SHARP $k$-ORDER SOBOLEV INEQUALITIES
IN THE HYPERBOLIC SPACE $\mathbb{H}^n$

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Dedicated to Professor Louis Nirenberg with admiration

Abstract. In this paper, the sharp $k$-order Sobolev inequalities in the hyperbolic space $\mathbb{H}^n$ are established for all $k = 1, 2, \cdots$. This solves a long-standing open problem posed by Aubin [A4, p.176-177] for $W^{k,2}(\mathbb{H}^n)$.

§1. Introduction

It is well-known that sharp Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. A striking example where they have played an essential role in the Riemannian context is given by the solution of the famous Yamabe problem (see [A1] and [S]).

The classical Sobolev inequality (also called Sobolev imbedding theorem, see [So]) says that given an integer $k \in \mathbb{N}$, if $2k < n$, then there is a constant $C(n,k)$, depending only upon $n$ and $k$, such that for all $u \in W^{k,2}(\mathbb{R}^n)$,

$$\|u\|_{L^{2n/(n-2k)}(\mathbb{R}^n)} \leq C(n,k)\|u\|_{W^{k,2}(\mathbb{R}^n)},$$

or equivalently, $W^{k,2}(\mathbb{R}^n)$ is continuously embedded in $L^{2n/(n-2k)}(\mathbb{R}^n)$.

For the case $k = 1$, Talenti [T] and Aubin [A2, A3] (see also [R], [Ro]) proved that for any $u \in W^{1,2}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u(x)|^{2n/(n-2)}dx\right)^{(n-2)/n} \leq \Lambda_1 \int_{\mathbb{R}^n} |\nabla u|^2dx,$$
where the best Sobolev constant is

\[ \Lambda_1 = \frac{4}{n(n-2)\omega_n^{2/n}}, \]

and \( \omega_n \) is the surface area of the sphere \( S^n = \{ x \in \mathbb{R}^{n+1} | |x| = \sqrt{x_1^2 + \cdots + x_{n+1}^2} = 1 \} \). The equality sign holds in (1.2) if and only if \( u \) has the form:

\[ u(x) = a \left[ \frac{\tau^2 + |x - x_0|^2}{2\tau} \right]^{1 - \frac{n}{2}}, \]

where \( a \in \mathbb{R}^1, 0 < \tau \in \mathbb{R}^1 \) and \( x_0 \in \mathbb{R}^n \).

By solving the Yamabe problem, Aubin proved (see [A1] and [A2]) that the above sharp inequality is equivalent to the sharp Sobolev inequality on the sphere \( S^n \), written as

\[ \left( \int_{\mathbb{S}^n} |u(y)|^{2n/(n-2)}dy \right)^{(n-2)/n} \leq \Lambda_1 \int_{\mathbb{S}^n} |\nabla u|^2dy + \omega_n^{-2/n} \int_{\mathbb{S}^n} |u(y)|^2dy \]

for any \( u \in W^{1,2}(S^n) \). The equality sign holds if and only if \( u \) has the form:

\[ u(y) = \frac{(\beta^2 - 1)(n-2)/4}{(\beta - \cos r)^{(n-2)/2}}, \quad \beta > 1, \]

where \( r \) is the distance from \( y \) to \( y_0 \) on \( S^n \), and \( y_0 \) is a fixed point on \( S^n \). Note that the inequality (1.4) is sharp because the two best constants \( \Lambda_1 \) and \( \omega_n^{-2/n} \) cannot be lowered (see [H1] or [H3]).

There has been much work on such inequalities (for \( k = 1 \)) and their applications. See, for example, Trudinger [Tr], Moser [M], Aubin [A2, A3], Talenti [T], Bliss [Bl], Lieb [L1, L2], Brezis and Nirenberg [BN], Cherrier [C], Brezis and Lieb [BL], Carleson and Chang [CC], Escobar [E], Carlen and Loss [CL], Becker [B], Adimurthi and Yadava [AY], Hebey and Vaugon [HV1, HV2], Hebey [H1, H2], Li and Zhu [LZ1, LZ2], Zhu [Z], Aubin, Druet and Hebey [ADH], Aubin and Li [AL], and the references therein.

For order \( k = 2 \), the sharp Sobolev inequality in \( \mathbb{R}^n \) was studied by Edmunds, Fortunato and Jannelli [EFJ] (also see Lions [Li]) and was explicitly given in [DHL]. On the sphere \( S^n \), the second-order sharp Sobolev inequality has also been established by Djadli, Hebey and Ledoux [DHL].

For higher order \( k \), Cotsiolis and Tavoularies [CT] (also see [L]) obtained the sharp Sobolev inequality (via Lieb’s sharp Hardy-Littlewood-Sobolev inequality): for any \( u \in W^{k,2}(\mathbb{R}^n) \),

\[ \left( \int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2k}}dx \right)^{\frac{n-2k}{n}} \leq \Lambda_k \int_{\mathbb{R}^n} |\triangle^{k/2} u|^2dx, \]
where
\[ |\Delta^{k/2} u|^2 = \begin{cases} |\Delta^{k/2} u|^2 & \text{when } k \text{ is even}, \\ |\nabla (\Delta^{(k-1)/2} u)|^2 & \text{when } k \text{ is odd}, \end{cases} \]
and the best Sobolev constant is
\[ \Lambda_k = \frac{2^{2k}\omega_n^{-(2k)/n}}{n \left[ n - 2k \right] \left[ n^2 - (2(k-1))^2 \right] \left[ n^2 - (2(k-2))^2 \right] \cdots \left[ n^2 - 2^2 \right]} . \]

The equality sign holds in (1.5) if and only if \( u \) has the form:
\[ u(x) = a \left[ \frac{\tau^2 + |x - x_0|^2}{2\tau} \right]^{k-\frac{2}{n}} \]
where \( a \in \mathbb{R}^1, 0 < \tau \in \mathbb{R}^1 \) and \( x_0 \in \mathbb{R}^n \).

In [L], the author proved the following total optimal Sobolev inequalities on the sphere \( S^n \): Let \((S^n, g)\) be the standard unit sphere of \( \mathbb{R}^{n+1}, n > 2k \), and let \( q = (2n)/(n - 2k) \). Assume that
\[ Q_k := \Delta_g^k + \sum_{m=0}^{k-1} A_{km} \Delta_g^m, \]
is the \( k \)-th standard operator on \( S^n \) whose coefficients \( A_{km} \) is given by \( Q_k = Q_1(Q_1 - 2) \cdots (Q_1 - k(k-1)) \) with \( Q_1 = \Delta_g + \frac{n(n-2)}{2} \). Then, for any \( u \in W^{k,2}(S^n) \),
\[ \left( \int_{S^n} |u|^q dV_g \right)^{2/q} \leq \Lambda_k \left[ \int_{S^n} \left( |\Delta_g^{k/2} u|^2 + \sum_{m=0}^{k-1} A_{km} |\Delta_g^{m/2} u|^2 \right) dV_g \right], \]
where \( \Lambda_k \) is the best Sobolev constant in \( \mathbb{R}^n \). The equality sign holds if and only if \( u \) has the form:
\[ u_\beta(r) = \frac{(\beta^2 - 1)^{\frac{n-2k}{4}}}{(\beta - \cos r)^{\frac{n-2k}{4}}}, \quad \beta > 1, \]
and
\[ Q_k (\omega_n^{-1/q} u_\beta) = \frac{1}{\Lambda_k} (\omega_n^{-1/q} u_\beta)^{q-1} \text{ on } S^n, \]
where \( r \) is the distance from \( y \) to \( y_0 \) on \( S^n \), and \( y_0 \) is a fixed point on \( S^n \). The best constants \( \Lambda_k A_{mk} \) in (1.8) are total optimal.

In this paper, for any order \( k = 1, 2, \cdots \), we shall establish the sharp \( k \)-order Sobolev inequality in the hyperbolic space \( \mathbb{H}^n \) of constant sectional curvature \(-1\). Our best Sobolev constants are also total optimal. This solves a long-standing open problem posed by Aubin [A4, p.176-177] for \( W^{k,2}(\mathbb{H}^n) \). In addition, we give an explicit representation (by an important recursive formula) for the high-order Paneitz operators \( P_k \) in \( \mathbb{H}^n \) \( (P_k \) are the conformal analogues of the power Laplacians \( \Delta^k \) on the flat \( \mathbb{R}^n \).)
We point out that the method here is more difficult than and quite different from that of [L] because there does not exist the extremal function for the sharp Sobolev inequality in \( \mathbb{H}^n \). Fortunately, we can still get total optimal Sobolev constants in \( \mathbb{H}^n \).

§2. High-order Paneitz operators in the hyperbolic space \( \mathbb{H}^n \)

The hyperbolic \( n \)-space \( \mathbb{H}^n \) \( (n \geq 2) \) is a complete simple connected Riemannian manifold having constant sectional curvature equal to \(-1\), and for a given dimensional number, any two such spaces are isometric ([W]). There are several models for \( \mathbb{H}^n \), the most important being the half-space model, the ball model, and the hyperboloid or Lorentz model, with the ball model being especially useful for questions involving rotational symmetry. We will only use the ball model in this paper.

Let \( \mathbb{B}_n = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n | (x_1^2 + \cdots + x_n^2)^{1/2} < 1 \} \) be the unit ball in the Euclidean space \( \mathbb{R}^n \). For \( \mathbb{B}_n \), if we endow with the Riemann metric

\[
        ds^2 := \frac{4|dx|^2}{(1 - |x|^2)^2},
\]

then the sectional curvature becomes the constant \(-1\). Furthermore, if we now define spherical coordinates about \( x = 0 \) by

\[
        x = t \zeta, \quad t = \tanh(r/2),
\]

where \( t \in [0, 1), r \in [0, \infty), \zeta \in \mathbb{S}^{n-1} \), then we obtain the metric

\[
        ds^2 = (dr)^2 + (\sinh^2 r) |d\zeta|^2.
\]

Note that for each \( x \in \mathbb{B}_n \), \( t \) and \( r \) are the Euclidean and the hyperbolic distances from 0 to \( x \), respectively. One easily sees that, in the ball model, the geodesics emanating from the origin are given by straight lines emanating from the origin, and their length to the boundary \( \mathbb{S}^{n-1} \) is infinite.

Let \( \Delta_h \) be the Laplacian on \( \mathbb{H}^n \) with the metric \( h \), and let \( F : \mathbb{H}^n \to \mathbb{R}^1 \in C^2 \), with

\[
        F(y(r, \zeta)) = f(r, \zeta).
\]

We have by direct calculation (see [Ch1, p.40]) that

\[
        (\Delta_h F)(y(r, \zeta)) = - (\sinh r)^{1-n} \frac{\partial}{\partial r} \left( (\sinh r)^{n-1} \frac{\partial f}{\partial r} \right) + (\sinh r)^{-2} \mathcal{L}_\zeta f,
\]

where, when writing \( \mathcal{L}_\zeta f \), we mean that \( f |_{\mathbb{S}(r)} \) is to be considered as a function on \( \mathbb{S}^{n-1} \) with associated Laplacian \( \mathcal{L} \).
If $f$ is a radial function on $(\mathbb{H}^n, h)$ (i.e., function that depends only on distance from 0 on $\mathbb{H}^n$), then the corresponding Laplacian takes the following simple form (see also [Ch2, p.180-181]):

$$\triangle_h f(r, \zeta) = -(\sinh r)^{1-n} \frac{\partial}{\partial r} \left[ (\sinh r)^{n-1} \frac{\partial f}{\partial r} \right].$$

We denote such a simple form as $\triangle_r$. Similarly, for any positive integer $m$ we can define the $m$th-iterated operator $\triangle^m_r$ on the set of radial functions as the following: For any radial function $f \in C^{2m}(\mathbb{H}^n)$,

$$\triangle^m_r f(r) = -(\sinh r)^{1-n} \frac{\partial}{\partial r} \left[ (\sin r)^{n-1} \frac{\partial(\triangle^{m-1}_r f)}{\partial r} \right], \quad m = 1, 2, \ldots .$$

Our first task is to determine the best Sobolev constants by the Euler-Lagrange equation (2.10) and a known radial function on $\mathbb{H}^n$.

