Abstract. We develop a framework to analyse invariant decompositions of elements of tensor product spaces. Namely, we define an invariant decomposition with indices arranged on a simplicial complex, and which is explicitly invariant under a group action. We prove that this decomposition exists for all invariant tensors after possibly enriching the simplicial complex. As a special case we recover tensor networks with translational invariance and the symmetric tensor decomposition. We also define an invariant separable decomposition and purification form, and prove similar existence results. Associated to every decomposition there is a rank, and we prove several inequalities between them. For example, we show by how much the rank increases when imposing invariance in the decomposition, and that the tensor rank is the largest of all ranks. Finally, we apply our framework to nonnegative tensors, where we define a nonnegative and a positive semidefinite decomposition on arbitrary simplicial complexes with group action. We show a correspondence to the previous ranks, and as a very special case recover the nonnegative, the positive semidefinite, the completely positive and the completely positive semidefinite transposed decomposition.

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1. **Introduction**

Tensor products appear prominently in almost all areas of mathematics, theoretical physics and numerous other branches of science. Multilinear maps, higher-order derivatives and homogeneous polynomials can be seen as tensors, for example. In quantum mechanics, the state space of a multi-particle quantum system is modelled as a tensor product of the individual state spaces. Tensors, that is, elements of tensor product spaces, are also used in electrical engineering, psychometrics, data analysis (see [5] and references therein), and in relation to machine learning (see [13] and references therein), to cite a few examples.

It is a basic fact that every element of a tensor product space

\[ v \in V = V_0 \otimes \cdots \otimes V_n \]

can be expressed as a finite sum of elementary tensors

\[ v[0] \otimes \cdots \otimes v[n] \]

where \( v[i] \in V_i \) for \( i = 0, \ldots, n \). A tensor network, often used in quantum information theory and condensed matter physics [18], is a certain way of arranging the indices in this sum: they are chosen to reflect the physical arrangement of the individual degrees of freedom in a given quantum system. For example, the indices could be arranged in a one-dimensional circle as in

\[ v = \sum_{\alpha_0, \ldots, \alpha_n = 1}^r v[0]_{\alpha_0, \alpha_1} \otimes v[1]_{\alpha_1, \alpha_2} \otimes \cdots \otimes v[n-1]_{\alpha_{n-1}, \alpha_n} \otimes v[n]_{\alpha_n, \alpha_0}. \]

Alternatively, there could be a single joint index,

\[ v = \sum_{\alpha = 1}^r v[0]_{\alpha} \otimes v[1]_{\alpha} \otimes \cdots \otimes v[n]_{\alpha}. \]

In either case, the smallest possible such \( r \) defines the corresponding rank of the element \( v \).

Symmetries play a central role in theoretical physics and mathematics. Characterising the symmetries of a system is both of fundamental importance, as they reveal the conserved quantities, and of practical importance, as symmetric systems have fewer degrees of freedom, and thus allow for more efficient parametrisations. In this paper we focus on external symmetries. That is, we assume that a group \( G \) acts on the set \( \{0, 1, \ldots, n\} \), and we consider the induced linear action of \( G \) on \( V \), i.e.

\[ g : v[0] \otimes \cdots \otimes v[n] \mapsto v[g[0]] \otimes \cdots \otimes v[g[n]]. \]
We then say that \( v \in V \) has an external symmetry (given by \( G \)) if it is fixed by this action. In contrast, in internal symmetries one typically has a representation \( U_g : V_i \rightarrow V_i \) of a group \( G \), and one says that \( v \) has an internal global symmetry if \( (U_g)^\otimes v = v \), or an internal local symmetry if \( (U_g)^\otimes_l v = v \) for a certain subset \( l \) of subsystems, as in a lattice gauge theory. Internal symmetries have been characterised in the context of tensor networks, e.g., in \([19, 21]\).

In the context of the tensor decompositions considered above, if \( v \) has an external symmetry, it is desirable to find a decomposition that makes this symmetry explicit, that is, to find an invariant decomposition. For example, if the system is arranged in a one-dimensional circle with cyclic symmetry, an invariant decomposition would be of the form

\[
v = \sum_{\alpha_0, \ldots, \alpha_n = 1}^r v_{\alpha_0, \alpha_1} \otimes v_{\alpha_1, \alpha_2} \otimes \cdots \otimes v_{\alpha_n, \alpha_0}.
\]

This is known as the translationally invariant matrix product operator form, and the minimal such \( r \) is called the t. i. operator Schmidt rank [8]. Another example is a single joint index with full symmetry, in which an invariant decomposition would be of the form

\[
v = \sum_{\alpha = 1}^r v_{\alpha} \otimes v_{\alpha} \otimes \cdots \otimes v_{\alpha},
\]

which is known as a symmetric tensor decomposition, and the minimal such \( r \) is called the symmetric tensor rank [6].

In this paper we develop a theoretical framework to study invariant tensor decompositions and their corresponding ranks, which specialises in particular to the above ones. Namely, we consider elements of tensor product spaces and express them as a sum of elementary tensor factors. The indices in the sum are arranged over a simplicial complex \( \Omega \), which is a well-studied object in topology. In addition, we consider a group \( G \) acting on the simplicial complex. We then define a corresponding invariant tensor decomposition, called the \((\Omega, G)\)-decomposition, and an associated rank as the minimal number of terms of that decomposition, called rank\((\Omega, G)\). We then address the following questions: Does every invariant element have an \((\Omega, G)\)-decomposition? Or, more precisely, what are the conditions on \( \Omega \) and \( G \) that guarantee that every invariant element has an \((\Omega, G)\)-decomposition?

Our main result is that such an invariant decomposition always exists, provided that the indices are chosen and grouped in the right way (Theorem 13). More precisely, we show that every invariant element has an \((\Omega, G)\)-decomposition, but only after possibly raising the weights of the facets of the simplicial complex, and refining the group action. This is the reason why we have to work with weighted simplicial complexes, instead of just simplicial complexes.
In addition, we define the separable \((\Omega, G)\)-decomposition and the \((\Omega, G)\)-purification form as a generalisation of the (translationally invariant) separable decomposition and (t.i.) purification form studied in [8], and prove similar existence results. These decompositions incorporate different notions of positivity into the local vectors.

We also study inequalities between the ranks. First, we prove several inequalities between the rank, the separable rank and the purification rank on arbitrary simplicial complexes and with arbitrary group actions. Second, we study how these ranks are modified when changing the group action. For example, we study how much the rank increases when transforming a non-invariant decomposition into an invariant one. Third, we study how the ranks change when the simplicial complex is modified.

Finally, we apply our framework to entry-wise nonnegative tensors. First we define the nonnegative and the positive semidefinite decomposition on arbitrary simplicial complexes with group action. These specialise to the nonnegative, positive semidefinite, completely positive and completely positive semidefinite transposed decompositions when the simplicial complex is an edge, and the group is trivial or the cyclic group of order 2. We then prove a correspondence with the previous decompositions (Theorem 43), thereby generalising the results of [8], and use it to prove inequalities for these new ranks.

This paper is organized as follows. In Section 2 we introduce the relevant notions related to simplicial complexes and group actions. In Section 3 we define the \((\Omega, G)\)-decomposition and prove our main existence results, among which Theorem 13 is the most important one. In Section 4 we prove inequalities between the ranks, and in Section 5 we apply our framework to several decompositions of nonnegative tensors. We close with the conclusions and outlook in Section 6.

2. Weighted simplicial complexes and group actions

In this section we introduce the relevant notions of weighted simplicial complexes with group actions, which provides the underlying topological structure on which we will consider tensor decompositions. Specifically, in Section 2.1 we define weighted simplicial complexes, and in Section 2.2 group actions. We write \([n]\) for the set \(\{0, \ldots, n\}\) and \(\mathcal{P}_n\) for its power set \(\mathcal{P}([n])\) throughout this paper.

2.1. Weighted simplicial complexes. We start by defining weighted simplicial complexes (see, for example, [7] for more information). Examples are provided in Section 2.2 below.

**Definition 1.** (i) A weighted simplicial complex (wsc) on \([n]\) is a function \(\Omega: \mathcal{P}_n \rightarrow \mathbb{N}\) such that \(S_1 \subseteq S_2\) implies that \(\Omega(S_1)\) divides \(\Omega(S_2)\). A wsc is called a simplicial complex (sc) if it only takes values 0 and 1.
(ii) A set \( S \in \mathcal{P}_n \) with \( \Omega(S) \neq 0 \) is called a simplex of \( \Omega \). We will assume throughout that each singleton \( \{i\} \) is a simplex, and call the elements \( i \in [n] \) the vertices of the complex. A maximal simplex (with respect to inclusion) is called a facet of \( \Omega \). We denote by

\[
\mathcal{F} := \{ F \in \mathcal{P}_n \mid F \text{ facet of } \Omega \}
\]

the set of all facets, and for each vertex \( i \in [n] \) by

\[
\mathcal{F}_i := \{ F \in \mathcal{F} \mid i \in F \}
\]

the set of facets that \( i \) is contained in.

The restriction of \( \Omega \) to \( \mathcal{F} \) and \( \mathcal{F}_i \) makes these sets multisets, for which we use the notation

\[
\tilde{\mathcal{F}} \quad \text{and} \quad \tilde{\mathcal{F}}_i.
\]

Each facet \( F \) is contained in \( \tilde{\mathcal{F}} \) precisely \( \Omega(F) \)-many times. There is the canonical collapse map

\[
c: \tilde{\mathcal{F}} \to \mathcal{F}, \quad c: \tilde{\mathcal{F}}_i \to \mathcal{F}_i,
\]

mapping all copies of a facet to the underlying facet.

(iii) Two vertices \( i, j \) are neighbors, if

\[
\mathcal{F}_i \cap \mathcal{F}_j \neq \emptyset \quad \text{(equivalently if } \tilde{\mathcal{F}}_i \cap \tilde{\mathcal{F}}_j \neq \emptyset)\]

Two vertices are connected if one can be reached from the other through a sequence of neighborly points. The wsc is connected if any two vertices are connected. \( \triangleq \)

Remark 2. (i) A sc \( \Omega \) is the characteristic function of a subset \( \mathcal{A} \subseteq \mathcal{P}_n \). By the definition of a sc, \( \mathcal{A} \) is closed under passing to subsets. This is precisely how an (abstract) simplicial complex is usually defined.

(ii) A wsc is a special case of a multihypergraph [4], in which all simplices of a facet are contained, and where the multiplicities satisfy condition (i) of Definition 1. Intuitively, one can think of a wsc as a well-formed multihypergraph.

For example, a multigraph without self-loops is a wsc in which every vertex has value 1 and every edge has the value given by its multiplicity in the multigraph. This applies in particular to every simple graph. Another example is a hypergraph [4] in which all simplices of a facet are contained: this is a sc in which the corresponding simplices have weight 1.

In fact, our framework could be formulated with multihypergraphs (see also Remark 12), but the slightly less general notion of a wsc is easier to define, digest and work with, in our opinion. \( \triangleq \)

2.2. Group actions. We start with some general definitions concerning group actions, and then consider actions on weighted simplicial complexes. We assume that the reader is familiar with the usual concept of a group acting on a set, as defined in [16] for example. The identity element of a group \( G \) will always be denoted by \( e \).
Definition 3. (i) Let $G$ be a group acting on the sets $X$ and $Y$. A function $f : X \to Y$ is called $G$-linear if
\[ f(gx) = gf(x) \]
holds for all $x \in X, g \in G$. In case that $G$ acts trivially on $Y$, we instead call $f$ $G$-invariant.

