Coherent state path integral and nonlinear sigma model for a condensate composed of fermions with precise, discrete steps in the time development and with transformations from Euclidean path integrations to spherical field variables

Bernhard Mieck

Abstract

A coherent state path integral is considered for fermions with precise, discrete time separation so that non-hermitian actions are obtained; the complex-conjugated, anti-commuting fields $\psi_{\vec{x},s}(t_p + \Delta t_p)$ always follow a discrete time step $\Delta t_p$ later on the non-equilibrium time contour than the corresponding fields $\psi_{\vec{x},s}(t_p)$ according to the chosen normal ordering within the original, second quantized Hamilton operator. We describe details of the derivation for a nonlinear sigma model of a pair condensate composed of fermions concerning the precise, discrete time step order which is usually abbreviated by the appealing (but in fact non-existent) hermitian form of the fields. The non-hermitian kind of actions with additional time shift $\Delta t_p$ in $\psi_{\vec{x},s}(t_p + \Delta t_p)$ (relative to $\psi_{\vec{x},s}(t_p)$) is necessary in order to avoid infinities which arise from the simultaneous action of field operators with their hermitian conjugates caused by the defining anti-commutators. This problem is ubiquitous in quantum many-body physics and already occurs in the original Dyson equation. However, one has only to include a few amendments concerning the precise, discrete time step development of coherent state path integrals, compared to previous, abridged approaches, so that one can accomplish the exact treatment and derivation of a spontaneous symmetry breaking (SSB) with a coset decomposition for coset matrices of pair condensates and sub-algebra elements of density parts. The involved Hubbard-Stratonovich transformation (HST) to self-energies is also stated more precisely regarding the separation to exact, discrete time steps with a few amendments for the matrix operations of hermitian conjugation and transposition of dyadic product related density matrices and hermitian self-energies. Finally, we define and give details for the transformation of path integrals from the Euclidean base manifold with Cartesian coordinates to spherical field variables so that inherent, rotational symmetries can be used to simplify non-perturbative calculations of path integrals. The involved metric tensor of spatial coordinates, considered as indices in addition to the spin indices, is therefore given as an extension of the metric tensor of internal degrees of freedom so that the supplementary invariant integration measure is achieved from the inverse square root of the determinant of the metric tensor with spatial indices.

Keywords: coherent states, coherent state path integral, spontaneous symmetry breaking, nonlinear sigma model, Keldysh time contour, many-particle physics.

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1 Introduction

1.1 The problem of discrete time steps in coherent state path integrals

Coherent states are a convenient tool to transform the time development of second quantized Hamilton operators to path integrals for quantum many particle problems. In general coherent states are determined by a coset decomposition of the underlying dynamical group where transformations with the maximal, commuting Cartan sub-algebra only change the reference state (as a vacuum or ground state) by redundant phases; on the contrary the cosets generate from the chosen reference state new states always being distinct up to redundant phase transformations by the subgroup elements [1]. In this paper we restrict to coherent states for the Heisenberg algebra of anti-commuting field operators in a spontaneous symmetry breaking (SSB,[2,3]) for pair condensates composed of fermionic constituents [4,5].

Previous articles of this problem with SSB in a coherent state path integral contain abbreviated, simplified relations and equations concerning the problem of precise, subsequent time steps of the time development and concerning the problem of Hubbard-Stratonovich transformations (HST,[6]) to self-energies for coset decompositions [7,8,9]. Therefore, we briefly describe in sections 2 to 3 various details of these transformations with appropriate time steps in a coherent state path integral on the non-equilibrium time contour. Despite of widespread use in many particle physics, these details of ‘correct time step limits’ of second quantized field operators are usually omitted for brevity, but are ubiquitous and have to be considered as soon as path integrations of coherent state fields are performed for quantum effects beyond classical approximations of the actions in the exponentials.

Although it is straightforward to obtain coherent state path integrals from insertion of overcomplete sets of coherent states for discrete, subsequently separated steps of the time development, there arises the problem of a suitable, discrete time-labelling of coherent state fields which substitute the field operators of second quantization in many particle systems. Apart from the derivation of the pair condensate of fermions in sections 2 to 3 we briefly illustrate a simpler problem for the second quantized Hamilton operator (1.2) with Fermi field operators $\hat{\psi}_\alpha$, $\hat{\psi}_\beta^\dagger$ (1.1) which are indexed by the first Greek letters $\alpha, \beta, \gamma, \ldots$ for various states as e. g. momentum space, coordinate space states or random states of disorder, etc. .

Aside from a one-particle operator $\hat{h}_{\alpha\beta}$, a real, symmetric interaction potential $\hat{V}_{\alpha\beta}$ is introduced for nontrivial contributions in a many particle problem

\[
\delta_{\alpha\beta} = \hat{\psi}_\alpha \hat{\psi}_\beta^\dagger + \hat{\psi}_\beta^\dagger \hat{\psi}_\alpha = \{\hat{\psi}_\alpha, \hat{\psi}_\beta^\dagger\} = \{\hat{\psi}_\alpha^\dagger, \hat{\psi}_\beta\} = 0 ; \quad (1.1)
\]

\[
\hat{H}(\hat{\psi}_\alpha^\dagger, \hat{\psi}_\beta) = \sum_{\alpha,\beta} \hat{\psi}_\alpha^\dagger \hat{h}_{\alpha\beta} \hat{\psi}_\beta + \frac{1}{2} \sum_{\alpha,\beta} \hat{\psi}_\alpha^\dagger \hat{\psi}_\beta \hat{V}_{\alpha\beta} \hat{\psi}_\beta \hat{\psi}_\alpha ; \quad (1.2)
\]

\[
\hat{h}_{\alpha\beta}^\dagger = \hat{h}_{\alpha\beta} ; \quad \hat{V}_{\alpha\beta}^T = \hat{V}_{\alpha\beta} \in \mathbb{R} . \quad (1.3)
\]

There exist various kinds of coherent state path integrals for the time development with a particular second quantized Hamilton operator as the normal ordering $\{\hat{\psi}_\alpha^\dagger \hat{\psi}_\beta\}$, the so called Weyl ordering $\{\hat{\psi}_\alpha^\dagger \hat{\psi}_\beta - \hat{\psi}_\beta^\dagger \hat{\psi}_\alpha\}/2$ or the anti-normal order $\{\hat{\psi}_\beta^\dagger \hat{\psi}_\alpha\}$ [10]; arbitrary mixtures between normal and anti-normal order can also be chosen with the Weyl-ordering as an intermediate choice between the two mentioned extremes of normal and anti-normal order. In this paper we restrict to a normal ordering and take for this special case the precise, discrete steps of the time development involving the derivation of a nonlinear sigma model with a coset decomposition. The normal order of the prevailing Hamilton operator for the coherent state path integral avoids infinities or ambiguities following from $\{\hat{\psi}_\alpha, \hat{\psi}_\beta^\dagger\} = \delta_{\alpha\beta}$. As one considers the corresponding coherent state path integral of (1.1)(1.3) with normal ordering between two coherent states $|\psi(T_{\text{ini}})\rangle$ and $|\psi(T_{\text{fin}})\rangle$ for the ‘initial’ and ‘final’ field configuration, one has to notice the occurrence of coherent state fields in combinations...
as \(\psi_{\alpha}^*(t + \Delta t) \hat{h}_{\alpha \beta} \psi_{\beta}(t)\) and \(\psi_{\alpha}^*(t + \Delta t) \psi_{\beta}^*(t + \Delta t) \frac{i}{2} \hat{V}_{\beta \alpha} \psi_{\beta}(t) \psi_{\alpha}(t)\) as the precise, discrete time steps instead of the more appealing 'hermitian' combinations \(\psi_{\alpha}^*(t) \hat{h}_{\alpha \beta} \psi_{\beta}(t)\) and \(\psi_{\alpha}^*(t) \psi_{\beta}^*(t) \frac{i}{2} \hat{V}_{\beta \alpha} \psi_{\beta}(t) \psi_{\alpha}(t)\). The latter combination (1.5) with equal times of \(\psi_{\alpha}^*(t), \psi_{\beta}(t)\) seems to be more preferable for HST’s to self-energies, coset decompositions and other transformations and approximations using an (indeed non-existent) hermitian property of the actions under subsequent separation into discrete time steps. We apply the definition \(|\psi(t)\rangle = \exp\{\sum_{\alpha} \psi_{\alpha}(t) \hat{\psi}_{\alpha}\}|0\rangle\) (1.6) (1.9) without the normalizing weight \(\exp\{\sum_{\alpha} \psi_{\alpha}^*(t) \psi_{\alpha}(t)\}\) for times \(T_{\text{fin}} > t > T_{\text{ini}}\) so that this weight has additionally to be included into the unit operator \(\hat{1}\) (1.8) (1.9) of overcomplete sets and into the definition of appropriate initial and final configurations of coherent states \(|\psi(T_{\text{ini}})\rangle, \langle \psi(T_{\text{fin}})\rangle\)

\[
\langle \psi(T_{\text{fin}})\rangle \exp\left\{ - \frac{i}{\hbar} \int_{T_{\text{ini}}}^{T_{\text{fin}}} dt \left( \hat{H} \hat{\psi}_{\alpha}^{\dagger} \hat{\psi}_{\beta} \right) \right\} |\psi(T_{\text{ini}})\rangle = \int d[\psi_{\alpha}^*(t), \psi_{\beta}(t)] \exp\left\{ - \frac{i}{\hbar} \int_{T_{\text{ini}}}^{T_{\text{fin}}} dt \sum_{\alpha} \psi_{\alpha}^*(t + \Delta t) (-i\hbar) \frac{\psi_{\alpha}(t + \Delta t) - \psi_{\alpha}(t)}{\Delta t} \right\} \times \\
\times \exp\left\{ - \frac{i}{\hbar} \int_{T_{\text{ini}}}^{T_{\text{fin}}} dt \sum_{\alpha, \beta} \left( \psi_{\alpha}^*(t + \Delta t) \left( \hat{h}_{\alpha \beta} - i \varepsilon_{+} \delta_{\alpha \beta} \right) \psi_{\beta}(t) + \frac{1}{2} \psi_{\alpha}^*(t + \Delta t) \psi_{\beta}^*(t + \Delta t) \hat{V}_{\beta \alpha} \psi_{\beta}(t) \psi_{\alpha}(t) \right) \right\} ; \quad (\varepsilon_{+} > 0) ;
\]

convenient, 'lax' hermitian form:

\[
\langle \psi(T_{\text{fin}})\rangle \exp\left\{ - \frac{i}{\hbar} \int_{T_{\text{ini}}}^{T_{\text{fin}}} dt \left( \hat{H} \hat{\psi}_{\alpha}^{\dagger} \hat{\psi}_{\beta} \right) \right\} |\psi(T_{\text{ini}})\rangle = \int d[\psi_{\alpha}^*(t), \psi_{\beta}(t)] \times \\
\times \exp\left\{ - \frac{i}{\hbar} \int_{T_{\text{ini}}}^{T_{\text{fin}}} dt \sum_{\alpha, \beta} \left( \psi_{\alpha}^*(t) \left( - i \hbar \frac{\partial}{\partial t} - i \varepsilon_{+} \delta_{\alpha \beta} + \hat{h}_{\alpha \beta} \right) \psi_{\beta}(t) + \frac{1}{2} \psi_{\alpha}^*(t) \psi_{\beta}^*(t) \hat{V}_{\beta \alpha} \psi_{\beta}(t) \psi_{\alpha}(t) \right) \right\} ;
\]

\[
|\psi(t)\rangle = \exp\left\{ \sum_{\alpha} \psi_{\alpha}(t) \hat{\psi}_{\alpha} \right\} |0\rangle ; \quad \hat{\psi}_{\alpha} |\psi(t)\rangle = \psi_{\alpha}(t) |\psi(t)\rangle ;
\]

\[
\langle \psi(t)\rangle = \langle 0 | \exp\left\{ \sum_{\alpha} \hat{\psi}_{\alpha} \psi_{\alpha}^*(t) \right\} \langle \psi(t) | \psi_{\alpha}^*(t) \rangle ;
\]

\[
\hat{1} = \int d[\psi_{\alpha}^*(t), \psi_{\alpha}(t)] \exp\left\{ - \sum_{\alpha} \psi_{\alpha}^*(t) \psi_{\alpha}(t) \right\} |\psi(t)\rangle \langle \psi(t)\rangle ;
\]

Insert unit operator \(\hat{1}\) (1.5) with fields \(\psi_{\alpha}(t), \psi_{\alpha}^*(t)\) for times between \((T_{\text{fin}} > t > T_{\text{ini}})!\).
sophisticated problem for a pair condensate creation from HST’s, coset decompositions and a gradient expansion in sections 2 to 3.

- As far as classical approximations are only taken from the variations $\delta/\delta \psi^*_\alpha(t + \Delta t)$, $\delta/\delta \psi^*_\alpha(t)$ of the actions in (1.4), (1.5), the precise, discrete time steps $'t + \Delta t'$ or $'t'$ do not matter if the chosen intervals $'\Delta t'$ are sufficiently small for the maximal energy range $\Delta E_{max}$ of the physical problem $(\Delta t \cdot \Delta E_{max} \ll 1 \cdot \hbar)$

$$\frac{\delta}{\delta \psi^*_\alpha(t + \Delta t)} \text{(Eq. (1.4))} \quad \Rightarrow \quad \frac{\delta}{\delta \psi^*_\alpha(t)} \text{(Eq. (1.5))}$$

$$\frac{\Delta}{\Delta t} \psi_\alpha(t + \Delta t) - \psi_\alpha(t) = \sum_\beta \left( \hat{h}_{\alpha\beta} \psi_\beta(t) + \psi^*_\alpha(t + \Delta t) \hat{V}_{\beta\alpha} \psi_\beta(t) \psi_\alpha(t) \right); \quad (1.10)$$

Hence, one should achieve physical results independent of the chosen discrete, time separation in the classical equations for sufficiently small time intervals $\Delta t$ adequate to the considered energy range of the classical many body problem.

- However, as one tries to include quantum effects (e.g. quadratic quantum fluctuations around classical field solutions of the actions), the precise form (1.4) of discrete time steps has to be used in the path integrations over coherent states. This is obvious because $\psi^*_\alpha(t + \Delta t)$ and $\psi^*_\alpha(t)$ are unrelated, completely independent, different integration variables in the coherent state path integral so that the path integration of the laxed form (1.5) $'\psi^*_\alpha(t) \hat{h}_{\alpha\beta} \psi_\beta(t) + \psi^*_\alpha(t) \frac{1}{2} \hat{V}_{\alpha\beta} \psi_\beta(t) \psi_\alpha(t)'$ fails to give correct physical results and the precise, discrete time steps (1.4) $'\psi^*_\alpha(t + \Delta t) \hat{h}_{\alpha\beta} \psi_\beta(t) + \psi^*_\alpha(t + \Delta t) \hat{V}_{\beta\alpha} \psi_\beta(t) \psi_\alpha(t)'$ should be taken into account instead. We have to point out that it is a very convenient standard form in literature to abbreviate the coherent state path integrals by their (non-existent) hermitian versions (1.5) of the actions in order to simplify the appearance of equations, but has always to keep in mind the precise, discrete separations of time steps (1.4) as soon as a path integration of coherent state fields is performed beyond classical approximations from variations of the actions.

Despite of the non-hermitian property of the interaction potential $'\psi^*_\alpha(t + \Delta t) \psi_\beta(t + \Delta t) \frac{1}{2} \hat{V}_{\alpha\beta} \psi_\beta(t) \psi_\alpha(t)'$ in (1.4), it is possible to transform to a hermitian self-energy density $\hat{\sigma}_{\alpha\beta}(t)$ by a small modification of the HST for the case with the lax, hermitian form of interaction. Apart from the imaginary increment $-\varepsilon_+ \delta_{\alpha\beta}$ of $\hat{h}_{\alpha\beta}$ for appropriate convergence properties of Green functions, we include a suitable imaginary increment in the real, symmetric interaction potential $\hat{V}_{\alpha\beta}$ for a convergent Gaussian integration of the self-energy $\hat{\sigma}_{\alpha\beta}(t)$

$$\hat{\sigma}_{\alpha\beta}^1(t) = \hat{\sigma}_{\alpha\beta}(t); \quad \hat{V}_{\alpha\beta} \in \mathbb{R}; \quad \hat{V}_{\alpha\beta} = \hat{V}_{\beta\alpha}; \quad \varepsilon_+ = \delta_{\alpha\beta} \varepsilon_+; \quad (\varepsilon_+ > 0); \quad (1.12)$$

$$1 := \int d[\hat{\sigma}_{\alpha\beta}(t)] \exp \left\{ -\frac{\varepsilon_+}{2 \hbar} \int_{T_{fin}}^t dt \sum_{\alpha\beta} \frac{1}{\hat{V}_{\alpha\beta} + \varepsilon_+} \times \right. \left. \hat{\sigma}_{\alpha\beta}(t) \hat{V}_{\beta\alpha} \psi_\alpha(t) \psi^*_\beta(t + \Delta t) \right\} \left( \hat{\sigma}_{\beta\alpha}(t) - \hat{V}_{\alpha\beta} \psi_\beta(t) \psi^*_\alpha(t + \Delta t) \right); \quad (1.13)$$
Note the precise, discrete time steps \( \psi^\alpha(t + \Delta t) \) in relation (1.14) which coincides with \( \psi^\alpha(t + \Delta t) \) in (1.14) along the first lower subdiagonal of matrix elements of the discrete time indices. Therefore, we define the matrix \( \hat{M}_{\beta\alpha}(t', t) \) for the states \( \beta, \alpha \) with the particular, discrete time steps \( t', t \) specified by Kronecker deltas \( \delta_{t',t} \simeq \delta(t'-t) \) and \( \delta_{t',t+\Delta t} \simeq \delta(t'-t - \Delta t) \) which replace the corresponding delta functions; however, we omit the complete labeling with time step indices for brevity \( (t, t') = (t_0 = T_{ini}), (t_1 = T_{ini} + \Delta t), \ldots, (t_{N-1} = T_{fin} - \Delta t), (t_N = T_{fin}) \)

\[
\Delta t \cdot \hat{M}_{\beta\alpha}(t', t) = \sum_{t'=t}^{t+\Delta t} \delta_{\beta\alpha} \frac{\Delta t}{\Delta t} + \frac{i}{\hbar} \left( \hat{\Delta}_{\beta\alpha} - i \hat{r} + \hat{\delta}_{\beta\alpha}(t) \right) \delta_{t',t+\Delta t} ;
\]

\[
\hat{M}_{\beta\alpha}(t', t) = \begin{pmatrix}
\hat{m}_{\beta\alpha}(t_0) - \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} \\
\hat{m}_{\beta\alpha}(t_1) - \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} \\
\hat{m}_{\beta\alpha}(t_2) & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} \\
\hat{m}_{\beta\alpha}(t_{N-1}) - \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha} & \hat{1}_{\beta\alpha}
\end{pmatrix}.
\]

After substitution of the HST (1.12,1.14) and the matrix \( \hat{M}_{\beta\alpha}(t', t) \) (1.15,1.16) into (1.14), we acquire the path integral (1.17) with bilinear coherent state fields which are to be removed by Gaussian integration with convergence generating imaginary factor \( -i \hat{r} + \hat{\delta}_{\beta\alpha} \) added to one particle operator \( \hat{h}_{\beta\alpha} \)

\[
\langle \psi(T_{fin}) \rangle \exp \left\{ - \frac{i}{\hbar} \int_{T_{ini}}^{T_{fin}} dt \hat{H}(\psi^\dagger, \psi^\dagger) \right\} = \langle \psi(T_{ini}) \rangle = \]

\[
= \int d[\tilde{\sigma}_{\alpha\beta}(t)] \exp \left\{ - \frac{i}{2 \hbar} \int_{T_{ini}}^{T_{fin}} dt \sum_{\alpha\beta} \frac{\tilde{\sigma}_{\alpha\beta}(t) \tilde{\sigma}_{\beta\alpha}(t)}{V_{\alpha\beta} + i \hat{r} + \hat{\delta}_{\beta\alpha}} \right\} \times 
\]

\[
\times \int d[\tilde{\psi}^\dagger_{\alpha}(t), \tilde{\psi}_{\beta}(t)] \exp \left\{ - \int_{T_{ini}}^{T_{fin}} dt \sum_{t''} \sum_{\alpha\beta} \tilde{\psi}^\dagger_{\alpha}(t) \tilde{\psi}_{\alpha}(t) \tilde{\psi}^\dagger_{\alpha}(t') \tilde{\psi}_{\alpha}(t') \hat{M}_{\beta\alpha}(t', t) \psi_{\alpha}(t) \right\} = 
\]

\[
= \int d[\tilde{\sigma}_{\alpha\beta}(t)] \exp \left\{ - \frac{i}{2 \hbar} \int_{T_{ini}}^{T_{fin}} dt \sum_{\alpha\beta} \frac{\tilde{\sigma}_{\alpha\beta}(t) \tilde{\sigma}_{\beta\alpha}(t)}{V_{\alpha\beta} + i \hat{r} + \hat{\delta}_{\beta\alpha}} \right\} \det \left[ \hat{M}_{\beta\alpha}(t', t) \right] ;
\]

\[
\det \left[ \hat{M}_{\beta\alpha}(t', t) \right] = \left[ \delta_{\beta\alpha} \frac{\delta_{\beta\alpha} - \delta_{\beta\alpha+\Delta t}}{\Delta t} + \frac{i}{\hbar} \left( \hat{h}_{\beta\alpha} - i \hat{r} + \hat{\delta}_{\beta\alpha}(t) \right) \delta_{\beta\alpha+\Delta t} \right].
\]
The problem of discrete time steps in coherent state path integrals

