BOOLEAN TERM ORDERS AND THE ROOT SYSTEM $B_n$

DIANE MACLAGAN

Abstract. A boolean term order is a total order on subsets of $[n] = \{1, \ldots, n\}$ such that $\emptyset \prec \alpha$ for all $\alpha \subseteq [n], \alpha \neq \emptyset$, and $\alpha \prec \beta \Rightarrow \alpha \cup \gamma \prec \beta \cup \gamma$ for all $\gamma$ with $\gamma \cap (\alpha \cup \beta) = \emptyset$. Boolean term orders arise in several different areas of mathematics, including Gröbner basis theory for the exterior algebra, and comparative probability.

The main result of this paper is that boolean term orders correspond to one-element extensions of the oriented matroid $\mathcal{M}(B_n)$, where $B_n$ is the root system $\{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq i < j \leq n\}$. This establishes boolean term orders in the framework of the Baues problem, in the sense of [10]. We also define a notion of coherence for a boolean term order, and a flip relation between different term orders. Other results include examples of noncoherent term orders, including an example exhibiting flip deficiency, and enumeration of boolean term orders for small values of $n$.

1. Introduction

Definition 1.1. A boolean term order is a total order on subsets of $[n] = \{1, \ldots, n\}$ such that:

1. $\emptyset \prec \alpha$ for all $\alpha \subseteq [n], \alpha \neq \emptyset$.
2. If $\alpha \prec \beta$, and $\gamma \cap (\alpha \cup \beta) = \emptyset$ then $\alpha \cup \gamma \prec \beta \cup \gamma$.

Where there can be no confusion, we will use the phrase term order.

Boolean term orders arise in several different areas of mathematics. The motivating example for this paper is term orders in the exterior algebra over a vector space of dimension $n$. A monomial $x^\alpha$ in the exterior algebra corresponds to the subset of $[n]$ given by $\text{supp}(\alpha) = \{i : x_i | x^\alpha\}$. The axioms above then correspond to the standard ones for Gröbner basis theory term orders (see [11]).

Another incarnation of boolean term orders is as antisymmetric comparative probability relations. The basic idea there is that we have a finite set $\Omega$ of events, and are interested in which events are more probable than others, as opposed to the exact probability of each event. We associate with a subset of $\Omega$ the (comparative) probability that at least one of the events in the subset occurs. If we demand that for any two subsets $A$ and $B$ of $\Omega$ either $A \prec B$ ($B$ is more probable than $A$), or $B \prec A$, then we have a boolean term order with $n = |\Omega|$. An overview of the theory of comparative probability can be found in [3].

The main result of this paper is that boolean term orders correspond to one-element extensions of the oriented matroid $\mathcal{M}(B_n)$, where $B_n$ is the root system $\{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq i < j \leq n\}$. This establishes boolean term orders in the framework of the Baues problem, in the sense of [10]. Section 2 contains precise definitions of term orders, including a notion of coherence. The connection to the Baues problem is strengthened in Section 3 with the introduction of a notion of a flip relation between different boolean term orders. Examples are given of term orders...
Lemma 2.2. Proof.

Definition 2.1. A generalized partial term order is an irreflexive partial order, \( \prec \), on the set of subsets of \([n]\) such that:

1. \( \alpha \prec \beta \iff \alpha \cup \gamma \prec \beta \cup \gamma \) for all \( \gamma \) with \( \gamma \cap (\alpha \cup \beta) = \emptyset \)
2. If \( \alpha \) and \( \beta \) are not comparable (written \( \alpha \sim \beta \)), then \( \{ \gamma : \gamma \prec \alpha \} = \{ \gamma : \gamma \prec \beta \} \), and \( \{ \gamma : \alpha \prec \gamma \} = \{ \gamma : \beta \prec \gamma \} \)

We write \( \alpha \lesssim \beta \) if \( \alpha \sim \beta \) or \( \alpha \prec \beta \). A partial boolean term order additionally satisfies \( \emptyset \prec \alpha \) for all \( \alpha \not= \emptyset \). A generalized term order is a generalized partial term order where \( \alpha \sim \beta \Rightarrow \alpha = \beta \). Note that a boolean term order is a generalized term order with the additional requirement that \( \emptyset \prec \alpha \) for all \( \alpha \not= \emptyset \).

Lemma 2.2. 1. The first condition of Definition 2.1 is equivalent to requiring that whenever \( a \prec b, c \prec d \) with \( a \cap c = \emptyset \) and \( b \cap d = \emptyset \), then \( a \cup c \prec b \cup d \).

2. Assuming the first condition of Definition 2.1, the second condition is equivalent to requiring that if either \( a \cup c \sim b \cup d \) or \( a \cup c = b \cup d \), with \( a \cap c = \emptyset \) and \( b \cap d = \emptyset \), and \( b \prec a \), then \( c \prec d \).

Proof. 1. Assume \( \prec \) satisfies the first condition of Definition 2.1 and write \( b = b' \cup l, c = c' \cup l \), where \( b', c' \), and \( l \) are all pairwise disjoint. Then \( a \cup c' \prec b \cup c' = b' \cup c \prec b' \cup d \), so \( a \cup c' \cup l \prec b' \cup d \cup l \). This proves one implication.

Conversely, assume that \( \prec \) is a total order on subsets of \([n]\) satisfying the condition of the lemma, but not satisfying the first condition of Definition 2.1, so there is a pair \( a \prec b \) with \( b \cup c \prec a \cup c \) for some \( c \) with \( c \cap (a \cup b) = \emptyset \). We may assume that \( a \cap b = \emptyset \), so the condition of the lemma implies \( a \cup b \cup c \prec a \cup b \cup c \), a contradiction.

2. Suppose \( \prec \) satisfies both conditions of Definition 2.1, \( b \prec a \), and either \( a \cup c \sim b \cup d \) or \( a \cup c = b \cup d \). If \( d \prec c \) we would have \( b \cup d \prec a \cup c \) from above. If \( c \sim d \), then write \( b = b' \cup l, c = c' \cup l \) as above. Then \( b' \cup c = b \cup c' \prec a \cup c' \). But \( b' \cup c \sim b' \cup d \), so \( b' \cup d \prec a \cup c' \). From this contradiction we conclude \( c \not\prec d \).

