Curvature spectra of simple Lie groups

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Abstract. The Killing form $\beta$ of a simple Lie group $G$ is a left-invariant pseudo-Riemannian Einstein metric. Let $\Omega$ denote the multiple of its curvature operator, acting on symmetric 2-tensors, with the factor chosen so that $\Omega \beta = 2 \beta$. We prove diagonalizability of $\Omega$ and describe its spectrum in each $G$. It turns out that 1 is not an eigenvalue of $\Omega$ unless $G$ is locally isomorphic to $SU(p,q)$, or $SL(n,\mathbb{R})$, or $SL(n,\mathbb{C})$ or, for even $n$ only, $SL(n/2,\mathbb{H})$, where $p \geq q \geq 0$ and $p + q = n > 2$. Due to this last conclusion, on simple Lie groups $G$ other the ones just listed, nonzero multiples of the Killing form $\beta$ are isolated among left-invariant pseudo-Riemannian Einstein metrics. Using the spectrum of $\Omega$ we also provide a proof of the known fact that a semisimple real or complex Lie algebra with no simple ideals of dimension 3 is essentially determined by its Cartan three-form.

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0. Introduction

Every Lie group $G$ carries a distinguished left-invariant torsionfree connection $D$, defined by requiring that $D_x y = [x,y]/2$ for all left-invariant vector fields $x$ and $y$. As a consequence of the Jacobi identity, the curvature tensor of $D$ is $D$-parallel. So is, consequently, the Ricci tensor of $D$, equal to a nonzero multiple of the Killing form $\beta$. Our convention about $\beta$ reads:

$$\beta(x,x) = \text{tr} [(\text{Ad} x)^2] \quad \text{for any } x \text{ in the Lie algebra } \mathfrak{g} \text{ of } G. \quad (0.1)$$

Thus, if $G$ is semisimple, $\beta$ constitutes a bi-invariant, locally symmetric, non-Ricci-flat pseudo-Riemannian Einstein metric on $G$, with the Levi-Civita connection $D$. We denote by $\Omega : [\mathfrak{g}^*]^{\otimes 2} \to [\mathfrak{g}^*]^{\otimes 2}$ a specific multiple of the curvature operator of the metric $\beta$, acting on symmetric 2-tensors:

$$[\Omega \sigma](x,y) = 2 \text{tr} [(\text{Ad} x)(\text{Ad} y)\Sigma] \quad \text{for } x,y \in \mathfrak{g}, \quad (0.2)$$

where $\sigma$ is any symmetric bilinear form on $\mathfrak{g}$, with $\Sigma : \mathfrak{g} \to \mathfrak{g}$ such that

$$\sigma(x,y) = \beta(\Sigma x,y) \quad \text{whenever } x,y \in \mathfrak{g}. \quad (0.3)$$

See Remark 1.4. The same formula (0.2) defines the operator $\Omega$ in a complex semisimple Lie group $G$. We then identify $\Omega$ with the analogous curvature operator for the ($\mathbb{C}$-bilinear) Killing form $\beta$, treating the latter as a holomorphic Einstein metric on the underlying complex manifold of $G$. 

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Our first result, about the structure of $\Omega$ in complex simple Lie groups, refers to their Lie algebras rather than the groups themselves.

**Theorem A.** For any complex simple Lie algebra $\mathfrak{g}$, the endomorphism $\Omega$ of $[\mathfrak{g}^*]^{\otimes 2}$ is diagonalizable and has the following systems $\text{Spec}[\mathfrak{g}]$ of eigenvalues and $\text{Mult}[\mathfrak{g}]$ of the corresponding multiplicities. In (c), $n$ is even and $n \geq 4$ while, in (d), $n = 7$ or $n \geq 9$.

(a) $\text{Spec}[\mathfrak{sl}_2] = (2, -1)$, $\text{Mult}[\mathfrak{sl}_2] = (1, 5)$. Also, $\text{Spec}[\mathfrak{so}_3] = (2, 1, -2/3)$, $\text{Mult}[\mathfrak{so}_3] = (1, 8, 27)$. (b) $\text{Spec}[\mathfrak{su}_n] = (2, 1, 2/n, -2/n)$, and $\text{Mult}[\mathfrak{su}_n]$ is the quadruple $(1, n^2 - 1, n^2(n - 3)(n + 1)/4, n^2(n + 3)(n - 1)/4)$.

(c) $\text{Spec}[\mathfrak{sp}_n] = (2, (n + 4)/(n + 2), -4/(n + 2), 2/(n + 2))$, while $\text{Mult}[\mathfrak{sp}_n] = (1, (n - 2)(n + 1)/2, n(n + 1)(n + 2)(n + 3)/24, n(n - 1)(n - 2)(n + 3)/12)$.

(d) $\text{Spec}[\mathfrak{so}_n] = (2, (n - 4)/(n - 2), 4/(n - 2), -2/(n - 2))$, and $\text{Mult}[\mathfrak{so}_n]$ equals $(1, (n + 2)/2, n(n - 1)(n - 2)(n - 3)/24, n(n + 1)(n + 2)(n + 3)/12)$.

(e) $\text{Spec}[\mathfrak{e}_6] = (2, 1/2, -1/6)$ and $\text{Mult}[\mathfrak{e}_6] = (1, 650, 2430)$.

(f) $\text{Spec}[\mathfrak{e}_7] = (2, 4/9, -1/9)$ and $\text{Mult}[\mathfrak{e}_7] = (1, 1539, 7371)$.

(g) $\text{Spec}[\mathfrak{e}_8] = (2, 2/5, -1/15)$ and $\text{Mult}[\mathfrak{e}_8] = (1, 3875, 27000)$.

(h) $\text{Spec}[\mathfrak{f}_4] = (2, 5/9, -2/9)$ and $\text{Mult}[\mathfrak{f}_4] = (1, 324, 1053)$.

(i) $\text{Spec}[\mathfrak{g}_2] = (2, 5/6, -1/2)$ and $\text{Mult}[\mathfrak{g}_2] = (1, 27, 77)$.

Note that all isomorphism types of complex simple Lie algebras are listed above.

We prove Theorem A in Sections 3–4. Its analog for real simple Lie algebras $\mathfrak{g}$ is Theorem 5.1, derived in Section 5 from the fact that, given any such $\mathfrak{g}$,

a) either $\mathfrak{g}$ is a real form of a complex simple Lie algebra $\mathfrak{h}$, or

b) $\mathfrak{g}$ arises by treating a complex simple Lie algebra $\mathfrak{h}$ as real. (0.4)

See [7, Lemma 4 on p. 173]. The Lie-algebra isomorphism types of real simple Lie algebras $\mathfrak{g}$ thus form two disjoint classes, characterized by (0.4.a) and (0.4.b).

