Beating the Harmonic lower bound for online bin packing

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Abstract

In the online bin packing problem, items of sizes in $(0, 1]$ arrive online to be packed into bins of size 1. The goal is to minimize the number of used bins. In this paper, we present an online bin packing algorithm with asymptotic competitive ratio of 1.5813. This is the first improvement in fifteen years and reduces the gap to the lower bound by 15%. Within the well-known SUPER HARMONIC framework, no competitive ratio below 1.58333 can be achieved.

We make two crucial changes to that framework. First, some of our algorithm’s decisions depend on exact sizes of items, instead of only their types. In particular, for each item with size in $(1/3, 1/2]$, we use its exact size to determine if it can be packed together with an item of size greater than $1/2$. Second, we add constraints to the linear programs considered by Seiden, in order to better lower bound the optimal solution. These extra constraints are based on marks that we give to items based on how they are packed by our algorithm. We show that for each input, there exists a single weighting function that can upper bound the competitive ratio on it.

We use this idea to simplify the analysis of SUPER HARMONIC, and show that the algorithm HARMONIC++ is in fact 1.58880-competitive (Seiden proved 1.58889), and that 1.5884 can be achieved within the SUPER HARMONIC framework. Finally, we give a lower bound of 1.5762 for our new framework.

1 Introduction

In the online bin packing problem, a sequence of items with sizes in the interval $(0, 1]$ arrive one by one and need to be packed into bins, so that each bin contains items of total size at most 1. Each item must be irrevocably assigned to a bin before the next item becomes available. The algorithm

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has no knowledge about future items. There is an unlimited supply of bins available, and the goal is to minimize the total number of used bins (bins that receive at least one item).

Bin packing is a classical and well-studied problem in combinatorial optimization. Extensive research has gone into developing approximation algorithms for this problem, e.g. [6, 8, 7, 13, 18, 10]. Such algorithms have provably good competitive ratio for any possible input and work in polynomial time. In fact, the bin packing problem was one of the first for which approximation algorithms were designed [11].

For bin packing, we are typically interested in the long-term behavior of algorithms: how good is the algorithm for large inputs, relative to the optimal solution? This ratio is often determined by very small inputs. To avoid such pathological instances, the asymptotic competitive ratio was introduced, which we now define. For a given input sequence \( \sigma \), let \( A(\sigma) \) be the number of bins used by algorithm \( A \) on \( \sigma \). The asymptotic competitive ratio for an algorithm \( A \) is defined to be

\[
\mathcal{R}_A^\infty = \limsup_{n \to \infty} \sup_{\sigma} \left\{ \frac{A(\sigma)}{OPT(\sigma)} \middle| OPT(\sigma) = n \right\}.
\]  

From now on, we only consider the asymptotic competitive ratio unless otherwise stated. For a given input, we typically consider a fixed optimal solution for the analysis.

Lee and Lee [14] presented an algorithm called HARMONIC, which partitions the interval \((0, 1]\) into \(m > 1\) intervals \((1/2, 1], (1/3, 1/2], \ldots, (0, 1/m]\). The type of an item is defined as the index of the interval which contains its size. Each type of items is packed into separate bins (\(i\) items per bin for type \(i = 1, \ldots, m - 1\); type \(m\) items are packed using NEXT Fit in dedicated bins). For any \(\varepsilon > 0\), there is a number \(m\) such that the HARMONIC algorithm that uses \(m\) types has a competitive ratio of at most \((1 + \varepsilon) \Pi_\infty\) [14], where \(\Pi_\infty \approx 1.69103\) for \(m \mapsto \infty\).

**Definition 1** We use the following adjectives for ranges of item sizes. Huge means \((2/3, 1]\), large means \((1/2, 2/3]\), medium means \((1/3, 1/2]\), and small means \((0, 1/3]\).

If we consider the bins packed by HARMONIC, then it is apparent that in bins with large items, nearly half the space can remain unused. It is better to use this space for items of other types. After a sequence of papers which used this idea to develop ever better algorithms [14, 16, 17], Seiden [19] presented a general framework called SUPER HARMONIC which captures all of these algorithms. We describe it in some detail, since we reuse many concepts, and in order to describe our modifications in a clear way.

**The SUPER HARMONIC framework** [19] The fundamental idea of all SUPER HARMONIC algorithms is to first classify items by size, and then pack an item according to its type (as opposed to letting the exact size influence packing decisions). For the classification of items, we use numbers \(t_1 = 1 \geq t_2 \geq \cdots \geq t_N > 0\) to partition the interval \([0, 1]\) into subintervals \(I_1, \ldots, I_N\). \(N\) is a parameter of the algorithm.) We define \(I_j = (t_{j+1}, t_j]\) for \(j = 1, \ldots, N - 1\) and \(I_N = (0, t_N]\). We denote the type of an item \(p\) by \(t(p)\), and its size by \(s(p)\). An item \(p\) has type \(j\) if \(s(p) \in I_j\). A type \(j\) item has size at most \(t_j\).

Each item receives a color when it arrives, red or blue; an algorithm in the SUPER HARMONIC framework defines parameters \(r_{dj} \in [0, 1]\) for each type \(j\), which denotes the fraction of items of
type $j$ that are colored red.

Blue items of type $j$ are packed using NEXT Fit. We use each bin until exactly $\text{bluefit}_j := \lfloor 1/t_j \rfloor$ items are packed into it. For each bin, smaller red items may be packed into the space of size $1 - \text{bluefit}_jt_j$ that remains unused. Red items are also packed using NEXT Fit, using a fixed amount of the available space in a bin. This space is chosen in advance from a fixed set $\text{REDSPACE} = \{ \text{redspace}_i \}_{i=1}^{K}$ of spaces, where $\text{redspace}_1 \leq \cdots \leq \text{redspace}_K$. If red items of type $j$ are packed into a space of size $\text{redspace}_i$, we pack $\text{redfit}_j := \lfloor \text{redspace}_i/t_j \rfloor$ red items into each bin. In the space not used by red items, the algorithm may pack blue items. There may be several types that the algorithm can pack into a bin together with red items of type $j$. Each bin will contain items of at most two different types. If a bin contains items of two types, it is called mixed. If it contains items of only one type, but items of another type may be packed into this bin later, it is called unmixed. A bin that will always contain items of one type is called pure blue.

A SUPER HARMONIC algorithm tries to minimize the number of unmixed bins, and to place red and blue items in mixed bins whenever possible. Seiden [19] showed that the SUPER HARMONIC algorithm HARMONIC++, which uses 70 intervals for its classification and has about 40 manually set parameters, achieves a competitive ratio of at most 1.58889.

The algorithm HARMONIC++ always packs only one red item in a bin, and Seiden exploits this fact in his analysis. However, a very minor technical change is sufficient to make his analysis more general. Since Seiden does mention the possibility of packing more than one red item in a bin, only deciding against it because he could not find good settings for the parameters, we do not see this as a new idea of our algorithm. By allowing more than one red item in a bin in the SUPER HARMONIC framework, a competitive ratio of 1.5884 can be achieved.

Ramanan et al. [16] gave a lower bound of $19/12 \approx 1.58333$ for all SUPER HARMONIC algorithms. It is based on critical bins (formally defined later) like the one shown in Fig. 1, which contain a medium item (size in $(1/3, 1/2]$) and a large item (size in $(1/2, 2/3]$). Both of these items arrive many times, and although they fit pairwise into bins, the algorithm does not combine them like this. In contrast, the optimal solution consists exclusively of critical bins.

**Our contribution** We avoid the lower bound construction of Ramanan et al. [16] by defining the algorithm so that it combines medium and large items whenever they fit together in a single bin. Essentially, we use ANY Fit to combine such items into bins (under certain conditions specified below). This is a generalization of the well-known algorithms FIRST Fit and BEST Fit [20, 8], which have been used in similar contexts before [3, 1]. For all other items, we essentially leave the structure of SUPER HARMONIC intact, although a number of technical changes are made, as we describe next. Each bin will still contain items from at most two types, and if there are two types in a bin, then the items of one type are colored blue and the others are colored red.

We extend the definitions of huge, large, medium and small items (Definition 1) to types in the natural way. In order to benefit from using ANY Fit, we need to ensure that for each medium type, as much as possible, the smallest items are colored red. Otherwise, we run into the same problems as SUPER HARMONIC, see Fig. 2 for an example. Our plan is therefore to initially give each medium item no color and pack it alone in a bin. After several items of some type $j$ have

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1. This parameter was called $\alpha_j$ by Seiden; we have made many changes to the (somewhat ad hoc) notation.
2. There will not be any types that contain 1/2, 1/3 or 2/3 as an inner point in their interval.
Figure 1: A critical bin. The item sizes are chosen such that a given interval classification algorithm (in this case, HARMONIC++) does not pack these items together. For any SUPER HARMONIC algorithm, such sizes can be found. The central idea of our new algorithm is that we limit the number of times that these critical bins can occur in the optimal solution. This is how we beat the ratio of 1.58333.

arrived, we color the smallest one red and the others blue. The next arriving blue items of this type (so-called late items) will be packed into the bins with single blue items. (See Fig. 3) In this way, at least the blue items which are first in their bins (the early items) are not smaller than the smallest red item. We thus have a lower bound on the size of half of the blue items (the early ones).

However, postponing coloring decisions like this is not always possible or even desirable. In fact there are exactly two cases where this will not be done upon arrival of a new medium item p.

1. If a bin with suitable small red items (say, of some type t) is available, and it is time to color p blue, we will pack p into that bin and color it blue, regardless of the precise size of p. In this case, in our analysis we will carefully consider how many small items of type t the input contains; knowing that there must be some. This implies that in the optimal solution, not all the bins can be critical. Moreover, our algorithm packs these small items very well, using almost the entire space in the bin.

2. If a bin with a large item is available, and p fits into such a bin, we will pack p in one such bin as a red item regardless of which color it was supposed to get. This is the best case overall, since finding combinations like this was exactly our goal! This helps to avoid the worst case instances for SUPER HARMONIC (Fig. 4). However, there is a technical problem with this, which we discuss below.

Overall, we have three different cases: medium items are packed alone initially (in which case we have a guarantee about the sizes of some of the blue items), medium items are combined with smaller red items (in which case these small items exist and must be packed in the optimal solution), or medium items are combined with larger blue items (which is exactly our goal).

The main technical challenge is to quantify these different advantages into one overall analysis. In order to do this, we introduce—in addition to and separate from the coloring—a marking of the medium items. The marking indicates whether the blue or red items of a given mark are in mixed or unmixed bins. This will bound the number of critical bins (Fig. 1) that can exist in the optimal solution, leading to better lower bounds for the optimal solution value than Seiden [19] used.
Figure 2: Packing of a sequence of medium items (all of the same type $i$) followed by one large item. The items arrive in the order indicated by the numbers. SUPER HARMONIC needs more bins than the optimal solution, as the red medium item is too large to be combined with the large item. (We assume $\text{red}_i = 1/9$ here.)

Figure 3: Illustration of the coloring in EXTREME HARMONIC. Hatched items are uncolored. In this example, $\text{red}_i = 1/9$, where $i$ is the type of all items depicted in this example. Note that the ratio of 1/9 does not hold (for the bins shown) at the time that the colors are fixed: 1/5 of the items are red at this point. The ratio 1/9 is achieved when all bins with blue items contain two blue items. The blue items which arrive in step (d) are called *late* items.

Figure 4: An example illustrating why it helps to occasionally color more than a $\text{red}_i$-fraction of the items of a medium type $i$ red. First, five large items arrive (numbered 1-5), then five medium items (numbered 6-10) of type $i$. We assume that $\text{red}_i = 1/5$ here.
Maintaining the fraction $\text{red}_j$ of red items for all marks separately is necessary for the analysis. As we have seen however, if many large items arrive first, we must pack medium items with them whenever possible, even if this violates the ratio $\text{red}_j$. If there are more than $\text{red}_j$ medium items of some type $j$ when the input ends, we call those items bonus items. Each bonus item is packed in a bin with a large item. After the input ends, we will (virtually) make some of those large items smaller so that they get type $j$ as well (see Fig. 5(a)). We then change the colors of the bonus items to ensure the proper fraction $\text{red}_j$ of red medium items. Hence we modify the input, but we only do this for the analysis and only once all the items have been packed. Clearly the number of bins in the optimal solution can only decrease as a result of making some items smaller.

However, there could be small red items (say, of type $t$) in separate bins that could have been packed in bins with two medium type $j$ items, had such bins been available at the time when the small red items arrived. Creating such bins after the input ends generates a packing that is not covered by our analysis (as this analysis assumes that such compatible items are packed together in one bin, not two; see Fig. 5b). To avoid this, we do not allow small items to be packed into new bins as red items as long as bins with large and medium items exist that may later be modified. Instead, in such a case, we count a single medium item in such a bin as a number of red small items of type $t$, and pack the incoming item of type $t$ as a blue item (Fig. 5(c)). This ensures (as we will show) that if suitable bins with blue items are available, red items of type $t$ are always packed in them, rather than in new bins.

At this point, we stress that our algorithm does not actually modify the input while it is packing it in any way. The only thing that changes is the internal accounting of the algorithm (in such a way that it thinks it has packed less total size than it actually has). We will show a number of properties of the packing that the algorithm produces, and we crucially show that all of these properties are maintained whenever a bonus item is counted as several small items, which is the only point in which the accounting of the algorithm is changed relative to the actual input. Thus we do not follow the perhaps more common approach of showing that the algorithm would have performed the same on the modified input; we believe that approach cannot be applied here, as we do not see how to define arrival times for the small items that are created.

Like Seiden [19] and many other authors [20, 14, 16], we use weighting functions to analyze the competitive ratio of our algorithm. A weighting function defines a weight for each item, depending on its type (and mark, in our case). By analyzing these, Seiden ended up with a set of mathematical programs that upper bounded the competitive ratio of $\text{SUPER HARMONIC}$ algorithms. These represented a kind of knapsack problems where each item has two different weights. Seiden used heuristics to solve these problems in reasonable time.

We instead split each mathematical program into two standard linear programs, and we add new constraints limiting how many critical bins there can be in the optimal solution, which can be deduced from the marks of the items. We solve the linear programs by creating a separation oracle for the dual, which solves a standard knapsack problem (with just one weight per item), making the results much easier to verify. The final weighting function we find depends on the input but does not depend on the marks anymore. The two dual programs of each pair of linear programs are symmetric, so it is sufficient to give a solution for one of them.

We implemented a computer program which quickly solves the knapsack problems and also
(a) If bonus items remain after the algorithm terminates, we transform some large items to medium items, re-establishing the correct ratio of red items for the medium items (in this example, this ratio is \(1/5\)).

(b) This situation **must not occur** in our algorithm: We shrink a large item packed with a bonus item but there are uncombined red small items compatible with the bonus item.

(c) In order to prevent the situation in Fig. 5b, we (virtually) resize and split the bonus item into small items when other small red items arrive. The new item becomes blue instead. Later, more small blue items can be packed with it.

Figure 5: Post-processing and change of the input for the analysis. Gray items denote bonus items.

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does the other necessary work, including the automated setting of many parameters like item sizes and values \(\text{red}_i\). As a result, our algorithm **SON OF HARMONIC** requires far less manual settings than **HARMONIC++**. We also provide a verifier program that checks the feasibility of these solutions; this verifier program should be easy to check by a reader. In addition, we also output the set of knapsack problems directly to allow independent verification.

This approach can also be applied to the original **SUPER HARMONIC** framework. Surprisingly, we find that the algorithm **HARMONIC++** is in fact 1.58880-competitive, using only one weighting function per input. A benefit of using our approach is that this result becomes more easily verifiable as well. Furthermore, we were able to improve and simplify the parameters of **HARMONIC++** to achieve a competitive ratio of 1.5884 within the **SUPER HARMONIC** framework.

Our second main contribution is a new lower bound for all algorithms of this kind. The fundamental property of all these algorithms is that they color a fixed fraction of all items red (for each type). We show that no such algorithm can be better than 1.5762-competitive. Thus fundamentally different ideas will be needed to get much closer to the lower bound of 1.54037, which we believe is closer to the true competitive ratio of this problem.

**Related Results**  The online bin packing problem was first investigated by Ullman [20]. He showed that the **FIRST FIT** algorithm has competitive ratio \(\frac{17}{10}\). This result was then published
in [8]. Johnson [12] showed that the NEXT FIT algorithm has competitive ratio 2. Yao showed that REVISED FIRST FIT has competitive ratio $\frac{5}{3}$, and further showed that no online algorithm has competitive ratio less than $\frac{3}{2}$ [22]. Brown and Liang independently improved this lower bound to $1.53635$ [5, 15]. The lower bound stood for a long time at $1.54014$, due to van Vliet [21], until it was improved to $\frac{248}{161} = 1.54037$ by Balogh et al. [4].

An improved upper bound of $1.5873$ by Balogh et al. [2] partially builds on our ideas but uses a different analysis. The comments about our paper in the arxiv version refer to a previous version and are outdated, as the issue it mentions has long been fixed.

The offline version, where all the items are given in advance, is well-known to be NP-hard [9]. This version has also received a great deal of attention, for a survey see [6].

2 The EXTREME HARMONIC framework

First of all, to facilitate the comparison to the new framework, we give a formal definition of the SUPER HARMONIC framework in Algorithm 1 and 2. It uses the following definitions. Let $n^i$ count the total number of items of type $i$, and $n^i_{\text{red}}$ count the number of red items of type $i$.

Algorithm 1 How the SUPER HARMONIC framework packs a single item $p$ of type $i \leq N - 1$. At the beginning, we set $n^i_{\text{red}} \leftarrow 0$ and $n^i \leftarrow 0$ for $1 \leq i \leq N - 1$.

1: $n^i \leftarrow n^i + 1$
2: if $n^i_{\text{red}} < \lfloor \text{red}_i n^i \rfloor$ then // pack a red item
3: $\text{PACSSENPLE}(p, \text{red})$
4: $n^i_{\text{red}} \leftarrow n^i_{\text{red}} + 1$
5: else // pack a blue item
6: $\text{PACSSENPLE}(p, \text{blue})$
7: end if

Algorithm 2 The algorithm PACKSIMPLE($p, c$) for packing an item $p$ of type $i$ with color $c \in \{\text{blue, red}\}$.

