SINGULAR FIBERS AND KODAIRA DIMENSIONS

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Abstract. Let \( f : X \to \mathbb{P}^1 \) be a non-isotrivial semi-stable family of varieties of dimension \( m \) over \( \mathbb{P}^1 \) with \( s \) singular fibers. Assume that the smooth fibers \( F \) are minimal, i.e., their canonical line bundles are semiample. Then \( \kappa(X) \leq \kappa(F) + 1 \). If \( \kappa(X) = \kappa(F) + 1 \), then \( s > \frac{m}{2} + 2 \). If \( \kappa(X) \geq 0 \), then \( s \geq \frac{m}{2} + 2 \). In particular, if \( m = 1 \), \( s = 6 \) and \( \kappa(X) = 0 \), then the family \( f \) is Teichmüller.

1. Introduction

We always work over the complex number \( \mathbb{C} \). Let \( f : S \to \mathbb{P}^1 \) be a nontrivial fibration of semi-stable curves of genus \( g \geq 1 \). It is a classical problem to determine the lower bound for the number \( s \) of singular fibers in the fibration \( f \), see [Bea81, Tan95, TTZ05, Tu07, Zam12, GLT13, LTXZ16]. In [Bea81], Beauville first proved that \( s \geq 4 \) and conjectured that \( s \geq 5 \) when \( g \geq 2 \). In [Tan95], the second author confirmed Beauville’s conjecture. Later, Tu, Zamora and the second author proved in [TTZ05] that \( s \geq 6 \) if \( S \) has non-negative Kodaira dimension. It is conjectured that \( s \geq 7 \) if \( S \) is of general type. The first purpose of this note is to confirm this conjecture.

Theorem 1.1. Let \( f : S \to \mathbb{P}^1 \) be a nontrivial semi-stable fibration \( f : S \to \mathbb{P}^1 \) of curves of genus \( g \geq 2 \) over \( \mathbb{P}^1 \) with \( s \) singular fibers. If \( S \) is of general type, then \( s \geq 7 \).

This conjecture has been verified for \( g \leq 5 \) ([TTZ05, Zam12] or \( g \geq 58 \) ([TTY], unpublished) by using the strict canonical class inequality established by the second author [Tan95]. Recently, the authors in [MSZ16] have also proved this conjecture under the condition that the family is birationally equivalent to a pencil of curves with only simple base points on the minimal model of \( S \).

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We can find in [TZZ05] the examples of surfaces of general type admitting a semi-stable fibration over $\mathbb{P}^1$ with 7 singular fibers.

It is an interesting phenomenon that when the number of singular fibers is minimal, the family is of very interesting arithmetic and geometric properties. When $g = 1$ and $s = 4$, Beauville [Bea82] proved that the family of curves must be modular, and there are exactly 6 such families. In [STZ04], the authors prove that for a non-isotrivial family of semi-stable K3 surfaces $f : X \to \mathbb{P}^1$ on a Calabi-Yau manifold $X$, then $s \geq 4$ and if $s = 4$, the family is modular.

**Theorem 1.2.** As in Theorem 1.1, if the Kodaira dimension of $S$ is zero and $s = 6$, then the family must be Teichmüller and $\omega_{S/\mathbb{P}^1}^2 = 6g - 6$.

Here a family of curves is said to be Teichmüller, if up to a suitable finite étale cover of $S$, it comes from a Techmüller curve.

For each type of surfaces of Kodaira dimension zero, Tu [Tu07] has constructed an example with a semi-stable family of curves over $\mathbb{P}^1$ admitting exactly 6 singular fibers.

When the Kodaira dimension of the surface is 1, the minimal number $s$ should be 6 or 7. We have not found examples with $s = 6$, and we tend to believe that there are no such examples. On the other hand, we give more precise description of such surfaces.

**Theorem 1.3.** With the notation as in Theorem 1.1, Suppose the Kodaira dimension of $S$ is 1 and $s = 6$. Then $S$ is simply connected, $p_g(S) = q(S) = 0$, the canonical elliptic fibration on $S$ admits exactly two multiple fibers, one of the multiplicities is 2, and the second one is $n = 3$ or 5.

1) If $n = 3$, then $6g - 5 \leq \omega_{S/\mathbb{P}^1}^2 \leq 6g - 3$.

