COLLISIONS OF FAT POINTS AND
APPLICATIONS TO INTERPOLATION THEORY

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ABSTRACT. We address the problem to determine the limit of the collision of fat points in \( \mathbb{P}^n \). We give a description of the limit scheme in many cases, in particular in low dimension and multiplicities.

The problem turns out to be closely related with interpolation theory, and as an application we exploit collisions to prove some new cases of Laface-Ugaglia Conjecture.

The study of linear systems on a projective variety is a fundamental problem in algebraic geometry. This kind of problems is classical, and it is studied since the beginning of 20th century. However, despite the efforts of many mathematicians, there is still much we do not know.

Interpolation theory deals with systems of divisors passing through a bunch of fixed points with prescribed multiplicities. Given such a linear system, it is natural to ask its dimension, and sometimes the naïf parameter counting does not give the correct answer. A challenging problem in interpolation theory is the study of special linear systems, that is, systems with dimension larger than the expected one. In general, this is a very hard problem, and we lack a complete classification of special linear systems even for the projective plane.

A standard approach to study the speciality of linear systems is via degenerations. By semicontinuity, a degeneration can not decrease the dimension of a system, so, if the degenerated system is nonspecial, then the original one is nonspecial as well. Typically, it is convenient to degenerate some of the assigned base points to a special configuration, for instance by sending them on a hyperplane in order to apply induction arguments. However, sometimes it can be useful to allow points not only to be in special position, but also to collide to the same point.

The degenerated linear system features a new singular base point, which is the limit of the collision we consider. So this degeneration strategy is useful only if we fully understand the limit scheme. This raises a fairly natural question, which is of interest in itself.

Question. Given \( n, m_1, \ldots, m_h \in \mathbb{N} \), what is the flat limit of \( h \) colliding fat points of multiplicities \( m_1, \ldots, m_h \) in \( \mathbb{P}^n \)? Or more generally on a smooth \( n \)-dimensional variety?

While it is easy to ask, this question has not a simple answer. Results in [11] and in [18] show the lack of a clean and complete solution even if \( n = 2 \).

Since the beginning of this work, we realized that there is a nice interplay between collisions of fat points and interpolation theory. On one hand, some basic properties of the limit of a collision can be stated in the language of linear systems. On the other hand, with this technique we can afford new ways to degenerate a bunch of singular points, so we have new tools to try to prove the nonspeciality of linear systems. We believe this connection to be worth of a deep analysis.

Our strategy to describe a collision is quite simple. Thanks to flatness, we know the degree of the limit scheme. First, we compute the multiplicity of the limit. This turns out to be a question about linear systems, see Proposition [13]. Once the multiplicity is determined, we try to get more information about this linear system, such as its base locus, in order to get further conditions on the limit. In this way we get a candidate scheme, and we compute its degree. Since this candidate is a subscheme of the limit, if their degree coincide, then they are the same scheme.
Once we are able to describe a limit, we can use it to degenerate linear systems. In Section 4, we use such degenerations to prove nonspeciality for some families of linear systems, and we compare our results with the known results in interpolation theory. For instance, the Laface-Ugaglia conjecture predicts that the linear system \( L_{3,d}(m') \) is nonspecial for \( d > \frac{1}{27}m + o(m) \). We will prove the following partial result in this direction.

**Theorem 1.** If \( r \leq 15 \) and \( d \geq 3m \), then \( L_{3,d}(m') \) is nonspecial. In particular, Laface-Ugaglia conjecture holds for these values \( d, m \) and \( r \).

We also show some results in higher dimension.

1. **Notations and preliminaries**

We work over the complex field \( \mathbb{C} \). Every scheme will be projective, unless we specify it is not. For a scheme \( X \) and a subscheme \( Y \subset X \), we will write \( \mathcal{I}_{Y,X} \) to denote the defining ideal of \( Y \) in \( X \). With abuse of notation, we will use the same symbol to indicate the associated ideal sheaf on \( X \). If no ambiguity is likely to arise, we will write simply \( \mathcal{I}_Y \) instead of \( \mathcal{I}_{Y,X} \).

We start by recalling some definitions and facts about 0-dimensional schemes.

**Definition 2.** Let \( X \) be a 0-dimensional scheme. The degree, or length, of \( X \), denoted by \( \deg X \), is the dimension of its ring of regular functions as a complex vector space.

If \( X \) is supported on a point \( p \), we define the multiplicity of \( X \), denoted by \( \mu X \), to be the largest natural number \( k \) such that \( X \) contains the \( k \)-tuple point supported on \( p \).

**Proposition 3.** Let \( X, Y \) be 0-dimensional schemes. If \( \deg X = \deg Y \), then \( X = Y \).

If \( X \subset \mathbb{P}^n \) is a 0-dimensional subscheme, then \( \deg X \) is the limit value of the Hilbert function of \( X \). In other words, if \( d \) is large enough, then \( X \) imposes \( \deg X \) independent linear conditions to degree \( d \) divisors of \( \mathbb{P}^n \).

Since we will deal with linear systems with assigned singularities, we now introduce the notations we are going to use.

**Definition 4.** Let \( V \) be a smooth variety, let \( p_1, \ldots, p_r \in V \). The linear system \( \mathcal{L}_{V,d}(m_1, \ldots, m_r)(p_1, \ldots, p_r) \subset \mathbb{P}(H^0\mathcal{O}_V(d)) \) is the projective space of divisors of \( V \) having multiplicities at least \( m_i \) at the point \( p_i \). If either the points \( p_1, \ldots, p_r \) are in general position, or no confusion is likely to arise, we indicate \( \mathcal{L}_{V,d}(m_1, \ldots, m_r) := \mathcal{L}_{V,d}(m_1, \ldots, m_r)(p_1, \ldots, p_r) \).

Moreover, if \( m_1 = \ldots = m_s = m \), then we indicate \( \mathcal{L}_{V,d}(m^s, m_{s+1}, \ldots, m_r) := \mathcal{L}_{V,d}(m_1, \ldots, m_r) \).

We will write \( \mathcal{L}_{n,d}(m_1, \ldots, m_r) \) instead of \( \mathcal{L}_{\mathbb{P}^n,d}(m_1, \ldots, m_r) \). Finally, we will use \( \mathcal{L}_{\mathbb{P}^1 \times \mathbb{P}^1, (a,b)}(m_1, \ldots, m_r) \) to indicate the system of bidegree \( (a, b) \) curves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with the prescribed singularities.

Again, if \( \mathcal{L} \) is a linear system, sometimes with abuse of notation we will use the same symbol to indicate the associated ideal sheaf.

**Definition 5.** The virtual dimension of such a linear system is \( \text{vdim} \mathcal{L}_{V,d}(m_1, \ldots, m_r) = h^0(\mathcal{O}_V(d)) - 1 - \sum_{i=1}^r \left( \frac{m_i - 1 + n}{n} \right) \).

The expected dimension is defined as \( \text{expdim} \mathcal{L}_{V,d}(m_1, \ldots, m_r) = \max \{ \text{vdim} \mathcal{L}_{V,d}(m_1, \ldots, m_r), -1 \} \), where expected dimension \(-1\) indicates that the linear system is expected to be empty. Note that \( \dim \mathcal{L}_{V,d}(m_1, \ldots, m_r) \geq \text{expdim} \mathcal{L}_{V,d}(m_1, \ldots, m_r) \).

When the linear conditions imposed by the base points are dependent, then previous inequality is strict, and the linear system is said to be special. On the other hand, if the
conditions are independent, then \( \dim \mathcal{L}_{V,d}(m_1, \ldots, m_r) = \expdim \mathcal{L}_{V,d}(m_1, \ldots, m_r) \) and the system is called nonspecial.

Not much is known about the speciality of linear systems \( \mathcal{L}_n,d(m_1, \ldots, m_r) \) for an arbitrary \( n \). The most important result in this direction is the celebrated Alexander-Hirschowitz theorem, proven in \( [1] \), that classifies all special linear systems under the assumption \( m_1 = \ldots = m_r = 2 \).

**Theorem 6** (Alexander-Hirschowitz). The linear system \( \mathcal{L}_n,d(2^h) \) is special if and only if \((n,d,h)\) is one of the following:

1. \((n,2,h)\) with \(2 \leq h \leq n\),
2. \((2,4,5), (3,4,9), (4,3,7), (4,4,14)\).

Let us introduce two very classical tools to deal with the computation of the dimension of a linear system. The first one is an useful exact sequence that will help us later.

**Definition 7.** Let \( \mathcal{L} \) be a linear system on a smooth variety \( V \), and let \( S \subset V \) be a hypersurface. Let \( \rho : \mathcal{L} \rightarrow \mathcal{L}|_S \) be the restriction map. Let \( \mathcal{L} - S := \ker(\rho) \), that is \( \mathcal{L} - S = \{0\} \cup \{D \in \mathcal{L} \mid D \supset S\} \). Then we have a short exact sequence of sheaves on \( V \)

\[
0 \rightarrow \mathcal{L} - S \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_S \rightarrow 0,
\]
called the restriction sequence or the Castelnuovo sequence.

By the Castelnuovo sequence, if both \( \mathcal{L} - S \) and \( \mathcal{L}|_S \) are nonspecial of non-negative virtual dimension, then \( \mathcal{L} \) is nonspecial.

Another thing we can do with a linear system \( \mathcal{L} \) is to degenerate it, namely we can pick \( q_1, \ldots, q_r \in V \) and move the singularities of \( \mathcal{L} \) from general position to the point we choose. In this way we have to deal with

\[ \mathcal{L}_0 := \mathcal{L}_{V,d}(m_1, \ldots, m_r)(q_1, \ldots, q_r) \]

instead of \( \mathcal{L} \). If we choose the points \( q_i \) wisely, hopefully we can say something on \( \mathcal{L}_0 \) (for instance, on its dimension) and use semicontinuity to get information about \( \mathcal{L} \). Now we want to make this intuitive notion more precise. The next definitions are based on \([9]\).

**Definition 8.** Let \( Y \) be a smooth variety. A degeneration is a morphism \( \pi : Y \rightarrow \Delta \), where \( \Delta \supseteq 0, 1 \) is a complex disk and \( \pi \) is proper and flat. For any \( t \in \Delta \), we denote by \( Y_t \) the fiber of \( \pi \) over \( t \). Let \( \sigma_i : \Delta \rightarrow Y \) be sections of \( \pi \) and let \( Z \) be a scheme with \( Z_{\text{red}} = \bigcup_i \sigma_i(\Delta) \). For \( t \in \Delta \), define \( Z_t := Z|_{Y_t} \), so that \( Z_0 \) is the flat limit of the schemes \( Z_t \). We say that \( Z_0 \) is a specialization of \( Z_t \).

For the sake of simplicity, sometimes we will say that \( Z_0 \) is a specialization of \( Z_1 \), instead of \( Z_t \), implying that \( 1 \) is any general point of \( \Delta \).

