EXISTENCE OF SOLUTIONS FOR A CLASS OF P-LAPLACIAN TYPE EQUATION WITH CRITICAL GROWTH AND POTENTIAL VANISHING AT INFINITY

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ABSTRACT. In this paper, we study the existence of positive solution for the following p-Laplacain type equations with critical nonlinearity
\[
\begin{aligned}
-\Delta_p u + V(x)|u|^{p-2}u &= K(x)f(u) + P(x)|u|^{p^*-2}u, & x \in \mathbb{R}^N, \\
&\quad u \in D^{1,p}(\mathbb{R}^N),
\end{aligned}
\]
where \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u),\) \(1 < p < N,\) \(p^* = \frac{Np}{N-p},\) \(V(x), K(x)\) are positive continuous functions which vanish at infinity, \(f\) is a function with a subcritical growth and \(P(x)\) is a bounded, nonnegative continuous function. By working in the weighted Sobolev spaces, and using variational method, we prove that the given problem has at least one positive solution.

1. Introduction and main results. In this paper, we study the existence of positive solution for the following p-Laplacain equation with critical nonlinearity
\[
\begin{aligned}
-\Delta_p u + V(x)|u|^{p-2}u &= K(x)f(u) + P(x)|u|^{p^*-2}u, & x \in \mathbb{R}^N, \\
&\quad u \in D^{1,p}(\mathbb{R}^N),
\end{aligned}
\]
where \(1 < p < N,\) \(p^* = \frac{Np}{N-p}\) is the critical Sobolev exponent, the potential \(V(x)\) and \(K(x) : \mathbb{R}^N \to \mathbb{R}\) are positive continuous functions vanishing at infinity, \(f : \mathbb{R} \to \mathbb{R}\) is a function with a subcritical growth and \(P(x) \geq 0\) is a bounded continuous function.

When \(p = 2,\) problem (1.1) appears in many interesting physical contexts. For example, the solutions of this class of problems are related to the existence of standing wave solutions \(\psi(x,t) = \exp(-iE_0t)v(x)\) for nonlinear Schrödinger equation
\[
\frac{\partial \psi}{\partial t} = -\Delta \psi + (V(x) + E_0)\psi - K(x)|\psi|^{-1}g(|\psi|)\psi, & x \in \mathbb{R}^N,
\]

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where \( E_0 \in \mathbb{R} \) and \( v(x) \) is a real function. Obviously, \( \psi \) satisfies (1.2) if and only if the function \( v(x) \) solves the semilinear scalar field equation
\[
- \Delta u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^N,
\]
(1.3) here we set \( f(u) = |u|^{p-1}g(|u|)u \). Problem of (1.3) has been extensively studied since 1970s, we would like to mention [6, 7, 16] to the readers. An important class of problems associated with (1.3) is the zero mass case, which occurs with the potentials \( V(x) \) vanishing at infinity, that is,
\[
\lim_{|x| \to +\infty} V(x) = 0.
\]
In [2], Ambrosetti, Felli and Malchiodi studied (1.3) with the zero mass case when
\[
f(s) = s^p \quad \text{with} \quad 2 < p < \frac{N + 2}{N - 2}
\]
and \( V, K \) satisfying the following assumptions:
\[
V, K: \mathbb{R}^N \to \mathbb{R} \text{ are smooth functions and there exist constants } \alpha, \beta, a, A, \kappa > 0 \text{ such that }
\]
\[
\frac{a}{1 + |x|^{\alpha}} \leq V(x) \leq A, \quad \text{and} \quad 0 < K(x) \leq \frac{\kappa}{1 + |x|^{\beta}}, \quad \forall x \in \mathbb{R}^N \quad (VK)
\]
and \( \alpha, \beta \) verifying
\[
\frac{N + 2}{N - 2} - \frac{4\beta}{\alpha(N - 2)} < p, \quad \text{if} \quad 0 < \beta < \alpha, \quad \text{or} \quad p > 1, \quad \text{when} \quad \beta \geq \alpha.
\]
The condition (VK) is interesting because Opic and Kufner in [19] have showed that it can be used to prove that the space \( E \) given by
\[
E = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \, dx < +\infty \right\}
\]
endowed with the norm
\[
\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx
\]
is compactly embedded into the weighted Lebesgue space
\[
L_{K}^{p+1}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^{p+1} \, dx < +\infty \right\}.
\]
Ambrosetti and Wang [4] also considered the condition (VK) but the inequality on \( V(x) \) is assumed only outside of a ball centered at origin. Alves and Souto [1] considered more general conditions on \( V(x) \) and \( K(x) \), by which the space \( E \) can compactly embedded into the weighted space \( L_{K}^{p+1}(\mathbb{R}^N) \) for certain \( p \in [1, 2^* - 1) \). Moreover, Bonheure and Schaffingten [8] introduced a new set of hypotheses on \( V(x) \) and \( K(x) \), by using the Marcinkiewicz spaces \( L^r,\infty(\mathbb{R}^N) \) for \( r > 1 \), which permitted them to show continuous and compact embeddings from \( E \) into the weighted space \( L_{K}^{q}(\mathbb{R}^N) \) for some \( q > 1 \).

The space \( E \) with fast increasing potentials also studied by many authors. For example, set \( K(x) = \exp(|x|^2/4) \), the space
\[
\widehat{E} = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} K(x)u^2 \, dx < +\infty \right\}
\]
endowed with the norm
\[
\|u\|_{\widehat{E}}^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + K(x)u^2) \, dx
\]
is used as the working space to deal with the following elliptic problem
\[-\text{div}(K(x)\nabla u) = K(x)f(u) + K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N.\]

For the reference, we would like to mention [9, 14, 15] and the reference therein. In particular, Escobedo and Kavian [14] have proved that the embedding \( \hat{E} \hookrightarrow L^q_{K}(\mathbb{R}^N) \) is continuous for all \( q \in [2, 2^*] \) and it is compact for all \( q \in [2, 2^*). \)

Over the last several decades, many authors have shown much interest in the second-order elliptic differential equations in unbounded domains with critical growth. For example, P. L. Lions [17, 18] established a Concentration-Compactness Principle for some nonlinear elliptic equations in \( \mathbb{R}^N \) and studied minimization problems associated with nonlinear elliptic equations in \( \mathbb{R}^N \) with critical growth. D. Smets [21] investigated the following problems
\[
\begin{cases}
-\Delta u - \mu \frac{u}{|x|^2} = K(x)u^{2^*-1}, \\
u > 0, \quad u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]
(1.4) where \( 2^* = \frac{2N}{N-2} \), \( K(x) \) satisfying some conditions. He established a complete non-compact analysis of (1.4) and through this analysis he obtained some existence results of the solutions for (1.4). In the non-compact analysis, he divided any Palais-Smale sequence \( \{u_n\} \) of the variational functional corresponding to (1.4) into two parts. One part is confined in a ball \( B(R) \) and the other part is confined in \( \mathbb{R}^N \setminus B(R) \). As for the part confined in a ball, he can treat it as the case that \( \{u_n\} \) is tight. For the other part, it can be transformed into one which confined in a ball by Kelvin transformation. The Kelvin transformation can be applied to problem (1.4) because there are no new blow up bubbles, except for the blow up bubbles caused by critical nonlinearity and Hardy term, to occur for the Palais-Smale sequences of the corresponding variational functional of (1.4).

