Deformed Complex Hermite Polynomials

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Abstract

We study a class of bivariate deformed Hermite polynomials and some of their properties using classical analytic techniques and the Wigner map. We also prove the positivity of certain determinants formed by the deformed polynomials. Along the way we also work out some additional properties of the (undeformed) complex Hermite polynomials and their relationships to the standard Hermite polynomials (of a single real variable).

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1 Introduction

The complex Hermite polynomials \( \{H_{m,n}(z_1, z_2)\} \) may be defined by

\[
H_{m,n}(z_1, z_2) = \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k} \binom{n}{k} z_1^{m-k} z_2^{n-k}.
\]
Their exponential generating function is

\[
\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m! n!} H_{m,n}(z_1, z_2) = e^{uz_1 + vz_2 - uv}.
\]

They satisfy the orthogonality relation, \[10\] and \[8\]

\[
\frac{1}{\pi} \int_{\mathbb{R}^2} H_{m,n}(x + iy, x - iy) H_{p,q}(x + iy, x - iy) e^{-x^2 - y^2} \, dx \, dy = m! n! \delta_{m,p} \delta_{n,q}.
\]

Many of their properties including a multilinear generating function are in \[13\] while their combinatorics have been studied in \[14\] and \[16\]. New proofs of the Kibble-Slepian formula for the Hermite and Complex Hermite polynomials are in \[15\]. The complex Hermite polynomials were introduced by Ito in \[17\] and many of their properties have been developed in \[2, 6, 8, 9, 13\], and \[22, 23\]. They are in a class of polynomials presented in the very recent book by Dunkl and Xu \[24\] Chapter 2.

In this paper we study the deformed complex Hermite polynomials \(\{H_{m,n}^{(g)}(z_1, z_2)\}\). They are defined through the generating function

\[
\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m! n!} H_{m,n}^{(g)}(z_1, z_2) = \exp((g_{1,1}u + g_{1,2}v)z_1 + (g_{2,1}u + g_{2,2}v)z_2 - (g_{1,1}u + g_{1,2}v)(g_{2,1}u + g_{2,2}v)),
\]

where \(g\) is the matrix

\[
g := \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix}.
\]

A version of these polynomials have been studied in \[22, 23, 24\]. In the setting adopted in this paper they were introduced in \[4\], where some preliminary properties, relating to orthogonality and growth were worked out as well as their relationship to a model of noncommutative quantum mechanics.

In Section 2 we derive some of the properties of the deformed complex Hermite polynomials including their orthogonality relation, Rodrigues formula, and a moment representation. In Section 3 we show that certain Hankel determinants formed by the polynomials \(\{H_{m,n}^{(g)}(z_1, z_2)\}\) are nonnegative. This is done along the same lines of \[3, 11\], which was motivated by the earlier works of Karlin \[18–19\], and the mammoth paper \[20\] by Karlin and Szegő.

## 2 Some properties of \(\{H_{m,n}^{(g)}(z_1, z_2)\}\)

**Theorem 2.1.** Let \(S^{(g)}\) be the operator defined by

\[
(S^{(g)}f)(z_1, z_2) = f(g_{1,1}z_1 + g_{1,2}z_2, g_{2,1}z_1 + g_{2,2}z_2).
\]

We have

\[
H_{m,n}^{(g)}(z_1, z_2) = e^{-\partial_{z_1} \partial_{z_2}} S^{(g)} e^{\partial_{z_1} \partial_{z_2}} H_{m,n}(z_1, z_2),
\]

\[
H_{m,n}^{(g)}(z_1, z_2) = e^{-\partial_{z_1} \partial_{z_2}} (g_{1,1}z_1 + g_{2,2}z_2)^m (g_{1,2}z_1 + g_{2,2}z_2)^n,
\]

\[
H_{m,n}^{(g)}(z_1, z_2) = \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} \binom{n}{k} g_{1,1}^{m-j} g_{1,2}^{j} g_{2,1}^{n-k} g_{2,2}^{k} H_{j+k,n+m-j-k}(z_1, z_2).
\]
Proof. Multiply the right-hand side by \( u^m v^n / (m! n!) \) and sum over \( m, n \geq 0 \) and use the generating function (1.2). The result is

\[
e^{-uv} e^{-\partial_1 \partial_2} S(g) e^{\partial_1 \partial_2 u z_1 + v z_2} = e^{-uv} e^{-\partial_1 \partial_2} S(g) e^{uv} e^{uz_1 + vz_2}
\]

\[
= e^{-\partial_1 \partial_2} \exp \left( u(g_{1,1} z_1 + g_{2,1} z_2) + v(g_{1,2} z_1 + g_{2,2} z_2) \right)
\]

\[
= \exp \left( - (g_{1,1} u + g_{1,2} v)(g_{2,1} u + g_{2,2} v) \right) \exp \left( z_1 (g_{1,1} u + g_{1,2} v) + z_2 (g_{2,1} u + g_{2,2} v) \right)
\]

and (2.2) follows. Similarly (2.3) follows from the generating function (1.2). Finally (2.4) follows from (2.3) and the binomial theorem.