From the introduction, by setting $\tau = \left( \frac{1-\beta}{1+\beta} \right)^{1/2}$ in (1.7) we know that for any integer $k$,

$$G_\beta(x) := \left[ \left( \frac{1-\beta}{1+\beta} \right)^{1/2} \left( \frac{1+\beta+|x|^2}{2} \right) \right]^{\frac{n-k}{2}} \quad (0 < \beta < 1)$$

are the extremal functions for the sharp $k$-order Sobolev inequality (1.5) in Euclidean space $\mathbb{R}^n$, and

$$\triangle^k \left( \omega_n^{-1/q} G_\beta \right) = \frac{1}{\Lambda_k} \left( \omega_n^{-1/q} G_\beta \right)^{q-1} \quad \text{in } \mathbb{R}^n,$$

where $q = \frac{2n}{n-2k}$. However, there is not extremal function for the sharp $k$-order Sobolev inequality in the unit ball $\mathbb{B}_n \subset \mathbb{R}^n$. In fact, the following Lemma holds

**Lemma 2.1.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n > 2k$, and let $q = \frac{2n}{n-2k}$. Assume that $\Xi_k$ is a constant defined by

$$\Xi_k = \inf_{u \in W^{k,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\triangle^{k/2} u|^2 \, dx}{\left( \int_{\Omega} |u|^{q} \, dx \right)^{2/q}}.$$

Then $\Xi_k = \Lambda_k$. Moreover, the infimum is not attained.

**Proof.** Standard rescaling arguments show that the infimum in the definition of $\Xi_k$ does not depend on $\Omega$. Since

$$\Lambda_k = \inf_{u \in W^{k,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\triangle^{k/2} u|^2 \, dx}{\left( \int_{\mathbb{R}^n} |u|^{q} \, dx \right)^{2/q}},$$

it immediately follows that $\frac{1}{\Lambda_k} \leq \frac{1}{\Xi_k}$. 
On the other hand, $\frac{1}{\Lambda_k} \geq \frac{1}{\Xi_k}$. For suppose $\frac{1}{\Lambda_k} < \frac{1}{\Xi_k}$. Then, there is a sequence $\{u_j\}$ in $C_0^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$\lim_{j \to \infty} \frac{\int_{\mathbb{R}^n} |\Delta^k/2 u_j|^2 dx}{\left( \int_{\mathbb{R}^n} |u_j|^q dx \right)^{2/q}} = \frac{1}{\Lambda_k}.$$  

Thus for large enough $j$, \( \frac{\int_{\mathbb{R}^n} |\Delta^k/2 u_j|^2 dx}{\left( \int_{\mathbb{R}^n} |u_j|^q dx \right)^{2/q}} < \frac{1}{\Xi_k} \), and this inequality contradicts the definition of $\Xi_k$ and its independence of $\Omega$. We conclude that $\Xi_k = \Lambda_k$.

Finally we show that the infimum in the definition of $\frac{1}{\Xi_k}$ is not attained. Suppose that there exists $u \in W_0^{k,2}(\Omega) \setminus \{0\}$ such that \( \frac{\int_{\Omega} |\Delta^k/2 u|^2 dx}{\left( \int_{\Omega} |u|^q dx \right)^{2/q}} = \frac{1}{\Xi_k} \). The extension of $u$ by zero outside $\Omega$ leads to a minimum of \( \frac{\int_{\mathbb{R}^n} |\Delta^k/2 u|^2 dx}{\left( \int_{\mathbb{R}^n} |u|^q dx \right)^{2/q}} \), but the minimizing function is not of the type (1.7). This is a contradiction, which proves the Lemma. $\Box$

It is easy to verify that

$$\lim_{\beta \to 1^-} \frac{\int_{\mathbb{B}_n} |\Delta^k/2 G_\beta|^2 dx}{\left( \int_{\mathbb{B}_n} |G_\beta|^2 dx \right)^{\frac{n-2k}{n}}} = \inf_{u \in W_0^{k,2}(\mathbb{B}_n) \setminus \{0\}} \frac{\int_{\mathbb{B}_n} |\Delta^k/2 u|^2 dx}{\left( \int_{\mathbb{B}_n} |u|^q dx \right)^{\frac{n-2k}{n}}} = \frac{1}{\Lambda_k}. \quad (2.5)$$

Let $\sigma$ be the conformal map from the Euclidean ball $\mathbb{B}_n$ to the hyperbolic space $\mathbb{H}^n$ defined by (2.1). Then the Jacobian of $\sigma$ is $J_\sigma = (\frac{2}{1-|r|^2})^n$. Suppose $f$ is a smooth function defined in the Euclidean ball $\mathbb{B}_n$. Lift it to the hyperbolic space $\mathbb{H}^n$ by formula:

$$(J_\sigma)^{1/q} F(y) = f(x), \quad x \in \mathbb{B}_n, \; y \in \mathbb{H}^n,$$

i.e.,

$$F(y) = \left( \frac{2}{1-|x|^2} \right)^{k-\frac{n}{2}} f(x), \quad x \in \mathbb{B}_n, \; y \in \mathbb{H}^n. \quad (2.6)$$

By this method, we can transform functions from the Euclidean ball $\mathbb{B}_n$ to $\mathbb{H}^n$.

It is not difficult to check that the above function $G_\beta(x)$ (defined in the Euclidean ball $\mathbb{B}_n$) is lifted to the function $(1 - (\beta^2)^{(n-2k)/4} (\cosh r - \beta)^{k-\frac{n}{2}}$ defined in $\mathbb{H}^n$. In fact, since $|x| = \tanh \frac{r}{2}$, we have

$$\frac{1 + |x|^2}{1 - |x|^2} = \frac{1 + \tanh^2 \frac{r}{2}}{1 - \tanh^2 \frac{r}{2}} = \frac{1}{\text{sech}^2 \frac{r}{2} + \tanh^2 \frac{r}{2}} = \cos^2 \frac{r}{2} + \sinh^2 \frac{r}{2} = 2 \cos^2 \frac{r}{2} - 1 = \cosh r.$$  

It follows that

$$\frac{1 + \frac{1+\beta}{1-\beta} |x|^2}{1 - |x|^2} = \frac{1}{1 - \beta} \left( \frac{1 + |x|^2}{1 - |x|^2} - \beta \right) = \frac{1}{1 - \beta} (\cosh r - \beta), \quad 0 < \beta < 1. \quad (2.7)$$
Thus
\[
\left( \frac{2}{1 - |x|^2} \right)^{k - \frac{n}{2}} G_\beta(x) = \left( \frac{2}{1 - |x|^2} \right)^{k - \frac{n}{2}} \left[ \left( \frac{1 - \beta}{1 + \beta} \right)^{1/2} \left( \frac{1 + \frac{1 + \beta}{1 - \beta} |x|^2}{2} \right) \right]^{k - \frac{n}{2}}
\]

\[
= \left[ \left( \frac{1 - \beta}{1 + \beta} \right)^{1/2} \frac{1 + \frac{1 + \beta}{1 - \beta} |x|^2}{1 - |x|^2} \right]^{k - \frac{n}{2}} \left(1 - \beta^2\right)^{-\frac{n}{2} \cdot \frac{1}{(1 - \beta)\left(1 + \frac{1 + \beta}{1 - \beta} |x|^2\right)^{k - \frac{n}{2}}} = (1 - \beta^2)^{(n-2k)/4}(cosh r - \beta)^{k-\frac{n}{2}}.
\]

(2.8)

Now, we calculate the coefficients of the following Euler-Lagrange equation (2.10) by the radial function \((1 - \beta^2)^{(n-2k)/4}(cosh r - \beta)^{k-\frac{n}{2}} \) \((0 < \beta < 1)\).

\textbf{Lemma 2.2.} Assume that \(k\) is a positive integer, and assume that \((\mathbb{H}^n, h)\) is the hyperbolic space, \(n > 2k\). Given \(\beta \in (0,1)\). Then by the radial function

\[
\psi_k(r) = (1 - \beta^2)^{(n-2k)/4}(cosh r - \beta)^{k-\frac{n}{2}}, \quad 0 \leq r < +\infty,
\]

and by the following differential equation

\[
\Delta_r^k \psi_k(r) + \sum_{m=0}^{k-1} a_{km} \Delta_r^m \psi_k(r) = b_k \psi_k(r)^{n+2k \over n-2k},
\]

we can uniquely determine the constants \(b_k\) and \(a_{k0}, a_{k1}, \ldots, a_{k,k-1}\). In particular,

\[
\begin{align*}
-a_{10} = b_1 &= \frac{n(n - 2)}{4}; \\
 a_{21} &= -\frac{n^2 - 2n - 4}{2}, \quad a_{20} = b_2 = \frac{n(n - 4)(n^2 - 2^2)}{2^4}; \\
 a_{32} &= -\frac{3n^2 - 6n - 32}{4}, \quad a_{31} = \frac{3n^4 - 12n^3 - 52n^2 + 128n + 192}{16}, \\
-a_{30} = b_3 &= \frac{n(n - 6)(n^2 - 4^2)(n^2 - 2^2)}{2^6}; \\
 a_{43} &= -(n^2 - 2n - 20), \quad a_{42} = \frac{3}{8} n^4 - \frac{3}{2} n^3 - \frac{27}{2} n^2 + 30n + 108, \\
 a_{41} &= -(\frac{n^6}{16} - \frac{3}{8} n^5 - 3n^4 + \frac{29}{2} n^3 + 39n^2 - 108n - 144), \\
 a_{40} = b_4 &= \frac{n(n - 8)(n^2 - 6^2)(n^2 - 4^2)(n^2 - 2^2)}{2^8}; \\
 a_{54} &= -\frac{5n^2 - 10n - 160}{4}, \ldots, \\
-a_{50} = b_5 &= \frac{n(n - 10)(n^2 - 8^2)(n^2 - 6^2)(n^2 - 4^2)(n^2 - 2^2)}{2^{10}},
\end{align*}
\]
and
\[(−1)^k a_{k0} = b_k \]
\[(2.11) \]
for \(k = 1, 2, \cdots, \)

**Proof.** We shall calculate the corresponding coefficients in (2.10) only for order \(k = 1, 2, 3.\) Other orders can similarly be done.

