Approximate Quantum Fourier Transform with $O(n \log(n))$ T gates

Yunseong Nam,¹,‡ Yuan Su,²† and Dmitri Maslov³∗

¹IonQ, College Park, MD 20740, USA
²Department of Computer Science, Institute for Advanced Computer Studies, and Joint Center for Quantum Information and Computer Science, University of Maryland, College Park, MD 20740, USA
³National Science Foundation, Alexandria, VA 22314, USA

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The ability to implement the Quantum Fourier Transform (QFT) efficiently on a quantum computer enables the advantages offered by a variety of fundamental quantum algorithms, such as those for integer factoring, computing discrete logarithm over Abelian groups, and phase estimation. The standard fault-tolerant implementation of an $n$-qubit QFT approximates the desired transformation by removing small-angle controlled rotations and synthesizing the remaining ones into Clifford+$\mathsf{T}$ gates, incurring the T-count complexity of $O(n \log^2(n))$. In this paper we show how to obtain approximate QFT with the T-count of $O(n \log(n))$. Our approach relies on quantum circuits with measurements and feedforward, and on reusing a special quantum state that induces the phase gradient transformation. We report asymptotic analysis as well as concrete circuits, demonstrating significant advantages in both theory and practice.

I. INTRODUCTION

Quantum Fourier Transform (QFT) is one of the most important operations in quantum computing. It can extract the periodicity encoded in the amplitudes of a quantum state, which is employed by an efficient algorithm for integer number factoring, widely known as Shor’s algorithm [1]. Shor’s integer factoring algorithm can be generalized (while still relying on the QFT) into a polynomial-time algorithm for the discrete logarithm problem over Abelian groups [1]. The importance of the above is witnessed through the threat such algorithms pose to modern public-key cryptosystems, such as the RSA or the ECC. Using the QFT as a subroutine, the eigenphase of a black-box unitary can be estimated up to an arbitrary precision [2], which may be used to estimate quantum amplitudes [3, 4], simulate quantum chemistry/dynamics [5], find the ground state/energy of a Hamiltonian [6], exponentiate unitaries [7], construct fractional powers of the QFT using constantly many copies of the controlled-QFT [7, 8], extract features of the solution of linear systems [9], and more. QFT has also been used in quantum arithmetics [10].

QFT can be implemented approximately by removing all rotation gates with angles smaller than a certain threshold value, resulting in the Approximate QFT (AQFT). In practice, it was shown that it suffices to apply AQFT with $\sim 5.3 \cdot 10^4$ controlled rotation gates to factor 2048-digit numbers (reflecting the de facto key size for today’s standard [11]) with a high expected algorithmic accuracy ($\geq 99.992\%$) [12]. AQFT has been studied extensively in the literature. The robustness of the quantum computer equipped with the AQFT was investigated in detail [13, 17]. A study of the optimal level of the approximation of the AQFT in the presence of certain errors may be found in [18]. Implementation of the QFT and its approximate version over restricted architectures was addressed in [13, 20]. An efficient approximate implementation of the AQFT that harnesses certain quantum hardware features was also investigated [21].

Quantum information is fragile, and it is generally accepted that the implementation of large quantum algorithms must rely on the fault-tolerant computations. Fault tolerance suppresses the errors at the cost of using multiple physical qubits to encode a single logical qubit. Fault tolerant computations must furthermore rely on a quantum gate library consisting of those gates that are constructible fault tolerant. A standard choice for such a computationally universal gate library is Clifford+$\mathsf{T}$. Within known fault tolerance approaches, Clifford gates can generally be implemented with the relative ease (frequently, transversally, i.e., mapping directly into physical-level gates). On the other hand, a non-Clifford gate typically does not admit such an implementation; for instance, a $\mathsf{T}$ gate may be implemented fault tolerant by distilling a certain quantum state and then teleporting it into the gate [22]. A $\mathsf{T}$ gate is indeed far more costly than any of the Clifford gates, and therefore efficient fault-tolerant circuits must minimize the T-count.

