Validity of Borodin & Kostochka Conjecture for a Class of Graphs
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Abstract: Borodin & Kostochka conjectured that if \( \Delta(G) \geq 9 \), then \( \chi(G) \leq \max\{\omega, \Delta - 1\} \). Here we prove that this Conjecture is true for \( \{P_3 \cup K_1\}\)-free graphs and \( \{K_2 \cup \overline{K_2}\}\)-free graphs.

Introduction:
In 1977, Borodin & Kostochka conjectured that if \( \Delta(G) \geq 9 \), then \( \chi(G) \leq \max\{\omega, \Delta - 1\} \) [1]. In 1999, Reed proved the conjecture for \( \Delta \geq 10^4 \) [2]. Also D. W. Cranston and L. Rabern [3] proved it for claw-free graphs and Medha Dhurandhar [4] proved it for \( 2K_2\)-free graphs. Here we prove that the conjecture is true for \( \{P_3 \cup K_1\}\)-free graphs and \( \{K_2 \cup \overline{K_2}\}\)-free graphs.

Notation: For a graph \( G \), \( V(G) \), \( E(G) \), \( \Delta \), \( \omega \), \( \chi \) denote the vertex set, edge set, maximum degree, size of a maximum clique, chromatic number of \( G \) resp. For \( u \in V(G) \), \( N(u) = \{v \in V(G) / uv \in E(G)\} \), and \( \overline{N(u)} = N(u) \setminus \{u\} \). If \( S \subseteq V \), then \( <S> \) denotes the subgraph of \( G \) induced by \( S \). If \( C \) is some coloring of \( V(G) \) and if \( u \in V(G) \) is colored \( m \) in \( C \), then \( u \) is called a \( m \)-vertex. Also if \( P \) is a path in \( G \) s.t. vertices on \( P \) are alternately colored say \( i \) and \( j \), then \( P \) is called an \( i-j \) path. All graphs considered henceforth are simple and undirected. For terms not defined herein, we refer to Bondy and Murty [5].

Let \( H = P_3 \cup K_1 \) and \( R = K_2 \cup \overline{K_2} \). Note that the only odd cycle in a \( H \)-free or \( R \)-free graph is \( C_5 \).

Main Result 1: If \( \Delta \geq 9 \), and \( G \) is \( H \)-free, then \( \chi \leq \max\{\omega, \Delta - 1\} \).
Proof: Let \( G \) be a smallest \( H \)-free with \( \Delta \geq 9 \) and \( \chi > \max\{\omega, \Delta - 1\} \). Then clearly as \( G \neq C_{2n+1} \) or \( K_{\frac{|V(G)|}{2}} \), \( \chi = \Delta > \omega \). Let \( u \in V(G) \). Then \( G-u \neq K_{\frac{|V(G)|}{2}} \) (else \( \chi = \omega \)). If \( \Delta(G-u) \geq 9 \), then by minimality \( \chi(G-u) \leq \max\{\omega(G-u), \Delta(G-u)-1\} \). Clearly if \( \omega(G-u) \leq \Delta(G-u)-1 \), then \( \chi(G-u) = \Delta(G-u)-1 \leq \Delta-1 \) and otherwise \( \chi(G-u) = \omega(G-u) < \Delta < \Delta-1 \). In all the cases, \( \chi(G-u) \leq \Delta-1 \). Also if \( \Delta(G-u) < 9 \), then as \( G-u \neq C_{2n+1} \) (else as \( G \) is \( H \)-free, \( G-u \sim C_3 \)), by Brook’s Theorem \( \chi(G-u) \leq \Delta(G-u) < 9 \leq \Delta \). Thus always \( \chi(G-u) \leq \Delta-1 \) and in fact, \( \chi(G-u) = \Delta-1 \) and \( \deg u \geq \Delta-1 \) \( \forall \ u \in V(G) \).

Let \( Q \subseteq V(G) \) be s.t. \( <Q> \) is a maximum clique in \( G \). Let \( u \in Q \) be s.t. \( \deg u = \max_{v \in Q} \deg v \). Let \( S = \{1,..., \Delta\} \) be a \( \Delta \)-coloring of \( V(G) \) s.t. \( u \) is colored \( \Delta \) and \( \{A_1, A_2, ..., A_{\Delta-2}\} \subseteq N(u) \) where \( A_i \) has color \( i \) for \( 1 \leq i \leq \Delta-2 \). If \( A_i \) is the only i-vertex in \( N(u) \), then \( A_i \) is said to have a unique color.

