AUTOMORPHISMS OF $\mathbb{C}^k$ AND ASSOCIATED COMPACT COMPLEX MANIFOLDS.

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Abstract. In this paper, we first construct $k$-dimensional compact complex manifolds from automorphisms of $\mathbb{C}^k$ which admit a fixed attracting point at infinity. Then, we characterize the fundamental group as well as the universal covering of the attracting basin of this fixed point thanks to a generalization of the method described by T. Bousch in his thesis [H].

1. Introduction

In the paper [3], the authors give the construction of surfaces which contain a global spherical shell (G.S.S) from the attracting basin of an Hénon automorphism that is an automorphism such that:

$$H : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}$$

The properties that the surfaces satisfy are consequences of results obtained by J. Hubbard and R.Oberste-Vorth in [5]. In this paper, the authors are interested in the dynamical system constituted by an Hénon mapping and the set $\mathbb{C}^2 \setminus K^+$ where

$$K^+ := \left\{ (x, y) \mid H^{\text{even}}(x, y) \not\to \infty \text{ when } n \to \pm \infty \right\}.$$ 

They give a topologic model as the projective limit of tori, then infer a holomorphic model. In his thesis, T. Bousch [H] considers the same dynamical system but tackles the problem in a different way. He finds the same topologic model as a consequence of various calculations among others, asymptotic developments. This method valid in dimension 2 also holds in dimension 3 as shown by the example of the automorphism $H(x, y, z) = (y, z, z^2 + x)$ given in his thesis.

Our aim is on the one hand to generalize to dimension $k$ the previous construction, from $H$ a polynomial automorphism of $\mathbb{C}^k$ of degree $d$ which, once extended to $\mathbb{P}^k$ given in homogeneous coordinates $[z_1 : z_2 : \cdots : z_k : t]$, contracts the hyperplane at infinity $\{t = 0\}$ minus the indeterminacy set onto a fixed attracting point $p$, in order to get a manifold with a global spherical shell, what we do in the first part; on the other hand to establish in a second part the structure of the attracting basin by using asymptotic developments.

In these two parts, we will study the class of automorphisms given by:

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with $\alpha_2 \neq 0$, $|\alpha_2| < d$ and $\alpha_3, \cdots, \alpha_k$ any constant in $\mathbb{C}$.

Let $U^+$ be the attracting basin at infinity of the automorphism we consider, that is the set:

$$U^+ := \{ w \in \mathbb{C}^k, \lim_{n \to \infty} H^n(w) = p \}.$$ 

The result we get is summed up in the following theorem:

**Theorem:**

$$\pi_1 (U^+) \simeq \mathbb{Z} \left[ \frac{1}{d} \right]$$

and $$\tilde{U}^+ = \mathbb{H} \times \mathbb{C}^{k-1}$$

where $\mathbb{H}$ denotes the Poincaré half-plane.

In the third part, we will deal with the particular case of quadratic automorphisms of $\mathbb{C}^3$ which have a fixed attracting point at infinity. The frame of this work is the classification of all these automorphisms into five classes established by Fornaess and Wu [4]. We give the list of all the quadratic automorphisms of $\mathbb{C}^3$ from which we can construct a threefold with a global spherical shell.

2. CONSTRUCTION OF A MANIFOLD WITH A GLOBAL SPHERICAL SHELL.

We can do in dimension $k$ the analogue of the construction of surfaces with global spherical shell as done in [2] and proceed as follows.

We extend the automorphism $H$ to $G$ birational, from $\mathbb{P}^k(\mathbb{C})$ to $\mathbb{P}^k(\mathbb{C})$ given once again in homogeneous coordinates by $[z_1 : \cdots : z_k : t]$.

$$G: \mathbb{P}^k(\mathbb{C}) \longrightarrow \mathbb{P}^k(\mathbb{C})$$

$$[z_1 : \cdots : z_k : t] \longrightarrow \left[ z_1^d + \sum_{j=2}^{k} \alpha_j z_j t^{d-1} : z_3 t^{d-1} : \cdots : z_k t^{d-1} : z_1 t^{d-1} : t^d \right].$$

Denote $I = \{ z_1 = t = 0 \}$ the set of indeterminacy; the hyperplane $\{ t = 0 \}$ minus $I$ is flattened onto the fixed point $p = [1 : 0 : \cdots : 0]$ which is superattracting.

Indeed, the birational mapping $G$ written in coordinates in $U_1 = \{ z_1 \neq 0 \}$, becomes

$$F(\zeta_2, \cdots, \zeta_{k+1}) = \left( \frac{\zeta_3^{d-1}}{D}, \cdots, \frac{\zeta_k^{d-1}}{D}, \frac{\zeta_{k+1}^{d-1}}{D}, \frac{\zeta_{k+1}^{d}}{D} \right)$$
with $D := 1 + \alpha_2 \zeta_2 \zeta_{k+1}^{d-1} + \cdots + \alpha_k \zeta_k \zeta_{k+1}^{d-1}$ and where 
$(\zeta_2, \cdots, \zeta_{k+1}) = \left(\frac{z_2}{z_1}, \cdots, \frac{z_k}{z_1}, \frac{t}{z_1}\right) = (0, \cdots, 0)$ corresponds to the point $p$.

The Jacobian matrix of $F$ at $(0, \cdots, 0)$ is

$$J = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

therefore the point $p$ is superattracting.

The map $F$ also satisfies the following:

**Lemma 2.1.** The germ $F : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}^k, 0)$ can be written as $\pi \eta$ with $\pi$ a composition of several blow-ups above the unit ball $B$ and $\eta$ a local biholomorphism.

**Proof:** Let $\eta$ be the application

$$\eta : (\zeta_2, \cdots, \zeta_k) \mapsto \left(\zeta_3, \cdots, \zeta_k, -\frac{\alpha_2 \zeta_2 - \cdots - \alpha_k \zeta_k}{\zeta_{k+1}}\right)$$

and let $\pi_j$ be the $2d-1$ blow-ups defined in local coordinates by:

$$\pi_1 : (u_1, \cdots, u_k) \mapsto (u_1 u_{k-1}, \cdots, u_{k-2} u_{k-1}, u_{k-1} u_k)$$

$$\pi_2 = \cdots = \pi_d : (u_1, \cdots, u_k) \mapsto (u_1, \cdots, u_{k-2}, u_{k-1} u_k)$$

$$\pi_{d+1} : (u_1, \cdots, u_k) \mapsto (u_1, \cdots, u_{k-2}, u_{k-1} u_k + 1, u_k)$$

$$\pi_{d+2} = \cdots = \pi_{2d-1} : (u_1, \cdots, u_k) \mapsto (u_1, \cdots, u_{k-2}, u_{k-1} u_k, u_k).$$

Let $\pi$ be the composition $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{2d-1}$. Then the germ $F$ can be written as $F = \pi \eta$ and it is easy to check that $\eta$ is a local biholomorphism. □

For a small $\varepsilon > 0$, let $B$ be $B_\varepsilon := \{w \in \mathbb{C}^k | ||w|| < 1 + \varepsilon\}$.

The map $\eta \pi : \pi^{-1}(B_\varepsilon) \rightarrow \pi^{-1}(B_\varepsilon)$ sends biholomorphically a neighbourhood of the boundary $\partial \pi^{-1}(B)$ of $\pi^{-1}(B)$ in $B_\varepsilon$, onto a neighbourhood of the boundary $\partial(\eta(B))$ of $\eta(B)$; by gluing up with $\eta \pi$ these holomorphic neighbourhoods, we get a compact complex manifold of dimension $k$ called $X$, with a global spherical shell.

**Remark 2.2.** Let $\Gamma$ be the maximal divisor of $X$, then $X \backslash \Gamma$ is isomorphic to the quotient $U^+ / \langle H \rangle$.

> From now on, we try to understand the structure of $U^+$, that is the fundamental group $\pi_1(U^+)$ and the universal covering $\tilde{U}^+$ of the attracting basin of the automorphism $H$.  

AUTOMORPHISMS 3
3. Structure of the attracting basin.

3.1. Fundamental group.

First, note that $U^+$ can be written in the form:

$$U^+ = \bigcup_n H^{-n}(V^+)$$

where $V^+ = \{(z_1, \cdots, z_k) \in \mathbb{C}^k \mid |z_1| > \text{Max}\{R, |z_2|, \cdots, |z_k|\}\}$,

where $R >> 0$.

