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To cite this article: D W W Ng et al 2018 J. Phys.: Conf. Ser. 1132 012080

View the article online for updates and enhancements.
The study of properties on generalized Beta distribution

D W W NG\textsuperscript{1}, S K KOH\textsuperscript{1}, S Z SIM\textsuperscript{1}, M C LEE\textsuperscript{2}

\textsuperscript{1}Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman (UTAR), Jalan Sungai Long, Bandar Sungai Long, 43000 Kajang, Selangor, Malaysia
\textsuperscript{2}School of Information Technology, Monash University, 47500 Subang Jaya, Selangor, Malaysia.

E-mail: dennisnw@gmail.com

Abstract. Statistical distributions such as Normal, Log-Normal, Gamma and Beta distributions have been studied to determine the fitting ability of the distributions in different field of studies. The objective of this research is to study the properties of the generalized Beta distribution which is a 6 parameter model from the Beta family as well as to evaluate the prediction level of the model. The advantages of the generalized Beta distribution is they can be very versatile and flexible where it will provide a good description to many different types of data, including unimodal, uniantimodal, increasing, decreasing, or bath-tub shape distribution depending on the parameter values. The distributions were fitted to rainfall data collected from Sg Lui, Selangor using Non-Linear Least-Squares Minimization package of a Python software with the maximum likelihood estimation as the parameter estimation method. Simulation was proceeded using Accept-Reject algorithm where their predictive level were evaluated using model selection criteria such as Mean Absolute Error (MAE) and Root Mean Square Error (RMSE).

1. Introduction

In [1], three new distributions were proposed to be fitted to rainfall data for hydrolysis analysis. Among the three distributions proposed, only the modified Beta distribution’s properties from the Beta family was managed to be derived. However, the other Beta family distribution, the generalized Beta distribution was not derived. Therefore, the objective of this research is to derive the properties of the generalized Beta distribution such as the cumulative distribution function (c.d.f), expected value ($E[X]$), general moment, variance ($Var[X]$), skewness ($skew(x)$) and kurtosis ($kurt(x)$) as well as to study the fitting of the distribution to rainfall volume.

Although present statistical distributions are commonly used in various field of studies, there is still a backdrop when it comes to flexibility where they are unable to be a good model to represent certain fields. The generalized Beta distribution is a generalized version for some of the mentioned distributions where it can be very versatile and flexible which might be able to provide a good description to many different types of data. It is related to other statistical distributions when some of the distribution’s parameters are substituted with certain values. There are many types of data distribution such as unimodal where the data distribution has
only one clear peak, uniantimodal which look like a bath-tub shape distribution, increasing and decreasing where the distribution shows and increasing and decreasing trend respectively as well as constant where there is no significant change in the distribution of the data.

Rainfall analysis is the analysis of rainfall trend, depth, volume and others to identify whether are there any significant changes in them as more environmental issues can be seen over the years. Aside from rainfall analysis, it is also possible to apply the distribution in other environmental studies such as light intensity or snow volume in some four seasons countries. Other possible fields such as agriculture and traffic queuing could also be applied to see how well does a flexible and versatile distribution fit to the data of those respective fields. Even insurance claims and finance data could be one of the fields to be studied for the proposed distribution.

In this research, aside from the introduction that was already mentioned in Section 1, literature review will be discussed in Section 2. Derivation of properties for the proposed generalized Beta distribution will be derived and explained in Section 3. The properties derived will be divided to their respective sub-sections and steps will be shown accordingly. Under Section 4, empirical studies on parameter estimation and simulation results will be discussed. The conclusions as well as future research work will be pointed out in Section 5.

2. Literature Review

2.1. Background

The method of proposing generalized classes of distributions has attracted theoretical and applied statisticians due to their flexible properties. The distributions are generalized either as a physical or statistical argument to explain the mechanism of generated data, an appropriate model that has previously been used successfully or a model whose empirical fit is good to the data [2]. In Section 2.2, the Beta family distributions will be discussed where distributions related to the generalized Beta distribution will be explained.