(i) For \(k = 1, \) since \(ψ_1(r) = (1 − β^2)^{(n−2)/4}(cosh r − β)^{1−\frac{n}{2}} \) we have

\[ \triangle r ψ_1 = −(sinh r)^{1−n} \frac{∂}{∂r} \left[ (sinh r)^{n−1} \frac{∂ψ_1}{∂r} \right] \]
\[=−(1 − β^2)^{(n−2)/4}(sinh r)^{1−n} \frac{∂}{∂r} \left[ (sinh r)^{n−1} \cdot (1 − \frac{n}{2})(cosh r − β)^{−\frac{n}{2}} sinh r \right] \]
\[=−(1 − β^2)^{(n−2)/4}(1 − \frac{n}{2})(sinh r)^{1−n} \frac{∂}{∂r} \left[ (cosh r − β)^{−\frac{n}{2}} sinh^n r \right] \]
\[=(1 − β^2)^{(n−2)/4}(1 − \frac{n}{2}) \left[ (\frac{n}{2})(cosh r − β)^{−\frac{n}{2}} − 1 sinh^2 r − n(cosh r − β)^{−\frac{n}{2}} cosh r \right]. \]

By setting
\[ \triangle r ψ_1 + a_{10} ψ_1 = b_1 ψ_1^{\frac{n+2}{n−2}}, \]
we get
\[ (1 − \frac{n}{2}) \left[ (\frac{n}{2})(cosh r − β)^{−\frac{n}{2}} − 1 sinh^2 r − n(cosh r − β)^{−\frac{n}{2}} cosh r \right] \]
\[+ a_{10}(cosh r − β)^{1−\frac{n}{2}} = b_1 (1 − β^2)(cosh r − β)^{−\frac{n}{2}} − 1, \]
i.e.,
\[ (1 − \frac{n}{2}) \left[ (\frac{n}{2}) sinh^2 r − n(cosh r − β) cosh r \right] \]
\[+ a_{10}(cosh r − β)^2 = b_1 (1 − β^2). \]

Thus
\[ \left[ − \frac{n}{2}(1 − \frac{n}{2}) + a_{10} \right] cosh^2 r + \left[ (1 − \frac{n}{2}) β − 2a_{10}β \right] cosh r \]
\[+ \left[ − (1 − \frac{n}{2}) \left( \frac{n}{2} \right) + a_{10}β^2 − b_1 (1 − β^2) \right] = 0. \]

This leads to the following system of equations with unknown constants \(a_{10}\) and \(b_1:\)

\[
\begin{aligned}
-\frac{n}{2}(1 - \frac{n}{2}) + a_{10} &= 0, \\
β \left[ n(1 - \frac{n}{2}) - 2a_{10} \right] &= 0, \\
-\frac{n}{2}(1 - \frac{n}{2}) + a_{10}β^2 - b_1 (1 - β^2) &= 0
\end{aligned}
\]
which has the unique solution
\[ a_{10} = -\frac{n(n - 2)}{4}, \quad b_1 = \frac{n(n - 2)}{4}. \]

(ii) For \( k = 2 \), by \( \psi_2(r) = (1 - \beta^2)^{(n-4)/4}(\cosh r - \beta)^{2-\frac{n}{2}} \) we have
\[
\triangle_r \psi_2 = - (\sinh r)^{1-n} \frac{\partial}{\partial r} \left[ (\sinh r)^{n-1} \frac{\partial \psi_2}{\partial r} \right] \\
= -(1 - \beta^2)^{\frac{n-4}{4}} (2 - \frac{n}{2}) (\sinh r)^{1-n} \frac{\partial}{\partial r} \left[ (\sinh r)^{n-1} (2 - \frac{n}{2})(\cosh r - \beta)^{1-\frac{n}{2}} \sinh r \right] \\
= -(1 - \beta^2)^{\frac{n-4}{4}} (2 - \frac{n}{2}) \left[ (1 - \frac{n}{2})(\cosh r - \beta)^{-\frac{n}{2}} \sinh^2 r + n(\cosh r - \beta)^{1-\frac{n}{2}} \cosh r \right].
\]
Furthermore,
\[
\Delta_r^2 \psi_2 = -(\sinh r)^{1-n} \frac{\partial}{\partial r} \left[ (\sinh r)^{n-1} \frac{\partial (\triangle_r \psi_2)}{\partial r} \right] \\
= (1 - \beta^2)^{\frac{n-4}{4}} (2 - \frac{n}{2}) (\sinh r)^{1-n} \frac{\partial}{\partial r} \left[ (1 - \frac{n}{2})(-\frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}-1} (\sinh r)^{n+2} \right. \\
+ (n + 2) (1 - \frac{n}{2})(\cosh r - \beta)^{-\frac{n}{2}} (\sinh r)^n \cosh r + n(\cosh r - \beta)^{1-\frac{n}{2}} (\sinh r)^n \right] \\
= (1 - \beta^2)^{\frac{n-4}{4}} (2 - \frac{n}{2}) \left[ (1 - \frac{n}{2})(-\frac{n}{2}) (-1)(\cosh r - \beta)^{-\frac{n}{2}-2} \sinh^4 r \right. \\
+ 2(n + 2)(1 - \frac{n}{2})(-\frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}-1} (\sinh r)^2 \cosh r \\
+ n(n + 2)(1 - \frac{n}{2})(\cosh r - \beta)^{-\frac{n}{2}} \cosh^2 r \\
+ 2(n + 1)(1 - \frac{n}{2})(\cosh r - \beta)^{-\frac{n}{2}} \sinh^2 r \\
+ n^2 (\cosh r - \beta)^{1-\frac{n}{2}} \cosh r \right].
\]
By putting
\[
\triangle_r^2 \psi_2 + a_{21} \triangle_r \psi_2 + a_{20} \psi_2 = b_2 \psi_2^{\frac{n+4}{4}},
\]
we get that
\[
(2 - \frac{n}{2}) \left[ -2(n + 1)(1 - \frac{n}{2})(\cosh r - \beta)^{-\frac{n}{2}} + (1 - \frac{n}{2})(n^2 + 4n + 2)(\cosh r - \beta)^{-\frac{n}{2}} \cosh^2 r \right. \\
+ n^2 (\cosh r - \beta)^{1-\frac{n}{2}} \cosh r + 2(n + 2)(1 - \frac{n}{2})(-\frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}-1} (\cosh^3 r - \cosh r) \\
+ (1 - \frac{n}{2})(-\frac{n}{2}) (-1)(\cosh^2 r - \beta)^{-\frac{n}{2}-2} (\cosh^4 r - 2 \cosh^2 r + 1) \right] \\
- a_{21} (2 - \frac{n}{2}) \left[ (1 - \frac{n}{2})(\cosh r - \beta)^{-\frac{n}{2}} \sinh^2 r + n(\cosh r - \beta)^{1-\frac{n}{2}} \cosh r \right] \\
+ a_{20}(\cosh r - \beta)^{2-\frac{n}{2}} = b_2 (1 - \beta^2)^2 (\cosh r - \beta)^{-\frac{n}{2}-1},
\]
i.e.,

\[
b_2(1 - \beta^2)^2 = \left(2 - \frac{n}{2}\right) \left[-2(n + 1) \left(1 - \frac{n}{2}\right) (\cosh r - \beta)^2 + (1 - \frac{n}{2}) (n^2 + 4n + 2) (\cosh r - \beta)^2 \cosh^2 r + n^2 (\cosh r - \beta)^3 \cosh r + 2(n + 2)(1 - \frac{n}{2})(-\frac{n}{2})(\cosh r - \beta)(\cosh^3 r - \cosh r) + (1 - \frac{n}{2})(-\frac{n}{2} - 1)(1 - \cosh^2 r)^2\right] - a_{21}(2 - \frac{n}{2}) \left[(1 - \frac{n}{2})(\cosh r - \beta)^2 (\cosh^2 r - 1) + n(\cosh r - \beta)^3 \cosh r\right] + a_{20}(\cosh r - \beta)^4.
\]

It follows that

\[
b_2(1 - \beta^2)^2 = a_{20}(\beta^4 - 4\beta^3 \cosh r + 6\beta^2 \cosh^2 r - 4\beta \cosh^3 r + \cosh^4 r)
\quad + (2 - \frac{n}{2})(n a_{21} - n^2) (\beta^3 \cosh r - 3\beta^2 \cosh^2 r + 3\beta \cosh^3 r - \cosh^4 r)
\quad + (2 - \frac{n}{2})(1 - \frac{n}{2})[a_{21} - 2(n + 1)] (\beta^2 - 2\beta \cosh r + \cosh^2 r)
\quad + (2 - \frac{n}{2})(1 - \frac{n}{2})[(n^2 + 4n + 2) - a_{21}] (\beta^2 \cosh^2 r - 2\beta \cosh^3 r + \cosh^4 r)
\quad + (2 - \frac{n}{2})(1 - \frac{n}{2})(-\frac{n}{2})(n + 2) \left[-\frac{1}{2} + 2\beta \cosh r - \cosh^2 r - 2\beta \cosh^3 r + \frac{3}{2} \cosh^4 r \right].
\]

Thus we have the following system of equations with unknown constants \(a_{21}, a_{20}\) and \(b_2\):

\[
\begin{align*}
-b_2(1 - \beta^2)^2 + \beta^4 a_{20} + (2 - \frac{n}{2})(1 - \frac{n}{2})[a_{21} - 2(n + 1)] \beta^2 & = 0, \\
-\frac{1}{2} (2 - \frac{n}{2})(1 - \frac{n}{2})(-\frac{n}{2}) (n + 2) & = 0, \\
-4\beta^3 a_{20} + \beta^3 (2 - \frac{n}{2}) [n a_{21} - n^2] - 2\beta(2 - \frac{n}{2})(1 - \frac{n}{2}) [a_{21} - 2(n + 1)] & = 0, \\
+2\beta (2 - \frac{n}{2})(1 - \frac{n}{2})(-\frac{n}{2}) (n + 2) & = 0, \\
6\beta^2 a_{20} - 3\beta^2 (2 - \frac{n}{2}) (n a_{21} - n^2) + (2 - \frac{n}{2})(1 - \frac{n}{2}) [a_{21} - 2(n + 1)] & = 0, \\
+\beta^2 (2 - \frac{n}{2})(1 - \frac{n}{2}) [(n^2 + 4n + 2) - a_{21}] & = 0, \\
-(2 - \frac{n}{2})(1 - \frac{n}{2})(-\frac{n}{2}) (n + 2) & = 0, \\
-4\beta a_{20} + 3\beta (2 - \frac{n}{2}) (n a_{21} - n^2) - 2\beta (2 - \frac{n}{2})(1 - \frac{n}{2}) [(n^2 + 4n + 2) - a_{21}] & = 0, \\
-2\beta (2 - \frac{n}{2})(1 - \frac{n}{2})(-\frac{n}{2}) (n + 2) & = 0, \\
a_{20} - (2 - \frac{n}{2})(n a_{21} - n^2) + (2 - \frac{n}{2})(1 - \frac{n}{2}) (n^2 + 4n + 2) - a_{21} & = 0, \\
+\frac{3}{2} (2 - \frac{n}{2})(1 - \frac{n}{2})(-\frac{n}{2}) (n + 2) & = 0.
\end{align*}
\]
from which we obtain the unique solution:

\[ a_{21} = -\frac{n^2 - 2n - 4}{2}, \quad a_{20} = b_2 = \frac{n(n - 4)(n^2 - 2)}{2^4}. \]

