Lectures on nonlinear sigma-models in projective superspace*

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Abstract

$\mathcal{N} = 2$ supersymmetry in four spacetime dimensions is intimately related to hyperkähler and quaternionic Kähler geometries. On one hand, the target spaces for rigid supersymmetric sigma-models are necessarily hyperkähler manifolds. On the other hand, when coupled to $\mathcal{N} = 2$ supergravity, the sigma-model target spaces must be quaternionic Kähler. It is known that such manifolds of restricted holonomy are difficult to generate explicitly. Projective superspace is a field-theoretic approach to construct general $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models, and hence to generate new hyperkähler and quaternionic Kähler metrics. Intended for a mixed audience consisting of both physicists and mathematicians, these lectures provide a pedagogical introduction to the projective-superspace approach.

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1. Introduction

The concept of supersymmetry in four spacetime dimensions was introduced in theoretical physics in the early 1970s [1–3]. It is a symmetry between bosons and fermions in relativistic theories (field theory, string theory, etc). The discovery of supersymmetry led, in a short period of time, to the appearance of new research directions in high-energy physics, due to remarkable properties of supersymmetric theories, including the following:

- Supersymmetry has nontrivial manifestations at the quantum level;
- Local supersymmetry implies gravity (supergravity [4]);
- One version of local supersymmetry (\(\mathcal{N} = 2\) supergravity [5]) fulfills Einstein’s dream of unifying gravity and electromagnetism;
- String theory requires supersymmetry.
These studies mostly involved the physics community. However, in the late 1970s and early 1980s, supersymmetry met complex geometry.

The year 1979, the Einstein centennial year, was special for physics and geometry. On the physics side, a work of Zumino [6] uncovered an intimate connection between supersymmetry and complex geometry. On the geometry side, Calabi [7] introduced the concept of hyperkähler geometry. The fact that the two discoveries took place in the same year was just a coincidence. However, what followed in the next 30 years was a remarkably fruitful interaction between supersymmetry and hyperkähler geometry. An example of this is the influential paper by Hitchin, Karlhede, Lindström and Roček [8]. These lectures will give an overview of some of these developments.

Nontrivial evidence for the existence of connections between supersymmetry and complex geometry comes from the consideration of supersymmetric nonlinear sigma-models. There are three relevant classic results:

- Kähler manifolds are target spaces for rigid supersymmetric sigma-models with four supercharges ($D \leq 4$) [6]. In four dimensions, $D = 4$, such sigma-models possess $\mathcal{N} = 1$ supersymmetry;
- Hyperkähler manifolds are target spaces for rigid supersymmetric sigma-models with eight supercharges ($D \leq 6$) [9]. In four dimensions, such sigma-models possess $\mathcal{N} = 2$ supersymmetry;
- Quaternionic Kähler manifolds are target spaces for locally supersymmetric sigma-models with eight supercharges ($D \leq 6$) [10].

Supersymmetric sigma-models generalize ordinary bosonic ones. It is pertinent here to recall that a bosonic nonlinear sigma-model is a field theory over a spacetime $X$ in which the fields take values in a $d$-dimensional Riemannian manifold $(M^d, g)$ (known as the target space). If $X$ is four-dimensional Minkowski space, $M^4$, the sigma-model action is

$$S = -\frac{1}{2} \int d^4x g_{\mu\nu}(\varphi) \partial^\mu \varphi^\nu \partial_\nu \varphi^\nu,$$

where $\varphi^\mu(x)$ are scalar fields on $M^4$ and local coordinates on $M^d$ (more precisely, the field $\varphi(x)$ takes its values in $M^d$).

Unlike Kähler metrics, the hyperkähler and quaternionic Kähler metrics are rather difficult to generate explicitly [11]. In this regard, it turns out that the sigma-model results of [9, 10] have an important implication that was not immediately recognized and appreciated. The idea is that off-shell $\mathcal{N} = 2$ supersymmetry, provided its power is properly elaborated, is a device to generate hyperkähler and quaternionic Kähler structures [8, 12–14]. More precisely, suppose it is possible to develop a formalism for constructing $\mathcal{N} = 2$ rigid supersymmetric sigma-models generated by a Lagrangian of reasonably general functional form (say, an arbitrary real analytic function of several variables). Then, for any choice of the Lagrangian, the target space metric must be hyperkähler. Any deformation of the Lagrangian will lead to a new $\mathcal{N} = 2$ sigma-model, and hence to a new hyperkähler metric.

It appears that the only way to make the above idea work is to develop $\mathcal{N} = 2$ superspace techniques for constructing general $\mathcal{N} = 2$ supersymmetric sigma-models. Indeed, superspace is known to provide unique opportunities to engineer supersymmetric theories. Two fully-fledged $\mathcal{N} = 2$ superspace approaches have been developed: (i) harmonic superspace [15, 16] and (ii) projective superspace [13, 17–20]. The former is more general [21, 22]; it is

1 Off-shell projective multiplets and their couplings can be obtained from those emerging within the harmonic-superspace approach via a singular truncation of multiplets [21] or, equivalently, by integrating out some auxiliary degrees of freedom [22].
also powerful in the context of quantum $\mathcal{N} = 2$ super Yang–Mills theories. However, it is the latter approach which is ideally designed for sigma-model constructions. These notes provide a pedagogical introduction to the projective-superspace approach.

It should be noted that the problem of generating quaternionic Kähler metrics can be reduced to that of hyperkähler ones. There exists a remarkable one-to-one correspondence between $4n$-dimensional quaternionic Kähler manifolds and $(4n+1)$-dimensional hyperkähler spaces possessing a homothetic conformal Killing vector, and hence an isometric action of $SU(2)$ rotating the complex structures [23] (see also [24]). Such hyperkähler spaces are called Swann bundles in mathematical literature [25] and hyperkähler cones in physics literature [26]. Hyperkähler cones are target spaces for $\mathcal{N} = 2$ rigid superconformal sigma-models [26, 27]. Therefore, it is sufficient to develop techniques to generate arbitrary $\mathcal{N} = 2$ rigid supersymmetric nonlinear sigma-models, and hence hyperkähler metrics.

These notes are organized as follows. In order to make our presentation reasonably self-contained and accessible to mathematicians, two introductory sections are included. Section 2 is devoted to algebraic aspects of supersymmetry (the $\mathcal{N}$-extended super-Poincaré group, its algebra and superspace), while section 3 presents elements of field theory in superspace. Section 4 describes the formulations of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models in terms of $\mathcal{N} = 1$ chiral superfields. In section 5, we introduce an extension of the $\mathcal{N} = 2$ conventional superspace by auxiliary bosonic directions,

\[ \mathbb{M}^{4|8} \longrightarrow \mathbb{M}^{4|8} \times \mathbb{C}P^1 \equiv \mathbb{M}^{4|8} \times S^2, \]

and give a brief introduction to the harmonic and projective superspace approaches. Off-shell projective supermultiplets and related constructions are discussed in section 6. In section 7, we present the most general $\mathcal{N} = 2$ off-shell supersymmetric sigma-models in projective superspace, and then review two versions of the Legendre transform construction: the generalized and linear ones. As an application of the methods developed, in sections 8 and 9 we review various aspects of the $\mathcal{N} = 2$ supersymmetric sigma-models on cotangent bundles of Kähler manifolds. Section 10 includes comments on the topics not covered. This review is concluded with two technical appendices. Appendix A is devoted to the $\mathcal{N}$-extended superconformal group in four spacetime dimensions. Appendix B contains essential information about canonical coordinates for Kähler manifolds.

Our notation and two-component spinor conventions correspond to those used in two textbooks [28, 29]. In particular, the Minkowski metric is chosen to be $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$. A brief summary of the two-component (iso)spinor conventions is given in appendix C.

2. Algebraic aspects of supersymmetry

In our presentation of the $\mathcal{N}$-extended super-Poincaré group and superspace, we follow the 1973 paper by Akulov and Volkov [30] in which these concepts were introduced for the first time.\(^2\)

\(^2\) The Akulov–Volkov paper [30] was submitted to the journal *Theoretical and Mathematical Physics* on 8 January 1973 and published in January 1974. It remains largely unknown, probably because it was published in a Russian journal. The concepts of the $\mathcal{N}$-extended super-Poincaré group and superspace have been discussed in many books and reviews; however, the pioneering approach of [30] is still one of the best.
2.1. Matrix realization of the Poincaré group

Denote by $\mathcal{P}(4)$ the universal covering group of the restricted Poincaré group $\text{ISO}_0(3, 1)$. The principle of relativistic invariance states that $\mathcal{P}(4)$ must be a subgroup of the symmetry group of any quantum field theory.

Traditionally, $\mathcal{P}(4)$ is realized as the group of linear inhomogeneous transformations on the space of $2 \times 2$ Hermitian matrices (with $\vec{\sigma}$ being the Pauli matrices)

$$x := x^m \sigma_m = x^1 = (x_\alpha), \quad \sigma_m = (1_2, \vec{\sigma}), \quad x^m \in \mathbb{R}^4$$

(2.1)

defined to act as follows:

$$x \to x' = x^m \sigma_m = M x M^\dagger + b, \quad b = b^m \sigma_m,$$

(2.2)

with

$$M = (M_\alpha^\beta) \in \text{SL}(2, \mathbb{C}), \quad b^m \in \mathbb{R}^4.$$  

(2.3)

Here $M^\dagger := M^T$ denotes the Hermitian conjugate of $M$, and $\hat{M} = (\hat{M}_\beta^\alpha)$ is the complex conjugate of $M$ with $\hat{M}_\alpha^\beta := \bar{M}^{\beta}_\alpha$.

The above realization of $\mathcal{P}(4)$ admits a useful equivalent form, as the group of linear inhomogeneous transformations on the space of $2 \times 2$ Hermitian matrices

$$\bar{x} := x^m \bar{\sigma}_m = \bar{x}^1 = (\hat{x}^{\beta\dagger}), \quad \bar{\sigma}_m = (1_2, -\bar{\sigma}), \quad \bar{x}^m \in \mathbb{R}^4$$

(2.4)

defined to act as follows:

$$\bar{x} \to \bar{x}' = x^m \bar{\sigma}_m = (M^{-1})^\dagger \bar{x} M^{-1} + \bar{b}, \quad \bar{b} = b^m \bar{\sigma}_m.$$  

(2.5)

The matrices $(\sigma_m)_{\alpha\beta}$ and $(\bar{\sigma}_m)^{\dagger\beta}$ turn out to be invariant tensors of the restricted Lorentz group $\text{SO}_0(3, 1)$, and they transform into each other under space reflection $x^m = (x^0, \vec{x}) \to \bar{x}^m := (x^0, -\vec{x})$.

For our subsequent consideration, it is advantageous to realize $\mathcal{P}(4)$ as a subgroup of the group $\text{SU}(2, 2)$, which is a $4$–$1$ covering of the conformal group in four spacetime dimensions, consisting of all block triangular matrices of the form

$$(M, b) := \begin{pmatrix} M & 0 \\ -ibM & (M^{-1})^\dagger \end{pmatrix} = (1_2, b)(M, 0),$$  

(2.6)

$$M \in \text{SL}(2, \mathbb{C}), \quad \bar{b} := b^m \bar{\sigma}_m = \bar{b}^\dagger, \quad b^m \in \mathbb{R}^4.$$  

It is well known that Minkowski space $\mathbb{M}^4 := \mathbb{R}^{3,1}$ is a homogeneous space of the Poincaré group and can be identified with the coset space $\text{ISO}_0(3, 1)/\text{SO}_0(3, 1)$. However, it can equivalently be realized as the coset space

$$\mathbb{M}^4 = \mathcal{P}(4)/\text{SL}(2, \mathbb{C}).$$  

(2.7)

Its points are naturally parametrized by the Cartesian coordinates $x^m \in \mathbb{R}^4$ corresponding to the coset representative

$$(1_2, x) = \begin{pmatrix} 1_2 & 0 \\ -ix \vec{\sigma} & 1_2 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 \\ -ix \vec{\sigma} & 0 \end{pmatrix}.$$  

(2.8)

From here one can read off the action of $\mathcal{P}(4)$ on $\mathbb{M}^4$:

$$(M, b)(1_2, x) = (1_2, x')(M, 0) \iff x'^m = (\Lambda(M))^{\alpha\beta} x^\beta + b^m.$$  

(2.9)

Here $\Lambda : \text{SL}(2, \mathbb{C}) \to \text{SO}_0(3, 1)$ is the doubly covering homomorphism defined by

$$(\Lambda(M))^{\alpha\beta} = -\frac{1}{2} \text{tr}(\bar{\sigma}^{\alpha\beta} M^\dagger M).$$  

(2.10)

The right-hand side of (2.9) coincides with the standard action of $\text{ISO}_0(3, 1)$ on the Minkowski space.
2.2. Matrix realization of the super-Poincaré group

Supersymmetry is the only consistent and nontrivial extension of the Poincaré symmetry that is compatible with the principles of quantum field theory [31].

Denote by \( \Psi(4|N) \) the \( N \)-extended super-Poincaré group. It can be realized as a subgroup of \( SU(2, 2|N) \), the \( N \)-extended superconformal group (see appendix \( A \) for its definition). Any element \( g \in \Psi(4|N) \) is a \((4 + N) \times (4 + N)\) supermatrix of the form

\[
g = s(b, \varepsilon) h(M), \quad \varepsilon := (\varepsilon^i_\alpha, \bar{\varepsilon}^j_\dot{\alpha}), \quad i = 1, \ldots, N \quad (2.11a)
\]

\[
s(b, \varepsilon) := \begin{pmatrix}
\mathbb{1}_2 & 0 & 0 \\
-i\bar{b}(\varepsilon) & \mathbb{1}_2 & 2\varepsilon \\
2\bar{\varepsilon} & 0 & \mathbb{1}_N
\end{pmatrix} = \begin{pmatrix}
\delta_{\alpha\dot{\beta}} & 0 & 0 \\
-i\bar{\varepsilon}^{\dot{\alpha}} & \delta^{\dot{\alpha}\dot{\beta}} & 2\varepsilon^{\dot{\alpha}j} \\
2\varepsilon_\alpha & 0 & \delta_{\alpha\beta}
\end{pmatrix}, \quad (2.11b)
\]

\[
h(M) := \begin{pmatrix}
M & \mathbb{1} & 0 \\
0 & (M^{-1})^\dagger & 0 \\
0 & 0 & \mathbb{1}_N
\end{pmatrix} = \begin{pmatrix}
M_{\alpha\beta} & 0 & 0 \\
0 & (M^{-1})_{\dot{\alpha}\dot{\beta}} & 0 \\
0 & 0 & \delta_{\alpha\beta}
\end{pmatrix}, \quad (2.11c)
\]

where \( M \in \text{SL}(2, \mathbb{C}) \) and

\[
b^m_\alpha := b^m_\alpha \pm i\varepsilon_\alpha \sigma^m \bar{\varepsilon}^i_\dot{\alpha} = b^m_\alpha \pm i\bar{\varepsilon}^i_\dot{\alpha} (\sigma^m)_{\alpha\dot{\alpha}}, \quad \bar{b}^m := b^m_\alpha. \quad (2.12)
\]

The group element \( s(b, \varepsilon) \) is generated by four commutating (or bosonic) real parameters \( b^m \), 2\( N \) anti-commutating (or fermionic) complex parameters \( \varepsilon_i^a \) and their complex conjugates \( \bar{\varepsilon}^{\dot{i}}_\dot{a} \). In supersymmetric quantum field theory, the various elements of \( \Psi(4|N) \) correspond to several different symmetries, specifically \( h(M) \) describes a Lorentz transformation, \( s(b, 0) \) a spacetime translation and \( s(0, \varepsilon) \) a supersymmetry transformation.

It is easy to check that the set of supermatrices \( \Psi(4|N) \) introduced is a group. This follows from the easily verified identities

\[
s(b, \varepsilon)s(c, \eta) = s(d, \varepsilon + \eta), \quad (2.13a)
\]

\[
h(M)s(b, \varepsilon)h(M^{-1}) = s(\Lambda(M)b, \hat{\varepsilon}), \quad (2.13b)
\]

where we have defined

\[
d^m := b^m + \varepsilon^m + i(\bar{\eta}_\alpha \sigma^m \bar{\varepsilon}^i_\dot{\alpha} - \varepsilon_i^a (\sigma^m)_{\alpha\dot{\alpha}}), \quad \hat{\varepsilon} := \begin{pmatrix}
M^{-1} & 0 \\
0 & M\end{pmatrix}. \quad (2.14)
\]

By definition, \( N \)-extended Minkowski superspace is the homogeneous space

\[
\mathbb{M}^{4|4N} = \Psi(4|N)/\text{SL}(2, \mathbb{C}), \quad (2.15)
\]

where \( \text{SL}(2, \mathbb{C}) \) is now identified with the set of all matrices \( h(M) \). The points of \( \mathbb{M}^{4|4N} \) can be parametrized by the variables

\[
z^M = (x^m, \theta^a_\alpha, \bar{\theta}^{\dot{a}}_\dot{\alpha}) \equiv (x, \Theta) \quad (2.16)
\]

which correspond to the following coset representative:

\[
s(z) := s(x, \Theta) = \begin{pmatrix}
\mathbb{1}_2 & 0 & 0 \\
-i\mathbb{1}^{(\alpha)} & \mathbb{1}_2 & 2\theta \\
2\bar{\theta} & 0 & \mathbb{1}_N
\end{pmatrix} = \exp \begin{pmatrix}
0 & 0 & 0 \\
-i\theta & 0 & 0 \\
2\theta & 0 & 0
\end{pmatrix}. \quad (2.17)
\]
The action of $\mathbb{P}(4|N)$ on $\mathbb{M}^{4|4N}$ is naturally defined by
\begin{equation}
g = s(b, \varepsilon) h(M) : s(z) \rightarrow s(z') := s(b, \varepsilon) h(M)s(z) h(M^{-1}). \tag{2.18}\end{equation}

Using this definition allows one to read off a Poincaré transformation associated with $g = s(b, \varepsilon) h(M)$:
\begin{equation}\begin{aligned}
x'^m &= (\Lambda^1(M))_m^n x^n + b^m, \\
n_i^\alpha &= \theta_i^\alpha (M - 1)_{\alpha}^\beta, \tag{2.19}\end{aligned}\end{equation}
as well as a supersymmetry transformation corresponding to $g = s(0, \varepsilon)$:
\begin{equation}\begin{aligned}
x'^m &= x^m + i(\theta_i^m \bar{e}^i - \varepsilon_i^m \bar{e}^i), \\
n_i^\alpha &= \theta_i^\alpha + \varepsilon_i^\alpha. \tag{2.20}\end{aligned}\end{equation}

2.3. The super-Poincaré algebra

We can represent group elements of $\mathbb{P}(4|N)$ in an exponential form:
\begin{equation}
s(b, \varepsilon) = \exp i\left\{ -b^m P_m + \varepsilon_i^\alpha Q_i^\alpha + \bar{\varepsilon}_i^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}} \right\}, \tag{2.21a}\end{equation}
\begin{equation}h(e^{\omega_{mn} \sigma_{mn}}) = \begin{pmatrix} 0 & 0 & 0 \\ e^{\omega_{mn} \sigma_{mn}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp \left\{ \frac{i}{2} \omega_{mn} J_{mn} \right\}, \tag{2.21b}\end{equation}
where $\omega_{mn} = -\omega_{nm}$ are real parameters and
\begin{equation}\begin{aligned}
\sigma_{mn} &:= -\frac{1}{2}(\sigma_m \sigma_n - \sigma_n \sigma_m), \\
\bar{\sigma}_{mn} &:= -\frac{1}{2}(\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m). \tag{2.22}\end{aligned}\end{equation}

Here $P_m$, $J_{mn}$, $Q_i^\alpha$ and $\bar{Q}_{\dot{\alpha}}$ are the generators of the Lie superalgebra $\mathfrak{p}(4|N)$ of $\mathbb{P}(4|N)$. In field-theoretic representations of $\mathfrak{p}(4|N)$, $P_m = (-E, \bar{P})$ is identified with the energy–momentum 4-vector, $J_{mn}$ the Lorentz generators, and $Q_i^\alpha$ and $\bar{Q}_{\dot{\alpha}}$ the supersymmetry generators.