2.2. Beta Family Distributions

In probability theory and statistics, Beta family distributions are continuous probability distributions that are normally bounded by the interval [0,1] and the parameters that appear as exponents of the random variable control the shape of the distribution. Beta-type distributions with finite range gives an advantage in modelling a dataset. Beta densities are versatile and are able to model many types of uncertainties since it can be unimodal, uniantimodal, increasing, decreasing or constant depending on the values of $\gamma$ and $q$ [3]. However, it is not preferred in some ways as the 2-parameter distribution only provides limited precision in fitting the data. It is preferable to have more parametrically flexible versions of Beta in order to provide a richer empirical description of the data while offering more structure rather than a non-parametric estimator. With that, many researches were done on different forms of generalized Beta distribution. In [4], a 3-parameter Beta-2 distribution which is a generalization of the normalized Beta distribution was studied and fitted to income distribution and inequality in China. [5] introduced a 5-parameter Beta distribution that was fitted to income distribution, stock returns and regression analysis where it is compared to other fewer parameters related distributions. Even generalized Beta-generated distributions were studied in [6] where generalized Beta distribution of the 1st kind as well as special models of the Beta-generated distributions such as generalized Beta Normal distribution were studied and fitted to various areas such as financial and voltage data. In this research, the generalized Beta distribution consisting of 6 parameters ($\gamma, \rho, \beta, \alpha, \sigma$ and $z$) expanded from the generalized Gauss Hypergeometric function was proposed and studied. Various Beta family distributions could be reduced from the generalized Beta distribution by setting it’s parameters with certain values as follows:
Gauss Hypergeometric distribution, $\alpha = \gamma; \rho = \beta + \theta$ and $z = -t$.

Generalized Beta of the 1st kind distribution, $z = 0; \alpha = \gamma$ and $\rho - \beta = q$

Kumaraswamy distribution, $z = 0; \alpha = \gamma; \rho - \beta = q; b = 1$ and $\gamma = 1$

Standard Arcsine distribution, $z = 0; \alpha = 0; \beta = 0; \gamma = 0.5$ and $\rho = 0.5$

3. Theoretical Studies on the Proposed Generalized Beta Distribution

3.1. Background

From [1], three new distributions were proposed. Studies were done on the modified Beta distribution where it’s properties were developed. However, no further studies were done on the other Beta family distribution (i.w. generalized Beta) where it’s properties are yet to be derived.

In this section, the derivation of the proposed generalized Beta distribution will be shown. The theory needed to derive the properties of the generalized Beta distribution will be stated in Section 3.2 Related Theories. In addition, a basic introduction of the probability density function (p.d.f) and the cumulative distribution function (c.d.f) will be explained in Section 3.3 Density Functions. Under Section 3.4 Moment Generating Function (m.g.f), the expected value (E[X]), skewness (skew(x)) and kurtosis (kurt(x)) properties will be derived in that respective order.

3.2. Related Theories

To derive the properties of the generalized Beta distribution, the $2F_1$ Hypergeometric contiguous relation function is needed. It could be found in [7] where two hypergeometric function with the same argument $z$ is said to be contiguous if their parameters $a, b$ and $c$ differ by integers. Further explanation on the reason will be explained in Section 3.4. Only Equation (22) from the research article mentioned is needed out of the list of functions available:

$$c_2F_1(a, b; c; z) - a_2F_1(a + 1, b; c + 1; z) + (a - c)2F_1(a, b; c + 1; z) = 0$$

$$\implies 2F_1(a, b; c; z) = \frac{a_2F_1(a + 1, b; c + 1; z) - (a - c)2F_1(a, b; c + 1; z)}{c}$$

3.3. Density Functions

3.3.1 Probability Density Function, p.d.f

$$f(x) = \frac{\Gamma(\gamma + \rho - \alpha)\Gamma(\gamma + \rho - \beta)}{\Gamma(\gamma + \rho)\Gamma(\gamma + \rho - \alpha - \beta)}(1 - z)^\rho x^{\gamma - 1}(1 - x)^{\rho - 1}(1 - zx)^{-\sigma}F(\alpha, \beta; \gamma; x)$$

$$3F_2(\rho, \sigma, \gamma + \rho - \alpha - \beta; \gamma + \rho - \alpha, \gamma + \rho - \beta; \frac{z}{z - 1})B(\gamma, \rho)$$

where $\sigma > 0, \gamma > 0, z < 0.5, (\gamma + \rho - \alpha - \beta) > 0$

$3F_2$ is a generalized Hypergeometric function defined by:

$$3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k} \frac{z^k}{k!}$$

and

$$F(\alpha, \beta; \gamma; x) = 2F_1(\alpha, \beta; \gamma; x)$$

which is the $2F_1$ Hypergeometric function defined by:

$$2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}$$
Given that
\[ \int_0^1 x^{\gamma-1} (1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \, dx = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)} (1-z)^{-\sigma} \times \]
\[ _3F_2(\rho, \sigma, \gamma + \rho - \alpha - \beta; \gamma + \rho - \alpha, \gamma + \rho - \beta; \frac{z}{z-1}) \]
where \( \Re \gamma > 0, \Re \rho > 0, \Re(\gamma + \rho - \alpha - \beta) > 0, |\arg(1-z) < \pi| \) [9].

