Asymptotic behavior of Nambu-Bethe-Salpeter wave functions for multi-particles in quantum field theories

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Abstract

We derive asymptotic behaviors of the Nambu-Bethe-Salpeter (NBS) wave function at large space separations for systems with more than 2 particles in quantum field theories. To deal with \( n \)-particles in the center of mass flame coherently, we introduce the Jacob coordinates of \( n \) particles and then combine their \( 3(n - 1) \) coordinates into the one spherical coordinate in \( D = 3(n - 1) \) dimensions. We parametrize on-shell \( T \)-matrix for \( n \)-particle system of scalar fields at low energy, using the unitarity constraint of the \( S \)-matrix. We then express asymptotic behaviors of the NBS wave function for \( n \) particles at low energy, in terms of parameters of \( T \)-matrix, and show that the NBS wave function carry the information of \( T \)-matrix such as phase shifts and mixing angles of the \( n \)-particle system in its own asymptotic behavior, so that the NBS wave function can be considered as the scattering wave of \( n \)-particles in quantum mechanics. This property is one of the essential ingredients of the HAL QCD scheme to define "potential" from the NBS wave function in quantum field theories such as QCD. Our results, together with an extension to systems with spin 1/2 particles, justify the HAL QCD’s definition of potentials for 3 or more nucleons (baryons) in terms the NBS wave functions.
I. INTRODUCTION

To understand hadronic interactions such as nuclear forces from the fundamental theory, Quantum Chromodynamics (QCD), non-perturbative methods such as the lattice QCD combined with numerical simulations are required, since the running coupling constant in QCD becomes large at hadronic scale. Conventionally the finite size method has been employed to extract the scattering phase shift in lattice QCD, but the method is so far limited to two-particle systems below the inelastic threshold, except a few extensions.

Recently an alternative method has been proposed and employed to extract the potential between nucleons below inelastic thresholds. This method has been extended, in order to investigate other more general hadronic interactions such as baryon-baryon interactions and meson-baryon interactions. See Refs. for reviews of recent activities.

In the method, called the HAL QCD method, a potential between hadrons is defined in quantum field theories such as QCD, through the equal-time Nambu-Bethe-Salpeter (NBS) wave function in the center of mass system, which is defined for two nucleons as

\[ \Psi_W(x) = \langle 0 \mid T \{ N(r, 0)N(r + x, 0) \} \mid NN, W \rangle_{in} \]

where \(|0\rangle = \text{out}\langle 0\rangle = \text{in}\langle 0|\) is the QCD vacuum (bra-)state, \(|NN, W\rangle_{in}\) is the two-nucleon asymptotic in-state at the total energy \(W = 2\sqrt{k^2 + m_N^2}\) with the nucleon mass \(m_N\) and the relative momentum \(k\), \(T\) represents the time-ordered product, and \(N(x)\) with \(x = (x, t)\) is the nucleon operator. As the distance between two nucleon operators, \(x = |x|\), becomes large, the NBS wave function satisfies the free Schrödinger equation,

\[ (E_W - H_0) \Psi_W(x) \simeq 0, \quad E_W = \frac{k^2}{2\mu}, \quad H_0 = -\frac{\nabla^2}{2\mu} \]

where \(\mu = m_N/2\) is the reduced mass. In addition, the asymptotic behavior of the NBS wave function is described in terms of the phase \(\delta\) determined by the unitarity of the S-matrix, \(S = e^{2i\delta}\), in QCD (or the corresponding quantum field theory). This has been shown for the elastic \(\pi\pi\) scattering, where the partial wave of NBS wave function for the orbital angular momentum \(L\) becomes

\[ \Psi_W^L \simeq A_L \frac{\sin(kx - L\pi/2 + \delta_L(W))}{kx} \]

as \(x \to \infty\) at \(W \leq W_{th} = 4m_\pi\) (the lowest inelastic threshold). The asymptotic behavior of the NBS wave function for the elastic \(NN\) scattering has been derived in Ref.
The HAL QCD method has also been applied to investigate three nucleon forces (3NF)\cite{23, 24}, even though asymptotic behaviors of NBS wave function for three nucleons have not been derived yet. The 3NF is necessary to explain the experimental binding energies of light nuclei \cite{25, 26} and high precision deuteron-proton elastic scattering data at intermediate energies \cite{27}. It may also play an important role for various phenomena in nuclear physics and astrophysics \cite{28–31}.

The purpose of this paper is to derive asymptotic behaviors of NBS wave functions for \(n\) particles with \(n \geq 3\) at large distances where separations among \(n\) operators become all large. To avoid complications due to non-zero spins of particles, we consider scalar fields in this paper. The results of this paper, together with an extension to spin 1/2 particles, fills the logical gap in the derivation of 3NF by the HAL QCD method\cite{23, 24}.

In Sec. II we explain our notations and definitions such as the modified Jacobi coordinate, the Lippmann-Schwinger equation, and the NBS wave function for \(n\) scalar particles. In Sec. III we parametrize on-shell \(T\)-matrix for \(n\) particles, by solving the unitarity constraint of \(S\)-matrix. For explicit calculations for \(n\)-particle systems, we introduce the spherical coordinates in \(D = 3(n - 1)\) dimensions, which is equal to a number of degrees of freedom for \(n\) particle in 3-dimensions in the center of mass frame, together with non-relativistic approximations. In Sec. IV using these techniques and results obtained in Sec. III we derive asymptotic behaviors of NBS wave functions for \(n\)-particles, in terms of phase shifts and mixing angles of the \(n\)-particle scattering. Conclusions and discussions are given in Sec. V. Some technical details are collected in three appendices.

II. SOME DEFINITIONS AND NOTATIONS

In this paper, to avoid complications arising from nucleon spins, we consider an \(n\)-scalar particle system which have the same mass \(m\) in the center of mass frame, whose coordinates and momenta are denoted by \(x_i, p_i\) \((i = 1, 2, \cdots, n)\) with \(\sum_{i=1}^{n} p_i = 0\). We introduce modified Jacobi coordinates and corresponding momenta as

\[
\begin{align*}
  r_k &= \sqrt{\frac{k}{k+1}} \times r_k^J, \\
  q_k &= \sqrt{\frac{k+1}{k}} \times q_k^J
\end{align*}
\]

where the standard Jacobi coordinates and momenta are given by

\[
\begin{align*}
  r_k^J &= \frac{1}{k} \sum_{i=1}^{k} x_i - x_{k+1}, \\
  q_k^J &= \frac{k}{k+1} \left( \frac{1}{k} \sum_{i=1}^{k} p_i - p_{k+1} \right)
\end{align*}
\]
for \( k = 1, 2, \cdots, n - 1 \). It is easy to see

\[
\sum_{i=1}^{n} p_i \cdot x_i = \sum_{i=1}^{n-1} q_i \cdot r_i, \quad E = \frac{1}{2m} \sum_{i=1}^{n} p_i^2 = \frac{1}{2m} \sum_{i=1}^{n-1} q_i^2.
\] (6)

