PRIMORDIAL GRAVITATIONAL WAVE DETECTABILITY WITH DEEP SMALL-SKY CMB EXPERIMENTS

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ABSTRACT

We use Bayesian estimation on direct T-Q-U CMB polarization maps to forecast errors on the tensor-to-scalar power ratio $r$, and hence on primordial gravitational waves, as a function of sky coverage $f_{\text{sky}}$. This T-Q-U matrix likelihood filters the quadratic pixel-pixel space into the optimal combinations needed for $r$ detection for cut skies, providing enhanced information over a first-step linear separation into a combination of $E$, $B$ and mixed modes, and ignoring the latter. With current computational power and for typical resolutions appropriate for $r$ detection, the large matrix inversions required are accurate and fast. Our simulations explore two classes of experiments, with differing bolometric detector numbers, sensitivities and observational strategies. One is motivated by a long duration balloon experiment like Spider, with pixel noise $\propto \sqrt{f_{\text{sky}}}$ for a specified observing period. This analysis also applies to ground-based array experiments. We find that, in the absence of systematic effects and foregrounds, an experiment with Spider-like noise concentrating on $f_{\text{sky}} \sim 0.02$--$0.2$ could place a $2\sigma \sim 0.014$ bound ($\sim 95\%$ CL), which rises to 0.02 with an $\ell$-dependent foreground residual left over from an assumed efficient component separation. We contrast this with a Planck-like fixed instrumental noise as $f_{\text{sky}}$ varies, which gives a Galaxy-masked ($f_{\text{sky}} = 0.75$) $2\sigma$, $\approx 0.015$, rising to $\approx 0.05$ with the foreground residuals. Using for a figure of merit the (marginalized) 1D Shannon information entropy of $r$, taken relative to the first 2003 WMAP1 CMB-only constraint, gives $-1.7$ bits from the 2010 WMAP7+ACT data, $-1.9$ bits from the 2011 WMAP7+SPT data, and forecasts of $-6$ bits from Spider (plus Planck); this compares with up to $-11$ bits for a CMBPol, CoRe and PIXIE post-Planck satellites and $-13$ bits for a perfectly noiseless cosmic variance limited experiment. We thus confirm the wisdom of the current strategy for $r$ detection of deeply probed patches covering the $f_{\text{sky}}$ minimum-error trough with balloon and ground experiments.

Subject headings: Cosmic background radiation – Cosmological parameters – Cosmology: theory – Methods: numerical

1. INTRODUCTION

Inflation, a period of accelerated expansion in the very early universe, is the most widely accepted scenario to solve the problems of the otherwise successful standard model of cosmology. In the simplest models the expansion is driven by an effective potential energy of a single scalar field degree of freedom, the inflaton. An unavoidable consequence is the quantum generation of scalar and tensor zero-point fluctuations in the spacetime metric. The former are 3-curvature perturbations, with associated density fluctuations that can grow via gravitational instability to create the cosmic web, with its rich observational characterization. The latter are gravity waves that induce potentially observable signatures in the spatial structure of the Cosmic Microwave Background (CMB), in particular in its polarization, the focus of this paper. Whereas curl-free $E$-modes of polarization can be produced both by tensor and scalar perturbations, divergence-free modes of CMB polarization ($B$-modes) would be induced on large scales by primordial gravitational waves but not by scalar curvature

fluctuations. Many experiments are in quest of this inflation signature, but the predicted signal, if detectable, is very small and subject to contamination by leakages from the total anisotropy $T$ and from the dominant $E$ polarization, as well as by other systematic effects, so extraordinary care is needed to analyze such data. At smaller scales, $B$ modes are induced from primordial $E$ modes through gravitational lensing distortions of the CMB polarization patterns, adding to the complexity of making a clean separation of the tensor-induced signal.

The primordial scalar and tensor power spectra (fluctuation variances per ln $k$) and their ratio $r(k)$ are often approximated by power laws in the 3D comoving wavenumber $k$,

\[
P_s(k) \approx A_s(k_{\text{tp}})(k/k_{\text{tp}})^{n_s(k_{\text{tp}})-1},
\]

\[
P_t(k) \approx A_t(k_{\text{tp}})(k/k_{\text{tp}})^{n_t(k_{\text{tp}})},
\]

\[
r(k) \equiv P_t(k)/P_s(k) \approx r(k/k_{\text{tp}})^{n_t(k_{\text{tp}})-n_s(k_{\text{tp}})+1},
\]

\[
r \equiv r(k_{\text{tp}}) \equiv P_t(k_{\text{tp}})/P_s(k_{\text{tp}}).
\]

The scalar and tensor pivots $k_{\text{tp}}$ and $k_{\text{tp}}$ about which the expansions occur are usually chosen to be different for scalars and tensors to reflect where the optimal signal weights come from. The main target of many of the current and coming CMB polarization experiments is, firstly, a one-parameter uniform $r$. An advantage of this ratio over $P_t(k_{\text{tp}})$ is that it removes a dominant near-degeneracy with the Thompson depth to Compton scattering $\tau$. The spectrum $r(k)$ also measures the inflation acceleration history $\epsilon(a)$, and can be directly related to

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the inflaton potential energy through this relation:
\[ r(k) \approx 16\epsilon (a \approx k/H), \quad \epsilon = -\frac{d\ln H}{d\ln a}, \]
\[ V \approx rM_p^4P,3/2(1-r/48) \sim (10^{18}\text{GeV})^4/0.1. \] (1)

Here \( M_p = 1/\sqrt{8\pi G} \) is the reduced Planck mass, with \( c \) and \( h \) set to unity. The relation \( k \approx H a \), of resolution \( k^{-1} \) to the dynamics encoded in the expansion and Hubble parameters, \( a \) and \( H \), is only approximate of course, but very useful, e.g., Bond (1996). A detection of \( r \approx 0.03 - 0.2 \) would provide a strong pointer to the specific inflation model. A tight upper bound, \( r \lesssim 0.03 \), would rule out a very large class of inflationary scenarios.

A bound that is achievable with the experiments we explore here. In this paper, we often use \( r_{\text{fid}} = 0.12 \) as a fiducial high-\( r \) case for tests, since it is near the 0.13 coming from the simplest \( V = m^2\phi^2/2 \) chaotic inflation model, and has an inflation energy scale \( V^{1/4} \) near \( 10^{16} \text{ GeV} \). We also explore the very small \( r_{\text{fid}} < 0.01 \) regime.

We would like to learn as much as we can about the full \( r(k) \), hence \( \epsilon(a) \), from CMB data. In addition to the deviations of the slopes from scale invariance (\( n_s = 0 \) and \( n_s - 1 = 0 \)), the slopes are expected to "run with \( k \)" just as the power does, although they may be approximately constant over the observable CMB range. The first order variations in \( \ln k \) define scalar and tensor "running" parameters, the first terms in polynomial expansions in higher order "running of running" parameters. In this paper \( n_s(k) \) is not our target, nor are high multipole CMB experiments which are necessary to get the long baseline needed to show whether \( n_s \) runs or not.

A consequence of the fall-off of the tensor-induced CMB signal beyond \( \ell \approx 150 \) is that only limited information can be obtained on \( n_s(k) \) — enough to allow a number of broad bands for \( r(k) \), but not enough for \( n_s(k) \), let alone \( n_t(k) \), to be determined with sufficient accuracy to test well the inflation consistency relation for gravity waves. In the limited 2-parameter tensor parameter space of \( r \) and uniform \( n_s \), this consistency condition is (e.g. Bond (1996))
\[ n_s \approx -r/8 \approx (1-r/16), \] (2)

so a convincing test would require an order of magnitude better determination of \( n_s \) than \( r \). Another complication in relating the experiments to inflation theory is that there is still observational room for subdominant scalar isocurvature perturbations in addition to the dominant curvature ones when multiple fields are dynamically important during or immediately after inflation; such fields are widely invoked for catalyzing the production of entropy at the end of inflation. Isocurvature perturbations with a nearly scale invariant primordial spectrum have significantly enhanced low-\( \ell \) CMB power because of the isocurvature effect Bond (1996), and that region, overlapping with the gravity wave induced CMB power, is where the constraint on the overall isocurvature amplitude comes from Sievers et al. (2007).

All CMB polarization experiments are limited in sky coverage by instrumental or Galactic foreground constraints. Thus, even though the \( B \) modes provide a unique \( r \)-signature and are orthogonal to the \( E \) modes over the full sky, realistically mode-mixing must always be dealt with, even though it may be larger for smaller \( f_{\text{sky}} \). Assessing the trade offs between shallow and deep small-sky observational strategies is the target of our investigation. Going for deep and small has the advantage that one can select the most foreground-free patches to target to decrease the high level of foreground subtraction. As well, the long waves which dominate foregrounds are naturally filtered. Ground-based or balloon-borne experiments using the deep and small-sky strategy are: BICEP and BICEP2, QUIET, PolarBear, EBBEX, Spider, KECK (Sheehy et al. 2011), ABS, PIPER (Chuss et al. 2010). Planck (and WMAP) are (relatively) shallow and large-sky. Proposed next-generation satellite experiments such as CorE, The COrE Collaboration (2011), PIXIE (Kogut et al. 2011) and LiteBIRD are deep and large-sky.

In this paper, we first review the general Bayesian framework for determining parameters to introduce the notations we use. We cast the quest for \( r \) into an information-theoretic language in which the forecasted outcomes of different experiments can be contrasted by considering the differences in their reduced \textit{a posteriori} Shannon entropies for \( r \), \( S_\text{R}(r|\text{expt}) \). We discuss the two basic approaches for constraining cosmological observables, such as those associated with inflation, and the relation of these to \( E-B \) mixing: (1) the \( t \)-space approach in which CMB maps are first compressed onto power spectrum parameters for \( TT-TE-EE \) and \( BB \), which are then compressed onto cosmic parameters; and (2) direct parameter extraction of \( r \) from map likelihoods. Our primary target is \( r \) and not the \( B \)-mode spectrum, hence the optimal one-step estimation from maps is preferred, provided it is computationally feasible — which it is for Spider-like experiments. The leakage between the \( E \) and \( B \) modes and its impact on \( r \) is quantified in § 3. In § 4 we present details of the method we use to bypass explicit \( E-B \) de-mixing and apply it to simulated data for realistic instrumental and foreground-residual noise levels for Spider-like and Planck-like experiments as \( f_{\text{sky}} \) varies. We end with our conclusions from this study.

2. BAYESIAN ANALYSIS OF MAPS, BANDPowers AND COSMIC PARAMETERS

As has become conventional in CMB analysis, the framework envisaged to reduce the information from Spider-like raw time ordered data to constraints on cosmic parameters, in particular our target \( r \), is one of a long Bayesian chain of conditional probabilities Bond (1996) Bond & Crittenden (2001). To introduce our notation, we review that framework with polarization. We also remark on how the associated conditional Shannon entropies decrease as we flow along the Bayesian chain, a novel way of looking at what is being done as the data is reduced to a precious set of parameter bits.

2.1. Reducing Noisy Data with Bayesian Chains
2.1.1. The Information Action in Bayesian Chains

In Bayesian analysis, we wish to construct the a posteriori probability distribution of parameters \( q = (q_1, \ldots, q_N) \), \( P(q|D, T) \), an update from the a priori probability \( P(q|T) \) on the theory space \( T \) of the parameters that is driven by the likelihood \( L(q|D, T) \equiv P(D|q, T) \) of the data \( D \) given \( q \):

\[
P(D|q, T) = P(D|q, T)P(q|T)/P(D|T)
\]

The prior may include theoretical prejudice, information derived from other data, and, at the very least, the specific measure adopted for the parameters. The evidence, \( P(D|T) \), a single normalization, is also needed to ensure the posterior integrates to unity. Its determination is derived from other data, and, at the very least, the spatial pixel number for the noise vector \( C \), a noise covariance matrix.

This case is a set of (generalized) pixel means \( \phi_q \), \( \phi \), \( q \), \( \phi \) are noise vectors that are incompatible with the templates \( \varphi \). (That projected-out information is a fertile residual space for searching for the signals of relevance for experimental systematics studies.)

