QCD effective potential with strong $U(1)_{em}$ magnetic fields

Sho Ozaki *

Institute of Physics and Applied Physics,
Yonsei University, Seoul 120-749, Korea

Abstract

We derive the analytic expression for the one-loop $SU(N_c)$ QCD effective potential including $N_f$ flavor quarks which nonlinearly interact with the chromomagnetic background field and the external $U(1)_{em}$ magnetic field. After the renormalization of couplings and fields, we obtain the correct one-loop $\beta$ functions of both QCD and QED, and the resulting effective potential satisfies the renormalization group equation. We investigate the effect of the magnetic field on the QCD vacuum by using the effective potential, in particular for the color $SU(3)$ case with the three flavors $(u,d,s)$. Our result shows that the chromomagnetic field prefers to be parallel to the external magnetic field. Furthermore, quark loop contributions to the effective potential with the magnetic field enhance gluonic contributions, and then the chromomagnetic condensate increases with an increasing magnetic field. This result supports the recent observed gluonic magnetic catalysis at zero-temperature in lattice QCD.

* e-mail: sozaki@yonsei.ac.kr
I. INTRODUCTION

Recently, quantum chromodynamics (QCD) which is the fundamental theory of the strong interaction, under an extremely strong magnetic field is extensively investigated. There are numerous studies related to QCD in the presence of the strong magnetic field, including the chiral magnetic effect [1,2], the magnetic catalysis [3,5], and hadron properties under the strong magnetic field [6,7], in terms of both effective models and lattice QCD. The question of how the QCD vacuum and hadron properties are affected by the strong magnetic field when the strength of the magnetic field approaches or exceeds the QCD scale $eB \gtrsim \Lambda_{QCD}^2$ is a very interesting one. Such a question is not academic but quite realistic. It has been recognized that very strong magnetic fields are generated in the relativistic heavy ion collision. The strength of the magnetic field would reach the scale of $\Lambda_{QCD}^2$ [1,8,9]. Furthermore it is a great theoretical advantage that the lattice QCD can simulate the strongly interacting quark and gluon system in the presence of strong magnetic fields without the sign problem that appeared at finite density.

In QCD under the strong magnetic field, two kinds of strong dynamics coexist, namely, strongly interacting quark and gluon dynamics which is governed by QCD and nonlinearly interacting quark and strong magnetic field dynamics which is governed by QED. Since gluons do not directly interact with the magnetic field (photon)–only quarks do in QCD–the effect of the magnetic field is reflected on QCD through the quarks. Therefore we need quarks that nonlinearly interact with electromagnetic fields as well as gluon fields. A technique to calculate the fermion propagator nonlinearly interacting with constant electromagnetic fields and the nonlinear effective action of QED in terms of the field theory was developed by J. Schwinger [10] and is called Schwinger’s proper time method. Using the technique, several nonlinear QED effects such as the Schwinger mechanism [10], vacuum birefringence [11,12], and photon splitting [12,14] are widely explored. Recently, the analytic expression for the vacuum polarization tensor of a photon within the one loop of a fermion including all order interaction with the external magnetic field, which is necessary for the description of vacuum birefringence, is obtained by the authors of [15]. Schwinger’s proper time method is also applied to evaluate the QCD effective potential [16,21] with the covariantly constant background field [22,24]. Although these analyses of the QCD effective potential are based on the one-loop calculation (but including all order interaction with gluon fields), they
qualitatively reproduce nonperturbative features of QCD, such as the static linear potential between two opposite color charges at zero temperature \[25, 27\] and the deconfinement transition at finite temperature \[28, 30\].

In this paper, we study the QCD vacuum in the presence of the strong magnetic field at zero temperature by means of the QCD effective potential. In order to investigate the effect of magnetic fields, we take into account the quark contributions which nonlinearly interact with the magnetic fields and the gluon fields. Diagrammatic contributions from the quark loop to the effective potential are depicted in Fig. 1. The calculation of these diagrams allows us to explore the sea quark effect with the magnetic field, the importance of which is emphasized in recent lattice studies \[31\] in the context of the (inverse) magnetic catalysis. Galilo and Nedelko have derived the integral form of the quark effective potential and numerically performed it \[32\]. In their results, the chromomagnetic(-electric) field prefers to be parallel (perpendicular) to the magnetic field. These behaviors are confirmed by recent SU(2) \[33, 34\] and SU(3) \[35\] lattice QCD simulations. Furthermore, Bali \textit{et al.} \[35\] have also reported the gluonic magnetic catalysis at zero temperature, which is an enhancement of the gluonic action density induced by the magnetic field. In these lattice studies \[33, 35\], a significant correlation between the chromomagnetic component of the plaquette energy and the external magnetic field is pointed out, and then the chromomagnetic component gives the larger contribution to the plaquette energy than the chromoelectric component. In this study, we focus on the chromomagnetic component of the gluon field and explore an influence of the external magnetic field on QCD through the quark loop contribution. We find that in the case of the pure chromomagnetic background, one can obtain the analytic expression for the QCD effective potential in the presence of the magnetic field. Then, we compare our analytic results with the previous study \[32\], in which the proper time integral is numerically performed, and also with current lattice QCD data \[33, 35\].

We study the QCD effective potential with the magnetic field, in particular for the color SU(3) case with the three flavors \((u, d, s)\). Our results show that the chromomagnetic field prefers to be parallel (or antiparallel) to the external magnetic field, which is consistent with the previous study \[32\] and recent lattice results \[33, 35\]. Furthermore, including the pure Yang-Mills (YM) part (gluon and ghost loops), we also obtain the correct one-loop \(\beta\) functions of both QCD and QED, and the total effective potential is renormalization-group invariant. When the magnetic field is turned off, the QCD effective potential as a
function of the chromomagnetic field has a minimum away from the origin. This minimum corresponds to the dynamical generation of the chromomagnetic condensate [16–21]. When the magnetic field is turned on, the sea quark effect with the magnetic field enhances gluonic (gluon and ghost loop) contributions, and then the chromomagnetic condensate increases with an increasing magnetic field. This result supports the gluonic magnetic catalysis at zero temperature reported by Bali et al.

This paper is organized as follows. In Sec. II, we derive the quark part of the QCD effective potential with the pure chromomagnetic background and the external $U(1)_{em}$ magnetic field. Including the YM part, we investigate the properties of the QCD effective potential in the presence of the magnetic field, especially for the color $SU(3)$ case with the three flavors ($u, d, s$) in Sec. III. Finally, we conclude our study in Sec. IV. In the Appendices, we provide a derivation of the one-loop effective potential for the $SU(N_c)$ pure Yang-Mills theory, and the relation between the dimensional regularization and the cutoff regularization.
II. ONE-LOOP EFFECTIVE POTENTIAL OF SU($N_c$) QCD WITH PURE CHROMOMAGNETIC BACKGROUND AND EXTERNAL U(1)$_{em}$ MAGNETIC FIELD

We shall start with the SU($N_c$) QCD Lagrangian with U(1)$_{em}$ electromagnetic fields which is given by

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \bar{q} i\gamma_\mu D^\mu - M_q q, \quad (1)$$

where the covariant derivative contains the photon field as well as the gluon field,

$$D_\mu = \partial_\mu - igA^A_\mu T^A - ieQ q^a_\mu, \quad (2)$$

and the field strengths are

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + gf^{ABC} A^B_\mu A^C_\nu, \quad (3)$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu.$$  

Here $A^A_\mu$ is the non-Abelian gauge (gluon) field, while $a^\mu$ is the U(1)$_{em}$ electromagnetic gauge (photon) field. The quark field $q$ has $N_c$ components for the color, $N_f$ components for the flavor and four components for the spinor. The quark mass and charge matrices are given by $M_q = \text{diag}(m_{q_1}, m_{q_2}, \ldots, m_{q_{N_f}})$ and $Q_q = \text{diag}(Q_{q_1}, Q_{q_2}, \ldots, Q_{q_{N_f}})$, respectively. In this study, we consider constant external electromagnetic fields, namely,

$$\partial f = 0, \quad (4)$$

and we do not consider a quantum fluctuation of photon. In particular, we will concentrate on constant magnetic fields later. We apply the background field method for the non-Abelian gauge field and decompose the gauge field into a background field and a quantum fluctuation as

$$A^A_\mu = \hat{A}^A_\mu + \mathcal{A}^A_\mu, \quad (5)$$

where $\hat{A}^A_\mu$ is a slowly varying classical background field corresponding to the low-energy mode of the gluon field, while $\mathcal{A}^A_\mu$ is a quantum fluctuation corresponding to the high-energy mode. We choose the covariantly constant field as a background field, which satisfies the following condition \[22,23\]

$$\hat{D}^{AC}_\rho \hat{F}^{C\mu\nu} = 0, \quad (6)$$

5
with $\hat{D}_\nu^{AC} = \partial_\nu \delta^{AC} + gf^{ABC} \hat{A}_\nu^B$ and $\hat{F}^{A}_{\mu\nu} = \partial_\mu \hat{A}_\nu^A - \partial_\nu \hat{A}_\mu^A + gf^{ABC} \hat{A}_\mu^B \hat{A}_\nu^C$. Here we suppose that $\hat{F}$ is a very slowly varying field so that $\partial \hat{F} = 0$ owing to the large wave length of the low-energy mode of the gluon field. The deviation from the homogeneity of the background field is treated as the quantum fluctuation $A$, as discussed in [32]. Then, $\hat{F}$ can be written as

$$\hat{F}^A_{\mu\nu} = F^A_{\mu\nu} \hat{n}^A,$$

where $F^A_{\mu\nu}$ and $\hat{n}$ are $x$ independent. $\hat{n}$ is a unit vector in the color space and normalized as $\hat{n}^A \hat{n}^A = 1$. The background gauge field is proportional to $\hat{n}$,

$$\hat{A}^A_\mu = A_\mu \hat{n}^A,$$

and $F_{\mu\nu}$ has an Abelian form,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Actually $\hat{A}^A_\mu$ obeys the condition (6). Then, the full non-Abelian field strength can be written as

$$F^A_{\mu\nu} = F^A_{\mu\nu} \hat{n}^A + \left( \hat{D}^A_{\mu} A^C_\nu - \hat{D}^{AC} A^C_\nu \right) + gf^{ABC} A^B_\mu A^C_\nu.$$