The integration measure for modified Jacobi momenta is given by

\[
\prod_{i=1}^{n} d^3 p_i \delta^{(3)} \left( \sum_{i=1}^{n} p_i \right) = \frac{1}{n^{3/2}} \prod_{i=1}^{n-1} d^3 q_i.
\] (7)

### A. Lippmann-Schwinger equation

As mentioned in the introduction, the asymptotic behavior of the NBS wave functions for a two-particle system has already been derived in Refs. [8, 20–22]. It is not straightforward, however, to extend their derivations to multi-particle systems. Instead, we utilize the Lippmann-Schwinger equation [32],

\[
|\alpha\rangle_{\text{in}} = |\alpha\rangle_0 + \int d\beta \frac{|\beta\rangle_0 T_{\beta\alpha}}{E_\alpha - E_\beta + i\varepsilon}, \quad T_{\beta\alpha} = \langle\beta|V|\alpha\rangle_{\text{in}},
\] (8)

which is found to be a powerful tool to study multi-particle systems. We assume in this paper that no bound state appears in two or more particle systems. Here the asymptotic in-state \( |\alpha\rangle_{\text{in}} \) satisfies

\[
(H_0 + V)|\alpha\rangle_{\text{in}} = E_\alpha |\alpha\rangle_{\text{in}},
\] (9)

whereas the non-interacting state \( |\alpha\rangle_0 \) satisfies

\[
H_0 |\alpha\rangle_0 = E_\alpha |\alpha\rangle_0.
\] (10)

The off-shell \( T \)-matrix element or the "potential" \( T_{\beta\alpha} = \langle\beta|V|\alpha\rangle_{\text{in}} \) is related to the on-shell \( S \)-matrix element as

\[
S_{\beta\alpha} \equiv \text{out} \langle\beta|\alpha\rangle_{\text{in}} \equiv \langle\beta|S|\alpha\rangle_0 = \delta(\beta - \alpha) - 2\pi i \delta(E_\alpha - E_\beta) T_{\beta\alpha}.
\] (11)

If we define \( S = 1 - iT \), we obtain

\[
\langle\beta|T|\alpha\rangle_0 = 2\pi \delta(E_\alpha - E_\beta) T_{\beta\alpha}.
\] (12)
B. NBS wave functions

The equal-time Nambu-Bethe-Salpeter (NBS) wave function for \( n \) scalar particles is defined by

\[
\Psi^n_\alpha([x]) = \text{in}\langle 0 | \varphi^n([x], 0) |\alpha\rangle_{\text{in}},
\]

(13)

where

\[
\varphi^n([x], t) = T\{ \prod_{i=1}^{n} \varphi_i(x_i, t) \},
\]

(14)

with the time-ordered product \( T, [x] = x_1, x_2, \cdots, x_n \), and \( i \) represents a "flavor" of scalar field. For simplicity, we regard all \( n \) scalar particles are different but have the same mass \( m \).

From the Lippmann-Schwinger equation (8), the vacuum instate is given by

\[
|0\rangle_{\text{in}} = |0\rangle_0 + \int d\gamma |\gamma\rangle_0 T\gamma_0 \frac{|\gamma\rangle_0 T\gamma_0}{E_0 - E_\gamma + i\varepsilon}.
\]

(15)

As shown in Appendix A, however, the contribution from the second term to the NBS wave function at large distances amounts to

\[
\text{in}\langle 0 | \varphi^n([x], 0) |\alpha\rangle_0 \simeq \frac{1}{Z_\alpha} \langle 0 | \varphi^n([x], 0) |\alpha\rangle_0,
\]

(16)

where \( Z_\alpha \) is the normalization factor whose deviation from the unity comes from the off-shell \( T \)-matrix \( T_{\alpha 0} \). Using this and the Lippmann-Schwinger equation (8), the NBS wave function can be written as

\[
\Psi^n_\alpha([x]) = \frac{1}{Z_\alpha} \langle 0 | \varphi^n([x], 0) |\alpha\rangle_0 + \int d\beta \frac{1}{Z_\beta} \langle 0 | \varphi^n([x], 0) |\beta\rangle_0 T_{\beta \alpha}. \]

(17)

To evaluate the above expression explicitly, we quantize all complex scalar fields in the Heisenberg representation at \( t = 0 \) as

\[
\varphi_i(x, 0) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_{ki}}} \left\{ a_i(k)e^{ik \cdot x} + b_i^\dagger(k)e^{-ik \cdot x} \right\}
\]

(18)

\[
|\alpha\rangle_0 \equiv |[k]_n\rangle_0 = \prod_{i=1}^{n} a_i^\dagger(k_i)|0\rangle_0, \quad E_{ki} = \sqrt{k_i^2 + m^2},
\]

(19)

where \( [k]_n = k_1, k_2, \cdots, k_n \) with \( \sum_{i=1}^{n} k_i = 0 \), and the full time evolution is given by

\[
\varphi^n([x], t) = e^{iHt} \varphi^n([x], 0)e^{-iHt} \quad \text{while} \quad H \rightarrow H_0 \quad \text{for the free field.}
\]

Our state normalization is given by

\[
\langle \beta_m |\alpha_n\rangle_0 = \delta(\beta_m - \alpha_n).
\]

(20)
Using the above, for the \( n \) particle system in the center of mass frame, we have

\[
o_0(0|\phi^n([x], 0)|k_n)_0 = \left( \frac{1}{\sqrt{(2\pi)^3}} \right)^n \prod_{i=1}^n \frac{1}{\sqrt{2E_{k_i}}} e^{ik_i \cdot x_i} = \left( \frac{1}{\sqrt{(2\pi)^3}} \right)^n \left( \prod_{i=1}^n \frac{1}{\sqrt{2E_{k_i}}} \right) \exp \left[ i \sum_{j=1}^{n-1} q_j \cdot r_j \right],
\]

where \( r_j \) and \( q_j \) are modified Jacobi coordinates and momenta, respectively.

### III. UNITARITY OF S-MATRIX AND PARAMETRIZATION OF T-MATRIX

The unitarity of \( S \)-matrix implies

\[
T^\dagger - T = iT^\dagger T.
\]

Defining

\[
0\langle [p^A]_n|T|[p^B]_n \rangle_0 = \delta(E^A - E^B)\delta(3)(P^A - P^B)T([q^A]_n, [q^B]_n)
\]

where \([p^X]_n = p^X_1, p^X_2, \ldots, p^X_n, [q^X]_n = q^X_1, q^X_2, \ldots, q^X_{n-1}\) with \( X = A, B \), and

\[
E^A \equiv \sum_{i=1}^n E_{p^A_i}, \quad E^B \equiv \sum_{i=1}^n E_{p^B_i}, \quad P^A \equiv \sum_{i=1}^n p^A_i, \quad P^B \equiv \sum_{i=1}^n p^B_i.
\]

Here we parametrize the \( T \)-matrix element in terms of modified Jacobi momenta \([q^A]_n\) and \([q^B]_n\). Note that \( T_{\beta\alpha} \), appeared in Lippmann-Schwinger equation, is expressed as

\[
T_{\beta\alpha} = \frac{1}{2\pi} \delta(3)(P^A - P^B)T([q^A]_n, [q^B]_n).
\]

Using the above expression, the unitarity constraint to \( T \)-matrix can be written as

\[
T^\dagger([q^A]_n, [q^B]_n) - T([q^A]_n, [q^B]_n) = \frac{i}{n^{3/2}} \int \prod_{i=1}^{n-1} d^3q^C_i \delta(E^A - E^C) \times T^\dagger([q^A]_n, [q^C]_n)T([q^C]_n, [q^B]_n).
\]

Our task is to solve this constraint.

