The ISS property for a feedback connection
of an ODE with a parabolic PDE

Sergey Dashkovskiy ∗ Vitalii Slyn’ko ∗

Abstract: We consider a linear non-autonomous ODE with an input signal, which is a solution of a linear non-autonomous parabolic PDE for which the solution of the ODE enters as an input. Moreover there are external disturbances entering through the boundary conditions of the parabolic equation. In this work we derive ISS estimates for this feedback connection under certain linear matrix inequality conditions. The derivation of results is based on the Lyapunov method.

Keywords: Stability, robustness, linear ODE-PDE interconnections, parabolic PDEs, Lyapunov method, input-to-state stability.

1. INTRODUCTION

Systems of interconnected ordinary and partial differential equations appear in mathematical modeling of different physical, chemical, biological and other processes as well as in dynamics of discrete mechanical systems including elastic and fluid objects. For example such a coupling appears in the modeling thermo-elasticity or thermo-plasticity processes with phase transformations, see Dashkovskiy and Narimanyan (2007) and Dachkovski and Böhm (2004), respectively, where the temperature distribution as well as elastic deformations are described by partial differential equations (PDE) and the evolution of solid-solid phase transformations is given in terms of ordinary differential equations (ODE). Stability properties of linear and nonlinear coupled ordinary and partial differential equations of the parabolic type were studied in many papers, see for example Bartyšev (1980); Kedyk (1991); Martynyuk and Slyn’ko (2008); Kang and Fridman (2017); Ahmed-Ali et al. (2017); Karafyllis and Krstić (2019), see also Mironchenko and Prieur (2020). To investigate different dynamical properties of a coupled system of a parabolic PDE and ODE Bartyšev (1980) applies the method of vector Lyapunov function, see Lakshmikantham et al. (1991). In the work of Kedyk (1991) stability problems of the monotone dynamic extensions of coupled ODE with PDE of the parabolic type were investigated. Also here the method of vector Lyapunov function is applied and hence in these works the asymptotic stability for each subsystem is assumed. Similar assumptions on the dynamics of subsystems are applied in the derivations of the small-gain type developed in the ISS framework for interconnected systems Karafyllis and Krstić (2019). Martynyuk and Slyn’ko (2008) propose a Lyapunov function for a coupled linear ODE-PDE parabolic system that allows to study stability in case when the ODE is unstable.

The results in Martynyuk and Slyn’ko (2008); Karafyllis and Krstić (2019); Bartyšev (1980) were developed for autonomous ODE-PDE couplings. Some special cases of non-autonomous ODE-PDE systems were considered in Kedyk (1991). The derivation of Lyapunov functions in the general non-autonomous case of a coupled ODE-PDE parabolic system is still an open problem. We also note that in most of works devoted to the investigations of the ISS-like properties for distributed parameter systems the case of autonomous systems only is considered. The ISS framework of non-autonomous infinite dimensional systems needs to be developed yet.

In this work we investigate the ISS properties of a linear periodic system of the coupled ODE-PDE parabolic system with input signals which enter at the boundary. Using the ideas from Slyn’ko (2019) we provide a time dependent Lyapunov function and derive conditions guaranteeing the ISS property including explicit estimations for solutions.

Notation. For a continuous function \( r \in C([0,1]; \mathbb{R}) \), \( r(z) > 0 \) by \( L^2(0,1) \) we denote the Banach weighted space of Lebesgue measurable and square integrable functions with weight \( r \) and norm given by \( \| f \|_{L^2(0,1)} = \left( \int_0^1 r(z)|f(s)|^2 \, ds \right)^{1/2} \). For a Banach space \( \mathcal{X} \) and the subset \( \mathcal{M} \subset \mathbb{R} \) by \( C(\mathcal{M}, \mathcal{X}) \) we denote the space of continuous functions \( \mathbb{R} \rightarrow \mathcal{X} \) and by \( C^1(\mathbb{R}, \mathcal{X}) \) the space of continuously differential functions with values in \( \mathcal{X} \). \( L^\infty(\mathbb{R}_+) \) denotes the space of essentially bounded Lebesgue measurable functions with the norm \( \| f \|_\infty = \text{ess sup}_{t \geq 0} |f(t)| \). In many definitions and estimates we will use the classes of comparison functions \( \mathcal{K}, \mathcal{L}, \mathcal{KL} \) as usually in the ISS framework, see Sontag (1989) and Liu et al. (2017).

