Modeling and simulation of rotary-rotary planer inverted pendulum

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Abstract. There exist different mechanical structures of inverted pendulum system. This paper presents mathematical modeling of rotary-rotary-planer inverted pendulum system (R-R-P inverted pendulum system), which is a highly nonlinear unstable system. In this planer inverted pendulum configuration, the pendulum is attached to its rotary-rotary actuating base with a pin joint. This configuration of planer inverted pendulum is taken into account, as the configuration can best describe the balancing of broomstick in one’s hand, by considering the shoulder and elbow as revolute joints and such configuration will also help in the study of underactuated robotic systems. The dynamical equation of R-R-P inverted pendulum is derived by using Lagrangian equation of motion. To validate the mathematical model, simulation of nonlinear mathematical model of the system is performed via MATLAB Simulink.

1. Introduction
The dynamics of a majority of multilink robotic manipulators are basically the dynamical equation of a number of linked pendula, therefore, modeling, stabilization in the vertically upward position and trajectory tracking of an inverted pendulum, which is eminently nonlinear and unstable in nature, is an eminent challenge for the modern control domain. The dynamics of balancing a pendulum can also be employed in applications concerning controlling of rocket thruster, aerial systems, marine systems, space vehicles, humanoid robot and human transporter segways [1].

The mechanical structure of inverted pendulum is based on two factors, the actuation method and degree of freedom (DOF) of the system. The actuation methods are linear and rotary. The simplest controllable structure of inverted pendulum system has two DOF, one for the base position and other for the pendulum angle. Examples of two DOF inverted pendulum systems are rotary inverted pendulum system and cart inverted pendulum system [2-3]. In case of cart inverted pendulum system the pendulum is attached to the cart by a pin joint and is free to rotate in the vertical plane and a motor is used to drive the cart of the system in the horizontal plane [4-9]. Rotary inverted pendulum (RIP) system is essentially a combination of three elements: a motor, an arm and a pendulum. In RIP one end of the arm is connected to the shaft of the motor and pendulum is attached to other end of the arm by a pin joint that is allowed to rotate freely in the vertical plane [10-13]. For higher degree of freedom inverted pendulum system configurations either the base is enabled to move in multiple dimensions or more links are added to the pendulum. Configurations, where the base is allowed to move in multiple directions, must then be actuated in two degree of freedom, which forms a horizontal plane and called planer inverted pendulum system [2, 14]. There can be three possible mechanical structure of planer inverted pendulum system: planer inverted
pendulum with linear-linear actuation, with rotary-rotary actuation and with a combination of linear and rotary actuation.

The literature covers plenty of papers on the modeling and construction of RIP and cart inverted pendulum, however very few are associated with the modeling of planer inverted pendulum system. Modeling using Lagrangian equation of motion of cart inverted pendulum system has been presented in [3-6]. Newton law based modeling of cart inverted pendulum system has been derived in [7]. In [11] the dynamics of the rotational inverted pendulum or Furuta pendulum incorporating full inertia tensor have been derived. To derive the nonlinear dynamics of RIP two techniques: a Lagrangian equation of motion and an iterative Newton-Euler formulation have been used. M. A. Cruz et al. in [12-13] presented, modeling of the Furuta pendulum followed by its construction. To design the mathematical model Lagrangian equation of motion has been used, and the software SolidWorks has been used to carry out the mechanical design of the Furuta pendulum. R. J. Wai and L.J. Chang [14] proposed, the nonlinear model of dual axis inverted pendulum system that includes dynamics of the actuator via the energy conservation principle. Theoretical analysis of a robot balancing a 2 Degree of freedom pendulum followed by its practical realization have been described in [15] by B. Spronger, L. Kucera and S. Mourad. That pendulum has been mounted on a SCARA robot with the help of a decoupled or weak coupled links. Some linear and nonlinear control laws have been implemented to stabilize, rotary inverted pendulum in vertically upward position in [16-21]. Elimination of limit cycle of Furuta pendulum has been presented in [22].

