Abstract

This is an expository paper about several sophisticated forcing techniques closely related to standard finite support iterations of ccc partial orders. We focus on the four topics of ultrapowers of forcing notions, iterations along templates, Boolean ultrapowers of forcing notions, and restrictions of forcing notions to elementary submodels.

Introduction

The method of finite support iteration (fsi) of ccc forcing, originally developed by Solovay and Tennenbaum to prove the consistency of Suslin’s hypothesis [ST], has since been used for a plethora of independence proofs, both in set theory proper and in other areas of mathematics like topology or algebra. One recurring theme has been its use for independence proofs about cardinal invariants of the continuum, that is, cardinal numbers describing the combinatorial structure of the Baire space $\omega^\omega$ or Cantor space $2^\omega$ and typically taking values between the first uncountable cardinal $\aleph_1$ and the size $c$ of the continuum. Examples of such cardinal invariants are the unbounding number $b$ and the dominating number $d$, the least size of a family $F$ of functions in $\omega^\omega$ such that no single function dominates modulo finite all functions in $F$ (such that all functions are dominated modulo finite by a member of $F$, respectively) The fact that the continuum can be made arbitrarily large adds to the versatility of the fsi method.

However, there are situations when an fsi cannot be used (or when it is not known whether it can be used). In such cases, a countable support iteration of proper forcing [Sh2] may be appropriate. This method, though, makes the continuum of size at most $\aleph_2$ and therefore is of no use for distinguishing three or more cardinal invariants. Therefore, a number of intricate and sophisticated methods, which are to some extent modifications of fsi, have been developed for solving specific problems, and later used for further results about cardinal invariants. The purpose of the present survey paper is to introduce four such methods with the hope of making them more accessible to researchers in the field. Specifically, we shall discuss

- ultrapowers of partial orders (Section 1),
- iterations along templates (Section 2),
- Boolean ultrapowers of partial orders (Section 3), and
- restrictions of partial orders to elementary submodels (Section 4).

The first two methods were introduced by Shelah [Sh4] to prove the consistency of $d < a$, first using a measurable cardinal and then in ZFC. Here $a$ is the almost disjointness number, that is, the least size of an infinite maximal almost disjoint (mad) family of infinite subsets of natural numbers, one of the most important cardinal invariants.

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Lemma 2. Assume of the continuum. The third was introduced in two papers, one, still unpublished, by Raghavan and Shelah [RS] dealing with the consistency of $\delta(\lambda) < \alpha(\lambda)$ where $\lambda$ is regular uncountable, and another, by Goldstern, Kellner, and Shelah [GKS] showing the consistency of Cichoń’s maximum, the statement that all cardinal invariants in Cichoń’s diagram can be simultaneously distinct. Both use large cardinal assumptions. The forth, then, was first used by Goldstern, Kellner, Mejia, and Shelah [GKMS] to prove the consistency of Cichoń’s maximum in ZFC.

For each of the four topics, we first describe the technique and prove a number of basic results in one or two subsections. In the next subsection, we present one proof obtained by the respective method in full detail and in the final subsection, we provide an overview of results obtained by the same method.

It should be noted that while only the template technique inherently is an fsi-style method, apart from the mentioned work by Raghavan and Shelah, the others so far have been used in an fsi context. For ultrapowers this seems to have to do with the fact that the rather complicated limit construction in the iteration only works in this case, but for the other two methods, Boolean ultrapowers and submodels, we expect more applications to higher cardinal invariants and, thus, e.g., to forcing notions which are $< \lambda$-closed and $\lambda^+$-cc for some uncountable regular $\lambda$.

0.1 Preliminaries

We assume basic knowledge about forcing theory (see [Je] and [Ku]), as well as some practice with cardinal invariants and their interplay with forcing (see [BJ], [B], and [Ha]).

We often freely switch between the partial order (p.o.) language and the complete Boolean algebra (cBa) language when dealing with forcing. We use $\mathbb{P} \prec \mathbb{Q}$ to denote that a p.o. $\mathbb{P}$ completely embeds into a p.o. $\mathbb{Q}$, and $\mathbb{P} * \mathbb{Q}$ is the two-step iteration of $\mathbb{P}$ with $\mathbb{Q}$. “limdir” denotes the direct limit of a directed system of forcing notions.

Let $\mathbb{P}$ and $\mathbb{Q}$ be complete Boolean algebras (cBAs) with $\mathbb{P} \subseteq \mathbb{Q}$. Then the projection mapping $h_\mathbb{P}^\mathbb{Q} : \mathbb{Q} \to \mathbb{P}$ is defined by $h_\mathbb{P}^\mathbb{Q}(q) = \prod \{ p \in \mathbb{P} : q \leq p \}$ for $q \in \mathbb{Q}$. Notice that in this context, $\mathbb{P} \prec \mathbb{Q}$ is equivalent to saying that for all $p \in \mathbb{P}$ and $q \in \mathbb{Q}$, if $p$ is compatible with $h_\mathbb{P}^\mathbb{Q}(q)$ in $\mathbb{P}$ then $p$ is compatible with $q$ in $\mathbb{Q}$.

A forcing notion $(\mathbb{P}, \leq)$ is said to be Suslin ccc if $\mathbb{P}$ is ccc and $\mathbb{P} \subseteq \omega^\omega$, $\leq \subseteq (\omega^\omega)^2$, as well as $\bot \subseteq (\omega^\omega)^2$ are $\Sigma_1^1$ sets. See [BJ] Section 3.6 for basic properties about Suslin ccc forcing. The following two basic lemmas are crucial and we therefore include the short proofs.

Lemma 1. If $\mathbb{P}$ is Suslin ccc, “$A \subseteq \mathbb{P}$ is a maximal antichain” is a $\Pi_1^1$ statement and therefore absolute between models of ZFC.

Proof. Let $A = \{ x_n : n \in \omega \} \subseteq \mathbb{P}$. “$A$ is a maximal antichain” iff

- $\forall n \neq m \forall y (y \notin \mathbb{P} \lor y \leq x_n \lor y \leq x_m)$
- $\forall y (y \notin \mathbb{P} \lor \exists n - (y \perp x_n))$

Both formulas are $\Pi_1^1$, and therefore $\Sigma_1^1$-absoluteness applies. \hspace{1cm} \square

Lemma 2. Assume $\mathbb{P}, \mathbb{P}'$ are partial orders with $\mathbb{P} \prec \mathbb{P}'$. Also let $\mathbb{Q}$ be a Suslin ccc forcing. Then $\mathbb{P} * \mathbb{Q} \prec \mathbb{P}' * \mathbb{Q}$ where the first $\mathbb{Q}$ is the $\mathbb{P}$-name for (the interpretation of the code of) $\mathbb{Q}$ in $V_\mathbb{P}$ and the second the corresponding $\mathbb{P}'$-name.

Proof. Assume $B = \{ (p_\alpha, \dot{q}_\alpha) : \alpha < \kappa \}$ is a maximal antichain in $\mathbb{P} * \mathbb{Q}$. We need to show $B$ is still maximal in $\mathbb{P}' * \mathbb{Q}$.

Assume $(p', q') \in \mathbb{P}' * \mathbb{Q}$. Let $G'$ be a $\mathbb{P}'$-generic filter over $V$ with $p' \in G'$. Then $G := G' \cap \mathbb{P}$ is $\mathbb{P}$-generic over $V$ and, in $V[G]$, $\{ \dot{q}_\alpha[G] : p_\alpha \in G \}$ is a maximal antichain in $\dot{Q}[G]$. (This means in particular that $\{ \alpha : p_\alpha \in G \}$ is at most countable.) By the previous lemma, this is still a maximal antichain in $\dot{Q}[G']$ in $V[G']$. Therefore, in $V[G']$, there is $\alpha$ with $p_\alpha \in G$ such that $\dot{q}_\alpha[G]$ and $\dot{q}'[G']$ are compatible. Hence there are $p'' \leq p_\alpha, p'$ in $\mathbb{P}'$ and a $\mathbb{P}'$-name $\dot{q}''$ for an element of $\mathbb{Q}$ such that $(p'', \dot{q}'') \leq (p_\alpha, \dot{q}_\alpha), (p', \dot{q}')$, as required. \hspace{1cm} \square

In Sections 1 and 2 we need the following basic notion from [Br3]. Let $\mathbb{P}_{0,1} \prec \mathbb{P}_j \prec \mathbb{P}_{0,1}$, $j \in \{ 0, 1 \}$, be cBAs. We say projections in the diagram

\[
\begin{array}{c}
\mathbb{P}_1 \\
\mathbb{P}_{0,1} \\
\mathbb{P}_0
\end{array}
\]

\[
\begin{array}{c}
\mathbb{P}_j \prec \mathbb{P}_{0\cup j} \prec \mathbb{P}_{0,1}
\end{array}
\]

in the diagram.
are correct if \( h_{0\alpha 1}^{0\gamma 1}(p_0) = h_{0\alpha 1}^0(p_0) \) for all \( p_0 \in P_0 \) iff \( h_{0\alpha 1}^{0\gamma 1}(p_1) = h_{0\alpha 1}^0(p_1) \) for all \( p_1 \in P_1 \) iff whenever \( h_{0\alpha 1}^0(p_0) = h_{0\alpha 1}^0(p_1) \) then \( p_0 \) and \( p_1 \) are compatible in \( P_{0\alpha 1} \). Notice this implies (but is not equivalent to) \( P_{0\alpha 1} = P_0 \cap P_1 \). A typical example for a diagram with correct projections is given by letting \( P_{0\alpha 1} = \{0, 1\} \) and \( P_{0\gamma 1} \) the usual product forcing, that is, the completion of \( (P_0 \setminus \{0\}) \times (P_1 \setminus \{0\}) \). Another important example is obtained by letting \( P_{0\alpha 1} \prec P_0 \) be arbitrary forcing notions and putting \( P_1 := P_{0\alpha 1} * Q \) and \( P_{0\gamma 1} := P_0 * Q \), where \( Q \) is a Suslin ccc forcing notion. In both cases, correctness is straightforward. More on correctness can be found in \[ Br5 \].

Correctness can be used to show complete embeddability between direct limits.

**Lemma 3.** Let \( K \) be a directed index set. Assume \( (P_k : k \in K) \) and \( (Q_k : k \in K) \) are systems of cBa’s such that \( P_k \prec P_\ell \), \( Q_k \prec Q_\ell \), and \( P_k \prec Q_k \) for any \( k \leq \ell \). Assume further projections in all diagrams of the form

\[
\begin{array}{c}
Q_k \\
\cap \leq\
\downarrow \\
\downarrow \\
P_k \prec \end{array}
\begin{array}{c}
Q_\ell \\
\cap \leq\
\downarrow \\
\downarrow \\
P_\ell \end{array}
\]

are correct for \( k \leq \ell \). Then \( P := \lim \text{dir}_{k \in K} P_k \) completely embeds into \( Q := \lim \text{dir}_{k \in K} Q_k \). Furthermore, correctness is preserved in the sense projections in all diagrams of the form

\[
\begin{array}{c}
Q_k \\
\cap \leq\
\downarrow \\
\downarrow \\
P_k \prec \end{array}
\begin{array}{c}
Q \\
\cap \leq\
\downarrow \\
\downarrow \\
P \end{array}
\]

for \( k \in K \) are correct.

**Proof.** Let \( A \subseteq \bigcup_{k \in K} P_k \) be a maximal antichain in \( P \). We have to show \( A \) is still maximal in \( Q \). Choose \( q \in Q \). Then \( q \in Q_k \) for some \( k \in K \). By maximality of \( A \) there is \( p \in A \) such that \( h_{0\alpha 1}^{0\gamma 1}(q) \) is compatible with \( p \) in \( P \) and thus with \( h_{0\alpha 1}^0(p) \) in \( P_k \). Find \( \ell \geq k \) such that \( p \in P_\ell \). By correctness, \( p \) and \( q \) are compatible in \( Q_\ell \) and thus in \( Q \), as required. Preservation of correctness is straightforward.

\[ \square \]

## 1 Ultrapowers

Assume \( P \) is a ccc partial order and \( \kappa \) is a measurable cardinal as witnessed by the \( \kappa \)-complete ultrafilter \( D \). Then the ultrapower \( P^\kappa / D \) is again a ccc partial order and \( P \) completely embeds into \( P^\kappa / D \) so that we may view \( P^\kappa / D \) as a two-step iteration of \( P \) and some remainder forcing (see Subsection 1.2 for details). \( P^\kappa / D \) shares many properties with \( P \) and some objects added by \( P \) will actually be preserved by the ultrapower, while on the other hand, if \( P \) forces that \( a \) has size at least \( \kappa \), then \( P^\kappa / D \) destroys all mad families of the intermediate extension \( V^P \). This simple and ingenious observation, due to Shelah, forms the basis of his consistency proofs of \( \mathfrak{d} < a \) and \( u < a \) \[ Sh4 \].

Typically, the ultrapower operation is applied to iterations \( P = \langle P_\gamma : \gamma \leq \mu \rangle \), and ultrapowers of iterations are again iterations. When iterating the process of taking such ultrapowers the question arises what to do in limit steps. One option is a direct limit (this has been used e.g. in Theorem 1.1.9), but often embedding the iterations obtained by taking ultrapowers into a larger iteration is necessary (e.g. for \( \mathfrak{d} < a \) and \( u < a \) ). Technical details of this are discussed in Subsection 1.2.

In Subsection 1.3 we present a complete proof of Shelah’s consistency of \( \mathfrak{d} < a \) from a measurable (Theorem 1.3), and in Subsection 1.4 we discuss further results obtained by the ultrapower method. Our exposition follows to some extent our earlier \[ Br4 \].

### 1.1 Ultrapowers of partial orders

Let \( \kappa \) be a measurable cardinal and let \( D \) be a \( \kappa \)-complete ultrafilter on \( \kappa \). For a p.o. \( P \) and \( f \in P^\kappa \),

\[
[f] = f / D = \{ g \in P^\kappa : \{ \alpha < \kappa : f(\alpha) = g(\alpha) \} \in D \}
\]

is the equivalence class of \( f \) modulo \( D \). The ultrapower \( P^\kappa / D \) of \( P \) consists of all such equivalence classes. It is partially ordered by

\[
[f] \leq [g] \text{ iff } \{ \alpha < \kappa : f(\alpha) \leq g(\alpha) \} \in D
\]
As usual, we identify \( p \in \mathbb{P} \) with the class \([f]\) of the constant function \( f(\alpha) = p, \alpha < \kappa \), and thus construe \( \mathbb{P} \) as a subset of \( \mathbb{P}^\kappa / \mathcal{D} \).

**Lemma 4.** If \( \mathbb{P} \) is \( \kappa \)-cc then \( \mathbb{P} < \circ \mathbb{P}^\kappa / \mathcal{D} \).

**Proof.** Fix \( \nu < \kappa \), and let \( A = \{ p_\gamma : \gamma < \nu \} \) be a maximal antichain in \( \mathbb{P} \). Given arbitrary \( f \in \mathbb{P}^\kappa \), for all \( \alpha < \kappa \) there is \( \gamma < \nu \) such that \( f(\alpha) = p_\gamma \) and \( p_\gamma \) are compatible. By \( \kappa \)-completeness of \( \mathcal{D} \), there is \( \gamma < \nu \) such that \( \{ \alpha < \kappa : f(\alpha) \text{ and } p_\gamma \text{ are compatible} \} \) belongs to \( \mathcal{D} \). Hence \([f]\) is compatible with \( p_\gamma \) in \( \mathbb{P}^\kappa / \mathcal{D} \), and \( A \) is still a maximal antichain in \( \mathbb{P}^\kappa / \mathcal{D} \). \( \square \)

Notice that the converse holds as well. If \( \mathbb{P} \) is not \( \kappa \)-cc, then there is a maximal antichain \( \{ p_\gamma : \gamma < \mu \} \subseteq \mathbb{P} \) with \( \mu \geq \kappa \), and \([f]\) given by \( f(\alpha) = p_\alpha \) for \( \alpha < \kappa \) is an element of \( \mathbb{P}^\kappa / \mathcal{D} \) incompatible with all \( p_\gamma \). Hence \( \mathbb{P} \) does not completely embed into \( \mathbb{P}^\kappa / \mathcal{D} \).

**Lemma 5.** If \( \mathbb{P} \) is \( \nu \)-cc for some \( \nu < \kappa \) then so is \( \mathbb{P}^\kappa / \mathcal{D} \).

**Proof.** Take arbitrary elements \( f_\gamma \in \mathbb{P}^\kappa, \gamma < \nu \). By the \( \nu \)-cc of \( \mathbb{P} \), for each \( \alpha < \kappa \), there are \( \gamma < \delta < \nu \) such that \( f_\gamma(\alpha) \) and \( f_\delta(\alpha) \) are compatible. By \( \kappa \)-completeness of \( \mathcal{D} \), there are \( \gamma < \delta < \nu \) such that \( \{ \alpha < \kappa : f_\gamma(\alpha) \text{ and } f_\delta(\alpha) \text{ are compatible} \} \) belongs to \( \mathcal{D} \). Thus \([f_\gamma]\) and \([f_\delta]\) are compatible. Therefore every antichain of \( \mathbb{P}^\kappa / \mathcal{D} \) has size less than \( \nu \). \( \square \)

On the other hand, if \( \mathbb{P} \) is not \( \nu \)-cc for any \( \nu < \kappa \), then \( \mathbb{P}^\kappa / \mathcal{D} \) is not \( \kappa \)-cc. Indeed, let \( A_\alpha = \{ p_{\alpha, \gamma} : \gamma < \nu_\alpha \} \) be a maximal antichain in \( \mathbb{P} \) of size \( \nu_\alpha \geq |\alpha| \) for each \( \alpha < \kappa \), and fix \( p \in \mathbb{P} \) arbitrarily. Define \( f_\gamma \in \mathbb{P}^\kappa \) for \( \gamma < \kappa \) by

\[
\begin{align*}
 f_\gamma(\alpha) &= \begin{cases} 
 p_{\alpha, \gamma} & \text{if } \nu_\alpha > \gamma \\
 p & \text{otherwise}
\end{cases}
\end{align*}
\]

It is easy to see that \( \{ [f_\gamma] : \gamma < \kappa \} \) is an antichain in \( \mathbb{P}^\kappa / \mathcal{D} \).

For the remainder of this section, we assume \( \mathbb{P} \) is ccc. Therefore \( \mathbb{P} < \circ \mathbb{P}^\kappa / \mathcal{D} \) and \( \mathbb{P}^\kappa / \mathcal{D} \) is ccc by the two previous lemmata.

We next describe the relationship between \( \mathbb{P} \)-names and \( \mathbb{P}^\kappa / \mathcal{D} \)-names for real numbers. First notice that given \( \kappa \) many maximal antichains \( \{ p_\alpha^\kappa : n \in \omega \}, \alpha < \kappa \), in \( \mathbb{P} \), letting \( f_n(\alpha) = p_\alpha^\kappa \) for \( \alpha < \kappa \) we obtain a maximal antichain \( \{ [f_n] : n \in \omega \} \) in \( \mathbb{P}^\kappa / \mathcal{D} \). Furthermore all maximal antichains of \( \mathbb{P}^\kappa / \mathcal{D} \) are of this form. Now recall that a \( \mathbb{P} \)-name \( \dot{x} \) for a real in \( \omega^\kappa \) is given by maximal antichains \( \{ p_\alpha^\kappa : n \in \omega \} \) and numbers \( \{ k_{n,i} : n \in \omega \}, i \in \omega \), such that

\[
p_{n,i} \forces_{\mathbb{P}} \dot{x}(i) = k_{n,i}.
\]

Therefore, a \( \mathbb{P}^\kappa / \mathcal{D} \)-name \( \dot{y} \) for a real is given by maximal antichains \( \{ p_\alpha^\kappa : n \in \omega \} \) and numbers \( \{ k_{n,i} : n \in \omega \}, i \in \omega \), such that, letting \( f_{n,i}(\alpha) = p_{n,i}^\kappa \),

\[
[f_{n,i}] \forces_{\mathbb{P}^\kappa / \mathcal{D}} \dot{y}(i) = k_{n,i}.
\]

Since \( \{ p_{n,i}^\kappa : n \in \omega \} \) and \( \{ k_{n,i} : n \in \omega \}, i \in \omega \), determine a \( \mathbb{P} \)-name \( \dot{x}^\alpha \) for a real, we may think if \( \dot{y} \) as the average or mean of the \( \dot{x}^\alpha \) and write \( \dot{y} = (\dot{x}^\alpha : \alpha < \kappa) / \mathcal{D} \). Notice that every \( \mathbb{P}^\kappa / \mathcal{D} \)-name for a real is of this form.

**Lemma 6.** Let \( \mathbb{P} \) be ccc. Assume \( \dot{A} \) is a \( \mathbb{P} \)-name for a mad family of size at least \( \kappa \). Then \( \mathbb{P}^\kappa / \mathcal{D} \) forces that \( \dot{A} \) is not maximal. In particular, if \( \mathbb{P} \) forces a \( \geq \kappa \), then no a.d. family of \( V^\mathbb{P} \) is maximal in \( V^\mathbb{P}^\kappa / \mathcal{D} \).

**Proof.** Assume \( \dot{A} = \{ \dot{A}^\gamma : \gamma < \nu \} \) where \( \nu \geq \kappa \) and all \( \dot{A}^\gamma \) are \( \mathbb{P} \)-names for infinite subsets of \( \omega \). Then \( \dot{A} = (\dot{A}^\alpha : \alpha < \kappa) / \mathcal{D} \) is a \( \mathbb{P}^\kappa / \mathcal{D} \)-name for an infinite subset of \( \omega \) by the preceding discussion.

Fix \( \gamma < \nu \). Since for all \( \alpha < \kappa \) with \( \alpha \neq \gamma \),

\[
\forces_{\mathbb{P}} |\dot{A}^\alpha \cap \dot{A}^\gamma| < \aleph_0,
\]

we see that \( \{ \alpha < \kappa : \forces_{\mathbb{P}} |\dot{A}^\alpha \cap \dot{A}^\gamma| < \aleph_0 \} \) belongs to \( \mathcal{D} \). Therefore

\[
\forces_{\mathbb{P}^\kappa / \mathcal{D}} |\dot{A} \cap \dot{A}^\gamma| < \aleph_0
\]

because \( \dot{A} \) is the average of the \( \dot{A}^\alpha \). Hence \( \dot{A} \) witnesses that \( \mathbb{P}^\kappa / \mathcal{D} \) forces non-maximality of \( \dot{A} \). \( \square \)
Lemma 7. Let $\mathbb{P}$ be ccc, $\mu \neq \kappa$ regular, and assume $\mathbb{P}$ adjoins a scale $\{d_\beta : \beta < \mu\}$. Then $\mathbb{P}^\kappa/\mathcal{D}$ forces that $\{d_\beta : \beta < \mu\}$ is still a scale. In particular, if $\mathbb{P}^\kappa/\mathcal{D} \cong \mathbb{P} \ast \dot{\mathcal{Q}}$, then $\mathbb{P}$ forces that $\dot{\mathcal{Q}}$ is an $\omega^\kappa$-bounding forcing notion.

Proof. Let $\dot{y}$ be a $\mathbb{P}^\kappa/\mathcal{D}$-name for a real in $\omega^\omega$. By the preceding discussion, there are $f_{n,i} \in \mathbb{P}^\kappa$ and $k_{n,i} \in \omega$, $n, i \in \omega$, such that $\dot{y}$ is determined by $\{f_{n,i} : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$. Letting $p_{n,i}^\alpha = f_{n,i}(\alpha)$, $p_{n,i}^\alpha : n \in \omega$, $i \in \omega$, determine $\mathbb{P}$-names $\dot{x}^\alpha$ for reals, and $\dot{y} = (\dot{x}^\alpha : \alpha < \kappa)/\mathcal{D}$.