\( \mathbb{R}^n \) denotes the \( n \)-dimensional space with the euclidean norm \( \| \cdot \| \), \( \mathbb{R}^{n \times m} \) is the linear space of \( n \times m \)-matrices, which forms a Banach algebra in case \( m = n \). Let the norm in \( \mathbb{R}^{n \times n} \) be induced by the norm in \( \mathbb{R}^n \), that is \( \| A \| = \sup_{\| x \|_1 = 1} \| Ax \| = \lambda_{\max}^{1/2}(A^T A) \). In case of the norm \( \| x \|_1 = \)}
\[ \sum_{l=1}^{n} |x_l| \text{ for } x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n, \text{ the noinduced norm in } \mathbb{R}^{n \times n} \text{ is given by } \|A\|_1 = \sup_{\|x\|_1 = 1} \|Ax\|_1. \] By \( S^n \) we denote the set of square symmetric matrices of the size \( n \). The partial order in \( S^n \) is given as follows: for \( P, Q \in S^n \) we say that \( P = Q \) meaning that \( P - Q \) is positive definite. For \( A \in S^n \) by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) we denote the minimal and maximal eigenvalue of \( A \). For \( A \in \mathbb{R}^{n \times n} \) by \( \sigma_r(A) \) we denote the spectral radius of \( A \).

### 2. PRELIMINARIES

The following notation and notions that we will use throughout the paper are borrowed from Magnus (1954). The commutator of matrices \( A, B \in \mathbb{R}^{n \times n} \) given by

\[ [A, B] = AB - BA \]

defines the Lie-Algebra structure in \( \mathbb{R}^{n \times n} \).

For matrix variables \( X, Y \) and \( Z \) let \( F(X, Y) \) be some formal series of the variables \( X, Y \) and for \( \lambda \in \mathbb{R} \) the polarization identity

\[ F(X + \lambda Z, Y) = F(X, Y) + \lambda F_1(X, Y, Z) + \lambda^2 F_2(X, Y, Z) + \cdots, \]

where \( F_1(X, Y, Z) \) and \( F_2(X, Y, Z) \) are certain formal series of the variables \( X, Y, Z \) defines the Hausdorff derivative \( \left( \frac{\partial}{\partial \lambda} \right) F(X, Y) := F_1(X, Y, Z) \).

The Lie-polynomials of matrix variables \( X, Y \) are defined recursively as follows (see Magnus (1954) for more details on Lie-elements).

\[ \{Y, X^0\} = Y, \quad \{Y, X^{l+1}\} = \left\{ [Y, X^l], X \right\}, \quad l \in \mathbb{Z}_+. \]

The following Hausdorff equalities will play an important role in the sequel

\[ \begin{align*}
(Y, \frac{\partial}{\partial X}) e^X &= e^X \left( Y + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \{Y, X^k\} \right), \\
(Y, \frac{\partial}{\partial X}) e^X &= \left( Y + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \{Y, X^k\} \right) e^X.
\end{align*} \]

As well we will need the chain rule for the differentiation of compositions: Let \( P(X) \) be a series of the matrix variable \( X \) and \( A : (a, b) \to \mathbb{R}^{n \times n} \) be a mapping differentiable at the point \( t_0 \in (a, b) \) such that \( B(t) := P(A(t)) \) is well defined in a neighborhood of \( t_0 \), then

\[ \frac{dB}{dt} \bigg|_{t=t_0} := \left( Y, \frac{\partial}{\partial X} \right) P(X), \quad Y := \frac{dA}{dt} \bigg|_{t=t_0}, X = A(t_0). \]

### 3. PROBLEM STATEMENT

Let us consider the following system of differential equations

\[ \begin{align*}
u(t) &= Au(t, z) + B(z, t)x(t), \\
\dot{x}(t) &= C(t)x(t) + \int_0^t D(z, t)u(z, t) dz,
\end{align*} \]

with initial

\[ \begin{align*}
x(0) &= x_0 \in \mathbb{R}^n, \quad u(z, 0) = \varphi(z), \quad \varphi(0) = 0, \\
\varphi &\in L^2_2([0, l]), \quad z \in (0, l),
\end{align*} \]

and boundary conditions

\[ \begin{align*}
u(0, t) &= d_1(t), \quad u(l, t) = d_2(t), \\
d_1 &\in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad t \in (0, +\infty),
\end{align*} \]

where \( A \) is the linear Sturm-Liouville operator given by

\[ A = \frac{1}{r(z)} \frac{d}{dz} \left( r(z) \frac{df}{dz} \right) + \frac{q(z)}{r(z)} f, \]

\[ D(A) = \{ f : f \in C^3([0, l]; \mathbb{R}), \quad f(0) = f(l) = 0 \}, \]

where \( r \in C([0, l], \mathbb{R}_+), \quad r(z) > 0, \quad q \in C([0, l], \mathbb{R}), \quad B \in C([-l], \mathbb{R} \times \mathbb{R}, \mathbb{R}^{2 \times n}), \quad \|B(t, l)\|_2 \in L^\infty. \]