Recently, Deng, Jin and Peng [11] established a complete non-compact expression for the Palais-Smale sequences of the variational functional corresponding to
\[
-\Delta u - \mu \frac{u}{|x|^2} + a(x)u = |u|^{2^*-2}u + f(x, u), \quad u \in H^1(\mathbb{R}^N),
\]
(1.5) which including all the blow up bubbles caused by critical exponents, Hardy term and unbounded domains. By using the non-compact expression for the Palais-Smale sequences of the variational functional corresponding to (1.5), the existence of positive solutions for problem (1.5) is obtained. But the potential \( a(x) \) should be non-vanishing at infinity. For more results about the existence of solutions for semilinear or quasilinear elliptic problems with critical growth and non-vanishing potential at infinity, the readers can refer to the papers [10, 12, 13, 16, 23, 26] and the reference therein.

However, there seems to be little progress on the existence of positive solution for a general elliptic equation, for example problem (1.1), with critical growth and the potential \( V(x) \) vanishing at infinity.

In this paper, we establish the existence of positive solution for problem (1.1) with critical nonlinearity and the potential \( V(x) \) vanishing at infinity. To this end, we need some assumptions on \( V(x), K(x), f(s) \) and \( P(x) \).

As in [1], we say \((V, K) \in \mathcal{K}\) if the following conditions hold:
(i) \( V(x), K(x) > 0, \forall x \in \mathbb{R}^N \) and \( K(x) \in L^\infty(\mathbb{R}^N) \).
(ii) If \{A_n\} ⊂ \mathbb{R}^N is a sequence of Borel sets such that |A_n| ≤ R, for all n and some R > 0, we have
\[
\lim_{r \to +\infty} \int_{A_n \cap B_r(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}.
\]

(K_1)

(iii) One of the following conditions occurs:
\[
\frac{K(x)}{V(x)} \in L^\infty(\mathbb{R}^N)
\]

(K_2)

or there is \( p_0 \in (p, p^*) \) such that
\[
\frac{K(x)}{|V(x)|^{\frac{p^* - p}{p - p_0}}} \to 0 \quad \text{as } |x| \to +\infty.
\]

(K_3)

Related to the function \( f \), we assume the following conditions:

\( (f_1) \)
\[
\limsup_{s \to 0} \frac{f(s)}{|s|^p} = 0 \quad \text{if } (K_2) \text{ holds},
\]
or
\[
\limsup_{s \to 0} \frac{f(s)}{|s|^{p_0 - 1}} < +\infty \quad \text{if } (K_3) \text{ holds}.
\]

\( (f_2) \) \( f \) has a subcritical growth, that is,
\[
\limsup_{s \to +\infty} \frac{f(s)}{s^{p^* - 1}} = 0.
\]

\( (f_3) \) There exists \( \theta \in (p, p^*) \) such that
\[
0 < \theta F(s) \leq sf(s), \quad \forall s > 0,
\]
where \( F(u) = \int_0^u f(t)dt \).

Moreover, for the function \( P(x) \), we assume that

\( (P_1) \) There is a point \( x_0 \), such that
\[
P(x_0) = \sup_{x \in \mathbb{R}^N} P(x).
\]

\( (P_2) \) For \( x \) close to \( x_0 \), we have
\[
P(x) = P(x_0) + O(|x - x_0|^p) \quad \text{as } x \to x_0.
\]

Our main result of this paper is as follows:

**Theorem 1.1.** Assume that \( (V, K) \in \mathcal{K} \), \( f \) satisfies \((f_1)-(f_3)\) and \( P(x) \) satisfies \((P_1)-(P_2)\), then problem \((1.1)\) has at least one positive solution if either \( N \geq p^2 \) or \( p < N < p^* \) and \( \theta > p^* - \frac{p}{p^* - 1} \).

There are serious difficulties in trying to find the nontrivial solutions in \( \mathcal{D}^{1,p}(\mathbb{R}^N) \) of \((1.1)\) by standard variational methods. Firstly, the potential \( V(x) \) vanishes at infinity, we cannot work on the usual Sobolev space \( W^{1,p}(\mathbb{R}^N) \). Secondly, the space \( \mathcal{D}^{1,p}(\mathbb{R}^N) \) can not embedding into \( L^r(\mathbb{R}^N) \) for \( p \leq r < p^* \) and the embedding \( \mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N) \) is not compact. In order to prove the existence result, we first define space \( E \), which is a subspace of \( \mathcal{D}^{1,p}(\mathbb{R}^N) \), and weight space \( L^q_{K}(\mathbb{R}^N) \). We then establish a Hardy-type inequality (see Lemma 2.1) involving \( V(x) \) and \( K(x) \) as in \([1]\). Since the embedding \( E \hookrightarrow L^{p^*}_{p}(\mathbb{R}^N) \) is still not compact, we imitate
the method in [7] by using the mountain pass theorem without (PS) condition, and the existence of positive solution in \( E \) (also in \( D^{1,p}(\mathbb{R}^N) \)) of (1.1) is proved.

The rest of this paper is organized as follows. In Section 2, we present some embedding results, which generalize the corresponding embedding results in [1]. In Section 3, we prove our main result.

2. Some preliminary lemmas. In order to prove the main results, first we introduce the space

\[
E := \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) |u|^p dx < \infty \right\}
\]

endowed with the norm

\[
\|u\|^p := \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p) dx.
\]

Denote by \( L^p_K(\mathbb{R}^N) \) the weighted Lebesgue space

\[
L^p_K(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^q dx < +\infty \right\}
\]

endowed with the norm

\[
\|u\|_{q,K}^q := \int_{\mathbb{R}^N} K(x)|u|^q dx.
\]

\( E \) and \( L^q_K(\mathbb{R}^N) \) are particular cases of weighted space and are discussed in [19]. The following two lemmas discuss the continuous and compact embedding for \( E \hookrightarrow L^q_K(\mathbb{R}^N) \). The proof is inspired by [1, 8].

