Discretizations of Stochastic Evolution Equations in Variational Approach Driven by Jump-Diffusion

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Abstract

Stochastic evolution equations with compensated Poisson noise are considered in the variational approach with monotone and coercive coefficients. Here the Poisson noise is assumed to be time homogeneous with $\sigma$-finite intensity measure on a metric space. By using finite element methods and Galerkin approximations, some explicit and implicit discretizations for this equation are presented and their convergence is proved. Polynomial growth condition and linear growth condition are assumed on the drift operator, respectively for the implicit and explicit schemes.

Keywords: Stochastic partial differential equations, polynomial growth, variational approach, Gelfand triple, numerical schemes, compensated Poisson measure.

1 Introduction

There exist many results for numerical approximations of stochastic evolution equations (SEEs) driven by both continuous and càdlàg martingales. In particular, for stochastic partial differential equations (SPDEs) driven by Wiener noise, results exist concerning numerical approximations in variational setting (see [22], [40], [53], [54], [55], [56], [57], [87], [120], [121]), approximations of linear stochastic partial differential equations (see e.g. [45], [46], [61] and references therein) and numerical approximation of SPDEs in the semigroup framework (see e.g. [1], [2], [19], [30], [34], [35], [36], [37], [50], [51], [71], [72], [73], [74], [86], [91], [93], [99], [104], [105], [119], [125], [127], [130], [132]).

More recently results concerning explicit schemes for equations with drift operator not satisfying the linear growth condition have been obtained for finite dimensional stochastic differential equations (SDEs) driven by Wiener noise in [13], [14], [58], [59], [60], [61], [65], [66], [69], [90], [95], [101], [102], [106], [117], [118], [122], [123], [126], [129], [133], [135] and for SDEs with jump-diffusive noise in [32], [89], [124].

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For semilinear SPDEs with super-linearly growing drift driven by Wiener or Gaussian noise, numerical schemes are considered in [7], [11], [27], [41], [65], [68], [70], [74], [75], [77], [78], [80], [81], [98], [112], [113], [131], [134].

Numerical schemes for SPDEs driven by Wiener/Gaussian noise, having super-linearly growing drift operator in variational setting are given in [22], [40], [54], [57], [87], [120], [121].

For special types of SPDEs with super-linearly growing drift with Gaussian noise, numerical schemes have been obtained e.g. for Navier-Stokes equations in [12], [18], [24], [28], [29], [38], [39], [92], [103], for incompressible Euler equations in [63], for equations of geophysical fluid dynamics in [48], for stochastic total variation flow in [116], for Ginzburg-Landau equations in [10], for Allen-Cahn equations in [8], [16], [20], [21], [83], [84], [100], [128], for heat equation in [9], for Cahn-Hilliard equations in [42], [82], for Burgers equations in [15], [17], [47], [67], [76], [79], for Schrödinger equations in [31], and for Manakov equations in [43], [44].

For stochastic partial differential equations driven by càdlàg martingales in the semigroup framework one should mention e.g. [4], [5], [6], [62], [85], [94], [114] (and references therein).

Numerical schemes for linear stochastic integro-differential equations of parabolic type arising in non-linear filtering of jump-diffusion processes have been obtained in [33]. Numerical approximation for stochastic Schrödinger type equations driven by martingale noise has been obtained in [26].

To the best of our knowledge, numerical schemes for (general) stochastic partial differential equations with Poisson noise/Lévy noise have not yet been considered in the variational setting.

In the article [54], I. Gyöngy and A. Millet studied discretizations of stochastic partial differential equations with Wiener noise in the variational setting. In this paper, we will generalize their approach by adding also compensated Poisson noise. More precisely, we consider the equation

\[ u_t = \xi + \int_0^t A_s(\bar{u}_s) \, ds + \int_0^t B_s(\bar{u}_s) \, dW_s + \int_0^t \int_E F_s(\bar{u}_s, \xi) \, \tilde{N}(ds, d\xi), \]

with respect to a Gelfand triple \( V \hookrightarrow H \hookrightarrow V^* \). We assume \( V \) to be a reflexive separable real Banach space which is continuously and densely embedded into the Hilbert space \( H \). \( V^* \) stands for the dual space of \( V \) containing \( H^* \), the dual of \( H \), as a dense subset. Identifying \( H \) with \( H^* \), we obtain the dense and continuous embeddings \( V \hookrightarrow H \hookrightarrow V^* \). A solution of the equation above is supposed to be a càdlàg \( H \)-valued stochastic process \( u \) taking values in \( V \) almost everywhere and therefore has a \( V \)-valued predictable modification \( \bar{u} \).

In the above equation \( W \) is a cylindrical Wiener process in a Hilbert space \( U \). We denote by \( N \) a Poisson random measure, independent of \( W \), which is considered to be time homogeneous with \( \sigma \)-finite intensity measure \( \nu \) on a metric space \( E \). \( \tilde{N} \) is the compensated Poisson random measure corresponding to \( N \). The coefficients \( A, B \) and \( F \) take values in \( V^* \), \( L_2(U, H) \) and \( H \) respectively. Here, \( L_2(U, H) \) is the space of Hilbert-Schmidt operators from \( U \) to \( H \). In this article, following [54], we assume hemicontinuity and growth condition on \( A \) (cf. (C4) and (C3) below), and monotonicity and coercivity conditions on \( (A, B, F) \) (cf. (C1) and (C2)).

Following [54], we provide explicit and implicit numerical schemes for the above equation. Here, the condition for convergence of explicit scheme is weaker than the corresponding
condition in [54]. We will consider numerical schemes with respect to equipartition of the
time interval $[0, T]$ into subintervals $[t_{i-1}, t_i]$ and the approximated value of $u$ at time $t_i$
will be calculated implicitly or explicitly. In the explicit scheme, the operators $A$, $B$ and $F$
at every time will be replaced with their integral means, taken on the previous time subinterval.
Orthogonal projection to a finite dimensional subspace of $V$ with respect to the inner product
of $H$ is also essential for this explicit scheme. Similarly, in the implicit scheme, the operators
$B$ and $F$ will be replaced with their integral means taken on the preceding time subintervals,
but $A$ will be replaced with its integral mean taken on the current time subinterval. The
orthogonal projection to finite dimensional subspaces is optional in the implicit method and
this will give us two types of implicit schemes.

For the mathematical background on the variational approach to stochastic evolution
equations we refer to [88], [97], [108], and [111]. For stochastic partial differential equations
driven by Lévy noise or more general Poisson random measures we refer to the books [3] and
[110].

2 About the Equation

Let $V$ be a reflexive separable Banach space, embedded densely and continuously in a Hilbert
space $H$. Its dual space $H^*$ is then densely and continuously embedded in $V^*$. Identifying
$H$ with its dual space $H^*$ using Riesz’ isometry, we obtain the Gelfand triple
$V \hookrightarrow H \hookrightarrow V^*$.

We denote by $\langle \cdot, \cdot \rangle$ the duality between $V$ and $V^*$ and by $(\cdot, \cdot)_H$ the inner product in $H$. We
will be interested in equations of the following type

$$u_t = \zeta + \int_0^t A_s(\bar{u}_s) \, ds + \int_0^t B_s(\bar{u}_s) \, dW_s + \int_0^t \int_E F_s(\bar{u}_s, \xi) \, \tilde{N} (ds, d\xi), \quad (1)$$

where $u$ is a càdlàg $H$-valued process and $\bar{u}$ is its predictable $V$-valued modification.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a complete probability space such that the filtration $\mathcal{F}_t$ satisfies the
usual conditions, i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

Let $W$ be an adapted cylindrical Wiener process in a Hilbert space $U$ such that for $t > s$,
$W_t - W_s$ is independent of $\mathcal{F}_s$. Let $N$ be an adapted time homogeneous Poisson random
measure on $([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{E})$, independent of $W$. Here $E$ is a metric space and $\mathcal{E}$ is its Borel $\sigma$-field. We assume that the
intensity measure $\nu$ of $N$ is $\sigma$-finite on the metric space $(E, \mathcal{E})$ where $E$ is countable union of
compact sets, and that $N ((s, t], \cdot)$ is independent of $\mathcal{F}_s$, as for the Wiener process. Finally,
let $\tilde{N} := N - dt \otimes \nu$ be the compensated Poisson random measure associated with $N$.

In the next step let us specify the measurability assumptions on the coefficients $A$, $B$, and $F$. Using the notation of [96], let $\mathcal{B}\mathcal{F}$ be the $\sigma$-field of progressively measurable sets on $[0, T] \times \Omega$, i.e.

$$\mathcal{B}\mathcal{F} = \{ A \subseteq [0, T] \times \Omega : \forall t \in [0, T], A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t \} .$$

Let $\mathcal{P}$ denote the predictable $\sigma$-field, i.e. the $\sigma$-field generated by all left continuous and $\mathcal{F}_t$-
adapted real-valued processes on $[0, T] \times \Omega$. We denote by $(L_2(U, H), \langle \cdot, \cdot \rangle_2, \| \cdot \|_2)$ the space
of Hilbert-Schmidt operators from the Hilbert space $U$ to $H$. Then we assume that

$$A : ([0, T] \times \Omega \times V, \mathcal{B}F \otimes \mathcal{B}(V)) \to (V^*, \mathcal{B}(V^*)),$$

$$B : ([0, T] \times \Omega \times V, \mathcal{B}F \otimes \mathcal{B}(V)) \to (L_2(U, H), \mathcal{B}(L_2(U, H))),$$

$$F : ([0, T] \times \Omega \times V \times E, \mathcal{P} \otimes \mathcal{B}(V) \otimes \mathcal{E}) \to (H, \mathcal{B}(H))$$

are measurable. Note that predictability of $F$ is required, since it is integrated with respect to the Poisson random measure. Now we shall make some assumptions on the operators $A$, $B$ and $F$ and the initial condition $\zeta$. Let $p \in [2, \infty)$ and $q$ be its conjugate i.e. $1/p + 1/q = 1$. Let $K_1, K_2$ be non-negative integrable functions on $[0, T] \times \Omega$ and $\lambda$ and $\mu$ be respectively positive and non-negative deterministic integrable functions on $[0, T]$. The following conditions will be needed throughout the paper:

(C1) Monotonicity condition on $(A, B, F)$: almost surely for all $x, y \in V$ and all $t \in [0, T],$

$$2 \langle A_t(x) - A_t(y), x - y \rangle + \|B_t(x) - B_t(y)\|_2^2$$

$$+ \int_E \|F_t(x, \xi) - F_t(y, \xi)\|^2_H \nu(d\xi) \leq 0.$$

(C2) Coercivity condition on $(A, B, F)$: almost surely for all $x \in V$ and all $t \in [0, T],$

$$2 \langle A_t(x), x \rangle + \|B_t(x)\|_2^2 + \int_E \|F_t(x, \xi)\|^2_H \nu(d\xi) + \lambda(t)\|x\|^p_V$$

$$\leq K_1(t) + \mu(t)\|x\|^2_H.$$

(C3) Growth condition on $A$: there exists $\alpha > 0$ such that almost surely for all $x \in V$ and all $t \in [0, T],$

$$\|A_t(x)\|^q \leq \alpha \lambda^q(t)\|x\|^p + K_2(t)\lambda^{q-1}(t).$$

(C4) Hemicontinuity of $A$: almost surely for all $x, y, z \in V$ and all $t \in [0, T],$

$$\lim_{\epsilon \to 0} \langle A_t(x + \epsilon y), z \rangle = \langle A_t(x), z \rangle.$$

(C5) $\zeta \in L^2(\Omega, \mathcal{F}_0; H)$.

(C6) There exists an increasing sequence $E^1 \subset E^2 \subset E^3 \subset \cdots$ of compact subsets of $E$, having finite $\nu$-measure, such that $\bigcup_{i=1}^{\infty} E^i = E.$

The monotonicity condition [C1] can be weakened as follows:

(C1’) Almost surely for all $x, y \in V$ and all $t \in [0, T],$

$$2 \langle A_t(x) - A_t(y), x - y \rangle + \|B_t(x) - B_t(y)\|_2^2$$

$$+ \int_E \|F_t(x, \xi) - F_t(y, \xi)\|^2_H \nu(d\xi) \leq K(t)\|x - y\|^2_H,$$

where $K$ is a non-negative integrable function on $[0, T]$. 

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Indeed, let $u_t$ be a solution to equation (1) and let $\gamma_t := \exp \left(-\frac{1}{2} \int_0^t K_s ds \right)$. Then $v_t = \gamma_t^{-1} u_t$ is a solution of
\[v_t = \zeta + \int_0^t \bar{A}_s (\bar{v}_s) \, ds + \int_0^t B_s (\bar{v}_s) \, dW_s + \int_0^t \int_E F_s (\bar{v}_s, \xi) \bar{N} (ds, d\xi),\]
where
\[\bar{A}_s (x) := \gamma_t^{-1} A_t (\gamma_t x) - \frac{1}{2} K_t x, B_t (x) := \gamma_t^{-1} B_t (\gamma_t x), F_t (x, \xi) := \gamma_t^{-1} F_t (\gamma_t x, \xi).\]
If $A, B, \text{and } F$ satisfy (C1), (C2), (C4) it follows that $\bar{A}, \bar{B}$ and $\bar{F}$ satisfy (C1), (C2), (C4). If $A$ satisfies (C3) and $K(t) \leq C \lambda(t), t \in [0, T]$ for some constant $C$, then $A$ satisfies (C3).

**Proposition 2.1.** Condition (C3) together with each of (C1) and (C2) gives respectively the following conditions on $(B, F)$:
\[
\|B_t(x) - B_t(y)\|_2^2 + \int_E \|F_t(x, \xi) - F_t(y, \xi)\|_H^2 \nu(d\xi)
\leq 3 \left( \alpha^{1/q} + \frac{1}{p} \right) \lambda(t) (\|x\|_V^p + \|y\|_V^p) + \frac{8}{q} K_2(t),
\]
(2)
\[
\|B_t(x)\|_2^2 + \int_E \|F_t(x, \xi)\|_H^2 \nu(d\xi)
\leq 6 \left( \alpha^{1/q} + \frac{1}{p} \right) \lambda(t) (\|x\|_V^p + \frac{16}{q} K_2(t) + 2 K_1(t)).
\]
(3)

**Proof.** We get by (C3) that
\[
\|A_t(x)\|_V, \leq \alpha^{1/q} \lambda(t) \|x\|_V^{p-1} + K_2^{1/q}(t) \lambda^{1/p}(t).
\]
So using Young’s inequality we have
\[
|\langle x, A_t(x) \rangle| \leq \alpha^{1/q} \lambda(t) \|x\|_V^p + K_2^{1/q}(t) \lambda^{1/p}(t) \|x\|_V
\leq \left( \alpha^{1/q} + \frac{1}{p} \right) \lambda(t) \|x\|_V^p + \frac{1}{q} K_2(t)
\]
and
\[
|\langle y, A_t(x) \rangle| \leq \alpha^{1/q} \lambda(t) \|x\|_V^{p-1} \|y\|_V + K_2^{1/q}(t) \lambda^{1/p} \|y\|_V
\leq \alpha^{1/q} \lambda(t) \left( \frac{p-1}{p} \|x\|_V^p + \frac{1}{p} \|y\|_V^p \right) + \frac{1}{q} \lambda(t) \|y\|_V^p + \frac{1}{q} K_2(t).
\]
Combining these inequalities with (C1) and (C2) yields
\[
\|B_t(x) - B_t(y)\|_2^2 + \int_E \|F_t(x, \xi) - F_t(y, \xi)\|_H^2 \nu(d\xi)
\leq 2 |\langle x, A_t(y) \rangle| + 2 |\langle y, A_t(x) \rangle| + 2 |\langle x, A_t(x) \rangle| + 2 |\langle y, A_t(y) \rangle|
\leq 3 \left( \alpha^{1/q} + \frac{1}{p} \right) \lambda(t) (\|x\|_V^p + \|y\|_V^p) + \frac{8}{q} K_2(t).
\]
and
\[ \|B_t(x)\|_2^2 + \int_E \|F_t(x, \xi)\|_H^2 \nu(d\xi) \leq 2\|B_t(x) - B_t(0)\|_2^2 + 2 \int_E \|F_t(x, \xi) - F_t(0, \xi)\|_H^2 \nu(d\xi) \]
\[ + 2\|B_t(0)\|_2^2 + 2 \int_E \|F_t(0, \xi)\|_H^2 \nu(d\xi) \leq 6 \left( \alpha^{1/q} + \frac{1}{p} \right) \lambda(t) \|x\|_V^p + \frac{16}{q} K_2(t) + 2K_1(t). \]

Now we are going to define the solution of equation (1). First we remind the notion of modification of a stochastic process.

**Definition 2.2.** Let \( z \) be a stochastic process. \( \bar{z} \) is called a modification of \( z \) if for \( dt \otimes P \)-almost all \((t, \omega) \in [0, T] \times \Omega\), \( \bar{z}(t, \omega) = z(t, \omega) \).

**Definition 2.3.** A càdlàg \( H \)-valued \((F_t)\)-adapted stochastic process \( u \) is a (strong) solution to the equation (1), if it has a predictable \( V \)-valued modification \( \bar{u} \) in
\[ L^p([0, T] \times \Omega, \lambda dt \otimes P; V) \cap L^2([0, T] \times \Omega, dt \otimes P; H) \]
such that the equation
\[ (u_t, v)_H = (\zeta, v)_H + \int_0^t \langle A_s(\bar{u}_s), v \rangle \, ds + \int_0^t \langle v, B_s(\bar{u}_s) dW_s \rangle_H \]
\[ + \int_0^t \int_E \langle F_s(\bar{u}_s, \xi), v \rangle_H \tilde{N}(ds, d\xi) \]
holds for all \( v \in V \) and \( dt \otimes P \)-almost every \((t, \omega) \in [0, T] \times \Omega\).

**Remark 2.4.** Suppose that \( z \) is an adapted càdlàg stochastic process in \( H \) that \( dt \otimes P \)-almost everywhere belongs to \( V \). Since \( V \) is a Borel subset of \( H \) and \( z(t-) \), \( t \in [0, T] \) is a predictable modification of \( z \) in \( H \), we have \( z(t-) \mathbb{1}_{\{z(t-) \in V\}} \) as a \( V \)-valued predictable modification of \( z \).

The following existence and uniqueness theorems hold (see [25], [49]).

**Theorem 2.5.** Let conditions (C1)-(C6) hold. Then equation (1) has a solution \( u \) that satisfies
\[ \sup_{t \in [0, T]} \mathbb{E} \|u_t\|_H^2 < \infty. \] (4)

**Theorem 2.6.** Assuming (C1)-(C6), the solution of (1) in the sense of Definition 2.3 is unique and satisfies almost surely, for all \( t \in [0, T] \),
\[ u_t = \zeta + \int_0^t A_s(\bar{u}_s) ds + \int_0^t B_s(\bar{u}_s) dW_s + \int_0^t \int_E F_s(\bar{u}_s, \xi) \tilde{N}(ds, d\xi). \] (5)
We will prove Theorem 2.5 by numerical approximations. To prove Theorem 2.6 we use the following theorem of [52] with the constant stopping time \( \tau \equiv T \).

**Theorem 2.7. [52, Theorem 1]** Let \( \Lambda \) be an increasing adapted real valued stochastic process with càdlàg trajectories. Assume \( z \) and \( y \) are respectively \( V \) and \( V^\ast \)-valued progressively measurable stochastic processes. Suppose that \( \|z(t)\|_V, \|y(t)\|_V, \) and \( \|z(t)\|_V \times \|y(t)\|_V^\ast \) are locally integrable with respect to \( d\Lambda_t \), i.e. their trajectories are almost surely integrable with respect to \( d\Lambda_t \). Let \( h(t) \) be an \( H \)-valued locally square integrable càdlàg martingale and \( \tau \) denote a stopping time. Set \( \tau \equiv T \) and \( \mathbb{P} \) almost every \((t, \omega) \in \Omega : t \leq \tau(\omega) \) and suppose that for all \( v \in V \), and \( d\Lambda_t \otimes \mathbb{P} \)-almost every \((t, \omega) \in D \) we have

\[
(v, z(t))_H = \int_0^t \langle v, y(s) \rangle \, d\Lambda_s + (v, h(t))_H .
\]

Then there exists a subset \( \Omega' \subseteq \Omega \) with \( \mathbb{P}(\Omega') = 1 \) and an adapted càdlàg \( H \)-valued process \( \tilde{z} \) that is equal to \( z \) for \( d\Lambda \otimes \mathbb{P} \) almost all \((t, \omega) \) and has the following property instead of (6):

\[
\forall \omega \in \Omega', \forall t \leq \tau(\omega), \forall v \in V : (v, \tilde{z}(t))_H = \int_0^t \langle v, y(s) \rangle \, d\Lambda_s + (v, h(t))_H ,
\]

and the following Itô formula for \( \|\cdot\|^2_H \) holds:

\[
\|\tilde{z}(t)\|^2_H = \|h(0)\|^2_H + 2 \int_0^t \langle z(s), y(s) \rangle \, d\Lambda_s + 2 \int_0^t (\tilde{z}(s) - h(s), dh(s))_H \\
- \int_0^t \|y(s)\|^2_H \Delta \Lambda(s) d\Lambda(s) + [h]_t ,
\]

where \([h]_t\) is the quadratic variation of \( h \) and \( \Delta \Lambda(s) \) is the value \( \Lambda(s) - \Lambda(s^-) \). If \( y(s) \notin H \), we set \( \|y(s)\|^2_H := \infty \).

