TOPOLOGICAL OBSTRUCTIONS TO EMBEDDING OF A MATRIX
ALGEBRA BUNDLE INTO A TRIVIAL ONE

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ABSTRACT. In the present paper we describe topological obstructions to embedding of a (complex)
matrix algebra bundle into a trivial one under some additional arithmetic condition on their
dimensions. We explain a relation between this problem and some principal bundles with structure
groupoid. Finally, we briefly discuss a relation of our results to the twisted K-theory.

1. A HOMOTOPIE DESCRIPTION OF OBSTRUCTIONS

1.1. Introduction. The starting point of our work was the following question. Let $X$ be (say)
a compact manifold, $A_k \to X$ a locally trivial bundle with fibre a complex matrix algebra $M_k(\mathbb{C})$
(so its “natural” structure group is $\text{Aut}(M_k(\mathbb{C})) \cong \text{PGL}_k(\mathbb{C})$). Then is $A_k$ a subbundle of a (finite
dimensional) trivial bundle $X \times M_n(\mathbb{C})$, i.e. is there a fiberwise map (in fact embedding)

\[
\begin{array}{ccc}
A_k & \xrightarrow{\mu} & X \times M_n(\mathbb{C}) \\
\downarrow & & \downarrow \\
X & \rightarrow & \\
\end{array}
\]

such that $\forall x \in X$ its restriction $\mu |_x$ embeds the fibre $(A_k)_x$ into $M_n(\mathbb{C})$ as a unital subalgebra?

It is natural to compare this question with the well-known fact that any vector bundle $\xi$ over a
compact base $X$ is a subbundle of a product bundle $X \times \mathbb{C}^n$.

Obviously, a unital homomorphism $M_k(\mathbb{C}) \to M_n(\mathbb{C})$ exists only if $n = kl$ for some $l \in \mathbb{N}$.
Clearly, as in the case of vector bundles $n$ should be large enough relative to $\dim(X)$; thus, the
initial question can be reformulated as follows: are there “stable” (i.e. non-vanishing when $l$
grows) obstructions to existence of embedding (1)?

It turns out that (taking into account the previous remark) the answer is positive if we do not
impose any additional condition on $l$. But if we require, say, $l$ to be relatively prime to $k$, then
stable obstructions arise.

It is convenient to replace the groups $\text{PGL}_n(\mathbb{C})$ by compact ones $\text{PU}(n)$ considering only $*$-
homomorphisms instead of all unital homomorphisms of matrix algebras. Since $\text{PU}(n)$ is a deforma-
tion retract of $\text{PGL}_n(\mathbb{C})$ this does not have any effect on the homotopy theory.

1.2. The main construction. The obstructions can be described more explicitly by reducing
the embedding problem (1) to a lifting problem for a suitable fibration. The next construction
can be regarded as a version of a “bijection” $\text{Mor}(X \times Y, Z) \to \text{Mor}(X, \text{Mor}(Y, Z))$ adapted to
the case of fibrations (“Mor” means “morphisms”).
So, let \( \text{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \) be the set of all unital \(*\)-homomorphisms \( M_k(\mathbb{C}) \to M_{kl}(\mathbb{C}) \). It follows from Noether-Skolem’s theorem \([7]\) that there is the representation
\[
\text{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \cong \text{PU}(kl)/(E_k \otimes \text{PU}(l))
\]
(here and below the tensor product symbol \( \otimes \) denotes the Kronecker product of matrices) in the form of homogeneous space of the group \( \text{PU}(kl) \). For short we denote this space by \( \text{Fr}_{k,l} \) (“Fr” refers to “frame”). Together with the Bott periodicity this representation allows us to compute the stable (i.e. low dimensional) homotopy groups of this space:
\[
\pi_r(\text{Fr}_{k,l}) \cong \mathbb{Z}/k\mathbb{Z} \quad \text{for } r \text{ odd and } \pi_r(\text{Fr}_{k,l}) = 0 \quad \text{for } r \text{ even.}
\]

Let \( A_{k}^{\text{univ}} \to \text{BPU}(k) \) be the universal \( M_k(\mathbb{C}) \)-bundle. Applying the functor (taking values in the category of topological spaces) \( \text{Hom}_{\text{alg}}(\ldots, M_{kl}(\mathbb{C})) \) to \( A_{k}^{\text{univ}} \) fiberwisely, we obtain the fibration
\[
\begin{array}{ccc}
\text{Fr}_{k,l} & \longrightarrow & \text{H}_{k,l}(A_{k}^{\text{univ}}) \\
& \downarrow^{p_{k,l}} & \\
& \text{BPU}(k).
\end{array}
\]

It is easy to see that there exists the canonical embedding of \( M_k(\mathbb{C}) \)-bundle \( p_{k,l}^{*}(A_{k}^{\text{univ}}) \to \text{H}_{k,l}(A_{k}^{\text{univ}}) \) into the product bundle \( \text{H}_{k,l}(A_{k}^{\text{univ}}) \times M_{kl}(\mathbb{C}) \).

Let
\[
\tilde{f}: X \to \text{BPU}(k)
\]
be a classifying map for \( A_{k} \), i.e. \( A_{k} = \tilde{f}^{*}(A_{k}^{\text{univ}}) \). Now it is easy to see that an embedding \([\dagger]\) with \( n = kl \) is the same thing as a lift \( \tilde{f} \) of the classifying map \( f \),
\[
\tilde{f}: X \to \text{H}_{k,l}(A_{k}^{\text{univ}}), \quad p_{k,l} \circ \tilde{f} = \tilde{f},
\]
and vice versa, such a lift defines an embedding. Thus we have the following proposition.

**Proposition 1.** There is a natural one-to-one correspondence between embeddings \([\dagger]\) of \( A_{k} = \tilde{f}^{*}(A_{k}^{\text{univ}}) \) and lifts \( \tilde{f} \) of its classifying map \( f \) in \([\dagger]\).

So the lift of \( f \) corresponding to an embedding \( \mu \) we denote by \( \tilde{f}_{\mu} \). Clearly, we also have a one-to-one correspondence between homotopy classes of embeddings and (fiberwise) homotopy classes of lifts given by \([\mu] \mapsto [\tilde{f}_{\mu}]\).

It turns out that the total space \( \text{H}_{k,l}(A_{k}^{\text{univ}}) \) of fibration \([\dagger]\) is homotopy equivalent to the so-called matrix Grassmannian \( \text{Gr}_{k,l} \), the homogeneous space parametrizing the set of \( k \)-subalgebras (i.e. unital \(*\)-subalgebras isomorphic \( M_k(\mathbb{C}) \)) in the algebra \( M_{kl}(\mathbb{C}) \). Note that it can be represented as
\[
\text{Gr}_{k,l} \cong \text{PU}(kl)/(\text{PU}(k) \otimes \text{PU}(l))
\]
according to Noether-Skolem’s theorem. The mentioned homotopy equivalence
\[
\tau_{k,l}: \text{H}_{k,l}(A_{k}^{\text{univ}}) \xrightarrow{\cong} \text{Gr}_{k,l}
\]
Remark is defined as follows: it takes a point \( h \in H_{k,l}(A^{univ}_k) \) such that \( p_{k,l}(h) = x \in BPU(k) \) to the \( k \)-subalgebra \( h((A^{univ}_k)_x) \subset M_{kl}(\mathbb{C}) \) (here we identify points in \( Gr_{k,l} \) with \( k \)-subalgebras in \( M_{kl}(\mathbb{C}) \)). Note that in fact \( \tau_{k,l} \) is a fibration with contractible fibers \( EPU(k) \) (the total space of the universal principal \( PU(k) \)-bundle).

The tautological \( M_k(\mathbb{C}) \)-bundle \( A_{k,l} \rightarrow Gr_{k,l} \) can be defined as a subbundle in the product bundle \( Gr_{k,l} \times M_{kl}(\mathbb{C}) \) consisting of all pairs \( \{(x, T) \mid x \in Gr_{k,l}, T \in M_{k,x} \subset M_{kl}(\mathbb{C})\} \), where \( M_{k,x} \) denotes the \( k \)-subalgebra corresponding to \( x \in Gr_{k,l} \). Clearly, the above constructed homotopy equivalence \( H_{k,l}(A^{univ}_k) \cong Gr_{k,l} \) identifies \( p^*_k(A^{univ}_k) \hookrightarrow H_{k,l}(A^{univ}_k) \times M_{kl}(\mathbb{C}) \) with \( A_{k,l} \hookrightarrow Gr_{k,l} \times M_{kl}(\mathbb{C}) \).

Remark 2. The matrix Grassmannians \( Gr_{k,l} \) classify equivalence classes of pairs \( (A_k, \mu) \) over finite \( CW \)-complexes \( X \), where \( A_k \rightarrow X \) is a locally trivial \( M_k(\mathbb{C}) \)-bundle over \( X \) and \( \mu \) is an embedding \( A_k \rightarrow X \times M_{kl}(\mathbb{C}) \) (see \( \text{(1)} \)). Two such pairs \( (A_k, \mu), (A'_k, \mu') \) are equivalent if \( A_k \cong A'_k \) and \( \mu \) is homotopic to \( \mu' \).

1.3. The first obstruction. Now let us give the promised description of obstructions to lifting in fibration \( \text{(1)} \). First, consider the first obstruction. According to the obstruction theory, it is a characteristic class \( A_k \mapsto \bar{\omega}_1(A_k) = \bar{f}^*(\bar{\omega}_1) \in H^2(X, \mathbb{Z}/k\mathbb{Z}) \), where \( \bar{\omega}_1 := \bar{\omega}_1(A^{univ}_k) \in H^2(BPU(k), \mathbb{Z}/k\mathbb{Z}) \).

**Theorem 3.** The first obstruction is the obstruction to the reduction (or lift) of the structure group \( PU(k) \) of the bundle \( A_k \xrightarrow{p_k} X \) to \( SU(k) \) (here we mean the exact sequence of groups \( 1 \rightarrow \rho_k \rightarrow SU(k) \xrightarrow{\delta} PU(k) \rightarrow 1 \), where \( \rho_k \) is the group of \( k \)-th roots of unity).

**Proof.** Note that in our case \( (k, l) = 1 \) the projective unitary groups in representation \( \text{(6)} \) can be replaced by special unitary ones, i.e. the matrix Grassmannian has the equivalent representation \( \text{(8)} \)

\[
Gr_{k,l} \cong SU(kl)/(SU(k) \otimes SU(l)).
\]

This follows from the obvious fact that if \( k \) and \( l \) are relatively prime, then the center of \( SU(kl) \) (which is the group \( \rho_{kl} \) of \( k \)-th roots of unity) is the product \( \rho_k \times \rho_l \) of centers of \( SU(k) \) and \( SU(l) \). Hence the structure group of \( M_k(\mathbb{C}) \)-bundles \( A_{k,l} \rightarrow Gr_{k,l} \) and \( p^*_k(A^{univ}_k) \rightarrow H_{k,l}(A^{univ}_k) \) is \( SU(k) \). From the other hand, if the structure group of \( A_k \) can be reduced to \( SU(k) \), then \( \omega_1(A_k) = 0 \) because \( BSU(k) \) is 3-connected. \( \square \)

Obviously, \( \bar{\omega}_1 \) is a generator of \( H^2(BPU(k), \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z} \).

Now assume that \( A_k \) has the form \( End(\xi_k) \), where \( \xi_k \rightarrow X \) is a vector \( \mathbb{C}^k \)-bundle (not every \( M_k(\mathbb{C}) \)-bundle can be represented in this form; the obstruction is the class \( \delta(\bar{\omega}_1(A_k)) \in Br(X) := H^3_{tors}(X, \mathbb{Z}) \), where \( \delta : H^2(X, \mathbb{Z}/k\mathbb{Z}) \rightarrow H^3(X, \mathbb{Z}) \) is the coboundary homomorphism corresponding to the coefficient sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/k\mathbb{Z} \rightarrow 0).
\]

**Theorem 4.** For \( A_k = End(\xi_k) \) the first obstruction is \( c_1(\xi_k) \mod k \in H^2(X, \mathbb{Z}/k\mathbb{Z}) \), where \( c_1 \) is the first Chern class.
Proof. Let $\xi_k^\text{univ} \to \text{BU}(k)$ be the universal $\mathbb{C}P^k$-bundle. Applying the functor $\text{Hom}_{\text{alg}}(\ldots, M_{kl}(\mathbb{C}))$ to the $M_k(\mathbb{C})$-bundle $\text{End}(\xi_k^\text{univ}) \to \text{BU}(k)$ fiberwisely, we obtain the fibration (cf. (11)):

$$
\begin{array}{ccc}
\text{Fr}_{k,l} & \longrightarrow & H_{k,l}(\text{End}(\xi_k^\text{univ})) \\
\downarrow \tilde{\rho}_{k,l} & & \downarrow \rho_{k,l} \\
\text{BU}(k). & & 
\end{array}
$$

(9)

Note that there is the canonical embedding $\tilde{\rho}_{k,l}(\text{End}(\xi_k^\text{univ})) \hookrightarrow H_{k,l}(\text{End}(\xi_k^\text{univ})) \times M_{kl}(\mathbb{C})$. Now it is easy to see that an embedding $\text{End}(\xi_k) \hookrightarrow X \times M_{kl}(\mathbb{C})$ is the same thing as a lift in (9) of the classifying map $f: X \to \text{BU}(k)$ for $\xi_k$.

