Kinetic-theory approach to Gluon Self-energy beyond Hard Thermal Loops *

Zheng Xiaoping Li Jiarong

a Department of Physics, Huazhong Normal University
b Institute of Particle Physics, Huazhong Normal University

March 25, 2022

Abstract

We compare the effective dynamics of soft fields, based on temperature field theory, with the mean field dynamics from non-Abelian kinetic theory. We derive the polarization tensor with the leading logarithmic factor \[ \log\left(\frac{gT}{\mu}\right) \] from the effective Boltzmann-Langevin equation given by Litim and Manuel. The tensor is identical with effective one-loop contributions within the hard thermal loop effective theory.

1 Introduction

It is predicted that nuclear matter will transform to a quark-gluon plasma (QGP) from hadron matter at high temperature or density.

The temperature field theory and the kinetic theory are universally used to investigate the collective dynamics. First of all, the effective theory so-called the "hard thermal loops" (HTL) for the soft gauge fields at the scale \( gT \), where \( g \) is gauge coupling and \( T \) is temperature, is constructed according to the fully quantum field theory techniques[1]. Soon afterwards, the effective action generating HTLs is straightforwardly derived from classical kinetic theory and the kinetic theory is compatible with the temperature field theory approaches in calculations of the polarization tensor[2, 3]. As

*This work is supported by NNSF of China under grant No. 10175026
well known, perturbation theory for non-Abelian gauge theories breaks down at the spatial momentum scale $g^2T$ even if the running gauge coupling $g$ is small\cite{g1, g2}. For static quantities, one can integrate out the high momentum modes ($p \gg g^2T$) in perturbation theory using dimensional reduction\cite{g3, g4} and then at leading order, the non-perturbative physics associated with the scale $g^2T$ can be treated in lattice simulations, of which the recent results\cite{g5} show this contribution is small. However, dynamics quantities, such as transport coefficients, are sensitive to the scale $g^2T$ and more difficult to deal with due to failure of dimensional reduction for the consideration of non-zero real frequencies. In recent years, one devoted to constructing effective theories of the ultrasoft gauge fields (momentum scale $\sim g^2T$) and fortunately obtained the effective theory by integrating out high momentum degrees of freedom using perturbation theory\cite{g6}. To leading logarithmic contributions, the diagrammatic explanations is given\cite{g7}. The results show that the contributions do not have an Abelian analogue since there are no hard thermal loop vertices in Abelian theories. Interestingly, there is another way to arrive at the same effective dynamics, that is so-called mean field dynamics from kinetic theory\cite{g8, g9}. Those furthermore demonstrate the connection between kinetic theory and temperature theory under specified conditions.

In this article, we will extend and apply the non-Abelian kinetic theory to compute the self-energy of the ultra-soft modes. The polarization tensor obtained by Bödeker method is from two effective one-loop contributions but they correspond to two 2-loop and a 3-loop diagrams in the original theory, which involves the sum over Matsubara frequencies while the sum is not easy to compute\cite{g10}. In comparison with diagrammatic approach, the kinetic-theory appears to be much technically easier in calculus because some of the intrinsic complications of a quantum field theoretical description can be avoided\cite{g11}. Especially, the kinetic-theory –the temperature field theory connection can further be uncovered by another representative example beyond the “hard thermal/density loops”, i.e. from soft field dynamics in non-perturbation domain. This article is organized as follows: Sect.2 briefly reviews the the effective one-loop contributions to the 4-point function within the hard thermal loop effective theory. Sect.3 introduces the non-Abelian kinetic theory to compute the polarization tensor involving ultra-soft fields. Sect.4 gives the summary and discussion.
2 The effective one-loop polarization tensor

It is well-known that the one-loop diagrams for large loop momentum ($\sim T$) and soft external momentum ($\sim gT$) are called hard thermal loops (HTL’s). The HTL 2-point function, which is the same in Abelian and non-Abelian gauge theories, easily reads [13]

$$\delta \Pi_{\mu\nu} = m_D^2 \left[ -g_{\mu0}g_{\nu0} + p_0 \int \frac{d\Omega_v}{4\pi} \frac{v_\mu v_\nu}{v \cdot P} \right]$$

(1)

where the factor $m_D$ is Debye mass, of which the square is equal to $(1/3)(N + N_f/2)g^2T^2$ for a SU($N$) gauge theory with $N_f$ fermions.