1: Try the following types of bins to place $p$ with color $c$ in this order:
2: • a pure blue, mixed, or unmixed $c$-open bin with items of type $i$ and color $c$
3: • an unmixed bin that is compatible with $p$ (the bin becomes mixed)
4: • a new unmixed bin (or pure blue bin, if $\text{leaves}(i) = 0$ and $c = \text{blue}$)

A SUPER HARMONIC algorithm uses a function $\text{leaves} : \{1, \ldots, N\} \rightarrow \{0, \ldots, K\}$ to map each item type to an index of a space in REDSPACE, indicating how much space for red items it leaves unused in bins with blue items of this type. Here $\text{leaves}(j) = 0$ means that no space is left for red items. The algorithm also uses a function $\text{needs} : \{1, \ldots, N\} \rightarrow \{0, \ldots, K\}$ to map how much space (given by an index of REDSPACE) red items of each type require. We define $\text{needs}(i) = 0$ if and only if $\text{red}_i = 0$ (i.e., there are no red items of this type).
For each type \( i \) such that \( \text{leaves}(i) = 0 \), the items of this type are packed in pure blue bins, that contain only blue items (only one type per bin). An unmixed bin is called unmixed blue or unmixed red depending on the color of the items in it.

A mixed bin with blue items of type \( i \) and red items of type \( j \) satisfies the following properties: \( \text{leaves}(i) > 0, \text{red}_j > 0, \text{redspace}_{\text{needs}(j)} \leq \text{redspace}_{\text{leaves}(i)} \). Note that the last inequality holds if and only if \( \text{needs}(j) \leq \text{leaves}(i) \). The blue items will use space at most \( 1 - \text{redspace}_{\text{leaves}(i)} \) and the red items will use space at most \( \text{redspace}_{\text{needs}(j)} \leq \text{redspace}_{\text{leaves}(i)} \). An unmixed blue bin with blue items of type \( j \) is compatible with a red item of type \( i \) if \( \text{needs}(i) \leq \text{leaves}(j) \). An unmixed red bin with red items of type \( j \) is compatible with a blue item of type \( i \) if \( \text{needs}(j) \leq \text{leaves}(i) \).

**Definition 2** A bin is red-open if it contains some red items but can still receive additional red items. We define blue-open analogously. A bin is open if it is red-open or blue-open.

Red-open bins with red items of type \( j \) contain at least one and at most \( \text{redfit}_j - 1 \) red items. Blue-open bins can be pure blue. Red-open and blue-open bins can be mixed or unmixed. Mixed bins can be red-open and blue-open at the same time. A bin with blue fit, items of type \( i \) but no red items is not considered open, even though red items might still be packed into it later.

For EXTREME HARMONIC, we extend the definition of compatible bins. As noted in the Introduction, some items will not receive a color when they arrive, but only later. The goal of having uncolored items is to try and make sure that relatively small items of each medium type become red in the end (to make it easier to combine them with large items).

**Definition 3** An unmixed bin is red-compatible with a newly arriving item \( p \) if

1. the bin contains blue or uncolored items\(^3\) of type \( i \), \( p \) is small and \( \text{leaves}(i) \geq \text{needs}(t(p)) \), or
2. the bin contains a (blue) large item of size \( x \), \( p \) is medium and \( s(p) \leq 1 - x \).

An unmixed bin is blue-compatible with a newly arriving item \( p \) if

1. the bin contains red items\(^3\) of type \( j \), \( p \) is medium or small and \( \text{leaves}(t(p)) \geq \text{needs}(j) \), or
2. the bin contains one red or uncolored medium item of size \( x \), \( p \) is large and \( s(p) \leq 1 - x \).

It follows that for checking whether a large item and a medium item can be combined in a bin, we ignore the values \( \text{leaves}(i) \) and \( \text{needs}(j) \) and use only the relevant parts 2 of Definition 3.

Like SUPER HARMONIC algorithms, an EXTREME HARMONIC algorithm first tries to pack a red (blue) item into a red-open (blue-open) bin with items of the same type and color; then it tries to find an unmixed compatible bin; if all else fails, it opens a new bin. Note that the definition of compatible has been extended compared to SUPER HARMONIC, but we still pack blue items with red items of another type and vice versa; there will be no bins with blue (or red) items of two different types. The new framework is formally described in Algorithms 3 and 4. Items of type \( N \) are packed using NEXT FIT as before. We discuss the changes from SUPER HARMONIC one by one:

\(^3\)We will see later that if an item has no color, it is the only item in its bin (Property 11).
one. All the changes stem from our much more careful packing of medium items. The algorithm
MARK AND COLOR called in line 28 of EXTREME HARMONIC will be presented in Section 2.2.
This algorithm will take care of assigning marks and colors to the items. In particular, this will
take care of fixing the color of medium items as described in Figure 3.

As can be seen in PACK (lines 2, 4 and 5), medium items that are packed into new bins are
initially packed one per bin and not given a color. We wait until enough of these items have
arrived, and then color the smallest one red using MARK AND COLOR (Fig. 3). Note that \( n_{\text{red}}^i \)
is increased in line 17 even though the item might not receive a color at this time. This means that
the value \( n_{\text{red}}^i \) does not always accurately reflect how many red items there currently are. We will
show that this is not an issue for the analysis (it will be accurate up to a constant).

Algorithm 3 How the EXTREME HARMONIC framework packs a single item \( p \) of type \( i \leq N - 1 \).
At the beginning, we set \( n_{\text{red}}^i \leftarrow 0 \), \( n_{\text{bonus}}^i \leftarrow 0 \) and \( n^i \leftarrow 0 \) for \( 1 \leq i \leq N - 1 \).

1: \( n_i^i \leftarrow n_i^i + 1 \) // pack a red item
2: if \( n_{\text{red}}^i < [\text{red}_i n^i] \) then
3: \hspace{1em} if \( n_{\text{bonus}}^i > 0 \) or \( \text{needs}(i) \leq 1/3 \land \exists j : n_{\text{bonus}}^j > 0 \) then
4: \hspace{2em} // special case: replace bonus item instead and pack the new item as blue; see Fig. 5c
5: \hspace{2em} if \( n_{\text{bonus}}^i > 0 \) then
6: \hspace{3em} Let \( b \) be a bonus item of type \( i \) // in this case, \( \text{redfit}_i = 1 \)
7: \hspace{3em} else
8: \hspace{4em} Let \( b \) be a bonus item of some type \( j \) with \( n_{\text{bonus}}^j > 0 \) // here \( i \) is a small type
9: \hspace{3em} end if
10: \hspace{2em} \( n_{\text{bonus}}^i \leftarrow n_{\text{bonus}}^i - 1 \) // count \( b \) as type \( i \) item(s) and color it/them red
11: \hspace{2em} Label \( b \) as type \( i \)
12: \hspace{2em} \( n_i^i \leftarrow n_i^i + \text{redfit}_i \) // \( b \) might have been of type \( i \) already, then \( \text{redfit}_i = 1 \)
13: \hspace{2em} \( n_{\text{red}}^i \leftarrow n_{\text{red}}^i + \text{redfit}_i \) // since we now have \( n_{\text{red}}^i \geq [\text{red}_i n^i] \) again
14: \hspace{2em} PACK(\( p \), blue) // The item is red or uncolored
15: \hspace{2em} else
16: \hspace{3em} PACK(\( p \), red)
17: \hspace{3em} \( n_{\text{red}}^i \leftarrow n_{\text{red}}^i + 1 \)
18: \hspace{3em} end if
19: else \hspace{1em} // pack a blue item
20: \hspace{2em} if \( p \) is medium, \( \text{red}_i > 0 \), and there exists a bin \( B \) that is red-compatible with \( p \) then
21: \hspace{3em} Place \( p \) in \( B \) and label it as bonus item. // special case: bonus item
22: \hspace{3em} \( n_i^i \leftarrow n_i^i - 1 \) // we do not count this item for type \( i \)
23: \hspace{3em} \( n_{\text{bonus}}^i \leftarrow n_{\text{bonus}}^i + 1 \) // Note that \( B \) contains a large item
24: \hspace{2em} else
25: \hspace{3em} PACK(\( p \), blue) // The item is blue or uncolored
26: \hspace{3em} end if
27: \hspace{2em} end if
28: Update the marks and colors using MARK AND COLOR (Section 2.2).
Algorithm 4 The algorithm \( \text{PACK}(p, c) \) for packing an item \( p \) of type \( i \) with color \( c \in \{ \text{blue}, \text{red} \} \).

1: Try the following types of bins to place \( p \) with (planned) color \( c \) in this order:
2:   • a pure blue, mixed, or unmixed \( c \)-open bin with items of type \( i \) and color \( c \)
3:   • a \( c \)-compatible unmixed bin (the bin becomes mixed, with fixed colors of its items)
4:   • a new unmixed bin (or pure blue bin, if \( \text{leaves}(i) = 0 \) and \( c = \text{blue} \))
5: If \( p \) was packed into a new bin, \( p \) is medium and \( \text{red}_i > 0 \), give \( p \) no color, else give it the color \( c \).

When an item arrives, in many cases, we cannot postpone assigning it a color, since a \( c \)-open or \( c \)-compatible bin is already available (see lines 2–3 of \( \text{PACK}(p, c) \)). Additionally, if we are about to color an item blue because currently \( n_{\text{red}}^i \geq \lfloor \text{red}_i n^i \rfloor \), we check whether a suitable large item has arrived earlier. We deal with this case in lines 20–23 of EXTREME HARMONIC. In this special case, we ignore the value \( \text{red}_j \). We pack the medium item with the large item as if it were red (no further item will be packed into this bin), but we do not count it towards the total number of existing medium items of its type; instead we label it a bonus item. Bonus items do not have a mark or color, but this can change later during processing in the following two cases.

1. Additional items of type \( i \) arrive which are packed as blue items. If enough of them arrive (so that it is time to color an item red again), we first check in line 3 of EXTREME HARMONIC if there is a bonus item of type \( i \) that we could color red instead. If there is, we will do so, and pack the new item as a blue item.

2. An item of some type \( j \) and size at most \( 1/3 \) arrives, that should be colored red. In this case, for our accounting, we view the bonus item as \( \text{redfit}_j \) red items of type \( j \), and adjust the counts accordingly in lines 11–13 of EXTREME HARMONIC. The new item of type \( j \) is packed as a blue item in line 14 of EXTREME HARMONIC.\(^5\)

It can be seen that blue items of size at most \( 1/3 \) are packed as in SUPER HARMONIC. For red items of size at most \( 1/3 \), we deal with existing bonus items in lines 11–13 of EXTREME HARMONIC, and in line 3 of \( \text{PACK}(p,c) \), an existing medium item may be colored red or blue (the opposite of the parameter \( c \)). Otherwise, the packing proceeds as in SUPER HARMONIC for these items as well.

2.1 Properties of EXTREME HARMONIC algorithms

All EXTREME HARMONIC algorithms are required to satisfy the following properties. The first two easy properties also hold for SUPER HARMONIC and the third and fourth property hold for HARMONIC++ (but not necessarily for all SUPER HARMONIC algorithms). Let \( \varepsilon = t_N \).

Property 1 (Lemma 2.1 in Seiden [19]) Each bin containing items of type \( N \), apart from possibly the last one, contains items of total size at least \( 1 - \varepsilon \).

\(^4\)Note that the meanings of \( i \) and \( j \) are switched in the description of the algorithm for reasons of presentation.

\(^5\)Strictly speaking, we only need this whole procedure if type \( j \) is compatible with the bonus item, to avoid the case in Figure 5b. Instead, we do it for all small items for simplicity.
Property 2 For any type $i$, if $\text{needs}(i) > 0$, then $\text{leaves}(i) < \text{needs}(i)$.

Proof If $\text{needs}(i) > 0$, then $\text{red}_i > 0$. If $\text{leaves}(i) \geq \text{needs}(i) > 0$, an additional item of type $i$ could be placed in the space $\text{redspace}_{\text{needs}(i)} \leq \text{redspace}_{\text{leaves}(i)}$, which means we could fit $\text{bluefit}_i + 1$ blue items of type $i$ into one bin, contradicting the definition of $\text{bluefit}_i$. □

Property 3 If $j$ is a small type with $\text{red}_j > 0$, $\text{redspace}_{\text{needs}(j)} \leq 1/3$. If $i$ is a medium type, then $\text{redspace}_{\text{leaves}(i)} < 1/3 < \text{redspace}_{\text{needs}(i)}$. If $i$ is a large or huge type, then $\text{red}_i = 0$, so $n^i_{\text{red}} = 0$ at all times.

Property 4 For $x > 1/3$, we have $x \in \text{REDSPACE}$ if and only if $\exists i : x = t_i$.

Property 5 We have $\text{red}_i < 1/3$ for all types $i$.

Property 6 We have $t_1 = 1$, $t_2 = 2/3$, $t_3 = 1/2$, and $\text{red}_1 = \text{red}_2 = \text{red}_N = 0$. All type 1 items (i.e., huge items) and type $N$ items are packed in pure blue bins. We have $1/3 \in \text{REDSPACE}$.

This property implies that $\text{leaves}(1) = 0$ and $\text{redspace}_{\text{leaves}(2)} = 1/3$. This means that an unmixed bin with a large item is never red-compatible with a medium item via Condition 1 of Definition 3 (so only Condition 2 is relevant for this combination). This furthermore implies that for a medium item of type $i$, the precise value $\text{needs}(i)$ is irrelevant for the algorithm (only the fact that $\text{redspace}_{\text{needs}(i)} > 1/3$ is relevant). It will nevertheless be useful for the analysis to have $t_i \in \text{REDSPACE}$ as required by Property 4.

Property 7 Let $t_i, t_{i+1}$ be two consecutive medium type thresholds of the algorithm. Then $t_i - t_{i+1} < t_N < 1/100$.

Property 8 For each type $i$ and color $c$, at any time, there is at most one $c$-open bin that contains items of type $i$ and no other type. For each pair of types and color $c$, at any time, there is at most one $c$-open bin with items of those types.

Proof All bonus items are in mixed bins; such a bin remains mixed if its bonus item gets labeled with a different type. Consider an item of type $i$ and color $c$. By the order in which $	ext{PACK}$ tries to place items into bins, we only open a new unmixed or pure blue bin of type $i$ if no $c$-open bin is available, so the first claim holds.

Now consider a pair of types. Say the blue items are of type $i$ and the red items are of type $j$. The only cases in which a mixed bin with such items is created are the following:

- A red item of type $j$ is placed into an unmixed bin $B$ with blue items of type $i$. In this case, there was no existing red-open mixed or unmixed bin with red items of type $j$.
- A blue item of type $i$ is placed into an unmixed bin $B$ with red items of type $j$. In this case, there was no existing blue-open mixed, unmixed or pure blue bin with blue items of type $i$.
- A bin receives a bonus item in line 21 of EXTREME HARMONIC and is now considered mixed.

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• A bonus item gets counted as items of type $j$ in lines 11-13.

At the beginning, there are zero open bins with items of type $i$ and type $j$. Such bins are only created via one of the cases listed above. In the last two cases, no open bins are created (note that only one medium and one large item can be packed together in a bin). In the second case, $B$ is the only red-open bin with these types (if $\text{redfit}_i > 1$, that is), and no red items of type $j$ are packed into unmixed bins with blue items until $B$ contains $\text{redfit}_i$ type $i$ items by line 3 of Algorithm 4. In the first case, similarly, no new blue item of type $i$ will be packed into unmixed bins with red items as long as $B$ remains blue-open. □

Property 9 At all times, for each type $i$, $n^i_{\text{red}} \geq \lfloor \text{red}_i n^i \rfloor - 1$. For each medium type $i$, $n^i_{\text{red}} \leq \lfloor \text{red}_i n^i \rfloor$.

Proof The first bound follows from the condition in line 2 of EXTREME HARMONIC and because $n^i$ increases by at most 1 in between two consecutive times that this condition is tested, unless a bonus item is labeled in lines 11-13 of EXTREME HARMONIC; but in that case, the fraction of red items of type $i$ only increases, because $n^i$ and $n^i_{\text{red}}$ increase by the same amount.

The upper bounds follow because for each medium type $i$, $n^i_{\text{red}}$ increases by at most 1 when $n^i_{\text{red}} < \lfloor \text{red}_i n^i \rfloor$ and a new item of this type arrives: either in line 13 ($\text{redfit}_i = 1$ for medium items) or in line 17. Furthermore, if $n^i_{\text{red}} = \lfloor \text{red}_i n^i \rfloor$, $n^i_{\text{red}}$ is not increased anymore. If a bonus item is created, $n^i$ and $n^i_{\text{red}}$ are unchanged (lines 1 and 22). For small items, $n^i_{\text{red}}$ increases by at most $\text{redfit}_i$ in one iteration (line 13), and this only happens if the ratio is too low (line 2).

Recall that $n^i_{\text{red}}$ is not always the true number of medium red items of type $i$, as some of these may not have a color yet. For a small type $i$, the value $n^i_{\text{red}}$ may also not be accurate, because it may include some bonus items. We will fix this in postprocessing, where we replace the bonus items by items of type $i$ to facilitate the analysis.

Property 10 At all times, for each type $i$ that is not medium, $n^i_{\text{bonus}} = 0$.

Property 11 Each bin with an uncolored item contains only that item.

Proof By line 3 of the PACK method, as soon as a bin becomes mixed, the colors of its items are fixed. By line 2 of the PACK method, an unmixed bin with an uncolored item does not receive a second item of the same type. □

In particular, no bin which contains an uncolored item is a mixed bin. The following important invariant generalizes a result for SUPER HARMONIC (which is easy to see for that algorithm).

Invariant 1 If there exists an unmixed bin with red items of type $j$, then for any type $i$ such that $\text{needs}(j) \leq \text{leaves}(i)$, there is no bin with a bonus item of type $i$ and no unmixed bin with blue items of type $i$.