For both real and complex semisimple Lie groups $G$, studying $\Omega$ can be further motivated as follows. Let ‘metrics’ on $G$ be, by definition, pseudo-Riemannian or, respectively, holomorphic, and $E$ denote the set of Levi-Civita connections of left-invariant Einstein metrics on $G$. Then, as shown in [6, Remark 12.3], whenever a semisimple Lie group $G$ has the property that 1 is not an eigenvalue of $\Omega$, the Levi-Civita connection $D$ of its Killing form $\beta$ is an isolated point of $E$. The converse implication holds except when $G$ is locally isomorphic to $SU(n)$, with $n \geq 3$. See [6, Theorems 22.2 and 22.3].

In a real/complex Lie algebra $\mathfrak{g}$, we define $\Lambda : [\mathfrak{g}^*]^{\otimes 2} \to [\mathfrak{g}^*]^{\wedge 4}$ by

$$(\Lambda \sigma)(x, y, z, z') = \sigma([x, y], [z, z']) + \sigma([y, z], [x, z']) + \sigma([z, x], [y, z']).$$

(0.5)

Thus, $\Lambda$ is a real/complex-linear operator, sending symmetric bilinear forms $\sigma$ on $\mathfrak{g}$ to exterior 4-forms on $\mathfrak{g}$. For the Killing form $\beta$ one has $\beta([x, y], [z, z']) = \beta([x, y], z, z')$, ad $z$ is $\beta$-skew-adjoint. By the Jacobi identity and (0.1) – (0.2),

i) $\Lambda \beta = 0$, ii) $\Omega \beta = 2\beta$. (0.6)
If, in addition, \( g \) is semisimple, there is also the operator \( \Pi : [g^*]^\otimes 4 \to [g^*]^\otimes 2 \) with
\[
\Pi(\xi \otimes \xi' \otimes \eta \otimes \eta') = \beta([x, x'], \cdot) \otimes \beta([y, y'], \cdot),
\]
for \( \xi, \xi', \eta, \eta' \in g^* \), where \( x, x', y, y' \in g \) are characterized by \( \xi = \beta(x, \cdot), \xi' = \beta(x', \cdot), \eta = \beta(y, \cdot), \eta' = \beta(y', \cdot) \). We have \( \Pi([g^*]^\otimes 4) \subset [g^*]^\otimes 2 \), cf. formula (2.1).

The next result, established in Section 2, relates \( \Omega \) to \( \Pi \Lambda : [g^*]^\otimes 2 \to [g^*]^\otimes 2 \), the composite of \( \Lambda \) and the restriction of \( \Pi \) to the subspace \([g^*]^\otimes 4 \subset [g^*]^\otimes 4\).

**Theorem B.** Let \( \Omega, \Lambda \) and \( \Pi \) be the operators defined by (0.2), (0.5) and (0.7) for a given semisimple real/complex Lie algebra \( g \). Then \( 2\Pi \Lambda = - (\Omega + \text{Id})(\Omega - 2 \text{Id}) \).

Our final application of Theorem A, using Theorem B as well, is the following description of \( \text{Ker} \Lambda \) for semisimple Lie algebras \( g \), obtained in Section 6. It provides a crucial step in our proof of Theorem D (see below).

**Theorem C.** Given a real/complex semisimple Lie algebra \( g \) with a decomposition \( g = g_1 \oplus \ldots \oplus g_s \) into simple ideals, \( s \geq 1 \), let \( \Lambda \) and \( \Lambda_i \) denote the operator defined by (0.5) for \( g \) and, respectively, its analog for the \( i \)th summand \( g_i \).

(a) \( \text{Ker} \Lambda = \text{Ker} \Lambda_1 \oplus \ldots \oplus \text{Ker} \Lambda_s \), with \([g_i^*]^\otimes 2 \subset [g^*]^\otimes 2 \) by trivial extensions.
(b) \( \Lambda = 0 \) if \( \dim g = 3 \).
(c) \( \dim \text{Ker} \Lambda = 12 \) if \( g \) is simple and \( \dim g = 6 \).
(d) \( \dim \text{Ker} \Lambda \in \{1, 2\} \) whenever \( g \) is simple and \( \dim g \notin \{3, 6\} \).

In (c), \( g \) is necessarily real and isomorphic to the underlying real Lie algebra of \( \mathfrak{sl}(2, \mathbb{C}) \), while \( \text{Ker} \Lambda \) consists of the real parts of all symmetric \( \mathbb{C} \)-bilinear functions \( g \times g \to \mathbb{C} \). In (d), the Killing form \( \beta \) spans \( \text{Ker} \Lambda \) if \( g \) is either complex or real of type (0.4.a), while otherwise \( \text{Ker} \Lambda \) is spanned by \( \text{Re} \beta^h \) and \( \text{Im} \beta^h \) for the Killing form \( \beta^h \) of a complex simple Lie algebra \( h \) with (0.4.b).

One defines the **Cartan three-form** \( C \in [g^*]^\wedge 3 \) of a Lie algebra \( g \) by
\[
C = \beta([\cdot, \cdot], \cdot), \quad \text{where } \beta \text{ denotes the Killing form. (0.8)}
\]

The last result has been known for decades, although no published proof of it seems to exist [3]. By an **isomorphism of the Cartan three-forms** we mean a vector-space isomorphism of the Lie algebras, sending one three-form onto the other.

**Theorem D.** Let \( g \) be a real/complex semisimple Lie algebra with a fixed direct-sum decomposition into simple ideals.

(i) **If** \( h \) is a real/complex Lie algebra such that \( g \) and \( h \) have isomorphic Cartan three-forms and, in the real case, \( g \) has no simple direct summands of dimension 3, **then** \( h \) is isomorphic to \( g \).

(ii) **If** \( g \) contains no simple ideals of dimension 3 or 6, **then** every automorphism of the Cartan three-form of \( g \) is a Lie-algebra automorphism of \( g \) followed by a linear automorphism that acts on each simple direct summand as a multiplication by a cubic root of 1.
Conversely, if a decomposition of \( g \) into simple ideals has \( k \) summands of dimension 3 and \( l \) summands of dimension 6, then the Lie-algebra automorphisms of \( g \) form a subgroup of codimension \( 5k + 12l \) in the automorphism group of the Cartan three-form.

We derive Theorem D from Theorem C, in Section 7.

1. Preliminaries

Consider the underlying real Lie algebra \( g \) of a complex Lie algebra \( h \). We denote by \( \beta \) the Killing form of \( g \), by \( \Lambda \) the operator in (0.5) associated with \( g \), and use the symbols \( \beta^h, \Lambda^h \) for their counterparts corresponding to \( h \). Obviously, whenever \( \sigma : g \times g \to \mathbb{C} \) is a symmetric \( \mathbb{C} \)-bilinear form,

\[
\begin{align*}
\text{i)} \quad & \beta = 2 \Re \beta^h, \\
\text{ii)} \quad & \Lambda(\Re \sigma) = \Re (\Lambda^h \sigma).
\end{align*}
\]

For (1.1.i), see also [6, formula (13.1)].

