TWISTED BUNDLES AND ADMISSIBLE COVERS

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1. Introduction

The purpose of this paper is twofold. First, we discuss and prove results on twisted covers announced without proofs in [AV], section 3 (with slightly modified notation). Second, we continue with some new results about nonabelian level structures. Some results related to ideas in this paper were also discovered independently by F. Wewers [W].

For the sake of motivation, we start with the problem of smooth moduli spaces of stable curves with level structures.

1.1. Moduli of curves - course and fine. The moduli space of smooth curves $M_g$ and its natural compactification by the moduli space of stable curves $\overline{M}_g$, are among the most illustrious successes of twentieth century mathematics. Yet they have a somewhat unpleasant feature - they are in general singular, where they really shouldn’t be - the deformation spaces of stable curves are all smooth, yet automorphisms prevent the space from being a fine moduli space, and often force the coarse moduli spaces to be singular. Nowadays one knows that this is in some sense an “optical illusion” - one should really work with the corresponding moduli stack, which is always nonsingular. Yet it is a bit dissatisfying to require the use of a specialized tool-kit such as algebraic stacks to see the smoothness in a moduli problem which should have been visibly smooth for the bare eyes.

A satisfactory solution for the “open” moduli space was proposed by Mumford - the space $M_g$ admist a finite Galois cover $M_g^{(m)}$, the moduli space of curves with level-$m$ structure, and as soon as $m \geq 3$ this is a smooth, fine moduli space.

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The search for a similar solution for $\overline{M}_g$ has taken several turns through the years. In [Mu] Mumford used the normalization $\overline{M}_g^{(m)}$ of $\overline{M}_g$ in $M_g^{(m)}$:

\[
\begin{array}{c}
M_g^{(m)} \subset \overline{M}_g^{(m)} \\
\downarrow \\
M_g \subset \overline{M}_g
\end{array}
\]

as a finite covering of $\overline{M}_g$ which carries a “tautological family” of stable curves. This covering is in general singular, it is not a fine moduli space of a natural moduli problem, yet the fact that its singularities are Cohen-Macaulay was useful for intersection theory.

Looijenga [Lo], and soon after Pikaart and De Jong [P-J] (see also [B-P]), used instead the normalizations $G\overline{M}_g$ of $\overline{M}_g$ in the moduli spaces of smooth curves with Teichmüller level structures $G\overline{M}_g$. They showed that, with careful appropriate choices of finite groups $G$, these spaces are smooth Galois covers of $\overline{M}_g$. Yet they did not write down a simple description of these spaces as fine moduli spaces of appropriate moduli problems.

It seems that the main reason a fine–moduli–space interpretation of $G\overline{M}_g$ was not given in [Lo], [P-J] is that this would require working with certain objects up to outer automorphisms. We describe this situation in this paper in terms of a process of rigidification of a stack. It follows that the spaces studied by Looijenga and Pikaart–De Jong are indeed fine moduli spaces of certain Teichmüller structures obtained by rigidification. This solves the problem, but one would prefer a solution which avoids the process of rigidification altogether.

In Section 7.5 we introduce a moduli space which entirely circumvents the need for rigidification. For any $g \geq 2$ we give a finite group $G$ such that the moduli space of connected admissible $G$ covers of genus $g$ is a smooth, fine moduli space, which is a Galois cover of $\overline{M}_g$. The proofs rely on methods introduced in [Lo] and [P-J], and on our theory of twisted $G$-covers, developed in the first few sections of the paper and summarized in 1.3 below.

1.2. Algebra-to-analysis translation table.

In this paper we systematically use the language of stacks. However, the results in this paper may be of interest for people studying moduli spaces from the point of view of differential geometry and analysis, some of whom may prefer the language of differential orbifolds. These people should be able to get by using the following rough translation table.
1.3. Summary of the paper. We fix a noetherian base scheme $S$. The analytically-inclined reader may very well assume that $S = \text{Spec } \mathbb{C}$ and work complex-analytically, perhaps using the translation table provided in section 1.2 above.

1.3.1. Twisted stable maps. Kontsevich’s stack $\overline{M}_{g,n}(X, \beta)$ of $n$-pointed stable maps of genus $g$ into a projective scheme $X$ with homology class $\beta$ (see [Ko]) have served as an extremely useful tool in enumerative geometry (see e.g. [F-P]) and as a construction technique (see e.g., [R-O2]). For similar reasons it is of interest to replace $X$ by a Deligne–Mumford stack. In the paper [R-V2], a proper stack $K_{g,n}(\mathcal{M}, d)$ of twisted stable maps was constructed for each tame Deligne-Mumford stack $\mathcal{M}$ admitting a projective coarse moduli scheme $\mathcal{M}$. As it turns out, the stack of “usual” stable maps into a stack $\mathcal{M}$ may fail to be proper - an example relevant to this paper is given in [R-V2], section 1.3. The main point of the paper [R-V2] was, that in order to have a proper stack of maps into the stack $\mathcal{M}$, it is natural to allow the curves at the source of the map to acquire orbispace structure at the nodes (and to have a complete picture, also along the markings). A particularly important open-and-closed substack of $K_{g,n}(\mathcal{M}, d)$ is the stack $K_{g,n}(\mathcal{M}, d)$ of balanced twisted stable maps, where the underlying twisted curves are smoothable.

Here we study the constructions of [R-V2] in the case where $\mathcal{M}$ is the classifying stack $BG$ of a finite group (or, more generally, a finite étale group scheme) $G$. (The assumption that $BG$ be tame translates to the requirement that for each field $k$ and point $p$ in $S(k)$, the degree of $G_p$ is invertible in $k$.)

1.3.2. Twisted stable maps to $BG$ via $G$-covers. There are several reasons why we find it interesting to pursue this special case. On the one hand, restricting to this special case allows us to give fairly explicit descriptions of the stacks of twisted stable maps, notably in terms of certain ramified Galois $G$-covers of “classical” stable pointed curves, with no reference to
orbifold curves of any kind. Constructing the stack of these $G$-covers directly is a standard procedure in moduli theory, and thus one can completely circumvent the delicate steps that were used in $\aleph$-V2 to construct the stack of twisted stable maps. One hopes that one might be able to use ideas of this paper to significantly simplify the constructions of $\aleph$-V2 in general. There is also some hope that further study of this case may shed light on possible constructions for a non-tame target stack $\mathcal{M}$, see $\aleph$-O2, [B-W]. Discussion of twisted stable maps in this special case of $BG$ also serves as an opportunity to study some properties of twisted curves and twisted stable maps which were not addressed in $\aleph$-V2.

1.3.3. Hurwitz stacks via twisted stable maps. On the other hand, having the tool-kit of twisted stable maps available allows us to shed new light on a time-honored topic in algebraic geometry: the stacks of balanced twisted $G$-covers are easily interpreted as a certain compactification of a Hurwitz-type stack of Galois covers of curves.

First, it is an easy exercise in deformation theory to show that this stack of balanced twisted covers is always nonsingular and of dimension $3g - 3 + n$, giving a “finite” (though in general not representable) flat cover of $\overline{M}_{g,n}$.

Restricting to the case where $G$ is the symmetric group, and using the usual correspondence between étale covers of degree $n$ and $S_n$-covers, we get a compactification of a “usual” Hurwitz-type stack (without group actions). We show that this is no other than the normalization of the Harris–Mumford stack of admissible covers (with arbitrary ramification type). Giving a moduli interpretation to this normalization has been a desirable goal since the appearance of [H-M], since the singularities of the stack of admissible covers are not natural for most applications.

1.3.4. Rigidification and Teichmüller structures. We proceed to consider the open-and-closed substack of $B_{g,n}^\text{bal}(G)$ consisting of connected balanced twisted $G$-covers of unpointed stable curves of genus $g$. A connected $G$-bundle over a smooth curve corresponds to a Teichmüller level structure in the sense of [D-M], Section 5, however the center $Z(G)$ of $G$ acts on the $G$-bundle fixing the level structure. We describe a fairly general procedure of removing an étale group action from the center of stabilizers in a stack, which we call rigidification. Thus the rigidification of $B_{g,n}^\text{bal}(G)$ obtained by removing the center $Z(G)$ is a smooth, proper Deligne–Mumford stack $B_{g,n}^{\text{tei}}(G)$, which serves as a natural compactification of the stack $G\mathcal{M}_g$ of Teichmüller level structures on smooth curves. We call this stack the stack of twisted Teichmüller structures. This stack deserves the notation $\overline{G\mathcal{M}}_g$, but unfortunately this notation has been wrongfully used for another stack by other authors (Deligne, Mumford, Pikaart, De Jong ... ).

1.3.5. A good compactification of $M_g^{(m)}$. We find that two special situations are of particular interest. First, if $G = (\mathbb{Z}/m\mathbb{Z})^{2g}$, then a Teichmüller $G$-structure on a smooth curve corresponds to a level-$m$ structure in the usual sense, and $B_{g}^{\text{tei}}(G)$ is a natural compactification of Mumford’s space of curves with level-$m$ structure. We give a detailed description of the type of objects that appear in the boundary: the precise “twisting” of the underlying twisted curves, and an interpretation of the twisted level-$m$ structure in terms of a trivialization of the étale cohomology with values in $\mathbb{Z}/m\mathbb{Z}$ of the twisted curve. If the structure sheaf of the base scheme contains the $m$-th roots of 1, then one can also define a symplectic structure on the étale cohomology group.

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1In the final version an appendix on such construction will be added
2In the final version, a discussion of symplectic structure will be added
Mumford considered a different compactification - the normalization $\overline{\mathcal{M}}_g^{(m)}$ of $\overline{\mathcal{M}}_g$ in the function field of the space $\mathcal{M}_g^{(m)}$ of smooth curves with level structures. Unlike $\mathcal{B}_g^{\text{tei}}(G)$, this space is singular, a fact which here we see resulting from the fact that a twisted level-$m$ structure on a singular curve may have automorphisms. Indeed $\overline{\mathcal{M}}_g^{(m)}$ is the coarse moduli space of $\mathcal{B}_g^{\text{tei}}(G)$. Our setup allows us to show, as a fairly easy consequence of Serre’s lemma, the well known fact that Mumford’s compactification does carry a “tautological family” of stable curves (albeit without a level structure), i.e. a morphism $\overline{\mathcal{M}}_g^{(m)} \to \overline{\mathcal{M}}_g$.

1.3.6. A projective fine moduli space of curves with level structure. The second special situation we consider is that of “rigid” nonabelian level structures, in particular those discussed in the papers [Lc] of E. Looijenga and [P-J] of M. Pikaart and A.J. de Jong. In [Lc] the group in question is the structure group of a $\left[\frac{2m}{2}\right]$-Prym level structure. In [P-J] the group $G$ is the maximal nilpotent quotient of exponent $n$ and nilpotence order $k$ of the fundamental group of a Riemann surface of genus $g$. Consider the normalization $\mathcal{O}_G\mathcal{M}_g$ in the space of smooth curves with Teichmüller level-$G$ structure $\mathcal{O}_G\mathcal{M}_g$. Looijenga and Pikaart and De Jong show in their respective cases that, for suitable values of the parameters $m$, respectively $(n,k)$, the space $\mathcal{O}_G\mathcal{M}_g$ is a nonsingular finite cover of $\overline{\mathcal{M}}_g$. In this case we are able to show that this space coincides with our stack $\mathcal{B}_g^{\text{tei}}(G)$; thus its non-singularity can be interpreted in terms of the fact that the automorphism group of a twisted Teichmüller $G$-structure is always trivial.

We improve on these results by introducing, for each genus $g$, a group $G$, such that moreover the automorphism group of every connected admissible $G$-cover is trivial (in particular $G$ has trivial center). The group $G$ can be quickly described as follows: if $p_1$ and $p_2$ are two distinct primes, let $G_1$ be the structure group of a $\left[p_1\right]$-Prym level structure, and let $G_2$ be the structure group of a $\left[p_2\right]$-Prym level structure. These groups admit natural homomorphisms to $(\mathbb{Z}/p_1p_2\mathbb{Z})^{2g}$, and $G$ is the fibered product.

With this $G$, our stack $\mathcal{B}_g^{\text{tei}}(G)$ is then a fine moduli space of connected, admissible $G$ covers, and the rigidification process is avoided.

1.4. Summary of notation.

- $\mathcal{S}$ — the base scheme.
- $\mathcal{C}$ — a twisted curve.
- $\pi : \mathcal{C} \to \mathcal{C}$ — the morphism to the coarse moduli scheme.
- $\mathcal{C}_\text{gen}$ — the generic locus of $\mathcal{C}$ (or $\mathcal{C}$).
- $\Sigma^C_i, \Sigma^C_i$ — the $i$-th marking of $\mathcal{C}$ and $\mathcal{C}$, the union of which is denoted $\Sigma^C$, respectively, $\Sigma^C$.
- $G$ — a finite group.
- $\mathcal{B}G$ — the classifying stack of $G$.
- $\mathcal{B}_{g,n}(G)$ — the stack of twisted $G$-covers of $n$-pointed curves of genus $g$.
- $\mathcal{B}_{g}(G)$ — the stack of twisted $G$-covers of (unpointed) curves of genus $g$.
- $\mathcal{B}_{g,n,d}^{\text{bal}}(G)$ — the stack of balanced twisted $G$-covers.
- $\mathcal{B}_{g,n,0}^{\text{tei}}(G)$ — the stack of twisted Teichmüller $G$-structures (of unpointed curves).
- $\mathcal{Adm}_{g,n,d}$ — the stack of generalized Harris–Mumford admissible covers of degree $d$ over $n$-pointed curves of genus $g$.
- $\mathcal{Adm}_{g,n}(G)$ — the stack of admissible $G$-covers.

1.5. What we mean by “the local picture”. We often need to “identify” a scheme, or a diagram connecting several schemes, locally in terms of explicit equations. To avoid repeated
mention of étale localizations or strict henselizations, we make the following agreement: we say that “the local picture of $X$ at a geometric point $p$ is the same as $U$ (at point $q$)” if the germ of $X$ at $p$ is isomorphic to the germ of $U$ at $q$, in other words:

1. in the algebraic situation: there is an isomorphism between the strict henselization $X^\text{sh}$ of $X$ at $p$ and the strict henselization $U^\text{sh}$ of $U$ at $q$. Often, however, it is enough to assume that there is an étale neighborhood $X'$ of $p$ and an isomorphism $X' \sim U'$ with an étale neighborhood $U'$ of $q$.
2. In the analytic situation: there is a small contractible neighborhood $X'$ of $p$ and an isomorphism $X' \sim U'$ with a neighborhood $U'$ of $q$.

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2. Terminology

In this section we recall basic facts about twisted stable maps, and introduce the notion of twisted $G$-covers.

2.1. Twisted curves and twisted stable maps. We follow the setup in $[\text{N-V2}]$ for defining twisted stable maps. The basic object underlying a twisted stable map is a twisted curve, i.e., a pointed nodal curve $C$ along with a Deligne–Mumford stack structure $C$ at its nodes and markings.

The local picture of a twisted curve at a geometric point $p$ can be explicitly described as follows:

2.1.1. At a marking. The local picture of $\mathcal{C} \to S$ is the same as $[U/\mu_r] \to T$, where

1. $T = \text{Spec } A$,
2. $U = \text{Spec } A[z]$, and
3. the action of $\mu_r$ is given by $z \mapsto \zeta_r \cdot z$.

2.1.2. At a node. If $p$ lies over a node, the local picture of $\mathcal{C} \to S$ is the same as $[U/\mu_r] \to T$, where

1. $T = \text{Spec } A$,
2. $U = \text{Spec } A[z, w]/(zw - t)$ for some $t \in A$, and
3. the action of $\mu_r$ is given by $(z, w) \mapsto (\zeta_r z, \zeta_r^a w)$, for some $a \in (\mathbb{Z}/r\mathbb{Z})^\times$.

This description is implicit in $[\text{N-V2}]$, Proposition 3.2.3.

2.1.3. Balanced actions, balanced curves. Note that, unless $a \equiv -1 \mod r$ in $[\text{2.1.2}]$ above, the element $t$ must vanish, and the node cannot be smoothly deformed; therefore the “locally smoothable” case $a \equiv -1 \mod r$ deserves special attention. An action with $a \equiv -1 \mod r$ is called balanced, a node presented via a balanced action is called balanced, and a twisted curve is said to be balanced when all its nodes are balanced.

