A NEW APPROACH ON HELICES IN EUCLIDEAN $n$–SPACE

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Abstract. In this work, we give some new characterizations for inclined curves and slant helices in $n$-dimensional Euclidean space $E^n$. Moreover, we consider the pre-characterizations about inclined curves and slant helices and reconfigure them.

1. Introduction

The helices share common origins in the geometries of the platonic solids, with inherent hierarchical potential that is typical of biological structures. The helices provide an energy-efficient solution to close-packing in molecular biology, a common motif in protein construction, and a readily observable pattern at many size levels throughout the body. The helices are described in a variety of anatomical structures, suggesting their importance to structural biology and manual therapy [9].

In [8], Özdamar and Hacısalihoğlu defined harmonic curvature functions $H_i$ ($1 \leq i \leq n - 2$) of a curve $\alpha$ in $n$-dimensional Euclidean space $E^n$. They generalized inclined curves in $E^3$ to $E^n$ and then gave a characterization for the inclined curves in $E^n$:

"A curve $\alpha$ is an inclined curve if and only if $\sum_{i=1}^{n-2} H_i^2 = \text{constant}$. (1.1)"

Then, Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of slant helices in Euclidean 3–space $E^3$ [6]. In 2008, Önder et al. defined a new kind of slant helix in Euclidean 4–space $E^4$ which is called $B_2$–slant helix and they gave some characterizations of these slant helices in Euclidean 4–space $E^4$ [7]. And then in 2009, Gök et al. defined a new kind of slant helix in Euclidean $n$–space $E^n$, $n > 3$, which they called $V_n$–slant helix and they gave some characterizations of these slant helices in Euclidean $n$–space [4]. The new kind of helix is generalization of $B_2$–slant helix to Euclidean 4–space $E^4$. On the other hand, Camcı et al. gave some characterizations for a non-degenerate curve to be a generalized helix by using its harmonic curvatures [2].

Since Özdamar and Hacısalihoğlu defined harmonic curvature functions, lots of authors have used them in their papers for characterization of inclined curves and slant helices. In these studies, they gave some characterizations similar to (1.1) for inclined curves and slant helices. But, Camcı et al. see for the first time that the characterization of inclined curves in (1.1) is true for the case necessity but not true for the case sufficiency and gave an example of inclined curve in order to show why the case sufficiency is not true [2]. Also, they gave a characterization of inclined curves (Theorem 3.3, pp.2594) with only necessary condition [2]. But, they did not obtain when the characterization has sufficiency case. And then, Gök et al. [1] corrected the characterization of $B_2$–slant helix (Theorem 3.1, pp.1436, in [7]) like the characterization in (1.1). But, they also did not give the answer of the question: When the characterization has sufficiency case? After them, Ahmad and Lopez gave the definition of $G_i$ ($1 \leq i \leq n$) functions and obtain a characterization of slant helices, that is, $V_2$–slant helix (Theorem 1.2, pp 2, in [1]).

In this paper, we investigate the answer of the following question with the similar method in Theorem 4.1 in [5]:

When the characterizations of inclined curves and slant helices in Euclidean $n$–space $E^n$ which are similar to (1.1) have a necessary and sufficient case?
2. Preliminaries

Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be an arbitrary curve in \( E^n \). Recall that the curve \( \alpha \) is said a unit speed curve (or parameterized by arclength functions) if \( \langle \alpha'(s), \alpha'(s) \rangle = 1 \), where \( \langle \cdot, \cdot \rangle \) denotes the standart inner product of \( \mathbb{R}^n \) given by

\[
\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i
\]

for each \( X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). In particular, the norm of a vector \( X \in \mathbb{R}^n \) is given by \( \|X\|^2 = \langle X, X \rangle \). Let \( \{V_1, V_2, \ldots, V_n\} \) be the moving Frenet frame along the unit speed curve \( \alpha \), where \( V_i \) (\( i = 1, 2, \ldots, n \)) denotes ith Frenet vector field. Then the Frenet formulas are given by

\[
\begin{bmatrix}
V_1' \\
V_2' \\
V_3' \\
\vdots \\
V_{n-2}' \\
V_{n-1}' \\
V_n'
\end{bmatrix} =
\begin{bmatrix}
0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_{n-2} \\
V_{n-1} \\
V_n
\end{bmatrix}
\]

where \( k_i (i = 1, 2, \ldots, n-1) \) denotes the ith curvature function of the curve \([5]\). If all of the curvatures \( k_i \) \( (i = 1, 2, \ldots, n-1) \) of the curve nowhere vanish in \( I \subset \mathbb{R} \), the curve is called non-degenerate curve.

**Definition 2.1.** Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be a curve in \( E^n \) with arc-length parameter \( s \) and let \( X \) be a unit constant vector of \( E^n \). For all \( s \in I \), if

\[
\langle V_1, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = \text{constant},
\]

then the curve \( \alpha \) is called a general helix or inclined curve (\( V_1 \)-slant helix) in \( E^n \), where \( V_1 \) is the unit tangent vector of \( \alpha \) at its point \( \alpha(s) \) and \( \varphi \) is a constant angle between the vector fields \( V_1 \) and \( X \) \([5]\).

