Squeezing in su(2) intelligent states

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**Abstract.** Angular momentum intelligent states are defined to satisfy
\[ \Delta J_x \Delta J_y = \frac{1}{2} |J_z|. \] They also share with angular momentum coherent states a number of features. In this paper we describe and illustrate the squeezing properties of angular momentum intelligent states. We analyze three classes of state: never squeezed, always squeezed and sometimes squeezed. Our conclusions are applicable for a broad range of definitions of the standard quantum limit for angular momentum systems.

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1. Introduction

Squeezing is an important tool of quantum information: spin-squeezed states are entangled and exhibit non-classical correlations [1]. Following the seminal work of Kitagawa and Ueda [2], and motivated in part by the results on squeezing for position and momentum [3], much effort has been devoted to studying spin squeezing obtained by transformations generated by ‘nonlinear’ operators quadratic in the $su(2)$ generators [4]. Using the formal equivalence between $su(2)$ operators and the quantum Stokes parameters, squeezing of $su(2)$ observables has experimentally been achieved as polarization-squeezed states [5, 6]. Spin squeezing has also been reported in atomic systems [7].

The objective of this paper is to investigate and discuss the squeezing properties of $su(2)$ intelligent states (or, alternatively, angular momentum intelligent states) [8]. By definition, the product $\Delta A \Delta B$ of the variances of two Hermitian operators $\hat{A}$ and $\hat{B}$, calculated using a (normalized) intelligent state, satisfies the strict equality

$$\Delta A \Delta B = \frac{1}{2} |\langle \psi (\alpha) | [\hat{A}, \hat{B}] | \psi (\alpha) \rangle|.$$  (1)

Here, $\alpha$ is a real parameter, which appears because $|\psi (\alpha)\rangle$ is a solution to the parameter-dependent eigenvalue equation

$$\hat{A} - i \alpha \hat{B}) |\psi (\alpha)\rangle = \lambda |\psi (\alpha)\rangle,$$  (2)

i.e. $|\psi (\alpha)\rangle$ is an eigenvector of the non-Hermitian operator $\hat{A} - i \alpha \hat{B}$. The range of $\alpha$ depends on the operators $\hat{A}$ and $\hat{B}$.

In view of equation (1), intelligent states represent a very natural class of states to study in connection with squeezing: they are an obvious extension to finite-dimensional systems of the concept of minimum uncertainty states of position and momentum. Intelligent states are closely related to coherent states [9] and so have been used in [10] to study the evolution of coherent
wave packets of angular momentum states. Angular momentum intelligent states have also been considered for applications in interferometry [11].

For angular momentum, equation (2) becomes

$$\hat{J}_x - i\alpha \hat{J}_y |\psi\rangle = \lambda |\psi\rangle.$$  

Equation (3) follows from the triangle inequality and the general requirement that

$$|\langle \psi(\alpha) | (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle) + (\hat{B} - \langle \hat{B} \rangle)(\hat{A} - \langle \hat{A} \rangle) |\psi(\alpha)\rangle| = 0.$$  

Combining equations (4) and (3) yields, after simple manipulations,

$$\alpha^2 = \frac{(\Delta J_x)^2}{(\Delta J_y)^2}.$$  

As pointed out in [2] and emphasized in [12], the ratio on the right of equation (5) cannot be used to define squeezing in angular momentum states as it is not SU(2)-invariant (we revisit this point later). As a consequence, the parameter $\alpha$ is not sufficient to determine if an intelligent state is squeezed or not. Indeed, we will show that almost all angular momentum intelligent states display squeezing for some range of values $\alpha$ but are also not squeezed over a finite range of this parameter.

Our paper is organized as follows. In section 2, we introduce squeezing and, in order to better highlight some features of angular momentum squeezing, we start with a brief review of the concept for $\hat{x}$ and $\hat{p}$. We next discuss SU(2)—invariant squeezing and some consequences of this definition (see also [13] for a discussion on how spin squeezing and $x-p$ squeezing can be connected).

Two separate methods of construction of intelligent states are presented in section 3. Although both can be used to (mathematically) construct intelligent states for any value of $\alpha$, they are ‘easily’ applicable in separate ranges of this parameter and so provide different insight and physical interpretations into different regimes of $\alpha$.

A first method is based on the work in [14] and presented in section 3.1. A possible implementation of this scheme was proposed in [15]. We also include a perturbative calculation limited to small $\alpha$ to clarify some calculations in this regime.

Our results show that most intelligent states exhibit no squeezing for sufficiently small $\alpha$. A construction scheme focusing on the region where $\alpha$ is not small was proposed in [16]. It is presented in section 3.3 and involves a projection rather than a nonlinear interaction. This projection is automatic if the states are constructed from polarized bosons [17], and so might lead to an alternate implementation using polarized light rather than polynomial Hamiltonians.

The core results are presented in section 4, where we use Wigner quasi–probability distributions on the sphere [19] to illustrate the resulting states. Positive and negative values of this distribution appear respectively as mountains or valleys above or below a sphere of radius $r$. The radius $r$ can be related to the angular momentum quantum number $\ell$, but we have elected to choose values of $r$ that allow each figure to best display the features we wish to illustrate. Final remarks and conclusions are found in section 5. Two appendices contain more technical results.

### 2. Some general properties of squeezing and squeezed states

Squeezing is a relative concept. It depends first and foremost on identifying a physically relevant threshold $(\Delta A)_{SQL}$ referred to as a standard quantum limit (henceforth SQL) for the variance of
an observable $\hat{A}$. Once this threshold has been established, a state $|\phi\rangle$ is considered squeezed if

$$\Delta A_{\langle \phi \rangle} < (\Delta A)_{\text{sql}}.$$  

(6)

Clearly, squeezing is useful in practice or in theory only to the extent that the threshold is remarkable, either theoretically or experimentally.

