The Diameter and Automorphism Group of Gelfand–Tsetlin Polytopes

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Abstract

Gelfand–Tsetlin polytopes arise in representation theory and algebraic combinatorics. One can construct the Gelfand–Tsetlin polytope \( GT_\lambda \) for any partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of weakly increasing positive integers. The integral points in a Gelfand–Tsetlin polytope are in bijection with semi-standard Young tableau of shape \( \lambda \) and parametrize a basis of the \( GL_n \)-module with highest weight \( \lambda \). The combinatorial geometry of Gelfand–Tsetlin polytopes has been of recent interest. Researchers have created new combinatorial models for the integral points and studied the enumeration of the vertices of these polytopes. In this paper, we determine the exact formulas for the diameter of the 1-skeleton, \( \text{diam}(GT_\lambda) \), and the combinatorial automorphism group, \( \text{Aut}(GT_\lambda) \), of any Gelfand–Tsetlin polytope. We exhibit two vertices that are separated by at least \( \text{diam}(GT_\lambda) \) edges and provide an algorithm to construct a path of length at most \( \text{diam}(GT_\lambda) \) between any two vertices. To identify the automorphism group, we study \( GT_\lambda \) using combinatorial objects called ladder diagrams and examine faces of co-dimension 2.

Keywords Gelfand–Tsetlin · Polytope · Automorphism group · Diameter

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1 Introduction and Statement of Results

Gelfand–Tsetlin (GT) polytopes are compact convex polytopes defined by a set of linear inequalities depending on a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) as shown in Fig. 1. The polytope \( \text{GT}_\lambda \) corresponds to all fillings of this triangular array with real numbers \( x_{ij} \) for \( 1 \leq j \leq i \leq n \) such that all rows and columns are weakly increasing.

GT polytopes arise from the study of representations of \( \text{GL}_n(\mathbb{C}) \) and have connections to areas of representation theory and algebraic geometry (see for example [3]). For any integer partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \), let \( n \) be the length of \( \lambda \) and let \( \text{GT}_\lambda \) denote the GT polytope associated to \( \lambda \). Then the integral points within \( \text{GT}_\lambda \) are in bijection with the number of semi-standard Young tableaux of shape \( \lambda \) with tableaux entries bounded by \( n \). Furthermore, the integral points of \( \text{GT}_\lambda \) parametrize a Gelfand–Tsetlin basis of the \( \text{GL}_n \)-module with highest weight \( \lambda \), so the number of integral points equals the dimension of this module. GT polytopes can also be viewed as the marked order polytope of a poset which is discussed in [1,5].

This paper describes the diameter of the 1-skeleton and the combinatorial automorphism group of GT polytopes. Since our results are purely combinatorial, it suffices to consider partitions of the form \( \lambda = (1^{a_1}, 2^{a_2}, \ldots, m^{a_m}) \) where \( a_i > 0 \) for all \( 1 \leq i \leq m \). This is explained in Sect. 2 and Remark 2.5.

The 1-skeleton of a polytope is the graph obtained by looking only at its vertices and edges (the 0- and 1-dimensional faces). The shortest path between two vertices is the path with the minimum number of edges. The diameter of a graph is the maximum over all pairs of vertices of the length of the shortest path between the pair of vertices. Our first result is a formula for the diameter of the 1-skeleton of \( \text{GT}_\lambda \) which we denote by \( \text{diam}(\text{GT}_\lambda) \).

We adopt the following notational conventions:

- The Kronecker delta function is defined as \( \delta_{x,y} := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases} \)
- For any \( a \in \mathbb{Z}_{>0} \), let \( S_a \) denote the symmetric group on a set of size \( a \).
- For \( \lambda = (1^{a_1}, 2^{a_2}, \ldots, m^{a_m}) \), its reverse partition is \( \lambda' := (m^{a_m}, 2^{a_{m-1}}, \ldots, 1^{a_1}) \). If \( a_i = a_{m+1-i} \) for all \( 1 \leq i \leq m \), then we say \( \lambda = \lambda' \) and call \( \lambda \) a reverse symmetric partition.
- We use the shorthand \( [n] \) for the set \( \{1, \ldots, n\} \).

\[
\begin{align*}
\lambda_1 \\
\lambda_2 \\
x_{2,1} \leq \lambda_2 \\
\vdots \\
x_{3,1} \leq x_{3,2} \leq \lambda_3 \\
\vdots \\
x_{4,1} \leq x_{4,2} \leq x_{4,3} \leq \lambda_4 \\
\vdots \\
x_{n,1} \leq x_{n,2} \leq \cdots \leq x_{n,n-1} \leq \lambda_n
\end{align*}
\]

Fig. 1 Inequality constraints of GT polytopes
Theorem 1.1 (Diameter of 1-skeleton) For any GT polytope $\text{GT}_\lambda$, $\text{diam}(\text{GT}_\lambda) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}$.

A polytope’s face poset (or face lattice) is formed by ordering its faces by inclusion. A combinatorial automorphism of a polytope is an automorphism of its face poset. Our second result is a description of the combinatorial automorphism group of $\text{GT}_\lambda$.

Theorem 1.2 ($m = 2$ Automorphisms) Suppose $\lambda = (1^{a_1}, 2^{a_2})$ and $a_1, a_2 \geq 2$. If $a_1 = a_2 = 2$, then

$$\text{Aut}(\text{GT}_\lambda) \cong D_4 \times \mathbb{Z}_2.$$ 

Otherwise,

$$\text{Aut}(\text{GT}_\lambda) \cong D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\delta_{a_1,a_2}},$$

where $D_4$ is the dihedral group of order 8 and $\mathbb{Z}_2$ is the cyclic group of order 2.

Theorem 1.3 ($m \geq 3$ Automorphisms) Suppose $\lambda = (1^{a_1}, \ldots, m^{a_m})$ and $m \geq 3$. Let $r_1$ be the number of $k$ such that $a_k, a_{k+1} \geq 2$. Let $r_2 = 1$ if $\lambda = \lambda'$ and let $r_2 = 0$ otherwise. Then

$$\text{Aut}(\text{GT}_\lambda) \cong (\mathbb{S}_{a_2}^{\delta_{1,a_1}} \times \mathbb{S}_{a_m-1}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{r_1+1}) \ltimes \varphi \mathbb{Z}_2^{r_2}$$

where if $r_2 = 1$, then $\varphi: \mathbb{Z}_2 \to \text{Aut}(\mathbb{S}_{a_2}^{\delta_{1,a_1}} \times \mathbb{S}_{a_m-1}^{\delta_{1,a_m}} \times \mathbb{Z}_2^{r_1+1})$ sends the nonidentity element of $\mathbb{Z}_2$ to the map sending $(\sigma_1, \sigma_2, z_1, \ldots, z_{r_1}, z_{r_1+1}) \mapsto (\sigma_2, \sigma_1, z_{r_1}, \ldots, z_1, z_{r_1+1})$.

Example 1.4 Figure 2 shows the Gelfand–Tsetlin polytope $\text{GT}_{(1,2,3)}$. It is clear that the diameter of $\text{GT}_{(1,2,3)}$ is 2 and the automorphism group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. One generator is given by rotating $180^\circ$ about the axis through vertex 1 and bisecting edge (4, 7). The other generator is given by fixing vertex 1 and interchanging vertex 2 with vertex 3, vertex 5 with vertex 6, and vertex 4 with vertex 7. These results are predicted by our theorems.

Fig. 2 Polytope $\text{GT}_{(1,2,3)}$
In Sect. 2, we review background on GT polytopes including how to model their face poset using combinatorial objects called ladder diagrams. Then we present a proof of Theorem 1.1 in Sect. 3. In Sect. 4, we identify the generators of Aut(GTλ) and present proofs of Theorems 1.2 and 1.3.

2 Preliminaries

In this section, we formally define GT polytopes and describe how to view their faces as combinatorial objects called ladder diagrams. These diagrams are the primary objects we will use to model the face lattice of GTλ.

2.1 Gelfand–Tsetlin Polytopes

A partition of s is a sequence of weakly increasing positive integers \( \lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n) \) such that \( \sum_{i=1}^{n} \lambda_i = s \). We will often use multiplicative notation for \( \lambda \) and write \( \lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_m^{a_m}) \) for \( a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0} \) to denote a partition with \( a_1 \) copies of \( \lambda_1 \), \( a_2 \) copies of \( \lambda_2 \), and so forth. We may omit writing the term \( \lambda_i^{a_i} \) if \( a_i = 0 \).

Definition 2.1 (GT Polytope) Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \), the Gelfand–Tsetlin polytope \( \text{GT}_\lambda \) is the set of points \( x = (x_{i,j})_{1 \leq j \leq i \leq n} \in \mathbb{R}^{\frac{n(n+1)}{2}} \) such that \( x_{i,j} = \lambda_i \) for \( 1 \leq i \leq n \) and such that the following inequalities are satisfied:

1. \( x_{i-1,j} \leq x_{i,j} \leq x_{i+1,j} \),
2. \( x_{i,j-1} \leq x_{i,j} \leq x_{i,j+1} \).

Suppose for some \( i < j \) that \( \lambda_i = \lambda_j \). Then for every \( i \leq i', j' \leq j \), we are forced to have \( x_{i',j'} = \lambda_i \). Whenever such a situation occurs, we say that the coordinate \( x_{i',j'} \) is fixed. In general, \( \text{GT}_\lambda \) will be a polytope in \( \mathbb{R}^d \) where \( d \) is at most \( n(n-1)/2 \).

These constraints can be visualized in a triangular array as shown in Fig. 1.

We adopt the following notational conventions:

- Let \( n \) denote the length of \( \lambda \). We specify a partition as \( \lambda = (\lambda_1, \ldots, \lambda_n) \). We typically only consider partitions of the form \( \lambda = (1^{a_1}, \ldots, m^{a_m}) \) where \( a_i > 0 \) for all \( 1 \leq i \leq m \).
- Let \( m \) denote the number of distinct values of \( \lambda \). This agrees with our notation for partitions of the form \( \lambda = (1^{a_1}, \ldots, m^{a_m}) \) where \( a_i > 0 \) for all \( 1 \leq i \leq m \).
- Let \( d \) denote the dimension of \( \text{GT}_\lambda \). Note that \( d = \left( \begin{array}{c} n \\end{array} \right) - \sum_{i=1}^{m} \left( \begin{array}{c} a_i \\end{array} \right) \).
- Let \( \mathcal{F}(\text{GT}_\lambda) \) denote the face poset of \( \text{GT}_\lambda \) ordered by inclusion.
- Let \( \mathcal{I}_n = \{(i, j) : 1 \leq j \leq i \leq n\} \) denote the triangular grid with shape shown in Fig. 1.

2.2 Ladder Diagrams

For every \( \lambda \), we define a graph \( \Gamma_\lambda \) such that faces of \( \text{GT}_\lambda \) correspond to subgraphs of \( \Gamma_\lambda \) with certain restrictions. These subgraphs are the ladder diagrams introduced in [2].
Fig. 3 Let $\lambda = (1^2, 2^1, 4^2, 7^3, 8^1)$. From left to right: $\Gamma_\lambda$ with the origin and terminal vertices as dots, ladder diagram for a 0-dimensional face (vertex), and ladder diagram for a 2-dimensional face.

