REMARKS ON TWO CONSTANTS OF POSITIVE CONFORMAL CLASSES OF RIEMANNIAN METRICS

MOHAMMED LABBI

Abstract. For a given Riemannian metric of positive scalar curvature on a compact $n$-manifold, we define two constants $Ein([g])$ and $ein([g])$ of the conformal class of $g$. Roughly speaking, the constant $Ein([g]) \in (0, n]$ measures how far away is the class $[g]$ from containing an Einstein metric of positive scalar curvature. The constant $ein([g]) \in (-\infty, 0)$ measures how far away is the class $[g]$ to possess a metric of nonnegative Ricci curvature and positive scalar curvature. We prove a vanishing theorem of the Betti numbers of a conformally flat manifold by assuming a lower bound on the first constant or an upper bound on the second one. As a consequence we were able to determine these constants for reducible conformally flat manifolds. In four dimensions we prove that if a manifold admits a positive conformally flat class then it admits a conformally flat class $[g]$ with $Ein([g]) \geq 3$ and $ein([g]) = -\infty$. We prove similar results for non-conformally flat classes but with an additional conformal condition.

1. Introduction

Let $(M, g)$ be a Riemannian manifold of dimension $n$. We denote as usual by $Ric$ and $Scal$ the Ricci and scalar curvatures. Recall that the $k$-th modified Einstein tensor on $M$ is defined by

$Ein_k(g) := Scal - kRic,$

Where $k < n$ is a constant. We define the following metric invariants

$Ein(g) := \sup\{k \in (0, n) : Ein_k(g) > 0\}$ and $ein(g) := \inf\{k < 0 : Ein_k(g) > 0\}.$

Where by $Ein_k > 0$, we mean that it is positive definite at each point of the manifold $M$. We set $Ein(g) = ein(g) = 0$ if the scalar curvature of $g$ is not positive and $Ein(g) = -\infty$ in case the corresponding set of $k$’s is unbounded below.

Recall that $Ein(g) = n$ if and only if the metric $g$ is Einstein with positive scalar curvature. Furthermore, $ein(g) = -\infty$ if and only if $g$ has nonnegative Ricci curvature and positive scalar curvature, see [3].

Let $\mathcal{M}$ denotes the space of all Riemannian metrics on $M$. We define two smooth invariants $Ein(M)$ and $ein(M)$ of $M$ as follows

$Ein(M) = \sup\{Ein(g) : g \in \mathcal{M}\}$ and $ein(M) = \inf\{ein(g) : g \in \mathcal{M}\}.$
The above invariants were studied in [9]. In this paper we restrict our study of the above constants to a given conformal class. We start by a definition

Definition 1.1. Let \([g]\) denotes the conformal class of the metric \(g\), we define the following two constants of \([g]\)

\[
\text{Ein}([g]) = \sup \{ \text{Ein}(g) : g \in [g] \} \quad \text{and} \quad \text{ein}([g]) = \inf \{ \text{ein}(g) : g \in [g] \}.
\]

We remark first that if the conformal class of \(g\) contains an Einstein metric with positive scalar curvature then \(\text{Ein}([g]) = n\). Similarly, if the conformal class of \(g\) contains a metric with non-negative Ricci curvature and with positive scalar curvature then \(\text{ein}([g]) = -\infty\).

1.1. Statement of the main results. The first result is a vanishing theorem

**Theorem A.** Let \((M, g)\) be a compact oriented conformally flat \(n\)-manifold and \(p\) an integer such that \(1 < p < n\). Let

\[
k_1 = \frac{(n-1)(2p-n)}{p-1} \quad \text{and} \quad k_2 = \frac{(n-1)(n-2p)}{n-p-1}.
\]

1. Suppose \(n > 2p\). If \(\text{Ein}([g]) > k_2\) or \(\text{ein}([g]) < k_1\) then the betti numbers \(b_p\) and \(b_{n-p}\) of \(M\) vanish.

2. Suppose \(n < 2p\). If \(\text{Ein}([g]) > k_1\) or \(\text{ein}([g]) < k_2\) then the betti numbers \(b_p\) and \(b_{n-p}\) of \(M\) vanish.