2.1. Review of the $\hat{x}$–$\hat{p}$ case

Let us briefly review the familiar case of position and momentum (or, alternatively, quadratures of the electromagnetic field).

Neither $\hat{x}$ nor $\hat{p}$ has normalizable eigenstates. Because their commutator is constant, the product $\Delta x \Delta p$ (when evaluated on any normalized state) is bounded from below by $\frac{\hbar}{2} (\hbar = 1$ throughout): this is the SQL and it is independent of the state. With $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, $\Delta x$ and $\Delta p$ are easily shown to satisfy

$$\langle \Delta x \rangle^2 = -\frac{\alpha}{2}, \quad \langle \Delta p \rangle^2 = -\frac{1}{2\alpha}, \quad |\alpha| = \frac{\Delta x}{\Delta p}.$$  

(7)

The parameter $\alpha$ is thus restricted to negative values. Specifying $\alpha$ is enough to completely determine the squeezing properties of a state, but not enough to determine the state itself. The solutions to equation (2) for $\hat{x}$ and $\hat{p}$ are given by

$$\psi(x; \alpha, \langle x \rangle, \langle p \rangle) = C e^{(1/\alpha)(x - \langle x \rangle)^2 - i \langle p \rangle x},$$  

(8)

and so depend not only on $\alpha$ but also on the ‘parameters’ $\langle x \rangle$ and $\langle p \rangle$ [18].

If $\alpha = -1$, there is no squeezing and the state of equation (8) is in fact a familiar coherent state $|\xi\rangle$ with $\text{Re}(\xi) = \langle x \rangle$ and $\text{Im}(\xi) = \langle p \rangle$. Thus, coherent states represent a family of states that can be used to obtain the SQL for the $\hat{x}$–$\hat{p}$ system. If $\alpha \neq -1$, there is squeezing in either $x$ or $p$. In other words, except at the special value $\alpha = -1$, squeezing occurs for every $\alpha$ irrespective of $\langle x \rangle$ and $\langle p \rangle$.

Because the rhs in equation (1) is constant, intelligent states are also minimum uncertainty states in the sense that $\Delta x \Delta p$ reaches its lower bound for states of the form of equation (8).

Note that, without loss of generality we can further limit $\alpha$ to be in the range $-1 \leq \alpha \leq 0$: values of $\alpha$ smaller than $-1$ can be obtained by interchanging the roles of $\hat{x}$ and $\hat{p}$ in equation (2) and redoing the analysis for $-1 \leq \alpha \leq 0$.

2.2. The $\hat{J}_x$–$\hat{J}_y$ case

In a system of spin $j$, we choose a set $\{|j, m\rangle, m = -j, \ldots, j\}$ of $2j + 1$ eigenstates of $\hat{J}_z$:

$$\hat{J}_z |j, m\rangle = m |j, m\rangle.$$  

(9)

In this basis, matrix elements of $\hat{J}_x$ and $\hat{J}_y$ are most easily calculated by introducing the usual ladder operators $\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$, with nonzero commutators

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z, \quad [\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm,$$  

(10)

and standard action on angular momentum kets $|j, m\rangle$:

$$\hat{J}_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \mp 1\rangle.$$  

(11)
If \( \hat{A} = \hat{J}_x \) and \( \hat{B} = \hat{J}_y \), the commutator \([\hat{J}_x, \hat{J}_y] = \hat{J}_z\) is now an operator and the rhs of equation (1) becomes state dependent. If we work in a finite-dimensional space, the eigenstates of equation (3) are all normalizable. The states for which equation (1) hold are no longer minimum uncertainty states in the strict sense of the word: the lower bound on the product of uncertainties is clearly 0 and reached if one chooses an eigenstate of either \( \hat{J}_x \) or \( \hat{J}_y \). Thus, whereas the concepts of minimum uncertainty states and intelligent states coincide in the \( \hat{x} - \hat{p} \) case, there is for angular momentum systems a distinction between the two concepts. There is no mathematical restriction on \( \alpha \), but it is convenient to limit the analysis to the range \(-1 \leq \alpha \leq 1\): for \( |\alpha| > 1 \), we interchange the role of \( \hat{J}_x \) and \( \hat{J}_y \) and use the range \( |\alpha| \leq 1 \).

Like the position–momentum intelligent states of equation (8), angular momentum intelligent states are specified by three parameters: \( \alpha \), the total angular momentum \( \ell \) and a ‘magnetic’-like parameter \( m \).

For \( \alpha = \pm 1 \), the lhs of equation (3) reduces to \( \hat{J}_\mp \). The intelligent states are \( |\ell, \mp \ell\rangle \): these two eigenstates of \( \hat{J}_z \) are in fact also two coherent states. For \( \alpha = 0 \), the intelligent states are eigenstates of \( \hat{J}_x \). However, not all eigenstates of \( \hat{J}_x \) are coherent.

In contradistinction with the \( x - p \) case, angular momentum coherent states of the form

\[
|\psi^j(\beta)\rangle = R_\beta |j, j\rangle
\]

(12)
correspond to \( |\alpha| \neq 1 \) unless \( \beta = \pi/2 \). However, as noted in [2] and as discussed in [12], the probability distribution of a coherent state is simply a copy of the distribution at \( \beta = 0 \): it has been displaced but not deformed by the rotation. More pictorially, a coherent state can be represented using Wigner functions [19] as a bulge on a sphere; as illustrated in figures 1(a) and (b), the rotation does not deform the distribution.

