Null Lagrangians and Invariance of Action: Exact Gauge Functions

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Abstract. Null Lagrangians and their gauge functions are derived for given standard and non-standard Lagrangians. The obtained standard null Lagrangians generalize those previously found but the non-standard null Lagrangians are new. The gauge functions are used to make the action invariant and introduce the exact null Lagrangians, which form a new family of null Lagrangians. The conditions required for the action to be invariant are derived for all null Lagrangians presented in this paper, thus, making them exact. As a specific application, the exact null Lagrangians are derived for a second-order ordinary differential equation. The application shows that the exact null Lagrangians can be obtained for any ordinary differential equation with known Lagrangian.

1. Introduction

In the calculus of variations, the action $A[x(t)]$ is a functional with $x(t)$ denoting an ordinary ($x : \mathcal{R} \to \mathcal{R}$) and smooth function with at least two continuous derivatives ($C^2$). The action $A[x(t)]$ is defined by an integral over a smooth function $L(\dot{x}, x, t)$ called Lagrangian, where $\dot{x} = dx/dt$. According to the principle of least action, or Hamilton’s principle [1,2], the action must be stationary (to have either a minimum or maximum or saddle point), which is mathematically expressed as $\delta A = 0$, where $\delta$ is the functional derivative of the action with respect to $x(t)$. The necessary condition that $L(\dot{x}, x, t)$ satisfies this principle is $\hat{E}L[L(\dot{x}, x, t)] = 0$, where the Euler-Lagrange (EL) operator is

$$\hat{E}L = \frac{d}{dt} \left( \frac{\partial}{\partial \dot{x}} \right) - \frac{\partial}{\partial x}. \quad (1)$$

In general, Lagrangians can be classified as standard, non-standard and null. Following the original work by Lagrange [3], standard Lagrangians (SLs) depend on $\dot{x}^2(t)$ and $x^2(t)$. However, non-standard Lagrangians (NSLs) do not contain squares of the dependent variable [4]. Null Lagrangians are those that solve identically the EL equation and can be expressed as the total derivative of a scalar function [5], also called a gauge function [6]. The existence of the SLs and NSLs is guaranteed by the Helmholtz conditions [7]; however, the existence of null Lagrangians is independent from
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these conditions. The process of finding a Lagrangian for a given differential equation is called the inverse problem of the calculus of variations, or the Helmholtz problem [8,9].

There are different methods to find the SLs, NSLs and null Lagrangians. The SLs can be found by using either the Jacobi last multiplier [10,11], or by fractional derivatives [12], or direct methods [13,14]; there are numerous applications of SLs to different physical systems. Methods to construct the NSLs were developed by many authors [13-17]. A novel form of NSLs was proposed by El Nabulsi [18], who used a fractional action approach to obtain the cosmological equations. Generalized forms of NSLs were also presented [19] and then applied to second-order ordinary differential equations (ODEs), whose solutions are special functions of mathematical physics [20]. Other generalizations of the NSLs and their applications to nonlinear ODEs were also made to the Riccati equation [21] and to a Linard-type nonlinear oscillator [22].

Null Lagrangians, also known as trivial Lagrangians, have been extensively investigated in mathematics. Specific studies involved structures and generation of these Lagrangians [5,23,24], geometric formulations [25], Cartan and Lepage forms and symmetries of Lagrangians [24,26,27], trivial Lagrangians in field theories [28,29], and Carathéodory’s theory of fields of extremals and integral invariants [1,24]. Null Lagrangians were also applied to elasticity, where they represent the energy density function of materials [30,31], and to demonstrate the Galilean invariance of the Newton law of inertia [32]. However, the role of null Lagrangians and their gauge functions in physics is not yet established.