We begin by constructing a holomorphic function which will be useful in what follows:

**Proposition 3.1.** There exists a holomorphic function $\varphi : V^+ \rightarrow \mathbb{C} \setminus \Delta$, with $\Delta$ the unit disk, such that:

1. For all $(z_1, \cdots, z_k) \in V^+$, $\varphi(H(z_1, \cdots, z_k)) = (\varphi(z_1, \cdots, z_k))^d$. (*)

2. $\varphi(z_1, \cdots, z_k) \sim z_1$ when $|z_1| \rightarrow +\infty$ in $V^+$. (**)

**Proof:** For a point $z = (z_1, \cdots, z_k)$, we write $H^{\circ n}(z)$ in the form

$$H^{\circ n}(z) = (f_{1,n}(z), \cdots, f_{k,n}(z)),$$

for $n \in \mathbb{Z}$.

We will show that, for each $n$, we can find a $d^n$th root called $\varphi_n(z)$ of $f_{1,n}(z)$ on $V^+$ and the sequence $\varphi_n(z)$ converges uniformly to the required holomorphic function $\varphi$. From the definition of $H$, we have:

$$f_{1,1} = z_1^d + \alpha_2 z_2 + \cdots + \alpha_k z_k \quad \text{or}$$

$$\frac{f_{1,1}}{z_1^d} = 1 + \alpha_2 \frac{z_2}{z_1^d} + \cdots + \alpha_k \frac{z_k}{z_1^d}.$$  

If $z \in V^+$, then $|f_{1,1}| > R$, hence $1 + \alpha_2 \frac{z_2}{z_1^d} + \cdots + \alpha_k \frac{z_k}{z_1^d}$ admits a logarithm $\beta(z)$ and $f_{1,1} = z_1^d \exp\left(\beta(z)\right)$.

The logarithm is of the form: $\beta(z) = (\alpha_2 z_2 + \cdots + \alpha_k z_k)O\left(\frac{1}{z_1^d}\right)$ and $\beta$ is bounded on $V^+$ because

$$\left|\frac{\alpha_2 z_2 + \cdots + \alpha_k z_k}{z_1^d}\right| < \frac{|\alpha_2||z_2| + \cdots + |\alpha_k||z_k|}{|z_1|^d} < \frac{|\alpha_2| + \cdots + |\alpha_k|}{|R|^{d-1}}.$$  

In the above equation, we replace $z$ by $H^{\circ n-1}(z) = (f_{1,n-1}(z), \cdots, f_{k,n-1}(z))$, and we obtain

$$f_{1,n}(z) = f_{1,1}\left(H^{\circ(n-1)}(z)\right) = f_{1,n-1}(z) \exp\left(\beta(H^{\circ(n-1)}(z))\right).$$  

Using similar relations for $H^{\circ n-2}, H^{\circ n-3}, \cdots$, we have
We can determine \( \phi \) function \( \phi \) function for which the series converges normally since \( \beta \) is bounded on \( V^+ \). The function \( \varphi(z) = \lim_{n \to \infty} \varphi_n(z) \) satisfies the two expected properties since

\[
\varphi_n(H(z)) = (\varphi_{n+1}(z))^d.
\]

We can determine \( \varphi \) more precisely:

\[
\varphi(w) = z_1 \exp \left( \frac{z_2}{z_1} + \cdots + \frac{z_k}{z_1} \right) + \cdots
\]

\[
= z_1 \left( 1 + \alpha_2 O \left( \frac{z_2}{z_1} \right) + \cdots + \alpha_k O \left( \frac{z_k}{z_1} \right) \right)
\]

\[
= z_1 + \alpha_2 O \left( \frac{z_2}{z_1} \right) + \cdots + \alpha_k O \left( \frac{z_k}{z_1} \right).
\]

\( \square \)

**Corollary 3.2.** Let \((z_0, \cdots, z_{-k+1})\) be an element of \( V^+ \), \( U := (\varphi(z_0, \cdots, z_{-k+1}))^d \)

and let the sequences \((z_{1,n})_n, \cdots, (z_{k,n})_n \) be defined by

\[
H^{\circ n}(z_0, \cdots, z_{-k+1}) := (z_{1,n}, \cdots, z_{k,n}).
\]

The sequence \((z_{1,n})_n \) admits as a first estimation the following development:

\[
z_{1,n} = U + O \left( U^{d-k+j-1+1-d} \right) \text{ if } \alpha_{j+1} = \cdots = \alpha_k = 0 \text{ and } \alpha_j \neq 0.
\]

**Proof:** Let \((z_0, \cdots, z_{-k+1})\) be in \( V^+ \) and let us consider:

\[
H^{\circ n}(z_0, \cdots, z_{-k+1}) = (z_{1,n}, \cdots, z_{k,n})
\]

we have

\[
\varphi(z_{1,n}, \cdots, z_{k,n}) = \varphi(H^{\circ n}(z_0, \cdots, z_{-k+1})) \overset{(\ast)}{=} \varphi(z_0, \cdots, z_{-k+1})^d = U.
\]

An equivalent of \( z_{1,n} \) is \( U \) thanks to property \((\ast)\) of the previous proposition. The first estimation is obtained thanks to the function \( \varphi(z) \). \( \square \)

**Proposition 3.3.** There exists a closed 1-form \( \omega \) on \( U^+ \) such that

\[
H^* \omega = d \omega.
\]

**Proof:** We define \( \omega := d(\log \varphi) \) or \( \frac{d \varphi}{\varphi} \) on \( V^+. \) We have :

\[
\varphi^*(H(z)) = \varphi^d(z)
\]

or

\[
H^* \varphi = \varphi^d
\]

by differentiating

\[
H^* d \varphi = d \varphi^{d-1} d \varphi
\]

by dividing (2) by (1)

\[
H^* \frac{d \varphi}{\varphi} = d \frac{d \varphi}{\varphi}
\]

thus

\[
H^* \omega = d \omega \text{ on } V^+.
\]
In general, we define \( \omega := \frac{1}{d^n}(H^n)^*\omega \) sur \( H^{-n}(V^+) \). □

This closed form \( \omega \) allows us to study from now on the fundamental group \( \pi_1(U^+) \) of \( U^+ \).

For a closed curve \( C \) in \( U^+ \), we set

\[
\alpha(C) := \frac{1}{2i\pi} \int_C \omega.
\]

Since \( \omega \) is a closed 1-form, the number \( \alpha(C) \) is determined by the homotopy class of \( C \).

**Proposition 3.4.** For any closed curve \( C \) in \( U^+ \), the following assertions holds:

1. \( \alpha(H(C)) = d \alpha(C) \),
2. \( \alpha(C) \) belongs to \( \mathbb{Z}\left[\frac{1}{d}\right] \),
3. \( C \) is zero-homotopic in \( U^+ \) if and only if \( \alpha(C) = 0 \),
4. for any \( r \in \mathbb{Z}\left[\frac{1}{d}\right] \), there exists a closed curve \( C \) in \( U^+ \) such that \( \alpha(C) = r \).

**Proof:**

1. \( \alpha(H(C)) = \frac{1}{2i\pi} \int_{H(C)} \omega = \frac{1}{2i\pi} \int_C H^*\omega = \frac{1}{2i\pi} \int_C \omega = d \alpha(C) \).

2. Assume \( C \) is in \( V^+ \).

In \( V^+ \), \( \varphi \) is in the form \( \varphi = z_1 e^{\gamma(z)} \) hence \( \frac{d\varphi}{\varphi} = \omega = \frac{dz_1}{z_1} + d\gamma \).

For a closed curve \( C \) in \( V^+ \),

\[
\alpha(C) = \frac{1}{2i\pi} \int_C \omega = \frac{1}{2i\pi} \int_C \frac{dz_1}{z_1} + \frac{1}{2i\pi} \int_C d\gamma
\]

since \( C \) is closed, \( \int_C d\gamma = 0 \), and \( \frac{1}{2i\pi} \int_C \omega = \frac{1}{2i\pi} \int_C \frac{dz_1}{z_1} \).

This vanishes if \( C \) is zero-homotopic in \( V^+ \).

Let \( C \) denote a closed curve in \( U^+ \). There exists an integer \( n \) such that \( H^n(C) \subset V^+ \) that is such that \( C \subset H^{-n}(V^+) \). We know that \( \alpha(H^n(C)) \) is an integer hence

\[
\alpha(C) = \frac{1}{d^n} \alpha(H^n(C)) \quad \text{belongs to} \quad \mathbb{Z}\left[\frac{1}{d}\right].
\]

If \( \alpha(C) = 0 \), then \( \alpha(H^n(C)) = 0 \) and hence \( H^n(C) \) is zero-homotopic in \( V^+ \).

Since \( H^n : H^{-n}(V^+) \rightarrow V^+ \) is a biholomorphism, \( C \) is zero-homotopic in \( H^{-n}(V^+) \).

