. Note that the bilinear form of the Hamiltonian \cite{1} corresponds to a linearized version of some more general interactions.

This model has been applied in nuclear physics \cite{5, 6} and for a rotating BEC (cf. Ref. \cite{7}). In condensed matter physics the Hamiltonian \cite{1} is used: i) to study the interaction between an atom and a radiative field \cite{8}; ii) as a starting point for analysis of electronic properties of two-dimensional quantum dots in a perpendicular magnetic field \cite{9, 10}. The dynamics of two-component Bose condensate trapped in a double-well potential can be also mapped on the time-evolution of two coupled-harmonic oscillators in the low excitation regime \cite{11}. In the simplest case, the model describes two levels of the condensed atoms which are coupled owing to the classical field of radiation \cite{12}. In this case the interaction constants ($g_1$ and $g_2$) in the Hamiltonian \cite{1} simulate the coupling between the electromagnetic field and the two-level (two-mode) boson system. Varying the interaction (the coupling) and the choice of initial states one can analyze various properties of this system.

The bilinear form of boson interaction enables one to investigate the system dynamics analytically. In particular, the exact solutions as well as the generation of squeezed states in this model have been analyzed in \cite{13}. The main aim of the present paper is to gain a better insight into the dynamical interplay between the strength of the interaction, entanglement and squeezing dynamics of the system. Here, we focus upon the dynamics of Gaussian states, which are of great practical importance.

By means of the Bogoliubov transformation

\begin{equation}
\hat{a}_k = \sum_{m=1}^{2} \left( A^k_m \hat{c}_m + B^k_m \hat{c}_m^{\dagger} \right), \quad (2)
\end{equation}

the Hamiltonian \cite{1} is reduced to the diagonal form \( \hat{H} = \sum_{k=1}^{2} \hbar \Omega_k \left( \hat{a}_k^{\dagger} \hat{a}_k + 1/2 \right) \) with the two eigenmodes

\begin{equation}
\Omega_k^2 = \left[ \eta_+ + 2(g_1^2 - g_2^2) + (-1)^{k+1} \Delta \right]/2, \quad (3)
\end{equation}

\begin{equation}
\Delta = \left[ \eta_+^2 - 4 \eta_+ (g_1^2 - g_2^2) + 8 \omega_1 \omega_2 (g_1^2 + g_2^2) \right]^{1/2}, \quad (4)
\end{equation}

where $\eta_{\pm} = \omega_1^2 \pm \omega_2^2$. The coefficients $A^k_m$ and $B^k_m$ of the transformation matrices in Eq. (2) are defined in Ref. \cite{13}.

By virtue of the inverse transformation and time evolution of the collective operators $\hat{a}_k(t) = \hat{a}_k(0)e^{-i \Omega_k t}$, one can determine the time dependence of the initial operators $\hat{c}_m$

\begin{equation}
\hat{c}_m(t) = \sum_{n=1}^{2} \left[ \alpha_{mn}(t) \hat{c}_n(0) + \beta_{mn}(t) \hat{c}_n^{\dagger}(0) \right], \quad (5)
\end{equation}

where

\begin{equation}
\alpha_{mn}(t) = \sum_{k=1}^{2} \left[ A^k_m A^k_n e^{-i \Omega_k t} - B^k_m B^k_n e^{i \Omega_k t} \right], \quad (6)
\end{equation}

\begin{equation}
\beta_{mn}(t) = \sum_{k=1}^{2} \left[ A^k_m B^k_n e^{-i \Omega_k t} - B^k_m A^k_n e^{i \Omega_k t} \right]. \quad (7)
\end{equation}
The matrix elements $\alpha_{mn}(t), \beta_{mn}(t)$ obey the relation
\begin{equation}
\sum_{n=1}^{2} \left| \alpha_{mn}(t) \right|^2 - \left| \beta_{mn}(t) \right|^2 = 1.
\end{equation}

In the most general case it is natural to expect that at the initial stage of the time evolution the fields can be found in the superposition of coherent and chaotic states. For example, the Bose condensate of “cold” atoms can be described by a coherent state. However, there is always a nonzero temperature which spoils the plain coherence and brings some chaoticity (the decoherence effects). Similar phenomenon occurs at the interaction of the laser beam with vibrational modes in media, which are characterized by some thermal distribution. Therefore, we assume that the initial density matrix can be presented in the following factorized form
\begin{equation}
\hat{\rho}(0) = \prod_{j=1,2} \frac{(n_j)\hat{b}_j^\dagger \hat{b}_j}{\langle n_j \rangle \hat{b}_j^\dagger + 1}.
\end{equation}

