Entropy solutions of an ultra-parabolic equation with the one-sided Dirac delta function as the minor term

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Abstract. The Cauchy-Dirichlet problem for the genuinely nonlinear ultra-parabolic equation with the piece-wise smooth minor term is considered. The minor term depends on a small positive parameter and collapses to the one-sided Dirac delta function as this parameter tends to zero. As the result, we arrive at the limiting initial-boundary value problem for the impulsive ultra-parabolic equation. The peculiarity is that the standard entropy solution of the problem for the impulsive equation generally is not unique. In this report, we propose a rule for selecting the ‘proper’ entropy solution, relying on the limiting procedure in the original problem incorporating the smooth minor term.

1. Introduction. On the mechanical genesis of impulsive conditions
In the present work, we continue a study of impulsive ultra-parabolic equations, initiated in [1] and further developed in [2]. More certainly, we extend the method presented in [2] to a broader case of impulsive sources. In view of possible applications, this method can be applied to the mechanical problems involving various phase transitions in micro-pores (for example, ‘liquid-liquid’ and ‘liquid-vapour’ transitions), since these transitions lead to fluctuations of concentration of components. These fluctuations must be taken into account at the microscopic level, i.e., at the characteristic scale of pores, while they can be ignored at the macroscopic level. In principle, they can be described with the help of stochastic differential equations [3]. But in this work, we deal only with a time-jump process that enables to treat merely impulsive conditions instead of stochastic ones, see [4]. Therefore, impulsive partial differential equations can be applied in order to encompass liquid-liquid phase transition in fluid transport through pores. Furthermore, in some cases such impulsive fluctuations can deform pores [5]. Possibility of this phenomenon gives an idea that our approach can be implemented to the problem of hydraulic fracturing. This is the subject of our future research. Concluding this introduction, we note that the degenerate (ultra-parabolic) type of the partial differential equation under study arises from the circumstance that, according to the idea presented in [6], we can ignore the dissipation process along a stream direction in a micro-pore.

2. Problem Πγ – the basic formulation
Let Ω be a bounded domain of spatial variables \( x \in \mathbb{R}^d \) with a smooth boundary \( \partial \Omega \) \( (\partial \Omega \in C^2) \). Let \( t \in [0,T] \) and \( s \in [0,S] \) be two independent time-like variables. Here \( T \) and \( S \) are given positive constants. Denote \( Q_{T,S} := \Omega \times (0,T) \times (0,S) \) and \( \Gamma_l := \partial \Omega \times (0,T) \times (0,S) \).
In this paper we study the following Cauchy-Dirichlet problem.

**Problem Πγ.** It is necessary to find a function \( u: Q_{T,S} \mapsto \mathbb{R} \) satisfying the quasilinear ultra-parabolic equation

\[
\partial_t u + \partial_x a(u) + \text{div}_x \varphi(u) - \Delta_x u = -K^+_\gamma(t; \tau)Z(x, s, u), \quad (x, t, s) \in Q_{T,S},
\]

the initial conditions with respect to the time-like variables \( t \) and \( s \):

\[
u|_{t=0} = u^{(1)}(x, s), \quad (x, s) \in \Omega \times (0, S), \tag{1b}
\]

\[
u|_{s=0} = u^{(2)}(x, t), \quad (x, t) \in \Omega \times (0, T), \tag{1c}
\]

and the homogeneous boundary condition

\[
u|_{\Gamma_1} = 0. \tag{1d}
\]

Here \( \gamma \in (0, 1] \) is a fixed small parameter, \( \tau \in (0, T) \) is a fixed value of variable \( t \),

\[
K^+_\gamma(t; \tau) = 1_{t > \tau} \frac{2}{\gamma} \omega \left( \frac{t - \tau}{\gamma} \right), \tag{2}
\]

\( \omega(\cdot) \) is the standard nonnegative even Friedrichs mollifier supported on \([-1, 1]\) such that

\[
\int_{\mathbb{R}} \omega(z) \, dz = 1. \tag{3}
\]

