1. Introduction

In this talk, manifolds, maps, group actions etc. are all assumed to be of class $C^\infty$. Throughout the talk, $M$ stands for a closed $C^\infty$ manifold, and $G$ for a connected and simply connected Lie group. Denote by $A^r$ the set of locally free right $G$-actions (of class $C^\infty$) endowed with the Whitney $C^r$-topology ($1 \leq r \leq \infty$).

An action $\varphi : M \times G \to M$ in $A^r$ is said to be $C^r$ locally rigid if there exists a neighbourhood $U$ of $\varphi$ in $A^r$ such that any $\psi \in U$ is smoothly conjugate to $\varphi$ by a diffeomorphism $F$ of $M$, up to an automorphism $A$ of $G$, i.e.

$$F(\varphi(x; g)) = \psi(F(x); A(g))$$

for any $x \in M$ and $g \in G$.

An action $\varphi$ is said to be globally rigid if (1) holds for any $\psi \in A^r$.

Of course this is an extremely strong property. For example any flow (an $\mathbb{R}$-action) on a manifold $M$ other than $S^1$ cannot be $C^1$ locally rigid. This follows from Pugh’s closing lemma in case the flow does not admit a periodic orbit. In the other case notice that the eigenvalues of the Poincaré map along a periodic orbit can be easily changed by a perturbation.

However for other Lie groups, there exist examples of even globally rigid actions. Let $GA$ be the Lie group of the orientation preserving affine transformations on the real line. Let $A \in SL(2; \mathbb{Z})$ be a hyperbolic automorphism of the 2-torus and denote by $T_A$ the mapping torus of $A$. Then the weak stable foliation of the suspension flow is the orbit foliation of a locally free $GA$ action. This action is known to be globally rigid ([GS, G]).

There are other examples. Let $\Gamma$ be a Fuchsian triangle group. Then on the manifold $\Gamma \setminus PSL(2; \mathbb{R})$, a locally free $GA$-action is defined by the right action of the elements

$$\begin{bmatrix} e^{t/2} & x \\ 0 & e^{-t/2} \\ 1 \end{bmatrix}.$$
The orbit foliation of this action is again a weak stable foliation of an Anosov flow. E. Ghys ([G]) showed that this action is also globally rigid.

The proofs of these facts consist of the studies of two independent phenomena. One is concerned about the $C^\infty$ rigidity of the orbit foliation, and the other is about the rigidity of the parametirization of the action. When A. Katok and Lewis ([KL]) showed the $C^1$ local rigidity for certain $\mathbb{R}^n$-actions, the same strategy was taken. So let us devide the local rigidity into two parts.

**Definition 1.1.** An action $\varphi \in A^r$ is said to be $C^r$ locally orbit rigid if there exists a neighbourhood $U$ of $\varphi$ such that the orbit foliation $O_\psi$ of any element $\psi \in U$ is smoothly conjugate to the orbit foliation $O_\varphi$ of $\varphi$, i.e. there exists a diffeomorphism $F$ of $M$ which sends each leaf of $O_\psi$ to a leaf of $O_\varphi$.

**Definition 1.2.** An action $\varphi \in A^r$ is said to be parameter rigid if for any action $\psi \in A^r$ such that $O_\psi = O_\varphi$, there exist a diffeomorphism $F$ which preserves leaves of these identical foliation and an automorphism $A$ of $G$ such that (1) holds.

The local version of parameter rigidity is not defined, simply because we have no intermediate example. In Sect. 2, 3 and 10, we will focus our attention on the parameter rigidity for abelian or solvable Lie group actions.

We will define the leafwise cohomology in Sect. 3 and discuss its close relation with the parameter rigidity when the group $G$ is abelian. After we introduced methods for the computation of the leafwise cohomology in Sect. 4, Sect. 5, 6, 8 and 9 are devoted to the computational results for various concrete foliations.

In Sect. 10, we raise examples of parameter rigid solvable group actions. The final Sect. 11 is devoted to the relation of the leafwise cohomology to the problem of the existence of Riemannian metric for which all the leaves are minimal surfaces.

2. REGIDITY OF FLOWS

An $\mathbb{R}$-action $\varphi$ is what is usually called a flow, and is associated with the vector field $X$ given by

$$X_x = \frac{d}{dt}|_{t=0}\varphi^t(x).$$

An $\mathbb{R}$-action $\varphi$ is locally free if and only if $X$ is nonsingular. As we have mentioned in Sect. 1, there is no $C^1$ locally rigid flow unless the
manifold is $S^1$. However there does exist a parameter rigid flow, which we shall explain below.

Given a real number $\alpha \in \mathbb{R}$, define a Kroneker flow $\varphi_\alpha$ on the 2-torus $T^2$ by

$$\varphi_t^\alpha(x, y) = (x + \alpha t, y + t).$$

If the slope $\alpha$ is rational, then the flow is periodic i.e. $\varphi_q^\alpha$ is the identity for some $q > 0$. All the orbits are closed and their periods are the same. The flow can never be parameter rigid, since one can change the periods of closed orbits so that they are not identical.

If $\alpha$ is irrational, then all the orbits are dense in $T^2$. It is also known that the flow is uniquely ergodic i.e. the $\varphi_\alpha$-invariant probability is unique. As for parameter rigidity, we have the following dichotomy.

**Proposition 2.1.** The Kronecker flow $\varphi_\alpha$ is parameter rigid if and only if the slope $\alpha$ is badly approximable.

**Definition 2.2.** A real number $\alpha$ is said to be badly approximable if there exist $C > 0$ and $\rho > 0$ such that

$$\|k\alpha\|_{S^1} > C|k|^{-\rho}, \quad \forall k \in \mathbb{Z} \setminus \{0\},$$

where $\| \cdot \|_{S^1}$ denotes the distance to 0 of the projected image in $S^1 = \mathbb{R}/\mathbb{Z}$.

The proof of Proposition 2.1 in one direction is in order. Assume $\alpha$ is badly approximable. Let $S^1$ be a circle in $T^2$ defined by $y = 0$. $S^1$ is a global cross section for $\varphi_\alpha$ and the first return map is the rotation by $\alpha$, $R_\alpha$. Let $\psi$ be the flow obtained by reparametrization of $\varphi_\alpha$, i.e. $O_\psi = O_{\varphi_\alpha}$.

