Fast identification of the pull-in voltage of a nano/micro-electromechanical system

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Abstract
The pull-in voltage is crucial in designing an optimal nano/micro-electromechanical system (N/MEMS). It is vital to have a simple formulation to calculate the pull-in voltage with relatively high accuracy. Two simple and effective methods are suggested for this purpose; one is an ancient Chinese algorithm and the other is an extension of He’s frequency formulation.

Keywords
Nano/micro-electromechanical system oscillator, pull-in instability, periodic solution, ancient Chinese mathematics

Introduction
The pull-in instability\textsuperscript{1–6} is an inherent property of a nano/micro-electromechanical system (N/MEMS) when the applied voltage reaches a threshold (see Figure 1), and it plays a significant role in electrostatically actuated sensors for their effective and reliable operation. The MEMS system opens a broad road for microfluidics,\textsuperscript{7} energy harvester,\textsuperscript{8} drug delivery device,\textsuperscript{9} timing and frequency control,\textsuperscript{10} and portable devices.\textsuperscript{11} The pull-in voltage can be easily obtained for the linear case; however, it is highly intricate and challenging for the nonlinear case.

Since the problem of static pull-in (as a consequence of a saddle-node bifurcation)\textsuperscript{12–16} and dynamic pull-in (as a consequence of a homoclinic bifurcation)\textsuperscript{17–20} in the MEMS structure is a highly decisive factor, fast estimation of the pull-in voltage is much needed in practical applications.

We consider the following dynamical pull-in problem\textsuperscript{21}

\[
x'' + x - \frac{\lambda}{1 - x} = 0, x'(0) = 0, x(0) = 0, \lambda > 0
\]

where \( x \) is the dimensionless distance as shown in Figure 1 and \( \lambda \) is a voltage-related parameter.

The pull-in behavior occurs when \( \lambda > \lambda^* \), where \( \lambda^* = 0.203632188 \); it is the solution of the following transcendental equation\textsuperscript{21}

\[
\left(\frac{1 + \sqrt{1 - 4\lambda}}{4}\right)^2 + 2\lambda \ln\left(1 - \frac{1 + \sqrt{1 - 4\lambda}}{2}\right) = 0
\]

This implicit function cannot clearly see the effect of the system’s parameters on the pull-in voltage. An analytical closed-form solution is much needed for MEMS applications.\textsuperscript{16} There are many experimental, numerical, and analytical...
methods for this purpose; for example, the finite-difference method,\textsuperscript{22} the finite element method,\textsuperscript{23} the modal expansion method,\textsuperscript{24} the generalized differential quadrature method,\textsuperscript{25} the Ritz method,\textsuperscript{26,27} and the variational iteration method.\textsuperscript{28,29}

To solve \( \lambda \) simply and effectively, we suggest two methods, one is the ancient Chinese method\textsuperscript{30,31} and the other is He’s frequency formulation.\textsuperscript{32,33}

### Ancient Chinese Algorithm

We begin with the well-known Newton iteration method. If we choose the initial guess as \( \lambda_0 = 0.25 \), the method becomes invalid entirely. The convergence depends strongly upon the initial guess; a good one always leads to an ideal result, while an inappropriate one might result in a divergent outcome.

Consider a simple example

\[
f(x) = \sin x = 0
\]  

We want to find a solution between 0 and \( 2\pi \), so we begin with \( x_0 = 3.14/2 \) and \( 3 \times 3.15/2 \), respectively; the Newton iteration processes are shown in Tables 1 and 2. Both cases cannot lead to the needed result.

To solve the transcendental equation given in equation (2), we introduce the ancient Chinese method.\textsuperscript{30,31} Consider an algebraic equation in the form

\[
f(x) = 0
\]  

The ancient Chinese algorithm begins with two trials,\textsuperscript{30} \( x_1 \) and \( x_2 \), which lead to two residuals \( f(x_1) \) and \( f(x_2) \). The approximate solution of equation (4) is\textsuperscript{30}

\[
x = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}
\]  

The iteration process of the ancient Chinese algorithm is illustrated in Figure 2. For the above example, the two residuals are

\[
f(x_1 = 3.14/2) = 0.999999683 \text{ and } f(x_2 = 3 \times 3.15/2) = -0.9999205;
\]

according to equation (5), we estimate that \( x = 3.14756246 \). This is an approximate solution with a relative error of 0.19%.