By ccc-ness of $\mathbb{P}$, for each $\alpha < \kappa$ there is $\beta_\alpha < \mu$ such that

$$\|\dot{x}^\alpha \leq^* \dot{d}_{\beta_\alpha}.$$

Since $\mu \neq \kappa$ are both regular, we obtain $\beta < \mu$ such that $\{\alpha < \kappa : \beta_\alpha \leq \beta\} \in \mathcal{D}$ (use the $\kappa$-completeness of $\mathcal{D}$ in case $\mu < \kappa$). Therefore

$$\|\mathbb{P}^\kappa/\mathcal{D} \dot{y} \leq^* \dot{d}_\beta,$$

because $\dot{y}$ is the average of the $\dot{x}^\alpha$. Hence $\{d_\beta : \beta < \mu\}$ remains a scale in the $\mathbb{P}^\kappa/\mathcal{D}$-extension. \qed

STRATEGY. These two simple lemmas provide us with a scenario for proving the consistency of $\mathfrak{d} < \mathfrak{a}$. Let $\kappa < \mu < \lambda$ be regular cardinals. Force $b = d = \mu$ with a ccc p.o. $\mathbb{P}$. Then keep taking ultrapowers of $\mathbb{P}$ for $\lambda$ many steps. By Lemma 7, $b = d = \mu$ should be preserved while, by the ZFC-inequality $b \leq \mathfrak{a}$, Lemma 8 tells us that mad families of size less than $\lambda$ will be destroyed so that $\mathfrak{a} = \lambda$ in the final model. The problem, however, is what to do in limit steps of the procedure of iteratively taking ultrapowers. To get a handle on this, we shall look at ultrapowers of iterations and iterations of ultrapowers in the next subsection.

1.2 Ultrapowers and iterations

We start with the discussion of two-step iterations.

Lemma 8. Assume $\mathbb{P} \ast \dot{\mathbb{Q}}$ are cBa’s. Then $\mathbb{P}^\kappa/\mathcal{D} \ast \dot{\mathbb{Q}}^\kappa/\mathcal{D}$ and, in fact, the projection mapping is given by $h_{\mathbb{P}^\kappa/\mathcal{D}}^\kappa/\mathcal{D}([f]) = \langle h_{\mathbb{P}}(\alpha) : \alpha < \kappa\rangle/\mathcal{D}$ for $f \in \mathbb{Q}^\kappa$. Furthermore, projections in the diagram $\langle \mathbb{P}, \mathbb{Q}, \mathbb{P}^\kappa/\mathcal{D}, \mathbb{Q}^\kappa/\mathcal{D} \rangle$ are correct.

Proof. Let $g \in \mathbb{P}^\kappa$ and notice that

$$[g] \geq h_{\mathbb{P}^\kappa/\mathcal{D}}^\kappa/\mathcal{D}([f]) \iff [g] \geq [f] \iff \{\alpha < \kappa : g(\alpha) \geq f(\alpha)\} \in \mathcal{D} \iff \{\alpha < \kappa : g(\alpha) \geq h_g^\kappa(f(\alpha))\} \in \mathcal{D} \iff [g] \geq \langle h_g^\kappa(f(\alpha)) : \alpha < \kappa\rangle/\mathcal{D}.$$

This shows equality. Thus

$$[g] \perp h_{\mathbb{P}^\kappa/\mathcal{D}}^\kappa/\mathcal{D}([f]) \iff [g] \perp \langle h_g^\kappa(f(\alpha)) : \alpha < \kappa\rangle/\mathcal{D} \iff \{\alpha < \kappa : g(\alpha) \perp h_g^\kappa(f(\alpha))\} \in \mathcal{D} \iff \{\alpha < \kappa : g(\alpha) \perp f(\alpha)\} \in \mathcal{D} \iff [g] \perp [f]$$

and we obtain complete embeddability. To see correctness, let $q \in \mathbb{Q}$, $p := h_{\mathbb{P}}^\kappa(q)$, $g(\alpha) = q$, and $f(\alpha) = p$ for all $\alpha$. Then

$$h_{\mathbb{P}^\kappa/\mathcal{D}}^\kappa/\mathcal{D}(q) = h_{\mathbb{P}^\kappa/\mathcal{D}}^\kappa/\mathcal{D}([g]) = \langle h_g^\kappa(g(\alpha)) : \alpha < \kappa\rangle/\mathcal{D} = \langle f(\alpha) : \alpha < \kappa\rangle/\mathcal{D} = [f] = p$$

as required. \qed

Lemma 9. Let $\mathbb{P}$ be a p.o. and let $\dot{\mathbb{Q}}$ be a Suslin ccc forcing notion. Then $(\mathbb{P} \ast \dot{\mathbb{Q}})^\kappa/\mathcal{D} \cong \mathbb{P}^\kappa/\mathcal{D} \ast \dot{\mathbb{Q}}$.

Note that the first $\dot{\mathbb{Q}}$ is a P-name while the second is a $\mathbb{P}^\kappa$-name.

Proof. Taking $(p_\alpha, \dot{x}_\alpha) \in \mathbb{P} \ast \dot{\mathbb{Q}}$, $\alpha < \kappa$, we see that an arbitrary condition in $(\mathbb{P} \ast \dot{\mathbb{Q}})^\kappa/\mathcal{D}$ is of the form $\langle (p_\alpha, \dot{x}_\alpha) : \alpha < \kappa\rangle/\mathcal{D}$. Therefore, letting $f \in \mathbb{P}^\kappa$ be defined by $f(\alpha) = p_\alpha$ and setting $\dot{y} = \langle \dot{x}_\alpha : \alpha < \kappa\rangle/\mathcal{D}$, we see that we can identify $\langle (p_\alpha, \dot{x}_\alpha) : \alpha < \kappa\rangle/\mathcal{D}$ with the condition $(\langle f \rangle, \dot{y}) \in \mathbb{P}^\kappa/\mathcal{D} \ast \mathbb{D}$. This obviously defines an isomorphism between (dense subsets of) the two partial orders. \qed
We now move to transfinite iterations. Let $\mu$ be an ordinal. Say a sequence of cBas $P = (P_\gamma : \gamma \leq \mu)$ is an iteration (see also [Br4]) if $P_\gamma \prec P_\delta$ for $\gamma < \delta$. Note that, for technical reasons which will become obvious later on (see Lemma 11), we do not require that $P_\delta$ is any kind of limit of $P_\gamma$, $\gamma < \delta$, for limit ordinals $\delta$. For $\gamma < \delta$, let $h_\delta^\gamma : P_\delta \to P_\gamma$ be the projection. The support of $p \in P_\mu$ is defined by

$$\text{supp}(p) = \{ \delta : \text{there is no } \gamma < \delta \text{ such that } h_\delta^\gamma(p) = h_\delta^\gamma(p) \}$$

Note that $\delta + 1 \in \text{supp}(p)$ iff $h_{\delta + 1}^\gamma(p) < h_\delta^\gamma(p)$. Similarly, for limit ordinals $\delta$, $\delta \in \text{supp}(p)$ iff $h_\delta^\gamma(p) < h_\delta^\gamma(p)$ for all $\gamma < \delta$.

An iteration $P$ has finite supports if $\text{supp}(p)$ is finite for all $p \in P_\mu$. While this is not the usual definition of a finite support iteration (fsi) the following simple lemma implies that an iteration with finite supports is equivalent to an fsi (we leave the details of this to the reader).

**Lemma 10.** Assume $P = (P_\gamma : \gamma \leq \mu)$ has finite supports. Let $\delta \leq \mu$ be a limit ordinal. Then $\limdir P_\gamma \prec P_\delta$ where, as usual, $\limdir$ denotes the direct limit of forcing notions.

**Proof.** Let $p \in P_\delta$. If $\delta \notin \text{supp}(p)$, then $p = h_0^\mu(p)$ for some $\gamma_0 < \delta$. Therefore $p \in P_{\gamma_0} \subseteq \bigcup_{\gamma < \delta} P_\gamma = \limdir P_\gamma$, and there is nothing to prove.

If $\delta \in \text{supp}(p)$, then $p = h_0^\mu(p) < h_\delta^\gamma(p)$ for all $\gamma < \delta$. However, since supports are finite, there is $\gamma_0 < \delta$ such that $h_0^\mu(p) = h_\delta^\gamma(p)$ for all $\gamma$ with $\gamma_0 \leq \gamma < \delta$. Set $p_0 = h_\gamma^\mu(p)$. So $p_0 \in P_{\gamma_0} \subseteq \bigcup_{\gamma < \delta} P_\gamma = \limdir P_\gamma$. We claim $p_0$ is a reduction of $p$ to $\limdir P_\gamma$. Indeed, suppose $q \leq p$ belongs to $\limdir P_\gamma$. Then there is $\gamma$ with $\gamma_0 \leq \gamma < \delta$ such that $q \in P_\gamma$. Since $h_\delta^\gamma(p) = p_0 \geq q$ in $P_\gamma$, $p$ and $q$ are compatible in $P_\delta$ (with common extension $p \cdot q$), as required.

We next investigate ultrapowers of iterations.

**Lemma 11.** Let $P = (P_\gamma : \gamma \leq \mu)$ be an iteration. Then $P^\kappa / D = (P_\gamma^\kappa / D : \gamma \leq \mu)$ also is an iteration. Moreover, if $P$ has finite supports then so does $P^\kappa / D$.

**Proof.** The first part is obvious by Lemma 8. So assume that $P$ has finite supports. Let $f \in (P_\mu^\kappa)^\kappa$. Then $\text{supp}(f(\alpha))$ is finite for all $\alpha$, say $\text{supp}(f(\alpha)) = \{ \gamma_0^\alpha < \gamma_1^\alpha < \ldots < \gamma_{n_\alpha}^\alpha \}$ for all $\alpha$. By completeness of $D$ there is $n \in \omega$ such that $\{ \alpha < \kappa : n_\alpha = n \}$ belongs to $D$. For each $i < n$ there is $\gamma_i \leq \mu$ such that

- either $\{ \alpha < \kappa : \gamma_i^\alpha = \gamma_i \} \in D$
- or $\{ \alpha < \kappa : \gamma_i^\alpha < \gamma_i \} \in D$ while $\{ \alpha < \kappa : \gamma_i^\alpha \leq \delta \} \notin D$ for all $\gamma < \delta$.

In the latter case we necessarily have $\text{cf}(\gamma_i) = \kappa$ by the $\kappa$-completeness of $D$.

We claim that $\text{supp}(f(\alpha)) = \{ \gamma_i : i < n \}$ and so is finite as required. (Therefore, $|\text{supp}(f(\alpha))| \leq n$, but equality does not necessarily hold because the $\gamma_i$ are not necessarily distinct.)

Indeed,

$$\gamma \in \text{supp}(f) \iff \forall \delta < \gamma (h_\delta^\gamma(f) < h_\delta^\gamma(f))$$

$$\iff \forall \delta < \gamma (\{ \alpha < \kappa : h_\delta^\gamma(f(\alpha)) < h_\delta^\gamma(f(\alpha)) \} \in D)$$

(by Lemma 8)

$$\iff \text{either } \{ \alpha < \kappa : \gamma \in \text{supp}(f(\alpha)) \} \in D$$

or $\text{cf}(\gamma) = \kappa$ and $\forall \delta < \gamma (\{ \alpha < \kappa : (\delta, \gamma) \cap \text{supp}(f(\alpha)) \neq \emptyset \} \in D)$ (second case)

$$\iff \gamma = \gamma_i \text{ for some } i < n$$

We note that even if $P_\delta = \limdir P_\gamma$ for all limits $\delta$, this is not necessarily the case for $(P_\delta)^\kappa / D$. Indeed, the proof above shows that for $\delta$ with $\text{cf}(\delta) = \kappa$, $\limdir (P_\gamma)^\kappa / D$ will be a strict complete suborder of $(P_\delta)^\kappa / D$. This is the reason for our notion of iteration with finite supports.

We want to iterate the procedure of taking ultrapowers - and thus obtain iterations of ultrapowers. To this end, we need to explain what to do in limit steps. The following lemma (Lemma 7 of [Br4]) is a special case of the amalgamated limit from [Br5].

**Lemma 12.** Let $\mu$ and $\lambda$ be limit ordinals. Assume $P^\alpha = (P_\gamma^\alpha : \gamma \leq \mu, \alpha < \lambda$, are iterations such that $P_\gamma^\alpha \prec P_\delta^\beta$ for $\alpha < \beta < \lambda$ and $\gamma \leq \mu$. Also assume $(P_\alpha^\lambda : \gamma < \mu)$ is an iteration such that $P_\gamma^\lambda \prec P_\delta^\lambda$ for $\alpha < \lambda$ and $\gamma < \mu$. Furthermore, assume that for all $\alpha < \beta \leq \lambda$ and $\gamma < \delta \leq \mu$ with $(\beta, \delta) \neq (\lambda, \mu)$, the projections in the diagram $P_\gamma^\lambda, P_\alpha^\mu, P_\delta^\mu$ are correct.

Then there is a p.o. $P_\mu^{\lambda\gamma}$ such that $P_\gamma^{\lambda\gamma} = (P_\gamma^\lambda : \gamma \leq \mu)$ is an iteration and $P_\gamma^\alpha \prec P_\delta^\lambda$ for all $\alpha < \lambda$. Moreover, correctness is preserved in the sense that the projections in the diagram $P_\gamma^{\lambda\gamma}, P_\alpha^\mu, P_\delta^\mu$ are correct as well.

Assume also all $P^\alpha$, $\alpha < \lambda$, and $(P_\alpha^\lambda : \gamma < \mu)$ have finite supports. Then so does $P^\lambda$. 
Lemma 13. Let \( P \) be a forcing notion. Then all preservation of correctness is obvious by the definition of \( P \). We next need to argue that the limit construction of the previous lemma preserves the ccc. This is far from obvious. In view of later applications (Theorem 14 and Subsection 1.3) we do this in the special situation when \( P^{\alpha+1} \) is the ultrapower of \( P^{\alpha} \) and the iterands in the \( P^{\alpha} \) are sufficiently simple.

We say a forcing notion \( (P, \leq) \) is Suslin \( \sigma \)-linked if it is Suslin ccc and \( P = \bigcup_n P_n \) where all \( P_n \) are linked (that is, any two elements of \( P_n \) have a common extension) \( \Sigma^1_1 \) sets. This implies \( P_n \) is linked" is a \( \Pi^1_1 \) statement and therefore absolute. (Indeed, linkedness of \( P_n \) is equivalent to \( \forall x,y (x \notin P_n \land y \notin P_n \lor (\exists \delta \in P_n)(\delta \land y)) \)."

Lemma 13. Let \( \mu \) and \( \lambda \) be limit ordinals. Let \( Q = \bigcup_n Q_n \) be a Suslin \( \sigma \)-linked forcing notion. Assume \( P = \langle P^\eta_\gamma : \gamma \leq \mu \rangle \) is iterations with finite support, \( \alpha < \lambda \), such that

- \( P^0 = \{0,1\} \) and \( P^0 = \lim \text{dir}_{<\delta} P^0 \) for limit \( \delta \).
- \( P^{\alpha+1}_\gamma = P^{\alpha}_\gamma \star \hat{Q} \) for all \( \alpha \) and \( \gamma \).
- \( P^{\alpha+1}_\gamma = (P^{\alpha}_\gamma)^{\gamma}/D \) for all \( \alpha \) and \( \gamma \).
- \( P^\eta_\gamma \) is built according to Lemma 14 for limit \( \beta \) and \( \gamma \).

Then all \( P^\eta_\gamma \) satisfy property \( K \) (and thus are ccc).

Let us note that by Lemma 14 there is no conflict between the second and third clauses in the assumption.

Proof. By recursion on \( \alpha \leq \lambda \), we define \( I^\alpha, D^\alpha_\gamma, \gamma \leq \mu \), and \( s^\alpha \), such that

1. the \( I^\alpha \) are linear orders, \( I^0 = \mu \), and \( I^\alpha \subseteq I^\beta \) for \( \alpha \leq \beta \),
2. the \( D^\alpha_\gamma \) are dense subsets of \( \mathbb{P}^\gamma_\delta \) and \( D^\alpha_\gamma \subseteq D^\beta_\delta \) for \( \alpha \leq \beta \) and \( \gamma \leq \delta \),
3. the \( s^\alpha \) are functions with domain \( D^\alpha_\mu \) and \( s^\alpha \subseteq s^\beta \) for \( \alpha \leq \beta \),
4. if \( p \in D^\alpha_\gamma, \gamma \leq \mu \), then \( s^\alpha(p) : I^\alpha \to \omega \) is a finite partial function with \( \text{dom}(s^\alpha(p)) \cap \mu = \{ \delta : \delta + 1 \in \text{supp}(p) \} \),
5. if \( p \in D^\beta_\gamma, \gamma \leq \delta \) with \( \delta + 1 \in \text{supp}(p) \), then \( h^\gamma_\delta \cap (p(\delta) \in \hat{Q} s^\alpha(p(\delta)) \),
6. if \( p, q \in D^\beta_\gamma, \gamma \leq \mu \), \( \delta < \gamma \), \( s^\alpha(p) \) and \( s^\alpha(q) \) agree on their common domain, and \( r_0 \leq h^\gamma_\delta(p), h^\gamma_\delta(q) \), then there is \( r \leq p, q \) in \( \mathbb{P}^\gamma_\delta \) such that \( h^\gamma_\delta(r) = r_0 \).
Note that for $\delta = 0$, the last clause means that $p$ and $q$ in $\mathbb{P}_\alpha$ are compatible if $s^\alpha(p)$ and $s^\alpha(q)$ agree on their common domain. By the $\Delta$-system lemma, it is then immediate that $\mathbb{P}_\alpha$ has property K and the ccc follows. So it suffices to carry out the recursion.

Basic step $\alpha = 0$. Let $I^0 := \mu$. Define $D^0_\gamma$ and $s^0$ with the required properties by recursion on $\gamma \leq \mu$.

If $\gamma = 0$ there is nothing to do.

Assume $\gamma = \delta + 1$ is successor, and $D^0_\gamma$ and $s^0 \upharpoonright D^0_\delta$ have been defined. Let

$$D^0_\gamma = D^0_\delta \cup \{ p = (p_0, \dot{x}) \in \mathbb{P}_\gamma : p_0 \in D^0_\delta \text{ and } p_0 \models \dot{x} \in \check{Q}_n \text{ for some } n \}.$$ 

Clearly, $D^0_\gamma$ is dense in $\mathbb{P}_\gamma$. Next, for such $p = (p_0, \dot{x}) \in D^0_\gamma$, notice that $\delta + 1 \in \text{supp}(p)$, let dom($s^0(p)$) = dom($s^0(p_0)$) $\cup \{ \delta \}$ and define $s^0(p)$ such that $s^0(p_0) \subseteq s^0(p)$ and $s^0(p)(\delta) = n$ where $p_0 \models \dot{x} \in \check{Q}_n$. Then (5) is satisfied and we need to show (6).

If $p = (p_0, \dot{x}), q = (q_0, \dot{y}) \in D^0_\gamma$, $s^0(p)$ and $s^0(q)$ agree on their common domain, and $r_0 \leq p_0, q_0$, then $r_0 \models \dot{x}, \dot{y} \in \check{Q}_n$, so that $r_0$ forces $\dot{x}, \dot{y}$ to be compatible and there is a name $\dot{z}$ such that $r_0 \models \dot{z} \leq \dot{x}, \dot{y}$. Letting $r = (r_0, \dot{z})$, we see that $r$ is as required.

Finally assume $\gamma$ is a limit ordinal. Since $\mathbb{P}_\gamma = \lim \text{dir}_{\delta < \gamma} \mathbb{P}_\delta$, $D^0_\gamma = \bigcup_{\delta < \gamma} D^0_\delta$ is dense in $\mathbb{P}_\gamma$, and properties (3) through (6) hold vacuously.

Successor step $\alpha = \beta + 1$. Then $\mathbb{P}_\alpha = (\mathbb{P}_\beta)^{\check{\mathbb{P}}_\beta}/D$ for all $\gamma \leq \mu$. We let $I^\gamma = (I^\beta)^{\check{\mathbb{P}}_\beta}/D$. Clearly, $I^\alpha$ is linearly ordered with $[v] \leq [w]$ if $\{ \xi < \kappa : v(\xi) \leq w(\xi) \} \in D$ for $v, w : \kappa \to I^\beta$, and $I^\beta \subseteq I^\alpha$. Next let

$$D^\gamma_\alpha = \{ [f] \in \mathbb{P}_\alpha : f : \kappa \to \mathbb{P}_\beta^{\check{\mathbb{P}}_\gamma} \text{ and } \{ \xi < \kappa : f(\xi) \in D^\gamma_\delta \} \in D \}.$$ 

Clearly $D^\gamma_\alpha$ is dense and $D^\gamma_\alpha \subseteq D^\gamma_\beta$ for $\delta \leq \gamma$. For such $f$, and for all $\xi$ with $f(\xi) \in D^\beta_\gamma$, dom($s^\beta(f(\xi))$) is finite, say dom($s^\beta(f(\xi))$) = $\{ i_0^\xi < \cdots < i_{m_\xi-1}^\xi \}$. By $\omega_1$-completeness of $\mathbb{D}$, there is $m$ such that $\{ \xi < \kappa : m_\xi = m \}$ belongs to $\mathbb{D}$. Define $v_j : \kappa \to I^\beta$ by

$$v_j(\xi) = \begin{cases} i_{j}^\xi & \text{if this is defined} \\ 0 & \text{otherwise} \end{cases}$$

for $j < m$. (Note that the first case occurs $\mathbb{D}$-almost everywhere.) Then $\{ [v_0] < \cdots < [v_{m-1}] \} \subseteq I^\alpha$ and we let dom($s^\alpha([f])$) = $\{ [v_j] : j < m \}$. Applying once more $\omega_1$-completeness of $\mathbb{D}$, we see that there are $n_j, j < m$, such that $\{ \xi < \kappa : s^\beta(f(\xi))(i_j^\xi) = n_j \} \in D$. Thus we let $s^\alpha([f])([v_j]) = n_j$ for $j < m$. Clearly $s^\beta \leq s^\alpha$.

To see (5), if $\delta < \gamma$ and $s^\alpha([f])(\delta) = n$, then $\{ \xi < \kappa : s^\beta(f(\xi))(\delta) = n \} \subseteq D$. Also,

$$h_{\delta+1,\alpha}^{\gamma}([f]) = (h_{\delta+1,\beta}^{\gamma}(f(\xi)) : \xi < \kappa)/D = ((h_{\delta+1,\beta}^{\gamma}(f(\xi)), \dot{\check{x}}^\xi : \xi < \kappa)/D = (h_{\delta+1,\alpha}^{\gamma}([f]), \dot{\check{x}}^\xi) \in (\mathbb{P}_\beta)^{\check{\mathbb{P}}_\beta}/D \ast \check{Q}_n$$

where $\dot{\check{x}}^\xi = (\xi : \xi < \kappa)/D$ (see Lemmata 8 and 10). By induction hypothesis (5) we know that $\{ \xi < \kappa : h_{\delta+1,\beta}^{\gamma}(f(\xi)) \models \dot{\check{x}}^\xi \in \check{Q}_n \} \in D$. Therefore $h_{\delta+1,\alpha}^{\gamma}([f]) \models \dot{\check{x}}^\xi \in \check{Q}_n$.