To implement an $n$-qubit AQFT fault-tolerantly, the standard approach is to approximate the desired transformation by removing small-angle controlled rotations to bring down the gate count from $O(n^2)$ [24, page 219] to $O(n \log(n))$, and then replace the remaining $O(n \log(n))$ controlled rotations with their Clifford+$\mathsf{T}$ implementations. The resulting
FIG. 1. AQFT with \( n = 6 \) and \( b = 3 \). Note that each of the \( n-1 \) sets of controlled-\( z^a \) gates are separated by the Hadamard gates.

FIG. 2. Ancilla-aided, measurement/feedforward-based fault-tolerant controlled-\( z^a \) gate.

circuit has the \( T \)-count of \( O(n \log^2 n) \). In this paper, we develop a more efficient implementation with the \( T \)-count complexity of \( O(n \log(n)) \), improving over the standard construction by a factor of \( \log n \). Since our implementation is more involved compared to the standard, we also make a separate effort to show that the constant factor blowup and small-order additive terms missing in the asymptotic considerations but otherwise present in our construction do not prevent it from achieving a significant practical advantage.

II. MAIN RESULT

We start with an \( n \)-qubit AQFT whose construction relies on \( O(nb) \) controlled-\( z^a \) gates with

\[
Z^a := \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi a} \end{bmatrix},
\]

where \( a \in \{1/2, 1/4, ..., 1/2^b\} \), for \( b := \lceil \log n \rceil \), and \( n \) Hadamard gates (see Figure 1 for an illustration with \( n = 6 \) and \( b = 3 \)). Such a choice of \( b \) implies a very specific approximation error \( \varepsilon \), whose analysis will be detailed in the coming subsection. We unite the individual controlled rotations into \( n-1 \) sets separated by the Hadamard gates, such as illustrated in Figure 1.

To implement a given controlled-\( z^a \) rotation, we map its real valued degree of freedom into that of the uncontrolled power of Pauli-\( Z \), such as shown in Figure 2. This implementation was developed by combining Kitaev’s trick [2] with Toffoli-measurement construction of Jones [24], over optimized Clifford+\( T \) implementation of the Margolus gate [23, page 183], and our own custom circuit simplifications. Our circuit improves over the one reported in [25, Figure 10] (note that the middle \( T \) gate can be replaced with the \( Z^a \) gate) by 4 \( T \) gates (8 \( \mapsto \) 4), 9 CNOT gates (12 \( \mapsto \) 3), and 1 Hadamard gate (4 \( \mapsto \) 3) at the cost of introducing 1 measurement and 1 classically-controlled controlled-\( z \) operation. Note that the fault tolerant cost of those operations introduced is significantly lower than that of a single \( T \) gate, as the construction of the \( T \) gate itself requires both a measurement and a classically controlled quantum correction [22].
dependence on the approximation error $\varepsilon$.

The circuit will be $O(\varepsilon)$ and the ideal AQFT is

We now group the uncontrolled $z^a$ rotations into one layer (time slice), as shown in Figure 3. This layer induces the transformation that was coined the phase gradient operation in [26]. Such a transformation can be implemented by a $(b + 1)$-qubit adder at the cost of $4b + O(1)$ T gates [27], so long as one has access to a special quantum state $|Sp\rangle := \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{2\pi i j / 2^n} |j\rangle$. The quantum state $|Sp\rangle$ can be reused to induce phase gradient transformations in all $n \cdot 4$ sets of controlled-$z^a$ rotations. A schematic circuit diagram of our AQFT implementation is shown in Figure 4.