In order to perform the functional integral of the gauge field fluctuation, we have to fix the gauge. Here we apply the background gauge:

$$\hat{D}^{AC} A^{C\mu} = 0.$$

We can obtain the effective action for $\hat{A}$ by performing the following functional integral

$$\exp \left[ i S_{eff}(\hat{A}) \right] = \int D\mathcal{A} Dc D\bar{c} Dq D\bar{q} \exp \left\{ i \int d^4 x \left[ -\frac{1}{4} \left( F_{\mu\nu} \hat{n}^A + \left( \hat{D}^A_{\mu} A^C_\nu - \hat{D}^{AC} A^C_\nu \right) \right) + gf^{ABC} A^B_\mu A^C_\nu \right] + \bar{c} \left( i\gamma_\mu D^\mu - M_q \right) c - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right\},$$

where $c$ is the ghost field, and $\xi$ is the gauge parameter. To calculate the one-loop effective action, we evaluate the functional integral for second-order field fluctuations and omit higher order terms. From the functional integrals of the second-order gluon, ghost, and quark field fluctuations, we get

$$\int D\mathcal{A} \exp \left\{ \int d^4 x -\frac{i}{2} \mathcal{A}^{\mu} \left[ -\left( \hat{D}^2 \right)^{AC} g^{\mu\nu} - 2gf^{ABC} \hat{F}^B_{\mu\nu} \right] \mathcal{A}^C_\nu \right\} = \det \left[ -\left( \hat{D}^2 \right)^{AC} g^{\mu\nu} - 2gf^{ABC} \hat{F}^B_{\mu\nu} \right]^{-1/2}.$$
\[
\int \mathcal{D}c\mathcal{D}\bar{c} \exp \left\{ i \int d^4 x \ \bar{c}^A \left[ -\left( \hat{D}^2 \right)^{AC} \right] c^C \right\} = \det \left[ -\left( \hat{D}^2 \right)^{AC} \right]^{+1},
\]
\[
\int \mathcal{D}q\mathcal{D}\bar{q} \exp \left\{ i \int d^4 x \ \bar{q} \left( i\gamma_\mu \hat{D}^\mu - M_q \right) q \right\} = \det \left[ i\gamma_\mu \hat{D}^\mu - M_q \right]^{+1}. \tag{13}
\]

Here we take the Feynman gauge, \( \xi = 1 \). The resulting effective action for gluon and ghost parts, namely, for the YM theory is well known (see [16–21], and we also provide a review for a derivation of the one-loop effective action for the YM theory in Appendix A). Integrating out the quantum fluctuation of the gluon field, we can focus on the low-energy mode of the gluon and obtain its effective action. In the quark part, the covariant derivative includes the background field (low-energy mode) of the gluon field and the external electromagnetic field,

\[
\hat{D}^\mu = \partial^\mu - ig\hat{A}_A^\mu T^A - ieQ_a \mu.
\tag{14}
\]

In this section, we analytically derive the effective potential of the quark part in which quarks nonlinearly interact with the low-energy mode of the gluon field and the external electromagnetic field. Together with the gluon and ghost parts in (13) which contain the gluon dynamics, the effective potential allows us to investigate how the low-energy mode of the gluon field such as a gluon condensate is influenced by the external electromagnetic fields through the quark loop.

### A. Quark part of QCD effective potential

The quark part of the QCD effective action is given as

\[
iS_q = \log \det \left[ i\gamma_\mu \hat{D}^\mu - M_q \right]. \tag{15}\]

To calculate this log det, we have to diagonalize the matrix \( \hat{n}^A T^A \). Since this matrix is Hermitian, the diagonalization is possible. From a certain color rotation, we can get

\[
U \hat{n}^A T^A U^\dagger = \text{diag}(w_1, w_2, \cdots, w_{N_c}). \tag{16}\]

For the \( SU(2) \) case, the diagonalization is straightforward, and we can easily obtain the eigenvalues \( w_1 = +1/2, \ w_2 = -1/2 \). We will discuss the eigenvalues for the \( SU(3) \) case in the next section. The eigenvalues \( w_a \) satisfy the following relations:

\[
\sum_{a=1}^{N_c} w_a^2 = \text{tr}(\hat{n}^A T_A \hat{n}^B T_B) = \hat{n}^A \hat{n}^B \text{tr} T^A T^B = \hat{n}^A \hat{n}^B \frac{1}{2} \delta^{AB} = \frac{1}{2}, \tag{17}\]
\[
\sum_{a=1}^{N_c} w_a = \text{tr}(\hat{n}^A T^A) = \hat{n}^A \text{tr} T^A = 0. \quad (18)
\]

These relations play an important role in the latter results. Using the eigenvalues \(w_a\), the quark part of the effective action can be written as
\[
iS_q = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \log \left[ \det (iD_{a,i} - m_{q_i}) \right],
\]
with \(D_{a,i}^\mu = \partial^\mu - iA_{a,i}^\mu\), where the fields \(A_{a,i}^\mu\) are linear combinations of the gluon field \(A^\mu\) and the photon field \(a^\mu\) as
\[
A_{a,i}^\mu = g_{w_a} A^\mu + eQ_{q_i} a^\mu. \quad (20)
\]

Since now the gluon field \(A^\mu\) is Abelian like a photon field \(a^\mu\), the linear combined gauge field \(A_{a,i}^\mu\) is also Abelian. Therefore, the field strength of \(A_{a,i}^\mu\) has the Abelian form
\[
F_{a,i}^{\alpha\beta} = \partial^\alpha A_{a,i}^\beta - \partial^\beta A_{a,i}^\alpha. \quad (21)
\]

Using the relations \(\log[\det(iD_{a,i} - m_{q_i})] = \frac{1}{2}\log[\det(D_{a,i}^2 + m_{q_i}^2)]\) and \(D_{a,i}^2 = -D_{a,i}^2 - \frac{1}{2}\sigma_{\alpha\beta} F_{a,i}^{\alpha\beta}\), we proceed to evaluate the quark effective action,
\[
iS_q = \frac{1}{2} \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \text{Tr} \left[ \log \left( -D_{a,i}^2 - \frac{1}{2}\sigma_{\alpha\beta} F_{a,i}^{\alpha\beta} + m_{q_i}^2 \right) \right]
\]
\[
= \frac{1}{2} \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \int d^4x \text{tr}(x) \left[ \log \left( -D_{a,i}^2 - \frac{1}{2}\sigma_{\alpha\beta} F_{a,i}^{\alpha\beta} + m_{q_i}^2 \right) \right] |x). \quad (22)
\]
Here we use the following identity,
\[
\log(\hat{M} - i\delta) = \frac{1}{\epsilon} - \frac{i\epsilon}{\epsilon \Gamma(\epsilon)} \int_0^\infty \frac{ds}{s^{1-\epsilon}} e^{-is(\hat{M} - i\delta)},
\]
(23)
in the limits of \(\epsilon \to 0\) and \(\delta \to 0\). The first divergent term can be omitted since this term does not depend on the fields. Furthermore, \(\epsilon \Gamma(\epsilon) = 1 + O(\epsilon)\). Therefore, the effective action becomes
\[
iS_q = -\frac{i\epsilon}{2} \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \int d^4x \int_0^\infty \frac{ds}{s^{1-\epsilon}} e^{-is(m_i^2 - i\delta)} \text{tr} \langle x | e^{-i\hat{H}_{a,i}s} | x \rangle.
\]
(24)
Here we have defined the Hamiltonian \(\hat{H}_{a,i} = -D^2_{a,i} - \frac{1}{2} \sigma_{a\beta} F_{a,i}^{\alpha\beta}\). Now, let us consider the matrix element \(\langle x' | e^{-i\hat{H}_{a,i}s} | x'' \rangle\). In this matrix element, \(e^{-i\hat{H}_{a,i}s}\) can be regarded as a time evolution operator, bringing about the time evolution of the system which is governed by the Hamiltonian \(\hat{H}_{a,i}\). The matrix element \(\langle x' | e^{-i\hat{H}_{a,i}s} | x'' \rangle\) gives an amplitude for a "particle" governed by \(\hat{H}_{a,i}\) to travel from \(x''\) to \(x'\) in a given time \(s\). Schwinger developed a technique to evaluate the matrix element in the case of the constant field of QED, called the Schwinger's proper time method [10]. Because in our case the combined gauge field \(A_{a,i}^{\mu}\) is Abelian and satisfies \(\partial F_{a,i} = 0\), we can apply the same technique as in QED to calculate the matrix element \(\langle x' | e^{-i\hat{H}_{a,i}s} | x'' \rangle\). Then, we find
\[
\langle x' | e^{-i\hat{H}_{a,i}s} | x'' \rangle = -\frac{i}{(4\pi)^2} \Psi_{a,i}(x', x'') e^{-L_{a,i}(s)} s^{-2} \times \exp \left[ \frac{i}{4} (x' - x'') F_{a,i} c \coth(F_{a,i}s)(x' - x'') \right] \exp \left[ \frac{i}{2} \sigma F_{a,i}s \right],
\]
(25)
where
\[
\Psi_{a,i}(x', x'') = \exp \left[ i \int_{x''}^{x'} A_{a,i}^{\mu}(x) + \frac{1}{2} F_{a,i}^{\mu\nu}(x - x') \right] dx_{\mu},
\]
\[
L_{a,i}(s) = \frac{1}{2} \text{tr} \log \left[ (F_{a,i}s)^{-1} \sinh(F_{a,i}s) \right].
\]
(26)
If we replace \(A_{a,i}^{\mu}\) and \(F_{a,i}^{\mu\nu}\) by \(ea^{\mu}\) and \(ef^{\mu\nu}\), the matrix element reproduces the result of QED [10, 36, 37]. Here the function \(\Psi_{a,i}(x', x'')\) is independent of the integration path, since the integrand \(A_{a,i}^{\mu}(x) + \frac{1}{2} F_{a,i}^{\mu\nu}(x - x') \) has a vanishing curl [10, 36, 37]. By restricting the integration path to be a straight line connecting \(x''\) to \(x'\), we can simply write
\[
\Psi_{a,i}(x', x'') = \exp \left[ i \int_{x''}^{x'} A_{a,i}^{\mu}(x) dx_{\mu} \right].
\]
(27)
When we take the coincidence limit: \( x'' \to x' \), this function becomes unity: \( \Psi_{a,i}(x', x'') \to 1 \). We can also evaluate the following quantities when we take the coincidence limit:

\[
\text{tr exp} \left[ \frac{i}{2} \sigma F_{a,i} s \right] = 4 \cos(a_{a,i}s) \cosh(b_{a,i}s),
\]

\[
e^{-L_{a,i}(s)} = \frac{(a_{a,i}s)(b_{a,i}s)}{\sin(a_{a,i}s) \sinh(b_{a,i}s)},
\]

as in the QED calculation [10, 36, 37]. Here \( a_{a,i} \) and \( b_{a,i} \) are related to the four eigenvalues \( \pm F^{(1)}_{a,i}, \pm F^{(2)}_{a,i} \) of the field strength tensor \( F^{\mu\nu}_{a,i} \) as [10]

\[
\pm F^{(1)}_{a,i} = \pm ia_{a,i}, \quad \pm F^{(2)}_{a,i} = \pm b_{a,i}
\]

with

\[
a_{a,i} = \frac{1}{2} \sqrt{F_{a,i}^4 + (F_{a,i} \cdot \tilde{F}_{a,i})^2 + F_{a,i}^2}, \quad b_{a,i} = \frac{1}{2} \sqrt{F_{a,i}^4 + (F_{a,i} \cdot \tilde{F}_{a,i})^2 - F_{a,i}^2}.
\]

The dual field strength tensor \( \tilde{F}_{a,i}^{\mu\nu} \) is defined as \( \tilde{F}_{a,i}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{a,i \alpha \beta} \). By using (21), \( F_{a,i}^2 \) and \( F_{a,i} \cdot \tilde{F}_{a,i} \) can be expressed as

\[
F_{a,i}^2 = 2 \left[ (gw_a)^2 (\tilde{H}_{c}^2 - \tilde{E}_{c}^2) + (eQ_i)^2 (\tilde{B} - \tilde{E})^2 + 2gw_a e Q_i (\tilde{H}_{c} \cdot \tilde{B} - \tilde{E} \cdot \tilde{E}) \right],
\]

\[
F_{a,i} \cdot \tilde{F}_{a,i} = -4 \left[ (gw_a)^2 \tilde{E}_{c} \cdot \tilde{H}_{c} + (eQ_i)^2 \tilde{E} \cdot \tilde{B} + gw_a e Q_i (\tilde{E}_{c} \cdot \tilde{B} + \tilde{E} \cdot \tilde{H}_{c}) \right].
\]