**Construction 9** (Specialization without collisions). Let \( m_1, \ldots, m_r \in \mathbb{N} \) and let \( V \) be a smooth variety. Let \( Y := V \times \Delta \) and let \( \pi : Y \rightarrow \Delta \) be the canonical projection. Fix \( r \) disjoint sections \( \sigma_1, \ldots, \sigma_r : \Delta \rightarrow Y \). Let

\[ Z := \bigcup_{i=1}^r \sigma_i(\Delta)^{m_i} \]

be the scheme supported on the sections with multiplicity \( m_i \) along \( \sigma_i(\Delta) \). Let

\[ \mathbb{L}_{n,d}(m_1, \ldots, m_r)(\sigma_1, \ldots, \sigma_r) \]

be the linear system on \( Y \) associated to degree \( d \) divisors having multiplicities at least \( m_i \) along \( \sigma_i(\Delta) \). Then, for any \( t \in \Delta \), the linear system

\[ \mathbb{L}_t := \mathbb{L}_{n,d}(m_1, \ldots, m_r)(\sigma_1(t), \ldots, \sigma_r(t)) \]

is

\[ \mathcal{L}_t := \mathcal{L}_{V,d}(m_1, \ldots, m_r)(\sigma_1(t), \ldots, \sigma_r(t)). \]

By semicontinuity, we have

\[ h^0(V_0, \mathcal{L}_0) \geq h^0(V_t, \mathcal{L}_t). \]

Therefore, in order to prove the nonspeciality of \( \mathcal{L}_t \), it is enough to produce a specialization such that \( \mathcal{L}_0 \) is nonspecial.
In this paper we are interested in a different kind of degeneration, namely we want to drop the hypothesis that $\sigma_1, \ldots, \sigma_n$ are disjoint. We now modify Construction 9 in order to allow the specialized points to collapse. Since the collision is a local technique, we can work with the affine space instead of a variety $V$. This idea is based on [10].

**Construction 10** (Specialization with $h$ collapsing points). Let $Y = \mathbb{A}^n \times \Delta$, with projections $\nu : Y \to \mathbb{A}^n$ and $\pi : Y \to \Delta$. Fix a point $q \in \mathbb{A}^n \times \{0\}$ and $h$ general sections $\sigma_1, \ldots, \sigma_h : \Delta \to Y$ such that $\sigma_i(0) = q$. Define

$$Z := \bigcup_i \sigma_i(\Delta)^{m_i}.$$ 

Let $X \to Y$ be the blow-up of $Y$ at the point $q$, with exceptional divisor $W$. Then we have a natural morphism $\nu_X : X \to \mathbb{A}^n$, a degeneration $\pi_X : X \to \Delta$, and sections $\sigma_{X,i} : \Delta \to X$. The fiber $X_0$ is reducible, and it is given by $W \cup \tilde{Y}_0$, where $\tilde{Y}_0$ is $\mathbb{A}^n$ blown up at one point and $W \cong \mathbb{P}^{n-1}$ be the exceptional divisor of this blow-up. We want to stress that, since the sections $\sigma_i$ are general, $\{\sigma_{X,i}(0)\}$ is a set of general points of $W$.

With these notations, we say that $Z_0 = Z(\nu_X)$ is the flat limit of $h$ collapsing points of multiplicities $m_1, \ldots, m_h$. Our problem will be to describe $Z_0$. Once we understand the limit, we may study the speciality of a linear system via its specializations with collapsing points, using the same technique described in Construction 9.

**Remark 11.**

(1) Since a collision is a local construction, our results about collisions on $\mathbb{A}^n$ hold on any smooth variety.

(2) When we consider degenerations as in Construction 9 or 10 by flatness we know that the length is preserved, that is, $\deg Z_0 = \deg Z_1$.

(3) One could give the same definitions without requiring that the sections $\sigma_i$ are general, but in this case the theory becomes more involved and less interesting for applications.

As a warm-up, we start with a very easy result that completely describes all collisions of flat points on smooth curves.

**Proposition 12.** Let $m_1, \ldots, m_h \in \mathbb{N}$ and let $m = m_1 + \ldots + m_h$. The limit of $h$ collapsing points of multiplicities $m_1, \ldots, m_h$ in $\mathbb{A}^1$ is an $m$-tuple point.

**Proof.** It is enough to observe that the only scheme of length $m$ supported on a point is the $m$-tuple point. \hfill $\square$

Now that the case $n = 1$ is settled, for the rest of this paper we assume $n \geq 2$ and we try to move to some more interesting cases in higher dimension. In order to understand what $Z_0$ is, the first problem to tackle is to compute its multiplicity. We will show that $\text{mult } Z_0$ does not depend on the choice of the sections $\sigma_i$, and we will give a method to compute it.

In [13] Theorem 2.6], it is proved that the multiplicity of the limit scheme $Z_0$ is at least the minimum integer $j$ such that the linear system $L_{m,j}(m_1, \ldots, m_h)$ is not empty. A different proof can be found in [15] Lemma 20. Now we want to improve this result, and show that the estimated value is actually achieved with equality.

**Proposition 13.** Define $k = \min\{j \in \mathbb{N} \mid H^0(\mathcal{I}_{Z_0}, m_1) \neq 0\}$. Then $\text{mult } Z_0 = k$. In particular, the multiplicity of the limit scheme does not depend on $\sigma_i$, as long as they are general.

**Proof.** It is enough to show that $\text{mult } Z_0 \leq k$. For $t \neq 0$, set $l = h^0(\mathcal{I}_{Z_t}(k))$. Since the base points of $Z_t$ are in general position, $l$ does not depend on $t$, and by hypothesis we know that $l \geq 1$. Let $P \subset A_t = \mathbb{A}^n$ be a set of $l - 1$ general points, and define $Z_t' = Z_t \cup P$. Observe that $Z_t' \supseteq Z_t$ for every $t$, and there is a unique degree $k$ divisor $D_t \subset A_t$ such that $D_t \supseteq Z_t$. Let $f_t$ be the polynomial defining $D_t$ as a divisor in $A_t$. Then $f_{0}$ defines a divisor $D_{0}$ of $A_{0}$ which is the flat limit of the $D_t$'s. The degree of $f_{0}$ is at most $k$ and $D_0 \supseteq Z'_{0} \supseteq Z_0$, so $\text{mult } Z_0 \leq k$. \hfill $\square$

In some cases the multiplicity is enough to compute the limit scheme.
Lemma 14. Let $m_1, \ldots, m_h, n \in \mathbb{N}$ such that
\[
\binom{m_1 + n - 1}{n} + \cdots + \binom{m_h + n - 1}{n} = \binom{m + n}{n}
\]
If $L_{n,m}(m_1, \ldots, m_h)$ is nonspecial, then the limit of $h$ colliding points of multiplicities $m_1, \ldots, m_h$ in $\mathbb{P}^n$ is a $(m + 1)$-tuple point.

Proof. Consider the scheme $Z_1 \subseteq \mathbb{A}^n$ made by $h$ fat points of multiplicities $m_1, \ldots, m_h$. By hypothesis, $L_{n,m}(m_1, \ldots, m_h)$ is nonspecial and it has expected dimension $-1$, so it is empty. On the other hand, $L_{n,m+1}(m_1, \ldots, m_h)$ has positive expected dimension, so it is not empty. By Proposition [13] the limit scheme $Z_0$ contains a $(m + 1)$-tuple point. We know that $\deg Z_0 = \deg Z_1$ by flatness, and by hypothesis the length of $Z_1$ coincides with that of a $(m + 1)$-tuple point. We conclude by Proposition [5].

The previous Lemma will be very useful for our purposes. Indeed, when we use limits to specialize a linear system, the most effective result would be a description of $Z_0$ as a fat point of some multiplicity. However, this can happen only if the hypothesis of Lemma [14] are satisfied. When the scheme we are specializing does not have the degree of a multiple point, this analysis is not enough to determine the limit scheme.

In the notations of Construction [10] consider the limit scheme $Z_0 \subseteq \mathbb{A}^n$. Let
\[
\Sigma_X = \bigcup_{i=1}^{h} \sigma_{X,i}(\Delta)
\]
be the smooth scheme associated to strict transform $Z_X$ of $Z$ on $X$. Let $X' \to X$ be the blow-up of the ideal sheaf $I_{X,i}$, with exceptional divisors $\mathcal{E}_1, \ldots, \mathcal{E}_h$, and let $\varphi : X' \to \Delta$ be the degeneration onto $\Delta$. Note that this blow-up is an isomorphism in a neighbourhood of $Y_0$. The central fiber is $X_0 := \varphi^{-1}(0) = P \cup Y_0$, where $P$ is the blow-up of $W$ at $h$ general points. As before, let $R = P \cap Y_0$. The linear systems we are interested in are $L := \mathcal{O}_X(-\sum 2\mathcal{E}_i - \mu P)$ and its restrictions $L_P$, $L_R$, to $P$ and $R$. The linear system $L$ is complete. However, the following example ([15] Example 2.10) shows that in general $L_R$ is not complete.

Example 15. In the case of 3 colliding double points in $\mathbb{A}^2$, the limit has multiplicity 3. On the other hand, a triple point has degree 6, while $\deg Z_1 = 9$, therefore the limit is not only the triple point. There are 3 more linear conditions the linear system has to satisfy. In order to understand them, observe that a plane cubic with 3 double points in general position is a union of 3 lines. These lines intersect the divisor $R$ in 3 points, and the missing linear conditions are exactly the passage through those 3 points.

It is worth to mention that, unlike $L_R$, the system $L_P$ is always complete, as proved in [15] Lemma 24].

Before we move to the first results on limits, it is important to have clear in mind what kind of characterization we want. In general, it will be way too complicated to determine the limit up to isomorphism. For instance, consider 14 collapsing simple points in $\mathbb{P}^2$. They lie on a unique plane quartic $C$, so mult $Z_0 = 4$. Be $L_{2,4}(1^{14}) = C$, so its restriction to the exceptional line $R$ consists of 4 simple points. Thus the candidate limit is a fourtuple point with 4 tangent directions, and has length
\[
\binom{3+2}{2} + 4 = 10 + 4 = 14 = \deg Z_1.
\]
Therefore, our candidate actually coincides with the limit. However, notice that if we change the sections $\sigma_1, \ldots, \sigma_{14}$, then we will have different tangent directions to the limit. Recall that two 4-tuples of points in $\mathbb{P}^2$ are not projectively equivalent in general, so the limits do not need to be isomorphic. Nonetheless, we will be satisfied to say that the limit is a fourtuple point with 4 infinitely near simple points.

We also want to stress that our analysis works as long as we make all points collide at once. If we collide some of them to a limit scheme $Z_1$ and then we collide the others together with $Z_1$, we are not guaranteed to obtain the same limit scheme as if we collide.
all of them at once. As an example, let $Z_1$ be the scheme consisting of a double point and 3 simple points in $\mathbb{P}^2$. If we make them collide, the multiplicity of the limit scheme $Z_0$ is 3 by Proposition 14. On the other hand, we could collide the 3 simple points to a double point, but the limit of 2 colliding double points has multiplicity 2. However, this kind of multi-staged collisions provides new legitimate ways to degenerate a linear system.