Making maps in this way is just one example of matched-filter processing of linear data models. The main ingredient in an optimal filter \( \psi \) constructed from the linear templates \( \varphi \) with weight \( w_{n\ell} \):

\[
d = \varphi q + n, \quad n = d - \varphi(q|d)
\]

The more elements there are in the chain, the more additive contributions there are to this energy. The evidence

\[
\ln P(D|T) = \int d^N q e^{-S(q)}.
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2.1.2. Reduction to Maps and Other Matched Filterings

Initially \( D \) is in the form of time-ordered data, and, typically, many flags about the data. The first step is to create maps from these, with \( q \) being the map data vector \( \Delta \), with components \( \Delta_{\text{exp}} \) labelled by frequency channel \( c \), Stokes polarization index \( x = T, Q, U, V \) and spatial pixel number \( p = 1, \ldots, N_{\text{pix}} \). The Stokes parameters \( Q, U, V \) are referred to a fixed polarization sky reference frame in real space. (Most experiments do not have simultaneous \( T, Q, U, V \) and \( V \) detectors.)

The solution of the parameter estimation problem in this case is a set of (generalized) pixel means \( \Delta \), and a noise covariance matrix \( C_N = \langle \delta \Delta \delta \Delta \rangle \), in terms of the noise vector \( \delta \Delta = \Delta - \Delta \). Henceforth, we do not use bold letters for the matrices, which are the most often used entities. The way one does this is to solve \( d = d_{\text{op}}(q) + n \), with the operator \( d_{\text{op}}(q) = \varphi q \), a linear data model with amplitudes \( q \) and templates \( \varphi \). Here \( d \) represents the time ordered data. The templates form an \( N_t \times N_{\text{pix}} \) matrix, where \( N_t \) and \( N_{\text{pix}} \) are the total number of digitized time observations and pixels (from all maps) respectively. This is a large compression of the data, by of order \( N_t/N_{\text{pix}} \), done by projecting out elements of the time-streams that are incompatible with the templates \( \varphi \). (That projected-out information is a fertile residual space for searching for the signals of relevance for experimental systematics studies.)

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\[
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\]

Initial signals, done by projecting out \( \delta \Delta \). In the sense that it minimizes the final correlation

\[
\ln P(D|T) = \int d^N q e^{-S(q)}.
\]

The more elements there are in the chain, the more additive contributions there are to this energy. The evidence

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laborization component maps derived from data were presented in the CBI papers (Reeves et al. 2000), of course with the BB a non-detection consistent with the noise. There has been much discussion about using variants of this approach for E-B separation (e.g. Lewis et al. 2002; Bunn 2002; Bunn et al. 2003; Bunn 2011).

From S_d, the statistics of the cosmic (and other) parameters the w_{d} in the prior depends upon are derived. This is our main focus here.

2.1.3. From Pixel Maps to E and B Maps

The map data vector Δ is composed of a number of signals s as well as the map noise n. The noise encompasses true instrumental noise, experimental systematic effects, and possibly, may draw terms from the signal side that are unwanted residuals on the sky, e.g., from foreground subtraction uncertainties. Each signal has a frequency dependence and polarization components, labeled by the Stokes parameter index x. The map components are generally considered to be linear in the sky signals,

\[ \Delta_{exp} = \sum_{x} s_{exp} + n_{exp}, \quad x \in \{T, Q, U, V\}, \]

\[ s_{exp} = \sum_{\ell m} \int_{\nu} F_{exp, jx\nu m} a_{jx\nu m}, \quad (6) \]

where the spherical harmonic signal amplitude for signal J is a_{jx\nu m}. The transformation from this natural multipole space for the signals to the map space is encoded in the filters F_{exp, jx\nu m}, which includes beam and pixelization information, the frequency response function for the channels, and a mask \mu_{\ell m}. The mask \mu could be a sharp cookie cutter or be more gently tapered.

The a_{jx\nu m} are the coefficients in the standard expansion of the CMB temperature and polarization fields in orthogonal mode functions, which are the spherical harmonics, spin-0 for T and spin-2 for polarization, with further linear combinations of the spin-2 expansion coefficients defining the E and B modes:

\[ T_{\nu}(\theta, \phi) = \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} a_{\nu T\ell m} Y_{\ell m}(\theta, \phi), \]

\[ (Q \pm iU)_{\nu}(\theta, \phi) = \sum_{\ell \geq 2} \sum_{m=-\ell}^{\ell} \pm 2a_{\nu Elm} [\pm 2Y_{\ell m}(\theta, \phi)], \]

\[ a_{\nu Elm} = -\frac{1}{i} (2a_{\nu J\ell m} + 2a_{J\ell m}) , \]

\[ a_{\nu B\ell m} = -\frac{1}{i} (2a_{\nu J\ell m} - 2a_{J\ell m}) . \]

The separation of the polarization into E and B-modes is useful because scalar perturbations only result in the E mode whereas the tensor perturbations generate both (Kamionkowski et al. 1997, Zaldarriaga & Seljak 1997)). Nonlinear transport effects associated with the weak lensing of the primary CMB fluctuations turn some scalar E mode into scalar B mode, mostly at higher ℓs than the tensor component gives, so separation for r detection can be done. Note that this lensing source has non-Gaussian features which means the power spectra are not enough to characterize that signal.

For Thompson scattering anisotropies, the V Stokes parameter associated with circular polarization vanishes, as it also does for most Galactic foregrounds contaminating the primary CMB signal, so we now drop it from our consideration. It would of course be of interest to show experimentally that there is indeed no circular polarization in the CMB data.

As we have noted above, eqs. [4] and [6] can be applied to the case in which the d are the maps \Delta_{exp}, the templates \varphi are the E and B mode function rotators and the parameters q are the E and B amplitudes in ℓm space. Since these compressed maps q and their variances contain complete statistical information for a Gaussian model, the q-power can be estimated from \langle q|d\rangle|\langle d|q\rangle + \langle \delta q\delta q\rangle. This is not the optimal determination of power. We adopt the CBIpol approach (Sievers 2001), that while such optimal E, B separation is good for checking robustness of results and for visualization of the polarization signals, the path to parameters (including bandpowers) is through the quadratic matrix methods, the mLikely approach of CBIpol, using eq [6].

2.1.4. Maps to Parameters with Matrix-based Likelihoods

For statistically isotropic signals there are generally six cross-spectra among the coefficients,

\[ \langle a_{x\ell m}a_{x'\ell' m'} \rangle = C_{X, 0\ell \ell'} \delta_{\ell \ell'}, \quad X = T, E, B, \quad \ell \in \{T, E, B\}, \quad \ell' \in \{T, E, B\}. \]

Typically the EB and TB power vanish (theoretically anyway) and only four are needed. However, EB and TB may be kept for systematics monitoring. For statistically homogeneous and isotropic 3D Gaussian initial conditions, the primary CMB T,Q,U are isotropic 2D Gaussian fields whose probability distribution depends only upon the power spectra C_{X, f}, or, equivalently the X-power per ln(ℓ + 1/2),

\[ C_{X, f} = \frac{\ell(\ell + 1)}{2\pi} C_{X, f}. \]

If there is no correlation between signal and noise, the components of the total covariance matrix C_{X, f} are given by the sum

\[ C_{f} = C_{N} + \sum_{JJ'} C_{S, JJ'}; C_{N, exp} c_{x} c_{x'} = \langle n_{exp} n_{exp} c_{x} c_{x'} \rangle, \]

\[ C_{S, exp} c_{x} c_{x'} c_{x''} = \langle s_{exp} s_{exp} c_{x} c_{x'} c_{x''} \rangle. \]

The goal of bandpower estimation is to radically-compress the map information onto ℓ-band power amplitudes the q^{X,β}, with templates \varphi of form C_{X,β, f}. With sufficiently fine ℓ-space banding, this stage of compression can be relatively lossless, allowing the cosmic parameters to be derived accurately. The inter-band shape of these templates may be crafted to look like theoretically expected shapes, or could just be flat, which imposes no prior prejudice. Both approaches have been effectively used. Usually the β-shapes have been chosen to be sharply truncated with no overlap in ℓ-space, but this is not at all necessary.

With cut-sky maps, bands are coupled even though they would not be for full sky observations with statistically homogeneous noise. The optimal method for estimating power spectra in the general case is the computationally expensive brute-force maximum likelihood (MLE) analysis (e.g. Bond et al. 1998), which iteratively corrects a quadratic expression for deviations δq^β.
of the various bandpowers from their initial values \( q_\ell^2 \) until the maximum likelihood \( \theta_\ell^2 \) is reached. The weight matrix \( C^{-1}_\ell(q) \) is adjusted at each step, until it settles into \( C^{-1}_\ell(q_m) \). The weight enters in two ways, one is quadratically in the likelihood-curvature matrix (approximately the Fisher matrix) and the other is in the force that drives the relaxation of the parameters to \( q_\ell^2 \). It turns out that one can think of the quadratic expression as describing the action of a matched filter on the pixel-pair data product, similar to the way linear filters acting on the data vector may be matched, as we show in §2.1.4.

Matrix methods for bandpower estimation were used by Boomerang (de Bernardis et al. 2001; Ruhl et al. 2003) and in all CBI papers. If the cosmic parameter of interest is linear, like \( r \), then it can be viewed as a single template big-band bandpower. Even with the fully nonlinear \( C_X(\ell) \), the amplitudes \( \delta q \) can be iteratively solved for using linear derivative templates, and, with convergence, the result is the same as a full nonlinear treatment gives.

2.1.5. Pseudo-\( C_{X,\ell} \) cf. Map-Matrix Methods

Several fast sub-optimal approximate methods have been developed to make the bandpower computations less computationally intense than in the map-matrix method: e.g., pseudo-\( C_{\ell} \) estimators (Hansen & Górski 2003; Chon et al. 2004), SPICE (Szapudi et al. 2001), MASTER (Hivon et al. 2002) and Xmaster (Contaldi et al. 2010; Rocha et al. 2012). Pseudo-\( C_{\ell} \)'s are constructed by direct spherical harmonic transform of the cut-sky maps, or more generally, taper-weighted CMB maps. The all-sky bandpower centered on a specific \( \ell \), \( q_\ell^2 \), is then related by an appropriate filtering which draws the pseudo-\( C_{X,\ell} \)'s from a wide swath of \( \ell \)'s determined by a mask-defined coupling matrix into the desired \( \ell \)-band. In spite of this \( \ell \)-space mixing, extensive testing has shown these methods to be accurate for temperature anisotropies for large pixel numbers where the matrix inversions of the iterated quadratic approach are prohibitively expensive computationally. They have also been applied effectively to polarization datasets such as Boomerang (Montrov et al. 2006; Placentini et al. 2006).

The pseudo-\( C_{X,\ell} \) for \( X = EE, BB \) suffer from \( E-B \) mixing in addition to the \( \ell \)-space mixing: the estimated \( C_{BB,\ell} \) receives contributions from both \( E \) and \( B \)-modes. The contamination coming from the \( E \)-mode can be removed from \( C_{BB,\ell} \) in the mean by having the estimators undergo a de-biasing step. However, there is still an extra contribution to the variance of estimators which is due to the dominance of the relatively large \( E \) signal mixed into the \( B \) measurement. This can limit the primordial gravitational wave detection to \( r \approx 0.05 \) for deep small sky surveys (covering about 1% of the sky) as shown by Challinor & Chon (2002). Lewis et al. (2002) show how to construct window functions that cleanly separate the \( E \) and \( B \) modes in harmonic space for azimuthally symmetric sky observations at the cost of some information loss due to the boundary of the patch. In another treatment of the \( E-B \) mixing problem, Bunn et al. (2003) show that the polarization maps can be optimally decomposed into three orthogonal components: pure \( E \), pure \( B \), and ambiguous modes. The ambiguous modes receive a non-restorable contribution from both \( E \) and \( B \) signals, and are dominated by \( E \) signal, thus should be removed in \( B \)-mode analysis. Based on this decomposition, a near-optimal pure pseudo-\( C_{\ell} \) estimator was proposed (Smith 2006) and developed (Smith & Zaldarriaga 2007, Grain et al. 2009) which ensures no \( E-B \) mixing. Recently Bunn (2011) has given a more efficient recipe for decomposing polarization data into \( E, B \) and ambiguous maps, although still along the lines of Bunn et al. (2003).

It is clear that if the full map-likelihood analysis can be done, then it should be done, since relevant information is not being thrown away. There are two drawbacks to the map-based approach. The first is that \( C_{\ell} \) should saturate all contributions to signal and noise since we are in quest of a small, essentially perturbative, component associated with \( r \) whose values can be biased by the missing components. This could be challenging in the presence of complex filtering resulting from time-ordered data processing. Also the computational cost of the required large matrix manipulations is high compared to the suboptimal methods. The matrix size depends upon the fraction of sky covered and the resolution. For example, for an experiment covering 25% of the sky analyzed at a Healpix resolution of \( N_{side} = 64 \), the sizes are \( 35K \times 35K \) and we find the likelihood calculation takes about 5 minutes on a node with 16 Dual-Core Power 6 CPU’s at 4.7 GHz (and theoretically capable of doing 600 GFLOPS/node).