As a consequence we are able to compute the \(\text{Ein}([g])\) and \(\text{ein}([g])\) constants for the product of two space forms of opposite signs as follows

**Corollary A.** Let \(n, d\) be positive integers such that \(d < \frac{n-2}{2}\). Let \((S^{n-d-1}, g_0)\) be the standard sphere of curvature \(+1\) and let \((M^{d+1}, g_1)\) be a compact space form of curvature \(-1\). We denote by \(g\) the Riemannian product of \(g_0\) and \(g_1\) on \(S^{n-d-1} \times M^{d+1}\). Then

\[
\text{Ein}([g]) = \text{Ein}(g) = (n-1) \frac{2d-n+2}{d-n+2} \quad \text{and} \quad \text{ein}([g]) = \text{ein}(g) = -(n-1) \frac{n-2d-2}{d}.
\]

In four dimensions, the following theorem shows that one can upgrade positive scalar curvature conformally flat metric to a metric of positive Ein\(k\) curvature for any \(k \in (-\infty, 3)\).

**Theorem B.** Let \((M, g_0)\) be a compact conformally flat 4-manifold with positive scalar curvature then \(M\) admits a conformally flat metric \(g_1\) such that

\[
\text{Ein}([g_1]) \geq 3 \quad \text{and} \quad \text{ein}([g_1]) = -\infty.
\]

In particular, we have \(\text{Ein}(M) \geq 3\) and \(\text{ein}(M) = -\infty\).

In the general case one needs extra conditions on the conformal class in order to be able to upgrade positive scalar curvature as in the following theorem

**Theorem C.** Let \((M, g)\) be a compact 4-dimensional Riemannian manifold with positive scalar curvature.
(1) If \( \int \sigma_2(A) \mu_g > 0 \) then \( \text{Ein}([g]) > 2 \) and \( \text{ein}([g]) = -\infty \).

(2) If \( \int \sigma_2(A) \mu_g = 0 \) then \( \text{Ein}([g]) \geq 2 \) and \( \text{ein}([g]) = -\infty \).

(3) If \( \int \sigma_2(A) \mu_g < 0 \) then

\[
\text{Ein}([g]) \geq \frac{4Y[g]}{\sqrt{(Y[g])^2 - 96 \int \sigma_2(A) \mu_g + Y[g]}} \quad \text{and} \quad \text{ein}([g]) \leq \frac{-4Y[g]}{\sqrt{(Y[g])^2 - 96 \int \sigma_2(A) \mu_g - Y[g]}}.
\]

Where \( Y[g] \) is the Yamabe constant of the conformal class \([g]\) and \( \sigma_2(A) \) is the \( \sigma_2 \)-curvature.

We note that the above Theorem C is merely a reformulation of a Theorem by Gursky and Viacklovsky [6] as will be shown in the last section of this paper.

2. \( \text{Ein}([g]) \) and \( \text{ein}([g]) \) constants of a conformally flat class

2.1. A vanishing theorem for manifolds with positive \( \text{Ein}_k \) curvature and applications.

In this subsection, we prove Theorem A and Corollary A stated in the introduction.

**Theorem 2.1.** Let \((M, g)\) be a compact oriented conformally flat \(n\)-manifold and \(p\) an integer such that \(1 < p < n\). Let \(k_1 = \frac{(n-1)(2p-n)}{p-1}\) and \(k_2 = \frac{(n-1)(n-2p)}{n-p-1}\).

1. Suppose \(n > 2p\). If \( \text{Ein}([g]) > k_2 \) or \( \text{ein}([g]) < k_1 \) then the betti numbers \(b_p\) and \(b_{n-p}\) of \(M\) vanish.

2. Suppose \(n < 2p\). If \( \text{Ein}([g]) > k_1 \) or \( \text{ein}([g]) < k_2 \) then the betti numbers \(b_p\) and \(b_{n-p}\) of \(M\) vanish.

**Proof.** The hypothesis of the theorem implies the existence of a metric \(g\) in the conformal class \([g]\) such that \(\text{Ein}_k(g) > 0\) where \(k\) is either \(k_1\) or \(k_2\) according to the cases. To prove the theorem we shall use the classical Weitzenböck formula for differential forms

\[
\Delta = \nabla^* \nabla + \mathcal{W},
\]

where \(\Delta\) is the Laplacian of differential forms, \(\nabla\) is the Levi-Civita connexion and \(\mathcal{W}\) is the Weitzenböck curvature term. We shall prove that under the theorem hypotheses the Weitzenböck curvature term is positive and the theorem follows from the previous formula. The curvature term \(\mathcal{W}\) once operating on \(p\)-forms take the form [11]

\[
\mathcal{W}_p = \frac{g^{p-2}}{(p-2)!} \left( \frac{g \text{Ric}}{p-1} - 2 \bar{R} \right).
\]

