Magnetothermal instabilities in magnetized anisotropic plasmas

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Abstract. Using the 16–moment transport equations for an ideal anisotropic collisionless plasma we analyze the influence of pressure anisotropy on the magnetothermal (MTI) and heat–flux–driven buoyancy (HBI) instabilities. We calculate the dispersion relation and the growth rates for these instabilities in the presence of a background heat flux and for configurations with static pressure anisotropy, finding that when the frequency at which heat conduction acts is much larger than any other frequency in the system (i.e. weak magnetic field) the pressure anisotropy has no effect on the MTI/HBI, provided the degree of anisotropy is small. In contrast, when this ordering of timescales does not apply the instability criteria depend on pressure anisotropy.

1. Introduction

In many magnetized dilute astrophysical plasmas thermal electron conduction occurs almost exclusively parallel to magnetic field lines. In this regime, the equations of ideal magnetohydrodynamics (MHD) that describe the plasma dynamics must be supplemented with anisotropic transport terms for energy and momentum due to the near free-streaming motions of particles along magnetic field lines [1]. [2, 3] have shown that anisotropic thermal conduction can fundamentally alter the Schwarzschild convective instability criterion in stratified atmospheres, in that convection sets in when a temperature gradient (as opposed to an entropy gradient) is present. This new convective instability is termed the magnetothermal instability (MTI) [2] or the heat–flux–driven buoyancy instability (HBI) [4] when the temperature decreases or increases with height, respectively. We note that the HBI only takes place if a background heat–flux is present. Essentially, the MTI and the HBI occur because the heat flux must follow the perturbed magnetic field lines, and thus there are regions in the plasma which are locally heated or cooled (see [2–6]).

The MTI and HBI couple the magnetic structure of the plasma to its thermal properties and can have important implications for galaxy clusters (see e.g.[5–8]). For example, in a weakly magnetized atmosphere local simulations have demonstrated that the MTI can amplify magnetic field acting as a magnetic dynamo and also drive strong turbulence.

Dilute magnetized plasmas not only exhibit anisotropic thermal conduction but also an anisotropic pressure tensor, which results from different kinetic temperatures for electron motion in the parallel and perpendicular direction to the magnetic field. For collisionless plasmas the
pressure does not isotropize, and hence if we insist on using a fluid description the standard magnetohydrodynamics theory (MHD) must be modified [9–13].

The purpose of this work is to study the MTI and the HBI in a dilute magnetized plasma including the effect of pressure anisotropy. To this end, we employ the theory of kinetic MHD (KMHD) as derived by the 16–moment method from the Vlasov equation [10–12]. This formalism extends the well–known double–adiabatic theory of Chew, Goldenberg and Low (CGL) [9] in that it allows for nonvanishing (parallel and perpendicular) heat fluxes, which are neglected in the CGL theory. Although we keep the discussion quite general throughout the paper, we have in mind an astrophysical magnetized collisionless plasma, and as a typical example we consider the ICM in section 5. We note that our study is limited to the case where pressure perturbations are zero, i.e. we only treat the case of a static background pressure anisotropy [5, 14].

2. Basic equations
In this section we describe the equations and set the stage for the analysis of wave instabilities given later on (More complete accounts of KMHD and similar approaches can be found in [11–13]).

The inadequacies of the CGL approach (e.g. in the criterion and nonlinear evolution of the fire hose and mirror instabilities [13, 15, 16]) stem from the unwarranted neglect of the third–order moment of Vlasov equation. The 16–moment method provides a consistent way of deriving from the Vlasov equation the correct fluid equations for a heat–conducting anisotropic magnetized plasma. If the Larmor radius is much smaller than the other characteristic lengths of the plasma the equations can be considerably simplified [10] (see also [11, 12]):

\[
\frac{d\rho}{dt} + \rho \vec{v} \cdot \vec{v} = 0
\]

\[
\rho \frac{d\vec{v}}{dt} + \vec{v} \left( p_\perp + \frac{B^2}{8\pi} \right) - \frac{1}{4\pi} (\vec{B} \cdot \vec{v}) \vec{B} = \rho \vec{g} + b\vec{\nabla} \Delta p
\]