Remark 1.1. With \( g \) and \( h \) as above, it is clear from (1.1.i) that \( \Re \beta^h \) and \( \Im \beta^h \) span the real space of symmetric bilinear forms \( \sigma \) on \( g \) arising via (0.3) from linear endomorphisms \( \Sigma \) which are complex multiples of \( \text{Id} \).

Lemma 1.2. Given a linear endomorphism \( \Omega \) of a real/complex vector space \( T \), subspaces \( W_i \subset T \), and mutually distinct scalars \( a_i \), \( i = 1, \ldots, m \), such that \( \dim T < \infty \) and \( W_i \subset \ker (\Omega - a_i \text{Id}) \) for all \( i \), let \( W_1, \ldots, W_m \) span \( T \).

Then \( \Omega \) is diagonalizable, has the eigenvalues \( a_i \), and the corresponding eigenspaces \( W_i \), with \( i \) ranging over the set \( \{ i : W_i \neq \{ 0 \} \} \).

Proof. This is clear from linear independence of a union of linearly independent subsets of eigenspaces of \( \Omega \) corresponding to distinct eigenvalues: the subsets in question are bases of the subspaces \( W_i \). \( \square \)

Let \( g \) now be a Lie algebra over the scalar field \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \). A fixed basis of \( g \) allows us to represent elements \( x, y \) of \( g \), symmetric bilinear forms \( \sigma \) on \( g \), and the Lie-algebra bracket operation \( [\, , \] \) by their components \( x^i, y^i, \sigma_{ij} \) and \( C_{ijk} \) (the structure constants of \( g \)), so that \( \sigma(x, y) = \sigma_{ij} x^i y^j \) and \( [x, y]^k = C_{ijk} x^j y^j \). Repeated indices are summed over. The Cartan three-form \( C \) with (0.8) has the components \( C_{ijk} = C_{ij}^r \beta_{kr} \), where \( \beta \) is the Killing form. The definition (0.1) of \( \beta \), its bi-invariance, and the Jacobi identity now read

\[
\begin{align*}
\text{i)} \quad & \beta_{ij} = C_{ip}^q C_{jq}^p, \\
\text{ii)} \quad & C_{ijk} \text{ is skew-symmetric in } i, j, k, \\
\text{iii)} \quad & C_{ij}^q C_{qk}^l + C_{jk}^q C_{qi}^l + C_{ki}^q C_{qj}^l = 0. 
\end{align*}
\]

In the remainder of this section \( g \) is also assumed to be semisimple. We can thus lower and raise indices using the components \( \beta_{ij} \) of the Killing form \( \beta \) and \( \beta^{ij} \) of its reciprocal: \( C^{ijk} = \beta^{ir} \beta^{js} C_{rs}^{\phantom{rs}k} \), and \( C_{ij}^{sp} = \beta^{sk} C_{kp}^{\phantom{kp}i} \). For any \( x, y, z \in g \), one has \( 2 \operatorname{tr} [(\text{Ad} x)(\text{Ad} y)(\text{Ad} z)] = C(x, y, z) \), that is,

\[
2 C_{ir}^{\phantom{ir}p} C_{jq}^{\phantom{jq}q} C_{kp}^{\phantom{kp}i} = C_{ijk}.
\]
In fact, by successively using the equalities \( C^k_p = q = C^p_qk \) and \( C^p_rT = -C^r_pT \) (both due to (1.2.ii)), then again (1.2.iii), and (1.2.i–ii), we see that

\[
2C^p_rT^rC^k_q = 2C^r_pT^rC^q_p = C^r_pT^r(C^q_prC^k_q + C^p_qrC^q_k) = C^r_pT^rC^q_pC^s_rC^k_s = \delta^s_pC^s_rC^k_s = C^s_r.
\]

Lowering the index \( k \), we obtain (1.3). Next, we introduce the linear operator

\[
T : [g^*]^{\otimes 2} \to [g^*]^{\otimes 2} \quad \text{with} \quad (T\sigma)_{ij} = T^{kl}_{ij}\sigma_{kl}, \quad \text{where} \quad T^{kl}_{ij} = 2C^p_{ip}C^r_{jpr}.
\]

**Lemma 1.3.** For \( T \) and the operator \( \Omega : [g^*]^{\otimes 2} \to [g^*]^{\otimes 2} \) given by (0.2),

(a) \( T \) leaves the subspaces \([g^*]^{\otimes 2}\) and \([g^*]\) invariant,

(b) \( \Omega \) coincides with the restriction of \( T \) to \([g^*]^{\otimes 2}\).

**Proof.** This is clear from (1.4) and the fact that, by (1.4), \( T\sigma \) is the same as \( \Omega\sigma \) in (0.2) – (0.3), except that now \( \sigma : g \times g \to \mathbb{F} \) need not be symmetric. \( \square \)

**Remark 1.4.** The curvature operator of a (pseudo)Riemannian metric \( \gamma \) on a manifold, acting on symmetric 2-tensors, has been studied by various authors [4], [2], [1, pp. 51–52]. It is given by \( 4R(x,y)z = [[x,y],z] \) for left-invariant vector fields \( x,y,z \), that is, \( 4R_{ij} = C_{ij}^pC^p_{kl} \). Lemma 1.3(b) now implies our claim, as \( T^{kl}_{ij} = -8\beta^{kp}R_{jps} \) due to (1.2.ii) and (1.4).

**2. Proof of Theorem B**

We use the component notation of Section 1. According to (0.5) and (0.7),

\[
\begin{align*}
(L\sigma)_{ijkl} &= \Lambda_{ijkl} \equiv r^s\sigma_{rs} \quad \text{with} \quad \Lambda_{ijkl} = C^r_{ij}C^s_{kl} + C^r_{jk}C^s_{il} + C^r_{ki}C^s_{jl}, \\
(P\gamma)_{pq} &= C^s_{ij}C^q_{kl}\zeta_{ijkl}, \quad \text{whenever} \quad \sigma \in [g^*]^{\otimes 2} \quad \text{and} \quad \zeta \in [g^*]^{\otimes 4}.
\end{align*}
\]

(2.1)

In any real/complex semisimple Lie algebra \( g \), for \( C^r_{ij}, T^{kl}_{ij} \) as in Section 1,

\[
2C^p_{ik}C^q_{jl}(C^r_{ij}C^s_{kl} + C^r_{jk}C^s_{li} + C^r_{ki}C^s_{jl}) = 2\delta^r_p\delta^q_s + T^{rs}_{ij} - T^{ik}_{ip}T^{js}_{js}.
\]

(2.2)