Where \(\bar{R}\) is the Riemann tensor and all the products are exterior products of double forms. Since the manifold is conformally flat then the Weyl tensor vanishes and we have \(\bar{R} = gA\) where

\[
A = \frac{1}{n-2} \left( \text{Ric} - \frac{\text{Scal}}{2(n-1)} g \right),
\]
is the Schouten tensor. Consequently, we can see that

\[
W_p = \frac{g^{p-2}}{(p-2)!} \left( \frac{g \text{Ric}}{p-1} - 2gA \right) = \frac{g^{p-1}}{(p-1)!} \left( \text{Ric} - (2p-2)A \right)
\]

(6)

Note that the last term \( \frac{g^{p-1}}{(p-1)!} \) (Ein\_k) is positive if and only if the sum of the lowest \( p \) eigenvalues of Ein\_k is positive. In particular, \( \frac{g^{p-1}}{(p-1)!} \) (Ein\_k) is positive if Ein\_k is positive. To complete the proof just notice that in a similar way the curvature term \( W \) once operating on \( n-p \)-forms takes the form

\[
W_{n-p} = \frac{n-p-1}{(n-1)(n-2)(n-p-1)!} g^{n-p-1} \text{(Ein} \_k).\]

\[\Box\]

Applying the previous theorem for the case \( p = 2 \) we get

**Corollary 2.2.** Let \( (M, g) \) be a compact conformally flat manifold of dimension \( n > 4 \). If \( \text{Ein}([g]) > \frac{(n-1)(n-4)}{n-3} \) or \( \text{ein}(g) < -(n-1)(n-4) \) then the betti numbers \( b_r \) of \( M \) vanish for \( 2 \leq r \leq n-2 \).

**Corollary 2.3.** Let \( p \geq 3 \), \((S^p, g_0)\) be the standard sphere of curvature \(+1\) and let \((M^{p-1}, g_1)\) be a compact space form of curvature \(-1\). We denote by \( g \) the Riemannian product of \( g_0 \) and \( g_1 \) on \( S^p \times M^{p-1} \). Then

\[
\text{Ein}([g]) = \text{Ein}(g) = 2 \quad \text{and} \quad \text{ein}([g]) = \text{ein}(g) = -\frac{2(p-1)}{p-2}.
\]

**Proof.** It is not difficult to check by a direct computation that \( \text{Ein}(g) = 2 \) and \( \text{ein}(g) = -\frac{2(p-1)}{p-2} \). Therefore \( \text{Ein}([g]) \geq 2 \) and \( \text{ein}([g]) \leq -\frac{2(p-1)}{p-2} \). Note that in our case \( n = 2p-1 < 2p \) and \( k_1 = 2 \) and \( k_2 = -\frac{2(p-1)}{p-2} \). On the other hand, The metric \( g \) is conformally flat and therefore any metric in the conformal class of \( g \) is conformally flat as well. Since \( b_{(p-1)}(S^p \times M^{p-1}) = b_{(p-1)}(M^{p-1}) \neq 0 \) then by Theorem 2.1 no conformally flat metric on \( S^p \times M^{p-1} \) can have \( \text{Ein}([g]) > k_1 = 2 \) or \( \text{ein}([g]) < k_2 = -\frac{2(p-1)}{p-2} \). \[\Box\]

More generally, we have the following

**Corollary 2.4.** Let \( n, d \) be positive integers such that \( d < \frac{n-2}{2} \). Let \((S^{n-d-1}, g_0)\) be the standard sphere of curvature \(+1\) and let \((M^{d+1}, g_1)\) be a compact space form of curvature \(-1\). We denote by \( g \) the Riemannian product of \( g_0 \) and \( g_1 \) on \( S^{n-d-1} \times M^{d+1} \). Then

\[
\text{Ein}([g]) = \text{Ein}(g) = (n-1) \frac{2d-n+2}{d-n+2} \quad \text{and} \quad \text{ein}([g]) = \text{ein}(g) = -(n-1) \frac{n-2d-2}{d}.
\]
Proof. The proof is completely similar to the above proof. A straightforward computation shows that for the standard product metric \( g \) one has \( \text{Ein}(g) = (n-1)\frac{2d-n+2}{d-n+2} \) and \( \text{ein}(g) = -(n-1)\frac{n-2d-2}{d} \). Therefore \( \text{Ein}([g]) \geq (n-1)\frac{2d-n+2}{d-n+2} \) and \( \text{ein}([g]) \leq -(n-1)\frac{n-2d-2}{d} \).