Let $f : S^1 \to \mathbb{R}$ be the first return time of $\psi$ for the cross section $S^1$; thus $\psi_f(x) = R_\alpha(x)$.

**Claim:** There exist $g \in C^\infty(S^1)$ and $c \in \mathbb{R}$ such that $f = g \circ R_\alpha - g + c$.

Let us show first that Claim is sufficient to show the parameter rigidity of $\varphi_\alpha$. For this, define a global cross section $C$ to the flow $\psi$ by

$$C = \{ \psi^{-g(x)}(x) \mid x \in S^1 \}.$$

Then we have

$$\psi_c(\psi^{-g(x)}(x)) = \psi^{-g(R_\alpha(x))}(\psi_f(x)) = \psi^{-g(R_\alpha(x))}(R_\alpha(x)).$$

This shows that the return time for the cross section $C$ is identically equal to $c$. Now it is easy to construct a diffeomorphism $F$ conjugating $\psi$ to $\varphi_\alpha$ up to time change $t \to ct$. ($F$ maps $C$ to $S^1$.)

Let us turn to the proof of Claim. We shall show it for complex valued functions. A square integrable function $f \in L^2(S^1)$ can be
expressed as
\[ f = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{2\pi i k t}, \quad \hat{f}_k \in \mathbb{C}, \quad \sum |\hat{f}_k|^2 < \infty. \]

Notice that \( f \in C^\infty(S^1) \) if and only if for any \( r > 0 \), there exists \( C_r > 0 \) such that \( |\hat{f}_k||k|^r < C_r \) for any \( k \in \mathbb{Z} \). Now the equation \( f = g \circ R_\alpha - g + c \) can be read off as
\[ \hat{g}_k = \frac{\hat{f}_k}{(e^{2\pi i k \alpha} - 1)}, \quad c = \hat{f}_0, \]
where \( \hat{g}_k \) is the Fourier coefficients of \( g \). Since \( \alpha \) is badly approximable and since
\[ |e^{2\pi i k \alpha} - 1| \geq C_1 \|k\alpha\|_{S^1} \]
for some \( C_1 > 0 \), we have
\[ \|\hat{g}_k\| \|k\|^r \leq C_1^{-1} \frac{|\hat{f}_k||k|^r}{\|k\alpha\|_{S^1}} \leq C_2 |\hat{f}_k||k|^r \|k\|^r \leq C_3. \]
This shows the smoothness of \( g \), and the proof of Claim is complete. We shall omit the proof of the other implication of Proposition 2.1.

Badly approximability of the slope can be defined for Kronecker flows on higher dimensional torus, and these flows are also shown to be parameter rigid. They constitute all the known examples. For related topics, see [AS, dS, LS].

Here is a criterion of the parameter rigidity of a flow, which will be useful for the study of dynamical properties of such flows.

**Proposition 2.3.** The nonsingular flow \( \varphi \) on \( M \) given by a vector field \( X \) is parameter rigid if and only if for any \( f \in C^\infty(M) \), there exist \( g \in C^\infty(M) \) and \( c \in \mathbb{R} \) such that \( f = X(g) + c \).

Let us show the only if part of the proposition. (The other part is just by an analogous argument, using the integration instead of the differentiation.) Choose \( f \in C^\infty(M) \). We are going to seek for \( g \) and \( c \) as in the proposition. It is no loss of generality to assume that \( f \) is positive. Let \( \psi \) be the flow defined by the vector field \((1/f)X\). Since \( \psi \) and \( \varphi \) has the same orbit foliation, there is a function \( \tau : \mathbb{R} \times M \to \mathbb{R} \) such that
\[ \varphi^t(x) = \psi^{\tau(t,x)}(x). \]
Take \( \frac{d}{dt}|_{t=0} \) of both sides, and we get
\[ \frac{d}{dt}|_{t=0} \tau(t, x) = f(x). \]
Now by the parameter rigidity of \( \varphi \), there is a diffeomorphism \( F \) and \( c \in \mathbb{R} \) such that \( \psi^{ct}(F(x)) = F(\varphi^t(x)) \). The diffeomorphism \( F \) preserves the orbits, and can be written as \( F(x) = \psi^{-g(x)}(x) \) for some \( g \in C^\infty(M) \). We have
\[
\psi^{ct-g(x)}(x) = \psi^{-g(\varphi^t(x))} (\varphi^t(x)), \quad \text{and thus}
\psi^{g(\varphi^t(x))-g(x)+ct}(x) = \varphi^t(x).
\]
That is,
\[
\tau(t, x) = g(\varphi^t(x)) - g(x) + ct.
\]
Taking \( \frac{d}{dt}|_{t=0} \), we obtain \( f = X(g) + c \), as is desired.

An important implication of this observation is:

**Corollary 2.4.** A parameter rigid flow is uniquely ergodic, and it leaves smooth volume form invariant.

Notice that a flow \( \varphi \) is uniquely ergodic if and only if for any \( C^\infty \) map \( f \), the orbit average
\[
\frac{1}{T} \int_0^T f(\varphi^t(x))dt
\]
converges to a constant function, uniformly on \( x \in M \). Thus Proposition 2.3 immediately implies the unique ergodicity.

For the latter statement, let \( \text{vol} \) be a Riemannian volume form on \( M \). Take the Lie derivative: \( L_X(\text{vol}) = f \cdot \text{vol} \). By Proposition 2.3, we have
\[
L_X(\text{vol}) = (X(g) + c)\text{vol}.
\]
The integration over \( M \) shows that \( c = 0 \). Thus we have
\[
X(e^{-g} \cdot \text{vol}) = 0,
\]
i. e. \( e^{-g} \cdot \text{vol} \) is an invariant volume form.

