FPT Algorithms for a Special Block-structured Integer Program with Applications in Scheduling

Hua Chen∗  Lin Chen †  Guochuan Zhang∗

November 15, 2021

Abstract

We consider integer programs (IPs) whose constraint matrix has a special block structure. More precisely, we consider IP: \( \min \{ f(x) : H_{com}x = b, \, l \leq x \leq u, \, x \in \mathbb{Z}^t B + nA \} \), in which the objective function \( f \) is separable convex and the constraint matrix \( H_{com} \) is composed of small submatrices \( A_i, B, C, D_i \) such that the first row of \( H_{com} \) is \( (C, D_1, D_2, \ldots, D_n) \), the first column of \( H_{com} \) is \( (C, B, B, \ldots, B)^\top \), the main diagonal of \( H_{com} \) is \( (C, A_1, A_2, \ldots, A_n) \), and the rest entries are 0. Furthermore, the rank of submatrix \( B \) is 1.

We study fixed parameter tractable (FPT) algorithms by taking as parameters the number of rows and columns of small submatrices, together with the largest absolute value over their entries.

We call the IP studied (almost) combinatorial 4-block \( n \)-fold IP. It generalizes the generalized \( n \)-fold IP and is meanwhile a special case of the generalized 4-block \( n \)-fold IP. In the literature, existing FPT algorithms for block-structured IP rely on bounding the \( \ell_1 \)- or \( \ell_\infty \)-norm of elements of the Graver basis. The existence of FPT algorithms for 4-block \( n \)-fold IP is a major open problem and Chen et al. [ESA 2020] showed that the \( \ell_\infty \)-norm of the Graver basis elements of 4-block \( n \)-fold IP is \( \Omega(n) \). This motivates us to study special cases of the generalized 4-block \( n \)-fold IP to find structural insights.

We show that, the \( \ell_\infty \)-norm of the Graver basis elements of combinatorial 4-block \( n \)-fold IP is also \( \Omega(n) \). However, there exists some FPT-value \( \lambda \) such that for any nonzero element \( g \in \{ x : H_{com}x = 0 \} \), \( \lambda g \) can always be decomposed into Graver basis elements in the same orthant whose \( \ell_\infty \)-norm is FPT-bounded (while \( g \) itself might not admit such a decomposition). This seems to exhibit an “intermediate” phenomenon. Based on this, we are able to bound the \( \ell_\infty \)-norm of Graver basis elements for combinatorial 4-block \( n \)-fold IP by \( O_{FPT}(n) \) and develop an \( O_{FPT}(n^4 \hat{L}^2) \)-time algorithm (here the \( O_{FPT} \) hides a multiplicative FPT-term, and \( \hat{L} \) denotes the logarithm of the largest number occurring in the input).

As applications, we show that combinatorial 4-block \( n \)-fold IP can be used to model important generalizations of the classical scheduling problems, including scheduling with rejection and bicriteria scheduling, which implies that our FPT algorithm establishes a general framework to settle the classical scheduling problems.

Keywords: 4-block \( n \)-fold IP, Fixed parameter tractable, Scheduling, Integer programming

*Zhejiang University, Hangzhou, China. chenhua.by@zju.edu.cn; zgc@zju.edu.cn
†Texas Tech University, Lubbock, TX, US. chenlin198662@gmail.com
1 Introduction

Integer programs (IPs) whose constraint matrix has a special block structure have received a considerable attention in recent years. As an important subclass of the general IP, it finds applications in a variety of optimization problems including scheduling [7, 24, 28], routing [7], stochastic integer multi-commodity flows [18], stochastic programming with second-order dominance constraints [15], etc.

First, we consider a block-structured IP as follows:

\[(IP)_{n,b,l,u,f} : \min \{ f(x) : H_{\text{com}} x = b, l \leq x \leq u, x \in \mathbb{Z}^{l n + nt}\}, \tag{1}\]

where \( f : \mathbb{R}^{l n + nt} \rightarrow \mathbb{R} \) is a separable convex function, and \( H_{\text{com}} \) consists of small submatrices \( A_i, B, C \) and \( D_i \) as follows:

\[H_{\text{com}} := \begin{pmatrix}
    C & D_1 & D_2 & \cdots & D_n \\
    B & A_1 & 0 & \cdots & 0 \\
    B & 0 & A_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots \\
    B & 0 & 0 & \cdots & A_n
\end{pmatrix}, \quad H := \begin{pmatrix}
    C & D_1 & D_2 & \cdots & D_n \\
    B_1 & A_1 & 0 & \cdots & 0 \\
    B_2 & 0 & A_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots \\
    B_n & 0 & 0 & \cdots & A_n
\end{pmatrix}. \tag{2}\]

Here, \( A_i \)'s (or \( B \) or \( C \) or \( D_i \)'s, resp.) are \( s_A \times t_A \) (or \( s_B \times t_B \) or \( s_C \times t_C \) or \( s_D \times t_D \), resp.) matrices, and furthermore, the rank of matrix \( B \) is 1.

Note that when \( C = B = 0 \), the above problem reduces to the generalized \( n \)-fold IP. In the meantime, IP [1] is a special case of the generalized \( 4 \)-fold \( n \)-fold IP [19] where the constraint matrix \( H \) consists of submatrices \( A_i, B_i, C \) and \( D_i \) as Eq [2]. It is worth mentioning that the overall structure of \( H \) implies that \( s_C = s_D, s_A = s_B, t_B = t_C \) and \( t_A = t_D \).

Let \( \Delta \) be the largest absolute value among all the entries of \( A_i, B, C \) and \( D_i \). The goal of this paper is to study FPT algorithms for combinatorial \( 4 \)-block \( n \)-fold IP by taking \( \Delta, s_A, s_B, s_C, s_D \) and \( t_A, t_B, t_C, t_D \) as parameters, i.e., we aim for an algorithm that runs polynomially in \( n \).

When \( s_B = s_A = 1 \), we call IP [1] combinatorial \( 4 \)-block \( n \)-fold IP (and \( H_{\text{com}} \) combinatorial \( 4 \)-block \( n \)-fold matrix) as it generalizes the combinatorial \( n \)-fold IP studied in [28] (combinatorial \( n \)-fold IP can be viewed as a special case where \( C = B = 0 \) and all the entries of \( A_i \)'s are 1).

To be consistently, when the rank of matrix \( B \) is 1, IP [1] is called almost combinatorial \( 4 \)-block \( n \)-fold IP. To tackle this problem, first we are focused on combinatorial \( 4 \)-block \( n \)-fold IP while \( s_B = s_A = 1 \). Then we show that all results achieved remain true for almost combinatorial \( 4 \)-block \( n \)-fold IP.

There are two facts that make (almost) combinatorial \( 4 \)-block \( n \)-fold IP an interesting subclass of the general block-structured IP.

From an application point of view, combinatorial \( 4 \)-block \( n \)-fold IP generalizes combinatorial \( n \)-fold IP and thus offers a stronger tool for optimization problems. In particular, Knop and Koutecký [27] modeled parallel machine scheduling problems \( R || C_{\text{max}} \) and \( R || \sum t \ell \ kC_\ell \) as \( n \)-fold IPs and developed FPT algorithms (parameters include the largest job processing time, different types of machines and different types of jobs). Utilizing combinatorial \( 4 \)-block \( n \)-fold IP, we are able to model a broader class of scheduling problems and derive similar FPT algorithms. Specifically, we consider two generalizations of the classical scheduling model. One is the bicriteria scheduling problem \( R || (\theta C_{\text{max}} + \sum t \ell \ kC_\ell) \), which considers the combination of two common scheduling objectives. The other is the scheduling problem with job rejection \( R || C_{\text{max}} + E \), where jobs can be rejected at a certain cost and the goal is to minimize the scheduling cost plus the total rejection costs.
cost. The reader may refer to Section 5 for the precise definitions of the two problems and the corresponding FPT algorithms.

From a theoretical point of view, combinatorial 4-block n-fold IP exhibits an interesting “intermediate” phenomenon in its Graver basis (see Section 2 for the definition). As we will provide more details later in the related work, FPT algorithms have been developed for several special cases of the generalized 4-block n-fold IP (see, e.g., [4, 9, 20, 21, 24, 29]). All of these algorithms rely on the fact that the $\ell_\infty$-norm (or even $\ell_1$-norm) of the Graver basis elements for these special cases are bounded by some FPT-value. Unfortunately, Chen et al. [6] showed very recently that the $\ell_\infty$-norm of Graver basis elements for 4-block n-fold IP is $\Omega(n)$. It thus becomes a challenging problem that without the boundedness of $\ell_\infty$-norm, what other properties can we expect from the Graver basis elements which may lead to an FPT algorithm? In this paper, we observe an interesting phenomenon: On the one hand, the $\ell_\infty$-norm of the Graver basis elements for combinatorial 4-block n-fold IP is still $\Omega(n)$ even if $s_D = 1$ (see Theorem 3). On the other hand, Graver basis elements whose $\ell_\infty$-norm is bounded by some FPT-value seem to be strong enough for the purpose of decomposition. More precisely, we have the following Theorem 1 which states that for some fixed $\lambda$ and any $g \in \ker_\mathbb{Z}(H_{com})$, $\lambda g$ can always be decomposed into the summation of Graver basis elements with $\ell_\infty$-norm bounded by some FPT-value. Interestingly, this $\lambda$ only depends on $t_B$ and $\Delta$.

Theorem 1. Let $H_{com}$ be a combinatorial 4-block n-fold matrix. Then there exists a positive integer $\lambda \leq 2^{2O(t_B \log(t_B \Delta))}$ (which is only dependent on $t_B$ and $\Delta$) such that for any $g \in \ker_\mathbb{Z}(H_{com})$, we have $\lambda g = g_1 + g_2 + \cdots + g_p$ for some $p \in \mathbb{Z}_{>0}$ and $g_j \in \ker_\mathbb{Z}(H_{com})$, and furthermore, $g_j \subseteq \lambda g$ and $\|g_j\|_\infty = 2^{2O(t_B \log \Delta + s_D t_D \log \Delta)} 2^{O(t_B \log \Delta)}$.

Here the upper bounds for $\lambda$ and $\|g_j\|_\infty$’s are triply exponential in the parameters.

Utilizing Theorem 1, we are able to show that $\|g\|_\infty = O_{FPT}(n)$ for any Graver basis element $g$, and develop an algorithm of running time $O_{FPT}(n^4 L^2)$ for combinatorial 4-block n-fold IP, where $O_{FPT}$ hides a multiplicative factor that only depends on $\Delta, s_A, s_B, s_C, s_D, t_A, t_B, t_C, t_D$, and $L$ denotes the logarithm of the largest number occurring in the input. The special feature implied by Theorem 1 as well as our techniques may be of separate interest for a broader class of IPs.

Remark. Theorem 1 and our FPT algorithm for combinatorial 4-block n-fold IP remain true for almost combinatorial 4-block n-fold IP. Such a generalization allows submatrices $A_i$’s to contain multiple rows subject to that these rows are “local constraints”. It is, however, not clear whether Theorem 1 still holds if we allow the $n$ submatrices $B \in \mathbb{Z}^{1 \times t_B}$ to be different.

Related work. The existence of FPT algorithms for the generalized 4-block n-fold IP (where the constraint matrix is given by $H$ in Eq. (2)) remains as one major open problem in the area of integer programming. However, important progress has been achieved in recent years on its special cases. In particular, extensive research has been carried out on three fundamental subclasses – 4-block n-fold IP, the generalized n-fold IP and the generalized two-stage stochastic IP.

When $A_i = A, B_i = B$ and $D_i = D$, the generalized 4-block n-fold reduces to 4-block n-fold IP, which has been studied before mainly by Hemmecke et al. [19] and Chen et al. [6]. In particular, Chen et al. [6] showed that the infinity norm of Graver basis elements for such 4-block n-fold IP is bounded by $\min\{n^{O(t_A)}, n^{O(s_D)}\}$, and developed a $\min\{n^{O(t_A t_B)}, n^{O(s_D)}\}$-time algorithm. Consequently, their results do not yield FPT algorithms for combinatorial 4-block n-fold IP. Very recently Chen et al. [5] studied 4-block n-fold IP when $\Delta$ is not part of the parameters, and proved
that when \( t_A = s_A + 1 \) and \( \text{rank}(A) = s_A \), 4-block \( n \)-fold IP can be solved in \( (t_A + t_B)^{O(t_A+t_B)} \cdot n^{O(t_A^2)} \cdot \text{poly}(\log \Delta) \) time.

When \( C = B_i = 0 \) for all \( i \), the generalized 4-block \( n \)-fold IP reduces to the generalized \( n \)-fold IP, and we denote the constraint matrix as \( H^{n \text{-fold}} \). This IP was initialized by De Loera et al. [11]. In 2013, Hemmecke et al. [20] developed the first FPT algorithm. Later on, a series of researches have been carried out to further improve its running time [3, 9, 12, 13, 24, 25]. Most recently, Csolvjecsek et al. [9] presented an algorithm of running time \( 2^{O(s_A^2 s_D)} (s_D s_A \Delta)^{O(s_A^2 s_D^2)} (n t_A)^{1+o(1)} \) for the generalized \( n \)-fold IP.

When \( C = D_i = 0 \) for all \( i \), the generalized 4-block \( n \)-fold IP reduces to the generalized two-stage stochastic IP, and we denote the constraint matrix as \( H^{\text{two-stage}} \). This IP was first studied by Hemmecke and Schultz [21] and Aschenbrenner and Hemmecke [4]. Their result was improved by in a series of subsequent papers [13, 23, 26, 29]. The current best-known algorithm for the generalized two-stage stochastic IP runs doubly exponential in the parameters \( \Delta, s_A, t_B \) by Klein [26].

2 Notations and Preliminaries

**Notations.** We write column vectors in boldface, e.g., \( \mathbf{x}, \mathbf{y} \), and their entries in normal font, e.g., \( x_i, y_i \). If \( \mathbf{x} \in \mathbb{Z}^d \) and \( \mathbf{y} \in \mathbb{Z}^d \), then we abuse the notation by using \( (\mathbf{x}, \mathbf{y}) \) to denote a column vector in \( \mathbb{Z}^{d_1+d_2} \). Recall that a solution \( \mathbf{x} \) for 4-block \( n \)-fold IP is a \( (t_B + nt_A) \)-dimensional column vector, and we write it into \( n+1 \) bricks, such that \( \mathbf{x} = (\mathbf{x}^0, \mathbf{x}^1, \cdots, \mathbf{x}^n) \) where \( \mathbf{x}^0 \in \mathbb{Z}^{t_B} \) and each \( \mathbf{x}^i \in \mathbb{Z}^d, 1 \leq i \leq n \). We call \( \mathbf{x}^i \) the \( i \)-th brick for \( 0 \leq i \leq n \). For a vector or a matrix, we write \( \| \cdot \|_\infty \) to denote the maximal absolute value of its elements. For two vectors \( \mathbf{x}, \mathbf{y} \) of the same dimension, \( \mathbf{x} \cdot \mathbf{y} \) denotes their inner product. We use \( [i] \) to represent the set of integers \( \{1, 2, \cdots, i\} \), and \( [i:] \) for \( \{i, i+1, \cdots, j\} \) where \( i < j \).

Two vectors \( \mathbf{x} \) and \( \mathbf{y} \) are called *sign-compatible* if \( x_i \cdot y_i \geq 0 \) holds for every pair of coordinates \( (x_i, y_i) \). Recall the matrix \( H_{\text{com}} \) in Eq. (2). We denote by \( H_{\text{com}}^{n \text{-fold}} \) the submatrix obtained from \( H_{\text{com}} \) by removing the first column \((C, B, B, \cdots, B)^\top\), and \( H_{\text{com}}^{\text{two-stage}} \) the submatrix obtained by removing the first row \((C, D_1, \cdots, D_n)\).