The mapping \( C : \mathbb{R} \to \mathbb{R}^{n \times n} \) is piece-wise continuous and periodic with period \( \theta > 0 \) and assumed to be such that the linear ODE

\[ \xi = C(t)\xi \]

is asymptotically stable. Moreover it is assumed that for any \( N \in \mathbb{N} \) there exist positive constants \( a_m, b_m, c_m, m = 0, \ldots, N - 1 \) such that for \( t \in (mh, (m + 1)h) \) and \( h = \frac{\theta}{n} \) the following holds

\[ \sup_{t \in (mh, (m+1)h)} \|C(t)\| \leq a_m, \quad \| \int_{mh}^{t} C(s) ds \| \leq c_m(t - mh), \]

\[ \left\| \int_{mh}^{t} C(t) dt \right\| \leq b_m(t - mh). \]

The function \( D \in C([0, l], \mathbb{R}_+, \mathbb{R}^{n \times n}) \) is assumed to be uniformly bounded, that is \( \|D(t)\| \in L^\infty \).

In this work we consider only such initial conditions and disturbances that the problem (3)-(5) has classical solutions \( (u, x) \in C^{2,1}([0, l] \times \mathbb{R} \times \mathbb{R} \times C^1(\mathbb{R}, \mathbb{R}^{n \times n}) \).

The questions of existence and uniqueness of such solutions were studied in Slyn’ko (2006); Karafyllis and Krstic (2019).

### Definition

The linear system (3)-(5) is called ISS, if there exist functions \( \beta_i \in K, \gamma_i \in K, i = 1, 2 \) such that for any initial state in (4) and any input functions \( d_1, d_2 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) the corresponding solution satisfies

\[ \|\varphi(\cdot, t, \varphi_0, x_0)\|_{L^2([0, l])} \leq \beta_1(\|\varphi_0\|_{L^2([0, l])}, \|x_0\|_2, t) + \gamma_1(d_1), \]

\[ \|x(t, \varphi_0, x_0)\| \leq \beta_2(\|\varphi_0\|_{L^2([0, l])}, \|x_0\|_2, t) + \gamma_2(d_2), \]

where \( d_1 \) and \( d_2 \) are defined in (4) as

\[ \sup_{(z, t) \in [0, l] \times (0, \infty)} |w(z, t)| \leq \max\{\|d_1\|_{\infty}, \|d_2\|_{\infty} \} = d_\infty. \]

### 4. AUXILIARY RESULTS

#### Lemma 1

Let \((u(t, z), x(t))\) be a pair of functions satisfying (3), then

\[ u(t, z) = w(z, t) + v(z, t), \]

where \((v(z, t), x(t))\) is the solution to the system

\[ \begin{align*}
u(t) &= Au(t, z) + B(z, t)x(t), \\
\dot{x}(t) &= C(t)x(t) + \int_0^t D(z, t)v(z, t) dz + \int_0^t D(z, t)w(z, t) dz
\end{align*} \]

with initial conditions

\[ \begin{align*}
x(0) &= x_0, \quad v(z, 0) = \varphi(z), \quad \varphi(0) = 0, \\
\varphi &\in L^2_2([0, l]), \quad z \in (0, l).
\end{align*} \]

and boundary conditions

\[ \begin{align*}
u(0, t) &= 0, \quad v(l, t) = 0, \\
\varphi(0) &= 0, \quad \varphi(l) = 0,
\end{align*} \]

and function \( w(z, t) \) satisfies the following estimation

\[ \sup_{(z, t) \in [0, l] \times (0, \infty)} |w(z, t)| \leq \max\{\|d_1\|_{\infty}, \|d_2\|_{\infty} \} = d_\infty. \]
Proof. Let us substitute \( u(z,t) = w(z,t) \) to the equations (3), then we get
\[
\begin{align*}
v_t(z,t) &= Av(z,t) + B(z,t)x(t) + \cdots, \\
\dot{x}(t) &= C(t)x(t) + \int_0^1 D(z,t)v(z,t)\,dz + \int_0^1 D(z,t)w(z,t)\,dz.
\end{align*}
\] (14)

Let function \( w(z,t) \) satisfy the following equations:
\[
\begin{align*}
w_t(z,t) - Aw(z,t) &= 0, \\
w(0,t) &= d_1(t), \\
w(l,t) &= d_2(t), \\
w(z,0) &= 0.
\end{align*}
\]

Then function \( v(z,t) \) needs to satisfy the following initial and boundary conditions
\[
v(0,t) = 0, \quad v(l,t) = 0, \quad v(z,0) = \varphi(z).
\]

By the maximum principle applied to the function \( w(z,t) \), we obtain the desired estimation (13). Then from (14) it follows that the function \( v(z,t) \) satisfies the system of linear differential equations (10) subject to (11) and (12), which completes the proof of the lemma.

