Extracting Phases from Aperiodic Signals

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There are some quantities which, although in principle exactly and uniquely defined, are hard to estimate from experimental data. Prominent examples are information theoretic quantities like algorithmic complexities (for which only upper bounds can be estimated [1]), Shannon entropies [2], and mutual entropies [3]. Opposed to these are quantities where even a unique definition is lacking, although most researchers believe that they have a decent common sense “definition”. Maybe the most important quantity in this category is the phase of a non-periodic signal.

For a pure sine wave signal the notion of phase is obvious and trivial, provided the sampling rate is high enough (which we will assume throughout the following). Things are nearly as clean for anharmonic periodic signals. There, we can map the orbit, e.g. by delay embedding, onto a closed loop in a plane, and we can define the phase by the angle of the vector from some point in the interior of the loop to the point corresponding to the actual state. This phase will of course depend on the central point and on the delay, but different choices will give equivalent phases, if the loop does not intersect itself: They will give the same average angular velocity \( \omega = \lim_{T \to \infty} (\phi(t + T) - \phi(t))/T \), and the difference between two phases defined that way will stay bounded with time. This “geometric” definition of phase can be generalized to aperiodic signals whenever there is a way – either via embeddings or using multivariate time series – to project the orbit into a plane in such a way that it always encircles some point. A typical example is the Rössler attractor for particular values of its parameters. But if the loop intersects itself such that its interior is divided into several domains, then central points chosen in different regions will lead to non-equivalent phases (see Fig.1).

In addition to this geometrical definition, another popular approach is via the Hilbert transform. Under mild restrictions on the signal \( x(t) \), the pair \( \{x(t), y(t) = (Hx)(t)\} \) form real and imaginary part of an analytic function, and the phase is defined as its argument in the complex plane. In general this gives a well-defined phase. Its value changes of course if the signal is shifted, \( x(t) \to x(t) + c \), but a unique phase is obtained after de-meaning, i.e. when \( c \) is such that \( \langle x(t) \rangle = 0 \) after the shift. Problems arise if the trajectory goes through the origin, \( x(t) = (Hx)(t) = 0 \) at some time \( t \).

A third situation where one might be tempted to see a “natural” way of defining a phase is when a signal arises from circular motion in some \( d \)-dimensional space with constant amplitude \( A \) and arbitrarily changing \( \phi(t) \),

\[
x(t) = A \cos(\phi(t))
\]

Certainly this is considered by many as the prototype of a situation where a phase is uniquely defined in an obvious way.

We now ask ourselves whether all three approaches give in general the same phase. If this is not the case, then each approach might be useful by itself, but it cannot claim to be really fundamental and universal.

In Fig.2 we show part of a signal. A phase portrait obtained by plotting \( x(t) \) against \( x(t + \tau) \) is shown in Fig.3, a similar phase portrait using the Hilbert transform in Fig.4. In neither case one sees a point around which the orbit circles, thus neither allows a clear and robust definition of phase. The spectrum, obtained with a Welch window, is shown in Fig.5. A prominent peak is seen, but this peak is not sharp and thus a unique angular frequency seems not obtainable. One can try several
other methods popular in signal analysis, but we argue that none of them will lead to a robust determination of a phase.

And yet – there is a simple and clear-cut phase that enters in this example. The signal shown in Fig. 2 is generated by a random process defined as

\[ x(t) = \cos(\phi(t)) \]  

with the phase performing a biased random walk (cf. \[10\]),

\[ \frac{d\phi(t)}{dt} = \omega + \eta(t) \]  

where \( \eta(t) \) is \( \delta \)-correlated white noise,

\[ \langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = D\delta(t-t'). \]  

The parameter values used in Figs. 2 to 5 are \( \omega = 1 \) Hz and \( D = 5 \), and the integration was made with step \( \delta t = 0.0001 \). The delay used in Fig. 3 was \( \tau = 0.015 \).

If the noise variance \( D \) were much smaller, we would not have much problems. The problems arise since we chose a rather large \( D \) such that the phase is not monotonically increasing. Instead there are large intervals during which the phase decreases, leading to “fake” loops in Figs. 3 and 4. Our point is not that presently popular methods for extracting phases cannot distinguish between such phase reversals (or even just sudden slowdowns of the instantaneous phase velocity) and “true” amplitude variations \[9\]. Rather, we want to stress that there is no way in principle to distinguish between them. Thus attempts to improve on phase extraction methods in similarly ambiguous situations \[10,11\] are likely to lead to ambiguous results, even if this ambiguity might be hidden.

Ways out of avoiding these ambiguities can be found only by restricting what we accept as a sensible phase definition. One could argue e.g. that a basic intuitive feature of a phase is its continuous temporal progression, i.e. positivity of the instantaneous phase velocity. Demanding this would mean that there is no possibility at all to define a phase for the above model, and the same would be true for a large class of signals.

Does this mean that such a requirement is too restric-
tive to be useful? We believe not. One traditional way out of the dilemma when phases should be defined for arbitrary signals is Fourier analysis. One decomposes the signal into harmonic components, and can then define phases for each component (or, when the signal is decomposed into frequency bands, for each band). What we propose is to decompose signals more generally into components with positive but not necessarily constant (as in a Fourier decomposition) phase velocities. This added freedom might allow much more physically relevant decompositions. Indeed, we do not have to invent any new example for this, since the best example demonstrating the power of such an approach is known since nearly four hundred years: progress in understanding planetary motion was only possible when Kepler replaced the decomposition into the harmonic epicycles of Ptolemaeus and Copernicus by a decomposition into elliptic motions, which are just of the type advocated by us [12]. Details of such a decomposition will of course depend on the problem at hand, and we cannot give any general algorithm. But the possibility and the eventual usefulness of such an approach should be kept in mind.

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[9] A hint at the actual structure of the present time sequence is obtained from the fact that it has many degenerate extrema: while there are many local maxima and minima with fluctuating amplitudes, the absolute extrema are all at \( x(t) = \pm 1 \). One might use such kind of information in special cases like the present one, but this will hardly lead to a universal and robust algorithm.
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[12] A general ansatz with monotonically increasing phases in the spirit of the above discussion would be \( x(t) = f(\phi_1(t), \ldots, \phi_n(t)) \) with \( \dot{\phi}_i(t) > 0 \) for \( i = 1, \ldots, n \).