Bivariate Discrete Inverse Weibull Distribution

M. S. Eliwa\textsuperscript{1} and M. El-Morshedy\textsuperscript{1,}\textsuperscript{*}

\textsuperscript{1}Mathematics Department, Faculty of Science, Mansoura University, Mansoura, Pin 35516, Egypt.

Abstract

In this paper, we propose a new class of bivariate distributions, called the bivariate discrete inverse Weibull (BDsIW) distribution, whose marginals are discrete inverse Weibull (DsIW) distributions. Some statistical and mathematical properties are presented. The maximum likelihood method is used for estimating the model parameters. Simulations are presented to verify the performance of the direct maximum likelihood estimation. Finally, two real data sets are analyzed to illustrative purposes.

Key words: Bivariate discrete distributions; discrete inverse Weibull distribution; maximum likelihood method.

AMS 2000 subject classification: 62F10; 62H10.

1 Introduction

The Weibull (W) distribution is one of the most popular and widely used distributions for failure time in reliability theory (see, Weibull (1951)). The cumulative distribution function (CDF) of W distribution is given by

\[ \Pi(x; \nu, \zeta) = 1 - e^{-\nu x^\zeta}; \quad x > 0, \]  

where \( \nu > 0 \) is scale parameter and \( \zeta > 0 \) is shape parameter. Clearly, the exponential (E) distribution and the Rayleigh (R) distribution are special cases for \( \zeta = 1 \) and \( \zeta = 2 \) respectively. Unfortunately, the shape of the hazard rate function (HRF) of W distribution can be only increasing, decreasing or constant. So, more modifications and generalizations of W distribution are presented in the statistical literature to describe various phenomena in different fields, because in many applications, empirical hazard rate curves often exhibit non-monotonic shapes such as a bathtub, unimodal and others. For example:

1. Keller et al. (1985) proposed inverse Weibull (IW) distribution. The CDF of IW distribution is given by

\[ \Pi(x; \nu, \zeta) = e^{-\nu x^{-\zeta}}; \quad x > 0. \]  

2. Lai et al. (2003) introduced modified Weibull (MW) distribution. The CDF of MW distribution is given by

\[ \Pi(x; \nu, \zeta, \lambda) = 1 - e^{-\nu x^\zeta e^{\lambda x}}; \quad x > 0, \]  

where \( \lambda > 0 \) is an accelerating parameter. The exponentiated MW distribution was proposed by Jalmar et al. (2008).

3. Bebbington et al. (2007) proposed flexible Weibull (FxW) distribution. The CDF of FxW distribution is given by

\[ \Pi(x; \nu, \gamma) = 1 - e^{-e^{\nu x^{-\gamma}}}; \quad x > 0, \]  

where \( \gamma > 0 \) is scale parameter. The exponentiated FxW distribution was presented by El-Gohary et al. (2015a), the inverse flexible Weibull (IFxW) distribution was proposed by El-Gohary et al. (2015b) and the exponentiated of it was studied by El-Morshedy et al. (2017).
4. Cordeiro et al. (2013) introduced exponential-Weibull (E-W) distribution. The CDF of E-W distribution is given by

\[ \Phi(x; \gamma, \nu, \beta) = 1 - e^{-\gamma x - \nu x^\beta}; \quad x > 0, \] \hspace{1cm} (5)

where \( \beta \in (0, \infty) - \{1\} \) is shape parameter.

5. Nadarajah et al. (2013) proposed exponentiated Weibull (EW) distribution. The CDF of EW distribution is given by

\[ \Phi(x; \nu, \zeta, \eta) = \left(1 - e^{-\nu x^\zeta}\right)^\eta; \quad x > 0, \] \hspace{1cm} (6)

where \( \eta > 0 \) is shape parameter.

6. El-Bassiouny et al. (2017) introduced exponentiated generalized Weibull-Gompertz (EGW-Gz) distribution. The CDF of EGW-Gz distribution is given by

\[ \Phi(x; \nu, \zeta, \lambda, \eta, \rho) = \left(1 - e^{-\nu x^\zeta(e^{\lambda x \eta} - 1)}\right)^\rho; \quad x > 0, \] \hspace{1cm} (7)

where \( \rho > 0 \) is shape parameter. The mixture of 2-EGW-Gz distribution was studied by El-Bassiouny et al. (2016).

Moreover, some discrete versions of E, R, W distributions and its generalizations have been presented in the literature because in several cases, lifetimes need to be recorded on a discrete scale rather than on a continuous analogue. So, discretizing continuous distributions has received much attention in the literature. For example:

1. Toshio and Shunji (1975) introduced discrete Weibull (DsW) distribution. The probability mass function (PMF) of DsW distribution is given by

\[ \pi(x; \theta, \zeta) = \theta^{x^\zeta} - \theta^{(x+1)^\zeta}; \quad x \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}, \] \hspace{1cm} (8)

where \( 0 < \theta < 1 \). Clearly, the discrete Rayleigh (DsR) distribution is a special case for \( \zeta = 2 \), which was presented by Dilip (2004).

2. Gómez-Déniz (2010) proposed generalization of geometric (GGo) distribution. The PMF of GGo distribution is given by

\[ \pi(x; \theta, \gamma) = \frac{\gamma \theta^x (1 - \theta)}{(1 - [1 - \gamma] \theta^{x+1}) (1 - [1 - \gamma] \theta^x)}; \quad x \in \mathbb{N}_0. \] \hspace{1cm} (9)

The GGo distribution reduces to geometric or discrete exponential (DsE) distribution when \( \gamma = 1 \).

3. Jazi et al. (2010) introduced DsIW distribution. The PMF of DsIW distribution is given by

\[ \pi(x; \theta, \zeta) = \theta^{(x+1)^\zeta} - \theta^{x^\zeta}; \quad x \in \mathbb{N}_0. \] \hspace{1cm} (10)

4. Vahid et al. (2013) proposed discrete generalized exponential type II (DsGE-T2) distribution. The PMF of DsGE-T2 distribution is given by

\[ \pi(x; \theta, \zeta) = \left(1 - \theta^{x+1}\right)^\zeta - \left(1 - \theta^x\right)^\zeta; \quad x \in \mathbb{N}_0. \] \hspace{1cm} (11)

5. Vahid and Hamid (2015a) introduced exponentiated discrete Weibull (EDsW) distribution. The PMF of EDsW distribution is given by

\[ \pi(x; \theta, \rho, \zeta) = \left(1 - \theta^{(x+1)^\rho}\right)^\zeta - \left(1 - \theta^{x^\rho}\right)^\zeta; \quad x \in \mathbb{N}_0. \] \hspace{1cm} (12)