We have the pullback diagram

$$
\begin{array}{ccc}
H_{k,l}(\text{End}(\xi_k^\text{univ})) & \xrightarrow{B\chi_k} & H_{k,l}(A_k^\text{univ}) \\
\downarrow \tilde{\rho}_{k,l} & & \downarrow \rho_{k,l} \\
\text{BU}(k) & \xrightarrow{B\chi_k} & \text{BPU}(k),
\end{array}
$$

(10)

where $B\chi_k$ is the map of classifying spaces $\text{BU}(k) \to \text{BPU}(k)$ induced by the group epimorphism $\chi_k: U(k) \to \text{PU}(k)$ (or equivalently by the classifying map for $\text{End}(\xi_k^\text{univ})$ as an $M_k(\mathbb{C})$-bundle). In particular, our obstruction is the class $\omega_1 := B\chi_k(\bar{\omega}_1) \in H^2(\text{BU}(k), \mathbb{Z}/k\mathbb{Z})$. We have to prove that $\omega_1 = c_1(\xi_k^\text{univ}) \bmod k$.

We claim that the structure group of $\text{End}(\xi_k)$ can be reduced to $\text{SU}(k)$ iff $c_1(\xi_k) \equiv 0 \bmod k$. Indeed, $c_1(\xi_k) \equiv 0 \bmod k \iff c_1(\xi_k) = k \alpha, \alpha \in H^2(X, \mathbb{Z})$. There is a line bundle $\zeta' \to X$ (which is unique up to isomorphism) such that $c_1(\zeta') = -\alpha$. Then $c_1(\xi_k \otimes \zeta') = c_1(\xi_k) + kc_1(\zeta') = 0$, i.e. $\xi_k \otimes \zeta'$ is an $\text{SU}(k)$-bundle. From the other hand, $\text{End}(\xi_k) = \text{End}(\xi_k \otimes \zeta')$. Conversely, the classifying map $B\chi_k: \text{BU}(k) \to \text{BPU}(k)$ for $\text{End}(\xi_k^\text{univ})$ as a $\text{PU}(k)$-bundle has the fiber $\mathbb{C}P^\infty$. It follows from the obstruction theory that $\text{End}(\xi_k) \cong \text{End}(\xi_k')$ as $M_k(\mathbb{C})$-bundles iff $\xi_k' = \xi_k \otimes \zeta'$ for some line bundle $\zeta' \to X$. Clearly, $c_1(\xi_k) \equiv c_1(\xi_k') \bmod k$ and $\xi_k'$ is an $\text{SU}(k)$-bundle $\iff c_1(\xi_k') = 0 \iff c_1(\xi_k) \equiv 0 \bmod k$. □

Remark 5. Let us describe the relation between two versions ("PU" and "U") of obstructions and the Brauer group $Br(X) = H^3_{\text{tors}}(X, \mathbb{Z})$. Consider the exact coefficient sequence

$$
0 \to \mathbb{Z} \xrightarrow{\lambda} k \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0
$$

and a piece of the corresponding cohomology sequence:

$$
H^2(X, \mathbb{Z}) \xrightarrow{\delta} H^2(X, \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\delta} H^3(X, \mathbb{Z}).
$$

Then $\delta(\bar{\omega}_1(\text{A}_k)) = 0 \iff \text{A}_k$ has the form $\text{End}(\xi_k)$ for some vector $U(k)$-bundle $\xi_k$ (Note that $\delta(\bar{\omega}_1(\text{A}_k)) \in H^2(X, \mathbb{Z})$ is exactly the class of $\text{A}_k$ in the Brauer group). If $\delta(\bar{\omega}_1(\text{A}_k)) = 0$, then $\bar{\omega}_1(\text{A}_k) = \lambda(c_1(\xi_k))$, where $\lambda$ is the reduction modulo $k$. But $\xi_k$ such that $\text{End}(\xi_k) = \text{A}_k$ is not unique: $\xi_k' = \xi_k \otimes \zeta'$ also suits. Clearly, $c_1(\xi_k') \equiv c_1(\xi_k) \bmod k$ and $c_1(\xi_k) \equiv 0 \bmod k \iff \xi_k' = \xi_k \otimes \zeta'$ is an $\text{SU}(k)$-bundle for some $\zeta'$. \[ \square \]
1.4. The second obstruction. Now assume that for the bundle $A_k \xrightarrow{p_k} X$ the first obstruction is equal to 0. We have shown that such a bundle has the form $\text{End}(\tilde{\xi}_k)$ for some vector $\mathbb{C}^k$-bundle $\tilde{\xi}_k$ with the structure group $\text{SU}(k)$. Equivalently, the classifying map $\bar{f}: X \to \text{BPU}(k)$ (5) can be lifted to $f: X \to \text{BSU}(k)$. It follows from standard facts of topological obstruction theory and given above (stable) homotopy groups of the space $\text{Fr}_{k,l} = \text{Hom}_{alg}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$ (see (3)) that the next obstruction belongs to $H^4(X, \mathbb{Z}/k\mathbb{Z})$.

**Theorem 6.** The second obstruction is $c_2(\tilde{\xi}_k) \mod k$, where $c_2$ is the second Chern class.

**Proof.** To show this, first note that the space $\text{Fr}_{k,l}$ has the universal covering

$$\rho_k \rightarrow \tilde{\text{Fr}}_{k,l} \rightarrow \text{Fr}_{k,l}.$$  

Hence $\pi_r(\tilde{\text{Fr}}_{k,l}) = \pi_r(\text{Fr}_{k,l})$ for $r \geq 2$ and $\pi_1(\tilde{\text{Fr}}_{k,l}) = 0$ (while $\pi_1(\text{Fr}_{k,l}) = \mathbb{Z}/k\mathbb{Z}$). Obviously, $\tilde{\text{Fr}}_{k,l} \cong \text{SU}(kl)/(E_k \otimes \text{SU}(l))$ (cf. (2)).

Now consider the following diagram:

$$\begin{array}{ccc}
\text{Fr}_{k,l} & \longrightarrow & \text{EPU}(k) \times_{\text{PU}(k)} \text{Fr}_{k,l} \\
\tilde{\text{Fr}}_{k,l} & \longrightarrow & \text{ESU}(k) \times_{\text{SU}(k)} \tilde{\text{Fr}}_{k,l} \\
 & \longrightarrow & \text{BPU}(k) \\
 & \longrightarrow & \text{BSU}(k),
\end{array}$$

where $p_{k,l}$ is fibration (11). Note that the homotopy equivalence $\tilde{\tau}_{k,l}: \text{ESU}(k) \times_{\text{SU}(k)} \tilde{\text{Fr}}_{k,l} \simeq \text{Gr}_{k,l}$ (cf. (7)) can easily be deduced from representation (8). $\pi_3(\tilde{\text{Fr}}_{k,l}) = \mathbb{Z}/k\mathbb{Z}$ ⇒ the “universal” obstruction is a characteristic class $\omega_2 \in H^4(\text{BSU}(k), \mathbb{Z}/k\mathbb{Z})$.

Let $\tilde{\xi}^{\text{univ}}_k \rightarrow \text{BSU}(k)$ be the universal SU($k$)-bundle. Since $c_2(\tilde{\xi}^{\text{univ}}_k) \mod k$ is a generator of $H^4(\text{BSU}(k), \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$, we see that

$$\omega_2 = \alpha c_2(\tilde{\xi}^{\text{univ}}_k) \mod k \in H^4(\text{BSU}(k), \mathbb{Z}/k\mathbb{Z}), \alpha \in \mathbb{Z}.$$  

We have the commutative diagram

$$\begin{array}{ccc}
\text{ESU}(k) \times_{\text{SU}(k)} \tilde{\text{Fr}}_{k,l} & \longrightarrow & \text{Gr}_{k,l} \\
\tilde{p}_{k,l} & \longrightarrow & \text{BPU}(k),
\end{array}$$

where $\lambda_{k,l}$ is the classifying map for $A_{k,l} \rightarrow \text{Gr}_{k,l}$ as an SU($k$)-bundle. Thus, the piece of the homotopy sequence for the “SU”-part of (12)

$$\pi_4(\tilde{\text{Fr}}_{k,l}) \to \pi_4(\text{ESU}(k) \times_{\text{SU}(k)} \tilde{\text{Fr}}_{k,l}) \to \pi_4(\text{BSU}(k)) \to \pi_3(\tilde{\text{Fr}}_{k,l}) \to \pi_3(\text{ESU}(k) \times_{\text{SU}(k)} \tilde{\text{Fr}}_{k,l})$$
is exactly
\[
0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z} \to 0
\]
⇒ the image \(\pi_4(ESU(k) \times \tilde{F}_k) \xrightarrow{\iota} \pi_4(BSU(k))\) is the subgroup of index \(k\).

Now take \(X = S^4\) and consider the group homomorphism \(\pi_4(BSU(k)) \to H^4(S^4, \mathbb{Z}/k\mathbb{Z}), \ [g] \mapsto g^*\omega_2\), where \(g: S^4 \to BU(k)\) and \([g] \in \pi_4(BSU(k))\) the corresponding homotopy class. If \(k \mid [g]\) in \(\pi_4(BSU(k)) \cong \mathbb{Z}\) then \(g^*\omega_2 \neq 0\) because \(g^*\omega_2\) is the unique obstruction to embedding in this case. Hence \(\alpha\) in \([13]\) is invertible modulo \(k\), in particular we can take \(\alpha = 1\). \(\square\)

Note that the obstructions are stable in the sense that they do not vanish when we take the direct limit over pairs \(\{k, l\}\) satisfying the condition \((k, l) = 1\).

1.5. **On “higher” obstructions.** In general, “higher” obstructions (in stable dimensions) are in \(H^{2r}(X, \mathbb{Z}/k\mathbb{Z})\), \(r \in \mathbb{N}\). But for \(r > 2\) they do not coincide with the Chern classes reduced modulo \(k\). To see this, take \(X = S^8\) and consider a 6-dimensional vector bundle \(\xi_6 \to S^8\). It is well-known \([4]\) that for \(S^{2r}\) the Chern classes of complex vector bundles form the subgroup of index \((r − 1)!\) in \(H^{2r}(S^{2r}, \mathbb{Z}) \cong \mathbb{Z}\). In particular, in our case \(r = 4, k = 6\) we have \(c_4(\xi_6) \equiv 0 \pmod{6}\), but it follows from the homotopy sequence of fibration \([4]\) (or \([12]\)) that not every such a bundle has a lift.

In order to go further, one can use the modification of Chern classes for connected covers of \(BU\). More precisely, let \(\iota: BU(2r) \to BU\) be the connective cover of \(BU\) whose first non-zero homotopy is in degree \(2r\) (thus \(BU(2) = BU, BU(4) = BU, \ldots\)). Then the image of the \(r\)’th Chern class under the pullback \(\iota^*: H^*(BU, \mathbb{Z}) \to H^*(BU(2r), \mathbb{Z})\) is divisible by \((r − 1)!\) \([6]\). Put \(\bar{c}_r := \frac{\iota^*(c_r)}{(r − 1)!}\).

The following theorem generalizes Theorems \([4]\) and \([6]\).

**Theorem 7.** For bundles classified by \(BU(2r)\) the first obstruction to the above lifting problem is \(\bar{c}_r \mod k\).

