Gelfand–Dickey hierarchy, generalized BGW tau-function, and $W$-constraints

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Abstract

Let $r \geq 2$ be an integer. The generalized Brezin–Gross–Witten (BGW) tau-function for the Gelfand–Dickey hierarchy of $(r - 1)$ dependent variables (aka the $r$-reduced Kadomtsev–Petviashvili hierarchy) is defined as a particular tau-function that depends on $(r - 1)$ constant parameters $d_1, \ldots, d_{r-1}$. In this paper we show that this tau-function satisfies a family of linear equations, called the $W$-constraints of the second kind. The operators giving rise to the linear equations also depend on $(r - 1)$ constant parameters. We show that there is a one-to-one correspondence between the two sets of parameters.

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1. Introduction

Let $r \geq 2$ be an integer, and let $n = r - 1$. The Gelfand–Dickey (GD) hierarchy with $n$ unknown functions is an infinite family of partial differential equations (PDEs), defined by

$$\frac{\partial L}{\partial t_i} = \left( L^{i/(r)} \right)_+, \quad i \in \mathbb{N}\setminus r\mathbb{N},$$

(1.1)

where

$$L := \partial^r + v_1 \partial^{-2} + \cdots + v_{r-1}$$

(1.2)

is the Lax operator, $L^{i/r}, i \in \mathbb{N}\setminus r\mathbb{N}$, denote the fractional powers of $L$ (see e.g. [14] for the definition), and $\partial$ is understood as $\partial_t$. This integrable hierarchy can also be viewed as a
reduction of the Kadomtsev–Petviashvili (KP) hierarchy (see e.g. [14] or section 2). There are many interesting solutions to the GD hierarchy. For example, in the study of Witten’s $r$-spin invariants [20, 23, 26, 35], the so-called topological solution [17, 19] to the GD hierarchy plays an important role. For example, for the case $r = 2$, the GD hierarchy is the celebrated Korteweg–de Vries (KdV) hierarchy, and the topological solution is famously known as the Witten–Kontsevich solution (see e.g. [18, 36]), governing the integrals of psi-classes over the moduli space of curves [28, 34]. The interest of this paper is on another solution to the GD hierarchy again for any $r$. Unlike the topological solution, the solution of interest of this paper will depend non-trivially on $r - 1$ arbitrary parameters. Again, let us look at the KdV case first (i.e. the case with $r = 2$). For this case, it is known that there exists a solution to the KdV hierarchy, called the generalized BGW solution, depending non-trivially on one arbitrary parameter [3, 18], having bispectral properties [18, 19], and possessing enumerative meanings [27, 32, 38]. This motivates us to generalize the generalized BGW solution to an arbitrary $r \geq 2$. Indeed, let $d_1, \ldots, d_n$ be arbitrarily given complex numbers, and define $v_{BGW}(t)$ as the unique solution in $\mathbb{C}[t]^{n}$ to the GD hierarchy, satisfying the initial condition
\[ v_{\alpha,BGW}(1, t_{2} = 0) = \frac{d_{\alpha}}{(1 - t_{1})^{\alpha + 1}}, \quad \alpha = 1, \ldots, r - 1, \]
where $t = (t_{i})_{i \in \mathbb{N} \setminus \{0\}}$. We call $v_{BGW}(t; d_{1}, \ldots, d_{n})$ the generalized BGW solution to the GD hierarchy. The Dubrovin–Zhang type tau-function of this solution (see [7, 9, 14, 18, 36]) will be called the generalized BGW tau-function, denoted by $\tau_{BGW} = \tau_{BGW}(t; d_{1}, \ldots, d_{n})$. We show in section 3 that the generalized BGW tau-function $\tau_{BGW}$ can be chosen such that
\[ \sum_{i \in \mathbb{N} \setminus \{0\}} \hat{h}_{i} \frac{\partial \tau_{BGW}}{\partial t_{i}} + \frac{d_{1}}{r} \tau_{BGW} = 0, \]
where $\hat{h}_{i} = t_{i} - \delta_{i,1}$, moreover, it is unique up to multiplying by a nonzero constant.

The above definition of the generalized BGW tau-function for the GD hierarchy was given in joint work by Dubrovin, Zagier and the first-named author of the present paper in a more general set up, i.e. for the Drinfeld–Sokolov (DS) hierarchy of $g$-type with $g$ any simple Lie algebra (a reduction to the so-called topological ODEs of the second kind is also obtained there for any $g$, and certain analogues of the triangle numbers on the constants manifold were observed there for the case when $g = A_{2}$). Here we remind the reader that the DS hierarchy of $g$-type means the DS hierarchy [16] associated to the untwisted affine Kac–Moody algebra $g^{(1)}$ with a particular choice of vertex in the Dynkin diagram [6, 7]. The GD hierarchy can be considered as the DS hierarchy associated to the $A_{n}$ simple Lie algebra under the Wronskian gauge. In [29], certain generalized BGW tau-functions were also given for the DS hierarchy associated to an affine Kac–Moody algebra.

For $r = 2$, the generalized BGW tau-function can also be identified with the solution to Virasoro constraints [3, 8, 10]. The goal of this paper is to show that for an arbitrary $r \geq 2$, the generalized BGW tau-function (defined above) satisfies a set of linear constraints, which will be called W-constraints of the second kind. To be precise, define a family of operators $W_{\alpha,q}^{red}$, $\alpha = 1, \ldots, n = r - 1, q > 0$, by
\[ W_{\alpha,q}^{red}(t) := \text{res}_{\lambda} \lambda^{\alpha+(q-\alpha)r} \left( \partial_{\mu}^{\alpha+1} (X_{GD}(t; \lambda, \mu)) \right) \bigg|_{\mu = \lambda} d\lambda, \]
where the residue is taken at $\lambda = \infty$, and $X_{GD}(t; \lambda, \mu)$ is given by
\[ X_{GD}(t; \lambda, \mu) := e^{\sum_{i \in \mathbb{N} \setminus \{0\}} t_{i} (\mu^{i} - \lambda^{i})} \circ e^{\sum_{i \in \mathbb{N} \setminus \{0\}} \left( \frac{1}{i} - \frac{1}{\mu^{i}} \right) X_{red}. \]
These operators were given e.g. in [2]; according to [2, 4, 22], they can be expressed by operators coming from the twisted module of the $W_A^q$-algebra [5] (cf [21, 23, 30]). We have the following theorem.

**Theorem 1.1.** There exist unique constants $\rho_1, \ldots, \rho_{r-1} \in \mathbb{C}$ such that

$$W_{\alpha, q}^{red}(\tau_{BGW}) = (-1)^{\alpha} \rho_\alpha \delta_{\alpha,q} \tau_{BGW}, \quad \alpha = 1, \ldots, r-1, \ q \geq \alpha. \quad (1.7)$$

Moreover, these constants $\rho_\alpha$ are polynomials of $d_1, \ldots, d_n$, having the form

$$\rho_\alpha = \frac{d_\alpha}{r} + \omega_\alpha(d_1, \ldots, d_{\alpha-1}). \quad (1.8)$$

We refer to equation (1.7) as $W$-constraints of the second kind. We note that the $W$-constraints for the topological tau-function [2, 4, 5, 10, 24, 42], being referred to as $W$-constraints of the first kind, start with $q = 0$ instead of $q = \alpha$ and have the dilaton shift at $t_{r-1}$ instead of at $t_1$.