Next, we are going to derive estimates for the solution to (10) by means of the direct Lyapunov method. The Lyapunov function will be constructed by the discretization method as follows: Let
\[
V(t,v(\cdot,t),x) = p(t)V_1(v(\cdot,t)) + V_2(t,x),
\]
with
\[
V_1(v(\cdot,t)) = \int_0^t r(z)v^2(z,t)\,dz = \|v(\cdot,t)\|_{L^2[0,t]}^2, \\
V_2(t,x) = x^T P(t)x
\]
where \( p : \mathbb{R} \to \mathbb{R}_+ \) and \( P : \mathbb{R} \to S^n \) are left continuous periodic functions with period \( \theta \).

Due to the periodicity it is enough to define the function \( P(t) \) on the interval \( (0,\theta] \). Let \( N \) be a natural number and \( h = \frac{\theta}{N} \) be the discretization step. First we define the following matrices
\[
\hat{C}_m(t) = \int_{mh}^{(m+1)h} C(s)\,ds, \\
\hat{C}_m = \frac{1}{h} \hat{C}_m((m+1)h) = \int_{mh}^{(m+1)h} C(s)\,ds, \\
\Phi = e^{\hat{C}_1} \cdots e^{\hat{C}_N}.
\]

By the asymptotic stability of the linear system (7) we can choose the number \( N \) such that \( r_\sigma(\Phi) < 1 \). Hence we can choose a positive definite matrix \( P_0 \) such that the following inequality is satisfied
\[
\Phi^T P_0 \Phi - P_0 \prec 0.
\]

Further we define the following positive definite matrices
\[
P_{m+1} = e^{-\hat{C}_m} P_m e^{-\hat{C}_m}, \quad m = 0,\ldots,N-1
\]
and finally the function \( P(t) \) is defined by
\[
P(t) = e^{-\hat{C}_m(t)} P_m e^{-\hat{C}_m(t)}, \quad t \in (mh, (m+1)h].
\]

Note that from \( P_0 > 0 \) follows that \( P(t) \succ 0 \) for \( t \geq 0 \).

Now let us estimate the derivative of \( V_1(v(\cdot,t)) \) along solutions of the system (10) for \( t \in (k\theta + mh, k\theta + (m+1)h] \), \( k \in \mathbb{Z}_+ \), \( m = 0,\ldots,N-1 \).

Let us denote for short \( r_{\min} = \min_{z \in [0,1]} r(z) \), \( r_{\max} = \max_{z \in [0,1]} r(z) \), \( q_{\max} = \max_{z \in [0,1]} q(z) \), \( \rho = \frac{r_{\min} \pi^2}{2} - q_{\max} \).

Now we can state our second technical lemma as follows.

**Lemma 2.** Let \( \rho > 0 \) and \( B_m := \sup_{k \in \mathbb{Z}_+} \sup_{t \in (k\theta + mh, k\theta + (m+1)h]} \|B(\cdot,t)\|_{L^2[0,t]} \).

Then for \( t \in (k\theta + mh, k\theta + (m+1)h] \) the following estimate holds
\[
\frac{dV_1(v(\cdot,t))}{dt} \leq -2\rho V_1(v(\cdot,t)) + 2B_m \int_0^t r(z)v(z,t)\,dz = 2 \int_0^t r(z)v(z,t)v_1(z,t)\,dz
\]
\[
= 2 \int_0^t r(z)v(z,t)Av(z,t)\,dz + 2 \int_0^t r(z)v(z,t)B(z,t)\,dx\,dt
\]
The first integral in last line can be estimated as follows
\[
\int_0^t r(z)v(z,t)Av(z,t)\,dz = \int_0^t v(z,t)\frac{\partial}{\partial z}(r(z)\frac{\partial v}{\partial z}(z,t))\,dz
\]
\[
= -\int_0^t r(z)v^2(z,t)\,dz + \int_0^t q(z)v^2(z,t)\,dz
\]
\[
\leq -r_{\min} \int_0^t v^2(z,t)\,dz + q_{\max} \int_0^t v^2(z,t)\,dz.
\]

With help of the Friedrichs inequality
\[
\int_0^t v^2(z,t)\,dz \leq \frac{t^2}{\pi^2} \int_0^t v^2(z,t)\,dz
\]
we get
\[
\int_0^t r(z)v(z,t)Av(z,t)\,dz
\]
\[
\leq \left( -\frac{r_{\min} \pi^2}{2} + q_{\max} \right) \int_0^t v^2(z,t)\,dz.
\]

From the obvious estimation
\[
V_1(v(\cdot,t)) = \int_0^t r(z)v^2(z,t)\,dz \leq r_{\max} \int_0^t v^2(z,t)\,dz
\]
we get the inequality
\[
\frac{dV_1(v(\cdot,t))}{dt} \leq -2\rho V_1(v(\cdot,t)) + 2 \int_0^t r(z)v(z,t)B(z,t)\,dx\,dt.
\]

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And finally by the Cauchy inequality we arrive at the desired estimate
\[
\frac{dV_1(v(\cdot, t))}{dt} \leq -\frac{2p}{r_{\max}} V_1(v(\cdot, t)) + 2\|v(\cdot, t)\|_{L^2[0, t]} B_m \|P^{-1/2}(t)\| \sqrt{V_2(t, x(t))}
\]
\[
\leq -\frac{2p}{r_{\max}} V_1(v(\cdot, t)) + 2B_m \|P^{-1/2}(t)\| p^{-1/2}(t) \sqrt{p(t)} V_1(v(\cdot, t)) V_2(t, x(t)),
\]
which proves the lemma.