It was Kitagawa and Ueda in [2] who noted that imposing SU(2) invariance to the probability distributions implied that the criterion of equation (6) should be modified. They observed that coherent states satisfy

\[
(\Delta J_\varphi)^2 = (\Delta J_\gamma)^2 = \frac{1}{2}|\langle \hat{J}_z \rangle|,
\]

(13)
where the rotated observables \( \hat{J}_k \) are obtained as

\[
\hat{J}_k = R(\Omega) \hat{J}_k R^{-1}(\Omega).
\]

(14)
The rotation \( R(\Omega) \) is given in terms of the Euler factorization:

\[
R(\Omega) = R_\varphi(\psi) R_\gamma(\theta) R_\zeta(\gamma) = e^{-i\varphi \hat{J}_z} e^{-i\theta \hat{J}_y} e^{-i\gamma \hat{J}_z}.
\]

(15)
Equation (13) is the backbone for the SU(2)-invariant definition of squeezing for an arbitrary state proposed in [2], in the sense that \( |\phi\rangle \) is squeezed in \( J_\varphi \) when

\[
(\Delta J_\varphi)^2 < \frac{1}{2}|\langle \hat{J}_z \rangle|.
\]

(16)
In (16), it is the state \( |\phi\rangle \) itself that is used to calculate the SQL.

Besides being manifestly SU(2) invariant, equation (16) has two other very desirable features. Firstly, it is independent of the SU(2) transformation properties of \( |\phi\rangle \): the state could belong to one or more representations of SU(2), or could even be replaced by a mixed state. Secondly, it reduces to the usual definition of squeezing if we go back to the \( x - p \) system.
Figure 1. (a) Wigner function for a coherent state with \( L = 7 \) located at the north pole. (b) Wigner function for a coherent state with \( L = 7 \) rotated about \( \hat{y} \) by \( 9\pi/31 \). The axes \( \hat{z}' \) and \( \hat{x}' \) are also indicated. The direction of \( \hat{z}' \) is given in equation (21). A simple rotation does not change the shape of the quasi-distribution function. An animation using 75 frames is also available at: stacks.iop.org/NJP/12/033037/mmedia.

Equation (16) follows naturally from equation (1) once these properties are accepted. The drawback of this definition is that it is algebraic (as it involves \( [\hat{J}_{x'}, \hat{J}_{y'}] \)): we assign some physical property to a state by comparing a fluctuation of one observable with the average value of another.

Other definitions, discussed for instance in [12], use rotated observables on the left of equation (16), as required by SU(2) invariance, but consider alternate choices of thresholds and so have a different rhs side to equation (16).

For instance, one could insist on obtaining the SQL using \( \Delta J_{x'} \) calculated from some appropriate reference state \(|\psi_{\text{ref}}\rangle\). How to unambiguously construct the reference state given the original state \(|\phi\rangle\), what ought to be the physically relevant properties of \(|\psi_{\text{ref}}\rangle\) and what the numerical value for \( \Delta J_{x'} \) would be is unclear in general, although one may wish for \(|\psi_{\text{ref}}\rangle\) to reduce to a coherent state when the state \(|\phi\rangle\) belongs to a single representation. In such a case, the threshold becomes \( \frac{1}{2} \ell \). Since \( \frac{1}{2} \langle \hat{J}_{z'} \rangle \leq \frac{1}{2} \ell \), the algebraic criteria of equation (16) might strictly exclude states that might be deemed squeezed on physical grounds.

An interesting distinction between the \( x-p \) and angular momentum systems can now be made. By the Stone–von Neumann theorem [20], there is only one unitary representation of the Heisenberg–Weyl group (up to equivalences). On the other hand, there are an infinite number of inequivalent SU(2) unitary irreps, each having different dimensions. Hence, it is intuitively easy to see how a uniformly valid criterion for squeezing can be established much more naturally in the \( x-p \) case than in the angular momentum case.
To find the correct rotation $R(\Omega)$ between the rotated and space-fixed observables, it was argued in [2] that the direction determined by the average angular momentum vector

$$\hat{z}' = \frac{\langle \hat{J}_x \rangle, \langle \hat{J}_y \rangle, \langle \hat{J}_z \rangle}{\sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_y \rangle^2 + \langle \hat{J}_z \rangle^2}} = (n_x, n_y, n_z), \quad (17)$$

calculated using $|\phi\rangle$ should be chosen as the quantization axis: after all, it is the only direction that can be sensibly deduced from the state $|\phi\rangle$. Thus we obtain

$$\hat{J}_z' = n_x \hat{J}_x + n_y \hat{J}_y + n_z \hat{J}_z. \quad (18)$$

In an obvious way, the observables $\hat{J}_x'$ and $\hat{J}_y'$ are obtained from $\hat{x}'$ and $\hat{y}'$, respectively.

Hence, if a rotation $R(\omega)$ takes the eigenket $|jj\rangle$, pictured classically as pointing along $\hat{z}$ (as $\langle \hat{J}_z \rangle = \langle \hat{J}_y \rangle = 0$), to the state $|\phi\rangle$ pointing along $\hat{z}'$, the active rotation $R^{-1}(\omega)$ will take $\{\hat{x}, \hat{y}, \hat{z}\}$ to $\{\hat{x}', \hat{y}', \hat{z}'\}$, respectively. The relation

$$\hat{z}' = R^{-1}(\omega)\hat{z} \quad (19)$$

between the average spin direction $\hat{z}'$ and the usual $\hat{z}$-axis implies a corresponding transformation of the observables

$$\hat{J}_z' \equiv R(\omega)\hat{J}_z R^{-1}(\omega). \quad (20)$$

It is shown in equation (A.3) that $\langle \hat{J}_y \rangle = 0$ for every angular momentum intelligent state. Thus, the direction vector $\hat{z}'$ of equation (17) is always of the form

$$\hat{z}' = \hat{z} \cos \theta + \hat{x} \sin \theta = R_x(\theta)\hat{z}, \quad (21)$$

with

$$\cos \theta = \frac{\langle \hat{J}_z \rangle}{\sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2}}, \quad \sin \theta = \frac{\langle \hat{J}_x \rangle}{\sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2}}. \quad (22)$$