**Definition 2.2.** Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be a curve in \( E^n \) with arc-length parameter \( s \) and let \( X \) be a unit constant vector of \( E^n \). For all \( s \in I \), if

\[
\langle V_2, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = \text{constant},
\]

then the curve \( \alpha \) is called a slant helix or \( V_2 \)-slant helix in \( E^n \), where \( V_2 \) is the 2th vector field of \( \alpha \) and \( \varphi \) is a constant angle between the vector fields \( V_2 \) and \( X \) \([1]\).

**Definition 2.3.** Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be a unit speed curve with nonzero curvatures \( k_i \) \( (1 \leq i \leq n-1) \) in \( E^n \) and let \( \{V_1, V_2, \ldots, V_n\} \) denote the Frenet frame of the curve \( \alpha \). We call that \( \alpha \) is a \( V_n \)-slant helix if the \( n \)th unit vector field \( V_n \) makes a constant angle \( \varphi \) with a fixed direction \( X \), that is,

\[
\langle V_n, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = \text{constant},
\]

along the curve \( \alpha \), where \( X \) is unit vector field in \( E^n \) \([1]\).

3. Inclined Curves and Their Harmonic Curvature Functions

In this section, we reconfigure some known characterizations by using harmonic curvatures for inclined curves.

**Definition 3.1.** Let \( \alpha \) be a unit curve in \( E^n \). The harmonic curvatures of \( \alpha \) are defined by \( H_i : I \rightarrow \mathbb{R} \), \( i = 0, 1, \ldots, n-2 \), such that

\[
H_0 = 0, H_1 = \frac{k_1}{k_2}, H_i = \left\{ H_{i-1} + k_i H_{i-2} \right\} \frac{1}{k_{i+1}}
\]

for \( 2 \leq i \leq n-2 \), where \( k_i \neq 0 \) for \( i = 1, 2, \ldots, n-1 \)[8].

**Lemma 3.1.** Let \( \alpha \) be a unit curve in \( E^n \) and let \( H_{n-2} \neq 0 \) for \( i = n-2 \). Then, \( H_1^2 + H_2^2 + \ldots + H_{n-2}^2 \) is a nonzero constant if and only if \( H_{n-2} = -k_{n-1}H_{n-3} \).
Proof. First, we assume that \( H_1^2 + H_2^2 + \ldots + H_{n-2}^2 \) is a nonzero constant. Consider the functions

\[
H_i = \left\{ H_{i-1} + k_i H_{i-2} \right\} \frac{1}{k_{i+1}}
\]

for \( 3 \leq i \leq n - 2 \). So, from the equality, we can write

\[
k_{i+1} H_i = H_{i-1} + k_i H_{i-2}, \quad 3 \leq i \leq n - 2.
\]  

(3.1)

Hence, in (3.1), if we take \( i + 1 \) instead of \( i \), we get

\[
H_i' = k_{i+2} H_{i+1} - k_{i+1} H_{i-1}, \quad 2 \leq i \leq n - 3
\]

(3.2)

together with

\[
H_1' = k_3 H_2.
\]

(3.3)

On the other hand, since \( H_1^2 + H_2^2 + \ldots + H_{n-2}^2 \) is constant, we have

\[
H_1 H_1' + H_2 H_2' + \ldots + H_{n-2} H_{n-2}' = 0
\]

and so,

\[
H_{n-2} H_{n-2}' = -H_1 H_1' - H_2 H_2' - \ldots - H_{n-3} H_{n-3}'.
\]

(3.4)

By using (3.2) and (3.3), we obtain

\[
H_1 H_1' = k_3 H_1 H_2
\]

(3.5)

and

\[
H_i H_i' = k_{i+2} H_{i+1} H_{i+1} - k_{i+1} H_{i-1} H_{i-1}, \quad 2 \leq i \leq n - 3.
\]

(3.6)

Therefore, by using (3.4), (3.5) and (3.6), a algebraic calculus shows that

\[
H_{n-2} H_{n-2}' = -k_{n-1} H_{n-3} H_{n-2}.
\]

Since \( H_{n-2} \neq 0 \), we get the relation \( H_{n-2}' = -k_{n-1} H_{n-3} \).