6. Vahid et al. (2015b) proposed discrete beta exponential (DsBE) distribution. The PMF of DsBE distribution is given by

\[ \pi(x; \theta, \gamma, \zeta) = c \theta^{\gamma (x-1)} (1 - \theta^x)^{\zeta-1}; \quad x \in \mathbb{N}_0 - \{0\}, \] \hspace{1cm} (13)

where \( c^{-1} = \sum_{j=0}^{\infty} \frac{(-\theta)^j}{1 - \theta^{j+1}} \frac{(\zeta-1)!(\zeta-2)! \cdots (\zeta-j)!}{j!} \).
On the other hand, in many practical situations, it is important to consider different bivariate continuous and discrete distributions that could be used to model bivariate lifetime data in many fields. So, several bivariate continuous and discrete distributions are available in the statistical literature. For example, Lee (1997), Karlis and Ntzoufras (2000), Wu and Yuen (2003), Yuen et al. (2006), Sarhan and Balakrishnan (2007), Kundu and Dey (2009), Morata (2009), Kundu and Gupta (2009), Ong and Ng (2013), Balakrishnan and Shiji (2014), Lee and Cha (2015), Rasool and Akbar (2016), Hiba (2016), El-Bassiouny et al. (2016), El-Gohary et al. (2016), Vahid and Kundu (2017), Mohamed et al. (2017), Kundu and Vahid (2018), El-Morsheyed and Khalili (2018) among others. An excellent encyclopedic survey of various continuous and discrete bivariate distributions can be found in Balakrishnan and Lai (2009) and Johnson et al. (1997) respectively.

In this regard, we focus the aim of this paper on presenting a flexible discrete bivariate distribution called BDsIW distribution, which can be usefully applied not only by statisticians, but also by data analysis in many different disciplines, such as sports, engineering, and medical applications. The proposed discrete model can be obtained from 3-independent DsIW distributions by using the maximization method as suggested by Lee and Cha (2015). The main reasons for introducing BDsIW distribution are:

1. The proposed model is a very flexible bivariate discrete distribution, and its joint PMF can take different shapes depending on the parameter values.
2. The generation from the proposed model is straightforward. So, the simulation experiments can be performed quite conveniently.
3. The marginals of the proposed model are DsIW distributions. Hence, the marginals are able to analyze the hazard rates in the discrete case.
4. The DsE and DsR distributions are special cases from the proposed model.

2 The BDsIW Distribution and Its Statistical Properties

2.1 Definition and interpretations

Suppose \( W_1 \sim \text{DsIW}(\theta_1, \zeta) \), \( W_2 \sim \text{DsIW}(\theta_2, \zeta) \) and \( W_3 \sim \text{DsIW}(\theta_3, \zeta) \) and they are independently distributed. If \( X_d = \max(W_d, W_3); d = 1, 2 \), then we can say that the bivariate vector \( X = (X_1, X_2) \) has a BDsIW distribution with the parameter vector \( \Psi = (\theta_1, \theta_2, \theta_1, \zeta)^T \). We will denote this discrete bivariate distribution by BDsIW(\( \theta_1, \theta_1, \theta_1, \zeta \)). If \( X \sim \text{BDsIW}(\theta_1, \theta_1, \theta_1, \zeta) \), then the joint CDF of \( X \) for \( x_1, x_2 \in \mathbb{N}_0 \) and for \( x_3 = \min\{x_1, x_2\} \) is given by

\[
F_{X_1, X_2}(x_1, x_2; \Psi) = \theta_1^{(x_1+1) \zeta} \theta_2^{(x_2+1) \zeta} \theta_3^{(x_3+1) \zeta} F_{\text{DsIW}}(x_1; \theta_1, \zeta) F_{\text{DsIW}}(x_2; \theta_2, \zeta) F_{\text{DsIW}}(x_3; \theta_3, \zeta)
\]

\[
= \begin{cases} 
F_{\text{DsIW}}(x_1; \theta_1, \zeta) F_{\text{DsIW}}(x_2; \theta_2, \zeta) F_{\text{DsIW}}(x_3; \theta_3, \zeta) & ; 0 < x_1 < x_2 < \infty \\
F_{\text{DsIW}}(x_1; \theta_1, \zeta) F_{\text{DsIW}}(x_2; \theta_2, \zeta) & ; 0 < x_2 < x_1 < \infty \\
F_{\text{DsIW}}(x; \theta_1 \theta_2 \theta_3, \zeta) & ; 0 < x_1 = x_2 = x < \infty
\end{cases}
\]  \hspace{1cm} (14)

The marginal CDFs for BDsIW distribution can be represented as follows

\[
F_{X_d}(x_d; \theta_d, \theta_3, \zeta) = P[\max(W_d, W_3) \leq x_d] = F_{\text{DsIW}}(x_d; \theta_d, \theta_3, \zeta).
\]  \hspace{1cm} (15)

The corresponding joint PMF of \( X \) for \( x_1, x_2 \in \mathbb{N}_0 \) is given by

\[
f_{X_1, X_2}(x_1, x_2; \Psi) = \begin{cases} 
f_1(x_1, x_2; \Psi) & ; 0 < x_1 < x_2 < \infty \\
f_2(x_1, x_2; \Psi) & ; 0 < x_2 < x_1 < \infty \\
f_3(x; \Psi) & ; 0 < x_1 = x_2 = x < \infty
\end{cases}
\]  \hspace{1cm} (16)

where

\[
f_1(x_1, x_2; \Psi) = F_{\text{DsIW}}(x_1; \theta_1 \theta_3, \zeta) F_{\text{DsIW}}(x_2; \theta_2, \zeta),
\]

\[
f_2(x_1, x_2; \Psi) = F_{\text{DsIW}}(x_1; \theta_1, \zeta) F_{\text{DsIW}}(x_2; \theta_2 \theta_3, \zeta),
\]

\[
f_3(x; \Psi) = F_{\text{DsIW}}(x; \theta_2, \zeta) F_{\text{DsIW}}(x; \theta_1 \theta_3, \zeta) - F_{\text{DsIW}}(x-1; \theta_2 \theta_3, \zeta) F_{\text{DsIW}}(x; \theta_1, \zeta).
\]
The expressions $f_1(x_1, x_2; \Psi)$, $f_2(x_1, x_2; \Psi)$ and $f_3(x; \Psi)$ for $x_1, x_2 \in \mathbb{N}_0$ can be easily obtained by using the relation
\[ f_{X_1,X_2}(x_1, x_2; \Psi) = F(x_1, x_2; \Psi) - F(x_1 - 1, x_2; \Psi) - F(x_1, x_2 - 1; \Psi) + F(x_1 - 1, x_2 - 1; \Psi). \] (17)

Figure 1 shows the plots of the joint PMF of BDsIW distribution for different parameter values.