The rotated observables $\hat{J}_x'$, $\hat{J}_y'$ and $\hat{J}_z'$ are then

$$\hat{J}_x' = \cos \theta \hat{J}_x + \sin \theta \hat{J}_z, \quad \hat{J}_y' = \hat{J}_y, \quad \hat{J}_x' = \cos \theta \hat{J}_x - \sin \theta \hat{J}_z. \quad (23)$$

with

$$\langle \hat{J}_x' \rangle = 0, \quad \langle \hat{J}_y' \rangle = 0, \quad \langle \hat{J}_z' \rangle = \sqrt{\langle \hat{J}_x' \rangle^2 + \langle \hat{J}_z' \rangle^2}. \quad (24)$$

Furthermore, provided $\langle \hat{J}_z' \rangle \neq 0$, we have

$$\langle \Delta J_z \rangle^2 = \frac{1}{\langle \hat{J}_z' \rangle^2} \langle (\langle \hat{J}_z \rangle \hat{J}_x - \langle \hat{J}_z \rangle \hat{J}_z)^2 \rangle. \quad (25)$$

3. Construction of angular momentum intelligent states

3.1. Intelligent states using non-unitary transformations

Following Rashid [14], we express intelligent states as a non-unitary transformation of an angular momentum ket $|\ell, m\rangle$ in the form

$$|\psi_m^\ell(\nu)\rangle \equiv e^{-i(\pi/2)\hat{J}^\dagger} e^{i\varphi \hat{J}_z} |\ell, m\rangle, \quad (26)$$
where $|\ell, m\rangle$ are the familiar angular momentum states. (The round ket denotes an un-normalized state.)

To show how $|\psi_m^\ell(\nu)\rangle$ is in fact intelligent, note that

$$\left(\hat{J}_x - i\alpha \hat{J}_y\right) e^{-i(\nu/2)\hat{J}_y} e^{i\nu \hat{J}_y} |\ell, m\rangle = e^{-i(\nu/2)\hat{J}_y} e^{i\nu \hat{J}_y} [(\cosh \nu - \alpha \sinh \nu) \hat{J}_x + i(\sinh \nu - \alpha \cosh \nu) \hat{J}_y] |\ell, m\rangle.$$  \hspace{1cm} (27)

Choosing $\nu$ such that the coefficient for $\hat{J}_y$ is zero,

$$\tanh \nu = \alpha$$ \hspace{1cm} (28)

yields

$$\left(\hat{J}_x - i\alpha \hat{J}_y\right) e^{-i(\nu/2)\hat{J}_y} e^{i\nu \hat{J}_y} |\ell, m\rangle = \frac{m}{\cosh \nu} e^{-i(\nu/2)\hat{J}_y} e^{i\nu \hat{J}_y} |\ell, m\rangle.$$ \hspace{1cm} (29)

(We will assume for the rest of our discussion that $|\alpha| < 1$; the treatment of $|\alpha| > 1$ is essentially equivalent as it interchanges the role of $\hat{J}_x$ and $\hat{J}_y$; none of our conclusions are affected by this restriction.) The non-unitary transformation does not preserve the norm, but this is easily calculated:

$$N^2 = \langle \ell, m | e^{i\nu \hat{J}_y} e^{-i(\nu/2)\hat{J}_y} e^{i(\nu/2)\hat{J}_y} e^{-i\nu \hat{J}_y} | \ell, m \rangle = d_{m,m}^{\ell}(2i\nu),$$

$$|\psi_m^\ell(\nu)\rangle = \frac{1}{N} e^{-i(\nu/2)\hat{J}_y} e^{i\nu \hat{J}_y} |\ell, m\rangle,$$ \hspace{1cm} (30) (31)

with $d_{m,m}^{\ell}(2i\nu)$ the usual SU(2) Wigner $d$-function \cite{21} but with imaginary argument.

Some matrix elements have already been evaluated in the basis of equation (30) in [15]. They, along with other matrix elements relevant for this paper, have been placed in appendix A for quick reference.

3.2. Perturbative solution

For small $\alpha$, we have $\nu \approx 0$: one can either expand equation (31) in powers of $\nu$ or start from the set $\{e^{-i\nu/2} |\ell, m\rangle\}$ of eigenstates of $\hat{J}_y$ and consider the term $-i\alpha \hat{J}_y$ of equation (3) as a perturbation. Straight perturbation theory produces a normalized ket as a series, which, up to and including second-order terms in $\alpha$, is given by

$$|\psi_m^\ell(\alpha)\rangle \approx e^{-i(\nu/2)\hat{J}_y} \left( \left( 1 - \frac{\alpha^2}{4} (\ell(\ell + 1) - m^2) \right) |\ell, m\rangle \right.$$

$$+ \frac{\alpha}{2} \sqrt{(\ell + m)(\ell - m + 1)} |\ell, m - 1\rangle$$

$$+ \frac{\alpha}{2} \sqrt{(\ell - m)(\ell + m + 1)} |\ell, m + 1\rangle$$

$$+ \frac{\alpha^2}{8} \sqrt{(\ell + m)(\ell + m - 1)(\ell - m + 1)(\ell - m + 2)} |\ell, m - 2\rangle$$

$$+ \frac{\alpha^2}{8} \sqrt{(\ell - m - 1)(\ell - m)(\ell + m + 1)(\ell + m + 2)} |\ell, m + 2\rangle \right).$$ \hspace{1cm} (32)
3.3. Intelligent states as coupled angular momentum coherent states