Proof As long as an unmixed red bin with items of some type $j$ exists, no unmixed blue bin with items of type $i$ for which $\text{needs}(j) \leq \text{leaves}(i)$ can be opened and vice versa (line 3 of PACK). Now assume for a contradiction that there is an unmixed red bin with red items of type $j$ (denote the first item in this bin by $f$) and a bin with a bonus item $b$ of type $i$. Assume $b$ arrived before
Consider the point in time where \( f \) arrived. After deciding that \( f \) should be colored red in line 2 of EXTREME HARMONIC, we would have found that the second part of the condition in line 3 of EXTREME HARMONIC is true, and as a consequence would have made \( b \) no longer be bonus, a contradiction to our assumption.

Now assume that \( f \) arrived before \( b \). In this case, either \( f \) or the large item \( L \) that is packed with \( b \) arrived first. (Note that \( b \) definitely arrived after \( L \), or it would not have been made bonus.) Now \( s(L) < 2/3 \) since \( L \) was packed with the medium item \( b \). But \( 1/3 > \text{redspace}_{\text{leaves}(i)} \geq \text{redspace}_{\text{needs}(j)} \) by Property 3 and the assumption of the lemma. Hence, regardless of which item among \( f \) and \( L \) arrived first, the algorithm does not pack them in different unmixed bins; the second arriving item would be packed at the latest by line 3 of PACK.

2.2 Marking the items

**Definition 4** A critical bin for an EXTREME HARMONIC algorithm is a bin used in the optimal solution that contains a pair of items, one of a medium type \( j \) \((t_j \in (1/3, 1/2])\) and one of a large type \( i \) \((t_i \in (1/2, 2/3])\) such that \( t_j + t_i > 1 \) but \( t_{j+1} + t_{i+1} < 1 \).

An example was given in Fig. 1. By marking the medium items, we keep track of how many red and blue items of a given type \( j \) are in mixed bins. Blue medium items in mixed bins imply the existence of compatible small items in the input (which need to be packed somewhere in the optimal solution). Red medium items in mixed bins means that the algorithm managed to combine at least some pairs of medium and large items together into bins. In both cases, we have avoided the situation where the offline packing consists only of critical bins, whereas the online algorithm did not create any bins which contain a large and a medium item. We use three different marks, which together cover all the cases. Our marking is illustrated in Fig. 6.

\( \mathcal{R} \) For any medium type \( j \), a fraction \( red_j \) of the items marked \( \mathcal{R} \) are red, and all of these red items are packed into mixed bins (i.e., together with a large item).

\( \mathcal{B} \) For any medium type \( j \), a fraction \( red_j \) of the items marked \( \mathcal{B} \) are red, and the blue items are packed into mixed bins (i.e., together with small red items).

\( \mathcal{N} \) For any medium type \( j \), a fraction \( red_j \) of the items marked \( \mathcal{N} \) are red, and none of the red and blue items marked \( \mathcal{N} \) are packed into mixed bins.

The algorithm MARK AND COLOR is defined in Algorithm 5. For a given type \( i \) and set \( \mathcal{M} \in \{ \mathcal{N}, \mathcal{B}, \mathcal{R} \} \), denote the number of red items by \( n^i_{\text{red}}(\mathcal{M}) \), and the total number of items by \( n^i(\mathcal{M}) \). Algorithm 5 is run every time after an item has been packed, and for every medium type \( i \) for which \( red_i > 0 \) separately. It divides the medium items into three sets \( \mathcal{N}, \mathcal{B} \) and \( \mathcal{R} \) (see Fig. 6). Once assigned, an item remains in a set until the end of the input (after which it may be reassigned, see Section 3). In many cases, the algorithm will have nothing to do, as none of the conditions hold. Therefore, some items will remain temporarily unmarked, in a set \( \mathcal{U} \). The set \( \mathcal{U} \) does not contain the bonus items (in fact none of the sets does).
(a) Items get mark \( \mathcal{R} \): uncolored items and a red item in a mixed bin. The bins with blue \( \mathcal{R} \)-items will receive an additional blue item of the same type before any new bin is opened for this type.

(b) Items get mark \( \mathcal{B} \): a single uncolored item and blue items (in pairs) in mixed bins.

(c) Items get mark \( \mathcal{N} \): a set of uncolored items. The bins with blue \( \mathcal{N} \)-items will receive an additional blue item of the same type before any new bin is opened for this type. See Fig. 3.

Figure 6: Marking the items. For simplicity, we have taken \( \text{red}_i = 1/9 \) here (where \( i \) is the type of the medium items).

Line 17 of \text{MARK AND COLOR} ensures the following property, which was the point of postponing the coloring. Recall that early items are blue \( \mathcal{N} \)-items which did not get their color immediately and were packed one per bin (each late item is packed in a bin that already contains an early item).

\textbf{Property 12} Each early \( \mathcal{N} \)-item is at least as large as the red \( \mathcal{N} \)-item that received its mark in the same iteration of \text{MARK AND COLOR}.

After all items have arrived and after some post-processing, we will have

\[ |n^i_{\text{red}}(\mathcal{M}) - n^i(\mathcal{M}) \cdot \text{red}_i| = O(1) \text{ for } \mathcal{M} \in \{\mathcal{N}, \mathcal{B}, \mathcal{R} \} \text{ and each medium type } i. \]  \( \text{(2)} \)

Each item will be marked according to the set to which it (initially) belongs. We will see that the values \( x_{\mathcal{R}}, x_{\mathcal{B}} \) and \( x_{\mathcal{N}} \) in \text{MARK AND COLOR} are calculated in such a way that \( n^i_{\text{red}}(\mathcal{M}) = [n^i(\mathcal{M}) \cdot \text{red}_i] \) holds just before any assignment to \( \mathcal{M} \in \{\mathcal{N}, \mathcal{B}, \mathcal{R} \} \). The proof is straightforward.

Note that \text{MARK AND COLOR} never changes the values \( n^i_{\text{red}} \) and \( n^i \). As we saw, the value \( n^i_{\text{red}} \) may be inaccurate for some types in any event. This will be fixed for small types in post-processing, whereas for medium types we will prove (2). Of course, \text{MARK AND COLOR} does change values \( n^i_{\text{red}}(\mathcal{M}) \) and \( n^i(\mathcal{M}) \) for \( \mathcal{M} \in \{\mathcal{R}, \mathcal{N}, \mathcal{B} \} \) in order to record how many items with each mark there are (and these values \textit{will} be accurate).
Algorithm 5 The algorithm MARK AND COLOR as applied to medium items of type $i$ for which $\text{red}_i > 0$.

1: if there is an unmarked item $p_1$ in a bin with a marked item $p_2$ then
2:   Give $p_1$ the same mark $M$ as $p_2$.
3:   $n_i^*(M) \leftarrow n_i^*(M) + 1$
4: end if
5: Let $x_R$ be the minimum integer value such that $[(n_i^*(R) + 2x_R + 1)\text{red}_i] > n_i^*_{\text{red}}(R)$.
6: if there exist $x_R$ uncolored non-bonus items and one unmarked red or bonus item in a mixed bin then
7:   Mark these $x_R + 1$ items $R$. If there is a choice of items in mixed bins, use a bonus item if possible and color it red. Color the (other) uncolored items blue.
8:   If a bonus item is used, $n_i^*_{\text{bonus}} \leftarrow n_i^*_{\text{bonus}} - 1$.
9: end if
10: Let $x_B$ be the minimum integer value such that $[(n_i^*(B) + x_B + 1)\text{red}_i] > n_i^*_{\text{red}}(B)$.
11: if there exists an uncolored non-bonus item and a set of mixed bins with two unmarked blue items each, which contains a number $x_B' \in \{x_B, x_B + 1\}$ of blue items in total, then
12:   Mark these $x_B'$ items $B$ and color the uncolored item red.
13:   $n_i^*(B) \leftarrow n_i^*(B) + x_B' + 1, n_i^*_{\text{red}}(B) \leftarrow n_i^*_{\text{red}}(B) + 1$
14: end if
15: Let $x_N$ be the minimum integer value such that $[(n_i^*(N) + 2x_N + 1)\text{red}_i] > n_i^*_{\text{red}}(N)$.
16: if there exist $x_N + 1$ uncolored non-bonus items then
17:   Mark the $x_N$ largest uncolored items and the single smallest uncolored item $p$ with the mark $N$. Color $p$ red and the other $x_N$ items blue.
18:   $n_i^*(N) \leftarrow n_i^*(N) + x_N + 1, n_i^*_{\text{red}}(N) \leftarrow n_i^*_{\text{red}}(N) + 1$
19: end if

Lemma 1 Let $M \in \{N, R\}$. Just before assignments of new items to $M$ in lines 7[18] or lines 17[18] for each medium type $i$ such that $\text{red}_i > 0$, we have $n_i^*_{\text{red}}(M) = [\text{red}, n_i^*(M)]$ and $x_M < 1/(2\text{red}_i) + 1/2$. Generally, we have $n_i^*_{\text{red}}(M) \in [[\text{red}, n_i^*(M)]]$, $[\text{red}, n_i^*(M)] + 1$.

Proof Call the assignment of new items to $M$ due to lines 7[18] or lines 17[18] early assignments.

At the beginning, we have $n_i^*_{\text{red}}(M) = n_i^*(M) = 0$. Thus the lemma holds at this time. When an early assignment takes place, $n_i^*(M)$ increases by $x_M + 1$, and $n_i^*_{\text{red}}(M)$ by 1. By minimality of $x_M$, just before any early assignment we have

\[
[(n_i^*(M) + 2(x_M - 1) + 1)\text{red}_i] \leq n_i^*_{\text{red}}(M) \quad (3)
\]

\[
\Rightarrow (n_i^*(M) + 2(x_M - 1) + 1)\text{red}_i < n_i^*_{\text{red}}(M) + 1
\]

\[
\Rightarrow (n_i^*(M) + 2x_M + 1)\text{red}_i < n_i^*_{\text{red}}(M) + 1 + 2\text{red}_i
\]

\[
\Rightarrow [(n_i^*(M) + 2x_M + 1)\text{red}_i] \leq n_i^*_{\text{red}}(M) + 1
\]

\[
\Rightarrow [(n_i^*(M) + 2x_M + 1)\text{red}_i] = n_i^*_{\text{red}}(M) + 1
\]
where we have used Property 5 and integrality in the penultimate line and the definition of $x_M$ in the last line. This immediately implies that right after an early assignment to $M$,

$$\lfloor (n^i(M) + x_M)\text{red}_i \rfloor = n^i_{\text{red}}(M).$$ (4)

There are then $x_M$ bins with one early blue medium item of type $i$. EXTREME HARMONIC will put the next arriving blue items of this type into these $x_M$ bins (one additional item per bin) before opening any new bins. All of these late blue items are assigned to $M$ and $n^i(M)$ is increased accordingly in lines 2 to 5 so eventually $n^i_{\text{red}}(M) = \lfloor \text{red}_i n^i(M) \rfloor$.

After that, $n^i(M)$ and $n^i_{\text{red}}(M)$ remain unchanged until the next early assignment of items to $M$. Hence before an early assignment of items to $M$, the first claimed equality holds.

This equality together with (3) gives $\lfloor n^i(M)\text{red}_i + (2(x_M - 1) + 1)\text{red}_i \rfloor \leq \lfloor n^i(M)\text{red}_i \rfloor$, which implies $(2(x_M - 1) + 1)\text{red}_i < 1$ and thus $x_M < 1/(2\text{red}_i) + 1/2 < 1/\text{red}_i$ since $\text{red}_i < 1$. This, together with (4), implies $n^i_{\text{red}}(M) < \lfloor (n^i(M)) + 1 \rfloor$ (note that $n^i_{\text{red}}(M)$ is largest relative to $n^i(M)$ right after an early assignment to $M$, i.e., when (4) holds).

\textbf{Corollary 1} After each execution of MARK AND COLOR and for each medium type $i$ such that $\text{red}_i > 0$, $n^i(\mathcal{U}) \leq 1/\text{red}_i$.

\textbf{Proof} We have $x_M < 1/\text{red}_i$ and $x_R < 1/\text{red}_i$ by Lemma 1, since $\text{red}_i < 1$, so at the latest when $1/\text{red}_i + 1$ uncolored non-bonus items exist, they are marked and colored. \qed

\textbf{Lemma 2} At all times and for each medium type $i$ such that $\text{red}_i > 0$, $n^i_{\text{red}}(\mathcal{B}) = \lfloor \text{red}_i n^i(\mathcal{B}) \rfloor$ and $x_B < 1/\text{red}_i$.

\textbf{Proof} We use similar calculations to the proof of Lemma 1. At the beginning, all counters are zero. When MARK AND COLOR is about to assign items to $B$, we have $\lfloor \text{red}_i (n^i(\mathcal{B}) + x_B + 1) \rfloor = \lfloor \text{red}_i (n^i(\mathcal{B}) + x_B + 2) \rfloor = n^i_{\text{red}}(\mathcal{B}) + 1$ by definition of $x_B$ and Property 5. This immediately implies that after each assignment, we have $\lfloor \text{red}_i n^i(\mathcal{B}) \rfloor = n^i_{\text{red}}(\mathcal{B})$. By minimality of $x_B$, we also conclude $\lfloor \text{red}_i (n^i(\mathcal{B}) + x_B) \rfloor = n^i_{\text{red}}(\mathcal{B})$, so $x_B \cdot \text{red}_i < 1$. \qed

3 \hspace{1em} Post-processing

Since we consider only the asymptotic competitive ratio in this paper, it is sufficient to prove that a certain ratio holds for all but a constant number of bins: such bins are counted in the additive constant. We will perform a sequence (of constant length) of removals of bins in this section. We will also change the marks of some items to better reflect the actual output, fix the type and color of any remaining bonus items and reduce the sizes of some items to match the values used by EXTREME HARMONIC in its accounting (see line 12 of Algorithm 3).

To begin with, we remove the at most $\sum_{i: \text{red}_i > 0} 1/\text{red}_i$ bins with unmarked medium items (Corollary 1), but not the bonus items. We also remove (at most $N - 1$) blue-open pure blue bins, as well as the single bin with items of type $N$ of total size at most $1 - \varepsilon$, if it exists (see Property 1). Additionally, we remove any bins with a single blue $N$- or $R$-item, as well as all bins that were
assigned to \( \mathcal{N} \) and \( \mathcal{R} \) at the same time as such bins (i.e., during one execution of lines 12 or 17 of \textsc{Mark and Color}). This is at most \( \sum_{i: \text{red}_i > 0} (1 + 1/\text{red}_i) \) bins by Lemma 1. Overall we have removed at most a constant number of bins so far. The packing now has the following property.

This and subsequent Packing Properties will continue to hold during post-processing and will be the basis of our proof of the competitive ratio.

\textbf{Packing Property 1} All medium non-bonus items are marked. Each blue item in \( \mathcal{N}, \mathcal{R} \) and \( \mathcal{B} \) is packed in a bin that contains two blue items, and \( |n_i^{\text{req}}(\mathcal{M}) - n_i(\mathcal{M}) \cdot \text{red}_i| = O(1) \) for \( \mathcal{M} \in \{\mathcal{N}, \mathcal{B}, \mathcal{R}\} \) and each medium type \( i \). All bins with blue \( \mathcal{B} \)-items or red \( \mathcal{R} \)-items are mixed. All bins with items of type \( \mathcal{N} \) are at least \( 1 - \varepsilon \) full.

\textbf{Proof} Lines 12 or 17 of \textsc{Mark and Color} are only executed if all blue items that were assigned to \( \mathcal{N} \) or \( \mathcal{R} \) in a previous run of \textsc{Mark and Color} are already packed into bins with two blue items, since Algorithm \textsc{Pack} prefers to pack a new blue item into an existing blue-open bin. Thus, when we remove all bins with single \( \mathcal{N} \)- or \( \mathcal{R} \)-items, this is only constantly many bins. The blue \( \mathcal{B} \)-items are packed two per bin by the rules of \textsc{Mark and Color}. The equality then follows from Lemmas 1 and 2. The penultimate line follows from the way \textsc{Mark and Color} selects the items to mark. The last line follows from Property 1. \( \square \)

\textbf{Final marking} An overview of our changes of marks and sizes is given in Fig. 7. We will change marks of some items to \( \mathcal{R} \) or \( \mathcal{B} \) if such marks are appropriate. To do this, we run Algorithm 6 for every medium type \( i \) separately. Note that seemingly wrongly marked items like the ones we look for in Algorithm 6 can indeed exist because while the algorithm is running we only mark each item once, when it is assigned to a set; other items could arrive later and be packed with it, invalidating its mark. Packing Property 1 is not affected by Algorithm 6, since we change marks in the correct proportions, and we only add items to \( \mathcal{R} \) and \( \mathcal{B} \) that satisfy Packing Property 1.

Instead of the process described in Algorithm 6, an easier approach might seem to be the following. For changing marks from \( \mathcal{N} \) to \( \mathcal{R} \), we could simply take the group of bins containing the \( \mathcal{N} \)-items that received their mark at the same time as the red \( \mathcal{N} \)-item in the mixed bin. The problem with this approach is that not all these groups have the same size in general, since \( x_N \) may vary. This means the ratio \( \text{red}_i \) would possibly not be maintained for \( \mathcal{R} \) (and then also not for \( \mathcal{N} \)).

\textbf{Packing Property 2} No bins with items in \( \mathcal{N} \) are mixed. No bins with red items in \( \mathcal{B} \) are mixed.

\textbf{Lemma 3} After running Algorithm 6 only constantly many bins need to be removed in order to ensure that Packing Property 2 holds. Packing Property 1 is maintained.