However even among those flows which satisfy the necessary conditions of the previous corollary, no examples of parameter rigid flows are found except the linear flows on higher dimensional torus with badly approximable slopes. Here are two typical negative results in this direction. Let \( F_\lambda : T^2 \to T^2 \) be a diffeomorphism defined by
\[
F_\lambda(x, y) = (x + y, y + \lambda),
\]
where \( \lambda \) is a real number, and let \( \varphi_\lambda \) be the suspension flow. For irrational \( \lambda \), the flow \( \varphi_\lambda \) satisfies the conditions of Corollary 2.4. However we have the following result found in [K].
Theorem 1. For any real number $\lambda$, the flow $\varphi_\lambda$ is not parameter rigid.

The other result is about the horocycle flows. Let us introduce them briefly. Let $M$ be the quotient $\Gamma \setminus SL(2; \mathbb{R})$ by a cocompact lattice $\Gamma$. On $M$ the right action of the one parameter subgroup

$$\left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

defines a flow, called horocycle flow. Horocycle flows are known to satisfy the conditions of Corollary 2.4 ([F]). The following theorem is due to L. Flaminio and G. Forni ([FF]).

Theorem 2. The horocycle flow on a compact manifold $M$ is not parameter rigid.

We will discuss these two theorems later in the next section after the leafwise cohomology of a foliation is introduced.

By the way, here is a very simple geometric proof of Theorem 2 when the manifold $M$ is not a rational homology sphere.

Let us take a basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ as follows.

$$Y = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These are left invariant vector fields on $SL(2; \mathbb{R})$ and induces vector fields on $M = \Gamma \setminus SL(2; \mathbb{R})$. The horocycle flow $\varphi$ is the one defined by the vector field $S$.

Let us denote by $\eta$, $\sigma$ and $u$ the left invariant 1-forms on $SL(2; \mathbb{R})$ which are dual to $Y$, $S$ and $U$. They also induce 1-forms on $M$.

Since the first Betti number of $M$ is nonzero, there is a closed 1-form $\omega$ such that the period map $[\omega] : \pi_1(M) \to \mathbb{R}$ is nontrivial and takes value in $\mathbb{Z}$. Let us write $\omega$ as

$$\omega = f\sigma + *\eta + *u \quad f, * \in C^\infty(M).$$

Assume for contradiction that $f = S(g) + c$. Let $\hat{M}$ be the cyclic covering of $M$ associated with the homomorphism $[\omega]$. Then the lift $\hat{\omega}$ of $\omega$ is exact and the primitive $\hat{h} \in C^\infty(\hat{M})$ is proper. Denote by $\hat{\varphi}$ the lift of the horocycle flow $\varphi$ to $\hat{M}$. For any $x \in M$, any lift $\hat{x}$ of $x$, and
any $T > 0$, we have
\[
\hat{h}(\hat{\phi}^T(\hat{x})) - \hat{h}(\hat{x}) = \int_{\hat{\phi}^0(T\hat{I}(\hat{x}))} \hat{\omega} = \int_{\phi^0(T\hat{x})} \omega = \int_0^T f(\phi^t(x)) dt
\]
\[
= \int_0^T (S(g)(\phi^t(x)) + c) dt = g(\phi^T(x)) - g(x) + cT.
\]

Now the lift of the horocycle flow $\hat{\phi}$ has a dense orbit (Hedlund). Take $\hat{x}$ from a dense orbit. Assume $c \neq 0$. Then there exists a sequence $T_i \to \pm \infty$ such that $\hat{\phi}^{T_i}(\hat{x}) \to \hat{x}$. This is a contradiction since the left hand side for $T_i$ tends to 0, while the right hand side tends to $\pm \infty$. Assume now $c = 0$. Then since the function $\hat{h}$ is proper, there exists $T_j'$ such that the left hand side for $T_j'$ tends to $\pm \infty$. Again a contradiction.

Here is a conjecture by A. Katok ([K]).

**Conjecture 2.5.** An arbitrary parameter rigid flow is smoothly conjugate to a linear flow on the torus of badly approximable slope.

This conjecture is known to be true if the manifold has cup length equal to the dimension, or if the manifold is 3-dimensional and has nonvanishing Betti number.

3. **Leafwise cohomology and parameter rigidity**

In this section, we define the leafwise cohomology of foliations, and discuss its relationship with the parameter rigidity when the foliation is given by a locally free $\mathbb{R}^p$-action.

Let $\mathcal{F}$ be a foliation on a manifold $N$, and denote by $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$. A leafwise $k$-form $\omega$ is a $C^\infty$ cross section of the homomorphism bundle $\text{Hom}(\bigwedge^k T\mathcal{F}; \mathbb{R})$. The space of leafwise $k$-forms is denoted by $\Omega^k(\mathcal{F})$. Thus given $k$ vector fields $X_1, X_2, \ldots, X_k$ tangent to $\mathcal{F}$ i.e. smooth cross sections of $T\mathcal{F}$, and $\omega \in \Omega^k(\mathcal{F})$, a smooth function $\omega(X_1, X_2, \ldots, X_k)$ is defined. The integrability condition for $T\mathcal{F}$ allows us to define the leafwise exterior derivative
\[
d_\mathcal{F} : \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})
\]
just as the usual exterior derivative. For example, for $\omega \in \Omega^1(\mathcal{F})$ we define
\[
(d_\mathcal{F}\omega)(X_1, X_2) = X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2]).
\]

Then $(\Omega(\mathcal{F}), d_\mathcal{F})$ constitutes a cochain complex, whose cohomology is called the leafwise cohomology of $\mathcal{F}$, denoted by $H^*(\mathcal{F})$. This cochain complex is not elliptic, and the leafwise cohomology can be infinite dimensional.
Example 3.1. The 0-dimensional leafwise cohomology $H^0(\mathcal{F})$ coincides with the vector space formed by basic functions i.e. smooth functions on $M$, constant along the leaves. Thus if $\mathcal{F}$ admits a dense leaf, then $H^0(\mathcal{F}) \cong \mathbb{R}$.