To prove (6), assume $\delta < \gamma$, $[f], [g] \in D^\alpha_\gamma$, $s^\alpha([f])$ and $s^\alpha([g])$ agree on their common domain, and $[h_0] \leq h_{\delta+1,\alpha}^{\gamma}([f]), h_{\delta,\alpha}^{\gamma}([g])$ in $\mathbb{P}_\gamma$. Let $\{ [v_0] < \cdots < [v_{n-1}] \}$ list the common domain of $s^\alpha([f])$ and $s^\alpha([g])$. Then

$$\{ \xi < \kappa : V(\xi) = \{ [v_0] < \cdots < [v_{n-1}] \} \} \subseteq D \ast \check{Q}_n$$

belongs to $\mathbb{D}$. Also, the set $\{ \xi < \kappa : h_0(\xi) \leq h_{\delta,\beta}^{\gamma}(f(\xi)), h_{\delta,\beta}^{\gamma}(g(\xi)) \} \subseteq D$. For $\xi$ which belong to both sets we find, by induction hypothesis (6), $h(\xi) \in D^\alpha_\gamma$ with $h(\xi) \leq f(\xi), g(\xi)$ and $h_{\delta,\beta}^{\gamma}(h(\xi)) = h_0(\xi)$. So $[h] \leq [f], [g]$ and $h_{\delta,\alpha}^{\gamma}([h]) = [h_0]$ as required.

Limit step $\alpha$. Let $I^\alpha = \bigcup_{\beta < \alpha} I^\beta$, equipped with the obvious ordering. As in the basic step, we define $D^\alpha_\gamma$ and $s^\alpha$ by recursion on $\gamma \leq \mu$.

The cases $\gamma = 0$ and $\gamma = \delta + 1$ are identical to the basic step. The only difference is that, this time, $D^\alpha_\gamma$ must contain all $D^\beta_\gamma$, and that $s^\alpha$ must extend all $s^\beta$, for $\beta < \alpha$.

So assume $\gamma$ is a limit ordinal, and $D^\alpha_\gamma$ and $s^\alpha \upharpoonright D^\alpha_\delta$ have been defined for $\delta < \gamma$. Since supports are finite (see also Lemmas 11 and 12), we know that $\mathbb{P}_\alpha = \mathbb{P}_\delta^{\check{\mathbb{P}}_\alpha}$ where $\mathbb{P}_\alpha := \lim \text{dir}_{\delta < \gamma} \mathbb{P}_\delta$, by Lemma 10. By the proof of Lemma 12 elements of $\mathbb{P}_\alpha$ are formal products $p \cdot \check{p}$ with $p \in \mathbb{P}_\beta$ for some $\beta < \alpha$, $\check{p} \in \mathbb{P}_\gamma$, and $h_{\gamma,\beta}^{\gamma}(\check{p}) = h_{\gamma,\beta}^{\gamma}(p)$ (where we use $\check{p}$ as an index for the direct limit $\check{\mathbb{P}}_\gamma$ of the $\mathbb{P}_\beta$, $\delta < \gamma$, which completely embeds into $\mathbb{P}_\gamma$, where $j = \alpha, \beta$). By strengthening $p$ and $\check{p}$, if necessary, we may assume $p \in D^\beta_\gamma$. By further strengthening $\check{p}$, we may
assume \( \vec{p} \in D^\alpha_\gamma = \bigcup_{\beta < \gamma} D^\alpha_\beta \). In general, we will then only have \( h^{\gamma,\alpha}_\beta(\vec{p}) \leq h^{\gamma,\beta}_\gamma(\vec{p}) \), but this does not concern us because the collection of formal products satisfying this weaker condition is obviously forcing equivalent with the original \( P^\alpha_\beta \). Hence, if we let \( D^\alpha_\gamma \) consist of formal products \( p \cdot \vec{p} \) with \( p \in D^\alpha_\beta \) for some \( \beta < \alpha, \vec{p} \in D^\alpha_\beta \), and \( h^{\gamma,\alpha}_\beta(\vec{p}) \leq h^{\gamma,\beta}_\gamma(\vec{p}) \), then \( D^\alpha_\gamma \) is dense in \( P^\alpha_\gamma \). Clearly \( D^\alpha_\gamma \subseteq D^\alpha_\delta \) and \( D^\beta_\gamma \subseteq D^\beta_\delta \) for \( \beta < \alpha \). For such \( p \cdot \vec{p} \in D^\alpha_\delta \), we define \( s^\alpha(p \cdot \vec{p}) \) by \( \dom(s^\alpha(p \cdot \vec{p})) = \dom(s^\alpha(\vec{p})) \cup \{ i \in \dom(s^\beta(p)) : i > \delta \text{ for all } \delta < \gamma \} \) and

\[
s^\alpha(p \cdot \vec{p})(i) = \begin{cases} 
    s^\alpha(\vec{p})(i) & \text{for } i \in \dom(s^\alpha(\vec{p})) \\
    s^\beta(p)(i) & \text{for } i \in \dom(s^\beta(p)) \text{ with } i > \delta \text{ for all } \delta < \gamma.
\end{cases}
\]

(5) is immediate by induction hypothesis (5) for \( \vec{p} \).

To prove (6), let \( \delta < \gamma, p \cdot \vec{p}, q \cdot \vec{q} \in D^\alpha_\gamma, s^\alpha(p \cdot \vec{p}) \) and \( s^\alpha(q \cdot \vec{q}) \) agree on their common domain, and \( r_0 \leq h^{\gamma,\alpha}_\delta(p \cdot \vec{p}), h^{\gamma,\alpha}_\delta(q \cdot \vec{q}) \in P^\alpha_\delta \). It is immediate that \( h^{\gamma,\alpha}_\delta(p \cdot \vec{p}) = h^{\gamma,\alpha}_\delta(p) \) and similarly for \( q \cdot \vec{q} \). Thus, by induction hypothesis (6), there is \( \vec{r} \leq \vec{p}, \vec{q} \in P^\alpha_\delta \) such that \( h^{\gamma,\alpha}_\delta(\vec{r}) = r_0 \). Let \( \beta_p, \beta_q < \alpha \) be such that \( p \in D^\beta_p \) and \( q \in D^\beta_q \). Without loss of generality \( \beta_p \leq \beta_q \), and we let \( \beta := \beta_q \). By correctness \( h^{\gamma,\beta}_\delta(p) = h^{\gamma,\beta}_\delta(\vec{p}) \). So we know that \( h^{\gamma,\alpha}_\delta(\vec{r}) \leq h^{\gamma,\beta}_\delta(\vec{p}) \leq h^{\gamma,\alpha}_\delta(p) \leq h^{\gamma,\beta}_\gamma(p) \) and \( h^{\gamma,\alpha}_\delta(\vec{r}) \leq h^{\gamma,\alpha}_\delta(\vec{q}) \leq h^{\gamma,\beta}_\gamma(\vec{q}) \). Thus by induction hypothesis (6) there is \( r \leq p, q \) in \( P^\alpha_\gamma \) such that \( h^{\gamma,\alpha}_\delta(r) = h^{\gamma,\alpha}_\delta(\vec{r}) \). Hence \( h^{\gamma,\alpha}_\delta(r \cdot \vec{r}) = h^{\gamma,\alpha}_\delta(\vec{r}) = r_0 \), and \( r \cdot \vec{r} \leq p \cdot \vec{p}, q \cdot \vec{q} \) as is required.

\[\square\]

1.3 The consistency of \( \mathfrak{d} < \mathfrak{a} \) from a measurable cardinal

Recall that Hechler forcing \( \mathcal{D} \) consists of \((s, x) \in \omega^{<\omega} \times \omega^\omega \) with \( s \subseteq x \) ordered by \((t, y) \leq (s, x) \) if \( t \supseteq s \) and \( y \supseteq x \) everywhere. \( \mathcal{D} \) is a Suslin \( \sigma \)-centered forcing adding a dominating real. In particular it is Suslin \( \sigma \)-linked.

We outline the proof of:

**Theorem 14 (Shelah [14]).** Assume ZFC + “there is a measurable cardinal” is consistent. Then so is \( \mathfrak{d} < \mathfrak{a} \). More explicitly, if GCH holds, \( \kappa \) is measurable and \( \lambda > \mu > \kappa \) are regular, then there is a ccc forcing extension satisfying \( b = \mathfrak{d} = \mu \) and \( \mathfrak{a} = \kappa = \lambda \).

**Proof.** As usual, let \( D \) be the \( \kappa \)-complete ultrafilter on \( \kappa \). By recursion on \( \alpha \leq \lambda \) we construct iterations (with finite supports) \( P^\alpha = (P^\alpha_\gamma : \gamma \leq \mu) \) such that

1. \( P^0_0 = \{ \{0, 1\} \) and \( P^0_\gamma = \lim \dir \gamma < \delta P^0_\delta \) for limit \( \delta \),
2. \( P^{\alpha+1}_\gamma = P^\alpha_\gamma \ast \mathcal{D} \) for all \( \alpha \) and \( \gamma \),
3. \( P^\alpha_\gamma = (P^\alpha_\gamma)^{\mathcal{D}} / D \) for all \( \alpha \) and \( \gamma \),
4. \( P^\beta_\gamma \) is built according to Lemma 12 for limit \( \beta \) and \( \delta \),
5. if \( \alpha \leq \beta \) and \( \gamma \leq \delta \), then \( P^\alpha_\gamma \leq P^\beta_\delta \),
6. projections in all diagrams \((P^\alpha_\gamma, P^\beta_\delta, P^\delta_\delta, P^{\alpha+1}_\gamma)\), \( \alpha < \beta \) and \( \gamma < \delta \), are correct.

For \( \beta = 0 \) construct \( P^0 \) according to the first two clauses, that is, \( P^0 = P^0_\alpha \) is a \( \mu \)-stage finite support iteration of Hechler forcing. If \( \beta = \alpha + 1 \), \( P^\beta_\gamma \) is the ultrapower of \( P^\alpha_\gamma \); see Lemma 11. Thus (3) is satisfied, and (5) and (6) follow from Lemmas 4 and 8. Furthermore, (2) also holds for by definition, the induction hypothesis for (2), and Lemma 9 we have

\[
P^\beta_\gamma = (P^{\alpha+1}_\gamma)^{\mathcal{D}} / D = (P^\alpha_\gamma \ast \mathcal{D}) = (P^\gamma_\delta)^{\mathcal{D}} = P^\beta_\delta \ast \mathcal{D}.
\]

Finally, for limit \( \beta \), we define \( P^\beta_\delta \) for successor \( \delta = \gamma + 1 \) by (2), and for limit \( \delta \), by (4), (5) and (6) follow from Lemma 2 and the comment after the definition of correctness (in Subsection 1.1.1) in the first case, and from Lemma 12 in the second case. This completes the recursive construction.

By Lemma 13 all \( P^\alpha_\gamma \) are ccc. In particular so is \( P^\beta_\delta \).

We next note that \( |P| = \lambda \). Indeed, by induction on \( \alpha \), we have that \( \max(|\alpha|, \mu) \leq |P^\alpha_\gamma| \leq \max(|\alpha|, \mu) \kappa \) for all \( \alpha \leq \lambda \): \( |P^0_\mu| = \mu = \mu^\kappa \) is obvious. For successor, the formula follows from \( |P^{\alpha+1}_\mu| \leq |P^\alpha_\mu| \kappa \), and, for limit, from \( |P^\beta_\mu| = \sum_{\alpha < \beta} |P^\alpha_\mu| \). In particular, a standard argument with nice names gives \( \kappa = \lambda \).
The last iteration $\mathcal{P}^\lambda = (\mathcal{P}_\gamma^\lambda : \gamma \leq \mu)$ cofinally often adds a dominating real (more explicitly, $\mathcal{P}_{\gamma+1}^\lambda$ adds a Hechler real dominating the reals of the $\mathcal{P}_\gamma^\lambda$-extension), and therefore $b = \emptyset = \mu$ holds in the extension. (In fact, the sequence of $\mu$ Hechler reals is already added by $\mathcal{P}^0$, see also Lemma 6).

To see $a \geq \lambda$, we use Lemma 6 assume $\mathcal{A}$ is an a.d. family of size $\nu$ for some $\nu < \lambda$ in the generic extension. If $\nu < \mu$ then $\mathcal{A}$ is not maximal because $b \leq a$ in ZFC. If $\nu \geq \mu > \kappa$, by the ccc and the regularity of $\lambda$, there is an $\alpha < \lambda$ such that $\mathcal{A}$ is a $\mathcal{P}_\mu^\alpha$-name. By Lemma 6 we then see that $\mathcal{P}_\mu^\alpha = (\mathcal{P}_\mu^\alpha)^{\mathcal{A}} / \mathcal{D}$ forces that $\mathcal{A}$ is not maximal. This completes the proof. \hfill \qed

1.4 Further results

We collect a number of results obtained by related methods.

In his original work [Sh4], Shelah also obtains the consistency of $u < a$ by basically the same method. Recall here that, given a nonprincipal ultrafilter $\mathcal{U}$ on $\omega$, $B \subseteq \mathcal{U}$ is a base of $\mathcal{U}$ if for every $A \in \mathcal{U}$ there is $B \in B$ such that $B \subseteq A$. If we only require that $B \subseteq [\omega]^\omega$, $B$ is called a $\pi$-base. The character $\chi(\mathcal{U})$ ($\pi$-character $\pi\chi(\mathcal{U})$, respectively) of $\mathcal{U}$ is the least size of a base ($\pi$-base, resp.) of $\mathcal{U}$. We define $u := \min\{\chi(\mathcal{U}) : \mathcal{U}$ is a nonprincipal ultrafilter on $\omega\}$, the ultrafilter number.

**Theorem 15** (Shelah [Sh4]). Assume ZFC + “there is a measurable cardinal” is consistent. Then so is $u < a$. More explicitly, if GCH holds, $\kappa$, $\mu$, and $\lambda > \mu > \kappa$ are regular, then there is a ccc forcing extension satisfying $u = b = \emptyset = \mu$ and $a = c = \lambda$.

This is proved by replacing Hechler forcing $\mathcal{D}$ with Laver forcing $\mathbb{L}_\mathcal{U}$ with an ultrafilter $\mathcal{U}$ in the iterated forcing construction of the proof of Theorem 13. For details see also [Br4].

In two subsequent papers ([Sh5], [Sh6]), Shelah used this construction for obtaining several results about non-convexity of the character spectrum $\text{Spec}(\chi) = \{\chi(\mathcal{U}) : \mathcal{U}$ is a nonprincipal ultrafilter on $\omega\}$.

**Theorem 16** (Shelah [Sh5]). Assume ZFC + “there is a measurable cardinal” is consistent. Then so is “$\text{Spec}(\chi)$ is not convex”.

More explicitly, if GCH holds, $\kappa$ is measurable and $\lambda > \kappa > \mu$ are uncountable regular, then there is a ccc forcing extension satisfying $u = b = \emptyset = \mu$, $c = \lambda$, $\{\mu, \lambda\} \subseteq \text{Spec}(\chi)$ and $\kappa \notin \text{Spec}(\chi)$.

Using two measurables, this can be combined with Theorem 13 [Sh5] Theorem 1.1]. Furthermore, there is an extension to a variant of the $\pi$-character spectrum [Sh5] Theorem 2.5]. Again, details can be found as well in [Br4].

**Theorem 17** (Shelah [Sh6]). Given two disjoint sets $\Theta_1$ and $\Theta_2$ of regular cardinals such that $\theta^{<\theta} = \theta$ for all $\theta \in \Theta_1$ and all members of $\Theta_2$ are measurable cardinals, there is a partial order forcing $\Theta_1 \subseteq \text{Spec}(\chi)$ and $\Theta_2 \cap \text{Spec}(\chi) = \emptyset$.

This is a generalization of Theorem 16 obtained by a product of a ccc forcing with iterated ultrapowers (to guarantee the measurables in $\Theta_2$ are not characters) and a forcing not adding reals but adjoining ultrafilters of character in $\Theta_1$.

**Theorem 18** (Shelah [Sh6]). Assuming the consistency of infinitely many strongly compact cardinals, it is consistent that for any $A \subseteq \omega \setminus \{0\}$, $\{\aleph_n : n \in A\}$ is the set of characters below $\aleph_\omega$.

This is proved by combining the previous theorem with a product of Levy collapses. Some of these results answer questions originally addressed in [BS].

A family $S \subseteq [\omega]^\omega$ is a splitting family if for all $A \in [\omega]^\omega$ there is $B \in S$ with $|A \cap B| = |A \setminus B| = \omega$ (we say $B$ splits $A$). The splitting number $s$ is the smallest cardinality of a splitting family. Iterating the ultrapower construction as in Subsection 1.2 in a matrix and also destroying splitting families while preserving an unbounded family added by the first iteration, but taking direct limits in the limit step (this makes the complex Lemmas 12 and 13 unnecessary) one obtains:

**Theorem 19** (Brendle and Fischer [BF]). Assume GCH holds, $\kappa$ is measurable and $\lambda > \mu > \kappa$ are regular cardinals. Then there is a ccc forcing extension in which $b = \mu < s = a = c = \lambda$.

This generalizes an old result of Shelah [Sh1] who proved that $b = \aleph_1 < s = a = c = \aleph_2$ is consistent with a countable support iteration of proper forcing.
2 Templates

In standard well-ordered iterations $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \mu \rangle$ where $\mu$ is an ordinal, the initial segments of the iteration $\mathbb{P}_\alpha$ and the iterands $\dot{\mathbb{Q}}_\alpha$ are handed down by the same recursion, with the latter being $\mathbb{P}_\alpha$-names, one defines $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \ast \dot{\mathbb{Q}}_\alpha$, and then one has to specify what to do in the limit step. However, while wellfoundedness is a necessity for recursively defining the initial segments in many cases, this is not so for the iterands, and they may well be indexed by a non-wellfounded structure. This is what iterations along templates do. The index set of their iterands is an arbitrary linear order $L$, and their initial segments are produced by specifying a wellfounded subfamily $\mathcal{I}$ of $\mathcal{P}(L)$ and then recursively defining the segment $\mathbb{P}[A]$ for $A \in \mathcal{I}$. Since we need to consider the same iterand $\dot{\mathbb{Q}}_x$ for $x \in L$ over various initial extensions, definability of such iterands is crucial, and iteration along templates can be seen as a generalization of finite support iteration of Suslin ccc forcing. See Subsection 2.2 for details (the exposition follows largely [Br3]).

As an application, we provide a complete proof of Shelah’s proof of the consistency of $\delta < \alpha$ in ZFC [Sh4] in Subsection 2.2 (Theorem 22), following roughly [Br1]. Subsection 2.3 presents further results obtained by the template method.

2.1 The template method

Let $\langle L, \leq_L \rangle$ be a linear order. For $x \in L$ let $L_x = \{y \in L : y <_L x\}$ be the initial segment determined by $x$.

Definition 20. A template is a pair $\langle L, \mathcal{I} = \{\mathcal{I}_x : x \in L\} \rangle$ such that $\langle L, \leq_L \rangle$ is a linear order, $\mathcal{I}_x \subseteq \mathcal{P}(L_x)$ for all $x \in L$, and

1. $\mathcal{I}_x$ contains all singletons and is closed under unions and intersections,
2. $\mathcal{I}_x \subseteq \mathcal{I}_y$ for $x <_L y$,
3. $\mathcal{I} := \bigcup_{x \in L} \mathcal{I}_x \cup \{L\}$ is wellfounded with respect to inclusion, as witnessed by the depth function $\text{Dp}_\mathcal{I} : \mathcal{I} \rightarrow \text{On}$.

If $A \subseteq L$ and $x \in L$, we define $\mathcal{I}_x|A := \{B \cap A : B \in \mathcal{I}_x\}$, the trace of $\mathcal{I}_x$ on $A$ and let $\mathcal{I}|A = \{\mathcal{I}_x|A : x \in A\}$.

$L$ is meant to be an index set for the iterands of an iteration while $\mathcal{I}$ describes in a sense the support. The wellfoundedness condition 3 is crucial because it allows for a recursive definition of the iteration. By clause 1, a real $r_y$ added at stage $y$ will be generic over a real $r_x$ added at stage $x$ for $x <_L y$ while by clause 2, if $r_x$ is generic over some initial stage of the iteration then so is $r_y$. Conditions 1 and 2 together imply that $\mathcal{I}$ is also closed under intersections and unions.

Note that $(A, \mathcal{I}|A)$ is a template as well and, if $B \in \mathcal{I}$ and $C = B \cap A \in \mathcal{I}|A$, then $\text{Dp}_{\mathcal{I}|A}(C) \leq \text{Dp}_{\mathcal{I}}(B)$. Thus, with every $A \subseteq L$ we can associate its depth, $\text{Dp}(A) := \text{Dp}_{\mathcal{I}|A}(A)$. $\text{Dp}$ has the following properties:

Lemma 21. 1. If $B \in \mathcal{I}|A$ then for any $C \subseteq B$, $C \in \mathcal{I}|B$ iff $C \in \mathcal{I}|A$.

2. If $B \in \mathcal{I}|A$, then $\text{Dp}_{\mathcal{I}|A}(B) = \text{Dp}(B)$.

3. If $B \subseteq A$ then $\text{Dp}(B) \leq \text{Dp}(A)$ and if additionally $B \in \mathcal{I}|A$ and $B \subseteq A$ then $\text{Dp}(B) < \text{Dp}(A)$.

4. If $B = B_0 \cup \{x\} \subseteq A$, $x \notin B_0$, and $B_0, B \in \mathcal{I}|A$ then $\text{Dp}(B) = \text{Dp}(B_0) + 1$.

Proof. 1) If $C \in \mathcal{I}|A$ then there is $D \in \mathcal{I}$ with $C = A \cap D$. Thus $C = B \cap D$ and $C \in \mathcal{I}|B$ follows.

On the other hand if $C \in \mathcal{I}|B$ then there is $D \in \mathcal{I}$ with $C = B \cap D$. Also there is $D' \in \mathcal{I}$ with $B = A \cap D'$. Thus $C = A \cap (D' \cap D)$ and therefore $C \in \mathcal{I}|A$.

2) is immediate by (1), and (3) is obvious.

4) $\text{Dp}(B) \geq \text{Dp}(B_0) + 1$ follows by (3), and $\text{Dp}(B) \leq \text{Dp}(B_0) + 1$ is shown by induction on $\text{Dp}(B_0)$. Suppose $\text{Dp}(B_0) = \alpha$. Let $C \subseteq B$ with $C \in \mathcal{I}|B$ and $x \in C$. Put $C_0 = C \cap B_0 = C \setminus \{x\}$. Then $C_0 \in \mathcal{I}|B_0$ and $C_0 \subseteq B_0$, and therefore $\text{Dp}(C_0) < \alpha$ by (3). Since also $C_0, C \in \mathcal{I}|A$, by induction hypothesis $\text{Dp}(C) \leq \text{Dp}(C_0) + 1 \leq \alpha$. Thus $\text{Dp}(B) \leq \alpha + 1$.

Call a Suslin ccc forcing notion $\mathbb{Q}$ correctness-preserving if for every diagram $\langle \mathbb{P}_i \rangle$ with correct projections, the projections in the diagram $\langle \mathbb{P}_i \ast \dot{\mathbb{Q}} \rangle$ are correct as well.