Throughout this paper, we use \( O_{FPT}(1) \) to represent a parameter that depends only on \( \Delta, s_A, s_B, s_C, s_D, t_A, t_B, t_C, t_D \) where \( \Delta \) is the maximal absolute value among all the entries of \( A_i, B_i, C, D_i \). In other words, \( O_{FPT}(1) \) is only dependent on the small matrices \( A_i, B_i, C, D_i \) and is independent of \( n \). For any computable function \( g(x) \), we write \( O_{FPT}(g) \) to represent a computable function \( g'(x) \) such that \( |g'(x)| \leq O_{FPT}(1) \cdot |g(x)| \).

**Graver basis.** We define \( \sqsubseteq \) to be the *conformal order* in \( \mathbb{R}^d \) such that \( \mathbf{x} \sqsubseteq \mathbf{y} \) if \( \mathbf{x} \) and \( \mathbf{y} \) are sign-compatible and \( |x_i| \leq |y_i| \) for each \( i = 1, \cdots, d \). Given any subset \( X \subseteq \mathbb{R}^d \), we say \( \mathbf{x} \) is a \( \sqsubseteq \)-minimal element of \( X \) if \( \mathbf{x} \in X \) and there does not exist \( \mathbf{y} \in X, \mathbf{y} \neq \mathbf{x} \) such that \( \mathbf{y} \sqsubseteq \mathbf{x} \). It is known that every subset of \( \mathbb{Z}^d \) has finitely many \( \sqsubseteq \)-minimal elements.

Then the *Graver basis* [16] of an integer matrix \( A \) is defined the finite set \( G(A) \), which consists of all \( \sqsubseteq \)-minimal elements of \( \ker_{\mathbb{Z}}(A) \setminus \{0\} \), where \( \ker_{\mathbb{Z}}(A) = \{ \mathbf{x} \in \mathbb{Z}^N \mid A \mathbf{x} = 0 \} \).

**Graver-best augmentation.** Consider a general IP

\[
\min \{ f(\mathbf{x}) : A \mathbf{x} = \mathbf{b}, 1 \leq \mathbf{x} \leq \mathbf{u}, \mathbf{x} \in \mathbb{Z}^d \},
\]  

(3)

We call \( \mathbf{x} \) a feasible solution if \( A \mathbf{x} = \mathbf{b} \) and \( 1 \leq \mathbf{x} \leq \mathbf{u} \). Given a feasible solution \( \mathbf{x} \) to IP (3), we call \( \mathbf{g} \) a *feasible step* if \( \mathbf{x} + \mathbf{g} \) is feasible for the IP. Furthermore, if \( f(\mathbf{x} + \mathbf{g}) < f(\mathbf{x}) \), a feasible step \( \mathbf{g} \) is called *augmenting*. An augmenting step \( \mathbf{g} \) and a step length \( \rho \in \mathbb{Z} \) form an \( \mathbf{x} \)-feasible step pair.
with respect to a feasible solution $x$ if $1 \leq x + \rho g \leq u$. An augmenting step $h$ with $\rho_0 \in \mathbb{Z}$ is a Graver-best step for $x$ if $f(x + \rho_0 h) \leq f(x + \rho g)$ for all $x$-feasible step pairs $(g, \rho) \in G(A) \times \mathbb{Z}$.

The Graver-best augmentation procedure for an IP and a given feasible solution $x_0$ works as follows:

1. If there is no Graver-best step for $x_0$, return it as optimal.
2. If a Graver-best step $(h, \rho)$ for $x_0$ exists, set $x_0 := x_0 + \rho h$ and go to 1.

The following Lemma 1 tells us that it is sufficient to focus all our attention on finding Graver-best steps.

**Lemma 1 ([10], implicit in Theorem 3.4.1).** Given a feasible solution $x_0$, and a separable convex function $f$, the Graver-best augmentation procedure finds the optimum in at most $(2n - 2)\log(f)$ steps, where $f = f(x_0) - f(x^*)$ for some integer optimum $x^*$.

**Lemma 2 ([31], Lemma 3.2).** Every integer vector $x \neq 0$ with $Ax = 0$ is a sign-compatible sum $x = \sum_i g_i$ of Graver basis elements $g_i \in G(A)$, with some elements possibly appearing with repetitions.

**Theorem 2 ([20], Theorem 2).** Let $g$ be a Graver element of a generalized two-stage stochastic IP with constraint matrix $H_{\text{two-stage}}$. Then $\|g\|_{\infty} \leq g_\infty(H_{\text{two-stage}})$, where $g_\infty(H_{\text{two-stage}})$ only depends on $s_B, t_B, \Delta$ and $g_\infty(H_{\text{two-stage}}) \leq 2^{O(s_B t_B^2 \log(s_B \Delta))}$.

By Lemma 2 and Theorem 2 we conclude that for the constraint matrix of a generalized two-stage stochastic IP, any one basis $x$ satisfying $H_{\text{two-stage}}x = 0$ can be decomposed into a sign-compatible sum of Graver bases; i.e., $x = \sum_i g_i$ where $g_i \in G(H_{\text{two-stage}})$, and $\|g_i\|_{\infty} \leq 2^{O(s_B t_B^2 \log(s_B \Delta))}$.

**The Steinitz Lemma ([17, 32]).** Let an arbitrary norm be given in $\mathbb{R}^d$, and let $x_1, \ldots, x_m \in \mathbb{R}^d$ with $\|x_i\| \leq \zeta$ for $i = 1, \ldots, m$. If $\sum_{i=1}^m x_i = x$, then there is a permutation $\pi$ such that for each $\ell \in \{1, \ldots, m\}$ the norm of the partial sum $\sum_{i=1}^\ell x_{\pi(i)} - \frac{\ell - d}{m} x \leq d\zeta$.

The above Steinitz Lemma is commonly used to bound the $\ell_\infty$-norm of Graver basis elements, and is also used in our paper. In particular, Lemma 3 follows from the Steinitz Lemma.

**Lemma 3 ([6]).** Let $x_1, x_2, \ldots, x_m$ be a sequence of vectors in $\mathbb{Z}^d$ such that $x = \sum_{i=1}^m x_i$, and $\|x_i\|_{\infty} \leq \zeta$. Then the set $[m]$ can be partitioned into $m'$ subsets $T_1, T_2, \ldots, T_m'$ satisfying that: $\bigcup_{j=1}^{m'} T_j = [m]$, and for every $1 \leq j \leq m'$ it holds that $\sum_{i \in T_j} x_i \subseteq x$, $|T_j| \leq (c\zeta)^d$ for some constant $c$. In particular, if $d = 1$, then $|T_j| \leq 6\zeta + 2$ for all $j$.

When applying combinatorial 4-block $n$-fold IP to solve optimization problems, we may establish IPs with the constraint being $H_{\text{com}}x \leq b$. The following observation ensures that such a constraint can be transformed to a standard form Eq (1) without destroying the structure of the constraint matrix.

**Observation 1.** Considering the IP $\min \{f(x) : H_{\text{com}}x \leq b, 1 \leq x \leq u, x \in \mathbb{Z}^{t_B + n_A}\}$, where $H_{\text{com}}$ is defined as in [2] and $B \in \mathbb{Z}^{1 \times t_B}$, we can make the constraint $H_{\text{com}}x \leq b$ tight $(H'_{\text{com}}x = b)$ by adding $n(s_B + 1)$ slack variables in total and keep the new constraint matrix $H'_{\text{com}}$ being in the form of [2]. Specifically, we write the constraints in [2] as follows:

$$C^0 + \sum_{i=1}^n D_i x^i \leq b^0$$

(4)

$$B^0 + A_i x^i \leq b^i, \quad \forall 1 \leq i \leq n$$

(5)
For Constraint (4), notice that there are in fact \( s_D \) inequalities, and for each inequality, we add \( n \) slack variables. Similarly, since \( B \) is a vector, Constraint (5) includes \( n \) inequalities, and for each inequality, we add 1 slack variable. Thus we have the new constraint matrix \( H'_{\text{com}} \) in (6).

\[
H_{\text{com}} = \begin{pmatrix}
C & D_1 & D_2 & \cdots & D_n \\
B & A_1 & 0 & 0 & \\
B & 0 & A_2 & 0 & \\
\vdots & \ddots & & \\
B & 0 & 0 & & A_n
\end{pmatrix}, \quad H'_{\text{com}} = \begin{pmatrix}
C & D'_1 & D'_2 & \cdots & D'_n \\
B & A'_1 & 0 & 0 & \\
B & 0 & A'_2 & 0 & \\
\vdots & \ddots & & \\
B & 0 & 0 & & A'_n
\end{pmatrix},
\]

where \( D'_i \) has dimension \( s_D \times (t_A + 1 + s_D) \) and \( A'_i \) has dimension \( 1 \times (t_A + 1 + s_D) \), and

\[
D'_i = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & & \ddots & \\
0 & 0 & 0 & & 1
\end{pmatrix}, \quad A'_i = (A_i, 1, 0, \ldots, 0).
\]

### 3 Structural Results for Combinatorial 4-block \( n \)-fold

The goal of this section is to prove Theorem 1, based on which in Section 4 we will be able to bound the \( \ell_\infty \)-norm of the Graver basis elements of combinatorial 4-block \( n \)-fold IP, and design an FPT algorithm using the iterative augmentation framework developed in a series of prior research works (see Graver-best augmentation in Section 2).

Towards the proof of Theorem 1, we first give an example.

**Example.** Let \( H_0 \) be a 4-block \( n \)-fold matrix where \( C = (-1, -1, -1) \), \( D = (5, 3) \), \( B = (0, -1, 1) \) and \( A = (3, 4) \). Let \( g = (g^0, g^1, \ldots, g^n) \) such that \( g^0 = (1, n - 1, n) \) and \( g^i = (1, -1) \) for all \( i \) (see the left side of Eq (7) where \( g \) is written explicitly). It is not difficult to verify that \( H_0 g = 0 \). Moreover, we are able to prove that \( g \) is a Graver basis element, thus proving Theorem 3 (see Appendix A.1 for the omitted proof).

**Theorem 3.** There exists a 4-block \( n \)-fold IP where \( s_B = s_D = 1 \) such that \( \|g\|_\infty = \Omega(n) \) for some Graver basis element \( g \).

Despite that the \( g \) constructed in the proof cannot be decomposed into “thin” kernel elements in the same orthant, we observe that, interestingly, by multiplying \( g \) with some small value (bounded by \( O_{\text{FPT}}(1) \)), such a decomposition follows. More precisely, we have the following.
Notice that there are in total \( n + 1 \) vectors on the right side of Eq \( \text{[7]} \) and let them be \( \mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_{n+1} \): Among \( \mathbf{g}_i \)'s, the first vector \( \mathbf{g}_1 = (0, 0, 11, 3, -5, \cdots, 3, -5, 7, -8) \) consists of \( \mathbf{g}_1^0 = (0, 0, 11), \) \( n - 1 \) copies of \( \mathbf{g}_1^1 = (3, -5) \) and one copy of \( \mathbf{g}_1^n = (7, -8) \). Each of \( \mathbf{g}_2 \) to \( \mathbf{g}_n \) consists of \( (0, 11, 11), \) one copy of \( (8, -6) \) and \( 0 \)'s. And the last vector \( \mathbf{g}_{n+1} \) consists of \( (11, 0, 0), \) one copy of \( (4, -3) \) and \( 0 \)'s. It is easy to verify that \( \mathbf{g}_j \in 11\mathbf{g} \) and \( H_0\mathbf{g}_j = 0 \).

**A high level overview on the proof of Theorem 1.** Recall that \( H_{\text{com}} \) is a combination of two submatrices, the first row \((C, D_1, \cdots, D_n)\) and a two-stage stochastic matrix \( H_{\text{com}}^{\text{two-stage}} \). Therefore, any \( \mathbf{g} \in \ker_{\Sigma}(H_{\text{com}}) \) also satisfies that \( \mathbf{g} \in \ker_{\Sigma}(H_{\text{com}}^{\text{two-stage}}) \), and by Theorem 2 for any \( \lambda \in \mathbb{Z}_{\geq 0} \) we have \( \lambda \mathbf{g} = \sum_{j=1}^{L} \xi_j \), where \( \|\xi_j\|_{\infty} = O_{\text{FPT}}(1) \), \( \xi_j \subseteq \lambda \mathbf{g} \) and \( \xi_j \in \ker_{\Sigma}(H_{\text{com}}^{\text{two-stage}}) \). Note that \((C, D_1, \cdots, D_n)\xi_j \) is not necessarily \( \mathbf{0} \) and hence \( \xi_j \)'s may not belong to \( \ker_{\Sigma}(H_{\text{com}}) \).

To show \( \lambda \mathbf{g} \) can be decomposed into sign-compatible elements of \( \ker_{\Sigma}(H_{\text{com}}) \) with bounded \( \ell_{\infty} \)-norm, it suffices to show that if \( L \) (and consequently \( \|\lambda \mathbf{g}\|_{\infty} \)) is too huge, then there exists some \( \eta \in \ker_{\Sigma}(H_{\text{com}}) \) such that \( \eta \subseteq \lambda \mathbf{g} \) and \( \|\eta\|_{\infty} = O_{\text{FPT}}(1) \). Afterwards, we proceed to decompose \( \lambda \mathbf{g} - \eta \). A natural idea to construct such an \( \eta \) is to select a subset \( S \) with an \( O_{\text{FPT}}(1) \) number of \( \xi_j \)'s such that \((C, D_1, D_2, \cdots, D_n)\sum_{j \in S} \xi_j = \mathbf{0} \). Unfortunately, the cardinality of \( S \) needs to be \( \Omega(n) \) to make \((C, D_1, D_2, \cdots, D_n)\sum_{j \in S} \xi_j = \mathbf{0} \) observed by Chen et al. [6]. To bypass this obstacle, \( \eta \) needs to be constructed in a way more “flexible” than a direct summation of \( \xi_j \)'s. Thus, we try to enable a “cross-position” construction, that is, we will allow each brick \( \eta^i \) to consist of bricks from different positions of \( \xi_j \)'s, e.g., \( \eta^i = \xi_{j_1}^{i_1} + \xi_{j_2}^{i_2} \) where \( i_1, i_2 \) may be different from \( i \). This will cause a critical problem. Suppose \( \eta^i = \xi_{j_1}^{i_1} + \xi_{j_2}^{i_2} \) and \( \eta^{i'} = \xi_{j_3}^{i_3} + \xi_{j_4}^{i_4} \), then how should we set the value of \( \eta^0 \) to ensure that \( B\eta^0 + A_i \eta^i = \mathbf{0} \)? We observe that, if the decomposition \( \lambda \mathbf{g} = \sum_{j=1}^{L} \xi_j \) satisfies that \( B\xi_j^0 \) equals the same value for all \( j \) (called the uniform condition), and additionally if it holds that \( A_i = A_{i_1} = A_{i_2} \) and \( A_i = A_{i_3} = A_{i_4} \), then by setting \( \eta^0 = \xi_{j_1}^0 + \xi_{j_2}^0 \) (or equivalently, \( \eta^0 = \xi_{j_3}^0 + \xi_{j_4}^0 \)) we have \( B\eta^0 + A_i \eta^i = B\xi_{j_1}^0 + A_{i_1} \xi_{j_1}^{i_1} + B\xi_{j_2}^0 + A_{i_2} \xi_{j_2}^{i_2} = \mathbf{0} \), and similarly \( B\eta^0 + A_{i'} \eta^{i'} = \mathbf{0} \). That means, “cross-position” construction is possible if the uniform condition is met. Unfortunately, the uniform condition is not necessarily true. Only for combinatorial 4-block \( n \)-fold IP and some suitably chosen \( \lambda = O_{\text{FPT}}(1) \) we can guarantee the uniform condition (nevertheless, our proof remains true for almost combinatorial 4-block \( n \)-fold IP, as we discuss at the end of Section 3.2).