Thus, the function \( K^+_\gamma \) is a weak* approximation of the ‘one-sided’ Dirac delta function:

\[
K^+_\gamma(t; \tau) \xrightarrow{\gamma \to 0+} \delta(t=\tau+0) \quad \text{weakly* in } \mathcal{M}(\mathbb{R}). \tag{4}
\]

In (4), \( \mathcal{M}(\mathbb{R}) \) is the space of Radon measures on \( \mathbb{R} \). By \( \delta(t=\tau+0) \) we denote the ‘one-sided’ Dirac delta function, which is concentrated on the ‘right hand side’ of the point \( t = \tau \in \mathbb{R} \), i.e., the relation \( \langle \delta(t=\tau+0), \phi \rangle = \phi(\tau+0) \) holds for any function \( \phi \), which is integrable in the neighborhood of the point \( t = \tau \in \mathbb{R} \) and is right-continuous at this point: \( \phi(\tau+0) = \lim_{t \to \tau} \phi(t) \).

The given functions \( a, \varphi_1, \ldots, \varphi_d, \) and \( Z \) and initial data \( u^{(1)}_0 \) and \( u^{(2)}_0 \) are as follows.

**Conditions on \( a, \varphi, \) and \( Z.** (i) The vector-function \( \varphi(\lambda) = (\varphi_1(\lambda), \ldots, \varphi_d(\lambda)) \), the function \( a = a(\lambda) \), and the function \( Z = Z(x, s, \lambda) \) satisfy the regularity conditions \( a \in C^2_{\text{loc}}(\mathbb{R}), a(0) = 0, \varphi \in C^2_{\text{loc}}(\mathbb{R})^d, \varphi(0) = (0, \ldots, 0), \) and \( Z \in C^0_0(\Omega \times [0, S] \times \mathbb{R}^d) \).

(ii) The function \( a \) is monotonously increasing, i.e., \( a'(\lambda) > 0, \forall \lambda \in \mathbb{R} \), and satisfies the following genuine nonlinearity condition.

The Lebesgue measure of the set \( \{ \lambda \in \mathbb{R} : \xi_1 + a'(\lambda)\xi_2 = 0 \} \) is equal to zero for any fixed pair \( (\xi_1, \xi_2) \in \mathbb{R}^2 \) such that \( \xi_1^2 + \xi_2^2 = 1 \), i.e., for any fixed \( (\xi_1, \xi_2) \in S^1 \).

**Conditions U1.** The regularity conditions \( u^{(1)}_0 \in C^{2+\alpha}(\Omega \times [0, S]) \) and \( u^{(2)}_0 \in C^{2+\alpha}(\Omega \times [0, T]) \) hold. Also \( u^{(1)}_0 \) and \( u^{(2)}_0 \) are compatible with the boundary condition (1d). Namely, \( u^{(1)}_0 \) vanishes in the neighborhood of \( \partial(\Omega \times (0, S)) \) and \( u^{(2)}_0 \) vanishes in the neighborhood of \( \partial(\Omega \times (0, T)) \).
3. Notion of entropy solutions of Problem $\Pi_\gamma$

Entropy solutions to Problem $\Pi_\gamma$ appear, as usually, as limiting points of the family of solutions of the standard strictly parabolic regularized formulation (for more details, see [1] or [2, Sec. 2, 3]). The notion of entropy solutions is as follows.

