Bracket Symmetries of the Classical N=1 String

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We show that it is possible to extend Moore’s analysis of the classical scattering amplitudes of the bosonic string to those of the N=1 superstring. Using the bracket relations we are able to show that all possible amplitudes involving both bosonic and fermionic string states at arbitrary mass levels can be expressed in terms of amplitudes involving only massless states. A slight generalization of Moore’s original definition of the bracket also allows us to determine the 4-point massless amplitudes themselves using only the bracket relations and the usual assumptions of analyticity. We suggest that this should be possible for the higher point massless amplitudes as well.
1. Introduction

In a recent pair of papers [1], [2] Moore showed that the classical S-matrix for the flat, twenty-six dimensional bosonic string is uniquely determined up to the string coupling constant. Using a natural algebraic structure on the space of physical string states, he first wrote down an infinite set of linear relations among the exact classical scattering amplitudes for all values of the kinematical invariants. He then used these relations to show that an arbitrary \( n \)-point amplitude could be written as a linear combination of \( n \)-point tachyon amplitudes at different values of the kinematical invariants. Finally he showed that when coupled with a number of assumptions about the analytic behavior of the scattering amplitudes, these relations could be used to determine the \( n \)-point tachyon amplitude itself, thus fixing all \( n \)-point amplitudes up to a choice of multiplicative constant \( c_n \). Since factorization further determines all of these potentially different \( c_n \) in terms of the string coupling, Moore actually succeeded in uniquely fixing the full S-matrix up to a choice of the string coupling. The algebraic structure used by Moore was a particular case of a more general one called the Gerstenhaber bracket, which exists for the full BRST cohomology of an arbitrary chiral operator algebra. This particular case is itself called the bracket, and the relations it generates are called bracket relations.

In this paper we will show that Moore’s techniques can be extended to the flat, 10-dimensional N=1 superstring. Because of the existence of different ghost pictures, we must begin by showing that bracket is actually well-defined. With this done we go on to generate a set of relations among \( n \)-point amplitudes involving only massless states. Since the four possible 4-point massless amplitudes can be calculated, these relations can be explicitly checked for \( n = 4 \), and they indeed hold. We next show that any \( n \)-point amplitude involving massive states can be expressed in terms of one of a finite number of massless amplitudes. Using the bracket relations generated by the massless fermionic operator—the generalization of the supersymmetry operator to non-zero momentum—the number of such amplitudes can be reduced to a maximal independent set. For \( n = 4 \) there are two such amplitudes, and we show that a generalization of Moore’s original definition of the bracket allows us to fix the value of both of these amplitude up to a constant. We suggest that this should be possible for the independent higher point massless amplitudes as well, and thus that the full S-matrix for the N=1 string should be uniquely determined up to a choice of coupling constant. Finally we show that the construction used by Moore in [2] to lift a restriction on \( n \) encountered in [1] can also be carried over to the \( N = 1 \) string.
2. Review of Moore’s formalism

We begin with a review of the formalism introduced by Moore [1]. For the purpose of this review we will restrict ourselves to the open bosonic string. The extension to the closed string can be found in [1]. It is known that the BRST cohomology of a general chiral operator algebra admits an operation called the Gerstenhaber bracket [3] that maps \{\cdot, \cdot\} : H^{g_1} \times H^{g_2} \to H^{g_1 + g_2 - 1}. We are interested in this bracket for the case of \(g_1 = g_2 = 1\), since \(H^1\) is just the space of physical string states. We have the following explicit contour integral representation of the bracket:

\[
\{O_1, O_2\}(z) = \oint \, dw \, (b_{-1} O_1)(w) O_2(z). \tag{2.1}
\]

Note that up to the factor of \(c\) implicitly contained in \(O_2\), this is nothing more than the commutator of two dimension-one operators. We also note that since all of the operators for the flat \(N = 1\) string contain a factor of the form \(e^{ip \cdot X}\), and \(e^{iq \cdot X(w)} e^{ip \cdot X(z)} \sim (w - z)^{q \cdot p} e^{i(q+p) \cdot X(z)} + \cdots\), this bracket is only defined for operators whose momenta satisfy \(q \cdot p \in \mathbb{Z}\).

The bracket allows us to find relations among scattering amplitudes as follows. We begin by choosing \(n + 1\) physical state operators \(V_i, i = 1, \ldots, n\), and \(J\), with momenta \(p_i\) and \(q\), respectively. We assume that the momenta satisfy \(q + \sum_i p_i = 0\) and \(q \cdot p_i \in \mathbb{Z}\), so that \(J\) is mutually local with respect to each of the \(V_i\). A different notation is used for the last operator since it will be used to generate the relations among (derivatives of) the other operators. Now consider the correlation function \(\langle 0 | V_1(z_1) V_2(z_2) \ldots V_n(z_n) \oint J | 0 \rangle\), where the contour is taken around a small circle that does not enclose any other operators. Since we are dealing with the open string, this contour should really be restricted to the upper-half plane, but for the moment let us imagine that \(J\) can be analytically continued so that the correlator is well-defined for a general contour. Since the contour does not enclose another operator, the correlator vanishes. However if we deform the contour back around infinity, we pick up a contribution from each of the operators equal to just the bracket of that operator with \(J\). Finally if we fix the positions of the first, second and

\footnote{Since the physical operators are elements of the level one BRST cohomology, each operator is actually an equivalence class of operators. For the theory at hand we can always choose a representative of the form \(cV\), where \(V\) is a dimension one operator.}
$n$-th operators at the points $\infty, 1$ and 0, respectively; and integrate a fixed ordering of the positions of the remaining $n - 3$ operators over the interval $[0, 1]$, we find

$$\sum_{i=1}^{n} (-1)^{q(p_i + \cdots + p_n)} A(V_1, \ldots, \{J, V_i\}, \ldots, V_n) = 0,$$

(2.2)

where $\{J, V_i\}$ is just the bracket we defined above. These are Moore’s bracket relations or finite difference relations, so-called because the amplitudes are evaluated at different values of the kinematic invariants.