Lemma 2.1. Assume that \((V, K) \in \mathcal{K}\). Then \( E \) is continuously embedded in \( L^q_K(\mathbb{R}^N) \) for all \( q \in [p, p^*] \) if \((K_2)\) holds. Moreover, \( E \) is continuously embedded in \( L^{p^*}_K(\mathbb{R}^N) \) if \((K_3)\) holds.

Proof. By assuming that \((K_2)\) is true, the proof is trivial if \( q = p \) or \( p^* \). Now, we fix \( q \in (p, p^*) \) and let \( \lambda = \frac{q - p}{p - p^*} \), then \( q = \lambda p + (1 - \lambda)p^* \). It follows that

\[
\int_{\mathbb{R}^N} K(x)|u|^q dx = \int_{\mathbb{R}^N} K(x)|u|^\lambda p^{(1-\lambda)p^*} dx \\
\leq (\int_{\mathbb{R}^N} |K(x)|^{1/\lambda}|u|^p dx)^\lambda (\int_{\mathbb{R}^N} |u|^{(1-\lambda)p^*} dx)^{1-\lambda} \\
\leq (\sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{1/\lambda}}) (\int_{\mathbb{R}^N} V(x)|u|^p dx)^\lambda (\int_{\mathbb{R}^N} |u|^{(1-\lambda)p^*} dx)^{1-\lambda} \\
\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{1/\lambda}} \right) (\int_{\mathbb{R}^N} V(x)|u|^p dx)^\lambda (\int_{\mathbb{R}^N} |\nabla u|^{p^*} dx)^{\frac{(1-\lambda)p^*}{p}} \\
\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{1/\lambda}} \right) (\int_{\mathbb{R}^N} |\nabla u|^p + V(x)|u|^p dx)^\lambda (\int_{\mathbb{R}^N} |\nabla u|^p + V(x)|u|^p dx)^{\frac{2}{p}}.
\]

Since \( K(x) \in L^\infty(\mathbb{R}^N) \) and \( \frac{K}{V} \in L^\infty(\mathbb{R}^N) \), we have that

\[
\|u\|_{L^q_K(\mathbb{R}^N)} \leq C\|u\|.
\]
Lemma 2.2. Assume that \( \lambda_0 = \frac{p^* - p_0}{p^* - p} \), we have that \( p_0 = \lambda_0 p + (1 - \lambda_0)p^* \). As above, we have

\[
\int_{\mathbb{R}^N} K(x)|u|^{p_0}dx = \int_{\mathbb{R}^N} K(x)|u|^{\lambda_0 p}|u|^{(1-\lambda_0)p^*}dx
\]

\[
\leq \left( \int_{\mathbb{R}^N} |K(x)|^{1/\lambda_0}|u|^pdx \right)^{\lambda_0} \left( \int_{\mathbb{R}^N} |u|^{p^*}dx \right)^{1-\lambda_0}
\]

\[
\leq \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda_0}} \right) \left( \int_{\mathbb{R}^N} V(x)|u|^pdx \right)^{\lambda_0} \left( \int_{\mathbb{R}^N} |u|^{p^*}dx \right)^{1-\lambda_0}
\]

\[
\leq C \left( \sup_{x \in \mathbb{R}^N} \frac{|K(x)|}{|V(x)|^{\lambda_0}} \right) \left( \int_{\mathbb{R}^N} |u|^pdx \right)^{\lambda_0} \left( \int_{\mathbb{R}^N} |u|^{p^*}dx \right)^{\lambda_0}.
\]

Since (K3) holds, we have \( \frac{K(x)}{|V(x)|^\lambda_0} \in L^\infty(\mathbb{R}^N) \), and hence

\[
\|u\|_{L^p_{\lambda_0}(\mathbb{R}^N)} \leq C\|u\|.
\]

This completes the proof of our lemma. \( \square \)

**Lemma 2.2.** Assume that \((V, K) \in K (K)\). Then \( E \) is compactly embedded in \( L^p_{\lambda_0}(\mathbb{R}^N) \) for all \( q \in (p, p^*) \) if (K2) holds. Moreover, \( E \) is compactly embedded in \( L^2_{\lambda_0}(\mathbb{R}^N) \) if (K3) holds.

**Proof.** The proof of this lemma will be divided into two parts.

Firstly, we consider the case when (K2) holds. For fixed \( q \in (p, p^*) \) and given \( \varepsilon > 0 \), there are \( 0 < s_0 < s_1 \) and \( C > 0 \) such that

\[
K(x)|s|^q \leq \varepsilon C(V(x)|s|^p + |s|^{p^*}) + CK(x)\chi_{[s_0, s_1]}(|s|)|s|^{p^*}, \quad \forall s \in \mathbb{R}. \tag{2.1}
\]

Hence,

\[
\int_{B_{\varepsilon}(0)} K(x)|u|^qdx \leq \varepsilon CQ(u) + C \int_{A \cap B_{\varepsilon}(0)} K(x)|u|^{p^*}dx, \quad \forall u \in E, \tag{2.2}
\]

where

\[
Q(u) = \int_{\mathbb{R}^N} V(x)|u|^pdx + \int_{\mathbb{R}^N} |u|^{p^*}dx,
\]

and

\[
A = \{ x \in \mathbb{R}^N : s_0 \leq |u(x)| \leq s_1 \}.
\]

If \( \{v_n\} \) is a sequence such that \( v_n \rightharpoonup v \) in \( E \), there is \( M_1 > 0 \) such that

\[
\int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx \leq M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p^*}dx \leq M_1, \quad \forall n \in \mathbb{N}
\]

which gives that \( \{Q(v_n)\} \) is bounded. On the other hand, setting

\[
A_n = \{ x \in \mathbb{R}^N : s_0 \leq |v_n(x)| \leq s_1 \},
\]

the last inequality implies that

\[
s_0^{p^*} |A_n| \leq \int_{A_n} |v_n|^{p^*}dx \leq M_1, \quad \forall n \in \mathbb{N},
\]

which shows that \( \sup_{n \in \mathbb{N}} |A_n| < +\infty \). Therefore, from (K1), there is an \( r > 0 \) such that

\[
\int_{A_n \cap B_{\varepsilon}(0)} K(x)dx < \frac{\varepsilon}{s_1^{p^*}} \quad \text{for all} \ n \in \mathbb{N}. \tag{2.3}
\]
which yields

\[
\lim_{n \to +\infty} \int_{B_{r}(0)} K(x) |v_n|^q dx = \int_{B_{r}(0)} K(x) |v|^q dx. \tag{2.5}
\]