In [16], a general method for constructing intelligent states of angular momentum \( \ell \) was introduced. It is based on the observation that, given two counter-rotated angular momentum coherent states \( R_y(\beta) |\ell_a, \ell_a\rangle \) and \( R_y(-\beta) |\ell_b, \ell_b\rangle \), their product

\[
|\psi_{\ell_a, \ell_b}(\beta)\rangle = \left[ R_y(\beta) |\ell_a, \ell_a\rangle \right] \otimes \left[ R_y(-\beta) |\ell_b, \ell_b\rangle \right]
\]

(33)

is also intelligent. Here, the angle \( \beta \) is given by

\[
\cos \beta = -\alpha.
\]

(34)

Because \( |\psi_{\ell_a, \ell_b}(\beta)\rangle \) is not obtained by a collective rotation of a state, it has pieces in every angular momentum occurring in the decomposition of the product

\[
\ell_a \otimes \ell_b = (\ell_a + \ell_b) \oplus (\ell_a + \ell_b - 1) \oplus \ldots |\ell_a - \ell_b|.
\]

(35)

To obtain the intelligent state of angular momentum \( \ell = \ell_a + \ell_b \) from equation (33), we need to project onto the states with \( \ell = \ell_a + \ell_b \):

\[
|\psi_{\ell_a, \ell_b}^\ell(\beta)\rangle = \sum_{\mu=\ell}^{\ell} |\ell, \mu\rangle \kappa_{\ell_a, \ell_b}^{\ell, \mu}(\beta) |\ell_a, \ell_b\rangle \otimes |\ell_b, \ell_b\rangle.
\]

(36)

It is important to mention that, without projection, there is no squeezing: equation (33) describes factorizable uncorrelated coherent states. It is the projection to a specific \( \ell \) subspace that forces correlations between the systems and gives rise to the resulting squeezing. One easily ascertains that the eigenvalue associated with \( |\psi_{\ell_a, \ell_b}^\ell(\beta)\rangle \) is \((\ell_a - \ell_b) \sin \beta\). Thus, we have, using equation (29),

\[
\frac{m}{\cosh v} = (\ell_a - \ell_b) \sin \beta \Rightarrow m = \ell_a - \ell_b, \quad v = -\arccosh(1/\sin \beta).
\]

(37)

(The negative sign is required to guarantee that both methods produce the same \( \langle \hat{J}_z \rangle \).) The normalized angular momentum intelligent states can be written as

\[
|\psi_{\ell_a, \ell_b}^\ell(\beta)\rangle = N_{\ell_a, \ell_b}^{\ell}(\beta) \sum_{\mu=\ell}^{\ell} |\ell, \mu\rangle \kappa_{\ell_a, \ell_b}^{\ell, \mu}(\beta),
\]

(38)

with \( \kappa_{\ell_a, \ell_b}^{\ell, \mu}(\beta) \) succinctly given in [16]. For our purposes, it is more useful to use a recursion relation found in appendix B. The first few \( \kappa_{\ell_a, \ell_b}^{\ell, \mu}(\beta) \)'s are easily found to be

\[
\kappa_{\ell_a, \ell_b}^{\ell, \ell-1}(\beta) = \sqrt{\frac{2}{\ell}} (\ell_a - \ell_b) \tan \frac{\beta}{2} \kappa_{\ell_a, \ell_b}^{\ell, \ell}(\beta),
\]

\[
\kappa_{\ell_a, \ell_b}^{\ell, \ell-2}(\beta) = -\sqrt{\frac{1}{\ell(2\ell - 1)}} (\ell - 2(\ell_a - \ell_b)^2) \left( \tan \frac{\beta}{2} \right)^2 \kappa_{\ell_a, \ell_b}^{\ell, \ell}(\beta).
\]

(39)

4. Squeezing in angular momentum intelligent states

Once we accept on physical grounds that results should be SU(2) invariant, we are inevitably led to rotated observables and thus to an analysis of \( \Delta \hat{J}_s \). Squeezing depends on how we choose...
Figure 2. The change of $\beta$ of $(\Delta J_x)^2$ (thin lines) and two squeezing thresholds: $\frac{1}{2}|\langle \hat{J}_z \rangle|$ (thick lines) and $\frac{1}{2}\ell$ (dashed lines). For the range $\pi/2 < \beta < \pi$, the graphs are reflection-symmetric about $\beta = \pi/2$. (a) $\ell_a = \ell_b = 1$. (b) $\ell_a = \ell_b = 9/2$. In both cases, $(\Delta J_x)^2 < \frac{1}{2}|\hat{J}_z|$ or $(\Delta J_x)^2 < \frac{1}{2}\ell$ for every value of $\beta$ (except possibly at $\beta = 0, \pi/2$ and $\pi$), indicating that the corresponding intelligent states are always squeezed.
Figure 3. The variation with $\beta$ of $(\Delta J_x)^2$ (thin lines) and two squeezing thresholds: $\frac{1}{2}|\hat{J}_z|$ (thick lines) and $\frac{1}{2}\ell$ (dashed lines). Here, $\ell_a = \ell_b = 3$. The dots on the $(\Delta J_x)^2$ curve are located at $\beta = \pi/4, 3\pi/8, 7\pi/16$ and $19\pi/40$. The corresponding Wigner functions are given in figure 4.

reflection symmetric about $\beta = \pi/2$, so the range $0 \leq \beta \leq \pi/2$ can be presented without loss of information.