We must stop here to note that the above procedure can only be carried out for scattering amplitudes involving at most twenty-six $V_i$. This restriction is easily seen as follows. Our ability to write down the bracket relations rests entirely upon our ability to choose momenta such that their sum is zero, all of the operators are on-shell, and $J$ is mutually local with respect to each of the $V_i$. If $d$ is the number of non-compact target space dimensions, then for $n \leq d$ the number of independent conditions on the momenta is equal to the number of independent kinematical invariants available to us. Thus we can always find the necessary momenta. However, because of the linear relations among $d + 1$ vectors in a $d$-dimensional space, when $n > d$ the number of conditions is larger than the number of invariants by $n - d$. Thus we are no longer assured of being able to choose the necessary momenta. In his second paper [2] Moore was able to overcome this restriction by embedding the ordinary twenty-six dimensional string in a string theory with $26 + 2m$ target space dimensions. We shall see later that this same trick works for the $N = 1$ string.

3. The N=1 String

3.1. The Spectrum

We begin by recalling the operator content of the $N = 1$ critical superstring [4]. The space $\mathcal{H}$ of physical operators is a direct sum of two subspaces, $\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f$, whose elements have either bosonic or fermionic target space statistics, respectively. Each of these subspaces carries both a discrete and a continuous grading, and may be written as

$$\mathcal{H}_* = \bigoplus_{n \in \mathbb{Z}_+} \int_{p \in \mathbb{R}^{1,9}} dp \mathcal{H}_*[p, n],$$

(3.1)

where $\mathcal{H}_*[p, n]$ is the space of operators of momentum $p$ at mass level $n$. These spaces are null unless $p^2 = -2n$; and when this condition is satisfied, $\dim \mathcal{H}_b[p, n] = \dim \mathcal{H}_f[p, n] = p_8(n)$, since the spectrum of the string is supersymmetric.
Before we write down any examples of physical operators, let us recall that the spectrum of the $N = 1$ string admits an infinite number of inequivalent, irreducible representations [5]. These representations are labeled by their charge with respect to the field $\phi$, where $\phi$ enters into the bosonization of the $(\beta, \gamma)$ system. An operator with charge $n$ is said to be in the $n$-th ghost picture, or to carry ghost charge $n$. Bosonic operators always carry integral ghost charge, and fermionic operators half-odd integral charge.

We may pass between the different representations using the picture changing operation: $O_{q+1} = [Q_{\text{BRST}}, 2\xi O_q]$. Since we will be only be interested in computing scattering amplitudes, it is only necessary to keep the part of $Q_{\text{BRST}}$ that conserves the $(b,c)$ ghost charge, namely $Q_{\text{BRST}} = \oint dz \frac{1}{2} \bar{\psi} \cdot \partial X e^\phi \eta(z)$. The $(-1)$- and $(-1/2)$-pictures are called the canonical pictures. Operators take on their simplest form when written in one of these pictures.

The massless spectrum consists of a 10-dimensional target space vector and a 10-dimensional target space spinor. These operators have representatives [5] in the canonical pictures of the form

$$V_{-1}(\zeta, p) = c\zeta \cdot \psi_{(-1)} e^{ip \cdot X} = c\zeta \cdot \psi e^{-\phi} e^{ip \cdot X}, \quad (3.2a)$$

$$V_{-1/2}(u, p) = cuS_{(-1/2)} e^{ip \cdot X} = cuSe^{-\phi/2} e^{ip \cdot X}. \quad (3.2b)$$

BRST invariance requires that $p^2 = 0$ for both operators, $\zeta \cdot p = 0$ and $\bar{u}p \cdot \gamma = 0$; and the GSO projection requires that $u$ be a chiral spinor. We will also need the form of the bosonic vertex in the 0-ghost picture. This is given by

$$V_0(\zeta, p) = -c\zeta \cdot (\partial X + ip \cdot \psi \psi) e^{ip \cdot X}. \quad (3.3)$$

The general form of the physical operators at the first massive level has been worked out by Koh et al. in [5]. We will not list their results here, but rather simply make use of them as the need arises.

