Cache-Aided Modulation for Heterogeneous Coded Caching over a Gaussian Broadcast Channel

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Abstract—Coded caching is an information theoretic scheme to reduce high peak hours traffic by partially prefetching files in the users local storage during low peak hours. This paper considers heterogeneous decentralized caching systems where users' caches and content library files may have distinct sizes. The server communicates with the users through a Gaussian broadcast channel. The main contribution of this paper is a novel joint coded caching and modulation strategy to map the multicast messages generated in the coded caching delivery phase to the symbols of a signal constellation, such that users can leverage their cached content to demodulate the desired symbols with higher reliability and for the sake of simplicity, in this paper we focus only on “uncoded” modulation and symbol-by-symbol error probability. However, our scheme in conjunction with multilevel coded modulation can be extended to channel coding over a larger block lengths.

I. INTRODUCTION

Coded caching, originally proposed by Maddah-Ali and Niesen (MAN) in their seminal work [1], leads to an additional coded multicast gain compared to the conventional uncoded caching. In the MAN model, a server has access to a library of N files and is connected to K users through an error-free shared link of unit capacity. Each user is equipped with a cache of size equivalent to M files. The MAN coded caching scheme consists of two phases: placement and delivery. During the placement phase, users partially store files from the library in their cache memories. The placement is agnostic of the future user demands. After the user demands are revealed, the server delivers a sequence of multicast messages. Each multicast message is simultaneously delivered to multiple users. Such messages are computed as a function of the user demands, of the library files, and of the user cache content, such that after receiving the multicast message, each user uses its own cache content to decode its requested subfile from the multicast message with zero error probability (or vanishing error probability in the limit of large file size). A coded caching system is called centralized [1] if the server assigns the files segments to the users as a function of the number of users in the system. In contrast, in the decentralized case [2], each user individually and independently of the others fills up its cache with bits from the library files without knowing how other users are in the system and which segment have been already cached by the other users.

In practice, it may be more realistic to consider the case where users and files have distinct sizes (heterogeneous caching systems). In [3], the authors proposed a decentralized coded caching scheme with varying cache sizes by applying zero-padding to subfiles of different length to enable their encoding in a joint multicast message. Further improvements on heterogeneous caching could be found in [4]–[7]. A common point of the existing heterogeneous caching schemes is that the delivery phases are based on clique-covering method, which is a direct extension of the MAN delivery. In our heterogeneous network user’s caches and content library files may have distinct sizes.

In this paper, we consider the implementation of a heterogeneous decentralized coded caching system over a Gaussian broadcast channel, which is a more realistic model for the actual communication physical layer than the error-free capacitated shared link. We consider that one multicast message is being transmitted from the server at any one time. In order to achieve the capacity region of the broadcast channel, superposition coding can be applied instead of sequential transmissions [8]. Our main novel scheme maps MAN-type XOR-ed messages to modulation symbols in PSK/QAM. Users can exploit their cache content for both improving demodulation and decoding their requested subfile from the multicast message. In our proposed scheme the contents stored in cache memory also provide side information to the demodulation block to achieve smaller symbol error rate.

Strategies to improve the demodulation error probability by exploiting side information have been proposed in [9]–[12] for index coding and by the authors in [13], who applied the scheme in [11] to coded caching with unequal channel rates of users. The baseline scheme in [9] coincides with the conventional modulation scheme with zero padding. Since the schemes in [12], [13] were designed for special cases of two users or particular modulation formats respectively, their extension to an arbitrary number of users and/or large families of signal constellations is not evident. Unlike the

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1 Each transmitted multicast message in the delivery phase is a binary sum of a set of subfiles and useful to a subset of users, where each corresponding user knows all subfiles in the sum except one such that it can decode the remaining subfile.
aforementioned scenarios our proposed scheme applies to any PSK/QAM modulation constellation and to an arbitrary number of users $K$.

In heterogeneous caching systems, the subfiles in each multicast message generated by a clique-covering method may have distinct sizes, i.e., there is some inherent redundancy in each multicast message. Our idea is to leverage this redundancy in the modulation/demodulation step, such that the average symbol error rate of users can be reduced.

- We propose a novel way to map coded caching multicast messages to modulation symbols, such that users can exploit the cache side information not only to decode their requested subfile in multicast message, but also to improve the symbol error probability.
- We show that the set partitioning labelling proposed in [14]–[17] is optimal (i.e., where the minimum distance is maximized) in our new scheme.
- We prove that the proposed cache-aided modulation scheme outperforms the conventional modulation scheme with zero padding.