Combining (2.4) and (2.5),

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x) |v_n|^q dx = \int_{\mathbb{R}^N} K(x) |v|^q dx. \tag{2.6}
\]

which yields

\[v_n \to v \text{ in } L^q_K(\mathbb{R}^N), \quad \forall q \in (p, p^*).\]

Now, we suppose that \((K_3)\) holds. First of all, it is important to observe that for each \(x \in \mathbb{R}^N\) fixed, the function

\[g(s) = V(x)s^{p-\rho_0} + s^{p^*-\rho_0}, \quad \forall s > 0\]

has \(C_{\rho_0} V^{p-\rho_0/p} (x)\) as its minimum value, where

\[C_{\rho_0} = \left( \frac{p^*-p}{p^*-\rho_0} \right) \left( \frac{\rho_0-p}{p^*-\rho_0} \right)^{\frac{p^*-\rho_0}{p^*-p}}.\]

Hence,

\[C_{\rho_0} V^{p-\rho_0/p} (x) \leq V(x)s^{p-\rho_0} + s^{p^*-\rho_0}, \quad \forall x \in \mathbb{R}^N \text{ and } s > 0.\]

Combining the last inequality with \((K_3)\), there is \(r > 0\) large enough, such that

\[K(x)|s|^\rho_0 \leq C_\varepsilon (V(x)|s|^p + |s|^{p^*}), \quad \forall s \in \mathbb{R} \text{ and } |x| \geq r,\]

which leads to

\[\int_{B_{r}(0)} K(x)|u|^\rho_0 dx \leq C_\varepsilon \int_{B_{r}(0)} (V(x)|u|^p + |u|^{p^*}) dx, \quad \forall u \in E.\]

If \(\{v_n\}\) is a sequence such that \(v_n \to v\) in \(E\), there is \(M_1 > 0\) such that

\[\int_{\mathbb{R}^N} V(x)|v_n|^p dx \leq M_1 \text{ and } \int_{\mathbb{R}^N} |v_n|^{p^*} dx \leq M_1, \quad \forall n \in \mathbb{N},\]

and so

\[\int_{B_{r}(0)} K(x)|v_n|^\rho_0 dx \leq 2C_\varepsilon M_1, \quad \forall n \in \mathbb{N}. \tag{2.7}\]

Since \(\rho_0 \in (p, p^*)\) and \(K\) is a continuous function, it follows from Sobolev embedding on bounded domain that

\[\lim_{n \to +\infty} \int_{B_{r}(0)} K(x)|v_n|^\rho_0 dx = \int_{B_{r}(0)} K(x)|v|^\rho_0 dx. \tag{2.8}\]

From (2.7) and (2.8),

\[\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x)|v_n|^\rho_0 dx = \int_{\mathbb{R}^N} K(x)|v|^\rho_0 dx, \tag{2.9}\]

which implies that

\[v_n \to v \text{ in } L^\rho_0_K(\mathbb{R}^N).\]

Then the proof of our lemma is completed. \(\square\)
Lemma 2.3. Suppose that $f$ satisfies (f$_1$)-(f$_2$) and $(V,K) \in \mathcal{K}$. Let $\{v_n\}$ be a sequence such that $v_n \to v$ in $E$, then

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x)f(v_n)dx = \int_{\mathbb{R}^N} K(x)f(v)dx, \quad (2.10)$$

and

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} K(x)f(v_n)v_n dx = \int_{\mathbb{R}^N} K(x)f(v)v dx. \quad (2.11)$$

Proof. We only give the proof of (2.10) and the proof of (2.11) can be proved in the same way.

We begin the proof by assuming that $(K_2)$ occurs. From (f$_1$)-(f$_2$), fixed $q \in (p, p^\ast)$ and given $\varepsilon > 0$, there is $C > 0$ such that

$$K(x)f(s) \leq \varepsilon C(V(x)|s|^p + |s|^{p^\ast}) + CK(x)|s|^q, \quad \forall s \in \mathbb{R}. \quad (2.12)$$

From lemma 2.2,

$$\int_{\mathbb{R}^N} K(x)|v_n|^q dx \to \int_{\mathbb{R}^N} K(x)|v|^q dx,$$

and there is $r > 0$ such that

$$\int_{B_r(0)} K(x)|v_n|^q dx < \varepsilon, \quad \forall n \in \mathbb{N}. \quad (2.13)$$

Since $\{v_n\}$ is bounded in $E$, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^p dx \leq M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p^\ast} dx \leq M_1, \quad \forall n \in \mathbb{N}.$$

Combining the last inequalities with (2.12) and (2.13),

$$\left| \int_{B_r^c} K(x)f(v_n)dx \right| < (2CM_1 + C)\varepsilon, \quad \forall n \in \mathbb{N}. \quad (2.14)$$

Now we assume $(K_3)$ holds. Repeating the same arguments explored in the proof of lemma 2.2, for given $\varepsilon > 0$ small enough, there is $r > 0$ large enough such that

$$K(x) \leq \varepsilon (V(x)|s|^{p^\ast - p_0} + |s|^{p^\ast - p_0}), \quad \forall s \in \mathbb{R} \quad \text{and} \quad |x| > r.$$

From (f$_1$) and (f$_2$), for the given $\varepsilon > 0$, we have

$$F(s) \leq C|s|^{p_0} + \varepsilon|s|^{p^\ast}, \quad \forall s \in I,$$

where $I = \{x \in \mathbb{R}^N : |s| < s_0 \quad \text{or} \quad |s| > s_1\}$.