Conversely, let \( \prec \) be a partial order on monomials satisfying the first condition of Definition 2.1 and such that whenever \( b \prec a \) and either \( a \cup c \sim b \cup d \) or \( a \cup c = b \cup d \), with \( a \cap c = \emptyset \) and \( b \cap d = \emptyset \), then \( c \not\prec d \). Suppose \( \alpha \sim \beta \), and \( \alpha \prec \gamma \). We can assume that \( \alpha \cap \beta \cap \gamma = \emptyset \). Write

\[
\alpha = \alpha' \cup \delta \cup \phi \\
\beta = \beta' \cup \delta \cup \psi \\
\gamma = \gamma' \cup \phi \cup \psi
\]

where \( \alpha', \beta', \gamma', \delta, \phi, \psi \) are all disjoint. Then \( \alpha \sim \beta \) implies \( \alpha \cup \gamma' \sim \beta \cup \gamma' \). Writing this out in full, we have \((\gamma' \cup \phi) \cup (\alpha' \cup \delta) \sim (\gamma' \cup \psi) \cup (\beta' \cup \delta) \). But \( \alpha \prec \gamma \) means \( \alpha' \cup \delta \prec \gamma' \cup \psi \), so \( \beta' \cup \delta \prec \gamma' \cup \phi \), and thus, multiplying by \( \psi \),
we get $\beta \prec \gamma$, as required by the definition. The case with $\gamma \prec \alpha$ is the same
with the inequalities reversed.

\[\Box\]

**Corollary 2.3.** If $\prec$ is a generalized partial term order with $a \prec b$ then $[n] \setminus b \prec [n] \setminus a$.

**Proof.** We have $[n] = a \cup ([n] \setminus a)$, and $[n] = b \cup ([n] \setminus b)$, so the result follows from part 2 of Lemma 1.2. $\Box$

This means that the second half of the term order is the complement of the first, in reverse order.

**Definition 2.4.** A boolean term order is **coherent** if there is a weight vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{N}^n$ such that

$$\alpha \prec \beta \iff \sum_{i \in \alpha} w_i < \sum_{j \in \beta} w_j$$

In the interpretation of boolean term orders as Gröbner basis term orders in the exterior algebra, a term order is coherent if it can be extended to a Gröbner basis term order on all the monomials in $n$ variables. This follows from the fact that a Gröbner basis term order in $n$ variables can be induced up to a given finite degree by an integral weight vector (see, for example, Chapter 15 of [3]). In the comparative probability language these are the comparative probability orders which have an agreeing probability measure, and are known as additive antisymmetric comparative probability orders. The question of whether all antisymmetric comparative probability orders were additive was first raised by de Finetti in 1951 [2], and first answered in 1959 by Kraft, Pratt, and Seidenberg [4].

**Example 2.5.** A noncoherent boolean term order for $n = 5$ is:

$$\emptyset \prec \{1\} \prec \{2\} \prec \{3\} \prec \{1, 2\} \prec \{5\} \prec \{1, 3\} \prec \{2, 3\} \prec \{1, 4\} \prec \{2, 4\} \prec \{2, 5\} \prec \{3, 4\} \prec \{1, 2, 3\} \prec \{1, 2, 4\} \prec \{3, 4, 5\} \prec \{1, 2, 3, 4\} \prec \{1, 2, 3, 5\} \prec \{1, 2, 4, 5\} \prec \{1, 3, 4, 5\} \prec \{2, 3, 4, 5\}$$

To see that this term order is not coherent, we notice that $\{4\} \prec \{1, 2\}$, $\{2, 3\} \prec \{1, 4\}$, and $\{1, 4\} \prec \{3, 5\}$. In the exterior algebra characterization, this is $x_4 \prec x_1x_2$, $x_2x_3 \prec x_1x_4$, $x_1x_5 \prec x_2x_4$, and $x_1x_2x_4 \prec x_3x_5$. Note that if $\prec$ extended to an order on all monomials, we could multiply all the left and right sides to get $x_4^2x_2x_3x_5^2 = x_1^2x_2x_3x_5^2x_4^2$, a contradiction. So there is no order on the polynomial ring extending $\prec$.

This method of giving a certificate for the noncoherency of a boolean term order has received attention recently in the work of Fishburn [6], [7]. An open problem is to give a sharp upper bound on the number of inequalities needed in such a certificate for term orders on subsets of $[n]$. 
3. FLIPS FOR TERM ORDERS

In this section we place boolean term orders in the framework of the Baues problem. Specifically, we define a notion of flip for a term order, analogous to bistellar flips for triangulations.

**Definition 3.1.** A primitive pair in a term order is a pair \( \alpha \prec \beta \), with \( \alpha \cap \beta = \emptyset \), which is adjacent, in the sense that there is no \( \gamma \) with \( \alpha \prec \gamma \prec \beta \).

**Example 3.2.** In the term order of Example 2.5, the primitive pairs are: \( \emptyset \prec \{1\} \), \( \{1\} \prec \{2\} \), \( \{2\} \prec \{3\} \), \( \{3\} \prec \{4\} \), \( \{4\} \prec \{1, 2\} \), \( \{1, 2\} \prec \{5\} \), \( \{5\} \prec \{1, 3\} \), \( \{2, 3\} \prec \{1, 4\} \), \( \{1, 5\} \prec \{2, 4\} \), \( \{2, 5\} \prec \{3, 4\} \), and \( \{1, 2, 4\} \prec \{3, 5\} \).

**Proposition 3.3.** The order on the primitive pairs determines the boolean term order, in the sense that if \( \prec_1 \) and \( \prec_2 \) are different term orders, then there is a primitive pair of \( \prec_1 \), \( \alpha \prec_1 \beta \), such that \( \beta \prec_2 \alpha \).

**Proof.** Suppose that \( \prec_1 \) and \( \prec_2 \) are two distinct term orders which have the same order on all the primitive pairs of \( \prec_1 \). Let \( \alpha \) be such that, for all \( \mu, \nu \subseteq [n] \), \( \mu \prec_1 \nu \prec_1 \alpha \Leftrightarrow \mu \prec_2 \nu \prec_2 \alpha \), and suppose that \( \alpha \) is the greatest subset with respect to \( \prec_1 \). Denote the next subset for \( \prec_1 \) by \( \beta \), and for \( \prec_2 \) by \( \gamma \). By assumption \( \beta \neq \gamma \).

Let \( \delta \) be the subset immediately preceding \( \gamma \) in \( \prec_1 \). We know that \( \delta \cap \gamma \neq \emptyset \), as otherwise \( \delta \prec_1 \gamma \) is a primitive pair of \( \prec_1 \), in which case from consideration of \( \prec_2 \) we see that \( \delta \) is at most \( \alpha \), and so \( \beta = \gamma \). Denote \( \delta \setminus \gamma \) and \( \gamma \setminus \delta \) by \( \delta' \) and \( \gamma' \) respectively. Looking at \( \prec_2 \), we see \( \gamma' \) is at most \( \alpha \), so \( \delta' \prec_2 \gamma' \), and thus \( \delta \prec_2 \gamma \), contradicting the adjacency of \( \alpha \) and \( \gamma \) with respect to \( \prec_2 \). \( \square \)

**Definition 3.4.** A flippable pair in a term order \( \prec \) is a primitive pair \( \alpha \prec \beta \) such that all multiples \( \alpha \cup l \prec \beta \cup l \) with \( l \cap (\alpha \cup \beta) = \emptyset \) are adjacent.