Here is a formal definition of twisted curves:
Definition 2.1.4. A twisted nodal $n$-pointed curve over a scheme $S$ is a diagram
\[
\begin{array}{c}
\Sigma^C \\
\downarrow \\
C \\
\downarrow \\
S
\end{array}
\]
where
1. $C$ is a tame Deligne-Mumford stack, proper over $S$, which étale locally is a nodal curve over $S$;
2. $\Sigma^C = \bigcup_{i=1}^n \Sigma^C_i$, where $\Sigma^C_i \subset C$ are disjoint closed substacks in the smooth locus of $C \to S$;
3. $\Sigma^C_i \to S$ are étale gerbes;
4. the morphism $C \to C$ exhibit $C$ as the coarse moduli scheme of $C$; and
5. $C \to C$ is an isomorphism over $C_{\text{gen}}$.

The notation $C_{\text{gen}}$ in the definition above stands for the generic locus, namely the complement of the nodes and markings on $C$. We denote the image marking of $\Sigma^C_i$ in $C$ by $\Sigma^C_{\text{inv}}$. The stabilizer of a geometric point $p$ over a node or marking will be denoted $\Gamma_p$.

2.1.5. Twisted stable maps. Fix a proper Deligne-Mumford stack $M$ over $S$, admitting a projective coarse moduli scheme $M$, on which we fix a very ample invertible sheaf. For arithmetic situations we need a bit more: as in [Â-V2], we assume that $M$ is tame, namely the order of the automorphism group of a geometric object of $M$ is prime to the residue characteristic of the field over which the object is defined.

We can now recall the definition of a twisted stable map:

Definition 2.1.6. A twisted stable $n$-pointed map of genus $g$ and degree $d$ over $S$
\[(C \to S, \Sigma^C \subset C, f : C \to M)\]
consists of a commutative diagram
\[
\begin{array}{c}
C \\
\downarrow \\
\Sigma^C \subset C \\
\downarrow \\
S
\end{array}
\]
along with a closed substack $\Sigma^C \subset C$, satisfying:
1. $C \to C \to S$ along with $\Sigma^C$ is a twisted nodal $n$-pointed curve over $S$;
2. $(C \to S, \Sigma^C, f : C \to M)$ is a stable $n$-pointed map of degree $d$; and
3. the morphism $C \to M$ is representable.

While the first two conditions are “classical”, the third is truly stack-theoretic: it means that the stabilizer of a geometric point in $C$ injects into the stabilizer of the image point in $M$. It can be viewed as a stability condition - it is there to guarantee that the moduli problem be separated. In effect, we allow $C$ to be just twisted enough to afford a morphism to $M$.

Twisted curves (and thus twisted stable maps) naturally form a 2-category, where the 1-morphisms are given by fiber diagrams. In [Â-V2] it was shown that the automorphism group of every 1-morphism is trivial, therefore this 2-category is equivalent to the associated category, obtained by replacing 1-morphisms by their 2-isomorphism classes. The category of twisted stable maps thus obtained is denoted $K_{g,n}(M, d)$. The main theorem of [Â-V2] is:
Theorem 2.1.7. 1. The category $K_{g,n}(M,d)$ is a proper algebraic stack with finite diagonal.
2. The coarse moduli space $K_{g,n}(M,d)$ of $K_{g,n}(M,d)$ is projective.
3. There is a commutative diagram
\[
\begin{array}{c}
K_{g,n}(M,d) \\ \\
\downarrow \\
K_{g,n}(M,d)
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
K_{g,n}(M,d)
\end{array}
\]
where the top arrow is proper, quasifinite, relatively of Deligne–Mumford type and tame, and the bottom arrow is finite. In particular, if $K_{g,n}(M,d)$ is a Deligne–Mumford stack, then so is $K_{g,n}(M,d)$.

2.1.8. Twisted objects. In the paper [KV2], a considerable effort was made to give an explicit realization of $K_{g,n}(M,d)$ as a category of twisted objects over “usual” curves, given using local charts.

Briefly, a twisted stable map gives rise to an object $\xi$ of $M(C_{\text{gen}})$. Also, étale locally we can present $C$ around a marking or a node by $[U/\Gamma]$, where $U$ is a (non-proper) marked curve and $\Gamma$ is a finite group whose action on $U$ is free on $U_{\text{gen}}$. Over $U$ we have a $\Gamma$-equivariant object $\eta \in M(U)$. The data $(U,\eta,\Gamma)$ is called a chart.

It was shown that the collection of such data $(U,\eta,\Gamma)$ are compatible charts in an atlas for a twisted $M$-valued object over $C$. Further a moduli category of twisted $M$-valued objects was defined. It was also shown that there is a base-preserving equivalence of categories between $K_{g,n}(M,d)$ and the stack of twisted objects. The category of twisted objects is closely related to the moduli problem described by Chen and Ruan in [C-R].

A slightly different, and somewhat simpler, realization is given below in this paper in the case where $M = BG$.

2.2. Twisted $G$-covers. We now introduce twisted $G$-covers, the main objects of our paper, in terms of twisted stable maps. The reader who finds this difficult to picture is encouraged to read Section 4.3, where a concrete realization is given.

Let $G$ be a finite group, or, more generally, a finite étale group scheme of constant degree. We assume that the degree is prime to all residue characteristics in the base scheme $S$.

Consider the classifying stack $BG$ of $G$. We recall that $BG$ is a category whose objects over a scheme $T$ are principal $G_T$ bundles $P \to T$, and morphisms are $G$-equivariant fiber diagrams of such principal bundles:

\[
\begin{array}{c}
P_1 \\ \\
\downarrow \\
T_1
\end{array}
\begin{array}{c}
P_2 \\ \\
\downarrow \\
T_2
\end{array}
\]

We stick with the tradition that $G$ acts on a principal bundle $P \to S$ on the right. On the other hand, we write $G$-automorphisms of $P \to S$ on the left: if $S$ is connected this allows us to identify these $G$-automorphisms as elements of $G$ rather than $G^\text{opp}$.

We have a presentation $BG = [S/G]$, with $G$ acting trivially on $S$. Also useful is the fact that the coarse moduli space of $BG$ is simply $S$.

Define a twisted $G$-cover of an $n$-pointed curve of genus $g$ to be an object of the stack of twisted stable maps $K_{g,n}(BG,0)$. According to Theorem 2.1.7 quoted above, this stack is a proper stack whose moduli space is projective over $S$. Also, since the coarse moduli scheme of $M = BG$ is the base scheme $M = S$, the stack $K_{g,n}(M,d) = \overline{M}_{g,n}$ is a Deligne–Mumford stack, and therefore $K_{g,n}(BG,0)$ is also a Deligne Mumford stack. From now on we will denote it by
\(B_{g,n}(G)\). We will also denote by \(B_{g,n}^{\text{bal}}(G)\) the open and closed substack \(K_{g,n}^{\text{bal}}(BG, 0)\) consisting of balanced stable maps.

In accordance to the name “twisted \(G\)-cover”, an object of \(B_{g,n}(G)\) over a scheme \(S\) will be represented by the associated principal bundle \(P \to C\), where \(C \to S\) is the underlying twisted curve. To avoid confusion, we will refer to the morphism \(C \to BG\) as the twisted stable map associated to the twisted \(G\)-cover \(P \to C\).

We discuss the structure of \(P \to C\) in some detail later in this paper. One useful fact we cite right away is the following:

**Lemma 2.2.1.** \(P \to S\) is a projective nodal curve.

**Proof.** Note that \(BG = [S/G]\). The morphism \(S \to BG\), the universal principal \(G\) bundle, is clearly étale and finite. It is also representable since \(S\) is. Also the morphism \(C \to BG\) is representable being a twisted stable map. Therefore \(P = C \times_{BG} S\) is representable. For the same reason \(P \to C\) is finite. Since \(C \to S\) is projective, it follows that \(P \to S\) is projective. Finally \(P \to S\) is a nodal curve since it is étale over the nodal twisted curve \(C\).

### 3. Deformation theory of twisted covers

We show that the stack of twisted \(G\) covers is unobstructed, therefore smooth, and we calculate its dimension. We note that, at least in the balanced case, this can be shown on the level of Galois admissible covers, see \([W]\). A somewhat less detailed version of this argument can be found in \([\aleph-J]\).

**Theorem 3.0.2.** The stack \(B_{g,n}(G)\) is smooth. Its dimension at a given twisted \(G\)-cover is 
\[3g + n - 3 - u,\]
where \(u\) is the number of nodal points at which the bundle is not balanced.

**Proof.** The cotangent complex \(\mathbb{L}_{BG/S}\) of \(BG\) is trivial, therefore deformations and obstructions of a twisted stable map \(C \to BG\) are identical to those of the underlying pointed twisted curves.

This can be seen a bit more explicitly, as follows. Let \(P \to C\) be the twisted \(G\)-cover associated to \(C \to BG\). As was seen in Lemma 2.2.1, the curve \(P\) is projective, and deforming \(C \to BG\) is equivalent to deforming \(P\) as a \(G\) space. Moreover, since \(P \to C\) is étale, this is equivalent to deforming \(C\).

#### 3.0.3. Obstructions

As in \([\aleph-V2]\), Lemma 5.3.3, obstructions of a pointed twisted curve \((C \to S, \Sigma^C_i)\) are the same as obstructions of the underlying \(C \to S\), given by \(\text{Ext}^2(\Omega^1_C, \mathcal{O}_C)\). To show that this group vanishes, we follow \([D-M]\) closely.

The local-to-global spectral sequence for \(\text{Ext}^* (\Omega^1_C, \mathcal{O}_C)\) involves, in degree 2, the three terms
\[H^2(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)), \quad H^1(C, \mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C)), \quad \text{and } H^0(C, \mathcal{E}xt^2(\Omega^1_C, \mathcal{O}_C)).\]

We treat each term separately.

1. By \([\aleph-V2]\), Lemma 2.3.4, we have
\[H^2(C, \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) = H^2(C, \pi_* \mathcal{H}om(\Omega^1_C, \mathcal{O}_C)) = 0,
\]
where \(\pi : C \to C\) is the canonical morphism to the moduli space.

2. Similarly, \(\mathcal{E}xt^1(\Omega^1_C, \mathcal{O}_C)\) is supported in dimension 0, hence the second term vanishes.

\[^3\text{Precise citation needed}\]
3. We claim that the sheaf $\mathcal{E}xt^2(\Omega^1_C, \mathcal{O}_C))$ vanishes. This follows since locally in $\mathcal{C}$, the sheaf $\Omega^1_C$ has a 2-term locally free resolution. For instance, at a node where the local picture for $\mathcal{C}$ is $[U/\mu_r]$ with $U = \text{Spec} k[z, w]/(zw)$, we have an $\mu_r$-equivariant exact sequence on $U$:}

$$0 \to \mathcal{O}_U \xrightarrow{(z, w)} \mathcal{O}_U \oplus \mathcal{O}_U \xrightarrow{(dw, dz)} \Omega^1_U \to 0,$$

with appropriate $\mu_r$-weights, giving a locally free resolution of $\Omega^1_C$.

Thus twisted curves are unobstructed.

3.0.4. Deformations: To calculate the dimension of $\mathcal{B}_{g,n}(G)$ we evaluate the dimension of its tangent space.

We denote by $\mathcal{N}_{\Sigma^C_i}$ the normal bundle of the $i$-th marking $\Sigma^C_i \subset \mathcal{C}$. As in [N-V2], Lemma 5.3.2, we have an exact sequence

$$A \to \text{Hom}(\Omega^1_C, \mathcal{O}_C) \to H^0(\mathcal{C}, \oplus \mathcal{N}_{\Sigma^C_i}) \to \text{Def} \to \text{Ext}^1(\Omega^1_C, \mathcal{O}_C) \to 0,$$

where $\text{Def}$ is the tangent space of the stack.

**Infinitesimal automorphisms:** the space $A$ on the left is the space of infinitesimal automorphisms of $(\mathcal{C}, \Sigma^C_i)$, or, equivalently, of the twisted cover. We claim that this space vanishes. This follows directly from the fact that $\mathcal{B}_{g,n}(G)$ is a Deligne-Mumford stack, since the Isom schemes are unramified.

We remark that this can also be computed on the level of twisted marked curves: the sheaf of infinitesimal automorphisms is the subsheaf $\mathcal{A}$ of $\mathcal{H}om(\Omega^1_C, \mathcal{O}_C)$ of homomorphisms vanishing along $\Sigma_i$. Considering its direct image in $\mathcal{C}$, a local calculation (similar to the one given below for the other terms) reveals that it is the same as the sheaf of infinitesimal automorphisms of $\mathcal{C}$ fixing the markings $\Sigma^C_i$, whose group of global sections vanishes by the stability assumption.

**The normal sheaf to a marking:** at points where $\Sigma^C_i$ is untwisted we have that $H^0(\mathcal{C}, \mathcal{N}_{\Sigma^C_i})$ has dimension 1. At twisted markings the normal space $\mathcal{N}_{\Sigma^C_i}$ has no nontrivial sections: the local picture of $\mathcal{C}$ is $[U/\mu_r]$, with $U = \text{Spec} k[z]$ with the standard action of $\mu_r$, therefore $\mu_r$ acts on a generator $\partial/\partial z$ of $\mathcal{N}_{\Sigma^C_i}$ via the nontrivial character $\zeta_r \mapsto \zeta_r^{-1}$, and therefore the space of invariants is trivial.

**The extension groups:** the group $\text{Ext}^1(\Omega^1_C, \mathcal{O}_C)$ is dual to $H^{1-i}(\mathcal{C}, \Omega^1_C \otimes \omega_C)$. By [N-V2], Lemma 2.3.4 this is the same as $H^{1-i}(\mathcal{C}, \pi_*(\Omega^1_C \otimes \omega_C))$ where $\pi : \mathcal{C} \to C$ is the natural map. Let us compare $\pi_*(\Omega^1_C \otimes \omega_C)$ with $\Omega^1_C \otimes \omega_C$. These are clearly isomorphic away from the twisted markings and the twisted nodes.

First consider a twisted marking $\Sigma^C_i$ where the local picture of $\mathcal{C}$ is the same as $[U/\mu_r]$ with $U$ as above. The action of $\mu_r$ on $dz$ is via the standard character, therefore the invariant quadratic differentials are generated by $z^r(dz/z)^2 = r^{-2}(x^2 - dw^2)/x$, where $x$ is a parameter on $C$. That is, locally near such a marking we have $\pi_*(\Omega^1_C \otimes \omega_C) = \Omega^1_C \otimes \omega_C(\Sigma^C_i)$.

Now consider a node on $\mathcal{C}$. The local picture of $\mathcal{C}$ is the same as $[U/\mu_r]$ with $U = \text{Spec} k[z, w]/(zw)$, and the action can be described via $(z, w) \mapsto (\zeta_r z, \zeta_r^a w)$ for some $a \in (\mathbb{Z}/l\mathbb{Z})^\times$. The sheaf $\omega_C$ has an invariant generator $\nu_*(dz/z - dw/w)$, where $\nu$ is the normalization. The sheaf $\Omega^1_C$ has sections $f(z)dz + g(w)dw + \alpha dw$. Invariant elements in $f(z)dz + g(w)dw$ are exactly $\Omega^1_C$ torsion, whereas $zdw$ is invariant if and only if $a = -1$, i.e. the node is balanced.

All in all we have $\chi_*(\pi_*(\Omega^1_C \otimes \omega_C)) = \chi_*(\Omega^1_C \otimes \omega_C(\sum \Sigma^C_j)/T)$, where the sum $\sum \Sigma^C_j$ is taken over the twisted markings, and the sheaf $T$ is the torsion subsheaf supported at unbalanced nodes. It follows that

$$\dim \text{Def} = \chi_*(\pi_*(\Omega^1_C \otimes \omega_C)) + H^0(\mathcal{C}, \mathcal{N}_{\Sigma^C_i}) = H^0(\mathcal{C}, \Omega^1_C \otimes \omega_C) + n - u.$$
The Proposition follows.

The case \( u = 0 \) corresponds to balanced twisted covers. We have:

**Corollary 3.0.5.** The morphism \( B_{g,n}^{\text{bal}}(G) \rightarrow \overline{\mathcal{M}}_{g,n} \) is flat, proper and quasi-finite.

**Proof.** The morphism is proper and quasi-finite by Theorem 2.1.7, since \( M = S \), and therefore \( K_{g,n}(M, d) = \overline{\mathcal{M}}_{g,n} \). To check that it is flat it suffices to look at the map of deformation spaces. But since the deformation spaces of source and target are of the same dimension, and the map is quasifinite, it is equidimensional. Since both are smooth, it follows from the local criterion for flatness that the map is flat.

4. **Twisted covers and admissible covers**

In this section we compare our notion of twisted \( G \)-covers with the notion of admissible covers.

4.1. **Admissible covers.** We recall the definition of an admissible cover of nodal marked curves. Let \( (C \rightarrow S, \Sigma_i) \) be an \( n \)-pointed nodal curve of genus \( g \), and let \( d \) be a positive integer smaller than all the residue characteristics of \( S \).

