On the Staruszkiewicz Modification of the Schrödinger Equation

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Abstract

We discuss Staruszkiewicz’s nonlinear modification of the Schrödinger equation. It is pointed out that the expression for the energy functional for this modification is not unique as the field-theoretical definition of energy does not coincide with the quantum-mechanical one. As a result, this modification can be formulated in three different ways depending on which physically relevant properties one aims to preserve. Some nonstationary one-dimensional solutions for suitably chosen potentials, including a KdV soliton, are presented, and the question of finding stationary solutions is also discussed. The analysis of physical and mathematical features of the modification leads to the conclusion that the Staruszkiewicz modification is a very subtle modification of the fundamental equation of quantum mechanics.

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1 Introduction

Some time ago Staruszkiewicz [1] put forward what seemed to be at the time unique for three dimensions a way of modifying the Schrödinger equation by adding to its action density a Galilean invariant term $A_{\text{mod}} = \gamma (\Delta S)^2/2$, historically motivated by his theory of the free electromagnetic phase [2, 3]. Here $S$ stands for the phase of the wave function $\Psi = R \exp(iS)$, originally identified with the electromagnetic one. The coupling constant $\gamma$ was found by some reasoning involving electromagnetic theory to be equal $1/16\pi^2$ in natural units [2]. This constant is dimensionless in the system of natural units in three dimensions, a property that distinguishes the Staruszkiewicz modification from any other proposed so far. However, neither the identification mentioned nor any other arguments based on electromagnetic theory are really necessary as the modification in question can simply be postulated on the grounds of physical consistency. In such a case $\gamma$ is an arbitrary constant. It is this approach that we adopt for our paper. It should be stressed that this is a purely field-theoretical approach. As of now the equations proposed by Staruszkiewicz have not been derived within any phenomenological scheme. Unfortunately, this is rather disadvantageous for the physical analysis of the modification as any insight one could derive from such a scheme cannot be gained here.

The purpose of this paper is to extend the original brief report [1] by providing a more elaborate analysis of some aspects of this interesting modification together with some solutions to it. To this end, in the next section, we review main points of the modification largely along the lines of [1] and discuss them in some detail. The following section contains a more detailed study of the modification, including a comment on the variational derivation of the equations of motion, a discussion of the problem of separability of noncorrelated subsystems, remarks on its semiclassical approximation and the Ehrenfest relations, as well as other comments on its characteristic properties. This section also addresses the problem of the energy definition, which for nonhomogeneous nonlinear modifications as the one in question is not unique [7]. Depending on how one chooses to approach this issue one can have two different formulations of the modification. Yet another formulation can be adopted to circumvent the problem of nonseparability in the approach of Białynicki-Birula and Mycielski [10]. In another section, concerned with solutions, guided by earlier observations, we find some characteristic and physically important solution, a coherent state. It is in this section that we also demonstrate how to reduce the continuity equation to the form of the Korteweg-de Vries equation and the entire Schrödinger equation to a soluble form for a suitably chosen but time-dependent potential. Here we also address the problem of additional, beyond those that satisfy the linear Schrödinger equation, stationary solutions for the one-dimensional case. Other solutions can be found when the equations of motion of the modification are cast into a dynamical system, a stationary point of which turns out to be a plane wave. The coherent state solution and the Gaussian wave packet are also identified within this framework. Our findings and observations are summarized in the last section, where a broader motivation for this study is also presented.

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1It was demonstrated in [5] that the Staruszkiewicz modification can be extended by allowing more Galilean invariant terms of the same property in three dimensions.
2In fact, as shown in [6] any fully electromagnetic extension of the Schrödinger equation based on the identification of these phases runs into conflict with the Galilean invariance of the modified equation.
2 Main Features of the Modification

To begin with, let us recall after [1] that with the term $\gamma(\Delta S)^2/2$ implemented the modified action for the Schrödinger equation

$$S_{\text{mod}}(\vec{r}, t) = -\int dt d^3x \left\{ \hbar R^2 \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m} \left[ (\vec{\nabla} R)^2 + R^2 \left( \vec{\nabla} S \right)^2 \right] + R^2 V + \frac{\gamma}{2} (\Delta S)^2 \right\}$$

(1)

leads to only one new equation

$$\hbar \frac{\partial R^2}{\partial t} + \frac{\hbar^2}{m} \vec{\nabla} \cdot \left( R^2 \vec{\nabla} S \right) - \gamma \Delta \Delta S = 0,$$

(2)

The other equation, that remains unchanged in this Schrödinger-Madelung representation, reads

$$\frac{\hbar^2}{m} \Delta R - 2\hbar R \frac{\partial S}{\partial t} - 2RV - \frac{\hbar^2}{m} R \left( \vec{\nabla} S \right)^2 = 0.$$

(3)

The first of these equations should be viewed as the continuity equation for the probability density $\rho = R^2$ and the current

$$\vec{j} = \frac{\hbar^2}{m} R^2 \vec{\nabla} S - \gamma \vec{\nabla} \Delta S.$$  

(4)

The solutions to (2) and (3) are supposed to be normalized in the sense $\int d^3x R^2 = 1$, unless they correspond to the continuous part of the energy spectrum, in which case the normalization by imposing the periodicity condition, or the normalization to the delta function is to be applied. Along with this we need to consider the energy functional which in the original formulation of Staruszkiewicz [1] was defined as

$$E_{FT} = \int d^3x \left\{ \frac{\hbar^2}{2m} \left[ (\vec{\nabla} R)^2 + R^2 \left( \vec{\nabla} S \right)^2 \right] + \frac{\gamma}{2} (\Delta S)^2 \right\}.$$  

(5)