(ii) If $G$ acts on $X$, then for any map $f : X \to Y$ and any $g \in G$ we define a new map
\[ g f : X \to Y \]
\[ x \mapsto f(g^{-1}x). \]
We have
\[ h(gf) = hg f \quad \text{and} \quad e f = f. \]
In particular, the mapping $f \mapsto gf$ is a bijection on the set of all functions from $X$ to $Y$. If $f$ is only defined on a subset $X' \subseteq X$, then $gf$ is defined on $gX' = \{gx \mid x \in X'\} \subseteq X$.

(iii) An action of $G$ on $X$ is free, if $\text{Stab}(x) = \{e\}$ for every $x \in X$, where
\[ \text{Stab}(x) := \{g \in G \mid gx = x\}. \]

(iv) An action of $G$ on $[n]$ is blending, if whenever $\{g_0, \ldots, g_n\} = [n]$ for certain $g_0, \ldots, g_n \in G$, then there is some $g \in G$ with $g_i = g_i$ for all $i = 0, \ldots, n$.

We now introduce the main notion of a group action on a wsc.

Definition 4. (i) A group action of $G$ on the wsc $\Omega$ consists of the following:

- An action of $G$ on $[n]$, such that $\Omega$ is $G$-invariant with respect to the induced action of $G$ on $\mathcal{P}_n$ (i.e. the action permutes vertices in such a way that simplices become simplices of the same weight). This then induces an action of $G$ on $\mathcal{F}$.
- An action of $G$ on $\widetilde{\mathcal{F}}$, such that the collapse map
\[ c : \widetilde{\mathcal{F}} \to \mathcal{F} \]
is $G$-linear (we also say the action of $G$ on $\widetilde{\mathcal{F}}$ refines the action of $G$ on $\mathcal{F}$).

(ii) An action of $G$ on the wsc $\Omega$ is called free if the action of $G$ on $\widetilde{\mathcal{F}}$ is free.

Remark 5. (i) Since a wsc has finitely many vertices, we will usually assume that the group $G$ is finite as well. Since groups always act by permutations, we could also assume that $G$ is a subgroup of the permutation group $S_{n+1}$, but sometimes it is much more convenient not to choose the latter representation.

(ii) A group action permutes the vertices $[n]$ of the wsc $\Omega$ in a way that preserves the weighted adjacency structure. The action induces an action
of $G$ on $\mathcal{F}$, where facets from the same orbit have the same weight. Each
$g \in G$ provides a weight-preserving bijection

$$g : F_i \mapsto F_{gi}$$

$$F \mapsto gF.$$

(iii) To obtain a group action on a wsc, one has to provide additional
information, namely how elements $g \in G$ permute the different copies of
facets when passing from a facet $F$ to the facet $gF$. Clearly, a group action
can always be refined (that is, defined compatibly on $\tilde{F}$), but there might be
more than one way to do so. For each vertex $i$ and each group element $g$, we
then obtain the following commutative diagram:

$$\begin{array}{ccc}
\tilde{F}_i & \xrightarrow{g} & \tilde{F}_{gi} \\
\downarrow c & & \downarrow c \\
F_i & \xrightarrow{F \mapsto gF} & F_{gi}
\end{array}$$

(iv) For a sc, the notion of group action precisely covers the usual notion
of a group acting by automorphisms.

(v) The notion of a blending group action just refers to the action of $G$
on $[n]$. It means that any possible permutation within any orbit is provided
by some group element.

(vi) The notion of a free group action on a wsc involves the action of $G$ on $\tilde{F}$. Note that an action of $G$ on $\Omega$ can be free, without the underlying action
of $G$ on $[n]$ or on $\mathcal{F}$ being free. In fact, any action of $G$ on $\Omega$ can be refined
to a free action, after possibly increasing the weights of the facets, as we
will show in Proposition 7. In combination with Theorem 13, this is why we
consider weighted simplicial complexes instead of just simplicial complexes
here.

(vii) An action $G$ on a set $X$ is free if and only if there is a $G$-linear map
$z : X \rightarrow G$
where $G$ acts on itself by left-multiplication (this action is clearly free). To
define $z$ for a free action, choose an element $x$ in each orbit and map $gx$ to
$g$. The other implication is clear. $\triangle$

**Example 6.** (i) The sc $\Sigma_n$ that maps each subset of $[n]$ to 1 is called the
$n$-simplex.

![n-simplex diagram]

It has only one (multi-)facet, i.e. $\mathcal{F} = \tilde{\mathcal{F}} = \{ [n] \}$. Any group action on $[n]$ is
a group action on $\Sigma_n$. The action of the full permutation group on $[n]$ is
blending. The only free action on $\Sigma_n$ is the action from the trivial group. However, if the weight of the (only) facet is raised to $|G|$, any action from $G$ on $[n]$ has a free refinement, as we will see in Proposition 7.

(ii) For $n \geq 1$, the complete graph $K_n$ is the sc with weight one on all sets $\{i, j\}$, for $i, j \in [n]$, and otherwise 0.

This sc has $\binom{n+1}{2}$ facets. Again, any group action on $[n]$ is a group action on $K_n$. The action of the full permutation group is blending but not free.

(iii) For $n \geq 1$, the line of length $n$ is the sc $\Lambda_n$ corresponding to the following graph:

The set $F = \tilde{F}$ has $n$ elements. The only non-trivial group action on $\Lambda_n$ is by the cyclic group with two elements $G = C_2$, where the generator inverts the order of vertices, i.e. vertex $i$ is sent to $n - i$. This action is free if and only if $n$ is even, and blending if and only if $n \leq 2$. For $n$ odd, the action admits a free refinement if the weight of the middle edge is increased to 2.

(iv) For $n \geq 3$, the circle of length $n$ is the sc $\Theta_n$ corresponding to the following graph:

It has $n$ facets. There is for example the canonical action of the cyclic group $G = C_n$. It is free but not blending.

(v) We have $\Sigma_1 = K_1 = \Lambda_1$, which is just the simple edge, having precisely one (multi)-facet. The only interesting group action is by $C_2 = S_2$, which is blending but not free (although the action on $\{0, 1\}$ is free!). The double edge is the wsc $\Delta$ on $P_1$ that assigns the value 1 to $\{0\}, \{1\}$ and the value 2 to $\{0, 1\}$.
In this case
\[ \mathcal{F}_0 = \mathcal{F}_1 = \mathcal{F} = \{\{0, 1\}\} \]
are singletons, but
\[ \tilde{\mathcal{F}}_0 = \tilde{\mathcal{F}}_1 = \tilde{\mathcal{F}} = \{a, b\} \]
are not. If \( C_2 \) also flips \( a \) and \( b \), the action is free.

(vi) Let \( G \neq \{e\} \) be a nontrivial finite group with generating set \( e \notin S \subseteq G \). We first define the Cayley graph
\[ \Gamma(G, S) = (V, E) \]
as the oriented graph with vertex set \( V = G \), where \((g, h) \in E\) is an edge if and only if there exists some \( s \in S \) with \( gs = h \). We now define the corresponding wsc \( \mathcal{C}(G, S) \), called the Cayley complex, on the vertex set \( V \) by assigning the value 1 to all vertices, and for \( g \neq h \in V \)
\[ \mathcal{C}(G, S)((g, h)) := \begin{cases} 
2 & : \{(g, h), (h, g)\} \subseteq E \\
1 & : \#((g, h), (h, g)) \cap E = 1 \\
0 & : \text{else.} 
\end{cases} \]
Since \( S \) is a generating set for \( G \), the wsc \( \mathcal{C}(G, S) \) is connected, and its facets are all of cardinality 2. Their weights indicate whether there are one or two oriented edges between the corresponding vertices in the Cayley graph. The set \( \tilde{\mathcal{F}}_g \) is identified with \( S \times \{\text{in}, \text{out}\} \), for each \( g \in V \), and the set \( \tilde{\mathcal{F}} \) is identified with \( E \).

The group \( G \) acts on itself by left-multiplication. This provides a free action of \( G \) on the wsc \( \mathcal{C}(G, S) \), by letting \( G \) act on the elements of \( \tilde{\mathcal{F}} = E \) entry-wise. Note that the double edge \( \Delta \) and the circle \( \Theta_n \) are special cases of this construction, where \( G = C_2 \) and \( C_n \), respectively, and the generating set is \( S = \{1\} \).

We now prove what we have already seen in Example 6 (i), (iii) and (v).

**Proposition 7.** Any action of the finite group \( G \) on the wsc \( \Omega \) has a free refinement after possibly increasing the weights of the facets of \( \Omega \).

**Proof.** Assume \( G = \{g_1, \ldots, g_r\} \) with \( r = |G| \). Let \( \tilde{\Omega} \) be the wsc obtained by multiplying the weights of all facets of \( \Omega \) by \( r \). Assume \( \Omega(F) = m \) for some \( F \in \mathcal{F} \). We denote the \( m \) copies of \( F \) in \( \tilde{\mathcal{F}} \) by \( F_1, \ldots, F_m \). For any \( g \in G \) we know that \( gF_1, \ldots, gF_m \) are the copies of \( gF \). Now label the \( rm \) copies of \( F \) in the new multiset \( \tilde{\mathcal{F}} \) by
\[ F_1^{g_1}, \ldots, F_r^{g_1}, \ldots, F_1^{g_r}, \ldots, F_m^{g_r}, \]
i.e. every copy \( F_i \in \tilde{\mathcal{F}} \) is replaced by \( r \) duplicates, indexed by the group elements. The collapse map \( \tilde{c} : \tilde{\mathcal{F}} \to \mathcal{F} \) factors through \( \tilde{\mathcal{F}} \) via the partial collapse map
\[ \tilde{c} : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}; F_i^g \mapsto F_i. \]
We now define
\[
g \cdot F^h_i := (gF_i)^{gh}
\]
as the \(gh\)-th duplicate of the facet \(gF_i\). This defines an action of \(G\) on \(\tilde{F}\) which makes \(\tilde{c}\) a \(G\)-linear map, i.e. the action refines the given action on \(\tilde{F}\). Since the action of \(G\) on itself by left-multiplication is free, this new action on \(\tilde{\Omega}\) is free.

Remark that we do not claim that this refinement is optimal. In the above construction, the weight of every facet is multiplied by the cardinality of \(G\), but there might another free refinement that increases less the multiplicity of each facet.

3. Invariant tensor decompositions and ranks

In this section we define and study several different tensor decompositions and tensor ranks on a wsc. Specifically, in Section 3.1 we prove the main result of this paper, namely the existence of invariant decompositions in many cases. In Section 3.2 we specialise to the separable decomposition, and in Section 3.3 to the purification form. We also prove subadditivity of the ranks and provide many examples along the way.

Throughout this section, we fix a wsc \(\Omega\) with an action from the group \(G\). For each \(i \in [n]\) we fix a \(\mathbb{C}\)-vector space \(V_i\) (called the *local vector space at site* \(i\)), and whenever \(i, j\) are in the same orbit, the spaces must coincide. Unless otherwise mentioned, we do not impose any further conditions on the spaces \(V_i\)—they can be, for example, infinite dimensional. We define the *global vector space* as
\[
V := V_0 \otimes \cdots \otimes V_n.
\]
Note that we always consider the algebraic tensor product, even if the spaces are infinite dimensional and have more structure, such as Hilbert spaces. So by the very definition, every element of \(V\) is a finite sum of elementary tensors.