Definition and Theorem 22. Assume $\langle L, \mathcal{I} \rangle$ is a template. Also assume $\langle \mathbb{Q}_x : x \in L \rangle$ is a family of correctness-preserving Suslin ccc forcing notions whose definition lies in the ground model. By recursion-induction on $\text{Dp}(A)$, $A \subseteq L$,
1. we define the partial order \( \mathbb{P}|A \),
2. we prove that \( \mathbb{P}|D \subseteq \mathbb{P}|A \) and \( \leq_{\mathbb{P}|D} \subseteq \leq_{\mathbb{P}|A} \) for \( D \subseteq A \),
3. we prove that \( \mathbb{P}|A \) is transitive,
4. we describe how \( \mathbb{P}|A \) is obtained from \( \mathbb{P}|B \) where \( B \subseteq A \) with \( B \in \mathcal{I}|A \) (so \( \text{Dp}(B) < \text{Dp}(A) \)),
5. we prove that \( \mathbb{P}|D \circ \mathbb{P}|A \) for \( D \subseteq A \),
6. we prove that for \( D \subseteq L \) with \( \text{Dp}(D) \leq \text{Dp}(A) \), we have \( \mathbb{P}|(A \cap D) = \mathbb{P}|A \cap \mathbb{P}|D \),
7. we prove correctness in the sense that if \( A', D \subseteq A \), and \( D' = A' \cap D \), then projections in the diagram \( \langle \mathbb{P}|D', \mathbb{P}|A', \mathbb{P}|D', \mathbb{P}|A \rangle \) are correct.

\( \mathbb{P}|L \) is called iteration along a template.

**Definition and Proof.** (1) First consider the case \( \text{Dp}(A) = 0 \). This is equivalent to \( A = \emptyset \). Let \( \mathbb{P}|\emptyset = \{\emptyset\} \).

So assume \( \text{Dp}(A) > 0 \). Let \( \mathbb{P}|A \) consist of all finite partial functions \( p \) with domain contained in \( A \) and such that, putting \( x = \max(\text{dom}(p)) \), there is \( B \in \mathcal{I}_x|A \) (so \( \text{Dp}(B) < \text{Dp}(A) \)) such that \( p|(A \cap L_x) \in \mathbb{P}|B \) and \( p(x) \) is a \( \mathbb{P}|B \)-name for a condition in \( \mathbb{Q}_x \) (where we construe \( \mathbb{Q}_x \) as a \( \mathbb{P}|B \)-name as well).

The ordering on \( \mathbb{P}|A \) is given as follows. \( q \leq_{\mathbb{P}|A} p \) if \( \text{dom}(q) \supseteq \text{dom}(p) \) and, putting \( x = \max(\text{dom}(q)) \), there is \( B \in \mathcal{I}_x|A \) such that \( q|(A \cap L_x) \in \mathbb{P}|B \) and

- either \( x \notin \text{dom}(p) \), \( p \in \mathbb{P}|B \), and \( q|(A \cap L_x) \leq_{\mathbb{P}|B} p \),
- or \( x \in \text{dom}(p) \), \( p|(A \cap L_x) \in \mathbb{P}|B \), \( q|(A \cap L_x) \leq_{\mathbb{P}|B} p|(A \cap L_x) \), and \( p(x) \) and \( q(x) \) are \( \mathbb{P}|B \)-names for conditions in \( \mathbb{Q}_x \) such that \( q|(A \cap L_x) \models_{\mathbb{P}|B} q(x) \leq_{\mathbb{Q}_x} p(x) \).

Concerning the first alternative here, note that it is easy to see that given \( p \in \mathbb{P}|A \) and \( x >_L \max(\text{dom}(p)) \) with \( x \in A \), there is \( B \in \mathcal{I}_x|A \) such that \( p \in \mathbb{P}|B \). (Indeed, let \( y = \max(\text{dom}(p)) \). Then there is \( B_y \in \mathcal{I}_y|A \) such that \( p|(A \cap L_y) \in \mathbb{P}|B_0 \) and \( p(y) \) is a \( \mathbb{P}|B_0 \)-name for a condition on \( \mathbb{Q}_y \). Now \( B = B_0 \cup \{y\} \in \mathcal{I}_x|A \) and \( p \in \mathbb{P}|B \).

(2) Let \( D \subseteq A \) and \( p \in \mathbb{P}|D \). Also let \( x = \max(\text{dom}(p)) \). There is \( E \in \mathcal{I}_x|D \) such that \( p|(D \cap L_x) \in \mathbb{P}|E \) and \( p(x) \) is a \( \mathbb{P}|E \)-name for a condition in \( \mathbb{Q}_x \). Let \( B \in \mathcal{I}_x|A \) be such that \( E = B \cap D \). Then \( E \subseteq B \) and, by induction hypothesis (2), \( p|(D \cap L_x) = p|(A \cap L_x) \in \mathbb{P}|B \). By induction hypothesis (5), \( p(x) \) is a \( \mathbb{P}|B \)-name as well. Therefore \( p \in \mathbb{P}|A \).

The inclusion for the order is proved similarly.

(3) We use completeness of the embeddings (induction hypothesis (5)) and closure of the template under unions.

Assume \( r \leq_{\mathbb{P}|A} q \leq_{\mathbb{P}|A} p \). Let \( y \) and \( x \) be the maximal elements of \( \text{dom}(r) \) and \( \text{dom}(q) \), respectively. There are \( B_y \in \mathcal{I}_y|A \) and \( B_x \in \mathcal{I}_x|A \) witnessing the order relationship. In particular, \( r|(A \cap L_y), q|(A \cap L_y) \in \mathbb{P}|B_y \), \( q|(A \cap L_x), p|(A \cap L_x) \in \mathbb{P}|B_x \), and \( r|(A \cap L_y) \leq_{\mathbb{P}|B_y} q|(A \cap L_y), q|(A \cap L_x) \leq_{\mathbb{P}|B_x} p|(A \cap L_x) \). Let \( B = B_y \cup B_x \in \mathcal{I}_y|A \). We check that \( B \) witnesses \( r \leq_{\mathbb{P}|A} p \).

If \( x < y \), we have \( x \in B \), (2) gives us \( q, p|(A \cap L_x) \in \mathbb{P}|B \) and therefore also \( p \in \mathbb{P}|B \), and \( r|(A \cap L_y) \leq_{\mathbb{P}|B} q \leq_{\mathbb{P}|B} p \) by induction hypothesis (3), \( r|(A \cap L_y) \leq_{\mathbb{P}|B} p \), and \( r \leq_{\mathbb{P}|A} p \) follows.

If \( x = y \), \( r|(A \cap L_y) \leq_{\mathbb{P}|B} q|(A \cap L_y) \leq_{\mathbb{P}|B} p|(A \cap L_y) \), and thus by induction hypothesis (3), \( r|(A \cap L_y) \leq_{\mathbb{P}|B} p \), so we are done if \( x \notin \text{dom}(p) \). Assume \( x \in \text{dom}(p) \). Then \( r(y) \) and \( q(y) \) are \( \mathbb{P}|B_y \)-names, and \( q(x) \) and \( p(x) \) are \( \mathbb{P}|B_x \)-names. By induction hypothesis (5), they are all \( \mathbb{P}|B \)-names and \( r|(A \cap L_y) \models_{\mathbb{P}|B} r(y) \leq_{\mathbb{Q}_y} q(y) \), as required.

(4) We consider several cases.

Case 1. There is \( x = \max(A) \) such that \( A_0 := A \cap L_x = A \setminus \{x\} \in \mathcal{I}_x|A \). Then \( \mathbb{P}|A \) is easily seen to be the standard two-step iteration \( \mathbb{P}|A_0 \ast \mathbb{Q}_x \) where \( \mathbb{Q}_x \) is a \( \mathbb{P}|A_0 \)-name, for if \( p \in \mathbb{P}|A \), then \( p|A_0 \in \mathbb{P}|A_0 \) and \( p(x) \) is a \( \mathbb{P}|A_0 \)-name for a condition in \( \mathbb{Q}_x \). In particular, \( \mathbb{P}|A_0 \circ \mathbb{P}|A \).

Case 2a. There is \( x = \max(A) \), but \( A_0 = A \cap L_x \notin \mathcal{I}_x|A \). Let \( p \in \mathbb{P}|A \). There is \( B_0 \in \mathcal{I}_x|A \) such that \( p|A_0 \in \mathbb{P}|B_0 \) and \( r \notin \text{dom}(p) \) or \( p(x) \) is a \( \mathbb{P}|B_0 \)-name. Note \( B_0 \subseteq A_0 \). Let \( B := B_0 \cup \{x\} \subseteq A \). Clearly \( \text{Dp}(B) < \text{Dp}(A) \). (This holds because \( \text{Dp}(B) = \text{Dp}(B_0) + 1 \) and \( \text{Dp}(A) \geq \text{Dp}(B_0) + \omega \) by parts 3 and 4 of Lemma [21]). Also \( B_0 = B \cap L_x \in \mathcal{I}_x|B \) and thus \( p \in \mathbb{P}|B \). Since \( \mathcal{I}_x|A \) is closed under unions, the collection of \( B \subseteq A \) with \( B \cap L_x \in \mathcal{I}_x|A \) is directed. Also note that by induction hypothesis (5), if \( B \subseteq B' \) are of this form, then \( \mathbb{P}|B \circ \mathbb{P}|B' \). Therefore \( \mathbb{P}|A \) is the direct limit of the \( \mathbb{P}|B \) with \( B \subseteq A \) and \( B \cap L_x \in \mathcal{I}_x|A \).
Case 2b. A has no maximum. Let \( p \in \mathbb{P}|A \) be a condition. There is \( x > \max(\text{dom}(p)) \) with \( x \in A \) and therefore, as remarked earlier, there is \( B \in \mathcal{I}|A \) such that \( p \notin \mathbb{P}|B \). Clearly \( \mathbb{D}p(B) < \mathbb{D}p(A) \). The collection of \( B \in \mathcal{I}|A \) with \( B \in \mathcal{I}|A \) for some \( x \in A \) is directed and therefore, using again induction hypothesis (5), we see that \( \mathbb{P}|A \) is the direct limit of the \( \mathbb{P}|B \) for such \( B \).

(5) Let \( D \subseteq A \). We split the proof into cases according to (4) for \( A \).

Case 1. Let \( D_0 := D \cap A_0 = D \cap L_x \in \mathcal{I}|D \). Since \( \mathbb{D}p(A_0) \), we may use the induction hypothesis (5) and see \( \mathbb{P}|D_0 \to \mathbb{P}|A_0 \). We know already \( \mathbb{P}|A \cong \mathbb{P}|A_0 \cup Q_x \). If \( x \notin D \), then \( D = D_0 \), and \( \mathbb{P}|D \to \mathbb{P}|A_0 \) follows. If \( x \in D \), then \( \mathbb{P}|D \cong \mathbb{P}|D_0 \cup Q_x \) where \( Q_x \) is a \( \mathbb{P}|D_0 \)-name. Since \( Q_x \) is Sulsic ccc, \( \mathbb{P}|D \to \mathbb{P}|A \) follows from Lemma 3.

Case 2a. Assume first \( D_0 = D \cap L_x \in \mathcal{I}|D \). So there is \( B_0 \in \mathcal{I}|A \) such that \( D_0 = D \cap B_0 \). Put \( B := B_0 \cup \{x\} \subset A \). Then \( D \subseteq B \) and \( \mathbb{P}|D \to \mathbb{P}|B \). \( \mathbb{P}|A \) where the first \( \to \) is by induction hypothesis (5) (because \( \mathbb{D}p(B) < \mathbb{D}p(A) \)) and the second, by Case 2a of (4) above.

So assume \( D_0 \notin \mathcal{I}|D \). Suppose first that \( x \notin D \). By Case 2a of (4) applied to \( D \) instead of \( A \), \( \mathbb{P}|D \) is the direct limit of the \( \mathbb{P}|E \) where \( E \subseteq D \) with \( E \cap L_x \in \mathcal{I}|D \). Each such \( E \) is of the form \( D \cap B \) where \( B \subseteq A \) and \( B \cap L_x \in \mathcal{I}|A \). Conversely, any \( D \cap B \) is such an \( E \). Using the inductive hypothesis for correctness (7), we see that projections in all diagrams of the form \( (\mathbb{P}|D \cap B, \mathbb{P}|B, \mathbb{P}|D \cap B', \mathbb{P}|B', \mathbb{P}|A) \), where \( B \subseteq B' \subseteq A \) with \( B \cap L_x, B' \cap L_x \in \mathcal{I}|A \). By correctness and the inductive hypothesis for (7), we can apply Lemma 3 and see that \( \mathbb{P}|D \to \mathbb{P}|A \).

Case 2b. If \( D \in \mathcal{I}|D \) for some \( x \in A \), we are done because then \( D \subseteq B \) for some \( B \in \mathcal{I}|A \), and \( \mathbb{P}|D \to \mathbb{P}|B \to \mathbb{P}|A \) by induction hypothesis (5) and Case 2b of (4) above.

So assume \( D \notin \mathcal{I}|D \) for any \( x \in A \). Again, we must be in Case 2 for \( D \) and, as in the last paragraph of Case 2a in (5), we see that \( \mathbb{P}|D \) is the direct limit of \( \mathbb{P}|E \) where \( E \in \mathcal{I}|D \) for some \( x \in A \). Using again Lemma 3, we conclude that \( \mathbb{P}|D \to \mathbb{P}|A \).

(6) \( \mathbb{P}|(A \cap D) \subseteq \mathbb{P}|A \cap \mathbb{P}|D \) is immediate by part (2). So assume \( p \in \mathbb{P}|A \cap \mathbb{P}|D \). Let \( x = \max(\text{dom}(p)) \). There are \( B \in \mathcal{I}|A \) and \( E \in \mathcal{I}|D \) such that \( p|L_x \in \mathbb{P}|B \cap \mathbb{P}|E \) and \( p(x) \) is both an \( \mathbb{P}|B \)-name and a \( \mathbb{P}|E \)-name. (To see the latter simply note that since \( p(x) \) is a name for a real, being an \( \mathbb{P}|B \)-name means that all Boolean values \( \mathbb{p}(x,i) = j \) belong to \( \mathbb{P}|B \) and similarly for \( \mathbb{P}|E \). Hence the Boolean values must belong to \( \mathbb{P}|B \cap \mathbb{P}|E \).) Since \( \mathbb{D}p(B) < \mathbb{D}p(A) \) and \( \mathbb{D}p(E) < \mathbb{D}p(D) \leq \mathbb{D}p(A) \), we may apply the induction hypothesis (6) and get \( \mathbb{P}|(B \cap E) \to \mathbb{P}|B \cap \mathbb{P}|E \). Note that \( B \cap E \in \mathcal{I}|A \cap D \). Therefore \( p \in \mathbb{P}|(A \cap D) \), as required.

(7) Again we split into cases according to (4) for \( A \).

Case 1. \( x = \max(A) \), \( A_0 = A \cap L_x \in \mathcal{I}|A \), and \( \mathbb{P}|A = \mathbb{P}|A_0 \cup Q_x \). If \( x \notin A' \), we get \( A' \subseteq A_0 \) and thus \( \mathbb{P}|A' \to \mathbb{P}|A_0 \) and correctness follows from induction hypothesis (7). Similarly if \( x \notin D \). So we may assume \( x \in A' \cap D = D' \). Let \( A_0' = A' \cap L_x \), \( D_0 = D \cap L_x \), \( D_0' = D' \cap L_x \), apply the induction hypothesis (7) to the diagram \( (\mathbb{P}|D_0', \mathbb{P}|A_0', \mathbb{P}|D_0, \mathbb{P}|A_0) \) and use that \( Q_x \) is correctness-preserving. (This is the only place where this assumption is needed.)

Case 2a. \( x = \max(A) \), \( A_0 = A \cap L_x \notin \mathcal{I}|A \), and \( \mathbb{P}|A \) is the direct limit of the \( \mathbb{P}|B \) where \( B \subseteq A \) and \( B \cap L_x \in \mathcal{I}|A \). Let \( p \in \mathbb{P}|A' \subseteq \mathbb{P}|A \). We need to show that the projections agree, that is, that \( h_{A \cap D}(p) = h_{A \cap D}(p) \).

First assume that \( D_0 = D \cap L_x \in \mathcal{I}|D \). Then, by the discussion in (5) (Case 2a), \( D \subseteq B \) for \( B \) as above and, enlarging \( B \) if necessary, we may assume \( p \in \mathbb{P}|B \). Let \( B' = A' \cap B \). By (6) we know that \( p \in \mathbb{P}|B' \). By \( \mathbb{D}p(B) < \mathbb{D}p(A) \) and induction hypothesis (7), \( h_{A \cap D'}(p) = h_{B \cap D'}(p) = h_{B \cap D}(p) = h_{A \cap D}(p) \), as required. If \( A_0' = A' \cap L_x \in \mathcal{I}|A' \), then, by the symmetry of correctness, the same argument works.

So assume \( D_0 \notin \mathcal{I}|D \) and \( A_0' \notin \mathcal{I}|A' \). By the discussion in (5) (Case 2a), we know that \( \mathbb{P}|D \to \mathbb{P}|A \) (respectively) is the direct limit of \( \mathbb{P}|(D \cap B) \) (resp. \( \mathbb{P}|A \cap B \), resp.) where \( B \) is as above. Again fix such \( B \) such that \( p \in \mathbb{P}|B \). (6) gives us \( p \in \mathbb{P}|(A \cap B) \). Using \( \mathbb{D}p(B) < \mathbb{D}p(A) \) and induction hypothesis (7), we see that \( h_{A \cap B \cap D}(p) = h_{B \cap D}(p) \). Again by induction hypothesis (7), we have that \( h_{B_0 \cap A \cap D}(p) = h_{B \cap D}(p) \) for any \( B \subseteq B_0 \subseteq A \) with \( B_0 \cap L_x \in \mathcal{I}|A \). Since \( \mathbb{P}|A \) and \( \mathbb{P}|D \) are the direct limits of such \( \mathbb{P}|B_0 \) and \( \mathbb{P}|(D \cap B_0) \), respectively, \( h_{A \cap D}(p) = h_{B \cap D}(p) \) follows (see Lemma 3). In case \( D_0' = D' \cap L_x \in \mathcal{I}|D' \), argue as...
in the previous paragraph to see that we may assume $D' \subseteq B$ and thus obtain $h_{A',D'}(p) = h_{A \cap B,D'}(p)$, while if $D'_0 \not\subseteq I_x[D']$, $h_{A',D'}(p) = h_{A \cap B,D' \cap B}(p)$ follows as in the previous sentence. In either case, $h_{A',D'}(p) = h_{AD}(p)$, and we are done.

Case 2b. Depending on whether $D \in I_x[D]$ or $A' \in I_x[A']$ for some $x \in A$, we repeat the previous argument, referring to Case 2b of (5).

While the definition of the iteration along a template looks complicated, clause (4) should be seen as saying that such iterations are recursively built up using the two simple operations of two-step iteration and direct limit – as are standard finite support iterations (fni). Note in this context that an fni is the special case where $L = \mu$ is an ordinal and $I_\alpha = \{ \beta \cup F: \beta \leq \alpha \text{ and } F \in [\alpha]^{<\omega} \}$ for $\alpha < \mu$.

Lemma 23. Assume $(L, \bar{I})$ is a template, and the $Q_x$, $x \in L$, are correctness-preserving Suslin $\sigma$-linked partial orders coded in the ground model, $Q_x = \bigcup_n Q_{x,n}$. Then, for any $A \subseteq L$, $P[A]$ is a ccc p.o.

Proof. We argue in three steps.

Step 1. By induction on $Dp(A)$, we show that given $p \in P[A]$, there is $q \leq_{P[A]} p$ such that for all $x \in \text{dom}(q)$ there are $B \in I_x[A]$ and $n = n_{q,x}$ such that $q|(A \cap I_x) \in P[B]$ and $q|(A \cap I_x) \models_{P[B]} q(x) \in Q_{x,n}$. Indeed, let $p \in P[A]$. Also let $x = \text{max}(\text{dom}(p))$. There is $B \in I_x[A]$ such that $q|(A \cap I_x) \in P[B]$ and $p(x)$ is a $P[B]$-name for a condition in $Q_x$. Thus we may find $r \in P[B]$ and $n \in \omega$ with $r \leq_{P[B]} p|(A \cap I_x)$ and such that $r \models_{P[B]} p(x) \in Q_{x,n}$. Since $Dp(B) < Dp(A)$, there is $q_0 \in P[B]$ with $q_0 \leq_{P[B]} r$ satisfying the induction hypothesis. Let $q \in P[A]$ be such that $\text{dom}(q) = \text{dom}(q_0) \cup \{x\}$, $q|(A \cap I_x) = q_0$ and $q(x) = p(x)$. Then $q$ is as required.

Step 2. Assume $p, q \in P[A]$ are as in Step 1, that is, the $n_{p,x}$ and $n_{q,x}$ exist for all $x \in \text{dom}(p)$ and $x \in \text{dom}(q)$, respectively. Also suppose that $n_{p,x} = n_{q,x}$ for all $x \in \text{dom}(p) \cup \text{dom}(q)$. Then $p$ and $q$ are compatible.

This is proved by building a common extension by recursion on $\text{dom}(p) \cup \text{dom}(q)$. For $x = \text{min}(\text{dom}(p) \cup \text{dom}(q))$, $r_x \in P[(A \cap I_x)]$ is the trivial condition. Assume $r_x \in P[(A \cap I_x)]$ has been produced for some $x \in \text{dom}(p) \cup \text{dom}(q)$. Let $y$ be the successor of $x$ in $\text{dom}(p) \cup \text{dom}(q)$ or let $y = \infty$ if $x = \text{max}(\text{dom}(p) \cup \text{dom}(q))$. In the latter case also let $\infty = L$. If $x \in \text{dom}(p) \setminus \text{dom}(q)$, let $r_y \in P[(A \cap L_y)]$ be such that $\text{dom}(r_y) = \text{dom}(r_x) \cup \{x\}$, $r_y|(A \cap L_x) = r_x$, and $r_y(x) = p(x)$. If $x \in \text{dom}(p) \setminus \text{dom}(q)$, define $r_y$ analogously. If $x \in \text{dom}(p) \cap \text{dom}(q)$, find $r_y|(A \cap I_x) \leq r_x$ and $r_y(x) = p(x)$. This is possible because $n_{p,x} = n_{q,x}$. Letting $r = r_\infty$, we see that $r$ is a common extension of $p$ and $q$.

Step 3. ccc-ness now follows by a straightforward $\Delta$-system argument.

Lemma 24. Let $(L, \bar{I})$ be a template. Also assume the $Q_x$ are as in the previous lemma. Let $A \subseteq L$.

1. If $p \in P[A]$, then there is a countable $C \subseteq A$ such that $p \in P[C]$.

2. If $\dot{f}$ is a $P[A]$-name for a real, then there is a countable $C \subseteq A$ such that $\dot{f}$ is a $P[C]$-name.

Proof. This is proved by a simultaneous induction on $Dp(A)$.

(1) Assume $p \in P[A]$. Let $x = \text{max}(\text{dom}(p))$. There is $B \in I_x[A]$ such that $p|(A \cap I_x) \in P[B]$ and $p(x)$ is a $P[B]$-name. By induction hypothesis (1), there is a countable $C_0 \subseteq B$ such that $p|(A \cap I_x) \in P[C_0]$. By induction hypothesis (2), since $p(x)$ is a name for a real, there is a countable $C_1 \subseteq B$ such that $p(x)$ is a $P[C_1]$-name. Let $C = C_0 \cup C_1 \cup \{x\}$. Then $C$ is countable and $p \in P[C]$.

