Moduli of products of curves

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Abstract

Some technical results on the deformations of varieties of general type and on permanence of semi-log-canonical singularities are proved. These results are applied to show that the connected component of the moduli space of stable surfaces containing the moduli point of a product of stable curves is the product of the moduli spaces of the curves, assuming the curves have different genera. An application of this result shows that even after compactifying the moduli space and fixing numerical invariants, the moduli spaces are still very disconnected.

The main result of this article is the construction of several connected, irreducible components of the moduli space of stable surfaces. These components parameterize products of stable curves. They are constructed from the corresponding moduli spaces of stable curves. The essential results are that products of stable curves are stable surfaces, and all infinitesimal deformations of a product of stable varieties (of any dimension) come from the deformations of the factors. In particular, this gives a proof that even after fixing certain topological invariants, the resulting moduli space may have arbitrarily many components.

These examples also show that these components of the moduli space of smooth minimal surfaces of general type are not joined in the moduli space of stable surfaces.

All schemes are defined over the field \( \mathbb{C} \). A variety will be a connected, reduced, and separated scheme of finite type, not necessarily assumed irreducible. A family will be a flat morphism of varieties. The deformation theory results used can be found in Vis; the base space of a miniversal deformation will be called the Kuranishi space, following the convenient terminology from analytic geometry.

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1 Semi-log-canonical singularities and moduli of stable surfaces

First a preliminary remark: if \( X \) is a \( S_2 \) variety which is Gorenstein in codimension 1, then the extension of the dualizing sheaf of the Gorenstein locus (which is locally free) is a reflexive sheaf on \( X \) which corresponds to a Weil divisor \( K_X \). \( X \) is called \( \mathbb{Q} \)-Gorenstein if some multiple of \( K_X \) is Cartier. Using these definitions, one may define the class of singularities to be studied.

Definition 1.1. A variety \( X \) is said to have semi-log-canonical (slc) singularities if

1. \( X \) is \( \mathbb{Q} \)-Gorenstein;
2. \( X \) is \( S_2 \);
3. \( X \) has at worst normal crossing singularities is codimension 1;
4. there exists a good desingularization \( f: Y \to X \) such that in the formula

\[
K_Y = f^*K_X + \sum a_i E_i
\]

all of the \( a_i \) are positive.

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The moduli space of stable surfaces with fixed Hilbert polynomial is difficult to define. In fact, there are many different definitions which make sense but lead to nonisomorphic moduli spaces. The original definition (as well as the definition of slc singularities) appeared in [KSB88]. Later Kollár amended the conditions that a family of stable varieties should satisfy in [Kol90].

In the case of Gorenstein varieties, these subtleties do not occur, and the moduli functors do not differ. However, it is only in special cases that the moduli space of minimal surfaces of general type can be compactified by adding stable surfaces with only Gorenstein singularities to the moduli problem. Since this suffices for the purposes of this paper, only this special case is considered.

**Definition 1.2.** The moduli functor of stable Gorenstein surfaces is a functor from schemes to sets which assigns to a scheme $B$ the set of isomorphism classes of flat, proper morphisms $X \rightarrow B$ whose fibers are Gorenstein schemes with slc singularities and whose relative dualizing sheaf $\omega_{X/B}$ is ample.

This article considers a smaller functor. Let $M_{g_1, g_2}$ be the functor which assigns to $B$ the set of isomorphism classes of flat proper morphisms $X \rightarrow B$ whose fibers are products of stable curves of genera $g_1$ and $g_2$. The results of this article will show that this functor is coarsely representable by a connected and projective variety and that it is an open and closed subfunctor of the moduli functor of stable Gorenstein surfaces.

## 2 Deformations of products

In this section, some general deformation-theoretic results are proved about products of varieties. These results are formal and primarily homological. The goal is to show that under some conditions on singularities, the small deformations of a product of varieties are obtained by deforming the factors.

Let $X = X_1 \times X_2$ be a variety which is the product of two local complete intersection varieties $X_1$ and $X_2$ of general type; let $\pi_i$ denote the projection map to $X_i$. This notation will be fixed throughout this section.

The following “rigidity lemma” will be useful:

**Lemma 2.1.** If $h : X_1 \times X_2 \rightarrow B_1 \times B_2$ is a surjective morphism of products of stable varieties, then after possibly renumbering, $h$ can be written as the product of maps $h_i : X_i \rightarrow B_i$, $i = 1, 2$.