II. SYSTEM MODEL AND PROBLEM SETTING

A. System model

We consider a content delivery system with a server having access to a library of $N$ independent files $\mathbf{W} = \{W_1, W_2, \ldots, W_N\}$ with distinct sizes. For each $i \in [N]$, File $W_i$ has $F_i B$ bits where $B = \sum_{j=1}^N |W_j|$ is the total library size in bits, and $F_i = |W_i|/B$. The server (e.g., a wireless base station) transmits a signal $x(t)$ to the users which receive $y_k(t) = \sqrt{\gamma_k} x(t) + \nu_k(t)$, where $\nu_k(t)$ is the Additive White Gaussian Noise (AWGN) at the $k$-th receiver, with unit power, $x(t)$ is also normalized to have unit power, and $\gamma_k$ denotes the receiver Signal-to-Noise Ratio (SNR). Without loss of generality we shall adopt the standard complex baseband discrete-time model and since we focus on symbol-by-symbol demodulation we can omit the discrete time index and simply write $y_k = \sqrt{\gamma_k} x + \nu_k$ for a generic symbol at user $k$ receiver, use $X$ and $Y_k$ to denote the whole transmit and receive sequences over many symbols. Each user $k$ has a cache memory with size $M_k$ bits where $M_k \in [0, B]$. We defined normalized cache sizes as $\mu_k = M_k/B$. The users have different cache sizes, without loss of generality, $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_K$.

The caching system comprises a placement and a delivery phase. In the placement phase users store contents from the library in a decentralized manner without any knowledge about demands. We define $\phi_k$ as the caching function for user $k$, which maps the library $\mathbf{W}$ to the cache content $Z_k \triangleq \phi_k(W_1, W_2, \ldots, W_N)$ for user $k$ with the content of all caches being denoted by $Z := (Z_1, Z_2, \ldots, Z_K)$. In the delivery phase, each user requests one file from the library. We denote the file demanded by user $k$ as $d_k$ and demands of all users by $d := (d_1, d_2, \ldots, d_K)$; Given $(d, Z)$, the server sends the codewords $X \in \mathcal{C}^L$, where $\mathcal{C}$ is a $q$-dimensional signal constellation (i.e., a discrete set of points in $\mathbb{C}^q$), and $L$ is the broadcast codeword length in terms of constellation symbols. Upon receiving $Y_k$, user $k$ needs to decode $W_{d_k}$ from $Y_k$ and $Z_k$.

Given the cache sizes of the users and the file sizes we design a shared link caching scheme to fill the users’ caches in the placement phase and to generate broadcast messages $P$ of total size $RB$ bits in the delivery phase. The broadcast messages are designed such that each requested file $W_{d_k}$ can be recovered from $(Z_k, P)$ for each $k \in [K]$. This caching scheme is agnostic of the users’ SNR and physical layer modulation. The transmitter then maps the coded bits in $P$ into a sequence of $q$-dimensional constellation points, by dividing $P$ into labels of $m$ bits each, and using these labels to index the $2^m$ points of the constellation $\mathcal{C}$, transmitted sequentially over the Gaussian broadcast channel defined before. The normalized load of the broadcast channel in terms of channel uses per $B$ bits is given by $\frac{K}{m q}$, where $R$ is the load of the coded caching scheme as defined above and $q/m$ is the spectral efficiency of the underlying physical layer modulation scheme, expressed in bits per complex signal dimension. Such spectral efficiency depends on the physical layer modulation used, which in turns should be optimized with respect to the expected typical receiver SNR.

The goal of the coded caching delivery scheme is to minimize $R$ while guaranteeing that each user demand can be satisfied, subject to correct decoding of the multicast messages.

The objective of the physical layer modulation is to minimize the average symbol error rate $\bar{T}$ among all users, where

$$\bar{T} = \frac{1}{K} \sum_{k \in [K]} \frac{S_k}{L_k},$$

where $S_k$ represents the number of symbols in $X$ which are useful to user $k$ and decoded wrongly by user $k$, and $L_k$ represents the number of symbols in $X$ which are useful to user $k$.