Example 3.2. Let $N = L \times T$ and let $\mathcal{F}$ be the foliation on $N$ whose leaves are $L \times \{t\}$, $t \in T$. Then we have

$$H^k(\mathcal{F}) = H^k(L; C^\infty(T)) = H^k(L; \mathbb{R}) \otimes C^\infty(T).$$

The cochain space $\Omega^k(\mathcal{F})$ is equipped with the Whitney $C^\infty$ topology, and the leafwise exterior derivative $d_F$ is continuous. Thus the cocycle space $\text{Ker}(d_F)$ is closed. But the coboundary space $\text{Im}(d_F)$ is not necessarily closed. The quotient of the cocycle space by the closure of the coboundary space is called the reduced leafwise cohomology and is denoted by $H^*(\mathcal{F})$.

J. Álvarez López and G. Hector ([AH]) have given sufficient conditions for the foliations to have infinite dimensional reduced cohomology, and raised a lot of examples. Their examples are for the most part foliations with dense leaves, definitely not like Example 3.2. But in this talk, we are mainly interested in such a foliation for which the leafwise cohomology is finite dimensional.

There are two ways, important for us, to produce elements of the leafwise cohomology. First notice that the restriction map

$$r : \text{Hom}(\bigwedge^\ast TM; \mathbb{R}) \to \text{Hom}(\bigwedge^\ast TF; \mathbb{R})$$

induces a cochain homomorphism (denoted by the same letter)

$$r : \Omega^*(M) \to \Omega^*(\mathcal{F}),$$

where $\Omega^*(M)$ denotes the de Rham complex of $M$. This induces a homomorphism of the cohomology groups

$$r_* : H^*(M; \mathbb{R}) \to H^*(\mathcal{F}).$$

The homomorphism $r_*$ is often nontrivial and yields elements of $H^*(\mathcal{F})$.

Secondly, suppose that the foliation $\mathcal{F}$ is given by a locally free right action of a Lie group $G$. Then by the differentiation, we get a Lie algebra homomorphism $\iota : \mathfrak{g} \to \mathcal{X}^\infty(\mathcal{F})$, where $\mathfrak{g} = T_eG$ is identified with the Lie algebra of the left invariant vector fields on $G$, and $\mathcal{X}^\infty(\mathcal{F})$ the Lie algebra of the vector fields of $M$ tangent to the foliation $\mathcal{F}$. Let $X_1, \cdots, X_p$ be the basis of $\mathfrak{g}$. Then a leafwise $k$-form $\omega$ is completely determined by $\omega(\iota X_{i_1}, \cdots, \iota X_{i_k})$ for $1 \leq i_1 < \cdots < i_k \leq p$, since for each
point \( x \in M \) the tangent space \( T_x(F) \) is spanned by \((\iota X_1)_x, \cdots (\iota X_p)_x\).

Thus given a left invariant \( k \)-form

\[
\omega : g \times \cdots \times g \to \mathbb{R},
\]

a leafwise \( k \)-form \( \iota \omega \) is defined by

\[
\iota \omega (\iota X_{i_1}, \cdots, \iota X_{i_k}) = \omega (X_{i_1}, \cdots, X_{i_k}).
\]

This induces a homomorphism

\[
\iota_* : H^*(g) \to H^*(F).
\]

**Proposition 3.3.** (1) The homomorphism \( \iota_* : H^1(g) \to H^1(F) \) is injective.

(2) If \( G = \mathbb{R}^p \), then \( \iota_* : H^i(\mathbb{R}^p) \to H^i(F) \) is injective for any \( i \geq 0 \).

**Remark 3.4.** The cohomology of the abelian Lie algebra \( \mathbb{R}^p \) is isomorphic to the exterior algebra of \( \mathbb{R}^p \), and hence to the cohomology of the \( p \)-torus:

\[
H^*(\mathbb{R}^p) \cong \bigwedge^* \mathbb{R}^p \cong H^*(T^p; \mathbb{R}).
\]

Let us give a proof of Proposition 3.3. Let \( \xi_1, \cdots, \xi_r \) be the closed left invariant 1-forms whose classes form a basis of \( H^1(g) \). They are of course linearly independent in the dual space \( g^* \) of \( g \). Therefore there exist elements \( X_1, \cdots, X_r \) of \( g \) such that \( \xi_i(X_j) = \delta_{ij} \). Assume

\[
a_1 \iota \xi_1 + \cdots + a_r \iota \xi_r = \frac{df}{f}
\]

for \( a_i \in \mathbb{R} \) and \( f \in C^\infty(M) \). Let \( \gamma_T : [0, T] \to M \) be an integral curve of the vector field \( \sum_i a_i \iota X_i \). Then we have

\[
f(\gamma_T(T)) - f(\gamma_T(0)) = \int_{\gamma_T} \frac{df}{f} = T \sum_i a_i^2.
\]

Since \( T \) can be arbitrarily large and the left hand side is bounded, all the coefficients \( a_i \)'s must vanish. This shows that the classes of \( \iota \xi_i \) are linearly independent in \( H^1(F) \).

The proof for the second part is basically similar and use the Stokes theorem on an embedding of the rectangle \([0, T] \times \cdots \times [0, T]\).

By Proposition 3.3, if \( F \) is the orbit foliation of a locally free \( \mathbb{R}^p \) action, then we have \( \dim H^1(F) \geq p \). The following proposition relates the leafwise cohomology of the orbit foliation to the parameter rigidity of the action.

**Proposition 3.5.** A locally free effective \( \mathbb{R}^p \)-action is parameter rigid if and only if its orbit foliation \( F \) satisfies \( \dim H^1(F) = p \).
An action is said to be effective if the isotropy subgroup of some point of the manifold is trivial. The proof for $p = 1$ is in order. Let $X$ be the vector field defining a nonsingular flow $\varphi$. Define a leafwise 1-form $\omega$ by $\omega(X) = 1$. An arbitrary leafwise 1-form (which is always closed by the dimension reason) is written as $f\omega$ for some $f \in C^\infty(M)$. Given $g \in \Omega^0(\mathcal{F}) = C^\infty(M)$, notice that $(d_{\mathcal{F}}g)(X) = X(g)$, and thus $d_{\mathcal{F}}g = X(g)\omega$. Therefore $\dim H^1(\mathcal{F}) = 1$ if and only if for any $f\omega$, there exist $g \in C^\infty(M)$ and $c \in \mathbb{R}$ such that

$$f\omega = X(g)\omega + c\omega,$$

which is equivalent to the parameter rigidity of the flow $\varphi$ by Proposition 2.3. The proof for $p \geq 2$ is found in [MM].