**Definition 4.1.1.** An admissible cover \( p : D \rightarrow C \) of degree \( d \) is a finite morphism satisfying the following assumptions:

1. \( D \rightarrow S \) is a nodal curve.
2. Every node of \( D \) maps to a node of \( C \).
3. The restriction of \( p : D \rightarrow C \) to \( C_{\text{gen}} \) is étale of constant degree \( d \).
4. The local picture of \( D \rightarrow C \rightarrow S \) at a point of \( D \) over a node of \( C \) is the same as that of \( D' \rightarrow C' \rightarrow S' \), with

\[
\begin{align*}
D' &= \text{Spec } A[\xi, \eta]/(\xi \eta - a) \\
C' &= \text{Spec } A[x, y]/(xy - a^e) \\
S' &= \text{Spec } A
\end{align*}
\]

for some integer \( e \geq 0 \), where \( p^*x = \xi^e \) and \( p^*y = \eta^e \).
5. For a geometric point over a marking of \( C \), there is an integer \( e \geq 0 \) and an analogous local picture

\[
\begin{align*}
D' &= \text{Spec } A[\xi] \\
C' &= \text{Spec } A[x] \\
S' &= \text{Spec } A
\end{align*}
\]

where \( p^*x = \xi^e \), and \( x \) is a local parameter for the marking.

We note that this generalizes the original definition of Harris and Mumford in three ways:

- the genus of \( C \) is arbitrary,
- the ramifications over \( \Sigma_i \) are not assumed to be simple, and
- the curve \( D \) is not required to have connected fibers.
Moduli of admissible covers in various degrees of generality were discussed previously by Mochizuki \[\text{[Mo]}\], Wewers \[\text{[W]}\], and \[\text{[N-O2]}\]. See also Bernstein \[\text{[Ber]}\], Bouw–Wewers \[\text{[B-W]}\].

For the rest of the section we fix \(\mathbb{S} = \text{Spec} \mathbb{Z}[1/d!]\). Admissible covers of degree \(d\) of stable \(n\)-pointed curves of genus \(g\) form a proper Deligne–Mumford stack \(\text{Adm}_{g,n,d} \to \text{Spec} \mathbb{Z}[1/d!]\) admitting a projective coarse moduli space, see \[\text{[Mo]}\]. See also \[\text{[N-O2]}\].

4.2. Twisted covers and the normalization of the Harris–Mumford stack. Since the appearance of \[\text{[H-M]}\], there has been some dissatisfaction with the stack of admissible covers, for two reasons. First, the original definition involves a description of families of admissible covers, as the moduli problem was not determined by the geometric objects it parametrized - a resolution of this issue using logarithmic structures is given in Mochizuki’s work \[\text{[Mo]}\]; our approach below uses twisted curves instead. Second, it follows from the description of the deformation spaces of admissible covers in \[\text{[H-M]}\] that \(\text{Adm}_{g,n,d}\) is in general not normal, but its normalization is always smooth. Below we exhibit this normalization as a stack of twisted covers.

We build on the usual equivalence of categories
\[
\{ \text{finite étale covers } D \to S \text{ of degree } d \} \leftrightarrow \{ \text{principal } S_d\text{-bundles } P \to S \},
\]
where \(S_d\) is the symmetric group on \(d\) letters. We briefly recall that to a principal \(S_d\)-bundle \(P \to S\) one associates the finite étale cover \(D \to S\) where \(D = P/S_{d-1}\); and given a finite étale cover \(D \to S\) of degree \(d\), we consider the complement \(P\) of all the diagonals in the \(d\)-th fibered power \(D^d\), which is easily seen to be a principal \(S_d\)-bundle.

Consider a balanced twisted \(S_d\) cover \(P \to C\). The schematic quotient \(D = P/S_{d-1}\) is not necessarily étale over \(C\). Instead we have the following:

**Lemma 4.2.1.** The morphism \(D \to C\) is an admissible cover of degree \(d\).

**Proof.** Indeed \(D = P/S_{d-1}\) is a nodal curve, being the quotient of a nodal curve by a group acting along the fibers. Since every node of \(D\) is the image of a node on \(P\), we have that its image in \(C\) is again a node. Since \(P_{\text{gen}} \to C_{\text{gen}}\) is a principal \(S_d\)-bundle, its quotient \(D_{\text{gen}}\) by \(S_{d-1}\) is finite and étale of degree \(d\). Finally, given a geometric point of \(D\) over a node of \(C\), we can choose a point \(z\)’ in the preimage in \(P\), which is nodal. The local picture is the same as \(\text{Spec} A[u,v]/(uv - b)\). The stabilizer in \(S_d\) of \(p\) is a cyclic group \(C_r\) which acts via \((u,v) \mapsto (\xi u, \zeta_r^{-1} v)\), in such a way that \(x = u^r\) and \(y = v^r\). If \(C_{r'} = C_r \cap S_{d-1}\), write \(\xi = u^{r'}\) and \(\eta = v^{r'}\), and the local picture of \(D\) is \(\xi \eta = b^{r'}\). Setting \(e = r/r'\), it follows that \(x = \xi^e\) and \(y = \eta^e\), as required. A similar (and simpler) argument gives the structure along a marking. ♣

The functor, which associates to a twisted \(S_d\)-cover \(P \to C\) the admissible cover \(P/S_{d-1} \to C\), is a morphism of stacks \(\phi : \mathcal{B}_{g,n}^{\text{bal}}(S_d) \to \text{Adm}_{g,n,d}\).

**Proposition 4.2.2.** The morphism \(\phi\) exhibits \(\mathcal{B}_{g,n}^{\text{bal}}(S_d)\) as the normalization of \(\text{Adm}_{g,n,d}\).

**Proof.** It suffices to show that \(\phi\) is finite and surjective, and there is an open dense substack \(\mathcal{A}_{g,n,d}^0\), whose inverse image is dense in \(\mathcal{B}_{g,n}^{\text{bal}}(S_d)\), over which \(\phi\) is an isomorphism.

The morphism \(\phi\) is finite since

1. \(\phi\) is representable: for this it suffices to show (see, e.g., \[\text{[N-V2]}\], Lemma 4.4.3) that any automorphism of a twisted \(S_d\)-bundle \(P \to C\) which acts trivially on the associated admissible cover \(D \to C\) is the identity. Indeed, such an isomorphism acts trivially on \(D_{\text{gen}} \to C_{\text{gen}}\), and by the equivalence of categories cited above it acts trivially on \(P_{\text{gen}}\) as

\[\text{[bal]}\]
well. By [R-V2, Theorem 4.4.1] (and also Remark 4.4.4) an automorphism of a twisted stale map \( \mathcal{C} \to \mathcal{M} \) is determined by its action on the generic object \( \mathcal{C}_\text{gen} \to \mathcal{M} \), and the claim follows.

2. \( \phi \) is proper, since \( \mathcal{B}^{\text{bal}}_{g,n}(\mathcal{S}_d) \) is proper.

3. \( \phi \) is quasifinite, since, by Theorem 2.1.7, the stack \( \mathcal{B}^{\text{bal}}_{g,n}(\mathcal{S}_d) \) is quasifinite over \( \overline{\mathcal{M}}_{g,n} \).

Consider the dense open substack \( \mathcal{A}dm^0_{g,n,d} \) of admissible covers of smooth curves \( \mathcal{C} \). Its inverse image in \( \mathcal{B}^{\text{bal}}_{g,n}(\mathcal{S}_d) \) is the dense open substack of balanced twisted covers over smooth twisted curves. We claim that over \( \mathcal{A}dm^0_{g,n,d} \) the morphism \( \phi \) is an isomorphism. It is easy to see that \( \mathcal{A}dm^0_{g,n,d} \) is smooth, therefore we need only construct an inverse functor of \( \phi \) for admissible covers over reduced base schemes. Consider an admissible cover \( D \to C \) with \( C \to S \) smooth and \( S \) reduced. Then \( D \) is a smooth curve branched only over the marked sections of \( C \). The restriction \( D_\text{gen} \to C_\text{gen} \) is a finite étale cover corresponding to a principal \( \mathcal{S}_d \)-bundle \( P_\text{gen} \to C_\text{gen} \). The tameness assumption and Abhyankar’s lemma (see [G-SGA1], exp. XIII section 5) imply that the normalization \( P \) of \( C \) in the structure sheaf of \( P_\text{gen} \) is again a smooth curve, the quotient \( C = [P/\mathcal{S}_d] \) is a smooth twisted pointed curve over \( S \), and \( P \to C \) is a twisted \( \mathcal{S}_d \)-bundle. This provides an inverse of \( \phi \) restricted over \( \mathcal{A}dm^0_{g,n,d} \).

Next we show that \( \phi \) is surjective. We give two arguments for this fact, since we feel they are instructive in different ways.

4.2.3. Surjectivity I. For the first argument, consider an admissible cover \( D_0 \to C_0 \) over an algebraically closed field. In [H-M] it is shown that its deformation space is reduced, and the locus of smooth admissible covers in it is nonempty. Thus \( \mathcal{A}dm^0_{g,n,d} \) is everywhere dense, and by properness \( \phi \) surjects on a dense closed substack. But since \( \mathcal{A}dm_{g,n,d} \) is reduced, \( \phi \) is surjective.

4.2.4. Surjectivity II. The second argument is longer, but more elementary, as it does not use deformation theory. Let \( D \to C \) be an admissible cover defined over an algebraically closed field; we will produce a twisted cover of \( C \) according to the following procedure. We think of \( \mathcal{B}S_d \) as the stack of étale covers of degree \( d \); the restriction of \( D \) to \( C_\text{gen} \) gives a generic object \( C_\text{gen} \to \mathcal{B}S_d \). Let us produce charts for this object in the sense of [R-V2, Section 3.2].

Let \( p \in C \) be a marked point, and let \( m \) be the least common multiple of all the ramification indices of points of \( D \) over \( C \). Let \( U \to C \) be a morphism from a smooth, but not necessarily complete, curve \( U \) such that

1. the image of \( U \) does not contain any special point of \( C \) except \( p \),
2. there is precisely one point \( q \in U \) over \( p \), and
3. there is an action of a cyclic group \( \Gamma \) of order \( m \) on \( U \), having \( q \) as a fixed point and leaving the morphism \( U \to C \) invariant, which is free outside of \( q \) and such that the induced morphism \( U/\Gamma \to C \) is étale.

The normalization \( \tilde{D} \) of the pullback of \( D \) to \( U \) is étale over \( U \), and the action of \( \Gamma \) on \( U \) lifts to an action on \( \tilde{D} \); this gives a chart around the point \( p \). It is easily checked that the quotient \( \tilde{D}/\Gamma \) is the pullback of \( D \) to \( U/\Gamma \).

If \( p \in C \) is a node, the procedure is similar. Let \( m \) be the least common multiple of all the ramification indices of points of \( D \) over \( C \). Let \( U \to C \) a morphism from a nodal, but not necessarily complete, curve \( U \) such that

1. the image of \( U \) does not contain any special point of \( C \) except \( p \),
2. there is precisely one point \( q \in U \) over \( p \),
3. \( U \) has two irreducible components \( U_1 \) and \( U_2 \), which are smooth and intersect only at \( q \),
4. there is a balanced action of a cyclic group $\Gamma$ of order $m$ on $U$, having $q$ as a fixed point and leaving the morphism $U \to C$ invariant, which is free outside of $q$ and such that the induced morphism $U/G \to C$ is étale.

Recall that the action of $\Gamma$ on $U$ is balanced when the two characters of $\Gamma$ describing the action of $\Gamma$ on the tangent spaces to $U_1$ and $U_2$ are opposite.

Let $\tilde{D}_1$ and $\tilde{D}_2$ be the normalizations of the pullbacks of $D$ to $U_1$ and $U_2$ respectively; then $\tilde{D}_i$ is smooth over $U_i$. The action of $\Gamma$ on $U_i$ lifts to an action on $\tilde{D}_i$. To obtain a chart, we choose a way of identifying the fiber of $\tilde{D}_1$ over $q$ with the fiber of $\tilde{D}_2$ over $q$; this gives a $\Gamma$-equivariant étale cover $\tilde{D} \to U$, and the quotient $\tilde{D}/\Gamma$ is precisely the pullback of $D$ to $U/\Gamma$.

5.3. Admissible $G$-covers and twisted covers.

**Definition 4.3.1.** Let $C$ be a nodal curve over a scheme $S$. An admissible $G$-cover $\phi: P \to C$ is an admissible cover, with an action of $G_S$ on $P$ leaving $\phi$ invariant, satisfying the following two conditions.

1. The restriction $P_{\text{gen}} \to C_{\text{gen}}$ is a principal $G_S$-bundle.

   Let $p$ be a geometric point of $P$. Notice that this first condition insures that the stabilizer $G_p$ of $p$ is a cyclic group. Then we also assume

2. for each geometric nodal point $p$ of $P$, with image point $s$ of the base $S$, the action of the stabilizer $G_p$ of $p$ on the fiber $P_s$ over $s$ of $P \to S$ is balanced.

Admissible $G$-covers form a category $\mathcal{A}dm_{g,n}(G)$ with arrows given by fiber diagrams.

In contrast with the case of plain admissible cover, we have the following result.

**Theorem 4.3.2.** There is a base-preserving equivalence of categories between $B_{g,n}^{\text{bal}}(G)$ and $\mathcal{A}dm_{g,n}(G)$.

So in particular $\mathcal{A}dm_{g,n}(G)$ is a Deligne–Mumford stack, isomorphic to $B_{g,n}^{\text{bal}}(G)$.\footnote{In the final version this should be proven directly in appendix.}

**Proof.** Let $C$ be a twisted curve over a scheme $S$, with moduli space $C$. If $P \to C$ is a balanced twisted $G$-cover, we have seen in 2.2 that $P$ is an algebraic space. We claim that the composition of $P \to C \to C$, with $C \to C$ the morphism to the moduli space, is an admissible $G$-cover. The fact that $P_{\text{gen}} \to C_{\text{gen}}$ is a principal bundle follows from the definition, since $C_{\text{gen}} = C_{\text{gen}}$ and $P \to C$ is a principal bundle. The fact that $P \to C$ is an admissible cover, as well as condition (2) above, are identical to the argument of Lemma 4.2.1.

Conversely, given an admissible $G$-cover $P \to C$ over a scheme $S$, consider the stack quotient $C = [P/G_S]$. Now $P \to C$ is a principal $G$-bundle, the morphism $C \to BG$ is representable since $P$ is, $C$ is nodal being the quotient of a nodal curve, $C$ is the moduli space of $C$, and the action of $G$ on $P$ is balanced, showing that $C$ is balanced.

It is easy to see that these correspondences are functorial, and that they are inverse to each other in the usual sense.

5. Rigidification and Teichmüller structures

In this section we define the notion of rigidification of a stack, and use it to define twisted Teichmüller $G$-structures.

\footnote{In the final version there should be a subsection about the singularities of $\phi: B_{g,n}(S_d) \to A_{g,n,d}$.}
5.1. Rigidification.

5.1.1. The example of line bundles. The classic instance of rigidification of a stack occurs when constructing the Picard scheme of an irreducible projective variety.

Consider the stack $BG_m$, the classifying stack of the multiplicative group. As it stands, its objects are $G_m$-bundles, but it is well known to be equivalent to the stack of line bundles: objects over a scheme $X$ are line bundles over $X$, and a morphisms between $L_1 \to X_1$ and $L_2 \to X_2$ are fiber squares

$$
\begin{array}{ccc}
L_1 & \to & L_2 \\
\downarrow & \square & \downarrow \\
X_1 & \to & X_2
\end{array}
$$

Of course every line bundle over $X$ has the group of invertible functions $O_X(X)^\times$ in its automorphism group, and in particular, if $X$ is a projective variety over a field $k$, the automorphism group contains $k^\times$. For such $X$ we can define the “Picard stack”, whose objects over a $k$-scheme $T$ are line bundles on $T \times X$. Clearly this stack is not representable, since every geometric object has automorphisms. How do we obtain the Picard scheme, say, of an irreducible projective variety $X$ out of $BG_m$?

The point is, that one can take the category of line bundles over $X$ and rigidify it by removing the multiplicative group from the automorphisms of any object. The traditional procedure (see [G-FGA]) is to take, as a first approximation, the category whose objects are line bundles over $T \times X$ and whose arrows are “isomorphisms up to twisting” $L_1 \to L_2 \otimes M$, where $M$ is a line bundle coming from $T$. This works étale-locally, and one needs to sheafify the resulting category in the étale topology. The result is precisely the Picard scheme. In effect, this procedure produces a stack whose geometric objects are still line bundles over $X$, but where the automorphisms given by the multiplicative group are “removed from the picture”.