Let $Q$ be the infinite graph corresponding to first quadrant of the Cartesian plane, i.e., let $Q$ have vertices $(i, j)$ for all $i, j \geq 0$ and edges $\{(i, j), (i + 1, j)\}$ and $\{(i, j), (i, j + 1)\}$. For the sake of convenience, define $a_0 := 0$ and $s_j := \sum_{i=0}^{j} a_i$ for $0 \leq j \leq m$.

**Definition 2.2 (\(\Gamma_\lambda\) and Ladder Diagrams)** For $\lambda = (1^{a_1}, \ldots, m^{a_m})$, the grid $\Gamma_\lambda$ is an induced subgraph of $Q$ constructed as follows. We identify the vertex at $(0, 0)$ as the origin and the vertices $t_j = (s_j, n - s_j)$ for $0 \leq j \leq m$ as terminal vertices. $\Gamma_\lambda$ consists of all vertices and edges appearing on any North–East path between the origin and some terminal vertex.

A ladder diagram is a subgraph of $\Gamma_\lambda$ such that
1. the origin is connected to every terminal vertex by some North–East path,
2. every edge is on a North–East path from the origin to some terminal vertex.

An example of the grid $\Gamma_\lambda$ and some of its ladder diagrams are shown in Fig. 3. All fixed coordinates are shaded. The terminal vertices lie along the main diagonal of $\Gamma_\lambda$. As standalone objects, the ladder diagrams of $\Gamma_\lambda$ form a poset ordered by inclusion: given two ladder diagrams $L_1, L_2$, we have $L_1 \leq L_2$ if $L_1$ is a subgraph of $L_2$.

**Definition 2.3 (Face Lattice of Ladder Diagrams)** Let $\mathcal{F}(\Gamma_\lambda)$ denote the set of all ladder diagrams of $\Gamma_\lambda$ ordered by inclusion. This may also be called the face lattice of $\Gamma_\lambda$.

**Theorem 2.4** [2, Thm. 1.9] $\mathcal{F}(GT_\lambda) \cong \mathcal{F}(\Gamma_\lambda)$.

**Proof** An isomorphism is given by taking a point in $GT_\lambda$ and drawing lines around adjacent groups of $x_{i,j}$ with equal value will produce a face of $\Gamma_\lambda$. For more details, see [2]. We also mention that [4] proves an analogous relation between $\mathcal{F}(GT_\lambda)$ and the poset of GT tilings, which is essentially equivalent to $\mathcal{F}(\Gamma_\lambda)$. \(\square\)

Note that $\mathcal{F}(\Gamma_\lambda)$ is graded by the number of bounded regions where $k$-dimensional faces correspond to ladder diagrams with $k$ bounded regions. In fact, given a point $x \in GT_\lambda$, we can determine the dimension of the minimal face containing $x$ by mapping $x$ to its corresponding ladder diagram. Then the number of bounded regions is the dimension of this minimal face (Fig. 4).
Remark 2.5 Note that the poset \( \mathcal{F}(\Gamma_\lambda) \) only depends on the multiplicities \( a_i \) and not on the values of \( \lambda_i \). So when examining the purely combinatorial properties of \( \text{GT}_\lambda \), it suffices to consider partitions of the form \( \lambda = (1^{a_1}, 2^{a_2}, \ldots, m^{a_m}) \).

3 Combinatorial Diameter

In this section, we prove Theorem 1.1 which gives an exact formula for the diameter \( \text{diam}(\text{GT}_\lambda) \) of the 1-skeleton of the Gelfand–Tsetlin polytope \( \text{GT}_\lambda \) for any partition \( \lambda \). As explained in Remark 2.5, it suffices to consider \( \lambda = (1^{a_1}, \ldots, m^{a_m}) \) where \( a_1, \ldots, a_m \in \mathbb{Z}_{>0} \). Our proofs will indirectly work with vertices and edges of \( \text{GT}_\lambda \) by manipulating their corresponding ladder diagrams. When there is risk of confusion, we refer to vertices and edges of the polytope \( \text{GT}_\lambda \) as \( p \)-vertices and \( p \)-edges, and vertices and edges of ladder diagrams as \( l \)-vertices and \( l \)-edges.

In order to study the diameters of the 1-skeleton of \( \text{GT}_\lambda \), we need to first understand how vertices correspond to ladder diagrams and under what conditions two vertices are connected by an edge.

Remark 3.1 Our formula for the diameter of the 1-skeleton depends on whether \( a_1 = 1 \) and \( a_m = 1 \). Readers may find it helpful to think of partitions with \( a_1, a_m \geq 2 \) as the general case and partitions with \( a_1 = 1 \) (and/or \( a_m = 1 \)) as corner cases.

Definition 3.2 Two paths in a ladder diagram from the origin to terminal vertices are noncrossing if they do not meet again after their first separation (Fig. 5).

In particular, vertices of \( \text{GT}_\lambda \) have ladder diagrams consisting of \( m + 1 \) noncrossing paths. Two \( p \)-vertices are connected by a \( p \)-edge iff the union of their ladder diagrams has exactly one bounded region, as shown in Fig. 6. We think of traveling from one \( p \)-vertex to another as moving a subpath of the first \( p \)-vertex’s ladder diagram. Note that such a move can alter more than one of the \( m + 1 \) noncrossing paths.

Lemma 3.3 Any two vertices of \( \text{GT}_\lambda \) are separated by at most \( 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m} \) edges.
Proof We present an algorithm to find a path between any two vertices \( v \) and \( w \) of \( \text{GT}_\lambda \) such that the path has length at most \( 2m - 2 - \delta_1.a_1 - \delta_1.a_m \). The key idea is that there exists some fixed vertex \( u \) (presented in Phase 2 below) such that it is reachable from any vertex \( v \) in at most \( m - 1 \) steps, showing an upper bound of \( 2m - 2 \) for the diameter. And in case of \( a_1 = 1 \) or \( a_m = 1 \), we can shorten this path by \( \delta_1.a_1 + \delta_1.a_m \) steps via some preliminary steps (shown in Phase 1 below).

Assume that in the ladder diagram representation, p-vertex \( v \) corresponds to noncrossing paths \( v_1, \ldots, v_{m-1} \) where \( v_j \) connects the origin \((0,0)\) to terminal l-vertex \( t_j = (s_j, n - s_j) \) where \( s_j = \sum_{i=0}^{j} a_i \). Similarly denote the noncrossing paths corresponding to \( w \) as \( w_1, \ldots, w_{m-1} \).

Essentially, we want to change \( v_1 \) to \( w_1 \), \( v_2 \) to \( w_2 \), \ldots, \( v_{m-1} \) to \( w_{m-1} \), making sure that the \( m - 1 \) paths we have are always noncrossing, and that the common refinement before and after changing some paths has exactly one bounded region. This ensures we are always traveling along p-edges in \( \text{GT}_\lambda \). Note that the two paths ending at \( t_0 \) and \( t_m \) are the same for every ladder diagram so we ignore them here.

Phase 1. If \( a_m = 1 \), then \( v_{m-1}, w_{m-1} \) are paths that go from \((0,0)\) to \((n-1,1)\). Therefore, there exists an index \( r_v \) such that path \( v_{m-1} \) passes through \((r_v,0)\) and \((r_v,1)\). In other words, \( r_v \) is the horizontal coordinate at which \( v_{m-1} \) passes from row 0 to row 1. We define \( r_w \) similarly. If \( r_v = r_w \), then we skip this phase. Otherwise, without loss of generality, assume \( r_v < r_w \) so path \( w_{m-1} \) is contained inside the lower right region bounded by \( v_{m-1} \). Then the ladder diagram consisting of \( v_1, v_2, \ldots, v_{m-1}, w_{m-1} \) has one bounded region and is a p-edge \( e \) with p-vertex \( v \) as one of its endpoints. Let \( v' \) be the other endpoint. Notice that l-edge \( \{(r_w,0), (r_w,1)\} \) must be in the ladder diagram of \( v' \) so the path to terminal vertex \( t_{m-1} \) equals \( w_{m-1} \). Similarly, if \( a_1 = 1 \), then we can move along one p-edge to make the paths to terminal
l-vertex $t_1$ equal. This results in two p-vertices with the same paths to $t_1$ and/or $t_{m-1}$ if $a_1 = 1$ and/or $a_{m-1} = 1$. Figure 7 shows an example of this process.

**Phase 2.** Now we describe an algorithm that takes $v'$ to some p-vertex $u$ in at most $m - 1 - \delta_{1,a_1} - \delta_{1,a_m}$ steps. The algorithm works as follows: for each $i = 1 + \delta_{1,a_1}, \ldots, m - 1 - \delta_{1,a_m}$ in this increasing order, change path $v_i$ so that it starts at terminal l-vertex $t_i$, goes horizontally to the left until it meets and merges with path $v_{i-1}$. Notice that by the time we change $v_i$, the paths $v_j$’s have been already modified for $j < i$, becoming horizontal from the terminal l-vertex $t_j$ until it touches the vertical axis. First, the ladder diagram after this change is clearly a p-vertex. Also, if we take the common refinement of the two ladder diagrams before and after the change, or equivalently, start with the old ladder diagram and add a new path $v'_i$ described above, then this new path simply cuts the tile bounded by $v_{i-1}$ and $v_i$ into two parts and thus there exists a p-n edge between these two p-vertices. Figure 8 shows an example of this algorithm.

Similarly, we can apply the same algorithm to $w'$ to get to the same p-vertex $u$ in the same number of steps. The total number of steps is at most

$$(\delta_{1,a_1} + \delta_{1,a_m}) + 2(m - 1 - \delta_{1,a_1} - \delta_{1,a_m}) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}.$$  

Next we move on to proving the lower bound. We will construct two specific vertices of $GT_\lambda$ and show that their distance achieves the desired bound.
**Definition 3.4 (Zigzag Vertices)** Let $\lambda = (1^{a_1}, \ldots, m^{a_m})$. If $a_1 > 1$, meaning that $(1, n - 1)$ is not a terminal vertex, we call $(1, n - 1)$ a virtual terminal vertex. Similarly, if $(n - 1, 1)$ is not an actual terminal vertex, meaning that $a_m > 1$, we call it a virtual terminal vertex.

We will consider the ladder diagram for a vertex of $\Gamma_\lambda$ as $(m - 1)$ southwest lattice paths from terminal vertices to the origin. For $j = 1, \ldots, m - 1$, define a horizontal zigzag path $h_j$ to be the path that starts at terminal vertex $t_j$, goes horizontally left until reaching a column where there exists a terminal vertex or a virtual terminal vertex on it, then goes vertically down until reaching a row where there exists a terminal vertex or a virtual terminal vertex, and so on and so forth until the path reaches column 0 or row 0. Similarly define a vertical zigzag path $v_j$ with the only difference that it will start vertically instead of horizontally. Finally, let $z_h$ be the p-vertex of $\Gamma_\lambda$ represented by $(h_1, \ldots, h_{m-1})$ and let $z_v$ be the p-vertex of $\Gamma_\lambda$ represented by $(v_1, \ldots, v_{m-1})$. We call $z_h$ and $z_v$ the horizontal and vertical zigzag vertices. Figure 9 shows an example.