This is a field-theoretical definition of energy. One arrives at it from a given Lagrangian density $L$, by identifying the energy density of a field configuration with the time-time component of the canonical energy-momentum tensor

$$T^\mu_\nu = \sum_i \left[ \frac{\delta L}{\partial \partial_t \phi_i} \partial_\nu \phi_i + \frac{\delta L}{\partial \partial_\alpha \phi_i} \partial_\nu \partial_\alpha \phi_i - \partial_\alpha \left( \frac{\delta L}{\partial \partial_\mu \partial_\alpha \phi_i} \partial_\nu \phi_i \right) - \delta_\nu^\mu L \right].$$

(6)

Here $L$ is assumed to depend on a set of fields $\phi_i$ and their first and second order derivatives. The total energy $E_{FT}$ is then given by a space integral over $T^0_0$, one example of which is formula (5). The energy-momentum tensor satisfies the conservation law,

$$\partial_\mu T^\mu_\nu = 0,$$

(7)

resulting in the energy being a constant of motion unless the parameters of $T^0_0$ (like the potential $V$) depend explicitly on time. In some cases though even such time-dependence does not prevent $E_{FT}$ from being a constant of motion.

The Staruszkiewicz modification can be put in the form that involves the entire wave function $\Psi$ and not only its parts $R$ and $S$. By observing that $\Delta S = -i\Delta \ln (\Psi^*/\Psi)/2$, one can write the Lagrangian density for the modification as

$$L_{\text{mod}}(\vec{r}, t) = \frac{i\hbar}{2} \left( \Psi^* \frac{\partial \Psi^*}{\partial t} - \frac{\partial \Psi^*}{\partial t} \Psi \right) - \frac{\hbar^2}{2m} \vec{\nabla} \Psi^* \vec{\nabla} \Psi - V \Psi^* \Psi + \frac{\gamma}{8} \left( \Delta \ln \left( \frac{\Psi^*}{\Psi} \right) \right)^2.$$  

(8)
The factor $\gamma/8$ in the last expression was chosen so as to reproduce the continuity equation in the form of (2). By altering the density in question so that the modification term is homogeneous in the wave function $\Psi$ similarly as the rest of the Lagrangian one arrives at yet another modification [8]. Comparing these two almost identical modifications can provide a good way to learn about the impact of nonhomogeneity. From (8) one obtains now the modified Schrödinger equation for $\Psi$

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \Psi - \frac{\gamma}{8} \left[ \Delta^2 \ln \left( \frac{\Psi^*}{\Psi} \right) \right] \Psi.$$  

(9)

As seen from this equation, the Hamiltonian for the Staruszkiewicz modification, $H_{SM}$, can also be couched in the hydrodynamic form as

$$H_{SM} = H_{LSE} + \frac{i\gamma \Delta^2 S}{2R^2},$$  

(10)

where $H_{LSE}$ is the Hamiltonian for the linear Schrödinger equation. Knowing that in quantum theory the energy of a quantum system is defined as the expectation value of the Hamiltonian $H$, that is, $E_{QM} = \langle \Psi | H | \Psi \rangle = \int d^3 x \Psi^* \dot{H} \Psi$, one finds that

$$E_{QM} = \int d^3 x \left\{ \frac{1}{2m} \left( \nabla R \right)^2 + R^2 \left( \nabla S \right)^2 \right\} + VR^2 + \frac{i\gamma}{2} \Delta^2 S.$$  

(11)

Now, even if these two energy functionals are equal for most solutions to the linear Schrödinger equation that are also solutions to this modification, they do drastically differ for ordinary Gaussian wave packets for which $\Delta S = g(t)$. Noting that $E_{QM}$ is not the same as $E_{FT}$, one is faced with a challenge of choosing the right form of the expression for energy. We will discuss this issue in the next section.

It is reasonable to demand that $E < \infty$, which besides the normalization condition constitutes another constraint on the solutions to (2) and (3). In fact, as just pointed out after [3], this constraint cannot always be satisfied, even for solutions as common to the Schrödinger equation as the Gaussian wave packets if $E_{FT}$ is adopted for the definition of energy.

As noted, the discussed modification does not introduce any new dimensional constants in the system of natural units in three dimensions. In an arbitrary system of units the dimension of the new

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3The continuity equation implies that for all those situations where $R^2$ is normalizable and the gradient of the phase decays fast enough at infinity, the imaginary term in (11) does not contribute at all by the virtue of probability conservation.
coupling constant $\gamma$ is the same as that of the product $\bar{h}c$ or the square of the electric charge. The coupling constant $\gamma$ can thus be represented as $\gamma = \gamma' \bar{h}c$, where $\gamma'$ is a dimensionless parameter in an arbitrary system of units and $c$ is the speed of light. In a way, if we are to paraphrase J. A. Wheeler once more, this product can be viewed as introducing the self-interacting electric charge without introducing any charge at all. However, the combinations $Gm^2$ and $\sqrt{G} \bar{h}cm$, where $G$ is Newton’s constant, have the same dimension as $\bar{h}c$. This suggests that $\gamma$ could be of the order of either of the discussed combinations, meaning that $\gamma'$ is of the order of 1, although diametrically different values of this parameter cannot be excluded by these purely dimensional considerations. In the case $\gamma$ is of the order of $Gm^2$ or $\sqrt{G} \bar{h}cm$, one can suspect that the Staruszkiewicz term $(\Delta S)^2$ is of a gravitational origin.