In this manner one has mapped the original coherent state path integral \[ \text{(1.4)} \] of Grassmann fields with precise, discrete time steps to a path integral \[ \text{(1.17,1.18)} \] with the hermitian self-energy integration variables \( \hat{\sigma}_{\beta\alpha}(t) \) \[ \text{(1.12,1.14)} \] by using the matrix \( \hat{M}_{\beta\alpha}(t',t) \) \[ \text{(1.15,1.16)} \] with one-particle operator \( \hat{h}_{\beta\alpha} \). If one only applies a lax kind of matrix as \( \hat{M}_{\beta\alpha}(t',t) \) \[ \text{(1.19)} \] with missing infinitesimal time shifts \( \Delta t \) as already given in relation \[ \text{(1.5)} \]

\[
\hat{M}_{\beta\alpha}(t',t) = \delta_{\beta\alpha} \frac{\delta\nu_{t} - \delta\nu_{t+\Delta t}}{\Delta t} + \frac{i}{\hbar} \left( \hat{h}_{\beta\alpha} - i \varepsilon_{+} + \hat{\sigma}_{\beta\alpha}(t) \right) \delta\nu_{t} =
\]

\[
\begin{pmatrix}
\hat{i}_{\beta\alpha} + \hat{m}_{\beta\alpha}(0) \\
\hat{1}_{\beta\alpha} - \hat{i}_{\beta\alpha} + \hat{m}_{\beta\alpha}(1) \\
\hat{1}_{\beta\alpha} + \hat{m}_{\beta\alpha}(t_{N-2}) \\
\hat{1}_{\beta\alpha} + \hat{m}_{\beta\alpha}(t_{N-1}) \\
\hat{1}_{\beta\alpha} + \hat{m}_{\beta\alpha}(t_{N})
\end{pmatrix}
\]

one has obviously made a mistake because the precise location of the self-energy as integration variable enters into the computation of the fermion determinant and path integral. This problem of the shift of \( \hat{m}_{\beta\alpha}(t) \) to its first lower sub-diagonal is also inherent in the original Dyson equation which utilizes the corresponding relation \[ \text{(1.20)} \] with the self-energy matrix \( \hat{\Sigma}_{\beta\alpha}(t',t) \) \[ \text{(1.22)} \], the total Green function \( \hat{G}_{\beta\alpha}(t',t) \) and the free Green function \( \hat{g}_{\beta\alpha}(t',t) \) \[ \text{(1.21)} \] instead of our path integral \[ \text{(1.17,1.18)} \]

\[
\hat{M}_{\beta\alpha}(t',t) = \hat{G}_{\beta\alpha}^{-1}(t',t) = \hat{g}_{\beta\alpha}^{-1}(t',t) + \hat{\Sigma}_{\beta\alpha}(t',t) ;
\]

\[
\hat{g}_{\beta\alpha}^{-1}(t',t) = \delta_{\beta\alpha} \frac{\delta\nu_{t} - \delta\nu_{t+\Delta t}}{\Delta t} + \frac{i}{\hbar} \left( \hat{h}_{\beta\alpha} - i \varepsilon_{+} \right) \underbrace{\delta\nu_{t+\Delta t}}_{\Delta t} ;
\]

\[
\hat{\Sigma}_{\beta\alpha}(t',t) = \frac{i}{\hbar} \hat{\sigma}_{\beta\alpha}(t) \underbrace{\delta\nu_{t+\Delta t}}_{\Delta t}.
\]

The exact form of the Dyson equation includes an infinitesimal shift \( \Delta t \) in the delta functions \( \delta(t' - t - \Delta t) \) in order to avoid the action of a field operator \( \hat{\psi}_{\alpha}(t) \) with its hermitian conjugate \( \hat{\psi}_{\alpha}^{\dagger}(t) \) at the same time point. (This essential time shift \( \Delta t \) is usually omitted for convenience in standard representations of the Dyson equation !). As one starts to solve the Dyson equation (e. g. by iteration) without these time shifts \( \Delta t \) in the delta functions, one fails to achieve correct, meaningful physical results. Therefore, the removal of these infinitesimal, inconspicuous time shifts \( \Delta t \) proves to be erroneous at the quantum level where one has to perform correct path integrations of the prevailing independent integration variables. However, we again emphasize that these shifts and limits with time interval \( \Delta t \) are assumed to be present in the physical literature of many particle physics even though they are usually omitted in the presentation of equations for brevity and clarity of mathematical relations.

In the remainder of this paper we briefly describe by various amendments the precise location of proper time shifts \( \Delta t_{p=\pm} \) on the non-equilibrium time contour; these brief amendments of the necessary, precise time shifts \( \Delta t_{p=\pm} \) at the quantum level are performed for the more sophisticated problem with a spontaneous symmetry breaking to a pair condensate composed of fermionic constituents where various HST’s and coset decompositions are involved in the derivation for coset matrices \( \hat{T}(\vec{x},t_{p}) \) consisting of anomalous terms. We distinguish between
the two cases of a spatially short-ranged $V_0\delta_{\vec{x},\vec{x}'}$ and long-ranged $V_{|\vec{x} - \vec{x}'|}$ interaction potential in sections 2.2-2.4 and 3.1,3.2 respectively. The short-ranged interaction only yields local self-energies after the HST whereas the general, long-ranged interaction case results into spatially nonlocal self-energies whose dimensions are extended by the total number $N_x$ (1.23) of spatial grid points for a D-dimensional spherical volume with discrete intervals $\Delta x$ in each dimension and radial length $L$ ($\Omega_D :=$ surface angle, $\Omega_{D=2} = 2\pi$, $\Omega_{D=3} = 4\pi$)

\[ N_x = \frac{\Omega_D}{\Delta x^D}. \]  

(1.23)

We emphasize that the total number of considered space points $N_x$ (1.23) can be applied as the relevant parameter for saddle point approximations and gradient expansions in particular for the bulk of an ordered (or even disordered) system where the (coupled) field variables of the self-energy at the various, different space points can be taken as equivalent in the case of an approximate translational space symmetry.

The parameter $N_x$ (1.23) with the maximum possible energy $\hbar \Omega = \hbar (1/|\Delta t_p|)$, due to discrete time steps, appears in the transformation of the determinant of an operator $\hat{O}$ to its exponential-trace-logarithm form because this transformation requires space and time integrals for the extraction of actions from the trace-logarithm term. The trace does not only comprise internal degrees of freedom as angular momentum or spin $s = \uparrow, \downarrow$, but has also to incorporate the discrete, spatial grid and involved discrete time points

\[ \text{DET}\{ \hat{O} \} = \exp \left\{ \text{TR} \ln \hat{O} \right\} = \exp \left\{ \int_C \frac{dt_p}{\hbar} \eta_p \sum_{\vec{x}} N \text{Tr} \ln \hat{O} \right\}; \]  

(1.24)

\[ \text{TR}[\ldots] \overset{\Delta}{=} \text{complete trace with summation over discrete space and time points with} \textit{out infinitesimal volume elements}; \]  

(1.25)

\[ \text{Tr}[\ldots] \overset{\Delta}{=} \text{trace over internal degrees of freedom}; \]  

(1.26)

\[ \sum_{\vec{x}} \ldots \overset{\Delta}{=} \int_{|\vec{x}|<L} (dDx/S^D) \ldots; \text{(space integral normalized by spherical volume} S^D); \]  

(1.27)

\[ \int_C dt_p \ldots \overset{\Delta}{=} \text{time contour integral}; \quad \eta_p = \text{time contour metric}; \quad (\text{cf. following Eqs. (2.8,2.9)}); \]  

(1.28)

\[ \mathcal{N} \overset{\Delta}{=} (\hbar\Omega) \cdot N_x; \quad \Omega = 1/|\Delta t_p|. \]  

(1.29)

In the final sections 4.1,4.2 we describe how to transform from ‘path integration field variables’ on an underlying Euclidean, ‘flat’ spatial grid to spherical coordinates (for $D = 2, 3$) in order to take into account rotational symmetries of the actions. This transformation is performed under the assumption that the coherent state path integrals in discrete form can be regarded as ordinary integrals of complex variables in multiple dimensions following from the $N_x$ (1.23) discrete grid points. Therefore, we start out from following metric of $(dS)^2$ (1.30) with complex (anti-commuting) fields $\psi_{\vec{x},s}(t_p), \psi_{\vec{x},s}(t_p)$ where the summations over the spatial grid points are included apart from internal degrees of freedom (e.g. the spin summation in following sections)

\[ (dS)^2 = \sum_{\vec{x}} \sum_{s=\uparrow,\downarrow} d\psi_{\vec{x},s}^*(t_p) \ d\psi_{\vec{x},s}(t_p). \]  

(1.30)

Since the inverse square root of the metric tensor of $(dS)^2$ (1.30) determines the invariant integration measure, one obtains the corresponding integration measure in spherical coordinates by transforming with the spherical basis functions and by regrouping in $(dS)^2$ the transformed fields with spherical coordinate ‘indices’.
2 Coherent state path integral for a pair condensate composed of fermions

2.1 Grassmann fields of fermionic operators and the non-equilibrium time contour

It is our main destination to supplement the various, essential time shifts of matrix elements analogous to $\hat{M}_{0\alpha}(t', t)$ (1.15, 1.16) in order to derive a pair condensate in terms of coset matrices $T(\vec{x}, t_p)$ after IHT's, coset decompositions and also a gradient expansion. We consider fermionic field operators $\hat{\psi}_{\vec{x}, s}$, $\hat{\psi}_{\vec{x}, s}^\dagger$ (2.1, 2.2) on a D-dimensional, Cartesian space grid, specified by the D-dimensional vectors $\vec{x}, \vec{x}', \vec{x}_1, \ldots, \vec{x}_2, \vec{x}_3, \ldots$, and include the spin 1/2 angular momentum $s, s', \ldots = \uparrow, \downarrow$ as internal degree of freedom

$$\langle \hat{\psi}_{\vec{x}, s}^\dagger, \hat{\psi}_{\vec{x}', s'} \rangle + = \delta_{\vec{x}, \vec{x}'} \delta_{ss'} ; \quad \{ \hat{\psi}_{\vec{x}, s}, \hat{\psi}_{\vec{x}', s'} \} + = 0 .$$

Aside from the real two-body potential $V_{[\vec{x}-\vec{x}]}$, the second quantized Hamilton operator $\hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t)$ (2.3) consists of the spin independent one-particle operator $\hat{h}(\vec{x})$ (2.4) with kinetic energy and external potential $u(\vec{x})$ which is shifted by the zero-temperature, chemical potential $\mu_0$ as a reference energy. Furthermore, the second quantized Hamilton operator $\hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t)$ (2.3) contains symmetry breaking source fields, as the anti-commuting fields $j_{\psi; s}(\vec{x}, t)$, $j_{\psi; s}^\dagger(\vec{x}, t)$ (2.5) for a coherent, macroscopic field $\langle \psi_{\vec{x}, s}(t) \rangle$ of fermions and the complex, even-valued, anti-symmetric field matrices $\hat{j}_{\psi; ss'}(\vec{x}, t)$, $\hat{j}_{\psi; ss'}^\dagger(\vec{x}, t)$ (2.6) for creating anomalous pair condensates as $\langle \psi_{\vec{x}, s}(t) \psi_{\vec{x}', s'}(t) \rangle$ and their hermitian conjugates $\langle \psi_{\vec{x}, s}^\dagger(t) \psi_{\vec{x}', s'}^\dagger(t) \rangle$

$$\hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t) = \sum_{\vec{x}} \sum_{s = \uparrow, \downarrow} \hat{h}(\vec{x}) \hat{\psi}_{\vec{x}, s}^\dagger \hat{\psi}_{\vec{x}, s} + \sum_{\vec{x}, \vec{x}', s, s' = \uparrow, \downarrow} \hat{\psi}_{\vec{x}, s}^\dagger \hat{\psi}_{\vec{x}', s'}^\dagger \hat{\psi}_{\vec{x}', s'} \hat{\psi}_{\vec{x}, s} + \sum_{\vec{x}} \sum_{s = \uparrow, \downarrow} \left( j_{\psi; s}(\vec{x}, t) \hat{\psi}_{\vec{x}, s}^\dagger + \hat{\psi}_{\vec{x}, s} j_{\psi; s}(\vec{x}, t) \right) + \frac{1}{2} \sum_{\vec{x}, s, s' = \uparrow, \downarrow} \text{tr} \left[ \hat{j}_{\psi; ss'}(\vec{x}, t) \hat{\psi}_{\vec{x}, s}^\dagger \hat{\psi}_{\vec{x}, s'}^\dagger + \hat{\psi}_{\vec{x}, s} \hat{\psi}_{\vec{x}, s'} \hat{j}_{\psi; ss'}(\vec{x}, t) \right] ;$$

$$\hat{h}(\vec{x}) = \frac{\vec{p}^2}{2m} + u(\vec{x}) - \mu_0 ;$$

$$j_{\psi; s}(\vec{x}, t), j_{\psi; s}^\dagger(\vec{x}, t) \in C_{\text{odd}} ; \quad \hat{j}_{\psi; ss'}(\vec{x}, t) = \hat{j}_{\psi; ss'}^T(\vec{x}, t) = -\hat{j}_{\psi; ss'}^\dagger(\vec{x}, t) ;$$

$$j_{\psi; ss'}(\vec{x}, t) = (\tau_2)_{ss'} j_{\psi; ss'}(\vec{x}, t) ; \quad j_{\psi; ss'}(\vec{x}, t) = [j_{\psi; ss'}(\vec{x}, t)] \exp(i \gamma(\vec{x}, t)) \, .$$

We point out the particular definition (2.7) for the complex conjugation of a product of anti-commuting fields $\xi_1 \ldots \xi_n$ so that the ’real’ term $\psi_{\vec{x}, s}^\ast(t) \psi_{\vec{x}', s'}(t)$ or $\xi_j^\ast \xi_j$ of coherent state fields is reproduced under the defined complex conjugated transformation

$$(\xi_1 \ldots \xi_n)^\ast = \xi_1^\ast \ldots \xi_n^\ast ; \quad \xi_j^\ast = \xi_j ; \quad (\xi_j^\ast \xi_j)^\ast = \xi_j^\ast \xi_j^\ast = \xi_j^\ast \xi_j .$$

In comparison to section 1.1 we introduce a forward and an additional backward time development (2.8) for the coherent state matrix elements of the propagator of the second quantized Hamilton operator $\hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t)$ (2.3); this guarantees the generating functional to be normalized to unity. The ’initial’ (’final’) coherent state field configurations are created from the vacuum by the source fields in the last two lines of $\hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t)$ (2.3). The forward ’+’ and backward ’−’ time development is combined by the non-equilibrium time contour $t_{p=\pm}, t_{q=\pm}$.
where one has to distinguish between time variables \( t_+ \), \( t_- \), \( t'_+ \), \( t'_- \) (indices \( p, q = \pm \)) for forward and backward propagation by using a time contour metric \( \eta_{p=\pm} = p = \pm 1 \)

\[
\int_C dt_p \ldots = \int_{T_{ini}}^{T_{fin}} dt_+ \ldots + \int_{T_{ini}}^{T_{fin}} dt_- \ldots = \int_{T_{ini}}^{T_{fin}} dt_+ \ldots - \int_{T_{ini}}^{T_{fin}} dt_- \ldots
\]

\[
(2.8)
\]

\[
= \sum_{p=\pm} \int_{T_{ini}}^{T_{fin}} |dt_p| \eta_p \ldots ; \quad \eta_{p=\pm} = p = \pm 1 ;
\]

\[
\Delta t_p = \left| \Delta t_p \right| \eta_p .
\]

\[
(2.9)
\]

Since we aim on the derivation of pair condensates with coset matrices, we have to perform an anomalous doubling of fields \( \psi_{\vec{x},s}(t_p) \) with their complex conjugates \( \psi_{\vec{x},s}^*(t_p) \) to the composed field \( \Psi_{\vec{x},s}^a(t_p) \) (2.10) where the first letters \( a, b, \ldots \) of the Latin alphabet specify one out of the two possible components of \( \Psi_{\vec{x},s}^a(t_p) \) (2.10) (and similar for its anomalous doubled, hermitian-conjugated form \( \bar{\Psi}_{\vec{x},s}^{ia}(t_p) \) (2.11))

\[
(1) : \text{'equal time', anomalous-doubled field :}
\]

\[
\Psi_{\vec{x},s}^{a(1/2)}(t_p) = \begin{pmatrix} \psi_{\vec{x},s}^a(t_p) \\ \psi_{\vec{x},s}^a(t_p) \end{pmatrix}^a ; \psi_{\vec{x},s}^\dagger(t_p) ; \psi_{\vec{x},s}^\dagger(t_p) \]

\[
(2.10)
\]

\[
(2) : \text{'hermitian-conjugation' \( \dagger \), of 'equal time', anomalous-doubled field :}
\]

\[
\Psi_{\vec{x},s}^{a(1/2)}(t_p) = \begin{pmatrix} \psi_{\vec{x},s}^a(t_p) \\ \psi_{\vec{x},s}^a(t_p) \end{pmatrix}^{a=1} ; \psi_{\vec{x},s}^\dagger(t_p) ; \psi_{\vec{x},s}^\dagger(t_p) \]

\[
(2.11)
\]

According to the analogous normal ordering of \( \hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t) \) (2.3) for the 'contour time' development '\( \mathcal{T}_p \)' similar to (1.1)

\[
\langle 0 | \mathcal{T}_p \exp \left\{ - \frac{i}{\hbar} \int_C dt_p \hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t) \right\} |0 \rangle \equiv 1 ,
\]

\[
(2.12)
\]

ordering operator on the time contour : \( \mathcal{T}_p \),

one has to add an infinitesimal, non-equilibrium time shift \( \Delta t_p = |\Delta t_p| \eta_p \) in the complex conjugated second part of \( \Psi_{\vec{x},s}^{a=2}(t_p) = \psi_{\vec{x},s}^a(t_p) \rightarrow \psi_{\vec{x},s}^a(t_p + \Delta t_p) \); however, the 'true' hermitian conjugation of \( \Psi_{\vec{x},s}^{a(1/2)}(t_p) \) (2.10) just yields 'equal time' fields in \( \Psi_{\vec{x},s}^{a=1/2}(t_p) \) (2.11) or under consideration of the just mentioned time shift \( \psi_{\vec{x},s}^a(t_p) \rightarrow \psi_{\vec{x},s}^a(t_p + \Delta t_p) \) in \( \Psi_{\vec{x},s}^{a=2}(t_p) \) (2.10) the incorrect, hermitian conjugated case where the complex conjugated field \( \psi_{\vec{x},s}^a(t_p) \) in the first part \( (a = 1) \) acts before its second part \( \psi_{\vec{x},s}(t_p + \Delta t_p) \) \( (a = 2) \) contrary to the normal ordering of the second quantized field operators in \( \hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t) \) (2.3) for the non-equilibrium time development. Therefore, we have to define an additional hermitian conjugation \( \bar{\Psi}_{\vec{x},s}^{a(1/2)}(t_p) \) (2.13) of a slightly modified, anomalous doubled field \( \bar{\Psi}_{\vec{x},s}^{a(1/2)}(t_p) \) (2.11) which is adapted to the normal ordering of \( \hat{H}(\hat{\psi}^\dagger, \hat{\psi}, t) \) (2.3) with the field creation operators \( \hat{\psi}_{\vec{x},s}^\dagger \) left to the annihilation operators \( \hat{\psi}_{\vec{x},s} \), thereby always acting the time interval \( \Delta t_p \) later on the time contour \( \mathcal{T}_p \):

\[
(1) : \text{'time shifted' \( \Delta t_p \), anomalous-doubled field '\( \bar{\cdot} \)' :}
\]

\[
\text{The fields } \bar{\Psi}_{\vec{x},s}^{a(1/2)}(t_p) \text{ or } \bar{\Psi}_{\vec{x},s}^{a(1/2)}(t_p) \text{, which include the additional time shifts '\( \Delta t_p \)' in the complex parts } \psi_{\vec{x},s}^a(t_p + \Delta t_p) \text{ relative to } \psi_{\vec{x},s}(t_p) \text{, are marked by the symbol '\( \bar{\cdot} \)' above the Greek, capital letter of the field and also above an analogous matrix.}
In later steps to the derivation of the pair condensates, we have to take dyadic products (2.15) and (2.16) of the anomalous doubled fields corresponding to (2.15) and to (2.16), respectively. The first case (2.15, 2.17-2.18) is completely hermitian by using the ordinary hermitian conjugation \(^{\dagger}\) of anomalous doubled fields and for the definition of density matrices whereas we have to list in relations (2.17, 2.18) and (2.19, 2.20) the two different cases of dyadic products \(\tilde{\Psi}_{x,s}^{a}(t_{p})\) (2.14) so that there arise two distinct forms of dyadic products

\[
\tilde{\Psi}_{x,s}^{a}(t_{p}) \quad \overset{\text{'hermitian-conjugation'}}{\Rightarrow} \quad \tilde{\Psi}_{x,s}^{a}(t_{p} + \Delta t_{p}) \quad \text{in the resulting complex part :}
\]

\[
\tilde{\Psi}_{x,s}^{a}(t_{p}) = \left( \begin{array}{c}
\psi_{x,s}^{a}(t_{p}) \\
\psi_{x,s}^{a}(t_{p} + \Delta t_{p})
\end{array} \right)^{T}; \quad (2.13)
\]

\[
\tilde{\Psi}_{x,s}^{a}(t_{p}) = \left( \begin{array}{c}
\psi_{x,s}^{a}(t_{p}) \\
\psi_{x,s}^{a}(t_{p} + \Delta t_{p})
\end{array} \right) \quad \text{with 'time shift correction' } \Delta t_{p}
\]

