Synchronization in delayed discrete-time complex networks*

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Abstract

In this paper, we study synchronization in the delayed discrete-time complex networks. Several criteria of synchronization stability for such networks are established. And illustrative examples are presented. The numerical simulations coincide with the theoretical analysis.

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1. Introduction

Time delays commonly exist in the real networks, some of them are trivial so can be ignored whilst some of can not be ignored, such as in the long-distance communication and traffic congestions, etc. So such networks with retard time attracts much attention. Recently, Li, et al., studied the following continuous network model with time delays [1],

\[ \dot{x}_i = f(x_i) + \varepsilon \sum_{j=1}^{N} c_{ij}(x_{j1}(t-\tau_1), x_{j2}(t-\tau_2), \ldots, x_{jn}(t-\tau_n))^T \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuously differentiable function, \( x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n \) are the state variables of node \( i \), \( \varepsilon > 0 \) represents the coupling strength, \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \) indicates inner-coupling between the elements of the node itself, while \( C = (c_{ij})_{N \times N} \) denotes the outer-coupling between the nodes of the whole network (it is often assumed that there is at most one connection between node \( i \) and another node \( j \), and that there are no isolated clusters, i.e., \( C \) is an irreducible matrix). The entries \( c_{ij} \) are defined as follows: if there is a connection between node \( i \) and node \( j (j \neq i) \), then we set \( c_{ij} = 1 \); otherwise \( c_{ij} = 0 (j \neq i) \), and the diagonal elements of \( C \) are defined by \( c_{ii} = -\sum_{j=1, j \neq i}^{N} c_{ij} \), \( i = 1, 2, \ldots, N \), \( \tau_i, i = 1, \ldots, n \), are the time delays.

In this paper, we study the discrete version of network (1), which is described as,

\[ x_i(k+1) = f(x_i(k)) + \varepsilon \sum_{j=1}^{N} c_{ij}A(x_{j1}(k-\tau_1), x_{j2}(k-\tau_2), \ldots, x_{jn}(k-\tau_n))^T \]

In model (2), the inner-coupling is linear since \( A \) is a matrix, a natural generalization is that the inner-coupling can be nonlinear [2]. Such a discrete-time network with retard time reads as,

\[ x_i(k+1) = f(x_i(k)) + \varepsilon \sum_{j=1}^{N} c_{ij}A_g(x_{j1}(k-\tau_1), x_{j2}(k-\tau_2), \ldots, x_{jn}(k-\tau_n))^T \]

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where \( f, \varepsilon, x_i, \tau_i (i = 1, 2, \cdots, N) \) and \( C \) have the same meanings as those in (2), \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a continuously differentiable nonlinear function.

The outer-coupling configuration \( C \) in networks (2) and (3) has following properties [3,4].

\textbf{Lemma 1:} Suppose that \( C = (c_{ij})_{N \times N} \) is a real symmetric and irreducible matrix, where \( c_{ij} \geq 0 \) \( (i \neq j) \), \( c_{ii} = -\sum_{j=1,j \neq i}^{N} c_{ij} \), then

\begin{enumerate}[(1)]
\item 0 is an eigenvalue of \( C \) with multiplicity 1, associated with eigenvector \((1,1,\cdots,1)^T\);
\item all the other eigenvalues of \( C \) are less than 0;
\item there exists a unitary matrix, \( \Phi = (\phi_1, \phi_2, \cdots, \phi_N) \) such that
\end{enumerate}

\[ C^T \phi_k = \lambda_k \phi_k, \quad k = 1, 2, \cdots, N, \]

where \( 0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N \) are the eigenvalues of \( C \).

In what follows, the definition of synchronization for networks (2) and (3) is introduced below [2].

\textbf{Definition 1:} Let \( \mathcal{A} \) be an attractor of the discrete-time dynamical system

\[ s(k+1) = f(s(k)). \]

We say that networks (2) and (3) are (asymptotically) synchronized to \( \mathcal{A} \), if for \( k \to +\infty \),

\[ x_i \to \mathcal{A}, \quad i = 1, \cdots, N. \]

In the rest of this paper, the criterions of synchronization stability for networks (2) and (3) are established in Section 2. And the numerical examples are presented in Section 3.

\section{2. Synchronization theorems}

By utilizing Lemma 1, one can derive the following theorem.