In practice, our matrices are smaller than this since the quest for \( r \) requires a relatively low resolution analysis and only a few other parameters that are correlated with it need to be carried along, as we show here. To include many more parameters standard Bayesian sampling algorithms such as MCMC and adaptive importance sampling (Wraith et al. 2009) can be used. If we need to cover small angular scales as well as large, the matrices become prohibitively large, and hybrid methods, with a map-based likelihood for large scales joined to an \( \ell \)-space-based likelihood for small scales, are needed.

2.2. The Downward Flow of Shannon Entropy from Data Compression onto Theory Subspaces

The Shannon entropy \( S_I \) of the final (posterior) probability distribution is an average of the log of the local phase space volume \( \langle \ln p_T^{-1} \rangle_i \) over the posterior probability distribution \( p_T \), and is considered to provide an estimate of the total information content in the final ensemble (see, e.g., Mackay (2003)):

\[
S_I(T|D) = - \int d^N q p_T \ln p_T = \langle \ln P(q|D, T)^{-1} \rangle_i
\]

\[
= \langle S_I \rangle_i + \ln \langle P(D|T) \rangle
\]

\[
\langle S_I \rangle_i = \int d^N q e^{-S_I} S_I / \int d^N q e^{-S_I}.
\]

The initial entropy is averaged over the initial ensemble: \( S_i = \langle \ln P(q|T)^{-1} \rangle_i \). For a uniform prior over a volume \( V_{q,i} \) in \( q \)-space, it is \( S_i = \ln V_{q,i} \). The final entropy can be thought of as having a contribution from (the log of) an effective phase space volume, reduced relative to the initial one because of the measurement, plus a term related to the average \( \chi^2 \) associated with the mean-squared-deviations of \( q \), usually just the number
of degrees of freedom unless the model is a very poor representation of the information content of the data.

It should not seem curious to say that the information entropy decreases as a result of measurements, but it may seem curious to word it as: the average information content decreases. That is because the fully random initial state has more information, in that the variables can take on a wider range of values. We think the reduced post-experiment information content is of higher quality. What constitutes Quality in information is subjective of course.

Consider the initial space of the $D$, the space of full time-ordered-information, replete with bolometer readouts, flagged glitches, housekeeping information, etc. The amount of information we begin with is therefore enormous. From this data optimal maps are constructed with map parameters $\mathbf{q}$ having channel and Stokes as well as pixel indices (the $\Delta_{exp}$ defined above) which defines the $T$-space for this leg of compression. As the iterations progress towards the maximum likelihood map, there is a mismatch between the noise power spectrum $w_{\text{nf}}$ on the prior iteration, and the noise variance on the posterior iteration: the latter will be less, hence so will its logarithm, hence so will the Shannon information, until it settles into its final converged value, the information entropy in the maps, $S_f$ (maps). Thus, $S$(maps) decreases substantially from the large available information in the uninform prior, but also decreases as the iterations converge, settling on $S_f$(maps) = $N_D/2 + N_D \ln(2\pi)/2 + \text{Tr} \ln \sigma_N^2$ /2. The new dimension for the reduced data is the number $N_D$ of generalized pixels: it is the total number of pixels from all channels times 3 for $T, Q, U$. The $N_D \times N_D$ noise matrix is $\sigma_N^2 = (\mathbf{q} \cdot \mathbf{w}_{\text{nif}} \cdot \mathbf{q})^{-1}$. The information per generalized pixel is not so large but there are lots of such pixels.

The standard noise assumption we make for our maps is that it is homogeneous and white, usually different for $T$ and $Q, U$. Given a total integrated noise power, $N_{\text{pix}} \sigma_{\text{pix}}^2$, the map entropy is maximum if the noise is white with the same $\sigma_{\text{pix}}$ for each pixel. We have included modest (yet realistic) inhomogeneity in the noise as well to test sensitivity to this assumption, but find that makes little difference to our results. The noise in a pixel of area $A_{\text{pix}}$ from observations over an observing time $T_{\text{obs}}$ is $\sigma_{\text{pix}}^2 \propto 4\pi f_{\text{sky}} / A_{\text{pix}} T_{\text{obs}}$. In that case, the entropy is $S$(maps) /pixel $= [1 + \ln(2 \pi \sigma_{\text{pix}}^2)] /2$. With a fixed $T_{\text{obs}}$ and pixel size, the total entropy difference is $\propto f_{\text{sky}} \ln f_{\text{sky}}$ times a large number, thus quite a bit higher for larger regions, and not just because there are more pixels: it is higher per pixel. The entropy in the map is more constrained if we focus our available resources on smaller regions, but of course only if the regions are of a size and resolution to be of relevance for our target cosmological parameter, e.g., $r$.

We can obviously use the maps rather than the Tol’s as our starting point since, by design, no information relevant to estimation of our target $r$ is lost in the compression. The pixel sizes are chosen so this is true. Most of the huge entropy store in the Tol’s is inaccessible to $r$. As we have discussed, the traditional approach is to further compress $D$, but in $D \otimes D$ space (actually in the symmetric $D \vee D$ space), by solving for bandpowers in the manner described above. The translation of the variables is: $d$ are now the map products $\Delta \Delta^1$, $\varphi$ is $C_{\delta \varphi \varphi}$ and $\bar{q}$ is the vector of (normalized) bandpowers. Because the bandpower likelihood surface $p(\bar{q})$ is quite complex, non-Gaussian and with band-to-band correlations, determining the information in the bandpowers requires a direct integration. As well, the $N_{\text{band}}$-band are generalized ones, indexed by channel number, polarization component (a number for $TT, TE, EE, BB, TB, EB$), as well as by $\ell$-band number. With maximum likelihood relaxations to the bandpowers and Fisher matrix determination of errors (such as is used in XFaster), we would get $S$(band) = $N_{\text{band}}/2 + N_{\text{band}} \ln(2\pi)/2 + \text{Tr} \ln(\sigma_{\text{band}}^{-1}) /2$.

An oft-used approximation to likelihood surfaces fully determines $P(q^3)$ for each band $\beta$ with amplitude $q^3$, but treats band-band correlations in a weakly coupled Gaussian approximation. For example, Boomerang and CBI and other CMB likelihood analyses used the offset log-normal approximation of $\text{Bond et al. (1998)}$: each $P(q^3)$ was fit by a Gaussian in the variable $z^3 = \ln(q^3 + q_N^3)$ which required an estimate of the noise in the band $q_N^3$ as well as the observational mean $\bar{q}^3$, with a posterior of form

$$- \ln P(q^3 | D, T) = \frac{1}{2} \delta z^3 F_2 \delta z + \frac{1}{2} N \ln(2\pi) + \frac{1}{2} \text{Tr} \ln F_z^{-1},$$

in terms of the fluctuation $\delta z = z - \bar{z}$ about the observational $z$-average $\bar{z}^3 = \ln(\bar{q}^3 + q_N^3)$. The transformed correlation matrix is $F_z^{-1} = \langle \delta z \delta z \rangle$. For WMAP, a correction to this treatment was used, and for Planck a much more accurate characterization of the likelihood surface is needed, and continues to be under active development (e.g., $\text{Rocha et al. (2010)}$). For both, the likelihood is a hybrid, map-based for the low $\ell$’s, and bandpower-based (with $\Delta f = 1$) for high $\ell$’s.

A fully-characterized bandpower likelihood surface can of course be used for $r$ estimation provided it is lossless. If only a few bands $\beta$ are used, we can use intra-band template shapes with amplitudes $r_{\beta \bar{q}}$, which are approximately lossless for $r$; a 2-band calculation is shown in § 4.7. Mostly we quote single-band results, the one-step leap from the maps, using full-matrix posteriors, as we explore how different expenditures of observational time for various experimental sensitivities lead to changes in the error. We primarily quote $2\sigma_r$, as our error figure of merit, determined as explained in § 4.3.

A better figure of merit than $2\sigma_r$ is the change in 1D Shannon entropy which tells us the average amount by which the log of the allowed volume in the $r$ parameter space shrinks in response to varying the experimental setups. It is 1D because we marginalize over all other $N - 1$ parameters, the cosmic ones of interest and any nuisance parameters deemed necessary for the analysis, such as those characterizing uncertainties in calibration, beams, bolometer $T-Q-U$ leakage, and foreground uncertainties. The final 1D a posteriori probability $p_1(r | D, T) dr = \delta(r_{\text{obs}} - r) dr = \exp[-S_1(r) - \ln P(D, T)] dr$ involves a 1D information action $S_1(r)$, the integration over all parameters except the operator $r_{\text{op}}$ whose value is constrained to be fixed at $r$. Here $D$ refers to data, e.g., the maps, $T$ refers to the overall theoretical framework, e.g., inflation-inspired tilted $\Lambda$CDM adiabatic with the usual basic six cosmic parameters plus $r$, and $T_r$ refers to $T$.
with the $r_{\text{sp}} = r$ constraint.

The 1D Shannon information entropy, $S_1(r) = \langle S_1(r) \rangle_t + \ln P(D, T)$, is best done by numerical integration over the $r$-grid. The result is very simple if we truncate the ensemble-averaged expansion of $S_1(r)$ at quadratic order:

$$S_1(r) \approx \frac{1}{2} + \frac{1}{2} \ln(2\pi) + \ln(\sigma_r) = \frac{1}{2} + \ln V_r,$$

where $V_r$ (defined by the equation) is the compressed phase space volume for $r$ after the measurements.

Although we have used the natural log to make the entropy expressions familiar for physicists, in information theory one often uses the binary logarithm, $\log_2$. With natural logs the information is in nats, but with $\log_2$ it is in bits. When expressing information differences in $\log_2$ we translate to bits. Since a full bit represents a factor of 2 improvement in the error bar, $\Delta S_1(r)$ may only be a fraction of a bit, trivial perhaps, but subtle too, given the mammoth information compression from raw data to this one targeted parameter degree of freedom.

2.3. 2$\sigma$ Calculation

We define $\sigma_{95}$ through:

$$\int_{r_0}^{r_0 + \sigma_{95}} \mathcal{L}(r)dr = 0.954 \int_{0}^{\infty} \mathcal{L}(r)dr \quad (7)$$

where $r_0$ is the best-fit value of $r$. The $\sigma_{95}$-limit is determined by numerically integrating the Gaussian-fitted 1D likelihood curve.

In most cases considered in this paper the likelihood curves turn out to be well approximated by Gaussians. Therefore, when there is a few $\sigma$ detection (e.g. for $r = 0.12$) or when $r \sim 0$, to a very good approximation $\sigma_{95} = 2\sigma$ where $\sigma$ is the width of the Gaussian fit. Thus, throughout this paper we will use the common notation of $2\sigma$ which represents $\sigma_{95}$ and has been calculated through eq. (7). The only exception to this way of determining $2\sigma$ is when it is being directly given by the inverse of the Fisher matrix, where $\sigma$ represents the width of the likelihood function, under the assumption of its Gaussianity.

3. CONSTRAINED CORRELATIONS AND LINEAR RESPONSE IN PIXEL-PAIR AND PARAMETER SPACE

3.1. Matched Filters in Quadratic Pixel-Pair Space and Maximum Likelihood Estimation

When $C_t(q) = C_N + C_S(q)$ depends in a nonlinear way on $q$, we can still explore the posterior space by a sequence of linearized steps $\delta q^a$ which converge to zero in the approach to the maximum likelihood; $C_t^\ast = C_t(q_\ast)$ evaluated at the prior step $q_\ast$ can be thought of as the new general noise matrix and $\partial C_S(q)/\partial q^a \delta q^a$ the new signal matrix in the linear model. The quadratic expression determining the step is the action of a matched filter on the pixel-pixel pair data $d = \Delta \Delta^\ast$, and has a form that can be unravelled from the general expression eq. (4) with a non-zero residual mean $\langle n \rangle = C_t\ast$. If instead of the value at the last iteration we take $q_\ast = 0$, we get the usual map noise $C_N$, and a generalized noise with some signal contribution to it if only some of the $q_\ast$ are non-zero (e.g., foreground residual parameters).