To solve equation (2), we introduce a residual function

\[
R(\lambda) = \left( \frac{1 + \sqrt{1 - 4\lambda}}{2} \right)^2 + 2\lambda \ln \left( \frac{1 + \sqrt{1 - 4\lambda}}{2} \right)
\]  

**Figure 1.** The MEMS system.
We choose two trials $\lambda_1 = 0.25$ and $\lambda_2 = 0.17$, the residuals are $R_1 = -0.096574$ and $R_2 = 0.0936174$, respectively, so the pull-in voltage can be calculated as

$$\lambda^* = \frac{R_2 \lambda_1 - R_1 \lambda_2}{R_2 - R_1} = \frac{0.0936174 \times 0.25 + 0.096574 \times 0.17}{0.0936174 + 0.096574} = 0.209378$$

This has a relative error of 2.82%. The accuracy can be improved using the other two trials $\lambda_1 = 0.25$ and $\lambda_2 = 0.209378$; and the two residuals are $R_1 = -0.096574$ and $R_2 = -0.014168$, respectively, using the ancient Chinese algorithm, we have $\lambda^* = 0.202394$ with a relative error of 0.60%. Alternatively, we use the two trials $\lambda_1 = 0.209378$ and $\lambda_2 = 0.17$, we have two residuals $R_1 = -0.014168$ and $R_2 = 0.093617$, and the pull-in voltage is $\lambda^* = 0.204202$ with a relative error of 0.27%. A modification of the ancient Chinese algorithm was given in Ref. 30, and it is called Chun-Hui He algorithm in the literature.31

**He’s frequency formulation**

He’s frequency formulation32,33 is to calculate the frequency of a nonlinear oscillator in the form

$$x'' + q(x) = 0$$

where $q(x)$ is the nonlinear function of $x$. The formulation is32,33
\[ \omega^2 = \frac{q(x_0)}{x_0} \]  

where \( \omega \) is the frequency and \( x_0 \) is a location point; generally, we recommend \( x_0 = 0.8 \) for fast estimation, where \( A \) is the amplitude. Equation (9) and its modifications were widely used in the literature to find a periodic solution of a nonlinear oscillator.34,35

For equation (1), we have

\[ \omega^2 = 1 - \frac{\dot{\lambda}}{x_0(1 - x_0)} \]  

The pull-in threshold is to change the periodic solution (\( \omega^2 > 0 \)) to the non-periodic one (\( \omega^2 < 0 \)), so the pull-in voltage can be determined from the condition

\[ \omega^2 = 0 \]  

That is

\[ \lambda^* = x_0(1 - x_0) \]  

We have a maximum \( \lambda^*_{\text{max}} = 1/4 \) when \( x_0 = 1/2 \), this leads to a relative error of 22.78%, too high to be used in practical applications. We choose multiple points to calculate \( \lambda^* \); then, an average is used. We choose \( x_0 = 1/2, 1/3, 1/4, \) and \( 1/5 \), and we can determine \( \lambda^* = 1/4, 2/9, 3/16, \) and \( 4/25 \), respectively. Its average value is

\[ \lambda^* = 0.2049 \]  

Now, the relative error reduces to 0.62%.

**Discussion and Conclusion**

Newton’s iteration method is sensitive to the initial guess. A good guess always leads to a good result, while a not-good guess results in a non-convergent solution, or the convergent solution is a wrong one. To overcome the shortcoming of the Newton’s iteration method, we suggest two simple but effective methods in this paper to determine the pull-in voltage. A simple method with relatively high accuracy is welcome in many practical applications.

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