Highly linked tournaments with large minimum out-degree

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Abstract

We prove that there exists a function \( f : \mathbb{N} \to \mathbb{N} \) such that for any positive integer \( k \), if \( T \) is a strongly 4\( k \)-connected tournament with minimum out-degree at least \( f(k) \), then \( T \) is \( k \)-linked. This makes progress towards resolving a conjecture of Pokrovskiy. Along the way, we show that a tournament with sufficiently large minimum out-degree contains a subdivision of a complete directed graph. This result may be of independent interest.

1 Introduction

Recall that a (undirected) graph is \( k \)-connected if it remains connected after the removal of any set of \( k - 1 \) vertices. Menger’s theorem provides a characterization of \( k \)-connected graphs. It states that a graph is \( k \)-connected if and only if between any two distinct vertices there exist \( k \) internally vertex disjoint paths. This easily implies that if \( X \) and \( Y \) are any two disjoint sets of \( k \) vertices, then there exist \( k \) vertex disjoint paths from \( X \) to \( Y \), and which are internally disjoint from \( X \cup Y \). However, we do not have control over the endpoints of these paths: we cannot specify that a given vertex of \( X \) must be joined by one of these paths to a given vertex of \( Y \). This leads to the notion of \( k \)-linkedness. A graph is \( k \)-linked if for any two disjoint sets of vertices \( \{x_1, \ldots, x_k\} \) and \( \{y_1, \ldots, y_k\} \) there are vertex disjoint paths \( P_1, \ldots, P_k \) such that \( P_i \) joins \( x_i \) to \( y_i \) for \( i = 1, \ldots, k \).

Clearly, \( k \)-linkedness is a stronger notion than \( k \)-connectivity. But how much stronger? Larman and Mani [6] and Jung [4] showed that there is an \( f(k) \) such that any \( f(k) \)-connected graph is \( k \)-linked. They used a theorem of Mader [8], which states that any sufficiently connected graph contains a topological complete graph \( TK_{3k} \) on \( 3k \) vertices, and noticed that any \( 2k \)-connected graph containing a \( TK_{3k} \) must be \( k \)-linked. However, their proof gives an exponential bound for \( f(k) \). Later, Bollobás and Thomason [2] showed that \( f(k) \) can be taken to be linear in \( k \): in particular, they showed that any \( 22k \)-connected graph is \( k \)-linked.

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The definitions for $k$-connectivity and $k$-linkedness carry over similarly for directed graphs. A directed graph is **strongly connected** if for any pair of distinct vertices $x$ and $y$ there is a directed path from $x$ to $y$, and is strongly $k$-connected if it remains connected upon removal of any set of at most $k-1$ vertices. In the sequel, we shall omit the use of the word ‘strongly’ with the understanding that we always mean strong connectivity. Menger’s theorem carries over in the directed case as well and asserts that a directed graph is $k$-connected if and only if for any two distinct vertices $x$ and $y$ there are $k$ internally vertex disjoint directed paths from $x$ to $y$. And finally, the notion of $k$-linkedness is the same for directed graphs with the condition that all paths must be directed. We can now ask for directed analogues of the questions addressed in the previous paragraph. However, directed graphs exhibit quite different behaviour. Indeed, Thomassen [15] constructed directed graphs with arbitrarily large connectivity which are not even 2-linked.

Since directed graphs behave so differently, it is natural to consider the situation for a restricted class of directed graphs, namely, **tournaments** (recall that a tournament is a directed graph such that for every pair of vertices $x$ and $y$, exactly one of $xy$ and $yx$ is an edge). And, indeed, Thomassen [14] proved that there is a $g(k)$ such that every $g(k)$-connected tournament is $k$-linked, with $g(k) = Ck!$, for some absolute constant $C$. Majorly improving Thomassen’s bound on $g(k)$, Kühn, Lapinskas, Osthus, and Patel [5] showed that one may take $g(k) = 10^4k \log k$ and still ensure $k$-linkedness. They went on to conjecture that $g(k)$ may be taken to be linear in $k$. Pokrovskiy [11] resolved this conjecture by showing that any $452k$-connected tournament is $k$-linked. Except for small $k$, an optimal bound for $g(k)$ is not known. Bang-Jensen [1] showed that any 5-connected tournament is 2-linked, and there exists a family of 4-connected tournaments which are not 2-linked. For general $k$, Pokrovskiy [11] constructed $(2k-2)$-connected tournaments with arbitrarily large out and in-degree which are not $k$-linked.

Going back to undirected graphs for a moment, if some density conditions are assumed on the graph, then Bollobás and Thomason’s $22k$ can be taken all the way down to $2k$, since Mader [9] proved that a graph with sufficiently large average degree contains a topological complete graph of order $3k$. Note that $2k$ is close to the theoretical minimum connectivity in any $k$-linked graph (a $k$-linked graph is necessarily $(2k-1)$-connected). Recently, Thomas and Wollan [12] showed that any $2k$-connected graph with average degree at least $10k$ is $k$-linked, greatly reducing the bound on the required average degree. Motivated by this result, Pokrovskiy [11] conjectured that a similar phenomenon should occur for tournaments with a natural ‘density’ condition: high minimum out-degree and in-degree. In particular, he conjectured that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that any $2k$-connected tournament with minimum out and in-degree at least $f(k)$ is $k$-linked. Here is our main result, which makes progress on this conjecture.

**Theorem 1.1.** For every positive integer $k$ there exists $f(k)$ such that every $4k$-connected tournament $T$ with $\delta^+(T) \geq f(k)$ is $k$-linked.

Note that we do not assume any lower bound on the minimum in-degree.

Recall that the complete directed graph $\overrightarrow{K}_k$ is the directed graph on $k$ vertices where, for every pair $x,y$ of distinct vertices, both $xy$ and $yx$ are present. In order to prove
Theorem 1.1 we shall show that large minimum out-degree allows us to embed subdivisions of the complete directed graph $\overrightarrow{K}_k$. As mentioned above, Mader [9] showed that for any positive integer $k$ there is $g(k)$ such that any graph with average degree at least $g(k)$ contains a subdivision of $K_k$. The following theorem can be viewed as an analogue of Mader’s result for tournaments, replacing ‘average degree’ with ‘minimum out-degree’, and may be of independent interest.

**Theorem 1.2.** For any positive integer $k$ there exists a $d(k)$ such that the following holds. If $T$ is a tournament with $\delta^+(T) \geq d(k)$, then $T$ contains a subdivision of $\overrightarrow{K}_k$.