**Proof of Theorem 2.6.** We first prove the uniqueness of the solution. Let \( u^{(1)} \) and \( u^{(2)} \) be two solutions for equation (1) with predictable \( V \)-valued modifications \( \bar{u}^{(1)} \) and \( \bar{u}^{(2)} \) respectively. Let us calculate \( \|u^{(1)}_t - u^{(2)}_t\|^2_H \) by means of the Itô formula from the previous theorem. For this purpose, we set \( z(t) = \bar{u}^{(1)}_t - \bar{u}^{(2)}_t \) and define \( d\Lambda_t := \lambda(t) dt \). We have

\[
(v, z(t))_H = \int_0^t \langle v, y(s) \rangle \, d\Lambda_s + (v, h(t))_H \otimes \mathbb{P}-a.e.
\]

where

\[
y(t) = \left( A_t \left( \bar{u}^{(1)}_t \right) - A_t \left( \bar{u}^{(2)}_t \right) \right) \lambda(t)^{-1} ,
\]

\[
h(t) = \int_0^t \left( B_s \left( \bar{u}^{(1)}_s \right) - B_s \left( \bar{u}^{(2)}_s \right) \right) dW_s \\
+ \int_0^t \int_E \left( F_s \left( \bar{u}^{(1)}_s, \xi \right) - F_s \left( \bar{u}^{(2)}_s, \xi \right) \right) \tilde{N}(ds, d\xi) .
\]

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Note that $d\Lambda_t \otimes \mathbb{P}$ and $dt \otimes \mathbb{P}$ are absolutely continuous with respect to each other and therefore $d\Lambda_t \otimes \mathbb{P}$-a.e. is equivalent to $dt \otimes \mathbb{P}$-a.e. By \cite{2} we obtain that $h$ is a càdlàg square integrable $H$-valued martingale. It is obvious that $y$ and $z$ are progressively measurable. From the assumptions of Theorem \cite{2.7}, it remains to check the local integrability of $\|z(t)\|_V \cdot \|y(t)\|_V$ and $\|z(t)\|_V \cdot \|y(t)\|_V$ with respect to $d\Lambda_t$. We have

$$
\int_0^T \|y(t)\|_{V^*}^q \, d\Lambda_t = \int_0^T \left\| A_t \left( u_t^{(1)} \right) - A_t \left( u_t^{(2)} \right) \right\|_{V^*}^q \lambda^{1-q}(t) \, dt \\
\leq 2^{q-1} \int_0^T \sum_{i=1}^2 \alpha \lambda(t) \left\| u_t^{(i)} \right\|_V^p \, dt + 2q \int_0^T K_2(t) \, dt < \infty \quad \text{a.s.,}
$$

and

$$
\int_0^T \|z(t)\|_V^p \, d\Lambda_t = \int_0^T \left\| u_t^{(1)} - u_t^{(2)} \right\|_V^p \lambda(t) \, dt \\
\leq 2^{p-1} \int_0^T \sum_{i=1}^2 \left\| u_t^{(i)} \right\|_V^p \lambda(t) \, dt < \infty \quad \text{a.s.}
$$

Applying Hölder’s inequality, we get

$$
\int_0^T \|y(t)\|_{V^*} \cdot \|z(t)\|_V \, d\Lambda_t \\
\leq \left( \int_0^T \|y(t)\|_V^q \, d\Lambda_t \right)^{1/q} \left( \int_0^T \|z(t)\|_V^p \, d\Lambda_t \right)^{1/p} < \infty \quad \text{a.s.}
$$

So the conditions of Theorem \cite{2.7} with $\tau = T$ hold and $u^{(1)} - u^{(2)}$ which is the càdlàg $H$-valued modification of $z$ satisfies

$$
\left\| u_t^{(1)} - u_t^{(2)} \right\|_H^2 = \int_0^t 2 \left\langle \tilde{u}_{s}^{(1)} - \tilde{u}_{s}^{(2)}, A_s \left( \tilde{u}_{s}^{(1)} \right) - A_s \left( \tilde{u}_{s}^{(2)} \right) \right\rangle ds \\
+ \int_0^t \left\| B_s \left( \tilde{u}_{s}^{(1)} \right) - B_s \left( \tilde{u}_{s}^{(2)} \right) \right\|_2^2 ds \\
+ \int_0^t \int_E \left\| F_s \left( \tilde{u}_{s}^{(1)}, \xi \right) - F_s \left( \tilde{u}_{s}^{(2)}, \xi \right) \right\|_H^2 \nu(d\xi) ds + m_t.
$$

where

$$
m_t = 2 \int_0^t \left( u_s^{(1)} - u_s^{(2)}, B_s \left( \tilde{u}_{s}^{(1)} \right) - B_s \left( \tilde{u}_{s}^{(2)} \right) \right)_H ds \\
+ 2 \int_0^t \int_E \left( u_{s-}^{(1)} - u_{s-}^{(2)}, F_s \left( \tilde{u}_{s}^{(1)}, \xi \right) - F_s \left( \tilde{u}_{s}^{(2)}, \xi \right) \right)_H \tilde{N}(ds, d\xi) \\
+ \int_0^t \int_E \left\| F_s \left( \tilde{u}_{s}^{(1)}, \xi \right) - F_s \left( \tilde{u}_{s}^{(2)}, \xi \right) \right\|_H^2 \tilde{N}(ds, d\xi)
$$

is a local martingale. The monotonicity condition [C1] gives

$$
\left\| u_t^{(1)} - u_t^{(2)} \right\|_H^2 \leq m_t.
$$
Let $\sigma_n \uparrow \infty$ be stopping times such that $m_{t \wedge \sigma_n}, t \geq 0$ are martingales. Using Fatou’s lemma, we have

$$
\mathbb{E} \left\| u_t^{(1)} - u_t^{(2)} \right\|_H^2 \leq \liminf_{n \to \infty} \mathbb{E} \left\| u_{t \wedge \sigma_n}^{(1)} - u_{t \wedge \sigma_n}^{(2)} \right\|_H^2 \leq \liminf_{n \to \infty} \mathbb{E}(m_{t \wedge \sigma_n}) = 0
$$

and therefore the uniqueness is proved.

Let $u$ be a solution to equation (1). In order to show equation (5) for solution $u$, we apply Theorem 2.7 for $z = \bar{u}$, $d\Lambda_t = \lambda(t)dt$, $y = A_t(\bar{u}_t)\lambda(t)^{-1}$, $\tau = T$ and

$$
h(t) = \zeta + \int_0^t B_s(\bar{u}_s) dW_s + \int_0^t \int_E F_s(\bar{u}_s, \xi) \tilde{N}(ds, d\xi).
$$

By definition 2.3, equation (6) holds. By (3), we obtain that $h$ is a càdlàg square integrable $H$-valued martingale. It can be shown that $y$ and $z$ are progressively measurable. From the assumptions of Theorem 2.7, it remains to check the local integrability of $\|z(t)\|_V, \|y(t)\|_V$, and $\|z(t)\|_V, \|y(t)\|_V$, with respect to $d\Lambda_t$. We have

$$
\int_0^T \|y(t)\|_V^q d\Lambda_t = \int_0^T \|A_t(\bar{u}_t)\|_{V^q}^q \lambda^1(t) dt \leq \int_0^T \alpha \lambda(t) \|\bar{u}_t\|_V^p dt + \int_0^T K_2(t) dt < \infty \quad a.s.,
$$

$$
\int_0^T \|z(t)\|_V^p d\Lambda_t = \int_0^T \|\bar{u}_t\|_V^p \lambda(t) dt < \infty \quad a.s.
$$

Applying Hölder’s inequality, we get

$$
\int_0^T \|y(t)\|_V \cdot \|z(t)\|_V d\Lambda_t \leq \left( \int_0^T \|y(t)\|^q d\Lambda_t \right)^{1/q} \left( \int_0^T \|z(t)\|^p d\Lambda_t \right)^{1/p} < \infty \quad a.s.
$$

So the conditions of Theorem 2.7 with $\tau = T$ hold. Therefore, we get almost surely, for all $t \in [0, T]$ and for all $v \in V$,

$$
(v, u_t)_H = \int_0^t \langle v, A_s(\bar{u}_s) \rangle ds + \langle v, \zeta \rangle_H + \left( v, \int_0^t B_s(\bar{u}_s) dW_s \right)_H + \left( v, \int_0^t \int_E F_s(\bar{u}_s, \xi) \tilde{N}(ds, d\xi) \right)_H = \left( v, \zeta + \int_0^t A_s(\bar{u}_s) ds + \int_0^t B_s(\bar{u}_s) dW_s + \int_0^t \int_E F_s(\bar{u}_s, \xi) \tilde{N}(ds, d\xi) \right)_H
$$

By using the fact that the space $V$ is a dense subset of $H$, this implies equation (5). $\square$

Suppose that $X$ is a separable Banach space, $\varphi$ is a positive function on $[0, T]$, and $p \in [1, \infty)$. For simplicity let us denote with $L^p_X(\varphi)$ the space

$$
L^p \left( [0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, \varphi dt \otimes \mathbb{P}; X \right).
$$
Denote with $\mathcal{L}^p_X(\varphi, BF)$ and $\mathcal{L}^p_X(\varphi, P)$, the subspaces of $\mathcal{L}^p_X(\varphi)$ consisting of respectively progressively measurable and predictable processes. When $\varphi \equiv 1$, we use the notations $\mathcal{L}^p_X$, $\mathcal{L}^p_X(BF)$ and $\mathcal{L}^p_X(P)$. Denote by $\mathcal{G}$ the following Banach space

$$\mathcal{G} := \left\{ y \in \mathcal{L}^p_{V}(\lambda) : \text{ess sup}_{t \in [0,T]} \mathbb{E} \|y_t\|^2_H < \infty \right\}$$

with the norm

$$|y|_\mathcal{G} = \left( \mathbb{E} \int_0^T \|y_t\|^p_{V, \lambda(t)} dt \right)^{1/p} + \left( \text{ess sup}_{t \in [0,T]} \mathbb{E} \|y_t\|^2_H \right)^{1/2}.$$ 

Let $\mathcal{G}_{BF}$ be subspace of $\mathcal{G}$, consisting of progressively measurable processes. $\mathcal{G}_{BF}$ is a Banach space too. Note that since for every $X$-valued adapted stochastic process like $z$, there exists a sequence of bounded continuous stochastic processes that converges to $z$ in $\mathcal{L}^p_X(\varphi)$, so $\mathcal{G}_{BF}$ is dense in $\mathcal{L}^p_X(BF)$.

Following [54], we characterize the solution of equation (1).

**Definition 2.8.** Denote by $\mathcal{A}$, the set consisting of quadruples $(\zeta, a, b, f)$ with the following conditions

(i) $\zeta \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$,

(ii) $a \in \mathcal{L}^q_{V,*}(\lambda^{1-q}, BF)$,

(iii) $b \in L^2_{L^2(U,H)}(\mathcal{B}F)$,

(iv) $f : ([0, T] \times \Omega \times E, \mathcal{P} \otimes \mathcal{E}) \to (H, B(H))$ satisfies

$$\mathbb{E} \int_0^T \int_E \|f(s, \xi)\|^2_H \nu(d\xi) ds < \infty,$$

(v) there exists $x \in \mathcal{G}_{BF}$ such that for all $v \in V$ and almost all $(t, \omega) \in [0, T] \times \Omega$,

$$(x_t, v)_H = (\zeta, v)_H + \int_0^t \langle a_s, v \rangle ds + \int_0^t \langle b_s dW_s, v \rangle_H$$

$$+ \int_0^t \int_E (f(s, \xi), v)_H \tilde{N}(ds, d\xi).$$

Let $(\zeta, a, b, f)$ belong to $\mathcal{A}$ and $x$ be the stochastic process as in part (v) of the above definition corresponding to the quadruple $(\zeta, a, b, f)$. For $y \in \mathcal{G}$, set

$$I_y(\zeta, a, b, f) := \mathbb{E} \int_0^T \left[ 2 \langle a_s - A_s(y_s), x_s - y_s \rangle + \| b_s - B_s(y_s) \|^2_2 ight.$$ 

$$\left. + \int_E \| f(s, \xi) - F_s(y_s, \xi) \|^2_H \nu(d\xi) \right] ds$$

The next theorem which is an analogue of [54, Theorem 2.7] characterizes the solution of equation (1) and will be used for the proofs of the approximation theorems.
Theorem 2.9. Assume conditions \((C1)-(C5)\). If for some \((\zeta, a, b, f) \in \mathcal{A}\), and every \(y \in \mathcal{G}_{BF}\),

\[ I_y(\zeta, a, b, f) \leq 0, \]

then the stochastic process \(x\) corresponding to \((\zeta, a, b, f)\) as in part \((v)\) of Definition 2.8, has an \(H\)-valued càdlàg modification which is a solution of equation (1) with initial condition \(\zeta\) in the sense of Definition 2.3.

Proof. By Theorem 2.7, \(x\) has an adapted càdlàg \(H\)-valued modification, so it is sufficient to prove \(a_s = A_s(x_s), b_s = B_s(x_s)\) and \(f(s, \xi) = F_s(x_s, \xi)\) almost everywhere. Set \(y = x + \epsilon z\) where \(z \in \mathcal{G}_{BF}\), so \(y \in \mathcal{G}_{BF}\). Then

\[
0 \geq I_y(\zeta, a, b, f) = \mathbb{E} \int_0^T \left[ 2 \langle a_s - A_s(x_s + \epsilon z_s), -\epsilon z_s \rangle + \|b_s - B_s(x_s + \epsilon z_s)\|_2^2 \
+ \int_E \|F_s(x_s + \epsilon z_s, \xi) - f(s, \xi)\|_H^2 \nu(d\xi) \right] ds \tag{9}
\]

By choosing \(\epsilon = 0\), it follows that \(b_s = B_s(x_s)\) and \(f(s, \xi) = F_s(x_s, \xi)\) almost everywhere. Then (9) yields

\[
\mathbb{E} \int_0^T \langle a_s - A_s(x_s + \epsilon z_s), -z_s \rangle ds \leq 0
\]

for all \(z \in \mathcal{G}_{BF}\). Hemicontinuity of \(A\) implies

\[
\lim_{\epsilon \to 0} \langle A_s(x_s + \epsilon z_s), z_s \rangle = \langle A_s(x_s), z_s \rangle,
\]

and growth condition \((C3)\) implies that the functions \(\langle A_s(x_s + \epsilon z_s), z_s \rangle\) are dominated by an integrable function on \([0, T] \times \Omega\). Hence, by dominated convergence,

\[
\mathbb{E} \int_0^T \langle a_s - A_s(x_s), -z_s \rangle ds \leq 0
\]

for every \(z \in \mathcal{G}_{BF}\). Substitute \(-z\) instead of \(z\), it is obvious that

\[
\mathbb{E} \int_0^T \langle a_s - A_s(x_s), z_s \rangle ds = 0.
\]

Since \(a_s - A_s(x_s)\) belongs to \(\mathcal{L}_V^q(\lambda^{1-q}, \mathcal{B}F)\) and \(\mathcal{G}_{BF}\) is dense in \(\mathcal{L}_V^p(\lambda, \mathcal{B}F)\), the dual space of \(\mathcal{L}_V^q(\lambda^{1-q}, \mathcal{B}F)\), we get \(a_t = A_t(x_t)\) for almost all \((t, \omega) \in [0, T] \times \Omega\).

Now we are going to discretize space and time and the \(\sigma\)-finite measure \(\nu\). Then we will apply these discretizations to the equation (1) and formulate explicit and implicit numerical schemes in the next section.
3 Discretizations

Let us first introduce our space discretization. Let $V_1 \subseteq V_2 \subseteq \cdots$ be an increasing sequence of finite dimensional subspaces of $V$ such that $\bigcup_{n=1}^{\infty} V_n$ is dense in $V$. Consider the orthogonal projection operator $\Pi_n$ from $H$ onto $V_n$. Extend its domain to the space $V^*$ such that the operator remains continuous and linear and denote the obtained operator again by $\Pi_n$. Let $\mathcal{B}_n = \{e_1, e_2, \ldots, e_{l_n}\}$ be a basis of $V_n$, orthonormal in $H$. Then $\Pi_n$ has the following form:

$$\forall x \in V^* \quad \Pi_n x = \langle e_1, x \rangle e_1 + \langle e_2, x \rangle e_2 + \cdots + \langle e_{l_n}, x \rangle e_{l_n}$$

**Proposition 3.1.** The following properties hold for $\Pi_n$:

(i) $\forall x \in V_n, \quad \Pi_n x = x$,

(ii) $\forall x \in V, y \in V^*, \quad \langle \Pi_n x, y \rangle = \langle x, \Pi_n y \rangle$,

(iii) $\forall h, k \in H, \quad (\Pi_n h, k)_H = (h, \Pi_n k)_H$,

(iv) $\forall h \in H, \quad \|\Pi_n h\|_H \leq \|h\|_H, \lim_{n \to \infty} \|h - \Pi_n h\|_H = 0$.

Let $\{g_1, g_2, \ldots\}$ be an orthonormal basis of $U$ and $\tilde{\Pi}_l$ be the orthogonal projection from $U$ to span $\{g_1, g_2, \ldots, g_l\}$. Set

$$W_l := \tilde{\Pi}_l W = \sum_{k=1}^{l} (W_l, g_k)_U g_k.$$ 

To approximate the compensated Poisson random measure, we use assumption [(C6)] which says there exists an increasing sequence $E^1 \subset E^2 \subset E^3 \subset \cdots$ of compact subsets of $E$, having finite $\nu$-measure, such that $\bigcup_{n=1}^{\infty} E^l = E$. For every $l \in \mathbb{N}$, let $D^l := \{E^l_1, E^l_2, \ldots, E^l_{r_l}\} \subset \mathcal{E}$ be a partition of $E^l$ finer than $\{E^1, E^2 \setminus E^1, \ldots, E^l \setminus E^{l-1}\}$ such that for every $1 \leq j \leq r_l$, the diameter of $E^l_j$ is less than $\varepsilon_l$. We suppose that $\varepsilon_l \downarrow 0$ as $l \to \infty$ for convergence of the numerical schemes.

Concerning time discretization, we divide the time interval $[0, T]$ into $m$ subintervals of equal length and set $\delta_m = T/m, t_i = i \delta_m, i \in \{0, 1, 2, \ldots\}$. Now we wish to use these discretizations of time and the spaces $U, H, V, E$ to present explicit and implicit numerical schemes for equation (1).

3.1 The Explicit Numerical Scheme

Let us first formulate $\tilde{A}^m, \tilde{B}^m$ and $\tilde{F}^{m,l}$ as approximations of operators $A, B$ and $F$. For all $x \in V$, all $\xi \in E$ and all $t \in (t_{i-1}, t_i]$ we set

$$\tilde{A}^m_t(x) := \tilde{A}^m_t(x), \quad \tilde{B}^m_t(x) := \tilde{B}^m_t(x), \quad \tilde{F}^{m,l}_t(x, \xi) := \tilde{F}^{m,l}_t(x, \xi),$$
If \( \xi \in E \setminus E^l \), we set for all \( x \in V \) and \( t \in [0, T] \),
\[
\tilde{F}^m_{t_i}(x, \xi) := 0.
\]
If for some \( 1 \leq j \leq r_1 \), \( \xi \in E^l_j \), we set
\[
\tilde{F}^m_{t_i}(x, \xi) := \frac{1}{\delta_m} \nu(E^l_j) \int_{t_{i-2}}^{t_{i-1}} \int_{E^l_j} F_s(x, \eta) \nu(d\eta)ds, \quad 2 \leq i \leq m.
\]

The explicit discretization scheme is as follows
\[
u_{m,i}(t_0) := 0, \quad \nu_{m,i}(t_1) := \Pi_n \zeta,
\]
\[
u_{m,i}(t_i) := \nu_{m,i}(t_{i-1}) + \delta_m \Pi_n A^{m}_{t_i} \left( \nu_{m,i}(t_{i-1}) \right) + \Pi_n B^{m}_{t_i} \left( \nu_{m,i}(t_{i-1}) \right) \left( W^l_{t_i} - \nu_W \right)
\]
\[
+ \int_E \Pi_n F^{m,l}_{t_i} \left( \nu_{m,l}(t_{i-1}), \xi \right) \tilde{N} \left( \left( t_{i-1}, t_i \right), d\xi \right), \quad 2 \leq i \leq m + 1,
\]
\[
t_{i-1} < t \leq t_i : \quad \nu_{m,i}(t) := \nu_{m,i}(t_i), \quad 1 \leq i \leq m.
\]

By using induction on \( i \), it is clear that \( \nu_{m,i}(t_i) \) is \( F_{t_i} \)-measurable. Every trajectory of \( \nu_{m,i} \) is a left continuous step function. When \( n, l \) and \( m \) tend to infinity, the stochastic processes \( \nu_{m,i} \) may not converge. Let us first introduce the following notation.