**Lemma 3.5** Zigzag vertices $z_h$ and $z_v$ (Definition 3.4) are separated by at least $2m - 2 - \delta_1,a_1 - \delta_1,a_m$ edges.

We first develop some terminology that will be useful in proving Lemma 3.5.

**Definition 3.6 (Path)** A path between two p-vertices $v$ and $w$ is a sequence of p-vertices $v = y_0, y_1, \ldots, y_\ell = w$ such that $y_{k-1}$ and $y_k$ are adjacent (connected by a p-edge in $\Gamma_\lambda$) for every $k = 1, \ldots, \ell$.

**Definition 3.7 (Lattice Path History)** Given a path $v = y_0, y_1, \ldots, y_\ell = w$ between $v$ and $w$, each transition $y_{k-1} \rightarrow y_k$ corresponds to traveling along a p-edge where the ladder diagram of this edge is the union of the ladder diagrams of $y_{k-1}$ and $y_k$. The lattice path history $X_1, \ldots, X_\ell$ is the sequence of index sets recording which paths are changed to obtain the ladder diagram of $y_k$ from the ladder diagram of $y_{k-1}$. More formally, we have $X_k := \{i : \text{the path from the origin to the } i\text{th terminal vertex changes in } y_{k-1} \rightarrow y_k\}$.

Our goal is to show that the lattice path history between $z_h$ and $z_v$ satisfies certain conditions (laid out in Proposition 3.9), which say that, at least when $a_1, a_m \geq 2$, each
path from the origin to a terminal vertex has a step when it starts being moved and stops being moved. These steps cannot coincide for any paths. Since there are \( m - 1 \) such paths, the lattice path history has length \( \geq 2m - 2 \).

**Lemma 3.8** Suppose we have a p-edge consisting of two p-vertices, \( v \) and \( w \), whose ladder diagrams consist of paths \( v_0, \ldots, v_m \) and \( w_0, \ldots, w_m \). Let \( P_v = \{ v_i : v_i \neq w_i \} \) and \( P_w = \{ w_i : w_i \neq v_i \} \) be paths in \( v \) and \( w \) that differ. Then all paths in \( P_v \) share an interior l-edge of \( \mathcal{T}_n \), all paths in \( P_w \) share an interior l-edge and all paths in \( P_v \cup P_w \) share an interior l-vertex of \( \Gamma_\lambda \).

In this section, “interior” means not lying on either \( x \)-axis or \( y \)-axis. In Sect. 4, we will re-define this term once our purposes change.

Notice that in each transition \( y_{k-1} \to y_k \), the connecting p-edge’s ladder diagram has one bounded region. It follows from Lemma 3.8 that each \( X_k \) consists of consecutive indices, i.e., is of the form \( \{i, i + 1, \ldots, j\} \) for some \( i \leq j \).

**Proof** Since \( v \) and \( w \) are connected by a p-edge, when we superimpose the ladder diagrams for \( v \) and \( w \) we get exactly one bounded region \( R \) (see Fig. 6 for a reference). This region \( R \) is cut out by two north-east paths from one l-vertex of \( \Gamma_\lambda \) to another, which we will refer to as the top and bottom path. Note that the top and bottom path must meet at an interior point \( A \) of \( \Gamma_\lambda \) with positive \( x \) and \( y \) coordinate, i.e., both the top and bottom path contain interior l-edges.

Consider the ladder diagram of \( v \) and \( w \) superimposed. As \( v \neq w \), \( P_v \neq \emptyset \) so let \( v_i \in P_v \) and \( w_i \in P_w \). This means that inside this ladder diagram there are two different north-east paths from the origin to \( t_i \), and this can only happen if these two north-east paths contain the top path and the bottom path respectively. Assume \( v_i \) contains the top path and \( w_i \) contains the bottom path. Now for any \( v_j \in P_v \), if \( v_j \) contained the bottom path, then the ladder diagram of \( v \), containing both \( v_i \) and \( v_j \), would create a bounded region \( R \), yielding a contradiction. As a result, all paths in \( P_v \) contain the top path and all paths in \( P_w \) contain the bottom path so each of these two sets of paths share some interior l-edges of \( \mathcal{T}_n \). Moreover, all paths in \( P_v \cup P_w \) share an interior l-vertex \( A \) as desired. \( \square \)

In the language of lattice path history, a part of Lemma 3.8 is saying that all paths in \( X_k \) share some interior l-edges in \( y_{k-1} \) (and in \( y_k \)).

**Proposition 3.9** If \( a_1, a_m \geq 2 \), the lattice path history \( X_1, \ldots, X_\ell \) (Definition 3.7) for any path \( z_h = y_0, \ldots, y_\ell = z_v \) between zigzag vertices (Definition 3.4) satisfies the following conditions:

1. For each \( s \in [m - 1]:= \{1, \ldots, m - 1\} \), \( s \) appears in at least two of the sets \( X_i \)'s. In other words, for each \( s \in [m - 1] \), there exists \( 1 \leq i < j \leq \ell \) such that \( s \in X_i \), \( s \in X_j \).
2. For each \( s \neq s' \in [m - 1] \), the index of the last set in which \( s \) appears is different from the index of the last set in which \( s' \) appears.
3. If \( X_k = \{i, i + 1, \ldots, j\} \), then at least \( j - i \) (i.e. all but at most one) of \( i, i + 1, \ldots, j \) must appear in some (possibly different) \( X_{k'} \) where \( k' < k \).
4. If \( X_k = \{i, i + 1, \ldots, j\} \) and it is the last set in which \( s \) appears, then each one of \( \{i, i + 1, \ldots, j\} \setminus \{s\} \) must appear in (possibly different) \( X_{k'} \) for \( k' < k \).
If $a_1 = 1$ (or/and $a_m = 1$), then $\{1\}$ (or/and $\{m − 1\}$) must appear as a singleton as one of the terms in $X_1, \ldots, X_\ell$. After we remove such singletons and delete $1$’s (or/and $m − 1$’s) from all sets, the remaining sequence satisfies the four conditions above for elements other than $1$ (or/and $m − 1$).

**Proof** Notice that if the last set in which $s$ appears is $X_k$, then the path $p_s$ is already the vertical zigzag path $v_s$ (Definition 3.4) in $y_k$, and is unchanged in the remaining transitions.

First assume that $a_1, a_m \geq 2$.

**Condition (1).** If for some $s \in [m − 1]$, it appears in only one set $X_k$, then it means when we go from $p$-vertex $y_{k−1}$ to $y_k$, the path $p_s$ is changed from $v_s$ to $h_s$ in exactly one step. But by construction, $h_s$ and $v_s$ meet at $t_s$. Let $t_{s−1} = (x_1, y_1)$ and $t_{s+1} = (x_2, y_2)$ be the terminal vertices, or virtual terminal vertices, directly north-west and south-east of $t_s$, respectively. Note that $x_i, y_i > 0$. By construction, $h_s$ and $v_s$ also meet at $(x_1, y_2)$. Since $h_s$ and $v_s$ also share no l-edges, this implies that $h_s \cup v_s$ bound at least two regions, so no p-vertex containing $h_s$ is connected to a p-vertex containing $v_s$ by a p-edge. Therefore, each $s \in [m − 1]$ appears in at least two sets.

**Condition (2).** If $X_k$ is the last time that both $s$ and $s'$ appear, then it means that from $y_{k−1}$ to $y_k$, paths $p_{s'}$ and $p_s$ are changed to $v_{s'}$ and $v_s$ simultaneously. However, $v_{s'}$ and $v_s$ do not have any intersection in the interior of $I_n$. So if we want to go back from $y_k$ to $y_{k−1}$, we have to change $v_{s'}$ and $v_s$ simultaneously, contradiction Lemma 3.8.

**Condition (3).** Suppose for contradiction that there are two paths $h_r, h_s$ that do not appear in any $X_{k'}, k' < k$. Again, trying to go from $y_k$ to $y_{k−1}$ would contradict Lemma 3.8 since $h_r$ and $h_s$ do not share any interior l-edges.

**Condition (4).** This condition is crucial and it justifies our choices for $z_h$ and $z_v$. Assume that $X_k = \{i, i + 1, \ldots, j\}$ is the last time that $s$ appears, meaning that when we go from vertex $y_{k−1}$ to $y_k$, we change $p_s$ to $v_s$. If there exists $s' \neq s \in X_k$ such that path $s'$ has not changed before, we know $p_{s'} = h_{s'}$. Notice that since we change paths $i, i + 1, \ldots, j$ simultaneously, all of these paths before and after the change will have an interior l-vertex in $\Gamma_\beta$ in common by Lemma 3.8. Therefore, $v_r$ and $h_{s'}$ must have a common interior vertex. Since $s \neq s'$, we must have $s' = s + 1$. As we have assumed $a_1, a_m \geq 2$, superimposing the ladder diagram of $y_{k−1}$ and $y_k$ will create at least two bounded regions, because of the definition of $v_s$ and $h_{s+1}$. Therefore, we have a contradiction and thus all $s' \neq s$ must have already appeared at least once.

Now we consider the cases where $a_1$ and/or $a_m$ may be 1. If $a_1 = 1$, then path $h_1$ has only one interior l-edge between $(0, n − 1)$ and $(1, n − 1)$. In order for it to merge with other paths, it must first change on its own, meaning that $\{1\}$ will appear, because $h_1$ and $p_2, \ldots, p_{m−1}$ will never have any interior intersection. If it never merges with other paths, then it must appear as $\{1\}$ at some point so that $p_1$ can be changed from $h_1$ to $v_1$. If $a_m = 1$ path $p_{m−1}$ must eventually be changed to $v_{m−1}$, which has only one interior l-edge. This change is recorded as $\{m − 1\}$ because $p_1, \ldots, p_{m−2}$ cannot have any interior intersection with $v_{m−1}$. Therefore, if $a_1 = 1$ (or/and $a_m = 1$), in the sequence $X_1, \ldots, X_\ell$, we can take out the singleton $\{1\}$ (or/and $\{m − 1\}$) and delete all other 1’s (or/and $m − 1$’s) in the sets. The remaining sequence will satisfy the four conditions above by the same reasoning. □
We are ready to prove the main lemma for lower bound.

**Proof of Lemma 3.5** We will first consider the case where \( a_1, a_m \geq 2 \) so that the idea of the proof can be shown clearly. Consider any lattice path history \( X_1, \ldots, X_\ell \) from \( z_h \) to \( z_v \). Our goal is to show that \( \ell \geq 2m - 2 \). Recall that \( X_1, \ldots, X_\ell \) satisfies the four conditions listed in Proposition 3.9. It thus suffices to show that any sequence of sets satisfying those four conditions has length \( \ell \geq 2m - 2 \).