We see that the chromoelectromagnetic fields and \( U(1)_{em} \) electromagnetic fields are coupled to each other through the quark loop. Since \( x' \to x'' = x \) in the local effective action calculation, the matrix element (25) can be simplified by taking the trace as

\[
\text{tr}(x') e^{-i H_{a,i}s} |x \rangle = -\frac{i}{4\pi^2 s^2} \frac{(a_{a,i}s)(b_{a,i}s)}{\sin(a_{a,i}s) \sinh(b_{a,i}s)} \cos(a_{a,i}s) \cosh(b_{a,i}s).
\]

The effective action of the quark part is then

\[
i S_q = \sum_{a=1}^{N_f} \sum_{i=1}^{N_c} \frac{i^{1+i}}{8\pi^2} \int d^4x \int_0^\infty ds \frac{d^3s}{e^{-is(m_{a,i} - i\delta)}} \frac{(a_{a,i}s)(b_{a,i}s)}{\sin(a_{a,i}s) \sinh(b_{a,i}s)} \cos(a_{a,i}s) \cosh(b_{a,i}s).
\]

Taking the Wick rotation for the proper time \( s \),

\[
\int_0^\infty ds = \int_0^{-i\infty} ds,
\]

and changing the integral variable \( s \to -is \), the action reads

\[
i S_q = -\sum_{a=1}^{N_f} \sum_{i=1}^{N_c} \frac{i}{8\pi^2} \int d^4x \int_0^\infty ds \frac{d^3s}{e^{-ms_{a,i}}} \frac{(a_{a,i}s)(b_{a,i}s)}{\sin(a_{a,i}s) \sinh(b_{a,i}s)} \cos(a_{a,i}s) \cosh(b_{a,i}s).
\]
Here we have taken $\delta \to 0$ after the rotation. With the effective action, the effective Lagrangian is defined as

$$
\mathcal{L}_q = \frac{S_q}{\int d^4x} = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m_{qi}^2s}}{s^{3-\epsilon}} \left( a_{a,i}s \right) \left( b_{a,i}s \right) \cosh(a_{a,i}s)\cos(b_{a,i}s). \tag{36}
$$

This is the same form as the integral representation in [32]. Now, we consider the pure chromomagnetic background and an external magnetic field, namely, $b_{a,i} \to 0$ and

$$
a_{a,i} = \sqrt{(gw_a\tilde{H}_c + eQ_q\vec{B})^2} = \sqrt{(gw_aH_c)^2 + (eQ_q\vec{B})^2 + 2gw_a eQ_q H_c B \cos \theta_{H_cB}}, \tag{37}
$$

where $H_c = \sqrt{\tilde{H}_c^2}$ and $B = \sqrt{\vec{B}^2}$. $\theta_{H_cB}$ stands for the angle between the chromomagnetic field $\tilde{H}_c$ and the $U(1)_{em}$ magnetic field $\vec{B}$. Consequently, the effective Lagrangian of the quark part is given as

$$
\mathcal{L}_q = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \frac{a_{a,i}}{8\pi^2} \int_0^\infty ds \frac{e^{-m_{qi}^2s}}{s^{2-\epsilon}} \coth(a_{a,i}s). \tag{38}
$$

Using the integral representation for the generalized zeta function [38],

$$
\int_0^\infty d\tau \, \tau^{\alpha-1} e^{-\beta\tau} \coth(\tau) = \Gamma(\alpha) \left[ 2^{1-\alpha} \zeta\left( \alpha, \frac{\beta}{2} \right) - \beta^{-\alpha} \right], \tag{39}
$$

we obtain

$$
\mathcal{L}_q = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ -\frac{m_{qi}^4}{16\pi^2} \left( \frac{1}{\epsilon - \gamma_E} \right) - \frac{a_{a,i}^2}{24\pi^2} \left( \frac{1}{\epsilon - \gamma_E} \right) + \frac{a_{a,i}^2}{24\pi^2} \left[ \log(2a_{a,i}) + 12\zeta'(-1, \frac{m_{qi}^2}{2a_{a,i}}) - 1 \right] - \frac{a_{a,i}m_{qi}^2}{8\pi^2} \log\left( \frac{2a_{a,i}}{m_{qi}^2} \right) + \frac{m_{qi}^4}{16\pi^2} \left[ \log(2a_{a,i}) - 1 \right] \right\}, \tag{40}
$$

where $\gamma_E = 0.577 \cdots$ is the Euler constant, and $\zeta(s, \lambda)$ is the generalized zeta function. The derivative of the generalized zeta function $\zeta'(s_0, \lambda)$ is defined as $\zeta'(s_0, \lambda) = d/ds \zeta(s, \lambda)|_{s=s_0}$. Using the relations between the dimensional regularization and the cutoff regularization (see
Appendix B), we replace the divergences in (40) by a ultraviolet cutoff $\Lambda$ as

\[
\mathcal{L}_q = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ -\frac{1}{16\pi^2} \left( \Lambda^4 - 2m_{q_i}^2\Lambda^2 + m_{q_i}^4\log\Lambda^2 \right) - \frac{a_{a,i}^2}{24\pi^2}\log\Lambda^2 \right. \\
+ \frac{a_{a,i}^2}{24\pi^2} \left[ \log \left( 2a_{a,i} \right) + 12\zeta'(-1, \frac{m_{q_i}^2}{2a_{a,i}}) - 1 \right] \\
- \frac{a_{a,i}m_{q_i}^2}{8\pi^2}\log \left( \frac{2a_{a,i}}{m_{q_i}^2} \right) + \frac{m_{q_i}^4}{16\pi^2}\left[ \log \left( 2a_{a,i} \right) - 1 \right] \left\} . \tag{41}
\]

The first divergent terms in the bracket are independent of any fields. Thus these terms do not contribute to the dynamics, and we can simply omit this part. We will see soon that the second logarithmic divergent term which is proportional to $a_{a,i}^2$ can be absorbed by the renormalization of couplings and fields. When $B = 0$, the effective Lagrangian (41) reduces to the quark part of the QCD effective Lagrangian [20, 21], whereas when $H_c = 0$ with $N_c = N_f = 1$ and $Q_q = 1$, the Lagrangian reduces to the Euler-Heisenberg Lagrangian of QED [39] with pure magnetic fields [36, 40], as we expected. In order to satisfy $\mathcal{L}_q \to 0$ at the zero field point $H_c = B = 0$, we add the following field-independent terms:

\[
N_c \times \sum_{i=1}^{N_f} \left\{ -\frac{m_{q_i}^4}{16\pi^2} \left( \log(m_{q_i}^2) - \frac{3}{2} \right) \right\} . \tag{42}
\]

Now, the effective potential of quark part $V_q = -\mathcal{L}_q$ can be written as

\[
V_q = V_q^{fin} + V_q^{div}, \tag{43}
\]

where

\[
V_q^{fin} = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ -\frac{a_{a,i}^2}{24\pi^2} \left[ \log \left( 2a_{a,i} \right) + 12\zeta'(-1, \frac{m_{q_i}^2}{2a_{a,i}}) - 1 \right] \\
+ \frac{a_{a,i}m_{q_i}^2}{8\pi^2}\log \left( \frac{2a_{a,i}}{m_{q_i}^2} \right) - \frac{m_{q_i}^4}{16\pi^2}\left[ \log \left( \frac{2a_{a,i}}{m_{q_i}^2} \right) + \frac{1}{2} \right] \right\} , \tag{44}
\]

and

\[
V_q^{div} = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \frac{a_{a,i}^2}{24\pi^2}\log\Lambda^2. \tag{45}
\]

In this form of the effective potential, we can take the massless limit $m_{q_i} \to 0$ without any infrared divergences. In the massless limit, the finite part of the effective potential becomes

\[
V_q^{fin} = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ -\frac{a_{a,i}^2}{24\pi^2} \left[ \log(a_{a,i}) - c_q \right] \right\} , \tag{46}
\]

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where \( c_q = -12 \zeta'(-1,0) + 1 - \log 2 = 12 \log G - \log 2 \). Here we have used \( \zeta'(-1,0) = \frac{1}{12} - \log G \), and \( G = 1.282427 \cdots \) is the Glaisher constant. It is worth mentioning that in the divergent part (45), the cross terms \( gw_a e Q_q H_c B \cos \theta_{H_c B} \) in \( a^2_{a,i} \) cancel after the summation of the color index thanks to the property of the eigenvalues \( w_a \) (18). Thus, the logarithmic divergences are proportional to either \( H^2_c \) or \( B^2 \) as

\[
V^{\text{div}}_q = \frac{N_f}{48\pi^2} (gH_c)^2 \log \Lambda^2 + \frac{N_c}{24\pi^2} \left( \sum_{i=1}^{N_f} Q^2_{Q_i} \right) (eB)^2 \log \Lambda^2, \tag{47}
\]

and then these divergent terms can be absorbed by the renormalization of couplings and fields with the tree level potential. We will renormalize these logarithmic divergences together with that of the YM part in the next subsection.

B. Total effective potential and renormalization

The one-loop effective potential of the pure Yang-Mills theory which includes gluon and ghost parts is given by (16–21) (see also Appendix A)

\[
\Re V = V^{\text{fin}}_Y + V^{\text{div}}_Y, 
\]

where

\[
V^{\text{fin}}_Y = \frac{11 N_c}{96\pi^2} (gH_c)^2 \left\{ \log (gH_c) - c_g + \frac{1}{N_c} \sum_{h=1}^{N_c^2-1} v^2_h \log |v_h| \right\}, 
\]

\[
V^{\text{div}}_Y = -\frac{11 N_c}{96\pi^2} (gH_c)^2 \log \Lambda^2, \tag{49}
\]

and

\[
\Im V_Y = -\frac{N_c}{16\pi^2} (gH_c)^2. \tag{50}
\]

Here \( c_g = (12 + 2\log 2 - 12 \log G)/11 = 0.94556 \cdots \). \( v_h \) is the eigenvalue of the Hermitian matrix \( (T_c)^{AC} = if^{ABC} \hat{n}^B \), which can be obtained by using a certain \( (N_c^2 - 1) \times (N_c^2 - 1) \) unitary matrix as

\[
U \mathcal{T} U^\dagger = \text{diag}(v_1, v_2, \cdots, v_{N_c^2-1}). \tag{51}
\]

For the \( SU(2) \) case, this diagonalization is also straightforward, and we can easily obtain the eigenvalues \( v_1 = +1 \), \( v_2 = -1 \) and \( v_3 = 0 \). We will discuss the eigenvalues \( v_h \) for the
SU(3) case in the next section. From the Cartan-Killing metric of $SU(N_c)$, we can get the following property of $v_h$:

$$\sum_{h=1}^{N^2-1} v_h^2 = \text{tr}(T_c T_c) = \delta^{AB} N_c \delta^{AB} = N_c.$$  

(52)

The imaginary part of the effective potential [50] corresponds to the Nielsen-Olesen (N-O) instability [18].