**Definition 3.2.** For the space \( V_n \), define
\[
\mathcal{C}(n) := \text{ess sup} \sup_{t \in [0, T]} \sup_{v \in V_n} \| A_{t}(v) \|^2_{H}, \quad \omega \in \Omega
\]

Now the following convergence theorem for the explicit scheme, with weaker condition than [54] Theorem 2.8, holds.

**Theorem 3.3.** Suppose conditions [C1]–[C6] with \( p = 2 \). If \( n, l \) and \( m \) tend to infinity such that
\[
\mathcal{C}(n) \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s)ds \to 0,
\]
then the sequence of stochastic processes \( \nu_{m,i} \) converges weakly in \( L^p_V(\lambda) \) to \( u \), a solution of equation [\ref{eq:1}] in \( \Omega \). In addition \( \nu_{m,i}(T) \) converges strongly in \( L^2(\Omega; H) \) to \( u(T) \).

In [54] Theorem 2.8, \( \sum_{k=1}^{t_n} |e_k|^2_{V} \) is considered instead of \( \mathcal{C}(n) \) which has bigger order (see section 5). For \( D = (0, 1), V = W^2_0(D), H = L^2(D), Au = \frac{\partial^2 u}{\partial x^2}, \) and \( e_n := \sin(n\pi \cdot), n \in \mathbb{N} \), the condition in [54] Theorem 2.8 reads as \( n^2 \to 0 \) and \( \int_0^T \lambda(s)ds \) reads as \( n^2/m \to 0 \). Note that by uniform continuity of the function \( [0, T] \ni t \mapsto \int_0^t \lambda(s)ds \), we have
\[
\max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s)ds \to 0 \quad \text{as } m \to \infty.
\]
3.2 The Implicit Numerical Schemes

Here we discretize the operators $B$ and $F$ in the same way as in the explicit scheme. But for the operator $A$ we set the value of its discrete approximation $A^m$ at time $t$ to the average of $A$ over the subinterval containing $t$, instead of its preceding subinterval. More precisely,

\[ A^m_{t_i}(x) := 0, \]
\[ A^m_{t_i}(x) := \frac{1}{\delta_m} \int_{t_{i-1}}^{t_i} A_s(x) \, ds, \quad 1 \leq i \leq m, \]
\[ t_{i-1} < t \leq t_i : \quad A^m_t := A^m_{t_i}, \quad 1 \leq i \leq m. \]

With respect to the above introduced discretization of time, space $U$ and the measure $\nu$ we then define the following scheme

\[ u^{m,l}(t_0) := \zeta, \]
\[ u^{m,l}(t_i) := u^{m,l}(t_{i-1}) + \delta_m A^m_{t_i} (u^{m,l}(t_{i-1})) + \tilde{B}^m_{t_i} (u^{m,l}(t_{i-1})) \left( W^l_{t_i} - W^l_{t_{i-1}} \right) \]
\[ + \int_E \tilde{F}^m_{t_i} (u^{m,l}(t_{i-1}), \xi) \tilde{N} ((t_{i-1}, t_i], d\xi), \quad 1 \leq i \leq m + 1, \]
\[ t_{i-1} < t \leq t_i : \quad u^{m,l}(t) := u^{m,l}(t_i), \quad 1 \leq i \leq m. \] (12)

Adding the projection $\Pi_n$, we get another implicit scheme:

\[ u^{n,m,l}(t_0) := \Pi_n \zeta, \]
\[ u^{n,m,l}(t_i) := u^{n,m,l}(t_{i-1}) + \delta_m \Pi_n A^m_{t_i} (u^{n,m,l}(t_{i-1})) \]
\[ + \Pi_n \tilde{B}^m_{t_i} (u^{n,m,l}(t_{i-1})) \left( W^l_{t_i} - W^l_{t_{i-1}} \right) \]
\[ + \int_E \Pi_n \tilde{F}^m_{t_i} (u^{n,m,l}(t_{i-1}), \xi) \tilde{N} ((t_{i-1}, t_i], d\xi), \quad 1 \leq i \leq m + 1, \]
\[ t_{i-1} < t \leq t_i : \quad u^{n,m,l}(t) := u^{n,m,l}(t_i), \quad 1 \leq i \leq m. \] (13)

Equations (12) and (13) have unique solutions $u^{m,l}(t_i)$ and $u^{n,m,l}(t_i)$ respectively, for $m$ sufficiently large. This fact is stated in the next theorem which is similar to [54, Theorem 2.9].

**Theorem 3.4.** Assume conditions [(C1)](C1)-(C6) with $p \in [2, \infty)$. Then there is a natural number $m_0 \geq 1$ such that for every $m \geq m_0$ and $l \geq 1$, equation (12) has a unique solution $u^{m,l}(t_i)$ that is $\mathcal{F}_{t_i}$-measurable and $\mathbb{E} \left\| u^{m,l}(t_i) \right\|_V^p < \infty$ for each $i = 1, 2, \ldots, m$. Similarly there exists a natural number $m_0$ such that for every $m \geq m_0$ and $n,l \geq 1$, equation (13) has a unique solution $u^{n,m,l}(t_i)$ that is $\mathcal{F}_{t_i}$-measurable and $\mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_V^p < \infty$ for each $i = 1, 2, \ldots, m$.

The convergence theorem for the implicit schemes, which is analogous to [54, Theorem 2.10], is given as follows:

**Theorem 3.5.** Assume [(C1)](C1)-(C6) with $p \geq 2$. If $m$ and $l$ converge to infinity, then $u^{m,l}$ converges weakly in $L^p_V(\lambda)$ to $u$, the solution of equation (1) and $u^{m,l}(T)$ converges strongly in $L^2(\Omega, H)$ to $u(T)$. Similarly, if $m,l$, and $n$ tend to infinity, $u^{n,m,l}$ converges weakly in $L^p_V(\lambda)$ to $u$ and $u^{n,m,l}(T)$ converges strongly in $L^2(\Omega, H)$ to $u(T)$. 

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4 Proof of Results

4.1 Convergence of the Explicit Scheme

First we obtain the integral form of equation (10). For $t_i < t \leq t_{i+1}, 0 \leq i \leq m - 1$ set
\[
\kappa_1(t) := t_i, \quad \kappa_2(t) := t_{i+1},
\]
and for $t_0, \kappa_1(t_0) = \kappa_2(t_0) = t_0$. Then
\[
\begin{align*}
u^m_n(t) = & \Pi_n \zeta(t > t_0) + \int_0^{\kappa_1(t)} \Pi_n A_s (u^m_n(s)) \, ds \\
& + \int_0^{\kappa_2(t)} \Pi_n \tilde{B}^m_s (u^m_n(\kappa_1(s))) \tilde{\Pi}_l dW_s \\
& + \int_0^{\kappa_2(t)} \int_E \Pi_n \tilde{F}^m_s (u^m_n(\kappa_1(s))) \tilde{N}(ds, d\xi).
\end{align*}
\]
(14)

We wish to prove boundedness of $u^m_n$ and the integrands of the above equation in some convenient reflexive Banach spaces. Then weakly compactness of bounded sequences in reflexive Banach spaces (see e.g. [23, Theorem 3.18]) implies weak convergence of some subsequences of $u^m_n$ and the integrands to some stochastic processes, like $\bar{u}_\infty, a_\infty, b_\infty$ and $f_\infty$, where for all $t \in [0, T]$ and all $z \in V$,
\[
(\bar{u}_\infty(t), z)_H = (\zeta, z)_H + \int_0^t (a_\infty(s), z) ds + \int_0^t (b_\infty(s) dW_s, z)_H \\
+ \int_0^t \int_E (f_\infty(s, \xi), z)_H \tilde{N}(ds, d\xi) \quad a.s.
\]

We will get that $(\zeta, a_\infty, b_\infty, f_\infty)$ belongs to the set $\mathcal{A}$ and $I_y(\zeta, a_\infty, b_\infty, f_\infty) \leq 0$ for all $y \in \mathcal{G}_{BF}$. So by Theorem 2.9, $\bar{u}_\infty$ will have a modification which is a solution of equation (1). This will complete the proof.

Proof of Theorem 3.3. The first step asserts the boundedness of $u^m_n$ and also the integrands of equation (14).

Step 1. If $0 < \gamma < 1$ and
\[
I_\gamma := \left\{ (n, m, l) : \alpha C_B(n) \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \leq \gamma \right\},
\]
then the following functions of $(n, m, l)$ are bounded on $I_\gamma$:
\[
\begin{align*}
(i) \quad & \sup_{t \in [0, T]} \mathbb{E} \left\| u^m_n(t) \right\|_H^2, \\
(ii) \quad & \mathbb{E} \int_0^T \left\| u^m_n(t) \right\|_V^2 \lambda(t) dt, \\
(iii) \quad & \mathbb{E} \int_0^T \left\| A_t (u^m_n(t)) \right\|_V^2 \lambda(t)^{-1} dt.
\end{align*}
\]
Proof of Step 1. By the definition of the explicit scheme, i.e. equation (10), for $1 \leq i \leq m$, we have

$\mathbb{E} \left[ \left\| u_{m,l}^n(t_i) \right\|^2_H \right] = \mathbb{E} \left[ \left\| u_{m,l}^n(t_{i-1}) \right\|^2_H + 2 \delta_m \mathbb{E} \left[ \left\| \Pi_n \tilde{A}_v^n (u_{m,l}^n(t_{i-1})) \right\|_H^2 \right] \right]$

\[+ \delta_m \mathbb{E} \left[ \left\| \Pi_n \tilde{B}_v^n (u_{m,l}^n(t_{i-1})) \tilde{\Pi}_l \right\|_2^2 \right] + \delta_m \mathbb{E} \left[ \left\| \Pi_n \tilde{F}_v^{m,l} (u_{m,l}^n(t_i), \xi) \right\|_H^2 \nu(d\xi) \right]$

\[+ 2 \delta_n \mathbb{E} \left[ \left\langle u_{m,l}^n(t_{i-1}), \Pi_n \tilde{A}_v^n (u_{m,l}^n(t_{i-1})) \right\rangle \right].\]

Note that, since $W_{t_i} - W_{t_{i-1}}$ and $\tilde{N}((t_{i-1}, t_i], d\xi)$ are independent of each other and independent of $\mathcal{F}_{t_{i-1}}$, the expectation of their cross product with any $\mathcal{F}_{t_{i-1}}$-adapted random variable is zero. By using the definition of the discretized operators $\tilde{A}_v^n, \tilde{B}_v^n$ and $\tilde{F}_v^{m,l}$, we get

$\mathbb{E} \left[ \left\| u_{m,l}^n(t_i) \right\|^2_H \right] \leq \mathbb{E} \left[ \left\| u_{m,l}^n(t_{i-1}) \right\|^2_H \right] + \mathbb{E} \left[ \int_{t_{i-2}}^{t_{i-1}} \Pi_n A_s (u_{m,l}^n(t_{i-1})) ds \right]$

\[+ \delta_m \mathbb{E} \left[ \int_{t_{i-2}}^{t_{i-1}} \Pi_n B_s (u_{m,l}^n(t_{i-1})) \tilde{\Pi}_l ds \right] + \delta_m \sum_{1 \leq j \leq r} \nu(E_j) \mathbb{E} \left[ \int_{t_{i-2}}^{t_{i-1}} \int_{E_j} \Pi_n F_s (u_{m,l}^n(t_{i-1}), \xi) \nu(d\xi) ds \right]$

\[+ 2 \mathbb{E} \left[ \left\langle u_{m,l}^n(t_{i-1}), \int_{t_{i-2}}^{t_{i-1}} A_s (u_{m,l}^n(t_{i-1})) ds \right\rangle \right].\]

and by using Definition 3.2 for $\left\| \Pi_n A_s (u_{m,l}^n(t_{i-1})) \right\|_H$ and the Cauchy-Schwartz’s inequality

$\mathbb{E} \left[ \left\| u_{m,l}^n(t_i) \right\|^2_H \right] \leq \mathbb{E} \left[ \left\| u_{m,l}^n(t_{i-1}) \right\|^2_H \right]$

\[+ \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \right) C(n) \mathbb{E} \left[ \int_{t_{i-2}}^{t_{i-1}} \left\| A_s (u_{m,l}^n(t_{i-1})) \right\|_{V_*} \lambda^{-1}(s) ds \right]$

\[+ \mathbb{E} \left[ \int_{t_{i-2}}^{t_{i-1}} \left\| \Pi_n B_s (u_{m,l}^n(t_{i-1})) \tilde{\Pi}_l \right\|_2^2 + \int_E \left\| \Pi_n F_s (u_{m,l}^n(t_{i-1}), \xi) \right\|_H^2 \nu(d\xi) \right]$

\[+ 2 \left\langle u_{m,l}^n(t_{i-1}), A_s (u_{m,l}^n(t_{i-1})) \right\rangle ds.\]
The coercivity condition \([C2]\) and the growth condition \([C3]\) yield that the right hand side of \((16)\) is less than or equal to

\[
\mathbb{E} \left\| u^n_{m,l}(t_{i-1}) \right\|^2_H + \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \right) C(n) \\
\times \mathbb{E} \int_{t_{i-2}}^{t_{i-1}} \left[ \alpha \lambda(s) \left\| u^n_{m,l}(t_{i-1}) \right\|^2_V + K_2(s) \right] ds \\
+ \mathbb{E} \int_{t_{i-2}}^{t_{i-1}} \left[ -\lambda(s) \left\| u^n_{m,l}(t_{i-1}) \right\|^2_V + \mu(s) \left\| u^n_{m,l}(t_{i-1}) \right\|^2_H + K_1(s) \right] ds.
\]

Now define \(\rho := 1 - \alpha \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \right) C(n)\). Since \((n,m,l) \in I_\gamma\), we have \(\rho > 0\) and

\[
\mathbb{E} \left\| u^n_{m,l}(t_i) \right\|^2_H + \rho \mathbb{E} \int_{t_{i-2}}^{t_{i-1}} \lambda(s) \left\| u^n_{m,l}(t_{i-1}) \right\|^2_V ds \\
\leq \mathbb{E} \left\| u^n_{m,l}(t_{i-1}) \right\|^2_H + \mathbb{E} \left\| u^n_{m,l}(t_{i-1}) \right\|^2_H \int_{t_{i-2}}^{t_{i-1}} \mu(s) ds \\
+ \mathbb{E} \int_{t_{i-2}}^{t_{i-1}} \left[ K_1(s) + K_2(s)/\alpha \right] ds.
\]

Summing up the above inequality for \(i = 1, 2, \ldots, k\), with \(1 \leq k \leq m + 1\), we get

\[
\mathbb{E} \left\| u^n_{m,l}(t_k) \right\|^2_H + \rho \mathbb{E} \int_0^{t_{k-1}} \lambda(s) \left\| u^n_{m,l}(s) \right\|^2_V ds \leq C + \sum_{i=1}^{k-1} \alpha_i \mathbb{E} \left\| u^n_{m,l}(t_{i-1}) \right\|^2_H, \tag{17}
\]

where \(\alpha_i = \int_{t_{i-1}}^{t_i} \mu(s) ds\) and \(C = \mathbb{E} \left\| \zeta \right\|^2_H + \mathbb{E} \int_0^T \left[ K_1(s) + K_2(s)/\alpha \right] ds\). Now we neglect the second term on the left hand side of inequality above, and using induction and the fact that

\[
\mathbb{E} \left\| u^n_{m,l}(t_0) \right\|^2_H = 0, \quad \mathbb{E} \left\| u^n_{m,l}(t_1) \right\|^2_H = \mathbb{E} \left\| \Pi_n \zeta \right\|^2_H \leq C,
\]

we get the following inequality for \(0 \leq k \leq m\)

\[
\mathbb{E} \left\| u^n_{m,l}(t_k) \right\|^2_H \leq C(1 + \alpha_1)(1 + \alpha_2) \cdots (1 + \alpha_{k-1}) \\
\leq C(1 + \alpha_1)(1 + \alpha_2) \cdots (1 + \alpha_m) \\
\leq C \left( 1 + \frac{\int_0^T \mu(s) ds}{m} \right)^m.
\]

The sequence \(C \left( 1 + \frac{\int_0^T \mu(s) ds}{m} \right)^m, m \in \mathbb{N}\) converges to the finite number

\[
\mathbb{E} \left\| u^n_{m,l}(t) \right\|^2_H < \infty.
\]

as \(m \to \infty\), so we have

\[
\sup_{(n,m) \in I_\gamma} \sup_{t \in [0,T]} \mathbb{E} \left\| u^n_{m,l}(t) \right\|^2_H < \infty.
\]
Equation (17) for \( k = m + 1 \) implies boundedness of (ii) over \( I_\gamma \), which together with the growth condition \((C3)\) of \( A \) implies

\[
\sup_{(n,m) \in I_\gamma} \mathbb{E} \int_0^T \left\| A_t \left( u_{m,l}^n(t) \right) \right\|_{V^*}^2 \lambda(t)^{-1} dt \\
\leq \sup_{(n,m) \in I_\gamma} \mathbb{E} \int_0^T \left[ \alpha \lambda(t) \left\| u_{m,l}^n(t) \right\|_V^2 + K_2(t) \right] dt < \infty,
\]

so (iii) is also bounded over \( I_\gamma \). By the definition of \( \tilde{B}^m \) and Cauchy-Schwartz’s inequality we have

\[
\mathbb{E} \int_0^T \left\| \Pi_n \tilde{B}_s^m \left( u_{m,l}^n(\kappa_1(s)) \right) \tilde{\Pi}_t \right\|_2^2 ds \\
= \delta_m^{-1} \sum_{i=2}^m \mathbb{E} \left\| \int_{t_{i-2}}^{t_{i-1}} \Pi_n B_s \left( u_{m,l}^n(t_{i-1}) \right) \tilde{\Pi}_t ds \right\|_2^2 \\
\leq \sum_{i=2}^m \mathbb{E} \int_{t_{i-2}}^{t_{i-1}} \left\| \Pi_n B_s \left( u_{m,l}^n(t_{i-1}) \right) \tilde{\Pi}_t \right\|_2^2 ds \\
\leq \mathbb{E} \int_0^T \left\| \Pi_n B_s \left( u_{m,l}^n(s) \right) \tilde{\Pi}_t \right\|_2^2 ds.
\]

Similarly

\[
\mathbb{E} \int_0^T \int_E \left\| \Pi_n \tilde{F}_t^m \left( u_{m,l}^n(\kappa_1(t)), \xi \right) \right\|_H^2 \nu(d\xi) dt \\
\leq \mathbb{E} \int_0^T \int_E \left\| \Pi_n F_t \left( u_{m,l}^n(t), \xi \right) \right\|_H^2 \nu(d\xi) dt.
\]

By using Proposition 2.1 we get

\[
\sup_{(n,m,l) \in I_\gamma} \mathbb{E} \int_0^T \left[ \left\| \Pi_n B_s \left( u_{m,l}^n(s) \right) \tilde{\Pi}_t \right\|_2^2 + \int_E \left\| \Pi_n F_s \left( u_{m,l}^n(s), \xi \right) \right\|_H^2 \nu(d\xi) \right] ds \\
\leq \sup_{(n,m,l) \in I_\gamma} \int_0^T \left[ 6 \left( \alpha^{1/q} + 1/p \right) \lambda(s) \left\| u_{m,l}^n(s) \right\|_V^2 + \frac{16}{q} K_2(s) + 2K_1(s) \right] ds \\
\leq \infty.
\]

Hence, (iv) and (v) are bounded too, and the proof of Step 1 is completed.