The $W$-constraints of the second kind seem to have common cases with the $W$-constraints given in [10] in an equivalent way. For certain special common cases, the solutions to the $W$-constraints of [10] were conjectured by Chidambaram, Garcia–Failde and Giacchetto in a recent letter to the authors of the present paper to be tau-functions for the GD hierarchy. Theorem 1.1 (cf also theorem 4.5) should lead to this conjecture; still, it will be interesting to investigate the explicit relationship between $\tau_{BGW}$ and the partition functions defined in [10].

In a subsequent publication, we will consider the analogous open extension of the generalized BGW tau-function for arbitrary $r \geq 2$ (see [41] for the $r = 2$ case; see also [9, 12]).

1.1. Organization of the paper

In section 2 we review some basics on KP and GD hierarchies. In section 3 we give the definition of the generalized BGW tau-function $\tau_{BGW}$ in more details. In section 4 we prove theorem 1.1. In section 5 we present some examples.

2. Preliminaries

In this section we review tau-functions and wave functions for the KP hierarchy and for the GD hierarchy.

Let $L_{KP}$ denote the pseudo-differential operator

$$L_{KP} := \partial + \sum_{k \geq 1} u_k \partial^{-k}. \quad (2.1)$$

Here $\partial := \partial_x$. Recall that the KP hierarchy [14] is the following commuting system of PDEs for the infinitely many dependent variables $u_1(t_{KP}), u_2(t_{KP}), \ldots$:

$$\frac{\partial L_{KP}}{\partial t_i} = \left[ (L_{KP})_+, L_{KP} \right], \quad i \geq 1. \quad (2.2)$$

Here $t_{KP} := (t_1, t_2, t_3, \ldots)$ denotes the infinite vector of times. The first equation in equation (2.2) reads

$$\frac{\partial u_k}{\partial t_1} = \frac{\partial u_k}{\partial x}, \quad k \geq 1.$$

Therefore we identify the time $t_1$ with $x$. We consider solutions to the KP hierarchy in $\mathbb{C}[t_{KP}]^N$, i.e. $u_k(t_{KP}) \in \mathbb{C}[t_{KP}], \ k \geq 1$. Denote for simplicity $u := (u_1, u_2, \cdots)$. It is known (see for
example [14]) that for an arbitrary power series solution \( u(t_{KP}) = (u_1(t_{KP}), u_2(t_{KP}), \ldots) \) to the KP hierarchy, there exists a pseudo-differential operator

\[
\Phi(t_{KP}) = 1 + \sum_{k \geq 1} \phi_k(t_{KP}) \partial^{-k}, \quad \phi_k(t_{KP}) \in \mathbb{C}[t_{KP}],
\]

(2.3)
called a dressing operator, satisfying

\[
L_{KP} = \Phi \circ \partial \circ \Phi^{-1},
\]

(2.4)
\[
\frac{\partial \Phi}{\partial t_i} = -(L_{KP}^i)_+ \circ \Phi, \quad i \geq 1.
\]

(2.5)
The dressing operator \( \Phi \) is uniquely determined by the solution \( u \) up to the right multiplication by an operator of the form

\[
1 + \sum_{k \geq 1} a_k \partial^{-k} \in \mathbb{C}[[\partial^{-1}]],
\]

where \( a_k, k \geq 1 \) are constants. The wave and dual wave functions \( \psi(t_{KP}; \lambda), \psi^*(t_{KP}; \lambda) \) associated to the solution \( u(t_{KP}) \) are elements in \( \mathbb{C}([\lambda^{-1}])[[t_{KP}]] \) defined by

\[
\psi(t_{KP}; \lambda) := \Phi(t_{KP}; \lambda) \left( e^{\xi(t_{KP}; \lambda)} \right), \quad \psi^*(t_{KP}; \lambda) = (\Phi^*(t_{KP}; \lambda))^{-1} \left( e^{-\xi(t_{KP}; \lambda)} \right),
\]

(2.6)
where \( \xi(t_{KP}; \lambda) := \sum_{i \geq 1} \lambda^i t_i \), and \( \Phi^* \) denotes the formal adjoint operator of \( \Phi \), i.e.

\[
\Phi^* := 1 + \sum_{k \geq 1} (-\partial)^{-k} \circ \phi_k.
\]

(2.7)
They satisfy

\[
L_{KP}(\psi) = \lambda \psi, \quad \frac{\partial \psi}{\partial t_i} = (L_{KP}^i)_+ (\psi),
\]

\[
L_{KP}^*(\psi^*) = \lambda \psi^*, \quad \frac{\partial \psi^*}{\partial t_i} = -\left( (L_{KP}^i)_+^\ast \right)_+ (\psi^*)
\]

with \( L_{KP} := -\partial + \sum_{k \geq 1} (-\partial)^k \circ u_k \). Introduce the following two operators:

\[
X(t_{KP}; \lambda) = e^{\sum_{i \geq 1} t_i \lambda^i} e^{-\sum_{i \geq 1} t_i \lambda^i}, \quad X^*(t_{KP}; \lambda) = e^{-\sum_{i \geq 1} t_i \lambda^i} e^{\sum_{i \geq 1} t_i \lambda^i}.
\]

(2.8)
It was proved in [14] that for an arbitrary solution \( u \) in \( \mathbb{C}[[t_{KP}]]^N \) to the KP hierarchy, there exists a power series \( \tau_{KP}(t_{KP}) \in \mathbb{C}[t_{KP}] \) with \( \tau_{KP}(0) \neq 0 \), satisfying

\[
\psi(t_{KP}; \lambda) = X(t_{KP}; \lambda) (\tau_{KP}(t_{KP})) \psi^*(t_{KP}; \lambda) = X^*(t_{KP}; \lambda) (\tau_{KP}(t_{KP})) \]

(2.9)
We call \( \tau_{KP}(t_{KP}) \) the tau-function of the solution \( u \) for the KP hierarchy. We also call \( (\Phi, \tau_{KP}) \) a dressing pair associated to \( u \). The dressing pair is uniquely determined by the solution \( u \) up to the transformation

\[
(\Phi, \tau_{KP}) \mapsto (\Phi \circ e^{-\sum_{i \geq 1} \frac{\theta_i}{\tau_{KP}}} \tau_{KP}^b e^{\sum_{i \geq 1} b_i \Omega_{t_{KP}}^i}, \quad b_0, b_1, b_2, \ldots \in \mathbb{C}).
\]

