Calculating Subgroups with GAP

Alexander Hulpke

Department of Mathematics, Colorado State University, 1874 Campus Delivery, Fort Collins, CO, 80523-1874, USA,

\texttt{hulpke@colostate.edu}, \texttt{http://www.math.colostate.edu/~hulpke}

Abstract. We survey group-theoretic algorithms for finding (some or all) subgroups of a finite group and discuss the implementation of these algorithms in the computer algebra system \texttt{GAP}.

One of the earliest questions posed for the development of group theoretic algorithms has been the determination of the subgroups of a finite group $G$, as well as the associated lattice structure.

Since $G$ acts on its subgroups an obvious storage improvement is to store the subgroups as conjugacy classes, representing each class by a subgroup $U$ and a transversal of coset representatives of $N_G(U)$ in $G$.

The purpose of this article is to survey the methods that are currently in use for such computations, not with an aim to supersede the original descriptions \cite{Neu60,Hul99,CCH01,Hul13a} or to give an implementable description, but to given an overview of the methods employed. This should allow the reader to understand the interplay of the methods employed, computational tools required, scope of calculations, and potential for adaption or modifications by users.

On the way we will indicate a number of open problems, whose solution would lead to improvements of theoretical or practical aspects of the algorithms.

While we shall point to the \texttt{GAP} functions that implement the respective functionality, we shall stop short of printing transcripts of system sessions, instead the reader is referred to the system documentation.

Neither is this paper intended as a complete survey of Computation Group theory over its history of at least 60 years. We thus do not aim to cite every relevant work, but give preference to handbooks or summary articles that are often easier accessible.
We will illustrate the scope of calculations by assuming a contemporary (as of 2017) standard desktop machine with a 3.5GHz processor (utilizing just a single core) and 8GB of memory.

1 Tools Required

In general we will represent a subgroup $S$ of the finite group $G$ by a set of generators, given as elements of $G$. One may think of $G$ as the group containing all transformations of a given kind — for example in the case of permutations a symmetric group $S_n$ or even the finitary symmetric group on positive integers. Similarly, in the case of matrices this group might be the full general linear group.

We thus need methods that allow us to determine for such a subgroup $S$:

- The order of $S$.
- Test whether an element of $G$ is contained in $S$, and if so:
  - Express an element of $S$ as a word in the given generators of $S$, thus enabling us to evaluate homomorphisms.
  - Write a presentation for $S$ in a given generating set, thus testing whether a map on generators is a homomorphism. (In practice one often does not use an arbitrary generating set, but a specific one that allows for a nicer presentation.)
  - Determine a composition series, a chief series and the radical $\text{Rad}(G)$ (the largest solvable normal subgroup) of $G$, as well as a representation of $G/\text{Rad}(G)$ as a permutation or matrix group.

For permutation groups, such functionality is obtained through a stabilizer chain [HEO05 Chapter 4], respectively [Ser03]. For matrix groups, such functionality is provided by the data structure of a composition tree [BHLGO15,NS06]. These tools can be extended to groups of other classes of invertible transformations of a finite object using the “black-box” paradigm [BBS09].

For solvable groups, polycyclic generating sets (that is a set of generators that is adapted to a composition series and allows for an effective normal form) provide such functionality [LNS84], see also [HEO05 Chapter 8].
1.1 Complexity

For solvable groups, polynomial time algorithms are known for all of these tasks.

For permutation groups, the known algorithms are proven to be polynomial time, as long as no composition factor of type $2G_2(q)$ occurs (in which case the result will still be correct, but the time bound is not known to hold.). In fact, the algorithms are almost linear (linear up to logarithmic factors) time in a Las Vegas probabilistic setting (see 1.2 below).

Open Problem 1 Show that the groups $2G_2(q)$ have a short presentation in the sense of [BGK+97]. Such a result will allow the removal of the qualifier in the previous paragraph.

The complexity situation for matrix groups [BBS09] is as with permutation groups with one further complication: $\text{GL}_n(q)$ contains cyclic subgroups (Singer cycles) of order $q^n - 1$, and calculations in these groups are equivalent to Discrete Logarithm problems. The proven complexity is therefore also up to a Discrete Logarithm “oracle”, that is the cost of discrete logarithm calculations is not accounted for.

These polynomial time algorithms for solvable and for permutation groups have been fully implemented in GAP and in Magma. The available implementations for the matrix group algorithms involve many, but not all, of the polynomial time methods. The reason for this is that there are number of algorithms for subtasks that perform better in practice than the generic black-box algorithm, but so far no proof of polynomial time has been found.

Arbitrary finitely presented groups will require the use of a faithful representation in one the representations discussed before.