For momenta $p_0, p \leq gT$, the resummation of the hard thermal loop can give the correction to the tree level kinetic term. However, the investigations found that the HTL approximation is not sufficient for obtaining the correct effective theory for the softer modes. The some higher loop contributions should be considered. The two- and three-diagrams by adding soft lines ($\sim gT$) in hard momenta ($\sim T$) (see fig 1 and fig 3 in [10]) are shown to be as large as $\delta \Pi(P)$ when the external momenta $p_0$ and $p$ are of order $g^2T$ or smaller. Fortunately, the higher loop diagrams can be regard as two effective one-loop diagrams with so-called hard thermal loop vertices (fig (a) and (b)).

Thus the one-loop polarization tensors corresponding respectively to two 2-loop (with a hard loop momentum) and a 3-loop (with two hard loop momenta) diagrams in original theory can be computed if ones use the HTL effective theory. Bödeker derived from those after complicated treatments

$$\Pi^{(2)}_{\mu\nu}(P) \simeq -i \frac{m_D^2}{4\pi} N g^2 T p_0 \int \frac{d\Omega_v}{4\pi} \frac{v_\mu v_\nu}{(v \cdot P)^2} \left[ \log \left( \frac{gT}{\mu} \right) + \text{finite} \right],$$

(2)

$$\Pi^{(3)}_{\mu\nu}(P) \simeq -i \frac{m_D^2}{\pi} N g^2 T p_0 \int \frac{d\Omega_{v_1}}{4\pi} \frac{v_{1\mu}}{v_1 \cdot P} \int \frac{d\Omega_{v_2}}{4\pi} \frac{v_{2\nu}}{v_2 \cdot P} \left[ \log \left( \frac{gT}{\mu} \right) \frac{(v_1 \cdot v_2)^2}{\sqrt{1 - (v_1 \cdot v_2)^2}} + \text{finite} \right],$$

(3)

where supperscripts 2 and 3 represent respectively the 2- and 3-loop contributions in original theory while $\mu$ in logarithmic term is a introduced scale, which denotes a separation of ultra-soft momenta from momenta of order $gT$, such that

$$g^2 T \ll \mu \ll gT.$$
Adding Eqs(2) and (3), the polarization tensor at leading logarithmic accuracy becomes

$$\Phi_{\mu\nu}^{LA}(P) = -\frac{i}{4\pi}m_D^2 N g^2 T \log \left( \frac{gT}{\mu} \right) p_0 \int \frac{d\Omega_{v_1}}{4\pi} \frac{v_{1\mu}}{v_1 \cdot P} \int \frac{d\Omega_{v_2}}{4\pi} \frac{v_{2\nu}}{v_2 \cdot P} I(v_1, v_2).$$  

(5)

This expression is identical with ref[10] except that a prefactor $-\frac{1}{4\pi}$ exists. It is because we here adopt Litim’s notation [11, 12]

$$\int \frac{d\Omega}{4\pi} I(v_1, v_2) = 0,$$

(6)

with

$$I(v_1, v_2) = \delta^{(2)}(v_1 - v_2) - \frac{4}{\pi} \frac{(v_1 \cdot v_2)^2}{\sqrt{1 - (v_1 \cdot v_2)^2}},$$

(7)

while it has $\int d\Omega I(v_1, v_2) = 0$ in ref[10] instead.

3 Calculus of gluon self-energy based on kinetic theory

The results obtained above are based on full field theory, which have been done in detail in[10]. As we have known, the leading kinetic equations closed to thermal equilibrium can derive the HTL’s self-energy as well as field theory method[11]. Hence we here hope to reproduce the polarization tensor beyond HTL in the framework of classical kinetic theory.