Note that the relation (2.3) is the Rodrigues formula for \( H_{m,n}^{(g)}(z_1, z_2) \). Also, as shown in [4], it is possible to rewrite (2.4) in a somewhat different form for fixed \( m, n \):

\[
H_{k,L-k}^{(g)}(z, \bar{z}) = \sum_{r=0}^{L} M(g, L)_{r,k} H_{r,L-r}(z, \bar{z}),
\]

where

\[
M(g, L)_{r,k} = \sum_{q = \max \{0, r+k-L\}}^{\min \{r,k\}} \binom{k}{q} \binom{L-k}{r-q} g_{11}^q g_{21}^{k-q} g_{12}^{r-q} g_{22}^{L-k+q-r}, \quad 0 \leq r, k \leq L.
\]

From (2.3) it is clear that

\[
H_{m,n}^{(g)}(z, \bar{z}) = H_{n,m}^{(h)}(z, \bar{z}), \text{where } h = g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We now establish the orthogonality relation (see also [4, 24]).

**Theorem 2.2.** The orthogonality relation

\[
\int_{\mathbb{R}^2} H_{m,n}^{(g)}(z, \bar{z}) \overline{H_{p,q}^{(h)}(z, \bar{z})} e^{-x^2-y^2} dxdy = 0
\]

if \( (m, n) \neq (p, q) \) holds if and only if

\[
h^* g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.
\]

When (2.3) holds then

\[
\int_{\mathbb{R}^2} H_{m,n}^{(g)}(z, \bar{z}) \overline{H_{p,q}^{(h)}(z, \bar{z})} e^{-x^2-y^2} dxdy = m! n! \lambda_1^m \lambda_2^n \delta_{m,p} \delta_{n,q}.
\]

**Proof.** To save space we denote rows 1 and 2 of

\[
\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix}
\]

\[
\begin{pmatrix} u \\ v \end{pmatrix}
\]
by \( g_1(u, v) \) and \( g_2(u, v) \), respectively. It is straightforward to use the generating function (1.4) and see that for real \( u_j, v_j, j = 1, 2 \) we have

\[
\sum_{m,n=0}^{\infty} \frac{u_1^m \bar{v}_1^n u_2^p \bar{v}_2^q}{m! n! p! q!} \int_{\mathbb{R}^2} H_{m,n}^{(g)}(z, \bar{z}) H_{p,q}^{(h)}(z, \bar{z}) e^{-x^2 - y^2} \, dx \, dy
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}^2} \exp(g_1(u_1, v_1)z + g_2(u_1, v_1)\bar{z} - g_1(u_1, v_1)g_2(u_1, v_1)) \exp(h_1(u_2, v_2)z - h_1(u_2, v_2)h_2(u_2, v_2)) e^{-x^2 - y^2} \, dx \, dy
\]

\[
= \exp(-g_1(u_1, v_1)g_2(u_1, v_1) - h_1(u_2, v_2)h_2(u_2, v_2))
\]

\[
\times \int_{\mathbb{R}^2} \exp((g_1(u_1, v_1) + \bar{h}_2(u_2, v_2))(g_2(u_1, v_1) + \bar{h}_1(u_2, v_2))\bar{z}) e^{-x^2 - y^2} \, dx \, dy.
\]

By evaluating the integral we see that the integral in the last line is

\[
\exp \left( (g_1(u_1, v_1) + \bar{h}_2(u_2, v_2))(g_2(u_1, v_1) + \bar{h}_1(u_2, v_2)) \right)
\]

We have orthogonality if and only if

\[
\sum_{m,n=0}^{\infty} \frac{u_1^m \bar{v}_1^n u_2^p \bar{v}_2^q}{m! n! p! q!} \int_{\mathbb{R}^2} H_{m,n}^{(g)}(z, \bar{z}) H_{p,q}^{(h)}(z, \bar{z}) e^{-x^2 - y^2} \, dx \, dy = f(u_1u_2, v_1v_2),
\]

for some function \( f \) of two variables. This is equivalent to the condition (2.8).