With the uniform condition, the construction of \( \eta \) still has two major challenges. One is that \( \eta \) must satisfy \((C, D_1, \cdots, D_n)\eta = \mathbf{0} \). We will generalize the Steinitz Lemma to a “colorful” variant to handle it (see Lemma 7). The other challenge is more fundamental and is due to “cross-position”
construction itself. Say, e.g., $\eta^i = \xi_{i1}^i + \xi_{i2}^i$. While we know $\xi_{i1}^i \subseteq \lambda g^i$ given that $\xi_{i1} \subseteq \lambda g$, it is not necessary that $\xi_{i2}^i \subseteq \lambda g^i$. How can we select the right bricks so that $\eta^i \subseteq \lambda g^i$ for all $i$? Indeed, is it even possible or not? Towards this, our rough idea is as follows: we consider every coordinate of $\lambda g^i$. If one coordinate is sufficiently large (larger than some threshold $\sigma = O_{FPT}(1)$), then the summation of any $O_{FPT}(1)$ bricks $\xi_j$’s should never exceed it. Otherwise, $\eta^i \not\subseteq \lambda g^i$ may be violated and this coordinate becomes critical. We will introduce a hierarchy over $\lambda g^i$’s depending on each of its coordinate being critical or not, and the “cross-position” construction will only be carried out for positions (e.g., $i_1$ and $i_2$ in $\eta^i = \xi_{i1}^i + \xi_{i2}^i$) in the same level under the hierarchy. We will show that, by doing so, if $\|\lambda g\|_{\infty}$ is sufficiently large, then $\eta \subseteq \lambda g$ can be guaranteed through a counting argument.

The remainder of this section is devoted to the proof of Theorem 1. Towards this, we first introduce some concepts.

Consider the generalized 4-block $n$-fold IP with constraint matrix $H$ and let $g \in \ker_2(H)$ be an arbitrary kernel element. A decomposition $g = \sum_{j=1}^{N} \eta_j$ is called uniform, if for all $j$ it holds that $\eta_j \subseteq g$, $H^{\text{two-stage}} \eta_j = 0$, and moreover, there is some fixed $q \in \mathbb{Z}^{s_B}$, $q \neq 0$, such that for any $j \in [N]$,

$$B_i \eta_j^0 = 0, \forall i \in [n] \quad \text{or} \quad B_i \eta_j^0 = q, \forall i \in [n].$$

That is, for all $i$ and $j$, $B_i \eta_j^0$ may only take two possible values. For each $j$, $B_i \eta_j^0$ must be the same for all $i$. We say $\eta_j$ is tier-0 if $B_i \eta_j^0 = 0$, and $\eta_j$ is tier-1 if $B_i \eta_j^0 = q$. Consequently, $A_i \eta_j^0 = 0$ for all $i$ or $A_i \eta_j^0 = -q$ for all $i$.

In case of combinatorial 4-block $n$-fold IP, $B_i = B$ and $s_B = 1$, and hence Eq (8) is simplified such that $B \eta_j^0$ is either 0 or $q$ for all $j$.  

Consider an arbitrary $\bar{g} \in \ker_2(H)$ that admits a uniform decomposition $\bar{g} = \sum_{j=1}^{\bar{N}} \bar{\eta}_j$ such that $\|\bar{\eta}_j\|_{\infty} \leq \bar{\eta}_{\max}$. As each $\bar{\eta}_j$ is either tier-0 or tier-1, we denote by $\bar{N}_0$ (or $\bar{N}_1$) the number of tier-0 (or tier-1) vectors among $\bar{\eta}_1$ to $\bar{\eta}_{\bar{N}}$. We say that the decomposition is $\omega$-balanced if $\bar{N}_0 \leq \omega \bar{N}_1$, and exact $\omega$-balanced if the equality holds. In particular, we define that $0$ admits an $\omega$-balanced uniform decomposition.

**Lemma 4.** For any $g \in \ker_2(H)$, if $\bar{g}$ admits a uniform decomposition $\bar{g} = \sum_{j=1}^{\bar{N}} \bar{\eta}_j$ where $\|\bar{\eta}_j\|_{\infty} \leq \bar{\eta}_{\max}$, then there exists $g \in \ker_2(H)$ such that $g \subseteq \bar{g}$, $B_i(g^0 - g^0) = 0$ for all $i \in [n]$, and $g$ admits an $\omega$-balanced uniform decomposition for $\omega \leq (\Delta t \bar{u}_{\max})^{O(s_B^2)}$. Moreover, if $\bar{g} - g \neq 0$, then we have $\bar{g} - g = g_1 + g_2 + \cdots + g_p$ for some $p \in \mathbb{Z}$ and $g_j \in \ker_2(H)$, and furthermore, $g_j \subseteq \bar{g} - g$ and $\|g_j\|_{\infty} \leq (\Delta t \bar{u}_{\max})^{O(s_B^2)}$.

**Remark.** If $\bar{g}^0 = 0$, then Lemma 4 holds for $g = 0$.

It suffices to focus on a balanced uniform decomposition. Further notice that if $\bar{\eta}_j_1$ is tier-1 and $\bar{\eta}_j_2$ is tier-0, then $\bar{\eta}_j_1 + \bar{\eta}_j_2$ is tier-1. Hence, we have the following.

**Lemma 5.** If $g = \sum_{j=1}^{\bar{N}} \eta_j$ is an $\omega$-balanced uniform decomposition where $\bar{u}_{\max} = \max_{j \in [\bar{N}]} \|\bar{\eta}_j\|_{\infty}$, then $g$ admits a uniform decomposition $g = \sum_{j=1}^{N} \eta_j$ such that every $\eta_j$ is tier-1, and $\eta_{\max} = \max_{j \in [N]} \|\eta_j\|_{\infty} \leq \omega \bar{u}_{\max}$.

We will prove the following Lemma 6 in Section 3.1. Then we show the existence of such decomposition for combinatorial 4-block $n$-fold IP, thus concluding Theorem 1 in Section 3.2.
Lemma 6. Suppose \( g \in \ker_2(H) \) admits a uniform decomposition \( g = \sum_{j=1}^{N} \eta_j \) such that \( \|\eta_j\|_\infty \leq \eta_{\max} \), and every \( \eta_j \) is tier-1. There exists \( \tau = (\Delta \eta_{\max})^{\Delta O(s_{AT}^A + s_{DT}^D)} \) such that if \( \|g\|_\infty > \tau \), then there exists \( \eta \in \ker_2(H) \) such that \( \eta \subseteq g \) and \( \|\eta\|_\infty \leq \tau \), and furthermore, \( \eta^0 = \sum_{j \in S} \eta_j^0 \) for some \( S \subseteq [N] \).

3.1 Proof of Lemma 6

3.1.1 A hierarchical structure over bricks of \( g \)

As we describe in the overview, we will construct \( \eta \) in Lemma 6 from the bricks \( \eta_j \)'s via “cross-position” construction. For each \( \eta_j \), there will be some restrictions regarding which brick \( \eta_j' \) can be used, indicated by the hierarchical structure we introduce in the following.

We first observe that as \( A_i \)'s and \( D_i \)'s are small submatrices with the largest coefficient bounded by \( \Delta \), there are in total at most \( \Delta O(s_{AT}^A + s_{DT}^D) \) different kinds of \( A_i \)'s and \( D_i \)'s, and hence \( \varphi \leq \Delta O(s_{AT}^A + s_{DT}^D) = O_{FPT}(1) \) different pairs of \((A_i,D_i)\). By re-indexing, we may divide \([n]\) into \( \varphi \) subsets as \([n] = \bigcup_{k=1}^{\varphi}[k_{k-1} + 1 : r_k] \) where \( 0 = r_0 < r_1 < r_2 < \cdots < r_{\varphi} = n \) such that \((A_i,D_i)'s\) are identical for every \( i \in [r_{k-1} + 1 : r_k] \). Let \( I_0 = \{0\} \) and \( I_k = [r_{k-1} + 1 : r_k] \). For simplicity we let \( D_{r_0} = D_0 = C \), then \((C,D_1,D_2,\cdots,D_\varphi)\) \( \eta_j = \sum_{k=0}^{\varphi} \sum_{i \in I_k} D_{r_k} \eta_j' \).

We define type and subtype for integer vectors. Let \( \sigma \) be some sufficiently large value (it suffices to take \( \sigma = (\Delta \eta_{\max})^{\Delta O(s_{AT}^A + s_{DT}^D)} \) as we will explain later). We classify each integer \( x \) into one of the five types:

- 0, if \( x = 0 \),
- close-positive, if \( 1 \leq x \leq \sigma \),
- faraway-positive, if \( x > \sigma \),
- close-negative, if \( -\sigma \leq x \leq -1 \), and
- faraway-negative, if \( x < -\sigma \).

We can further classify all integers into \( 2\sigma + 3 \) subtypes by sub-dividing the type close-positive (or close-negative) into \( \sigma \) categories, that is, \( x \) is called of subtype-x if \( -\sigma \leq x \leq \sigma \).

We now extend the definitions of types and subtypes to vectors. All \( d \)-dimensional vectors can be classified into \( 5^d \) types (or \( (2\sigma + 3)^d \) subtypes) such that two vectors \( x \) and \( y \) belong to the same type (or subtype) as a vector if and only if for every \( 1 \leq \ell \leq d \), the \( \ell \)-th coordinate of \( x \) and \( y \) have the same type (or subtype) as an integer.

Now we classify the indices \( 0 \leq i \leq n \) based on \( g \) as follows:

- Megazone. Each \( I_k \), \( 0 \leq k \leq \varphi \) is called a megazone. There are \( \varphi + 1 = \Delta O(s_{AT}^A + s_{DT}^D) \) megazones.
- Zone. A megazone is sub-divided into zones such that indices \( i,i' \) belong to the same zone if and only if they belong to the same megazone and \( g_i \) and \( g_{i'} \) have the same type. There are at most \( 1 + 5^{t^A \varphi} = \Delta O(s_{AT}^A + s_{DT}^D) \) different zones. For \( 0 \leq \nu \leq 5^{t^A \varphi} \), let \( \beta_\nu \in \mathbb{Z}_{ \geq 0} \) be the number of indices belonging to zone-\( \nu \).
- Subzone. A zone is sub-divided into subzones so that indices \( i,i' \) belong to the same subzone if and only if they belong to the same zone and \( g_i \) and \( g_{i'} \) have the same subtype. There are at most \( (2\sigma + 3)^{t^A \varphi} \cdot (1 + 5^{t^A \varphi}) \) subzones. For \( 0 \leq \ell \leq (2\sigma + 3)^{t^A}(1 + 5^{t^A \varphi}) - 1 \), let \( \gamma_\ell \in \mathbb{Z}_{ \geq 0} \) be the number of indices belonging to subzone-\( \ell \).
• Slot. Every index 0 ≤ i ≤ n is called a slot. There are n + 1 slots.

Figure 1 in Appendix [A.3] illustrates the relationships among megazones, zones and subzones. It is remarkable that the number of zones, 1 + 5^k, is independent of the value of σ. σ only comes into play at subzone level, which is crucial to our proof. Further, note that megazone-0 only contains one zone, and this zone contains one subzone, and this subzone contains one slot, which is slot-0. For simplicity, we let slot-0 be in subzone-0 and zone-0.

For ease of description, we will take a viewpoint of the Scheduling problem. We view each brick ηj as a job and there are N(n + 1) jobs. We assume there are n + 1 machines (from machine 0 to machine n), and think of each job ηj as a job originally scheduled on machine i. Machines can be divided into megazones, zones and subzones based on their indices. A job (brick) that is originally scheduled on a machine in megazone-k (or zone-ν or subzone-ι, resp.) is called a megazone-k (or zone-ν or subzone-ι, resp.) job (brick). We add up jobs on each machine just like adding up vectors, whereas the load of machine i is the capacity of machine i. If the summation of several jobs equals x ⊑ g, then we say the jobs fit machine i.

To prove Lemma 6, we need to construct a partial schedule η such that (i) H^two-stage η = 0, (ii) (C, D1, ···, Dn)η = 0 and (iii) η ⊑ g. In the following Subsection 3.1.2, Subsection 3.1.3 and Subsection 3.1.4 we will identify the conditions for the partial schedule to satisfy each property respectively, and finalize the proof of Lemma 6 in Subsection 3.1.5.

3.1.2 Selecting jobs to satisfy property (i) - H^two-stage η = 0

Recall that A_i’s are the same for i in each megazone (and hence in each zone). For ν ≥ 1, let machine i be an arbitrary zone-ν machine and η_i be an arbitrary zone-ν job. Then A_iη_i = -q by the definition in Eq [8]. If we put one zone-ν job η_i on machine i and meanwhile put one zone-0 job η_0 on machine 0, then it holds that B_iη_i + A_iη_i = 0. Hence, we have the following observation.

Observation 2. Let h be an arbitrary non-negative integer. Let η = (η_0, η_1, ···, η^n) be a partial schedule where we assign arbitrary h jobs in zone-ν to each zone-ν machine (i.e., for every i in zone-ν, η_i is the summation of h zone-ν jobs). Then H^two-stage η = 0.

3.1.3 Selecting jobs to satisfy property (ii) - (C, D1, ···, Dn)η = 0

Recall that D_0 = C and (D_0, D_1, ···, D_n) Σ_j=1 η_j = 0, which is a long sequence of addition consisting of (n+1)N summands. We are interested in a subsequence whose sum is 0 and meanwhile respects Observation 2 that is, we want to select exactly h Jobs from zone-ν such that their sum (after multiplying corresponding D_i’s) is 0 (while recall that there are exactly β_ν zone-ν machines). Towards this, we first prove the following lemma, which gives a “colorful” version of the Steinitz Lemma.

Lemma 7. Let x_1, ···, x_M ∈ Z^d be a sequence of vectors such that ∥x_i∥∞ ≤ ζ for some ζ ≥ 1 and every i = 1, ···, M. Furthermore, there are μ colors, and each vector x_i is associated with one color. There are in total α_j m vectors of color j where α_j, m ∈ Z ≥ 0 and Σ_j=1 α_j = α, M = α m. Suppose that Σ_i=1 M x_i = 0 and M is sufficiently large (i.e., M > (2dζ + 2μζ + 1)^d+µ α + α + d + µ),
then among \( x_1, \ldots, x_M \) we can find \( \alpha_j m \) vectors of each color \( j \) such that their summation is \( 0 \), and \( m \leq (2d\zeta + 2\mu\zeta + 1)^{d+\mu} \).

By the Steinitz Lemma, it is easy to see the existence of a subset of vectors that add up to \( 0 \). Lemma 7 further indicates that the number of vectors of each color in this subset is proportional to their number in the whole set of \( M \) vectors. Notice that \( m \) and \( m \) are independent with each other. \( m \) may be very large, while \( m \) can be bounded by an FPT-value. See the proof in Appendix A.4.

Now we apply Lemma 7 to the equation \( (D_0, D_1, \ldots, D_n) \sum_{j=1}^N \eta_j = \sum_{i,j,l} D_i \eta_j^l = 0 \) as follows. If \( i \) belongs to some zone-\( \nu \) (which further belongs to some megazone-\( k \)), then we take each summand \( D_i \eta_j^l \) as a vector in \( \mathbb{Z}^{t_D} \) of color \( \nu \). Consequently, we have in total \( 1 + 5^{t_A}C = \Delta^{O(s A + s D t_D)} \) different colors, and \( M = (n+1)N \) vectors where the number of vectors in each color \( \nu \) is \( N \beta_\nu \). Further notice that \( \|D_i \eta_j^l\|_\infty \leq t_D \Delta \eta_{\max} \). Hence, as long as \( M = (n+1)N > \rho(n+1) \) for \( \rho = (\Delta \eta_{\max})^{\Delta^{O(s A + s D t_D)}} \), we can always find out \( M \beta_\nu \) summands in color \( \nu \) (corresponding to \( M \beta_\nu \) jobs in zone-\( \nu \)) such that \( m \leq (\Delta \eta_{\max})^{\Delta^{O(s A + s D t_D)}} \), and they sum up to \( 0 \). Moreover, Lemma 7 can be applied iteratively until there are fewer than \( \rho(n+1) \) jobs left. Our argument above implies the following.