**Definition 1.** Function $u \in L^\infty(Q,T,S) \cap L^2((0,T) \times (0,S); W_{1}^{1}(\Omega))$ is an entropy solution of Problem $\Pi_\gamma$ if it satisfies the entropy inequality

$$\partial_t \eta(u) + \partial_s q_\varphi(u) + \text{div}_x q_\psi(u) + K_\gamma^+(t;\tau)Z(x,s,u)\eta'(u) - \Delta_x \eta(u) \leq -\eta''(u)|\nabla_x u|^2,$$

the maximum principle

$$\|u\|_{L^\infty(Q,T,S)} \leq M_0 \overset{\text{def}}{=} C_1(T, \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))}, \|u_0^{(2)}\|_{L^\infty(\Omega \times (0,T))}),$$

and the initial conditions

$$\text{ess lim}_{t \to 0^+} \int_0^S \int_\Omega \left| u(x,t,s) - u_0^{(1)}(x,s) \right| \, dx \, ds = 0$$

and

$$\text{ess lim}_{s \to 0^+} \int_0^T \int_\Omega \left| u(x,t,s) - u_0^{(2)}(x,t) \right| \, dx \, dt = 0.$$

In (5a), $\eta \in C^2(\mathbb{R})$ is an arbitrary convex test function: $\eta''(\lambda) \geq 0 \ \forall \ \lambda \in \mathbb{R}$, and $(\eta,q_\varphi,q_\psi)$ is a convex entropy flux triple: $q_\varphi'(\lambda) = a'(\lambda)\eta'(\lambda)$, $q_\psi'(\lambda) = \varphi'(\lambda)\eta'(\lambda)$, $\lambda \in \mathbb{R}$.

In (5b), the value $C_1 = C_1(T, \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))}, \|u_0^{(2)}\|_{L^\infty(\Omega \times (0,T))})$ is defined by the formula

$$C_1(t, \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))}, \|u_0^{(2)}\|_{L^\infty(\Omega \times (0,T))}) \overset{\text{def}}{=} \xi_t \max \left\{ \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))}, \|u_0^{(2)}\|_{L^\infty(\Omega \times (0,T))} \right\}, \forall t \in (0,T),$$

where

$$\xi_t = \begin{cases} 0 & \text{for } b_1 \leq \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))} - 1, \\ 2 & \text{for } b_1 > \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))} - 1, \\ \frac{2 - \tau - \gamma_0}{\ln \left( \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))} \right)} & \text{for } b_1 \leq \|u_0^{(1)}\|_{L^\infty(\Omega \times (0,S))} - 1, \end{cases}$$

$$b_1 \overset{\text{def}}{=} \max_{\Pi \times [0,S] \times \mathbb{R}_+} |\partial_\lambda Z(x,s,\lambda)|.
$$

The entropy inequality (5a) is understood in the sense of distributions.

The following assertion on well-posedness of Problem $\Pi_\gamma$ is valid.

**Proposition 1. (Existence, uniqueness, and stability of entropy solutions.)** Assume Conditions on $a\varphi\&Z$ and Conditions U1 hold; then for an arbitrarily fixed small $\gamma > 0$ there exists a unique entropy solution $u = u(x,t,s)$ of Problem $\Pi_\gamma$ in the sense of Definition 1.

Moreover, this solution is stable in the following sense.

Let $u_1$ and $u_2$ be two entropy solutions corresponding to two sets $(u_{1,0}, u_{1,0})$ and $(u_{2,0}, u_{2,0})$ of initial data, respectively. Then the estimate

$$\|u_1(\cdot,t,\cdot) - u_2(\cdot,t,\cdot)\|_{L^1(\Omega \times (0,S))} \leq e^{\Phi_\epsilon(t;\tau)} \left( \|u_{1,0} - u_{2,0}\|_{L^1(\Omega \times (0,S))} + \max_{\lambda \in [-M(t),M(t)]} |a'(\lambda)| \int_0^t e^{-\Phi_\epsilon(t';\tau')} \|u_{1,0}^{(2)}(\cdot,t') - u_{2,0}^{(2)}(\cdot,t')\|_{L^1(\Omega)} \, dt' \right), \forall t \in (0,T),$$

where

$$\Phi_\epsilon(t;\tau) = \begin{cases} (1 - \epsilon) & \text{for } t = 0, \\ \epsilon t + 1 & \text{for } t > 0, \end{cases}$$

and $M(t)$ is the maximum of $\Phi_\epsilon(t;\tau)$ over all $t \geq 0$ and $\tau = 0$.
holds true, where

\[ \varpi_\gamma(t; \tau) = \begin{cases} \frac{1}{\omega} & \text{for } t > \tau, \\ 0 & \text{for } t \leq \tau, \end{cases} \]  