The latter dyadic product \(\tilde{\Psi}_{x,s}^{a}(t_{p})\) (2.16) is more appropriate for the correct time ordering in the anomalous doubled coherent state path integrals and for the definition of density matrices whereas we have to apply the first, 'equal time' version \(\Psi_{x,s}^{a}(t_{p}) \otimes \Psi_{x,s}^{b}(t_{p})\) (2.15) for the definition of a total, hermitian self-energy matrix. In consequence, we list in relations (2.17, 2.18) and (2.19, 2.20) the two different cases of dyadic products of anomalous doubled fields corresponding to (2.15) and to (2.16), respectively. The first case (2.15, 2.17, 2.18) is completely hermitian by using the ordinary hermitian conjugation \(^{\dagger}\) for defining a hermitian self-energy whereas the second case (2.16, 2.19, 2.20) takes into account the shifts with the time interval \(\Delta t_{p}\) for \(\psi_{x,s}^{a}(t_{p} + \Delta t_{p})\) (compared to \(\psi_{x,s}(t_{p})\)) in order to avoid incorrect, 'equal time' combinations as \(\psi_{x,s}^{a}(t_{p}) \ldots \psi_{x,s}(t_{p})\) in the coherent state path integral. The two different kinds of dyadic products of anomalous doubled fields can be considered as spatially nonlocal order parameters \(\tilde{\Phi}_{x,s,z',s'}^{ab}(t_{p})\) (2.17), \(\tilde{\Phi}_{x,s,z',s'}^{ab}(t_{p})\) (2.19) with density terms on the block diagonals \((a = b)\) and anomalous pair condensates on the off-diagonal blocks \((a \neq b)\)

\[
\tilde{\Phi}_{x,s,z',s'}^{ab}(t_{p}) = \Psi_{x,s}^{a}(t_{p}) \otimes \Psi_{x,s}^{b}(t_{p}) = \left( \begin{array}{c}
\psi_{x,s}^{a}(t_{p}) \\
\psi_{x,s}^{a}(t_{p} + \Delta t_{p})
\end{array} \right) \otimes \left( \begin{array}{c}
\psi_{x,s}^{b}(t_{p}) \\
\psi_{x,s}^{b}(t_{p} + \Delta t_{p})
\end{array} \right) \quad (2.17)
\]

\[
\tilde{\Phi}_{x,s,z',s'}^{ab}(t_{p}) = \Psi_{x,s}^{a}(t_{p}) \otimes \Psi_{x,s}^{b}(t_{p}) = \left( \begin{array}{c}
\psi_{x,s}^{a}(t_{p}) \\
\psi_{x,s}^{a}(t_{p} + \Delta t_{p})
\end{array} \right) \otimes \left( \begin{array}{c}
\psi_{x,s}^{b}(t_{p}) \\
\psi_{x,s}^{b}(t_{p} + \Delta t_{p})
\end{array} \right) \quad (2.19)
\]

\[
\tilde{\Phi}_{x,s,z',s'}^{ab}(t_{p}) = \left( \begin{array}{c}
\langle \tilde{\Psi}_{x,s}(t_{p}) \tilde{\Psi}_{x,s}^{\dagger}(t_{p}) \rangle \\
\langle \tilde{\Psi}_{x,s}(t_{p}) \tilde{\Psi}_{x,s}^{\dagger}(t_{p}) \rangle
\end{array} \right) = \left( \begin{array}{c}
\tilde{\Phi}_{x,s,z',s'}^{11}(t_{p}) \\
\tilde{\Phi}_{x,s,z',s'}^{21}(t_{p})
\end{array} \right) \quad (2.18)
\]

\[
\tilde{\Phi}_{x,s,z',s'}^{ab}(t_{p}) = \left( \begin{array}{c}
\langle \tilde{\Psi}_{x,s}(t_{p}) \tilde{\Psi}_{x,s}^{\dagger}(t_{p}) \rangle \\
\langle \tilde{\Psi}_{x,s}(t_{p}) \tilde{\Psi}_{x,s}^{\dagger}(t_{p}) \rangle
\end{array} \right) = \left( \begin{array}{c}
\tilde{\Phi}_{x,s,z',s'}^{11}(t_{p}) \\
\tilde{\Phi}_{x,s,z',s'}^{21}(t_{p})
\end{array} \right) \quad (2.20)
\]
\begin{align}
\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t') = \hat{\Phi}_{\vec{x},s,t'}^{aa}(\vec{x},s,t') = \hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t') = -\hat{\Phi}_{\vec{x},s,t'}^{11,T}(\vec{x},s,t') = -\hat{\Phi}_{\vec{x},s,t'}^{12,T}(\vec{x},s,t') = -\hat{\Phi}_{\vec{x},s,t'}^{22,T}(\vec{x},s,t').
\end{align}

Apart from the ordinary hermitian conjugation and transposition relations (2.18) between the various blocks of \(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\), we extend to the combined transposition, trace and hermitian conjugation (2.21,2.23) of all four involved blocks of \(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\) in its entity

\begin{align}
\left(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\right)^T &= \left(\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t') \hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')\right)^T = \left(\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t')^T \hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')^T\right)^T, \tag{2.21}
\end{align}

\begin{align}
\text{Tr}_{a,s,b,s'} \left[\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\right] &= \text{tr}_{s,s'=\uparrow,\downarrow} \left[\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t') + \text{tr}_{s,s'=\uparrow,\downarrow} \left[\hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')\right]\right], \tag{2.22}
\end{align}

\begin{align}
\left(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\right)\dagger &= \left(\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t') \hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')\right)\dagger = \left(\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t')^\dagger \hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')^\dagger\right)^\dagger, \tag{2.23}
\end{align}

In a similar manner we have defined an order parameter \(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\) in (2.19,2.20) appropriate for anomalous doubled density matrices and take into account the proper time ordering of the normal ordered Hamilton operator \(\hat{H}(\psi^\dagger,\psi,t)\) with inclusion of time shifts \(\Delta t_p\). Therefore, one has to adapt the ordinary hermitian conjugation \(^\dagger\) (2.10,2.11) to the introduced hermitian conjugation \(\dagger\) (2.13,2.14) which comprises an additional time shift correction \(\Delta t_p\) in such a manner that the resulting complex parts \(\hat{\psi}_{\vec{x},s,t'}^{a,b}(\vec{x},s,t')\) follow a particular time step \(\Delta t_p\) later from \(\hat{\psi}_{\vec{x},s}(t_p)\) on the non-equilibrium time contour \(t_p\). In analogy to relations (2.21,2.23), we specify transposition, traces and the time shifted, hermitian conjugation \(\dagger\) in relations (2.24,2.26) for the total dyadic product \(\hat{\Psi}_{\vec{x},s}(t_p) \otimes \hat{\Psi}_{\vec{x},s}(t_p)^\dagger\) (2.16) related order parameter \(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\) (2.19). As already mentioned, this kind of order parameter with the appropriate time shifts \(\Delta t_p\) has to be used for the density matrices in the time development on the non-equilibrium time contour

\begin{align}
\text{Tr}_{a,s,b,s'} \left[\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\right] &= \text{tr}_{s,s'=\uparrow,\downarrow} \left[\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t') + \text{tr}_{s,s'=\uparrow,\downarrow} \left[\hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')\right]\right], \tag{2.25}
\end{align}

\begin{align}
\left(\hat{\Phi}_{\vec{x},s,t'}^{ab}(\vec{x},s,t')\right)^z &= \left(\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t') \hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')\right)^z = \left(\hat{\Phi}_{\vec{x},s,t'}^{11}(\vec{x},s,t')^z \hat{\Phi}_{\vec{x},s,t'}^{22}(\vec{x},s,t')^z\right)^z, \tag{2.26}
\end{align}

Using the above definitions and notations, we can state the following path integral (2.27) of coherent states with the precise, appropriate subsequent steps of the time development on the time contour. Note the described, essential time shifts \(\Delta t_p\) in the complex parts \(\hat{\psi}_{\vec{x},s,t'}^{a,b}(t_p + \Delta t_p)\) and the additional, anomalous doubled source matrix \(\hat{\delta}_{\vec{x},s',t'}^{ab}(\vec{x},s',t_p)\) for generating bilinear or higher even order correlation functions with \(\hat{\Psi}_{\vec{x},s,t'}^{a,b}(t_p')\) \(\hat{\Phi}_{\vec{x},s}(t_p)\). The source fields \(\hat{j}_{\psi,s}(\vec{x},t_p)\) (2.5), \(\hat{j}_{\psi,s}(\vec{x},t_p)\) (2.6), which are extended to a dependence on the time contour for generating observables, are set to equivalent values on the two branches of the time contour at the final end after complete transformations (2.29,2.30) so that these eventually act as condensate seeds for a macroscopic wavefunction \(\langle \hat{\psi}_{\vec{x},s,t'}^{a,b}(t_p) \psi_{\vec{x},s}(t_p)\rangle\) and for a non-vanishing pair condensate correlation function \(\langle \hat{\psi}_{\vec{x},s,t'}^{a,b}(t_p) \psi_{\vec{x},s}(t_p)\rangle\) of coherent states.
2.2 Self-energy with anomalous terms and coset decomposition into density and pair condensate parts

\[ \langle \tilde{E}_{\rho} - \hat{\varepsilon}_{\rho} + \hat{h}(\vec{x}) \rangle = -i \hbar \frac{\partial}{\partial \tilde{t}_{\rho}} - i \frac{\vec{p}^2}{2m} + u(\vec{x}) - \mu_{0}; \]

(2.28)

\[ \int_{C} dt_{\rho} \sum_{\vec{x},s=1} \psi_{\vec{x},s}^{*}(t_{\rho} + \Delta t_{\rho}) \tilde{H}_{\rho}(\vec{x}, t_{\rho}) \psi_{\vec{x},s}(t_{\rho}) = \]

\[ \int_{C} dt_{\rho} \sum_{\vec{x},s=1} \left[ -i \hbar \psi_{\vec{x},s}^{*}(t_{\rho} + \Delta t_{\rho}) \frac{\psi_{\vec{x},s}(t_{\rho} + \Delta t_{\rho}) - \psi_{\vec{x},s}(t_{\rho})}{\Delta t_{\rho}} + \psi_{\vec{x},s}(t_{\rho} + \Delta t_{\rho}) \left( \hat{h}(\vec{x}) - i \hat{\varepsilon}_{\rho} \right) \psi_{\vec{x},s}(t_{\rho}) \right] \]

\[ \int_{C} dt_{\rho} dt_{\rho}' \sum_{\vec{x},s=1} \psi_{\vec{x},s}(t_{\rho}') \delta_{q_{p}} \eta_{q_{p}} \left( -i \hbar \frac{\delta_{q_{p}} t_{\rho} - \delta_{q_{p}} t_{\rho} + \Delta t_{\rho}}{\Delta t_{\rho}} \right) \left( \hat{h}(\vec{x}) - i \hat{\varepsilon}_{\rho} \right) \delta_{q_{p}} t_{\rho} \psi_{\vec{x},s}(t_{\rho}) \]

\[ \int_{C} dt_{\rho} dt_{\rho}' \sum_{\vec{x},s=1} \psi_{\vec{x},s}(t_{\rho}') \delta_{q_{p}} \eta_{q_{p}} \left( -i \hbar \frac{\delta_{q_{p}} t_{\rho} - \delta_{q_{p}} t_{\rho} + \Delta t_{\rho} - \delta_{q_{p}} t_{\rho}}{\Delta t_{\rho}} \right) \left( \hat{h}(\vec{x}) - i \hat{\varepsilon}_{\rho} \right) \delta_{q_{p}} t_{\rho} \psi_{\vec{x},s}(t_{\rho}) ; \]

(2.29)

(2.30)

2.2 Self-energy with anomalous terms and coset decomposition into density and pair condensate parts

In order to achieve a HST for anomalous doubled, quartic fields of the interaction, we consider following transformation from the ordinary, 'appropriate' time shifted density (2.31) to its anomalous doubled density form with metric \( \hat{S}_{4 \times 4} \) (2.32). One has to apply the fields \( \hat{\Psi}_{\vec{x},s}^{a}(t_{\rho}) \) with the hermitian conjugation \( \dagger \) (2.132.14) instead of the 'equal time' fields \( \Psi_{\vec{x},s}^{a}(t_{\rho}) \) with 'equal time' hermitian conjugation \( \dagger \) (2.10,2.11) so that the additional time shift \( \Delta t_{\rho} \) is always preserved in the complex parts \( \psi_{\vec{x},s}^{\ast}(t_{\rho} + \Delta t_{\rho}) \) relative to \( \psi_{\vec{x},s}(t_{\rho}) \)

\[ \psi_{\vec{x},s}^{T}(t_{\rho} + \Delta t_{\rho}) \psi_{\vec{x},s}(t_{\rho}) = \frac{1}{2} \left( \psi_{\vec{x},s}^{T}(t_{\rho} + \Delta t_{\rho}) \psi_{\vec{x},s}(t_{\rho}) - \psi_{\vec{x},s}(t_{\rho}) \psi_{\vec{x},s}^{T}(t_{\rho} + \Delta t_{\rho}) \right) \]

(2.31)

\[ \hat{S}_{4 \times 4} = \left\{ \begin{array}{l} \hat{1}_{2 \times 2}; \quad \hat{1}_{2 \times 2} \end{array} \right\}; \quad \hat{1}_{2 \times 2} = \hat{1}_{ss'}; \quad s, s' = \uparrow, \downarrow . \]

(2.32)
Straightforward application of (2.31,2.32) for dyadic products (2.16) transforms the quartic interaction part to its anomalous doubled kind (2.33) with anomalous doubled density matrix $\hat{\Sigma}_{x,x',s',s}(t_p)$ (2.34-2.35) having the appropriate, subsequent time shifts in all the complex parts $\psi_{x,s}^*(t_p + \Delta t_p)$ (relative to $\psi_{x,s}(t_p)$)

$$\sum_{x,x'} \sum_{s,s'=1,\downarrow} \psi_{x,s}^{*T}(t_p + \Delta t_p) \psi_{x',s'}^{*T}(t_p + \Delta t_p) V_{|x-x'|} \psi_{x,s}(t_p) \psi_{x',s'}(t_p) =$$

$$= \frac{1}{4} \sum_{x,x'} \sum_{s,s'=1,\downarrow} \left( \psi_{x,s}^{+a}(t_p) \hat{S} \psi_{x,s}^{+b}(t_p) \hat{S} \psi_{x',s'}^{*b}(t_p) \hat{S} \psi_{x',s'}^{*a}(t_p) \right) V_{|x-x'|} =$$

$$= \frac{1}{4} \sum_{x,x'} \text{Tr} \left[ \hat{S} \left( \psi_{x,s}^{+a}(t_p) \otimes \psi_{x',s'}^{*b}(t_p) \right) \hat{S} \left( \psi_{x',s'}^{*b}(t_p) \otimes \psi_{x,s}^{*a}(t_p) \right) \right] V_{|x-x'|} =$$

$$= \frac{1}{4} \sum_{x,x'} \text{Tr} \left[ \hat{S} \hat{R}_{ab, x,s; x', s'}(t_p) \hat{R}_{ba, x', s'; x, s}(t_p) \right] V_{|x-x'|} ;$$

$$\hat{R}_{ab, x,s; x', s'}(t_p) = \left( \begin{array}{cc} \hat{R}^{11}_{ab, x,s; x', s'}(t_p) & \hat{R}^{12}_{ab, x,s; x', s'}(t_p) \\ \hat{R}^{21}_{ab, x,s; x', s'}(t_p) & \hat{R}^{22}_{ab, x,s; x', s'}(t_p) \end{array} \right)^{ab}$$

(2.34)

$$\hat{R}^{11}_{ab, x,s; x', s'}(t_p) = \psi_{x,s}(t_p) \psi_{x',s'}(t_p + \Delta t_p) = \hat{R}^{11}_{ab, x,s; x', s'}(t_p) ;$$

$$\hat{R}^{22}_{ab, x,s; x', s'}(t_p) = \psi_{x',s'}(t_p + \Delta t_p) \psi_{x,s}(t_p) = \hat{R}^{22}_{ab, x,s; x', s'}(t_p) ;$$

$$\hat{R}^{12}_{ab, x,s; x', s'}(t_p) = \psi_{x,s}(t_p) \psi_{x',s'}(t_p) = -\hat{R}^{12}_{ab, x,s; x', s'}(t_p) ;$$

$$\hat{R}^{21}_{ab, x,s; x', s'}(t_p) = -\hat{R}^{11}_{ab, x,s; x', s'}(t_p) ;$$

(2.35)

The doubled density matrix $\hat{R}_{ab, x,s; x', s'}(t_p)$ (2.34,2.35) with pair condensate terms in the off-diagonal blocks ($a \neq b$) is constructed according to the dyadic product (2.10) and order parameter $\hat{\Phi}_{ab, x,s; x', s'}(t_p)$ (2.19,2.20,2.24,2.26); however, the self-energy matrix has to comply with the hermitian, anomalous doubled order parameter $\hat{\Phi}_{ab, x,s; x', s'}(t_p)$ (2.17,2.18,2.21,2.23) or dyadic product (2.15) with solely equal time fields $\hat{\Psi}_{x,s}(t_p)$ and equal time hermitian conjugation $^{\dagger}$ (2.10,2.11). In order to emphasize symmetries, we simplify to the case of a short-ranged interaction potential

$$V_{|x-x'|} \approx \delta_{x,x'} V_0 ,$$

(2.36)

which will be extended to the general case of arbitrary long-ranged interaction potentials $V_{|x-x'|} \neq \delta_{x,x'} V_0$ in section 3. We therefore introduce the anomalous doubled self-energy matrix $\hat{\Sigma}_{ss'}(\bar{x}, t_p)$ (2.37) which consists of the block diagonal, hermitian density field $\sigma^{(0)}_{D,s,s'}(\bar{x}, t_p)$ and the additional self-energy $\delta \hat{\Sigma}_{ab}(\bar{x}, t_p)$ (2.41), having hermitian density blocks $\hat{\Sigma}_{ss'}^{11}(\bar{x}, t_p), \hat{\Sigma}_{ss'}^{12}(\bar{x}, t_p), \hat{\Sigma}_{ss'}^{21}(\bar{x}, t_p)$ as subalgebra elements and anti-hermitian related coset parts $\imath \delta \hat{\Sigma}_{ss'}^{12}(\bar{x}, t_p), \imath \delta \hat{\Sigma}_{ss'}^{21}(\bar{x}, t_p)$ in the off-diagonal blocks (which is pointed out by the tilde $\sim$ above $\delta \hat{\Sigma}_{ss'}^{ab}(\bar{x}, t_p)$). We take the analogous notation of Ref. 8 for the super-symmetric, ortho-symplectic case $\text{Osp}(S, S|2L) / \text{U}(L) \otimes \text{U}(L|S)$, but reduce relations to the even, fermion-fermion blocks in order to underline the appropriate, precise steps in the time development. Thus, the coset matrices for anomalous parts are denoted by $\bar{T}(\bar{x}, t_p)$ and the remaining block diagonal densities are labeled by $\delta \hat{\Sigma}_{qq}^{aa}(\bar{x}, t_p)$ in the coset decomposition (2.42)

$$\hat{\Sigma}_{ss'}^{ab}(\bar{x}, t_p) = \sigma^{(0)ab}_{D,s,s'}(\bar{x}, t_p) \delta_{ab} + \delta \hat{\Sigma}_{ab}(\bar{x}, t_p)$$

(2.37)
2.2 Self-energy with anomalous terms and coset decomposition into density and pair condensate parts

\[
\hat{\Sigma}_{ss'}(\vec{x}, t) = \begin{pmatrix}
\hat{\Sigma}_{11}^{11}(\vec{x}, t, p) & i \hat{\Sigma}_{11}^{12}(\vec{x}, t, p) \\
-i \hat{\Sigma}_{11}^{21}(\vec{x}, t, p) & \hat{\Sigma}_{22}^{22}(\vec{x}, t, p)
\end{pmatrix};
\]

\[
\hat{\Sigma}_{ss'}(\vec{x}, t) = \sigma_{D:ss'}(\vec{x}, t) + \delta \hat{\Sigma}_{ss'}(\vec{x}, t); \quad (2.38)
\]

\[
\hat{\Sigma}_{ss'}(\vec{x}, t) = \sigma_{D:ss'}(\vec{x}, t) + \delta \hat{\Sigma}_{ss'}(\vec{x}, t); \quad (2.39)
\]

\[
\sigma_{(0)}^{(0)}_{D:ss'}(\vec{x}, t) = \sigma_{(0)}^{(0)}_{D:ss'}(\vec{x}, t); \quad \sigma_{(0)}^{(0)}_{D:ss'}(\vec{x}, t) = \sigma_{(0)}^{(0)}_{D:ss'}(\vec{x}, t); \quad (2.40)
\]

\[
\delta \hat{\Sigma}_{ab}(\vec{x}, t) = \begin{pmatrix}
\delta \hat{\Sigma}_{11}^{11}(\vec{x}, t, p) & i \delta \hat{\Sigma}_{11}^{12}(\vec{x}, t, p) \\
-i \delta \hat{\Sigma}_{11}^{21}(\vec{x}, t, p) & \delta \hat{\Sigma}_{22}^{22}(\vec{x}, t, p)
\end{pmatrix};
\]

\[
\delta \hat{\Sigma}_{ss'} = \begin{pmatrix}
\sigma_{D:ss'}(\vec{x}, t) + \delta \hat{\Sigma}_{11}^{11}(\vec{x}, t, p) & i \delta \hat{\Sigma}_{11}^{12}(\vec{x}, t, p) \\
-i \delta \hat{\Sigma}_{11}^{21}(\vec{x}, t, p) & \sigma_{D:ss'}(\vec{x}, t) + \delta \hat{\Sigma}_{22}^{22}(\vec{x}, t, p)
\end{pmatrix}; \quad (2.42)
\]

\[
\delta \Sigma_{ss'}^{11}(\vec{x}, t) = \delta \hat{\Sigma}_{11}^{11}(\vec{x}, t, p); \quad \delta \Sigma_{ss'}^{22}(\vec{x}, t) = \delta \hat{\Sigma}_{22}^{22}(\vec{x}, t, p); \quad (2.43)
\]

\[
\delta \Sigma_{ss'}^{12}(\vec{x}, t) = -\delta \hat{\Sigma}_{12}^{12}(\vec{x}, t, p); \quad (2.44)
\]

\[
\delta \Sigma_{ss'}^{21}(\vec{x}, t) = -\delta \hat{\Sigma}_{21}^{21}(\vec{x}, t, p); \quad (2.46)
\]

\[
\delta \Sigma_{ss'}^{21}(\vec{x}, t) = -\delta \hat{\Sigma}_{21}^{21}(\vec{x}, t, p) = \begin{pmatrix}
0 & -\delta \hat{\Sigma}_{21}^{21}(\vec{x}, t, p) \\
0 & \delta \hat{\Sigma}_{21}^{21}(\vec{x}, t, p)
\end{pmatrix}; \quad (2.47)
\]

\[
\delta \Sigma_{ss'}^{12}(\vec{x}, t) = -\delta \hat{\Sigma}_{12}^{12}(\vec{x}, t, p); \quad (2.45)
\]

\[
\delta \Sigma_{ss'}^{ab}(\vec{x}, t) \simeq \begin{pmatrix}
\delta \Sigma_{ss'}^{11}(\vec{x}, t) & i \delta \Sigma_{ss'}^{12}(\vec{x}, t) \\
-i \delta \Sigma_{ss'}^{21}(\vec{x}, t) & \delta \Sigma_{ss'}^{22}(\vec{x}, t)
\end{pmatrix} \in \text{so}(4); \quad (2.48)
\]

\[
\delta \Sigma_{ss'}(\vec{x}, t) = -\delta \Sigma_{ss'}^{11}(\vec{x}, t) \in \text{u}(2). \quad (2.49)
\]