5.1.2. Level structures and rigidified covers. Suppose one is interested in studying full level-$m$ structures on smooth curves of genus $g > 1$. A structure of full level-$m$ on a smooth curve $C$ of genus $g$ over an algebraically closed field is a basis for the $\mathbb{Z}/m\mathbb{Z}$-module $H^1(X, \mathbb{Z}/m\mathbb{Z})$. This basis corresponds to an element of $H^1(X, (\mathbb{Z}/m\mathbb{Z})^2g)$, with the property that the associated $(\mathbb{Z}/m\mathbb{Z})^2g$-principal bundle $E \to C$ is connected. One would be tempted to identify this level structure with the principal bundle $E \to C$, thought of as an element of $BG_{g,0}((\mathbb{Z}/m\mathbb{Z})^2g)$, but this would be an error, because the group $(\mathbb{Z}/m\mathbb{Z})^2g$ acts on the bundle $E \to C$, while a level structure should have no nontrivial automorphisms fixing $C$.

Considering twisted $G$-covers in general, the center $Z(G) \subset G$ always acts on any twisted $G$-cover; we wish to rigidify the twisted covers by making this action trivial. We now describe a procedure for this in some generality.

5.1.3. Rigidification in terms of a presentation. The idea of rigidification can be seen explicitly with a presentation.

Let $H$ be a flat finitely presented separated group scheme over a base scheme $S$, and $X$ an algebraic stack over $S$. Take a smooth map of finite presentation $U \to X$, and set $R = U \times_X U$, so that $R \to U$ is a smooth presentation for $X$.

Assume that there is an action of $H_{U \times_S U}$ on $R$, leaving the two projections $R \to U$ invariant. Then there exists a quotient smooth groupoid $R/H \to U$; this is a smooth presentation of a stack $X^H$, which is the rigidification of $X$, removing $H$ from the stabilizers.

We now describe a natural situation where such a picture occurs.
5.1.4. The rigidification setup. Again, let $H$ be a flat finitely presented separated group scheme over a base scheme $S$, $\mathcal{X}$ an algebraic stack over $S$. Assume that for each object $\xi \in \mathcal{X}(S)$ there is an embedding

$$\iota_\xi : H(S) \hookrightarrow \text{Aut}_S(\xi),$$

which is compatible with pullback, in the following sense: given two objects $\xi \in \mathcal{X}(S)$ and $\eta \in \mathcal{X}(T)$, and an arrow $\phi : \xi \to \eta$ in $\mathcal{X}$ over a morphism of schemes $f : S \to T$, the natural pullback homomorphisms

$$\phi^* : \text{Aut}_T(\eta) \to \text{Aut}_S(\xi)$$

and

$$f^* : H(T) \to H(S)$$

commute with the embeddings, that is, $\iota_\xi f^* = \phi^* \iota_\eta$.

This condition can also be expressed as follows. Let $\phi : \xi \to \eta$ be an arrow in $\mathcal{X}$ over a morphism of schemes $f : S \to T$, and $g \in H(T)$. Then the diagram

$$\begin{array}{ccc}
\xi & \xrightarrow{\phi} & \eta \\
\downarrow{f^*g} & & \downarrow{g} \\
\xi & \xrightarrow{\phi} & \eta
\end{array}$$

commutes. In particular, by taking $\xi = \eta$ and $\phi$ to be in $\text{Aut}_S(\xi)$, we see that $H(S)$ must be in the center of $\text{Aut}_S(\xi)$. In particular, $H$ must be commutative. (One might consider a more general situation, where the element $g$ on the right differs from the element appearing on the left, but this is not crucial for our purposes.)

**Theorem 5.1.5.** Let $\mathcal{X} \to S$ be an algebraic stack, $H \to S$ a flat finitely presented group scheme over $S$, and assume that for every object $\xi \in \mathcal{X}(T)$ there is an embedding $H_T \subset \text{Aut}_T(\xi)$ compatible with pullbacks. Then there is a smooth surjective finitely presented morphism of algebraic stacks $\mathcal{X} \to \mathcal{X}^H$ satisfying the following properties:

1. For any object $\xi \in \mathcal{X}(T)$ with image $\eta \in \mathcal{X}^H(T)$, we have that $H(T)$ lies in the kernel of $\text{Aut}_T(\xi) \to \text{Aut}_T(\eta)$.
2. The morphism $\mathcal{X} \to \mathcal{X}^H$ is universal for morphisms of stacks $\mathcal{X} \to \mathcal{Y}$ satisfying (1) above.
3. If $T$ is the spectrum of an algebraically closed field, then in (1) above, $\text{Aut}_T(\eta) = \text{Aut}_T(\xi)/H(T)$.
4. A moduli space for $\mathcal{X}$ is also a moduli space for $\mathcal{X}^H$.

Furthermore, if $\mathcal{X}$ is a Deligne–Mumford stack, then $\mathcal{X}^H$ is also a Deligne–Mumford stack and the morphism $\mathcal{X} \to \mathcal{X}^H$ is étale.

People familiar with the theory of $n$-stacks might recognize this rigidification as the stack associated to the quotient $[\mathcal{X}/\mathcal{B}H]$ of $\mathcal{X}$ by the action of the group-stack $\mathcal{B}H$. Our proof below takes a slightly more concrete view.

5.1.6. The action on $\text{Hom}$-sheaves. Given an object $\xi$ of $\mathcal{X}$ over a scheme $S$, the embeddings $\iota_\xi$ define a categorically injective morphism of $S$-group schemes of the pullback $H_S$ of $H$ to $S$ to the group scheme $\text{Aut}_S(\xi)$ of automorphisms of $\xi$.

With these hypotheses, if $\xi$ and $\eta$ are objects of $\mathcal{X}$ over two schemes $S$ and $T$ respectively, and $f : S \to T$ is a morphism of schemes, there is an action of $H(T)$ on the set $\text{Hom}_f(\xi, \eta)$ of arrows in $\mathcal{X}$ lying over $f$, defined by setting $g \cdot \phi = g \circ \phi = \phi f^* g$ for each $g \in H$ and each $\phi \in \text{Hom}_{\mathcal{X}}(\xi, \eta)$. If $f : S \to T$ and $f' : T \to U$ are morphisms of schemes, and $\xi$, $\eta$ and $\zeta$ are object of $\mathcal{X}$ over $S$, $T$ and $U$ respectively, there is a composition map

$$\text{Hom}_f(\xi, \eta) \times \text{Hom}_{f'}(\eta, \zeta) \to \text{Hom}_{f'f}(\xi, \zeta);$$
it is easy to see that map passes to the quotient, yielding a map
\[ \text{Hom}_f(\xi, \eta)/H(T) \times \text{Hom}_f(\eta, \zeta)/H(U) \to \text{Hom}_{f,f}(\xi, \zeta)/H(U). \]

This action of \( H(T) \) on \( \text{Hom}_f(\xi, \eta) \) induces a right action of the group scheme \( H_T \) on the
sheaf \( \text{Hom}_f(\xi, \eta) \), which sends each scheme \( T' \) over \( T \) to the set of arrows in \( \mathcal{X} \) from the pullback
of \( \xi \) to \( S \times_T T' \) to the pullback of \( \eta \) to \( T' \) lying over the projection \( S \times_T T' \to T' \).

5.1.7. Rigidification in categorical terms. Consider the quotient sheaf
\[ \text{Hom}_f^H(\xi, \eta) = \text{Hom}_f(\xi, \eta)/H_T: (\text{Sch}/T)^\text{opp} \to (\text{Sets}), \]
that is, the quotient sheaf associated to the presheaf which sends each \( T' \) to \( \text{Hom}_f(\xi, \eta)(T')/H(T') \). We define \( \text{Hom}_f^H(\xi, \eta) = \text{Hom}_f(\xi, \eta)(T) \) to be the set of global sections of
this sheaf. The composition map above induces a morphism of sheaves of sets on \( U \)
\[ f^*\text{Hom}_f^H(\xi, \eta) \times \text{Hom}_f^H(\eta, \zeta) \to \text{Hom}_{f,f}^H(\xi, \zeta) \]
and hence a map
\[ \text{Hom}_f^H(\xi, \eta) \times \text{Hom}_f^H(\eta, \zeta) \to \text{Hom}_{f,f}^H(\xi, \zeta). \]

We define a category \( \mathcal{X}_{\text{pre}}^H \), in which the objects are objects of \( \mathcal{X} \), and an arrow from \( \xi \) to
\( \eta \) consist of an element of \( \text{Hom}_f^H(\xi, \eta) \) for some \( f: S \to T \), where \( S \) and \( T \) are the schemes
underlying \( \xi \) and \( \eta \) respectively, and the composition is defined by the map above. This \( \mathcal{X}_{\text{pre}}^H \) is
a prestack([L-MB], Définition 3.1), but not a stack, in general; we define the category \( \mathcal{X}^H \) to
be the stack associated to \( \mathcal{X}_{\text{pre}}^H \), as in [L-MB], Lemme 3.2. We note that the process of taking
the stack associated to a prestack has the property that \( \mathcal{X}^H(T) = \mathcal{X}_{\text{pre}}^H(T) \) whenever \( T \) is the
spectrum of an algebraically closed field.

There are obvious functors from \( \mathcal{X} \) to \( \mathcal{X}^H \) and from \( \mathcal{X}^H \) to the category of schemes over \( S \).
It follows from the construction that if \( T \) is the spectrum of an algebraically closed field, and \( \xi \) and \( \eta \) are objects of \( \mathcal{X} \), then the set of isomorphisms of \( \xi \) and \( \eta \) in \( \mathcal{X}^H \) is the set of isomorphisms of
\( \xi \) and \( \eta \) in \( \mathcal{X}(T) \) divided by the natural action of \( H(T) \).

It is easily checked that \( \mathcal{X}^H \) is a stack fibered in groupoids over \( S \), and that it has properties (1), (2) and (3) of the theorem. Property (4) follows immediately from property (3).

We claim that \( \mathcal{X}^H \) is in fact an algebraic stack. First of all let us show that the diagonal of
\( \mathcal{X}^H \) is representable, finitely presented and separated. Let \( X \) and \( Y \) be schemes, \( X \to \mathcal{X}^H \) and \( Y \to \mathcal{X}^H \) two morphisms corresponding to objects \( \xi \in \mathcal{X}^H(X) \) and \( \eta \in \mathcal{X}^H(Y) \). We need
to show that the fiber product \( X \times_{\mathcal{X}^H} Y \) is representable, separated and of finite presentation
over \( X \times_S Y \). This is a local question in the flat topology over \( X \) and \( Y \), so we may suppose
that \( \xi \) and \( \eta \) are objects of \( \mathcal{X}(X) \) and \( \mathcal{X}(Y) \) respectively. Then \( X \times_{\mathcal{X}^H} Y \) corresponds to the
functor \( \text{Isom}_{X \times_S Y}(\text{pr}_X^*\xi, \text{pr}_Y^*\eta) \), which is by definition equal to the quotient of the algebraic
space \( X \times_{\mathcal{X}} Y = \text{Isom}_{X \times_S Y}(\text{pr}_X^*\xi, \text{pr}_Y^*\eta) \) by the action of \( H_U \). It is easy to see that this action
is free, in the sense that the morphism
\[ \text{Isom}_{X \times_S Y}(\text{pr}_X^*\xi, \text{pr}_Y^*\eta) \times_T H_T \to \text{Isom}_{X \times_S Y}(\text{pr}_X^*\xi, \text{pr}_Y^*\eta) \times_T \text{Isom}_{X \times_S Y}(\text{pr}_X^*\xi, \text{pr}_Y^*\eta) \]
which is the action on one component and the projection on the other is categorically injective. By a result of M. Artin ([L-MB], Corollaire 10.4.1), the quotient \( X \times_{\mathcal{X}^H} Y \) is an algebraic space
over \( X \times_S Y \). It is easily checked that it is separated and of finite presentation.

Next we’ll prove that the morphism \( \mathcal{X} \to \mathcal{X}^H \) is smooth, surjective and of finite presentation. If \( X \to \mathcal{X} \) is a smooth surjective map locally of finite presentation from a scheme \( X \), we claim
that the composition \( X \to \mathcal{X} \to \mathcal{X}^H \) is smooth, surjective and locally of finite presentation, which implies that \( \mathcal{X}^H \) is algebraic.
To check this, it is enough to show that given a morphism $T \to \mathcal{X}$, where $T$ is a scheme, the fiber product $\mathcal{X} \times_{\mathcal{X}^H} T$ smooth, surjective and of finite presentation on $T$. By passing to a flat cover of $X$, we may assume that this morphism $T \to \mathcal{X}^H$ factors through $\mathcal{X}$; in other words, it is enough to show that the projection $\mathcal{X} \times_{\mathcal{X}^H} \mathcal{X} \to \mathcal{X}$ is smooth, surjective and of finite presentation. An object of $\mathcal{X} \times_{\mathcal{X}^H} \mathcal{X}$ is given by a scheme $T$ over $\mathbb{S}$, two objects $\xi$ and $\eta$ of $\mathcal{X}$ over $T$, and an isomorphism $\alpha$ of $\xi$ with $\eta$ in $\mathcal{X}^H(T)$. Consider the principal $H$-bundle $\operatorname{Isom}_T^X(\xi, \eta) \to \operatorname{Isom}_T^{X^H}(\xi, \eta)$; by pulling it back to $T$ via the morphism $T \to \operatorname{Isom}_T^{X^H}(\xi, \eta)$ given by $\alpha$ we get a principal $H$-bundle $P \to T$. There is an obvious functor $\mathcal{X} \times_{\mathcal{X}^H} \mathcal{X} \to \mathcal{X} \times_{\mathbb{S}} B_S H$ sending the object $(\xi, \eta, \alpha)$ to $(\xi, P)$. I claim that this is an isomorphism. Let us define the inverse functor $\mathcal{X} \times_{\mathbb{S}} B_S H \to \mathcal{X} \times_{\mathcal{X}^H} \mathcal{X}$. Take an object $(\xi, P)$ of $\mathcal{X} \times_{\mathbb{S}} B_S H$ over a scheme $T$, and consider the pullback $\xi_P$ of $\xi$ to $P$; the embedding $H_P \hookrightarrow \operatorname{Aut}_P(\xi)$ defines an action of $H_T$ on $\xi_P$, giving descent data to descend $\xi_P$ to another object $\eta$ of $\mathcal{X}$ over $T$. By definition, this $\eta$ comes equipped with a canonical isomorphism $\alpha: \xi \simeq \eta$ in $\mathcal{X}^H(T)$. We define the image of the object $(\xi, P)$ to be the object $(\xi, \eta, \alpha)$ of $\mathcal{X} \times_{\mathcal{X}^H} \mathcal{X}$. We leave it to the reader to define the action of this functor on arrows, and to check that it gives an inverse to the functor above.

So the projection $\mathcal{X} \times_{\mathbb{S}} \mathcal{X} \to \mathcal{X}$ is isomorphic to the projection $\mathcal{X} \times_{\mathbb{S}} B_S H \to \mathcal{X}$. Obviously $B_S H$ is surjective and of finite presentation over $\mathbb{S}$; we only need to check that it is smooth. This is obvious when $H$ is smooth over $\mathbb{S}$; in general the morphism $S \to B_S H$ given by the trivial torsor is flat and surjective, but not smooth. The result follows from the following lemma.

**Lemma 5.1.8.** Let $\mathcal{X}$ be an algebraic stack flat of finite presentation over a scheme $S$, and assume that there exists a flat surjective morphism $U \to \mathcal{X}$, where $U$ is smooth over $S$. Then $\mathcal{X}$ is smooth over $S$.

**Proof.** The statement is local in the smooth topology on $\mathcal{X}$, so we may assume that $\mathcal{X}$ is a scheme; in this case the result is standard.

To conclude the proof of the theorem, we assume that $\mathcal{X}$ is Deligne–Mumford, and show that the morphism $\mathcal{X} \to \mathcal{X}^H$ is étale. As we saw above, it is enough to check that the projection $\mathcal{X} \times_{\mathcal{X}^H} \mathcal{X} \to \mathcal{X}$ is étale. Take a morphism $T \to \mathcal{X}$, where $T$ is a scheme; we have seen that the fiber product $\mathcal{X} \times_{\mathcal{X}^H} T$ is isomorphic to $T \times_{\mathbb{S}} B_S H = B_T H_T$, so we only need to prove that $B_T H_T$ is étale over $T$, or, equivalently, that $H_T$ is étale over $T$. But this is clear, because if $\xi \in \mathcal{X}(T)$ is the object corresponding to the given morphism $T \to \mathcal{X}$, then there is an embedding $H_T \hookrightarrow \operatorname{Hom}_T(\xi)$, and $\operatorname{Hom}_T(\xi)$ is unramified over $T$, because $\mathcal{X}$ is Deligne–Mumford.