**Proof.** For the connective cover \(\iota_k: BU(k)(2r) \to BU(k), k > r\) consider the Fr\(_k, r\)-fibration
\[
i_k^*(H_k(\text{End}(\xi_k^{\text{univ}}))) \to BU(k)(2r)
\]
induced from \([11]\). Clearly, the first obstruction to lifting in this fibration is a characteristic class \(\omega_r \in H^{2r}(BU(k)(2r), \pi_{2r−1}(Fr_k)) = H^{2r}(BU(k)(2r), \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}\).

It follows from the homotopy sequence of fibration \([14]\) that \(\pi_{2r}(i_k^*(H_k(\text{End}(\xi_k^{\text{univ}})))) \cong \mathbb{Z}\), the homomorphism \(\pi_{2r}(i_k^*(H_k(\text{End}(\xi_k^{\text{univ}})))) \to \pi_{2r}(BU(k)(2r))\) is injective and its image is the subgroup of index \(k\) in \(\pi_{2r}(BU(k)(2r)) \cong \mathbb{Z}\).

Now using the same argument as in the proof of Theorem \([6]\) with \(S^{2r}\) in place of \(S^4\), we see that for the bundle \(\xi_k \to S^{2r}\) corresponding to the generator \(1 \in \pi_{2r}(BU(k)) \cong \mathbb{Z}\) the class \(\omega_r\) is a generator of \(H^{2r}(S^{2r}, \mathbb{Z}/k\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}\), i.e. we can put \(\omega_r = \bar{c}_r \mod k\), as claimed. \(\square\)

\(^1\)I am grateful to Professor Thomas Schick for bringing this result to my attention.
The obtained result can also be reformulated as follows. Let $\xi_k \to X$ be a vector $\mathbb{C}^k$-bundle such that $c_1(\xi_k) = 0$, $c_2(\xi_k) = 0$. Then its classifying map $f_ξ: X \to \text{BSU}(k)$ can be lifted to $\text{BU}(k)(6)$:

$$
\begin{array}{c}
\cdots \to \text{BU}(k)(8) \xrightarrow{K(\mathbb{Z},5)} \text{BU}(k)(6) \xrightarrow{K(\mathbb{Z},3)} \text{BU}(k)(4) = \text{BSU}(k)
\end{array}
$$

(the upper row in the diagram is the Whitehead tower for $\text{BSU}(k)$). In fact, the space $\text{BU}(k)(6)$ represents some refined theory of bundles with $c_2 = 0$ and we can regard the lift $f_ξ^{(3)}$ as a classifying map for some bundle $ξ_k^{(3)}$ of this type. Thus, we have the characteristic class $\tilde{c}_3(ξ_k^{(3)}) := f_ξ^{(3)*}(\tilde{c}_3) \in H^6(X, \mathbb{Z})$. If $\tilde{c}_3(ξ_k^{(3)}) = 0$, then we choose a lift $f_ξ^{(4)}$ (see the above diagram) corresponding to some bundle $ξ_k^{(4)}$ (of even more subtle type of bundles with $\tilde{c}_2 = 0$, $\tilde{c}_3 = 0$) with the characteristic class $\tilde{c}_4(ξ_k^{(4)}) \in H^8(X, \mathbb{Z})$, etc. Suppose that starting with the bundle $ξ_k \to X$ we obtain a sequence of bundles $ξ_k^{(i)}$, $i \leq r$, $\tilde{c}_i(ξ_k^{(i)}) = 0$ for $i < r$. Then the first obstruction for embedding $μ: \text{End}(ξ_k^{(r)}) \to X \times M_{kl}(\mathbb{C})$, $(k, l) = 1$ is $\tilde{c}_r(ξ_k^{(r)})$ mod $k$.

We can also describe higher obstructions without extra conditions on integer characteristic classes. Put $κ_2 := c_2 \mod k \in H^4(\text{BSU}(k), \mathbb{Z}/k\mathbb{Z})$. Let $κ_2: \text{BSU}(k) \to K(\mathbb{Z}/k\mathbb{Z}, 4)$ be also the corresponding map, $F(κ_2)$ its homotopic fiber. Clearly, $(κ_2)_* : π_4(\text{BSU}(k))(\cong \mathbb{Z}) \to π_4(K(\mathbb{Z}/k\mathbb{Z}, 4))(\cong \mathbb{Z}/k\mathbb{Z})$ is onto, and it follows from the homotopy sequence of the fibration

$$
(15) \quad F(κ_2) \to \text{BSU}(k) \xrightarrow{κ_2} K(\mathbb{Z}/k\mathbb{Z}, 4)
$$

that $π_4(F(κ_2)) \cong \mathbb{Z}$ and the homomorphism $π_4(F(κ_2)) \to π_4(\text{BSU}(k))$ is $\mathbb{Z} \to \mathbb{Z}$, $1 \mapsto k$. Clearly, the fiber inclusion $F(κ_2) \to \text{BSU}(k)$ induces isomorphisms $π_r(F(κ_2)) \cong π_r(\text{BSU}(k))$ for $r \neq 4$.

Now consider the diagram

$$
(16)
\begin{array}{c}
\begin{array}{ccc}
\tilde{\text{Fr}}_{k,l} & \to & \text{Fr}_{k,l}^{[2]} \\
\downarrow & & \downarrow \\
\text{Gr}_{k,l} & \to & \text{Gr}_{k,l}^{[2]}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccc}
\tilde{\text{Fr}}_{k,l} & \to & \text{Fr}_{k,l}^{[2]} \\
\downarrow & & \downarrow \\
\text{Gr}_{k,l} & \to & \text{Gr}_{k,l}^{[2]}
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccc}
\tilde{\text{Fr}}_{k,l} & \to & \text{Fr}_{k,l}^{[2]} \\
\downarrow & & \downarrow \\
\text{Gr}_{k,l} & \to & \text{Gr}_{k,l}^{[2]}
\end{array}
\end{array}
\end{array}
\end{array}
$$

containing three fibrations. The map $p_{k,l}^{[2]}$ exists because $κ_2 \circ \tilde{p}_{k,l} \simeq *$ and $\text{Fr}_{k,l}^{[2]}$ is the homotopic fiber of $p_{k,l}^{[2]}$.

Consider the following piece of the morphism of homotopic sequences of fibrations (16):

$$
\begin{array}{cccccccc}
\pi_5(F(κ_2)) & \to & π_4(Fr_{k,l}^{[2]}) & \to & π_4(\text{Gr}_{k,l}) & \to & π_4(F(κ_2)) & \to & π_3(Fr_{k,l}^{[2]}) & \to & π_3(\text{Gr}_{k,l}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
π_5(\text{BSU}(k)) & \to & π_4(\text{Fr}_{k,l}) & \to & π_4(\text{Gr}_{k,l}) & \to & π_4(\text{BSU}(k)) & \to & π_3(\text{Fr}_{k,l}) & \to & π_3(\text{Gr}_{k,l})
\end{array}
$$
i.e.

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \pi_4(\text{Fr}^{[2]}_{k,l}) & \longrightarrow & \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} & \longrightarrow & \pi_3(\text{Fr}^{[2]}_{k,l}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & = \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/k\mathbb{Z} & \longrightarrow & 0
\end{array}
\]

which shows that \(\pi_3(\text{Fr}^{[2]}_{k,l}) = 0 = \pi_4(\text{Fr}^{[2]}_{k,l})\). One can easily verify that the map \(\text{Fr}^{[2]}_{k,l} \to \widetilde{\text{Fr}}_{k,l}\) (see (16)) induces isomorphisms \(\pi_r(\text{Fr}^{[2]}_{k,l}) \cong \pi_r(\widetilde{\text{Fr}}_{k,l})\) for \(r \neq 3\). In particular, \(\pi_{2r+1}(\text{Fr}^{[2]}_{k,l}) \cong \mathbb{Z}/k\mathbb{Z}\) for \(r \geq 2\) and 0 otherwise.

Now assume that for \(\text{SU}(k)\)-bundle \(\xi_k \to X \kappa_2(\xi_k) = 0\). Suppose we have chosen a lift \(f^{[2]}_\xi: X \to \text{F}(\kappa_2)\) of its classifying map \(f_\xi: X \to \text{BSU}(k)\). The next obstruction is a characteristic class \(\kappa_3 \in H^6(\text{F}(\kappa_2), \pi_5(\text{Fr}^{[2]}_{k,l})) = H^6(\text{F}(\kappa_2), 6)\). Consider the corresponding fibration (cf. (15)):

\[
\text{F}(\kappa_3) \to \text{F}(\kappa_2) \xrightarrow{\kappa_3} K(\mathbb{Z}/k\mathbb{Z}, 6)
\]

and the corresponding diagram (cf. (15)):

\[
\begin{array}{ccccc}
\text{Fr}^{[2]}_{k,l} & \longrightarrow & \text{Fr}^{[3]}_{k,l} \\
\downarrow & & \downarrow \\
\text{Gr}_{k,l} & \longrightarrow & \text{Gr}_{k,l}
\end{array}
\]

\[
\begin{array}{ccccc}
\text{F}(\kappa_3) & \longrightarrow & \text{F}(\kappa_2) & \longrightarrow & K(\mathbb{Z}/k\mathbb{Z}, 6)
\end{array}
\]

We claim that \((\kappa_3)_*: \pi_6(\text{F}(\kappa_2)) = \mathbb{Z} \to \pi_6(K(\mathbb{Z}/k\mathbb{Z}, 6)) = \mathbb{Z}/k\mathbb{Z})\) is onto. Indeed, take \(\varphi: S^6 \to \text{F}(\kappa_2)\) such that \(k \nmid [\varphi] \in \pi_6(\text{F}(\kappa_2))\) \(\Rightarrow\) \(\varphi\) does not have a lift \(\tilde{\varphi}: S^6 \to \text{Gr}_{k,l}\) (because \((p^{[2]}_{k,l})_*: \pi_6(\text{Gr}_{k,l}) \to \pi_6(\text{F}(\kappa_2))\) is the homomorphism \(\mathbb{Z} \to \mathbb{Z}, 1 \mapsto k\) \(\Rightarrow\) \(\varphi^*(\kappa_3) \neq 0 \in H^6(S^6, \mathbb{Z}/k\mathbb{Z})\) (because this is the unique obstruction in this case).

It follows that the homotopic fiber inclusion \(\text{F}(\kappa_3) \to \text{F}(\kappa_2)\) induces the homomorphism \(\pi_6(\text{F}(\kappa_3)) \to \pi_6(\text{F}(\kappa_2)), \mathbb{Z} \to \mathbb{Z}, 1 \mapsto k\) in dimension 6 and isomorphisms \(\pi_r(\text{F}(\kappa_3)) \xrightarrow{\kappa_3} \pi_r(\text{F}(\kappa_2))\) in other dimensions \(r \neq 6\). Using the above argument one can show that \(\pi_{2r+1}(\text{Fr}^{[3]}_{k,l}) \cong \mathbb{Z}/k\mathbb{Z}\) for \(r \geq 3\) and 0 otherwise. In the same way we obtain the characteristic classes \(\kappa_4 \in H^8(\text{F}(\kappa_3), \mathbb{Z}/k\mathbb{Z})\), \(\kappa_5 \in H^{10}(\text{F}(\kappa_4), \mathbb{Z}/k\mathbb{Z})\), etc., each of which is defined on the kernel of the predecessor. Note that \((p^{[3]}_{k,l})_*: \pi_r(\text{Gr}_{k,l}) \to \pi_r(\text{F}(\kappa_3))\) is an isomorphism for \(r \leq 2t + 1\).

The obtained results can be summarized by the following theorem. Suppose \(\xi_k \to X\) is an \(\text{SU}(k)\)-bundle with the classifying map \(f_\xi: X \to \text{BSU}(k)\), \(\text{dim } X < 2 \min\{k, l\}\).
Theorem 8. There is a lift \( \hat{f}_\xi : X \to \text{Gr}_{k,1} \) of \( f_\xi \) iff there is a sequence of maps \( f_{\xi}^{[i-1]} \), \( 3 \leq i \leq (\dim X)/2 \) making the diagram \((i \geq 2)\)

\[
\begin{array}{ccc}
\cdots & \longrightarrow & F(\kappa_{i+1}) \\
\downarrow p_{k,l}^{[i+1]} & | & \downarrow p_{k,l}^{[i]} \\
F(\kappa_i) & \longrightarrow & F(\kappa_{i-1}) \\
\downarrow f_{\xi}^{[i+1]} & | & \downarrow f_{\xi}^{[i]} \\
X & \longrightarrow & \text{Gr}_{k,l}
\end{array}
\]

commutative \((F(\kappa_1) := \text{BSU}(k), \ p_{k,l}^{[1]} = \tilde{p}_{k,l}, \ f_{\xi}^{[1]} = f_\xi)\) and such that \((f_{\xi}^{[i-1]})^* (\kappa_i) = 0, \ 2 \leq i \leq (\dim X)/2, \) where \( \kappa_i \in H^{2i}(F(\kappa_{i-1}), \mathbb{Z}/k\mathbb{Z}) \) are the above defined characteristic classes. (Note that by definition we put \( F(\kappa_1) := \text{BSU}(k) \), but it is more natural to define \( \kappa_1 := c_1 \mod k \) and put \( F(\kappa_1) \) to be the homotopy fiber of \( \text{BU}(k) \xrightarrow{\text{iso}} K(\mathbb{Z}/k\mathbb{Z}, 2) \)). In other words, a lift \( \hat{f}_\xi : X \to \text{Gr}_{k,l} \) can be constructed step by step and the obstruction for the \( i \)th step is \((f_{\xi}^{[i-1]})^* (\kappa_i) \in H^{2i}(X, \mathbb{Z}/k\mathbb{Z})\).