#### A. \( n = 2 \)

Let me consider the simplest case, \( n = 2 \). In this case, we can parametrize \( T \)-matrix, in terms of the spherical harmonic functions \( Y_{lm} \) as follows.

\[
T(q^A, q^B) = \sum_{l,m} T_l(q^A, q^B) Y_{lm}(\Omega_{q^A})Y_{lm}(\Omega_{q^B})
\]
where \( q^{A,B} = |q^{A,B}| \) and \( \Omega_q \) is the solid angle of the vector \( q \). Using orthogonal property of \( Y_{lm} \), the constraint becomes

\[
\overline{T}_l(q, q) - T_l(q, q) = \frac{i}{2^{3/2}} \int (q^C)^2dq^C \delta(E - E_C)\overline{T}_l(q, q^C)T_l(q^C, q)
\]  

(28)

where \( q = q^A = q^B \), \( E = E^A = E^B = 2\sqrt{m^2 + q^2/2} \) and \( E^C = 2\sqrt{m^2 + (q^C)^2/2} \). After \( q^C \) integral, the constraint now becomes

\[
\overline{T}_l(q, q) - T_l(q, q) = \frac{i q_E^2}{2 \times 2^{3/2}} \overline{T}_l(q, q)T_l(q, q),
\]

(29)

which can be solved as

\[
T_l(q) \equiv T_l(q, q) = -\frac{4 \times 2^{3/2}}{qE} e^{i\delta_l(E)} \sin \delta_l(E),
\]

(30)

where \( \delta_l(q) \) is the phase shift for the partial wave with the angular momentum \( l \) at energy \( E = 2\sqrt{m^2 + q^2/2} \).

**B. General \( n \)**

For general \( n \) case, we introduce the non-relativistic approximation for the energy in the delta-function as

\[
E^A - E^C \simeq \frac{(p^A)^2 - (p^C)^2}{2m} = \frac{(q^A)^2 - (q^C)^2}{2m}
\]

(31)

where \( (q^{A,C})^2 = \sum_{i=1}^{n-1}(q^{A,C}_i)^2 \) for modified Jacobi momenta \([q^{A,C}]_n\). To perform 3 dimensional momentum integral \((n - 1)\) times, we consider \( D = 3(n - 1) \) dimensional space. Denoting \( s = |s| \) is a \( D \)-dimensional hyper-radius and \( \Omega_s \) are angular variables for the vector \( s \) in \( D \) dimensions, the Laplacian operator is decomposed as

\[
\nabla^2 = \frac{\partial^2}{\partial s^2} + \frac{D - 1}{s} \frac{\partial}{\partial s} - \frac{\hat{L}^2}{s^2}
\]

(32)

where \( \hat{L}^2 \) is angular-momentum in \( D \)-dimensions. The hyper-spherical harmonic function [33], an extension of spherical harmonic function in 3-dimension to general \( D \)-dimensions satisfies

\[
\hat{L}^2 Y_{[l]}(\Omega_s) = L(L + D - 2) Y_{[l]}(\Omega_s)
\]

(33)
where \([L] = L, M_1, M_2, \cdots\) are a set of "quantum" numbers specifying the hyper-spherical harmonic function. The hyper-spherical harmonic function is orthogonal such that

\[
\int d\Omega_s \overline{Y_{[L]}(\Omega_s)} Y_{[L']} (\Omega_s) = \delta_{[L][L']}
\]  

(34)

and complete

\[
\sum_{[L]} \overline{Y_{[L]}(\Omega_s)} Y_{[L]} (\Omega_s) \delta(s - t) = s^{D-1} \delta(D)(s - t),
\]  

(35)

so that an arbitrary function \(f(s)\) of \(s \in \mathbb{R}^D\) can be expanded as

\[
f(s) = \sum_{[L]} f_{[L]}(s) Y_{[L]} (\Omega_s).
\]  

(36)

Using the hyper spherical function, we expand the \(T\)-matrix as

\[
T([q^A]_n, [q^B]_n) \equiv T(Q_A, Q_B) = \sum_{[L],[K]} T_{[L][K]}(Q_A, Q_B) Y_{[L]}(\Omega_{Q_A}) \overline{Y_{[K]}(\Omega_{Q_B})}
\]  

(37)

where \(Q_X = (q_X^1, q_X^2, \cdots, q_X^{n-1})\) for \(X = A, B\) is a momentum vector in \(D = 3(n - 1)\) dimensions. \(^1\)

With the non-relativistic approximation and orthogonal property, the unitarity relation eq. (26) after \(\Omega_{Q^C}\) integration leads to

\[
T_{[L][K]}(Q_A, Q_A) - T_{[L][K]}(Q_A, Q_A) = \frac{i}{n^{3/2}} \int Q^{D-1} dQ \delta(E_A - E) T_{[L][N]}(Q_A, Q) T_{[N][K]}(Q, Q_A)
\]

\[
= \frac{i m(Q_A)^{D-2}}{n^{3/2}} \sum_{[N]} T_{[L][N]}(Q_A, Q_A) T_{[N][K]}(Q_A, Q_A)
\]  

(38)

where \(Q_A = Q_B\) is used. By diagonalizing \(T\) with an unitary matrix \(U\) as

\[
T_{[L][K]}(Q, Q) = \sum_{[N]} U_{[L][N]}(Q) T_{[N]}(Q) U_{[N][K]}^\dagger(Q),
\]  

(39)

the above constraint can be solved as

\[
T_{[L]}(Q) = -\frac{2n^{3/2}}{m Q^{3n-5}} e^{i\delta_{[L]}(Q)} \sin \delta_{[L]}(Q),
\]  

(40)

\(^1\) For \(n \geq 3\), the \(T\)-matrix can have singularities at particular on-shell values of external momenta\(^{34}-^{36}\), which are expressed in terms of delta functions and principles values with the \(i\varepsilon\) prescription for propagators. Even for such cases, however, our expansion of \(T\)-matrix in eq. (37) is still valid in the sense of distributions, and these singularities originate from a sum over infinite terms\(^{37}\). We would like to thank Prof. S. R. Sharpe and Dr. M. T. Hansen for pointing out this problem and relevant references.
where $\delta_{[L]}(Q)$ is a real phase, which depends on $Q$ and $[L]$ in $D = 3(n - 1)$ dimensions. This is a main result of this section. Unfortunately, a relation of the phase shifts in the hyper-spherical coordinates with physical observables for $n$-particles in the standard Jacobi coordinates is not transparent. Therefore it will be an important task in the future to make the relation between them clear.