Furthermore, since any morphism from $\mathcal{X}$ to an algebraic space factors uniquely though $\mathcal{X}^H$, we see that $\mathcal{X}$ and $\mathcal{X}^H$ share the same moduli space.

**Definition 5.1.9.** Let $\mathcal{X}$ and $H$ be as in the theorem. Then we call $\mathcal{X} \to \mathcal{X}^H$ the rigidification of $\mathcal{X}$ along $H$. Objects of $\mathcal{X}^H$ are called $H$-rigified objects of $\mathcal{X}$.

### 5.2. Teichmüller structures

In this section we assume $n = 0$

**Definition 5.2.1.** Denote $B_g^{rig}(G) = (B_g(G))^{Z(G)}$, the stack of $Z(G)$-rigified twisted $G$-covers. Let $B_g^{rig}(G) \subset B_g^{rig}(G)$ be the open-and-closed substack whose geometric objects correspond to connected, balanced, rigidified twisted $G$-covers. We call this the stack of twisted Teichmüller $G$-structures. We denote by $B_g^{rig}(G)^0$ the open substack of smooth Teichmüller $G$-structures, namely twisted Teichmüller structures over smooth curves.

**Lemma 5.2.2.** The morphism $B_g^{rig}(G)^0 \to \mathcal{M}_g$ is finite and étale.
Proof. This morphism is étale since the deformation space of a smooth Teichmüller structure coincides with that of the underlying curve. We need to show that this morphism is representable, which, by a well known result (see [K-V2], Lemma 4.4.3) means that the induced maps of automorphism groups of geometric objects is injective. But the automorphism group \( \text{Aut}_C P \) of a connected principal bundle \( P \to C \) over an algebraically closed field fixing \( C \) is the center of the structure group, therefore, when we rigidify the automorphism group becomes trivial.

\[ \clubsuit \]

In [P-J], a stack of “Techmüller \( G \)-level structures over smooth curves” \( G\mathcal{M}_g \) is defined for any finite group \( G \). This generalizes the treatment of Deligne and Mumford in [D-M], Section 5. For the benefit of the reader familiar with their construction, we compare it with our stack. We note that this result is not necessary for understanding the rest of this paper.

**Proposition 5.2.3.** Assume \( G \) is a constant finite group. The stack \( \mathcal{B}_g^{\text{tei}}(G)^0 \) is isomorphic to the stack \( G\mathcal{M}_g \) of Teichmüller level structures.

**Proof.** Given an object of \( \mathcal{B}_g^{\text{tei}}(G)^0(S) \), then étale locally on the base \( S \) we have a principal \( G \)-bundle \( P \to C \) as well as a section \( s : S \to C \). By definition this yields a surjective homomorphism \( \pi_1(C/S, s) \to G \). It is straightforward to verify that this only depends on the original object and yields a global section of the sheaf \( \mathcal{H}om^\text{ext}(\pi_1(C/S), S) \), namely an object of \( G\mathcal{M}_g(S) \). It is also easy to check that this is functorial.

Both stacks \( \mathcal{B}_g^{\text{tei}}(G)^0 \) and \( G\mathcal{M}_g \) are finite and étale over \( \mathcal{M}_g \). In order to check that this functor gives an isomorphism it suffices to check that it is bijective on geometric points. Now, if \( C \) is a curve over an algebraically closed field, then there is a one-to-one correspondence between isomorphism classes of connected \( G \)-covers and surjective homomorphisms \( \pi_1(C, s) \to G \) up to conjugacy.

\[ \blacklozenge \]

Pikaart and De Jong (again generalizing Deligne and Mumford) proceed to define a proper stack \( G\mathcal{M}_g \) by normalizing \( \mathcal{M}_g \) in \( G\mathcal{M}_g \). There is still a functor \( \mathcal{B}_g^{\text{tei}}(G) \to G\mathcal{M}_g \), which is in general not an isomorphism: for instance, \( G\mathcal{M}_g \) is in general singular, and the morphism \( G\mathcal{M}_g \to \mathcal{M}_g \) is always representable. In contrast, \( \mathcal{B}_g^{\text{tei}}(G) \) is always nonsingular, and in general \( \mathcal{B}_g^{\text{tei}}(G) \to \mathcal{M}_g \) is not representable\[ \square \].

### 6. Abelian twisted level structures

**Convention.** Fix a positive integer \( m \), and set \( S = \text{Spec } \mathbb{Z}[1/m] \). Throughout this section \( G = (\mathbb{Z}/m\mathbb{Z})^{2g} \).

In this section we study in detail the case \( G = (\mathbb{Z}/m\mathbb{Z})^{2g} \). By elementary covering theory, a smooth Teichmüller \( G \)-structure on a curve \( C \) consists of a basis for \( H^1(C, \mathbb{Z}/m\mathbb{Z}) \), which is what often one calls a level-\( m \) curve. Below we extend this description to stable curves: a twisted Teichmüller \( G \)-structure over a base \( S \) is equivalent to a twisted level-\( m \) curve, which is a pre-level-\( m \) twisted curve \( h : C \to S \) along with a basis of \( R^1h_*\mathbb{Z}/m\mathbb{Z} \) which is a local system.

#### 6.1. Pre-level-\( m \) curves.

**Definition 6.1.1.** A balanced twisted nodal curve \( \mathcal{C} \) whose moduli space \( C \) is stable is said to be a pre-level-\( m \) curve if for each geometric fiber, the stabilizer at each separating node is trivial and the stabilizer at a non-separating node is cyclic of order \( m \).

\[ ^7 \text{In the final version some examples of these singularities are due} \]
Proposition 6.1.2. The underlying twisted curve of an object of $B^\text{tei}_g((\mathbb{Z}/m\mathbb{Z})^{2g})$ is a pre-level-$m$ curve.

Proof. By definition, it is enough to consider a geometric object, and such an object is the rigidification of a connected twisted $G$-cover $P \rightarrow \mathcal{C}$. The claim is obvious for a smooth curve, therefore we may assume $\mathcal{C}$ is nodal. The stack of unpointed balanced twisted $G$-covers is quasifinite over $\mathcal{M}_g$ and of pure dimension $3g - 3$, therefore we can smooth the nodes independently, in particular, for each node $z$ of $\mathcal{C}$ the twisted $G$-cover $P \rightarrow \mathcal{C}$ deforms to a curve $P_z \rightarrow C_z$ with exactly one node $z_z$ having $z$ in its closure. Since the index of a twisted curve at a node is invariant under specialization, it is enough to consider the case where $\mathcal{C}$ has exactly one node.

Case 1: A separating node. Write $\mathcal{C} = C_1 \cup C_2$, where $C_i$ are the irreducible components, and let $p_i \in C_i$ be the points over the node. Consider the restriction of $P \rightarrow \mathcal{C}$ to the general locus $\mathcal{C}_{\text{gen}}$. Since every tame abelian étale cover of $C_i - p_i$ extends to an étale cover of $C_i$, we have that $P \rightarrow \mathcal{C}$ is unramified, therefore $\mathcal{C} = \mathcal{C}_{\text{gen}}$.

Case 2: A nonseparating node. This case is especially simple in case $m$ is prime: we have $H^1(C, \mathbb{Z}/m\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z})^{2g - 1}$, thus $\mathcal{C}$ admits no connected $G$-cover. Since $\mathcal{C}$ does have a connected $G$-cover, it is not isomorphic to $\mathcal{C}$, and therefore the node must be twisted. Since the order of the stabilizer at the node divides the exponent $m$ of $G$, the assumption that $m$ is prime implies that this order is precisely $m$.

In general, consider the Leray spectral sequence of étale cohomology groups for $\pi : \mathcal{C} \rightarrow C$:

$$H^i(C, R^j\pi_* (\mathbb{Z}/m\mathbb{Z})) \implies H^{i+j}(\mathcal{C}, \mathbb{Z}/m\mathbb{Z})$$

Note that $\pi_* (\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$, therefore we have an exact sequence

$$0 \rightarrow H^1(C, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^1(C, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^0(C, R^1\pi_* (\mathbb{Z}/m\mathbb{Z}))$$

The existence of the twisted $G$-cover $P \rightarrow \mathcal{C}$ shows that $G = (\mathbb{Z}/m\mathbb{Z})^{2g} \subset H^1(C, \mathbb{Z}/m\mathbb{Z})$, therefore the order of $H^1(C, \mathbb{Z}/m\mathbb{Z})$ is at least $m^{2g}$. We have $H^1(C, \mathbb{Z}/m\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z})^{2g-1}$. By Proposition [A.0.7] in the appendix, $R^1\pi_* (\mathbb{Z}/m\mathbb{Z})$ is a sheaf concentrated at the node, whose stalk is $H^1(\Gamma, \mathbb{Z}/m\mathbb{Z})$, where $\Gamma$ is the stabilizer of a geometric point of $\mathcal{C}$ over the node. Note that $\Gamma$ is a cyclic group of order $m'$ dividing the exponent of $G$, namely $m$. So the order of $H^1(\Gamma, \mathbb{Z}/m\mathbb{Z})$ is precisely $m'$. The exact sequence above becomes

$$0 \rightarrow (\mathbb{Z}/m\mathbb{Z})^{2g-1} \rightarrow H^1(C, \mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{Z}/m'\mathbb{Z},$$

therefore the order of $H^1(C, \mathbb{Z}/m\mathbb{Z})$ is at most $m^{2g-1}m'$. Combining the two inequalities, we have that the order $m'$ of $\Gamma$ is precisely $m$.

\[ \Box \]

6.2. The local system.

Proposition 6.2.1. Given a pre-level-$m$ curve $h : \mathcal{C} \rightarrow S$, the sheaf $R^1h_* \mathbb{Z}/m\mathbb{Z}$ is a local system.

Lemma 6.2.2. A pre-level-$m$ curve over a strictly henselian ring has a twisted Teichmüller $(\mathbb{Z}/m\mathbb{Z})^{2g}$-structure.

Proof. Let $\mathcal{C} \rightarrow S$ be a pre-level-$m$ curve over the spectrum of a strictly henselian ring, with closed fiber $\mathcal{C}_0$. By the Proper-Base-Change Theorem for tame stacks (A.0.8 in the appendix) we have $H^1(\mathcal{C}, G) = H^1(\mathcal{C}_0, G)$, implying that any twisted $G$-cover on $\mathcal{C}_0$ extends to a twisted
Theorem 6.2.4. The category $\mathcal{M}_g^{(m)}$ of twisted curves with level $m$ structure is an algebraic stack isomorphic to $\mathcal{B}_g^{\text{tei}}((\mathbb{Z}/m\mathbb{Z})^{2g})$.

---

8 Little explanation on the morphism below is needed.
Proof. Given an object of $\mathcal{B}^\text{tei}_g(G)(S)$ over $\mathcal{C} \to S$, we have by Proposition 6.1.2 that $\mathcal{C} \to S$ is a pre-level-$m$ curve. By Lemma 6.2.3 we have an isomorphism $(\mathbb{Z}/m\mathbb{Z})^{2g}_S \to H^1_{\text{et}}(\mathcal{C}, \mathbb{Z}/m\mathbb{Z})$, giving an object of $\mathcal{M}^{(m)}_g(S)$.

In the other direction, let $\mathcal{C} \to S$ be a pre-level-$m$ curve, along with an isomorphism $(\mathbb{Z}/m\mathbb{Z})^{2g}_S \to H^1_{\text{et}}(\mathcal{C}, \mathbb{Z}/m\mathbb{Z})$. There is an étale surjective map $S' \to S$ such that this isomorphism comes from a group homomorphism $(\mathbb{Z}/m\mathbb{Z})^{2g} \to H^1(C', \mathbb{Z}/m\mathbb{Z})$, where $C' = \mathcal{C} \times_S S'$. This corresponds to a principal $G$-bundle $P \to C'$ over $x$. Denote $S'' = S' \times_S S'$ and $C'' \to S''$ the pullback. The two pullbacks $P_i \to C''$ of $P$ become isomorphic on some étale cover $T'' \to S''$. An isomorphism over $T''$ descends to an isomorphism of $P_1$ with $P_2$ in the category of twisted Teichmüller structures $\mathcal{B}^\text{tei}_g(G)$, giving descent data for $P \to C'$ to an object of $\mathcal{B}^\text{tei}_g(G)(S)$. \[\square\]

7. Automorphisms of twisted $G$-covers

We start this section with a concrete description of the group of automorphisms of a twisted curve $\mathcal{C}$ acting trivially on the coarse curve $C$. We then turn to automorphisms of $G$-covers, and show that, in case $\mathcal{C}$ surjects to $(\mathbb{Z}/m\mathbb{Z})^{2g}$, $m \geq 3$, every $G$-automorphism of a $G$-cover acts trivially on the coarse curve $C$. We give some structure results on this automorphism group in case $G$ is a characteristic quotient, and construct some fine moduli spaces of twisted Teichmüller $G$-structure and of twisted $G$-covers.

7.1. Automorphisms of twisted curves. Let $\mathcal{C}$ be a twisted curve over an algebraically closed field, and let $C$ be its moduli space. For each node $x \in C_{\text{sing}}$ denote by $\Gamma_x$ the stabilizer of a geometric point of $\mathcal{C}$ over $x$, which is a cyclic group.

Proposition 7.1.1. Denote by $\text{Aut}_\mathcal{C}(\mathcal{C})$ the automorphism group of $\mathcal{C}$ over $C$ in the category of twisted curves. There is an isomorphism

$$\text{Aut}_\mathcal{C}(\mathcal{C}) \simeq \prod_{x \in C_{\text{sing}}} \Gamma_x.$$ 

In other words, every node contributes exactly $\Gamma_x$ to this automorphism group (and the markings do not).

We use the following lemma, whose statement and proof are identical to the classical case:

Lemma 7.1.2. Let $P \to \mathcal{C}$ be a connected étale $G$-cover of a twisted curve. Then $\text{Aut}_\mathcal{C}P = G$ and $\text{Aut}_\mathcal{C}^G P = Z(G)$.

Proof of the Lemma. Clearly $\text{Aut}_\mathcal{C}P \supset G$. An automorphism $\phi \in \text{Aut}_\mathcal{C}P$ (which we write acting on the right) pulls back to an automorphism of the trivial cover $P \times C P \to P$ commuting with the Galois group of the base change $P \to \mathcal{C}$, which contains $G$ (which, to keep things compatible, acts on the left). Restricting to a geometric point and identifying the fiber with $G$, we get an element of the permutation group of the set $G$ commuting with the action of $G$ on the left, which therefore is an element of $G$ acting on the right, giving the first claim.

If, moreover, $\phi$ commutes with the action of $G$ on the right, then it is an element of the center $Z(G)$, giving the second claim. \[\square\]

Proof of the Proposition. We can view $\text{Aut}_\mathcal{C}\mathcal{C}$ as the group of global sections of the étale sheaf of relative automorphisms - see Lemma 7.1.3 below. Over $C_{\text{gen}}$ the map $\mathcal{C} \to C$ is an isomorphism, and the sheaf is trivial. Therefore the automorphism group is a product of

\[\text{Need to do arrows.}\]

\[\text{Need to clarify } G \text{ vs. } G^{\text{opp}}\]
contributions form small étale neighborhoods of twisted nodes and markings. Focusing on one of these twisted points \( x \), we may replace \( C \) by an affine twisted curve having the same local picture at a twisted geometric point, that is, \( C = [U/\Gamma_x] \) where \( U \) is either \( \text{Spec} \ k[z] \) or \( \text{Spec} \ k[z,w]/(zw) \), with \( \Gamma_x = \mu_r \) acting as described in 2.1.1 or 2.1.2, respectively.

Consider the exact sequence
\[
1 \longrightarrow \text{Aut}_{[U/\Gamma_x]} U \longrightarrow \text{Aut}_C(U \to [U/\Gamma_x]) \longrightarrow \text{Aut}_C C.
\]

**Claim.** We have \( \text{Aut}_{[U/\Gamma_x]} U = \Gamma_x \).

**Proof of claim.** This follows from Lemma 7.1.2, since \( U \) is a connected étale \( \Gamma_x \)-cover of \([U/\Gamma_x]\). ♣

**Claim.** The canonical inclusion \( \text{Aut}_C(U \to [U/\Gamma_x]) \subset \text{Aut}_C U \) is an isomorphism.

**Proof of claim.** In case \( U = \text{Spec} \ k[z] \), we have \( \text{Aut}_C(U \to [U/\Gamma_x]) \subset \text{Aut}_C U = \Gamma_x \). The exact sequence (2) implies that equality holds.