In fact, the first of the three terms naturally arising on the left-hand side of (2.2) equals \( 2\delta^r_p\delta^q_s \) since, by (1.2.i–ii), \( C^{ij}_{ip}C^{kl}_{ij} = -\delta^r_p \) and \( C^{kl}_{ij}C^{rs}_{kl} = -\delta^s_q \). The other two terms coincide (as skew-symmetry of \( C^{ij}_{ip} \) in \( i,j \) gives \( C^{ij}_{ip}C^{kl}_{ij} = -C^{ij}_{ip}C^{kl}_{ji} = C^{ij}_{pq}C^{qk}_{ji} \)), and so they add up to \( 4C^{kl}_{ij}C^{r}_{ip}C^{s}_{js} \), that is, \( 4C^{kl}_{ij}C^{r}_{ip}C^{s}_{js} = 4C^{kl}_{ij}C^{r}_{ip} + 4C^{kl}_{ij}C^{s}_{ip}\).
3. Theorem A for exceptional Lie algebras

**Lemma 3.1.** For the operators $\Omega : [\mathfrak{g}^*]^{\otimes 2} \to [\mathfrak{g}^*]^{\otimes 2}$ and $T : [\mathfrak{g}^*]^{\otimes 2} \to [\mathfrak{g}^*]^{\otimes 2}$ with (0.2) and (1.4), in a real/complex semisimple Lie algebra $\mathfrak{g}$ of dimension $d$,

(i) the restriction of $T$ to $[\mathfrak{g}^*]^{\otimes 2}$ is diagonalizable, cf. Lemma 1.3(a), with the eigenvalues 0 and 1 of multiplicities $d(d-3)/2$ and $d$,

(ii) $\text{tr } T = 0$, $\text{tr } T^2 = 4d$, $\text{tr } T^3 = 2d$,

(iii) $\text{tr } \Omega^0 = \dim([\mathfrak{g}^*]^{\otimes 2}) = d(d+1)/2$, and $\text{tr } \Omega = -d$, $\text{tr } \Omega^2 = 3d$, $\text{tr } \Omega^3 = d$.

**Proof.**

Next, the scalars $T_{kl}^{ij}$ representing $T$ as in (1.4) give rise to analogous components $(T^2)^{ij}_{kl} = T^p_q T^k_l = (T^3)^{ij}_{kl} = T^r_p T^r_{pq}$ for $T^2$ and $T^3$. Using (1.2) and (1.4), we now obtain (ii).

Specifically, $T_{kl}^{ij} = 2C_{kp}^l C_{ip}^j = 0$ as well as $(T^2)^{kl}_{ij} = T_{kl}^p T^p_{pq} = 4 C^k_{rs} C^l_{pq} C^j_{rs} C^s_{lj} = 4 C^k_{rs} C^l_{pq} (C^j_{rs} C^s_{lj}) = 4 \beta_{rs} \beta_{lj} = 2d$ by (1.2),

while (1.3) and (1.2) yield $(T^3)^{kl}_{ij} = T_{kl}^p T^p_{pq} T^q_{rs} T^q_{pq} = 8 C^k_{ij} C^l_{ij} C^m_{io} C^m_{jo} = -8(C_{ij} C^m_{ij} C^m_{ij} C^m_{ij}) = -2 C_{ij} C^m_{ij} = 2 \beta_{ij} = 2d$.

Finally, as a consequence of (i), the trace of the restriction to $[\mathfrak{g}^*]^{\otimes 2}$ of every positive power of $T$ equals $d$. Thus, by Lemma 1.3, $\text{tr } T^m = d + \text{tr } \Omega^m$ for every integer $m > 0$, and (iii) follows from (ii).

According to [5, p. ...], for the exceptional complex simple Lie algebras $\mathfrak{g}$ appearing in the lines $(e_8) - (g_2)$ of Theorem A, the triples $\text{Mult} [\mathfrak{g}] = (1, l, m)$ of multiplicities listed in each line consists precisely of the dimensions of the three irreducible direct summands $\mathcal{V}_1, \mathcal{V}_l, \mathcal{V}_m$ of $[\mathfrak{g}^*]^{\otimes 2}$ for the adjoint representation of $\mathfrak{g}$. Every ad-invariant subspace of $[\mathfrak{g}^*]^{\otimes 2}$ must therefore be the direct sum of some subset of $\{\mathcal{V}_1, \mathcal{V}_l, \mathcal{V}_m\}$. This applies, in particular, to the subspace spanned by the Killing form $\beta$ (which, consequently, coincides with $\mathcal{V}_1$), and to any eigenspace of $\Omega$. Furthermore, $\Omega$ is $\beta$-self-adjoint [6, Lemma 11.2(ii)], and so, as each of the exceptional Lie algebras in question has a compact real form [7, Lemma .. on p. ...], $\Omega$ must be diagonalizable. As a result, $\mathcal{V}_1 \subset \ker (\Omega - 2 \text{Id})$, cf. (0.6.ii), while $\mathcal{V}_l \subset \ker (\Omega - \lambda \text{Id})$ and $\mathcal{V}_m \subset \ker (\Omega - \mu \text{Id})$ for some $\lambda, \mu \in \mathbb{C}$.

Setting $N = l + m - (d - 1)(d + 2)/2$, $P = l \lambda + m \mu + (2 + d)$, $Q = l \lambda^2 + m \mu^2 + (4 - 3d)$, $R = l \lambda^3 + m \mu^3 + (8 - d)$, where $d = \dim \mathfrak{g}$, and then $K = (\lambda + \mu)P - Q - \lambda \mu N$, $L = \lambda \mu P - (\lambda + \mu)Q + R$, $M = (d - 1)L + 2K$, we easily obtain $M = d(d - 3)(3(\lambda + \mu) - 1)$. However, Lemma 3.1(iii) states that
$N = P = Q = R = 0$. Thus, $K = L = M = 0$. As $d(d-3)\lfloor (3\lambda+\mu-1) \rfloor = M = 0$, we have $\lambda + \mu = 1/3$. Equating $P - l(\lambda + \mu - 1/3)$ and $P - m(\lambda + \mu - 1/3)$ with 0, we now see that $\lambda = (d+2-m/3)/(m-1)$ and $\mu = (d+2-l/3)/(l-m)$, as claimed in Theorem A; $d$ in the lines $(e_6) - (g_2)$ equals 78, 133, 248, 52 and 14, which is clear since $N = 0$.