**Proof.** This follows from the fact that the tangent space to the scheme $\text{Hom}(X_i, B_j)$ at the equivalence class $[f]$ of a morphism is $H^0(X_i, f^*T_{B_j})$ which vanishes due to the stability assumption. A morphism $h$ as in the hypothesis which is not a product would be a non-trivial deformation of some morphism $f : X_i \rightarrow B_j$.

In particular, it follows that, up to renumbering, a product of curves of general type can be written as a product of curves in a unique way. This depends on the general type assumption, as there exist abelian surfaces which can be written in distinct ways as the product of elliptic curves.

The assumption that the varieties is this section are local complete intersections implies that the space $T^1(X)$ of first-order infinitesimal deformations of such a variety $X$ is given by $\text{Ext}^1_X(\Omega_X, \mathcal{O}_X)$. See [Vie] for details. Without the local complete intersection hypothesis, the sheaf $\Omega_X$ is replaced with the cotangent complex and Ext is taken to be the hyperext in the derived category: $T^1(X) = \text{Ext}^1(X, \mathcal{O}_X)$.

**Theorem 2.2.** Every first-order deformation of $X$ is the product of a first order deformation of $X_1$ with a first order deformation of $X_2$ if $X_1$ and $X_2$ are of general type.

**Proof.** Let $L_X$, $L_{Y_1}$, and $L_{Y_2}$ denote the cotangent complexes of $X$, $Y_1$, and $Y_2$, respectively. Denote by Ext the hyperext groups. We need to show:

$$\text{Ext}^1_X(L_X, \mathcal{O}_X) \cong \text{Ext}^1_{Y_1}(L_{Y_1}, \mathcal{O}_{Y_1}) \oplus \text{Ext}^1_{Y_2}(L_{Y_2}, \mathcal{O}_{Y_2})$$

by [Hart77, III.1.2.0].

By [Hart77, II.2.2.3],

$$\text{Ext}^1(L_X, \mathcal{O}_X) \cong \text{Ext}^1(\pi_1^*L_{Y_1} \oplus \pi_2^*L_{Y_2}, \mathcal{O}_X) \cong \text{Ext}^1(\pi_1^*\mathcal{O}_{Y_1} \oplus \pi_2^*\mathcal{O}_{Y_2}),$$

and

$$\text{Ext}^1(L_X, \mathcal{O}_X) \cong \text{Ext}^1(\pi_1^*L_{Y_1} \oplus \pi_2^*\mathcal{O}_{Y_2}),$$

(2)
The following computation finishes the proof:

\[ \text{Ext}^1(\pi_Y^*L_Y, \pi_Y^*O_Y) \cong H^1[\text{RHom}(\pi_Y^*L_Y, \pi_Y^*O_Y)] \]
\[ \cong H^1[\text{R}(X, \text{RHom}(\pi_Y^*L_Y, \pi_Y^*O_Y))] \]
\[ \cong H^1[\text{R}(X, \pi_Y^*\text{RHom}(L_Y, O_Y))] \]
\[ \cong H^1[\text{R}(Y_2, \text{RHom}(L_Y, O_Y))] \]
\[ \cong H^1(\text{R}(Y_2, O_{Y_2} \otimes \text{R}(Y_1, \text{RHom}(L_Y, O_Y)))) \]
\[ \cong H^1(\text{R}(Y_2, O_{Y_2} \otimes \text{RHom}(L_Y, O_Y))) \]
\[ \cong [H^0(\text{R}(Y_2, O_{Y_2})) \otimes H^1(\text{RHom}(L_Y, O_Y))]
\[ \oplus [H^1(\text{R}(Y_2, O_{Y_2})) \otimes H^0(\text{RHom}(L_Y, O_Y))] \]
\[ \cong \text{Ext}^1(L_Y, O_Y) \oplus [H^1(Y_2, O_{Y_2}) \otimes \text{Der}(O_{Y_1}, O_{Y_1})] \]
\[ \cong \text{Ext}^1(L_Y, O_Y) \]

The steps are justified as follows: the composition of derived functors rule ([Har66], II.5.3) justifies steps (4), (6), and part of (8). Step (5) follows from the flatness of \( \pi_Y \) using [Har66], II.5.8. Step (7) is [Har66], II.5.12. Step (8) follows from [Har66], II.5.16. Step (9) is the Künneth formula. Step (10) follows from properness of \( Y_2 \) and [Hit71], II.1.2.4.3. Step (11) follows from the fact that varieties of general type have no infinitesimal automorphisms, so \( \text{Der}(O_{Y_1}, O_{Y_1}) \) vanishes.