(iii) For \( k = 3 \) and \( \psi_3(r) = (1 - \beta^2)^{(n-6)/4} [\cosh r - \beta]^{3-\frac{n}{2}} \), we have that

\[ \Delta_r \psi_3 = -(1 - \beta^2)^{(n-6)/4} (3 - \frac{n}{2}) \left[ (2 - \frac{n}{2}) (\cosh r - \beta)^{1-\frac{n}{2}} \sinh^2 r + n (\cosh r - \beta)^{2-\frac{n}{2}} \cosh r \right], \]

\[ \Delta_r^2 \psi_3 = (1 - \beta^2)^{(n-6)/4} (3 - \frac{n}{2}) \left[ (2 - \frac{n}{2}) \left( \frac{1}{2} - \frac{n}{2} \right) (\cosh r - \beta)^{-\frac{n}{2} - 1} \sinh^4 r \right. \]

\[ + 2(n + 2) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}} (\sinh r)^2 \cosh r \]

\[ + n (n + 2) (2 - \frac{n}{2}) (\cosh r - \beta)^{1-\frac{n}{2}} \cosh^2 r \]

\[ + 2(n + 1) (2 - \frac{n}{2}) (\cosh r - \beta)^{1-\frac{n}{2}} \sinh^2 r + n^2 (\cosh r - \beta)^{2-\frac{n}{2}} \cosh r \right], \]

and

\[ \Delta_r^3 \psi_3 = (1 - \beta^2)^{\frac{n-6}{2}} (3 - \frac{n}{2}) \left[ (2 - \frac{n}{2}) \left( \frac{1}{2} - \frac{n}{2} \right) \left( \frac{n}{2} + 1 \right) (\cosh r - \beta)^{-\frac{n}{2} - 3} \sinh^6 r \right. \]

\[ - 3(n + 4) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) \left( \frac{n}{2} + 1 \right) (\cosh r - \beta)^{-\frac{n}{2} - 2} (\sinh r)^4 \cosh r \]

\[ - 3(n + 2) (n + 4) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) \left( \frac{1}{2} - \frac{n}{2} \right) (\cosh r - \beta)^{-\frac{n}{2} - 1} (\sinh r)^2 \cosh^2 r \]

\[ + (6n + 14) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) \left( \frac{n}{2} + 1 \right) (\cosh r - \beta)^{-\frac{n}{2} - 1} \sinh^4 r \]

\[ - (6n + 14) (n + 2) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}} (\sinh r)^2 \cosh r \]

\[ - n (n + 4) (n + 2) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}} \cosh^3 r \]

\[ - (n + 2) (3n + 2) \left( 2 - \frac{n}{2} \right) (1 - \frac{n}{2}) (\cosh r - \beta)^{-\frac{n}{2}} (\sinh r)^2 \cosh r \]

\[ - n (n + 2) (3n + 2) \left( 2 - \frac{n}{2} \right) (\cosh r - \beta)^{1-\frac{n}{2}} \cosh^2 r \]

\[ - 4(n + 1)^2 \left( 2 - \frac{n}{2} \right) (\cosh r - \beta)^{1-\frac{n}{2}} \sinh^2 r + n^3 (\cosh r - \beta)^{2-\frac{n}{2}} \cosh r \right]. \]

By substituting these results into

\[ \Delta_r^3 \psi_3 + a_{32} \Delta_r^2 \psi_3 + a_{31} \Delta_r \psi_3 + a_{30} \psi = b_3 \psi^{\frac{n+6}{2}}, \]
we have

\[
(1 - \beta^2)^3 b_3 = a_{30} (\cosh r - \beta)^6 - (3 - \frac{n}{2}) [a_{31} - n^2 a_{32} + n^3] (\cosh r - \beta)^5 \cosh r \\
- (3 - \frac{n}{2}) (2 - \frac{n}{2}) [a_{31} - 2(n + 1)a_{32} + 4(n + 1)^2] (\cosh r - \beta)^4 (\cosh^2 r - 1) \\
+ n(n + 2)(3 - \frac{n}{2}) (2 - \frac{n}{2}) [a_{32} - (3n + 2)] (\cosh r - \beta)^4 \cosh^2 r \\
+ (n + 2)(3 - \frac{n}{2}) (2 - \frac{n}{2}) (1 - \frac{n}{2}) [2a_{32} - (3n + 2)](\cosh r - \beta)^3 (\cosh^3 r - \cosh r) \\
+ (3 - \frac{n}{2}) (2 - \frac{n}{2}) (1 - \frac{n}{2}) (- \frac{n}{2}) [a_{32} - 2(3n + 7)] (\cosh r - \beta)^2 (\cosh^2 r - 1)^2 \\
- (n + 2)(3 - \frac{n}{2}) (2 - \frac{n}{2}) (1 - \frac{n}{2}) \left[ - \frac{n}{4} (\frac{n}{2} + 2) (\cosh^2 r - \beta)^3 \right] \\
+ \frac{3}{4} n(n + 4)(\cosh^2 r - \beta \cosh r)(1 - \cosh^2 r)^2 \\
+ \frac{3}{2} n(n + 4)(\cosh r - \beta)^2(\cosh^2 r - \cosh^4 r) \\
+ 2(3n + 7)(\cosh r - \beta)^3 (\cosh^3 r - \cosh r) + n(n + 4)(\cosh r - \beta)^3 \cosh^3 r] .
\]

Letting the coefficients of $1, \cosh r, \cdots, \cosh^6 r$ are respectively zero, and then solving the corresponding system of equations, we obtain the unique solution:

\[
A_{32} = -\frac{3n^2 - 6n - 32}{4}, \quad a_{31} = \frac{3n^4 - 12n^3 - 52n^2 + 128n + 192}{16}, \\
- a_{30} = b_3 = \frac{n(n - 6)(n^2 - 4^2)(n^2 - 2^2)}{2^6} .
\]

(iv) By repeating the same process as in (i), (ii) and (iii), we can get all the constants $b_k$ and $a_{km}$, $(k = 1, 2, \cdots; 0 \leq m \leq k - 1)$. For example, we get that

\[
a_{43} = -(n^2 - 2n - 20), \quad a_{42} = \frac{3}{8} n^4 - \frac{3}{2} n^3 - \frac{27}{2} n^2 + 30n + 108, \\
a_{41} = -\frac{n^6}{16} + \frac{3}{8} n^5 + 3n^4 - \frac{29}{2} n^3 - 39n^2 + 108n + 144, \\
a_{40} = b_4 = \frac{n(n - 8)(n^2 - 6^2)(n^2 - 4^2)(n^2 - 2^2)}{2^8}, \\
a_{54} = \frac{-5n^2 + 10n + 160}{4}, \cdots, \cdots, \\
- a_{50} = B_5 = \frac{n(n - 10)(n^2 - 8^2)(n^2 - 6^2)(n^2 - 4^2)(n^2 - 2^2)}{2^{10}},
\]

and

\[
(-1)^k a_{k0} = b_k = \frac{n[n - 2k][n^2 - (2(k - 1))^2][n^2 - (2(k - 2))^2] \cdots [n^2 - 2^2]}{2^{2k}}
\]
for all $k = 1, 2, \ldots$. □

Recall that for the ball model of the hyperbolic space $\mathbb{H}^n$, its standard metric is

$$h := \left( \frac{2}{1 - |x|^2} \right)^2 \delta,$$

where $\delta$ is the Euclidean metric of $\mathbb{R}^n$. Now, we introduce the following operator on $(\mathbb{H}^n, h)$ by

$$P_k := \Delta_h + \sum_{m=0}^{k-1} a_{km} \Delta_h^m,$$

where the constants $a_{k,k-1}, \ldots, a_{k0}$ are given by Lemma 2.2, and $\Delta_h$ is the Laplacian on $\mathbb{H}^n$ with the metric $h$ (which has the local representation:

$$\Delta_h u = -\frac{1}{h^{1/2}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( h^{1/2} h^{ij} \frac{\partial u}{\partial x_j} \right)$$

for every $u \in C^\infty(\mathbb{H}^n)$). We shall call $P_k$ the $k$-th standard operator on the hyperbolic space $(\mathbb{H}^n, h)$, and $a_{km}$ $(0 \leq m < k - 1)$ the coefficients of $P_k$. It is easy to see that $P_1$ is just the Yamabe operator $\Delta_h \frac{n(n-2)}{4}$ on $\mathbb{H}^n$ (see [A1], [A2], [LP] and [Cha]), and $P_2$ simply is the Paneitz-Branson operator on $\mathbb{H}^n$ (see [DHL] or [Cha]). The general Yamabe and Paneitz-Branson operators are defined as follows (cf.[Pa], [DHL] and [Cha]): Given $(M, g)$ a smooth $n$-dimensional Riemannian manifold, and let $S_g$ be the scalar curvature of $g$, and let $Ric_g$ be the Ricci curvature of $g$. Then the Yamabe operator $L_g$, the second order operator whose expression is given by

$$Y_g = \Delta_g u + \frac{n-2}{4(n-1)} S_g u.$$ 

The fourth order Paneitz-Branson operator $Z_g$ is given by

$$Z_g u = \Delta_g^2 u - div_g [ (\tau_n S_g + \eta_n Ric_g) du ] + \frac{n-4}{2} Q_g u,$$

where

$$Q_g = \frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S^2 - \frac{2}{(n-2)^2} |Ric_g|^2$$

and

$$\begin{cases}
\tau_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \\
\eta_n = -\frac{n-4}{n-2}.
\end{cases}$$

Note that the $k$-th standard operator $P_k$ on $(\mathbb{H}^n, h)$ is a $2k$-order linear differential operator defined on $\mathbb{H}^n$. Surprisingly, we have the following recursive formula, which shows that the $k$-th standard operator $P_k$ on $\mathbb{H}^n$ can be calculated by the first standard operator $P_1$: 
Theorem 2.3. Assume that \((\mathbb{H}^n, h)\) is the hyperbolic space. Let \(P_1 = \Delta_h - \frac{n(n-2)}{4}\) be the first standard operator on \(\mathbb{H}^n\), and let \(P_k\) be the \(k\)-th standard operator on \(\mathbb{H}^n\) whose coefficients is given by Lemma 2.2. Then

\[
(2.13) \quad P_k = P_{k-1} [P_1 + k(k-1)], \quad k = 2, 3, \ldots ,
\]

and therefore

\[
(2.14) \quad P_k = P_1 [P_1 + 2] [P_1 + 6] \cdots [P_1 + k(k-1)]
\]

for all \(k = 1, 2, \ldots\).

Proof. (i) For \(k = 2\), we have

\[
P_1 (P_1 + 2) = \left[ \Delta_h - \frac{n(n-2)}{4} \right] \left[ \Delta_h - \left( \frac{n(n-2)}{4} - 2 \right) \right]
= \Delta_h^2 - \left( \frac{2n(n-2) - 8}{4} \right) \Delta_h + \frac{n(n-2)(n^2 - 2n - 8)}{16}
= \Delta_h^2 - \left( \frac{n^2 - 2n - 4}{2} \right) \Delta_h + \frac{n(n-4)(n^2 - 2^2)}{2^4} = P_2.
\]