Since $K(x) \in L^\infty(\mathbb{R}^N)$, then for all $s \in I$ and $|x| > r$, we have

$$K(x)|F(s)| \leq CK(x)|s|^{p_0} + \varepsilon K(x)|s|^{p^\ast} \leq C\varepsilon(V(x)|s|^{p^\ast - p_0} + |s|^{p^\ast - p_0})|s|^{p_0} + \varepsilon k(x)||_{L^\infty(\mathbb{R}^N)}|s|^{p^\ast} \leq \varepsilon C(V(x)|s|^{p} + |s|^{p^\ast}).$$

Therefore, for any $u \in E$, we have the following estimate

$$\int_{B_r^c(0)} K(x)f(u)dx \leq \varepsilon Q(u) + C \int_{A \cap B_r^c(0)} K(x)dx$$

where

$$Q(u) = \int_{\mathbb{R}^N} V(x)|u|^p dx + \int_{\mathbb{R}^N} |u|^{p^\ast} dx,$$
and
\[ A = \{ x \in \mathbb{R}^N : s_0 \leq |u(x)| \leq s_1 \}. \]
Since \( \{ v_n \} \) is bounded in \( E \), there is \( M_1 > 0 \) such that
\[ \int_{\mathbb{R}^N} V(x)|v_n|^p \, dx \leq M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^{p'} \, dx \leq M_1. \]
Thus,
\[ \int_{B_r^+(0)} K(x)F(v_n) \, dx \leq 2CM_1 \varepsilon + C \int_{A_n \cap B_r^+(0)} K(x) \, dx \]
where
\[ A_n = \{ x \in \mathbb{R}^N : s_0 \leq |v_n(x)| \leq s_1 \}. \]
Repeating the same arguments used in the proof of lemma 2.2, it follows that
\[ \int_{A_n \cap B_r^+(0)} K(x) \, dx \to 0 \quad \text{as} \quad r \to +\infty, \]
and so, for \( n \) large enough
\[ \left| \int_{B_r^+(0)} K(x)F(v_n) \, dx \right| \leq C(2M_1 + 1) \varepsilon. \]
To conclude the proof, we need to show that
\[ \lim_{n \to +\infty} \int_{B_r^+(0)} K(x)F(v_n) \, dx = \int_{B_r^+(0)} K(x)F(v) \, dx, \]
However, this limit follows by using the compactness lemma of Strauss [see \[22\], Compactness lemma 2, p.156]: \( B_r(0) \) is a bounded domain, \( |v_n|_{L^{p'}(B_r(0))} \) is bounded and \( (f_2) \), together with the convergence almost everywhere imply the limits as required.

3. The proofs of our main results. In this section, we prove the existence of positive solutions for problem (1.1) by variation technique. In order to use mountain pass lemma \[7\] to obtain the solution, some arguments of this proof were adapted from the articles \[7, 12, 13\].

Since we intend to find positive solutions, we assume that
\[ f(s) = 0, \quad \forall s \in (-\infty, 0]. \quad (3.1) \]
The variational functional associated with (1.1) is given by
\[ I(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p) \, dx - \int_{\mathbb{R}^N} K(x)F(u) \, dx - \frac{1}{p'} \int_{\mathbb{R}^N} P(x)|u|^{p'} \, dx, \quad (3.2) \]
for all \( u \in E \). From the conditions on \( f(s) \) and Lemma 2.1, the functional \( I \) is well defined and \( I \in C^1(E, \mathbb{R}) \). Its Gateaux derivative is given by
\[ I'(u)v = \int_{\mathbb{R}^N} \left( |\nabla u|^{p-2} \nabla u \nabla v + V(x)|u|^{p-2} uv \right) \, dx - \int_{\mathbb{R}^N} K(x)f(u)v \, dx \]
\[ - \int_{\mathbb{R}^N} P(x)|u|^{p-2} uv \, dx, \quad \text{for all} \quad u, v \in E. \]
Then, it is easy to check that the critical points of \( I \) are weak solutions of (1.1).

Since \( E \) can be embedded into \( L^{p'}_K(\mathbb{R}^N) \) continuously for some \( q \) (see Lemma 2.1), we can verify that the functional \( I \) exhibits the Mountain-Pass geometry.
Lemma 3.1. The functional $I$ satisfies

(i) there exist $\alpha, \rho > 0$, such that $I(u) > \alpha$ for all $\|u\| = \rho$;

(ii) there exists an $e \in E$, such that $\|u\| > \rho$ and $I(e) < 0$.

As a consequence of Lemma 3.1 and the mountain pass lemma (see [3]), for the constant

$$c_0 = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)) > 0,$$

where

$$\Gamma = \{ \gamma \in C([0,1], E), \gamma(0) = 0, \gamma(1) \neq 0, I(\gamma(1)) < 0 \},$$

there exists a $(PS)_c$ sequence $\{u_n\}$ in $E$ at the level $c_0$, that is,

$$I(u_n) \to c_0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

Lemma 3.2. The sequence $\{u_n\}$ in (3.4) is bounded in $E$.

Proof. From $(f_3)$ we have

$$I(u_n) - \frac{1}{\theta} I'(u_n)u_n = \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|^p + \frac{1}{\theta} \int_{\mathbb{R}^N} K(x)(f(u_n)u_n - \theta F(u_n))dx$$

$$+ \left( \frac{1}{\theta} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} |u_n|^{p^*} dx \geq \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|^p.$$  

Since $I(u_n) \to c_0$ and $I'(u_n) \to 0$ as $n \to +\infty$, we deduce that $\{u_n\}$ is bounded in $E$.

Let $\{u_n\}$ be a $(PS)_{c_0}$-sequence given by (3.4). By Lemma 3.2, we deduce that $\{u_n\}$ is bounded. Using the standard argument (see [5, 12, 16, 25, 26]), up to a subsequence, there is $u \in E$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad E,$$

$$u_n \to u \quad \text{in} \quad L^r_{\text{loc}}(\mathbb{R}^N) \quad \text{for all} \quad p \leq r < p^*,$$

$$u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N,$$

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{a.e.} \quad \mathbb{R}^N,$$

$$|\partial_i |u_n|^{p-2} u_n| \rightharpoonup |\partial_i |u|^{p-2} u| \quad \text{in} \quad (L^p(\mathbb{R}^N))^*.$$  

In the following, we show that $u$ is a positive solution of (1.1). To this end, we exploit the fact that the critical equation

$$-\Delta_p u = |u|^{p^*-2} u \quad \text{in} \quad \mathbb{R}^N,$$

has positive solutions

$$u_\varepsilon(x) = k_\varepsilon^{(N-p)/(p^*-p)} (\varepsilon^{p^*-1} + |x-x_0|^{p/p-1})^{(p-N)/p}.$$  

We can choose $k > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p dx = \int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} dx = S_\varepsilon^N,$$

where $S$ denote the best constant for the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, namely

$$S := \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p dx, \int_{\mathbb{R}^N} |u|^{p^*} dx = 1 \right\}.$$
Let $R > 0$ be small enough that $B_{2R}(x_0) \subset \mathbb{R}^N$, and $\psi(x)$ be a piecewise smooth function with support in $B_{2R}(x_0)$ such that $\psi(x) \equiv 1$ in $B_R(x_0)$, $0 \leq \psi(x) \leq 1$ in $B_{2R}(x_0)$ and $|\nabla \psi| \leq \frac{C}{R}$.