In decentralized coded caching, during the placement phase user $k$ independently fills his memory with $\mu_k F_i B$ bits of file $W_i$. For each $S \subseteq [K]$ and each $i \in [N]$, $W_{i,S}$ represents the set of bits of $W_i$ which are uniquely cached by users in $S$. Since $B$ is large enough, by the law of the large number, we have

$$|W_{i,S}| = F_i B \left( \prod_{j \in S} \mu_j \prod_{k \notin S} (1 - \mu_k) \right).$$

In the delivery phase, for every non-empty subset of users $S \subseteq [K]$, the server transmits the following coded multicast messages

$$P_S = \bigoplus_{k \in S} W_{d_k, S \setminus \{k\}},$$

In this paper, for the sake of simplicity, we assume the modulation is uncoded, i.e., we let $q = 1$ and consider classical QAM/PSK signal constellations.

In general, the value of $m/q$ can be adapted depending on the worst-case user SNR $\min \gamma_k$. \n
\footnote{In this paper, for the sake of simplicity, we assume the modulation is uncoded, i.e., we let $q = 1$ and consider classical QAM/PSK signal constellations.}
of length $|P_S| = \max_k |W_{d_k, S \setminus k}|$, where enough zeros are added to the shorter subfiles to make their length to \(\max_k |W_{d_k, S \setminus k}|\) in [3]. In addition, for all distinct demands \(d_k\) the server must also broadcast directly to all users having requested file \(d_k\) the subfiles \(W_{d_k, 0}\), which are requested but not cached at any user. Also, notice that the subfiles indexed by \(S = [K]\) is cached by everybody; therefore, there is no need to transmit subfiles \(W_{d_k, [K]}\) to users. In this paper, for the simplicity of illustration, we use the decentralized MAN caching scheme. We will propose a new cache-aided modulation scheme, which can be concatenated with any caching scheme based on clique-covering method.

### III. CACHED-AIDED MODULATION

The main idea of the proposed modulation scheme is that the different lengths of the subfiles in each multicast message \(P_S\) provide some side information to the users with larger cache size to demodulate subfiles. We use a toy example to illustrate the idea.

**Example 1:** Consider a library of two files \(A = (101001010)\) of length 9 bits and \(B = (111001)\) of length 6 bits. Let \(K = 2\) with cache memory \(M_1 = M_2 = 5\) bits. Two users randomly cache \(\mu_1 = \mu_2 = \frac{5}{15} = \frac{1}{3}\) of each file, and suppose that the cache realizations is such that the subfile division is \(A = \{A_0, A_{11}, A_{12}, A_{13}\} = (101, 001, 010, 0)\) and \(B = \{B_0, B_{11}, B_{12}\} = (11, 10, 01, 0)\), i.e., the cache contents are \(Z_1 = \{A_{11}, B_{11}, A_{12}, B_{12}\} = \{101, 10\}\) and \(Z_2 = \{A_{12}, B_{12}, A_{1,2}, B_{1,2}\} = \{001, 01\}\). Let’s assume the demands \(d_1 = A\) and \(d_2 = B\), and we focus on the coded multicast message

\[
P_{12} = A_{[2]} \oplus B_{[1]} = 010 \oplus 010 = 000, \tag{4}
\]

where the symbol \(\oplus\) is used to denote a “blank” position due to the difference in length of the two subfiles. Suppose that the 8PSK modulation constellation is used with labeling as shown in Fig. 1, such that \(P_{12}\) is mapped directly onto the constellation point indexed by the 3-bit label \(P_{12}\). User 2 has subfile \(A_{[2]} = (010)\) in its cache memory and wishes to decode subfiles \(B_{[1]} = (10)\). Since \(|A_{[2]}| > |B_{[1]}|\), user 2 knows the first bit in the label of the transmitted modulation symbol, which must be equal to the first bit of \(A_{[2]}\). Hence, it can demodulate the symbol by considering only the subconstellation of points whose first label bit is 0 (the blue points in Fig. 1). On the other hand, user 1 does not know any bit in the symbol label because its known subfile \(B_{[1]}\) is shorter. Therefore, user 1 must decode the symbol considering the whole constellation. The minimum distance of the 8PSK constellation is \(2\sin(\pi/16)\) while the minimum distance of the “blue” subconstellation is \(2\sin(\pi/4)\). It follows that user 2 has a lower decoding error for the coded multicast message \(P_{12}\) than user 1.