1.2 Random Elements

Some of the algorithms utilize random selections of elements. It thus seems appropriate to briefly address this issue.

First, on the computer random selection is always based on a random number generator, and thus is inherently pseudo-random.
Secondly, once we can test membership of elements, the underlying data structures allow us to construct a bijection between $G$ and the numbers $1,\ldots,|G|$ and thus select elements of the same random quality as the random number generator provides.

Some of the functions to build basic data structures also utilize pseudo-random elements which are obtained as pseudo-random products of generators and inverses \cite{CLGM95,BP04}. All of these calculations then involve verification steps that ensure the returned result is always correct, regardless of the random choices or the quality of randomness.

As far as complexity is concerned, any such algorithm then lies in a class denoted by “Las Vegas”: That is the algorithm will always return a correct result and will, with a user-chosen probability $0 < \varepsilon < 1$, terminate in the given time. However with probability $1 - \varepsilon$ the calculation will take longer (but will eventually terminate with a correct result).

### 1.3 Mid-level tools

Building on these tools, a number of mid-level tools obtain structural group-theoretic information:

- For $S \leq G$, representatives of the cosets of $S$ in $G$ \cite{DM88}.
- The centralizer $C_G(g)$ of elements $g \in G$ as well as conjugating elements $x$ that for given $g, h \in G$ satisfy $g^x = h$ (if they exist). (For permutation groups this is a backtrack search, following \cite{Leo91}.
- The normalizer $N_G(S)$ of a subgroup $S \leq G$ as well as conjugating elements $x$ that for given $S, T \leq G$ satisfy that $S^x = T$ \cite{Leo97}.
- Representatives of the conjugacy classes of elements of $G$ \cite{MN89,CS97,Hu100,CH06,Hu13b}.
- Representatives of Sylow subgroups of $G$ for a chosen prime.
- For a normal subgroup $N \unlhd G$, representatives of the $G$-classes of complements to $N$ in $G$, provided that $N$ is solvable \cite{CNW90}. This algorithm is based on cohomology through a presentation for $G/N$.
- If $G/N$ is solvable, complements can be computed in a combination of cohomology and reduction to subgroups \cite{Hu13a}.
– Determine an effective\(^1\) isomorphism between two groups \(G\) and \(H\) (or show that no such isomorphism can exist) [O'B94,CH03].

These algorithms are typically not of polynomial, but exponential worst case time complexity. However in most cases of practical interest they tend to work well, allowing for them to be used as building blocks for larger calculations.

\[
\text{2 The Basic Structure}
\]

The basic structure underlying most subgroup calculations and the one we shall use is based on the solvable radical (or trivial fitting) paradigm [BB99,Hol97,CS97], as depicted in Figure 1.

Let \(G\) be a finite group, \(R = \text{Rad}(G)\) and \(\varphi: G \to G/R =: F\). Then \(S = \text{Soc}(F) = \prod T_i\) must be the direct product of nonabelian simple groups \(T_i\). We thus can assume that \(F\) is represented as a subgroup of \(\text{Aut(Soc}(F))\); that is as a subgroup of a direct product of groups of the form \(\text{Aut}(T_i) \wr S_{m_i}\) for \(T_i\) simple and \(\sum m_i\) the number of simple factors of \(S\).

The action of \(F\) on the socle factors has a kernel denoted by \(P\text{ker}\), the factor \(P\text{ker}/S\) is a direct product of subgroups of outer automorphisms. We denote by \(S\) and \(P\text{ker}\) the full preimages of these subgroups in \(G\).

We now determine subgroups in the following way.

1. Subgroups of the simple socle factors \(T_i\).
2. Combine these to subgroups of \(\text{Soc}(F)\).
3. Calculate the subgroups of \(F/\text{Soc}(F)\) (which will be a much smaller group than \(F\)).
4. Extend the subgroups of \(\text{Soc}(F)\) to subgroups of \(F\) by using the subgroups of \(F/\text{Soc}(F)\).
5. Determine a series of normal subgroups \(R = R_0 > R_1 > R_2 > \cdots > R_k = \langle 1 \rangle\) with \(R_i \lhd G\) and \(R_i/R_{i+1}\) elementary abelian.
6. Determine subgroups of \(G/R_{i+1}\) from subgroups of \(G/R_i\) (initialized for \(i = 0\) with \(G/R_0 = F\)) and the \(G\)-module action on \(R_i/R_{i+1}\). Iterate.