We remark that this theorem does not hold if we replace $T$ by a general digraph, as was shown by Mader [10]. This also follows from a result of Thomassen [13]. Indeed, he showed that for every integer $n$ there exist digraphs on $n$ vertices with minimum out-degree at least $\frac{1}{2} \log n$ which do not contain an even directed cycle. But since any subdivision of $\overrightarrow{K}_3$ must contain an even directed cycle, these digraphs do not contain any subdivision of a complete directed graph.

In order to prove Theorem 1.1, we shall need a little more than Theorem 1.2. Roughly speaking, we shall first embed in $T$ a subdivided $\overrightarrow{K}_k$, and then attach a few additional paths to it (see Section 2).

### 1.1 Organization and Notation

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.2 which allows us to embed subdivisions of a complete directed graph and related structures in tournaments with high minimum out-degree. In Section 3, we shall prove one preparatory lemma and then finish our proof of Theorem 1.1. Our final section concludes with some open problems.

Our notation is standard. Thus, for a directed graph $D$ we use $N^+(x), N^-(x), d^+(x), d^-(x)$ to denote the out-neighbourhood, in-neighbourhood, out-degree, and in-degree of a vertex $x$, respectively. We use $\delta^+(D)$ to denote the minimum out-degree of $D$. A directed path $P = x_1 \ldots x_\ell$ in $D$ is a sequence of distinct vertices such that $x_i x_{i+1}$ is an edge for every $i = 1, \ldots, \ell - 1$. We call $x_1$ the initial vertex and $x_\ell$ the terminal vertex of $P$. The length of $P$ is the number of its directed edges. We say that $P$ is *internally disjoint* from some subset $X \subset V(D)$ if $\ell \geq 3$ and $\{x_2, \ldots, x_{\ell-1}\} \cap X = \emptyset$. If $A$ and $B$ are subsets of $V(D)$, then we shall write $A \to B$ if every edge with one endpoint in $A$ and the other endpoint in $B$ is directed from $A$ to $B$. Lastly, if $\mathcal{P}$ is a family of directed paths in a digraph, then we use $\bigcup \mathcal{P}$ to denote the set $\bigcup_{P \in \mathcal{P}} V(P)$.

### 2 Proof of Theorem 1.2

The original proof that graphs with sufficiently large connectivity are $k$-linked uses a result of Mader, which allows one to embed a subdivision of a complete graph in a graph with sufficiently large average degree. Our proof of Theorem 1.1 follows a similar strategy. In
order to proceed, we need a directed analogue of Mader’s result for tournaments, a proof of which is our main aim in the present section. We shall use the following simple lemma of Lichiardopol [7], which was independently rediscovered by Havet and Lidický [3]. We include the short proof for convenience of the reader.

**Lemma 2.1.** Every tournament with minimum out-degree at least $k$ has a subtournament with minimum out-degree $k$ and order at most $3k^2$.

**Proof.** Let $T$ be a tournament with minimum out-degree at least $k$, and let $T'$ be a vertex-minimal subtournament of $T$ such that $\delta^+(T') \geq k$. Denote by $L$ the collection of vertices in $T'$ with out-degree $k$ in $T'$, and let $|T'| = t$ and $|L| = \ell$. By minimality, for every vertex $v \in T'$ we have $\delta^+(T' \setminus \{v\}) \leq k - 1$. Hence, every vertex in $T' \setminus L$ has an in-neighbour in $L$, and so there are at least $t - \ell$ edges from $L$ to $T' \setminus L$. On the other hand, the number of such edges is exactly $\ell k - \binom{\ell}{2}$, and so $t - \ell \leq \ell k - \ell^2/2 + \ell/2$. It follows that

$$\ell^2 - \ell(2k + 3) + 2t \leq 0,$$

implying the bound $(2k + 3)^2 - 8t \geq 0$. In other words, $t \leq \frac{1}{8}(2k + 3)^2$, so since $t$ must be an integer we get $t \leq \frac{1}{8}((2k + 3)^2 - 1) = k^2/2 + 3k/2 + 1 \leq 3k^2$, as required. 

We are now ready to prove Theorem 1.2. In the following, for a positive integer $k$ and nonnegative integer $m \leq 2\binom{k}{2}$, an $m$-partial $\overrightarrow{K}_k$ is any spanning subdigraph of $\overrightarrow{K}_k$ with precisely $m$ directed edges present.

**Proof of Theorem 1.2.** For a positive integer $k$ and nonnegative integer $m \leq 2\binom{k}{2}$, let $d(k, m)$ denote the smallest positive integer such that any tournament with $\delta^+(T) \geq d(k, m)$ contains a subdivision of an $m$-partial complete directed graph on $k$ vertices. We shall show that if $m < 2\binom{k}{2}$, then $d(k, m + 1) \leq 7d(k, m)^2$. We use induction on $k$, and for each fixed $k$, induction on $m$. For $k = 1$ there is nothing to show and we can take $d(1, 0) = 1$. So let us assume $k \geq 2$ is given and that we can embed a subdivision of an $m$-partial $\overrightarrow{K}_k$ in any tournament with minimum out-degree at least $d(k, m)$, and let $T$ be a tournament with $\delta^+(T) \geq 7d(k, m)^2$.

**Claim 2.2.** We may assume that there is a subdivision of an $m$-partial $\overrightarrow{K}_k$ contained in the out-neighbourhood of some vertex of $T$, and which spans at most $3d(k, m)^2$ vertices.

**Proof.** Since certainly we have $\delta^+(T) \geq d(k, m)$, by Lemma 2.1 we may find a subtournament $T'$ of size at most $3d(k, m)^2$ and with minimum out-degree at least $d(k, m)$. By induction we may embed in $T'$ a subdivision of an $m$-partial $\overrightarrow{K}_k$. Denote this subdivision by $\overrightarrow{K}$. We wish to add a missing directed edge, say $xy$. In other words, we must find a directed
path from $x$ to $y$ in $T$ such this path is internally disjoint from $V(K)$. Let $T'' = T \setminus T'$ and partition it into strongly connected subtournaments $T'' = S_1 \cup \cdots \cup S_\ell$ such that $S_i \to S_j$ for all $1 \leq i < j \leq \ell$ (unless, of course, $T''$ itself is strongly connected). Observe that since $d^+(x) \geq 7d(k,m)^2$ and $|T'| \leq 3d(k,m)^2$, we have that $x$ has an out-neighbor in $T''$. Therefore, if some vertex of $S_\ell$ is joined to $y$ we are done, as we can find a directed path from $x$ to $y$ outside of $T'$. So we may assume that $S_\ell \subseteq N^+(y)$. Now, as $|T'| \leq 3d(k,m)^2$ and no vertex of $S_\ell$ is joined to any vertex of $S_i$ for $i < \ell$, we have that

$$\delta^+(S_\ell) \geq 7d(k,m)^2 - 3d(k,m)^2 \geq d(k,m),$$

Applying Lemma 2.1 to $S_\ell$, we find a subtournament $S \subseteq S_\ell$ such that $\delta^+(S) \geq d(k,m)$ and with size at most $3d(k,m)^2$. It follows by induction that we may embed a subdivision of an $m$-partial $\overrightarrow{K}_k$ in $S$. But since $S \subseteq S_\ell \subseteq N^+(y)$ and $|S| \leq 3d(k,m)^2$, the claim holds.