3.2. The Bracket

We now want to extend the bracket to the $N = 1$ string. Here the existence of an infinite number of inequivalent ghost pictures immediately raises the following question: does the bracket depend on which picture we use? Consider the derivation of the bracket relations, in particular the correlator $\langle 0| V_1(z_1)V_2(z_2) \ldots V_n(z_n) \oint J|0 \rangle$. It is always possible
to redistribute the ghost charges inside a correlation function, so let us imagine shifting one unit of ghost charge from some $V_i$ to a different $V_j$, both before and after we perform the contour deformation. Since the resulting relations should be the same, this suggests that taking the bracket of an operator $O_1$ with the picture changed version of another operator $O_2$ should be the same as the picture changed version of $\{O_1, O_2\}$. For a more direct proof of this, consider the expression $[Q_{BRST}, 2 \xi \{O_1, O_2\}]$. Since neither $O_1$ nor $O_2$ contains $\eta$, we can move $2 \xi$ inside the bracket next to $O_2$. Furthermore since $O_1$ is always assumed to be a physical operator, $Q_{BRST}$ sees only $2 \xi O_2$. Thus we can move $Q_{BRST}$ inside the bracket, leaving us with the desired expression $\{O_1, [Q_{BRST}, 2 \xi O_2]\}$. Using the relation $\{O_1, O_2\} = -(-1)^{q \cdot p} \{O_2, O_1\}$, we can also write this as $\{[Q_{BRST}, 2 \xi O_1], O_2\}$. Thus the picture changing operation commutes with the bracket. Along these same lines, the argument that says we can redistribute the ghost charges inside a correlation function also tells us that we can redistribute the ghost charge inside the bracket. These two properties tell us that the bracket is independent of the ghost pictures of the operators involved. We will see below that this is indeed borne out by example.

Having shown that the bracket is well-defined, we want to compute the brackets needed to write down the relations among scattering amplitudes involving only massless operators. Because the bracket is picture independent, we will choose the pictures such that the brackets take on a simple form. The bracket of two massless bosonic operators, with one in the $(-1)$-picture and the other in the 0-picture, whose momenta $q$ and $p$ satisfy $q \cdot p = 0$, is given by

$$\{V_{-1}(\zeta, q), V_0(\zeta', p)\} = -\int dw \, \zeta \cdot \psi_{(-1)} e^{iq \cdot X(w)} c_{\zeta'} \cdot (\partial X + ip \cdot \psi \psi) e^{ip \cdot (z)}
$$

$$= i c (\zeta \cdot p \zeta' - \zeta' \cdot q \zeta - \zeta \cdot \zeta' p) \cdot \psi_{(-1)} e^{i(q+p) \cdot X(z)}
$$

$$= i \zeta \cdot p V_{-1}(\zeta', q + p) - i \zeta' \cdot q V_{-1}(\zeta, q + p)
$$

$$- i \zeta \cdot \zeta' V_{-1}(p, q + p).$$

To verify that this result is indeed picture independent, let us compute the bracket with both operators in the $(-1)$-picture.

$$\{V_{-1}(\zeta, q), V_{-1}(\zeta', p)\}(z) = \int dw \, \zeta \cdot \psi_{(-1)} e^{iq \cdot X(w)} c_{\zeta'} \cdot \psi_{(-1)} e^{iq \cdot X(z)}
$$

$$= c (\zeta \cdot \psi \zeta' \cdot \psi - i \zeta' \cdot \zeta' q \cdot \partial X) e^{-2\phi} e^{i(q+p) \cdot X(z)}.\]
It is easy to check that the picture changed version of this result is the same as above. Because of this independence we will write
\[
\{ V_b(\zeta, q), V_b(\zeta', p) \} = i\zeta \cdot p V_b(\zeta', q + p) - i\zeta' \cdot q V_b(\zeta, q + p) \\
- i\zeta \cdot \zeta' V_b(p, q + p),
\]
with the understanding that the ghost picture will be chosen so that the ghost charges balance.

The mixed case of the bracket between a massless boson and a massless fermion of momenta \( q \) and \( p \), again with \( q \cdot p = 0 \), is
\[
\{ V_b(\zeta, q), V_f(u, p) \} = -\frac{i}{2} V_f(\bar{\mathbf{q}} + \mathbf{p}) u \cdot q.
\]
It is easy to check that both \( \{ V_0(\zeta, q), V_{-1/2}(u, p) \} \) and \( \{ V_{-1}(\zeta, q), V_{1/2}(u, p) \} \), and the picture changed version of \( \{ V_{-1}(\zeta, q), V_{-1/2}(u, p) \} \) all give this answer. Finally the bracket between two massless fermionic operators, again with \( q \cdot p = 0 \), is given by
\[
\{ V_f(v, q), V_f(u, p) \} = -\frac{1}{\sqrt{2}} V_b(\bar{v} \gamma u, q + p).
\]

3.3. The Massless Relations

Now let us use these brackets to derive relations among scattering amplitudes involving only massless operators. We begin with the relation generated by \( n + 1 \) massless bosonic operators \( \{ V_b(\zeta_1, p_1), \ldots, V_b(\zeta_n, p_n); V_b(\zeta, q) \} \), whose momenta satisfy \( q \cdot p_i = 0 \) for all \( p_i \). Again since the bracket is picture independent, we will always specify the operators by giving the spinors, polarization tensors, etc. needed to characterize them in the canonical picture, with the understanding that the ghost charges will be chosen so that they sum to \( -2 \). The relation associated to these operators is
\[
\sum_{i=1}^n \zeta \cdot p_i A_{b \ldots b} \left( \begin{array}{ccc} \zeta_i & \cdots & s_{ij} \\ \cdots & q + p_i & \cdots \\ \cdots & s_{ij} \end{array} \right) + \sum_{i=1}^n \zeta_i \cdot q A_{b \ldots b} \left( \begin{array}{ccc} \zeta & \cdots & s_{ij} \\ \cdots & q + p_i & \cdots \\ \cdots & s_{ij} \end{array} \right) \\
+ \sum_{i=1}^n \zeta \cdot \zeta_i A_{b \ldots b} \left( \begin{array}{ccc} p_i & \cdots & s_{ij} \\ \cdots & q + p_i & \cdots \\ \cdots & s_{ij} \end{array} \right) = 0.
\]