\[
+ \Delta p \left[ b(\vec{v} \cdot \hat{b}) + \nabla \parallel \hat{b} \right]
\]

\[
\frac{d\vec{B}}{dt} + \vec{B} (\nabla \cdot \vec{v}) - (\vec{B} \cdot \nabla) \vec{v} = 0
\]

\[
\rho T \parallel \frac{d}{dt} s = -\nabla \parallel \vec{Q}
\]

\[
\rho T \perp \frac{d}{dt} s = -\nabla \perp \vec{Q}
\]

where

\[
\Delta p = p_\perp - p_\parallel, \quad \hat{b} = \vec{B}/B, \quad \nabla \parallel = \hat{b} \cdot \nabla \quad \text{and} \quad \vec{Q}_\parallel, \vec{Q}_\perp \quad \text{are the parallel and perpendicular heat fluxes,}
\]

which we assume for simplicity to be given by Braginskii’s approximation [1] \( \vec{Q}_\parallel, \vec{Q}_\perp = -\hat{b} \chi \nabla T \||\perp \)

where \( \chi \) is the electron thermal diffusivity (assumed to be a constant [4]).

The specific entropies associated with parallel and perpendicular motion are given by [15]

\[
s_\parallel = \frac{cv}{3} \ln \left( \frac{p_\parallel B^2}{\rho^3} \right) \quad \text{and} \quad s_\perp = \frac{2cv}{3} \ln \left( \frac{p_\perp}{\rho B} \right)
\]
where $c_V$ is the specific heat. Note that the total entropy

$$s = s_{\|} + s_\perp = c_V \ln \left( \frac{p_{\|}^{1/3} p_{\perp}^{2/3}}{\rho^{5/3}} \right)$$

reduces to the ordinary MHD expression for the specific entropy in the isotropic case $p_{\|} = p_{\perp}$.

### 3. MTI/HBI in the presence of pressure anisotropy

We consider a thermally stratified plasma in the presence of gravity $\vec{g} = -g \hat{z}$, so that in equilibrium $dp/dz = -\rho g$. The magnetic field of the equilibrium state is assumed to be homogeneous and in the $(x, z)$ plane $\vec{B} = B_x \hat{x} + B_z \hat{z}$ so that there is a background heat flux given by $\vec{Q}_{\|,\perp} = -\chi (b_x b_z \hat{x} + b_z^2 \hat{z}) dT_{\|,\perp}/dz$. As in [4], we assume a steady–state initial equilibrium so that $\nabla \cdot \vec{Q} = 0$, which implies a linear temperature profile in $\hat{z}$. Note that $p_{\perp} = p_{\|} + \text{const.}$ and that we consider a vanishing initial velocity.

Under these circumstances, putting $\delta v_z \sim \exp(-i\omega t + ik \cdot \vec{r})$ and similarly for other quantities the linearly perturbed equations can be written as

$$\omega \delta \rho - \rho k_{\nu} = 0$$

$$\omega [k_{\nu} \vec{b} - k^2 \delta \vec{v}] = (v_A^2 - \Delta p/\rho) k^2 \delta \vec{B} - \delta B_k + \delta B_{\nu} (\delta p_{\perp} - \delta p_{\|}) + i(k_x \vec{b} - k^2 \hat{z}) g \delta \rho/\rho = 0$$

$$\omega \delta B - \vec{B} k_{\nu} + (\vec{k} \cdot \vec{B}) \delta \vec{v} = 0$$

$$i \frac{c_V}{3} \omega \left[ \frac{1}{p_{\|}} (\delta p_{\|} + 2 \delta p_{\perp}) \right] - \delta \frac{\vec{B}}{B} (1 - \alpha) + \delta \frac{\rho}{\rho} (3 + 2 \alpha)$$