However, the dimensionless nature of the coupling constants of the Staruszkiewicz modification is by no means universal for it is only characteristic of three dimensions in the natural system of units. One can also argue whether the choice of the system of units in which $c = 1$ is physically justifiable in a modification of the nonrelativistic equation and whose nonrelativistic status is sealed by its Galilean invariance. We believe that similarly as the original electromagnetic underpinnings of the modification cannot be maintained for this would violate the $U(1)$-gauge invariance and the Galilean invariance of the scheme [3], so it is more consistent to treat the Staruszkiewicz model without referring to any special system of units, particularly that this is not invariant. Let us note at this point that there exist quantities that are dimensionless in any system of units. Such quantities, to name as an example the fine structure constant, can be defined as ratios of other quantities having the same dimensions. Yet, the coupling constant in the Staruszkiewicz proposal is not of this nature. These shortcomings of the original motivation of the Staruszkiewicz proposal can easily be avoided. To this end, we suggest that this proposal be reformulated as the simplest nonhomogeneous phase modification that derives from a local Galilean invariant Lagrangian. The assumption that it is the Lagrangian that should be Galilean invariant and not only the equations of motion together with the assumption of simplicity leads to a unique modification with a dimensionless coupling constant in the natural system of units in three space dimensions. It should be noted that in order for the simplicity in question to become apparent, the modification needs to be put in the framework of hydrodynamic representation of the Schrödinger equation. Reformulated in this way, the Staruszkiewicz modification can truly stand on its own merits.

3 Further Analysis of the Modification

It may be instructive to see how the equations of motion for $R$ and $S$ are derived for the modification under study. The nontrivial aspect of this derivation consists in the fact that the action functional (1) from which these equations are obtained contains derivatives of the second order and because of this it yields a new boundary term, coming from the $(\Delta S)^2$ part of the action. Varying (1) with respect to $R$ and $S$, one arrives at $\delta S_{\text{mod}} = I_1 + I_2$ with

$$I_1 = \int_D \left[ \left\{ -\hbar \frac{\partial R^2}{\partial t} - \frac{\hbar^2}{m} \vec{\nabla} \cdot \left( R^2 \vec{\nabla} S \right) + \gamma \Delta S \right\} \delta S + \left\{ \frac{\hbar^2}{m} \Delta R - 2 \hbar R \frac{\partial S}{\partial t} - 2RV - \frac{\hbar^2}{m} \left( \vec{\nabla} S \right)^2 \right\} \delta R \right],$$

$$I_2 = \int_{\partial D} \left[ \left( \hbar R^2 + \frac{\hbar^2}{m} \vec{n} \cdot \left( R^2 \vec{\nabla} S \right) - \gamma \vec{n} \cdot \vec{\nabla} (\Delta S) \right) \delta S + \frac{\hbar^2}{m} \vec{n} \cdot \left( \vec{\nabla} R \right) \delta R \right] + \int_{\partial D} \gamma \Delta S \vec{n} \cdot \delta \left( \vec{\nabla} S \right),$$
where $D$ stands for the domain of integration, being the four-dimensional space-time, and $\partial D$ for its boundary which is spatial infinity in the limit $|t| \to \infty$. The second integral in the last expression is the new boundary term. To ensure that $\delta S_{\text{mod}} = 0$ one needs to assume either that $\delta(\vec{\nabla} S) = 0$ on the boundary or that $\Delta S$ vanishes there.\footnote{The other boundary terms in $I_2$ are handled in the standard way by assuming that variations of both $R$ and $S$ vanish at spatial infinity for $|t| \to \infty$.} The latter choice seems to be more attractive, as one would expect it to promote solutions with finite energy. Unfortunately, even if $\Delta S$ falls off sufficiently rapidly for large $|\vec{x}|$ or large $t$, the energy functional (5) or (11) can still have singularity at the origin. Therefore, in general it seems to be unavoidable to formally impose $\delta(\vec{\nabla} S)|_{\partial D} = 0$, which in fact may constitute a constraint on the class of acceptable functions for $S$. In any case, in order to arrive at the Staruszkiewicz modification it is necessary to ensure that the discussed integral does not contribute to the variation in question. Of course, if the equations of motion for the modification are simply postulated and not derived from a Lagrangian density, the above analysis does not apply.

The fact that the Staruszkiewicz modification is not homogeneous causes it to be nonseparable in the sense first discussed by Białynicki-Birula and Mycielski [10], also referred to as the weak separability. This approach is valid only for noncorrelated subsystems. That this is so was demonstrated for its extended version in [13]. The issue of separability is certainly physically important. The lack of separability means that even in the absence of interactions the motion of one wave packet can affect the other in a system consisting of these two packets. This question has recently been reformulated by Czachor [11] in a novel “effective” way which, instead of pure states, uses density matrices as basic object subjects to a nonlinear quantum evolution. It has been shown that a much larger class of nonlinearities is allowed than in the Białynicki-Birula and Mycielski approach. In particular, all nonlinearities of the form $F(|\psi(x)|)$, as for instance the cubic nonlinear Schrödinger equation, are acceptable. As opposed to the weak separability that assumes that the total wave function of the system is factorizable to reflect the fact that its subsystems are noncorrelated, the wave functions in Czachor’s approach do not have to obey this condition. For this reason, the approach in question belongs to the category of strong separability. As asserted by Czachor [11], the Staruszkiewicz modification is separable in this framework.

As noted in the preceding section, the energy definition for the Staruszkiewicz modification exhibits a curious ambiguity that is absent in the energy formulation for the linear Schrödinger equation. Since quantum mechanics is a probabilistic theory, it is certainly reasonable to adhere to the probabilistic interpretation of energy as a quantity that is defined only in terms of averages. Therefore, the correct form is given by the expression for $E_{QM}$ which as the mean value of the Hamiltonian represents the energy according to the standard interpretation of quantum theory. In keeping with this approach, one dismisses the integral of motion derived from the conservation of energy-momentum tensor as a viable candidate for energy in the quantum-mechanical framework, although its predictive power can still be useful for systems of finite $E_{FT}$. As a consequence, since $E_{FT}$ is not really an observable in quantum theory unless it matches the expectation value of the Hamiltonian, one should be not concerned if this quantity is infinite, even though any infinity in a physical theory is rather disturbing. A less radical alternative approach would consist in reconciling these two different objects.