Having reviewed the literature, it has been found that there are very few authors who address the modeling of planer inverted pendulum. Thus to give contribution in this direction, In this paper the nonlinear mathematical modeling of rotary-rotary-planer inverted pendulum system (R-R-P inverted pendulum system), where the pendulum is attached with its rotary-rotary actuating base with a pin joint has been proposed, unlike [15], where the pendulum is mounted on the base with decoupled links. This configuration of planer inverted pendulum is taken into account, as the configuration can best describe the balancing of broomstick in one’s hand, by considering the shoulder and elbow as revolute joints and such configuration will also help in the study of underactuated robotic systems. The dynamical equation of R-R-P inverted pendulum is derived by using Lagrangian equation of motion. To validate the mathematical model, numerical simulation of nonlinear model of the system is performed via MATLAB Simulink.

The remaining organization of the paper is as follows. In section 2, the structure of the R-R-P inverted pendulum is presented, section 3 deal with the mathematical modeling of the system using Lagrangian equation of motion. The simulation results of the model are presented in section 4. Finally, in section 5 some concluding remarks are given.

2. Structure of R-R-P Inverted Pendulum System

The structure of rotary-rotary-planer inverted pendulum is presented in figure 1. The structure basically consists, three main elements: arm 1, arm 2 and pendulum. Arm 1 is connected with the shaft of a d.c. motor and allowed to move in the horizontal plane. Arm 1 and arm 2 are fastened with revolute joint and a servo motor can be attached to the joint to provide a torque to rotate it in horizontal plane. The pendulum is linked with arm 2 by a pin joint and is free to move in vertical plane.

In the figure 2, $q_1$ is the arm 1 angular position and it is measured from an arbitrary position, $q_2$ is the arm 2 angular position measured with respect to arm 1 position and $q_3$ is pendulum position measured with respect to vertical upright position. All the angular displacement is taken positive in counterclockwise direction. $m_1$, $m_2$ and $m_3$ are masses of arm 1, arm 2 and pendulum respectively. The two arms have length $L_1$, $L_2$ and pendulum has length $L_3$. The lengths of arm 1, arm 2 and pendulum, from the point of rotation of the arms and pendulum to there center of mass are $l_1$, $l_2$ and $l_3$. 
Electric motors are used to apply a torque $\tau_1$ and a torque $\tau_2$ to arm 1 and arm 2. The link between arm 2 and pendulum is not actuated and free to rotate. $J_1$, $J_2$ and $J_3$ are inertia tensors of arm 1, arm 2 and pendulum (about its center of masses) respectively. The viscous damping coefficients of the joints are $b_1$, $b_2$ and $b_3$.

### 3. Lagrangian formulation of R-R-P Inverted Pendulum

The R-R-P inverted pendulum is a three degree of freedom (DOF) system and the Lagrangian formulation of the system dynamics is governed by three Lagrange equation of motion as follows [23]:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q_1} \right) - \frac{\partial L}{\partial q_1'} = \tau_1 - b_1 \dot{q}_1$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q_2} \right) - \frac{\partial L}{\partial q_2'} = \tau_2 - b_2 \dot{q}_2$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q_3} \right) - \frac{\partial L}{\partial q_3'} = -b_3 \dot{q}_3$$

where $L$ is the Lagrangian of the system and it is difference in kinetic and potential energies,

$$L = E_k - E_p$$

being $E_k$ and $E_p$ the kinetic energy and potential energy of the R-R-P inverted pendulum system respectively.

The kinetic energy $E_k$ is the summation of kinetic energies of the two arms and the pendulum, and $E_p$ is the potential energies of the two arms and the pendulum. Since arm 1 arm 2 are moved on horizontal plane their potential energies are constant and therefore assumed to be zero, i.e.

$$E_{p1} = E_{p2} = 0$$

From figure 3-(b), the potential energy of the pendulum is given by

$$E_{p3} = m_3 g l_3 (\cos(q_3) - 1)$$

Combining the potential energies of arm 1, arm 2 and pendulum, the total potential energy of the system is given by

$$E_p = E_{p1} + E_{p2} + E_{p3}$$

The kinetic energy of arm 1 is,

$$E_{k1} = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} J_1 \dot{q}_1^2$$
where \( v_1 \) is the linear velocity of the arm 1 center of mass and hence, an analysis of the R-R-P inverted pendulum system kinematics is needed to study. From Figure 3, center of mass location of arm 1 is defined by

\[
\mathbf{O}_1 = [x_1, y_1, z_1]^T
\]