(2) Assume $\dot{f}$ is a $P[A]$-name. For $i \in \omega$, let $\{p_{n,i}: n \in \omega\}$ be a maximal antichain of conditions deciding $\dot{f}(i)$. This uses the ccc-ness proved in the previous lemma. By part (1), there are countable $C_{n,i} \subseteq A$ such that $p_{n,i} \in P[C_{n,i}]$. Let $C = \bigcup_{n,i} C_{n,i}$. Then $\dot{f}$ is a $P[C]$-name.

Corollary 25. Let $(L, \bar{I})$ be a template. Also assume the $Q_x$ are as in Lemma 23. Then $P[L]$ is the direct limit of the $P[A]$ where $A \subseteq L$ is countable.

Proof. By the previous lemma, $P[L] = \bigcup\{P[A]: A \subseteq L \text{ is countable}\}$. Since the collection of countable subsets of $L$ is directed, $P[L] = \lim\text{dir}\{P[A]: A \subseteq L \text{ is countable}\}$ follows.

An easy consequence of this is for example that the limit of an fni of Suslin ccc partial orders can be represented as the direct limit of its countable fragments\footnote{When iterating along a wellorderer, ccc-ness is preserved, so the $\sigma$-linkedness of Lemma 23 is not needed.} More explicitly, if $\{\bar{Q}_\alpha : \alpha < \mu\}$ is such an iteration, then $P_\mu = \lim\text{dir}\{P_A: A \subseteq \mu \text{ is countable}\}$ where $P_A$ is obtained by only iterating the $\bar{Q}_\alpha$ with $\alpha \in A$.\footnote{When iterating along a wellorderer, ccc-ness is preserved, so the $\sigma$-linkedness of Lemma 23 is not needed.}
2.2 The consistency of \( \mathfrak{d} < \mathfrak{a} \) in ZFC

For showing the consistency of \( \mathfrak{d} < \mathfrak{a} \) in ZFC, the ultrapower argument is replaced by an isomorphism-of-names argument. Recall the following folklore result (see e.g. [171] Proposition 3.1] for a proof).

**Proposition 26.** Assume CH, and let \( \lambda = \lambda^2 \) be a cardinal. In the forcing extension obtained by adding \( \lambda \) Cohen reals, every mad family has either size \( \aleph_1 \) or size \( \varepsilon = \lambda \).

**STRATEGY.** The point of the proof of Proposition 26 is that using CH and a \( \Delta \)-system argument, if \( N_2 \leq \kappa < \lambda \), and \( \{ A_\alpha : \alpha < \kappa \} \) is a name for an a.d. family, then one can produce another name \( A_\kappa \) isomorphic to \( N_2 \) many \( A_\alpha \) and such that \( A_\kappa \) is a.d. from all \( A_\alpha \). By global homogeneity of Cohen forcing, the isomorphism producing \( A_\kappa \) comes from an automorphism of the whole forcing, but this is more than what is needed. To obtain the consistency of \( \mathfrak{d} < \mathfrak{a} \), it suffices to build a partial order forcing \( b = \mathfrak{d} = \aleph_2 \) and having sufficient local homogeneity to allow for the construction of \( A_\kappa \). This is exactly what the template method achieves. Mad families of size \( \aleph_1 \) are ruled out in this scenario by \( b = \aleph_2 \).

**Lemma 27.** Hechler forcing is a correctness-preserving Suslin ccc forcing notion: assume projections in \( \langle P_i \rangle \) are correct. Then so are projections in \( \langle P_1 \ast \bar{D} \rangle \).

**Proof.** We may assume without loss of generality that \( P_{01} = \{0, 1\} \). Let \( q_1 = (p_1, (s_1, f_1)) \in P_1 \ast \bar{D} \) and fix \( (s, f) \leq h_{01}(q_1) \). We may suppose that \( s \supseteq s_1 \). Then, given any \( t \supseteq s \) such that \( t \) dominates \( f \) on its domain, there is \( p'_1 \leq p_1 \) forcing \( t \supseteq f_1 \) on its domain. (This is so because \( (t, g) \leq (s, f) \) with \( g(|t|, \infty) = f(|t|, \infty) \) is compatible with \( q_1 \).)

Now assume \( q_0 = (p_0, (s_0, f_0)) \in P_0 \ast \bar{D} \) extends \( (s, f) \). Thus \( s_0 \supseteq s \) and \( s_0 \) dominates \( f \) on its domain. By the previous paragraph find \( p'_1 \leq p_1 \) forcing \( s_0 \supseteq f_1 \) on its domain. By correctness, \( p_0 \) and \( p'_1 \) are compatible in \( P_{01} \) and, clearly, the common extension \( p_0 \ast p'_1 \) forces that \( (s_0, f_0) \) and \( (s_1, f_1) \) are compatible. Thus \( (s, f) \leq h_{01}(q_1) \) in \( P_0 \ast \bar{D} \).

We now introduce the template for the proof of the main theorem (Theorem 29).

Let \( \mu \) and \( \lambda \) be cardinals. As usual, \( \lambda^* \) denotes (a disjoint copy of) \( \lambda \) with the reverse ordering. Elements of \( \lambda \) will be called positive, and members of \( \lambda^* \) are negative. Choose a partition \( \lambda^* = \bigcup_{\alpha < \omega} S^\alpha \) such that each \( S^\alpha \) is coinitial in \( \lambda^* \). Define \( L = L(\mu, \lambda) \) as follows. Elements of \( L \) are non-empty finite sequences \( x \) such that \( x(0) \in \mu \) and \( x(n) \in \lambda^* \cup \lambda \) for \( n > 0 \). The order is naturally given by \( x < y \) if

- either \( x \subseteq y \) and \( y(|x|) \in \lambda \),
- or \( y \subset x \) and \( x(|y|) \in \lambda^* \),
- or \( x(0) < y(0) \),
- or, letting \( n := \min\{|m : x(m) \neq y(m)\} > 0 \), \( x(n) < y(n) \) in the natural ordering of \( \lambda^* \cup \lambda \).

It is immediate that this is indeed a linear ordering. We identify sequences of length one with their range so that \( \mu \subseteq L \) is cofinal. Say \( x \in L \) is relevant if \( |x| \geq 3 \) is odd, \( x(n) \) is negative for odd \( n \) and positive for even \( n \), \( x(|x| - 1) < \omega_1 \), and whenever \( n < m \) are even such that \( x(n), x(m) < \omega_1 \), then there are \( \beta < \alpha \) such that \( x(n - 1) \in S^\alpha \) and \( x(m - 1) \in S^\beta \). For relevant\( x \), set \( J_x = [x(|x| - 1), x) \), the interval of nodes between \( x(|x| - 1) \) and \( x \) in the order of \( L \). Notice that if \( x < y \) are relevant, then either \( J_x \cap J_y = 0 \) or \( J_x \subseteq J_y \) (in which case we also have \( |y| \leq |x|, x(|y| - 1) = y(|y| - 1) \) and \( x(|y| - 1) \leq y(|y| - 1) \)).

For \( x \in L \), let \( \mathcal{I}_x \) consist of finite unions of \( L_\alpha \), where \( \alpha \leq x \) and \( \alpha \in \mu \), of \( J_y \), where \( y \leq x \) is relevant, and of singletons.

**Lemma 28.** \( (L, \bar{I} = \{I_x : x \in L\}) \) is a template.

**Proof.** By definition \( I_x \) contains singletons, is closed under unions, and \( I_x \subseteq I_y \) for \( x \leq y \). Closure under intersections follows easily from the discussion immediately preceding the definition of \( I_x \). Hence it suffices to show that \( \mathcal{I} := \bigcup_{x \in L} I_x \cup \{L\} \) is wellfounded.

Assume \( A_n \), \( n \in \omega \), is a decreasing chain from \( \mathcal{I} \). Let \( \alpha_n \) be such that \( L_{\alpha_n} \) occurs in \( A_n \) as a component. The \( \alpha_n \) must be decreasing and therefore eventually constant. This means it suffices to consider the \( J_x \) components of the \( A_n \) and we may as well assume without loss of generality that \( A_0 = J_x \), and that there is a finitely branching tree \( T \subseteq \omega^{<\omega} \) such that \( A_n = \bigcup_{\sigma \in T \cap \omega} J_{\sigma^*} \cup F_n \) where the \( F_n \subseteq L \) are finite, such that \( \sigma \subseteq \tau \) implies \( J_{\sigma^*} \subseteq J_{\sigma^*} \), and such that the \( J_{\sigma^*} \), \( \sigma \in T \cap \omega^n \), are pairwise disjoint. Now note that if \( f \in |T| \) is a branch, then
the sequence \( \{x^f \cap \alpha \cap \omega \} \) must eventually stabilize. Indeed, if \( |x^f| \to \infty \), then \( \{\alpha : x^f|n| - 2 \} \in S^\alpha \) for some \( n \) would constitute a decreasing sequence of ordinals, by the definition of “relevant”, a contradiction. Therefore \( |x^f| \) is eventually constant. But then the decreasing sequence \( x^f|n| - 1 \) must be eventually constant as well, and so must be \( x^f \). Since \( T \) is a finitely branching tree this means that the total number of \( x^\omega \) is finite which in turn implies that the sequence of the \( A_n \) eventually stabilizes.

Note that, ordered by inclusion, \( L \) is a tree of countable height. Countable subtrees \( A, B \subseteq L \) are called isomorphic if there is a bijection \( \varphi = \varphi_{A,B} : A \to B \) such that for all \( x, y \in A \) and all \( n \in \omega \),

- \( |\varphi(x)| = |x| \),
- \( \varphi(x)|n = \varphi(x)|n \),
- \( x < y \) iff \( \varphi(x) < \varphi(y) \),
- \( x(n) \) is positive iff \( \varphi(x)(n) \) is positive,
- \( Q_x = Q_{\varphi(x)} \), and
- \( \varphi \) maps \( I \upharpoonright A \) to \( I \upharpoonright B \).

Since the trace of \( I \) on any countable set is countable, there are at most \( c \) many isomorphism types of trees. Note that, in view of the last two clauses, if \( A \) and \( B \) are isomorphic, then so are \( P|A \) and \( P|B \), for the partial order only depends on the structure of the template and on the iterands. If only the first four clauses hold, we call the trees weakly isomorphic.

Theorem 29 (Shelah [34]). Assume CH. Let \( \lambda > \mu > \aleph_0 \) be regular cardinals with \( \lambda^\omega = \lambda \). Then there is a ccc forcing extension satisfying \( b = \emptyset = \mu \) and \( n = \aleph_0 = \lambda \).

Proof. Take the template \( (L, \bar{I}) \) introduced above. Let \( \bar{P} = \bar{P}|I \bar{I} \) be the iteration of Hechler forcing along this template, that is, \( Q_x = \bar{P} \) for all \( x \in L \) in Definition and Theorem 22. Using the description of \( \bar{P}|A \) as a two-step iteration or direct limit in \( \bar{P}|I \bar{I} \) of the latter, it is easy to prove by induction on \( D \), for \( A \subseteq L \), that \( \bar{P}|A \) has size \( |A|^\omega \) and that there are \( |A|^\omega \) many \( \bar{P}|A \)-names for reals. Thus \( |\bar{P}| = \lambda^\omega = \lambda \) and \( \bar{P} \) forces \( \lambda \leq \mu \).

Also, letting \( \bar{d}_\alpha, \alpha < \mu \), be the \( \bar{P} \)-name of the Hechler generic added at stage \( \alpha \), we see that the \( \bar{d}_\alpha \) form a scale of length \( \mu \). Indeed, if \( \alpha < \beta \), then since \( \bar{d}_\beta \) is generic over \( \bar{P}|L_\beta \) and \( \alpha < \beta \), \( \bar{d}_\beta \) dominates \( \bar{d}_\alpha \). Furthermore, if \( \bar{x} \) is an arbitrary name for a real, by Lemma 24 there is a countable \( A \subseteq L \) such that \( \bar{x} \) is a \( \bar{P}|A \)-name. Choosing \( \alpha \) such that \( A \subseteq \bar{L}_\alpha \) and recalling \( \bar{L}_\alpha \in I \bar{I} \), we see that \( \bar{d}_\alpha \) dominates \( \bar{x} \). Thus \( b = \emptyset = \mu \) follows.

We are left with showing \( \alpha \leq \mu \). Since \( b \leq a \) in ZFC, we already know \( a \leq \mu \). Thus let \( \bar{A} \) be a name for an almost disjoint family of size \( \lambda = \mu > \alpha \) and \( \sigma \), say \( \bar{A} = \{A^\alpha : \alpha < \kappa \} \) where \( \kappa \geq \omega_2 \cdot 2 \) (the latter choice is for later pruning arguments). By Lemma 23 there are countable \( \bar{B}^\alpha \subseteq L \) such that the \( A^\alpha \) are \( \bar{B}^\alpha \)-names. More explicitly, letting \( \{p_{n,i}^\alpha : n \in \omega \}, \ i \in \omega \), be maximal antichains and \( \{k_{n,i}^\alpha \in \{0,1\} : i, n \in \omega \} \) be such that \( p_{n,i}^\alpha \models i \in \bar{A}^\alpha \) if \( k_{n,i}^\alpha = 1 \) and \( p_{n,i}^\alpha \models i \notin \bar{A}^\alpha \) if \( k_{n,i}^\alpha = 0 \), we have \( \{p_{n,i}^\alpha : i, n \in \omega \} \subseteq \bar{B}^\alpha \). We may also assume all \( B^\alpha \)'s are trees. Letting \( B := \bigcup_{\alpha < \kappa} B^\alpha \) we see that \( |B| < \lambda \). By CH and the \( \Delta \)-system lemma we may also assume that \( \{B^\alpha : \alpha < \omega_2 \} \) forms a \( \Delta \)-system with root \( R \) and that

- \( \varphi_{\alpha,\beta} : B^\alpha \to B^\beta \) is an isomorphism of trees (as defined above) fixing \( R \),
- the induced isomorphism \( \psi_{\alpha,\beta} : \bar{P}|B^\alpha \to \bar{P}|B^\beta \) maps \( p_{n,i}^\alpha \) to \( p_{n,i}^\beta \),
- there are numbers \( k_{n,i}^\alpha \) such that \( k_{n,i}^\alpha = k_{n,i} \) for all \( \alpha < \omega_2 \),
- there is some \( \theta_0 < \omega_1 \) such that whenever \( \alpha < \omega_2 \), \( x \in B^\alpha \), \( j \) odd, and \( x(j) \in \lambda^\alpha \), then \( x(j) \in \lambda^\theta \) for some \( \theta < \theta_0 \).

Note that the second and third clauses immediately imply that \( \psi_{\alpha,\beta} \) also maps the name \( \bar{A}^\alpha \to \bar{A}^\beta \).

For \( \alpha < \omega_2 \), write \( B^\alpha = \{x^\alpha_s : s \in T \} \) where \( T \subseteq (\omega^\alpha \cup \omega) \) is the canonical tree weakly isomorphic to any \( B^\alpha \). This means in particular that \( |s| = |x^\alpha_s| \), that \( s(n) \) is positive iff \( x^\alpha_s(n) \) is positive, and that \( \varphi_{\alpha,\beta}(x^\alpha_s) = x^\beta_s \). Let \( S \subseteq T \) be the subtree corresponding to the root \( R \), that is, \( s \in S \) if \( x^\alpha_s \in R \) for any
\(\alpha < \omega_2\). So, for \(\alpha \neq \beta\), \(x_s^\alpha = x_s^\beta\) iff \(s \in S\). List the immediate successors of \(S\) in \(T\) as \(\{t_n : n \geq 1\}\), i.e., \(\{t_n : n \geq 1\} = \{t \in T \setminus S : t|t-1 \in S\}\). For \(\alpha < \beta < \omega_2\) define

\[
F(\{\alpha, \beta\}) = \begin{cases} 
  n & \text{if either } t_n|t_n - 1 \in \omega_1 \text{ and } x_t^\alpha(t_n - 1) > x_t^\beta(t_n - 1) \\
  or \ t_n(t_n - 1) \in \omega_1 \text{ and } x_t^\alpha(t_n - 1) < x_t^\beta(t_n - 1) \\
  0 & \text{if such } n \text{ exists and is minimal with this property} 
\end{cases}
\]

Note that, by wellfoundedness of the ordinals, for every \(n \geq 1\), any subset of \(\omega_2\) homogeneous in color \(n\) must be finite. Hence, by the Erdős-Rado Theorem, we obtain a subset of size \(\omega_2\) homogeneous in color \(0\) and may as well assume that \(\omega_2\) itself is \(0\)-homogeneous. Using further pruning arguments, we may additionally suppose that if \(s \in S\) and \(\xi, \eta \in \omega_1 \cup \omega_1\) with \(s^\xi, s^\eta \in T \setminus S\) (so \(s^\xi = t_n, s^\eta = t_m\), for some \(n \neq m \geq 1\), then for all \(\alpha < \beta < \omega_1\),

- if \(\xi\) is positive, then \(x_s^{\alpha}(|s|) < x_s^{\beta}(|s|)\), all \(x_s^{\alpha}(|s|)\) are larger than \(\omega_1\), and if \(\xi < \eta\) then
  - either \(x_s^{\beta}(|s|) < x_s^{\alpha}(|s|)\) (this is the case when \(\sup_{\alpha < \omega} x_s^{\alpha}(|s|) < \sup_{\alpha < \omega} x_s^{\beta}(|s|)\)),
  - or \(x_s^{\alpha}(|s|) < x_s^{\beta}(|s|)\) (this is the case when \(\sup_{\alpha < \omega} x_s^{\alpha}(|s|) = \sup_{\alpha < \omega} x_s^{\beta}(|s|)\)),

- if \(\xi\) is negative, then \(x_s^{\alpha}(|s|) > x_s^{\beta}(|s|)\), and if \(\xi > \eta\) then
  - either \(x_s^{\beta}(|s|) > x_s^{\alpha}(|s|)\) (this is the case when \(\inf_{\alpha < \omega} x_s^{\alpha}(|s|) > \inf_{\alpha < \omega} x_s^{\beta}(|s|)\)),
  - or \(x_s^{\alpha}(|s|) > x_s^{\beta}(|s|)\) (this is the case when \(\inf_{\alpha < \omega} x_s^{\alpha}(|s|) = \inf_{\alpha < \omega} x_s^{\beta}(|s|)\)).

Define \(x_s^\alpha \in L\) by recursion on the length of \(s \in T\), as follows. If \(s \in S\), then let \(x_s^\alpha = x_s^\beta\) for any \(\alpha < \omega_1\) (in particular, \(|x_s^\alpha| = |x_s^\beta| = |s|\)). If \(s \in S\) and \(s^\xi \notin S\), we will have \(|x_s^{\alpha}| = |s^\xi| + 2\). First let \(x_s^{\alpha}(|s|)\) be the limit of \(x_s^{\alpha}(n)\) (so it is either the sup or the inf, depending on whether \(\xi\) is positive or negative). Next find \(\gamma < \lambda < \omega_1\) and \(\gamma^* \in S^\gamma, \) such that for all \(s\) and \(\xi\),

- if \(x_s^{\alpha}(|s|) = \sup_{\alpha < \omega} x_s^{\alpha}(|s|)\), then for all \(y \in B\) with \(|y|(|s| + 1) = x_s^{\alpha}(|s|)|(|s| + 1)\), we have \(|y|(|s| + 1) > \gamma^*\),

- if \(x_s^{\alpha}(|s|) = \inf_{\alpha < \omega} x_s^{\alpha}(|s|)\), then for all \(y \in B\) with \(|y|(|s| + 1) = x_s^{\alpha}(|s|)|(|s| + 1)\), we have \(|y|(|s| + 1) < \gamma^*\).

It is clear that such a \(\gamma^*\) exists because \(\lambda > |B|\) is regular. In the first case, let \(x_s^{\alpha}(|s| + 1) = \gamma^*\), and in the second case, \(x_s^{\alpha}(|s| + 1) = \gamma\). To complete the definition of \(x_s^{\alpha}\) define

\[
x_s^{\alpha}(|s| + 2) = \begin{cases} 
  x_s^{\alpha}(|s|) & \text{if } |s| > 0 \\
  \xi + 2n + 1 & \text{if } |s| = 0 \text{ and } \xi = \xi + n \text{ with } \xi \text{ limit} 
\end{cases}
\]

Finally, for the remaining \(t \in T\), stipulate again that \(|x_t^\alpha| = |t| + 2\), find \(s \subseteq t\) with \(s \in S\) maximal, put \(x_t^\alpha(|s| + 3) = x_t^\alpha(|s|)\), and \(x_t^\alpha(j + 2) = x_t^\alpha(j)\) for \(j > |s|\).

Let \(B^\alpha = \{x_s^\alpha : s \in T\}\). Notice that \(B^\alpha\), though very tree-like, is not a tree as the \(B^\alpha\)'s. For \(\alpha < \omega_1\) define \(\varphi_{\alpha, \kappa} : B^\alpha \to B^\kappa\) via \(\varphi_{\alpha, \kappa}(x_s^\alpha) = x_s^{\kappa}\) for \(s \in T\). We proceed to show that \(\varphi_{\alpha, \kappa}\) maps \(T|B^\alpha\) to \(T|B^\kappa\) and that \(\mathbb{P}|B^\alpha\) and \(\mathbb{P}|B^\kappa\) are isomorphic by the induced map \(\psi_{\alpha, \kappa}\). It suffices to consider the case \(\alpha = 0\).

Clearly, \(\varphi = \varphi_{0, \kappa}\) is order-preserving.

First fix \(\beta\) and consider \(L_\beta\). Note that there is \(\beta_0 \leq \beta\) such that \(\varphi(L_{\beta_0} \cap B^0) = L_{\beta_0} \cap B^\kappa = L_\beta \cap B^\kappa\). For any \(s \in T\) with \(x_s^0 \in L_\beta\) yet \(x_s^0 \notin L_\beta\), we must have \(x_s^0(0) - \beta \geq \beta_0\) and \(x_s^0(0) = \sup_{\gamma < \omega} x_s^{\gamma}(0)\). In particular, for all such \(s\), \(x_s^0(0)\) must have the same value, say \(\gamma_0\). Also \(x_s^0(1) = \gamma^*\) and \(x_s^0(2) = \xi + 2n + 1 < \omega_1\) where \(s(0) = \xi + n\) with \(\xi\) limit. If, for some \(s \in T\), \(x_s^0(0) = \beta\), let \(\eta = \xi + 2n + 1\) where \(s(0) = \xi + n\) with \(\xi\) limit. If there is no such \(\eta = \xi + n < \sup \{s(0) + 1 : x_s^0(0) < \beta\}\), \(\xi\) limit, then \(\eta = \xi + 2n\). Then we see that \(L_\beta \cap B^0\) is mapped to \((L_\beta \cup J_\beta) \cap B^\kappa\) via \(\varphi\), where \(|x| = 3, x(0) = \gamma_0, x(1) = \gamma^*\), and \(x(2) = \eta\) (note that this \(x\) is indeed relevant).