The basis for the Lagrange formalism is the jet-bundle theory and the null Lagrangians are typically studied within the frame of this theory [28,29]. Let $T$ and $X$ be differentiable manifolds of dimensions $m$ and $M + m$, respectively, and let $\pi : X \to T$ be a fibred bundle structure, with $\pi$ being the canonical projection of the fibration. Let $J^r_m(X) \to T$ be the $r$-th jet bundle, where $t \in T$ and $x \in X$, with $r \in IN$. Then, an ODE of order $q$ is called locally variational (or the E-L type) if, and only if, there exists a local real function $L$ constrained by the condition $q \leq r$. For the ODEs with $q = 2$ the resulting local Lagrangian depends on $\dot{x}(t)$, $x(t)$ and $t$, and it can be written as $L(\dot{x}, x, t)$. Such local Lagrangians are not unique as other Lagrangians may also exist and they would give the same original equations when substituted into the E-L equations.

A general second-order ODE is of the form $\hat{D}x(t) = 0$, where $\hat{D} = A(t)d^2 dt + B(t)d dt + C(t)$ is a linear operator and $A(t)$, $B(t)$ and $C(t)$ are ordinary and smooth functions to be specified. In order to demonstrate a role that the standard and non-standard ENLs may play in the theory of ODEs, the simplest second-order ODE, $\hat{D}_o x(t) = 0$ with $\hat{D}_o = d^2 dt$ is considered as an example; the obtained results suggest that derivation of the ENLs for $\hat{D}x(t) = 0$ is straightforward.

The main objective of this paper is to develop methods to find null Lagrangians for given standard and non-standard Lagrangians, and introduce a requirement that allows defining a new class of null Lagrangians called here the exact null Lagrangians (ENLs). All null Lagrangians derived in this paper are the ENLs and their gauge functions are
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also obtained. Typically, the SLs and NSLs are found for a given ordinary or partial differential equation by solving the inverse (or Helmholtz) problem of the calculus of variations. In this paper, our approach is different, namely, we specify forms of the SLs and NSLs that have been commonly used and then present methods of finding exact null Lagrangians for the given SLs and NSLs. Since the forms of the SLs and NSLs are very different, the resulting standard and non-standard ENLs are also distinct.

The outline of the paper is as follows: in Section 2, the standard and non-standard null Lagrangians and their gauge functions are constructed; in Section 3, invariance of the action is described and used to define the exact null Lagrangians; application of the obtained results to a simple second-order ordinary differential equation is presented in Section 4; and the conclusions are given in 5.

2. Null Lagrangians and their gauge functions

2.1. Standard null Lagrangians

The standard Lagrangians \( L_s[\dot{x}(t), x(t)] \) considered here are those that depend on the squares of the dependent variable \( x^2(t) \), and its derivative \( \dot{x}^2(t) \), and they are of the form

\[
L_s[\dot{x}(t), x(t)] = \frac{1}{2}[\alpha \dot{x}^2(t) - \beta x^2(t)] ,
\]

where \( \alpha \) and \( \beta \) are either functions of the independent variable or they are constants; in both cases, they must be determined once a mathematical or physical problem is specified; in the original Lagrange formulation \([3]\), \( \alpha = \alpha_0 = \text{const} \) and \( \beta = \beta_0 = \text{const} \).

In the previous work \([5,23,24,32]\), null Lagrangians were constructed for a given differential equation. However, here we propose a method to construct null Lagrangians for the SL given by Eq. \eqref{eq:2}. The basis of this method is the requirement that the constructed null Lagrangian contains only terms with the dependent or independent variables or their combination, which are of lower order than those present in \( L_s[\dot{x}(t), x(t)] \). Following \([32]\), the following null Lagrangian is obtained

\[
L_{s,n}[\dot{x}(t), x(t), t] = c_1 \dot{x}(t)x(t) + c_2 [\dot{x}(t)t + x(t)] + c_3 \dot{x}(t) + c_4 ,
\]

and its gauge function

\[
\Phi_{s,n}[x(t), t] = \frac{1}{2} c_1 x^2(t) + c_2 x(t)t + c_4 x + c_6 t ,
\]

where the subscript ’n’ stands for null, and \( c_1, c_2, c_3 \) and \( c_4 \) are constants to be determined that can be expressed in terms of \( \alpha \) and \( \beta \). In the previous work, \( L_{s,n}[\dot{x}(t), x(t), t] \) was obtained first and then its gauge function \( \Phi_{n}[x(t), t] \) was derived. Here, we generalize these results by replacing the constants in either \( L_{s,n}[\dot{x}(t), x(t), t] \) or \( \Phi_{s,n}[x(t), t] \) by ordinary and smooth functions. The resulting null Lagrangians and their gauge functions are presented in Propositions 1 and 2.