Let \( C_0 \) denote the closed curve in \( V^+ \) given by:

\[
C_0 : [0, 1] \rightarrow V^+ \quad t \rightarrow (2Re^{2\pi t}, 0, \ldots, 0)
\]
we have
\[ \alpha(C_0) = \frac{1}{2i\pi} \int_{C_0} \omega = 1 \quad \text{et} \quad \alpha(mC_0) = m. \]

Let \( r := \frac{m}{d^n} \in \mathbb{Z}[\frac{1}{d}] \), we choose a closed curve \( C := H^{-n}(mC_0) \); it is such that
\[ \alpha(C) = \alpha(H^{-n}(mC_0)) = \frac{1}{d^n} \alpha(mC_0) = \frac{m}{d^n} = r. \]
This proves the proposition. \( \square \)

The proposition shows that \( \alpha \) is an isomorphism of \( \pi_1(U^+) \) onto \( \mathbb{Z}[\frac{1}{d}] \) and the fundamental group is determined.

**Remark 3.5.** Note that we could have used the Green functions to find \( \varphi \), but the estimation of \( z_{1,n} \) would not have been precise enough for our needs in the following paragraph.

### 3.2. Fundamental group of \( U^+ \).

¿From now on, we are looking for the universal covering of \( U^+ \). We apply the method of the asymptotic development introduced by T.Bousch in his thesis. We have to distinguish two cases according the degree \( d \) equals 2 or is greater than 3.

We show the following proposition:

**Proposition 3.6.** There exists one and only one holomorphic function
\[ G : (\mathbb{C}\setminus \Delta) \times \mathbb{C}^{k-1} \rightarrow \mathbb{C}^k \setminus K^+ = U^+ \]
for which the following properties hold:

1. \( G \) is a locally trivial bundle with discrete countable fibers.

2. Let \( \omega : (\mathbb{C}\setminus \Delta) \times \mathbb{C}^{k-1} \rightarrow (\mathbb{C}\setminus \Delta) \times \mathbb{C}^{k-1} \) be defined by
\[
\omega(v, s_1, \cdots, s_{k-1}) = \left( v^d, \Lambda_1(s_1 + \frac{v^{(d^k-1)d^{k-2}}}{k-1} - \sum_{i=3}^{k} \frac{\alpha_i}{d(k-1)} v^{(-d^{-3} + d^{i+k-4} + (d-1)d^{k-2})}) \right)
, \cdots , \Lambda_{k-1}(s_{k-1} + \frac{v^{(d^k-1)d^{k-2}}}{k-1} - \sum_{i=3}^{k} \frac{\alpha_i}{d(k-1)} v^{(-d^{-3} + d^{i+k-4} + (d-1)d^{k-2})})
\]
(where \( \Lambda_1, \cdots, \Lambda_{k-1} \) are solutions of \( dr^{k-1} + \alpha_2 = 0 \), in the case \( d \geq 3 \);

or, if \( d = 2 \), be defined by
\[ \omega(v, s_1, \cdots, s_{k-1}) = \left( v^2, \Lambda_1(s_1 + \frac{v^{(2^k-1)2^{k-1}}}{k-1} - \sum_{l=3}^k \frac{\alpha_l}{2(k-1)} v(-2^{l-2}+2^{k-3}+2^{k-1}) \right) + \frac{\alpha_3\alpha_k}{4(k-1)} v, \cdots, \Lambda_{k-1}(s_{k-1} + \frac{v^{(2^k-1)2^{k-1}}}{k-1} - \sum_{l=3}^k \frac{\alpha_l}{2(k-1)} v(-2^{l-2}+2^{k-3}+2^{k-1}) + \frac{\alpha_3\alpha_k}{4(k-1)} v) ) \]

(\text{where } \Lambda_1, \cdots, \Lambda_{k-1} \text{ are solutions of } 2 \cdot r^{k-1} + \alpha_2 = 0)

then the following diagram is commutative:

\[
\begin{array}{ccc}
(C \setminus \Delta) \times \mathbb{C}^{k-1} & \overset{\omega}{\longrightarrow} & (C \setminus \Delta) \times \mathbb{C}^{k-1} \\
G \downarrow & & \downarrow G \\
C \setminus K^+ = U^+ & \overset{H}{\longrightarrow} & C \setminus K^+ = U^+
\end{array}
\]

Let us set, for \( n \in \mathbb{Z} \), \((z_{1,n}, z_{2,n}, \cdots, z_{k,n}) := H^n(z_0, z_{-1}, \cdots, z_{-k+1}) \). To study the sequence of the iterates of \((z_0, z_{-1}, \cdots, z_{-k+1})\) under \(H\), it is sufficient to study the sequence \((z_{1,n})_n\) which satisfies an induction formula of the type:

\[ z_{1,n+1} = z_{1,n}^d + \alpha_2 z_{1,n-k+1} + \alpha_3 z_{1,n-k+2} + \cdots + \alpha_k z_{1,n-1}. \]

The idea of the demonstration of the proposition lies on the asymptotic development of \(z_{1,n}\). In what follows, the calculations are purely formal. We will deal with the convergence later. We suppose that \(\alpha_k \neq 0\), if not, we replace \(k\) by \(k-1\) for instance.

We begin with a preliminary lemma:

**Lemma 3.7.** \(z_{1,n}\) admits the following asymptotic development for \(d \geq 3\):

\[
\begin{align*}
z_{1,n} &= \left( \frac{(d-1)d^{k-1}}{U_1 - d^{k-1}} \left[ \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{d} \right)^j \frac{1 - d^k}{U_1 - d^{k-1}} \cdot (d^{1-k})^j \right] \\
&- \sum_{l=3}^k \frac{\alpha_l}{d} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{d} \right)^j \frac{d^{1-k} - d^{2} - (d-1)}{U_1 - d^{k-1}} \cdot (d^{1-k})^j \right) + \cdots
\end{align*}
\]
and for $d = 2$:

$$z_{1,n} = \frac{2^{k-1}}{U^{1-2^{k-1}}} \left[ \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{2} \right)^j U 1 - 2^{k-1} \cdot (2^{1-k})^j \right]$$

$$- \sum_{i=3}^{k} \frac{\alpha_i}{2} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{2} \right)^j \frac{2^{l-k-1} - 2^{l-2} - 1}{1 - 2^{k-1}} \cdot (2^{1-k})^j$$

$$+ \frac{\alpha_3 \alpha_k}{4} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{2} \right)^j \frac{-1}{U \left( 1 - 2^{k-1} \right) \cdot 2^{k-1}} \cdot (2^{1-k})^j \right] + \ldots$$

**Proof:** We prove the lemma for $d$ greater than 3. To establish the asymptotic development of $z_{1,n}$ thanks to the first estimation, we isolate $z_{1,n}$ in the induction formula then we inject the developments of $z_{1,n+1}$, $z_{1,n-1}$, $\cdots$, $z_{1,n-k+1}$ in the relation which gives $z_{1,n}$; we suppose $\alpha_k \neq 0$.

We have:

$$z_{1,n} = \left( z_{1,n+1} - \alpha_2 z_{1,n-k+1} - \cdots - \alpha_k z_{1,n-1} \right)^{\frac{1}{d}}.$$

We refine the first estimation of the asymptotic development given by the corollary. We begin with the estimation of $z_{1,n}$ and we use the fact that from $n$ to $n + 1$, $U$ is changed into $U^d$:

$$z_{1,n} = U + O( U^{\frac{1}{d} - d + 1} )$$

**donc** \( z_{1,n+1} = U^d + O( U^{1-d^2+d} ) \)

$$z_{1,n-1} = U^{\frac{1}{d^2}} + O( U^{\frac{1-d^2-d}{d^2}} )$$

$$\vdots$$

$$\vdots$$

$$z_{1,n-k+2} = U^{\frac{1}{d^k}} + O( U^{\frac{1-d^2+2d}{d^k}} )$$

$$z_{1,n-k+1} = U^{\frac{1}{d^k-1}} + O( U^{\frac{1-d^2+d}{d^k-1}} )$$

thus
\[ z_{1,n} = \left( U^d - \alpha_2 U^{d_{k-1}} - \alpha_3 U^{d_{k-2}} - \cdots - \alpha_k U^{d_{k-1}} + O\left( U^{1-d_2} \right) \right)^{\frac{1}{2}} \]

\[ = U \left( 1 - \alpha_2 U^{1-d_{k-1}} - \alpha_3 U^{1-d_{k-2}} - \cdots \right. \]

\[ - \alpha_k U^{1-d_{k-1}} + O\left( U^{1-d_2} \right) \right) \]

\[ = U \left( 1 - \frac{\alpha_2}{d} U^{1-d_{k-1}} - \frac{\alpha_3}{d} U^{1-d_{k-2}} - \cdots \right. \]

\[ - \frac{\alpha_k}{d} U^{1-d_{k-1}} + O\left( U^{1-d_2} \right) \right) \]

\[ = U - \frac{\alpha_2}{d} U^{1-d_{k-1}} - \frac{\alpha_3}{d} U^{1-d_{k-2}} - \cdots \]

\[ - \frac{\alpha_k}{d} U^{1-d_{k-1}} + O\left( U^{1-d_2} \right) \]

\[ = U - \frac{\alpha_2}{d} U^{1-d_{k-1}} - \frac{\alpha_3}{d} U^{1-d_{k-2}} - \cdots \]

\[ - \frac{\alpha_k}{d} U^{1-d_{k-1}} + O\left( U^{1-d_2} \right) \]

\[ = U - \frac{\alpha_2}{d} U^{1-d_{k-1}} - \frac{\alpha_3}{d} U^{1-d_{k-2}} - \cdots \]

\[ - \frac{\alpha_k}{d} U^{1-d_{k-1}} + O\left( U^{1-d_2} \right) \]