It is time to move to the description of the limit $Z_0$, and we will start with the limit of a bunch of colliding double points. While in some sense it is simpler, the study of collisions of simple points requires a different and peculiar treatment, and will be addressed in another paper.

2. Double points

In this section we assume that all the collapsing points have multiplicity 2. When dealing with linear systems with double points, we are repeatedly using Theorem 6.

First we deal with the cases satisfying the hypothesis of Lemma 14.

Proposition 16. Let $m \geq 2$, $(n, m) \notin \{(2, 3), (2, 5), (4, 4), (4, 5)\}$. Define $h = \frac{(n+m-1)}{n+1}$. If $h \in \mathbb{N}$, then the limit of $h$ colliding double points in $\mathbb{P}^n$ is an $m$-tuple point.

Proof. By our numerical assumption, together with Theorem 6, $L_{n,m-1}(2^h)$ is nonspecial. Now the thesis follows by Lemma 14.

As we already noticed, in most cases the limit is not just a point with multiplicity. As Example 15 shows, once we understand the minimum degree of a divisor containing $Z_1$, we need information on the base locus of such divisors.

When we deal with double points, it is convenient to work in the case $h > n$. Indeed, $h \leq n$ yields mult $Z_0 = 2$, and $L_{n,2}(2^h)$ has a nonreduced base locus, so it is difficult to describe the conditions imposed on the limit linear system. On the other hand, if $h > n$ then we have mult $Z_0 = 3$, at least for $n$ big enough, and the base locus of cubics with assigned double points is very well understood.

First we need a technical result.

Lemma 17. Let $n \geq 2$ and $l \leq n + 2$. Let $A = \{a_1, \ldots, a_l\}$ be a set of $l$ general points in $\mathbb{P}^n$ and let $R$ be a hyperplane such that $A \cap R = \emptyset$. Let

$$B = \{p_{ij} : a_i \cap R \mid 1 \leq i < j \leq l\}.$$ 

Then $L_{n-1,2}(B)$ and $L_{n-1,3}(B)$ are nonspecial, that is, the points of $B$ impose independent conditions to quadrics and cubics of $R$.

Proof. It is enough to prove the claim for $l = n + 1$ for quadrics and $l = n + 2$ for cubics. $L_{n-1,2}(B)$ is nonspecial by 15, Lemma 25.

Now assume that $l = n + 2$. We prove that the points of $B$ are general for cubics by induction on $n$. It is easy to check that the thesis holds for $n = 2$, so we assume $n \geq 3$. Specialize $a_1, \ldots, a_{n+1}$ on a general hyperplane $L \subset \mathbb{P}^n$. Define

$$B_1 = \{p_{ij} : 1 \leq i < j \leq n + 1\} \text{ and } B_2 = \{p_{1,n+2}, \ldots, p_{n+1,n+2}\}.$$ 

Observe that the points of $B_2$ are in general position on $R$, and $B = B_1 \cup B_2$. Let $H = L \cap R = \mathbb{P}^{n-2}$. Castelnuovo exact sequence reads

$$0 \to I_{B_2,R}(2) \to I_{B,R}(3) \to I_{B_1,R}(3) \to 0.$$ 

Since $B_2$ is a set of general points of $R$, $h^1 I_{B_2,R}(2) = 0$. If we set $A_1 = \{a_1, \ldots, a_{n+1}\},$

then $A_1$ is a set of general points in $L$ and $H$ is an hyperplane of $L$ such that $A_1 \cap H = \emptyset$. By induction hypothesis, $h^1 I_{B_1,R}(3) = 0$. Hence $h^1 I_{B,R}(3) = 0$ and so $B$ imposes independent conditions on cubics of $R$.

Remark 18. Note that even if $B$ imposes independent conditions, the points of $B$ are not in linear general position. For every choice of $t$ points of $A$, their span is a $\mathbb{P}^{t-1}$, so the corresponding $\binom{l}{t}$ points of $B$ lie on a $\mathbb{P}^{l-t}$.

The next two Propositions, proven in 15, solve the cases $h = n + 1$ and $h = n + 2$. 

3. Simple points
Proposition 19. The limit of \( n + 1 \) collapsing double points in \( \mathbb{P}^n \) is a triple point with \( \binom{n+1}{2} \) tangent directions. The infinitely near simple points are in the special position described by Remark \[14\].

Proposition 20. Let \( Z_0 \) be the limit of \( n + 2 \) collapsing double points.

1. If \( n = 2 \), then \( Z_0 \) is a 4-tuple point, together with the involution described in [10] Proposition 3.1.
2. If \( n = 3 \), then \( Z_0 \) is a 4-tuple point.
3. If \( n \geq 4 \), then \( Z_0 \) is a triple point with \( \binom{n+2}{2} \) tangent directions. In this case the infinitely near simple points are in the special position described by Remark \[14\].

The proofs rely on Proposition \[13\] to compute the multiplicity. Then we determine the base locus of the linear system and we apply Lemma \[17\] to check that the tangent directions give independent conditions.

Despite the previous results, the limit scheme can be more complicated than a fat point with a bunch of infinitely near points. Such problems may occur even in low dimension and when all the multiplicities are 2. For instance, the limit of 5 colliding double points in \( \mathbb{P}^2 \) is described in [10] Proposition 3.1 as a fourtuple point with a pair of infinitely near tacnodal points.

We could try to apply the argument of Propositions \[14\] and \[20\] to an higher number of colliding double points. Anyway, we can not expect the same proof to work, because Lemma \[17\] does not hold for \( l \geq n + 3 \).

As an example, let us work out one of the exceptions of Theorem \[6\].

Example 21. Consider a set of general points \( A = \{a_1, \ldots, a_7\} \subset \mathbb{P}^4 \). As in the setting of Lemma \[17\] let \( R \) be a hyperplane such that \( A \cap R = \emptyset \) and

\[ B = \{p_{ij} := \langle a_i, a_j \rangle \cap R \mid 1 \leq i < j \leq 7\}. \]

Indeed, in that case \( |B| = 21 \) and \( h^0 \mathcal{O}_R(3) = 20 \). We know there is exactly one cubic \( C \) singular at \( a_1, \ldots, a_7 \). \( C \) contains all the lines joining pairs of points of \( A_0 \), so in particular \( C \cap R \supset B \). Moreover, consider the Castelnuovo exact sequence

\[ 0 \to \mathcal{L}_{4,2}(2^7) \to \mathcal{L}_{4,3}(2^7) \to \mathcal{I}_{B,R}(3) \to 0. \]

Since \( \mathcal{L}_{4,2}(2^7) \) has no global sections, the restriction \( \mathcal{L}_{4,3}(2^7) \to \mathcal{I}_{B,R}(3) \) is injective and therefore \( \mathcal{L}_{4,3}(2^7) \subset \mathcal{I}_{B,R}(3) \). This means there is at least one cubic of \( R \) containing \( B \). Since \( h^0 \mathcal{O}_R(3) = 20 \), the 21 points of \( B \) impose at most 19 independent conditions on cubics of \( R \). An easy software-aided computation shows that \( B \) actually imposes exactly 19 independent conditions.

More generally, let \( Z_1 \) be a scheme of \( n + 3 \) double points, with \( n \geq 5 \). Observe that \( \deg Z_1 = (n + 1)(n + 3) \) and \( \mult Z_0 = 3 \). It is easy to see that \( \text{Bs} \mathcal{L}_{n,3}(2^{n+3}) \) consists of the double points and of the \( \binom{n+3}{2} \) lines joining the pair of points. Then we have \( \binom{n+3}{2} \) simple points infinitely near to the limit triple point. However, these simple points do not give independent conditions. Indeed, if they did, then

\[ \deg Z_0 \geq \binom{n+2}{2} + \binom{n+3}{2} = n^2 + 4n + 4 = 1 + \deg Z_1. \]

Hence those \( \binom{n+3}{2} \) simple points impose at most \( \binom{n+3}{2} - 1 \) conditions on \( Z_0 \). On the other hand, at least \( \binom{n+2}{2} \) of the simple points impose independent conditions by Lemma \[17\].

How can we give a description of the limit is these cases?

Remark 22. Let \( Z \) be an \( m \)-tuple point with an infinitely near simple point, and let \( l \) be the line through \( Z \) corresponding to the infinitely near point. The restriction of \( Z \) to a general line is an \( m \)-tuple point, while \( Z|_l \) has multiplicity \( m + 1 \). This suggest a possible description of the limit of \( n + k \) collapsing double points. Assume that \( \mult Z_0 = 3 \), and let \( l_1, \ldots, l_{\binom{n+k}{2}} \) be the base lines, all passing through the limit point \( q \). Let \( S_i^k \) be the multiplicity 4 subscheme of \( l_i \), supported at \( q \). We know that \( Z_0 \) contains the union of the \( S_i^k \)'s, and we conjecture that they coincide. Now we want to precisely formulate the problem and to provide a solution for small \( k \).
Definition 23. Let $n, m \geq 2$, and let $l_1, \ldots, l_t \subset \mathbb{A}^n$ be lines meeting at the origin. Let $S_m^l$ be the 0-dimensional degree $m$ subscheme of $l_t$ supported at the origin, and let $I_{n,S_m^l}$ be the ideal defining $S_m^l$ in $\mathbb{A}^n$. Define $Z_n(l_1, \ldots, l_t)$ to be the union scheme defined by the ideal

$$I_{n,S_m^l} = I_{n,S_m^l} \cap \cdots \cap I_{n,S_m^l}. $$

If $l_1, \ldots, l_t$ are general lines through the origin and $m = 4$, then we define

$$Z_{n,t} = Z_n(l_1, \ldots, l_t) $$

$$I_{n,t} = I_{n,l_1, \ldots, l_t}.$$

When mult $Z_{n,t} = 3$, we can think of this scheme as a triple point with $t$ infinitely near simple points, representing the directions corresponding to $l_1, \ldots, l_t$.

Remark 24. Consider $n + k$ colliding double points in $\mathbb{A}^n$ and assume the limit has multiplicity 3. Then the limit triple point has $\binom{n + k}{2}$ infinitely near simple points, in special position, giving possibly dependent conditions on cubics. Nevertheless, the restriction of the limit scheme to one of the $\binom{n + k}{2}$ corresponding lines $l_1, \ldots, l_{n+k}$ has degree strictly greater than 3. In particular the limit scheme contains $Z_n\{l_1, \ldots, l_{\binom{n+k}{2}}\}$. So if we prove that they have the same degree, then we get an explicit description of the limit scheme.

We aim to identify the limit of a bunch of colliding double points with a scheme of the form $Z(l_1, \ldots, l_t)$. For this reason, our next task is to study such schemes. First we compute the multiplicity of the scheme $Z_n(l_1, \ldots, l_t)$.

Lemma 25. Let $R = \mathbb{A}^{n-1}$ be a general hyperplane in $\mathbb{A}^n$, and $p_i = l_i \cap R$. Define $P = \{p_1, \ldots, p_t\}$ and set

$$k = \min\{m \in \mathbb{N} \mid H^0 P_R(m) \neq 0\}. $$

Then mult $Z(l_1, \ldots, l_t) = \min(4, k)$.

