Research Article

Study of a Forwarding Chain in the Category of Topological Spaces between $T_0$ and $T_2$ with respect to One Point Compactification Operator

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In the following text, we want to study the behavior of one point compactification operator in the chain $\Xi := \{\text{Metrizable, Normal, } T_2, \text{KC, SC, US, } T_1, \text{T}_D, \text{T}_{UD}, T_0, \text{Top}\}$ of subcategories of category of topological spaces, Top (where we denote the subcategory of Top, containing all topological spaces with property $P$, simply by $P$). Actually we want to know, for $P \in \Xi$ and $X \in P$, the one point compactification of topological space $X$ belongs to which elements of $\Xi$. Finally we find out that the chain $\{\text{Metrizable, } T_2, \text{KC, SC, US, } T_1, \text{T}_D, \text{T}_{UD}, T_0, \text{Top}\}$ is a forwarding chain with respect to one point compactification operator.

1. Introduction

The concept of forwarding and backwarding chains in a category with respect to a given operator has been introduced for the first time in [1] by the first author. The matter has been motivated by the following sentences in [1]: “In many problems, mathematicians search for theorems with weaker conditions or for examples with stronger conditions. In other words they work in a subcategory $D$ of a mathematical category, namely, $C$, and they want to change the domain of their activity (theorem, counterexample, etc.) to another subcategory of $C$ like $H$ such that $H \subseteq D$ or $D \subseteq H$ according to their need.” Most of us have the memory of a theorem and the following question of our professors: “Is the theorem valid with weaker conditions for hypothesis or stronger conditions for result?” The concept of forwarding, backwarding, or stationary chains’ concept with respect to one point compactification or Alexandroff compactification operator.

We want to study the chain $\{\text{Metrizable, Normal, } T_2, \text{KC, SC, US, } T_1, \text{T}_D, \text{T}_{UD}, T_0, \text{Top}\}$ of subcategories of Top in the point of view of forwarding, backwarding, and stationary chains’ concept with respect to one point compactification or Alexandroff compactification operator.

Remark 1. Suppose $\leq$ is a partial order on $A$. We call $B \subseteq A$

(i) a chain, if for all $x, y \in B$, we have $x \leq y \lor y \leq x$;

(ii) cofinal, if for all $x \in A$, there exists $y \in B$ such that $x \leq y$.

In the following text, by a chain of subcategories of category $C$, we mean a chain under “$\subseteq$” relation (of subclasses of $C$). We recall that if $\mathcal{M}$ is a chain of subcategories of category $C$ such that $\bigcup \mathcal{M}$ is closed under (multivalued) operator $\psi$, then we call $\mathcal{M}$

(i) a forwarding chain with respect to $\psi$; if for all $C \in \mathcal{M}$, we have $\psi((\bigcup \mathcal{M}) \setminus C) \cap C = \emptyset$ (i.e., $\psi((\bigcup \mathcal{M}) \setminus C) \subseteq ((\bigcup \mathcal{M}) \setminus C)$;
(ii) a full-forwarding chain with respect to \( \psi \); if it is a forwarding chain with respect to \( \psi \) and for all distinct \( C_1, C_2, C_3 \in \mathcal{M} \), we have
\[
C_1 \subseteq C_2 \subseteq C_3 \Rightarrow (\exists X \in C_2 \setminus C_1 \psi(X) \in C_3 \setminus C_2),
\]
where, for multivalued function \( \psi \), by \( \psi(X) \in C_2 \setminus C_1 \), we mean that at least one of the values of \( \psi(X) \) belongs to \( C_3 \setminus C_2 \).

(iii) a backwarding chain with respect to \( \psi \); if for all \( C \in \mathcal{M} \), we have \( \psi(C) \subseteq C \);

(iv) a full-backwarding chain with respect to \( \psi \); if it is a backwarding chain with respect to \( \psi \) and for any distinct \( C_1, C_2, C_3 \in \mathcal{M} \), we have
\[
C_1 \subseteq C_2 \subseteq C_3 \Rightarrow (\exists X \in C_3 \setminus C_1 \psi(X) \in C_2 \setminus C_1),
\]
where, for multivalued function \( \psi \), by \( \psi(X) \in C_3 \setminus C_1 \), we mean that at least one of the values of \( \psi(X) \) belongs to \( C_2 \setminus C_1 \);

(v) a stationary chain with respect to \( \psi \) if it is both forwarding and backwarding chains with respect to \( \psi \).

Basic properties of forwarding, backwarding, full-forwarding, full-backwarding, and stationary chains with respect to given operations have been studied in [1]. We refer the interested reader to [2] for standard concepts of the Category Theory.

We recall that by \( \mathbb{N} \) we mean the set of all natural numbers \{1, 2, ...\}; also \( \omega = \{0, 1, 2, ...\} \) is the least infinite ordinal (cardinal) number and \( \Omega \) is the least infinite uncountable ordinal number. Here ZFC and GCH (generalized continuum hypothesis) are assumed (note: by GCH for infinite cardinal number \( \alpha \), there is not any cardinal number \( \beta \) with \( \alpha < \beta < 2^\alpha \), i.e., \( 2^\alpha = 2^\omega \)).

We call a collection \( \mathcal{F} \) of subsets of \( X \) a filter over \( \emptyset \) if \( \emptyset \notin \mathcal{F} \); for all \( A, B \in \mathcal{F} \) we have \( A \cap B \in \mathcal{F} \); for all \( A \in \mathcal{F} \) and \( B \subseteq X \) with \( A \subseteq B \) we have \( B \in \mathcal{F} \). If \( \mathcal{F} \) is a maximal filter over \( X \) (under \( \subseteq \) relation), then we call it an ultrafilter over \( X \). If for all \( A \in \mathcal{F} \), we have \( \text{card}(A) = \text{card}(X) \); then we call \( \mathcal{F} \) a uniform ultrafilter over \( X \).

We end the section by the following two examples.

(I) For \( n \in \mathbb{N} \), let \( C_n := [0, 1/n] \) and \( \psi : [0, 1] \rightarrow [0, 1] \) with \( \psi(0) = 0 \) and \( \psi(x) = n^n x + 1 - n \) for \( x \in C_n \), \( C_{n+1} = (1/(n + 1), 1/n] \). Then \( |C_n : n \in \mathbb{N}| = 2^n \) is full-forwarding with respect to \( \psi \) [1, Example 2.2].

(II) Let ON denote the class of all ordinal numbers; CN denotes the class of all cardinal numbers; for every set \( A \) with \( |A| \) we mean cardinal number of \( A \), and for each cardinal number \( \alpha \in \text{ON} \), \( D_\alpha = \{y \in \text{ON}: |y| = \alpha\} \), \( C_\alpha = \{y \in \text{ON}: |y| < \alpha\} \). Define \( \psi : \text{ON} \rightarrow \text{ON} \) with \( \psi(y) = y - \alpha \) for \( y \in D_\alpha \) and \( \alpha \in \text{CN} \). Then \( |C_\alpha : \alpha \geq \omega| = 2^\omega \) is a full-backwarding chain of subclasses of \( \text{ON} \) with respect to \( \psi \) [1, Example 2.3].

2. Basic Definitions in Separation Axioms

In this section we bring our basic definitions in Top.