Making use of equation (2.13a), one can derive the (anti-)commutation relations:
\begin{equation}\begin{aligned}
[P_m, P_n] &= 0, \tag{2.23a} \\
[P_m, Q_i^\alpha] &= [P_m, \bar{Q}_{\dot{\alpha}}] = 0, \tag{2.23b} \\
\{Q_i^\alpha, Q_j^\beta\} &= \{\bar{Q}_{\dot{i}}^\alpha, \bar{Q}_{\dot{j}}^{\dot{\beta}}\} = 0, \tag{2.23c} \\
\{Q_i^\alpha, \bar{Q}_{\dot{j}}^{\dot{\beta}}\} &= 2\delta_i^j (\sigma_m)_{\alpha\beta} P_m. \tag{2.23d}\end{aligned}\end{equation}

In conjunction with commutation relations involving the Lorentz generators, which can be readily derived with the aid of (2.21b), the above (anti-)commutation relations constitute the $N$-extended super-Poincaré algebra. The $N = 1$ super-Poincaré algebra was discovered in 1971 by Golfand and Likhtman [1].

2.4. Adding the R-symmetry group

The super-Poincaré algebra $\mathfrak{p}(4|N)$ has a nontrivial group of outer automorphisms that is isomorphic to $\text{U}(N)$ and is known as the $R$-symmetry group. The $N$-extended super-Poincaré group $\mathbb{P}(4|N)$ can be generalized to include the $R$-symmetry group. The resulting supergroup
is denoted by $\mathcal{P}_A(4|\mathcal{N})$. Any element $g \in \mathcal{P}_A(4|\mathcal{N})$ is a $(4 + \mathcal{N}) \times (4 + \mathcal{N})$ supermatrix of the form [30]

$$g = s(b, \varepsilon)h(M, U), \quad \varepsilon := (\epsilon_i^a, \tilde{\epsilon}^a_i), \quad i = 1, \ldots, \mathcal{N} \quad (2.24a)$$

$$s(b, \varepsilon) := \begin{pmatrix}
0 & i\tilde{b}_{i+1} \\
-i\tilde{b}_i & 0
\end{pmatrix}, \quad b \in \mathbb{R}, \quad (2.24b)$$

$$h(M, U) := \begin{pmatrix}
0 & (M^{-1})^{ij} & 0 \\
0 & 0 & U
\end{pmatrix}, \quad U = (U^{ij}) \in U(\mathcal{N}). \quad (2.24c)$$

The $\mathcal{N}$-extended Minkowski superspace is the homogeneous space

$$\mathbb{M}^{4|4\mathcal{N}} = \mathcal{P}_A(4|\mathcal{N})/\text{SL}(2, \mathbb{C}) \times U(\mathcal{N}). \quad (2.25)$$

In the case $\mathcal{N} > 1$, the super-Poincaré algebra can be further generalized to include central charges [31]. Such a generalization was not considered in [30].

3. Field theory in superspace

This section is a mini-introduction to supersymmetric field theory. It contains only those concepts and results that we consider absolutely essential for the subsequent discussion of supersymmetric nonlinear sigma-models. Comprehensive reviews of supersymmetric field theory can be found, e.g., in textbooks [28, 29, 32].

3.1. A brief review of the coset construction

Here we succinctly review the salient points of Cartan’s coset construction. From the point of view of a theoretical physicist, this is a procedure to develop a field theory on a homogeneous space $X = \{x\}$ of a Lie group $G$. The homogeneous space can always be realized as a left coset space

$$X = G/H \quad (3.1)$$

for some closed subgroup $H$ of $G$. We denote by $\pi$ the natural projection, $\pi : G \rightarrow G/H$, defined by $\pi(g) = gH$, for any $g \in G$.

For simplicity, we assume the existence\(^3\) of a global cross-section (also known as the coset representative) $s(x) : X \rightarrow G$ such that

$$\pi \circ s = \text{id} \iff \pi(s(x)) = x, \quad \forall x \in X. \quad (3.2)$$

We then have the following unique decomposition in the Lie group $G$: for any group element $g \in G$ there exist unique $x \in X$ and $h \in H$ such that

$$g = s(x)h. \quad (3.3)$$

Now, the fact that $G$ acts on $X = G/H$ can be expressed as follows:

$$gs(x) = s(g \cdot x)h(g, x) = s(x')h(g, x), \quad h(g, x) \in H. \quad (3.4)$$

\(^3\) Quite often, no global cross-section exists, and then one has to restrict the consideration to local coordinate charts. For example, this happens if $X = S^2 = SU(2)/U(1)$ and $G = SU(2)$. However, in some cases of interest, one can construct such a global cross-section. This is indeed the case if $X = \mathbb{M}^{4|\mathcal{N}}$ and $G$ coincides with $\mathcal{P}_A(4|\mathcal{N})$. 

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where \( h(g, x) \) obeys the property (see, e.g., [33])

\[
h(g_1 g_2, x) = h(g_1, g_2 x) h(g_2, x).
\]

Let \( R \) be a finite-dimensional representation of \( H \) on a vector space \( V \). We can then define a representation \( T \) of \( G \) acting on a linear space of fields \( \varphi(x) \) over \( V \) with values in \( V \), \( \varphi : X \to V \), by the rule

\[
[T(g)\varphi](g \cdot x) = \varphi'(x') = R(h(g, x))\varphi(x).
\]

The representation \( T \) is called induced (more precisely, the representation of \( G \) induced by the representation \( R \) of the subgroup \( H \)); see, e.g., [33] for more details. The notion of induced representation is indispensable to quantum field theory. The point is that all relativistic fields we deal with in physics are examples of this construction.

The notion of induced representation can be reformulated in a way that requires no use of \( h(g, x) \) [33]. Consider a linear space of \( V \)-valued functions on \( G \), \( \phi(g) \), such that \( \phi(gh^{-1}) = R(h)\phi(g) \) for arbitrary \( g \in G \) and \( h \in H \). On this space, we can define a representation \( T \) of \( G \) by

\[
[T(g)\phi](g_0) = \phi(g^{-1}g_0),
\]

which can be seen to be equivalent to the induced one. Indeed, the construction under consideration reduces to that considered above by introducing \( \varphi(x) := \phi(s(x)) \).

Our next task is to learn how to differentiate fields \( \varphi(x) \) over \( X \) in a \( G \)-covariant way. Denote by \( \mathcal{G} \) and \( \mathcal{H} \) the Lie algebras of \( G \) and \( H \), respectively. Suppose that there exists a complement \( \mathcal{K} \) of \( \mathcal{H} \) in \( \mathcal{G} \) which is invariant under the adjoint representation of \( H \) on the Lie algebra \( \mathcal{G} \). Thus, we have

\[
\mathcal{G} = \mathcal{K} \oplus \mathcal{H}, \quad [\mathcal{H}, \mathcal{H}] \in \mathcal{H}, \quad [\mathcal{H}, \mathcal{K}] \in \mathcal{K}.
\]

Let \( \{T_a\} \) be a basis of \( \mathcal{K} \), and \( \{T_I\} \) a basis for \( \mathcal{H} \). Introduce the left-invariant Maurer–Cartan one-form

\[
\begin{align*}
s^{-1} ds &= E + \Omega, \\
E &= dx^\mu E_\mu(x) T_a \equiv E^a T_a, \\
\Omega &= dx^\mu \Omega^i_\mu(x) T_I \equiv E^a \Omega^i_\mu T_I.
\end{align*}
\]

Here \( x^\mu \) are local coordinates on \( X \), the one-forms \( \{E^a\} \) constitute the vielbein and \( \Omega \) is called the connection. Associated with a group element \( g \in G \) is the transformation

\[
x \to x' = g \cdot x \iff s(x) \to s(x') = gs(x)h^{-1}(g, x)
\]

which leads to \( s^{-1} ds \to h(s^{-1} ds)h^{-1} - dh \ h^{-1} \), and hence

\[
E \to h E h^{-1}, \quad \Omega \to h \Omega h^{-1} - dh h^{-1}.
\]

The vielbein is seen to transform covariantly under \( G \), while the transformation law of \( \Omega \) includes an inhomogeneous piece typical of gauge fields.

Let \( \varphi(x) \) be a field over \( X \) with the group transformation law

\[
\varphi(x) \to \varphi'(x') = h(g, x)\varphi(x),
\]

where, for simplicity of notation, \( h(g, x) \) stands for \( R(h(g, x)) \). The covariant derivative of \( \varphi \) is defined as follows:

\[
\nabla \varphi := (d + \Omega)\varphi = E^a \nabla_a \varphi, \quad \nabla_a \varphi := (E_a + \Omega_a)\varphi.
\]
Here \( \{ \mathcal{E}_\alpha = E_{\alpha}^\mu(x) \partial_\mu \} \) is the dual basis of \( \{ E^\mu = dx^\mu E_{\alpha}^\mu(x) \} \), that is,
\[
E_{\alpha}^\mu(x) E_{\beta}^\nu(x) = \delta_\alpha^\beta \iff E_{\nu}^\mu(x) E_{\alpha}^\nu(x) = \delta_\alpha^\mu.
\]

It should be remarked that the coset representative \( s(x) \) is not uniquely defined. The intrinsic freedom in its choice is described by gauge transformations
\[
s(x) \rightarrow \tilde{s}(x) = s(x) \ell(x), \quad \ell(x) \in H,
\]
with \( \ell(x) \) completely arbitrary. Under such a transformation, the geometric objects and fields change as follows:
\[
h(g, x) \rightarrow \tilde{h}(g, x) = \ell(g \cdot x) h(g, x) \ell^{-1}(x),
\]
\[
E \rightarrow \tilde{E} = \ell E \ell^{-1}, \quad \Omega \rightarrow \tilde{\Omega} = \ell \Omega \ell^{-1} - d\ell \ell^{-1},
\]
\[
\varphi \rightarrow \tilde{\varphi} = \ell \varphi.
\]

### 3.2. Flat superspace geometry

We can now apply the general formalism developed above to the case of the \( N \)-extended superspace \( M^{4|N} \) using the following correspondence:

| \( X \) | \( G \) | \( H \) | \( g \) | \( h \) | \( x^\mu \) | \( s(x) \) | \( h(g, x) \) |
|---|---|---|---|---|---|---|---|
| \( M^{4|N} \) | \( \mathbb{P}(4|N) \) | \( \text{SL}(2, \mathbb{C}) \) | \( s(z) h(M) \) | \( h(M) \) | \( z^M \) | \( s(z) \) | \( h(M) \) |

Here we have denoted \( z^M = (x^m, \bar{\theta}_\mu, \theta_\mu) \). It is important to point out that \( h(M) \), which corresponds to \( h(g, x) \) in the case under consideration, has no explicit dependence on the superspace coordinates \( z^M \). It only remains to identify elements of the super-Poincaré algebra that correspond to the generators \( T_a \) and \( T_i \):
\[
T_a \rightarrow T_A := (P_a, Q^i, \tilde{Q}^\dot{i}), \quad T_i \rightarrow J_{ab}.
\]

As a result, we can read off the Maurer–Cartan form [30]
\[
s^{-1} ds = \begin{pmatrix}
0 & 0 & 0 \\
-\imath e & 0 & 2d\bar{\theta}
\end{pmatrix}, \quad e^\alpha := dx^\alpha + \imath(\bar{\theta}^\beta \sigma^a \partial \bar{\theta}^\dot{a} - d\theta^\beta \sigma^a \partial \theta^\dot{a}).
\]

In particular, for the vielbein and connection, we get
\[
e^A = dz^M e_M^A(z) = (e^\alpha, d\bar{\theta}^\dot{a}, d\theta^\beta), \quad \Omega = 0.
\]

The components of the vielbein, \( e^A \), comprise the supersymmetric one-forms, i.e. those one-forms which are invariant under the supersymmetry transformations.

Following (3.13), for the covariant derivatives, we obtain
\[
\mathcal{D} = d \equiv dz^M \frac{\partial}{\partial z_M} = e^A D_A, \quad D_A = (\partial_a, D^i_a, \tilde{D}^{\dot{i}}_a),
\]
where the spinor covariant derivatives have the form
\[
D^i_a = \frac{\partial}{\partial \theta^\beta} + \imath(\sigma^b)_{\beta\gamma} \bar{\theta}^{\dot{b}} \partial_b \gamma, \quad \tilde{D}_{\dot{i}} = \frac{\partial}{\partial \bar{\theta}^{\dot{a}}} - \imath(\sigma^a)_{\beta\dot{a}} \theta^\beta \partial_a.
\]

In the \( \mathcal{N} = 1 \) case, we denote \( z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}}) \) and \( D_A = (\partial_a, D_a, \tilde{D}^{\dot{i}}_a) \).
3.3. Superfields

In accordance with (3.6), a tensor superfield \( \mathcal{W}(z) \), with all indices suppressed, is defined to transform under the super-Poincaré group as follows:

\[
g = s(b, \varepsilon)h(M) : \quad \mathcal{W}(z) \longrightarrow \mathcal{W}'(z') = R(M)\mathcal{W}(z),
\]

with \( R \) a finite-dimensional representation of \( \text{SL}(2, \mathbb{C}) \). The important concept of superfields was introduced by Salam and Strathdee [34].

In the case of an infinitesimal supersymmetry transformation \( g = s(0, \varepsilon) \), equation (3.20) gives

\[
\delta \mathcal{W} := \mathcal{W}'(z) - \mathcal{W}(z) = i(\bar{\epsilon}^a_{\dot{a}} Q^a_{\dot{a}} + \bar{\epsilon}^i_{\dot{i}} \bar{Q}^i_{\dot{i}})\mathcal{W},
\]

where the supersymmetry generators have the form

\[
Q^a_{\dot{a}} = \frac{\partial}{\partial \bar{\theta}^\dot{a}} + (\sigma^b)_{\dot{a}\dot{b}} \bar{\theta}^\dot{b} \partial_b, \quad \bar{Q}^i_{\dot{i}} = -i \partial^i (\sigma_{\dot{i}})_{\dot{i} \dot{a}} \theta^\dot{a} \partial_c.
\]

If \( \mathcal{W}(z) \) is a tensor superfield, then \( D_A \mathcal{W}(z) \) is also a tensor superfield. This implies that the covariant derivatives commute with the supersymmetry transformations

\[
[D_A, \epsilon^a_{\dot{a}} Q^a_{\dot{a}}] = [D_A, \bar{\epsilon}^i_{\dot{i}} \bar{Q}^i_{\dot{i}}] = 0.
\]

The spinor covariant derivatives obey the following anti-commutation relations:

\[
\{ D'_a, D^a_{\dot{a}} \} = \{ \bar{D}_a, \bar{D}^a_{\dot{a}} \} = 0, \quad \{ D'_a, \bar{D}^a_{\dot{a}} \} = -2i \delta^a_{\dot{a}} (\sigma^c)_{\dot{a} \dot{b}} \theta^\dot{b} \partial_c.
\]

3.4. Chiral superfields

Let us return to the coset representative (2.17) and consider its first \((4 + N) \times 2\) block-column

\[
\mathcal{C}(x_{(+)}, \theta) = \begin{pmatrix} 1 \ & \ 0 \\ -i \bar{\theta}^i \ & \ 2\theta \end{pmatrix}, \quad x_{(+)}^m := x^m + i\theta_i \sigma^m \bar{\theta}^i.
\]

In accordance with (2.18), the super-Poincaré transformation law of \( \mathcal{C}(x_{(+)}, \theta) \) is

\[
\mathcal{C}(x_{(+)}, \theta) \rightarrow \mathcal{C}(x'_{(+)}, \theta') := g \mathcal{C}(x_{(+)}, \theta)M^{-1}, \quad g = s(b, \varepsilon)h(M).
\]

It follows that the variables \( x^m_{(+)} \) and \( \theta^\alpha \) transform amongst themselves (that is, they do not mix with \( \bar{\theta}^\dot{a} \)) under \( \mathcal{P}(4|N) \). This means that all superfields, which depend on \( x^m_{(+)} \) and \( \theta^\alpha \) only, preserve this property under the super-Poincaré group:

\[
\Phi(z) := \phi(x_{(+)} , \theta) \implies \Phi'(z) = R(M)\Phi(g^{-1} \cdot z) = \phi'(x_{(+)} , \theta).
\]

Such superfields are singled out by the following first-order differential constraints

\[
\bar{D}_{\dot{a}i} \Phi = 0 \iff \Phi(x, \theta, \bar{\theta}) = e^{i \theta_i \sigma^m \bar{\theta}^i} \phi(x, \theta)
\]

and are called chiral.

The \( N = 1 \) chiral scalar supermultiplet was discovered by Wess and Zumino [3] in a component form, and some time later re-cast in superspace. Chiral superfields are indispensable in the context of \( N = 1, 2 \) supersymmetric theories.
3.5. Supersymmetric action principle

In order to construct supersymmetric field theories, we have to learn how to generate supersymmetric invariants. For this, an indispensable mathematical concept is that of the Berezin integral [35].

Consider a function $f(\theta)$ of one Grassmann variable $\theta$ or, equivalently, a function over $\mathbb{R}^{0|1}$. Integration over $\mathbb{R}^{0|1}$ is defined by

$$\int d\theta f(\theta) := \frac{d}{d\theta} f(\theta) \bigg|_{\theta=0}. \quad (3.29)$$

This definition can be immediately generalized to define integration over $\mathbb{R}^{0|q}$. Finally, in conjunction with the standard notion of integration over $\mathbb{R}^p$, we can define integration over a superspace $\mathbb{R}^{p|q}$ as a multiple integral. A detailed discussion can be found, e.g., in [29].

In the case of $\mathcal{N}=1$ supersymmetry, the construction of the most general supersymmetric actions turns out to be almost trivial. Let $L(z)$ be a real scalar superfield. Then

$$S := \int d^4x \, d^2\theta \, d^2\bar{\theta} L \quad (3.30)$$

is invariant under the $\mathcal{N}=1$ super-Poincaré group. To prove the invariance of $S$, we note that it can be represented in the following equivalent forms:

$$S = \frac{1}{16} \int d^4x \, D^\alpha \bar{D}^2 D_\alpha L \quad (3.31)$$

where we have made use of the identity

$$D^\alpha \bar{D}^2 D_\alpha = \bar{D} \bar{D}^2 \bar{D} \bar{D} \quad (3.32)$$

The proof goes as follows:

$$\delta_{\text{SUSY}} S = \frac{i}{16} \int d^4x \, D^\alpha \bar{D}^2 D_\alpha (\epsilon Q + \bar{\epsilon} \bar{Q}) L \bigg|_{\theta=0}$$

$$= \frac{i}{16} \int d^4x (\epsilon Q + \bar{\epsilon} \bar{Q}) D^\alpha \bar{D}^2 D_\alpha L \bigg|_{\theta=0}$$

$$= -\frac{1}{16} \int d^4x (\epsilon D + \bar{\epsilon} \bar{D}) D^\alpha \bar{D}^2 D_\alpha L \bigg|_{\theta=0} = \int d^4x \partial_m f^m = 0 \quad (3.33)$$

for some field $f^m(x)$. Here we have made use of (i) the explicit form of the spinor covariant derivatives (3.19) and the supersymmetry generators (3.22), as well as (ii) the anti-commutation relations (3.24).

Along with the representations (3.31), the action (3.30) can also be written as

$$S = \frac{1}{16} \int d^4x \, D^2 \bar{D} \bar{D} L \bigg|_{\theta=0} = -\frac{1}{4} \int d^4x \, D^2 L \bigg|_{\theta=0}, \quad L_c := \frac{1}{4} \bar{D} \bar{D} L \quad (3.34)$$

The superfield $L_c$ introduced can be seen to be chiral. This exercise leads to a new procedure to construct $\mathcal{N}=1$ supersymmetric invariants. Given a chiral scalar $L_c$, $D^aL_c = 0$, the functional

$$S_c := \int d^4x \, d^2\theta \, L_c \quad (3.35)$$

is invariant under the $\mathcal{N}=1$ super-Poincaré group.

The above simple rules of constructing supersymmetric invariants can be readily generalized to the case $\mathcal{N} > 1$. However, it turns out that this does not allow one to obtain the most interesting actions.
4. Nonlinear sigma-models in $\mathcal{N} = 1$ superspace

In four spacetime dimensions, nonlinear sigma-models can possess two types of supersymmetry: (i) $\mathcal{N} = 1$ supersymmetry or (ii) $\mathcal{N} = 2$ supersymmetry\(^4\). Here we review their formulations in terms of $\mathcal{N} = 1$ chiral superfields. In what follows, for the Grassmann integration measure, we will use the notation $d^4\theta := d^2\theta \, d^2\bar{\theta}$.