Proof. First note that mult $Z_n(l_1, \ldots, l_t)$ is nondecreasing with respect to $t$. Moreover, mult $Z_n(l_1, \ldots, l_t)$ is nondecreasing with respect to $t$. Moreover, mult $Z_n(l_1, \ldots, l_t) \leq 4$ by construction. Indeed, once multiplicity 4 is reached, the restriction to any line has degree at least 4, so by adding another $S^1_l$ we do not change anything. Now let $D \subset R$ be a degree $m$ divisor containing $p_1, \ldots, p_t$. The cone $C$ over $D$ with vertex the origin is a degree $m$ divisor in $\mathbb{A}^n$ containing $l_1, \ldots, l_t$, and therefore $C \supset S^1_l \cup \cdots \cup S^1_t$. Hence the ideal of $Z_n(l_1, \ldots, l_t)$ contains a generator of degree $m$ and so mult $Z_n(l_1, \ldots, l_t) \leq m$. This implies mult $Z_n(l_1, \ldots, l_t) \leq \min(4, k)$.

On the other hand, if mult $Z_n(l_1, \ldots, l_t) = 4 \geq \min(4, k)$, then there is nothing else to prove. Suppose that $m := \text{mult} Z_n(l_1, \ldots, l_t) \in \{1, 2, 3\}$. Then it is contained in a degree $m$ divisor $C \subset \mathbb{A}^n$. Since it has an $m$-tuple point, $C$ is a cone. Moreover the restriction of $Z_n(l_1, \ldots, l_t)$ to each $l_i$ has degree $4 > m$ so $C$ contains each $l_i$, and in particular $C|_R$ is a degree $m$ divisor in $R$ containing $p_1, \ldots, p_t$.

Corollary 26. Let $t \in \mathbb{N}$ and let $R = \mathbb{A}^{n-1}$ be a general hyperplane in $\mathbb{A}^n$. Set

$$k = \min\{m \in \mathbb{N} \mid h^0 O_R(m) > t\}. $$

If $l_1, \ldots, l_t$ are general lines, then mult $Z_{n,t} = \min(4, k)$.

Proof. Apply Lemma 25 in the case $p_1, \ldots, p_t \in R$ are general.

Now we want to determine the length of $Z_n(l_1, \ldots, l_t)$. The next Lemma provides a way to compute it inductively.

Lemma 27. Let $n \geq 2$. Then

1. $\deg Z_n(l_1) = 4$.
2. $\deg Z_n(l_1, \ldots, l_i, l_{i+1}) = \deg Z_n(l_1, \ldots, l_i) + 4 - \deg (Z_n(l_1, \ldots, l_i)l_{i+1})$.
3. $\deg Z_{n+1} = \deg Z_n + 4 - \text{mult } Z_{n,t}$.

Proof. (1) The length of $Z_n(l_1)$ does not depend on the immersion. Regarding $Z_n(l_1) = S^1_l$ as a divisor in $l_1 = \mathbb{P}^1$, it has degree 4 by construction.
(2) Let $\mu = \deg (Z_n(l_1, \ldots, l_t)_{l_{t+1}})$. Of course $Z_n(l_1, \ldots, l_t) \supset S^l_{t+1}$, so

$$Z_n(l_1, \ldots, l_t) = S^l_{t+1} \cup \ldots \cup S^l_1 = S^l_1 \cup \ldots \cup S^l_t \cup S^\mu_{t+1}.$$ 

Hence the difference $\deg Z_n(l_1, \ldots, l_t, l_{t+1}) - \deg Z_n(l_1, \ldots, l_t)$ coincides with the difference $\deg S^l_{t+1} - \deg S^\mu_{t+1} = 4 - \mu$.

(3) When $l_1, \ldots, l_t, l_{t+1}$ are general, the restriction of $Z_n,t$ to $l_{t+1}$ has degree equal to $\mult Z_n,t$, so it is enough to apply (2).

Example 28. Corollary 26 and Lemma 27 allow us to compute multiplicity and degree of the scheme $Z_n,t$ for every $n$ and $t$. As an example, here is the table for $n = 2$.

| $t$ | $\deg Z_2,t$ | $\mult Z_2,t$ |
|-----|---------------|----------------|
| 1   | 4             | 1              |
| 2   | 7             | 2              |
| 3   | 9             | 3              |
| $t \geq 4$ | 10          | 4              |

Now we consider what happens when the lines are not general, in particular when they have the configuration described in Remark 18.

Definition 29. Let $\{l_{ij} \mid 1 \leq i < j \leq m\}$ be a set of $\binom{m}{2}$ lines in $\mathbb{A}^n$ meeting at the origin, such that $l_{ab}, l_{bc}$ and $l_{ac}$ lie on the same plane for every $\{1 \leq a < b < c \leq m\}$.

Define $\tilde{Z}_{n,(\binom{m}{2})} = Z_n(l_{ij} \mid 1 \leq i < j \leq m)$.

Remark 30. Let $n, m \geq 2$. We start with the following simple observations.

1. $Z_2(\binom{m}{2}) = \tilde{Z}_2(\binom{m}{2})$.
2. $\tilde{Z}_{n,1} = Z_{2,1}$ and $\tilde{Z}_{n,3} = Z_{2,3}$.
3. More generally, if $n \geq m$, then $\langle l_1, \ldots, \binom{m}{2} \rangle = \mathbb{A}^{m-1}$. Thus $\mult \tilde{Z}_{n,(\binom{m}{2})} = 1$ and $\tilde{Z}_{n,(\binom{m}{2})} = \tilde{Z}_{m-1,(\binom{m}{2})}$.

We are now ready to compute the multiplicity and degree of $\tilde{Z}_{n,(\binom{m}{2})}$. By Remark 30 we know the multiplicity and degree of $\tilde{Z}_{2,(\binom{m}{2})}$ from Lemma 27. Now we tackle the cases $n = 3$ and $n = 4$.

Example 31. The next table shows the values of $\deg \tilde{Z}_{3,(\binom{m}{2})}$ and $\mult \tilde{Z}_{3,(\binom{m}{2})}$.

| $m$ | $t$ | $\deg \tilde{Z}_{3,t}$ | $\mult \tilde{Z}_{3,t}$ |
|-----|-----|-------------------------|-------------------------|
| 2   | 1   | 4                       | 1                       |
| 3   | 3   | 9                       | 1                       |
| 4   | 6   | 16                      | 3                       |
| $m \geq 5$ | $t \geq 10$ | 20                     | 4                       |

Degrees and multiplicities of $\tilde{Z}_{4,(\binom{m}{2})}$ are presented in the following one.

| $m$ | $t$ | $\deg \tilde{Z}_{4,t}$ | $\mult \tilde{Z}_{4,t}$ |
|-----|-----|-------------------------|-------------------------|
| 2   | 1   | 4                       | 1                       |
| 3   | 3   | 9                       | 1                       |
| 4   | 6   | 16                      | 1                       |
| 5   | 10  | 25                      | 3                       |
| 6   | 15  | 30                      | 3                       |
| 7   | 21  | 34                      | 3                       |
| $m \geq 8$ | $t \geq 28$ | 35                     | 4                       |

In order to compute the multiplicities, it is enough to apply Remark 30 and Lemma 26 together with Lemma 17. After that, Lemma 27 allows us to compute the degree. We only have to pay attention for $(n, m) = (4, 7)$. Indeed, this is an exception of Theorem 6 and we already considered it in Example 21.
If we look at \( \tilde{Z}_{4,6} \) and \( \tilde{Z}_{4,10} \), we see that their multiplicities and degrees are consistent with the cases of 4 and 5 collapsing double points in \( \mathbb{A}^3 \). In the same way, the numbers we found about \( \tilde{Z}_{4,10} \) and \( \tilde{Z}_{4,15} \) are consistent with the case of 5 and 6 colliding double points in \( \mathbb{A}^4 \).

We will try now to find a general statement about the degree and the multiplicity of \( \tilde{Z}_{n,\binom{n}{2}} \). The situation is easy when \( m \leq n \).

**Proposition 32.** Let \( n \geq 2 \). If \( 3 \leq m \leq n \), then
\[
\text{mult} \tilde{Z}_{n,\binom{n}{2}} = 1 \quad \text{and} \quad \deg \tilde{Z}_{n,\binom{n}{2}} = m^2.
\]

**Proof.** If \( m \leq n \), then \( \text{mult} \tilde{Z}_{n,\binom{n}{2}} = 1 \) by Remark 30.3.

We prove the statement about the degree by induction on \( m \). We saw that \( \deg \tilde{Z}_{n,3} = 9 \).

Let us assume \( \deg \tilde{Z}_{n,\binom{n}{2}} = m^2 \) and let us compute \( \deg \tilde{Z}_{n,\binom{m+1}{2}} \). \( \tilde{Z}_{n,\binom{m+1}{2}} \) is obtained from \( \tilde{Z}_{n,\binom{n}{2}} \subset \mathbb{A}^m = H \) by adding \( S_{1,m+1}, \ldots, S_{m,m+1} \). Observe that \( S_{1,m+1} \not\subset H \), so it increases the degree by 3; the resulting scheme is contained in some \( P = \mathbb{A}^{m+1} \), and by adding \( S_{1,m+1}, \ldots, S_{m-1,m+1} \), we remain inside \( P \). As a subscheme of \( P \), \( \tilde{Z}_{n,\binom{m+1}{2}} \) has multiplicity 2, because there are only \( \binom{m}{2} + m - 1 < h^0 \mathcal{O}_{\mathbb{A}^{m+1}}(2) \) lines. But we know that, even if they are in special position, they are general for quadrics, so each new addition of \( S_{4,m+1}, \ldots, S_{m,m+1} \) increases the degree by \( 4 - 2 = 2 \). Hence
\[
\deg \tilde{Z}_{n,\binom{m+1}{2}} = \deg \tilde{Z}_{n,\binom{m}{2}} + 3 + 2(m - 1) = m^2 + 2m + 1 = (m + 1)^2,
\]
and therefore the thesis holds. \( \square \)

Before we move to the more interesting case \( m > n \geq 5 \), we need some technical results. We already observed that Lemma 17 does not hold in the case of more than \( n + 2 \) points in \( \mathbb{P}^n \), so our next goal is to understand what happens with larger numbers of points. In particular, we are looking for a suitable generalization of Lemma 17.

**Lemma 33.** For \( k \in \mathbb{N} \), define
\[
n_k = \min \left\{ t \geq 2 \mid \frac{\binom{t+3}{3}}{t+1} - n > k \right\}.
\]

For every \( n \geq n_k \) and every \( r \in \mathbb{N} \), let \( A_r = \{a_1, \ldots, a_r\} \subset \mathbb{P}^n \) be a set of \( r \) general points, and let \( R \subset \mathbb{P}^n \) be a hyperplane such that \( A_r \cap R = \emptyset \). Let
\[
B_r = \{(a_i, a_j) \cap R \mid 1 \leq i < j \leq r\}.
\]

Assume that \( B_{n+k} \) imposes \( \binom{n+k}{2} - \binom{k+1}{2} \) independent conditions to cubics of \( R \). Then \( B_{n+k} \) impose exactly \( \binom{n+k}{2} - \binom{k+1}{2} \) independent conditions to cubics of \( R \) for every \( n \geq n_k \).