Convention 1. Henceforth in the topological space \( X \) suppose \( \emptyset \notin \mathcal{F} \). So (see [3, 4])
\[
\mathcal{B} := \{ \emptyset \subseteq X : U \text{ is an open subset of } X \}
\]
\[
\cup \{ V \cup \{ \emptyset \} : V \subseteq X, X \setminus V \text{ is compact and closed} \}
\]
is a topological basis on \( X \cup \{ \emptyset \} \). The space \( X \cup \{ \emptyset \} \) with topological basis \( \mathcal{B} \) is called one point compactification or Alexandroff compactification of \( X \).

Let
\[
A(X) := \begin{cases} 
\text{one point compactification of } X & \text{if } X \text{ is not compact,} \\
X & \text{if } X \text{ is compact.} 
\end{cases}
\]

By the operator \( A \), in this text we mean the above mentioned operator.

Remark 2. We call a topological space \( X \) (if \( A \subseteq X \), by \( A' \), we mean the set of all limit points of \( A \) in \( X \))

(i) \( T_0 \); if for all distinct \( x, y \in X \), there exist open neighborhood \( U \) of \( x \) and open neighborhood \( V \) of \( y \) such that \( x \notin V \) and \( y \notin U \);

(ii) \( T_{UD} \); if for all \( x \in X, \{ x \} \) is a union of closed subsets of \( X \);

(iii) \( T_D \); if for all \( x \in X, \{ x \} \) is a closed subset of \( X \);

(iv) \( T_1 \); if for all \( x \in X, \{ x \} \) is a closed subset of \( X \);

(v) \( US \); if any convergent sequence has a unique limit;

(vi) \( SC \); if for any convergent sequence \( (x_n : n \in \omega) \) to \( x \in X, \{ x_n : n \in \omega \} \cup \{ x \} \) is a closed subset of \( X \);

(vii) \( KC \); if any compact subset of \( X \) is closed;

(viii) \( T_2 \) (or Hausdorff); if for all distinct \( x, y \in X \), \( x \neq y \) there exist open neighborhood \( U \) of \( x \) and open neighborhood \( V \) of \( y \) with \( U \cap V = \emptyset \);

(ix) normal; if it is \( T_2 \) and for every disjoint closed subsets \( A, B \) of \( X \), there exist disjoint open subsets \( U, V \) of \( X \) with \( A \subseteq U \) and \( B \subseteq V \);

(x) \( k \)-space; if for all \( A \subseteq X \), \( A \) is closed if and only if for all closed compact subset \( K \) of \( X \), \( A \cap K \) is closed.

Regarding [5], we have \( T_2 \subseteq KC \subseteq SC \subseteq US \subseteq T_1 \). Also by [6] we have \( T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0 \) therefore.

Metrizable \( \subseteq \) Normal \( \subseteq T_2 \subseteq KC \subseteq SC \subseteq US \subseteq T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0 \subseteq Top \).

In this section, we want to study the operator \( A \) on the above chain. However, it has been proved in [1, Lemma 3.1 and Corollary 3.2] that the chain \( T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0 \) is
stationary with respect to the operator $A$; therefore, the main interest is on Metrizable $\subseteq$ Normal $\subseteq$ T$_2$ $\subseteq$ KC $\subseteq$ SC $\subseteq$ US $\subseteq$ T$_1$.

Note 1. A topological space $X$ is KC if and only if $\{U \subseteq X : U$ is an open subset of $X\}$ is a basis of $X$.

Note 2. A topological space $X$ is KC if and only if $\{K \subseteq X : K$ is a closed subset of $X\}$ is a topology on $X$.

Remark 3. Suppose $X$ is a noncompact topological space, then $\{\emptyset\}$ is a topology on $X$.

Lemma 4. Let $X_1$, $X_2$ be disjoint compact subsets of $X$. Then $\bigcap_{i=1}^{2} X_i = \emptyset$.

Proof. Suppose $X_1 \cap X_2 = \emptyset$ and $U$ is an open subset of $A(X_1 \cup X_2) = X_1 \cup X_2$ containing $\emptyset$. Using the following cases, $U$ is an open subset of $X_1 \cup A(X_2)(= X_1 \cup X_2 \cup \{\infty\})$ too.

(i) Consider $\infty \notin U$. In this case, $U := U \cap X_1 \cap X_2$ is an open subset of $X_1 \cup A(X_2)$. Since $X_1 \cap X_2 = \emptyset$, $U \cap X_1 \cap X_2$ is an open subset of $X_1 \cup A(X_2)$.

(ii) Consider $\infty \in U$. In this case, $(X_1 \cup X_2) \cap U$ is a closed compact subset of $X_1 \cup X_2$. Since $X_1$ and $X_2$ are two closed subsets of $X_1 \cup X_2$, $(X_1 \cup X_2) \cap U \cap X_1 \cap X_2$ is a closed subset of $X_1 \cup X_2$. Since $X_1 \cap X_2 = \emptyset$, $U \cap X_1 \cap X_2$ is an open subset of $X_1 \cup A(X_2)$.

Remark 5. Suppose $X$ is a noncompact topological space, then $\{\emptyset\}$ is a topology on $X$.

Lemma 5. If $Y$ is a closed subset of $X$, then $A(X) = X \cup \{\infty\}$ is an embedding of $A(Y)$.

Proof. If $Y$ is compact, then $A(Y) = Y$ and by Remark 3(4) we are done. If $Y$ is not compact, $X \setminus Y$ is an open subset of $X$ and $A(X)$; thus $Y = Y \cup \{\infty\}$ is a compact subset of $A(X)$. Suppose $F \subseteq Y \cup \{\infty\}$; we prove that $F$ is a closed subset of $Y^* := Y \cup \{\infty\}$ as a subspace of $A(X)$ if and only if $F$ is a closed subset of $A(Y) = Y \cup \{\infty\}$ as one point compactification of $Y$. However, we mention that $Y \cup \{\infty\}$ in both topologies is an embedding of $Y$ by Remark 3(4).

First, suppose $F$ is a closed subset of $Y^*$. Using the following two cases, $F$ is a closed subset of $A(Y)$ too.

(i) Consider $\infty \in F$. In this case, $U := Y^* \setminus F = Y \setminus F$ is an open subset of $Y$; therefore it is an open subset of $A(Y)$, so $F = A(Y) \setminus U$ is a closed subset of $A(Y)$.

(ii) Consider $\infty \notin F$. In this case, $F$ is a closed subset of $A(X)$ since it is a closed subset of $Y^*$ and $Y^*$ is closed in $A(X)$. Therefore, $U := A(X) \setminus F$ is an open subset of $A(X)$ with $\infty \in U$. So $A(X) \setminus U$ is a closed compact subset of $X$. Therefore, $(A(X) \setminus U) \cap Y^* = F$ is a closed compact subset of $Y^*$. Since $(A(X) \setminus U) \cap Y^* = (A(X) \setminus U) \cap Y^* = F$ is a closed compact subset of $Y^*$, so $F$ is closed in $Y$. Conversely, suppose $F$ is a closed subset of $A(Y)$. Using the following two cases, $F$ is a closed subset of $Y^*$ too.

(iii) Consider $\infty \in F$. In this case, $U := A(Y) \setminus F = Y \setminus F$ is an open subset of $Y$; therefore, there exists an open subset $V$ of $X$ with $V \cap Y = U$. $V$ is an open subset of $A(X)$ too; thus $V \cap Y^* = (V \cap Y) \setminus U = F$ is a closed subset of $Y^*$.