The action of \(G\) on \([n]\) induces a linear action on \(V\), by permuting the tensor factors. An element \(v \in V\) is called \(G\)-invariant if it is invariant under this action. The subspace of invariant elements is denoted \(V_{\text{inv}}\).

For two sets \(X\) and \(Y\), the set \(Y^X\) contains, by definition, all functions from \(X\) to \(Y\). If \(X\) is finite, such a function is often written as the tuple of its values. For example, for any set \(I\) a function \(\alpha \in I^{\tilde{F}}\) could be written as a tuple with entries from \(I\), indexed over the finite set \(\tilde{F}\). However, the functional point of view turns out to be much more flexible and less technical.
in most of our proofs below. For example, when $\alpha$ is seen as a function, for $i \in [n]$ the restriction of $\alpha$ to $\widetilde{F}_i$ 

$$\alpha|_{\widetilde{F}_i} \in \mathcal{I}$$

is easily defined in the obvious way. We also call it the restriction of $\alpha$ to $i$ and write $\alpha_i$ instead. In tuple notation this means that we erase those entries of the tuple whose indices do not belong to $\widetilde{F}_i$, thus making it possibly shorter. We will stick to the functional viewpoint whenever possible, and use the tuple notation only in some explicit example.

3.1. The invariant decomposition. We now define the basic invariant tensor decomposition, called the $(\Omega,G)$-decomposition. Explicit examples of such decompositions are provided in Example 10 below. Since the definition might look quite abstract at the beginning, let us first explain the main idea behind it. We consider sums of elementary tensors, where these indices take finitely many values. The summation indices will be placed on the facets of $\Omega$. Each elementary tensor in such a sum is composed of vectors from the local spaces, and each local vector is indexed by just those facets that contain the corresponding vertex. The reader might want to have a look at Example 10 first to get an idea on how the decompositions look like.

**Definition 8.** (i) For $v \in V$, an $(\Omega,G)$-decomposition of $v$ consists of a finite set $I$ and families 

$$V[i] = (v[i])_{\beta \in I}$$

with all $v[i]_{\beta} \in V_i$, for all $i \in [n]$, such that:

(a) We have 

$$v = \sum_{\alpha \in I} v[0]_{\alpha[0]} \otimes v[1]_{\alpha[1]} \otimes \cdots \otimes v[n]_{\alpha[n]}.$$ 

(b) For all $i \in [n], g \in G$ and $\beta \in I_{\widetilde{F}_i}$ we have 

$$v[i]_{\beta} = v[g\beta]$$

where $g\beta$ is the function defined in Definition 3 (ii).

(ii) The smallest cardinality of an index set $I$ among all $(\Omega,G)$-decompositions of $v$ is called the $(\Omega,G)$-rank of $v$, denoted 

$$\text{rank}_{(\Omega,G)}(v).$$

(iii) For the trivial group action we call an $(\Omega,G)$-decomposition just $\Omega$-decomposition, and write $\text{rank}_\Omega(v)$ for the rank.

**Remark 9.** (i) The family $V[i]$ from Definition 8 (i) is called the local tensor at site $i$, and its elements are called the local vectors at site $i$. Condition (a) specifies how the local vectors need to be combined in order to obtain $v$. The wsc $\Omega$ hereby determines which indices to use at the different vertices, and thus how different sites interact locally. Condition (b) takes into account
invariance with respect to the group action, by specifying how local vectors along orbits must coincide. The \((\Omega,G)\)-rank expresses the minimal number of local vectors needed to express \(v\).

(ii) We adopt the convention to set \(\text{rank}_{(\Omega,G)}(v) = \infty\) if \(v\) does not admit an \((\Omega,G)\)-decomposition. \(\triangle\)

Example 10. (i) Consider the \(n\)-simplex \(\Sigma_n\), for \(n \geq 1\). Since 
\[
\tilde{F}_i = \tilde{F} = \{0, 1, \ldots, n\}
\]
is a singleton for each \(i \in [n]\), a \(\Sigma_n\)-decomposition only uses one index and is thus of the form
\[
v = \sum_{a=1}^{r} v_a^{[0]} \otimes \cdots \otimes v_a^{[n]}.
\]
Thus \(\text{rank}_{\Sigma_n}(v)\) is the smallest number of elementary tensors needed to obtain \(v\) as their sum, which is also known as the tensor rank of \(v\).

Now assume that the action of a group \(G\) on \([n]\) is transitive, i.e. there is only one orbit. Then condition (b) requires that the \((\Sigma_n,G)\)-decomposition of \(v\) uses the same local vectors everywhere. It is thus of the form
\[
\sum_{a=1}^{r} v_a \otimes \cdots \otimes v_a,
\]
which is also known as a symmetric tensor decomposition, and \(\text{rank}_{(\Sigma_n,G)}(v)\) is also called the symmetric tensor rank of \(v\) (see for example [6]). We will come back to this point in Remark 12.

(ii) Consider the complete graph \(K_3\). A \(K_3\)-decomposition of \(v\) is of the form 
\[
v = \sum_{i,j,k,l,m,n=1}^{r} v_{ijk}^{[0]} \otimes v_{ilm}^{[1]} \otimes v_{jln}^{[2]} \otimes v_{kmn}^{[3]}.
\]
If \(S_3\) is the full permutation group acting on \(\{0, 1, 2, 3\}\), a \((K_3,S_3)\)-decomposition of \(v\) is 
\[
v = \sum_{i,j,k,l,m,n=1}^{r} v_{ijk} \otimes v_{ilm} \otimes v_{jln} \otimes v_{kmn},
\]
with the additional property that the local tensor 
\[
(v_{ijk})_{i,j,k=1}^{r}
\]
is fully symmetric. Note that this decomposition reveals the same symmetry as (i), namely invariance under the full symmetry group \(S_3\), but the simplicial complex is different. This illustrates how in an \((\Omega,G)\)-decomposition both elements are important: the simplicial complex and the group action.

(iii) For \(n \geq 1\) consider \(\Lambda_n\), the line of length \(n\), where a \(\Lambda_n\)-decomposition of \(v\) has the form
\[
v = \sum_{a_0,\ldots,a_{n-1} = 1}^{r} v_{a_0}^{[0]} \otimes v_{a_{0},a_1}^{[1]} \otimes \cdots \otimes v_{a_{n-2},a_{n-1}}^{[n-1]} \otimes v_{a_{n-1}}^{[n]}.
\]
This is also called a matrix product operator form of \( v \), and the \( \Lambda_n \)-rank is also called the operator Schmidt rank—see for example [8] and references therein.

For \( n = 2 \), with action of \( C_2 \) (acting as \( 0 \mapsto 2 \), \( 1 \mapsto 1 \), \( 2 \mapsto 0 \)), a \( (\Lambda_2, C_2) \)-decomposition is

\[
v = \sum_{\alpha,\beta=1}^{r} v_\alpha \otimes w_{\alpha,\beta} \otimes v_\beta
\]

with the additional property that \( w_{\alpha,\beta} = w_{\beta,\alpha} \) for all \( \alpha, \beta \) (i.e. the local tensor at site 1 is symmetric).

(iv) For \( n \geq 3 \) consider the circle \( \Theta_n \) of length \( n \). A \( \Theta_n \)-decomposition has the form

\[
v = \sum_{\alpha_0, \ldots, \alpha_{n-1}=1}^{r} v_{\alpha_0,0}^{[0]} \otimes v_{\alpha_1,1}^{[1]} \otimes \cdots \otimes v_{\alpha_{n-1},0}^{[n-1]},
\]

This is almost the same as the decomposition from (iii), but with closed (i.e. periodic) boundary conditions. Now let the cyclic group \( C_n \) act on \( \Theta_n \). A \( (\Theta_n, C_n) \)-decomposition then is

\[
v = \sum_{\alpha_0, \ldots, \alpha_{n-1}=1}^{r} v_{\alpha_0,1}^{[0]} \otimes v_{\alpha_1,0}^{[1]} \otimes \cdots \otimes v_{\alpha_{n-1},1}^{[n-1]},
\]

which is called translational invariant matrix product operator form in [8], and the corresponding rank is called the t.i. operator Schmidt rank.

(v) On the simple edge \( \Sigma_1 = K_1 = \Lambda_1 \), the decomposition is

\[
v = \sum_{\alpha=1}^{r} v_\alpha^{[0]} \otimes v_\alpha^{[1]}
\]

and the \( C_2 \)-invariant decomposition is

\[
v = \sum_{\alpha=1}^{r} v_\alpha \otimes v_\alpha.
\]

On the double edge \( \Delta \), with free action as in Example 6 (v), the corresponding decompositions are

\[
v = \sum_{\alpha,\beta=1}^{r} v_{\alpha,\beta}^{[0]} \otimes v_{\beta,\alpha}^{[1]}
\]

and

\[
v = \sum_{\alpha,\beta=1}^{r} v_{\alpha,\beta} \otimes v_{\beta,\alpha}.
\]

The difference between the simple edge and the double edge has been observed and examined in [8], but without developing the theoretical foundations, as we do here.
Our first result on the existence of decompositions does not involve a group action yet:

**Theorem 11.** For every connected wsc $\Omega$ and every $v \in \mathcal{V}$, we have $\text{rank}_\Omega(v) < \infty$.

**Proof.** We start with a decomposition

$$v = \sum_{j \in \mathcal{I}} w[j]_0 \otimes \cdots \otimes w[j]_n$$

of $v$ as a finite sum of elementary tensors. For $i \in [n]$ and $\beta \in \tilde{\mathcal{F}}_i$ we then define

$$v[i]_{\beta} := \begin{cases} w[j]_i & \text{if } \beta \text{ takes the constant value } j \in \mathcal{I} \\ 0 & \text{else.} \end{cases}$$

Since $\Omega$ is connected, for $\alpha \in \tilde{\mathcal{F}}$ the functions $\alpha|_i$ are constant only if $\alpha$ is constant. This implies

$$\sum_{\alpha \in \tilde{\mathcal{F}}} v[\alpha|_0]_0 \otimes \cdots \otimes v[\alpha|_n]_n = \sum_{j \in \mathcal{I}} w[j]_0 \otimes \cdots \otimes w[j]_n = v,$$

which proves the claim. \qed

**Remark 12.** Clearly not every $v \in \mathcal{V}$ admits an $(\Omega, G)$-decomposition, since such a decomposition for example requires $G$-invariance:

$$g \cdot v = \sum_{\alpha \in \tilde{\mathcal{F}}} v[\alpha|_0]_0 \otimes \cdots \otimes v[\alpha|_n]_n$$

$$= \sum_{\alpha \in \tilde{\mathcal{F}}} v[(s^{-1}\alpha)|_0]_0 \otimes \cdots \otimes v[(s^{-1}\alpha)|_n]_n$$

$$= \sum_{\alpha \in \tilde{\mathcal{F}}} v[0]_0 \otimes \cdots \otimes v[n]_0$$

$$= \sum_{\alpha \in \tilde{\mathcal{F}}} v[0]_0 \otimes \cdots \otimes v[n]_0 = v.$$

For the third equation we have used condition (b) from Definition 8, and for the fourth that $s^{-1}\alpha$ runs through $\tilde{\mathcal{F}}$ if $\alpha$ does.