Here we should comment on the N-O instability. The origin of the N-O instability is well known as follows. After the diagonalization of the gluon in the color space, the quantum fluctuations $A$ correspond to the off-diagonal gluons. The diagonal parts of $A$ do not couple to the background field $\hat{A}$ and thus do not contribute to the dynamics [21]. Without loss of generality, we can choose the orientation of $\vec{H}_c$ as the $z$ direction. Then, the energy spectra of the massless off-diagonal gluon in the presence of the background chromomagnetic field are given by

$$E_n = \sqrt{p_z^2 + 2|gv_h H_c|(n + 1/2)} - 2gv_h H_c S_z$$

where $n = 0, 1, 2 \cdots$ and $S_z = \pm 1$. The lowest Landau level (LLL) $n = 0$ with $S_z = +1$ gives the tachyonic mode for $gv_h H_c > 0$, because $E_0 = \sqrt{p_z^2 - |gv_h H_c|}$ become pure imaginary when $p_z^2 < |gv_h H_c|$. This is the origin of the imaginary part of the effective potential, namely, the N-O instability. Now, in the low energy (strong-coupling) region, the true ground state (vacuum) of the YM theory should be stable, so the N-O instability obtained by the one-loop calculation could be stabilized by nonperturbative dynamics of the YM theory.  

Although the N-O instability in the strong-coupling region would be a nontrivial problem as well as the Schwinger mechanism in the strong-coupling region [43], there is one possibility to stabilize the vacuum. In the strong-coupling region, quenched lattice QCD simulations observe the large mass of the off-diagonal gluon $M_A \gtrsim 1 \text{GeV}$ in the maximally Abelian gauge [44–46] and also in the Cho-Faddeev-Niemi decomposition [47]. Since there is no mass term in the original YM Lagrangian, it can be regarded as a dynamical generation of the off-diagonal gluon mass caused by nonperturbative gluodynamics. Then, the large off-diagonal gluon mass could shift the energy spectrum of the LLL to

$$E_0 = \sqrt{p_z^2 + M_A^2 - |gv_h H_c|}$$

and stabilize

1 However in high-energy (weak-coupling) regions, the N-O instability (pair creation of gluons) would be a physical phenomenon because in such energy regions the one-loop analysis should be justified. For instance, in the early stages of relativistic heavy ion collisions, extremely strong chromomagnetic fields (as well as chromoelectric fields) are generated. This chromomagnetic field could decay into gluons through the N-O instability as discussed in [41, 42].
the YM vacuum, as discussed in [48–50]. On the other hand, the real part of the one-loop effective potential (known as the leading log model) is qualitatively in agreement with the nonperturbative analyses for the effective potential, such as the quenched lattice QCD [51] and the functional renormalization group [52]. Thus, we expect that only the real part has physical meanings in the low-energy region. Furthermore, the quark loop contributions (43) nonlinearly interacting with chromomagnetic fields and $U(1)_{em}$ magnetic fields are always real within the one-loop level. Therefore, in this study, we concentrate on the real part of the effective potential and investigate the effect of the magnetic field on the QCD vacuum.

Including the tree level potential and the quark part of the effective potential, the total QCD effective potential is given by

$$V_{\text{eff}} = \frac{H_c^2}{2} + \frac{B^2}{2} + V^{\text{fin}} + V^{\text{div}},$$

where

$$V^{\text{fin}} = \frac{11N_e}{96\pi^2} (gH_c)^2 \left\{ \log (gH_c) - c_g + \frac{1}{N_c} \sum_{h=1}^{N^2_c-1} v_h^2 \log |v_h| \right\}$$

$$+ \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ -\frac{a_{a,i}^2}{24\pi^2} \left[ \log (2a_{a,i}) + 12\zeta'(-1, \frac{m_{q_i}^2}{2a_{a,i}}) - 1 \right] - \frac{a_{a,i}m_{q_i}^2}{8\pi^2} \log \left( \frac{2a_{a,i}}{m_{q_i}^2} \right) - \frac{m_{q_i}^4}{16\pi^2} \left[ \log \left( \frac{2a_{a,i}}{m_{q_i}^2} \right) + \frac{1}{2} \right] \right\},$$

and

$$V^{\text{div}} = \frac{1}{2} \left\{ -\frac{1}{(4\pi)^2} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right) \right\} (gH_c)^2 \log \Lambda^2$$

$$+ \frac{1}{2} \frac{N_c}{12\pi^2} \sum_{i=1}^{N_f} Q_{q_i}^2 \, (eB)^2 \log \Lambda^2.$$ (55)

Now, the divergent part of the effective potential $V^{\text{div}}$ can be absorbed by the renormalization of couplings and fields with the tree level potential. Replacing the couplings and fields in the effective potential by bare couplings $g_0$, $e_0$ and bare fields $H_{c0}$, $B_0$, the effective potential becomes

$$V_{\text{eff}} = \frac{H_{c0}^2}{2} + \frac{B_0^2}{2} + V_0^{\text{div}} + V_0^{\text{fin}},$$

where $V_0^{\text{fin}}$ and $V_0^{\text{div}}$ are the same functions as (54) and (55) with replacing $gH_c$ and $eB$ by $g_0H_{c0}$ and $e_0B_0$. Following a conventional renormalization procedure of the gauge theory,
we rescale the couplings and fields as
\[ g = Z_{3,QCD}^{1/2} g_0, \quad e = Z_{3,QED}^{1/2} e_0, \quad (57) \]
and
\[ H_{c0} = Z_{3,QCD}^{1/2} H_c, \quad B_0 = Z_{3,QED}^{1/2} B, \quad (58) \]
where the renormalization rescaling factors are given by
\[ Z_{3,QCD} = 1 + \delta_{3,QCD}, \quad Z_{3,QED} = 1 + \delta_{3,QED}. \quad (59) \]
Here \( \delta_{3,QCD} \) and \( \delta_{3,QED} \) are counterterms. Using these renormalized couplings \( g, e \) and fields \( H_c, B \), the effective potential can be written as
\[ V_{\text{eff}} = \frac{1}{2} Z_{3,QCD} H_c^2 + \frac{1}{2} Z_{3,QED} B^2 + V^\text{div} + V^\text{fin}, \quad (60) \]
where \( V^\text{fin} \) and \( V^\text{div} \) are the same functions as (54) and (55) with \( g H_c \) and \( e B \) because \( g_0 H_{c0} = g H_c \) and \( e_0 B_0 = e B \). Choosing the counterterms so that the logarithmic divergences in \( V^\text{div} \) cancel,
\[ \delta_{3,QCD} = - \frac{g^2}{(4\pi)^2} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right) \log \left( \frac{\Lambda^2}{\mu^2} \right), \]
\[ \delta_{3,QED} = - \frac{e^2}{12\pi^2} N_c \left( \sum_{i=1}^{N_f} Q_{q_i}^2 \right) \log \left( \frac{\Lambda^2}{\mu^2} \right), \quad (61) \]
we finally obtain the finite renormalized effective potential
\[ V_{\text{eff}} = \frac{H_c^2}{2} + \frac{B^2}{2} + V_{\text{YM}} + V_q, \quad (62) \]
with
\[ V_{\text{YM}} = \frac{11 N_c}{96\pi^2} (g H_c)^2 \left\{ \log \left( \frac{g H_c}{\mu^2} \right) - c_g + \frac{1}{N_c} \sum_{h=1}^{N_c^2-1} v_h^2 \log |v_h| \right\}, \quad (63) \]
\[ V_q = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ - \frac{a_{a,i}^2}{24\pi^2} \left[ \log \left( \frac{2 a_{a,i}}{\mu^2} \right) + 12 \zeta'(-1, \frac{m_{q_i}^2}{2 a_{a,i}}) - 1 \right] + \frac{a_{a,i} m_{q_i}^2}{8\pi^2} \log \left( \frac{2 a_{a,i}}{m_{q_i}^2} \right) - \frac{m_{q_i}^4}{16\pi^2} \left[ \log \left( \frac{2 a_{a,i}}{m_{q_i}^2} \right) + \frac{1}{2} \right] \right\}. \quad (64) \]
Here we have introduced an arbitrary renormalization scale point \( \mu \) in the counterterms (61), and thus the final expression of the effective potential explicitly contains \( \mu \). However the
effective potential (62) should be independent of an arbitrary renormalization scale point \( \mu \). Actually, the effective potential is \( \mu \)-independent. To see this, we consider the following renormalization group (RG) equation,

\[
\mu \frac{d}{d\mu} V(g, H_c, e, B, \mu) = 0,
\]

(65)

where couplings and fields depend on \( \mu \), namely, running couplings and fields. Since in this study we do not take into account the quark mass renormalization, \( m_q \) is independent of \( \mu \). The equation (65) gives

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta_{QCD} \frac{\partial}{\partial g} - 2\gamma_{H_c} H_c^2 \frac{\partial}{\partial H_c^2} + \beta_{QED} \frac{\partial}{\partial e} - 2\gamma_B B^2 \frac{\partial}{\partial B^2} \right] V(g, H_c, e, B, \mu) = 0, \tag{66}
\]

where \( \beta_{QCD} \) and \( \beta_{QED} \) are \( \beta \) functions of QCD and QED, defined as \( \beta_{QCD} = \mu \partial g / \partial \mu \) and \( \beta_{QED} = \mu \partial e / \partial \mu \). By using the equations (57) and the renormalization rescaling factors (59), we can evaluate these \( \beta \) functions as

\[
\beta_{QCD} = \mu \frac{\partial g}{\partial \mu} = \frac{1}{2} g \mu \frac{1}{Z_{3,QCD}} \frac{\partial Z_{3,QCD}}{\partial \mu} = -\frac{g^3}{(4\pi)^2} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right), \tag{67}
\]

\[
\beta_{QED} = \mu \frac{\partial e}{\partial \mu} = \frac{1}{2} e \mu \frac{1}{Z_{3,QED}} \frac{\partial Z_{3,QED}}{\partial \mu} = \frac{e^3}{12\pi^2} N_c \left( \sum_{i=1}^{N_f} Q_{q_i}^2 \right). \tag{68}
\]

We obtain the correct one-loop \( \beta \) functions of both QCD and QED. \( \gamma_{H_c} \) and \( \gamma_B \) are the anomalous dimensions of fields, obtained as

\[
\gamma_{H_c} = -\mu \frac{\partial H_c}{H_c \partial \mu} = \frac{1}{2} \mu \frac{1}{Z_{3,QCD}} \frac{\partial Z_{3,QCD}}{\partial \mu} = \frac{1}{g} \beta_{QCD}, \tag{69}
\]

\[
\gamma_B = -\mu \frac{\partial B}{B \partial \mu} = \frac{1}{2} \mu \frac{1}{Z_{3,QED}} \frac{\partial Z_{3,QED}}{\partial \mu} = \frac{1}{e} \beta_{QED}. \tag{70}
\]

We can easily verify that the effective potential (62) satisfies the RG equation (66). Accordingly, the effective potential is independent of the renormalization scale point \( \mu \), provided that we take into account appropriate running couplings and fields which can be obtained by solving the differential equations (67)–(70).