We use this new development to refine a little bit more and we get \( z_{1,n} \) in the form of a series (we are not confronted with the problem of convergence since the calculations are formal):

\[ z_{1,n} = \frac{(d-1)d^{k-1}}{U} \sum_{l=3}^{\infty} \left( -\frac{\alpha_2}{d} \right)^{j} \frac{1 - d^k}{1 - d^{k-1}} \cdot \left( d^{1-k} \right)^j \]

\[ - \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{d} \right)^{j} \frac{d^{l-k-1} - d^{l-2} - (d-1)}{1 - d^{k-1}} \cdot \left( d^{1-k} \right)^j \]

For \( d = 2 \), a new term appears, the one associated with the coefficient \( \alpha_k \) in the development of \( z_{1,n-2} \), this is the reason why a term with a coefficient equal to \( \alpha_3 \alpha_k \) appears. 

\[ \square \]

**Remark 3.8.** The development of \( z_{1,n} \) depends not only on \( U \). Moreover, several sequences \( (z_{1,n}) \) which have the same \( U \), differ by an error which is about

\[ \frac{(d-1)d^{k-1}}{U} \]

To estimate the error, we write once again \( z_{1,n} \):
We take once more the induction formula which gives $z_{1,n}$ and we compare the coefficients.

Up to $\frac{d - 1}{1 - d^{k-1}}$, the coefficients exactly compensate. When one compares the coefficients of $U \cdot \frac{d - 1}{1 - d^{k-1}}$, one obtains the induction formula:

$$d v_n + \alpha_2 v_{n-k+1} = 0.$$  

The sequences which satisfy this relation form a vector space of dimension $k - 1$, generated by the solutions of

$$d r^{k-1} + \alpha_2 = 0,$$

a basis of which is noted $\Lambda_1, \cdots, \Lambda_{k-1}$. We set:

$$z_{1,n} = \frac{(d-1)d^{k-1}}{U \cdot 1 - d^{k-1}} \left[ \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{d} \right)^j \frac{1 - d^k}{U \cdot 1 - d^{k-1}} \cdot (d^{1-k})^j \right]$$

$$- \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{d} \right)^j \frac{d^{l-k-1} - d^{l-2} - (d - 1)}{U \cdot 1 - d^{k-1}} \cdot (d^{1-k})^j \right]$$

$$+ \frac{(d - 1)d^{k-1}}{U \cdot 1 - d^{k-1}} \cdot \left( Q_1 + \cdots + Q_{k-1} \right)$$

where $Q_j := (\Lambda_j)^n q_{j,0}$.  

For $d = 2$, the induction formula which gives the error is

$$2 v_n + \alpha_2 v_{n-k+1} = 0.$$  

The sequences which satisfy this relation form a $(k - 1)$-dimensional vector space generated by the solutions of $2r^{k-1} + \alpha_2 = 0$, we denote $\Lambda_1, \cdots, \Lambda_{k-1}$ a basis. We set
\[ z_{1,n} = \frac{2^{k-1}}{U^{1-2^{k-1}}} \left[ \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{2} \right)^j \frac{1-2^k}{U^{1-2^{k-1}}} \cdot (2^{1-k})^j \right] \]

\[ - \sum_{l=3}^{k} \frac{\alpha_1}{2} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{2} \right)^j \frac{2^{l-k-1} - 2^{l-2} - 1}{1-2^{k-1}} \cdot (2^{1-k})^j \]

\[ + \frac{\alpha_3 \alpha_k}{4} \sum_{j=0}^{\infty} \left( -\frac{\alpha_2}{2} \right)^j \frac{-1}{U^{1-2^{k-1}}} \cdot 2^{k-1} \cdot (2^{1-k})^j \]

\[ + \frac{2^{k-1}}{U^{1-2^{k-1}}} \cdot (Q_1 + \cdots + Q_{k-1}) \text{ where } Q_j := (\Lambda_j)^n q_{j,0}. \]

Once we have established the development of \( z_{1,n} \) as well as the sequences \((Q_1, \cdots, Q_{k-1})\), we need to determinate the universal covering of \( U^+ \). That’s what we do from now on through the resolution of a functional equation. We can already state that \( \widetilde{U^+} = \mathbb{H} \times \mathbb{C}^{k-1} \).

We have just seen that \( z_{1,n} \) only depends on \( k \) parameters \( U, Q_1, \cdots, Q_{k-1} \) which become, when one passes from \( n \) to \( n+1 \), \( (U^d, \Lambda_1 Q_1, \cdots, \Lambda_{k-1} Q_{k-1}) \). This transformation will be used as a model and helps us to find a map:

\[
f : \left( \mathbb{C} \wr \Delta \right) \times \mathbb{C}^{k-1} \rightarrow \mathbb{C}
\]

\[
(u, q_1, \cdots, q_{k-1}) \mapsto f(u, q_1, \cdots, q_{k-1})
\]

such that the sequence \( \left( f(u^d, \Lambda_1^n q_1, \cdots, \Lambda_{k-1}^n q_{k-1}) \right)_n \) satisfies the relation:

\[
\forall (u, q_1, \cdots, q_{k-1}) \in \left( \mathbb{C} \wr \Delta \right) \times \mathbb{C}^{k-1},
\]

\[
f(u^d, \Lambda_1 q_1, \cdots, \Lambda_{k-1} q_{k-1}) = \left( f(u, q_1, \cdots, q_{k-1}) \right)^d + \alpha_2 f \left( u^{\frac{1}{\Lambda_1}}, \frac{q_1}{\Lambda_1}, \cdots, \frac{q_{k-1}}{\Lambda_{k-1}} \right)
\]

\[ + \cdots + \alpha_k f \left( u^{\frac{1}{\Lambda_k}}, \frac{q_1}{\Lambda_1}, \cdots, \frac{q_{k-1}}{\Lambda_{k-1}} \right). \]

We begin with
\[ f_0(u, q_1, \ldots, q_{k-1}) = \frac{(d-1)d^{k-1}}{u \left( 1 - d^{k-1} \right)} \left[ \sum_{j=0}^{\infty} \left( -\frac{\alpha^2}{d} \right)^j \frac{1 - d^j}{u \left( 1 - d^{k-1} \right)} \cdot (d^{1-k})^j \right] \]

\[ - \sum_{l=3}^{k} \frac{\alpha^l}{d} \sum_{j=0}^{\infty} \left( -\frac{\alpha^2}{d} \right)^j \frac{d^j - 1 - d^{j-2} - (d-1) \cdot (d^{1-k})^j}{1 - d^{k-1}} \left( \frac{d}{u} \right)^j \cdot \left( q_1 + \cdots + q_{k-1} \right) \]

and we make several approximations by rewriting the functional equation as follows (this new writing makes the fractional exponents, which are not well defined on \((\mathbb{C}\setminus\mathbb{D})\), disappear):

\[ f(u^{d^k}, \Lambda_1 q_1, \ldots, \Lambda_{k-1} q_{k-1}) = \left( f(u^{d^{k-1}}, \Lambda_1^{k-1} q_1, \ldots, \Lambda_{k-1}^{k-1} q_{k-1}) \right)^d + \alpha_2 f(u, q_1, \ldots, q_{k-1}) + \cdots + \alpha_k f(u^{d^{k-2}}, \Lambda_1^{k-2} q_1, \ldots, \Lambda_{k-1}^{k-2} q_{k-1}). \]

We set

\[ \omega : \ (\mathbb{C}\setminus\mathbb{D}) \times \mathbb{C}^{k-1} \rightarrow \ (\mathbb{C}\setminus\mathbb{D}) \times \mathbb{C}^{k-1} \]

\[ (u, q_1, \ldots, q_{k-1}) \rightarrow (u^d, \Lambda_1 q_1, \ldots, \Lambda_{k-1} q_{k-1}), \]

and we are led to study a simpler functional equation:

\[ f = \frac{1}{\alpha^2} \left[ f \circ \omega^k - (f \circ \omega^{(k-1)})^d - \alpha_3 f \circ \omega - \cdots - \alpha_k f \circ \omega^{(k-2)} \right]. \]

We get a formal series \( f(u, q_1, \ldots, q_{k-1}) \) which satisfies this functional equation, but we cannot cope with the problem of convergence. The formula giving \( f_0 \) with \( u \in (\mathbb{C}\setminus\mathbb{D}) \) et \( (q_1, \ldots, q_{k-1}) \in \mathbb{C}^{k-1} \) has sense as a formal series but converges only if \(|\alpha^2| < d\).