An expression for $\langle \hat{J}_z \rangle$ valid for all values of $\beta$ is easily obtained from equation (A.1) by noting that $d_{\ell 0}^0(2i\nu) = P_\ell(\cosh(2\nu))$, with $P_\ell$ a Legendre polynomial. Thus:

$$\langle \hat{J}_z \rangle = -\frac{1}{2} \frac{d}{d\nu} \log(P_\ell(\cosh \nu)).$$

(40)

We can obtain a qualitative understanding of some features of $\langle \hat{J}_z \rangle$ and thus of $\Delta J_x$ as follows. Near $\beta = 0$, we use equation (39) with $m = 0$ and expand to obtain

$$\frac{1}{2} \langle \hat{J}_z \rangle \approx \frac{1}{2} \ell - \frac{\ell}{16(2\ell - 1)} \beta^4,$$

(41)

$$(\Delta J_x)^2 \approx \frac{1}{2} \ell - \frac{\ell}{4} \beta^2 + \cdots.$$ Note how $(\Delta J_x)^2$ curves downward much faster than $\frac{1}{2} \langle \hat{J}_z \rangle$, which is very flat with $\beta$ for small $\beta$s.

For $\beta \approx \frac{1}{2}\pi$, we have $\nu \approx 0$, so we use instead equation (31) and note that

$$\langle \hat{J}_z \rangle \approx -\langle \ell, 0 | (1 + 2\nu \hat{J}_x) \hat{J}_x | \ell, 0 \rangle = -\frac{1}{2} \nu \ell (\ell + 1).$$

(42)

The graph is then linear with $\nu$, with a negative slope increasing quadratically with $\ell$. Obviously, the value of $|\langle \hat{J}_z \rangle|$ will reach its maximum value of $\frac{1}{2}\ell$ very quickly as $\ell$ increases: this leads to the general broadening with $\ell$ of the region in which $|\langle \hat{J}_z \rangle|$ is essentially constant, as can be seen by comparing figures 2(a) and (b).

Some of these features can be illustrated with the help of Wigner distributions. For discussion purposes, we focus on the case with $\ell_a = \ell_b = 3$ and some specific angles: $\beta = \pi/4, 3\pi/8, 7\pi/16$ and $19\pi/40$. Also included will be the special case of $\beta = \pi/2$.

We give in figure 3 the curves for $(\Delta J_x)^2$ and $\frac{1}{2}|\langle \hat{J}_z \rangle|$. The dots on the curve $(\Delta J_x)^2$ correspond to the previous values of $\beta$. The Wigner functions for $\ell_a = \ell_b = 3$ at selected angles are presented in figures 4(a)–(e).
(a) $\beta = \pi/4$
(b) $\beta = 3\pi/8$
(c) $\beta = 7\pi/16$
(d) $\beta = 19\pi/40$
(e) $\beta = \pi/2$

**Figure 4.** Wigner functions for the intelligent state $\ell_a = \ell_b = 3$ at various angles. The angles correspond to the dots in figure 3. An animation using 75 frames is also available at stacks.iop.org/NJP/12/033037/mmedia.

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In figures 4(a) and (b), we are still in the ‘flat’ part of the $\langle \hat{J}_z \rangle$ curve. The initial bulge is slowly ‘pinched’ in the $\hat{x}$-direction of the $xz$-plane, and this pinching is in line with the intuitive idea of squeezing along $\hat{x}$. As seen in the $yz$-plane, the deformation of the bulge is symmetric, thus guaranteeing that $\langle \hat{J}_z \rangle = 0$. The slow deformation of the bulge is compatible with the slow decrease in $\langle \hat{J}_z \rangle$. In the range where $\langle \hat{J}_z \rangle$ varies rapidly with $\beta$, wrinkles develop very rapidly, as illustrated in figures 4(c) and (d). It is difficult to qualitatively understand how this still leads to a decrease in $\Delta J_x$ as the Wigner function extends beyond the meridional slice in which it was initially located. At $\beta = \pi / 2$, the wrinkling is maximal. The intelligent state is an eigenstate of $\hat{J}_x$: $\langle \hat{J}_z \rangle = 0$ but so does $\Delta J_x = 0$.

4.3. The case $m \neq 0$ (or $\ell_a \neq \ell_b$)

This case differs from the $m = 0$ one in that the rotated observables are not aligned with the space-fixed observables. For small values of $\beta$, we can expand the intelligent state

$$|\psi_{\ell_a, \ell_b}^\ell(\beta)\rangle = |\ell, \ell\rangle + \sqrt{2} (\ell_a - \ell_b) \tan \frac{1}{2} \beta |\ell, \ell - 1\rangle + \cdots. \quad (43)$$

so

$$\langle \hat{J}_z \rangle = \sqrt{\langle \hat{J}_a \rangle^2 + \langle \hat{J}_b \rangle^2} \approx \ell - c_{\ell,m} \beta^4,$$

where the constant $c_{\ell,m}$ is positive, indicating that the curve is again very flat with $\beta$.

For $\beta \approx \frac{1}{2} \pi$ (or $\alpha \approx 0$), we use equation (32):

$$\langle \hat{J}_z \rangle = -\frac{1}{2} v (\ell(\ell + 1) - m^2) + \cdots, \quad (45)$$

$$\langle \hat{J}_x \rangle = \frac{m}{\cosh v}, \quad (46)$$

$$\langle \hat{J}_c \rangle = \sqrt{\langle \hat{J}_a \rangle^2 + \langle \hat{J}_b \rangle^2} = |m| + \frac{(\ell(\ell + 1) - m^2)^2 - m^2}{2|m|} v^2 + \cdots. \quad (47)$$

The coefficient of $v^2$ is always positive, indicating that $\langle \hat{J}_c \rangle$ is quadratic concave in $v$ with the minimum at $\beta = \pi / 2$. The minimum of $\hat{J}_c$ is $|m| = |\ell_a - \ell_b|$. These features are illustrated in figure 5, with the quadratic behavior of $\langle \Delta J_c \rangle^2$ near $\beta = \pi / 2$ clearly visible in figure 5(a).

