On the Dispersion of Sparse Grids

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Abstract

For any natural number \(d\) and positive number \(\varepsilon\), we present a point set in the \(d\)-dimensional unit cube \([0,1]^d\) that intersects every axis-aligned box of volume greater than \(\varepsilon\). These point sets are very easy to handle and in a vast range for \(\varepsilon\) and \(d\), we do not know any smaller set with this property.

1 The Result

The dispersion of a point set \(P\) in \([0,1]^d\) is the volume of the largest axis-aligned box in \([0,1]^d\) which does not intersect \(P\). Point sets with small dispersion already proved to be useful for the uniform recovery of rank one tensors [4] and for the discretization of the uniform norm of trigonometric polynomials [9]. Recently, great progress has been made in the question for the minimal size for which there exists a point set whose dispersion is at most \(\varepsilon > 0\), see Dumitrescu and Jiang [3], Aistleitner, Hinrichs and Rudolf [1], Rudolf [7] and Sosnowiec [8]. In this note, we want to provide such point sets. They should be both small and easy to handle. To that end, we define the point sets

\[
P(k,d) = \bigcup_{|j|=k} M_{j_1} \times \cdots \times M_{j_d}
\]

of order \(k \in \mathbb{N}_0\) and dimension \(d \in \mathbb{N}\), generated by the one-dimensional sets

\[
M_j = \left\{ \frac{1}{2j+1}, \frac{3}{2j+1}, \ldots, \frac{2^{j+1}-1}{2j+1} \right\} \quad \text{for} \quad j \in \mathbb{N}_0.
\]

You can find a picture of the set of order 3 in dimension 2 in Figure 1. These point sets are particular instances of a sparse grid as widely used for high-dimensional numerical integration and approximation. We refer to Novak and Woźniakowski [5] and the references therein. Here, we will prove the following result.
Theorem. Let \( k(\varepsilon) = \lceil \log_2 (\varepsilon^{-1}) \rceil - 1 \) for any \( \varepsilon \in (0, 1) \) and let \( d \geq 2 \). Then the dispersion of \( P(k(\varepsilon), d) \) is at most \( \varepsilon \) and

\[
|P(k(\varepsilon), d)| = 2^{k(\varepsilon)} \left( \frac{d + k(\varepsilon) - 1}{d - 1} \right).
\]

The formula for the size of \( P(k(\varepsilon), d) \) may be simplified. On the one hand, we have

\[
|P(k(\varepsilon), d)| \leq \varepsilon^{-1} \lceil \log_2 (\varepsilon^{-1}) \rceil^{d-1},
\]

which shows that the size grows almost linearly in \( 1/\varepsilon \) for a fixed dimension \( d \). On the other hand,

\[
|P(k(\varepsilon), d)| \leq (2d)^{k(\varepsilon)},
\]

which shows that the size only grows polynomially in \( d \) for a fixed error tolerance \( \varepsilon \). Although very simple, \( P(k(\varepsilon), d) \) is the smallest explicitly known point set in \([0, 1]^d\) with dispersion at most \( \varepsilon \) for many instances of \( \varepsilon \) and \( d \), see Section 3.

2 The Proof

Let us introduce the notation used in this section. We write \([d] = \{1, \ldots, d\}\) for each \( d \in \mathbb{N} \) and \( |\mathbf{j}| = j_1 + \ldots + j_d \) for \( \mathbf{j} \in \mathbb{N}^d \). The vector \( \mathbf{e}_\ell \in \mathbb{R}^d \) has the entry 1 in the \( \ell \)th coordinate and 0 in all other coordinates. A set \( B \subset \mathbb{R}^d \) is called a box, if it is the Cartesian product of \( d \) open intervals. Its volume \( |B| \) is the product of their lengths. If \( M \) is a finite set, \( |M| \) denotes the number of its elements. We start with computing the number of elements in \( P(k, d) \) for \( k \in \mathbb{N}_0 \) and \( d \in \mathbb{N} \).

Lemma 1.

\[
|P(k, d)| = 2^k \left( \frac{d + k - 1}{d - 1} \right).
\]

Proof. Note that \( |M_j| = 2^j \) for all \( j \in \mathbb{N} \) and also for \( j = 0 \). The identity

\[
|P(k, d)| = \sum_{|\mathbf{j}|=k} |M_{j_1} \times \ldots \times M_{j_d}| = \sum_{|\mathbf{j}|=k} 2^{j_1 + \ldots + j_d} = 2^k \left| \left\{ \mathbf{j} \in \mathbb{N}_0^d : |\mathbf{j}| = k \right\} \right|
\]

yields the statement of the lemma.