$$+ \left( \frac{ds_{\|}}{dz} + \alpha \frac{ds_{\perp}}{dz} \right) \delta v_z - i \chi \vec{b} \frac{D}{\rho} \left( \frac{d \ln T_{\|}}{dz} + \alpha \frac{d \ln T_{\perp}}{dz} \right)$$

$$+ \chi \rho \left( \frac{6 T_{\|}}{T_{\|}} + \alpha \frac{6 T_{\perp}}{T_{\perp}} \right) = 0$$

$$i \frac{c_V}{3} \omega \left[ \frac{1}{p_{\|}} (\delta p_{\|} - 2 \delta p_{\perp}) \right] - \delta \frac{\vec{B}}{B} (1 + \alpha) + \delta \frac{\rho}{\rho} (3 - 2 \alpha)$$

$$+ \left( \frac{ds_{\|}}{dz} - \alpha \frac{ds_{\perp}}{dz} \right) \delta v_z - i \chi \vec{b} \frac{D}{\rho} \left( \frac{d \ln T_{\|}}{dz} - \alpha \frac{d \ln T_{\perp}}{dz} \right)$$

$$+ \chi \rho \left( \frac{6 T_{\|}}{T_{\|}} - \alpha \frac{6 T_{\perp}}{T_{\perp}} \right) = 0$$

where we have used that

$$\delta \vec{Q}_{\|,\perp} = -\chi \delta b \nabla T_{\|,\perp} - \chi \hat{b} (\delta b \cdot \nabla T_{\|,\perp}) - i \chi \hat{b} \delta b \delta T_{\|,\perp},$$

$v_A = B/(4\pi\rho)^{1/2}$ is the Alfvén speed, and we have defined $\hat{b} = \vec{k} \cdot \vec{b}$, $k_{\nu} = \vec{k} \cdot \delta \vec{v}$, $D = (\delta B_{\perp} - 2 \delta B)/B$, and $\alpha = T_{\perp}/T_{\|}$.
4. Neglecting pressure perturbations

We will now consider the simpler case in which pressure perturbations are neglected. The growing modes of interest have growth times much longer than the sound crossing time of the perturbation, so it is sufficient to work in this approximation. From (9)–(13) we get the following dispersion relation

\[ iA_3\omega^3 + A_2\omega^2 + (iA_1 + \tilde{A}_1)\omega + A_0 = 0 \]  

where

\[ A_0 = (\omega_A^2 - \omega_s^2)A_2 - gKp_\parallel\omega_c,\parallel(1 + \alpha)\frac{d\ln T}{dz} \]
\[ A_1 = \frac{2}{5}\rho c v_T(1 + \frac{2}{3}\alpha)(\omega_A^2 - F) - \rho N^2\frac{k_\perp^2}{k_\parallel^2} \]
\[ \tilde{A}_1 = \frac{4}{15}gT_c v(1 - \alpha)\frac{b}{k_\perp^2}(b_xk_z - b_zk_x) \]
\[ A_2 = p_\parallel(1 + \alpha)\omega_c,\parallel \]
\[ A_3 = \frac{2}{5}\rho c v_T\left(1 + \frac{2}{3}\alpha\right) \]

and \( \omega_A = \vec{k} \cdot \vec{v}_A \), \( N^2 = (2/5)gd(s_\parallel + \alpha s_\perp)/dz \), and \( \omega_c,\parallel = (2/5)\chi_0^2T_\parallel/p_\parallel \) are the Alfvén, Brunt–Väisälä and conduction frequencies, respectively, suitably modified by the anisotropy parameter, and \( F = (c_\perp^2 - c_\parallel^2)b^2 \) with \( c_\parallel^2 = p_\parallel/\rho \). In (15) we have put \( K = \frac{1}{\tau^2}[(1-2b_x^2)k_\perp^2 + 2b_xb_zk_xk_z] \) for notational simplicity. To derive (15) we have used that \( \delta p/\rho = -\delta T_\parallel/T_\parallel = -\delta T_\perp/T_\perp \) (see [2, 4]), and also that \( d\ln T_\parallel/dz = d\ln T_\perp/dz = d\ln T/dz \), where \( T = (T_\parallel + 2T_\perp)/3 \) is the total temperature, which follows since \( \alpha \) is assumed to be a constant.