However, an $(\Omega, G)$-decomposition might imply an even stronger symmetry than $G$-invariance of $v$. In Example 10 (i) we have seen that all transitive group actions on the $n$-simplex lead to the same $(\Sigma_n, G)$-decomposition, which is the fully symmetric decomposition. So if, for example, $v$ is only invariant under the cyclic group $C_n$, it cannot have a $(\Sigma_n, C_n)$-decomposition.

One way around this problem would be to use the multiset $\tilde{\mathcal{S}}$ of all simplices instead of $\tilde{\mathcal{F}}$ in all definitions, since the action of $G$ on $\tilde{\mathcal{S}}$ determines the action on $[n]$. However, this would not allow us to cover the fully symmetric tensor decomposition from Example 10 (i) anymore.
Another way around this problem, which is the one we have chosen here, is to raise the weights of the facets so that the action becomes free (Proposition 7). Freeness suffices to prove that every invariant element has an invariant decomposition, as we will see in Theorem 13.

The following is our main result on the existence of invariant tensor decompositions. In combination with Proposition 7 it shows that a $G$-invariant decomposition exists for every invariant tensor, after possibly enriching the underlying topological structure (or, in a less complicated formulation, by using more indices at each site).

**Theorem 13 (Main result).** Let the action of $G$ on the connected wsc $\Omega$ be free. Then for every $v \in \mathcal{V}_{\text{inv}}$ we have $\text{rank}_{(\Omega,G)}(v) < \infty$. Moreover, for every decomposition

$$v = \sum_j w_j^{[0]} \otimes \cdots \otimes w_j^{[n]}$$

of $v$ as a sum of elementary tensors, there is an $(\Omega,G)$-decomposition of $v$, which uses only nonnegative multiples of the $w_j^{[i]}$ as its local vectors.

Note the theorem does not say anything about the value of $\text{rank}_{(\Omega,G)}(v)$. It says that, provided the action of $G$ on $\Omega$ is free, an $(\Omega,G)$-decomposition exists, and the proof provides a (generally non-optimal) way to obtain it.

**Proof.** Using Remark 5 (vii), we fix a $G$-linear map $z: \tilde{F} \to G$. For $v \in \mathcal{V}_{\text{inv}}$ we first choose an $\Omega$-decomposition, whose existence we have proven in Theorem 11 (we can choose the local vectors from any initial tensor decomposition, as it is clear from the proof). The local tensors from the $\Omega$-decomposition are denoted

$$W^{[i]} = \left( w^{[i]}_\beta \right)_{\beta \in \tilde{F}_i}$$

for $i \in [n]$. We define the new index set

$$\tilde{I} := I \times G$$

and consider the projection maps

$$p_1: \tilde{I} \to I, \quad p_2: \tilde{I} \to G.$$

For each $i \in [n]$ and $\beta \in \tilde{I}_i$, we now define

$$v^{[i]}_\beta := \begin{cases} w^{[i]}_{\beta(p_1 \circ \beta)} & : p_2 \circ \beta = (g^{-1}z)_i, \\ 0 & : \text{else.} \end{cases}$$

Note that if such a $g$ exists for $p_2 \circ \beta$, it is uniquely determined, since $z$ is $G$-linear and the action of $G$ on itself by left-multiplication is free. Also note that the new local vectors are all among the initial vectors.
The arising local tensors now fulfill (b) from Definition 8 (i), as one easily checks. We now compute
\[
\sum_{\tilde{\alpha} \in \tilde{I}} v^{[0]}_{\tilde{\alpha} | 0} \otimes \cdots \otimes v^{[n]}_{\tilde{\alpha} | n} = \sum_{z \in G^{\tilde{F}}} \sum_{\alpha \in \tilde{I}} u^{[g_0]}_{\alpha|g_0} \otimes \cdots \otimes u^{[g_n]}_{\alpha|g_n}.
\]
\[\forall i \exists g_i: z_i = (g_i^{-1} z)_i,\]
Since \(\Omega\) is connected, \(z\) is \(G\)-linear, and the action of \(G\) on itself is free, we immediately obtain \(g_i = g_j =: g\) for all \(i, j\), if \(z\) fulfills the above conditions. So for each fixed \(z\), the sum simplifies to
\[
\sum_{\alpha \in \tilde{I}} u^{[g_0]}_{\alpha|g_0} \otimes \cdots \otimes u^{[g_n]}_{\alpha|g_n}
\]
for some \(g\) depending on \(z\). But this is just \(g \cdot v\), and since \(v\) is \(G\)-invariant it is in fact \(v\). So the total sum yields a positive multiple of \(v\) (the sum is not empty, since at least \(z = z\) fulfills the conditions). Since a positive scaling factor can be absorbed into the local vectors, this proves the claim. □

**Remark 14.** (i) Freeness of the action of \(G\) on \(\Omega\) in Theorem 13 is necessary to obtain an \((\Omega, G)\)-decomposition for every invariant vector. For example, if a group action on the \(n\)-simplex is transitive on \([n]\), an \((\Omega, G)\)-decomposition requires full symmetry, which is stronger than \(G\)-invariance in general. However, such an action is never free.

(ii) Even if an \((\Omega, G)\)-decomposition exists for all invariant vectors, in general the local vectors cannot be chosen from any initial tensor decomposition. This also requires freeness of the action, that is, freeness is also necessary for the second statement of Theorem 13. One example is the simple edge with action from \(C_2\), for which an \((\Omega, G)\)-decomposition exists for each invariant vector (by Theorem 17 below), but one cannot choose the local vectors from any initial tensor decomposition. This is only possible on the double edge, where the action is indeed free. This will be studied in Section 3.2. △

**Example 15.** (i) The cyclic action of \(C_n\) on the circle \(\Theta_n\) is free, so every invariant vector admits a \((\Theta_n, C_n)\)-decomposition, or, in the words of [8], a translational invariant matrix product operator form.

(ii) More generally, whenever \(G\) is a finite group with generating set \(S\) as in Example 6 (vi), the action of \(G\) on \(C(G, S)\) is free, and the invariant decomposition thus exists for every invariant vector. △

We will prove another existence result for decompositions below, for which we need the following basic inequalities:

**Proposition 16.** Let \(G\) act on the connected wsc \(\Omega\). Then for all \(v, w \in V\) the following is true:

(i) \(\text{rank}_{(\Omega, G)}(v + w) \leq \text{rank}_{(\Omega, G)}(v) + \text{rank}_{(\Omega, G)}(w)\).
(ii) If all $V_i$ are algebras, then $\text{rank}_{(\Omega,G)}(vw) \leq \text{rank}_{(\Omega,G)}(v) \text{rank}_{(\Omega,G)}(w)$.

Proof. Both statements are clearly true if either $v$ or $w$ does not admit an $(\Omega,G)$-decomposition. So let

$$V[i] = \left( v[i]_{\beta} \right)_{\beta \in \mathcal{I}^i}, \quad W[i] = \left( w[i]_{\beta} \right)_{\beta \in \mathcal{J}^i},$$

be the local tensors from $(\Omega,G)$-decompositions of $v$ and $w$.

For (i) we take the direct sum of the local tensors to obtain an $(\Omega,G)$-decomposition for $v + w$. In detail, we define the new index set $L := \mathcal{I} \sqcup \mathcal{J}$ as the disjoint union of $\mathcal{I}$ and $\mathcal{J}$, set

$$x[i]_\beta := \begin{cases} v[i]_{\beta} : \beta \text{ takes values only in } \mathcal{I} \\ w[i]_{\beta} : \beta \text{ takes values only in } \mathcal{J} \\ 0 : \text{ else} \end{cases}$$

for $\beta \in \mathcal{L}^i$, and obtain new local tensors

$$X[i] := V[i] \oplus W[i] := \left( x[i]_\beta \right)_{\beta \in \mathcal{L}^i}.$$

Then condition (b) from Definition 8 is clearly fulfilled, and using connectedness of $\Omega$ one immediately checks

$$\sum_{\alpha \in \mathcal{L}^i} x[0]_\alpha \otimes \cdots \otimes x[n]_\alpha = v + w.$$ 

Since $|L| = |\mathcal{I}| + |\mathcal{J}|$, the statement is proven.

For (ii) we take the tensor product of the local tensors. In detail, consider the new index set $L := \mathcal{I} \times \mathcal{J}$ with the two projections $p_1 : L \to \mathcal{I}$, $p_2 : L \to \mathcal{J}$, and define

$$x[i]_\beta := v[i]_{p_1(\beta)} w[i]_{p_2(\beta)}$$

for $\beta \in \mathcal{L}^i$. The new local tensors

$$X[i] := V[i] \otimes W[i] := \left( x[i]_\beta \right)_{\beta \in \mathcal{L}^i}$$

then provide an $(\Omega,G)$-decomposition for $vw$, with $|L| = |\mathcal{I}| \cdot |\mathcal{J}|$. □

The following is our second result on the existence of invariant decompositions. It is a consequence of the symmetric decomposition of symmetric tensors in finite dimension [6].

**Theorem 17.** Let the action of $G$ on the connected wsc $\Omega$ be blending. Then for every $v \in \mathcal{V}_{nw}$ we have $\text{rank}_{(\Omega,G)}(v) < \infty$.

Proof. We start with a decomposition

$$v = \sum_{j \in \mathcal{I}} w[j]_0 \otimes \cdots \otimes w[j]_n$$
of $v$ as a finite sum of elementary tensors. We then choose complex numbers $d_i^{[\ell]}$ for $i=0,\ldots,n$ and $\ell=1,\ldots,r$ (for some large enough $r$), such that the following holds:

$$\sum_{\ell=1}^r d_i^{[\ell]} \cdots d_n^{[\ell]} = \begin{cases} 1: & \{i_0,\ldots,i_n\} = [n] \\ 0: & \text{else.} \end{cases}$$  (2)

This is in fact just a symmetric tensor decomposition of the symmetric tensor defined by the right hand side, so the existence of such numbers follows from [6, Lemma 4.2]. For $i \in [n], \ell = 1,\ldots,r$ and $\beta \in \mathcal{I}_{\tilde{F}}$ we now define

$$v_{i,\ell,\beta} := \begin{cases} \sum_{g \in G} d_i^{[\ell]} w_j^{[g]} : \beta \text{ takes the constant value } j \in \mathcal{I} \\ 0: & \text{else.} \end{cases}$$

For each fixed $\ell$, these vectors fulfill condition (b) of Definition 8 (i) and thus provide an $(\Omega,G)$-decomposition of a certain element $v_\ell \in \mathcal{V}$. We now compute

$$v_1 + \cdots + v_r = \sum_{\ell=1}^r \sum_{\alpha \in \mathcal{I}_{\tilde{F}}} v_{\ell,\alpha_{i_0}}^{[0]} \otimes \cdots \otimes v_{\ell,\alpha_n}^{[n]}$$

$$= \sum_{g_0,\ldots,g_n \in G} \sum_{\ell=1}^r d_i^{[g_0]} \cdots d_n^{[g_n]} \sum_{j \in \mathcal{I}} w_j^{[g_0]} \otimes \cdots \otimes w_j^{[g_n]}.$$

For the last equation we have again used that $\Omega$ is connected, so if $\alpha_i$ is constant for all $i$, then $\alpha$ is constant. By the choice of the $d_i^{[\ell]}$ and the fact that the group action is blending, this simplifies further (where $\sim$ stands for “some positive multiple of”):

$$v_1 + \cdots + v_r = \sum_{g_0,\ldots,g_n \in G, (g_0,\ldots,g_n) = [n]} \sum_{j \in \mathcal{I}} w_j^{[g_0]} \otimes \cdots \otimes w_j^{[g_n]}$$

$$\sim \sum_{g \in G} \sum_{j \in \mathcal{I}} w_j^{[g]} \otimes \cdots \otimes w_j^{[g]}$$

$$= \sum_{g \in G} g \cdot v$$

$$\sim v.$$

Now Proposition 16 (i) implies $\text{rank}_{(\Omega,G)}(v) \leq r|\mathcal{I}| < \infty$. \hfill \square

**Example 18.** Any fully symmetric tensor admits a symmetric tensor decomposition

$$v = \sum_{\alpha=1}^r v_\alpha \otimes \cdots \otimes v_\alpha.$$  

This is the statement of Theorem 17 for the full symmetric group acting on the $n$-simplex $\Sigma_n$. This is indeed not very surprising, since we have used
the result for finite-dimensional local spaces in our proof. But Theorem 17 shows that the result also holds for infinite-dimensional spaces.