Denote by \( A_u := A_{u,0} \left[ \partial^k (u_k); i, k \geq 1 \right] \) the ring of differential polynomials of \( u \), where \( A_{u,0} \) denotes the ring of holomorphic functions of \( u \) on an open set of \( \mathbb{C} \). For a pseudo-differential operator of the form \( a = \sum_{i \geq 1} a_i \partial^i \), define \( \text{res} \partial a = a_{-1} \). Define a family of differential polynomials in \( u \) by [14]

\[
\Omega_{ij}^{t_{KP}} = \Omega_{ij}^{t_{KP}}(u, u_s, \ldots) := \partial^{-1} \left( \frac{\partial}{\partial t_j} \text{res} \partial L_{KP} \right) \in A_u, \quad i, j \geq 1.
\]

(2.10)
Let \( \tau_{\text{KP}} \) be the tau-function of the KP hierarchy. Then the following formulae hold true:

\[
\frac{\partial^2 \log \tau_{\text{KP}}}{\partial t_i \partial t_j} = \Omega_{i,j}^{\text{KP}}, \quad \forall i,j \geq 1.
\]  

(2.11)

**Proof.** Let \((\Phi, \tau_{\text{KP}})\) be the dressing pair associated to \(u\), and let \(\psi\) be the corresponding wave function. It was shown in \([14]\) that for given \(i,j \geq 1\),

\[
\frac{\partial^2 \log \tau_{\text{KP}}}{\partial t_i \partial t_j} = \text{res}_{z^0} \left( -z^{-1} \sum_{\ell \geq 1} \partial_{t_\ell} + \frac{\partial}{\partial z} \right) \left( -\frac{(L^j_{\text{KP}})_+ (\psi)}{\psi} \right).
\]  

(2.12)

Note that \((L^j_{\text{KP}})_-\) can be rewritten into the form

\[
(L^j_{\text{KP}})_- = \sum_{k \geq 1} d_{j,k} L^{-k}_{\text{KP}},
\]  

(2.13)

where \(d_{j,k} \in \mathcal{A}_u\) satisfy \(a_{j,k}|_{u=a_{\lambda} \cdots a_{\mu}} = 0 \). Combining equation (2.12) with equation (2.13) we find that

\[
\frac{\partial^2 \log \tau_{\text{KP}}}{\partial t_i \partial t_j} = ia_{j,1} + \sum_{k=1}^{i-1} \partial_{t_{i-k}}.
\]

In particular, observing that \(a_{j,1} = \text{res} \partial L^1_{\text{KP}}\), we have

\[
\frac{\partial^2 \log \tau}{\partial t_i \partial t_j} = \text{res} \partial L^1_{\text{KP}}.
\]

Taking the derivative with respect to \(t_i\) on the both sides of the above identity, and then by using the definition (2.10), the lemma is proved.

Introduce the following operator:

\[
X(t_{\text{KP}}; \lambda, \mu) := e^{\sum_{i \geq 1} t_i (\mu - \lambda)} \circ e^{\sum_{i \geq 1} \left( \frac{1}{i} \right) \frac{\partial}{\partial t_i}} \circ \Omega_{\text{KP}}.
\]  

(2.14)

It follows from formulae (1.8.2), (1.8.4) of \([13]\) that

\[
\text{res}_\nu X(t_{\text{KP}}; \nu) \circ X(t_{\text{KP}}; \lambda, \mu) (\tau_{\text{KP}}(t_{\text{KP}})) X^*(t_{\text{KP}}; \nu) (\tau_{\text{KP}}(t_{\text{KP}}')) \frac{d\nu}{\tau_{\text{KP}}(t_{\text{KP}}')} = (\lambda - \mu) \psi(t_{\text{KP}}; \mu) \psi^*(t_{\text{KP}}'; \lambda),
\]  

(2.15)

where \(t_{\text{KP}} = (t_1, t_2, \ldots)\) and \(t_{\text{KP}}' = (t'_1, t'_2, \ldots)\).

Following \([1]\) and \([33]\), introduce the following operator:

\[
M := \Phi \circ \left( \sum_{i \geq 1} i t_i \partial^{i-1} \right) \circ \Phi^{-1} \in \mathbb{C}((\partial^{-1}))[[t_{\text{KP}}]].
\]  

(2.16)

By using formula (1.10) of \([2]\) (cf also formula (2.12) of \([24]\) and lemma 6.2.5 in \([14]\)), we have

\[
\text{res} \partial M^k \circ L^1_{\text{KP}} = \text{res} \lambda^k \psi^*(t_{\text{KP}}; \lambda) \partial^k \psi(t_{\text{KP}}; \lambda), \quad \forall i, k \geq 0.
\]  

(2.17)
It follows from equations (2.15) and (2.17) that
\[
\text{res}_\partial M^\prime \circ L_{\text{KP}}^\prime = \frac{1}{i+1} \partial \left( \frac{\text{res}_\lambda \lambda^i \partial^{i+1} \circ X(t_{\text{KP}}; \lambda, \mu) (\tau_{\text{KP}}(t_{\text{KP}})) |_{\mu=\lambda}}{\tau_{\text{KP}}(t_{\text{KP}})} \right). \tag{2.18}
\]

Denote by \( t = (t_i)_{i \in \mathbb{N}, r \in \mathbb{N}} \) the infinite time vector for the GD hierarchy. For an arbitrary solution \( v(t) = (v_1(t), \ldots, v_{r-1}(t)) \) in \( \mathbb{C}[[t]]^{r-1} \) to the GD hierarchy, we associate to it an infinite sequence of power series in \( \mathbb{C}[[t]] \) defined by
\[
u_k = u_k(t) := \text{res}_\partial \left( L^{1/r} \circ \partial^{k-1} \right), \quad k \geq 1.
\]
In other words,
\[
\partial + \sum_{k \geq 1} u_k \partial^{-k} = L^{1/r}. \tag{2.19}
\]
Obviously, \( u = (u_1, u_2, \ldots) \) satisfies the KP hierarchy, namely, for all \( i \in \mathbb{N} \),
\[
\frac{\partial L^{1/r}}{\partial t_i} = \left( \left( L^{1/r} \right)_i + L^{1/r} \right).
\tag{2.20}
\]
Let \( \tau_{\text{KP}} \) be the tau-function of the solution \( u \) to the KP hierarchy. By the definition (2.10), we know \( \Omega^{\text{KP}}_{i,j} = 0 \) for \( i, j \geq 1 \) when \( L_{\text{KP}} = L^{1/r} \). It then follows from lemma 2.1 that \( \tau_{\text{KP}} \) satisfies
\[
\frac{\partial^2 \log \tau_{\text{KP}}}{\partial t_i \partial t_j} = 0, \quad i, j \geq 1.
\]
This means that, there exist constants \( a_1, a_2, \ldots \), such that
\[
\frac{\partial}{\partial t_i} \left( \sum_{j \geq 1} a_j t_j + \log \tau_{\text{KP}} \right) = 0, \quad i \geq 1.
\]
Let \( \tau = \tau(t) := \tau_{\text{KP}}(t_{\text{KP}}) \exp \left( \sum_{j \geq 1} a_j t_j \right) \). Then \( \tau \) is still a KP tau function. We call this particular chosen \( \tau \) the tau-function of the solution \( v \) for the GD hierarchy reduced from the KP hierarchy.

Let us proceed to give a second definition of tau-function for the GD hierarchy.