**Example 3.5.** In the term order of example 2.5, the flippable pairs are \( \{4\} \prec \{1, 2\} \), \( \{2, 3\} \prec \{1, 4\} \), \( \{1, 5\} \prec \{2, 4\} \), \( \{2, 5\} \prec \{3, 4\} \), and \( \{1, 2, 4\} \prec \{3, 5\} \).

**Definition 3.6.** Given a boolean term order \( \prec_1 \) and a flippable pair \( \alpha \prec_1 \beta \), with \( \alpha \neq \emptyset \), we construct a new total order \( \prec_2 \) by exchanging the order of \( \alpha \cup l \) and \( \beta \cup l \) for all \( l \subseteq [n] \setminus (\alpha \cup \beta) \). We say that \( \prec_2 \) is obtained from \( \prec_1 \) by flipping across \( \alpha \prec_1 \beta \).

**Proposition 3.7.** The \( \prec_2 \) constructed above is a boolean term order.

**Proof.** Since \( \emptyset \) is still the smallest element in \( \prec_2 \), we need only check that the order satisfies the second condition of Definition 3.2. Suppose it does not, so there exist \( a, b, c \subseteq [n] \) with \( c \cap (a \cup b) = \emptyset \) such that \( a \prec_2 b \), but \( b \cup c \prec_2 a \cup c \). The only pairs whose orders have changed from \( \prec_1 \) are the multiples of \( \alpha \prec_1 \beta \), so one of \( a \prec_2 b \), and \( b \cup c \prec_2 a \cup c \) must be such a multiple. But if one pair is a multiple of \( \beta \prec_2 \alpha \), so is the other, and so their order is still consistent, as it is the reverse of the consistent order under \( \prec_1 \). Thus \( a \prec_2 b \Leftrightarrow a \cup c \prec_2 b \cup c \), so \( \prec_2 \) is a boolean term order. \( \square \)

**Remark 3.8.** Note that \( \prec_1 \) and \( \prec_2 \) agree on every pair of disjoint sets except for the pair \( \{\alpha, \beta\} \).
Example 3.9. Flipping across \{4\} \prec \{1, 2\} in the term order of Example 2.3 gives:
\[
0 \prec \{1\} \prec \{2\} \prec \{3\} \prec \{1, 2\} \prec \{4\} \prec \{5\} \prec \{1, 3\} \prec \\
\{2, 3\} \prec \{1, 4\} \prec \{1, 5\} \prec \{2, 4\} \prec \{2, 5\} \prec \{1, 2, 3\} \prec \\
\{3, 4\} \prec \{1, 2, 4\} \prec \{3, 5\} \prec \ldots
\]
Only the first half of the order is given, as the second half can be deduced from the first by Corollary 2.3. This is a coherent term order, given by the weight vector (7, 10, 16, 20, 22).

Remark 3.10. The analogue of Proposition 3.3 for flippable pairs is false. The following noncoherent term order in six variables, with the flippable pairs marked by \prec, has the same order on its flippable pairs as the coherent term order given by the vector (6, 14, 15, 18, 28, 38). It is thus not determined by the order on its flippable pairs.

\[
0 \prec \{1\} \prec \{2\} \prec \{1, 2\} \prec \{3\} \prec \{1, 3\} \prec \{4\} \prec \{2, 3\} \prec \{1, 4\} \prec \\
\{1, 2, 3\} \prec \{2, 4\} \prec \{1, 2, 4\} \prec \{3, 4\} \prec \{1, 5\} \prec \{2, 5\} \prec \\
\{6\} \prec \{1, 3, 4\} \prec \{2, 3, 4\} \prec \{1, 2, 5\} \prec \{3, 5\} \prec \{1, 6\} \prec \\
\{2, 6\} \prec \{1, 2, 3, 4\} \prec \{1, 3, 5\} \prec \{4, 5\} \prec \{2, 3, 5\} \prec \{1, 2, 6\} \prec \\
\{1, 4, 5\} \prec \{3, 6\} \prec \{1, 2, 3, 5\} \prec \{1, 3, 6\} \prec \{2, 4, 5\} \ldots
\]

Remark 3.11. The central pair is flippable in every term order. By Corollary 2.3 the \(2^{n-1}+1\)st term is the complement of the \(2^n-1\)th term, so there are no nontrivial multiples to consider.

4. Hyperplane Arrangements and the Root System \(B_n\)

A hyperplane arrangement in \(\mathbb{R}^n\) partitions \(\mathbb{R}^n\) into relatively open regions of points such that in each region all the points lie either on, or on the same side of, each hyperplane. Let \(\mathcal{H}_n\) denote the hyperplane arrangement consisting of all the hyperplanes with normals in \(\{0, 1, -1\}^n \setminus \{0^n\}\). The equivalence classes of real weight vectors \(w\) which determine the same coherent generalized partial term order correspond to regions of \(\mathcal{H}_n\).

Lemma 4.1. 1. The \(n\)-dimensional regions of \(\mathcal{H}_n\) are in bijection with coherent generalized term orders.
2. Flipping across a flippable pair from one coherent term order to another coherent term order corresponds geometrically to passing from one \(n\)-dimensional region of \(\mathcal{H}_n\) to an adjacent region.

Proof. Part 1 is immediate from the definition of \(\mathcal{H}_n\). Part 2 follows from Remark 2.8 since if two regions of \(\mathcal{H}_n\) are on the same side of all but one hyperplane, as is the case for the regions corresponding to term orders connected by a flip, then the two regions must be adjacent.