(iv) Consider $\infty \notin F$. In this case, $F$ is a closed compact subset of $A(Y)$ with $\infty \notin F$; thus $F$ is a closed compact...
Lemma 6. Suppose \( \mathcal{C} \in \{ \text{Metrizable, Normal, } T_2, \text{ KC, SC, US, } T_4, \text{ T}_{2\text{D}}, \text{ T}_{2\text{UD}}, \text{ T}_{2\text{p}}, \text{ Top} \} \); also consider topological spaces \( X, Y \). We have the following.

1. \( X \cup Y \in \mathcal{C} \) if and only if \( X, Y \in \mathcal{C} \).
2. Consider two closed subsets \( A, B \) of \( X \) with \( A \cup B = X \) and \( A \cap B = \{ t \} \). So \( A, B \in \mathcal{C} \) if and only if \( Y \in \mathcal{C} \).

Proof. (1) has a formal proof, so we deal with (2). If \( X \in \mathcal{C} \) and \( E \) is a closed subspace of \( X \), then \( E \in \mathcal{C} \). Suppose \( A, B \in \mathcal{C} \); \( A, B \) are closed subspaces of \( X \) with \( A \cap B = \{ t \} \) and \( A \cup B = X \). We prove \( X \in \mathcal{C} \).

First, note the fact that if \( V \) is an open subset of \( A \) (resp. \( B \)) with \( t \notin V \), then \( V \) is an open subset of \( X \), since \( V \) is an open subset of \( A \cap B = \{ t \} \) and \( A \cup B = X \) is an open subset of \( X \). Now consider the following cases for \( \mathcal{C} \).

(i) Consider \( \mathcal{C} = \text{Metrizable} \). If \( A, B \) are metrizable subspaces of \( X \), then there exist metrics \( d_1, d_2 \), respectively, on \( A, B \) such that \( d_1, d_2 \leq 1 \), the metric topology induced from \( d_1 \) on \( A \) is subspace topology on \( A \) induced from \( X \), and the metric topology induced from \( d_2 \) on \( B \) is subspace topology on \( B \) induced from \( X \). Define \( d : X \times X \to [0, +\infty) \) with

\[
d(x, y) = \begin{cases} 
  d_1(x, y) & x, y \in A, \\
  d_2(x, y) & x, y \in B, \\
  2 & \text{otherwise.}
\end{cases}
\]

Then the metric topology induced from \( d \) on \( X \) coincides with \( X \)'s original topology.

(ii) Consider \( \mathcal{C} = T_2 \). Suppose \( A, B \) are Hausdorff subspaces of \( X \) and \( x, y \in X \) are two distinct points of \( X \). Consider the following cases:

1. \( x \in X \setminus A = B \setminus \{ t \} \) and \( y \in X \setminus B = A \setminus \{ t \} \); in this case, \( B \setminus \{ t \} \) and \( A \setminus \{ t \} \) are disjoint open neighborhoods of, respectively, \( x \) and \( y \);
2. \( x, y \in A \); there exist disjoint open subsets \( U_1, U_2 \) of \( A \) with \( x \in U_1 \) and \( y \in U_2 \); suppose \( t \notin U_1 \); thus \( U_1 \) is an open subset of \( X \). There exists an open subset \( U \) of \( X \) with \( U \cap A = U_2 \). Hence, \( U_1, U \) are disjoint open subsets of \( X \) with \( x \in U_1 \) and \( y \in U \).

Using the above cases, \( X \) is Hausdorff.

(iii) Consider \( \mathcal{C} = \text{Normal} \). If \( A, B \) are normal subspaces of \( X \), then \( A, B \) are Hausdorff and, using the case \( \mathcal{C} = T_2 \), \( X \) is Hausdorff. Now suppose \( E, F \) are disjoint closed subsets of \( X \); also we may suppose \( t \notin E \). Let \( E_A := E \cap A \), \( E_B := E \cap B \), \( F_A := F \cap A \), and \( F_B := F \cap B \). There are disjoint open subsets \( U_E, U_F \) of \( A \) containing, respectively, \( E_A, F_A \). Also there are disjoint open subsets \( V_E, V_F \) of \( B \) containing, respectively, \( E_B, F_B \). There are open subsets \( U, V \) of \( X \) with \( U_E = A \cap U \) and \( V_F = V \cap B \). Let \( W_E := (U_E \setminus \{ t \}) \cup (V_E \setminus \{ t \}) \) and \( W_F := U \cup V \); then \( W_E, W_F \) are disjoint open subsets of \( X \) containing, respectively, \( E, F \).

(iv) Consider \( \mathcal{C} = \text{KC} \). Suppose \( A, B \) are KC and \( K \) is a compact subset of \( X \). Since \( A, B \) are closed, \( A \cap K, B \cap K \) are compact too. Since \( A \cap K \) is a compact subset of \( A \) and \( K \) is a closed subset of \( A \). Since \( A \cap K \) is a closed subset of \( A \) and \( A \cup K \) is closed subset of \( X \). Similarly, \( B \cap K \) is a closed subset of \( X \). Thus \( K = (A \cap K) \cup (B \cap K) \) is a closed subset of \( X \) and \( X \) is KC.

(v) Consider \( \mathcal{C} = \text{SC} \). Suppose \( A, B \) are SC and \( (x_n : n \in \omega) \) is a sequence in \( X \) converging to \( x \). Using the following cases, \( (x_n : n \in \omega) \cup \{ x \} \) is a closed set of \( X \).

1. Consider \( x \neq t \). Suppose \( x \in A \setminus \{ t \} \). In this case, \( A \setminus \{ t \} \) is an open neighborhood of \( x \) in \( X \), so there exists \( N \in \omega \) such that \( x_n \in A \setminus \{ t \} \) for all \( n \geq N \). Hence \( (x_n : n \geq N) \) is a converging sequence to \( x \) in \( A \). Since \( A \) is SC, \( (x_n : n \geq N) \cup \{ x \} \) is a closed subset of \( A \). Therefore, \( (x_n : n \geq N) \cup \{ x \} \) is a closed subset of \( X \). For each \( n \in \omega \) if \( x_n \in B \) (resp. \( x_n \in A \)), \( (x_n) \) is a closed subset of \( B \) (resp. \( A \)). Since \( B \) (resp. \( A \)) is SC and in particular \( T_1 \). Thus for all \( n \in \omega \), \( \{ x_n \} \) is a closed subset of \( X \). By closeness of \( (x_n : n \leq N) \) and \( (x_n : n \geq N) \cup \{ x \} \) in \( X \), the set \( \{ x_n : n \in \omega \} \cup \{ x \} \) is closed in \( X \).

2. Consider \( x = t \) and there exists \( N \in \omega \) such that \( (x_n : n \geq N) \subseteq A \) or \( (x_n : n \geq N) \subseteq B \). Suppose there exists \( N \in \omega \) with \( (x_n : n \geq N) \subseteq A \). If this case \( (x_n : n \geq N) \) is a converging sequence to \( x \) in \( A \), and, using the same argument as in the second paragraph of the case \( x \neq t \), \( (x_n : n \in \omega) \cup \{ x \} \) is closed in \( X \).