\textbf{Proof} Let us fix a medium type \( i \). After the first loop is finished, there can be at most constantly many red \( \mathcal{N} \)-items and \( \mathcal{B} \)-items in mixed bins, since these sets of items are both colored with the correct proportion of red items by Packing Property 1 and we move a maximal subset of items with the correct proportion to \( \mathcal{R} \). After Algorithm 6 completes, there are at most constantly many blue \( \mathcal{N} \)-items in mixed bins for the same reason. We can remove all of these bins at the end if needed. This does not affect Packing Property 1. \( \square \)

The following lemma helps us to bound the optimal solution later.
Figure 7: Reassigning marks after the input is complete and changing some items to get rid of bonus items. Items are sorted into their correct sets whenever possible, updating the marks that they received while the algorithm was running. Some item sizes are reduced (!). The bins next to the arrows indicate what sets of bins are being reassigned.

**Lemma 4** Let the smallest medium red item of type $i$ in $\mathcal{N}$ be $\tau_i$. It is packed alone in a bin. At most $\frac{1 - \text{red}_i}{2} (n^i(\mathcal{R}) + n^i(\mathcal{N})) + O(1)$ items in $\mathcal{N}$ have size less than $s(\tau_i)$.

**Proof** Item $\tau_i$ is packed alone by Packing Property 2. Each red $\mathcal{N}$-item of type $i$ has size at least $s(\tau_i)$ by definition of $\tau_i$. Furthermore, each *early* blue $\mathcal{N}$-item of type $i$ has size at least $s(p)$, where $p$ is the red $\mathcal{N}$-item that got its mark at the same time (Property 12). However, it is possible that the bin containing $p$ received an additional (large, blue) item later. In that case, after post-processing, the item $p$ does not have mark $\mathcal{N}$ anymore, so it is not considered when determining $\tau_i$, and may in fact be smaller than $\tau_i$. In Algorithm 6, we therefore take care to always select the bins with the smallest early blue $\mathcal{N}$-items (line 1).

We now give an upper bound for the number of early blue $\mathcal{N}$-items that can be smaller than $\tau_i$ but still have mark $\mathcal{N}$ after Algorithm 6 completes. Let $z = |\text{red}_i T|$ be the number of red $\mathcal{N}$-items in mixed bins that receive the mark $\mathcal{R}$ in line 4 of Algorithm 6. Then the total number of early items that got their mark $\mathcal{N}$ at the same time as these $z$ items is upper bounded by $z/(2\text{red}_i) + z/2$ by
Algorithm 6 Final marking for items of type $i$ in EXTREME HARMONIC algorithms. Again we only consider items of medium type $i$.

1: Sort the bins with two blue $N$-items in order of increasing size of the early $N$-items in these bins.
2: for $M = \{N, B\}$ do
3: Let $T$ be the largest integer value such that there exist
   • $\lfloor \text{red}, T \rfloor$ red $M$-items in mixed bins (one per bin) and
   • $(T - \lfloor \text{red}, T \rfloor)/2$ bins with (two) blue $M$-items (so $(T - \lfloor \text{red}, T \rfloor)/2 \in \mathbb{N}$)
4: Assign the $\lfloor \text{red}, T \rfloor$ largest red $M$-items in mixed bins and the blue $M$-items in the first $(T - \lfloor \text{red}, T \rfloor)/2$ bins in the sorted order to $R$
5: end for
6: Let $T$ be the largest integer value such that there exist
   • $\lfloor \text{red}, T \rfloor$ red $N$-items
   • $(T - \lfloor \text{red}, T \rfloor)/2$ mixed bins with (two) blue $N$-items
7: Assign the $\lfloor \text{red}, T \rfloor$ largest red $N$-items and the blue $N$-items in the first $(T - \lfloor \text{red}, T \rfloor)/2$ mixed bins in the sorted order that were not yet reassigned to $R$, to $B$

Lemma 1 We transfer in total $(T - z)/2$ early items from $N$ to $R$. The number of early items that do not get transferred and are potentially smaller than $r_i$ is therefore at most $z/(2\text{red}_i) + z - T/2 \leq z$ since $T \geq z/\text{red}_i$. Clearly, since we move $z$ red $N$-items to $R$, $z$ is at most $\text{red}_i n^i(R)$ afterwards.

Finally, we give an upper bound for the number of late blue $N$-items. There are $n^i(N) - n^i_{\text{red}}(N) = n^i(N) - \lfloor \text{red}, n^i(N) \rfloor$ blue items in $N$ (using Lemma 1). Half of them are packed into existing bins (i.e., as late items). We have $\frac{1}{2}(n^i(N) - \lfloor \text{red}, n^i(N) \rfloor) \leq \frac{1 - \text{red}_i}{2} n^i(N) + \frac{1}{2}$.

Since $\text{red}_i < \frac{1 - \text{red}_i}{2}$ by Property 5, the lemma follows.

A modification of the input In line 21 of EXTREME HARMONIC, bonus items are created. These are medium items which are packed as red items (each such item is in a bin with a large blue item) but violate the ratio $\text{red}_i$. Some of them may still be bonus when the algorithm has finished. Also, some of them may be labeled with a different type than the type they belong to according to their size. We call such items reduced items. Note that EXTREME HARMONIC treated each reduced item as small red items in its accounting (but had in fact packed the larger bonus item). All reduced items are in mixed bins. They are not counted as bonus items.

After EXTREME HARMONIC has finished, and the steps previously described in this section have been applied, we modify the packing that it outputs as described in Algorithm 7. Again we run this algorithm for every medium type $i$. The post-processing is illustrated in Fig. 5; the process in lines 2–6 is illustrated in Fig. 5a; the process in lines 8–10 in Fig. 5c.

Lemma 5 Denote the set of items in a given packing $P$ by $\sigma$. Denote the set of items after applying Algorithm 7 to the packing $P$ by $\sigma'$. Then $P$ induces a valid packing for $\sigma'$, and $\text{OPT}(\sigma') \leq \text{OPT}(\sigma)$. 

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Algorithm 7 Modifying the input after packing all items

1: Let the number of bonus items of type \( i \) be \( T \). \hfill \text{ // These are not reduced items}
2: Color \( \left\lfloor \frac{2 \text{red}_i T}{1 + \text{red}_i} \right\rfloor \) of these items red and the others blue. Mark them all \( \mathcal{R} \).
3: Reduce the size of blue large items in the bins with (now) blue medium items of type \( i \) to \( t_i \).
4: Mark all of these items \( \mathcal{R} \) as well.
5: \( n^i(\mathcal{R}) \leftarrow n^i(\mathcal{R}) + 2T - \left\lfloor \frac{2 \text{red}_i T}{1 + \text{red}_i} \right\rfloor \).
6: \( n^i_{\text{red}}(\mathcal{R}) \leftarrow n^i_{\text{red}}(\mathcal{R}) + \left\lfloor \frac{2 \text{red}_i T}{1 + \text{red}_i} \right\rfloor \).
7: for each reduced item \( p \) do
8:    Let \( j \) be the type with which \( p \) is labeled.
9:    Split up \( p \) into \( \text{redfit}_j \) red items of size \( s(p)/\text{redfit}_j \).
10:   Reduce the size of the newly created items until they belong to type \( j \).

Proof In line 3 of Algorithm 7, items are only made smaller. In line 9, a medium item of type \( i \) is split into \( \text{redfit}_j \) items of some type \( j \). The condition for an item to be labeled with type \( j \) in line 3 of EXTREME HARMONIC is that \( j \) is a small type.

By definition of \( \text{redfit}_j \) and \( \text{redspace}_{\text{needs}}(j) \), we have that \( \text{redfit}_j \) items of type \( j \) have total size at most \( \text{redspace}_{\text{needs}}(j) \). Since \( j \) is a small type, this value is less than \( 1/3 \) by Property 5. This means the newly created items occupy less space than the medium item that they replace. Hence, in both cases we do not increase the amount of occupied space in any bin.

The inequality follows by choosing \( P \) to be an optimal packing for \( \sigma \).

Lemma 6 Lemma 4 still holds after executing Algorithm 7

Proof Algorithm 7 only creates new \( \mathcal{R} \)-items. Therefore, the number of “problematic items” that we want to upper bound, that is, the number of \( N \)-items of size less than \( s(t_i) \), does not increase. As we only increase \( n^i(\mathcal{R}) \) in Algorithm 7 the upper bound in Lemma 4 is not decreased.

Theorem 1 For a given input \( \sigma \), denote the result of all the post-processing done in this section by \( \sigma' = \{p_1, \ldots, p_n\} \). Packing Properties 7 and 2 as well as Invariant 7 still hold after post-processing. For any type \( i \), at the end we have \( |n^i_{\text{red}} - \text{red}_i n^i| = O(1) \), where \( n^i_{\text{red}} \) counts the (correct) total number of red items of type \( i \) after postprocessing. There are no bonus items, and the optimal cost to pack the input did not increase in post-processing.

Proof Let \( P_0 \) be the packing of \( \sigma \) that is output by \( \mathcal{A} \). Let \( P_1 \) be the packing after running Algorithm 6, and let the items packed into \( P_1 \) be \( \sigma_1 \). Packing Properties 1 and 2 hold for \( P_1 \) by Lemma 3. Since in Algorithm 7 we colored the bonus items in the right proportions, the ratio \( \text{red}_i \) of red items holds for each medium type \( i \) up to a constant number of items by Packing Property 1. For a small type \( i \), we have \( |n^i_{\text{red}} - \text{red}_i n^i| = O(1) \) by Property 9. The only effect of postprocessing is that afterwards, \( n^i_{\text{red}} \) counts the actual number of red items of type \( i \) in \( \sigma_1 \). (Some of these red items replace bonus items in \( \sigma_1 \), but the algorithm already counted them in the value \( n^i_{\text{red}} \).) All bonus items were removed, and Packing Properties 1 and 2 remain unaffected.
Consider some medium type $i$. Invariant 1 is not affected by any change of marks or removal of bins. The effect of lines 2–6 of Algorithm 7 is that some bins with bonus items of type $i$ are replaced with unmixed bins with blue type $i$ items. This does not affect the validity of Invariant 1.

To get from $P_0$ to $P_1$, we only removed some bins (and changed marks, which are irrelevant for the optimal solution). Hence $OPT(\sigma_1) \leq OPT(\sigma)$. We can now apply Lemma 5 to the optimal packing for $\sigma_1$ to get the final claim.

\[ \square \]

### 4 Weights

Let $A$ be an EXTREME HARMONIC algorithm. For analyzing the competitive ratio of $A$, we will use the well-known technique of weighting functions. The idea of this technique is the following. We assign weights to each item such that the number of bins that our algorithm uses in order to pack a specific input is equal (up to an additive constant) to the sum of the weights of all items in this input. Then, we determine the average weight that can be packed in a bin in the optimal solution. This average weight for a single bin gives us an upper bound on the competitive ratio. In order to use this technique, we now define a set of weighting functions.

Fix an input sequence $\sigma$. Denote the result of post-processing $\sigma$ by $\sigma' = \{p_1, \ldots, p_n\}$. Let $P$ be the packing of $\sigma$ that is output by $A$. Let $P'$ be the packing of $\sigma'$ induced by $P$ (Lemma 5).

From this point on, our analysis is purely based on the structural properties of the packing $P'$ that we established in Theorem 1. We view $\sigma'$ only as a set of items and not as a list. We prove in Theorem 2 below that this is justified. In particular, we do not make any statement about $A(\sigma')$, since the post-processing done in Algorithm 6 means that some items (e.g., the ones introduced in lines 8–10) do not have clearly defined arrival times, and it is not obvious how to define arrival times for them in order to ensure that $A(\sigma') = A(\sigma)$.

The class of an item of type $t$ is $\text{leaves}(t)$, if it is blue, and $\text{needs}(t)$ if it is red. The class of an item $p$ indicates how much space is reserved for red items in the bin containing $p$ (both if $p$ is red and if $p$ is blue), namely $\text{redspace}_i$ space if the class is $i$.

**Lemma 7** For $k \in \{1, \ldots, K\}$, red items of class $k$ are either all medium or all small. If they are medium, they are of the unique type $t$ such that $k = \text{needs}(t)$.

**Proof** By Property 3, if for a red item of type $t$ we have $\text{redspace}_{\text{needs}(t)} > 1/3$, then it is a medium item; in this case, type $t$ is the only type such that $\text{needs}(t) = k$ since each medium type is in a different class by Property 4 and for each small item $p$ we have $\text{redspace}_{\text{needs}(t(p))} \leq 1/3$. \[ \square \]

The class of a bin with red items is the class of those red items. This is well-defined, as each bin contains red items of only one type.

**Definition 5** Let $k$ be the minimum class of any unmixed red bin. Let $\nu$ be a smallest item in the unmixed red bins of class $k$. If all red items are in mixed bins, we define $k = K + 1$ (and $\nu$ is left undefined).

If $k \in \{1, \ldots, K\}$, then by this definition we have $k = \text{needs}(t(\nu))$. If $\text{redspace}_k \leq 1/3$, there may be several red items in one bin, as in SUPER HARMONIC (in HARMONIC++, there is
always at most one red item per bin). Also, there can be several types \( t \) such that \( k = \text{needs}(t) \). If \( \text{redspace}_k > 1/3 \), there is only one type \( t \) such that \( k = \text{needs}(t) \), and this is a medium type; it is only in this case that we need to consider the item \( r \) and in particular its exact size.

We follow Seiden’s proof, adapting it to take the marks into account. In order to define the weight functions, it is convenient to introduce some additional types. Note that the algorithm does not depend on the weight functions in any way. It is also unaware of the added type thresholds. First of all, for each \( i \) such that \( 1/3 < \text{redspace}_i < 1/2 \), we add a threshold \( 1 - \text{redspace}_i \) between \( t_2 = 2/3 \) and \( t_3 = 1/2 \) (see Property 6). For a type \( t \) with upper bound \( 1 - \text{redspace}_i \) we define leaves\((t) = i \). We furthermore add a threshold \( 1 - s(r) \) in case \( r \) is medium. This splits an existing type into two types. For the new type \( t^1 \) with upper bound \( 1 - s(r) \), we define leaves\((t^1) = k \), where \( k = \text{needs}(t(r)) \). For the new type \( t^2 \) with lower bound \( 1 - s(r) \), we define leaves\((t^2) = k - 1 \). To maintain consistency with the rest of the paper, we add negative indices for the types to maintain \( t_3 = 1/2 \). That is, if there are \( a \) values in \( \text{REDSPACE} \) in the range \((1/3, 1/2) \), the corresponding values \( 1 - \text{redspace}_i \) and the threshold \( 1 - s(r) \) (if \( r \) is medium) are stored in ascending order in the values \( t_2, t_1, \ldots, t_{2-a}, \) and \( t_{1-a} = 2/3, t_{-a} = 1 \).

For large items, the value of leaves is only used by the algorithm to check whether small items can be combined with them. Moreover, for small items, the only relevant piece of information is that at least \( 1/3 \) of space is left by large items. An EXTREME HARMONIC algorithm defines leaves\((2) \) such that \( \text{redspace}_{\text{leaves}(2)} = 1/3 \) (and then ignores this value when considering to pack a medium item with a large item). The additional types simply make the function leaves more accurate, in particular with the threshold \( 1 - s(r) \), which the algorithm does not know. It can be seen that the definition of \( k \) (and \( r \)) is not affected by these new types, as only types of large (i.e., blue) items are changed, and \( k \) and \( r \) are defined based on unmixed red bins.

The weights of an item \( p \) will depend on \( s(r) \), the class of the red and blue items of type \( t(p) \) relative to \( k \), and the mark of \( p \). This means we essentially define them for every possible input sequence separately. The value of \( k \) and \( s(r) \) (and the marks) become clear by running the algorithm. We do not write the dependence on \( \sigma \) explicitly since we have fixed \( \sigma \) in this section.

The two weight functions of an item of size \( x \), type \( t \) and mark \( \mathcal{M} \) are given by Table 1. Recall that \( \varepsilon = t_N \). Regarding \( w_k(p) \), non-medium items have no mark and are handled under the case \( \mathcal{M} \neq R \). (Unmarked medium items were removed in the previous section). Note that \( w_k(p) \) does not depend on \( s(r) \) or the added types, as \( \text{red}_r = 0 \) and \( \text{needs}(t) = 0 \) for all items larger than \( 1/2 \). In contrast, \( v_{k,s(r)}(p) \) depends on \( s(r) \), as the value of \( \text{leaves}(t) \) changes at the threshold \( 1 - s(r) \) if \( r \) is medium as described above.

Note that \( w_k \) counts all blue items, and \( v_{k,s(r)} \) counts all red items. By definition of \( k \) and Packing Property 2 we have \( \mathcal{M} = \mathcal{R} \) for all items with type \( t \) such that \( \text{needs}(t) < k \). For simplicity, we ignore the markings for any type \( t \) with \( \text{needs}(t) > k \), essentially assuming that there are no items of such types that are marked \( \mathcal{R} \). It is clear that this assumption can only increase the weight of any item.

Define \( v_k(p) = v_{k,t(p)}(p) \). Note that for any item \( p \), we have \( v_k(p) \geq v_{k,s(r)}(p) \) since \( 1 - s(r) \geq 1 - t_{t(r)} \), and this is the point at which the \( \text{leaves} \) function drops below \( k \).
Table 1: Weighting functions of class $k$ for an item $p$ of size $x$, type $t$ and mark $M$.

| $w_k(p)$ | $v_{k,s}(p)$ |
|-----------------|-----------------|
| $w_k(x, t, M)$ | $v_{k,s}(x, t)$ |
| $\frac{1-\text{red}_k}{\text{bluefit}_t} + \frac{\text{red}_k}{\text{redfit}_t}$ | $\frac{1-\text{red}_k}{\text{bluefit}_t} + \frac{\text{red}_k}{\text{redfit}_t}$ |
| if $t \leq N$, needs($t$) > $k$ or needs($t$) = 0 | if $t \leq N$, leaves($t$) < $k$ |
| $\frac{1-\text{red}_k}{\text{bluefit}_t} + \frac{\text{red}_k}{\text{redfit}_t}$ | $\frac{\text{red}_k}{\text{redfit}_t}$ |
| if $t \leq N$, needs($t$) = $k$, $M \neq R$ | if $t \leq N$, leaves($t$) $\geq k$ |
| $\frac{1-\text{red}_k}{\text{bluefit}_t}$ | $\frac{1}{1-\varepsilon} x$ |
| if $t \leq N$, 0 < needs($t$) < $k$ | if $t = N$ |
| $\frac{1}{1-\varepsilon} x$ |   |

**Theorem 2**  For any input $\sigma$ and EXTREME HARMONIC algorithm $A$, defining $k$ as above we have

$$\mathcal{A}(\sigma) \leq \min \left\{ \sum_{i=1}^{n} w_k(p_i), \sum_{i=1}^{n} v_k(p_i) \right\} + O(1)$$ \hspace{0.5cm} (5)$$

**Proof** Our goal is to upper bound $\mathcal{A}(\sigma)$ by the weights of the items $p_1, \ldots, p_n$, which are the items in $\sigma'$. We will show that the number of bins in the packing $P'$ is upper bounded by the first term in (5), with the additive constant $O(1)$ corresponding to the bins removed in post-processing. We follow the line of the corresponding proof in Seiden [19].