The block diagonal densities \(\delta \Sigma_{ss'}^{aa}(\vec{x}, t, p)\) \(\text{2.42, 2.50, 2.53}\) are further decomposed into eigenvalues \(\delta \Lambda_{ss'}^{aa}(\vec{x}, t, p)\), \(\delta \hat{\Lambda}_{ss'}(\vec{x}, t, p)\) \(\text{2.41, 2.55}\) and diagonalizing matrices \(\hat{\Sigma}_{ss'}^{aa}(\vec{x}, t, p)\) \(\text{2.56, 2.58}\) with generator \(\hat{\mathcal{F}}_{D:ss'}(\vec{x}, t, p)\) and complex parameter \(\mathcal{F}_{ss'}(\vec{x}, t, p)\) \(\text{2.50, 2.60}\). The details of the parameters and generators are listed in Eqs. \(\text{2.50 to 2.60}\) and are in complete accordance with the given parameters and generators of the more general ortho-symplectic case \([4]\) which is just restricted to the even, fermion-fermi on parts in order to point out the problem of the appropriate, precise discrete time steps in the coset decomposition

\[
\delta \Sigma_{D:ss'}^{ab}(\vec{x}, t, p) = \delta_{ab} \delta \Sigma_{D:ss'}^{aa}(\vec{x}, t, p); \quad (2.50)
\]
\[
\delta \Sigma_{D;ss'}^a(x, t_p) = \delta \Sigma_{D;ss'}^a \uparrow(x, t_p); \\
\delta \Sigma_{D;ss'}^{22}(x, t_p) = -\delta \Sigma_{D;ss'}^{11,T}(x, t_p); \\
\delta \Sigma_{D;ss';11}^{aa}(x, t_p) = \tilde{Q}^{-1}_{ss'}^{aa}(x, t_p) \tilde{\delta} \Sigma_{ss';ss'}^{aa}(x, t_p) \tilde{Q}_{ss'}^{aa}(x, t_p); \\
\delta \dot{\lambda}_{s's}(x, t_p) = \delta \sigma_s \text{ diag}\left(\frac{\delta \lambda_s(x, t_p)}{a=1} ; -\frac{\delta \lambda_s(x, t_p)}{a=2}\right); \\
\dot{\lambda}_s(x, t_p) = \left(\dot{\lambda}_{s=1}(x, t_p) , \dot{\lambda}_{s=2}(x, t_p)\right); \\
\dot{Q}_{ss'}^{ab}(x, t_p) = \delta_{ab} \left(\begin{array}{c}
\tilde{Q}_{ss'}^{11}(x, t_p) \\
\tilde{Q}_{ss'}^{22}(x, t_p)
\end{array}\right)_{ss'}; \\
\hat{\Sigma}_{D;ss'}^{11}(x, t_p) = \exp\left\{i \hat{F}_{D;ss'}(x, t_p)\right\}_{ss'}; \\
\hat{\Sigma}_{D;ss'}^{22}(x, t_p) = \exp\left\{-i \hat{F}_{D;ss'}^T(x, t_p)\right\}_{ss'}; \\
\hat{F}_{D;ss'}(x, t_p) \equiv 0; \\
\hat{F}_{D;ss'}(x, t_p) = \hat{F}_{D;ss'}(x, t_p) = \left(\begin{array}{c}
0 \\
\hat{F}_{11}(x, t_p)
\end{array}\right); \\
\hat{F}_{11}(x, t_p) = \left|\hat{F}_{11}(x, t_p)\right| \exp\{i \varphi(x, t_p)\}; \\
\varphi \equiv \varphi((x, t_p), (\hat{F}_{11}(x, t_p))) \text{, (2 real parameters)}.
\]

As one follows the described parameters in Eqs. (2.37, 2.60), one has performed a coset decomposition (2.61, 2.62) of a general SO(4) symmetry or so(4) self-energy generator \(\delta \Sigma_{ss'}^{ab}(x, t_p)\) (2.41, 2.48) into the unitary sub-algebra parts \(u(2)\) (2.49) for the block diagonal densities \(\delta \Sigma_{D;ss'}^{aa}(x, t_p)\) (2.50, 2.52) and into the coset algebra elements with matrices \(\hat{T}(x, t_p)\)

\[
\left(\begin{array}{c}
\delta \Sigma_{ss'}^{ab} \\
\delta \Sigma_{ss'}^{ab} \\
\delta \Sigma_{ss'}^{aa} \\
\delta \Sigma_{ss'}^{aa} \\
\delta \Sigma_{D;ss'}^{12} \\
\delta \Sigma_{D;ss'}^{12}
\end{array}\right)_{SO(4)/U(2) \otimes U(2)} \simeq \left(\begin{array}{c}
\delta \Sigma_{11}^{12} \\
\delta \Sigma_{11}^{12}
\end{array}\right)_{SO(4)/U(2) \otimes U(2)}; \\
\left(\begin{array}{c}
\text{6 parameters} \\
\text{4 parameters}
\end{array}\right) \simeq \left(\begin{array}{c}
\delta \Sigma_{11}^{12}(x, t_p), \dot{\delta} \Sigma_{11}^{12}(x, t_p)
\end{array}\right) . \\
\text{2 remaining parameters}
\]

The parametrization (2.37, 2.62) and coset decomposition (2.42, 2.61, 2.62) of the self-energy matrix \(\delta \Sigma_{ss'}^{ab}(x, t_p)\) (2.41) has only ‘equal time’ fields and ‘equal time’ hermitian conjugation operations without any time shifts \(\Delta t_p\) as in the anomalous doubled density matrices \(\tilde{R}_{ss'}^{ab}(x, t_p)\) (2.34, 2.35). The coset matrix \(\hat{T}_{ss'}^{ab}(x, t_p)\) (2.64) consists of the coset generator \(\hat{T}_{ss'}^{ab}(x, t_p)\) (2.64) with further, anti-symmetric, complex-valued sub-matrices \(\hat{X}_{ss'}(x, t_p), \hat{X}_{ss'}(x, t_p)\) (2.63, 2.64) in the off-diagonal blocks. We use the anti-symmetric Pauli-matrix (\(\hat{\tau}_2\)) in order to define the complex pair condensate fields \(f(x, t_p), f^*(x, t_p)\) with absolute value \(|f(x, t_p)|\) and phase \(\phi(x, t_p)\). The modified coset matrix \(\hat{T}_{0}(x, t_p)\) (2.67) is introduced for convenience because this combination of coset matrix \(\hat{T}(x, t_p)\) and diagonalizing density matrix \(\tilde{Q}_{ss'}^{aa}(x, t_p)\) appears in the
calculation of the corresponding invariant integration measure \( \text{SO}(4) / \text{U}(2) \otimes \text{U}(2) \)

\[
\tilde{T}_{ss'}^{ab}(x, t_p) = \left( \exp \left\{ -\tilde{Y}_{ss'}^{ab}(x, t_p) \right\} \right)_{ss'}^{ab};
\]

\[
\tilde{\psi}_{ss'}^{a \neq b}(x, t_p) = \left( \tilde{X}_{ss'}^{dagger}(x, t_p) \right)_{ss'}^{ab};
\]

\[
\tilde{X}_{ss'}^{dagger}(x, t_p) = -\tilde{X}_{ss'}^{T}(x, t_p) = (\tilde{\tau}_2)_{ss'} f(x, t_p) = \tilde{t} \begin{pmatrix}
0 & -f(x, t_p) \\
-f(x, t_p) & 0
\end{pmatrix};
\]

\[
\tilde{X}_{ss'}^{+}(x, t_p) = (\tilde{\tau}_2)_{ss'} f^*(x, t_p); \quad f(x, t_p) = \left| f(x, t_p) \right| \exp \left\{ \tilde{t} \phi(x, t_p) \right\};
\]

\[
\tilde{\hat{T}}(x, t_p) = \tilde{T}(x, t_p) \tilde{Q}^{-1}(x, t_p).
\]

We briefly collect above relations \([2.33, 2.68]\) with incorporation of analogous results for the ortho-symplectic case in Ref. \([8]\) and list the corresponding, anomalous doubled HST \([2.68]\) for the quartic interaction of fields, but under inclusion of the precise, essential, subsequent time shifts in the complex parts \(\tilde{\psi}_{s,s}^{a}(t_p + \Delta t_p)\).

Furthermore, the total HST has to encompass imaginary increments with extension of the approximating potential parameter \([2.36]\) \(V_0 \rightarrow V_0^{ab}\) to a matrix \(V_0^{ab}\) in order to achieve converging Gaussian integrations for the self-energy matrices \(\delta \Sigma_{ss'}^{ab}(\vec{x}, t_p)\) \([2.41]\) and for the hermitian self-energy density field \(\sigma_{D;ss'}^{(0)}(\vec{x}, t_p)\). (We again emphasize that it is a convenient standard to simplify Eqs. \([2.33, 2.35, 2.68]\) of this HST to solely ‘equal field’ fields and relations without auxiliary, imaginary increments as e.g. described in Refs. \([8, 9, 11]\)).

\[
\exp \left\{ -\frac{\tilde{t}}{\hbar} \int dt_p \sum_{x,x', s,s'=\dagger,1} \psi_{s,s}^{a}(t_p + \Delta t_p) \psi_{x,x'}^{a}(t_p + \Delta t_p) V_{x-x'} \psi_{x', s'}^{a}(t_p) \psi_{x,s}^{a}(t_p) \right\}_{x-x' \approx V_0}^{V_{x-x'} \approx V_0}
\]

\[
\int \left\{ -\frac{\tilde{t}}{2\hbar} \int dt_p \sum_{x,x', s,s'=\dagger,1} \frac{\sigma_{D;ss'}^{(0)}(\vec{x}, t_p)}{V_0 + \tilde{t} \varepsilon_p} \delta \Sigma_{s,s}^{ab}(\vec{x}, t_p) \delta \Sigma_{s,s}^{ba}(\vec{x}, t_p) \right\}
\]

\[
\times \left\{ \frac{\tilde{t}}{2\hbar} \int dt_p \sum_{x,x', s,s'=\dagger,1} \text{Tr}_{a,b} \left[ \left( \tilde{R}_{s,s'}^{11}(\vec{x}, t_p) \tilde{R}_{s,s'}^{12}(\vec{x}, t_p) \right)^{ab} \right] \right\}
\]

\[
\times \left\{ \frac{\tilde{t}}{2\hbar} \int dt_p \sum_{x,x', s,s'=\dagger,1} \text{Tr}_{a,b} \left[ \left( \tilde{R}_{s,s'}^{21}(\vec{x}, t_p) \tilde{R}_{s,s'}^{22}(\vec{x}, t_p) \right)^{ab} \right] \right\}
\]

\[
\times \left\{ \delta \Sigma_{s,s}^{11}(\vec{x}, t_p) \delta \Sigma_{s,s}^{11}(\vec{x}, t_p) - \delta \Sigma_{s,s}^{11}(\vec{x}, t_p) \delta \Sigma_{s,s}^{11}(\vec{x}, t_p) \right\}
\]

\[
\hat{\psi}_{s,s}^{ab} = V_0 + \tilde{t} \varepsilon_p \left( \delta_{a=b} - \delta_{a \neq b} \right); \quad \varepsilon_p = \eta_p \varepsilon_+; \quad \varepsilon_+ > 0.
\]

A similar anomalous doubling of the one-particle part \(\psi_{s,s}^{a}(t_p + \Delta t_p) \tilde{H}_{s,s}(t_p) \psi_{s,s}^{a}(t_p)\) as for the interaction part \([2.33, 2.35, 2.68]\) leads to relation \([2.69]\) with the anomalous doubled, one-particle operator \(\tilde{\hat{T}}_{s,s'}^{ba}(\vec{x}, t_p)\) \([2.70]\) which includes the transpose \(\tilde{\hat{T}}_{s,s'}^{ba}(\vec{x}, t_p)\) \([2.72]\) apart from \(\tilde{\hat{T}}_{s,s'}^{ab}(\vec{x}, t_p)\). Note that one has to use the doubled fields \(\tilde{\psi}_{s,s}^{b}(t_p)\) \([2.13]\) with the hermitian conjugate \(\tilde{\psi}_{s,s}^{b}(t_p)\) \([2.14]\) in order to maintain the ubiquitous time shifts ‘\(\Delta t_p\)’ in the complex part \(\tilde{\psi}_{s,s}^{a}(t_p + \Delta t_p)\) (relative to \(\tilde{\psi}_{s,s}^{a}(t_p)\)). We also list a ‘lax’ kind
for the anomalous doubled one-particle operator with \( \hat{\Psi}_{x,s}^{a} (t', t_p) \) in relations (2.73)(2.75)

\[
\sum_{\vec{x}} \sum_{s=\uparrow, \downarrow} \psi_{\vec{x},s}^{T} (t_p + \Delta t_p) \ \hat{H}_{p} (\vec{x}, t_p) \ \psi_{\vec{x},s} (t_p) = \sum_{\vec{x}} \sum_{s=\uparrow, \downarrow} \psi_{\vec{x},s}^{T} (t_p) (-) \ \hat{H}_{p}^{T} (\vec{x}, t_p) \ \psi_{\vec{x},s} (t_p + \Delta t_p) = \quad (2.69)
\]

\[
\int_{C} dt' \sum_{\vec{x}, \vec{x}', s, s'} \frac{1}{2} \hat{\Psi}_{\vec{x},s}^{a} (t'_q) \ \hat{\Psi}_{\vec{x}',s'}^{a} (t'_q, t_p) \ ; \ (2.70)
\]

\[
\hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) = \text{diag} \left( \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) ; \hat{\Psi}_{\vec{x}',s'}^{a} (t'_q, t_p) \right) ; \ (2.71)
\]

\[
\hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) = \delta_{pq} \eta_{p} \left( -i \ h \ \frac{\delta_{q'_p} - \delta_{q_p}}{\Delta t_p} + (\hat{h}(\vec{x}') - i \ \varepsilon_{p}) \delta_{q'_p, \vec{x}} \right) \delta_{x, \vec{x}} ; \quad (2.72)
\]

\[
\int_{C} dt' \sum_{\vec{x}, \vec{x}', s, s'} \frac{1}{2} \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) \ ; \ (2.73)
\]

\[
\hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) \cong \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) ; \quad (2.74)
\]

\[
\hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) = \delta_{pq} \eta_{p} \text{diag} \left( \hat{H}_{p} (\vec{x}', t'_q) \hat{1}_{N \times N} ; \hat{H}_{p}^{T} (\vec{x}', t'_q) \hat{1}_{N \times N} \right) \delta_{(t'_q - t_p)} \delta_{x, \vec{x}} ; \quad (2.75)
\]

In correspondence to the interaction (2.33)(2.35)(2.38) and one-particle part (2.69)(2.75), one has to perform an anomalous doubling of the source fields \( j_{\psi,s}^{a} (\vec{x}, t_p) \), \( j_{\psi,s' s''}^{a} (\vec{x}, t_p) \) to \( j_{\psi,s}^{a} (\vec{x}, t_p) \) (2.76) and \( j_{\psi,s' s''}^{a} (\vec{x}, t_p) \) (2.77)(2.78) in the 'equal time' form (2.10) without any time shifts of the complex parts so that the 'equal time' hermitian conjugation operations with \( \dagger \) (2.11) have to be applied. This anomalous doubled, 'equal time' form of the source fields \( j_{\psi,s}^{a} (\vec{x}, t_p) \) and \( j_{\psi,s' s''}^{a} (\vec{x}, t_p) \) has to incorporate the symmetry relations (2.37)(2.39) of the 'equal time' restricted, hermitian self-energies

\[
J_{\psi,s}^{a} (1/2) (\vec{x}, t_p) = \begin{pmatrix} j_{\psi,s}^{a} (\vec{x}, t_p) ; j_{\psi,s}^{a} (\vec{x}, t_p) \end{pmatrix}^{T} ; \quad (2.76)
\]

\[
J_{\psi,s' s''}^{a} (\vec{x}, t_p) = \begin{pmatrix} 0 & j_{\psi,s' s''}^{a} (\vec{x}, t_p) \\ j_{\psi,s' s''}^{a} (\vec{x}, t_p) & 0 \end{pmatrix} ; \quad (2.77)
\]

\[
\hat{J}_{\psi,s' s''} (\vec{x}, t_p) = -\hat{J}_{\psi,s' s''} (\vec{x}, t_p) = \xi_{2} j_{\psi,s} (\vec{x}, t_p) ; \quad (2.78)
\]

\[
\hat{J}_{\psi,\vec{x}} (\vec{x}, t_p) = |j_{\psi,\vec{x}} (\vec{x}, t_p)| \exp \{ i \gamma (\vec{x}, t_p) \} . \quad (2.79)
\]

Further collection of the above HST and the anomalous doubled one-particle and source field parts results into the path integral (2.79) with only bilinear, anomalous doubled fields which fulfill the requirement of additional time shifts \( \Delta t_{p} \) in the complex parts \( \psi_{\vec{x},s}^{a} (t_p + \Delta t_p) \) (relative to \( \psi_{\vec{x},s} (t_p) \)). This bilinear part of fields \( \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) \), \( \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) \) comprises the matrix \( \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) \) (2.80) with the sum of anomalous doubled one particle operator, source matrix \( \hat{\Psi}_{\vec{x},s}^{a} (t'_q, t_p) \) and self-energies \( \sigma_{D_{s's's''}}^{a} (\vec{x}, t_p) \), \( \delta_{s's''}^{ab} (\vec{x}, t_p) \). In order to accomplish the coherent state path integral (2.79)(2.80), one has to transform by the metric (2.82)(2.83) with the imaginary units in


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the second part \((a = 2)\) satisfying \(\hat{I} \cdot \hat{I} = \hat{S}\)

\[
Z[\hat{J}, \hat{J}_\psi, t, \hat{J}_\psi] = \int d[\sigma^{(0)}_{D,s,s'}(\vec{x}, p)] \exp \left\{ -\frac{i}{2h} \int_C dt_p \sum_{\vec{x}, t, s, s' = \uparrow, \downarrow} \left[ \sigma^{(0)}_{D,s,s'}(\vec{x}, p) \frac{\sigma^{(0)}_{D,s',s}(\vec{x}, p)}{V_0 + i \varepsilon_p} \right] \right\} \quad (2.79)
\]

\[
\times \int d[\delta \Sigma(\vec{x}, p)] \exp \left\{ -\frac{i}{4h} \int_C dt_p \sum_{\vec{x}, t, s, s'} \text{Tr} \left[ \delta \Sigma^{ab}_{s,s'}(\vec{x}, p) \delta \Sigma^{ba}_{s',s}(\vec{x}, p) \right] \right\} \times \int d[\psi^{a}_{\vec{x},s}(t_p), \psi^{a}_{\vec{x},s}(t_p)] \times \exp \left\{ -\frac{i}{2h} \int_C dt_p \sum_{\vec{x}, t, s, s' = \uparrow, \downarrow} \left[ \delta \Sigma^{ba}_{s,s'}(\vec{x}, p) \hat{\Sigma}^{ba}_{s,s'}(t_p) \hat{\Sigma}^{ba}_{s,s'}(t_p) \right] \right\};
\]

\[
|\Delta t'_q| \tilde{N}^{ba}_{\vec{x},s,s'}(\vec{x'}, t'_p, t_p) = \tilde{j}^{ba}_{\vec{x},s,s'}(\vec{x'}, t'_p, t_p) + \eta_q \hat{I} \hat{S} |\Delta t'_q| \hat{\Sigma}^{ba}_{s,s'}(t'_p, t_p) \hat{S} \hat{I} \eta_p + \quad (2.80)
\]

\[
+ \left( \sigma^{(0)11}_{D,s,s'}(\vec{x}, p) \delta_{ba} - \eta \hat{J}^{ba}_{\psi;\psi}(\vec{x}, p) + \frac{\delta \Sigma^{a1}_{s,s'}(\vec{x}, p)}{\delta \Sigma^{ba}_{s,s'}(\vec{x}, p)} \right) \hat{\Sigma}^{ba}_{s,s'}(t'_p, t_p); \quad (2.81)
\]

\[
\hat{J}^{1a}_{4\times4} = \left\{ \begin{array}{ccc} 1 \times 2 & 1 \times 2 \end{array} \right\}; \quad \hat{J}^{2a}_{4\times4} = \langle 1 \times 2 \rangle ; \quad s, s' = \uparrow, \downarrow; \quad (2.82)
\]

\[
\hat{I}^{1a}_{4\times4} \cdot \hat{I}^{1a}_{4\times4} = \hat{S}^{4\times4}. \quad (2.83)
\]