\textbf{Theorem 1:} Consider the delayed discrete-time network (2), let \( 0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \) be the eigenvalues of the coupling configuration matrix \( C \). If the following \( N-1 \) systems of \( n \)-dimensional linear delayed equations are asymptotically stable about their zero solutions,

\[ \eta(k+1) = Df(s(k))\eta(k) + \varepsilon \lambda_i A \cdot \eta(k-\tau_i), \quad i = 2, \cdots, N, \]

where \( Df(s(k)) \in \mathbb{R}^{n \times n} \) is the Jacobian of \( f(x(k)) \) at \( s(k) \), \( \eta(k) \in \mathbb{R}^n, \eta(k-\tau_i) = (\eta_1(k-\tau_i), \cdots, \eta_n(k-\tau_i))^T \in \mathbb{R}^n, \) \( s(k) \) is the orbit of attractor \( \mathcal{A} \) of equation \( s(k+1) = f(s(k)) \), then network (2) is synchronized to the attractor \( \mathcal{A} \).

\textbf{Proof:} Linearizing (2) at \( s(k) \) yields,

\[ e_i(k+1) = Df(s(k))e_i(k) + \varepsilon \sum_{j=1}^{N} c_{ij} A \cdot e_j(k-\tau), \quad 1 \leq i \leq N, \]

where \( e_i(k) \) denotes the deviation from the state \( s(k) \), i.e., \( e_i(k) = x_i(k) - s(k), \quad e_j(k-\tau) = (e_{j1}(k-\tau_1), \cdots, e_{jn}(k-\tau_n))^T \in \mathbb{R}^n. \) This linearized system can be rewritten as

\[ e_i(k+1) = Df(s(k))e_i(k) + \varepsilon A \cdot (e_1(k-\tau), e_2(k-\tau), \cdots, e_N(k-\tau))((c_{11}, \cdots, c_{1N}), \cdots, (c_{i1}, \cdots, c_{iN}))^T. \]

Let \( e(k) = (e_1(k), e_2(k), \cdots, e_N(k)) \in \mathbb{R}^{n \times N} \), the above equation can be expressed in a compact form,

\[ e(k+1) = Df(s(k))e(k) + \varepsilon A \cdot e(k-\tau) C^T, \]
where \( \overline{e(k - \tau)} = (e_1(k - \tau), e_2(k - \tau), \cdots, e_N(k - \tau)) \in \mathbb{R}^{n \times N}. \)

From Lemma 1, there exists a nonsingular matrix \( \Phi \), such that \( C^T \Phi = \Phi \Gamma \), \( \Gamma = \text{diag}(\lambda_1, \cdots, \lambda_N) \). If one sets \( e(k) \Phi = v(k) = (v_1(k), v_2(k), \cdots, v_N(k)) \in \mathbb{R}^{n \times N} \), then the above equation can be transformed into the following matrix equation

\[
v(k + 1) = Df(s(k))v(k) + \varepsilon A \overline{v(k - \tau) \Gamma},
\]

which immediately follows that,

\[
v_i(k + 1) = Df(s(k))v_i(k) + \varepsilon \lambda_i A \overline{v_i(k - \tau)}, \quad i = 1, \cdots, N,
\]

where \( \overline{v_i(k - \tau)} = (v_{i1}(k - \tau_1), \cdots, v_{iN}(k - \tau_n))^T \in \mathbb{R}^n. \)

Note that \( \lambda_1 = 0 \) corresponds to the linearized system of the isolate equation \( s(k + 1) = f(s(k)) \). If the following \( N - 1 \) pieces of \( n \)-dimensional linear time-varying delayed equations

\[
v_i(k + 1) = Df(s(k))v_i(k) + \varepsilon \lambda_i A \overline{v_i(k - \tau)}, \quad i = 2, \cdots, N,
\]

are asymptotically stable around their zero solutions, then \( e(k) \) will tend to zero, which shows that the conclusion holds. This completes the proof.