Conversely, we assume that

\[
H_{n-2}' = -k_{n-1} H_{n-3}
\]

(3.7)

By using (3.7) and \( H_{n-2} \neq 0 \), we can write

\[
H_{n-2} H_{n-2}' = -k_{n-1} H_{n-2} H_{n-3}
\]

(3.8)

From (3.6), we have

for \( i = n - 3 \), \( H_{n-3} H_{n-3}' = k_{n-1} H_{n-3} H_{n-2} - k_{n-2} H_{n-4} H_{n-3} \)

for \( i = n - 4 \), \( H_{n-4} H_{n-4}' = k_{n-2} H_{n-4} H_{n-3} - k_{n-3} H_{n-5} H_{n-4} \)

for \( i = n - 5 \), \( H_{n-5} H_{n-5}' = k_{n-3} H_{n-5} H_{n-4} - k_{n-4} H_{n-6} H_{n-5} \)

\[
\vdots
\]

\[
\vdots
\]

for \( i = 2 \), \( H_2 H_2' = k_4 H_2 H_3 - k_3 H_1 H_2 \)

and from (3.5), we have

\[
H_1 H_1' = k_3 H_1 H_2.
\]

So, an algebraic calculus show that

\[
H_1 H_1' + H_2 H_2' + \ldots + H_{n-3} H_{n-3}' + H_{n-4} H_{n-4}' + H_{n-5} H_{n-5}' + H_{n-2} H_{n-2}' = 0.
\]

(3.9)

And, by integrating (3.9), we can easily say that

\[
H_1^2 + H_2^2 + \ldots + H_{n-2}^2
\]

is a non-zero constant. This completes the proof. \( \square \)

**Theorem 3.1.** Let \( \alpha \) be an inclined curve and let \( X \) be a axis of \( \alpha \). Then,

\[
(V_{i+2}, X) = H_i (V_i, X), \quad 1 \leq i \leq n - 2,
\]

where \( \{V_1, V_2, \ldots, V_n\} \) denote the Frenet frame of the curve \( \alpha \) and \( \{H_1, H_2, \ldots, H_{n-2}\} \) denote the harmonic curvature functions of \( \alpha \) \( \mathbb{R}^3 \) or \( \mathbb{R}^4 \).
**Theorem 3.2.** Let \( \{V_1, V_2, ..., V_n\} \) be the Frenet frame of the curve \( \alpha \) and let \( \{H_1, H_2, ..., H_{n-2}\} \) be the harmonic curvature functions of \( \alpha \). Then, \( \alpha \) is an inclined curve (with the curvatures \( k_i \neq 0, \) \( i = 1, 2, ..., n - 1 \)) in \( E^n \) if and only if \( \sum_{i=1}^{n-2} H_i^2 = \text{constant} \) and \( H_{n-2} \neq 0 \).

**Proof.** Let \( \alpha \) be a inclined curve. According to the Definition 2.1,

\[
\langle V_1, X \rangle = \cos(\varphi) = \text{constant},
\]

where \( X \) the axis of \( \alpha \). And, from Theorem (3.1),

\[
\langle V_{i+2}, X \rangle = H_i \langle V_1, X \rangle
\]

for \( 1 \leq i \leq n - 2 \). Moreover, from (3.10) and Frenet equations, we can write \( \langle V_2, X \rangle = 0. \) Since the orthonormal system \( \{V_1, V_2, ..., V_n\} \) is a basis of \( \mathcal{N}(E^n) \) (tangent bundle), \( X \) can be expressed in the form

\[
X = \sum_{i=1}^{n} \langle V_i, X \rangle V_i.
\]

Hence, by using the equations (3.10), (3.11) and (3.12), we obtain

\[
X = \cos(\varphi)V_1 + \sum_{i=1}^{n-2} H_i \cos(\varphi)V_{i+2}.
\]

Since \( X \) is a unit vector field (see Definition 2.1),

\[
\cos^2(\varphi) + \sum_{i=1}^{n-2} H_i^2 \cos^2(\varphi) = 1
\]

and so

\[
\sum_{i=1}^{n-2} H_i^2 = \tan^2(\varphi) = \text{constant}.
\]

Now, we are going to show that \( H_{n-2} \neq 0 \). We assume that \( H_{n-2} = 0 \). Then, for \( i = n - 2 \) in Theorem 3.1,

\[
\langle V_n, X \rangle = H_{n-2} \langle V_1, X \rangle = 0.
\]

So, \( \langle DT V_n, X \rangle = \langle -k_{n-1} V_{n-1}, X \rangle = 0 \). We deduce that \( \langle V_{n-1}, X \rangle = 0. \) On the other hand, for \( i = n - 3 \) in Theorem 3.1,

\[
\langle V_{n-1}, X \rangle = H_{n-3} \langle V_1, X \rangle.
\]

And, since \( \langle V_{n-1}, X \rangle = 0, H_{n-3} = 0. \) Continuing this process, we get that \( H_1 = 0. \) Let us recall that \( H_1 = k_1/k_2, \) thus we have a contradiction because all the curvatures are nowhere zero. Consequently \( H_{n-2} \neq 0. \)

Conversely, we assume that \( \sum_{i=1}^{n-2} H_i^2 = \tan^2(\varphi) = \text{constant} \) and \( H_{n-2} \neq 0. \) Then, consider the vector field

\[
X = \cos(\varphi)V_1 + \sum_{i=3}^{n} H_{i-2} \cos(\varphi)V_i.
\]