Next we use the notations of the theorem, let \( p = n - d - 1 \) then \( 2p - n = n - 2d - 2 > 0 \) and \( k_1 = \frac{(n-1)(2p-n)}{p+1} = (n-1)\frac{2d-n+2}{d-n+2} \) and \( k_2 = \frac{(n-1)(n-2p)}{n-p-1} = -(n-1)\frac{n-2d-2}{d} \).

The metric \( g \) being conformally flat then any metric in the conformal class of \( g \) is conformally flat as well. Now since \( b_{(d+1)}(S^{n-d-1} \times M^{d+1}) = b_{(d+1)}(M^{d+1}) \neq 0 \) then by Theorem 2.7 no conformally flat metric on \( S^{n-d-1} \times M^{d+1} \) can have \( \text{Ein}([g]) > k_1 \) or \( \text{ein}([g]) < k_2 \). This completes the proof. \( \square \)

2.2. Nayatani metric and the \( \text{Ein}([g]) \) constant of a conformally flat class. In this section, let \((M, g)\) be a smooth compact connected locally conformally flat Riemannian \( n \)-manifold with positive scalar curvature of dimension \( n \geq 3 \), we denote by \( M \) its universal cover. Schoen and Yau [14], see also [8], proved then that the developing map \( \Phi : \tilde{M} \to S^n \) is a conformal embedding, \( \pi_1(M) \) is isomorphic to a discrete subgroup \( \Gamma \) of \( \text{Conf}(M) \), \( \Phi(M) \) is a domain \( \Omega \) in \( S^n \), it coincides with the complement \( \Omega(\Gamma) \) of the limit set \( \Lambda(\Gamma) \) of the action of \( \pi_1(M) \approx \Gamma \) on the sphere \( S^n \). In other words, \((M, g)\) is conformally equivalent to the Kleinian manifold \( \Omega/\Gamma \).

Let \( \delta := \delta(\Gamma) \) denotes the critical exponent of the Kleinian group \( \Gamma \), see [14]. It turns out that \( \delta \) depends only on the conformal class of \( g \), precisely \( \delta \) coincides with the Schoen-Yau conformal invariant \( d(M, [g]) \) of the conformal class \([g]\) of \( g \). In addition, if the Kleinian group \( \Gamma \) is not elementary, that is to say that its limit set \( \Lambda(\Gamma) \) is infinite, then \( \delta \) coincides with the Hausdorff dimension of the limit set.

We are now ready to state and prove the following

**Proposition 2.5.** Let \((M, g_0)\) be a smooth compact connected locally conformally flat Riemannian \( n \)-manifold with positive scalar curvature of dimension \( n \geq 3 \). Then

\[
\text{Ein}(\{g_0\}) \geq (n-1)\frac{2\delta - n + 2}{\delta - n + 2}.
\]

Where \( \delta \) is the conformal invariant of the class \([g_0]\) as above.

**Proof.** It follows from the above discussion that \( M = \Omega/\Gamma \) is a Kleinian manifold. We suppose that \( \delta(\Gamma) > 0 \). Nayatani [13] constructed a canonical conformally flat metric \( g \in [g_0] \) on \( M = \Omega/\Gamma \) whose Ricci curvature is given by

\[
\text{Ric} = -(n - 2)(\delta + 1)A + (n - 2 - \delta)\text{tr}_g A g,
\]

where \( A \) is a non-negative tensor whose trace \( \text{tr}_g A \) is strictly positive as we supposed the scalar curvature is positive, see [13].

Consequently, one has

\[
\text{Ein}_k(g) = (\frac{(n-1)(n-2-2\delta) - k(n-2-\delta)}{k}) \text{tr}_g A g + k(n-2)(\delta + 1)A.
\]
This is clearly positive if \( k < (n - 1) \frac{2d - n + 2}{d - n + 2} \).