(8)

\[ M_1(t) = \max \{ C_1(t, \|u_1^{(1)}\|_{L^\infty(\Omega \times (0,S))}, \|u_1^{(2)}\|_{L^\infty(\Omega \times (0,T))}), \]  

\[ C_1(t, \|u_2^{(1)}\|_{L^\infty(\Omega \times (0,S))}), \|u_2^{(2)}\|_{L^\infty(\Omega \times (0,T))} \} \]  

and \( C_1 \) is defined by (6).

**Proof.** This proposition is proved in [2, Theorem 3.1] in the case when the nonlinear function \( a = a(\lambda) \) is not necessarily monotonous and, at the same time, the system of two entropy boundary conditions [2, formulas (3.3d)&(3.3e)] takes place in the definition of entropy solution, instead of the initial condition (5d). It is easy to see that, in the case when \( a \) is strictly monotonously increasing, the above mentioned two entropy boundary conditions from [2] reduce to just one initial condition (5d). Consequently, the assertion of Proposition 1 holds true for entropy solutions in the sense of Definition 1. \( \square \)

**Remark 1.** On the strength of the condition (3) imposed on the mollifier \( \omega \), for any fixed value \( \gamma > 0 \) the exponent \( \varpi_\gamma(t; \tau) \) (see in (8)) is monotonously non-decreasing absolutely continuous function of \( t \) satisfying the inequality \( 0 \leq \varpi_\gamma(t; \tau) \leq 2b_1 \), \( \forall t \geq 0, \forall \gamma > 0 \).

4. Impulsive equation

Due to (4), we have that the function \( (x, t, s, \lambda) \mapsto K_\gamma^+(t; \tau)Z(x, s, \lambda), \) (with \( \gamma \) and \( \tau \) being parameters) satisfies the limiting relation \( \omega(t) \lim_{\gamma \to 0} (K_\gamma^+ Z) = \delta(t=\tau+0)Z \). Substituting \(-\delta(t=\tau+0)Z\) for \(-K_\gamma^+ Z\) in (1a), we technically arrive at the following formulation of the initial-boundary value problem for the impulsive ultra-parabolic equation.

**Problem \( \Pi_0 \).** It is necessary to find a function \( u: Q_{T,S} \mapsto \mathbb{R} \) satisfying the quasilinear ultra-parabolic equation

\[ \partial_t u + \partial_s a(u) + \text{div}_x \varphi(u) = \Delta_x u, \quad (x, t, s) \in QT,S \setminus \{ t = \tau \}, \]  

(9a)

the impulsive condition

\[ u(x, \tau + 0, s) + Z(x, s, u(x, \tau + 0, s)) = u(x, \tau - 0, s), \quad (x, s) \in \Omega \times [0, S], \]  

(9b)

the initial conditions (1b) and (1c), and the homogeneous boundary condition (1d).

**Remark 2.** In the sense of distributions, system (9a)-(9b) is equivalent to the equation

\[ \partial_t u + \partial_s a(u) + \text{div}_x \varphi(u) - \Delta_x u = -\delta(t=\tau+0)Z(x, s, u), \quad (x, t, s) \in QT,S. \]

We introduce the notion of entropy solution of Problem \( \Pi_0 \) analogously to Definition 1, with the natural modifications.