Let $\text{TINY}$ be the total size of the items of type $N$ in $\sigma'$. Let $\text{UNMIXEDRED}$ be the number of unmixed red bins in $P'$. Let $B_i$ and $R_i$ be the number of bins in $P'$ containing blue items of class $i$ and type less than $N$, and red items of class $i$, respectively. Note that this means that mixed bins are counted twice.

If $\text{UNMIXEDRED} = 0$, every red item is placed in a bin with one or more blue items, and $k = K + 1$. In this case, the total number of bins in $P'$ is exactly the total number of bins containing blue items. Each bin containing items of type $N$ contains at least a total size of $1 - \varepsilon$ due to Packing Property [1]. The bins used to pack $\text{TINY}$ are pure blue and $\sum_{t(p_i) = N} w_{K+1}(p_i) = \sum_{t(p_i) = N} v_{K+1}(p_i) = \text{TINY}/(1-\varepsilon)$. For each item $p$ of type $t < N$, we have $w_{K+1}(p) = \frac{1-\text{red}_k}{\text{bluefit}_t} < v_{K+1}(p)$. We see that $w_{K+1}$ counts all the bins with blue items, and $\mathcal{A}(\sigma) \leq \frac{\text{TINY}}{1-\varepsilon} + \sum_{i=0}^{K} B_i \leq \sum_{i=1}^{n} w_{K+1}(p_i)$ (since $B_0$ does not include bins with items of type $N$).

If $\text{UNMIXEDRED} > 0$, then $k = \text{needs}(t(v))$, and there is an unmixed red bin of class $k$. By Invariant[1] all bins with a blue item of class $i \geq k$ must be mixed bins. These are the bins which contain blue items of any type $j$ such that leaves($j$) $\geq k$; if $r$ is medium, this means exactly the large items with size at most $1 - s(r)$. We conclude

$$\text{UNMIXEDRED} \leq \sum_{i=1}^{K} R_i - \sum_{i=k}^{K} B_i.$$ \hspace{0.5cm} (6)$$

Let $R_k(\neg R)$ be the number of bins in $P'$ containing red items of class $k$ that are not marked $R$. If items of class $k$ are not medium, then $R_k(\neg R) = R_k$. This is a well-defined condition by Lemma [7]. Let $R^+_i$ be the number of unmixed bins in $P'$ containing red items of class $i$. Since every red
item with class less than \( k \) (that is, red items of any type \( j \) such that \( \text{needs}(j) < k \)) is placed in a mixed bin by definition of \( k \), we have

\[
\text{UNMIXEDRED} \leq \sum_{i=k+1}^{K} R_i^* + R_k(-\mathcal{R}) \leq \sum_{i=k+1}^{K} R_i + R_k(-\mathcal{R}).
\]  

(7)

The first inequality holds because the red items marked \( \mathcal{R} \) are in mixed bins by Packing Property \( \mathbb{1} \) (If \( r \) is not medium, \( R_k(-\mathcal{R}) = R_k \), so it also holds.) By combining (6) and (7), we have

\[
\text{UNMIXEDRED} \leq \min \left\{ \sum_{i=k+1}^{K} R_i + R_k(-\mathcal{R}), \sum_{i=1}^{K} R_i - \sum_{i=k}^{K} B_i \right\}. \quad \text{So if UNMIXEDRED} > 0,
\]

the total number of bins in \( P' \) is at most

\[
\frac{1}{1-\varepsilon} \text{TINY} + \text{UNMIXEDRED} + \sum_{i=0}^{K} B_i + O(1)
\]

\[
\leq B_0 + \frac{1}{1-\varepsilon} \text{TINY} + \min \left\{ \sum_{i=k+1}^{K} R_i + R_k(-\mathcal{R}) + \sum_{i=1}^{K} B_i, \sum_{i=1}^{K} R_i + \sum_{i=1}^{k-1} B_i \right\} + O(1). \quad \text{(8)}
\]

Let \( J \) be the set of types whose blue items are packed in pure blue bins, including type 1 and type \( N \). For each item \( p \) of type \( t \neq N \), \( t \in J \), we have \( \text{leaves}(t) = 0 < k \), so \( v_{k,s(t)}(p) = \frac{1-\text{red}_{\text{bluefit}}}{1-\varepsilon} \). Furthermore, for all \( t \neq N \) we have \( w_k(p) \geq \frac{1-\text{red}_{\text{bluefit}}}{1-\varepsilon} \). We conclude \( \sum_{j \in J} \sum_{t(p_i) = j} v_{k,s(t)}(p_i) \geq \sum_{j \in J} \sum_{t(p_i) = j} v_{k,s(t)}(p_i) \geq B_0 + \frac{\text{TINY}}{1-\varepsilon} \).

In the first term of the minimum in (8), we count all bins with blue items except the pure blue bins, all bins with red items of classes above \( k \), and the bins with red items of class \( k \) that are not marked \( \mathcal{R} \). (If red items of class \( k \) are small, this means all red items of this class.) This term is therefore upper bounded by \( \sum_{j \notin J} \sum_{t(p_i) = j} w_k(p_i) \). In the second term of the minimum in (8), we count all bins with red items, as well as bins with blue items of class at least 1 and at most \( k - 1 \). The second term is therefore upper bounded by \( \sum_{j \notin J} \sum_{t(p_i) = j} v_{k,s(t)}(p_i) \). As noted above Theorem \( 2 \) this is at most \( \sum_{j \notin J} \sum_{t(p_i) = j} v_{k}(p_i) \).

\[ \square \]

Note \quad In his proof, Seiden \cite{19} defines an item \( e \) as the smallest red item in an indeterminate red group bin, and proceeds to argue using the class of \( e \). This only works because there is one red item in each bin, so there could not be a larger red item of a smaller class that is in an indeterminate group bin. The proof structure above (defining first \( k \) and then \( r \)) allows \text{SUPER HARMONIC} algorithms to pack multiple red items in one bin as well.

Seiden expresses the upper bound as a maximum over \( k \), even though for a fixed input sequence, the value of \( k \) is of course fixed. While the resulting expression is correct, we prefer the easier and more direct formulation in Theorem \( 2 \) above.

5 The offline solution

Having derived an upper bound for the total cost of an \text{EXTREME HARMONIC} algorithm in Theorem \( 2 \) in order to calculate the asymptotic competitive ratio \( 1 \), we now need to lower bound the
optimal cost of a given input after post-processing. This will again depend on what the value of \( k \) is. There are two main cases if \( k \in \{ 1, \ldots, K \} \): \( \tau \) is medium and \( \tau \) is small. The case \( k = K + 1 \) is much easier, because \( w_{K+1}(p) \leq v_{K+1}(p) \) for each item \( p \), so \( \sum_{i=1}^{n} w_{K+1}(p) \) upper bounds the cost of \( A \) by Theorem 2 \( \text{and this sum does not depend on any marks of items. We can therefore use a standard knapsack search as in Seiden [19] (for this case) and other papers.} \)

For \( k \in \{ 1, \ldots, K \} \), we will be interested in the weights of items for a fixed value of \( k \). It can be seen that in the range \( (1/2, 1] \), the function \( v_k(p) \) changes at most once (viewed as a function of the size of \( p \)), namely at the threshold \( 1 - t_{t(\tau)} \), where \( \text{leaves}(k) \) drops below \( k \) if \( \tau \) is medium. On the other hand \( w_k(p) = 1 \) in the entire range \( (1/2, 1] \). For a fixed value of \( k < K + 1 \), we therefore reduce the number of types again as follows. Recall that \( t_3 = 1/2 \), and \( \tau \) is determined by \( k \).

**Case 1: \( \tau \) is medium** We set \( t_2 = 1 - t_{t(\tau)} \), \( t_1 = 2/3 \) and \( t_0 = 1 \). We set \( \text{leaves}(2) = k \), \( \text{leaves}(1) < k \) such that \( \text{redspace}_{\text{leaves}(1)} = 1/3 < t_{t(\tau)} \), and \( \text{leaves}(0) = 0 \).

**Case 2: \( \tau \) is small** We set \( t_2 = 2/3 \) and \( t_1 = 1 \) as in EXTREME HARMONIC itself (Property 6). We have \( \text{redspace}_{\text{leaves}(2)} = 1/3 \), and \( \text{leaves}(1) = 0 \).

After these changes, Theorem 2 remains valid for any fixed \( k \), as \( w_k \) and \( v_k \) remain unchanged (given \( k \)). This holds even though if \( \tau \) is medium, the types do not match the types used by EXTREME HARMONIC; the important property is that they match the behavior of EXTREME HARMONIC for any fixed value of \( k < K + 1 \).

We now define patterns for the two main cases. Intuitively, a pattern describes the contents of a bin in the optimal offline solution. If \( \tau \) is medium, a *pattern of class \( k \)* is an integer tuple \( q = (q_0, q_1, \ldots, q_{N-1}, (q_{t(\tau)}^N, q_{t(\tau)}^B, q_{t(\tau)}^R)) \) where \( q_i \in \mathbb{N} \cup \{ 0 \} \), \( q_{t(\tau)}^M \in \mathbb{N} \cup \{ 0 \} \) for \( M \in \{ \mathcal{N}, \mathcal{B}, \mathcal{R} \} \), \( q_{t(\tau)}^N + q_{t(\tau)}^B + q_{t(\tau)}^R = q_{t(\tau)} \) and

\[
\sum_{i=0}^{N-1} q_i t_{i+1} < 1. \tag{9}
\]

The values \( q_i \) describe how many items of type \( i \) are present in the bin. The value \( q_{t(\tau)}^M \) counts the number of items of type \( t(\tau) \) and mark \( M \). It can be seen that any feasible packing of a bin can be described by a pattern: the only quantity that is not fixed by a pattern is the total size of the items of type \( N \), which we will call sand. However, by (9), there can be at most \( 1 - \sum_{i=0}^{N-1} q_i t_{i+1} \) of sand in a bin packed according to pattern \( q \). Conversely, for each pattern, a set of items matching the pattern that fit into a bin can be found by choosing the size of each item close enough (from above) to the lower bound \( t_{i+1} \) for its type; then (9) guarantees the items will fit.

If \( \tau \) is small, we define a pattern of class \( k \) as an integer tuple \( q = (q_1, \ldots, q_{N-1}) \) where \( q_i \in \mathbb{N} \cup \{ 0 \} \) and (9) holds using \( q_0 = 0 \) (note that the values \( t_1 \) and \( t_2 \) depend on whether \( \tau \) is medium or small, but the definition of \( t(\tau) \) is consistent across these two cases).

There are only finitely many patterns for each value of \( k \). Denote this set by \( Q_k \) for \( k = 1, \ldots, K \). If \( \tau \) is small or \( k = K + 1 \), \( Q_k \) is a fixed set, denoted by \( Q \).

For a given weight function \( w \) of class \( k \), we define \( w(q) \) for some pattern \( q \) as the sum of the weights of the non-sand items in it plus \( w(1 - \sum_{i=0}^{N-1} q_i t_{i+1}, N, \emptyset) \). As noted, \( 1 - \sum_{i=0}^{N-1} q_i t_{i+1} \) is
an upper bound for the amount of sand in a bin packed according to pattern \(q\); this value is not necessarily in the range \((0, t_N]\). If \(t\) is medium, \(q_0 = 0\). Pattern \(q\) specifies all the information we need to calculate \(w(q)\), as \(w\) does not depend on the precise size of non-sand items, and for class \(k\) we know exactly how many items there are (if any) for each mark.

We can describe the solution of an offline algorithm for a given post-processed input \(\sigma'\) by a distribution \(\chi\) over the patterns, where \(\chi(q)\) indicates which fraction of the items in it.

Proof

We ignore additive constants in this proof, as we will divide by \(\text{OPT}(\sigma')\) at the end to achieve our result. The pattern \(q^1\) contains an \(N\)-item that is strictly smaller than \(t\). We apply Lemma 4 for \(i = t(\tau)\) (ignoring the additive constant) to get

\[
\frac{\chi(q^1) \text{OPT}(\sigma')}{2} \leq \frac{1}{2} \left( \chi(q^1) + \sum_{q \neq q^1} \chi(q) q^1_B(q) \right) \text{OPT}(\sigma'),
\]

and the bound in the lemma follows. \(\square\)
Lemma 9  In $q^2$, the $B$-item $p$ of type $t(r)$ is blue.

Proof  EXTREME HARMONIC did not pack $p$ alone in a bin as a red item, since it is smaller than $r$. But by Packing Property 2, $p$ also was not packed in a mixed bin as a red $B$-item.

Lemma 10  If $r$ is medium, then

$$\frac{1}{2} \chi(q^2) \leq \sum_{j:0 < \text{needs}(j) \leq \text{leaves}(t(r))} \sum_{q} \text{red}_j \frac{\chi(q)q_j(q)}{\text{redfit}_j}.$$  

Proof  Again, we ignore additive constants. There are $\chi(q^2) \text{OPT}(\sigma')$ bins packed with pattern $q^2$, meaning that $\sigma'$ contains at least $\chi(q^2) \text{OPT}(\sigma')$ blue $B$-items of type $t(r)$ by Lemma 9. So in the packing $P'$, there exist at least $\frac{1}{2} \chi(q^2) \text{OPT}(\sigma')$ bins with two blue $B$-items of type $t(r)$ and red items. The red items are red-compatible with those $B$-items. That is, each such red item is of a type $j$ such that $0 < \text{needs}(j) \leq \text{leaves}(t(r))$.

The number of items of type $j$ in $\sigma'$ is given by $\sum_q \chi(q)q_j(q) \cdot \text{OPT}(\sigma')$. By Theorem 1, the number of red items of type $j$ is $\text{red}_j \sum_q \chi(q)q_j(q) \cdot \text{OPT}(\sigma')$. We place $\text{redfit}_j$ red items together in each bin. This means that the number of bins in $P$ with red items of type $j$ is $\text{red}_j \sum_q \chi(q)q_j(q) \cdot \text{OPT}(\sigma')$. Summing over all types $j$ with $0 < \text{needs}(j) \leq \text{leaves}(t(r))$, we find that

$$\frac{1}{2} \chi(q^2) \text{OPT}(\sigma') \leq (\text{number of bins in } P' \text{ with two blue } B\text{-items of type } t(r) \text{ and red items})$$

$$\leq (\text{number of bins in } P' \text{ with red items that fit with items of type } t(r))$$

$$= \left( \sum_{j:0 < \text{needs}(j) \leq \text{leaves}(t(r))} \sum_{q} \text{red}_j \frac{\chi(q)q_j(q)}{\text{redfit}_j} \right) \cdot \text{OPT}(\sigma').$$

5.1 Linear program

Maximizing the minimum in (11) is the same as maximizing the first term under the condition that it is not larger than the second term—except that this condition might not be satisfiable, in which case we need to maximize the second term. For each value of $k \in \{1, \ldots, K\}$, we will therefore consider two linear programs, and furthermore these linear programs will differ depending on whether $r$ is medium or small, so that in total we get four different LPs which we will call $P_w^k, P_w^k, P_v^k, P_v^k$ (we will use the notation $P_w^k (P_v^k)$ whenever we want to refer to both $P_w^k$ and $P_v^k$). Let $Q_k = \{q^1, \ldots, q^{|Q_k|}\}$ and let $\chi_i = \chi(q^i), w_{ik} = w_k(q^i), v_{ik} = v_k(q^i), n_{ij} = q_j(q^i), m_i = q_{t(v)}(q^i)$. If $r$ is medium, $P_w^k$ is the following linear program.

28
\[
\begin{align*}
\text{max} & & \sum_{i=1}^{\left|Q_k\right|} \chi_i w_{ik} \\
\text{s.t.} & & \chi_1 - \frac{1 - \text{red}_{t(r)}}{1 + \text{red}_{t(r)}} \sum_{i=3}^{\left|Q_k\right|} \chi_i m_i \leq 0 \\
& & \frac{1}{2} \chi_2 - \sum_{j: \text{needs}(j) \leq \text{leaves}(t(r))}^{\left|Q_k\right|} \frac{\text{red}_j}{\text{redfit}_j} \chi_i n_{ij} \leq 0 \\
& & \sum_{i=3}^{\left|Q_k\right|} \chi_i (w_{ik} - v_{ik}) \leq 0 \\
& & \sum_{i=1}^{\left|Q_k\right|} \chi_i \leq 1 \\
& & \chi(q) \geq 0 \quad \forall q \in Q_k 
\end{align*}
\]

\((P_{w, \text{med}}^k)\)

\(P_{w, \text{med}}^k\) has a very large number of variables but only four constraints (apart from the nonnegativity constraints). Constraint (13) is based on Lemma 8, where we have used that \(q^2\) does not contain any item marked \(N\) or \(R\), implying \(m_2 = 0\). Constraint (14) is based on Lemma 10, using that \(q^1\) and \(q^2\) do not contain non-sand items of size less than \(1/3\), so \(n_{1j} = 0\) and \(n_{2j} = 0\) for all \(j\) for which \(\text{needs}(j) \leq \text{leaves}(t(r))\). Constraint (15) says simply that the objective function must be at most \(\sum_{i=1}^{\left|Q_k\right|} \chi_i v_{ik}\) (using that \(w_{ik} = v_{ik}\) for \(i = 1, 2\), which we will prove in Lemma 11): if this does not hold, we should be solving the linear program \(P_{w, \text{med}}^k\), which has objective function \(\sum_{i=1}^{\left|Q_k\right|} \chi_i v_{ik}\), instead. The final constraints (16) and (17) say that \(\chi\) is a distribution.