**Lemma 8.** There exist some \( m, \rho = (\Delta \eta_{\max})^{\Delta^{O(s A + s D t_D)}} \) such that if \( N > \rho \), then all the \( (n+1)N \) jobs (bricks) can be divided into \( \lfloor \frac{N-2}{m} \rfloor + 1 := \psi + 1 \) groups such that

- Except the last group, each group consists of \( \beta_\nu m \) zone-\( \nu \) jobs for all \( \nu \).
- The last group consists of \( \beta_\nu m' \) zone-\( \nu \) jobs where \( m' \leq \rho + m \).
- If we evenly distribute jobs in every group to machines such that a zone-\( \nu \) machine is assigned \( m \) jobs (or \( m' \) jobs if it is the last group), then the partial schedule \( \eta \) satisfies that \( (C, D_1, \ldots, D_n) \eta = 0 \).

**Remark.** Note that the number of zones, and thus \( m, \rho, \psi \), are all independent of \( \sigma \). We pick \( \sigma \geq (\rho + m) \eta_{\max} \) which guarantees that when we evenly distribute jobs in each group to machines, the infinity norm of their sum never exceeds \( \sigma \).

Notice that since we assign the same number of zone-\( \nu \) jobs to zone-\( \nu \) machines, by Observation 2 the partial schedule \( \eta \) in Lemma 8 also satisfies that \( H^{\text{two-stage}} \eta = 0 \), and hence \( H \eta = 0 \).

### 3.1.4 Selecting jobs to satisfy property (iii) - \( \eta \subseteq g \)

According to Lemma 8 by evenly distributing jobs to machines in each zone, every group of jobs induces a partial schedule \( \eta \). We show in this subsection that if there are sufficiently many groups, then there must be a group which induces \( \eta \subseteq g \). For simplicity we ignore the last group and focus on remaining groups.

We first briefly argue why evenly distributing jobs to machines in each zone in an arbitrary way may generate a partial schedule that is \( \not\subseteq g \). Note that when we apply Lemma 7 to divide jobs into groups, we can only guarantee there are \( \beta_\nu m \) jobs from each zone-\( \nu \) (and hence every machine in zone-\( \nu \) can get exactly \( m \) jobs in zone-\( \nu \)), but we cannot guarantee there are \( \gamma_\nu m \) jobs from each subzone-\( \iota \). Hence, when we evenly distribute jobs, some machine in subzone-\( \iota_1 \) may get jobs from subzone-\( \iota_2 \). As the subtypes of \( g^{\iota_1} \) and \( g^{\iota_2} \) are different, a job that fits a subzone-\( \iota_2 \) machine does not necessarily fit a subzone-\( \iota_1 \) machine.

Note that megazone-0 only contains one zone (and one subzone). Thus all megazone-0 jobs (and thus megazone-0 jobs in each group), fit machine 0. From now on we only consider machine 1 to machine \( n \), and only consider groups of jobs which are not the last group.

Consider machines and jobs in each zone-\( \nu \). Since in each zone \( g^i \)'s have the same type, we know if some coordinate, say, the \( h \)-th coordinate of \( g^i \) is 0, then the \( h \)-th coordinate of any zone-\( \nu \)
job is also 0. Recall that we have set \( \sigma \geq (m + \rho)\eta_{\max} \) to be sufficiently large such that if we add any \( m \) jobs, the absolute value of each coordinate of the sum is no more than \( \sigma \). Hence, when we distribute jobs to machines in each zone-\( \nu \), if the sum of \( m \) jobs does not fit machine \( i \) (i.e., \( \not\subseteq g^i \)), then the violation must occur at some coordinate of \( g^i \) which is close-positive or close-negative (i.e., with a value in \([1, \sigma] \cup [-\sigma, -1]\)). We call all close-positive or close-negative coordinates of each \( g^i \) as critical coordinates. Recall that \( g^i \)'s in the same zone share the same type, and hence the same critical coordinates. Let \( CI_\nu = \{ h_{1}^\nu, h_{2}^\nu, \cdots, h_{|CI_\nu|}^\nu \} \) be the set of critical coordinates for zone-\( \nu \), that is, for any \( i \) in zone-\( \nu \), the \( h_{\nu}^\nu \)-th coordinate of \( g^i \) falls in \([1, \sigma] \cup [-\sigma, -1]\).

We consider the \( h_{\nu}^\nu \)-th coordinate of every job in zone-\( \nu \). We say a job is good if its \( h_{\nu}^\nu \)-th coordinate is 0 for all \( 1 \leq \ell \leq |CI_\nu| \), and is bad otherwise (i.e., its \( h_{\nu}^\nu \)-th coordinate is nonzero for some \( \ell \)). It is clear that good jobs never cause trouble in the sense that any \( m \) good jobs in zone-\( \nu \) fit a zone-\( \nu \) machine. It suffices to consider the scheduling of bad jobs.

Recall there are \( \gamma_{\iota} \) slots (and hence \( \gamma_{\iota} \) machines) in each subzone-\( \iota \). We say a group is bad in subzone-\( \iota \) if it contains more than \( \gamma_{\iota} \) bad jobs in subzone-\( \iota \), and is good if it is not a bad group in any subzone. We have the following lemmas regarding good and bad groups.

**Lemma 9.** If a group is good and is not the last group in Lemma 8, then there is an assignment of jobs to machines such that the partial schedule \( \eta \) satisfies that \( H\eta = 0 \), \( \|\eta\|_\infty \leq m\eta_{\max} \) and \( \eta \subseteq g \).

**Proof.** Notice that a good group does not necessarily contain exactly \( m\gamma_{\iota} \) jobs in each subzone-\( \iota \), but it contains no more than \( \gamma_{\iota} \) bad jobs in each subzone-\( \iota \). Hence, we reschedule jobs to obtain a partial schedule such that every machine in subzone-\( \iota \) is assigned 1 or 0 bad job in subzone-\( \iota \), together with \( m - 1 \) or \( m \) good jobs in zone-\( \nu \) (that contains subzone-\( \iota \)). We claim that, this partial schedule \( \eta \) satisfies Lemma 9. First, by Lemma 8, jobs in every zone-\( \nu \) is evenly distributed among machines in zone-\( \nu \), and hence \( H\eta = 0 \). Next, by the definition, a subzone-\( \iota \) machine is originally scheduled on a subzone-\( \iota \) machine, and hence in the rescheduling it either stays at the original machine or moves to another subzone-\( \iota \) machine. By the definition of a subzone all machines in subzone-\( \iota \) share the same value on critical coordinates. This means, a single bad job in subzone-\( \iota \) fits any machine in subzone-\( \iota \). Recall that the critical coordinate of a good job always has value 0, so \( m \) good jobs, or a bad job with \( m - 1 \) good jobs fit any machine in subzone-\( \iota \). Hence, \( \eta \subseteq g \). \( \square \)

In the meantime, there are not too many bad groups as implied by the following lemma.

**Lemma 10.** The total number of bad groups is bounded by \((2\sigma + 3)^{t_A}(1 + 5^{t_A}\varphi)\sigma t_A\).

**Proof.** Consider any slot \( i \) in a subzone-\( \iota \) contained in zone-\( \nu \), and there are \(|CI_\nu|\) critical coordinates. Let \( g^i = (g^i[1], g^i[2], \cdots, g^i[t_A]) \). Recall there are \( \gamma_{\iota} \) slots (indices) in subzone-\( \iota \). Consider the summation of absolute value over critical coordinates of \( g^i \)'s in each subzone-\( \iota \), we have

\[
\sum_{i \in \text{subzone-}\iota} \sum_{h \in CI_\nu} |g^i[h]| \leq |CI_\nu|\sigma\gamma_{\iota} \leq \sigma\gamma_{\iota} t_A.
\]

Note that every bad job in subzone-\( \iota \) lies in the same orthant with nonzero value at some critical coordinate, and must thus contribute at least 1 to the above value. Recall that a bad group must be bad in at least one subzone, and any bad group in subzone-\( \iota \) contains more than \( \gamma_{\iota} \) bad jobs in subzone-\( \iota \). Hence, a bad group in subzone-\( \iota \) contributes at least \( \gamma_{\iota} \) in total, which implies that there can be at most \( \sigma t_A \) bad groups in subzone-\( \iota \). Given that there are \((2\sigma + 3)^{t_A}(1 + 5^{t_A}\varphi)\) subzones, there can be at most \((2\sigma + 3)^{t_A}(1 + 5^{t_A}\varphi)\sigma t_A \) bad groups, and Lemma 10 is proved. \( \square \)
3.1.5 Finalizing the proof of Lemma 6

By Lemma 8, except the last group, there are \( \psi = \lceil \frac{N-2}{m} \rceil \) groups, where each group is either bad or good. By Lemma 10 there are at most \((2\sigma + 3)tA(1 + 5tA\varphi)\sigma tA = (\Delta\eta_{\max})^2O(\varphi)\sigma tA + 1\) bad groups. Hence if \( \frac{N-2}{m} \geq (2\sigma + 3)tA(1 + 5tA\varphi)\sigma tA + 1 \), there will be at least one good group, and by Lemma 9 it induces some \( \eta \) such that \( \eta \subseteq g \), \( \|\eta\|_{\infty} \leq m\eta_{\max} \leq \tau \) and \( H\eta = 0 \). Further notice that only zone-0 jobs will be put on machine 0, and thus \( \eta^0 \) is the summation of some \( \eta_j^0 \)'s. Therefore Lemma 6 is proved.

3.2 Proof of Theorem 1

Now we are ready to prove Theorem 1. Consider an arbitrary \( g \in ker_{Z}(H_{com}) \). As \( g \in ker_{Z}(H_{comm}^{two-stage}) \), there exists a decomposition \( g = \sum_j \xi_j \) where \( \xi_j \in ker_{Z}(H_{comm}^{two-stage}) \), \( \|\xi_j\|_{\infty} = O_{FPT}(1) \) and \( \xi_j \subseteq g \). But when can we guarantee that this can lead to a uniform decomposition? We observe that \( B\xi_j^0 \)'s are integers such that \( s_B = 1 \), and \( B\xi_j^0 + A\xi_j^1 = 0 \). If we aim for a uniform decomposition by merging \( \xi_j^1 \)'s, then the question becomes whether we can partition \( \xi_j^1 \)'s into different groups such that \( B\xi_j^0 \)'s within each group sum up to the same value (bounded by \( O_{FPT}(1) \)).

An even partition does not need to exist, but we have the following sufficient condition.

**Lemma 11.** Let \( x_1, x_2, \ldots, x_m \in \mathbb{Z} \) and \( \zeta \in \mathbb{Z}_{\geq 0} \) be integers such that \( |x_i| \leq \zeta \) for \( i \in [m] \) and \( \sum_{i=1}^{m} x_i = x \). If \( x \) is a multiple of \((6\zeta^2 + 2\zeta + 1)!\), then the \( m \) integers can be partitioned into \( m' \) subsets \( T_1, T_2, \ldots, T_{m'} \) such that \( \bigcup_{i=1}^{m'} T_k = [m] \), and for all \( k \in [m'] \) it holds that \( |T_k| \leq 2^{O(\zeta^2 \log \zeta)} \), \( \sum_{i \in T_k} x_i \in \{0, \text{sgn}(x) \cdot (6\zeta^2 + 2\zeta + 1)!\} \) where \( \text{sgn} \) denotes the standard sign function such that \( \text{sgn}(x) = 1 \) if \( x > 0 \), \( \text{sgn}(x) = -1 \) if \( x < 0 \), and \( \text{sgn}(x) = 0 \) if \( x = 0 \).

With Lemma 11 we are able to prove the following.

**Lemma 12.** Let \( g \in ker_{Z}(H_{com}) \). Let

\[
\lambda = (6\lambda_0^2 + 2\lambda_0 + 1)! = 2^{2^{\Theta(\log \Delta)}} , \quad \text{where } \lambda_0 := \Delta t_Bg_{\infty}(H_{com}^{two-stage}) = 2^{2^{\Theta(\log \Delta)}} .
\]

If \( Bg^0 \) is a multiple of \( \lambda \), then \( g \) admits a uniform decomposition \( g = \sum_{j=1}^{N} \eta_j \) such that \( \|\eta_j\|_{\infty} \leq 2^{2^{\Theta(\log \Delta)}} \). Furthermore, \( B\eta_j^0 \) is a multiple of \( \lambda \) for all \( j \).

Now we are ready to prove our main theorem.

**Proof of Theorem 1.** Consider any \( g \in ker_{Z}(H_{com}) \). Clearly \( B(\lambda g^0) \) is a multiple of \( \lambda \), thus by Lemma 12 \( \lambda g \) admits a uniform decomposition \( \lambda g = \sum_{j=1}^{N} \eta_j \) where \( \|\eta_j\|_{\infty} \leq \eta_{\max} = 2^{2^{\Theta(\log \Delta)}} \) and every \( B\eta_j^0 \) is a multiple of \( \lambda \).

If this decomposition is not \( \omega \)-balanced for \( \omega \leq (\Delta t_D\eta_{\max})^{O(\log \Delta)} \), then by Lemma 4 we obtain \( \eta \subseteq \lambda g \) with \( \|\eta\|_{\infty} \leq (\Delta t_D\eta_{\max})^{O(\log \Delta)} \), \( \eta \in ker_{Z}(H_{com}) \) and \( B(\lambda g^0 - \eta^0) = 0 \). \( B\eta^0 \) is a multiple of \( \lambda \). Otherwise this decomposition is \( \omega \)-balanced. By Lemma 5 we can obtain a uniform decomposition \( \lambda g = \sum_{j=1}^{N'} \eta'_j \) such that \( \max_j \|\eta'_j\| \leq \omega\eta_{\max} \) and all \( \eta'_j \)'s are tier-1. According to Lemma 6 if \( \lambda\|g\|_{\infty} > \tau \) for \( \tau = (\omega\Delta\eta_{\max})^{2^{\Theta(\log \Delta)}} = 2^{2^{\Theta(\log \Delta)}} \), then we are able to find some \( \eta \subseteq \lambda g \) such that \( H_{com}\eta = 0 \), \( \|\eta\|_{\infty} = O_{FPT}(1) \) and \( \eta^0 = \sum_{j \in S} \eta_j^0 \) for some \( S \subseteq [N] \). As every \( B\eta_j^0 \) is a multiple of \( \lambda \), \( B\eta^0 \) is also a multiple of \( \lambda \). In both cases, we find \( \eta \subseteq \lambda g \) where \( B\eta^0 \) is a multiple of \( \lambda \).
Remark. Theorem 1 is also true for almost combinatorial 4-block n-fold IP. Now B is not a 1 × tB matrix, but rather an sB × tB matrix with rank 1. For such a matrix B, we can always transform it into B, in which the first row is r1, and all the other rows are 0. It implies that when rank(B) = 1, it is sufficient to consider such a case B = (r1, 0, ..., 0)\top, where r1 ≠ 0. Then we observe that for any x ∈ \mathbb{Z}^tB, Bx = (r1 · x, 0, ..., 0). Hence, our argument in the proof above applies directly, i.e., Theorem 1 holds for almost combinatorial 4-block n-fold IP (see Appendix A.7 for a formal proof). In other words, Theorem 1 and our FPT algorithm for combinatorial 4-block n-fold IP remain true for almost combinatorial 4-block n-fold IP. Such a generalization allows submatrices Ai’s to contain multiple rows subject to that these rows are “local constraints”.

4 Algorithms

Using Theorem 1, we are able to bound the ℓ∞-norm of the Graver basis elements:

**Theorem 4.** Let g ∈ G(H\text{com}) be a Graver basis element, then \|g\|_∞ = g_{∞}(H\text{com}) where g_{∞}(H\text{com}) ≤ 2\tilde{O}(t_A \log Δ + \max\{t_D \log Δ\}) \cdot n = O_{FPT}(n).

Utilizing Theorem 4 and the iterative augmentation framework (see Section 2), we are able to prove the following theorem.