It is straightforward to transform the contour time integrals in (2.79) to the appropriate, discrete time step version having the essential time shifts \(\Delta t_p\) in the complex fields \(\psi_{\vec{x},s}(t_p + \Delta t_p)\) (relative to \(\psi_{\vec{x},s}(t_p)\)) so that the original normal ordering of the second quantized Hamilton operator \(\hat{H}_{\vec{x},s}(t_p)\) is taken into account and the action of field operators with its hermitian conjugate at the same time contour point is avoided. However, we have to mention the following detail for a somewhat modified time contour integration path '\(\hat{C}\)' in (2.79) with \(\Delta t'_p\) instead of '\(\hat{C}\)' \((2.79)\) :

- In order to include the precise boundary and discrete 'time step' conditions, one has to extend the time contour integrations \((2.8)\) \((2.9)\) at the end points '\(t_{p=+} = T_{fin}\)' and '\(t_{p=-} = T_{ini}\)' of the two '\(p = \pm\)' branches by an additional single time step (where we omit the further straightforward specification of the discrete time steps between \(T_{ini}\) and \(T_{fin}\) for brevity)

\[
\int_C dt_p \ldots = \int_{T_{fin} - |\Delta t_p|}^{T_{fin}} dt_+ \ldots - \int_{T_{ini} - |\Delta t_p|}^{T_{ini}} dt_- \ldots \quad (2.84)
\]
and has to set the following fields and sources at the extended time boundary points \( t_p^+=T_{\text{ini}} - |\Delta t_p| \), \( t_p^-=T_{\text{ini}} - |\Delta t_p| \) of the two branches identical to zero

\[
\psi_{x,s}(t_p^\pm=T_{\text{ini}} - |\Delta t_p|) = \psi_{x,s}^*(t_p^\pm=T_{\text{ini}} - |\Delta t_p|) \equiv 0; \quad (2.85)
\]

\[
J^a_{\psi,s}(\vec{x}, t_p^\pm=T_{\text{ini}} - |\Delta t_p|) = 0; \quad J^b_{\psi,s}(\vec{x}, t_p^\pm=T_{\text{ini}} - |\Delta t_p|) \equiv 0; \quad (2.86)
\]

\[
\delta \Sigma_{ab}^b(\vec{x}, t_p^\pm=T_{\text{ini}} - |\Delta t_p|) = 0; \quad \sigma_D^{(0)}(\vec{x}, t_p^\pm=T_{\text{ini}} - |\Delta t_p|) \equiv 0. \quad (2.87)
\]

- This only amounts to an extension from the anomalous doubled field \( \tilde{\Psi}_{x,s}^a(t_p) = (\psi_{x,s}(t_p), \psi_{x,s}^*(t_p+\Delta t_p))^T \) with non-zero entries for times \( T_{\text{ini}} \geq t_p \geq T_{\text{ini}} \) to the interval \( T_{\text{ini}} \geq t_p \geq T_{\text{ini}} - |\Delta t_p| \) so that one also regards the complex conjugated field \( \psi_{x,s}^a(t_p=T_{\text{ini}}) \) in the second part (\( a=2 \)) of \( \tilde{\Psi}_{x,s}^a(t_p = T_{\text{ini}} - |\Delta t_p|) = (\psi_{x,s}(t_p = T_{\text{ini}} - |\Delta t_p|) \equiv 0, \psi_{x,s}^*(t_p=T_{\text{ini}}))^T \) with a vanishing first component at \( t_p = T_{\text{ini}} - |\Delta t_p| \), according to (2.85)-(2.87). Similar amendments have to be performed for the anomalous doubled one-particle operator \( \hat{\Sigma}_{ab}^b(t_q, t_p) \) (2.70)-(2.72) concerning this time point \( t_p, t_q=T_{\text{ini}} - |\Delta t_p| \).

Since one has the following relation between \( \tilde{\Psi}_{x,s}^a(t_p) \) and \( \tilde{\Psi}_{x,s}^b(t_p) \) with Pauli matrix \( \hat{\tau}_1 \) for the exchange of the two components \( 'a=1,2' \)

\[
\tilde{\Psi}_{x,s}^b(t_p) = (\hat{\tau}_1)^{ba} \tilde{\Psi}_{x,s}^a(t_p) \quad (2.88)
\]

the bilinear, anomalous doubled anti-commuting fields \( \tilde{\Psi}_{x,s}^a, \tilde{\Psi}_{x,s}^b \) in (2.79) can be removed by integration so that one obtains the square root of the anomalous doubled fermion determinant with matrix \( \tilde{N}_{x,s}^{ba}(t'_q, t'_p) \) (2.80) and the propagator \( N_{x,s}^{ba}(t'_q, t'_p) \) weighted by the anomalous doubled source fields \( J_{\psi,s}(\vec{x}, t'_p), J_{\psi,s}(\vec{x}, t'_p) \). The exchange matrix \( (\hat{\tau}_1)^{ba} \) of (2.88) can be omitted in the path integral (2.89) because its combined appearance on the two branches of the time contour does not affect the final weighting for observables, neither within the determinant nor with the propagator of \( \tilde{N}_{x,s}^{ba}(t'_q, t'_p) \)

\[
Z[\hat{J}, \hat{\psi}, \hat{J}_{\psi}] = \int d[\sigma_D^{(0)}(\vec{x}, t_p)] \exp \left\{ -\frac{i}{2\hbar} \int_C dt_p \sum_{x,s,\hat{s}=\downarrow,\uparrow} \frac{\sigma_D^{(0)}(\vec{x}, t_p)}{V_0 + t \varepsilon_p} \right\} \times \right.
\]

\[
\times \int d[\delta \Sigma(\vec{x}, t_p)] \exp \left\{ -\frac{i}{4\hbar} \int_C dt_p \sum_x \text{Tr}_{a,s,b,\hat{s}} \left[ \delta \Sigma_{ab}^{(0)}(\vec{x}, t_p) \delta \Sigma_{ab}^{0}(\vec{x}, t_p) \right] \right\} \times \right.
\]

\[
\times \left\{ \text{DET}_C \left[ (\hat{\tau}_1)^{ba} \tilde{M}_{x,s}^{ba}(t'_q, t'_p) \right] \right\}^{1/2} \times \right.
\]

\[
\times \exp \left\{ \frac{i}{2\hbar} \Omega^2 \int_C dt_p dt'_q \sum_{x,x'} \sum_{s,s'=\downarrow,\uparrow} J_{\psi,s'}^{a} \hat{J}_{\psi,s}(\xi') \tilde{M}_{x,x'}^{-1} \hat{J}_{\psi,s'}(\xi') \right\} \]

According to the particular relation (2.88) for the exchange between \( \tilde{\Psi}_{x,s}^b(t_p) \) and \( \tilde{\Psi}_{x,s}^a(t_p) \) with Pauli-matrix \( (\hat{\tau}_1)^{ba} \), we have applied the standard relation (2.90) for anti-commuting variables \( \xi_j \) \( (j = 1, \ldots, 2N) \) in order to remove the bilinear fields \( \tilde{\Psi}_{x,s}^a(t'_q) \) and \( \tilde{\Psi}_{x,s}^a(t_p) \) in (2.79)(2.80) by Gaussian integration. It has to be pointed out that the symmetric part of the matrix \( \tilde{M}_{ji} \) in (2.90) cancels in the exponent with the anti-commuting variables so that the determinant ‘DET’ in (2.90) only contains the anti-symmetric part of the considered matrix \( \tilde{M}_{ji} \)

\[
\int d[\xi] \exp \left\{ -\sum_{i,j=1}^{2N} \xi_j^T \tilde{M}_{ji} \xi_i \right\} = \left\{ \text{DET}[\tilde{M}_{ji}] \right\}^{1/2}. \quad (2.90)
\]
The coset decomposition into densities and anomalous parts with matrix $\hat{T}(\vec{x}, t_p)$ yields the path integral (2.91) where the change of the Jacobian is already incorporated from the 'flat' Euclidean self-energy $\delta \Sigma_{ss'}(\vec{x}, t_p)$ to densities and to the independent parameters of pair condensate fields of cosets within $d[\hat{T}^{-1}(\vec{x}, t_p) d\hat{T}(\vec{x}, t_p)]$ (see appendix [A]). Apart from the 'source action' $A_{\hat{J}_{\psi\psi} [\hat{T}]}$ for pair condensates, there appears a logarithmic action $A_{DET}[\hat{T}, \sigma_D(0); \delta \hat{T}]$ (2.92) from the determinant $(DET[\hat{N}^{ba}_{\vec{x}', s'; \vec{x}, s}(t'_q, t_p)])^{1/2}$ and a propagator $\delta \Sigma_{ab}(\vec{x}, t_p)$ weighted by the source fields $J^b_{\psi; s}(\vec{x}', t'_q)$, $J^a_{\psi; s}(\vec{x}, t_p)$ and additionally by the coset matrices $\hat{T}_{s' s}^{ab}(\vec{x}', t'_q)$, $\hat{T}_{s s'}^{a b}(\vec{x}, t_p)$. The two actions $A_{DET}[\hat{T}, \sigma_D(0); \delta \hat{T}]$, $A_{J_{\psi; s}}[\hat{T}, \sigma_D(0); \delta \hat{T}]$ (2.93) are determined by the operator $\delta \Sigma_{ab}(\vec{x}', t'_q)$ (2.94) which consists of the density part $\delta \hat{T}$ with one-particle operator $\hat{H}$ (2.70 2.72) and gradient term $\delta \hat{T}(\hat{T}^{-1}, \hat{T}) = \hat{T}^{-1}(\hat{H} + \sigma_D(0)) \hat{T} - (\hat{H} + \sigma_D(0))$ of coset matrices aside from the source term for generating correlation functions

\[
Z[\delta \hat{T}, J_{\psi; s}, \delta \hat{J}_{\psi\psi}] = \int d[\sigma_D(0); \delta \hat{T}] \exp \left\{ - \frac{1}{2\hbar} \int \frac{dt_p}{t_p} \sum_{\vec{x}, s, s'} \frac{\sigma_D(0, s'; s, \vec{x}, t_p)}{V_0 + t \varepsilon_p} \right\} \times \int d[\hat{T}^{-1}(\vec{x}, t_p) \hat{T}(\vec{x}, t_p)] \exp \left\{ i A_{J_{\psi; s}}[\hat{T}] \right\} \times \exp \left\{ i A_{DET}[\hat{T}, \sigma_D(0); \delta \hat{T}] \right\};
\]

\[
A_{DET}[\hat{T}, \sigma_D(0); \delta \hat{T}] = \frac{1}{2} \frac{\Omega^2}{\hbar} \int \frac{dt_p}{t_p} \sum_{\vec{x}, \vec{x}', s, s'} N_x \sum_{a, b, \delta, \sigma} \sum_{a, b, \delta, \sigma} \times \left[ J^b_{\psi; s'}(\vec{x}', t'_q) \hat{T}_{s' s}^{bb}(\vec{x}', t'_q) \hat{T}_{s s'}^{a \sigma}(\vec{x}, t_p) \hat{T}_{s' s}^{1 a \sigma}(\vec{x}, t_p) \delta \Sigma_{ab}(\vec{x}, t_p) \right]^{ba}_{\vec{x}', s'; \vec{x}, s};
\]

\[
\delta \hat{T}(\hat{T}^{-1}, \hat{T}) = \left[ \hat{T}^{-1}(\hat{H} + \sigma_D(0)) \hat{T} - (\hat{H} + \sigma_D(0)) \right] = \left[ \hat{T}^{-1}(\hat{H} + \sigma_D(0)) \hat{T} - (\hat{H} + \sigma_D(0)) \right].
\]

2.3 Determination of the 'pair condensate seed' action $A_{\hat{J}_{\psi\psi} [\hat{T}]}$

The 'pair condensate seed' action $A_{\hat{J}_{\psi\psi} [\hat{T}]}$ follows from the quadratic term of self-energies originating from the HST of the quartic interaction of anti-commuting fields. A shift (2.95) of the self-energy matrix is additionally performed in $\hat{N}^{ba}_{\vec{x}', s'; \vec{x}, s}(t'_q, t_p)$ (2.80) so that the source matrix $J^{ba}_{\psi; s}(\vec{x}, t_p)$ of pair condensate fields only appears in the quadratic term $A_{2}[\hat{T}, \delta \hat{\Sigma}; \delta \hat{J}_{\psi\psi}]$ (2.99) of the self-energy remaining from the HST

\[
\delta \hat{\Sigma}_{a'(s)}(\vec{x}, t_p) \rightarrow \delta \hat{\Sigma}_{a'(s)}(\vec{x}, t_p) + i J^{ba}_{\psi; s}(\vec{x}, t_p).
\]
The coset decomposition combined with the change of the integration measure leads to relations (2.96)(2.102) with block diagonal self-energy densities $\delta \Sigma_{ss'}^{a\alpha}(\vec{x}, t_p)$ (2.100) and parameters (2.101)(2.102) which are further decomposed into eigenvalues $\delta \lambda_s(\vec{x}, t_p)$ and diagonalizing matrices $\hat{Q}_{ss'}^{a\alpha}(\vec{x}, t_p)$

$$\exp \left\{ i A_J \hat{\psi} \left[ \hat{T} \right] \right\} = \int d[\delta \hat{\Sigma}_D(\vec{x}, t_p)] \mathcal{P}(\delta \hat{\Sigma}(\vec{x}, t_p)) \exp \left\{ i A_2 \left[ \hat{T}, \delta \hat{\Sigma}_D; i \hat{J}_\psi \right] \right\}$$

(2.96)

$$\det \left\{ \delta \hat{\Sigma}_{D,ss'}^{11}(\vec{x}, t_p) - \delta \lambda \delta_{ss'} \right\} = 0 \quad \det \left\{ \delta \hat{\Sigma}_{D,ss'}^{ab}(\vec{x}, t_p) - \delta_{ss'} \delta_{ss'} \right\} = 0$$

(2.97)

$$\delta \hat{\Sigma}_{D,ss'}^{11}(\vec{x}, t_p) = \left( \hat{Q}^{-1;11}(\vec{x}, t_p) \delta \hat{\lambda}(\vec{x}, t_p) \hat{Q}^{11}(\vec{x}, t_p) \right)_{ss'}^{11};$$

(2.100)

$$\delta \hat{\Sigma}_{D,ss'}^{11}(\vec{x}, t_p) = \left( \delta \Sigma_{D:11}(\vec{x}, t_p) \delta \Sigma_{D:11}(\vec{x}, t_p) \right)_{ss'}^{11};$$

(2.101)

$$\delta \hat{\Sigma}_{D,ss'}^{11}(\vec{x}, t_p) = \frac{1}{2} \left[ (\hat{\tau}_0)_{ss'} (\delta \lambda_1(\vec{x}, t_p) + \delta \lambda_1(\vec{x}, t_p)) + (\hat{\tau}_3)_{ss'} \cos \left( 2 |\mathcal{F}_{11}(\vec{x}, t_p)| \right) \right]$$

(2.102)

As one collects the various parameters of the total self-energy $\delta \Sigma_{ss'}^{ab}(\vec{x}, t_p)$ with pair condensate field $f(\vec{x}, t_p)$ and density fields $\delta \lambda(\vec{x}, t_p)$, $\delta \lambda_1(\vec{x}, t_p)$, $\mathcal{F}_{11}(\vec{x}, t_p)$ (2.103)(2.106), one finally achieves the quadratic term (2.107) of the self-energy which is caused by the HST in terms of six real field variables of the so(4) generators with four real density variables for the u(2) generators and two real field degrees of freedom within the coset generators so(4) / u(2) or pair condensate terms

$$\delta \Sigma_{ss'}^{ab}(\vec{x}, t_p) = \left( \hat{T}(\vec{x}, t_p) \right)_{ss'}^{a1} \delta \Sigma_{ss'}^{a1} = \left( \hat{T}(\vec{x}, t_p) \right)_{ss'}^{a2} \delta \Sigma_{ss'}^{a2}$$

(2.103)

$$\delta \Sigma_{ss'}^{22}(\vec{x}, t_p) = -\left( \delta \Sigma_{ss'}^{11}(\vec{x}, t_p) \right)_{ss'}^T; \quad \delta \Sigma_{ss'}^{21}(\vec{x}, t_p) = -\left( \delta \Sigma_{ss'}^{22}(\vec{x}, t_p) \right)_{ss'}^T$$

(2.104)

$$\delta \Sigma_{ss'}^{11}(\vec{x}, t_p) = \frac{1}{2} \left[ (\hat{\tau}_0)_{ss'} \cos \left( 2 |\mathcal{F}_{11}(\vec{x}, t_p)| \right) \delta \lambda_1(\vec{x}, t_p) + \delta \lambda_1(\vec{x}, t_p) \right] +$$

(2.105)
\[ \delta \Sigma_{ss}(x, t_p) = \frac{1}{2}(\hat{\theta}_2)_{ss} \sinh(2|f(x, t_p)|) \exp\{i \phi(x, t_p)\} \left( \delta \lambda_1(x, t_p) + \delta \lambda_1(x, t_p) \right); \]  
(2.106)

\[ i A_2[\hat{T}, \hat{Q}^{-1}_i \hat{\delta} \hat{Q}; i \hat{j}_{\psi \psi}] = -\frac{i}{4\hbar} \int_C d\tau_p \sum_{\vec{\varepsilon}} \times \]  
(2.107)

\[ \times \left( \frac{-i \varepsilon_+ n_P + V_0}{\varepsilon_+^2 + V_0^2} \left[ \cosh(2|f(x, t_p)|) \left( \delta \lambda_1(x, t_p) + \delta \lambda_1(x, t_p) \right)^2 + \left( \delta \lambda_1(x, t_p) - \delta \lambda_1(x, t_p) \right)^2 \right] + \right. \]  

\[ - 4 \frac{i \varepsilon_+ n_P + V_0}{\varepsilon_+^2 + V_0^2} \sinh(2|f(x, t_p)|) \sin(\phi(x, t_p) - \gamma(x, t_p)) \left| j_{\psi \psi}(x, t_p) \right| \left( \delta \lambda_1(x, t_p) + \delta \lambda_1(x, t_p) \right) + \]  

\[ - 4 \frac{i \varepsilon_+ n_P + V_0}{\varepsilon_+^2 + V_0^2} \left| j_{\psi \psi}(x, t_p) \right|^2. \]  
(2.108)

According to the imaginary increments \( i \varepsilon_+ (\delta_{ab} - \delta_{a \neq b}) \) of the parameter \( V_0 \) for an effective, short-ranged interaction, the integrations of \( \delta \lambda_s(x, t_p) \), \( Q_{\psi \psi s}(x, t_p) \) or \( \delta \Sigma_{aa}(x, t_p) \) can be performed in \( 2.96 \) so that the 'pair condensate seed' action \( \exp\{i A_{j_{\psi \psi}}[\hat{T}]\} \) simplifies to the relation \( 2.108 \) with remaining pair condensate field degrees of freedom \( f(x, t_p) = |f(x, t_p)| \exp\{i \phi(x, t_p)\} \). One has to note from the integration of the density variables the appearance of the nontrivial factors \( \cosh^{-2}(2|f(x, t_p)|) \) which have additionally to be considered in the coset integration measure of pair condensate field variables

\[
\exp\{i A_{j_{\psi \psi}}[\hat{T}]\} = \prod_{(x,t_p)} \left( a^2 \left( \frac{4 h N_x |V_0|^3}{|\Delta t_p|} \right)^3 \frac{1}{\cosh^3(2|f(x, t_p)|)} \right) \exp\{i \int_C d\tau_p \sum_{\vec{\varepsilon}} \left| j_{\psi \psi}(x, t_p) \right|^2 \}
\times \exp\{i \alpha_{j_{\psi \psi}}[\hat{T}]\} \left( 1 + 2i \alpha_{j_{\psi \psi}}[\hat{T}] \right); \]  
(2.108)

\[
a_{j_{\psi \psi}}[\hat{T}] = \int_C d\tau_p \sum_{\vec{\varepsilon}} \left| j_{\psi \psi}(x, t_p) \right|^2 \tanh^2(2|f(x, t_p)|) \sin^2(\phi(x, t_p) - \gamma(x, t_p)). \]  
(2.109)