It can be easily shown that
\[ \int_0^1 \frac{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}{\Gamma(\gamma+\rho-\alpha-\beta)} (1-z)^{\sigma} x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \, dx = 1 \]

3.3.2 Cumulative Distribution Function, c.d.f

In the e-handbook of [8], it was mention that the cumulative distribution function of the Beta distribution is also called the incomplete Beta function ratio denoted by \( I_x \) defined by:
\[ F(x) = I_x(p, q) = \int_0^x \frac{t^{\rho-1}(1-t)^{q-1}}{B(p, q)} \, dt \quad 0 \leq x \leq 1; p, q > 0 \] where \( B \) is the Beta function.

The incomplete function is present in all of the Beta family distributions which includes the generalized Beta distribution where it can be found in the integration part as follows:
\[ f(x) = h x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \]

Therefore, the c.d.f general form of the generalized Beta distribution could not be derived.

3.4. Moment Generating Function

To ensure that the derivations are systematically shown, a constant \( h \) is assigned to represent the parameters that do not contain the \( x \)-value as they will not be affected by the integration. Let
\[ h = \frac{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}{\Gamma(\gamma+\rho-\alpha-\beta)} (1-z)^{\sigma} \]
\[ _3F_2(\rho, \sigma, \gamma + \rho - \alpha - \beta; \gamma + \rho - \alpha, \gamma + \rho - \beta; \frac{z}{z-1}) B(\gamma, \rho) \]
\[ f(x) = hx^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \]

3.4.1 Expected Value, \( E[X] \)
\[ E[X] = \int_0^1 hx \cdot x^{\gamma-1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \, dx \]
\[ \Rightarrow E[X] = \int_0^1 hx^{(\gamma-1)+1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \, dx \quad (2) \]

From Equation (2), it can be seen that the \( \gamma \) parameter is affected when the \( x \)-variable is multiplied per expected value definition. From the modified Beta distribution shown in [1], the properties were derived based on a table of equations by [9]. Direct comparison could be made on the modified Beta distribution to derive the properties. However for the generalized Beta distribution, the same approach could not be followed as the \( \gamma \) parameter also exists in the \( _2F_1 \) Hypergeometric function, \( F(\alpha, \beta; \gamma; x) \). In order to derive the properties, the contiguous relation function mentioned in Section 3.2 is needed where the following is obtained:
E[X] = \int_0^1 h x^{(\gamma-1)+1}(1-x)^{\rho-1}(1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) \, dx \\
= \int_0^1 h x^{(\gamma-1)+1}(1-x)^{\rho-1}(1-zx)^{-\sigma} \frac{\alpha_2 F(\alpha + 1, \beta; \gamma + 1; x)}{\gamma} \, dx \\
= h \int_0^1 \frac{\alpha}{\gamma} x^{(\gamma-1)+1}(1-x)^{\rho-1}(1-zx)^{-\sigma} \frac{\alpha_2 F(\alpha + 1, \beta; \gamma + 1; x)}{\gamma} \, dx - \\
h \int_0^1 \frac{\alpha - \gamma}{\gamma} x^{(\gamma-1)+1}(1-x)^{\rho-1}(1-zx)^{-\sigma} \frac{\alpha_2 F(\alpha + 1, \beta; \gamma + 1; x)}{\gamma} \, dx \\

From here, it can be seen that adjustments have been made to the \( \gamma \) parameter of the \( 2F_1 \) Hypergeometric Function. With this, the table of equations by [9] can be used to obtain the properties of the generalized Beta distribution which is as follows:

\[ E[X] = \frac{h \alpha (\Gamma(\gamma+1)\Gamma(\gamma+\rho-\alpha, \beta))}{\Gamma(\gamma+1+\rho-\alpha, \beta)} (1-z)^{-\sigma} F_2(\rho, \sigma, \gamma + \rho - \alpha, \beta; \gamma + \rho - \alpha, \gamma + 1 + \rho - \beta; \frac{z}{z-1}) \]

Through the substitution of \( h \), the following equation is then expanded:

\[ E[X] = \frac{\alpha}{\gamma} \frac{\Gamma(\gamma+1)\Gamma(\gamma+\rho-\alpha, \beta)}{\Gamma(\gamma+1+\rho-\alpha, \beta)} F_2(\rho, \sigma, \gamma + \rho - \alpha, \beta; \gamma + \rho - \alpha, \gamma + 1 + \rho - \beta; \frac{z}{z-1}) \]