2) If $n = 5$, then $\omega_{S/\mathbb{P}^1}^2 = 6g - 3$.

When the stability assumption is dropped, then it is only known that $s \geq 3$ for any non-isotrivial fibration of curves over $\mathbb{P}^1$, even if we require that two of the singular fibers be semi-stable [Bea81, GLT13].

Our method works also for the higher dimensional cases.

**Theorem 1.4.** Let $f : X \to \mathbb{P}^1$ be a non-isotrivial semi-stable family of varieties of dimension $m$ over $\mathbb{P}^1$ with $s$ singular fibers. Assume that the smooth fibers $F$ are minimal, i.e., their canonical line bundles are semiample. Then $\kappa(X) \leq \kappa(F) + 1$.

1) If $\kappa(X) \geq 0$, then $s \geq \frac{4}{m} + 2$. In particular, $s \geq 6$ when $m = 1$, and $s \geq 4$ when $m = 2$ or 3.
2) If $\kappa(X) = \kappa(F) + 1$, then $s > \frac{4}{m} + 2$. In particular, $s \geq 7$ when $m = 1$, $s \geq 5$ when $m = 2$, and $s \geq 4$ when $m = 3$ or $4$.

Note that when $X$ is of general type, then $F$ must be also of general type and the equality $\kappa(X) = \kappa(F) + 1$ holds. Hence the lower bound $s > \frac{4}{m} + 2$ holds in this case.

This note is organized as follows. Theorems 1.1, 1.2 and 1.3 are proved in section 2, and Theorem 1.4 is proved in section 3.

2. Variations of the Hodge structures

In this section, we would like to prove Theorems 1.1-1.3. The main technique is based on the variation of the Hodge structures attached to a semi-stable family of curves, especially to a Teichmüller family.

2.1. Preliminaries. In this subsection, we give a brief recall about the Teichmüller curve and the associated variation of the Hodge structures, and derive some inequalities. For more details, we refer to [Möl06, Mö13, EKZ14].

Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus $g$, and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the bundle of pairs $(F, \omega)$, where $\omega \neq 0$ is a holomorphic one-form on $F \in \mathcal{M}_g$. Let $\Omega\mathcal{M}_g(m_1, \ldots, m_k) \subseteq \Omega\mathcal{M}_g \to \mathcal{M}_g$ be the stratum of pairs $(F, \omega)$ such that $\omega$ admits exactly $k$ distinct zeros of order $m_1, \ldots, m_k$ respectively. There is a natural action of $SL_2(\mathbb{R})$ on each stratum $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$. Each orbit projects to a complex geodesics in $\mathcal{M}_g$. When the projection of such an orbit is closed, it gives a so-called Teichmüller curve. After a suitable unramified cover and compactification of a given Teichmüller curve, one gets a universe family $f : S \to B$, which is a semi-stable family of curves of genus $g$. Moreover, there exist disjoint sections $D_1, \ldots, D_k$ of $f$ such that the restriction $(\sum_{i=1}^{k} m_i D_i)|_F$ to each fiber $F$ is just the zero locus of $\omega$.

Denote by $s$ the number of singular fibers contained in $f$. Then the Hodge bundle $f_*\omega_{S/B}$ for a Teichmüller curve contains a line subbundle $\mathcal{L} \subseteq f_*\omega_{S/B}$ with maximal slope:

$$2 \deg(\mathcal{L}) = 2g(B) - 2 + s.$$  

Consider the logarithmic Higgs bundle $(f_*\omega_{S/B} \oplus R^1 f_*\mathcal{O}_S, \theta)$ associated to the fibration $f$, which corresponds to the weight-one local system $R^1 f_*\mathcal{O}_{S^0}$; here $f : S^0 \to B^0$ is the smooth part of $f$. The Higgs field $\theta$ is simply the edge morphism

$$f_*\omega_{S/B} \cong f_*\Omega_{S/B}(\log \Upsilon) \longrightarrow R^1 f_*\mathcal{O}_S \otimes \Omega_B^{\log}(\log \Delta)$$
of the tautological sequence
\[ 0 \rightarrow f^*\Omega^1_B(\log \Delta) \rightarrow \Omega^1_S(\log \Upsilon) \rightarrow \Omega^1_{S/B}(\log \Upsilon) \rightarrow 0, \]
where $\Upsilon \rightarrow \Delta$ is denoted to be the singular locus of $f$. By [VZ03], the existence of a line subbundle $L \subseteq f_*\omega_{S/B}$ with maximal slope is equivalent to the existence of a rank two Higgs subbundle $(L \oplus L^{-1}, \theta)$ with maximal Higgs field contained in the logarithmic Higgs bundle $(f_*\omega_{S/B} \oplus R^1f_*\mathcal{O}_S, \theta)$ associated to the fibration $f$.