We briefly list the involved integration measure for density terms and the pair condensate with corresponding parameters which are specified in Eqs. \( 2.110,2.111 \), respectively (compare appendix A)

\[
\mathcal{P}(\delta \lambda(x, t_p)) = \prod_{(x,t_p)} \left| \delta \lambda_1(x, t_p) + \delta \lambda_1(x, t_p) \right|^2; \]  
(2.110)

\[
d[\delta \Sigma_D(x, t_p)] \mathcal{P}(\delta \lambda(x, t_p)) = d[\hat{Q}(x, t_p) \hat{Q}^{-1}(x, t_p); \delta \lambda(x, t_p)] \mathcal{P}(\delta \lambda(x, t_p)) = \]  
(2.111)

\[
= \prod_{(x,t_p)} \left\{ 8 d(\delta \lambda_1(x, t_p)) d(\delta \lambda_1(x, t_p)) \left| \delta \lambda_1^2(x, t_p) - \delta \lambda_1^2(x, t_p) \right|^2 \times \right. \]  

\[ \times \left. d(\mathcal{F}(x, t_p)) \sin(|\mathcal{F}(x, t_p)|) \cos(|\mathcal{F}(x, t_p)|) \right) d\varphi(x, t_p) \right\}; \]  
(2.111)

\[
d[\hat{T}^{-1}(x, t_p) d\hat{T}(x, t_p)] = \prod_{(x,t_p)} \left\{ 8 d(|f(x, t_p)|) \sinh(|f(x, t_p)|) \cosh(|f(x, t_p)|) d\phi(x, t_p) \right\}. \]  
(2.112)
The combination of (2.112) with the factors \( \cosh^{-3}(2|f(\vec{x},t_p)|) \) of (2.108-2.109) finally results into the total integration measure (2.113) for the pair condensate field variables

\[
d[\hat{T}^{-1}(\vec{x},t_p)] \times \prod_{\{\vec{x},t_p\}} \left( \frac{1}{\cosh^3(2|f(\vec{x},t_p)|)} \right) = (2.113)
\]

\[
= \prod_{\{\vec{x},t_p\}} \left\{ 8 \frac{d(|f(\vec{x},t_p)|)}{\sinh(|f(\vec{x},t_p)|) \cosh(|f(\vec{x},t_p)|)} \right\} \prod_{\{\vec{x},t_p\}} \left( \frac{1}{\cosh^3(2|f(\vec{x},t_p)|)} \right) = (2.108-2.109)
\]

\[
= \prod_{\{\vec{x},t_p\}} \left\{ 4 \frac{d(|f(\vec{x},t_p)|)}{\cosh^3(2|f(\vec{x},t_p)|)} \right\} = \prod_{\{\vec{x},t_p\}} \left\{ -d \left( \cosh^{-2} \left( 2|f(\vec{x},t_p)| \right) \right) \right\} = (2.113)
\]

It is instructive to point out the combination \( \exp\left\{ i \, \alpha_j^{(\psi)} [\hat{T}] \right\} \) in (2.108-2.109) which can also be obtained from transforming the term \( |\delta \lambda^2(\vec{x},t_p) - \delta \lambda^2(\vec{x},t_p)|^2 \) to a Vandermonde determinant with Hermite polynomials and Gaussian weights for orthogonal basis functions [2]. The above mentioned term with action \( \alpha_j^{(\psi)} [\hat{T}] \) (2.109) can therefore be regarded as a Hermite polynomial \( H_2(x) \) with argument \( x^2 = -i \, \alpha_j^{(\psi)} [\hat{T}] \) and corresponding Gaussian weight

\[
\exp \left\{ i \, \alpha_j^{(\psi)} [\hat{T}] \right\} \propto \exp\left\{ -x^2 \right\} \propto \exp\left\{ x^2 \right\} H_2(x) \quad x^2 = -i \, \alpha_j^{(\psi)} [\hat{T}] \quad (2.114)
\]

2.4 Gradient expansion of the fermion determinant

The separation of the integration measure from the 'flat' Euclidean self-energy \( \delta \Sigma_{ss'}^{ab}(\vec{x},t_p) \) to the product of a density and pair condensate part can be merged with a division of the actions \( A_{DET}[\hat{T},\hat{\sigma}_D^{(0)};\hat{j}] \) (2.92), \( A_{\psi}[\hat{T},\hat{\sigma}_D^{(0)};\hat{j}] \) (2.93) into a pure 'density path integral' \( Z[\hat{\sigma}_D^{(0)};\hat{j}_\psi] \) (2.115) with the hermitian self-energy field variables \( \sigma_{D,s,s'}^{(0)}(\vec{x},t_p) \) and remaining coset parts \( A'_{DET}[\hat{T};\hat{j}], A'_{\psi}[\hat{T};\hat{j}] \) (see following Eqs. (2.117-2.119))

\[
Z[\hat{\sigma}_D^{(0)};\hat{j}_\psi] = \int d[\sigma_{D,s,s'}^{(0)}(\vec{x},t_p)] \exp \left\{ -\frac{i}{2\hbar} \int_C dt_p \sum_{\vec{x},s,s'=1,\downarrow} \left( \sigma_{D,s,s'}^{(0)}(\vec{x},t_p) \sigma_{D,s',s}^{(0)}(\vec{x},t_p) \right) \right\}
\]

\[
\times \exp \left\{ \int_C \frac{dt_p}{\hbar} \sum_{\vec{x},s,s'=1,\downarrow} \left( \sigma_{D,s,s'}^{(0)}(\vec{x},t_p) \sigma_{D,s',s}^{(0)}(\vec{x},t_p) \right) \right\}
\]

\[
\times \exp \left\{ \frac{\Omega}{\hbar} \int_C dt_p dt'_p \sum_{\vec{x},s,s'=1,\downarrow} \left( \sum_{\vec{x}',s',s'=1,\downarrow} \sum_{\vec{x},s,s'=1,\downarrow} j_{\psi;ss'}(\vec{x}',t'_p) \right) \right\}
\]

\[
= \left\{ Z[\hat{\sigma}_D^{(0)};\hat{j}_\psi] \times \int \prod_{\{\vec{x},t_p\}} \left[ d[\tanh^2(2|f(\vec{x},t_p)|)] \right] \right\}
\]

One can use a saddle point approximation in (2.116) of the pure 'density path integral' \( Z[\hat{\sigma}_D^{(0)};\hat{j}_\psi] \) (2.115) for extracting classical, complex fields \( \langle \sigma_{D,s,s'}^{(0)}(\vec{x},t_p) \rangle \) whose imaginary parts of the corresponding eigenvalues comply with the imaginary increment \( i \, \bar{\varepsilon}_p \) of the anomalous doubled one-particle operator \( \hat{\sigma} \) (2.70-2.72). This classical approximation \( \langle \sigma_{D,s,s'}^{(0)}(\vec{x},t_p) \rangle \) can be inserted into \( A'_{DET}[\hat{T};\hat{j}], A'_{\psi}[\hat{T};\hat{j}] \) so that the independent parameters of the coset matrix \( T(\vec{x},t_p) \) are the only remaining 'path integration fields' (2.117-2.119)

\[
Z[\hat{j},J,\omega] = \left\{ Z[\hat{\sigma}_D^{(0)};\hat{j}_\psi] \times \int \prod_{\{\vec{x},t_p\}} \left[ d[\tanh^2(2|f(\vec{x},t_p)|)] \right] \right\}
\]

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\[ Z[\hat{J}, J_\psi, \hat{J}_\psi] \approx \int \prod_{\{x, t\}} \left( d[\tanh^2(2|f(x, t)|)] d[\phi(x, t)] \right) \]

\[ \times \exp \left\{ i a_{\psi \psi}[\hat{T}] \right\} \left( 1 + 2i a_{\psi \psi}[\hat{T}] \right) \exp \left\{ i \int_C dt_p \sum_{x} |j_{\psi \psi}(x, t_p)|^2 \right\} \]

\[ \times \exp \left\{ A_{DET}[\hat{T}, \hat{\sigma}^{(0)}_D; \hat{J}] - \int_C \frac{dt_p}{\hbar} \eta_p \sum_{x} N_s \text{ tr} \ln [\hat{H}^{11} + \hat{\sigma}^{(0)11}_D] \right\} \]

\[ \times \exp \left\{ 2.4 \text{ Gradient expansion of the fermion determinant} \right\} \]

\[ \times \exp \left\{ j^\dagger_{\psi; x'}(x', t'_q) (\hat{H}^{11}_{x,x'} + \hat{\sigma}^{(0)11}_{x,x'})^{-1} (t'_q, t_p) \hat{j}_{\psi; x}(x, t_p) \right\} \]

\[ \Rightarrow \text{ saddle point approximation for } \langle \sigma^{(0)}_{D,ss'}(x, t_p) \rangle \Rightarrow \]

\[ A'_{DET}[\hat{T}; \hat{J}] = \frac{1}{2} \int_C \frac{dt_p}{\hbar} \eta_p \sum_{x} N_s \text{ Tr}[\hat{H}^{11} + \hat{\sigma}^{(0)11}_D]^{-1} \hat{H}^{11}_{x,x'} + \hat{j}_{\psi; x}(x, t_p) \hat{j}_{\psi; x'}(x', t'_q) \hat{j}_{\psi; x}(x, t_p) \]

\[ \text{This allows to extract an effective Lagrangian of coset fields restricted to finite order gradients. If the combination } (\hat{H} + \langle \hat{\sigma}^{(0)}_D \rangle) \text{ of one-particle operator } \hat{\mathcal{H}} \text{ with the classical density field } \langle \sigma^{(0)}_{D,ss'}(x, t_p) \rangle \text{ contains comparable momentum values as the gradient term } \hat{\mathcal{H}}^{(T^{-1}, \hat{T})}, \text{ one can apply following identities for the logarithm and the inverse of an operator } \hat{\mathcal{O}} \]

\[ (\ln \hat{\mathcal{O}}) = \left( \int_0^{+\infty} dv \frac{\exp(-v \hat{1}) - \exp(-v \hat{\mathcal{O}})}{v} \right); \]

\[ (\hat{\mathcal{O}}^{-1}) = \left( \int_0^{+\infty} dv \exp(-v \hat{\mathcal{O}}) \right); \]

so that one also achieves valid expansions of the term

\[ \hat{\mathcal{O}} = \hat{1} + \hat{\mathcal{H}}^{(T^{-1}, \hat{T})} \left( \hat{H} + \langle \hat{\sigma}^{(0)}_D \rangle \right)^{-1} = \hat{1} + \left( T^{-1} (\hat{H} + \langle \hat{\sigma}^{(0)}_D \rangle) \hat{T} - (\hat{H} + \langle \hat{\sigma}^{(0)}_D \rangle) \right) \left( \hat{H} + \langle \hat{\sigma}^{(0)}_D \rangle \right)^{-1} \]
we can simply multiply the operator $\hat{\mathcal{O}}$ always results into positive eigenvalues in the hermitian part of the total operator $\hat{\mathcal{O}}$ (as e.g. in $\ln \hat{\mathcal{O}} = \ln [\hat{1} + (\hat{\mathcal{O}} - \hat{1})]$ or $\hat{\mathcal{O}}^{-1} = [\hat{1} + (\hat{\mathcal{O}} - \hat{1})]^{-1}$).

One might infer that the hermitian part of the rather general operator $\hat{\mathcal{O}}$, defined in terms of the coset matrices and saddle point fields (2.122), could have negative eigenvalues so that the given relations (2.120,2.121) do not converge for the logarithm and the inverse of the operator $\hat{\mathcal{O}}$ (2.122); although there are no restrictions onto the values of the hermitian part of $\hat{\mathcal{O}}$ (2.122), the anti-hermitian part of $\hat{\mathcal{O}}$ (2.122) has to comply with the infinitesimal imaginary increment $-i \varepsilon_p$ in the one-particle operator $\hat{\mathcal{H}}$. It determines the sign of the eigenvalues of the anti-hermitian part in $\hat{\mathcal{O}}$ (2.122) to be negative valued ($\times i$) because this infinitesimal imaginary increment $-i \varepsilon_p$ chooses a time direction in an otherwise time-reversal invariant system and therefore also fixes the sign of the imaginary parts of eigenvalues within the saddle point solution $\langle \sigma^{(0)}_{D:ssv}(\vec{x}, t_p) \rangle$. Since the determinant and the corresponding logarithm of the total operator $\hat{\mathcal{O}}$ (2.122) follow from bilinear integration with the anomalous doubled, anti-commuting fields over the one-particle operator with imaginary increment $-i \varepsilon_p$, one has to require the identical sign of eigenvalues of the anti-hermitian part of $\hat{\mathcal{O}}$; if this requirement fails to hold at any step of derivations and involved approximations, one has obtained an unstable system due to increasing field values within the time development. This sign convention, following from $-i \varepsilon_p$ within $\hat{\mathcal{H}}$, has also to be fulfilled for fermionic systems apart from bosonic systems with negative powers of the boson-determinant because the chosen imaginary increment $-i \varepsilon_p$ assigns a time direction and time development from small, real to larger, real time values on physical grounds. Therefore, the imaginary, infinitesimal increment can be regarded as an artificial, dissipative part in an otherwise hermitian system where subsequent approximations (as e.g. the saddle point solution $\langle \sigma^{(0)}_{D:ssv}(\vec{x}, t_p) \rangle$ or even the total operator $\hat{\mathcal{O}}$ (2.122)) have to stay in accordance with $' -i \varepsilon_p'$. Since the eigenvalues of the anti-hermitian part of $\hat{\mathcal{O}}$ (2.122) are determined to be negative valued ($\times i$), we can simply multiply the operator $\hat{\mathcal{O}}$ by the imaginary unit operator modified with the contour time metric, which has unit determinant and appropriate inverse, so that the change of the operator $\hat{\mathcal{O}}$ (2.122) to the operator

$$\hat{\mathcal{O}} \rightarrow (i \hat{1} \eta_p) \hat{\mathcal{O}},$$

always results into positive eigenvalues in the hermitian part of the total operator $'(i \hat{1} \eta_p) \hat{\mathcal{O}}'$. Therefore, this modified operator can be replaced in (2.120,2.121) for convergent integral relations

$$\text{DET}\{ (i \hat{1} \eta_p) \} \equiv 1; \quad (i \hat{1} \eta_p)^{-1} \times (i \hat{1} \eta_p) = (-i \hat{1} \eta_p) \times (i \hat{1} \eta_p) \equiv 1; \quad (2.124)$$

$$\left( \ln (i \hat{1} \eta_p) \hat{\mathcal{O}} \right) \equiv \left( \int_0^{+\infty} dv \exp \{ -v \hat{1} \hat{\mathcal{O}} \} - \exp \{ -v (i \hat{1} \eta_p) \hat{\mathcal{O}} \} \right); \quad (2.125)$$

$$\left( \hat{\mathcal{O}} \right)^{-1} \equiv \left( (i \hat{1} \eta_p) \hat{\mathcal{O}} \right)^{-1} \left( i \eta_p \right) = \left( \int_0^{+\infty} dv \exp \{ -v (i \hat{1} \eta_p) \hat{\mathcal{O}} \} \right) \left( i \beta \eta_p \right). \quad (2.126)$$

Further simplifications for a finite or infinite order gradient expansion are attained if the total number $N_x$ (2.123) of involved grid points is properly taken into account for an expansion.
3.1 Field variables and dimensions of the spatially nonlocal self-energy

In contrast to sections 2.2 to 2.4 with approximation $V_{|x-x'|} \approx \delta_{x,x'} V_0 (2.36)$, we extend to the case of an arbitrary long-ranged interaction potential $V_{|x-x'|}$ and specify the independent field variables of the self-energy with adaption of the coset decomposition and corresponding parameters. Since we start out from a Hamiltonian $(2.3)(2.4)$ without an ensemble average for disorder, the extended self-energy depends on a single time contour variable $t_p$ so that the nonlocal self-energy \[ \Sigma_{ab}^{(0)}(x_{s},s';t_p) \] only has to be specified by adding vector 'indices' $\vec{x}, \vec{x}'$ apart from the spin indices $s, s'$ with $s = \uparrow, \downarrow$. We also split the total self-energy $\Sigma_{ab}^{(0)}(x_{s},s';t_p)$ into a hermitian, density related self-energy $\Sigma_{ab}^{(0)}(x_{s},s';t_p)$ as in sections 2.2-2.4 and deviations $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$ with anti-hermitian pair condensate field degrees of freedom which are regarded by the coset matrices $\Sigma_{ab}^{(0)}(x_{s},s';t_p)$ and local eigenvalue densities $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$ in its anomalous doubled kind, and into off-diagonal $(a \neq b)$, anti-hermitian pair condensate field degrees of freedom which are regarded by the coset matrices $(\tilde{T}(t_p))_{\langle \vec{x},s_{1},s_{1}' \rangle}^a_{b}$, $(\tilde{T}^{-1}(t_p))_{\langle \vec{x}',s_{2},s_{2}' \rangle}^a_{b}$ with extension to the spatially nonlocal case. Note that the symmetries require extension of the hermitian conjugation and transpose between the various block parts of $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$ so that the spatial vectors $\vec{x}, \vec{x}'$ have to be incorporated into these matrix operations aside from the spin space

\[
\begin{align*}
\Sigma_{ab}^{(0)}(x_{s},s';t_p) &= \sigma_{D:ss'}^{(0)}(\vec{x}, \vec{x}', t_p) \delta_{ab} + \delta \Sigma_{ab}^{(0)}(x_{s},s';t_p) = \left( \begin{array}{cc}
\delta \Sigma_{11}^{(0)}_{x,s;\vec{x}', t_p} & \delta \Sigma_{12}^{(0)}_{x,s;\vec{x}', t_p} \\
\delta \Sigma_{21}^{(0)}_{x,s;\vec{x}', t_p} & \delta \Sigma_{22}^{(0)}_{x,s;\vec{x}', t_p}
\end{array} \right)_{ab} ; (3.1)
\end{align*}
\]

In the remainder the summation convention over multiply occurring spatial vectors is always implied for the presented case of nonlocal self-energies, unless the spatial vectors as indices are set into parenthesis (as e.g. $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$). In analogy to $(2.42)(2.47)$ for the short-ranged interaction $\delta_{x,x'} V_0 (2.36)$, we take the factorization of $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$ into block diagonal densities $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$, with further diagonalizing matrices $\tilde{\Sigma}_{11}^{(0)}_{x_{s},s';t_p}$ and local eigenvalue densities $\delta \lambda_{ab}^{(0)}(\vec{x}, t_p)$ in its anomalous doubled kind, and into off-diagonal $(a \neq b)$, anti-hermitian pair condensate field degrees of freedom which are regarded by the coset matrices $(\tilde{T}(t_p))_{\langle \vec{x},s_{1},s_{1}' \rangle}^a_{b}$, $(\tilde{T}^{-1}(t_p))_{\langle \vec{x}',s_{2},s_{2}' \rangle}^a_{b}$ with extension to the spatially nonlocal case. Note that the symmetries require extension of the hermitian conjugation and transpose between the various block parts of $\delta \Sigma_{ab}^{(0)}(x_{s},s';t_p)$ so that the spatial vectors $\vec{x}, \vec{x}'$ have to be incorporated into these matrix operations aside from the spin space

\[
\begin{align*}
\Sigma_{ab}^{(0)}(x_{s},s';t_p) &= \left( \begin{array}{cc}
\sigma_{D:ss'}^{(0)}(\vec{x}, \vec{x}', t_p) + \delta \Sigma_{11}^{(0)}_{x,s;\vec{x}', t_p} \\
\delta \Sigma_{21}^{(0)}_{x,s;\vec{x}', t_p}
\end{array} \right)_{ab} + \left( \begin{array}{cc}
\delta \Sigma_{12}^{(0)}_{x,s;\vec{x}', t_p} & \delta \Sigma_{12}^{(0)}_{x,s;\vec{x}', t_p}
\end{array} \right)_{ab} ; (3.6)
\end{align*}
\]
\[
\delta \tilde{\Sigma}^{11}_{\vec{x},s;\vec{x}',s'}(t_p) = \delta \tilde{\Sigma}^{11,\uparrow}_{\vec{x},s;\vec{x}',s'}(t_p) - \delta \tilde{\Sigma}^{11,\downarrow}_{\vec{x},s;\vec{x}',s'}(t_p) ;
\]
\[
\delta \tilde{\Sigma}^{22}_{\vec{x},s;\vec{x}',s'}(t_p) = \delta \tilde{\Sigma}^{22,\uparrow}_{\vec{x},s;\vec{x}',s'}(t_p) - \delta \tilde{\Sigma}^{22,\downarrow}_{\vec{x},s;\vec{x}',s'}(t_p) ;
\]
\[
\delta \tilde{\Sigma}^{21}_{\vec{x},s;\vec{x}',s'}(t_p) = \delta \tilde{\Sigma}^{21,\uparrow}_{\vec{x},s;\vec{x}',s'}(t_p) - \delta \tilde{\Sigma}^{21,\downarrow}_{\vec{x},s;\vec{x}',s'}(t_p) .
\]

According to the nonlocal form of \(\delta \tilde{\Sigma}^{ab}_{\vec{x},s;\vec{x}',s'}(t_p)\), the dimensions of the coset decomposition increase from the \(so(4)\) generators and \(u(2)\) densities \((2.45)\) with factoring into \(SO(4) \simeq SO(4)/U(2) \otimes U(2)\) \((2.51)\) for the local case to the extended dimensions of \(so(4N_x)\) generators and \(u(2N_x)\) densities \((3.12)\) with corresponding factoring into \(SO(4N_x)/U(2N_x) \otimes U(2N_x)\) \((3.13)\). We emphasize the appearance of the total number of spatial grid points \(N_x\) \((3.20)\) which determine the number of summands involved over the summations of the spatial vector 'indices'
\[
\delta \hat{\lambda}_s(\vec{x}, t_p) = \text{diag}\left(\delta \hat{\lambda}_{s=1}(\vec{x}, t_p), \delta \hat{\lambda}_{s=1}(\vec{x}, t_p)\right); \\
\hat{Q}_{s,s'; s', t'(t_p)}^{ab} = \delta_{ab} \left(\hat{Q}_{s,s'; s', t'(t_p)}^{11} \hat{Q}_{s,s'; s', t'(t_p)}^{22}\right); \\
\hat{Q}_{s,s'; s', t'(t_p)}^{11} = \left(\exp\left\{i \hat{J}_{D; s_1; s_2}(t_p)\right\}\right)_{s,s'; s', t'}; \\
\hat{Q}_{s,s'; s', t'(t_p)}^{22} = \left(\exp\left\{-i \hat{J}_{D; s_1; s_2}(t_p)\right\}\right)_{s,s'; s', t'}; \\
\hat{J}_{D; s, s'; s, t'(t_p)} \equiv 0; \quad \hat{J}_{D; s, s'; s, t'(t_p)} = \hat{J}_{D; s, s'; s, t'(t_p)}^{11}; \\
\hat{J}_{D; s, s'; s, t'(t_p)} \triangleq u(2N_x) / \left(\prod_{(N_x)}(u(1) \otimes u(1))\right) \triangleq \hat{J}_{D; s, s'; s, t'(t_p)}^{11}(\hat{J}_{D; s, s'; s, t'(t_p)})^T, \quad (2N_x)^2 - 2N_x \text{ real parameters}.
\]