**Theorem 5.** Consider combinatorial 4-block n-fold IP with a separable convex objective function f mapping \mathbb{Z}^{tn_A} to \mathbb{Z}. Let P be the set of feasible integral points, and let \hat{f} := \max_{x,y \in P} (f(x) − f(y)). Then it can be solved in 2\tilde{O}(t_A \log Δ + \max\{t_D \log Δ\}) \cdot n^4 \hat{L}^2 \log^2 (\hat{f}) = O_{FPT}(n^4 \hat{L}^2 \log^2 (\hat{f})) time, where \hat{L} denotes the logarithm of the largest number occurring in the input.

The running time can be improved to O_{FPT}(n^{5+o(1)}) if the objective function is linear. See Appendix B.1 for the proof.

5 Applications in Scheduling with High Multiplicity

It has been shown by Knop and Koutecký [27] that the classical scheduling problems \text{R||C}_{max} and \text{R||} \sum \ell w_\ell C_\ell can be modeled as n-fold IPs, based on which FPT algorithms can be developed. However, we can try to model more sophisticated scheduling problems, especially scheduling with rejection \text{R||C}_{max} + E or bicriteria scheduling \text{R||θC}_{max} + \sum \ell w_\ell C_\ell, we run into 4-block n-fold IP. This is because for these problems, C_{max} needs to be taken as a variable in the IP, while for \text{R||C}_{max} we can use binary search on C_{max} and hence n-fold IP is sufficient.

We formally describe the scheduling problem. Given are m machines and k different types of jobs, with N_j jobs of type j. A job of type j has a processing time of p_j^i ∈ \mathbb{Z}_{≥0} if it is processed by machine i.

For scheduling with rejection \text{R||C}_{max} + E, every job of type j also has a rejection cost u_j. A job is either processed on one of the machine, or is rejected. The goal is to minimize the makespan C_{max} plus the total rejection cost E.

**Theorem 6.** \text{R||C}_{max} + E can be solved in m^{5+o(1)} \cdot 2^{\tilde{O}(k^2 \log p_{max})} \cdot 2^{\tilde{O}(\log p_{max})} + |I| time, where |I| denotes the length of the input.
More precisely, $|I|$ is bounded by $O(k p_{\text{max}} \max\{\log N_{\text{max}}, \log u_{\text{max}}\})$ where $N_{\text{max}} = \max_j N_j$ and $u_{\text{max}} = \max_j u_j$. One may suspect that the problem can be solved through the generalized $n$-fold IP by guessing out the value of $C_{\text{max}}$. However, this will require $p_{\text{max}} \cdot \max_j N_j$ enumerations. See Appendix C.1 for a detailed proof of Theorem 6.

For bicriteria scheduling $R||\theta C_{\text{max}} + \sum_{\ell} w_{\ell} C_{\ell}$, each job $\ell$ of type $j$ has a weight $w_j$, and the goal is to find an assignment of jobs to machines such that $\theta C_{\text{max}} + \sum_{\ell} w_{\ell} C_{\ell}$ is minimized, where $C_{\ell}$ is the completion time of job $\ell$, and $\theta$ is a fixed input value.

**Theorem 7.** $R||\theta C_{\text{max}} + \sum_{\ell} w_{\ell} C_{\ell}$ can be solved in $m^4 2^{O(k^2 \log p_{\text{max}})} 2^{O(\log p_{\text{max}})} |I|^4$ time, where $|I|$ denotes the length of the input.

More precisely, $|I|$ is bounded by $O(k p_{\text{max}} \max\{\log N_{\text{max}}, \log w_{\text{max}}\})$ where $N_{\text{max}} = \max_j N_j$, $w_{\text{max}} = \max_j w_j$. See Appendix C.2 for a detailed proof of Theorem 7.

For identical machines, $k \leq p_{\text{max}}$ and we obtain FPT algorithms parameterized by $p_{\text{max}}$. 

14
References

[1] A. Allahverdi and F. S. Al-Anzi. The two-stage assembly flowshop scheduling problem with bicriteria of makespan and mean completion time. The International Journal of Advanced Manufacturing Technology, 37(1):166–177, 2008.

[2] A. Allahverdi and T. Aldowaisan. No-wait flowshops with bicriteria of makespan and total completion time. Journal of the Operational Research Society, 53(9):1004–1015, 2002.

[3] K. Altmanová, D. Knop, and M. Koutecký. Evaluating and tuning n-fold integer programming. Journal of Experimental Algorithmics, 24(1):1–22, 2019.

[4] M. Aschenbrenner and R. Hemmecke. Finiteness theorems in stochastic integer programming. Foundations of Computational Mathematics, 7(2):183–227, 2007.

[5] L. Chen, H. Chen, and G. Zhang. Block-structured integer programming: Can we parameterize without the largest coefficient? arXiv preprint arXiv:2011.02826, 2020.

[6] L. Chen, M. Koutecký, L. Xu, and W. Shi. New bounds on augmenting steps of block-structured integer programs. In Proceedings of the 28th Annual European Symposium on Algorithms (ESA), volume 173 of LIPIcs, pages 33:1–33:19, 2020.

[7] L. Chen and D. Marx. Covering a tree with rooted subtrees–parameterized and approximation algorithms. In Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 2801–2820. SIAM, 2018.

[8] M. Cheng, P. R. Tadikamalla, J. Shang, and B. Zhang. Two-machine flow shop scheduling with deteriorating jobs: minimizing the weighted sum of makespan and total completion time. Journal of the Operational Research Society, 66(5):709–719, 2015.

[9] J. Cslovjecsek, F. Eisenbrand, C. Hunkenschröder, L. Rohwedder, and R. Weismantel. Block-structured integer and linear programming in strongly polynomial and near linear time. In Proceedings of the 32nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1666–1681. SIAM, 2021.

[10] J. A. De Loera, R. Hemmecke, and M. Köppe. Algebraic and geometric ideas in the theory of discrete optimization. SIAM, 2012.

[11] J. A. De Loera, R. Hemmecke, S. Onn, and R. Weismantel. N-fold integer programming. Discrete Optimization, 5(2):231–241, 2008.

[12] F. Eisenbrand, C. Hunkenschröder, and K. M. Klein. Faster algorithms for integer programs with block structure. arXiv preprint arXiv:1802.06289, 2018.

[13] F. Eisenbrand, C. Hunkenschröder, K. M. Klein, M. Koutecký, A. Levin, and S. Onn. An algorithmic theory of integer programming. arXiv preprint arXiv:1904.01361, 2019.

[14] D. W. Engels, D. R. Karger, S. G. Kolliopoulos, S. Sengupta, R. N. Uma, and J. Wein. Techniques for scheduling with rejection. Journal of Algorithms, 49(1):175–191, 2003.

[15] R. Gollmer, U. Gotzes, and R. Schultz. A note on second-order stochastic dominance constraints induced by mixed-integer linear recourse. Mathematical Programming, 126(1):179–190, 2011.
[16] J. E. Graver. On the foundations of linear and integer linear programming I. *Mathematical Programming*, 9(1):207–226, 1975.

[17] V. S. Grinberg and S. V. E. Sevast’yanov. Value of the steinitz constant. *Funktsional’nyi Analiz i ego Prilozheniya*, 14(2):56–57, 1980.

[18] R. Hemmecke, M. Köppe, and R. Weismantel. A polynomial-time algorithm for optimizing over n-fold 4-block decomposable integer programs. In *Proceedings of the 14th International Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pages 219–229. Springer, 2010.

[19] R. Hemmecke, M. Köppe, and R. Weismantel. Graver basis and proximity techniques for block-structured separable convex integer minimization problems. *Mathematical Programming*, 145(1-2):1–18, 2014.

[20] R. Hemmecke, S. Onn, and L. Romanchuk. N-fold integer programming in cubic time. *Mathematical Programming*, 137(1-2):325–341, 2013.

[21] R. Hemmecke and R. Schultz. Decomposition of test sets in stochastic integer programming. *Mathematical Programming*, 94(2-3):323–341, 2003.

[22] H. Hoogeveen, M. Skutella, and G. J. Woeginger. Preemptive scheduling with rejection. *Mathematical Programming*, 94(2):361–374, 2003.

[23] K. Jansen, K. M. Klein, and A. Lassota. The double exponential runtime is tight for 2-stage stochastic ILPs. In *Proceedings of the 22nd International Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pages 297–310. Springer, 2021.

[24] K. Jansen, K. M. Klein, M. Maack, and M. Rau. Empowering the configuration-IP-new PTAS results for scheduling with setups times. In *Proceedings of the 10th Innovations in Theoretical Computer Science Conference (ITCS)*, 2019.

[25] K. Jansen, A. Lassota, and L. Rohwedder. Near-linear time algorithm for n-fold ILPs via color coding. *SIAM Journal on Discrete Mathematics*, 34(4):2282–2299, 2020.

[26] K. M. Klein. About the complexity of two-stage stochastic IPs. *Mathematical Programming*, pages 1–19, 2021.

[27] D. Knop and M. Koutecký. Scheduling meets n-fold integer programming. *Journal of Scheduling*, 21(5):493–503, 2018.

[28] D. Knop, M. Koutecký, and M. Mnich. Combinatorial n-fold integer programming and applications. *Mathematical Programming*, 184(1):1–34, 2020.

[29] M. Koutecký, A. Levin, and S. Onn. A parameterized strongly polynomial algorithm for block structured integer programs. In *Proceedings of the 45th International Colloquium on Automata, Languages, and Programming (ICALP)*, volume 107 of *LIPIcs*, pages 85:1–85:14, 2018.

[30] M. Mnich and A. Wiese. Scheduling and fixed-parameter tractability. *Mathematical Programming*, 154(1):533–562, 2015.

[31] S. Onn. Nonlinear discrete optimization. *Zurich Lectures in Advanced Mathematics, European Mathematical Society*, 2010.
[32] E. Steinitz. Bedingt konvergente reihen und konvexe systeme. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1913(143):128–176, 1913.

[33] M. Sviridenko and A. Wiese. Approximating the configuration-LP for minimizing weighted sum of completion times on unrelated machines. In *Proceedings of the 16th International Conference on Integer Programming and Combinatorial Optimization (IPCO)*, pages 387–398. Springer, 2013.

[34] E. Tardos. A strongly polynomial algorithm to solve combinatorial linear programs. *Operations Research*, 34(2):250–256, 1986.

[35] F. Xiong and K. Xing. Meta-heuristics for the distributed two-stage assembly scheduling problem with bi-criteria of makespan and mean completion time. *International Journal of Production Research*, 52(9):2743–2766, 2014.
A  Omitted contents in Section 3

A.1 Proof of Theorem 3

Theorem 3. There exists a 4-block n-fold IP where \( s_B = s_D = 1 \) such that \( \|g\|_\infty = \Omega(n) \) for some Graver basis element \( g \).

Proof. Consider the 4-block n-fold IP where its constraint matrix is defined by \( H_0 \) in which \( C = (-1,-1,-1), \quad D = (5,3), \quad B = (0,-1,1) \) and \( A = (3,4) \). Let \( g = (g^0, g^1, \cdots, g^n) \) where \( g^0 = (1,n-1,1), \) and \( g^i = g^2 = \cdots = g^n = (1,-1). \)

It is easy to see that \( \|g\|_\infty = n \). Meanwhile we have the following:

\[
Cg^0 + D \sum_{i=1}^{n} g^i = (-1,-1,-1) \cdot (1,n-1,n) + (5,3) \cdot (n,-n) = 0,
\]

\[
Bg^0 + Ag^i = (0,-1,1) \cdot (1,n-1,n) + (3,4) \cdot (1,-1) = 0, \quad \forall 1 \leq i \leq n,
\]

which means that \( H_0 g = 0 \). In what follows, we prove that \( g \) is a Graver basis element, i.e., there does not exist any non-zero \( \eta \subseteq g \) such that \( H_0 \eta = 0 \). Towards this, we assume on the contrary that there exists a vector \( \eta \subseteq g \) such that \( H_0 \eta = 0 \). Consequently, we have

\[
C\eta^0 + D \sum_{i=1}^{n} \eta^i = 0, \tag{9}
\]

\[
B\eta^0 + A\eta^i = 0, \quad \forall 1 \leq i \leq n. \tag{10}
\]

Let \( \eta^0 = (a,b,c) \) and \( \eta^i = (a_i,b_i) \). We first make the following claim.

Claim 1. \( B\eta^0 \neq 0 \).

Proof. Suppose on the contrary that \( B\eta^0 = 0 \). Given that \( \eta^i = (a_i,b_i) \subseteq (1,-1) \), we get \( a_i \in \{0,1\} \) and \( b_i \in \{0,-1\} \). By \[10\], \( A\eta^i = 3a_i + 4b_i = 0 \). Consequently, we have \( a_i = b_i = 0 \). Plug \( \eta^i = (0,0) \) into the Eq \[9\], we get \( C\eta^0 = 0 \). That is, \( (-1,-1,-1) \cdot \eta^0 = -a - b - c = 0 \). Since \( \eta^0 \subseteq (1,n-1,n) \), we get \( a,b,c \geq 0 \), implying that \( a = b = c = 0 \). Hence, \( \eta = 0 \), which is a contradiction. Thus, \( B\eta^0 \neq 0 \) and the claim follows. \( \square \)

Recall that \( a_i \in \{0,1\}, b_i \in \{0,-1\} \) for every \( i \). It is easy to see that there are three possibilities regarding the value of \( B\eta^0 \): i) \( B\eta^0 = 4 \) if \( a_i = 0, b_i = -1 \) for some \( i \). ii) \( B\eta^0 = -3 \) if \( a_i = 1, b_i = 0 \) for some \( i \). iii) \( B\eta^0 = 1 \) if \( a_i = 1, b_i = -1 \) for some \( i \). Consequently, in each case all \( a_i \)'s (\( b_i \)'s) must take the same value, i.e., there are three possibilities regarding the values of \( a_i \)'s and \( b_i \)'s:

- \( a_i = 0, b_i = -1 \) for all \( i \). Then we have \( C\eta^0 + D \sum_{i=1}^{n} \eta^i = (-1,-1,-1) \cdot \eta^0 + (5,3) \cdot (0,-n) = -(a + b + c) - 3n \). Since \( \eta^0 \subseteq (1,n-1,n) \), \( a,b,c \geq 0 \), whereas \( C\eta^0 + D \sum_{i=1}^{n} \eta^i < 0 \), contradicting Eq \[9\].

- \( a_i = 1, b_i = 0 \) for all \( i \). Then we have \( C\eta^0 + D \sum_{i=1}^{n} \eta^i = (-1,-1,-1) \cdot \eta^0 + (5,3) \cdot (n,0) = -(a + b + c) + 5n \). Using that \( \eta^0 \subseteq (1,n-1,n) \), \( a \leq 1, b \leq n-1 \) and \( c \leq n \), whereas \( C\eta^0 + D \sum_{i=1}^{n} \eta^i \geq -2n + 5n > 0 \), contradicting Eq \[9\].

- \( a_i = 1, b_i = -1 \) for all \( i \). Then we have \( C\eta^0 + D \sum_{i=1}^{n} \eta^i = (-1,-1,-1) \cdot \eta^0 + (5,3) \cdot (n,-n) = -(a + b + c) + 2n \). Using that \( a \in [0,1], b \in [0,n-1] \) and \( c \in [0,n], -(a + b + c) + 2n = 0 \) if and only if \( a = 1, b = n-1 \) and \( c = n \). Hence, \( \eta^0 = (1,n-1,n) \).

