We study the problem of quantum thermalization from a very recent perspective: via discrete interactions with thermalized systems. We thus extend the previously introduced scattering thermalization program by studying not only a specific channel but allowing any possible one. We find a channel that solves a fixed point condition using the Choi matrix approach that is in general non-trace-preserving. We also find a general way to complement the found channel so that it becomes trace-preserving. Therefore we find a general way of characterizing a family of channels with the same desired fixed point. From a quantum computing perspective, the results thus obtained can be interpreted as a condition for quantum error correction that also reminds of quantum error avoiding.

1 Introduction

Thermalization is an ubiquitous phenomenon in the universe. Just as ubiquitous is the applicability of quantum physics. This implies that a concept of quantum thermalization should exist. It would mean that there must be a process where a physical system thermalizes even when quantum physics is the most accurate description of it. This field of study enters the wider subject of quantum thermodynamics [1]. However, where is the limit of applicability of the term thermalization? When does the description as an aggregate of quantum particles should be abandoned so that a collection of classical particles make more sense?

There has been work in this direction, specifically, taking into account terminology from quantum information theory [2, 3]. Their approach involves having a Hamiltonian that dictates the evolution of a quantum state and observing when the system is close to a thermal state. Using general arguments of typicality [4] one can find conditions where thermalization occurs and why assumptions of statistical mechanics arise naturally [2]. This line of thought can be called the program of dynamical typicality [3].

Not all systems fall into this scheme for thermalization. A situation that is not strictly within the scope of the dynamical typicality program is presented in the study of Jacob et. al. [5, 6] where thermalization in quantum systems is reached through a scattering process. They consider a reservoir that contains particles thermalized to a specific temperature. This reservoir shoots particles spaced through time intervals into a central system that scatters them as shown in Fig. (1). Many questions can be asked in this setting about the thermalization process of the scatterer.
Once the interaction $V$ is specified, the Hamiltonian $H$ implies a unitary evolution with the interaction and then without it. If we define the map $\rho_Y = \exp[-itH_Y/h] \rho \exp[itH_Y/h]$, we can write the dynamics at step $n$ as

$$\rho_Y^{(n)} = \varepsilon_{\tau_n} \circ S \circ \ldots \circ \varepsilon_{\tau_2} \circ S \circ \varepsilon_{\tau_1} \circ S \rho_Y^{(0)}. \quad (5)$$

From the analysis of a scattering map that acts on mixed states we can divide the wave packets into those that are broad in the momentum variance and those who are narrow. One of the main results from Ref. [5] is that a necessary condition for thermalization in the scattering scenario is that the wave packets have to be narrow. Another central condition is microscopic reversibility, which is the reason that there are two baths shooting into a central system.

Summarizing, we have two conditions for thermalization when considering quantum particles in this scheme:

- Narrow wave packets.
- Microscopic reversibility.

Now, a question arises, why is condition 1 necessary? If one is presented with wide wave packets, as would require a natural quantum-mechanical description, thermalization should occur, as it is an ubiquitous phenomenon in the universe. That is, thermalization should be independent from the wavelength.

Notice however that the scattering thermalization program uses a specific channel (see equation (4)) we call it the scattering channel. The approach of Jacob et. al. can be seen as an investigation of thermalization in discrete applications of this channel. Here, we extend the scattering thermalization program to consider any possible channel. We fix the desired final state and ask for a channel that reaches such a state. In this sense, our approach is dual to Jacob et. al..

## 2 Iterations of a quantum channel

In our problem we know the state the source is producing, what we don’t know is the initial target state. We ask for a channel that through iterations on a system would change it to a desired state regardless of the initial one. The mathematical problem can be stated explicitly: find a quantum channel that produces the desired thermal state starting from another given state after many iterations, in other words, find a quantum channel with a desired fixed point. This topic is thus related to the stabilizer formalism used to prove the Knill-Gottesman theorem [8, 9].

Suppose we are given a thermal state that depends on a Hamiltonian $H$ as $\rho_{th}[H]$, it is then relevant to ask if there exists a (nontrivial) channel $\Phi$ such that it is the stabilizer channel of $\rho_{th}[H]$ in the sense that

$$\Phi(\rho_{th}[H]) = \rho_{th}[H]. \quad (6)$$

The Choi representation of channels [10] will be useful for addressing the existence of a nontrivial channel that fulfills equation (6). Then, the simplest nontrivial channel $\Phi$ that fulfills equation (6) is obtained as a solution to the semidefinite program (SDP) [11, 10, 12]