From Figure 1, it is clear that the joint PMF can take different shapes depending on the model parameter values. Assume $X \sim \text{BDsIW}(\theta_1, \theta_2, \theta_3, \zeta)$, then the joint reliability function of $X$ can be expressed as
\[
R_{X_1,X_2}(x_1, x_2; \Psi) = 1 - F_{X_1}(x_1; \theta_1, \theta_3, \zeta) - F_{X_2}(x_2; \theta_2, \theta_3, \zeta) + F_{X_1,X_2}(x_1, x_2; \Psi)
\]
\[
= \begin{cases} 
R_1(x_1, x_2; \Psi) & ; \ 0 < x_1 < x_2 < \infty \\
R_2(x_1, x_2; \Psi) & ; \ 0 < x_2 < x_1 < \infty \\
R_3(x; \Psi) & ; \ 0 < x_1 = x_2 = x < \infty,
\end{cases}
\] (18)

where
\[
R_1(x_1, x_2; \Psi) = 1 - F_{\text{DaIW}}(x_1; \theta_1, \theta_3, \zeta) - F_{\text{DaIW}}(x_2; \theta_2, \theta_3, \zeta) + F_{\text{DaIW}}(x_1; \theta_1, \theta_3, \zeta) F_{\text{DaIW}}(x_2; \theta_2, \zeta),
\]
\[
R_2(x_1, x_2; \Psi) = 1 - F_{\text{DaIW}}(x_1; \theta_1, \theta_3, \zeta) - F_{\text{DaIW}}(x_2; \theta_2, \theta_3, \zeta) + F_{\text{DaIW}}(x_1; \theta_1, \zeta) F_{\text{DaIW}}(x_2; \theta_2 \theta_3, \zeta),
\]
and
\[
R_3(x; \Psi) = 1 - F_{\text{DaIW}}(x; \theta_1 \theta_3, \zeta) - F_{\text{DaIW}}(x; \theta_2 \theta_3, \zeta) + F_{\text{DaIW}}(x; \theta_1 \theta_2 \theta_3, \zeta).
\]

Moreover, the bivariate hazard rate function (BHRF) of $X$ can be represented as
\[
r_{X_1,X_2}(x_1, x_2; \Psi) = \begin{cases} 
r_1(x_1, x_2; \Psi) & ; \ 0 < x_1 < x_2 < \infty \\
r_2(x_1, x_2; \Psi) & ; \ 0 < x_2 < x_1 < \infty \\
r_3(x; \Psi) & ; \ 0 < x_1 = x_2 = x < \infty,
\end{cases}
\] (19)

where $r_j(\cdot; \Psi) = \frac{f_j(\cdot; \Psi)}{R_j(\cdot; \Psi)}; j = 1, 2, 3$. Figure 2 shows the plots of the BHRF of BDsIW distribution for different parameter values.
Similarly, when $X_2 < X_1$, then

$$r^*(X_1|X_2 > x_2) = \frac{\zeta(x_2 + 1)^{-\zeta - 1}}{R_2(x_1, x_2; \Psi)} \{F_{\text{DsIW}}(x_2; \theta_2, \zeta) - 1\} \ln (\theta_1 \theta_3), \quad (21)$$

and the HRF of the conditional distribution $X_2$ given $X_1 > x_1$ is given by

$$r^*(X_2|X_1 > x_1) = \frac{\zeta(x_1 + 1)^{-\zeta - 1} F_{\text{DsIW}}(x_1; \theta_1, \zeta)}{R_2(x_1, x_2; \Psi)} \{F_{\text{DsIW}}(x_1; \theta_1 \theta_3, \zeta) - F_{\text{DsIW}}(x_1; \theta_2, \zeta) \ln (\theta_1 \theta_3)\}. \quad (22)$$

Similarly, when $X_2 < X_1$, then

$$r^{**}(X_1|X_2 > x_2) = \frac{\zeta(x_1 + 1)^{-\zeta - 1}}{R_2(x_1, x_2; \Psi)} \{F_{\text{DsIW}}(x_1; \theta_1, \zeta) - 1\} \ln (\theta_1 \theta_3). \quad (23)$$

On the other hand, assume a parallel system contains 2-component. Then, we can defined the BHRF as a vector which is useful to measure the total life span of a 2-component as follows

$$r(x) = (r(x), r_{12}(x_1|x_2), r_{21}(x_2|x_1)), \quad (24)$$

where $r(x)$ gives the HRF of the system using the information that the 2-component has survived beyond $x$, $r_{12}(x_1|x_2)$ gives the HRF span of the first component given that it has survived to an age $x_1$ and the other has failed at $x_1$. Similar argument holds for $r_{21}(x_2|x_1)$, (see Cox (1972)). If $X \sim \text{BDsIW}(\theta_1, \theta_3, \zeta)$, then

$$r(x)_{X=\min(x_1, x_2)} = \frac{F_{\text{DsIW}}(x - 1; \theta_3, \zeta)}{R_3(x; \Psi)} \left[-F_{\text{DsIW}}(x - 1; \theta_1, \zeta) - F_{\text{DsIW}}(x - 1; \theta_2, \zeta) + F_{\text{DsIW}}(x - 1; \theta_1 \theta_2, \zeta)\right]$$

$$+ \frac{F_{\text{DsIW}}(x; \theta_3, \zeta)}{R_3(x; \Psi)} \left[F_{\text{DsIW}}(x; \theta_1, \zeta) + F_{\text{DsIW}}(x; \theta_2, \zeta) - F_{\text{DsIW}}(x; \theta_1 \theta_2, \zeta)\right],$$

$$r_{12}(x_1|x_2)_{X_1 > X_2} = \zeta(x_1 + 1)^{-\zeta - 1} [1 - F_{\text{DsIW}}(x_1; \theta_1, \zeta)]^{-1} \ln (\theta_1),$$

and

$$r_{21}(x_2|x_1)_{X_1 < X_2} = \zeta(x_2 + 1)^{-\zeta - 1} [1 - F_{\text{DsIW}}(x_2; \theta_2, \zeta)]^{-1} \ln (\theta_2).$$

The following shock model and maintenance model interpretations can be provided for BDsIW distribution.
1. Shock model: Consider a system has 2-component, and it is assumed that the amount of shocks is measured in a discrete unit. Each component is subjected to individual shocks say $W_1$ and $W_2$ respectively. The system faces an overall shock $W_3$, which is transmitted to both the component equally, independent of their individual shocks. So, the observed shocks at the 2-component are $X_1 = \max(W_1, W_3)$ and $X_2 = \max(W_2, W_3)$ respectively.