We want to verify that \( X \) is a constant along \( \alpha \), i.e. \( D_{V_i} X = 0. \) So,

\[
D_{V_i} X = D_{V_i} (\cos(\varphi)V_1) + \sum_{i=3}^{n} D_{V_i} (H_{i-2} \cos(\varphi)V_i)
\]

\[
= \cos(\varphi) D_{V_i} V_1 + \sum_{i=3}^{n} (H_{i-2} \cos(\varphi)V_i + H_{i-2} \cos(\varphi) D_{V_i} V_i)
\]

\[
= \cos(\varphi) \left( k_1 V_2 + \sum_{i=3}^{n-1} (H_{i-2} V_i - k_{i-1} H_{i-2} V_{i-1} + k_i H_{i-2} V_{i+1}) + H_{n-2} V_n - k_{n-1} H_{n-2} V_{n-1} \right)
\]

On the other hand, by using (3.2), we can write

\[
H_{i-2} = k_i H_{i-1} - k_{i-1} H_{i-3}
\]

for \( 4 \leq i \leq n - 1 \) together with (3.3). Moreover, from Lemma 3.1, we know that

\[
H_{n-2} = -k_{n-1} H_{n-3}
\]
Therefore, by using (3.3), (3.13) and (3.14), an algebraic calculus shows that $D_v \alpha = 0$. Since

$$\|X\| = \cos^2(\varphi) + \sum_{i=3}^{n} H_i^2 \cos^2(\varphi)$$

$$= \cos^2(\varphi) \left( 1 + \sum_{i=1}^{n-2} H_i^2 \right)$$

$$= \cos^2(\varphi) (1 + \tan^2(\varphi))$$

$$= 1.$$ 

$X$ is a unit vector field. Furthermore, $(V_1, X) = \cos(\varphi)$ =constant. Hence, we deduce that $\alpha$ is an inclined curve.

\[\square\]

**Remark 3.1.** The following corollary is the reconfiguration of the Theorem 3.4 in [2].

**Corollary 3.1.** Let $\{V_1, V_2, ..., V_n\}$ be the Frenet frame of the curve $\alpha$ and let $\{H_1, H_2, ..., H_{n-2}\}$ be the harmonic curvature functions of $\alpha$. Then, $\alpha$ is an inclined curve (with the curvatures $k_i \neq 0$, $i = 1, 2, ..., n-1$) in $E^n$ if and only if $H_{n-2}^{*} = -k_{n-1} H_{n-3}^{*}$ and $H_{n-2}^{*} \neq 0$.

**Proof.** It is obvious by using Lemma (3.1) and Theorem (3.2). \[\square\]

4. $V_n$-slant helices and their harmonic curvature functions

In this section, we reconfigure some known characteristics by using harmonic curvatures for $V_n$-slant helices.

**Definition 4.1.** Let $\alpha : I \subset \mathbb{R} \rightarrow E^n$ be a unit speed curve with nonzero curvatures $k_i$ ($i = 1, 2, ..., n-1$) in $E^n$. Harmonic curvature functions of $\alpha$ are defined by $H_i^*: I \subset \mathbb{R} \rightarrow \mathbb{R}$,

$$H_0^* = 0, H_1^* = \frac{k_{n-1}}{k_{n-2}}, H_i^* = \left\{k_{n-i} H_{i-2}^{*} - H_{i-1}^{*}\right\} \frac{1}{k_{n-(i+1)}}$$

for $2 \leq i \leq n-2$ [1].

**Lemma 4.1.** Let $\alpha$ be a unit curve in $E^n$ and let $H_{n-2}^{*} \neq 0$ be for $i = n-2$. Then, $H_1^2 + H_2^2 + ... + H_{n-2}^2$ is a nonzero constant if and only if $H_{n-2}^{*} = k_{1} H_{n-3}^{*}$.

**Proof.** First, we assume that $H_1^2 + H_2^2 + ... + H_{n-2}^2$ is a nonzero constant. Consider the functions

$$H_i^* = \left\{k_{n-i} H_{i-2}^{*} - H_{i-1}^{*}\right\} \frac{1}{k_{n-(i+1)}}$$

for $3 \leq i \leq n-2$. So, from the equality, we can write

$$k_{n-(i+1)} H_i^* = k_{n-i} H_{i-2}^{*} - H_{i-1}^{*}, \quad 3 \leq i \leq n-2.$$ (4.1)

Hence, in (4.1), if we take $i + 1$ instead of $i$, we get

$$H_i^{*} = k_{n-(i+1)} H_{i-1}^{*} - k_{n-(i+2)} H_{i+1}^{*}, \quad 2 \leq i \leq n-3$$ (4.2)

together with

$$H_1^{*} = -k_{n-3} H_2^{*}. \quad \text{(4.3)}$$

On the other hand, since $H_1^2 + H_2^2 + ... + H_{n-2}^2$ is constant, we have

$$H_1^{*} H_1^{*} + H_2^{*} H_2^{*} + ... + H_{n-2}^{*} H_{n-2}^{*} = 0$$

and so,

$$H_{n-2}^{*} H_{n-2}^{*} = -H_{1}^{*} H_1^{*} - H_{2}^{*} H_2^{*} - ... - H_{n-3}^{*} H_{n-3}^{*}. \quad \text{(4.4)}$$