It is interesting to analyze the special case of a weak magnetic field, in which there exists an ordering in frequencies given by [4]

\[ \omega_c \gg \omega_d = \left(\frac{g}{H}\right)^{1/2} \gg \omega_A \]  

where \( H \) is the local scale-height of the system and \( \omega_d \) its dynamical frequency. Besides, if \( \alpha \) is sufficiently close to 1 then one can safely put \( |F| < \omega_c \). In this limit the dispersion relation becomes

\[ \omega^2 \approx gK \frac{d\ln T_\parallel}{dz} = gK \frac{d\ln T}{dz} \]  

This expression is identical to the result of [4], which shows that when the magnetic field is sufficiently weak so that the timescale ordering given in (21) holds and \( \alpha \sim 1 \) (as occurs e.g. in the ICM), the M1 and the HBI become independent of the pressure anisotropy.

Following [3], we will now analyze the stability of solutions to (15) using the Routh–Hurwitz criteria. It should be noted that the polynomial (15) has complex coefficients and therefore the generalized Routh–Hurwitz theorem must be used (for details see e.g.[17]). Applying this procedure to the polynomial in (15) we find the stability criteria \( A_0 > 0, A_2 > 0, \) and \( A_2(A_1A_2 - A_0) > \tilde{A}_1^2 \). Comparing this result to the isotropic case studied in [3], we see that the stability criteria get modified by pressure anisotropy in two ways. Not only do the expressions for the coefficients \( A_n \) contain \( \alpha \), but also an extra term \( A_1 \) appears which vanishes if \( \alpha = 1 \) or if \( b_xk_z = b_zk_x \). The first effect does not change in a significant way the quantitative analysis of [3], so we shall not pursue it here. However, the appearance of \( \tilde{A}_1 \) in the third stability criteria implies that when pressure anisotropy is taken into account \( A_1A_2 - A_0 \) cannot be arbitrarily small, otherwise an instability develops. This represents a qualitative change with respect to the isotropic case, where \( A_1A_2 - A_0 > 0 \) for stable solutions. Note that this modification only affects transverse perturbations, since \( \tilde{A}_1 = 0 \) if \( k_\perp = 0 \).
Figure 1. (Color online) Growth time for the MTI/HBI as a function of $\alpha$ for $k_z = k_\perp = 1/$kpc and $k_z = k_\perp = 5/$kpc. The inset is a zoom for $0.85 < \alpha < 1.15$, which is a closer range to the values expected in the ICM.

5. Application to the ICM

As an illustrative case, we will now present numerical results corresponding to the ICM. As typical values we take $\rho \sim 10^{-3}$ cm$^{-3}$, $B \sim 10^{-6}$ G, $T \sim 10^{7}$ K, and $P \sim 10^{-10}$ ergs cm$^{-3}$. The value of $g$ is estimated as the average of $g(r) = GM/r^2$ from $r = r_c$ to $r = 2r_c$, where $G$ is the gravitational constant, $M = 10^{14} M_\odot$ is the mass of the ICM and $r_c = 290$ kpc. This gives $g \sim 7 \times 10^{-7}$ cm s$^{-2}$. For the entropy gradient we take $\frac{3d\ln s}{dz} = -\frac{d\ln T}{dz}$, and taking the Brunt–Väisälä frequency to be $\sim 10^{-12}$ s$^{-1}$ [8] we get $\frac{d\ln T}{dz} \sim 10^{-23}$ cm$^{-1}$. The anisotropy parameter in the ICM can be written as $\alpha = 1/(1 + \varphi)$, where $\varphi = |p_\perp - p_\parallel|/p_\perp \sim 1/\sqrt{Re}$ and $Re$ is the Reynolds number [13]. To estimate the lower and upper bounds for the values of $\alpha$ we will consider $Re > 50$, so we get $\alpha \in [0.87, 1.16]$. Note that in the ICM $Re \gg 50$ for convective motions, so that actually even lower values for $|1 - \alpha|$ are expected in this environment. However, in order to show the physical behavior of this instability as a function of the anisotropy parameter it is convenient to consider a broader, though unphysical, range for $\alpha$ (as we do in figure 1). In what follows we take a tangled magnetic field corresponding to $b_x = 0.3$.