### III. PROPERTIES OF COLOR \( SU(3) \) QCD EFFECTIVE POTENTIAL WITH EXTERNAL \( U(1)_{em} \) MAGNETIC FIELD

Let us now focus on the color \( SU(3) \) QCD with the external \( U(1)_{em} \) magnetic field. In order to obtain the color \( SU(3) \) QCD effective potential, we need the eigenvalues of \( \hat{n}^{AT} \)
FIG. 2: Diagrammatic representations of eigenvalues $w_a$ and $\pm \lambda_a$. Left panel: the rotated weight diagram. Each $w_a$ is given by the $x$ coordinate of each vertex of the $\Theta$-rotated triangle. Right panel: the rotated root diagram. Each $\pm \lambda_a$ is given by the $x$ coordinate of vertex of the $\Theta$-rotated hexagon.

and $T_c$ for the color $SU(3)$ case. First, we consider the eigenvalues of $\hat{n}^A T^A$. To get the eigenvalues, we evaluate the determinant of the $3 \times 3$ matrix $(\hat{n}^A T^A - w I_{3 \times 3})$ and find

$$\det(\hat{n}^A T^A - w I_{3 \times 3}) = -w^3 + Aw^2 - Bw + C = (w_1 - w)(w_2 - w)(w_3 - w),$$

(71)

where

$$A = 0, \quad B = -\frac{1}{4} \hat{n}^2, \quad C = \frac{1}{12} [d_{ABC} \hat{n}^A \hat{n}^B \hat{n}^C].$$

(72)

Therefore, the eigenvalues should satisfy the following equations:

$$w_1 + w_2 + w_3 = A,$$

$$w_1 w_2 + w_1 w_3 + w_2 w_3 = B,$$

$$w_1 w_2 w_3 = C.$$  

(73)

The first equation is the same as (18). We need to solve Eqs. (73) to get the eigenvalues. Now, since $\hat{n}^A T^A$ is the $3 \times 3$ matrix and it is traceless, the diagonalized matrix of $\hat{n}^A T^A$ and thus the solutions of (73) may be expressed in terms of the two diagonal matrices $T^3$ and $T^8$ which are traceless as

$$U \hat{n}^A T^A U^\dagger = T^3 \cos \Theta - T^8 \sin \Theta = \text{diag}(w_1, w_2, w_3),$$

(74)
where
\[ w_1 = \frac{1}{\sqrt{3}} \cos\left(\Theta + \frac{\pi}{6}\right), \quad w_2 = -\frac{1}{\sqrt{3}} \cos\left(\Theta - \frac{\pi}{6}\right), \quad w_3 = \frac{1}{\sqrt{3}} \sin\Theta. \] (75)

\( \Theta \) is related to the second Casimir invariant \( C_2 = [d_{ABC} \hat{n}^A \hat{n}^B \hat{n}^C]^2 \) as
\[ \sin^2 3\Theta = 3C_2. \] (76)

Here we follow the notation of [53]. The relation of the notations to [54, 55] is discussed in [53]. The eigenvalues (75) indeed satisfy the equations (73) and also (17). The author of [53] discussed the diagrammatic interpretation of the eigenvalues (75) as depicted in the left panel of Fig. 2. Each \( w_a \) corresponds to the \( x \) coordinate of each vertex of the rotated weight diagram. \( \Theta \) is the rotating angle of the weight diagram.

Next, we shall consider the eigenvalues of \( \mathcal{T}_c = i f^{ABC} \hat{n}^B \). In order to obtain the eigenvalues, one needs to evaluate the determinant of the \( 8 \times 8 \) matrix \( (i f^{ABC} \hat{n}^B - v \delta^{AC}) \) as [55]
\[ \det(i f^{ABC} \hat{n}^B - v \delta^{AC}) = v^2(v^6 - A'v^4 + B'v^2 - C') \]
\[ = v^2(v^2 - \lambda_1^2)(v^2 - \lambda_2^2)(v^2 - \lambda_3^2), \] (77)
where
\[ A' = \frac{3}{2} \hat{n}^2, \quad B' = \frac{A'^2}{4}, \quad C' = \frac{1}{16}([\hat{n}^A \hat{n}^A]^3 - 3C_2). \] (78)

There are two zero eigenvalues, and other eigenvalues \( \lambda_a \) are all paired and satisfy
\[ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = A', \]
\[ \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 = B', \]
\[ \lambda_1^2 \lambda_2^2 \lambda_3^2 = C'. \] (79)

The first equation of (79) corresponds to the condition (52). To get paired eigenvalues \( \pm \lambda_a \), we need to solve these equations (79). Now, since two of the eigenvalues are zero and other eigenvalues are all paired, the number of independent variables is three. Furthermore, the matrix \( \mathcal{T}_c \) satisfies \( \text{tr}(\mathcal{T}_c^2) = 3 \) from (52). Then, the diagonalized matrix of \( \mathcal{T}_c \) and thus the solutions of (79) may be expressed in terms of two diagonal matrices which satisfy (52) as
\[ U \mathcal{T}_c U^\dagger = T_d^3 \cos \Theta - T_d^8 \sin \Theta, \] (80)
where $T_d^3 = \text{diag}(+1, -1, 0, +1/2, -1/2, -1/2, +1/2, 0)$ and $T_d^8 = \text{diag}(0, 0, 0, \sqrt{3}/2, -\sqrt{3}/2, \sqrt{3}/2, -\sqrt{3}/2, 0)$ are diagonalized matrices of $if^{A3C}$ and $if^{A8C}$, respectively. $T_d^3$ and $T_d^8$ have at least two zero eigenvalues, and other eigenvalues are all paired. From (80), the diagonalized matrix of $T_c$ can be written as

$$ U T U^\dagger = \text{diag}(\lambda_1, -\lambda_1, 0, \lambda_2, -\lambda_2, \lambda_3, -\lambda_3, 0), $$

(81)

where

$$ \lambda_1^2 = \cos^2\Theta, $$
$$ \lambda_2^2 = \cos^2\left(\Theta + \frac{\pi}{3}\right), $$
$$ \lambda_3^2 = \cos^2\left(\Theta + \frac{2\pi}{3}\right). $$

(82)

One can easily verify that these eigenvalues satisfy Eqs. (79) when $\Theta$ obeys (76). These results (82) coincide with the eigenvalues obtained in [55] if we use the relation of the notations mentioned in [53]. Moreover, these eigenvalues have a diagrammatic interpretation as depicted in the right panel of Fig. 2. Each $\pm \lambda_a$ corresponds to the $x$ coordinate of each vertex of the rotated root diagram. $\Theta$ is the rotating angle of the root diagram. From the symmetries of the diagrams in Fig. 2, it is sufficient to consider the angle range $-\pi/6 \leq \Theta \leq \pi/6$.

Now, let us investigate the properties of the color $SU(3)$ QCD effective potential with the magnetic field. We take into account the three flavors for quarks with $Q_u = +2/3$ and $Q_d = Q_s = -1/3$. The quark masses are taken as $m_u = m_d = 5$ MeV and $m_s = 140$ MeV. Through this section, we take the strong and the electromagnetic coupling constants so that $\alpha_s = 1$ and $\alpha_{E.M.} = 1/137$ at the renormalization scale point $\mu = 1$ GeV which is chosen as a typical hadron scale. Since the effective potential is renormalization group invariant, the following results are $\mu$-independent, provided that we take into account appropriate running couplings and fields.

Figure 3 shows the quark part of the effective potential $V_q$ as a function of $\theta_{H_eB}$ with a fixed value of $g_{H_c}$. Here we set $g_{H_c} = 0.2$ GeV. From Fig. 3, we see that $V_q$ is symmetric under the simultaneous transformations $\theta_{H_eB} \rightarrow -\theta_{H_eB} + \pi$ and $\Theta \rightarrow -\Theta$. This symmetry can be understood from the factor $a_{a,i} = \sqrt{(gw_aH_c)^2 + (eQ_{q_i}B)^2 + 2gw_a eQ_{q_i} H_c B \cos\theta_{H_eB}}$ in $V_q$ and the transformation law of $w_a$, $(w_1, w_2, w_3) \rightarrow (-w_2, -w_1, -w_3)$ under $\Theta \rightarrow -\Theta$. Another important observation in Fig. 3 is that the minima of the effective potential appear at $\theta_{H_eB} = \ldots$
FIG. 3: Quark effective potential $V_q$ as a function of $\theta_{H_cB}$ with a fixed value of the chromomagnetic field: $gH = 0.2$ GeV$^2$ and various values of the magnetic field: $eB = 0.1, 0.2, 0.3, 0.4$ GeV$^2$ and $\Theta$: $\Theta = 0, \pi/12, \pi/6, -\pi/12, -\pi/6$.

$0$ (or $\pi$) with any strengths of the magnetic field. This means that the chromomagnetic field $\vec{H}_c$ prefers to be parallel (or antiparallel) to the external $\vec{B}$-field. This result is consistent with the previous result [32] in which the proper time integral is numerically performed and also with the recent lattice results [33–35]. In what follows, we shall take $\theta_{H_cB} = 0$ with $-\pi/6 \leq \Theta \leq \pi/6$.

Next we incorporate the gluon and ghost contributions into the potential. Here we concentrate on the real part of the effective potential, as discussed in the previous section. Figure 4 shows the QCD effective potential as a function of $gH_c$ with $B = 0$. In this figure, we set $\Theta = 0$. It is well known that the one-loop YM effective potential $H_c^2/2 + V_{YM}$ has a minimum away from the origin. This minimum corresponds to the dynamical generation of the chromomagnetic condensate [16][21]. As shown in Fig. 4, the quark loop contribution
$V_q$ attenuates the gluonic contribution $V_{YM}$, owing to the opposite sign of $V_{YM}$. Then, the minimum of the total effective potential $V_{eff}$ shifts to the left-hand side from the minimum of $H^2_c/2 + V_{YM}$. We investigate how the chromomagnetic condensate behaves in the presence of the magnetic field.

Here, we have employed the strength of the strong coupling $\alpha_s = 1$ at the renormalization scale point $\mu = 1$ GeV. Another choice of the coupling strength and the renormalization scale point will give a different position of the minimum of the effective potential. However, the qualitative behavior of the effective potential is independent of the choice. In particular, the tendency of the effective potential in the presence of the magnetic field does not depend on the choice of the coupling strength and renormalization scale point.