At $n = 2$, we have

$$T_{[L]}(Q) = -\frac{2 \times 2^{3/2}}{mQ} e^{i\delta_{[L]}(Q)} \sin \delta_{[L]}(Q),$$

which agrees with eq. (30) under the non-relativistic approximation that $E \simeq 2m$, together with the replacement that $Q \rightarrow q$ and $[L] \rightarrow l$ and $U \rightarrow 1$.

IV. ASYMPTOTIC BEHAVIORS OF NBS WAVE FUNCTIONS FOR $n$ PARTICLES

In this section, we derive the asymptotic behaviors of NBS wave functions for multi-particle systems, using expressions (17)

$$\Psi^n_{\alpha}(\mathbf{x}) = \frac{1}{Z_{\alpha}} 0\langle 0 | \varphi^n(\mathbf{x}, 0) | \alpha \rangle_0 + \int d \beta \frac{1}{Z_{\beta}} \frac{0\langle 0 | \varphi^n(\mathbf{x}, 0) | \beta \rangle_0 T_{\beta \alpha}}{E_{\alpha} - E_{\beta} + i \varepsilon},$$

and (21)

$$0\langle 0 | \varphi^n(\mathbf{x}, 0) | \mathbf{k}_n \rangle_0 = \left( \frac{1}{\sqrt{(2\pi)^3}} \right)^n \left( \prod_{i=1}^n \frac{1}{\sqrt{2E_{k_i}}} \right) \exp \left[ i \sum_{j=1}^{n-1} \mathbf{q}_j \cdot \mathbf{r}_j \right].$$

A. $n = 2$

As an exercise, let us first consider the $n = 2$ case, whose result is already known. Using $\mathbf{r} = (\mathbf{x}_2 - \mathbf{x}_1)/\sqrt{2}$, $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{q}/\sqrt{2}$ and $E_\mathbf{q} = \sqrt{m^2 + q^2/2}$, the NBS wave function at $n = 2$ is given by

$$\Psi^2_q(\mathbf{r}) = \frac{1}{2E_qZ_q} \left[ \frac{e^{i\mathbf{q} \cdot \mathbf{r}}}{(2\pi)^3} + \int \frac{d^3k}{2^{3/2}(2\pi)^3} \frac{Z_qE_q}{Z_kE_k} \frac{e^{i\mathbf{k} \cdot \mathbf{r}} T(\mathbf{k}, \mathbf{q})}{4\pi(E_q - E_k + i \varepsilon)} \right],$$

where $\mathbf{k}$ is also the modified Jacobi momentum. Using expansions that

$$e^{i\mathbf{q} \cdot \mathbf{r}} = 4\pi \sum_{lm} i^l j_l(qr) Y_{lm}(\Omega_r) \overline{Y_{lm}(\Omega_q)},$$

$$\Psi^2_q(\mathbf{r}) = \sum_{lm} \mathcal{I} \Psi^2_l(r, q) Y_{lm}(\Omega_r) \overline{Y_{lm}(\Omega_q)}.$$
where \( j_l(x) \) is the spherical Bessel function of the first kind, together with eq. (27), and integrating over \( \Omega_k \), we obtain

\[
\Psi^2_l(r, q) = \frac{4\pi}{(2\pi)^3 2 E_q Z_q} \left[ j_l(qr) + \int_0^\infty \frac{k^2 dk}{2 \pi^{3/2}} \frac{Z_q E_q j_l(kr) T_l(k, q)}{2 Z_k E_k 2(E_q - E_k + i\varepsilon)} \right].
\]  

(47)

Since \( E_q \) is below inelastic thresholds, we assume that \( T_l(q, k) \) does not have any poles in the positive real axis. Under this assumption, we perform the \( k \) integral using the formula

\[
\int_0^\infty \frac{k^2 dk}{q^2 - k^2 + i\varepsilon} F_l(k) \simeq -\frac{\pi q^2}{2q} F_l(q) [n_l(qr) + ij_l(qr)]
\]

(48)

for \( r \gg 1 \) \cite{2000}, where \( F_l(k) \) does not have any poles in the positive real axis and satisfies

\[
\int k^{-1} j_0(kr) F_l(k) k^2 dk \simeq 0,
\]

which follows from \((\nabla^2 + q^2)\Psi^2_q(r) \simeq 0\), for large \( r \) \cite{2000}, and \( n_l(x) \) is the spherical Bessel function of the second kind. After the \( k \) integral using this formula, the second term in eq. (47) becomes

\[
- [n_l(qr) + ij_l(qr)] \frac{q E_q}{2 \times 2^{3/2}} T_l(q, q) = [n_l(qr) + ij_l(qr)] e^{i\delta_l(q)} \sin \delta_l(q),
\]

(49)

where the unitarity constraint \((30)\) for \( T_l(q, q) \) is used to obtain the last equality. We then obtain

\[
\Psi^2_l(r, q) = \frac{4\pi}{(2\pi)^3 2 E_q Z_q} e^{i\delta_l(q)} [j_l(qr) \cos \delta_l(q) + n_l(qr) \sin \delta_l(q)]
\]

\[
\simeq \frac{4\pi}{(2\pi)^3 2 E_q Z_q} e^{i\delta_l(q)} \frac{\sin(qr - l\pi/2 + \delta_l(q))}{qr}
\]

(50)

(51)

for \( r \gg 1 \), where asymptotic behaviors that \( j_l(x) \simeq \sin(x - l\pi/2)/x \) and \( n_l(x) \simeq \cos(x - l\pi/2)/x \) are used. The phase of the S-matrix, \( \delta_l(q) \), can be interpreted as the scattering phase shift of the NBS wave function for the \( n = 2 \) case.

**B. General \( n \)**

The NBS wave function for general \( n \) is expressed as

\[
\Psi^n(R, Q_A) = C(Q_A) \left[ e^{iQ_A R} + \frac{n-3/2}{2\pi} \int d^D Q C(Q) C(Q_A) \frac{e^{iQ R}}{E_{Q_A} - E_Q + i\varepsilon} T(Q, Q_A) \right]
\]

(52)

where \( R = (r_1, r_2, \cdots, r_{n-1}) \) and \( Q_A^{(A)} = (q_1, q_2, \cdots, q_{n-1})^{(A)} \) are the modified Jacobi coordinates and momenta in \( D = 3(n-1) \) dimensions,

\[
C(Q_A) = \frac{1}{Z(Q_A)} \prod_{j=1}^n \frac{1}{\sqrt{(2\pi)^3 2 E_{\nu_j}^A}}.
\]