3.4.2 General Moment

To obtain the 2nd, 3rd, 4th moment and etc., multiple recursions are needed. As it only affects the \( 2F_1 \) Hypergeometric function, a generalized formed could be derived. From the expansion done, a certain pattern could be identified. Constants 1, 1 ; 1, 2,1 and 1, 3, 3, 1 show that a Binomial expansion pattern can be seen. After some derivations, a generalized version is formed:

\[ _2F_1(a, b; c; z) = \sum_{k=1}^{n+1} \binom{n}{k-1} (-1)^{n-k+1} \frac{(a-c+n+k)_{n-k+1}}{c_n} _2F_1(a+k-1, b; c+n; z) \] (4)

By placing Equation (4) into the general moment function, the following equation is obtained:

\[ E[X^n] = \frac{\Gamma(\gamma+\rho-\alpha, \beta) \Gamma(\gamma+\rho-\alpha, \beta) (1-z)^{-\sigma}}{\Gamma(\gamma+1+\rho-\alpha, \beta)} \frac{B(\gamma, \rho) \int_0^1 x^{\gamma-1+n}(1-x)^{\rho-1}(1-zx)^{-\sigma}}{3F_2(\rho, \sigma, \gamma + \rho - \alpha, \beta; \gamma + \rho - \alpha, \gamma + 1 + \rho - \beta; \frac{z}{z-1})} \]

\[ \sum_{k=1}^{n+1} \binom{n}{k-1} (-1)^{n-k+1} \frac{(a-c+n+k)_{n-k+1}}{c_n} _2F_1(a+k-1, b; c+n; z) \] (5)
3.4.3 Variance, \( \text{Var}[X] \); Skewness, \( \text{skew}(x) \) and Kurtosis, \( \text{kurt}(x) \)

The general formula of the variance, skewness and kurtosis are as follows:

\[
\text{Var}[X] = E[X^2] - (E[X])^2 \tag{6}
\]

\[
\text{Skew}(x) = E[(x - \mu)^3] = \frac{E[X^3] - 3E[X]E[X^2] + 2E[X]^3}{(E[X^2] - E[X])^2} \tag{7}
\]

\[
\text{Kurt}(x) = E[(x - \mu)^4] = \frac{E[X^4] - 4E[X]E[X^3] + 6E[X]^2E[X^2] - 3E[X]^4}{(E[X^2] - E[X])^2} \tag{8}
\]

To obtain Equations (6), (7) and (8), the 2nd, 3rd and 4th moments are needed respectively. They can be obtained by substituting \( n \) with the \( nth \) moment needed. For example, to obtain the 2nd moment, substitute \( n = 2 \) and then expand the summation. Once the respective moments are derived, the general form of the properties could be obtained. However, due to the length and complexity of the equation, it will not be shown in this paper.

4. Empirical Studies on the Proposed Generalized Beta Distribution

The empirical studies were conducted through distribution fitting to estimate the parameters of the generalized Beta distribution as well as to simulate data to test for significance. In this research, the dataset will be on rainfall volume collected from Sg Lui, Hulu Langat, Selangor within the years of 2002 to 2012. The trend of rainfall volume data is illustrated in Figure 1 and also the histogram of Figure 2. The parameter were estimated using Non-Linear Least-Squares Minimization package due to its consideration of the parameters’ constraints with a Python software [10]. The method used to estimate the parameters is the Maximum Likelihood Estimation (MLE) where the distribution was \( ln \) then multiplied with \( (-1) \) to transform the minimization process to maximization. Simulation was done using Accept-Reject algorithm [11].

The whole procedure begins by splitting the rainfall data to train data consisting of the first \( n\% \) of the whole data set and test data with the remaining \( (100 - n)\% \); \( [n\% : (100 - n)\%] \). Three different sets were tested in this research which are \( [70:30] \), \( [80:20] \) and \( [90:10] \). The parameters were estimated using the train data which will then be used to simulate a set of data which has the same count as the test data. However, the maximum rainfall volume is 136.1 mm and the proposed Beta distribution’s \( x \)-variable constraint is between 0 and 1 \( (0 < x < 1) \). Therefore, the data needs to be rescaled to extend the range of \( x \) to be fitted into the dataset. As the maximum value is 136.1 mm, it is reasonable to cap at 150 mm per day. To rescale the data, the entire dataset is divided by 150 where the maximum amount is now 0.90733. Besides that, the distribution could not accept zero-value data where the zero-values in the dataset represent no rainfall. To solve this issue, a small value of 0.000001 is added to the dataset. The addition would not affect the empirical result as the value added is only an insignificant amount.