Next assume \(x\) is relevant and consider \(J_\beta\). Assume that \(J_\beta \cap B^0 \neq \emptyset\). Then there must be \(s \in T\) such that \(|s| = |x| - 1\) and \(x_s^0 = x(|x| - 1)\). In case \(s \in S\), we have \(x_s^0 = x_s^0\) and \(J_s \cap B^0\) is mapped to \(J_s \cap B^\kappa\) via \(\varphi\) because, by construction, we must have \(y \in R\) for any \(y \in B^0\) with \(|y| = |x|, y(|x| - 1) = x_s^0\) and \(|y| - 1) = x(|x| - 1) < \omega_1\). In case \(s \in T \setminus S\), let \(j_0 < |s|\) be maximal with \(s|j_0 \in S\). Define \(y\) by \(|y| = |x| + 2, y(|x| - 1) = x_s^0\) and \(|y| - 1) = x(|x| - 1) = \omega_1\). And note that \(J_s \cap B^0\) gets mapped to \(J_y \cap B^\kappa\) via \(\varphi\) provided we can show that \(y\) is relevant. In case \(j_0 > 0\), this follows because whenever \(x_s^0(j) > \omega_1\) where \(j \geq j_0\) is even then
also \( x_n^\kappa(j + 2) = x_n^\kappa(j) > \omega_1 \), and, if \( j_0 \) is even, we additionally have \( x_n^\kappa(j_0) = \sup_{\alpha < \omega_1} x_n^\kappa(j_0) > \omega_1 \) while, if \( j_0 \) is odd, we additionally have \( x_n^\kappa(j_0 + 1) = \gamma > \omega_1 \). In case \( j_0 = 0 \) this is true because \( x_n^\kappa(1) \in S^{\theta_0} \) and \( \theta_0 \) is larger than all the \( \theta \) for which \( x_n^\kappa(j) \in S^\theta \) where \( j > 1 \) is odd. We leave it to the reader to verify that similar arguments show that if \( J_\kappa \cap B^\kappa \neq \emptyset \) then there is a relevant \( y \) such that \( \varphi(J_\kappa \cap B^\kappa) = J_\kappa \cap B^\kappa \).

As mentioned already this means that \( \psi_{\alpha, \kappa} \) is an isomorphism of \( \mathbb{P}^A \) and \( \mathbb{P}^B \), and we can define \( A^\kappa \) as the \( \psi_{\alpha, \kappa} \)-image of \( A^\alpha \) (where \( \alpha < \omega_1 \) is arbitrary). More explicitly, \( p_n^\kappa = \psi_{\alpha, \kappa}(p_n^\alpha) \), and \( p_n^\kappa \models i \in A^\kappa \) iff \( k_{n,i} = 1 \) and \( p_n^\kappa \models i \notin A^\kappa \) iff \( k_{n,i} = 0 \).

By construction, it is then also clear that if \( \beta < \kappa \) is arbitrary, we can find \( \alpha < \omega_1 \) such that \( B^\alpha \cap B^\beta \) and \( B^\kappa \cap B^\beta \) are order isomorphic via the mapping \( \varphi' \) fixing nodes of \( B^\beta \) and sending the \( x_n^\alpha \) to the corresponding \( x_n^\kappa \) via \( \varphi_{\alpha, \kappa} \) (in fact, this is true for all but countably many \( \alpha \)). Also \( \varphi' \) maps \( I^B(B^\alpha \cup B^\beta) \) to \( I^B(B^\kappa \cup B^\beta) \). Thus the induced map \( \psi' : \mathbb{P}^B(B^\alpha \cup B^\beta) \to \mathbb{P}^B(B^\kappa \cup B^\beta) \) is an isomorphism fixing the name \( A^\beta \) and mapping the name \( A^\alpha \) to the name \( A^\kappa \). Since \( \mathbb{P}^B(B^\alpha \cup B^\beta) \) forces that \( A^\alpha \cap A^\beta \) is finite, \( \mathbb{P}^B(B^\kappa \cup B^\beta) \) forces that \( A^\kappa \cap A^\beta \) is finite. As this is true for any \( \beta, A \) is not maximal, and the proof is complete.

Note that the template framework is in a sense more general than the framework of Subsection 1.2 for the proof of Theorem 14 in Subsection 1.3 can also be done using templates. See [Br2, Section 4] for details. For example: 

2.3 Further results

By modifying the template of Theorem 29 Shelah also proved \( a \) may be a singular cardinal of uncountable cofinality.

**Theorem 30** (Shelah [Sh4]). Assume GCH. Let \( \mu > \aleph_1 \) be regular and \( \lambda > \mu \) singular of uncountable cofinality. Then there is a ccc forcing extension satisfying \( b = \emptyset = \mu \) and \( a = \kappa = \lambda \).

Embedding Hechler’s forcing for adding a mad family of size \( \aleph_\omega \) into the template framework, the author obtained a model in which \( a = \aleph_\omega \) has countable cofinality.

**Theorem 31** (Brendle [Br2]). Assume CH and let \( \lambda \) be a singular cardinal of countable cofinality. Then there is a ccc forcing extension satisfying \( a = \lambda \). In particular, \( a = \aleph_\omega \) is consistent.

A subgroup \( G \) of Sym(\( \omega \)) is called cofinitary if any non-identity member of \( G \) fixes only finitely many numbers. The cardinal invariant \( \alpha \) is the minimal size of a maximal cofinitary group, is a relative of \( a \), and similar results about it can be proved with the same techniques. For example:

**Theorem 32** (Fischer and Törnquist [FT]). Assume CH and let \( \lambda \) be a singular cardinal of countable cofinality. Then there is a ccc forcing extension satisfying \( a = \lambda \). In particular, \( a = \aleph_\omega \) is consistent.

More recently, the template technique has been used to obtain several consistency results about the 1/2-independence number \( i_{1/2} \) in [BHKLS] (see there for a definition), e.g.:

**Theorem 33** (Brendle, Halbeisen, Klausner, Lischka, and Shelah [BHKLS]). Assume CH and let \( \lambda \) be a singular cardinal of countable cofinality. Then there is a ccc forcing extension satisfying \( i_{1/2} = \lambda \). In particular, \( i_{1/2} = \aleph_\omega \) is consistent.

The interest of these three results stems from the fact that for most cardinal invariants of the continuum, it is known that they may have uncountable cofinality, and apart from the above and some more variations of \( a \), the only cardinal which is known to consistently have countable cofinality is \( \text{cov}(\mathcal{N}) \) [Sh3]. It is open whether the independence number \( i \) or the reaping number \( r \) (see [B1] for definitions) can have countable cofinality.

Replacing Hechler forcing in the template framework by other Suslin ccc forcings, one obtains a number of related consistency results about the order relationship of cardinal invariants of the continuum. See [Br2, Section 4] for details. For example:

**Theorem 34** (Brendle [Br2]). Assume CH. Let \( \lambda > \mu > \aleph_1 \) be regular cardinals with \( \lambda^\omega = \lambda \). Then there is a ccc forcing extension satisfying \( \text{add}(\mathcal{N}) = \text{cof}(\mathcal{N}) = \mu \) and \( a = \kappa = \lambda \).

In all the template models discussed so far \( s = \aleph_1 \) (this is so because iterations of Suslin ccc forcing notions keep \( s \) small, see [B1, Theorem 3.6.21]), and the question arose as to whether one could also increase \( s \) in the template framework. Incorporating the ultrafilter construction from Section 1 [Mejía 2018] introduced iterations of non-definable ccc partial orders along templates and proved:
Theorem 35 (Mejía [Me]). Assume GCH and let $\theta < \kappa < \mu < \lambda$ be uncountable regular cardinals with $\kappa$ measurable. Then there is a ccc p.o. forcing $s = \theta$, $b = d = \mu$, and $a = c = \lambda$.

The large cardinal assumption in fact can be removed:

Theorem 36 (Fischer and Mejía [FM]). Assume GCH and let $\theta < \mu < \lambda$ be uncountable regular cardinals. Then there is a ccc p.o. forcing $s = \theta$, $b = d = \mu$, and $a = c = \lambda$.

As remarked in Subsection 1.4 (see Theorem 15), Shelah also used the technique of iterating ultrapowers of ccc forcing notions to obtain the consistency of $\aleph_1 < u < a$, assuming the consistency of a measurable cardinal. It is not known whether this can be done in ZFC alone. The problem is that while non-definable $B$ cannot be incorporated into the matrix-like framework discussed in Subsection 1.2, it is not clear how to do this with the more complex template framework in Subsection 2.1. However, using a countable support iteration of proper forcing, Guzmán and Kalajdzievski [GK] recently proved the consistency of $u = \aleph_1 < a = c = \aleph_2$. On the other hand, whether $d = \aleph_1 < a$ is consistent is a famous old open problem of Roitman’s from the seventies.

3 Boolean ultrapowers

Assume $P$ is a ccc partial order, $\kappa$ is a strongly compact cardinal, and $B$ is a $\kappa^+$-cc and $\kappa$-distributive cBa. Given a $\kappa$-complete ultrafilter $D$ on $B$ we may form the Boolean ultrapower $\text{Ult}_D(P, B)$. This is again a ccc partial order, $P$ completely embeds into $\text{Ult}_D(P, B)$, and much of the basic theory is very similar to the ultrapowers of Section 1 (see Subsection 3.1). In particular Boolean ultrapowers of iterations are again iterations. Since there is considerable freedom in selecting both the cBa $B$ and the ultrafilter $D$, this method turns out to be more powerful and there is a lot of control as to what can be achieved by just taking the Boolean ultrapower once. Accordingly, all results obtained with this method (Theorems 50 through 56) are obtained by finitely many Boolean ultrapowers, and sophisticated limit constructions as in Subsection 1.2 become unnecessary.

In Subsection 3.3 we present a proof of the result, due to Goldstern, Kellner, and Shelah [GKS], saying that $\aleph_1 < u < a < c = \aleph_2$ is consistent. Further results using Boolean ultrapowers can be found in Subsection 3.4.

3.1 Boolean ultrapowers of partial orders

Assume $\kappa$ is a strongly compact cardinal. Let $B$ be a $\kappa^+$-cc and $\kappa$-distributive cBa. Then every $\kappa$-complete filter on $B$ can be extended to a $\kappa$-complete ultrafilter. Let $D$ be a $\kappa$-complete ultrafilter on $B$. For a p.o. $P$ define

$$F = F(P, B) = \{ f : \text{dom}(f) \text{ is a maximal antichain in } B, \text{ran}(f) \subseteq P \}$$

For $f, g \in F$, the Boolean value of $f = g$ is defined by

$$[f = g] = \bigvee \{ b \in B : \exists a_f \in \text{dom}(f), a_g \in \text{dom}(g) \ (b \leq a_f, a_g \text{ and } f(a_f) = g(a_g)) \}$$

Similarly we define Boolean values of other statements, e.g. $[f \leq g]$ etc. For $f \in F$,

$$[f] = f/D = \{ g \in F : [f = g] \in D \}$$

is the equivalence class of $f$ modulo $D$. The Boolean ultrapower $\text{Ult}_D(P, B)$ consists of all such equivalence classes. It is partially ordered by $[f] \leq [g]$ iff $[f \leq g] \in D$.

As in the discussion of ultrapowers in Section 1, we identify $p \in P$ with the class $[f]$ of the constant function $f(1) = p$ and think of $P$ as a subset of $\text{Ult}_D(P, B)$.

Lemma 37. If $P$ is $\kappa$-cc then $P \circ \text{Ult}_D(P, B)$.

Proof. Like the proof of Lemma 4.

Lemma 38. If $P$ is $\nu$-cc for some $\nu < \kappa$ then so is $\text{Ult}_D(P, B)$. 


Proof. This is like the proof of Lemma 5 but we provide the argument for the sake of completeness. Let \( f_\gamma \in \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \), \( \gamma < \nu \). By \( \kappa \)-distribution of \( \mathcal{B} \), the maximal antichains \( \text{dom}(f_\gamma) \) have a common refinement \( A \) and we may assume \( \text{dom}(f_\gamma) = A \) for all \( \gamma < \nu \). By the \( \nu \)-cc of \( \mathbb{P} \), for all \( a \in A \) there are \( \gamma < \delta < \nu \) such that \( f_\gamma(a) \) and \( f_\delta(a) \) are compatible. By \( \kappa \)-completeness of \( \mathcal{D} \), there are \( \gamma < \delta \) such that \( \bigvee \{ a \in A : f_\gamma(a) \text{ and } f_\delta(a) \text{ are compatible} \} \) belongs to \( \mathcal{D} \). Thus \( [f_\gamma] \) and \( [f_\delta] \) are compatible, as required.

For the remainder of this section, assume \( \mathbb{P} \) is ccc. Then so is \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \) and \( \mathbb{P} \) completely embeds into \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \). As in Section 1 we obtain a natural description of \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \)-names for reals in terms of \( \mathbb{P} \)-names for reals. Let \( A \) be a maximal antichain in \( \mathbb{B} \), and let \( \{ p^a_n : n \in \omega \}, a \in A \), be \( |A| \) many maximal antichains in \( \mathbb{P} \). Defining \( f_n : A \to \mathbb{P} \) by \( f_n(a) = p^n_a \) for \( a \in A \) we obtain a maximal antichain \( \{ [f_n] : n \in \omega \} \) in \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \). Furthermore, by distributivity of \( \mathcal{B} \), all maximal antichains of \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \) are of this form. Next assume we have \( |A| \) many \( \mathbb{P} \)-names \( \dot{x}^a \) for reals in \( \omega^\omega \), \( a \in A \), given by maximal antichains \( \{ p^a_n : n \in \omega \} \) and numbers \( \{ k^a_n : n \in \omega \}, i \in \omega \) and \( a \in A \), such that

\[ p^a_n \|_\mathbb{P} \dot{x}^a(i) = k^a_{n,i}. \]

Then, letting \( f_{n,i}(a) = p^a_{n,i} \) and defining \( k_{n,i} \) to be the unique \( \ell \) such that \( \bigvee \{ a \in A : k^a_{n,i} = \ell \} \in \mathcal{D} \), we obtain an \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \)-name \( \dot{y} \) for a real given by

\[ [f_{n,i}] \|_{\text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B})} \dot{y}(i) = k_{n,i}. \]

This is the average or mean of the \( \dot{x}^a \), and we will usually write \( \dot{y} = \langle \dot{x}^a : a \in A \rangle / \mathcal{D} \). Using again the distributivity of \( \mathcal{B} \) we see that every \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \)-name for a real is of this form.

**Lemma 39.** Assume \( \mathbb{P} \subset \mathbb{Q} \). Then \( \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \subset \text{Ult}_\mathcal{D}(\mathbb{Q}, \mathbb{B}) \).

*Proof.* Like the proof of Lemma 8 \( \square \)

The following result is not needed, but we include it to show that much of the theory can be developed like for ultrapowers (Section 1).

**Lemma 40.** Let \( \mathbb{P} \) be a p.o. and let \( \mathbb{Q} \) be a Suslin ccc forcing notion. Then \( \text{Ult}_\mathcal{D}(\mathbb{P} \star \mathbb{Q}, \mathbb{B}) \cong \text{Ult}_\mathcal{D}(\mathbb{P}, \mathbb{B}) \star \mathbb{Q} \).

*Proof.* Like the proof of Lemma 9 \( \square \)

**Lemma 41.** Let \( \mathcal{P} = \langle \mathcal{P}_\gamma : \gamma \leq \mu \rangle \) be an iteration. Then \( \text{Ult}_\mathcal{D}(\mathcal{P}, \mathbb{B}) = \langle \text{Ult}_\mathcal{D}(\mathcal{P}_\gamma, \mathbb{B}) : \gamma \leq \mu \rangle \) also is an iteration. Moreover, if \( \mathcal{P} \) has finite supports then so does \( \text{Ult}_\mathcal{D}(\mathcal{P}, \mathbb{B}) \).

*Proof.* Like the proof of Lemma 10 \( \square \)

In the main result of this section (Theorem 50), we will apply the Boolean ultrapower operation (finitely often) to an iteration. By the previous lemma, the result is again an iteration, though this is not really relevant for us.

### 3.2 The properties COB and EUB

We introduce and present the basic properties of two principles, COB and EUB, which are important for preservation of cardinal invariants. They will be used again in Section 4.

Suppose we have a binary Borel relation \( R \) on the Baire space \( \omega^\omega \) (or the Cantor space \( 2^\omega \)) such that

- for all \( x \in \omega^\omega \) there is \( y \in \omega^\omega \) with \( xRy \),
- for all \( y \in \omega^\omega \) there is \( x \in \omega^\omega \) with \( \neg(xRy) \).

If \( xRy \) holds, we say that \( y \) \( R \)-dominates \( x \), and if \( \neg(xRy) \), \( x \) is \( R \)-unbounded over \( y \). We associate two cardinals with this relation \( R \), the unbounding number

\[ b(R) := \min\{|F| : F \subseteq \omega^\omega \text{ is not } R\text{-dominated by a single } y \in \omega^\omega\} \]

and the dominating number

\[ d(R) := \min\{|F| : F \subseteq \omega^\omega \text{ and all } x \in \omega^\omega \text{ are } R\text{-dominated by a member of } F\} \]
A typical example is when \( \mathbb{R} = \mathbb{R}^* \), the eventual domination ordering: say \( x \leq_n y \) if for all \( k \geq n \), \( x(k) \leq y(k) \) holds. \( \leq = \bigcup_n \leq_n \), and \( b(\leq^*) = b(\{\leq^*\}) = 0 \), respectively is the usual unbounding (dominating, resp.) number. We shall see more examples shortly.

Given such a relation \( R \), a ccc partial order \( \mathbb{P} \), and cardinals \( \lambda \leq \nu \) with \( \lambda \) regular, we say \( \mathbb{P} \) forces a \( \lambda \)-directed \( R \)-cone of bounds of size \( \nu \), \( \text{COB}(R, \mathbb{P}, \lambda, \nu) \) in symbols, if there are a \( \lambda \)-directed partial order \( \langle S, \leq \rangle \) of size \( \nu \) and \( \mathbb{P} \)-names \( \langle \dot{z}_s : s \in S \rangle \) for reals such that for every \( \mathbb{P} \)-name \( \dot{y} \) for a real there is \( s \in S \) such that for all \( t \geq s \),

\[ \forces \dot{y} R \dot{z}_t \]

The connection between \( \text{COB} \) and the values of the cardinals \( b(R) \) and \( d(R) \) in the forcing extension is given by:

**Lemma 42.** \( \text{COB}(R, \mathbb{P}, \lambda, \nu) \) implies that \( b(R) \geq \lambda \) and \( d(R) \leq \nu \) in the \( \mathbb{P} \)-generic extension.

**Proof.** Clearly the \( \langle \dot{z}_s : s \in S \rangle \) form a witness for \( d(R) \) in the extension. On the other hand, by \( \lambda \)-directedness, any \( F \subseteq \omega^\omega \) of size \( \lambda \) will be bounded. \( \square \)

We next discuss the relationship between \( \text{COB} \) for a partial order and its ultrapower.

**Lemma 43.** Assume \( \nu \leq \lambda \) are regular and \( \text{COB}(R, \mathbb{P}, \lambda, \nu) \).

1. If \( \kappa < \lambda \) or \( \nu < \kappa \), then \( \text{COB}(R, \text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B}), \lambda, \nu) \).

2. If \( \lambda < \kappa \) and \( \kappa \leq \nu \), then \( \text{COB}(R, \text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B}), \lambda, \max(\nu, \mu)^\mathbf{\omega}) \) where \( \mu = |\mathcal{B}| \).

**Proof.** (1) This is similar to Lemma 7 but we sketch the argument. Let \( \langle \dot{z}_s : s \in S \rangle \) be \( \lambda \)-directed and cofinal of size \( \nu \) in the \( \mathbb{P} \)-generic extension. We show this property is preserved in the \( \text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B}) \)-generic extension. To see this, let \( \dot{y} = \langle \dot{x}_a : a \in A \rangle / \mathcal{D} \) be a \( \text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B}) \)-name for a real in \( \omega^\omega \). For each \( a \in A \), find \( s(a) \in S \) such that for all \( t \geq s(a) \),

\[ \forces \dot{y} R \dot{x}_a \dot{z}_t \]

If \( \kappa < \lambda \), directness of \( S \) gives us \( s \in S \) bigger than all \( s(a) \). If \( \nu < \kappa \), from the completeness of \( \mathcal{D} \) we obtain \( s \in S \) such that \( \forall a \in A : s(a) = s \in \mathcal{D} \). In either case, for all \( t \geq s \),

\[ \forces_{\text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B})} \dot{y} R \dot{x}_a \dot{z}_t \]

follows easily.

(2) Again let \( \langle \dot{z}_s : s \in S \rangle \) be \( \lambda \)-directed and cofinal of size \( \nu \) in the \( \mathbb{P} \)-generic extension. Letting \( U = \text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B}) \) we easily see that \( U \) is \( \lambda \)-directed of size \( \leq \max(\nu, \mu)^\mathbf{\omega} \). For \( [u] \in U \), let \( \dot{y}_{[u]} = \langle \dot{z}_{u(a)} : a \in A \rangle / \mathcal{D} \) where \( A \subseteq \mathcal{B} \) is a maximal antichain and \( u : A \to S \). We claim that \( \langle \dot{y}_{[u]} : [u] \in U \rangle \) is cofinal in the \( \text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B}) \)-generic extension. Taking \( \dot{x} = \langle \dot{x}_a : a \in A \rangle / \mathcal{D} \) arbitrarily, there is \( u : A \to S \) such that for all \( a \in A \) and all \( t \geq u(a) \),

\[ \forces_{\text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B})} \dot{x} R \dot{z}_t \]

In particular, if \( [v] \in U \) with \( [v] \geq [u] \) (i.e. \( [v \geq u] \in \mathcal{D} \)), we see that \( \forall a \in A : \forces_{\text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B})} \dot{x} R \dot{z}_{v(a)} \) \( \in \mathcal{D} \) and therefore

\[ \forces_{\text{Ult}_\mathbb{D}(\mathbb{P}, \mathcal{B})} \dot{x} R \dot{y}_{[v]} \]

as required. \( \square \)

Given a Borel relation \( R \), a ccc partial order \( \mathbb{P} \), and a limit ordinal \( \nu \), we say \( \mathbb{P} \) forces an eventually \( R \)-unbounded sequence of length \( \nu \), \( \text{EUB}(R, \mathbb{P}, \nu) \) in symbols, if there are \( \mathbb{P} \)-names \( \langle \dot{x}_\alpha : \alpha < \nu \rangle \) for reals such that for all \( \mathbb{P} \)-names \( \dot{y} \) for reals there is \( \alpha < \nu \) such that for all \( \beta \geq \alpha \),

\[ \forces \neg (\dot{x}_\beta R \dot{y}) \]

(i.e. \( \dot{y} \) does not \( R \)-dominate \( \dot{x}_\beta \)).

Note that for every Borel relation \( R \) on the Baire space, we have the dual relation, \( R^\perp \), given by

\[ x R^\perp y \iff \neg (y Rx) \]

It is well-known and easy to see that \( b(R^\perp) = d(R) \) and \( d(R^\perp) = b(R) \). Using duality we see that \( \text{EUB} \) is a special case of \( \text{COB} \).
Lemma 44. COB\((R^+,\mathbb{P},\nu,\nu)\) and EUB\((R,\mathbb{P},\nu)\) are equivalent.

Proof. Indeed, if \(S\) is \(<\nu\)-directed of size \(\nu\), then \(S\) has a cofinal subset isomorphic to \(\nu\), and we may as well assume \(S = \nu\). Now, COB\((R^+,\mathbb{P},\nu,\nu)\) means that there are \(\mathbb{P}\)-names \((\dot{x}_\alpha : \alpha < \nu)\) for reals such that for every \(\mathbb{P}\)-name \(\dot{y}\) for a real there is \(\alpha < \nu\) such that for all \(\beta \geq \alpha\),

\[\models_{\mathbb{P}} \neg(\dot{x}_\beta \dot{R}\dot{y})\]

which is exactly EUB\((R,\mathbb{P},\nu)\).