**Step 2.** Let \((n,m,l)\) be a sequence from \( I_\gamma \) for some \( \gamma \in (0,1) \), such that \( m, n \) and \( l \) converge to infinity. Then it contains a subsequence, denoted also by \((n,m,l)\), such that

(i) \( u_{m,l}^n \) converges weakly in \( \mathcal{L}_V^2(\lambda) \) to some progressively measurable process \( \bar{u}_\infty \),

(ii) \( u_{m,l}^n(T) \) converges weakly in \( L^2(\Omega; H) \) to some random variable \( u_{T\infty} \),
(iii) $A(u_{m,l}^n(t))$ converges weakly in $\mathcal{L}_*^2(\lambda^{-1})$ to some progressively measurable process $a_\infty$,

(iv) $\Pi_n\tilde{B}_m^m(u_{m,l}(\kappa_1(\cdot))) \tilde{\Pi}_l$ converges weakly in $\mathcal{L}_2^2(U,H)(\mathcal{B}\mathcal{F})$ to some process $b_\infty$,

(v) $\Pi_n\tilde{F}_m^m(u_{m,l}(\kappa_1(\cdot)), \star)$ converges weakly in $L^2([0, T] \times \Omega \times E, \mathcal{P} \otimes \mathcal{E}, dt \otimes \mathcal{P} \otimes \nu; H)$ to some process $f_\infty$,

(vi) $(\zeta, a_\infty, b_\infty, f_\infty) \in \mathcal{A}$ and for all $z \in V$, and $dt \otimes \mathbb{P}$-almost all $(t, \omega)$ we have

$$
(u_\infty(t), z)_H = (\zeta, z)_H + \int_0^t \langle a_\infty(s), z \rangle ds + \int_0^t \langle b_\infty(s)dW_s, z \rangle_H + \int_0^t \int_E (f_\infty(s, \xi), z)_H \tilde{N}(ds, d\xi)
$$

and for all $z \in V$, almost surely

$$
(u_{T,T}, z)_H = (\zeta, z)_H + \int_0^T \langle a_\infty(s), z \rangle ds + \int_0^T \langle b_\infty(s)dW_s, z \rangle_H + \int_0^T \int_E (f_\infty(s, \xi), z)_H \tilde{N}(ds, d\xi).
$$

Proof of Step 2. The convergences in (i)-(v) can be immediately concluded from Step 1, except the fact that $\bar{u}_\infty$ and $a_\infty$ are progressively measurable. Note that $\bar{u}_\infty(t)$ and $a_\infty(t)$ are $\mathcal{F}_{t+\delta_m}$-adapted processes for each $m \geq 1$, so they are $\mathcal{F}_t$-adapted and also $\mathcal{B}(\{0, T\}) \otimes \mathcal{F}$-measurable. Hence they have progressively measurable modifications, that will replace them in the following (see e.g. [107]). It remains to prove (vi). Fix $N \in \mathbb{N}$. It is sufficient to verify (vi) for $z \in V_N$ because $\bigcup_{N=1}^\infty V_N$ is dense in $V$. Both sides of (18) belong to the Hilbert space $\mathcal{L}_*^2$. Therefore to verify (18), it is sufficient to prove that the inner products of both sides and any $\varphi \in \mathcal{L}_*^2$ are the same, i.e.

$$
\mathbb{E} \int_0^T (\bar{u}_\infty(t), z)_H \varphi(t) dt = \mathbb{E} \int_0^T (\zeta, z)_H \varphi(t) dt + \mathbb{E} \int_0^T \left( \int_0^t \langle a_\infty(s), z \rangle ds \right) \varphi(t) dt + \mathbb{E} \int_0^T \left( \int_0^t \langle b_\infty(s)dW_s, z \rangle_H \right) \varphi(t) dt + \mathbb{E} \int_0^T \left( \int_0^t \int_E (f_\infty(s, \xi), z)_H \tilde{N}(ds, d\xi) \right) \varphi(t) dt.
$$

Note that the integral form of the explicit scheme (14) yields for $z \in V_N$ and $n \geq N$

$$
(u_{m,l}^n(t), z)_H = (\zeta, z)_H \mathbf{1}_{\{t > t_0\}} + \int_0^{\kappa_1(t)} \langle A_s(u_{m,l}^n(s)), z \rangle ds + \int_0^{\kappa_2(t)} \langle z, \Pi_n\tilde{B}_m^m(u_{m,l}^n(\kappa_1(s))) dW_s^l \rangle_H + \int_0^{\kappa_2(t)} \int_E \left( \Pi_n\tilde{F}_m^m(u_{m,l}^n(\kappa_1(s))), z \right)_H \tilde{N}(ds, d\xi) \quad a.s.
$$

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Taking the inner products of both sides and \( \varphi \), we get for \( n \geq N \),

\[
\mathbb{E} \int_0^T (u_{m,l}^n(t), z)_H \varphi(t) dt = \mathbb{E} \int_0^T (\zeta, z)_H \varphi(t) dt + J_1 + J_2 + J_3 - R_1 - R_2 - R_3,
\]

where

\[
J_1 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^{\tau(t)} \langle A_s(u_{m,l}^n(s)), z \rangle ds \right) dt,
\]

\[
J_2 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^{\kappa_2(t)} \left( \Pi_n \tilde{B}_s^m(u_{m,l}(\kappa_1(s))) \tilde{\Pi}_l dW_s, z \right)_H ds \right) dt,
\]

\[
J_3 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^{\kappa_2(t)} \left( \Pi_n \tilde{F}_s^m(u_{m,l}(\kappa_1(s))), \xi, z \right)_H \tilde{N}(ds, d\xi) \right) dt,
\]

and

\[
R_1 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^{\kappa_1(t)} \langle A_s(u_{m,l}(s)), z \rangle ds \right) dt,
\]

\[
R_2 = \mathbb{E} \int_0^T \varphi(t) \left( \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{B}_s^m(u_{m,l}(\kappa_1(s))) \tilde{\Pi}_l dW_s, z \right)_H ds \right) dt,
\]

\[
R_3 = \mathbb{E} \int_0^T \varphi(t) \left( \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{F}_s^m(u_{m,l}(\kappa_1(s))), \xi, z \right)_H \tilde{N}(ds, d\xi) \right) dt.
\]

Our goal is to identify the limits of \( J_i \)'s and \( R_i \)'s. For \( J_1 \), consider the linear operator

\[
S_1 : \mathcal{L}^q_V(\lambda^{1-q}) \to \mathcal{L}^q_{\mathbb{R}},
\]

defined by

\[
S_1(g)(t) := \int_0^t \langle g(s), z \rangle ds
\]

for all \( g \in \mathcal{L}^q_V(\lambda^{1-q}) \). \( S_1 \) is bounded, because by Hölder’s inequality we have that

\[
\mathbb{E} \int_0^T |S_1(g)(t)|^q dt
\]

\[
= \mathbb{E} \int_0^T \left| \int_0^t \langle g(s), z \rangle ds \right|^q dt
\]

\[
\leq \mathbb{E} \int_0^T \left( \int_0^t |\langle g(s), z \rangle|^q \lambda(s)^{-q/p} ds \right) \left( \int_0^t \lambda(s) ds \right)^{q/p} dt
\]

\[
\leq \|z\|_V^q \mathbb{E} \int_0^T \left( \int_0^T \|g(s)\|_{\mathcal{V}^*} \lambda(s)^{1-q} ds \right) \left( \int_0^T \lambda(s) ds \right)^{q/p} dt
\]

\[
\leq \|z\|_V^q T \left( \int_0^T \lambda(s) ds \right)^{q/p} \|g\|_{\mathcal{L}^q_V(\lambda^{1-q}, \mathbb{R})}. \]
So, $S_1$ is continuous with respect to the weak topologies. Thus by (iii), i.e.,

$$A_n(u_{m,t}(\cdot)) \rightarrow a_\infty \quad \text{in } L^q_{V_*}(\lambda^{1-q}),$$

we obtain that

$$S_1(A_n(u_{m,t}(\cdot))) \rightarrow S_1(a_\infty) \quad \text{in } L^q_R,$$

therefore

$$J_1 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^t \langle A_s(u_{m,t}(s)), z \rangle ds \right) dt \rightarrow \mathbb{E} \int_0^T \varphi(t) \left( \int_0^t \langle a_\infty(s), z \rangle ds \right) dt.$$

Now for $J_2$, take $S_2$, the bounded linear operator as follows:

$$S_2 : L^2_{L_2(U,H)}(B\mathcal{F}) \rightarrow L^2_R$$

$$S_2(g)(t) = \int_0^t (z, g(s)dW_s)_H.$$

The boundedness of $S_2$ yields that $S_2$ is continuous with respect to the weak topologies. Therefore, by using

$$\Pi_n \tilde{B}^m_n(u_{m,t}(\kappa_1(\cdot))) \Pi_t \rightarrow b_\infty \quad \text{in } L^2_{L_2(U,H)}(B\mathcal{F}),$$

we obtain that

$$J_2 = \mathbb{E} \int_0^T \varphi(t) \int_0^t \left( \Pi_n \tilde{B}^m_n(u_{m,t}(\kappa_1(s))) \Pi_t dW_s, z \right)_H dt \rightarrow \mathbb{E} \int_0^T \varphi(t) \int_0^t (b_\infty(s)dW_s, z)_H dt.$$

Similarly, let us define the linear operator $S_3$ as

$$S_3 : L^2([0,T] \times \Omega \times \mathcal{E}, \mathcal{P} \otimes \mathcal{E}, dt \otimes \mathbb{P} \otimes \nu; H) \rightarrow L^2_R$$

$$S_3(g)(t) := \int_0^t \int_E (g(s, \xi), z)_H N(ds, d\xi).$$

We have by Itô-Lévy’s isometry

$$\mathbb{E} \int_0^T \|S_3(g)(t)\|^2 dt \leq \mathbb{E} \int_0^T \left\| \int_0^t \int_E (g(s, \xi), z)_H N(ds, d\xi) \right\|^2 dt \leq \mathbb{E} \int_0^T \int_0^t \int_E (g(s, \xi), z)_H^2 N(ds, d\xi) ds dt \leq \|z\|^2_H T \mathbb{E} \int_0^T \int_E \|g(s, \xi)\|^2_H \nu(d\xi) ds.$$
So $S_3$ is bounded linear operator and therefore it is continuous with respect to the weak topologies. Since

$$\Pi_n \tilde{F}_m^l \left( u_{m,l}^n (\kappa_1 (\cdot)), * \right) \rightharpoonup f_\infty,$$

we get

$$J_3 = E \int_0^T \varphi(t) \left( \int_0^t \int_E \left( \Pi_n \tilde{F}_m^l \left( u_{m,l}^n (\kappa_1 (s)), \xi \right), \tilde{N}(ds, d\xi) \right) \right) dt$$

$$\rightarrow E \int_0^T \varphi(t) \left( \int_0^t \int_E \left( f_\infty (s, \xi), \tilde{N}(ds, d\xi) \right) \right) dt.$$

Now we wish to prove that the "$R_i$"s tend to zero. Applying Cauchy-Schwartz’s inequality yields

\[
R_i^2 = \left| E \int_0^T \varphi(t) \left( \int_{\kappa_i(t)}^t \left( A_s \left( u_{m,l}^n (s) \right), z \right) ds \right) dt \right|^2
\]

\[
\leq \| \varphi \|_{L^2}^2 E \int_0^T \left| \int_{\kappa_i(t)}^t \left( A_s \left( u_{m,l}^n (s) \right), z \right) ds \right|^2 dt
\]

\[
\leq \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \right)
\times \| z \|_V^2 \| \varphi \|_{L^2}^2 \int_0^T \int_{\kappa_i(t)}^t \| A_s \left( u_{m,l}^n (s) \right) \|_{V^*, \lambda^{-1}(s)}^2 ds dt
\]

\[
= \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \right)
\times \| z \|_V^2 \| \varphi \|_{L^2}^2 T \times E \int_0^T \| A_s \left( u_{m,l}^n (s) \right) \|_{V^*, \lambda^{-1}(s)}^2 ds.
\]
Itô-Lévy’s isometry, the computation for (iv) of step 1, \( R \)
Here, we used the fact that \( m \)
Hence, by (iii) of step 1, we get \( R \)
\[ R^2_2 = \left| \mathbb{E} \int_0^T \varphi(t) \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{B}^m_s \left( u^m_{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t dW_s, z \right)_H dt \right|^2 \]
\[ \leq \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T \left| \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{B}^m_s \left( u^m_{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t \right) dW_s, z \right|_H^2 dt \]
\[ \leq \| z \|^2_{\mathcal{V}} \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T \left| \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{B}^m_s \left( u^m_{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t \right) dW_s, z \right|_H^2 dt \]
\[ = \| z \|^2_{\mathcal{V}} \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T (\kappa_2(t) - t) \left| \Pi_n \tilde{B}^m_t \left( u^m_{n,m,l}(\kappa_1(t)) \right) \tilde{\Pi}_t \right|_H^2 dt \]
\[ \leq \delta_m \| z \|^2_{\mathcal{V}} \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T \left| \Pi_n \tilde{B}^m_t \left( u^m_{n,m,l}(\kappa_1(t)) \right) \tilde{\Pi}_t \right|_H^2 dt. \]

Here, we used the fact that \( \left| \Pi_n \tilde{B}^m_t \left( u^m_{n,m,l}(\kappa_1(t)) \right) \tilde{\Pi}_t \right|_H^2 \) is constant for \( s \in (t, \kappa_2(t)) \). Thus by (iv) of step 1, \( R_2 \to 0 \) when \( m \to \infty, n \geq N \) and \( (n, m, l) \in I_\gamma \) for \( \gamma \in (0, 1) \). Finally, using Itô-Lévy’s isometry, the computation for \( R_3 \) is as follows:

\[ R^2_3 = \left| \mathbb{E} \int_0^T \varphi(t) \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{F}^m_{s,l} \left( u^m_{n,m,l}(\kappa_1(s)), \xi \right), z \right)_H \tilde{N}(ds, d\xi) dt \right|^2 \]
\[ \leq \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T \left( \Pi_n \tilde{F}^m_{s,l} \left( u^m_{n,m,l}(\kappa_1(s)), \xi \right), z \right)_H \tilde{N}(ds, d\xi) \|_H^2 dt \]
\[ \leq \| z \|^2_{\mathcal{H}} \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T \left( \Pi_n \tilde{F}^m_{s,l} \left( u^m_{n,m,l}(\kappa_1(s)), \xi \right) \|_H^2 \nu(d\xi) ds dt \]
\[ \leq \delta_m \| z \|^2_{\mathcal{H}} \| \varphi \|^2_{L^2} \mathbb{E} \int_0^T \left( \Pi_n \tilde{F}^m_{s,l} \left( u^m_{n,m,l}(\kappa_1(s)), \xi \right) \|_H^2 \nu(d\xi) dt \]

where we used the fact that \( \left| \Pi_n \tilde{F}^m_{s,l} \left( u^m_{n,m,l}(\kappa_1(s)), \xi \right) \|_H^2 \) is constant for \( s \in (t, \kappa_2(t)) \). This together with (v) of step 1 implies that \( R_3 \to 0 \), when \( m \to \infty, n \geq N \) and \( (n, m, l) \in I_\gamma \) for \( \gamma \in (0, 1) \). Now we have proven that the limit of the right hand side of equation \( 21 \) is the right hand side of equation \( 20 \). By the fact that \( u^m \to \bar{u}_\infty \) we deduce the similar result for the left hand side, so equation \( 18 \) is obtained. It remains to prove \( 19 \). Both sides of this equation belong to the Hilbert space \( L^2(\Omega; \mathbb{R}) \), so it is sufficient to prove that the inner product of both sides with \( \psi \in L^2(\Omega; \mathbb{R}) \) and \( z \in V_N \) are the same. Thus we wish to verify
the following equality
\[
\begin{align*}
\mathbb{E} \left[ \psi (u_{T\infty}, z)_H \right] &= \mathbb{E} \left[ \psi (\zeta, z)_H \right] + \mathbb{E} \left[ \psi \int_0^T \langle a_\infty(s), z \rangle \, ds \right] \\
&\quad + \mathbb{E} \left[ \psi \int_0^T (b_\infty(s) dW_s, z)_H \right] \\
&\quad + \mathbb{E} \left[ \psi \int_0^T \int_E (f_\infty(s, \xi), z)_H \tilde{N}(ds, d\xi) \right], 
\end{align*}
\]
for \( \psi \in L^2(\Omega; \mathbb{R}) \) and \( z \in V_N \). Fix \( N \in \mathbb{N} \) and \( z \in V_N \). By equation (14), we get for \( n \geq N \) that
\[
(u^n_{m,l}(T), z)_H = (\zeta, z)_H 1_{\{t \geq t_0\}} + \int_0^{T-\delta_m} \langle A_s (u^n_{m,l}(s)), z \rangle \, ds \\
+ \int_0^T \left( z, \Pi_n \tilde{B}^m_s (u^n_{m,l}(\kappa_1(s))) \tilde{\Pi}_l dW_s \right)_H \\
+ \int_0^T \int_E \left( \Pi_n \tilde{F}^{m,l}_s (u^n_{m,l}(\kappa_1(s))), z \right)_H \tilde{N}(ds, d\xi).
\]
Taking the inner products of both sides and \( \psi \), we get for \( n \geq N \)
\[
\mathbb{E} \left[ \psi (u^n_{m,l}(T), z)_H \right] = \mathbb{E} \left[ \psi (\zeta, z)_H \right] + \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 - \tilde{R}_1, \tag{22}
\]
where
\[
\begin{align*}
\tilde{J}_1 &= \mathbb{E} \left[ \psi \int_0^T \langle A_s (u^n_{m,l}(s)), z \rangle \, ds \right], \\
\tilde{J}_2 &= \mathbb{E} \left[ \psi \int_0^T \left( \Pi_n \tilde{B}^m_s (u^n_{m,l}(\kappa_1(s))) \tilde{\Pi}_l dW_s, z \right)_H \right], \\
\tilde{J}_3 &= \mathbb{E} \left[ \psi \int_0^T \int_E \left( \Pi_n \tilde{F}^{m,l}_s (u^n_{m,l}(\kappa_1(s))), z \right)_H \tilde{N}(ds, d\xi) \right], \\
\tilde{R}_1 &= \mathbb{E} \left[ \psi \int_{T-\delta_m}^T \langle A_s (u^n_{m,l}(s)), z \rangle \, ds \right].
\end{align*}
\]
Assume that \( \tilde{S}_1 \) is the linear operator from \( L^q_{V,*,}(\lambda^{1-q}, \mathcal{B}F) \) to \( L^q(\Omega; \mathbb{R}) \), defined by
\[
\tilde{S}_1(g) := \int_0^T \langle g(s), z \rangle \, ds,
\]
for all \( g \in L^q_{V,*,}(\lambda^{1-q}, \mathcal{B}F) \). \( \tilde{S}_1 \) is bounded, so it is continuous with respect to the weak topologies. This continuity gives that
\[
\tilde{J}_1 = \mathbb{E} \left[ \psi \int_0^T \langle A_s (u^n_{m,l}(s)), z \rangle \, ds \right] \rightarrow \mathbb{E} \left[ \psi \int_0^T \langle a_\infty(s), z \rangle \, ds \right].
\]
For $\tilde{J}_2$, take the linear operator $\tilde{S}_2 : L^2_H(B\mathcal{F}) \to L^2(\Omega; \mathbb{R})$ as
\[
\tilde{S}_2(g) := \int_0^T (g(s)dW_s, z)_H
\]
and deduce
\[
\tilde{J}_2 = \mathbb{E} \left[ \psi \int_0^T \left( \Pi_n \tilde{B}_s^m \left( u_{m,l}(\kappa_1(s)) \right) \right) \tilde{\Pi}_t dW_s, z \right]_H
\]
\[
\to \mathbb{E} \left[ \psi \int_0^T (b_{\infty}(s)dW_s, z)_H \right].
\]
Similarly, the linear operator
\[
\tilde{S}_3 : L^2([0, T] \times \Omega \times E, \mathcal{P} \otimes \mathcal{E}, dt \otimes \mathcal{P} \otimes \nu; H) \to L^2(\Omega; \mathbb{R}),
\]
\[
\tilde{S}_3(g) := \int_0^T \int_E g(s, \xi) \tilde{N}(ds, d\xi)
\]
is bounded and continuous with respect to the weak topologies. Therefore
\[
\tilde{J}_3 = \mathbb{E} \left[ \psi \int_0^T \int_E \left( \Pi_n \tilde{F}_s^m \left( u_{m,l}(\kappa_1(s)) \right), \xi \right)_H \tilde{N}(ds, d\xi) \right]
\]
\[
\to \mathbb{E} \left[ \psi \int_0^T \int_E (f_{\infty}(s, \xi), z)_H \tilde{N}(ds, d\xi) \right].
\]
It is easy to check that $\tilde{R}_1 \to 0$ as $m \to \infty$. By using $u_{m,l}(T) \rightharpoonup u_{T\infty}$, equation (19) is obtained.