\(^1\) Meaning that it, and its inverse can be applied to group elements to obtain the image.
Typically we will store not all subgroups of a group $G$, but only representatives of the conjugacy classes under $G$, since this saves substantially on the memory requirements. This enumeration up to conjugacy can be translated for each of these steps to conjugacy under suitable actions. For example in step 1 it is conjugacy by the subgroup of Aut($T_i$) induced through the action of $N_F(T_i)$. Finding representatives up to conjugacy can in general mean that we have to do explicit subgroup conjugacy tests. In some steps of the algorithm (say when calculating complements by cohomological methods) such tests can be preempted or reduced by using other equivalences amongst the objects constructed.

In the following more detailed description we shall focus on the task of finding all groups rather than the elimination of conjugates.

Methods similar to section 4.1 can then be used to determine the incidence structure of the full subgroup lattice.

3 The steps of the algorithm

We now describe the different steps of the algorithm in more detail:
3.1 Factor Groups

A fundamental paradigm of the approach is to work in homomorphic images. This raises the question of how to represent factor groups of $G$ in a suitable way. While this is difficult in general, for the particular factor groups required here effective solutions exist:

- It has been shown \cite{LS97, Hol97} that for permutation groups $G$, the factor $G/\mathrm{Rad}(G)$ can be (constructively) represented with permutation degree not exceeding that of $G$. (In \textsc{GAP} this is a call to \texttt{NaturalHomomorphismByNormalSubgroup(G,\text{RadicalGroup}(G))}. More generically, the special structure of $G/\mathrm{Rad}(G)$ as a subgroup of a direct product of wreath products allows for a representation of moderate degree, using imprimitive wreath products.

- By Schreier’s conjecture (as proven in \cite{Fei80}), for a simple group $T$ the factor $\mathrm{Aut}(T)/T$ is small. Thus $F/\mathrm{Soc}(F)$ (which embeds into a direct product of groups of the form $(\mathrm{Aut}(T_i)/T_i) \wr S_m$) is comparatively small and can be easily represented in an ad-hoc way.

- In many cases it is not necessary to represent a factor group $G/N$ faithfully, but it is sufficient to use representatives of elements and full preimages of subgroups. In particular, we can use this to perform linear algebra with coefficient vectors for the abelian factors $R_i/R_{i+1}$ of the radical.

The question of the minimal permutation degree of factor groups of permutation groups has been studied also theoretically, and one can ask for other classes of normal subgroups for which such degree bounds hold:

**Open Problem 2** Extending the work of \cite{EP88}, describe (constructively) cases in which for permutation groups or matrix groups $G$ and $N \triangleleft G$ one can represent the factor group $G/N$ in degree not exceeding that of $G$.

3.2 Subgroups of simple groups

Step 1 (from page 5) asks us to determine the subgroups of a simple group $T$. 
The basic method for this is the “cyclic extension” algorithm, dating back to [Neu60]: A subgroup $S \leq T$ is either perfect, or there is a smaller subgroup $S' \leq U < S$ such that $S = \langle U, n \rangle$ with $n \in N_G(U)$. Thus:

a) Initialize the perfect subgroups of $T$. This requires a precomputed list of isomorphism types of perfect groups such as [HP89] for groups of order at most $10^6$. (By now, due to the rapid progress in computer engineering, the same methods would allow us to build such lists for larger orders.)

Then, in an approach close to isomorphism test algorithms, search for isomorphic copies of each of these groups as subgroups of $T$. In GAP such a list is obtained using the operation `RepresentativesPerfectSubgroups`.

b) For every subgroup $U$ listed so far, classify the $U$-orbits of elements of $N_G(U)$ outside $U$. If for an orbit representative $n$ the group $\langle U, n \rangle$ is not yet known (i.e. not conjugate to a known group) then add it to the list. Iterate.

To allow for an efficient storage/comparison of subgroups, the algorithm maintains a list of cyclic subgroups of prime power order (called zuppos by their German acronym). It then represents every subgroup as a bit list indicating which zuppos it contains.

Simple groups tend to have relatively few subgroups, enabling the calculation of subgroups even for large group orders. The assumed standard computer will calculate the subgroups of a simple group of order $10^5$ in under a minute, order $10^6$ about 5-10 minutes and (provided the potential perfect subgroups are available) order $10^7$ about 90 minutes. (This is assuming that the group is given as a permutation group of minimal degree.)

The algorithm of course also will work for groups that are not simple, but in this case is often not competitive.

In GAP, this algorithm is implemented by the command `LatticeByCyclicExtension`.

In practice, we can (using this algorithm) create a database of subgroups of simple groups $T$ up to a certain order limit once, and then store them. If the algorithm then is called for one of these simple groups, one then simply can fetch subgroups from the database.

---

2 “Zyklische Untergruppen von Primzahl-Potenz Ordnung”
GAP does exactly this, the databases used to obtain subgroup information is the library of tables of marks, provided by the tomlib package (which will be loaded automatically, if available). As of writing, this library contains full subgroup data for most of the simple groups in the ATLAS of order roughly up to $10^7$. Some information about maximal subgroups of symmetric and alternating groups is also obtained through the library of primitive groups.

