Enriched purity and presentability in Banach spaces

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ABSTRACT

The category $\text{Ban}$ of Banach spaces and linear maps of norm $\leq 1$ is locally $\mathbb{N}_1$-presentable but not locally finitely presentable. We prove, however, that $\text{Ban}$ is locally finitely presentable in the enriched sense over complete metric spaces. Moreover, in this sense, pure morphisms are just ideals of Banach spaces. We characterize classes of Banach spaces approximately injective with respect to sets of morphisms having finite-dimensional domains and separable codomains.

1. Introduction

A model theoretical approach to Banach spaces (and other structures from functional analysis) is well established (see, e.g., [14]). Our aim is to place this approach into the framework of the theory of locally presentable and accessible categories. The first steps in this direction were done in [27] and [4]. Accessible categories were introduced by M. Makkai and R. Paré [23] as a foundation of categorical model theory. Their theory was further developed in [3] where, in particular, the importance of injectivity and purity have been understood. To deal with structures from functional analysis, we need this theory to be enriched over complete metric spaces. For instance, [4] showed that an analytical concept of approximate injectivity coincides with enriched injectivity. [4] also showed that finite-dimensional Banach spaces are finitely generated in the enriched sense; here, morphisms of Banach spaces are linear maps of norm $\leq 1$. This provides a new perspective to the construction of a Gurarii space.

Our first result improves [4] by showing that finite-dimensional Banach spaces are even finitely presentable in the enriched sense. This implies that the category of Banach spaces is locally finitely presentable as an enriched category. Let us add that, as an ordinary category, it is only locally $\mathbb{N}_1$-presentable. We then prove that pure morphisms of Banach spaces in the enriched sense coincide with the well-established concept of ideals of Banach spaces [25]. We also show that pure morphisms have the same model-theoretical meaning as pure morphisms in accessible categories when we use the logic of positive bounded formulas of [13, 14]. This complements the results of [27] where purity in metric enriched categories was introduced. Using purity, we characterize classes of Banach spaces approximately injective with respect to sets of morphisms having finite-dimensional domains and separable codomains.

2. Preliminaries

We denote by $\text{CMet}$ the category of complete metric spaces and nonexpanding maps as morphisms where we allow distances to be $\infty$. This category is symmetric monoidal closed where the tensor product $X \otimes Y$ puts the $+\text{-metric}$

$$d((x, y), (x', y')) = d(x, x') + d(y, y')$$
on the product $X \times Y$. The internal hom provides the hom-set $\text{CMet}(X, Y)$ with the sup-metric $d(f, g) = \sup\{d(fx, gx) \mid x \in X\}$. Moreover, $\text{CMet}$ is locally $\aleph_1$-presentable (see [4, 2.3(2)]). A category $\mathcal{K}$ is enriched over $\text{CMet}$ if hom-sets $\mathcal{K}(A, B)$ are complete metric spaces and the composition maps $\mathcal{K}(B, C) \otimes \mathcal{K}(A, B) \to \mathcal{K}(A, C)$ are nonexpanding. Our principal examples are the category $\text{CMet}$ itself and the category $\text{Ban}$ of Banach spaces and linear maps of norm $\leq 1$. A functor $F : \mathcal{K} \to \mathcal{L}$ between $\text{CMet}$-enriched categories is enriched if the mapping $\mathcal{K}(A, B) \to \mathcal{L}(FA, FB)$ is nonexpansive for all objects $A$ and $B$ in $\mathcal{K}$. An adjunction $U \dashv F$ is enriched if $\mathcal{L}(FK, L) \cong \mathcal{K}(K, UL)$ is an isomorphism of metric spaces for all objects $K$ in $\mathcal{K}$ and $L$ in $\mathcal{L}$. Consult [3] for the concept of a locally presentable category and [15] for the theory of enriched categories. Enriched categories over $\text{CMet}$ were studied in [27] and [4] from where we recall the following concepts.

Given morphisms $f, g : A \to B$ in a $\text{CMet}$-enriched category $\mathcal{K}$ and $\varepsilon \geq 0$, we say that $f \sim_{\varepsilon} g$ if $d(f, g) \leq \varepsilon$ in the metric space $\mathcal{K}(A, B)$. An $\varepsilon$-commutative square

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{f} D
\end{array}
\]

is a square such that $\bar{f} g \sim \varepsilon \, \bar{g} f$.

**Definition 2.1.** [27, 2.2] Let $\varepsilon \geq 0$. An $\varepsilon$-commutative square

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{f} D
\end{array}
\]

is called an $\varepsilon$-pushout if for every $\varepsilon$-commutative square

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{f'} D'
\end{array}
\]

there is a unique morphism $t : D \to D'$ such that $t \bar{f} = f'$ and $t \bar{g} = g'$.

0-commutative squares are commutative squares and pushouts are 0-pushouts.

**Remark 2.2.** An $\varepsilon$-pushout in $\text{Ban}$ is a square

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{f} D
\end{array}
\]
where \( D \) is the coproduct \( B \oplus C \) endowed with the norm
\[
\| (x, y) \| = \inf \{ \| b \| + \| c \| + \varepsilon \| a \| ; x = b + f(a), y = c - g(a) \}
\]
(see [7, 6.5]). Hence an \( \varepsilon \)-pushout of finite-dimensional (or separable) Banach spaces is finite-dimensional (separable resp.).

**Remark 2.3.** (1) A morphism \( f : A \to B \) in a \textbf{CMet}-enriched category \( \mathcal{K} \) is called an isometry if for every \( u, v : C \to A \) we have \( d(u, v) = d(fu, fv) \). Isometries in \textbf{CMet} or \textbf{Ban} are the usual isometries.

(2) Let \( \mathcal{K} \) have pushouts. We say that isometries in \( \mathcal{K} \) are stable under pushouts if

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow g & & \downarrow g \\
C & \xrightarrow{f} & D
\end{array}
\]
is a pushout and \( f \) an isometry then \( \bar{f} \) is an isometry. Isometries in \textbf{Ban} are stable under pushouts (see [5, A.19]).

Following [4, 3.19]), isometries in \textbf{Ban} are also stable under \( \varepsilon \)-pushouts.

(3) A morphism \( f : A \to B \) in a \textbf{CMet}-enriched category \( \mathcal{K} \) is called an \( \varepsilon \)-isometry provided that there are isometries \( g : B \to C \) and \( h : A \to C \) such that \( gf \sim_\varepsilon h \). If \( \mathcal{K} \) have \( \varepsilon \)-pushouts for all \( \varepsilon \geq 0 \) and isometries are stable under pushouts then, following [4, 3.22], \( f : A \to B \) is an \( \varepsilon \)-isometry if and only if in the following \( \varepsilon \)-pushout

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \text{id}_A & & \downarrow f_\varepsilon \\
A & \xrightarrow{f} & P
\end{array}
\]
\( \bar{f} \) is an isometry. (The stability of isometries under pushouts is missing in [4, 3.22]).

**Lemma 2.4.** Let \( \mathcal{K} \) be a \textbf{CMet}-enriched category with \( \varepsilon \)-pushouts for all \( \varepsilon \geq 0 \). Assume that isometries in \( \mathcal{K} \) are stable under pushouts. Then \( f \) is an \( \varepsilon \)-isometry if and only if there is an isometry \( h \) and a morphism \( g \) such that \( gf \sim_\varepsilon h \).

**Proof.** Necessity is evident and sufficiency follows from the proof of [4, 3.22]). \( \Box \)

**Lemma 2.5.** Let \( \mathcal{K} \) be a \textbf{CMet}-enriched category with \( \varepsilon \)-pushouts for all \( \varepsilon \geq 0 \). Assume that isometries in \( \mathcal{K} \) are stable under pushouts. Let \( f \sim_\varepsilon g \) where \( g \) is an isometry. Then \( f \) is an \( \varepsilon \)-isometry.