Although the dataset is now within the \( x \)-variable constraint, a transformation process is needed on the proposed distribution to ensure that a consistent result is obtained. To transform, let

\[
x = \frac{y}{150} + 0.000001 \quad \Rightarrow \quad dx = \frac{1}{150} dy \quad \text{then},
\]

\[
\int_{0}^{x} f(x) \, dx = \int_{0}^{\frac{x}{150}} \frac{1}{150} f\left(\frac{y}{150}\right) \, dy, \quad \text{where} \quad 0 \leq y \leq 150
\]

With the transformation process completed, model selection criteria such as root mean square error (RMSE) and mean absolute error (MAE) were calculated to test the accurateness of the simulated data with the test data where the formulas of MAE and RMSE are as follows:
\[ MAE = \sum_{i=1}^{n} \frac{|y_i - x_i|}{n} \quad RMSE = \sqrt{\frac{\sum_{i=1}^{n} (y_i - x_i)^2}{n}} \]

where \( y_i \) is the simulated value and \( x_i \) is the actual value.

From the formulas of MAE and RMSE, it could be understood that the values show the average difference between the simulated values and the actual values (i.e. test data). The values represent the accuracy of the model to simulate values related to the field of study.

Table 1. Summary of Empirical Results of generalized Beta distribution.

| Samples | Parameters          | MAE   | RMSE  |
|---------|---------------------|-------|-------|
| [70:30] | \( \gamma = 0.7117, \rho = 4.160, \beta = 0.9471, \) \( \alpha = 0.09267, \sigma = 0.5135, z = -6.6366 \) | 0.4269 | 0.5029 |
| [80:20] | \( \gamma = 0.7135, \rho = 6.2308, \beta = 1.9940, \) \( \alpha = 0.8292, \sigma = 0.4935, z = -8.2744 \) | 0.4232 | 0.5024 |
| [90:10] | \( \gamma = 0.7106, \rho = 4.4080, \beta = 0.6789, \) \( \alpha = 0.2474, \sigma = 0.5051, z = -5.6153 \) | 0.4302 | 0.5082 |

Table 1 shows the results of the parameters’ estimated as well as the RMSE and MAE values. Based on the values shown, the MAE and RMSE calculated are 0.4269 and 0.5029 for the [70:30] dataset, 0.4232 and 0.5029 for the [80:20] as well as 0.4302 and 0.5802 for the [90:10] dataset respectively. From the formula of the MAE and RMSE, it could be understood that the values show how far does the simulated values differ from the actual values on average. It could be seen that the results for all the 3 samples are close where the MAE is approximately 0.4 and RMSE 0.5. This concludes that the estimation and simulation method used are consistent methods. As mentioned before, the maximum and minimum rainfall volume are 0.9073 and 0.000001 respectively. With the MAE and RMSE values being approximately 0.4 and 0.5, this implies that the simulated values differ widely from the actual ones which is about half the range of the data. Further proof could be illustrated graphically in Figure 2 where it can be seen that the distribution does not fit well to the dataset for all the 3 samples. Therefore, it can be concluded that the generalized Beta distribution is not a good model to represent rainfall volume.

5. Conclusion
With the properties of generalized Beta distribution derived, the theoretical aspect of the proposed distribution is completed and better understanding could be achieved. As the distribution contain the incomplete Beta function ratio [8], the c.d.f general form is the only property that could not be derived. In this research, the empirical studies show that the generalized Beta distribution do not fit well to rainfall volume. Although it was only applied to rainfall analysis in this research and in the work of [1], it could be applied to other fields as well.

For future research work, empirical studies could be done on various fields such as insurance claims, income, stock returns or even other environmental phenomena to observe how well does the proposed distribution fit to different area of studies through parameter estimation, simulation and model selection criteria. The accuracy of the simulations could be tested using other various model selection criteria such as the Akaike Information Criterion (AIC) and Kolmogorov-Smirnov (K-S) test to provide more conclusive evidence towards the hypothesis involved. Other estimation methods such as method of moments or even non-linear least squares methods could
be used and compared with the maximum likelihood estimation to identify the best method that can produce the most accurate estimates. Besides that, parameters re-calibrations could also be considered for future work as they are common practices in the banking industry and studies were done by [12–15]. Comparison could be done to identify whether the re-calibrated distribution result would be more accurate compared to the non-recalibrated ones.

In conclusion, the generalized Beta distribution’s properties are completely derived and when the distribution is fitted to rainfall volume, the results show that it is not a good fit where it could not be a good model to represent rainfall volume. Further studies on the application of the distribution towards other areas of research could be done as future work.

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