In case \( \delta = 0 \), then \((M, g_0)\) admits a conformal metric \( g \) with which \( M \) is locally isometric to \( S^{n-1} \times \mathbb{R} \). Therefore, in this case one has \( \text{Ein}([g_0]) \geq \text{Ein}([g]) = n - 1 \). Finally, If \( \delta = -1 \), that is \( \pi_1(M) \) is finite, one has \( \text{Ein}([g_0]) = n \).

\[\square\]

**Example.** Let \( S^d \) be a round \( d \)-sphere in the \( n \)-sphere \( S^n \) with \( 1 \leq d \leq n - 3 \). Let \( \Omega = S^n \setminus S^d \) and \( G = \text{Conf}(\Omega) := \{ f \in \text{Conf}(S^n) : f(S^d) = S^d \} \). It turns out that there exists a \( G \)-invariant conformally flat metric, say \( g_0 \), on \( \Omega \) that makes it isometric to the standard product \( S^{n-d-1} \times H^{d+1} \), see [13].

Let now \( \Gamma \subset G \) be a Kleinian group with limit set \( \Lambda(\Gamma) = S^d \). The manifold \( M = \Omega / \Gamma \) is conformally flat and here \( \delta = \delta(\Gamma) \) coincides with the Hausdorff dimension of the limit set \( S^d \) is \( d \). The above proposition shows that

\[ \text{Ein}([g_0]) \geq (n - 1) \frac{2d - n + 2}{d - n + 2}. \]

In fact, we have even equality, see Corollary [2.3].

### 2.3. Connected sums of conformally flat manifolds of positive Ein\(_k\) curvature

The positivity of the Ein\(_k\) curvatures tensor are preserved under surgeries of certain codimensions, see [9]. In particular, we have the following special case

**Theorem 2.6 ([9]).**

1. For \( k \in (-\infty, 2) \), the connected sum of two manifolds each one of positive Ein\(_k\) curvature and with dimension \( \geq 3 \) has a metric of positive Ein\(_k\) curvature.
2. For \( k \geq 2 \), the connected sum of two manifolds each one of positive Ein\(_k\) curvature and with dimension \( > k + 1 \) has a metric of positive Ein\(_k\) curvature.

The next theorem is a refinement of the above result

**Theorem 2.7.**

1. For \( k \in (-\infty, 2) \), the connected sum of two conformally flat manifolds each one of positive Ein\(_k\) curvature and with dimension \( \geq 3 \) admits a conformally flat metric of positive Ein\(_k\) curvature.
2. For \( k \geq 2 \), the connected sum of two conformally flat manifolds each one of positive Ein\(_k\) curvature and with dimension \( > k + 1 \) admits a conformally flat metric of positive Ein\(_k\) curvature.

**Proof.** This theorem is actually a special case of a more general theorem due to Hoelzel, see Theorem 6.1 in [7]. We shall use the same notations as in [7]. Let \( C_B(\mathbb{R}^n) \) denote the vector space of algebraic curvature operators \( \Lambda^2 \mathbb{R}^n \to \Lambda^2 \mathbb{R}^n \) satisfying the first Bianchi identity and endowed with the canonical inner product. Let

\[ C_{\text{Ein}_k>0} := \{ R \in C_B(\mathbb{R}^n) : \text{Ein}_k(R) > 0 \}, \]
where for a unit vector $u$, $\text{Ein}_k(R)(u) = \text{Scal}(R)u - k\text{Ric}(u)$. Here Ric and Scal($R)$ denote respectively as usual the first Ricci contraction and the full contraction of $R$. The subset $C_{\text{Ein}_k>0}$ is clearly open, convex, star shaped with respect to the origin and it is an $O(n)$-invariant cone. Furthermore, it is easy to check that $\text{Ein}_k(S^{n-1} \times \mathbb{R}) > 0$ for $n - 1 > k$. The theorem follows then from Theorem 6.1 in [7].

□

3. $\text{Ein}([g])$ and $\text{ein}([g])$ Constants of Conformally Flat Metrics in Four Dimensions:

Proof of Theorem B

The following theorem (this is Theorem B stated in the introduction) shows that one can upgrade positive scalar curvature conformally flat metric to positive Ein_k curvature for any $k \in (-\infty, 3)$.

**Theorem 3.1.** Let $(M, g_0)$ be a compact conformally flat 4-manifold with positive scalar curvature then $M$ admits a conformally flat metric $g_1$ such that

$$\text{Ein}([g_1]) \geq 3 \quad \text{and} \quad \text{ein}([g_1]) = -\infty.$$  

In particular, we have $\text{Ein}(M) \geq 3$ and $\text{ein}(M) = -\infty$.