The finite difference relation among scattering amplitudes involving only massless fermions can be derived from \( \{ V_f(u_1, p_1), \ldots, V_f(u_n, p_n); V_b(\zeta, q) \} \), where again \( q \cdot p_i = 0 \).
\[
\sum_{i=1}^n A_{f \ldots f} \left( \begin{array}{ccc} \bar{\mathbf{q}} + \mathbf{p_i} & \zeta u_i & \cdots \\ \cdots & q + p_i & \cdots \\ \cdots & s_{ij} \end{array} \right) = 0.
\]
Finally we can find relations among scattering amplitudes involving both massless bosons and fermions by choosing the generating operator to be fermionic. For example the relation associated with \( \{ V_f(u_1, p_1), V_f(u_2, p_2), V_f(u_3, p_3), V_b(\zeta_4, p_4); V_f(v, q) \} \) is

\[
A_{fff} \left( \begin{array}{ccc} u_1 & u_2 & u_3 \\ p_1 & p_2 & p_3 \\ (\bar{q} + p_4) & q + p_4 \end{array} \right) |s, t\rangle \\
+ A_{fbb} \left( \begin{array}{ccc} u_1 & u_2 & \bar{\nu}\gamma u_3 \\ p_1 & p_2 & q + p_3 \\ \zeta_4 & p_4 \end{array} \right) |s, t\rangle \\
+ A_{bfb} \left( \begin{array}{ccc} \bar{\nu}\gamma u_1 & u_2 & u_3 \\ q + p_1 & p_2 & p_3 \\ \zeta_4 & p_4 \end{array} \right) |s, t\rangle = 0.
\]

Since \( V_f \) is the supersymmetry operator at non-vanishing momentum, this should be thought of as a generalized supersymmetry relation. For general \( n \), there are \( [(n + 1)/2] \) such relations. Since the four possible 4-point scattering amplitudes have all been computed \( [4] \), it is possible to check these relations for \( n = 4 \), and indeed they hold. This provides a non-trivial check on our formalism.

### 3.4. The Massive Relations

Having shown that Moore's ideas can be extended to the \( N = 1 \) string, we want to prove that one of his non-trivial assertions, namely that we can express any \( n \)-point amplitude in terms of the \( n \)-point tachyon amplitude, has a counterpart in the superstring. Consider the set of \( n \) operators \( V_b^l = V_b(\zeta, lq), l = 1, \ldots, n \), where \( \zeta \cdot q = q^2 = 0 \). Combining the bracket with these operators, we have the maps

\[
\{ V_b^l, * \} : \mathcal{H}_*[p - (n - l)q, n - l] \rightarrow \mathcal{H}_*[p - nq, n],
\]

where we have assumed \( q \cdot p = 0 \). If we then note that these maps are equivalent to the action of the bosonic DDF operators \( \zeta \cdot A_l \), then the no-ghost theorem implies that the map had by summing over \( l \) is onto \( [4] \). This means that given, say, any \( m \)-point bosonic scattering amplitude involving string states at levels \( n_1, \ldots, n_m \), we can always find an \( l \) such that the relation generated by \( \{ V_b(n_1 - l), \ldots, V_b(n_m); V_b^l \} \) expresses our original amplitude in terms of amplitudes involving at least one string states at a strictly lower mass level. Repeating this process we can express any bosonic \( m \)-point function in terms of the \( m \)-point amplitude for the massless bosonic states. This argument easily generalizes to amplitudes involving an even number of fermionic states. Thus we can say that an arbitrary amplitude can be expressed in terms of one of a finite number of massless amplitudes. For the first few cases \( n = 4, 5, 6, 7 \) there are naïvely \( 4, 4, 8, 10 \) amplitudes. Using the generalized supersymmetry relations, we can reduce this to \( 2, 2, 5, 7 \) independent amplitudes.
4. Recursion Relations

4.1. An Initial Attempt

We would next like to derive a set of recursion relations for the massless scattering amplitudes that would allow us to evaluate them without ever having to go through the usual conformal field theory calculations. We will restrict ourselves to the simplest case of 4-point amplitudes. Following the pattern of Moore’s original work, we are tempted to proceed as follows. We choose the generating current to be $V_b(\zeta, q)$. This guarantees that we will not change the number of either bosons or fermions. We then choose the momenta such that one of the amplitudes involves a massive string state. To see why this is necessary, note that in the massless relations derived above, all of the amplitudes are evaluated at the same values of $s$ and $t$. These sort of relations are useless if we want to have a recursive method for finding the dependence of the scattering amplitudes on the kinematical invariants. Choosing the momenta such that $q \cdot p_i \neq 0$ for some $p_i$ guarantees that we will have different values of $s$ and $t$. Finally we try to choose the polarization vectors and spinors such that the massive state is BRST trivial, leaving us again with a relation among only massless amplitudes.