It would thus be interesting to know the number of regions of \(\mathcal{H}_n\). Unfortunately this does not appear to be a simple combinatorial function. One way to compute the number of regions of a hyperplane arrangement is via its characteristic polynomial \(\chi_{\mathcal{H}_n}(x)\), which is defined in terms of the lattice of intersections of the arrangement.
A result of Zaslavsky (Theorem 2.68 in [9]) states that $|\chi_{\mathcal{H}_n}(-1)|$ is the number of regions of the arrangement. For a particular class of hyperplane arrangements, known as free arrangements, the characteristic polynomial is known to have integer roots. In general, $\mathcal{H}_n$ is not a free arrangement, as can be seen from the following table.

| $n$ | $\chi_{\mathcal{H}_n}(x)$ |
|-----|-----------------------------|
| 1   | $x - 1$                     |
| 2   | $(x - 1)(x - 3)$            |
| 3   | $(x - 1)(x - 5)(x - 7)$     |
| 4   | $(x - 1)(x - 11)(x - 13)(x - 15)$ |
| 5   | $(x - 1)(x - 29)(x - 31)(x^2 - 60x + 971)$ |
| 6   | $(x - 1)(x^5 - 363x^4 + 54310x^3 - 4182690x^2 + 165591769x - 2691439347)$ |
| 7   | $(x - 1)(x^6 - 1092x^5 + 518385x^4 - 136815000x^3 + 21151739259x^2 - 1814252700708x + 67379577529235)$ |

From part 2 of Lemma 4.1 we know that the number of facets of a particular $n$-dimensional region corresponds to the number of coherent neighbors the corresponding term order has. The arrangements in dimension two and three are simplicial (all $n$-dimensional regions are simplices), but this is no longer the case in dimension four.

**Example 4.2.** The following term order has five flippable pairs. As has long been known (see [8]), and can be seen from enumerative results (see Section 6), all term orders are coherent in dimension four, so this term order corresponds to a cone with five facets. This is the maximal number of facets of a region of $\mathcal{H}_4$.

$$
\emptyset \prec \{1\} \prec \{2\} \prec \{3\} \prec \{1, 2\} \prec \{1, 3\} \prec \{2, 3\} \\
\prec \{4\} \prec \{1, 2, 3\} \prec \ldots
$$

The comparisons marked $\prec$ are flippable pairs.

**Remark 4.3.** Since every $n$-dimensional region of $\mathcal{H}_n$ has at least $n$ facets, coherent term orders have at least $n$ flippable pairs. The same is not true for noncoherent term orders.

**Theorem 4.4.** For $n \geq 6$ there are term orders with fewer than $n$ flippable pairs. These are examples of flip deficiency in the sense of [10].

**Proof.** The term order on subsets of $\{1, 2, 3, 4, 5, 6\}$ in Remark 3.10 of Section 3 has only five flippable pairs. Using this as a base, we can construct term orders on subsets of $[n]$ with $n - 1$ flippable pairs by setting $[k] \prec \{k + 1\}$ for $6 \leq k \leq n - 1$. \qed

We now develop the connection between $\mathcal{H}_n$ and the root system $B_n$.

**Definition 4.5.** The root system $B_n$ is the collection of vectors $\{e_i : 1 \leq i \leq n\} \cup \{e_i - e_j, e_i + e_j : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$, where $e_i$ is the $i$th standard basis vector.
Lemma 4.6. \( \mathcal{H}_n \) is the discriminantal arrangement of \( B_n \). In other words, \( \mathcal{H}_n \) is the collection of hyperplanes \( \{ H : H \text{ is spanned by a subset of } B_n \} \).

Proof. Let \( H \) be a hyperplane in \( \mathcal{H}_n \), with normal vector \( v \). Let \( P = \{ i : v_i > 0 \} \), \( N = \{ i : v_i < 0 \} \), and \( Z = \{ i : v_i = 0 \} \). Then \( H = \text{span}(\{e_i| i \in Z\} \cup \{e_i - e_j| i, j \in P\} \cup \{e_i - e_j| i, j \in N\} \cup \{e_i + e_j| i \in P, j \in N\}) \). For the other inclusion, let \( u \) be the normal vector of a hyperplane spanned by a subset \( V \) of \( B_n \). For all \( e_i - e_j \in V \) we have \( v_i = v_j \), and for all \( e_i + e_j \in V \) we have \( v_i = -v_j \). Also, whenever \( e_i \in V \), we have \( v_i = 0 \). Suppose there exists \( i, j \) with \( v_i, v_j \neq 0 \) such that \( v_i \neq \pm v_j \). Then let \( w \) be the vector with \( w_k = v_k \) when \( v_k = \pm v_k \), and \( w_k = 0 \) otherwise, and let \( u \) be the vector with \( u_k = v_k \) when \( u_k = \pm v_k \) and \( u_k = 0 \) otherwise. Then each vector in \( V \) is perpendicular to both \( u \) and \( w \), so lies in a codimension two subspace, and thus \( V \) does not span a hyperplane. \( \square \)

Let \( \mathcal{M}(B_n) \) denote the oriented matroid of the vector configuration \( B_n \). A wealth of information about oriented matroids can be found in [1], along with many equivalent definitions. For the purposes of the next result, we use the following chirtope definition.

Definition 4.7. Given a vector configuration \( V \) of \( d > n \) vectors in \( \mathbb{R}^n \), the oriented matroid \( \mathcal{M}(V) \) is the function mapping ordered subsets of \( V \), each consisting of \( n \) vectors, to \{+, 0, −\}, where each subset is mapped to the sign of its determinant. We identify the function with the sign vector of length \( n! \binom{d}{n} \) encoding the image of this map.

We say an oriented matroid \( \mathcal{M}(V) \) is projectively unique if whenever \( \mathcal{M}(W) = \mathcal{M}(V) \) for some vector configuration \( W \) in \( \mathbb{R}^d \) there is a projective transformation (linear transformation plus scaling individual vectors) taking \( W \) to \( V \).

Lemma 4.8. \( \mathcal{M}(B_n) \) is projectively unique for \( n \geq 3 \).

Proof. Suppose \( \mathcal{M}(V) = \mathcal{M}(B_n) \). We denote the vectors of \( V \) by \( v_e \), where \( e \) is the vector in \( B_n \) corresponding to \( v_e \).

We first apply a linear transformation of \( V \) to move \( v_e \) to \( e_i \) for \( 1 \leq i \leq n \). Note that for any vector \( e \in B_n \) the determinant of the \( n \) vectors \( \{v_e\} \cup \{e_i : i \neq j\} \) is \((-1)^{d-1}(v_e)j\), while the determinant of the \( n \) vectors \( \{e\} \cup \{e_i : i \neq j\} \) is \((-1)^{d-1}(e)j\). So \( v_e \) is zero in exactly the same coordinates as \( e \), and has the same sign in its nonzero coordinates.

We can now do further combinations of linear transformations and scaling of vectors to also move \( v_{e_i+e_{i+1}} \) to \( e_i + e_{i+1} \), for \( 1 \leq i \leq n-1 \). Lastly, we scale the other vectors \( v_{e_i \pm e_j} \) so they are of the form \( e_i \pm ae_j \) for some \( a \) depending on \( i, j \) and the sign. The lemma will follow if we can now show that in each case \( a = 1 \).