Let us now argue that compared to $E_{FT}$, $E_{QM}$ seems to be more acceptable on the grounds of physical consistency. In the Staruszkiewicz modification the phase is decoupled from the amplitude of the wave function for, as seen from the equations of motion, it remains a physically determined quantity, having a life of its own even if $R = 0$! In linear quantum mechanics, except for the nodal
points that constitute a discrete set of points, a situation like that does not take place. In fact, it seems to be meaningless to talk about the phase when the amplitude vanishes in a finite region of space and so do the equations of motions leaving no room to determine the evolution in this region. However, in the Staruszkiewicz modification the phase can, in principle, be determined through the equations of motion even if it is not accompanied by the amplitude. To see what consequences this can possibly entail let us consider a one dimensional wave packet moving freely in space until it encounters a totally impenetrable wall, an infinitely large potential barrier. Now, according to the standard interpretation of quantum mechanics, the amplitude of the packet has to vanish beyond the wall, but this does not apply to the phase which can be nonzero there. In linear quantum mechanics, the energy associated with the phase beyond such an infinite wall is zero as long as its gradient is finite which is to be expected as the probability current $\vec{j} = R^2 \nabla S$ should vanish there, too. This is not so in the Staruszkiewicz modification unless one requires that the energy contribution due to the nonlinear part is zero in the region beyond the wall. The vanishing of the probability current does not entail this if one identifies the energy with $E_{FT}$. Therefore, one is faced with a transmission of energy through an infinitely large barrier that may not be accompanied by a flow of the probability current. It is clear that this situation is not physically sound, but fortunately it can be amended if one defines energy as $E_{QM}$, which seems to be the first argument in favor of this definition of energy. Now, assuming that the current is a continuous function and vanishes on the wall implies that it vanishes also beyond it as $\Delta^2 S$ is zero everywhere beyond the wall by the virtue of the equations of motion or just the continuity equation. In other words, $\Delta^2 S = 0$ implies that the current is constant beyond the wall, but owing to its continuity it can only be zero there. Moreover, the energy associated with the phase beyond the wall is zero as is the energy of a “quantum mechanical system” in the entirely free space in the absence of the amplitude if identified with $E_{QM}$. Unless the phase alone is really proven to be endowed with energy, this observation once again supports $E_{QM}$ as a more physically reasonable expression for the energy of a quantum-mechanical system. Still, even if no energy can be associated with the phase in the limit of vanishing amplitude, it is not out of the question that it can be detected by some diffraction or interference phenomena similar to the Aharonov-Bohm effect [13]. To summarize our reasoning, electing $E_{QM}$ over $E_{FT}$ stems from a very simple yet physically respectable requirement that no information, energy in particular, is allowed to be transmitted through an infinitely large potential barrier.

The energy functional $E_{QM}$ contains an imaginary component. Since the energy is supposed to be a real quantity one might want to require that this part does not contribute to the total energy, which imposes a constraint on physically acceptable states in this particular nonlinear model of quantum mechanics. That this constraint is not necessarily very restrictive has already been demonstrated for the Gaussian wave packets. Since for such packets $E_{FT}$ is infinite, they would have to be excluded from legitimate solutions to the modification if this energy definition were employed. Yet, they are perfectly fine on the energetic grounds if one uses the quantum-mechanical definition of energy. Moreover, the continuity equation coupled with the assumption of probability conservation implies the vanishing of this term for a large class of physically plausible situations. The proposed condition of real energy is equivalent to the selection of observables in nonlinear quantum mechanics for which contributions of nonlinear parts vanish on normalized states $[12]$. On the other hand, complex energy is not in the least foreign to linear quantum mechanics and therefore it is certainly justifiable to entertain the possibility of retaining it in the modification under discussion. In linear quantum mechanics, complex energy can appear as a result of implementing complex or “optical” potentials usually invoked to phenomenologically model absorption in scattering processes. The complex potentials have also been
used to describe decoherence [14]. Since the Staruszkiewicz modification introduces complex potential-
like terms in a natural manner, one finds it tempting to allow states with complex energy if only in
the hope that they can mediate in the process of decoherence. Such states, however, would have to be
rather exotic because, as noted, in the majority of physically reasonable situations one expects the
integral over the imaginary term in question to vanish.

As we see, the properties of observables depend on the space of states in which they are defined,
which also applies to linear quantum mechanics, where, for instance, the self-adjointness of an operator
depends on its domain. In line with this approach, one can attempt to reconcile $E_{QM}$ with $E_{FT}$ by
choosing a domain in which they are equal for each function in the domain. This alternative seems
to be coming as close as possible to the original Staruszkiewicz formulation by retaining $E_{FT}$ as a
physically relevant object, but it is probably more restrictive than the formulation just proposed for
it requires not only that $E_{FT}$ be finite and $E_{QM}$ be real, but, in addition, that these two quantities
be equal. In what follows, for the sake of exploring all possibilities, we will consider both of these
formulations. In fact, none of them seems to be decidedly more convincing than the other. We will
refer to the alternative in question as maximal to contrast it with a moderate formulation that is
concerned only with the expectation value of a Hamiltonian treating $E_{FT}$ as useful but not a crucial
quantity in the quantum-mechanical scheme it adheres to. As we will see, all the nontrivial solutions
that we have found belong to the moderate formulation, although one of them is shared by the
original formulation as well. This lack of uniqueness in the definition of energy clearly marks the
difference between linear quantum mechanics and its nonlinear models where uniqueness seems to be
an exception rather than a rule.