By Claim 2.2, choose a vertex $z$ with the smallest possible minimum out-degree satisfying the property that there is a subdivision of an $m$-partial $\overrightarrow{K}_k$ contained in $N^+(z)$ spanning at most $3d(k,m)^2$ vertices. Denote by $N$ the out-neighborhood of $z$ and $K_z$ the subdivision with $K_z \subseteq N$. We wish to add one more directed edge to this subdivision, say $uv$ with $u, v \in K_z$. From $N$ remove all vertices of $K_z$ except for $u$ and $v$ and call this set $N'$. If $T[N']$ is strongly connected then we are done; otherwise, partition $T[N']$ into strongly connected sub-tournaments, say $T[N'] = S'_1 \cup \cdots \cup S'_t$ where $S'_i \to S'_j$ for all $1 \leq i < j \leq t$. Suppose that some vertex $w \in S'_i$ is joined to a vertex $w' \in N^-(z)$. Then since there is a directed path $P$ from $u$ to $w$ in $T[N']$ we have that $uPww'zv$ is a directed path from $u$ to $v$ which avoids $K_z \setminus \{u, v\}$. Hence we may assume that every vertex of $N^-(z)$ dominates $S'_i$. But then, since $|K_z| \leq 3d(k,m)^2$ and there are no edges from $S'_i$ to $S'_j$ for $i < t$, one has that $\delta^+(S'_i) \geq 7d(k,m)^2 - 3d(k,m)^2 = 4d(k,m)^2$. So we can repeat the argument in Claim 2.2 to $S'_i$ with minimum out-degree $4d(k,m)^2$ instead of $7d(k,m)^2$ (observe that we need $4d(k,m)^2 - 3d(k,m)^2 \geq d(k,m)$ to hold, which is clearly true). Accordingly, there is a vertex $q \in S'_i$ such that $N^+(q)$ contains a subdivision of an $m$-partial $\overrightarrow{K}_k$ spanning at most $3d(k,m)^2$ vertices. However, since $\bigcup_{i \leq i \leq t} S'_i \neq \emptyset$ (as $T[N']$ is not strongly connected), and $q$ is not joined to any vertex of $\bigcup_{i \leq i \leq t} S'_i \cup N^-(z)$, we have $d^+(q) < d^+(z)$, a contradiction to the minimality of $z$. This completes the proof of Theorem 1.2, as we may take $d(k) = d(k, 2^k(2^k))$. 

We now need to embed a slightly more complicated structure in $T$. In particular, we shall need to attach a few special paths to our subdivided complete directed graph. Say a subdivision $S$ is **minimal** in a tournament $T$ if all of its paths have minimal length. This implies that every path in $S$ is **backwards transitive**: if $x_1 \ldots x_t$ is a path in $S$ between branch vertices, then $x_i x_j \notin E(T)$ whenever $i \in [t-2]$ and $i < j + 1$. Let $K^\text{min}$ denote a minimal subdivision of a $\overrightarrow{K}_r$. Since any subdivision of $\overrightarrow{K}_r$ contains a minimal subdivision, Theorem 1.2 allows us to find a $K^\text{min}$ in tournaments with sufficiently large out-degree. If $U$ denotes the set of branch vertices of this subdivision, then for every $u, v \in U$, $K^\text{min}$ consists of directed paths $P_{uv}, P_{vu}$ going from $u$ to $v$ and from $v$ to $u$, respectively. Since $T$ is a tournament and $K^\text{min}$ is minimal, precisely one of these paths is a directed edge.
Now we define our augmented subdivision, denoted by $\mathcal{K}_r^*$, as follows. Let $\mathcal{K}$ denote a copy of $\mathcal{K}_r^{\text{min}}$ in $T$. The branch vertices of $\mathcal{K}_r^*$ are precisely the branch vertices of $\mathcal{K}$; denote this set by $U$. We form $\mathcal{K}_r^*$ by adding a collection $\mathcal{L}$ of special ‘loop’ paths in the following manner. For each pair $u, v \in U$, if, say, $P_{uv}$ is the path between $u$ and $v$ in $\mathcal{K}$ of length at least two, then each of $u$ and $v$ has an associated directed path from $\mathcal{L}$: one directed path $L^u_{uv}$ going from the second vertex of $P_{uv}$ to $u$, and another directed path $L^v_{uv}$ going from $v$ to the penultimate vertex of $P_{uv}$; we require that these paths are internally disjoint from $V(\mathcal{K})$. We also impose that the paths in $\mathcal{L}$ are minimal and hence backwards transitive. For $u \in U$, we let $\mathcal{L}_u$ denote the collection of paths in $\mathcal{L}$ which contain $u$. Note that $\mathcal{K}_r^*$ and $\mathcal{K}_r^{\text{min}}$ really denote families of subdigraphs which depend on the underlying tournament $T$. When we speak of ‘a $\mathcal{K}_r^*$’ we really mean ‘a member of $\mathcal{K}_r^*$ in $T$’; we hope this usage of notation does not cause confusion, but we think that it is simpler. Now the proof of the existence of a $\mathcal{K}_r^*$ follows exactly in the same way as the proof of Theorem 1.2, namely by induction on the number of ‘loops’. We state it as a corollary and provide only a sketch of the proof.

**Corollary 2.3.** For any positive integer $k$ there exists a $d^*(k)$ such that the following holds. If $T$ is a tournament with $\delta^+(T) \geq d^*(k)$, then $T$ contains a $\mathcal{K}_k^*$.