This approach requires an isomorphism between the concrete simple group $T$ and its incarnation $D$ in the database. Such an isomorphism can be facilitated in many cases through the use of so-called standard generators \cite{Wil96}: For a simple group $T$, this is a pair of elements $a, b \in T$ such that:

- $T = \langle a, b \rangle$. (By \cite{AG84} every finite simple group can be generated by two elements.)
- The pair $(a, b)$ (that is its $\text{Aut}(T)$-orbit) is characterized by simple relations, such as orders of $a$ and $b$ or short product expressions in $a$ and $b$, or $T$-class memberships of $a$ and $b$. This implies that if $T_1 \cong T_2 \cong T$ an isomorphism $T_1 \to T_2$ is obtained by finding instances of standard generators $a_1, b_1 \in T_1$ and $a_2, b_2 \in T_2$ and constructing the homomorphism that maps $a_1$ to $a_2$ and $b_1$ to $b_2$.
- In a given instance of $T$, such a pair $(a, b)$ can be found quickly by only using basic group operations such as product and inverse (thus allowing for pseudo-random elements) and element order. A typical property achieving this is if the elements lie in small conjugacy classes that are powers of large conjugacy classes: A (pseudo-)random element will likely lie in a large class, by powering we get an element in the small class and only few conjugates to consider.

For example $|a| = 2$, $|b| = 3$, $|ab| = 5$ could be used as such a generating set for $A_5$.

Such standard generators have been defined for all sporadic groups and many groups of Lie type of small order.

**Open Problem 3** Generalize “standard generators” to all quasisimple groups of Lie type.

The concept of standard generators can be generalized to constructive recognition, that is the task to find an isomorphism from a
simple group $T$ to its stored database incarnation $D$, without relying on the need to find specific generators, but rather “rebuilding” natural combinatorial structures from within the group. For example, if the group $T$ is a matrix group isomorphic to $A_n$, one might want to find a subspace of the natural module that has an orbit of length $n$ under $T$, thus providing such an isomorphism through the action on the subspaces in the orbit. See the survey [DLGO15] for formal definitions and details.

3.3 Subdirect products

Step 2 combines the subgroups of direct factors to those of a direct product. By induction it is sufficient to consider the case of a direct product of two groups, $G \times H$. Let $S \le G \times H$ and denote the projection from $S$ to $G$ by $\alpha$ and that from $S$ to $H$ by $\beta$. The image groups $A = S^\alpha$ and $B = S^\beta$ then are subgroups of $G$, respectively $H$.

Given such subgroups $A$ and $B$, the construction of a subdirect product (which dates back at least to [Rem30]) then allows to construct all groups $S$ (see Figure 2):

\[
\begin{array}{c}
A/D \xrightarrow{\chi} S \xrightarrow{\beta} B \xrightarrow{\sigma} B/E \\
\langle 1 \rangle \xleftarrow{\alpha} D \xrightarrow{\beta} E \xrightarrow{\langle 1 \rangle} \\
\langle 1 \rangle \xleftarrow{\alpha - \ker \alpha} \xrightarrow{\ker \beta - \beta} \langle 1 \rangle \\
\langle 1 \rangle
\end{array}
\]

Fig. 2. Subdirect product construction

Denote by $D \triangleleft A$ the image of $\ker \beta$ under $\alpha$ and by $E \triangleleft B$ the image of $\ker \alpha$ under $\beta$. Then by the isomorphism theorem

\[ A/D \cong S/\langle \ker \alpha, \ker \beta \rangle \cong B/E. \]
If $\chi: A/D \to B/E$ is this isomorphism, and we denote the natural homomorphisms by $\varrho: A \to A/D$ and $\sigma: B \to B/E$, then

$$S = \{(a, b) \in G \times H \mid a \in A, b \in B, (a \varrho)^\chi = b \sigma\}.$$ 

To construct all subdirect products $S$ corresponding to the pair $A, B$, we thus classify pairs of normal subgroups $D \triangleleft A$, $E \triangleleft F$ together with isomorphisms $\chi: A/D \to B/E$.

Conjugacy of subgroups by $N_G(A) \times N_G(B)$ will induce equivalences on the normal subgroups and amongst the isomorphisms.

In the case we consider – subgroups of $\text{Soc}(F)$ – furthermore there may be a conjugation action of $F$ on the direct factors of its socle that causes further fusion of subgroups.