We will begin with $\{V_f(u_1, p_1), \ldots, V_f(u_4, p_4); V_b(\zeta, q)\}$, with the momenta chosen such that $p_1 \cdot q = 1$, $p_2 \cdot q = p_3 \cdot q = 0$ and $p_4 \cdot q = -1$. This requires computing $\{V_{-1}(\zeta, q), V_{-1/2}(u, p)\}$ for $q \cdot p = -1$. This is given by

$$\frac{1}{\sqrt{2}}(Y \cdot \partial X + R \cdot \psi \gamma)S_{(-3/2)}e^{i(q+p) \cdot X(z)}, \quad (4.1)$$

where we have defined $Y = iq\bar{u}\gamma$ and $R = \frac{1}{4}\bar{u}\zeta - \frac{1}{72}\bar{u}\gamma$. Note that this is the vertex operator for a massive string state. Now we want to choose $\zeta$ and possibly a number of the $u_i$ such that that massive vertex is BRST trivial. One of the two possible sets of conditions that the render the vertex trivial is given by the pair of equations [8]

$$8R + Y(q+p) + R \gamma = 0,$$
$$Y - R(q+p) - 4R(q+p) - \frac{1}{9}Y \gamma = 0. \quad (4.2)$$

For our particular $Y$ and $R$, the first of these simplifies to $2\bar{u}\zeta + q\bar{u}\gamma(q+p) = 0$. The only solution of this equation is $\zeta = q$, and in this case all of the other states generated by $V_b(\zeta, q)$ vanish identically. Thus we fail to generate a nontrivial relation. There is another
set of conditions on $Y$ and $R$ that also yields BRST trivial states, but these require that $Y \propto q + p$, so we can never write our state as one of these.

Let us try again with \( \{ V_b(\zeta_1, p_1), \ldots, V_b(\zeta_4, p_4); V_b(\zeta, q) \} \), $p_1 \cdot q = 1$, $p_2 \cdot q = p_3 \cdot q = 0$ and $p_4 \cdot q = -1$. We begin by computing \( \{ V_{-1}(\zeta, q), V_{-1}(\zeta', p) \} \) for $q \cdot p = -1$:

\[
\left\{ \zeta \cdot \psi \zeta' \cdot \psi i q \partial X - \frac{1}{2} \zeta \cdot \zeta' \left( iq \cdot \partial^2 X + (i q \cdot \partial X)^2 \right) + \zeta \cdot \partial \psi \zeta' \cdot \psi \right\} e^{-2\phi} e^{i(q+p) \cdot X}
\]

If we apply the picture changing operator to this we find

\[
(\alpha_{\mu\nu\rho} \psi^{\mu} \psi^{\nu} \psi^{\rho} + (\sigma_{\mu\nu} + \alpha_{\mu\nu}) \partial X^{\mu} \psi^{\nu} + \sigma_{\mu} \partial \psi^{\mu}) e^{-\phi} e^{i(q+p) \cdot X} \tag{4.3}
\]

where

\[
\alpha_{\mu\nu\rho} = -\frac{i}{6} q_{[\mu} \zeta_{\nu]} \zeta_{\rho]}, \\
\sigma_{\mu\nu} + \alpha_{\mu\nu} = -\zeta \cdot p \zeta'_{\mu} q_{\nu} + \zeta \cdot q \zeta'_{\mu} q_{\nu} - \zeta \cdot \zeta'_{\mu} q_{\nu} - \zeta_{\mu} \zeta_{\nu}, \\
\sigma_{\mu} = i \zeta \cdot \zeta'_{\mu} q_{\nu} - i \zeta' \cdot q \zeta_{\mu}
\tag{4.4}
\]

Here $\sigma_{\mu\nu}$ denotes the symmetric piece and $\alpha_{\mu\nu}$ the antisymmetric. If we choose $\zeta = q$, then this vertex operator vanishes. However, as in the case of four fermions, all of the vertices generated from $V_b(\zeta, q)$ in this case will be BRST trivial. Since we cannot set $\zeta = q$, we must have a nontrivial $\sigma_{\mu}$. From [6] we know that $\sigma_{\mu}$ contributes only to the BRST trivial part of a vertex. Thus we should separate out the contribution from $\sigma_{\mu}$ and check whether there is any remaining physical piece that cannot be removed. First of all $\sigma_{\mu}$ must satisfy $\sigma \cdot (p + q) = 0$. This condition can be written as $\zeta \cdot \zeta' + \zeta' \cdot q \zeta \cdot p = 0$. The BRST trivial piece of $\sigma_{\mu\nu}$ is $\sigma^{\text{trivial}}_{\mu\nu} = (q + p)_{(\mu} \sigma_{\nu)}$. Subtracting this from the above expression for $\sigma_{\mu\nu}$, we find

\[
\sigma^{\text{phys}}_{\mu\nu} = -\frac{1}{2} \zeta \cdot p (\zeta'_{\mu} q_{\nu} + q_{\mu} \zeta'_{\nu}) - \frac{1}{2} (\zeta'_{\mu} \zeta_{\nu} + \zeta_{\mu} \zeta'_{\nu}) \\
+ \frac{1}{2} \zeta \cdot \zeta' (p_{\mu} q_{\nu} + q_{\mu} p_{\nu}) - \frac{1}{2} \zeta' \cdot q (p_{\mu} \zeta_{\nu} + \zeta_{\mu} p_{\nu}). \tag{4.5}
\]

With the help of the the above condition we can show that $\sigma^{\text{phys}}_{\mu\nu}$ satisfies $(q + p)^{\mu} \sigma^{\text{phys}}_{\mu\nu} = 0$ and $\eta^{\mu\nu} \sigma^{\text{phys}}_{\mu\nu} = 0$, so that $\sigma^{\text{phys}}_{\mu\nu}$ is indeed a physical state [6]. The upshot of this is that we cannot arrange things such that the massive vertex is BRST trivial, and we again fail to find a recursion relation.
4.2. A Generalized Bracket