2. Maintenance model: Consider a system has 2-component, and it is assumed that each component has been maintained independently and also there is an overall maintenance. Due to component maintenance, assume the lifetime of the individual component is increased by $W_i$ amount and because of the overall maintenance, the lifetime of each component is increased by $W_3$ amount. Here, $W_1$, $W_2$ and $W_3$ are all measured in a discrete unit. So, the increased lifetimes of the 2-component are $X_1 = \max(W_1, W_3)$ and $X_2 = \max(W_2, W_3)$ respectively.

### 2.2 Some statistical properties

Assume $X \sim \text{BDsIW} \left( \theta_1, \theta_2, \theta_3, \zeta \right)$, then $X_1$ and $X_2$ are positive quadrant dependent (PQD) where

$$ F_{X_1, X_2}(x_1, x_2; \Psi) \geq F_{X_1}(x_1; \theta_1, \theta_3, \zeta) F_{X_2}(x_2; \theta_2, \theta_3, \zeta). \quad (25) $$

Furthermore, for every pair of increasing functions $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$, we get $\text{Cov} \left\{ f_{X_1}(X_1), f_{X_2}(X_2) \right\} \geq 0$; see for example Nelsen (2006). Let us recall that, the function $k(u, v) : R \times R \to R$, is said to have the total positivity of order two ($TP-O_2$) property if $k(u, v)$ satisfies

$$ k(u_1, v_1)k(u_2, v_2) \geq k(u_2, v_1)k(u_1, v_2), \quad (26) $$

for all $u_1, v_1, u_2, v_2 \in R$. It is consider a very strong and an important property in lifetime testing, see for example Hu et al. (2003). Assume $x_{11}, x_{21}, x_{12}, x_{22} \in N_0$ and $x_{11} < x_{21} < x_{12} < x_{22}$ from $X \sim \text{BDsIW} \left( \theta_1, \theta_2, \theta_3, \zeta \right)$, then the joint reliability function of $X$ satisfies the $TP-O_2$ property where

$$ \frac{R_{X_1, X_3}(x_{11}, x_{21}) R_{X_1, X_3}(x_{12}, x_{22})}{R_{X_1, X_3}(x_{12}, x_{21}) R_{X_1, X_3}(x_{11}, x_{22})} \geq 1. \quad (27) $$

Similarly, when $x_{11} = x_{21} < x_{12} < x_{22}, x_{21} < x_{11} < x_{12} < x_{22}$ etc. Now, we present some statistical properties of the proposed model in a form of results.

**Result 1.** If the bivariate vector $X = (X_1, X_2)$ has the $\text{BDsIW} \left( \theta_1, \theta_2, \theta_3, \zeta \right)$, then

1. $\max \{ X_1, X_2 \} \sim \text{DslW} \left( \theta_1, \theta_2, \theta_3, \zeta \right)$.

2. The stress-strenght probability is given by

$$ P[X_1 < X_2] = \sum_{x=0}^{\infty} \left( \theta_1 \theta_3 \right)^{(x+1)-x} \left[ \left( \theta_2 \theta_3 \right)^{(x+1)-x} - \left( \theta_2 \theta_3 \right)^{-x} \right]. \quad (28) $$

3. The median of $X_1$ and $X_2$ is given by

$$ M_{X_d} = \left\{ \log \frac{\theta_d \theta_3}{U} \right\}^{-1}; \quad d = 1, 2, \quad (29) $$

where $U$ has a uniform $U(0, 1)$ distribution.

4. The coefficient of median correlation between $X_1$ and $X_2$ is given by

$$ M_{X_1, X_2} = \left\{ \begin{array}{ll} 4F_{\text{DslW}} \left( M_{X_1}; \theta_1 \theta_3, \zeta \right) F_{\text{DslW}} \left( M_{X_2}; \theta_2, \zeta \right) - 1 & ; \ x_1 < x_2 \\
4F_{\text{DslW}} \left( M_{X_1}; \theta_1, \zeta \right) F_{\text{DslW}} \left( M_{X_2}; \theta_2 \theta_3, \zeta \right) - 1 & ; \ x_2 \leq x_1. \end{array} \right. \quad (30) $$

5. The conditional PMF of $X_1$ given $X_2 = x_2$ is given by

$$ f_{X_1 \mid X_2}(x_1 \mid x_2) = \left\{ \begin{array}{ll} f_{X_1 \mid X_2=x_2}^{(1)}(x_1 \mid x_2) & \text{if} \quad 0 < x_1 < x_2 < \infty \\
f_{X_1 \mid X_2=x_2}^{(2)}(x_1 \mid x_2) & \text{if} \quad 0 < x_2 < x_1 < \infty \\
f_{X_1 \mid X_2=x_2}^{(3)}(x_1 \mid x) & \text{if} \quad 0 < x_1 = x_2 = x < \infty, \quad (31) \end{array} \right. $$

6. The...
where

\[
\begin{align*}
    f_{X_1|X_2 = x_2}^{(1)}(x_1 | x_2) &= \frac{f_{\text{BDsIW}}(x_1; \theta_1 \theta_2, \zeta) f_{\text{BDsIW}}(x_2; \theta_2 \zeta)}{f_{\text{BDsIW}}(x_2; \theta_2 \zeta)}, \\
    f_{X_1|X_2 = x_2}^{(2)}(x_1 | x_2) &= f_{\text{BDsIW}}(x_1; \theta_1 \zeta),
\end{align*}
\]

and

\[
f_{X_1|X_2 = x_2}^{(3)}(x_1 | x) = \frac{f_{\text{BDsIW}}(x; \theta_2 \zeta) f_{\text{BDsIW}}(x; \theta_1 \theta_3 \zeta) - f_{\text{BDsIW}}(x-1; \theta_2 \theta_3 \zeta) f_{\text{BDsIW}}(x; \theta_1 \zeta)}{f_{\text{BDsIW}}(x; \theta_2 \theta_3 \zeta)}.
\]

6. The conditional CDF of $X_1$ given $X_2 \leq x_2$, is given by

\[
F_{X_1|X_2 = x_2}(x_1 | x_2) = \begin{cases} 
    \frac{F_{\text{BDsIW}}(x_1; \theta_1 \theta_3 \zeta)}{F_{\text{BDsIW}}(x_2; \theta_2 \zeta)} & \text{if } 0 < x_1 < x_2 < \infty \\
    \frac{F_{\text{BDsIW}}(x_1; \theta_1 \zeta)}{F_{\text{BDsIW}}(x_2; \theta_2 \zeta)} & \text{if } 0 < x_2 < x_1 < \infty \\
    \frac{F_{\text{BDsIW}}(x_1; \theta_1 \zeta)}{F_{\text{BDsIW}}(x_1; \theta_1 \zeta)} & \text{if } 0 < x_1 = x_2 = x < \infty.
\end{cases}
\]