To do this, let \( k \) be the largest index \( i \) such that \( X_i \) is not a singleton. If \( X_k \) is the last set that some \( s \in [m - 1] \) appears, then we claim that changing \( X_k \) to \( \{s\} \) will still satisfy all four conditions. According to Condition (4), for each \( s' \neq s \) that is in \( X_k \), \( s' \) must have appeared before. According to Condition (2), for each \( s' \neq s \), this is not the last time that \( s' \) appears so \( s' \) will appear sometime later. Therefore, condition (1) still holds after changing \( X_k \) to \( \{s\} \). Condition (2) also holds because this change does not modify the indices of the sets where each \( s' \in [m - 1] \) appears last. Condition (3) and (4) hold trivially because we have less non-singleton sets to worry about.

Another case is that \( X_k \) is not the last time that any element appears last. According to Condition (3), there exists \( s \in X_k \) such that each one of \( X_k \setminus \{s\} \) has appeared before. Similarly, we claim that all these four conditions will hold after changing \( X_k \) to \( \{s\} \). For each one of \( s' \in X_k \setminus \{s\} \), as it appears before and \( X_k \) is not the last time that it appears, we know that \( s' \) will appear at least twice even after this change. The number of appearances of \( s \) does not change. Therefore, Condition (1) is satisfied. Condition (2) holds because each one of \( s' \in X_k \) will appear sometime later. Conditions (3) and (4) hold trivially because similarly we have less non-singleton sets to worry about.

Continuing this procedure inductively, we will eventually end up with a sequence of sets \( Y_1, \ldots, Y_\ell \) where each one is a singleton. As Condition (1) still holds, we conclude that \( \ell \geq 2m - 2 \) as desired.

Now we consider the cases where \( a_1 \) and/or \( a_m \) may be 1. According to Proposition 3.9, we can take out the singleton \( \{1\} \) (or/and \( \{m - 1\} \)) and delete all other 1’s (or/and \( m - 1 \)’s) in the sets. The remaining sequence still satisfies the four conditions. We have taken out at least \( \delta_{1,a_1} + \delta_{1,a_m} \) singletons of \( \{1\} \) and \( \{m - 1\} \) and there are \( m - 1 - \delta_{1,a_1} - \delta_{1,a_m} \) paths remaining. So by the induction argument, the total length of the sequence is

\[
\ell \geq (\delta_{1,a_1} + \delta_{1,a_m}) + 2(m - 1 - \delta_{1,a_1} - \delta_{1,a_m}) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m}.
\]

Lemmas 3.3 and 3.5 together prove Theorem 1.1.

**Theorem 1.1** (Diameter of 1-skeleton) \( \text{diam}(\text{GT}_\lambda) = 2m - 2 - \delta_{1,a_1} - \delta_{1,a_m} \).

## 4 Combinatorial Automorphisms

In this section, we prove Theorems 1.2 and 1.3 which together completely describe the combinatorial automorphism group \( \text{Aut}(\text{GT}_\lambda) \) of the Gelfand–Tsetlin polytope \( \text{GT}_\lambda \) for any partition \( \lambda \). A combinatorial automorphism of a polytope is a permutation of its faces that preserves inclusion. Such a permutation can be thought of as an
automorphism of the face lattice. For the sake of brevity, we will refer to combinatorial automorphisms simply as automorphisms. As explained in Remark 2.5, it suffices to consider \( \lambda = (1^{a_1}, \ldots, m^{a_m}) \) where \( a_1, \ldots, a_m \in \mathbb{Z}_{>0} \).

The proofs of Theorems 1.2 and 1.3 each consist of two parts: first we give a “lower bound” for the automorphism group by explicitly identifying automorphisms of \( \Gamma^\lambda \) which generate a subgroup of the automorphism group. Then we “upper bound” the automorphism group by upper bounding the size of the automorphism group by the size of this explicit subgroup. Therefore, the automorphisms we identify generate the entire automorphism group.

In Sect. 4.1, we give an outline of the main ideas of the proof. In the remaining sections, we present the lower bound and upper bound proofs in full detail: in Sect. 4.2 we prove our lower bound by presenting explicit automorphisms. In Sect. 4.3 we define and study facet chains which allow us to obtain upper bounds in Sect. 4.4 using the Orbit–Stabilizer theorem. These bounds will show that the explicit automorphisms identified in Sect. 4.2 generate the entire automorphism group.

### 4.1 Overview

Recall that a facet of a polytope is a face of codimension one. We will bound the number of automorphisms by studying the different ways an automorphism can act on the facets of \( \Gamma^\lambda \). This is motivated by two observations, that we make precise below. First, any automorphism of \( \Gamma^\lambda \) is determined by where it sends the facets of \( \Gamma^\lambda \), and second, there is an explicit bijection between facets of \( \Gamma^\lambda \) and interior edges of \( \Gamma^\lambda \).

**Lemma 4.1** An automorphism of \( \Gamma^\lambda \) is determined by where it sends the facets of \( \Gamma^\lambda \) or equivalently, where it sends the ladder diagrams of facets.

**Proof** This follows from the fact that for a general polytope \( P \), every face of \( P \) can be written as an intersection of the facets of \( P \). Thus specifying the image of each facet suffices to specify the image of any face.

**Definition 4.2** We define the interior edges of \( \Gamma^\lambda \) to be all edges of the form \( \{(s_j, n-s_{j+1}), (s_j, n-s_{j+1}+1)\} \) or \( \{(s_j, n-s_{j+1}), (s_{j+1}, n-s_{j+1})\} \) and all edges lying inside \( \Gamma^\lambda \). All other edges of \( \Gamma^\lambda \) are considered boundary edges (Fig. 10).

**Proposition 4.3** The facets of \( \Gamma^\lambda \) are in bijection with the interior edges of \( \Gamma^\lambda \).

**Proof** Since ladder diagrams in \( \mathcal{F}(\Gamma^\lambda) \) are graded by number of bounded regions, any facet will correspond to a ladder diagram with all possible edges except one, which we claim will be an interior edge. To see this, note that the boundary edges are exactly those edges such that any North–East path meeting the South–West vertex of that edge must actually contain that edge, so removing that edge does not produce a valid ladder diagram. For each interior edge, it is possible for a North–East path to meet the South–West vertex and not contain the edge, so its removal produces a valid ladder diagram. The bijection is then given by mapping a facet \( F \) to the single interior
We will often represent a facet $F \in \Gamma_{\lambda}$ by its corresponding interior edge which we denote by $e(F)$. A face is not contained in facet $F$ iff its ladder diagram contains edge $e(F)$.

In Sect. 4.2, we will list elements of the automorphism group, describe how they act on ladder diagrams and facets, and write down the defining relations between. This will give us a lower bound on the size of the automorphism group, as well give us the isomorphism type of the group that these elements generate. The goal of Sect. 4.4 will be to show that this is in fact the entire group, and our strategy is as follows. For any group $G$ acting on the set of facets of $\Gamma_{\lambda}$, we use the following notation:

- For any set of facets $X$ and subgroup $H \subseteq G$, we denote the point-wise stabilizer of $X$ in $H$ by $H(X) := \{ h \in H : h \cdot x = x \ \forall x \in X \}$.
- For any facet $x$ and any subgroup $H \subseteq G$, we denote the orbit of $x$ with respect to $H$ by $O_H(x)$.

The Orbit–Stabilizer theorem will be our main tool, which states that $|G| = |O_G(x)||G(\{x\})|$.
In the proofs of Theorems 1.2 and 1.3, the goal is to construct a decreasing chain of subgroups

\[ \text{Aut}(\Gamma_{\lambda}) = G_0 \supset G_1 \supset \ldots \supset G_l = \text{Id}, \]

where \( G_{i+1} = G_i(\{f_i\}) \) for some facet \( f_i \). We will carefully choose each \( f_i \) so that we can upper bound \(|O_{G_i}(f_i)|\) based on our study of facet chains in Sect. 4.3. (In fact, the description of the orbits in Sect. 4.2 will show that we actually have equality at each step.) By the Orbit–Stabilizer theorem, our bound \(|O_{G_i}(f_i)|\) gives us the same bound for \(|G_i|/|G_{i+1}|\). Our work in Sect. 4.3 will also allow us to prove that \( G_i \) is in fact the identity, allowing us to compute \(|G|\) as a product of sizes of orbits:

\[ |\text{Aut}(\Gamma_{\lambda})| = |G_0| = \left( \prod_{i=0}^{l-1} \frac{|G_i|}{|G_{i+1}|} \right) |G_l| = \left( \prod_{i=0}^{l-1} |O_{G_i}(f_i)| \right) \cdot 1. \]

### 4.2 Automorphisms

We begin by explicitly identifying automorphisms of \( \Gamma_{\lambda} \) which we will later prove form a set of generators for the automorphism group. By Theorem 2.4, to show that a map is an automorphism of \( \Gamma_{\lambda} \), it suffices to show that the map is an automorphism of \( F(\Gamma_{\lambda}) \).

**Proposition 4.4** (The Corner Symmetry) For any \( \lambda \), there is an order 2 automorphism \( \mu \) of \( F(\Gamma_{\lambda}) \) which acts by exchanging the facets corresponding to edges \{\( (0, 1) \), \( (1, 1) \)\} and \{\( (1, 0)(1, 1) \)\}, fixing all other facets.

**Proof** Recall that each point \( x \in \Gamma_{\lambda} \) consists of coordinates \((x_i, j)\) \(1 \leq j \leq i \leq n\) as labeled in Fig. 1. Let \( \mu : \mathbb{R}^{n(n+1)/2} \rightarrow \mathbb{R}^{n(n+1)/2} \) be the linear map that sends \( x_{n,1} \mapsto x_{n,2} + x_{n-1,1} - x_{n,1} \) and acts as the identity on all other \( x_{i,j} \). Since \( x_{n-1,1} \leq x_{n-1,1} + x_{n,2} - x_{n,1} \leq x_{n,2} \), all necessary inequalities are satisfied so \( \mu(\Gamma_{\lambda}) \subset \Gamma_{\lambda} \). Since \( \mu^2 = \text{Id} \), the previous inclusion is actually an equality. Combining this fact with the fact that \( \mu \) is an affine transformation, we see \( \mu \) induces a combinatorial automorphism on \( \Gamma_{\lambda} \), which we shall abuse notation and call \( \mu \). If \( x \) is a point in the facet corresponding to the edge \{\( (1, 0) \), \( (1, 1) \)\}, then \( x_{n,1} = x_{n,2} \). We see that \( \mu(x)_{n,1} = \mu(x)_{n-1,1} \), and conclude \( \mu(x) \) is contained in the facet corresponding to the edge \{\( (1, 0)(1, 1) \)\} (Fig. 12). \( \square \)

The next map is very similar to the Corner Symmetry \( \mu \) except it occurs at the \( k \)th terminal vertex instead of at the origin. The argument is very similar to Proposition 4.4, so we will omit some details.