**Proof (Sketch).** Similarly as in Theorem 1.2, for a positive integer $k$ and nonnegative integer $m \leq 2\binom{k}{2}$, an $m$-partial $\mathcal{K}_k^*$ is any minimal subdivision of $\overrightarrow{K}_k^2$ with precisely $m$ loop paths present. Let $d^*(k, m)$ denote the smallest positive integer such that any tournament with $\delta^+(T) \geq d^*(k, m)$ contains a subdivision of an $m$-partial $\mathcal{K}_k^*$. We show, as before, that if $m < 2\binom{k}{2}$, then $d^*(k, m + 1) \leq 7d^*(k, m)^2$. For $k = 1$ there is nothing to show and we can take $d^*(1, 0) = 1$. So assume $k \geq 2$ is given. Then $d^*(2, 0)$ exists by Theorem 1.2 (i.e., we can embed a subdivision of $\overrightarrow{K}_2$ which contains a minimal such subdivision). Thus let $m \geq 1$ and suppose we can embed an $m$-partial $\mathcal{K}_k^*$ in any tournament with minimum out-degree at least $d^*(k, m)$. Let $T$ be a tournament with $\delta^+(T) \geq 7d^*(k, m)^2$. Then the same proof used to show Theorem 1.2 gives that we may attach one more loop path, which we may assume has minimal length. Therefore we can embed an $(m + 1)$-partial $\mathcal{K}_k^*$ in $T$, as claimed. \[ \square \]

### 3 Proof of the main theorem

In this section we finish the proof of Theorem 1.1. The structure of the proof is as follows. First, assuming the minimum degree of our tournament is sufficiently large, we shall embed in $T$ a copy $\mathcal{S}$ of $\mathcal{K}_r^*$ where $r = r(k)$ is sufficiently large. If $x_1, \ldots, x_k$, $y_1, \ldots, y_k$ are the vertices we want to link, then we shall show that there exists a collection of $k$ directed paths going from the $x_i$’s to the branch vertices of $\mathcal{S}$, and a collection of $k$ directed paths going from the branch vertices of $\mathcal{S}$ to the $y_i$’s, all of these paths being pairwise vertex disjoint. Here we only use the assumption that $T$ is $4k$-connected (see Lemma 3.1 below). Finally, we show that, provided one chooses these paths appropriately, one can link each $x_i$ to $y_i$ by rerouting the paths through $\mathcal{S}$. The rerouting step is more complicated than one might
expect, and we shall see that we do need the slightly richer structure $\mathcal{K}_r^*$ rather than just a subdivided complete directed graph.

We need a small bit of terminology first before proceeding. If $X$ and $Y$ are two disjoint sets of vertices in a directed graph, then we say that there is an out-matching (resp., in-matching) of $X$ to $Y$ if there is a matching from $X$ into $Y$ such that all matching edges are directed from $X$ to $Y$ (resp., directed from $Y$ to $X$).

**Lemma 3.1.** Let $T$ be a $4k$-connected tournament. Suppose $A, B \subset V(T)$ are two disjoint subsets of size $k$, and let $L \subset V(T)$ be a set of $4k$ vertices disjoint from $A \cup B$. Then there are $k$ directed paths from $A$ to $L$, and $k$ directed paths from $L$ to $B$, all these paths pairwise vertex disjoint and internally disjoint from $L$.

**Proof.** Choose two disjoint subsets $W_A, W_B$ disjoint from $A \cup B \cup L$ with maximum size subject to the following properties:

- Every vertex in $W_A$ has at least $2k$ out-neighbours in $L$, and every vertex in $W_B$ has at least $2k$ in-neighbours in $L$.

- There is an in-matching $\mathcal{M}_A$ from $W_A$ to $A$, and an out-matching $\mathcal{M}_B$ from $W_B$ to $B$.

We shall assume, without loss of generality, that $|W_A| \leq |W_B|$. Let $A'$ denote the set of $|W_A|$ vertices in $A$ that are incident with an edge of $\mathcal{M}_A$, and let $A'' = A \setminus A'$. Let $B', B''$ denote the analogous sets of vertices in $B$. As $T$ is $4k$-connected, we can find pairwise vertex disjoint directed paths from some $k - |W_B|$ vertices of $L$ to $B''$ avoiding $A \cup W_A \cup B' \cup W_B$. Choose a collection of such paths $\mathcal{P}$ which minimizes $|\bigcup \mathcal{P}|$, and subject to that, maximizes the number of paths whose second vertex has at least $2k$ in-neighbours in $L$. Partition $\mathcal{P}$ into sets $\mathcal{P}', \mathcal{P}''$ where the former denotes the collection of paths in $\mathcal{P}$ whose second vertex has at least $2k$ in-neighbours in $L$, and the latter denotes the collection of remaining paths. Denote by $X'$ the set of all second and third vertices on paths in $\mathcal{P}'$, and denote by $X''$ the set of all first and second vertices on paths in $\mathcal{P}''$. Consider the set $Y := A' \cup W_A \cup X' \cup X'' \cup B \cup W_B$ and note that we can bound the size of $Y$ as

$$|Y| \leq 2|W_A| + 3(k - |W_B|) + 2|W_B|.$$  

We now find $k - |W_A|$ disjoint directed paths from the vertices in $A''$ to some subset of $L$, avoiding $Y$. This is possible since $T$ is $4k$-connected and

$$4k - |Y| \geq 4k - (2|W_A| + 3(k - |W_B|) + 2|W_B|)$$

$$= k - 2|W_A| + |W_B| \geq k - |W_A|,$$

where the last inequality holds since we are assuming that $|W_A| \leq |W_B|$. Therefore, choose a collection $\mathcal{Q}$ of pairwise disjoint directed paths from $A''$ to $L$ avoiding $Y$ with $|\bigcup \mathcal{Q}|$ as small as possible. We claim that these new paths do not intersect any path from $\mathcal{P}$:

**Claim 3.2.** No path from $\mathcal{Q}$ intersects a path from $\mathcal{P}$.
Proof. Suppose that some path \( Q \in \mathcal{Q} \) intersects a path \( P \in \mathcal{P} \). Let \( P = x_1 \ldots x_s \) and \( Q = y_1 \ldots y_t \), and let \( L_A = (\bigcup \mathcal{Q}) \cap L \) and similarly \( L_B = (\bigcup \mathcal{P}) \cap L \). We consider two cases, according to whether \( P \in \mathcal{P}' \) or \( P \in \mathcal{P}'' \). Suppose first the former holds, and let \( y_i \ (i \geq 2) \) be the first vertex of \( Q \) that intersects \( P \). We may assume that \( y_i \neq x_1 \); indeed, if \( y_i = x_1 \), then \( |L_A \cup L_B| \leq 2k - 1 \), and since \( P \in \mathcal{P}' \), we have that \( x_2 \) has at least \( 2k \) in-neighbours in \( L \). Therefore, we may choose some in-neighbour \( x' \) disjoint from \( L_A \cup L_B \) and replace \( P \) with \( P' := x'x_2 \ldots x_s \). Moreover, since the paths in \( Q \) avoid \( \{x_2, x_3\} \) we may assume that \( y_1 = x_4 \). Consider \( y_i - 1 \) and pick any vertex \( z \in L \setminus (L_A \cup L_B) \). If \( y_i - z \in E(T) \), then we may replace \( Q \) with the shorter directed path \( y_1 \ldots y_i - 1z \), contradicting the minimality of \( |\bigcup \mathcal{Q}| \). So we have \( zy_i - 1 \in E(T) \). But then as long as \( i \geq 3 \) we may replace \( P \) with the shorter path \( zy_i - 1x_4 \ldots x_s \), contradicting the initial minimal choice of \( |\bigcup \mathcal{P}| \). It remains to consider when \( i = 2 \). In this case, \( zy_2 \notin E(T) \) for every \( z \in L \setminus (L_A \cup L_B) \), since otherwise we can replace \( P \) with a shorter directed path. Thus \( y_2 \) has at least \( 2k \) out-neighbours in \( L \), and we can add \( y_1y_2 \) to the matching \( \mathcal{M}_A \), a contradiction to the maximality of this matching. It follows that \( P \cap Q = \emptyset \) for \( P \in \mathcal{P}' \).

So let us assume that \( P \in \mathcal{P}'' \). Since the paths in \( Q \) avoid \( \{x_1, x_2\} \), we may assume in this case that \( y_i = x_3 \). The same argument as in the previous paragraph shows that we may assume \( i \geq 3 \) (otherwise, we obtain a larger matching than \( \mathcal{M}_A \)). Also, as before, if \( z \in L \setminus (L_A \cup L_B) \), then \( y_i - z \notin E(T) \); otherwise we can replace \( Q \) with the shorter path \( y_1 \ldots y_i - z \). Hence \( y_i - 1 \) has at least \( |L| - |L_A \cup L_B| \geq 2k \) in-neighbours in \( L \). Choose one of these in-neighbours \( u \) (disjoint from \( L_A \cup L_B \)) and consider the path \( P^* := uy_i - 1x_3 \ldots x_s \). Then \( P^* \) has the same length as \( P \) and its second vertex has at least \( 2k \) in-neighbours in \( L \), so we could replace \( P \) with \( P^* \), contradicting the maximality of \( \mathcal{P}' \). Therefore, we must have \( P \cap Q = \emptyset \), and the proof of Claim 3.2 is complete.

\[ \square \]

Armed with Claim 3.2, the proof of Lemma 3.1 is essentially complete. Indeed, every vertex in \( W_A \) has at least \( 2k \) out-neighbours in \( L \), and so each of these vertices has at least

\[ 2k - |L_A \cup L_B| = |W_A| + |W_B|, \]

out-neighbours in \( L \setminus (L_A \cup L_B) \). So for each vertex in \( W_A \) we may select a distinct out-neighbour in \( L \setminus (L_A \cup L_B) \). Then every vertex in \( W_B \) has at least \( |W_B| \) in-neighbours from the remaining vertices of \( L \), so we can pick a distinct in-neighbour for every vertex of \( W_B \). The paths of length 2 using vertices of \( W_A \cup W_B \) together with \( \mathcal{P} \) and \( \mathcal{Q} \) form the required collection of paths.

\[ \square \]

We can now finish the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( k \geq 2 \) be an integer and let \( f(k) := d^*(12k^2) + 2k \), where \( d^* : \mathbb{N} \to \mathbb{N} \) is the function provided by Corollary 2.3. Suppose that \( T \) is a \( 4k \)-connected tournament with minimum out-degree at least \( f(k) \), and let \( X = \{x_1, \ldots, x_k\}, Y = \{y_1, \ldots, y_k\} \) be two disjoint \( k \)-sets of vertices. We wish to find pairwise vertex disjoint directed paths going from \( x_i \) to \( y_i \) for each \( i \in [k] \). Remove \( X \cup Y \) from \( T \); the tournament induced on
We denote the set of edges of \( S \) by \( E(S) \). For example, whenever we speak of distances in \( S \), we insist that they are computed using only these directed edges. Let \( P \) and \( Q \) be any two collections of pairwise disjoint directed paths such that every path in \( P \) goes from \( U \) to \( Y \), every path in \( Q \) goes from \( X \) to \( U \), and all of these paths are internally vertex disjoint from \( U \); by Lemma 3.1, such collections exist. We say that a pair \( (u, x) \) \( \in U \times V(S) \) is at in-distance \( d \) in \( S \) if \( d \) is the smallest integer such that there is a directed path \( P' \) of length \( d \) using only edges of \( S \), and such that \( P' \) goes from \( u \) to \( x \). We shall also sometimes say that \( x \) has in-distance \( d \) in \( S \) from \( u \). Similarly, we say that \( (u, x) \) \( \in U \times V(S) \) is at out-distance \( d \) in \( S \) if \( d \) is the smallest integer such that there is a directed path \( Q' \) of length \( d \) using only edges of \( S \), and such that \( Q' \) goes from \( x \) to \( u \) in \( S \); we shall also sometimes say that \( x \) has out-distance \( d \) in \( S \) from \( u \). We denote in-distance by \( d^{\text{in}}(u, x) \) and out-distance by \( d^{\text{out}}(u, x) \) (where we have suppressed the dependence on \( S \)).

**Observation 3.3.** Let \( x \in V(S) \setminus U \). Then \( x \) is at in-distance (or out-distance) at least 3 from every vertex of \( U \), except possibly the branch vertex (or vertices) belonging to the path of \( S \) containing \( x \).

**Proof.** If \( x \in V(S) \setminus U \), then either \( x \in P_{uv} \) for some \( u, v \in U \) or \( x \in L^u_{uv} \in \mathcal{L}_u \) (or possibly both). Let \( w \in U \setminus \{u, v\} \). In order to get from \( w \) to \( x \) using only edges of \( S \), we must first reach either \( u \) or \( v \). However, recall that the single edge paths in \( K \) are not edges of \( S \), so the path from \( w \) to \( u \) or \( v \) in \( S \) has length at least 2. Therefore, \( x \) has in-distance at least 3 from \( w \), as required. A symmetric argument shows that the observation remains true with ‘out-distance’ instead of ‘in-distance’.