The behavior of $\langle \Delta J_c \rangle^2$ with $\beta$ is more complicated. We first see that there is a region extending from $\beta = 0$ to some angle $\beta_c$ for which squeezing occurs, whereas for $\beta$ between $\beta_c$ and $\pi / 2$ there is no squeezing. Using equation (32) and recalling that $m \neq 0$, one can show (after tedious but straightforward algebra) that, for $0 \approx \alpha = (\beta - \pi / 2)$,

$$\langle \Delta J_c \rangle^2 \approx \frac{1}{2} (\ell(\ell + 1) - m^2) - \alpha^2 c_{\ell,m} + \cdots, \quad (48)$$

with $c_{\ell,m}$ given by a complicated expression that is always positive. Clearly the maximum $\frac{1}{2} (\ell(\ell + 1) - m^2)$ is greater than the maximum for $\frac{1}{2} \langle \hat{J}_c \rangle$. With the coefficient of $\alpha^2$ always negative, $\langle \Delta J_c \rangle^2$ decreases (quite abruptly for small $m/\ell \neq 0$) near $\beta = \pi / 2$, in agreement with both graphs in figure 5.

We have not been able to find a closed form expression for $\beta_c$. It is possible to combine equations (48) and (47) to obtain an estimate of $\beta_c$. Qualitatively, the $\langle \Delta J_c \rangle^2$ curve essentially always intersects $\frac{1}{2} |\langle \hat{J}_c \rangle|$ in this region where the latter is no longer flat. For fixed $\ell$, this flat
region increases as $|m|$ decreases from $\ell$ to $|\ell_a - \ell_b|$, and the minimum of $\Delta J_z'$ also decreases. These two factors combine to gradually push the intersection point towards the right as the label $|m|$ decreases. The upward curve in $(\Delta J_z')^2$ can also be understood qualitatively if we observe that $\langle \hat{J}_z' \rangle$ appears in the denominator of equation (25), so that a decrease in $\langle \hat{J}_z' \rangle$ favors an increase in $(\Delta J_z')^2$.

Wigner functions for $\ell_a = 6$, $\ell_b = 5/2$ at selected angles are presented in figures 6(a)–(e). The new feature of these is a drift of the quantization axis $\hat{z}'$ with increasing $\beta$, with the change in orientation between figures 6(d) and (e) being particularly sudden. The angle between $\hat{z}$ and $\hat{z}'$ is always less than $\beta$, indicating that $\hat{z}'$ always lags the position it would have if the state were coherent. Note that, despite the wrinkles in frames 6(d) and (e), these states are not squeezed.

Wigner functions for $\ell_a = 9/2$, $\ell_b = 4$ at selected angles are presented in figures 7(a)–(e). The drift of $\hat{z}'$ is smaller than that of the $\ell_a = 6$, $\ell_b = 5/2$, $m = 7/2$ state. States with higher values of $|m|$ are qualitatively ‘closer’ to coherent states than states with smaller values of $|m|$: the $\hat{z}'$-axis differs from $\hat{z}$ by a rotation angle closer to the parameter angle $\beta$ for larger values of $|m|$. Moreover, equations (22) and (A.3) together show that the orientation of $\hat{z}'$ is determined in part by $m = \ell_a - \ell_b$: small-$m$ states exhibit a much smaller drift in $\hat{z}'$ than higher-$m$ states. (Recall that, for $m = 0$, the rotated and space-fixed axes coincide.)

5. Conclusion

Angular momentum intelligent states exhibit squeezing properties that are quite different from those in $x$–$p$ systems. The criterion to establish squeezing in angular momentum systems is also different from that of the $x$–$p$ system.

The criterion used in this paper is given in equation (16) and satisfies some very natural requirements: it is SU(2) invariant, is not limited to states having good angular momentum $\ell$,
(a) $\beta = \pi/4$  
(b) $\beta = 3\pi/8$  
(c) $\beta = 7\pi/16$  
(d) $\beta = 19\pi/40$  
(e) $\beta = \pi/2$

**Figure 6.** The Wigner functions for the intelligent state $\ell_a = 6, \ell_b = 5/2$ for $\beta = \pi/4, 3\pi/8, 7\pi/16, 19\pi/40$ and $\pi/2$. These values match the dots in figure 5(a). An animation showing using 75 frames is also available at stacks.iop.org/NJP/12/033037/mmedia.
Figure 7. The Wigner functions for the intelligent state $\ell_a = 9/2, \ell_b = 4$ for $\beta = \pi/4, 3\pi/8, 7\pi/16, 19\pi/40$ and $\pi/2$. These values match the dots in figure 5(b). An animation showing using 75 frames is also available at stacks.iop.org/NJP/12/033037/mmedia.
can be applied to pure and mixed states, and reduces to the usual \(x-p\) definition under substitutions \(\hat{J}_x \rightarrow \hat{x}, \hat{J}_y \rightarrow \hat{p}\). States squeezed under our criteria are also squeezed according to other criteria discussed elsewhere [12]. Moreover, extensive numerical calculations also show that angular momentum intelligent states satisfy an in-betweenness condition
\[
(\Delta J'_x)^2 \leq \frac{1}{2}|\langle \hat{J}_z \rangle| \leq (\Delta J'_y)^2,
\] (49)
that is the explicitly SU(2)-invariant version of the criteria used in [6].

A surprising feature is the behavior of \(\langle \hat{J}_z \rangle\) with \(\beta\). Calculations show that, unless \(\beta \approx \pi/2\) (or \(\alpha \approx 0\)), \(\langle \hat{J}_z \rangle\) remains remarkably close to the value of \(1/2\ell\) for a properly oriented coherent state. This is illustrated in figures 2, 3 and 5, and follows from the very weak \(\beta^4\)-dependence of \(\langle \hat{J}_z \rangle\). Whereas \(\Delta J'_x\) remains constant for a coherent state, this quantity is variable for an intelligent state and decreases much faster than \(\langle \hat{J}_z \rangle\), at least for those values of \(\beta\) not too close to \(\pi/2\): squeezing results from this difference in rates.