4. The remaining part of Theorem A

Throughout this section $\mathcal{V}$ is a vector space of dimension $n \geq 3$ over the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $h$ denotes a nondegenerate $\mathbb{F}$-bilinear form $\mathcal{V} \times \mathcal{V} \to \mathbb{F}$, assumed in addition to be symmetric or skew-symmetric:

$$h(w, v) = \pm h(v, w) \quad \text{for all } v, w \in \mathcal{V}, \quad (4.1)$$

with a fixed sign $\pm$. The Lie algebra $\mathfrak{gl}(\mathcal{V})$ of all linear endomorphisms $\mathcal{V} \to \mathcal{V}$ contains the Lie subalgebra $\mathfrak{g}$ associated with the group of $h$-preserving linear automorphisms of $\mathcal{V}$. Explicitly,

$$\mathfrak{g} = \{ x \in \mathfrak{gl}(\mathcal{V}) : x^\ast = -x \}, \quad (4.2)$$

where $x^\ast \in \mathfrak{gl}(\mathcal{V})$ is the $h$-adjoint of $x$, characterized by $h(xv, w) = h(v, x^\ast w)$. Note that $x^{\ast\ast} = x$ and $(xy)^\ast = y^\ast x^\ast$ whenever $x, y \in \mathfrak{gl}(\mathcal{V})$.

A basis $e_i$ of $\mathfrak{g}$, with $i = 1, \ldots, n$, allows us to represent vectors $v, w \in \mathcal{V}$, bilinear forms $m$ on $\mathcal{V}$, and linear endomorphisms $x \in \mathfrak{gl}(\mathcal{V})$, including elements of $\mathfrak{g}$, by their components $v^i, w^i, m_{ij}$ and $x^i_j$, so that $v = v^i e_i$, $m(v, w) = m_{ij} v^i w^j$ and $(xy)^i_j = x^i_k v^k_j$. Repeated indices are summed over. The reciprocal tensor of $h$ has the components $h^{ij}$ forming the inverse matrix of $[h_{ij}]$. Thus, $h^{ik} h_{kj} = \delta^i_j$, where $\delta^i_j$ (the Kronecker delta) equals $x^i_j$ for $x = \text{Id}$. We use $h^{ij}$ and $h_{ij}$ to raise and lower indices:

\begin{align*}
\text{Lemma 4.1.} \quad \text{For } & \mathfrak{V}, n, \mathbb{F}, h, \pm, \mathfrak{g} \text{ as above, any } x, y \in \mathfrak{gl}(\mathcal{V}), \text{ and the linear endomorphisms } z \mapsto xzy \text{ and } z \mapsto xz^\ast y \text{ of } \mathfrak{gl}(\mathcal{V}). \\
&(i) \quad \text{tr } \{ z \mapsto xzy \} = (\text{tr } x) \text{ tr } y \text{ and } \text{tr } \{ z \mapsto xz^\ast y \} = \pm \text{ tr } x y^\ast, \\
&\text{(ii) } \beta(x, y) = (n + 2) \text{ tr } xy \text{ if } x, y \in \mathfrak{g} \text{ and } \beta \text{ is the Killing form of } \mathfrak{g}, \text{ with } (0.1). 
\end{align*}

Proof. Let $a,b \in \mathfrak{gl}(n, \mathbb{F})$. As $(\text{Ad } a)\text{Ad } b$ sends $v \in \mathfrak{gl}(n, \mathbb{F})$ to $v \mapsto abv - ava + bva$, its $\mathfrak{gl}(n, \mathbb{F})$-trace is $2n(a, b) - 2(\text{tr } a) \text{ tr } b$, cf. (4.3). By Remark 4.2(i), if $a, b \in \mathfrak{sl}(n, \mathbb{F})$, this equals the $\mathfrak{sl}(n, \mathbb{F})$-trace of $(\text{Ad } a)\text{Ad } b$, which proves our claim in the case where $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$.

Our assertion for $\mathfrak{g} = \mathfrak{su}(l,j)$ now follows from (??) applied to $\mathfrak{g} = \mathfrak{su}(l,j)$ and $\mathfrak{g}^\mathbb{F} = \mathfrak{sl}(n, \mathbb{F})$, cf. (??). \qed
\[ g \text{ as above, any } v, w \in g, \text{ let } I F = \mathbb{R} \text{ or } I F = \mathbb{C}. \] For the linear endomorphism \[ v \mapsto uvw \] of \( gl(n, I R) \),

the \( gl(n, I F) \)-trace of

\[ (uvw)_q^s = u_p^s v_p^r w_q^r. \]

**Remark 4.2.** Let \( V \) be a vector space with \( \dim I F V < \infty \).

(i) If \( A(\tilde{V}) \subset \tilde{V} \) for a linear endomorphism \( A \) of \( V \) and a subspace \( \tilde{V} \subset V \),

then the trace of \( A : V \to V \) equals the trace of its restriction to \( \tilde{V} \).

(ii) Given a linear functional \( \alpha \in V^* \), and a vector \( w \in V \), one clearly has

\[ \text{tr} (\alpha \otimes w) = \alpha(w), \]

where \( \alpha \otimes w \) acts on \( v \in V \) by \( v \mapsto \alpha(v)w \).

**Remark 4.3.** In the next two sections traces of linear endomorphisms \( A \) of \( sl(n, I F) \) will be evaluated as follows:

(a) write \( A \) as an endomorphism \( gl(n, I F) \), valued in \( sl(n, I F) \),

(b) find the \( gl(n, I F) \)-trace of the latter, from either (4.3) or Remark 4.2(ii),

(c) note that, by Remark 4.2(i), this is also the \( sl(n, I F) \)-trace of \( A \).

All vector spaces discussed here are assumed real or complex and finite-dimensional. An inner product in a vector space over \( I F = \mathbb{R} \text{ or } I F = \mathbb{C} \) is an \( I F \)-valued nondegenerate symmetric \( I F \)-bilinear form. Real inner-product spaces are often referred to as *pseudo-Euclidean*.

By an algebraic curvature tensor [1, p. 46] in an inner-product space \( V \) over \( I F = \mathbb{R} \text{ or } I F = \mathbb{C} \) we mean any \( I F \)-valued quadrilinear form \( \sigma : V \times V \times V \times V \to I F \), skew-symmetric both in the first and last pair of arguments, and satisfying the Bianchi identity (in the sense that it yields 0 when summed cyclically over the first three arguments). These properties are well-known [8, p. 54] to imply symmetry with respect to the switch of the first and last pair of arguments: \( \sigma(v, v', w, w') = \sigma(w, w', v, v') \).

Parts (a) and (b) were proved in [6, Lemma 17.1]. For the final clause of Theorem A, see [9, pp. 8 and 77].

Let \( \mathfrak{g} = sl(V) \). The \( S^2 \mathfrak{g} \) consists of tensors \( F_{ij}^{kl} \) such that \( F_{ij}^{li} = 0 \) and \( F_{ij}^{lj} = F_{ji}^{lk} \).