In the following, we shall always assume that any family \( F \) of directed paths in \( T \) between \( X \cup Y \) and \( U \) are internally disjoint from \( U \). We also denote by \( U_T \) the set \( U \cap (\bigcup F) \). Our first claim asserts that we may assume the paths in one of the collections \( P \), \( Q \) contains few vertices which are ‘close’ in \( S \) to a vertex in \( U \).
Lemma 3.4. We may choose either $\mathcal{P}$ or $\mathcal{Q}$ such that there are at most $8k^2 + 4k$ vertices $u \in U \setminus U_\mathcal{P}$ (resp., $U \setminus U_\mathcal{Q}$) with $d^{in}(u, x) \leq 2$ (resp., $d^{out}(u, x) \leq 2$) for some $x \in \bigcup \mathcal{P} \setminus U_\mathcal{P}$ (resp., for some $x \in \bigcup \mathcal{Q} \setminus U_\mathcal{Q}$).

Proof. Apply Lemma 3.1 with $A = X$, $B = Y$, and $L = U$. Using the proof and notation of Lemma 3.1, assume that $|W_X| \leq |W_Y|$. Then recall that we may choose the paths from $U$ to $Y$ first minimally (with respect to the number of vertices used) upon the removal of $W_X \cup W_Y$, a set of at most $2k$ vertices. Recall also that each such path which uses a vertex of $W_X \cup W_Y$ has length two. Suppose there is a set $U_0 \subset \bigcup \mathcal{P} \setminus U_\mathcal{P}$ of more than $8k^2 + 4k$ vertices such that for every $u \in U_0$ there is $x \in \bigcup \mathcal{P} \setminus U_\mathcal{P}$ with $d^{in}(u, x) \leq 2$. We claim that this contradicts minimality. Indeed, by pigeonhole there is a set $U_0' \subset U_0$ of size more than $8k + 4$, and a path $P \in \mathcal{P}$ such that for each $u \in U_0'$ there is some $x \in P$ with $d^{in}(u, x) \leq 2$.

From Observation 3.3, it follows that for each interior vertex $v$ of $P$ there are at most two vertices of $U_0'$ that are at in-distance 2 from $v$. Therefore $P$ must have more than two edges so does not intersect $W_X \cup W_Y$. For each vertex $u \in U_0'$, pick some vertex $v_u \in P$ at in-distance exactly 2 from $u$, and denote by $D$ the set containing all such vertices $v_u$. Note that $P$ contains at most one vertex at in-distance 1 from a vertex in $U \setminus U_\mathcal{P}$, as otherwise we may reroute $P$ and obtain a shorter path avoiding $W_X \cup W_Y$. Using Observation 3.3 again, there is a set $D'$ of at least $\frac{1}{2}(8k + 4) = 4k + 2$ vertices in $D$ corresponding to distinct vertices of $U_0'$. Let $P = p_0 \ldots p_t$, where $p_0 \in U$ and $p_t \in X$, $F := D' \setminus \{p_1, p_2\}$.

For each $p_j \in F$, we may choose vertex disjoint directed paths $u_jm_jp_j$ of length 2 in $S$, where $u_j \in U_0'$. Accordingly, there are at least $4k$ ‘middle vertices’ $m_j$, at least $2k$ of which are disjoint from $W_X \cup W_Y$; let $M$ denote the set of middle vertices disjoint from $W_X \cup W_Y$. Now, suppose some $m_j \in M$ does not intersect any path in $\mathcal{P}$. Then we may replace $P$ with the path $u_jm_jp_jP$, which is shorter and still avoids $W_X \cup W_Y$, a contradiction. Thus, each middle vertex in $M$ belongs to some member of $\mathcal{P}$ and so by pigeonhole there is a path $P'$ which contains at least two vertices of $M$. But both of these vertices are at in-distance 1 from a vertex in $U \setminus U_\mathcal{P}$, which, as noted before, is a contradiction. Hence at most $8k^2 + 4k$ vertices in $U \setminus U_\mathcal{P}$ have the stated property, as claimed. A symmetric argument shows that we may choose $\mathcal{Q}$ with the stated property in the event that $|W_Y| \leq |W_X|$. This completes the proof of the lemma.

Suppose $\mathcal{F}$ is a collection of pairwise disjoint directed paths from $U$ to $Y$ (internally disjoint from $U$), and let $P = p_0 \ldots p_t$ be any path in $\mathcal{F}$. We call the pairs $(p_0, p_1)$ and $(p_0, p_2)$ trivial if they have in-distance at most 2 in $S$; any other pair with in-distance at most 2 is nontrivial. For a subset $U' \subseteq U$ we shall say that $\mathcal{F}$ is $U'$-good if no nontrivial pair of vertices from $U' \times (\bigcup \mathcal{F} \setminus U_\mathcal{F})$ is at in-distance at most 2 in $S$. In particular, each path $P \in \mathcal{F}$ intersects $U'$ in at most one vertex, namely its initial vertex. Suppose that $\mathcal{F}$ satisfies the property stated in Lemma 3.4. Then we have the following:

Claim 3.5. There exists a subset $U' \subset U \setminus U_\mathcal{F}$ of size at least $2k$ such that $\mathcal{F}$ is $U'$-good.

Proof. This follows immediately from the previous lemma. Indeed, remove from $U$ every vertex in $U_\mathcal{F}$ and every vertex in $U \setminus U_\mathcal{F}$ at in-distance at most 2 in $S$ from some vertex of $\bigcup \mathcal{F} \setminus U_\mathcal{F}$; let $U'$ denote the remaining set of vertices. By Lemma 3.4, we have removed at
most $8k^2 + 5k$ vertices. As $|U| = 12k^2$ we have $|U'| \geq 12k^2 - (8k^2 + 5k) \geq 2k$, since $k \geq 2$. Clearly $F$ is $U'$-good.