Conversely, one has the following theorem, which is due to Möller [Möl06].

**Theorem 2.1.** Let $f : S \rightarrow B$ be a semi-stable fibration of curves of genus $g \geq 2$ over a smooth projective curve with $s$ singular fibers. Suppose that there exists a line subbundle $L \subseteq f_*\omega_{S/B}$ satisfying the equality (2.1) above. Then the family $f$ comes from a Techmuller curve; that is, the induced map $B^0 \rightarrow \mathcal{M}_g$ is a finite unramified cover of a Techmuller curve. Here $f : S^0 \rightarrow B^0$ is the smooth part of $f$.

Since the relative canonical sheaf of a fibration of curves over a Techmuller curve has a very special form (see (2.3) below), we can derive the following upper bound on $\omega^2_{S/B}$.

**Proposition 2.2.** Let $f : S \rightarrow B$ be a semi-stable fibration of curves as in Theorem 2.1 and assume also that there exists a line subbundle $L \subseteq f_*\omega_{S/B}$ with the equality (2.1). Then
\[ (2.2) \quad \omega^2_{S/B} \leq \frac{3}{2}(g - 1)(2g(B) - 2 + s). \]

**Proof.** By Theorem 2.1 the induced map $B^0 \rightarrow \mathcal{M}_g$ is finite unramified covering of a Techmuller curve. Hence after a suitable unramified base change, there exist disjoint sections $D_1, \cdots D_k$ of $f$ such that the relative canonical sheaf $\omega_{S/B}$ has the form (cf. [EKZ14])
\[ (2.3) \quad \omega_{S/B} \cong f^*\mathcal{L} \otimes \mathcal{O}_S\left(\sum_{i=1}^k m_i D_i\right), \]
where $\mathcal{L} \subseteq f_*\omega_{S/B}$ is the line subbundle satisfying the equality (2.1). Note that the inequality (2.2) is invariant under any finite unramified base change. Thus we may assume that $\omega_{S/B}$ already has the form as above.

As $D_i$'s are disjoint sections, it follows that $D_i \cdot D_j = 0$ for $i \neq j$, and that
\[ (\omega_{S/B} + D_i) \cdot D_i = 0, \quad \forall \ 1 \leq i \leq k. \]
Combining these with (2.3), one gets that
\[ D_i^2 = -\frac{1}{m_i+1} \cdot \deg \mathcal{L}. \]
Note also that \( \sum_{i=1}^{k} m_i = \deg \omega_F = 2g - 2, \) where \( \omega_F \) is the canonical sheaf on a general fiber of \( f. \) Hence by (2.3) again, we obtain that
\[
\omega_{S/B}^2 = 4(g-1) \cdot \deg \mathcal{L} + \sum_{i=1}^{k} m_i^2 D_i^2
\]
\[ = \left( 4(g-1) - \sum_{i=1}^{k} \frac{m_i^2}{m_i+1} \right) \deg \mathcal{L}. \]

As \( \sum_{i=1}^{k} m_i = 2g - 2, \) one gets easily that
\[
\sum_{i=1}^{k} \frac{m_i^2}{m_i+1} \geq \sum_{i=1}^{k} \frac{m_i}{2} = g - 1.
\]
Therefore,
\[
\omega_{S/B}^2 \leq 3(g-1) \cdot \deg \mathcal{L} = \frac{3(g-1)(2g(B) - 2 + s)}{2}.
\]
This completes the proof. \( \Box \)

In the case when \( f : S \to \mathbb{P}^1 \) is a semi-stable fibration of curves of genus \( g \geq 2 \) over \( \mathbb{P}^1 \) with \( s = 6 \) singular fibers, we have the following easy criterion when \( f \) comes from a Techm"uller curve.