**Proof.** Take the \( \varepsilon \)-pushout from Remark 2.3(3). Since \( g \cdot \text{id}_A \sim_\varepsilon \text{id}_B \cdot f \), there is \( t : P \to B \) such that \( t\bar{f} = g \). Since isometries are left cancellable, \( \bar{f} \) is an isometry. Following Remark 2.3(3), \( f \) is an \( \varepsilon \)-isometry. \( \Box \)

**Lemma 2.6.** Let \( \mathcal{K} \) be a \textbf{CMet}-enriched category with \( \varepsilon \)-pushouts for all \( \varepsilon \geq 0 \). Assume that isometries in \( \mathcal{K} \) are stable under pushouts. If \( f : A \to B \) is an \( \varepsilon \)-isometry and \( g : B \to C \) an \( \delta \)-isometry then \( gf \) is an \((\varepsilon + \delta)\)-isometry.
Proof. There are isometries \( h_1 : B \to D_1, h_2 : A \to D_1, h_3 : C \to D_2 \) and \( h_4 : B \to D_2 \) such that \( h_1 f \sim_\varepsilon h_2 \) and \( h_3 g \sim_\delta h_4 \). Consider a pushout

\[
\begin{array}{ccc}
B & \xrightarrow{h_1} & D_1 \\
\downarrow{h_4} & & \downarrow{\bar{h}_4} \\
D_2 & \xrightarrow{h_1} & D
\end{array}
\]

Then

\[\bar{h}_1 h_3 g f \sim_\delta \bar{h}_1 h_4 f = \bar{h}_4 h_1 f \sim_\varepsilon \bar{h}_4 h_2,\]

and thus \( \bar{h}_1 h_3 g f \sim_{\varepsilon + \delta} \bar{h}_4 h_2 \). Since \( \bar{h}_1 h_3 \) and \( \bar{h}_4 h_2 \) are isometries, \( gf \) is an \( (\varepsilon + \delta) \)-isometry. \( \square \)

**Remark 2.7.** (1) A linear mapping \( f : A \to B \) between Banach spaces is called an \( \varepsilon \)-isometry if

\[
(1 - \varepsilon) \| x \| \leq \| fx \| \leq (1 + \varepsilon) \| x \|.
\]

This is equivalent to

\[
\| \| fx \| - \| x \| \| \leq \varepsilon
\]

for every \( x \in A, \| x \| \leq 1 \) (see [4, 7.1(2)]). \( \varepsilon \)-isometries of norm \( \leq 1 \) are precisely \( \varepsilon \)-isometries in the sense of Remark 2.3(3) (see [4, 7.2]).

(2) There is a stronger concept of an \( \varepsilon \)-isometry on Banach spaces, namely a linear mapping \( f : A \to B \) such that

\[
(1 + \varepsilon)^{-1} \| x \| \leq \| fx \| \leq (1 + \varepsilon) \| x \|.
\]

We will call this a strong \( \varepsilon \)-isometry. Since \((1 - \varepsilon) \leq (1 + \varepsilon)^{-1}\), every strong \( \varepsilon \)-isometry is an \( \varepsilon \)-isometry.

(3) For \( 0 < \varepsilon < 1 \), an \( \varepsilon \)-isometry is a strong \( \frac{\varepsilon}{1 - \varepsilon} \)-isometry. Indeed, \((1 + \frac{\varepsilon}{1 - \varepsilon})^{-1} = 1 - \varepsilon \) and \( \varepsilon \leq \frac{\varepsilon}{1 - \varepsilon} \).

(4) Every \( \varepsilon \)-isometry is injective. Indeed, if \( \| fx \| = 0 \) then \( \| x \| = 0 \).

For a \( \text{CMet} \)-enriched category \( K \), the arrow category \( K^- \) is a \( \text{CMet} \)-enriched category with the hom-space \( K^-(f, g) \) of \( f : A \to B \) and \( g : C \to D \) defined by the following pullback in \( \text{Met} \)

\[
\begin{array}{ccc}
K^-(f, g) & \xrightarrow{K(A, g)} & K(A, C) \\
\downarrow & & \downarrow \\
K(B, D) & \xrightarrow{K(f, D)} & K(A, D)
\end{array}
\]

Recall that objects of \( K^- \) are morphisms \( f : A \to B \) of \( K \) and morphisms \((u, v) : f \to g\) are given by commutative squares

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{u} & & \downarrow{v} \\
C & \xrightarrow{g} & D
\end{array}
\]
3. Finite presentability

In an ordinary category $\mathcal{K}$, an object $A$ is finitely presentable if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \text{Set}$ preserves directed colimits. If $\mathcal{K}$ is enriched over $\mathbf{CMet}$, an object $A$ is finitely presentable (in the enriched sense) if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \mathbf{CMet}$ preserves directed colimits. While only the trivial Banach space $[0]$ is finitely presentable in the ordinary sense (see [4, 2.7(1)]), we prove that finite-dimensional Banach spaces are finitely presentable (in the enriched sense). This improves [4, 7.6] which showed that hom-functors $\mathcal{K}(A, -) : \text{Ban} \to \mathbf{CMet}$ with $A$ finite-dimensional preserve directed colimits of isometries.

**Theorem 3.1.** Finite-dimensional Banach spaces are finitely presentable in the enriched sense.

**Proof.** Let $A$ be finite-dimensional and $(k_{ij} : K_i \to K_j)_{i \leq j \in I}$ be a directed diagram in $\text{Ban}$ with a colimit $(k_i : K_i \to K)_{i \in I}$. Following [4, 2.5], we have to prove the following

1. Given $f : A \to K$ and $\varepsilon > 0$ then there are $i \in I$ and $g : A \to K_i$ such that $k_{ig} \sim_\varepsilon f$.
2. Given $f, g : A \to K_i$ then

$$\| k_{if} - k_{ig} \| = \inf_{j \geq i} \| k_{jf} - k_{ijg} \| .$$

We can assume that $I$ is a well-ordered chain (see [3, 1.7]). If the cofinality $\text{cof}(I)$ of $I$ is $\aleph_0$ then $I$ is $\aleph_1$-directed and the result follows from the $\aleph_1$-presentability of finite-dimensional Banach spaces [4, 2.7(2)]. Assume that $\text{cof}(I) = \aleph_0$ and let $i_0, i_1, \ldots, i_t, \ldots$ be a cofinal chain in $I$.

The proof of (1): Every morphism $f : A \to K$ has a factorization $f = f_j f_i$ where $f_i : B \to K$ is an isometry and $f_j : A \to B$ is surjective. Thus, $B$ is finite-dimensional provided that $A$ is finite-dimensional and therefore, we can assume, without a loss of generality, that $f$ is an isometry. At first, we prove

(*) For every $\varepsilon > 0$ there exist $i \in I$ and an $\varepsilon$-isometry $f^* : A \to K_i$ (not necessarily of norm $\leq 1$) such that $f \sim_\varepsilon f^*$.

Let $e_1, \ldots, e_n$ be a basis of $A$. Since any two norms on a finite-dimensional Banach space are equivalent, there is a number $r$ such that

$$\sum_{0 < m \leq n} |a_m| \leq r \sum_{0 < m \leq n} a_m e_m$$

for all $a_1, \ldots, a_n \in \mathbb{C}$. Let $\delta = \frac{\varepsilon}{2r}$. Following [4, 2.5(4)], there are elements $u_1, \ldots, u_n \in K_i$, $i < \mu$, such that

$$\| k_{i0} u_m - f e_m \| \leq \delta$$

for $m = 1, \ldots, n$. We can assume that $i = i_0$. Let $f^* : A \to K_i$ be the linear mapping such that $f^* e_m = u_m$ for $m = 1, \ldots, n$. We have

$$\| (k_{i0} f^* - f)(\sum_{0 < m \leq n} a_m e_m) \| \leq \sum_{0 < m \leq n} |a_m| \| (k_{i0} f^* - f)(e_j) \| \leq \sum_{0 < m \leq n} |a_m| \delta$$

$$\leq \frac{\varepsilon}{2} \| \sum_{0 < m \leq n} a_m e_m \| .$$

Hence $k_{i0} f^* \sim_\frac{\varepsilon}{2} f$.

Since $\| k_{i0} f^* \|$ converges pointwise to $\| k_{i0} f^* \|$ (see [4, 2.5(4)]) and the closed unit ball $A_1$ of $A$ is compact, the convergence is uniform on $A_1$ following Dini’s theorem (see [8, p. 165]). Hence

$$\| k_{i0} f^* \| - \| k_{i0} f \| \leq \frac{\varepsilon}{2}$$

for some $t$. We will show that $f^* = k_{i0} f^* : A \to K_i$ is an $\varepsilon$-isometry. Since $f^*$ does not need to have norm $\leq 1$, we cannot use Lemma 2.4 but we will apply Remark 2.7(1).
Let \( a = \sum_{0 < j \leq n} a_m e_m \in A_1 \) and \( \| a \| \leq 1 \). We have

\[
\| f^* a \| - \| a \| = \| k_{ii} f' a \| - \| k_{ij} f' a \| + \| k_{if} a \| - \| a \| \\
\leq \| k_{ii} f' a \| - \| k_{ij} f' a \| + \| k_{if} a \| - \| a \| \\
\leq \frac{\varepsilon}{2} + \| k_{if} a \| - \| a \| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \| a \| \leq \varepsilon.
\]

Hence \( f^* \) is an \( \varepsilon \)-isometry and

\[
f \sim' \frac{1}{2} k_{if} = k_{ii} k_{ij} f' = k_{ii} f^*.
\]

This proves (\( \ast \)).