### 3.4 Normal Subgroups and Complements

In steps 4 and 6 of the calculation, we have a normal subgroup $N \triangleleft G$ and know the subgroups of $G/N$ as well as the subgroups of $N$. (In step 6 the normal subgroup $N$ is a vector space whose subgroups are easily enumerated.) From these we want to construct the subgroups of $G$.

We first analyze the situation: Let $S \leq G$ and set $A = \langle N, S \rangle$ and $B = S \cap N \triangleleft B$. (See Figure 3 left.)

![Fig. 3. Complement situations for subgroups](image-url)
A) Abelian Normal subgroup  We consider first the case that $N$ is abelian (which arises in step 6). Then $B \triangleleft N$ and thus $B \triangleleft \langle S, N \rangle = A$. 

Thus $S/B$ is a complement to $N/B$ in $A/B$. As $N/B$ is elementary abelian, such complements can be obtained through cohomological methods, following [CNW90]. The input to such a computation is the linear action of $A$ on $N/B$, together with a presentation for $A/N$.

To find all subgroups of $G$, we iterate through all $A$ (as subgroups of $G/N$) and for each $A$ determine candidates for $B$ as submodules of $N$ under the action of $A$ [LMR94].

As step 6 then iterates over a series, a crucial step towards efficiency is to extend a presentation for $A/N$ to a presentation for $S$, if $S/B$ is such a complement. This is easy, as $B$ is elementary abelian.

B) Nonabelian Normal subgroup  If $N$ is not abelian (as it will be in step 4), the situation is more complicated, as $B$ is not necessarily normal in $A$, and there is no algorithm to easily determine complementing subgroups. In this case, following [Hul13a], we iterate through the possible subgroups $B \leq N$ and for each such $B$ determine the groups $S$ such that $S \cap N = B$:

As $N \leq \langle N, S \rangle = A$, we have that $N_N(B) \leq N_A(B) = \langle S, N_N(B) \rangle \leq N_G(B)$. Furthermore, $N_G(B)/N_N(B)$ is isomorphic to a subgroup of $G/N$. (See Figure 3 right.) In this situation $S/B$ is a complement to $N_N(B)/B$ in $N_A(B)/B$.

Given a subgroup $B \leq N$, we thus determine the subgroups of $N_G(B)/N_N(B)$ (e.g. from the subgroups of $G/N$) and for each subgroup $N_A(B)/N_N(B)$ determine the candidates for $S/B$ as complements. If $N_N(B)/B$ is solvable, this again can be done using cohomology calculations.

The group $N_N(B)/B$ does not need to be solvable – if the factor group however is solvable (which will be the case unless Soc$(F)$ contains a single simple factor at least quintuply, in which case there will be storage problems already for the subgroups of Soc$(F)$), [Hul13a] describes an approach for complements that reduces to $p$-groups, corresponding to a chief series of the factor.
GAP contains a function \texttt{ComplementClasses Representatives\,(G,N)} that determines representatives of the classes of complements to \(N\) in \(G\), up to conjugacy by \(G\), provided that \(N\) or \(G/N\) are solvable.

In the case that neither \(G\) and \(G/N\) are solvable, no algorithm for complements exists yet:

\textbf{Open Problem 4} \textit{Find a good algorithm for determining complements if both normal subgroup and factor groups are not solvable. This also has relevance to maximal subgroup computations [CH04].}

### 3.5 Implementation

In GAP the algorithm described in the previous sections (with some variants depending on the representation of the groups) is obtained through the operation \texttt{ConjugacyClassesSubgroups}. It takes as argument a group and returns a list of conjugacy classes of subgroups. For each class \texttt{Representative} will return one subgroup; \texttt{AsList} applied to a class will return all subgroups in this class, thus

\[
\text{Concatenation(List(ConjugacyClassesSubgroups(G),AsList))};
\]

returns all subgroups of a group \(G\). In general such an enumeration of all subgroups is not recommended as it is very costly in terms of memory.

It is also possible to visualize the full lattice of subgroups of a group \(G\). For this, the command

\[
\text{DotFileLatticeSubgroups(LatticeSubgroups(G),"filename.dot")};
\]

produces a text file, called \texttt{filename.dot} (or whatever file name is given) that describes the incidence structure of the subgroup lattice in the \texttt{graphviz} format (see \url{www.graphviz.org} for a description and for viewer programs for this format. There also are programs to convert this format into others, e.g. \texttt{dot2tex} converts to \texttt{TikZ} or \texttt{PSTricks} format.

Figure 4 illustrates the result in the example of the symmetric group \(S_4\). Rectangles represent normal subgroups, circles ordinary subgroups and their conjugates. A number \(a – b\) indicates group number \(b\) in class \(a\) (there is no \(b\)-part if the group is normal, as it will default to \(b = 1\)).
Fig. 4. Subgroup Lattice for $S_4$

This group can be obtained in GAP then as $c1[a][b]$ (that is $c1[8][3]$ for $a = 8$ and $b = 3$) where $c1:=ConjugacyClassesSubgroups(G)$.