3. Consider none of the above two cases. In this case, \( (x_n : n \in \omega) \) converges to \( t \) and it has two subsequences \( (x_{n_k} : k \in \omega) \) and \( (x_{m_k} : k \in \omega) \) such that \( (x_{n_k} : k \in \omega) \subseteq A \), \( (x_{m_k} : k \in \omega) \subseteq B \), and \( (n_k : k \in \omega) \cup (m_k : k \in \omega) = \omega \). Using item (2), \( (x_{n_k} : k \in \omega) \cup \{ x \} \) and \( (x_{m_k} : k \in \omega) \cup \{ x \} \) are two closed subsets of \( X \); thus \( \{ x_n : k \in \omega \} \cup \{ x \} = \{ x_{n_k} : k \in \omega \} \cup \{ x \} \cup \{ x_{m_k} : k \in \omega \} \cup \{ x \} \) is a closed subset of \( X \).

(vi) Consider \( \mathcal{C} = \text{US} \). If \( A, B \) are US and \( X \) is not US, consider converging sequence \( (x_n : n \in \omega) \) in \( X \) to \( x, y \) with \( x \neq y \). Let \( x \neq t \); we may suppose \( x \in A \). The set \( A \setminus \{ t \} = X \setminus B \) is an open neighborhood of \( x \). Thus there exists \( N \in \omega \) with \( (x_n : n \geq N) \subseteq A \setminus \{ t \} \) and \( y \in (x_n : n \geq N) \subseteq A \). So \( (x_n : n \geq N) \) is a
converging sequence to $x, y$ in $A$ and $x \neq y$; thus $A$ is not US which is a contradiction.

(vii) Consider $\emptyset = T_1$. Suppose $A$ and $B$ are $T_1$; let $x \in X$. We may suppose $x \in A$. Since $A$ is $T_1$, $\{x\}$ is a closed subset of $A$. Since $A$ is a closed subset of $X$ and $\{x\}$ is a closed subset of $A$, $\{x\}$ is a closed subset of $X$.

(viii) Use similar methods for the rest of the cases of $\emptyset$. □

Lemma 7. Suppose $\emptyset \in \{\text{Metrizable, Normal, } T_2, \text{ KC, SC, US, } T_1, T_D, T_{UD}, T_0, \text{ Top}\}$; also consider topological spaces $X, Y$. We have $A(X \cup Y) \in \emptyset$ if and only if $A(X), A(Y) \in \emptyset$.

Proof. By Lemma 6 and Lemma 4, it is clear if $X$ or $Y$ is compact. So we may suppose $X$ and $Y$ are two disjoint noncompact topological spaces. Since $X$ and $Y$ are two open subset of $X \cup Y$, two sets $X^* := A(X \cup Y) \setminus \{x \in X \cup \{\infty\}\}$ and $Y^* := A(X \cup Y) \setminus \{y \in Y \cup \{\infty\}\}$ are two closed subsets of $A(X \cup Y)$ with $X^* \cap Y^* = A(X \cup Y)$. By Lemma 6, $A(X \cup Y) \in \emptyset$ if and only if $X^*, Y^* \in \emptyset$. By Lemma 5, $X^*$ is homeomorphic to $A(X)$ and $Y^*$ is homeomorphic to $A(Y)$; hence $A(X \cup Y) \in \emptyset$ if and only if $A(X), A(Y) \in \emptyset$. □

Lemma 8. In topological space $X$, if $X$ is SC, then $A(X)$ is US.

Proof. Let $X$ be a noncompact SC space. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X) = X \cup \{\infty\}$ converging to $x, y \in A(X)$. We have the following cases.

(i) Consider $x, y \in X$. In this case, $X$ is an open neighborhood of $x, y$ in $A(X)$; hence there exists $N \in \omega$ such that $x_n \in X$ for all $n \geq N$. Therefore, $(x_n : n \geq N)$ is a converging sequence in $X$ to $x, y$. Since $X$ is SC, $X$ is US and $x = y$.

(ii) Consider $x \in X, y = \infty$. In this case, there exists $N \in \omega$ such that $x_n \in X$ for all $n \geq N$. Therefore, $(x_n : n \geq N)$ is a converging sequence in $X$ to $x$. Thus $(x_n : n \geq N) \cup \{x\}$ is a closed subset of $X$. So $\{x_n : n \geq N\} \cup \{x\}$ is a compact closed subset of $X$ and $V := A(X) \setminus \{(x_n : n \geq N) \cup \{x\}\}$ is an open neighborhood of $\infty(= y)$ which is a contradiction by $x_n \notin V$ for all $n \geq N$ and by converging $(x_n : n \in \omega)$ to $y$. So this case does not occur.

Using the above cases, we have $x = y$, and $A(X)$ is US. □

3. The Main Table

See Figure 1; then we have Table 1 which we prove in this Section and where:

The mark "√" indicates that in the corresponding case, there exists $X \in P$ such that $A(X) \in Q$, and the mark "—" indicates that in the corresponding case for all $X \in P$ we have $A(X) \notin Q$.

Let

$$E := \{C_1, C_2 \setminus C_1, C_3, C_4, C_6, C_7 \setminus C_8, C_9 \setminus C_{10}, C_{11}, C_{12}\},$$

$$F := \{C_8 \setminus C_7, C_9, C_{10} \setminus C_9, \ldots, C_{10} \setminus C_{11}, C_{12}\}. (6)$$

By Remark 3(7) in Table 1, the mark "—" for cases in which "$P \in E, Q \in F'$" or "$P \in F, Q \in E'$" is evident. However, it has been proved in [1, Lemma 3.1 and Corollary 3.2] that the chain $T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0$ is stationary with respect to the operator $A$, so corresponding marks of the cases in which $Q \in F$ are obtained. Thus it remains to discuss cases in which $P, Q \in E$.

Since the subspace of a metrizable (resp. $T_2$, SC, and US) space is metrizable (resp. $T_2$, SC, and US) using Remark 3(4) and (5), if $A(X)$ is, respectively, metrizable $T_2$, SC, or US, then $X$ is too. Hence we obtain "—" for the following cases too (choose $P$ and $Q$ from the same rows of Table 2).

Proof (proof of the rest of the cells of Figure 1).

First Row. Here we have $P = C_1$ and the following cases for $Q$.

(i) Consider $Q = C_1$. Consider two spaces $X := (0, 1)$ (with induced metric from Euclidean space $\mathbb{R}$) and $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (with induced metric from Euclidean space $\mathbb{R}^2$); then $A(X)$ is homeomorphic to $S^1$; moreover, $X, S^1 \in C_1$; therefore $X, A(X) \in C_1$.

(ii) Consider $Q = C_2 \setminus C_1$. Consider $X := (0, 1)$ with discrete topology. $A(X)$ is compact Hausdorff, so it is normal. If $\mathcal{D}$ is a topological basis for $A(X)$, then for all $t \in (0, 1)$ we have $\{t\} \in \mathcal{D}$. Therefore $\mathcal{D}$ is uncountable and compact space $A(X)$ is not metrizable. Thus $X \in C_1$ and $A(X) \in C_2 \setminus C_1$.

(iii) Consider $Q = C_3 \setminus C_2$. Use Remark 3(3).

(iv) Consider $Q = C_4 \setminus C_3$. Consider $X$ as the set of all rational numbers as a subspace of Euclidean space $\mathbb{R}$. Since $X$ is not locally compact, by Remark 3(3), $A(X)$ is not Hausdorff. Suppose $M$ is a compact subset of $A(X)$; in order to show that $A(X)$ is KC, we show $M$ is a closed subset of $A(X)$.