**Proposition 1:** Let \( f_1(t), f_2(t), f_3(t) \) and \( f_4(t) \) be ordinary and smooth functions of the following test-Lagrangian

\[
L_{s,t}[\dot{x}(t), x(t), t] = f_1(t)\dot{x}(t)x(t) + f_2(t)[\dot{x}(t)t + x(t)] + f_3(t)\dot{x}(t) + f_4(t) .
\]
Then, \( L_{s,t}[\dot{x}(t), x(t), t] \) is a null Lagrangian if, and only if, \( f_1(t) = c_1, f_2(t) = c_2 \) and \( f_3(t) = c_3 \), where \( c_1, c_2 \) and \( c_3 \) are constants, and \( f_4(t) \) is arbitrary.

**Proof:** The necessary condition that \( L_{s,t}[\dot{x}(t), x(t), t] \) is a null Lagrangian is \( \tilde{E}L\{L_{s,t}[\dot{x}(t), x(t), t]\} = 0 \), which gives

\[
\dot{f}_1(t)x(t) + \dot{f}_2(t)t + \dot{f}_4(t) = 0. \tag{6}
\]

This condition is only satisfied when \( f_1(t) = c_1, f_2(t) = c_2 \) and \( f_3(t) = c_3 \), and it does not depend on \( f_4(t) \), which implies that this function can be arbitrary.

Comparison of \( L_{s,t}[\dot{x}(t), x(t), t] \) obtained in Proposition 1 to \( L_{s,n}[\dot{x}(t), x(t), t] \) of Eq. \( \text{(3)} \) can be summarized as follows.

**Corollary 1:** Any arbitrary constant or arbitrary function of independent variable may be added to any null Lagrangian without changing the properties of this Lagrangian.

**Proposition 2:** Let the general null Lagrangian be

\[
L_{s,gn}[\dot{x}(t), x(t), t] = [f_1(t)\dot{x}(t) + \frac{1}{2}\dot{f}_1(t)x(t)]x(t) + [f_2(t)\dot{x}(t) + \dot{f}_2(t)x(t)]t + f_2(t)x(t) + [f_3(t)\dot{x}(t) + \dot{f}_3(t)x(t)] + [f_4(t) + \dot{f}_4(t)]. \tag{7}
\]

where \( f_1(t), f_2(t), f_3(t) \) and \( f_4(t) \) are ordinary and smooth functions of the independent variable. Then, the null Lagrangian \( L_{s,gn}[\dot{x}(t), x(t), t] \) is obtained if, and only if, the gauge function \( \Phi_{s,n}[x(t), t] \) (see Eq. \( \text{(4)} \)) is generalized to

\[
\Phi_{s,gn}[x(t), t] = \frac{1}{2}f_1(t)x(t) + f_2(t)x(t)t + f_3(t)x(t) + f_4(t)t, \tag{8}
\]

which becomes the general gauge function.

**Proof:** In order for \( L_{s,gn}[\dot{x}(t), x(t), t] \) to be the general null Lagrangian, it is required that \( \tilde{E}L\{L_{s,gn}[\dot{x}(t), x(t), t]\} = 0 \) and

\[
L_{s,gn}[\dot{x}(t), x(t), t] = \frac{d\Phi_{s,gn}[x(t), t]}{dt} = \frac{\partial\Phi_{s,gn}[x(t), t]}{\partial t} + \dot{x}(t)\frac{\partial\Phi_{s,gn}[x(t), t]}{\partial x}. \tag{9}
\]

Substitution of Eq. \( \text{(8)} \) into Eq. \( \text{(9)} \) gives the general null Lagrangian.