From what we have seen above, it would appear impossible to determine any of the 4-point amplitudes using only the bracket relations. What we want to show now is that we have been too narrow in our thinking. Since the bracket was originally conceived as a special case of the Gerstenhaber bracket, we assumed that it could only be applied to the physical operators of the $N = 1$ string. However there is no reason we cannot consider $\{O_1, O_2\}$, where $O_2$ is a physical operator but $O_1$ is just a dimension one chiral operator constructed from the fields of our theory, as long as the result is again a physical operator. $O_1$ must still have dimension one, since otherwise our contour deformation arguments would not go through. To show that there are indeed examples of such operators, let us compute the bracket of a massless fermion and what can be thought of as an on-shell, bosonic string tachyon.

\[
\{V_T(q), V_{-1/2}(u, p)\} = \oint dw e^{i q \cdot X(w)} c \bar{\upsilon} S_{(-1/2)} e^{ip \cdot X(z)} = V_{-1/2}(u, q + p), \tag{4.6}
\]

where we have assumed that their momenta satisfy $q \cdot p = -1$. If we replace the fermion by a boson, we find the analogous result

\[
\{V_T(q), V_{-1}(\zeta, p)\} = V_{-1}(\zeta, q + p), \tag{4.7}
\]

where again $q \cdot p = -1$.

Since the effect of the tachyon is to shift the momentum of the fermion, this would appear to be just what we need to generate recursion relations. Thus let us consider the relation associated with $\{V_{-1/2}(u_1, p_1), V_{-1/2}(u_2, p_2), V_{-1/2}(u_3, p_3), V_{-1/2}(u_4, p_4); V_T(q)\}$, where $q \cdot p_1 = q \cdot p_2 = q \cdot p_3 = -q \cdot p_4 = 1$. We find

\[
A_{ffff}(s - 1, t - 1) = A_{ffff}(s - 1, t) + A_{ffff}(s, t - 1). \tag{4.8}
\]

This is just the kind of relation found by Moore for the 4-point tachyon amplitude for the bosonic string \[1\]. As before it is easy to show that this relation is satisfied. Note that in all of the above computations, we never used the picture independent notation of section 3, but rather made it clear that we were always working in the canonical ghost pictures. This was done for two reasons. First of all our arguments for picture independence do not go through if both operators in the bracket are not physical. Thus we had to restrict
ourselves to some definite picture. Second of all, had we worked in say the (+1/2)- and 0-pictures, we would have found that the tachyon did not map physical states into physical states. The tachyon can be used in the canonical pictures because there the massless states are the product of a tachyon and something that the tachyon does not see.

As with the tachyon amplitude, equation (4.8) is not enough to solve for the 4-point fermionic amplitude. Thus let us continue along the path followed by Moore, and consider the bracket of a massless fermionic operator and what would be a physical photon of the bosonic string.

\[
\{ V_\gamma(\zeta, q), V_{-1/2}(u, p) \} = \oint d\zeta \cdot \partial X e^{iq \cdot X(w)} \bar{u} S_{-1/2} e^{ip \cdot X(z)}
\]

\[
= \begin{cases} 
- i(\zeta + (\zeta \cdot p)q) \cdot p V_{-1/2}(u, q + p) & q \cdot p = 0 \\
(\zeta + (\zeta \cdot p)q) \cdot \partial X \bar{u} S_{(-1/2)} e^{i(q+p) \cdot X} & q \cdot p = -1 
\end{cases}
\]

(4.9)

Since \(q^2 = 0\), we can define \(\zeta' = \zeta + (\zeta \cdot p)q\) and still preserve the relation \(\zeta' \cdot q = 0\). This greatly simplifies the bracket relations that follow. Now consider the relation generated by \(\{ V_{-1/2}(u_1, p_1), V_{-1/2}(u_2, p_2), V_{-1/2}(u_3, p_3), V_{-1/2}(u_4, p_4); V_\gamma(q) \}\), where \(q \cdot p_1 = q \cdot p_2 = 0\) and \(-q \cdot p_3 = q \cdot p_4 = 1\).

\[
A_{fff}(s, t) = -i\zeta \cdot p_1 A_{fff}(s, t) - i\zeta \cdot p_2 A_{fff}(s, t - 1),
\]

(4.10)

where the subscript \(\zeta\) on the left hand side of the relation denotes the massive state \(\zeta \cdot \partial X \bar{u} S_{(-1/2)} e^{i(q+p) \cdot X}\). If by some convenient choice of \(\zeta\) we could render this state BRST trivial, equation (4.10) would reduce to the recursion relation necessary to determine \(A_{fff}\). But according to Koh et al. there is no value of \(\zeta\) for which this state is BRST trivial. Thus we seem to be in the same straits as before. However since the form of this operator is so much simpler than what we previously encountered for massive states, let us blindly proceed for the moment.

We begin by imposing the BRST invariance conditions for the massive state. These imply that \(\zeta \cdot (p_3 + q) = \bar{u}_3 \xi = \bar{u}_3 \eta = 0\). Recall that we also have the initial conditions \(q \cdot p_3 = -1, q^2 = p_3^2 = \zeta \cdot q = \bar{u}_3 \gamma_3 p_3 = 0\). Now \(A_{fff}(s, t)\) must be of the form

\[
f(s, t)\bar{u}_1 \gamma u_2 \cdot \bar{u}_3 \gamma u_4 + g(s, t)\bar{u}_2 \gamma u_3 \cdot \bar{u}_4 \gamma u_1
\]

(4.11)

since \(\gamma_{\alpha\beta} \cdot \gamma_{\delta\epsilon}\) and \(\gamma_{\beta\delta} \cdot \gamma_{\epsilon\alpha}\) are the only two independent \(SO(10)\) invariants with four spinor indicies. Let us set \(u_2 = u_3\). This restricts our attention to the unknown function \(f(s, t),\)
and also implies that $\bar{u}_3 p_2 = 0$. Using $\bar{u}_3 \zeta = 0$ along with the other conditions on $u_3$, we find that the most general form of $\zeta$ is $\zeta = q + \alpha p_2 + \beta p_3$. Using $\zeta \cdot (q + p_3) = 0$ we can then solve for $\beta$ to find $\zeta = q + \alpha p_2 + (\alpha t - 1)p_3$. Thus we are left with a one-parameter family of massive states. Now let us recall that we want to get rid of the massive state. Since we cannot choose $\zeta$ such that the massive state is BRST trivial, perhaps we can choose $\alpha$ such that the amplitude involving the massive state vanishes.