Now, we prove (1). We take \( f^* \) from (\( \ast \)) for \( \varepsilon' = \frac{\varepsilon}{2} \). Let \( \| \sum_{0 < j \leq n} a_j e_j \| = 1 \). Since \( f^* \) is an \( \varepsilon' \)-isometry, following Remark 2.7(1), we have

\[
\| f^* (\sum_{0 < j \leq n} a_j e_j) \| - 1 \leq \varepsilon'.
\]

If \( \| f^* (\sum_{0 < j \leq n} a_j e_j) \| \geq 1 \) then \( \| f^* (\sum_{0 < j \leq n} a_j e_j) \| \leq 1 + \varepsilon' \). If \( \| f^* (\sum_{0 < j \leq n} a_j e_j) \| \leq 1 \) then, again, \( \| f^* (\sum_{0 < j \leq n} a_j e_j) \| \leq 1 + \varepsilon' \). We have proved that \( \| f^* \| \leq 1 + \varepsilon' \). Hence \( g = \frac{1}{1 + \varepsilon'} f^* \) is a morphism in \( \text{Ban} \).

For \( a = \sum_{0 < j \leq n} a_j e_j \) we have

\[
\| f^* a - g a \| = \frac{1}{1 + \varepsilon'} \| (1 + \varepsilon') f^* a - f^* a \| = \frac{\varepsilon}{1 + \varepsilon'} \| f^* a \| \leq \frac{\varepsilon}{1 + \varepsilon'} (1 + \varepsilon') \| a \| = \varepsilon' \| a \|.
\]

Hence \( \| f^* - g \| \leq \varepsilon' \) and thus \( \| k_{ij} f^* - k_{ij} g \| \leq \varepsilon' \). Since \( \| k_{ij} f^* - f \| \leq \varepsilon' \), we have \( \| k_{ij} g - f \| \leq \varepsilon \).

The proof of (2): We have to show that

\[
\| k_i f - k_i g \| = \inf_{j \neq i} \| k_{ij} f - k_{ij} g \|
\]

for \( f, g : A \rightarrow K_i \). This means that

\[
\| k_{ij} f - k_{ij} g \| = \lim_{t \rightarrow 1} \| k_{ij} f - k_{ij} g \|.
\]

Following [4, 2.5(4)], \( k_{ii}(f - g) \) converges pointwise to \( k_i(f - g) \) on the closed unit ball \( A_1 \) of \( A \). Using Dini's theorem, we get that the convergence is uniform. Hence

\[
\| k_i(f - g) \| = \lim_{t \rightarrow 1} \| k_{ii}(f - g) \|.
\]

Locally finitely presentable enriched categories were introduced in [16]. Since \( \text{CMet} \) is not locally finitely presentable as an ordinary category, [16] does not fully apply to \( \text{CMet} \)-enriched categories.

We say that a \( \text{CMet} \)-enriched category is \( \text{locally finitely presentable} \) if it has all weighted colimits and a set \( \mathcal{A} \) of finitely presentable objects (in the enriched sense) such that every object is a directed colimit of objects of \( \mathcal{A} \).

These categories are locally finitely presentable in the sense of [16] because they have a strong generator consisting of finitely presentable objects. However, \( \text{CMet} \) is locally finitely presentable in the sense of [16] (see (3.4) in this paper) but not locally finitely presentable in our sense.

**Corollary 3.2.** \( \text{Ban} \) is locally finitely presentable as a category enriched over \( \text{CMet} \).

**Proof.** \( \text{Ban} \) has all weighted colimits (see [4, 4.5(3)]). Thus the result follows from Theorem 3.1 and the fact that every Banach space is a directed colimit of finite-dimensional spaces. \( \Box \)
Remark 3.3. [4, 7.7] characterizes Banach spaces $A$ such that $\text{Ban}(A, -) : \text{Ban} \to \text{CMet}$ preserves directed colimits of isometries as those admitting for every $\varepsilon > 0$ a morphism $u : A \to A_0$ to a finite-dimensional Banach space with $r : A_0 \to A$ such that $ru \sim_\varepsilon \text{id}_A$. These Banach spaces do not need to be finite-dimensional and satisfy (1) above for every directed colimit $(k_i : K_i \to K)$. Indeed, for $f : A \to K$ and $\varepsilon > 0$, we take $u : A \to A_0$ and $r : A_0 \to A$ with $A_0$ finite-dimensional and $ru \sim_\varepsilon \text{id}_A$. Following Theorem 3.1, there is $g : A_0 \to K_i$ such that $k_ig \sim_\varepsilon fr$. Then

$$k_igu \sim_\varepsilon fru \sim_\varepsilon f.$$

Hence $k_igu \sim_\varepsilon f$.

We do not know whether these Banach spaces $A$ can satisfy (2) as well. So, we do not know whether finitely presentable Banach spaces are finite-dimensional.

Proposition 3.4. Let $U : \mathcal{K} \to \mathcal{L}$ and $F : \mathcal{L} \to \mathcal{K}$ be an enriched adjunction (where $F$ is a left adjoint) such that $U$ preserves directed colimits. Then $F$ preserves finitely presentable objects (in the enriched sense).

Proof. Let $A$ be finitely presentable in $\mathcal{L}$. Since

$$\mathcal{K}(FA, -) \cong \mathcal{L}(A, U-),$$

and $\mathcal{L}(A, U-)$ preserves directed colimits, $FA$ is finitely presentable. $\square$

4. Purity

A morphism $f : K \to L$ in an ordinary locally finitely presentable category $\mathcal{K}$ is pure if in every commutative square

$$\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}$$

with $A$ and $B$ finitely presentable there exists $t : B \to K$ such that $tg = u$. Pure morphisms are precisely directed colimits of split monomorphisms in $\mathcal{K}$ (see [3, 2.30]). Recall that a monomorphism $f : K \to L$ is split if there exists $s : L \to K$ such that $sf = \text{id}_K$.

Since $\text{Ban}$ is not locally finitely presentable as an ordinary category, one cannot expect that pure morphisms can be defined in this way.

Definition 4.1. A morphism $f : K \to L$ in $\text{Ban}$ is pure if for every $\varepsilon > 0$ and every commutative square

$$\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}$$

with $A$ and $B$ finite-dimensional, there exists $t : B \to K$ such that $tg \sim_\varepsilon u$.

In [27, 5.2], these morphisms were called barely ap-pure.
Lemma 4.2. A morphism \( f : K \to L \) in Ban is pure if and only if for every \( \varepsilon > 0 \) and every \( \varepsilon \)-commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}
\]

with \( A \) and \( B \) finite-dimensional, there exists \( t : B \to K \) such that \( tg \sim_{2\varepsilon} u \).

Proof. Sufficiency is evident. Let \( f \) be pure and consider an \( \varepsilon \)-commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}
\]

with \( A \) and \( B \) finite-dimensional. Consider an \( \varepsilon \)-pushout

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{id_A} & & \downarrow{g\varepsilon} \\
A & \xrightarrow{g} & B
\end{array}
\]

There is a unique morphism \( t : C \to L \) such that \( t\bar{g} = fu \) and \( t\bar{g}\varepsilon = v \). Thus we get a commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{t} \\
A & \xrightarrow{g} & C
\end{array}
\]

Following Remark 2.2, \( C \) is finite-dimensional and thus there exists \( w : C \to K \) such that \( w\bar{g} \sim_{\varepsilon} u \). Hence

\[
w\bar{g}\varepsilon g \sim_{\varepsilon} w\bar{g} \sim_{\varepsilon} u.
\]

Hence \( w\varepsilon g \sim_{2\varepsilon} u \).

Remark 4.3. In the terminology of [27], this means that barely \( \lambda \)-pure and weakly \( \lambda \)-pure morphisms coincide. The same holds for barely \( \lambda \)-pure and weakly \( \lambda \)-pure morphisms.
Lemma 4.4. A morphism \( f : K \to L \) in \( \text{Ban} \) is pure if and only if for every \( \varepsilon > 0 \) and every commutative square

\[
\begin{array}{c}
K \\
\downarrow u \\
A \\
\downarrow g \\
A_0 \\
\end{array} \quad \begin{array}{c}
L \\
\downarrow v \\
B \\
\downarrow v \\
B_0 \\
\end{array}
\]

with \( A \) and \( B \) finitely presentable in the enriched sense, there exists \( t : B \to K \) such that \( tg \sim_\varepsilon u \).