Note that the general type hypothesis is essential: suppose the \( Y_i \) were both smooth elliptic curves. Then one may replace all of the \( \text{Ext}^1(\Omega, \mathcal{O}) \) on \( X \) and the \( Y_i \) with \( H^1(T) \). The tangent sheaf of an abelian variety is trivial, so \( h^1(Y_1, \mathcal{T}_{Y_1}) + h^1(Y_2, \mathcal{T}_{Y_2}) = h^1(Y_1, \mathcal{O}_{Y_1}) + h^1(Y_2, \mathcal{O}_{Y_2}) = 2 \), but by Hodge theory, \( h^1(X, \mathcal{T}_X) = 2h^1(X, \mathcal{O}_X) = \dim \mathbb{C} H^1(X, C) = 4 \).

**Corollary 2.3.** The Kuranishi space of a product of finitely many stable curves is smooth.

**Proof.** This follows from the fact that the deformations of stable curves are unobstructed, and from the above result shows that the only infinitesimal deformations of the product come from the factors, and are consequently unobstructed.

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### 3 Products of stable varieties

An essential advantage in using the compactified moduli space becomes apparent when one can determine all of the stable degenerations of a class of varieties. This section is dedicated to proving that products of smooth curves degenerate to products of stable curves, although the result is slightly more general. Higher-dimensional versions of such a result would depend on a deeper study of obstructions which appear. The proof given here also uses the normality of the moduli spaces of stable curves, which follows from unobstructedness.

**Proposition 3.1.** The product of stable curves is a stable surface. More specifically, the product of stable curves has only normal crossings and degenerate cusps as singular points.

**Proof.** The question is analytically local. The singularities of stable curves are nodes. Since the product of a smooth point on a curve with a node is a normal crossing singularity, it suffices to check that the product of a node with itself is slc.

One must compute a semi-resolution of the scheme \( \text{Spec } \mathbb{C}[x, y, w, z]/(xy, wz) \). This scheme is the affine cone over a cycle of rational curves, so blowing up the cone point is a semi-resolution with exceptional locus begin a cycle of rational curves. Therefore the singularity at the cone point is a degenerate cusp. All of the other singular points are plainly normal crossings.

Having checked that the singularities are slc, the stability assertion is simply the ampleness of the canonical bundle, which follows from the ampleness of the canonical bundles of the factors.

This is a special case of the following more general result, but the proof with coordinate rings is retained to see exactly what singularities occur in the case of products of stable curves.

**Theorem 3.2.** Let \( Y_1 \) and \( Y_2 \) be smoothable stable varieties. Then \( Y_1 \times Y_2 \) is a smoothable stable variety.
Exceptional divisors of the product morphism $X \to Y_1 \times Y_2$ are rational Gorenstein. It remains to verify that products of slc singularities are slc. First, the conditions of $Q$-Gorenstein, $S_2$ and normal crossings in codimension 1 are clearly preserved under taking products. Let $f : X \to Y_1$ be a desingularization. Then write

$$K_X = f^* K_{Y_1} + \sum a_i E_i$$

where the $E_i$ are exceptional. The $a_i$ are all greater than or equal to -1 since $Y_1$ is slc. Therefore the exceptional divisors of the product morphism $X \times Y_2 \to Y_1 \times Y_2$ occur with coefficient greater than or equal to -1. Since $X$ is smooth and $Y_2$ is slc, $X \times Y_2$ is slc. Therefore the discrepancies of a resolution of $X \times Y_2$ are all greater than or equal to -1, so $Y_1 \times Y_2$ is slc, since a resolution of $X \times Y_2$ is also a resolution of $Y_1 \times Y_2$. 

A stronger version of this theorem which depends on minimal model hypotheses, and which we will not use here is in [vO03]. Precisely, the total space of a flat family over a base with only slc singularities whose special fiber has only slc singularities has only slc singularities.