**Proof.** The proofs are quite standard and the details are avoided.

**Result 2.** Assume $(X_{1i}, X_{2i}) \sim \text{BDsIW}(\theta_{1i}, \theta_{2i}, \theta_{3i}, \zeta)$ for $i = 1, 2, \ldots, n$ and they are independently distributed. If

\[
Z_s = \max(X_{1s}, X_{2s}, \ldots, X_{ns}): s = 1, 2 \implies (X_{1s}, X_{2s}) \sim \text{BDsIW} \left( \prod_{i=1}^{n} \theta_{1i}, \prod_{i=1}^{n} \theta_{2i}, \prod_{i=1}^{n} \theta_{3i}, \zeta \right).
\]

**Proof.** It is easy to prove that using the joint CDF.

**Result 3.** If the bivariate vector $X \sim \text{BDsIW}((\theta_{1}, \theta_{2}, \theta_{3}, \zeta)$, then the joint probability generating function (PGF) of $X_1$ and $X_2$ can be written as infinite mixtures,

\[
G_{X_1, X_2}(y_1, y_2) = \sum_{j=0}^{\infty} \sum_{i=0}^{j-1} \left[ (\theta_1 \theta_3)^{(i+1)} - (\theta_1 \theta_3)^{i} \right] \left[ (\theta_2 \theta_3)^{(j+1)} - (\theta_2 \theta_3)^{j} \right] y_1^i y_2^j
\]

\[
+ \sum_{j=0}^{\infty} \sum_{i=j+1}^{\infty} \left[ (\theta_1 \theta_3)^{(i+1)} - (\theta_1 \theta_3)^{i} \right] \left[ (\theta_2 \theta_3)^{(j+1)} - (\theta_2 \theta_3)^{j} \right] y_1^i y_2^j
\]

\[
+ \sum_{i=0}^{\infty} \theta_1^{(i+1)} - (\theta_1 \theta_3)^{i} \left[ (\theta_2 \theta_3)^{(i+1)} - (\theta_2 \theta_3)^{i} \right] y_1^i y_2^j
\]

\[
- \sum_{i=0}^{\infty} \theta_1^{(i+1)} - (\theta_1 \theta_3)^{i} \left[ (\theta_2 \theta_3)^{(i+1)} - (\theta_2 \theta_3)^{i} \right] y_1^i y_2^j;
\]

\[
|y_1|, |y_2| < 1.
\]

**Proof.** The proof can be easily obtained by using the fact that

\[
G_{X_1, X_2}(y_1, y_2) = E \left( y_1^{X_1} y_2^{X_2} \right) = \sum_{j=0}^{\infty} P [X_1 = i, X_2 = j] y_1^i y_2^j.
\]

Hence, different moments and product moments of BDsIW distribution can be obtained, as infinite series, using the joint PGF.

# 3 Statistical Inference

## 3.1 Maximum likelihood estimation (MLE)

In this section, we use the maximum likelihood method to estimate the unknown parameters $\theta_1, \theta_2, \theta_3$ and $\zeta$ of BDsIW distribution. Suppose that, we have a sample of size $n$, of the form $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})\}$ from BDsIW
distribution. We use the following notations: \( I_1 = \{ x_{1j} < x_{2j}, I_2 = \{ x_{2j} < x_{1j} \}, I_3 = \{ x_{1j} = x_{2j} = x_j \}, I = I_1 \cup I_2 \cup I_3, |I_1| = n_1, |I_2| = n_2, |I_3| = n_3 \) and \( n = \sum_{k=1}^3 n_k \). Based on the observations, the likelihood function is given by

\[
l(\Psi) = \prod_{j=1}^{n_1} f_1(x_{1j}, x_{2j}) \prod_{j=1}^{n_2} f_2(x_{1j}, x_{2j}) \prod_{j=1}^{n_3} f_3(x_j).
\]  

(34)

The log-likelihood function becomes

\[
L(\Psi) = \sum_{j=1}^{n_1} \ln \left( \phi_1(x_{1j}; \theta_1, \zeta) \right) + \sum_{j=1}^{n_2} \ln \left( \phi_1(x_{2j}; \theta_2, \zeta) \right) + \sum_{j=1}^{n_3} \ln \left( \phi_1(x_j; \theta_3, \zeta) \right) + \sum_{j=1}^{n_3} \ln \left( \phi_2(x_j; \theta_3, \zeta) \right) + \sum_{j=1}^{n_3} \ln \left( \phi_3(x_j; \theta_3, \zeta) \right).
\]

(35)

where \( \phi_1(x; \theta, \zeta) = \theta^{x+1-\zeta} - \theta^{-\zeta} \). The MLEs of the model parameters can be obtained by computing the first partial derivatives of Equation (35) with respect to \( \theta_1, \theta_2, \theta_3 \) and \( \zeta \) and then putting the results equal to zero. We get the likelihood equations as in the following form

\[
\frac{\partial L}{\partial \theta_1} = \sum_{j=1}^{n_1} \frac{\theta_3 \phi_2(x_{1j} + 1; \theta_1, \theta_3, \zeta) - \theta_3 \phi_2(x_{1j}; \theta_1, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_1, \theta_3, \zeta)} + \sum_{j=1}^{n_2} \frac{\theta_3 \phi_2(x_{1j} + 1; \theta_1, \theta_3, \zeta) - \theta_3 \phi_2(x_{1j}; \theta_1, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_1, \theta_3, \zeta)} + \sum_{j=1}^{n_3} \frac{\theta_3 \phi_2(x_{1j} + 1; \theta_1, \theta_3, \zeta) - \theta_3 \phi_2(x_{1j}; \theta_1, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_1, \theta_3, \zeta)}
\]

(36)

\[
\frac{\partial L}{\partial \theta_2} = \sum_{j=1}^{n_1} \frac{\phi_2(x_{1j} + 1; \theta_2, \zeta) - \phi_3(x_{1j}; \theta_2, \zeta) - \phi_2(x_{1j}; \theta_2, \zeta)}{\phi_1(x_{1j}; \theta_2, \zeta)} + \sum_{j=1}^{n_2} \frac{\phi_2(x_{1j} + 1; \theta_2, \zeta) - \phi_3(x_{1j}; \theta_2, \zeta) - \phi_2(x_{1j}; \theta_2, \zeta)}{\phi_1(x_{1j}; \theta_2, \zeta)} + \sum_{j=1}^{n_3} \frac{\phi_2(x_{1j} + 1; \theta_2, \zeta) - \phi_3(x_{1j}; \theta_2, \zeta) - \phi_2(x_{1j}; \theta_2, \zeta)}{\phi_1(x_{1j}; \theta_2, \zeta)}
\]