**Lemma 2.2.** For an arbitrary solution \( v = v(t) \) in \( \mathbb{C}[[t]]^{r-1} \) to the GD hierarchy (1.1), there exists a power series \( \tau_{\text{DZ}} \in \mathbb{C}[[t]] \) satisfying
\[
\frac{\partial^2 \log \tau_{\text{DZ}}}{\partial t_i \partial t_j} = \Omega_{i,j}^{\text{GD}}, \quad \forall i, j \in \mathbb{N} \setminus r\mathbb{N}, \tag{2.21}
\]
where \( \Omega_{i,j}^{\text{GD}} \) are differential polynomials in \( v \) defined as
\[
\Omega_{i,j}^{\text{GD}} = \Omega_{i,j}^{\text{GD}}(v, v_1, \ldots) := \partial^{-1} \left( \text{res}_\partial L^{1/r} \right). \tag{2.22}
\]
Here \( \partial^{-1} \) is again fixed by the no-integration-constant rule. We call \( \tau_{\text{DZ}} \) the Dubrovin–Zhang type tau-function of the solution \( v \) to the GD hierarchy.

Note that \( \tau_{\text{DZ}} \) is uniquely determined by \( v \) up to multiplying by a factor the form
\[
\exp \left( b_0 + \sum_{i \in \mathbb{N} \setminus r\mathbb{N}} b_i t_i \right), \tag{2.23}
\]
where \( b \)'s are arbitrary constants.
Theorem 2.3. Let \( v \) be an arbitrary solution in \( C[[t]]^{r-1} \) to the GD hierarchy, and \( \tau_{\text{DZ}} \) and \( \tau \) be the Dubrovin–Zhang type tau-function and the tau-function reduced from the KP hierarchy of \( v \), respectively. Then there exist constants \( b_0, b_1, b_2, \ldots \) such that

\[
\tau = \tau_{\text{DZ}} \exp \left( b_0 + \sum_{i \in \mathbb{N} \setminus r\mathbb{N}} b_i t_i \right).
\]

(2.24)

Proof. It suffices to show \( \tau \) satisfies equation (2.21). Let \( u \) be the solution to the KP hierarchy determined by \( v \) via equation (2.19). By the definition of \( \Omega_{i,j}^{\text{KP}} \), we have

\[
\Omega_{i,j}^{\text{KP}}(u, u_\times, \ldots) = \Omega_{i,j}^{\text{GD}}(v, v_\times, \ldots), \quad \forall i, j \in \mathbb{N} \setminus r\mathbb{N}.
\]

On the other hand, by using lemma 2.1 and using the definition of \( \tau \), we know \( \tau \) satisfies

\[
\partial^2 \log \tau / \partial t_i \partial t_j = \Omega_{i,j}^{\text{KP}}(u, u_\times, \ldots), \quad \forall i, j \in \mathbb{N} \setminus r\mathbb{N},
\]

where \( \Omega_{i,j}^{\text{KP}} \) is given by equation (2.10).

3. The generalized BGW tau-function for the GD hierarchy

In this section we give more details about the definition of the generalized BGW tau-function.

Introduce a gradation on \( \mathcal{A}_v \) by assigning the degree:

\[
\deg \partial^k (v_\alpha) = \alpha + 1 + k, \quad \alpha = 1, \ldots, r_1, \quad k \geq 0.
\]

It is easy to verify that

\[
\deg \text{res}_\partial \left( L^{i-j} \partial^{-k} \right) = i - k, \quad i \geq 1, \quad k \leq i - 2.
\]

(3.1)

This implies the GD hierarchy (1.1) has the form

\[
\frac{\partial v_\alpha}{\partial t_i} = X'_\alpha(v, v_\times, \ldots), \quad \alpha = 1, \ldots, r-1, \quad i \in \mathbb{N} \setminus r\mathbb{N},
\]

with \( X'_\alpha = X'_\alpha(v, v_\times, \ldots) \in \mathcal{A}_v \), having the degree

\[
\deg X'_\alpha = \alpha + i + 1.
\]

(3.2)

Proposition 3.1. The generalized BGW solution \( v_{\text{BGW}} \) satisfies the following linear equations:

\[
\sum_{i \in \mathbb{N} \setminus r\mathbb{N}} h_i \frac{\partial v_{\alpha, \text{BGW}}}{\partial t_i} + (\alpha + 1) v_{\alpha, \text{BGW}} = 0.
\]

(3.3)

Proof. For simplicity, we denote \( v_\alpha = v_{\alpha, \text{BGW}} \), and denote

\[
f_\alpha(t) := \sum_{i \in \mathbb{N} \setminus r\mathbb{N}} h_i \frac{\partial v_\alpha(t)}{\partial t_i} + (\alpha + 1) v_\alpha(t).
\]

(3.4)

We are to show \( f_\alpha = 0 \). Firstly, from the initial condition (1.3), it is easy to see that

\[
f_\alpha(t_1 = x, t_2 = 0, \ldots) = (x - 1) \partial_x (v_\alpha(x, 0)) + (\alpha + 1) v_\alpha(x, 0) = 0.
\]

(3.5)

Taking the derivative of equation (3.4) with respect to \( t_j \), we have
The generalized BGW tau-function

By using equations (3.1), one can verify that

\[
\deg \Omega_{j_1,j_2}^{GD} = j_1 + j_2, \quad j_1, j_2 \in \mathbb{N} \setminus \mathbb{N}.
\]

Then it follows from proposition 3.1 that

\[
\sum_{i \in \mathbb{N} \setminus \mathbb{N}} \hat{u}_i \frac{\partial \Omega_{j_1,j_2}^{GD}}{\partial t_i} = \sum_{i \in \mathbb{N} \setminus \mathbb{N}} \hat{u}_i \sum_{\alpha=1 k \geq 1}^{r-1} \partial_\alpha \left( v^{(k)}_\beta \right) \frac{\partial \Omega_{j_1,j_2}^{GD}}{\partial v^{(k)}_\alpha} = - \sum_{\alpha=1 k \geq 1}^{r-1} (\alpha + k + 1) v^{(k)}_\alpha \frac{\partial \Omega_{j_1,j_2}^{GD}}{\partial v^{(k)}_\alpha} = -(j_1 + j_2) \Omega_{j_1,j_2}^{GD}.
\]