It is clear from (2.9) that we can rescale \( H_{m,n}^{(g)}(z, \bar{z}) \) and \( H_{m,n}^{(h)}(z, \bar{z}) \) to make \( \lambda_1 = \lambda_2 = 1 \), which we now assume. Therefore we assume that

\[
h = (g^*)^{-1}.
\]

Thus we have the orthogonality relation

\[
\int_{\mathbb{R}^2} H_{m,n}^{(g)}(z, \bar{z}) H_{p,q}^{(h)}(z, \bar{z}) e^{-x^2 - y^2} \, dx \, dy = m!n! \delta_{m,p} \delta_{n,q},
\]

where \( h = (g^*)^{-1} \).

**Theorem 2.3.** Let \( z = x + iy \). The polynomials \( \{ H_{m,n}^{(g)}(z_1, z_2) \} \) have the integral representation

\[
H_{m,n}^{(g)}(iz, i\bar{z}) = \frac{i^{m+n}}{\pi} \int_{\mathbb{R}^2} \left( g_1(z_1) + g_2(z_2) \right)^m (g_1(z_1) - g_2(z_2))^n e^{-(r-x)^2 - (s-y)^2} \, dr \, ds,
\]

where \( \zeta := r + is \).

**Proof.** First replace \( \zeta \) by \( \zeta + z \) in the right-hand side of (2.12) then multiply the right-hand side of (2.12) by \( u^m v^n / (m! n!) \) and add the terms for \( m, n \geq 0 \). This sum equals

\[
\frac{1}{\pi} \exp \left( iu(g_1(z_1) + g_2(z_2)) + iv(g_1(z_1) + g_2(z_2)) \right)
\]

\[
\times \int_{\mathbb{R}^2} \exp \left( -r^2 - s^2 + iu(g_1(z_1) + g_2(z_2)) + iv(g_1(z_1) + g_2(z_2)) \right) \, dr \, ds.
\]
The integral in the above expression is given by
\[
\int_{\mathbb{R}^2} \exp \left( -r^2 - s^2 + iur(g_{1,1} + g_{2,1}) + ivr(g_{1,2} + g_{2,2}) + us(g_{2,1} - g_{1,1}) + vs(g_{2,2} - g_{1,2}) \right) \, dr \, ds
\]
\[
= \exp \left( \left( u(g_{2,1} - g_{1,1}) + v(g_{2,2} - g_{1,2}) \right)^2 / 4 - (u(g_{1,1} + g_{2,1}) + v(g_{1,2} + g_{2,2}))^2 / 4 \right)
\]
\[
= \exp \left( (ug_{1,1} + vg_{1,2})(ug_{2,1} + vg_{2,2}) \right).
\]
Therefore the exponential generating function of the right-hand side of (2.12) simplifies to the exponential generating function of the left-hand side as in (1.1) and the proof is complete. □

For many applications of moment techniques to special functions we refer the reader to [12].

3 A positivity result

In this section we establish the positivity of a Hankel determinant formed by the \( H_{m,n}^g \) polynomials.

**Theorem 3.1.** Assume that \( g_{1,2} = \overline{g_{2,1}} \) and \( g_{2,2} = \overline{g_{1,1}} \). Then the determinant formed by \((-i)^{m+n}(-1)^s H_{m+s,n+s}^i(iz, i\bar{z}) : 0 \leq m, n \leq N \) is positive for all \( N \geq 0 \).

**Proof.** Let \( \Delta_N \) be the determinant whose elements are
\[
(-i)^{m+n+2s} \pi H_{m+s,n+s}^i(iz, i\bar{z}) : 0 \leq m, n < N.
\]
It is convenient to set
\[
U_k = g_{1,1} \zeta_k + g_{2,1} \overline{\zeta_k}, \quad V_k = \overline{U_k} = g_{1,2} \zeta_k + g_{2,2} \overline{\zeta_k}.
\]