3.1 Classical kinetic equations in non-Abelian plasma

The definite trajectories described with $x(\tau), p(\tau)$ and $Q(\tau)$ for color classical particles are govern by Wong equations [14]

$$m \frac{dx^\mu}{d\tau} = p^\mu, \quad m \frac{dp^\mu}{d\tau} = gQ^a F^{\mu\nu}_a p_\nu, \quad m \frac{dQ^a}{d\tau} = -gp_\mu f^{abc} A^\mu_b Q_c$$

(8)

From Liouville’s theorem $\frac{df}{d\tau} = 0$, we can immediately get

$$p^\mu [\partial_\mu - g f^{abc} A^\mu_b Q_c \partial_a^Q - gQ a F^{\mu a}_{\nu} \partial^\nu_p] f(x, p, Q) = 0,$$

(9)
where, $f_{abc}$ are the structure constants of SU($N$), $A^a_{\mu}$ denotes the microscopic vector gauge field, $F^a_{\mu\nu}[A] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$ is the corresponding microscopic field strength govern by Yang-Mills field equations

$$D_\mu F^{\mu\nu}(x) = J^\nu(x).$$

(10)

It couples to the kinetic equation (9) by the currents

$$J^\mu_a(x) = g \int dP dQ p^{\mu\nu} Q_a f(x, p, Q).$$

(11)

with the integral measure $dP \equiv d\tilde{P} \frac{d\Omega}{4\pi}$, where $d\tilde{P} = 4\pi dp_0 dp \|p\|^2 \Theta(p_0) \delta(p^2)$

In principle, the set of equations can be applied to all collective dynamics except quantum effects. When ones expand the distribution function $f(x, p, Q)$ in powers of $g$, a polarization tensor formulated as eq(1) can easily be derived from the Boltzmann equation for $f^{(1)}[2, 3]$, the distribution function at leading order in $g$.

### 3.2 Beyond HTL

The deviation $f^{(1)}$ from equilibrium distribution function is thought of as a small fluctuation closed to equilibrium state. Therefore the excitation induced by the fluctuation is of order $gT$. This just corresponds to the situation of HTL approximation. However the ultra-soft field(or mean field called in[11, 12]) dynamics can be constructed in the framework of kinetic theory if we take statistical average over Eqs(9),(10)and (11). This effective transport theory gives a detail description in[12]. Here we give a slightly different simplified version in comparison with Litim-Manuel theory. The quantities $A^a_{\mu}, f, J^\mu_a$ are decomposed into mean terms and fluctuation terms

$$A^a_{\mu} = \bar{A}^a_{\mu} + a^a_{\mu}, f = \bar{f} + \delta f, J^\mu_a = \bar{J}^\mu_a + \delta J^\mu_a,$$

(12)

Keeping only the terms including the largest contributions, we have the equations for mean fields and fluctuations

$$p^{\mu}(\bar{D}_\mu - gQ_a \bar{F}_{\mu\nu}^{\alpha} \partial_{\nu}^{\alpha})\bar{f} = \bar{\xi},$$

(13)

$$p^{\mu} \tilde{D}_\mu \delta f = gp^{\mu} Q_a f^{\alpha}_{\mu\nu} \partial_{\nu}^{\alpha} \bar{f} + gp^{\mu} a_{\mu\nu} f^{\alpha\beta\gamma} Q_a \partial_{\alpha}^{\beta} \bar{f},$$

(14)

$$D_{\mu} f^\mu_{\alpha} = \delta J^\alpha,$$

(15)
where $\bar{D} \equiv D[A], D^f \equiv D[a], \bar{F} \equiv F[A], f \equiv f[a]$ and $\xi$ reads

$$\xi = gp^\mu f^{abc} Q^c \delta^Q a_\mu \delta f. \quad (16)$$

For convenience, we use the Litim-manuel’s notion to define a new quantity $J$ by the following relation

$$J(x) = \int \frac{d\Omega}{4\pi} J(x, v). \quad (17)$$

It now is well reasoned that the equations (13) and (14) transform into the ones for currents $\mathcal{J}$ and $\delta \mathcal{J}$ by the integration of Eqs (13) and (14) over $d\bar{P}dQ$