**Theorem 2:** Assume that all eigenvalues of the matrix \( C \) in (2) are listed in order, \( 0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \). If there exists a positive-definite matrix \( P > 0 \) such that

\[
\begin{bmatrix}
Df(s(k))^T P Df(s(k)) - P + I & \varepsilon \lambda_i A \overline{v_i(k - \tau)} \\
\varepsilon \lambda_i A^T P Df(s(k)) & \varepsilon^2 \lambda_i^2 A^T PA - I
\end{bmatrix} < 0, \quad i = 2, 3, \cdots, N,
\]

where \( I \) is the identity matrix, \( s(k) \) is the orbit of attractor \( A \) of the equation \( s(k + 1) = f(s(k)) \), then network (2) is synchronized to \( A \) for any fixed delay \( \tau_i \in \mathbb{Z}^+ \) \( (i = 1, 2, \cdots, n) \).

**Proof:** Select a Lyapunov functional as

\[
V(\eta(k)) = \eta(k)^T P \eta(k) + \sum_{i=1}^n \sum_{\sigma = k - \tau_i}^{k-1} \eta_i(\sigma)^T \eta_i(\sigma),
\]

in which \( \eta(k) = (\eta_1(k), \eta_2(k), \cdots, \eta_n(k))^T \). Then, along the solution of the \( i \)-th \( (i = 2, 3, \cdots, N) \) equation in system (4), one gets

\[
\Delta V(\eta(k)) = V(\eta(k + 1)) - V(\eta(k))
= \left[ Df(s(k))\eta(k) + \varepsilon \lambda_i A \cdot \overline{\eta(k - \tau)} \right]^T P \left[ Df(s(k))\eta(k) + \varepsilon \lambda_i A \cdot \overline{\eta(k - \tau)} \right] \\
- \eta(k)^T P \eta(k) + \eta(k)^T \eta(k) - \eta(k - \tau)^T \eta(k - \tau)
= \begin{bmatrix} \eta(k) \\ \eta(k - \tau) \end{bmatrix}^T \begin{bmatrix}
Df(s(k))^T P Df(s(k)) - P + I & \varepsilon \lambda_i Df(s(k))^T PA \\
\varepsilon \lambda_i A^T P Df(s(k)) & \varepsilon^2 \lambda_i^2 A^T PA - I
\end{bmatrix} \begin{bmatrix} \eta(k) \\ \eta(k - \tau) \end{bmatrix}
\]

By using linear matrix inequality (LMI) (5), one has \( \Delta V(\eta(k)) < 0 \) for all \( i = 2, 3, \cdots, N \), which implies that the zero solutions of systems (4) are asymptotically stable. From Theorem 1, network (2) is synchronized to \( A \). The proof is finished.

In the following, one can similarly obtain theorems 3 and 4 for network (3).
Theorem 3: Consider the delayed discrete-time network (3), let \( 0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \) be the eigenvalues of the coupling configuration matrix \( C \). If the following \( N - 1 \) systems of \( n \)-dimensional linear delayed equations are asymptotically stable about their zero solutions,

\[
\eta(k + 1) = Df(s(k))\eta(k) + \varepsilon \lambda_i A Dg(s(k - \tau)) \cdot \eta(k - \tau), \quad i = 2, \cdots, N,
\]

where \( Df(s(k)), Dg(s(k - \tau)) \in \mathbb{R}^{n \times n} \) are the Jacobians of \( f(x(k)), g(x(k)) \) at \( s(k) \) and \( s(k - \tau) \), \( \eta(k) \in \mathbb{R}^n, s(k - \tau) = (s_1(k - \tau_1), \cdots, s_n(k - \tau_n))^T \in \mathbb{R}^n, \eta(k - \tau) = (\eta_1(k - \tau_1), \cdots, \eta_n(k - \tau_n))^T \in \mathbb{R}^n \), \( s(k) \) is the orbit of attractor \( \mathcal{A} \) of the equation \( s(k + 1) = f(s(k)) \), then network (3) is synchronized to \( \mathcal{A} \).

Theorem 4: Assume that all eigenvalues of the matrix \( C \) in (3) are listed in order, \( 0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \). If there exists a positive-definite matrix \( P > 0 \) such that

\[
\begin{bmatrix}
  Df(s(k))^T P Df(s(k)) - P + I & \varepsilon \lambda_i Df(s(k))^T P A Dg(s(k - \tau)) \\
  \varepsilon \lambda_i Dg(s(k - \tau))^T A^T P Df(s(k)) & \varepsilon^2 \lambda_i^2 Dg(s(k - \tau))^T A^T P A Dg(s(k - \tau)) - I
\end{bmatrix} < 0, \quad i = 2, \cdots, N,
\]

where \( I \) is the identity matrix, \( s(k) \) is the orbit of attractor \( \mathcal{A} \) of equation \( s(k + 1) = f(s(k)) \), then network (3) is synchronized to \( \mathcal{A} \) for any fixed delay \( \tau_i \in \mathbb{Z}^+ \) \((i = 1, 2, \cdots, n)\).