Remark 9. The described results indicate that the “stable” obstructions depend only on the bundle \( A_k \), not on the choice of \( l \) which is relatively prime to \( k \). In fact, this is true.

It turns out that the lifting in fibration \([1]\) is equivalent to the “reduction” of the structure group \( \text{PU}(k) \) to the group \( \Omega_{\text{SU}(k) \odot \text{SU}(l)}(\text{SU}(kl)) \) of paths in \( \text{SU}(kl) \) with origin in the subgroup \( \text{SU}(k) \odot \text{SU}(l) \subset \text{SU}(kl) \) and end in the unit element \( e \); moreover, \( \text{Gr}_{k,l} \) is its classifying space \([10], [11]\).

One can also describe the set of mutually nonhomotopic embeddings of form \([1]\) in terms of fibration \([4]\). Namely, there is a natural bijection between it and the set of fibrewise homotopy classes of sections of the pullback fibration \( \tilde{f}^* (H_{k,l}(A_{k}^{\text{inv}})) \to X \) (see \([5]\)). In particular, if \( A_k \) is the product bundle \( X \times M_k(\mathbb{C}) \), then this is just the set of homotopy classes \([X, \text{Hom}_{\text{alg}}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))]\).

2. AN APPROACH VIA GROUPOIDS

It turns out that above considered spaces and bundles (like \( \text{Gr}_{k,l}, H_{k,l}(A_{k,l}), H_{k,l}(A_{k}^{\text{inv}}) \) etc.) can naturally be interpreted in terms of some groupoid \( \mathcal{G}_{k,l} \) of matrix subalgebras in the fixed matrix algebra \( M_{kl}(\mathbb{C}) \).

2.1. Groupoids \( \mathcal{G}_{k,l} \). Let \( M_{kl}(\mathbb{C}) \) be the complex matrix algebra. Recall that unital \(*\)-subalgebras in \( M_{kl}(\mathbb{C}) \) isomorphic to \( M_k(\mathbb{C}) \) we call \( k\)-subalgebras.

Define the following category \( C_{k,l} \). Its objects \( \text{Ob}(C_{k,l}) \) are \( k\)-subalgebras in the fixed \( M_{kl}(\mathbb{C}) \), i.e. actually points of the matrix grassmannian \( \text{Gr}_{k,l} \).

For two objects \( M_{k,\alpha}, M_{k,\beta} \in \text{Ob}(C_{k,l}) \) the set of morphisms \( \text{Mor}_{C_{k,l}}(M_{k,\alpha}, M_{k,\beta}) \) is just the space \( \text{Hom}_{\text{alg}}(M_{k,\alpha}, M_{k,\beta}) \) of all unital \(*\)-homomorphisms of matrix algebras (i.e. actually isometric isomorphisms).

Put

\[
\mathcal{G}^0_{k,l} := \text{Ob}(C_{k,l}), \quad \mathcal{G}_{k,l} := \bigcup_{\alpha, \beta \in \text{Ob}(C_{k,l})} \text{Mor}_{C_{k,l}}(M_{k,\alpha}, M_{k,\beta}).
\]

Clearly, \( \mathcal{G}_{k,l} \) is a topological groupoid (in fact, even a Lie groupoid).
Remark 10. Note that we do not fix an extension of a homomorphism from \( \text{Hom}_{\text{alg}}(M_{k,\alpha}, M_{k,\beta}) \) to an automorphism of the whole algebra \( M_{k}(\mathbb{C}) \), so it is not the action groupoid corresponding to the action of \( \text{PU}(k) \) on \( \text{Ob}(C_{k,l}) \).

It is interesting to note that if \( \mathcal{G}_{k,l} \) would be an action groupoid for some topological group \( H \) acting on \( \mathcal{G}_{k,l}^0 \), then \( H \cong \text{Fr}_{k,l} \). This result follows from the homotopy equivalence \( B\mathcal{G}_{k,l} \cong \text{BPU}(k) \) (see below) and the fact that for action groupoid \( \mathcal{G} := X \times H \) corresponding to an action of \( H \) on \( X \) the classifying space \( B\mathcal{G} \) is homotopy equivalent to \( X \times EH \) [3].

As a topological space \( \mathcal{G}_{k,l} \) can be represented as follows. Applying fiberwisely the functor \( \text{Hom}_{\text{alg}}(\ldots, M_{k}(\mathbb{C})) \) (see Subsection 1.2) to the tautological \( M_{k}(\mathbb{C}) \)-bundle \( A_{k,l} \to \text{Gr}_{k,l} \) we obtain the space \( H_{k,l}(A_{k,l}) \) which is exactly \( \mathcal{G}_{k,l} \).

Being a groupoid, \( \mathcal{G}_{k,l} \) has canonical morphisms: source and target \( s, t: \mathcal{G}_{k,l} \to \mathcal{G}_{k,l}^0 \), composition \( m: \mathcal{G}_{k,l} \times \mathcal{G}_{k,l} \to \mathcal{G}_{k,l} \), identity \( e: \mathcal{G}_{k,l}^0 \to \mathcal{G}_{k,l} \) and inversion \( i: \mathcal{G}_{k,l} \to \mathcal{G}_{k,l} \).

Let us describe first two of them in terms of topological spaces \( \text{Gr}_{k,l} \sim \mathcal{G}_{k,l}^0 \), and \( H_{k,l}(A_{k,l}) \sim \mathcal{G}_{k,l} \). The source morphism \( s: H_{k,l}(A_{k,l}) \to \text{Gr}_{k,l} \) is just the bundle projection (recall that \( H_{k,l}(A_{k,l}) \) is obtained from the bundle \( A_{k,l} \to \text{Gr}_{k,l} \) by the fiberwise application of the functor \( \text{Hom}_{\text{alg}}(\ldots, M_{k}(\mathbb{C})) \)). The target morphism \( t: H_{k,l}(A_{k,l}) \to \text{Gr}_{k,l} \) is the map \( h \mapsto h((A_{k,l})_{alpha}) \), where \( h \in H_{k,l}(A_{k,l}) \), \( s(h) = \alpha \in \text{Gr}_{k,l} \) and as usual we identify the \( k \)-subalgebra \( h((A_{k,l})_{alpha}) \subset M_d(\mathbb{C}) \) with the corresponding point in \( \text{Gr}_{k,l} \).

There are also analogous descriptions of maps \( e: \text{Gr}_{k,l} \to H_{k,l}(A_{k,l}) \), \( i: H_{k,l}(A_{k,l}) \to H_{k,l}(A_{k,l}) \) and

\[
m: H_{k,l}(A_{k,l}) \times H_{k,l}(A_{k,l}) \to H_{k,l}(A_{k,l}).\]

Note that there are bifunctors \( C_{k,l} \times C_{m,n} \to C_{km,ln} \) induced by the tensor product of matrix algebras and therefore the corresponding morphisms of topological groupoids

\[
\mathcal{G}_{k,l} \times \mathcal{G}_{m,n} \to \mathcal{G}_{km,ln}.
\]

They cover the maps \( \text{Gr}_{k,l} \times \text{Gr}_{m,n} \to \text{Gr}_{km,ln} \) [12].

Remark 11. Note that one can define an “SU”-analog of the groupoid \( \mathcal{G}_{k,l} \) replacing \( \text{PU}(k) \) by \( \text{SU}(k) \). This is a \( k \)-fold covering of \( \mathcal{G}_{k,l} \) (cf. Subsection 1.4).

Note that for any \( \alpha \in \text{Ob}(C_{k,l}) \) we have the (full) subcategory with one object \( \alpha \). The corresponding groupoid morphism \( \text{PU}(k) \to \mathcal{G}_{k,l} \) is a Morita morphism, i.e. the diagram

\[
\begin{array}{ccc}
\text{PU}(k) & \longrightarrow & \mathcal{G}_{k,l} \\
\downarrow & & \downarrow \\
\alpha & \longrightarrow & \text{Gr}_{k,l} \times \text{Gr}_{k,l}
\end{array}
\]

is a Cartesian square. It turns out (see the next subsection) that this Morita morphism induces a homotopy equivalence of the classifying spaces \( \text{BPU}(k) \cong B\mathcal{G}_{k,l} \).
2.2. Groupoids $\tilde{G}_{k,l}$. Define a new category $\tilde{C}_{k,l}$ whose objects $\text{Ob}(\tilde{C}_{k,l}) = \text{Ob}(C_{k,l})$ but morphism from $\alpha \in \text{Ob}(\tilde{C}_{k,l})$ to $\beta \in \text{Ob}(\tilde{C}_{k,l})$ is the set of all pairs $(\lambda, \mu)$, where $\lambda: M_{k,\alpha} \to M_{k,\beta}$ and $\mu: M_{l,\alpha} \to M_{l,\beta}$ are $*$-isomorphisms, where $M_{l,\alpha} \cong M_l(\mathbb{C})$, $M_{l,\beta} \cong M_l(\mathbb{C})$ are centralizers (in $M_{kl}(\mathbb{C})$) of $M_{k,\alpha}$ and $M_{k,\beta}$ respectively.

Let $\tilde{G}_{k,l}$ be the set of all morphisms in $\tilde{C}_{k,l}$. Clearly, it is again a topological (even a Lie) groupoid. As a topological space it can also be described as the total space of some $\text{PU}(k) \times \text{PU}(l)$-bundle over $\text{Gr}_{k,l} \times \text{Gr}_{k,l}$ (the projection is given by $s \times t: \tilde{G}_{k,l} \to \text{Gr}_{k,l} \times \text{Gr}_{k,l}$).

We also have the map $\hat{\vartheta}: \tilde{G}_{k,l} \to \text{PU}(kl)$, $(\lambda, \mu) \mapsto \hat{\vartheta}(\lambda, \mu)$, where $\hat{\vartheta}(\lambda, \mu): M_{kl}(\mathbb{C}) \to M_{kl}(\mathbb{C})$ is the unique automorphism induced by $(\lambda, \mu)$.

Remark 12. In fact, $\tilde{G}_{k,l}$ is an action groupoid $\text{Gr}_{k,l} \times \text{PU}(kl)$ related to the action of $\text{PU}(kl)$ on $\text{Gr}_{k,l}$.

We have the natural groupoid morphism $\pi: \tilde{G}_{k,l} \to G_{k,l}$, $(\lambda, \mu) \mapsto \lambda$. The fiber of $\pi$ is clearly $\text{PU}(l)$. Thus, we have the groupoid extension

\begin{equation}
\begin{CD}
\text{PU}(l) @>>> \tilde{G}_{k,l} @>{\pi}>> G_{k,l}.
\end{CD}
\end{equation}

Remark 13. Note that $G_{k,l}$ can also be regarded as an extension of the pair groupoid $\text{Gr}_{k,l} \times \text{Gr}_{k,l}$ by $\text{PU}(k)$.

2.3. Universal principal groupoid $\mathbf{E}_{k,l}$-bundle. In this subsection we shall show that our previous construction (see Subsection 1.2) which to an $M_k(\mathbb{C})$-bundle $A_k \to X$ associates $\text{Fr}_{k,l}$-bundle $H_{k,l}(A_k) \to X$ is nothing but the extension functor from the structure group $\text{PU}(k)$ to the structure groupoid $G_{k,l}$. Moreover, it turns out that $H_{k,l}(A_{k,\text{univ}}^\text{univ}) \to \text{BPU}(k)$ is the universal principal $G_{k,l}$-bundle, in particular, the classifying spaces $\text{BPU}(k)$ and $\mathbf{B}G_{k,l}$ are homotopy equivalent. Consequently, every $G_{k,l}$-bundle can be obtained from some $M_k(\mathbb{C})$-bundle in this way.