(37)

\[
\frac{\partial L}{\partial \theta_3} = \sum_{j=1}^{n_1} \frac{\phi_2(x_{1j} + 1; \theta_1, \theta_3, \zeta) - \phi_2(x_{1j}; \theta_1, \theta_3, \zeta) - \phi_2(x_{1j} + 1; \theta_1, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_1, \theta_3, \zeta)} + \sum_{j=1}^{n_2} \frac{\phi_2(x_{1j} + 1; \theta_2, \theta_3, \zeta) - \phi_2(x_{1j}; \theta_2, \theta_3, \zeta) - \phi_2(x_{1j} + 1; \theta_2, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_2, \theta_3, \zeta)} + \sum_{j=1}^{n_3} \frac{\phi_2(x_{1j} + 1; \theta_3, \theta_3, \zeta) - \phi_2(x_{1j}; \theta_3, \theta_3, \zeta) - \phi_2(x_{1j} + 1; \theta_3, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_3, \theta_3, \zeta)}
\]

(38)

and

\[
\frac{\partial L}{\partial \zeta} = \sum_{j=1}^{n_1} \frac{\phi_3(x_{1j}; \theta_1, \theta_3, \zeta) - \phi_3(x_{1j} + 1; \theta_1, \theta_3, \zeta)}{\phi_1(x_{1j}; \theta_1, \theta_3, \zeta)} + \sum_{j=1}^{n_2} \frac{\phi_3(x_{2j}; \theta_2, \theta_3, \zeta) - \phi_3(x_{2j} + 1; \theta_2, \theta_3, \zeta)}{\phi_1(x_{2j}; \theta_2, \theta_3, \zeta)} + \sum_{j=1}^{n_3} \frac{\phi_3(x_j; \theta_3, \theta_3, \zeta) - \phi_3(x_j + 1; \theta_3, \theta_3, \zeta)}{\phi_1(x_j; \theta_3, \theta_3, \zeta)}
\]

(39)

where \( \phi_2(x; \theta, \zeta) = x^{-\zeta} \theta e^{-\zeta} - 1 \) and \( \phi_3(x; \theta, \zeta) = x^{-\zeta} \theta e^{-\zeta} - 1 \ln(x) \ln(\theta) \). The MLEs of the parameters \( \theta_1, \theta_2, \theta_3 \) and \( \zeta \) can be obtained by solving the above system of four non-linear equations from Equation (36) to Equation (39). The solution of these equations is not easy to solve, so we need a numerical technique to get the MLEs.
3.2 Simulation results

In this section, we introduce some simulation results to show how the proposed MLE performs for different sample sizes and for different parameter values. So, we have taken two sets of parameter values: $\theta_1 = 0.8, \theta_2 = 0.4, \theta_3 = 0.4, \zeta = 0.5$ and $\theta_1 = 0.6, \theta_2 = 0.25, \theta_3 = 0.3, \zeta = 0.9$. The population parameters are generated using software "Mathcad prime 3" package. The sampling distributions are obtained for different sample sizes $n = [50, 100, 150, 250, 400]$ from $N = 500$ replications. In each case we have generated a random sample from the BDsIW($\theta_1, \theta_2, \theta_3, \zeta$) with the given sample size and the parameter values. Tables 1 and 2 obtain the average estimates (AvE) and the mean squared errors (MSEs) of the different parameters.

| Size | $\theta_1$ | MSE | $\theta_2$ | MSE | $\theta_3$ | MSE | $\zeta$ | MSE |
|------|------------|-----|------------|-----|------------|-----|--------|-----|
| 50   | 0.765      | 0.0311 | 0.424      | 0.0229 | 0.399      | 0.0147 | 0.515   | 0.0038 |
| 100  | 0.770      | 0.0307 | 0.412      | 0.0226 | 0.398      | 0.0129 | 0.504   | 0.0017 |
| 150  | 0.771      | 0.0303 | 0.414      | 0.0215 | 0.399      | 0.0119 | 0.497   | 0.0010 |
| 250  | 0.774      | 0.0299 | 0.413      | 0.0194 | 0.402      | 0.0102 | 0.503   | 0.0005 |
| 400  | 0.788      | 0.0284 | 0.410      | 0.0193 | 0.401      | 0.0100 | 0.499   | 0.0004 |

Table 1. The AvE and MSE values for the BDsIW(0.8, 0.4, 0.4, 0.5).

| Size | $\theta_1$ | MSE | $\theta_2$ | MSE | $\theta_3$ | MSE | $\zeta$ | MSE |
|------|------------|-----|------------|-----|------------|-----|--------|-----|
| 50   | 0.667      | 0.0340 | 0.285      | 0.0321 | 0.295      | 0.0190 | 0.878   | 0.0070 |
| 100  | 0.663      | 0.0326 | 0.283      | 0.0304 | 0.295      | 0.0157 | 0.882   | 0.0055 |
| 150  | 0.661      | 0.0311 | 0.283      | 0.0291 | 0.297      | 0.0137 | 0.884   | 0.0032 |
| 250  | 0.660      | 0.0284 | 0.280      | 0.0212 | 0.293      | 0.0136 | 0.887   | 0.0023 |
| 400  | 0.653      | 0.0202 | 0.279      | 0.0204 | 0.290      | 0.0135 | 0.890   | 0.0017 |

Table 2. The AvE and MSE values for the BDsIW(0.6, 0.25, 0.3, 0.9).

Based on the simulation results, it is observed that as $n$ increases, the MSE decreases. Moreover, the AvE and initial values are approximately equal. So; the MLE can be used quite effectively for data analysis purposes.

3.3 Data analysis

In this section, we explain the experimental importance of BDsIW distribution using two applications to real data sets. In each data, we shall compare the fits of BDsIW distribution with some competitive models. The tested distributions are compared using some criteria namely, the maximized log-likelihood ($-L$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). Further, we can use the Pearson’s chi-square goodness-of-fit test for grouped data to test the goodness of fit of a proposed bivariate distribution. But the sample size must be sufficiently large in order to apply this test. For this reason, we did not use this test in the two data sets analyzed here.