Another decomposition for fully symmetric tensors comes from the complete graph $K_n$, as illustrated in Example 10 (ii) for $n = 3$. This decomposition also exists for invariant vectors, but is obviously weaker than the one on the simplex (it can be constructed in an obvious way from the simplex-decomposition). △

3.2. The invariant separable decomposition. Separability and its negation, entanglement, are central notions in quantum information theory. We will now formulate and study separable invariant tensor decompositions in our framework. Throughout this section we thus assume that each local space

$$V_i = B(H_i)$$

is the space of bounded operators on some (not necessarily finite dimensional) Hilbert space $H_i$, and again consider the global space

$$V = V_0 \otimes \cdots \otimes V_n = B(H_0) \otimes \cdots \otimes B(H_n).$$

**Definition 19.** (i) An element $\sigma \in V$ is separable if it admits a decomposition

$$\sigma = \sum_{j=1}^{r} \sigma_0^{[0]} \otimes \cdots \otimes \sigma_n^{[n]}$$

where all $\sigma_i^{[i]} \in B(H_i)$ are positive semidefinite (psd) operators.

(ii) For $\sigma \in V$, a separable $(\Omega, G)$-decomposition is an $(\Omega, G)$-decomposition

$$\sigma = \sum_{\alpha \in I} \sigma_0^{[0]} |_0 \otimes \cdots \otimes \sigma_n^{[n]} |_n$$

in which all local operators $\sigma_i^{[i]}$ are psd.

(iii) The smallest cardinality of the index set $I$ among all separable $(\Omega, G)$-decomposition of $\sigma$ is called the separable $(\Omega, G)$-rank, denoted

$$\text{sep-rank}_{(\Omega, G)}(\sigma).$$

(iv) In case of the trivial group action, we call a separable $(\Omega, G)$-decomposition just separable $\Omega$-decomposition, and write sep-rank$_{\Omega}(\sigma)$ for the separable rank.

**Remark 20.** (i) A separable $(\Omega, G)$-decomposition of $\sigma$ can clearly exist only if $\sigma$ is separable and $G$-invariant.

(ii) In the case of the line $\Lambda_n$, a separable $\Lambda_n$-decomposition is called the separable decomposition in [8]. For a circle $\Theta_n$ with cyclic group action $C_n$, a separable $(\Theta_n, C_n)$-decomposition is called a translational invariant separable decomposition in [8]. △

We now easily obtain our main result on the existence of separable $(\Omega, G)$-decompositions.
Theorem 21. Let the action of $G$ on the connected wsc $\Omega$ be free. Then for every separable $\sigma \in \mathcal{V}_{\text{inv}}$ we have $\text{sep-rank}_{(\Omega,G)}(\sigma) < \infty$.

Proof. This is a direct consequence of Theorem 13, when starting with a decomposition of $v$ as a sum of psd elementary tensors. □

Example 22. (i) For every wsc $\Omega$, every separable $v \in \mathcal{V}$ has a separable $\Omega$-decomposition. This is true since the action of the trivial group is free.

(ii) On the circle $\Theta_n$ (for $n \geq 3$), every cyclically invariant and separable $\rho$ admits a separable $(\Theta_n, C_n)$-decomposition.

(iii) On the simple edge with action from $C_2$, not every separable $v \in \mathcal{V}_{\text{inv}}$ admits a separable $(\Lambda_1, C_2)$-decomposition, which would be of the form

$$\sigma = \sum_{\alpha=1}^{r} \sigma_{\alpha} \otimes \sigma_{\alpha}$$

with all $\sigma_{\alpha}$ psd. This was shown in [8], using the fact that not every symmetric and entry-wise nonnegative matrix has a completely positive factorization (this also follows from the results in Section 5). This shows that, even if an $(\Omega, G)$-decomposition exists for all invariant vectors, the second statement of Theorem 13 (concerning how the local vectors can be chosen) needs freeness of the action. On the double edge, the action is free and the separable decomposition thus exists—namely, it is of the form

$$\sigma = \sum_{\alpha, \beta=1}^{r} \sigma_{\alpha,\beta} \otimes \sigma_{\beta,\alpha}.$$ 

(iv) More generally, the separable decomposition exists for each invariant separable tensor on the Cayley complex $\mathcal{C}(G, S)$ from Example 6 (vi). △

Proposition 23. Let $G$ act on the connected wsc $\Omega$. Then for $\rho, \sigma \in \mathcal{V}$ we have

$$\text{sep-rank}_{(\Omega,G)}(\rho + \sigma) \leq \text{sep-rank}_{(\Omega,G)}(\rho) + \text{sep-rank}_{(\Omega,G)}(\sigma).$$

Proof. Just follow the proof of Proposition 16 (i). □

Remark 24. Instead of a separable $(\Omega, G)$-decomposition, one could define decompositions where the local vectors must be taken from certain specified cones in each space $\mathcal{V}_i$. The separable decomposition just corresponds to the case of the cone of psd operators. Theorem 13 ensures that such invariant decompositions exist for all invariant elements that admit a standard tensor decomposition with such vectors, since scaling with positive reals is compatible with cones. In Section 5, when defining the nonnegative decomposition for finite-dimensional tensors, we will use make use of this fact. △

3.3. The invariant purification form. The separable $(\Omega, G)$-decomposition, which reveals the positivity of the element, exists only for separable elements. The purification form is another decomposition that reveals the
positivity of the element, and which exists for all positive elements. We will now introduce an \((\Omega, G)\)-purification form and study it in our framework. As in the last section we assume that each local space

\[ \mathcal{V}_i = B(\mathcal{H}_i) \]

is the space of bounded operators on some Hilbert space \(\mathcal{H}_i\). We again consider the global space

\[ \mathcal{V} = \mathcal{V}_0 \otimes \cdots \otimes \mathcal{V}_n = B(\mathcal{H}_0) \otimes \cdots \otimes B(\mathcal{H}_n) \subseteq B(\mathcal{H}_0 \otimes \cdots \otimes \mathcal{H}_n). \]

With the inclusion on the right we can define what it means for an element from \(\mathcal{V}\) to be psd. We will denote the set of bounded linear operators from \(\mathcal{H}_i\) to \(\mathcal{H}'_i\) by \(B(\mathcal{H}_i, \mathcal{H}'_i)\).

**Definition 25.** (i) For \(\sigma \in \mathcal{V}\) an \((\Omega, G)\)-purification is an element 

\[ \xi \in B(\mathcal{H}_0, \mathcal{H}'_0) \otimes \cdots \otimes B(\mathcal{H}_n, \mathcal{H}'_n) \]

with

\[ \sigma = \xi^* \xi \quad \text{and} \quad \operatorname{rank}_{(\Omega, G)}(\xi) < \infty, \]

where \(*\) indicates the adjoint. Here, the \(\mathcal{H}'_i\) can be arbitrary Hilbert spaces.

(ii) The smallest \((\Omega, G)\)-rank among all \((\Omega, G)\)-purifications of \(v\) is called the \((\Omega, G)\)-purification rank of \(\sigma\), denoted

\[ \operatorname{puri-rank}_{(\Omega, G)}(\sigma). \]

(iii) In case of the trivial group action, we again just say \(\Omega\)-purification and \(\Omega\)-purification rank, denoted \(\operatorname{puri-rank}_\Omega(\sigma)\).

\[ \triangle \]

**Remark 26.** An \((\Omega, G)\)-purification of \(\sigma\) can clearly exist only if \(\sigma\) is positive semidefinite and \(G\)-invariant. Positive semidefiniteness is obvious, since Hermitian squares are always psd. Invariance follows from the fact that the set of invariant elements is a \(*\)-subalgebra of \(\mathcal{V}\).

We now easily obtain a result on the existence of invariant purifications in many cases:

**Theorem 27.** Let \(G\) act on the wsc \(\Omega\), and assume \(\operatorname{rank}_{(\Omega, G)}(\xi) < \infty\) holds for every \(\xi \in \mathcal{V}_\text{inv}\). Further assume that all \(\mathcal{H}_i\) are finite-dimensional. Then for every positive semidefinite \(\sigma \in \mathcal{V}_\text{inv}\) we have \(\operatorname{puri-rank}_{(\Omega, G)}(\sigma) < \infty\).

**Proof.** If all \(\mathcal{H}_i\) are finite dimensional, then \(\mathcal{V} = B(\mathcal{H}_0 \otimes \cdots \otimes \mathcal{H}_n)\), and thus the positive semidefinite operator \(\sigma \in \mathcal{V}\) admits a (unique) positive semidefinite square root \(\xi \in \mathcal{V}\). This \(\xi\) is in fact a polynomial expression in \(\sigma\), and thus also \(G\)-invariant. By assumption, \(\xi\) admits an \((\Omega, G)\)-decomposition and is thus an \((\Omega, G)\)-purification of \(\sigma\). \(\square\)

Note that the proof uses the square root of \(\sigma\), which is a special case of a purification (see also [8]).
Proposition 28. Let $G$ act on the connected wsc $\Omega$. Then for $\rho, \sigma \in V$ we have

$$\text{puri-rank}_{(\Omega,G)}(\rho + \sigma) \leq \text{puri-rank}_{(\Omega,G)}(\rho) + \text{puri-rank}_{(\Omega,G)}(\sigma).$$

Proof. We can assume that both $\rho$ and $\sigma$ admit $(\Omega,G)$-purifications

\begin{align*}
\xi \in & \mathcal{B}(\mathcal{H}_0, \mathcal{H}'_0) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n, \mathcal{H}'_n) \\
\chi \in & \mathcal{B}(\mathcal{H}_0, \mathcal{H}''_0) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n, \mathcal{H}''_n).
\end{align*}

We understand both $\xi$ and $\chi$ as elements from

\begin{align*}
\mathcal{B}(\mathcal{H}_0, \mathcal{H}'_0 \times \mathcal{H}''_0) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n, \mathcal{H}'_n \times \mathcal{H}''_n),
\end{align*}

by letting the local operators from $(\Omega,G)$-decompositions map to the first/second components, respectively. This way we obtain

\begin{align*}
(\xi + \chi)^* (\xi + \chi) &= \xi^* \xi + \chi^* \chi = \rho + \sigma,
\end{align*}

and to $\xi + \chi$ we can apply Proposition 16 (i). \hfill \Box

4. SOME INEQUALITIES

In this section we derive some inequalities, first between the several ranks we have introduced (Section 4.1), then between different group actions on the same complex (Section 4.2), and finally between different complexes (Section 4.3).