The signal coefficients $q_{X,\beta}$ would be the isotropic power spectra bandpowers for the sets $TT, EE, BB, TE, TB, EB$. If the bands are of width $\Delta \ell = 1$ consisting of a single multipole, but all $m$, the $\varphi$ are the filters for defining the pixel-pixel correlation matrices in terms of an $\ell, m$ expansion of the total and polarization fluctuations, expressed in terms of the filters of eq. (2.1.3) $\sim \sum_m \mathcal{F}_{\ell,m} X_{\ell,m} \mathcal{F}_{\ell',m}^{\ast} X_{\ell',m}^{\ast}$. Or we can choose just one $\ell$ band with a template shape for each of the 6 $X$ cases with 6 amplitudes $r_X$ multiplying these. An example of this approach is shown in § 2.4.3. Or we could choose just one set of shapes for all 6, with only one amplitude multiplier, $q$ which we can normalize to be $r$. Template consistency is therefore assumed, and this gives the maximum leverage for teasing out the best determination for $r$ from the data, although it is of course heavily conditioned by the assumptions that go into the template construction (namely the values assumed for the other cosmological parameters which fix the structure of the templates).

The pixel-pair residual fluctuation weight, $W \equiv w_{ij} = (\delta n \delta n^{-1})$ is, for Gaussian models of $\Delta$, expressible as quadratic combinations of $w = \langle n \rangle^{-1} = C_t^{-1}$:

$$W_{ij}(kl) = [w_{ik}w_{jl} + w_{jk}w_{il} + w_{ij}w_{kl}]/4.$$

The inverse is

$$W_{ij}^{-1}(kl) = C_t^{-1,ik}C_t^{-1,jl} + C_t^{-1,jk}C_t^{-1,il} + C_t^{-1,ij}C_t^{-1,kl},$$

related so that

$$W_{ij}(kl)W_{ij}(mn) = \delta_{ij}(mn) = (\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in})/2.$$

(We use the Einstein summation convention, that like indices are to be summed.) When we reorganize the $\varphi^\ast W$ projector on the right hand side and the $\varphi^\ast W\varphi$ inverse residual matrix on the left hand side, we obtain the familiar Fisher expression for the parameter response $\delta q^a$ driven by the pixel-pair deviation $\delta C_{1O} = \Delta \Delta^\ast - C_{t\ast}$ of the raw observational correlation function $C_{1O}$ from its current estimate $C_t$:

$$F_{\alpha\beta}\langle \delta q^a \rangle\delta C_{1O} = \frac{1}{2} Tr[C_t^{-1}\partial C_t/\partial q^a C_t^{-1}(\Delta \Delta^\ast - C_{t\ast})]$$

$$= [\varphi^\ast W]^\ast [\Delta \Delta^\ast - C_{t\ast}]_{ij},$$

$$F_{\alpha\beta} = \frac{1}{2} Tr[C_t^{-1}\partial C_t/\partial q^a C_t^{-1}\partial C_t/\partial q^b]$$

$$= [\varphi^\ast W\varphi]_{\alpha\beta}. \quad (8)$$

This expression shows that the $\delta q^a$-adjustment is through a matched filter based on the templates $\varphi_{X,ij}^{\ast} \equiv \left( \partial C_{1O}/\partial q^{X,ij} \right)_{X,ij}$. The weighting in pixel-pair space shown is essential for it to be optimal. In § 2.4.2 and 2.4.3 we replace $\delta C_{1O}$ by other pixel-pair deviations to show how the single tensor template-based bandwidth, namely $r$, responds to individual $E$ and $B$ multipoles i.e., a window function showing where the $\ell$ power that $r$ is sensitive to lies. With the noise and sky fraction embedded in the weights and in $\varphi$, these window functions vary from experimental setup to experimental setup.

Equation (8) is an exact one following from a $\chi^2$ minimization of the linear expansion of $C_t$ in $\delta q$, albeit to be iteratively corrected. The more data-related path is to expand the information action associated with the posterior $p_t(q + \delta q)$ about the starting point $q$, to second
The fluctuation $F - F_0$ of the curvature metric $F_{\alpha\beta}$ from the Fisher matrix $F_{\alpha\beta}$ of eq. (8) has the two terms shown. Both are associated with the residual $\delta C_{10}$ mismatch since the parameter space correlations may not be able to fully saturate the data correlations. If the theory (including noise) is a good approximation to those components of $\delta C_{10}$ which survive the heavy matched-filtering, then these terms disappear with ensemble-averaging over all realizations. A caution is of course that we only inhabit a single realization. (The first subdominant second order term depends upon $\partial C_t/\partial q^\alpha \partial q^\beta$, hence vanishes in a linear expansion model.) Each $\delta q^\alpha = \left[ F^{-1} \right]_{\alpha\beta} p_{\beta}$ drives the system towards the $p_{\alpha}(q_* + \delta q) = 0$ "equilibrium", but corrective steps are needed to fully relax to $q_m^\alpha$. In practice, using $F_{\alpha\beta}$ in place of $F_{\alpha\beta}$ is usually adequate, and indeed often preferred. In cases with structure-less likelihood functions, a few iterations usually suffice to take us as close to the peak as required.

Since the entire statistics, given the validity of the Gaussian approximation for both signal and noise, is fully specified by the likelihood expression together with the prior probability defining the measure on the parameter space, no issue explicitly arises about mixing the $EB$-modes. The optimal quadratic filter to obtain the maximum likelihood for $r$ takes into account all aspects of the polarization. We can operate in the $QU$ polarization space, with specific spatial axes chosen for the polarization basis vectors, or we can do a transformation to
Tegmark & de Oliveira-Costa (2001) used similar window functions to those produced for the Boomerang "B98" results (Lange et al. 2001). These were used in likelihood evaluations in the 2000 release of the Bolometers Only Experiment (BOOMERAN). They constructed window functions to be consistent with the language used by Tegmark & de Oliveira-Costa (2001) (and other cosmological parameter estimation procedures). We have verified numerically that a full sky observation using the matrix methods gives uncorrelated modes $W_{\ell\ell'} = \delta_{\ell\ell'} C_{\ell\ell'}$. Figure 1 shows the increase in mode correlation with decreasing $f_{\text{sky}}$ for a fixed observation time. The observed patches are in the form of spherical caps. We plot an $\ell = 100$ stimulus for $f_{\text{sky}} = 0.07$ (at $N_{\text{side}} = 64$, pixel size \(\approx 56''\)) and $f_{\text{sky}} = 0.007$ (at $N_{\text{side}} = 128$, pixel size \(\approx 28''\)). (Figure 2 shows the associated beam and pixel window functions along with the polarization power spectra.)

Although the EE and BB responses are localized around the input $\ell = 100$, they are spread over $\ell$ and leak into the other $X$-mode. By contrast, the cross-filters ($W_{EE,100}\ell$ and $W_{BB,100}\ell$) are not localized. Note that they are substantially smaller than $W_{BB,100,10}$ and $W_{EE,100,10}$. We also see that the relative contribution of the EE signal to the contamination of BB is about 3 orders of magnitude larger than the contamination in EE due to BB. We can conclude that EE power uncertainties from a large range of scales will affect the BB measurement. The width of the oscillation $\Delta \ell \sim \theta_{\text{patch}}^{-1}$ is related to the cap size, narrowing as $f_{\text{sky}}$ goes up. The leakage is larger for smaller $r$, hence must be well characterized for highly sensitive $B$-mode experiments to avoid a false detection.

3.3. Linear Response of $r$ to $C_{BB\ell}$ and $C_{EE\ell}$

In this section, we use quadratic matched-filters to quantify the leakage of CMB power among the different $C_{\ell\ell'}$ spectra. These are "susceptibilities", relating the linear response of a target variable to the stimulus of a driver variable. We also refer to them as window functions to be consistent with the language used for bandpowers, in which the driver is the $C_{S,\ell\ell}$, and the response is the bandpower. The window function attached to each bandpower "gathers" in $\ell$-space from a given $C_{S,\ell\ell}$ the bandpower. There is a long history of making such windows publicly available. They were used in likelihood evaluations in the 2000 release of the Boomerang "B98" results (Lange et al. 2001). Tegmark & de Oliveira-Costa (2001) used similar window functions in the quest for a best quadratic estimator.

If we treat $C_{S,\ell\ell}$ as our variable and replace $C_{\ell\ell}^{-1} \delta C_{0\ell}$ by its ensemble average, we have

$$ \langle \delta C_{0\ell} \rangle = \delta C_{\ell} = \sum_{X,\ell} C_{S_{XX},\ell\ell} \delta C_{X,\ell}, $$

hence

$$ \delta C_{\ell} = \sum_{X,\ell} W_{X,\ell} \delta C_{X,\ell}, $$

$$ W_{X,\ell} = C_{X,\ell} \sum_{\beta} [F_{\beta X,\ell}^{-1}]^{-1} F_{\beta X,\ell}^{-1} = C_{X,\ell} \sum_{\beta} [F_{\beta X,\ell}^{-1}]^{-1} F_{\beta X,\ell}^{-1}. $$

The filter for a Spider-like experiment with a fiducial $r = 0.12$ is shown in Figure 3 as $f_{\text{sky}}$ varies (as does the pixel size). The red, purple, blue and green curves correspond to $f_{\text{sky}} = 0.75, 0.25, 0.07$ and 0.007, calculated at $N_{\text{side}} = 16, N_{\text{side}} = 32, N_{\text{side}} = 64$ and $N_{\text{side}} = 128$ respectively. As expected, the figures show that the measured $r$ is more sensitive to BB than to EE on most scales.

4. SIMULATION METHODS AND CALCULATIONAL RESULTS

In this section, we use the map-based $TQU$ likelihood procedure of §2 to compute the posterior $P(q, f_{\text{sky}}, D, T)$ in parameter subspaces and, by marginalization, the 1D
sian beam of width $\sigma_b = 0.425\theta_{\text{FWHM}}$, and pixelization windows $W_{\text{pix},\ell}$, an isotropized approximation to finite pixel size effects. (Timestream digitization filters are also generally required, but are swamped by these two filters.) The product is used to construct the symmetric $3 \times 3$ theoretical pixel-pixel signal covariance matrices, with 6 independent sub-matrices, $C_{E,X}$, $X \in \{TT, TQ, TU, QQ, QU, UU\}$. We assume experimental noise is Gaussian and usually take it to be white, so $C_{N,T} = \sigma_n^2 I$ for the temperature block and $C_{N,Q,U} = \sigma_{n,\text{pol}}^2 I$ for the polarization block of the covariance matrix, where we usually have $\sigma_{n,\text{pol}} \sim \sqrt{2}\sigma_{n,T}$. Here the $\sigma_n$ are effective noises per pixel, an amalgamation of the noises coming from different frequency channels. $I$ is the identity matrix. We neglect leakage from $T$ to $Q,U$.

Since we are forecasting the uncertainties in $r$ from different experimental setups, and not analyzing actual CMB maps, we can bypass creating a large ensemble of simulated CMB maps by replacing the observed correlation matrix $C_{10}$ by its ensemble average:

$$\text{Tr}C_{\ell}^{-1}\Delta\Delta^\dagger \rightarrow \text{Tr}(C_{\ell}^{-1}\Delta\Delta^\dagger) = \text{Tr}C_{\ell}^{-1}\bar{C}_{10}.$$  

Here $\bar{C}_{10}$ is the ensemble-averaged "pixel-pair data", namely the covariance matrix of the input fiducial signal model together with the instrument noise, and $C_{1}(q)$ is the signal-pixel-pixel covariance matrix for the parameters $q$ plus the various noise contributions, instrumental and otherwise. An advantage of this approach is that the recovered values of the parameters are what the ensemble average of sky realizations would yield, and will not move hugely due to the chance strangeness of any one realization (as the real sky may provide for us). Note that while sample variance does not impact the location of the maximum likelihood in this ensemble-averaged approach, it is fully reflected in the width of the posterior distribution from which our uncertainties are derived.

We mask out the part of the Galaxy falling in the observed patch (the P06 WMAP-mask [Page et al. (2007)]), assuming it to be too foreground-dominated for useful parameter extraction. We also project out modes larger than the fundamental mode of the observed patch since, due to time-domain filtering, information is not usually recoverable on such large scales. For instance, if the mask has the shape of a spherical cap extending from the north pole to $\theta = \theta_{\text{patch}}$, we add a very large noise to the modes with $2\ell + 1 < [2\pi/\varpi]$ where $[.]$ takes the integer part and $\varpi = 2\sin(\theta_{\text{patch}}/2)$ is the flat 2D radius of the disk with an area equal to the solid angle of the cap. This makes the likelihood insensitive to any information at and beyond the patch scale. This large scale mode cut is especially important to include for larger values of $f_{\text{sky}}$, where the low $\ell$ modes contribute significantly to $r$ measurement through the reionization bump. In real large sky experiments it will not be easy to draw such modes from the maps.