We first show this for \( v_{e_i+e_j} \) when \( j - i \) is odd, and \( v_{e_i-e_j} \) when \( j - i \) is even. Consider \( v_{e_i+e_j} = e_i + ae_j \), with \( j - i \) odd. The determinant of \( \{e_i : l < i\} \cup \{e_l + e_{l+1} : i \leq l < j - 1\} \cup \{v_{e_i+e_j}\} \cup \{e_l : l > j\} \), which should be zero, is \( a-1 \), so we see that \( v_{e_i+e_j} = e_i + e_j \). Considering the analogous determinant for \( v_{e_i-e_j} \), where \( j - i \) is even we conclude that \( v_{e_i-e_j} = e_i - e_j \) for these \( i \) and \( j \).

Now consider the determinant of \( \{v_{e_i-e_2}, v_{e_2+e_3}, v_{e_1+e_3}\} \cup \{e_i : 4 \leq i \leq n\} \), which should be zero. If we write \( v_{e_1-e_2} = e_1 - ae_2 \), and \( v_{e_1+e_3} = e_1 + be_3 \), then this determinant is \( b-a \), so \( b = a \). Notice that we get the same result if we switch replace \( v_{e_1-e_2} \) and \( v_{e_2+e_3} \) by \( v_{e_1+e_3} \) and \( v_{e_2-e_3} \). Considering the analogous subdeterminants with similar adjacent subdeterminants, we can conclude that there exists a single
a such that \( v_{e_i-e_{i+1}} = e_i - ae_{i+1} \) for all \( 1 \leq i \leq n - 1 \). The determinant of \( \{v_{e_1-e_3}, v_{e_1-e_2}, v_{e_2-e_3}\} \cup \{e_i : 4 \leq i \leq n\} \), which again should be zero, is now \( a^2 - 1 \), so we see that \( a = \pm 1 \). The case \( a = -1 \) is ruled out by consideration of the determinant \( \{v_{e_1+e_2}, v_{e_1-e_2}\} \cup \{e_i : 3 \leq i \leq n\} \), which is \(-a - 1\), and should be negative, so \( a = 1 \), and thus \( v_{e_i-e_{i+1}} = e_i - e_{i+1} \) for \( 1 \leq i \leq n - 1 \).

From consideration, when \( j - i \) is even, of the determinant of \( \{e_i : l < i\} \cup \{v_{e_i-e_{i+1}} \cup \{v_{e_i+e_{i+1}} : i+1 \leq l \leq j - 1\} \cup \{v_{e_i+e_j}\} \cup \{e_l : l > j\} \), which should be zero, but is \(-a - 1\), we see that \( v_{e_i+e_j} = e_i + e_j \). Finally, the analogous determinant for \( v_{e_i-e_j} \), when \( j - i \) is odd, yields \( v_{e_i-e_j} = e_i - e_j \).

Thus each vector \( v_e \in B_n \) has been moved to the corresponding vector \( e \), so there is a projective transformation moving \( V \) to \( B_n \), and thus we conclude that \( M(B_n) \) is projectively unique.

\[\square\]

**Corollary 4.9.** There is a bijection between realizable one-element extensions of \( M(B_n) \) and coherent generalized partial term orders.

**Proof.** The implication of Lemma 4.8 is that realizable one-element extensions of \( M(B_n) \) correspond exactly to regions of the discriminantal arrangement of \( B_n \) for \( n \geq 3 \). This is also true by inspection for \( n = 2 \). Thus from Lemma 4.6 we get a bijection between realizable one-element extensions of \( M(B_n) \) and regions of \( H_n \), and so Lemma 4.1 gives a bijection between realizable one-element extensions of \( M(B_n) \) and coherent generalized partial term orders.

\[\square\]

### 5. Oriented Matroids

In the previous section we saw that coherent generalized partial term orders correspond to realizable one-element extension of \( M(B_n) \). In this section we expand on this, showing that all generalized partial term orders correspond to one-element extensions of \( M(B_n) \). We assume more familiarity with oriented matroids.

The content of Lemma 4.6 was that the cocircuits of \( M(B_n) \) are in bijection with the hyperplane normals in the set \( \{+1, 0, -1\}^n \setminus 0^n \). We will represent these normals by their sign vectors, which will be denoted by capital letters, such as \( X \). The corresponding cocircuit will be denoted by \( \overline{X} \). The passage from \( X \) to \( \overline{X} \) is as follows: \( \overline{X}_{e_i} = X_i \), \( \overline{X}_{e_i+e_j} = X_i + X_j \), and \( \overline{X}_{e_i-e_j} = X_i - X_j \), where + and - applied to \( \{+, 0, -\} \) evaluate to the sign of the corresponding operation on \( \{+1, 0, -1\} \).

For example, if \( n = 3 \), and \( X = (+ + 0) \), then \( \overline{X} = (+ + 0 + + + 0 + +) \) where the coordinates of \( \overline{X} \) are listed in the following order: the \( e_i \), then the \( e_i + e_j \) in lexicographic order, and finally the \( e_i - e_j \), also in lexicographic order.

**Definition 5.1.** \( X^+ \) is the set \( \{i : X_i = +\} \). \( X^- \) is the set \( \{i : X_i = -\} \). Given two sign vectors \( X \) and \( Y \) we say that a sign vector \( Z \) is an *elimination candidate* for \( X \) and \( Y \) if \( Z^+ \subseteq \overline{X}^+ \cup \overline{Y}^+ \) and \( Z^- \subseteq \overline{X}^- \cup \overline{Y}^- \)
We first describe some $Z$ which are elimination candidates for given $X$ and $Y$. We can decompose $X^\pm, Y^\pm$ by writing

\[

to b, Z
\]

We are now ready for the main theorem. A generalized partial term order

In addition, if $m = p = \emptyset$, then $(b \cup d, a \cup c)$ is also an elimination candidate.

Lemma 5.2. With decomposition as above, the following pairs $(Z^+, Z^-)$ are elimination candidates for $X$ and $Y$:

\[
(p, m)
\]

This coding is reversible, as for all these pairs $(Z^+, Z^-)$, and also for $(Z^+, Z^-) \in \{(X^+, X^-), (Y^+, Y^-)\}$, we have $(b \cup d \cup p) \cap Z^- = \emptyset$, and $A \subseteq Z^+ \Rightarrow B \subseteq Z^-$, for $(A, B) \in \{(b, a), (c, d), (p, m), (x, y), (y, x)\}$. In this example, when we pass to $(p, m)$, this can be encoded as $(0, 0, +, 0, +, 0, +, 0, +, 0, +, 0, -)$, with the coordinates ordered first $b, d, p, x$, then sums in lexicographic order, and finally differences in lexicographic order. The elimination candidate condition is then equivalent to requiring that, under this encoding, whenever \( \overline{S}_e = + \), either \( \overline{X}_e = + \) or \( \overline{Y}_e = + \), and similarly whenever \( \overline{S}_e = - \) either \( \overline{X}_e = - \) or \( \overline{Y}_e = - \).