Given \( \phi, \Phi_i^j, S_{ij}^{kl}, \) and \( A_{ij}^{kl} \) such that \( \phi \) is a number, \( \Phi_i^j = 0, S_{ij}^{kl} = S_{ji}^{lk}, \)

\( S_{ij}^{kl} = 0, \) and \( A_{ij}^{kl} = -A_{ji}^{lk}, A_{kj}^{lj} = 0 \) we can construct

\[ F_{ij}^{kl} = \phi \delta_i^k \delta_j^l + \Phi_i^j \delta_j^l + \Phi_j^i \delta_i^k + S_{ij}^{kl} + A_{ij}^{kl} \in S^2 \mathfrak{g} \]

and such a decomposition is unique.

In other words \( S^2 \mathfrak{g} = 1 \oplus \mathfrak{g} \oplus A \oplus S \), where \( A \) and \( S \) are spaces of tensors of the above symmetry. Note that for \( n = 3 \) the space \( A \) is empty.

\[ \mathfrak{g} = so_n, \quad n \geq 5 \]
Let $\mathfrak{g} = \mathfrak{so}(V, g)$. The $S^2\mathfrak{g}$ consists of tensors $F_{ijkl}$ such that $F_{ijkl} = -F_{jikl} = F_{klij}$.

Given $\phi, \Phi_{ij}, A_{ijkl},$ and $S_{ijkl}$ such that $\phi$ is a number, $\Phi_{ij} = \Phi_{ji}, \Phi_{ij}g^{ij} = 0,$ $S_{ijkl} = -S_{jikl} = S_{klij}, S_{ijkl} + S_{kijl} + S_{ijkl} = 0, S_{ijkl}g^{jk} = 0,$ and $A_{ijkl} = -A_{jikl} = -A_{ikjl} = -A_{ijlk},$ we can construct

$$F = \phi g * g + \Phi * g + A + S \in S^2\mathfrak{g},$$

where $(X * Y)_{ijkl} = X_{ik}Y_{jl} + X_{jl}Y_{ik} - X_{il}Y_{jk} - X_{jk}Y_{il}$ and such a decomposition is unique.

In other words, $\mathfrak{g} = \Lambda^2 V$ and $S^2\mathfrak{g} = 1 \oplus S^2_0 V \oplus \Lambda^4 V \oplus W$ is the decomposition into irreducibles.

Let $\mathfrak{g} = \mathfrak{sp}(V, \omega)$. The $S^2\mathfrak{g}$ consists of tensors $F_{ijkl}$ such that $F_{ijkl} = F_{jikl} = F_{klij}$.

Given $\phi, \Phi_{ij}, S_{ijkl},$ and $A_{ijkl}$ such that $\phi$ is a number, $\Phi_{ij} = -\Phi_{ji}, \Phi_{ij}\omega^{ij} = 0, S_{ijkl} = S_{jikl} = S_{kijl} = S_{ijlk},$ and $A_{ijkl} = A_{jikl} = A_{klij}, A_{ijkl} + A_{ikjl} + A_{ijlk} = 0, A_{ijkl}\omega^{jk} = 0,$ we can construct

$$F = \phi \omega * \omega + \Phi * \omega + S + A \in S^2\mathfrak{g},$$

where $(X * Y)_{ijkl} = X_{ik}Y_{jl} + X_{jl}Y_{ik} + X_{il}Y_{jk} + X_{jk}Y_{il}$ and such a decomposition is unique.

In other words, $\mathfrak{sp}(V) = S^2V$ and $S^2\mathfrak{g} = 1 \oplus \Lambda^2_0 V \oplus S^4V \oplus W$ is the irreducible decomposition.

5. The spectrum of $\Omega$ in real simple Lie algebras

Theorem 5.1. Let $\Omega$ denote the operator with (0.2) corresponding to a fixed real simple Lie algebra $\mathfrak{g}$, and $\Omega^\mathfrak{h}$ its analog for $\mathfrak{h}$ chosen so as to satisfy (0.4).

(i) $\Omega$ is always diagonalizable.

(ii) In case (0.4.a), $\Omega$ has the same spectrum as $\Omega^\mathfrak{h}$, including the multiplicities.

(iii) In case (0.4.b), the spectrum of $\Omega$ arises from that of $\Omega^\mathfrak{h}$ by first doubling the original multiplicities, and then including 0 as an additional eigenvalue with the required complementary multiplicity. Note that, according to Theorem A, 0 is not an eigenvalue of $\Omega^\mathfrak{h}$.

(iv) The eigenspace $\text{Ker} (\Omega - 2\text{Id})$ is spanned in case (0.4.a) by $\beta$, and in case (0.4.b) by $\text{Re} \beta^\mathfrak{h}$ and $\text{Im} \beta^\mathfrak{h}$, for the Killing forms $\beta$ and $\beta^\mathfrak{h}$ of $\mathfrak{g}$ and $\mathfrak{h}$.

Proof. By [6, Lemma 14.3(ii) and formulae (14.5) – (14.7)], if $\mathfrak{g}$ is of type (0.4.a), the complexification of $[\mathfrak{g}^\mathfrak{c}]^\otimes 2$ may be naturally identified with its (complex) counterpart $[\mathfrak{h}^\mathfrak{c}]^\otimes 2$ for $\mathfrak{h}$, in such a way that $\Omega^\mathfrak{h}$ becomes the unique $\mathfrak{c}$-linear extension of $\Omega$ and the Killing form $\beta$ and $\beta^\mathfrak{h}$ coincide. Now Theorem A, which clearly implies that
(v) **in complex simple Lie algebras** 2 is an eigenvalue of \( \Omega \) with multiplicity 1, combined with (0.6.ii) yields (i), (ii) and (iv) in case (0.4.a).

For \( g \) of type (0.4.a), Lemma 13.1 of [6] states the following. First, \( [g^*]^2 \) is the direct sum of two \( \Omega \)-invariant subspaces: one formed by the real parts of \( C \)-bi-linear symmetric functions \( \sigma : h \times h \rightarrow \mathbb{C} \), the other by the real parts of functions \( \sigma : h \times h \rightarrow \mathbb{C} \) which are antilinear and Hermitian. Secondly, \( \Omega \) vanishes on the “Hermitian” summand, and its action on the “symmetric” summand is equivalent, via the isomorphism \( \sigma \mapsto \text{Re} \sigma \), to the action of \( \Omega^h \) on \( \mathbb{C} \)-bilinear symmetric functions \( \sigma \). With diagonalizability of \( \Omega^h \) again provided by Theorem A, this proves our remaining claims. (The multiplicities are doubled since the original complex eigenspaces are viewed as real, while the eigenspace \( \Omega^h \) for the eigenvalue 2 consists, by (v) and (0.6.ii), of complex multiples of \( \beta^h \), the real parts of which are precisely the real linear combinations of \( \text{Re} \beta^h \) and \( \text{Im} \beta^h \).)