Now let us return to the flows in Theorems 1 and 2. In fact what is proven respectively by A. Katok, and L. Flaminio and G. Forni are much stronger than stated there.

**Theorem 3.** Let $\mathcal{F}_\lambda$ be the orbit foliation of the flow $\varphi_\lambda$ in Theorem 1. Then the leafwise cohomology $H^1(\mathcal{F}_\lambda)$ is infinite dimensional. Moreover if $\lambda$ is badly approximable, then we have $\mathcal{H}^1(\mathcal{F}_\lambda) = H^1(\mathcal{F}_\lambda)$, i.e. the space of the coboundaries is closed.

**Theorem 4.** Let $\mathcal{F}$ be the orbit foliation of the horocycle flow. Then $H^1(\mathcal{F})$ is infinite dimensional and we have $H^1(\mathcal{F}) = \mathcal{H}^1(\mathcal{F})$.

4. **How to compute leafwise cohomology**

Here we will show some methods to compute the leafwise cohomology of a foliation.

Let $\pi : M \to B$ be a foliated bundle i.e. a fiber bundle equipped with a foliation $\mathcal{F}$ on $M$ which is transverse to each fiber. Thus the dimension of the leaves of $\mathcal{F}$ is greater than or equal to the dimension of the base $B$.

An important class of foliated bundles is obtained by a construction called suspension. Let $(F, \mathcal{G})$ be a foliated manifold. Denote by $\text{Diff}(F, \mathcal{G})$ the group of diffeomorphisms of $F$ which leaves the foliation $\mathcal{G}$ invariant. Let $h : \Gamma \to \text{Diff}(F, \mathcal{G})$ be a homomorphism from a group $\Gamma$. Let $B$ be a manifold such that $\pi_1(B) \cong \Gamma$. Then a foliated manifold $(M, \mathcal{F})$ called the suspension of $h$ is defined as follows. On the product $F \times \tilde{B}$, where $\tilde{B}$ stands for the universal covering of $B$, there is defined a foliation $\tilde{\mathcal{F}}$ whose leaves are $L \times \tilde{B}$, where $L$ is a leaf of $\mathcal{G}$. Consider the diagonal action of $\Gamma$ on $F \times \tilde{B}$, on the first factor through $h$ and on the second by deck transformation. This action clearly preserves the foliation $\tilde{\mathcal{F}}$ and as the quotient we have a foliated manifold $(M, \mathcal{F})$. 
The following fundamental result is due to A. El Kacimi and A. Tihami ([ET]).

**Theorem 5.** For the suspension foliation, there is a spectral sequence such that

\[ E_2^{p,q} = H^p(B; H^q(G)) \]

which converges to \( H^{p+q}(\mathcal{F}) \), where \( H^q(G) \) is a system of local coefficients on which \( \Gamma = \pi_1(B) \) acts through the homomorphism \( h \).

This shows for example that if \( H^q(G) \) is finite dimensional for any \( 0 \leq q \leq n \), then the leafwise cohomology \( H^n(\mathcal{F}) \) is finite dimensional.

Let us consider a special case where the foliation \( G \) on the fiber is a point foliation. In this case we call \( \mathcal{F} \) the suspension of a point foliation by a homomorphism \( h : \Gamma \to \text{Diff}(F) \). We have \( H^k(G) = 0 \) unless \( k = 0 \). Thus the spectral sequence of Theorem 5 collapses and we have:

**Corollary 4.1.** If \( \mathcal{F} \) is the suspension of a point foliation by a homomorphism \( h : \Gamma = \pi_1(B) \to \text{Diff}(F) \) and if \( \widetilde{B} \) is contractible, then we have:

\[ H^p(\mathcal{F}) = H^p(\Gamma; C^\infty(F)), \]

for any \( p \geq 0 \), where \( \Gamma \) acts on \( C^\infty(F) \) through the homomorphism \( h : \Gamma \to \text{Diff}(F) \).

Here is another way for computing the leafwise cohomology, completely different from the above, due to A. Haefliger ([H]).

**Theorem 6.** Let \( \mathcal{F} \) be the suspension of a point foliation by a homomorphism \( h : \Gamma \to \text{Diff}(F) \). Then there is an isomorphism

\[ \int_{\mathcal{F}} : H^{\text{dim}\mathcal{F}}(\mathcal{F}) \to H_0(\Gamma, C^\infty(F)). \]

The group \( H_0(\Gamma, C^\infty(F)) \) is by definition the quotient space of \( C^\infty(F) \) by the subspace spanned by the elements \( f \circ h(\gamma) - f \) for \( f \in C^\infty(F) \) and \( \gamma \in \Gamma \).

In sections below, we shall show computational results for some concrete foliations.

5. **Linear foliations**

Let us consider the suspension foliation of a point foliation, where the group \( \Gamma \) is \( \mathbb{Z}^p \), the fiber \( F \) is \( T^q \), and the homomorphism \( h \) is given by

\[ h : \mathbb{Z}^p \to T^q \subset \text{Diff}(T^q), \]
where $T^q$ is to be the group of the translations of $T^q$. Let $B$ be a real valued $p \times q$ matrix.

**Definition 5.1.** The matrix $B$ is called *badly approximable* if there exist $C > 0$ and $\rho > 0$ such that for any $k \in \mathbb{Z}^q \setminus \{0\}$, we have

$$
\|Bk\|_{T^p} \geq C|k|^{-\rho}.
$$

Given a matrix $B$, a homomorphism

$$
h_B : \mathbb{Z}^p \to \text{Diff}(T^q)
$$

is defined by

$$
h_B(n)(x) = x + nB.
$$

As the suspension of $h_B$ over the base manifold $T^p$, we get a linear foliation $F_B$ on $T^{p+q}$. The matrix $B$ is called the *slope matrix* of the foliation $F_B$.