**Step 3.** Let $(n, m, l)$ be a subsequence which satisfies items (i)-(vi) of Step 2. Here $n, m, l$ tend to infinity and $C(n)\max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s)ds$ converges to zero. Then for all $y \in \mathcal{G}_{BF}$
\[
I_y(\zeta, a_{\infty}, b_{\infty}, f_{\infty}) + \liminf \mathbb{E} \| u_{m,l}(T) \|^2_H - \mathbb{E} \| u_{T\infty} \|^2_H \leq 0.
\]

**Proof of Step 3.** Define a bounded linear operator $\mathcal{T}_m$ from $\mathcal{H} = L^2([0, T] \times \Omega \times E, B\mathcal{F} \otimes \mathcal{E}, dt \otimes \mathcal{P} \otimes \nu; H)$ or $L^2_{L_2(U,H)}(B\mathcal{F})$ into themselves as
\[
\mathcal{T}_m(Z)(t) := \begin{cases} 
\delta_m^{-1} \int_{t_{i-2}}^{t_{i-1}} Z(s)ds & t_{i-1} < t_i, 2 \leq i \leq m,
0 & \text{otherwise}.
\end{cases}
\]
(23)
It is easy to check that the induced operator norm of $\mathcal{T}_m$ is equal to 1. We know from functional analysis that for every $Z \in \mathcal{H}$ or $Z \in L^2_{L_2(U,H)}(B\mathcal{F})$, when $m \to \infty$, $\mathcal{T}_m(Z)$ converges to $Z$ in $\mathcal{H}$ or $L^2_{L_2(U,H)}(B\mathcal{F})$, respectively. Define the operator $\mathcal{S}_i$ on $\mathcal{H}$ as
\[
\mathcal{S}_i(Z)(t, \xi) := \begin{cases} 
(\nu(E_1^i))^{-1} \int_{E_1^i} Z(t, \eta)\nu(d\eta) & \xi \in E_j^i, 1 \leq j \leq r_i,
0 & \text{otherwise},
\end{cases}
\]
(24)
with the convention $0/0 := 0$. The operator norm of $\mathcal{S}_l$ is also $1$. Since $\nu \otimes dt$ is $\sigma$-finite Borel measure, there is a sequence of random functions $Z_n(\omega) \in C([0, T] \times E), \omega \in \Omega$, having a support contained in $[0, T] \times E_n$, such that $Z_n \to Z$ in $\mathcal{H}$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary. For large enough $n \in \mathbb{N}$, we have

$$
\mathbb{E} \int_0^T \int_E \|Z(t, \xi) - \mathcal{S}_l(Z)(t, \xi)\|_H^2 \nu(d\xi) dt \leq \mathbb{E} \int_0^T \int_{E_n} \|Z_n(t, \xi) - \mathcal{S}_l(Z_n)(t, \xi)\|_H^2 \nu(d\xi) dt + \varepsilon.
$$

Since the diameter of $E_j^l, 1 \leq j \leq r_l$ is less than $\varepsilon_l$ and $\varepsilon_l \to 0$ as $l \to \infty$, we get by continuity of $Z_n$ that

$$
\lim_{l \to \infty} \mathbb{E} \int_0^T \int_E \|Z(t, \xi) - \mathcal{S}_l(Z)(t, \xi)\|_H^2 \nu(d\xi) dt \leq \lim_{l \to \infty} \mathbb{E} \int_0^T \int_{E_n} \|Z_n(t, \xi) - \mathcal{S}_l(Z_n)(t, \xi)\|_H^2 \nu(d\xi) dt + \varepsilon = \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, this implies that $\mathcal{S}_l(Z)$ converges to $Z$ in $\mathcal{H}$ as $l \to \infty$.

We observe that

$$
\mathcal{T}_m(\Pi_n B_0 (u_{m,l}(\cdot)) \tilde{\Pi}_l) = \Pi_n \tilde{B}_m^n (u_{m,l}(\kappa_1(\cdot))) \tilde{\Pi}_l
$$

$$
\mathcal{S}_l \circ \mathcal{T}_m(\Pi_n F_0 (u_{m,l}(\cdot), *)) = \Pi_n \tilde{F}_m^n (u_{m,l}(\kappa_1(\cdot)), *)
$$

and for $y \in \mathcal{G}_{BF}$,

$$
\mathcal{T}_m(\Pi_n B(y)) \to B(y) \quad \text{in } \mathcal{L}_2^2(\nu, H)(BF),
$$

$$
\mathcal{S}_l \circ \mathcal{T}_m(\Pi_n F(y, *)) \to F(y, *) \quad \text{in } \mathcal{H}.
$$

Define

$$
P^{n}_{m,l}(y) := \mathbb{E} \int_0^T \left[ 2 \left\langle A_s (u_{m,l}^n(s)) - A_s(y_s), u_{m,l}^n(s) - y_s \right\rangle + \left\| \Pi_n \tilde{B}_m^n (u_{m,l}^n(\kappa_1(s))) \tilde{\Pi}_l - \mathcal{T}_m(\Pi_n B(y))(s) \right\|^2_2
\right.
$$

$$
+ \left. \int_{E_n} \left\| \Pi_n \tilde{F}_m^n (u_{m,l}^n(\kappa_1(\xi))), \xi) - \mathcal{S}_l \circ \mathcal{T}_m(\Pi_n F(y, *))(s, \xi) \right\|^2_H \nu(d\xi) \right] ds.
$$

Since the operator norms of $\mathcal{T}_m$ and $\mathcal{S}_l$ equal one, we have

$$
P^{n}_{m,l}(y) \leq \mathbb{E} \int_0^T \left[ 2 \left\langle A_s (u_{m,l}^n(s)) - A_s(y_s), u_{m,l}^n(s) - y_s \right\rangle + \left\| \Pi_n B_s (u_{m,l}^n(s)) \tilde{\Pi}_l - \Pi_n B_s(y_s) \right\|^2_2
\right.
$$

$$
+ \left. \int_{E_n} \left\| \Pi_n F_s (u_{m,l}^n(s), \xi) - \Pi_n F_s(y_s, \xi) \right\|^2_H \nu(d\xi) \right] ds.
$$
Using the monotonicity condition \( [C1] \), it is obvious to see that \( I_{m,l}(y) \leq 0 \). By taking the sum of inequality \( (15) \) over \( 2 \leq i \leq m \), we get

\[
\mathbb{E} \left\| u_{m,l}^n(T) \right\|_H^2 \\
\leq \mathbb{E} \left\| \psi \right\|_H^2 + \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) \, ds \right) C(n) \mathbb{E} \int_0^{T-\delta_m} \left\| A_s \left( u_{m,l}^n(s) \right) \right\|_{Y^*}^2 \lambda(s)^{-1} \, ds \\
+ \mathbb{E} \int_0^T \left[ 2 \left\langle u_{m,l}^n(s), A_s \left( u_{m,l}^n(s) \right) \right\rangle + \left\| \Pi_n \tilde{E}_s^m \left( u_{m,l}^n(\kappa_1(s)) \right) \Pi_l \right\|_2^2 \\
+ \int_E \left\| \Pi_n \tilde{F}_s^{m,l} \left( u_{m,l}^n(\kappa_1(s)), \xi \right) \right\|_H^2 \nu(d\xi) \right] \, ds \\
- \mathbb{E} \int_{T-\delta_m}^T 2 \left\langle u_{m,l}^n(s), A_s \left( u_{m,l}^n(s) \right) \right\rangle \, ds.
\]

Thus

\[
0 \geq I_{m,l}(y) \geq \mathbb{E} \left\| u_{m,l}^n(T) \right\|_H^2 - \mathbb{E} \left\| \psi \right\|_H^2 + \mathbb{E} \int_0^T 2 \left\langle A_s(y_s), y_s \right\rangle \, ds \\
+ J_1 + J_2 - 2(J_3 + J_4 + J_5 + J_6) - R_1 + 2R_2,
\]

where

\[
J_1 := \mathbb{E} \int_0^T \left\| \mathcal{T}_m(\Pi_n B.(y.))(s) \right\|_2^2 \, ds \rightarrow \mathbb{E} \int_0^T \left\| B_s(y_s) \right\|_2^2 \, ds,
\]

\[
J_2 := \mathbb{E} \int_0^T \int_E \left\| S_t \circ \mathcal{T}_m(\Pi_n F.(y.,*)(s, \xi)) \right\|_H^2 \nu(d\xi) \, ds \\
\rightarrow \mathbb{E} \int_0^T \int_E \left\| F_s(y_s, \xi) \right\|_H^2 \nu(d\xi) \, ds,
\]

\[
J_3 := \mathbb{E} \int_0^T \left\langle A_s \left( u_{m,l}^n(s) \right), y_s \right\rangle \, ds \rightarrow \mathbb{E} \int_0^T \left\langle a_\infty(s), y_s \right\rangle \, ds,
\]

\[
J_4 := \mathbb{E} \int_0^T \left\langle A_s(y_s), u_{m,l}^n(s) \right\rangle \, ds \rightarrow \mathbb{E} \int_0^T \left\langle A_s(y_s), \bar{u}_\infty(s) \right\rangle \, ds,
\]

\[
J_5 := \mathbb{E} \int_0^T \left\langle \mathcal{T}_m(\Pi_n B.(y.))(s), \Pi_n \tilde{E}_s^m \left( u_{m,l}^n(\kappa_1(s)) \right) \Pi_l \right\|_2 \, ds \\
\rightarrow \mathbb{E} \int_0^T \left\langle B_s(y_s), b_\infty(s) \right\|_2 \, ds,
\]

\[
J_6 := \mathbb{E} \int_0^T \int_E \left( S_t \circ \mathcal{T}_m(\Pi_n F.(y.,*)(s, \xi)) \right) \left( \Pi_n \tilde{F}_s^{m,l} \left( u_{m,l}^n(\kappa_1(s)), \xi \right) \right) \nu(d\xi) \, ds \\
\rightarrow \mathbb{E} \int_0^T \int_E \left( F_s(y_s, \xi), f_\infty(s, \xi) \right)_H \nu(d\xi) \, ds.
\]
and

\[ R_1 := \left( \max_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \lambda(s) ds \right) C(n) \mathbb{E} \int_0^{T-\delta_m} \| A_s \left( u_{m,l}^n(s) \right) \|_{V^*}^2 \lambda(s)^{-1} ds \rightarrow 0, \]
\[ R_2 := \mathbb{E} \int_T^{T-\delta_m} \langle u_{m,l}^n(s), A_s \left( u_{m,l}^n(s) \right) \rangle ds. \]

Now it remains to identify the limit of \( R_2 \). By parts (i) and (iii) of Step 1 and Cauchy-Schwartz’s inequality, we get that

\[ R_2^2 \leq \left( \mathbb{E} \int_T^{T-\delta_m} \| u_{m,l}^n(s) \|_H^2 \lambda(s) ds \right) \left( \mathbb{E} \int_0^T \| \Pi_n A_s \left( u_{m,l}^n(s) \right) \|_H^2 \lambda^{-1}(s) ds \right) \]
\[ \leq \left( \max_{1 \leq i \leq m} \int_H^{t_i} \lambda(s) ds \right) C(n) \sup_{0 \leq i \leq m} \mathbb{E} \| u_{m,l}^n(t_i) \|_H \| A_s \left( u_{m,l}^n(\cdot) \right) \|_{V^*}^2 \lambda^{-1} \]
\[ \rightarrow 0. \]

Now the limits of all terms in the inequality (25), except \( \mathbb{E} \| u_{m,l}^n(T) \|_H^2 \), have been identified. Since \( u_{m,l}^n(T) \rightarrow u_{T\infty} \), we can write

\[ d := \lim \inf \mathbb{E} \| u_{m,l}^n(T) \|_H^2 - \mathbb{E} \| u_{T\infty} \|_H^2 \geq 0. \quad (26) \]

By Theorem 2.7, \( \bar{u}_\infty \) has an adapted càdlàg \( H \)-valued modification, denoted by \( u_\infty \), such that almost surely for all \( t \in [0, T] \)

\[ u_\infty(t) = \zeta + \int_0^t a_\infty(s) ds + \int_0^t b_\infty(s) dW_s + \int_0^t \int_E f_\infty(s, \xi) \tilde{N}(ds, d\xi). \]

This equation for \( t = T \), together with the equation (19), implies that \( u_\infty(T) = u_{T\infty} \) a.s. By Itô’s formula (see Theorem 2.7), we have that

\[ \mathbb{E} \| u_{T\infty} \|_H^2 \]
\[ = \mathbb{E} \| \zeta \|_H^2 + \mathbb{E} \int_0^T \left[ 2 \langle a_\infty(s), \bar{u}_\infty(s) \rangle + \| b_\infty(s) \|_2^2 + \int_E \| f_\infty(s, \xi) \|_{H^*}^2 \nu(d\xi) \right] ds. \quad (27) \]

Therefore, by taking limit inferior of inequality (25) and using (26) and (27), we get

\[ 0 \geq d + \mathbb{E} \int_0^T \left[ 2 \langle a_\infty(s), \bar{u}_\infty(s) \rangle + \| b_\infty(s) \|_2^2 + \int_E \| f_\infty(s, \xi) \|_{H^*}^2 \nu(d\xi) \right] ds \]
\[ + \mathbb{E} \int_0^T \left[ 2 \langle A_s(y_s), y_s \rangle + \| B_s(y_s) \|_2^2 + \int_E \| F_s(y_s, \xi) \|_{H^*}^2 \nu(d\xi) \right] ds \]
\[ - 2 \mathbb{E} \int_0^T \left[ \langle a_\infty(s), y_s \rangle + \langle A_s(y_s), \bar{u}_\infty(s) \rangle \right] ds \]
\[ - 2 \mathbb{E} \int_0^T \left[ \langle B_s(y_s), y_s \rangle + \| B_s(y_s) \|_2^2 + \int_E \langle F_s(y_s, \xi), f_\infty(s, \xi) \rangle_H \nu(d\xi) \right] ds, \]
\[ = d + I_y(\zeta, a_\infty, b_\infty, f_\infty). \]

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Hence by Theorem 2.9, \( u_\infty \) must be a solution of equation (1) which is unique. Setting \( y = u_\infty \), we get \( d = 0 \) which implies that a subsequence of \( u_{m,l}^n(T) \) converges strongly in \( L^2(\Omega, \mathbb{P}; H) \) to \( u_\infty(T) \). Since there exist unique limits for any convergent subsequences of \( u_{m,l}^n \) and \( u_{m,l}^n(T) \), and on the other hand each subsequence of these sequences has a convergent subsequence, the whole sequences \( u_{m,l}^n \) and \( u_{m,l}^n(T) \) converge. Thus the proof of theorem is complete.

4.2 Convergence of the Implicit Schemes

The proof of Theorem 3.4 is based on [54, Proposition 3.4].

Proposition 4.1. [54] Let \( D : V \to V^* \) be such that:

(i) \( D \) is monotone, i.e. for every \( x, y \in V \), \( \langle D(x) - D(y), x - y \rangle \geq 0 \).

(ii) \( D \) is hemicontinuous, i.e. \( \lim_{\varepsilon \to 0} \langle D(x + \varepsilon y), z \rangle = \langle D(x), z \rangle \) for every \( x, y, z \in V \).

(iii) \( D \) satisfies the growth condition, i.e. there exists \( K > 0 \) such that for every \( x \in V \),

\[
\|D(x)\|_{V^*} \leq K(1 + \|x\|_V^{p-1}).
\]

(iv) \( D \) is coercive, i.e. there exist constants \( C_1 > 0 \) and \( C_2 \geq 0 \) such that

\[
\langle D(x), x \rangle \geq C_1 \|x\|_V^p - C_2, \quad \forall x \in V
\]

Then for every \( y \in V^* \), there exists \( x \in V \) such that \( D(x) = y \) and

\[
\|x\|_V^p \leq \frac{C_1 + 2C_2}{C_1} + \frac{1}{C_1^2} \|y\|_{V^*}^2.
\]

If there exists a positive constant \( C_3 \) such that

\[
\langle D(x_1) - D(x_2), x_1 - x_2 \rangle \geq C_3 \|x_1 - x_2\|_{V^*}^2, \quad x_1, x_2 \in V,
\]

then for any \( y \in V^* \), the equation \( D(x) = y \) has a unique solution \( x \in V \).

Proof of Theorem 3.4. Let \( I_d : V \to V \) and \( I_{d_n} : V_n \to V_n \) be the identity maps. It is easy to check that for large enough \( m \), the operators

\[
D = I_d - \int_{t_{i-1}}^{t_i} A_s(x)ds,
\]

\[
D_n = I_{d_n} - \int_{t_{i-1}}^{t_i} \Pi_n A_s(x)ds,
\]

have the properties (i)-(iv) stated in Proposition 4.1 and satisfy (28). Using Proposition 4.1, each of equations \( D(x) = y \) and \( D_n(x) = y \) has unique solution. According to (28), \( D^{-1} \) and \( D_n^{-1} \) are continuous and hence they are measurable. So the existence of unique \( \mathcal{F}_{t_i} \)-measurable solutions to (12) and (13) follows inductively. We use induction on \( i \) to prove
that $\mathbb{E}\|u^{n,m,l}(t_i)\|_V^p < \infty$. The proof of $\mathbb{E}\|u^{m,l}(t_i)\|_V^p < \infty$ is the same. According to the previous proposition, we get that

$$\|u^{n,m,l}(t_i)\|_V^p \leq C'_1 + C'_2 \|y_i\|_{V^*}^2,$$

where

$$y_i := u^{n,m,l}(t_{i-1}) + \Pi_n \tilde{B}^m_{t_{i}} (u^{n,m,l}(t_{i-1})) \left( W^l_{t_i} - W^l_{t_{i-1}} \right) + \int_E \Pi_n \tilde{F}^{m,l}_{t_{i}} (u^{n,m,l}(t_{i-1}), \xi) \tilde{N} ((t_{i-1}, t_i), d\xi).$$

For $i = 1$, $y_1 = \Pi_n \zeta$ and $\mathbb{E}\|y_1\|_{V^*}^2 \leq C\mathbb{E}\|\zeta\|_H^2 < \infty$, so we get $\mathbb{E}\|u^{n,m,l}(t_1)\|_V^p < \infty$. Let us estimate $\mathbb{E}\|y_i\|_{V^*}^2$ for $i \geq 2$. Using Proposition 2.1 and $p \geq 2$, we have

$$\mathbb{E}\|y_i\|_{V^*}^2 \leq C\mathbb{E}\|y_i\|_H^2 \leq C\mathbb{E}\|u^{n,m,l}(t_{i-1})\|_H^2 + C\delta_m \mathbb{E}\|\Pi_n \tilde{B}^m_{t_{i}} (u^{n,m,l}(t_{i-1}))\|_2^2 + C\delta_m \int_E \mathbb{E}\|\Pi_n \tilde{F}^{m,l}_{t_{i}} (u^{n,m,l}(t_{i-1}), \xi)\|_H^2 \nu(d\xi)$$

$$\leq C\mathbb{E}\|u^{n,m,l}(t_{i-1})\|_H^2 + C\delta_m^{-1} \mathbb{E}\left[ \left( \nu(E_j^i) \right)^{-1} \left\| \int_{t_{i-2}}^{t_{i-1}} \Pi_n F_s (u^{n,m,l}(t_i), \xi) \nu(d\xi) ds \right\|_H^2 \right] + C\delta_m^{-1} \sum_{1 \leq j \leq r_i} \left( \nu(E_j^i) \right)^{-1} \left\| \int_{t_{i-2}}^{t_{i-1}} \int_E \Pi_n F_s (u^{n,m,l}(t_i), \xi) \nu(d\xi) ds \right\|_H^2$$

$$\leq C\mathbb{E}\|u^{n,m,l}(t_{i-1})\|_H^2 + C\int_{t_{i-2}}^{t_{i-1}} \left[ \|B_s (u^{n,m,l}(t_{i-1}))\|_2^2 + \int_E \|F_s (u^{n,m,l}(t_{i-1}), \xi)\|_H^2 \nu(d\xi) \right] ds$$

$$\leq C\mathbb{E}\|u^{n,m,l}(t_{i-1})\|_H^2 + C\int_{t_{i-2}}^{t_{i-1}} \left[ 6 (\alpha^{1/q} + 1/p) \left\| u^{n,m,l}(t_{i-1}) \right\|_V^p \lambda(s) + \frac{16}{q} K_2(s) + 2K_1(s) \right] ds$$

$$\leq C \left( 1 + \mathbb{E}\left\| u^{n,m,l}(t_{i-1}) \right\|_V^p \right).$$

So, if $\mathbb{E}\|u^{n,m,l}(t_{i-1})\|_V^p < \infty$, then $\mathbb{E}\|u^{n,m,l}(t_i)\|_V^p < \infty$ too, and the proof of Theorem 3.4 is complete. \hfill \Box

**Proof of Theorem 3.5.** The integral form of the implicit scheme is

$$u^{n,m,l}(t) = \Pi_n \zeta + \int_0^{\kappa_2(t)} \Pi_n A_s (u^{n,m,l}(s)) ds$$

$$+ \int_0^{\kappa_2(t)} \Pi_n \tilde{B}^m_s (u^{n,m,l}(\kappa_1(s))) dW^l_s$$

$$+ \int_0^{\kappa_2(t)} \int_E \Pi_n \tilde{F}^{m,l}_{s} (u^{n,m,l}(\kappa_1(s)), \xi) \tilde{N}(ds, d\xi).$$

(29)
Step 1. There exist a natural number $m_0$ and a constant $L > 0$ such that for every $m \geq m_0$ and $n,l \geq 1$ the value of
\[
\left\| u^{n,m,l} \right\|_{L^\infty([0,T],d\nu;L^2(\Omega,H))} + \left\| u^{n,m,l} \right\|_{\mathcal{L}_2^q(\lambda)} + \left\| A_\cdot (u^{n,m,l}(\cdot)) \right\|_{\mathcal{L}_1^q(\lambda^{-1})} \\
+ \left\| \Pi_n B^m (u^{n,m,l}(\kappa_1(\cdot))) \tilde{\Pi}_l \right\|_{\mathcal{L}_2^2(\nu,H)} \\
+ \left\| \Pi_n \bar{E}^m (u^{n,m,l}(\kappa_1(\cdot)),\cdot) \right\|_{L^2([0,T]\times\Omega\times E;H)}
\]
(30)
is bounded by $L$. The same is true for $u^{m,l}$.

