Nonlinear evolution equations for degenerate transverse waves in anisotropic elastic solids

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Abstract.

Transverse elastic waves behave differently in nonlinear isotropic and anisotropic media. While in the former the quadratically nonlinear coupling in the evolution equations for wave amplitudes is not possible, such a coupling may occur for certain directions in anisotropic materials. We identify the expression responsible for the coupling and we derive coupled canonical evolution equations for transverse wave amplitudes in the case of the two-fold and three-fold symmetry acoustic axes. We illustrate our considerations by examples for a cubic crystal.

Keywords: Acoustic axes, degenerate waves, coupled nonlinear evolution equations.
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1. INTRODUCTION

It is known that transverse elastic waves cannot interact with each other on the quadratically nonlinear level in an isotropic material [5]. On the other hand it is also known that in anisotropic materials this is not the case [1]. There are certain directions in crystals for which quadratically nonlinear interactions do occur between (quasi-) shear elastic waves. This manifests itself by the presence of couplings in the evolution equations for pairs of wave amplitudes. We show that the two-fold and three-fold symmetry acoustic axes are examples of these special directions along which such a coupling takes place in crystals. We identify the quantities which when nonzero are responsible for the occurrence of the coupling between nonlinear elastic shear waves. Then we use the perturbation method of weakly nonlinear geometric acoustics and we derive coupled canonical evolution equations for shear wave amplitudes of plane waves propagating along two-fold and three-fold symmetry acoustic axes. We illustrate our considerations by examples for cubic crystals.

2. PLANE WAVE EQUATIONS

Let us consider the system of equations describing plane waves propagating in \textbf{n} direction in an arbitrary nonlinear anisotropic hyperelastic material in which both geometrical and physical nonlinearities are taken into account. This system of equations can be written as follows [1, 2, 3, 4]:

\[
\mathbf{w}_{,t} + \mathbf{A}(\mathbf{w}, \mathbf{n}) \mathbf{w}_{,x} = 0,
\]

(1)

Nonlinear evolution equations for degenerate transverse waves in anisotropic elastic solids
with
\[ w = \begin{bmatrix} v(x,t) \\ m(x,t) \end{bmatrix}, \quad A(w, n) = -\begin{pmatrix} 0 & \frac{1}{\rho_0} B(m, n) \\ I_0 & 0 \end{pmatrix}, \] (2)
where
\[ B = \Lambda + \Psi m + \cdots. \] (3)
Here \( \rho_0 \) is a constant density\(^1\), \( v \) is a velocity, \( m \) is a 1-D strain, \( n \) is the direction of the plane wave propagation, \( I \) is a 3 \( \times \) 3 identity matrix and \( B \) is a 3 \( \times \) 3 symmetric matrix, which we restrict to consist of zero and first order terms in strains \( \Lambda \) and \( \Psi \), respectively.

Alternatively, in components,
\[ B_{ac} = c_{abcd} n_b n_d + c_{abcdef} n_b n_d n_f m_e + \cdots. \] (4)
where \( c_{abcd} \) and \( c_{abcdef} \) are second and third order elastic constants and \( n_a \) and \( m_a \) are components of the vectors \( n \) and \( m \), respectively. By the standard reduction of indices we represent the second order constants with two indices and third order constants with three indices.

We assume that the Christoffel tensor has the form
\[ \Lambda = \alpha_1 k_1 \otimes k_1 + \alpha_2 k_2 \otimes k_2 + \alpha_3 k_3 \otimes k_3, \] (5)
where \( \alpha_j > 0 \) and \( \{k_1, k_2, k_3\} \) is an orthonormal triad of vectors. The six eigenvalues of \( A(0) \) are the characteristic speeds of a pair of longitudinal and two pairs of shear waves.

\[ \lambda_1 = -\sqrt{\alpha_1} = -\lambda_2, \] (6)
\[ \lambda_3 = -\sqrt{\alpha_2} = -\lambda_4, \] (7)
\[ \lambda_5 = -\sqrt{\alpha_3} = -\lambda_6. \] (8)
and the right and left eigenvectors are, respectively,
\[ r_{2j-1} = \begin{pmatrix} -\lambda_{2j-1} k_j \\ k_j \end{pmatrix}, \quad r_{2j} = \begin{pmatrix} -\lambda_{2j} k_j \\ k_j \end{pmatrix}, \] (9)
\[ l_{2j-1} = \frac{1}{2} \begin{pmatrix} -\lambda_{2j-1}^{-1} k_j \\ k_j \end{pmatrix}, \quad l_{2j} = \frac{1}{2} \begin{pmatrix} -\lambda_{2j}^{-1} k_j \\ k_j \end{pmatrix}. \]
We have \( l_i \cdot r_j = \delta_{ij} \). We are interested in cases when the matrix \( A(0) \) has coincident eigenvalues corresponding to the speeds of shear waves. Such waves will be called degenerate. Hence suppose that \( \alpha_2 = \alpha_3 \), so \( \lambda_s = \lambda_{s+2} \) for \( s = 3, 4 \). In such a case \( n \) is an acoustic axis (see [7], or [10]).