**Lemma 2.3.** Let \( f : S \to \mathbb{P}^1 \) be a semi-stable fibration of curves of genus \( g \geq 2 \) over \( \mathbb{P}^1 \) with \( s = 6 \) singular fibers. If the geometric genus \( p_g(S) := \dim H^0(S, \omega_S) > 0, \) then there exists a line subbundle \( \mathcal{L} \subseteq f_\ast \omega_{S/B} \) satisfying the equality (2.1), and hence \( f \) comes from a Techm"uller curve.

**Proof.** As a locally free sheaf on \( \mathbb{P}^1, \) the direct image sheaf \( f_\ast \omega_{S/\mathbb{P}^1} \) is isomorphic to a direct sum of invertible sheaves:
\[
f_\ast \omega_{S/\mathbb{P}^1} \cong \bigoplus_{i=1}^{g} O_{\mathbb{P}^1}(d_i).
\]
Note that \( d_i \geq 0 \) due to the semi-positivity of the direct image sheaf \( f_\ast \omega_{S/\mathbb{P}^1} \) (cf. [Fuj78]), and that \( d_i \leq \frac{1}{2}(2g(B) - 2 + s) = 2 \) due to the Arakelov type inequality (cf. [VZ03]). Without loss of generality, we assume that \( 0 \leq d_1 \leq \cdots \leq d_g \leq 2. \) By [Fuj78, Theorem 3.1], we obtain that
\[
d_1 = \cdots = d_{q(S)} = 0, \quad \text{and} \quad d_i > 0 \quad \forall \ i \geq q(S) + 1,
\]
where \( q(S) := \dim H^1(S, \omega_S) \) is the irregularity of \( S \). Hence
\[
\deg f_* \omega_{S/\mathbb{P}^1} = \sum_{i=q(S)+1}^g d_i.
\]
On the other hand, it is well-known that
\[
\deg f_* \omega_{S/\mathbb{P}^1} = \chi(\omega_S) - (g - 1)(g(\mathbb{P}^1) - 1) = g + p_g(S) - q(S).
\]
Therefore, \( d_g = 2 \) once \( p_g(S) > 0 \). In other word, the line subbundle \( \mathcal{O}_{\mathbb{P}^1}(d_g) \subseteq f_* \omega_{S/\mathbb{P}^1} \) satisfies the equality (2.1).

\[\Box\]

**Corollary 2.4.** Let \( f : S \to \mathbb{P}^1 \) be a semi-stable fibration of curves of genus \( g \geq 2 \) over \( \mathbb{P}^1 \) with \( s = 6 \) singular fibers. If the geometric genus \( p_g(S) > 0 \), then \( f \) comes from a Techm"uller curve and
\[
\omega_{S/\mathbb{P}^1}^2 \leq 6(g - 1).
\]

**Proof.** This is a combination of Lemma 2.3 and Proposition 2.2. \( \Box \)

2.2. **Proof of Theorem 1.1.** By [TTZ05, Theorem 0.1], \( s \geq 6 \) if \( S \) is of general type (actually, the inequality \( s \geq 6 \) holds once the Kodaira dimension of \( S \) is non-negative). To complete the proof, it suffices to deduce a contradiction if \( s = 6 \).

Since \( S \) is of general type, we may assume that \( g \geq 5 \) by [TTZ05, Theorem 0.1(2)], and according to [TTZ05, Theorem 0.2] one has
\[
\omega_{S/\mathbb{P}^1}^2 \geq 6g - 6 + \frac{1}{2} \left( \omega_X^2 + \sqrt{\omega_X^2 \omega_S^2 + 8g - 8} \right),
\]
where \( X \) is the minimal model of \( S \). Hence we may assume that \( p_g(S) = 0 \) by Corollary 2.4. It then follows that
\[
\deg f_* \omega_{S/\mathbb{P}^1} = \chi(\omega_S) - (g - 1)(g(\mathbb{P}^1) - 1) = g - q(S).
\]
Let \( \delta_f \) be the number of nodes contained in the fibers of \( f \). Then by Noether’s formula, one has
\[
\delta_f = 12 \deg f_* \omega_{S/\mathbb{P}^1} - \omega_{S/\mathbb{P}^1}^2 = 12(g - q(S)) - \omega_{S/\mathbb{P}^1}^2.
\]
According to [Tan95], for any integer \( e \geq 2 \), we have the following inequality
\[
\omega_{S/\mathbb{P}^1}^2 \leq (2g - 2) \left( 2g(\mathbb{P}^1) - 2 + \frac{(e - 1)s}{e} \right) + \frac{3\delta_f}{e^2},
\]
\[
= (2g - 2) \left( 4 - \frac{6}{e} \right) + \frac{3}{e^2} \left( 12(g - q(S)) - \omega_{S/\mathbb{P}^1}^2 \right).
\]
Hence
\[ \omega_{S/P}^2 \leq \frac{e^2}{e^2 + 3} (2g - 2) \left( 4 - \frac{6}{e} \right) + \frac{36(g - q(S))}{e^2 + 3}. \]