4.1. Inequalities between different ranks. To compare the different ranks, we fix an action of $G$ on the connected wsc $\Omega$. We further assume that $V_i = \mathcal{B}(\mathcal{H}_i)$ for each $i \in [n]$. The following statement is a generalisation of the results in [8]:

Proposition 29. For each $\sigma \in V$ we have

\begin{enumerate}
  \item \text{rank}_{(\Omega,G)}(\sigma) \leq \text{sep-rank}_{(\Omega,G)}(\sigma)
  \item \text{puri-rank}_{(\Omega,G)}(\sigma) \leq \text{sep-rank}_{(\Omega,G)}(\sigma)
  \item \text{rank}_{(\Omega,G)}(\sigma) \leq \text{puri-rank}_{(\Omega,G)}(\sigma)^2.
\end{enumerate}

Proof. Since a separable $(\Omega,G)$-decomposition is a special case of an $(\Omega,G)$-decomposition, (i) is clear. For (ii) assume a separable $(\Omega,G)$-decomposition for $\sigma$ over the index set $I$ exists. Denote the local psd operators by $\sigma^{[i]} \in V_i$ and define

$$\tau^{[i]}_\beta := \sqrt{\sigma^{[i]}_\beta} \in V_i.$$

Then consider the bounded operator

$$\xi^{[i]}_\beta : \mathcal{H}_i \rightarrow \mathcal{H}'_i := \mathcal{H}_i \times \cdots \times \mathcal{H}_i$$

$$h \mapsto (0, \ldots, 0, \tau^{[i]}_\beta h, 0, \ldots, 0)$$
where the product runs over \( \mathcal{I}' \), and \( \tau^i_\beta \) appears in the entry indexed by \( \beta \). These local operators fulfill (b) from Definition 8 and thus provide an \((\Omega, G)\)-decomposition of some \( \xi \in \mathcal{V} \), with \( \operatorname{rank}_{(\Omega, G)}(\xi) \leq |\mathcal{I}'| \). By construction we have

\[
\left( \xi^i_\beta \right)^* \xi^i_\gamma = \delta_{\beta, \gamma} \cdot \sigma^i_\beta
\]

where \( \delta_{\beta, \gamma} \) is the Kronecker delta. This immediately implies \( \xi^* \xi = \sigma \), so we obtain \( \operatorname{puri-rank}_{(\Omega, G)}(\sigma) \leq |\mathcal{I}'| \), the desired result. For (iii) we let \( \xi \) be an \((\Omega, G)\)-purification of \( \sigma \) and compute

\[
\operatorname{rank}_{(\Omega, G)}(\sigma) = \operatorname{rank}_{(\Omega, G)}(\xi^* \xi) \leq \operatorname{rank}_{(\Omega, G)}(\xi^*) \operatorname{rank}_{(\Omega, G)}(\xi) = \operatorname{rank}_{(\Omega, G)}(\xi)^2
\]

where we use the construction from Proposition 16 (ii).

\section*{Remark 30}

(i) The purification rank cannot be upper bounded by a function of the rank alone, in general. This happens already in the case \( \Lambda_1 \), the simple edge, without group action. Similarly, the separable rank cannot be upper bounded by a function of the purification rank, and thus of the rank, in general. This is true on the simple edge again, with and without a group action. This was shown in [8, 9, 14], using a connection to factorisations of nonnegative matrices, which is generalised in Theorem 43 below.

(ii) There exist upper bounds on the different ranks in terms of the dimension of the global space \( \mathcal{V} \). We refer the reader to [8]; the generalisations to the more general framework from this work are straightforward.

\section*{4.2. Changing the group}

We now compare the ranks with respect to different group actions. So throughout this section we fix an action of \( G \) on the connected wsc \( \Omega \), and let \( H \subseteq G \) be a subgroup. Then the restricted action is an action of \( H \) on \( \Omega \), and we can compare the ranks with respect to \( G \) and \( H \). Since an \((\Omega, G)\)-decomposition is also an \((\Omega, H)\)-decomposition, we clearly have

\[
\operatorname{rank}_{(\Omega, H)}(v) \leq \operatorname{rank}_{(\Omega, G)}(v)
\]

for all \( v \in \mathcal{V} \), and the same is true for the separable rank and the purification rank. The first nontrivial result is about free actions as in Theorem 13:

\textbf{Proposition 31.} Let the action of \( G \) on \( \Omega \) be free and let \( H \) be a normal subgroup of \( G \). Then for every \( G \)-invariant \( v \in \mathcal{V} \) we have

\[
\operatorname{rank}_{(\Omega, G)}(v) \leq |G/H| \cdot \operatorname{rank}_{(\Omega, H)}(v),
\]

and in particular

\[
\operatorname{rank}_{(\Omega, G)}(v) \leq |G| \cdot \operatorname{rank}_\Omega(v).
\]

The same inequalities also hold for the separable rank.

\textbf{Proof.} The proof is almost the same as the one of Theorem 13. This time start with an \((\Omega, H)\)-decomposition of \( v \), define \( \mathcal{I}' = \mathcal{I} \times G/H \) and use the \( G \)-linear map

\[
\zeta': \tilde{\mathcal{F}} \xrightarrow{\mathcal{I}' \cdot \mathcal{I}} G \xrightarrow{\mathcal{I}} G/H
\]
instead of \( z \), where \( \text{pr} \) denotes the canonical projection map. All constructions are well-defined since \( H \) is a normal subgroup, and result in an \((\Omega, G)\)-decomposition of \( v \). From

\[|\tilde{I}| = |I \times G/H| = |I| \cdot |G/H|\]

the result follows. \( \square \)

**Example 32.** (i) For cyclically invariant elements on the circle \( \Theta_n \) we obtain that the \((\Theta_n, C_n)\)-rank is always at most \( n \) times the \( \Theta_n \)-rank. This statement is given in [8, Proposition 60].

(ii) More generally, for a finite group \( G \) with generating set \( S \) as in Example 6 (vi), and for \( G \)-invariant elements on \( C(G, S) \), we obtain that the \( G \)-invariant rank is always at most \( |G| \) times the rank. \( \triangle \)

The following result concerns blending group actions and the trivial subgroup. For the proof we need a strengthening of the property of blending, called strong blending. We say that the action of \( G \) on \( \Omega \) is strongly blending if whenever \( \{g_0, \ldots, g_n\} = [n] \) there exists some \( g \in G \) such that \( g_i = g_i \) and \( g_i \) act identically on \( \tilde{F}_i \), for all \( i \in [n] \). On the \( n \)-simplex, for example, this is equivalent to blending.

**Proposition 33.** There is a function \( C : \mathbb{N} \to \mathbb{N} \) such that whenever the action of a group \( G \) is strongly blending on the wsc \( \Omega \), we have

\[\text{rank}_{(\Omega, G)}(v) \leq C(n) \cdot \text{rank}_\Omega(v)\]

for all \( G \)-invariant \( v \in V \).

Recall that the \( n \) on the right hand side of the equation refers to the vertex set of \( \Omega \), which is \([n]\).

**Proof.** The proof is similar to the one of Theorem 17. One starts with an \( \Omega \)-decomposition of \( v \), with local vectors \( w^{[i]}_\beta \), and defines

\[ u^{[i]}_{\ell, \beta} := \sum_{g \in G} d^{[g]}_\ell w^{[g]}_\beta, \]

with the same numbers \( d^{[g]}_\ell \) as in the proof of Theorem 17. The rest of the proof is similar. Note that to replace

\[ w^{[g_i]}_{g_i(\alpha_i)} \]

in the last step of the proof, we need strong blending of the action (this is also why the proof of Theorem 17 did not start with an \( \Omega \)-decomposition of \( v \), but with a decomposition as a sum of elementary tensors).

Note that \( C(n) \) is called \( r \) in the proof of Theorem 17, and is simply the symmetric tensor rank of the tensor defined in the right hand side of (2). \( \square \)

**Remark 34.** (i) For the \( n \)-simplex \( \Sigma_n \) and the full permutation group \( G \), the \( \Sigma_n \)-rank is the tensor rank and the \((\Sigma_n, G)\)-rank is the symmetric tensor rank. Until recently is was unknown whether these two ranks always coincide
for symmetric tensors (this was known as Comon’s conjecture). It was then shown in [20] that the symmetric tensor rank can be strictly larger than the tensor rank. It seems that not much is known about bounds of the symmetric tensor rank in terms of the tensor rank in general. Proposition 33 applies to this case and gives a bound on the symmetric tensor rank in terms of the tensor rank and \(n\). For example, in the case \(n = 2\) (i.e. three-partite tensors) this bound is \(C(2) = 3\), as one easily checks.

We conjecture that \(C(n) = n + 1\) for all \(n\). In view of [23] it would be enough to show that the tensor rank of tensor defined on the right hand side of (2) is at most \(n + 1\). We have verified this with Mathematica for up to \(n = 9\).

(ii) It is unclear whether Proposition 33 also holds for the purification rank. On the other hand, it clearly does not hold for the separable rank, since a separable \((\Omega, G)\)-decomposition might not even exist for a \(G\)-invariant element, even if a separable \(\Omega\)-decomposition exists.

\(\triangle\)

4.3. Changing the simplicial complex. Throughout this section let

\(\Omega, \Psi : \mathcal{P}_n \to \mathbb{N}\)

be two weighted simplicial complexes on \([n]\). We want to compare the \(\Omega\)-rank to the \(\Psi\)-rank of elements \(v \in \mathcal{V}\). For simplicity, we do not employ a group action here. The following general construction will be used in all of the below results. It is in essence a generalisation of the idea of the proof of Theorem 11.

Construction 35. (i) We denote by \(\tilde{\mathcal{F}}(\Omega)\) and \(\tilde{\mathcal{F}}(\Psi)\) the multisets of facets of \(\Omega\) and \(\Psi\), respectively. Assume we are given \(v \in \mathcal{V}\) and a \(\Psi\)-decomposition using local vectors \(w_i^{[\beta]} \in \mathcal{V}_i\) for \(i \in [n]\) and

\[\beta : \tilde{\mathcal{F}}(\Psi)_i \to \mathcal{I}.