The integral representation (2.12) implies that \( \Delta_N \) has the integral representation, with \( \zeta_k = r_k + is_k \),
\[
\int_{\mathbb{R}^{2N}} \begin{vmatrix}
1 & V_1 & \cdots & (V_1)^{N-1} \\
U_2 & U_2 V_2 & \cdots & U_2 (V_2)^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
(U_N)^{N-1} & (U_N)^{N-1} V_N & \cdots & (U_N)^{N-1} (V_N)^{N-1} \\
\end{vmatrix}
\]
\[
\times \prod_{k=1}^{N} (U_k)^s (V_k)^s \prod_{j=1}^{N} e^{-(r_j-x)^2-(s_j-y)^2} \, dr_j \, ds_j
\]
\[
= \int_{\mathbb{R}^{2N}} \prod_{k=1}^{N} |U_k|^{2s} \prod_{j=1}^{N} |U_j|^{j-1} \begin{vmatrix}
1 & V_1 & \cdots & (V_1)^{N-1} \\
1 & V_2 & \cdots & (V_2)^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & V_N & \cdots & (V_N)^{N-1} \\
\end{vmatrix}
\]
\[
\times \prod_{j=1}^{N} e^{-(r_j-x)^2-(s_j-y)^2} \, dr_j \, ds_j.
\]
Now apply a permutation on the tuples \((r_j, s_j)\). If reorder the Vandermonde determinant with indices in increasing order we should multiply the answer by the sign of the permutation, say
sign(σ). We then sum over σ in the symmetric group $S_N$ and divide by the size of $S_N$, that is we divide by $N!$. Therefore

$$
\Delta_N = \frac{1}{N!} \int_{\mathbb{R}^{2N}} \prod_{k=1}^{N} |U_k|^{2\sigma} \left[ \prod_{1 \leq j < k \leq N} |U_j - U_k|^2 \right] \prod_{j=1}^{N} ds_j d\sigma.
$$

(3.2)

**Corollary 3.2.** Set $z = x + iy$ and let $\Delta_N$ be the determinant whose elements are

$$
(-i)^{m+n} \pi H_{m,n}(iz, i\bar{z}) : 0 \leq m, n < N.
$$

Then $\Delta_N$ is given by (3.2).

4 The Wigner map and the polynomials $\{H_{m,n}\}$

To proceed further, we write the (undeformed) polynomials $H_{m,n}$ as basis elements of the Hilbert space $L^2(\mathbb{C}, d\nu(z, \bar{z}))$, where $d\nu(z, \bar{z}) = e^{-|z|^2} \frac{d^2x dy}{\pi}$. It is well known (see, for example, [8]) that the vectors $h_{m,n} = \frac{H_{m,n}}{\sqrt{m!n!}}$ form an orthonormal basis of $L^2(\mathbb{C}, d\nu(z, \bar{z}))$ and they may be obtained by the action of two differential operators on the ground state vector $h_{0,0} \in L^2(\mathbb{C}, d\nu(z, \bar{z}))$, which is the constant function, taking the value 1 everywhere:

$$
h_{m,n}(z, \bar{z}) = \frac{(z - \partial_z)^m (\bar{z} - \partial_{\bar{z}})^n}{\sqrt{m!n!}} h_{0,0}, \quad m, n = 0, 1, 2, \ldots, \infty.
$$

(4.1)

Furthermore, the operators

$$
a_1 = \partial_z, \quad a_1^\dagger = z - \partial_{\bar{z}}, \quad a_2 = \partial_{\bar{z}}, \quad a_2^\dagger = \bar{z} - \partial_z,
$$

(4.2)

satisfying the commutation relations,

$$
[a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2, \quad [a_1, a_2] = 0,
$$

(4.3)

form an irreducible set on $L^2(\mathbb{C}, d\nu(z, \bar{z}))$. In terms of these operators,

$$
h_{m,n}(z, \bar{z}) = \frac{(a_1^\dagger)^m (a_2^\dagger)^n}{\sqrt{m!n!}} h_{0,0}, \quad m, n = 0, 1, 2, \ldots, \infty.
$$

(4.4)

We next define the **Wigner map** (see, for example, [3]) in an abstract setting. Let $\mathfrak{H}$ be an infinite dimensional, abstract, separable Hilbert space over the complexes and let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of it. Define the two (annihilation and creation) operators $a, a^\dagger$,

$$
a\phi_n = \sqrt{n}\phi_{n-1}, \quad a\phi_0 = 0, \quad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}.
$$

(4.5)

Then,

$$
\phi_n = \frac{(a^\dagger)^n}{\sqrt{n!}} \phi_0, \quad n = 0, 1, 2, \ldots, \infty.
$$

(4.6)
For \( z \in \mathbb{C} \) define the unitary displacement operator
\[
D(z, \bar{z}) = e^{za^\dagger - \bar{z}a} = e^{za^\dagger}e^{-|z|^2/2} = e^{-\bar{z}a}e^{za^\dagger}e^{-|z|^2/2}.
\]