Now we define the normalized effective potential as

$$\bar{V}(H_c, B) = V_{eff}(H_c, B) - V_{eff}(0, B),$$

so that $\bar{V}(H_c, B)$ becomes zero at $H_c = 0$. This normalized effective potential is also renormalization group invariant. The second term does not affect the minimum position. The left panel of Fig. 5 shows the magnetic field dependence of the normalized effective potential with $\Theta = 0$. The minimum shifts to the right-hand side as the magnetic field increases. This behavior is independent of $\Theta$. The right panel of Fig. 5 shows the chromomagnetic condensate $(gH_c)^2_{min}$ as a function of the magnetic field. In the small magnetic field region, $(gH_c)^2_{min}$ slowly increases. In the case of massless limit $m_{q_i} \to 0$, which is actually a good
approximation for light quarks, we can obtain the analytic form of \((gH_c)^2_{\text{min}}\) with \(eB = 0\) by calculating \(\partial V_{\text{eff}} / \partial H_c = 0\),

\[
(gH_c)^2_{\text{min},0} = \mu^4 \exp \left\{ -\frac{8\pi}{b_0\alpha_s} - 1 + \frac{2}{b_0} \left( \frac{11N_c}{3} c'_g - \frac{2N_f}{3} c'_q \right) \right\}, \quad b_0 = \frac{11N_c}{3} - \frac{2N_f}{3}, \tag{84}
\]

where \(c'_g = c_g - \frac{1}{N_c} \sum_{a=1}^{N_c} \lambda_a^2 \log \lambda_a^2\) and \(c'_q = c_q - \sum_{a=1}^{N_c} w_a^2 \log w_a^2\). Using the \((gH_c)^2_{\text{min},0}\) we find the \(eB\) dependence of the chromomagnetic condensate \((gH_c)^2_{\text{min}}\) for the small \(eB\) region \(((gH_c)_{\text{min}} \gg eB)\) as

\[
(gH_c)^2_{\text{min}} = (gH_c)^2_{\text{min},0} + \frac{(4\pi)^2}{b_0} \frac{N_c}{12\pi^2} \sum_{i=1}^{N_f} Q_i^2 eB^2, \tag{85}
\]

with \(N_c = 3\) and \(N_f = 3\) (\(u, d, s\)) in our case. In this expression, \((gH_c)^2_{\text{min}}\) quadratically increases with respect to \(eB\). We note that the coefficient of the second term is the ratio of the coefficients of \(\beta_{\text{QCD}}\) \((67)\) and \(\beta_{\text{QED}}\) \((68)\). In the large \(eB\) region, \(eB > (gH_c)_{\text{min}}\), \((gH_c)^2_{\text{min}}\) still monotonically increases as the magnetic field increases. These behaviors are quite similar to the recent lattice QCD result \([35]\) with \(N_f = 1 + 1 + 1\) staggered quarks of physical masses in which the authors insist an enhancement of the gluonic action density in the presence of the magnetic field at zero temperature, called the gluonic magnetic catalysis.

**FIG. 5:** Left panel: the magnetic field dependence of the QCD effective potential with \(\Theta = 0\). Right panel: the magnetic field dependence of the chromomagnetic condensate with various values of \(\Theta\): \(\Theta = 0, \pi/12, \pi/6, -\pi/12, -\pi/6\).

In our results, quark loop contributions which correspond to the sea quark effect discussed in \([31]\) should be important, since only \(V_q\) has \(B\) dependence. To see the importance of the
sea quark effect, we define the following quantity with the pure chromomagnetic background:

\[
\Delta \bar{V}(H_c, B) = \bar{V}(H_c, B) - \bar{V}(H_c, 0) 
= V_q(H_c, B) - V_q(0, B) - V_q(H_c, 0) 
= i \int \frac{d^4x}{d^4x} \log \left[ \frac{\det(i \hat{D}(H_c, B) - M_q)}{\det(i \hat{D}(H_c, 0) - M_q) \det(i \hat{D}(0, B) - M_q)} \right].
\] (86)

This quantity \(\Delta \bar{V}\) indicates the change of the normalized effective potential \(\bar{V}\) measured from the one at \(B = 0\), namely, the change from the uppermost (black) line to other lower (colored) lines in the left panel of Fig. 5. Since the final line of (86) contains only quark determinants, this change \(\Delta \bar{V}\) is purely a sea quark effect. A similar quantity is also calculated in the lattice study to investigate the sea quark effect with nonzero magnetic fields in terms of the reweighting technique [31]. Furthermore, \(\Delta \bar{V}\) is renormalization group invariant since it satisfies the RG equation (66). We have numerically verified that the quantity \(\Delta \bar{V}\) is always negative with any values of \(gH_c, eB\) and \(\Theta\) and monotonically decreasing as either \(eB\) or \(gH_c\) increases. Figure 6 shows \(\Delta \bar{V}(H, B)\) as a function of \(gH_c\) and \(eB\) in the case of \(\Theta = 0\) as an example. \(\Delta \bar{V}\) is negative in the whole region of \(gH_c-eB\) plane and monotonically decreasing. Now, when \(B = 0\), the quark loop contribution \(V_q\) attenuates the gluonic contribution \(V_{YM}\) in the total effective potential \(V_{eff} = H_c^2/2 + V_{YM} + V_q\), owing to the opposite sign of \(V_{YM}\) as we have seen in Fig. 4. From (86), we can rewrite the normalized effective potential \(\bar{V}(H_c, B)\) as

\[
\bar{V}(H_c, B) = \bar{V}(H_c, 0) + \Delta \bar{V}(H_c, B) 
= \frac{H_c^2}{2} + V_{YM} + [V_q(H_c, 0) + \Delta \bar{V}(H_c, B)].
\] (87)

Here \(\Delta \bar{V}(H_c, B)\) can be regarded as the \(B\)-dependent part of the quark loop contribution, whereas \(V_q(H_c, 0)\) as the \(B\)-independent part. Since \(\Delta \bar{V}\) is always negative and monotonically decreasing, the \(B\)-dependent part of the quark loop contribution enhances the gluonic contribution \(V_{YM}\), which plays a completely opposite role of the \(B\)-independent part \(V_q\). Thanks to the properties of the \(B\)-dependent part \(\Delta \bar{V}\) of the quark loop (sea quark) contribution, the chromomagnetic condensate \((gH_c)^2_{min}\) monotonically increases with an increasing magnetic field, as we have seen in Fig. 5. This property of the sea quark supports the gluonic magnetic catalysis at the zero temperature, observed in current lattice data [35].

Although our analysis is based on the one-loop calculation but containing all order interaction with the chromomagnetic field and the external magnetic field, our results are
FIG. 6: \( \Delta \tilde{V} \) as a function of \( g_{H_c} \) and \( eB \) with \( \Theta = 0 \).

qualitatively in agreement with recent lattice results. Thus we expect that our analysis captures the essence of the actual physics situation.

Finally, we mention the dynamical breaking of chiral symmetry and our future works. In this paper, we do not take into account the dynamical breaking of chiral symmetry and thus the magnetic catalysis [3–5] (an enhancement of the dynamical quark mass induced by the magnetic field). However, we expect that the magnetic catalysis also contributes to the gluonic magnetic catalysis since dynamical quark masses \( M_q^*(B) \) would suppress the quark loop, especially in large \( eB \) regions \( eB > M_q^{*2}(B = 0), (g_{H_c})_{min} \). In order to incorporate the dynamical chiral symmetry breaking and the magnetic catalysis into our framework, we should take into account a higher-order term of the quark field, namely, interaction between quarks. In a future work, we will take the interaction between quarks such as the Nambu-Jona-Lasinio-type interaction into our effective potential and explore the effects of the dynamical chiral symmetry breaking and the magnetic catalysis. It is also intriguing how the situation changes at finite temperature, and whether we obtain the (gluonic) \textit{inverse} magnetic catalysis from the sea quark effect in particular near the pseudocritical temperature \( T \sim T_c \), as reported in [31, 35]. This interesting issue at finite temperature will also be addressed in our next project.
IV. SUMMARY AND CONCLUSION

In this paper, we derive the analytic expression for the one-loop $SU(N_c)$ QCD effective potential including $N_f$ flavor quarks which nonlinearly interact with the pure chromomagnetic background field and the external $U(1)_{em}$ magnetic field. After the renormalization of couplings and fields, we obtain the correct one-loop $\beta$ functions of both QCD and QED. The resulting effective potential is renormalization group invariant, namely, independent of the renormalization scale point $\mu$. We investigate the effect of the magnetic field on the QCD effective potential in particular for the color $SU(3)$ case with the three flavors $(u,d,s)$. We find that the chromomagnetic field prefers to be parallel (or antiparallel) to the external magnetic field. This result is consistent with the previous results [32] in which the proper time integral is numerically performed and also with the recent lattice results [33–35]. Furthermore, our result shows that quark loop contributions to the effective potential (sea quark effect) with the magnetic field enhance the gluonic contributions, and thus the chromomagnetic condensate $(gH_c)^2_{\text{min}}$ monotonically increases with an increasing magnetic field. This result supports the recent observed gluonic magnetic catalysis at zero temperature in lattice QCD [35].

In a future work, we will incorporate the effects of the dynamical breaking of chiral symmetry and magnetic catalysis into the effective potential. Furthermore, we will investigate the properties of the sea quark effect at finite temperature, especially near the pseudocritical temperature $T \sim T_c$. The sea quark will play an important role for the (gluonic) inverse magnetic catalysis, as recently pointed out in the lattice QCD study [31].

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Appendix A: The one-loop effective potential of $SU(N_c)$ Yang-Mills theory with a pure chromomagnetic background

We derive the one-loop effective potential of the $SU(N_c)$ YM theory (see [16–21] for the original works). From [13], the one-loop effective action of the YM theory is given by

$$iS_{YM} = iS_g + iS_c,$$

(A1)

where

$$iS_g = \log \det \left[-(\hat{D}^2)^{AC} g_{\mu\nu} - 2gf^{ABC} \hat{F}^B_{\mu\nu}\right]^{-1/2},$$

(A2)

for the gluon part and

$$iS_c = \log \det \left[-(\hat{D}^2)^{AC}\right],$$

(A3)

for the ghost part, respectively. Using the eigenvalues $v_h$ of the matrix $T_c^{AC} = if^{ABC} \tilde{\sigma}^B$, the effective action of the gluon part becomes

$$iS_g = -\frac{1}{2} \sum_{h=1}^{N_c^2-1} \log \det \left[-D_{v_h}^2 g_{\mu\nu} + 2igv_h F_{\mu\nu}\right]$$

$$= -\frac{1}{2} \sum_{h=1}^{N_c^2-1} \log \det \left(-D_{v_h}^2 - 2g|v_h|a\right) \left(-D_{v_h}^2 + 2g|v_h|a\right)$$

$$\times \left(-D_{v_h}^2 + 2ig|v_h|b\right) \left(-D_{v_h}^2 - 2ig|v_h|b\right),$$

(A4)

where $D_{v_h} = \partial_\mu - igv_h A_\mu$, and

$$a = \frac{1}{2} \sqrt{F^4 + (F \cdot \tilde{F})^2 + F^2}, \quad b = \frac{1}{2} \sqrt{F^4 + (F \cdot \tilde{F})^2 - F^2}$$

(A5)

are related to the eigenvalues $\pm F^{(1)}$ and $\pm F^{(2)}$ of the field strength tensor $F_{\mu\nu}$ as [10]

$$\pm F^{(1)} = \pm ia, \quad \pm F^{(2)} = \pm b.$$  

(A6)

Here $F^2$ and $F \cdot \tilde{F}$ can be expressed in terms of the chromomagnetic field $\vec{H}_c$ and the chromoelectric fields $\vec{E}_c$ as

$$F^2 = 2(\vec{H}_c^2 - \vec{E}_c^2), \quad F \cdot \tilde{F} = -4\vec{E}_c \cdot \vec{H}_c.$$  

(A7)
The absolute values of $v_h$ in (A4) appear when we explicitly calculate the eigenvalues of the matrix $-D^2_{v_h}g_{\mu\nu} + 2igv_hF_{\mu\nu}$. We shall introduce $\rho_{v_h}$, which stands for any one of $\pm 2g|v_h|^a$, $\pm 2i|v_h|^b$, and evaluate the following action:

$$iS_{\rho_{v_h}} = -\frac{1}{2}\log \det(-D^2_{v_h} + \rho_{v_h})$$

$$= -\frac{1}{2}\Tr \log(-D^2_{v_h} + \rho_{v_h})$$

$$= -\frac{1}{2}\int d^4x \log((-D^2_{v_h} + \rho_{v_h})x).$$

(A8)