**Proposition 4.5** (The \( k \)-Corner Symmetry) Let the \( k \)th terminal vertex be \( t_k = (n-i, i) \) and suppose that \( a_k, a_{k+1} \geq 2 \). Then there is an order 2 automorphism \( \mu_k \) which acts by exchanging the facets \{\( (n-i-1, i-1) \), \( (n-i-1, i) \), \( (n-i-1, i-1) \), \( (n-i, i-1) \)\}, fixing all other facets.
Proof Recalling the labeling in Fig. 1, let $x_{i',j'}$ be the coordinate which is immediately adjacent to the values $\lambda_k$ and $\lambda_{k+1}$. Let $\mu_k : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the linear map that sends $x_{i',j'} \mapsto x_{i',j'-1} + x_{i'+1,j'} - x_{i',j'}$ and acts as the identity on all other $x_i,j$. Since $a_k, a_{k+1} \geq 2$, we have

$$\lambda_k \leq x_{i',j'-1} \leq x_{i',j'-1} + x_{i'+1,j'} - x_{i',j'} \leq x_{i'+1,j'} \leq \lambda_{k+1},$$

so all four constraints on the coordinate $x_{i',j'}$ are satisfied. Since $\mu_k^2 = 1$, the same argument as in Proposition 4.4 gives an automorphism $\mu_k$ with the desired properties (Fig. 13).

**Proposition 4.6** (Symmetric Group Symmetries) If $a_1 = 1$, then $\mathcal{G}_{a_2} \subset \text{Aut}(\mathcal{F}(\Gamma_{\lambda}))$, acts on edges in the following manner. Number the topmost $a_2$ horizontal edges in the first column of $\Gamma_{\lambda}$ from 1 through $a_2$. Let $\sigma$ be any element in $\mathcal{G}_{a_2}$. Then $\sigma$ acts on facets by permuting the corresponding edges, fixing all other facets. Similarly if $a_m = 1$, then $\mathcal{G}_{a_{m-1}} \subset \text{Aut}(\mathcal{F}(\Gamma_{\lambda}))$, acts on the rightmost $a_{m-1}$ vertical edges, fixing all other facets.

**Proof** We will define how $\mathcal{G}_{a_2}$ acts on ladder diagrams, then show that this action gives a valid automorphism of $\mathcal{F}(\Gamma_{\lambda})$. Let $\sigma \in \mathcal{G}_{a_2}$. Then $\sigma$ acts on each ladder diagram by permuting the horizontal edges in the first column. Specifically, for a given ladder diagram, if there is a path passing through the $i$th horizontal edge and $\sigma(i) = j$, then $\sigma$ acts on this ladder diagram by changing this path to go through the $j$th horizontal edge. Note that there is only one way to change the vertical edges of the modified path to yield a valid ladder diagram. Again, $\sigma$’s action on $\mathcal{F}(\Gamma_{\lambda})$ has finite order so $\sigma$ is a bijection on the underlying set of $\mathcal{F}(\Gamma_{\lambda})$. Furthermore, $\sigma$ preserves inclusions of ladder diagrams so $\sigma$ is an automorphism of $\mathcal{F}(\Gamma_{\lambda})$. (Fig. 14).
Proposition 4.7 (The Flip Symmetry) Suppose that \( \lambda = \lambda' \). Then there is an order 2 automorphism \( \rho \) of \( \mathcal{F}(\Gamma \lambda) \) which maps a facet corresponding to an edge to the facet corresponding to that edge reflected around the line \( y = x \).

Proof Recall that if \( \lambda = (1^{a_1}, 2^{a_2}, \ldots, m^{a_m}) \), then \( \lambda' := (1^{a_m}, 2^{a_{m-1}}, \ldots, m^{a_1}) \). So if \( \lambda = \lambda' \), we have \( a_i = a_{m-i+1} \) for \( 1 \leq i \leq m \). In other words, the terminal vertices of \( \Gamma \lambda \) are symmetric about the line \( y = x \). Thus reflecting about the line \( y = x \) sends ladder diagrams to valid ladder diagrams, denote this map \( \rho \). Since \( \rho^2 = 1 \) and \( \rho \) preserves inclusions of ladder diagrams, \( \rho \) is an automorphism of \( \mathcal{F}(\Gamma \lambda) \) (Fig. 15). \( \square \)

Remark 4.8 When \( \lambda = (1^{a_1}, 2^{a_2}, \ldots, m^{a_m}) \), \( \rho \) can be induced by the affine map \( f(x) = -Px + (m + 1) \cdot 1 \), where \( P \) is a permutation matrix and \( 1 \) is the all-ones vector. However, for \( \lambda \) of a different form, this is no longer true, even in small cases. For example when \( \lambda = (1, 2, 4) \), a straightforward computation shows that \( \rho \) cannot be induced by any affine transformation.

Proposition 4.9 (The \( m = 2 \) Rotation Symmetry) Suppose that \( m = 2 \). There is an order 2 automorphism \( \tau \) on \( \mathcal{F}(\Gamma \lambda) \) that rotates all paths from \( (0, 0) \) to \( t_1 \) by \( 180^\circ \) about \( \left( \frac{a_1}{2}, \frac{a_2}{2} \right) \) to produce paths from \( t_1 \) to \( (0, 0) \), and has the corresponding action on facets.

Proof The map \( \tau \) takes ladder diagrams to valid ladder diagrams. Since \( \tau^2 = 1 \) and \( \tau \) preserves inclusions of ladder diagrams, \( \tau \) is an automorphism of \( \mathcal{F}(\Gamma \lambda) \) (Fig. 16). \( \square \)

Proposition 4.10 (The \( m = 2 \) Vertex Symmetry) When \( m = 2 \), there are two facets \( f_1 \) and \( f_2 \) corresponding to the rightmost and topmost interior edges (note when \( m = 2 \)
there is exactly one of each. There is a symmetry \( \alpha \) which exchanges these two facets, fixing all others.

**Proof** \( \alpha \) will act on ladder diagrams as follows. There are exactly two valid paths from the origin to the terminal vertex \( t_1 \) that turn exactly once, call them \( p_1 \) and \( p_2 \). If a ladder diagram contains \( p_1 \) and not \( p_2 \), remove all edges in \( p_1 \) that are not contained in other paths, and insert all missing edges of \( p_2 \), and vice versa if a ladder diagram contains \( p_2 \) and not \( p_1 \). If a ladder diagram contains both \( p_1 \) and \( p_2 \) or neither, then \( \alpha \) fixes it. Similar to the proofs above, \( \alpha^2 = 1 \), and it preserves inclusion, thus is an automorphism of \( \mathcal{F}(GT_\lambda) \) (Fig. 17).

It is clear that the group formed by these possible generators is contained in \( \text{Aut}(GT_\lambda) \). Note that if \( m = 1 \), then the polytope is a single point with only the trivial automorphism. If \( m = 2 \) and either \( \lambda = (1, 2^{a_2}) \) or \( \lambda = (1^{a_1}, 2) \), then the polytope is a simplex and its automorphism group is the symmetric group. In all other cases, either \( m = 2 \) and \( a_1, a_2 \geq 2 \), or \( m \geq 3 \). For both of these cases, we will describe the group generated by the automorphisms listed above and finish proving that this group is the entire automorphism group in Sect. 4.4.

**Theorem 1.2** \( (m=2 \text{ Automorphisms}) \) Suppose \( \lambda = (1^{a_1}, 2^{a_2}) \) and \( a_1, a_2 \geq 2 \). If \( a_1 = a_2 = 2 \), then

\[
\text{Aut}(GT_\lambda) \cong D_4 \times \mathbb{Z}_2.
\]

Otherwise,

\[
\text{Aut}(GT_\lambda) \cong D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2^{\delta_{a_1,a_2}}.
\]
Proof of inclusion Using the notation for the automorphisms defined above, we will show that

- if $a_1 = a_2 = 2$, then $\langle \mu, \mu_1, \tau, \alpha, \rho \rangle \cong D_4 \times \mathbb{Z}_2 \subseteq \text{Aut}(\Gamma_T)$,
- if $a_1 \neq a_2$, then $\langle \mu, \mu_1, \tau, \alpha \rangle \cong D_4 \times \mathbb{Z}_2 \subseteq \text{Aut}(\Gamma_T)$,
- if $a_1 = a_2 \geq 3$, then $\langle \mu, \mu_1, \tau, \alpha, \rho \rangle \cong D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq \text{Aut}(\Gamma_T)$.

Suppose $a_1 \neq a_2$. Collecting the applicable generators discussed previously, the subgroup of automorphisms which they generate is $\langle \mu, \mu_1, \tau, \alpha \rangle$. Note that these generators satisfy the following relations: $\mu^2 = \mu_1^2 = \tau^2 = \alpha^2 = 1$, $\mu \tau = \tau \mu_1$, and all other pairs of generators commute with each other. The subgroup $\langle \tau \mu, \mu \rangle$ is isomorphic to $D_4$. The generator $\alpha$ commutes with all other generators, so the entire subgroup is isomorphic to $D_4 \times \mathbb{Z}_2$.

Next suppose $a_1 = a_2 \geq 3$. Then the subgroup of applicable automorphisms is $\langle \mu, \mu_1, \tau, \alpha, \rho \rangle$. Note that $\rho$ commutes with all of these generators, so this subgroup is isomorphic to $D_4 \times \mathbb{Z}_2^2$.

Finally if $a_1 = a_2 = 2$, then $\alpha = \rho \mu \mu_1$ so the subgroup of applicable automorphisms is isomorphic to $D_4 \times \mathbb{Z}_2$.

We take care in specifying how to write down the composition of elements of $\text{Aut}(\Gamma_T)$ as a tuple. This is not as straightforward as in the $m = 2$ case because the Flip Symmetry does not act locally and does not commute with the other symmetries.

In the statement of Theorem 1.3, elements of $\text{Aut}(\Gamma_T)$ are tuples $(\sigma_1, \sigma_2, z_1, \ldots, Z_{r_1}, z_{r_1+1}, z_{r_1+2})$. Here $\sigma_1 \in \mathcal{S}_{a_2}^{\delta_1, a_1}$ and $\sigma_2 \in \mathcal{S}_{a_m}^{\delta_1, a_m}$ correspond to the Symmetric Group Symmetries and $z_1, \ldots, Z_{r_1} \in \mathbb{Z}_2$ correspond to the $k$-Corner Symmetries. Finally $z_{r_1+1} \in \mathbb{Z}_2$ corresponds to the Corner Symmetry, $z_{r_1+2} \in \mathbb{Z}_2$ corresponds to the Flip Symmetry. Let $g \in \mathcal{S}_{a_2}^{\delta_1, a_1} \times \mathcal{S}_{a_m}^{\delta_1, a_m} \times \mathbb{Z}_2^{r_1+1}$ be such that $(\sigma_1, \sigma_2, z_1, \ldots, Z_{r_1}, z_{r_1+1}, z_{r_1+2}) = (g, z_{r_1+2})$. Then $(g, z_{r_1+2}) \cdot (g', z'_{r_1+2}) = (g \varphi(z_{r_1+2})(g'), z_{r_1+2} + z'_{r_1+2})$ where $\varphi(0)$ is the identity map on $\mathcal{S}_{a_2}^{\delta_1, a_1} \times \mathcal{S}_{a_m}^{\delta_1, a_m} \times \mathbb{Z}_2^{r_1+1}$ and $\varphi(1)$ is the map sending $(\sigma_1, \sigma_2, z_1, \ldots, Z_{r_1}, z_{r_1+1}) \mapsto (\sigma_2, \sigma_1, Z_{r_1}, \ldots, Z_{r_1}, z_{r_1+1})$. This is formally stated in Theorem 1.3.