3.3.1 The first data: Football data

This data is reported in Lee and Cha (2015), and it represents a football match score in Italian football match (Serie A) during 1996 to 2011, between ACF Fiorentina($X_1$) and Juventus($X_2$). This data is reported in Table 3.
Table 3. The score data between ACF Fiorentina and Juventus.

| Obs. | Match Date   | $X_1$ | $X_2$ | Obs. | Match Date   | $X_1$ | $X_2$ |
|------|--------------|-------|-------|------|--------------|-------|-------|
| 1    | 25th Oct. 2011 | 1     | 2     | 14   | 16th Feb. 2002 | 1     | 2     |
| 2    | 17th Apr. 2011 | 0     | 0     | 15   | 19th Dec. 2001 | 1     | 1     |
| 3    | 27th Nov. 2010 | 1     | 1     | 16   | 12th May. 2001 | 1     | 3     |
| 4    | 06th Mar. 2010 | 1     | 2     | 17   | 06th Jan. 2001 | 3     | 3     |
| 5    | 17th Oct. 2009 | 1     | 1     | 18   | 21st Apr. 2000 | 0     | 1     |
| 6    | 24th Jan. 2009 | 0     | 1     | 19   | 18th Dec. 1999 | 1     | 1     |
| 7    | 31st Aug. 2008 | 1     | 1     | 20   | 24th Apr. 1999 | 1     | 2     |
| 8    | 02nd Mar. 2008 | 3     | 2     | 21   | 12th Dec. 1998 | 1     | 0     |
| 9    | 07th Oct. 2007 | 1     | 1     | 22   | 21st Feb. 1998 | 3     | 0     |
| 10   | 09th Apr. 2006 | 1     | 1     | 23   | 04th Oct. 1997 | 1     | 2     |
| 11   | 04th Dec. 2005 | 1     | 2     | 24   | 22nd Feb. 1997 | 1     | 1     |
| 12   | 09th Apr. 2005 | 3     | 3     | 25   | 28th Sept. 1996 | 0     | 1     |
| 13   | 10th Nov. 2004 | 0     | 1     | 26   | 23rd Mar. 1996 | 0     | 1     |

We shall compare the fits of BDsIW distribution with some competitive models like BDsE, BDsR, BDsW, bivariate Poisson with minimum operator (BPo$_{\text{min}}$), bivariate Poisson with 3-parameter (BPo-3P), independent bivariate Poisson (IBPo), BDsIE, and BDsIR distributions. Before trying to analyze the data using BDsIW distribution, we fit at first the marginals $X_1$ and $X_2$ separately and the min($X_1, X_2$) on this data. The MLEs of the parameters $\theta$ and $\zeta$ of the corresponding DsIW distribution for $X_1$, $X_2$ and min($X_1, X_2$) are (0.237, 2.798), (0.095, 2.601) and (0.310, 3.103) respectively. Moreover, the $-L$ values are 30.86, 33.73 and 28.02 respectively. Figure 3 shows the estimated PMF plots for the marginals $X_1$, $X_2$ and min($X_1, X_2$) using this data.

![Figure 3](image.png)

Figure 3. The estimated PMF for the marginals $X_1$, $X_2$ and min($X_1, X_2$) using football data set.

From Figure 3, it is clear that DsIW distribution fits the data for the marginals. Now, we fit BDsIW distribution on this data. The MLEs, $-L$, AIC, CAIC, BIC, and HQIC values for the tested bivariate models are reported in Table 4.
Test 1: $H_{01} : \zeta = 1$ (BDsIE) against $H_{11} : \zeta \neq 1$ (BDsIW).

Test 2: $H_{02} : \zeta = 2$ (BDsIR) against $H_{12} : \zeta \neq 2$ (BDsIW).

The likelihood ratio test statistics ($\Lambda$), d.f and p-values for BDsIE and BDsIR distributions are given in Table 5.

From Table 4, it is clear that BDsIW distribution provides a better fit than the other tested distributions, because it has the smallest values among $-L$, AIC, CAIC, BIC and HQIC. Since, BDsIE and BDsIR distributions are special cases from BDsIW distribution. Hence, we want to perform the following two tests:

We can conclude that $H_{01}$ and $H_{02}$ are rejected with 5% level of significance. Hence, BDsIE and BDsIR distributions cannot be used for this data set. So, we prefer BDsIW distribution for analyzing this data. Figure 4 shows the estimated joint PMF for BDsIW, BDsIE and BDsIR distributions using this data, which support the results of Table 5.

| Model      | MLEs          | $-L$ | AIC   | CAIC  | BIC   | HQIC  |
|------------|---------------|------|-------|-------|-------|-------|
| BDsE       | $\hat{\theta}_1 = 0.652, \hat{\theta}_2 = 0.812, \hat{\theta}_3 = 0.713$ | 75.35 | 156.70 | 157.79 | 160.47 | 157.79 |
| BDsR       | $\hat{\theta}_1 = 0.790, \hat{\theta}_2 = 0.872, \hat{\theta}_3 = 0.905$ | 63.99 | 133.98 | 135.07 | 137.75 | 135.07 |
| BDsW       | $\hat{\theta}_1 = 0.807, \hat{\theta}_2 = 0.882, \hat{\theta}_3 = 0.917, \hat{\zeta} = 2.125$ | 63.89 | 133.78 | 134.87 | 137.55 | 134.87 |
| BPo_{min}  | $\hat{\theta}_1 = 1.36, \hat{\theta}_2 = 2.10, \hat{\theta}_3 = 2.27$ | 64.22 | 134.44 | 135.53 | 138.21 | 135.53 |
| BPo-3P     | $\hat{\alpha}_1 = 1.08, \hat{\alpha}_2 = 1.38, \hat{\alpha}_3 = 0.70$ | 64.92 | 135.83 | 136.93 | 139.61 | 136.93 |
| IBPo       | $\hat{\lambda}_1 = 1.08, \hat{\lambda}_2 = 1.38$ | 67.60 | 139.21 | 139.72 | 141.72 | 139.92 |
| BDsIE      | $\hat{\theta}_1 = 0.669, \hat{\theta}_2 = 0.388, \hat{\theta}_3 = 0.514$ | 78.54 | 163.07 | 163.99 | 167.28 | 164.42 |
| BDsIR      | $\hat{\theta}_1 = 0.493, \hat{\theta}_2 = 0.212, \hat{\theta}_3 = 0.561$ | 64.10 | 134.21 | 135.29 | 137.98 | 135.29 |
| BDsIW      | $\hat{\theta}_1 = 0.420, \hat{\theta}_2 = 0.141, \hat{\theta}_3 = 0.587, \hat{\zeta} = 2.738$ | 61.96 | 131.82 | 133.82 | 136.95 | 133.37 |

Table 4. The MLEs, $-L$, AIC, CAIC, BIC, and HQIC values.