(53)
with $p_j^{(A)}$ is the momentum of the $j$-th particle. In the non-relativistic limit that

$$C(Q_A) \to C(Q_A) = \frac{1 + cQ_A^2}{(2\pi)^3 2m^{n/2}}, \quad \frac{C(Q)}{C(Q_A)} \to \frac{C(Q)}{C(Q_A)}, \quad (E_{Q_A} - E_Q) \to \frac{Q_A^2 - Q^2}{2m},$$

with some constant $c$, we have

$$\Psi^n(R, Q_A) = C(Q_A) \left[ e^{iQ_A \cdot R} + \frac{2m}{2\pi n^{3/2}} \int d^D Q \frac{C(Q)}{C(Q_A)} \frac{e^{iQ \cdot R}}{Q_A^2 - Q^2 + i\epsilon} T(Q, Q_A) \right]. \quad (55)$$

In $D$-dimensions, we have

$$e^{iQ \cdot R} = (D - 2)!! \frac{2\pi^{D/2}}{\Gamma(D/2)} \sum_{[L]} i^L j^D_L(QR) Y_{[L]}(\Omega_R) \overline{Y_{[K]}(\Omega_Q)}, \quad (56)$$

which is the generalization of the $D = 3$ formula in eq. (45), where $j^D_L$ is the hyperspherical Bessel function of the first kind defined by

$$j^D_L(x) = \frac{\Gamma(D/2)}{(D - 2)!!} \frac{2^{D/2} x^{D/2 - 4}}{x^{D/2 - 2}} J_L(x), \quad (57)$$

with $L_D = L + \frac{D-2}{2}$ and the Bessel function of the first kind, $J_L(x)$.

Using an expansion that

$$\Psi^n(R, Q_A) = \sum_{[L], [K]} \Psi^n_{[L], [K]}(R, Q_A) Y_{[L]}(\Omega_R) \overline{Y_{[K]}(\Omega_Q)}, \quad (58)$$

with eqs. (57) and (56), and performing $d\Omega_Q$ integral, we obtain

$$\Psi^n_{[L], [K]}(R, Q_A) = C(Q_A) j^L(2\pi)^{D/2} (Q_AR)^{D/2} \left[ J_{L_D}(QAR) \delta_{LK} + \int dQ \frac{J_{L_D}(QR)}{Q_A^2 - Q^2 + i\epsilon} H_{[L], [K]}(Q, Q_A) \right] \quad (59)$$

where

$$H_{[L], [K]}(Q, Q_A) = \frac{m}{\pi n^{3/2}} \frac{C(Q)}{C(Q_A)} Q^{D/2} Q_A^{D/2 - 1} T_{[L], [K]}(Q, Q_A). \quad (60)$$

We now perform the $Q$ integral, assuming that $T_{[L], [K]}(Q, Q_A)$ does not have any poles on the positive real axis at $Q_A$ below inelastic thresholds. We consider $n = 2k$ and $n = 2k + 1$ cases separately.
1. \( n = 2k \) case

In this case,

\[
J_{L_{D}}(x) = j_{L_{k}}(x) \sqrt{\frac{2}{\pi}} x^{1/2}
\]

(61)

where \( L_{k} = L + 3(k - 1) \) and \( j_{L_{k}} \) is the spherical Bessel function of the first kind. Using eq. (48), the second term in eq. (59) can be evaluated as

\[
\int dQ \frac{j_{L_{k}}(QR)}{Q_{A}^{2} - Q^{2} + i\varepsilon} \sqrt{\frac{2}{\pi}} (QR)^{1/2} H_{[L],[K]}(Q,Q_{A})
\]

\[
\approx -[n_{L_{k}}(Q_{A}R) + ij_{L_{k}}(Q_{A}R)] \frac{\pi}{2Q_{A}} \sqrt{\frac{2}{\pi}} (Q_{A}R)^{1/2} H_{[L],[K]}(Q_{A},Q_{A})
\]

\[
= [N_{L_{D}}(Q_{A}R) + iJ_{L_{D}}(Q_{A}R)] \sum_{[N]} U_{[L],[N]}(Q_{A}) e^{i\delta_{[N]}(Q_{A})} \sin \delta_{[N]}(Q_{A}) U_{[N],[K]}^{\dagger}(Q_{A})
\]

(62)

for \( R \gg 1 \), where the unitarity constraint to \( T \) in eq. (40) is used to obtain the last line, and \( J_{L_{D}} \) and \( N_{L_{D}} \) are Bessel functions of the first and second kinds, respectively.

2. \( n = 2k + 1 \) case

In this case, \( L_{D} = L + 3k - 1 \) is an integer, and for large \( R \), \( J_{L_{D}}(x) \) becomes

\[
J_{L_{D}}(x) \simeq \sqrt{\frac{2}{\pi x}} \sin(x - \Delta_{L}), \quad N_{L_{D}}(x) \simeq \sqrt{\frac{2}{\pi x}} \cos(x - \Delta_{L}), \quad \Delta_{L} = \frac{2L_{D} - 1}{4\pi}.
\]

(63)

Using this asymptotic behavior, the \( Q \) integral in eq. (59) can be performed, and we obtain for \( R \gg 1 \)

\[
I \equiv \int dQ \frac{J_{L_{D}}(QR)}{Q_{A}^{2} - Q^{2} + i\varepsilon} H_{[L],[K]}(Q,Q_{A})
\]

\[
\approx -\sqrt{\frac{2}{\pi Q_{A}R}} \left[ \frac{\pi e^{i(Q_{A}R - \Delta_{L})}}{2Q_{A}} H_{[L],[K]}(Q_{A},Q_{A}) + O \left( R^{3-D}/2 \right) \right]
\]

(64)

\[
\approx [N_{L_{D}}(Q_{A}R) + iJ_{L_{D}}(Q_{A}R)] \sum_{[N]} U_{[L],[N]}(Q_{A}) e^{i\delta_{[N]}(Q_{A})} \sin \delta_{[N]}(Q_{A}) U_{[N],[K]}^{\dagger}(Q_{A}),
\]

(65)

where, in the last line, the \( O(1/R) \) contribution is neglected for large \( R \) and the unitarity condition for \( T \) in eq. (40) is used, and \( e^{i(Q_{A}R - \Delta_{D})} \) is replaced by the asymptotic behaviors of \( J_{n} \) and \( H_{n} \). The detailed calculation of the \( Q \) integral is given in Appendix B.
C. Asymptotic behavior

For both \( n = 2k \) and \( n = 2k + 1 \), we finally obtain

\[
\Psi_n^{[L],[K]}(R,Q_A) \simeq C_i^L \frac{(2\pi)^{D/2}}{(Q_AR)^{D/2}} \sum_{[N]} U_{[L][N]}(Q_A) e^{i\delta_{[N]}(Q_A)} U_{[N][K]}^\dagger(Q_A) \\
\quad \times \left[ J_{LD}(Q_AR) \cos \delta_{[N]}(Q_A) + N_{LD}(Q_AR) \sin \delta_{[N]}(Q_A) \right] \\
\simeq C_i^L \frac{(2\pi)^{D/2}}{(Q_AR)^{D/2}} \sum_{[N]} U_{[L][N]}(Q_A) e^{i\delta_{[N]}(Q_A)} U_{[N][K]}^\dagger(Q_A) \\
\quad \times \sqrt{\frac{2}{\pi}} \sin \left( Q_AR - \Delta_L + \delta_{[N]}(Q_A) \right)
\]

(66)

(67)

for \( R \gg 1 \), which agrees with eq. (51) at \( n = 2 \). Eq. (67) is the main result of this paper, which tells us that the NBS wave function of \( n \)-particles for large \( R \) can be considered as the generalized scattering wave of \( n \) particles, whose generalized scattering phase shift \( \delta_{[N]}(Q_A) \) is nothing but the phase of the \( S \)-matrix in QCD, determined in eq. (40) by the unitarity.