Remark 14. Note that $\mathbf{B} \tilde{G}_{k,l} \simeq \text{BPU}(k) \times \text{BPU}(l)$ because $\tilde{G}_{k,l}$ is an action groupoid (cf. Remarks 10 and 12).

In Subsection 1.2 (see 7) we defined the map $\tau_{k,l}: H_{k,l}(A_{k,\text{univ}}^\text{univ}) \to \text{Gr}_{k,l}$, $h \mapsto h((A_{k,\text{univ}}^\text{univ})_x) \subset M_{kl}(\mathbb{C})$, where $x \in \text{BPU}(k)$ and $h \in p_{k,l}^{-1}(x)$ which is a fibration with contractible fibres; in particular, it is a homotopy equivalence.

There is the free and proper action

$$
\varphi: G_{k,l} \times_{\text{Gr}_{k,l}} H_{k,l}(A_{k,\text{univ}}^\text{univ}) \to H_{k,l}(A_{k,\text{univ}}^\text{univ})
$$

($\tau := \tau_{k,l}$) defined by the compositions of algebra homomorphisms. More precisely, for $g \in G_{k,l}$, $h \in p_{k,l}^{-1}(x)$, $x \in \text{BPU}(k)$ such that $s(g) = \tau_{k,l}(h)$ we put $\varphi(g, h) := g(h((A_{k,\text{univ}}^\text{univ})_x)) \subset M_{kl}(\mathbb{C})$ (in particular, $\tau_{k,l}(\varphi(g, h)) = t(g)$).

Theorem 15. The base space of the principal groupoid $G_{k,l}$-bundle $(H_{k,l}(A_{k,\text{univ}}^\text{univ}), G_{k,l}, \varphi)$ is $\text{BPU}(k)$ (see 11).

Proof. It is easy to see that the map

$$
G_{k,l} \times_{\text{Gr}_{k,l}} H_{k,l}(A_{k,\text{univ}}^\text{univ}) \to H_{k,l}(A_{k,\text{univ}}^\text{univ}) \times_{\text{BPU}(k)} H_{k,l}(A_{k,\text{univ}}^\text{univ}), (g, p) \mapsto (gp, p)
$$

is a homotopy equivalence.
is a homeomorphism. □

Thus, the action $\varphi$ turns the fibration (11) into a principal groupoid $G_{k,l}$-bundle. Moreover, it is the universal $G_{k,l}$-bundle because (as we have already noticed) $\tau_{k,l}: H_{k,l}(A^\text{univ}_k) \to \text{Gr}_{k,l}$ has contractible fibers. Therefore there is a homotopy equivalence $B G_{k,l} \simeq \text{BPU}(k)$.

**Remark 16.** The last result (in particular, that the homotopy type of $B G_{k,l}$ does not depend on $l$) can be explained using the notion of Morita equivalence for groupoids (see [5]). Take a positive integer $m$ and define $G_{k,l} - G_{k,m}$-bimodule $M_{k,l}^{\ k,m}$ as follows. $M_{k,l}^{\ k,m}$ consists of all unital $*$-homomorphisms from $k$-subalgebras in $M_{km}(C)$ to $k$-subalgebras in $M_{kl}(C)$. Clearly, $M_{k,l}^{\ k,m}$ is an equivalence bimodule [5]. If we take $m = 1$ we obtain the homotopy equivalence $B G_{k,l} \simeq \text{BPU}(k)$ directly.

**Remark 17.** It is easy to see that for the SU-analog of the groupoid $G_{k,l}$ (see Remark 11) the classifying space is homotopy equivalent to $\text{BSU}(k)$ (cf. [12]).

Note that the groupoid $G_{k,l}$ itself is (the total space of) a principal $G_{k,l}$-bundle with the base space $\text{Gr}_{k,l} = G_{k,l}^0$. This bundle is called unit [8]. A principal groupoid $G_{k,l}$-bundle $H_{k,l}(A_k) \to X$ (we have already noticed that every principal $G_{k,l}$-bundle is of this form) is called trivial w.r.t. a map $f: X \to G_{k,l}^0$ if it is the pullback of the unit bundle via this map [8]. In particular, the unit bundle is trivial with respect to the identity map $\text{id}: G_{k,l}^0 \to G_{k,l}^0$. Thus, in general, there are non isomorphic trivial bundles over the same base space. Note that a $G_{k,l}$-bundle $H_{k,l}(A_k) \to X$ is trivial iff it has a section, i.e. there is an embedding (11) (with $n = kl$) iff $H_{k,l}(A_k) \to X$ is a trivial principal groupoid $G_{k,l}$-bundle.

**Remark 18.** Let us return to the functor $(A_k, \mu) \mapsto A_k$ (see Remark 2) corresponding to the map of classifying spaces $\text{Gr}_{k,l} \to \text{BPU}(k)$. Now we see that it can be interpreted as the factorization by the action of the groupoid $G_{k,l}$ (cf. Subsection 3.2 below).

### 2.4. A remark about stabilization.

Note that maps (18) induce maps of classifying spaces

$$
\begin{array}{cccc}
H_{k,l}(A^\text{univ}_k) \times H_{m,n}(A^\text{univ}_m) & \longrightarrow & H_{km,ln}(A^\text{univ}_{km}) \\
\downarrow & & \downarrow \\
\text{BPU}(k) \times \text{BPU}(m) & \longrightarrow & \text{BPU}(km)
\end{array}
$$

(we should restrict ourself to the case $(km, ln) = 1$), cf. [12]. In the direct limit we obtain the $H$-space homomorphism

$$
\text{Gr} \to \lim_{k} \text{BPU}(k),
$$

where $\text{Gr} := \lim_{(k, l)=1} \text{Gr}_{k,l}$ [12], maps in the direct limits are induced by the tensor product and we use the homotopy equivalences $H_{k,l}(A^\text{univ}_k) \simeq \text{Gr}_{k,l}$. Since there is an $H$-space isomorphism $\text{Gr} \cong \text{BSU}_\otimes$ [12], we see that (20) is the composition of the localization map

$$
\text{BSU}_\otimes \to \prod_{n \geq 2} K(\mathbb{Q}, 2n)
$$
and the natural inclusion
\[
\prod_{n \geq 2} K(\mathbb{Q}, 2n) \hookrightarrow K(\mathbb{Q}/\mathbb{Z}, 2) \times \prod_{n \geq 2} K(\mathbb{Q}, 2n) \simeq \lim_{k} \text{BPU}(k).
\]

Consider the abelian group
\[
(21) \quad \text{coker}\{[X, \text{Gr}] \to [X, \lim_{k} \text{BPU}(k)]\},
\]
where the homomorphism of the groups of homotopy classes is induced by (20). It admits the following “geometric” description. We call an \(M_k(\mathbb{C})\)-bundle embeddable if there is an embedding \(\mu: A_k \hookrightarrow X \times M_k(\mathbb{C})\) as above for some \(l\), \((k, l) = 1\). We say that \(M_k(\mathbb{C})\) and \(M_m(\mathbb{C})\)-bundles \(C_k, D_m\) over \(X\) are equivalent if there are embeddable bundles \(A_l, B_n\) such that \(C_k \otimes A_l \cong D_m \otimes B_n\).

The set of such equivalence classes over the given base space \(X\) is a group with respect to the operation induced by the tensor product. Clearly, this group is the cokernel (21). In particular, for every even-dimensional sphere \(S^{2n}\) it is \(\mathbb{Q}/\mathbb{Z}\) (and 0 for every odd-dimensional one).

**Remark 19.** Since BSU\(\otimes\) is an infinite loop space [9], this invariant can be interpreted in terms of the coefficient sequence for the corresponding cohomology theory.

### 3. Some constructions

#### 3.1. Partial isomorphisms

Let \(A_k \to X\) be an \(M_k(\mathbb{C})\)-bundle over \(X\) and \(\mu: A_k \hookrightarrow X \times M_k(\mathbb{C})\) \(((k, l) = 1)\) a bundle map which is a unital \(*\)-algebra homomorphism on each fiber as above. So every fiber \((A_k)_x, x \in X\) can be identified with the corresponding \(k\)-subalgebra \(\mu|_x((A_k)_x) \subset M_k(\mathbb{C})\) and we have the triple \((A_k, \mu, X \times M_k(\mathbb{C}))\). Let \((A'_k, \mu', X \times M_k(\mathbb{C}))\) be another triple of such a kind. Assume that the bundles \(A_k\) and \(A'_k\) are isomorphic and choose some \(*\)-isomorphism \(\vartheta: A_k \cong A'_k\).

Note that embeddings \(\mu, \mu'\) define the corresponding maps to the matrix Grassmannian \(f_\mu, f_{\mu'}: X \to G_{k,1}\) and, moreover \(\vartheta, \mu\) and \(\mu'\) define a map \(\nu: X \to G_{k,1}\) such that \(s \circ \nu = f_\mu, t \circ \nu = f_{\mu'}\) and \(\nu|_x = \mu' \circ \vartheta|_x \circ \mu^{-1}: \mu((A_k)_x) \to \mu'((A'_k)_x)\).

Conversely, a map \(\nu: X \to G_{k,1}\) gives us some maps \(f_\mu := s \circ \nu\) and \(f_{\mu'} := t \circ \nu: X \to G_{k,1}\) that come from some triples \((A_k, \mu, X \times M_k(\mathbb{C})), (A'_k, \mu', X \times M_k(\mathbb{C}))\), and an isomorphism \(\vartheta: A_k \cong A'_k\). Such a \(\nu\) will be called a partial isomorphism from \((A_k, \mu, X \times M_k(\mathbb{C}))\) to \((A'_k, \mu', X \times M_k(\mathbb{C}))\) or just a partial automorphism of the trivial bundle \(X \times M_k(\mathbb{C})\). Partial isomorphisms that can be lifted to “genuine” automorphisms of the trivial bundle \(X \times M_k(\mathbb{C})\) (i.e. to genuine bundle maps \(\vartheta: X \times M_k(\mathbb{C}) \to X \times M_k(\mathbb{C})\) such that the diagram
\[
\begin{array}{ccc}
A_k & \xrightarrow{\vartheta} & A'_k \\
\mu \downarrow & & \downarrow \mu' \\
X \times M_k(\mathbb{C}) & \xrightarrow{\vartheta} & X \times M_k(\mathbb{C})
\end{array}
\]
commutes) are just called isomorphisms.
Remark 20. An extension of a partial isomorphism \( \nu: X \to \mathfrak{g}_{k,l} \) to a genuine isomorphism is equivalent to the choice of a lift \( \tilde{\nu}: X \to \tilde{\mathfrak{g}}_{k,l} \) of \( \nu \) in [19] (to show this one can use the map \( \tilde{\vartheta}: \tilde{\mathfrak{g}}_{k,l} \to \text{PU}(k,l) \) introduced in Subsection 2.2).

Now we claim that there are partial isomorphisms that are not isomorphisms. To show this, take \( X = \text{Fr}_{k,l} \). The map \( \nu: \text{Fr}_{k,l} \to \mathfrak{g}_{k,l} \) is defined as follows. Fix \( \alpha \in \text{Gr}_{k,l} \) and consider all \(*\)-isomorphisms from \( M_{k,\alpha} \subseteq M_{kl}(\mathbb{C}) \) to \( M_{k,\beta} \subseteq M_{kl}(\mathbb{C}) \), where \( \beta \) runs over all \( k \)-subalgebras in \( M_{kl}(\mathbb{C}) \). Clearly, this defines a subspace in \( \mathfrak{g}_{k,l} \) homeomorphic to \( \text{Fr}_{k,l} \). In our case \( \mathcal{A}_k \cong \mathcal{A}'_k = \text{Fr}_{k,l} \times M_k(\mathbb{C}) \), but \( \mu \) and \( \mu' \) are different.