Proof of Step 1. We prove this step only for $u^{n,m,l}$. The proof for $u^{m,l}$ can be done similarly. Since $\mathcal{F}_{t_i-1}$, $W_{t_i} - W_{t_{i-1}}$, and $\tilde{N}((t_{i-1},t_i],d\xi)$ are independent, we get from [13] that
\[
\begin{align*}
\mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 &- 2\delta_m \mathbb{E} \left\langle \Pi_n A_t^m \left( u^{n,m,l}(t_i) \right), u^{n,m,l}(t_i) \right\rangle \\
&\leq \mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 - \delta_m \mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 \\
&\leq \mathbb{E} \left\| u^{n,m,l}(t_{i-1}) \right\|_H^2 + \delta_m \mathbb{E} \left\| \Pi_n \tilde{B}^m (u^{n,m,l}(t_{i-1})) \tilde{\Pi}_l \right\|_2^2 \\
&\quad + \delta_m \int_{t_i}^{t_{i-1}} \mathbb{E} \left\| \Pi_n \tilde{E}^m (u^{n,m,l}(t_{i-1}),\xi) \right\|_H^2 \nu(d\xi)
\end{align*}
\]
(31)
So
\[
\begin{align*}
\mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 &\leq \mathbb{E} \left\| u^{n,m,l}(t_{i-1}) \right\|_H^2 + \mathbb{E} \int_{t_i}^{t_{i-1}} \left\langle u^{n,m,l}(t_i), A_\cdot \left( u^{n,m,l}(t_i) \right) \right\rangle ds \\
&\quad + \mathbb{E} \int_{t_{i-1}}^{t_{i-2}} \left[ \left\| \Pi_n B_s \left( u^{n,m,l}(t_{i-1}) \right) \tilde{\Pi}_l \right\|_2^2 \\
&\quad + \int_{E} \left\| \Pi_n F_s \left( u^{n,m,l}(t_{i-1}),\xi \right) \right\|_H^2 \nu(d\xi) \right] ds.
\end{align*}
\]
(32)
Summing up the above inequalities with respect to $i$ and using the coercivity condition we obtain that
\[
\begin{align*}
\mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 &\leq \mathbb{E} \left\| \zeta \right\|_H^2 + \mathbb{E} \int_0^{t_i} \left[ 2 \left\langle u^{n,m,l}(s), A_\cdot \left( u^{n,m,l}(s) \right) \right\rangle \\
&\quad + \left\| \Pi_n B_s \left( u^{n,m,l}(s) \right) \tilde{\Pi}_l \right\|_2^2 + \left\| \Pi_n F_s \left( u^{n,m,l}(s) \right) \right\|_H^2 \right] ds \\
&\quad \leq \mathbb{E} \left\| \zeta \right\|_H^2 + \mathbb{E} \int_0^{t_i} \left[ -\lambda(s) \left\| u^{n,m,l}(s) \right\|_{V'}^p \\
&\quad + \mu(s) \left\| u^{n,m,l}(s) \right\|_H^2 + K_1(s) \right] ds.
\end{align*}
\]
It is evident that there exists a natural number \( m_1 \) such that for every \( m \geq m_1 \):

\[
\sup_{1 \leq i \leq m} \int_{t_{i-1}}^{t_i} \mu(s)ds \leq \frac{1}{2}.
\]

So by taking \( \alpha_i := \int_{t_{i-1}}^{t_i} \mu(s)ds \) for \( 1 \leq i \leq m \), one obtains that

\[
\frac{1}{2} \mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 + \mathbb{E} \int_{t_{i-1}}^{t_i} \lambda(s) \left\| u^{n,m,l}(s) \right\|_V^p ds \leq C + \sum_{k=1}^{i-1} \alpha_k \mathbb{E} \left\| u^{n,m,l}(t_k) \right\|_H^2.
\]  

(33)

By using induction on \( i \), it is easy to check that

\[
\mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 \leq 2C(1 + 2\alpha_1)(1 + 2\alpha_2) \cdots (1 + 2\alpha_{i-1}).
\]

So

\[
\mathbb{E} \left\| u^{n,m,l}(t_i) \right\|_H^2 \leq 2C(1 + 2\alpha_1) \cdots (1 + 2\alpha_m) \leq 2C \left( 1 + \frac{2\int_0^T \mu(s)ds}{m} \right)^m,
\]

and this inequality implies that

\[
\sup_{n,m,l \geq 1, m \geq m_1} \sup_{s \in [0,T]} \mathbb{E} \left\| u^{n,m}(s) \right\|_H^2 < \infty.
\]

Therefore by inequality (33), the boundedness of \( u^{n,m,l} \) in \( \mathcal{L}_p^b(\lambda) \) follows. The boundedness of \( A(\lambda^{1-q}) \) in \( \mathcal{L}_q^c(\lambda^{1-q}) \) is obtained from the growth condition \((C3)\). The boundedness of the fourth and the fifth summand in relation (30) can be obtained in the same way as the proof of parts (iv) and (v) of Step 1 in the proof of Theorem 3.3.

**Step 2.** Let \((n,m,l)\) be a sequence such that \(m,n\) and \(l\) converge to infinity. Then it contains a subsequence, denoted also by \((n,m,l)\), such that

(i) \( u^{n,m,l} \) converges weakly in \( \mathcal{L}_p^b(\lambda) \) to some progressively measurable process \( \bar{u}_\infty \),

(ii) \( u^{n,m,l}(T) \) converges weakly in \( L^2(\Omega; H) \) to some random variable \( u_{T\infty} \),

(iii) \( A(\cdot) \) converges weakly in \( \mathcal{L}_q^c(\lambda^{1-q}) \) to some progressively measurable process \( a_\infty \),

(iv) \( \Pi_n \bar{B}^m_\Pi \) converges weakly in \( \mathcal{L}^2_{L^2(U,H)}(\mathcal{B}\mathcal{F}) \) to some process \( \bar{b}_\infty \),

(v) \( \Pi_n \bar{F}^m_\Pi \) converges weakly in \( L^2([0,T] \times \Omega \times E, \mathcal{P} \otimes \mathcal{E}, dt \otimes \mathcal{P} \otimes \nu; H) \) to some process \( \bar{f}_\infty \),

(vi) \( (\zeta, a_\infty, b_\infty, f_\infty) \in \mathcal{A} \) and for all \( z \in \mathcal{V} \), and \( dt \otimes \mathcal{P} \)-almost all \((t, \omega)\) we have

\[
(\bar{u}_\infty(t), z)_H = (\zeta, z)_H + \int_0^t \langle a_\infty(s), z \rangle ds + \int_0^t \langle b_\infty(s)dW_s, z \rangle_H + \int_0^t \int_E (f_\infty(s, \xi), z)_H \tilde{N}(ds, d\xi)
\]

(34)
and for all \( z \in V \), almost surely

\[
(u_{T\infty}, z)_H = (\zeta, z)_H + \int_0^T \langle a_{\infty}(s), z \rangle ds + \int_0^T \langle z, b_{\infty}(s)dW_s \rangle_H + \int_0^T \int_E (f_{\infty}(s, \xi), z)_{H} \tilde{N}(ds, d\xi).
\]

(35)

**Proof of Step 2.** The convergences in (i)-(v) directly follow from (30). The fact that \( \bar{u}_{\infty} \) and \( a_{\infty} \) are progressively measurable can be shown similarly to the proof of the corresponding statements in Step 2 of the proof of Theorem 3.3. It remains to prove (vi). Fix \( N \in \mathbb{N} \). It is sufficient to verify (vi) for \( z \in V_N \), because \( \bigcup_{N=1}^{\infty} V_N \) is dense in \( V \). To verify (34), it is sufficient to prove that for any \( \varphi \in L^\infty_{\mathbb{R}} \),

\[
\mathbb{E} \int_0^T (u_{\infty}(t), z)_H \varphi(t)dt = \mathbb{E} \int_0^T (\zeta, z)_H \varphi(t)dt + \mathbb{E} \int_0^T \left( \int_0^t \langle a_{\infty}(s), z \rangle ds \right) \varphi(t)dt + \mathbb{E} \int_0^T \left( \int_0^t \langle b_{\infty}(s)dW_s, z \rangle_H \right) \varphi(t)dt + \mathbb{E} \int_0^T \left( \int_0^t \int_E (f_{\infty}(s, \xi), z)_{H} \tilde{N}(ds, d\xi) \right) \varphi(t)dt.
\]

(36)

Taking the inner products of both sides and \( \varphi \), we get for \( n \geq N \),

\[
\mathbb{E} \int_0^T (u^{n,m,l}(t), z)_H \varphi(t)dt = \mathbb{E} \int_0^T (\zeta, z)_H \varphi(t)dt + J_1 + J_2 + J_3 - R_1 - R_2 - R_3,
\]

(37)
where
\[
J_1 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^t \langle A_s(u^{n,m,l}(s)), z \rangle \, ds \right) \, dt \\
\quad \to \mathbb{E} \int_0^T \varphi(t) \left( \int_0^t \langle a_\infty(s), z \rangle \, ds \right) \, dt,
\]
\[
J_2 = \mathbb{E} \int_0^T \varphi(t) \int_0^t \left( \Pi_n \bar{B}^m_s(u^{n,m,l}(\kappa_1(s))) \right) \bar{N}_t dW_s, z) \, dt \\
\quad \to \mathbb{E} \int_0^T \varphi(t) \int_0^t \left( b_\infty(s) dW_s, z \right) \, dt,
\]
\[
J_3 = \mathbb{E} \int_0^T \varphi(t) \left( \int_0^t \int_E \left( \Pi_n \bar{B}^m_s(u^{n,m,l}(\kappa_1(s)), \xi), z \right) \, dN(s, d\xi) \big) \, dt \\
\quad \to \mathbb{E} \int_0^T \varphi(t) \left( \int_0^t \int_E \left( f_\infty(s, \xi), z \right)_H \, dN(ds, d\xi) \big) \, dt,
\]
and
\[
R_1 = \mathbb{E} \int_0^T \varphi(t) \left( \int_t^{t_{\kappa_2(t)}} \langle A_s(u^{n,m,l}(s)), z \rangle \, ds \right) \, dt,
\]
\[
R_2 = \mathbb{E} \int_0^T \varphi(t) \int_t^{t_{\kappa_2(t)}} \left( \Pi_n \bar{B}^m_s(u^{n,m,l}(\kappa_1(s))) \right) \bar{N}_t dW_s, z) \, dt,
\]
\[
R_3 = \mathbb{E} \int_0^T \varphi(t) \int_t^{t_{\kappa_2(t)}} \int_E \left( \Pi_n \bar{B}^m_s(u^{n,m,l}(\kappa_1(s)), \xi) \right), z) \, dN(ds, d\xi) \, dt.
\]

The limits of $J_i$, $i = 1, 2, 3$ can be obtained similar to the limits of $J_i$, $i = 1, 2, 3$ of (21). Now we wish to prove that the “$R_i$’s tend to zero. Hölder’s inequality yields
\[
|R_1| \leq \|\varphi\|_{L^q_V} \left( \mathbb{E} \int_0^T \left| \int_t^{t_{\kappa_2(t)}} \langle A_s(u^{n,m,l}(s)), z) \rangle \, ds \right|^q \, dt \right)^{1/q} \\
\quad \leq \|\varphi\|_{L^q_V} \| z \|_V \\
\quad \times \left[ \mathbb{E} \int_0^T \left| \int_0^T \| A_s(u^{n,m,l}(s)) \|_{V}^q \, ds \cdot \int_t^{t_{\kappa_2(t)}} \lambda(s) \, ds \right|^{q/p} \, dt \right]^{1/q} \\
\quad \leq \|\varphi\|_{L^q_V} \| z \|_V \left( \int_{t_1}^{T} \lambda(s) \, ds \right)^{1/p} \\
\quad \times \left( \mathbb{E} \int_0^T \| A_s(u^{n,m,l}(s)) \|_{V}^q \, ds \cdot \lambda(s)^{1-q} \, ds \right)^{1/q} \to 0.
\]
For $R_2$, using Cauchy-Schwartz’s inequality we have

$$R_2^2 = \left| \mathbb{E} \int_0^T \varphi(t) \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{B}_s^m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t dW_s, z \right) \bigg|_H dt \right|^2$$

$$\leq \| \varphi \|_{L_2^\infty}^2 \mathbb{E} \int_0^T \left| \int_t^{\kappa_2(t)} \left( \Pi_n \tilde{B}_s^m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t dW_s, z \right) \bigg|_H \right|^2 dt$$

$$\leq \| \varphi \|_{L_2^\infty}^2 \| z \|_H^2 \mathbb{E} \int_0^T \left| \Pi_n \tilde{B}_t^m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t \bigg|_2^2 ds dt$$

$$\leq \delta_m \| \varphi \|_{L_2^\infty}^2 \| z \|_H^2 \mathbb{E} \int_0^T \left| \Pi_n \tilde{B}_t^m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t \bigg|_2^2 dt,$$

since $\| \Pi_n \tilde{B}_s^m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_t \|_2^2$ is constant for $s \in (t, \kappa_2(t))$. Thus by Step 1, $R_2 \to 0$ when $m \to \infty$. Finally the computation for $R_3$ is as follows:

$$R_3^2 = \left| \mathbb{E} \int_0^T \varphi(t) \int_t^{\kappa_2(t)} \int_E \left( \Pi_n \tilde{F}_s^{m,l} \left( u^{n,m,l}(\kappa_1(s)), \xi \right), z \right)_H \tilde{\nu}(ds, d\xi) dt \right|^2$$

$$\leq \| \varphi \|_{L_2^\infty}^2 \mathbb{E} \int_0^T \left| \int_t^{\kappa_2(t)} \int_E \left( \Pi_n \tilde{F}_s^{m,l} \left( u^{n,m,l}(\kappa_1(s)), \xi \right), z \right)_H \tilde{\nu}(ds, d\xi) \right|^2 dt$$

$$\leq \| \varphi \|_{L_2^\infty}^2 \| z \|_H^2 \mathbb{E} \int_0^T \left| \Pi_n \tilde{F}_t^{m,l} \left( u^{n,m,l}(\kappa_1(s)), \xi \right) \bigg|_H^2 \nu(d\xi) ds dt$$

$$\leq \delta_m \| \varphi \|_{L_2^\infty}^2 \| z \|_H^2 \mathbb{E} \int_0^T \left| \Pi_n \tilde{F}_t^{m,l} \left( u^{n,m,l}(\kappa_1(s)), \xi \right) \bigg|_H^2 \nu(d\xi) dt$$

In the last inequality, we used the fact that $\| \Pi_n \tilde{F}_s^{m,l} \left( u^{n,m,l}(\kappa_1(s)), \xi \right) \|_H^2$ is constant for $s \in (t, \kappa_2(t))$. So by Step 1, $R_3 \to 0$ when $m \to \infty$. Now we have proven that the limit of right hand side of equation (37) is the right hand side of equation (36). By the fact that $u^{n,m,l} \to \tilde{u}_\infty$, we deduce the similar result for the left hand side, so equation (36) is proved. It remains to prove (35). Both sides of this equation belong to the space $L^1(\Omega; \mathbb{R})$, So it is sufficient to prove that for every $\psi \in L^{\infty}(\Omega; \mathbb{R})$ and for $z \in V_N$

$$\mathbb{E} \left[ \psi \left( u_{T,\infty}, z \right)_H \right] = \mathbb{E} \left[ \psi(\zeta, z)_H \right] + \mathbb{E} \left[ \psi \int_0^T \langle a_\infty(s), z \rangle ds \right]$$

$$+ \mathbb{E} \left[ \psi \int_0^T \langle b_\infty(s)dW_s, z \rangle_H \right]$$

$$+ \mathbb{E} \left[ \psi \int_0^T \int_E \langle f_\infty(s, \xi), z \rangle_H \tilde{\nu}(ds, d\xi) \right].$$

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Fix $N \in \mathbb{N}$ and $z \in V_N$. Using equation (29), we get for $n \geq N$ that

$$
(u^{n,m,l}(T), z)_H = (\zeta, z)_H 1_{\{t > t_0\}} + \int_0^T \langle A_s(u^{n,m,l}(s)), z \rangle_H \, ds \\
+ \int_0^T \left( z, \Pi_n \tilde{B}_s^m (u^{n,m,l}(\kappa_1(s))) \tilde{\Pi}_t dW_s \right)_H \\
+ \int_0^T \int_E \left( \Pi_n \tilde{F}_s^{m,l} (u^{n,m,l}(\kappa_1(s))), z \right)_H \tilde{N}(ds, d\xi).
$$

Taking the inner products of both sides and $\psi$, we get for $n \geq N$

$$
E \left[ \psi(u^{n,m,l}(T), z)_H \right] = E [\psi(\zeta, z)_H] + \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3,
$$

where

$$
\tilde{J}_1 = E \left[ \psi \int_0^T \langle A_s(u^{n,m,l}(s)), z \rangle \, ds \right] \rightarrow E \left[ \psi \int_0^T \langle a_\infty(s), z \rangle \, ds \right],
$$

$$
\tilde{J}_2 = E \left[ \psi \int_0^T \left( \Pi_n \tilde{B}_s^m(u^{n,m,l}(\kappa_1(s))) \tilde{\Pi}_t dW_s, z \right)_H \right]
\rightarrow E \left[ \psi \int_0^T (b_\infty(s) dW_s, z)_H \right],
$$

$$
\tilde{J}_3 = E \left[ \psi \int_0^T \int_E \left( \Pi_n \tilde{F}_s^{m,l}(u^{n,m,l}(\kappa_1(s)), \xi), z \right)_H \tilde{N}(ds, d\xi) \right]
\rightarrow E \left[ \psi \int_0^T \int_E \left( f_\infty(s, \xi), z \right)_H \tilde{N}(ds, d\xi) \right].
$$

The limits are obtained similarly as the limits of the corresponding terms in (22). So (35) is proved.