Theorem 1.3 $(m \geq 3$ Automorphisms) Suppose $\lambda = 1^{a_1} \ldots m^{a_m}$ and $m \geq 3$. Let $r_1$ be the number of $k$ such that $a_k, a_{k+1} \geq 2$. Let $r_2 = 1$ if $\lambda = \lambda'$ and let $r_2 = 0$ otherwise. Then

$$\text{Aut}(\Gamma_T) \cong \left( \mathcal{S}_{a_2}^{\delta_1, a_1} \times \mathcal{S}_{a_m}^{\delta_1, a_m} \times \mathbb{Z}_2^{r_1+1} \right) \rtimes \varphi \mathbb{Z}_2^{r_2},$$

where if $r_2 = 1$, then $\varphi: \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{S}_{a_2}^{\delta_1, a_1} \times \mathcal{S}_{a_m}^{\delta_1, a_m} \times \mathbb{Z}_2^{r_1+1})$ sends the nonidentity element of $\mathbb{Z}_2$ to the map sending $(\sigma_1, \sigma_2, z_1, \ldots, Z_{r_1}, z_{r_1+1}) \mapsto (\sigma_2, \sigma_1, z_{r_1}, \ldots, z_1, z_{r_1+1})$.

Proof of inclusion We show that $(\mathcal{S}_{a_2}^{\delta_1, a_1} \times \mathcal{S}_{a_m}^{\delta_1, a_m} \times \mathbb{Z}_2^{r_1+1}) \rtimes \varphi \mathbb{Z}_2^{r_2} \subseteq \text{Aut}(\Gamma_T)$.

By the previous propositions, the relevant generators are $\mu, \mu_1, \ldots, \mu_{m-1}, \mathcal{S}_{a_2}, \mathcal{S}_{a_m}$ with $\mathcal{S}_{a_2}$ or $\mathcal{S}_{a_m}$ possibly omitted depending on $\lambda$. Whichever symmetries are present commute with each other since they act on disjoint sets of edges in the ladder diagrams.
Fig. 18  The gray boxes indicate coordinates $x_{i,j}$ that are equal to each other on each facet. The dashed box indicates the coordinate forced to be equal to the other three. In the third figure, the coordinates are equal to some $\lambda_j$ fixed by the partition.

Fig. 19  Examples of facets that are not dependent

If $\lambda$ is reverse symmetric, then we also have the generator $\rho$. Note that since $\rho$ flips every ladder diagram about the line $y = x$, $\rho$ satisfies the following commutation relations: $\rho \mu = \mu \rho$, $\rho \mu_i = \mu_{m-i} \rho$, and $\rho \sigma = \sigma' \rho$ for $\sigma \in S_{a_2}$ and $\sigma' \in S_{a_{m-1}}$ that correspond to each other ($a_2$ and $a_{m-1}$ must be equal for $\lambda$ to be reverse symmetric) or vice versa. By these relations, the subgroup generated by $\mu, \mu_1, \ldots, \mu_{m-1}, S_{a_2}, S_{a_{m-1}}$ is isomorphic to $(\mathbb{Z}_{a_2}^{\delta_{a_1} a_1} \times \mathbb{Z}_{a_{m-1}}^{\delta_{a_{m-1}} a_{m-1}} \times \mathbb{Z}_2^{r_1+1}) \rtimes \mathbb{Z}_2^{r_2}$.

In the following sections, we finish the proofs of Theorems 1.2 and 1.3 by bounding the order of $\text{Aut}(\Gamma_\lambda)$ by the order of the group generated by the symmetries defined above. This bound comes from examining the action of any combinatorial automorphism on the facets of $\mathcal{F}(\Gamma_\lambda)$ and applying the Orbit–Stabilizer theorem. First, we develop ways to classify and partition the facets of $\Gamma_\lambda$.

4.3 Chains of Facets

In this section, we define the facet chains of $\Gamma_\lambda$, and discuss how an automorphism can act on facet chains. As mentioned in Sect. 4.1, our goal is use facet chains to upper bound the orbits of facets under elements of the automorphism group.

Definition 4.11  Two facets are called dependent if their intersection is a $d - 3$-dimensional face.

Given two facets $F_1$ and $F_2$, a necessary condition for $F_1$ and $F_2$ to be dependent is for them to be arranged as shown in Fig. 18. As shown in Fig. 19, this is not always sufficient if there are already forced equalities amongst the four shaded coordinates.

Facet dependencies naturally partition the facets of $\Gamma_\lambda$ into facet chains which are easy to represent visually in ladder diagrams.

Definition 4.12  (Facet Chains) A facet chain $C = (F_1, \ldots, F_\ell)$ of length $\ell$ is an ordered list of facets $F_1, \ldots, F_\ell$ such that $F_j$ is dependent on $F_{j-1}$ and all facets on
which $F_1$ and $F_\ell$ are dependent are in $C$. Visually, a chain is a set of edges $e(F_j)$ of $\Gamma_\lambda$ forming a zigzag pattern. The facets $F_1, \ldots, F_\ell$ are ordered such that $e(F_1)$ has the smallest $x$-coordinate.

Every facet is part of some maximal facet chain, in the sense that no facet in the chain is dependent on a facet not in the chain. Let $C$ denote the set of maximal facet chains of $\Gamma_\lambda$. Then $C$ partitions the interior edges of $\Gamma_\lambda$. After we have this partition, it is then natural to study the relations between these maximal facet chains. For the rest of our arguments, we will never consider non-maximal chains, and so will refer to maximal chains as just chains. We define the notion of adjacency between facet chains as follows.

**Definition 4.13 (Adjacent Chains)** Two different chains $C, C' \in C$ are adjacent if there exist $F_i, F_{i+1} \in C$ and $J_j, J_{j+1} \in C'$ such that $F_i \cap F_{i+1} = J_j \cap J_{j+1}$. We say $C$ and $C'$ are adjacent at $k$ points if there are $k$ distinct sets of facets $F_i, F_{i+1}, J_j, J_{j+1}$ with $F_i \cap F_{i+1} = J_j \cap J_{j+1}$.

Visually, two chains are adjacent iif one chain sits directly to the North–East of the other chain as shown in Figs. 20 and 21.

We are now ready to study how each automorphism $\phi \in \text{Aut}(\Gamma_\lambda)$ acts on facet chains. Specifically, the action of an automorphism $\phi \in \text{Aut}(\Gamma_\lambda)$ can be extended to sets of facets. For any sets of facets $X_1$ and $X_2$, we say $\phi(X_1) = X_2$ if the restriction of $\phi$ to $X_1$ is a bijection between sets $X_1$ and $X_2$. In particular, we will often abuse notation and write $\phi(C_1) = C_2$, thinking of chains $C_1$ and $C_2$ as sets of facets. We
now state a few simple lemmas that will be essential to our proofs of Theorems 1.2 and 1.3.

**Lemma 4.14** Let $C$ be a facet chain and $\phi \in \text{Aut}(\text{GT}_\lambda)$. Then $\phi(C)$ is a facet chain of the same length as $C$.

**Lemma 4.15** Let $C$ and $C'$ be adjacent facet chains and $\phi \in \text{Aut}(\text{GT}_\lambda)$. Then $\phi(C)$ and $\phi(C')$ are also adjacent facet chains.

**Proof of Lemmas 4.14 and 4.15** Dependency between facets and adjacency between facet chains can be realized as properties of coatomes, i.e., elements covered by the maximum, in the face lattice of the GT polytope. Thus, the lemmas follow directly from the definition of an automorphism.

Given a notion of adjacency between facet chains, it is natural to examine the adjacency graph and how an automorphism can act on it.

**Definition 4.16** (Adjacency Graph) For any $\lambda$, let $G_\lambda$ be the adjacency graph of the chains of $\Gamma_1\lambda$ with length at least 2. More precisely, the nodes of $G_\lambda$ are the chains in $\mathcal{C}$ of length at least 2 and there is an edge between two nodes iff their corresponding chains are adjacent. Note that $G_\lambda$ is connected. We make $G_\lambda$ a rooted planar tree by letting its root be $C_0$, the chain of length two near the origin, and giving its nodes the ordering from their corresponding chains in $\Gamma_1\lambda$.

Note that length 2 chains are the leaves of $G_\lambda$ since longer chains are adjacent to some chain above and some chain below.

Unless specified otherwise, when talking about the action of an automorphism on nodes of $G_\lambda$, we are treating chains as unordered sets of facets so we will not specify how the automorphism acts on the facets within a chain. So if we say that an automorphism fixes a node of $G_\lambda$, we are not specifying whether the automorphism flips the chain.

**Lemma 4.17** Given facet chains $C = (F_1, \ldots, F_k)$ and $C' = (F'_1, \ldots, F'_k)$ and $\phi \in \text{Aut}(\text{GT}_\lambda)$ such that $\phi(C) = C'$, either $\phi(F_i) = F'_i$ for all $1 \leq i \leq k$ or $\phi(F_i) = F'_{k+1-i}$ for all $1 \leq i \leq k$.

**Proof** If $\phi(F_i) = F'_i$, the chain of dependencies of the $F_i$ will determine the image of each $F_i$. More specifically, $\phi(F_2)$ must be dependent on $\phi(F_1) = F'_1$, so we must have $\phi(F_2) = F'_2$, and so forth. Else if $\phi(F_1) \neq F'_1$, we must have $\phi(F_1) = F'_k$, since $F_1$ and $F_k$ are the only facets that are dependent on exactly one other facet. Again the chain of dependencies implies that $\phi(F_2)$ must be dependent on $\phi(F_1) = F'_k$, so we must have $\phi(F_2) = F'_{k-1}$, and so forth.

Under the assumptions of Lemma 4.17, if $\phi(F_i) = F'_{k+1-i}$ for all $1 \leq i \leq k$, then we say $C$ is mapped to the flip of $C'$. In particular if $\phi(C) = C$, then $\phi$ either flips or does not flip $C$. We call this the orientation of chain $C$ under $\phi$.

**Lemma 4.18** Suppose $C$ and $C'$ are adjacent facet chains with distinct facets $F_1, F_2, F_3, F_4 \in C$ and $J_1, J_2, J_3, J_4 \in C'$ such that $F_1 \cap F_2 = J_1 \cap J_2$ and $F_3 \cap F_4 = J_3 \cap J_4$. 

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Fig. 22 The correspondence between an interior point of $\Gamma_\lambda$ and the ladder diagram for a $d-3$-dimensional face.

$F_3 \cap F_4 = J_3 \cap J_4$. Equivalently, $C$ and $C'$ are adjacent at at least two points. Given an automorphism $\phi \in \text{Aut}(\text{GT}_\lambda)$ such that $\phi(C) = C$ and $\phi(C') = C'$, $C$ is flipped under $\phi$ iff $C'$ is flipped under $\phi$.