**Definition 2.** Function \( u \in L^\infty(Q_{T,S}) \cap L^2((0,T) \times (0,S); \overset{\circ}{W}^{1/2}_2(\Omega)) \) is an entropy solution of Problem \( \Pi_0 \) if it satisfies the entropy inequality?

\[ \partial_t \eta(u) + \partial_s q_\alpha(u) + \text{div}_x q_\ell(u) - \Delta_x \eta(u) \leq -\eta''(u)|\nabla_x u|^2, \quad (x, t, s) \in QT,S \setminus \{ t = \tau \}, \]  

(10a)

the maximum principle

\[ \|u\|_{L^\infty(Q_{T,S})} \leq M_2, \]  

(10b)

the initial conditions (5c) and (5d), and the impulsive condition (9b).
In (10a) \((\eta, q_0, q_p)\) is a convex entropy triple, which is defined in Definition 1. The constant \(M_2\) depends only on the input data of the problem. In view of (5b) and (9b), we can take \(M_2 = 2M_0 + \|Z\|_{C([\Omega \times [0,S] \times \mathbb{R}_3])}\). The entropy inequality (10a) is understood in the sense of distributions. The impulsive condition (9b) should be satisfied almost everywhere in \(\Omega \times (0,S)\).

Directly from Proposition 1, we deduce the following existence result for Problem \(\Pi_0\) by rather simple arguments.

**Theorem 1.** Assume Conditions on \(a\&\varphi\&Z\) and Conditions U1 hold; then there exists at least one entropy solution \(u = u(x, t, s)\) of Problem \(\Pi_0\) in the sense of Definition 2.

**Proof.** In order to construct an entropy solution of Problem \(\Pi_0\) in the sense of Definition 2, it suffices to fulfill the following three steps.

At first, notice that equations (1a) and (9a) coincide in the subdomain \(Q_{r,S} = \Omega \times (0, \tau) \times (0, S)\), since \(K_p^+ (t; \tau)Z(x, s, u) = 0\) there. Hence the formulations of Problems \(\Pi_0\) and \(\Pi_1\) coincide on the segment \(\{0 < t < \tau\}\) and therefore, by Proposition 1, the both problems have the same unique entropy solution \(u = u(x, t, s)\), and this solution has the trace \(u(x, \tau - 0, s)\).

At second, consider the equation

\[
v + Z(x, s, v) = u(x, \tau - 0, s)
\]

for the sought function \(v \in \mathbb{R}\) for almost all points \((x, s) \in \Omega \times (0, S)\). More precisely, these points \((x, s)\) are the Lebesgue points of the function \((x, s) \mapsto u(x, \tau - 0, s)\). Since the image of the function \(v \mapsto v + Z(x, s, v)\) is the whole space \(\mathbb{R}\) for all \((x, s) \in \Omega \times [0, S]\) and the function \(u = u(x, \tau - 0, s)\) is bounded due to the maximum principle (5b), equation (11) has at least one solution \(v = v_{x,s} \in \mathbb{R}\), which depends on \(x\) and \(s\) parametrically. Set \(u(x, \tau + 0, s) = v_{x,s}\).

By construction, \(u(x, \tau + 0, s)\) along with \(u(x, \tau - 0, s)\) satisfies the impulsive condition (9b). Since the function \(Z = Z(x, s, u)\) is continuous and the function \((x, s) \mapsto u(x, \tau - 0, s)\) is measurable, we conclude that the function \((x, s) \mapsto u(x, \tau + 0, s)\) is also measurable. From the maximum principle (5b) and the finiteness and continuity of the function \(Z\) it also follows that \(u(\cdot, \tau + 0, \cdot)\) belongs to \(L^\infty(\Omega \times (0, S))\). Furthermore, the following bound holds true:

\[
\|u(\cdot, \tau + 0, \cdot)\|_{L^\infty(\Omega \times (0, S))} \leq M_0 + \|Z\|_{C([\Omega \times [0,S] \times \mathbb{R}_3])}.
\]