The above argument implies that \( \eta = (\eta^0, \eta^1, \cdots, \eta^n) \) where \( \eta^0 = (1,n-1,n) \) and \( \eta^i = (1,-1) \) for all \( i \), and thus \( \eta = g \), implying that \( g \) is a Graver basis element. Hence, Theorem 3 is proved. \( \square \)
A.1.1 Proof of Lemma 4

**Lemma 4.** For any $g \in \ker Z(H)$, if $g$ admits a uniform decomposition $\bar{g} = \sum_{j=1}^{N} \bar{\eta}_j$ where $\|\bar{\eta}_j\|_\infty \leq \bar{\eta}_{\max}$, then there exists $g \in \ker Z(H)$ such that $g \subseteq \bar{g}$, $B_\ell(g^0 - \bar{g}^0) = 0$ for all $i \in [n]$, and $g$ admits an $\omega$-balanced uniform decomposition for $\omega \leq (\Delta t D \bar{\eta}_{\max})^{O(s_D^2)}$. Moreover, if $\bar{g} - g \neq 0$, then we have $\bar{g} - g = g_1 + g_2 + \cdots + g_p$ for some $p \in \mathbb{Z}$ and $g_j \in \ker Z(H)$, and furthermore, $g_j \subseteq \bar{g} - g$ and $\|g_j\|_\infty \leq (\Delta t D \bar{\eta}_{\max})^{O(s_D^2)}$.

**Proof.** For simplicity let $D_0 = C$ and we consider the equation below:

\[ 0 = (D_0, D_1, \ldots, D_n) \sum_{j=1}^{N} \bar{\eta}_j = \sum_{i=0}^{n} \sum_{j=1}^{N} D_i \bar{\eta}_j. \tag{11} \]

Obviously each summand on the right side of Eq (11) is an $s_D$-dimensional vector such that $\|D_i \bar{\eta}_j\|_\infty \leq \Delta t D \bar{\eta}_{\max}$. We say $D_i \bar{\eta}_j$ is a tier-0 (or tier-1) summand if $\bar{\eta}_j$ is tier-0 (or tier-1). Consequently, there are $(n + 1)\bar{N}_0$ tier-0 summands and $(n + 1)\bar{N}_1$ tier-1 summands. According to Lemma 3 all the summands can be divided into $m'$ subsets $T_1, T_2, \ldots, T_{m'}$ such that each subset contains at most $\eta := (\Delta t D \bar{\eta}_{\max})^{O(s_D^2)}$ summands, and consequently $m' \leq (n + 1)\bar{N}_1/\eta$. It is easy to see that if $(n + 1)\bar{N}_1/\eta > (n + 1)\bar{N}_1$ (or equivalently, $\bar{N}_0 > (u - 1)\bar{N}_1$), then $m' > (n + 1)\bar{N}_1$, and by Pigeonhole principle there exists some $T_k$ such that $T_k$ does not contain any tier-1 summand. Consider such $T_k$ and let it contain summands $D_{i_1} \bar{\eta}_{j_1}^{i_1}$ to $D_{i_u} \bar{\eta}_{j_u}^{i_u}$ where every $\bar{\eta}_{j_i}$ is tier-0.

Now we let $\eta = (\eta^0, \ldots, \eta^n)$ be such that $\eta^i = \sum_{\ell: i_\ell = i} \bar{\eta}_{j_\ell}^{i_\ell}$ (specifically, $\eta^i = 0$ if $i_\ell \neq i$ for all $\ell$). Then it follows directly that $(D_0, D_1, \ldots, D_n)\eta = 0$. Furthermore, by the definition of tier-0, for $i_\ell = 0$ we have $B_\ell \bar{\eta}_{j_\ell} = 0$ for all $i \in [n]$, and for any $i_\ell \geq 1$ we have $A_{i_\ell} \bar{\eta}_{j_\ell} = 0$. Hence, $H^{12}\eta = 0$. Consequently, $H \eta = 0$ and $\eta \subseteq \bar{g}$. As $\eta$ consists of at most $u$ bricks, $\|\eta\|_\infty \leq u \bar{\eta}_{\max} = (\Delta t D \bar{\eta}_{\max})^{O(s_D^2)}$.

To summarize, as long as $\bar{N}_0 > (u - 1)\bar{N}_1$ for $u = (\Delta t D \bar{\eta}_{\max})^{O(s_D^2)}$ we can find $\eta \in \ker Z(H)$ satisfying that $H \eta = 0$, $\eta \subseteq \bar{g}$ and $\|\eta\|_\infty \leq u \bar{\eta}_{\max} = (\Delta t D \bar{\eta}_{\max})^{O(s_D^2)}$. Hence, we can iteratively apply our argument above to decompose $\bar{g}$ until it becomes $(u - 1)$-balanced, and Lemma 4 is proved.

A.2 Proof of Lemma 5

**Lemma 5.** If $g = \sum_{j=1}^{N} \eta_j$ is an $\omega$-balanced uniform decomposition where $\eta_{\max} = \max_{j \in [N]} \|\eta_j\|_\infty$, then $g$ admits a uniform decomposition $g = \sum_{j=1}^{N} \eta_j$ such that every $\eta_j$ is tier-1, and $\eta_{\max} = \max_{j \in [N]} \|\eta_j\|_\infty \leq \omega \eta_{\max}$.

**Proof.** Consider the $\omega$-balanced uniform decomposition $g = \sum_{j=1}^{N} \eta_j$ and suppose $(\omega' - 1)N_1 \leq N_0 \leq \omega' N_1$ for $1 \leq \omega' \leq \omega$. Then we pick $N_0 - (\omega' - 1)N_1 \leq N_1$ tier-0 vectors, and merge each of them with a distinct tier-1 vector. By doing so we obtain an exact $(\omega' - 1)$-balanced uniform decomposition. Next, we merge each tier-1 vector with exactly $\omega' - 1$ distinct tier-0 vectors. Then we obtain a uniform decomposition with only tier-1 vectors. It is easy to see that at most $\omega' \leq \omega$ vectors are merged together, and thus the infinity norm increases by at most $\omega$ times. Hence, Lemma 5 is true.

A.3 A figure in Section 3.1.1

Notice that two adjacent bricks in the same megazone in a column are not necessarily belonging to the same zone. Two adjacent bricks in the same zone in a column are not necessarily belonging to the same subzone.
A.4 Proof of Lemma 7

Lemma 7. Let \( x_1, \ldots, x_M \in \mathbb{Z}^d \) be a sequence of vectors such that \( \|x_i\|_\infty \leq \zeta \) for some \( \zeta \geq 1 \) and every \( i = 1, \ldots, M \). Furthermore, there are \( \mu \) colors, and each vector \( x_i \) is associated with one color. There are in total \( \alpha \sum_j \mu \) vectors of color \( j \) where \( \alpha \sum_j \mu = \alpha, M = \alpha \mu \).

Supposing that \( \sum_{i=1}^{M} x_i = 0 \) and \( M \) is sufficiently large (i.e., \( M > (2d\zeta + 2\mu\zeta + 1)^{d+\mu} + \alpha + d + \mu \)), then among \( x_1, \ldots, x_M \) we can find \( \alpha \mu \) vectors of each color \( j \) such that their summation is \( 0 \), and \( m \leq (2d\zeta + 2\mu\zeta + 1)^{d+\mu} \).

Proof. We lift the vectors in \( \mathbb{Z}^d \) to \( \mathbb{Z}^{d+\mu} \) such that if \( x_i \) is of color \( j \), then it is mapped to \( y_i = (x_i, e_j) \) where \( e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \) is the vector with its \( j \)-th coordinate being 1. Given that there are \( \alpha \mu \) vectors of color \( j \), we have:

\[
\sum_{i=1}^{M} y_i = (0, 0, \ldots, 0, \alpha_1 \sum_j \mu, \alpha_2 \sum_j \mu, \ldots, \alpha_\mu \sum_j \mu).
\]

Denote by \( y \) the right side of the above equation. Note that \( \|y_i\|_\infty \leq \zeta \). Applying the Steinitz Lemma, then there exists a permutation \( \pi \) such that for every \( \ell \in [M] \) we have

\[
\|\sum_{i=1}^{\ell} y_{\pi(i)} - \frac{\ell - d - \mu}{M} y\|_\infty \leq (d + \mu)\zeta.
\]

Notice that \( M = \alpha \sum_j \mu \), and we have

\[
\frac{\ell - d - \mu}{M} y = (\ell - d - \mu) \cdot (0, 0, \ldots, 0, \frac{\alpha_1}{\alpha}, \frac{\alpha_2}{\alpha}, \ldots, \frac{\alpha_\mu}{\alpha}),
\]

(12)
and note that if \( \ell - d - \mu \) is a multiple of \( \alpha \), then the right side is an integral vector. Consider 
\[ \ell_k = ka + d + \mu \text{ for } k \in \mathbb{Z}_{> 0}, \]
so by Eq (12) 
\[ z_k := \sum_{i=1}^{\ell_k} y_{\pi(i)} - \frac{\ell_k - d - \mu}{M} y \] is a \((d + \mu)\)-dimensional integral vector whose \( \ell_\infty \)-norm is bounded by \((d + \mu)\). Hence, there are at most \((2d\zeta + 2\mu\zeta + 1)^{d+\mu}\) distinct \(z_k\)'s, which implies that if \( M \) is large enough (and thus induces sufficiently many \(z_k\)'s), i.e., 
\[ M > ((2d\zeta + 2\mu\zeta + 1)^{d+\mu + 1}) \alpha + d + \mu, \]
then there must exist two integers \(k_1, k_2 \leq (2d\zeta + 2\mu\zeta + 1)^{d+\mu + 1}\) such that 
\[ z_{k_1} = z_{k_2}, \]
and consequently 
\[ \sum_{i=\lfloor k_1 \alpha + d + \mu \rfloor}^{k_2 \alpha + d + \mu} y_{\pi(i)} = \frac{(k_2 - k_1)\alpha}{M} y = (k_2 - k_1) \cdot (0, 0, \ldots, 0, \alpha_1, \alpha_2, \ldots, \alpha_\mu). \]

This means, we have found a subset of \(x_i\)'s with at most \((2d\zeta + 2\mu\zeta + 1)^{d+\mu}\) vectors which add up to 0, and furthermore, the total number of vectors of each color \(j\) is proportional to \(\alpha_j\). \(\square\)

### A.5 Proof of Lemma 11

To prove Lemma 11 we need the following lemma.

**Lemma 13.** Let \(x_1, x_2, \ldots, x_m \in \mathbb{Z}_{> 0}\) with \(x_i \leq \zeta\) for \(i \in [m]\) and \(\sum_{i=1}^{m} x_i = x\). If \(x\) is a multiple of \((\zeta + 1)\), then the \(m\) integers can be partitioned into \(m' = x / (\zeta + 1)!\) subsets \(T_1, T_2, \ldots, T_{m'}\) such that \(\bigcup_{k=1}^{m'} T_k = [m]\), and \(\sum_{i \in T_k} x_i = (\zeta + 1)!\) for all \(k \in [m']\).

**Proof.** Since \(x\)’s can only take at most \(\zeta\) distinct values, we let \(u_j\) be the total number of \(x_i\)’s taking the value \(j \leq \zeta\). We can divide the \(u_j\) numbers into \(\lceil \frac{u_j}{\zeta} \rceil\) groups, with all except 1 group containing \(\frac{\zeta}{\zeta}\) numbers, and one group containing \(\text{res}_j = u_j - \frac{\zeta}{\zeta} \lceil \frac{u_j}{\zeta} \rceil\) numbers. Consequently, we obtain a grouping of \(x_i\)’s such that there are \(\zeta\) groups where the summation of numbers inside is \(j \cdot \text{res}_j < \zeta\), together with \(h\) other groups where the summation of numbers inside any group is exactly \(\zeta\). Notice that 
\[ x = \sum_{j=1}^{\zeta} j \cdot \text{res}_j + \zeta \cdot h, \]
and \(x\) is a multiple of \(\zeta\), hence \(\sum_{j=1}^{\zeta} j \cdot \text{res}_j \leq \zeta \cdot \zeta!\) is also a multiple of \(\zeta\), and we let it be \(a\zeta!\) for \(a \leq \zeta\). Agglomerating these \(\zeta\) groups, we obtain \(h + 1\) groups, where the summation of numbers within 1 group (called extra group) is \(a\zeta!\) and the summation of numbers within any other group (called regular group) is exactly \(\zeta\). Given that \(x = a\zeta! + h\zeta!\) is a multiple of \((\zeta + 1)!\), we can further agglomerate the extra group with \(\zeta + 1 - a\) regular groups, that is, 
\[ x = [a\zeta! + (\zeta + 1 - a)\zeta!] + (h - \zeta - 1 + a)\zeta!, \]
and then the remaining regular groups are evenly divided into subsets such that each subset contains \(\zeta + 1\) regular groups. This is possible since \(x - (\zeta + 1)!\) is a multiple of \((\zeta + 1)!\). It is easy to see that now the numbers within every agglomerated group sum up to \((\zeta + 1)!\), and Lemma 13 is proved. \(\square\)

Now we are ready to prove Lemma 11.

**Lemma 11.** Let \(x_1, x_2, \ldots, x_m \in \mathbb{Z}\) and \(\zeta \in \mathbb{Z}_{>0}\) be integers such that \(|x_i| \leq \zeta\) for \(i \in [m]\) and \(\sum_{i=1}^{m} x_i = x\). If \(x\) is a multiple of \((6\zeta^2 + 2\zeta + 1)!\), then the \(m\) integers can be partitioned into \(m'\) subsets \(T_1, T_2, \ldots, T_{m'}\) such that \(\bigcup_{k=1}^{m'} T_k = [m]\), and for all \(k \in [m']\) it holds that \(|T_k| \leq 2^{O(\zeta^2 \log \zeta)}\), 
\[ \sum_{i \in T_k} x_i \in \{0, \text{sgn}(x) \cdot (6\zeta^2 + 2\zeta + 1)!\} \]
where \(\text{sgn}\) denotes the standard sign function such that \(\text{sgn}(x) = 1\) if \(x > 0\), \(\text{sgn}(x) = -1\) if \(x < 0\), and \(\text{sgn}(x) = 0\) if \(x = 0\).
Proof. Without loss of generality, we assume \( x \geq 0 \) (If \( x < 0 \), we simply apply the argument below to the sequence of \( -x_i \)’s). Notice that \( x_i \)’s do not necessarily lie in the same orthant. We first apply Lemma 3 to the sequence of \( x_i \)’s, and obtain a partition of \([m]\) into \( m_1 \) subsets \( T_1', T_2', \cdots, T_{m_1}' \) such that for every \( k \in [m_1] \), \( \sum_{i \in T_j' \atop x_i} x_i \subseteq x \) and \( |T_j'| \leq 6\zeta + 2 \). Let \( y_j = \sum_{i \in T_j'} x_i \). If \( x = 0 \), then \( y_j = 0 \) for all \( j \). Lemma 11 is proved. Otherwise \( x > 0 \), and it follows that \( y_j \geq 0 \) for all \( j \). Consider all \( y_j \)’s which are positive. Without loss of generality, let them be \( y_1, y_2, \cdots, y_{m_2} \). We know that \( y_j > 0 \) for \( j \in [m_2] \), \( y_j \leq \zeta(6\zeta + 2) \) and \( \sum_j y_j = x \) where \( x \) is a multiple of \((6\zeta^2 + 2\zeta + 1)!\). Applying Lemma 13, we can obtain a partition of \([m_2]\) into \( m_3 \) subsets \( T_1'', T_2'', \cdots, T_{m_3}'' \) such that \( \sum_{i \in T_k''} y_j = (6\zeta^2 + 2\zeta + 1)! \) for all \( k \in [m_3] \). Given that \( y_j = \sum_{i \in T_j'} x_i \), we let \( T_k = \{ i : \text{there exists some } j \in T_k' \text{ such that } i \in T_j' \} \), then it is clear that \( \sum_{i \in T_k} x_i = (6\zeta^2 + 2\zeta + 1)! \). Further, \( y_j = 0 \) for \( j > m_2 \). We simply let \( T_j' \) be \( T_{j-m_2+m_3} \). Now it is easy to verify that we obtain a partition of \([m]\) into \( m' = m_1 - m_2 + m_3 \) subsets \( T_k \)'s such that \( |T_k| \leq (6\zeta^2 + 2\zeta + 1)! \). \((6\zeta^2 + 2\zeta + 1)! = 2^{O(\zeta^2 \log \zeta)} \), and \( \sum_{i \in T_k} x_i \in \{0, (6\zeta^2 + 2\zeta + 1)!\} \). Hence, Lemma 11 is proved.