Let \( \mathcal{B}_2(\mathcal{H}) \) denote the set of all Hilbert-Schmidt operators on \( \mathcal{H} \). It is well known that this is a Hilbert space under the scalar product
\[
\langle X | Y \rangle_{\mathcal{B}_2} = \text{Tr}[X^*Y], \quad X, Y \in \mathcal{B}_2(\mathcal{H}),
\]
and the vectors \( \{|\phi_m\rangle\}_{m=0}^\infty \), form an orthonormal basis of \( \mathcal{B}_2(\mathcal{H}) \). The Wigner map is then defined as the linear transformation, \( \mathcal{W} : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(\mathbb{C}, d\nu(z, \bar{z})) \):
\[
\mathcal{W}(z)(x, \bar{z}) = e^{|z|^2/2} \text{Tr} [D(z, \bar{z})^*X] , \quad X \in \mathcal{B}_2(\mathcal{H}).
\]
This map is well known to be a linear isometry, i.e.
\[
\|\mathcal{W}X\|_{L^2}^2 = \int \|\mathcal{W}(X)(z, \bar{z})\|^2 d\nu(z, \bar{z}) = \text{Tr}[X^*X] = \|X\|^2_{\mathcal{B}_2},
\]
being first defined on finite rank elements \( X \) and then extended by continuity.

For any operator \( B \) on \( \mathcal{H} \), let us define the two commuting operators, \( B_{\ell}, B_r \), on \( \mathcal{B}_2(\mathcal{H}) \):
\[
B_{\ell}(X) = BX, \quad B_r(X) = XB^*, \quad X \in \mathcal{B}_2(\mathcal{H});
\]
we shall be interested in their Wigner transformed versions, \( \mathcal{W}\{B_{\ell}, B_r\}\mathcal{W}^* \) on the Hilbert space \( L^2(\mathbb{C}, d\nu(z, \bar{z})) \). Using (4.7) to write (4.8) in the form
\[
\mathcal{W}(X)(z, \bar{z}) = \text{Tr}[e^{-za^\dagger}Xe^{\bar{z}a}] e^{\bar{z}z} = \text{Tr}[e^{\bar{z}a}Xe^{-za^\dagger}]
\]
differentiating with respect to \( z \) and \( \bar{z} \), we get for the Wigner transformed versions of the operators \( a_{\ell}, a_{\ell}^\dagger, a_r, a_r^\dagger \), corresponding to the \( a, a^\dagger \) in (4.5),
\[
\mathcal{W}a_{\ell}\mathcal{W}^* = \partial_{\bar{z}} = a_2 , \quad \mathcal{W}a_{r}^\dagger\mathcal{W}^* = \bar{z} - \partial_z = a_2^\dagger , \quad \mathcal{W}a_{r}\mathcal{W}^* = -\partial_z = -a_1 , \quad \mathcal{W}a_{r}^\dagger\mathcal{W}^* = -z + \partial_{\bar{z}} = -a_1^\dagger .
\]
From this, and since \( \mathcal{W}(|\phi_0\rangle\langle\phi_0|) = h_{0,0} \), we easily get
\[
\mathcal{W}(|\phi_m\rangle\langle\phi_m|) = (-a_2^\dagger)^m (a_2)^m/\sqrt{m!} n! h_{0,0} = (-1)^m h_{m,n} \]

The Hilbert space \( \mathcal{H} \) was taken to be an abstract space. As a concrete realization let us take it to be the space \( L^2(\mathbb{R}, \frac{e^{-x^2}dx}{\sqrt{\pi}}) \), on which \( a = \frac{1}{\sqrt{2}} \partial_x \) and \( a^\dagger = \frac{1}{\sqrt{2}} (2x - \partial_x) \). Also, \( \phi_0 \) is the constant vector which is equal to one everywhere and
\[
\sqrt{2^n n!} \phi_n(x) = (2x - \partial_x)^n \phi_0(x) = H_n(x)
\]
where the \( H_n(x) \) are the real Hermite polynomials, [12],
\[
H_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2}.
\]
Furthermore, $B_2(\mathfrak{S})$ can now be identified with $L^2(\mathbb{R}^2, \frac{e^{-x^2-y^2}}{\pi} \, dx \, dy)$, so that

\[
(\langle \phi_n \rangle \langle \phi_m \rangle)(x, y) = \frac{1}{\sqrt{2^{n+m} \cdot n! \cdot m!}} H_m(x) H_n(y).
\]

Thus, from (4.8) and (4.11)

\[
H_{m,n}(z, \bar{z}) = \frac{(-1)^n}{\sqrt{2^{m+n}}} \mathcal{W}(\langle H_n \rangle \langle H_m \rangle)(z, \bar{z})
\]

\[
= \frac{(-1)^n e^{\frac{iz^2}{2}}}{\sqrt{2^{m+n}}} \text{Tr}[D(z, \bar{z})^* |H_n\rangle \langle H_m|]
\]