V. CONCLUSION AND DISCUSSION

In this paper, we have investigated the asymptotic behaviors of the NBS wave functions at large separations for \( n \) complex scalar fields. We have first solved the unitarity constraint of the \( S \)-matrix for \( n \geq 3 \), using the \( D = 3(n - 1) \) coordinate space and employing the hyper-spherical harmonic function, together with the non-relativistic approximation for the energy. The results are summarized in eqs. (39) and (40). We then have calculated the asymptotic behaviors of the NBS wave functions at large separations for \( n \geq 3 \), using again the hyper-spherical harmonic function, which is found to be quite useful for this purpose. We finally obtain eq. (67), which is the main result in this paper. In appendix C, we generalize our results to the coupled channels, where the particle mixing occurs during the scattering.

Using the results in this paper, we can generalize the HAL QCD method to hadron interactions for the \( n \)-particle system with \( n \geq 3 \). This give a firm theoretical background to the extraction of interactions among many hadrons by the HAL QCD method, in particular, the three nucleon force \([23,24]\), together with an extension to systems with spin 1/2 particles, which is a straightforward but much more complicated task in future. Moreover, combining it with the results in our previous paper [38], which shows that non-local but energy independent potentials can be constructed from the NBS wave functions above the
inelastic threshold, the HAL QCD method can be extended to hadronic interactions above
the inelastic threshold energy, where particle productions such as \( NN \to NN\pi \) can occur.

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\[ \text{Appendix A: Contribution from vacuum} \]

In this appendix, we show eq. (16). Assuming each flavor is conserved, \( 0\langle \gamma | \) which contributes in eq. (16) is a sum of the following form.

\[
o_{I_{k}} = 0\langle 0| \prod_{i \in I_{k}} a_{i}(k_{i}^{A})b_{i}(k_{i}^{B}) \quad (A1)
\]

with \( \sum_{i \in I_{k}}(k_{i}^{A} + k_{i}^{B}) = 0 \), where \( k \leq n \) and \( I_{k} = \{i_{1}, i_{2}, \ldots, i_{k}\} \) with \( 1 \leq i_{1} < i_{2} < \cdots < i_{k} \leq n \). Note that the operator \( a_{i}b_{i} \) creates a particle-antiparticle pair with flavor \( i \). Using this notation, we have

\[
in\langle 0| \varphi^{n}([x], 0)|[k]n\rangle_{0} = 0\langle 0| \varphi^{n}([x], 0)|[k]n\rangle_{0} + \sum_{k=1}^{n} \sum_{I_{k}} \prod_{i \in I_{k}} \int d^{3}k_{i}^{A} d^{3}k_{i}^{B} \times \delta^{(3)}(\sum_{i \in I_{k}}(k_{i}^{A} + k_{i}^{B})) \frac{T_{0;I_{k}}}{E_{0} - E_{I_{k}} + i\varepsilon} 0\langle I_{k}| \varphi^{n}([x], 0)|[k]n\rangle_{0}, \quad (A2)
\]

where \( E_{0} = 0 \) for the vacuum.

Using

\[
(2\pi)^{3n/2}0\langle I_{k}| \varphi^{n}([x], 0)|[k]n\rangle_{0} = \prod_{i \in I_{k}} \frac{e^{-i k_{i}^{B} x_{i}}}{\sqrt{2 E_{k_{i}^{B}}}} \delta^{(3)}(k_{i} - k_{i}^{A}) \prod_{j \in I_{k}} \frac{e^{i k_{j} x_{j}}}{\sqrt{2 E_{k_{j}}}}, \quad (A3)
\]

where \( \bar{I}_{k} \cup I_{k} = \{1, 2, 3, \ldots, n\} \) and \( \bar{I}_{k} \cap I_{k} = \phi \), the second term in eq. (A2) becomes

\[
C_{n} \sum_{k=1}^{n} \sum_{I_{k}} \prod_{i \in I_{k}} \int \frac{d^{3}k_{i}^{B}}{\sqrt{2 E_{k_{i}^{B}}}} e^{-i k_{i}^{B} x_{i}} \delta^{(3)}(\sum_{i \in I_{k}}(k_{i} + k_{i}^{B})) \prod_{j \in I_{k}} \frac{e^{i k_{j} x_{j}}}{\sqrt{2 E_{k_{j}}}} \times \frac{T_{0;I_{k}}(0; [k, k^{B}])}{E_{0} - E_{[k, k^{B}]}}. \quad (A4)
\]
where \( C_n = (2\pi)^{-\frac{3n}{2}} \), \( E_{[k, k^B]} = \sum_{i \in I_k} \left( \sqrt{k_i^2 + m^2} + \sqrt{(k_i^B)^2 + m^2} \right) \), \( T_{0; k}^\dagger (0; [k, k^B]) \) is the off-shell T-matrix from vacuum to \( 2k \) particles, and \( [k, k^B] = \{k_{i_1}, k_{i_2}, k_{i_3}, \cdots, k_{i_k}, k_{i_k}^B\} \).

We first show that terms at \( k \geq 2 \) in eq. (A4) do not contribute at large distances. After \( k^B_{i_k} \) integral, the factor in the first exponential is written as
\[
-\sum_{i \in I_{k-1}} x_{i_k}^i - x_{i_k}^i + i \sum_{i \in I_{k}} k_i x_{i_k}^i,
\]
where \( I_{k-1} = \{i_1, i_2, \cdots, i_{k-1}\} \). Since \( I_{k-1} \neq \phi \) for \( k \geq 2 \), we perform the \( k^B_{i_k} \) integral in eq. (A4). Using the same method which leads to eq. (48) from eq. (44) and noticing the fact that there is no real poles for the \( k^B_{i_k} \) integral in eq. (A4), it is clear that the contribution is suppressed exponentially in large \( |x_{i_k}^i - x_{i_k}^i| \). This means that terms at \( k = 1 \) only contribute in eq. (A4) and other terms at \( k \geq 2 \) are suppressed asymptotically at large distances.

The term at \( k = 1 \) is easily evaluated as
\[
C_n \prod_{j=1}^{n} e^{i x_{j_i}^i} \frac{T_{0; i, -i}^\dagger (0; [k_{i}, -k_{i}])}{\sqrt{2E_{k_j}}} \frac{n}{-2 \sqrt{k_i^2 + m^2}},
\]
(A5)
where \( T_{0; i, -i}^\dagger \) is the off-shell T-matrix from vacuum to a pair of particle-antiparticle with the flavor \( i \).