It follows from [9] that the dispersion of \( P(k, d) \) decays like \( 2^{-k} \), if \( d \) is fixed and \( k \) tends to infinity. In fact, it can be computed precisely. In dimension \( d = 1 \), it is easy to see that the dispersion of \( P(k, d) \) equals \( 2^{-k} \) for \( k \geq 1 \) and \( 1/2 \) for \( k = 0 \). In dimension \( d \geq 2 \), we obtain the following.
Lemma 2. For any $k \in \mathbb{N}_0$ and $d \geq 2$, the dispersion of $P(k, d)$ is $2^{-(k+1)}$.

Proof. We first observe that there are many boxes of volume $2^{-(k+1)}$ which do not intersect with $P(k, d)$. For instance, the box

$$(0, 2^{-(k+1)}) \times (0, 1) \times \cdots \times (0, 1)$$

has these properties. Now, let $B = I_1 \times \cdots \times I_d$ be any box in $[0, 1]^d$ that does not intersect with $P(k, d)$. The set

$$P = \bigcup_{m \in \mathbb{N}} P(m, d) = \left\{ \frac{\alpha}{2^\beta} \mid \beta \in \mathbb{N} \text{ and } \alpha \in [2^\beta - 1] \right\}^d$$

is dense in $[0, 1]^d$. Without loss of generality, we assume that the interior of $B$ is nonempty. Therefore, $B$ intersects with $P$ and hence with $P(m, d)$ for some $m \in \mathbb{N}$. Let $m$ be minimal with this property. Since $B$ does not intersect with $P(k, d)$, we either have $m > k$ or $m < k$. Let $x \in P(m, d) \cap B$. This means that there is some $j \in \mathbb{N}_0^d$ with $|j| = m$ and

$$x_\ell \in M_{j_\ell} \cap I_\ell$$

for all $\ell \in [d]$. We observe that the numbers $x_\ell \pm \frac{1}{2^{|j_\ell|}}$ are either contained in $\{0, 1\}$ or in $M_j$ for some $j < j_\ell$. Hence, they are not contained in $I_\ell$, because $I_\ell$ is
a subset of \((0, 1)\) and \(m\) is minimal. We obtain that
\[
I_\ell \subseteq \left( x_\ell - \frac{1}{2^{j_\ell + 1}}, x_\ell + \frac{1}{2^{j_\ell + 1}} \right),
\]
and hence
\[
|B| \leq \prod_{\ell \in [d]} 2^{-j_\ell} = 2^{-m}.
\]
In the case \(m > k\), this yields the statement. In the case \(m < k\), we observe that the numbers \(x_\ell \pm \frac{1}{2^{k-m+j_\ell+1}}\) cannot be contained in \(I_\ell\) for any \(\ell \in [d]\), since otherwise the points
\[
x \pm \frac{e_\ell}{2^{k-m+j_\ell+1}}
\]
would be both in \(B\) and in \(P(k, d)\). This means that
\[
I_\ell \subseteq \left( x_\ell - \frac{1}{2^{k-m+j_\ell+1}}, x_\ell + \frac{1}{2^{k-m+j_\ell+1}} \right).
\]
We obtain
\[
|B| \leq \prod_{\ell \in [d]} 2^{m-k-j_\ell} = 2^{dm-dk-m} \leq 2^{-(k+1)},
\]
where we used that \(d \geq 2\), and the statement is proven.

For any \(\varepsilon \in (0, 1)\), the smallest number \(k \in \mathbb{N}_0\) that satisfies \(2^{-(k+1)} \leq \varepsilon\) is obviously given by
\[
k(\varepsilon) = \lceil \log_2 \left( \frac{1}{\varepsilon-1} \right) \rceil - 1.
\]
This yields the statement of our theorem.