It is possible to adopt yet another approach that takes care of the weak nonseparability problem
and the problem of uniqueness of energy at the same time. This is accomplished by considering
as legitimate only those solutions to the modified Schrödinger equation that satisfy the equation
$\Delta^2 S = 0$. Even if the strong separability seems to offer a more all-encompassing approach, it is
the weak separability that is a more stringent condition and for this reason one can use it as a
criterion for the formulation discussed. We will call this formulation minimal for the domain of its
solutions is expected to be the smallest of all the formulations considered. It overlaps with that of the
maximal formulation in the case $\Delta S = 0$ and is entirely contained within the domain of the moderate
formulation. Despite its name, this approach ensures the maximum of physically desirable properties,
including, as we will soon see, the standard classical limit of quantum theory.

Let us now comment on the semiclassical approximation for the modification. Formally, one
performs this approximation by rescaling $S$ to $S/\hbar$, followed by expanding $S$ in a series $S = S_0 +
\hbar S_1 + \hbar^2 S_2 +...$ One obtains in this way an expansion of the Schrödinger equation in the powers of
the Planck constant. The lowest order approximation term to the continuity equation of the linear
Schrödinger equation is proportional to $\hbar$. This, however, is not the case for its modified counterpart
whose expansion in the powers of $\hbar$ starts with $\hbar^{-1}$. The next leading term is proportional to $\hbar^0$.
As a result, we obtain that $\Delta^2 S_0 = \Delta^2 S_1 = 0$. In practice, this means that, in a very good first
approximation, the solutions to (2) can be sought for among wave functions whose phase $S$ satisfies
the condition $\Delta^2 S = 0$. One could hope to find in this way even exact solutions to (2) and (3), with,
perhaps, some additional but physically well justified potential $V$. As we will see in the next section,
this hope turns out to be justified.

The classical limit of this modification in the sense of the Ehrenfest theorem may not always exist
since the standard Ehrenfest theorem of linear quantum mechanics is altered by nonlinear corrections.
As worked out in [5], the general form of the modified Ehrenfest relations is

\begin{align}
    m \frac{d}{dt} \langle \vec{r} \rangle &= \langle \vec{p} \rangle + F_1, \\
    \frac{d}{dt} \langle \vec{p} \rangle &= -\langle \vec{\nabla} V \rangle + F_2,
\end{align}

where

\begin{align}
    F_1 &= \frac{2m}{\hbar} \int d^3x \vec{r}R^2H_I, \\
    F_2 &= \int d^3x R^2 \left( 2H_I \vec{\nabla} S - \vec{\nabla} H_R \right).
\end{align}

$H_R$ and $H_I$ represent the real and imaginary parts of the nonlinear part of the Hamiltonian of the modified Schrödinger equation, respectively. In the derivation of the last formula it was assumed that $\int d^3x \vec{\nabla} (R^2H_I) = 0$ and because of that this term was discarded. Since $H_I = \gamma \Delta^2 S/2R^2$, this is tantamount to assuming that $|\vec{x}|^2 \Delta^2 S$ vanishes at infinity. This, in turn, is implied by the condition that the energy $E_{QM}$ be finite, and it is satisfied for all those configurations that are square-integrable as can be seen from the continuity equation given the probability is conserved. Had $\int d^3x \vec{\nabla} (R^2H_I)$ been nonzero, it would introduce an imaginary component to $F_2$. Now, since $H_R = 0$ and $H_I = \gamma \Delta^2 S/2R^2$, the Ehrenfest relations for the modification in question read

\begin{align}
    m \frac{d}{dt} \langle \vec{r} \rangle &= \langle \vec{p} \rangle + \gamma m \frac{\hbar}{\hbar} \int d^3x \vec{r} \Delta^2 S, \\
    \frac{d}{dt} \langle \vec{p} \rangle &= -\langle \vec{\nabla} V \rangle + \gamma \int d^3x \Delta^2 S \vec{\nabla} S.
\end{align}

These relations are Galilean covariant and their nonlinear contributions are also Galilean invariant for the wave functions that are normalizable and do not violate the probability conservation.

We see that, in general, the nonlinear corrections to the Ehrenfest relations do not vanish, leading to a different classical limit than in linear theory. This feature of the modification is shared by other nonlinear generalizations of the Schrödinger equation, the only notable exception from this rule being the Bialynicki-Birula and Mycielski modification. It suggests that the Staruszkiewicz modification is not linearizable and as such may describe phenomena that cannot be captured by the linear Schrödinger equation. On the other hand, perhaps due to its simplicity, the relevance of the Ehrenfest theorem tends to be overestimated. As argued in [13], this theorem is neither sufficient nor necessary to characterize the classical regime of quantum theory.

However, one can also entertain the possibility that the classical limit of quantum world does not always exist in the form suggested by classical theory. This might offer an explanation of some anomalous or otherwise unexplained phenomena some of which owing to their strangeness remain outside the realm of accepted exact science. As a matter of fact, since even the quantization procedure is sometimes ambiguous, a unique classical limit of quantum theory is something that is enforced by rather than naturally emerges from it. What is remarkable in the modification discussed, it is that the departure from the standard classical limit is very subtle and, at the same time, pretty elusive due to the very elusive nature of the phase. The departure in question, unlike, for instance, in some other modification that possesses the same property [8], is also entirely defined by the phase.\footnote{Other modifications, as for instance the Doebner-Goldin [18] modification, can also have a non-standard classical limit. However, since the Doebner-Goldin modification describes only irreversible and dissipative quantum systems it cannot be treated on the same footing as the Staruszkiewicz modification. Moreover, it is also less unique for it involves as many as 6 dimensional parameters.}
Finally, let us make two more comments utilizing yet another rather straightforward modification of the Schrödinger equation, a relativistically modified Schrödinger equation which follows when the expression for the energy of a single particle in terms of its momentum is taken in the leading relativistic approximation

\[ E = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}, \]  

and the first quantization is performed. The Schrödinger equation for this modification,