**Proof.** We prove the statement by induction on \( n \geq n_k \). The first step of induction is granted by hypothesis, so we suppose that \( n > n_k \). In order to lighten the notation, throughout this proof we will write \( A \) and \( B \) instead of \( A_{n+k} \) and \( B_{n+k} \). Specialize \( a_1, \ldots, a_{n+k-1} \) on \( L = \mathbb{P}^{n-1} \). Define
\[
B_1 = \{p_{ij} \mid 1 \leq i < j \leq n + k - 1\} \quad \text{and} \quad B_2 = \{p_{1,n+k}, \ldots, p_{n+k-1,n+k}\}.
\]

Let \( H = L \cap R = \mathbb{P}^{n-2} \). When we restrict to \( H \), Castelnuovo exact sequence reads
\[
0 \rightarrow \mathcal{I}_{B_2,R}(2) \rightarrow \mathcal{I}_{B,R}(3) \rightarrow \mathcal{I}_{B_1,H}(3) \rightarrow 0.
\]

First observe that the points of \( B_2 \) are general on \( R \), so
\[
h^0 \mathcal{I}_{B_2,R}(2) = \binom{n+1}{2} - (n + k - 1) \quad \text{and} \quad h^1 \mathcal{I}_{B_2,R}(2) = 0.
\]

Now we want to compute the dimension of the right hand side of the sequence. Note that \( A_1 := \{a_1, \ldots, a_{n+k-1}\} \) is a set of general points in \( L = \mathbb{P}^{n-1} \), \( H \) is a hyperplane of \( L \) with \( A_1 \cap H = \emptyset \) and \( B_1 = \{(a_i, a_j) \cap H \mid 1 \leq i < j \leq n + k - 1\} \), so by induction hypothesis
\[
h^0 \mathcal{I}_{B_1,H}(3) = \binom{n+1}{3} - \binom{n+k-1}{2} + \binom{k-1}{2}.
\]
Therefore
\[ h^0 L_{B,R}(3) = h^0 L_{B_2,R}(2) + h^0 L_{B_1,R}(3) \]
\[ = \left( \frac{n+1}{2} \right) - (n+k-1) + \left( \frac{n+1}{3} \right) - \left( \frac{n+k-1}{2} \right) + \left( k - \frac{1}{2} \right). \]

Since the points of $B$ impose $\binom{n+k}{2} - \binom{k-1}{2}$ conditions in this specialized configuration, they impose at least $\binom{n+k}{2} - \binom{k-1}{2}$ conditions in the original configuration. We already noticed they can not impose more than $\binom{n+k}{2} - \binom{k-1}{2}$ conditions. \qed

Lemma 33 provides an inductive way to prove that $B$ imposes the suitable number of conditions on cubic of $R$. However, in order to apply it we need the first step of induction for every $k$. While we are not able to prove this first step in general, we believe this is the right way to compute the number of independent conditions imposed by $B$.

**Conjecture 34.** Assume $0 \leq k < \frac{(n+3)}{n+1} - n$. Let $A = \{a_1, \ldots, a_{n+k}\}$ be a set of $n+k$ general points in $\mathbb{P}^n$ and $R$ a hyperplane such that $A \cap R = \emptyset$. Let
\[ B = \{(a_i, a_j) \cap R \mid 1 \leq i < j \leq n+k\}. \]

Then the points of $B$ impose exactly $\binom{n+k}{2} - \binom{k-1}{2}$ independent conditions to cubics of $R$.

By applying Lemma 33 it is easy to prove that Conjecture 34 holds for $k \in \{0, 1, 2\}$, and in this way we recover some of the results of Lemma 17. Moreover, the software Macaulay2 ([16]) allows us to prove the first step for $k \leq 4$ as well.

**Remark 35.** Assume Conjecture 34 is true. Then we have a way to compute degree and multiplicity of $\tilde{Z}_{n,(\frac{n+k}{2})}$ for every $n$ and $m$. Indeed, in this case it is possible to prove that
\[ \text{mult } \tilde{Z}_{n,(\frac{n+k}{2})} = 3 \quad \text{and} \quad \deg \tilde{Z}_{n,(\frac{n+k}{2})} = (n+1)(n+k). \]

Therefore, under this assumption, the limit of $n + k$ collapsing double points in $\mathbb{A}^n$ is $\tilde{Z}_{n,(\frac{n+k}{2})}$. Since we know that Conjecture 34 holds for small values of $k$, that improves Propositions 19 and 20. However, this approach only works in the range
\[ 1 \leq k < \frac{(n+3)}{n+1} - n. \]

When $k \leq 0$, the limit scheme has multiplicity 2. As we already pointed out, the linear system $L_{n,2}(2^{n+k})$ has nonreduced base locus, and this makes it difficult to understand the first order neighbourhood of the limit point. On the other hand, when $n + k \geq \frac{(n+3)}{n+1}$, the limit scheme has multiplicity at least 4 and the base locus may not give us information. It is enough to consider $(n,k) = (3,3)$ to bump into the linear system $L_{3,4}(2^n)$, which has no base locus outside the imposed singularities. Our work on infinitesimally near points gives us no clue in this type of cases.

One could argue in a similar way with higher multiplicities, and hope to find other cases in which there are base lines. For instance, we could work with triple points, and we know that the lines joining a pair of triple points are in the base locus of quintics. Unfortunately, this strategy works only if we know the degree of the linear system we are dealing with. By Proposition 18 this is equivalent to compute the smallest degree of a divisor in $\mathbb{P}^n$ containing a bunch of general multiple points. This is a hard problem, and the answer is unknown in its generality even in the planar case.

It is also worth to mention that, given a scheme $X \subset V$ made by a triple point with $t$ tangent directions, in general we can not produce $X$ as a limit of double points. Indeed, first we need that $t = \binom{n+k}{2}$ for some $k$ in the range $[1]$. Moreover, the tangent directions have to be in the special position described in Remark 18. It is legitimate to wonder if
there are more conditions to be met in order to express $X$ as a limit of double points. In other words, can we lift $X$ to a bunch of double points in such a way that $X$ is the limit of those colliding points, under the previous assumptions?

We will now give a positive answer to this question. Remark 18 describes the configurations of the points in the exceptional divisor and suggests the following definition.

**Definition 36.** Let $n \geq 2$ and $t \geq 3$. Define
\[
W_{n,t} = \left\{ (x_{ij})_{1 \leq i < j \leq t} \in (\mathbb{P}^n)^t \mid x_{bc} \in \langle x_{ab}, x_{ac} \rangle \forall 1 \leq a < b < c \leq t \right\}.
\]
If we look at $R = \mathbb{P}^n$ as a general hyperplane in $\mathbb{P}^{n+1}$, there is a rational map
\[
\pi_{n,t} : (\mathbb{P}^{n+1})^t \to W_{n,t} \subset (\mathbb{P}^n)^t
\]
defined by sending $(p_1, \ldots, p_t)$ to $(x_{ij})_{1 \leq i < j \leq t}$, where $x_{ij}$ is the intersection of the line $(p_i, p_j)$ with $R$.

For $1 \leq k \leq 4$, we know that the limit of $n + k$ double points in $\mathbb{P}^n$ is a triple point with $\binom{n+2}{k}$ infinitely near simple points. The simple points form a $\binom{n}{2}$-tuple $(x_{ij})_{1 \leq i < j \leq n+k} \in W_{n,n+k}$. We want to understand whether all such schemes can be obtained as limits of double points. This is equivalent to ask if $\pi_{n,n+1}$ is dominant, and our next task is to give a positive answer, by proving the following result.

**Theorem 37.** $\pi_{n,t}$ is dominant for every $n \geq 2$ and every $t \geq 3$. The general fiber has dimension $n + 2$.

Let us start with some simple observations.

**Observation 38.**
1. We have $\dim W_{n,t} = n(t - 1) + t - 2$. Indeed, one can choose freely $t - 1$ general points $x_{12}, \ldots, x_{1t} \in \mathbb{P}^n$. Then, for $i \in \{3, \ldots, t\}$, it is possible to choose the $t - 2$ points $x_{2i}$ general on $\langle x_{12}, x_{1i} \rangle$. After that, for $3 \leq j < k \leq t$, the other points $x_{jk}$ are defined by $\langle x_{1j}, x_{1k} \rangle \cap \langle x_{2j}, x_{2k} \rangle$.

2. Assume that $n \geq 3$ and let $(x_{ij})_{1 \leq i < j \leq t} \in W_{n,t}$. For $1 \leq a < b < c \leq t$, let $l_{abc}$ be the line containing $x_{ab}, x_{ac}, x_{bc}$. Note that $l_{abc}$ and $l_{abd}$ meet at $x_{bc}$, so they span a plane containing $l_{abc}$ and $l_{abd}$ as well. This plane therefore passes through the 6 points $\{x_{ij} \mid i, j \in \{a, b, c, d\}, i < j\}$. By the same argument, for every choice of $m$ indexes $1 \leq i_1 < \cdots < i_m \leq t$, the $\binom{n}{m}$ points $\{x_{ij} \mid 1 \leq i_1 < \cdots < i_m \leq t\}$ lie on the same $\mathbb{P}^{m-2}$.

3. In particular, if $t \leq n+1$, then $p_{i_1}, \ldots, p_{i_t} \in \mathbb{P}^{n+1}$ lie on a linear subspace $L = \mathbb{P}^{t-1}$. Hence the $\binom{t}{2}$ points $(p_i, p_j) \cap R$ all lie on $L \cap R = \mathbb{P}^{t-2}$. Then $W_{n,t} = W_{t-2,1}$, and $\pi_{n,t}$ restricts to $\pi_{t-2,1} : L^t = (\mathbb{P}^{t-1})^t \to W_{t-2,1}$. For this reason, from now on we will assume $t \geq n + 2$.

The next Lemma is the first step towards the proof of Theorem 37.

**Lemma 39.** $\pi_{n,n+2} : (\mathbb{P}^{n+1})^{n+2} \to W_{n,n+2}$ is dominant for every $n$. The general fiber has dimension $n + 2$.

**Proof.** Let $x = (x_{ij})_{1 \leq i < j \leq n+2} \in W_{n,n+2}$ be general. For $i \in \{1, \ldots, n+2\}$, let $L_i = \langle x_{jk} \mid j, k \neq i \rangle$ be the dimension $n - 1$ linear subspace of $R = \mathbb{P}^n$ obtained by choosing all indexes except $i$. Let $\Pi_i \subset \mathbb{P}^{n+1}$ be a general hyperplane containing $L_i$. For $j \in \{1, \ldots, n+2\}$, define the point
\[
P_j = \bigcap_{i \neq j} \Pi_i.
\]
If $k, h \in \{1, \ldots, n+2\}$ and $h \neq k$, then $p_k$ and $p_h$ are distinct points of the line $\bigcap_{i \neq k, h} \Pi_i$, so
\[
(p_k, p_h) \cap R = \bigcap_{i \neq k, h} \Pi_i \cap R = \bigcap_{i \neq k, h} L_i,
\]
which is one of the $x_{ij}$’s. Then, up to reorder, $(p_1, \ldots, p_{n+2})$ is a preimage of $(x_{ij})_{1 \leq i < j \leq n+2}$.