In order to show this, define the \( M_l(\mathbb{C}) \)-bundle \( \mathcal{B}_{k,l} \to \text{Gr}_{k,l} \) as the centralizer of the tautological subbundle \( \mathcal{A}_{k,l} \subseteq \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \) (for more details see e.g. [11]). Clearly, \( f_{\mu}: \text{Fr}_{k,l} \to \text{Gr}_{k,l} \) is the map to the point \( \alpha \in \text{Gr}_{k,l} \), while \( f_{\mu'}: \text{Fr}_{k,l} \to \text{Gr}_{k,l} \) is (the projection of) the principal \( \text{PU}(k) \)-bundle \( \text{PU}(k) \to \text{Fr}_{k,l} \to \text{Gr}_{k,l} \). Clearly, both bundles \( \mathcal{A}_k = f_{\mu}^{*}(\mathcal{A}_{k,l}) \) and \( \mathcal{A}'_k = f_{\mu'}^{*}(\mathcal{A}_{k,l}) \) are trivial, as we have already asserted (note that \( \mathcal{A}'_k = f_{\mu'}^{*}(\mathcal{A}_{k,l}) \) is trivial because \( f_{\mu'}: \text{Fr}_{k,l} \to \text{Gr}_{k,l} \) is the frame bundle for \( \mathcal{A}_{k,l} \to \text{Gr}_{k,l} \)). The bundle \( f_{\mu}'(\mathcal{B}_{k,l}) \) is also trivial, while \( f_{\mu'}(\mathcal{B}_{k,l}) \) is nontrivial (because it is associated with the principal bundle \( \text{PU}(l) \to \text{PU}(k,l) \to \text{Fr}_{k,l} \)). This shows that for chosen \( \nu: \text{Fr}_{k,l} \to \mathfrak{g}_{k,l} \) \( \vartheta \) can not be extended to an automorphism of \( \text{Fr}_{k,l} \times M_{kl}(\mathbb{C}) \) (because such an automorphism induces an isomorphism not only between the subbundles \( \mathcal{A}_k \), \( \mathcal{A}'_k \), but also between their centralizers).

In particular, we see that the analog of Noether-Skolem’s theorem is not true for matrix algebras \( \Gamma(X, X \times M_{kl}(\mathbb{C})) = M_{kl}(C(X)) \) over \( C(X) \).

3.2. An action on fibers of a forgetful functor. Consider the forgetful functor given by the assignment \((A_k, \mu, X \times M_{kl}(\mathbb{C})) \mapsto A_k \) corresponding to the map of representing spaces \( \text{Gr}_{k,l} \to \text{BPU}(k) \) (whose homotopy fiber is \( \text{Fr}_{k,l} \)). We claim that our previous construction can be regarded as an action of the groupoid on its fibres.

First, let us recall some of the previous results. Applying fiberwisely the functor \( \text{Hom}_{\text{alg}}(\ldots, M_{kl}(\mathbb{C})) \) to the universal \( M_{k}(\mathbb{C}) \)-bundle \( \mathcal{A}_{k}^{\text{univ}} \to \text{BPU}(k) \) we obtain the fibration

\[
\begin{array}{ccc}
\text{Fr}_{k,l} & \longrightarrow & H_{k,l}(A_{k}^{\text{univ}}) \\
\downarrow & & \\
\text{BPU}(k) & & \\
\end{array}
\]

with fiber \( \text{Fr}_{k,l} := \text{Hom}_{\text{alg}}(M_{k}(\mathbb{C}), M_{kl}(\mathbb{C})) \). We have the map \( \tau_{k,l}: H_{k,l}(A_{k}^{\text{univ}}) \to \text{Gr}_{k,l}, h \mapsto h((A_{k}^{\text{univ}})_x) \subseteq M_{kl}(\mathbb{C}) \), where \( x \in \text{BPU}(k) \) and \( h \in \tilde{p}_k^{-1}(x) \), which is a fibration with contractible fibres, i.e. a homotopy equivalence.

Moreover, there is the free and proper action

\[
\varphi: \mathfrak{g}_{k,l} \times \text{Fr}_{k,l} \to H_{k,l}(A_{k}^{\text{univ}}) \to H_{k,l}(A_{k}^{\text{univ}})
\]

which turns the fibration (22) into the universal principal groupoid \( \mathfrak{g}_{k,l} \)-bundle.
We have also shown that for a map \( \tilde{f} : X \to \text{BPU}(k) \) the choice of its lift \( \hat{f} : X \to \text{H}_{k,l}(A_k^\text{univ}) \) (if it exists) is equivalent to the choice of an embedding \( \mu : \hat{f}^*(A_k^\text{univ}) \to X \times M_{kl}(\mathbb{C}) \). Such a lift we denoted by \( \hat{f}_\mu \).

Given \( \nu : X \to \mathfrak{G}_{k,l} \) such that \( s \circ \nu = \tau_{k,l} \circ \hat{f}_\mu = f_\mu \), \( t \circ \nu = f_{\mu'} : X \to \text{Gr}_{k,l} \) we define the composite map \( \hat{f}_{\mu'} : \)

\[
X \xrightarrow{\text{diag}} X \times X \xrightarrow{\nu \times \hat{f}_\mu} \mathfrak{G}_{k,l} \times_{\text{Gr}_{k,l}} \text{H}_{k,l}(A_k^\text{univ}) \xrightarrow{\nu} \text{H}_{k,l}(A_k^\text{univ})
\]

which is (in general) another lift of \( \tilde{f} \) \((p_{k,l} \circ \hat{f}_\mu = \tilde{f} = p_{k,l} \circ \hat{f}_{\mu'})\), i.e. it corresponds to another (homotopy nonequivalent in general) embedding \( \mu' : \hat{f}^*(A_k^\text{univ}) \to X \times M_{kl}(\mathbb{C}) \), i.e. \( f_{\mu'} = \tau_{k,l} \circ \hat{f}_{\mu'} : X \to \text{Gr}_{k,l} \). Clearly, this action is transitive on homotopy classes of such embeddings.

### 3.3. A remark about groupoid cocycles

In this subsection we sketch an approach to groupoid bundles via local trivializing data and 1-cocycles. The reader can find the general results in [S], but we hope that our groupoids provide an instructive illustration of the general theory.

In Subsection 2.3 we have already seen that a trivial \( \mathfrak{G}_{k,l} \)-bundle \( \text{H}_{k,l}(A_k) \to X \) is the pullback of the unit bundle \( \text{H}_{k,l}(A_k) \to \text{Gr}_{k,l} \) via some map \( f : X \to \text{Gr}_{k,l} \). Moreover, such a map \( f \) is nothing but a trivialization of \( \text{H}_{k,l}(A_k) \to X \). Such a trivialization can also be thought of as a triple \((A_k, \mu, X \times M_{kl}(\mathbb{C})) \) (see Subsection 3.1), where \( \mu : A_k \to X \times M_{kl}(\mathbb{C}) \) is a fiberwise embedding as above, because \( f = f_\mu \) is its classifying map.

For a topological group \( G \) the group of automorphisms of a trivial \( G \)-bundle over \( X \) can be identified with the group of continuous maps \( X \to G \) which take one trivialization to another. The analogous maps \( \nu : X \to \mathfrak{G}_{k,l} \) to the groupoid \( \mathfrak{G}_{k,l} \) were called partial isomorphisms in Subsection 3.1. Recall that such \( \nu \) defines two compositions \( s \circ \nu \) and \( t \circ \nu : X \to \text{Gr}_{k,l} \) which give rise to some triples as above and therefore to some trivializations.

Let \( X \) be a compact manifold, \( \mathcal{U} := \{U_\alpha\}_{\alpha \in A} \) its open covering. A \( \mathfrak{G}_{k,l} \)-1-cocycle can be defined as a groupoid homomorphism (more precisely, as a functor) from the Čech groupoid to \( \mathfrak{G}_{k,l} \). So we get the following unfolded form of this definition.

**Definition 21.** A groupoid \( \mathfrak{G}_{k,l} \)-1-cocycle \( \{g_{\alpha \beta}\}_{\alpha, \beta \in A} \) is a collection of continuous maps \( g_{\alpha \beta} : U_\alpha \cap U_\beta \to \mathfrak{G}_{k,l} \) such that

1. \( g_{\alpha \beta} \) and \( g_{\beta \gamma} \) are composable on \( U_\alpha \cap U_\beta \cap U_\gamma \), i.e. \( \forall x \in U_\alpha \cap U_\beta \cap U_\gamma \ t(g_{\alpha \beta}(x)) = s(g_{\beta \gamma}(x)) \), where \( s \) and \( t \) are the source and target maps for \( \mathfrak{G}_{k,l} \);

2. \( g_{\alpha \beta}g_{\beta \gamma} = g_{\alpha \gamma} \) on \( U_\alpha \cap U_\beta \cap U_\gamma \) (in particular, \( g_{\alpha \alpha} \in e \), \( g_{\alpha \beta} = i(g_{\alpha \beta}) \), where \( e \) and \( i \) are the identity and the inversion for the groupoid \( \mathfrak{G}_{k,l} \), see Subsection 2.1).

In the same way one can define a groupoid \( \hat{\mathfrak{G}}_{k,l} \)-1-cocycle \( \{\hat{g}_{\alpha \beta}\}_{\alpha, \beta \in A} \).

**Remark 22.** Note that a trivial bundle over \( X \) of our kind with a trivialization \((A_k, \mu, X \times M_{kl}(\mathbb{C})) \) corresponds to the trivial \( \mathfrak{G}_{k,l} \)-1-cocycle. Indeed, two maps \( f_\mu, f_{\mu'} : X \to \text{Gr}_{k,l} \) give rise to the identity partial isomorphism (see Subsection 3.1) \( \nu : X \to \mathfrak{G}_{k,l} \), \( \nu|_x = \text{id}_{\mu(A_k)}(x) \) which is the trivial groupoid \( \mathfrak{G}_{k,l} \)-1-cocycle as claimed.
Now the gluing of a groupoid bundle using local data can be described as follows. So we start with an open covering \( U = \{ U_\alpha \}_{\alpha \in A} \) and trivial groupoid bundles \((A_k, \mu_\alpha, U_\alpha \times M_k(\mathbb{C}))\) over \( U_\alpha, \alpha \in A \). Suppose we are given a groupoid \( \mathfrak{G}_{k, l} \) 1-cocycle \( \{g_{\alpha \beta}\}_{\alpha, \beta \in A} \) (over the same open covering \( U \)) such that \( s(g_{\alpha \beta}) \equiv f_{\mu_\alpha}|_{U_\alpha \cap U_\beta} \) and \( t(g_{\alpha \beta}) \equiv f_{\mu_\beta}|_{U_\alpha \cap U_\beta} \) \( \forall \alpha, \beta \in A \). (In our previous notation it is natural to denote it by \( \{\nu_{\alpha \beta}\} \)). The groupoid \( \mathfrak{G}_{k, l} \) 1-cocycle \( \{g_{\alpha \beta}\}_{\alpha, \beta \in A} \) defines partial isomorphisms (see Subsection 3.1) from \((U_\alpha \cap U_\beta, \nu_{\alpha \beta}\) maps such that \( s\nu_{\alpha \beta} = f_{\mu_\alpha}|_{U_\alpha \cap U_\beta} \) and \( t\nu_{\alpha \beta} = f_{\mu_\beta}|_{U_\alpha \cap U_\beta} \) \( \forall \alpha, \beta \in A \). We have \( \nu_{\alpha \beta} \) satisfies the cocycle conditions on triple intersections. The idea is to regard the map \( f_\alpha : U_\alpha \rightarrow \text{Gr}_{k, l} \) (corresponding to the “identity” \( g_{\alpha \alpha} \)) as a local trivialization and \( \nu_{\alpha \beta} : U_\alpha \cap U_\beta \rightarrow \mathfrak{G}_{k, l} \) for \( \alpha \neq \beta \) as a gluing of different trivializations over the double intersection \( U_\alpha \cap U_\beta \). Thus, we have maps

\[
f : Y \rightarrow \text{Gr}_{k, l}, \quad g : Y \times_{X} Y \rightarrow \mathfrak{G}_{k, l}
\]

such that \( s \circ g = f \circ \pi_1|_{Y \times Y}, \ t \circ g = f \circ \pi_2|_{Y \times Y} \), where \( \pi_i : Y \times Y \rightarrow Y \) are the projections onto \( i \)th factor.

There is a natural equivalence relation on the set of groupoid 1-cocycles generalizing the equivalence relation on group 1-cocycles. As in the case of usual bundles constructed by means of group \( G \) 1-cocycles, we have:

1) equivalence of 1-cocycles over the same open covering \( U \);
2) equivalence of 1-cocycles related to the refinement of the open covering.