Our simulations cover two observational cases: an all-sky experiment with Planck-like white noise levels, and a partial sky experiment with Spider-like white noise levels, each with two frequency channels, assuming other frequencies are used for subtracting foregrounds. We have also made the simplifying assumption that in each ex-

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**Fig. 3.** -- Window functions $W_{\chi,X}$ for $X \subset \{EE, BB\}$ for different sky cuts show that, as expected, all-sky experiments are nicely sensitive to reionization BB bump, but smaller sky experiments are not, although they pick up well the BB signal. We show that such a naive approach does not work particularly well in predicting the errors: if properly handled, $C_{\text{TE}}$, $C_{\text{TT}}$, and $C_{\text{QQ}}$ are more due to the onset of experimental noise than to the onset of experimental noise. To illustrate the role of this, a CMBpol-like experiment with $f_{\text{sky}} = 0.007$ can punch out a robust detection from 50 to 150 in $\ell$, and though 0.07 loses out a bit (relatively) at 150, its detection would come from a wider stretch in ln $\ell$, out to $\ell \sim 20$ before falling off. Only at $f_{\text{sky}} > 0.25$ does one begin to pick up the reionization bump. The curious drop in the all-sky $N_{\text{side}} = 16$ red line at the top is due to the Spider-like noise for higher $\ell$ being heavily enhanced because all of the sky is covered in the same amount of observing time. To illustrate the role of this, a CMBpol-like experiment with $C_{\chi} \sim 1000$ is plotted, with $N_{\text{side}} = 16$ (dashed straight line) and $N_{\text{side}} = 64$ (triple-dot-dashed line). The reason all three are offset from one another is because the normalizing $\sigma_n^2$ depends upon the amount the filter captures of the total $r$ signal.

We neglect instrumental leakage from $T$ to $Q,U$. The posterior $P(r|f_{\text{sky}},D,T)$ as a function of $f_{\text{sky}}$. Although we avoid explicit $E$-$B$ decomposition with this method, we do make identical calculations to the $TQ$ matrix ones in $\ell$-space using $TT$, $TE$, $EE$ and $BB$, and assuming no mixing. We show that such a naive approach does quite well in predicting the errors: if properly handled, polarization-mode-mixing is not a significant error source in most cases. Of course for either method to be successful, all generalized noise sources need to be identified including instrumental leakage from $T$ to $Q,U$.

**4.1. Calculation of Ensemble-Averaged Posteriors on Parameter Grids**

We calculate the posterior distribution on a gridded parameter space, a method mostly applicable to low dimensional parameter spaces. At each point of the parameter grid the $C_{\chi}$’s are calculated using the public code CAMB [13]. These are then multiplied by beam windows, $B_\ell^2 = e^{-\ell(\ell+1)\sigma_n^2}$, assuming a Gaussian beam of $0.425\theta_{\text{FWHM}}$. The product is used to construct the symmetric $3 \times 3$ theoretical pixel-pixel signal covariance matrices, with 6 independent sub-matrices, $C_{E,X}$, $X \in \{TT, TQ, TU, QQ, QU, UU\}$. These are then multiplied by beam windows, $B_\ell^2 = e^{-\ell(\ell+1)\sigma_n^2}$, assuming a Gaussian beam of $0.425\theta_{\text{FWHM}}$, and pixelization windows $W_{\text{pix},\ell}$, an isotropized approximation to finite pixel size effects. (Timestream digitization filters are also generally required, but are swamped by these two filters.) The product is used to construct the symmetric $3 \times 3$ theoretical pixel-pixel signal covariance matrices, with 6 independent sub-matrices, $C_{E,X}$, $X \in \{TT, TQ, TU, QQ, QU, UU\}$. We assume experimental noise is Gaussian and usually take it to be white, so $C_{N,T} = \sigma_n^2 I$ for the temperature block and $C_{N,Q,U} = \sigma_{n,\text{pol}}^2 I$ for the polarization block of the covariance matrix, where we usually have $\sigma_{n,\text{pol}} \sim \sqrt{2}\sigma_{n,T}$. Here the $\sigma_n$ are effective noises per pixel, an amalgamation of the noises coming from different frequency channels. $I$ is the identity matrix. We neglect leakage from $T$ to $Q,U$.

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$$\text{Tr}C_{\ell}^{-1}\Delta\Delta^\dagger \rightarrow \text{Tr}(C_{\ell}^{-1}\Delta\Delta^\dagger) = \text{Tr}C_{\ell}^{-1}\bar{C}_{10}.$$  

Here $\bar{C}_{10}$ is the ensemble-averaged "pixel-pair data", namely the covariance matrix of the input fiducial signal model together with the instrument noise, and $C_{1}(q)$ is the signal-pixel-pixel covariance matrix for the parameters $q$ plus the various noise contributions, instrumental and otherwise. An advantage of this approach is that the recovered values of the parameters are what the ensemble average of sky realizations would yield, and will not move hugely due to the chance strangeness of any one realization (as the real sky may provide for us). Note that while sample variance does not impact the location of the maximum likelihood in this ensemble-averaged approach, it is fully reflected in the width of the posterior distribution from which our uncertainties are derived.

We mask out the part of the Galaxy falling in the observed patch (the P06 WMAP-mask [Page et al. (2007)]), assuming it to be too foreground-dominated for useful parameter extraction. We also project out modes larger than the fundamental mode of the observed patch since, due to time-domain filtering, information is not usually recoverable on such large scales. For instance, if the mask has the shape of a spherical cap extending from the north pole to $\theta = \theta_{\text{patch}}$, we add a very large noise to the modes with $2\ell + 1 < [2\pi/\varpi]$ where $[.]$ takes the integer part and $\varpi = 2\sin(\theta_{\text{patch}}/2)$ is the flat 2D radius of the disk with an area equal to the solid angle of the cap. This makes the likelihood insensitive to any information at and beyond the patch scale. This large scale mode cut is especially important to include for larger values of $f_{\text{sky}}$, where the low $\ell$ modes contribute significantly to $r$ measurement through the reionization bump. In real large sky experiments it will not be easy to draw such modes from the maps.

Our simulations cover two observational cases: an all-sky experiment with Planck-like white noise levels, and a partial sky experiment with Spider-like white noise levels, each with two frequency channels, assuming other frequencies are used for subtracting foregrounds. We have also made the simplifying assumption that in each ex-
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4.2. Residual Foreground-Subtraction "Noise"

No study of gravitational wave detectability by B-mode experiments can ignore the impact of polarized foreground emission. Component separation is a major industry in itself. Various techniques have been utilized with CMB data up to now - often involving template parameter marginalization of one sort or another. We have been lucky so far in that the foregrounds have been manageable for TT, TE and EE. The level of subtraction needed to unearth the very tiny gravity wave induced B-signal is rather daunting, especially since the foregrounds are largest at the low $\ell$. Thus, although we may wrestle the generalized noise from the detectors and from experimental systematics to levels allowing small $r$ to be detectable, the foregrounds will need to be well addressed before any claim of primordial detection will be believable. Although we have learned much already about the TT foregrounds and, from WMAP, the synchrotron EE, we do not know the $\ell$-shape or the amplitude of the polarization for dust. In O'Dea et al. (2011b), the polarization emission from thermal dust is based on a three-dimensional model of dust density and two-component Galactic magnetic field. It is assumed that the degree of polarization has a quadratic dependence on the magnetic field strength and its direction is perpendicular to the component of the local magnetic field in the plane of the sky, similar to the model assumed by WMAP in Page et al. (2007). In forecasting for proposed post-Planck satellite experiments, simple approximations for thermal dust and synchrotron emission have been made (e.g., Baumann et al. (2004), and references therein). The dusty $\ell$-structure in this model is similar to the O'Dea et al. (2011b) form: $C_X \sim \ell^{-0.5}$ for $X = EE, BB$. We follow this Baumann et al. (2009) approach here, but apply it to our pixel-based analysis.

We therefore assume that the maps are already foreground-subtracted, possibly with the wider Planck

| Experiment | Freq  | FWHM | num. of det. | $\Delta T$ | $\Delta T$ | obs. time |
|------------|-------|------|--------------|-----------|-----------|-----------|
| Spider-like | 96    | 50'  | 768          | 3.2       | 4.5       | 580 hr    |
| Spider-like | 150   | 32'  | 960          | 2.7       | 3.8       | 580 hr    |
| Planck-like | 100   | 10'  | 8            | 3.8       | 6.1       | 2.5 yr    |
| Planck-like | 143   | 7'   | 8            | 2.4       | 4.6       | 2.5 yr    |
| CMBPol (mid-cost) | 100 | 8'   | 1.05       | 0.18     | 0.26      | -         |
| CMBPol (mid-cost) | 150  | 5'   | 0.19        | 0.27      | -         | -         |

$^a$ nK, the instrument sensitivity divided by $\sqrt{\text{total observation time}}$.

$^b$ These Spider-like specifications which are used as the default in this paper are different from the more recently proposed ones in Fraisse et al. (2011) with two 20 day flights. The first flight uses three 90 and three 150 GHz receivers each with 288 and 512 detectors respectively. In the second flight, two 280 GHz receivers replace one 90 and one 150 GHz telescope, leaving the configuration of the flight identical to the first one. The detector sensitivity as proposed in Fraisse et al. (2011) is 150, 150 and 380 $\mu$K $\sqrt{\text{Hz}}$ at 90, 150 and 280 GHz, respectively. The performance of the default Spider-like experiment in this paper and the more recent proposal as in Fraisse et al. (2011) are very close (see Figure 12).

$^c$ http://www.rssd.esa.int/index.php?project=planck

$^d$ For a mid-cost full-sky CMBPol experiment based on table 13 of Baumann et al. (2011). We are using 100 and 150 GHz channels in our simulations. Adding more channels, in the unrealistic case of no foreground contamination we simulate, would not affect the limits on $r$, since with these low instrument noise levels, either lensing or cosmic variance, depending on how small $r$ is, would be the dominant source of uncertainty.

For the Spider-like case we keep the flight time constant so that the observation gets deeper as $f_{\text{sky}}$ decreases, while for the Planck-like experiment the pixel noise is assumed constant for different values of $f_{\text{sky}}$. The latter case, with small values of $f_{\text{sky}}$, is used to illustrate how well a strategy of only analyzing the lowest foreground sky could work, if for example, foreground removal turns out to be prohibitive over much of the sky. If foregrounds can be well-removed from Planck, then full sky is appropriate.

We calculate the constraints on targeted cosmological parameters for different $f_{\text{sky}}$'s, assuming the observed patches are spherical caps from $\theta = 0$ to $\theta = \theta_{\text{patch}}$, corresponding to $\theta = \cos^{-1}(1 - 2f_{\text{sky}})$. We perform the analysis at different resolutions for different sky cuts to minimize the effect of pixelization for small $f_{\text{sky}}$ on the one hand, and to keep the computational time reasonable for large $f_{\text{sky}}$ on the other hand. We use $N_{\text{side}} = 32$, $N_{\text{side}} = 64$ and $N_{\text{side}} = 128$ for $f_{\text{sky}} > 0.25$, $0.007 < f_{\text{sky}} < 0.25$, and $f_{\text{sky}} \leq 0.007$, respectively. We checked the results for two neighbour resolutions at resolution switches. For the low $f_{\text{sky}}$ switch, results are not sensitive to the change of resolution while for the switch at larger $f_{\text{sky}}$ we are about 10% - 15% pessimistic in the results by choosing the lower resolution, specifically for a Planck-like case (with small beam) and for a higher value of $r$, e.g., $r = 0.12$. In these cases, lensing starts to dominate at higher multipoles and choosing a high enough resolution for the analysis would improve the errors on $r$ by resolving the primordial gravity waves at relatively high multipoles.