Moreover, in order to obtain a series whose exponents are integer, we gather the fractional exponents by setting, in the case \( d \geq 3 \):
\[ s_1 = q_1 + \frac{1}{k - 1} \sum_{j=1}^{\infty} \Lambda_j^1 \frac{1 - d^k}{1 - d^{k-1}} \cdot d^{-j} \]

\[ - \frac{1}{k - 1} \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=1}^{\infty} \Lambda_j^l \frac{d^{l-k-1} - d^{l-2} - (d - 1)}{1 - d^{k-1}} \cdot d^{-j} \]

\[ \vdots \]

\[ s_{k-1} = q_{k-1} + \frac{1}{k - 1} \sum_{j=1}^{\infty} \Lambda_j^{k-1} \frac{1 - d^k}{1 - d^{k-1}} \cdot d^{-j} \]

\[ - \frac{1}{k - 1} \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=1}^{\infty} \Lambda_j^{k-1} \frac{d^{l-k-1} - d^{l-2} - (d - 1)}{1 - d^{k-1}} \cdot d^{-j} \]

We put the previous expressions of \( q_1, \ldots, q_{k-1} \) into \( z_{1,n} \), this allows us to gather the fractional exponents.

\[ z_{1,n} = \frac{(d - 1)d^{k-1}}{u} \frac{1 - d^k}{1 - d^{k-1}} \left[ \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=1}^{\infty} \Lambda_j^l \frac{d^{l-k-1} - d^{l-2} - (d - 1)}{1 - d^{k-1}} \cdot d^{-j} \right] \]

\[ + \sum_{j=1}^{\infty} \left( -\frac{\alpha_2}{d} \right)^j \frac{1 - d^k}{u} \frac{1 - d^{1-k}}{1 - d^{k-1}} \cdot (d^{1-k})^j \]

\[ - \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=1}^{\infty} \left( -\frac{\alpha_2}{d} \right)^j \frac{d^{l-k-1} - d^{l-2} - (d - 1)}{1 - d^{k-1}} \cdot (d^{1-k})^j \]

\[ + s_1 + \cdots + s_{k-1} - \frac{1}{k - 1} \sum_{j=1}^{\infty} \left( \Lambda_j^1 + \cdots + \Lambda_j^{k-1} \right) \frac{1 - d^k}{u} \frac{1 - d^{k-1}}{1 - d^{k-1}} \cdot d^{-j} \]

\[ + \frac{1}{k - 1} \sum_{l=3}^{k} \frac{\alpha_l}{d} \sum_{j=1}^{\infty} \left( \Lambda_j^l + \cdots + \Lambda_j^{k-1} \right) \frac{d^{l-k-1} - d^{l-2} - (d - 1)}{1 - d^{k-1}} \cdot d^{-j} \]
\[
\begin{align*}
\omega & = \frac{(d-1)d^{k-1}}{u \cdot 1 - d^{k-1}} \left[ \sum_{l=3}^{k} \frac{\alpha_l}{d} \frac{d^{l-k-1} - d^{l-2} - (d-1)}{1 - d^{k-1}} \right] \\
& \quad + \frac{1 - d^k}{u \cdot 1 - d^{k-1}} \left[ \sum_{j=1}^{\infty} \left( -\frac{\alpha_2}{d^j} \right) \frac{d^{j-k-1} - d^{j-2} - (d-1)}{1 - d^{k-1}} \cdot d^{-j} \right]
\end{align*}
\]

The change of variable \( v := \frac{1}{u \cdot (d^{k-1} - 1) \cdot d^{k-2}} \), \( \omega \) becomes

\[
\omega : (\mathbb{C} \setminus \Delta) \times \mathbb{C}^{k-1} \rightarrow (\mathbb{C} \setminus \Delta) \times \mathbb{C}^{k-1}
\]

\[
\omega(v, s_1, \ldots, s_{k-1}) = \left( v^d \cdot \Lambda_1 \left( s_1 + \frac{v(d^{k-1}) \cdot d^{k-2}}{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{d(k-1)} v \left( -d^{l-3} + d^{l+k-4} + (d-1)d^{k-2} \right) \right) \right)
\]

\[
, \ldots, \Lambda_{k-1} \left( s_{k-1} + \frac{v(d^{k-1}) \cdot d^{k-2}}{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{d(k-1)} v \left( -d^{l-3} + d^{l+k-4} + (d-1)d^{k-2} \right) \right)
\]

In variables \( (v, s_1, \ldots, s_{k-1}) \), the function \( f \) renamed \( g \) becomes
$g(v, s_1, \cdots, s_{k-1}) = v^{(d^{k-1}-1)-d^{k-2}} + v^{(1-d)-d^{2k-3}} (s_1 + \cdots + s_{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{d^l} u^l (-d^{l-3} - d^{l+k-4} - (d-1)d^{k-2})) + \cdots$

This provides us with the first term $g_0 = v^{(d^{k-1}-1)-d^{k-2}} + v^{(1-d)-d^{2k-3}} (s_1 + \cdots + s_{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{d^l} u^l (-d^{l-3} + d^{l+k-4} + (d-1)d^{k-2}))$

of a sequence of functions $(g_n)_n$ which is given by the induction formula:

$$g_{n+1} = \frac{1}{\alpha_2} \left[ g_n \circ \omega^k - (g_n \circ \omega^{(k-1)})^d - \alpha_3 g_n \circ \omega - \cdots - \alpha_k g_n \circ \omega^{(k-2)} \right].$$

In the case $d = 2$, by gathering the fractional exponents, we obtain after calculations:

$$z_{1,n} = \frac{2^{k-1}}{u^{1-2k-1}} \left[ s_1 + \frac{1-2^k}{u^{1-2k-1}} + \sum_{l=3}^{k} \frac{\alpha_l}{2(k-1)} u^l \frac{2^{l-k-1} - 2^{l-2} - 1}{1-2k-1} u^{1-2k-1} \right. + \frac{\alpha_3 \alpha_k}{4(k-1)} u^{(1-2k-1) \cdot 2k-1} + \cdots + s_{k-1} + \frac{1-2^k}{u^{1-2k-1}} \left. - \sum_{l=3}^{k} \frac{\alpha_l}{2(k-1)} u^l \frac{2^{l-k-1} - 2^{l-2} - 1}{1-2k-1} + \frac{\alpha_3 \alpha_k}{4(k-1)} u^{(1-2k-1) \cdot 2k-1} \right]$$

With the change of variable $v := u^{2^{k-1} - 1} \cdot d^{k-1}$, $\omega$ becomes

$$\omega : (\mathbb{C} \setminus \Delta) \times \mathbb{C}^{k-1} \rightarrow (\mathbb{C} \setminus \Delta) \times \mathbb{C}^{k-1}$$

$$\omega(v, s_1, \cdots, s_{k-1}) = \left( v^2, \Lambda_1 \left( s_1 + \frac{v(2^k-1)-2^{k-1}}{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{2(k-1)} v \left( -2^{l-2} + 2^{l+k-3} + 2^{k-1} \right) \right) \right. + \frac{\alpha_3 \alpha_k}{4(k-1)} v^{(1-2k-1) \cdot 2k-1} + \cdots + \Lambda_{k-1} \left( s_{k-1} + \frac{v(2^k-1)-2^{k-1}}{k-1} \right) - \sum_{l=3}^{k} \frac{\alpha_l}{2(k-1)} v \left( -2^{l-2} + 2^{l+k-3} + 2^{k-1} \right) + \frac{\alpha_3 \alpha_k}{4(k-1)} v^{(1-2k-1) \cdot 2k-1} \left. \right)$$

In variables $(v, s_1, \cdots, s_{k-1})$, the function $f$ renammed $g$ becomes in the case $d = 2$. 