And the last lemma that we need for the estimation of the time derivative of the function \(V_2\) is as follows.

**Lemma 3.** Let \(D_m = \sup_{k \in \mathbb{Z}} \sup_{t \in [k \theta + mh, k \theta + (m+1)h]} \|D(\cdot, t)\|_{L^2[0, t]}, \|
\eta_m = b_m \sum_{k=1}^{2c_m h} \frac{(2c_m h)^{k-1}}{(k+1)!} \).

Then for \(t \in (k \theta + mh, k \theta + (m+1)h)\) the following inequality holds
\[
\frac{dV_2(t, x(t))}{dt} \leq 2e^{2c_m h} \sqrt{\frac{\lambda_{\max}(P_m)}{\lambda_{\min}(P_m)}} \eta_m V_2(t, x)
\]
\[
+ 2p^{-1/2}(t) D_m \|P^{1/2}(t)\| \sqrt{p(t)} V_1(v(\cdot, t)) V_2(t, x(t))
\]
\[
+ 2\sqrt{r_{\max} d_{\infty}} D_m \|P^{1/2}(t)\| \sqrt{V_2(t, x(t))}.
\]

**Proof.** Let us first consider the expression \(\dot{P}(t) + C^T(t) P(t) + P(t) C(t)\) to calculate the derivative \(\dot{P}(t)\) we first derive an expression for the derivative \(\frac{d}{dt} e^{-C_m(t)}\). By the chain rule we have
\[
\frac{d}{dt} e^{-C_m(t)} = \left(\frac{dC_m(t)}{dt} \frac{\partial}{\partial X} \right) e^X \bigg|_{X=-C_m(t)}
\]
\[
= - \left( C(t) \frac{\partial}{\partial X} \right) e^X \bigg|_{X=-C_m(t)}.
\]
Then by the Haussdorff equality we get
\[
\frac{d}{dt} e^{-C_m(t)} = -e^{-C_m(t)} \left( C(t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \{C(t), C_m(t)\} \right).
\]
Hence we can write
\[
\frac{d}{dt} e^{-C_m(t)} = - \left( C(t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \{C(t), C_m(t)\} \right)^T e^{-C_m(t)}
\]
Finally we obtain
\[
\dot{P}(t) + C^T(t) P(t) + P(t) C(t)
\]
\[
= - \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \{C(t), C_m(t)\} \right)^T P(t)
\]
\[
- P(t) \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \{C(t), C_m(t)\} \right)
\]
Taking into account that for all \(t \in (k \theta + mh, k \theta + (m+1)h)\) we have
\[
\|\{C(t), C_m(t)\}\| \leq b_m (2c_m h)^{k-1} (t - k \theta - mh)^k,
\]
\[
\left| \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \{C(t), C_m(t)\} \right| \leq b_m \sum_{k=1}^{\infty} \frac{(2c_m h)^{k-1} (t - k \theta - mh)^k}{(k+1)!} \leq \eta_m h,
\]
we can estimate
\[
x^T (\dot{P}(t) + C^T(t) P(t) + P(t) C(t)) x
g \leq 2p^{-1/2}(t) \|P^{1/2}(t)\| \|P(t) C(t)\| x
\]
\[
g \leq 2p^{-1/2}(t) \|P^{1/2}(t)\| \|P(t) C(t)\| x.
\]
Then from the inequalities
\[
\|P^{1/2}(t)\| = \lambda_{\max}(P(t)) = \|P(t)\|^{1/2}
\]
\[
eq \|e^{-C_m(t)} P_m e^{-C_m(t)}\|^{1/2} \leq e^{c_m (t-k \theta - mh)} \|P_m\|^{1/2},
\]
\[
\lambda_{\min}(P(t)) = \min_{\|x\|=1} x^T P(t) x
\]
\[
= \min_{\|x\|=1} (e^{-C_m(t)} P_m e^{-C_m(t)}) x
\]
\[
\geq \lambda_{\min}(P_m) \min_{\|x\|=1} \|e^{-C_m(t)} x\|^2
\]
\[
\geq \min_{\|x\|=1} (e^{c_m (t-k \theta - mh)} \|x\|^2
\]
\[
= \lambda_{\min}(P_m) e^{c_m (t-k \theta - mh)}
\]
and
\[
\|P^{1/2}(t)\| = \lambda_{\max}^{1/2}(P(t)) = \lambda_{\min}^{1/2}(P(t))
\]
\[
\leq \lambda_{\min}(P_m) e^{c_m (t-k \theta - mh)},
\]
for all \(t \in (k \theta + mh, k \theta + (m+1)h)\).