Proof. Sufficiency follows from Theorem 3.1. Assume that \( f \) is pure and consider a commutative square

\[
\begin{array}{c}
K \\
\downarrow u \\
A \\
\downarrow g \\
A_0 \\
\end{array} \quad \begin{array}{c}
L \\
\downarrow v \\
B \\
\downarrow v \\
B_0 \\
\end{array}
\]

with \( A \) and \( B \) finitely presentable in the enriched sense. Let \( \varepsilon > 0 \) and take \( u_A : A \to A_0, r_A : A_0 \to A \), \( u_B : B \to B_0 \) and \( r_B : B_0 \to B \) from Remark 3.3 such that \( r_A u_A \sim_\varepsilon \text{id}_A \) and \( r_B u_B \sim_\varepsilon \text{id}_B \). Then the square

\[
\begin{array}{c}
K \\
\downarrow u_A \\
A_0 \\
\downarrow u_B g r_A \\
B_0 \\
\end{array} \quad \begin{array}{c}
L \\
\downarrow v r_B g r_A \\
B \\
\downarrow v r_B g r_A \\
B_0 \\
\end{array}
\]

\( \varepsilon_4 \)-commutes because

\[ v r_B u_B g r_A \sim_\varepsilon v g r_A = f u_A. \]

Following Lemma 4.2, there exists \( t : B_0 \to K \) such that \( t u_B g r_A \sim_\varepsilon u_A \). Then

\[ t u_B g \sim_\varepsilon t u_B g r_A u_A \sim_\varepsilon u_A u_A \sim_\varepsilon u \]

Hence \( t u_B g \sim_\varepsilon u \). \( \square \)

Remark 4.5. Ordinary purity can be expressed in the following way. Let \( Q(g, f) \) be the image of the projection \( p_1 : \mathcal{K}^\rightarrow (g, f) \to \mathcal{K}(A, K) \) sending \((u, v)\) to \( u \). Hence \( Q(g, f) \) consists of those \( u : A \to K \) for which there exists \( v : B \to L \) such that \( v g = f u \). Then \( f \) is pure if and only if the mapping

\[ q : \mathcal{K}(B, K) \to Q(g, f) \]

sending \( t : B \to K \) to \( tg \) is surjective for every \( g : A \to B \) with \( A \) and \( B \) finitely presentable.

The approach of [19] suggests how to define enriched purity. We replace the factorization system (surjective, injective) in \( \text{Set} \) by the factorization system (dense, isometry) in \( \text{CMet} \) (see [4, 3.16(2)]). Then \( Q(g, f) \) is given by the (dense, isometry) factorization of \( p_1 \) and we demand that \( q \) above is dense. Using Lemma 4.4, this leads to the just defined pure morphisms.

Proposition 4.6. Pure morphisms are closed under directed colimits of in \( \text{Ban}^\rightarrow \).
Proof. Let \((k_i, l_i) : (f_i \rightarrow f)_{i \in I}\) be a directed colimit of pure morphisms \(f_i : K_i \rightarrow L_i\) in \(\text{Ban}^\to\). This means that \(k_i : K_i \rightarrow K\) and \(l_i : L_i \rightarrow L\) are directed colimits in \(\text{Ban}\) and \(fk_i = l_i f_i\) for every \(i \in I\). Consider a commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\uparrow{u} & & \uparrow{v} \\
A & \xrightarrow{g} & B
\end{array}
\]

where \(A\) and \(B\) are finite-dimensional and take \(\varepsilon > 0\). Following Theorem 3.1(1), there is \(i \in I\) and \(u' : A \rightarrow K_i\) and \(v' : B \rightarrow L_i\) such that \(k_i u' \sim \frac{\varepsilon}{2} u\) and \(l_i v' \sim \frac{\varepsilon}{2} v\). Since

\[l_i f_i u' = f k_i u' = f u = v g = l_i v' g,\]

Theorem 3.1(2) implies that there is \(j \geq i\) such that \(f_j u' \sim \frac{\varepsilon}{2} v' g\). Following Lemma 4.2, there is \(t : B \rightarrow K_j\) such that \(tg \sim \frac{\varepsilon}{2} u'\). Hence

\[k_j t g \sim \frac{\varepsilon}{2} k_j u' \sim \frac{\varepsilon}{2} u.\]

Thus \(k_j t g \sim \varepsilon u\). □

Remark 4.7. (1) Split monomorphisms \(f : K \rightarrow L\) in \(\text{Ban}\) coincide with projections \(p : L \rightarrow L\) of norm one on \(L\). Indeed, if \(f\) is split by \(s\) (i.e., \(sf = \text{id}_K\)) then \(p = fs\). Conversely, given a projection \(p : L \rightarrow L\) of norm one then the embedding \(p|L| \rightarrow L\) is split by \(p\) (see [6, 2.2.9]). Equivalently, \(L\) is a biproduct of \(p|L|\) and \(\ker(p)\).

(2) Every split monomorphism in \(\text{Ban}\) is pure. It suffices to take \(t = vs\).

Proposition 4.8. A morphism \(f\) in \(\text{Ban}\) is pure if and only if for every \(\varepsilon > 0\) there are \(h_1, h_2\) such that \(h_1 f \sim \varepsilon h_2\) and \(h_2\) is a directed colimit of split monomorphisms in \(\text{Ban}^\to\).

Proof. Assume that, for every \(\varepsilon > 0\), there are \(h_1, h_2\) such that \(h_1 f \sim \varepsilon h_2\) where \(h_2\) is a directed colimit of split monomorphisms in \(\text{Ban}^\to\). Consider a commutative square

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\uparrow{u} & & \uparrow{v} \\
A & \xrightarrow{g} & B
\end{array}
\]

where \(A\) and \(B\) are finite-dimensional and take \(\varepsilon > 0\) and \(h_1 : L \rightarrow M, h_2 : K \rightarrow M\) above for \(\frac{\varepsilon}{2}\). Then the square

\[
\begin{array}{ccc}
K & \xrightarrow{h_2} & M \\
\uparrow{u} & & \uparrow{h_1 v} \\
A & \xrightarrow{g} & B
\end{array}
\]

\(\frac{\varepsilon}{2}\)-commutes. Since split monomorphisms are pure, Proposition 4.6 implies that \(h_2\) is pure. Following Lemma 4.2, there is \(t : B \rightarrow K\) such that \(tg \sim \varepsilon u\). Thus \(f\) is pure.

Conversely, let \(f : K \rightarrow L\) be pure and \(\varepsilon > 0\). Express \(K\) as a directed colimit \((k_i : K_i \rightarrow K)_{i \in I}\) of isometries \(k_i t : K_i \rightarrow K\) between finite-dimensional Banach spaces. Similarly, let \((l_j : L_j \rightarrow L)_{j \in J}\) be
a directed colimit of isometries between finite-dimensional Banach spaces. Following Theorem 3.1 (or [4, 7.6]), for every $i \in I$ there is $f_i : K_i \to L_i$ such that $l_j f_i \sim \varepsilon f_k$. Following Lemma 4.2, for every $i \in I$ there is $t_i : L_i \to K$ such that $t_i f_i \sim \varepsilon f_i$. Like in the proof of [27, 3.11], in the $\varepsilon$-pushouts

\[
\begin{array}{c}
K \xrightarrow{f_i} L_i \\
\uparrow k_i \quad \uparrow k_i \\
K_i \xrightarrow{f_i} L_i
\end{array}
\]

every $\tilde{f}_i$ is a split monomorphism. Since

\[
\tilde{f}_i k_i = \tilde{f}_i k_i \tilde{k}_i^\prime = \tilde{k}_i^\prime f_i k_i, \\
\tilde{k}_i^\prime l_j f_i = \tilde{k}_i^\prime l_j f_i
\]

there is a unique morphism $\tilde{k}_i^\prime : L_i \to \tilde{L}_i$ such that $\tilde{k}_i^\prime f_i = \tilde{k}_i^\prime f_i$ and $\tilde{k}_i^\prime k_i = \tilde{k}_i^\prime l_j f_i$.

We get a directed diagram $(\text{id}_K, \tilde{k}_i^\prime) : f_i \to \tilde{f}_i$ of split monomorphisms whose colimit $\tilde{f} : K \to \tilde{L}$ is given by the $\varepsilon$-pushout

\[
\begin{array}{c}
K \xrightarrow{h_2} \tilde{L} \\
\uparrow \text{id}_K \quad \uparrow h_1 \\
K \xrightarrow{f} L
\end{array}
\]

Remark 4.9. In the proof above, $\tilde{k}_i^\prime$ are isometries (see Remark 2.3(2)). Hence $h_2$ can be taken as a directed colimit of split monomorphisms and isometries in $\text{Ban}^{\rightarrow}$.

Lemma 4.10. Pure morphisms in $\text{Ban}$ are left-cancellable, i.e., $f_2 f_1$ pure implies that $f_1$ is pure.

Proof. It follows from Proposition 4.8.

Lemma 4.11. Pure morphisms in $\text{Ban}$ are closed under composition.

Proof. Let $f_1 : K \to L$ and $f_2 : L \to M$ be pure. Consider a commutative diagram

\[
\begin{array}{c}
K \xrightarrow{f_1} L \xrightarrow{f_2} M \\
\uparrow u \quad \uparrow v \\
A \xrightarrow{g} B
\end{array}
\]

and $\varepsilon > 0$. Since $f_2$ is pure, there is $t : B \to L$ such that $t g \sim \varepsilon f_1 u$. Following Lemma 4.2, there is $s : B \to K$ such that $s g \sim \varepsilon u$. Thus $f_2 f_1$ is pure.

Lemma 4.12. Pure morphisms in $\text{Ban}$ are isometries.