\(^1\) which, in what follows, we assume to be equal to 1 for simplicity.
3. COUPLED EVOLUTION EQUATIONS

Let us consider the initial-value problem for the system of plane waves elastodynamics (1), (2) with perturbed initial data which are of compact support ($\varepsilon$ is a small parameter):

\[
\begin{align*}
\{ & w_t^\varepsilon + A(w^\varepsilon, n)w_{t,x}^\varepsilon = 0, \\
& w^\varepsilon|_{t=0} = \varepsilon w_1(x,x/\varepsilon).
\end{align*}
\]  

\[ \text{We apply the asymptotic method of weakly nonlinear geometric acoustics. The method combines the use of a multiscale perturbation with a small amplitude, high frequency asymptotics [1]. Assume that the asymptotic expansion is of the form} \]

\[
w(t,x) = \varepsilon \left( \sigma_s(t,x,\eta)r_s + \sigma_{s+2}(t,x,\eta)r_{s+2} \right) + O(\varepsilon^2), \quad s = 3, 4, \quad (11)
\]

\[ \text{with } \theta = \varepsilon^{-1}(x-\lambda t), \lambda = \lambda_s = \lambda_{s+2}, \text{ and where } \sigma_s, \sigma_{s+2} \text{ are the unknown amplitudes. Inserting (11) into (10) and applying the method of multiple-scale asymptotics (see [4]), we obtain the following coupled evolution equations for the amplitudes } \sigma_s \text{ and } \sigma_{s+2}: \]

\[
\begin{cases}
(\sigma_s)_t + \lambda_s(\sigma_s)_x + \Gamma_s \sigma_s(\sigma_s)_\theta + \Gamma_s^{s+2}(\sigma_s\sigma_{s+2})_\theta = 0, \\
(\sigma_{s+2})_t + \lambda_{s+2}(\sigma_{s+2})_x + \Gamma_s^{s+2} \sigma_s(\sigma_{s+2})_\theta + \Gamma_s^{s+2} \sigma_{s+2}(\sigma_{s+2})_\theta = 0.
\end{cases}
\]

\[ \text{Here we denote } \Gamma^s_{s,s} \equiv \Gamma_s, \text{ and we use the relations which hold for elastodynamics} \]

\[
\Gamma_s^{s+2} \equiv \Gamma_s^{s+2} = \Gamma_{s,s+2}, \quad \Gamma_s^{s+2} \equiv \Gamma_{s+2,s+2} = \Gamma_{s+2,s+2}, \quad (13)
\]

\[ \text{where the interaction coefficients are defined} \]

\[ \Gamma^i_{p,q} = l_p \cdot (\nabla w A(w r_p r_q))|_{w=0}. \quad (14) \]

3.1. Two-fold symmetry acoustic axis

\[ \text{In the case of a two-fold symmetry acoustic axis, the evolution equations further simplify and it turns out (see [4]) that only two coefficients } \Gamma_s \text{ and } \Gamma_{s+2}^s \text{ characterize the nonlinear terms in the coupled evolution equations (12) for a pair of amplitudes of (quasi-)shear waves, in that case.} \]

3.2. Three-fold symmetry acoustic axis

\[ \text{In the case of a three-fold symmetry acoustic axis only one coefficient } \Gamma_s \text{ characterizes the nonlinear terms in the evolution equations for a pair of amplitudes of plane degenerate shear waves, and the coupled evolution equations are as follows:} \]

\[
\begin{cases}
(\sigma_s)_t + \lambda_s(\sigma_s)_x + \Gamma_s \left[ \sigma_s(\sigma_s)_\theta - \sigma_{s+2}(\sigma_{s+2})_\theta \right] = 0, \\
(\sigma_{s+2})_t + \lambda_{s+2}(\sigma_{s+2})_x - \Gamma_s (\sigma_{s+2})_\theta = 0.
\end{cases}
\]

\[ \text{Nonlinear evolution equations for degenerate transverse waves in anisotropic elastic solids} \]
4. EXAMPLES

In the examples below we consider a cubic crystal of class \textit{m}3\textit{m} in which the strain energy \( W \) is characterized by three second order and six third order elastic constants (see [1, 2, 3, 4])

\[
W = W(c_{11}, c_{12}, c_{44}, c_{111}, c_{112}, c_{144}, c_{1123}, c_{166}, c_{456}).
\]

**Example 1.** Here we present the case where the propagation direction of the plane wave is a two-fold symmetry axis \( \mathbf{n} = \frac{1}{\sqrt{2}}[1 1 0] \) which is not an acoustic axis. The speeds of shear waves are (see [1, 2, 3, 4])

\[
\lambda_3 = -\sqrt{\frac{c_{11} - c_{12}}{2}} = -\lambda_4, \quad \lambda_5 = -\sqrt{c_{44}} = -\lambda_6,
\]

hence \( \lambda_3 \neq \lambda_4 \neq \lambda_5 \neq \lambda_6 \), and \( \Gamma_s = 0 \) for \( s = 3, 4, 5, 6 \). It turns out that there is no coupling and moreover one can show that decoupled evolution equations with cubic nonlinearity describe the propagation of shear waves in this case (see [1]).

**Example 2.** Consider now the case of a three-fold symmetry axis \( \mathbf{n} = \frac{1}{\sqrt{3}}[1 1 1] \) which is the acoustic axis. In this case we obtain (see [1, 2, 3, 4]) coupled equations (15) with

\[
\lambda_3 = \lambda_5 = -\sqrt{\frac{c_{11} - c_{12} + c_{44}}{3}} = -\lambda_4 = -\lambda_6,
\]

\[
\Gamma_3 = -\frac{1}{18\sqrt{2}} \left[ \frac{c_{111} + 2c_{123} - 2c_{456} - 3(c_{112} - c_{144} + c_{166})}{\lambda_3} \right] = -\Gamma_4.
\]

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