The coset matrix \(\hat{T}_{s,s'; s', t'(t_p)}^{ab}(3.27)\) of the pair condensates is specified by the spatially nonlocal generators \(\hat{Y}_{s,s'; s', t'(t_p)}^{a \neq b}(3.28)\), with complex-valued, anti-symmetric sub-generators \(\hat{X}_{s,s'; s', t'(t_p)}(3.29)\) in the off-diagonal blocks \((a \neq b)\), each having \((2N_x)^2 - 2N_x\) remaining, real parameters. We have also included the modified coset matrix \(\hat{T}_{s,s'; s', t'(t_p)}^{ab}(3.30)\) for the calculation of the invariant coset integration measure whose computation follows in an analogous manner from the local case of appendix A or can be taken from the more general case of an ortho-symplectic integration measure with super-symmetry described in detail in Ref. [8].

\[
\hat{T}_{s,s'; s', t'(t_p)}^{ab} = \left(\exp\left\{-\hat{Y}_{s,s'; s', t'(t_p)}^{a \neq b}\right\}\right)_{s,s'; s', t'}; \\
\hat{Y}_{s,s'; s', t'(t_p)}^{a \neq b} = \left(\begin{array}{cc} 0 & \hat{X}_{s,s'; s', t'(t_p)} \\ -\hat{X}_{s,s'; s', t'(t_p)}^T & 0 \end{array}\right); \\
\hat{X}_{s,s'; s', t'(t_p)} = -\hat{X}_{s,s'; s', t'(t_p)}^T; \quad \hat{X}_{s,s'; s, t'(t_p)} \equiv 0; \\
\left(\hat{T}_{0}(t_p)\right)^{ab}_{s,s'; s', t'(t_p)} = \hat{T}_{s,s'; s', t'(t_p)}^{ab} \hat{Q}_{s,s'; s', t'(t_p)}^{11}.
\]

### 3.2 HST and separation into actions of density and pair condensate terms

In correspondence to the precise, discrete time steps within (2.68), one can accomplish the HST with nonlocal self-energies where we exactly distinguish between 'equal time', anomalous doubled fields and 'equal time' hermitian conjugation \(\hat{\psi}^{s}_{s,s'; s', t'(t_p)}\) and between the hermitian conjugation \(\hat{\psi}^{s}_{s,s'; s', t'(t_p)}\) with 'time shift' correction \(\Delta t_p\) in the resulting complex part (2.13-2.14). The latter kind has to be applied for the transformation to density and pair condensate matrices following from dyadic products of anomalous doubled fields whereas the 'equal time' form has to be used for self-energies and its coset decomposition

\[
\exp\left\{-\frac{i}{\hbar} \int_{C} dt_p \sum_{\vec{x}, \vec{x}', s,s'; s', t'} \psi_{s,s'}^{s}(t_p + \Delta t_p) \psi_{s,s'}^{s}(t_p + \Delta t_p) V_{[\vec{x} - \vec{x}']} \psi_{s,s'}(t_p) \psi_{s,s'}(t_p)\right\} = \\
\int d[\sigma_{D; s,s'}^{(0)}(\vec{x}, \vec{x}', t_p)] \exp\left\{-\frac{i}{2\hbar} \int_{C} dt_p \sum_{\vec{x}, \vec{x}', s,s'; s', t'} \frac{\sigma_{D; s,s'}^{(0)}(\vec{x}, \vec{x}', t_p) \sigma_{D; s', s}^{(0)}(\vec{x}', \vec{x}, t_p)}{V_{[\vec{x} - \vec{x}']} + i \varepsilon_p}\right\}
\]
\begin{align*}
\times \int d[\delta \Sigma_{a,b}^{ab}(\vec{x},s',s',t_p)] \exp \left\{ -\frac{i}{4\hbar} \int_C dt_p \sum_{\vec{x},\vec{x}' , a,b,s,s'} \text{Tr} \left[ \delta \Sigma_{a,b}^{ab}(\vec{x},s',s',t_p) \frac{\delta \Sigma_{a,b}^{ab}(\vec{x},s',s',t_p)}{V_{|\vec{r} - \vec{x}|} + i \varepsilon_p (\delta_{a=b} - \delta_{a\neq b})} \right] \right\} \\
\times \exp \left\{ \frac{i}{2\hbar} \int_C dt_p \sum_{\vec{x},\vec{x}' , a,b,s,s'} \text{Tr} \left[ \left( \frac{\tilde{R}_{a,b}^{11}(\vec{x},\vec{x}',t_p)}{V_{|\vec{r} - \vec{x}|}} + i \varepsilon_p (\delta_{a=b} - \delta_{a\neq b}) \right) \tilde{S} \times \frac{i=1}{V_{|\vec{r} - \vec{x}|} + i \varepsilon_p (\delta_{a=b} - \delta_{a\neq b})} \right] \\
\times \left( \frac{\sigma^{(0)}_{D,s'}(\vec{x}',\vec{x},t_p) + \delta \Sigma_{a,b}^{ab}(\vec{x},\vec{x}',s',s',t_p)}{V_{|\vec{r} - \vec{x}|} + i \varepsilon_p (\delta_{a=b} - \delta_{a\neq b})} \right) \right\} ; \\
\hat{V}_{ab}^{\alpha \neq b} = \left( \begin{array}{cc} 0 & \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \\ \hat{J}_{\bar{\psi},\psi,s,s'}(t_p) & 0 \end{array} \right)^{ab} ; \\
\hat{J}_{\psi,\bar{\psi},s',s'}(t_p) = \left( \begin{array}{c} \hat{J}_{\psi,s}(\vec{x},t_p) \\ \hat{J}_{\bar{\psi},s}(\vec{x},t_p) \end{array} \right) ; \\
\hat{J}_{\bar{\psi},\psi,s,s'}(t_p) = \left( \begin{array}{c} \hat{J}_{\bar{\psi},s}(\vec{x},t_p) \\ \hat{J}_{\psi,s}(\vec{x},t_p) \end{array} \right)^T .
\end{align*}

Since the precise, anomalous doubling of one-particle parts \( (2.69, 2.75) \) is not affected by the extension to nonlocal self-energies, we have only to generalize the source matrix of pair condensates to its nonlocal kind \( \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \) with anti-symmetric sub-matrices \( \hat{J}_{\psi,\bar{\psi},s,s'}(t_p) \), \( \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \) whereas the anti-commuting source field \( \hat{J}_{\bar{\psi},\psi,s,s'}(t_p) \) exactly remains for a coherent, macroscopic wavefunction in its manner

\begin{align*}
\hat{J}_{\psi,\bar{\psi},s',s'}(t_p) &= \left( \begin{array}{c} \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \\ \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \end{array} \right) ; \\
\hat{J}_{\psi,\bar{\psi},s,s'}(t_p) &= -\frac{i}{\hbar} \tilde{\Sigma}_{D,s,s'}^{(0)}(\vec{x},\vec{x}',t_p) ; \\
\hat{J}_{\psi,\bar{\psi},s,s'}(t_p) &= \left( \begin{array}{c} \hat{J}_{\psi,s}(\vec{x},t_p) \\ \hat{J}_{\psi,s}(\vec{x},t_p) \end{array} \right)^T .
\end{align*}

After collection of terms with nonlocal self-energies, one finally achieves relation \( (3.35) \) with bilinear, anomalous doubled fields \( \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(\vec{r},t_p) \), \( \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(\vec{r},t_p) \) with exact discrete time steps and overall matrix \( \hat{\Sigma}_{D,s,s'}^{(0)}(t_q',t_p) \) \( (3.36) \) which is only modified by the nonlocal self-energies \( \delta \Sigma_{D,s,s'}^{(0)}(t_q',t_p) \) and source matrix \( \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \) in comparison to the local case \( (2.70, 2.83) \)

\begin{align*}
Z[\hat{\mathcal{G}}, \hat{J}, \hat{\psi}] &= \int d[\sigma^{(0)}_{D,s,s'}(\vec{x},\vec{x}',t_p)] \exp \left\{ -\frac{i}{2\hbar} \int_C dt_p \sum_{\vec{x},\vec{x}',a,s,s',t_p} \frac{\sigma^{(0)}_{D,s,s'}(\vec{x},\vec{x}',t_p)}{V_{|\vec{r} - \vec{x}|} + i \varepsilon_p (\delta_{a=b} - \delta_{a\neq b})} \right\} \left( \begin{array}{c} \hat{J}_{\psi,\bar{\psi},s',s'}(t_p) \\ \hat{J}_{\psi,\bar{\psi},s,s'}(t_p) \end{array} \right) ; \\
\times \int d[\delta \Sigma_{D,s,s'}^{ab}(t_p)] \exp \left\{ -\frac{i}{4\hbar} \int_C dt_p \sum_{\vec{x},\vec{x}',a,b,s,s'} \text{Tr} \left[ \delta \Sigma_{D,s,s'}^{ab}(t_p) \frac{\delta \Sigma_{D,s,s'}^{ab}(t_p)}{V_{|\vec{r} - \vec{x}|} + i \varepsilon_p (\delta_{a=b} - \delta_{a\neq b})} \right] \right\} ; \\
\times \int d[\hat{\Psi}_{\psi,\bar{\psi}}^{ab}(t_p), \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(t_p)] \exp \left\{ -\frac{i}{2\hbar} \int_C dt_p dt_q \sum_{\vec{x},\vec{x}',s,s',|s,s'|,a,b=1,2} \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(t_p) \tilde{\Sigma}_{\psi,\bar{\psi}}^{ab}(t_q',t_p) \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(t_p) \right\} ; \\
\times \exp \left\{ -\frac{i}{2\hbar} \int_C dt_p \sum_{\vec{x}} \sum_{s=1,1} \left( J_{\psi,s}(\vec{x},t_p) \tilde{S} \hat{\Psi}_{\psi,s}(t_p) + \hat{\Psi}_{\psi,s}(t_p) \tilde{S} J_{\psi,s}(\vec{x},t_p) \right) \right\} ; \\
\times \left| \Delta t_q \right| \tilde{\mathcal{N}}_{\psi,\bar{\psi}}^{ab}(t_q',t_p) = \left( \begin{array}{c} \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(t_q',t_p) \\ \hat{\Psi}_{\psi,\bar{\psi}}^{ab}(t_q',t_p) \end{array} \right) + \eta_q \tilde{S} \left| \Delta t_q \right| \tilde{\mathcal{N}}_{\psi,\bar{\psi}}^{ab}(t_q',t_p) \tilde{S} \tilde{S} \tilde{S} \eta_q .
\end{align*}
HST and separation into actions of density and pair condensate terms

\[ \sigma_{D;ss'}(\bar{x}', \bar{x}, t_p) \delta_{ba} - \frac{1}{N_x} \int_{\psi_{\bar{x},\bar{x}',s,s'}(t_p)} \sigma_{D;ss'}(\bar{x}', \bar{x}, t_p) \] \[ = \sigma_{D;ss'}(\bar{x}', \bar{x}, t_p); \quad \sigma_{D;ss'}(\bar{x}', \bar{x}, t_p) = \sigma_{D;ss'}(\bar{x}', \bar{x}, t_p); \] \[ \hat{I}_{1 \times 4} = \int \left\{ \frac{i}{2} \Omega^2 \int dt_p dt'_q \sum_{x,x'} N_x \sum_{s,s'=\uparrow, \downarrow} J_{\psi,s'}(x', t_q) \hat{I}_{1 \times 4} \right\} \] \[ = \hat{S}_{1 \times 4}. \]
\[ \times J_{\Psi;32}^{+,b}(\vec{x}', t'_q) \hat{I} \left( \hat{T}_{\vec{x}_a;\vec{x}_d;\vec{x}_s;\vec{x}_t}(t'_q) \hat{\Omega}_{\vec{x}_a;\vec{x}_s;\vec{x}_t}(t'_q, t_p) \hat{T}_{\vec{x}_1;\vec{x}_3;\vec{x}_4}(t_p) \right)^{ba}_{\vec{x},s',s} \hat{I} \hat{J}_{\Psi}^{a}(\vec{x}, t_p) ; \]

\[ \hat{\Omega}_{\vec{x}_a;\vec{x}_s;\vec{x}_t}(t_p, t'_q) = \left( \frac{1}{|\Delta t'_q|} \right) \left[ (\hat{\mathcal{H}} + \delta^0_D) + (\hat{T}^{-1}(t_p)(\hat{\mathcal{H}} + \delta^0_D) \hat{T}(t'_q) - (\hat{\mathcal{H}} + \delta^0_D) \hat{T}(t'_q) \right]^{ab}_{\vec{x},s',s} + \]

\[ + \hat{T}^{-1,a\alpha}_{\vec{x},s;\vec{x}_1;\vec{x}_s}(t_p) \hat{I} \hat{S} \eta_p \frac{\Delta t'_q}{N_a} \eta_q \hat{S} \hat{T}_{\vec{x}_a;\vec{x}_d;\vec{x}_s;\vec{x}_t}(t'_q) \right]^{ab}_{\vec{x},s',s'} . \]

In analogy to relations in section 2.3, the pair condensate ‘seed’ action term \( \exp \{ \imath A_{\Psi;\Psi}[^\hat{T}] \} \) is given by Eqs. (3.45) to (3.49) and can be computed by application of Vandermonde determinants for the integration measure of the eigenvalues \( \delta \lambda_\alpha(\vec{x}, t_p) \) with the orthogonal properties of Hermite polynomials and corresponding Gaussian weights

\[ \exp \{ \imath A_{\Psi;\Psi}[^\hat{T}] \} = \int d[\delta \Sigma_{D;\vec{x},s;\vec{x}_t}(\vec{x}, t_p)] \mathcal{P}(\delta \lambda(\vec{x}, t_p)) \exp \{ \imath A_{\Sigma}[^\hat{T}, \delta \Sigma_{D}; \imath \hat{J}_{\Psi}] \} = \quad (3.45) \]

\[ = \int d[\delta \Sigma_{D;\vec{x},s;\vec{x}_t}(\vec{x}, t_p)] \mathcal{P}(\delta \lambda(\vec{x}, t_p)) \exp \{ \imath A_{\Sigma}[^\hat{T}, \delta \Sigma_{D}; \imath \hat{J}_{\Psi}] \} ; \]

\[ \text{det} \left\{ \delta \Sigma_{D;\vec{x},s;\vec{x}_t}(\vec{x}, t_p) - \delta \lambda \delta \Sigma_{\vec{x},\vec{x}_t} \delta s_\alpha \right\} = 0 ; \quad \text{det} \left\{ \delta \Sigma_{D;\vec{x},s;\vec{x}_t}(\vec{x}, t_p) - \delta \Sigma_{\vec{x},\vec{x}_t} \delta s_\alpha \right\} = 0 ; \quad (3.46) \]

\[ \delta \Sigma_{D;\vec{x},s;\vec{x}_t}(\vec{x}, t_p) = -\delta \Sigma_{D;\vec{x},s;\vec{x}_t}(\vec{x}, t_p) ; \quad (3.47) \]

\[ \imath A_{\Sigma}[^\hat{T}, \delta \Sigma_{D}; \imath \hat{J}_{\Psi}] = \quad (3.48) \]

\[ = -\frac{\imath}{4\hbar} \int_C dt_p \sum_{\vec{x},\vec{x}_t} \text{Tr} \left[ \frac{\delta \Sigma_{ab;\vec{x},s;\vec{x}_t}(t_p) + \imath \hat{J}_{\Psi}^{a\neq b}[\psi;\vec{x}_s;\vec{x}_t;\vec{x}_s](t_p)}{V[\vec{x}_t - \vec{x}]} \right] + \imath \varepsilon_p (\delta_{a=b} - \delta_{a\neq b}) \]

\[ + \left\{ \text{Tr} \left[ \frac{\hat{\mathcal{J}}_{ab}^{a\neq b}[\psi;\vec{x}_s;\vec{x}_t;\vec{x}_s](t_p)}{V[\vec{x}_t - \vec{x}]} \right] + \imath \varepsilon_p (\delta_{a=b} - \delta_{a\neq b}) \right\} ; \]

\[ \imath A_{\Sigma}[^\hat{T}, \delta \Sigma_{D}; \imath \hat{J}_{\Psi}] = -\frac{\imath}{4\hbar} \int_C dt_p \sum_{\vec{x},\vec{x}_t} \times \]

\[ \left\{ \text{Tr} \left[ \frac{\hat{\mathcal{J}}_{ab}^{a\neq b}[\psi;\vec{x}_s;\vec{x}_t;\vec{x}_s](t_p)}{V[\vec{x}_t - \vec{x}]} \right] + \imath \varepsilon_p (\delta_{a=b} - \delta_{a\neq b}) \right\} ; \quad (3.49) \]

\[ \times \left\{ \text{Tr} \left[ \frac{\hat{\mathcal{J}}_{ab}^{a\neq b}[\psi;\vec{x}_s;\vec{x}_t;\vec{x}_s](t_p)}{V[\vec{x}_t - \vec{x}]} \right] + \imath \varepsilon_p (\delta_{a=b} - \delta_{a\neq b}) \right\} ; \]

\[ \text{Tr} \left[ \frac{\hat{\mathcal{J}}_{ab}^{a\neq b}[\psi;\vec{x}_s;\vec{x}_t;\vec{x}_s](t_p)}{V[\vec{x}_t - \vec{x}]} \right] + \imath \varepsilon_p (\delta_{a=b} - \delta_{a\neq b}) \right\} ; \quad (3.49) \]
The nonlocal self-energy reaction procedure is obvious. The confinement potential, inserted into the classical Lagrangian and action, should incorporate a confining potential embedded in a higher dimensional, Euclidean base space where the quantization problem of the mentioned ambiguous transitions from the classical to a quantum system. Since we only use the discrete time step development of the quantized Hamilton operator which already has the transformation to curvilinear coordinates, and spaces with nontrivial curvature which keep this property under integration fields in a curved manifold of spacetime. This turns out in particular because the path integral resembles the discrete time step development of the quantized Hamilton operator which already has the problem of the mentioned ambiguous transitions from the classical to a quantum system. Since we only use Euclidean, spatial base manifolds, we extend the invariant length \((ds_{SO(4)})^2\) of internal degrees of freedom to the invariant length \((dS)^2\) of coordinates which introduces the additional summation over the space 'sum of' with \(|x| < L\). Therefore, we have to include two independent, spatial summations in the case of the nonlocal self-energy

\[
\sum_{x} |L| \delta_{ab} \langle \psi_{x,s}(t_p), \psi_{x,s}(t_p) \rangle = \sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]

\[
\sum_{x} \sum_{s=1,1} d\psi_{x,s}^* \psi_{x,s}(t_p) ;
\]
then cause the restriction to the physical sub-space with the required curvature even after a quantization. The wavefunctions of the quantized system may extend above the entire, higher dimensional, Euclidean space, however, the confining potential has to be chosen in such a manner that the wavefunctions are squeezed to the lower dimensional, physical sub-space with the nontrivial curvature. This seems to be very feasible for a constant, static curvature by static confinement potentials; but, one has to find the appropriate dynamics of the confining potential in the nonstatic case, e. g. if one requires the confinement potential to restrict the higher dimensional, Euclidean system to the dynamics of the Einstein field equations in lower sub-dimensions in a classical limit. We can simply introduce a confinement potential for the \( S^1 \)-(\( S^2 \))-sphere embedded in a two-(three-) dimensional Euclidean space in order to obtain quantization into unique coherent state path integrals with the appropriate definition of the invariant lengths given in Eqs. \((4.1-4.3)\). In this case the definition \((4.1-4.3)\) is not affected for deriving the integration measure because the nontrivial curvature of the \( S^1 \)-(\( S^2 \))-sphere only originates from the confinement potential which restricts the two-(three-) dimensional Euclidean space in order to obtain quantization into unique coherent state path integrals with the appropriate definition of the invariant lengths given in Eqs. \((4.1-4.3)\). In this case the definition \((4.1-4.3)\) is not affected for deriving the integration measure because the nontrivial curvature of the \( S^1 \)-(\( S^2 \))-sphere only originates from the confinement potential which restricts the two-(three-) dimensional Euclidean space to the physically relevant, lower dimensional spheres.