In the following, we provide the general description of our proposed cache-aided modulation scheme. We define \(\ell_S = \max_{k \in S} |W_{d_k, S \setminus k}|\). To transmit requested subfiles, we need to transmit \(n_S = \frac{\ell_S}{m}\) constellation symbols.\(^4\) First we divide each subfile \(W_{d_k, S \setminus k}\) into \(n_S\) pieces, each of which is denoted by \(W_{d_k, S \setminus k}^i\) where \(i \in [n_S]\).\(^5\) We generate one coded block

\[
P_S^j = \bigoplus_{k \in S} W_{d_k, S \setminus k}^j,
\]

for all \(i \in [n_S]\) and then transmit each block. Notice that each code block has size \(m\) and therefore can be mapped directly onto a modulation point. We define \(n_{S, k}\) as the number of useful symbols to user \(k\) among the symbols for \(P_S\). Notice that, in the proposed scheme, \(n_{S, k} = n_S\) for all \(k\). On the other hand, the difference between the conventional zero padding scheme and our proposed scheme is the way to partition each subfile into blocks. In the delivery phase of the zero padding scheme, we pad enough zero to the end of \(W_{d_k, S \setminus k}\) to make it same length as the longest subfiles \(\max_k |W_{d_k, S \setminus k}|\) subfiles and divide it into \(n_S\) partitions denoted as \(W_{d_k, S \setminus k}^i, i \in [n_S]\).

Recall that \(n_{S, k}\) is the number of useful symbols among \(n_S\) symbols for \(P_S\). Notice that, in the zero padding scheme, \(n_{S, k}\) might be different for users in \(S\) and in general we have \(n_{S, k} \leq n_{S}\).

In Fig. 2, comparison between zero padding scheme and proposed scheme is illustrated through an example. In coded multicast message we would like to transmit subfiles \(A_{[2]}\) of length 9 bits and subfiles \(B_{[1]}\) of length 3 bits. Consider \(m = 3\), for our proposed scheme we first divide \(A_{[2]}\) and \(B_{[1]}\) into 3 blocks of equal size and then pad with zeros to create blocks of length 3. After this per symbol padding, we encode them together as messages \(P_{12}^1\) and \(P_{12}^2\) and \(P_{12}^3\). In conventional zero padding, we pad with zeros the whole subfile \(B_{[1]}\) to create a subfile of size 9 bits. Then we divide both subfiles into 3 blocks and encode each block as messages \(P_{12}^1\) and \(P_{12}^2\) and \(P_{12}^3\). In one word, the proposed scheme uses a zero padding on a “symbol level”, while the conventional scheme uses a zero padding on a “subfile level”.

\(^4\)Since \(B\) is large enough, we assume that \(m\) divides \(\ell_S\).

\(^5\)We also assume that \(n_S\) divides \(|W_{d_k, S \setminus k}|\). Because of this assumption, for user \(k\) and \(S \in [K]\) we have \(|W_{d_k, S \setminus k}^i| = |W_{d_k, S \setminus k}|\) for \(i, j \in [n_S]\).
Fig. 2: proposed and zero padding scheme (most left bit is consider first bit in demodulation)

Fig. 3: Set partitioning labelling for 16-QAM

### A. Derivation of the uncoded symbol error rate

The error probability for $2^m$-PSK is bounded as follows [18]

$$P_{e,2^m-PSK} \leq 2Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right), \tag{5}$$

where $d_{\text{min}}$ is the minimum distance between any two data symbols in a signal constellation and the Q-function is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du. \tag{6}$$

Since with the proposed binary labeling some users will have some of the most significant bits (leftmost in the label arranged from left to right) known, as seen in the example, we wish to use a binary labeling such that the $d_{\text{min}}$ of the sub constellation indexed by the label with fixed first $n$ most significant bits is maximized for any configuration of the $n$ bits. This is known to be the labeling by set partitioning, well-known in the coded modulation literature [14]–[17]. In Appendix A and B, show that the set partitioning labelling is optimal labelling for our proposed scheme. For this labeling for the $2^m$-PSK modulation we have

$$d_{\text{min},n} = 2\sin\left(\frac{\pi}{2^{m-n}}\right) \sqrt{E}, \tag{7}$$

where $n$ is number of known bits at receiver and $P$ is power.