This is easy to check. We have the following encoding of $X$ and $Y$:

\[
\overline{X} = (+, 0, +, +, +, 0, +, -) = (0, +, +, +, +, +, +, +, +, +, -)
\]

From this we can see that $(p, m)$ is an elimination candidate. The rest of the proof is listing the remaining four vectors.

In the case where $m = p = \emptyset$, we have the reduced encoding of $X$ and $Y$ as $(+, 0, -)$ and $(0, +, +)$, leaving out the $p$th coordinate. The condition to be an elimination candidate remains the same, so from the encoding of $(b \cup d, a \cup c)$ as $(+, +, 0, +, +, +, 0, +, +)$ we can see that it is an elimination candidate. \( \square \) \( \square \)

We are now ready for the main theorem. A generalized partial term order $\prec$ induces a function $\mu$ from $\{+, 0, -\}^n$ to $\{+, 0, -\}$ by

\[
\mu(X) = \begin{cases} 
+ & \text{if } X^- \prec X^+ \\
- & \text{if } X^+ \prec X^- \\
0 & \text{otherwise}
\end{cases}
\]

Let $\sigma$ be the induced function on the cocircuits of $\mathcal{M}(B_n)$ given by $\sigma(\overline{X}) = \mu(X)$. 

BOOLEAN TERM ORDERS AND THE ROOT SYSTEM $B_n$
Theorem 5.3. The map \( \sigma \) is a localization, and so the set of generalized partial term orders is in bijection with a subset of the set of one-element extensions of \( \mathcal{M}(B_n) \).

Proof. Since different generalized partial term orders give different functions \( \mu \), it suffices to prove that \( \sigma \) is a localization. By Corollary 7.1.9 of [1], this is equivalent to showing that the set \( \sigma^{-1}(\{+,0\}) \) satisfies the weak cocircuit elimination axiom. This says that for any \( X, Y \) with \( X \neq \overline{Y} \) and \( \mu(X), \mu(Y) \in \{+,0\} \) such that \( X_e = + \) and \( Y_e = - \) for some \( e \in B_n \), there is some \( Z \) with \( Z_e = 0 \) and \( \mu(Z) \in \{+,0\} \) which is an elimination candidate for \( X \) and \( Y \).

We first show that if \( \mu(X), \mu(Y) \in \{+,0\} \) then there is an elimination candidate, \( Z \), for \( X \) and \( Y \) with \( \mu(Z) \in \{+,0\} \). We decompose \( X, Y, Z \) as above, and write the candidate \( Z \) as \( (Z^+, Z^-) \). Parentheses will be omitted at times to simplify notation. Note that the condition that \( \mu(X), \mu(Y) \in \{+,0\} \) means \( a \cup m \cup x \leq b \cup p \cup y \) and \( c \cup m \cup y \leq d \cup p \cup x \), and also that \( X \neq \overline{Y} \). The argument divides into two cases. We make repeated use of Lemma 2.2.

Case I: \( m \preceq p \). Then \( \mu(p,m) = \{+,0\} \).

Case II: \( p < m \) or \( m = p = \emptyset \). Then \( a \cup x \leq b \cup y \), and \( c \cup y \leq d \cup x \). There are three further cases.

1. Case (a): \( x < y \). Then \( c \cup m \preceq d \cup p \), so \( \mu(d \cup p, c \cup m) = + \). Also \( c \preceq d \), so \( \mu(b \cup d \cup p \cup y, a \cup c \cup m \cup x) = + \). Note that these are both nonempty pairs in the case \( m = p = \emptyset \).

2. Case (b): \( y < x \). Then \( \mu(b \cup p, a \cup m) = + \) and \( a \preceq b \), so \( \mu(b \cup d \cup p \cup x, a \cup c \cup m \cup y) = + \). Again, these are both nonempty pairs.

3. Case (c): \( x \sim y \) or \( x = y = \emptyset \). We break into two further cases. [1]
   - Case (i): \( p < m \). Then \( a \cup m \preceq b \cup p \), so \( \mu(b \cup p, a \cup m) \in \{+,0\} \), and similarly \( \mu(d \cup p, c \cup m) \in \{+,0\} \). Also \( c \preceq d \), so \( \mu(b \cup d \cup p \cup y, a \cup c \cup m \cup x) = + \).
   - Case (ii): \( m = p = \emptyset \). Then \( \mu(b \cup d, a \cup c), \mu(b \cup d \cup y, a \cup c \cup x), \mu(b \cup d \cup x, a \cup c \cup y) \in \{+,0\} \). Since \( X \neq \overline{Y} \), these are all nonempty pairs.

So we have the diagram of cases shown in Figure 1. Note that in each branch the pair is nonempty, so it does represent a cocircuit. By Lemma 5.2 the pair in each branch is an elimination candidate for \( X \) and \( Y \). So we have shown that if \( \mu(X), \mu(Y) \in \{+,0\} \) then there is an elimination candidate, \( Z \), for \( X \) and \( Y \) with \( \mu(Z) \in \{+,0\} \). We are now ready to show that \( \sigma^{-1}(\{+,0\}) \) satisfies the weak cocircuit elimination condition. Recall that this involves showing for all \( X, Y, Z \in \mu^{-1}(\{+,0\}) \) with \( X \neq \overline{Y} \), and \( e \in B_n \) such that \( X_e = + \) and \( Y_e = - \), there exists a \( Z \in \mu^{-1}(\{+,0\}) \) which is an elimination candidate for \( X \) and \( Y \) with \( Z_e = 0 \). To do this, it suffices to show that if \( X_e = + \) and \( Y_e = - \) then on every branch of Figure 1 there is a \( Z \) with \( Z_e = 0 \). This is again a consideration of cases, depending on the form of \( e \). The following tables enumerate the cases, with the rightmost column indicating which \( Z \) is chosen for each branch of the above diagram. We read the nodes of the diagram in the order the corresponding cases appear above: that is, top to bottom then left to right. The numbers are those in the diagram, so 11111 represents the choice of \( (p,m), (d \cup p, c \cup m) \), \( (b \cup p, a \cup m) \), \( (b \cup p, a \cup m) \), and \( (b \cup d, a \cup c) \) in the corresponding branches.