Proof. Since isometries are closed under directed colimits ([4, 2.5(4)])) and split monomorphisms are isometries, $h_2$ in Proposition 4.8 are isometries. Following Proposition 4.8 and Lemma 2.4, a pure morphism $f$ is an $\varepsilon$-isometry for every $\varepsilon > 0$. Hence $f$ is an isometry (see Remark 2.7(1)).
Recall that a closed subspace $K$ of a Banach space $L$ is called an ideal if the annihilator of $K$ in the dual space $L^*$ is the kernel of norm one projection in $L^*$. Ideals $K$ in $L$ are characterized as closed subspaces $K$ such that for every finite-dimensional subspace $B$ of $L$ and every $\varepsilon > 0$, there exists a linear map $t : B \to K$ such that $tx = x$ for $x \in K \cap B$ and $\| t \| \leq 1 + \varepsilon$ (see [22, Theorem 1]).

**Proposition 4.13.** A closed subspace $K$ of a Banach space $L$ is an ideal if and only if the embedding $K \to L$ is pure.

**Proof.** I. Let $f : K \to L$ be an embedding of an ideal. Consider a commutative square

$$
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}
$$

with $A$ and $B$ finite-dimensional and $\varepsilon > 0$. Let $v = v_2v_1$ be a (dense, isometry) factorization of $v$. Let

$$
\begin{array}{ccc}
K & \xrightarrow{\bar{f}} & L \\
\downarrow{\bar{v}} & & \downarrow{v_2} \\
C & \xrightarrow{\bar{f}} & B_1
\end{array}
$$

be a pullback. There is $w : A \to C$ such that $\bar{v}w = u$ and $\bar{f}w = v_1g$. Take $\delta = \frac{\varepsilon}{1 + \varepsilon}$. There is a linear operator $t : B_1 \to K$ such that $t\bar{f} = \bar{v}$ and $\| t \| \leq 1 + \delta$. Then $t' = \frac{1}{1+\delta}t$ has norm $\leq 1$. For $x \in C$, $\| x \| \leq 1$, we have

$$
\| t'\bar{f}x - \bar{v}x \| = \| \frac{t\bar{f}x}{1 + \delta} - \bar{v}x \| = \| \frac{\bar{v}x}{1 + \delta} - \bar{v}x \| = \frac{\delta}{1 + \delta} \| \bar{v}x \| \leq \frac{\delta}{1 + \delta} = \varepsilon.
$$

Hence $t'\bar{f} \sim_{\varepsilon} \bar{v}$. Therefore $t'v_1g = t'\bar{f}w \sim_{\varepsilon} \bar{v}w = u$. We have proved that $f$ is pure.

II. Let $f : K \to L$ be pure. Consider a finite-dimensional subspace $B$ of $L$ and $\varepsilon > 0$. Let $v : B \to L$ be the embedding and take the pullback

$$
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{\bar{v}} & & \downarrow{v} \\
A & \xrightarrow{\bar{f}} & B
\end{array}
$$

Then $\bar{f} : K \cap B = A \to B$ is the embedding of a finite-dimensional subspace of $B$. Let $e_1, \ldots, e_m, \ldots, e_n$ be a basis of $B$ such that $e_1, \ldots, e_m$ is a basis of $A$. Since any two norms on a finite-dimensional Banach space are equivalent, there is a number $r \geq 1$ such that

$$
\sum_{0 < i \leq n} |a_i| \leq r \sum_{0 < i \leq n} a_i e_i.
$$

Let $\delta = \frac{\varepsilon}{r}$. Since $f$ is pure, there exists $t' : B \to K$ such that $t'\bar{f} \sim_{\varepsilon} \bar{v}$. Let $t : B \to K$ be defined as follows:

$t(e_i) = e_i$ for $i = 1, \ldots, m$ and $t(e_i) = t'(e_i)$ for $i = m + 1, \ldots, n$. Consider $a \in B$, $\| a \| \leq 1$. Then

$$
\| t(a) \| = \| \sum_{0 < i \leq m} a_i e_i + \sum_{m < i \leq n} a_i t'(e_i) \| = \| \sum_{0 < i \leq m} a_i (e_i - t'(e_i)) + \sum_{0 < i \leq n} a_i t'(e_i) \|
$$
\[
\leq \| \sum_{0<i\leq m} a_i(e_i - t'(e_i)) \| + \| t'(a) \| \leq \| \sum_{0<i\leq m} a_i(e_i - t'(e_i)) \| + \| a \|
\]
\[
\leq \sum_{0<i\leq n} |a_i| \delta + \| a \| \leq \sum_{0<i\leq n} |a_i| \delta + \| a \| \leq r \| a \|
\]
\[
= \varepsilon \| a \| + \| a \| = (1 + \varepsilon) \| a \|.
\]
Hence \( \| t \| \leq 1 + \varepsilon \). Since \( t(x) = x \) for \( x \in A \), we have proved that \( K \) is an ideal in \( L \).

**Corollary 4.14.** A morphism in \( \text{Ban} \) is pure if and only if it is an embedding of an ideal.

**Proof.** It follows from Lemma 4.12 and the fact that an isometry \( f : K \to L \) makes \( f[K] \) a closed subspace of \( L \).

**Proposition 4.15.** Pure morphisms in \( \text{Ban} \) are stable under \( \varepsilon \)-pushouts for every \( \varepsilon > 0 \).

**Proof.** Consider an \( \varepsilon \)-pushouts

\[
\begin{array}{ccc}
K & \xrightarrow{\tilde{f}} & \tilde{L} \\
\uparrow{u} & & \uparrow{\tilde{u}} \\
K & \xrightarrow{f} & L
\end{array}
\]

where \( f \) is pure. Following Proposition 4.8, there are \( g, h \) such that \( g\tilde{f} \sim_{\varepsilon} h \), with \( h \) being a directed colimit of split monomorphisms. Take the pushout

\[
\begin{array}{ccc}
\tilde{K} & \xrightarrow{\tilde{h}} & \tilde{M} \\
\uparrow{u} & & \uparrow{\tilde{u}} \\
K & \xrightarrow{h} & M
\end{array}
\]

Since \( \tilde{u}g\tilde{f} \sim_{\varepsilon} \tilde{uh} = \tilde{h}u \), there is a unique \( \tilde{g} : \tilde{L} \to \tilde{M} \) such that \( \tilde{g}\tilde{f} = \tilde{h} \) and \( \tilde{g}\tilde{u} = \tilde{u}g \). Since split monomorphisms are stable under pushouts and pushouts commute with directed colimits, \( \tilde{h} \) is a directed colimit of split monomorphisms. Then Proposition 4.8 implies that \( f \) is pure.

**Proposition 4.16.** Let \( F : \text{Ban} \to \text{Ban} \) be an enriched functor preserving directed colimits. Then \( F \) preserves pure morphisms.

**Proof.** \( F \) clearly preserves directed colimits of split monomorphisms. Since \( F \) is enriched \( f \sim_{\varepsilon} g \) implies \( Ff \sim_{\varepsilon} Fg \). Thus the result follows from Proposition 4.8.

**Remark 4.17.** The projective tensor product functor \( K \otimes - : \text{Ban} \to \text{Ban} \) is enriched and preserves all colimits. Hence, by Proposition 4.16, it preserves pure morphisms. Thus it preserves ideals, which was proved in [25, Theorem 1]. Add that it does not need to preserve isometries (see [28], p.18).

**Remark 4.18.** Following [25, Lemma 1(i)], the canonical embedding \( d_X : X \to X^{**} \) is an ideal. Recall that \( X^* \) is the space \( \{X, \mathbb{C}\} \) of bounded linear mappings \( X \to \mathbb{C} \).

Recall that a Banach space \( K \) is injective with respect to \( h : A \to B \) in \( \text{Ban} \) if for every \( f : A \to K \) there exists \( g : B \to K \) such that \( gh = f \).
Definition 4.19. Banach spaces injective with respect to pure morphisms will be called pure-injective.

Remark 4.20. Injective Banach spaces, i.e., injective with respect to isometries, are pure-injective.

Lemma 4.21. Banach spaces $K^*$ are pure-injective.

Proof. Let $h : A \to B$ be pure and $f : A \to K^*$. Since $K \otimes -$ is left adjoint to $\{K, -\}$, we get the adjoint transpose $\tilde{f} : K \otimes A \to C$ and the pure morphism $K \otimes h : K \otimes A \to K \otimes B$ (see Proposition 4.17). Since $C$ is injective, there is $g : K \otimes B \to C$ such that $g(K \otimes h) = \tilde{f}$. Then the adjoint transpose $\tilde{g} : B \to K^*$ satisfies $\tilde{g}h = f$. Hence $K^*$ is pure-injective.

Remark 4.22. Ban has enough pure-injectives because every Banach space admits a pure morphism $K \to K^{**}$ and $K^{**}$ is pure-injective.

5. Model theory

Pure morphisms in locally finitely presentable categories have a model-theoretic characterization as morphisms elementary with respect to positive-primitive formulas (see [3, 5.15]). We will prove an analogical result for Banach spaces. The needed logic is closely related to the logic of positive bounded formulas (see [13] and [14]).