(9)

where \( x_1, y_1 \) and \( z_1 \) are defined by using free body diagram presented in Figure 3(a) as follows:

\[
x_1 = l_1 \cos(q_1), \quad y_1 = l_1 \sin(q_1), \quad z_1 = 0.
\]

Thus, \( v_1 \) is given by

\[
v_1 = [x_1, y_1, z_1]^T
\]

(10)

being

\[
x_1 = -q_1 l_1 \sin(q_1), \quad y_1 = q_1 l_1 \cos(q_1), \quad z_1 = 0.
\]

Substitution of equation (10) in equation (8) and arranging the expression, gives:

\[
E_k = \frac{1}{2} m_1 l_1^2 \dot{q}_1^2 + \frac{1}{2} J_1 \dot{q}_1^2
\]

(11)

Figure 3. R-R-P Inverted Pendulum free body diagram. (a) Projection of the system in horizontal plane. (b) Projection of the system in vertical plane.

The kinetic energies of arm 2 and pendulum are denoted by \( E_{k2} \) and \( E_{k3} \), and is derived in the similar way as we have derived it for arm 1. The obtained expressions for them are:

\[
E_{k2} = \frac{1}{2} m_2 l_2^2 \dot{q}_2^2 + \frac{1}{2} m_2 l_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2 L_1 l_2 \dot{q}_1 \dot{q}_2 \cos(q_2) + \frac{1}{2} J_2 (\dot{q}_1 + \dot{q}_2)^2
\]

(12)

\[
E_{k3} = \frac{1}{2} l_3^2 \dot{q}_3^2 + \frac{1}{2} m_3 [L_1^2 \dot{q}_1^2 + l_3^2 \dot{q}_3^2 + (\dot{q}_1 + \dot{q}_2)^2 l_3^2 \sin^2(q_3) + (\dot{q}_1 + \dot{q}_2)^2 L_2^2 

-2 \dot{q}_1 \dot{q}_2 l_3 L_2 \cos(q_3) - 2 \dot{q}_1 \dot{q}_3 \cos(q_2) \cos(q_3) + 2 \dot{q}_1 \dot{q}_2] \]

(13)

Combining the kinetic energies of arm 1, arm 2 and pendulum, the total kinetic energy of the system is given by

\[
E_k = E_{k1} + E_{k2} + E_{k3}
\]

(14)

Substituting the values of kinetic and potential energy in equation (4) from previous terms, and reducing the resulting expression, gives the expression of Lagrangian as:

\[
\mathcal{L} = \frac{1}{2} m_1 \dot{q}_1^2 \left[ J_1 + m_1 l_1^2 \right] + \frac{1}{2} l_2 (\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2} m_2 [L_1^2 \dot{q}_1^2 + l_2^2 (\dot{q}_1 + \dot{q}_2)^2]
\]
\[ M_{11} = f_1 + f_2 + 2m_lL_1^2 + m_2L_2^2 + m_3L_3^2 + 2m_2L_1L_2 \cos(q_2) + 2m_3L_3^2 \sin^2(q_3) + 2m_3L_1L_3 \sin(q_2) \sin(q_3) + 2m_3L_1L_2 \cos(q_2), \]
\[ M_{12} = f_3 + 2m_2L_1L_2 \sin(q_2) + q_3m_3L_3^2 \sin(2q_2) + 2q_2m_3L_3L_1 \cos(q_2) \sin(q_3), \]
\[ M_{22} = f_3 + 2m_2L_1L_2 \sin(q_2) + q_3m_3L_3^2 \sin(2q_2) - 2q_2m_3L_3L_1 \cos(q_2) \sin(q_3), \]
\[ M_{33} = 2m_3L_3L_1 \sin(q_3) + q_3m_3L_3L_1 \sin(q_3) \cos(q_3), \]
\[ C_{11} = -2q_2m_2L_1L_2 \sin(q_2) + q_3m_3L_3^2 \sin(q_2) \sin(q_3) - 2q_2m_3L_3L_1 \sin(q_2) \cos(q_3), \]
\[ C_{12} = q_3m_3L_3L_1 \cos(q_3) \sin(q_3) - q_2m_3L_3L_1 \sin(q_2), \]
\[ C_{22} = q_3m_3L_3^2 \sin(q_2), \]
\[ C_{33} = q_3m_3L_3L_1 \sin(q_3) + q_3m_3L_3L_1 \sin(q_3) \cos(q_3), \]
\[ G_1 = 0, \quad G_2 = 0, \quad G_3 = -m_3gL_3 \sin(q_3), \quad F_{v1} = b_1q_1, \quad F_{v2} = b_2q_2, \quad F_{v3} = b_3q_3. \]