Therefore, by using equation (2.21), we have

\[
\frac{\partial^2}{\partial t_{j_1} \partial t_{j_2}} \left( \sum_{i \in \mathbb{N} \setminus \mathbb{N}} \hat{u}_i \frac{\partial \log \tau_{BGW}}{\partial t_i} \right) = 0, \quad j_1, j_2 \in \mathbb{N} \setminus \mathbb{N}.
\]
Hence there exist constants $a_0$ and $a_i, i \in \mathbb{N}\setminus\mathbb{N}$, such that
\[
\sum_{i \in \mathbb{N}\setminus\mathbb{N}} a_i \frac{\partial \log \tau_{BGW}}{\partial t_i} = \sum_{i \in \mathbb{N}\setminus\mathbb{N}} a_i t_i + a_0.
\]
Let us modify $\tau_{BGW}$ as $\tau_{BGW} \exp \left( - \sum_{i \in \mathbb{N}\setminus\mathbb{N}} a_i t_i \right)$. Then $\tau_{BGW}$ is still a Dubrovin–Zhang type tau-function, and satisfies
\[
\sum_{i \in \mathbb{N}\setminus\mathbb{N}} a_i \frac{\partial \log \tau_{BGW}}{\partial t_i} - a_0 = 0.
\]
It remains to show $a_0 = -d_1/r$. The above equation (3.6) implies
\[
\frac{\partial^2 \log \tau_{BGW}}{\partial t_i^2} = \frac{\partial \log \tau_{BGW}}{\partial t_i} = 0.
\]
Hence by using equations (2.21), (2.22) and the initial condition (1.3), we have
\[
\frac{\partial^2 \log \tau_{BGW}}{\partial t_i^2} (0) = \Omega_{1,1}^{GD} |_{t=0} = \frac{v_1(0)}{r} = \frac{d_1}{r}.
\]
The theorem is proved. \qed

**Remark 3.3.** We note that, as we mentioned in section 1, the generalized BGW tau-function satisfying equation (1.4) is unique up to multiplying by a nonzero constant, which can be deduced with the help of lemma 2.2 and the equation (1.4). We also note that the parameters $C_1, \ldots, C_n$ in the generalized BGW tau-function are nontrivial; the case with $n = 1$ can be seen explicitly in e.g. [18] (see also e.g. [41] for a similar phenomenon in a certain extension).

4. $W$-constraints of the second kind

In this section, we show that the generalized BGW tau-function $\tau_{BGW}$ for the GD hierarchy satisfies the $W$-constraints of the second kind. Let $v_{BGW}$ be the generalized BGW solution to the GD hierarchy, and $u_{BGW}$ the corresponding solution to the KP hierarchy (cf equation (2.19)). By theorem 2.3 we know that $\tau_{BGW}$ can be regarded as a tau-function of $u_{BGW}$ to the KP hierarchy. Let
\[
L_{KP} = \partial + u_{1,BGW} \partial^{-1} + u_{2,BGW} \partial^{-2} + \cdots
\]
be the Lax operator for $u_{BGW}$, and let $\Phi$ be the dressing operator for $u_{BGW}$ such that $\Phi$ and $\tau_{BGW}$ form a dressing pair. The identities in the following lemma are analogues of equation (2.18).

**Lemma 4.1.** The following identities hold true: $\forall i, k \geq 0$,
\[
\text{res}_\partial (M - 1)^i \circ L^k = \frac{1}{i+1} \partial \left( \text{res}_\lambda \lambda^k \partial_{\lambda}^{i+1} \circ X(\mathbf{i}_{KP}; \lambda, \mu) \big|_{\mu=\lambda} \frac{\tau_{BGW}}{\tau_{BGW}} \right),
\]
where the operator $M$ is given by equation (2.16), and $\mathbf{i}_{KP} = (t_1 - 1, t_2, t_3, \ldots)$.

**Proof.** Recalling the definition (2.14), we have
\[
\partial_{\lambda}^{i+1} \circ X(\mathbf{i}_{KP}; \lambda, \mu) = \sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^{i+1-j} \partial_{\lambda}^j \circ X(\mathbf{i}_{KP}; \lambda, \mu).
\]
By using equation (2.18), one can then write the right-hand side of equation (4.1) into

\[
\frac{1}{i+1} \sum_{j=1}^{i+1} \binom{i+1}{j} (-1)^{i+1-j} M^{i-j} \circ L_{KP}^k = \text{res}_\delta (M - 1)^i \circ L_{KP}^k. \tag{4.3}
\]

By noticing that \( L_{KP} = L^1 \), the lemma is proved. \( \square \)

**Lemma 4.2.** The following identities hold true:

\[
\left( (M - 1)^i \circ L^{k+\frac{1}{i}} \right)_- = 0, \quad \forall i, k \geq 0.
\tag{4.4}
\]

**Proof.** The case \( i = 0 \) is obvious. For \( i \geq 1 \), let us first show \( \left( (M - 1)L^{1/r} \right)_- = 0 \). Observe that

\[
\sum_{j \geq 1} \int_{\mathbb{R}} \frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_j} + \frac{d_1}{r} = \frac{\lambda}{2} \partial^2 X(t_{KP}; \lambda, \mu) - \lambda \partial_x X(t_{KP}; \lambda, \mu) + \frac{d_1}{\lambda r}
\]

\[
=: G(t_{KP}).
\]

By identity (1.4), we have \( G(t_{KP})(\tau_{BGW}) = 0 \). Therefore,

\[
0 = \int_{\mathbb{R}} \frac{X(t_{KP}; \nu) \circ G(t_{KP})(\tau_{BGW}(t_{KP}))}{\tau_{BGW}(t_{KP})} \frac{X^*(t_{KP}'; \nu) (\tau_{BGW}(t_{KP}'))}{\tau_{BGW}(t_{KP})} \, d\nu
\]

\[
= -\text{res}_\lambda (\lambda \partial_x (\psi(t_{KP}; \lambda)) \psi^*(t_{KP}'; \lambda) - \lambda \psi(t_{KP}; \lambda) \psi^*(t_{KP}'; \lambda)) \, d\lambda. \tag{4.6}
\]

Here the second equality used the identity (2.15). By using the facts

\[
M(\psi) = \partial_x (\psi), \quad L_{KP}(\psi) = \lambda \psi,
\]

as well as the following identity [13]: for arbitrary \( x, x' \) and arbitrary \( t_2 = t_2', t_3 = t_3', \ldots \),

\[
(U(t_{KP})V(t_{KP}''))_-(\delta(x - x'))
\]

\[
= -\text{res}_\lambda U(t_{KP}) \left( e^{iz + \sum_{j \geq 1} C_j t_j} \right) V^*(t_{KP}') \left( e^{-iz' - \sum_{j \geq 1} C_j t_j} \right) \, dz \, H(x - x'). \tag{4.7}
\]

with \( U(t_{KP}), V(t_{KP}') \) being arbitrary pseudo-differential operators whose coefficients are power series of their arguments, we have

\[
0 = -\text{res}_\lambda (\lambda \partial_x (\psi(t_{KP}; \lambda)) \psi^*(t_{KP}'; \lambda) - \lambda \psi(t_{KP}; \lambda) \psi^*(t_{KP}'; \lambda)) \, d\lambda \, H(x - x')
\]

\[
= (M \circ L_{KP} - L_{KP})_-(\delta(x - x'))
\]

\[
= \sum_{i \geq 0} \text{Coef}_{\delta x}(M \circ L_{KP} - L_{KP}) \frac{(x - x')^{i-1}}{(i-1)!} H(x - x'). \tag{4.8}
\]

Here \( \delta(x) \) is the Dirac delta function, and \( H(x) \) is the Heaviside unit step function. Therefore, \( \left( (M - 1) \circ L_{KP} \right)_- = 0 \).