**Proof:** Appendix A

A similar reasoning applied to QAM constellations of size $2^m$ obtained by carving $2^m$ from the infinite squared grid on the complex plane. In this case, we have

$$P_{e,2^m-QAM} \leq \left(1 - \left[1 - 2Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right)\right]\right)^2 \leq 4Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right), \tag{8}$$

and the labeling by set partitioning yields the subconstellation minimum distance

$$d_{\text{min},n} = (\sqrt{2})^n \sqrt{\frac{6mP}{2^m - 1}}, \tag{9}$$

where $\gamma_k$ denotes the SNR at receiver of user $k$ and $i_0 = 1$. Notice that $|W_{d_k,S\setminus k}^i|$ is invariant regarding to index $i$. In other words, $|W_{d_k,S\setminus k}^i| = |W_{d_k,S\setminus k}^{i_0}|$ where $\forall i \in [n_S]$. After calculating $P_{e,k,S}$ for $\forall i \in [n_S,k]$ The total number of useful symbols for user $k$ is given by

$$L_k = \sum_{S:k \in S} n_{S,k}. \tag{11}$$

The average total number of symbol errors among the symbols useful to user $k$ is give by

$$S_k = \sum_{S:k \in S} n_{S,k} P_{e,k,S}. \tag{12}$$

For each user $k$ average symbol error rate is defined as

$$\tilde{T}_k = \frac{S_k}{L_k}. \tag{13}$$
Finally, $T$ in (1) is obtained by using above equations.

**Lemma 1:** The proposed cache-aided modulation achieves lower or equal symbol error rate $T_k$ (13) for all user $k$ than conventional zero-padding scheme.

**Proof:** By considering file sizes are very large, first we derive $T_k$ for user $k$ for zero padding scheme. In zero padding scheme, for any $S$ and any user $k \in S$ we have two cases: first case if $i \in [n_{S,k}]$ we have $|W_{d_i,S,k}| = m$ and second case if $i \in [n_S] \setminus [n_{S,k}]$ we have $|W_{d_i,S,k}| = 0$. In any useful symbol for user $k$ with indexes in $[n_{S,k}]$ the number of known bits for user $k$ is zero, i.e. $\max_j |S_{d_{io}}| = \min |S_{d_{io}}| = 0$. In zero padding scheme, for all user $k$ for all $S \subseteq [K]$ we have $P_{e,k} = P_{e,k,S}$, where $P_{e,k} = 2Q \left( \sqrt{2\gamma_k \sin^2(\pi)} \right)$. This implies $T_k = P_{e,k}$, where $T_k$ is denoted the symbol error rate in (13) for user $k$. $T_k$ is denoted the symbol error rate for user $k$ for proposed scheme is given by

$$T_k^p = \frac{\sum_{S,k \in S} n_{S,k} P_{e,k,S}}{\sum_{S,k \in S} n_{S,k}}$$

(14a)

$$\leq \max_S P_{e,k,S}$$

(14b)

$$\leq 2Q \left( \sqrt{2\gamma_k \sin^2(\pi)} \right)$$

(14c)

$$= T_k^p.$$  

(14d)

For any received signal in any receiver for proposed scheme the number of known bits and $d_{min}$ are at least as big as the number of known bits for zero padding ones. 

**IV. Simulation results**

In Fig. 4 and Fig. 5, $T_k$ for each user $k$ in (13) and $T$ in (1) versus SNR are plotted. In Fig. 4 we compare $T_k$ in (13) for different users for proposed and zero padding scheme. User 1 has lower cache size among other users, which implies that for this user our scheme does not have any improvement compare to zero padding scheme, the symbol error rate of user 1 with $\mu_1 = 1/5$ for both case are same. User 2 has larger cache size $\mu_2 = 1/3$, symbol error rate of proposed scheme improves compare to zero padding one. Our proposed algorithm achieves noticeable gain for user 3 with cache size $\mu_3 = 1/2$ who has biggest cache size among other users.