To construct the special $(b + 1)$-qubit state $|Sp\rangle$, we first apply H gates to the quantum register $|00...0\rangle$ and then exercise the gates $z, z^{-1/2}, ..., z^{-1/2^n}$. The latter step is accomplished via approximating each $z^a$ by RUS circuits [27]. Specifically, we approximate complex number $e^{i\pi a}$ by $z^a / z$, where $z \in \mathbb{Z}[\omega]$ with $\omega := e^{i\pi/4}$ being the cyclotomic integer obtained from the PSLQ Algorithm [28]. We choose $r \in \mathbb{Z}[\sqrt{2}]$ randomly and search the solution $y \in \mathbb{Z}[\omega]$ of the norm equation $|y|^2 = 2^L - |rz|^2$ with $L = \lceil \log(|rz|^2) \rceil$ [29], such that $V := \frac{1}{\sqrt{2^L}} \left( \begin{smallmatrix} rz & y \\ -y^* & rz^* \end{smallmatrix} \right)$ is a unitary. We exactly synthesize the two-qubit gate $\left( \begin{smallmatrix} V & 0 \\ 0 & V^\dagger \end{smallmatrix} \right)$ into a Clifford+T circuit [27, 30]. Upon measuring the second qubit and obtaining 0, the gate $z^a$ is successfully implemented. Otherwise, a z error takes place and can be reversed at zero cost in the T gate count. The expected number of repetitions until success is $2^L / |rz|^2$. We resorted to using this more complex algorithm as opposed to the simpler one given by [29, 30], as we already use quantum circuits with measurements and feedforward elsewhere in our constructions, and the RUS approach results in about 2.5-fold improvement [27] in the number of the T gates required to obtain the desired $z^a$.

A. Complexity analysis

The total T-count in our implementation of the AQFT is $8nb + O(b \log(b/\varepsilon))$. This is because each of the $nb - b(b + 1)/2 = nb + O(b^2)$ controlled-$z^a$ gates consumes 4 T gates to be first mapped into an uncontrolled $z^a$ and another 4 T gates for the $z^a$ to be implemented as a part of the adder circuit. The construction of the special state $|Sp\rangle$ requires implementation of $O(b)$ $z^a$ rotations, and we approximate each rotation with $O(\log(b/\varepsilon))$ T gates [27] to achieve accuracy $\varepsilon/b$ per rotation.

There are two sources of approximation errors in our construction. Our circuit differs from the ideal AQFT circuit only in the preparation of the special state $|Sp\rangle$. Therefore, the spectral norm distance between our AQFT circuit and the ideal AQFT is $O(b \cdot \varepsilon/b) = O(\varepsilon)$. If we choose $b = O(\log(n/\varepsilon))$, the spectral norm error of the ideal AQFT circuit will be $O(\varepsilon)$. Due to the triangle inequality, the total error can be upper bounded by adding the error of the Clifford+T synthesis and the error of AQFT, which is still $O(\varepsilon)$.

The above error analysis shows that for all effective purposes (specifically, when $\varepsilon > n/2^n$) we can drop the dependence on the approximation error $\varepsilon$, resulting in the claimed T-count of $O(n \log n)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{aqft.png}
\caption{A 4-qubit example of the layer of controlled-$z^a$ gates. The uncontrolled rotations are grouped together to induce the phase gradient operation [26].}
\end{figure}
We implemented our improved fault-tolerant construction as described above in software. We synthesized the RUS circuits for $z^a$ gates with $a \in \{-1/2^1, -1/2^4, \ldots, -1/2^{13}\}$, motivating the choice of the smallest angle $\pi/2^b$ by that sufficient to launch a quantum attack on the classically-infeasible instance of the integer factoring problem corresponding to cracking the RSA-2048. Our implementation can be straightforwardly scaled to generate an $n$-qubit AQFT with arbitrary small rotation angle $\pi a = \pi/2^b$ so as to obtain any desired accuracy at the algorithmic level.

For a concrete comparison with the previous state of the art \cite{25,31} at the gate-by-gate level, we chose the overall error that arises from the gate synthesis to be below $1.1 \cdot 10^{-4}$ for all sizes of the AQFT ($n \leq 4096$ and $b = 13$) we considered. In particular, we chose the error $10^{-5}$ per $z^a$ gate approximation for our improved construction. This amounts to the gate-synthesis error budget of $\sim 10^{-5}/n$ per rotation for the previous state-of-the-art AQFT circuit. The improvement of the accuracy per $z^a$ gate is justified by the fact that our implementation of the AQFT requires the approximation of only $O(b)$ rotations instead of $O(nb)$ in the previous constructions.