A.6 Proof of Lemma 12

Lemma 12 Let \( g \in \ker_Z(H_{\text{com}}) \). Let

\[
\lambda = (6\lambda_0^2 + 2\lambda_0 + 1)! = 2^{2^2O(\lambda_0^2 \log \Delta)}, \quad \text{where } \lambda_0 := \Delta t_B g_\infty(H_{\text{two-stage}}^{\text{com}}) = 2^{2^2O(\lambda_0^2 \log \Delta)}.
\]

If \( Bg^0 \) is a multiple of \( \lambda \), then \( g \) admits a uniform decomposition \( g = \sum_{j=1}^N \eta_j \) such that \( \|\eta_j\|_\infty \leq 2^{2^2O(\lambda_0^2 \log \Delta)} \). Furthermore, \( B\eta_j^0 \) is a multiple of \( \lambda \) for all \( j \).

Proof. As \( g \in \ker_Z(H_{\text{com}}^{\text{two-stage}}) \), there exists some integer \( L' \) and \( \xi_j \)’s such that: \( g = \xi_1 + \xi_2 + \cdots + \xi_{L'} \), where for all \( j \) it holds that \( H_{\text{com}}^{\text{two-stage}} \xi_j = 0 \), \( \|\xi_j\|_\infty \leq g_\infty(H_{\text{com}}^{\text{two-stage}}) \), and \( \xi_j \subseteq g \). Consequently, \( B\xi_j^0 \leq \Delta t_B g_\infty(H_{\text{com}}^{\text{two-stage}}) = \lambda_0 \). Consider the sequence \( B\xi_1^0, B\xi_2^0, \cdots, B\xi_{L'}^0 \). It is clear that \( |B\xi_j^0| \leq \lambda_0 \) and \( \sum_j |B\xi_j^0| = Bg^0 \) is a multiple of \( \lambda \). According to Lemma 11, we can partition \([L']\) into \( m' \) subsets \( T_1, T_2, \cdots, T_{m'} \) such that \( \sum_{k=1}^{m'} T_k = [m] \), and for all \( k \in [m'] \) it holds that \( |T_k| \leq 2^O(\lambda_0^2 \log \lambda_0) \). \( \sum_{i \in T_k} B\xi_j^0 \in \{0, \lambda \cdot \text{sgn}(Bg^0)\} \). Let \( \eta_j = \sum_{j \in T_k} \xi_j \). According to the definition in Eq (8), we get that \( g = \sum_{j=1}^{m'} \eta_j \) is a uniform decomposition.

A.7 Extension of Theorem 1 to almost combinatorial 4-block \( \nu \)-fold IP

The extension of Theorem 1 to almost combinatorial 4-block \( \nu \)-fold IP is straightforward. For the completeness of the paper, we give the formal proof below.

When considering such an \( s_B \times t_B \) matrix \( B \) with rank 1, we can always transform \( B \) into \( \bar{B} \), in which the first row is \( r_1^\top \), and all the other rows are \( 0 \). It implies that when \( \text{rank}(B) = 1 \), it is sufficient to consider such a case \( B = (r_1, 0, \ldots, 0)^\top \), where \( r_1 \neq 0 \).

Thus, without loss of generality, we assume that all almost combinatorial 4-block \( \nu \)-fold matrices always have the common feature that \( B = (r_1, 0, \ldots, 0)^\top \), where \( r_1 \neq 0 \). From now on we denote by \( \bar{H}_{\text{com}} \) an almost combinatorial 4-block \( \nu \)-fold matrix, and by \( \bar{H}_{\text{two-stage}} \) the two-stage stochastic matrix obtained by removing \((C, D_1, \cdots, D_\nu)\) from \( \bar{H}_{\text{com}} \).

We first have a similar result to Lemma 12.

Lemma 14. Let \( g \in \ker_Z(\bar{H}_{\text{com}}) \). Let

\[
\lambda = (6\lambda_0^2 + 2\lambda_0 + 1)! = 2^{2^2O(\lambda_0^2 \log \Delta)}, \quad \text{where } \lambda_0 := \Delta t_B g_\infty(\bar{H}_{\text{two-stage}}^{\text{com}}) = 2^{2^2O(\lambda_0^2 \log \Delta)}.
\]
Let \(B^g_0 = (Bg^0[1], 0, \cdots, 0)\). If \(B^g_0[1]\) is a multiple of \(\lambda\), then \(g\) admits a uniform decomposition \(g = \sum_{j=1}^N \eta_j\) such that \(\|\eta_j\|_{\infty} \leq 2^{2^{O(t_B \log \Delta)}} \). Furthermore, \(B^\eta_0 = (B^g_j[1], 0, \cdots, 0)\) where \(B^g_j[1]\) is a multiple of \(\lambda\) for all \(j\).

**Proof.** As \(g \in \ker_H\), there exists some integer \(L'\) and \(\xi_j\)'s such that: \(g = \xi_1 + \xi_2 + \cdots + \xi_{L'}, \) where for all \(j\) it holds that \(\hat{H}_{\text{com}} \xi_j = 0, \) \(\|\xi_j\|_{\infty} \leq g_{\infty}(\hat{H}_{\text{com}})\), and \(\xi_j \subseteq g\). Consequently, \(B^0_j = (B^g_0[1], 0, \cdots, 0)\) where \(B^\eta_j[1] \leq \Delta t_B g_{\infty}(\hat{H}_{\text{com}}) = \lambda_0\). Consider the sequence \(B\xi_0^0[1], B\xi_2^0[1], \cdots, B\xi_j^0[1]\). It is clear that \(|B\xi_j^0[1]| \leq \lambda_0\) and \(\sum_j B\xi_j^0[1] = B^g_0[1]\) is a multiple of \(\lambda\). According to Lemma 8, we can partition \([L']\) into \(m'\) subsets \(T_1, T_2, \cdots, T_{m'}\) such that \(\bigcup_{k=1}^{m'} T_k = [m]\), and for all \(k \in [m']\) it holds that \(|T_k| \leq 2^{O(\lambda_0^2 \log \lambda_0)}, \) \(\sum_{j \in T_k} B\xi_j^0[1] \in \{0, \lambda \cdot \text{sgn}(B^g_0)\}\). Let \(\eta_j = \sum_{j \in T_k} \xi_j\). According to the definition in Eq. (8), we get that \(g = \sum_{j=1}^{m'} \eta_j\) is a uniform decomposition.

**Theorem 8.** Let \(\hat{H}_{\text{com}}\) be an almost combinatorial 4-block \(n\)-fold matrix. Then there exists a positive integer \(\lambda \leq 2^{2^{O(t_B^2 \log t_B \Delta)}}\) (which is only dependent on \(t_B\) and \(\Delta\)) such that for any \(g \in \ker_H\), we have \(\lambda g = \sum_{j=1}^N \eta_j\) and \(\|g\|_{\infty} = 2^{2^{O(\lambda^2 \log \Delta)}} \cdot 2^{2^{O(t_B^2 \log \Delta)}} \cdot 2^{O(\lambda^2 \log \Delta)}\).

**Proof.** Consider any \(g \in \ker_H\). Clearly \(B(\lambda g^0) = \lambda B^0 = (\lambda B^g_0[1], 0, \cdots, 0)\) where \(\lambda B^g_0[1]\) is a multiple of \(\lambda\), thus by Lemma 14 \(\lambda g\) admits a uniform decomposition \(\lambda g = \sum_{j=1}^N \eta_j\) where \(\|\eta_j\|_{\infty} \leq \eta_{\text{max}} = 2^{2^{O(t_B^2 \log \Delta)}}\) and every \(B\eta_j^0 = (B\eta_j^0[1], 0, \cdots, 0)\) where \(B\eta_j^0[1]\) is a multiple of \(\lambda\).

If this decomposition is not \(\omega\)-balanced for \(\omega \leq (\Delta t_B \eta_{\text{max}})^{O(s_B^0)}\), then by Lemma 4 we obtain \(\eta \subseteq \lambda g\) with \(\|\eta\|_{\infty} \leq (\Delta t_B \eta_{\text{max}})^{O(s_B^0)}\), \(\eta \in \ker_H\) and \(B(\lambda g - \eta) = (\lambda g - \eta)^0 = 0\). \(B\eta^0[1]\) is a multiple of \(\lambda\). Otherwise this decomposition is \(\omega\)-balanced. By Lemma 5 we can obtain a uniform decomposition \(\lambda g = \sum_{j=1}^N \eta_j\) such that \(\max_j \|\eta_j\|_{\infty} \leq \omega \eta_{\text{max}}\) and all \(\eta_j\)'s are \(1\)-tie. According to Lemma 6 if \(\lambda \|g\|_{\infty} > \tau\) for \(\tau = (\omega \Delta \eta_{\text{max}})^{\Delta^O(s_B^0 + \Delta t_B)}\), then we are able to find some \(\eta \subseteq \lambda g\) such that \(\hat{H}_{\text{com}} \eta = 0, \) \(\|\eta\|_{\infty} = \mathcal{O}_{\text{FPT}}(1)\) and \(\eta^0 = \sum_{j \in S} \eta_j^0\) for some \(S \subseteq [N]\). As every \(B\eta^0_j = (B\eta^0_j[1], 0, \cdots, 0)\) satisfies that \(B\eta^0_j[1]\) is a multiple of \(\lambda\), we know \(B\eta^0 = (B\eta^0_0[1], 0, \cdots, 0)\) where \(B\eta^0_0[1]\) is also a multiple of \(\lambda\). In both cases, we find \(\eta \subseteq \lambda g\) where \(B\eta^0_0 = (B\eta^0_0[1], 0, \cdots, 0)\), and \(B\eta^0_0[1]\) is a multiple of \(\lambda\).

Now consider \(\lambda g - \eta\). Obviously \(\lambda g - \eta \in \ker_H\). It is easy to see \(B(\lambda g - \eta) = (x, 0, \cdots, 0)\) where \(x\) is a multiple of \(\lambda\). Thus, if \(\|\lambda g - \eta\|_{\infty} > \tau\) we can continue to decompose \(\lambda g - \eta\) using our argument above. Hence, Theorem 8 is proved.

**B Omitted contents in Section 4**

The goal of this section is to develop algorithms for combinatorial 4-block \(n\)-fold IP. Towards this, we first bound the infinity norm of Graver basis elements.

**Theorem 4.** Let \(g \in \mathcal{G}(H_{\text{com}})\) be a Graver basis element, then \(\|g\|_{\infty} = g_{\infty}(H_{\text{com}})\) where \(g_{\infty}(H_{\text{com}}) \leq 2^{2^{O(\lambda \log \Delta + \Delta t_B \log \Delta)}} \cdot 2^{2^{O(t_B^2 \log \Delta)}} \cdot n = \mathcal{O}_{\text{FPT}}(n)\).

**Proof.** According to Theorem 1 we know for \(\lambda = \mathcal{O}_{\text{FPT}}(1)\) there exist \(g_j \in \ker_H\) such that \(\lambda g = \sum_{j=1}^p g_j, g_j \subseteq \lambda g\) and \(\|g\|_{\infty} = \mathcal{O}_{\text{FPT}}(1)\). To show \(\|g\|_{\infty} = \mathcal{O}_{\text{FPT}}(n)\), it suffices to show that \(p = \mathcal{O}_{\text{FPT}}(n)\). Note that if any \(g_j \not\subseteq g\), then it will violate the fact that \(g\) is a Graver basis.
element. Let \( x[h] \) denote the \( h \)-th coordinate of a vector \( x \). We know \( g_j \not\subseteq g \) implies that there exists some \( h_j \)-th coordinate such that \( |g_j[h_j]| \geq |g[h_j]| \), and we call \( h_j \) as the critical coordinate of \( g_j \). If there are multiple critical coordinates, we pick an arbitrary one. Now we have a list of critical coordinates \( h_1, h_2, \ldots, h_p \) where \( 1 \leq h_j \leq t_B + n_A \). We claim that every index \( k \in [t_B + n_A] \) can occur at most \( \lambda \) times in the list. Supposing on the contrary some index \( k \) appears \( \lambda + 1 \) or more times, then there exist \( g_{j_1}, g_{j_2}, \ldots, g_{j_{\lambda+1}} \) where everyone’s \( k \)-th coordinate has an absolute value no less than \( |g[k]| \). However, \( \sum_{\ell=1}^{\lambda+1} g_{j_{\ell}} \subseteq \lambda g \) implies that the summation of the absolute value of their \( k \)-th coordinates is bounded by \( |\lambda g[k]| \), which is a contradiction. Hence, every index occurs at most \( \lambda \) times in the list, implying that \( p \leq \lambda (t_B + n_A) \). Hence, Theorem 4 is proved. More precisely,
\[
g_\infty(H_{\text{com}}) \leq 2^{2O(t_A \log \Delta + sD + \log \Delta)} \cdot 2^{O(t_B \log \Delta)} \cdot n.
\]

\[ \square \]

**Remark.** As Theorem 7 remains true for almost combinatorial 4-fold IP where \( \text{rank}(B) = 1 \), Theorem 4 is also true for almost combinatorial 4-fold IP. Denote by \( H_{\text{com}} \) the constraint matrix of almost combinatorial 4-fold IP, and denote by \( g_\infty(H_{\text{com}}) \) the upper bound on its Graver basis elements, then we have
\[
g_\infty(H_{\text{com}}) \leq 2^{2O(s_A^2 \log \Delta + sD + \log \Delta)} \cdot 2^{O(t_B^2 \log \Delta)} \cdot n.
\]

Now we are ready to design FPT algorithms for combinatorial 4-fold IP using the iterative augmentation framework.

### B.1 Linear Objective Functions

**Theorem 9.** Combinatorial 4-fold IP with a linear objective function \( f(x) = wx \) can be solved in time:
\[
2^{2O(t_A \log \Delta + sD + \log \Delta)} \cdot 2^{O(t_B^2 \log \Delta)} \cdot n^{5+o(1)} = O_{\text{FPT}}(n^{5+o(1)}).
\]

**Proof.** Utilizing the idea of approximate Graver-best oracle introduced by Altmanová et al. [3] and implicitly by Eisenbrand et al. [12], it is sufficient that for every \( \rho = 2^0, 2^1, 2^2, \ldots, 2^h \) where \( h = O(n^{1+o(1)} \log \Delta) \) we find out an augmentation of the form \( \rho y \) which is no worse than \( \rho g \) for any Graver basis element \( g \) (i.e., \( \rho y \) gives an improvement to the objective value larger than or equal to any \( \rho g \)). Observing that \( Bg^0 \in \mathbb{Z} \) and \( |Bg^0| \leq t_B \Delta g_\infty(H_{\text{com}}) \) for every Graver basis element, we consider the following IP(\( \rho, \phi \)) for every fixed \( \rho \) and \( \phi \in [-t_B \Delta g_\infty(H_{\text{com}}) : t_B \Delta g_\infty(H_{\text{com}})] \):
\[
\min \{ w \cdot y : H_{\text{com}}y = 0, 1 \leq x_0 + \rho y \leq u, By^0 = \phi, y \in \mathbb{Z}^{t_B n + n_A} \}.
\]

(13)

It is clear that the optimal solution \( y^*(\rho, \phi) \) to IP (13) is no worse than \( \rho g \) for any Graver basis element satisfying that \( Bg^0 = \phi \). Taking the best solution out of all \( y^*(\rho, \phi) \) gives the desired augmentation.

We write down explicitly the constraints of IP(\( \rho, \phi \)) as follows:
\[
Cy^0 + \sum_{i=1}^{n} D_i y^i = 0
\]
\[
By^0 = \phi
\]
\[
A_i y^i = -\phi, \quad \forall 1 \leq i \leq n
\]

\[1\]Here \( h \leq \log \|u - I\|_\infty \). However, utilizing the techniques of Tardos [31], Koutecký et al. [29] showed that without loss of generality \( \|b\|_\infty, \|l\|_\infty, \|u\|_\infty \leq 2^{O(n \log n)} \Delta^{O(n)} \).