By using (4.2) and (4.3), we obtain

$$H_1^{*} H_1^{*} = -k_{n-3} H_1^{*} H_1^{*} \quad \text{(4.5)}$$

and

$$H_i^{*} H_i^{*} = k_{n-(i+1)} H_{i-1}^{*} H_{i-1}^{*} - k_{n-(i+2)} H_{i+1}^{*} H_{i+1}^{*}, \quad 2 \leq i \leq n-3. \quad \text{(4.6)}$$

Therefore, by using (4.4), (4.5) and (4.6), a algebraic calculus shows that

$$H_{n-2}^{*} H_{n-2}^{*} = k_{1} H_{n-3}^{*} H_{n-3}^{*}.$$ 

Since $H_{n-2}^{*} \neq 0$, we get the relation $H_{n-2}^{*} = k_{1} H_{n-3}^{*}$.
Conversely, we assume that
\[ H^*_n = k_1 H^*_{n-1}. \] 
(4.7)
By using (4.7) and \( H^*_{n-2} \neq 0 \), we can write
\[ H^*_{n-2} H^*_{n-2} = k_1 H^*_{n-2} H^*_{n-3} \] 
(4.8)
From (4.6), we have
\[
\begin{align*}
\text{for } i &= n - 3, \quad H^*_{n-3} H^*_{n-3} = k_2 H^*_{n-4} H^*_{n-3} - k_1 H^*_{n-3} H^*_{n-2} \\
\text{for } i &= n - 4, \quad H^*_{n-4} H^*_{n-4} = k_3 H^*_{n-5} H^*_{n-4} - k_2 H^*_{n-4} H^*_{n-3} \\
\text{for } i &= n - 5, \quad H^*_{n-5} H^*_{n-5} = k_4 H^*_{n-6} H^*_{n-5} - k_3 H^*_{n-5} H^*_{n-4} \\
\end{align*}
\]
and from (4.5), we have
\[ H^*_{n-1} H^*_{n-1} = -k_{n-3} H^*_{n-1} H^*_{n-2}. \]
So, an algebraic calculus show that
\[ H^*_{1} H^*_{1} + H^*_{2} H^*_{2} + \ldots + H^*_{n-5} H^*_{n-5} + H^*_{n-4} H^*_{n-4} + H^*_{n-3} H^*_{n-3} + H^*_{n-2} H^*_{n-2} = 0. \] 
(4.9)
And, by integrating (4.9), we can easily say that
\[ H^*_{1} + H^*_{2} + \ldots + H^*_{n-2} \]
is a nonzero constant. This completes the proof. \( \square \)

**Proposition 4.1.** Let \( \alpha : I \subset \mathbb{R} \to E^n \) be an arc-lengthed parameter curve in \( E^n \) and \( X \) a unit constant vector field of \( \mathbb{R}^n \). \( \{V_1, V_2, \ldots, V_n\} \) denote the Frenet frame of the curve \( \alpha \) and \( \{H^*_1, H^*_2, \ldots, H^*_n\} \) denote the harmonic curvature functions of the curve \( \alpha \). If \( \alpha : I \subset \mathbb{R} \to E^n \) is an \( V_n \)-slant helix with \( X \) as its axis, then we have for all \( i = 0, 1, \ldots, n-2 \)
\[ \langle V_{n-(i+1)}, X \rangle = H^*_i \langle V_n, X \rangle. \]

**Remark 4.1.** The following Theorem is the new version of the Theorem 4 in [4] with addition sufficiency case.

**Theorem 4.1.** Let \( \{V_1, V_2, \ldots, V_n\} \) be the Frenet frame of the curve \( \alpha \) and let \( \{H^*_1, H^*_2, \ldots, H^*_n\} \) be the harmonic curvature functions of \( \alpha \). Then, \( \alpha \) is an \( V_n \)-slant helix (with the curvatures \( k_i \neq 0 \), \( i = 1, 2, \ldots, n-1 \)) in \( E^n \) if and only if \( \sum_{i=1}^{n-2} H^*_i = \text{constant} \) and \( H^*_{n-2} \neq 0 \).

**Proof.** Let \( \alpha \) be a \( V_n \)-slant helix. According to the Definition 2.3,
\[ \langle V_n, X \rangle = \cos(\varphi) = \text{constant}, \] 
(4.10)
where \( X \) the axis of \( \alpha \). And, from Proposition 4.1.,
\[ \langle V_{n-(i+1)}, X \rangle = H^*_i \langle V_n, X \rangle. \] 
(4.11)
for \( 1 \leq i \leq n-2 \). Moreover, from (4.10) and Frenet equations, we can write \( \langle V_{n-1}, X \rangle = 0 \). Since the orthonormal system \( \{V_1, V_2, \ldots, V_n\} \) is a basis of \( \mathcal{Z}(E^n) \) (tangent bundle), \( X \) can be expressed in the form
\[ X = \sum_{i=1}^{n} \langle V_i, X \rangle V_i. \] 
(4.12)
Hence, by using the equations (4.10), (4.11) and (4.12), we obtain
\[ X = \cos(\varphi)V_n + \sum_{i=1}^{n-2} H^*_i \cos(\varphi)V_{n-(i+1)}. \]
Since \( X \) is a unit vector field (see Definition 2.3),
\[ \cos^2(\varphi) + \sum_{i=1}^{n-2} H^*_i \cos^2(\varphi) = 1 \]
and so
\[ \sum_{i=1}^{n-2} H_i^2 = \tan^2(\varphi) = \text{constant}. \]