Taking \( e = 3 \), one obtains
\[ \omega_{S/P}^2 \leq 6g - 3 - 3q(S) \leq 6g - 3. \]

Combining this with (2.5), one obtains that
\[ \sqrt{\omega_X^2} \sqrt{\omega_X^2 + 8g - 8} \leq 6 - \omega_X^2, \]
\[ \Rightarrow \quad \omega_X^2 (\omega_X^2 + 8g - 8) \leq (6 - \omega_X^2)^2, \]
\[ \Rightarrow \quad \omega_X^2 \leq \frac{9}{2g + 1} < 1, \quad \text{since} \quad g \geq 5. \]

This gives a contradiction. \( \square \)

2.3. Proof of Theorem 1.2. If \( S \) is either an abelian surface or a K3 surface, then \( p_g(S) > 0 \), and hence the conclusion follows directly from Corollary 2.4 and [TTZ05, Theorem 0.2].

In the remaining cases, \( S \) must be either an Enriques surface or a bielliptic surface according to the classification of surfaces with Kodaira dimension equal to zero. Let \( K_X \) be the canonical divisor on the minimal model \( X \) of \( S \). Then there exists an \( n > 1 \) such that \( nK_X \equiv 0 \). Hence one can construct a finite étale cover \( \pi : \tilde{S} \to \tilde{S} \) such that \( p_g(\tilde{S}) > 0 \) and that the Kodaira dimension of \( \tilde{S} \) is still zero. Moreover \( \tilde{f} := f \circ \pi : \tilde{S} \to \mathbb{P}^1 \) is still a semi-stable fibration with 6 singular fibers by [Bea81, Lemma 3].

Since \( \pi \) is finite étale,
\[ \omega_{\tilde{S}/\mathbb{P}^1}^2 = \deg(\pi) \cdot \omega_{\tilde{S}/\mathbb{P}^1}^2, \quad \tilde{g} - 1 = \deg(\pi) \cdot (g - 1), \]
where \( \tilde{g} \) is the genus of a general fiber of \( \tilde{f} \). By construction, \( \tilde{S} \) is either an abelian surface or a K3 surface, so \( p_g(\tilde{S}) = 1 \). Hence the family \( \tilde{f} \) comes from a Teichmüller curve and \( \omega_{\tilde{S}/\mathbb{P}^1}^2 = 6(\tilde{g} - 1) \) by the above argument. Therefore, the family \( f \) is Teichmüller. Moreover, Together with (2.7), we obtain \( \omega_{S/P}^2 = 6g - 6 \) as required. \( \square \)
2.4. **Proof of Theorem 1.3.** Because the Kodaira dimension of $S$ is 1, by [TTZ05, Theorem 0.2] we have

\[(2.8) \quad \omega_{S/P_1}^2 \geq 6g - 5.\]

Hence $p_g(S) = 0$ by Corollary 2.4. Similar to the proof of Theorem 1.1, the inequality (2.6) holds. Thus $q(S) = 0$ by (2.8) and (2.6).

As the Kodaira dimension of $S$ is 1, the minimal model $X$ of $S$ admits an elliptic fibration

\[h : X \to C.\]

Since $q(X) = q(S) = 0$, it follows that $C \cong \mathbb{P}^1$. Let $\{n_1 \Gamma_1, \ldots, n_r \Gamma_r\}$ be the set of multiple fibers of $h$ with $2 \leq n_1 \leq \cdots \leq n_r$. Then the canonical sheaf of $X$ is given by (cf. [GH94, § IV-5])

\[(2.9) \quad \omega_X = h^* \left( \mathcal{O}_{\mathbb{P}^1}(-1) \right) \otimes \mathcal{O}_X \left( \sum_{i=1}^r (n_i - 1) \Gamma_i \right).\]

We claim first that $r = 2$. Indeed, it is clear that $r \geq 2$ by (2.9) since $\kappa(X) = 1$, and that $r < 3$, since otherwise by an unramified cover one can construct a new surface $\tilde{S}$ with $p_g(\tilde{S}) > 0$. Moreover, similar to the proof of Theorem 1.2, one shows that $\tilde{S}$ is still a semi-stably fibred over $\mathbb{P}^1$ with 6 singular fibers. This is a contradiction by the above argument.