**Lemma 11** \(v_{1k} = w_{1k} = w_{2k} = v_{2k}\).

**Proof** Recall that \(q^1\) contains one \(N\)-item of type \(t(r)\), i.e. the same type as \(r\), and one item larger than \(1 - s(r)\). Call the \(N\)-item \(t'(r)\) and the large one \(L\); note that \(t(L) = 2\). We have that \(w_k(q^1) = w_k(t'(r)) + w_k(L) + S\), where \(S\) is an upper bound for the weight of the sand, and \(v_k(q^1) = v_k(t'(r)) + v_k(L) + S\) (the maximum possible amount of sand and hence also its weight is equal in the two cases). As \(\text{red}_2 = 0\) (\(L\) is larger than \(1/2\) and such items are never red), and \(L\) is too large to be combined with \(r\), we have \(w_k(L) = v_k(L) = 1/\text{bluefit}_2 = 1\).

For \(w_k(t'(r))\), consider that \(t'\) and \(r\) have the same type, and as the mark of \(t'\) is \(N\), we get \(w_k(t') = \frac{1 - \text{red}_{t(r)}}{\text{bluefit}_{t(r)}} + \frac{\text{red}_{t(r)}}{\text{redfit}_{t(r)}}\). For type \(t(r)\), we have that \(\text{leaves}(t(r)) < \text{needs}(t(r))\) (Property 2). Therefore, \(v_k(t') = \frac{1 - \text{red}_{t(r)}}{\text{bluefit}_{t(r)}} + \frac{\text{red}_{t(r)}}{\text{redfit}_{t(r)}} = w_k(t')\). This shows that \(v_{1k} = w_{1k}\).

The pattern \(q^2\) contains one \(B\)-item of type \(t(r)\) (denoted by \(t''\)) and one item larger than \(1 - r\) (again denoted by \(L\)). We have \(w_k(t'') = w_k(t')\) since the weight \(w_k\) is the same for \(N\)- and \(B\)-items.

\[\text{We also have } n_{3j} = 0, \text{ but we keep the term for } i = 3 \text{ in (14) to make the dual easier to write down.}\]
of the same class. As above, we find \( w_k(\mathcal{E}) = v_k(\mathcal{E}) = 1 \) and \( v_k(\mathcal{E}) = 1 - \frac{\text{red}_{i(t)}}{\text{bluefit}_{i(t)}} + \frac{\text{red}_{i(t)}}{\text{redfit}_{i(t)}} = w_k(\mathcal{E}) \).

This shows \( w_{2k} = v_{2k} \) and \( w_{1k} = w_{2k} \). □

For the case when \( r \) is small, we do not have conditions (13) and (14), and the linear program \( P_{w}^{k, sm} \) looks as follows. Here we denote the set of patterns simply by \( Q \) since it is the same for all values of \( k \) for which \( \text{redspace}_k \leq 1/3 \). In this setting, \( q^1, q^2, q^3 \) do not have a special meaning.

\[
\begin{align*}
&P_{w}^{k, sm} \\
\text{max} & \quad \sum_{i=1}^{\mid Q \mid} \chi_i w_{ik} \\
\text{s.t.} & \quad \sum_{i=1}^{\mid Q \mid} \chi_i (w_{ik} - v_{ik}) \leq 0 \\
& \quad \sum_{i=1}^{\mid Q \mid} \chi_i \leq 1 \\
& \quad \chi(q) \geq 0 \quad \forall q \in Q
\end{align*}
\]

\textbf{Intermezzo} It is useful to consider the value of \( w_{1k} \) (etc.). We have not discussed the values of the parameters yet. However, as an example, for the algorithm HARMONIC++, two of the types are \((341/512, 1)\) and \((1/3, 171/512)\) (types 1 and 18). Let us consider the case where at the end of the input, an item of type 18 is alone in a bin, and no smaller items are alone in bins. For this case, for HARMONIC++, the two weighting functions for the pattern which contains types 1 and 18 both evaluate to

\[
1 + \frac{1 - 0.176247}{2} + \frac{0.176247}{1} + \frac{1}{1 - \frac{50}{1536}} \cdot \frac{1}{1536} \approx 1.58879.
\]

In other words, a distribution \( \chi \) consisting only of this one pattern immediately gives a lower bound of 1.58879 on the competitive ratio of HARMONIC++.

Our improved packing of the medium items and our marking of them ensures that this distribution, where the optimal solution uses critical bins exclusively, can no longer be used, since it is not a feasible solution to \( P_{w}^{k} \). This is the key to our improvement over HARMONIC++.

\section{5.2 Dual program}

Our general idea is as follows: We consider the duals of the linear programs given above. These dual LPs have variables \( y_1, \ldots, y_4 \) or \( y_3, y_4 \), respectively. Any feasible solution for the dual (which is a minimization problem) is an upper bound on the competitive ratio of our algorithm by duality and by (11). We are interested in feasible dual solutions with objective value \( c \), where \( c \) is our target competitive ratio.

\textbf{Case 1:} \( r \) is small The dual of \( P_{w}^{k, sm} \) is the following.

\[
\begin{align*}
&P_{w}^{k, sm} \\
\text{min} & \quad y_4 \\
\text{s.t.} & \quad (w_{ik} - v_{ik}) y_3 + y_4 \geq w_{ik} \quad i = 1, \ldots, \mid Q \mid \\
& \quad y_i \geq 0 \quad i = 3, 4
\end{align*}
\]
If the constraint (22) does not hold for pattern \(q_i\) and a given dual solution \(y^*\), we have

\[
(1 - y_3^*) w_{ik} + y_3^* v_{ik} > y_4^* \tag{23}
\]

We need to determine if there is a pattern such that (23) holds. For \(y_3^* \in [0, 1]\), the left hand side of (23) represents a weighted average of the weights \(w_{ik}\) and \(v_{ik}\). We add the condition \(y_3 \leq 1\) to \(D_{w,sml}^k\). A feasible solution with objective value \(c\) and \(y_3 \leq 1\) exists for \(D_{w,sml}^k\) if and only if a feasible solution with objective value \(c\) and \(y_3 \leq 1\) exists for \(D_{v,sml}^k\), as (23) is now symmetric in \(w\) and \(v\). This means that feasibility of \(D_{w,sml}^k\) and \(D_{v,med}^k\) with \(y_3 \leq 1\) can be checked at the same time. Again, note that it is sufficient for our purposes to find a feasible solution.

We define \(\omega_k(p) = (1 - y_3^*) w_k(p) + y_3^* v_k(p)\) for each item \(p\). Since \(r\) is small, there are no marked items of type \(t(r)\), so \(\omega_k(p)\) depends only on the type and size of \(p\). The problem of determining \(W = \max_{q \in Q} \omega_k(q)\) for a given value of \(y_3^*\) is a simple knapsack problem, which is straightforward to solve using dynamic programming.

All that remains to be done is to determine a value for \(y_3^*\) for given \(k\) such that \(W \leq c\). In order to do this, we use a binary search in the interval \([0, 1]\). We start by setting \(y_3^* = 1/2\) and compute \(W\). If \(W \leq y_3^*\), \(D_{w,sml}^k\) and \(D_{v,sml}^k\) have objective value at most \(y_3^*\) and we are done. Else, the dynamic program returns a pattern \(q\) such that \(\omega_k(q) > y_4^*\). For this pattern \(q\), we compare its weights according to \(w\) and \(v\). If \(w_{ik} > v_{ik}\), we increase \(y_3^*\), else we decrease it (halving the size of the interval we are considering). If after 20 iterations we still have no feasible solution, we return infeasible. This may be incorrect (it depends on how long we search), but our claimed competitive ratio depends only on the correctness of feasible solutions.

Summarizing the above discussion, if \(r\) is small, proving that an EXTREME HARMONIC algorithm is \(c\)-competitive can be done by running the binary search for \(k = \text{needs}(t^*)\) using \(y_3^* = c\). If \((y_3^*, y_4^*)\) is a feasible solution for \(D_{w,sml}^k\), then \((1 - y_3^*, y_4^*)\) is a feasible solution for \(D_{v,sml}^k\).

**Case 2: \(r\) is medium**  
For the more interesting case when \(r\) is medium, the dual \(D_{w,med}^k\) of the program \(P_{w,med}^k\) is the following.

\[
\begin{align*}
\min & \quad y_4 \\
\text{s.t.} & \quad y_1 + y_4 \geq w_{1k} \tag{24} \\
& \quad \frac{1}{2} y_2 + y_4 \geq w_{2k} \tag{25} \\
& \quad \left(\frac{1 - \text{red}_{j(t)} m_{ij}}{1 + \text{red}_{t(r)}} - \sum_{j:0 < \text{needs}(j) \leq \text{leaves}(t(r))} \frac{\text{red}_j}{\text{redfit}_j} n_{ij} \right) y_1 - y_2 + \left(\frac{1}{3} - \frac{\text{red}_{j(t)} m_{ij}}{1 + \text{red}_{t(r)}} \right) y_2 + (w_{ik} - v_{ik}) y_3 + y_4 \geq w_{ik} \tag{26} \\
& \quad y_i \geq 0 \quad \text{for } i = 1, 2, 3, 4 \tag{27}
\end{align*}
\]

Again we restrict ourselves to solutions with \(y_3^* \in [0, 1]\). If the value \(y_4^* = c \geq w_{1k} = w_{2k}\), the conditions (24) and (25) are automatically satisfied by (27). In this case we can set \(y_3^* = 0\) and \(y_2^* = 0\). In effect, this reduces \(D_{w,med}^k\) to \(D_{w,sml}^k\) for which we already know how to find a feasible
value for \( y_3^* \). We therefore ignore the entire marking done by the algorithm and set the weight for each item to be the weight for the case that its mark is not \( \mathcal{R} \). Then weights again do not depend on marks and we apply the method from Case 1.

Let us now consider the case \( y_3^* = c < w_{1k} \). For given \( y_3^* \) we need to determine if \( D_{v,k}^{\text{med}} \) and \( D_{w,k}^{\text{med}} \) are feasible; this requires finding suitable values for \( y_1, y_2 \) and \( y_3 \). If a solution vector \( y^* \) is feasible for \( D_{w,k}^{\text{med}} \) (or \( D_{v,k}^{\text{med}} \)), \( y_3^* < w_{1k} = v_{1k} = w_{2k} = v_{2k} \), and constraint (24) or (25) is not tight, then we can decrease \( y_1^* \) and/or \( y_2^* \) and still have a feasible solution. We therefore restrict our search to solutions for which (24) and (25) are tight, and \( y_3^* < w_{1k} \). Then

\[
\begin{align*}
  y_1^* &= w_{1k} - y_4^* > 0 \quad (28) \\
  y_2^* &= 2(w_{1k} - y_4^*) > 0. \quad (29)
\end{align*}
\]

This means that given \( y_3^* < w_{1k} \), we know the values of \( y_1^* \) and \( y_2^* \). We can therefore prove \( y_4^* \) is a feasible objective value for \( D_{w,k}^{\text{med}} \) by giving \( y_3^* \)-values that make the linear program feasible. If constraint (26) does not hold for pattern \( q_i \) \((i \geq 3)\) and a given dual solution \( y^* \), we have the following by some simple rewriting:

\[
(1 - y_3^*)w_{ik} + y_3^*v_{ik} + \frac{1 - \text{red}_{l(t)}}{1 + \text{red}_{l(t)}} m_i y_1^* + y_2^* \sum_{j:0 < \text{needs}(j) \leq \text{leaves}(l(t))} \frac{\text{red}_j}{\text{redfit}_j} n_{ij} > y_4^* \quad (30)
\]

If this holds for some pattern \( q \) that contains an \( \mathcal{R} \)-item, then it obviously also holds if we replace that \( \mathcal{R} \)-item by an \( \mathcal{N} \)-item of the same type. This gives a pattern with the same values \( m_i = q_i^{l(t)}(q^i) \) and \( n_{ij} = q_j(q^i) \) but a higher value for \( w_{ik} \). It is therefore sufficient to check the patterns with \( \mathcal{N} \)-items. The only exception to this is if replacing the \( \mathcal{R} \)-item by an \( \mathcal{N} \)-item would give pattern \( q^i \), which does have weight larger than \( y_4^* \) and therefore violates (30) (but constraint (26) does not involve pattern \( q^i \)). We therefore check pattern \( q^j \) separately.

We have \( w_{3k} = 1 + \frac{1 - \text{red}_{l(t)}}{2} + \frac{1}{1 - \varepsilon} \), \( v_{3k} = 1 + \frac{1}{1 - \varepsilon} \), \( m_3 = 1 \), \( n_{3j} = 0 \) for all \( j \). Hence the left hand side of (30) is at most \( \frac{1}{2} + \frac{1}{1 - \varepsilon} + y_3^* \leq 1.516 \) for \( \varepsilon < 0.01 \), \( y_3^* \leq 0.005 \), and \( y_3^* \leq 1/2 \) using Property 7. All our solutions will satisfy these constraints and thus we can ignore \( \mathcal{R} \)-items in the knapsack problem. (For completeness, we check pattern \( q^i \) separately in our program.)

We define a new weighting function \( \omega(p) \) for the items as given in Table 2, which depends only on types and sizes (and not on marks).

| \( t(p) \) | \( \omega(p) \) |
|---|---|
| \( t(r) \) | \( (1 - y_3^*) \left( \frac{1 - \text{red}_{l(t)}}{\text{bluefit}_{l(t)}} + \frac{\text{red}_{l(t)}}{\text{redfit}_{l(t)}} \right) + y_3^* v_k(p) + \frac{1 - \text{red}_{l(t)}}{1 + \text{red}_{l(t)}} y_1^* \) |
| \( j, 0 < \text{needs}(j) \leq \text{leaves}(t(r)) \) | \( (1 - y_3^*) w_k(p) + y_3^* v_k(p) + y_2^* \frac{\text{red}_j}{\text{redfit}_j} y_2^* \) |
| else | \( (1 - y_3^*) w_k(p) + y_3^* v_k(p) \) |

In order to prove that an EXTREME HARMONIC algorithm is \( c \)-competitive if \( r \) is medium and \( c < w_{1k} \), it is sufficient to verify that there exists a value \( y_3^* \in [0, 1] \) such that \( \max_{q \in Q_k} \omega(q) \leq c. \)
Values for $y^*_3$ that satisfy this can again be found in Appendix A. Finding these values was done again by a binary search for each value of $k$ for which $\text{redspace}_k > 1/3$, each time setting $y^*_4 = c$ and using (28) and (29).

Summary Overall, our approach is as follows: We first fix a target competitive ratio $c$. We do the following for every value of $k \in \{1, \ldots, K\}$.

Consider the value for $y^*_3$ (for our algorithm SON OF HARMONIC, these values are specified in Table 4). If $r$ is small, we check that $D_{w,sml}^k$ is feasible for $y^*_4 = c$ and this $y^*_3$. If $r$ is medium, we compute $w_{1k}$ and check whether $w_{1k} \leq c$ or $w_{1k} > c$. In the latter case, we again check that $D_{w,sml}^k$ is feasible for $y^*_4 = c$ and the given value of $y^*_3$. In the former case, we check that $D_{w,med}^k$ is feasible for $y^*_4 = c$ and the given value of $y^*_3$.

Finally, for $k = K + 1$, it is sufficient to count blue bins, and we solve a single knapsack problem based on $w_k$ alone, checking that the heaviest pattern is not heavier than $y^*_4 = c$.

5.3 Solving the knapsack problems

In order to prove our competitive ratio $c = 1.5813$, we prove feasibility of the discussed dual linear programs, which amounts to solving knapsack problems and comparing the maximum weight of a pattern to our target competitive ratio. We will now describe how our implementation of this knapsack solving works, given a set of item types as described at the beginning of Section 5 and a corresponding weight function $w$ (one weight per type).

We use two main heuristics to speed up the computation. First, for each type $i$, we define the expansion $\text{exp}_i$ of type $i$ as the weight according to function $w$ divided by $t_{i+1}$. Now we sort the types in decreasing order of expansion; call this permutation of types $\pi$. When constructing a pattern with high weight, we try to add items in the order of this permutation. Note that $\pi$ will not contain types that have expansion below that of sand: Such types will not be part of a maximum weight pattern, as the pattern with sand instead of these items has no smaller weight.

Second, we use branch and bound. We use a variable maxFound that will store the maximum weight of a pattern found so far, and give this the initial value $c - 1/1000$. Whenever the current pattern cannot be extended to a pattern with weight more than maxFound (based on the expansion of the next item in the ordering $\pi$ that still fits), we stop the calculation for this branch. Initializing maxFound with a value close to $c$ immediately eliminates many patterns.

The process works as follows. We start with type $t = \pi(1)$ (i.e., the type with the largest expansion) and an empty pattern. For current type $t = \pi(j)$ and current pattern $q$ that contains items of total size $S$ and total weight $w(q)$ (counting only the non-sand items in the calculation of the weight), we compute an upper bound on the weight that this pattern $q$ can at most get by adding items of types $\pi(j), \pi(j+1), \ldots, \pi(N)$, as follows. We find the first type $i$ in this order that still fits with the items of $q$ and compute $u = w(q) + (1 - S)\text{exp}_i$. This is an upper bound for the weight of any bin which contains the items from $q$. If this upper bound is already smaller than maxFound, we immediately cancel the further exploration of this pattern $q$.

Otherwise, if we have no more types to add (i.e., we reached the end of list of types in $\pi$), set maxFound to the weight of $q$ (now including the sand) and store $q$ as the heaviest pattern so far. If we still have more types to explore, find out how many items of the next type can fit maximally...
into $q$; call this number $m$ (if adding an item of the next type would create pattern $q^1$ or $q^2$ and we are considering the dual program $D_{w}^{k,med}$, we set $m = 0$ as we do not need to consider these patterns). Now recursively call this procedure with type $\pi(j+1)$ and patterns $q_0, \ldots, q_m$ where $q_i$ is obtained from $q$ by adding $i$ items of type $t$.