The following Corollary extends (without a proof) the results of Proposition 2 to all null Lagrangians of this category.

**Corollary 2:** The Lagrangian \( L_{s,gn}[\dot{x}(t), x(t), t] \) is the most general null Lagrangian among all null Lagrangians that can be constructed from terms with the lowest orders of the dependent variable and its derivative.

**Corollary 3:** For the special cases of \( f_1(t) = c_1, f_2(t) = c_2, f_3(t) = c_3 \) and \( f_4(t) = c_4 \), the general null Lagrangian \( L_{s,gn}[\dot{x}(t), x(t), t] \) reduces to that given by Eq. \( \text{(3)} \) or the one presented in \([32]\).

To determine the arbitrary functions in the standard null Lagrangians, additional conditions are necessary; such conditions are introduced in Section ??, where it is shown how the coefficients \( \alpha \) and \( \beta \) in the standard Lagrangian \( L_s[\dot{x}(t), x(t)] \) are related to the arbitrary functions in \( L_{s,gn}[\dot{x}(t), x(t), t] \).
2.2. Non-standard null Lagrangians

Any Lagrangian different than \( L_s[\dot{x}(t), x(t)] \) (see Eq. (2)) is called a non-standard Lagrangian (NSL). Among known NSLs, the most commonly used is

\[
L_{ns}[\dot{x}(t), x(t), t] = \frac{1}{g_1(t)\dot{x}(t) + g_2(t)x(t) + g_3(t)},
\]

where \( g_1(t) \), \( g_2(t) \) and \( g_3(t) \) are ordinary and smooth functions to be determined in such a way that a second-order ODE derived by using \( L_s[\dot{x}(t), x(t), t] \) is identical to that obtained when \( L_{ns}[\dot{x}(t), x(t), t] \) is substituted into the EL equation. As shown by Eq. (2), this derivation of the ODE depends on the form of the coefficients \( \alpha \) and \( \beta \), and whether these coefficients are constants or functions of the independent variable. In other words, a prior knowledge of the ODE resulting from the standard Lagrangian is needed in order to determine the forms of the functions \( g_1(t) \), \( g_2(t) \) and \( g_3(t) \) [13,14,16,17]; see Section 4 for details.

However, for the purpose of finding non-standard null Lagrangians, determination of \( g_1(t) \), \( g_2(t) \) and \( g_3(t) \) is not required. Since there are no non-standard null Lagrangians published in the literature, in the following we present our first such Lagrangians and demonstrate their association with \( L_{ns}[\dot{x}(t), x(t), t] \). To find these Lagrangians, we impose two conditions. First, a form of any non-standard null Lagrangian must be similar to that of \( L_{ns}[\dot{x}(t), x(t), t] \), and second, the order of the dependent variable and its derivative must not exceed their order in the NSL given by Eq. (10). The obtained non-standard null Lagrangians are presented in Propositions 3 and 4, and the Corollaries that follow them.

**Proposition 3:** Let \( a_1, a_2, a_3 \) and \( a_4 \) be constants in the following non-standard test-Lagrangian

\[
L_{ns,t}[\dot{x}(t), x(t), t] = \frac{a_1\dot{x}(t)}{a_2x(t) + a_3t + a_4}.
\]

Then, \( L_{ns,t}[\dot{x}(t), x(t), t] \) is a null Lagrangian if, and only if, \( a_3 = 0 \).

Substitution of this Lagrangian into the EL equation, \( \dot{E}L\{L_{ns,t}[\dot{x}(t), x(t), t]\} = 0 \) that gives \( a_1a_3 = 0 \); since \( a_1 \neq 0 \), then \( a_3 = 0 \).

**Corollary 4:** Let \( L_{ns,n}[\dot{x}(t), x(t)] \) be the non-standard null Lagrangian with \( a_3 = 0 \). Then, its gauge function is given by \( \Phi_{ns,n}[x(t)] = (a_1/a_2) \ln |a_2x(t) + a_4| \).