24
The constraint matrix $H_{(1)}$ is as follows:

$$H_{(1)} = \begin{pmatrix}
C & D_1 & D_2 & \cdots & D_n \\
B & 0 & 0 & \cdots & 0 \\
0 & A_1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & & & A_n & 0
\end{pmatrix}.$$ 

Hence, IP$(\rho, \phi)$ is a generalized $n$-fold IP. Using the algorithm of Cslovjecsek et al. [9], it can be solved in time $2^{O(t_A \log D + s \log A)}(n^{2+o(1)})$. We assume that the objective function $f$ is separable if there are convex functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \sum_{i=0}^n f_i(x_i) = \sum_{j=1}^{t_B} f_j(x_j^0) + \sum_{i=1}^{t_A} \sum_{j=1}^{t_A} f_j(x_j^i)$. Henceforth, we consider the problem

$$\min \{ f(x) : H_{\text{com}} x = b, 1 \leq x \leq u, x \in \mathbb{Z}^{t_B + n t_A} \},$$

(14)

Using the same argument but replacing $g_{\infty}(H_{\text{com}})$ with $g_{\infty}(\hat{H}_{\text{com}})$, we have the following:

**Corollary 1.** Almost combinatorial 4-block $n$-fold IP with a linear objective function $f(x) = wx$ can be solved in time:

$$2^{2^{O(t_A \log D + s \log A)}(\log \|D\| + \log \|A\|)} \cdot n^{5+o(1)} = O_{FPT}(n^{5+o(1)}).$$

### B.2 Separable Convex Objective Functions

We consider a separable convex objective function. A convex function $f : \mathbb{R}^{t_B + n t_A} \rightarrow \mathbb{R}$ is called separable if there are convex functions $f_j^i : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \sum_{i=0}^n f_i^j(x_i) = \sum_{j=1}^{t_B} f_j^0(x_j^j) + \sum_{i=1}^{t_A} \sum_{j=1}^{t_A} f_j^i(x_j^j)$. Henceforth, we consider the problem

$$\min \{ f(x) : H_{\text{com}} x = b, 1 \leq x \leq u, x \in \mathbb{Z}^{t_B + n t_A} \},$$

(14)

We assume that the objective function $f$ is presented by an evaluation oracle that, when queried on a vector $x$, returns the values $f^i(x^i)$ for all $i = 0, 1, \ldots, n$. The time complexity now measures the number of arithmetic operations and oracle queries.

**Theorem 5.** Consider combinatorial 4-block $n$-fold IP with a separable convex objective function $f$ mapping $\mathbb{Z}^{t_B + n t_A}$ to $\mathbb{Z}$. Let $P$ be the set of feasible integral points, and let $\tilde{f} := \max_{x \in P}(f(x) - f(y))$. Then (14) can be solved in $O_{FPT}(n^4 \tilde{L}^2 \log^2(\tilde{f}))$ time. More specifically, the running time is

$$2^{2^{O(t_A \log D + s \log A)}(\log \|D\| + \log \|A\|)} \cdot n^4 \tilde{L}^2 \log^2(\tilde{f}),$$

where $\tilde{L}$ denotes the logarithm of the largest number occurring in the input.

**Proof.** We use a similar idea as Theorem 9. It has been shown by Eisenbrand et al. [12] that for a separable convex function, it is still sufficient that for every $\rho = 2^0, 2^1, 2^2, \ldots, 2^{O(\log \|u - l\|_\infty)}$, we find out an augmentation of the form $\rho y$ which is no worse than $\rho g$ for any Graver basis element.
Hence, similarly, for every $\rho$ and every $\phi \in [-t_B \Delta g, t_B \Delta g]$, we solve the following:

$$\min\{f(x_0 + \rho y) - f(x_0) : H_{\text{com}} y = 0, 1 \leq x_0 + \rho y \leq u, By^0 = \phi, y \in \mathbb{Z}^{t_B + nt_A}\}.$$ 

The above is a generalized $n$-fold IP with a separable convex objective function, which can be solved in time of $n^2 t_B^2 \log(f) \tilde{O}(s \Delta)$ \cite{12}, where $t = \max\{t_A, t_B\}$.

The number of augmenting steps can be bounded by $(2n - 2) \log(\hat{f})$ \cite{10}. Hence, the overall running time is

$$2^{2O(t_A \log \Delta + s \Delta)} \cdot 2^{O(t_B \log \Delta)} \cdot n^4 \hat{L}^2 \log^2(\hat{f}) = O_{FPT}(n^4 \hat{L}^2 \log^2(\hat{f})).$$

Using the same argument but replacing $g(H_{\text{com}})$ with $g(\tilde{H}_{\text{com}})$, we have the following:

**Corollary 2.** Consider almost combinatorial 4-block $n$-fold IP with a separable convex objective function $f$ mapping $\mathbb{Z}^{t_B + nt_A}$ to $\mathbb{Z}$. Let $P$ be the set of feasible integral points for (14), and let $\hat{f} := \max_{x,y \in P} (f(x) - f(y))$. Then (14) can be solved in $O_{FPT}(n^4 \hat{L}^2 \log^2(\hat{f}))$ time. More specifically, the running time is

$$2^{2O(t_A \log \Delta + s \Delta)} \cdot 2^{O(t_B \log \Delta)} \cdot n^4 \hat{L}^2 \log^2(\hat{f}),$$

where $\hat{L}$ denotes the logarithm of the largest number occurring in the input.

## C Omitted contents in Section 5

Scheduling is a fundamental problem in operations research and computer science. The classical scheduling problem as well as its generalizations have been studied extensively in the literature. In particular, approximation algorithms have been developed for scheduling with rejection cost (see, e.g., \cite{14, 22, 33}), and scheduling with the bicriteria of makespan and (weighted) total completion time (see, e.g., \cite{1, 2, 8, 35}). In recent years, FPT algorithms have been developed for the classical scheduling problems \cite{27, 30}. However, not much is known regarding how these algorithms can be generalized to deal with more sophisticated scheduling models. In particular, FPT algorithms have been developed for single machine scheduling with rejection cost \cite{30}, while FPT algorithms for parallel machines are still unknown. FPT algorithms for bicriteria scheduling are also unknown.

In this section, we show that combinatorial 4-block $n$-fold IP offers a strong tool for dealing with these generalizations on the classical scheduling problems.

### C.1 Scheduling with rejection

We restate our problem $R||C_{\text{max}} + E$ here. Given are $m$ machines and $k$ different types of jobs, with $N_j$ jobs of type $j$. A job of type $j$ has a processing time of $p_{ij} \in \mathbb{Z}_{\geq 0}$ if it is processed by machine $i$. Every job of type $j$ also has a rejection cost $u_j$. A job is either processed on one of the machines, or is rejected. The goal is to minimize the makespan $C_{\text{max}}$ plus the total rejection cost $E$, where makespan denotes the largest job completion.

FPT algorithms for scheduling with rejection has been considered by Mnich and Wiese \cite{30}. However, they considered single machine scheduling with rejection. We are not aware of FPT algorithms for parallel machine scheduling with job rejection cost.

The goal of this subsection is to prove the following.
Theorem 6. $R||C_{\text{max}} + E$ can be solved in $m^{5+o(1)}2^{O(k^2 \log p_{\text{max}})}2^{O(\log p_{\text{max}})} + |I|$ time, where $|I|$ denotes the length of the input.

Proof. We model the scheduling problem with rejection cost as a combinatorial 4-block $n$-fold IP to solve it. Let $x^i_j \in \mathbb{Z}_{\geq 0}$ denote the total number of jobs of type $j$ assigned to machine $i$ in a schedule, and $C_{\text{max}}$ be the makespan. Then we have the following IP_schel:

$$\min \quad C_{\text{max}} + \sum_{j=1}^k u_j (N_j - \sum_{i=1}^m x^i_j)$$

$$\sum_{i=1}^m x^i_j \leq N_j, \quad \forall 1 \leq j \leq k \quad (15)$$

$$\sum_{j=1}^k p^i_j x^i_j - C_{\text{max}} \leq 0, \quad \forall 1 \leq i \leq m \quad (16)$$

$$x^i_j \in \mathbb{Z}_{\geq 0}$$

Here Constraint (15) indicates that the total number of type-$j$ jobs being processed is at most $N_j$. Constraint (16) indicates that the total job processing time on every machine is bounded by the makespan $C_{\text{max}}$. Let all variables be ordered as a vector $x = (x^1, x^2, \ldots, x^m)$, where $x^i = (x^1_i, x^2_i, \ldots, x^m_i)$. It is easy to see that IP_schel has the following constraint matrix:

$$H(3) = \begin{pmatrix}
0 & I & I & I & \cdots & I \\
-1 & p^1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & p^2 & 0 & \cdots & 0 \\
-1 & 0 & 0 & p^3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & 0 & \cdots & p^m
\end{pmatrix},$$

where $p^i = (p^i_1, p^i_2, \ldots, p^i_k)$. Hence, IP_schel is a combinatorial 4-block $n$-fold IP with a linear objective function. IP_schel can be solved in $m^{5+o(1)}2^{O(k^2 \log p_{\text{max}})}2^{O(\log p_{\text{max}})} + |I|$ time by using Theorem 9, where $|I|$ denotes the input size of the given problem. More precisely, $|I|$ is bounded by $O(kp_{\text{max}}(\max\{\log N_{\text{max}}, \log u_{\text{max}}\}))$ where $N_{\text{max}} = \max_j N_j$ and $u_{\text{max}} = \max_j u_j$. \qed

Remark. One may suspect that IP_schel can be solved through the generalized $n$-fold IP by guessing out the value of $C_{\text{max}}$. However, this will require $p_{\text{max}} \cdot \max_j N_j$ enumerations.

C.2 Scheduling with the objective of minimizing weighted completion time plus makespan

We restate our problem $R||\theta C_{\text{max}} + \sum_{\ell} w_{\ell} C_{\ell}$ here. Given are $m$ machines and $k$ different types of jobs, with $N_j$ jobs of type $j$. A job of type $j$ has a processing time of $p^i_j \in \mathbb{Z}_{\geq 0}$ if it is processed by machine $i$. Each job $\ell$ of type $j$ also has a weight $w_{\ell}$, and the goal is to find an assignment of jobs to machines such that $\theta C_{\text{max}} + \sum_{\ell} w_{\ell} C_{\ell}$ is minimized, where $C_{\ell}$ is the completion time of job $\ell$, $C_{\text{max}}$ is the largest job processing time, and $\theta$ is a fixed input value.

FPT algorithms for $R||C_{\text{max}}$ and $R||\sum_{\ell} w_{\ell} C_{\ell}$ have been developed by Knop and Koutecky [27]. However, their technique does not generalize to bicriteria as the natural IP formulation becomes 4-block $n$-fold, as we will show below.

The goal of this subsection is to prove the following.
Theorem 7. $R | \theta C_{\text{max}} + \sum_\ell w_\ell C_\ell$ can be solved in $m^4 2^{O(k^2 \log P_{\text{max}})} 2^{O(\log P_{\text{max}})} |I|^4$ time, where $|I|$ denotes the length of the input.

Proof. Again we model the scheduling problem with combinatorial 4-block $n$-fold IP. Towards this, we need to transform the objective function to a separable convex function. Such a transformation has been achieved by Knop and Koutecký [27]. For completeness of the paper, we briefly recap their transformation here.

Consider jobs scheduled on each machine $i$. Assume a set of jobs $J^i := \{J_1, \ldots, J_h\}$ will be scheduled on the machine $i$ such that $\delta_i(q) \geq \delta_i(q + 1)$ for all $1 \leq q \leq h - 1$, where $\delta_i(q) := w_q/p_q^i$. We denote $\delta_i(h + 1) = 0$. It is clear that these jobs will be scheduled according to the Smith rule, and thus in the sequence of $J_1, J_2, \ldots, J_h$. Denote by $C_q^i$ the completion time of job $J_q$ on this machine $i$. The following observation has been made in Lemma 2 of [27],

$$\sum_{q=1}^{h} w_q C_q^i = \sum_{q=1}^{h} \left[ \frac{1}{2} p^i(\{J_1, \ldots, J_q\})^2 (\delta_i(q) - \delta_i(q + 1)) + \frac{1}{2} w_q p_q^i \right],$$

where $p^i(S) = \sum_{J_q \in S} p_q^i$.

Now we are ready to set up an IP. We use $x^i_j$ to represent the number of jobs of type $j$ $(1 \leq j \leq k)$ that are scheduled on machine $i$ $(1 \leq i \leq m)$, then the following holds:

Lemma 15 ([27], Corollary 1). Given $x^i_1, \ldots, x^i_k$ representing the number of jobs of each type scheduled to run on machine $i$, a permutation $\pi_i : [k] \rightarrow [k]$ such that $\delta_i(\pi_i(j)) \geq \delta_i(\pi_i(j + 1))$ for all $1 \leq j \leq k - 1$ and $\delta_i(\pi_i(k + 1)) = 0$, then $f^i(x^i) = \frac{1}{2} \sum_{j=1}^{k} (\sum_{h=1}^{j} p_h^i x_h^i)^2 (\delta_i(\pi_i(j)) - \delta_i(\pi_i(j + 1))) + \sum_{J_q \in \pi_i^{-1}(\pi_i(J^i))} w_q p_q^i x_q^i$.

We introduce new variables as $z^i_j := \sum_{h=1}^{j} p_h^i x_h^i$, then the objective function can be written as

$$\theta C_{\text{max}} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{k} (z^i_j)^2 (\delta_i(\pi_i(j)) - \delta_i(\pi_i(j + 1))) + \sum_{J_q \in \pi_i^{-1}(\pi_i(J^i))} w_q p_q^i x_q^i$$. Note that $f^i(x^i, z^i)$ is separable convex for any $i$ $(1 \leq i \leq m)$ [27].

To summarize, we have the following IP$_{\text{sche2}}$:

$$\min \quad \theta C_{\text{max}} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{k} (z^i_j)^2 (\delta_i(\pi_i(j)) - \delta_i(\pi_i(j + 1))) + \sum_{J_q \in \pi_i^{-1}(\pi_i(J^i))} w_q p_q^i x_q^i$$

$$\sum_{j=1}^{k} x_j^i = N_j, \quad \forall 1 \leq j \leq k$$

$$\sum_{j=1}^{k} p_j^i x_j^i - C_{\text{max}} \leq 0, \quad \forall 1 \leq i \leq m$$

$$\sum_{h=1}^{j} p_h^i x_h^i = z_j^i, \quad \forall 1 \leq i \leq m, 1 \leq j \leq k$$

$$x_j^i \in \mathbb{Z}_{\geq 0}$$

It is easy to verify that the constraint matrix is as follows:

$$H_{(4)} = \begin{pmatrix}
0 & D_1 & D_2 & \cdots & D_m \\
B & A_1 & 0 & 0 \\
B & 0 & A_2 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
B & 0 & 0 & A_m \\
\end{pmatrix}$$
where

\[
D_i = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix},
A_i = \begin{pmatrix}
p_i^1 & p_i^2 & \cdots & p_i^k & 0 & 0 & \cdots & 0
\end{pmatrix},
\]

and \(B = (-1, 0, \ldots, 0)^T\).

This is an almost combinatorial 4-block \(n\)-fold IP. Using Theorem 5, the above IP can be solved in time \(m^{42^{O(k^2 \log p_{\max})}} \cdot 2^{2^{O(\log p_{\max})}} |I|^4\), where \(|I|\) denotes the length of the input, which is bounded by \(O(kp_{\max}(\max\{\log N_{\max}, \log w_{\max}\}))\) where \(N_{\max} = \max_j N_j, w_{\max} = \max_j w_j\). \(\square\)