(ii) For \(k = 3\), it is easy to verify that

\[
P_2 [P_1 + 6] = \left[ \Delta_h^2 - \frac{n^2 - 2n - 4}{2} \right] \left[ \Delta_h - \frac{n(n-2)}{4} + 6 \right]
= \Delta_h^3 - \left[ \frac{n(n-2) - 24}{4} + \frac{n^2 - 2n - 4}{2} \right] \Delta_h^2
+ \left[ \frac{(n^2 - 2n - 4)(n^2 - 2n - 24)}{8} + \frac{n(n-4)(n^2 - 2^2)}{16} \right] \Delta_h
- \frac{n(n-4)(n^2 - 2^2)(n^2 - 2n - 24)}{64}
= \Delta_h^3 - \frac{3n^2 - 6n - 32}{4} \Delta_h^2 + \frac{3n^4 - 12n^3 - 52n^2 + 128n + 192}{16} \Delta_h
- \frac{n(n-6)(n^2 - 4^2)(n^2 - 2^2)}{2^6}
= P_3.
\]
(iii) For \( k = 4 \), we have

\[
P_3[P_1 + 12] = \left[ \frac{3n^2 - 6n - 32}{4} \Delta_h^3 + \frac{3n^4 - 12n^3 - 52n^2 + 128n + 192}{16} \Delta_h \right. \\
- \frac{n(n-6)(n^2 - 4^2)(n^2 - 2^2)}{2^6} \times \left[ \triangle - \left( \frac{n(n-2)}{4} - 12 \right) \right] \\
= \Delta_4^4 - \left[ \frac{n^2 - 2n - 48}{4} + \frac{3n^2 - 6n - 32}{4} \right] \Delta_h^3 \\
+ \left[ \frac{(3n^2 - 6n - 32)(n^2 - 2n - 48)}{16} + \frac{3n^4 - 12n^3 - 52n^2 + 128n + 192}{16} \right] \Delta_h^2 \\
- \frac{(3n^4 - 12n^3 - 52n^2 + 128n + 192)(n^2 - 2n - 48) + n(n-6)(n^2 - 4^2)(n^2 - 2^2)}{2^6} \Delta_h \\
+ \frac{n(n-6)(n^2 - 4^2)(n^2 - 2^2)(n^2 - 2n - 48)}{2^8} \\
= \Delta_4^4 - \left( n^2 - 2n - 20 \right) \Delta_h^3 + \left( \frac{3}{8}n^4 - \frac{3}{2}n^3 - \frac{27}{2}n^2 + 30n + 108 \right) \Delta_h^2 \\
- \left( \frac{n^6}{16} - \frac{3}{8}n^5 - 3n^4 + \frac{29}{2}n^3 + 39n^2 - 108n - 144 \right) \Delta_h \\
+ \frac{n(n-8)(n^2 - 6^2)(n^2 - 4^2)(n^2 - 2^2)}{2^8} \\
= P_4.
\]

(iv) By checking the results that are obtained from Lemma 2.2 for all \( k \), we get the desired conclusion. \( \square \)

**Remark 2.4.** Now we can easily calculate the coefficients \( a_{km} \) of the \( k \)-th standard operator \( P_k \) on \( \mathbb{H}^n \) by the recursive formula (2.14) instead of that complicated method as in Lemma 2.2.

The following theorem give a basic transformation law for \( P_k \):

**Theorem 2.5.** Let \((\mathbb{H}^n, h)\) be the hyperbolic \( n \)-space, and let \( P_k \) be defined as above. Suppose \( \sigma : \mathbb{B}_n \to \mathbb{H}^n \) is the conformal transformation defined by (2.1). Assume that \( \xi_k(x) = \left( \frac{2}{1-|x|^2} \right)^{(n-2k)/2} \). Then for any \( u \in \mathcal{D}(\mathbb{H}^n) \),

\[
\xi_k \frac{n+2k}{2^k} [P_k (u \circ \sigma)] = \Delta_k (\xi_k \cdot (u \circ \sigma)), \quad \text{for} \ x \in \mathbb{B}^n,
\]

where \( \mathcal{D}(\mathbb{H}^n) \) denotes the space of smooth functions with compact support in \( \mathbb{H}^n \), and \( \Delta_k \) is the standard \( k^{th} \)-iterated Laplacian in \( \mathbb{R}^n \) as in Introduction (see, §1).
Proof. We shall prove this theorem by principle of mathematical induction. For convenience, we shall simply write $P_k(u \circ \sigma)$ as $P_k u$ in $B_n$. For $k = 1$, noting that
\[
\sqrt{h} = \left( \frac{2}{1 - |x|^2} \right)^n,
\]
we have
\[
\triangle_h u = - \frac{1}{\sqrt{h}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{h} h^{ij} \frac{\partial u}{\partial x_j} \right)
\]
\[
= - \sum_{i,j=1}^{n} \left[ h^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{1}{\sqrt{h}} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} \left( \sqrt{h} h^{ij} \right) \right]
\]
\[
= - \sum_{i,j=1}^{n} \left[ \left( \frac{1 - |x|^2}{2} \right)^2 \frac{\partial^2 u}{\partial x_i \partial x_j} \delta_{ij} + \left( \frac{1 - |x|^2}{2} \right)^n \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i} \left( \frac{2}{1 - |x|^2} \right)^{n-2} \delta_{ij} \right]
\]
\[
= - \sum_{i,j=1}^{n} \left[ \left( \frac{1 - |x|^2}{2} \right)^2 \frac{\partial^2 u}{\partial x_i \partial x_j} \delta_{ij} + (n-2) \left( \frac{1 - |x|^2}{2} \right) x_i \frac{\partial u}{\partial x_j} \delta_{ij} \right]
\]
\[
= \left( \frac{1 - |x|^2}{2} \right)^2 \Delta u + (n-2) \left( \frac{1 - |x|^2}{2} \right) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i},
\]
so that
\[
\xi_1^{n+2} (P_1 u) = \left( \frac{2}{1 - |x|^2} \right)^{n+2} \left( \Delta_h u - \frac{n(n-2)}{4} u \right)
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2} \left[ \left( \frac{1 - |x|^2}{2} \right)^2 \Delta u + (n-2) \left( \frac{1 - |x|^2}{2} \right) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} - \frac{n(n-2)}{4} u \right]
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2} \Delta u + (n-2) \left( \frac{2}{1 - |x|^2} \right) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} - \frac{n(n-2)}{4} \left( \frac{2}{1 - |x|^2} \right)^{n+2} u.
\]

On the other hand,
\[
\frac{\partial \xi_1}{\partial x_i} = \left( \frac{n}{2} - 1 \right) \left( \frac{2}{1 - |x|^2} \right)^{\frac{n}{2}} x_i, \quad i = 1, 2, \ldots, n
\]
and
\[
\frac{\partial^2 \xi_1}{\partial x_i^2} = \left( \frac{n}{2} - 1 \right) \left( \frac{2}{1 - |x|^2} \right)^{\frac{n}{2}} + \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} \right) \left( \frac{2}{1 - |x|^2} \right)^{\frac{n+2}{2}} x_i^2,
\]
so that
\[
\triangle \xi_1 = - \sum_{i=1}^{n} \frac{\partial^2 \xi_1}{\partial x_i^2} = \frac{n(n-2)}{4} \left( \frac{2}{1 - |x|^2} \right)^{n+2}.
\]
It follows that
\[
\nabla (u \xi) = - \sum_{i=1}^{n} \frac{\partial^2 (u \xi)}{\partial x_i^2} = \xi \nabla u - 2 \nabla u \cdot \nabla \xi + u \nabla \xi
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n-2} \nabla u + (n-2) \left( \frac{2}{1 - |x|^2} \right)^2 \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}
\]
\[
- \frac{n(n-2)}{4} \left( \frac{2}{1 - |x|^2} \right)^{n+2} u.
\]
Thus
\[
\xi_{k+1}^{n+2 (k+1)} (P_{k+1} u) = \nabla (u \xi),
\]
i.e., (2.15) holds for \(k = 1\). We now suppose that for \(k \geq 1\), the law (2.15) holds, that is,
\[
(2.16) \quad \left( \frac{2}{1 - |x|^2} \right)^{n+2k} (P_k u) = \nabla^k \left( \left( \frac{2}{1 - |x|^2} \right)^{n-k} u \right).
\]
Our purpose is to prove that (2.15) still holds for \(k + 1\). By (2.15) and (2.16), we have that
\[
\xi_{k+1}^{n+2 (k+1)} (P_{k+1} u) = \left( \frac{2}{1 - |x|^2} \right)^{n+2 (k+1)} (P_{k+1} u)
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2 (k+1)} \left[ P_k (P_1 + k(k + 1)) u \right]
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2k} \left[ P_k (P_1 + k(k + 1)) u \right]
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2k} \nabla^k \left[ \left( \frac{2}{1 - |x|^2} \right)^{n-2k} (P_1 + k(k + 1)) u \right]
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2k} \nabla^k \left[ \left( \frac{2}{1 - |x|^2} \right)^{n-2k} (P_1 u) + k(k + 1) \left( \frac{2}{1 - |x|^2} \right)^{n-2k} u \right]
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n+2k} \nabla^k \left[ \left( \frac{2}{1 - |x|^2} \right)^{n-2k} \nabla^{k-1} \left( \left( \frac{2}{1 - |x|^2} \right)^{n-2k} u \right) + k(k + 1) \left( \frac{2}{1 - |x|^2} \right)^{n-2k} u \right]
\]
On the other hand, we can verify, by a direct calculation, that for any positive integer \(k \geq 1\),
\[
\nabla^{k+1} \left[ \left( \frac{2}{1 - |x|^2} \right)^{n-2 (k+1)} u \right]
\]
\[
= \left( \frac{2}{1 - |x|^2} \right)^{n-2 (k+1)} \nabla^{k+1} \left[ \left( \frac{2}{1 - |x|^2} \right)^{n-2k} u \right] + k(k + 1) \left( \frac{2}{1 - |x|^2} \right)^{n-2k} u.
\]
Lemma 3.1. Let $\sigma$ be constants $b(3.1)$ where $q$ as bilinear forms. Choosing $\epsilon$ such that there exist $b_0$ and radius $y_i$) The identity (2.15) can also be written as

$$\left( P_k u \right) \circ \sigma = J_{\sigma}^{-n+2k} \Delta^k \left[ J_{\sigma}^{-n} u \circ \sigma \right], \quad \text{for } u \in D(\mathbb{H}^n),$$

where $J_{\sigma}$ is the Jacobian of $\sigma$.

(ii) The $k$-th standard operator $P_k$ on the hyperbolic space $(\mathbb{H}^n, h)$ is essentially the power $\Delta^k$ of the positive $\mathbb{R}^n$ Laplacian lifted to the hyperbolic space via the conformal transformation $\sigma$, using the identity (2.15). We can also call $P_k$ the $k$-th intertwining operator on $(\mathbb{H}^n, h)$.

§3. Sharp $k$-order Sobolev inequality in the hyperbolic space $\mathbb{H}^n$

Lemma 3.1. Let $(\mathbb{H}^n, h)$ be the hyperbolic space, $n > 2k$. Suppose that there exist real constants $b'_k$ and $\{a'_{km}\}_{0 \leq m \leq k-1}$ such that for any $u \in D(\mathbb{H}^n)$,

$$b'_k \left( \int_{\mathbb{H}^n} |u|^q \, dV_h \right)^{2/q} \leq \int_{\mathbb{H}^n} |\Delta_h^{k/2} u|^2 \, dV_h + \sum_{m=0}^{k-1} a'_{km} \int_{\mathbb{H}^n} |\Delta_h^{m/2} u|^2 \, dV_h,$$

where $q = (2n)/(n - 2k)$. Then $b'_k \leq \frac{1}{\Lambda_k}$, where $\Lambda_k$ is the best $k$-order Sobolev constant in $\mathbb{R}^n$.