**Corollary 5:** Another null Lagrangian that can be constructed is \( L_{tn}(t) = b_1/(b_2t + b_3) \) with its gauge function \( \Phi_{tn}(t) = (b_1/b_2) \ln |b_2t + b_3| \); however, this Lagrangian and its gauge function do not obey the first condition, thus, they will not be further considered.

Our generalization of the gauge function \( \Phi_{ns,n}(t) \) given in Proposition 3 is now presented in the following proposition.

**Proposition 4:** Let \( h_1(t), h_2(t) \) and \( h_4(t) \) be ordinary and smooth functions in the following general, non-standard, test-gauge function

\[
\Phi_{ns,gf}[\dot{x}(t), x(t), t] = \left[\frac{h_1(t)}{h_2(t)}\right] \ln[h_2(t)x(t) + h_4(t)] .
\]
This gauge function gives a null Lagrangian if, and only if, \( h_4(t) = a_4 = \text{const.} \)

**Proof:** Using Eq. (10), the gauge function \( P_{ns,gt}[\dot{x}(t), x(t), t] \) gives the general non-standard test-Lagrangian

\[
L_{ns,gt}[\dot{x}(t), x(t), t] = \frac{h_1(t)\dot{x}(t)}{h_2(t)x(t) + h_4(t)} + \frac{h_1(t)\dot{h}_2(t)x(t) + \dot{h}_4(t)}{h_2(t)h_2(t)x(t) + h_4(t)}
\]

\[
+ \left[ \frac{h_1(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2^2(t)} \right] \ln |h_2(t)x(t) + h_4(t)| . \tag{13}
\]

Verification of \( E\tilde{L}\{L_{ns,gt}[\dot{x}(t), x(t), t]\} = 0 \) leads to

\[
\frac{\dot{h}_4(t)}{h_2(t)x(t) + h_4(t)} = 0 , \tag{14}
\]

which shows that \( \dot{h}_4(t) = 0 \) or \( h_4(t) = a_4 = \text{const.} \). Thus, the final form of the general, non-standard, null Lagrangian is

\[
L_{ns,gn}[\dot{x}(t), x(t), t] = \frac{h_1(t)\dot{x}(t)}{h_2(t)x(t) + a_4} + \frac{h_1(t)\dot{h}_2(t)x(t)}{h_2(t)h_2(t)x(t) + a_4}
\]

\[
+ \left[ \frac{h_1(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2^2(t)} \right] \ln |h_2(t)x(t) + a_4| , \tag{15}
\]

and this is the general non-standard null Lagrangian, whose gauge function is given by Eq. (12).

According to the results of Proposition 4, the general, non-standard gauge function for \( L_{ns,gn}[\dot{x}(t), x(t), t] \) is

\[
\Phi_{ns,gn}[\dot{x}(t), x(t), t] = \left[ \frac{h_1(t)}{h_2(t)} \right] \ln [h_2(t)x(t) + a_4] . \tag{16}
\]

**Corollary 6:** The Lagrangian \( L_{ns,gn}[\dot{x}(t), x(t), t] \) reduces to \( L_{ns,n}[\dot{x}(t), x(t), t] \), and the gauge function \( \Phi_{ns,gn}[\dot{x}(t), x(t), t] \) becomes \( \Phi_{ns,n}[\dot{x}(t), x(t), t] \) when the functions \( h_1(t) \) and \( h_2(t) \) are replaced by the constants \( a_1 \) and \( a_2 \), respectively.

The derived \( L_{ns,gn}[\dot{x}(t), x(t), t] \) and its gauge function represent the most general non-standard null Lagrangian and the gauge function that can be obtained for the NSL given by Eq. (10). This is based on the condition that the order of the dependent variable in the null Lagrangian is either the same or lower than that displayed in the NSL.