We want to turn this into an \(\Omega\)-decomposition of \(v\) while keeping track of the index set. To this end, we assume there is an index set \(\mathcal{J}\) and maps \(\pi, \pi_i\) that make each of the following diagrams commute:

\[
\begin{array}{ccc}
\mathcal{J} & \overset{\pi}{\longrightarrow} & \tilde{\mathcal{F}}(\Psi) \\
\downarrow & & \downarrow \\
\mathcal{J} \tilde{\mathcal{F}}(\Omega)_i & \overset{\pi_i}{\longrightarrow} & \tilde{\mathcal{F}}(\Psi)_i
\end{array}
\]

In addition, we assume:

- \(\pi_{\ast}\) and \(\pi_i\) are defined on subsets \(\mathcal{D}\) and \(\mathcal{D}_i\), respectively.
- For \(\alpha \in \mathcal{J} \tilde{\mathcal{F}}(\Omega)\) we have: \(\alpha \in \mathcal{D} \iff \alpha_i \in \mathcal{D}_i\) for all \(i \in [n]\).
- \(\pi : \mathcal{D} \to \tilde{\mathcal{F}}(\Psi)\) is surjective and all fibers (that is, preimages of single elements) have the same cardinality.
We will see below that these properties can often be found. So, given this, for $\beta: \tilde{F}(\Omega)_i \to J$ we define

$$v^{[i]}_\beta := \begin{cases} w_{\pi_1(\beta)}^{[i]} & : \beta \in D_i \\ 0 & : \beta \notin D_i. \end{cases}$$

We now compute

$$\sum_{\alpha \in J} v^{[0]}_\alpha \cdot \cdots \cdot v^{[n]}_\alpha = \sum_{\alpha \in D} w^{[0]}_{\pi_0(\alpha_{i_0})} \otimes \cdots \otimes w^{[n]}_{\pi_n(\alpha_{i_n})} \sim \sum_{\alpha \in I} w^{[0]}_{\alpha_{i_0}} \otimes \cdots \otimes w^{[n]}_{\alpha_{i_n}} = v.$$

For the first equation we have used that $\alpha \in D$ if and only if all $\alpha_{i_1} \in D_i$. For the second equation we have used surjectivity of $\pi$, and that all fibers have the same cardinality. We have thus provided an $\Omega$-decomposition of $v$ over the index set $J$. Note that the local vectors from the arising $\Omega$-decomposition are among the ones from the initial $\Psi$-decomposition. So this construction also transforms a separable $\Psi$-decomposition into a separable $\Omega$-decomposition.

(ii) A special case of the construction in (i) is the following. Assume there is a set $X$ and compatible embeddings

$$\tilde{F}(\Psi) \hookrightarrow X \times \tilde{F}(\Omega)$$

For any index set $I$ define

$$J := I^X$$

and obtain an induced commutative diagram of restriction maps

$$\begin{array}{ccc} J\tilde{F}(\Omega) & \xrightarrow{\pi} & I\tilde{F}(\Psi) \\
\downarrow & & \downarrow \\
J\tilde{F}(\Omega)_i & \xrightarrow{\pi_i} & I\tilde{F}(\Psi)_i. \end{array}$$

The above conditions are then obviously fulfilled. Note that

$$|J| \leq |I|^{|X|}$$

holds in this case.

$\triangle$

**Proposition 36.** If $\Omega$ is connected, then for every other wsc $\Psi$ on $[n]$ and every $v \in \mathcal{V}$ we have

$$\text{rank}_\Omega(v) \leq \text{rank}_\Psi(v)|\tilde{F}(\Psi)|.$$

The same is true for the separable rank and the purification rank.
Proof. We apply Construction 35 (i) with \( J := \mathcal{I} \tilde{F}(\Psi) \), \( D \) and \( D_i \) the sets of constant mappings, and \( \pi, \pi_i \) the mappings that send a constant function to its image. All conditions from Construction 35 (i) are easily checked to hold. From
\[
|J| = |I| \tilde{F}(\Psi)
\]
the result follows.

Example 37. For every connected wsc \( \Omega \) we have
\[
\text{rank}_{\Omega}(v) \leq \text{rank}_{\Sigma_{n}}(v).
\]
This is clear since \( \tilde{F}(\Sigma_{n}) \) is a singleton. The rank on the right is just the usual tensor rank. This is exactly what was shown in the proof of Theorem 11. \( \triangle \)

One can improve upon Proposition 36 if the two complexes are not completely independent of each other. We first consider the case \( \Psi = m\Omega \) for some \( m \in \mathbb{N} \).

**Proposition 38.** For every wsc \( \Omega \), \( m \in \mathbb{N} \) and \( v \in \mathcal{V} \) we have
\[
\text{rank}_{m\Omega}(v) \leq \left\lceil \text{rank}_{\Omega}(v)^{1/m} \right\rceil \quad \text{and} \quad \text{rank}_{\Omega}(v) \leq \text{rank}_{m\Omega}(v)^m.
\]
The same is true for the separable rank and the purification rank.

**Proof.** The multiset \( \tilde{F}(m\Omega) \) arises from \( \tilde{F}(\Omega) \) by raising the multiplicity of each element by a factor of \( m \). So the first inequality is obtained via Construction 35 (i) by choosing an index set \( J \) with \( I \hookrightarrow J \) and taking as \( D, D_i \) those functions that only take values in \( I \). The second inequality is obtained via Construction 35 (ii) by choosing \( X \) as a set with \( m \) elements. \( \square \)

Finally, for the next result, let \( G \neq \{e\} \) be a finite group with two generating sets \( S, T \) as in Example 6 (vi), giving rise to the Cayley complexes \( \mathcal{C}(G,S) \) and \( \mathcal{C}(G,T) \).

**Proposition 39.** For every \( v \in \mathcal{V} \) we have
\[
\text{rank}_{\mathcal{C}(G,S)}(v) \leq \text{rank}_{\mathcal{C}(G,T)}(v)^{\left\lfloor \frac{|G|}{|T|} \right\rfloor}.
\]
The same inequality is true for the separable rank and the purification rank.

**Proof.** This is obtained through Construction 35 (ii) by choosing a set \( X \) such that there is an injective mapping \( \varphi = (\varphi_1, \varphi_2) : T \hookrightarrow X \times S \). The inclusions from Construction 35 (ii) are then obtained as
\[
(g, gt) \mapsto (\varphi_1(t), (g, g\varphi_2(t))),
\]
where we regard the multiedges as directed edges in the Cayley graph. \( \square \)

**Example 40.** We consider the group \( C_5 \) with the two generating sets \( S = \{1\} \) and \( T = \{1, 2\} \).
We obtain
\[ \text{rank}_{\mathcal{C}(C_5,T)}(v) \leq \text{rank}_{\mathcal{C}(C_5,S)}(v) \leq \text{rank}_{\mathcal{C}(C_5,T)}(v)^2 \]
for all \( v \in \mathcal{V} \).

5. Applications to nonnegative tensors

In this section we consider the global space
\[ \mathcal{V} = \mathbb{C}^{d_0} \otimes \cdots \otimes \mathbb{C}^{d_n}, \]
and in there the convex cone of tensors with nonnegative entries. For \( n = 1 \) we have
\[ \mathcal{V} = \text{Mat}_{d_0,d_1}(\mathbb{C}) \]
and we thus study nonnegative matrices. For nonnegative matrices, several factorisations and corresponding ranks have been proposed and studied in the literature (see, e.g., [2]), with applications to areas such as discrete geometry, lifting techniques in optimisation and quantum theory (see for example [8] and references therein). Among them are the rank, the nonnegative rank [22], the psd-rank [12, 11], the completely positive rank [1] and the completely positive semidefinite (transposed) rank [17, 8]. We will now generalise these ranks to the multipartite case, and prove a correspondence to the ranks considered in Section 3. This will allow us to obtain inequalities for the new ranks as easy corollaries of the already proven inequalities.

Any \( M \in \mathcal{V} = \mathbb{C}^{d_0} \otimes \cdots \otimes \mathbb{C}^{d_n} \) can be written as
\[ M := \sum_{i_0,\ldots,i_n} m_{i_0 \ldots i_n} e_{i_0} \otimes \cdots \otimes e_{i_n}, \]
where \( e_j \) is the vector with a 1 in position \( j \) and 0 else, and where the complex numbers \( m_{i_0 \ldots i_n} \) are uniquely defined. By definition \( M \) is nonnegative if all \( m_{i_0 \ldots i_n} \) are nonnegative real numbers. To such \( M \) we now associate a diagonal matrix
\[ \sigma \in \text{Mat}_{d_0}(\mathbb{C}) \otimes \cdots \otimes \text{Mat}_{d_n}(\mathbb{C}) \cong \text{Mat}_{d_0 \cdots d_n}(\mathbb{C}) \]
by the formula
\[ \sigma = \sum_{i_0,\ldots,i_n} m_{i_0 \ldots i_n} E_{i_0 i_0} \otimes \cdots \otimes E_{i_n i_n}, \]
where \( E_{jk} \) is the matrix with a 1 in position \( (j,k) \) and 0 everywhere else. Clearly \( \sigma \) is positive semidefinite if and only if \( M \) is nonnegative, and in this case \( \sigma \) is automatically separable. We now define different types of
nonnegative decompositions of $M$, and relate them to decompositions of $\sigma$ considered in Section 3. This is a generalisation of the results in [8, Section 4] to the multipartite case on a general wsc. So throughout this section we let $\Omega$ be a connected wsc on $[n]$ equipped with an action from the group $G$. Clearly, $G$-invariance of $\sigma$ is equivalent to $G$-invariance of $M$.

**Definition 41.** Let $M \in \mathbb{C}^{d_0} \otimes \cdots \otimes \mathbb{C}^{d_n}$ be given.

(i) An $(\Omega, G)$-decomposition of $M$ is defined exactly as in Definition 8 (we repeat the definition here to obtain a consistent numbering).

(ii) A nonnegative $(\Omega, G)$-decomposition of $M$ is an $(\Omega, G)$-decomposition of $M$ in which all local vectors have real nonnegative entries. The corresponding rank is called the nonnegative $(\Omega, G)$-rank of $M$, denoted $\text{nn-rank}_{(\Omega, G)}(M)$.

(iii) A positive semidefinite $(\Omega, G)$-decomposition of $M$ consists of positive semidefinite matrices

$$0 \leq E_j^{[i]} \in \text{Mat}_{2^{\tilde{I}_i}}(\mathbb{C})$$

for $i = 0, \ldots, n$ and $j = 1, \ldots, d_i$, such that

$$(E_j^{[g]})_{g \beta, g \beta'} = (E_j^{[g']})_{\beta, \beta'}$$

for all $i, g, j, \beta, \beta'$, and

$$m_{i_0 \ldots i_n} = \sum_{\alpha, \alpha' \in \tilde{I}^{\tilde{I}}} (E_i^{[0]})_{\alpha_i \alpha'} \cdots (E_i^{[n]})_{\alpha_i \alpha'}$$

for all $i_0, \ldots, i_n$. The smallest cardinality of such an index set $I$ is called the positive semidefinite $(\Omega, G)$-rank of $M$, denoted $\text{psd-rank}_{(\Omega, G)}(M)$.

**Remark 42.** (i) In the case $n = 1$ and $\Omega = \Lambda_1 = \Sigma_1$, i.e. the simple edge, the different ranks of $M$ are precisely the matrix ranks listed in [8, Section 4.2]. Without a group action, the $\Lambda_1$-rank is the rank, the nonnegative $\Lambda_1$-rank is the nonnegative rank, and the positive semidefinite $\Lambda_1$-rank is the psd-rank. With action from $G = C_2$, the $(\Lambda_1, C_2)$-rank is the symmetric rank, the nonnegative $(\Lambda_1, C_2)$-rank is the cp-rank, and the positive semidefinite $(\Lambda_1, C_2)$-rank is the cpsdt-rank.

(ii) For $M$ to have a nonnegative $(\Omega, G)$-decomposition, it is necessary that $M$ is $G$-invariant and that every entry of $M$ is nonnegative. If the action of $G$ on the connected wsc $\Omega$ is free, $G$-invariance and nonnegativity is also sufficient for a nonnegative $(\Omega, G)$-decomposition. This follows directly from Remark 24, and also from Theorem 43 (ii) in combination with Theorem 21.