**Caveat:** The ordering (both $a$ and $b$-parts) of subgroups can involve ad-hoc choices within the algorithm. When creating the group $G$ a second time with the same generators, it is possible that a different numbering is chosen. It thus is not safe to use the $a-b$ indices for specifying a concrete subgroup outside a particular run of GAP.

### 3.6 Practicality and Modifications

With the construction process proceeding through layers, in each step proceeding through all subgroups found in the previous step, the limiting factor to calculation is (as timings in [Hul13a](#) indicate) the total number of subgroups, rather than the group order.

If only some subgroups are desired, and calculation of the full lattice is infeasible, it might be possible to restrict the calculations to certain subgroups, as long as a filter can be defined that is appropriate to the construction process and will iterate the construction only for subgroups with certain properties. (For example, the cyclic
extension algorithm might be instructed to not calculate subgroups larger than a prescribed limit.)

At the moment, GAP provides options to define such filters in a few cases (see the manual for details):

- The general algorithm, as described, is implemented by the operation \texttt{LatticeViaRadical}. If given two groups as argument it calculates subgroups of the second group up to conjugacy by the first group.
- \texttt{LatticeByCyclicExtension} allows for limiting the extension step to subgroups with a particular property.
- \texttt{SubgroupsSolvableGroup}, an implementation of the algorithm described for the case of solvable groups (in which case only step 6 is needed) allows to limit the determination of complements to specified cases, depending on properties of \(A\), \(N\) and \(B\).
- In a different restriction, \texttt{SubgroupsSolvableGroup} also allows for determination of only those subgroups that are fixed (as subgroups) under a prescribed set of automorphisms, generalizing the concept of submodules [Hu99].

4 Maximal, Low-Index and Intermediate Subgroups

A different class of algorithms is obtained by considering maximal subgroups.

If \(M \leq G\) is a maximal subgroup, the action of \(G\) on the cosets of \(M\) is primitive. The classification of primitive groups under the label O’Nan-Scott theorem [Sco80] (see [LPS88] for a full proof with corrections) thus can be used to describe possible maximal subgroups – one needs to search for quotient groups of \(G\) that have the correct structure to allow a primitive action, the point stabilizers for these actions will be maximal subgroups.

An approach to determine representatives of the conjugacy classes of maximal subgroups of a finite group, using this idea, is described in [EH01,CH06]. The fundamental ingredients of these calculations again are the simple factors of \(\text{Soc}(F)\), and complements.

Taking again a series as described in Section 2 the algorithm then identifies factor groups of \(G\) that can have a faithful primitive
representation. This is done via the socle of these subgroups, that is chief factors (or combinations of chief factors) of $G$:

- Maximal subgroups intersecting the radical lead to primitive actions of affine type and thus are obtained as complements. This is the only case of a solvable socle.
- Nonsolvable chief factors are obtained as part of $\text{Soc}(F)$. Isomorphisms between the simple factors can be used to construct the different types of primitive actions, according to the diagonal and product action cases of the O’Nan-Scott theorem.
- The base case is maximal subgroups of simple groups, for which classifications exist in [KL90] and (far more explicitly) [BHRD13].

**Open Problem 5** *Extend the concrete classification of maximal subgroups in [BHRD13] to larger degrees.*

As in the case of using stored tabulated information about subgroups, an explicit isomorphism needs to be found using constructive recognition or standard generators.

In **GAP**, representatives of the classes of maximal subgroups can be obtained using the function `MaximalSubgroupClassReps`. (Be aware that while `MaximalSubgroups` also exists, it will enumerate all maximal subgroups, often at significant cost.) Again tabulated information about maximal subgroups of simple groups is used.

### 4.1 Small index and intermediate subgroups

The maximal subgroup functionality can be used to determine the maximal subgroups of a subgroup, thus obtaining maximal inclusion. (This also is used in general to provide the maximality relations required for the subgroup lattice structure.)

Iterating maximal subgroups can be used to find subgroups that have bounded index [CHSS05], or simply to iterate the computation of maximal subgroups for all subgroups obtained so far to find subgroups that are $k$-step maximal in $G$. To reduce the cost it will be natural to fuse conjugates under the action of the whole group.

In **GAP**, such latter functionality will be provided (starting with the 4.9 release) by a function `LowLayerSubgroups` that for a given group $G$ and step limit $k$ determines the subgroups of $G$, up to
conjugacy, that are at most $k$-step maximal in $G$. It is possible to limit the calculation to obtain only subgroups of specified bounded index.