$$\text{minimize}_{Z} \quad \text{Tr}[Z]$$
$$\text{subject to} \quad \text{tr}_{H_2}[Z(1_{H_1} \otimes \sigma^T)] \geq \sigma \quad (7)$$
$$Z \geq 0.$$
Observe that the solution of the SDP $Z$ corresponds to the Choi matrix of the channel $\Phi$. The minimization of the trace of $Z$ is relevant, as it eliminates any (unnecessary) orthogonal element. This means, suppose that $K = R + Z$ fulfills the conditions of the SDP and $Z$ is the optimal solution, the minimization assures that

$$\text{tr} \nu_2 [R(1_H \otimes \rho_{th}[H^T])] = 0 \quad (8)$$

for $R \geq 0$. If \( \sigma = \rho_{th}[H] \) in the SDP (7) then we have solved our problem at hand. Fortunately, this SDP can be explicitly solved.

**Theorem 1.** The SDP (7) has the solution $1/\lambda_{\text{max}}$ where $\lambda_{\text{max}}$ is the maximum eigenvalue of $\sigma$.

**Proof.** We build the dual program to (7),

$$\max \quad \text{Tr}[W\sigma]$$

subject to $W^T \otimes \sigma \leq 1 \quad (9)$

$W \geq 0$.

Let

$$\sigma = \sum_i \lambda_i |v_i \rangle \langle v_i| \quad (10)$$

be the spectral decomposition of $\sigma$. We define

$$Z_\sigma := \frac{\sigma \otimes (|v_{\text{max}} \rangle \langle v_{\text{max}}|)^T}{\lambda_{\text{max}}} \quad \text{and} \quad W_\sigma := \frac{1}{\lambda_{\text{max}}} 1.$$  

where $|v_{\text{max}}\rangle$ is the correspondent eigenvector for $\lambda_{\text{max}}$. Observe that $Z_\sigma$ and $W_\sigma$ belong in the primal and dual feasible sets of their respective programs. They also yield the same value for the figure of merit, so strong duality is always fulfilled. The value is thus the optimal one because of Slater’s theorem for semidefinite programs [10].

The optimal solution channel $Z_\sigma$ is not trace-preserving, observe that an arbitrary state $\rho$ would result in

$$\Phi(\rho) = \frac{\langle v_{\text{max}} | \rho | v_{\text{max}} \rangle}{\lambda_{\text{max}}} \sigma. \quad (12)$$

The channel is trace-preserving only when

$$\langle v_{\text{max}} | \rho | v_{\text{max}} \rangle = \lambda_{\text{max}}. \quad (13)$$

Also, notice that the solution $Z_\sigma$ fulfills the condition of the stabilizer channel exactly, not only solving the SDP (7).

We can define the operators $A_{ij}$ as

$$A_{ij} := \sqrt{\frac{\lambda_i}{\lambda_{\text{max}}}} |v_i \rangle \langle v_{\text{max}}|. \quad (14)$$

Observe that $A_{ij}$ has two sub-indices and can therefore be ordered. We thus define the operator

$$A := \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_d \end{pmatrix} \quad (15)$$

This operator acts on the state with an ancilla with a Hilbert space $Z$ of dimension $d$,

$$\Phi(\sigma) = \text{tr} [A(1_d \otimes \sigma)A^\dagger]. \quad (16)$$

The actual implementation of the channel (12) requires a quantification of the resources needed. Such a question corresponds to resource theories [13, 14].

However, we can study the cases where the equality (13) is fulfilled. Observe that this happens when $\rho = U_{v_{\text{max}}}DU_{v_{\text{max}}}^\dagger$ and

$$D = \begin{pmatrix} \lambda_{\text{max}} & 0 \\ 0 & (1 - \lambda_{\text{max}}) \Lambda_{d-1} \end{pmatrix} \quad (17)$$

$\lambda_{\text{max}} \geq 1/2$ w.l.g. and $\Lambda_{d-1}$ is a $(d-1) \times (d-1)$ positive diagonal matrix with $\text{tr} \Lambda_{d-1} = 1$. $U_{v_{\text{max}}}$ is a unitary matrix of dimension $d$ which includes $|v_{\text{max}}\rangle$ as a column that corresponds to $\lambda_{\text{max}}$. These states denote the set of states that are stabilized by channel (12) without loss.