Case 1. If $\infty \notin M$, then $M$ is a compact subset of $X$; since $X$ is a metric space, $M$ is a closed subset of $X$ too. Therefore, $A(X) \setminus M$ is an open subset of $A(X)$. Hence, $M$ is a closed subset of $A(X)$.

Case 2. If $\infty \in M$, we claim that $X \setminus M$ is an open subset of $X$ and so an open subset of $A(X)$; otherwise (since $X$ is metrizable) there exists a one-to-one sequence $(x_n : n \in \omega)$ in $X \setminus (X \setminus M)$ converging to a point $x \in X \setminus M$ (in metric space $X$). For all $m \in \omega, x_n : n \geq m) \setminus \{x\}$ is a compact closed subset of $X$, and $U_m := A(X) \setminus \{(x_n : n \geq m) \setminus \{x\}\}$ is an open subset of $A(X)$. Since $x \notin M, M \subseteq \bigcup_{m \geq 1} U_m$. Using the compactness of $M$, there exists $m \geq 1$ such that $M \subseteq U_0 \cup U_1 \cup \cdots \cup U_m$. Since $x_n \in M \setminus (U_0 \cup U_1 \cup \cdots \cup U_m) = M \setminus U_m$, we have $M \notin U_0 \cup U_1 \cup \cdots \cup U_m$ which
Table 1

| P   | C_1 | C_2 \ C_1 | C_3 \ C_2 | C_4 \ C_3 | C_5 \ C_4 | C_6 \ C_5 | C_7 \ C_6 | C_8 \ C_7 | C_9 \ C_8 | C_10 \ C_9 | C_11 \ C_10 |
|-----|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| C_1 | ✓   | ✓         | —         | ✓         | ✓         | ✓         | ✓         | —         | —         | —         | —         |
| C_2 \ C_1 | —   | ✓         | ✓         | ✓         | ✓         | ✓         | ✓         | —         | —         | —         | —         |
| C_3 \ C_2 | —   | —         | ✓         | ✓         | ✓         | ✓         | ✓         | —         | —         | —         | —         |
| C_4 \ C_3 | —   | —         | —         | ✓         | ✓         | ✓         | ✓         | —         | —         | —         | —         |
| C_5 \ C_4 | —   | —         | —         | —         | ✓         | ✓         | ✓         | —         | —         | —         | —         |
| C_6 \ C_5 | —   | —         | —         | —         | —         | ✓         | ✓         | —         | —         | —         | —         |
| C_7 \ C_6 | —   | —         | —         | —         | —         | —         | ✓         | —         | —         | —         | —         |
| C_8 \ C_7 | —   | —         | —         | —         | —         | —         | —         | ✓         | —         | —         | —         |
| C_9 \ C_8 | —   | —         | —         | —         | —         | —         | —         | —         | ✓         | —         | —         |
| C_10 \ C_9 | —   | —         | —         | —         | —         | —         | —         | —         | —         | ✓         | —         |
| C_11 \ C_10 | —   | —         | —         | —         | —         | —         | —         | —         | —         | —         | ✓         |

Table 2

| P   | Q   | Reason of omitting this case |
|-----|-----|------------------------------|
| C_2 \ C_1, C_3 \ C_2, C_4 \ C_3, C_5 \ C_4, C_6 \ C_5, C_7 \ C_6 | C_1 | If A(X) is metrizable, then X is metrizable too |
| C_2 \ C_1, C_1 \ C_2 | C_1 | If A(X) is T_2, then X is T_2 too |
| C_3 \ C_2, C_4 \ C_3, C_5 \ C_4, C_6 \ C_5, C_7 \ C_6 | C_2 | If A(X) is KC, then X is KC too |
| C_4 \ C_3, C_5 \ C_4, C_6 \ C_5, C_7 \ C_6 | C_3 | If A(X) is SC, then X is SC too |
| C_5 \ C_4, C_5 \ C_4, C_6 \ C_5, C_7 \ C_6 | C_4 | If A(X) is US, then X is US too |

Figure 1: Consider the previous classes of topological spaces (in order to be more convenient, note the right diagram).
is a contradiction. Thus $X \setminus M$ is an open subset of $X$, and $M = A(X) \setminus (X \setminus M)$ is a closed subset of $A(X)$.

Finally we have $X \in C_1$ and $A(X) \in C_4 \setminus C_3$.

(v) Consider $Q = C_3 \setminus C_4$. If $X$ is a metric space, then it is a $k$-space and, by Remark 3(2), $A(X)$ is KC; hence $A(X) \notin C_5 \setminus C_4$.

(vi) Consider $Q = C_0 \setminus C_1$ or $C_2 \setminus C_0$. We claim that if $X$ is a metric space, then $A(X)$ is SC. First, note the fact that, by Remark 3(1), $A(X)$ is US and hence $T_1$. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X)$ converging to $x \in A(X)$; we show that $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$. Consider the following cases.

Case 1. $x \neq \omega; \infty$; in this case, $X$ is an open neighborhood of $x$, thus there exists $m \in \omega$ such that $x_n \in X$ for all $n \geq m$ and $(x_n : n \geq m)$ converges to $x$ in metric space $X$ (by Remark 3(4), $X$ as a subspace of $A(X)$ has its original topology). Thus $\{x_n : n \geq m\} \cup \{x\}$ is a compact subset of $X$; therefore $A(X) \setminus (\{x_n : n \geq m\} \cup \{x\})$ is an open subset of $A(X)$. Finally $\{x_n : n \geq m\} \cup \{x\}$ is a closed subset of $A(X)$ and since $A(X)$ is $T_1$, $\{x_n : n \geq m\} \cup \{x\}$ is a closed subset of $A(X)$ too.

Case 2. $x = \omega$ and $\{x_n : n \in \omega\}$ is finite. In this case, $\{x_n : n \in \omega\} \cup \{x\}$ is a finite subset of $T_1$ space $A(X)$ and it is closed.

Case 3. $x = \omega$ and $\{x_n : n \in \omega\}$ is infinite. In this case, we may assume $x_n \in X$ for all $n \in \omega$. If $\{x_n : n \in \omega\}$ is not a closed subset of $X$, then there exists a subsequence $(x_n : k \in \omega)$ of $(x_n : n \in \omega)$ converging to $y \in X \setminus \{x_n : n \in \omega\}$. Thus $(x_n : k \in \omega)$ converges to $y$ in $A(X)$ too (use Remark 3(4)). Since $(x_n : k \in \omega)$ converges to $y$, $x(= \omega)$ and $A(X)$ is US, we have $y = x$ which is a contradiction with $y \in X = A(X) \setminus \{\omega\}$. Therefore $\{x_n : n \in \omega\}$ is a closed subset of $X$, so

$$X \setminus \{x_n : n \in \omega\} = X \setminus (\{x_n : n \in \omega\} \cup \{x\})$$

is an open subset of $X$ and $A(X)$. Finally $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$.

Using the above three cases, $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$ and we are done.

Second Row. Here we have $P = C_2 \setminus C_1$ and the following cases for $Q$.

(i) Consider $Q = C_2 \setminus C_1$. Suppose $\Omega$ is the least uncountable ordinal number. Consider $X = \Omega + 1$ (with order topology). Since $X$ is well ordered, it is normal. However, $X$ is not metrizable and $A(X) = X$.

(ii) Consider $Q = C_1 \setminus C_2$. If $X$ and $A(X)$ are $T_2$, then, by Remark 3(3), $A(X)$ is normal.