\[ i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \Delta - \frac{\hbar^4}{8m^3c^2} \Delta^2 \right) \Psi = 0, \]

when put in the Madelung representation involves additional contributions to the continuity equation in both \( S \) and \( R \). It is one of the reasons that this particular modification can be considered more general than the Staruszkiewicz modification. However, despite its generality, the relativistic modification does not contain Staruszkiewicz’s modification as a special case. This can be seen by comparing the dimensions of the coupling constants of these modifications. Moreover, unlike the Staruszkiewicz modification, the relativistic modification is homogeneous of degree one in the wave function and is linear. The other comment consists in noting that the Gaussian wave packets, which are not solutions to the Staruszkiewicz modification for energetic reasons\(^6\), are also not solutions to (19), which indicates that it is rather difficult to preserve wave packet solutions in other extensions of this equation as well. Therefore, the modification under discussion in its original formulation should not necessarily be viewed as pathological in this respect. However, what is really unacceptable here is that \( E_{FT} \neq E_{QM} \). This causes the wave packets to be excluded from the maximal formulation of the Staruszkiewicz modification.

4 Solutions

It is not easy to find solutions to the modification discussed here. The main obstacle to this stems from the nonhomogeneity of (2). Even if such solutions are found they may not necessarily yield to a clear physical interpretation. Therefore, it should not come as a surprise that the physically best understood solution we discovered so far represents a case when the nonhomogeneity does not interfere.

As suggested in the previous section, one should look for the solutions to the Staruszkiewicz modification among the phenomena for which the square of the Laplacian of the phase \( S \) vanishes. We do not intend to present here a complete solution to the equation \( \Delta^2 S = 0 \) together with a discussion of its applicability to this modification, reserving it for another publication \(^9\). Instead, we prefer to show the simplest possible and physically important solution to the equation \( \Delta S = 0 \) that, obviously, satisfies the previous one and ensures the finite unique energy in each of the discussed formulations of this modification. Such a solution is provided by a coherent state \(^4\). It is easy to check that indeed the coherent state described by

\[ R^2 = \frac{1}{\sqrt{\pi}x_0} \exp \left[ -\frac{(x - x_0\sqrt{2}\cos(\omega t - \delta))^2}{x_0^2} \right] \]  

\(^6\)As found in \(^1\), due to a slow convergence of their phase in the infinity, the energy \( E_{FT} \) is not finite for the Gaussian wave packets.
and
\[ S = - \left( \frac{\omega t}{2} - \frac{|\alpha|^2}{2} \sin 2(\omega t - \delta) + \frac{\sqrt{2}|\alpha|x}{x_0} \sin (\omega t - \delta) \right) \] (21)

represents a solution to equations (2) and (3) in one dimension in the potential of a simple harmonic oscillator \( V = m\omega^2x^2/2 \). Here \( x_0 = (m\omega)^{-1/2} \), while \( \alpha \) and \( \delta \) are arbitrary constants, complex and real, respectively. We also put \( \hbar = 1 \). The three dimensional case can be worked out as a product of one-dimensional amplitudes with \( \omega \) and the phase \( \delta \) appropriately redefined.

Another physically interesting solution to (2) in one dimension can be found for a general negative \( \gamma = -|\gamma| \). In this case, (2) becomes
\[ \hbar \frac{\partial R^2}{\partial t} + \frac{\hbar^2}{m} \left( R^2 S_x \right)_x + |\gamma| S_{xxxx} = 0. \] (22)

We will now show that this equation can be reduced to one of the standard forms of the celebrated Korteweg-de Vries equation
\[ u_t + 3(u^2)_x + u_{xxx} = 0. \] (23)

To this end, let us make identifications \( S_x = R^2 = du(x) \), where \( d \) is a free constant whose dimension is to be worked out. Because the dimensions of \( S_x \) and \( R^2 \) are equal, the dimension of \( t \) has to be meter\(^3\). This makes it impossible to maintain the natural system of units if we insist that \( \gamma \) be dimensionless to preserve the fundamental idea of the modification in its original formulation. We can assume that either \( \hbar = 1 \) or \( c = 1 \) but not both. Such a compromise is unavoidable in one dimension. We will assume that \( \hbar = 1 \) which implies that the speed of light has the dimension of meter\(^{-2} \) and the dimension of \( m \) has to be that of meter. Now, it is certainly in the spirit of this modification to make the coefficient of \( (u^2)_x \) dimensionless and, in fact, it has to be so for (22) to reduce to the form of (23). Therefore, \( d/m|\gamma| = 3 \) which does not impose any constraints on \( |\gamma| \) because of the freedom of choosing \( d \), but fixes the dimension of \( d \) to be that of meter. Moreover, \( t \) gets rescaled to \( t' = |\gamma|t \).

The most physically attractive solution to the Korteweg-de Vries equation is known to be a soliton of the form
\[ u_{\text{sol}}(\theta; k) = \frac{k^2}{2 \cosh^2 \frac{\theta}{2}}, \] (24)

where \( \theta = k\xi + \tau \), \( \xi = x - |\gamma|k^2t \), and \( \tau \) is an arbitrary phase. The dimension of \( k^2 \) is the same as the dimension of \( u(x) \), meter\(^{-2} \). This solution is normalizable and the normalization condition, \( \int dx R^2(x) = d \int dx u(x) = 1 \), implies that \( dk = 1/2 \). Consequently, since \( d/m|\gamma| = 3 \), the “speed” of the soliton is \( v_x = |\gamma|/4d^2 = 1/36|\gamma|m^2 \). From (3), we find out that the entire Schrödinger-Madelung system of equations can support the KdV soliton only in the presence of the potential
\[ V(\theta; k) = \frac{1}{8m \cosh^4 \frac{\theta}{2}} \left( k^2 + 4Em \right) \cosh^2 \frac{\theta}{2} + 2k^2 \left( 2md|\gamma|k^2 - 1 \right) \cosh^2 \frac{\theta}{2} - d^2k^4 \], (25)

where \( E \) is a constant of the same dimension as energy, meter\(^{-3} \). In this potential \( E_{FT} \) is finite (and also constant!), while \( E_{QM} \) is real and constant as well, however they are not equal. It is not clear how this potential can be physically realized. The solution under consideration, similarly as the Gaussian and coherent state wave packets, does not alter the standard Ehrenfest relations.