Now, we are going to show that \( H_{n-2}^* \neq 0 \). We assume that \( H_{n-2}^* = 0 \). Then, for \( i = n-2 \) in (4.11),
\[ \langle V_1, X \rangle = H_{n-2}^* \langle V_n, X \rangle = 0. \]
So, \( \langle D_T T, X \rangle = \langle k_1 V_2, X \rangle = 0 \). We deduce that \( \langle V_2, X \rangle = 0 \). On the other hand, for \( i = n-3 \) in (4.11),
\[ \langle V_2, X \rangle = H_{n-3}^* \langle V_n, X \rangle. \]
And, since \( \langle V_2, X \rangle = 0 \), \( H_{n-3}^* = 0 \). Continuing this process, we get that \( H_i^* = 0 \). Let us recall that \( H_i^* = k_{n-i}/k_n \), thus we have a contradiction because all the curvatures are nowhere zero. Consequently \( H_{n-2}^* \neq 0 \).

Conversely, we assume that \( \sum_{i=1}^{n-2} H_i^2 = \tan^2(\varphi) = \text{constant} \) and \( H_{n-2}^* \neq 0 \). Then, consider the vector field
\[ X = \cos(\varphi) V_n + \sum_{i=3}^{n} H_i^* \cos(\varphi) V_{n-(i-1)}. \]

We want to verify that \( X \) is a constant along \( \alpha \), i.e. \( D_{V_1} X = 0 \). So,
\[ D_{V_1} X = D_{V_1} (\cos(\varphi) V_n) + \sum_{i=3}^{n} D_{V_1} (H_i^* \cos(\varphi) V_{n-(i-1)}) \]
\[ = \cos(\varphi) D_{V_1} V_n + \sum_{i=3}^{n} (H_i^{**} \cos(\varphi) V_{n-(i-1)} + H_{i-2}^* \cos(\varphi) D_{V_1} V_{n-(i-1)}) \]
\[ = \cos(\varphi) (-k_{n-i} V_{n-1} + \sum_{i=3}^{n-1} (H_i^{**} V_{n-(i-1)} - k_{n-i} H_{i-2}^* V_{n-1} + k_{n-(i-1)} H_{i-2}^* V_{n-(i-2)}) + H_{n-2}^* V_1 + k_1 H_{n-2}^* V_2). \]

On the other hand, by using (4.2), we can write
\[ H_{i-2}^{**} = k_{n-(i-1)} H_{i-3}^* - k_{n-i} H_{i-1}^* \quad \text{(4.13)} \]
for \( 4 \leq i \leq n-1 \) together with (4.3). Moreover, from Lemma 4.1, we know that
\[ H_{n-2}^{**} = k_1 H_{n-3}^*. \quad \text{(4.14)} \]
Therefore, by using (4.3), (4.13) and (4.14), an algebraic calculus shows that \( D_{V_1} X = 0 \). Since
\[ \|X\| = \cos^2(\varphi) + \sum_{i=3}^{n} H_i^2 \cos^2(\varphi) \]
\[ = \cos^2(\varphi) \left( 1 + \sum_{i=1}^{n-2} H_i^2 \right) \]
\[ = \cos^2(\varphi) \left( 1 + \tan^2(\varphi) \right) \]
\[ = 1, \]
\( X \) is a unit vector field. Furthermore, \( \langle V_n, X \rangle = \cos(\varphi) = \text{constant} \). Hence, we deduce that \( \alpha \) is a \( V_n \)-slant helix.

**Remark 4.2.** The following corollary is the reconfiguration of the Theorem 2 in [4].

**Corollary 4.1.** Let \( \{V_1, V_2, ..., V_n\} \) be the Frenet frame of the curve \( \alpha \) and let \( \{H_1^*, H_2^*, ..., H_{n-2}^*\} \) be the harmonic curvature functions of \( \alpha \). Then, \( \alpha \) is a \( V_n \)-slant helix (with the curvatures \( k_i \neq 0 \), \( i = 1, 2, ..., n-1 \)) in \( E^n \) if and only if \( H_{n-2}^* = k_1 H_{n-3}^* \) and \( H_{n-2}^* \neq 0 \).

**Proof.** It is obvious by using Lemma 4.1. and Theorem 4.1. □
5. Slant helices and their $G_i$ differentiable functions

In this section, we reconfigure some known characterizations of slant helices by using $G_i$ differentiable functions which is similar to harmonic curvature functions.