**Proof.** First, note that in dimension 4, a positive scalar curvature conformally flat metric has positive isotropic curvature. Next, according to the classification of compact four manifolds with positive isotropic curvature by Chen-Tang-Zhu [2], such a manifold must be either diffeomorphic to the sphere $S^4$, the real projective $\mathbb{RP}^4$, quotients of $S^3 \times \mathbb{R}$ or a connected sum of these. All the previous manifolds and their connected sums admit conformally flat metrics with positive Ein_k curvature for any $k \in (-\infty, 3)$ by Theorem [2.7]. This completes the proof. □

**Remark.** It is an open question to decide whether one can choose the metric $g_1$ in the class $[g_0]$ of the initial metric $g_0$, that is to decide whether $\text{Ein}([g_0]) \geq 3$ and $\text{ein}([g_0]) = -\infty$.

4. $\text{Ein}([g])$ and $\text{ein}([g])$ Constants in Four Dimensions

4.1. **Proof of Theorem C.** Let $(M, g)$ be a compact Riemannian 4-manifold and let $A$ denotes its Schouten tensor, we denote as usual by $\sigma_1(A)$ and $\sigma_2(A)$ respectively the trace of $A$ and second elementary symmetric function in the eigenvalues of $A$. We recall two important conformal invariants of $[g]$. The first one is the celebrated Yamabe invariant $Y[g]$ and is defined by

$$Y[g] = \inf_{\bar{g} \in [g]} \frac{1}{\text{Vol}(\bar{g})^{1/2}} \int_M \text{Scal}(\bar{g})\mu_{\bar{g}}.$$  

The second one is $\int \sigma_2(A)\mu_g$, that is the integral over $M$ of $\sigma_2(A)$.

In four dimensions, the positivity of $\sigma_1(A)$ and $\sigma_2(A)$ imply simultaneously the positivity of the Einstein tensor and the positivity of the Ricci tensor [3]. Consequently they imply $\text{Ein}([g]) > 2$ and
ein([g]) = −∞. The previous simple algebraic property was generalized in [1] and [6] to conformally invariant properties of the conformal class of g, see the main Theorem of [6] and Theorem A in [1]. As a consequence of their results we have the following

**Theorem 4.1.** Let $(M, g)$ be a compact 4-dimensional Riemannian manifold with positive scalar curvature.

1. If $\int \sigma_2(A) \mu_g > 0$ then $\text{Ein}([g]) > 2$ and $\text{ein}([g]) = −\infty$.
2. If $\int \sigma_2(A) \mu_g = 0$ then $\text{Ein}([g]) \geq 2$ and $\text{ein}([g]) = −\infty$.
3. If $\int \sigma_2(A) \mu_g < 0$ then

$$\text{Ein}([g]) \geq \frac{4Y[g]}{\sqrt{(Y[g])^2 - 96 \int \sigma_2(A) \mu_g + Y[g]}}$$

and

$$\text{ein}([g]) \leq \frac{-4Y[g]}{\sqrt{(Y[g])^2 - 96 \int \sigma_2(A) \mu_g - Y[g]}}.$$

**Proof.** The first part is a direct consequence of Corollary B to Theorem A in [1]. The remaining two parts are a consequence of the main Theorem in [6] which asserts that under the hypothesis

$$4 \int_M \sigma_2(A) \mu_g + \frac{\alpha(\alpha + 1)}{6} (Y[g])^2 > 0$$

for an arbitrary positive constant $\alpha$ the existence of a conformal metric $\bar{g} \in [g]$ with

(9)\[\text{Ein}_{\frac{\alpha}{\alpha + 1}}(\bar{g}) > 0 \text{ and } \text{Ein}_{\frac{\alpha + 1}{\alpha}}(\bar{g}) > 0.\]