The solutions discussed above were non-stationary. As observed in [1], the Staruszkiewicz modification, as opposed to its extension [3] and some other notable nonlinear modifications of the Schrödinger
equation \([10, 18]\), does not affect stationary states of quantum mechanical systems determined by the Schrödinger equation. This means that wave functions describing these states are also solutions to the modified equation with unchanged energy levels. It is this particular property that makes the Staruszkiewicz modification stand out. As a function of time and position, the phase of these states is given by

\[ S = -\frac{Et}{\hbar} + \sigma(\vec{x}) \]

but its position dependence encoded in \(\sigma(\vec{x})\) is physically inconsequential. Usually, \(\sigma(\vec{x})\) is at most a linear function of spatial coordinates. However, it is conceivable that there exist other stationary states for which the phase is also a physically nontrivial function of position in the sense that, for instance, the phase affects the energy of the system. We will now show how one can approach the problem of finding such solutions in the one-dimensional case.

To find stationary solutions to the Staruszkiewicz modification one starts with the condition

\[ \frac{\partial R^2}{\partial t} = 0. \quad (26) \]

This condition when applied to (2) and (3) together with the assumption of time-independent potential implies that \(\partial S/\partial t = \text{const} = -E/\hbar\), where \(E\) is to be identified with the energy of a quantum-mechanical system. For one-dimensional stationary problems, assuming \(\hbar = 1\), equations (2) and (3) reduce to

\[ u'' + aR^2u + c = 0, \quad (27) \]
\[ R'' + 2m \left( E - V - bu^2 \right) R = 0, \quad (28) \]

where \(u = S_x\), \(a = -1/\gamma m\), \(b = 1/2m\), and \(c\) is a free constant that can be zero.

One can seek solutions to this system in two different ways. One is by selecting a normalizable \(R\) representing a solution to the linear Schrödinger equation in some potential \(V\). Treating this as an Ansatz one attempts to solve the first of the above equations for \(u\). This method will usually lead to a complicated equation for \(u\) that may not be solvable in an analytical manner even for a simple \(R\). Once \(u\) is determined, the elected \(R\) can be interpreted as the solution to the linear Schrödinger equation, but in the effective potential \(V_{\text{eff}} = V - bu^2\). The other way boils down to picking \(u\) that ensures that (27) leads to a normalizable and positive \(R^2\). Inserting now \(R\) and \(u\) into (28) gives us the potential that supports this configuration. Nevertheless, the solutions found in these ways may not necessarily yield to a simple physical interpretation. For instance, by employing the second method for \(c = 0\), \(a = -2\), and \(u(x) = 1 + x^2\) one obtains \(R^2 = 1/(1 + x^2)\). This configuration is obviously well localized and thus normalizable. It exists in the effective potential \(V_{\text{eff}} = (2x^2 - 1)/2(x^2 + 1)^2 - \gamma (1 + x^2)^2\) which is unbounded from below as \(\gamma\) is positive. The energy \(E_{\text{QM}}\) of this stationary solution is finite and equal zero even if the potential \(V_{\text{eff}}\) can be negative in the whole domain for a sufficiently large \(\gamma\). However, its field-theoretical energy is infinite. The configuration in question does not affect the standard Ehrenfest relations.

Similar solutions appear in the linear equation as well. For example, one can obtain exactly the same solution as considered above in the linear case if \(c \neq 0\). In the modification discussed, the solutions in question might, however, play more important and perhaps a novel role due to a more pronounced contribution of the phase to the equations of motion. The major problem with these solutions in both linear and nonlinear theory is that they seem to be vulnerable to generic small perturbations. The perturbed configurations can have infinite energy and therefore cannot be considered physical. This implies that the functional domain of stability of the solutions discussed is very small. In fact, they may not be stable at all, that is, an infinitely small perturbation can destroy them. Such a statement would require a formal proof, but since we cannot provide it here, we offer
it as a conjecture. Consequently, it is safe to say that no other normalizable stationary solutions of physical interest beyond those to the linear equation appear in the Staruszkiewicz modification. In principle, one can also address the issue under consideration as a spectral problem of equation (9). However, in the case of nonlinear equations, each spectral problem requires a separate mathematical analysis as, unlike for linear operators on Hilbert spaces, there is not any general theory that could tackle such problems.

One of the most general frameworks in which to seek solutions to the modified Schrödinger equation is that of dynamical systems. For simplicity, we will employ it here only for the one-dimensional case, in which one has

\[ \omega' = \frac{1}{\gamma} \left[ \frac{\partial \epsilon}{\partial t} + \frac{1}{m} (2 \zeta \eta + \nu) \right] , \]

\[ \zeta' = 2m \left( \frac{\partial S}{\partial t} + V(x,t) + \frac{1}{2m} \eta^2 \right) - \zeta^2 , \]

\[ \epsilon' = 2 \epsilon \zeta , \]

\[ \eta' = \nu , \]

\[ \nu' = \omega , \]

with the variables \( \eta, \epsilon, \zeta, \nu, \) and \( \omega \) defined as \( \eta = S' = dS/dx, \epsilon = R^2, \zeta = R'/R, \nu = S'', \) and \( \omega = S''' . \) It should be noted that the dynamical system (29-33) “evolves” in \( x, \) the actual time variable \( t \) being a parameter.