**Definition 5.1.** Let $\alpha : I \to E^n$ be a unit speed curve (with the curvatures $k_i \neq 0$, $i = 1, 2, \ldots, n - 1$) in $E^n$. Define the functions

$$G_1 = \int k_1(s)ds, \quad G_2 = 1, \quad G_3 = \frac{k_1}{k_2}G_1, \quad G_i = \frac{1}{k_{i-1}}[k_{i-2}G_{i-2} + G'_{i-1}]$$

where $4 \leq i \leq n$. \[\square\]

**Lemma 5.1.** Let $\alpha$ be a unit curve in $E^n$ and let $G_n \neq 0$ be for $i = n$. Then, $G_1^2 + G_2^2 + \cdots + G_n^2$ is a nonzero constant if and only if $G_n = -k_{n-1}G_{n-1}$.

**Proof.** First, we assume that $G_1^2 + G_2^2 + \cdots + G_n^2$ is a nonzero constant. Consider the functions

$$G_i = \frac{1}{k_{i-1}}[k_{i-2}G_{i-2} + G'_{i-1}]$$

for $5 \leq i \leq n$. So, from the equality, we can write

$$k_{i-1}G_i = G_{i-1} + k_{i-2}G_{i-2}, \quad 5 \leq i \leq n.$$ \[\text{(5.2)}\]

Hence, in (5.2), if we take $i + 1$ instead of $i$, we get

$$G'_i = k_iG_{i+1} - k_{i-1}G_{i-1}, \quad 4 \leq i \leq n - 1.$$ \[\text{(5.3)}\]

On the other hand, since $G_1^2 + G_2^2 + \cdots + G_n^2$ is constant, we have

$$G_1G'_1 + G_2G'_2 + \cdots + G_nG'_n = 0$$

and so,

$$G_nG'_n = -G_1G'_1 - G_2G'_2 - \cdots - G_{n-1}G'_{n-1}.$$ \[\text{(5.5)}\]

By using (5.3) and (5.4), we obtain

$$G_2G'_2 = 0 \quad \text{and} \quad k_3G_3G_4 = G_1G'_1 + G_3G'_3$$ \[\text{(5.6)}\]

and

$$G_1G'_1 = k_3G_3G_4 - k_{i-1}G_{i-1}G_i, \quad 4 \leq i \leq n - 1.$$ \[\text{(5.7)}\]

Therefore, by using (5.5), (5.6) and (5.7), a algebraic calculus shows that

$$G_nG'_n = -k_{n-1}G_{n-1}G_n.$$ \[\text{(5.8)}\]

Since $G_n \neq 0$, we get the relation $G'_n = -k_{n-1}G_{n-1}$.

Conversely, we assume that

$$G_n = -k_{n-1}G_{n-1}.$$ \[\text{(5.9)}\]

By using (5.8) and $G_n \neq 0$, we can write

$$G_nG'_n = -k_{n-1}G_{n-1}G_n.$$ \[\text{(5.10)}\]

From (5.7), we have

- for $i = n - 1$, $G_{n-1}G'_{n-1} = k_{n-1}G_{n-1}G_n - k_{n-2}G_{n-2}G_{n-1}$
- for $i = n - 2$, $G_{n-2}G'_{n-2} = k_{n-2}G_{n-2}G_{n-1} - k_{n-3}G_{n-3}G_{n-2}$
- for $i = n - 3$, $G_{n-3}G'_{n-3} = k_{n-3}G_{n-3}G_{n-2} - k_{n-4}G_{n-4}G_{n-3}$

and so, from (5.9) and the last system, we have

$$G_4G'_4 + G_5G'_5 + \cdots + G_nG'_n = -k_3G_3G_4$$ \[\text{(5.10)}\]
by doing an algebraic calculus. On the other hand, from (5.6), we know that
\[ G_2 G'_2 = 0 \quad \text{and} \quad k_3 G_3 G_4 = G_1 G'_1 + G_3 G'_3 . \] (5.11)
Finally, from (5.10) and (5.11), we obtain
\[ G_1 G'_1 + G_2 G'_2 + \ldots + G_n G'_n = 0 . \] (5.12)
And, by integrating (5.12), we can easily say that
\[ G_1^2 + G_2^2 + \ldots + G_n^2 \]
is a nonzero constant. This completes the proof. \( \square \)

**Corollary 5.1.** Let \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) be an arc-lengthed parameter curve with nonzero curvatures \( k_i \) \( (1 \leq i \leq n-1) \) in \( E^n \) and \( X \) a unit constant vector field of \( \mathbb{R}^n \). \( \{V_1, V_2, \ldots, V_n\} \) denote the Frenet frame of the curve \( \alpha \). If \( \alpha : I \subset \mathbb{R} \rightarrow E^n \) is an \( V_2 \)-slant helix with \( X \) as its axis, then we have for all \( i = 1, \ldots, n \)
\[ (V_i, X) = G_i (V_2, X) . \]