$$v^\mu D_\mu \mathcal{J}^\rho + m_\mu^2 v^\rho v^\nu F_{\mu \nu} = \bar{\xi}^\rho, \quad (18)$$

$$(v^\mu \bar{D}_\mu \delta \mathcal{J}^\rho)_a = -m_\mu^2 v^\rho v^\mu f_{a, \mu 0} - gf_{abc} v^\mu v^\nu \partial_\mu \mathcal{J}^{c, \rho}, \quad (19)$$

and hence we have

$$\xi_\rho^a = -gf_{abc} v^\mu v^\nu \partial_\mu \delta \mathcal{J}^{c, \rho}. \quad (20)$$

From these, we can reproduce the equations found in [11, 12] follows as

$$v^\mu D_\mu \mathcal{J}^\rho(x, v) + m_\mu^2 v^\rho v^\nu F_{\mu \nu}(x) = -\gamma v^\rho \int \frac{d\Omega}{4\pi} I(v, v') \mathcal{J}^0(x, v') + \zeta^\rho(x, v), \quad (21)$$

Here we remove the notion bar denoting the mean fields, the $\zeta^\rho$ represents the stochastic noise, the coefficient $\gamma$ reads

$$\gamma = \frac{g^2}{4\pi} NT \log \left( \frac{gT}{\mu} \right). \quad (22)$$

Combining the equations and the Yang-Mills equations of mean fields together, Litim and Manuel derived the color conductivity with the logarithmic effect $\log \left( \frac{gT}{\mu} \right)$.

### 3.3 Self-energy beyond HTL approximation

We can now proceed to do calculations from eq (21) to obtain the polarization tensor beyond HTL approximation. We adopted the iterating method in [12] to solve the equation (21). The distinct scale parameters involving (21) are well separated

$$g^2 T \ll \gamma \ll gT \ll T. \quad (23)$$
Then we expand $\mathcal{J}^\mu(x, v)$ in term of $\frac{v}{v_D}$ close to the scale of $gT$

$$\mathcal{J}^\mu(x, v) = \sum_{n=0}^\infty \mathcal{J}^\mu_{(n)}(x, v),$$

(24)

Therefore, for momenta about the Debye mass, it immediately has

$$v^\mu D_\mu \mathcal{J}^\rho_{(0)}(x, v) = -m_D^2 v^\rho F_{\mu 0}(x) + \zeta^\rho(x, v),$$

(25)

while for momenta below the Debye mass, the damping term of collision integral is important,

$$v^\mu D_\mu \mathcal{J}^\rho_{(n)}(x, v) = -\gamma v^\rho \int \frac{d\Omega v'}{4\pi} I(v, v') \mathcal{J}^\rho_{(n-1)}(x, v').$$

(26)

The current $\mathcal{J}^\rho_{(0)}$ obeying the equation (25) coincides with the HTL current besides the noise term. In local limit approximation, the equation becomes in momentum space

$$-iv \cdot K \mathcal{J}^\rho_{(0)}(K, v) = -m_D^2 v^\rho v^\mu F_{\mu 0}(K),$$

(27)

Taking $\rho = 0$, we have

$$\mathcal{J}^0_{(0)}(K, v) = -im_D^2 \frac{v^\mu}{v \cdot K} F_{\mu 0}(K) = m_D^2 \left[ -g^{0\mu} + k_0 \frac{v^\mu}{v \cdot K} \right] A_\mu,$$

(28)

Substituting this into the equation for $n = 1$ in (26), we get in momentum space

$$-iv^\mu K_\mu \mathcal{J}^\rho_{(1)}(K, v) = -\gamma v^\rho \int \frac{d\Omega v'}{4\pi} I(v, v') \mathcal{J}^\rho_{(0)}(K, v').$$

(29)

The current reads

$$\mathcal{J}^\rho_{(1)}(K, v) = -im_D^2 \gamma k_0 \frac{v^\rho}{v \cdot K} \int \frac{d\Omega v'}{4\pi} \frac{v'^\mu}{v' \cdot K} I(v, v') A_\mu,$$