Proof. This proof follows from the lines of [A1] (also see [H1]). Suppose by contradiction that there exist $b'_k$ and $\{a'_{km}\}$ satisfying $b'_k > \frac{1}{\Lambda_k}$, such that inequality (3.1) holds for any $u \in D(\mathbb{H}^n)$. Let $y \in \mathbb{H}^n$. It is easy to see that for any $\epsilon > 0$ there exists a chart $(U, \phi)$ of $\mathbb{H}^n$ at $y$, and there exists $\tau > 0$ such that $\phi(U) = B_\tau(0)$ (here $B_\tau(0)$ is the Euclidean ball of center 0 and radius $\tau$ in $\mathbb{R}^n$), and such that the components $h_{ij}$ of $h$ in this chart satisfy

$$(1 - \epsilon)\delta_{ij} \leq h_{ij} \leq (1 + \epsilon)\delta_{ij}$$

as bilinear forms. Choosing $\epsilon$ small enough we can get by (3.1) that there exist $\tau_0 > 0$, $b''_k$ and $\{a''_{km}\}$ satisfying $b''_k > \frac{1}{\Lambda_k}$ such that for any $\tau \in (0, \tau_0)$ and any $u \in D(B_\tau(0))$,

$$b''_k \left( \int_{\mathbb{R}^n} |u|^q \, dx \right)^{2/q} \leq \int_{\mathbb{R}^n} |\Delta_h^{k/2} u|^2 \, dV_h + \sum_{m=0}^{k-1} a''_{km} \int_{\mathbb{R}^n} |\Delta_h^{m/2} u|^2 \, dx.$$

Applying Nirenberg’s lemma (see [N] or [ADN, Lemma 14.1]), we find that there exists a constant $c$ depending only on $\tau$ such that for any $0 < m \leq k - 1$,

$$\int_{\mathbb{R}^n} |\Delta_h^{m/2} u|^2 \leq \tau \int_{\mathbb{R}^n} |\Delta_h^{k/2} u|^2 \, dx + c \int_{\mathbb{R}^n} |u|^2 \, dx.$$
Noting that $1/q = 1/2 - k/n$, it follows from Hölder’s inequality that

$$
\int_{B_\tau(0)} |u|^2 dx \leq |B_\tau(0)|^{2k/n} \left( \int_{B_\tau(0)} |u|^q \right)^{2/q},
$$

where $|B_\tau(0)|$ denotes the volume of the ball $B_\tau(0)$ in $\mathbb{R}^n$. Therefore by choosing $\tau$ small enough, we obtain that there exist $\tau > 0$ and $b'''_k > \frac{1}{\Lambda_k}$ such that for any $u \in \mathcal{D}(B_\tau(0))$,

$$
b'''_k \left( \int_{\mathbb{R}^n} |u|^q dx \right)^{2/q} \leq \int_{\mathbb{R}^n} |\Delta^{k/2} u|^2 dx.
$$

For any $u \in \mathcal{D}(\mathbb{R}^n)$, let us set $u_\eta(x) = u(\eta x)$, $\eta > 0$. Take $\eta$ large enough, $u_\eta \in \mathcal{D}(B_\tau(0))$. Thus,

$$
b'''_k \left( \int_{\mathbb{R}^n} |u_\eta|^q dx \right)^{2/q} \leq \int_{\mathbb{R}^n} |\Delta^{k/2} u_\eta|^2 dx.
$$

But

$$
\left( \int_{\mathbb{R}^n} |u_\eta|^q dx \right)^{2/q} = \eta^{-(2n)/q} \left( \int_{\mathbb{R}^n} |u|^q dx \right)^{2/q}
$$

while

$$
\int_{\mathbb{R}^n} | \Delta^{k/2} u_\eta |^2 dx = \eta^{2k-n} \int_{\mathbb{R}^n} | \Delta^{k/2} u |^2 dx.
$$

In view of $1/q = 1/2 - k/n$, we have that for any $u \in \mathcal{D}(\mathbb{R}^n)$,

$$
b'''_k \left( \int_{\mathbb{R}^n} |u|^q dx \right)^{1/q} \leq \int_{\mathbb{R}^n} |\Delta^{k/2} u|^2 dx.
$$

Since $b'''_k > \frac{1}{\Lambda_k}$, such an inequality is contradiction with (1.5). This completes proof of the lemma. □

We have the following sharp $k$-order Sobolev inequality on $\mathbb{H}^n$:

**Theorem 3.2.** Let $(\mathbb{H}^n, h)$ be the hyperbolic $n$-space, $n > 2k$, and let $q = (2n)/(n - 2k)$. Assume that $P_k$ is the $k$-th standard operator on $\mathbb{H}^n$, that is, $P_k = P_1(P_1 + 2) \cdots (P_1 + k(k-1))$ with $P_1 = \Delta_h - \frac{n(n-2)}{2}$. Then, for any $u \in \mathcal{D}(\mathbb{H}^n)$,

$$
(3.2) \quad \left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q} \leq \Lambda_k \int_{\mathbb{H}^n} (P_k u) u dV_h,
$$

or equivalently,

$$
(3.3) \quad \left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q} \leq \Lambda_k \left[ \int_{\mathbb{H}^n} \left( |\Delta^{k/2}_h u|^2 + \sum_{m=0}^{k-1} a_{km} |\Delta^{m/2}_h u|^2 \right) dV_h \right],
$$

where $a_{km}$ are coefficients determined by the geometry of $\mathbb{H}^n$. □
where $\Lambda_k$ is the best $k$-order Sobolev constant in $\mathbb{R}^n$ (see, §1) and $a_{km}$ are the coefficients of the standard operator $P_k$. Moreover, for any $0 < \beta < 1$, if

$$u_\beta(r) = \frac{(1 - \beta^2)^{n-2k}}{(\cosh r - \beta)^{n-2k}}, \quad r \in [0, +\infty),$$

where $r$ is the distance from 0 on $\mathbb{H}^n$, then

$$\lim_{\beta \to 1^-} \frac{\int_{\mathbb{H}^n} (P_k u_\beta) u_\beta \, dV_h}{\left( \int_{\mathbb{H}^n} |u_\beta|^q dV_h \right)^{2/q}} = \inf_{u \in \mathcal{D}(\mathbb{H}^n) \setminus \{0\}} \frac{\int_{\mathbb{H}^n} (P_k u) u \, dV_h}{\left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q}},$$

and

$$P_k \left( \omega_n^{-1/q} u_\beta(r) \right) = \frac{1}{\Lambda_k} \left( \omega_n^{-1/q} u_\beta(r) \right)^{q-1}, \quad 0 \leq r < +\infty.$$

**Proof.** Recall that

$$dV_h = \left( \frac{2}{1 - |x|^2} \right)^n \, dx,$$

where $dx$ is the volume element of the Euclidean space $\mathbb{R}^n$. It is clear that for every $u \in \mathcal{D}(\mathbb{H}^n)$,

$$\frac{\int_{\mathbb{H}^n} (P_k u) u \, dV_h}{\left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q}} = \frac{\int_{\mathbb{H}^n} \left( |\Delta_h^{k/2} u|^2 + \sum_{m=0}^{k-1} a_{km} |\Delta_h^{m/2} u|^2 \right) \, dV_h}{\left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q}},$$

and

$$\frac{\int_{\mathbb{H}^n} (P_k u) u \, dV_h}{\left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q}} = \frac{\int_{\mathbb{B}_n} (P_k u) u \, dV_h}{\left( \int_{\mathbb{B}_n} |u|^q dV_h \right)^{2/q}},$$

where $u$ and $P_k u$ in $\mathbb{B}_n$ mean $u \circ \sigma$ and $P_k (u \circ \sigma)$ respectively, and $\sigma : \mathbb{B}_n \to \mathbb{H}^n$ is the conformal mapping defined by (2.1). By Theorem 2.5 we have

$$\xi_k^{n-2k} (P_k u) = \triangle^k (\xi_k u) \quad \text{in} \quad \mathbb{R}^n,$$

where $\xi_k = \left( \frac{2}{1 - |x|^2} \right)^{n-2k}$, and $\triangle^k$ is the $k$th-iterated standard Laplacian in $\mathbb{R}^n$ as in §1. Multiplying $\xi_k u$ to equation (3.8) and then integrating the result in $\mathbb{B}_n$ (here $\mathbb{B}_n$ is the n-dimensional unit ball in $\mathbb{R}^n$), we get

$$\int_{\mathbb{B}_n} \left( \frac{2}{1 - |x|^2} \right)^n (P_k u) u \, dx = \int_{\mathbb{B}_n} (\xi_k u) \left( \triangle^k (\xi_k u) \right) \, dx.$$
Note that

(3.10) \[ \int_{B_n} \left( \frac{2}{1 - |x|^2} \right)^n (P_k u) u \, dx = \int_{B_n} (P_k u) u \, dV_h. \]

By applying integration by parts to the right-hand side of (3.9), we have

(3.11) \[ \int_{B_n} (\xi_k u) (\Delta^k (\xi_k u)) \, dx = \int_{B_n} |\Delta^{k/2} (\xi_k u)|^2 \, dx. \]

It follows from (3.7)—(3.11) that

(3.12) \[ \frac{\int_{H^n} (P_k u) u \, dV_h}{\left( \int_{H^n} |u|^q \, dV_h \right)^{2/q}} = \frac{\int_{B_n} |\Delta^{k/2} (\xi_k u)|^2 \, dx}{\left( \int_{B_n} |\xi_k u|^q \, dx \right)^{2/q}}. \]

From Lemma 3.1, we see that

\[ \inf_{u \in \mathcal{D}(H^n) \setminus \{0\}} \frac{\int_{H^n} \left( |\Delta^{k/2} u|^2 + \sum_{k=0}^{k-1} a_{km} |\Delta_h^{m/2} u|^2 \right) \, dV_h}{\left( \int_{H^n} |u|^q \, dV_h \right)^{2/q}} \leq \frac{1}{\Lambda_k}. \]

Suppose by contradiction that

\[ \inf_{u \in \mathcal{D}(H^n) \setminus \{0\}} \frac{\int_{H^n} \left( |\Delta^{k/2} u|^2 + \sum_{k=0}^{k-1} a_{km} |\Delta_h^{m/2} u|^2 \right) \, dV_h}{\left( \int_{H^n} |u|^q \, dV_h \right)^{2/q}} < \frac{1}{\Lambda_k}, \]

and let \( u_0 \in \mathcal{D}(H^n) \), \( u_0 \neq 0 \) satisfy

(3.13) \[ \frac{\int_{H^n} |\Delta_h^{k/2} u_0|^2 \, dV_h + \sum_{k=0}^{k-1} a_{km} |\Delta_h^{m/2} u_0|^2 \, dV_h}{\left( \int_{H^n} |u_0|^q \, dV_h \right)^{2/q}} < \frac{1}{\Lambda_k}. \]

By (3.6), (3.12) and (3.13), we have

\[ \frac{\int_{B_n} |\Delta^{k/2} (\xi_k u_0)|^2 \, dx}{\left( \int_{B_n} |\xi_k u_0|^q \, dx \right)^{2/q}} < \frac{1}{\Lambda_k}. \]

Clearly, \( \xi_k u_0 \in \mathcal{D}(B_n) \). But

\[ \frac{\int_{B_n} |\Delta^{k/2} (\xi_k u_0)|^2 \, dx}{\left( \int_{B_n} |\xi_k u_0|^q \, dx \right)^{2/q}} \geq \inf_{w \in \mathcal{D}(B_n) \setminus \{0\}} \frac{\int_{B_n} |\Delta^{k/2} w|^2 \, dx}{\left( \int_{B_n} |w|^q \, dx \right)^{2/q}} \geq \frac{1}{\Lambda_k}. \]
This is a contradiction, which shows

\[ \inf_{u \in \mathcal{D}(\mathbb{H}^n) \setminus \{0\}} \frac{\int_{\mathbb{H}^n} \left( |\nabla_h^{k/2} u|^2 + \sum_{k=0}^{k-1} a_{km} |\nabla_h^m u|^2 \right) dV_h}{\left( \int_{\mathbb{H}^n} |u|^q dV_g \right)^{2/q}} = \frac{1}{\Lambda_k}. \]

Hence we get (3.3).