4.1. $\mathcal{N} = 1$ supersymmetric nonlinear sigma-models

In 1979, Zumino \cite{6} put forward the following $\mathcal{N} = 1$ supersymmetric theory:

$$
S = \int d^4x \, d^4\theta \, K(\Phi^a, \bar{\Phi}^\dot{b}), \quad \bar{D}_a \Phi^a = 0
$$

(4.1)

with the dynamical variables being $n$ chiral scalar superfields, $\Phi^a(z)$, and their complex conjugates $\bar{\Phi}^\dot{b}(z)$. The above action is obtained from that describing $n$ free massless scalar multiplets \cite{3} by replacing its quadratic Lagrangian, $K_0(\Phi, \bar{\Phi}) = \delta_{\dot{a}b} \Phi^a \bar{\Phi}^b$, with an arbitrary real analytic function. A key result of the work of \cite{6} is the geometric interpretation of the theory \cite{4} it provided. It demonstrated that the Lagrangian $K(\Phi^a, \bar{\Phi}^\dot{b})$ can be interpreted as the Kähler potential of a Kähler manifold $\mathcal{M}$, parametrized by local complex coordinates $\Phi^a$, with the following Kähler metric:

$$
g_{ab}(\Phi, \bar{\Phi}) = \frac{\partial^2 K}{\partial \Phi^a \partial \bar{\Phi}^b} = K_{ab}, \quad g_{ab} = g_{\dot{a}\dot{b}} = 0.
$$

(4.2)

Here and in what follows, we use the notation

$$
K_{a_1 \ldots a_p b_1 \ldots b_q} := \frac{\partial^{p+q} K}{\partial \Phi^{a_1} \ldots \partial \Phi^{a_p} \partial \bar{\Phi}^{b_1} \ldots \partial \bar{\Phi}^{b_q}}.
$$

(4.3)

As is well known, the metric on $\mathcal{M}$ can locally be expressed in terms of a single function, equation (4.2), due to the fact that the Kähler form

$$
\omega = ig_{ab} \, d\Phi^a \wedge d\bar{\Phi}^b
$$

(4.4)

is closed, $d\omega = 0$. The metric (4.2) does not change under a Kähler transformation

$$
K(\Phi, \bar{\Phi}) \rightarrow K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}),
$$

(4.5)

with $\Lambda(\Phi)$ an arbitrary holomorphic function. For the above interpretation of the theory (4.1) to be correct, the action functional must be invariant under arbitrary Kähler transformations. This indeed follows from the facts that (i) the action can be represented, due to (3.32), as

$$
S = \int d^4x \, \mathcal{L}, \quad 16\mathcal{L} := \mathcal{D}^a \mathcal{D}_a K |_{\theta = 0} = \mathcal{D}_a \mathcal{D}^a K |_{\theta = 0} = \mathcal{D}_a \Lambda(\Phi) = 0.
$$

(4.6)

and (ii) the space of chiral superfields has a ring structure, that is,

$$
\mathcal{D}_a \Phi^a = 0 \rightarrow \mathcal{D}_a \Lambda(\Phi) = 0.
$$

(4.7)

The above analysis actually shows that the component Lagrangian, $\mathcal{L}$, in (4.6) is invariant under arbitrary Kähler transformations (4.5). This property allows us to demonstrate that the theory is independent of a choice of local coordinates in the target space. Specifically, if $\{U_{(ij)}, \Phi_{(ij)}\}$ is an atlas on $\mathcal{M}$, and $K_{(ij)}(\Phi_{(ij)}, \bar{\Phi}_{(ij)})$ is the local Kähler potential corresponding to the chart $U_{(ij)}$, then one and the same point $p \in \mathcal{M}$ can belong to several charts. In the intersections of two charts, $U_{(ij)}$ and $U_{(kl)}$, we have

$$
K_{(ij)}(\Phi_{(ij)}, \bar{\Phi}_{(ij)}) = K_{(kl)}(\Phi_{(kl)}, \bar{\Phi}_{(kl)}) + [\Lambda(\Phi_{(kl)}) + \text{c.c.}], \quad \Phi_{(ij)}^a = f^a(\Phi_{(kl)}),
$$

(4.8)

\(^4\) Only in these cases one can define a scalar supermultiplet comprising fields of spin 0 and 1/2. The $\mathcal{N} = 2$ scalar supermultiplet is also called a hypermultiplet.
for some holomorphic functions $\Lambda(\Phi)$ and $f^a(\Phi)$. From here we can see that the Lagrangian $\mathcal{L}$ is indeed independent of the choice of $K_{(i)}$ made.

Let us turn to computing the component Lagrangian. Introduce the component fields of $\Phi^a(z)$ by the rule

$$\Phi^a(x, \theta, \bar{\theta}) = e^{\theta \sigma^a \bar{\hat{h}}_{bc}} \{ \psi^a(x) + \theta \bar{\psi}^a(x) + \theta^2 F^a(x) \}. \quad (4.9)$$

Here $\psi^a$ and $F^a$ are complex scalar fields, while $\bar{\psi}^a$ is a spinor field. Direct calculations lead to

$$\mathcal{L} = -g_{ab}(\varphi, \bar{\varphi}) \left( \partial_m \psi^a \partial_m \bar{\psi}^b + \frac{1}{4} \psi^a \sigma^m \bar{\psi}^b \nabla_m \bar{\psi}^b \right) + g_{ab}(\varphi, \bar{\varphi}) F^a F^b.$$  

where $\nabla_m \psi^a$ denotes the covariant derivative of $\psi^a$:

$$\nabla_m \psi^a := \partial_m \psi^a + (\partial_m \varphi^b) \Gamma^a_{bc}(\varphi, \bar{\varphi}) \psi^c.$$  

and we also define

$$F^a := F^a - \frac{1}{4} \Gamma^a_{bc}(\varphi, \bar{\varphi}) \psi^b \psi^c.$$  

Finally, $\Gamma^a_{bc}(\varphi, \bar{\varphi})$ and $R_{abc\bar{d}}(\varphi, \bar{\varphi})$ denote the Christoffel symbols and the Riemann tensor associated with the Kähler metric $g_{ab}(\varphi, \bar{\varphi})$:

$$\Gamma^a_{bc} = \bar{g}^{ad} K_{bd}, \quad R_{abc\bar{d}} = K_{ac\bar{d}} - g_{cf} \Gamma^e_{ac} \Gamma^d_{ef}.$$  

The equations of motion for the $F^a$ are

$$F^a = 0 \iff F^a = \frac{1}{4} \Gamma^a_{bc}(\varphi, \bar{\varphi}) \psi^b \psi^c.$$  

The fields $F^a$ and their conjugates $\bar{F}^a$ appear in the action without derivatives. When their equations of motion hold, they become functions of other fields. Their sole role is to have supersymmetry linearly realized. Such fields are called auxiliary.

### 4.2. $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models

How to construct $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models? A possible approach is to work in terms of $\mathcal{N} = 1$ superfields. In such a setting, one starts from the general $\mathcal{N} = 1$ supersymmetric nonlinear sigma-model [6]

$$S = \int d^4x \, d^4 \theta \bar{\delta}(\Phi^a, \bar{\Phi}^\dagger), \quad \bar{D}_a \Phi^a = 0,$$  

which is associated with some Kähler manifold $\mathcal{M}$, and look for those target space geometries which are compatible with an additional hidden supersymmetry.

As a first step, it is necessary to make an educated guess regarding the explicit form of a second supersymmetry. A correct ansatz was proposed in [12, 36]. It is defined modulo an irrelevant trivial symmetry transformation (that is proportional to the equations of motion), such as

$$\delta \phi^i = \Gamma^{ij} \delta S[\phi], \quad \Gamma^{ij} = -\Gamma^{ji},$$  

that any theory $S[\phi]$ of bosonic fields $\phi^i$ possesses. The second supersymmetry is

$$\delta \Phi^a = \frac{1}{2} D^2 (\epsilon(\theta) \Omega^a), \quad \delta \bar{\Phi}^\dagger = \frac{1}{2} D^2 (\bar{\epsilon}(\bar{\theta}) \Omega^a),$$  

for some functions $\Omega^a = \Omega^a(\Phi, \bar{\Phi})$ associated with the Kähler manifold $\mathcal{M}$. Here the transformation parameter $\epsilon(\theta)$ has the form

$$\epsilon(\theta) = \tau + \epsilon^a \theta_a, \quad \tau, \epsilon^a = \text{const},$$  

14
where $\epsilon^\alpha$ is the supersymmetry parameter, while $\tau$ generates a central charge transformation. Actually, the latter transformation should be a trivial symmetry, for it is not present in the supersymmetry algebra $(2.23c)$ and $(2.23d)$. However, it is natural to keep it in $(4.18)$, because such a transformation is generated, off the mass shell, by commuting the first and second supersymmetries.

There are two simple observations to justify the fact that the ansatz $(4.17)$ is indeed general. First, since $\delta/\Phi^a_1$ must be chiral, it can be represented as $\delta/\Phi^a = \bar{D}^2(\cdots)$. Second, we can assign dimension zero to $\bar{\Omega}^a_1$ and then any function $\bar{\Omega}^a_1(\Phi, \bar{\Phi})$ is also dimensionless. The mass dimensions of $\epsilon^\alpha$, $\theta^\alpha$ and $\bar{D}^2$ are, respectively, $-1/2$, $-1/2$ and $+1$. These observations lead to $(4.17)$. To be more precise, it is possible to deform the variation $\delta/\Phi^a_1$ given in $(4.17)$ by adding a term proportional to $\epsilon^\alpha \theta^\alpha \bar{D}^2 \bar{\Omega}^a_1$. However, the latter proves to generate a trivial symmetry (see, e.g., [37] for more details), and therefore can be ignored. In section 9, we show that the transformation law $(4.17)$ naturally follows from an off-shell formulation for $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models.

The next steps should be to analyze the implications of the requirements that (i) the action $(4.15)$ be invariant under the transformations $(4.17)$ and (ii) the first and second supersymmetry transformations form the $\mathcal{N} = 2$ super-Poincaré algebra on the mass-shell. This analysis was carried out in [36], and here we only summarize the results obtained.

• The action $(4.15)$ is invariant under the transformations $(4.17)$ if the following conditions hold:

$$\omega_{bc} := g_{a\bar{c}} \bar{\Omega}^a_{\bar{c}}, \quad -\omega_{\bar{c}b}. \tag{4.19}$$

and

$$\omega_{bc,\bar{a}} := \partial_\bar{a} \omega_{bc} = \nabla_\bar{a} \omega_{bc} = 0, \tag{4.20a}$$

$$\nabla_\bar{a} \omega_{bc} = 0. \tag{4.20b}$$

It can be shown that $\omega_{bc} (\Phi)$ is a globally defined holomorphic two-form on $\mathcal{M}$. Equations $(4.20a)$ and $(4.20b)$ mean that the two-form $\omega_{bc}$ is covariantly constant, and therefore the target space $\mathcal{M}$ is a manifold of restricted holonomy.

• The first and the second supersymmetries form the $\mathcal{N} = 2$ super-Poincaré algebra (with $i, j = 1, 2$, where the values of isospinor indices are underlined for later convenience)

$$\{Q^a_i, Q^\beta_j\} = \{\bar{Q}^\alpha_i, \bar{Q}^\beta_j\} = 0, \quad \{Q^a_i, \bar{Q}^\beta_j\} = 2\delta^a_i (\sigma_\alpha)_{\alpha\beta} P^\beta, \tag{4.21}$$

on the equations of motion if

$$\bar{\Omega}^a_{\bar{c}}, \Omega^2_{\bar{b}} = -\delta^a_{\bar{b}}. \tag{4.22}$$

A detailed derivation of the above results can be found in [37].

Denote by $J_3$ the complex structure chosen on the target space $\mathcal{M}$,

$$J_3 = \begin{pmatrix} \bar{D}^a_{\bar{b}} & 0 \\ 0 & -\bar{D}^a_{\bar{b}} \end{pmatrix}. \tag{4.23}$$

It follows from the previous results that there exist two more covariantly constant complex structures

$$J_1 := \begin{pmatrix} 0 & \bar{g}^{a\bar{c}} \omega_{\bar{c}b} \\ \bar{g}^{a\bar{c}} \omega_{\bar{c}b} & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & ig^{a\bar{c}} \omega_{\bar{c}b} \\ -ig^{a\bar{c}} \omega_{\bar{c}b} & 0 \end{pmatrix}. \tag{4.24}$$

This follows from the fact that, on the mass shell, the variations $\delta \Phi^a$ and $\delta \bar{\Phi}^a$ in $(4.17)$ should constitute a vector field on $\mathcal{M}$.

5 This follows from the fact that, on the mass shell, the variations $\delta \Phi^a$ and $\delta \bar{\Phi}^a$ in $(4.17)$ should constitute a vector field on $\mathcal{M}$.
such that (i) $M$ is Kähler with respect to each of them and (ii) the operators $J_A = (J_1, J_2, J_3)$ form the quaternionic algebra:

$$J_A J_B = -\delta_{AB} \mathbb{1} + \varepsilon_{ABC} J_C.$$  

(4.25)

Therefore, the target space $M$ is a hyperkähler manifold.

Given a hyperkähler space $(M, g, J_A)$, we pick one of its complex structures, say $J_3$, and introduce complex coordinates $\phi^a$ compatible with it. In these coordinates, $J_3$ has the form (4.23). Then, two other complex structures, $J_1$ and $J_2$, are given by equation (4.24). The matrix elements of $J_1$ and $J_2$ are determined by the holomorphic two-form, equation (4.19), from which we cannot directly read off the functions $\Omega^a$ and $\Omega^b$ appearing in (4.17), but only their partial derivatives. Reference [36] presented the following explicit expression for $\Omega^a$:

$$\bar{\Omega}^a = \omega^{ab}(\Phi) \bar{\rho}_b(\Phi, \bar{\Phi}), \quad \bar{\rho}_b(\Phi, \bar{\Phi}) := \frac{\partial}{\partial \Phi^b} \bar{\rho}(\Phi, \bar{\Phi}).$$  

(4.26)

Although $\bar{\Omega}^a$ changes under the Kähler transformations as

$$\bar{\rho}(\Phi, \bar{\Phi}) \to \bar{\rho}(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}),$$

$$\omega^{ab}(\Phi) \bar{\rho}_b(\Phi, \bar{\Phi}) \to \omega^{ab}(\Phi) \bar{\rho}_b(\Phi, \bar{\Phi}) + \omega^{ab}(\Phi) \Lambda_b(\Phi),$$  

(4.27)

the supersymmetry transformation $\delta \Phi^a = \frac{1}{2} \bar{D}^2(\varepsilon \bar{\Omega}^a)$ remains invariant.

The Lagrangian of the $N = 2$ supersymmetric sigma-model, equation (4.15), is the hyperkähler potential of $M$.

5. $N = 2$ superspace with auxiliary dimensions

As with the component ($N = 0$) formulation for general $N = 2$ supersymmetric nonlinear sigma-models [9, 10], their formulation in terms of $N = 1$ superfields described above is just an existence theorem. The $N = 1$ formulation has two major drawbacks.

- It is not suitable from the point of view of generating $N = 2$ supersymmetric nonlinear sigma-models.
- It provides no insight from the point of view of constructing $N = 2$ superconformal nonlinear sigma-models.

To overcome the drawbacks of the $N = 1$ formalism is hardly possible without making use of $N = 2$ superspace techniques. However, in the early 1980s there emerged a conceptual problem concerning such techniques. It was realized that standard multiplets defined in the conventional $N = 2$ superspace $\mathbb{M}^{4|8}$ are not suitable (say, too long) for sigma-model constructions. A way out was to look for an extension of the conventional superspace.

The correct superspace setting was found in 1983–1984 independently by three groups who pursued somewhat different goals [13, 15, 38]. It is $\mathbb{M}^{4|8} \times \mathbb{C}P^1 = \mathbb{M}^{4|8} \times S^2$.  

(5.1)

Below, we will briefly discuss each of the three approaches mentioned.

5.1. Isotwistor superspace

In order to introduce the construction given in [38], we should start from the algebra of $N = 2$ spinor covariant derivatives ($i, j = \frac{1}{2}, \frac{3}{2}$):

$$\{D^i_a, D^j_b\} = 0, \quad \{\bar{D}^i_a, \bar{D}^j_b\} = 0, \quad \{D^i_a, \bar{D}^j_b\} = 2i \delta^{ij} (\sigma^m)_{ab} \partial_m.$$  

(5.2)
Following the work\textsuperscript{6} of [38], introduce an isowitzor $v^i \in \mathbb{C}^2 \setminus \{0\}$ and define\textsuperscript{7}
\begin{equation}
\mathcal{D}_a := v_i D^i_a, \quad \mathcal{D}_a := v_i D^i_a, \quad v_i := \varepsilon_{ij} v^j.
\end{equation}
Then, the anti-commutation relations (5.2) imply that
\begin{equation}
\{\mathcal{D}_a, \mathcal{D}_b\} = \{\mathcal{D}_a, \mathcal{D}_b\} = \{\mathcal{D}_a, \mathcal{D}_b\} = 0.
\end{equation}
These relations allow us to introduce a new type of superfields obeying the (Grassmann) analyticity constraints
\begin{equation}
\mathcal{D}_a \phi = \tilde{\mathcal{D}}_a \phi = 0, \quad \phi = \phi(z, v, \bar{v}), \quad \bar{v}_i := (v^i)^*.
\end{equation}
Such a superfield depends of half of the Grassmann coordinates.

It should be pointed out that the operators $\mathcal{D}_a$ and $\tilde{\mathcal{D}}_a$, equation (5.3), are not complex conjugate of each other. However, they turn out to be conjugate with respect to the generalized conjugation defined in subsection 6.2.

The constraints $\mathcal{D}_a \phi = \tilde{\mathcal{D}}_a \phi = 0$ do not change if we replace $v^i \to c v^i$, with $c \in \mathbb{C}^*$, in the definition of $\mathcal{D}_a$ and $\tilde{\mathcal{D}}_a$. It is natural to restrict our attention to those superfields which (i) obey the constraints $\mathcal{D}_a \phi = \tilde{\mathcal{D}}_a \phi = 0$ and (ii) only scale under arbitrary re-scalings of $v$:
\begin{equation}
\phi(z, cv, \bar{v}, C) = e^{\alpha v} \phi(z, v, \bar{v}), \quad C \in \mathbb{C}^*,
\end{equation}
for some parameters $n_\alpha$ such that $n_+ - n_-$ is an integer. By redefining $\phi(z, v, \bar{v}) \to \phi(z, v, \bar{v})/(v^i v^i)^{n_+}$, we can always choose $n_- = 0$. Any superfield with the homogeneity property
\begin{equation}
\phi^{(n)}(z, cv, \bar{v}, C) = e^{\alpha v} \phi^{(n)}(z, v, \bar{v}), \quad C \in \mathbb{C}^*,
\end{equation}
is said to have weight $n$. A weight-$n$ isowitzor superfield is defined to obey the following properties:
\begin{equation}
\mathcal{D}_a \phi^{(n)} = \tilde{\mathcal{D}}_a \phi^{(n)} = 0, \quad \phi^{(n)}(z, cv, \bar{v}, C) = e^{\alpha v} \phi^{(n)}(z, v, \bar{v}), \quad C \in \mathbb{C}^*.
\end{equation}

We see that the isowitzor $v^i \in \mathbb{C}^2 \setminus \{0\}$ is defined modulo the equivalence relation $v^i \sim c v^i$, with $c \in \mathbb{C}^*$; hence, it parametrizes $\mathbb{C}P^1$. The isowitzor superfields introduced live in the space $\mathbb{M}^{1|2} \times \mathbb{C}P^1$ which was called isowitzor superspace by Rosly and Schwarz [40].

Given an isowitzor superfield $\phi^{(n)}(v^i, \bar{v}_j)$, its complex conjugate $\bar{\phi}^{(n)}(\bar{v}_i, v^j)$ is
\begin{equation}
\bar{\phi}^{(n)}(\bar{v}_i, v^j) := \phi^{(n)}(v^i, \bar{v}_j)
\end{equation}
is no longer isowitzor, for it satisfies neither the constraints (5.5) nor the homogeneity condition (5.7). This is completely similar to the situation with the chiral superfields. However, there is a fundamental difference between the isowitzor and chiral superfields: for the former one can define a modified conjugation that maps any isowitzor superfield $\phi^{(n)}(v^i, \bar{v}_j)$ into an isowitzor one $\tilde{\phi}^{(n)}(v^j, \bar{v}_i)$ defined as a composition of the complex conjugation with the antipodal mapping\textsuperscript{8}
\begin{equation}
\phi^{(n)}(v^i, \bar{v}_j) \to \tilde{\phi}^{(n)}(\bar{v}_i, v^j) \to \tilde{\phi}^{(n)}(\bar{v}_i \to -v_i, \bar{v}_j \to -v^j) =: \tilde{\phi}^{(n)}(v^j, \bar{v}_i).
\end{equation}
The weight-$n$ isowitzor superfield $\tilde{\phi}^{(n)}(v^j, \bar{v}_i)$ is said to be the smile-conjugate of $\phi^{(n)}(v^i, \bar{v}_j)$.
One can check that
\begin{equation}
\tilde{\phi}^{(n)}(v, \bar{v}) = (-1)^n \phi^{(n)}(v, \bar{v}).
\end{equation}
Therefore, if the weight $n$ is even, real isowitzor superfields can be defined, $\tilde{\phi}^{(2m)}(v, \bar{v}) = \phi^{(2m)}(v, \bar{v})$.