(iii) Consider $Q = C_1 \setminus C_3$. Consider $X$ as disjoint union of $X_1 = \Omega + 1$ with order topology and $X_2 = Q$ as the set of all rational numbers with induced metric from Euclidean space $\mathbb{R}$. The topological space $X$ is normal since $X_1, X_2$ are normal. Moreover, $X$ is disjoint union of $X_1$ and $X_2$, so $X$ has nonmetrizable subspace $X_1$, thus $X$ is nonmetrizable and $X \in C_2 \setminus C_1$. Since $X_1$ is compact, by Lemma 4 we have $A(X) = X_1 \cup A(X_2)$. Considering case “$P = C_1$, $Q = C_4 \setminus C_3$”, we have $A(X) = C_4 \setminus C_3$. Using Lemma 6, we have $X_1 \cup A(X_2) \in C_4 \setminus C_3$; thus $A(X) \in C_4 \setminus C_3$.

(iv) Consider $Q = C_5 \setminus C_4$. Let $X$ be an uncountable set and $b \in X$. Consider $X$ under Fortissimo topology with particular point $b$, that is, under the topology $\{U \subseteq X \setminus \{b\} : b \notin U \cup (X \setminus U)$ is countable$\}$ (see [4, counterexample 25]). For distinct $x, y \in X$, suppose $x \neq b$. Then $U = \{x\}, V = X \setminus \{x\}$ are disjoint open subsets of $X$ and $X$ is Hausdorff. On the other hand, if $E, F$ are disjoint closed subsets of $X$, suppose $b \notin E$, then $E$ and $X \setminus E(\geq F)$ are disjoint open subsets of $X$ containing $E$ and $F$. Thus $X$ is normal.

Moreover, if $W$ is an open neighborhood of $b$, then $X \setminus W$ is countable; therefore $W$ is uncountable and $W \cap (X \setminus \{b\}) \neq \emptyset$. Therefore, $b \in X \setminus \{b\}$. Let $(x_n : n \in \omega)$ be a sequence of elements of $X \setminus \{b\}$. The sequence $(x_n : n \in \omega)$ does not converge to $b$, since $X \setminus \{x_n : n \in \omega\}$ is an open neighborhood of $b$. The space $X$ is not metrizable since $b \in X \setminus \{b\}$ and there is not any sequence in $X \setminus \{b\}$ converging to $b$. So $X \in C_2 \setminus C_1$.

On the other hand, using the definition of one point compactification, any subset $M$ of $A(X)$ containing $b$ is a compact subset of $A(X)$. Therefore, $A(X) \setminus \{b\}$ is a compact subset of $A(X)$, but it is not the closed subset of $A(X)$; thus $A(X)$ is not KC. We claim that $A(X)$ is SC. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X)$ converging to $w$. We have the following cases.

Case 1. Consider $w \in X \setminus \{b\}$. In this case, $\{w\}$ is an open neighborhood of $w$ and there exists $N \geq 1$ such that for all $n \geq N$ we have $x_n = w$. Using Remark 3(1), $A(X)$ is $T_1$; thus $\{x_n : n \in \omega\} \cup \{w\} = \{x_0, \ldots, x_N, w\}$ is a closed subset of $A(X)$.

Case 2. Consider $w = b$. The set $X \setminus \{x_n : n \in \omega, x_n \neq b\}$ is an open neighborhood of $w$; thus there exists $N \geq 1$ such that for all $n \geq N$ we have $x_n \in (X \setminus \{x_n : n \in \omega, x_n \neq b\})$ (thus for all $n \geq N$ we have $x_n = b$). Using Remark 3(1), $A(X)$ is $T_1$; thus $\{x_n : n \in \omega\} \cup \{w\} = \{x_0, \ldots, x_N, w\}$ is a closed subset of $A(X)$.

Case 3. Consider $w = \infty$. In this case, $A(X) \setminus \{(x_n : n \in \omega) \cup \{w\} = X \setminus \{x_n : n \in \omega\}$ is an open subset of $X$; therefore it is an open subset of $A(X)$. Thus $\{x_n : n \in \omega\} \cup \{w\}$ is a closed subset of $A(X)$.

Using the above three cases, $\{x_n : n \in \omega\} \cup \{w\}$ is a closed subset of $A(X)$, and $A(X)$ is SC.

Since $A(X)$ is SC and it is not KC, $A(X) \in C_2 \setminus C_4$.

(v) Consider $Q = C_0 \setminus C_5$. Suppose $\mathcal{F}$ is a uniform filter over $\mathbb{N}$. Consider $X = \mathbb{N} \cup \{0\}(= \omega)$ under topology $\{A \subseteq X \setminus \{0\} : b \notin A \cup \{0\} \in \mathcal{F}\}$. If $x, y \in X$ are distinct with $x \neq 0$, then $\{x\}, X \setminus \{x\}$ are disjoint open subsets of $X$ containing $x, y$ and $X$ is Hausdorff. If $E, F$ are disjoint closed subsets of $X$, suppose $0 \notin E$. Therefore, $E, X \setminus E(\geq F)$ are disjoint open subsets of $X$ and $X$ is normal. Since $\mathcal{F}$ is a uniform filter over $\mathbb{N}$, it does not contain any finite subset of $X$. Since all of the elements of $\mathcal{F}$ are infinite, $0$ is a limit point of $X$ and $0 \in X \setminus \{0\}$. Consider a sequence $(x_n : n \in \omega)$ in $X \setminus \{0\}$. We have the following cases.
Case 1. \((x_n : n \in \omega)\) has a constant subsequence like \((x_{n_k} : k \in \omega)\). Since \(X\) is Hausdorff and every sequence converges to at most one point, \((x_{n_k} : k \in \omega)\) converges to its constant value and does not converge to 0. Thus \((x_n : n \in \omega)\) does not converge to 0.

Case 2. \((x_n : n \in \omega)\) does not have any constant subsequence. Suppose \((x_{n_k} : k \in \omega)\) is a one-to-one subsequence of \((x_n : n \in \omega)\). Since \(\mathcal{F}\) is an ultrafilter over \(X \setminus \{0\} (= \mathbb{N})\), and \([x_{n_k} : k \in \omega] \cap [x_{n'_k} : k \in \omega] = \varnothing\), we have \([x_{n_k} : k \in \omega] \notin \mathcal{F}\) or \([x_{n'_k} : k \in \omega] \notin \mathcal{F}\). Suppose \([x_{n_k} : k \in \omega] \notin \mathcal{F}\). Since \(\mathcal{F}\) is an ultrafilter over \(\mathbb{N}\), \(\mathbb{N} \setminus [x_{n_k} : k \in \omega] \in \mathcal{F}\). Therefore \((\mathbb{N} \setminus [x_{n_k} : k \in \omega]) \cup \{0\}\) is an open neighborhood of 0 and \((x_n : n \in \omega)\) does not converge to 0.

Since \(0 \in X \setminus \{0\}\) and by the above two cases, there is not any sequence in \(X \setminus \{0\}\) converging to 0; \(X\) is not metrizable. Thus \(X \in C_2 \cap C_1\).

Now pay attention to the following claims.