Using the identity (23), we get

$$iS_{\rho_{v_h}} = \frac{i}{2}\int d^4x \int_0^\infty \frac{ds}{s^{1-\epsilon}} e^{-i(\rho_{v_h} - i\delta)s} \langle x | e^{-is(-D^2_{v_h})} | x \rangle.$$  \hspace{1cm} (A9)

As mentioned in Sec. II, we can apply the Schwinger’s proper time method [10] to evaluate the matrix element $\langle x' | e^{-is(-D^2_{v_h})} | x'' \rangle$ as in QED, since the gauge field $A_{\mu}$ is now Abelian like a photon field. Defining the Hamiltonian $H_{v_h} = -D^2_{v_h}$, we obtain the matrix element as

$$\langle x' | \exp (-iH_{v_h}s) | x'' \rangle = \frac{i}{4\pi^2} \Psi_{v_h}(x', x'') e^{-L_{v_h}(s)s^{-2}}$$

$$\times \exp \left[ \frac{i}{4}(x' - x'')(gv_hF)\coth(gv_hFs)(x' - x'') \right],$$  \hspace{1cm} (A10)

where

$$\Psi_{v_h}(x', x'') = \exp \left[ igv_h \int_{x''}^{x'} A_{\mu} dx'^{\mu} \right],$$

$$L_{v_h}(s) = \frac{1}{2} \Tr \log \left[ (gv_hFs)^{-1}\sinh(gv_hFs) \right].$$  \hspace{1cm} (A11)

In the case of the local effective action, we take $x' \rightarrow x'' = x$ and thus (A9) becomes

$$iS_{\rho_{v_h}} = -\frac{i^{1+\epsilon}}{32\pi^2} \int d^4x \int_0^\infty \frac{ds}{s^{3-\epsilon}} e^{-L_{v_h}(s)e^{-i(\rho_{v_h} - i\delta)s}}.$$  \hspace{1cm} (A12)

Replacing $\rho_{v_h}$ by $\pm 2g|v_h|^a$ and $\pm 2i|v_h|^b$ and gathering all the contributions, the effective action of the gluon part can be written as

$$iS_g = \sum_{h=1}^{N^2-1} -\frac{i^{1+\epsilon}}{32\pi^2} \int d^4x \int_0^\infty \frac{ds}{s^{3-\epsilon}} e^{-L_{v_h}(s)}$$

$$\times \left\{ e^{-i(-2g|v_h|^a - i\delta)s} + e^{-i(+2g|v_h|^a - i\delta)s} + e^{-i(-2i|v_h|^b - i\delta)s} + e^{-i(+2i|v_h|^b - i\delta)s} \right\}. \hspace{1cm} (A13)$$
We can evaluate $e^{-L_{v_h}(s)}$ by using the second identity of [28], and then the action reads

$$iS_g = \sum_{h=1}^{N^2-1} \frac{g^3}{32\pi^2} \int d^4x \int_0^\infty ds \frac{(g|v_h|as)(g|v_h|bs)}{s^{3-\epsilon} \sin(g|v_h|as) \sinh(g|v_h|bs)} \times \left\{ e^{-i(-2g|v_h|a-i\delta)s} + e^{-i(2g|v_h|a-i\delta)s} + e^{-i(-2g|v_h|b+i\delta)s} + e^{-i(2g|v_h|b+i\delta)s} \right\}.$$  \hspace{1cm} (A14)

Similarly, we can evaluate the effective action of the ghost part [A3]. Including the ghost part, the resulting effective action of the Yang-Mills theory is given as

$$iS_{YM} = \int d^4x \frac{-i^{1+\epsilon}}{32\pi^2} \sum_{h=1}^{N^2-1} \int_0^\infty ds \frac{g|v_h|Hc_{s}}{s^{3-\epsilon} \sin(g|v_h|Hc_{s})} e^{-\delta s} \left\{ e^{-2ig|v_h|Hc_{s}} + e^{-2g|v_h|Hc_{s}} \right\}.$$  \hspace{1cm} (A15)

The last term $-2$ in the bracket corresponds to the ghost contribution. Now, let us consider the pure chromomagnetic background, $a = \sqrt{H_c} = H_c$ and $b \to 0$. Then, we get

$$iS_{YM} = \int d^4x \frac{-i^{1+\epsilon}}{32\pi^2} \sum_{h=1}^{N^2-1} \int_0^\infty ds \frac{g|v_h|Hc_{s}}{s^{3-\epsilon} \sin(g|v_h|Hc_{s})} e^{-\delta s} \left\{ e^{-2ig|v_h|Hc_{s}} + e^{-2g|v_h|Hc_{s}} \right\}.$$  \hspace{1cm} (A16)

The effective Lagrangian can be obtain in terms of $S_{YM}$ as

$$\mathcal{L}_{YM} = \frac{S_{YM}}{i\epsilon^{4+\epsilon}} = \mathcal{L}^{stab}_{YM} + \mathcal{L}^{unstab}_{YM},$$  \hspace{1cm} (A17)

where we define the stable and unstable parts of the effective Lagrangian as [18]

$$\mathcal{L}^{stab}_{YM} = -i^{1+\epsilon} \sum_{h=1}^{N^2-1} \frac{g|v_h|Hc_{s}}{16\pi^2} \int_0^\infty ds \frac{e^{-\delta s}}{s^{2-\epsilon}} \left\{ e^{-3ig|v_h|Hc_{s}} + e^{-ig|v_h|Hc_{s}} \right\},$$

$$\mathcal{L}^{unstab}_{YM} = -i^{1+\epsilon} \sum_{h=1}^{N^2-1} \frac{g|v_h|Hc_{s}}{16\pi^2} \int_0^\infty ds \frac{e^{-\delta s}}{s^{2-\epsilon}} e^{+ig|v_h|Hc_{s}}.$$  \hspace{1cm} (A18)

First, we shall consider the stable part of the effective Lagrangian. Taking the Wick rotation of the proper time $s$, the stable part reads

$$\mathcal{L}^{stab}_{YM} = -\sum_{h=1}^{N^2-1} \frac{g|v_h|Hc_{s}}{16\pi^2} \int_0^{-i\infty} ds \frac{e^{-\delta s}}{s^{2-\epsilon}} \left( e^{-3ig|v_h|Hc_{s}} + e^{-ig|v_h|Hc_{s}} \right).$$  \hspace{1cm} (A19)

Now we change the integral variable as $s \to -is$ and take $\delta \to 0$. Then, the stable part becomes

$$\mathcal{L}^{stab}_{YM} = -\sum_{h=1}^{N^2-1} \frac{g|v_h|Hc_{s}}{16\pi^2} \int_0^{-i\infty} ds \frac{e^{-\delta s}}{(is)^{2-\epsilon}} \left( e^{-3g|v_h|Hc_{s}} + e^{-g|v_h|Hc_{s}} \right)$$

$$= \sum_{h=1}^{N^2-1} \frac{g|v_h|Hc_{s}}{16\pi^2} \int_0^\infty ds \frac{1}{\sinh(g|v_h|Hc_{s})} - e^{-g|v_h|Hc_{s}}.$$  \hspace{1cm} (A20)
Here we consider the integral of the first term
\[
I_1 = \frac{g|v_h|H_c}{16\pi^2} \int_0^\infty \frac{ds}{s^{2-\epsilon}} \frac{1}{\sinh(g|v_h|H_cs)}
\]
\[
= \frac{g|v_h|H_c}{8\pi^2} \int_0^\infty \frac{ds}{s^{2-\epsilon}} \frac{e^{-g|v_h|H_cs}}{1 - e^{-2g|v_h|H_cs}}.
\] (A21)

Applying the following representation of the generalized zeta function [38],
\[
\zeta(s, \lambda) = \sum_{n=0}^\infty \frac{1}{(n+\lambda)^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \exp(-\lambda x) \frac{dx}{1 - \exp(-x)},
\] (A22)
the integral \(I_1\) can be obtained as
\[
I_1 = \left( \frac{g|v_h|H_c}{4\pi^2} \right)^2 \left\{ - \left( \frac{1}{\epsilon} - \gamma_E + 1 \right) \zeta(-1, 1/2) + \log(2g|v_h|H_c) \zeta(-1, 1/2) - \zeta'(-1, 1/2) \right\}.
\] (A23)

Next we evaluate the integral of the second term in (A20)
\[
I_2 = -\frac{g|v_h|H_c}{16\pi^2} \int_0^\infty \frac{ds}{s^{2-\epsilon}} e^{-g|v_h|H_cs}
\]
\[
= -\frac{g|v_h|H_c}{16\pi^2} \frac{\Gamma(\epsilon-1)}{\Gamma(\epsilon)}
\]
\[
= -\frac{(g|v_h|H_c)^2}{16\pi^2} \left\{ - \left( \frac{1}{\epsilon} - \gamma_E + 1 \right) + \log(g|v_h|H_c) \right\}.
\] (A24)

Now, let us consider the unstable part. Taking the Wick rotation with the upper contour, which is a different contour from the stable part calculation, we get
\[
\mathcal{L}_{YM}^{unstab} = \sum_{h=1}^{N^2-1} -\frac{g|v_h|H_c i^{1+\epsilon}}{16\pi^2} \int_0^{+i\infty} \frac{ds}{s^{2-\epsilon}} e^{ig|v_h|H_cs}.
\] (A25)

After the Wick rotation, we have taken \(\delta \rightarrow 0\). We change the integral variable as \(s \rightarrow +is\), and then the unstable part becomes
\[
\mathcal{L}_{YM}^{unstab} = \sum_{h=1}^{N^2-1} -\frac{g|v_h|H_c}{16\pi^2} (-1)^\epsilon \int_0^\infty \frac{ds}{s^{2-\epsilon}} e^{-g|v_h|H_cs}
\]
\[
= \sum_{h=1}^{N^2-1} -\frac{(g|v_h|H_c)^2}{16\pi^2} \left\{ - \left( \frac{1}{\epsilon} - \gamma_E + 1 \right) + \log(g|v_h|H_c) - i\pi \right\}.
\] (A26)
The imaginary part appears in the last term. Gathering the both stable and unstable parts, we obtain the total effective Lagrangian of the YM theory as

\[ L_{YM} = \sum_{h=1}^{N_c^2-1} \left\{ \frac{(g v_h H_c)^2}{4 \pi^2} \left[ - \left( \frac{1}{\epsilon} - \gamma_E + 1 \right) \zeta(-1, 1/2) + \log(2g|v_h|H_c)\zeta(-1, 1/2) - \zeta'(-1, 1/2) \right] \right\} \\
- \frac{(g v_h H_c)^2}{8 \pi^2} \left[ - \left( \frac{1}{\epsilon} - \gamma_E + 1 \right) + \log(g|v_h|H_c) \right] \right\} \\
+ i \sum_{h=1}^{N_c^2-1} \frac{(g v_h H_c)^2}{16 \pi}. \] (A27)

Using (52), the imaginary part of the Lagrangian can be written as

\[ \text{Im} L_{YM} = \frac{N_c}{16 \pi} (g H_c)^2. \] (A28)