To compute the trace in the above formula explicitly, we note that (see, e.g., [3]) for any $f \in L^2(\mathbb{R}, \frac{e^{-x^2}}{\sqrt{\pi}})$,

\[
[D(z, \bar{z})^* f](u) = e^{-i(x+\sqrt{2}u)} f(u + \sqrt{2}x), \quad z = x + iy.
\]

We thus have the result,

**Theorem 4.1.** The complex Hermite polynomials are Wigner transforms of bilinear products of the real Hermite polynomials:

\[
H_{m,n}(z, \bar{z}) = \frac{(-1)^n}{\sqrt{2^{m+n}}} \mathcal{W}(\langle H_n \rangle \langle H_m \rangle)(z, \bar{z})
\]

\[
= \frac{(-1)^n}{\sqrt{2^{m+n}}} \int_{\mathbb{R}} e^{-(u + \frac{z}{\sqrt{2}})^2} H_m(u) H_n(u + \frac{1}{\sqrt{2}}(z + \bar{z})) \, \frac{du}{\sqrt{\pi}}.
\]

In fact we have the more general result

\[
H_{m,n}(z, \bar{z}) = \frac{(-1)^n}{2^{(m+n)/2}} \int_{\mathbb{R}} e^{-(u+a)^2} H_m(u + b) H_n(u + c) \frac{du}{\sqrt{\pi}},
\]

where

\[
b = a + z/\sqrt{2}, \quad a - c = \bar{z}/\sqrt{2}.
\]

**Proof.** We give another proof using generating functions. Consider the integral

\[
I(m, n) = \frac{1}{2^{(m+n)/2}} \int_{\mathbb{R}} e^{-(u+a)^2} H_m(u + b) H_n(u + c) \frac{du}{\sqrt{\pi}}.
\]

Clearly

\[
\sum_{m,n=0}^{\infty} I(m, n) \frac{s^m t^n}{m! \cdot n!} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp \left(-(u+a)^2 + \sqrt{2}s(u+b) + \sqrt{2}t(u+c) - (s^2 + t^2)/2\right) \, du
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \exp \left(-(u+a - (s+t)/\sqrt{2})^2 + \sqrt{2}sb + \sqrt{2}tc + st - \sqrt{2}a(s+t)\right)
\]

\[
= \exp \left(\sqrt{2}s(b-a) + \sqrt{2}t(c-a) + st\right).
\]

We now make the choices indicated in the theorem and prove (4.14). □
From (2.5) it is clear that (4.14) can be extended to deformed polynomials in the manner

\[(4.17)\ H_{k,L-k}^{(g)}(z,\bar{z}) = \frac{(-1)^n}{2^{n+m+n/2}} \sum_{r=0}^{L} M(g,L)_{rk} \int_{\mathbb{R}} e^{-(u+b)^2} H_r(u+b)H_{L-r}(u+c) \frac{du}{\sqrt{\pi}}.\]

The integral representation (4.14) suggests a possible extension to two variables. Indeed, we have the following result.

**Theorem 4.2.** For \(a \in \mathbb{C}\), we have the integral representation

\[(4.18)\ \int_{\mathbb{R}^2} H_{m,n}(z+a,\bar{z}+\bar{a})H_{p,q}(z-a,\bar{z}-\bar{a})e^{-z\bar{z}}dxdy = (-1)^{p+q}\pi H_{m,q}(a,\bar{a})H_{n,p}(a,\bar{a}).\]

**Proof.** Let \(f(m,n,p,q)\) denote the right-hand side of (4.18). The generating function (1.2) implies

\[
\sum_{m,n,p,q=0}^{\infty} f(m,n,p,q) \frac{u^m v^n \xi^p \eta^q}{m! n! p! q!} = e^{au+\bar{a}v+a\xi+\bar{a}\eta-uv-\xi\eta} \int_{\mathbb{R}^2} \exp \left( (x(u+v+\xi+\eta) + iy(u-v+\xi-\eta) e^{-x^2-y^2} dxdy ight) = \pi \exp \left( (a(u-\xi) + \bar{a}(v-\eta) - uv - \xi\eta + (u+\xi)(v+\eta) \right) = \pi \exp \left( (au + \bar{a}(-\eta) + a(-\xi) + \bar{a}v + v\xi + u\eta) \right).
\]

In view of the generating function (1.2), the above expression is

\[
\pi \sum_{m,q=0}^{\infty} H_{m,q}(a,\bar{a}) \frac{u^m(-\eta)^q}{m! q!} \sum_{n,p=0}^{\infty} H_{n,q}(a,\bar{a}) \frac{v^n(-\xi)^p}{n! p!}
\]

This establishes the theorem.