**Remark 5.2.** We will need the well-known fact [9, p. 30] that, up to isomorphisms, \( \mathfrak{sl}(n, \mathbb{R}) \) as well as \( \mathfrak{su}(p,q) \) with \( p + q = n \) and, if \( n \) is even, \( \mathfrak{sl}(n/2, \mathbb{H}) \), are the only real forms of \( \mathfrak{sl}(n, \mathbb{C}) \).

**Lemma 5.3.** The only complex, or real, simple Lie algebras of dimensions less than 7 are, up to isomorphisms, \( \mathfrak{sl}(2, \mathbb{C}) \) or, respectively, \( \mathfrak{sl}(2, \mathbb{R}) \), \( \mathfrak{su}(2) \), \( \mathfrak{su}(1,1) \) and \( \mathfrak{sl}(2, \mathbb{C}) \), the last one being both complex three-dimensional and real six-dimensional. Consequently,

(i) a complex simple Lie algebra cannot be six-dimensional,

(ii) there is just one isomorphism type of a complex or, respectively, real simple Lie algebra of dimension 3 or, respectively, 6, both represented by \( \mathfrak{sl}(2, \mathbb{C}) \),

(iii) \( \dim g \notin \{1, 2, 4, 5 \} \) for every real or complex simple Lie algebra \( g \).

**Proof.** By the final clause of Theorem A, in the complex case \( \mathfrak{sl}(2, \mathbb{C}) \) is the only possibility. For real Lie algebras, one can use Remark 5.2 and (0.4). \( \square \)

**Remark 5.4.** Using Theorem 5.1, we can now justify the claim, made in [6, Remark 12.3], that 1 is not an eigenvalue of \( \Omega \) in any real or complex simple Lie algebra except the ones isomorphic to

\[
\mathfrak{sl}(n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}), \mathfrak{su}(p,q) \quad \text{or, for even \( n \) only,} \quad \mathfrak{sl}(n/2, \mathbb{H}),
\]

where \( n = p + q \geq 3 \). In fact, by Theorem A and parts (ii) – (iii) of Theorem 5.1, the only real or complex simple Lie algebras in which \( \Omega \) has the eigenvalue 1 are, up to isomorphisms, \( \mathfrak{sl}(n, \mathbb{C}) \) for \( n \geq 3 \) and their real forms. According to Remark 5.2, these are all included in the list (5.1).

6. **Proof of Theorem C**

Let \( \sigma \in [g^*]^2 \) and \( \Delta \sigma = 0 \). Hence, by (0.5), \( \sigma([u,v],[w,w']) + \sigma([v,w],[u,w']) + \sigma([w,u],[v,w']) = 0 \) for all \( u,v,w,w' \) in \( g \). Thus, \( \sigma([u,v],[w,w']) = 0 \) whenever \( u,v \in h_i \) and \( w,w' \in h_j \) with \( j \neq i \). The summands \( h_i \) and \( h_j \), being simple, are
spanned by such brackets \([u, v]\) and \([w, w']\), so that \(h_i\) is \(\sigma\)-orthogonal to \(h_j\). As this is the case for any two summands, we obtain (a), the right-to-left inclusion being obvious. Next,

\[
\text{Ker} (\Omega - 2 \text{Id}) \subset \text{Ker} \Lambda \subset \text{Ker} (\Omega - 2 \text{Id}) \oplus \text{Ker} (\Omega + \text{Id}). \tag{6.1}
\]

In fact, the second inclusion is obvious from Theorem B; the first, from Theorem 5.1(iv), (0.6.i) and (1.1.ii) applied to complex multiples \(\sigma\) of \(\beta h\).

Part (b) of Theorem C is immediate, as \([g^*]^\wedge 3 = \{0\}\) when \(\dim g = 3\). Also, if \(g\) is simple and \(\dim g = 6\), Lemma 5.3(iii) implies that \(g\) is real and isomorphic to \(sl(2, \mathbb{C})\). From (1.1.ii), with \(\Lambda h\sigma = 0\) due to (b), one in turn obtains \(F \subset \text{Ker} \Lambda\) for \(F = \{\text{Re } \sigma : \sigma \in [g^*]^{\odot 2}\}\), where \([g^*]^{\odot 2}\) denotes the space of all symmetric \(\mathbb{C}\)-bilinear forms \(\sigma : g \times g \rightarrow \mathbb{C}\). As \(\text{Re } \sigma\) uniquely determines such \(\sigma\), that is, the operator \(\sigma \mapsto \text{Re } \sigma\) is injective, we have \(\dim_{\mathbb{R}} F = 12\). The second inclusion in (6.1) is therefore an equality, and \(F = \text{Ker} \Lambda\), for dimensional reasons:

\[
\text{Ker} \Lambda \text{ contains the subspace } F \text{ of real dimension 12, equal, in view of part (a) of Theorem A and Theorem 5.1(iii), to } \dim_{\mathbb{R}} [\text{Ker} (\Omega - 2 \text{Id}) \oplus \text{Ker} (\Omega + \text{Id})].
\]

This yields assertion (c) in Theorem C and the final comment about (c).

Let \(g\) now be simple, with \(\dim g \not\in \{3, 6\}\). Due to Theorems A and 5.1(ii)-(iii), \(-1\) is not an eigenvalue of \(\Omega\). Thus, \(\text{Ker} (\Omega + \text{Id}) = \{0\}\), and the inclusions in (6.1) are equalities which, by Theorem 5.1(iv), completes the proof.

7. Proof of Theorem D

For a real/complex Lie algebra \(g\), let the mapping \(\Phi : [g^*]^\wedge 3 \times g^{\odot 2} \rightarrow [g^*]^\wedge 4\) be defined by declaring \([\Phi(C, \mu)]((u, v, w, w')) = \mu(C(u, v), C(w, w')) + \mu(C(v, w), C(u, w'))\) whenever \(\mu \in g^{\odot 2}\) is treated as a symmetric real/complex-bilinear form on \(g^*\), and \(C(u, v)\) stands for the element \(C(u, v, \cdot)\) of \(g^*\). If, in addition, \(g\) is semisimple, the isomorphic identification \(g \approx g^*\) provided by the Killing form \(\beta\) induces an isomorphism \([g^*]^{\odot 2} \rightarrow g^{\odot 2}\), which we write as \(\sigma \mapsto \sigma^\sharp\). The definition (0.5) of \(\Lambda\) then gives, for the Cartan three-form \(C\) with (0.8),

\[
\Phi(C, \sigma^\sharp) = \Lambda \sigma \quad \text{whenever } \sigma \in g^{\odot 2}. \tag{7.1}
\]

Theorem D is a trivial consequence of the following lemma combined with Lemma 5.3(ii) and the fact that, by multiplying a Lie-algebra bracket operation \([, ,]\) by a nonzero scalar, one obtains a Lie-algebra structure isomorphic to the original one.