We claim also that $n_1 \nmid n_2$. Suppose $n_1$ divides $n_2$, one can construct an unramified cover $S''$ over $S$, which is still semi-stably fibred over $\mathbb{P}^1$ with 6 singular fibers. Moreover, the minimal model of $S''$ admits an elliptic fibration with only one multiple fiber. This is again a contradiction by the above argument.

Let $F$ be a general fiber of $f$ and $F_0$ its image in $X$. Let $\Gamma$ be a general fiber of $h$ and $d = \gcd(n_1, n_2)$. Then there exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 n_1 + m_2 n_2 = d$. Let $\Gamma_0 = m_2 \Gamma_1 + m_1 \Gamma_2$. Then numerically,

\[\Gamma_0 = \frac{m_2}{n_1} \cdot n_1 \Gamma_1 + \frac{m_1}{n_2} \cdot n_2 \Gamma_2 \sim_{\num} \left( \frac{m_2}{n_1} + \frac{m_1}{n_2} \right) \Gamma = \frac{d}{n_1 n_2} \cdot \Gamma.\]

Moreover, by (2.9), one has the following numerical equivalence:

\[\omega_X \sim_{\num} \left( 1 - \frac{1}{n_1} - \frac{1}{n_2} \right) \Gamma \sim_{\num} \left( \frac{n_1 n_2}{d} - \frac{n_1}{d} - \frac{n_2}{d} \right) \Gamma_0.\]
According to the proof of [TTZ05, Theorem 2.1], one has
\[
\omega_{S/P}^2 \geq 6g - 6 + \omega_X \cdot F_0
\]
\[
= 6g - 6 + \left( \frac{n_1 n_2}{d} - \frac{n_1}{d} - \frac{n_2}{d} \right) \Gamma_0 \cdot F_0
\]
\[
\geq 6g - 6 + \left( \frac{n_1 n_2}{d} - \frac{n_1}{d} - \frac{n_2}{d} \right).
\]

From \( n_1 \not| n_2 \) and (2.6), we see that there are only two possibilities as stated in Theorem 1.3.

It remains to show that \( S \) is simply connected. Since \( \chi(O_X) = 1 > 0 \), it follows from Noether’s formula that the elliptic fibration \( h \) admits at least one singular fiber. Moreover, we have shown that \( h \) has exactly two multiply fibers whose multiplicities are coprime. From [Moi77, § II.2-Theorem 10], it follows that \( X \), and hence also \( S \), are both simply connected. This completes the proof. \( \square \)

3. Arakelov type inequality

In this section, we generalize our results to the high dimension cases, i.e., we prove Theorem 1.4. The technique uses the Arakelov type inequality, which is deduced from the variation of the Hodge structures attached to such families.

The Arakelov type inequality for the direct image of the relative pluri-canonical sheaves goes back to Viehweg and the last author [VZ06, Zuo08]. This kind of inequality is generalized in the recent work [LTZ16]. The following form can be found in [Zuo08, Theorem 4.4] and [LTZ16, Prop 3.1 & Remark 3.2], which is the key to our proof.

**Theorem 3.1.** Let \( f : X \to B \) be a semi-stable family of varieties of relative dimension \( m \geq 1 \) over a smooth projective curve of genus \( g(B) \) with \( s \) singular fibers. Assume that the smooth fibers \( F \) are minimal, i.e., their canonical line bundles are semiample. Let \( \omega_{X/B} \) be the relative canonical sheaf, and \( \mathcal{E} \subseteq f_*(\omega_{X/B}^k) \) be any non-zero subsheaf. Then the slope \( \mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\text{rank} \mathcal{E}} \) satisfies that

\[
\mu(\mathcal{E}) \leq \frac{mk(2g(B) - 2 + s)}{2}.
\]