To determine the dimension of the general fiber, we can either note that for each of the $n + 2$ points $p_i$ we chose a hyperplane $\Pi_i$ in the pencil of those containing $L_i$, or we can compute the difference $\dim(\mathbb{P}^{n+1})^{n+2} - \dim W_{n,n+2}$. \qed
It is worth to note that one could give the definition of $W_{1,t}$ and $\pi_{1,t}$ as well. However, we are computing limits under the assumption that $n \geq 2$. Moreover $W_{1,t}$ coincides with $(\mathbb{P}^1)^{(l)}$, so the case $n = 1$ is not very interesting for our purpose.

We are now ready to prove the result we claimed.

Proof of Theorem 37. As we noticed in Observation 38, we may assume that $t \geq n + 2$. We argue by induction on $t$. The case $t = n + 2$ is the content of Lemma 39 so we focus on the case $t > n + 2$.

Let $(x_{ij})_{1 \leq i < j \leq t} \in W_{n,t}$ be general. By induction hypothesis exist $t - 1$ general points $p_1, \ldots, p_{t-1} \in \mathbb{P}^{n+1}$ such that $(p_i, p_j) \cap R = x_{ij}$. Define

$$ p_i = (p_1, x_{1i}) \cap (p_2, x_{2i}). $$

In order to conclude, we have to make sure that $(p_i, p_l)$ meets $R$ at $x_{il}$ for every $i \in \{3, \ldots, t - 1\}$. First observe

$$ (x_{1i}, x_{1l}) = (p_1, x_{1i}) \cap R = (p_1, p_i, p_l) \cap R, $$

because $p_i \in (p_1, x_{1l})$ by construction. Hence

$$ (p_i, p_l) \cap R = ((p_1, p_i) \cap (p_2, p_l)) \cap R $$

$$ = (p_1, p_i) \cap R \cap (p_2, p_l) \cap R $$

$$ = (x_{1i}, x_{1l}) \cap (x_{2i}, x_{2l}) = x_{il}. $$

The general fiber has dimension $\dim (\mathbb{P}^{n+1})^{t} - \dim W_{n,t} = n + 2$. \qed

In terms of collision, this means that if $t \in \{n + 1, \ldots, n + 4\}$, then every scheme in $\mathbb{P}^n$ made by a triple point with $(\binom{t}{2})$ infinitely near simple points $x_{ij}$ such that $(x_{ij})_{1 \leq i < j \leq t}$ is a general point of $W_{n,t}$ can be obtained as a limit of $t$ collapsing double points in $\mathbb{P}^{n+1}$. If Conjecture 34 is true, the same holds for the collision of $t$ points, where

$$ n + 1 \leq t < \frac{(n+3)}{4} \frac{n}{n + 1}. $$

3. Other collisions

Degenerations are widely used in interpolation theory to compute the dimension of linear systems. The most studied cases are dimension 2 and 3, where there are conjectures about the reasons why a linear system is special. For $n \in \{2, 3\}$, all known special linear systems $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_r)$ have a base locus containing a particular variety, with precise properties. Roughly speaking, what those conjectures state is that the only known geometric reason for a linear system to be special is the existence of such a special effect variety in its base locus. The precise definition about special effect varieties can be found in [1]. Some examples of special effect varieties are known, see [5] and [6], and the hard problem is to classify all of them. We will not look into special effect varieties, but we will exploit some of the results of interpolation theory to try to describe some limits of colliding multiple points.

As a first example, we can easily extend Proposition 10 to higher multiplicity.

Proposition 40. Let $l, m \in \mathbb{N}$, let $n \in \{2, 3\}$ and define

$$ h = \frac{(m+n-1)}{(l+n-1)} $$

Assume that $h \in \mathbb{N}$.

1. If $l, m \leq 42$ and $h \geq 10$, then the limit of $h$ collapsing $l$-tuple points in $\mathbb{A}^2$ is an $m$-tuple point.

2. If $l \leq 5$ and $m \geq 2l + 1$, then the limit of $h$ collapsing $l$-tuple points in $\mathbb{A}^3$ is an $m$-tuple point.

Proof. (1) Since $\mathcal{L}_{2,m-1}(n^h)$ is a system of plane curves whose $h \geq 10$ multiplicities are all the same and they do not exceed 42, it is nonspecial by [12] and [5] Theorem 4.9. Now we conclude by Lemma 14.
(2) By Lemma [14] it is enough to prove that \( L_{3,m}(l^h) \) is nonspecial. If \( l \leq 4 \), this is true by [2]. If \( l = 5 \), this is true by [3].

\[ \]

Observe that in point (2) the assumption \( m \geq 2l + 1 \) is necessary. Indeed, if we consider \( l = 4, m = 8 \) and \( h = 6 \), then by applying Cremona transformations we can check

\[ L_{3,7}(4^6) \cong L_{3,5}(4^2,2^4) \cong L_{3,3}(2^4) \]

is not empty, so \( \text{mult} Z_0 = 7 < m \).

Our approach relies on a length counting, so it needs a good behaviour of the numbers involved. Therefore it is difficult to prove general results. However, there are other specific cases in which it is easy to compute the limit on smooth threefolds.

**Example 41.** Here we present some more examples in \( \mathbb{A}^3 \).

1. The limit of four 4-tuple points is a 6-tuple point with 6 infinitely near double points.
2. The limit of five 5-tuple points is a 9-tuple point with 10 infinitely near simple points.
3. The limit of five 8-tuple points is a 14-tuple point with 10 infinitely near double points.

At the infinitely near points are in the special position described by Remark [15]. Anyway, for case (3) we still need to prove that 10 double points in this special position impose independent conditions on degree 14 plane curves. Let \( p_{ij} = R \cap (p_i, p_j) \) and let \( C \) be a smooth conic through \( p_{15}, p_{25}, p_{35}, p_{45} \). Consider the Castelnuovo exact sequence

\[ 0 \to L - C \to L \to L_C \to 0. \]

We can see \( C \) as the 2-Veronese embedding of \( \mathbb{P}^1 \), so \( L_C = L_{1,6}(2^4) \) is nonspecial and effective. On the other hand, \( L - C = L_{2,6}(2^5,1^4) \). It is easy to see that the 6 double points in this position impose independent conditions on degree 6 plane curves. To show that the simple points are not base points, define \( l_{ijk} \) to be the line through \( p_{ij}, p_{ik}, p_{jk} \).

Now it suffices to consider \( l_{123} + l_{124} + l_{134} + l_{234} + B \), where \( B \) is a conic.

Up to now, we mostly considered collisions of points with the same multiplicities, but of course there are many other cases in which we can try to determine the limit \( Z_0 \). For instance, when one of the collapsing points has multiplicity much larger than the others, it is easy to compute the limit scheme.

**Proposition 42.** Let \( m, m_1, \ldots, m_s \in \mathbb{N} \). Assume that \( L_{n-1,m}(m_1, \ldots, m_s) \) is not empty and nonspecial. Then the limit scheme of \( s + 1 \) collapsing points of multiplicity \( m, m_1, \ldots, m_s \) in \( \mathbb{A}^n \) is an \( m \)-tuple point with \( s \) infinitely near points of multiplicity \( m_1, \ldots, m_s \).

**Proof.** Clearly \( L_{n,m-1}(m_1, \ldots, m_s) \) is empty. On the other hand,

\[ L_{n,m}(m_1, \ldots, m_s) \cong L_{n-1,m}(m_1, \ldots, m_s) \]

is not empty by hypothesis, so \( \text{mult} Z_0 = m \) by Proposition [13]. The base locus of \( L_{n,m}(m, m_1, \ldots, m_s) \) contains the \( s \) lines joining the \( m \)-tuple point with each of the others, counted with multiplicities \( m_1, \ldots, m_s \). They cut \( s \) general points on \( R \), of multiplicities \( m_1, \ldots, m_s \), that give independent conditions on degree \( m \) divisors of \( R \). To conclude, observe that the scheme made by an \( m \)-tuple point with \( s \) infinitely near points of multiplicity \( m_1, \ldots, m_s \) has the same length as \( Z_1 \).

The next two Propositions will deal with the case of a fat point colliding together with a bunch of low multiplicity points.

**Proposition 43.** Let \( m, n \geq 2 \) and let \( s = \binom{n+m-1}{m} \). Then the limit of \( s \) simple points and a point of multiplicity \( m \) colliding in \( \mathbb{A}^n \) is an \( (m+1) \)-tuple point.

**Proof.** By our assumption on \( s \), \( L_{n,m}(m, 1^h) \) is empty, while \( L_{n,m+1}(m, 1^h) \) is not, so \( \text{mult} Z_0 = m + 1 \) by Proposition [13]. To conclude, observe that a \( (m+1) \)-tuple point has degree \( \binom{n+m}{m} = \text{deg} Z_1 \), so \( Z_0 \) is a \( (m+1) \)-tuple point.
Proposition 44. Let $m, n \geq 3$ and $(m, n) \not\in \{(4, 3), (3, 5)\}$. Suppose that $h = \frac{m+n-1}{n} \in \mathbb{N}$. Then the limit of $h$ double points and a point of multiplicity $m$ colliding in $\mathbb{A}^n$ is a $(m+1)$-tuple point with $h$ infinitely near simple points.

Proof. By hypothesis

$$\text{vdim } L_{n,m+1}(m, 2^h) = \binom{n + m + 1}{n} - \binom{n + m - 1}{n} - h(n + 1) = \left(\frac{n + m + 1}{n}\right) - \left(\frac{n + m - 1}{n}\right) - \frac{h(n + 1)}{n} = \frac{(n + m + 1)!}{n!(m + 1)!} - \frac{(n + m - 1)!}{n!(m - 1)!} - \frac{(m + n - 1)!}{(m + 1)! m!} = \frac{(n + m + 1)!}{n!(m - 1)!} \left[\frac{(n + m + 1)(n + m)}{(m + 1)mn} - 1\right] > 0,$$

hence $L_{n,m+1}(m, 2^h)$ is not empty. On the other hand,

$$\binom{n + m - 1}{n} - h = \binom{n + m - 1}{n - 1} - \frac{h(n + 1)}{n + 1} = 0,$$

so $L_{n,m}(m, 2^h) \not\subseteq L_{n,m+1}(m, 2^h)$ is expected to be empty. The latter is nonspecial by Theorem [5] so mult $Z_0 = m + 1$ by Proposition [13]. The $h$ lines joining the $m$-tuple point and one of the double points are contained in the base locus of $L_{n,m+1}(m, 2^h)$, and they cut $h$ general simple points on $R$. The candidate limit scheme is a $(m+1)$-tuple point with $h$ infinitely near simple points, which has length $(\binom{n+m-1}{n} + h = \deg Z_1$.