A further variant is to determine the intermediate subgroups $U < V < G$ for a given subgroup $U \leq G$ [Hul17]: Instead of choosing an arbitrary representative $M$ for each class of maximal subgroups, we determine in each step which conjugates of $M$ contain the chosen subgroup $U$ and then iterate.

This variant is implemented in GAP by the function \texttt{IntermediateSubgroups} (again this will see a significant performance improvement with the 4.9 release).

5 Summary

We have described the various methods that can be used in GAP to determine the subgroups of a given finite group. Different approaches provide different options to adapt the calculation. The methods also rely on a significant framework for basic operations that is essentially invisible to a user who does not look into the inner workings. While a calculation of subgroups is mostly limited by the size of the output set, there are still open research problems whose solution would improve this (and other) group theoretic algorithms.

Acknowledgment

The author’s work has been supported in part by Simons Foundation Collaboration Grant 244502.

References

AG84. M. Aschbacher and R. Guralnick. Some applications of the first cohomology group. J. Algebra, 90(2):446–460, 1984.

BB99. László Babai and Robert Beals. A polynomial-time theory of black box groups. I. In C. M. Campbell, E. F. Robertson, N. Ruskuc, and G. C. Smith, editors, Groups St Andrews 1997 in Bath, volume 260/261 of London Mathematical Society Lecture Note Series, pages 30–64. Cambridge University Press, 1999.
Alexander Hulpke

BBS09. László Babai, Robert Beals, and Ákos Seress. Polynomial-time theory of matrix groups. In Proceedings of the 41st Annual ACM Symposium on Theory of Computing, STOC 2009, Bethesda, MD, USA, pages 55—64. ACM Press, 2009.

BGK+97. László Babai, Albert J. Goodman, William M. Kantor, Eugene M. Luks, and Péter P. Pálfy. Short presentations for finite groups. J. Algebra, 194:97–112, 1997.

BHLGO15. Henrik Baärnhielm, Derek Holt, C. R. Leedham-Green, and E. A. O'Brien. A practical model for computation with matrix groups. J. Symbolic Comput., 68(part 1):27–60, 2015.

BHRD13. John N. Bray, Derek F. Holt, and Colva M. Roney-Dougal. The maximal subgroups of the low-dimensional finite classical groups, volume 407 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2013. With a foreword by Martin Liebeck.

BP04. László Babai and Igor Pak. Strong bias of group generators: an obstacle to the “product replacement algorithm”. J. Algorithms, 50(2):215–231, 2004. SODA 2000 special issue.

CCH01. John Cannon, Bruce Cox, and Derek Holt. Computing the subgroup lattice of a permutation group. J. Symbolic Comput., 31(1/2):149–161, 2001.

CH03. John Cannon and Derek Holt. Automorphism group computation and isomorphism testing in finite groups. J. Symbolic Comput., 35(3):241–267, 2003.

CH04. John Cannon and Derek Holt. Computing maximal subgroups of finite groups. J. Symbolic Comput., 37(5):589–609, 2004.

CH06. John J. Cannon and Derek F. Holt. Computing conjugacy class representatives in permutation groups. J. Algebra, 300(1):213–222, 2006.

CHSS05. John J. Cannon, Derek F. Holt, Michael Slattery, and Allan K. Steel. Computing subgroups of bounded index in a finite group. J. Symbolic Comput., 40(2):1013–1022, 2005.

CLGM+95. Frank Celler, Charles R. Leedham-Green, Scott H. Murray, Alice C. Niemeyer, and E. A. O'Brien. Generating random elements of a finite group. Comm. Algebra, 23(13):4931–4948, 1995.

CNW90. Frank Celler, Joachim Neubüser, and Charles R. B. Wright. Some remarks on the computation of complements and normalizers in soluble groups. Acta Appl. Math., 21:57–76, 1990.

CS97. John Cannon and Bernd Souvignier. On the computation of conjugacy classes in permutation groups. In Wolfgang Küchlin, editor, Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, pages 392–399. The Association for Computing Machinery, ACM Press, 1997.

DLGO15. Heiko Dietrich, C. R. Leedham-Green, and E. A. O’Brien. Effective blackbox constructive recognition of classical groups. J. Algebra, 421:460–492, 2015.

DM88. John D. Dixon and Abdul Majeed. Coset representatives for permutation groups. Portugal. Math., 45(1):61–68, 1988.