For the case of one qubit the freedom that equation (12) allows is null: specifying one eigenvalue specifies the second one, also specifying a vector specifies its orthogonal one (on the opposite side of Bloch’s sphere). Imagine now that we have two qubits, the effective dimension of the Hilbert space is four. The freedom here is much more interesting, we can specify the eigenvalue matrix following equation (17) and the $U_{\text{max}}$ can be specified as follows

$$U_{\text{max}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & v_1^1 & v_2^1 & v_3^1 \\ 0 & v_1^2 & v_2^2 & v_3^2 \\ 0 & v_1^3 & v_2^3 & v_3^3 \end{pmatrix} \quad (18)$$

where $v_j^i$ represents the $j$th entry of the $i$th eigenvector. Generalizing, $n$ qubits imply that there is a Hilbert space of dimension $2n - 1$ where we can
choose freely a state which represents the allowed errors.

Remember that the SDP (7) required the minimum possible expression for the channel in question, there is still the necessity of characterizing the whole channel. Observe that in general there is an infinite number of channels corresponding to a single fixed point. We have to observe the remaining part of the channel so that it becomes a trace-preserving one. This characterization is done in the following theorem.

**Theorem 2.** Given a state \( \sigma \) we can describe a trace-preserving separable family of channels with fixed point \( \sigma \) in terms of its Choi matrix \( \mathcal{C} \) as follows

\[
\mathcal{C}[\sigma, B] = \sigma \otimes \left( \frac{|V_{\text{max}}\rangle \langle V_{\text{max}}|}{\lambda_{\text{max}}} + B \otimes (I - \frac{|V_{\text{max}}\rangle \langle V_{\text{max}}|}{\lambda_{\text{max}}}) \right),
\]

(19)

\( \lambda_{\text{max}} \) is the maximum eigenvalue of \( \sigma \) and \( |V_{\text{max}}\rangle \) its correspondent eigenvector. \( B \) is a state. This description is valid for \( \langle V_{\text{max}}| B |V_{\text{max}}\rangle \leq \lambda_{\text{max}} \) and any input state \( \langle V_{\text{max}}| \rho |V_{\text{max}}\rangle \leq \lambda_{\text{max}} \).

Equation (19) implies that there is a relevant positive semidefinite part of a channel is given by the operator

\[
P_\sigma \equiv \sigma \otimes \left( \frac{|V_{\text{max}}\rangle \langle V_{\text{max}}|}{\lambda_{\text{max}}} \right).
\]

(20)

**Proof.** Because of theorem 1 the operator with minimum trace with fixed point \( \sigma \) has to be \( P_\sigma \). In the case that \( \langle V_{\text{max}}| \rho |V_{\text{max}}\rangle < \lambda_{\text{max}} \) the channel \( P_\sigma \) would be non-trace-preserving. To have trace preservation we need to add an additional operator. We consider an operator \( B' \) such that the Choi matrix of \( \Phi \) is

\[
\mathcal{C} = P_\sigma + B'.
\]

(21)

The fixed point condition \( \Phi[\sigma] = \sigma \) requires that

\[
\text{tr}_{\mathcal{H}_2}[B'(I \otimes \sigma^T)] = 0.
\]

(22)

For trace preservation, we have that for any state \( \rho \),

\[
\text{tr}[B'(I \otimes \rho^T)] = 1 - \frac{\langle V_{\text{max}}| \rho |V_{\text{max}}\rangle}{\lambda_{\text{max}}}.
\]

(23)

Also, by theorem 2.26 of [10], for trace-preserving maps

\[
\text{tr}_{\mathcal{H}_1}[C] = I.
\]

(24)

This is achieved by an operator of the form

\[
B' = B \otimes (I - \frac{|V_{\text{max}}\rangle \langle V_{\text{max}}|}{\lambda_{\text{max}}})
\]

(25)

with \( \text{tr}[B] = 1 \). This defines a family of channels, explicitly, those that can be written as in Eq. (25).

Observe that the resulting operator yields the channel acting one and two times on an operator \( \rho \) as follows

\[
\Phi[\rho] = \sigma \frac{\langle V_{\text{max}}| \rho |V_{\text{max}}\rangle}{\lambda_{\text{max}}} + B(1 - \frac{\langle V_{\text{max}}| \rho |V_{\text{max}}\rangle}{\lambda_{\text{max}}})
\]

(26)

\[
\Phi^2[\rho] = \sigma \frac{\langle V_{\text{max}}| B |V_{\text{max}}\rangle}{\lambda_{\text{max}}} (1 - \frac{\langle V_{\text{max}}| \rho |V_{\text{max}}\rangle}{\lambda_{\text{max}}}) + B(1 - \frac{\langle V_{\text{max}}| \rho |V_{\text{max}}\rangle}{\lambda_{\text{max}}}) \times
\]

\[
(1 - \frac{\langle V_{\text{max}}| \rho |V_{\text{max}}\rangle}{\lambda_{\text{max}}}).
\]

(27)

We observe that the proportion of \( \sigma \) grows with each iteration and the proportion of \( B \) decreases. As we asked for trace-preserving channels then \( \Phi^n[\rho] \rightarrow \sigma \) for \( n \rightarrow \infty \).