Claim 1. The sequence \((n : n \geq 1)\) converges to \(0\) in \(A(X)\). Suppose \(U\) is an open neighborhood of \(0\) in \(A(X)\). Since \(X = T_1, A(X) = T_1\) too. Therefore, \(V := U \setminus \{0\}\) is an open neighborhood of \(0\) in \(X\); thus \(A(X) \cap V = X \setminus V \subseteq X \setminus \{0\}\) is a compact (and closed) subset of \(X\). Also \(A(X) \cap V = X \setminus \{0\}\) is finite, since \(X \setminus \{0\}\) is discrete and \(A(X) \cap V\) is a compact subset of \(X \setminus \{0\}\) (use Remark 3(4)). Suppose \(N = \max(A(X) \cap V)\). For all \(n > N\), we have \(n \in V \subseteq U\). Hence \((n : n \geq 1)\) converges to \(0\).

Claim 2. \((n : n \geq 1)\cup \{0\}\) is not a closed subset of \(A(X)\). Using the fact that \(0 \in X \setminus \{0\}\), we have \([n : n \geq 1] \cup \{0\} = A(X)\); hence \([n : n \geq 1] \cup \{0\}\) is not a closed subset of \(A(X)\).

Regarding Claims 1 and 2, \(A(X)\) is not SC. Since \(X\) is normal, it is KC; so using Remark 3(1), \(A(X)\) is US. Therefore, \(A(X) \in C_2 \cap C_3\).

(vi) Consider \(Q = C_2 \cap C_6\). If \(X\) is Normal, then it is KC and by Remark 3(1), \(A(X)\) is US. Thus \(A(X) \notin C_2 \cap C_6\).

Third Row. Here we have \(P = C_3 \cap C_2\) and the following cases for \(Q\).

(i) Consider \(Q = C_3 \cap C_1\). Suppose \(X\) is a Hausdorff locally compact nonnormal topological space. Since \(X\) is not normal, it is not metrizable and \(X \in C_2 \cap C_3\). By Remark 3(3), \(A(X)\) is normal. By Remark 3(4), \(A(X)\) is not metrizable. Hence \(A(X) \in C_2 \cap C_1\). Moreover \(X = \{(\omega + 1) \times (\Omega + 1) \setminus \{\omega, \Omega\}\), where \(\omega + 1\) and \(\Omega + 1\) have their order topology and \((\omega + 1) \times (\Omega + 1)\) equipped with product topology (deleted 

(ii) Consider \(Q = C_3 \cap C_2\). Use a similar method described for \(P = C_2 \cap C_1\) and \(Q = C_3 \cap C_2\).

(iii) Consider \(Q = C_3 \cap C_2\). Consider \(X = \mathbb{R}\) under topology \((\mathbb{O} \setminus B : O\) is an open subset of \(\mathbb{R}\) in its Euclidean topology and \(B \subseteq \{1/n : n \in \mathbb{N}\}\)). then \(X \in C_3 \cap C_2\) (Smirnov’s deleted sequence topology \[4, counterexample 64\]). Also \(X\) is first countable; therefore it is a \(k\)-space by Remark 3(6).

(iv) Consider \(Q = C_3 \cap C_4\). Consider \(X\) as disjoint union of \(X_1\) and \(X_2\), where

\(1\) \(X_1\) is an uncountable set under Fortissimo topology with particular point \(b \in X_1\), that is, under topology \([U \subseteq X_1 : b \notin U \setminus X_1 \cup \{0\}\) is countable\] (see \[4, counterexample 25\] and proof of Table 1 regarding case “\(P = C_2 \setminus C_1, Q = C_3 \cap C_4\)”;

\(2\) \(X_2 = \mathbb{R}\) under the topology \((\mathbb{O} \setminus B : O\) is an open subset of \(\mathbb{R}\) in its Euclidean topology and \(B \subseteq \{1/n : n \in \mathbb{N}\}\)) (see Smirnov’s deleted sequence topology \[4, counterexample 64\] and proof of Table 1 regarding case “\(P = C_3 \setminus C_2, Q = C_4 \setminus C_3\)”.

Since \(X_1 \in C_2 \setminus C_1\) and \(X_2 \in C_3 \setminus C_2\), we have \(X = X_1 \cup X_2 \in C_2 \setminus C_3\) by Lemma 6(1). Moreover \(A(X_1) \in C_3 \setminus C_4\) and \(A(X_2) \in C_4 \setminus C_3\) lead us to \(A(X) = A(X_1) \cup A(X_2) \in C_3 \setminus C_4\) by Lemma 7.

(v) Consider \(Q = C_6 \setminus C_5\). Consider \(X\) as disjoint union of \(X_1\) and \(X_2\), where we have the following.

\(1\) Suppose \(\mathcal{F}\) is a uniform ultrafilter over \(\mathbb{N}\). Consider \(X_1 = \mathbb{N} \cup \{0\} (= \omega)\) under topology \([A \subseteq X : 0 \notin A \cap V \setminus \{0\} \in \mathcal{F}\) (see proof of Table 1 regarding case “\(P = C_2 \setminus C_1, Q = C_6 \setminus C_5\)”.

\(2\) \(X_2\) is Smirnov’s deleted sequence topological space (see proof of Table 1 regarding case “\(P = C_3 \setminus C_2, Q = C_4 \setminus C_3\)”.

Then \(X \in C_3 \setminus C_2\) by \(X_1 \in C_3 \setminus C_2\) and \(X_2 \in C_3 \setminus C_2\) and Lemma 6(1). Also \(A(X) \in C_6 \setminus C_5\) by \(A(X_1) \in C_6 \setminus C_5\), \(A(X_2) \in C_4 \setminus C_3\), and Lemma 7.

(vi) Consider \(Q = C_7 \cap C_6\). If \(X\) is T$_3$, then it is KC and by Remark 3(1), \(A(X)\) is US. Thus \(A(X) \notin C_7 \cap C_6\).

Fourth Row. Here we have \(P = C_4 \setminus C_3\) and the following cases for \(Q\).

(i) Consider \(Q = C_4 \setminus C_3\). Consider \(W\) as the set of all rational numbers as a subspace of Euclidean space \(\mathbb{R}\). Using the case “\(P = C_1, Q = C_4 \setminus C_3\)” for \(X := A(W)\), we have \(X \in C_4 \setminus C_3\). Since \(X\) is compact, we have \(A(X) = X \in C_1 \setminus C_3\).

(ii) Consider \(Q = C_4 \setminus C_2\). Consider uncountable set \(X\) with countable complement topology \([U \subseteq X : U = \varnothing \cap V \setminus \{0\}\) is countable\] \[4, counterexamples 20 and 21\]. Since every two nonempty open subsets of \(X\) have nonempty intersection, \(X\) is not Hausdorff. It is clear that \(X\) is T$_1$. Moreover, \(M\) is a compact subset of \(X\) if and only if \(M\) is finite. Therefore, every compact subset of \(X\) is closed and \(X\) is KC. So \(X \in C_3 \setminus C_2\).

Now suppose \(E\) is an uncountable subset of \(X\) with uncountable complement. So \(E\) is not closed. For all compact subset \(M\) of \(X\), the set \(E \cap M\) is finite and closed. Therefore,
X is not a k-space. Using Remark 3(2), A(X) is not KC. Using Remark 3(1), A(X) is US; we claim that A(X) is SC. Suppose \((x_n : n \in \omega)\) is a sequence in A(X), converging to \(x \in A(X)\). We have the following cases.