I. Every \( A_i \) with a unique color has at the most one repeat color and two vertices of that color (else \( N(A_i) \) has a color say \( r \) missing. Then color \( A_i \) by \( r \) and \( u \) by \( i \)).
II. Now if \( \deg u = \Delta-1 \), then all vertices in \( N(u) \) have unique colors (else some color \( r \) is missing in \( N(u) \)). Color \( u \) by \( r \) and if \( \deg u = \Delta \), then \( N(u) \) has \( \Delta-2 \) vertices with unique colors and two vertices with the same color. Thus \( N(u) \) has at least seven vertices with unique colors.
III. Every \( A_i \) with a unique color has a j-vertex \( \forall j \neq i \) (else color \( A_i \) by \( j \), \( u \) by \( i \)).

Claim: All vertices in \( N(u) \) with unique colors are adjacent to each other.
Let if possible say \( A_i, A_s \notin E(G) \) where \( A_i \) and \( A_s \) have unique colors. Then clearly \( \exists A_r-A_s \) path = \( \{A_r, B, C, A_s\} \) s.t. \( B, C \) have colors \( s, r \) resply. Now \( B \) is the only \( s \)-vertex in \( V(G) \)-\( N(u) \) (else let \( B' \) be another \( s \)-vertex. If \( A_r B \notin E(G) \), then \( <u, A_r, B, B'> = H \) and if \( A_i B \in E(G) \), then \( <A_s, B, B', A_r,> = H \).
Hence the claim holds.

Thus \( \deg u = \Delta > \omega \geq \Delta - 1 \Rightarrow \omega = \Delta - 1 \). Let \( Q = \{ u, A_1, A_2, \ldots, A_{\omega - 1} \} \) and \( A_{\omega}, A_{\omega + 1} \in N(u) - Q \) have the same color \( \omega \).

As \( G \) is \( H \)-free, \( V(G) - \overline{N(u)} \) has no \( \omega \)-vertex. Hence \( A_{\omega} \cup A_{\omega + 1} \in E(G) \) \( \forall i, 1 \leq i \leq \omega - 1 \) (else color \( A_i \) by \( \omega \), \( u \) by \( i \)). W.l.g. let \( A_{\omega}A_2 \notin E(G) \Rightarrow A_2A_{\omega - 1} \in E(G) \) and let \( A_{\omega}A_2 \notin E(G) \Rightarrow A_1A_{\omega} \in E(G) \).

Now by \( III \), \( A_1 \) (\( A_2 \)) has an \( i \)-vertex \( \forall i \neq 1 \). As \( \Delta \geq 9 \), w.l.g. let \( A_2 \neq A_i, A_3 \) be adjacent to \( A_{\omega} \). Again \( A_{\omega} \) has at least one repeat color (else \( A_{\omega} \) has some color missing. Color \( A_{\omega} \) by the missing color, \( A_1 \) by \( \omega \), \( u \) by \( 1 \)). Hence w.l.g. let \( A_3, A_4 \) be the only 3-vertex, 4-vertex of \( A_{\omega} \). Also w.l.g. let \( A_3 \) be the only 3-vertex of \( A_2 \). Now if \( A_2A_{\omega - 1} \notin E(G) \), then \( A_3 \) by \( \omega \), \( A_4 \) by \( 3 \), \( A_2 \) by \( 3, u \) by \( 2 \) and if \( A_2A_{\omega - 1} \in E(G) \), then \( A_1 \) is the only 1-vertex of \( A_3 \) and hence color \( A_3 \) by \( 1 \), \( A_4 \) by \( \omega \), \( A_3 \) by \( 3, A_2 \) by \( 3, u \) by \( 2 \), a contradiction in both the cases.

This proves Result 1.

**Main Result 2**: If \( \Delta \geq 9 \), and \( G \) is \( H \)-free, then \( \chi \leq \max \{ \omega, \Delta - 1 \} \).