Let $L$ be the language consisting of a binary operation symbol $+$, a nullary operation symbol $0$ and unary operation symbols $r \cdot -$ where $r$ is a rational complex number. Moreover, for every rational number $M$, we have a unary relation symbol $\| - \| \leq M$. Terms are given in a usual way using operation symbols. Atomic formulas are either $t_1 = t_2$ where $t_1$ and $t_2$ are terms or $\| t \| \leq M$. The notation $\varphi(x_1, \ldots, x_n)$ for an atomic formula means that all variables in $\varphi$ are among $x_1, \ldots, x_n$.

Positive-primitive formulas are

$$\varphi(x_1, \ldots, x_n) = (\exists y_1, \ldots, y_m)(\bigwedge_{i<\omega} \psi_i(x_1, \ldots, x_n, y_1, \ldots, y_m))$$

where $\psi_i$ are atomic formulas. Approximations of $\varphi$ are formulas $\varphi_\varepsilon$, $\varepsilon > 0$ where every occurrence of an atomic formula $\| t \| \leq M$ is replaced by $\| t \| \leq M + \varepsilon$. In particular,

$$\varphi_\varepsilon(x_1, \ldots, x_n) = (\exists y_1, \ldots, y_m)(\bigwedge_{i<\omega} (\psi_i)_\varepsilon(x_1, \ldots, x_n, y_1, \ldots, y_m)).$$

For a Banach space $K$ and $a_1, \ldots, a_n \in K$, the meaning of $K \models \varphi[a_1, \ldots, a_n]$
Following Remark 5.1, there exists a linear map $f : A \to K$ such that $f(e_i) = a_i$ for $i = 1, \ldots, n$ has norm $\leq 1$. In analogy to [3, 5.5] we call $\pi_A$ the presentation formula of $A$. Similarly,

$$K \models (\pi_A)_{\varepsilon}[a_1, \ldots, a_n]$$

if the linear map $f : A \to K$ such that $f(e_i) = a_i$ for $i = 1, \ldots, n$ has norm $\leq 1 + \varepsilon$.

**Proposition 5.2.** A closed subspace $K$ of a Banach space $L$ is an ideal if and only if for each positive-primitive formula $\varphi(x_1, \ldots, x_n)$ and each assignment of values $a_1, \ldots, a_n \in K$ we have

$$K \models_{ap} \varphi[a_1, \ldots, a_n] \iff L \models_{ap} \varphi[a_1, \ldots, a_n].$$

**Proof.** Let $K$ be a closed subspace of $L$, $\varphi(x_1, \ldots, x_n)$ a positive-primitive bounded formula and $a_1, \ldots, a_n \in K$. Obviously,

$$K \models_{ap} \varphi[a_1, \ldots, a_n] \Rightarrow L \models_{ap} \varphi[a_1, \ldots, a_n].$$

Thus, it is only the reverse implication that matters.

I. Let $K$ satisfy the reverse implication and consider a finite-dimensional subspace $B$ of $L$ and $\varepsilon > 0$. Let $e_1, \ldots, e_n$ be a basis of $A = B \cap K$ and $e_1, \ldots, e_n, \ldots, e_{n+m}$ a basis of $B$. Consider the formula

$$\varphi(x_1, \ldots, x_n) = (\exists x_{n+1}, \ldots, x_{n+m}) \pi_B(x_1, \ldots, x_{n+m})$$

where $\pi_B$ is the presentation formula of $B$. Since $L \models \varphi[e_1, \ldots, e_n]$, we have

$$K \models_{ap} \varphi[e_1, \ldots, e_n].$$

Following Remark 5.1, there exists a linear map $f : B \to K$ of norm $\leq 1 + \varepsilon$ such that $f(e_i) = e_i$ for $i = 1, \ldots, n$. Hence $K$ is an ideal in $L$.

II. For the converse, let $K$ be an ideal in $L$ and consider $a_1, \ldots, a_n \in K$ and a positive-primitive formula

$$\varphi(x_1, \ldots, x_n) = (\exists y_1, \ldots, y_m) \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

such that $L \models_{ap} \varphi[a_1, \ldots, a_n]$. Let

$$\psi(x_1, \ldots, x_n, y_1, \ldots, y_m) = \bigwedge_{i < \omega} \psi_i(x_1, \ldots, x_n, y_1, \ldots, y_m)$$
where $\psi_i(x_1, \ldots, x_m, y_1, \ldots, y_m)$ are atomic formulas. Thus there are $b_1, \ldots, b_m \in L$ such that $L \models \psi_i[a_1, \ldots, a_n, b_1, \ldots, b_m]$ for every $\varepsilon_1 > 0$. Let $B$ be the subspace of $L$ generated by $a_1, \ldots, a_n, b_1, \ldots, b_m$.

Since $K$ is an ideal in $L$, for every $\varepsilon_2 > 0$, there is a linear map $f : B \to K$ such that $\| f \| \leq 1 + \varepsilon_2$ and $f(a_i) = a_i$ for every $j = 1, \ldots, n$. Now, let $\varepsilon > 0$ and take $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$. For $\psi_i = \| t \| \leq M$, we have

$$\| t(a_1, a_n, \ldots, f(b_1), \ldots, f(b_m)) \| \leq \| t(a_1, a_n, b_1, \ldots, b_m) \| + \varepsilon 2 - M + \frac{\varepsilon}{2} = M + \varepsilon.$$

Hence $K \models \psi[a_1, \ldots, a_n]$.

A morphism $f : K \to L$ in $\text{Ban}$ is a $u$-extension if there exists an ultrapower $K^{\text{UL}}$ and an isometry $g : L \to K^{\text{UL}}$ such that $gK = dK$ where $dK : K \to K^{\text{UL}}$ is the canonical embedding of $K$ to its ultrapower (see [30, 6.3]). Concerning ultraproducts of Banach spaces one can also consult [14, p. 7] or [12, 14.1]. Let $K$ be a closed subspace of a Banach space $L$. Following [30, 6.7] (see also [12, 14.2.7]), the embedding $f : K \to L$ is a $u$-extension if and only if for every finite-dimensional subspace $B$ of $L$ and every $\varepsilon > 0$, there exists a strong $\varepsilon$-isometry $t : B \to K$ such that $tx = x$ for every $x \in K \cap B$. Hence $u$-extensions are ideals. In [1, 1.3], $u$-extensions are called almost isometric ideals.

Remark 5.3. (1) Following Remark 2.7(3), $f : K \to L$ is a $u$-extension if and only if for every finite-dimensional subspace $B$ of $L$ and every $\varepsilon > 0$, there exists an $\varepsilon$-isometry $t : B \to K$ such that $tx = x$ for every $x \in K \cap B$.

(2) Let $A$ be a finite-dimensional Banach space with a basis $e_1, \ldots, e_n$. Following [11, 2.1], let $M = \max |a_i|$ where $a = \sum a_i e_i$ for $\| a \| \leq 1$. Let $\delta = \frac{\varepsilon}{nM}$. Then for every Banach space $K$, for every linear map $f : A \to X$ the following implication holds

$$\max _{i \leq n} \| f(e_i) \| \leq \delta \Rightarrow \| f \| \leq \varepsilon.$$

Lemma 5.4. Let $f : K \to L$ be an isometry in $\text{Ban}$ such that for every $\varepsilon > 0$ and every commutative square of isometries

$$\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\leftarrow \downarrow & & \uparrow \mathrel{v} \\
A & \xrightarrow{g} & B
\end{array}$$

with $A$ and $B$ finite-dimensional, there exists an isometry $t : B \to K$ such that $tg \sim_\varepsilon u$. Then $f$ is a $u$-extension.

Proof. Take $\varepsilon > 0$. Let $e_1, \ldots, e_m, \ldots, e_n$ be a basis of $B$ such that $e_1, \ldots, e_m$ is a basis of $A$. Choose $\delta$ from Remark 5.3(2) for the basis $e_1, \ldots, e_n$ and $\varepsilon$. Let $t' : B \to K$ be an isometry such that $t'g \sim_\delta u$. Take $t : B \to K$ such that $t(e_i) = e_i$ for $i = 1, \ldots, m$ and $t(e_i) = t'(e_i)$ for $i = m + 1, \ldots, n$. Since $\| (t' - t')(e_i) \| \leq \delta$ for every $i = 1, \ldots, n$, Remark 5.3(2) implies that $\| t - t' \| \leq \varepsilon$. Hence

$$\| t(x) \| - \| x \| = \| t(x) - t'(x) \| \leq \| t(x) - t'(x) \| \leq \delta$$

for every $x$ such that $\| x \| \leq 1$. Following Remark 2.7(1), $t$ is an $\varepsilon$-isometry and the result follows from Remark 5.3(1).

Remark 5.5. (1) Concerning a converse, in I. of the proof of Proposition 4.13, $t$ is injective (see Remark 2.74). But we would need that it is an isometry.