Let $g(v, s_1, \ldots, s_{k-1}) = v^{(2^{k-1} - 1)2^{k-1}} + v^{-2^{2k-2}}(s_1 + \cdots + s_{k-1}) - \frac{\alpha_t}{2}v^{-(2^{l-2} - 2^l + k - 3 - 2^{k-1})} + \frac{\alpha_3\alpha_k}{4}v + \cdots$.

This provides us with the first term

$$g_0 = v^{(2^{k-1} - 1)2^{k-1}} + v^{-d_{2k-2}}(s_1 + \cdots + s_{k-1}) - \frac{\alpha_t}{2}v^{-d_{2k-2}}(2^{l-2} + 2^{l+k-3} + 2^{k-1}) + \frac{\alpha_3\alpha_k}{4}v$$

of a sequence of functions $(g_n)_n$ given by the induction formula:

$$g_{n+1} = \frac{1}{\alpha_2}[g_n \circ \omega^{o(k)} - (g_n \circ \omega^{o(k-1)})^2 - \alpha_3g_n \circ \omega - \cdots - \alpha_kg_n \circ \omega^{o(k-2)}].$$

The sequence $(g_n)_n$ converges uniformly towards the function $g$ on a domain of the kind:

$$\begin{cases} 
|v| \geq 1 + \varepsilon \\
|s_1| \leq K_1|v|(d^{k-1} - 1)\cdot d^{k-2} \\
\vdots \\
|s_{k-1}| \leq K_{k-1}|v|(d^{k-1} - 1)\cdot d^{k-2}
\end{cases}$$

or

$$\begin{cases} 
|v| \geq 1 + \varepsilon \\
|s_1| \leq K_1|v|(2^{k-1} - 1)\cdot 2^{k-1} \\
\vdots \\
|s_{k-1}| \leq K_{k-1}|v|(2^{k-1} - 1)\cdot 2^{k-1}
\end{cases}$$

Let us set

$$G = \begin{bmatrix}
g \circ \omega^{o(k-1)} \\
g \\
g \circ \omega \\
\vdots \\
g \circ \omega^{o(k-2)}
\end{bmatrix}$$
then $G$ is such that: $G \circ \omega = H \circ G$.

Let us give now the proof of the proposition. We only give it in the case $d \geq 3$. The proof would be almost the same for $d = 2$.

**Proof:** Let $(z_{1,n})_n$ and $(\zeta_{1,n})_n$ be two sequences which tend to infinity and are such that:
\[
\begin{cases}
  z_{1,n+1} = z_{1,n}^d + \alpha_2 z_{1,n-k+1} + \cdots + \alpha_k z_{1,n-1} \\
  \zeta_{1,n+1} = \zeta_{1,n}^d + \alpha_2 \zeta_{1,n-k+1} + \cdots + \alpha_k \zeta_{1,n-1}
\end{cases}
\]
then either $z_{1,n}$ and $\zeta_{1,n}$ do not have the same $\varphi$ and in this case they have nothing to do one another, or they have the same $\varphi$ and in this case,
\[
z_{1,n} - \zeta_{1,n} \sim \text{cste} \left(\frac{(d-1)\alpha_{k-1}}{1 - d^{k-1}}\right) \sim \text{cste} \left(\frac{(d-1)\alpha_{k-1}}{1 - d^{k-1}}\right).
\]

We are led to set the following definitions:

**Definition 3.9.** Let $(z_{1,n})_n$ be a sequence such that $z_{1,n}$ tends to infinity with $n$ and the quotient $\frac{z_{1,n+1}}{z_{1,n}} \to +\infty$.

i. $(z_{1,n})_n$ is said to be class $(k, d)$ if for any $n$ in $\mathbb{Z}$, it satisfies
\[
-z_{1,n+1} + z_{1,n}^d + \alpha_2 z_{1,n-k+1} + \cdots + \alpha_k z_{1,n-1} = 0.
\]

ii. $(z_{1,n})_n$ is said to be almost class $(k, d)$ if for any $n$, it satisfies:
\[
-z_{1,n+1} + z_{1,n}^d + \alpha_2 z_{1,n-k+1} + \cdots + \alpha_k z_{1,n-1} = oo\left(\frac{1}{z_{1,n}}\right)
\]
where $u_n := oo(v_n) \iff \sum_{n_0}^{\infty} \left|\frac{u_n}{v_n}\right| < +\infty$ for $n_0 >> 1$.

**Definition 3.10.** Let $(z_{1,n})_n$ and $(\zeta_{1,n})_n$ be almost class $(k, d)$. The two sequences are:

- neighbour if $z_{1,n} - \zeta_{1,n} = O\left(\frac{1}{z_{1,n}}\right)$,
- twin if $z_{1,n} - \zeta_{1,n} = o\left(\frac{1}{z_{1,n}}\right)$.

These are equivalence relations.

**Proposition 3.11.** Let $(z_{1,n})_n$ be an almost class $(k, d)$ sequence. Then there exists one and only one sequence $(y_{1,n})_n$ class $(k, d)$ twin with respect to $(z_{1,n})_n$. 

We introduce the distance $j_{no}(z_{1,n}, \zeta_{1,n}) = \sup_{n \geq n_0} |z_{1,n} - \zeta_{1,n}| \frac{(d-1)\alpha_{k-1}}{z_{1,n}^{d-1} - 1}$.
The symmetry follows from the fact we have an equivalence relation. The distance equals 0 only when \((z_{1,n})_n\) and \((\zeta_{1,n})_n\) are equal from a certain rank but it does not satisfy the triangular inequality because of the non-linearity with respect to \(d_{k-1} \frac{(d-1)d^{k-1}}{d^k-1} - 1\).

Let \(Z = (z_{1,n})\) be an almost class \((k, d)\) sequence. We associate \(F(Z) = \zeta_{1,n}\) where

\[
\zeta_{1,n} = \frac{1}{\alpha_2} (z_{1,n+k} - z_{1,n+k-1}^{d} - \alpha_3 z_{1,n+1} - \cdots - \alpha_k z_{1,n-k+2})
\]

The fixed points of \(F\) are class \((k, d)\) sequences. \(Z_p\) are almost class \((k, d)\) and twin. There remains to see that \((Z_p)\) converges uniformly on a domain of the kind \([n_0, \infty)\).

\[
\sum_{l=p}^{p+v-1} z_{1,l}^{d-1} (z_{1,l}^{d} + \alpha_2 z_{1,l-k+1} + \cdots + \alpha_k z_{1,l-1} - z_{1,l+1}) < \infty
\]

hence \((Z_p)\) is a Cauchy sequence for \(j_{n_0}\) so it converges uniformly for \(n \geq n_0\) towards a class \((k, d)\) sequence for \(n \geq n_0\).

**Surjectivity of \(G\).**

If \((z_{1,n})\) is a class \((k, d)\) sequence then there exists \((v, s_1, \cdots, s_{k-1}) \in \mathbb{C} \setminus \Delta \times \mathbb{C}^{k-1}\) such that \(\mu_n = g_0 \circ \omega^{n}(v, s_1, \cdots, s_{k-1})\) is twin with respect to \((z_{1,n})\).

We can suppose

\[
\begin{pmatrix}
  z_{1,0} \\
  z_{1,-1} \\
  \vdots \\
  z_{1,-k+1}
\end{pmatrix} \in V^+ \text{ and take } \varphi
\begin{pmatrix}
  z_{1,0} \\
  z_{1,-1} \\
  \vdots \\
  z_{1,-k+1}
\end{pmatrix} = v^{(d^{k-1}-1)d^{k-2}}
\]

\[
\omega(v, 0, \cdots, 0) = \left( v^d, \Lambda_1 \left( \frac{v^{(d-1)d^{k-2}}}{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{d(k-1)} v^{(-d^{l-3}+d^{l+k-4}+(d-1)d^{k-2})} \right) \right)
\]

\[
\cdots, \Lambda_{k-1} \left( \frac{v^{(d-1)d^{k-2}}}{k-1} - \sum_{l=3}^{k} \frac{\alpha_l}{d(k-1)} v^{(-d^{l-3}+d^{l+k-4}+(d-1)d^{k-2})} \right)
\]

and by iterating, we obtain
\[ \omega^{on}(v, 0, \ldots, 0) = \left( v^{d^n}, \frac{1}{k-1} \sum_{j=1}^{n} \Lambda_j \left( v^{(d^k-1) \cdot d^{k-2} \cdot d^{n-j}} - \sum_{l=3}^{k} \frac{\alpha_l}{d} \left( -d^{l-3} + d^{l+k-4} + (d-1)d^{k-2} \right) \cdot d^{n-j} \right) \right) \]

\]

Let us set

\[ \chi_n := g_0 \circ \omega^{on}(v, 0, \ldots, 0) \]

then, \((z_{1,n} - \chi_n) \cdot \chi_n^{(d-1) \cdot d^{2k-3}}\) converges to \(w\).