So if $\int_M \sigma_2(A) = 0$ and since $\alpha > 0$ is arbitrary we immediately get the conclusions $\text{Ein}([g]) \geq 2$ and $\text{ein}([g]) = −\infty$. Finally, if $\int_M \sigma_2(A) < 0$ denote by $c$ the following positive constant

$$c = \frac{-24 \int_M \sigma_2(A) \mu_g}{(Y[g])^2}.$$

The aforementioned theorem guarantees under the condition $\alpha > \frac{1}{2} (\sqrt{4c + 1} - 1)$ the existence of a conformal metric $\bar{g} \in [g]$ that has the property [8] that is $\text{ein}([g]) < \frac{2}{\alpha}$ and $\text{Ein}([g]) > \frac{2}{\alpha + 1}$. In other words,

$$\alpha > \frac{1}{2} (\sqrt{4c + 1} - 1) \implies \alpha > \frac{-2}{\text{ein}([g])}$$

and

$$\alpha > \frac{1}{2} (\sqrt{4c + 1} - 1) \implies \alpha > \frac{2}{\text{Ein}([g]) - 1}.$$

From which we conclude that

$$\text{ein}([g]) \leq \frac{-4}{\sqrt{4c + 1} - 1} \text{ and } \text{Ein}([g]) \geq \frac{4}{\sqrt{4c + 1} + 1}.$$

This completes the proof. □
4.2. **The positivity of Paneitz operator and the condition** $\mathrm{Ein}([g]) > 1$. For a Riemannian manifold $(M, g)$ of dimension 4, the Paneitz operator $P_g$ is a fourth order generalization of the usual Laplacian $\Delta$ and is defined by

$$P_g(\phi) = \Delta^2 \phi + \frac{2}{3} \delta(\mathrm{Ein}_3) d\phi.$$ 

Where $\mathrm{Ein}_3 = \mathrm{Scal}_g - 3\mathrm{Ric}$. An important future of $P_g$ is that it is conformally invariant. Precisely, if $\bar{g} = e^{-2u}g$ then

$$P_{\bar{g}} = e^{4u}P_g.$$ 

In particular, the positivity of the Paneitz operator is a conformally invariant property and its kernel is conformally invariant as well. The following theorem is a slightly weaker form of a Theorem due to Eastwood and Singer, see Theorem 5.5 in [4] and also due to Gursky and Viacklovsky, see Proposition 6.1 in [6]. It provides a sufficient condition for the non-negativity of Paneitz operator.

**Theorem 4.2.** If a compact Riemannian 4-manifold $(M, g)$ has the property $\mathrm{Ein}([g]) > 1$ then its Paneitz operator $P_g$ is non negative. Furthermore, the kernel of $P_g$ consists only of constant functions.

Note that the condition $\mathrm{Ein}([g]) > 1$ implies by definition the existence of a metric $g_1$ in the conformal class $[g]$ such that $\mathrm{Ein}_k(g_1) > 0$ for some $k > 1$ and the theorem follows from Theorem 5.5 in [4]. Also, since $\mathrm{Ein}_k(g_1) > 0 \Rightarrow \mathrm{Ein}_1(g_1) > 0$ the theorem follows from Proposition 6.1 in [6]. For the seek of completeness we provide a proof of this theorem

**Proof.** Let $\lambda$ be an eigenvalue of $P_g$ and $u$ a corresponding eigenfunction. Using the fact that the Laplacian operator is self adjoint we get

$$\int \langle \lambda u, u \rangle \mu_g = \int \langle Pu, u \rangle \mu_g = \lambda \int u^2 \mu_g$$

$$= \int \left\{ \langle \Delta^2 u, u \rangle + \left( \frac{2}{3} \mathrm{Scal}_g - 2\mathrm{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g$$

$$= \int \left\{ (\Delta u)^2 + \left( \frac{2}{3} \mathrm{Scal}_g - 2\mathrm{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g$$

$$= \int \left\{ \frac{1}{3} (\Delta u)^2 + \frac{4}{3} (\Delta u)^2 + \left( \frac{2}{3} \mathrm{Scal}_g - 2\mathrm{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g.$$ 

The Bochner formula shows that

$$\frac{4}{3} \int (\Delta u)^2 \mu_g = \frac{4}{3} \int \{ |\mathrm{Hess} u|^2 + \mathrm{Ric} (\nabla u, \nabla u) \} \mu_g$$

Hence, we deduce that
\begin{equation}
\lambda \int u^2 \mu_g = \int \left\{ \frac{-1}{3} (\Delta u)^2 + \frac{4}{3} |\text{Hess } u|^2 + \left( \frac{2}{3} \text{Scal } g - \frac{2}{3} \text{Ric} \right) (\nabla u, \nabla u) \right\} \mu_g \\
\geq \frac{2}{3} \int \left\{ (\text{Ein}_1) (\nabla u, \nabla u) \right\} \mu_g.
\end{equation}