Define

$$w_{\varepsilon}(x) = \psi(x)u_{\varepsilon}(x) \quad (3.6)$$

and

$$v_{\varepsilon}(x) = w_{\varepsilon}(x) \left[ \int_{\mathbb{R}^N} P(x)w_{\varepsilon}^p(x)dx \right]^{-\frac{1}{p'}} \quad (3.7)$$

Denote

$$V_{\text{max}} := \max_{x \in B_{2R}(x_0)} V(x),$$

and

$$K_{\text{min}} := \min_{x \in B_{2R}(x_0)} K(x).$$

Since $\partial u_{\varepsilon}/\partial \bar{n} \leq 0$, we have that

$$\int_{B_R(x_0)} |\nabla w_{\varepsilon}|^p dx = \int_{B_R(x_0)} |\nabla u_{\varepsilon}|^p dx \leq \int_{B_R(x_0)} |u_{\varepsilon}|^p dx,$$

and by the assumption ($P_2$) we also have

$$P(x_0) \int_{B_R(x_0)} |u_{\varepsilon}|^p dx \leq \int_{B_R(x_0)} P(x)|u_{\varepsilon}|^p dx + O(\varepsilon^p).$$

Simple calculations as [12] gives that

$$\int_{\mathbb{R}^N \setminus B_R(x_0)} |u_{\varepsilon}|^p dx = O(\varepsilon^{N/(p-1)}),$$

$$A_{\varepsilon} := \int_{\mathbb{R}^N \setminus B_R(x_0)} |\nabla w_{\varepsilon}|^p dx = O(\varepsilon^{(N-p)/(p-1)}),$$

$$\int_{\mathbb{R}^N} |v_{\varepsilon}|^p dx = \begin{cases} 
 k\varepsilon^p + O(\varepsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\
 k\varepsilon^p |\ln \varepsilon| + O(\varepsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\
 O(\varepsilon^{(N-p)/(p-1)}), & \text{if } N < p^2,
\end{cases} \quad (3.8)$$

as $\varepsilon \to 0$, where $k$ is a positive constant. Therefore, we have

$$\int_{\mathbb{R}^N} |\nabla w_{\varepsilon}|^p dx = \int_{B_R(x_0)} |\nabla w_{\varepsilon}|^p dx + A_{\varepsilon}$$

$$\leq \int_{B_R(x_0)} |u_{\varepsilon}|^p dx + A_{\varepsilon} \leq S \left[ \int_{B_R(x_0)} |u_{\varepsilon}|^p dx \right]^{\frac{p}{p'}} + A_{\varepsilon}$$

$$\leq S(\|P(x)\|_{L^\infty(\mathbb{R}^N)})^{-\frac{p}{p'}} \left[ \int_{B_R(x_0)} P(x)|u_{\varepsilon}|^p dx \right]^{\frac{p}{p'}} + O(\varepsilon^p) + O(\varepsilon^{(N-p)/(p-1)}).$$

Set $V_{\varepsilon} \equiv \int_{\mathbb{R}^N} |\nabla v_{\varepsilon}|^p dx$, since for small $\varepsilon > 0$, say $\varepsilon \leq \varepsilon_0$, it is easy to see that

$$\int_{B_R(x_0)} P(x)|w_{\varepsilon}|^p dx \geq C_{\varepsilon_0}$$

for some positive constant $C_{\varepsilon_0}$. The definition of $V_{\varepsilon}$ and the last two inequalities imply that

$$V_{\varepsilon} \leq S \left( \|P(x)\|_{L^\infty(\mathbb{R}^N)} \right)^{-\frac{p}{p'}} + O(\varepsilon^p) + O(\varepsilon^{(N-p)/(p-1)}). \quad (3.9)$$
Lemma 3.3. Assume that \((V, K) \in \mathcal{K}, f\) satisfies \((f_1)-(f_3)\) and \(P(x)\) satisfies \((P_1)-(P_2)\). Then there exists a \(u_0 \in E \setminus \{0\}\) such that

\[
0 < \sup_{t \geq 0} I(tu_0) = \frac{1}{N} S_{\frac{N}{p}}^{\frac{p}{p-N}} \left[ \|P(x)\|_{L^\infty(\mathbb{R}^N)} \right]^{\frac{p-N}{p}}
\]

if either \(N \geq p^*\) or \(p < N < p^*\) and \(\theta > p^* - \frac{p}{p^* - p}\).

Proof. We consider now

\[
I(t \varepsilon) = \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + V(x)|v_\varepsilon|^p) dx - \int_{\mathbb{R}^N} K(x) F(t \varepsilon) dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} P(x)|v_\varepsilon|^{p^*} dx
\]

\[
= \frac{t^p}{p} \int_{B_{2R}(x_0)} |\nabla v_\varepsilon|^p dx + \frac{t^p}{p} \int_{B_{2R}(x_0)} V(x)|v_\varepsilon|^p dx - \int_{B_{2R}(x_0)} K(x) F(t \varepsilon) dx
\]

\[
- \frac{t^{p^*}}{p^*} \int_{B_{2R}(x_0)} P(x)|v_\varepsilon|^{p^*} dx.
\]

Clearly, \(\lim_{t \to +\infty} I(t \varepsilon) = -\infty\) for all \(\varepsilon > 0\), by \((f_1)\) and \((f_2)\), \(\sup_{t \geq 0} I(t \varepsilon) > 0\) is attained by some \(t_\varepsilon > 0\).

We claim that there are two positive constants \(A_1\) and \(A_2\) independent of \(\varepsilon\) such that \(A_1 < t_\varepsilon < A_2\) if \(\varepsilon > 0\) is sufficiently small.