We then finally obtain
\[
\langle 0 | \varphi^n([x], 0) | [k]_n \rangle_0 \approx \frac{1}{Z([k]_n) \varphi^n([x], 0) | [k]_n \rangle_0}
\]
(A6)
with
\[
\frac{1}{Z([k]_n) = 1 + \sum_{i=1} T_{0; i, -i}^\dagger (0; k_{i}, -k_{i})} \frac{x_{i_k}^i}{-2 \sqrt{k_i^2 + m^2}},
\]
(A7)
which proves eq. (16) with \( Z_{\alpha} = Z([k]_n) \).

Appendix B: \( Q \) Integrals

In this appendix, we evaluate the \( Q \) integral in the following form
\[
I = \int_0^{\infty} dQ J_{LD}(Q R) \frac{J_{LD}(Q R)}{Q^2 A - Q^2 + i \varepsilon} H_{[L], [K]}(Q, Q_A),
\]
(B1)
for large \( R \), assuming that \( H_{[L], [K]}(Q, Q_A) \) have no poles in the real axis at \( Q \geq 0 \). Using the asymptotic form of \( J_{LD}(x) \) at large \( R \) given in eq. (63), we write
\[
I \approx \sqrt{\frac{2}{\pi R}} \frac{1}{2i} (I_+ - I_-), \quad R \to \infty
\]
(B2)
We evaluate $I_+$ and $I_-$ separately. For $I_+$, we consider an integration in the complex $Q$ plane on a closed path $C = [0, \infty] \oplus C_\theta \oplus i[\infty, 0]$ in Fig. 1 which leads to

$$I_+ + I_1 + I_2 = \int_C \frac{e^{i(QR - \Delta L)}}{Q^2_A - Q^2 + i\varepsilon} f(Q) = -\frac{\pi i}{Q_A} e^{i(QA R - \Delta L)} f(Q_A) + O(e^{-cR}), \quad (B4)$$

where

$$I_1 \equiv \lim_{q \to \infty} \int_{\pi/2}^{\pi} q e^{i\theta} i d\theta \frac{e^{i(QR - \Delta L)}}{Q^2_A - Q^2 + i\varepsilon} f(Q) \bigg|_{Q = q e^{i\theta}}, \quad (B5)$$

$$I_2 \equiv \int_{\infty}^{i0} dQ \frac{e^{i(QR - \Delta L)}}{Q^2_A - Q^2 + i\varepsilon} f(Q) = -\int_{0}^{\infty} idq \frac{e^{-qR - i\Delta L}}{Q^2_A + q^2 + i\varepsilon} f(iq), \quad (B6)$$
and the term $O(e^{-cR})$ with $c > 0$ represents the contributions from complex poles inside $C$.

It is easy to show that $I_1$ vanishes as

$$|I_1| \leq \lim_{q \to \infty} \frac{q}{(q^2 - Q^2_A)^{\frac{1}{2}}} \max_{0 \leq \theta \leq \pi/2} \left| f(qe^{i\theta}) \right| \int_0^{\pi/2} d\theta e^{-qR \sin \theta} \leq \lim_{q \to \infty} F(q) \int_0^{\pi/2} d\theta e^{-2qR\theta/\pi} = \lim_{q \to \infty} F'(q) \frac{\pi}{2qR} (1 - e^{-qR}) \to 0,$$

where we assume that $\max_{0 \leq \theta \leq \pi/2} |f(qe^{i\theta})|$ does not grow as fast as $q^2$ in the large $q^2$ limit. Similarly, we estimate

$$|I_2| \leq \int_0^\infty dq \frac{e^{-qR}}{Q^2_A + q^2} |f(iq)| \leq \frac{1}{Q^2_A} \max_{0 < q} |f(iq)| \int_0^\infty dq e^{-qR} = \frac{1}{Q^2_A} \max_{0 < q} |f(iq)| \frac{1}{R},$$

for $Q_A \neq 0$, which vanishes $1/R$ for large $R$ as long as $\max_{0 < q} |f(iq)| < \infty$. If some poles happen to exist on the positive imaginary axis, we can modify the path a little to avoid poles, so that the above estimate still holds. We indeed have a more stronger bound of $|I_2|$ for all $Q_A$ including $Q_A = 0$ as shown below at $n \geq 3$. (At $n = 2$, we can evaluate $I$ by the different method.) Since we can write $f(Q) = Q^{(D-1)/2} g(Q)$ with $|g(0)| < \infty$ from eqs. (60) and (B3), we have

$$|I_2| \leq \max_{0 < q} |g(iq)| \int_0^\infty dq q^{(D-5)/2} e^{-qR} = \max_{0 < q} |g(iq)| R^{(3-D)/2} \int_0^\infty dt t^{(D-5)/2} e^{-t},$$

which vanishes as $R^{(3-D)/2}$ for large $R$ at $n \geq 3$ ( $D \geq 6$ ), as long as $\max_{0 < q} |g(iq)| < \infty$. (Again we can modify the path if poles exits on the positive imaginary axis.) Altogether we obtain

$$I_+ \simeq -\frac{\pi i}{Q_A} e^{i(QAR-\Delta L)} f(Q) + O(R^{(3-D)/2}).$$

For $I_-$, we take another closed path $C' = [0, \infty] \oplus C_\theta' \oplus i[-\infty, 0]$ in Fig. 1. Since poles at $Q = \pm(Q_A + i\varepsilon)$ are not contained in this closed path, we have

$$I_+ + I_1' + I_2' = \int_{C'} \frac{e^{-i(QR-\Delta L)}}{Q^2_A - Q^2 + i\varepsilon} f(Q) = O(e^{-c'R})$$

with $c' > 0$, where

$$I_1' \equiv \lim_{q \to \infty} \int_{0}^{\pi/2} q^2 e^{i\theta} i d\theta \frac{e^{-i(QR-\Delta L)}}{Q^2_A - Q^2 + i\varepsilon} f(Q) \bigg|_{Q=qe^{i\theta}},$$

$$I_2' \equiv \int_{-i\infty}^{0} dQ \frac{e^{-i(QR-\Delta L)}}{Q^2_A - Q^2 + i\varepsilon} f(Q) \equiv \int_0^\infty idq \frac{e^{-qR+i\Delta L}}{Q^2_A + q^2 + i\varepsilon} f(-iq).$$
As in the case before, it is easy to show that

\[ |I_1'| = 0, \quad |I_2'| = O(R^{(3-D)/2}), \quad (B14) \]

which leads to \( I_- = O(R^{(3-D)/2}). \)

Combining these, we finally obtain

\[ I = -\sqrt{\frac{2}{\pi Q_A R}} \left[ \frac{\pi e^{i(Q_AR-\Delta L)}}{2Q_A} H_{[L],[K]}(Q_A, Q_A) + O(R^{(3-D)/2}) \right], \quad (B15) \]

which proves eq. (B14).