**Case 1.** Consider \(x \in X\). In this case, \(U := (X \setminus \{x_n : n \in \omega\}) \cup \{x\}\) is an open neighborhood of \(x\) in \(A(X)\). So there exists \(N \geq 1\) such that for all \(n \geq N\) we have \(x_n \in U\) and \(x_n = x\). Therefore, \(\{x_n : n \in \omega\} \cup \{x\} = \{x_0, x_1, \ldots, x_N, x\}\) is a (finite) closed subset of \(X\).

**Case 2.** Consider \(x = \infty\). Since \(A(X) \setminus (\{x_n : n \in \omega\} \cup \{x\}) = X \setminus \{x_n : n \in \omega\}\) is open in \(X\), it is open in \(A(X)\) too. Thus \(\{x_n : n \in \omega\} \cup \{x\}\) is closed in \(A(X)\).

By the above cases, \(\{x_n : n \in \omega\} \cup \{x\}\) is closed in \(A(X)\) and \(A(X) \in SC\). Hence \(A(X) \in C_5 \setminus C_4\).

(iii) Consider \(Q = C_6 \setminus C_5\). Consider \(X\) as disjoint union of \(X_1\) and \(X_2\), where we have the following.

1. Suppose \(\mathcal{F}\) is a uniform ultrafiltration over \(\mathbb{N}\). Consider \(X_1 = \mathbb{N} \setminus \{0\} = \omega\) under topology \(\{A \subseteq X : 0 \notin A \cup \omega : A \in \mathcal{F}\}\) (see proof of Table 1 regarding case “\(P = C_2 \setminus C_1, Q = C_6 \setminus C_5\)”).

2. \(X_2\) is an uncountable set with countable complement topology \(\{U \subseteq X_2 : U = \emptyset \cup \{x_2 : U \text{ is countable}\}\}\) (see [4, counterexamples 20 and 21]) and proof of Table 1 regarding case “\(P = C_3 \setminus C_4, Q = C_6 \setminus C_5\”\).

Then \(X \in C_4 \setminus C_3\) by \(X_1 \in C_2 \setminus C_1, X_2 \in C_4 \setminus C_3\), and Lemma 6(1). Also \(A(X) \in C_6 \setminus C_5\) by \(A(X_1) \in C_6 \setminus C_5, A(X_2) \in C_6 \setminus C_5\), and Lemma 7.

(iv) Consider \(Q = C_7 \setminus C_6\). If \(X\) is KC and by Remark 3(1), \(A(X)\) is US. Thus \(A(X) \notin C_7 \setminus C_6\).

**Fifth Row.** Here we have \(P = C_5 \setminus C_4\) and the following cases for \(Q\).

(i) Consider \(Q = C_5 \setminus C_4\). Consider \(X\) as \(\Omega + 1\) with doubling \(2\) [8]; that is, if \(p \notin \Omega + 1\), let \(X = (\Omega + 1) \cup \{p\}\) under topological basis \(\{\alpha, \beta\} \in (\alpha, \beta)\) are ordinal numbers with \(\alpha, \beta \in \Omega\) \(\cup\) \(\{0, \alpha\} =\) an ordinal number with \(\alpha < \omega\) \(\cup\) \(\{\alpha, \Omega\} =\) an ordinal number with \(\alpha < \omega\) \(\cup\) \(\{\alpha : \alpha\) is an ordinal number with \(\alpha < \omega\) \(\cup\) \(\{\alpha, \Omega\} \cup \{p\} =\) is an ordinal number with \(\alpha < \omega\). Then \(X \in C_5 \setminus C_4\) and \(X\) is compact which leads to \(A(X) \notin C_5 \setminus C_4\) too.

(ii) Consider \(Q = C_6 \setminus C_5\). Consider \(X\) as disjoint union of \(X_1\) and \(X_2\), where we have the following.

1. \(X_1 = \Omega + 1\) with doubling \(2\) as in case “\(P = C_6 \setminus C_5, Q = C_6 \setminus C_5\)” where \(X_1 = A(X_1) \in C_6 \setminus C_5\).

2. For uniform ultrafiltration \(\mathcal{F}\) over \(\mathbb{N}\), consider \(X_2 = \mathbb{N} \setminus \{0\} = \omega\) equipped with topology \(\{A \subseteq \{0, \infty\} : 0 \notin A \cup \omega : A \in \mathcal{F}\}\). Using case “\(P = C_6 \setminus C_5, Q = C_6 \setminus C_5\)” we have \(X_2 \in C_3 \setminus C_2\) and \(A(X_2) \in C_6 \setminus C_5\). By \(X_1 \in C_6 \setminus C_5, X_2 \in C_3 \setminus C_2\), and Lemma 6(1), we have \(X = X_1 \cup X_2 \in C_5 \setminus C_4\). By \(A(X_1) \in C_3 \setminus C_2\), \(A(X_2) \in C_5 \setminus C_3\), and Lemma 7, we have \(A(X) = A(X_1 \cup X_2) \in C_6 \setminus C_5\).

(iii) Consider \(Q = C_6 \setminus C_5\). Use Lemma 8.

| \(P\) | \(T_2\) | \(KC\setminus T_2\) | \(SC\setminus KC\) | \(T_1\setminus SC\) |
|---|---|---|---|---|
| \(T_2\) | √ | √ | √ | √ |
| \(KC\setminus T_2\) | — | √ | √ | √ |
| \(SC\setminus KC\) | — | — | √ | — |
| \(T_1\setminus SC\) | — | — | — | √ |

**Sixth Row.** Here we have \(P = C_6 \setminus C_5\) and the following cases for \(Q\).

(i) Consider \(Q = C_6 \setminus C_5\). Consider \(X\) as \((\omega + 1) \cup \{\mathcal{F}\}\), such that \(\omega + 1\) has its usual order topology, \(\mathcal{F}\) is a uniform ultrafiltration over \(\omega\), and \(\{\mathcal{F}\} \cup \{U : U \in \mathcal{F}\}\) is an open neighborhood basis for \(\mathcal{F}\) [8, example 1.2]; then \(X\) is compact and \(A(X) = X \in C_6 \setminus C_5\).

(ii) Consider \(Q = C_7 \setminus C_6\). According to [7, example 5], there exists a US topological space \(X\) such that \(A(X)\) is not US. By Remark 3(7), \(A(X) = T_1\). Hence \(A(X) \in C_7 \setminus C_6\). Moreover, by Lemma 8, \(X\) is not SC, thus \(X \notin C_6 \setminus C_5\).

**Seventh Row.** Here we have \(P = C_7 \setminus C_6\) and \(Q = C_7 \setminus C_6\). Suppose \(X\) as an infinite set with finite complement topology \(\{U \subseteq X : U = \emptyset \cup \{x : U \text{ is finite}\}\}\) [4, counterexamples 18 and 19]. Then \(X\) is compact and \(A(X) = X \in C_7 \setminus C_6\).

**4. Some Observations in Figure 1**

Using Figure 1, we have the following results.

(i) The collection \(\{T_2, KC, SC, T_1\}\) is a full-forwarding chain with respect to \(A\). In other words, Table 3 is valid.

In Table 3, the mark “√” indicates that in the corresponding case there exists \(X \in P\) such that \(A(X) \in Q\), and the mark “—” indicates that in the corresponding case for all \(X \in P\) we have \(A(X) \notin Q\).

(ii) The collection \{Metrizable, T_2, KC, SC, US, T_1, T_D, T_UD, T_0, Top\} is a forwarding chain with respect to \(A\). The collection \(T_1, T_D, T_UD, T_0, Top\) is a stationary chain with respect to \(A\).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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