The piece of the scattering amplitude involving only $X$ and its derivatives is given by

$$\langle e^{ip_1 \cdot X(z_1)} e^{ip_2 \cdot X(z_2)} \zeta \cdot \partial X e^{ip_3 \cdot X(z_3)} e^{ip_4 \cdot X(z_4)} \rangle$$

$$= \left( \frac{\zeta \cdot p_1}{z_3 - z_1} + \frac{\zeta \cdot p_2}{z_3 - z_2} + \frac{\zeta \cdot p_4}{z_3 - z_4} \right) \langle e^{ip_1 \cdot X(z_1)} e^{ip_2 \cdot X(z_2)} e^{ip_3 \cdot X(z_3)} e^{ip_4 \cdot X(z_4)} \rangle$$

If we fix the $SL(2,\mathbb{C})$ symmetry as usual, then this reduces to

$$\left( -\frac{\zeta \cdot p_2}{1 - x} + \frac{\zeta \cdot p_4}{x} \right) x^s(1 - x)^t,$$

where we have also included the contributions of the $(b, c)$ ghost system. If we note that the remaining contribution to the scattering amplitude does not contain any explicit factors of $s$ or $t$, then we can think of the denominators $x$ and $1 - x$ as effectively shifting $s$ and $t$; and requiring this amplitude to vanish implies that $f(s, t)$ must satisfy the additional relation

$$\zeta \cdot p_2 f(s, t - 1) = \zeta \cdot p_4 f(s - 1, t) \quad (4.12)$$

Since $\zeta$ depends on only a single parameter, with this equation we should have enough relations both to find the needed $\alpha$ and to solve for $f(s, t)$.

Assuming that the desired $\alpha$ does indeed exist, equation (4.10) becomes

$$\zeta \cdot p_1 f(s, t) = -\zeta \cdot p_2 f(s, t - 1). \quad (4.13)$$

Combining this with equation (4.12) we find

$$\zeta \cdot p_1 f(s, t) = -\zeta \cdot p_4 f(s - 1, t). \quad (4.14)$$

If we use these equations to write the right hand side of (4.8) in terms of $f(s - 1, t - 1)$, the resulting equation allows us to solve for $\alpha$, with the result $\alpha = 1/(s + t)$. With this value of $\alpha$ equations (4.13) and (4.14) become

$$f(s, t) = \frac{t}{s + t} f(s, t - 1) = \frac{s - 1}{s + t} f(s - 1, t). \quad (4.15)$$
Using these recursion relations we find

$$f(s, t) = c_4 \frac{\Gamma(s) \Gamma(t + 1)}{\Gamma(s + t + 1)}$$

(4.16)

in agreement with the known result. Of course our relations only allow us to determine $f(s, t)$ for integer $s$ and $t$, but as in [1] we can analytically continue our result to the entire complex plane by using two mild assumptions about the analyticity of the amplitudes.

In order to solve for the unknown function $g(s, t)$, we simply repeat the entire process, setting $u_1 = u_2$ rather than $u_2 = u_3$. This yields $g(s, t) = c'_4 \frac{\Gamma(s) \Gamma(t + 1)}{\Gamma(s + t + 1)}$, again in agreement with the known result. Finally we can use the cyclical symmetry of the amplitude to fix $c'_4 = c_4$. Our generalized bracket can also be used to compute $A_{bbff}$, and then the two relations of the form (3.9) can be used to determine the two remaining amplitudes $A_{bbff}$ and $A_{bbbb}$. This completes the solution of the 4-point massless amplitudes using only the bracket.

4.3. Extension to $n > 4$

Let us begin with the $2n$-point massless fermionic amplitude for $n > 3$. The generalizations of equation (4.8) are generated by

$$\{V_{+1/2}(p_1), \ldots, V_{+1/2}(p_{n-3}), V_{-1/2}(p_{n-2}), \ldots, V_{-1/2}(p_{2n}); V_T(q)\}$$

with $q \cdot p_i = 1$ for $i = 1, \ldots, n - 1$ and $q \cdot p_i = -1$ for $i = n, \ldots, 2n$, and its various permutations. Note that we omit the polarization tensors for simplicity’s sake. These relations should serve as the analogues of Moore’s triangle relations [4], which serve to reduce the $2n$-point amplitude from a function of $n(2n - 3)$ variables to one of $2n - 3$ variables. The generalizations of (4.10) are the relations generated by

$$\{V_{+1/2}(p_1), \ldots, V_{+1/2}(p_{n-3}), V_{-1/2}(p_{n-2}), \ldots, V_{-1/2}(p_{2n}); V_\gamma(q)\}$$

with $q \cdot p_i = 1$ for $i = 1, \ldots, n$ and $q \cdot p_i = -1$ for $i = n + 1, \ldots, 2n$, and its permutations, along with the relations constraining the amplitudes involving massive states to vanish. These would then be used to solve for the dependence on the remaining variables. The six point amplitude has in fact been worked out in [1] using a rather involved technique developed in [8] for computing correlators. It would be an interesting exercise to try to use the above bracket relations to reproduce their results.