\textsuperscript{6} Rosly’s approach [38] was inspired by earlier ideas due to Witten [39].
\textsuperscript{7} See appendix C for our convention to raise and lower isowitzor indices.
\textsuperscript{8} The smile conjugation is similar to the Dirac conjugation of four-component spinors defined as follows: $\Psi \to \bar{\Psi} := \Psi^\dagger y^0$. 

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5.2. Harmonic superspace approach

We turn to a very brief discussion of the harmonic superspace approach pioneered by Galperin, Ivanov, Kalityn, Ogievetsky and Sokatchev [15]. A detailed account can be found, e.g., in the monograph [16].

One can use the equivalence relation \( u^i \sim c u^i \), with \( c \in \mathbb{C}^* \), to switch to a description in terms of the following normalized isotwistors:

\[
u^i = \frac{u^i}{\sqrt{v^i v}}, \quad u_j = \frac{\bar{v}_j}{\sqrt{v^i v}} = \frac{u^i}{\sqrt{v^i v}} \quad \Rightarrow \quad (u_i^-, u_i^+) \in \text{SU}(2).
\]

The \( u_i^\pm \) are called harmonics. They are defined modulo the equivalence relation \( u_i^\pm \sim \exp(\pm i \alpha) u_i^\pm \) with \( \alpha \in \mathbb{R} \). It is clear that the harmonics parametrize the coset space \( \text{SU}(2)/\text{U}(1) \cong S^2 \).

Associated with an isotwistor superfield \( \phi^{(i)}(z, v, \bar{v}) \) is the superfield

\[
\psi^{(i)}(z, u_i^+, u_i^-) := \frac{1}{(\sqrt{v^i v})^n} \phi^{(i)}(z, v, \bar{v})
\]

obeying the homogeneity condition

\[
\psi^{(i)}(z, e^{i\alpha} u_i^+, e^{-i\alpha} u_i^-) = e^{i\alpha n} \psi^{(i)}(z, u_i^+, u_i^-).
\]

The \( \psi^{(i)}(z, u_i^\pm) \) is said to have \( \text{U}(1) \) charge \( n \).

Within the harmonic superspace approach, \( \psi^{(i)}(z, u_i^\pm) \) is required to be a smooth charge-\( n \) function over \( \text{SU}(2) \) or, equivalently, a smooth tensor field over the two-sphere \( S^2 \). Such a superfield is called analytic. It can be represented, say for \( n \geq 0 \), by a convergent Fourier series (see, e.g., [41]):

\[
\psi^{(i)}(z, u_i^\pm) = \sum_{p=0}^{\infty} \psi^{(i, \ldots, i \pm (p+1))} (z) u_i^{p+1} \cdots u_i^{p+1} \bar{u}_j^{p+1} \cdots \bar{u}_j^{p+1},
\]

in which the coefficients \( \psi^{(i, \ldots, i \pm (p+1))} (z) \) are ordinary \( N = 2 \) superfields obeying first-order differential constraints that follow from (5.8). The beauty of this approach is that the power of harmonic analysis can be used.

To construct supersymmetric theories, a supersymmetric action principle is required. In harmonic superspace, it includes integration over \( S^2 \) in addition to that over the spacetime and (half of) Grassmann variables. Let \( L^{(4)}(z, u_i^\pm) \) be a real analytic superfield of \( \text{U}(1) \) charge +4, and

\[
\mathcal{L}^{(4)}(z, v, \bar{v}) := (v^i v)^2 L^{(4)}(z, u_i^+, u_i^-)
\]

the corresponding weight-\( n \) isotwistor superfield. Associated with \( \mathcal{L}^{(4)} \) is the following \( N = 2 \) supersymmetric invariant:

\[
S := \int d^4x \int d^2\mu \Delta^{(-4)} \mathcal{L}^{(4)}(z, v, \bar{v}) \bigg|_{\theta = \bar{\theta} = 0}.
\]

Here,

\[
d^2\mu := \frac{i}{2\pi} \frac{v_j du^j \wedge \bar{v}^j d\bar{v}^j}{(v^i v)^2} = \frac{i}{2\pi} \frac{v_j du^j \wedge \bar{v}^j d\bar{v}^j}{(\bar{v}_k v_k)^2}
\]

can be recognized as the usual measure on \( S^2 \). Indeed, introducing a complex (inhomogeneous) coordinate \( \zeta \) in the north chart of \( \mathbb{C}P^1 \) as

\[
v^i = v^1(1, \zeta), \quad \zeta := \frac{v_1}{v_2}, \quad i = 1, 2.
\]
one obtains
\[ d^2 \mu = \frac{i}{2\pi} \frac{d\zeta \wedge d\bar{\zeta}}{(1 + \zeta \bar{\zeta})^2}. \]  
(5.20)

The operator \( \Delta^{(-4)} \) in (5.17) is
\[ \Delta^{(-4)} := \frac{1}{16} \nabla^a \nabla_\beta \bar{\nabla}_\beta \bar{\nabla}_\delta, \quad \nabla_a := \frac{1}{v^i} v^i D^i_a, \quad \bar{\nabla}_\alpha := \frac{1}{v^i} v^i \bar{D}_\beta^i. \]  
(5.21)

5.3. Projective superspace approach

The formation of the projective superspace approach [13, 17–20] has taken several years, from 1984 to 1990, although its key elements already appeared in the work by Karlhede, Lindström and Roček [13] on self-interacting \( \mathcal{N} = 2 \) tensor multiplets. The name ‘projective superspace’ was coined in 1990 [19]. Modern projective-superspace terminology appeared in the 1998 work [20] mostly devoted to quantum aspects, along with important formal developments, of the approach.

In this approach, off-shell supermultiplets are described in terms of weight-\( n \) isotwistor superfields \( Q^{(n)}(z, v) \):
\[ \mathcal{D}_a Q^{(n)} = \mathcal{D}_z Q^{(n)} = 0, \quad Q^{(n)}(z, cv) = e^n Q^{(n)}(z, v), \quad c \in \mathbb{C}^*, \]  
which are holomorphic over an open domain of \( \mathbb{C}P^1 \),
\[ \frac{\partial}{\partial \bar{v}_i} Q^{(n)} = 0. \]  
(5.23)

Such a superfield is called weight-\( n \) projective superfield\(^9\). There is no need to require \( Q^{(n)}(z, v) \) to be holomorphic over \( \mathbb{C}P^1 \) for such a requirement is not essential for the construction of projective-superspace actions.

The \( \mathcal{N} = 2 \) supersymmetric action principle is formulated in terms of a Lagrangian \( \mathcal{L}^{(2)}(z, v) \) which is a real weight-2 projective superfield. The action functional includes a closed contour integral in \( \mathbb{C}P^1 \), along with integration over Minkowski space and half of the Grassmann variables:
\[ S := \frac{1}{2\pi} \oint \gamma v_i \, dv^i \int d^4x \, \Delta^{(-4)} \mathcal{L}^{(2)}(z, v) \bigg|_{\theta = \bar{\theta} = 0}. \]  
(5.24)

Here \( \gamma \) denotes a closed contour in \( \mathbb{C}P^1 \), \( v^i(t) \), parametrized by an evolution parameter \( t \). The action makes use of the following fourth-order differential operator:
\[ \Delta^{(-4)} := \frac{1}{16} \nabla^a \nabla_\beta \bar{\nabla}_\beta \bar{\nabla}_\delta, \quad \nabla_a := \frac{1}{(v, u)} u_i D^i_a, \quad \bar{\nabla}_\beta := \frac{1}{(v, u)} u_i \bar{D}_\beta^i, \]  
(5.25)

where \( (v, u) := v^i u_i \). Here \( u_i \) is a fixed isotwistor chosen to be arbitrary modulo the condition \( (v, u) \neq 0 \) along the integration contour.

Making use of the analyticity constraints obeyed by \( \mathcal{L}^{(2)}(z, v) \), one can show that the action is invariant under the \( \mathcal{N} = 2 \) super-Poincaré group. The proof is analogous to that considered earlier in the \( \mathcal{N} = 1 \) case, equation (3.33). The supersymmetry transformation acts on \( \mathcal{L}^{(2)} \) as follows:
\[ \delta_{\text{SUSY}} \mathcal{L}^{(2)} = \frac{i}{2} (\epsilon^\alpha Q^\alpha_a + \bar{\epsilon}^\alpha \bar{Q}_a^\alpha) \mathcal{L}^{(2)} = i (\epsilon^i Q^i + \bar{\epsilon}^i \bar{Q}_i) \mathcal{L}^{(2)}, \]  
(5.26)

\(^9\) The terminology ‘weight-\( n \) projective superfield’ appears to be more appropriate than ‘degree-\( n \) projective superfield’ because in the superconformal case the parameter \( n \) coincides with the superconformal weight of \( Q^{(n)}(z, v) \) [37].
compare with equation (3.21). Since the supersymmetry generators anti-commute with the spinor covariant derivatives, the variation of the actions is

\[
\delta_{\text{SUSY}} S = \frac{i}{2\pi} \oint \gamma v_i d\gamma^i \int d^4x (\epsilon_i Q^i + \bar{\epsilon}^{i} \bar{Q}) \Delta^{(-4)} \mathcal{L}^{(2)}(z, v) \bigg|_{\theta = \bar{\theta} = 0},
\]

where we have made use of the explicit form of the supersymmetry generators and spinor covariant derivatives. Using the completeness relation

\[
\delta_i^j = v^i u_j - v^j u_i, \quad (v, u),
\]

the first term on the right can be transformed as follows:

\[
\epsilon_i D^i = v^i \epsilon_i \nabla - \frac{1}{(v, u)} u^i \epsilon_i \mathcal{D}.
\]

Here the first term does not contribute to (5.27), since \(\nabla \Delta^{(-4)} = \bar{\nabla} \Delta^{(-4)} = 0\). As for the second term, the operator \(\mathcal{D} \) can be pushed through \(\Delta^{(-4)}\) in (5.27) until it hits \(\mathcal{L}^{(2)}\), which gives zero, due to (5.22). In the process of pulling \(\mathcal{D} \) to the right, contributions appear proportional to spacetime derivatives, due to the identity

\[
\{\mathcal{D}_a, \bar{\nabla}_\beta\} = -2i (\sigma^m)_{\alpha\beta} \partial_m,
\]

which do not contribute to the action. This completes the proof.

An important property of the action (5.24) is its invariance under arbitrary projective transformations of the form

\[
(u_i, v_i) \rightarrow (u_i, v_i) R, \quad R = \begin{pmatrix} a(t) & 0 \\ b(t) & c(t) \end{pmatrix} \in \text{GL}(2, \mathbb{C}),
\]

where the matrix elements of \(R\) obey the first-order equations

\[
\dot{a} = b \frac{\dot{v}}{(v, u)}, \quad \dot{b} = -b \frac{\dot{v}}{(v, u)}, \quad \dot{\psi} := \frac{d\psi(t)}{dt},
\]

along the integration contour in order to keep the transformed isotwistor \(u_i \) \(t\)-independent. This invariance allows one to make \(u_i \) arbitrary modulo the constraint \((v, u) \neq 0\), and therefore the action is independent of \(u_i\).

\[
\frac{\partial}{\partial u_i} S = 0.
\]

The projective-superspace action was originally given in [13] in a form that differs slightly from (5.24). The latter representation appeared first in [42].

6. Off-shell projective supermultiplets

We now turn to a systematic study of projective supermultiplets.

6.1. Projective superfields in the north chart of \(\mathbb{C}P^1\)

Introduce the inhomogeneous complex coordinate, \(\zeta\), on \(\mathbb{C}P^1 - \{\infty\}\) defined by equation (5.19). Given a weight-\(n\) projective superfield \(Q^{(n)}(z, v)\), we can associate with it a new object \(\tilde{Q}^{(n)}(z, \zeta)\) defined as

\[
\tilde{Q}^{(n)}(z, \zeta) \rightarrow Q^{(n)}(z, \zeta) \propto Q^{(n)}(z, v), \quad \frac{\partial}{\partial \zeta} Q^{(n)} = 0.
\]
The explicit form of $Q^{[n]}(z, \zeta)$ for various projective multiplets will be given later on. The superfield introduced can be represented by a series

$$Q^{[n]}(z, \zeta) = \sum_{p}^{q} Q_{k}(z) \zeta^{k}, \quad -\infty \leq p < q \leq +\infty, \quad (6.2)$$

with $Q_{k}(z)$ some ordinary $N = 2$ superfields. Here $p$ and $q$ are invariants of the supersymmetry transformations.

In the north chart of $\mathbb{CP}^{1}$, the analyticity constraints

$$D_{\alpha}Q^{[n]}(\zeta) = \bar{D}_{\bar{\alpha}}Q^{[n]}(\zeta) = 0, \quad D_{\alpha} := v_{i}D_{a}, \quad \bar{D}_{\bar{\alpha}} := \bar{v}_{i}\bar{D}_{\bar{a}} \quad (6.3)$$

take the form

$$D_{\alpha}^{2}Q^{[n]}(\zeta) = \zeta D_{\alpha}^{1}Q^{[n]}(\zeta), \quad \bar{D}_{\bar{\alpha}}^{2}Q^{[n]}(\zeta) = -\frac{1}{\zeta} \bar{D}_{\bar{\alpha}}^{1}Q^{[n]}(\zeta). \quad (6.4)$$

These relations can be interpreted as follows. The dependence of the component superfields $Q_{k}$ of $Q^{[n]}(\zeta)$ on $\theta_{\alpha}^{1} \equiv \theta_{\alpha}$ and $\bar{\theta}_{\bar{\alpha}}^{1} \equiv \bar{\theta}_{\bar{\alpha}}$, which can be identified with the Grassmann coordinates of $N = 1$ superspace parametrized by $z^{M} = (x^{m}, \theta^{a}, \bar{\theta}_{\bar{a}})$.

### 6.2. Smile conjugation

The notion of smile conjugation was introduced in subsection 5.1. As formulated, the definition directly applies to $Q^{[n]}(z, v)$. Now we wish to re-express it in terms of $Q^{[n]}(z, \zeta)$.

Consider a projective superfield

$$Q(z, \zeta) \equiv Q^{[n]}(z, \zeta) = \sum_{-p}^{q} Q_{k}(z) \zeta^{k}. \quad (6.5)$$

It is constrained as in equation (6.4). Let $\tilde{Q}(z, \tilde{\zeta})$ be the complex conjugate of $Q(z, \zeta)$:

$$\tilde{Q}(z, \tilde{\zeta}) = \sum_{-p}^{q} \tilde{Q}_{\tilde{k}}(z) \tilde{\zeta}^{\tilde{k}}, \quad \tilde{Q}_{\tilde{k}}(z) := \overline{Q_{k}(z)}. \quad (6.6)$$

It is not a projective superfield for it satisfies the conditions

$$D_{\tilde{a}}^{2} \tilde{Q}(\tilde{\zeta}) = -\frac{1}{\zeta} D_{a}^{2} \tilde{Q}(\tilde{\zeta}), \quad \bar{D}_{\bar{\tilde{a}}}^{2} \tilde{Q}(\tilde{\zeta}) = \bar{\zeta} \bar{D}_{\bar{\tilde{a}}}^{2} \tilde{Q}(\tilde{\zeta}), \quad (6.7)$$

which do not coincide with the analyticity constraints. However, the object

$$\tilde{Q}(z, \zeta) := \tilde{Q}(z, -\frac{1}{\zeta}) = \sum_{-p}^{q} (-1)^{k} \tilde{Q}_{-\tilde{k}}(z) \zeta^{k} \quad (6.8)$$

does obey the analyticity constraints, and therefore it is a projective superfield. $\tilde{Q}(\zeta)$ is called the smile-conjugate of $Q(\zeta)$.

A real projective superfield is characterized by the properties

$$\tilde{Q}(z, \zeta) = Q(z, \zeta) = \sum_{-p}^{q} Q_{k}(z) \zeta^{k}, \quad \tilde{Q}_{\tilde{k}}(z) = (-1)^{k} Q_{-\tilde{k}}(z). \quad (6.9)$$

10 Compared with (6.2), we have changed $p \rightarrow -p$ in equation (6.5).
6.3. \( N = 2 \) supersymmetric action in \( N = 1 \) superspace

Consider the \( N = 2 \) supersymmetric action

\[
S := \frac{1}{2\pi} \oint_{\gamma} v_i \, dv^i \int d^4x \Delta^{(-4)} \mathcal{L}^{(2)}(z, v) \bigg|_{\theta_i = \bar{\theta}_i = 0}. \tag{6.10}
\]

We recall that \( \mathcal{L}^{(2)}(z, v) \) is a real weight-2 projective superfield,

\[
\Delta^{(-4)} := \frac{1}{16} \nabla^2 \nabla^2, \quad \nabla_a := \frac{1}{(v, u)} u_i \nabla^i_a, \quad \nabla_{\bar{\beta}} := \frac{1}{(v, u)} u_i \nabla^i_{\bar{\beta}}, \tag{6.11}
\]

and \( u_i \) is a fixed isotwistor such that \((v, u) \neq 0\) at each point of \( \gamma \). As demonstrated in subsection 5.3, the action is independent of \( u_i \).

Without loss of generality, we can assume that the integration contour \( \gamma \) does not pass through the ‘north pole’ \( v^i \sim (0, 1) \). We can then introduce the inhomogeneous complex coordinate, \( \zeta \), on \( \mathbb{C}P^1 - \{ \infty \} \) defined by \( v_1 = v^2(1, \zeta) \). Since the action, \( S \), is independent of \( u_i \), the latter can be chosen to be \( u_i = (1, 0) \), such that \((v, u) = v^2 \neq 0\). We also represent the Lagrangian in the form

\[
\mathcal{L}^{(2)}(z, \zeta) = i v_1 v_2 \mathcal{L}(z, \zeta), \quad \bar{\mathcal{L}}^{(2)}(z, \zeta) = -\frac{1}{\zeta} \bar{\mathcal{L}}(z, \zeta), \tag{6.12}
\]

It is important to remark that \( \mathcal{L}(z, \zeta) \) is a real projective superfield in the sense of equation (6.9). Now, the action takes the form

\[
S = \frac{1}{16} \oint \frac{d\zeta}{2\pi i} \int d^4x \zeta (D_1^2 \bar{D}_2^2)^2 \mathcal{L}(z, \zeta) \bigg|_{\theta_i = \bar{\theta}_i = 0}. \tag{6.13}
\]

Finally, if we make use of the analyticity of \( \mathcal{L} \),

\[
D_2^2 \mathcal{L}(\zeta) = \zeta D_2^2 \mathcal{L}(\zeta), \quad \bar{D}_2^2 \mathcal{L}(\zeta) = -\frac{1}{\zeta} \bar{D}_2^2 \mathcal{L}(\zeta), \tag{6.14}
\]

the action turns into

\[
S = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^4x \left\{ \frac{1}{16} (D_1^2 \bar{D}_2^2)^2 \mathcal{L}(z, \zeta) \bigg|_{\theta_i = \bar{\theta}_i = 0} \right\} \tag{6.15}
\]

In the final expression for \( S \), the integration is carried out over the \( N = 1 \) superspace. The action is now formulated entirely in terms of \( N = 1 \) superfields. At the same time, by construction, it is off-shell \( N = 2 \) supersymmetric! This is one of the most powerful features of the projective superspace approach.

6.4. Projective multiplets and constrained \( N = 1 \) superfields

There is an important feature of projective multiplets that has to be specially emphasized. Consider a projective multiplet

\[
Q^{[q]}(z, \zeta) = \sum_p \sum_{p < q} Q_k(z) \zeta^k, \quad -\infty < p < q < +\infty. \tag{6.16}
\]

In terms of \( Q_k \), the analyticity conditions are

\[
D_2^2 Q_k = D_2^2 Q_{k-1}, \quad \bar{D}_2^2 Q_k = -\bar{D}_2^2 Q_{k-1} \tag{6.17}
\]

In what follows, the bar-projection in expressions like the second line in (6.15) is omitted.
Suppose that the series (6.16) terminates from below, that is, \( p > -\infty \). Then \( Q_p \) and \( Q_{p+1} \) can be seen to be constrained \( \mathcal{N} = 1 \) superfields. The corresponding constraints are
\[
\bar{D}^\dot{\alpha} Q_{\dot{p}} = 0, \quad \bar{D}^\dot{\alpha} Q_{\dot{p}+1} = 0, \quad \bar{D}^\dot{\alpha} := \bar{D}^\dot{\alpha}_J.
\] (6.18)
Thus, \( Q_p \) is chiral, while \( Q_{p+1} \) is said to be linear.