TABLE 1

Specifications of Spider-like, Planck-like and CMBPol (mid-cost) experiments for simulations.
frequency coverage used in conjunction with the Spider maps, with the CMB-component having a residual uncertainty, which we incorporate in our analysis as an additional large-scale (inhomogeneous) noise component \( C_{\text{N}}^{\ell}(\nu) \). We assume the power spectrum of the foreground residuals has the same shape as the original foreground spectrum, but with only a few percent of the amplitude:

\[
C_{X\ell} \to C_{X\ell} + \sum_{fg=S,D} \frac{C_{X\ell}^{\ell}(\nu)}{C_{X\ell}^{\ell}(\nu)} \frac{C_{X\ell}^{fg}(\nu)}{C_{X\ell}^{fg}(\nu)}, \quad X = EE, BB,
\]

with the sum over synchrotron \( S \) and dust \( D \) emissions. The tunable removal-efficiency parameters \( \epsilon^{fg}(\nu) \) are taken to be 5% in our plots. The shapes are:

\[
\text{synchrotron : } C_{X\ell}^{\text{S}}(\nu) = \frac{\ell(\ell + 1)}{2\pi} A_{S} \frac{\nu}{\nu_{0}}^{\alpha_{S}} \left( \frac{\ell}{\ell_{0}} \right)^{\beta_{S}}
\]

\[
\text{dust : } C_{X\ell}^{\text{D}}(\nu) = \frac{\ell(\ell + 1)}{2\pi} A_{D} \frac{\nu}{\nu_{0}}^{\alpha_{D}} \left( \frac{\ell}{\ell_{0}} \right)^{\beta_{D}} \times \left[ \frac{e^{\frac{\nu_{0}}{kT}} - 1}{e^{\frac{\nu}{kT}} - 1} \right]^{2}.
\]

The dust polarization fraction, \( p \), is assumed to be around 5%. The values for the other parameters taken from [Baumann et al. 2003] are listed in Table 2. They were chosen to give agreement with WMAP, DASI, and IRAS observations (and the Planck sky model, which is based on these). Although this model provides only a rough guide to the impact incomplete foreground subtraction will have on r-estimation, it does include the crucial large-scale dependence which differentiates it so much from the structure of the instrumental noise.

A natural question when considering deep small sky observations is how many patches there are on the sky with low foregrounds so the requisite cleaning is at a minimum. The Planck Sky Model for the polarized foreground emission ([Leach et al. 2008, Delabrouille et al. 2011]) is similar to the one we have adopted. Using a code developed by Miville-Deschenes, we have calculated for patches of radius \( R \) the pixel-averaged variance at pixel \( p \), \( \sigma_{\text{pol},fg}^{2}(p, R) = \langle (p - \bar{P}(< R))^{2} \rangle \) of the polarization intensity \( P = \sqrt{Q^{2} + U^{2}} \) about the patch-average \( \bar{P} \) arising from the synchrotron and dust foregrounds. We compare this with the \( \sigma_{\text{pol},gw}^{2}(p, R) \) we obtained for each patch in a single tensor-only primordial polarization realization (which is proportional to \( r^{2} \)). The patches are sorted in decreasing order of the "signal-to-noise" ratio \( \sigma_{\text{pol},gw}(p, R)/\sigma_{\text{pol},fg}(p, R) \). The next pixel on the list is included in a patch list if it has no overlap with the patches in the previously-detected higher-signal-to-noise list. A patch is considered to be r-clean if this polarization signal-to-noise exceeds unity, a rather strong criterion. At 100 GHz, we found no "r=0.01"-clean patches, seven "r=0.05"-clean patches and ten "r=0.1"-clean patches with \( f_{\text{sky}} \gtrsim 0.007 \) (\( R = 10^\circ \)). There are one "r=0.05"-clean patch and two "r=0.1"-clean patches for \( f_{\text{sky}} \gtrsim 0.03 \) (\( R = 20^\circ \)). At 150 GHz, we found no "r=0.05"-clean patches and one "r=0.1"-clean patch with \( f_{\text{sky}} \gtrsim 0.007 \) but no r=0.1-clean patches for \( f_{\text{sky}} \gtrsim 0.03 \).

The non-overlapping criterion is quite severe. Another measure of r-cleanliness is to determine the fraction of sky with \( \sigma_{\text{pol},gw}(p, R)/\sigma_{\text{pol},fg}(p, R) \) above unity. The "r"-clean fraction is clearly \( \sim 0 \) for those values of \( r \) and \( R \) with no corresponding clean patches (as stated above). Here only the non-zero values are reported. At 100 GHz, the "r=0.05"-clean fraction is \( \sim 0.14 \) (\( R = 10^\circ \)) and the "r=0.1"-clean fraction is \( \sim 0.24 \) (\( R = 10^\circ \)). For both values of \( r \), there is no appreciable decrease in the sky fraction by increasing the patch sizes to \( R = 20^\circ \). At 150 GHz, the "r=0.1"-clean fraction is \( \sim 0.04 \) (\( R = 10^\circ \)). It should be noted that as these sky fractions do not necessarily correspond to contiguous regions, the sky fraction of interest for small-sky B-mode experiments is in principle smaller. The Planck Sky Model at the lower frequencies agrees with the (extrapolated) synchrotron emission from WMAP, but the higher frequency polarized dust emission really requires better observations, and awaits the release of the Planck mission results.

4.3. Correlations of r with Other Cosmic Parameters

Either detecting \( r \) or placing a tight upper bound is crucial for progress in inflation studies. Correlations of \( r \) with other parameters \( q^\alpha \) must be properly accounted for, since they are marginalized in the reduction to the 1D \( r \)-posterior. The relative importance of the various \( q^\alpha \) is determined by calculating the posterior-averaged cross-correlations \( \langle \delta r | \delta q^\alpha \rangle \), which depend upon the experimental configuration and its noise. Within the quadratic approximation for the posterior information action, the correlations can be estimated from the inverse components, \( F^{-1}\alpha\alpha \), using the Fisher matrix equation 5, with lensing as well as instrumental noise included in the generalized noise matrix. Small steps in the main parameters of the standard ΛCDM model (\( \ln(\Omega_b h^2), \ln(\Omega_c h^2), H_0, n_s, \tau, r \)) from the fiducial WMAP7 values 14 were taken to determine \( F \) by numerical differentiation. The scalar amplitude \( A_s \) is treated as a normalization parameter here, so it is not included in the parameter list. We use two different fiducial values for \( r \), 0.2 and 0.01, and three values of \( f_{\text{sky}} \), 0.007, 0.07, and 0.75, for a Spider-like experiment. We use a Gaussian prior on all parameters \( q^\alpha \) but \( r \) with \( q^\alpha = \frac{1}{\sigma_{\text{WMAP7}}} | F_{\text{prior}} \), given by the WMAP7 best-fit parameters. We choose \( F_{\text{prior}} | \alpha\beta = \sigma_{\text{WMAP7}} | \delta_{\alpha\beta} \), which gives a weaker prior than the true WMAP7 results would give. In the quadratic approximation to the posterior information action, the total Fisher matrix is \( F = F_F + F_{\text{prior}} \).

The average deviation in \( r \), \( \langle \delta r | \delta q^\alpha \rangle \), and its variance, \( \langle \Delta r \delta r | \delta q^\alpha \rangle \), driven by given fluctuations in the other

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14 http://lambda.gsfc.nasa.gov/product/map/dr4/params/lcdm_sz_lens_wmap7.cfm
parameters, $\delta q^\alpha$, are
\[\delta r \equiv \langle \delta r | \delta q^\alpha \rangle = \langle F^{-1}_t \rangle^{\tau,\alpha} (F_t)_{\alpha\beta} \delta q^\beta.
\]
\[\langle (\delta r - \delta r)^2 | \delta q^\alpha \rangle = \langle F^{-1}_t \rangle^{\tau,\alpha} - \langle F^{-1}_t \rangle^{\tau,\alpha} (F_t)_{\alpha\beta} \langle F^{-1}_t \rangle^{\tau,\beta}.
\]
If we let only one $\delta q^\alpha$ at a time differ from zero, and normalize the deviations to their 1-sigma values, we can express the result in terms of a dimensionless measure $\rho_{\tau \alpha}$ of the degree of correlation:
\[\rho_{\tau \alpha} \equiv \langle (\delta r | \delta q^\alpha) / \sigma_r / \delta q^\alpha \rangle / \sigma_{\tau \alpha} \approx F^{-1}_t \langle (F^{-1}_t)_{\alpha\beta} (F_t)_{\alpha\beta} \rangle^{1/2}.
\]
The variance is $(\delta r - \delta r)^2 | \delta q^\alpha \rangle \approx \sigma_{\tau \alpha}^2 (1 - \rho_{\tau \alpha}^2)$. 

For the full sky case, we find the largest $\rho_{\tau \alpha}$ for $\tau$ and $n_s$, with $\rho_{\tau \tau}$ and $\rho_{n_s\tau}$ both $\approx 0.25$. For smaller sky coverage, the degeneracy between $r$ and $\tau$ disappears since the main constraints on $\tau$ come from the large scale polarization, which small cut-sky cases are not sensitive to. The dominant correlations of $r$ are with the matter density parameters $\Omega_c h^2$ and $\Omega_b h^2$, at the 0.1–0.2 level, a consequence of the gravitational lensing induced $BB$ noise. Even in the 25% case for $\rho$, the constrained error diminishes only by 3%.

Thus we should be able to safely estimate the error on $r$ with all or none of the basic six parameters held fixed. We verified this explicitly by comparing the 2D uncertainties calculated from the full 2D $r - \tau$-grid with the full 6D uncertainties calculated from the inverse Fisher matrix, in $\ell$-space and in pixel-based space. 

| method      | param space | $N_{\text{side}} = 32$ | $N_{\text{side}} = 64$ | $N_{\text{side}} = 128$ |
|-------------|-------------|------------------------|------------------------|------------------------|
| Fisher      | 1 param     | 0.022                  | 0.118                  | 0.037                  |
| $\ell$-space| 2 param     | 0.023                  | 0.018                  | 0.037                  |
|             | 6 param     | 0.025                  | 0.020                  | 0.037                  |
| Fisher      | 1 param     | 0.022                  | 0.019                  | 0.034                  |
| pixel-space | 2 param     | 0.023                  | 0.019                  | 0.034                  |
|             | 6 param     | 0.025                  | 0.020                  | 0.035                  |
| grid-based  | 2 param     | 0.021                  | 0.018                  | 0.036                  |

Table 3: Reduced parameter spaces gives robust results, independent of cap sizes, here for $f_{\text{sky}} = 1, 0.07, 0.007$.

In this section, we use $\tau$ as well as $r$ to make our 2D parameter space since it has a direct impact on the $BB$ reionization bump. We fix the overall $C_l$ normalization for each parameter pair to the WMAP $TT$ measurement at $\ell = 220$. This is equivalent to having $A_s$ as an adjustable parameter. If not otherwise stated, lensing has been included in all of the following simulations with a fixed noise template, linearly scaled with $A_s$ accordingly. Treating lensing in the noise covariance completely takes into account its effect on sample variance. It may be possible for it to be partly removed in the patch using delensing algorithms, (see e.g., Smith et al. (2008) and references therein), leading to a reduced variance in the same way that we are treating a foreground residual. However, treating lensing as a noise source is a good assumption for our purposes here.

The $2\sigma_{r, (F_{\text{sky}})}$ plots in Figures 4 and 5 are our main results. Shown are two fiducial models with $r_{\text{fid}} = 0.12, 0.001$, both having $\tau_{\text{fid}} = 0.09$. The $f_{\text{sky}}$ in the

![Plot of uncertainties in r for different sky coverages](https://via.placeholder.com/150)
plots is the sky coverage before the Galaxy is masked. The Galaxy cut starts coming into the observed patch for $\theta_{\text{patch}} \sim 40^\circ$.

The results are compared to the expected error bars on $r$ from a simplified $\ell$-space analysis. Counting modes properly is a difficulty in the $\ell$-space approximation for cut-skies. (This differs from the full pixel-pixel covariance matrix analysis in which all modes are naturally taken care of.) For the $\ell$-space approximation, we have taken the mode number to be the naive $[f_{\text{sky}}(2\ell + 1)]$ where $[\ldots]$ indicate the integer part. This imposes a low $\ell$-cut on the modes by demanding $[f_{\text{sky}}(2\ell + 1)] \geq 1$ which overrides the $\ell$-cut from the fundamental mode of the patch, $2\ell + 1 = [2\pi/2\sin(\theta_{\text{patch}}/2)]$, up to $\theta \approx 30^\circ$.