With \( \zeta(-1, 1/2) = \frac{1}{24} \) and \( \zeta'(-1, 1/2) = -\frac{1}{24} \log 2 - \frac{1}{24} + \frac{1}{2} \log G \), the real part reads

\[ \text{Re} L_{YM} = \frac{11 N_c}{96 \pi^2} (g H_c)^2 \left( \frac{1}{\epsilon} - \gamma_E \right) \]

\[ - \frac{11 N_c}{96 \pi^2} (g H_c)^2 \left\{ \log(g H_c) - \epsilon g + \frac{1}{N_c} \sum_{h=1}^{N_c^2-1} v_h^2 \log|v_h| \right\}, \] (A29)

where \( \epsilon g = (12 + 2 \log 2 - 12 \log G)/11 \approx 0.94556 \cdots \). Therefore, the total effective potential of the Yang-Mills theory \( V_{YM} = -L_{YM} \) is given as

\[ V_{YM} = \text{Re} V_{YM} + \text{Im} V_{YM}, \] (A30)

where the real part is

\[ \text{Re} V_{YM} = V_{YM}^{\text{fin}} + V_{YM}^{\text{div}}, \] (A31)

with

\[ V_{YM}^{\text{fin}} = + \frac{11 N_c}{96 \pi^2} (g H_c)^2 \left\{ \log(g H_c) - \epsilon g + \frac{1}{N_c} \sum_{h=1}^{N_c^2-1} v_h^2 \log|v_h| \right\}, \] (A32)

and the imaginary part is

\[ \text{Im} V_{YM} = - \frac{N_c}{16 \pi} (g H_c)^2. \] (A33)

The resulting one-loop effective potential of the \( SU(N_c) \) YM theory coincides with the result of [20, 21].
Appendix B: Relation between dimensional regularization and cutoff regularization

Considering the weak field expansion of the (38), we obtain

\[ \mathcal{L}_q = - \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} a_{a,i} \int_0^\infty \frac{ds}{s^{2-\varepsilon}} e^{-m^2_i s} \coth(a_{a,i} s) \]

\[ \rightarrow - \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^{3}} e^{-m^2_i s} \left( 1 + \frac{1}{3} (a_{a,i} s)^2 \right). \]  

(B1)

We expect that these divergent integrals in the second line correspond to the divergent terms in (40). In order to estimate the divergences, we consider the following integrals:

\[ I_1 = \int_0^\infty \frac{ds}{s^{3}} e^{-m^2 s}, \quad I_2 = \int_0^\infty \frac{ds}{s} e^{-m^2 s}. \]  

(B2)

First, we calculate the \( I_2 \) by using the dimensional regularization and the cutoff regularization, and we get

\[ I_2^{(d)} = \int_0^\infty \frac{ds}{s^{1-\varepsilon}} e^{-m^2 s} = \frac{1}{\varepsilon} - \gamma_E - \log m^2 + O(\varepsilon), \]

\[ I_2^{(c)} = \int_0^\infty \frac{ds}{s} e^{-m^2 s} = \log \Lambda^2 - \log m^2 + O\left( \frac{m^2}{\Lambda^2} \right). \]  

(B3)

Since the \( \log m^2 \) term is common, we find the relation between dimensional regularization and the cutoff regularization as

\[ \frac{1}{\varepsilon} - \gamma_E \leftrightarrow \log \Lambda^2. \]  

(B4)

We have used the relation in the replacement of (40) by (41). We note the \( \log m^2 \) term does not appear in the results of (40) and (41). Therefore we can take the massless limit in (41) without any infrared divergences.

Next we consider the \( I_1 \). We evaluate the \( I_1 \) by using the dimensional regularization and the cutoff regularization, and the results are given as

\[ I_1^{(d)} = \int_0^\infty \frac{ds}{s^{3-\varepsilon}} e^{-m^2 s} = \frac{m^4}{2} \left( \frac{1}{\varepsilon} - \gamma_E \right) + \frac{3}{4} m^4 - \frac{m^4}{2} \log m^2 + O(\varepsilon), \]

\[ I_1^{(c)} = \int_0^\infty \frac{ds}{s^{3}} e^{-m^2 s} = \frac{1}{2} \left( \Lambda^4 - 2m^2 \Lambda^2 + m^4 \log \Lambda^2 \right) + \frac{3}{4} m^4 - \frac{m^4}{2} \log m^2 + O\left( \frac{m^2}{\Lambda^2} \right). \]  

(B5)

Therefore, we read off the following relation:

\[ m^4 \left( \frac{1}{\varepsilon} - \gamma_E \right) \leftrightarrow \left( \Lambda^4 - 2m^2 \Lambda^2 + m^4 \log \Lambda^2 \right). \]  

(B6)
We have also used the relation in the replacement of (40) by (41). However, these divergent terms do not depend on any fields in (41), so we have omitted these divergences.

[1] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A 803, 227 (2008) [arXiv:0711.0950 [hep-ph]].
[2] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D 78, 074033 (2008) [arXiv:0808.3382 [hep-ph]].
[3] H. Suganuma and T. Tatsumi, Annals Phys. 208, 470 (1991).
[4] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, Phys. Rev. Lett. 73, 3499 (1994) [Erratum-ibid. 76, 1005 (1996)] [hep-ph/9405262].
[5] V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, Phys. Lett. B 349, 477 (1995) [hep-ph/9412257].
[6] M. N. Chernodub, Phys. Rev. D 82, 085011 (2010) [arXiv:1008.1055 [hep-ph]].
[7] Y. Hidaka and A. Yamamoto, Phys. Rev. D 87, 094502 (2013) [arXiv:1209.0007 [hep-ph]].
[8] V. Skokov, A. Y. Illarionov and V. Toneev, Int. J. Mod. Phys. A 24, 5925 (2009) [arXiv:0907.1396 [nucl-th]].
[9] A. Bzdak and V. Skokov, Phys. Lett. B 710, 171 (2012) [arXiv:1111.1949 [hep-ph]].
[10] J. S. Schwinger, Phys. Rev. 82, 664 (1951).
[11] E. Brezin and C. Itzykson, Phys. Rev. D 3, 618 (1971).
[12] Z. Bialynicka-Birula and I. Bialynicki-Birula, Phys. Rev. D 2, 2341 (1970).
[13] S. L. Adler, J. N. Bahcall, C. G. Callan and M. N. Rosenbluth, Phys. Rev. Lett. 25, 1061 (1970).
[14] S. L. Adler, Annals Phys. 67, 599 (1971).
[15] K. Hattori and K. Itakura, Annals Phys. 330, 23 (2013) [arXiv:1209.2663 [hep-ph]].
[16] G. K. Savvidy, Phys. Lett. B 71, 133 (1977).
[17] S. G. Matinyan and G. K. Savvidy, Nucl. Phys. B 134, 539 (1978).
[18] N. K. Nielsen and P. Olesen, Nucl. Phys. B 144, 376 (1978).
[19] H. Leutwyler, Nucl. Phys. B 179, 129 (1981).
[20] W. Dittrich and M. Reuter, Phys. Lett. B 128, 321 (1983).
[21] E. Elizalde and J. Soto, Annals Phys. 162, 192 (1985).
[22] I. A. Batalin, S. G. Matinyan and G. K. Savvidy, Sov. J. Nucl. Phys. 26, 214 (1977) [Yad. Fiz. 26, 407 (1977)].
[23] M. Gyulassy and A. Iwazaki, Phys. Lett. B 165, 157 (1985).
[24] N. Tanji and K. Itakura, Phys. Lett. B 713, 117 (2012) [arXiv:1111.6772 [hep-ph]].
[25] H. Pagels and E. Tombouli, Nucl. Phys. B 143, 485 (1978).
[26] S. L. Adler, Phys. Rev. D 23, 2905 (1981) [Erratum-ibid. D 24, 1063 (1981)].
[27] S. L. Adler and T. Piran, Phys. Lett. B 113, 405 (1982) [Erratum-ibid. B 121, 455 (1983)].
[28] J. I. Kapusta, Nucl. Phys. B 190, 425 (1981).
[29] M. Engelhardt and H. Reinhardt, Phys. Lett. B 430, 161 (1998) [hep-th/9709115].
[30] H. Gies, Phys. Rev. D 63, 025013 (2001) [hep-th/0005252].
[31] F. Bruckmann, G. Endrodi and T. G. Kovacs, JHEP 1304, 112 (2013) [arXiv:1303.3972 [hep-lat]].
[32] B. V. Galilo and S. N. Nedelko, Phys. Rev. D 84, 094017 (2011) [arXiv:1107.4737 [hep-ph]].
[33] E. -M. Ilgenfritz, M. Kalinowski, M. Muller-Preussker, B. Petersson and A. Schreiber, Phys. Rev. D 85, 114504 (2012) [arXiv:1203.3360 [hep-lat]].
[34] E. -M. Ilgenfritz, M. Muller-Preussker, B. Petersson and A. Schreiber, arXiv:1310.7876 [hep-lat].
[35] G. S. Bali, F. Bruckmann, G. Endrodi, F. Gruber and A. Schaefer, JHEP 1304, 130 (2013) [arXiv:1303.1328 [hep-lat]].
[36] W. Dittrich and M. Reuter, Lect. Notes Phys. 220, 1 (1985).
[37] W. Dittrich and H. Gies, Springer Tracts Mod. Phys. 166, 1 (2000).
[38] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press (1965); M. Abramowitz and I. Stegun, Handbook of Mathematical Functions, Dover, New York (1970)
[39] W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936) [physics/0605038].
[40] G. V. Dunne, In *Shifman, M. (ed.) et al.: From fields to strings, vol. 1* 445-522 [hep-th/0406216].
[41] H. Fujii and K. Itakura, Nucl. Phys. A 809, 88 (2008) [arXiv:0803.0410 [hep-ph]].
[42] H. Fujii, K. Itakura and A. Iwazaki, Nucl. Phys. A 828, 178 (2009) [arXiv:0903.2930 [hep-ph]].
[43] K. Hashimoto and T. Oka, JHEP 1310, 116 (2013) [arXiv:1307.7423 [hep-th]].
[44] K. Amemiya and H. Suganuma, Phys. Rev. D 60, 114509 (1999) [hep-lat/9811035].
[45] S. Gongyo, T. Iritani and H. Suganuma, Phys. Rev. D 86, 094018 (2012) [arXiv:1207.4377 [hep-lat]].

[46] S. Gongyo and H. Suganuma, Phys. Rev. D 87, 074506 (2013) [arXiv:1302.6181 [hep-lat]].

[47] A. Shibata, S. Kato, K. -I. Kondo, T. Murakami, T. Shinohara and S. Ito, Phys. Lett. B 653, 101 (2007) [arXiv:0706.2529 [hep-lat]].

[48] K. -I. Kondo, Phys. Lett. B 600, 287 (2004) [hep-th/0404252].

[49] K. -I. Kondo, Phys. Rev. D 74, 125003 (2006) [hep-th/0609166].

[50] K. -I. Kondo, arXiv:1309.2337 [hep-th].

[51] J. Ambjorn, V. K. Mitrjushkin and A. M. Zadorozhnyi, Phys. Lett. B 245, 575 (1990).

[52] A. Eichhorn, H. Gies and J. M. Pawlowski, Phys. Rev. D 83, 045014 (2011) [Erratum-ibid. D 83, 069903 (2011)] [arXiv:1010.2153 [hep-ph]].

[53] N. Tanji, Annals Phys. 325, 2018 (2010) [arXiv:1002.3143 [hep-ph]].

[54] G. C. Nayak, Phys. Rev. D 72, 125010 (2005) [hep-ph/0510052].

[55] G. C. Nayak and P. van Nieuwenhuizen, Phys. Rev. D 71, 125001 (2005) [hep-ph/0504070].