(iii) For $M$ to have a positive semidefinite $(\Omega, G)$-decomposition, it is necessary that $M$ be $G$-invariant and that all entries of $M$ be nonnegative. The upcoming Theorem 43 (iii), together with Theorem 27, ensures sufficiency of these conditions in many cases. $\triangle$
The following is a generalisation of [8, Theorem 38], which is recovered in the case that $\Omega = \Sigma_1 = \Lambda_1$, and $G$ is the trivial group or $C_2$.

**Theorem 43.** Let $G$ act on the connected wsc $\Omega$, and let $M$ and $\sigma$ be defined as above. We then have:

(i) $\text{rank}_{(\Omega,G)}(M) = \text{rank}_{(\Omega,G)}(\sigma)$

(ii) $\text{nn-rank}_{(\Omega,G)}(M) = \text{sep-rank}_{(\Omega,G)}(\sigma)$

(iii) $\text{psd-rank}_{(\Omega,G)}(M) = \text{puri-rank}_{(\Omega,G)}(\sigma)$.

**Proof.** For (i) start with an $(\Omega, G)$-decomposition

$$M = \sum_{\alpha \in I^\mathbb{F}} w_{\alpha_0}^{[0]} \otimes \cdots \otimes w_{\alpha_n}^{[n]}$$

with all $w_{\beta}^{[i]} \in \mathbb{C}^{d_i}$. Define

$$v_{\beta}^{[i]} := \text{diag}(w_{\beta}^{[i]}) \in \text{Mat}_{d_i}(\mathbb{C})$$

and immediately check that the $v_{\beta}^{[i]}$ provide an $(\Omega, G)$-decomposition of $\sigma$.

Conversely, if

$$\sigma = \sum_{\alpha \in I^\mathbb{F}} w_{\alpha_0}^{[0]} \otimes \cdots \otimes w_{\alpha_n}^{[n]}$$

is an $(\Omega, G)$-decomposition of $\sigma$, the vectors $v_{\beta}^{[i]} \in \mathbb{C}^{d_i}$ that contain the diagonal elements of $w_{\beta}^{[i]}$ provide an $(\Omega, G)$-decomposition of $M$.

(ii) is proven exactly the same way, using that nonnegative diagonal matrices are psd and that psd matrices have a nonnegative diagonal.

For (iii) let $E_{\beta}^{[i]}$ be the psd matrices from a positive semidefinite $(\Omega, G)$-decomposition of $M$. Write a Gram decomposition

$$E_{\beta}^{[i]} = \left( a_{j,\beta}^{[i]} a_{j,\beta}^{[i]^*} \right)_{\beta,\beta'}$$

with column vectors $a_{j,\beta}^{[i]} \in \mathbb{C}^{d_i \mathbb{F}}$ such that

$$a_{j,\beta}^{[i]} a_{j,\beta}^{[i]^*} = a_{j,\beta}^{[i]}.$$

This is for example possible by taking the columns of $\sqrt{E_{\beta}^{[i]}}$ as the $a_{j,\beta}^{[i]}$. Now set

$$\tau_{\beta}^{[i]} := \sum_{j=1}^{d_i} a_{j,\beta}^{[i]} \otimes E_{jj}$$

and easily check that these local vectors provide an $(\Omega, G)$-decomposition of a purification $\xi$ of $\sigma$. Conversely, let

$$\xi \in \text{Mat}_{d_0 \times d_0}(\mathbb{C}) \otimes \cdots \otimes \text{Mat}_{d_n \times d_n}(\mathbb{C})$$
be an \((\Omega, G)\)-purification of \(\sigma\). Denote the local matrices from an \((\Omega, G)\)-decomposition of \(\xi\) by \(\tau_{i}^{[i]}\) and define

\[
E_{j}^{[i]} := \left(\tau_{j}^{[i]} \, \tau_{ij}^{[i]} \right)_{\beta, \beta'}.
\]

It is now easily verified that this provides a positive semidefinite \((\Omega, G)\)-decomposition of \(M\).

The following corollary generalises several known inequalities on matrix ranks to the multipartite setup.

**Corollary 44.** Let \(G\) act on the connected wsc \(\Omega\). Then for all \(M \in \mathbb{C}^{d_{0}} \otimes \ldots \otimes \mathbb{C}^{d_{n}}\) we have:

(i) \(\text{rank}_{(\Omega, G)}(M) \leq \text{nn-rank}_{(\Omega, G)}(M)\)

(ii) \(\text{psd-rank}_{(\Omega, G)}(M) \leq \text{nn-rank}_{(\Omega, G)}(M)\)

(iii) \(\text{rank}_{(\Omega, G)}(M) \leq \text{psd-rank}_{(\Omega, G)}(M)^{2}\).

**Proof.** This is immediate from Theorem 43 and Proposition 29.

---

6. Conclusions and Outlook

We have considered an element \(v\) in a tensor product space and have defined and studied an \((\Omega, G)\)-decomposition thereof. The weighted simplicial complex \(\Omega\) determines the arrangement of the indices in the sum of elementary tensor factors, and the decomposition is explicitly invariant under the group action of \(G\).

We have shown that every group action can be refined to become free (Proposition 7), and that if the group action is free then an \((\Omega, G)\)-decomposition exists for all invariant \(v\) (Theorem 13). These two results prove the existence of an \((\Omega, G)\)-decomposition after possibly increasing the weights of the facets in \(\Omega\). In addition, we have shown that if the action of \(G\) is blending, then every invariant \(v\) has an \((\Omega, G)\)-decomposition as well (Theorem 17).

We have also defined the separable \((\Omega, G)\)-decomposition and the \((\Omega, G)\)-purification form, and have shown that they exist if the action of \(G\) is free (Theorem 21), or free or blending (Theorem 27), respectively.

We have also provided many inequalities between the different ranks, of which we would like to highlight the following. First we have generalised the relations between the \(\text{rank}_{(\Omega, G)}(M)\) and its separable and purification counterpart (Proposition 29), then we have shown how much the rank increases when imposing invariance under a certain group action (Proposition 31), and finally we have shown that the tensor rank is the largest of all ranks (Example 37).

Finally, we have applied our framework to nonnegative tensors, where we have first defined a nonnegative and a positive semidefinite \((\Omega, G)\)-decomposition, and then proved a correspondence with the decompositions above (Theorem 43), and a generalisation of the results of [8] (Corollary 44).
We remark that all our existence results are constructive, meaning that they can be used to transform the tensor decomposition of (1) into an \((\Omega, G)\)-decomposition. Only in the proof of Theorem 17 we have used a symmetric decomposition of a specific finite-dimensional tensor, but this can also be found algorithmically, see for example [5] (and it has to be found only once, independently of the vector in consideration).

An interesting open question is to find effective algorithms that produce the optimal \((\Omega, G)\)-decomposition, and to analyse their computational complexity. On the line, such a procedure is described in [8, Remark 6], the symmetric case on the \(n\)-simplex is covered in [5], and many other cases are analysed in [15], but the general case is open.

A further question concerns approximate decompositions, which are often of great importance. The simplest example are low-rank approximations of matrices, which have numerous applications in data science and modelling, but for higher-dimensional tensors the best low-rank approximation might not even exist [10]. It would be interesting to study the existence and hardness of approximate \((\Omega, G)\)-decompositions.

The latter could also be studied from another angle: Instead of fixing a rank and looking for the best approximation not exceeding this rank, one could fix a neighborhood of the tensor in consideration, and search for the tensor of lowest rank in this neighborhood. This is related to the notions of border rank and border tensors. While for matrices the usual rank can only decrease in the limit of a sequence, this fails to be true for higher-order tensors [3]. It would be interesting to examine under which conditions on the simplicial complex and the group action this phenomenon appears.

References

[1] A. Berman, M. Dür, and N. Shaked-Monderer. Open problems in the theory of completely positive and copositive matrices. Electron. J. Linear Algebra, 29:46, 2015.
[2] A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. SIAM, 1994.
[3] D. Bini, G. Lotti, and F. Romani. Approximate solutions for the bilinear form computational problem. SIAM J. Comput., 9:692, 1980.
[4] B. Bollobas. Modern Graph Theory. Springer, 1998.
[5] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas. Symmetric tensor decomposition. Linear Algebra Appl., 433:1851, 2010.
[6] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain. Symmetric tensors and symmetric tensor rank. SIAM J. Matrix Anal. Appl., 30:1254, 2008.
[7] R. J. M. Dawson. Homology of weighted simplicial complexes. Cahiers Topologie Geom. Differentielle Categ., 31:229, 1990.
[8] G. De las Cuevas and T. Netzer. Mixed states in one spatial dimension: decompositions and correspondence with nonnegative matrices. arXiv:1907.03664, 2019.
[9] G. De las Cuevas, N. Schuch, D. Perez-Garcia, and J. I. Cirac. Purifications of multipartite states: limitations and constructive methods. New J. Phys., 15:123021, 2013.
[10] V. de Silva and L.-H. Lim. Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. Appl., 30:1084, 2008.
[11] H. Fawzi, J. Gouveia, P. A. Parrilo, R. Z. Robinson, and R. R. Thomas. Positive semidefinite rank. Math. Program., 153:133, 2015.
[12] S. Fiorini, S. Massar, S. Pokutta, H. R. Tiwary, and R. de Wolf. Exponential lower bounds for polytopes in combinatorial optimization. Journal of the ACM, 62:17, 2015.
[13] I. Glasser, R. Sweke, N. Pancotti, J. Eisert, and J. I. Cirac. Expressive power of tensor-network factorizations for probabilistic modeling, with applications from hidden Markov models to quantum machine learning. arXiv:1907.03741, 2019.
[14] J. Gouveia, P. A. Parrilo, and R. R. Thomas. Lifts of convex sets and cone factorizations. Math. Oper. Res., 38:248, 2013.
[15] C. J. Hillar and L.-H. Lim. Most Tensor Problems Are NP-Hard. Journal of the ACM, 60:1, 2013.
[16] S. Lang, Algebra. Graduate Texts in Mathematics 211, third edition. Springer-Verlag, New York, 2002.
[17] M. Laurent and T. Piovesan. Conic approach to quantum graph parameters using linear optimization over the completely positive semidefinite cone. SIAM J. Optim., 25:2461, 2015.
[18] R. Orus. Tensor networks for complex systems. Nat. Rev. Phys., 2018.
[19] N. Schuch, D. Perez-Garcia, and J. I. Cirac. PEPS as ground states: Degeneracy and topology. Ann. Phys., 325:2153, 2010.
[20] Y. Shitov. A counterexample to Comon’s conjecture. SIAM J. Appl. Algebra Geom., 2:428, 2018.
[21] S. Singh, R. N. C. Pfeifer, and G. Vidal. Tensor network decompositions in the presence of a global symmetry. Phys. Rev. A, 82:050301, 2010.
[22] M. Yannakakis. Expressing combinatorial optimization problems by linear programs. J. Comput. Syst. Sci., 43:441, 1991.
[23] X. Zhang, Z.-H. Huang, and L. Qi. Comon’s conjecture, rank decomposition, and symmetric decomposition of symmetric tensors. SIAM J. Matrix Anal. Appl., 37:1719, 2016.