We now focus on $\mathbb{P}^3$. Recall that 8 is the maximum $r$ such that we know the full classification of special linear systems $L_{3,d}(m_1, \ldots, m_r)$, see [11]. The following Proposition will be useful to get some results beyond this bound.

Proposition 45. The limit of the collision of 8 $m$-tuple points and $m + 1$ simple points in $\mathbb{P}^3$ is a point of multiplicity $2m + 1$.

Proof. It is immediate to check that

$$\frac{8(m + 2)}{3} + m + 1 = \binom{2m + 3}{3}.$$

By Lemma [13] it is enough to prove that $L_{3,2m}(m^8, 1^{m+1})$ is nonspecial. Since general simple points always give independent conditions, it suffices to show that $L_{3,2m}(m^8)$ is nonspecial. The latter is true by [11] Theorem 5.3).

Remark 46. In this Section, our arguments to describe limits of collisions of fat points rely on known results on nonspecial linear systems of $\mathbb{P}^2$ and $\mathbb{P}^3$. With a similar approach, other results can be applied in the same way to prove more statements about collisions. Examples on $\mathbb{P}^3$ include [11] Theorem 5.3 and [6] Theorem 5.8. On $\mathbb{P}^2$ there is [13] Theorem 34, as well as the results contained in [19] and in the survey [5].

While most of our knowledge of special linear systems is concentrated in low dimensional varieties, there is also something we can say about any $\mathbb{P}^n$. As an example, there are the results contained in [5].

Proposition 47. The limit of the collision of 6 triple points and 36 simple points in $\mathbb{P}^4$ is a point of multiplicity 6.
Proof. The proof works as in Proposition 43. We check that
\[
\binom{6}{4} + 36 = \binom{9}{4}.
\]
By Lemma 13 it is enough to prove that \(\mathcal{L}_{4,5}(3^6, 1^{36})\) is nonspecial. Again, general simple points always give independent conditions, so we only have to show that \(\mathcal{L}_{5,2m}(3^6)\) is nonspecial. The latter is true by [11, Corollary 4.8].

Next Proposition has a slightly different flavour. It states that, up to add a bunch of simple points, we can always turn two fat points into a unique fat point.

**Proposition 48.** Let \(n \geq 2\), \(m_1, m_2 \in \mathbb{N}\). Then exist \(h, m \in \mathbb{N}\), depending on \(n, m_1, m_2\), such that the limit of two points of multiplicity \(m_1\) and \(m_2\) and \(h\) simple points in \(\mathbb{P}^n\) is an \(m\)-tuple point.

**Proof.** Define
\[
m := m(n, m_1, m_2) = m_1 + m_2 + 1
\]
and
\[
h := h(n, m_1, \ldots, m_s) = \left(\frac{n + m_1 + 1}{n} - \left(\frac{m_1 + n - 1}{n}\right) - \left(\frac{m_2 + n - 1}{n}\right) - 1.
\]
By construction \(\text{vdim} \mathcal{L}_{n,m}(m_1, m_2, 1^h) \geq 0\), hence \(\mathcal{L}_{n,m}(m_1, m_2, 1^h)\) is not empty. Since an \(m\)-tuple point has degree equal to the length of the starting scheme, it is enough to show that \(\mathcal{L}_{n,m-1}(m_1, m_2, 1^h)\) is nonspecial, and therefore empty. Since the \(h\) simple points always give independent conditions, it suffices to prove that \(\mathcal{L}_{n,m-1}(m_1, m_2)\) is nonspecial. By [11, Corollary 4.8], such system is linearly nonspecial, so in order to prove nonspeciality we just need to observe that there are no base linear cycles. □

4. APPLICATIONS TO INTERPOLATION THEORY

The first example of an application of collisions to interpolation theory is [15], where Proposition 16 allows the authors to solve a long-standing problem about Waring decompositions of polynomials.

It is also important to mention [13], in which the author uses a suitable collision of fat points in \(\mathbb{P}^2\) to prove nonspeciality for an infinite family of linear systems of plane curves.

More generally, Proposition 13 shows that, in order to determine the multiplicity of the limit, we need to understand the speciality of the systems of divisors containing the starting scheme \(Z_1\), or equivalently its Hilbert function \(h_{Z_1}\). Indeed, in Section 3 we used known results in interpolation theory to provide clues about what the limit is. Therefore it is just fair to try to return the favour, using the limits we constructed as tools to specialize linear systems in order to prove their nonspeciality or nonemptiness.

We begin with our contribution to Laface-Ugaglia conjecture (see [17, Conjecture 4.1] and [8, Conjecture 5.1]). When all multiplicities are the same, the conjecture predicts that \(\mathcal{L}_{3,d}(m^r)\) is nonspecial whenever
\[
(d + 1)^2 > 9 \left(\frac{m + 1}{2}\right) \quad \text{and} \quad d \geq 2m - 1.
\]
The most restrictive between these two conditions is the first one, which reads
\[
d > -1 + 3 \sqrt{\frac{m^2 + m}{2}} = \frac{3}{\sqrt{2}}m + o(m).
\]
As we stated in Theorem 4 for systems with at most 15 base points we are able to prove nonspeciality under the stronger assumption \(d \geq 3m\).

**Proof of Theorem 4.** We only have to prove the statement for the case \(r = 15\). Since \(\text{vdim} \mathcal{L}_{3,d}(m_{15}^{15}) > m + 1\), it is enough to prove that \(\mathcal{L}_{3,d}(1^{15}, 1^{m+1})\) is nonspecial. We apply Proposition 15 to degenerate \(\mathcal{L}_{3,d}(m_{15}^{15}, 1^{m+1})\) to \(\mathcal{L}_{3,d}(2m + 1, m^5)\), and the latter is nonspecial by [11, Theorem 5.3]. □

Proposition 14 allows us to prove nonspeciality for another family of linear systems.
Proposition 49. Let $m_1 \geq \ldots \geq m_8$ be non-negative integers. Assume that

1. $\text{vdim } \mathcal{L}_{3,d}(m_1^8, \ldots, m_8^8) > 8 + m_1 + \ldots + m_8$;
2. $d \geq 2(m_1 + m_2) + 1$.

Then $\mathcal{L}_{3,d}(m_1^8, \ldots, m_8^8)$ is nonspecial.

Proof. By assumption (1), it is enough to prove that $\text{vdim } \mathcal{L}_{3,d}(m_1^8, \ldots, m_8^8, 1^{8+m_1+\ldots+m_8})$ is nonspecial. We apply Proposition 40 to degenerate $\mathcal{L}_{3,d}(m_1^8, \ldots, m_8^8, 1^{8+m_1+\ldots+m_8})$ to $\mathcal{L}_{3,d}(2m_1 + 1, \ldots, 2m_8 + 1)$. The latter is nonspecial by (2) and [11, Theorem 5.3].

We now aim to provide further examples of nonspecial linear systems on low dimensional smooth varieties.

Proposition 50. Let $n \in \{2, 3\}$, and let $V$ be a smooth $n$-dimensional variety. Let $l, m \in \mathbb{N}$, and assume that

$$h := \frac{(m+n-1)}{(l+n-1)}$$

is a natural number.

1. If $n = 2$, $l, m \leq 42$ and $h \geq 10$, then $\mathcal{L}_{V,d}(l^n)$ is nonspecial and $\mathcal{L}_{V,d}(l^1)$ is nonspecial whenever $\text{vdim } \mathcal{L}_{V,d}(l^1) \geq (h-t)(l+1)$.
2. If $n = 3$, $l \leq 5$ and $m \geq 2l+1$, then $\mathcal{L}_{V,d}(l^n)$ is nonspecial and $\mathcal{L}_{V,d}(l^1)$ is nonspecial whenever $\text{vdim } \mathcal{L}_{V,d}(l^1) \geq (h-t)(l+2)$. Furthermore, $\mathcal{L}_{V,d}(m_1, \ldots, m_7, l^8)$ is nonspecial under the assumption that $d > a + b$ for every $a, b \in \{m_1, \ldots, m_7, m\}$.

Proof. (1) We apply Proposition 49 to degenerate $\mathcal{L}_{V,d}(l^n)$ to $\mathcal{L}_{V,d}(m)$, which is always nonspecial. For the second part, observe that $\mathcal{L}_{V,d}(l^1)$ is not empty, so the nonspeciality of $\mathcal{L}_{V,d}(l^1)$ is implied by the nonspeciality of $\mathcal{L}_{V,d}(l^8)$.

(2) For the first two statements we apply Proposition 40 to degenerate the system, and then we argue as in point (1). For the last part, we can degenerate $\mathcal{L}_{V,d}(m_1, \ldots, m_7, l^8)$ to $\tilde{L} := \mathcal{L}_{V,d}(m_1, \ldots, m_7, m)$. By [11, Theorem 5.3], $\tilde{L}$ is nonspecial and therefore $\mathcal{L}_{V,d}(m_1, \ldots, m_7, l^8)$ is nonspecial.

Let us point out that are cases in which we can compute the limit with less assumptions on the number involved, and therefore we can still apply this degeneration. For instance, on surfaces the hypothesis $h \geq 10$ can be relaxed.

Example 51. Pick $l = 5$, $m = 14$, and $h = 7$. By a sequence of Cremona transformation, it is easy to check that $\mathcal{L}_{2,13}(5^1)$ is empty and therefore nonspecial, so the limit of 7 collapsing 5-tuple points in $\mathbb{P}^2$ is a 14-tuple point by Lemma 14.

In [7, Example 3.13], the authors apply their results to compute the Hilbert function of 5 points of multiplicity 5 in $\mathbb{P}^1 \times \mathbb{P}^1$. Their method works for curves of bidegree $a, b$, except for $7 \leq a, b \leq 14$, where they need explicit software-aided computation. In all cases in which this computation indicates the system is nonspecial, Proposition 50 gives a theoretical proof of nonspeciality, the only exception being $a = b = 9$. For this specific system, we can prove nonspeciality by projecting it to $\mathbb{P}^3$, obtaining $\mathcal{L}_{\mathbb{P}^3 \times \mathbb{P}^1, (9,9)}(5^7) \cong \mathcal{L}_{2,18}(9^2, 5^5)$, and by applying a sequence of Cremona transformations to decrease the degree.

$\mathcal{L}_{\mathbb{P}^3 \times \mathbb{P}^1, (9,9)}(5^7) \cong \mathcal{L}_{2,18}(9^2, 5^5) \cong \mathcal{L}_{2,13}(5^4, 4^2)$$\cong \mathcal{L}_{2,11}(5, 4^2, 3^3) \cong \mathcal{L}_{2,9}(3^4, 2^2)$.

By Lemma 14 the limit of 3 triple points and a double point in $\mathbb{A}^2$ is a 6-tuple point, so we can degenerate $\mathcal{L}_{2,9}(3^4, 2^2)$ to $\mathcal{L}_{2,6}(6, 3, 2) \cong \mathcal{L}_{2,7}(4, 1)$, which is clearly nonspecial.