### 2.1 Jacob et. al. example

For example, in Jacob et. al. problem they consider a qubit state as the central one, remember from Fig. (1). The structure of the problem is a quantum channel over a central system \( \rho_{H_2} \) given in equation (4)

\[
\Phi_T[\rho_Y] := \text{tr}_X[S(\rho_X \otimes \rho_Y)S^\dagger],
\]

(28)

with \( S \) a unitary matrix that contains the interactions and the Hamiltonian. With a use of a state \( \rho_X \) each time the channel acts on \( \rho_Y \). Expanding the state of the first Hilbert space like

\[
\rho_X = \sum_i r_i |r_i\rangle\langle r_i|,
\]

(29)

we get a Stinespring representation of the quantum channel,

\[
\Phi_T[\rho_Y] = \sum_{ij} r_j^{1/2} \langle i| S |r_j\rangle \rho_Y r_j^{1/2} \langle r_j| S^\dagger |r_i\rangle.
\]

(30)
This channel has the correspondent Choi matrix [10],
\[ J(\Phi) = \sum_{ij} \text{Vec}(r_j^{1/2} \langle r_i | S | r_j \rangle) \text{Vec}(r_j^{1/2} \langle r_j | S^\dagger | r_i \rangle)^\dagger. \] (31)

Suppose that this system has a Hamiltonian \( H \).

The thermal state associated to this state is
\[ \rho_c = e^{-\beta H} Z. \] (32)

Observe then that theorem 2 restricts the unitary operations \( S \) that thermalize to this state into those with Choi matrix,
\[ J(\Phi_T) = C[\rho_c, B], \] (33)
as defined by Eqs. (31) and (19). The problem is now to interpret this restriction in physical terms, which is beyond the scope of this paper.

2.2 The relationship with error-correction

The initial condition (6) is the same used in quantum error correction [15]. Nevertheless, this approach is for mixed states meanwhile most approaches use pure states. The construction of a stabilizer channel for mixed states has been already discussed, in theorem 4.8 of [10], which states that a stabilizer channel that uses information that leaks into the environment to correct the state in question can be built as a mixed-unitary channel. However, we present here the same problem from a different angle, as our approach is less ambitious which allows us to get to the relation (13) for error correctability.

There are two approaches to deal with errors for quantum technologies: first, correct them using quantum error correction codes (QECC), and second, avoid errors using quantum error avoiding codes (QEAC). QECC is related to error-correcting codes for pure states, its approach is to construct the stabilizer operator for a state. On the other hand, QEAC is more focused on mixed states and finding strategies so that the quantum information is preserved in decoherence-free subspaces. Our approach seems something in the middle: it uses mixed states and a channel that presupposes openness to an environment and asks for the preservation of a subsystem, however, it also is based on a stabilizer operator. Notice however, that our requirement is very mild, we only ask for the conservation of

the eigenvector correspondent to the maximum eigenvalue and the maximum eigenvalue as well. However, we would have to extend (or restrict) the conditions to allow the final state to be not a specific one, as we do here but a corrected state.

3 Discussion

We address the problem of thermalization by studying a stabilizer channel that, given any initial state of dimension \( d < \infty \) it yields the desired thermal state after many actions on a physical system.

Our approach yields several questions to be answered. First of all, there is the question of resources to build the channel. Therefore, how to quantify the resources to build the channel that we want, i.e. the one that corresponds to \( Z_\sigma \)? This is a question that demands the use of tools and terminology from resource theories. Also, to characterize the possible states \( B \) to complete the channel in theorem (2), so that it becomes a trace-preserving channel.

Second, due to the original motivation of this research, a question of interpretation arises. Jacob et. al. [5] proposed a “natural” channel for thermalizing a system towards a given state: the scattering process. Given the negative answer to this natural assumption, we ask if there is another channel that does thermalize. We find there is, however, we do not know if this channel is natural in some way. To be more precise, the channel studied by Jacob et. al. had a clear physical interpretation whereas the channel that we study here has no such interpretation. Future research should be geared toward interpreting the stabilizer channel studied here.

Finally, there is a relationship between the channel in question and quantum error correction. Specifically, it is related to the stabilizer formalism. One of the great arguments against quantum computation is the difficulty of taming the errors from the environment [16]. Further work using the formalism developed here should address these arguments. Specifically, to have stabilizer channels that correct errors with respect to desired symmetries.
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