(2) The property in Lemma 5.4 says that $f$ is pure in the category $\text{Ban}_\text{iso}$ of Banach spaces with isometries as morphisms.
Proposition 5.6. Let $\mathcal{U}$ be an ultrafilter on a set $I$, $K$ a Banach space and $d_K : K \to K^\mathcal{U}$ the canonical embedding of $K$ to its ultrapower. Then $d_K$ is pure.

Proof. The ultrapower $K^\mathcal{U}$ is a directed colimit of powers $K^U$, $U \in \mathcal{U}$ (cf. [3, 5.17]) and $d_K$ is a directed colimit of diagonals $d_U : K \to K^U$. Since $d_U$ are split monomorphisms, Proposition 4.8 implies that $d_K$ is pure. \qed

Lemma 5.7. Ultrapowers preserve pure morphisms.

Proof. Let $f : K \to L$ be pure. We have to show that $f^\mathcal{U} : K^\mathcal{U} \to L^\mathcal{U}$ is pure. It is easy to see that $f^U : K^U \to L^U$ is pure for every $U \in \mathcal{U}$. Indeed, consider a commutative square

$$
\begin{array}{ccc}
K^U & \xrightarrow{f^U} & L^U \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{g} & B
\end{array}
$$

with $A$ and $B$ finite-dimensional. Let $p^i : K^U \to K$ and $q^i : L^U \to L$ be projections for $i \in U$. Let $\varepsilon > 0$. Then, for every $i \in U$, there is $t_i : B \to K$ such that $t_ig \sim \varepsilon p_iu$. Then the induced morphism $t : B \to K^U$ satisfies $tg \sim \varepsilon u$.

Since $f^\mathcal{U}$ is a directed colimit of $f^U$, the result follows from Proposition 4.6. \qed

Remark 5.8. The category $\text{Mod}(R)$ of modules over a ring $R$ is locally finitely presentable. Pure-injective modules (i.e., modules injective with respect to pure morphisms) coincide with algebraically compact ones (see [24, 4.3.11]). There is a set $I$ and an ultrafilter $\mathcal{U}$ on $I$ such that $K^\mathcal{U}$ is algebraically compact for every module $K$ (cf. [24, 4.2.18]). Hence pure morphisms and $u$-extensions coincide.

Indeed, let $f : K \to L$ be pure. Like in Lemma 5.7, $f^\mathcal{U}$ is pure. Since $K^\mathcal{U}$ is pure-injective, $f^\mathcal{U}$ is split by $s$. Then $sd_L$ makes $f$ an $u$-extension. Conversely, in the same way as in Proposition 5.6, one proves that the canonical embedding $d_K : K \to K^\mathcal{U}$ is pure. Since pure morphisms are left-cancellable, every $u$-extension is pure.

6. Injectivity

Definition 6.1. Given a class $\mathcal{H}$ of morphisms in a $\text{CMet}$-enriched category $K$, an object $K$ is called approximately injective with respect to $\mathcal{H}$ if for every $h : A \to B$ in $\mathcal{H}$, every $\varepsilon > 0$ and every morphism $f : A \to K$ there exists a morphism $g : B \to K$ such that $gh \sim \varepsilon f$.

The class of objects approximately injective with respect to $\mathcal{H}$ will be denoted as $\text{Inj}_{ap} \mathcal{H}$.

Proposition 6.2. Let $U : K \to L$ and $F : L \to K$ be an enriched adjunction. Then $UK$ is approximately injective with respect to $\mathcal{H}$ iff $K$ is approximately injective with respect to $F(\mathcal{H})$.

Proof. $UK$ is approximately injective with respect to $\mathcal{H}$ iff for every $h : A \to B$ in $\mathcal{H}$ and every morphism $f : A \to UK$ there exists a morphism $g : B \to UK$ such that $gh \sim \varepsilon f$. This is the same as $\tilde{g}Fh = \tilde{g}h \sim \varepsilon \tilde{f}$, which is equivalent to $K$ being approximately injective with respect to $F(\mathcal{H})$. \qed

Remark 6.3. Lindenstrauss spaces are Banach spaces which are ideals in every superspace [1, 4.1]. Hence $K$ is a Lindenstrauss space if and only if every isometry $K \to L$ is pure. This implies the well-known characterization of Lindenstrauss spaces as Banach spaces approximately injective with respect to isometries between finite-dimensional Banach spaces.
Indeed, let $K$ be Lindenstrauss, $h : A \to B$ be an isometry between finite-dimensional Banach spaces and $f : A \to K$. Consider the pushout

\[
\begin{array}{c}
K \\
\downarrow h \\
A \\
\end{array} \quad \begin{array}{c}
\bar{h} \\
\downarrow f \\
L \\
\end{array} \quad \begin{array}{c}
\bar{f} \\
\downarrow \bar{f} \\
B \\
\end{array}
\]

Since $\bar{h}$ is an isometry, it is pure and thus, for every $\varepsilon > 0$, there is $g : B \to K$ such that $gh \sim_{\varepsilon} f$. Hence $L$ is approximately injective with respect to $h$. The converse is evident.

**Lemma 6.4.** Let $\mathcal{H}$ be a set of morphisms between finite-dimensional Banach spaces. Then $\text{Inj}_{\text{ap}} \mathcal{H}$ is closed under products, directed colimits and ideals.

**Proof.** Following [27, 4.3], $\text{Inj}_{\text{ap}} \mathcal{H}$ is closed under products. Let $(k_i : K_i \to K)_{i \in I}$ be a directed colimit of $K_i \in \text{Inj}_{\text{ap}} \mathcal{H}$. Consider $h : A \to B$ in $\mathcal{H}, f : A \to K$ and take $\varepsilon > 0$. Following Theorem 3.1, there exists $i \in I$ and $f' : A \to K_i$ such that $k_if' \sim_{\varepsilon} f$. There is $g : B \to K_i$ such that $gh \sim \frac{\varepsilon}{2} f'$. Hence

$k_ig \sim \frac{\varepsilon}{2} k_if' \sim \frac{\varepsilon}{2} f$

Thus $(k_ig)h \sim_{\varepsilon} f$. Hence $K \in \text{Inj}_{\text{ap}}$.

The closure if $K$ under ideals follows from [27, 4.5], [27, 3.6(2)] and Proposition 4.13.

**Remark 6.5.** (1) In locally finitely presentable categories, classes of objects closed under products, directed colimits and pure submodules are called *definable* (see [17]). They coincide with classes of objects injective with respect to a set of morphisms between finitely presentable objects (see [3, 4.11]). We do not know whether this holds in $\text{Ban}$.

(2) Let $\lambda$ be an uncountable regular cardinal. In [27, 5.5], there are characterized classes $\text{Inj}_{\text{ap}} \mathcal{H}$ of Banach spaces where the domains and the codomains of morphisms in $\mathcal{H}$ have density character $< \lambda$. They are classes closed under products, $\lambda$-directed colimits and $\lambda$-pure subspaces. Here, $\lambda$-pure morphisms are defined as in Definition 4.1 with $A$ and $B$ of density character $< \lambda$. Following Remark 4.3, they coincide with weakly $\lambda$-ap-pure morphisms.

**Definition 6.6.** A morphism $f : K \to L$ in $\text{Ban}$ will be called $(\omega, \omega_1)$-pure if in every commutative square

\[
\begin{array}{c}
K \\
\downarrow f \\
A \\
\end{array} \quad \begin{array}{c}
L \\
\downarrow v \\
B \\
\end{array} \quad \begin{array}{c}
u \\
g \\
\end{array}
\]

where $A$ is finite-dimensional and $B$ separable and for every $\varepsilon > 0$ there exists $t : B \to K$ such that $tg \sim_{\varepsilon} u$.

**Theorem 6.7.** Classes of Banach spaces closed under products, directed colimits and $(\omega, \omega_1)$-pure subspaces are precisely classes of Banach spaces approximately injective with respect to a set of morphisms having finite-dimensional domain and separable codomain.

**Proof.** I. Like in Lemma 6.4, $\text{Inj}_{\text{ap}} \mathcal{H}$ is closed under products and directed colimits. The closure under $(\omega, \omega_1)$-pure subspaces follows from the proof of [27, 4.5]. Indeed, let $f : K \to L$ be $(\omega, \omega_1)$-pure and $L \in \text{Inj}_{\text{ap}} \mathcal{H}$. Consider $u : A \to K, h : A \to B$ in $\mathcal{H}$ and $\varepsilon > 0$. There exists $v : B \to L$ such that $vh \sim_{\varepsilon} fu$. Following Remark 4.3, there is $t : B \to K$ such that $th \sim_{\varepsilon} u$. Hence $K \in \text{Inj}_{\text{ap}} \mathcal{H}$. 
II. Let $\mathcal{L}$ be closed under products, directed colimits and $(\omega, \omega_1)$-pure subspaces. Like in the proof of [27, 4.8], $\mathcal{L}$ is weakly reflective in $\textbf{Ban}$. Hence every Banach space $K$ comes with a morphism $r_K : K \to K^*$, $K^* \in \mathcal{L}$ such that every object of $\mathcal{L}$ is injective with respect to $r_K$. Let $\mathcal{H}$ consist of all morphisms $\tilde{f} : A \to B$ such that $A$ is finite-dimensional, $B$ is separable and every object of $\mathcal{L}$ is approximately injective with respect to $f$. We have to prove that $\text{Inj}_\mathcal{L} \mathcal{H} \subseteq \mathcal{L}$. For this, it suffices to show that, for $K \in \text{Inj}_\mathcal{L} \mathcal{H}$, any weak reflection $r : K \to K^*$ is $(\omega, \omega_1)$-pure.