Because squeezing is inherently non-classical, it remains a challenge to understand clearly the origin of these trends in \(\langle \hat{J}_z \rangle\) and \(\Delta J'_x\), not only from calculations but also from qualitative information presented, for instance, by the Wigner function on the sphere. States may be squeezed even if their Wigner functions are delocalized: in such cases, it is legitimate to ask whether fluctuations in quantities like \(\hat{J}_x\), which geometrically lies in the plane tangent to \(\langle \hat{J} \rangle\), represent a physically meaningful way of ‘encapsulating’ information about the noise in the system.

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Appendix A. Some matrix elements

Briefly, one rapidly verifies [15] that
\[
\langle \hat{J}_z \rangle = -\frac{1}{2d_{\ell,m,m}(2i\nu)} \frac{d}{d\nu}[d_{\ell,m,m}(2i\nu)].
\] (A.1)
Since \(|\psi_{m}(\nu)\rangle\) is an eigenstate of \(\hat{J}_x - i\alpha \hat{J}_y\), we also have
\[
\frac{m}{\cosh \nu} = \langle \hat{J}_x \rangle - i\alpha \langle \hat{J}_y \rangle,
\] (A.2)
which yields
\[
\langle \hat{J}_x \rangle = \frac{m}{\cosh \nu}, \quad \langle \hat{J}_y \rangle = 0.
\] (A.3)
For intelligent states, the deviations of the unrotated observables satisfy
\[
\alpha^2(\Delta J_y)^2 = (\Delta J_x)^2
\] (A.4)
so that, using the definition \(\Delta J_x^2 (\Delta J_y)^2 = \frac{1}{4} \langle \hat{J}_z \rangle^2\), we can write
\[
(\Delta J_y)^2 = \left| \frac{\alpha}{2} \right| \langle \hat{J}_z \rangle.
\] (A.5)
Appendix B. Recursion relation for $\kappa_{\ell_a, \ell_b}^{\ell m}(\beta)$

We start with the definition
\[ \kappa_{\ell_a, \ell_b}^{\ell m}(\beta) = \langle \ell, m | \left[ R_y(\beta) |\ell_a, \ell_a\rangle \right] \otimes \left[ R_y(-\beta) |\ell_b, \ell_b\rangle \right] \] (B.1)
and recall that, by construction [16],
\[ \langle \ell_x - i\alpha \hat{J}_y \rangle \left[ R_y(\beta) |\ell_a, \ell_a\rangle \right] \otimes \left[ R_y(-\beta) |\ell_b, \ell_b\rangle \right] \]
\[ = (\ell_a - \ell_b) \sin \beta \left[ R_y(\beta) |\ell_a, \ell_a\rangle \right] \otimes \left[ R_y(-\beta) |\ell_b, \ell_b\rangle \right]. \] (B.2)

Thus,
\[ \langle \ell, m | \langle \hat{J}_x - i\alpha \hat{J}_y \rangle \left[ R_y(\beta) |\ell_a, \ell_a\rangle \right] \otimes \left[ R_y(-\beta) |\ell_b, \ell_b\rangle \right] \]
\[ = \kappa_{\ell_a, \ell_b}^{\ell m}(\beta) (\ell_a - \ell_b) \sin \beta. \] (B.3)

On the other hand,
\[ \hat{J}_x - i\alpha \hat{J}_y = \left( \frac{1 - \alpha}{2} \right) \hat{J}_+ + \left( \frac{1 + \alpha}{2} \right) \hat{J}_-, \] (B.4)
so
\[ \langle \ell, m | \langle \hat{J}_x - i\alpha \hat{J}_y \rangle = \cos^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell + m)(\ell - m + 1)} \langle \ell, m - 1 | + \sin^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell - m)(\ell + m + 1)} \] (B.5)
where $\alpha = -\cos \beta$ has been used. Hence
\[ \langle \ell, m | \langle \hat{J}_x - i\alpha \hat{J}_y \rangle \left[ R_y(\beta) |\ell_a, \ell_a\rangle \right] \otimes \left[ R_y(-\beta) |\ell_b, \ell_b\rangle \right] \]
\[ = \cos^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell + m)(\ell - m + 1)} \kappa_{\ell_a, \ell_b}^{\ell m-1}(\beta) + \sin^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell - m)(\ell + m + 1)} \kappa_{\ell_a, \ell_b}^{\ell m+1}(\beta), \] (B.6)
from which
\[ \kappa_{\ell_a, \ell_b}^{\ell m}(\beta)(\ell_a - \ell_b) \sin \beta = \cos^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell + m)(\ell - m + 1)} \kappa_{\ell_a, \ell_b}^{\ell m-1}(\beta) \]
\[ + \sin^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell - m)(\ell + m + 1)} \kappa_{\ell_a, \ell_b}^{\ell m+1}(\beta) \] (B.7)
follows. Using the trigonometric identity $\sin(\beta) = 2 \sin(\beta/2)\cos(\beta/2)$ and rearranging yields
\[ 2\kappa_{\ell_a, \ell_b}^{\ell m}(\beta)(\ell_a - \ell_b) \tan \left( \frac{\beta}{2} \right) \]
\[ = \sqrt{(\ell + m)(\ell - m + 1)} \kappa_{\ell_a, \ell_b}^{\ell m-1}(\beta) + \tan^2 \left( \frac{\beta}{2} \right) \sqrt{(\ell - m)(\ell + m + 1)} \kappa_{\ell_a, \ell_b}^{\ell m+1}(\beta). \] (B.8)
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