The heuristics described in this section are still not enough to be able to examine all possible patterns in reasonable time. We explain in the next section how to reduce the set of patterns further (by reducing the number of small types) and how to ensure that larger items are more important than smaller items (by making sure the expansion of small items is monotonically nondecreasing in the size, that is, larger (but still small) items do not have smaller expansions than smaller items).

### 6 The algorithm SON OF HARMONIC

For our algorithm SON OF HARMONIC we have set initial values as follows. The right part of Table 3 below contains item sizes and corresponding $red_i$ values that were set manually. Some numbers of the form $1/i$ until the value $t_N$ are added automatically by our program if they are not listed below (see below for details on how these are selected).

| Parameter | Value |
|-----------|-------|
| $c$       | $15813/10000$ |
| $t_N$     | $4000$ |
| $\Gamma$  | $\frac{7}{1} (\text{starting from } \frac{1}{14})$ |
| $T$       | $5/18$, $7/27$, $1/4$ |

| Item size | $red_i$ | Item size | $red_i$ |
|-----------|---------|-----------|---------|
| $33345/100000$ | 0 | $8/39$ | $8/100$ |
| $33340/100000$ | 0 | $1/5$ | $93/1000$ |
| $33336/100000$ | 0 | $3/17$ | $3/100$ |
| $33334/100000$ | 0 | $1/6$ | $8/100$ |
| $5/18$ | $2/100$ | $3/20$ | 0 |
| $7/27$ | $105/1000$ | $29/200$ | 0 |
| $1/4$ | $1061/10000$ | $1/7$ | $16/100$ |

The remaining values $red_i$ are set automatically using heuristics designed to speed up the search and minimize the resulting upper bound. In the range $[1/3, 1/2]$, we automatically generate item sizes (with corresponding values $red_i$ and redspace,) that are less than $t_N$ apart to ensure uniqueness of $q^1$ and $q^2$: no non-sand item can be packed into any bin of pattern $q^1$ or $q^2$. The value $\Gamma$ specifies an upper bound on how much room is used by red items of size at most $1/14$; larger items ($\leq 1/3$) use at most $1/3$ room. Since we have this bound $\Gamma$, we also add size thresholds of the form $\Gamma/i$ for $i = 1, 2, 3, 4$, to ensure that items just below this threshold can be packed without leaving much space unused.

The last parameter is some item size $T = t_j$. Above this size, we generate all item sizes of the form $1/i$ for $i > 3$. Below this size, we skip some item sizes as described below.

Our program uses an exact representation of fractions, with numerators and denominators of potentially unbounded size, in order to avoid rounding errors. The source code and the full list of all types and parameters as determined by the program can be found at [https://sheydrich.](https://sheydrich.)
In Appendix A we provide an alternative set of parameters, which give a competitive ratio of 1.583 with a much smaller set of knapsack problems to check.

Additionally, in Table 4 we provide the $y_3^*$-values that certify the competitive ratio of our algorithm. Note that only two different values for $y_3^*$ were used.

### Table 4: $y_3^*$-values used to certify that SON OF HARMONIC is 1.5813-competitive.

| $y_3^*$ | range of $k$       | $y_3^*$ | range of $k$       |
|---------|--------------------|---------|--------------------|
| $\frac{3}{32}$ | $k \leq 4,$       | $\frac{3}{16}$ | $k = 5,$ |
|         | $6 \leq k \leq 7,$ |         | $8 \leq k \leq 43,$ |
|         | $44 \leq k \leq 49$ |         | $k > 49$               |

#### Automatic generation of item sizes

We start by generating all item sizes of the form $1/i$ for $i$ between 2 and $T$ (if they are not already present in the parameter file). After that, we generate types above $1/3$ in steps of size $t_N$. By choosing this step size, we make sure that no non-sand items can be added to the patterns $q^1, q^2, q^3$. The value $\text{red}_j$ for such a type $j$ is chosen such that the pattern containing an item $r'$ of type $j$ and a large item $L$ of type 2 (i.e., $t_{i+1} = 1/2$) has as weight exactly our target competitive ratio if $k = K + 1$. That is, we consider the weighting function $w_{K+1}$. We have $w_{K+1}(r') = \frac{1-\text{red}_j}{2}$, $w_{K+1}(L) = 1$, and an upper bound for the amount of sand that fits with these items is $1/2 - t_{j+1}$. Therefore, $\text{red}_j$ is defined as the solution of the equation

$$1 + \frac{1 - \text{red}_j}{2} + \frac{1}{1 - \varepsilon} \left( \frac{1}{2} - t_{j+1} \right) = c = 1.5813, \quad (31)$$

as long as this value is positive. We stop generating types as soon as it becomes negative. To be precise, our highest value $t_{j+1}$ is defined by taking $\text{red}_j = 0$ in (31).

We have now generated all item sizes above $T$. We generate large types as described in Section 5.3. In the range $(T, t_N)$, we do not generate all $1/i$ types, but we skip some (to speed up the knapsack search) if this can be done without a deterioration in the competitive ratio. We do this by considering the expansion of such items, that is, the weight divided by the infimum size. We will ensure that the expansion of smaller items is smaller than that of larger items, so that they are irrelevant (or less relevant) for the knapsack problem.

Let us consider how we test whether a certain type $(1/j, x]$ is required (where $x$ is the next larger type, i.e. either the last type generated before we started this last phase or the last type generated in this phase), and which $\text{red}_i$ we should choose. Denote by $s_i := 1/j$ the value we want to check. We compute a lower and upper bound $\text{red}_i, \text{red}_i$ for the $\text{red}_i$-value of this type as follows: We can compute $\text{bluefit}_i$ and $\text{redfit}_i$, only depending on the upper bound of the size of items of this type, i.e. depending on $x$, the lower bound of the next larger item size. First, we require $\frac{1-\text{red}_i}{\text{bluefit}_i, t_{i+1}} \leq 1$, which gives $\text{red}_i \geq 1 - s_i \cdot \text{bluefit}_i =: \text{red}_i$. Second, we want to make sure that the maximum expansion of the current type is not larger than the expansion of the previous (next larger) type (since that might slow down the search), $\text{exp}_{i-1} \cdot \text{red}_i, \text{red}_i, s_i, s_i \leq$
and $r$ Seiden used the following weighting functions, but presented them in a different way. Define $7$ Super Harmonic revisited

This type is at least $f$ is that it ensures that the “small expansion” of these items, where we count only the blue items of $t$ is added as a redspace-value and for every type $i$ such that $2 \cdot t_{i+1} \in [1/6, 1/3]$, $2 \cdot t_{i+1}$ is added as a redspace-value. Additionally, we make sure that for each medium type we have a redspace-value equal to $x$ and one equal to $1 - 2x$ where $x$ is the lower bound of the size of items of this type.

After computing the functions leaves and needs, we then eliminate redspace-values that are unused and less than $1/3$, i.e., if there is no pair of types $i, j$ such that needs$(i) = \text{leaves}(j) = l$, redspace$_i < 1/3$, then redspace$_i$ is removed from the list. This reduces the number of knapsack problems that need to be solved.

Computation and adjustment of values red$_i$ For each item type $i$ that has size at most $1/6$ and at least $T$, we adjust the value red$_i$ such that $\frac{1-\text{red}_i}{\text{bluefit}_i - \text{redfit}_i} \geq f$ where $f = \frac{95}{100}$ if $t_{i+1} \leq \frac{1}{13}$ and $t_{i+1} > T$ and $f = 1$ otherwise. To be precise, we set red$_i = 1 - t_{i+1}\text{bluefit}_i$. The reason for this is that it ensures that the “small expansion” of these items, where we count only the blue items of this type, is at least $f$. This is a heuristic; it does not seem to help to make red$_i$ larger than this.

### Computation of redspace-values

The redspace-values are completely auto-generated, in contrast to Seiden’s paper, where these values are given by hand. For every type $i$ such that $t_{i+1} \in [1/6, 1/3]$, $t_{i+1}$ is added as a redspace-value and for every type $i$ such that $2 \cdot t_{i+1} \in [1/6, 1/3]$, $2 \cdot t_{i+1}$ is added as a redspace-value. Additionally, we make sure that for each medium type we have a redspace-value equal to $x$ and one equal to $1 - 2x$ where $x$ is the lower bound of the size of items of this type.

After computing the functions leaves and needs, we then eliminate redspace-values that are unused and less than $1/3$, i.e., if there is no pair of types $i, j$ such that needs$(i) = \text{leaves}(j) = l$, redspace$_i < 1/3$, then redspace$_i$ is removed from the list. This reduces the number of knapsack problems that need to be solved.

### Computation and adjustment of values red$_i$

For each item type $i$ that has size at most $1/6$ and at least $T$, we adjust the value red$_i$ such that $\frac{1-\text{red}_i}{\text{bluefit}_i - \text{redfit}_i} \geq f$ where $f = \frac{95}{100}$ if $t_{i+1} \leq \frac{1}{13}$ and $t_{i+1} > T$ and $f = 1$ otherwise. To be precise, we set red$_i = 1 - t_{i+1}\text{bluefit}_i$. The reason for this is that it ensures that the “small expansion” of these items, where we count only the blue items of this type, is at least $f$. This is a heuristic; it does not seem to help to make red$_i$ larger than this.

### 7 Super Harmonic revisited

Seiden used the following weighting functions, but presented them in a different way. Define $h$ and $r$ as in Definition 5. The two weight functions of an item of type $i$ are given by Table 5.

| $w_k(i)$ | $v_k(i)$ |
|----------|----------|
| $\frac{1-\text{red}_i}{\text{bluefit}_i} + \frac{\text{red}_i}{\text{redfit}_i}$ if needs$(i) \geq k$ or needs$(i) = 0$ | $\frac{1-\text{red}_i}{\text{bluefit}_i} + \frac{\text{red}_i}{\text{redfit}_i}$ if leaves$(i) < k$ |
| $\frac{1-\text{red}_i}{\text{bluefit}_i}$ if needs$(i) < k$ | $\frac{1-\text{red}_i}{\text{bluefit}_i}$ if leaves$(i) \geq k$ |

Using these weight functions, he shows that (11) with $c = 1.58889$ holds for SUPER HARMONIC algorithms. Instead of the mathematical program that Seiden considers, we use $P_w^{k,sml}$ and its dual $D_w^{k,sml}$. We use the method described in Section 5.2 (a binary search for a weighted average of weights) to check for feasibility of the dual linear programs for all values of $k$, including the cases where $r$ is medium. This is a significantly easier method than the one Seiden used, since it is based on solving standard knapsack problems.

A small modification of our computer program can be used to verify Seiden’s result. Surprisingly, it shows that HARMONIC++ is in fact 1.58880-competitive. In contrast to Seiden’s heuristic program, which took 36 hours to prove HARMONIC++’s competitive ratio, our program terminates.
in a few seconds. Of course, this was over fifteen years ago, but we believe the algorithmic improvement explains a significant part of the speedup. The fast running time of our approach also allowed us to improve upon HARMONIC++ within the SUPER HARMONIC framework (at least as long as we allow multiple red items per bin): Using improved red_i values, we can show a 1.5884-competitive SUPER HARMONIC-algorithm. Our values are also simpler than the ones Seiden used (which were optimized up to precision \(1/2 \cdot 10^{-7}\)); they can be found in the appendix.

## 8 Lower bound

We prove a lower bound for any EXTREME HARMONIC algorithm. We will consider inputs consisting of essentially four different item sizes: \(1/2 + \varepsilon\), \(1/3 + \varepsilon\), \(1/4 + \varepsilon\), and \(1/7 + \varepsilon\) (we also speak of types 1 through 4). Here \(\varepsilon\) is a very small number. However, there will be many different item sizes in the range \((1/3, 1/3 + \varepsilon]\). The value of \(\varepsilon\) is chosen small enough that the algorithm puts all these sizes in the same type. Note that the algorithm has not much choice about how many red items of types 2 and 3 can be packed in one bin: only one such item can be packed, else larger blue items could not be added anymore. For type 4, between 1 and 3 red items could be packed in one bin, and we will give lower bound constructions for each of these three cases.

Consider the case that the algorithm packs red type 4 items pairwise into bins. In Table 6, we give four different inputs that together will prove a lower bound of 1.5762 for this case. A pattern \((a, b, c, d)\) denotes a set of items containing \(a\) items of type 1, \(b\) items of type 2 and so on. Note that our types defined here do not necessarily correspond to size thresholds used by the algorithm; nevertheless, each item gets a single type assigned by the algorithm, and if we use notation such as redfit_i for type \(i\) as defined here, we mean the redfit-value of the item type the algorithm assigns to such an item. The other two columns of the table are explained below.

The first three lines of the table represent three different inputs to the algorithm, and the last three lines together represent the final input used in the lower bound. We construct the first three inputs as follows. For each pattern in the table, items arrive in order from small to large. Each item in the pattern arrives \(N\) times. In addition, we get \(N\) times some amount of sand per bin, that fills up the bin completely. Based on each pattern and the values \(\text{red}_i\) and \(\text{redfit}_i\), we can calculate exactly how much space (represented as fractions of bins) the online algorithm needs to pack each item in the pattern on average. To do this, we assume that if red small items can be packed with
larger blue ones, the algorithm will always do this (this is a worst-case assumption). The result of this calculation is shown in the column Space.

To illustrate this approach, let us consider an input based on the pattern \((0, 0, 3, 1)\) in the manner described above. As we assumed that \(\text{redfit}_4 = 2\), we know that items of types 3 and 4 will not be combined by the algorithm, as \(3/4 + 2/7 > 1\). Thus, the algorithm will not be able to combine the red items of both types with any other items. The number of bins used for blue type 3 items is at least \(3 \cdot (1 - \text{red}_3) N\), the number of bins for red type 3 items is at least \(3 \cdot \text{red}_3 N\). Analogously, we need at least \(1 - \text{red}_3\) bins for blue type 4 items and at least \(\text{red}_4 / 2\) bins for red type 4 items. Finally, sand of total volume arbitrarily close to \((1 - 3/4 - 1/7) N = \frac{3}{28} N\) arrives, which is packed in at least as many bins by the online algorithm. Thus, on average the items in this pattern need \(3 \cdot (1 - \text{red}_3) + 3 \cdot \text{red}_3 + \frac{1 - \text{red}_4}{6} + \frac{\text{red}_4}{2} + \frac{3}{28} = \frac{31}{28} + 2\text{red}_3 + \frac{1 - \text{red}_4}{6} + \frac{\text{red}_4}{2}\) bins to be packed. The space needed for the second and third patterns can be calculated in the same way.

![Figure 8](image-url)

Figure 8: Fourth input for our lower bound construction. The three patterns used in the optimal solution are depicted on the left. The shaded area in the first pattern denotes sand. The algorithm produces the five types of bins depicted on the right, plus bins that only contain sand (not depicted here).

The fourth input (based on pattern \((0,2,1,0)\)) requires more explanation; see also Fig. 8. For this input, we consider a combination of three patterns that arrive in the distribution given in the last column of the table. Items of type 2 have size \(1/3 + \epsilon\) (according to the table above) and some of them end up alone in bins. We extend the input in this case by a number of items of size almost \(2/3\), where this number is calculated as explained below. All these large items will be placed in

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new bins by the online algorithm. In order for this to hold, the items of type 2 must have slightly different sizes - not all exactly $1/3 + \varepsilon$. We therefore pick $\varepsilon$ small enough so that the interval $(1/3, 1/3 + \varepsilon]$ is contained in a single type according to the classification done by the algorithm. The first item of this type will have size $1/3 + \varepsilon/2$. The sizes of later items depend on how it is packed:

- If the item is packed in a new bin, all future items will be smaller (in the interval $(1/3, 1/3 + \varepsilon/2]$)
- If the item is packed into a bin with an existing item of type 2 or 3, all future items will be larger (in the interval $(1/3 + \varepsilon/2, 1/3 + \varepsilon]$)

We use the same method for all later items of the same type, each time dividing the remaining interval in two equal halves. By induction, it follows that whenever an item is placed in a new bin, all previous items that were packed first into their bins are larger, and all previous items that were packed into existing bins are smaller. Therefore, after all items of this type have arrived, let $x$ be the size of the last item that was placed into a new bin. (Since the algorithm maintains a fixed fraction of red items of type 2, there can be only constantly many items that arrived after this item; we ignore such items.) We have the following.

- All items of size more than $x$ are packed either alone into bins or are the first item in a bin with two medium but no small red items; and
- All items of size less than $x$ are in bins with items of type 3 or were packed as the second item of their type in an existing bin.

We now let items of size exactly $1 - x$ arrive. For every bin with red type 3 items and blue type 2 items, two such items arrive, which will be packed in $q^2$-bins. Assume that we have $N$ bins with pattern $q^0 = (0, 2, 1, 0)$, then we create exactly $\text{red}_3 N$ such bins, i.e., we let $2\text{red}_3 N$ large items arrive for these. For every bin with a pair of blue medium items but no red items, one such $1 - x$ item arrives. The number of these bins is harder to calculate. Let $M$ be the total number of medium items in the input. Then the number of such bins is $\frac{1}{2} M - \text{red}_3 N$. Now, we want to express $M$ in terms of $N$: Observe that $N$ is half the number of medium items larger than $x$ (as only these end up in $q^0$-bins). The number of those items is equal to the number of bins with medium items (which is $\text{red}_2 M$) plus the number of bins with two blue medium but no red items (which is $\frac{1}{2} M - \text{red}_3 N$). This shows that $M = \frac{4 + 2\text{red}_3}{1 + \text{red}_2} N$. Finally, we conclude that we can send $\frac{1}{2} M - \text{red}_3 N = \frac{1}{2} \cdot 4 + 2\text{red}_3 N - \text{red}_3 N = 2(1 - \text{red}_2 - \text{red}_3) N$ many large items and thus get this many $q^1$-bins.