The case 1) concerns to the different choices of trivializations over open subsets \( U_\alpha \). We have already noticed that such a trivialization is actually a map \( f_{\mu_\alpha} : U_\alpha \rightarrow \text{Gr}_{k, l} \) and two such trivializations \( f_{\mu_\alpha}, f_{\mu_\alpha}' \) are related by the map \( \nu_\alpha : U_\alpha \rightarrow \mathfrak{G}_{k, l} \) (such that \( s \circ \nu_\alpha = f_{\mu_\alpha}, \ t \circ \nu_\alpha = f_{\mu_\alpha}' \)).

**Remark 23.** Note that using groupoid \( \mathfrak{G}_{k, l} \) 1-cocycle one can glue some global \( M_k(\mathbb{C}) \)-bundle \( A_k \rightarrow X \) such that \( A_k|_{U_\alpha} = A_k, \alpha \). It agrees with the proved above homotopy equivalence \( B \mathfrak{G}_{k, l} \simeq \text{BPU}(k) \). In other words, the groupoid bundle glued by the 1-cocycle is \( H_{k, l}(A_k) \rightarrow X \). Note that
The case of the groupoid $\hat{G}_{k,l}$ can be described in the similar way. In this case a map $f: X \to Gr_{k,l}$ (a “trivialization”) can be regarded as the product bundle $X \times \text{Id}(\mathbb{C})$ together with a chosen decomposition into the tensor product $A_k \otimes B_l$ of its $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$-subbundles $A_k \to X$ and $B_l \to X$ respectively.

Note that using a $\hat{G}_{k,l}$-cocycle $\{\hat{g}_{a,b}\}_{a,b \in A}$ as above one can glue some $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$-bundles $A_k$, $B_l$ over $X$. It relates to the existence of a homotopy equivalence $B\hat{G}_{k,l} \simeq B\text{PU}(k) \times B\text{PU}(l)$ (cf. Remarks 10 and 12).

The relation between $\hat{G}_{k,l}$ and $G_{k,l}$-groupoid bundles follows from the exact sequence (19).

4. On the $K$-theory automorphisms

4.1. The case of line bundles. First, consider the case of line bundles. The classifying space of $K$-theory can be taken to be $\text{Fred}(\mathcal{H})$, the space of Fredholm operators on Hilbert space $\mathcal{H}$. It is known [2] that for a compact space $X$ the action of the Picard group $\text{Pic}(X)$ on $K(X)$ is induced by the conjugation action

$$\gamma: \text{PU}(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \to \text{Fred}(\mathcal{H}), \quad \gamma(g, T) = gTg^{-1}$$

of $\text{PU}(\mathcal{H})$ on $\text{Fred}(\mathcal{H})$. The more precise statement is given in the following theorem (recall that $\text{PU}(\mathcal{H}) \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$).

**Theorem 24.** If $f_\xi: X \to \text{Fred}(\mathcal{H})$ and $\varphi_\zeta: X \to \text{PU}(\mathcal{H})$ represent $\xi \in K(X)$ and $\zeta \in \text{Pic}(X)$ respectively, then the composite map

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{\varphi_\zeta \times f_\xi} \text{PU}(\mathcal{H}) \times \text{Fred}(\mathcal{H}) \xrightarrow{\gamma} \text{Fred}(\mathcal{H})$$

represents $\zeta \otimes \xi \in K(X)$.

**Proof** see [2].

If we want to restrict ourself to the action of line bundles corresponding to elements of finite order in the group $\text{Pic}(X)$ we have to consider the subgroups $\text{PU}(k) \subset \text{PU}(\mathcal{H})$. Let us describe this inclusion.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on the separable Hilbert space $\mathcal{H}$, $M_k(\mathcal{B}(\mathcal{H})) := M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$ the matrix algebra over $\mathcal{B}(\mathcal{H})$ (of course, it is isomorphic to $\mathcal{B}(\mathcal{H})$). Let $U_k(\mathcal{H}) \subset M_k(\mathcal{B}(\mathcal{H}))$ be the corresponding unitary group (isomorphic to $U(\mathcal{H})$). It acts on $M_k(\mathcal{B}(\mathcal{H}))$ by conjugations (which are $*$-algebra automorphisms). Moreover, the kernel of the action is the center, i.e. the subgroup of scalar matrices $U(1)$. The corresponding quotient group we denote by $\text{PU}_k(\mathcal{H})$ (it is isomorphic to $\text{PU}(\mathcal{H})$).

$M_k(\mathbb{C}) \otimes \text{Id}_{\mathcal{B}(\mathcal{H})}$ is a $k$-subalgebra (i.e. a unital $*$-subalgebra isomorphic to $M_k(\mathbb{C})$) in $M_k(\mathcal{B}(\mathcal{H}))$. Then $\text{PU}(k) \subset \text{PU}_k(\mathcal{H})$ is the subgroup of automorphisms of the mentioned $k$-subalgebra. Thus we have defined the injective group homomorphism

$$\Psi_k: \text{PU}(k) \hookrightarrow \text{PU}_k(\mathcal{H}),$$
Proposition 25. For a line bundle $\zeta \to X$ satisfying the condition
\begin{equation}
\zeta \oplus k \sim X \times \mathbb{C}^k
\end{equation}
the classifying map $\varphi_\zeta: X \to PU_k(\mathcal{H}) \cong PU(\mathcal{H})$ can be lifted to a map $\widetilde{\varphi}_\zeta: X \to PU(k)$ such that $\Psi_k \circ \widetilde{\varphi}_\zeta \simeq \varphi_\zeta$.

Proof. Consider the exact sequence of groups
\begin{equation}
1 \to U(1) \to U(k) \xrightarrow{\chi_k} PU(k) \to 1
\end{equation}
and the fibration
\begin{equation}
PU(k) \xrightarrow{\psi_k} BU(1) \xrightarrow{\omega_k} BU(k)
\end{equation}
obtained by its extension to the right. In particular, $\psi_k: PU(k) \to BU(1) \simeq \mathbb{C}P^\infty$ is a classifying map for the $U(1)$-bundle (25). Clearly, $\Psi_k \simeq \psi_k$ under the homotopy equivalence $PU(\mathcal{H}) \simeq BU(1)$.

(Indeed, it follows from the map of $U(1)$-bundles $U(k) \to U(k)$ given by $g \mapsto g \otimes \text{Id}_{B(\mathcal{H})}$.)

The choice of a lift $\widetilde{\varphi}_\zeta$ corresponds to the choice of a trivialization (24): two lifts differ by a map $X \to U(k)$. Therefore the subgroup in $Pic(X)$ formed by line bundles satisfying condition (24) is isomorphic to $\text{im}\{\psi_k: [X, PU(k)] \to [X, \mathbb{C}P^\infty]\} = [X, PU(k)]/(\chi_k[X, U(k)])$ (cf. the definition of the “finite” Brauer group).

4.2. The general case. In [1], M. Atiyah and G. Segal wrote: “The group $Fred_1$ is a product
\[ Fred_1 \simeq \mathbb{P}^\infty \times SFred_1, \]
where $SFred_1$ is the fibre of the determinant map
\[ Fred_1 \cong BU \to BT \cong \mathbb{P}^\infty, \]
and the twistings of this paper are those coming from $(\pm 1) \times \mathbb{P}^\infty$. We do not know any equally geometrical approach to the more general ones.”

In what follows we are going to describe the action of the group of (isomorphism classes of) SU-bundles of finite order on $K(X)$. It corresponds to the torsion subgroup in $SFred_1$ (our notation differs from the one in [1]). We hope that this construction would provide a geometric approach to more general twistings in $K$-theory.

As we have seen in the previous subsection, the group $PU(\mathcal{H})$ from one hand acts on the representing space $Fred(\mathcal{H})$ of the $K$-theory and from the other hand it is the base space of the
universal line bundle. These two facts lead to the result that the action of $\text{PU}(\mathcal{H})$ on $K(X)$ corresponds to the tensor product by elements of the Picard group $\text{Pic}(X)$ (i.e. by line bundles). This action can be restricted to the subgroups $\text{PU}(k) \subset \text{PU}(\mathcal{H})$ which classify the elements of finite order $k$, $k \in \mathbb{N}$.

In what follows the role of subgroups $\text{PU}(k)$ will play the spaces $\text{Fr}_{k,l}$. From one hand, they in some sense “act” on the $K$-theory (more precisely, we have the action of their direct limit which is an $H$-space with respect to the natural operation), from the other hand they are bases of some $l$-dimensional bundles whose classes have order $k$. We shall show that the “action” of $\text{Fr}_{k,l}$ on $K(X)$ corresponds to the tensor product by those $l$-dimensional bundles (see Theorem \ref{thm:action}).

**Proposition 26.** A map $X \to \text{Fr}_{k,l}$ is the same thing as an embedding

$$\mu: X \times M_k(\mathbb{C}) \hookrightarrow X \times M_{kl}(\mathbb{C})$$

whose restriction to every fiber is a unital $*$-algebra homomorphism.

**Proof** follows directly from the bijection $\text{Mor}(X \times M_k(\mathbb{C}), M_{kl}(\mathbb{C})) \to \text{Mor}(X, \text{Mor}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$. $\square$

For the $\text{PU}(k)$-bundle projection $\pi_{k,l}: \text{Fr}_{k,l} \to \text{Gr}_{k,l}$ ($\pi_{k,l} = f_{\mu'}$ in the notation of Subsection 3.1) the pull-back $\pi_{k,l}^\ast(A_{k,l})$ has the canonical trivialization (because $\pi_{k,l}$ is the frame bundle for $A_{k,l}$). In general $\mu$ in (27) is a nontrivial embedding, i.e. not equivalent to the choice of a constant $k$-subalgebra in $X \times M_{kl}(\mathbb{C})$ (equivalently, the homotopy class of $X \to \text{Fr}_{k,l}$ is nontrivial). In particular, the subbundle of centralizers $B_l$ for $\mu(X \times M_k(\mathbb{C})) \subset X \times M_{kl}(\mathbb{C})$ can be nontrivial as an $M_l(\mathbb{C})$-bundle.

The fibration

$$\text{PU}(l) \xrightarrow{E_k \oplus -} \text{PU}(kl) \xrightarrow{\chi_{k,l}} \text{Fr}_{k,l}$$

(cf. (2)) can be extended to the right

$$\text{Fr}_{k,l} \xrightarrow{\psi^l_{k,l}} \text{BPU}(l) \xrightarrow{\omega^l_{k,l}} \text{BPU}(kl),$$

where $\psi^l_{k,l}$ is the classifying map for the $M_l(\mathbb{C})$-bundle $\tilde{B}_{k,l} := \pi_{k,l}^\ast(B_{k,l}) \to \text{Fr}_{k,l}$ (see the end of Subsection 3.1) which is associated with principal $\text{PU}(l)$-bundle (28).

**Proposition 27.** (Cf. Proposition \[26\]) For $M_l(\mathbb{C})$-bundle $B_l \to X$ such that

$$[M_k] \otimes B_l \cong X \times M_{kl}(\mathbb{C})$$

(cf. (24)), where $[M_k]$ is the trivial $M_k(\mathbb{C})$-bundle over $X$, a classifying map $\varphi_{B_l}: X \to \text{BPU}(l)$ can be lifted to a map $\tilde{\varphi}_{B_l}: X \to \text{Fr}_{k,l}$ (i.e. $\psi^l_{k,l} \circ \tilde{\varphi}_{B_l} = \varphi_{B_l}$ or $B_l = \tilde{\varphi}_{B_l}(\tilde{B}_{k,l})$).

**Proof** follows from fibration (29). $\square$

Moreover, the choice of such a lift corresponds to the choice of trivialization (30) and we return to the interpretation of a map $X \to \text{Fr}_{k,l}$ given in Proposition \[26\]. We stress that a map $X \to \text{Fr}_{k,l}$ is not just an $M_l(\mathbb{C})$-bundle but an $M_l(\mathbb{C})$-bundle together with a particular choice of trivialization (30).
It follows from our previous results that the bundle $B_l \to X$ has the form $	ext{End}(\eta_l)$ for some (unique up to isomorphism) $\mathbb{C}'$-bundle $\eta_l \to X$ with the structure group $\text{SU}(l)$ (that’s because we assumed that $(k, l) = 1$). Let $\zeta \to \text{Fr}_{k,l}$ be the line bundle associated with universal covering and $\zeta' \to X$ its pullback via $\tilde{\varphi}_{B_l}$. Put $\eta_l' = \eta_l \otimes \zeta'$.

Let $\text{Fred}_n(\mathcal{H})$ be the space of Fredholm operators in $M_n(\mathcal{B}(\mathcal{H}))$. Clearly, $\text{Fred}_n(\mathcal{H}) \cong \text{Fred}(\mathcal{H})$.