In case \( U = \text{Spec} \ k[z,w]/(zw) \), we have \( \text{Aut}_C(U \to [U/\Gamma_x]) \subset \text{Aut}_C U = \mu_r^2 \), where the action of \((\zeta_1, \zeta_2) \in \mu_r^2\) is via
\[
(z, w) \mapsto (\zeta_1 z , \zeta_2 w).
\]
This action clearly commutes with the action of \( \Gamma_x \), which means that \((\zeta_1, \zeta_2)\) acts on \( U \to [U/\Gamma_x] \). ♣

**Claim.** The morphism on the right in the sequence (2) is surjective.

**Proof of Proposition assuming the claim.**

- In case \( U = \text{Spec} \ k[z] \), \( \text{Aut}_C C = \Gamma_x/\Gamma_x \) is trivial.
- In case \( U = \text{Spec} \ k[z,w]/(zw) \), we have \( \text{Aut}_C C = \mu_r^2/\Gamma_x \simeq \Gamma_x \), as required. ♣

**Proof of the claim.** Let \( \phi \in \text{Aut}_C [U/\Gamma_x] \). This comes from a functor \([U/\Gamma_x] \to [U/\Gamma_x]\) preserving \( U/\Gamma_x \).

The canonical morphism \( U \to [U/\Gamma_x] \) corresponds to the diagram
\[
U \times \Gamma_x \longrightarrow U \\
\downarrow \\
U
\]
where the vertical map is the projection on the first factor and the horizontal map is the action of \( \Gamma_x \) on \( U \). The automorphism \( \phi_U \) gives another principal \( \Gamma_x \) bundle
\[
P \longrightarrow U \\
\downarrow \\
U
\]
Since \( U \) has trivial tame fundamental group, we may choose a section \( U \to P \), and composing with the horizontal map \( P \to U \) we obtain an automorphism of \( U \) over \( U/\Gamma_x \). This gives a lifting of \( \phi \) to \( \text{Aut}_C U \), which by the previous claim is the same as \( \text{Aut}_C(U \to C) \). ♣

In the proof we used the following lemma:

**Lemma 7.1.3.** Let \( \mathcal{X} \) be a separated Deligne–Mumford stack over a scheme \( S \). Suppose that there is an open and scheme-theoretically dense substack of \( \mathcal{X} \) which is an algebraic space. Then the functor that sends each étale morphism of finite type \( U \to S \) to the group of automorphisms \( \text{Aut}_U(\mathcal{X}_U) \) is a sheaf on the small étale site of \( S \).
The group Aut(U(X)) is the group of base-preserving equivalences of categories of X with itself, modulo isomorphism. Recall that the groupoid of base-preserving equivalences of categories of X with itself is in fact equivalent to a group ([N-V], Lemma 4.2.3). In other words, no such equivalence has nontrivial automorphisms.

Proof. First of all, let us check that if \{U_i \to U\} is an étale cover, F, G: XU \to XU are base-preserving equivalences, such that their pullbacks F_{U_i}, G_{U_i}: X_{U_i} \to X_{U_i} are all isomorphic, then F and G are isomorphic.

This follows immediately from the fact that the isomorphisms of pullbacks of F and G to étale morphisms into U form a sheaf in the small étale topology of U. In concrete terms, given isomorphisms \(\phi_i: F_{U_i} \simeq G_{U_i}\), the pullbacks of \(\phi_i\) and \(\phi_j\) to \(F_{U_{ij}} \simeq G_{U_{ij}}\) must coincide, because of the unicity of isomorphisms, therefore \{\phi_i\} satisfies the cocycle condition, and, by the stack axioms, the \(\phi_i\) descend to an isomorphism of F with G.

Now assume that you are given a collection \(F_i: X_{U_i} \to X_{U_i}\) of base preserving equivalences, such that \((F_i)_{U_{ij}}\) and \((G_i)_{U_{ij}}\) are isomorphic. Let \(\phi_{ij}: (F_i)_{U_{ij}} \simeq (F_i)_{U_{ij}}\) be an isomorphism; the unicity of isomorphisms insures that the cocycle condition

\[\phi_{ij}\phi_{jk} = \phi_{ik}: (F_i)_{U_{ij}} \simeq (F_i)_{U_{ij}}\]

is satisfied.

Let \(T \to U\) be a morphism, and set \(T_i = T \times_U U_i\) and \(T_{ij} = T \times_U U_{ij}\). Suppose that \(\xi\) is an object of \(X(T)\). Since we have \(F_i(\xi_{T_{ij}}) = F_j(\xi_{T_{ij}})\), the isomorphism \(\phi_{ij}(\xi_{T_{ij}}): F_j(\xi_{T_{ij}}) \simeq F_i(\xi_{T_{ij}})\) yield isomorphisms \(\psi_{ij}: F_j(\xi_{T_{ij}})_{T_{ij}} \simeq F_i(\xi_{T_{ij}})_{T_{ij}}\). The cocycle condition on the \(\phi_{ij}\) says that these isomorphisms \(\psi_{ij}\) give descent data; therefore we obtain an object \(F(\xi)\) of \(X(T)\), together with isomorphisms \(F(\xi)_U \simeq F_i(\xi_{U_i})\).

It is a simple matter to check that if \(f: \xi \to \eta\) is an arrow in \(X(U)\), then the restrictions \(f_i: F_i(\xi_{U_i}) \to F_i(\eta_{U_i})\) glue together to yield an arrow \(F(f): F(\xi) \to F(\eta)\); therefore we obtain a functor \(F: X_U \to X_U\), which lifts to the \(F_i\), as desired. \(\diamondsuit\)

7.1.4. Automorphisms and deformations. In case \(\mathcal{C}\) is balanced and unmarked, one can read off the automorphism group from the deformation space. Again, the problem is local, so we may focus on the case where \(\mathcal{C}\) has a unique twisted node. We note that the deformation space Def\(_C\) of \(\mathcal{C}\) surjects to the deformation space Def\(_C\) of \(C\). Also, since there is a unique isomorphism class of twisted curves with given twisting having coarse moduli space \(C\), we have that Def\(_C\) = Def\(_C\)/Aut\(_C\)\(_\mathcal{C}\). Moreover, since the generic curve is smooth and since there are no markings, it is untwisted. This means that the action of Aut\(_C\)\(_\mathcal{C}\) on Def\(_C\) is effective.

From the analysis of the sequence \([\text{I}]\) in section \([\text{II}]\) we see that we only need compare the deformation spaces of the nodes on \(\mathcal{C}\) and \(C\). Clearly the map from the deformation space Spec \(k[[t]]\) of \(U = \{zw = 0\}\) (having versal family \(zw = t\)) to the deformation space Spec \(k[[s]]\) of \(U/\mu_r = \{xy = 0\}\) (having versal family \(xy = s\)) is given by \(s = t^r\), and its Galois group is \(\mu_r = \Gamma_x\).

7.2. Serre’s lemma and existence of tautological families. In the rest of this section we assume \(n = 0\), that is, the curves have no markings. We fix \(G\), a finite group, and write \(S = \text{Spec} \mathbb{Z}[1/\#G]\).

The following is an interpretation of a well known lemma of Serre:

Lemma 7.2.1. Assume \(G\) admits a surjection onto \((\mathbb{Z}/m\mathbb{Z})^g\), for some \(m \geq 3\). Then every \(G\)-automorphism of a balanced, connected twisted \(G\)-cover \(P \to \mathcal{C}\) acts trivially on the coarse
curve \( C \). In other words,
\[
\text{Aut}^G(P \to C) = \text{Aut}^G_C(P \to C).
\]

**Proof.** Let \( Q \to C' \) be the twisted \((\mathbb{Z}/m\mathbb{Z})^{\text{tw}}\)-cover associated to \( P \to C \) obtained using the functoriality result \( [\text{N-V}] \), Corollary 9.1.2, applied with \( \mathcal{M} = BG \) and \( \mathcal{M}' = B(\mathbb{Z}/m\mathbb{Z})^{\text{tw}} \).

An automorphism of the twisted \( G \)-cover \( P \to C \) gives an automorphism of \( Q \to C' \) over an automorphism \( \phi : C' \to C' \). Recall that \( Q \to C' \) gives a basis for \( H^1(C', \mathbb{Z}/m\mathbb{Z}) \) (see Lemma 5.2.3). Since \( \phi^* Q \cong Q \), we have that \( \phi^* : H^1(C', \mathbb{Z}/m\mathbb{Z}) \to H^1(C', \mathbb{Z}/m\mathbb{Z}) \) is the identity. But \( H^1(C, \mathbb{Z}/m\mathbb{Z}) \subset H^1(C', \mathbb{Z}/m\mathbb{Z}) \), therefore \( \phi \) induces the identity on \( H^1(C, \mathbb{Z}/m\mathbb{Z}) \). By Serre’s Lemma for stable curves (see, e.g., \( [\text{N-O}] \), Lemma 3.5.) it follows that \( \phi \) induces the identity on \( C \).

We deduce the following well known corollary:

**Corollary 7.2.2.** Assume \( G \) admits a surjection onto \((\mathbb{Z}/m\mathbb{Z})^{\text{tw}}\), for some \( m \geq 3 \). Then the morphism of coarse moduli spaces \( B_{g^p}^g(G) \to \overline{M}_g \) admits a lifting \( B_{g^p}^g(G) \to \overline{M}_g \). In other words, \( B_{g^p}^g(G) \) carries a tautological family of stable curves.

**Proof.** Let \( C_0 \) be a twisted curve over an algebraically closed field admitting a twisted Teichmüller \( G \)-structure corresponding to a connected balanced \( G \)-cover \( P \to C_0 \). Denote \( A^G = \text{Aut}^G(P \to C_0) \). The local picture of \( B_{g^p}^g(G) \) at the point corresponding to \( P \to C_0 \) is \( [\text{Def}_{P\to C_0}/A^G] \), and the local picture of the universal curve is \( [C/A^G] \), where \( C \to \text{Def}_{P\to C_0} \) is the twisted curve underlying the universal deformation. Consequently, on the level of coarse moduli spaces, we have that the local picture of \( B_{g^p}^g(G) \) at the point corresponding to \( P \to C \) is the scheme \( \text{Def}_{P\to C}/A^G \), and the local picture of the coarse moduli space of the universal curve is \( C/A^G \), where \( C \) is the coarse curve underlying the universal deformation. Since the action of \( A^G \) on \( C_0 \) is trivial, we have that \( C/A^G \to \text{Def}_{P\to C}/A^G \) is a stable curve, as required.

### 7.3. Structure of automorphisms of connected twisted \( G \) covers

Let \( P \to C \) be a connected twisted \( G \) cover over an algebraically closed field. Given a node \( x \) of \( C \), we denote by \( r_x \) the index of \( C \) at \( x \).

We wish to have some understanding the \( G \)-automorphism group \( \text{Aut}^G_C(P \to C) \) of the twisted \( G \)-cover \( P \to C \) fixing \( C \). It is easy to see that \( \text{Aut}^G_C(P \to C) = \text{Aut}^G_C P \), since \( C \) can be recovered as \( [P/G] \). One may try to study it via its natural embedding as the centralizer of \( G \) in the group \( \text{Aut}_C P \), but the latter group is in general too big - the action of an element of \( \text{Aut}_C P \) is in general not compatible with local charts for the twisted cover \( P \to C \).

We denote \( A = \text{Aut}_C(P \to C) \), the automorphism group of the morphism \( P \to C \), fixing \( C \) (but not necessarily commuting with \( G \)). This is precisely the group of automorphisms of \( P \) over \( C \) preserving the charts of \( P \to C \) as a twisted cover. Then the \( G \)-equivariant automorphisms are \( A^G \). These in turn can be thought of as the \( G \)-automorphisms of \( P \) over \( C \).

Set \( M = \prod_{x \in C_{\text{sing}}} \Gamma_x \).

**Lemma 7.3.1.** We have an exact sequence
\[
1 \to G \to A \to M
\]
which, when taking \( G \)-invariants, gives an exact sequence
\[
1 \to Z(G) \to A^G \to M
\]
Proof. Consider the natural sequence
\[ 1 \longrightarrow \text{Aut}_C P \longrightarrow \text{Aut}_C(P \to C) \longrightarrow \text{Aut}_C C. \]

The group \( \text{Aut}_C P \) is naturally isomorphic to \( G \) since \( P \) is connected and \( P \to C \) is a principal bundle (Lemma 7.1.2). The term on the right is \( \prod_{x \in C_{\text{sing}}} \Gamma_x \) by Proposition 7.1.1.

Recall that the stack of balanced twisted \( G \)-covers is nonsingular and flat over \( \bar{\mathcal{M}}_g \). This implies that any object \( P \to C \) can be deformed to a smooth object in characteristic 0. It also implies that \( P \to C \) can be deformed to an object in characteristic 0 preserving the topological type of \( C \).

Definition 7.3.2. We say that a connected balanced twisted \( G \)-cover \( P \to C \) is characteristic if for some deformation to a smooth \( G \) cover \( P' \to C' \) in characteristic 0, and some choice of base point \( s \) in \( C' \), the kernel of the corresponding epimorphism \( \pi_1^{\text{geom}}(C', s) \to G \) is a characteristic subgroup of \( \pi_1^{\text{geom}}(C', s) \).

Since the stack \( \mathcal{B}_{\text{bal}}^g(G) \) is smooth over \( S \), and since the property of \( \pi_1^{\text{geom}}(C', s) \to G \) being characteristic is Galois invariant, this property is independent of the choice of deformation, and thus it is an invariant of the connected component of the stack.

Lemma 7.3.3. Assume that \( P \to C \) is characteristic. Then the homomorphism \( A \to M \) is surjective, giving an exact sequence
\[ 1 \longrightarrow G \longrightarrow A \longrightarrow M \longrightarrow 1. \]

Furthermore, when the base field is \( \mathbb{C} \), then for any deformation \( \mathcal{C}_\Delta \to \Delta \) of \( \mathcal{C} \) with smooth generic fiber, and each node \( x \in C \), there is a generator \( \sigma_x \in \Gamma_x \subset M \) with a lifting \( \delta_x \in A \), whose action on \( G \) via conjugation is obtained by the action of a Dehn twist \( D_x \) along the vanishing cycle of the node, on the fundamental group of a nearby smooth curve in the deformation \( \mathcal{C}_\Delta \to \Delta \).

We remark that this exact sequence can be shown to split. Our proof relies on topological considerations over \( \mathbb{C} \).

7.3.4. Reduction to \( \mathbb{C} \). We claim that it suffices to show the Lemma when \( k = \mathbb{C} \). First consider the case of characteristic 0. Note that \( A \), being the group of points of a finite group scheme, is invariant under extensions of algebraically closed field. It follows that if the statement holds in over \( \mathbb{C} \) then it holds in characteristic 0.

If \( P_0 \to C_0 \) is in positive characteristic, we choose a deformation with constant topological type \( P \to C \) on a discrete valuation ring \( R \) of mixed characteristic, with fraction field \( K \). We may assume \( R \) contains all the roots of unity of order dividing the order of \( G \). The sequence above comes from a sequence of group schemes \( 1 \longrightarrow \bar{G} \longrightarrow \bar{A} \longrightarrow \bar{M} \). Here \( \bar{G} \) is assumed constant, and \( \bar{M} = \prod \mu_{r_x} \) is constant since the deformation has constant topological type and \( R \) contains the roots of unity of order \( r_x \). An element \( m \in M \) is therefore also an element of \( M(K) \). This element lifts to an element of \( \bar{A}(K) \), since \( K \) has characteristic 0, which is the case we settled above. This implies that after replacing \( R \) by a finite extension, the element \( m \) lifts to \( \bar{A}(K) \). Since \( \bar{A} \) is a finite group scheme, the element in \( \bar{A}(K) \) specializes to an element \( a \) of \( A \).
7.3.5. Topologically trivial parametrization. Now consider the case $k = \mathbb{C}$. Let $P_\Delta \to C_\Delta \to \Delta$ be a small analytic deformation of $P \to C$ with smooth generic fiber, where $\Delta$ is the disc around the origin in $\mathbb{C}$. We assume that the only singular fiber lies over the origin in $\Delta$. Let $C_1$ be the fiber over a general point $t_1 \in \Delta^*$. Consider the parametrized line segment $\beta : [0, 1] \to \Delta$ given by $\beta(t) = tt_1$, connecting 0 with $t_1$.

Claim. There is a family of continuous maps $\psi_t : C_1 \to C_{\beta(t)}$, such that $\psi_1$ is the identity, $\psi_t$ is a homeomorphism whenever $t \neq 0$, and $\pi \circ \psi_0 : C_1 \to C_0 = C$ is the contraction of the vanishing cycles.

Proof of claim. First recall a typical construction of a continuous map $\phi_t : C_1 \to C_{\beta(t)}$ to the coarse curve, using polar coordinates. Write $C_\Delta = C' \cup \bigcup V_i$, where $C'$ is an open set which is topologically trivial over $\Delta$, and $V_i$ are small neighborhoods of the nodes of $C_0$. We may assume that we have an analytic isomorphism $V_i \simeq \{(x_i, y_i, u) : |x_i|, |y_i| \leq 1, u \in \Delta, x_i y_i = h_i(u)\}$, for some analytic function $h_i$ on $\Delta$. We note that the existence of $C_\Delta$ implies that $h_i = g_i^t$.