**Proof** Suppose $\phi$ does not flip $C_i$, so $\phi(F_1) = F_1$ and $\phi(F_2) = F_2$. Since $F_1 \cap F_2 = J_1 \cap J_2$, and the intersection of any pair of facets in a chain is unique, we must have $\phi(J_1), \phi(J_2) \in \{J_1, J_2\}$. Suppose for contradiction that $\phi(J_3) = J_2$ implying $C_j$ is flipped. Note flipping a chain can preserve at most one adjacent pair of facets in that chain, so we must have $\phi((J_3, J_4)) \neq \{J_3, J_4\}$. Again the intersection of two facets in a chain is unique, implying $\phi(J_3) \cap \phi(J_4) \neq J_3 \cap J_4$, but $\phi(J_3) \cap \phi(J_4) = \phi(F_3) \cap \phi(F_4) = F_3 \cap F_4 = J_3 \cap J_4$, a contradiction. The reverse direction is similar. \(\square\)

**Lemma 4.19** If an automorphism $\phi$ fixes every node of $G_\lambda$, then all nodes corresponding to chains of length greater than 2 have the same orientation under $\phi$.

**Proof** By Lemma 4.18, it only remains to prove that if two adjacent chains of length greater than 2 are adjacent at one point, then they have the same orientation under $\phi$. Notice that here when we talk about a point at which two adjacent chains are adjacent, we actually mean a $d-3$-dimensional face as the intersection. See Fig. 22 for the natural correspondence.

Consider the set of points at which two chains with length greater than 2 can be adjacent. These are the points inside $\Gamma_\lambda$, excluding the corner points near terminal vertices, whose nearby parts $a_i, a_{i+1}$ have size at least 2. We partition these points into sets as follows: two points $p_1, p_2$ are in the same set if they lie along the same diagonal and if all of the points lying on this diagonal and lying between $p_1, p_2$ are inside $\Gamma_\lambda$. See Fig. 23 for an example.

If two chains are adjacent, then the points at which they are adjacent correspond to one of these sets. So we are only concerned with adjacencies corresponding to sets of size 1. There only exist sets of size 1 corresponding to adjacencies between chains of length greater than 2 if either $a_1 = 2$ or $a_n = 2$. See Fig. 24 for an example.

WLOG assume $a_1 = 2$. Let $D_1, \ldots, D_{a_2-2}$ denote the length 3 chains such that $D_1$ is adjacent to the length 2 chain near $t_1$ and $D_k$ is adjacent to $D_{k+1}$ for $1 \leq k \leq a_2 - 3$. Once the facets in $D_1$ are fixed, the intersection of the two leftmost facets in $D_2$...
is fixed since this equals the intersection of the two rightmost facets in $D_1$. So $D_1$
 could not have been flipped, and $D_2$ is also not flipped. Similarly, we can argue that
$D_3, \ldots, D_{a_2-2}$ are not flipped. If $D_{a_2-2}$ is adjacent to a chain of length greater than
2, repeating this argument shows that this chain cannot be flipped. In particular, all of
these chains have the same orientation.

\[ \square \]

4.4 The Automorphism Group

In this section, we finish the proofs of Theorems 1.2 and 1.3. Recall that in Sect. 4.2,
we showed that the desired groups of automorphisms are contained in $\text{Aut}(\Gamma_{\lambda})$. We
now show equality by bounding the size of $\text{Aut}(\Gamma_{\lambda})$, recalling the notation of Sect.
4.1, setting $G = \text{Aut}(\Gamma_{\lambda})$.

Proof of Theorem 1.2 cont. In Sect. 4.2, we showed that $D_4 \times Z_2 \times Z^{a_1,a_2} \subseteq \text{Aut}(\Gamma_{\lambda})$.
Now we show that the order of $\text{Aut}(\Gamma_{\lambda})$ is at most the order of this group. We can verify by a finite computation that our theorem holds for the case of $\lambda = (2, 2)$, so we
assume this is not the case.

There are two facets in length 1 chains, corresponding to the right-most and top-most
interior facets (see Fig. 17). Denote these facets as $F_1$ and $F_2$, and let $X_1 := \{F_1, F_2\}$. Let $C_0 := (C_{0,1}, C_{0,2})$ denote the length 2 chain near the origin, let $C_1 := (C_{1,1}, C_{1,2})$
denote the length 2 chain near \( t_1 \), and let \( H = (H_1, H_2, H_3, H_4) \) be the chain directly North–East of \( C_0 \). Let \( X_2 := X_1 \cup \{C_{0,1}, C_{0,2} \} \) and \( X_3 := X_2 \cup \{C_{1,1}, C_{1,2} \} \).

We will argue that we have a descending chain

\[
G \supset (\{F_1\}) = G(X_1) \supset G(X_1 \cup C_{0,1}) = G(X_2) \supset G(X_2 \cup \{C_{1,1}\}) = G(X_3) \supset G(X_3 \cup H_0) = 1
\]

while computing the sizes of the relevant orbits.

Suppose \( \phi \in G \). Since there are only two facets in length 1 chains, we apply Lemma 4.14 to deduce that \( |O_G(F_1)| \leq 2 \), and furthermore if \( \phi \) fixes \( F_1 \), then \( \phi \) also fixes \( F_2 \), so \( G(\{F_1\}) = G(X_1) \). Lemma 4.14 shows that \( |O_G(X_1)(C_{0,1})| \leq 4 \), since there are only four facets in length 2 chains. If \( \phi \in G(X_1) \) fixes \( C_{0,1} \), it must also fix \( C_{0,2} \), since they are both in the same length 2 chain, so \( G(X_1 \cup \{C_{0,1}\}) = G(X_2) \).

Now if \( \phi \in G(X_2) \), then \( \phi \) can map \( C_{1,1} \) to either \( C_{1,1} \) or \( C_{1,2} \) so \( |O_G(X_2)(C_{1,1})| \leq 2 \).

Again, if \( C_{1,1} \) is fixed, then \( C_{1,2} \) is fixed so

\[
G(X_2 \cup C_{1,1}) = G(X_3).
\]

If \( \phi \in G(X_3) \), \( \phi \) fixes \( C_0 \) which is adjacent to \( H \), so Lemmas 4.15 and 4.17 imply that \( \phi \) fixes \( H \) setwise, and either flips it or not, mapping \( H_0 \) to either \( H_0 \) or \( H_3 \). Note that the tree \( G_\lambda \) is a path. Applying Lemma 4.18 and inducting from \( H \) along the path, we see that every chain of length greater than 2 is fixed and has the same orientation as \( H \). Therefore \( \phi \) can only flip \( H \) if \( a_1 = a_2 \), so we conclude that \( |O_G(X_3)(H_0)| \leq 2^{a_1,a_2} \).

Finally, we claim that \( G(X_3 \cup H_0) \) consists of the identity map. This is already argued above: if \( \phi \in G(X_3 \cup H_0) \), then \( \phi \) fixes every facet in a chain of length \( \leq 2 \) since it acts identically on facets in \( X_3 \). Also \( \phi \) does not flip \( H \), so by Lemma 4.18 and induction, every chain of length greater than 2 is not flipped. Therefore \( \phi \) fixes every facet and Lemma 4.1 implies that \( \phi \) is the identity map. We have produced the the desired descending chain of subgroups, so arguing as in Sect. 4.1, we have

\[
|G| = |O_G(F_1)| \times |O_G(X_1)(C_{0,1})| \times |O_G(X_2)(C_{1,1})| \times |O_G(X_3)(H_0)| \times |G(X_3 \cup H_0)|
\leq 2 \cdot 4 \cdot 2 \cdot 2^{a_1,a_2}.
\]

\( \square \)

**Remark 4.20** It is natural to ask how the automorphism group of \( \Gamma_\lambda \) is related to the automorphism group of the 1-skeleton of \( \Gamma_\lambda \). Clearly \( \text{Aut}(\Gamma_\lambda) \) is contained within the automorphism group of the 1-skeleton, but we do not always have equality. If \( \lambda = (3, 3) \), then \( \text{Aut}(\Gamma_\lambda) \) has size 16 while the automorphism group of the 1-skeleton has size 32. Numerical computations suggest that we may have equality in all other cases when \( m = 2 \).

The proof of Theorem 1.3 uses similar arguments. We begin by establishing some useful lemmas about the image of chains of length 1 or 2 under any automorphism.

We label the length 1 chains in \( \Gamma_\lambda \) as follows: let \( D_1 \) and \( D_2 \) be the sets of length 1 chains occurring below \( t_0 \) and to the left of \( t_m \) respectively. Figure 25 shows all length
1 chains. Occasionally we will abuse terminology and refer to the length 1 chains in $D_i$ as facets.

We label the length 2 chains in $\Gamma_\lambda$ as follows: let $C_0$ denote the length 2 chain near the origin. Let $C_2, C_3, \ldots, C_{2m-2}$ denote the length 2 chains that occur along the border of $\Gamma_\lambda$ where each even index chain $C_{2k}$ (which may not exist) occurs near terminal vertex $t_k$ and each odd index chain $C_{2k-1}$ occurs at the corner of the $k$th shaded triangular subgrid corresponding to the coordinates that are fixed by $k^{a_k}$ in $\lambda$. We call all odd index chains $C_{2k-1}, k = 2, \ldots, m-1$, the type A chains and all even index chains $C_{2k}, k = 1, \ldots, m-1$, (excluding $C_0$) the type B chains. As shown in Fig. 26, the type B chain $C_{2k}$ will not exist if $a_k = 1$ or $a_{k+1} = 1$ while all type A chains will exist. Our notation implicitly assumes that all of $C_2, \ldots, C_{2m-2}$ exist but all of our arguments hold with reference to the chains that actually exist for any given $\lambda$.

**Definition 4.21** We say that a vertex $v$ of $GT_\lambda$ contains a facet $F$ if the ladder diagram corresponding to $v$ contains $e(F)$, the edge corresponding to $F$.

With this definition, a vertex $v$ contains a facet $F$ iff $v \notin F$, i.e., this is a combinatorial property of the polytope. We can use this to distinguish type A and type B chains; for each type A chain, there is a vertex containing both facets in the chain. There is no vertex containing both facets in a type B chain. In particular, the image of a type A chain under an automorphism is a type A chain, and the same goes for type B chains. This idea is made more precise in Lemma 4.22.
Lemma 4.22 If $\phi \in \text{Aut}(\Gamma \lambda)$ then

1. $\phi(C_0) = C_0$
2. The chains $D_1$, $D_2$, $C_2$, $C_3$, $\ldots$, $C_{2m-2}$ can be ordered into a sequence $D_1$, $C_2$, $C_3$, $\ldots$, $C_{2m-2}$, $D_2$ such that any $\phi \in \text{Aut}(\Gamma \lambda)$ either preserves the ordering within this sequence or reverses the order of the sequence.

Figure 27 depicts the sequence of chains mentioned in Lemma 4.22.

Proof By Lemma 4.14, $\phi$ preserves lengths of chains. We say two length 1 chains are incompatible if there does not exist a vertex containing both of the facets in the chains and we say they are compatible otherwise. Note that all chains in $D_1$ are incompatible with each other but compatible with any chain in $D_2$. The set of chains that is the image of $D_1$ under $\phi$ must all be incompatible with each other, so $\phi$ must map all chains of $D_1$ to chains in $D_1$ or it must map all chains of $D_1$ to chains in $D_2$. We will now show that if the latter happens, $\phi$ must reverse the order of the sequence. (Lemma 4.23 will show that this occurs only if $\lambda$ is reverse-symmetric).