In [1], by using “matrix measure” [5,6], Li, et al., discussed the synchronization of network (1) and the following network (6),

\[
\dot{x}_i = f(x_i) + \varepsilon \sum_{j=1}^{N} c_{ij} A(x_j(t - \tau_{j1}), x_j(t - \tau_{j2}), \cdots, x_j(t - \tau_{jn}))^T \triangleq f(x_i) + \varepsilon \sum_{j=1}^{N} c_{ij} A \cdot x_j(t - \tau_j), \quad (6)
\]

where \( f, \varepsilon, x_i \) \((i = 1, 2, \cdots, N)\), \( C \) and \( A \) have the same meanings as those in (1), the sole difference is that in (6) a different node \( j \) has a different time-delay vector, \((\tau_{j1}, \tau_{j2}, \cdots, \tau_{jn})\).

We can also apply “LMIs” presented here to establishing synchronization theorems for delayed continuous-time networks [1] and [6]. In the following, the definition of synchronization is given.

Definition 2 [1,2]: Assume that \( \mathcal{B} \) is an attractor of the continuous dynamical system

\[
\dot{s} = f(s).
\]

We say that networks (1) and (6) are (asymptotically) synchronized to \( \mathcal{B} \), if for \( t \to +\infty \),

\[
x_i \to \mathcal{B}, \; i = 1, \cdots, N.
\]

Here, we list the synchronization theorems of networks (1) and (6) just for reference later on. For details of the proofs, see [7].

Theorem 5: Consider the delayed continuous network (1), let \( 0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \) be the eigenvalues of the coupling configuration matrix \( C \). If the following \( N - 1 \) systems of \( n \)-dimensional linear time-varying delayed differential equations are asymptotically stable about their zero solutions,

\[
\dot{\eta}(t) = Df(s(t))\eta(t) + \varepsilon \lambda_i A \cdot \eta(t - \tau), \quad i = 2, \cdots, N,
\]

4
where $Df(s(t)) \in \mathbb{R}^{n \times n}$ is the Jacobian of $f(x(t))$ at $s(t)$, $\eta(t) \in \mathbb{R}^{n}$, $\eta(t-\tau) = (\eta_1(t-\tau_1), \cdots, \eta_n(t-\tau_n))^T \in \mathbb{R}^{n}$, $s(t)$ is an orbit of attractor $B$ of equation $\dot{s} = f(s)$, then network (1) is synchronized to $B$.

**Theorem 6:** Assume that all eigenvalues of the matrix $C$ in (1) are listed in order, $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$. If there exists a positive-definite matrix $P > 0$ such that

$$
\begin{bmatrix}
PDf(s(t)) + Df(s(t))^T P + I & \varepsilon \lambda_i P A \\
\varepsilon \lambda_i A^T P & -I
\end{bmatrix} < 0, \ i = 2,3,\cdots,N,
$$

where $I$ is the identity matrix, $s(t)$ is an orbit of the attractor $B$ of the equation $\dot{s} = f(s)$, then network (1) is synchronized to $B$ for any fixed delay $\tau_k > 0 (k = 1,2,\cdots,n)$.

**Theorem 7:** Consider the delayed dynamical network (6), let $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of the coupling configuration matrix $C$. If the following $N - 1$ systems of $n$-dimensional linear time-varying delayed differential equations are asymptotically stable about their zero solutions,

$$
\dot{\eta}(t) = Df(s(t))\eta(t) + \varepsilon \lambda_i A \cdot \eta(t-\tau_i), \ i = 2,\cdots,N,
$$

where $Df(s(t)) \in \mathbb{R}^{n \times n}$ is the Jacobian of $f(x(t))$ at $s(t)$, $\eta(t) \in \mathbb{R}^{n}$, $\eta(t-\tau) = (\eta_1(t-\tau_1), \cdots, \eta_n(t-\tau_n))^T \in \mathbb{R}^{n}$, $s(t)$ is the stable equilibrium $B_0$ of equation $\dot{s} = f(s)$, then network (6) is synchronized to $B_0$.