EH01. Bettina Eick and Alexander Hulpke. Computing the maximal subgroups of a permutation group I. In William M. Kantor and Ákos Seress, editors, Proceedings of the International Conference at The Ohio State University, June 15–19, 1999, volume 8 of Ohio State University Mathematical Research Institute Publications, pages 155–168, Berlin, 2001. de Gruyter.
Calculating Subgroups with GAP

EP88. David Easdown and Cheryl E. Praeger. On minimal faithful permutation representations of finite groups. Bull. Austral. Math. Soc., 38:207–220, 1988.

Fei80. Walter Feit. Some consequences of the classification of finite simple groups. In The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), volume 37 of Proc. Sympos. Pure Math., pages 175–181. Amer. Math. Soc., Providence, R.I., 1980.

FK97. Larry Finkelstein and William M. Kantor, editors. Groups and Computation II, volume 28 of DIMACS: Series in Discrete Mathematics and Theoretical Computer Science, Providence, RI, 1997. American Mathematical Society.

HEO05. Derek F. Holt, Bettina Eick, and Eamonn A. O'Brien. Handbook of Computational Group Theory. Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, FL, 2005.

Hol97. Derek F. Holt. Representing quotients of permutation groups. Quart. J. Math. Oxford Ser. (2), 48(191):347–350, 1997.

HP89. Derek F. Holt and W. Plesken. Perfect groups. Oxford University Press, 1989.

Hul99. Alexander Hulpke. Computing subgroups invariant under a set of automorphisms. J. Symbolic Comput., 27(4):415–427, 1999. (ID jsco.1998.0260).

Hul00. Alexander Hulpke. Conjugacy classes in finite permutation groups via homomorphic images. Math. Comp., 69(232):1633–1651, 2000.

Hul13a. Alexander Hulpke. Calculation of the subgroups of a trivial-fitting group. In ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, pages 205–210. ACM, New York, 2013.

Hul13b. Alexander Hulpke. Computing conjugacy classes of elements in matrix groups. J. Algebra, 387:268–286, 2013.

Hul17. Alexander Hulpke. Finding intermediate subgroups. Portugal. Math., 74(3), 2017.

KL90. Peter Kleidman and Martin Liebeck. The subgroup structure of the finite classical groups, volume 129 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1990.

Leo91. Jeffrey S. Leon. Permutation group algorithms based on partitions, I: theory and algorithms. J. Symbolic Comput., 12:533–583, 1991.

Leo97. Jeffrey S. Leon. Partitions, refinements, and permutation group computation. In Finkelstein and Kantor [FK97], pages 123–158.

LMR94. Klaus Lux, Jürgen Müller, and Michael Ringe. Peakword Condensation and Submodule Lattices: An Application of the Meat-Axe. J. Symbolic Comput., 17:529–544, 1994.

LNS84. Reinhard Laue, Joachim Neubüser, and Ulrich Schoenwaelder. Algorithms for finite soluble groups and the SOGOS system. In Michael D. Atkinson, editor, Computational group theory (Durham, 1982), pages 105–135. Academic press, 1984.

LPS88. Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl. On the O’Nan-Scott theorem for finite primitive permutation groups. J. Austral. Math. Soc. Ser. A, 44:389–396, 1988.

LS97. Eugene M. Luks and Ákos Seress. Computing the fitting subgroup and solvable radical for small-base permutation groups in nearly linear time. In Finkelstein and Kantor [FK97], pages 169–181.
MN89. M. Mecky and J. Neubüser. Some remarks on the computation of conjugacy classes of soluble groups. *Bull. Austral. Math. Soc.*, 40(2):281–292, 1989.

Neu60. Joachim Neubüser. Untersuchungen des Untergruppenverbandes endlicher Gruppen auf einer programmgesteuerten elektronischen Dualmaschine. *Numer. Math.*, 2:280–292, 1960.

NS06. Max Neunhöffer and Ákos Seress. A data structure for a uniform approach to computations with finite groups. In *ISSAC 2006*, pages 254–261. ACM, New York, 2006.

O'B94. E. A. O'Brien. Isomorphism testing for $p$-groups. *J. Symbolic Comput.*, 17:133–147, 1994.

Rem30. Robert Remak. Über die Darstellung der endlichen Gruppen als Untergruppen direkter Produkte. *J. Reine Angew. Math.*, 163:1–44, 1930.

Sco80. Leonard L. Scott. Representations in characteristic $p$. In Bruce Cooperstein and Geoffrey Mason, editors, *The Santa Cruz conference on finite groups*, volume 37 of *Proc. Sympos. Pure Math.*, pages 318–331, Providence, RI, 1980. American Mathematical Society. Corrigendum in [LPS88].

Ser03. Ákos Seress. *Permutation Group Algorithms*. Cambridge University Press, 2003.

Wil96. Robert A. Wilson. Standard generators for sporadic simple groups. *J. Algebra*, 184(2):505–515, 1996.