It is clear from (1.1) that \(H_{m,n}(0,0) = (-1)^n n! \delta_{m,n}\). Thus, the case of \(a = 0\) in (4.18) gives the orthogonality relation. Furthermore, using (2.5) we could extend (4.18) to the case of deformed Hermite polynomials in a straightforward manner.

5 A second representation

A second representation of the \(H_{m,n}\) in terms of the real Hermite polynomials can be obtained by noting that \(L^2(\mathbb{C}, d\nu(z,\bar{z})) \simeq L^2(\mathbb{R}, \frac{e^{-x^2}}{\sqrt{\pi}} dx) \otimes L^2(\mathbb{R}, \frac{e^{-y^2}}{\sqrt{\pi}} dy)\). Then writing

\[
z - \partial_z = \frac{1}{2}(2x - \partial_x) + \frac{i}{2}(2y - \partial_y), \quad \bar{z} - \partial_z = \frac{1}{2}(2x - \partial_x) - \frac{i}{2}(2y - \partial_y),
\]
in (4.11) we get
\[
H_{m,n}(z,\bar{z}) = \frac{1}{2^{m+n}}[(2x - \partial_x) + i(2y - \partial_y)]^m \left[(2x - \partial_x) - i(2y - \partial_y)\right]^n h_{0,0}.
\]
Expanding the binomials and comparing with (4.12),
\[
(5.1) \quad H_{m,n}(z,\bar{z}) = \frac{1}{2^{m+n}} \sum_{k=0}^{m} \sum_{\ell=0}^{n} (-1)^{n-\ell} (i)^{m+n-k-\ell} \binom{m}{k} \binom{n}{\ell} H_{k+\ell}(x) H_{m+n-k-\ell}(y).
\]
This result has also been obtained in [15], using a different technique.

It is useful to rewrite the above expression in a somewhat different manner. Fixing \(m+n=L\) and writing \(k+\ell=q\) the above equation can be written as (see (2.6))
\[
(5.2) \quad H_{m,L-m}(z,\bar{z}) = \frac{1}{2^L} \sum_{r=0}^{L} \sum_{k=\max\{0,r+m-L\}}^{\min\{r,m\}} (-1)^{L-r+k} (i)^{L-r} \binom{m}{k} \binom{L-m}{r-k} \times H_r(x) H_{L-r}(y).
\]
Defining the \((L+1) \times (L+1)\) matrix \(M(L)\), with elements
\[
M(L)_{rm} = \frac{1}{2^L} \sum_{k=\max\{0,r+m-L\}}^{\min\{r,m\}} (-1)^{L-r+k} (i)^{L-r} \binom{m}{k} \binom{L-m}{r-k},
\]
we can also write (5.2) as
\[
(5.3) \quad H_{m,L-m}(z,\bar{z}) = \sum_{r=0}^{L} M(L)_{rm} H_r(x) H_{L-r}(y).
\]
Once again, using (2.3) we could extend the above relation to the case of deformed Hermite polynomials.

We can interpret Eq.(5.3) in the following manner. Both sets, \(H_m(x) H_n(y)\), \(m, n = 0, 1, 2, \ldots \infty\), and \(H_{m,n}(z,\bar{z})\), \(m, n = 0, 1, 2, \ldots \infty\), form bases in the Hilbert space
\[
L^2(\mathbb{R}, \frac{e^{-x^2-y^2} \, dx \, dy}{\pi}) \simeq L^2(\mathbb{R}, \frac{e^{-x^2} \, dx}{\sqrt{\pi}}) \otimes L^2(\mathbb{R}, \frac{e^{-y^2} \, dy}{\sqrt{\pi}})
\]
For fixed \(L\), the set of vectors \(H_r \otimes H_{L-r}, r = 0, 1, 2, \ldots L\), span an \((L+1)\)-dimensional subspace, \(\mathcal{H}(L)\), of this Hilbert space. Two subspaces, \(\mathcal{H}(L)\) and \(\mathcal{H}(L')\) are orthogonal if \(L \neq L'\). The matrix \(M(L)\) simply effects a basis change on this subspace, to the new basis \(H_{r,L-r}, r = 0, 1, 2, \ldots L\). The integral transform (4.13) effects a basis change on the entire Hilbert space. Denoting this transform by the operator \(M\), so that \(M(H_m \otimes H_n) = H_{m,n}\), \(m, n = 0, 1, 2, \ldots \infty\), we see that it decomposes into the orthogonal direct sum \(M = \bigoplus_{L=0}^{\infty} M(L)\). General maps of the type \(M(L)\) have been studied in [4] and shown to be useful in the construction of pseudo fermions in [1].
References

[1] S.T. Ali, F. Bagarello and Jean Pierre Gazeau, Extended pseudo-fermions from noncommutative bosons, J.Math. Phys. 54 (2013), 073516, 13 pp; doi: 10.1063/1.4815935.