**Theorem 8:** Assume that all eigenvalues of the matrix $C$ in (6) are listed in order, $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$. If there exists a positive-definite matrix $P > 0$ such that

$$
\begin{bmatrix}
PDf(s(t)) + Df(s(t))^T P + I & \varepsilon \lambda_i P A \\
\varepsilon \lambda_i A^T P & -I
\end{bmatrix} < 0, \ i = 2,3,\cdots,N,
$$

where $I$ is the identity matrix, $s(t)$ is the equilibrium $B_0$ of equation $\dot{s} = f(s)$, then network (6) is synchronized to $B_0$ for any fixed delay $\tau_{kl} > 0 (k,l = 1,2,\cdots,n)$.

**Remark 1:** Obviously, $B_0 \subsetneq B$. $B$ also contains other stable limit sets, e.g., stable (quasi-)periodic orbit, strange attractor. Theorems 7 and 8 hold only for $B_0$. For the other cases, the studies are not easy but need further consideration [8].

3. Illustrative examples

In this section, we consider a five-node network, in which each node is a simple 2-dimensional Hénon map [9,10], described by

$$
\begin{bmatrix}
f_1(x_1,x_2) \\
f_2(x_1,x_2)
\end{bmatrix} =
\begin{bmatrix}
1 + x_2 - ax_1^2 \\
bx_1
\end{bmatrix}
$$

(6)

If $a = 0.5$, $b = 0.3$, (6) has a period solution [9].
We use the coupled configuration matrix \((c_{ij})_{N \times N}\) [11], which is
\[
C = \begin{bmatrix}
-2 & 1 & 0 & 0 & 1 \\
1 & -3 & 1 & 1 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 1 & 1 & -3 & 1 \\
1 & 0 & 0 & 1 & -2 \\
\end{bmatrix}
\]
whose eigenvalues are \(\lambda_1 = 0\), \(\lambda_2 = -1.382\), \(\lambda_3 = -2.382\), \(\lambda_4 = -3.168\), \(\lambda_4 = -4.168\), and use the inner-coupling matrix \(A = \text{diag}(1, 1)\).

At first, the time-delay vector \((\tau_1, \tau_2) = (1, 2)\) is considered. Here, set the coupling strength \(\varepsilon = 0.015\). By using the MATLAB LMI Toolbox, one can obtain that there exists a common positive-definite matrix,
\[
P = \begin{bmatrix}
5.3626 & 0 \\
0 & 9.1633 \\
\end{bmatrix}
\]
such that the condition of Theorem 2 is satisfied. From Theorem 2, we know that this delayed network is synchronized to the stable periodic state of the isolated Hénon map. The following numerical simulations are also in line with the above theoretical analysis.

In Fig 1, we plot the curves of the synchronization errors between the states of node \(i\) and the 1st node, that is, \(e_{ij} = x_{ij}(k) - x_{1j}(k)\) for \(i = 2, \cdots, 5, j = 1, 2\).

Fig 1. Synchronization errors for the coupled network with time-delay vector \((\tau_1, \tau_2) = (1, 2)\). (a) \(j = 1\) (b) \(j = 2\).

Secondly, we choose the nonlinear inner-coupling function as
\[
\begin{bmatrix}
g_1(x_1, x_2) \\
g_2(x_1, x_2)
\end{bmatrix} = \begin{bmatrix}
\varepsilon x_1 \\
\sin x_2
\end{bmatrix}.
\]  
(7)

By using the MATLAB LMI Toolbox again, we can find there exists a positive-definite matrix
\[
P = \begin{bmatrix}
5.3649 & 0 \\
0 & 9.1714 \\
\end{bmatrix}
\]
such that the condition of Theorem 4 is satisfied. So the five-node Hénon network with time delay \((τ_1, τ_2) = (1, 2)\) is synchronized to the stable periodic state of the isolated Hénon map. The numerical simulation below shows this point of view.

In Fig 2, we plot the curves of the synchronization errors between the states of node \(i\) and node 1, i.e., \(e_{ij} = x_{ij}(k) - x_{1j}(k)\) for \(i = 2, \cdots, 5, j = 1, 2\).

![Fig 2. Synchronization errors for the five-node Hénon network. (a) \(j = 1\) (b) \(j = 2\).](image)

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