In fact, since \(I(t \varepsilon) = \sup_{t \geq 0} I(t \varepsilon)\) and hence \(\frac{d}{dt} I(t \varepsilon)\big|_{t = t_\varepsilon} = 0\), we have that

\[
(t_\varepsilon)^{p-1} \int_{B_{2R}(x_0)} (|\nabla v_\varepsilon|^p + V(x)|v_\varepsilon|^p) dx
\]

\[
= \int_{B_{2R}(x_0)} K(x) f(t_\varepsilon v_\varepsilon) v_\varepsilon dx + (t_\varepsilon)^{p^*-1} \int_{B_{2R}(x_0)} P(x)|v_\varepsilon|^{p^*} dx.
\]

If there is a sequence \(t_{\varepsilon_n} \to +\infty\), as \(\varepsilon_n \to 0^+\), by the above equality, we get

\[
(t_{\varepsilon_n})^{p-1} \int_{B_{2R}(x_0)} (|\nabla v_{\varepsilon_n}|^p + V(x)|v_{\varepsilon_n}|^p) dx \geq (t_{\varepsilon_n})^{p^*-1} \int_{B_{2R}(x_0)} P(x)|v_{\varepsilon_n}|^{p^*} dx,
\]

which gives a contradiction since \(p^* > p\).

Similarly, we suppose there is a sequence \(t_{\varepsilon_n}' \to 0\) as \(\varepsilon_n \to 0^+\). Firstly, if \((K_2)\) holds, from \((f_1)\) and \((f_2)\), \(\forall \delta > 0\), \(\exists C_\delta > 0\), such that

\[
\int_{\mathbb{R}^N} K(x) f(t_{\varepsilon_n}' v_{\varepsilon_n}) v_{\varepsilon_n} dx
\]

\[
\leq \delta (t_{\varepsilon_n}')^{p-1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^p dx + C_\delta (t_{\varepsilon_n}')^{p^*-1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{p^*} dx
\]

\[
\leq \delta C (t_{\varepsilon_n}')^{p-1} \int_{\mathbb{R}^N} (|\nabla v_{\varepsilon_n}|^p + V(x)|v_{\varepsilon_n}|^p) dx + C_\delta (t_{\varepsilon_n}')^{p^*-1} \int_{\mathbb{R}^N} K(x)|v_{\varepsilon_n}|^{p^*} dx.
\]

Taking \(\delta = \frac{1}{2C}\), we obtain, from (3.11) that

\[
\int_{\mathbb{R}^N} (|\nabla v_{\varepsilon_n}|^p + V(x)|v_{\varepsilon_n}|^p) dx
\]
\[ \leq C_\delta (t\varepsilon_n')^{p-1} \int_{\mathbb{R}^N} K(x) |v_{\varepsilon_n}|^{p^*} \, dx + (t\varepsilon_n')^{p-1} \int_{\mathbb{R}^N} P(x) |v_{\varepsilon_n}|^{p^*} \, dx. \]

This is also impossible because \( p^* > p \).

Next, we suppose that \( (K_3) \) holds. By \((f_1), (f_2)\), there is a constant \( \tilde{C} > 0 \), such that

\[ \int_{\mathbb{R}^N} K(x)f(t\varepsilon_n v_{\varepsilon_n}) v_{\varepsilon_n} \, dx \]

\[ \leq (t\varepsilon_n')^{p_0-1} \int_{\mathbb{R}^N} K(x) |v_{\varepsilon_n}|^{p_0} \, dx + \tilde{C}(t\varepsilon_n')^{p-1} \int_{\mathbb{R}^N} K(x) |v_{\varepsilon_n}|^{p^*} \, dx. \]

It follows from (3.11) that

\[ (t\varepsilon_n')^{p-1} \int_{\mathbb{R}^N} (|\nabla v_{\varepsilon_n}|^p + V(x) |v_{\varepsilon_n}|^p) \, dx \]

\[ \leq (t\varepsilon_n')^{p_0-1} \mu \int_{\mathbb{R}^N} K(x) |v_{\varepsilon_n}|^{p_0} \, dx + \tilde{C}(t\varepsilon_n')^{p-1} \mu \int_{\mathbb{R}^N} K(x) |v_{\varepsilon_n}|^{p^*} \, dx \]

\[ + (t\varepsilon_n')^{p-1} \int_{\mathbb{R}^N} P(x) |v_{\varepsilon_n}|^{p^*} \, dx. \]

This is also impossible because \( p_0 > p \) and \( p^* > p \). So we complete the proof of our claim.

Since \( A_1 < t \varepsilon < A_2 \), together with the definition of \( V_{\text{max}} \) and \( K_{\text{min}} \), we have,

\[ I(t\varepsilon) \]

\[ = \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v_{\varepsilon}|^p + V(x) |v_{\varepsilon}|^p) \, dx - \int_{\mathbb{R}^N} K(x) F(t\varepsilon) \, dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} P(x) |v_{\varepsilon}|^{p^*} \, dx \]

\[ = \frac{t^p}{p} V_{\varepsilon} + \frac{t^p}{p} \int_{\mathbb{B}_{2R}(x_0)} V(x) |v_{\varepsilon}|^p \, dx - \int_{\mathbb{B}_{2R}(x_0)} K(x) F(t\varepsilon) \, dx - \frac{t^{p^*}}{p^*} \]

\[ \leq \frac{(t\varepsilon)^p}{p} V_{\varepsilon} + \frac{(t\varepsilon)^p}{p} \int_{\mathbb{B}_{2R}(x_0)} V(x) |v_{\varepsilon}|^p \, dx - \int_{\mathbb{B}_{2R}(x_0)} K(x) F(t\varepsilon) \, dx - \frac{(t\varepsilon)^{p^*}}{p^*} \]

\[ \leq \frac{(t\varepsilon)^p}{p} V_{\varepsilon} + \frac{(t\varepsilon)^p}{p} V_{\max} \int_{\mathbb{B}_{2R}(x_0)} |v_{\varepsilon}|^p \, dx - K_{\min} \int_{\mathbb{B}_{2R}(x_0)} F(t\varepsilon) \, dx - \frac{(t\varepsilon)^{p^*}}{p^*} \]