Looking for stationary points of this system, we dismiss \( \epsilon = 0 \) as unphysical, which leaves us with an alternative \( \zeta = \omega = \nu = 0 \) and \( \partial \epsilon / \partial t = 0, \) and \( \partial S / \partial t + V(x,t) + \eta^2 / 2m = 0. \) As can easily be verified, the plane wave described by \( S = -Et + kx \) and \( R = 1, \) with the normalization condition realized by imposing the periodic boundary conditions, is a stationary point of this dynamical system. In fact, it is probably the only such a point.

As a way to demonstrate the capacity of this method let us now find out how the other solutions to the nonlinear Schrödinger equation discussed in the present paper can be identified within this framework. The coherent wave packet with \( R^2 = \epsilon \) and \( S \) given by (20-21) corresponds to \( \eta = -\sqrt{2} |\alpha| \sin(\omega t - \delta)/x_0, \zeta = - \left( x - x_0 \sqrt{2} \cos(\omega t - \delta) \right)/x_0^2, \) and \( \nu = \omega = 0. \) One deals here with an effectively two-dimensional phase space (for any fixed \( t \)) spanned by \( \zeta \) and \( \epsilon \) as they are the only variables allowed to change in “time” \( x. \) For comparison, the Gaussian wave packet with \( R \) and \( S \) specified by

\[ R = \left[ \frac{mt_0}{\pi (t^2 + t_0^2)} \right]^{1/4} \exp \left[ -\frac{mt_0 x^2}{2 (t^2 + t_0^2)} \right] \]

and

\[ S = \frac{mtx^2}{2 (t^2 + t_0^2)} - \frac{1}{2} \arctan \frac{t}{t_0} , \]

lives in an effectively three-dimensional phase space. Indeed, in this case the motion takes place in variables \( \eta = mt|x|/(t^2 + t_0^2), \zeta = -mt_0|x|/(t^2 + t_0^2), \) and \( \epsilon, \) the other variables being \( \nu = mt / (t^2 + t_0^2) \)

\[ \text{Since the solutions in question appear in both linear and nonlinear theory, we will discuss them in greater detail in a separate paper hoping to resolve this issue in it \( \text{[16]} \).} \]

\[ \text{8This observation was first made by A. Z. Górska and P. O. Mazur in 1984 \( \text{[17]} \).} \]
and $\omega = 0$. They both are constants for the actual time fixed. With the time allowed to run, the manifolds on which the evolutions of these packets occur are three- and four-dimensional, correspondingly. This comparison of ordinary wave packets with the coherent states is just another way of demonstrating that the phase space of the latter is smaller than that of the former. It is due to this very fact that the coherent wave packets are also called “the minimal phase space packets”. The KdV soliton does not represent any particular plane in this phase space.

5 Conclusions

The paper presented was aimed at recalling and a further discussion of the nonlinear modification of the Schrödinger equation proposed by Staruszkiewicz. Historically motivated by a longitudinal modification of the electromagnetic action, here it has been treated as postulated on the grounds of physical consistency and certain attractiveness similarly as other notable nonlinear modifications of this equation \[10, 18\]. Unlike these, the Staruszkiewicz modification does not introduce any new dimensional constants in the system of natural units as long as it is formulated in three dimensions. As argued, due to the energy ambiguity typical of nonhomogeneous modifications of the Schrödinger equation, depending on which physically relevant properties one decides to maintain, one can come up with three different formulations of the Staruszkiewicz modification, although we believe that the arguments presented favor the moderate formulation which assumes that the correct definition of energy is given by the expectation value of the Hamiltonian and rejects any other definitions as spurious. It is within this formulation that the largest number of solutions can be found, including the Gaussian wave packet that could not be incorporated in the original version of the modification. The formulation in question admits also some other physically interesting solutions, among them the coherent state for the potential of harmonic oscillator, shared though by all the formulations, and the KdV soliton for another suitably chosen but time-dependent potential. None of these solutions alters the standard Ehrenfest relations.

One can speculate that Staruszkiewicz’s modification describes some part of reality that due to rather an elusive nature of the phase of wave function has escaped our realization. In fact, until relatively recently the phase had been treated as an object of rather secondary importance to the understanding of quantum-mechanical systems. However, it turned out that in a number of physical situations such a position could not be maintained for it would lead to an incomplete physical description. The phase of the wave function encodes relevant information on the quantum evolution as manifested in the Aharonov-Bohm effect \[13\] and other phenomena of similar nature \[19, 20\] (see also \[21\] for a more complete list of references). Another motivation to study this modification stems from the long-standing problem of the collapse of wave function. Since it is rather reasonable to expect that the phase plays an important part in this process, one hopes that the modification discussed can offer some insight into the physics of this phenomenon.

In all its formulations, the Staruszkiewicz modification introduces very minimal changes to linear quantum theory. Not only are the stationary states and thus also the atomic structure unchanged, but also the energy of nonstationary phenomena remains the same in most if not all physically relevant cases. Despite these, the modification in question differs from linear quantum mechanics in its classical limit. It is, however, not out of the question that to solve the riddle of the measurement problem and perhaps to understand other phenomena currently viewed as anomalous, the idea of modifying the classical limit of quantum theory must be seriously considered. Being a fundamental theory of nature,
it is the quantum theory that should tell us what the world is like in the classical limit. Imposing such a limit on the basis of our apparently “classical” collective experience may deprive us of the understanding of phenomena that are not so common to this experience and thus, sometimes, not even widely regarded as real. As noted by Einstein, it is theory that is supposed to tell us what is observable. The Staruszkiewicz modification offers the subtlest conceivable proposal to extend the classical limit. Moreover, as partially indicated by this non-standard limit, it is very plausible that this modification is not linearizable and therefore it could describe phenomena that cannot be captured by the linear theory.

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