Summary of the resulting quantum resource cost estimates of our improved AQFT implementation is shown in Table I. We included comparison of the gate costs of our implementation to those circuits known previously: first set relying on \cite{25, Figure 10} to implement controlled-$z^a$ gates in the AQFT and the second set resulting from an automated AQFT circuit optimization \cite{31}. For both implementations, we used Gridsynth algorithm \cite{29} to synthesize $z^a$ gates. Note that our implementation carries a significant practical advantage, saving quantum resource cost in the form of the T-count by a factor as large as 11 (AQFT$_{4096}$ with $b = 13$). The slight increase in the number of qubits and the CNOT gate costs are completely offset by the savings in the T-count in the fault-tolerant regime.

Our improved implementation of AQFT$_n$ requires $n_q = 3 \min(b, n-1) + n + 1$ qubits, the CNOT gate count of $n-1 \cdot 8 \min(b, l) + 16 \min(b, n-1)$, and the T-count of $n-1 \cdot 2^{b+1}$. When considering $z^{1/2^b}$ gate, we used $C_T = 1$ for both RUS and Gridsynth, $C_{\text{CNOT}}(\text{RUS}_2) = 0$, and $p_2 = 1$.

III. CONCLUSION

Before our contribution, the best known approximation of the $n$-qubit QFT to an error $\epsilon$ by a quantum fault-tolerant Clifford+$\tau$ circuit featured the T-count of $O\left(n \log(n/\epsilon) \log\left(\frac{n \log(n/\epsilon)}{\epsilon}\right)\right)$, with the term $O(n \log(n/\epsilon))$ originating from the standard AQFT construction using controlled rotations, and term $O\left(\log\frac{n \log(n/\epsilon)}{\epsilon}\right)$ coming from the fault-tolerance overhead. In this paper we reported an improved approximation of the QFT by a quantum Clifford+$\tau$ circuit with the T-count of $O\left(n \log(n/\epsilon) + \log(n/\epsilon) \log\left(\frac{n \log(n/\epsilon)}{\epsilon}\right)\right)$. Our improvement is twofold: first, we reduce the dependence on $n$ from $O(n \log^2 n)$ to $O(n \log n)$, and second, we moved the dependence on $\epsilon$ from the leading term into a lower order additive term. This means that the smaller the desired approximation error the more efficient our construction is compared to those known previously.

Our implementation includes constant factor improvements that are not captured by the asymptotics. We report
TABLE 1. Quantum resource counts for implementing an $n$-qubit AQFT with $b = 13$. $n_q$ denotes the number of qubits required to execute the corresponding circuit. Columns CNOT and $T$ report the number of respective gates in the circuits.

| Circuit | $n_q$ | CNOT | $T$ | $n_q$ | CNOT | $T$ |
|---------|-------|------|-----|-------|------|-----|
| AQFT$_8$ | 30 | 548 | 318 | 9 | 336 | 1.083 |
| AQFT$_{16}$ | 56 | 2,160 | 1,257 | 17 | 1,404 | 6,309 |
| AQFT$_{32}$ | 72 | 5,424 | 2,921 | 33 | 3,900 | 19,261 |
| AQFT$_{64}$ | 104 | 11,952 | 6,249 | 65 | 8,892 | 47,999 |
| AQFT$_{128}$ | 168 | 25,008 | 12,900 | 129 | 18,876 | 106,631 |
| AQFT$_{256}$ | 296 | 51,120 | 26,217 | 257 | 38,844 | 229,729 |
| AQFT$_{512}$ | 552 | 103,344 | 52,841 | 513 | 78,780 | 476,873 |
| AQFT$_{1024}$ | 1,064 | 207,792 | 106,089 | 1,025 | 158,652 | 993,727 |
| AQFT$_{2048}$ | 2,088 | 416,688 | 212,585 | 2,049 | 318,396 | 2,084,983 |
| AQFT$_{4096}$ | 4,136 | 834,480 | 425,577 | 4,097 | 637,884 | 4,316,993 |

**Complexity** $O(n) O(n \log(n)) O(n \log(n)) O(n) O(n \log(n)) O(n \log^2(n)) O(n) O(n \log(n)) O(n \log^2(n))$

significant practical advantages from applying our construction, as is evidenced by the numbers in Table 1 showing the improvement by a factor of 9 to 11 in the $T$-count for values of $n$ of the size expected in practical applications of quantum computers. This shows that our result carries both theoretical and practical value.

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