**Appendix C: Coupled channel cases**

In this appendix, we extend our investigation to the case where \( l \to n \) scatterings with \( l \neq n \) can occur.

1. **Unitarity constraint to T-matrix**

The unitarity relation to \( T \)-matrix in eq. (26) can be generalized to

\[ T^\dagger(Q_n, Q_l) - T(Q_n, Q_l) = \sum_k^i \int dQ_k \delta(E_{Q_n} - E_{Q_k}) \times T^\dagger(Q_n, Q_k)T(Q_k, Q_l) \quad (C1) \]

for general \( n, l \), where the energy conservation that \( E_{Q_n} = E_{Q_l} \) is always satisfied.

As in the case of the single channel, we expand \( T \) in term of the hyper-spherical harmonic function as

\[ T(Q_n, Q_l) = \sum_{[N_n],[L_l]}^\sum T_{[N_n],[L_l]}(Q_n, Q_l)Y_{[N_n]}(\Omega_{Q_n})Y_{[L_l]}(\Omega_{Q_l}), \quad (C2) \]

where \( Q_n^2 - Q_l^2 = 2m^2(l-n) \) in the non-relativistic approximation. Putting this into eq. (C1), we have

\[ T^\dagger_{[N_n],[L_l]}(Q_n, Q_l) - T_{[N_n],[L_l]}(Q_n, Q_l) = \sum_{k,[K_k]} \frac{mQ_k^{D_k-2}}{k^{3/2}} T^\dagger_{[N_n],[K_k]}(Q_n, Q_k)T_{[K_k],[L_l]}(Q_k, Q_l) \quad (C3) \]
where \( D_k = 3(k - 1) \) and \( Q_n^2 - Q_k^2 = 2m^2(k - n) \). Defining and diagonalizing \( \hat{T} \) as

\[
T_{[N_n],[L_l]}(Q_n, Q_l) = \frac{Q_n^{D_n/2-1}T_{[N_n],[L_l]}(Q_n, Q_l)Q_l^{D_{3/2-1}}}{B_{3/4}} = \sum_{k,[K_k]} U_{[N_n],[K_k]}(Q_k)\hat{T}_{[K_k]}(Q_k)U^\dagger_{[K_k],[L_l]}(Q_k), \tag{C4}
\]

where \( Q_n^2 - Q_k^2 = 2m^2(k - n) \), eq. \((C3)\) leads to

\[
\hat{T}_{[K_k]}(Q_k) = -\frac{2}{m}e^{i\delta_{[K_k]}(Q_k)} \sin \delta_{[K_k]}(Q_k). \tag{C5}
\]

This gives us the final result,

\[
T_{[N_n],[L_l]}(Q_n, Q_l) = -\frac{2n^{3/4}l^{3/4}}{mQ_n^{D_n/2-1}Q_l^{D_{3/2-1}}} \sum_{k,[K_k]} U_{[N_n],[K_k]}(Q_k)e^{i\delta_{[K_k]}(Q_k)} \sin \delta_{[K_k]}(Q_k) \times U^\dagger_{[K_k],[L_l]}(Q_k), \tag{C6}
\]

which reproduces eq. \((40)\) for the single channel at \( n = l = k \).

2. Asymptotic behavior of the NBS wave function

For the coupled channel, the NBS wave function corresponding to eq. \((55)\) in the non-relativistic approximation becomes

\[
\Psi^{nl}(R_n, Q_l) = C_n \left[ \delta_{nl} e^{iQ_l R_n} + \frac{2m}{2\pi n^{3/2}} \int dP \frac{e^{iP_n R_n}T(P_n, Q_l)}{Q_l^2 - P_n^2 + 2m^2(l - n) + i\varepsilon} \right], \tag{C7}
\]

where \( C_n = ((2\pi)^32m)^{-n/2} \). (We here omit irrelevant \( \frac{Q_n^2}{m} \) contributions.) Expanding the NBS wave function in terms of the hyper-spherical function as

\[
\Psi^{nl}(R_n, Q_l) = \sum_{[N_n],[L_l]} \Psi_{[N_n],[L_l]}(R_n, Q_l)Y_{[N_n]}(\Omega_{R_n})Y^*_{[L_l]}(\Omega_{Q_l}), \tag{C8}
\]

together with Eq. \((56)\), we have

\[
\Psi_{[N_n],[L_l]}(R_n, Q_l) = C_n i^{N_n} \left( \frac{2\pi}{Q_n R_n} \right)^{D_n/2-1} \left[ J_{N_n}(Q_n R_n)\delta_{nl} \delta_{[N_n],[L_l]} \right. \nonumber
\]
\[
+ \left. \int dP \frac{J_{N_n}(P_n R_n)}{Q_l^2 - P_n^2 + 2m^2(l - n) + i\varepsilon} H_{[N_n],[L_l]}(P_n, Q_l) \right] \tag{C9}
\]

where \( \tilde{N}_n = N_n + (3n - 5)/2 \) and

\[
H_{[N_n],[L_l]}(P_n, Q_l) = \frac{m}{\pi n^{3/2}}P_n^{D_n/2}Q_l^{D_{3/2-1}}T_{[N_n],[L_l]}(P_n, Q_l). \tag{C10}
\]
As before, after $P_n$ integral, the second term in eq. (C9) for large $R_n$ is given by

$$\Psi_{|N_n|,|L_l|}(R_n, Q_l) \simeq C_n i^{N_n} \frac{(2\pi)^{D_n/2}}{(Q_n R_n)^{D_n/2-1}} \left( \frac{l}{n} \right)^{3/4} \frac{Q_n^{D_n/2-1}}{Q_l^{D_l/2-1}} \sum_{k,[K_k]} U_{|N_n|,[K_k]}(Q_k) e^{i\delta_{[K_k]}(Q_k)}$$

$$\times \left[ J_{N_n}(Q_n R_n) \cos \delta_{[K_k]}(Q_k) + H_{N_n}(Q_n R_n) \sin \delta_{[K_k]}(Q_k) \right] U_{[K_k],[L_l]}^\dagger(Q_k)$$

$$\simeq C_n i^{N_n} \frac{(2\pi)^{D_n/2}}{(Q_n R_n)^{D_n/2-1}} \left( \frac{l}{n} \right)^{3/4} \frac{Q_n^{D_n/2-1}}{Q_l^{D_l/2-1}} \sum_{k,[K_k]} U_{|N_n|,[K_k]}(Q_k) e^{i\delta_{[K_k]}(Q_k)}$$

$$\times \sqrt{\frac{2}{\pi}} \sin(Q_n R_n - \Delta_{N_n} + \delta_{[K_k]}(Q_k)) U_{[K_k],[L_l]}^\dagger(Q_k)$$

where $\Delta_{N_n} = (2N_n - 1)\pi/4$, which correctly reproduces eq. (67) in the single channel case at $n = l = k$.

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