Now let us estimate the time derivative of \(V_2(t, x(t))\) along solutions of the system (10) for \(t \in [k \theta + mh, k \theta + (m+1)h]\). First we calculate that
\[
\frac{dV_2(t, x(t))}{dt} = x^T (\dot{P}(t) + C^T(t) P(t) + P(t) C(t)) x(t)
\]
\[
+ 2 \int_0^l v(z, t) D^T(t, z) dz P(t) x(t)
\]
\[
+ 2 \int_0^l v(z, t) D(t, z) dz P(t) x(t).
\]
Then by the Cauchy inequality we get that for \(t \in (k \theta + mh, k \theta + (m+1)h)\)
\[
\|v(\cdot, t)\|_{L^2[0, l]} D_m \|P^{1/2}(t)\| \sqrt{V_2(t, x(t))}
\]
\[
\leq p^{-1/2}(t) D_m \|P^{1/2}(t)\| \sqrt{p(t)} V_1(v(\cdot, t)) V_2(t, x(t))
\]
and hence from the inequality (9),
\[
\int_0^l v(z, t) D^T(t, z) dz P(t) x(t)
\]
\[
\leq \sqrt{r_{\max} d_{\infty}} D_m \|P^{1/2}(t)\| \sqrt{V_2(t, x(t))}
\]
\[
\leq \sqrt{r_{\max} d_{\infty}} D_m \|P^{1/2}(t)\| \sqrt{V(t, v(\cdot, t), x(t))}
\]
the statement of the lemma follows.

5. MAIN RESULTS

Let us introduce the following matrices \(\Gamma_m = (\gamma_{ij}(m))_{i,j=1,2}, m = 0, \ldots, N-1\).
Then we can state our first main result as follows.

**Theorem 1.** Let $P_0$ be positive definite and satisfy the linear matrix inequality (17). If $\alpha \in (0, 1)$ and $\rho = \frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} > 0$, then the linear system (3)-(5) satisfies the ISS property and, in particular, the following estimations for its solution are valid

$$
\|x(t)\| \leq \frac{\lambda_{\text{max}}^h/2}{\lambda_\alpha \sqrt{q}} e^{\frac{\lambda_{\text{max}}}{\lambda_\alpha \sqrt{q}} t} \sqrt{e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} \|\varphi_0\|_{L_2}^2} + \lambda_{\text{max}}(P_0) \|x_0\|^2} + K \frac{1}{\lambda_\alpha} d_{\infty},
$$

$$
\|u(t)\|_{L_2} \leq \frac{\lambda_{\text{max}}^h/2}{\sqrt{q}} e^{\frac{\lambda_{\text{max}}}{\sqrt{q}} t} \sqrt{\|\varphi_0\|_{L_2}^2 + e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} \lambda_{\text{max}}(P_0) \|x_0\|^2} + (\sqrt{r_{\text{max}}} + K e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} \lambda_{\text{max}}(P_0)}) d_{\infty}.
$$

**Proof.** For the calculation of the time derivative of the Lyapunov function $V(t, v(\cdot), x(t))$ along solutions of the system (9) the Lemmas 2 and 3 can be applied, which imply that the following inequality holds

$$
\dot{V}(t, v(\cdot), x(t)) \leq \left(\dot{p}(t) - \frac{2\rho}{\tau_{\text{max}}} p(t)\right) V_1(v(\cdot), t) + 2e^{c_{\text{max}} h} (B_m \lambda_{\text{min}}^{-1/2}(P_m)) p(t)^{1/2}(t)
$$

$$
+ p^{1/2}(t) D_m \lambda_{\text{max}}^{1/2}(P_m) \sqrt{p(t) V_1(v(\cdot), t) V_2(t, x)} + 2e^{c_{\text{max}} h} \frac{\lambda_{\text{max}}(P_m)}{\lambda_{\text{min}}(P_m)} h \eta_m V_2(t, x)
$$

$$
+ 2\sqrt{r_{\text{max}}} D_m \|\varphi(t)\|_{L_2} \sqrt{V(t, v(\cdot), x(t)) d_{\infty}}
$$

for all $t \in (k \theta + mh, k \theta + (m + 1)h)$. Let us choose the function $p(t)$ so that

$$
\dot{p}(t) = -\frac{2\rho}{\tau_{\text{max}}} p(t) = 0,
$$

$$
p(k \theta + mh + 0) = p_m.
$$

Then $p(t) = e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} (t - mh - k \theta)} p_m$ for $t \in (k \theta + mh, k \theta + (m + 1)h)$, where $p_m = p_0 e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} mh}$. The constant $p_0$ we choose as follows: $p_0 = e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} h}$. Then

$$
\dot{V}(t, v(\cdot), x(t)) \leq 2\gamma_{12} e^{c_{\text{max}} h} \sqrt{p(t) V_1(v(\cdot), t) V_2(t, x)} + \gamma_{22} V_2(t, x) + 2\delta_m d_{\infty} \sqrt{V(t, v(\cdot), x(t))}
$$

$$
\leq \lambda_{\text{max}}(\Gamma_m) V(t, v(\cdot), x(t)) + 2\delta_m d_{\infty} \sqrt{V(t, v(\cdot), x(t))}.
$$