1. \( e = e_i \). Then \( i \in x \cup y \) and we require \( Z_i = 0 \), so we take 11111.
2. \( e = e_i + e_j \). Then we require \( Z_i = -Z_j \)
$m \preceq p$

(1) $(p, m)$

$p \prec m$ or

$m = p = \emptyset$

$x \prec y$

(1) $(d \cup p, c \cup m)$

(2) $(b \cup d \cup p \cup y, a \cup c \cup m \cup x)$

$p \bowtie m$

(1) $(b \cup p, a \cup m)$

(2) $(d \cup p, c \cup m)$

(3) $(b \cup d \cup p \cup y, a \cup c \cup m \cup x)$

$m \preceq p = \emptyset$

(1) $(b \cup d, a \cup c)$

(2) $(b \cup d \cup y, a \cup c \cup x)$

(3) $(b \cup d \cup x, a \cup c \cup y)$

$y \sim x$ or

$x = y = \emptyset$

Figure 1. $\mu$-nonnegative vectors

| $X_i$ | $X_j$ | $Y_i$ | $Y_j$ | $i \in j \in Z$ |
|-------|-------|-------|-------|-----------------|
| +     | +     | -     | -     | $y \quad y$ 11111 |
| +     | +     | -     | 0     | $y \quad b$ 11223 |
| +     | +     | 0     | -     | $b \quad y$ 11223 |
| +     | 0     | -     | -     | $y \quad c$ 12112 |
| +     | 0     | -     | 0     | $y \quad 0$ 11111 |
| +     | 0     | 0     | -     | $b \quad c$ 12231 |
| 0     | +     | -     | -     | $c \quad y$ 12112 |
| 0     | +     | -     | 0     | $c \quad b$ 12231 |
| 0     | +     | 0     | -     | $0 \quad y$ 11111 |

3. $e = e_i - e_j$. Then we require $Z_i = Z_j$.

| $X_i$ | $X_j$ | $Y_i$ | $Y_j$ | $i \in j \in Z$ |
|-------|-------|-------|-------|-----------------|
| +     | -     | -     | +     | $y \quad x$ 11111 |
| +     | -     | -     | 0     | $y \quad a$ 11223 |
| +     | -     | 0     | +     | $b \quad x$ 11223 |
| +     | 0     | -     | +     | $y \quad d$ 12112 |
| +     | 0     | -     | 0     | $y \quad 0$ 11111 |
| +     | 0     | 0     | +     | $b \quad d$ 12231 |
| 0     | -     | -     | +     | $c \quad x$ 12112 |
| 0     | -     | -     | 0     | $c \quad a$ 12231 |
| 0     | -     | 0     | +     | $0 \quad x$ 11111 |

This shows that $\sigma^{-1}(\{+, 0\})$ satisfies the weak cocircuit elimination axiom, so $\sigma$ is a localization. □
Corollary 5.4. A noncoherent generalized partial term order determines a nonrealizable one-element extension of $\mathcal{M}(B_n)$.

Proof. This follows directly from Theorem 5.3 and Corollary 4.9. □

We will show in Remark 5.6 that there are one-element extensions of $\mathcal{M}(B_n)$ which are not induced by a generalized partial term order in this way. To this end we characterize those one-element extensions which are in the image of the bijection.

Proposition 5.5. Let $\mu$ be a function from $\{+, -, 0\}^n \setminus \{0^n\}$ to $\{+, 0, -\}$. It is induced by a generalized partial term order if and only if it satisfies the following criteria:

1. $\mu(-x) = -\mu(x)$
2. (First Addition Condition) If $\mu(x) = \mu(y) = +$, and $x_i \neq y_i$ whenever $x_i \neq 0$ for $1 \leq i \leq n$, then $\mu(z) = +$, where
   $$ z_i = \begin{cases} x_i & y_i = 0 \\ y_i & x_i = 0 \\ 0 & x_i = -y_i \neq 0 \end{cases} $$
3. (Second Addition Condition) If $\mu(x) = 0$, $\mu(y) = +$, and $x_i \neq y_i$ whenever $x_i \neq 0$ for $1 \leq i \leq n$, then $\mu(z) = +$, where $z$ is as above.

Proof. The necessity of the first condition is immediate from the way $\mu$ is induced by a term order. Lemma 2.3 implies the second and third conditions.

For sufficiency, let $\mu$ be a function satisfying the above conditions. Given two subsets of $[n]$, define their order to be the one induced on the pair of subsets obtained by removing their intersection from each subset. Suppose that this does not define a partial order on the set of all subsets of $[n]$. Then there is some string of inequalities $b_0 \prec b_1 \prec b_2 \prec \ldots \prec b_n \prec b_0$. We can reduce this to a string of three distinct monomials. If $b_i \prec b_{i+2}$, we can remove $b_{i+1}$ from the chain and repeat this procedure with the shorter chain. Otherwise we have the three-element chain $b_i \prec b_{i+1} \prec b_{i+2} \preceq b_i$. Denote this three-element chain $\alpha \prec \beta \preceq \gamma \preceq \alpha$. We may assume that $\alpha \cap \beta \cap \gamma = \emptyset$. Write

$$ \alpha = \alpha' \cup x \cup y $$
$$ \beta = \beta' \cup x \cup z $$
$$ \gamma = \gamma' \cup y \cup z $$

where $\alpha'$, $\beta'$, $\gamma'$, $x$, and $y$ are pairwise disjoint. Then we have $\alpha' \cup y \prec \beta' \cup z$, $\beta' \cup x \prec \gamma' \cup y$, and $\gamma' \cup z \preceq \alpha' \cup x$. But the condition demands $\alpha' \cup x \prec \gamma' \cup z$ so $\mu$ determines a partial order on subsets of $[n]$.

Suppose the partial order does not satisfy the condition on incomparable elements, so there is $\alpha \sim \beta$, $\alpha \prec \gamma$, but $\gamma \preceq \beta$. Decompose $\alpha, \beta,$ and $\gamma$ as above. Then $\alpha' \cup y \sim \beta' \cup z$, and $\alpha' \cup x \prec \gamma' \cup z$ but $\gamma' \cup y \preceq \beta' \cup x$. Then $\mu(\alpha' \cup y, \beta' \cup z) = 0$, $\mu(\gamma' \cup z, \alpha' \cup x) = +$, and $\mu(\gamma' \cup y, \beta' \cup x) \in \{0, -\}$. However, the second addition condition demands $\mu(\gamma' \cup y, \beta' \cup x) = +$, so the order satisfies the condition on incomparable elements.