4. Simulation of R-R-P Inverted Pendulum model

In this section, the nonlinear dynamical equation of the R-R-P inverted pendulum system, that is, (16), has been simulated in MATLAB environment. By simulation, the system variables, \( q_1 \), \( q_2 \) and \( q_3 \) will be estimated, which will allow to study the behavior of the R-R-P inverted pendulum system.

Keeping in mind the parameters of basic RIP system, experimental set up of which is widely available in labs, the values of the parameters of the R-P inverted pendulum are assumed and given in Table 1.
Table 1. Numerical Parameters of R-R-P Inverted Pendulum System.

| Symbol | Value     | Units  |
|--------|-----------|--------|
| $g$    | 9.81      | m/s$^2$|
| $L_1$  | 0.1414    | M      |
| $L_2$  | 0.09267   | M      |
| $L_3$  | 0.56      | M      |
| $l_1$  | $L_1/2$   | M      |
| $l_2$  | $L_2/2$   | M      |
| $l_3$  | $L_3/2$   | M      |
| $m_1$  | 0.065     | Kg     |
| $m_2$  | 0.04259   | Kg     |
| $m_3$  | 0.038     | Kg     |
| $J_1$  | $1.083 \times 10^{-4}$ | kg.m$^2$ |
| $J_2$  | $3.0479 \times 10^{-5}$ | kg.m$^2$ |
| $J_3$  | $9.9307 \times 10^{-4}$ | kg.m$^2$ |
| $b_1$  | 0.0001    | Nms    |
| $b_2$  | 0.00018   | Nms    |
| $b_3$  | 0.00028   | Nms    |

![Graphs](a), (b), (c), (d)
In the numerical simulation initially, the pendulum is considered in the vertically upward position and represented by 0 radian line in figure 4-e. A torque of 0.17 Nm is applied to the actuating arm 1 and a torque of 0.13 Nm is applied to the actuating arm 2 of the system (i.e. $\tau_1 = 0.17 \text{Nm}$ and $\tau_2 = 0.13 \text{Nm}$), for a time period of 1 sec. The application of the torque moves the arms from an arbitrary position and brings the pendulum from its vertically upward position to a position around 1.57 radian. The simulation results are presented in Figure 4.

Figure 4(a) and 4(b) represent the driving torques to rotating arm 1 and arm 2 of the system. In figure 4(c) and 4(d) the movement of arm 1 and arm 2 with application of the torque is presented. As there is no any stabilization control acting on the pendulum, the movement of arms bring the pendulum from its vertical unstable equilibrium position to a position around 1.57 radian and is presented in figure 4(e).

5. Conclusion

Mathematical modeling and simulation of rotary-rotary-planer inverted pendulum system have been presented in this paper. The mathematical modeling is derived using Lagrangian equation of motion.

To visualize the position of arms and pendulum in the derivation of mathematical model, projection in the horizontal plane and vertical plane have been presented geometrically.

Simulation of the nonlinear model has been carried out using Matlab-Simulink. The pendulum position of this planer inverted pendulum system is observed same as in case of rotary inverted pendulum system, presented in [11], and therefore it ensures the successfulness of the mathematical modeling of the R-R-P inverted pendulum system derived in this paper.

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