## 3 A Comparison with Known Results

Let \(d \geq 2\) be an integer and \(\varepsilon \leq 1/4\) be a positive number. Let us call a point set in \([0, 1]^d\) admissible, if its dispersion is at most \(\varepsilon\). In 2015, Aistleitner, Hinrichs and Rudolf [1] proved that any admissible point set contains at least
\[
(4\varepsilon)^{-1}(1 - 4\varepsilon) \log_2 d \tag{1}
\]
points. At that time, the smallest known admissible point set was a Halton-Hammersley set of size
\[
\left[2^{d-1} \pi_d \varepsilon^{-1} \right], \tag{2}
\]
where $\pi_d$ is the product of the first $(d - 1)$ primes. This was proven by Rote and Tichy [6], see also Dumitrescu an Jiang [3] for more details. The size of this set grows as slowly as possible as $\varepsilon$ tends to zero, if $d$ is fixed. However, it grows super-exponentially with $d$. Gerhard Larcher realized that also certain $(t, m, s)$-nets that contain only
\[ 2^{7d+1\varepsilon^{-1}}, \] (3)
points are admissible. The proof is included in [1]. This number is smaller than (2) for $d \geq 54$. However, its exponential growth with respect to $d$ for fixed $\varepsilon$ is still far away from the logarithmic growth of the lower bound (1). In the beginning of 2017, Rudolf [7] significantly narrowed this gap. In line with results of Blumer, Ehrenfeucht, Haussler and Warmuth [2], he obtained the existence of an admissible point set of size
\[ 8d\varepsilon^{-1}\log (33\varepsilon^{-1})]. \] (4)
Very recently, the remaining gap was completely closed by Sosnovec [8], who proved the existence of an admissible point set of size
\[ q^{q^{2+2}(1 + 4 \log q) \cdot \log d}, \] (5)
where $q = \lceil 1/\varepsilon \rceil$. Just like the lower bound (1), this number only grows logarithmically with $d$. On the other hand, it now depends super-exponentially on $1/\varepsilon$. This can most likely be improved, but up to now, (4) is still the smallest known upper bound for the minimal size $N_*(\varepsilon, d)$ of admissible point sets for a vast majority of parameters $\varepsilon \geq 10^{-12}$ and $d \leq 10^{12}$. However, the upper bounds (4) and (5) are both nonconstrucive in the sense that no admissible point set was provided. Here, we proved the existence of an admissible point set of size
\[ 2^{k(\varepsilon)} \left( \frac{d + k(\varepsilon) - 1}{d - 1} \right) \] (6)
where $k(\varepsilon) = \lceil \log_2 (\varepsilon^{-1}) \rceil - 1$, which does not grow exponentially with $d$ either, but we also provided such a set, namely $P(k(\varepsilon), d)$. Already for modest dimensions, this number is much smaller than the cardinalities (2) and (3). As an example, we consider the parameters $d \in \{2, \ldots, 100\}$ and $\varepsilon \in \{1/4, 1/5, \ldots, 1/100\}$ in Figure 2. The dark gray area represents the parameters for which the author does not know any smaller admissible set than $P(k(\varepsilon), d)$, although Rudolf [7] proved their existence. There are also some parameters, where $P(k(\varepsilon), d)$ improves
on all the mentioned upper bounds for $N_*(\varepsilon, d)$, constructive or nonconstructive. These parameters are indicated by the black area. In the light gray area, the Halton-Hammersley point set is smaller.

References

[1] C. Aistleitner, A. Hinrichs, D. Rudolf: *On the Size of the Largest Empty Box amidst a Point Set*. Discrete Applied Mathematics **230**, 146–150, 2017.

[2] A. Blumer, A. Ehrenfeucht, D. Haussler, M. Warmuth: *Learnability and the Vapnik-Chervonenkis dimension*. J. Assoc. Comput. Mach. **36**(4), 929–965, 1989.
[3] A. Dumitrescu, M. Jiang: *On the Largest Empty Axis-Parallel Box amidst n Points*. Algorithmica **66**(2), 225–248, 2013.

[4] E. Novak, D. Rudolf: *Tractability of the Approximation of High-Dimensional Rank One Tensors*. Constructive Approximation **43**, 1–13, 2016.

[5] E. Novak, H. Woźniakowski: *Tractability of Multivariate Problems. Volume II: Standard Information for Functionals*. EMS, Zürich, 2010.

[6] G. Rote, R. F. Tichy. *Quasi-Monte Carlo Methods and the Dispersion of Point Sequences*. Math. Comput. Modelling, **23**(8-9), 9–23, 1996.

[7] D. Rudolf: *An upper Bound on the Minimal Dispersion via Delta Covers*. Festschrift for Ian Sloan’s 80th Birthday, to appear. Available at ArXiv e-prints, arXiv:1701.06430 [cs.CG].

[8] J. Sosnovec: *A Note on the Minimal Dispersion of Point Sets in the Unit Cube*. ArXiv e-prints, 2017, arXiv:1707.08794 [cs.CG].

[9] V. N. Temlyakov: *Universal Discretization*. ArXiv e-prints, 2017, arXiv:1708.08544 [math.NA].