**Step 3.** Let $(n, m, l)$ be a subsequence which satisfies items (ii),(vi) of the previous step. Then for all $y \in \mathcal{G}_{BF}$

$$
I_y(\zeta, a_\infty, b_\infty, f_\infty) + \lim \inf E \|u^{n,m,l}(T)\|^2_H - E \|u_{T\infty}\|^2_H \leq 0.
$$

**Proof of Step 3.** Let $T_m$ and $S_l$ be the operators defined in (23) and (24) respectively. Then

$$
T_m(\Pi_n B(u_{n,m,l}(\cdot)) \tilde{\Pi}_t) = \Pi_n \tilde{B}_s^m(u_{n,m,l}(\kappa_1(\cdot))) \tilde{\Pi}_t,
$$

$$
S_l \circ T_m(\Pi_n F(u_{n,m,l}(\cdot), *)) = \Pi_n \tilde{F}_s^{m,l}(u_{n,m,l}(\kappa_1(\cdot)), *)
$$

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Define

\[ I^{n,m,l}(y) := \mathbb{E} \int_0^T \left[ \langle A_s \left( u^{n,m,l}(s) \right) - A_s(y_s), u^{n,m,l}(s) - y_s \rangle ight. \]
\[ + \left\| \Pi_n \tilde{B}_m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_l - \mathcal{T}_m \left( \Pi_n B. (y.) \right) (s) \right\|_2^2 \]
\[ + \int_{\mathcal{E}} \left\| \Pi_n \tilde{F}_m \left( u^{n,m,l}(\kappa_1(s)), \xi \right) - \mathcal{F}_m \left( \Pi_n F. (\cdot, \cdot) \right) (s, \xi) \right\|_H^2 \nu(d\xi) \bigg] ds. \]

Since the operator norms of \( \mathcal{T}_m \) and \( \mathcal{S}_l \) equal one, we get

\[ I^{n,m,l}(y) \leq \mathbb{E} \int_0^T \left[ \langle A_s \left( u^{n,m,l}(s) \right) - A_s(y_s), u^{n,m,l}(s) - y_s \rangle \right. \]
\[ + \left\| \Pi_n B_s \left( u^{n,m,l}(s) \right) \tilde{\Pi}_l - \Pi_n B_s(y_s) \right\|_2^2 \]
\[ + \int_{\mathcal{E}} \left\| \Pi_n \tilde{F}_s \left( u^{n,m,l}(s), \xi \right) - \Pi_n F_s(y_s, \xi) \right\|_H^2 \nu(d\xi) \bigg] ds. \]

The monotonicity condition \([\text{C}1] \) implies that \( I^{n,m,l}(y) \leq 0 \). Summing up inequality \((31)\) with respect to \( i, 1 \leq i \leq m \), we get

\[ \mathbb{E} \left\| u^{n,m,l}(T) \right\|_H^2 \leq \mathbb{E} \left\| \zeta \right\|_H^2 + \mathbb{E} \int_0^T \left[ 2 \left\langle u^{n,m,l}(s), A_s \left( u^{n,m,l}(s) \right) \right\rangle \right. \]
\[ + \left\| \Pi_n \tilde{B}_m \left( u^{n,m,l}(\kappa_1(s)) \right) \tilde{\Pi}_l \right\|_2^2 + \int_{\mathcal{E}} \left\| \Pi_n \tilde{F}_m \left( u^{n,m,l}(\kappa_1(s)), \xi \right) \right\|_H^2 \nu(d\xi) \bigg] ds. \]

Thus

\[ 0 \geq I^{n,m,l}(y) \geq \mathbb{E} \left\| u^{n,m,l}(T) \right\|_H^2 - \mathbb{E} \left\| \zeta \right\|_H^2 + \mathbb{E} \int_0^T 2 \left\langle A_s(y_s), y_s \right\rangle ds \]
\[ + J_1 + J_2 - 2(J_3 + J_4 + J_5 + J_6), \quad (38) \]
where

\[ J_1 := \mathbb{E} \int_0^T \| T_m (\Pi_n B_\cdot (y_\cdot)) (s) \|_2^2 \, ds \rightarrow \mathbb{E} \int_0^T \| B_s(y_s) \|_2^2 \, ds, \]

\[ J_2 := \mathbb{E} \int_0^T \int_E \| S_t \circ T_m (\Pi_n F_\cdot (y_\cdot, \cdot)) (s, \xi) \|_H^2 \nu(d\xi) \, ds \]

\[ \rightarrow \mathbb{E} \int_0^T \int_E \| F_s(y_s, \xi) \|_H^2 \nu(d\xi) \, ds, \]

\[ J_3 := \mathbb{E} \int_0^T \langle A_s(u^{n,m,l}(s)), y_s \rangle \, ds \rightarrow \mathbb{E} \int_0^T \langle a_\infty(s), y_s \rangle \, ds, \]

\[ J_4 := \mathbb{E} \int_0^T \langle A_s(y_s), u^{n,m,l}(s) \rangle \, ds \rightarrow \mathbb{E} \int_0^T \langle A_s(y_s), \tilde{u}_\infty(s) \rangle \, ds, \]

\[ J_5 := \mathbb{E} \int_0^T \left\langle T_m (\Pi_n B_\cdot (y_\cdot)) (s), \Pi_n \tilde{B}^m_s (u^{n,m,l}(\kappa_1(s))) \tilde{\Pi}_l \right\rangle_2 \, ds \]

\[ \rightarrow \mathbb{E} \int_0^T \int_E \langle B_s(y_s), b_\infty(s) \rangle_2 \, ds, \]

\[ J_6 := \mathbb{E} \int_0^T \int_E \left( S_t \circ T_m (\Pi_n F_\cdot (y_\cdot, \cdot)) (s, \xi), \Pi_n F^m_s (u^{n,m,l}(\kappa_1(s)), \xi) \right)_H \nu(d\xi) \, ds \]

\[ \rightarrow \mathbb{E} \int_0^T \int_E \langle F_s(y_s, \xi), f_\infty(s, \xi) \rangle_H \nu(d\xi) \, ds. \]

Since \( u^{n,m,l}(T) \rightharpoonup u_{T_\infty} \), we can write

\[ d := \lim \inf \mathbb{E} \| u^{n,m,l}(T) \|_H^2 - \mathbb{E} \| u_{T_\infty} \|_H^2 \geq 0. \] (39)

By Step 2 and Theorem 2.7, \( \tilde{u}_\infty \) has an adapted càdlàg \( H \)-valued modification, denoted with \( u_\infty \), such that almost surely for all \( t \in [0, T] \)

\[ u_\infty(t) = \zeta + \int_0^t a_\infty(s) \, ds + \int_0^t b_\infty(s) \, dW_s + \int_0^t \int_E f_\infty(s, \xi) \tilde{N}(ds, d\xi). \]

This equation for \( t = T \) together with the equation (35) implies \( u_\infty(T) = u_{T_\infty} \) a.s.. By Itô’s formula (see Theorem 2.7), we have

\[ \mathbb{E} \| u_{T_\infty} \|_H^2 = \mathbb{E} \| u_\infty(T) \|_H^2 \]

\[ = \mathbb{E} \| \zeta \|_H^2 + \mathbb{E} \int_0^T \left[ 2 \langle a_\infty(s), \tilde{u}_\infty(s) \rangle + \| b_\infty(s) \|_2^2 + \int_E \| f_\infty(s, \xi) \|_H^2 \nu(d\xi) \right] \, ds. \] (40)
Therefore, by taking limit inferior of inequality (38) and using (39) and (40), we conclude that

\[ 0 \geq d + \mathbb{E} \int_0^T \left[ 2 \langle a_\infty(s), \bar{u}_\infty(s) \rangle + \| b_\infty(s) \|_2^2 + \int_E \| f_\infty(s, \xi) \|_H^2 \nu(d\xi) \right] ds \\
+ \mathbb{E} \int_0^T \left[ 2 \langle A_s(y_s), y_s \rangle + \| B_s(y_s) \|_2^2 + \int_E \| F_s(y_s, \xi) \|_H^2 \nu(d\xi) \right] ds \\
- 2 \mathbb{E} \int_0^T \left[ (a_\infty(s), y_s) + \langle A_s(y_s), \bar{u}_\infty(s) \rangle \right] ds \\
- 2 \mathbb{E} \int_0^T \left[ \langle B_s(y_s), b_\infty(s) \rangle_2 + \int_E (F_s(y_s, \xi), f_\infty(s, \xi))_H \nu(d\xi) \right] ds, \\
= d + I_y(\zeta, a_\infty, b_\infty, f_\infty). \]

Hence by Theorem 2.9, \( u_\infty \) must be a solution of equation (1) which is unique. By setting \( y = u_\infty \) one obtains that \( d = 0 \). The weak convergence of \( u^{n,m,l}(T) \), combined with the fact that \( d = 0 \), implies that a subsequence of \( u^{n,m,l}(T) \) converges strongly in \( L^2(\Omega, \mathbb{P}; H) \) to \( u_\infty(T) \). Since there exist unique limits for any convergent subsequences of \( u^{n,m,l} \) and \( u^{n,m,l}(T) \), and on the other hand each subsequence of these sequences has a convergent subsequence, the whole sequences \( u^{n,m,l} \) and \( u^{n,m,l}(T) \) converge. Thus the proof of theorem is complete. \( \square \)

5 Example: Stochastic parabolic equations

In this section, we illustrate an example for the provided set-up and we conduct a simulation using both the explicit and implicit schemes. We consider the equation

\[ du_t = \left( \frac{1}{2} \partial_{xx} u_t - u_t \right) dt + \theta \partial_x u_t dW_t \\
+ \int_{[1,\infty)^2} \frac{u_t}{\xi_1 \xi_2} \tilde{N}(dt, d\xi_1, d\xi_2) \]  

(41)

with given some initial condition \( u_0(x) \) for \( x \in [0, 1] \).

To establish the provided framework, first we introduce the Gelfand triple \( V = H_0^{1,2}(0, 1) \subset \) \( H := L^2(0, 1) \subset V^* = H^{-1,2}(0, 1) \) (see [109]). Here, \( W \) is standard one-dimensional Brownian motion, \( \theta \in (-1,1) \), and \( \tilde{N} \) is compensated Poisson measure on \( [0, T] \times [0, 1] \times [1, \infty)^2 \) with Lebesgue intensity measure. Now we define the corresponding operators

\[ A : H_0^{1,2}(0, 1) \rightarrow H^{-1,2}(0, 1), \quad Au := \frac{1}{2} u'' - u, \]

\[ B : H_0^{1,2}(0, 1) \rightarrow L^2(0, 1), \quad Bu := \theta u', \]

\[ F : H_0^{1,2}(0, 1) \times [1, \infty)^2 \rightarrow L^2(0, 1), \quad F(u, \xi_1, \xi_2) := \frac{u}{\xi_1 \xi_2}. \]
Let $V_n$ be the subspace of $V$ consisting of continuous piecewise linear functions like $v$ which are affine linear on $[(k - 1)/n, k/n]$ for all $k \in \{1, 2, 3, \ldots, n\}$ and $v(0) = v(1) = 0$. Consider the basis $\{e_1, \ldots, e_{n-1}\}$ for $V_n$, where
\[ e_k(x) := 1_{[\frac{k-1}{n}, \frac{k}{n}]}(x) [1 - |nx - k|]. \]
So for $v \in V_n$, $[v] := [v(1/n), \ldots, v((n - 1)/n)]$ is the representation of $v$ with respect to the basis $\{e_1, \ldots, e_{n-1}\}$.

We have
\[ (e_i, e_j)_H = \begin{cases} 2 & i = j \\ \frac{2}{3 \times n} & |i - j| = 1 \\ \frac{1}{6 \times n} & |i - j| > 1 \end{cases}, \quad (e'_i, e'_j)_H = \begin{cases} 2n & i = j \\ -n & |i - j| = 1 \\ 0 & |i - j| > 1 \end{cases} \]

To represent this inner product in matrix form, we define a matrix $M = T(\frac{2}{3}, 1, \frac{1}{6})$ where
\[ T(a, b, c) := \begin{pmatrix} a & b \\ c & a & b \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & b & c \\ c & a \end{pmatrix} \]
is tridiagonal $(n - 1) \times (n - 1)$ matrix. Thus, for $u, v \in V_n$
\[ (u, v)_H = \frac{1}{n} [u] M [v]^T \]

The columns of $\sqrt{n}M^{-1}[e_1, \ldots, e_{n-1}]^T$ consist an orthonormal basis for $V_n$ with respect to inner product in $H$. So
\[ \sum_{k=1}^{n-1} \|e_k\|_V^2 = n^2 \text{tr} \left( M^{-1}T(2, -1, -1) \right) \simeq Cn^3 \]
and the condition for convergence in [54, Theorem 2.8] reads as $n^3/m \to 0$.

We need to obtain $\Pi_n v$, $\Pi_n v''$, $\Pi_n v'$. For $v \in V^*$ and $k \in \{1, 2, 3, \ldots, n - 1\}$, we have
\[ \frac{1}{n} [\Pi_n v] M [e_k]^T = (\Pi_n v, e_k)_H = \langle v, e_k \rangle \]
\[ = \int_{(k-1)/n}^{(k+1)/n} v(r)e_k(r)dr \]
Since $[e_k]_i = 1_{k=i}$, we get
\[ [\Pi_n v] = n[\langle v, e_1 \rangle, \langle v, e_2 \rangle, \ldots, \langle v, e_{n-1} \rangle] M^{-1} \]
For \( v \in V_n \), we have
\[
\langle v', e_k \rangle = \int_0^1 v'(r)e_k(r)dr = \frac{v((k+1)/n) - v((k-1)/n)}{2}.
\]
and
\[
\langle v'', e_k \rangle = \int_0^1 v''(r)e_k(r)dr = -\int_0^1 v'(r)e'_k(r)dr
- n\int_{k/n}^{(k+1)/n} v'(r)dr + n\int_{k/n}^{(k-1)/n} v'(r)dr
= n \left[ v((k-1)/n) - 2v(k/n) + v((k+1)/n) \right]
\]
So for \( v \in V_n \), we obtain the matrix representations of \( \Pi_n v' \) and \( \Pi_n v'' \) as follows,
\[
[\Pi_n v'] = \frac{n^2}{2} [v]T (0, -1, 1) M^{-1} =: [v]M'.
\]
and
\[
[\Pi_n v''] = n^2[v]T(-2, 1, 1)M^{-1} =: [v]M''
\]
Let us estimate \( C(n) \). We have for \( v \in V_n \),
\[
\|\Pi_n A(v)\|_H^2 = \frac{1}{n} \left( \frac{1}{2} [v]M'' - [v] \right) M \left( \frac{1}{2} M''[v]^T - [v]^T \right)
\leq C n^3 [v]T(-2, 1, 1)M^{-1}T(-2, 1, 1)[v]^T
\]
and for \( u \in V_n \),
\[
\langle v'', u \rangle = \sum_{k=1}^{n-1} n \left[ v((k-1)/n) - 2v(k/n) + v((k+1)/n) \right] u(k/n).
\]
For \( u \in V_n \) with \( [u] = [v]T(-2, 1, 1) \), we have
\[
\langle A(v), u \rangle = \left\langle \frac{1}{2} v'' - v, u \right\rangle = \frac{n}{2} [u][u]^T + [v]T(2, -1, -1)[v]^T \geq \frac{n}{2} [u][u]^T,
\]
\[
\|u\|_V^2 = n[u]T(2, -1, -1)[u]^T \leq 4n[u][u]^T.
\]
So
\[
\|A(v)\|_V^2 \geq \frac{\|\langle A(v), u \rangle \|^2}{\|u\|_V^2} \geq \frac{n}{4} [u][u]^T
\geq \frac{\|M\|n^3[u]M^{-1}[u]^T}{4n^2}
= \frac{\|M\|}{4Cn^2} \|\Pi_n A(v)\|_H^2
\]
Figure 1: Simulated solution of equation [41] for (a) explicit scheme with \( n = 50 \) and \( m = 2500 \) (b) explicit scheme with \( n = 200 \) and \( m = 40000 \) (c) implicit scheme with \( n = 50 \) and \( m = 2500 \) (d) implicit scheme with \( n = 200 \) and \( m = 40000 \). Plots (e) to (h) show the simulated solution without the Poisson noise (only Brownian noise) with the same parameters. We have set the initial condition \( u_0(x) = 10 \sin(2\pi x) \) and \( \theta = .5 \). In all cases we set \( l = 10 \).

Therefore, \( C(n) \leq 4Cn^2/\|M\| \) and the condition in Theorem 3.3 reads as \( n^2/m \to 0 \) which is significantly weaker than the condition in [54, Theorem 2.8].

We also set \( E^l := [1, l] \times [1, l] \) and partition \( E^l \) by dividing \([1, l]\) into equal subintervals of length \( \frac{1}{l} \).

So explicit scheme becomes

\[
[u_{m,l}^n(t_0)] = [\Pi_n u_0(x)],
\]

\[
[u_{m,l}^n(t_{k+1})] = [u_{m,l}^n(t_k)] \left( I + \delta_m \left( \frac{1}{2}M'' - I \right) + \theta \sqrt{\delta_m}N M' + \tilde{Z}^l I \right)
\]

where \( N \) is standard normal random variable and

\[
\tilde{Z}^l = \sum_{j=1}^{r_l} \tilde{F}^l_j (Z^l_j - \delta_m |E^l_j|)
\]

and \( Z^l_j \) is a Poisson random variable with parameter \( \delta_m |E^l_j| \) and \( \tilde{F}^l_j = \int_{E^l_j} \frac{d\xi_1 d\xi_2}{\xi_1 \xi_2} \).

For the implicit scheme, in each step we have to solve the following equation

\[
[u_{m,l}^n(t_{k+1})] = [u_{m,l}^n(t_{k+1})] \delta_m \left( \frac{1}{2}M'' - I \right) + [u_{m,l}^n(t_k)] \left( I + \theta \sqrt{\delta_m}N M' + \tilde{Z}^l I \right)
\]

which leads to

\[
[u_{m,l}^n(t_{k+1})] = [u_{m,l}^n(t_k)] \left( I + \theta \sqrt{\delta_m}N M' + \tilde{Z}^l I \right) \left( I - \delta_m \left( \frac{1}{2}M'' - I \right) \right)^{-1}
\]

Figure 1 shows the graph of the solutions resulting from the explicit and implicit schemes for various values of parameters for both with and without the Poissonian noise.
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References

[1] Abdulle, Assyr and Bréhier, Charles-Edouard and Vilmart, Gilles. Convergence analysis of explicit stabilized integrators for parabolic semilinear stochastic PDEs. arXiv preprint:2102.03209, 2021.

[2] Adamu, Iyabo Ann. Numerical approximation of SDEs and stochastic Swift-Hohenberg equation. PhD thesis, Heriot-Watt University, 2011.

[3] Applebaum, David. Lévy Processes and Stochastic Calculus. Cambridge university press, 2009.

[4] Barth, Andrea and Lang, Annika. Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises. Applied Mathematics & Optimization, 66(3):387–413, 2012.

[5] Barth, Andrea and Stein, Andreas. Stochastic Transport with Lévy Noise–Fully Discrete Numerical Approximation. arXiv preprint:1910.14657, 2019.

[6] Barth, Andrea and Stüwe, Tobias. Weak convergence of Galerkin approximations of stochastic partial differential equations driven by additive Lévy noise. Mathematics and Computers in Simulation, 143:215–225, 2018.

[7] Beccari, Matteo and Hutzenthaler, Martin and Jentzen, Arnulf and Kurniawan, Ryan and Lindner, Felix and Salimova, Diyora. Strong and weak divergence of exponential and linear-implicit Euler approximations for stochastic partial differential equations with superlinearly growing nonlinearities. arXiv preprint:1903.06066, 2019.

[8] Becker, Sebastian and Gess, Benjamin and Jentzen, Arnulf and Kloeden, Peter E. Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations. arXiv preprint:1711.02423, 2017.

[9] Becker, Sebastian and Gess, Benjamin and Jentzen, Arnulf and Kloeden, Peter E. Lower and upper bounds for strong approximation errors for numerical approximations of stochastic heat equations. BIT Numerical Mathematics, 60(4):1057–1073, 2020.

[10] Becker, Sebastian and Jentzen, Arnulf. Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg–Landau equations. Stochastic Processes and their Applications, 129(1):28–69, 2019.
[11] Bessaih, Hakima and Hausenblas, Erika and Randrianasolo, Tsiry AvisoA and Razafimandimby, Paul André. Numerical approximation of stochastic evolution equations: convergence in scale of Hilbert spaces. *Journal of Computational and Applied Mathematics*, 343:250–274, 2018.

[12] Bessaih, Hakima and Millet, Annie. On strong L2 convergence of numerical schemes for the stochastic 2d Navier-Stokes equations. *arXiv preprint:1801.03548*, 2018.

[13] Beyn, Wolf-Jürgen and Isaak, Elena and Kruse, Raphael. Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes. *Journal of Scientific Computing*, 67(3):955–987, 2016.

[14] Beyn, Wolf-Jürgen and Isaak, Elena and Kruse, Raphael. Stochastic C-stability and B-consistency of explicit and implicit Milstein-type schemes. *Journal of Scientific Computing*, 70(3):1042–1077, 2017.

[15] Blomker, Dirk and Jentzen, Arnulf. Galerkin approximations for the stochastic Burgers equation. *SIAM Journal on Numerical Analysis*, 51(1):694–715, 2013.

[16] Blömker, Dirk and Kamrani, Minoo. Numerically computable a posteriori-bounds for the stochastic Allen–Cahn equation. *BIT Numerical Mathematics*, 59(3):647–673, 2019.

[17] Blömker, Dirk and Kamrani, Minoo and Hosseini, S Mohammad. Full discretization of the stochastic Burgers equation with correlated noise. *IMA Journal of Numerical Analysis*, 33(3):825–848, 2013.

[18] Breckner, Hannelore. Galerkin approximation and the strong solution of the Navier-Stokes equation. *Journal of Applied Mathematics and Stochastic Analysis*, 13(3):239–259, 2000.

[19] Bréhier, Charles-Edouard. Influence of the regularity of the test functions for weak convergence in numerical discretization of SPDEs. *Journal of Complexity*, 56:101424, 2020.

[20] Bréhier, Charles-Edouard and Cui, Jianbo and Hong, Jialin. Strong convergence rates of semidiscrete splitting approximations for the stochastic Allen–Cahn equation. *IMA Journal of Numerical Analysis*, 39(4):2096–2134, 2019.

[21] Bréhier, Charles-Edouard and Goudenège, Ludovic. Analysis of some splitting schemes for the stochastic Allen-Cahn equation. *Discrete & Continuous Dynamical Systems - Series B*, 24(8):4169–4190, 2019.

[22] Breit, Dominic and Hofmanová, Martina and Loisel, Sébastien. Space-Time Approximation of Stochastic p-Laplace-Type Systems. *SIAM Journal on Numerical Analysis*, 59(4):2218–2236, 2021.

[23] Brezis, Haim. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2010.
[24] Brzeźniak, Zdzislaw and Carelli, Erich and Prohl, Andreas. Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing. *IMA Journal of Numerical Analysis*, 33(3):771–824, 2013.

[25] Brzeźniak, Zdzislaw and Liu, Wei and Zhu, Jiahui. Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise. *Nonlinear Analysis: Real World Applications*, 17:283–310, 2014.