Thus, given $K \in \text{Inj}_\mathcal{L} \mathcal{H}$ and a weak reflection $r : K \to K^*$ in $\mathcal{L}$, we are to prove that in any commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow u & & \downarrow v \\
K & \xrightarrow{r} & K^*
\end{array}
$$

with $A$ finite-dimensional and $B$ separable, the morphism $u$ $\varepsilon$-factors through $h$.

**Claim:** For every $\varepsilon > 0$, there is an $\varepsilon$-factorization $u \sim_\varepsilon u_2 \cdot u_1$ and an $\varepsilon$-homomorphism $(u_1, v_1) : h \to \tilde{r}$ (i.e., $v_1 h \sim_\varepsilon \tilde{r} u_1$) where $\tilde{r} : \tilde{K} \to \tilde{K}^*$ is a weak reflection into $\mathcal{L}$ of a finite-dimensional $\tilde{K}$.

**Proof of claim.** Following the proof of [27, 4.8], for every $\varepsilon > 0$ there is a factorization $u = u_2 \cdot u_1$ and an $\varepsilon$-homomorphism $(u_1, v_1) : h \to \tilde{r}$ where $\tilde{r} : \tilde{K} \to \tilde{K}^*$ is a weak reflection into $\mathcal{L}$ of a separable $\tilde{K}$. Assume that the claim is false for $\varepsilon > 0$. Then it is false for $\frac{\varepsilon}{6}$. Take an $\frac{\varepsilon}{6}$-homomorphism $(u_1, v_1)$ from the claim with $\tilde{K}$ separable. We express $\tilde{K}$ as a colimit of a smooth chain $k_{ij} : K_i \to K_j$ ($i \leq j < \omega$) of finite-dimensional subspaces $K_i$. This provides weak reflections $r_i : K_i \to K_i^*$ into $\mathcal{L}$ such that their colimit $r_\omega : \tilde{K} \to K_\omega^*$ factorizes through $\tilde{r}$, i.e., $r_\omega = s \tilde{r}$ for some $s : \tilde{K}^* \to K_\omega^*$. Since $\tilde{r} u_1 \sim_\varepsilon \tilde{r} v_1 h$, we have $r_\omega u_1 = s \tilde{r} u_1 \sim_\varepsilon s v_1 h$, so that $(u_1, v_1) : h \to r_\omega$ is an $\frac{\varepsilon}{6}$-homomorphism. Following Theorem 3.1, there is $n < \omega, u' : A \to K_n$ and $v' : B \to K_n^*$ such that $k_n u' \sim_\varepsilon u_1$ and $k_n^* v' \sim_\varepsilon v_1$. Hence

$$
k_n^* v' h \sim_\varepsilon s v_1 h \sim_\varepsilon r_\omega u_1 \sim_\varepsilon r_\omega k_n u' = k_n^* r_n u'.
$$

Hence $k_n^* v' h \sim_\varepsilon k_n^* r_n u'$. Following Theorem 3.1, there is $n \leq m < \omega$ such that

$$
k_n^* v' h \sim_\varepsilon k_m^* v' k_n^* r_n u' = r_m k_n u'.
$$

Since

$$
u_1 \sim_\varepsilon k_n u' = k_m k_n u',
$$

we get the claim for $\varepsilon$, which yields a contradiction.

We are ready to prove that $u$ $\varepsilon$-factors through $h$. Let us consider an $\frac{\varepsilon}{3}$-factorization and an $\frac{\varepsilon}{3}$-homomorphism $(u_1, v_1) : h \to \tilde{r}$ as in the above claim for. Let us express $\tilde{K}$ as a $\omega_1$-directed colimit of separable Banach spaces $K_i$, $i \in I$, with a colimit cocone $k_i : K_i \to K^*$. Since $\tilde{K}$ is finite-dimensional and $B$ separable, there is $i \in I$, $\tilde{r}$ and $\tilde{v}_1$ such that $k_i \tilde{v}_1 = v_1$ and $k_i \tilde{r} = \tilde{r}$.
Since all objects of $\mathcal{L}$ are injective with respect to $\tilde{r}$, $\tilde{r} \in \mathcal{H}$. Thus $K$ is approximate injective with respect to $\tilde{r}$. Choosing $t : K_1 \to K$ with $u_2 \sim \frac{\varepsilon}{3} t \tilde{r}$ we obtain

$$u \sim \frac{\varepsilon}{3} u_2 u_1 \sim \frac{\varepsilon}{3} t \tilde{r} u_1 \sim \frac{\varepsilon}{3} t \tilde{y}_1 h.$$  

Thus $u \sim \frac{\varepsilon}{3} t \tilde{y}_1 h$. Hence, $r$ is $(\omega, \omega_1)$-pure.

**Remark 6.8.** We do not know whether classes of Banach spaces closed under products, directed colimits and pure subspaces are precisely classes of Banach spaces approximately injective with respect to a set of morphisms having finite-dimensional domains and codomains.

### 7. Saturation

**Definition 7.1.** Given a class $\mathcal{H}$ of isometries in a $\textbf{CMet}$-enriched category $\mathcal{K}$, an object $K$ is called approximately saturated with respect to $\mathcal{H}$ if for every $h : A \to B$ in $\mathcal{H}$, every $\varepsilon > 0$ and every isometry $f : A \to K$ there exists an isometry $g : B \to K$ such that $gh \sim \varepsilon f$.

The class of objects approximately saturated with respect to $\mathcal{H}$ will be denoted as $\text{Sat}_{ap} \mathcal{H}$.

**Lemma 7.2.** If $K$ has conical colimits, $K_0$ is wellpowered and $\mathcal{H}$ is stable under pushouts then every objects approximately saturated with respect to $\mathcal{H}$ is approximately injective with respect to $\mathcal{H}$.

**Proof.** Consider $h : A \to B$ in $\mathcal{H}$, $\varepsilon > 0$ and $f : A \to K$. Following [4, 3.14], $f = f_2 f_1$ where $f_2$ is an isometry. Let

$$
\begin{array}{cccc}
A & \xrightarrow{h} & B \\
\downarrow{f_1} & & \downarrow{\tilde{f}_1} \\
A_1 & \xrightarrow{h} & B_1
\end{array}
$$

be a pushout. Since $\tilde{h} \in \mathcal{H}$, there is an isometry $g_2 : B_1 \to K$ such that $g_1 \tilde{h} \sim \varepsilon f_2$. Then $g_2 \tilde{f}_1 h \sim \varepsilon f$. □

**Remark 7.3.**

1. In order to get Proposition 6.2 for approximately saturated objects, we would need that the adjoint isomorphism $L(h, UK) \cong K(Fh, K)$ preserves and reflects isometries.

2. If $\mathcal{H}$ consists of isometries between finite-dimensional Banach spaces then a Banach space $K$ is approximately saturated with respect to $\mathcal{H}$ iff it is of almost universal disposition for $\mathcal{H}$ in the sense of [5, 3.1] (see [18, (H)]). These spaces are called Gurari spaces.

3. Like in Remark 6.3, $K$ is Gurarii if and only if every isometry $K \to L$ satisfies the condition from Lemma 5.4. Following [1, 4.2], $K$ is Gurarii if and only if every isometry $K \to L$ is a u-extension.

4. Lemma 7.2 yields a well-known fact that every Gurarii space is a Lindenstrauss space (see [1, 4.1 and 4.3]).

**Remark 7.4.** In [4, 5.23(1)], a morphism $f : K \to L$ was called approximately split if for every $\varepsilon > 0$ there is $s : L \to K$ such that $sf \sim \varepsilon f$. Since an approximately split morphism is pure, it is an isometry.

**Lemma 7.5.** A Banach space $K$ is Lindenstrauss if and only if it has an approximately split morphism $K \to L$ to a Gurarii space $L$.

**Proof.** Following Remark 7.3(4), every Gurarii space is Lindenstrauss. Since approximately split morphisms are pure (see Remark 7.4), Lemma 6.4 implies that approximately split subspaces of a Gurarii space are Lindenstrauss.

Conversely, let $K$ be a Lindenstrauss space. Following [10, 3.6], there is an isometry $f : K \to L$ where $L$ is Gurarii. Then Remark 6.3 and Proposition 4.13 imply that $f$ approximately splits. □
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