Hence our Lyapunov function satisfies the following differential inequality for all $t \in (k \theta + mh, k \theta + (m + 1)h)$

$$
\dot{V}(t, v(\cdot), x(t)) \leq \lambda_{\text{max}}(\Gamma_m) V(t, v(\cdot), x(t)) + 2\delta_m d_{\infty} \sqrt{V(t, v(\cdot), x(t))}.
$$

Hence by the comparison principle we get the following estimate for $t \in (k \theta + mh, k \theta + (m + 1)h)$

$$
V(t, v(\cdot), x(t)) \leq u(t, mh + k \theta, V_{nk}),
$$

where

$$
V_{nk} = V(k \theta + mh + 0, v(\cdot), k \theta + mh + 0), x(k \theta + mh + 0) = V(k \theta + mh - 0, v(\cdot), k \theta + mh - 0), x(k \theta + mh - 0),
$$

for $m = 1, \ldots, N - 1$, $V_{0k} = V(k \theta + 0, v(\cdot), k \theta + 0), x(k \theta + 0)$,

$$
V_{nk} = V((k + 1) \theta - 0, v(\cdot), (k + 1) \theta - 0), x((k + 1) \theta - 0),
$$

and $u(t, mh + k \theta, x(t))$, $t \in (k \theta + mh, k \theta + mh + (m + 1)h)$ is the solution to the initial value problem

$$
\dot{u}(t) = \lambda_{\text{max}}(\Gamma_m) u(t) + 2\delta_m d_{\infty} \sqrt{u(t)},
$$

$$
u(t) = u(t, mh + k \theta, V_{nk}).
$$

Solving this initial value problem we get

$$
\sqrt{V_{m+1,k}} \leq \frac{\lambda_{\text{max}}^2(h_\gamma)}{\lambda_{\text{max}}(\Gamma_m)} \sqrt{V_{m,k} + 2 \delta_m e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} \lambda_{\text{max}}(P_0) - 1}} d_{\infty},
$$

$$
m = 0, \ldots, N - 1
$$

And in particular

$$
\sqrt{V_{Nk}} \leq Q_{N-1} \sqrt{V_{0k}} + Q_{N-1} \sum_{l=0}^{N-1} Q_{l}^{-1} \zeta_{ld_{\infty}}.
$$

Or equivalently

$$
\sqrt{V((k + 1) \theta, v(\cdot), (k + 1) \theta), x((k + 1) \theta)) \leq Q_{N-1} \sqrt{V(k \theta + 0, v(\cdot), k \theta + 0), x((k + 1) \theta))}
$$

$$
+ Q_{N-1} \sum_{l=0}^{N-1} Q_{l}^{-1} \zeta_{ld_{\infty}}.
$$

Since functions $p(t)$ are $P(t)$ are $\theta$-periodic we can write

$$
V((k + 1) \theta + 0, v(\cdot), (k + 1) \theta + 0), x((k + 1) \theta + 0) = p((k + 1) \theta + 0) V_1(v(\cdot), (k + 1) \theta + 0))
$$

$$
+ x^T ((k + 1) \theta + 0) P((k + 1) \theta + 0) x((k + 1) \theta + 0)
$$

$$
= p(t_0) V_1(v(\cdot), (k + 1) \theta + 0) x((k + 1) \theta + 0)
$$

$$
+ e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} \lambda_{\text{max}}(P_0) - 1} (P_{N-1}^2 P_{N-1} P_{N-1}^2 (x((k + 1) \theta))
$$

$$
\leq e^{-\frac{\tau_{\text{max}}}{\tau_{\text{max}} - q_{\text{max}}} \lambda_{\text{max}}(P_0) - 1} v_2((k + 1) \theta)) + \lambda_{\text{max}}(P_{N-1}) V_2((k + 1) \theta), x((k + 1) \theta))
$$

$$
\leq q_0 \lambda_{\text{max}}(P_0) - 1) V_2((k + 1) \theta)) + V_2((k + 1) \theta), x((k + 1) \theta))
$$

$$
= q_0 V((k + 1) \theta), v(\cdot), (k + 1) \theta), x((k + 1) \theta)).
$$

From the last estimate it follows that

$$
\sqrt{V((k + 1) \theta + 0, v(\cdot), (k + 1) \theta + 0), x((k + 1) \theta + 0)) \leq \sqrt{q_0 Q_{N-1}} \lambda_{\text{max}}(P_{N-1}) V_2((k + 1) \theta + 0, v(\cdot), (k + 1) \theta + 0), x((k + 1) \theta + 0))
$$