We shall assume without loss of generality that we may choose the paths from $U$ to $Y$ with the property stated in Lemma 3.4. So the previous two claims show that we may find collections of vertex disjoint directed paths $\mathcal{P}, \mathcal{Q}$ which are internally disjoint from $U$ and such that the paths in $\mathcal{P}$ go from $U$ to $Y$, the paths in $\mathcal{Q}$ go from $X$ to $U$, and $\mathcal{P}$ is $U'$-good for some $U' \subset U \setminus U_P$ with $|U'| \geq 2k$. Conditioned on this, we assume that $\mathcal{P} \cup \mathcal{Q}$ minimizes the number of edges outside of $S$, and again conditioned on this, we take such a pair with $|\bigcup \mathcal{P}| + |\bigcup \mathcal{Q}|$ as small as possible. Let $U'' = U' \setminus U_\mathcal{Q}$ so that $|U''| \geq k$ and it is disjoint from $U_\mathcal{P} \cup U_\mathcal{Q}$; we may assume that $U'' = \{u_1, \ldots, u_k\}$ has precisely $k$ elements. We now show that one can reroute the paths in $\mathcal{P} \cup \mathcal{Q}$ through $U''$ in order to create the desired paths linking $x_i$ to $y_i$ for each $i \in [k]$. Let $U_\mathcal{P} = \{z_1, \ldots, z_k\}$ and $U_\mathcal{Q} = \{w_1, \ldots, w_k\}$ so that $z_i$ is the initial vertex in $U$ of the path $P_i \in \mathcal{P}$ with terminal vertex $y_i \in Y$, and $w_i$ is the terminal vertex in $U$ of the path $Q_i \in \mathcal{Q}$ with initial vertex $x_i \in X$. Recall that for every pair of branch vertices $u, v \in U$, $P_{uv}$ and $P_{wu}$ denotes the path in $K$ from $u$ to $v$, and from $v$ to $u$, respectively. The following sequence of claims show that we can control intersections of paths in $\mathcal{P} \cup \mathcal{Q}$ with appropriate paths in $S$ in order to link each $x_i$ to $y_i$.

**Claim 3.6.** Suppose some path $Q \in \mathcal{Q}$ intersects $L_{w_iu_i} \in L_u$, for some $i \in [k]$. Let $z$ be the first vertex of $L_{w_iu_i}$ in the intersection. Then one of the following holds: $z$ is the terminal vertex of $L_{w_iu_i}$ and $z \in Q_i$, or $z$ is the second vertex of $L_{w_iu_i}$.

**Proof.** Suppose $z$ is not the second vertex of $L_{w_iu_i}$. If $z$ is an interior point of $L_{w_iu_i}$, then $zu_i \in E(T)$ by minimality of the path $L_{w_iu_i}$. Note that if $Q$ has an edge which is not in $E(S)$ after $z$ then we have a contradiction: indeed replacing $Q$ with $Qzu_i$ yields a collection of paths with fewer edges outside of $E(S)$. Otherwise, $Q = Q_i$ and it must use at least 2 edges after $z$, so we obtain a contradiction to the minimality of $|\bigcup \mathcal{P}| + |\bigcup \mathcal{Q}|$ by rerouting the path as before. Therefore, $z$ must be the terminal vertex of $L_{w_iu_i}$. Finally, $z$ must belong to $Q_i$, otherwise we may similarly reroute $Q$ through $u_i$, decreasing the number of edges used outside $E(S)$.

**Claim 3.7.** No path in $\mathcal{P}$ intersects $P_{w_iu_i}$. Moreover, if $q_i$ denotes the last vertex in $P_{w_iu_i}$ which occurs as the intersection of some path in $\mathcal{Q}$, then $q_i \in Q_i$.

**Proof.** No path in $\mathcal{P}$ intersects $\{u_i, w_i\}$, so it suffices to show that no such path intersects the interior of $P_{w_iu_i}$. Therefore, we may assume that $P_{w_iu_i}$ has length at least 2. Suppose first that some $P \in \mathcal{P}$ contains a vertex $v$ in the interior. Note that $v$ must be the penultimate vertex of $P_{w_iu_i}$. Otherwise, $u_iv \in E(T) \cap E(S)$ by the minimality of the subdivision $K$, and this contradicts the fact that $\mathcal{P}$ is $U'$-good. Consider the loop path $L = L_{w_iu_i} \in L_u$, at $u_i$ ending at $v$, and recall that the edges of $L$ are edges of $S$. Let $z$ be the first vertex in $L_{w_iu_i}$ belonging to some path $P' \in \mathcal{P}$ such a vertex and path exist since we may take $z = v$ and $P' = P$. Let $L'$ be the initial segment of the path $L_{w_iu_i}$ ending at $z$.

Suppose first that no path in $Q \in \mathcal{Q}$ intersects $L'$, and replace $P$ with $P''' = u_iL'zP'$. Since $P'$ cannot intersect $u_i$ or $w_i$ it must have an edge which is not in $E(S)$ before $z$. It follows
that \( P'' \) has fewer edges outside of \( S \). This is a contradiction to our choice of \( P \cup Q \), provided \( P'' := (P \setminus \{P'\}) \cup \{P''\} \) is \( U' \)-good. To see this, observe that any vertex of \( L \setminus \{v\} \) is at in-distance at least 3 from \( w_1 \). Moreover, if \( w_1 \in U' \), and \( z = v \) (and hence \( P' = P \)), then \( z \) is also at in-distance at least 3 from \( w_1 \). Accordingly, if \( w_1 \in U' \), then every vertex of \( P'' \) is still at in-distance at least 3 from \( w_1 \). By the minimality of \( L \), every vertex in the interior of \( L \) (except the second) is directed towards \( u_i \); thus, the only vertices at in-distance at most 2 from \( u_i \) are the second and third vertices of \( L \), say \( x \) and \( y \), respectively. But the pairs \((u_i, x)\) and \((u_i, y)\) are trivial pairs, and thus do not contradict \( U' \)-goodness. Lastly, by Observation 3.3 every vertex of \( P'' \) (except possibly \( u_i \)) is at in-distance at least 3 from every vertex of \( U' \setminus \{u_i, w_1\} \). It follows that \( P'' \) is \( U' \)-good, which is a contradiction to our choice of \( P \cup Q \).