[26] Brzeźniak, Zdzislaw and Millet, Annie. On the splitting method for some complex-valued quasilinear evolution equations. In *Stochastic Analysis and Related Topics*, pages 57–90. Springer, 2012.

[27] Campbell, Stuart and Lord, Gabriel. Adaptive time-stepping for Stochastic Partial Differential Equations with non-Lipschitz drift. *arXiv preprint:1812.09036*, 2018.

[28] Carelli, Erich. *Numerical Analysis of the Stochastic Navier-Stokes Equations*. PhD thesis, Universitätsbibliothek Tübingen, 2012.

[29] Carelli, Erich and Prohl, Andreas. Rates of Convergence for Discretizations of the Stochastic Incompressible Navier–Stokes Equations. *SIAM Journal on Numerical Analysis*, 50(5):2467–2496, 2012.

[30] Cox, Sonja and van Neerven, Jan. Pathwise Hölder convergence of the implicit-linear Euler scheme for semi-linear SPDEs with multiplicative noise. *Numerische Mathematik*, 125(2):259–345, 2013.

[31] Cui, Jianbo and Hong, Jialin and Liu, Zhihui and Zhou, Weien. Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations. *Journal of Differential Equations*, 266(9):5625–5663, 2019.

[32] Dareiotis, Konstantinos and Kumar, Chaman and Sabanis, Sotirios. On tamed Euler approximations of SDEs driven by Lévy noise with applications to delay equations. *SIAM Journal on Numerical Analysis*, 54(3):1840–1872, 2016.

[33] Dareiotis, Konstantinos and Leahy, James-Michael. Finite difference schemes for linear stochastic integro-differential equations. *Stochastic Processes and their Applications*, 126(10):3202–3234, 2016.

[34] De Bouard, Anne and Debussche, Arnaud. Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear schrodinger equation. *Applied Mathematics and Optimization*, 54(3):369–399, 2006.

[35] Debbi, Latifa and Dozzi, Marco. On a space discretization scheme for the fractional stochastic heat equations. *arXiv preprint:1102.4689*, 2011.

[36] Debussche, Arnaud. Weak approximation of stochastic partial differential equations: the nonlinear case. *Mathematics of Computation*, 80(273):89–117, 2011.
[37] Debussche, Arnaud and Printems, Jacques. Weak order for the discretization of the stochastic heat equation. *Mathematics of computation*, 78(266):845–863, 2009.

[38] Dörsek, Philipp. Semigroup splitting and cubature approximations for the stochastic Navier–Stokes equations. *SIAM Journal on Numerical Analysis*, 50(2):729–746, 2012.

[39] Duan, Yuanyuan and Yang, Xiaoyuan. The finite element method of a Euler scheme for stochastic Navier-Stokes equations involving the turbulent component. *International Journal of Numerical Analysis & Modeling*, 10(3), 2013.

[40] Emmrich, Etienne and Šiška, David. Nonlinear stochastic evolution equations of second order with damping. *Stochastics and Partial Differential Equations: Analysis and Computations*, 5(1):81–112, 2017.

[41] Feng, Xiaobing and Li, Yukun and Zhang, Yi. Strong convergence of a fully discrete finite element method for a class of semilinear stochastic partial differential equations with multiplicative noise. *Journal of Computational Mathematics*, 39(4), 2021.

[42] Furihata, Daisuke and Kovács, Mihály and Larsson, Stig and Lindgren, Fredrik. Strong Convergence of a Fully Discrete Finite Element Approximation of the Stochastic Cahn–Hilliard Equation. *SIAM Journal on Numerical Analysis*, 56(2):708–731, 2018.

[43] Gazeau, Maxime. Strong order of convergence of a semidiscrete scheme for the stochastic Manakov equation. *arXiv preprint:1308.1576*, 2013.

[44] Gazeau, Maxime. Probability and pathwise order of convergence of a semidiscrete scheme for the stochastic Manakov equation. *SIAM Journal on Numerical Analysis*, 52(1):533–553, 2014.

[45] Gerencsér, Máté and Gyöngy, István. Finite difference schemes for stochastic partial differential equations in Sobolev spaces. *Applied Mathematics & Optimization*, 72(1):77–100, 2015.

[46] Gerencsér, Máté and Gyöngy, István. Localization errors in solving stochastic partial differential equations in the whole space. *Mathematics of Computation*, 86(307):2373–2397, 2017.

[47] Ghayebi, Bakhtiyar and Hosseini, S Mohammad and Blömker, Dirk. Numerical solution of the Burgers equation with Neumann boundary noise. *Journal of Computational and Applied Mathematics*, 311:148–164, 2017.

[48] Glatt-Holtz, Nathan and Temam, Roger and Wang, Chuntian. Time discrete approximation of weak solutions to stochastic equations of geophysical fluid dynamics and applications. *Chinese Annals of Mathematics, Series B*, 38(2):425–472, 2017.

[49] Gyöngy, István. On stochastic squations with respect to semimartingales III. *Stochastics*, 7(4):231–254, 1982.
[50] Gyöngy, István. Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations Driven by Space-Time White Noise I. Potential Analysis, 1(9):1–25, 1998.

[51] Gyöngy, István. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise II. Potential Analysis, 11(1):1–37, 1999.

[52] Gyöngy, István and Krylov, Nicolai V. On stochastics equations with respect to semi-martingales II. Itô formula in Banach spaces. Stochastics, 6(3-4):153–173, 1982.

[53] Gyöngy, István and Martínez, Teresa. Solutions of stochastic partial differential equations as extremals of convex functionals. Acta Mathematica Hungarica, 109(1-2):127–145, 2005.

[54] Gyöngy, István and Millet, Annie. On discretization schemes for stochastic evolution equations. Potential Analysis, 23(2):99–134, 2005.

[55] Gyöngy, István and Millet, Annie. Rate of convergence of implicit approximations for stochastic evolution equations. Stochastic Differential Equations: Theory and Applications: A Volume in Honor of Professor Boris L Rozovskii, pages 281–310, 2007.

[56] Gyöngy, István and Millet, Annie. Rate of convergence of space time approximations for stochastic evolution equations. Potential Analysis, 30(1):29–64, 2009.

[57] Gyöngy, István and Sabanis, Sotirios and Šiška, David. Convergence of tamed Euler schemes for a class of stochastic evolution equations. Stochastics and Partial Differential Equations: Analysis and Computations, 4(2):225–245, 2016.

[58] Halidias, Nikolaos. A novel approach to construct numerical methods for stochastic differential equations. Numerical Algorithms, 66(1):79–87, 2014.

[59] Halidias, Nikolaos. Construction of positivity preserving numerical schemes for some multidimensional stochastic differential equations. Discrete & Continuous Dynamical Systems-B, 20(1):153–160, 2015.

[60] Halidias, Nikolaos and Stamatiou, Ioannis S. On the numerical solution of some non-linear stochastic differential equations using the semi-discrete method. Computational Methods in Applied Mathematics, 16(1):105–132, 2016.

[61] Hall, Eric Joseph. Accelerated spatial approximations for time discretized stochastic partial differential equations. SIAM Journal on Mathematical Analysis, 44(5):3162–3185, 2012.

[62] Hausenblas, Erika. Finite element approximation of stochastic partial differential equations driven by Poisson random measures of jump type. SIAM Journal on Numerical Analysis, 46(1):437–471, 2008.
[63] Hong, Jialin and Sheng, Derui and Zhou, Tau. A splitting semi-implicit method for stochastic incompressible Euler equations on $\mathbb{T}^2$. arXiv preprint:2102.01482, 2021.

[64] Hutzenthaler, Martin and Jentzen, Arnulf. Numerical Approximations of Stochastic Differential Equations with Non-Globally Lipschitz Continuous Coefficients. American Mathematical Soc., 2015.

[65] Hutzenthaler, Martin and Jentzen, Arnulf. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients. The Annals of Probability, 48(1):53–93, 2020.

[66] Hutzenthaler, Martin and Jentzen, Arnulf and Kloeden, Peter E. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. The Annals of Applied Probability, 22(4):1611–1641, 2012.

[67] Hutzenthaler, Martin and Jentzen, Arnulf and Lindner, Felix and Pušnik, Primož. Strong convergence rates on the whole probability space for space-time discrete numerical approximation schemes for stochastic Burgers equations. arXiv preprint:1911.01870, 2019.

[68] Hutzenthaler, Martin and Jentzen, Arnulf and Salimova, Diyora. Strong convergence of full-discrete nonlinearity-truncated accelerated exponential euler-type approximations for stochastic Kuramoto–Sivashinsky equations. Communications in Mathematical Sciences, 16(6):1489–1529, 2018.

[69] Hutzenthaler, Martin and Jentzen, Arnulf and Wang, Xiaojie. Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations. Mathematics of Computation, 87(311):1353–1413, 2018.

[70] Jentzen, Arnulf. Pathwise numerical approximations of SPDEs with additive noise under non-global Lipschitz coefficients. Potential Analysis, 31(4):375–404, 2009.

[71] Jentzen, Arnulf. Higher order pathwise numerical approximations of SPDEs with additive noise. SIAM Journal on Numerical Analysis, 49(2):642–667, 2011.

[72] Jentzen, Arnulf and Kloeden, Peter and Winkel, Georg. Efficient simulation of nonlinear parabolic SPDEs with additive noise. The Annals of Applied Probability, 21(3):908–950, 2011.

[73] Jentzen, Arnulf and Kloeden, Peter E. The numerical approximation of stochastic partial differential equations. Milan Journal of Mathematics, 77(1):205–244, 2009.

[74] Jentzen, Arnulf and Pušnik, Primož. Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities. IMA Journal of Numerical Analysis, 40(2):1005–1050, 2020.

[75] Jentzen, Arnulf and Röckner, Michael. A Milstein scheme for SPDEs. Foundations of Computational Mathematics, 15(2):313–362, 2015.
[76] Jentzen, Arnulf and Salimova, Diyora and Welti, Timo. Strong convergence for explicit space–time discrete numerical approximation methods for stochastic Burgers equations. *Journal of Mathematical Analysis and Applications*, 469(2):661–704, 2019.

[77] Kamrani, Minoo. Numerical solution of stochastic partial differential equations using a collocation method. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 96(1):106–120, 2016.

[78] Kamrani, Minoo and Blömker, Dirk. Pathwise convergence of a numerical method for stochastic partial differential equations with correlated noise and local Lipschitz condition. *Journal of Computational and Applied Mathematics*, 323:123–135, 2017.

[79] Kamrani, Minoo and Hosseini, S Mohammad. Spectral collocation method for stochastic Burgers equation driven by additive noise. *Mathematics and Computers in Simulation*, 82(9):1630–1644, 2012.

[80] Kamrani, Minoo and Hosseini, S Mohammad and Hausenblas, Erika. Implicit Euler method for numerical solution of nonlinear stochastic partial differential equations with multiplicative trace class noise. *Mathematical Methods in the Applied Sciences*, 323:1245–1260, 2011.

[81] Kloeden, Peter E and Lord, Gabriel J and Neuenkirch, Andreas and Shardlow, Tony. The exponential integrator scheme for stochastic partial differential equations: pathwise error bounds. *Journal of Computational and Applied Mathematics*, 235(5):1245–1260, 2011.

[82] Kossioris, Georgios T and Zouraris, Georgios E. Finite element approximations for a linear Cahn-Hilliard-Cook equation driven by the space derivative of a space-time white noise. *arXiv preprint:1205.4314*, 2012.

[83] Kovács, Mihály and Larsson, Stig and Lindgren, Fredrik. On the backward Euler approximation of the stochastic Allen-Cahn equation. *Journal of Applied Probability*, 52(2):323–338, 2015.

[84] Kovács, Mihály and Larsson, Stig and Lindgren, Fredrik. On the discretisation in time of the stochastic Allen–Cahn equation. *Mathematische Nachrichten*, 291(5-6):966–995, 2018.

[85] Kovács, Mihály and Lindner, Felix and Schilling, René L. Weak convergence of finite element approximations of linear stochastic evolution equations with additive Lévy noise. *SIAM/ASA Journal on Uncertainty Quantification*, 3(1):1159–1199, 2015.

[86] Kruse, Raphael. *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*. Springer, 2013.

[87] Kruse, Raphael and Weiske, Rico. The BDF2-Maruyama Scheme for Stochastic Evolution Equations with Monotone Drift. *arXiv preprint:2105.08767*, 2021.
Krylov, Nicolai V and Rozovskii, Boris L. Stochastic evolution equations. *Journal of Mathematical Sciences*, 16(4):1233–1277, 1981.

Kumar, Chaman and Sabanis, Sotirios. On Tamed Milstein Schemes of SDEs Driven by Lévy Noise. *arXiv preprint:1407.5347*, 2014.

Kumar, Chaman and Sabanis, Sotirios. On Milstein approximations with varying coefficients: the case of super-linear diffusion coefficients. *BIT Numerical Mathematics*, 59(4):929–968, 2019.

Kurniawan, Ryan. *Weak convergence rates for numerical approximations of parabolic stochastic partial differential equations with non-additive noise*. PhD thesis, ETH Zurich, 2018.

Li, Xiaocui and Yang, Xiaoyuan. Error estimates of finite element methods for fractional stochastic Navier–Stokes equations. *Journal of inequalities and applications*, 2018(1):1–15, 2018.

Lindgren, Fredrik. *On weak and strong convergence of numerical approximations of stochastic partial differential equations*. Chalmers Tekniska Hogskola (Sweden), 2012.

Lindner, Felix and Schilling, René L. Weak order for the discretization of the stochastic heat equation driven by impulsive noise. *Potential Analysis*, 38(2):345–379, 2013.

Liu, Wei and Mao, Xuerong. Strong convergence of the stopped Euler–Maruyama method for nonlinear stochastic differential equations. *Applied Mathematics and Computation*, 223:389–400, 2013.

Liu, Wei and Röckner, Michael. SPDE in Hilbert space with locally monotone coefficients. *Journal of Functional Analysis*, 259(11):2902–2922, 2010.

Liu, Wei and Röckner, Michael. *Stochastic Partial Differential Equations*. Springer, 2015.

Liu, Zhilinui and Qiao, Zhonghua. Strong approximation of monotone stochastic partial differential equations driven by white noise. *IMA Journal of Numerical Analysis*, 40(2):1074–1093, 2020.

Lord, Gabriel J and Tambue, Antoine. Stochastic exponential integrators for a finite element discretization of SPDEs. *arXiv preprint:1005.5315*, 2010.

Ma, Ting and Zhu, Rong Chan. Convergence Rate for Galerkin Approximation of the Stochastic Allen—Cahn Equations on 2D Torus. *Acta Mathematica Sinica, English Series*, 37(3):471–490, 2021.

Mao, Xuerong. The truncated Euler–Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 290:370–384, 2015.
[102] Mao, Xuerong. Convergence rates of the truncated Euler-Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 296:362–375, 2016.

[103] Mazzonetto, Sara. Strong convergence for explicit space-time discrete numerical approximation for 2D stochastic Navier-Stokes equations. *arXiv preprint:1809.01937*, 2018.

[104] Milstein, G. and Tretyakov, M. Solving parabolic stochastic partial differential equations via averaging over characteristics. *Mathematics of computation*, 78(268):2075–2106, 2009.

[105] Mishura, Yu S and Shevchenko, Georgii M. Approximation schemes for stochastic differential equations in Hilbert space. *Theory of Probability & Its Applications*, 51(3):442–458, 2007.

[106] Ngo, Hoang-Long and Luong, Duc-Trong. Strong rate of tamed Euler-Maruyama approximation for stochastic differential equations with Hölder continuous diffusion coefficient. *Brazilian Journal of Probability and Statistics*, pages 24–40, 2017.

[107] Ondreját, Martin and Seidler, Jan. On existence of progressively measurable modifications. *Electronic Communications in Probability*, 18(20):1–6, 2013.

[108] Pardoux, Étienne. *Equations aux dérivées partielles stochastiques non lineaires monotones: Etude de solutions fortes de type Ito*. PhD thesis, Université Paris-Sud XI - Orsay, 1975.

[109] Pardoux, Étienne. *Stochastic Partial Differential Equations*. Fudan Lecture Notes, Fudan University Press, 2007.

[110] Peszat, Szymon and Zabczyk, Jerzy. *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach*, volume 113. Cambridge University Press, 2007.

[111] Prévôt, Claudia and Röckner, Michael. *A Concise Course on Stochastic Partial Differential Equations*. Springer, 2007.

[112] Printems, Jacques. On the discretization in time of parabolic stochastic partial differential equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 35(6):1055–1078, 2001.

[113] Pušnik, Primož. *Strong convergence rates for full-discrete numerical approximations of stochastic partial differential equations with non-globally Lipschitz continuous non-linearities*. PhD thesis, ETH Zurich, 2020.

[114] Quer-Sardanyons, Lluís. *The stochastic wave equation: study of the law and approximations*. Universitat de Barcelona, 2005.
[115] Quer-Sardanyons, Lluís and Sanz-Solé, Marta. Space semi-discretisations for a stochastic wave equation. *Potential Analysis*, 24(4):303–332, 2006.

[116] Röckner, Michael and Wilke, André and others. Convergent numerical approximation of the stochastic total variation flow. *Stochastics and Partial Differential Equations: Analysis and Computations*, 9(2):437–471, 2021.

[117] Sabanis, Sotirios. A note on tamed Euler approximations. *Electronic Communications in Probability*, 18:1–10, 2013.

[118] Sabanis, Sotirios. Euler approximations with varying coefficients: the case of super-linearly growing diffusion coefficients. *The Annals of Applied Probability*, 26(4):2083–2105, 2016.

[119] Salimova, Diyora. *Numerical approximation results for semilinear parabolic partial differential equations*. PhD thesis, ETH Zurich, 2019.

[120] Sauer, Martin and Stannat, Wilhelm. Lattice approximation for stochastic reaction diffusion equations with one-sided Lipschitz condition. *Mathematics of Computation*, 84(292):743–766, 2015.

[121] Sauer, Martin and Stannat, Wilhelm. Analysis and approximation of stochastic nerve axon equations. *Mathematics of Computation*, 85(301):2457–2481, 2016.

[122] Song, MH and Lu, YL and Liu, MZ. Convergence of the tamed Euler method for stochastic differential equations with piecewise continuous arguments under non-global Lipschitz continuous coefficients. *Numerical Functional Analysis and Optimization*, 39(5):517–536, 2018.

[123] Szpruch, Lukasz and Zhāng, Xīlíng and others. V-Integrability, Asymptotic Stability And Comparison Theorem of Explicit Numerical Schemes for SDEs. *arXiv preprint:1310.0785*, 2013.

[124] Tambue, Antoine and Mukam, Jean Daniel. Strong convergence of the tamed and the semi-tamed Euler schemes for stochastic differential equations with jumps under non-global Lipschitz condition. *arXiv preprint:1510.04729*, 2015.

[125] Tambue, Antoine and Ngnotchouye, Jean Medard T. Weak convergence for a stochastic exponential integrator and finite element discretization of stochastic partial differential equation with multiplicative & additive noise. *Applied Numerical Mathematics*, 108:57–86, 2016.

[126] Tretyakov, Michael V and Zhang, Zhongqiang. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM Journal on Numerical Analysis*, 51(6):3135–3162, 2013.

[127] Wagner, Tim. *Optimal One-Point Approximation of Stochastic Heat Equations with Additive Noise*. PhD thesis, Technische Universität, 2008.
[128] Wang, Xiaojie. An efficient explicit full-discrete scheme for strong approximation of stochastic Allen–Cahn equation. *Stochastic Processes and their Applications*, 130(10):6271–6299, 2020.

[129] Wang, Xiaojie and Gan, Siqing. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Difference Equations and Applications*, 19(3):466–490, 2013.

[130] Wrzosek, Monika. Newton’s method for stochastic functional evolution equations in Hilbert spaces. *Mathematika*, 65(3):542–556, 2019.

[131] Yang, Li and Zhang, Yanzhi. Convergence of the spectral Galerkin method for the stochastic reaction–diffusion–advection equation. *Journal of Mathematical Analysis and Applications*, 446(2):1230–1254, 2017.

[132] Yang, Xiaoyuan and Li, Xiaocui and Qi, Ruisheng and Zhang, Yinghan. Full-discrete finite element method for stochastic hyperbolic equation. *Journal of Computational Mathematics*, pages 533–556, 2015.

[133] Zhang, Zhongqiang. New explicit balanced schemes for SDEs with locally Lipschitz coefficients. *arXiv preprint:1402.3708*, 2014.

[134] Zhang, Zhongqiang and Karniadakis, George Em. *Numerical Methods for Stochastic Partial Differential Equations with White Noise*. Springer, 2017.

[135] Zong, Xiaofeng and Wu, Fuke and Huang, Chengming. Convergence and stability of the semi-tamed Euler scheme for stochastic differential equations with non-Lipschitz continuous coefficients. *Applied Mathematics and Computation*, 228:240–250, 2014.