(30)

here we apply integral (6). The physical current can be obtained by integral over angle

$$J^\rho_{(1)} = \int \frac{d\Omega v}{4\pi} \mathcal{J}^\rho_{(1)}(K, v) = -im_D^2 \gamma k_0 \int \frac{d\Omega v}{4\pi} \frac{v^\rho}{v \cdot K} \int \frac{d\Omega v'}{4\pi} \frac{v'^\mu}{v' \cdot K} I(v, v') A_\mu.$$
Compared this with the definition

\[ J^\mu = \Pi^{\mu\nu} A_\nu, \quad (32) \]

a polarization tensor is arrived at

\[ \Pi^{\mu\nu}_\gamma(K) = -i m_D^2 \gamma k_0 \int \frac{d\Omega_v}{4\pi} \frac{v^\rho}{v \cdot K} \int \frac{d\Omega_{v'}}{4\pi} \frac{v'^\mu}{v' \cdot K} I(v, v'). \quad (33) \]

This result evidently coincides with the \( \Pi(LA) \) formulated in (5).

Continuous iteration, the higher currents are easily express in powers of \( v \cdot K \)

\[ J^{\rho}_{(n)} = -i m_D^2 k_0 \int \left[ \prod_{j=1}^{n} \frac{d\Omega_{v_j}}{4\pi} \frac{v_j}{v \cdot K} I(v_j+1, v_j) \right] v_n^\rho v_j^\mu A_\mu. \quad (34) \]

4 Summary and Discussion

The link between the temperature field theory and the kinetic theory for QCD plasma is one of the probing subjects. Ones have realized that the field theory is identical with the classical kinetic theory at level of HTL’s approximation. Recently, ones found respectively effective dynamics of soft fields or mean field dynamics from the diagrammatic approach and the kinetic theory. However the self-energy beyond HTL is given from the field theory but not from the kinetic theory.

Here we firstly compare the field theory with the kinetic theory method and find the calculations of polarization tensor in classical kinetic theory is simpler than the field theory. Secondly, we re-analyze the mean field dynamics from the kinetic theory based on the understanding of our simplified version. We follow the Litim-Manuel’s philosophy to expand the effective Boltzmann equation[12] and slightly proceed to derive the polarization tensor containing the logarithmic factor \( \log \frac{\mu}{gT} \). Our result fully coincides with the diagrammatic approach based on the full field theory. Remarkably, successively integrating out the hard and semi-hard(or called soft) degrees of freedom must be adopted in both calculations to obtain the dynamics of ultra-soft fields, but the advantage in the kinetic theory is able to directly extract the polarization tensor while much complicated treatments have to be done in the diagrammatic approach.
References

[1] F. Abe et al., Phys. Rev. Lett. 71, 3421 (1993).

[2] P. F. Kelly, Q. Liu, C. Lucchesi, C. Manuel, Phys. Rev. Lett., 72, 3461 (1994).

[3] P. F. Kelly, Q. Liu, C. Lucchesi, C. Manuel, Phys. Rev., D50, 4209 (1994).

[4] A. D. Linde, Phys. Lett. B96, 289 (1980).

[5] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[6] K. Farakos, K. Kajantie, K. Rummukainen and M. Shaposhnikov, Nucl. Phys. B425, 67 (1994).

[7] E. Braaten, A. Nieto, Phys. Rev. D51, 6990 (1995).

[8] K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, Phys. Rev. Lett. 86, 10 (2001).

[9] D. Bödeker, Phys. Lett. B426, 351 (1998).

[10] D. Bödeker, Nucl. Phys. B566, 402 (2000).

[11] D. F. Litim and C. Manuel, Phys. Rev. Lett. 82, 4981 (1999).

[12] D. F. Litim and C. Manuel, Nucl. Phys. B562, 237 (1999).

[13] See, e.g., H. A. Weldon, Phys. Rev. D26, 1394 (1982).

[14] S. Wang, Nuovo Cimento 65A, 689 (1970).