### 3. Invariance of action and exact null Lagrangians

Our main results derived in the previous section are the general, standard, null Lagrangian given by Eq. (7), and the general, non-standard, null Lagrangian given by Eq. (13), and their gauge functions. The Lagrangians and the gauge functions depend on arbitrary functions that must be ordinary and smooth. These functions must obey certain constraints, which are now specified.
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In general, the action is defined as

\[ A[x(t); t_e, t_o] = \int_{t_o}^{t_e} (L + L_{\text{null}}) dt = \int_{t_o}^{t_e} L dt + \int_{t_o}^{t_e} \left[ \frac{d\Phi_{\text{null}}(t)}{dt} \right] dt \]

where \( t_o \) and \( t_e \) denote the initial and final times, \( L \) is a Lagrangian that can be either any SL or any NSL, \( L_{\text{null}} \) is any null Lagrangian and \( \Phi_{\text{null}} \) is its gauge function. Since both \( \Phi_{\text{null}}(t_e) \) and \( \Phi_{\text{null}}(t_o) \) are constants, they do not affect the Hamilton Principle that requires \( \delta A[x(t)] = 0 \). However, the requirement that \( \Delta \Phi_{\text{null}} = \Phi_{\text{null}}(t_e) - \Phi_{\text{null}}(t_o) = \text{const} \) adds this constant to the value of the action. In other word, the value of the action is affected by the gauge function.

Let us now define a new category of null Lagrangians, which have one additional characteristic, namely, they make the action gauge invariant, or more precisely independent from the gauge function.

**Definition 1:** A null Lagrangian whose \( \Delta \Phi_{\text{null}} = 0 \) at the end conditions is called the exact null Lagrangian.

**Definition 2:** A gauge function with properties that \( \Delta \Phi_{\text{null}} = 0 \) is called the exact gauge function.

The condition \( \Delta \Phi_{\text{null}} = 0 \) is satisfied when \( \Phi_{\text{null}}(t_e) = 0 \) and \( \Phi_{\text{null}}(t_o) = 0 \). Thus, the exact null Lagrangians (ENLs) are those whose exact gauge functions (EGFs) are zero at the end conditions.

We use these conditions to establish constraints on arbitrary functions that the null Lagrangians and their gauge functions depend on. We begin with \( L_{s,gn}[\dot{x}(t), x(t), t] \) (see Eq. 7) and its gauge function \( \Phi_{s,gn}[x(t), t] \) (see Eq. 8), and this Lagrangian is exact if, and only if, \( \Phi_{s,gn}(t_e) = 0 \) and \( \Phi_{s,gn}(t_o) = 0 \), or explicitly

\[ \frac{1}{2} f_1(t_e)x_e^2 + f_2(t_e)x_e t_e + f_3(t_e)x_e + f_4(t_e) t_e = 0 \]

and

\[ \frac{1}{2} f_1(t_o)x_o^2 + f_2(t_o)x_o t_o + f_3(t_o)x_o + f_4(t_o) t_o = 0 \]

with \( x_e = x(t_e) \) and \( x_o = x(t_o) \) at the end points. If the arbitrary functions satisfy these conditions, then the action remains invariant, and \( L_{s,gn}[\dot{x}(t), x(t), t] \) is the general, standard, exact null Lagrangian. It is seen that the relationships \( f_3(t_e) = -f_1(t_e)x_e/2 \) and \( f_4(t_e) = -f_2(t_e)x_e \) solve the first condition, and similar relationships for \( t_o \) solve the second condition; this shows how these functions can be related to each other.

Applying the same procedure to \( L_{ns,gn}[\dot{x}(t), x(t), t] \) and its gauge function \( \Phi_{ns,gn}[\dot{x}(t), x(t), t] \), we obtain

\[ \frac{[h_1(t_e)]}{h_2(t_e)} \ln[h_2(t_e)x_e + a_4] = 0 \]

and

\[ \frac{[h_1(t_o)]}{h_2(t_o)} \ln[h_2(t_o)x_o + a_4] = 0 \]

where \( h_1, h_2, a_4 \) are arbitrary functions.

\[ [h_1(t)] = \text{arbitrary function} \]
Since ln[\(h_2(t_e)x(t) + a_4\)] ≠ 0, these conditions set up stringent limits on the function \(h_1(t)\), whose end values must be \(h_1(t_e) = 0\) and \(h_1(t_o) = 0\); however, this procedure does not impose any constraint on \(h_2(t)\).