**Theorem 7. [AS]** If the slope matrix $B$ is badly approximable, then the homomorphism

$$
\iota_* : H^i(\mathbb{R}^p) \to H^i(F_B)
$$

is an isomorphism for any $i \geq 0$.

Recalling the definition of the homomorphism $\iota_*$, we get the following corollary, which will be useful in Sect. 11.

**Corollary 5.2.** Under the same assumption as above, the homomorphism

$$
r_* : H^i(T^{p+q}) \to H^i(F_B)
$$

is a surjection for any $i \geq 0$.

As is easily shown the foliation $F_B$ is the orbit foliation of an $\mathbb{R}^p$ action. Since $H^1(F_B) \cong \mathbb{R}^p$, this action is parameter rigid if the action is effective.

In fact a bit wider class of $\mathbb{R}^p$-actions for which $\iota_*$ is an isomorphism are found in [Lu].

6. **Orbit foliation of transversely hyperbolic $\mathbb{R}^p$ actions**

Abundant examples of $C^1$ locally rigid (hence parameter rigid) $\mathbb{R}^p$-actions are presented by A. Katok and others ([K], [HK], [KL], [KS1], [KS2]). Here let us mention only one type among them. Let us denote by $Aff_+(T^{p+1})$ the group of the orientation preserving affine transformations of $T^{p+1}$, with respect to the standard affine structure of $T^{p+1}$. Let us consider a homomorphism

$$
h : \mathbb{Z}^p \to Aff_+(T^{p+1})
$$
with the following property;

(*) The derivative $h_* : \mathbb{Z}^p \to SL(p+1; \mathbb{Z})$ is injective and the image is generated by hyperbolic elements.

In this case, the image $h_*(\mathbb{Z}^p)$ is shown to be simultaneously diagonalizable in $SL(p+1; \mathbb{R})$.

**Theorem 8.** [KL] If $p \geq 2$, the cohomology $H^1(\mathbb{Z}^p; C^\infty(T^{p+1}))$ associated to $h$ above is isomorphic to $\mathbb{Z}^p$.

The suspension foliation $\mathcal{F}_h$ of $h$ over $T^p$ is the orbit foliation of an $\mathbb{R}^p$ action $\varphi_h$. The above theorem implies that $H^1(\mathcal{F}_h) \cong \mathbb{R}^p$, via Corollary 4. That is, by Proposition 3.5 this action is parameter rigid. Moreover it is shown to be $C^1$ locally rigid ([KL]).

As for the higher cohomology group, however, this foliation does not exhibit such rigid property as the foliation $\mathcal{F}_B$ does in Theorem 7.

**Proposition 6.1.** For $p \geq 1$, the top dimensional cohomology $H^p(\mathcal{F}_h)$ of the foliation $\mathcal{F}_h$ is infinite dimensional and we have $H^p(\mathcal{F}_h) = H^p(\mathcal{F}_h)$.

The proof uses Theorem 6. For $p = 1$ this result is classical. For the related topics for Anosov flows, see [L, GK, dLMM].

Let us state a rigidity result about hyperbolic $\mathbb{R}^p$ actions, which is global in nature. An $\mathbb{R}^p$ action $\varphi$ on a closed $(2p+1)$-dimensional manifold $M$ is called split hyperbolic if there is a continuous splitting of the tangent bundle

$$TM = T\mathbb{R}^p \oplus E_1 \oplus \cdots \oplus E_{p+1},$$

where $T\mathbb{R}^p$ is the tangent bundle of the orbit foliation and each $E_i$ is a 1-dimensional subbundle invariant under the differential of the action. Furthermore we assume that there exist $\xi_1, \cdots, \xi_{p+1} \in \mathbb{R}^p$ such that the flow $\varphi(\cdot, t\xi_i)$ is expanding along $E_i$ and contracting along $E_j$ ($j \neq i$).

**Theorem 9.** [M] A split hyperbolic $\mathbb{R}^p$ action is $C^\infty$ conjugate to the suspension action $\varphi_h$ of some affine representation

$$h : \mathbb{Z}^p \to Aff_+(T^{p+1})$$

with the property (*), up to an automorphism of $\mathbb{R}^p$. 
7. KÜNNETH FORMULA

We shall interrupt the computation of the leafwise cohomology and state a general theorem, which has an interesting application for the parameter rigidity of $\mathbb{R}^n$ actions.

A. El Kacimi and A. Tihami (ET) followed the arguments of R. Bott and L. W. Tu (BT) in the framework of leafwise forms, to obtain e. g. the Mayer-Vietoris theorem or the spectral sequence theorem (Theorem 5). One can further pursue this line to obtain the following Künneth formula for the leafwise cohomology. Let $F$ (resp. $G$) be a foliation on a manifold $M$ (resp. $N$). Then on the product manifold $M \times N$, products of leaves of $F$ and $G$ form a foliation called the product foliation, denoted by $F \times G$.

**Theorem 10.** Assume $\dim H^j(G) < \infty$ for $0 \leq j \leq k$. Then we have an isomorphism:

$$\sum_{i+j=k} H^i(F) \otimes H^j(G) \cong H^k(F \times G).$$

Here is an immediate corollary concerning $\mathbb{R}^n$ actions. Given two actions $\varphi : M \times \mathbb{R}^p \to M$ and $\psi : N \times \mathbb{R}^q \to N$, an $\mathbb{R}^{p+q}$ action $\varphi \times \psi$ on $M \times N$ called the product action is defined by

$$(\varphi \times \psi)((x,y),(s,t)) = (\varphi(x,s), \psi(y,t)),$$

where $x \in M$, $y \in N$, $s \in \mathbb{R}^p$ and $t \in \mathbb{R}^q$.

**Theorem 11.** Let $M$ and $N$ be closed manifolds. Assume that $\varphi$ and $\psi$ are parameter rigid actions of connected abelian Lie groups and that the product action $\varphi \times \psi$ is effective. Then $\varphi \times \psi$ is parameter rigid.

For example the products of the linear actions of Sect. 5 and the transversely hyperbolic actions of Sect. 6 can be parameter rigid. It seems that this theorem is difficult to prove without resorting to an algebraic topological argument.