Since we have only considered Euclidean coordinate systems in this paper, one can easily transform to curvilinear coordinates from the invariant lengths \((4.1-4.3)\). In order to preserve identical scaling and renormalization procedures, one has to require that the identical number of independent integration variables at the \( N_x \) \((1.23)\) spatial grid points of spheres with radial length \( L \) stays invariant under transformation to the \( D = 2 \) or \( D = 3 \) spherical coordinates. We distinguish the discrete coordinates of vectors \( \vec{x}_{ij} \) and \( \vec{x}_{ijk} \) by the indices \( 'i, j' \) and \( 'i, j, k' \), respectively

\[
 N_x = \frac{\Omega_D}{D} \left( \frac{L}{\Delta x} \right)^D; \quad \Omega_{D=2} = 2\pi; \quad \Omega_{D=3} = 4\pi; \quad (4.4)
\]

\( D=2 \) case: \( \vec{x}_{ij} = \{ x_i^{(1)}, x_j^{(2)} \} = \{ i \cdot \Delta x, j \cdot \Delta x \}; \) \( (4.5) \)

\[
 r_{ij} = \sqrt{(x_i^{(1)})^2 + (x_j^{(2)})^2} < L; \quad \tan \varphi_{ij} = \frac{x_j^{(2)}}{x_i^{(1)}}; \quad (4.6)
\]

\( D=3 \) case: \( \vec{x}_{ijk} = \{ x_i^{(1)}, x_j^{(2)}, x_k^{(3)} \} = \{ i \cdot \Delta x, j \cdot \Delta x, k \cdot \Delta x \}; \) \( (4.7) \)

\[
 r_{ijk} = \sqrt{(x_i^{(1)})^2 + (x_j^{(2)})^2 + (x_k^{(3)})^2} < L; \quad \tan \varphi_{ijk} = \frac{x_j^{(2)}}{x_i^{(1)}}; \quad \cos \theta_{ijk} = \frac{x_k^{(3)}}{r_{ijk}}; \quad (4.8)
\]

\[
 N_x = N_r \cdot N_{\Omega_D}; \quad N_r = \frac{L}{\Delta x}; \quad N_{\Omega_D} = \frac{\Omega_D}{D} \left( \frac{L}{\Delta x} \right)^{D-1}. \quad (4.9)
\]

If one chooses in the radial directions the same discrete interval as in the \( D = 2 \) and \( D = 3 \) Cartesian systems, one remains with \( N_{\Omega_D} \) grid points for the spherical degrees of freedom, although other decompositions of the total number \( N_x \) of grid points may also be meaningful. Therefore, we introduce following transformations to spherical coordinates in \( D = 2 \) \((4.10-4.13)\) and \( D = 3 \) \((4.14-4.17)\) for anti-commuting fields and for local, nonlocal self-energies, respectively, where the number of independent spherical integration variables is adapted to \( N_{\Omega_D} \) with \( N_r = L/\Delta x \) radial points of identical, discrete intervals \( \Delta x \) as in the Cartesian systems \( (\vartheta(x) := \text{Heaviside step function with } \vartheta(x) = 1 \text{ for } x \geq 0 \text{ and } \vartheta(x) = 0 \text{ for } x < 0) \)

\[
 d\psi_{\vec{x}_{ij},s}(t_p) = \sum_{n_r=0}^{N_r} \sum_{m=-m_0}^{+m_0} d\psi_{n_r,m,s}(t_p) \exp \left\{ i \varphi_{ij,m} \right\} \times \sqrt{2\pi} \times \left[ \vartheta(\sqrt{i^2 + j^2} - (n_r - \frac{1}{2})) - \vartheta(\sqrt{i^2 + j^2} - (n_r + \frac{1}{2})) \right]; \quad (4.10)
\]
After we have substituted the transformed variables into the defining invariant lengths \(4.1-4.3\) and have reordered summations, in particular above \(i, j\) in \(D = 2\) and \(i, j, k\) in \(D = 3\), one eventually achieves following metric tensors \([4.21, 4.25]\) in the transformed, spherical integration variables:

\[
\left( dS(\psi_{ij}^s, (t_p)) \right)^2 = 2 \sum_{n_r=0}^{N_r} \sum_{m=m=-m_0}^{+m_0} d\psi_{n_r, m, s}(t_p) g_{n_r^*, m^*, s}^{(D=2)} d\psi_{n_r, m, s}(t_p); \]

\[
\left( dS(\delta\Sigma_{ss'}(\vec{x}, t_p)) \right)^2 = 2 \sum_{n_r, n_r'} \sum_{m, m'=-m_0}^{+m_0} \text{Tr} \left[ d(\delta\Sigma_{ss'}(n_r, m', t_p)) d(\delta\Sigma_{ss'}(n_r, m, t_p)) \right] g_{n_r^*, m^*, s}^{(D=2)} g_{n_r', m', s}^{(D=2)} g_{n_r', m', s}^{(D=2)}; \]

\[
\left( dS(\delta\Sigma_{ss'}(\vec{x}, t_p)) \right)^2 = 2 \sum_{n_r, n_r'} \sum_{m, m'=-m_0}^{+m_0} \sum_{n_r, n_r'} \sum_{m, m'=-m_0}^{+m_0} g_{n_r^*, m^*, s}^{(D=2)} g_{n_r', m', s}^{(D=2)} g_{n_r', m', s}^{(D=2)} \]
\[
\frac{1}{2\pi} \sum_{i,j} \exp \{ \varphi_{ij} (m - m') \} \times \\
\left[ \theta \left( \sqrt{i^2 + j^2} - \left( n_r - \frac{1}{2} \right) \right) - \theta \left( \sqrt{i'^2 + j'^2} - \left( n_r' - \frac{1}{2} \right) \right) \right] \times \\
\left[ \theta \left( \sqrt{i'^2 + j'^2} - \left( n_r' - \frac{1}{2} \right) \right) - \theta \left( \sqrt{i^2 + j^2} - \left( n_r + \frac{1}{2} \right) \right) \right] \\
\left( dS \left( \psi_{\bar{s},s}(tp), \psi_{\bar{s},s}(tp) \right) \right) \quad \text{(4.23)}
\]

The Jacobian for the change to spherical coordinates is given by the (inverse) square roots of the determinant of the metric tensor. The nonlocal self-energies are weighted by the square of the integration measure compared to the local self-energies because of the square of independent spatial integration variables.

4.2 Generic use of the described coherent state path integral with precise, discrete time steps

In this paper we have mainly described technical details of coherent state path integrals; however, it has to be pointed out that the necessary limits and time step correction in the resulting complex fields are usually omitted for brevity. Some references 11, 13 even state insurmountable problems with coherent state path integrals, due to ambiguous choices of discrete time steps; however, as we have already mentioned in sections 11 and 21, \ifnum\count0=0\textit{\else with nontrivial SSB, one should not be guided by the appealing property for hermitian actions in coherent state path integrals because this implies simultaneous action of field operators with their hermitian conjugated operators at the same spacetime point which involves infinities from the (anti-)commutators. Therefore, the}\fi
chosen normal ordering in this paper is particularly suited for the coherent state path integrals and causes the hermitian conjugated field operators to transform to complex conjugated fields \( \hat{\psi}^\dagger_{x,s}(t_p + \Delta t_p) \) acting a discrete time step \( \Delta t_p \) later than the corresponding fields \( \hat{\psi}_{x,s}(t_p) \) which are located on the right-hand side of the hermitian conjugated operators. However, since we aim to achieve a coset decomposition of self-energies into block diagonal density and off-diagonal pair condensate parts, one has to take the 'equal time' hermitian and transposition operations for the self-energy which follow from the symmetries of the 'equal time' dyadic products of anomalous doubled, anti-commuting fields. In comparison to the 'lax', sloppy, hermitian form of coherent state path integrals, the essential changes only result from the originally defined (2.13), 'time shifted' \( \Delta t_p \), anomalous doubled field \( \hat{\Psi}^a_{x,s}(t_p) \) with the hermitian conjugation \( \hat{\Psi}^\dagger_{x,s}(t_p) \) and time shift correction in the resulting complex part. Straightforward use of these time shift corrections \( \Delta t_p \) only causes a few amendments so that the abridged 'lax' hermitian forms can be conveyed to the precise, discrete steps of the time development. Therefore, one can directly read from previous articles \[4, 7, 8, 9, 11, 14\] the exact, precise non-hermitian forms with the correct, discrete time steps.

The precise, discrete time steps of this paper can also be transferred to the case of disordered systems with an ensemble average of a random potential \[9\]. Further applications of coherent state path integrals are possible for quantum systems with fixed particle- and symmetry quantum-numbers where we take the trace with an ensemble average of a random potential \[9\]. Further applications of coherent state path integrals are possible for quantum systems with fixed particle- and symmetry quantum-numbers where we take the trace with coherent states over delta functions of second quantized operators which determine the maximal commuting set of symmetry transformations

\[
\Phi(E, l, s; n_0, l_z, s_z) = \text{Coherent state path integral of the trace of} \tag{4.26}
\]

\[
= \text{Tr} \left[ \delta \left( \hat{h} s_z - \hat{S}_z (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( \hat{h} l_z - \hat{L}_z (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( n_0 - \hat{N} (\hat{\psi}^\dagger, \hat{\psi}) \right) \times \\
\times \delta \left( \hat{h}^2 s(s + 1) - \hat{S} (\hat{\psi}^\dagger, \hat{\psi}) \cdot \hat{S} (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( \hat{h}^2 l(l + 1) - \hat{L} (\hat{\psi}^\dagger, \hat{\psi}) \cdot \hat{L} (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( E - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z) \right) \right] ;
\]

\[
\Phi(E_1, E_2, l, s; n_0, l_z, s_z) = \text{disordered system with ensemble average for} \ B_z \tag{4.27}
\]

\[
= \text{Tr} \left[ \delta \left( \hat{h} s_z - \hat{S}_z (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( \hat{h} l_z - \hat{L}_z (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( n_0 - \hat{N} (\hat{\psi}^\dagger, \hat{\psi}) \right) \times \\
\times \delta \left( \hat{h}^2 s(s + 1) - \hat{S} (\hat{\psi}^\dagger, \hat{\psi}) \cdot \hat{S} (\hat{\psi}^\dagger, \hat{\psi}) \right) \delta \left( \hat{h}^2 l(l + 1) - \hat{L} (\hat{\psi}^\dagger, \hat{\psi}) \cdot \hat{L} (\hat{\psi}^\dagger, \hat{\psi}) \right) \times \\
\times \delta \left( E_2 - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z) \right) \delta \left( E_1 - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z) \right) \right].
\]

The delta functions have standard representations with the Dirac identity and integrations over exponentials where one has to take into account the described, precise steps of the time development in this article

\[
\lim \frac{1}{E - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z) + \pm \varepsilon_+} = \text{Principal value} \frac{\pm \pi}{E - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z)} ; \tag{4.28}
\]

\[
\delta (E - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z)) = \lim_{|\varepsilon_p E| \to 0} \lim_{T(E) \to +\infty} \sum_{p_\pm = \pm} \int_0^{T(E)} \frac{d\rho_p(E)}{2\pi \hbar} \times \\
\times \exp \left\{ - \eta \rho_p \frac{\rho_p(E)}{\hbar} \left( E - \hat{H} (\hat{\psi}^\dagger, \hat{\psi}; B_z) - \varepsilon_p \right) \right\} . \tag{4.29}
\]

However, as we have verified in previous sections, one has only to amend a few changes in order to accomplish the exact time development for the case of fixed particle- and symmetry quantum numbers. This allows to
derive nonlinear sigma models from coherent state path integrals (even for the relativistic case) with the given description of the HST and coset decomposition for constrained cases of second quantized operators.

A Jacobian and integration measure \( \text{SO}(4) / \text{U}(2) \otimes \text{U}(2) \)

In this appendix we outline a few details for deriving the integration measure of a coset decomposition of \( \text{so}(4) \) generators and corresponding fields. The reader is referred for more details to Ref. 8, 15, 16 where the more general case of an ortho-symplectic super-group is investigated and the super-symmetric integration measure is attained from the square root of the super-determinant of the \( \text{Osp}(S,S|2L) / \text{U}(L|S) \otimes \text{U}(L|S) \) metric tensor \( S=\text{even integer}, \ L=\text{odd integer}, \) both related to angular momentum degrees of freedom of fermions and bosons, respectively.

We begin with the ‘flat’, Euclidean form of the self-energy of \( \text{so}(4) \) generators and their fields and define an invariant length \( (ds_{\text{SO}(4)})^2 \) of internal degrees of freedom which is accomplished by taking the trace over spin space and the anomalous doubling of fields

\[
(d_{\text{SO}(4)})^2 = - \text{Tr}_{a,s,b,s'} [d(\delta \Sigma) d(\delta \Sigma)] = - \text{Tr} [d(\hat{T} \delta \hat{\Sigma}^a_{B} \hat{T}^{-1}) d(\hat{T} \delta \hat{\Sigma}^b_{B} \hat{T}^{-1})]
\]

\( (A.1) \)

Using the definition (A.2) and the relation (A.3) for the self-energy, we transform the coset decomposition \( \text{SO}(4) / \text{U}(2) \otimes \text{U}(2) \), already inserted into (A.1), to the traces of the eigenvalues, to the traces of block diagonal '11', '22' density parts as \( \text{u}(2) \) sub-algebra elements and to the traces '12', '21' of coset generators and pair condensate terms. Note that the coset part of the complete \( \text{SO}(4) \) integrations also contains products of the eigenvalues which are finally to be shifted to the \( \text{u}(2) \) density part of the integration measure

\[
(d_{\text{SO}(4)})^2 = - \text{Tr}_{a,s,b,s'} \left[ (\hat{T}^{-1}_0 d\hat{T}_0, \delta \hat{\Lambda}) + d(\delta \hat{\Lambda}) \right]^2 = -2 \text{Tr}_{s,s'=1,1} [d(\delta \lambda_s) d(\delta \lambda_s)] + \]

\( (A.4) \)

The infinitesimal variation \( \hat{T}^{-1} d\hat{T} \) of coset matrices consists of off-diagonal \( \text{so}(4) / \text{u}(2) \) coset generators for pair condensates and block diagonal \( \text{u}(2) \) sub-algebra generators for the density part of the self-energy; however, the latter \( \text{u}(2) \) density and sub-algebra part within \( \hat{T}^{-1} d\hat{T} \) can be absorbed into \( \hat{Q}^{-1} d\hat{Q} \) of \( \text{u}(2) \) sub-algebra generators with the corresponding, independent density field variables of the self-energy

\[
\hat{T}^{-1}_0 d\hat{T}_0 = \hat{Q} \left( \hat{T}^{-1} d\hat{T} - \hat{Q}^{-1} d\hat{Q} \right) \hat{Q}^{-1}.
\] \( (A.5) \)
Therefore, we can reduce the block diagonal density summation with '11' (or '22') of \((\hat{T}_0^{-1} d\hat{T}_0)_{s s'}^{11}\) to \((\hat{Q}^{-1} d\hat{Q})_{s s'}^{11}\) within following relation

\[
(ds_{SO(4)})^2 = -2 \begin{array}{c}
\text{tr} \left[ d(\delta \lambda_0) (d(\delta \lambda_0)] + 2 \begin{array}{c}
\text{tr} \left[ (\hat{Q}^{-1} d\hat{Q})_{s s'}^{11} (\hat{Q}^{-1} d\hat{Q})_{s s'}^{11} (\delta \lambda_{s'} - \delta \lambda_0)^2 \right] \\
+ 2 \begin{array}{c}
\text{tr} \left[ (\hat{T}^{-1} d\hat{T})_{s s'}^{12} (\hat{T}^{-1} d\hat{T})_{s s'}^{21} (\delta \lambda_{s'} + \delta \lambda_0)^2 \right].
\end{array}
\end{array}
\end{array}
\]

As one applies the introduced, independent parameter fields of diagonalizing matrices \(\hat{Q}_{s s'}^{11}(\vec{x}, t_p)\) (A.7 A.10) and coset matrices \(\hat{T}(\vec{x}, t_p)\) (A.11 A.12) (compare sections 2), one can eventually compute the invariant length \((ds_{SO(4)})^2\) in terms of density fields and pair condensate terms of the so(4) self-energy

\[
\begin{align*}
\hat{Q}_{s s'}^{11}(\vec{x}, t_p) &= \begin{pmatrix} 
\cos \left(|\hat{F}_{11}(\vec{x}, t_p)|\right) & \frac{1}{2} \sin \left(|\hat{F}_{11}(\vec{x}, t_p)|\right) F_{s s'}^{11}(\vec{x}, t_p) \\
\frac{1}{2} \sin \left(|\hat{F}_{11}(\vec{x}, t_p)|\right) & \cos \left(|\hat{F}_{11}(\vec{x}, t_p)|\right)
\end{pmatrix} ; \\
\hat{T}(\vec{x}, t_p) &= \begin{pmatrix} 
\hat{1}_{s s'} & \cosh \left(|f(\vec{x}, t_p)|\right) \\
-\sinh \left(|f(\vec{x}, t_p)|\right) & \hat{1}_{s s'}
\end{pmatrix} ; \\
\hat{T}(\vec{x}, t_p) &= \begin{pmatrix} 
\hat{1}_{s s'} & -e^{i \phi_{\vec{x}, t_p}}/F_{s s'}^{ab}(\vec{x}, t_p) \\
e^{-i \phi_{\vec{x}, t_p}}/F_{s s'}^{ab}(\vec{x}, t_p) & \hat{1}_{s s'}
\end{pmatrix} ;
\end{align*}
\]

\[
(d\Sigma_{SO(4)})^2 = -2 \left[ (d(\delta \lambda_1))^2 + (d(\delta \lambda_2))^2 \right] - 4 \left[ (d(\phi))^2 + (d(\phi))^2 \sin^2 (|\phi|) \cos^2 (|\phi|) \right] \delta \lambda_1 + \delta \lambda_2 + 8 \left[ (d(\phi))^2 + (d(\phi))^2 \sin^2 (|\phi|) \cos^2 (|\phi|) \right] \delta \lambda_1 + \delta \lambda_2^2 .
\]

We shift the eigenvalue self-energy densities in terms of a polynomial \(\mathcal{P}(\hat{\delta}{\lambda}(\vec{x}, t_p))\) (A.14) from the coset and pair condensate part to the density part \(d[\delta \Sigma_D(\vec{x}, t_p)]\), resulting from the U(2) sub-group integration measure, and achieve the invariant integration measure of densities (A.15) and pair condensates (A.16) by performing the square root of the determinant of the metric tensors within \((ds_{SO(4)})^2\) (A.13)

\[
\mathcal{P}(\hat{\delta}{\lambda}(\vec{x}, t_p)) = \prod_{\{\vec{x}, t_p\}} |\delta \lambda_1(\vec{x}, t_p) + \delta \lambda_1(\vec{x}, t_p)|^2 ; \\
d[\delta \Sigma_D(\vec{x}, t_p)] \mathcal{P}(\hat{\delta}{\lambda}(\vec{x}, t_p)) = d[\hat{Q}(\vec{x}, t_p) \hat{Q}^{-1}(\vec{x}, t_p); \hat{\delta}{\lambda}(\vec{x}, t_p)] \mathcal{P}(\hat{\delta}{\lambda}(\vec{x}, t_p)) = \prod_{\{\vec{x}, t_p\}} \left[ 8 d(\delta \lambda_1(\vec{x}, t_p)) d(\delta \lambda_1(\vec{x}, t_p)) |\delta \lambda_1^2(\vec{x}; t_p) - \delta \lambda_1^2(\vec{x}; t_p)|^2 \times \right]
\]

\[
(\hat{T}_0^{-1} d\hat{T}_0)_{s s'}^{11} \mathcal{P}(\hat{\delta}{\lambda}(\vec{x}, t_p)) = d[\hat{Q}(\vec{x}, t_p) \hat{Q}^{-1}(\vec{x}, t_p); \hat{\delta}{\lambda}(\vec{x}, t_p)] \mathcal{P}(\hat{\delta}{\lambda}(\vec{x}, t_p)) = \prod_{\{\vec{x}, t_p\}} \left[ 8 d(\delta \lambda_1(\vec{x}, t_p)) d(\delta \lambda_1(\vec{x}, t_p)) |\delta \lambda_1^2(\vec{x}; t_p) - \delta \lambda_1^2(\vec{x}; t_p)|^2 \times \right] 
\]
\[
\times \ d\left(|\mathcal{F}(\vec{x}, t_p)|\right) \ \sin \left(|\mathcal{F}(\vec{x}, t_p)|\right) \ \cos \left(|\mathcal{F}(\vec{x}, t_p)|\right) \ d\varphi(\vec{x}, t_p) \right) ;
\]

\[
d\left[\hat{T}^{-1}(\vec{x}, t_p) \ d\hat{T}(\vec{x}, t_p)\right] = \prod_{\{\vec{x}, t_p\}} \left\{ 8 \ d\left(|f(\vec{x}, t_p)|\right) \ \sin \left(|f(\vec{x}, t_p)|\right) \ \cosh \left(|f(\vec{x}, t_p)|\right) \ d\phi(\vec{x}, t_p) \right\}. \tag{A.16}
\]

Furthermore, we remind that the source term \(\exp\{i \mathcal{A}_{j_{\psi}} [\hat{T}]\} \tag{2.108}\) contains the correction \(\cosh^{-3}(2|f(\vec{x}, t_p)|)\) for the integration measure \(\text{(A.16)}\) so that the total, complete integration measure of pair condensates is in fact given by

\[
d\left[\hat{T}^{-1}(\vec{x}, t_p) \ d\hat{T}(\vec{x}, t_p)\right] \times \prod_{\{\vec{x}, t_p\}} \left( \frac{1}{\cosh^3(2|f(\vec{x}, t_p)|)} \right) = \tag{A.17}
\]

\[
= \prod_{\{\vec{x}, t_p\}} \left\{ 8 \ d\left(|f(\vec{x}, t_p)|\right) \ \sin \left(|f(\vec{x}, t_p)|\right) \ \cosh \left(|f(\vec{x}, t_p)|\right) \ d\phi(\vec{x}, t_p) \right\} \times \prod_{\{\vec{x}, t_p\}} \left( \frac{1}{\cosh^3(2|f(\vec{x}, t_p)|)} \right)
\]

\[
= \prod_{\{\vec{x}, t_p\}} \left\{ 4 \ d\left(|f(\vec{x}, t_p)|\right) \ \frac{\sinh(2|f(\vec{x}, t_p)|)}{\cosh(2|f(\vec{x}, t_p)|)} \ d\phi(\vec{x}, t_p) \right\} = \prod_{\{\vec{x}, t_p\}} \left\{ -d\left( \cosh^{-2} \left(2|f(\vec{x}, t_p)|\right) \right) \ d\phi(\vec{x}, t_p) \right\}
\]

\[
= \prod_{\{\vec{x}, t_p\}} \left[ \tanh^2(2|f(\vec{x}, t_p)|) - 1 \right] \ d\left[\phi(\vec{x}, t_p)\right] = \prod_{\{\vec{x}, t_p\}} \left[ \tanh^2(2|f(\vec{x}, t_p)|) \right] \ d\left[\phi(\vec{x}, t_p)\right].
\]

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