In Eq. (7) we introduced new operators $\hat{b}_1, \hat{b}_2 = \hat{c}_1, \alpha_{1,2} - \hat{c}_1, \alpha_{1,2}$, where $\alpha_{1,2}$ are initial coherent amplitudes of the fields $\hat{c}_1, \hat{c}_1$ and the averages $\langle n_j \rangle$ are associated with mean numbers of the bosons in the corresponding chaotic states.

To trace the time evolution of the quantum state of the fields $\hat{c}_1$ and $\hat{c}_1$ governed by the Hamiltonian $\hat{H}$, it is convenient to introduce the Wigner function, instead of the density matrix. It can be done by dint of the symmetric characteristic function
\begin{equation}
\chi(\mu_1, \mu_2; t) = \text{Tr} \left\{ \hat{\rho}(0) \exp \left[ \sum_{m=1}^{2} (\mu_m \hat{c}_m^\dagger(t) - \mu_m^* \hat{c}_m(t)) \right] \right\}.
\end{equation}

It is straightforward to show with aid of Eq. (8) that the Wigner function obeys the relation
\begin{equation}
W(\nu_1, \nu_2; t) = \frac{1}{\pi} \int d^2 \mu_1 d^2 \mu_2 \chi(\mu_1, \mu_2; t) \exp D
= \frac{1}{\pi} \int d^2 \mu_1 d^2 \mu_2 \chi(t, \mu_1, \mu_2; 0) \exp D_1
= W(\nu_1(t), \nu_2(t); 0).
\end{equation}

Here $D = \sum_{m=1}^{2} (\mu_m \mu_m^* - \nu_m \nu_m^*), \quad D_1 = \sum_{m=1}^{2} (\mu_m^* \nu_m - \mu_m \nu_m^*), \quad D_2 = \sum_{m=1}^{2} (\alpha_m \xi_m - \beta_m \xi_m^*).$

The Gaussian character of the system state is well suited to introduce the entanglement measure within the approach proposed in Ref. The measure of entanglement between the fields can be calculated analytically in the form of the logarithmic negativity via the symplectic spectrum of the partial transpose of the covariance matrix. To proceed along this line we present the bilinear in fields part of the Wigner function by dint of the relation $F_1 = \hat{F}^\dagger \hat{V}^{-1} \hat{F}$. Here, the inverse variance matrix of the form
\begin{equation}
\hat{V}^{-1} = \begin{pmatrix}
X_1 & Y
\end{pmatrix}
\begin{pmatrix}
X_1 & Y
\end{pmatrix}^T
\end{equation}
is determined by the matrices

\[
\begin{align*}
X_i &= \begin{pmatrix} A_i - 2\text{Re}B_i & -2\text{Im}B_i \\ -2\text{Im}B_i & A_i + 2\text{Re}B_i \end{pmatrix}, \quad i = 1, 2, \quad (16) \\
Y &= \begin{pmatrix} \text{Re}(B_1 - B_3) & -\text{Im}(B_1 + B_3) \\ -\text{Im}(B_3 - B_4) & \text{Re}(B_4 + B_3) \end{pmatrix}. \quad (17)
\end{align*}
\]

And \( \zeta^T = (g_1, p_1, q_2, p_2) \) is a transposed four-vector whose elements are the quadrature-component variables defined as \( c_i = (q_i + i p_i)/\sqrt{2} \).

As a result, the logarithmic negativity

\[ E = -\frac{1}{2} \log_2(\mathcal{A}), \quad (18) \]

where

\[
\begin{align*}
\mathcal{A} &= \mathcal{B} - \sqrt{\mathcal{B}^2 - \det \mathbf{V}}, \\
\mathcal{B} &= \frac{\det \mathbf{X}_1 + \det \mathbf{X}_2}{2} - \det \mathbf{Y}, \quad (19, 20)
\end{align*}
\]
determines the strength of the entanglement for \( E > 0 \). Here, \( \mathbf{X}_1 \) and \( \mathbf{Y} \) are diagonal and nondiagonal block matrices of the variance matrix \( \mathbf{V} \), respectively. For \( E \leq 0 \) the composite state is separable according to the Peres-Horodecki criterion \[13\] (see also Refs.\[16\]). This measure enables us to trace the evolution of non-classical correlations in the system and the conditions for their amplification.