Suppose the series terminates from above, that is, \( q < \infty \). Then, the \( \mathcal{N} = 1 \) superfields \( Q_q \) and \( Q_{q-1} \) are constrained by
\[
D_\alpha Q_q = 0, \quad D_\alpha^2 Q_{q-1} = 0, \quad D_\alpha := D^\alpha_\bar{J}.
\] (6.19)
Thus, \( Q_q \) is antichiral, while \( Q_{q-1} \) is said to be antilinear.

There is a very special case: \( q - p = 2 \). Here, the \( \mathcal{N} = 1 \) superfield components are constrained by the rule
\[
D_\alpha Q_p = 0, \quad D_\alpha^2 Q_{p+1} = D_\alpha^2 Q_{p+2} = 0.
\] (6.20)
We see that \( Q_{p+1} \) is both linear and antilinear.

### 6.5. Off-shell realizations of the hypermultiplet

We now review off-shell projective multiplets that can be used to describe the \( \mathcal{N} = 2 \) scalar multiplet, also known as the hypermultiplet, comprising four spin-0 and two spin-1/2 fields. The \( \mathcal{N} = 2 \) supersymmetric nonlinear sigma-models can be viewed as models for self-interacting massless hypermultiplets.

Our first example is the so-called real \( \mathcal{O}(2n) \) multiplet \[18, 44\], \( n = 2, 3, \ldots \), which is described by a real weight-2 projective superfield \( H^{(2n)}(\zeta, \nu) \) of the form
\[
H^{(2n)}(\zeta, \nu) = H_{\bar{j}_1 \ldots \bar{j}_n}(\zeta) \zeta^{\bar{j}_1} \ldots \zeta^{\bar{j}_n} = \tilde{H}^{(2n)}(\zeta, \nu).
\] (6.21)
The analyticity constraints (6.3) are equivalent to
\[
D_\alpha (H_{\bar{j}_1 \ldots \bar{j}_n}) = \bar{D}_{\overline{\alpha}}(H_{\bar{j}_1 \ldots \bar{j}_n}) = 0.
\] (6.22)
The reality condition \( \tilde{H}^{(2n)} = H^{(2n)} \) is equivalent to
\[
\tilde{H}_{\bar{j}_1 \ldots \bar{j}_n} = H^{\bar{j}_1 \ldots \bar{j}_n} = \varepsilon^{i_1 j_1} \ldots \varepsilon^{i_n j_n} H_{j_1 \ldots j_n}.
\] (6.23)
Associated with \( H^{(2n)}(\zeta, \nu) \) is the superfield \( H^{(2n)}(\zeta, \xi) \) defined by
\[
H^{(2n)}(\zeta, \xi) = (\nu \zeta^{\bar{n}})^m H^{(2n)}(\zeta, \zeta) = (\nu \zeta^{\bar{n}})^m H^{(2n)}(\zeta, \zeta),
\] (6.24)
\[
H^{(2n)}(\zeta, \zeta) = \sum_{k=0}^{n} H_k(\zeta) \xi^{\bar{k}}, \quad \tilde{H}_k = (-1)^k H_{-k}.
\]
The \( H^{(2n)}(\zeta, \xi) \) is real in the sense of (6.9). Its two lowest components in the expansion (6.24), \( H_{-n} \) and \( H_{-n+1} \), are constrained \( \mathcal{N} = 1 \) superfields, chiral and linear, respectively:
\[
\bar{D}_{\overline{\alpha}} H_{-n} = 0, \quad \bar{D}^\dot{\alpha} H_{-n+1} = 0.
\] (6.25)

In the family of multiplets considered above, we intentionally did not include the real \( \mathcal{O}(2) \) multiplet \[13\] described by
\[
\eta(\zeta, \xi) = \frac{1}{\xi} \psi(z) + G(z) - \zeta \bar{\psi}(z), \quad \bar{G} = G, \quad \bar{D}_\zeta \psi = \bar{D}^\dot{\alpha} G = 0.
\] (6.26)
The point is that this multiplet is very special, for it corresponds to the \( \mathcal{N} = 2 \) tensor multiplet \[45\] in which one of the four spin-0 states is described by a gauge antisymmetric second rank tensor field.
All of the $O(2n)$ multiplets, with $n = 1, 2, \ldots$, are proved to define holomorphic tensor fields over $\mathbb{C} \mathbb{P}^1$. We now turn to introducing projective multiplets that are not globally defined on $\mathbb{C} \mathbb{P}^1$. By definition, the arctic multiplet [18] is described by a series

$$\Upsilon(z, \zeta) = \sum_{k=0}^{\infty} \Upsilon_k(z) \zeta^k, \quad D_\zeta \Upsilon_0 = 0, \quad D_\zeta^2 \Upsilon_1 = 0. \quad (6.27)$$

Its smile-conjugate, $\tilde{\Upsilon}(z, \zeta)$, is called an antarctic multiplet:

$$\tilde{\Upsilon}(z, \zeta) = \sum_{k=0}^{\infty} (-1)^k \tilde{\Upsilon}_k(z) \frac{1}{\zeta^k}. \quad (6.28)$$

The superfields $\Upsilon(z, \zeta)$ and $\tilde{\Upsilon}(z, \zeta)$ constitute a polar multiplet. This terminology, (ant)arctic and polar, was coined in [20] and appears to be quite natural, since several practitioners of projective superspace come from the Nordic country of Sweden.

Among the projective multiplets considered, the polar multiplet has two unique properties. First of all, it is the only multiplet which can be used to describe a charged hypermultiplet, since the structure of the arctic multiplet allows for phase transformations:

$$\Upsilon(\zeta) \rightarrow e^{i\alpha} \Upsilon(\zeta), \quad \alpha \in \mathbb{R}. \quad (6.29)$$

Second, the space of arctic superfields allows for a ring structure: for any arctic superfields $\Upsilon_A(\zeta)$ and $\Upsilon_B(\zeta)$, their product

$$\Upsilon_A(\zeta) \cdot \Upsilon_B(\zeta) = \Upsilon_C(\zeta) \quad (6.30)$$

is also arctic.

7. Sigma-models in projective superspace

We are finally prepared to write down general off-shell $N = 2$ supersymmetric nonlinear sigma-models.

7.1. General off-shell $N = 2$ supersymmetric sigma-models

Suppose we have a dynamical system described by a set of $N = 2$ tensor multiplets. Then, their most general $N = 2$ supersymmetric sigma-model couplings are realized by actions of the form [13, 17]

$$S_{\text{tensor}} = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^4x \int d^4\theta \mathcal{L}(\eta(\zeta); \zeta), \quad (7.1)$$

with $\eta(\zeta)$ given by equation (6.26). Upon evaluation of the contour integral, the action can be shown to reduce to that constructed originally in the $N = 1$ superspace setting in [12]$_{12}$.

Similarly, in the case of $O(2n)$ multiplets defined by equations (6.24) and (6.25), their general $N = 2$ supersymmetric sigma-model couplings are described by actions of the form [18, 44]

$$S_{O} = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^4x \int d^4\theta \mathcal{L}(H^{[-]}(\zeta); \zeta). \quad (7.2)$$

$^{12}$ Incidentally, using the general results on self-interacting $N = 2$ tensor multiplets obtained in [12], the representation (7.1) could have been discovered already in 1983, if the authors of [12] had used a classical formula of Whittaker for harmonic functions in $\mathbb{R}^3$ [43].
In the case of polar multiplets defined by equations (6.27) and (6.28), their general sigma-model couplings are described by actions of the form [18]

$$S_{\text{polar}} = \frac{1}{2\pi i} \oint d\zeta \int d^4 x \, d^4 \theta L(\Upsilon(\zeta), \bar{\Upsilon}(\zeta); \zeta).$$

(7.3)

Finally, the most general off-shell $\mathcal{N} = 2$ supersymmetric sigma-models describe couplings of tensor multiplets, $\mathcal{O}(2n)$ multiplets and polar multiplets:

$$S_{\text{general}} = \frac{1}{2\pi i} \oint d\zeta \int d^4 x \, d^4 \theta L(\eta(\zeta), H[\cdots](\zeta), \Upsilon(\zeta), \bar{\Upsilon}(\zeta); \zeta).$$

(7.4)

In all of the off-shell $\mathcal{N} = 2$ supersymmetric sigma-models introduced, the Lagrangian may depend explicitly on $\zeta$. Each of the Lagrangians

$$L(\eta; \zeta), \quad L(H[\cdots]; \zeta), \quad L(\Upsilon, \bar{\Upsilon}; \zeta), \quad L(\eta, H[\cdots], \Upsilon, \bar{\Upsilon}; \zeta)$$

should be an analytic function of its arguments, but otherwise arbitrary, modulo a reality condition with respect to the smile conjugation.

### 7.2. Generalized Legendre transform construction

The action (7.4) provides us with the most general off-shell $\mathcal{N} = 2$ supersymmetric sigma-models that can be constructed in projective superspace. The Lagrangian in (7.4) can be chosen at will, modulo mild restrictions discussed earlier. Different choices of the Lagrangian will lead, in general, to different hyperkähler metrics in target space. So, it is natural to ask: does projective superspace offer us a free lunch? In other words, can we immediately read off the target space metric from (7.4)? The answer in general is ‘no’. Except in the very special case of tensor models (7.1), which will be discussed separately, one has to go through a technical procedure known as the \textit{generalized Legendre transform construction}, originally sketched in [18], in order to derive a hyperkähler metric from (7.4). It is called ‘generalized’ because it is an extension of the so-called \textit{linear Legendre transform construction} [8, 12, 13] to be discussed in the next subsection.

To fix the ideas, consider an $\mathcal{N} = 2$ supersymmetric nonlinear sigma-model described either by a single $\mathcal{O}(2n)$ multiplet ($n \geq 2$) or by a polar multiplet. Upon evaluation of the contour integral, the action becomes

$$S = \int d^4 x \, d^4 \theta L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{U}_i)$$

(7.5)

for some Lagrangian $L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{U}_i)$. The dynamical variables of the theory consist of (i) two physical superfields $\Phi$ and $\Sigma$ and their conjugates $\bar{\Phi}$ and $\bar{\Sigma}$, and (ii) some number of auxiliary superfields $\mathcal{U}_i$. Here the index $i$ may take a finite ($2n - 3$, in the case of $\mathcal{O}(2n)$ multiplet) or infinite (in the case of polar multiplet) number of values. The physical superfields $\Phi$ and $\Sigma$ are chiral and complex linear:

$$\bar{D}_\alpha \Phi = 0, \quad D^2 \Sigma = 0,$$

(7.6)

while the auxiliary superfields $\mathcal{U}_i$ are \textit{unconstrained}. The $\mathcal{U}_i$s are auxiliary, for their Euler–Lagrange equations are algebraic:

$$\frac{\partial}{\partial \mathcal{U}_i} L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{U}_i) = 0.$$  

(7.7)

Under reasonable regularity conditions on the Lagrangian, these equations uniquely determine the auxiliary superfields as functions of the physical ones:

$$\mathcal{U}_i = \mathcal{U}_i(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}).$$

(7.8)
This leads to an action formulated in terms of the physical superfields:

\[ S = \int d^4x \, d^4\theta L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}), \]

\[ L := L_{\text{off-shell}}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}, \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})). \]

This action is of course \( \mathcal{N} = 2 \) supersymmetric; however, only one of the two supersymmetries is manifest. Since the auxiliaries have been eliminated, the first and second supersymmetry transformations form the \( \mathcal{N} = 2 \) super-Poincaré algebra only on the mass shell.

Even though the action (7.9) is formulated only in terms of the physical superfields, the Lagrangian

\[ L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \]

is not a hyperkähler potential, since the dynamical variable \( \bar{\Sigma} \) is complex linear. As discussed in subsection 4.2, the Lagrangian coincides with the hyperkähler potential of the target space, provided the theory is formulated in terms of chiral superfields and their conjugates only. Is it possible to develop such a (re)formulation for the theory (7.9)?

The answer is affirmative under reasonably general conditions, due to the existence of a duality between chiral and complex linear superfields that was noticed for the first time by Zumino [46].

It has been known for 30 years [47] that the chiral and complex linear superfields provide different off-shell descriptions of the free \( \mathcal{N} = 1 \) scalar multiplet, which are known as the minimal and non-minimal scalar multiplet models, respectively. They are described by the following actions:

\[ S_{\text{minimal}} = \int d^4x \, d^4\theta \Psi \bar{\Psi}, \quad \bar{D}_a \Psi = 0, \]  

\[ S_{\text{non-minimal}} = -\int d^4x \, d^4\theta \Sigma \bar{\Sigma}, \quad \bar{D}^2 \Sigma = 0. \]

It is easy to read off the corresponding equations of motion\(^\text{13}\). On the mass shell, the dynamical superfields must obey the off-shell constraints and the equations of motion. It is convenient to combine them in a simple table:

|                  | Off-shell constraint | Equation of motion |
|------------------|---------------------|--------------------|
| Minimal          | \( \bar{D}_a \Psi = 0 \) | \( D^2 \Psi = 0 \) |
| Non-minimal      | \( D^2 \Sigma = 0 \)  | \( \bar{D}_a \Sigma = 0 \) |

One can see that the two models (7.10a) and (7.10b) are dynamically equivalent. Moreover, these models are dual to each other. This means that they are related to each other through the use of a first-order action. Such an action can be chosen [46] to be

\[ S_{\text{first-order}} = \int d^4x \, d^4\theta \{-\bar{\Gamma} \Gamma + \Psi \bar{\Gamma} + \bar{\Psi} \bar{\Gamma}\}. \]

(7.11)

Here \( \Gamma \) is complex unconstrained, while \( \Psi \) is chiral, \( \bar{D}_a \Psi = 0 \). Varying this action with respect to \( \Psi \) gives \( \Gamma = \Sigma \), and then \( S_{\text{first-order}} \) reduces to (7.10b). On the other hand, the equation of motion for \( \Gamma \) implies \( \bar{\Gamma} = \Psi \), and then \( S_{\text{first-order}} \) reduces to (7.10a).

Let us generalize the simple example analyzed above. Consider a theory of self-interacting complex linear superfields \( \Sigma^a \) and their conjugates \( \bar{\Sigma}^\alpha \) described by an action of the form

\[ S = \int d^4x \, d^4\theta \mathcal{L}(\Sigma^a, \bar{\Sigma}^\alpha), \]

(7.12)

\(^\text{13}\) In deriving the equations of motion for \( \Psi \) and \( \Sigma \), it is useful to represent \( \Psi = \bar{D} \hat{R} \) and \( \Sigma = \bar{D}_a \hat{\xi}^a \), for unconstrained superfields \( \hat{R} \) and \( \hat{\xi}^a \).
where the Lagrangian $\mathcal{L}(\Sigma, \bar{\Sigma})$ is a real analytic function of the dynamical superfields. By analogy with (7.11), we can associate with (7.12) the following first-order action:

$$S_{\text{first-order}} = \int d^4 x \ d^4 \theta \left\{ \mathcal{L}(\Gamma, \bar{\Gamma}) + \Psi_a \Gamma^a + \bar{\Psi}_a \bar{\Gamma}^a \right\}, \quad (7.13)$$

where $\Gamma^a$ are complex unconstrained, while $\Psi_a$ chiral, $\bar{D}_a \Psi_a = 0$. This theory is equivalent to (7.12). Indeed, varying (7.13) with respect to $\Psi_a$ gives $\Gamma^a = \Sigma^a$, and then (7.13) reduces to (7.12). Now, consider the equations of motion for $\Gamma^a$ and $\bar{\Gamma}^a$:

$$\frac{\partial}{\partial \Gamma^a} \mathcal{L}(\Gamma, \bar{\Gamma}) + \Psi_a = 0, \quad \frac{\partial}{\partial \bar{\Gamma}^a} \mathcal{L}(\Gamma, \bar{\Gamma}) + \bar{\Psi}_a = 0. \quad (7.14)$$

These equations allow one to express $\Gamma^a$ and $\bar{\Gamma}^a$ in terms of $\Psi_a$ and $\bar{\Psi}_a$ provided

$$\det \left( \begin{array}{cc} \mathcal{L}_{ab} & \mathcal{L}_{ab} \\ \mathcal{L}_{ab} & \mathcal{L}_{ab} \end{array} \right) \neq 0. \quad (7.15)$$

Then, the action (7.13) turns into

$$S_{\text{dual}} = \int d^4 x \ d^4 \theta K(\Psi_a, \bar{\Psi}_b), \quad (7.16)$$

where we have defined

$$K(\Psi, \bar{\Psi}) := \left\{ \mathcal{L}(\Gamma, \bar{\Gamma}) + \Psi_a \Gamma^a + \bar{\Psi}_a \bar{\Gamma}^a \right\} |_{\Gamma = \Gamma(\Psi, \bar{\Psi})}. \quad (7.17)$$

It is clear that $K(\Psi, \bar{\Psi})$ is (up to a trivial sign difference) the Legendre transform of $\mathcal{L}(\Gamma, \bar{\Gamma})$. Standard properties of the Legendre transformation now imply

$$\frac{\partial}{\partial \Psi_a} K(\Psi, \bar{\Psi}) - \Gamma^a = 0, \quad \frac{\partial}{\partial \bar{\Psi}_a} K(\Psi, \bar{\Psi}) - \bar{\Gamma}^a = 0 \quad (7.18)$$

as well as

$$\det \left( \begin{array}{cc} \mathcal{K}_{ab} & \mathcal{K}_{ab} \\ \mathcal{K}_{ab} & \mathcal{K}_{ab} \end{array} \right) \neq 0. \quad (7.19)$$

It is natural to interpret the Lagrangian in (7.16) as the Kähler potential of a Kähler manifold. For such an interpretation to be consistent, it must hold that

$$\det \left( \frac{\partial K}{\partial \Psi_a \bar{\Psi}_b} \right) \neq 0. \quad (7.20)$$

Then, due to (7.15) and (7.19), we must also have

$$\det \left( \frac{\partial \mathcal{L}}{\partial \Sigma_a \bar{\Sigma}_b} \right) \neq 0. \quad (7.21)$$

The latter condition is equivalent to the fact that, say, the first equation in (7.14) can be solved to express the variables $\Gamma^a$s as functions of $\Psi_a$s and $\bar{\Psi}_a$s.

Our consideration shows that the requirements (7.15) and (7.21) are essential for the theory (7.12) to provide a dual description of $\mathcal{N} = 1$ supersymmetric nonlinear sigma-models.

Before returning to the theory of our interest, equation (7.9), it is worth mentioning another important aspect concerning the dual theories (7.12) and (7.16). One can develop a dual version of (7.16) by considering a first-order action of the form

$$S_{\text{first-order}} = \int d^4 x \ d^4 \theta \left\{ K(U, \bar{U}) - \Sigma_a U_a - \bar{\Sigma}^b \bar{U}_b \right\}. \quad (7.22)$$
where the superfields $U_a$ are complex unconstrained, and $\Sigma^a$ complex linear. The variables $U$ and $\bar{U}$ can be integrated out, due to (7.19). If $K$ coincides with (7.17), one then ends up with (7.12). However, the Lagrangian in (7.16) is defined modulo Kähler transformations

$$K(\Psi, \bar{\Psi}) \rightarrow \tilde{K}(\Psi, \bar{\Psi}) = K(\Psi, \bar{\Psi}) + \Lambda(\Psi) + \bar{\Lambda}(\bar{\Psi}),$$

(7.23)

with $\Lambda(\Psi)$ an arbitrary holomorphic function. If one replaces $K \rightarrow \tilde{K}$ in (7.22), and then integrates out the variables $U$ and $\bar{U}$, the resulting theory will be described by a Lagrangian $\tilde{L}(\Sigma^a, \bar{\Sigma}^\dagger)$ that differs from that appearing in (7.12). Actually, applying Kähler transformations may lead to quite a bizarre situation. The point is that the transformed Kähler potential, $\tilde{K}(\Psi, \bar{\Psi})$, always obeys the inequality (7.20). However, equation (7.19) may not hold for $\tilde{K}$, and then the procedure of integrating out the variables $U$ and $\bar{U}$ from (7.22) becomes more involved.