The main idea of proving Theorem 1.4 is to compute the plurigenera by applying Riemann-Roch theorem for the direct image sheaves \( f_*(\omega_X^k) \) on the base curve. Combining with the asymptotic behavior of the plurigenera, we complete the proof.
Proof of Theorem 1.4. Since the base is a rational curve $\mathbb{P}^1$, it follows that
\[ \omega_X = \omega_{X/\mathbb{P}^1} \otimes f^* \omega_{\mathbb{P}^1} = \omega_{X/\mathbb{P}^1} \otimes f^* \mathcal{O}_{\mathbb{P}^1}(-2). \]
Hence for any $k \geq 1$, one has
\[ (3.1) \quad f_*(\omega^\otimes_X) = f_*(\omega^\otimes_{X/\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1}(-2k). \]
Let $\mathcal{E} \subseteq f_*(\omega^\otimes_X)$ be any subsheaf. Then by (3.1), $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2k)$ is a subsheaf of $f_*(\omega^\otimes_{X/\mathbb{P}^1})$. Thus by Theorem 3.1, one obtains
\[ \mu(\mathcal{E}) + 2k = \mu(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2k)) \leq \frac{mk}{2} \cdot (s - 2); \]
equivalently, we have
\[ (3.2) \quad \mu(\mathcal{E}) \leq \frac{k}{2} (m(s - 2) - 4). \]
As a locally free sheaf on $\mathbb{P}^1$, the direct image sheaf $f_*(\omega^\otimes_X)$ is isomorphic to a direct sum of invertible sheaves,
\[ f_*(\omega^\otimes_X) \cong \bigoplus_{i=1}^{r_k} \mathcal{O}_{\mathbb{P}^1}(d_i), \quad r_k = \text{rank } f_*(\omega^\otimes_X). \]
By (3.2), we have
\[ d_i \leq \frac{k}{2} (m(s - 2) - 4), \quad i = 1, 2, \ldots, r_k. \]
Hence
\[ \dim H^0(X, \omega^\otimes_X) = \dim H^0(\mathbb{P}^1, f^*(\omega^\otimes_X)) = \sum_{d_i \geq 0} (d_i + 1) \leq \max \left\{ 0, \left\lfloor \frac{k}{2} (m(s - 2) - 4) \right\rfloor + 1 \right\} \cdot \text{rank } f_*(\omega^\otimes_X). \]
Here $\lfloor \cdot \rfloor$ stands for the integral part.

According to the definition of the Kodaira dimension of a variety, when $k$ is sufficiently large, one has
\[ \left\{ \begin{array}{l}
\text{rank } f_*(\omega^\otimes_X) = \dim H^0(F, \omega^\otimes_F) \sim k^{\kappa(F)}; \\
\dim H^0(X, \omega^\otimes_X) \sim k^{\kappa(X)}. 
\end{array} \right. \]
Hence $\kappa(X) \leq \kappa(F) + 1$. Moreover, if $\kappa(X) \geq 0$, then
\[ \frac{1}{2} (m(s - 2) - 4) \geq 0, \quad \text{i.e., } s \geq \frac{4m}{m} + 2; \]
and if $\kappa(X) = \kappa(F) + 1$, then
\[ \frac{1}{2} (m(s - 2) - 4) > 0, \quad \text{i.e., } s > \frac{4m}{m} + 2. \]
This completes the proof. □

Remark 3.2. Recall that the volume of a projective variety $X$ is defined to be

$$\text{Vol}(X) = \limsup_k \frac{(\dim X)! \cdot \dim H^0(X, \omega_X^\otimes k)}{k^{\dim X}}.$$

The above proof shows also that for a variety of general type semi-stably fibred over $\mathbb{P}^1$ with $s$ singular fibers, one has

$$\text{Vol}(X) \leq \frac{(m + 1)(m(s - 2) - 4)}{2} \text{Vol}(F),$$

where $F$ is a general fiber of $f$. In particular, when $X$ is of general type and $f : X \to \mathbb{P}^1$ is a semi-stable fibration of curves of genus $g \geq 2$ with $s$ singular fibers, one computes that

$$\text{Vol}(X) = \omega_{X_0}^2, \quad \text{Vol}(F) = 2g - 2,$$

where $X_0$ is the minimal model of $X$. Hence the above proof shows that in this case,

$$\omega_{X_0}^2 \leq 2(s - 6)(g - 1).$$

 REFERENCES

[Bea81] Arnaud Beauville, *Le nombre minimum de fibres singulières d’une courbe stable sur $\mathbb{P}^1$*, Astérisque 86 (1981), 97–108 (French).