Further constraints on all arbitrary functions that appear in the general standard and non-standard ENLs can be imposed by considering symmetries of these Lagrangians and the resulting dynamical equations. In general, Lagrangians possess less symmetry than the equations they generate [33]. Among different symmetries, Noether and non-Noether symmetries are identified [34-36]. The presence of null Lagrangians does not affect the Noether symmetries [33,36]; however, it may have effects on the non-Noether symmetries [32,37]. Studies of these symmetries will give new constraints on the functions and they will be investigated elsewhere.

4. Application of exact null Lagrangians

To demonstrate applications of the ENLs to ODEs, we select the simplest second-order ODE, and present the general standard and non-standard ENLs for this equation. The selection of this ODE has an important advantage because it represents the first Newtonian law of dynamics, also known as the law of inertia, which requires that motion of any classical body is always rectilinear and uniform with respect to any inertial frame of reference. Let \((x, y, z)\) be a Cartesian coordinate system and \(t\) be the same time in all inertial frames, then the one-dimensional motion of the body in one inertial frame is given by \(\dot{D}_o x(t) = 0\) with the end conditions \(t_o = 0\) and \(t_e = 1\), and with the initial conditions: \(x(0) = x_o = 1\), \(x(1) = x_1 = 2\), and \(\dot{x}(0) = u_o\). Then, the solution to the equation is \(x(t) = u_o t + 1\).

In the following, we derive standard and non-standard exact null Lagrangians and their corresponding exact gauge functions, and the obtained results will be valid for for both the simplest second-order ODE and the law of inertia. Our results significantly generalize the standard null Lagrangian derived in [32]; however, the obtained non-standard exact null Lagrangians are presented for the first time in the literature.

Taking \(\alpha = 1\), the SL (see Eq. 2) for \(\dot{D}_o x(t) = 0\) is

\[
L_s[\dot{x}(t), x(t)] = \frac{1}{2} \dot{x}^2(t) ,
\]

and the general standard ENL is given by Eq. (7) with its gauge function

\[
\Phi_{s, gn}[x(t), t] = \frac{1}{2} f_1(t) x^2(t) + f_2(t) x(t) t + f_3(t) x + f_4(t) t .
\]

For the null Lagrangian to be exact, the action must be invariant, which requires that \(\Phi_{s, gn}(1) = 0\) and \(\Phi_{s, gn}(0) = 0\). Using these conditions, the following relationships (see Eqs. 18 and 19) between the arbitrary functions are obtained

\[
f_1(1) + f_2(1) + f_3(1) + f_4(1) = 0 ,
\]

and

\[
f_3(0) = -\frac{1}{2} f_1(0) .
\]
If the functions satisfy these relationships, then $L_{s,gn}[\dot{x}(t), x(t), t]$ is the ENL, and the gauge function $\Phi_{s,gn}[x(t), t]$ is the EGF.

Let us now use Eq. (10) to obtain the non-standard Lagrangian for $\dot{D}_o x(t) = 0$ with the initial conditions specified above. Since the functions $g_1(t)$, $g_2(t)$ and $g_3(t)$ depend on the form of the considered ODE, they must be independently derived for each ODE. General conditions on the functions in $L_{ns}[\dot{x}(t), x(t), t]$ derived in [16,19] can be simplified to

$$\frac{g_2(t)}{g_1(t)} + \frac{1}{3} \frac{\dot{g}_1(t)}{g_1(t)} = 0,$$

$$\dot{g}_2(t) - \frac{1}{2} \frac{\dot{g}_1(t)}{g_1(t)} g_2(t) = 0,$$

and

$$\dot{g}_3(t) - \frac{1}{2} \frac{\dot{g}_1(t)}{g_1(t)} g_3(t) + \frac{g_3(t)}{2} g_2(t) = 0.$$ 