Now in the next two sections, we will be back to the computation of the leafwise cohomology.

8. THE WEAK STABLE FOLIATION OF THE SUSPENSION ANOSOV FLOW

Let $\mathcal{W}^s$ be the weak stable foliation of the suspension flow of a hyperbolic toral automorphism $A : T^n \to T^n$. Let $\mathcal{V}^s$ be the linear foliation on $T^n$ whose leaves are parallel translations of the stable eigenspace $E^s$. The automorphism $A$ leaves $\mathcal{V}^s$ invariant, and as is well known
the slope matrix of the linear foliation $V^s$ is badly approximable. By Theorem [7] we have

$$H^s(V^s) \cong H^s(\mathbb{R}^p) \cong \bigwedge^p \mathbb{R},$$

where $p$ denotes the dimension of the foliation $V^s$. The foliation $W^s$ in question is the suspension of $(T^n, V^s)$ by a homomorphism $h : \mathbb{Z} \to \text{Diff}(T^n, V^s)$ given by $h(1) = A$. In the case of foliated bundles over $S^1$, Theorem [5] gives rise to the Wang exact sequence;

$$H^{k-1}(V^s) \to H^k(W^s) \to H^k(V^s) \to H^k(V^s) \to H^k(V^s).$$

The first and the last arrows are $A^* - \text{Id}$. Notice that $A^* : H^i(V^s) \to H^i(V^s)$ is the $i$-th exterior product of $A|_{E^s}$, under the identification given by $\iota_* : \bigwedge^i E^s \cong H^i(\mathbb{R}^p) \cong H^i(V^s)$. Thus we get:

**Proposition 8.1.**

$$H^i(W^s) = \mathbb{R} \text{ for } i = 0, 1, \text{ and } H^i(W^s) = 0 \text{ for } i \geq 2.$$  

We shall return to this result in Sect. 11.

**9. Stable foliations of geodesic flows**

There are foliations for which neither Corollary [4.1] nor Theorem [6] apply but still we can compute the leafwise cohomology by a geometric method.

Let $M = \Gamma \setminus \text{SL}(2; \mathbb{R})$ where $\Gamma$ is a cocompact lattice and let $F^s$ be the foliation defined by the action of the subgroup

$$\left\{ \begin{array}{c}
e^{t/2} & x \\
0 & e^{-t/2} \end{array} \right| \begin{array}{c}t, x \in \mathbb{R}\end{array}. $$

This subgroup is isomorphic to the Lie group $GA$ of the orientation preserving affine action of the real line and we have $H^1(\mathfrak{g}a) \cong \mathbb{R}$. The foliation $F^s$ is the weak stable foliation of an Anosov flow.

**Theorem 12.** The homomorphism

$$(r_*, \iota_*): H^1(M; \mathbb{R}) \oplus \mathbb{R} \to H^1(F^s)$$

is an isomorphism.

Since there are many “leafwise 1-currents” given by closed orbits of the Anosov flow, it is not difficult to show that $(r_*, \iota_*)$ is injective. To show the surjectivity, we use the following lemma.
Lemma 9.1. Given a $C^\infty$ function $k$, the equation

\begin{equation}
   k = Y(g) + g
\end{equation}

has a $C^\infty$ solution $g$ if and only if there exists a $C^\infty$ function $l$ fulfilling

\[ S(k) = Y(l). \]

Here $Y$, $S$ and $U$ are the vector fields on $M$ induced by the action of the elements of the Lie algebra $\mathfrak{sl}(2; \mathbb{R})$;

\[ Y = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

The integration along the orbit of the $Y$-flow from $t = -\infty$ gives a continuous solution $g$ of (2), which is $C^\infty$ along $Y$ and $U$ directions. The condition $S(k) = Y(l)$ transforms the difference of $g$ between nearby two points on an $S$-leaf to the sum of the difference of the images of these points by time $-T$ of the $Y$-flow and the integration of $k$ along this long $S$-orbit. Then the unique ergodicity of $S$-flow (13) ensures that $g$ is $C^\infty$ along the $S$-direction.

Here is also a result about 2-dimensional cohomology, which is a translation of a result of A. Haefliger found in [H]. An application of Theorem 6.

Theorem 13. The homomorphism $r_* : H^2(M; \mathbb{R}) \rightarrow H^2(F^s)$ is an isomorphism.

Unfortunately the above theorem is only about the reduced leafwise cohomology $H^2(F^s)$. The honest leafwise cohomology $H^2(F^s)$ is not known. We will be back to this problem in Sect. 11.

10. Parameter rigidity of certain solvable Lie group actions

First of all let us state a criterion for the parameter rigidity of the action of a general Lie group, which takes form of a nonlinear equation if the Lie group is nonabelian.

Let $\varphi : M \times G \rightarrow M$ be a locally free action of a connected and simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Denote by $F$ the orbit foliation. As is explained in Sect. 3, the vector field $\iota(X) \in TF$ is defined for each $X \in \mathfrak{g}$. A $\mathfrak{g}$-valued leafwise 1-form $\omega_0$ associated with the action $\varphi$ is defined by

\[ \omega_0(\iota(X)) = X. \]

It satisfies

\[ d_F \omega_0 + [\omega_0 \wedge \omega_0] = 0, \]
where \([\omega_0 \wedge \omega_0]\) is a leafwise 2-form defined by
\[
[\omega_0 \wedge \omega_0](X, Y) = [\omega_0(X), \omega_0(Y)].
\]

**Proposition 10.1.** Assume the action \(\varphi\) is effective. The action \(\varphi\) is parameter rigid if for any \(g\)-valued leafwise 1-form \(\omega\) such that
\[
d_{\mathcal{F}}\omega + [\omega \wedge \omega] = 0,
\]
there exist an endomorphism \(\Phi\) of \(G\) and a smooth map \(b : M \to G\) such that
\[
\omega = b^{-1}\Phi_*(\omega_0)b + b^{-1}db.
\]

As an application let us consider the foliation of Sect. 8. Computation based on the result of Theorem 12 leads to an alternative proof of the following theorem by E. Ghys (\cite{G}).