To pack $N$ copies of a given pattern, the online algorithm needs $N$ times the space calculated in Table 6 while the optimal solution needs exactly $N$ bins. In order to calculate the final lower bound, for each of the four inputs, we simply calculate the space of the pattern(s), in the last case the weighted (in proportion to the distribution) sum of the three patterns’ spaces. All four cases yield a lower bound of at least 1.5762, which is achieved if $\text{red}_1 = 0, \text{red}_2 = 0.1800, \text{red}_3 = 0.1276, \text{red}_4 = 0.1428$. Whenever an algorithm has a smaller or larger value for some $\text{red}_i$ value,
the space needed by one of the patterns (or the weighted sum of the spaces needed by the three patterns of the last case) increases and thus gives a lower bound above 1.5762.

Constructions for the other two cases \( \text{redfit}_4 = 1 \) and \( \text{redfit}_4 = 3 \) can be found below in Tables 8 and 7. The analysis is completely analogous to the first case. For the case \( \text{redfit}_4 = 1 \), the best values the online algorithm can use are \( \text{red}_1 = 0, \text{red}_1 = 0.19, \text{red}_2 = 0.0872 \). The analysis for the case \( \text{redfit}_4 = 3 \) is particularly simple, as the given distribution requires 100/63 bins on average (independent of \( \text{red}_2 \) and \( \text{red}_3 \)), implying a lower bound of 100/63 \( \approx 1.5873 \).

Table 7: Inputs for lower bound 1.5788 in case \( \text{redfit}_4 = 1 \).

| Pattern | Space for \( \varepsilon \to 0 \) | Distribution \( \chi \) |
|---------|----------------------------------|---------------------|
| 1 1 1   | \( 1 + \frac{1-\text{red}_2}{2} + \frac{1-\text{red}_3}{6} + \frac{1}{42} \) | 1                   |
| 0 0 6   | \( 6 \cdot \frac{1-\text{red}_3}{6} + 6\text{red}_3 + \frac{1}{7} \) | 1                   |
| 0 2 2   | \( 2 \cdot \frac{1+\text{red}_2}{2} + 2 \cdot \frac{1-\text{red}_3}{6} + \frac{1}{21} \) | 1 (scaled)          |
| \( q^1 \) | \( 1 + \frac{1-\text{red}_2}{2} + \text{red}_2 \) | \( \frac{4(1-\text{red}_2-\text{red}_3-\text{red}_3)}{4+\text{red}_2} \) |
| \( q^2 \) | \( 1 + \frac{1-\text{red}_2}{2} + \text{red}_2 \) | \( \frac{4\text{red}_3}{4+\text{red}_2} \) |

Table 8: Inputs for lower bound 1.5872 in case \( \text{redfit}_4 = 3 \).

| Pattern | Space for \( \varepsilon \to 0 \) | Distribution \( \chi \) |
|---------|----------------------------------|---------------------|
| 1 1 1   | \( 1 + \frac{1-\text{red}_2}{2} + \frac{1-\text{red}_3}{6} + \frac{1}{42} \) | 2/3                 |
| 0 2 2   | \( 2 \cdot \frac{1+\text{red}_2}{2} + 2 \cdot \frac{1+\text{red}_3}{6} + \frac{1}{21} \) | 1/3                 |

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References

[1] Luitpold Babel, Bo Chen, Hans Kellerer, and Vladimir Kotov. Algorithms for on-line bin-packing problems with cardinality constraints. *Discrete Appl. Math.*, 143(1-3):238–251, 2004.

[2] János Balogh, József Békési, György Dósa, Leah Epstein, and Asaf Levin. A new and improved algorithm for online bin packing. *CoRR*, abs/1707.01728, 2017. To appear in ESA 2018.

[3] János Balogh, József Békési, György Dósa, Jirí Sgall, and Rob van Stee. The optimal absolute ratio for online bin packing. In Piotr Indyk, editor, *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, pages 1425–1438. SIAM, 2015.

[4] János Balogh, József Békési, and Gábor Galambos. New lower bounds for certain classes of bin packing algorithms. *Theor. Comput. Sci.*, 440-441:1–13, 2012.
[5] Donna J. Brown. A lower bound for on-line one-dimensional bin packing algorithms. Technical Report R-864, Coordinated Sci. Lab., Urbana, Illinois, 1979.

[6] Edward G. Coffman, Michael R. Garey, and David S. Johnson. Approximation algorithms for bin packing: A survey. In D. Hochbaum, editor, Approximation algorithms. PWS Publishing Company, 1997.

[7] Wenceslas Fernandez de la Vega and George S. Lueker. Bin packing can be solved within \( 1 + \varepsilon \) in linear time. Combinatorica, 1:349–355, 1981.

[8] Michael R. Garey, Ronald L. Graham, and Jeffrey D. Ullman. Worst-case analysis of memory allocation algorithms. In Proceedings of the Fourth Annual ACM Symposium on Theory of Computing, pages 143–150. ACM, 1972.

[9] Michael R. Garey and David S. Johnson. Computers and Intractability: A Guide to the theory of NP-Completeness. Freeman and Company, San Francisco, 1979.

[10] Michel X. Goemans and Thomas Rothvoß. Polynomiality for bin packing with a constant number of item types. In Chandra Chekuri, editor, Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, pages 830–839. SIAM, 2014.

[11] David S. Johnson. Near-optimal bin packing algorithms. PhD thesis, MIT, Cambridge, MA, 1973.

[12] David S. Johnson. Fast algorithms for bin packing. J. Comput. Systems Sci., 8:272–314, 1974.

[13] Narendra Karmarkar and Richard M. Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In Proceedings of the 23rd Annual Symposium on Foundations of Computer Science, pages 312–320, 1982.

[14] Chung-Chieh Lee and D. T. Lee. A simple online bin packing algorithm. J. ACM, 32:562–572, 1985.

[15] Frank M. Liang. A lower bound for online bin packing. Inform. Process. Lett., 10:76–79, 1980.

[16] Prakash V. Ramanan, Donna J. Brown, Chung-Chieh Lee, and D. T. Lee. Online bin packing in linear time. J. Algorithms, 10:305–326, 1989.

[17] Michael B. Richey. Improved bounds for harmonic-based bin packing algorithms. Discrete Appl. Math., 34:203–227, 1991.

[18] Thomas Rothvoß. Approximating bin packing within \( O(\log \text{OPT} * \log \log \text{OPT}) \) bins. In 54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, Berkeley, CA, USA, pages 20–29. IEEE Computer Society, 2013.

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Table 9: Parameters and item types.

| Parameter | Value                  |
|-----------|------------------------|
| c         | 1583                   |
| $t_N$     | $\frac{1}{1000}$       |
| $\Gamma$  | $\frac{3}{70}$ (starting from $\frac{1}{12}$) |
| $\mathcal{T}$ | $\frac{1}{30}$ |

(a) Parameters

| Item size | $\text{red}_i$ |
|-----------|----------------|
| 335/1000  | 0              |
| 334/1000  | 0              |
| 5/18      | 2/100          |
| 7/27      | 105/1000       |
| 1/4       | 106/1000       |
| 8/39      | 8/100          |
| 1/5       | 93/1000        |

| Item size | $\text{red}_i$ |
|-----------|----------------|
| 3/17      | 3/100          |
| 1/6       | 8/100          |
| 3/20      | 0              |
| 29/200    | 0              |
| 1/7       | 135/1000       |
| 1/13      | 1/10           |
| 1/14      | 1/13           |

(b) Size lower bounds and initial values $\text{red}_i$

Table 10: $\text{redspace}_i$-values below $1/3$ in the 1.583-competitive algorithm.

| index $i$ | $\text{redspace}_i$ |
|-----------|---------------------|
| 0         | 0                   |
| 1         | 1/6                 |
| 2         | 3/17                |
| 3         | 1/5                 |
| 4         | 11/50               |
| 5         | 2/9                 |
| 6         | 6/25                |
| 7         | 13/50               |
| 8         | 7/25                |
| 9         | 3/10                |
| 10        | 8/25                |
| 11        | 33/100              |

Table 10: $\text{redspace}_i$-values below $1/3$ in the 1.583-competitive algorithm.

[19] Steve S. Seiden. On the online bin packing problem. *J. ACM*, 49(5):640–671, 2002.

[20] Jeffrey D. Ullman. The performance of a memory allocation algorithm. Technical Report 100, Princeton University, Princeton, NJ, 1971.

[21] André van Vliet. An improved lower bound for online bin packing algorithms. *Inform. Process. Lett.*, 43:277–284, 1992.

[22] Andrew C. C. Yao. New algorithms for bin packing. *J. ACM*, 27:207–227, 1980.

A Alternative parameters for a competitive ratio of 1.583

We give a list of item types together with their parameters in Table 11. Please note that type 2 is only defined for the definition of the knapsack problem in case $\tau$ is medium. EXTREME HARMONIC algorithms, in contrast to SUPER HARMONIC algorithms, treat all items larger than 1/2 as a single type (thus it sees types 1 and 2 as a single type). Between type 6 and type 12, the values $t_i$ are 1/100 apart. Between type 39 and type 101, the types are of the form $1/i$ for some values $i \in \{14, \ldots, 100\}$ (below 1/30, we skip some values). The values $\text{red}_i$ for these types are computed as described in Sections 6. The parameters are auto-generated from the input in Table 9.

We give a list of all $\text{redspace}_i$-values that are at most 1/3 in Table 10. The $\text{redspace}_i$-values above 1/3 are equal to the $t_i$-values above 1/3. Finally, there were only two different $y_k^3$-values used to establish the feasibility of the dual LPs: 9/32 for the cases $k = 2, 3, 4, 6, 7, 9, 10, 11$ and 3/16 in all other cases.
| Type $i$ | $t_i$     | $\text{red}_i$ | $\text{bluefit}_i$ | $\text{redfit}_i$ | needs($i$) | leaves($i$) |
|---------|-----------|----------------|---------------------|-------------------|------------|-------------|
| 1       | 1         | 0              | 2                   | 0                 | 0          | 0           |
| 2       | 41783/100000 | 87/5500 $\approx$ 0.0158 | 2                   | 1                 | 23         | 0           |
| 3       | 41/100    | 1783/49500 $\approx$ 0.0360 | 2                   | 1                 | 22         | 2           |
| 4       | 2/5       | 253/4500 $\approx$ 0.0562 | 2                   | 1                 | 21         | 3           |
| 5       | 39/100    | 1261/16500 $\approx$ 0.0764 | 2                   | 1                 | 20         | 4           |
| 6       | 19/50     | 4783/49500 $\approx$ 0.0966 | 2                   | 1                 | 19         | 6           |
| 7       | 37/100    | 5783/49500 $\approx$ 0.1168 | 2                   | 1                 | 18         | 7           |
| 8       | 9/25      | 2261/16500 $\approx$ 0.1370 | 2                   | 1                 | 17         | 8           |
| 9       | 7/20      | 7783/49500 $\approx$ 0.1572 | 2                   | 1                 | 16         | 9           |
| 10      | 17/50     | 251/1500 $\approx$ 0.1673  | 2                   | 1                 | 15         | 10          |
| 11      | 67/200    | 8383/49500 $\approx$ 0.1694 | 2                   | 1                 | 14         | 11          |
| 12      | 167/500   | 25349/148500 $\approx$ 0.1707 | 2                   | 1                | 13         | 11          |
| 13      | 1/3       | 0              | 3                   | 0                 | 0          | 0           |
| 14      | 29/90     | 0              | 3                   | 0                 | 0          | 0           |
| 15      | 11/36     | 1/50 $\approx$ 0.0200  | 3                   | 1                 | 10         | 0           |
| 16      | 5/18      | 21/200 $\approx$ 0.1050 | 3                   | 1                 | 8          | 1           |
| 17      | 7/27      | 53/500 $\approx$ 0.1060 | 3                   | 1                 | 7          | 5           |
| 18      | 1/4       | 2/25 $\approx$ 0.0800  | 4                   | 1                 | 7          | 0           |
| 19      | 8/39      | 93/1000 $\approx$ 0.0930 | 4                   | 1                 | 4          | 2           |
| 20      | 1/5       | 3/100 $\approx$ 0.0300  | 5                   | 1                 | 3          | 0           |
| 21      | 3/17      | 2/25 $\approx$ 0.0800  | 5                   | 1                 | 2          | 0           |
| 22      | 1/6       | 1/30 $\approx$ 0.0333  | 6                   | 1                 | 1          | 0           |
| 23      | 29/180    | 1/12 $\approx$ 0.0833  | 6                   | 2                 | 11         | 0           |
| 24      | 11/72     | 1/10 $\approx$ 0.1000  | 6                   | 2                 | 10         | 0           |
| 25      | 3/20      | 13/100 $\approx$ 0.1300 | 6                   | 2                 | 9          | 0           |
| 26      | 29/200    | 1/7 $\approx$ 0.1429  | 6                   | 2                 | 9          | 0           |
| 27      | 1/7       | 1/8 $\approx$ 0.1250  | 7                   | 2                 | 9          | 0           |
| 28      | 1/8       | 1/9 $\approx$ 0.1111  | 8                   | 2                 | 7          | 0           |
| 29      | 1/9       | 1/30 $\approx$ 0.0333  | 9                   | 2                 | 5          | 0           |
| 30      | 29/270    | 1/12 $\approx$ 0.0833  | 9                   | 3                 | 11         | 0           |
| 31      | 11/108    | 1/10 $\approx$ 0.1000  | 9                   | 3                 | 10         | 0           |
| 32      | 1/10      | 1/11 $\approx$ 0.0909  | 10                  | 3                 | 9          | 0           |
| 33      | 1/11      | 1/12 $\approx$ 0.0833  | 11                  | 3                 | 8          | 0           |
| 34      | 1/12      | 1/30 $\approx$ 0.0333  | 12                  | 3                 | 7          | 0           |
| 35      | 29/360    | 8/65 $\approx$ 0.1231  | 12                  | 3                 | 7          | 0           |
| 36      | 1/13      | 163/2880 $\approx$ 0.0566 | 13                  | 3                 | 6          | 0           |
| 37      | 11/144    | 33/280 $\approx$ 0.1179 | 13                  | 3                 | 6          | 0           |
| 38      | 1/14      | 17/150 $\approx$ 0.1133 | 14                  | 4                 | 9          | 0           |
|         | ...       | ...            | ...                | ...               | ...       | ...        | 0           |
| 122     | 1/98      | 7/660 $\approx$ 0.0106 | 98                  | 28                | 9          | 0           |
| 123     | 1/99      | 21/2000 $\approx$ 0.0105 | 99                 | 28                | 9          | 0           |

Table 11: Parameters used by 1.583-algorithm. The values $t_4$ to $t_{13}$ are in REDSPACE.
B Parameters for an improved SUPER HARMONIC algorithm

With the parameters listed below in Table 12 using the same item types used by Seiden, we are able to achieve a SUPER HARMONIC algorithm (with more than one red item per bin) with competitive ratio 1.5884.

Note that the redfit_i-values are computed differently than in HARMONIC++. For types i such that red_i = 0 or t_i > redspace_K, we have redfit_i = 0. Otherwise, we have redfit_i = ⌊24/83 t_i⌋. The value 24/83 in this expression is related to the item threshold 12/83. By using this bound, two items of size slightly larger than 1/7 can be packed together in one bin.

Table 12: Parameters used for our improvement of HARMONIC++.

| i  | t_i | red_i | redfit_i | i  | t_i | red_i | redfit_i |
|----|-----|-------|----------|----|-----|-------|----------|
| 1  | 1   | 0     | 0        | 22 | 13/63 | 1/10 = 0.1 | 1 |
| 2  | 341/512 | 0   | 0        | 23 | 1/5  | 7/200 = 0.035 | 1 |
| 3  | 511/768 | 0   | 0        | 24 | 15/88 | 83/1000 = 0.083 | 1 |
| 4  | 85/128 | 0    | 0        | 25 | 1/6  | 789/10000 = 0.0789 | 1 |
| 5  | 127/192 | 0   | 0        | 26 | 12/83 | 13/100 = 0.13 | 2 |
| 6  | 21/32 | 0    | 0        | 27 | 1/7  | 29/2000 = 0.0145 | 2 |
| 7  | 31/48 | 0    | 0        | 28 | 11/83 | 71/1000 = 0.071 | 2 |
| 8  | 5/8  | 0    | 0        | 29 | 1/8  | 1191/20000 = 0.05955 | 2 |
| 9  | 7/12 | 0    | 0        | 30 | 1/9  | 1/20 = 0.05 | 2 |
| 10 | 1/2  | 0    | 0        | 31 | 1/10 | 9/200 = 0.045 | 2 |
| 11 | 5/12 | 9/100 = 0.09 | 1 | 32 | 1/11 | 4/125 = 0.032 | 3 |
| 12 | 3/8  | 267/2000 = 0.1335 | 1 | 33 | 1/12 | 11/500 = 0.022 | 3 |
| 13 | 17/48 | 311/2000 = 0.1555 | 1 | 34 | 1/13 | 71/2000 = 0.0355 | 3 |
| 14 | 11/32 | 829/5000 = 0.1658 | 1 | 35 | 1/14 | 17/2000 = 0.0085 | 4 |
| 15 | 65/192 | 107/625 = 0.1712 | 1 | 36 | 1/15 | 1/100 = 0.01 | 4 |
| 16 | 43/128 | 87/500 = 0.174 | 1 | 37 | 1/16 | 1/100 = 0.01 | 4 |
| 17 | 257/768 | 7/40 = 0.175 | 1 | 38 | 1/17 | 1/100 = 0.01 | 4 |
| 18 | 171/512 | 877/5000 = 0.1754 | 1 | 39 | 1/18 | 0 | 0 |
| 19 | 1/3  | 0    | 0        | 40 | :    | :     | :     |
| 20 | 13/48 | 37/400 = 0.0925 | 1 | 41 | :    | :     | :     |
| 21 | 1/4  | 46/625 = 0.0736 | 1 | 42 | :    | :     | :     |