Let now \(\mu_n\) be \(\mu_n := g_0 \circ \omega^{on}(v, w, \ldots, w)\).

The sequences \((z_{1,n})\) and \((\mu_n)\) are twin because

\[ z_{1,n} - \mu_n = O \left( \frac{(d-1)d^{k-1}}{1 - d^{k-1}} \right) \]

hence \(G(u, w, \ldots, w) = (z_{1,0}, z_{1,-1}, \ldots, z_{1,-k+1})\). The map \(G\) is surjective.

As for the fibers: if \((v, s_1, \ldots, s_{k-1})\) and \((v', s'_1, \ldots, s'_{k-1})\) are such that for any \(n \geq 0\), \(\omega^{on}(v, s_1, \ldots, s_{k-1}) \neq \omega^{on}(v', s'_1, \ldots, s'_{k-1})\) then the sequences \(g_0 \circ \omega^{on}(v, s_1, \ldots, s_{k-1})\) and \(g_0 \circ \omega^{on}(v', s'_1, \ldots, s'_{k-1})\) are not twin and consequently

\[ G(v, s_1, \ldots, s_{k-1}) \neq G(v', s'_1, \ldots, s'_{k-1}) \]

hence the fiber of \(G(v, s_1, \ldots, s_{k-1})\) is made of \(k\)-uplets \((v', s'_1, \ldots, s'_{k-1})\) where \(v'\) is a \(d^n\)-th root of unity and where for \(i\) from 1 to \(k-1\):

\[ \Lambda_i s_i + \sum_{j=1}^{n} \frac{\Lambda_j}{k-1} \left( v^{(d^k-1) \cdot d^{k-2} \cdot d^{n-j}} - \sum_{l=3}^{k} \frac{\alpha_l}{d} \left( -d^{l-3} + d^{l+k-4} + (d-1)d^{k-2} \right) \cdot d^{n-j} \right) \]

\[ = \Lambda_i s'_i + \sum_{j=1}^{n} \frac{\Lambda_j}{k-1} \left( v'^{(d^k-1) \cdot d^{k-2} \cdot d^{n-j}} - \sum_{l=3}^{k} \frac{\alpha_l}{d} v' \left( -d^{l-3} + d^{l+k-4} + (d-1)d^{k-2} \right) \cdot d^{n-j} \right) \]

or as well
\[
\Delta_i (v, v') := s'_i - s_i
\]
\[
= \sum_{j=1}^{n} \frac{\Lambda_i^{j-n}}{k-1} \left( v^{(d^k-1)-d^{k-2},d^{n-j}} - v'^{(d^k-1)-d^{k-2},d^{n-j}} \right)
- \sum_{l=3}^{k} \frac{\alpha_l}{d} v^{(-d^{l-3}+d^l+k-4+(d-1)d^k-2),d^{n-j}}
+ \sum_{l=3}^{k} \frac{\alpha_l}{d} v'^{(-d^{l-3}+d^l+k-4+(d-1)d^k-2),d^{n-j}}
\]
\[
= \sum_{m=0}^{n-1} \frac{\Lambda_i^{-m}}{k-1} \left( v^{(d^k-1)-d^{k-2},d^m} - v'^{(d^k-1)-d^{k-2},d^m} \right)
- \sum_{l=3}^{k} \frac{\alpha_l}{d} v^{(-d^{l-3}+d^l+k-4+(d-1)d^k-2),d^m}
+ \sum_{l=3}^{k} \frac{\alpha_l}{d} v'^{(-d^{l-3}+d^l+k-4+(d-1)d^k-2),d^m}
\]

finally, the fiber is countable.

What does it look like? The term \(v'\) describes a dense part of the cercle with a radius equal to \(|v|\). The fiber is discrete. Indeed, when \(v' = e^{2\pi p \cdot d^{-n}} \cdot v\) with a very big \(n\), \(\Delta_i (v, v')\) is very big (if \(p\) is such that \(2p \cdot d^{-n} \cdot (d^k - 1) \cdot d^{k-2} \cdot d^{n-1}\) is odd).

\[
\Delta_i (v, v') \quad \text{big, } p \text{-chosen } \quad \frac{\Lambda_i^{1-n}}{k-1} \left( v^{(d^k-1)-d^{k-2},d^{n-1}} - v'^{(d^k-1)-d^{k-2},d^{n-1}} \right)
\]
\[
\sim \quad 2\frac{\Lambda_i^{1-n}}{k-1} \cdot v^{(d^k-1)-d^{k-2},d^{n-1}}
\]

The action of \(\mathbb{Z} \left[ \frac{1}{d} \right] / \mathbb{Z}\) defined by:

\[
\mathcal{F}: \quad \left[ \frac{1}{d} \right] / \mathbb{Z} \times \left( \mathbb{C} \setminus \Delta \right) \times \mathbb{C}^{k-1} \rightarrow \left( \mathbb{C} \setminus \Delta \right) \times \mathbb{C}^{k-1}
\]
\[
(\theta, v, s_1, \cdots, s_{k-1}) \quad \mapsto \quad \mathcal{F} \left( \theta, v, s_1, \cdots, s_{k-1} \right)
\]

\[
\left\{ \begin{array}{l}
\quad \quad \quad v' = v \cdot e^{2\pi i \theta} \\
\quad \quad \quad s'_i = s_i + \Delta_i (v, v')
\end{array} \right.
\]

whose orbits are the fibers of \(G\), is free and effective.

The fibers are homeomorphic to \(\mathbb{Z} \left[ \frac{1}{d} \right] / \mathbb{Z}\).

The bundle is locally trivial as a consequence of \(\varphi\) being locally defined. We will deal later in the appendix the problems of convergence. \qed
Finally, we have established the following theorem:

**Theorem 3.12.**

\[ \pi_1(U^+) \simeq \mathbb{Z} \left\lfloor \frac{1}{d} \right\rfloor \]

\[ \tilde{U}^+ = \mathbb{H} \times \mathbb{C}^{k-1} \]

where \( \mathbb{H} \) denotes the Poincaré half-plane.

**Remark 3.13.** The attraction basin \( U^+ \) is a domain of holomorphy. Moreover, thanks to the automorphisms obtained from the action of \( \pi_1(U^+) \) on \( \mathbb{H} \times \mathbb{C}^{k-1} \), we note that the action of \( \mathbb{C}^{k-1} \) on \( \mathbb{H} \times \mathbb{C}^{k-1} \) induces an action of \( \mathbb{C}^{k-1} \) on \( U^+ \). The orbits of this action give a foliation of codimension 1 on \( U^+ \).

### 3.3. Appendix.

In this part, we want to solve the problems of convergence which have appeared in our proofs. In the proof of the lemma 3.7, we said that the sequence of functions defined by \( g_0 \) and the induction relation given by:

\[ g_{n+1} = \frac{1}{\alpha_2} \left[ g_n \circ \omega^k - (g_n \circ \omega^{(k-1)}) d - \alpha_3 g_n \circ \omega - \cdots - \alpha_k g_n \circ \omega^{(k-2)} \right] \]

was formally convergent. We have to explicit the notion of convergence.

Let \( K \) be a field or an integral ring. We set \( K \lceil X \rceil \) the space of the formal linear combinations \( \sum_{\kappa \in A} k_\kappa X^\kappa \) where \( A \) is a closed part of \( \mathbb{R} \), lowerly bounded and discrete on the right that is to say

\[ \forall \kappa \in A , \exists \varepsilon > 0 , \exists \kappa , \kappa + \varepsilon \cap A = \emptyset. \]

We can define the sum and the product of two elements of \( K[X] \) providing this latter with a structure of \( K \)-algebra. We could also define \( K[X] \) by identifying \( K[X^{-1}] \) and \( K[X] \).

Let \( f \) be in \( K[X], f \neq 0 \).

Let us note \( f = \sum_{\kappa \in A} k_\kappa X^\kappa \) then the number \( \min\{\kappa \in \mathbb{R}, k_\kappa \neq 0\} \) exists and is finite. We call it \( v_X(f) \) and we set \( v_X(0) = +\infty \).

By extending this notion, \( v_X(f) = \frac{1}{\kappa} v_X(f) \). \( v_X \) has the properties of a valuation.