**Theorem 14.** The action of the Lie group \(G\) in Sect. 8 is parameter rigid, if the manifold \(M\) is a rational homology sphere.

E. Ghys also showed that if the manifold \(M\) has nonvanishing first Betti number, then the action of \(G\) is parameter rigid among those actions which preserve the volume form. Our method using the leafwise cohomology just yields the same conclusion. Nothing more, except we can show that the sufficient condition of Proposition 10.1 are not satisfied in this case. So the question of the parameter rigidity of the \(G\)-actions for 3-manifolds with nonvanishing first Betti number is still open.

Let us raise other examples. Let \(A\) be a hyperbolic automorphism of the torus \(T^n\) \((n \geq 2)\) and let \(\mathcal{W}^s\) be the weak stable foliation of the suspension flow. By Proposition 8.1 we have \(H^1(\mathcal{W}^s) \cong \mathbb{R}\). If the matrix \(A\) has no negative eigenvalues of absolute value smaller than 1, then the foliation \(\mathcal{W}^s\) is the orbit foliation of a locally free action of a two step solvable group \(G\).

**Theorem 15.** If the characteristic polynomial of \(A\) is irreducible over \(\mathbb{Q}\), then the \(G\) action is parameter rigid.

Together with the rigidity result of the orbit foliations established by A. El Kacimi and M. Nicolau (\cite{EN}), we obtain the following.

**Theorem 16.** If furthermore all the eigenvalues of \(A\) are positive, then the \(G\) action is \(C^\infty\) rigid.
11. Minimizable foliation

A foliation $\mathcal{F}$ on a manifold $M$ is called *minimizable* if there is a Riemannian metric on $M$ for which all the leaves of $\mathcal{F}$ are minimal surfaces.

**Proposition 11.1.** A foliation $\mathcal{F}$ on the total space $E$ of a fiber bundle $\pi : E \to B$ transverse to each fiber and of complementary dimension is minimizable.

In fact a Riemannian metric on $B$ can be lifted to a leafwise Riemannian metric of $\mathcal{F}$ and extended to a Riemannian metric of $E$ in such a way that each fiber is orthogonal to each leaf. Clearly any leaf is totally geodesic for this metric.

Thus most of the examples of foliations so far listed are minimizable. However there is an exception.

**Proposition 11.2.** Horocycle flows on closed 3-manifolds are not minimizable.

In fact a foliation having a positive holonomy invariant measure which is the boundary of an invariant 1-current is not minimizable ([S]).

Here is a criterion for the minimizability due to H. Rummel and D. Sullivan ([R, S]). Assume all the foliations in this section are oriented.

**Theorem 17.** Let $g_0$ be a leafwise Riemannian metric of a $p$-dimensional foliation $\mathcal{F}$. It extends to a Riemannian metric of the whole manifold $M$ for which all the leaves are minimal surfaces if and only if the leafwise volume form $\omega_0$ induced by $g_0$ extends to a form $\omega$ of $M$ such that

$$d\omega(\xi_1, \ldots, \xi_{p+1}) = 0$$

whenever the first $p$ of the vector fields $\xi_i$’s are tangent to the leaves.

Now we call a foliation $\mathcal{F}$ *totally minimizable* if any leafwise Riemannian metric extends to a Riemannian metric for which all the leaves are minimal.

**Corollary 11.3.** A $p$-dimensional foliation $\mathcal{F}$ on $M$ is totally minimizable if the homomorphism

$$r_* : H^p(M; \mathbb{R}) \to H^p(\mathcal{F})$$

is surjective. If $\text{codim}\mathcal{F} = 1$, then the converse is also true.

In fact if $r_*$ is surjective, then the leafwise volume form $\omega_0$ of any leafwise Riemannian metric can be expressed as

$$\omega_0 = r(\omega) + d_F(\eta_0),$$
for a closed \( p \)-form \( \omega \) and a leafwise \( (p - 1) \)-form \( \eta_0 \). But since \( r : \Omega^{p-1}(M) \to \Omega^{p-1}(F) \) is surjective, we have \( \eta_0 = r(\eta) \) for some \( (p - 1) \)-form \( \eta \). It follows then that

\[
\omega_0 = r(\omega + d\eta),
\]

for a closed \( p \)-form \( \omega + d\eta \), whence the criterion of Theorem 17 is satisfied.

To show the converse statement for codimension one foliation, notice first of all that since \( \dim(M) = p + 1 \), the condition for the \( (p + 1) \)-form \( d\omega \) in Theorem 17 is equivalent to saying that \( d\omega = 0 \). Next notice that any leafwise \( p \)-form is the difference of the leafwise volume forms of two leafwise Riemannian metrics.

As applications, Corollary 5.2 implies that the linear foliations on the torus with badly approximable slope matrices are totally minimizable. Also the foliation \( W^s \) of Sect. 8. is totally minimizable by Proposition 8.1.

A foliation \( F \) is said to be almost totally minimizable if any leafwise Riemannian metric is approximated arbitrarily close in the \( C^\infty \) topology by the restriction of Riemannian metrics for which all the leaves are minimal.

One example of such foliations are \( p \)-dimensional linear foliations \( F \) on the torus \( T^n \) whose leaves are all dense. This is a consequence of the surjectivity of the map \( r_* : H^p(T^n) \to \mathcal{H}^p(F) \). This result is best possible if the slope matrix of the foliation \( F \) is not badly approximable. (Otherwise \( F \) is totally minimizable, as we mentioned above.)

Likewise for the foliation \( F^s \) of Sect. 9, the homomorphism \( r_* : H^2(M; \mathbb{R}) \to \mathcal{H}^2(F^s) \) is an isomorphism (Theorem 13). This implies that the foliation \( F^s \) is almost totally minimizable (\[H, HL\]). Unlike the previous example of linear flows, we do not know if this is best possible or not. It is known that the subspaces \( \{Y(f)|f \in C^\infty(M)\} \) and \( \{S(f)|f \in C^\infty(M)\} \) are closed in \( C^\infty(M) \) (\[GK\] or Proposition 6.1 and \[FF\] or Theorem 2). The problem is equivalent to deciding if the sum of these two subspaces is closed.

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