Where in the last step we used Newton-Maclaurin’s identity as follows

\[(\sigma_1(\text{Hess } u))^2 \geq \frac{8}{3} \sigma_2(\text{Hess } u) = 4/3 \left( (\sigma_1(\text{Hess } u))^2 - |\text{Hess } u|^2 \right) .\]

The first part of the result follows directly. If \(\lambda = 0\) then clearly \(\nabla u = 0\) and then the function \(u\) is constant. \(\square\)

Associated with the Paneitz operator is the \(Q\)-curvature defined by

\begin{equation}
Q_g = -\frac{1}{12} \Delta \text{Scal}_g + 2\sigma_2 (A_g).
\end{equation}

As a corollary of the above proposition we have

**Corollary 4.3.** If a compact Riemannian 4-manifold \((M, g)\) has the property \(\text{Ein}(\int g) > 1\) then it admits a conformal metric with constant \(Q\)-curvature.

**Proof.** First note that the previous proposition guarantees that the kernel of \(P_g\) consists of constant functions.

Next, since the condition \(\text{Ein}(\int g) > 1\) implies the positivity of the Yamabe constant of the conformal class \([g]\), Theorem B of Gursky [5] shows that \(\int Q_g \mu_g \leq 8\pi^2\) with equality if and only if the manifold is conformally equivalent to the sphere. In the case \(\int Q_g \mu_g < 8\pi^2\), a theorem of Djadli-Malchiodi [3] guarantees the existence of a metric with constant \(Q\)-curvature. \(\square\)

**Remarks.**

1. M. Lai proved in [12] an equivalent version the above theorem and corollary under the condition of 3-positive Ricci curvature. It is easy to show that a metric \(g\) has 3-positive Ricci curvature if and only if \(\text{Ein}_1 (g) > 0\).

2. For higher dimensions \(n \geq 4\), let \(k = \frac{2(n+1)}{3n-1} \). It is proved in [9] that \(\text{Ein}_k > 0 \Rightarrow \Gamma_2(A) > 0\), that is positive scalar curvature and positive \(\sigma_2\) curvature, in particular it implies the positivity of the integral of the \(Q\)-curvature.

**References**

[1] S.-Y. A. Chang, M. J. Gursky and P. C. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, Ann. of Math. (2) 155(3) (2002) 709-787.

[2] Chen B-L., Tang S-H. and Zhu X-P., Complete classification of compact four manifolds with positive isotropic curvature, J. Differential Geometry, 91 (2012), 41-80.

[3] Djadli Z. and Malchiodi A., Existence of conformal metrics with constant \(Q\)-curvature, Ann. of Math. (2) 168 (2008), no. 3, 813-858.

[4] Eastwood M. G. and Singer M., The Frohlicher spectral sequence on a twistor space, Journal of Differential Geometry, 38 (1993) 653-669.
[5] Gursky M., The Principal Eigenvalue of a Conformally Invariant Differential Operator, with an Application to Semilinear Elliptic PDE, Commun. Math. Phys. 207, 131-143 (1999).
[6] Gursky M., Viacklovsky J. A., A fully nonlinear equation on four-manifolds with positive scalar curvature, J. differential geometry 63 (2003) 131-154
[7] Hoelzel S., Surgery stable curvature conditions, Math. Ann. 365, 13-47 (2016).
[8] Izeki H., Limit sets of Kleinian groups and conformally flat Riemannian manifolds, Invent. math., 122, 603-625 (1995).
[9] Labbi M., On modified Einstein tensors and two smooth invariants of compact manifolds, arXiv preprint.
[10] M. L. Labbi, Sur les nombres de Betti des variétés conformément plates, Comptes rendus de l’Académie des sciences. Série 1, Mathématique 319 (1), 77-80 (1994).
[11] Labbi M., On Weitzenbock curvature operators, Mathematische Nachrichten 288 (4), 402-411 (2015).
[12] Lai M., A remark on the nonnegativity of the Paneitz operator, Proc. Amer. Math. Soc. 143 (2015), 4893-4900.
[13] Nayatani S., Patterson-Sullivan measure and conformally flat metrics, Math. Z. 225, 115-131 (1997).
[14] Schoen R. and Yau S. T., Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. math. 92, 47-71 (1988)

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, UNIVERSITY OF BAHRAIN, 32038, BAHRAIN.

E-mail address: mlabbi@uob.edu.bh