Using the earlier results about COB, we infer:

Corollary 45. 1. Assume \(\nu\) is regular and EUB\((R,\mathbb{P},\nu)\). Then \(\mathbb{P}\) forces that \(b(R) \leq \nu\) and \(\mathfrak{d}(R) \geq \nu\).

2. Assume \(\nu\) is regular and EUB\((R,\mathbb{P},\nu)\). If \(\nu \neq \kappa\), then EUB\((R, Ult_\kappa(\mathbb{P},\mathbb{B}),\nu)\).

We end this subsection with a couple of relations which we shall use in the next subsection as well as in Section 4.

- Say a function \(\varphi : \omega \to [\omega]^<\omega\) is a slalom if \(|\varphi(n)| = n\) for all \(n\). The slaloms can be identified with the Baire space. For a slalom \(\varphi\) and \(x \in \omega^\omega\), let

\[xR_1 \varphi\] if for all but finitely many \(n\), \(x(n) \in \varphi(n)\)

It is well-known [BJ, Theorem 2.3.9] that \(b(R_1) = \text{add}(\mathcal{N})\) and \(\mathfrak{d}(R_1) = \text{cof}(\mathcal{N})\).

- Let \(x \in \omega^\omega\). There is a canonical way to associate a null \(G_\delta\) set \(N_x\) with \(x\). More explicitly, let \((U^n_i : i \in \omega)\) list all clopen subsets of \(2^\omega\) of measure \(\leq 2^{-n}\), and put \(N_x = \bigcap_{m} \bigcup_{n \geq m} U^n_x\). For \(x, y \in \omega^\omega\), let

\[xR_2 y\] if \(y \notin N_x\)

Then clearly \(b(R_2) = \text{cov}(\mathcal{N})\) and \(\mathfrak{d}(R_2) = \text{non}(\mathcal{N})\).

- For \(x, y \in \omega^\omega\), let

\[xR_3 y\] if \(x \leq^* y\)

Then clearly \(b(R_3) = b\) and \(\mathfrak{d}(R_3) = d\).

- For \(x, y \in \omega^\omega\), let

\[xR_4 y\] if \(x \neq^* y\) if for all but finitely many \(n\), \(x(n) \neq y(n)\)

It is well-known [BJ, Theorems 2.4.1 and 2.4.7] that \(b(R_4) = \text{non}(\mathcal{M})\) and \(\mathfrak{d}(R_4) = \text{cov}(\mathcal{M})\).

These cardinals can be displayed in Cichoń’s diagram (see [BJ, Chapter 2] or [Bi, Section 5] for details), where cardinals grow as one moves up or right.

\[\begin{align*}
\text{cov}(\mathcal{N}) & \quad b(R_4) = \text{non}(\mathcal{M}) & \quad \text{cof}(\mathcal{M}) & \quad \mathfrak{d}(R_1) = \text{cof}(\mathcal{N}) & \quad \mathfrak{c} \\
\text{b} & \quad \mathfrak{d} \\
\aleph_1 & \quad b(R_1) = \text{add}(\mathcal{N}) & \quad \text{add}(\mathcal{M}) & \quad \mathfrak{d}(R_4) = \text{cov}(\mathcal{M}) & \quad \text{non}(\mathcal{N})
\end{align*}\]

Cichoń’s diagram

The following result forms the basis for the main results of both Subsections 4.3 and 4.2.

Theorem 46 (Goldstern, Mejía, and Shelah [GMS]). Assume GCH, and let \(\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5\) be uncountable regular cardinals. There is a ccc p.o. \(\mathbb{P}^{pre}\), the preparatory forcing, such that for \(1 \leq i \leq 4\),
• $\text{EB}(R_i, \mathbb{P}^{\text{pre}}, \nu)$ for every $\nu$ with $\lambda_i \leq \nu \leq \lambda_5$,
• $\text{COB}(R_i, \mathbb{P}^{\text{pre}}, \lambda_i, \lambda_5)$

In particular, $\mathbb{P}^{\text{pre}}$ forces

$$\text{add}(N) = \lambda_1 \leq \text{cov}(N) = \lambda_2 \leq \text{add}(M) = b = \lambda_3 \leq \text{non}(M) = \lambda_4 \leq \text{cov}(M) = c = \lambda_5$$

Proof Sketch. We first note that the values for the cardinal invariants follow from Lemma 42 and part 1 of Corollary 44.

We sketch the proof for the particular case $\lambda_4 = \lambda_5$ and then make some comments on the general case.

Make a finite support iteration $(\mathbb{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \lambda_5)$ of length $\lambda_5$ of ccc partial orders, going through

1. eventually different reals forcing $E$ cofinally often, as well as through
2. all subforcings of localization forcing of size $< \lambda_1$,
3. all subforcings of random forcing of size $< \lambda_2$, and
4. all subforcings of Hechler forcing of size $< \lambda_3$

using a book-keeping argument. Then $\mathbb{P}^{\text{pre}} = \mathbb{P}_{\lambda_5}$. For definitions of the particular forcing notions and their properties, see [BJ, Chapter 3 and 7.4.B]. Standard arguments show that the family of partial generics added by item $i + 1$ guarantees $\text{COB}(R_i, \mathbb{P}^{\text{pre}}, \lambda_i, \lambda_5)$ for $1 \leq i \leq 3$. For example, the partial Hechler generics form a witness for $\text{COB}(R_3, \mathbb{P}^{\text{pre}}, \lambda_3, \lambda_5)$. Similarly, the eventually different reals witness $\text{COB}(R_4, \mathbb{P}^{\text{pre}}, \lambda_4, \lambda_5)$ (since $\lambda_4 = \lambda_5$).

Fix regular uncountable $\nu \leq \lambda_5$. Let $(\dot{c}_\alpha : \alpha < \nu)$ be the sequence of Cohen reals added in the limit stages of the initial segment $\mathbb{P}_\alpha$ of the iteration. They clearly witness $\text{EB}(R_i, \mathbb{P}_\alpha, \nu)$ for all $i$. If $\lambda_i \leq \nu$, then standard preservation arguments show that $\text{EB}(R_i, \mathbb{P}_\alpha, \nu)$ holds for all $\alpha \geq \nu$, and $\text{EB}(R_i, \mathbb{P}^{\text{pre}}, \nu)$ follows. If $i = 4$ this is trivial (by $\lambda_4 = \lambda_5$). Moreover, preservation for limit ordinals $\alpha$ is a standard argument. For successor ordinals $\alpha$, if $i = 1$, use the fact that all $\mathcal{Q}_\alpha$ either are of size $< \lambda_1$ or carry a finitely additive measure (see [Ka] for why $\sigma$-centered forcings and subforcings of random forcing carry such a measure), and that this preserves $\text{EB}(R_1, \mathbb{P}_\alpha, \nu)$. If $i = 2$, use that all $\mathcal{Q}_\alpha$ are either of size $< \lambda_2$ or $\sigma$-centered and thus preserve $\text{EB}(R_2, \mathbb{P}_\alpha, \nu)$. If $i = 3$, use that all $\mathcal{Q}_\alpha$ either are of size $< \lambda_3$ or are $E$, which preserves $\text{EB}(R_3, \mathbb{P}_\alpha, \nu)$ by a compactness argument (see [Mi]). This completes the argument in the special case.

In case $\lambda_4 < \lambda_5$, one would like to go through all subforcings of eventually different reals forcing of size $< \lambda_4$ instead. It is not clear, however, why this should preserve $\text{EB}(R_3, \mathbb{P}_\alpha, \nu)$ for $\alpha \geq \nu$. For this reason the subforcings of $E$ have to be very carefully chosen in a sophisticated argument, see [GMS] for details.

By this theorem, all cardinal invariants on the left-hand side of Cichoń’s diagram can be separated simultaneously. It is harder to separate the dual cardinals on the right-hand side. We shall present two methods for doing this, in Subsections 3.3 and 4.2 using the fact that we already achieved separation on the left-hand side.

3.3 Compact cardinals and Cichoń’s maximum

Assume $\mu > \kappa$ is a regular cardinal (where $\kappa$ is strongly compact as before). Let $\mathbb{B}$ be the completion of $\text{Fn}(\mu, \kappa, < \kappa)$, forcing with partial functions from $\mu$ to $\kappa$ of size $< \kappa$. Note that $\mathbb{B}$ is $\kappa^+$-cc (because $2^{< \kappa} = \kappa$) and $< \kappa$-distributive. Let $A \subseteq \text{Fn}(\mu, \kappa, < \kappa)$ be a maximal antichain and $w : A \to \kappa$. Let $\text{supp}(A) = \bigcup \{ \text{dom}(a) : a \in A \}$, the support of $A$. Clearly $|\text{supp}(A)| \leq \kappa$. If the maximal antichain $A'$ refines $A$ we canonically extend $w$ to $A'$ by letting $w(a') = w(a)$ where $a$ is the unique element of $A$ above $a'$, for $a' \in A'$. If $w : A \to \kappa$ and $w' : A' \to \kappa$ are two such functions we get the Boolean value

$$[w < w'] = \bigvee \{ a'' \in A'' : w(a'') < w'(a'') \}$$

where $A''$ is a common refinement of $A$ and $A'$. For $\delta < \mu$ let $A_\delta$ be the maximal antichain of singleton partial functions $\{ (\delta, \xi) \}$, $\xi < \kappa$, and define $v_\delta : A_\delta \to \kappa$ by $v_\delta((\delta, \xi)) = \xi$.

Lemma 47. The Boolean values $[v_\delta > w]$ with $\delta > \text{sup}(\text{supp}(\text{dom}(w)))$ form a $\kappa$-complete filter on $\mathbb{B}$ and therefore can be extended to a $\kappa$-complete ultrafilter $\mathcal{D}$.

---

This special case has been known at least since the 90’s.
Proof. Let \( \nu < \kappa \) and \((w_\xi, \delta_\xi), \xi < \nu, \) be pairs such that \( \delta_\xi > \sup(\text{supp}(\text{dom}(w_\xi))) \), and let \( A_\xi = \text{dom}(w_\xi) \). We need to show that \( \bigwedge_{\xi < \nu} [v_\xi > w_\xi] \neq 0 \). Enumerate \( \{\delta_\xi : \xi < \nu\} \) in increasing order and without repetitions as \( \{\delta^\xi : \xi < \gamma\} \) for some \( \gamma \leq \nu \). Let \( C^\xi = \{\xi : \delta_\xi = \delta^\xi\} \). Construct a decreasing chain \( \{q^\xi : \xi < \gamma\} \) of conditions in \( \text{Fn}(\mu, \kappa, \nu, \kappa) \) as follows. \( q^0 \) is the trivial condition and for limit \( \xi, q^\xi \) is the union of the \( q^\eta, \eta < \xi \). Assume \( q^\xi \) has been constructed such that \( \text{dom}(q^\xi) \subseteq \delta^\xi \) and let \( q^\xi+1 \) be an extension such that \( \text{dom}(q^\xi+1) \subseteq \delta^\xi + 1 \). \( q^\xi+1|\delta^\xi \) extends an element \( a_\xi \in A_\xi \) for each \( \xi \in C^\xi \) and \( q^\xi+1(\delta^\xi) = \sup\{w_\xi(a_\xi) : a_\xi \in C^\xi\} + 1 \). Clearly \( q^\gamma \) is an extension of \( \bigwedge_{\xi < \nu} [v_\xi > w_\xi] \).

By strong compactness of \( \kappa \), we can now extend this filter base to a \( \kappa \)-complete ultrafilter on \( B \). \( \square \)

We assume from now on that \( D \) is constructed as in this lemma.

**Lemma 48.** Assume \( \text{EUB}(R, \mathbb{P}, \kappa) \). Then \( \text{EUB}(R, \text{Ult}_D(\mathbb{P}, B), \mu) \).

Proof. This is like the proof of Lemma 43 but we additionally need to use the special property of the ultrafilter \( D \) given by Lemma 47. Let \( (\dot{x}_a : a < \kappa) \) be the eventually unbounded sequence forced by \( \mathbb{P} \). For \( \delta < \mu \) let \( \dot{y}_\delta = (\dot{x}._{v_\delta(a)} : a \in A_\delta) / D \). We claim that \( \text{Ult}_{D}(\mathbb{P}, B) \) forces \( (\dot{y}_\delta : \delta < \mu) \) is an eventually unbounded sequence. Indeed, let \( \dot{z} = (\dot{z}_a : a \in A) / D \) be a \( \text{Ult}_{D}(\mathbb{P}, B) \)-name for a real in \( \omega^\omega \) and let \( \delta > \text{sup}(A) \). There is a function \( w : A \rightarrow \kappa \) such that for all \( a \in A \) and all \( \beta \geq w(a) \),

\[ \Vdash_{p} \neg \dot{x}_\beta R \dot{z}_a \]

By Lemma 47, we know that \( [v_\delta > w] \in D \). A fortiori \( \bigvee \{a \in A' : \Vdash_{p} \neg \dot{x}_{v_\delta(a)} R \dot{z}_a\} \in D \) where \( A' \) is a common refinement of \( A_\delta \) and \( A \) and therefore,

\[ \Vdash_{\text{Ult}_{D}(\mathbb{P}, B)} \neg \dot{y}_\delta R \dot{z} \]

as required. \( \square \)

\( \text{EUB} \) and \( \text{COB} \) are tools to compute the cardinal invariants \( b(R) \) and \( d(R) \) in \( \text{Ult}_{D}(\mathbb{P}, B) \)-generic extensions.

**Corollary 49.** Let \( \lambda \leq \nu \) be regular and assume \( \text{EUB}(R, \mathbb{P}, \lambda) \) and \( \text{COB}(R, \mathbb{P}, \lambda, \nu) \) hold.

1. If \( \lambda < \kappa \leq \nu \leq \mu = \mu^+ \) and, additionally, \( \text{EUB}(R, \mathbb{P}, \kappa) \) holds, then \( \text{EUB}(R, \text{Ult}_D(\mathbb{P}, B), \lambda), \text{EUB}(R, \text{Ult}_D(\mathbb{P}, B), \mu), \) and \( \text{COB}(R, \text{Ult}_D(\mathbb{P}, B), \lambda, \mu) \), and therefore \( b(R) = \lambda \) and \( d(R) = \mu \) in the \( \text{Ult}_{D}(\mathbb{P}, B) \)-generic extension.

2. If \( \kappa < \lambda \), and, additionally, \( \text{EUB}(R, \mathbb{P}, \nu) \) holds, then these properties are preserved in \( \text{Ult}_{D}(\mathbb{P}, B) \), and \( b(R) = \lambda \) and \( d(R) = \nu \) in the \( \text{Ult}_{D}(\mathbb{P}, B) \)-generic extension.

Proof. This is immediate by Lemmas 42, 43 and Corollary 45. \( \square \)

**STRATEGY.** Assume \( \mathbb{P} \) forces \( b(R) < d(R) \) as witnessed by \( \text{EUB} \) and \( \text{COB} \). For the case the strongly compact cardinal \( \kappa \) of the ground model lies between these two values, the first part of this corollary describes a method for further increasing \( d(R) \) while keeping the value of \( b(R) \) by taking the Boolean ultrapower of \( \mathbb{P} \). For the case \( \kappa \) is below the smallest cardinal, the Boolean ultrapower will preserve both cardinals by the second part of the corollary. This provides us with a scenario for obtaining a model in which all cardinals in Cichoń’s diagram are distinct. Namely, force the left-hand cardinals to be distinct, with strongly compact cardinals in between them, and then keep stretching the right-hand cardinals while keeping the ones on the left, by repeatedly taking Boolean ultrapowers.

**Theorem 50** (Goldstern, Kellner, and Shelah [GKS]). Assume the existence of four strongly compact cardinals is consistent. Then so is the statement that all cardinals in Cichoń’s diagram are distinct. More explicitly, assume \( \text{GCH} \) and let \( \kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 < \kappa_5 < \kappa_6 < \kappa_7 < \kappa_8 < \kappa_9 \) be regular cardinals such that the \( \kappa_i \) are strongly compact. Then there is a \( \text{ccc} \) p.o. \( \mathbb{P} \) forcing

\[ \text{add}(N) = \lambda_1 < \text{cov}(N) = \lambda_2 < \text{add}(M) = b = \lambda_3 < \text{non}(N) = \lambda_4 < \text{cov}(M) = \lambda_5 < d = \text{cof}(M) = \lambda_6 < \text{non}(N) = \lambda_7 < \text{cof}(N) = \lambda_8 < c = \lambda_9 \]

Proof. Assume \( B_j \) is the completion of \( \text{Fn}(\lambda_j, \kappa_j, \kappa_j) \) for \( 6 \leq j \leq 9 \), and let \( D_j \) be the \( \kappa_j \)-complete ultrafilter on \( B_j \) obtained from Lemma 47.

Let \( \mathbb{P}^j := \mathbb{P}^j \cap \mathbb{P} \) be the ccc partial order from Theorem 46. Next let \( \mathbb{P}^j = \text{Ult}_{D_j}(\mathbb{P}^{j-1}, B_j) \) for \( 6 \leq j \leq 9 \). We claim that \( \mathbb{P} := \mathbb{P}^9 \) is as required by the theorem. Since the proof is the same for the four relations \( R_i \), \( 1 \leq i \leq 4 \), we do it for \( i = 3 \), that is, we show that \( \mathbb{P}^9 \) forces \( b = b(R_3) = \lambda_3 \) and \( d = d(R_3) = \lambda_6 \). First, using part 2 of Lemma 43 part 2 of Corollary 45 and Lemma 48, we obtain
• \( \text{COB}(R_3, P^6, \lambda_3, \lambda_6) \)
• \( \text{EUB}(R_3, P^6, \lambda_3) \) and \( \text{EUB}(R_3, P^6, \lambda_6) \)

Then, using part 1 of Lemma 43 and part 2 of Corollary 45, we get

• \( \text{COB}(R_3, P^j, \lambda_3, \lambda_6) \)
• \( \text{EUB}(R_3, P^j, \lambda_3) \) and \( \text{EUB}(R_3, P^j, \lambda_6) \)

for \( 7 \leq j \leq 9 \). By Lemma 12 and part 1 of Corollary 15 \( b(R_3) = \lambda_3 \) and \( d(R_3) = \lambda_6 \) follow.

With more work resulting in a somewhat different preparatory forcing, the large cardinal assumption can be reduced to three strongly compact cardinals instead of four, see [BCM]. For a simple proof of a weaker version of Theorem 50 based on the preparatory forcing with \( \lambda_4 = \lambda_5 \) whose proof is sketched above (Theorem 46), using three strongly compact cardinals, and showing the consistency of

\[ \aleph_1 < \text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < b < d < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}) < \mathfrak{c} \]

we refer the reader to [KTT].

### 3.4 Further results

The method of the previous subsection can be used to obtain some other results where many cardinal invariants simultaneously assume distinct values.

**Theorem 51** (Kellner, Shelah, and Tănase [KST]). *Assume GCH and let \( \aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9 \) be regular cardinals such that the \( \kappa_i \) are strongly compact. Then there is a ccc p.o. forcing

\[
\begin{align*}
\text{add}(\mathcal{N}) &= \lambda_1 < \text{add}(\mathcal{M}) = b = \lambda_2 < \text{cov}(\mathcal{N}) = \lambda_3 < \text{non}(\mathcal{M}) = \lambda_4 < \\
&< \text{cov}(\mathcal{M}) = \lambda_5 < \text{non}(\mathcal{N}) = \lambda_6 < d = \text{cof}(\mathcal{M}) = \lambda_7 < \text{cof}(\mathcal{N}) = \lambda_8 < \mathfrak{c} = \lambda_9
\end{align*}
\]

This is based on a different, more involved, preparatory forcing, see also [KST].

Mixing the technique of [BCM] with small partial orders forcing specific values to \( m \) (the smallest cardinal for which Martin’s Axiom fails) and to the pseudointersection, distributivity, and groupwise density numbers, \( p, h, \) and \( g \) (see [B] Section 6) for definitions), in the iteration leading to the preparatory forcing, and then taking ultrapowers, one obtains:

**Theorem 52** (Goldstern, Kellner, Mejía, and Shelah [GKMS3]). *Assume GCH and let \( \aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \kappa_9 < \lambda_4 < \kappa_8 < \lambda_5 < \kappa_7 < \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \lambda_9 \leq \lambda_{10} \leq \lambda_{11} \leq \lambda_{12} \) be cardinals such that the \( \kappa_i \) are strongly compact, the \( \lambda_i \) are regular for \( i \neq 9, 12 \), and \( \text{cf}(\lambda_9) \geq \lambda_6 \) and \( \text{cf}(\lambda_{12}) \geq \lambda_3 \). Then there is a p.o. preserving cofinalities and forcing

\[
\aleph_1 \leq m = \lambda_1 \leq p = \lambda_2 \leq h = g = \lambda_3 < \text{add}(\mathcal{N}) = \lambda_4 < \text{cov}(\mathcal{N}) = \lambda_5 < \text{add}(\mathcal{M}) = b = \lambda_6 \leq \text{non}(\mathcal{M}) = \lambda_7 \leq \\
\leq \text{cov}(\mathcal{M}) = \lambda_8 \leq d = \text{cof}(\mathcal{M}) = \lambda_9 \leq \text{non}(\mathcal{N}) = \lambda_{10} \leq \text{cof}(\mathcal{N}) = \lambda_{11} \leq \mathfrak{c} = \lambda_{12}
\]

Composing this with collapses one gets for example:

**Theorem 53** (Goldstern, Kellner, Mejía, and Shelah [GKMS3]). *Assume GCH and there are three strongly compact cardinals. Then there is a p.o. forcing

\[
\aleph_1 < m = \aleph_2 < p = \aleph_3 < \text{add}(\mathcal{N}) = \aleph_4 < \text{cov}(\mathcal{N}) = \aleph_5 < \text{add}(\mathcal{M}) = b = \aleph_6 < \text{non}(\mathcal{M}) = \aleph_7 < \\
< \text{cov}(\mathcal{M}) = \aleph_8 < d = \text{cof}(\mathcal{M}) = \aleph_9 < \text{non}(\mathcal{N}) = \aleph_{10} < \text{cof}(\mathcal{N}) = \aleph_{11} < \mathfrak{c} = \aleph_{12}
\]

In still unpublished work [RS], Raghavan and Shelah have obtained a number of consistency results about higher cardinal invariants, that is, cardinal invariants describing the higher Baire space \( \lambda^\lambda \) for regular uncountable \( \lambda \) like \( b(\lambda) \) or \( d(\lambda) \), using the Boolean ultrapower technique.
Theorem 54 (Raghavan and Shelah [RS]). For any regular \( \lambda > \omega \), \( \mathfrak{d}(\lambda) < \mathfrak{a}(\lambda) \) is consistent relative to a supercompact cardinal. More specifically, suppose that \( \kappa_0 < \lambda = \lambda^{+} < \kappa \) and that \( \kappa \) is supercompact. Then there is a forcing extension in which \( \kappa < \mathfrak{b}(\lambda) = \mathfrak{d}(\lambda) < \mathfrak{a}(\lambda) \).