4 Main results

The main theorems below are stated and proved in the case of the product of two surfaces for ease of notation. However, the proofs generalize to the product of finitely many curves. Denote by $M_g$ the moduli functor of stable curves of genus $g$. This functor is known to be coarsely representable by a projective variety.

Theorem 4.1. Let $g_1, g_2 \geq 2$. If $g_1 \neq g_2$, then $M_{g_1, g_2}$ is isomorphic to $M_{g_1} \times M_{g_2}$.

Proof. Taking fibered products gives a natural transformation $M_{g_1} \times M_{g_2} \to M_{g_1, g_2}$. This natural transformation is relatively representable. By [22] it is étale. By [21] it is injective on geometric points, that is $M_{g_1}(k) \times M_{g_2}(k) = M_{g_1, g_2}(k)$ when $k$ is an algebraically closed field. The natural transformation is proper since $M_{g_1} \times M_{g_2}$ is proper. It follows that the functors are isomorphic and that $M_{g_1, g_2}$ is coarsely representable.

A similar argument proves:

Theorem 4.2. $M_{g, g}$ is isomorphic to the symmetric square of the functor $M_g$ if $g \geq 2$.

Corollary 4.3. Assume the minimal model program. Let $n > 1$. Given $m > 0$, there exists a Hilbert polynomial such that the moduli space of stable Gorenstein varieties of dimension $n$ with this Hilbert polynomial has at least $m$ components.

Proof. Given $m > 0$, there exists a positive integer $N$ which factors in at least $m$ distinct ways as a product of two distinct factors. Choose $m$ pairs $(a_i, b_i)$ such that $(a_i - 1)(b_i - 1) = N$. Let $C_{a_i}$ and $C_{b_i}$ be smooth curves of genus $a_i$ and $b_i$, respectively for each $i$.

Let $g_1, \ldots, g_{n-2}$ be distinct integers greater than 1 which are also distinct from all of the $a_i$ and $b_i$ and for each $j = 1, \ldots, n-2$, let $C_{g_j}$ be a smooth curve of genus $g_j$. Then the products

$$C_{g_1} \times \cdots \times C_{g_{n-2}} \times C_{a_1} \times C_{b_1}$$

$$\vdots$$

$$C_{g_1} \times \cdots \times C_{g_{n-2}} \times C_{a_m} \times C_{b_m}$$

have the same numerical invariants, since these can be computed from the invariants of the curves, and for a product of two curves, $\chi$ and $K^2$ are both multiples of $(a_i - 1)(b_i - 1)$. However, these curves belong to different components of the moduli space since the genera chosen are distinct.

One could also draw several easy corollaries of the theorem from the deep results in [HM08] concerning the moduli spaces of stable curves; in particular:

Corollary 4.4. $M_{g_1, g_2}$ is of general type if $g_1$ and $g_2$ are distinct and both greater than 23.
Also, the rational Picard group is not as simple as in the case of curves.

**Corollary 4.5.** Let \( g_1 \) and \( g_2 \) be distinct integers greater than 2. Then \( \text{Pic } M_{g_1,g_2} \otimes \mathbb{Q} \cong (\text{Pic } M_{g_1} \otimes \mathbb{Q}) \times (\text{Pic } M_{g_2} \otimes \mathbb{Q}) \).

**Proof.** The moduli spaces of curves are integral schemes of finite type. Furthermore, \( H^1(M_g, \mathbb{C}) = 0 \) (see, e.g. \[AC98\]). Their singularities are at worst finite quotient singularities, since the Kuranishi spaces for curves are smooth and stable curves have a finite automorphism group. Since finite quotient singularities are DuBois, the results of \[DB81\] imply that \( H^1(M_g, \mathcal{O}) \to H^1(M_g, \mathcal{O}_{M_g}) \) is surjective, so the latter group is zero. The result that the Picard group of the product decomposes as the product of Picard groups under these hypotheses is \[Har77\] ex. III.12.6.

Specifically, for the moduli spaces of curves, the rational Picard group is freely generated by the Hodge class and the classes of the components of the boundary divisor \[AC87\]. The rational Picard group of the product has too high a rank for the same to be true. This is not surprising, since the cycle structure of the moduli spaces of surfaces is not as simple as that for curves. In general, the boundary is not likely a divisor, and there will be other “geometric” classes which occur, for example, the closure of the locus of surfaces whose canonical model has rational double points.

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