The considered model provides various possibilities to study cumulative effects produced by a different strength of coupling between the fields prepared in the initial states of different degree of chaoticity (different temperatures). In the present paper we discuss a few interesting cases of the dynamical control of the entanglement by dint of different choices of the initial states and coupling constants. For illustration we consider the case \( \omega_1 = 1, \omega_2 = 2 \) (in relative units). Note, that at a given energy \( \hbar \omega_i \) of the \( \hat{c}_i \)-field initial state, the temperature \( T_i \) fixes the mean number of bosons \( \langle n_i \rangle \), i.e., \( \langle n_i \rangle = \langle \exp \left[ \hbar \omega_i/(k_B T_i) \right] - 1 \rangle^{-1} \). All frequencies and coupling constants are determined in units of the frequency \( \omega_1 \) and the time is scaled by inverted \( \omega_1 \).

At zero temperature, when both the fields are in coherent (vacuum) initial states, the interaction with equal coupling constants \( g_1 = g_2 \) produces the entangled system state (see Fig. 1a). However, the degree of entanglement is not monotonic with a time. It displays an oscillatory character with the distinctive alteration of the entanglement maxima and minima that can be considered to be revivals and collapses. In this regime the fields \( \hat{c}_1 \) and \( \hat{c}_2 \) were shown to be evolved into squeezed states \[13\] that is a consequence of such quantum correlations.

In case the system is associated with the interaction of a light with ion oscillations in a crystal (optical phonons), a suppression of zero-point optical vibrations (squeezing) may affect the electron-phonon interaction in nanostructures. We speculate that the nonmonotonic behavior of the entanglement and the squeezing results in the interaction strength which would fluctuate in time. Note that this interaction being essential to the understanding of electron-spin decoherence effects in quantum systems is assumed to be independent on time (cf \[17\]).

FIG. 1: Entanglement Eq. \[13\] between the modes being initially in coherent states. Model parameters are \( \omega_1 = 1, \omega_2 = 2 \) and (a) \( g_1 = g_2 = 0.5 \), (b) \( g_1 = 0, g_2 = 0.5 \).

The entanglement also occurs for the interaction regime \( g_1 = 0 \) and \( g_2 \neq 0 \) \( (k_B T = 0) \). In this case the interaction produces the picket-fence effect for the entanglement of equal maxima without revivals (see Fig. 1b) but with the periodical alteration of its minima. How-
ever, in this regime the fields $\hat{c}_1$ and $\hat{c}_2$ remain in coherent states, i.e., the squeezing does not take place \cite{13}. It appears that the interaction that creates (annihilates) the boson pair (associated with the strength $g_2$) is enough to produce the entanglement. On the other hand, both interactions (associated with the strengths $g_1$ and $g_2$) are indispensable in the Hamiltonian \cite{11} in order to produce the squeezing of the coherent initial states. The interaction of the $g_1 \neq 0$ and $g_2 = 0$ type does not yield an entangled state for the system at all, as well as the single-mode squeezing for the coherent states.

Thermal fluctuations are expected to attenuate the entanglement. In particular, when both the fields are initially in chaotic states, the degree of their entanglement in the regime of equal coupling constants becomes very small (see Fig. 2a). Moreover, there is a critical temperature above which the entanglement disappears and the system becomes separable (for equal coupling constants in our choice of the initial energies $\omega_{1,2}$ it corresponds to $\langle n_{1,2} \rangle \approx 0.63$). However, if one of the initial states is found in a coherent state while the other is in a chaotic one, the entanglement evolves at regular intervals with a periodical alteration of maxima (see Fig. 2b). With a proper combination of the initial energies of the system and a choice of number of bosons in the chaotic (initial) state, one may maintain the entanglement at a noticeable level, even at high temperatures.

In conclusion, the considered two-mode boson model demonstrates that the entanglement can be controlled in time by appropriate choice of the interaction at different temperature regimes. The obtained results show that this plain system exhibits a rich dynamics from the viewpoint of the quantum information theory and its possible physical applications.

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