By using the fact that

\[
(M - 1)^{i+1} \circ L^{k+\frac{1}{i}} = M^i \circ L^{k+\frac{1}{i}} \circ (M - 1) \circ L \circ (kr + i) \circ (M - 1)^i \circ L^{k+\frac{1}{i}}, \quad \forall i, k \geq 0,
\]

we can prove by induction that identities (4.4) are true. \( \square \)
From lemmas 4.1 and 4.2, it follows that, for $\alpha = 1, 2, \ldots, r - 1$ and $q \geq \alpha$,
\[
\partial_t \left( \frac{\text{res}_{\lambda} \lambda^{\alpha+q-\alpha-r} \left( \partial_{\mu}^{\alpha+1} \circ X(t_{\lambda}; \lambda, \mu) \right) (\tau_{BGW}(t)) \bigg|_{\mu=\lambda} }{\tau_{BGW}(t)} \right) = 0. \quad (4.9)
\]

We have that
\[
\partial_{\mu}^{\alpha+1} \circ X_{GD}(t; \lambda, \mu) \circ e^{\sum_{j \geq 1} \frac{t}{j} \left( \frac{x_j}{\tau} \right)} = \partial_{\mu}^{\alpha+1} \circ X(t_{\lambda}; \lambda, \mu) = \sum_{k_1=0}^{\alpha+1} \left( \begin{array}{c} \alpha+1 \\ k_1 \end{array} \right) \partial_{\mu}^{k_1} \left( e^{-\sum_{j \geq 1} \frac{t}{j} (\mu^j - x_j^j)} \right) \partial_{\mu}^{\alpha+1-k_1} \circ X(t_{\lambda}; \lambda, \mu) = \sum_{k_1=0}^{\alpha+1} \sum_{p \geq 1} f_{k_1, p} r_{\mu}^p \partial_{\mu}^{\alpha+1-k_1} \circ X(t_{\lambda}; \lambda, \mu),
\]
where $f_{k_1, p} = f_{k_1, p}(t_1, t_2, \ldots) \in \mathbb{C}[t_1, t_2, \ldots]$. By using this identity, one can obtain that
\[
\frac{1}{\tau_{BGW}(t)} \text{res}_{\lambda} \lambda^{\alpha+q-\alpha-r} \left( \partial_{\mu}^{\alpha+1} \circ X_{GD}(t; \lambda, \mu) \left( \tau_{BGW}(t) \right) \right) \bigg|_{\mu=\lambda} \frac{d\lambda}{\tau_{BGW}(t)} = \sum_{j=0}^{\alpha+1} \sum_{p \geq 1} f_{j, p} \text{res}_{\lambda} \lambda^{-j+q-\alpha+p} \left( \partial_{\mu}^{\alpha+1-j} \circ X(t_{\lambda}; \lambda, \mu) \left( \tau_{BGW}(t) \right) \right) \bigg|_{\mu=\lambda} \frac{d\lambda}{\tau_{BGW}(t)}.
\]
Together with equation (4.9), it follows that
\[
\partial_t \left( \frac{\text{red}_{\alpha,q}(t) (\tau_{BGW}(t))}{\tau_{BGW}(t)} \right) = 0, \quad \alpha = 1, \ldots, r - 1, \quad q \geq \alpha, \quad (4.10)
\]
where the operators $\text{red}_{\alpha,q}(t)$ are defined in equation (1.5).

**Proof of theorem 1.1.** From formula (4.10), we know that there exist power series $c_{\alpha,q}(t)$, independent of $x = t_1$, such that
\[
\text{red}_{\alpha,q}(t) (\tau_{BGW}(t)) = c_{\alpha,q}(t) \tau_{BGW}(t). \quad (4.11)
\]
By using theorem 3.2 and the fact that
\[
\text{red}_{\alpha,q}(t) = \sum_{i \in \mathbb{N} \cap \mathbb{N}} \frac{i!}{\partial^{i}},
\]
we have $\rho_1 = d_1 / r$. By the definition (1.5), one can verify that
\[
\left[ \text{red}_{1,1}(t), \text{red}_{\alpha,q}(t) \right] = -(q - \alpha) r \text{red}_{\alpha,q}(t).
\]
Applying the both sides of this identity onto $\tau_{BGW}(t)$, we obtain that
\[
\text{red}_{1,1}(t) (c_{\alpha,q}(t)) = -(q - \alpha) r c_{\alpha,q}(t).
\]
This implies that \( c_{\alpha,q}(t) \) are power series with non-positive degrees if we let the degree of \( t_i \) be assigned with \( i, i \in \mathbb{Z}_{\geq 2} \backslash \{0\} \). So \( c_{\alpha,q}(t) \) must be constants, moreover, these constants vanish if \( q > \alpha \).

Let us proceed to prove the property (1.8). To this end, we first prove the following lemma.

**Lemma 4.3.** Denote

\[
S_{\alpha,q} := \frac{1}{\alpha + 1} \text{res} \lambda^{\alpha + (q-\alpha)r} \left( \sum_{j \in \mathbb{N} \cup \{0\}} j \lambda^{r_j} + \sum_{j \in \mathbb{N} \cup \{0\}} \lambda^{-j-1} \frac{\partial}{\partial \lambda} \right)^{\alpha + 1} : \lambda, \quad (4.12)
\]

where \( \alpha = 1, \ldots, n, \ q \geq 0, \) and “; ;” denotes the normal ordering (defined by putting the operators \( \frac{\partial}{\partial \lambda} \) on the right of operators \( t_i \)). The constraints (1.7) can be equivalently written as

\[
S_{\alpha,q}(\tau_{BGW}) = (-1)^{\alpha} \sigma_{\alpha} \delta_{\alpha,q}\tau_{BGW}, \quad \alpha = 1, \ldots, r - 1, \ q \geq \alpha, \quad (4.13)
\]

where \( \sigma_1, \ldots, \sigma_{r-1} \) are certain polynomials of \( \rho_1, \ldots, \rho_{r-1} \).

**Proof.** We denote

\[
a(t; \lambda) := \sum_{i \in \mathbb{N} \cup \{0\}} \lambda^{t_i}, \quad b(t; \lambda) := - \sum_{i \in \mathbb{N} \cup \{0\}} \frac{1}{i!} \frac{\partial}{\partial \lambda} t_i,
\]

and denote

\[
P_i(t; \lambda) := \frac{\partial^{i+1}}{\partial \lambda^i} \left( \left. e^{a(t; \lambda) - a(t; \lambda) - b(t; \lambda)} \right|_{\mu=\lambda} \right), \quad i \geq 0. \quad (4.14)
\]

It is easy to see that \( P_i(t; \lambda) \) satisfy following the recursion relations:

\[
P_i(t; \lambda) = \frac{\partial a(t; \lambda)}{\partial \lambda} \circ P_i-1(t; \lambda) + P_{i-1}(t; \lambda) \circ \frac{\partial b(t; \lambda)}{\partial \lambda} + \frac{\partial P_{i-1}(t; \lambda)}{\partial \lambda}. \quad (4.15)
\]