This $\ell$-space $\sigma_r(f_{\text{sky}})$ is a lower bound since it ignores the mode mixing on the cut sky. Still, in the absence of systematic errors and for the simplified noise assumed here, the errors we find are near the true (matrix) values, as Figure 4 confirms for $r_{\text{fid}} = 0.01$. A similar measurement with $r_{\text{fid}} = 0.2$ shows the same thing, though with a more-flattened curve for $\sigma_r(f_{\text{sky}})$ for the Spider-like case and with foregrounds playing a smaller role. $E - B$ mixing does not seem to be a serious impediment, at least down to $f_{\text{sky}} \approx 0.01$. For the Spider-like experiment, the error minimum is $2\sigma_r = 0.035$ for $r_{\text{fid}} = 0.12$, at $f_{\text{sky}} \approx 0.15$, but the trough is broad. For the low $r_{\text{fid}} = 0.001$, for which only an upper limit can be expected, Figure 5 shows the agreement in $\sigma_r(f_{\text{sky}})$ between $\ell$-space and pixel-space is not quite as good, especially for $f_{\text{sky}} \approx 0.25 - 0.5$ for which considerable observation time is expended on the $\ell \approx 12 BB$ valley (see Figure 2) where there is little signal. The naive $\ell$-
Fig. 7.—1σ and 2σ \( r - \tau \) contours with and without foregrounds for a Spider-like experiment with different sky cuts and for a Planck-like Galaxy-masked experiment with effective \( f_{\text{sky}} \approx 0.75 \). In the two right panels the contours for the combined Spider-like and Planck-like experiments are also plotted. The black plus signs denote the input \( \tau_{\text{fid}} = 0.12 \) and \( \tau_{\text{id}} = 0.09 \). Expanding Spider-like observing time on large sky coverage would not improve much the Planck forecasted \( \tau \) error, but would decrease the combined \( \tau \) error, suggesting the deep small-sky option is better.

Space approximation underestimates this, but agreement with pixel-space is regained in runs with the reionization bump removed, by setting \( \tau = 0 \); for this case the monotonic rise in \( \sigma_r(f_{\text{sky}}) \) with increasing \( f_{\text{sky}} \) continues to full sky.

Extending to the full Galaxy-masked sky improves the upper limit on \( r \) since the window function captures the low-\( \ell \) bump. The \( \ell \)-space and pixel-space calculations disagree slightly, but when the Galaxy mask is removed, the estimates agree.

At small \( f_{\text{sky}} \), \( 2\sigma_r \) increases due to lensing which dominates the total \( BB \) spectrum at small scales. The competition between avoiding contamination by lensing and avoiding the \( \ell \approx 12 \) valley produces a weak minimum in \( \sigma_r \) at \( f_{\text{sky}} \approx 0.15 \) for \( r = 0.12 \), when a detection is expected, and at \( f_{\text{sky}} \approx 0.03 \) for \( r = 0.001 \), when an upper limit is expected. The full sky is weakly optimal for setting an upper limit in the absence of foregrounds.

The Planck-like measurements in the lower plots of Figures 4 and 5 show a rise in \( 2\sigma_r \), as \( f_{\text{sky}} \) drops since the information on the large scales are lost while the pixel noise stays unchanged. The dashed lines in these plots show the approximate \( 2\sigma_r \) for a full-sky Galaxy-masked Planck-like experiment if the large-scale modes are filtered e.g., by time-domain filtering or due to high foreground contamination and thus the observed region is considered to be a combination of smaller patches (adding up to the full sky in total observed area).

Not surprisingly, we see that foregrounds mostly affect experiments with larger \( f_{\text{sky}} \), and for fiducial models with smaller \( r \). We also see that deep observations of quite small patches seem to do as well as larger patches (observed less deeply) and even much better if \( r \) is small (for which the sample variance is very small and instrument noise plays the dominant role).

Figure 6 shows how different components contribute to the error on \( r \) calculated using the Fisher matrix for various \( r_{\text{id}} \) and \( f_{\text{sky}} = 0.007 \) and 0.07. As before the mode mixing is ignored in the \( \ell \)-space calculation. If there were no lensing and no mode-mixing, in the limit of no instrument noise, the only source of error would be the sample variance, which is, as expected, proportional to \( r \). The solid black lines show the minimum irreducible errors due to sample variance and lensing. We contrast this with calculations in both pixel and \( \ell \)-space of two Spider-like experiments. One has 10 times less noise than the fiducial Spider case, a noise level that can be seen to give almost no contribution to the errors for these sky cuts since lensing noise is dominant. The other has our standard Spider-like noise, which can be seen to significantly add to the error. The neglect of mode-mixing in determining \( \sigma_r \) vanishes as \( r \) increases, since sample variance dominates the error, as a comparison of the curves from the pixel-space and \( \ell \)-space analyses shows. The overplotted symbols represent the errors from measuring the likelihood curve in a gridded 2D parameter space (as explained earlier). The \( 2\sigma_r \)'s from the full method and the Fisher matrix approximation are close. The small difference is because the \( r \)-likelihood curve is not a perfect Gaussian.

Figure 7 shows the 2D \( r - \tau \) contours for 3 different values of sky coverage for a Spider-like experiment compared to a full-sky Planck-like experiment (with Galaxy mask cut) with and without foreground contamination. As expected, \( \tau \) is unconstrained as \( f_{\text{sky}} \) decreases for the Spider-like experiment since \( \tau \)-constraints come from the largest angular scales: what is optimal for \( r \) detection is awful for \( \tau \) determination, for which all-sky is best.

4.5. Results in \( r-n_s \) Space

In Figure 8 we have plotted the \( r-n_s \) contours for an \( f_{\text{sky}} = 0.08 \) Spider-like experiment and for a full-sky Planck-like survey, with and without foregrounds, using the model discussed in § 4.4. This shows almost no correlation between the two parameters for these experimental cases, as expected from the discussion in § 4.4. It also shows the remarkable set of inflation constraints that may arise from Planck and Spider-like experiments.

4.6. Results in \( r-n_t \) Space

Although detecting \( r \) would provide an invaluable measure of the mean acceleration parameter (and energy scale) of inflation, we want more, the shape of the tensor power embodied in the tensor tilt \( n_t \), which we explore here in a 2D space by fixing \( \tau, n_s \) and the other cosmic parameters. Figure 9 shows the 2D contours for \( r-n_t \)
4.7. Breaking $r$ up into $r_{X\beta}$-Shape Parameters: A Tensor Consistency Check

Because $r$ is essentially a linear parameter (for given $A_s$), we are effectively determining a single (very) broadband power amplitude multiplying a collection of fiducial $X$-template shapes $c_{X\ell}^{(g)}$ given by the gravitational wave powers. It is natural to test this locked-in monolithic parameterization by introducing a collection of parameters $r_{X\beta}$ multiplying individual $X$ and $\ell$-band templates:

$$
C_{EE\ell} = c_{EE\ell}^{(s)} + r_{EE\beta}(\ell)c_{EE\ell}^{(g)}
$$

$$
C_{BB\ell} = c_{BB\ell}^{(s)} + r_{BB\beta}(\ell)c_{BB\ell}^{(g)}.
$$

Here $c_{EE\ell}^{(s)}$ is the scalar part of $C_{EE\ell}$, including lensing, and $c_{BB\ell}^{(g)}$ is the lensed BB power. The overall normalization is arranged so that $r_{X\beta} = r$ is the tensor consistency condition. The $\chi_\beta(\ell)'s$ are the $\beta$-windows. These have often been taken to be top-hats satisfying a saturation property $\sum_\beta \chi_\beta(\ell) = 1$ and an orthogonality property $\chi_\beta(\ell)\chi'_\beta(\ell) = \delta_{\beta\beta}$ in bandpower work. However, the modes could also be quite overlapping as long as saturation and the $r_{X\beta} = r$ normalization are satisfied.

This is a reasonable path to finding the tensor bandpowers for BB and EE but, given the $\frac{4.4}{4.6}$ result on $n_s$, we will content ourselves with a 2D example using one $\ell$-band $\beta$ and two $X$ parameters, $r_{EE}$ and $r_{BB}$. For this study, we keep $A_s$ fixed (cf. $\frac{4.4}{4.6}$). The contours in Figure 11 show the degree to which the tensor consistency encoded in the $r_{EE} = r_{BB}$ line, can be checked. The contours confirm the expectation that the $B$-modes are the most influential source of information about primordial tensor perturbations, since the large scalar contribution to EE swamps the tiny tensor signal, infating the error bars. Using checks like these for showing consistency have had a long history. In the first EE polarization detection papers, the EE amplitude was shown to be consistent with the amplitude expected from TT parameters (Kovac et al. 2002; Sievers 2004). In the first lensing detections in the TT power spectra, the deviations from lens-free results were shown ro be consistent with expectations from the parameters determined from the primary TT data (Reichardt et al. 2008; Dunkley et al. 2010).

4.8. Breaking $f_{sky}$ into Many Fields

Using multiple (foreground-minimized) fields to make up a total $f_{sky}$ is an approach that has been advocated for ground-based strategies (e.g., for ABS, 14). In Figure 11 we show the impact of splitting $f_{sky}$ into four patches, while keeping the total integration time and the instrument noise constant. One does not lose that much as long as the total probe is a few percent of the sky, a consequence of the broad single-patch $\sigma_r(f_{sky})$ minimum. The number of polarization-foreground-clean patches is of course still to be determined. We also varied the patch geometry; e.g., for an $f_{sky} \sim 0.08$ rectangular region with $r_{halo} = 0.12$, we get $2\sigma_r = 0.048$ without foregrounds, in good agreement with the cap result $2\sigma_r = 0.050$.

http://www.princeton.edu/physics/research/cosmology-experiment/abs-experiment/
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Fig. 9.—$1\sigma$ and $2\sigma$ $r$-$n_\ell$ contours for a Spider-like experiment with different sky cuts and for a Planck-like Galaxy-masked $f_{sky}=0.75$ experiment. The contours for a CMBPol-like experiment as well as those for the combined Planck-like and Spider-like experiments are plotted for comparison. The black line is the inflation consistency line and the black plus sign is the fiducial input, $r = 0.12$ and $n_\ell = -0.015$. Even with this CMBpol, inflation consistency is not that well tested.

Fig. 10.—$1\sigma$ and $2\sigma$ contours in the $r_{EE}$-$r_{BB}$ plane for a Spider-like experiment with different sky cuts and for a Planck-like experiment with $f_{sky}=0.75$. The black solid lines show the tensor consistency curves $r_{EE} = r_{BB}$ and the plus signs show the fiducial $r_{EE} = r_{BB} = 0.12$ input model. As expected, $r_{BB}$ is better determined than $r_{EE}$ and this tensor consistency is not well tested.

5. SUMMARY AND CONCLUSIONS

In this paper, we applied a full matrix likelihood analysis to multifrequency $Q$-$U$ polarization maps and $T$-maps of forecasted data to determine the posterior probability distribution of $r$.

5.1. Leakage Levels and Leakage Avoidance

This method avoids the explicit linear $E$-$B$ decomposition of the polarization maps before doing the likelihood analysis and gives the best possible determination of $r$, provided that systematic errors are correctly modeled. For realistic cut-sky observations, we measured the level of $BB$ contamination from the inevitable mode-mixing from the much larger $EE$ power. In addition, there is leakage from instrumental effects, in particular with $T$ seeping into $Q$ and $U$, which has to be included in any approach. We have left the investigation of this issue to future work.

5.2. Computational Feasibility of Exact Likelihoods

It is often the case in CMB cosmology that the shear number of pixels precludes a direct full map-based likelihood procedure, with an intermediate power spectrum determination done before parameter estimation. However, for Spider and similar ground and balloon experiments targeting $r$, relatively low resolution and restricted sky coverage are all that is really needed for detection. The result is a total pixel number that allows computationally feasible inverse and determinant calculations of the large signal-plus-noise correlation matrices $C_t = C_N + C_S(q)$ — with contributions from both the parameter-dependant signal covariance $C_S(q)$ and the generalized noise $C_N$, which includes uncertainties from the foreground subtraction as well as from instrumental and systematic noise in the maps.

Matrix methods have had a long history, dating from the earliest CMB data sets, e.g., Bond & Crittenden [2001]. For example, they were used for COBE, Saskatoon, Boomerang, and CBI analyses. Often compression was used, e.g., to signal-to-noise eigenmodes (Bond 1995; Bond & Crittenden [2001]) or by coarse-grained gridding (Myers et al. 2003), to make the matrix manipulations tractable. With Boomerang, an important aspect was to make sure all issues regarding data-filtering, inhomogeneous and aspherical beams, transfer functions, striping etc. were properly included. Invariably, a Monte Carlo simulator of each experiment has been built, in which simulated timestreams have as many effects from systematics and data processing as one can think of included.
Fig. 11.—When one patch covering $f_{sky}$ is broken up into four $f_{sky}/4$ cap-patches, but the noise and observing time remain constant, the ($\tau$- marginalized) $\tau$-errors remain similar except at very small $f_{sky}$. We also show that factors of two changes in the noise swamp this effect. The calculations were done with $r_{norm}=0.12$ in the pixel-space except for the highest sky coverages where the pixel and $\ell$-space analysis are in excellent agreement. The effect of foreground contamination and Galaxy cut has not been taken into account here.