Proof: Let \( G \) be a smallest \( R \)-free graph with \( \Delta \geq 9 \) and \( \chi > \max \{ \omega, \Delta - 1 \} \). Then clearly as \( G \neq C_{2\omega + 1} \) or \( K_{\omega \mid |G|} \), \( \chi = \Delta > \omega \). Let \( u \in V(G) \). Then \( G - u \neq K_{\omega \mid |G|} \) (else \( \chi = \omega \)). If \( \Delta(G - u) \geq 9 \), then by minimality \( \chi(G - u) \leq \max \{ \omega(G - u), \Delta(G - u) - 1 \} \). Clearly if \( \omega(G - u) \leq \Delta(G - u) - 1 \), then \( \chi(G - u) = \Delta(G - u) - 1 \leq \Delta - 1 \) and otherwise \( \chi(G - u) = \omega(G - u) \leq \omega < \Delta \). In all the cases, \( \chi(G - u) \leq \Delta - 1 \). Also if \( \Delta(G - u) < 9 \), then as \( G - u \neq C_{2\omega + 1} \) (else as \( G \) is \( R \)-free, \( G - u \sim C_3 \)), by Brook’s Theorem \( \chi(G - u) \leq \Delta(G - u) < 9 \leq \Delta \). Thus always \( \chi(G - u) \leq \Delta - 1 \) and in fact \( \chi(G - u) = \Delta - 1 \) and \( \deg v \geq \Delta - 1 \ \forall v \in V(G) \).

Let \( Q \subseteq V(G) \) be s.t. \( \langle Q \rangle \) is a maximum clique in \( G \). Let \( u \in Q \) be s.t. \( \deg u = \max_{v \in Q} \deg v \). Let \( S = \{ 1, \ldots, \Delta \} \) be a \( \Delta \)-coloring of \( V(G) \) s.t. \( u \) is colored \( \Delta \) and \( \{ A_1, A_2, \ldots, A_{\Delta - 2} \} \subseteq N(u) \) where \( A_i \) has color \( i \) for \( 1 \leq i \leq \Delta - 2 \). If \( A_i \) is the only \( i \)-vertex in \( N(u) \), then \( A_i \) is said to have a unique color.

**I.** Every vertex \( A_i \) of \( N(u) \) with a unique color has at most two vertices of the same color (else \( N(A_i) \) has a color say \( r \) missing. Then color \( v \) by \( r \) and \( u \) by \( i \)).

**II.** Also as \( G \) is \( R \)-free, \( V(G) - \overline{N(u)} \) has no repeat color (else if \( V, W \in V(G) - \overline{N(u)} \) have color say \( i \), then \( \langle u, A_i, V, W \rangle = R \)).

**III.** Every \( A_i \) with a unique color has a \( j \)-vertex \( \forall j \neq i \) (else color \( A_i \) by \( j \), \( u \) by \( i \)).

**IV.** Now if \( \deg u = \Delta - 1 \), then all vertices in \( N(u) \) have unique colors (else some color \( r \) is missing in \( \overline{N(u)} \). Color \( u \) by \( r \) and if \( \deg u = \Delta \), then \( N(u) \) has \( \Delta - 2 \) vertices with unique colors and two vertices with the same color. Thus \( N(u) \) has at least seven vertices with unique colors.

**Claim:** All vertices in \( N(u) \) with unique colors are adjacent to each other.

Let if possible say \( A_1A_2 \notin E(G) \) where \( A_1, A_2 \) have unique colors. Then clearly \( \exists A_1, A_2 \) s.t. \( B, C \) have colors 2, 1 resp. Now by **II**, \( B \) is the only 2-vertex of \( A_1 \). Also \( A_2 \) is non-adjacent to the most one more \( A_k \in N(u), k \neq 1 \) (else if \( A_2A_k, A_2A_1 \notin E(G) \), then \( B \) has two, 1, k, 1 vertices and hence has some color \( r \) missing in \( \overline{N(B)} \). Color \( B \) by \( r, A_1 \) by 2, \( u \) by 1). By **IV**, w.l.g. let...
A_3,...,A_5 be vertices in N(u) with unique colors s.t. A_1A_i, A_2A_j ∈ E(G) for 3 ≤ i ≤ 5. Also by I, w.l.g. let A_3 be the only 3 vertex of A_1, A_2. Again by I, A_3 has either a unique 1-vertex or 2-vertex. W.l.g. let A_3 be the only 1-vertex of A_3. Then color A_3 by 1, A_1 and A_2 by 3, u by 2, a contradiction.