[2] S. T. Ali, F. Bagarello, and G. Honnouvo, Modular structures on trace class operators and applications to Landau levels, J. Phys. A 43 (2010), 105202, 17 pp; doi:10.1088/1751-8113/43/10/105202.

[3] S.T. Ali, J.-P. Antoine and J.-P. Gazeau, Coherent States, Wavelets and their Generalizations, Springer-Verlag, New York, 2nd Ed. 2014.

[4] F. Balogh, N.M. Shah and S.T. Ali, Some biorthogonal families of polynomials arising in noncommutative quantum mechanics, Concordia University preprint ((2013); arXiv:1309.4163v1 [math-ph] 17 Sep 2013.

[5] A. Baricz and M. E. H. Ismail, Turán type inequalities for Tricomi confluent hypergeometric functions, Constructive Approximation 37 (2013), 195–221.

[6] N. Cotfas, J. P. Gazeau, and K. Gorska, Complex and real Hermite polynomials and related quantizations, J. Phys. A 43(2010), 305304, 14 pp.

[7] C. Dunkl and Y. Xu, Orthogonal Polynomials of Several Variables, second edition, Cambridge University Press, Cambridge, Cambridge, 2014.

[8] A. Ghanmi, A class of generalized complex Hermite polynomials, J. Math. Anal. Appl. 340 (2008), 1395–1406.

[9] A. Ghanmi, Operational formulae for the complex Hermite polynomials $H_{p,q}(z,\bar{z})$, Integral Transforms and Special Functions 24 (2013), 884–895.

[10] A. Intissar and A. Intissar, Spectral properties of the Cauchy transform on $L^2(\mathbb{C};e^{-|z|^2}dz)$, J. Math. Anal. Appl. 31 (2006), 400-418.

[11] M. E. H. Ismail, Determinants with orthogonal polynomial entries, J. Comp. Appl. Math. 178 (2005), 255-266.

[12] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in one Variable, paperback edition, Cambridge University Press, Cambridge, 2009.

[13] M. E. H. Ismail, Analytic properties of complex Hermite polynomials, Trans. Amer. Math. Soc., to appear.

[14] M. E. H. Ismail and P. Simeonov, Complex Hermite Polynomials: Their Combinatorics and Integral Operators, submitted to Proc. Amer. Soc, to appear.

[15] M.E.H. Ismail and R. Zhang, The Kibble-Slepian formula, to appear.

[16] M.E.H. Ismail and J. Zeng, Combinatorial interpretations of the 2D-Hermite and 2D-Laguerre polynomials with applications, to appear.

[17] K. Ito, Complex multiple Wiener integral, Japan J. Math. 22(1952), 63–86.
[18] S. Karlin, *Determinants of eigenfunctions of Sturm-Liouville equations*, J. d’Anal. Math. **9** (1961/62), 365–397.

[19] S. Karlin, *Sign regularity of classical orthogonal polynomials*, in *Orthogonal Polynomials and Their Continuous Analogues*, D. Haimo, editor, Southern Illinois University Press, Carbondale, 1967, 55–74.

[20] S. Karlin and G. Szegő, *On certain determinants whose elements are orthogonal polynomials*, J. d’Anal. Math. **8** (1960/61), 1–157 reprinted in *Gabor Szegő Collected Papers*, vol. 3, R. Askey, ed., Birkhäuser, Boston, 1982, 605–761.

[21] K. Thirulogasanthar, G. Honnouvo, and A. Krzyzak, *Coherent states and Hermite polynomials on quaterionic Hilbert spaces*, J. Phys. A **43** (2010) 385205, 13 pp.

[22] A. Wünsche, *Laguerre 2D-functions and their applications in quantum optics*, J. Phys. A **31** (1998), 8267–8287.

[23] A. Wünsche, *Transformations of Laguerre 2D-polynomials and their applications to quasiprobabilities*, J. Phys. A **21** (1999), 3179–3199.

[24] A. Wünsche, *General Hermite and Laguerre two-dimensional polynomials*, J. Phys. A **33** (2000), 1603–1629.