We focus on one of the open sets $V_i$ and drop the subscripts $i$ for simplicity of notation.

For $t \in (0, 1]$ write $h(\beta(t)) = \tau(t) e^{2\pi i \alpha(t)}$, where $\tau(t) \in \mathbb{R}_{\geq 0}$ and $\alpha(t) \in \mathbb{R}/\mathbb{Z}$. Since $\beta$ is linear, this extends continuously to $t = 0$ as well.

We use coordinates $(\xi, \eta, \theta)$ on $\mathbb{R}^2_{\geq 0} \times \mathbb{R}/\mathbb{Z}$. Consider $\tilde{V} \subset \mathbb{R}^2_{\geq 0} \times \mathbb{R}/\mathbb{Z}$, the inverse image of $V$ by the polar coordinates map

$$
\begin{align*}
x &= \xi e^{2\pi i \alpha} \\
y &= \eta e^{2\pi i (\alpha(t) - \theta)}.
\end{align*}
$$

The inverse image of the point $x = y = 0$ is a circle representing the vanishing cycle.

The fibration $\tilde{V} \to [0, 1]$ is topologically trivial: all the fibers are homeomorphic to a cylinder.

Gluing each $\tilde{V}_i$ into $\beta^*(C_\Delta)$ instead of $V_i$, we get a topologically trivial fibration $C_{\text{polar}} \to [0, 1]$, and choosing a trivialization we have a continuous map $C_1 \times [0, 1] \to C_\Delta$.

We now lift this map to the stack $C_\Delta$. Locally near a node, $C_\Delta$ is given by $U : \{zw = g(t)\}$, where $x = z^r, y = w^r$ and $h(t) = g(t)^r$. Over $\tilde{V}$ we have a corresponding étale covering given by $(\zeta, \omega, \gamma)$, where

$$
\begin{align*}
\xi &= \zeta^r \\
\eta &= \omega^r \\
\theta &= r \cdot \gamma
\end{align*}
$$

which maps down to $U$ via

$$
\begin{align*}
z &= \zeta e^{2\pi i \gamma} \\
w &= \omega e^{2\pi i (\alpha(t)/r - \gamma)}.
\end{align*}
$$

and this map is clearly $\mu_r$-equivariant. This gives the desired lifting $\psi_t : C_1 \to C_\Delta$.

It should be remarked that, using polar coordinates on both $\Delta$ and $C_\Delta$, one can define a topologically locally trivial fibration $C_{\Delta}^{\text{polar}} \to \Delta^{\text{polar}}$. This is a morphism of real-analytic manifolds with corners, which is an instance of the logarithmic space associated to the natural logarithmically smooth structure on $C_\Delta \to \Delta$, see [K-N][1].

We continue the proof of the lemma. The pullback by $\psi_t$ of $P_t \to C_t$ gives a topological principal bundle over $C_t \times [0, 1]$, which therefore must be constant. This means that $\psi_0^* P_0$ is isomorphic as a topological $G$-bundle to $P_1$, and therefore is characteristic.

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[1] Precise citation needed
7.3.6. Making space around the vanishing cycle. Since Dehn twists are not analytic in nature, and since we want to keep some analytic properties, we make some space around every vanishing cycle in $C_1$ by inserting a cylinder, inside which all the non-analytic activities will occur.

Fix a node $x \in C_{\text{sing}}$. A small neighborhood $W_x$ of the vanishing cycle $(\pi \circ \psi_0)^{-1}x$ is homeomorphic to an open cylinder $S^1 \times (-\epsilon, \epsilon)$. We replace this by

$$W^\text{cyl}_x = S^1 \times \left( (-\epsilon, 0] \cup [0, 1] \cup [1, 1 + \epsilon) \right) = S^1 \times (-\epsilon, 1 + \epsilon).$$

Doing this at each node, we obtain a topological surface $C^\text{cyl}$ with a continuous map $\eta : C^\text{cyl} \to C_1$ shrinking each $S^1 \times [0, 1]$ to the cycle $S^1 \times \{0\}$. Denote $\psi_0 \circ \eta = \psi^\text{cyl} : C^\text{cyl} \to C_0$ and let $P^\text{cyl} = (\psi^\text{cyl})^* P_0$. This is a characteristic topological cover, in the sense that the corresponding subgroup $\pi_1(P^\text{cyl}) \subset \pi_1(C^\text{cyl})$ is characteristic.

7.3.7. The Dehn twist. Fix one node $x \in C$. Define the following homeomorphism $D_x : C^\text{cyl} \to C^\text{cyl}$. On the complement of $S^1 \times [0, 1] \subset W^\text{cyl}_x$ it is defined to be the identity. On $S^1 \times [0, 1]$ it is defined by $D_x(z, t) = (e^{2\pi i z}, t)$. The induced action of $D_x$ on $\pi_1(C_{\text{cyl}})$ is precisely the Dehn twist associated to the vanishing cycle over $x$.

Consider the pullback $D^* P^\text{cyl}$. Since $P^\text{cyl}$ is a characteristic topological cover, this pullback is isomorphic to $P^\text{cyl}$ as a topological cover, so $D_x$ lifts to a homeomorphism $\delta^\text{cyl}_x : P^\text{cyl} \to P^\text{cyl}$.

7.3.8. The action on $C$. We claim that $\delta^\text{cyl}_x$ descends to an automorphism $\delta_x : P \to P$ whose image in $M$ is a generator of $\Gamma_x$.

First we claim that there is a commutative diagram of topological stacks

$$
\begin{array}{ccc}
C^\text{cyl} & \xrightarrow{\psi^\text{cyl}} & C \\
\downarrow D_x & & \downarrow \sigma_x \\
C^\text{cyl} & \xrightarrow{\psi^\text{cyl}} & C
\end{array}
$$

for a suitable automorphism $\sigma_x$ corresponding to a generator of $\Gamma_x \subset \text{Aut}_C C$. To see this, consider a chart $(U, \Gamma_x)$ for $C$ at $x$ with a lifting $W^\text{cyl}_x \to [U/\Gamma]$ of $\psi^\text{cyl} : C^\text{cyl} \to C$, and let

$$
\begin{array}{ccc}
\tilde{W}^\text{cyl}_x & \longrightarrow & U \\
\downarrow & & \downarrow \\
W^\text{cyl}_x & \longrightarrow & [U/\Gamma_x]
\end{array}
$$

be the cartesian diagram. Then $D_x$ lifts to an automorphism of the cylinder $\tilde{W}^\text{cyl}_x$ which is trivial on $S^1 \times (-\epsilon, 0]$, acts on $S^1 \times [0, 1]$ by $(w, t) \mapsto (e^{2\pi i r_x w}, t)$, and rotates $S^1 \times [1, 1 + \epsilon]$ by $e^{2\pi i r_x}$ where $r_x$ is the order of $\Gamma_x$.

It follows that $\psi^\text{cyl} \circ D_x = \sigma_x \circ \psi^\text{cyl}$, where $\sigma_x \in \text{Aut}_C C$ acts as the identity on the branch of $U$ under $S^1 \times (-\epsilon, 0]$ and as $e^{2\pi i r_x}$ on the branch of $U$ under $S^1 \times [1, 1 + \epsilon]$.

7.3.9. The action on $P$. Now it is clear that $\delta^\text{cyl}_x$ descends to a homeomorphism $\delta_x : P \to P$. This homeomorphism is analytic outside of the nodes, and by continuity it is analytic everywhere. Now we have a commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\delta_x} & P \\
\downarrow & & \downarrow \\
C & \xrightarrow{\sigma_x} & C
\end{array}
$$

which is what was required.
7.3.10. An algebraic proof. Here is a sketch of an algebraic proof proposed by Johan de Jong. It is presented geometrically, but one can easily make it algebraic, for instance using the methods of [P-J].

Let $P_0 \to C_0$ be a connected balanced stable twisted $G$-cover over $\mathbb{C}$, and let $C_0$ be the coarse stable curve. Let $\overline{C} \to \overline{\Delta}$ be the universal deformation of $C_0$, and let $P \to C \to \Delta$ be the universal deformation of $P_0 \to C_0$. We denote by $C$ the coarse moduli space of $\mathcal{C}$.

The deformation $C \to \Delta$ of $C_0$ induces a morphism $\Delta \to \overline{\Delta}$. By Lemma 7.1.4 we have that $\Delta \to \overline{\Delta}$ is Galois with Galois group $M = \text{Aut}_{C_0} C_0$, where $M$ acts on $C \to \Delta$ by the universal property of the deformation.

Since $C = \overline{C} \times _{\overline{\Delta}} \Delta$, the action of $M$ on $\Delta$ lifts as a product action to $C$. This is nothing but the action induced by the action of $M$ on $\mathcal{C}$.

Fix $\sigma \in M$ and let $t$ be a geometric point of $\Delta$ in the smooth locus of $C \to \Delta$. The smooth curve $C_t$ carries two covers $P_t \to C_t$ and $P_{\sigma(t)} = (\sigma^*P)_t \to C_t$. Since the kernel of $\pi_1(P_t) \to \pi_1(C_t)$ (with appropriate base points) is a characteristic subgroup, these two covers are isomorphic, so there exists an isomorphism

$$\delta_t : P_t \to P_{\sigma(t)}$$

lying over

$$\sigma : C_t \to C_{\sigma(t)},$$

and, moreover, $\delta_t$ sends an element of $G$ (acting on $P_t$) to an element of $G$ (acting on $P_{\sigma(t)}$).

Since the stack of stable curves of genus $g(P)$ is separated, this reduces to an automorphism $\delta : P_0 \to P_0$ (lying over $\sigma : C_0 \to C_0$, which is the identity). Moreover $\delta$ acts on $G \subset \text{Aut}_{C_0} P_0$, giving an automorphism of $[P_0/G] = \mathcal{C}$ which clearly coincides with $\sigma$. Therefore the diagram

$$\begin{array}{ccc}
P_0 & \xrightarrow{\delta} & P_0 \\
\downarrow & & \downarrow \\
C_0 & \xrightarrow{\sigma} & C_0
\end{array}$$

is commutative, giving the surjectivity of $A \to M$.

Denote by $\Delta^*, \overline{\Delta}^*$ the loci where the curves are smooth, and by $P^* \to C^* \to \Delta$, respectively $\overline{C}^* \to \overline{\Delta}$ the curve fibrations. On the level of fundamental groups, consider the diagram with exact rows and injective columns

$$\begin{array}{cccccc}
1 & \to & \pi_1(P_t) & \to & \pi_1(P^*) & \to & \pi_1(\Delta^*) & \to & 1 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \to & \pi_1(C_t) & \to & \pi_1(\overline{C}^*) & \to & \pi_1(\overline{\Delta}^*) & \to & 1.
\end{array}$$

We have $M = \pi_1(\overline{\Delta}^*) / \pi_1(\Delta^*)$, so an element of $M$ lifts to an element $D \in \pi_1(\overline{C}^*)$, whose action on $\pi_1(C_t)$ is the Dehn twist (corresponding on the level of outer automorphisms to its image in the monodromy group $\pi_1(\overline{\Delta}^*)$). Since $\pi_1(P_t) \subset \pi_1(C_t)$ is characteristic, we have $\pi_1(P_t) \subset \pi_1(\overline{C}^*)$ normal, so $D$ acts on the quotient $G = \pi_1(C_t) / \pi_1(P_t)$ as a Dehn twist.

Incidentally, it can be shown that $\pi_1(P^*) \subset \pi_1(\overline{C}^*)$ is normal with quotient group $A$.

Lemma 7.3.3 implies that if $P \to C$ is characteristic, there is a canonical induced homomorphism of groups from $M$ to the group $\text{Out}(G)$ of outer automorphism of $G$. We can now summarize our results as follows:

**Proposition 7.3.11.** Assume $P \to C$ is a characteristic $G$-cover over an algebraically closed field. Then


1. if the homomorphism $M \to \text{Out}(G)$ is injective, then the subgroup of the group of automorphisms of $P \to C$ as a twisted Teichmüller structure, consisting of elements acting trivially on $C$, is trivial.

2. If, moreover, $G$ admits a surjection to $(\mathbb{Z}/m\mathbb{Z})^2$ for some $m \geq 3$, then the whole group of automorphisms of $P \to C$ as a twisted Teichmüller structure is trivial.

3. If in addition the center $Z(G)$ is trivial then the group of $G$-automorphisms $\text{Aut}^G(P \to C)$ is trivial.

Proof. The first statement follows from the exact sequence in Lemma 7.3.3 since the assumption implies that $A^G = Z(G)$. The second statement follows from Serre’s Lemma. The last statement follows from the same exact sequence since then $A^G$ is trivial.

7.4. The groups of Looijenga and Pikaart–De Jong. We now describe two cases where the first two statements of the proposition above hold. We note that condition (1) can be verified by checking it for twisted covers over $\mathbb{C}$. We denote by $\Pi_g$ be the fundamental group of a Riemann surface of genus $g$.

7.4.1. Looijenga’s groups. Fix an integer $m \geq 3$. Choose a Riemann surface $C$ of genus $g$, and let $C_2 \to C$ be the maximal abelian étale cover of exponent $2$. The Galois group $G_2$ of $C_2$ over $C$ is $H_1(C, \mathbb{Z}/2\mathbb{Z})$, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Let $C_2^{[m]} \to C_2$ be the maximal abelian étale cover of exponent $2m$. The curve $C_2^{[m]}$ is a Galois cover of $C$. Denote its Galois group by $G_2^{[m]}$. This is clearly a characteristic quotient of $\Pi_g$.

Let $C_0$ be a nodal curve and let $P_0 \to C_0$ be a connected admissible $G_2^{[m]}$-cover. We denote by $P_0 \to C_0$ the corresponding twisted $G_2^{[m]}$-cover.

Lemma 7.4.2. 1. If $x$ is a separating node of $C_0$ then $\Gamma_x \cong \mu_{2m}$.

2. If $x$ is a non-separating node of $C_0$ then $\Gamma_x \cong \mu_{4m}$.

Proof. There is an intermediate curve $D_0$ which is a connected admissible $(\mathbb{Z}/2\mathbb{Z})^2$ cover of $C_0$. According to [Ld], Proposition 2, every node of $D_0$ is a non-separating node. If $x$ is a separating node of $C_0$, then it follows from Proposition 6.1.2 that $D_0 \to C_0$ is unramified at $x$, $P_0 \to D_0$ has ramification index $2m$, and therefore $P_0 \to C_0$ has ramification index $2m$. If $x$ is non-separating, then it follows from the same proposition that $D_0 \to C_0$ has ramification index 2 at $x$, and $P_0 \to D_0$ has again index $2m$, therefore $P_0 \to C_0$ has index $4m$. The lemma follows.

Following Looijenga, let $E_0$ be the set of separating nodes of $C_0$ and $E_1$ the set of nonseparating nodes. Denote by $T = \prod_{x \in C_{\text{sing}}} \mathbb{Z}$ the group of Dehn twists. According to Looijenga [Ld], Proposition 3, the kernel of the natural homomorphism $T \to \text{Out}(G_2^{[m]})$ is precisely the subgroup

$$T_0 = \prod_{x \in E_1} 2m\mathbb{Z} \times \prod_{x \in E_2} 4m\mathbb{Z}.$$ 

It follows from Lemma 7.3.3 that we have an isomorphism $T/T_0 \cong \prod_{x \in C_{\text{sing}}} \Gamma_x$. This implies that $M \to \text{Out}(G_2^{[m]})$ is injective.

As an immediate outcome we have

Theorem 7.4.3. Suppose $m \geq 3$. Then the moduli stack $B_{g,1}^{\text{et}}(G_2^{[m]})$ is a smooth projective scheme over $\mathbb{Z}[1/2m]$ admitting a finite flat morphism to $\overline{M}_g$. 

7.4.4. The groups of Pikaart-De Jong. Following [P-J] we inductively define $\Pi_1^{(k)} = \Pi_g$ and $\Pi_1^{(k+1)} = [\Pi_1^{(k)}, \Pi_g]$, the group of $k$-th order commutators. We denote by $\Pi_1^{(k),n} = \Pi_1^{(k)} \cdot \Pi_g^n$, where $\Pi_g^n$ is the subgroup generated by $n$-th powers, and $G_g^{(k),n} = \Pi_1^{(k),n} / \Pi_1^{(k),n}$. Since the exponent of $G_g^{(k),n}$ divides $n$, we have that for every node $x$ of a curve $C$ underlying a twisted $G_g^{(k),n}$-cover, the order of $\Gamma_x$ divides $n$.