Since \( \frac{t^p}{p} V_{\varepsilon} - \frac{t^{p^*}}{p^*} \leq \frac{1}{N} V_{\varepsilon}^{\frac{N}{p}} \) for all \( t \geq 0 \), the estimate (3.9) on \( V_{\varepsilon} \) and the above inequality imply that

\[ \sup_{t \geq 0} I(t\varepsilon) = I(t\varepsilon) \]

\[ \leq \frac{1}{N} \int_{\mathbb{R}^N} (|P(x)|_{L^\infty(\mathbb{R}^N)})^{\frac{p-N}{p}} + O(\varepsilon^p) + O(\varepsilon^{(N-p)/(p-1)}) \]

\[ + \frac{(t\varepsilon)^p}{p} V_{\max} \int_{\mathbb{B}_{2R}(x_0)} |v_{\varepsilon}|^p \, dx - K_{\min} \int_{\mathbb{B}_{2R}(x_0)} F(t\varepsilon) \, dx \]

\[ \leq \frac{1}{N} \int_{\mathbb{R}^N} (|P(x)|_{L^\infty(\mathbb{R}^N)})^{\frac{p-N}{p}} - K_{\min} \int_{\mathbb{B}_{2R}(x_0)} F(t\varepsilon) \, dx + O(\varepsilon^p) \]
Then inequality (3.14) also follows from (3.13) if we choose \( \varepsilon \) small. From (3.2)-(3.4), we have

\[
\text{The conditions for the Mountain Pass Lemma [7] are satisfied by Lemma 3.1. From (3.2)-(3.4), we have}
\]

\[
I(u_n) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u_n|^p + V(x)|u_n|^p)\,dx - \int_{\mathbb{R}^N} K(x)F(u_n)\,dx - \frac{1}{p} \int_{\mathbb{R}^N} P(x)|u_n|^p\,dx
\]

\[
= c_0 + o_n(1),
\]

(3.15)
and

\[ I'(u_n)u_n = \int_{\mathbb{R}^N} (|\nabla u_n|^p + V(x)|u_n|^p)dx - \int_{\mathbb{R}^N} K(x)f(u_n)u_n dx - \int_{\mathbb{R}^N} P(x)|u_n|^{p^*}dx \\
= o_n(1)\|u_n\|. \tag{3.16} \]

Denote \( v_n = u_n - u \), then from (3.5), Lemma 2.3 and Brezis-Lieb Lemma [5],

\[ I(u_n) = I(u) + \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx - \frac{1}{p^*} \int_{\mathbb{R}^N} P(x)|v_n|^{p^*}dx \tag{3.17} \]

and

\[ I'(u_n)u_n = \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx - \int_{\mathbb{R}^N} K(x)f(u)u dx - \int_{\mathbb{R}^N} P(x)|u|^{p^*}dx \\
= \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx - \int_{\mathbb{R}^N} K(x)f(u)u dx - \int_{\mathbb{R}^N} P(x)|u|^{p^*}dx \\
+ \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx - \int_{\mathbb{R}^N} P(x)|v_n|^{p^*}dx + o_n(1). \tag{3.18} \]

Since \( I'(u_n) \to 0 \) as \( n \to \infty \), and by (3.5) again, we have

\[ I'(u_n)u = \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx - \int_{\mathbb{R}^N} K(x)f(u)u dx - \int_{\mathbb{R}^N} P(x)|u|^{p^*}dx + o_n(1). \tag{3.19} \]

By (3.18) and (3.19), we get

\[ \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx - \int_{\mathbb{R}^N} P(x)|v_n|^{p^*}dx \to 0 \quad \text{as} \quad n \to \infty \tag{3.20} \]

and

\[ I(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx - \int_{\mathbb{R}^N} K(x)f(u)dx - \frac{1}{p^*} \int_{\mathbb{R}^N} P(x)|u|^{p^*}dx \\
= \frac{1}{p} \int_{\mathbb{R}^N} K(x)f(u)dx - \int_{\mathbb{R}^N} K(x)f(u)dx + \left( \frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} P(x)|u|^{p^*}dx \\
\geq 0. \tag{3.21} \]

Without loss of generality we can suppose

\[ \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)dx \to \ell \quad \text{as} \quad n \to \infty, \tag{3.22} \]

and from (3.20) we have that

\[ \int_{\mathbb{R}^N} P(x)|v_n|^{p^*}dx \to \ell \quad \text{as} \quad n \to \infty. \tag{3.23} \]

We also have by Sobolev’s inequality that

\[ \int_{\mathbb{R}^N} |\nabla v_n|^pdx \geq S \left( \int_{\mathbb{R}^N} |v_n|^pdx \right)^{p/p^*} \]
\[ \geq S \left( \|P(x)\|_{L^\infty(\mathbb{R}^N)} \right)^{-p/p^*} \left( \int_{\mathbb{R}^N} P(x)|v_n|^{p^*}dx \right)^{p/p^*}. \tag{3.24} \]

Combining (3.22), (3.23) and (3.24), if \( \ell > 0 \), we have

\[ \ell \geq S \frac{N}{p} \left( \|P(x)\|_{L^\infty(\mathbb{R}^N)} \right)^{(p-N)/p}. \tag{3.25} \]
Taking the limit in (3.17) as \( n \to +\infty \), we have
\[
c_0 \geq \frac{1}{N} \ell \geq \frac{1}{N} S_\omega^\frac{N}{p} \left[ \left\| P(x) \right\|_{L^\infty(\mathbb{R}^N)} \right]^{(p-N)/p}. \tag{3.26}
\]
On the other hand, from (3.3) and Lemma 3.3, we have
\[
c_0 < \frac{1}{N} S_\omega^\frac{N}{p} \left[ \left\| P(x) \right\|_{L^\infty(\mathbb{R}^N)} \right]^{(p-N)/p}.
\]
A contradiction shows that \( \ell = 0 \) and thus
\[
I(u) = c_0 > 0 \quad \text{and} \quad I'(u) = 0,
\]
which gives that \( u \) is a nontrivial solution of (1.1). From (3.1) we may assume that \( u \geq 0 \). The strictly positivity of \( u \) is a consequence of Harnack-type inequality of Serrin ([20], Theorem 5) applied to an arbitrary ball in \( \mathbb{R}^N \). Tolksdorf’s theorem ([24], Theorem 1) implies the local \( C^{1,\alpha} \)-regularity of the solution. The proof is completed.

Remark 1. Applying Theorem 1.1 to the case when \( P(x) = 1 \), we can obtain the following corollary:

Corollary 1. Assume that \((V, K) \in K\) and \( f \) satisfies \((f_1)-(f_3)\). Then problem
\[
\begin{align*}
-\Delta_p u + V(x) |u|^{p-2} u &= K(x)f(u) + |u|^{p^*-2} u, \quad x \in \mathbb{R}^N, \\
u &\in D^{1,p}(\mathbb{R}^N),
\end{align*}
\]
has at least one positive solution \( u \) such that \( 0 < I(u) < \frac{1}{N} S_\omega^\frac{N}{p} \) if either \( N \geq p^2 \) or \( p < N < p^2 \) and \( \theta > p^* - \frac{p}{p^*-1} \).

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