Eliminating $g_2(t)$ from Eqs. (26) and (27), and defining $u(t) = \dot{g}_1(t)/g_1(t)$, we obtain

$$\dot{u}(t) + \frac{1}{3} u^2(t) = 0,$$

which is a special form of the Riccati equation. Following [20], the solution to Eq. (29) is

$$u(t) = 3 \frac{\dot{v}(t)}{v(t)},$$

with $v(t)$ representing a solution to $\ddot{v}(t) = 0$ which is the auxiliary condition [19,20]. Using the initial conditions $v(t = 0) = v_o$ and $\dot{v}(t = 0) = a_o$, the solution becomes $v(t) = a_o t + v_o$, and it gives

$$g_1(t) = C_1 (a_o t + v_o)^3,$$

where $C_1$ is an integration constant. Having obtained $g_1(t)$, we get

$$g_2(t) = -C_1 a_o (a_o t + v_o)^2.$$ 

Finally, $g_3(t)$ can be found by eliminating $g_1(t)$ and $g_2(t)$ from Eq. (28) and solving it. The solution is

$$g_3(t) = C_1 C_2 (a_o t + v_o)^2,$$

where $C_2$ is an integration constant.

The final form of the non-standard Lagrangian for $\dot{D}_o x(t) = 0$ is

$$L_{ns}[\dot{x}(t), x(t), t] = \frac{1}{C_1 (a_o t + v_o)^2} \left( \frac{1}{(a_o t + v_o)\dot{x}(t) - a_o x(t) + C_2} \right).$$

Despite its complexity, as compared to the standard Lagrangian given by Eq. (22), $L_{ns}[\dot{x}(t), x(t), t]$ gives $\dot{D}_o x(t) = 0$ when substituted into the EL equation.

The general non-standard null Lagrangian (Eq. (15) and its general gauge function (Eq. (16)) become exact if, and only if, the action remains invariant. To assure the invariance of the action, the conditions given by Eqs. (20) and (21) must be satisfied. This requires that $h_1(1) = 0$ and $h_1(0) = 0$; however, no contraints apply to the function $h_2(t)$, which can be of any form as long as it is ordinary and smooth.
5. Conclusions

General null Lagrangians of the lowest order of the dependent variable and its derivative are derived for given standard and non-standard Lagrangians. The standard null Lagrangians represent a generalization of those previously found; however, the presented non-standard null Lagrangians are new. For all null Lagrangians, their gauge functions are also obtained. The gauge functions are used to make the action invariant, which allows introducing a new family of exact null Lagrangians. The conditions required for the action to be invariant are established for all derived null Lagrangians. As a result, all presented null Lagrangians are exact.

The obtained results are used to derive the exact null Lagrangians for the simplest second-order ODE, which also is the main ODE of the Newtonian law of inertia. The derived standard exact null Lagrangians and their corresponding exact gauge functions significantly generalize the previously obtained results [32]. However, all non-standard exact null Lagrangians are new. Comparison between the standard and non-standard exact null Lagrangians shows significant differences between them. The presented applications demonstrate that the exact null Lagrangians can be obtained for any ODE whose standard and non-standard Lagrangians are known, and the results would also be valid for any law of physics represented by this equation.

Finally, let us briefly summarize important open problems resulting from this paper. The presented methods of finding standard and non-standard exact null Lagrangians and their exact gauge functions can be extended to other ODEs both homogeneous and inhomogeneous. A possible extension of this approach to partial differential equations (PDEs) would make it applicable to many problems of applied mathematics and theoretical physics. Since there are other families of non-standard Lagrangians, it would be interesting to explore methods of finding exact null Lagrangians for each family. More constraints on the arbitrary functions in the exact null Lagrangians and their exact gauge functions are necessary, and they can be obtained by investigating symmetries and their underlying groups, thus, Lie groups associated with the derived exact null Lagrangians, and their exact gauge functions must be determined [38].

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