The canonical evaluation map
\begin{equation}
\text{Fr}_{k,l} \times M_k(\mathbb{C}) \to M_{kl}(\mathbb{C}), \quad (h, T) \mapsto h(T) \tag{31}
\end{equation}
(recall that $\text{Fr}_{k,l} = \text{Hom}_{alg}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$) induces the map
\begin{equation}
\gamma_{k,l}': \text{Fr}_{k,l} \times \text{Fred}_{k,l}(\mathcal{H}) \to \text{Fred}_{kl}(\mathcal{H}). \tag{32}
\end{equation}

\textbf{Remark 28.} Note that map \eqref{eq:31} can be decomposed into the composition
\[
\text{Fr}_{k,l} \times M_k(\mathbb{C}) \to \text{Fr}_{k,l} \times M_k(\mathbb{C}) = \mathcal{A}_{k,l} \to M_{kl}(\mathbb{C}),
\]
where the last map is the tautological embedding $\mu: \mathcal{A}_{k,l} \to \text{Gr}_{k,l} \times M_{kl}(\mathbb{C})$ followed by the projection onto the second factor.

Now suppose $f_\xi: X \to \text{Fred}_{k,l}(\mathcal{H})$ represents some element $\xi \in K(X)$.

\textbf{Theorem 29.} (Cf. Theorem 23). In the above notation the composite map (cf. \eqref{eq:33})
\[
X \xrightarrow{\text{diag}} X \times X \xrightarrow{\varphi_{B_l} \times f_\xi} \text{Fr}_{k,l} \times \text{Fred}_{k,l}(\mathcal{H}) \xrightarrow{\gamma_{k,l}'} \text{Fred}_{kl}(\mathcal{H})
\]
represents $\eta_l' \otimes \xi \in K(X)$.

\textbf{Proof} (cf. [2], Proposition 2.1). By assumption the element $\xi \in K(X)$ is represented by a family $F = \{F_x\}$ of Fredholm operators in a Hilbert space $\mathcal{H}^k$. Then $\eta_l' \otimes \xi \in K(X)$ is represented by the family $\{\text{Id}_{B_l} \otimes F_x\}$ of Fredholm operators in the Hilbert bundle $\eta_l' \otimes (\mathcal{H}^k)$ (recall that $\text{End}(\eta_l) = B_l \Rightarrow \text{End}(\eta_l') = B_l$). A trivialization $\eta_l' \otimes (\mathcal{H}^k) \cong \mathcal{H}^{kl}$ is the same thing as a lift $\tilde{\varphi}_{B_l}: X \to \text{Fr}_{k,l}$ of the classifying map $\varphi_{B_l}: X \to \text{BPU}(l)$ for $B_l$ (see \eqref{eq:29}). $\square$

\textbf{Remark 30.} In order to separate the “SU”-part of the “action” $\gamma_{k,l}'$ from its “line” part, one can consider the analogous “action” of the space $\text{Fr}_{k,l}$ in place of $\text{Fr}_{k,l}$. Then one would have the representing map for $\eta_l \otimes \xi \in K(X)$ in the statement of Theorem 29.

The commutative diagram
\[
\text{Fr}_{k,l} \times \text{Fr}_{k,l} \times M_{k^2}(\mathbb{C}) \xrightarrow{\text{Fr}_{k^2,l^2} \times \text{Fr}_{k^2,l^2}} \text{Fr}_{k^2,l^2} \times M_{k^2}(\mathbb{C}) \xrightarrow{(h_2, h_1; x_1 \otimes x_2)} (h_2 \otimes h_1; x_1 \otimes x_2)
\]
(\text{Fr}_{k,l} \times M_{k^2}(\mathbb{C}) \xrightarrow{(h_2; h_1(x_1) \otimes x_2)} (h_2(x_1) \otimes h_2(x_2))
\]
(where $h_2 \in \text{Fr}_{k,l} = \text{Hom}_{alg}(M_k(\mathbb{C}), M_{kl}(\mathbb{C}))$, $x_i \in M_k(\mathbb{C})$, $M_{k^2}(\mathbb{C}) = M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$) gives rise to the “associativity” condition for the “action” $\gamma_{k,l}'$. Note that the map $\text{Fr}_{k,l} \times \text{Fr}_{k,l} \to \text{Fr}_{k^2,l^2}$ corresponds to the tensor product of corresponding $M_l(\mathbb{C})$-bundles. In fact, the maps $\text{Fr}_{k^m,l^m} \times \text{Fr}_{k^n,l^n} \to \text{Fr}_{k^{m+n},l^{m+n}}$ (induced by the tensor product of matrix algebras) define the
structure of an $H$-space on $\text{Fr}_{k, \infty} := \lim_{n} \text{Fr}_{k, l, n}$. Moreover, $\text{Fr}_{k, \infty}$ is an infinite loop space, because it is the fiber of the localization $\text{BU}_\otimes \to \text{BU}_\otimes \left[\frac{1}{k}\right]$ (and $\text{BU}_\otimes$ is an infinite loop space according [9]).

Note that the “action” (32) is not invertible because we take the tensor product of $K(X)$ by some $l$-dimensional bundle. Therefore it makes sense to consider the localization

$$\text{Fred}(\mathcal{H})_{(l)} := \lim_{n} \text{Fred}_{l, n}(\mathcal{H}),$$

where the direct limit is taken over the maps induced by the tensor product, so $l$ becomes invertible and the index takes values in $\mathbb{Z}[\frac{1}{l}]$. (In fact, our construction is independent of the choice of $l$, $(k, l) = 1$, so we can consider a pair of such numbers in order to avoid the localization.)

The idea is to associate a $\text{Fred}(\mathcal{H})_{(l)}$-bundle with the universal $\text{Fr}_{k, \infty}$-bundle $\text{Fr}_{k, \infty} \to \text{EFr}_{k, \infty}$ using the action $\lim_{n} \gamma_{k, l, n} : \text{Fr}_{k, \infty} \times \text{Fred}(\mathcal{H})_{(l)} \to \text{Fred}(\mathcal{H})_{(l)}$. This $\text{Fred}(\mathcal{H})_{(l)}$-bundle in our version of the twisted $K$-theory should play the same role as the $\text{Fred}(\mathcal{H})$-bundle associated (by the action $\gamma$) with the universal $\text{PU}(\mathcal{H})$-bundle in the “usual” version of the twisted $K$-theory.

4.3. Some speculations. We have already noticed (see Remark [10]) that $\mathfrak{S}_{k, l}$ is not an action groupoid related to an action of some Lie group on $\text{Gr}_{k, l}$. But in the direct limit it is an “action groupoid”. More precisely, consider $\mathfrak{S} := \lim_{(k, l) = 1} \mathfrak{S}_{k, l}$ (the maps are induced by the tensor product).

Since $\text{Fr} := \lim_{(k, l) = 1} \text{Fr}_{k, l}$ is an $H$-space (even an infinite loop space [9]), we see that $\mathfrak{S}$ corresponds to the action of $\text{Fr}$ on $\text{Gr}$ (see Subsection 2.4). Moreover, in this situation the map (20) can be extended to the fibration

$$(33) \quad \text{Gr} \to \lim_{k} \text{BPU}(k) \to \text{BFr} \quad \text{i.e.} \quad \text{BSU}_\otimes \to \text{K}(\mathbb{Q}/\mathbb{Z}, 2) \times \prod_{n \geq 2} \text{K}(\mathbb{Q}, 2n) \to \text{BFr}$$

(cf. Subsection 2.4). Note that we can also define $\tilde{\text{Fr}} := \lim_{(k, l) = 1} \tilde{\text{Fr}}_{k, l}$ (see (11)) and consider the corresponding fibration

$$(34) \quad \text{BSU}_\otimes \to \prod_{n \geq 2} \text{K}(\mathbb{Q}, 2n) \to \text{B} \tilde{\text{Fr}}.$$

In fact, $\text{BFr} = \text{K}(\mathbb{Q}/\mathbb{Z}, 2) \times \text{B} \tilde{\text{Fr}}$. We also have the “unitary” version

$$(35) \quad \text{BU}_\otimes \to \prod_{n \geq 1} \text{K}(\mathbb{Q}, 2n) \to \text{BFr},$$

where recall $\text{BU}_\otimes \cong \mathbb{C}P^\infty \times \text{BSU}_\otimes$ and therefore it splits as follows:

$$\mathbb{C}P^\infty \times \text{BSU}_\otimes \to \text{K}(\mathbb{Q}, 2) \times \prod_{n \geq 2} \text{K}(\mathbb{Q}, 2n) \to \text{K}(\mathbb{Q}/\mathbb{Z}, 2) \times \text{B} \tilde{\text{Fr}}.$$

The part

$$\mathbb{C}P^\infty \to \text{K}(\mathbb{Q}, 2) \to \text{K}(\mathbb{Q}/\mathbb{Z}, 2)$$
corresponds to the “usual” finite Brauer group $H^3_{\text{tors}}(X, \mathbb{Z}) (= \text{coker}\{H^2(X, \mathbb{Q}) \to H^3(X, \mathbb{Q}/\mathbb{Z})\} = \text{im}\delta: \{H^2(X, \mathbb{Q}/\mathbb{Z}) \to H^3(X, \mathbb{Z})\}$, cf. Remark 5. Therefore using the fibration \[ \text{(34)} \] one can define a “noncommutative” analog of the Brauer group of $X$ as $\text{coker}\{[X, \prod_{n \geq 2} K(Q, 2n)] \to [X, B\tilde{\text{Fr}}]\}$.

In this connection note that $\text{BSU}_\otimes$ represents the group of virtual SU-bundles of virtual dimension 1 with respect to the tensor product while $\mathbb{C}P^\infty$ represents the Picard group, i.e. the group of line bundles with respect to the tensor product too. The Picard group acts on $K(X)$ by group homomorphisms \[ \text{(22)} \] and this leads to the “usual” twisted $K$-theory.

Comparing two previous subsections we see that the “action” of $\text{Fr}_{\text{virt}}$ on $K(X)$ (strictly speaking, on the localization $K(X)[1/\ell]$) is an analog of the action of $\text{PU}(k^{\infty}) := \lim_{\to n} \text{PU}(k^n)$ on $K(X)$ which leads to the $k$-primary component of $\text{Br}(X)$. So the idea is to show that $\gamma'$ (see the previous subsection) gives rise to the action of the $H$-space $\text{Fr}$ on the $K$-theory spectrum and using this action to associate the corresponding $\text{Fred}(\mathcal{H})$-bundle with the universal $\text{Fr}$-bundle over $B\text{Fr}$ (as in the case of the “usual” twisted $K$-theory one associates a $\text{Fred}(\mathcal{H})$-bundle with the universal $\text{PU}(\mathcal{H})$-bundle over $B\text{PU}(\mathcal{H})$ using the action $\gamma$ (see Subsection 4.1)).

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**References**

1. M. Atiyah, G. Segal Twisted K-theory // arXiv:math/0407054v2 [math.KT]
2. M. Atiyah, G. Segal Twisted K-theory and cohomology // arXiv:math/0510674v1 [math.KT]
3. R. Brown: From Groups to Groupoids: A Brief Survey. *Bull. London Math. Soc.* 19, 113-134, 1987.
4. M. Karoubi: K-theory. An Introduction. *Springer Verlag*, 1978.
5. R. Meyer: Morita Equivalence In Algebra And Geometry, [http://citeseer.ist.psu.edu/meyer97morita.html](http://citeseer.ist.psu.edu/meyer97morita.html)
6. F.P. Peterson: Some remarks on Chern classes, *Ann. Math.* 69 (1959) 414-420.
7. R.S. Pierce: Associative Algebras. *Springer Verlag*, 1982.
8. C.A. Rossi: Principal bundles with groupoid structure: local vs. global theory and nonabelian Čech cohomology // arXiv:math/0404449v1 [math.DG]
9. G.B. Segal: Categories and cohomology theories. *Topology* 13 (1974).
10. A.V. Ershov: Theories of bundles with additional structures. *Fundamentalnaia i prikladnaia matematika*, vol. 13 (2007), no. 8, pp. 7798.
11. A.V. Ershov: Theories of bundles with additional homotopy conditions // arXiv:0804.1119v3 [math.KT]
12. A.V. Ershov: A generalization of the topological Brauer group // Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology, Volume 2, Special Issue 03, December 2008, pp 407-444

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