Hence the claim holds.

Thus ω ≥ Δ - 1 > ω - 1 ⇒ ω = Δ - 1 ⇒ deg u = Δ. Let Q = {u, A_1, A_2, A_3, A_4} where A_i has a unique color i and X, Y ∈ N(u) be colored ω. Further let YA_1 ∈ E(G). Then XA_1 E(G) (else by III, A_1 has a ω-vertex Z and <A_1, Z, X, Y> = R). Again w.l.g. let XA_2 E(G). Then YA_2 E(G).

Case 2.1: V(G)-N(u) has no ω-vertex.

By III, every A_i is adjacent to X or Y. Also X (Y) is the only ω-vertex of A_1 (A_2). Further as Δ ≥ 9, w.l.g. let XA_i E(G) for 3 ≤ i ≤ 5. Now X (Y) has a k-vertex ∀ k ≠ ω (else color X (Y) by the missing color, A_i(A_2) by ω, u by 1 (2)). Hence X and Y have at the most one repeat color. W.l.g. let A_3 be the unique 3-vertex of X and A_2. If XA_3 E(G), then color A_3 by ω, X by 3, A_2 by 3, u by 2 and if YA_3 E(G), then as A_3 has the unique 2-vertex A_2, color A_3 by 2, X by 3, A_2 by 3, A_1 by ω, u by 1, a contradiction in both the cases.

Case 2.2: V(G)-N(u) has a ω-vertex Z.

Then as G is R-free, every A_i has two ω-vertices and hence a unique j-vertex A_i, j≠i.

Case 2.2.1: X has no r-vertex for some r, 2 ≤ r ≤ ω - 1.

Then XA_i E(G) ⇒ YA_1 E(G). Clearly ZA_1, ZA_2 ∈ E(G). Now Z has another 1-vertex (else color X by r, A_j by ω, Z by 1, u by 1). Also Y has an i-vertex ∀ i ≠ ω (else color X by r, Y by i, u by ω). If ∃ s ≠ r, s.t. YA_i E(G) and A_i is the only s-vertex of Y, then color X by r, A_i by 1, A_j by s, Y by s, u by ω, a contradiction. Hence whenever YA_i E(G), s ≠ r, Y has another s-vertex. Hence Y is adjacent to at the most one A_j, s ≠ r. As Δ ≥ 9, w.l.g. let XA_1, ZA_j E(G) for say i = 3 ≤ i ≤ 6. As Z has two 1-vertices, Z has at the most one more repeat color (else N(Z) has some color t missing, color X by r, Z by t, A_i by ω, u by 1) and hence w.l.g. let Z have the unique 3-vertex A_3. Color X by r, Z by 3, A_3 by ω, u by 3, a contradiction.

Case 2.2.2: X, Y have an i-vertex ∀ i ≠ ω.

Clearly ZA_1, ZA_2 E(G). As Δ ≥ 9, w.l.g. XA_i E(G) for i = 3, 4, 5. Also as X has at the most one repeat color, w.l.g. let A_j be the only i-vertex of X for i = 3, 4. Now Z has another 1-vertex (else color A_j by 2, A_j by 3, A_1 by ω, Z by 1, u by 1) and Z has no color missing (else color Z by the missing color t, A_3 by 2, A_2 by 3, X by 3, A_1 by ω, u by 1). Hence Z has at the most one more repeat color other than 1 ⇒ ZA_3 E(G) or ZA_4 E(G) (else Z has a unique say 4-vertex A_4. Color X by 3, A_3 by 2, A_2 by 3, A_1 by ω, Z by 4, u by 4). W.l.g. let ZA_3 E(G) ⇒ YA_1 E(G). If A_2 is the unique 2-vertex of Y, then color A_j by 2, A_2 by 3, A_1 by 1, X by 3, Y by 2, u by ω and if A_1 is the unique 3-vertex of Y, then color A_3 by 1, A_3 by 3, Y by 3, A_4 by 2, A_2 by 4, X by 4, u by ω, a contradiction in both the cases.

This proves Result 2.

References

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