By using the above equation (4.15), one can prove the following identity by induction:

\[
P_i(t; \lambda) = \sum_{j=0}^{i-1} \frac{\partial^j}{\partial \lambda^j} \left( \left. (\partial_{\lambda} (a(t; \lambda)) + \partial_{\lambda} (b(t; \lambda)))^{i+1-j} \right|_{\mu=\lambda} \right), \quad i \geq 1. \quad (4.16)
\]

Then we have that, for \( \alpha = 1, \ldots, r - 1 \) and \( q \geq \alpha \),

\[
\text{res} \lambda^{\alpha + (q-\alpha)r} : \left( \partial_\lambda a(t; \lambda) + \partial_\lambda b(t; \lambda) \right)^{\alpha+1} : \lambda \nabla
\]

\[
= \sum_{j=1}^{\alpha-1} \text{res} \lambda^{\alpha + (q-\alpha)r} : \partial_\lambda (\partial_\lambda a(t; \lambda) + \partial_\lambda b(t; \lambda))^{\alpha+1-j} : \lambda \nabla
\]

\[
= \sum_{j=1}^{\alpha-1} (-1)^{(\alpha + (q-\alpha)r)} \text{res} \lambda^{\alpha + (q-\alpha)r} : (\partial_\lambda a(t; \lambda) + \partial_\lambda b(t; \lambda))^{\alpha+1-j} : \lambda \nabla.
\]

By using the definition (4.12) and by noticing that \( W_{\alpha,q}^{\text{red}}(t) \) can be rewritten as

\[
W_{\alpha,q}^{\text{red}}(t) = \frac{1}{\alpha + 1} \text{res} \lambda^{\alpha + (q-\alpha)r} P_\alpha(t; \lambda), \quad (4.17)
\]
we have
\[ S_{\alpha,q} = W_{\alpha,q}^{\text{red}} - \sum_{j=1}^{\alpha-1} \frac{(-1)^j (\alpha - j + 1)(\alpha + (q - \alpha)r)!}{(\alpha + (k - \alpha)r - j)!} S_{\alpha-j,q-j}. \] (4.18)

Therefore, by using equation (1.7) we obtain equation (4.13) (vice versa), where the constants \( \sigma_1, \ldots, \sigma_{r-1} \) can be uniquely determined by
\[ \sigma_\alpha = \rho_\alpha + \frac{\alpha}{\alpha + 1} \sum_{j=1}^{\alpha-1} \frac{(-1)^j}{j!} \sigma_j, \quad \alpha = 1, \ldots, r - 1. \] (4.19)

The lemma is proved.

The following lemma will also be needed, and will also have other important applications. For simplicity, we denote
\[ \langle \tau_i \cdots \tau_k \rangle^\bullet := \left. \frac{\partial^k \tau_{\text{BGW}}}{\partial t_{i_1} \cdots \partial t_{i_k}} \right|_{t=0}, \quad \langle \tau_i \cdots \tau_k \rangle := \left. \frac{\partial^k \log \tau_{\text{BGW}}}{\partial t_{i_1} \cdots \partial t_{i_k}} \right|_{t=0}. \]

**Lemma 4.4.** The system (1.7) (or equivalently (4.13)) has a unique solution in \( \mathbb{C}[\![t]\!] \) with initial value 1.

**Proof.** The existence of the solution is already proved. To show the uniqueness, we use the argument similar to that in [2, 10, 30]. By using equation (4.12), we know that
\[ S_{\alpha,q} = \sum_{j=0}^{\alpha} \frac{\alpha!}{j!(\alpha + 1 - j)!} \sum_{k_1, \ldots, k_{\alpha-j}} k_1 \cdots k_{\alpha-j} \frac{\partial^{\alpha-j}}{\partial t_{k_1+1} \cdots \partial t_{k_{\alpha-j+1}}} \frac{\partial^{\alpha-j-p}}{\partial t_{k_{p+1}} \cdots \partial t_{k_{\alpha-j+1}}} \right|_{t=0}, \]
where \( \alpha = 1, \ldots, r - 1 \) and \( q \geq \alpha \). Hence equation (4.13) can be recast to
\[ \left. \frac{\partial \tau_{\text{BGW}}}{\partial t_{\alpha+(q-\alpha)r}} \right|_{t=0} = \sigma_\alpha \delta_{q,\alpha} \tau_{\text{BGW}} + \sum_{j=0}^{\alpha} \left( \begin{array}{c} \alpha \\ j \end{array} \right) \frac{\partial^{\alpha-j}}{\partial t_{k_1} \cdots \partial t_{k_{\alpha-j+1}}} \right|_{t=0}. \] (4.20)

In terms of \( \langle \tau_i \cdots \tau_k \rangle^\bullet \), we have the recursion relations:
\[ \langle \tau_{A \alpha+(m-\alpha)r} \rangle^\bullet = \langle \tau_A \rangle^\bullet + \sum_{j=0}^{\alpha} \left( \begin{array}{c} \alpha \\ j \end{array} \right) \frac{\partial^{\alpha-j}}{\partial t_{k_1} \cdots \partial t_{k_{\alpha-j+1}}} \right|_{t=0} \]
\[ \times \sum_{p=0}^{\alpha-j} \sum_{k_{p+1}, \ldots, k_{\alpha-j+1}} p! k_1 \cdots k_p \langle \tau_{A \setminus \{k_1, \ldots, k_p\} \cup \{k_{p+1}, \ldots, k_{\alpha-j+1}\}} \rangle^\bullet. \] (4.21)

Here \( \tau_A := \tau_{a_1} \cdots \tau_{a_k} \) for \( A = \{a_1, \ldots, a_k\} \). In this way it is clear that all the coefficients of \( \langle \tau_i \cdots \tau_k \rangle^\bullet \) can be uniquely determined. The lemma is proved.
(The uniqueness statement in the above lemma 4.4 can also be proved directly from equation (1.7).)

By using lemma 4.3 and formula (4.21), we obtain that
\[ \langle \tau_1 \tau_\alpha \rangle = \langle \tau_\alpha \rangle = c_\alpha(\sigma_1, \ldots, \sigma_\alpha) = \sigma_\alpha + \gamma_\alpha(\sigma_1, \ldots, \sigma_{\alpha-1}), \]
where \( c_\alpha \) are certain polynomials of \( \sigma_1, \ldots, \sigma_\alpha \). The property (1.8) then follows from equation (4.19) and the relation
\[ \frac{\partial \log \tau_{BGW}}{\partial t_i \partial t_\alpha} = \Omega_{1,\alpha}(\nu_{BGW}, \nu_{BGW}, \ldots) = \text{res}_\beta L^\alpha \].

The theorem is proved.

We note that, the constants \( c_\alpha \) in the above proof are initial values of the normal coordinates \( \Omega_{1,\alpha} = \text{res}_\beta L^{\alpha/r} \) for the GD hierarchy, i.e. of the corresponding Dubrovin–Zhang hierarchy [36] (see also [9]).

By using theorem 1.1, lemmas 4.3 and 4.4, we arrive at the following theorem.