5.3. Matrix Estimation from Monte Carlo Noise and Signal Simulations and Relation to Master/XFaster

The Master/XFaster approach encodes this in isotropized $\ell$ space filters and rotationally symmetrized masks which allow one to relate the underlying all-sky $C_{N, cX\ell}$ to the filtered cut sky. Similarly an isotropized noise $C_{N, cX\ell}$ is also determined by taking processed noise timestreams, creating maps with them, $Y_{tm}$ transforming them, then forming a quadratic average over noise samples $J_{\ell}$, $C_{N, cX\ell} = \sum_{J_{\ell}} [a_{\ell}, c_{X\ell, tm}]^2 / (2\ell + 1)$.

When one has a large number of detectors, using only cross-correlations and no auto-correlations has an advantage, namely that the cross-noise is small, from systematic effects in the arrays and instrument as a whole. Precise modelling of the auto-noise is not easy. However, any operation that can be done for Master or XFaster can also be done to estimate the noise matrices, using noise sample sums. (Getting convergence of small off-diagonal components may require many samples.) Matrices have the advantage that they naturally allow for anisotropic and inhomogeneous components, in the noise maps - including striping effects - and in the beam maps and in the foreground maps. There are issues about optimal estimation of the generalized pixel-pixel matrices that one would like to tune, but there are no fundamental obstacles to making the $C_N$ and $C_S$ matrices highly accurate for parameter estimation.

WMAP used a matrix-based likelihood for low $\ell$, connected to an isotropized $\ell$-space likelihood covering the high $\ell$‘s. Planck is doing the same. We expect such a hybridized likelihood code will also be used for Spider-like experiments for routine parameter estimation, even though we think one can get away with a full matrix likelihood code.

If simulated timestreams are used for $C_N$ and $C_S$ estimation, generalized pixels may prove preferable to the usual spatial pixels. The Cosmic Background Imager CBI (Myers et al. 2003; Sievers 2004) used the reciprocal space pixels for the primary construction, rather natural for an interferometry experiment where the transform analog is a set of visibilities. ACT and QUaD also have done their power spectrum estimation in the Fourier transform space of spatial maps.

5.4. The CBIpol Approach as a Guide for Small Deep-sky Analyses

The use of matrix likelihood codes does not mean that $E$ and $B$ maps will not be constructed, just that parameters would not be extracted from them. The CBI example of how such $E, B$ maps were made and used, and why bandpower and parameter estimations did not use $E, B$ maps serves as a paradigm for how things could proceed for Spider-like data. The CBI data were compressed (via a GRIDR code) onto a discrete (reciprocal) lattice of wavenumbers by projecting measured interferometer visibilities onto a gridded 2D $K$-space. A direct unitary transformation takes such a basis of "momentum" modes into a basis of spatial modes in real space where $Q-U$ is a more appropriate representation. An important point is that the polarization map estimators evaluated on the discrete wavenumbers of the lattice are linear combinations of the continuous wavenumbers, the mode-coupling of finite maps which also leads to an $E-B$ mixing.

In the lattice representation, the resulting size of the correlation matrices for CBI were quite tractable for direct inversion and the full likelihood was evaluated (via an mLikely code) to determine bandpowers for $TT$, $EE$, $BB$, and $TE$, without separation of the Fourier maps into $E$ and $B$.

An optimal linear map reconstruction of $E$ and $B$ was done for visualization purposes, with real-space and momentum-space maps showing the CBI $E$ and $B$ Wiener-filtered means, accompanied by a few maps showing typical fluctuation maps about the mean maps. These were contour maps, since the usual headless vector polarization plots are of length the polarization degree, $\sqrt{Q^2 + U^2}$, tilted at an angle arctan$(U/Q)/2$.

For Spider-like bolometer-based experiments for which the raw data are bolometer time-streams from which $QU$ maps are constructed, the compression step leads to tractable matrices as in the CBIpol case, although in the first instance the pixelization choice may be in real space rather than in wavenumber space or in a generalized-pixel space. Just as with CBIpol, parameters and bandpowers would be determined with direct likelihood calculations, yet Wiener-filtered $EB$ maps would still be made for visualization.

5.5. Exact 2D Likelihood Computation

Given the matrix construction method, we determined the posterior probabilities on reduced 2D-grids consisting of $r$ and one other cosmic parameter, in many cases the Thomson scattering depth to reionization $\tau$. The grid could be extended to higher dimensions, as they
were in early CMB analyses of COBE, Boomerang, CBI and ACBAR. More efficiently, MCMC chains could be used to explore the posterior probability surface. Since, as we have shown, r is relatively weakly correlated with the other standard cosmic parameters, our use of a reduced dimensionality is accurate. We targeted τ for a second parameter, although it too is weakly correlated for Spider-like experiments probing modest $f_{\text{sky}}$, because of its importance for the reionization bump in BB which is picked by large $f_{\text{sky}}$ experiments such as Planck. We showed that as long as the input value $r_{\text{fid}}$ is reasonably larger than the error $\sigma_r$, e.g., $\sim 0.1$, $r_{\text{fid}}$ can be well-recovered by our methods.

5.6. The Inflation and Tensor Consistency Checks

We have used r and $n_t$ for our reduced 2D parameter space to see how well the inflation consistency condition, $n_t \approx -r/8$, can be tested. For example, with $r_{\text{fid}} = 0.12$ and the consistency value $n_t_{\text{fid}} = -0.015$, we obtain $2\sigma_r \approx 0.036$ and $2\sigma_{n_t} \approx 0.28$. The large 1-sigma error on $n_t$ is what one might have expected given the relatively small $\ell$-baseline (reminiscent of the $\pm 0.2$ limit on $n_t$ from the even smaller baseline COBE DMR data). Thus, although breaking up r into bands will be useful, the $n_t$ slope that follows will be not be powerful enough to test consistency. With CMBpol and at $N_{\text{side}} = 512$, the errors are $2\sigma_r \approx 0.014$ and $2\sigma_{n_t} \approx 0.07^\circ$, still too large. A more prosaic internal consistency check was done to show that what one thinks is r from the total BB agrees with what one gets from the less-tensor-sensitive total EE.

5.7. Relation to Planck

We based our Planck-like case on the Blue Book detector specifications. The actual in-flight performance is quite similar (Planck HFI Core Team et al. 2010; Mennella et al. 2011). It is encouraging that five performances are quite similar (Planck HFI Core Team et al. 2010; Mennella et al. 2011). We showed that splitting $f_{\text{sky}}$ into four patches with fixed integration time and the instrument noise results in only a small loss in r-sensitivity because $\sigma_r(f_{\text{sky}})$ has a relatively wide single-patch minimum. How many polarization-foreground-clean patches there are is still to be determined.

Although the specifications we chose for "Spider-like" was motivated by a bolometer array experiment feasible with current technology, our forecasts should not be taken as realistic mocks of the true Spider which is under development, and for which a number of campaigns are envisaged (see the footnote under Spider-like in Table 1). The techniques used here have, however, already been applied in Spider forecast papers using more realistic statistically inhomogeneous noise, scanning strategies and observational durations, e.g., in Filippini et al. (2011) and Fraisse et al. (2011). On an $f_{\text{sky}} \approx 0.1$, $r_{\text{fid}} = 0.01$ simulations, we compared (Fraisse et al. 2011) non-uniform noise modulated spatially by the scanning strategy's number-of-hits-per-pixel with uniform white noise with the same integrated noise power. Although the deviation in the standard deviation of the noise $\sigma_r$ was about a factor of two times the mean noise $\sigma_r$, with largest impact near the scanning boundaries, we found very similar results for the posterior, showing this paper's conclusions are insensitive to our use of uniform white noise. (Of course the foreground noise radically alters the whiteness, and this of course has been included by us, but only in a statistically isotropic way — the Galactic latitude dependence breaks this isotropy just as the pixel hits do.) In § 4.4 we showed that in the absence of foregrounds our Spider-like case could achieve $2\sigma_r \approx 0.02$ over a broad range of $f_{\text{sky}}$. The r-posterior shown in Figure 12 were made with the numerical codes described here, for the Spider experiment as envisaged in Fraisse et al. (2011) (labeled as "Spider" in the plot), and for an even more ambitious campaign of subsequent flights of the Spider instrument, as proposed for SCIP. We see that the performance of the experiment with Spider-like specifications used in this paper is very close to the actual Spider. A different foreground model used in Fraisse et al. (2011) for $f_{\text{sky}} \approx 0.1$ led to a similar $\sim 50\%$ error degradation.

5.9. History and Forecasts of r Constraints

When the large angle CMB anisotropies were first detected with COBE DMR, the broad-band TT power amplitude ($\ell \leq 20$), with wavenumbers $k^{-1} \approx 1000$ Mpc, was related to the linear density power spectrum amplitude at the radically different $k^{-1} \approx 6$ Mpc, assuming a nearly scale-invariant primordial spectrum: $\sigma_8 \approx 0.85e^{-(r-0.1)}/\sqrt{1 + 0.6^\circ \times 10^3 f_{\text{sky}}}$ for typical $\Lambda$CDM parameters popular in mid nineties, $\Omega_a \sim 2/3, h \sim 0.7$ (Bond 1996), rather similar to the values now. Requiring $\sigma_8 > 0.7$ to get reasonable cluster abundances at zero redshift – a venerable cosmological requirement from the 80s – gives a rough constraint on r from the COBE data in conjunction with large scale structure (LSS) data: $2\sigma_r < 1$ for current r values – but r only had an upper limit until WMAP1, with a more accurate determination waiting until WMAP3.

The first 2003 WMAP1 constraint on r from TT and TE CMB-only data (with weak priors) was $2\sigma_r < 0.81$, reducing to $2\sigma_r < 0.64$ with the WMAP3 TT, TE and EE data, and other TT CMB data available in 2005. It decreased to 0.31 with the LSS data of the time.

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History and Forecasts of r Constraints

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we get $2\sigma_r \approx 10^{-4}$ for $N_{\text{side}} = 128$ for tiny $r_{\text{fid}}$. It is unclear at this time how much inexact foreground subtraction and lensing noise will limit $r$ determinations in these ideal cases.

5.10. The 1D Shannon Entropy of $r$

We have described another way to cast the improvements expected in $r$-estimation as experiments attain higher and higher sensitivity, the marginalized 1D Shannon entropy $\Delta S_{1f}(r)$ for $r$. This measures the (phase-space) volume of $r$-space that the measurement allows. It is obtained by direct integration over the normalized 1D likelihood for $r$, with all non-Gaussian features in the likelihood properly included. We have found in practice that $\Delta S_{1f}(r) \approx \Delta \ln[\sigma_r \sqrt{2\pi}]$, with $\sigma_r$ determined by the forced Gaussianization described in the paper, works quite well, so in a way we are just restating the error improvements in the information theoretic language of bits.

We use the WMAP7+ACT $TT, TE$ and $EE$ $2\sigma_r \sim 0.25$ (Dunkley et al. 2010) constraint for our baseline. The first WMAP1 constraint in 2003 (Spergel et al. 2003), with $\Delta S_{1f}(r) = 1.70$ bits had, of course, higher information entropy. Here, as in the abstract, we have translated from nats to bits. The recent WMAP7+SPT results (Keisler et al. 2011) with $2\sigma_r \sim 0.21$ give a slight decrease in the entropy ($\Delta S_{1f}(r) = -0.25$) compared to the baseline. The asymptotic perfect noise-less all-sky experiment gives (the somewhat $r$-dependent) $\Delta S_{1f}(r) \approx -11$ bits, the limit on obtainable knowledge from the CMB. The proposed post-Planck CoRe, PIXIE and CMBPol-like experiments claim up to -9 bits. For the Spider-like experiments forecasted here, the foreground-free decrease is -4.2 bits (and -3.6 bits with a 95% effective component separation). Thus balloon-borne and ground-based experiments with large arrays making deep surveys focussing on a relatively clean few-percent of the sky yield tensor information at least comparable to shallow and wide surveys and are a powerful step towards a near-perfect deep and wide satellite future.

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