For a proof sketch, as in the previous subsection, let \( \mathcal{B} \) be the completion of \( \text{Fn}(\mu, \kappa, < \kappa) \) for some regular \( \mu > \kappa^{+} \). Using the fact that \( \kappa \) is supercompact, one builds a \( \kappa \)-complete optimal ultrafilter \( D \) on \( \mathcal{B} \). Let \( \mathbb{P} \) be the \( \kappa^{+} \)-stage iteration of \( \lambda \)-Hechler forcing with supports of size less than \( \lambda \). This is the canonical p.o. for forcing \( \mathfrak{b}(\lambda) = \mathfrak{d}(\lambda) = \kappa^{+} \). Next, let \( \mathbb{Q} = \text{Ult}(\mathbb{P}, \mathcal{B}) \). Forcing with \( \mathbb{Q} \) preserves \( \mathfrak{b}(\lambda) = \mathfrak{d}(\lambda) = \kappa^{+} \) (this is basically the same argument as the proof of Lemma 7 see also part 2 of Corollary 19 and makes \( \mathfrak{a}(\lambda) = \mu \) (this uses the combinatorial properties of \( D \) and is the core of the argument).

Further results of theirs include:

Theorem 55 (Raghavan and Shelah [RS]). Suppose that \( \kappa_0 < \lambda = \lambda^{+} < \kappa \) and that \( \kappa \) is supercompact. Then there is a forcing extension in which \( \kappa < \mathfrak{b}(\lambda) < \mathfrak{d}(\lambda) < \mathfrak{a}(\lambda) \).

Theorem 56 (Raghavan and Shelah [RS]). Suppose that \( \lambda < \kappa \), that \( \kappa \) is supercompact, and that \( \lambda \) is Laver indestructible supercompact. Then there is a forcing extension in which \( \lambda \) is still supercompact and \( \kappa < \mathfrak{u}(\lambda) < \mathfrak{a}(\lambda) \).

4 Submodels

Assume \( \mathbb{P} \) is a ccc partial order, \( \kappa \) is a regular uncountable cardinal, and \( N \) is a \( \kappa \)-closed elementary substructure of \( H(\chi) \) containing \( \kappa \) and \( \mathbb{P} \), where \( \chi \) is a large enough regular cardinal. Then the restriction of \( \mathbb{P} \) to \( N, \mathbb{P} \cap N \), is again a ccc partial order, and \( \mathbb{P} \cap N \) completely embeds into \( \mathbb{P} \). In fact, if we choose \( N \) sufficiently carefully (typically \( N \) is the union of a chain of submodels), then \( \mathbb{P} \cap N \) reflects the combinatorial properties of \( \mathbb{P} \), albeit with possibly different cardinals as witnesses. We will introduce this method and discuss its effect on COB and EUB in Subsection 4.2.4

In Subsection 4.2 we present a proof of the result, due to Goldstern, Kellner, Mejía, and Shelah [GKMS], saying that on the basis of ZFC it is consistent that all cardinal invariants in Cichoń’s diagram simultaneously assume distinct values (Theorem 61). Further results using the submodel method are presented in Subsection 4.3.

4.1 The submodel method

Assume (for the whole subsection) \( \kappa \) is a regular uncountable cardinal, \( \mathbb{P} \) is a \( \kappa \)-cc partial order, and \( N \preceq H(\chi) \) is \( \kappa \)-closed with \( \mathbb{P}, \mathbb{P} \cap N \), where \( \chi \) is a large enough regular cardinal. Then:

Lemma 57. 1. For every antichain \( A \subseteq \mathbb{P}, A \subseteq N \) if and only if \( A \subseteq N \).

2. \( \mathbb{P} \cap N \) is \( \kappa \)-cc.

3. \( \mathbb{P} \cap N \subsetneq \mathbb{P} \).

Proof. (1) First let \( A \subseteq N \). Since \( |A| < \kappa \leq |N| \) by the assumptions on \( \mathbb{P} \) and \( N \), \( A \subseteq N \) follows. If, on the other hand, \( A \subseteq N \), then by the \( \kappa \)-cc, \( |A| < \kappa \) and by \( \kappa \)-closure \( A \subseteq N \).

(2) Let \( A \subseteq \mathbb{P} \cap N \) be an antichain. By elementarity we see \( A \) is an antichain of \( \mathbb{P} \), and \( |A| < \kappa \) follows.

(3) Let \( A \subseteq \mathbb{P} \cap N \) be a maximal antichain. Again, \( A \) is an antichain of \( \mathbb{P} \), and by (1) \( A \subseteq N \). Clearly, \( N \) thinks that \( A \) is a maximal antichain of \( \mathbb{P} \), and therefore \( A \) is maximal in \( \mathbb{P} \) by elementarity.

By (3), any \( \mathbb{P} \cap N \)-generic filter \( G \) over \( V \) (or, equivalently, \( H(\chi) \)) can be extended to a \( \mathbb{P} \)-generic filter \( G^+ \) over \( V \) \( (H(\chi)) \), respectively. By elementarity \( G^+ \) is \( \mathbb{P} \)-generic over \( N \) as well, and \( N[G^+] \preceq H(\chi)^{V[G^+]} \). Using this we can establish a correspondence between \( \mathbb{P} \cap N \)-names \( \dot{x} \in V \) for reals and \( \mathbb{P} \)-names \( \dot{y} \in N \) for reals such that \( \dot{x}[G] = \dot{y}[G^+] \) and for all \( p \in \mathbb{P} \cap N \) and sufficiently absolute (e.g. Borel) \( \varphi \),

\[ p \vdash \varphi(\dot{y}) \text{ iff } \not\vdash_{\mathbb{P} \cap N} \varphi(\dot{x}) \]

In particular, \( N[G^+] \cap \omega^\omega = V[G] \cap \omega^\omega \).

To see this, recall that a \( \mathbb{P} \cap N \)-name \( \dot{x} \) for a real in \( \omega^\omega \) is given by maximal antichains \( \{p_{n,i} : n \in \lambda_i\} \subseteq \mathbb{P} \cap N \) and numbers \( \{k_{n,i} : n \in \lambda_i\} \), \( i \in \omega \) and \( \lambda_i < \kappa \), such that

\[ p_{n,i} \vdash_{\mathbb{P} \cap N} \dot{x}(i) = k_{n,i} \]
Since \( N \) is \( \kappa \)-closed, \( \{ p_{n,i} : n \in \lambda_i \}, \{ k_{n,i} : n \in \lambda_i \} : i \in \omega \) \( \in N \), and by \( P \cap N \prec P \), \( \hat{x} \) can be construed as a \( P \)-name in \( N \). On the other hand, if \( \hat{x} \in N \) is a \( P \)-name, that is, \( \{ p_{n,i} : n \in \lambda_i \}, \{ k_{n,i} : n \in \lambda_i \} : i \in \omega \) \( \in N \), then by part 1 of Lemma 57 \( \{ p_{n,i} : n \in \lambda_i \} \subseteq N \) for all \( i \) and \( \hat{x} \) is a \( P \cap N \)-name.

We now investigate how \( \text{COB} \) and \( \text{EUB} \) for \( P \cap N \) relate to \( \text{COB} \) and \( \text{EUB} \) for \( P \). For a partial order \( \langle S, \leq \rangle \), let \( \text{comp}(S) \), the completeness of \( S \), be the least \( \lambda \) such that \( S \) is not \( \lambda \)-directed.

**Lemma 58.** 1. Assume \( \text{COB}(R, P, \lambda, \nu) \) as witnessed by the partial order \( \langle S, \leq \rangle \in N \). Then \( \text{COB}(R, P \cap N, \lambda', \nu') \) whenever \( \lambda' \leq \text{comp}(S \cap N) \) and \( \nu' \geq \text{cof}(S \cap N) \).

2. Assume \( \text{EUB}(R, P, \nu) \) with \( \nu \in N \). Then \( \text{EUB}(R, P \cap N, \text{cof}(\nu \cap N)) \).

**Proof.** (1) Assume \( \{ z_s : s \in S \} \) witnesses \( \text{COB}(R, P, \lambda, \nu) \). It suffices to show that \( \{ z_s : s \in S \cap N \} \) witnesses \( \text{COB}(R, P \cap N, \lambda', |S \cap N|) \). For then we can replace \( S \cap N \) by any \( < \lambda' \)-directed superset of a cofinal subset. Let \( \hat{y} \) be a \( P \cap N \)-name for a real. By the previous discussion, we know that \( \hat{y} \in N \) may be construed as a \( P \)-name for a real. Hence, by \( \text{COB}(R, P, \lambda, \nu) \) and elementarity, there is \( s \in S \cap N \) such that, in \( N \), for all \( t \geq s \),

\[ \models \text{EUB(R, P, s)} \]

Again by the previous discussion, this means that for all \( t \geq s \) in \( N \) we have

\[ \models \text{EUB}(R, P \cap N, s) \]

as required.

(2) This follows from (1) and Lemma [44] \( \square \)

**Lemma 59.** Assume \( \kappa \leq \theta \leq \mu, \theta \) regular. Let \( \lambda \leq \nu \) with \( \lambda \) regular. Next let \( \langle N_i : i < \theta \rangle \) be an increasing sequence of \( < \theta \)-closed elementary submodels of \( H(\chi) \) with \( |N_i| = \mu, N_i \in N_{i+1} \), and \( \mu \cup \{ \mu, \lambda, \nu, P \} \in N_0 \). Put \( N = \bigcup N_i \).

1. Assume \( \text{EUB}(R, P, \nu) \). Then:
   
   (a) If \( \nu \leq \mu \) then \( \text{EUB}(R, P \cap N, \nu) \).
   
   (b) If \( \nu > \mu \) then \( \text{EUB}(R, P \cap N, \theta) \).

2. Assume \( S \in N_0 \) witnesses \( \text{COB}(R, P, \lambda, \nu) \). Then \( \text{comp}(S \cap N) \geq \min \{ \theta, \lambda \} \) and \( \text{cof}(S \cap N) \leq \min \{ \mu, \nu \} \) and therefore \( \text{COB}(R, P \cap N, \min \{ \theta, \lambda \}, \min \{ \mu, \nu \}) \). In particular:

   (a) If \( \nu \leq \mu \) then \( \text{COB}(R, P \cap N, \lambda, \nu) \).

   (b) If \( \mu < \lambda \) then \( \text{comp}(S \cap N) = \text{cof}(S \cap N) = \theta \) and thus \( \text{COB}(R, P \cap N, \theta, \theta) \).

**Proof.** (1) Notice that if \( \nu > \mu \) is regular, then \( \text{cof}(\nu \cap N) = \theta \). Hence this follows from part 2 of Lemma 58.

(2) \( \text{comp}(S \cap N) \geq \min \{ \theta, \lambda \} \) holds because if \( A \subseteq S \cap N \) with \( |A| < \min \{ \theta, \lambda \} \), then \( A \in N \) by \( < \theta \)-closure of \( N \), and \( N \) thinks that \( A \) has an upper bound by elementarity. \( \text{cof}(S \cap N) \leq |S \cap N| = \min \{ \mu, \nu \} \) is obvious. \( \text{COB}(R, P \cap N, \min \{ \theta, \lambda \}, \min \{ \mu, \nu \}) \) then follows from part 1 of Lemma 58.

(a) If \( \nu \leq \mu \), then \( S \in S \cap N \) and \( \text{COB}(R, P \cap N, \lambda, \nu) \) is immediate.

(b) Assume \( \mu < \lambda \). We know already \( \text{comp}(S \cap N) \geq \min \{ \theta, \lambda \} = \theta \). By elementarity and \( N_i \in N_{i+1} \), for every \( i < \theta \) there is \( s \in S \cap N_{i+1} \) such that \( s \geq t \) for all \( t \in S \cap N_i \). Hence \( \text{cof}(S \cap N) \leq \theta \) follows, and we must actually have \( \text{comp}(S \cap N) = \text{cof}(S \cap N) = \theta \).

**Lemma 60.** Assume additionally to the assumptions of part 2 of the previous lemma that \( \mu < \lambda \), that \( \theta' \geq \theta \) and \( \mu' > \mu \) are regular cardinals in \( N_0 \), and that \( \{ M_j : j < \theta' \} \in N_0 \) is a family of \( < \mu' \)-closed elementary submodels of \( H(\chi) \) with \( |M_j| = \mu' \). Put \( M = \bigcup M_j \). Then \( \text{COB}(R, P \cap M \cap N, \theta, \theta') \).

**Proof.** \( \text{comp}(S \cap M \cap N) \geq \theta \) is straightforward. So it suffices to show \( \text{cof}(S \cap M \cap N) \leq \theta' \).

Clearly \( \text{comp}(S \cap M_j) \geq \min \{ \mu', \lambda \} \) and \( \text{cof}(S \cap M_j) \leq \min \{ \mu', \nu \} \), and therefore \( \text{COB}(R, P \cap M_j, \min \{ \mu', \lambda \}, \min \{ \mu', \nu \}) \). By part 2 (b) of the previous lemma we see that \( \text{comp}(S \cap M_j \cap N) = \text{cof}(S \cap M_j \cap N) = \theta \). Let \( T_j \subseteq S \cap M_j \cap N \) be cofinal in \( S \cap M_j \cap N \) of size \( \theta \) and let \( T = \bigcup_{j < \theta'} T_j \). Then \( |T| = \theta' \), and \( T \) is cofinal in \( S \cap M \cap N \). Thus \( \text{cof}(S \cap M \cap N) \leq \theta' \), and \( \text{COB}(R, P \cap M \cap N, \theta, \theta') \) follows.

\( \square \)

\( ^3 \text{COB and EUB were originally defined for ccc forcing in Subsection } 5.2 \text{ but this does not really matter.} \)
**Strategy.** Assume $P$ forces $b(R) < d(R) = \nu$ as witnessed by $EUB$ and $COB$. Let $\theta < \theta' < \mu < \mu' := b(R)$ be arbitrary regular cardinals. Building first $(N_i : i < \theta')$ of size $\mu'$ according to Lemma 59 and then analogously $(N_j : j < \theta)$ of size $\mu$ such that $(N_i : i < \theta') \in N_\mu$, we see by the two previous lemmata that $P \cap N \cap N$ forces $b(R) = \theta$ and $d(R) = \theta'$. If we then further intersect $P$ with $N''$ such that $(N_i : i < \theta'), (N_i : i < \theta) \in N''$ with $\mu'' > \theta'$, we will not change these values anymore. This provides us with a scenario for obtaining a model for Cichoń’s maximum: force the left-hand cardinals to be distinct, of large enough value, and then “collapse” dual pairs of cardinals to a priori given values, by repeatedly restricting to appropriate elementary submodels.

4.2 Cichoń’s maximum in ZFC

We are ready to present a ZFC-proof of the consistency result of Theorem 50.

**Theorem 61** (Goldstern, Kellner, Mejía, and Shelah [GKMS1]). Assume GCH and $(\lambda_i : 1 \leq i \leq 9)$ is a $\leq$-increasing sequence of uncountable cardinals with $\lambda_i$ regular for $i \leq 8$ and $\lambda_9$ of uncountable cofinality. Then there is a ccc partial order $P$ forcing that

$$N_1 \leq \text{add}(N) = \lambda_1 \leq \text{cov}(N) = \lambda_2 \leq \text{add}(M) = b = \lambda_3 \leq \text{non}(M) = \lambda_4 \leq \text{cov}(M) = \lambda_5 \leq \mathcal{D} = \text{cov}(M) = \lambda_6 \leq \text{non}(N) = \lambda_7 \leq \text{cov}(N) = \lambda_8 \leq \epsilon = \lambda_9$$

**Proof.** Fix an increasing sequence of cardinals

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_8 \leq \lambda_9$$

such that all cardinals with the possible exception of $\theta_8$ are regular and $\text{cf}(\theta_8) \geq \aleph_1$. Next let $\lambda_i := \lambda_{8-i}$ for $1 \leq i \leq 4$. Let $P_{\text{pre}}$ be the ccc partial order from Theorem 49 for $\lambda_i$, $1 \leq i \leq 5$.

We will construct a complete subforcing $P = P_{\text{pre}} \cap N^*$ of $P_{\text{pre}}$ which forces $(b(R)), (d(R)) = (\theta_{9-2i}, \theta_{8-2i})$ for $1 \leq i \leq 4$ and $\epsilon = \theta_8$.

Fix $N_{\alpha, \nu}$ for $0 \leq \nu \leq 7$ and $\alpha < \theta$, $N_\alpha := \bigcup_{\alpha < \theta} N_{\alpha, \nu}$, as well as $N_{\theta}$ such that

- all $N_{\alpha, \nu}$ as well as $N_{\theta}$ are elementary submodels of $H(\chi)$ containing the sequence of cardinals, $P_{\text{pre}}$, as well as the witnesses $S_i$ of $\text{COB}(R_i, P_{\text{pre}}, \lambda_i, \lambda_5)$ (from Theorem 49 for $1 \leq i \leq 4$)
- $N_{\alpha, \nu}$ contains $(N_{m, \beta} : m < \eta, \beta < \theta_m)$ and $(N_{n, \beta} : \beta < \alpha)$, and $N_{\theta}$ contains $(N_{m, \beta} : m \leq 7, \beta < \theta_m)$
- the $N_{\alpha, \nu}$ are $< \mu_\nu$-closed of size $\mu_\nu$, and $N_\theta$ is $< \mu_1$-closed of size $\theta_8$

Let $N^* := \bigcap_{0 \leq \nu \leq 8} N_{\nu}$. For $0 \leq \nu \leq 8$, let $P_{\nu} := P_{\text{pre}} \cap \bigcap_{0 \leq \nu \leq 8} N_{\nu}$, and let $P := P_{\theta} := P_{\text{pre}} \cap N^*$. Note that $\bigcap_{0 \leq \nu \leq 8} N_{\nu}$ is again an elementary submodel of $H(\chi)$, and therefore each $P_{\nu}$ is a complete subforcing of $P_{\text{pre}}$.

We first show that for all $i$ with $1 \leq i \leq 4$, we have $EUB(R_i, P_{\nu}, \theta_{9-2i})$ and $EUB(R_i, P_{\nu}, \theta_{8-2i})$. Since the proof is the same for all $i$, we do it for $i = 3$. By Theorem 49 we have

- $EUB(R_3, P_{\text{pre}}, \mu_2)$ and $EUB(R_3, P_{\text{pre}}, \mu_1)$

By part 1 of Lemma 59 we successively get

- $EUB(R_3, P_{i}, \mu_2)$ and $EUB(R_3, P_{i}, \mu_1)$ for $i = 0, 1$,
- $EUB(R_3, P_{2}, \mu_2)$ and $EUB(R_3, P_{2}, \theta_2)$,
- $EUB(R_3, P_{j, \theta_3})$ and $EUB(R_3, P_{j, \theta_2})$ for $3 \leq j \leq 8$, as required.

Next, we prove that for all $i$ with all $1 \leq i \leq 4$, we have $\text{COB}(R_i, P, \theta_{9-2i}, \theta_{8-2i})$. Again we consider only the case $i = 3$. By Theorem 49 we have $\text{COB}(R_3, P_{\text{pre}}, \mu_2, \lambda_5)$. We first show $\text{COB}(R_3, P_{3}, \mu_3, \theta_2)$: for $s \in \theta_0 \times \theta_1 \times \theta_2$ let $M_s := N_{s, 0} \cap N_{s, 1} \cap N_{s, 2}$. Then $M_s$ is a $< \mu_2$-closed elementary submodel of $H(\chi)$ of size $\mu_2$. Also $(M_s : s \in \theta_0 \times \theta_1 \times \theta_2) \in N_{3, 0}, M := \bigcup_{s \in \theta_0 \times \theta_1 \times \theta_2} M_s = N_0 \cap N_1 \cap N_2$, and $\theta_0 \times \theta_1 \times \theta_2 = \theta_2$. Therefore we may apply Lemma 59 with $\theta = \theta_3, \theta'' = \theta_2, \mu = \mu_3, \lambda = \mu_2, \nu = \lambda_5, N_{\alpha} = N_{s, \alpha}$ for $\alpha < \theta_3$, and $N = N_3$ to obtain $\text{COB}(R_3, P_{3}, \mu_3, \theta_2)$ (note here that $P_3 = P_{\text{pre}} \cap M \cap N$). An easy application of part 2 (a) of Lemma 59 yields that $\text{COB}(R_3, P_{3}, \mu_3, \theta_2)$ still holds.

By Lemma 12 and part 1 of Corollary 45 $P$ forces $b(R_i) = \theta_{9-2i}$ and $d(R_i) = \theta_{8-2i}$.

Finally note that $|P| = |N^*| = |N_\theta| = \theta_8 = \theta_{8^6}$ and therefore by standard arguments $P$ forces $\epsilon \leq \theta_8$. On the other hand, there is a sequence $(\xi : \xi < \lambda_5)$ of $P_{\text{pre}}$-names for distinct reals belonging to $N^*$. Hence $(\xi : \xi \in \lambda_5 \cap N^*)$ is a sequence of $P$-names for distinct reals. Since $|\lambda_5 \cap N^*| = \theta_8$, $P$ forces $\epsilon \geq \theta_8$. □
4.3 Further results

Using the preparatory forcing from [KST] (cf Theorem 51) together with the submodel technique, one obtains:

**Theorem 62** (Goldstern, Kellner, Mejía, and Shelah [GKMS1]). Assume GCH and $(\lambda_i : 1 \leq i \leq 9)$ is a $\leq$-increasing sequence of uncountable cardinals with $\lambda_i$ regular for $i \leq 8$ and $\lambda_9$ of uncountable cofinality. Then there is a ccc partial order forcing that

$$\kappa_1 \leq \text{add}(N) = \lambda_1 \leq \text{add}(M) = b = \lambda_2 \leq \text{cov}(N) = \lambda_3 \leq \text{non}(M) = \lambda_4 \leq$$

$$\leq \text{cov}(M) = \lambda_5 \leq \text{non}(N) = \lambda_6 \leq d = \text{cof}(M) = \lambda_7 \leq \text{cov}(N) = \lambda_8 \leq c = \lambda_9$$

Further cardinal invariants can be included in the picture (cf Theorem 52):

**Theorem 63** (Goldstern, Kellner, Mejía, and Shelah [GKMS2]). Assume GCH and $(\lambda_i : 1 \leq i \leq 12)$ is a $\leq$-increasing sequence of uncountable cardinals with $\lambda_i$ regular for $i \leq 11$ and $\lambda_{12}$ of uncountable cofinality. Then there is a cofinality-preserving partial order forcing that

$$\kappa_1 \leq m = \lambda_1 \leq p = \lambda_2 \leq h = g = \lambda_3 \leq \text{add}(N) = \lambda_4 \leq \text{cov}(N) = \lambda_5 \leq \text{add}(M) = b = \lambda_6 \leq \text{non}(M) = \lambda_7 \leq$$

$$\leq \text{cov}(M) = \lambda_8 \leq d = \text{cof}(M) = \lambda_9 \leq \text{non}(N) = \lambda_{10} \leq \text{cov}(N) = \lambda_{11} \leq c = \lambda_{12}$$

**Theorem 64** (Goldstern, Kellner, Mejía, and Shelah [GKMS4]). In the previous result, letting $\lambda_1 = \kappa_1$ and $\lambda_9$ and $\lambda_\tau$ two regular cardinals with $\lambda_3 \leq \lambda_9 \leq \lambda_\tau$, $\lambda_9 \leq \lambda_\tau \leq \lambda_{12}$, and $\lambda_9 \in [\lambda_{3+i}, \lambda_{4+i}]$ iff $\lambda_\tau \in [\lambda_{11-i}, \lambda_{12-i}]$, there is a cofinality-preserving partial order forcing the given distribution together with $s = \lambda_9$ and $r = \lambda_\tau$.

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