$$
+ \sqrt{q_0 Q_{N-1}} \sum_{l=0}^{N-1} Q_{l}^{-1} \zeta_{ld_{\infty}}.
$$
and
\[
\sqrt{V(k\theta + 0, v(\cdot, k\theta + 0), x(k\theta + 0))} \\
\leq \alpha^k \sqrt{V(0 + 0, v(\cdot, 0 + 0), x(0 + 0))} + \frac{\beta}{1 - \alpha} d_\infty.
\]
Let us denote \( k_0 = \left\lfloor \frac{t}{h} \right\rfloor \) and \( m_0 = \left\lfloor \frac{t - \|\theta\|k_0}{h} \right\rfloor \), then \( t = k_0 \theta + m_0 h + t_1, t_1 \in [0, h) \).

From the inequality (20) we get
\[
\sqrt{V_{m_0,k_0}} \leq Q_{N-1} \sqrt{V_{0,k_0}} + Q_{N-1} \sum_{i=0}^{N-1} Q_i^* q_i d_\infty
\]
\[
= \frac{1}{\sqrt{q_0}} (\alpha \sqrt{V_{0,k_0}} + \beta d_\infty),
\]
and from the inequality (19) we get
\[
\sqrt{V(t,v(\cdot,t),x(t))} \leq e^{\lambda_{\text{max}}(\max(|v_0|)|/2)} \sqrt{V_{m_0,k_0}} + \zeta_{\text{max}} d_\infty
\]
Hence
\[
\sqrt{V(t,v(\cdot,t),x(t))} \leq e^{\lambda_{\text{h}}/2} \sqrt{V_{m_0,k_0}} + \zeta d_\infty
\]
\[
\leq \frac{\alpha \lambda_{\text{h}}/2}{\sqrt{q_0}} \sqrt{V_0,k_0} + \left( \frac{\beta e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} + \zeta \right) d_\infty.
\]
Then from (21) we obtain
\[
\sqrt{V(t,v(\cdot,t),x(t))} \leq \frac{\alpha e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} \left( \alpha^k \sqrt{V(0 + 0, v(\cdot, 0 + 0), x(0 + 0))} + \frac{\beta}{1 - \alpha} d_\infty \right) + \frac{\beta e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} d_\infty + \frac{\alpha \beta e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} d_\infty + \frac{\beta e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} d_\infty + \zeta d_\infty
\]
\[
\leq \frac{e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} \left( \frac{\ln \alpha}{\alpha} \sqrt{V(0 + 0, v(\cdot, 0 + 0), x(0 + 0))} + K d_\infty \right).
\]
Hence,
\[
\frac{e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} \leq e^{\lambda_{\text{h}}/2} p(t/2)\|v(\cdot,t)\|_{L_2[0,t]} + \lambda_{\text{max}}(p_0)\|x_0\|^2 + K d_\infty.
\]
Since \( p(t) \geq p_0 \) for all \( t \geq 0 \), then
\[
\frac{e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} \leq e^{\lambda_{\text{h}}/2} p(t/2)\|v(\cdot,t)\|_{L_2[0,t]} + \lambda_{\text{max}}(P_0)\|x_0\|^2 + K e^{\lambda_{\text{h}}/2} d_\infty.
\]
and as well
\[
\frac{e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} \leq \frac{e^{\lambda_{\text{h}}/2}}{\sqrt{q_0}} \left( \frac{\ln \alpha}{\alpha} \sqrt{V(0 + 0, v(\cdot, 0 + 0), x(0 + 0))} + K e^{\lambda_{\text{h}}/2} d_\infty \right),
\]
Now from (18) we obtain the estimate
\[
\|x(t)\| \leq \frac{e^{\lambda_{\text{h}}/2}}{\lambda e^{\sqrt{\theta}} e^{\frac{\lambda_{\text{h}}}{2} \sqrt{q_0}}} e^{\lambda_{\text{h}}/2} \left( \frac{\ln \alpha}{\alpha} \sqrt{V(0 + 0, v(\cdot, 0 + 0), x(0 + 0))} + \lambda_{\text{max}}(P_0)\|x_0\|^2 + K e^{\lambda_{\text{h}}/2} d_\infty \right)
\]
and the statement of the theorem follows immediately from Lemma 1.

6. CONCLUSION

We have established ISS properties for classical solutions for a feedback connection of an ODE with a parabolic PDE and provided explicit ISS estimates for the solutions. Connections of the considered class of systems was not discussed in the ISS framework to the best of authors knowledge. It should be mentioned that the ISS property can be rather restrictive in some applications. The study of such couplings can be performed similarly under such weaker properties as integral ISS or local ISS.

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