Finally, let us return to our sigma-model (7.9). It is equivalent to the following first-order action:

$$S_{\text{first-order}} = \int d^4x \ d^4\theta \{ L(\Phi, \bar{\Phi}; \Gamma, \bar{\Gamma}) + \Psi \Gamma + \bar{\Psi} \bar{\Gamma} \}. \quad (7.24)$$

Integrating out $\Gamma$ and $\bar{\Gamma}$ leads to an action of the form

$$S_{\text{dual}} = \int d^4x \ d^4\theta H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}), \quad (7.25)$$

where $H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ is the Legendre transform of $L(\Phi, \bar{\Phi}; \Sigma, \bar{\Sigma})$, with $\Phi$ and $\bar{\Phi}$ being treated as parameters. The resulting Lagrangian, $H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$, is the Kähler potential of a hyperkähler manifold.

### 7.3. Linear Legendre transform construction

Here we briefly review the famous linear Legendre transform construction [8, 12, 13]. Our presentation is intentionally brief, for it is hardly possible to present this construction better than it has already been done in [8].

The most general $\mathcal{N} = 2$ supersymmetric sigma-model coupling of several tensor multiplets $\eta^i(\zeta)$ is described by an action of the form [13]

$$S = \frac{1}{2\pi} \oint \frac{d\zeta}{\zeta} \int d^4x d^4\theta L(\eta^i(\zeta); \zeta), \quad (7.26)$$

where the dynamical variables are

$$\eta^i(\zeta) = \frac{1}{\zeta} \psi^i + G^i - \zeta \bar{\psi}^i; \quad \tilde{D}_a \psi^i = 0, \quad \tilde{D}^2 G^i = \tilde{G}^i - G^i = 0. \quad (7.27)$$

Unlike the multiplets considered in the previous subsection, the tensor multiplet requires no $\mathcal{N} = 1$ auxiliary superfields. Suppose we have evaluated the contour integral in (7.26). Then, the action turns into

$$S = \int d^4x d^4\theta L(\psi^i, \bar{\psi}^i, G^i). \quad (7.28)$$

Here the Lagrangian cannot yet be identified with a hyperkähler potential, for the superfields $G^i$ are real linear. In order to derive the hyperkähler potential of the target space, we have to dualize each $\mathcal{N} = 1$ tensor multiplet, $G^i$, into a chiral superfield $\Psi_i$ and its conjugates $\bar{\Psi}_i$. It is worth studying in some more detail how such a duality works.

As an example, consider $K(\Psi, \bar{\Psi}) = \bar{\Psi}\Psi$ and choose $\tilde{K}(\Psi, \bar{\Psi}) = \bar{\Psi}\Psi + (\alpha/2)(\psi^2 + \bar{\psi}^2)$ with $\alpha$ a constant parameter. Equation (7.19) does not hold for $\tilde{K}$ if $\alpha = \pm 1$. 

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The $\mathcal{N} = 1$ tensor multiplet [48] provides a variant off-shell realization of the massless scalar multiplet in which one of the two scalar fields is dualized into a gauge antisymmetric tensor field. Consider the models for free chiral and real linear superfields:

$$S_{\text{scalar}} = \frac{1}{2} \int d^4x \ d^4\theta (\Psi + \bar{\Psi})^2, \quad \bar{D}_a \Psi = 0, \quad (7.29a)$$

$$S_{\text{tensor}} = -\frac{1}{2} \int d^4x \ d^4\theta G^2, \quad \bar{D}^2 G = G - \bar{G} = 0. \quad (7.29b)$$

Here the action (7.29a) can be seen to coincide with (7.10a). The constraint on $G$ is solved [48] by introducing a chiral spinor prepotential $\eta_{\alpha}$ by the rule

$$G = D^\alpha \eta_{\alpha} + \bar{D}_{\dot{\alpha}} \bar{\eta}^\dot{\alpha}, \quad \bar{D}_{\dot{\alpha}} \eta_{\alpha} = 0. \quad (7.30)$$

The prepotential is defined modulo gauge transformations of the form

$$\delta \eta_{\alpha} = i \bar{D}^2 D_{\alpha} V, \quad V = \bar{V}. \quad (7.31)$$

The theories (7.29a) and (7.29b) are dynamically equivalent, as can be seen from the following table:

|                | Off-shell constraint | Equation of motion |
|----------------|---------------------|--------------------|
| Scalar multiplet | $\bar{D}^2 D_a (\Psi + \bar{\Psi}) = 0$ | $\bar{D}^2 (\Psi + \bar{\Psi}) = 0$ |
| Tensor multiplet  | $\bar{D}^2 G = 0$ | $\bar{D}^2 D_a G = 0$ |

Moreover, the theories (7.29a) and (7.29b) are dual to each other, because they are related to each other by the first-order action [48]

$$S_{\text{first-order}} = -\int d^4x \ d^4\theta \left\{ \frac{1}{2} \Pi^2 - \Pi (\Psi + \bar{\Psi}) \right\}. \quad (7.32)$$

Here $\Pi$ is a real unconstrained superfield.

As a generalization of the above example, consider a model of $n$ self-interacting tensor multiplets [48]

$$S = \int d^4x \ d^4\theta \mathcal{L}(G^i). \quad (7.33)$$

It proves to be dual to a nonlinear sigma-model described by chiral scalars $\Psi_i$ and their conjugates $\bar{\Psi}_i$ [12]. To construct the latter, one introduces the first-order action [12]

$$S_{\text{first-order}} = \int d^4x \ d^4\theta \{ \mathcal{L}(\Pi^i) + \Pi^i (\Psi_i + \bar{\Psi}_i) \}. \quad (7.34)$$

where $\Pi^i$ are real unconstrained superfields. Varying $\Psi_i$ gives $\Pi^i = G^i$, and then $S_{\text{first-order}}$ reduces to (7.33). On the other hand, one can integrate out $\Pi$s using their equations of motion

$$\frac{\partial}{\partial \Pi^i} \mathcal{L}(\Pi) + \Psi_i + \bar{\Psi}_i = 0. \quad (7.35)$$

to end up with

$$S = \int d^4x \ d^4\theta \mathcal{K}(\Psi^i + \bar{\Psi}^i), \quad \mathcal{K}(\Psi + \bar{\Psi}) := \mathcal{L}(\Pi) + \Pi^i (\Psi_i + \bar{\Psi}_i). \quad (7.36)$$

The Kähler potential, $\mathcal{K}$, is the Legendre transform of $\mathcal{L}$. Since $\mathcal{K}$ depends on $\Psi$s and $\bar{\Psi}$s only via combinations $(\Psi + \bar{\Psi})$, the $2n$-dimensional target spaces possess at least $n \ U(1)$ isometries.
The Legendre transformation considered generalizes to any number of tensor multiplets interacting with matter [12]. In particular, it can be applied to the model of our interest, equation (7.28). This requires considering the following first-order action:

$$S_{\text{first-order}} = \int d^4x d^4\theta \left\{ L(\phi^i, \bar{\phi}^i, \Pi^i) + \Pi^i (\Psi_i + \bar{\Psi}_i) \right\}. \quad (7.37)$$

Here $\Pi^i$ is real unconstrained, and $\Psi_i$ is chiral, $\bar{D}_\alpha \Psi_i = 0$. Integrating out the variables $\Pi$s leads to an action of the form

$$S_{\text{dual}} = \int d^4x d^4\theta H(\phi^i, \bar{\phi}^i, \Psi_j + \bar{\Psi}_j), \quad (7.38)$$

where $H(\phi, \bar{\phi}, \Psi + \bar{\Psi})$ is the Legendre transform of $L(\phi, \bar{\phi}, G)$. It is the Kähler potential of a hyperkähler manifold.

To construct a dual of (7.38), with respect to the variables $\Psi$s and $\bar{\Psi}$s, one could again use a first-order action of the type (7.22), that is,

$$S_{\text{first-order}} = \int d^4x d^4\theta \left\{ H(\phi^i, \bar{\phi}^i, U_j + \bar{U}_j) - \Sigma^i U_i - \bar{\Sigma}^i \bar{U}_i \right\}, \quad (7.39)$$

where the superfields $U_i$ are complex unconstrained and $\Sigma^i$ complex linear. However, the equations of motion for $U$s and $\bar{U}$s imply that $\Sigma^i = \bar{\Sigma}^i = G^i$. Therefore, the duality transformation can be performed using the following first-order action:

$$S_{\text{first-order}} = \int d^4x d^4\theta \left\{ H(\phi^i, \bar{\phi}^i, V_j) - G^i V_i \right\} \quad (7.40)$$

with the variables $V_i$ real unconstrained.

### 7.4. Universality of polar multiplet sigma-models

In general, off-shell $\mathcal{N} = 2$ supersymmetric $\sigma$-models can describe couplings of tensor multiplets, $\mathcal{O}(2n)$ multiplets and polar multiplets:

$$S = \frac{1}{2\pi i} \int d\zeta \int d^4x d^4\theta \mathcal{L}(\eta(\zeta), H^{1/4}(\zeta), \Upsilon(\zeta), \bar{\Upsilon}(\zeta); \zeta). \quad (7.41)$$

However, it is always possible, in principle, to dualize any tensor multiplet into a polar multiplet and also any $\mathcal{O}(2n)$ multiplet into a polar one [18, 20]. As a result, the most general $\mathcal{N} = 2$ $\sigma$-model can in principle be described by polar multiplets only, using the action [18]

$$S = \frac{1}{2\pi i} \int d\zeta \int d^4x d^4\theta \mathcal{L}(\Upsilon(\zeta), \bar{\Upsilon}(\zeta); \zeta). \quad (7.42)$$

Different choices of $\mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta)$ may lead to one and the same hyperkähler geometry. The point is that a polar multiplet can be dualized into a polar one [49, 50], and the dual Lagrangian differs, in general, from the original one.

**Example.** For any real parameter $\alpha \in \mathbb{R}$, $\alpha \neq \pm 1$, the Lagrangian

$$\mathcal{L}_\alpha(\Upsilon, \bar{\Upsilon}; \zeta) = \frac{1}{1 - \alpha^2} \left\{ \dot{\Upsilon} \Upsilon + \frac{\alpha}{2} \left( \frac{1}{\zeta^2} \Upsilon^2 + \zeta^2 \dot{\Upsilon}^2 \right) \right\} \quad (7.43)$$

is equivalent (dual) to the free polar multiplet Lagrangian

$$\mathcal{L}(\Upsilon, \bar{\Upsilon}) = \dot{\Upsilon} \Upsilon. \quad (7.44)$$
8. \( \mathcal{N} = 2 \) supersymmetric sigma-models on cotangent bundles of Kähler manifolds

As discussed earlier, the most general sigma-model couplings of polar multiplets, equation (7.42), were introduced in 1988 by Lindström and Roček [18]. For some 10 years, this theory remained a purely formal construction, because no technique existed to eliminate the auxiliary superfields contained in the arctic multiplet, except in the case of Lagrangians quadratic in \( \Upsilon \) and \( \bar{\Upsilon} \). This situation changed in the late 1990s when [21, 49, 51] identified a subclass of models (7.42), possessing interesting geometric properties. For these models, one can develop a simple procedure to eliminate the auxiliaries in perturbation theory, and in some cases exactly. They are described by \( \mathcal{N} = 2 \) supersymmetric actions of the form

\[
S[\Upsilon, \bar{\Upsilon}] = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta} \int d^4x \ d^4\theta K(\Upsilon^I(\zeta), \bar{\Upsilon}^J(\zeta)),
\]

where \( \gamma \) denotes a closed contour around the origin, and \( K(\Phi^I, \bar{\Phi}^J) \) is the Kähler potential of a real-analytic Kähler manifold \( \mathcal{M} \).

As usual, the dynamical variables \( \Upsilon^I(\zeta) \) and \( \bar{\Upsilon}^J(\zeta) \) in (8.1) are arctic and antarctic multiplets, respectively:

\[
\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \Upsilon_I^n \zeta^n = \Phi^I + \Sigma^I \zeta + O(\zeta^2), \quad \bar{\Upsilon}^J(\zeta) = \sum_{n=0}^{\infty} \bar{\Upsilon}_J^n (-\zeta)^{-n}.
\]

Here \( \Phi^I \) is chiral, \( \Sigma^I \) complex linear,

\[
D_a \Phi^I = 0, \quad D^2 \Sigma^I = 0,
\]

and the remaining component superfields are complex unconstrained. The above theory is a minimal \( \mathcal{N} = 2 \) extension of the general \( \mathcal{N} = 1 \) supersymmetric nonlinear sigma-model [6]

\[
S[\Phi, \bar{\Phi}] = \int d^4x \ d^4\theta K(\Phi^I, \bar{\Phi}^J).
\]

8.1. Geometric properties

Let us turn to discussing the geometric properties of theory (8.1). What distinguishes the Lagrangian in (8.1) from that appearing in the most general case, equation (7.42), is that the former has no explicit dependence on \( \zeta \). As is well known from classical mechanics, the mathematical realization of the principle of the homogeneity of time is that the Lagrangian of a closed dynamical system has no explicit dependence on the time variable. Given such a Lagrangian, \( L(q, \dot{q}) \), the action is invariant under arbitrary time translations. The Lagrangian in (8.1) has no explicit dependence on \( \zeta \) which can be viewed as a complex evolution parameter. It is easy to see that (8.1) is invariant under U(1) transformations:

\[
\Upsilon(\zeta) \mapsto \Upsilon(e^{i\alpha} \zeta) \iff \Upsilon_n(z) \mapsto e^{in\alpha} \Upsilon_n(z).
\]

Transformations \( \zeta \mapsto e^{i\alpha} \zeta \) can be interpreted as time translations along \( \gamma \). This becomes manifest if the integration contour \( \gamma \) is chosen to be \( \zeta(t) = R e^{it} \).

The \( \mathcal{N} = 2 \) supersymmetric sigma-model inherits all the geometric features of its \( \mathcal{N} = 1 \) predecessor, specifically.

- **Kähler invariance**

\[
\mathcal{N} = 1 \text{ case : } \quad K(\Phi, \bar{\Phi}) \quad \mapsto \quad K(\Phi, \bar{\Phi}) + A(\Phi) + \bar{A}(\bar{\Phi}),
\]

\[
\mathcal{N} = 2 \text{ case : } \quad K(\Upsilon, \bar{\Upsilon}) \quad \mapsto \quad K(\Upsilon, \bar{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon});
\]
• Holomorphic reparametrizations of the Kähler manifold

\( N = 1 \) case :
\[ \Phi^I \longrightarrow \Phi'^I = f^I(\Phi), \]  
(8.7a)

\( N = 2 \) case :
\[ \Upsilon^I(\zeta) \longrightarrow \Upsilon'^I(\zeta) = f^I(\Upsilon(\zeta)). \]  
(8.7b)

Therefore, the physical superfields of the \( N = 2 \) theory
\[ \Upsilon^I(\zeta) \bigg| \zeta = 0 = \Phi^I, \quad \frac{d\Upsilon^I(\zeta)}{d\zeta} \bigg| \zeta = 0 = \Sigma^I. \]  
(8.8)

should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point. We conclude that the variables \((\Phi^I, \Sigma^I)\) parametrize the holomorphic tangent bundle \(T\mathcal{M}\) of the Kähler manifold \([21]\).

8.2. Tangent-bundle and cotangent-bundle formulations

To describe the theory in terms of the physical superfields \(\Phi^I\) and \(\Sigma^I\) only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion
\[ \oint d\zeta \zeta^p \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \Upsilon^I} = \oint d\zeta \zeta^p \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \bar{\Upsilon}^I} = 0, \quad n \geq 2. \]  
(8.9)

Let \(\Upsilon^I_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})\) denote their unique solution subject to the initial conditions
\[ \Upsilon^I_*(0) = \Phi^I, \quad \bar{\Upsilon}^I_*(0) = \Sigma^I. \]  
(8.10)

For a general Kähler manifold \(\mathcal{M}\), the auxiliary superfields \(\Upsilon^I_2, \Upsilon^I_3, \ldots\), and their conjugates, can be eliminated using perturbation theory only, with the aid of the following ansatz \([52]\):
\[ \Upsilon^I_n = \sum_{p=0}^{\infty} G^I_{J_1 \ldots J_p} \Phi^{J_1} \ldots \Phi^{J_p} \Sigma^{L_1} \ldots \bar{\Sigma}^{L_p}, \quad n \geq 2. \]  
(8.11)

This is the most general ansatz compatible with the U(1) symmetry (8.5). The only essential assumption to justify the use of perturbation theory is the requirement that the Kähler potential \(K(\Phi, \bar{\Phi})\) is real analytic. Determining step by step the coefficients \(G^I_{J_1 \ldots J_p} \Phi^{J_1} \ldots \Phi^{J_p}\), we can completely reconstruct the required solution \(\Upsilon^I_*(\zeta)\).

In some cases, the solution \(\Upsilon^I_*(\zeta)\) can be determined exactly. Let \(\mathcal{M}\) be a Hermitian symmetric space, and hence its curvature tensor is covariantly constant:
\[ \nabla_L R_{i_j k_j i_j} = \bar{\nabla}_L R_{i_j k_j i_j} = 0. \]  
(8.12)

Then, the curve \(\Upsilon_*(\zeta)\) turns out to obey the generalized geodesic equation \([49]\):
\[ \frac{d^2 \Upsilon^I_*(\zeta)}{d\zeta^2} + \Gamma^I_{JK}(\Upsilon_*(\zeta), \bar{\Upsilon}_*(\zeta), \Phi) \frac{d\Upsilon^J_*(\zeta)}{d\zeta} \frac{d\Upsilon^K_*(\zeta)}{d\zeta} = 0. \]  
(8.13)

An derivation of this result will be given below. It follows from (8.13) that only the term with \(p = 0\) in (8.11) is non-zero in the case of Hermitian symmetric spaces.

Suppose that all the auxiliary superfields have been eliminated, and the \(\Upsilon_*(\zeta)\) is known explicitly. The next technical problem to address is the evaluation of the contour integral
\[ S_{tb}[\Phi, \Sigma] := \frac{1}{2\pi i} \oint d\zeta \oint d^4 x d^4 \theta K(\Upsilon_*(\zeta), \bar{\Upsilon}_*(\zeta)). \]  
(8.14)
This is only a technical issue, though rather complicated in practice. However complicated, the outcome should be an action of the form

\[
S_0[\Phi, \Sigma] = \int d^4x \, d^4\theta \{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \}.
\]

(8.15)

Here \( \mathcal{L}_{IJ} = -g_{IJ}(\Phi, \bar{\Phi}) \) and the coefficients \( \mathcal{L}_{I_1 \ldots I_n} \) for \( n > 1 \) are tensor functions of the Kähler metric \( g_{IJ}(\Phi, \bar{\Phi}) = \partial_I \partial_J K(\Phi, \bar{\Phi}) \), the Riemann curvature \( R_{IJKL}(\Phi, \bar{\Phi}) \) and its covariant derivatives. Each term \( \mathcal{L}^{(n)} \) in the Lagrangian contains equal powers of \( \Sigma \) and \( \bar{\Sigma} \), since the original action is invariant under the rigid U(1) transformations (8.5).

It is instructive to reproduce here the explicit expressions for several functions \( \mathcal{L}^{(n)} \) appearing in (8.15). Direct calculations give

\[
\mathcal{L}^{(1)} = -g_{IJ} \Sigma^I \bar{\Sigma}^J, \\
\mathcal{L}^{(2)} = \frac{1}{4} R_{I_1J_1I_2J_2} \Sigma^{I_1} \bar{\Sigma}^{J_1} \bar{\Sigma}^{J_2}, \\
\mathcal{L}^{(3)} = -\frac{1}{12} \left\{ \frac{1}{6} \{ \nabla_{I_1}, \bar{\nabla}_{J_1} \} R_{I_1J_1I_2J_2} + R_{I_1J_1I_2} R_{LJ_2J_3L} \right\} \Sigma^{I_1} \ldots \Sigma^{I_2} \bar{\Sigma}^{J_1} \ldots \bar{\Sigma}^{J_2}. 
\]

(8.16a)

(8.16b)

(8.16c)

The expression for \( \mathcal{L}^{(4)} \) is given in [53].