On the other hand, if some path \( Q' \in Q \) intersects \( L' \) in some vertex \( r \), then by Claim 3.6 \( r \) must be the second vertex of \( L_{u_i, z_i} \). Note that by \( U' \)-goodness, no path in \( P \) contains the third vertex \( r_1 \) of \( L_{u_i, u_i} \), hence we can replace \( Q' \) by \( Q'r_1u_i \) thus decreasing the number of edges outside \( E(S) \). Therefore we conclude that no path in \( P \) can intersect \( L_{u_i, u_i} \). Let us now show the second part of the claim. Suppose that \( q_j \in Q_j \) for some \( j \neq i \). Since \( Q_j \) must avoid \( \{u_i, w_1\} \) it contains an edge which is not in \( E(S) \) after \( q_j \). Replace \( Q_j \) with \( Q' = Q_j u_{w_i, u_i} \). Then by the previous paragraph, no path in \( P \) intersects \( Q' \) and the resulting collection of paths has fewer edges outside of \( S \), a contradiction. This completes the proof of the claim.

\[ \square \]

It remains to establish the analogous claims for the path \( P_{u_i z_i} \), namely that intersections of paths in \( P \cup Q \) with \( P_{u_i z_i} \) and \( L_{u_i, z_i} \) behave as one expects. The arguments are similar to those in the previous two claims. Theorem 1.1 will then be an immediate consequence.

**Claim 3.8.** For every \( i \in [k] \), no path in \( P \) intersects \( L_{u_i, z_i} \) in \( L_{u_i, z_i} \).

**Proof.** Suppose some \( P \in P \) intersects \( L_{u_i, z_i} \) in a vertex \( z \). Then \( z \) cannot be the first vertex of \( L_{u_i, z_i} \), as this would contradict the fact that \( P \) is \( U' \)-good. Therefore, if \( z' \) denotes the vertex preceding \( z \) in \( L_{u_i, z_i} \), then by the minimality of paths in \( L \), we have \( u_i z' \in E(T) \cap E(S) \). But then \( z \) is at in-distance 2 from \( u_i \), contradicting \( U' \)-goodness.

\[ \square \]

**Claim 3.9.** Let \( p_i \) denote the first vertex in \( P_{u_i z_i} \) which occurs as the intersection of some path in \( P \). Then no path in \( Q \) intersects \( P_{u_i z_i} \) and \( p_i \in P_i \).

**Proof.** As before, it suffices to show that no path in \( Q \) intersects the interior of \( P_{u_i z_i} \), so we may assume that \( P_{u_i z_i} \) has length at least 2. Suppose some \( Q \in Q \) intersects the interior of \( P_{u_i z_i} \) at \( v \). Note that since \( Q \) does not meet \( \{u_i, z_i\} \), it must leave \( S \) at some time after \( v \). If \( v \) is not the second vertex of \( P_{u_i z_i} \), then \( v u_i \in E(T) \cap E(S) \), and so we may replace \( Q \) with \( Q v u_i \). This path has fewer edges outside of \( S \) than \( Q \), and this contradicts our minimal choice of \( P \cup Q \). If \( v \) is the second vertex, then let \( L = L_{u_i, z_i} \in L_{u_i} \) be the loop path at \( u_i \) directed from \( v \) to \( u_i \). Let \( z \) be the last vertex of \( L \) which occurs as the intersection of some path \( Q' \in Q \) (\( z \) and \( Q' \) exist since we may take \( z = v \) and \( Q' = Q \)), and let \( L' \) be
the subpath of \( L \) from \( z \) to \( u_i \). By Claim 3.8, no path in \( P \) intersects \( L' \), so replace \( Q \) with \( Q'zL'u_i \). Again, the edges of \( L' \) are in \( E(S) \) so this path has fewer edges outside \( S \) than \( Q' \), a contradiction. It follows that no path in \( Q \) intersects \( P \) as claimed. For the second part of the claim, suppose that \( p_i \in P_j \) for some \( j \neq i \). Then \( P_j \) avoids \( \{u_i, z_i\} \) and therefore leaves \( S \) at some time before \( p_i \). Now, no path in \( P \cup Q \) intersects the interior of the subpath \( P_{u_i z_i}P_i \) so replace \( P_j \) with \( P'' = P_{u_i z_i}P_i P_j \). This path has fewer edges outside of \( S \). We claim that \( P'' = (P \setminus \{P_j\}) \cup \{P''\} \) is \( U' \)-good. Indeed, note that since \( P \) is \( U' \)-good, the subpath \( P_{u_i z_i}P_i \) has length at least 3. Also, for every \( v \in P_{u_i z_i} \) we have that \( vu_i \in E(T) \) by the minimality of \( K \). So the only pairs at in-distance at most 2 in \( U' \times (\bigcup P'' \setminus U_P) \) are the trivial pairs \( (u_i, x) \) and \( (u_i, y) \), where \( x, y \) are the second and third vertices, respectively, of \( P_{u_i z_i} \). But these pairs, by definition, do not contradict \( U' \)-goodness. It follows that \( j = i \), and the claim is proved.

By Claims 3.7 and 3.9, the directed paths \( Q_i q_i P_{w_i u_i} u_i P_{u_i w_i} P_i \), for each \( i \in [k] \), are pairwise vertex disjoint and link \( x_i \) to \( y_i \). This completes the proof of Theorem 1.1.

\[ \square \]

4 Final remarks and open problems

The most obvious open problem is to reduce our bound of \( 4k \) on the connectivity in Theorem 1.1. We remark that an improvement on the connectivity bound in Lemma 3.1 translates directly into a better bound in Theorem 1.1. Unfortunately, we could not go beyond \( 4k \). It is possible that Lemma 3.1 holds for every \( 2k \)-strongly-connected tournament (with possibly a bigger \( L \)). Aside from improving our bound of \( 4k \) on the connectivity and resolving completely Pokrovskiy’s conjecture, there are a few other open problems of interest. For example, what is the smallest function \( d(k) \) such that Theorem 1.2 holds?

**Problem 4.1.** Determine the smallest function \( d : \mathbb{N} \rightarrow \mathbb{N} \) such that any tournament \( T \) with \( \delta^+(T) \geq d(k) \) contains a subdivision of the complete directed graph \( K_k \).

Note that our proof gives a doubly exponential bound on \( d(k) \). Indeed, it is easy to check that \( d(k) \leq 2^{O(k^2)} \). Finally, while the conclusion of Theorem 1.2 does not hold if we replace \( T \) with a general digraph, can we embed subdivisions of acyclic digraphs in digraphs of large minimum out-degree? We end by recalling the following beautiful conjecture of Mader [10] from 1985.

**Conjecture 4.2.** For every positive integer \( k \), there exists a function \( f(k) \) such that every digraph with minimum out-degree at least \( f(k) \) contains a subdivision of the transitive tournament of order \( k \).

Of course, since every acyclic digraph is contained in the transitive tournament of the same order, this conjecture (if true) would give an affirmative answer to the preceding question.
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