Finally, by the conformal map \( \sigma \) from \( (\mathbb{B}^n, \delta) \) to \( (\mathbb{H}^n, h) \), the function \( G_\beta(x) \) (see (2.3)) defined in \( \mathbb{B}^n \) is lifted to \( u_\beta(r) \) (i.e., \( \psi_k(r) \)), and (3.4) is true. (2.10) of Lemma 2.2 also shows that equation (3.5) holds. \( \square \)

It is easily seen from Theorem 3.2 that for any constants \( \tau_{km}, m = 0, 1, \ldots, k \) satisfying \( \tau_{kk} \geq \Lambda_k \) and \( \tau_{km} \geq \Lambda_k a_{km} \) \( (m = 0, 1, \ldots, k - 1) \), we have

\[ \left( \int_{\mathbb{H}^n} |u|^q dV_h \right)^{2/q} \leq \int_{\mathbb{H}^n} \left( \sum_{m=0}^{k} \tau_{km} |\nabla_h^m u|^2 \right) dV_h \]

for any \( u \in \mathcal{D}(\mathbb{H}^n) \). Lemma 3.1 implies that in order to make the above inequality hold, it is necessary that \( \tau_{kk} \geq \Lambda_k \), i.e., \( \Lambda_k \) can’t be lowered. Combining this fact and the following theorem, we shall conclude that \( \Lambda_k, \Lambda_k a_{k,k-1}, \ldots, \Lambda_k a_{k0} \) are the best constants since they can’t be lowered.

**Theorem 3.3.** Let \( (\mathbb{H}^n, h) \) be the hyperbolic space, \( n > 4k - 2 \), and let \( q = (2n)/(n - 2k) \). Assume that \( T_i \) \( (i = 0, 1, \ldots, k - 1) \) is the operator defined by

\[ T_i = \sum_{m=i+1}^{k} a_{km} \nabla_h^m + \sum_{m=0}^{i} \tau_{km} \nabla_h^m, \quad i = 0, 1, \ldots, k - 1, \]

with \( a_{kk} = 1 \). For each fixed \( i \in \{0, 1, \ldots, k - 1\} \), there exist real numbers \( \tau_{k0}, \ldots, \tau_{ki} \) such that for all \( u \in \mathcal{D}(\mathbb{H}^n) \),

\[ \|u\|_{L^q(\mathbb{H}^n)}^2 \leq \Lambda_k \int_{\mathbb{H}^n} (T_i u) u dV_h \]

(3.14) \[ = \Lambda_k \left[ \sum_{m=i+1}^{k} a_{km} \int_{\mathbb{H}^n} |\nabla_h^m u|^2 dx + \sum_{m=0}^{i} \tau_{km} \int_{\mathbb{H}^n} |\nabla_h^m u|^2 dx \right]. \]

if and only if \( \tau_{ki} \geq a_{ki} \), where \( \{a_{km}\} \) are the coefficients of the \( k \)-th standard operator \( P_k \), and \( \Lambda_k \) is the best Sobolev constant in \( \mathbb{R}^n \).

**Proof.** For any fixed \( i \in \{0, 1, \ldots, k - 1\} \), if \( \tau_{ki} \geq a_{ki} \), the result immediately follows from (3.3) of Theorem 3.2 because we may take \( \tau_{km} = a_{km}, m = 0, 1, \ldots, i - 1 \).
Suppose, on the contrary, that there exist $\tau_{k_0}, \ldots, \tau_{k_l}$ satisfying $\tau_{k_l} < a_{k_l}$ such that (3.14) holds for all $u \in D(\mathbb{H}^n)$. For $0 < \beta < 1$ real, and $r$ the distance to the origin on $\mathbb{H}^n$, we let $u_\beta(r)$ be as in Theorem 3.2, that is

$$u_\beta(r) = \frac{(1 - \beta^2)^{\frac{n-2k}{4}}}{(\cosh r - \beta)^{\frac{n-2k}{2}}}.$$  

By (2.8), $u_\beta(r)$ is lifted by the function

$$G_\beta(x) = \left[\left(1 - \beta \frac{1}{1 + \beta}\right)^{1/2} \left(1 + \frac{1 + \beta}{1 - \beta} |x|^2 \right)\right]^{k - \frac{n}{2}},$$

and

$$\int_{\mathbb{H}^n} |u_\beta(r)|^q dV_h = \int_{\mathbb{H}^n} |G_\beta(x)|^q dx = \int_{\mathbb{H}^n} \left(1 - \beta \frac{1}{1 + \beta}\right)^{-n/2} \left(1 + \frac{1 + \beta}{1 - \beta} |x|^2 \right)^{-n} dx$$

$$= \omega_{n-1} \int_0^1 \left(1 - \beta \frac{1}{1 + \beta}\right)^{-n/2} \left(1 + \frac{1 + \beta}{1 - \beta} s^2 \right)^{-n} s^{n-1} ds = 2^n \omega_{n-1} \int_0^{\sqrt{1 + \beta}} (1 + z^2)^{-n} z^{n-1} dz.$$  

Hence

$$\lim_{\beta \to 1^-} \int_{\mathbb{H}^n} |u_\beta(r)|^q dV_h = \lim_{\beta \to 1^-} 2^n \omega_{n-1} \int_0^{\sqrt{1 + \beta}} (1 + z^2)^{-n} z^{n-1} dz$$

$$= 2^n \omega_{n-1} \int_0^{+\infty} (1 + z^2)^{-n} z^{n-1} dz = 2^{n-1} \omega_{n-1} \frac{\Gamma(n/2)\Gamma(n/2)}{\Gamma(n)} = \omega_{\mathbb{H},}$$

which also implies that $\int_{\mathbb{H}^n} |u_\beta(r)|^q dV_h$ is increasing with respect to $\beta$, $0 < \beta < 1$. According to (2.10), we have

$$\Delta_r^k u_\beta(r) + \sum_{m=0}^{k-1} a_{km} \Delta_r^m u_\beta(r) = b_k u_\beta(r)^{\frac{n-2k}{n-2k}},$$

i.e.,

$$P_k u_\beta(r) = b_k u_\beta^{q-1}, \quad 0 \leq r < +\infty,$$

where $P_k$ is the $k$-th standard operator on $\mathbb{H}^n$. Thus

$$P_k \left(\omega_n^{-1/q} u_\beta(r)\right) = \frac{1}{\Lambda_k} \left(\omega_n^{-1/q} u_\beta(r)\right)^{q-1}, \quad 0 \leq r < +\infty.$$  

It follows that for any $\beta \in (0, 1),$

$$\int_{\mathbb{H}^n} (P_k(\omega_n^{-1/q} u_\beta(r))) (\omega_n^{-1/q} u_\beta(r)) dV_h = \Lambda_k^{-1} \int_{\mathbb{H}^n} (\omega_n^{-1/q} u_\beta(r))^q dV_h$$

$$= \frac{1}{\Lambda_k} \left(\int_{\mathbb{H}^n} (\omega_n^{-1/q} u_\beta(r))^q dV_h\right)^{1/2} < \frac{1}{\Lambda_k}.$$
Since
\[ T_\iota u_\beta(r) = P_\kappa u_\beta(r) + \sum_{m=0}^{\iota} (\tau_{km} - a_{km}) \Delta^m_h (u_\beta(r)), \]
we obtain that for any \(0 < \beta < 1\),
\[
\int_{\mathbb{H}^n} \left( T_\iota \omega_n^{-1/q} u_\beta(r) \right) \left( \omega_n^{-1/q} u_\beta(r) \right) dV_h
\]
\[
\frac{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}
+ \frac{\int_{\mathbb{H}^n} \left( \sum_{m=0}^{\iota} (\tau_{km} - a_{km}) \Delta^m_h (\omega_n^{-1/q} u_\beta(r)) \right) \left( \omega_n^{-1/q} u_\beta(r) \right) dV_h}{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}
\]
\[
< \frac{1}{\Lambda_k} + \frac{\int_{\mathbb{H}^n} \left( \sum_{m=0}^{\iota} (\tau_{km} - a_{km}) \Delta^m_h (\omega_n^{-1/q} u_\beta(r)) \right) \left( \omega_n^{-1/q} u_\beta(r) \right) dV_h}{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}.
\]
(3.16)

Clearly, if \(i = 0\), then (3.16) implies
\[
\int_{\mathbb{H}^n} \left( T_\iota \omega_n^{-1/q} u_\beta(r) \right) \left( \omega_n^{-1/q} u_\beta(r) \right) dV_h
\]
\[
\frac{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}
< \frac{1}{\Lambda_k}.
\]

This contradicts (3.14) because we can choose a function \(v_\beta \in D(\mathbb{H}^n)\) (see below) such that
\[
\int_{\mathbb{H}^n} \left( T_\iota v_\beta \right) v_\beta dV_h
\]
\[
\frac{\|v_\beta\|_{L^q(\mathbb{H}^n)}^2}{\|v_\beta\|_{L^q(\mathbb{H}^n)}^2}
< \frac{1}{\Lambda_k}.
\]

This proves the conclusion of the theorem.

Now we consider the case \(i \geq 1\). It is obvious that
\[
\lim_{\beta \to 1^-} \frac{\partial^m u_\beta(r)}{\partial r^m} = 0, \quad m = 0, 1, \ldots, i - 1.
\]

Therefore, for \(\beta\) sufficiently close to 1, by (3.15)
\[
\int_{\mathbb{H}^n} \left( \sum_{m=0}^{i} (\tau_{km} - a_{km}) \Delta^m_h (\omega_n^{-1/q} u_\beta(r)) \right) \left( \omega_n^{-1/q} u_\beta(r) \right) dV_h
\]
\[
\frac{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}{\|\omega_n^{-1/q} u_\beta(r)\|_{L^q(\mathbb{H}^n)}^2}
= \sum_{m=0}^{i} (\tau_{km} - a_{km}) \int_{\mathbb{H}^n} |\Delta^{m/2}_h (\omega_n^{-1/q} u_\beta(r))|^2 dV_h + f_1(\beta),
\]

where

$$\lim_{\beta \to 1^-} \frac{f_1(\beta)}{\sum_{m=0}^i (\tau_{km} - a_{km}) \int_{\mathbb{H}^n} |\nabla_{h}^{m/2} (\omega_{-1/q} u_{\beta(r)})|^2 dV_{h}} = 0.$$  

For each $m \in \{0, 1, \ldots, i - 1\}$, it follows from Nirenberg’s lemma (see [N] or [ADN, Lemma 14.1]) that for any $\varrho > 0$, there exists a constant $c_m$ depending only on $\varrho$, $m$ and $i$ such that

$$\int_{\mathbb{H}^n} |\nabla_{h}^{m/2} u|^2 dV_{h} \leq \int_{\mathbb{H}^n} |\nabla_{h}^{i/2} u|^2 dV_{h} + c_m \int_{\mathbb{H}^n} |u|^2 dV_{h}$$

for all $u \in W^{k,2}(\mathbb{H}^n)$. We can choose $\varrho > 0$ small enough such that

$$\tau_{ki} - a_{ki} + \varrho \sum_{m=1}^{i-1} |\tau_{km} - a_{km}| < 0.$$

Thus

$$\int_{\mathbb{H}^n} (T_i(\omega_{-1/q} u_{\beta(r)}) (\omega_{-1/q} u_{\beta(r)}) dV_{h} \frac{\|\omega_{-1/q} u_{\beta(r)}\|_{L^q(\mathbb{H}^n)}}{\|\omega_{-1/q} u_{\beta(r)}\|_{L^2(\mathbb{H}^n)}}$$

$$< \frac{1}{\Lambda_k} + \left( \tau_{ki} - a_{ki} + \varrho \sum_{m=1}^{i-1} |\tau_{km} - a_{km}| \right) \int_{\mathbb{H}^n} |\nabla_{h}^{i/2} (\omega_{-1/q} u_{\beta(r)})|^2 dV_{h} + f_2(\beta)$$

$$+ c_i' \int_{\mathbb{H}^n} |\omega_{-1/q} u_{\beta(r)}|^2 dV_{h},$$

where

$$\lim_{\beta \to 1^-} \frac{f_2(\beta)}{\left( \tau_{ki} - a_{ki} + \varrho \sum_{m=1}^{i-1} |\tau_{km} - a_{km}| \right) \int_{\mathbb{H}^n} |\nabla_{h}^{i/2} (\omega_{-1/q} u_{\beta(r)})|^2 dV_{h}} = 0,$$

and $c_i'$ is a constant depending only on $\varrho$ and $i$.