To construct the dual formulation of (8.15), we follow the general scheme of subsection 7.2 and consider the first-order action

\[
S_{\text{first-order}} = \int d^4x \, d^4\theta \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}) + \Psi_I \Gamma^I + \bar{\Psi}_J \bar{\Gamma}^J \right\}.
\]

(8.17)

Here the tangent vector \( \Gamma^I \) is now complex unconstrained, while the one-form \( \Psi_I \) is chiral, \( \bar{D}_\theta \Psi_I = 0 \). Varying \( \Psi_I \) gives \( \bar{D}^2 \Gamma^I = 0 \), that is \( \Gamma^I = \Sigma^I \), and then (8.17) reduces to the original action. On the other hand, varying \( \Gamma^I \) gives

\[
\frac{\partial}{\partial \Gamma^I} \mathcal{L}(\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}) + \Psi_I = 0.
\]

(8.18)

Eliminating the \( \Gamma \)s and their conjugates\(^{15} \) leads to the dual action

\[
S_{\text{dual}}[\Phi, \bar{\Psi}] = \int d^4x \, d^4\theta \{ K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \}, \\
\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \ldots I_n} \Psi_{I_1} \ldots \Psi_{I_n} \bar{\Psi}_{J_1} \ldots \bar{\Psi}_{J_n},
\]

(8.19)

with

\[
\mathcal{H}^{I_1 \ldots I_n}(\Phi, \bar{\Phi}) = g^{I_1 \ldots I_n}(\Phi, \bar{\Phi}).
\]

(8.20)

The fact that each term in the expansion of \( \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \) contains equal powers of \( \Psi \) and \( \bar{\Psi} \), follows from the invariance of (8.17) under the rigid U(1) transformations

\[
\Phi^I(z) \rightarrow \Phi^I(z), \quad \Gamma^I(z) \rightarrow e^{i\omega} \Gamma^I(z), \quad \Psi_I(z) \rightarrow e^{-i\omega} \Psi_I(z).
\]

(8.21)

\(^{15} \)Since \( \mathcal{L} = -g_{IJ} \Sigma^I \bar{\Sigma}^J + O(\Sigma^4) \), both requirements (7.15) and (7.21) hold in a neighborhood of the zero section of the tangent bundle, \( TM \), of \( M \).
In the dual formulation of the $N = 2$ supersymmetric sigma-model, the target space is (an open neighborhood of the zero section of) the cotangent bundle $T^*M$ of the Kähler manifold $M$ \[49, 51\]. It is therefore a hyperkähler space, and

$$
\| (\Phi, \overline{\Phi}, \Psi, \overline{\Psi}) := K(\Phi, \overline{\Phi}) + \mathcal{H}(\Phi, \overline{\Phi}, \Psi, \overline{\Psi})
$$

the corresponding hyperkähler potential. Since

$$
\mathcal{H}(\Phi, \overline{\Phi}, \Psi, \overline{\Psi}) = g^I\overline{J}(\Phi, \overline{\Phi})\Psi_I\overline{\Psi}_J + O(|\Psi|^4),
$$

the hyperkähler metric is nonsingular in a neighborhood of the zero section of $T^*M$. These results agree with those derived independently in the mathematical literature by purely geometric means \[54, 55\].

### 8.3. Hermitian symmetric spaces: method I

If the Kähler manifold $M$ is Hermitian symmetric, then the $N = 2$ supersymmetric sigma-model on $T^*M$ can be derived in a closed form, as was first sketched in [49]. To carry out such a construction, there have been developed two alternative methods that are based on the use of conceptually different ideas and tools:

- Method I \[49, 51, 56, 57\] makes use of the properties that (i) $M$ is a homogeneous space, $M = G/H$; (ii) the group $G$ acts on $M$ by holomorphic isometries.
- Method II \[58, 59\] makes use of (i) the covariant constancy of the curvature; (ii) extended supersymmetry.

We now turn to discussing the first method. Method II will be reviewed in the next subsection.

As before, denote by $\Upsilon_\epsilon(\zeta) \equiv \Upsilon_\epsilon(\zeta; \Phi, \overline{\Phi}, \Sigma, \overline{\Sigma})$ the unique solution of the auxiliary field equations (8.9) under the initial conditions (8.10). Using the canonical coordinates \[60, 61\] for the Hermitian symmetric space $M$, which are defined in appendix B, we can find a part of the solution

$$
\Upsilon_0(\zeta) \equiv \Upsilon_{p_0}(\zeta) := \Upsilon_\epsilon(\zeta; \Phi = 0, \overline{\Phi} = 0, \Sigma_0, \overline{\Sigma}_0), \quad \Upsilon_{p_0}(0) = p_0,
$$

with $\Sigma_0$ a tangent vector at $p_0 \in M$, the origin of the canonical coordinate system. It is

$$
\Upsilon_0(\zeta) = \Sigma_0\zeta, \quad \Upsilon_{0}(\zeta) = -\overline{\Sigma}_0 \zeta.
$$

As a next step, we can construct a curve

$$
\Upsilon_p(\zeta), \quad \Upsilon_p(0) = p \in M
$$

obtained from $\Upsilon_{p_0}(\zeta)$ by applying an isometry transformation $g \in G$ such that $g \cdot p_0 = p$. The holomorphic isometry transformations leave the auxiliary field equations invariant.

Let $U \subset M$ be the neighborhood on which the canonical coordinate system is defined. We can construct a coset representative, $S: U \rightarrow G$, with the following property: associated with a point $p \in U$ is the holomorphic isometry $S[p] \in G$ of $M$, $q \rightarrow S[p] \cdot q \in M$, for any $q \in M$, such that

$$
S[p] \cdot p_0 = p.
$$

In local coordinates, $S[p] = S[\Phi, \overline{\Phi}]$, and it acts on a generic point $q \in U$ parametrized by complex variables $(\Xi^I, \overline{\Xi}^I)$ as follows:

$$
\Xi \rightarrow \Xi' = f(\Xi; \Phi, \overline{\Phi}), \quad f(0; \Phi, \overline{\Phi}) = \Phi.
$$
Now, applying the group transformation $S(\Phi, \bar{\Phi})$ to $\Upsilon_0(\zeta)$ gives
\[
\Upsilon_0(\zeta) \rightarrow \Upsilon_\ast(\zeta) = f(\Upsilon_0(\zeta); \Phi, \bar{\Phi}) = f(\Sigma_0; \Phi, \bar{\Phi}), \quad \Upsilon_\ast(0) = \Phi. \tag{8.28}
\]
Imposing the second initial condition, $\Upsilon_\ast(0) = \Sigma$, gives
\[
\Sigma' = \Sigma_0 \left. \frac{d}{d\zeta} f'(\zeta; \Phi, \bar{\Phi}) \right|_{\zeta=0}, \tag{8.29}
\]
and thus $\Sigma_0$ can be uniquely expressed in terms of $\Sigma$ and $\Phi, \bar{\Phi}$. As a result, the desired curve by $\Upsilon_\ast(\zeta)$ has been constructed.

As pointed out earlier, the curve $\Upsilon_\ast(\zeta)$ obeys the generalized geodesic equation (8.13). Now we are in a position to justify the claim. In the canonical coordinate system, the curve
\[
\Upsilon_0(\zeta) = \Sigma_0 \zeta, \quad \bar{\Upsilon}_0(\zeta) = -\Sigma_0 \bar{\zeta}, \tag{8.30}
\]
is characterized by
\[
\frac{d^2\Upsilon_0^I(\zeta)}{d\zeta^2} = \frac{d^2\bar{\Upsilon}_0^I(\zeta)}{d\bar{\zeta}^2} + \Gamma_{JK}^I(\Upsilon_0(\zeta), \Phi = 0) \frac{d\Upsilon_0^J(\zeta)}{d\zeta} \frac{d\Upsilon_0^K(\zeta)}{d\zeta} = 0. \tag{8.31}
\]
Since the equation
\[
\frac{d^2\Upsilon^I(\zeta)}{d\zeta^2} + \Gamma_{JK}^I(\Upsilon(\zeta), \Phi) \frac{d\Upsilon^J(\zeta)}{d\zeta} \frac{d\Upsilon^K(\zeta)}{d\zeta} = 0 \tag{8.32}
\]
is invariant under holomorphic isometries, we conclude that $\Upsilon_\ast(\zeta)$ indeed obeys the generalized geodesic equation (8.13).

Equation (8.13) leads to a simple corollary that is of special importance for the method to be discussed in the next subsection. Repeatedly using equation (8.13) allows us to compute the Taylor coefficients $\Upsilon_2, \Upsilon_3, \ldots$, in the expansion
\[
\Upsilon_\ast(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n^I \zeta^n = \Phi^I + \Sigma^I \zeta + \Upsilon_2^I \zeta^2 + O(\zeta^3). \tag{8.33}
\]
In particular, we derive
\[
\Upsilon_2^I = -\frac{1}{2} \Gamma_{JK}^I(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K. \tag{8.34}
\]

8.4. The Eguchi–Hanson metric and its non-compact cousin

As an instructive application of the method described, we consider the two-sphere $\mathcal{M} = \text{SU}(2)/\text{U}(1) \cong \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. In the north chart of $\mathbb{C}P^1$, the Kähler potential and metric are
\[
K(z, \bar{z}) = r^2 \ln(1 + z\bar{z}), \quad g_{z\bar{z}}(z, \bar{z}) = r^2 (1 + z\bar{z})^{-2}, \tag{8.35}
\]
with $1/r^2$ being proportional to the curvature. The coordinate system under consideration is canonical in the sense of appendix B.

Fractional linear (isometry) transformation
\[
z \rightarrow S[z, \Phi, \bar{\Phi}] = \frac{z + \Phi}{-\Phi z + 1}, \quad S[z, \Phi, \bar{\Phi}](0) = \Phi, \tag{8.36}
\]
induces
\[
\Upsilon_\ast(\zeta) = \frac{\Phi(1 + \Phi \bar{\Phi}) + \zeta \Sigma}{1 + \Phi \bar{\Phi} - \zeta \Phi \Sigma}. \tag{8.37}
\]
and then

\[ K(\bar{\Upsilon}_s(\zeta), \bar{\Upsilon}_s(\zeta)) = r^2 \ln \left\{ (1 + \Phi \Sigma) \left( 1 - \frac{\Sigma \Sigma}{(1 + \Phi \Phi)} \right) \right\} + \zeta \Lambda(\zeta) - \frac{1}{\zeta} \Lambda(-1/\zeta), \]  

(8.38)

with \( \Lambda(\zeta) \) some holomorphic function. The action becomes

\[ S[\bar{\Upsilon}_s, \bar{\Upsilon}_s] = \int d^4x \, d^2\theta \left\{ K(\Phi, \bar{\Phi}) + r^2 \ln \left( 1 - \frac{1}{r^2} g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \Sigma \right) \right\} \]  

(8.39)

and is well defined under the global restriction

\[ g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \Sigma < r^2. \]  

(8.40)

To construct the dual formulation of (8.39), we should introduce the corresponding first-order action (8.17). The equation of motion for \( \Gamma \), (8.18), in our case becomes

\[ \frac{g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Gamma}{r^2 - g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Gamma} = \frac{1}{r^2} \Psi, \quad g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Gamma < r^2. \]  

(8.41)

This equation and its conjugate allow us to express \( \Gamma \) in terms of \( \Psi \) and its conjugate, without any restriction on \( \Psi \):

\[ \Psi \bar{\Psi} < \infty. \]  

(8.42)

As a result, the target space of the dual theory coincides with \( T^* \mathbb{C}P^1 \) parametrized by local complex variables (\( \Phi, \bar{\Phi} \)). The Lagrangian of the dual theory, equation (8.19), is the hyperkähler potential

\[ \mathcal{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \Phi) + \mathcal{H}(\Phi, \Phi, \Psi, \bar{\Psi}), \]

\[ \mathcal{H}(\Phi, \Phi, \Psi, \bar{\Psi}) = r^2 \kappa, \quad r^2 \kappa = g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Psi \bar{\Psi}, \]  

(8.43a)

\[ \mathcal{F}(x) := \frac{1}{x} \left\{ \sqrt{1 + 4x} - 1 - \ln \frac{1 + \sqrt{1 + 4x}}{2} \right\}, \quad \mathcal{F}(0) = 1, \]  

(8.43b)

corresponding to the Eguchi–Hanson metric in the form given by Calabi [7].

It is of some interest to repeat the above analysis for the complex hyperbolic line \( \mathcal{M} = \text{SU}(1, 1)/U(1) \equiv \mathbb{H} \), which is a non-compact cousin of \( \mathbb{C}P^1 \). The Kähler potential and metric of \( \mathbb{H} \) are

\[ K(z, \bar{z}) = -r^2 \ln (1 - z \bar{z}), \quad g_{z \bar{z}}(z, \bar{z}) = r^2 (1 - z \bar{z})^{-2}. \]  

(8.44)

Instead of the action (8.39), we now have

\[ S[\bar{\Upsilon}_s, \bar{\Upsilon}_s] = \int d^4x \, d^2\theta \left\{ K(\Phi, \bar{\Phi}) - r^2 \ln \left( 1 + \frac{1}{r^2} g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \Sigma \right) \right\}. \]  

(8.45)

The action is defined on \( T\mathbb{H} \), and no restriction on the tangent variable \( \Sigma \) occurs. However, the dual formulation of the theory is well defined under the restriction

\[ g_{\Phi \bar{\Phi}}(\Phi, \bar{\Phi}) \Sigma \Sigma < r^2. \]  

(8.46)

As a result, the hyperkähler structure is defined on the open disk bundle in the cotangent bundle \( T^* \mathbb{H} \).

It is known that any compact Riemann surface, \( \mathcal{M}_g \), of genus \( g \geq 1 \) can be obtained from \( \mathbb{H} \) by factorization with respect to a discrete subgroup of \( \text{SU}(1, 1) \), see, e.g., [62]. Using the hyperkähler metric constructed on the open disk bundle in \( T^* \mathbb{H} \), we can generate a hyperkähler structure defined on an open neighborhood of the zero section of \( T^* \mathcal{M}_g \). The obtained hyperkähler metric is not complete.

\[ \text{Relations (8.40) and (8.42) are analogous to those appearing in special relativity. For a massive particle, its velocity, \( \bar{v} \), is constrained by } |\bar{v}| < c; \text{ however, the th-ee-momentum, } \bar{p}, \text{ can take arbitrary values, } |\bar{p}| < \infty. \]  

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8.5. Hermitian symmetric spaces: method II

Method I was successfully applied to the four series of compact Hermitian symmetric spaces:

\[
\begin{align*}
\text{U}(m + n), & \quad \text{Sp}(n), \quad \text{SO}(2n), \quad \text{SO}(n + 2) \quad \text{U}(m) \times \text{U}(n), \quad \text{U}(n), \\
\text{U}(m) \times \text{U}(n), & \quad \text{Sp}(n), \quad \text{SO}^*(2n), \quad \text{SO}_0(n, 2) \quad \text{SO}(n) \times \text{SO}(2) \times \text{SO}(2) \times \text{SO}(2),
\end{align*}
\]

(8.47)

as well as to their non-compact versions:

\[
\begin{align*}
\text{U}(m, n), & \quad \text{Sp}(n, \mathbb{R}), \quad \text{SO}^*(2n), \quad \text{SO}_0(n, 2) \quad \text{SO}(n) \times \text{SO}(2). \\
\text{U}(m) \times \text{U}(n), & \quad \text{U}(n), \quad \text{U}(n) \times \text{U}(n), \quad \text{U}(n).
\end{align*}
\]

(8.48)

on case by case basis. This construction was finalized in [57]. The general results are as follows.

- If the Hermitian symmetric space \( \mathcal{M} \) is compact, then the hyperkähler structure is defined on the whole \( T^* \mathcal{M} \).
- If \( \mathcal{M} \) is non-compact, then the hyperkähler structure is defined on a neighborhood of the zero section of \( T^* \mathcal{M} \), and cannot be extended to the whole cotangent bundle.

A detailed discussion of these related properties can be found in [57].

Although method I worked well for the regular series listed, it turned out to be impractical in the case of the exceptional Hermitian symmetric spaces:

\[
\begin{align*}
\text{E}_6 \quad \text{SO}(10) \times \text{U}(1), & \quad \text{E}_7 \quad \text{E}_6 \times \text{U}(1).
\end{align*}
\]

(8.49)

In order to work out these cases, an alternative method was developed in [58].

The method introduced in [58] is based on the use of extended supersymmetry. Let us start from the \( \mathcal{N} = 2 \) supersymmetry transformation of the arctic multiplet

\[
\delta \Upsilon^I(\zeta) = i \left( \epsilon^\alpha_Q Q^\alpha + \bar{\epsilon}^{\dot{\alpha}}_{\bar{Q}} \right) \Upsilon^I(\zeta)
\]

(8.50)

considered as an \( \mathcal{N} = 2 \) superfield. As a next step, we reduce this transformation to \( \mathcal{N} = 1 \) superspace. Then, the second hidden supersymmetry proves to act on the physical superfields \( \Phi \) and \( \Sigma \) as

\[
\delta \Phi^I = \bar{\epsilon}^\alpha \bar{D}^\alpha \Upsilon^I, \quad \delta \Sigma^I = -\epsilon^\mu D_\mu \Phi^I + \bar{\epsilon}^{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Upsilon^I.
\]

(8.51)

Upon elimination of the auxiliary superfields, the component \( \Upsilon^I_2 \) becomes a function of the physical superfields

\[
\Upsilon^I_2 = -\frac{1}{2} \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K.
\]

(8.52)

The tangent-bundle action

\[
S_{\text{tb}}[\Phi, \Sigma] = \int d^4x \, d^4\theta \{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \},
\]

(8.53)

has to be invariant under the above supersymmetry transformation. This is a highly nontrivial requirement. Indeed, by making use of the fact the the Riemann curvature is covariantly constant,

\[
\nabla_L R_{i_1 \ldots i_n \lambda} = \bar{\nabla}_L R_{i_1 \ldots i_n \lambda} = 0,
\]

(8.54)

and hence

\[
\nabla_L \mathcal{L}_{i_1 \ldots i_n \lambda} = \bar{\nabla}_L \mathcal{L}_{i_1 \ldots i_n \lambda} = 0,
\]

(8.55)

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we are able to show that the action is indeed supersymmetric provided
the following *linear* differential equation:
\[ \frac{1}{2} \sum_k \Sigma^k \Sigma^L R_{KL}^I L_I + \ell + g_{ij} \Sigma^j = 0, \quad \ell_I := \frac{\partial}{\partial \Sigma^I} \ell. \] (8.56)

A general solution of (8.56), which is compatible with the series expansion (8.53), was found in [58]. It is
\[ \ell = g_{ij} \Sigma^j + \Sigma^I \frac{R_{\Sigma^I}}{R_{\Sigma^I}}, \] (8.57)

In the dual, cotangent bundle formulation
\[ S_{\text{ctb}}[\Phi, \Psi] = \int d^dx d\theta [K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})], \]
\[ \mathcal{H} = \sum_{n=1}^{\infty} \mathcal{H}^{(n)}(\Phi, \bar{\Phi}) \Psi_1 \cdots \Psi_n \bar{\Psi}_1 \cdots \bar{\Psi}_n, \] (8.58)
the ‘Hamiltonian’ \( \mathcal{H} \) proves to obey the *nonlinear* differential equation
\[ \mathcal{H}^I g_{IJ} - \frac{1}{2} \mathcal{H}^I R_{KL}^J \Psi_I = \bar{\Psi}_J, \quad \mathcal{H}^I = \frac{\partial}{\partial \Psi_I} \mathcal{H}. \] (8.59)

It can be deduced from (8.56) by making use of the properties of the Legendre transformation. Using the above results, the case of \( E_6/\text{SO}(10) \times U(1) \) was worked out for the first time in [58].

A closed-form expression for \( \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \) was not obtained in [58]. It was derived in [59]. The same work also provided an alternative closed-form expression for \( \ell(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \). We reproduce here only the results obtained in [59].