At third, on the strength of Proposition 1, in the subdomain \(Q_{(\tau,T),S} = \Omega \times (\tau, T) \times (0, S)\) there exists a unique entropy solution of the initial-boundary value problem for equation (9a) supplemented by the initial data (1c) and \(u|_{t=\tau} = u(x, \tau + 0, s), (x, s) \in \Omega \times (0, S)\), and the homogeneous boundary condition (1d). The same as at the first step, here we remark that equations (1a) and (9a) coincide for \(Z \equiv 0\). Finishing the proof of the theorem, we notice that estimate (10b) appears as the combination of the estimate (5b) on the segment \(\{0 < t < \tau\}\), the analogous estimate on the segment \(\{\tau < t < T\}\), and estimate (12). Theorem 1 is proved.

**Remark 3.** It is plain to see that the constructed in the proof of Theorem 1 entropy solution is not unique, in general, since equation (11) may have two or even more solutions \(v_{x,s}\) in the case when the function \(v \mapsto v + Z(x, s, v)\) is not monotonous.

5. Limiting passage in Problem \(\Pi_\gamma\) as \(\gamma \to 0^+\)

In view of Remark 3, the question about construction of a ‘proper’ unique solution arises.

The following theorem is the main results of this article. In this theorem, we establish one rule for selecting a unique entropy solution on the basis of the limiting passage in the family of entropy solutions of Problem \(\Pi_\gamma\) as \(\gamma \to 0^+\).
Theorem 2. (i) Assume all assumptions of Theorem 1 hold; then the family \( \{u = u_\gamma\}_{\gamma > 0} \) of entropy solutions of Problem \( \Pi_\gamma \) has the unique limit \( u_* \in L^\infty(Q_{T,S}) \cap L^2((0,T) \times (0,S); W^1_2(\Omega)) \) such that

\[
\lim_{\gamma \to 0^+} u_\gamma \quad \text{strongly in} \quad L^1(Q_{T,S}) \quad \text{and weakly in} \quad L^2((0,T) \times (0,S); W^{\frac{1}{2}}_2(\Omega)).
\]

(13)

Besides, \( u_* \) is an entropy solution of Problem \( \Pi_0 \) in the sense of Definition 2.

(ii) Moreover, assume \( u_*1 \) and \( u_*2 \) are two entropy solutions of Problem \( \Pi_0 \) that satisfy the initial data \( (u^{(1)}_{1,0}, u^{(2)}_{1,0}) \) and \( (u^{(1)}_{2,0}, u^{(2)}_{2,0}) \), respectively, and are the limits of the families of solutions of Problem \( \Pi_\gamma \). Then they satisfy the inequality

\[
\|u_*1 - u_*2\|_{L^1(Q_{T,S})} \leq e^{2b_1} \left( T\|u^{(1)}_{1,0} - u^{(1)}_{2,0}\|_{L^1(\Omega \times (0,S))} \right. \\
\left. + \max_{\lambda \in [-M_2, M_2]} |a'(\lambda)| \|u^{(2)}_{1,0} - u^{(2)}_{2,0}\|_{L^1(\Omega \times (0,T))} \right). 
\]

(14)

Proof. Justification of assertion (i) in the theorem is similar to justification of assertion 1 of Theorem 9.1 in [2], except for the proof of the property of the uniqueness of the limit.

The uniqueness of the limit follows from estimate (14). In turn, due to the limiting relation (13), this estimate follows immediately from estimate (7), as \( \gamma \to 0^+ \). Thus the both assertions (i) and (ii) hold true. This observation completes the proof of Theorem 2.

Remark 4. On the strength of assertion (ii) of Theorem 2, we conclude that there can be at most one ‘concentrated’ entropy solution of Problem \( \Pi_0 \) in the sense of Definition 2. By the ‘concentrated’ entropy solution we mean the entropy solution, which is the limiting function of the family \( \{u_\gamma\}_{\gamma > 0} \) of entropy solutions of Problem \( \Pi_\gamma \), as \( \gamma \to 0^+ \). Thus, the limiting relation (13) serves as a selector of the unique solution.

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