With some extra effort, we can employ a sequence of collisions to prove nonspeciality in more examples.

Proposition 52. Let $m, n_1, \ldots, n_s \in \mathbb{N}$. For $i \in \{1, \ldots, s\}$, set $h_i = m \frac{(m+1)}{(n_i(n_i+1))}$. Assume $h_i \in \mathbb{N}$ and $h_i \geq 10$ for every $i \in \{1, \ldots, s\}$. If $m, n_1, \ldots, n_s \leq 42$, then $\mathcal{L}_{2,d}(m^k, n_1^{h_1}, \ldots, n_s^{h_s})$
is nonspecial for every $t_1, \ldots, t_s \in \mathbb{N}$.

**Proof.** By Proposition [10] we can collapse $h_1$ of the $n_1$-tuple points into an $m$-tuple point, thereby degenerating $L_{2,d}(m^k, n_1^{t_1 h_1}, \ldots, n_s^{t_s h_s})$ to $L_{2,d}(m^{k+1}, n_1^{(t_1-1) h_1}, \ldots, n_s^{t_s h_s})$.

By performing $t_1$ of these collisions, we obtain the system $L_{2,d}(m^{k+t_1}, n_2^{t_2 h_2}, \ldots, n_s^{t_s h_s})$.

Then we apply Proposition [10] again to collapse $h_2$ of the $n_2$-tuple points into an $m$-tuple point. By performing $t_2$ of these collisions, we specialize the system to $L_{2,d}(m^{k+t_1+t_2}, n_3^{t_3 h_3}, \ldots, n_s^{t_s h_s})$.

We iterate the argument till the $s$-th step. At the end we are dealing with the specialized system $L_{2,d}(m^{k+\cdots+t_s}, n_s^{t_s h_s})$. The latter is nonspecial by [12], and this implies $L_{2,d}(m^k, n_1^{t_1 h_1}, \ldots, n_s^{t_s h_s})$ is nonspecial.

□

The next result shows how collisions can prove nonspeciality when the system has a very large expected dimension. For simplicity, we work with a quasi-homogeneous linear system, but it is possible to suitably modify the hypothesis to prove a similar result even if the multiplicities are all different.

**Proposition 53.** Let $d, m, k \in \mathbb{N}$ be such that $k > m$ and

\[
\left( \frac{d+3}{3} \right) \geq \left( \frac{(m+1)(a-8) + k + 4}{3} \right) + 8 \left( \frac{m+2}{3} \right).
\]

Then $L := L_{3,d}(k, m^a)$ is nonspecial.

**Proof.** If $a \leq 7$, then $L$ is nonspecial by [11] Theorem 5.3], because our assumption implies $d \geq 2m - 1$. If $a = 8$, then $L$ is nonspecial by [8] Theorem 3.1], because $d \geq 2m$ and $k > m$.

Assume then $a \geq 9$. Now we will exploit the large expected dimension of $L$. First observe that, for every $h \in \mathbb{N}$, it suffices to show that $L := L_{3,d}(k, m^a, 1^h)$ is nonspecial.

Set $L_0 = L$, $k_0 = k$ and $k_1 = k + m + 1$. By Proposition [35] exists $h_0$ such that the limit of an $m$-tuple point, a $k$-tuple point and $h_0$ simple points is a $k_1$-tuple point. We need $\expdim L_{3,d}(k, m^a) \geq h_0$ to guarantee $L'_0 := L_{3,d}(k, m^a, 1^{h_0})$ is not empty. If so, by Proposition [35] we can degenerate $L'_0$ to $L_1 := L_{3,d}(k_1, m^{a-1})$. Then we apply the same argument to $L_1$. Set $k_2 = k_1 + m + 1$, we know there exists $h_1$ such that we can make a $k_1$-tuple point, an $m$-tuple point and $h_1$ simple points to a $k_2$-tuple point. This time we need to guarantee that $L'_1 := L_{3,d}(k_1, m^{a-1}, 1^{h_1})$ is not empty. If so, by Proposition [35] we can degenerate $L'_1$ to $L_2 := L_{3,d}(k_2, m^{a-2})$. We keep iterating until we are left with less than 9 points. At the $i$-th step we require that

\[
\expdim L_{3,d}(k_i, m^{a-i}, 1^{h_i}) \geq 0.
\]

The most restrictive among all these requirements is the last one, that is

\[
\left( \frac{d+3}{3} \right) \geq \left( \frac{k_{a-8} + m + 4}{3} \right) + 8 \left( \frac{m+2}{3} \right) = \left( \frac{(a-8)(m+1) + k + 4}{3} \right) + 8 \left( \frac{m+2}{3} \right).
\]

After $a-8$ steps we obtain the specialized linear system $L_{a-8} = L_{3,d}(k_{a-8}, m^8)$ which again is nonspecial by [8] Theorem 3.1].

□

The bound provided by Proposition [35] is far from being sharp. Anyway, the method can be useful in several specific examples.

Up to now, we could benefit from known results about nonspecial systems on $\mathbb{P}^2$ and $\mathbb{P}^3$. In these two cases, there are very precise conjectural classifications of special systems, and such conjectures are known to hold in many cases. However, for $n \geq 4$ not even a conjectural solution of the problem is known. For this reason, our results on fourfolds are
limited to triple points, but they still provide hints to understand an almost unexplored topic.

**Proposition 54.** If \( d \geq 8 \), then \( L_{d,4}(3^4) \) is nonspecial for every \( r \leq 11 \). Moreover, if \( d \geq 11 \), then \( L_{d,4}(3^5) \) is nonspecial for every \( r \leq 66 \).

**Proof.** For the first part, assume that \( d \geq 8 \). We only have to prove that \( L_{d,4}(3^{11}) \) is nonspecial. Since \( \text{vdim} L_{d,4}(3^{11}) > 36 \), it is enough to prove that \( L_{d,4}(3^{11},1^{36}) \) is nonspecial. We apply Proposition \ref{prop:reduce} to degenerate \( L_{d,4}(3^{11},1^{36}) \) to \( L_{d,4}(6,3^5) \). By using reducible divisors, it is easy to show that \( L_{d,4}(6,3^5) \) has a 0-dimensional base locus, and therefore it is nonspecial by \cite[Corollary 4.8]{ref1}.

For the second part, assume that \( d \geq 11 \). We only have to prove the case \( r = 66 \). Since \( \text{vdim} L_{d,4}(3^{66}) > 216 \), it is enough to prove that \( L_{d,4}(3^{66},1^{216}) \) is nonspecial. Again, we use Proposition \ref{prop:reduce} to degenerate \( L_{d,4}(3^{66},1^{216}) \) to \( L_{d,4}(6^6) \). By using reducible divisors, we see that \( L_{d,4}(6^6) \) has a 0-dimensional base locus, and therefore it is nonspecial by \cite[Corollary 4.8]{ref1}.

Actually, something stronger holds. Proposition \ref{prop:reduce} can be generalized, by proving that the collision of \( n + 2 \) triple points and a bunch of simple points in \( \mathbb{P}^n \) give a point of multiplicity 6. Thus we can repeat the argument of Proposition \ref{prop:nonspecial} to show that \( L_{n,8}(3^{2n+3}) \) is nonspecial. In a similar fashion, \( L_{4,11}(4^{11}) \) is nonspecial. However, these linear system have a large virtual dimension, so we feel that the most interesting results are the ones stated in Proposition \ref{prop:reduce}.

**References**

1. J. Alexander, A. Hirschowitz, *The blown-up Horace method: application to fourth-order interpolation*, Invent. Math. 107 (1992), no. 3, 585–602.
2. E. Ballico, M. C. Brambilla, *Postulation of general quartuple fat point schemes in \( \mathbb{P}^3 \)*, J. Pure Appl. Algebra 213 (2009), no. 6, 1002-1012.
3. E. Ballico, M. C. Brambilla, F. Caruso, M. Sala, *Postulation of general quintuple fat point schemes in \( \mathbb{P}^3 \)*, J. Algebra 363 (2012), 113-139.
4. C. Bocci, *Special effect varieties and (−1)-curves*, Rocky Mountain J. Math. 40 (2010), n. 2, 397-419.
5. M. C. Brambilla, O. Dumitrescu, E. Postinghel, *On a notion of speciality on linear system of \( \mathbb{P}^n \)*, Transactions of the American Mathematical Society 367 (2015), n. 8, 5447-5473.
6. M. C. Brambilla, O. Dumitrescu, E. Postinghel, *On linear systems of \( \mathbb{P}^3 \) with nine base points*, Annali di Matematica Pura ed Applicata 195 (2015) n. 5, 1551-1574.
7. E. Carlini, M. V. Catalisano, A. Oneto, *On the Hilbert function of general fat points in \( \mathbb{P}^1 \times \mathbb{P}^1 \)*, preprint.
8. C. Ciliberto, *Geometric aspects of polynomial interpolation in more variables and of Waring’s problem*, Proceedings of the European Congress of Mathematics 1, Barcelona (2000), Progress in Math. 201, Birkhäuser, Basel (2001), 289-316.
9. C. Ciliberto, R. Miranda, *Degenerations of planar linear systems*, Jurnal für die reine und angewandte Math. 501 (1998), 191-220.
10. C. Ciliberto, R. Miranda, *Matching conditions for degenerating plane curves and applications*, in Projective varieties with unexpected properties, Walter de Gruyter GmbH and Co. KG, Berlin (2005), 177–197.
11. C. De Volder, A. Laface, *On linear systems of \( \mathbb{P}^3 \) through multiple points*, J. Algebra 310 (2007), no. 1, 207-217.
12. M. Dunning, *Cutting diagram method for systems of plane curves with base points*, Ann. Polon. Math. 90 (2007), 131-143.
13. C. Dunning, W. Jarnicki, *New effective bounds on the dimension of a linear system in \( \mathbb{P}^2 \)*, J. Symbolic Comput. 42 (2007), 621-645.
14. L. Evain, *La fonction de Hilbert de la reunion de \( d^h \) points generiques de \( \mathbb{P}^2 \) de meme multiplicity*, J. of Alg. Geom. 8 (1999), n. 4, 787-796.
15. F. Galuppi, M. Mella, *Identifiability of homogeneous polynomials and Cremona Transformations*, Journal für die reine und angewandte Mathematik (2017), DOI 10.1515/crelle-2017-0043.
16. D. Grayson, M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at [www.math.uiuc.edu/Macaulay2](http://www.math.uiuc.edu/Macaulay2).
17. A. Laface, L. Ugaglia, *On a class of special linear systems on \( \mathbb{P}^3 \)*, Trans. Amer. Math. Soc. 358 (2006), no. 12, 5485-5500.
18. M. Nesi, *Collisions of fat points*, PhD thesis, Università Roma III (2009).
19. J. Roš, *Maximal rank for schemes of small multiplicity by Evain’s differential Horace method*, Trans. Amer. Math. Soc. 366 (2014), 857-874.

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