**Tangent bundle formulation**
\[ \ell(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = -\frac{1}{2} \sum^g \ln \left( 1 + \frac{R_{\Sigma^I}}{R_{\Sigma^I}} \right) \Sigma^I, \quad \Sigma^I := \left( \frac{\Sigma^I}{\Sigma^J} \right). \] (8.60)

Here we have defined
\[ g := \begin{pmatrix} 1 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}, \quad R_{\Sigma^I} := \begin{pmatrix} 1 & (R_{\Sigma^I})^J \\ (R_{\Sigma^I})^J & 0 \end{pmatrix}, \]
\[ (R_{\Sigma^I})^J := \frac{1}{2} R_{KL}^J \Sigma^K \Sigma^L, \quad (R_{\Sigma^I})^J := \frac{1}{2} R_{KL}^J \Sigma^K \Sigma^L. \] (8.61)

**Cotangent bundle formulation**
\[ \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{2} \Psi^J g^{-1} \mathcal{F}(R_{\Psi^J}, \bar{\Psi}) \Psi, \quad \Psi := \begin{pmatrix} \Psi_I \\ \bar{\Psi}_I \end{pmatrix}, \] (8.62)
where the function \( \mathcal{F}(x) \) is given by equation (8.43b). The operator \( R_{\Psi^J} \) is defined as\n\[ R_{\Psi^J} := \begin{pmatrix} 1 & (R_{\Psi^J})^I \\ (R_{\Psi^J})^I & 0 \end{pmatrix}, \]
\[ (R_{\Psi^J})^I := R_{\Psi^J} g^{IJ} \mathcal{F}(\Psi_I, \bar{\Psi}_I), \quad (R_{\Psi^J})_{KL} := \frac{1}{2} R_{KL}^J \Psi_I \bar{\Psi}_J. \] (8.63)

As a result, for any Hermitian symmetric space \( M \), the hyperkähler potential on \( T^*M \) is
\[ \mathcal{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + \frac{1}{2} \Psi^I g^{-1} \mathcal{F}(R_{\Psi^J}, \bar{\Psi}) \Psi. \] (8.64)
In the mathematical literature, there exists a different representation for the hyperkähler potential derived in [63]. The Biquard–Gauduchon representation is

$$\mathcal{H}(\Phi, \Phi, \Psi, \bar{\Psi}) = \Psi^I \hat{g}^{-1} \mathcal{F}(-\mathcal{R}_{\Psi, \Psi}) \Psi,$$  \hspace{1cm} (8.65)

where

$$( \mathcal{R}_{\Psi, \Psi} )_{IJ} := \frac{1}{2} R_{IKL} \Psi_L \bar{\Psi}_K$$ \hspace{1cm} (8.66)

and $\hat{g}$ denotes an off-diagonal block of the Kähler metric

$$g := \begin{pmatrix} 0 & g_{IJ} \\ g_{IJ} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \hat{\gamma} \\ \hat{\gamma} & 0 \end{pmatrix}.$$ \hspace{1cm} (8.67)

The above unified formula was derived by Biquard and Gauduchon with the aid of purely algebraic means involving the root theory for Hermitian symmetric spaces, in conjunction with some guesswork based on the use of the Calabi metrics for $T^* CP^n$ [7]. In the supersymmetric setting described above, the results were obtained by making use of a regular procedure. No guesswork was needed.

9. The case of an arbitrary Kähler manifold $\mathcal{M}$

In the previous section, we used the power of supersymmetry to determine the hyperkähler potential on the cotangent bundle $T^* \mathcal{M}$ of an arbitrary Hermitian symmetric space. It is natural to wonder how much information can be extracted by using similar supersymmetry considerations in the case when $\mathcal{M}$ is an arbitrary real-analytic Kähler space. This problem was analyzed in [37]. Here we give a brief review of the results obtained.

Upon elimination of the auxiliary superfields, the second hidden supersymmetry becomes

$$\delta \Phi^I = \bar{\epsilon}_a \hat{D}^a \Sigma^I, \hspace{1cm} \delta \Sigma^I = -\bar{\epsilon}^a D_a \Phi^I + \bar{\epsilon}_a \hat{D}^a \Upsilon^I_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}),$$ \hspace{1cm} (9.1)

where the general form for $\Upsilon^I_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ is as follows:

$$\Upsilon^I_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = -\frac{1}{2} \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K + G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}),$$ \hspace{1cm} (9.2)

$$G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) := \sum_{p=1}^{\infty} G^I_{J_1,\ldots,J_p} \Sigma^{J_1} \ldots \Sigma^{J_p} \Sigma^I + O(\Sigma^4 \bar{\Sigma}^2).$$ \hspace{1cm} (9.3)

Here $G^I_{J_1,\ldots,J_p} \Sigma^{J_1} \ldots \Sigma^{J_p}$ are tensor functions of the Kähler metric, the Riemann curvature $R_{IJKL}(\Phi, \bar{\Phi})$ and its covariant derivatives.

The above representation for $\Upsilon^I_2(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ follows from the structure of the transformation laws with respect to holomorphic reparametrizations of the Kähler manifold $\mathcal{M}$:

$$\Phi^I \longrightarrow \Phi'^I = f^I(\Phi),$$ \hspace{1cm} (9.3a)

$$\Sigma^I \longrightarrow \Sigma'^I = \frac{\partial f^I(\Phi)}{\partial \Phi^J} \Sigma^J,$$ \hspace{1cm} (9.3b)

$$\Upsilon^I_2 \longrightarrow \Upsilon'^I_2 = \frac{1}{2} \frac{\partial^2 f^I(\Phi)}{\partial \Phi^J \partial \Phi^K} \Sigma^J \Sigma^K \Sigma^I + \frac{\partial f^I(\Phi)}{\partial \Phi^J} \Upsilon^J_2.$$ \hspace{1cm} (9.3c)

One can check that

$$G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \frac{1}{6} \nabla_J R_{IJL} \Phi^I \Sigma^L \Sigma^H \Sigma^J \Sigma^K \Sigma^L + O(\Sigma^4 \bar{\Sigma}^2).$$ \hspace{1cm} (9.4)
Our next step is to require the tangent-bundle action, equation (8.15), to be supersymmetric. This proves to imply that $L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ and $G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ obey the following equations:

\[
\frac{\partial L}{\partial \Sigma^I} \frac{\partial G^J}{\partial \Sigma^I} = \frac{\partial \Xi}{\partial \Sigma^I}, \tag{9.5a}
\]
\[
\nabla_\Sigma^I L + \frac{\partial L}{\partial \Sigma^I} \frac{\partial G^J}{\partial \Sigma^I} = \frac{\partial \Xi}{\partial \Sigma^I}, \tag{9.5b}
\]
\[
\frac{1}{2} \bar{R}_KL^J \frac{\partial L}{\partial \Sigma^I} \Sigma^K \Sigma^L + \frac{\partial L}{\partial \Sigma^I} + g_{IJ} \Sigma^J = \frac{\partial L}{\partial \Sigma^I} \nabla_\Sigma^J G^J = -\nabla_\Sigma^I \Xi, \tag{9.5c}
\]

where $\Xi$ turns out to be

\[
\Xi = \Sigma^I \nabla_\Sigma^I L + 2G^I \frac{\partial L}{\partial \Sigma^I}. \tag{9.6}
\]

We also define

\[
\nabla_\Sigma^I L := \sum_{n=1}^{\infty} \left( \nabla_\Sigma^I \chi_{L_1 L_2 \cdots L_n} (\Phi, \bar{\Phi}) \Sigma^{L_1} \cdots \Sigma^{L_n} \right) = \frac{\partial L}{\partial \Phi^I} - \frac{\partial L}{\partial \Sigma^K} \Gamma^K_{IJ} \Sigma^J, \tag{9.7}
\]

and similarly for $\nabla_\Sigma^I G^J$.

It is natural to analyze how the above relations simplify in the special case when $\mathcal{M}$ is Hermitian symmetric. It holds that

\[
\nabla_\Sigma^I R_{l_1 l_2} = \nabla_\Sigma^I \chi_{L_1 L_2} = 0 \implies \nabla_\Sigma^I L = G^I = \Xi = 0. \tag{9.8}
\]

In the cotangent-bundle formulation, equation (8.19), the chiral variables $(\Phi^I, \Psi_I)$ are local complex coordinates on the cotangent bundle $T^*\mathcal{M}$, and the hyperkähler potential of $T^*\mathcal{M}$ is

\[
\mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) := K(\Phi, \bar{\Phi}) + \mathcal{G}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}). \tag{9.9}
\]

Within this formulation, the second supersymmetry can be shown to take the form

\[
\delta \Phi^I = \frac{1}{2} D^2 \left\{ \epsilon^\theta \frac{\partial K}{\partial \Psi^I} \right\}, \quad \delta \Psi_I = -\frac{1}{2} D^2 \left\{ \epsilon^\theta \frac{\partial \mathbb{K}}{\partial \Phi^I} \right\}. \tag{9.10}
\]

Introduce the condensed notation

\[
\phi^a := (\Phi^I, \Psi_I), \quad \bar{\phi}^a := (\bar{\Phi}^I, \bar{\Psi}_I), \tag{9.11}
\]

as well as the symplectic matrix $\mathbb{J} = (\mathbb{J}^{ab})$, its inverse $\mathbb{J}^{-1} = (-\mathbb{J}_{ab})$ and their complex conjugates,

\[
\mathbb{J}^{ab} = \mathbb{J}^{ba} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad \mathbb{J}_{ab} = \mathbb{J}^{ab} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{9.12}
\]

Then, the supersymmetry transformation can be rewritten as

\[
\delta \phi^a = \frac{1}{2} \mathbb{J}^{ab} D^2 \left\{ \epsilon^\theta \frac{\partial \mathbb{K}}{\partial \phi^b} \right\} = \frac{1}{2} D^2 \left[ \epsilon^\theta \Omega^\phi \right], \quad \Omega^\phi := \mathbb{J}^{ab} \frac{\partial \mathbb{K}}{\partial \phi^b}. \tag{9.13}
\]

These results can now be linked to the general analysis of $\mathcal{N} = 2$ sigma-models in $\mathcal{N} = 1$ superspace [36] reviewed in subsection 4.2. First of all, we see that the supersymmetry transformations agree with the ansatz (4.17).
Using equation (9.13), we can read off the expression for the holomorphic two-form on $T^*M$. By definition, the anti-holomorphic two-form is
\[
\bar{\omega}_{\bar{a}\bar{b}} = g_{\bar{a}\bar{b}} \bar{\Omega}^\bar{c},
\]
with $g_{\bar{a}\bar{b}}$ the Kähler metric\(^{17}\):
\[
g_{\bar{a}\bar{b}} = \frac{\partial^2 K}{\partial \phi^a \partial \bar{\phi}^b}.
\]
(9.14)
Recalling the explicit form of $\bar{\Omega}^\bar{c}$, equation (9.13), we can immediately see that $\bar{\omega}_{\bar{a}\bar{b}}$ is indeed antisymmetric:
\[
\bar{\omega}_{\bar{a}\bar{b}} = g_{\bar{a}\bar{c}} J^\bar{c}_{\bar{b}}, \quad \omega_{ab} = g_{a\bar{c}} J^\bar{c}_{\bar{b}}.
\]
(9.15)
Direct calculations show that
\[
\omega_{ab} = J_{ab} + O(\bar{\Psi}_I). \quad \omega_{\bar{a}\bar{b}} = J_{\bar{a}\bar{b}} \implies \omega^{ab} = g^{a\bar{c}} g^{\bar{b}d} \bar{\omega}_{\bar{c}\bar{d}} = J^{ab}.
\]
(9.16)
As a result, the holomorphic symplectic two-form $\omega^{(2,0)}$ of $T^*M$ coincides with the canonical holomorphic symplectic two-form
\[
\omega^{(2,0)} := \frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b = d\Phi^I \wedge d\Psi_I.
\]
(9.17)

10. Topics not covered

These lectures, which reflect the author’s interests, have not touched upon several important topics concerning sigma-models in projective superspace. Here we would like to make a few comments about some of these developments.

There is a large body of research literature on sigma-model couplings of $\mathcal{N} = 2$ tensor multiplets, including the pioneering papers [8, 12, 13]. A complete list of references can be found in [50]. Self-couplings of $O(2n)$ multiplets, with $n \geq 2$, are less studied, see [50] for a review.

Off-shell 4D $\mathcal{N} = 2$ superconformal multiplets in projective superspace and their general sigma-model couplings were presented in [53]. In the case of $\mathcal{N} = 2$ tensor multiplets, their superconformal couplings were described much earlier in [13, 26, 64]. The most general $\mathcal{N} = 2$ superconformal sigma-models can be realized in terms of polar multiplets [53]. They are described by the action (8.1) in which the Kähler potential obeys the homogeneity condition\(^{18}\)
\[
\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}).
\]
(10.1)
The geometric interpretation of such sigma-models, albeit realized in a somewhat different form, was given in [65]. Their formulation in terms of $\mathcal{N} = 1$ chiral superfields, which is obtained upon elimination of the polar multiplet auxiliaries, was presented in [37].

The projective superspace approach was extended to six [66, 67] and five [52, 68] dimensions (with [68] devoted to 5D off-shell superconformal sigma-models).

\(^{17}\) We use a bold-face notation for the Kähler metric on $T^*M$ in order to distinguish it from the metric on $M$.

\(^{18}\) The action (8.4) with the Kähler potential obeying the homogeneity condition (10.1) defines a general $\mathcal{N} = 1$ superconformal sigma-model; see [37] for more details.
General off-shell $\mathcal{N} = 2$ locally supersymmetric nonlinear sigma-models were constructed in [69]. Their properties remain largely unexplored.

Interesting geometric aspects of sigma-models in projective superspace were uncovered in [50].

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Appendix A. Superconformal group

This appendix contains a summary of the 4D $\mathcal{N}$-extended superconformal group SU(2, 2|\mathcal{N}). Any element $g \in$ SU(2, 2|\mathcal{N}) is a $(4 + \mathcal{N}) \times (4 + \mathcal{N})$ supermatrix of the form

$$g^\dagger \Omega g = \Omega, \quad \text{Ber} = 1, \quad \Omega = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathbb{1}_\mathcal{N} \end{pmatrix},$$

where Ber $A$ stands for the superdeterminant of a supermatrix $A$ [35, 70]. In a neighborhood of the unit element of SU(2, 2|\mathcal{N}), every group element can be represented in the exponential form:

$$g = e^L, \quad L^\dagger \Omega + \Omega L = 0, \quad \text{str}L = 0,$$

with $L$ an element of the superconformal algebra, su(2, 2|\mathcal{N}). The most general expression for $L$ is as follows:

$$L = \begin{pmatrix} \omega^a_{\alpha \beta} - \sigma \delta^a_{\alpha \beta} & -ia^a_{\alpha \beta} & 2\eta^i_{\alpha} \\ -ib^a_{\alpha \beta} & -\omega^a_{\alpha \beta} + \bar{\sigma} \delta^a_{\alpha \beta} & 2\tilde{\eta}^{\bar{\beta}}_\alpha \\ 2\epsilon^i_{\alpha} & 2\bar{\eta}^i_{\bar{\beta}} & \frac{2}{\mathcal{N}} (\bar{\sigma} - \sigma) \delta^i_{\bar{\beta}} + \Lambda^i_{\bar{\beta}} \end{pmatrix},$$

where

$$\sigma = \frac{1}{2} \left( \mathcal{N} \frac{r}{N - 4} \phi \right), \quad \Lambda^i = -\Lambda, \quad \text{tr} \Lambda = 0.$$

Here the matrix elements, which are not associated with the super-Poincaré transformations (2.11b) and (2.21b), correspond to a special conformal transformation $a = (a_{ab^\dagger}) = a^\dagger$, $S$-supersymmetry $(\eta_{a\bar{\beta}}, \bar{\eta}_{\alpha a})$, combined scale and chiral transformation $\sigma$, and chiral SU(\mathcal{N}) transformation $\Lambda_{\bar{\beta}}$. The case \mathcal{N} = 4 requires a special consideration.

The 4D \mathcal{N} = 1 superconformal group was introduced by Wess and Zumino [3].

Appendix B. Canonical coordinates for Kähler manifolds

In this appendix, we recall the concept of canonical coordinates for Kähler manifolds [60].
Given a Kähler manifold $\mathcal{M}$, for any point $p_0 \in \mathcal{M}$ there exists a neighborhood of $p_0$ such that holomorphic reparametrizations and Kähler transformations can be used to choose coordinates with the origin at $p_0$ in which the Kähler potential is

$$K(\Phi, \bar{\Phi}) = g_{I\bar{J}}|\Phi^I\bar{\Phi}^J| + \sum_{m,n \geq 2} K^{(m,n)}(\Phi, \bar{\Phi}),$$

$$K^{(m,n)}(\Phi, \bar{\Phi}) := \frac{1}{m!n!} K_{l_1 \ldots l_m j_1 \ldots j_n} |\Phi^{l_1} \ldots \Phi^{l_m} \bar{\Phi}^{j_1} \ldots \bar{\Phi}^{j_n}|.$$

Such a coordinate system in the Kähler manifold is called canonical. It was first introduced by Bochner [60] and extensively used by Calabi [61]. There still remains freedom to perform linear reparametrizations which can be used to set the metric at the origin, $p \in \mathcal{M}$, to $g_{I\bar{J}} = \delta_{I\bar{J}}$. It turns out that the coefficients $K_{l_1 \ldots l_m j_1 \ldots j_n}$ are tensor functions of the Kähler metric $g_{I\bar{J}}$, the Riemann curvature $R_{IJK\ell}$ and its covariant derivatives, evaluated at the origin. In particular, one finds

$$K^{(2,2)} = \frac{1}{4} R_{I\bar{J}K\ell} |\Phi^I \bar{\Phi}^J \Phi^K \bar{\Phi}^\ell|,$$  

(B.2a)

$$K^{(3,2)} = \frac{1}{12} \nabla_I R_{I\bar{J}K\ell} |\Phi^I \ldots \Phi^3 \bar{\Phi}^J \bar{\Phi}^K \bar{\Phi}^\ell|,$$  

(B.2b)

$$K^{(4,2)} = \frac{1}{48} \nabla_I \nabla_J R_{I\bar{J}K\ell} |\Phi^I \ldots \Phi^4 \bar{\Phi}^J \bar{\Phi}^K \bar{\Phi}^\ell|,$$  

(B.2c)

$$K^{(3,3)} = \frac{1}{12} \left\{ \nabla_I \bar{\nabla}_J R_{I\bar{J}K\ell} |\Phi^I \ldots \Phi^3 \bar{\Phi}^J \bar{\Phi}^K \bar{\Phi}^\ell| + R_{I\bar{J}K\ell} |L_{\ell IJ} | \right\} |\Phi^I \ldots \Phi^3 \bar{\Phi}^J \bar{\Phi}^K \bar{\Phi}^\ell|. $$  

(B.2d)

In the modern literature on supersymmetric sigma-models, some authors, being unaware of the work of [60], refer to the canonical coordinates introduced as a normal gauge [32] or Kähler normal coordinates [71].

If $\mathcal{M}$ is Hermitian symmetric, then

$$\nabla_I R_{I\bar{J}K\ell} = \bar{\nabla}_L R_{IL\bar{J}\bar{K}\bar{\ell}} = 0 \quad \implies \quad K^{(m,n)} = 0, \quad m \neq n. $$

(B.3)

This follows from the fact that, for Hermitian symmetric spaces, there exists a closed-form expression for the Kähler potential in the canonical coordinates [59]:

$$K(\Phi, \bar{\Phi}) = -\frac{1}{2} \Phi^T g \ln(\mathbb{I} - R_{\Phi,\bar{\Phi}}) \Phi, \quad \Phi := \left(\Phi^I \right).$$  

(B.4)

Here we have introduced

$$g := \begin{pmatrix} 0 & g_{I\bar{J}} \\ g_{I\bar{J}} & 0 \end{pmatrix}, \quad R_{\Phi,\bar{\Phi}} := \begin{pmatrix} 0 & (R_{\Phi})^J \bar{\ell} \\ (R_{\Phi})^I J & 0 \end{pmatrix},$$

$$\begin{aligned} (R_{\Phi})^J \bar{\ell} & := \frac{1}{2} R_{K\ell\bar{J}} \Phi^K \Phi^\ell, \\
(R_{\Phi})^I J & := \frac{1}{2} R_{K\ell\bar{I}} \Phi^K \Phi^\ell. \end{aligned}$$

(B.5)

**Appendix C. Two-component (iso)spinor conventions**

In the case of two-component undotted spinors, such as $\psi_\alpha$ and $\psi^\alpha$, their indices are raised and lowered by the rule

$$\begin{aligned} \psi^\alpha &= \epsilon^{\alpha\beta} \psi_\beta, \\
\psi_\alpha &= \epsilon_{\alpha\beta} \psi^\beta, \end{aligned}$$

(C.1)
where $\varepsilon^{\alpha \beta}$ and $\varepsilon_{\alpha \beta}$ are $2 \times 2$ antisymmetric matrices normalized as
\[ \varepsilon^{12} = \varepsilon_{21} = 1. \] (C.2)

The same conventions are used for dotted spinors ($\bar{\psi}^\alpha$ and $\bar{\psi}^\beta$), and for SU(2) isospinors ($v_i$ and $v^i$), in particular
\[ v^i = \varepsilon^{ij} v_j, \quad v_i = \varepsilon_{ij} v^j. \] (C.3)

The SL(2, $\mathbb{C}$) invariant antisymmetric tensors $\varepsilon^{\alpha \beta}$ and $\varepsilon_{\alpha \beta}$ and the SU(2) invariant antisymmetric tensors $\varepsilon^{ij}$ and $\varepsilon_{ij}$ are normalized as in (C.2).

Lorentz-invariant spinor bi-products are defined by
\[ \psi \lambda = \psi^\alpha \lambda_\alpha, \quad \psi^2 = \psi \psi, \quad \bar{\psi} \lambda = \bar{\psi}^\alpha \lambda_\alpha, \quad \bar{\psi}^2 = \bar{\psi} \bar{\psi}. \] (C.4)

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