**V. Conclusion**

In this paper, for heterogeneous caching systems, we proposed a new scheme to map the requested subfiles generated by the clique-cover delivery method onto physical layer modulation symbols such that users with larger subfiles can take advantage of the known bits in the constellation labels, effectively restricting their detection problem to a sub-constellation of increased minimum distance. We showed that our scheme achieves better or equal symbol error rate for all users with respect to the conventional zero-padding scheme, which does not use cache-aided side information for the demodulation process to any user. In addition, it can be seen that the best labeling for our scheme is the well-known binary set partitioning labeling, widely used in standard coded modulation techniques. Finally, it is possible to extend the proposed scheme to constellation constructions of longer dimension $q$ using the technique of multilevel coded modulation [19], and replace uncoded error rate with coded block error rate using finite-length coding results [20]. In this paper we have focused on the uncoded case for simplicity and for the sake of space limitation, while the full characterization of the achievable tradeoff between coding rate and block error rate is work in progress.

**APPENDIX A**

**Proof of (7)**

A $2^m$-PSK constellation contains $2^m$ points with $B$ denoting the set of these codewords. There are $2^{m-n}$ codewords which share the same prefix $b_1 b_2 \ldots b_n$ and we denote the set of these codewords by $A$. In order to maximise $d_{min}$, we should map these $2^{m-n}$ codewords in possible locations within our constellation in such a way that for any two nearest members of $A$ there should be $2^n - 1$ codewords possessing a different prefix (i.e. from the set $B \setminus A$). We
prove this statement by contradiction as follows. Consider that the statement is false: If there are more than $2^n - 1$ codeword points from the set $B \setminus A$ between any two nearest codeword points from $A$, then in total there should be more than $2^{(m-n)} \times (2^n - 1 + 1) = 2^m$ codewords, which is contradictory to our assumption of having $2^m$ points in the constellation. Having proven that there are $2^n - 1$ points from $B \setminus A$ between any two nearest points in $A$ it becomes trivial to derive (7) by using trigonometry.

**APPENDIX B**

**PROOF OF (9)**

We prove this statement by induction. The proof is structured according to whether $n$ is an even or odd number. This is accounted for in our induction step, where we first show that if the statement holds for $n = 2\hat{n}$ bits, then it also holds for $n = 2\hat{n} + 1$ bits and then proceed to make our proof exhaustive by showing that if the statement is true for $n = 2\hat{n} + 1$ then it follows that it holds for $n = 2\hat{n} + 2$ bits.

In the basis step of our proof, we first show that (9) holds for $n = 1$. Fig. 6a illustrates a sub-constellation of 4 neighbors comprising a square with vertices A, B, C, D. Consider the triangle with vertices A, B, C. According to the Pigeonhole principle, which states that if $n$ pigeons are put into $m$ holes, with $n > m$ there is at least one hole containing more than one pigeon, it follows that two of these codewords should have the same value for their first bit. In order to maximize $d_{\min,1}$, the only possible option is to assign A and C the same value for their first bit, while also giving B’s first bit a different value. By applying the same logic we also come to the conclusion that B and D should have same value for their first bit. It follows that $d_{\min,1} = \sqrt{2d}$, where $d = \sqrt{\frac{m!}{2^{m-1}}}$.

In the $2\hat{n}$-th step, we already assigned $2\hat{n}$ bits for all points within the constellation. Fig. 6a illustrates the neighboring points that share the same prefix of length $2\hat{n}$ bits and the constellation that they form. Similarly to step $n = 1$, the only possible option for maximizing the minimum distance is to assign A and C the same value for their $2\hat{n} + 1$-th bit, while also giving B and D’s $2\hat{n} + 1$-th bit a different value. Assigning the vertices on the diagonal the same value for their $(2\hat{n}+1)$-th bit provides $d_{\min,2\hat{n}+1} = \sqrt{2d_{\min,2\hat{n}}}$. In the $2\hat{n} + 1$-th step, we already assigned $2\hat{n} + 1$ bits for all points within the constellation. The neighboring points possessing the same prefix for their first $2\hat{n} + 1$ bits create the sub constellation Fig. 6b. When observing the triangle A,B,E in Fig. 6b in, it is clear that the vertices A and B should the have same value for their $2\hat{n} + 2$-th bit and that it must differ from E, which in turn should also be different than C and D. It follows that A,B,C,D should have same value for their $2\hat{n} + 2$-th bit and therefore $d_{\min,2\hat{n}+2} = \sqrt{2d_{\min,2\hat{n}+1}} = 2^{\hat{n}+1}d$.

We proved that $d_{\min,n} = \sqrt{2d_{\min,n-1}}$ and $d_{\min,1} = \sqrt{2d}$ and therefore we can conclude that $d_{\min,n} = (\sqrt{2})^n d$.

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