**Theorem 4.5.** A power series \( \tau \in \mathbb{C}[[t]] \) satisfies equation (1.7) if and only if \( \tau \) is the tau-function for the GD hierarchy satisfying equation (1.4).

## 5. Examples

In this section, we use theorem 1.1 (in particular lemma 4.3) to compute \( \tau_{BGW} \) and \( \log \tau_{BGW} \).

**Example 5.1.** For the case with \( r = 2 \), the constraints (4.13) give the following relations:
\[
\langle \prod_{i=1}^N \tau_{2a_i+1} \tau_{2m+1} \rangle = \sum_{i=1}^N (2a_i + 1) \langle \tau_{2a_i+1} \prod_{j \neq i} \tau_{2a_j+1} \rangle \frac{1}{2} \sum_{k_1+k_2=m-1} \langle \tau_{2a_1+1} \tau_{2a_2+1} \prod_{i=1}^N \tau_{2a_i+1} \rangle \]
\[ + \delta_{m,0} c_1 \langle \prod_{i=1}^N \tau_{2a_i+1} \rangle, \]
where \( N, a_1, \ldots, a_N, m \geq 0 \). The constants \( c_1, d_1, \sigma_1, \rho_1 \) are related by
\[ \rho_1 = \sigma_1 = c_1, \quad d_1 = 2c_1. \]

We have
\[ \langle \tau_1 \rangle = c_1, \quad \langle \tau_2 \rangle = c_1(c_1 + 1), \quad \langle \tau_3 \rangle = \frac{1}{2} c_1(c_1 + 1), \quad \langle \tau_4 \rangle = c_1(c_1 + 1)(c_1 + 2). \]

**Example 5.2.** For the case with \( r = 3 \), the constants \( c_\alpha, d_\alpha, \sigma_\alpha, \rho_\alpha \) are related by
\[ \rho_1 = \sigma_1 = c_1, \quad d_1 = 3c_1, \]
\[ \rho_2 = c_2 + \frac{2}{3} c_1, \quad \sigma_2 = c_2, \quad d_2 = \frac{3}{2} c_2 + 3c_1. \]
We have
\[
\langle \tau_1 \rangle^* = c_1, \quad \langle \tau_2 \rangle^* = c_2, \quad \langle \tau_1^2 \rangle^* = c_1 (c_1 + 1), \\
\langle \tau_2 \tau_1 \rangle^* = (c_1 + 2) \frac{c_2}{2}, \quad \langle \tau_1^3 \rangle^* = c_1 (c_1 + 1) (c_1 + 2), \\
\langle \tau_4 \rangle^* = c_2 (c_1 + 2), \quad \langle \tau_2^2 \rangle^* = c_2^2 - 2c_1 (c_1 + 1), \\
\langle \tau_2 \tau_1^2 \rangle^* = c_2 (c_1 + 2) (c_1 + 3), \quad \langle \tau_4^2 \rangle^* = c_1 (c_1 + 1) (c_1 + 2) (c_1 + 3), \\
\langle \tau_1 \rangle = c_1, \quad \langle \tau_2 \rangle = c_2, \quad \langle \tau_1^2 \rangle = c_1, \quad \langle \tau_2 \tau_1 \rangle = 2c_2, \quad \langle \tau_1^3 \rangle = 2c_1, \\
\langle \tau_4 \rangle = c_2 (c_1 + 2), \quad \langle \tau_2^2 \rangle = -2c_1 (c_1 + 1), \quad \langle \tau_2 \tau_1^2 \rangle = 6c_2, \quad \langle \tau_4^2 \rangle = 6c_1.
\]

**Example 5.3.** Similarly, for the case with \( r = 4 \), we have
\[
\rho_1 = \sigma_1 = c_1, \quad d_1 = 4c_1, \\
\rho_2 = c_2 + \frac{2}{3} c_1, \quad \sigma_2 = c_2 \quad d_2 = 4c_2 + 8c_1, \\
\rho_3 = c_3 - \frac{3}{4} c_2 - \frac{3}{2} c_1^2, \quad \sigma_3 = c_3 - \frac{3}{2} c_1^2 - \frac{3}{2} c_1 \quad d_3 = \frac{4}{3} c_3 + 3c_2 + 2c_1^2 + 10c_1,
\]
and
\[
\langle \tau_1 \rangle^* = c_1, \quad \langle \tau_2 \rangle^* = c_2, \quad \langle \tau_1^2 \rangle^* = c_1 (c_1 + 1), \quad \langle \tau_3 \rangle^* = c_3, \\
\langle \tau_2 \tau_1 \rangle^* = c_2 (c_1 + 2), \quad \langle \tau_1^3 \rangle^* = c_1 (c_1 + 1) (c_1 + 2), \\
\langle \tau_3 \tau_1 \rangle^* = c_3 (c_1 + 3), \quad \langle \tau_2^2 \rangle^* = 4c_3 + c_2^2 - 2c_1 (c_1 + 1), \\
\langle \tau_2 \tau_1^2 \rangle^* = c_2 (c_1 + 2) (c_1 + 3), \quad \langle \tau_4 \rangle^* = c_1 (c_1 + 1) (c_1 + 2) (c_1 + 3), \\
\langle \tau_1 \rangle = c_1, \quad \langle \tau_2 \rangle = c_2, \quad \langle \tau_1^2 \rangle = c_1, \quad \langle \tau_3 \rangle = c_3, \quad \langle \tau_2 \tau_1 \rangle = 2c_2, \quad \langle \tau_1^3 \rangle = 2c_1, \\
\langle \tau_3 \tau_1 \rangle = 3c_3, \quad \langle \tau_2^2 \rangle = 4c_3 - 2c_1 (c_1 + 1), \quad \langle \tau_2 \tau_1^2 \rangle = 6c_2, \quad \langle \tau_4 \rangle = 6c_1.
\]

**Remark 5.4.** Define
\[
\langle \tau_1 \cdots \tau_n \rangle_\infty = \lim_{r \to \infty} \langle \tau_1 \cdots \tau_n \rangle.
\]  
(5.2)

We call \( \langle \tau_1 \cdots \tau_n \rangle_\infty \) the stabilized generalized BGW correlators. For example,
\[
\langle \tau_1 \rangle_\infty = c_1, \quad \langle \tau_2 \rangle_\infty = c_2, \quad \langle \tau_1^2 \rangle_\infty = c_1, \quad \langle \tau_3 \rangle_\infty = c_3, \\
\langle \tau_2 \tau_1 \rangle_\infty = 2c_2, \quad \langle \tau_1^3 \rangle_\infty = 2c_1, \quad \langle \tau_4 \rangle_\infty = c_4, \quad \langle \tau_3 \tau_1 \rangle_\infty = 3c_3, \\
\langle \tau_2^2 \rangle_\infty = 4c_3 - 2c_1 (c_1 + 1), \quad \langle \tau_2 \tau_1^2 \rangle_\infty = 6c_2, \quad \langle \tau_4 \rangle_\infty = 6c_1.
\]

The partition function of these stabilized correlators and its relation to the KP hierarchy deserve a further study.

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