Similar relations to these can also be generated for amplitudes involving either one or two bosonic operators. However, when we try to use the tachyon to generate relations for amplitudes with three or more bosons, we find that it is impossible to do so. Now as we saw in the previous subsection, one of the problems with using the generalized bracket is
that it is not picture independent. For the particular operators we chose to use, \( V_T \) and \( V_\gamma \), we had to restrict ourselves to the canonical picture. When we couple this restriction with the necessity of having to choose representatives of operators whose ghost charges sum to \(-2\), we find that the two requirements are sometimes incompatible. For example in the case of three or more bosons, no matter how we choose the ghost pictures and the momenta, we end up having to compute the bracket of the tachyon with a state in a picture other than the canonical picture, which as we mentioned does not give us back a physical state. When we have four or more bosons, we can no longer even use the photon. One possible way of avoiding these restrictions would be to use higher mass states from the bosonic string to generate relations. Another possibility would be to allow the tachyon and photon to generate higher mass states of the \( N = 1 \) string. While the former approach seems quite tenable, the latter is hampered by the fact that the BRST analysis for the \( N = 1 \) string has not yet been perfomed at even the second massive level.

5. Lifting the Restriction \( n \leq 10 \)

In his original work [1] Moore found that he could only apply his bracket relations to scattering amplitudes involving at most twenty-six string states. The origin of this restriction was explained in section 2. In [2] he showed how this restriction could effectively be lifted. This was done as follows. The general form of the restriction on \( n \) is that it be no greater than the number of uncompactified target space dimensions. Thus the idea that immediately presents itself is to embed our theory in another with a larger target space. The problem with this is that doing so takes us off criticality. However we can restore the central charge to zero by tensoring our enlarged theory with a number of ghost systems.

To be more exact, let us denote by \( C(M) \) the conformal field theory with target space \( M \), e.g. \( C(\mathbb{R}^{1,25}) \) is the theory that underlies the ordinary critical bosonic string. Moore embedded the theory based on \( C(\mathbb{R}^{1,25}) \) into that based on \( C(\mathbb{R}^{1+E,25+E}) \bigotimes_{i=1}^{E} \langle \xi_i, \eta_i \rangle \cap \ker(\oint \eta_i) \), where \( \langle \xi_i, \eta_i \rangle \) is a spin \((0,1)\) fermionic ghost system. This allowed him to compute amplitudes involving up to \( 26 + 2E \) string states, and since \( E \) is arbitrary, this effectively removed the restriction on \( n \). Here we want to show that the same procedure also works for the \( N = 1 \) string if we extend the ghost fields to \( N = 1 \) superfields. We denote by \( SC(M) \) the superconformal field theory (SCFT) with target space \( M \). Since the \( N = 1 \) string is a SCFT with target space \( \mathbb{R}^{1,9} \), in order to apply Moore’s construction we need to extend not only the bosonic target space fields but also their fermionic partners.
To cancel the contribution to the central charge from these extra fields, we will tensor our theory with an additional $E$ copies of a spin $(1/2,1/2)$ bosonic ghost system $<\hat{\xi},\hat{\eta}>$. Note that this system does not have a $U(1)$ anomaly, so there is no need to restrict ourselves to a subspace of this theory. We can combine these two ghost systems into a single superghost system described by $\Xi = \xi + \theta \hat{\xi}$ and $H = \eta + \theta \eta$, and thus our full theory may be written as $SC(\mathbb{R}^{1+1},9+9) \otimes_{i=1}^{E} [<\Xi_i,H_i> \cap \ker(\oint \int d\theta H_i)]$.

Once we have solved for the amplitudes in this extended theory, how do we pull out the amplitudes for our original theory? We begin by restricting ourselves to amplitudes involving states built only from the ordinary (supersymmetric) matter fields. Since the bracket is closed with respect to such states, we need not worry about states involving ghost fields propagating in any of the intermediate channels; after all the coefficients of the bracket relations are nothing more than the three-point functions. Having computed these amplitudes, which live in a $10+2E$ dimensional target space, we simply continue our results to momenta whose last $2E$ components vanish.

6. Discussion

In this paper we have succeeded in extending most of Moore’s original analysis to the $N=1$ string, thus doing the second of his things that “we should do.” This required a slight extension of the notion of the bracket to include the bracket between a physical operator and an ordinary dimension one chiral field. Another way of thinking about this extension is to say that the $N=1$ string admits an algebra of external operators, or that the states of the $N=1$ string (in the canonical picture) form a module over this external algebra. The most glaring shortcoming of the present work is a lack of a general proof that the massless amplitudes are computable entirely in terms of the bracket. However, being that the 4-point amplitude is computable, we have every confidence that the general amplitude should be so as well.

Having shown that the bracket can be extended to the $N=1$ string, we can now begin to ask questions about more interesting theories such as the type II superstring and the heterotic string. Furthermore the bracket should be applicable not only to the flat backgrounds studied up until now, but also to more general conformal field theories as long as they contain some noncompact sector. In particular we could consider the above mentioned theories, but compactified on some non-trivial, internal conformal field theory;
or we could even consider non-critical theories. In any of these examples it would be interesting to see to what extent the bracket determines the S-matrix.

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