Any $\phi \in \text{Aut}(\Gamma \lambda)$ must send $C_0$ to itself because it is the only length 2 chain such that for either of its facets, there exist vertices containing this facet and any facet in a type A chain. Note that $\phi$ may send the facets of $C_0$ to each other.

Chains in $D_i$ and a type A chain are incompatible if there does not exist a vertex containing the facet in $D_i$ and the facets of the A chain. Visually, the type A chain closest to the length 1 chains in $D_i$ will be the only type A chain incompatible with it. Two type A chains are incompatible if there does not exist a vertex containing all four facets in these chains. Visually, two type A chains are incompatible iff they occur at the corners of adjacent shaded triangular subgrids (these are the subgrids corresponding to the fixed entries of $\mathbb{T}_n$). We can form a sequence starting with $D_1$ where two consecutive elements in this sequence are incompatible. Visually, this sequence corresponds to reading $D_1$, the type A chains from left to right, and then $D_2$.

Now we include type B chains in this sequence. A type B chain and $D_i$ will be compatible if there exists a vertex containing one of the facets in the type B chain and a facet in $D_i$. A type B chain and a type A chain are compatible if there exists a vertex containing at least one of the facets of the type B chain and both facets of the type A chain. If two chains are not compatible, we say they are incompatible. Visually, a type
B chain is incompatible only with the chains adjacent to it. So we can insert the type B chains into the sequence to obtain the sequence consisting of $D_1$, all length 2 chains from left to right, followed by $D_2$. Every two consecutive elements in this sequence are incompatible.

Note that $\phi \in \text{Aut}(\Gamma_\lambda)$ must preserve these relations of incompatibility so $\phi$ must preserve or flip the sequence $D_1, C_2, C_3, \ldots, C_{2m-2}, D_2$, flipping iff $\phi(D_1) = D_2$. We note that even if $\phi$ takes a length two chain to itself, it is possible that $\phi$ exchanges the two facets in the chain.

\[ \text{Lemma 4.23} \quad \text{If the sequence of chains } D_1, C_2, C_3, \ldots, C_{2m-2}, D_2 \text{ is flipped by some } \phi \in \text{Aut}(\Gamma_\lambda), \text{ then } \lambda = \lambda'. \]

\textbf{Proof} Assume this sequence of chains is flipped. Recalling the definition of the rooted tree $G_\lambda$, we can define the depth of a node as one plus the number of edges between it and the root. If we embed the grid $\Gamma_\lambda'$ into the Cartesian plane with the origin at $(0,0)$, then each chain $C$ lies above a line $y = -x + d$ where $d$ is the depth of $C$ (this follows inductively after observing that the root $C_0$ lies above $y = -x + 1$ and has depth 1).

Since the terminal vertices lie on line $y = -x + n$, the depth of $C_{2k-1}$ is $n - a_k$. Recall that $\phi \in \text{Aut}(\Gamma_\lambda)$ must send type A chains to type A chains. Since $\phi$ is an automorphism of $G_\lambda$ that fixes the root $C_0$, it preserves the depth of each node so if $\phi$ flips the sequence of length 2 chains, then it sends $C_{2k-1}$ to $C_{2m-2k+1}$ so $a_k = a_{m-k+1}$ for $2 \leq k \leq m - 1$, so we just need to show that $a_1 = a_m$.

Note $|D_1| = \begin{cases} a_2 & a_1 = 1 \\ 1 & \text{else} \end{cases}$ so if $a_1 = 1$ and $a_2 > 1$, the fact that $\phi(D_1) = D_2$ allows us to conclude that $a_m = 1$, so let us assume this is not the case, i.e. $|D_1| = 1$. Let $D_1'$ be the facet corresponding to the length 1 chain in $D_1$. Define the distance between $D_1'$ and $C_3$ (which always exists) to be the cardinality of the minimal set of facets $S$, containing no other facets from this sequence, such that the intersection of the facet in $D_1'$, some facet from $C_3$, and the facets in $S$ is empty (this occurs when $S$ is chosen to force some $x_{ij}$ to equal both 1 and 2). We give examples of such sets in Fig. 28. This distance is $a_1 + a_2 - 2$ since $S$ consists of facets on the path visually separating the coordinates fixed as 1 and 2 by $\lambda$. Similarly, letting $D_2'$ be the rightmost chain in $D_2$, the distance between $D_2'$ and $C_{2m-3}$ is $a_{m-1} + a_m - 2$. Note that $\phi$ must preserve this distance, as this property can be realized in the face lattice of the polytope. So if $\phi$ flips the sequence of chains, then $a_1 + a_2 = a_{m-1} + a_m$. Since we know $a_2 = a_{m-1}$, we conclude that $a_1 = a_m$, and hence $\lambda = \lambda'$.

Now we finish the proof of Theorem 1.3.

\textbf{Proof of Theorem 1.3 cont.} In Sect. 4.2, we showed that $(G_{a_2}^{a_1,a_1} \times G_{a_{m-1}}^{a_1,a_{m-1}} \times \mathbb{Z}^{r_1+1}_2) \mathrel{\triangleright} \mathbb{Z}^{r_2}_2 \subseteq \text{Aut}(\Gamma_\lambda)$. Now we show that the order of $\text{Aut}(\Gamma_\lambda)$ is at most the order of this group. Set $G = \text{Aut}(\Gamma_\lambda)$. As in the statement of Theorem 1.3, let $r_1$ be the number $k$ such that $a_k, a_{k+1} \geq 2$ (note this is equal to the number of type $B$ chains that exist) and $r_2 = 1$ if $\lambda = \lambda'$, 0 otherwise.

Let $X_1$ denote the set of facets in length 1 chains, and $X_2$ denote the facets in length 2 chains. Following the argument of Sect. 4.1, we want to show that $|G|/|G(X_1 \cup X_2)| \leq (a_2)^{a_1,a_1}(a_{m-1})^{a_m,1}2^{r_1+1}2^{r_2}$ and that $|G(X_1 \cup X_2)| = 1$, i.e. $G(X_1 \cup X_2) = 1$.
We will bound $|G|/|G(X_1)|$ in the case $a_1 = a_m = 1$. The other cases are similar, and easier. Let $f_{1,1}, \ldots, f_{1,a_2}$ be the facets that form the length 1 chains in $D_1$, and $f_{2,1}, \ldots, f_{2,a_{m-1}}$ be the facets that form the length 1 chains in $D_2$. By Lemma 4.23, $|O_G(f_{1,1})| \leq a_2 \cdot 2^r_2$, since $f_{1,1}$ can only map to $f_{i,j}$ since these are all the facets in length 1 chains, and $f_{1,1}$ maps to $f_{2,j}$ implies that $\lambda = \lambda'$. Set $G_{1,1} := G([f_{1,1}])$ and let $\phi \in G_{1,1}$. Since $\phi(f_{1,1}) = f_{1,1}$, Lemma 4.22 implies that $\phi(D_1) = D_1$, and thus $|O_{G_{1,1}}(f_{1,2})| \leq a_2 - 1$. Inductively set $G_{1,k} := G_{1,k-1}([f_{1,k}])$, and use the same argument to conclude that $|O_{G_{1,k-1}}(f_{1,k})| \leq a_2 - k$. Set $G_{2,1} := G_{1,a_2}(f_{2,1})$, $G_{2,j} := G_{2,j-1}(f_{2,j})$ for $2 \leq j \leq a_{m-1}$, and similarly conclude that $|O_{G_{2,j-1}}(f_{2,j})| \leq a_{m-1} - j$. Note that $G_{2,a_m} = G(X_1)$ and we have a chain of subgroups

$$G := G_{1,0} \supset G_{1,1} \supset \cdots \supset G_{1,a_2} \supset G_{2,1} \supset \cdots \supset G_{2,a_{m-1}} = G(X_1)$$

so following the computation in Sect. 4.1,

$$|G| = \prod_{k=0}^{a_2-1} |O_{G_{1,k}}(f_{1,k+1})| \times |O_{G_{1,a_2}}(f_{2,1})| \times \prod_{j=1}^{a_{m-1}-1} |O_{G_{2,j}}(f_{2,j+1})| \times |G(X_1)|$$

$$\leq 2^r a_2! a_{m-1}! |G(X_1)|$$

As mentioned, the cases when $a_2 > 1$ or $a_m > 1$ are virtually identical, only needing some re-indexing of the $f_{i,j}$.

So now we want to show that $|G(X_1)|/|G(X_1) \cup X_2| \leq 2^{r_1+1}$. If $\phi \in G(X_1)$ we know that $\phi(D_1) = D_1$, so by Lemma 4.22, $\phi$ must map every length 2 chain to itself. By a similar argument as above, we just have to argue that each facet in a type A chain is fixed (since there are $r_1 + 1$ length 2 chains that are not type A, which are free to flip). Note that each facet in $D_1$ is fixed. Consider the type A chain incompatible with the chains in $D_1$. There does not exist a vertex containing a facet in $D_1$ and the vertical facet in this type A chain, while the same is not true for the horizontal facet in the type A chain. Similarly, if the facets in this type A chain are fixed, then the facets in the next type A chain incompatible to it must also be fixed. Arguing inductively, we see that the facets in all type A chains must be fixed.
So now it suffices to show that $G(X_1 \cup X_2) = 1$. Consider any automorphism $\phi \in G(X_1 \cup X_2)$. We can argue up the rooted tree $G_{\lambda}$ to show that all of its vertices are fixed under $\phi$. Specifically, $\phi$ must preserve adjacencies between chains so it is an isomorphism of the tree fixing the leaves of $G_{\lambda}$ and hence must act trivially on the nodes. Furthermore, Lemma 4.19 implies that the orientation of any two non-leaves must be the same. If $G_{\lambda}$ has at least two leaves, then the common ancestor of these two leaves cannot be flipped by $\phi$ or else the direct children of this node cannot be fixed. Otherwise, $G_{\lambda}$ is a chain of nodes. If $m \geq 4$, then there are always at least two type A chains and hence at least two leaves. Similarly, if $m = 3$ and either $a_1, a_2 \geq 2$ or $a_2, a_3 \geq 2$, then there will be one type A chain and at least one type B chain. So $G_{\lambda}$ can only be a chain of nodes if $\lambda = (1, 2^{a_2}, 3)$ or $\lambda = (1^{a_1}, 2, 3^{a_3})$. In the first case, $G_{\lambda}$ contains two nodes, both of which have length 2. In the second case, WLOG assume that $a_1 \geq 2$. Consider the chain ending at a facet $F$ that shares a vertex of $\Gamma_{1\lambda}$ with the facet in $D_1$. An example of this chain is shown in Fig. 29. If this chain were flipped, then there exists a vertex containing the image of $F$ and the facet in $D_1$, but there does not exist a vertex containing $F$ and the facet in $D_1$. Hence all of the non-leaves in $G_{\lambda}$ are not flipped, allowing us to conclude that $G(X_1 \cup X_2) = 1$ and that $\text{Aut}(G_{\lambda}) \cong (\mathbb{S}_{a_2}^{\delta_1, a_1} \times \mathbb{S}_{a_m-1}^{\delta_1, a_m} \times \mathbb{Z}_{r_1+1}^{r_1}) \ltimes \varphi \mathbb{Z}_2^{r_2}$. \hfill \square

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