Recall that under the conformal map $\sigma$, we have

$$\cosh r - 1 = \frac{2|x|^2}{1 - |x|^2}, \quad r \in [0, +\infty), \quad x \in \mathbb{B}_n.$$  

Since

$$u_{\beta}(r) = (1 - \beta^2) \frac{a - 2k}{4} \left[ \cosh r - \beta \right]^{k - \frac{n}{2}}$$

$$= (1 - \beta^2) \frac{a - 2k}{4} \left[ (1 - \beta) + (\cosh r - 1) \right]^{k - \frac{n}{2}},$$
we find that the function $u_\beta(r)$ can be written as:

$$u_\beta(r) = (1 - \beta^2)^{\frac{n-2k}{4}} \left[ (1 - \beta) + \frac{2|x|^2}{1 - |x|^2} \right]^{k - \frac{n}{2}}, \quad x \in \mathbb{B}_n.$$  

It follows that

$$\int_{\mathbb{H}^n} |u_\beta(r)|^2 dV_h = \int_{\mathbb{B}_n} (1 - \beta^2)^{\frac{n-2k}{2}} \left[ (1 - \beta) + \frac{2|x|^2}{1 - |x|^2} \right]^{2k - n} \left( \frac{2}{1 - |x|^2} \right)^n dx$$

$$= \omega_{n-1}(1 - \beta^2)^{\frac{n-2k}{2}} \int_0^1 \left[ (1 - \beta) + \frac{2s^2}{1 - s^2} \right]^{2k - n} \left( \frac{2}{1 - s^2} \right)^n s^{n-1} ds$$

$$= 2^n \omega_{n-1}(1 - \beta^2)^{\frac{n-2k}{2}} \int_0^1 [(1 - \beta)(1 - s^2) + 2s^2]^{2k - n} \left( \frac{1}{1 - s^2} \right)^{2k} s^{n-1} ds.$$

Making the change of variables $t = \sqrt{(1 + \beta)/(1 - \beta)} s$, we have that

$$\int_{\mathbb{H}^n} |u_\beta(r)|^2 dV_h$$

$$= 2^n \omega_{n-1}(1 - \beta^2)^{\frac{n-2k}{2}} \int_0^1 \sqrt{\frac{1 + \beta}{1 - \beta}} \left[ (1 - \beta)(1 + t^2) \right]^{2k - n} \left( \frac{1 - \beta^2}{1 - \beta t^2} \right)^{2k} \left( \frac{1 - \beta}{1 + \beta} \right)^{\frac{n}{2}} t^{n-1} dt$$

$$= 2^n \omega_{n-1}(1 - \beta^2)^{\frac{n-2k}{2}} \int_0^1 \sqrt{\frac{1 + \beta}{1 - \beta}} \frac{1 - \beta^2}{1 - \beta t^2} \frac{1 - \beta}{1 + \beta} \frac{t^{n-1} dt}{(1 + t^2)^{n-2k}}$$

Since

$$\frac{\partial u_\beta(r)}{\partial x_i} = (1 - \beta^2)^{\frac{n-2k}{4}} \left( k - \frac{n}{2} \right) \left[ (1 - \beta) + \frac{2|x|^2}{1 - |x|^2} \right]^{k - \frac{n}{2} - 1} \frac{4x_i}{(1 - |x|^2)^2},$$

it follows that

$$|\nabla_h u_\beta(r)|^2$$

$$= \left( \frac{1 - |x|^2}{2} \right)^4 \cdot (1 - \beta^2)^{\frac{n-2k}{2}} \left( k - \frac{n}{2} \right)^2 \left[ (1 - \beta) + \frac{2|x|^2}{1 - |x|^2} \right]^{2k - n - 2} \left( \frac{2}{1 - |x|^2} \right)^4 |x|^2,$$

so that

$$\int_{\mathbb{H}^n} |\nabla_h u_\beta(r)|^2 dV_h$$

$$= (1 - \beta^2)^{\frac{n-2k}{2}} \left( k - \frac{n}{2} \right)^2 \int_{\mathbb{B}_n} \left[ (1 - \beta) + \frac{2|x|^2}{1 - |x|^2} \right]^{2k - n - 2} |x|^2 \left( \frac{2}{1 - |x|^2} \right)^n dx$$

$$= 2^n (1 - \beta^2)^{\frac{n-2k}{2}} \left( k - \frac{n}{2} \right)^2 \omega_{n-1}$$

$$\times \int_0^1 \left[ (1 - \beta)(1 - s^2) + 2s^2 \right]^{2k - n - 2} \left( \frac{1}{1 - s^2} \right)^{2k - 2} s^{n+1} ds.$$
Again, by the substitution
\[ t = \sqrt{(1 + \beta)/(1 - \beta)} \, s, \]
we have
\[ \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} u_{\beta}(r)|^2 dV_{\mathbb{H}} = 2^n (k - \frac{n}{2})^2 \omega_{n-1} (1 - \beta^2)^{-k-1} (1 - \beta)^{2k} \]
\[ \times \int_{0}^{\sqrt{\frac{1 + \beta}{1 - \beta} \frac{1 - 1-\beta^2}{1 + \beta}}} \left( \frac{1}{1 - \frac{1-\beta^2}{1 + \beta}} \right)^{2k-2} \frac{t^{n+1} dt}{(1 + t^2)^{n-2k+2}}. \]

It follows from the dominated convergence theorem, if \( n > 4k - 2 \),
\[ \lim_{\beta \to 1^-} \int_{0}^{\sqrt{\frac{1 + \beta}{1 - \beta} \frac{1 - 1-\beta^2}{1 + \beta}}} \left( \frac{1}{1 - \frac{1-\beta^2}{1 + \beta}} \right)^{2k-2} \frac{t^{n+1} dt}{(1 + t^2)^{n-2k+2}} = 0, \]
while
\[ \lim_{\beta \to 1^-} \int_{0}^{\sqrt{\frac{1 + \beta}{1 - \beta} \frac{1 - 1-\beta^2}{1 + \beta}}} \left( \frac{1}{1 - \frac{1-\beta^2}{1 + \beta}} \right)^{2k-2} \frac{t^{n+1} dt}{(1 + t^2)^{n-2k+2}} = \int_{0}^{\infty} \frac{t^{n+1} dt}{(1 + t^2)^{n-2k+2}}. \]
The latter integral on the right-hand side is a finite positive constant for \( n > 4k - 2 \). On the other hand, since
\[ \lim_{\beta \to 1^-} \frac{2^n \omega_{n-1} (1 - \beta^2)^{-k-1} (1 - \beta^2)^{2k}}{2^n (k - \frac{n}{2})^2 \omega_{n-1} (1 - \beta^2)^{-k-1} (1 - \beta)^{2k}} = \frac{1}{(k - \frac{n}{2})^2}, \]
it follows that
\[ (3.19) \quad \lim_{\beta \to 1^-} \frac{\int_{\mathbb{H}^n} |u_{\beta}(r)|^2 dV_{\mathbb{H}}}{\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} u_{\beta}(r)|^2 dV_{\mathbb{H}}} = 0, \]
which implies
\[ (3.20) \quad \lim_{\beta \to 1^-} \frac{\int_{\mathbb{H}^n} |\omega_{n-1}^{-1/q} u_{\beta}(r)|^2 dV_{\mathbb{H}}}{\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} (\omega_{n-1}^{-1/q} u_{\beta}(r))|^2 dV_{\mathbb{H}}} = 0. \]

By taking \( m = 1 \) and replacing \( u \) by \( u_{\beta}(r) \) in (3.17), we have
\[ 1 \leq \theta \int_{\mathbb{H}^n} |\Delta_{\mathbb{H}}^{i/2} (\omega_{n-1}^{-1/q} u_{\beta}(r))|^2 dV_{\mathbb{H}} \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} (\omega_{n-1}^{-1/q} u_{\beta}(r))|^2 dV_{\mathbb{H}} + c_1 \int_{\mathbb{H}^n} |\omega_{n-1}^{-1/q} u_{\beta}(r)|^2 dV_{\mathbb{H}}. \]

Letting \( \beta \to 1^- \), we find by this and (3.20) that
\[ (3.21) \quad \lim_{\beta \to 1^-} \frac{\int_{\mathbb{H}^n} |\Delta_{\mathbb{H}}^{i/2} (\omega_{n-1}^{-1/q} u_{\beta}(r))|^2 dV_{\mathbb{H}}}{\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} (\omega_{n-1}^{-1/q} u_{\beta}(r))|^2 dV_{\mathbb{H}}} \geq \frac{1}{\theta}. \]
Combining (3.18), (3.20) and (3.21), we obtain that if $\tau_{ki} < a_{ki}$ and $n > 4k - 2$, then for $\beta$ sufficiently close to 1,

$$
\left(\tau_{ki} - a_{ki} + \varrho \sum_{m=0}^{i-1} |\tau_{km} - a_{km}| \right) \int_{\mathbb{H}^n} |\triangle_h^{i/2} (\omega_n^{-1/q} u_{\beta}(r))|^2 dV_h + c_i \int_{\mathbb{H}^n} |\omega_n^{-1/q} u_{\beta}(r)|^2 dV_h < 0.
$$

From (3.18), it follows that for $\beta < 1$ sufficiently close to 1,

$$
\int_{\mathbb{H}^n} \left( T_i (\omega_n^{-1/q} u_{\beta}(r)) \right) (\omega_n^{-1/q} u_{\beta}(r)) dV_h < \frac{1}{\Lambda_k} \|\omega_n^{-1/q} u_{\beta}(r)\|_{L^q(\mathbb{H}^n)}^2.
$$

Finally, for $\beta$ sufficiently close to 1, since $u_{\beta}(r) \in W^{k,2}(\mathbb{H}^n)$, we can choose a sequence of smooth functions $w_j$ with compact support in $\mathbb{H}^n$ such that

$$
\lim_{j \to +\infty} \|u_{\beta} - w_j\|_{W^{k,2}(\mathbb{H}^n)} = 0.
$$

Thus we can find an integer $j_0$ such that

$$
\int_{\mathbb{H}^n} \left( T_i (w_{j_0}) \right) (w_{j_0}) dV_h < \frac{1}{\Lambda_k} \|w_{j_0}\|_{L^q(\mathbb{H}^n)}^2.
$$

Let us denote $w_{j_0}$ by $v_\beta$. This is in contradiction with (3.14), which proves the theorem when $n > 4k - 2$. $\square$

**Remark 3.4.**

i) For order $k = 1$, the inequality (3.3) had been given by Hebey in $[H_1, H_3]$.

ii) As argument before, there is not extremal function for the sharp Sobolev inequality (3.2) (i.e., (3.3)) on $\mathbb{H}^n$.

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