In order to illustrate the dependence of the stability of solutions to (15) on pressure anisotropy, we show in figure 1 the growth time $\tau = 1/|\text{Im}(\omega)|$ as a function of $\alpha$ for $k_z = k_\perp = 1/$kpc and $k_z = k_\perp = 5/$kpc.

It is seen from figure 1 that in the case of $\alpha = 1$ the growth time, $\tau$, is of the order of $\sim 100$ My. On the other hand, the presence of anisotropy causes a strong decrease in $\tau$, which
Figure 2. (Color online) Growth time for the MTI/HBI as a function of $k_\perp$ for $k_z = 1/kpc$ and $\alpha = 0.87; 1; 1.16$.

is considerably larger for smaller scales. The dependence of $\tau$ with $\alpha$ is similar in both cases ($\alpha < 1$ or $\alpha > 1$). The inset shows the behavior of $\tau$ in the range $0.85 < \alpha < 1.15$, which as mentioned is a closer range to the values expected in the ICM. It is seen that for scales $\sim 0.2$ kpc the growth time in the anisotropic case can be almost one order of magnitude smaller than the one corresponding to $\alpha = 1$. For larger scales ($> 1$ kpc) $\tau$ decreases by a factor of $\sim 2$ or less.

We now go over to analyze the dependence of $\tau$ with wave vector in the range $[0.1, 10] \times 1/kpc$ (magnetic tension stabilizes the instabilities on shorter scales). Figure 2 shows $\tau$ as a function of $k_\perp$ for $k_z = 1/kpc$ and for three values of $\alpha$. We note that for other values of $k_z$ in the studied range $[0.1, 10] \times 1/kpc$ the results are similar. It is seen that for any value of $k_\perp$ the growth time of the isotropic case is larger than the corresponding ones to $\alpha < 1$ and $\alpha > 1$. The difference in $\tau$ can reach one order of magnitude. As happened in the previous case, the behavior of $\tau$ is similar for both anisotropic cases. Interestingly, the decrease in $\tau$ with the anisotropy is significantly smaller in the kpc range. We will now briefly discuss some possible implications of our findings to the ICM (in the case where the timescale ordering of (21) does not apply). In this connection, the most important results of this work are that: (i) even with small pressure anisotropy the growth time of the MTI/HBI can become almost an order of magnitude smaller than the one corresponding to the isotropic case; (ii) the decrease of $\tau$ with anisotropy is smaller for scales $\sim \text{kpc}$; and (iii) the effect of anisotropy on $\tau$ is larger at smaller scales. As mentioned in the Introduction, the MTI/HBI can be a significant source of magnetic field amplification
[5, 6]. Our results imply that in the ICM the magnetic field amplification due to the MTI/HBI could be faster than what is expected in the isotropic case. Although we expect this effect to be small in the ICM because the magnetic field is weak and thus the timescale ordering holds, it could have some important consequences in the subsequent plasma dynamics following the linear regime.

6. Conclusions
We have found that if the magnetic field is sufficiently weak so that the dynamical frequency is larger than any other frequency of the system, the MTI and HBI do not depend on pressure anisotropy (provided the latter is small). On the other hand, if this timescale ordering does not apply the stability criteria for the MTI and the HBI will depend on pressure anisotropy. For the ICM, the growth time of the instability in the anisotropic case can be almost one order of magnitude smaller than the isotropic one.

The linear analysis of the MTI/HBI in anisotropic plasmas presented in this work is largely idealized, since we neglected viscosity and radiative cooling and we considered static pressure anisotropy configurations. In spite of this shortcomings, we believe that our analysis provides some insight into the effect of pressure anisotropy on the MTI/HBI, as well as on some possible implications for the dynamics of the ICM and similar astrophysical environments.

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