[Bea82] Arnaud Beauville, *Les familles stables de courbes elliptiques sur $\mathbb{P}^1$ admettant 4 fibres singulières*, C. R. Acad. Sc. Paris 294 (1982), 657-660

[EKZ14] Alex Eskin, Maxim Kontsevich, and Anton Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, Publ. Math. Inst. Hautes Études Sci. 120 (2014), 207–333. MR 3270590

[Fuj78] Takao Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan 30 (1978), no. 4, 779–794. MR 513085 (82h:32024)

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original. MR 1288523

[GLT13] Cheng Gong, Xin Lu, and Sheng-Li Tan, *Families of curves over $\mathbb{P}^1$ with 3 singular fibers*, C. R. Math. Acad. Sci. Paris 351 (2013), no. 9-10, 375–380. MR 3072164

[Kov97] Sándor J. Kovács, *On the minimal number of singular fibres in a family of surfaces of general type*, J. Reine Angew. Math. 487 (1997), 171–177. MR 1454264

[LTZ16] Jun Lu, Sheng-Li Tan, and Kang Zuo, *Canonical class inequality for fibred spaces*, to appear in Math. Ann., 2016.

[LTXZ16] Xin Lu, Sheng-Li Tan, Wan-Yuan Xu, and Kang Zuo, *On the minimal number of singular fibers with non-compact Jacobians for families of curves over $\mathbb{P}^1$*, J. Math. Pures Appl. (9) 105 (2016), no. 5, 724–733. MR 3479189
A. Huitrado-Mora, M. Castaneda-Salazar, and A. G. Zamora, Toward a conjecture of Tan and Tu on fibered general type surfaces, arXiv:1604.00050, 2016.

Boris Moishezon, Complex surfaces and connected sums of complex projective planes, With an appendix by R. Livne. Lecture Notes in Mathematics, Vol. 603. Springer-Verlag, Berlin-New York, 1977.

Martin Möller, Variations of Hodge structures of a Teichmüller curve, J. Amer. Math. Soc. 19 (2006), no. 2, 327–344. MR 2188128

Martin Möller, Teichmüller curves, mainly from the viewpoint of algebraic geometry, Moduli spaces of Riemann surfaces, IAS/Park City Math. Ser., vol. 20, Amer. Math. Soc., Providence, RI, 2013, pp. 267–318. MR 3114688

Xiaotao Sun, Sheng-Li Tan, and Kang Zuo, Families of K3 surfaces over curves reaching the Arakelov-Yau type upper bounds and modularity, Math. Res. Lett. 10 (2003), no. 2–3, 323–342.

Sheng-Li Tan, The minimal number of singular fibers of a semistable curve over \( \mathbb{P}^1 \), J. Algebraic Geom. 4 (1995), no. 3, 591–596. MR 1325793

Sheng-Li Tan, Yuping Tu, and Fei Yu, On semistable families of curves over \( \mathbb{P}^1 \) with a small number of singular curves, (unpublished)

Sheng-Li Tan, Yuping Tu, and Alexis G. Zamora, On complex surfaces with 5 or 6 semistable singular fibers over \( \mathbb{P}^1 \), Math. Z. 249 (2005), no. 2, 427–438. MR 2115452

Yuping Tu, Surfaces of Kodaira dimension zero with six semistable singular fibers over \( \mathbb{P}^1 \), Math. Z. 257 (2007), no. 1, 1–5. MR 2318565

Eckart Viehweg and Kang Zuo, On the isotriviality of families of projective manifolds over curves, J. Algebraic Geom. 10 (2001), no. 4, 781–799. MR 1838979

Kang Zuo, Yau’s form of Schwarz lemma and Arakelov inequality on moduli spaces of projective manifolds, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, pp. 659–676. MR 2483376

Alexis G. Zamora, Semistable genus 5 general type \( \mathbb{P}^1 \)-curves have at least 7 singular fibres, Note Mat. 32 (2012), no. 2, 1–4. MR 3071788

Kang Zuo, Yau’s form of Schwarz lemma and Arakelov inequality on moduli spaces of projective manifolds, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, pp. 659–676. MR 2483376
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