Table 5. The $\Lambda$, d.f and p-values.

| Model | $H_0$ | $\Lambda$ | d.f. | p-values |
|-------|-------|-----------|------|----------|
| BDsIE | $\zeta = 1$ | 33.152 | 1 | < 0.01 |
| BDsIR | $\zeta = 2$ | 4.288 | 1 | 0.0384 |

We can conclude that $H_{01}$ and $H_{02}$ are rejected with 5% level of significance. Hence, BDsIE and BDsIR distributions cannot be used for this data set. So, we prefer BDsIW distribution for analyzing this data. Figure 4 shows the estimated joint PMF for BDsIW, BDsIE and BDsIR distributions using this data, which support the results of Table 5.

Figure 4. The estimated joint PMF for BDsIW, BDsIE and BDsIR distributions using football data set.
3.3.2 The second data: Nasal drainage severity score

This data is reported in Davis (2002), and it represents the efficacy of steam inhalation in the treatment of common cold symptoms (0 = no symptoms; 1 = mild symptoms; 2 = moderate symptoms; 3 = severe symptoms). This data is presented in Table 6.

Table 6. Nasal drainage severity score.

| Obs. | Day 1 ($X_1$) | Day 2 ($X_2$) | Obs. | Day 1 ($X_1$) | Day 2 ($X_2$) |
|------|---------------|---------------|------|---------------|---------------|
| 1    | 1             | 1             | 16   | 2             | 1             |
| 2    | 0             | 0             | 17   | 1             | 1             |
| 3    | 1             | 1             | 18   | 2             | 2             |
| 4    | 1             | 1             | 19   | 3             | 1             |
| 5    | 0             | 2             | 20   | 1             | 1             |
| 6    | 2             | 0             | 21   | 2             | 1             |
| 7    | 2             | 2             | 22   | 2             | 2             |
| 8    | 1             | 1             | 23   | 1             | 1             |
| 9    | 3             | 2             | 24   | 2             | 2             |
| 10   | 2             | 2             | 25   | 2             | 0             |
| 11   | 1             | 0             | 26   | 1             | 1             |
| 12   | 2             | 3             | 27   | 0             | 1             |
| 13   | 1             | 3             | 28   | 1             | 1             |
| 14   | 2             | 1             | 29   | 1             | 1             |
| 15   | 2             | 3             | 30   | 3             | 3             |

We shall compare the fits of BDsIW distribution with some competitive models like bivariate Poisson with 4-parameter (BPo-4P), IBPo, BDsE, BDsIE and BDsIR distributions. We fit at first the marginals $X_1$ and $X_2$ separately and the $\min(X_1, X_2)$ on this data. The MLEs of the parameters $\theta$ and $\zeta$ of the corresponding DsIW distribution for $X_1$, $X_2$ and $\min(X_1, X_2)$ are (0.065, 2.505), (0.115, 2.524) and (0.181, 2.699) respectively. Moreover, the $-L$ values are 40.99, 39.83 and 36.68 respectively. Figure 5 shows the estimated PMF plots for the marginals $X_1$, $X_2$ and $\min(X_1, X_2)$ using this data.

![Figure 5](image1.png)  
![Figure 5](image2.png)  
![Figure 5](image3.png)

Figure 5. The estimated PMF for the marginals $X_1$, $X_2$ and $\min(X_1, X_2)$ using nasal drainage severity score.

From Figure 5, it is clear that DsIW distribution fits the data for the marginals. Now, we fit BDsIW distribution on
this data. The MLEs, $-L$, AIC, CAIC, BIC, and HQIC values for the tested bivariate models are reported in Table 7.

| Model | MLEs | $-L$ | AIC | CAIC | BIC | HQIC |
|-------|------|------|-----|------|-----|------|
| BP-4P | $\lambda_1 = 0.262, \hat{\alpha}_1 = 0.165, \lambda_2 = 0.405, \hat{\alpha}_2 = 2.97$ | 77.66 | 163.33 | 164.93 | 168.93 | 164.66 |
| IBP | $\hat{\lambda}_1 = 1.499, \hat{\lambda}_2 = 1.367$ | 92.48 | 190.96 | 191.88 | 195.16 | 192.30 |
| BDsE | $\hat{\theta}_1 = 0.846, \hat{\theta}_2 = 0.792, \hat{\theta}_3 = 0.693$ | 88.00 | 182 | 182.92 | 186.20 | 183.34 |
| BDsIE | $\hat{\theta}_1 = 0.501, \hat{\theta}_2 = 0.622, \hat{\theta}_3 = 0.383$ | 92.48 | 190.96 | 191.88 | 195.16 | 192.30 |
| BDsIR | $\hat{\theta}_1 = 0.262, \hat{\theta}_2 = 0.405, \hat{\theta}_3 = 0.363$ | 78.66 | 163.32 | 164.24 | 167.52 | 164.66 |
| BDsIW | $\hat{\theta}_1 = 0.192, \hat{\theta}_2 = 0.337, \hat{\theta}_3 = 0.360, \hat{\rho} = 2.453$ | $76.51$ | $161.02$ | $162.62$ | $166.62$ | $162.81$ |

From Table 7, it is clear that BDsIW distribution provides a better fit than the other tested distributions. Table 8 shows the $\Lambda$ and p-values for BDsIE and BDsIR distributions using nasal drainage severity score data set.

| Model | $H_0$ | $\Lambda$ | d.f. | p-values |
|-------|------|----------|-----|----------|
| BDIE | $H_0$ | 31.94 | 1 | < 0.01 |
| BDIR | $H_0$ | 4.3 | 1 | 0.0381 |

From Table 8, we can conclude that $H_{01}$ and $H_{02}$ are rejected with 5% level of significance. So, we prefer BDsIW distribution for analyzing this data. Figure 6 shows the estimated joint PMF for BDsIW, BDsIE and BDsIR distributions using this data, which support the results of Table 8.

**Figure 6.** The estimated joint PMF for BDsIW, BDsIE and BDsIR distributions using nasal drainage severity score.

### 4 Conclusions

In this paper, we presented a flexible bivariate discrete distribution called BDsIW distribution. The proposed model has the marginals, which are DsIW distributions. The joint CDF and joint PMF have simple forms; therefore, this new discrete model can be easily used in practice for modelling bivariate discrete data. Some statistical and mathematical properties of the proposed discrete model are studied. Moreover, the simulation results indicated that the MLE works quite satisfactorily and it can be used to estimate the model parameters. Also, we analyzed two real data sets and showed through goodness-of-fit tests that BDsIW distribution works quite well in practice in different fields.

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