*Proof.* It is obvious by using the proof of Theorem 1.2 in [1]. \( \square \)

**Remark 5.1.** The following Theorem is the new version of the Theorem 1.2 in [1].

**Theorem 5.1.** Let \( \{V_1, V_2, \ldots, V_n\} \) be the Frenet frame of the curve \( \alpha \). Then, \( \alpha \) is a \( V_2 \)-slant helix (with the curvatures \( k_i \neq 0, i = 1, 2, \ldots, n-1 \)) in \( E^n \) if and only if \( \sum_{i=1}^{n} G_i^2 = \text{constant} \) and \( G_n \neq 0 \). Here,
\[ G_1 = \int k_1 (s) ds \quad G_2 = 1 \quad G_3 = \frac{k_1}{k_2} G_1 \quad G_i = \frac{1}{k_{i-1}} [k_{i-2}G_{i-2} + G'_{i-1}] \]
where \( 4 \leq i \leq n \).

*Proof.* Let \( \alpha \) be a \( V_2 \)-slant helix. According to the Definition 2.2,
\[ (V_2, X) = \cos(\varphi) = \text{constant}, \] (5.13)
where \( X \) the axis of \( \alpha \). And, from Corollary 5.1.,
\[ (V_i, X) = G_i (V_2, X) \] (5.14)
for \( 1 \leq i \leq n \). Since the orthonormal system \( \{V_1, V_2, \ldots, V_n\} \) is a basis of \( \mathcal{H}(E^n) \) (tangent bundle), \( X \) can be expressed in the form
\[ X = \sum_{i=1}^{n} (V_i, X) V_i . \] (5.15)
Hence, by using the equations (5.13), (5.14) and (5.15), we obtain
\[ X = \sum_{i=1}^{n} G_i \cos(\varphi) V_i . \]
Since \( X \) is a unit vector field (see Definition 2.2),
\[ \cos^2(\varphi) \left( \sum_{i=1}^{n} G_i^2 \right) = 1 \]
and so,
\[ \sum_{i=1}^{n} G_i^2 = \frac{1}{\cos^2(\varphi)} = \text{constant} . \]
Now, we are going to show that \( G_n \neq 0 \). We assume that \( G_n = 0 \). Then, for \( i = n \) in (5.14),
\[ (V_n, X) = G_n (V_2, X) \neq 0 . \]
So, \( (D_T V_n, X) = (-k_{n-1} V_{n-1}, X) = 0 \). We deduce that \( (V_{n-1}, X) = 0 \). On the other hand, for \( i = n-1 \) in (5.14),
\[ (V_{n-1}, X) = G_{n-1} (V_2, X) . \]
And, since \( (V_{n-1}, X) = 0 \), \( G_{n-1} = 0 \). Continuing this process, we get that \( G_3 = 0 \). Let us recall that \( G_3 = \frac{k_1}{k_2} \int k_1 (s) ds \), thus we have a contradiction because all the curvatures are nowhere zero. Consequently \( G_n \neq 0 \).
Conversely, we assume that \( \sum_{i=1}^{n} G_i^2 = \frac{1}{\cos^2(\varphi)} \) =constant and \( G_n \neq 0 \). Then, consider the vector field
\[
X = \sum_{i=1}^{n} G_i \cos(\varphi)V_i.
\]
Then, by taking account
\[
G_1 = \int k_1(s)ds, \quad G_2 = 1, \quad G_3 = \frac{k_1}{k_2}G_1, \quad G_i = \frac{1}{k_{i-1}} \left[ k_{i-2}G_{i-2} + G_{i-1} \right], \quad 4 \leq i \leq n
\]
and Frenet equations, an algebraic calculus shows that \( D\nu_i X = 0 \). That is, \( X \) is a constant along \( \alpha \).

Also, since
\[
\|X\| = \sum_{i=1}^{n} G_i^2 \cos^2(\varphi)
\]
\[
= \cos^2(\varphi) \left( \sum_{i=1}^{n} G_i^2 \right)
\]
\[
= \cos^2(\varphi) \frac{1}{\cos^2(\varphi)}
\]
\[
= 1,
\]
\( X \) is a unit vector field. Furthermore, \( \langle V_2, X \rangle = \cos(\varphi) \) =constant. Hence, we deduce that \( \alpha \) is a \( V_2 \)-slant helix.

\[\square\]

**Remark 5.2.** The following corollary is the reconfiguration of the Theorem 3.1 in [1].

**Corollary 5.2.** Let \( \{V_1, V_2, ..., V_n\} \) be the Frenet frame of the curve \( \alpha \). Then, \( \alpha \) is a \( V_2 \)-slant helix in \( E^n \) if and only if \( G_n = -k_{n-1}G_{n-1} \) and \( G_n \neq 0 \), where the functions \( \{G_1, G_2, ..., G_n\} \) defined in (5.1).

**Proof.** It is obvious by using Lemma 5.1. and Theorem 5.1. \( \square \)

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