Lemma 7.1. In a real or complex semisimple Lie algebra \(g\), the Cartan three-form and the vector-space structure of \(g\) uniquely determine each of the following:

(i) the vector subspaces constituting the simple direct summand ideals of \(g\),
(ii) up to multiplications by cubic roots of \(1\), the restrictions of the Lie-algebra bracket of \(g\) to all such summands of dimensions other than \(3\) or \(6\),
(iii) the Lie-algebra isomorphism types of all summand ideals as above except those of real dimension \(3\).
Proof. By (7.1), \( \ker \Delta = \{ \sigma^* : \sigma \in \ker \Lambda \} \) for the real/complex-linear operator \( \Delta : \mathfrak{g}^{\otimes 2} \to (\mathfrak{g}^*)^\wedge^4 \) with \( \Delta \mu = \Phi(C, \mu) \), where \( C \) is the Cartan three-form. Then, with “minimality” referring to inclusion between the images \( \mu(\mathfrak{g}^*) \subset \mathfrak{g} \) of \( \mu \in \ker \Delta \subset \mathfrak{g}^{\otimes 2} \) viewed as linear operators \( \mu : \mathfrak{g}^* \to \mathfrak{g} \),

(iv) the simple direct summands of \( \mathfrak{g} \) are precisely the minimal elements of the set

\[ \mathbf{S} = \{ \mu(\mathfrak{g}^*) : \mu \in \ker \Delta, \ \text{and} \ \dim \mu(\mathfrak{g}^*) = 3 \ \text{or} \ \dim \mu(\mathfrak{g}^*) \geq 6 \}. \]

In fact, \( \mathbf{S} \) is formed by the images of linear endomorphisms \( \Sigma : \mathfrak{g} \to \mathfrak{g} \) with rank \( \Sigma \notin \{ 0, 1, 2, 4, 5 \} \), corresponding via (0.3) to elements \( \sigma \in \ker \Lambda \) of the same ranks. To describe all such \( \Sigma \), we use (a) – (d) in Theorem C. Specifically, by (a), these \( \Sigma \) are direct sums of linear endomorphisms \( \Sigma_i \) of the simple direct summands \( \mathfrak{h}_i \) of \( \mathfrak{g} \), while \( \Sigma_i \) are themselves subject to just two restrictions: one due to the exclusion of ranks 0, 1, 2, 4 and 5, the other depending, in view of (b) – (d), on \( l_i = \dim \mathfrak{h}_i \), as follows. If \( l_i = 3 \), (b) states that \( \Sigma_i \) is only required to be \( \beta \)-self-adjoint (to account for symmetry of \( \sigma_i \) related to \( \Sigma_i \) as in (0.3), with \( \mathfrak{g} \) replaced by \( \mathfrak{h}_i \)). Similarly, from (d) and the final comment about (d) in Theorem C, combined with Remark 1.1,

\[ \Sigma_i \] is a nonzero scalar multiple of \( \text{Id} \) when \( l_i \notin \{ 3, 6 \} \),

the scalar field being \( \mathbb{C} \) if \( \mathfrak{g} \) is either complex or real of type (0.4.a), and \( \mathbb{R} \) if \( \mathfrak{g} \) is real of type (0.4.a).

In the remaining case, \( l_i = 6 \) due to Lemma 5.3(iii); by (c) and the final clause of Theorem C, \( \Sigma_i \) is then complex-linear and \( \beta \)-self-adjoint, cf. Remark ...., but otherwise arbitrary.

The image \( \Sigma(\mathfrak{g}) \) of any \( \Sigma \) as above is the direct sum of the images of its summands \( \Sigma_i \), so it can be minimal only if there exists just one \( i \) with \( \Sigma_i \neq 0 \). For this \( i \), minimality of \( \Sigma(\mathfrak{g}) = \Sigma_i(\mathfrak{h}_i) \) further implies that \( \Sigma(\mathfrak{g}) = \mathfrak{h}_i \). In fact, as a consequence of the last paragraph, the cases \( l_i = 3 \) and \( l_i \notin \{ 3, 6 \} \) are obvious (the former since rank \( \Sigma_i \geq 3 \)) while, if \( l_i = 6 \), complex-linearity of \( \Sigma_i \) precludes, besides 0, 1, 2, 4 and 5, also 3 from being the value of its real rank.

We thus obtain one of the inclusions claimed in (iv): every minimal element of \( \mathbf{S} \) equals some summand \( \mathfrak{h}_i \). Conversely, let \( \mathfrak{h}_i \) be a fixed summand. First, \( \mathfrak{h}_i \) is an element of \( \mathbf{S} \), realized by \( \Sigma \) with \( \Sigma_i = \text{Id} \) and \( \Sigma_j = 0 \) for all \( j \neq i \). Secondly, minimality of \( \mathfrak{h}_i \) is clear from (7.2) if \( l_i \notin \{ 3, 6 \} \), while for \( l_i = 3 \) or \( l_i = 6 \) it follows from the restriction on rank \( \Sigma \) combined, in the latter case, with complex-linearity of \( \Sigma_i \). This proves (iv).

Assertion (i) is now obvious as \( \Delta \) and \( \mathbf{S} \) depend only on \( C \) and the vector-space structure of \( \mathfrak{g} \). Next, let us fix \( i \) with \( l_i \notin \{ 3, 6 \} \). Elements \( \mu \) of \( \ker \Delta \) having \( \mu(\mathfrak{g}^*) = \mathfrak{h}_i \) correspond, via (0.3) followed by the assignment \( \sigma \mapsto \mu = \sigma^i \), to endomorphisms \( \Sigma \) of \( \mathfrak{g} \) which satisfy (7.2) and are equal to 0 on \( \Sigma_j \) for \( j \neq i \). Any such \( \mu \), now treated as a symmetric bilinear form on \( \mathfrak{g}^* \), is therefore obtained from a symmetric bilinear form \( \mu_i \) on \( \mathfrak{h}_i^* \) by the trivial extension to \( \mathfrak{g}^* \), that is,
pullback under the obvious restriction operator $\mathfrak{g}^* \to \mathfrak{h}_i^*$. The forms $\mu_i$ in question are, according to (7.2), nonzero multiples of the reciprocal of the Killing form of $\mathfrak{h}_i$. Thus, the set of such multiples is uniquely determined by $C$ and the vector-space structure of $\mathfrak{g}$, which yields (ii) since, in any semisimple Lie algebra, $[,]$ depends bilinearly on the Cartan three-form $C$ and the reciprocal of the Killing form $\beta$. (A cubic root of 1 must be allowed as a factor: the multiplication of $[,]$ by a scalar $r$ results in multiplying $\beta$ and $C$ by $r^2$ and $r^3$.)

Finally, (ii) and Lemma 5.3(ii) easily yield (iii).

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