We can define a norm on \( K[X] \) given by: \( ||f|| = \exp[-v_X(f)] \); it is a norm because

\[ ||f|| = 0 \iff -v_X(f) = -\infty \iff v_X(f) = \infty \iff f = 0 \]

as for the triangular inequality, it is a consequence of the convexity of the function exponential. \( K[X] \) with this norm is complete.

Moreover, we have the equality: \( v_X(fg) = v_X(f) + v_X(g) \) as a consequence of \( K \) being integral. Finally, \( K[X] \) is integral.

\[ ||fg|| = \exp[-v_X(fg)] = \exp[-v_X(f)] - v_X(g) \]

\[ = \exp[-v_X(f)] \exp[-v_X(g)] = ||f|| ||g|| \]
If $K$ is a field, so is $K[X]$ and $x \mapsto x^{-1}$ is continuous.

In the space $\mathbb{C}[s_1, \ldots, s_{k-1}][u^{-1}] = \mathbb{C}[s_1, \ldots, s_{k-1}][u]$, the formula which defines $g_0$ takes sense and the functions $g_n$ are defined by the induction formula. We show that

$$g_1 = g_0 + o\left(u^{\frac{(d-1)d^{k-1}}{d^k-1}-1}\right)$$

and consequently

$$g_1 = g_0 + O\left(u^{\frac{(d-1)d^{k-1}}{d^k-1}+\varepsilon}\right).$$

We transform the sequence into a series by setting $\Delta_k = g_k - g_{k-1}$. Then, we have:

$$\Delta_{k+1} = g_{k+1} - g_k$$

$$= \frac{1}{\alpha_2}\left[g_k(u^d) - (g_k(u^{d^k-1}))^d - \alpha_3 g_k(u^d) - \cdots - \alpha_k g_k(u^{d^{k-2}})\right]$$

$$- \frac{1}{\alpha_2}\left[g_{k-1}(u^d) - (g_{k-1}(u^{d^{k-1}}))^d - \alpha_3 g_{k-1}(u^d) - \cdots - \alpha_k g_{k-1}(u^{d^{k-2}})\right]$$

$$= \frac{1}{\alpha_2} \left[\Delta_k(u^d) - \Delta_k(u^{d^{k-1}})(g_{k-1}^{d-1} + \cdots + g_{k-1}^{d-1}) - \alpha_3 \Delta_k(u^d) - \cdots - \alpha_k \Delta_k(u^{d^{k-2}})\right].$$

We prove by induction that $v_{u-1}(\Delta_k) \geq \frac{(d-1)d^{k-1}}{d^k-1} + d^k\varepsilon$.

One has $v_{u-1}(g_k^{d-1} + \cdots + g_{k-1}^{d-1}) = 1 - d$.

$$v_{u-1}(\Delta_k(u^d)) = d^k\left(\frac{(d-1)d^{k-1}}{d^k-1} + d^k\varepsilon\right) > \frac{(d-1)d^{k-1}}{d^k-1} + d^k\varepsilon.$$

$$v_{u-1}\left((g_k^{d-1} + \cdots + g_{k-1}^{d-1})\Delta_k(u^{d^{k-1}})\right) = v_{u-1}(g_k^{d-1} + \cdots + g_{k-1}^{d-1}) + v_{u-1}(\Delta_k(u^{d^{k-1}}))$$

$$> \frac{(d-1)d^{k-1}}{d^k-1} + d^k\varepsilon.$$

$$v_{u-1}(\Delta_k(u^d)) = d\left(\frac{(d-1)d^{k-1}}{d^k-1} + d^k\varepsilon\right)\left(\frac{(d-1)d^{k-1}}{d^k-1} + d^k\varepsilon\right).$$

The series $\Delta_k$ is convergent because the general term tends to 0 thus so does the sequence of $(g_k)$. It converges towards an element of $\mathbb{C}[s_1, \ldots, s_{k-1}][u^{-1}]$. □

4. The Case of Dimension $3$.

Now, we deal with the particular case when the dimension $k$ equals 3 and the degree $d$ equals 2. The quadratic automorphisms have been classified by Fornaess and Wu [4]. We are able to list all the automorphisms of this classification for which the method described before applies.

**Theorem 4.1.** The quadratic automorphisms of $\mathbb{C}^3$ in the list of Fornaess-Wu [4] for which the method described previously works are:
(1) in the first class, those of the form

\[ H_1(x, y, z) = \begin{cases} 
\alpha x^2 + bx + ay + \gamma \\
x + \varepsilon \\
\nu z 
\end{cases} \]

with \( a \neq 0, |a| < 2, \nu \neq 0, \alpha \neq 0 \) and \( b, \gamma, \varepsilon \) any constant.

For this class, there are also those of the form

\[ H_1(x, y, z) = \begin{cases} 
ax^2 + ay + p_1(z)x + p_2(z) \\
\varepsilon + x, a \neq 0, |a| < 2, \alpha \neq 0 \\
z 
\end{cases} \]

\( p_1 \) and \( p_2 \) are polynomials with degrees \( \text{deg}(p_1) \leq 1, \text{deg}(p_1) \leq 2 \).

(2) In the second class, those of the form

\[ H_2(x, y, z) = \begin{cases} 
ax + P(y, z) \\
\alpha y^2 + \beta y + bz + c, a \neq 0, \alpha \neq 0, b \neq 0, |b| < 2 \\
y 
\end{cases} \]

\( P \) is a polynomial with a degree smaller than 2, \( \beta \) and \( c \) are any constant.

(3) In the third class, those of the form

\[ H_3(x, y, z) = \begin{cases} 
\alpha x^2 + \mu x + \nu z + ay + \delta \\
\varepsilon x + z + \rho, a \neq 0, |a| < 2, \alpha \neq 0 \\
x 
\end{cases} \]

(4) In the fourth class, those of the form

\[ H_4(x, y, z) = \begin{cases} 
\alpha x^2 + \gamma x + \gamma' y + az + \delta \\
x + \rho, a \neq 0, |a| < 2, \alpha \neq 0 \\
y 
\end{cases} \]

there is no condition on the other constants.

There are also those of the form:
AUTOMORPHISMS

\[
H_4(x, y, z) = \begin{cases} 
\gamma y + az + \varepsilon \\
\alpha y^2 + \nu y + x + \delta \\
y
\end{cases}, \ a \neq 0, \ |a| < 2, \ \alpha \neq 0
\]

There is no condition on the other constants.

(5) In the fifth class, those of the form

\[
H_5(x, y, z) = \begin{cases} 
\alpha x^2 + \nu x + \delta + az \\
\beta x^2 + \gamma x + \rho + by \\
x
\end{cases}, \ a \neq 0, \ |a| < 2, \ \alpha \neq 0
\]

There is no condition on the other constants.

Each of these automorphisms admits an attracting fixed point at infinity whose attracting basin \(U^+\) is such that:

the fundamental group \(\pi_1(U^+) \simeq \mathbb{Z} \left[ \frac{1}{2} \right]\)

and his covering \(\tilde{U}^+ \simeq \mathbb{H} \times \mathbb{C}^2\)

where \(\mathbb{H}\) denotes the Poincaré half-plane.

Proof: Let us set, for \(n \in \mathbb{Z}\), \((x_n, y_n, z_n) := H^n(x, y, z)\). In order to study the sequence of the iterates of \((x, y, z)\) by \(H\), all we need is to study one of the sequences \((x_n)_n\), \((y_n)_n\) or \((z_n)_n\). If the induction relation that gives the \(n+1\)-th indexed term only depends on the \(n\)-th and \(n-1\)-th ones, we will say that the automorphism can be considered as an Hénon. Otherly, the induction formula depends on the terms of indexes \(n, n-1\) and \(n-2\).

We distinguish two cases. The automorphisms that can be considered as Hénon ones and that can be dealt with, in the same way as in T.Bousch’s thesis \(\mathbb{H}\); the others that can be dealt with in the same way as what precedes.

As far as the automorphisms that can be considered as Hénon are concerned, there are, in the previous list, those of the 1st, 2nd and 5th classes. Those of the 3rd and 4th classes belong to the second category. \(\square\)

Remark 4.2. We have to notice that those automorphisms satisfy the property 3.6.
In particular, \(\omega\) is given explicitly.

We end up with a question:

Question: Let \(f\) be \(f: \mathbb{C}^k \to \mathbb{C}^k\) a polynomial automorphism of \(\mathbb{C}^k\) which admits a fixed attracting point at infinity and let \(U^+\) be its attracting basin, do we always have:

\[
\tilde{U}^+ \simeq \mathbb{H} \times \mathbb{C}^{k-1}?
\]
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