The following is a restatement of a result of Pikaart and De Jong ([P-J], Theorem 3.1.3 part (4)).

**Proposition 7.4.5.** Let $G = G_g^{(k),n}$ with $k \geq 4$ and $\gcd(n, 6) = 1$. Then the group homomorphism $M \to \text{Out}(G)$ is injective.

As an immediate outcome we have

**Theorem 7.4.6.** Suppose $k \geq 4$ and $\gcd(n, 6) = 1$. Then the moduli stack $\mathcal{B}_{\text{stei}}^{G_g^{(k),n}}$ is a smooth projective scheme over $\mathbb{Z}[1/n]$ admitting a flat finite morphism to $\overline{\mathcal{M}}_g$.

We remark that, in this case, the space $\mathcal{B}_{\text{stei}}^{G_g^{(k),n}}$ coincides with the space $G\overline{\mathcal{M}}_g$ (and the stack $G\overline{\mathcal{M}}_g$) of Pikaart and De Jong ([P-J], Section 2.3.5).

7.5. A fine moduli space of $G$ covers. We are now ready to describe a particular type of finite group $G$ for which the stack of connected admissible $G$-covers is representable; in particular the center of $G$ is trivial. The construction is closely related to that of Looijenga [L2], and we rely on some of his arguments in our proofs.

We start with some auxiliary constructions.

Let $p_1, p_2$ be two distinct primes. Fix a smooth complex curve $C$ of genus $g > 1$. Let $C_{p_1} \to C$ be the maximal finite abelian étale cover of exponent $p_1$. The Galois group $G_{p_1}$ of $C_{p_1}$ over $C$ is $H_1(C, \mathbb{Z}/p_1 \mathbb{Z})$, which is isomorphic to $(\mathbb{Z}/p_1 \mathbb{Z})^{2g}$. The genus $g_{p_1}$ of $C_{p_1}$ is $p_1^{2g} (g - 1) + 1$. Let $C_{[p_2]}^{[p_1]} \to C_{p_1}$ be the maximal finite abelian étale cover of exponent $p_2$. The Galois group $H_{[p_2]}^{[p_1]}$ of $C_{[p_2]}^{[p_1]} \to C_{p_1}$ is $H_1(C_{p_1}, \mathbb{Z}/p_2 \mathbb{Z})$, which is isomorphic to $(\mathbb{Z}/p_2 \mathbb{Z})^{2g_{p_1}}$. The Galois group $G_{[p_2]}^{[p_1]}$ of the characteristic cover $C_{[p_2]}^{[p_1]} \to C$ sits in an exact sequence:

$$1 \to H_{[p_2]}^{[p_1]} \to G_{[p_2]}^{[p_1]} \to G_{p_1} \to 1.$$}

Since $p_1 \neq p_2$, this is in fact a semidirect product. Moreover, there is a canonical $G_{p_1}$-equivariant direct sum decomposition

$$H_{[p_2]}^{[p_1]} = H_1^{[p_2]} \oplus H_2^{[p_1]},$$

where the first factor consists of the $G_{p_1}$-invariants, and the second of non-invariants. Explicitly, for $h \in H_{[p_2]}^{[p_1]}$, we have the decomposition $h = h^i + h^n$ where $h^i = \frac{1}{p_1} \sum_{g \in G_{p_1}} h^g$. The subgroup $H_2^{[p_1]}$ clearly lies in the center of $G_{[p_2]}^{[p_1]}$. Denote the quotient by $G_{[p_2]}^{[p_1]}$, and the corresponding covering

$$C_{[p_2]}^{[p_1]} \to C.$$

**Lemma 7.5.1.**

1. The action of $G_{p_1}$ on $H_{[p_2]}^{[p_1]}$ is effective.

2. The center of $G_{[p_2]}^{[p_1]}$ is $H_1^{[p_2]}$, and the center of $G_{[p_2]}^{[p_1]}$ is trivial.

3. The group $H_2^{[p_1]}$ is canonically isomorphic to $G_{p_2}$. 
4. We have a diagram with cartesian squares:

\[
\begin{array}{ccc}
C^{[p_2]}_{[p_1]} & \rightarrow & C_{p_1} \times C_{p_2} \rightarrow C_{p_2} \\
\downarrow & & \downarrow \\
C^{n}_{[p_2]} & \rightarrow & C_{p_1} \rightarrow C
\end{array}
\]

Proof. Clearly \(H^n_{[p_2]} \subseteq G^{[p_2]}_{[p_1]}\) is a normal subgroup, and the quotient is abelian with exponent \(p_1 p_2\). Thus its \(p_2\)-Sylow subgroup \(H^n_{[p_2]}\) has order \(\leq p_2^{2g}\). This implies that \(H^n_{[p_2]}\) is nontrivial. In particular this means that some element of \(G_{p_1}\) acts nontrivially on \(H^{[p_2]}_{[p_1]}\).

Since the covering is characteristic, the automorphisms group of the fundamental group of \(C\) acts on \(G^{[p_2]}_{[p_1]} \rightarrow G_{p_1}\). This automorphism group contains the Teichmüller group, and it is well known\(^{12}\) that the image of this group in \(\text{Aut} G_{p_1}\) is the symplectic group, which is transitive on nonzero elements. It follows that every nonzero element of \(G_{p_1}\) acts nontrivially on \(H^{[p_2]}_{[p_1]}\), giving (1). It also follows that an element of the center maps to \(0 \in G_{p_1}\), therefore it is in \(H^{[p_2]}_{[p_1]}\), and since it commutes with \(G_{p_1}\) it is in \(H^{[p_2]}_{[p_1]}\), giving (2).

The curve \(C_{p_1} \times_C C_{p_2}\) is an abelian connected \(G_{p_2}\)-covering of \(C_{p_1}\), which means that \(H^{[p_2]}_{[p_1]}\) has order \(\geq p_2^{2g}\). Combined with the previous inequality we get equality, giving (3) and (4).

By switching the roles of \(p_1\) and \(p_2\) we can extend the fiber diagram as follows:

\[
\begin{array}{ccc}
C_{(p_1,p_2)} & \rightarrow & C^{[p_2]}_{[p_1]} \rightarrow C^{n}_{[p_2]} \\
\downarrow & & \downarrow \\
C'_{[p_2]} & \rightarrow & C_{p_1} \times C_{p_2} \rightarrow C_{p_2} \\
\downarrow & & \downarrow \\
C^{n}_{[p_2]} & \rightarrow & C_{p_1} \rightarrow C
\end{array}
\]

The resulting curve \(C_{(p_1,p_2)}\) is connected since the degrees in the top left square are relatively prime. We denote the Galois group of the covering \(C_{(p_1,p_2)} \rightarrow C\) by \(G_{(p_1,p_2)}\). It is clearly a characteristic quotient of the fundamental group, isomorphic to \(G^n_{[p_2]} \times G^n_{[p_1]}\). Its exponent is \(p_1 p_2\).

We use the following Proposition:

**Proposition 7.5.2.** Let \(C\) be a stable curve of genus \(g > 1\) and let \(C_{p_1} \rightarrow C\) be a connected admissible \(G_{p_1}\)-cover. Then no two nodes of \(C_{p_1}\) separate the curve.

*Proof. The argument of Looijenga in [Lo], Proposition 2, works word for word.*

**Lemma 7.5.3.** Let \(C\) be a stable curve of genus \(g > 1\) and let \(C_{(p_1,p_2)} \rightarrow C\) be a connected admissible \(G_{(p_1,p_2)}\)-cover. Then at every node of \(C\), the covering has index \(p_1 p_2\).

*Proof. The Proposition shows in particular that every node of \(C_{p_1}\) is nonseparating. Our Proposition 6.12 implies that the cover \(C^{[p_2]}_{[p_1]} \rightarrow C_{p_1}\) has index \(p_2\). Similarly \(C^{n}_{[p_1]} \rightarrow C_{p_2}\) has index \(p_1\). This implies that \(p_1 p_2\) divides the index of \(C_{(p_1,p_2)} \rightarrow C\). On the other hand the index divides the exponent \(p_1 p_2\) of \(G_{(p_1,p_2)}\), giving equality.*

\(^{12}\)Precise citation needed!
Lemma 7.5.4. Let $C$ be a stable curve of genus $g > 1$ and let $C_{p_1,p_2} \to C$ be a connected admissible $G_{(p_1,p_2)}$ cover. Consider the homomorphism $\partial : \prod_{e \in C_{\text{sing}}} \mathbb{Z} \to \text{Out}(G_{(p_1,p_2)})$ via Dehn twists. If the image of an element $u \in \prod_{e \in C_{\text{sing}}} \mathbb{Z}$ in $\text{Out}(G_{(p_1,p_2)})$ is trivial, then $u \in p_1p_2 \prod_{e \in C_{\text{sing}}} \mathbb{Z}$.

Proof. Denote by $E_0$ the set of separating nodes of $C$ and by $E_1$ the nonseparating nodes. We now follow the notation of [Lo]: we write $e \in C_{\text{sing}}$ for a node of $C$ (instead of $x$ used earlier); for such a node we denote the corresponding component of $u$ by $u_e$; a node of $C_{p_1}$ is denoted $\hat{e}$, and we indicate the condition that it lie over $e$ by $\hat{e}/e$. We denote by $[\hat{e}]$ the class of the corresponding vanishing cycle in $H_1(C_{p_1}^*, \mathbb{Z})$ (with some choice of orientation), where $C_{p_1}^*$ is a nearby smooth fiber in a deformation of $C_{p_1}$.

According to Looijenga’s discussion in [Lo], p. 287-288 the action of the element $p_1u$ on $v \in H_1(C_{p_1}^*, \mathbb{Z})$ is

$$v \mapsto v + \sum_{e \in E_0} p_1u_e \sum_{\hat{e}/e} (v, [\hat{e}])\hat{e} + \sum_{e \in E_1} u_e \sum_{\hat{e}/e} (v, [\hat{e}])\hat{e}.$$ 

If $\partial u$ is trivial, then $u$ acts on $H_1(C_{p_1}^*, \mathbb{Z}/p_2\mathbb{Z})$ as an element of $G_{p_1}$, therefore $p_1u$ acts trivially on $H_1(C_{p_1}^*, \mathbb{Z}/p_2\mathbb{Z})$. Proposition 7.3.2 says that Baclawski’s Lemma ([Lo], p. 286) applies, therefore $p_2|_{p_1u_e}$ for $e \in E_0$ and $p_2|_{u_e}$ for $e \in E_1$. Since $p_2 \neq p_1$ it follows that $p_2|_{u_e}$ for all $e$, i.e. $u \in p_2 \prod_{e \in C_{\text{sing}}} \mathbb{Z}$. Reversing the roles of $p_1$ and $p_2$ we get the Lemma.

Theorem 7.5.5. The automorphism group of any connected admissible $G_{(p_1,p_2)}$ cover is trivial.

Proof. By Lemma 7.3.3 we have a surjection $\delta : \prod_{e \in C_{\text{sing}}} \mathbb{Z} \to M$ compatible with the action by outer automorphisms, in other words, the homomorphism $\delta : \prod_{e \in C_{\text{sing}}} \mathbb{Z} \to \text{Out}(G_{(p_1,p_2)})$ factors through $\delta$. Thus

$$\text{Ker } \delta \subset \text{Ker } \partial.$$ 

By Lemma 7.5.3 we have

$$\text{Ker } \delta = p_1p_2 \prod_{e \in C_{\text{sing}}} \mathbb{Z}.$$ 

By lemma 7.5.4 we have that

$$\text{Ker } \partial \subset p_1p_2 \prod_{e \in C_{\text{sing}}} \mathbb{Z}.$$ 

Combining the statements we get equality. Thus the automorphism group of the cover is the center of the group, which by Lemma 7.5.1 is trivial.

Remark 7.5.6. We note that similar results can be obtained for covers of pointed curves, for instance using the reduction methods of [BE].

Appendix A. Some remarks on étale cohomology of Deligne–Mumford stacks

When we refer to a sheaf on a stack or algebraic space $\mathcal{M}$, we will always mean a sheaf in the small étale site of $\mathcal{M}$, whose objects are étale morphism locally of finite type $U \to \mathcal{M}$, where $U$ is a scheme.

In final version we should add the example of a connected $G$-cover with $Z(G)$ trivial and $A^G$ nontrivial. Maybe also discuss infinitely twisted curves.
Proposition A.0.7. Let $\mathcal{M}$ be a separated tame finitely presented Deligne–Mumford stack over a scheme, with moduli space $\pi : \mathcal{M} \to M$. Let $p : \text{Spec} \Omega \to \mathcal{M}$ be a geometric point of $\mathcal{M}$, $\Gamma$ its stabilizer, $q = \pi \circ p : \text{Spec} \Omega \to M$ its image in $M$. Let $F$ be a sheaf on $\mathcal{M}$; then there is a canonical isomorphism of groups between the stalk $(R^i\pi_*F)_q$ of the $i$th higher direct image sheaf of $F$ at $p$ with the $i$th cohomology group $H^i(\Gamma, F_p)$.

Proof. This statement is local in the étale topology on $M$, so we can make an étale base change and assume that $\mathcal{M}$ is a quotient $[U/\Gamma]$, where $U$ is a connected scheme and $p$ is a geometric point which is fixed by $\Gamma$; then the sheaf $F$ is a $\Gamma$-equivariant sheaf on $U$, and $M = U/\Gamma$. Let $\text{Spec} \Omega \to V \to U/\Gamma$ be an étale neighborhood of $q$ in $U/\Gamma$; then $[V \times_{U/\Gamma} U/\Gamma] = V \times_{U/\Gamma} [U/\Gamma]$. There is a spectral sequence

$$E^{ij}_2 = H^i(\Gamma, H^j(V \times_{U/\Gamma} U, F)) \implies H^{i+j}(V \times_{U/\Gamma} [U/\Gamma], F)$$

which is the Čech-to-global cohomology spectral sequence for the covering $U \to [U/\Gamma]$, as in $[\text{Mi}]$, Proposition 2.7, with the same proof. Now let us go to the limit over all étale neighborhoods $\text{Spec} \Omega \to V \to U/\Gamma$; since $\Gamma$ is a finite group, its cohomology groups commute with direct limits, so we have

$$\lim_V H^j(\Gamma, H^j(V \times_{U/\Gamma} U, F)) = H^j(\Gamma, \lim_V H^j(V \times_{U/\Gamma} U, F)).$$

But $\lim_V H^j(V \times_{U/\Gamma} U, F)$ is 0 for $j > 0$ and is $F_p$ for $j = 0$, while the limit of the abutment of the spectral sequence is precisely $(R^i\pi_*F)_q$.

Now we prove the proper base change theorem for tame Deligne–Mumford stacks.

A sheaf $F$ on a stack $\mathcal{M}$ is torsion if for any étale morphism $U \to \mathcal{M}$ from a quasicompact scheme $U$ the group $F(U)$ is torsion.

Theorem A.0.8. Let $f : \mathcal{M} \to S$ be a proper morphism from a tame Deligne–Mumford stacks to a scheme, and let

$$\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{\psi} & \mathcal{M} \\
\downarrow f' & & \downarrow f \\
S' & \xrightarrow{\phi} & S
\end{array}$$

be a cartesian diagram. Let $F$ be a torsion sheaf on $\mathcal{M}$. Then the natural base change homomorphism of sheaves $\phi^* R^i f_* F \to R^i f'_* \psi^* F$ is an isomorphism.

Proof. When $\mathcal{M}$ is a scheme, this is the usual proper base change theorem for étale cohomology, as in $[\text{Mi}]$, Corollary 2.3. This also holds when $\mathcal{M}$ is an algebraic space, with the same proof. We will reduce the general case to the case of algebraic spaces.

First of all, if $S$ is the moduli space of $\mathcal{M}$, the statement follows easily from A.0.7. In general, factoring through the moduli spaces we get a cartesian diagram

$$\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{\psi} & \mathcal{M} \\
\downarrow \pi' & & \downarrow \pi \\
M' & \xrightarrow{\rho} & M \\
\downarrow g' & & \downarrow g \\
S' & \xrightarrow{\phi} & S
\end{array}$$

such that $g \circ \pi = f$ and $g' \circ \pi' = f'$. The base change formula holds for $pi$ and $g$, so we have

$$\phi^* R^i g_* R^j \pi_* F = R^i g'_* \rho^* R^j \pi'_* F = R^i g'_* R^j \pi'_* \psi^* F.$$
We also have a morphism of $E_2$ spectral sequences
\[
\phi^* g_i^* R^j \pi_* F \Rightarrow \phi^* R^{i+j} f_* F
\]
\[
R^i g_i^* R^j \pi_* ^* F \Rightarrow R^{i+j} f'_* \psi^* F
\]
where the columns are base change maps. Since the left hand column is an isomorphism, so is the right hand column.

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