Since the partial order satisfies the first condition of Definition 2.3 (the multiplicative condition) by construction, it is a generalized partial term order, and so $\mu$ is induced by a generalized partial term order. □
Remark 5.6. The following list of hyperplanes in $\mathbb{R}^3$ satisfies the weak cocircuit elimination axiom when considered as cocircuits of $B_3$:

$$(- + 0), (0 - +), (+0-), (00), (0 + 0), (0 + +), (+0+), (+ + 0), (+ + +), (+ + -), (+ - +), (- + +)$$

Thus when we set $\sigma(\Xi) = +$ for each hyperplane $\Xi$, this determines a one-element extension of $\mathcal{M}(B_3)$. There is no term order, however, which induces this $\sigma$, as the set does not satisfy the first addition condition. Explicitly, if these were the positive cocircuits from some term order, then from the first three we have $x \preceq y$, $y \preceq z$, and $z \preceq x$, which is a contradiction. So not all one-element extensions of $\mathcal{M}(B_n)$ are induced by generalized partial term orders.

6. Numerical Results and Examples

In [4] Fine and Gill give crude bounds on the number of antisymmetric comparative probability relations, and the number of antisymmetric additive comparative probability relations, and the first terms of each sequence.

With improved computer speeds it is now possible to evaluate a few more terms in each sequence. The values calculated are displayed in the following table. The numbers in the second and third column are divided by $n!$, taking into account the action of the symmetric group.

| $n$ | Number of term orders$/n!$ | Number of coherent term orders$/n!$ |
|-----|---------------------------|-----------------------------------|
| 1   | 1                         | 1                                 |
| 2   | 1                         | 1                                 |
| 3   | 2                         | 2                                 |
| 4   | 14                        | 14                                |
| 5   | 546                       | 516                               |
| 6   | 169444                    | 124187                            |
| 7   | 560043206                 | 214580603                         |

These were calculated with the aid of programs written by Michael Kleber and Josh Levenberg. To calculate the total number of term orders we used a recursive procedure, calculating for each term order on subsets of $[n-1]$ the number of ways the subsets involving $n$ could be shuffled in. To calculate the number of coherent term orders we enumerated the regions of the corresponding hyperplane arrangement $\mathcal{H}_n$. These numbers have since been checked directly for $n \leq 6$.

The following table enumerates the number of flippable pairs each term order has for orders on subsets of $\{1, 2, 3, 4, 5, 6\}$.
the linear program obtained by seeking a weight vector with all 
≺
If it were less than a coherent partial term order other than ˆ
⪯
is not adjacent to any coherent term order:
(2 
Definition 6.1. The Baues poset for term orders has as its elements all generalized partial term orders. The order relation is that a term order ≺₁ is less than another term order ≺₂ if ≺₁ is a refinement of ≺₂. This poset has a 1, the generalized partial term order with all subsets unrelated.

Proposition 6.2. There exists a noncoherent boolean term order which lies below no coherent partial term order in the Baues poset except 1.

Proof. The following order is such an example:

\[
\begin{align*}
\emptyset & \prec \{1\} \prec \{2\} \prec \{1, 2\} \prec \{3\} \prec \{1, 3\} \prec \{2, 3\} \prec \{1, 2, 3\} \prec \\
\{4\} & \prec \{1, 4\} \prec \{2, 4\} \prec \{1, 2, 4\} \prec \{3, 4\} \prec \{5\} \prec \{1, 3, 4\} \prec \\
\{2, 3, 4\} & \prec \{1, 5\} \prec \{2, 5\} \prec \{1, 2, 3, 4\} \prec \{1, 2, 5\} \prec \{3, 5\} \prec \\
\{1, 3, 5\} & \prec \{2, 3, 5\} \prec \{6\} \prec \{1, 2, 3, 5\} \prec \{1, 6\} \prec \{4, 5\} \prec \\
\{1, 4, 5\} & \prec \{2, 6\} \prec \{1, 2, 6\} \prec \{3, 6\} \prec \{1, 3, 6\} \ldots
\end{align*}
\]

If it were less than a coherent partial term order other than 1 in the Baues poset, the linear program obtained by seeking a weight vector with all \(\prec\) above relaxed to \(\leq\) would have a nonzero solution. This is not the case. \(\square\)

Note that the example in the proof above is adjacent to a coherent order, however. Flipping over \(\{5\} \prec \{1, 3, 4\}\) yields the coherent order given by the vector \((2, 9, 12, 28, 48, 70)\).

The following term order in six variables, with flippable pairs marked with \(\prec_*\), is not adjacent to any coherent term order:

\[
\begin{align*}
\emptyset & \prec_* \{1\} \prec \{2\} \prec \{1, 2\} \prec \{3\} \prec \{1, 3\} \prec \{4\} \prec \{1, 4\} \prec \{5\} \prec \\
\{1, 5\} & \prec \{2, 3\} \prec \{1, 2, 3\} \prec \{6\} \prec \{1, 6\} \prec \{2, 4\} \prec \{1, 2, 4\} \prec \\
\{3, 4\} & \prec \{1, 3, 4\} \prec \{2, 5\} \prec \{1, 2, 5\} \prec \{2, 6\} \prec \{1, 2, 6\} \prec_* \\
\{3, 5\} & \prec \{1, 3, 5\} \prec \{3, 6\} \prec \{1, 3, 6\} \prec_* \{4, 5\} \prec \{1, 4, 5\} \prec \\
\{2, 3, 4\} & \prec \{1, 2, 3, 4\} \prec \{2, 3, 5\} \prec \{1, 2, 3, 5\} \prec_* \{4, 6\} \ldots
\end{align*}
\]

This is the case because \(\{5\} \prec \{2, 3\}, \{3, 4\} \prec \{2, 5\}, \{2, 6\} \prec \{3, 5\}\), and \(\{2, 3, 5\} \prec \{4, 6\}\), which, in monomial notation, gives \(x_2^2x_3^2x_4x_5^2x_6 < x_2^2x_3^2x_4x_5^2x_6\).
implying that $\prec$ is noncoherent. Flipping across any of the flippable pairs does not change any of these four inequalities, so none of the neighbors of $\prec$ are coherent.

This term order was constructed by taking a “lexicographic product” of the noncoherent term order of Example 2.3 with $\emptyset \prec \{1\}$. Using this product construction we can construct boolean term orders in $n$ variables which are at least $2^{n-5}$ flips from a coherent boolean term order.

7. Questions

The following questions are natural in the context of a Baues problem.

- Is the space of term orders connected by flips? This has been experimentally verified for $n \leq 6$.
- What is the homotopy type of the poset of (generalized) partial term orders? In particular, is it spherical? The subposet of coherent generalized partial term orders is easily seen to be spherical.
- What is the limit of the ratio of the number of coherent term orders to the total number of term orders as $n$ increases? Is it zero?
- What is an upper bound for the number of coherent neighbors for a coherent term order? In other words, how many facets do regions of the hyperplane arrangement have?
- Can we give a lower bound for the total number of flippable pairs in all term orders? A good upper bound?

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E-mail address: maclagan@math.berkeley.edu

Diane Maclagan, Department of Mathematics, University of California, Berkeley, CA 94720