Intersection theory and Chern classes on normal varieties

Adrian Langer

September 27, 2023

ADDRESS:
Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland
e-mail: alan@mimuw.edu.pl

Abstract

We study intersection theory and Chern classes of reflexive sheaves on normal varieties. In particular, we define generalization of Mumford’s intersection theory on normal surfaces to higher dimensions. We also define and study the second Chern class for reflexive sheaves on normal varieties. We use these results to prove some Bogomolov type inequalities on normal varieties in positive characteristic. We also prove some new boundedness results on normal varieties in positive characteristic.

Introduction

Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$. It is well known how to intersect one Weil divisor $D$ with a collection of $(n - 1)$ line bundles $L_1, ..., L_{n-1}$. The intersection number is simply the degree of the 0-cycle $c_1(L_1) \cap ... \cap c_1(L_{n-1}) \cap [D]$. Mumford introduced also a $\mathbb{Q}$-valued intersection theory for two Weil divisors in case $X$ is a surface (see [Fu, Examples 7.1.16 and 8.3.11]). To do so he passes to resolution of singularities $f : \tilde{X} \to X$, defines the pullback $f^*D$ of a Weil divisor $D$ as a $\mathbb{Q}$-divisor and then intersects the pullbacks of Weil divisors on $\tilde{X}$.
M. Enokizono in [En] Appendix] used a similar method for higher dimensional varieties considering very special pullback taking into account only exceptional divisors with codimension 2 centers. His approach uses existence of resolution of singularities in the characteristic zero case and de Jong’s alterations [dJ] in positive characteristic.

We give a different approach to this result working directly on the singular variety. We make some computations in the Grothendieck group of $X$, similar to Kleiman’s approach to intersection numbers for Cartier divisors (see [Ko, VI.2]). Using some asymptotic Riemann–Roch formulas, we also reconstruct Mumford’s intersection theory on surfaces without passing to resolution of singularities. The final outcome is the following result (see Section 2 for more precise results).

**Theorem 0.1.** For any Weil divisors $D_1$ and $D_2$ on $X$ there exists a unique $\mathbb{Z}$-multilinear symmetric form $\text{Pic} \ X \times (n-2) \to \mathbb{Q}, \ (L_1, \ldots, L_{n-2}) \to D_1.D_2.L_1\ldots L_{n-2}$ such that:

1. If $D_1$ and $D_2$ are Cartier divisors then

$$D_1.D_2.L_1\ldots L_{n-2} = \int_X c_1(\mathcal{O}_X(D_1)) \cap c_1(\mathcal{O}_X(D_2)) \cap c_1(L_1) \cap \ldots \cap c_1(L_{n-2}) \cap [X].$$

2. If $D_2$ is a Cartier divisor then

$$D_1.D_2.L_1\ldots L_{n-2} = \int_X c_1(\mathcal{O}_X(D_2)) \cap c_1(L_1) \cap \ldots \cap c_1(L_{n-2}) \cap [D_1] \in \mathbb{Z}.$$

3. If $L_1, \ldots, L_{n-2}$ are very ample then for a general complete intersection surface $S \in |L_1| \cap \ldots \cap |L_{n-2}|$ we have

$$D_1.D_2.L_1\ldots L_{n-2} = (D_1)_S.(D_2)_S.$$

The main aim of the paper is to see whether one can similarly generalize the theory of Chern classes of vector bundles to reflexive sheaves on normal varieties. More precisely, if $X$ is smooth then we have a well defined Chern character from the Grothendieck group of vector bundles to the Chow ring. This is a homomorphism of rings that satisfies various remarkable properties like the Riemann–Roch theorem (see [Fu, Chapter 15]). Using operational Chern classes, the Riemann–Roch theorem can be also proven for vector bundles on singular varieties (see [BFM]) but there is no ring structure on the direct sum of Chow groups. However, if $X$ is a normal projective surface then one can use Mumford’s intersection theory...
to define rational Chow ring structure. In that case one can ask wether there exists a well defined Chern character from the Grothendieck group of reflexive sheaves on X to this rational (or real) Chow ring. We provide a definition of such Chern character, that conjecturally is also a ring homomorphism (see Subsection 5.5). To do so we revise the theory of Chern classes of reflexive sheaves on normal surfaces that was studied in [La1] in the complex analytic setup. Here we develop an algebraic approach that works in an arbitrary characteristic. In particular, we prove the following theorem (see Definition 3.1 and Proposition 4.2).

**Theorem 0.2.** For any normal proper algebraic surface X defined over an algebraically closed field k and any coherent reflexive \(O_X\)-module \(E\) on X one can define its second Chern class \(c_2(E) \in A_0(X) \otimes \mathbb{R}\) so that the following conditions are satisfied:

1. If \(E\) has rank 1 then \(c_2(E) = 0\).

2. If \(E\) is a vector bundle on X then this class coincides with \(c_2(E) \cap [X]\).

3. If \(\pi : Y \to X\) is a finite morphism from a normal surface Y then

\[
\int_Y c_2(\pi^*[E]) = \deg \pi \cdot \int_X c_2(E),
\]

where \(\pi^*[E]\) is the reflexive hull of \(\pi^*E\).

In the above theorem \(A_0(X)\) stands for the group of 0-dimensional cycles modulo rational equivalence on X. Our approach to the above theorem is along similar lines as in [La1] but we give simplified and more detailed versions of several proofs. We also obtain the following result about additional terms in the Riemann–Roch theorem (see Theorems 3.12 and 4.4).

**Theorem 0.3.** Let X be a normal proper algebraic surface defined over an algebraically closed field. Then there exists a constant C depending only on X such that for every rank r coherent reflexive \(O_X\)-module \(E\) on X we have

\[
\left| \chi(X, E) - \left( \frac{1}{2} c_1(E) \cdot (c_1(E) - K_X) - \int_X c_2(E) + r \chi(X, O_X) \right) \right| \leq Cr^2.
\]

We use this result to construct a good theory for the second Chern class (or character) for reflexive sheaves in higher dimensions. This works well in case of
positive characteristic or for varieties with at most quotient singularities in codimension \( \geq 2 \) in characteristic 0. In particular, our theory in the characteristic zero case generalizes the one developed in [GKPT, Section 3].

Here we state one of the main results in positive characteristic (see Theorem 5.2). Let us mention that till now the theory of Chern classes of reflexive sheaves on singular varieties that are well-behaved under coverings (e.g., Mumford’s theory of Chern classes of \( \mathbb{Q} \)-sheaves) was always restricted to the characteristic zero case.

**Theorem 0.4.** Assume that \( k \) has positive characteristic. For any normal projective variety \( X/k \) and for any coherent reflexive \( \mathcal{O}_X \)-module \( \mathcal{E} \) on \( X \) there exists a \( \mathbb{Z} \)-multilinear symmetric form \( \int_X c_2(\mathcal{E}) : \text{Pic}^*(X) \to \mathbb{R} \) such that:

1. If \( \mathcal{E} \) is a vector bundle on \( X \) then
   \[
   \int_X c_2(\mathcal{E}) L_1 \cdots L_{n-2} = \int_X c_2(\mathcal{E}) \cap c_1(L_1) \cap \ldots \cap c_1(L_{n-2}) \cap [X].
   \]

2. If \( k \subset K \) is an algebraically closed field extension then
   \[
   \int_{X_K} c_2(\mathcal{E}_K)(L_1)_K \cdots (L_{n-2})_K = \int_X c_2(\mathcal{E}) L_1 \cdots L_{n-2}.
   \]

3. If \( n > 2 \) and \( L_1 \) is very ample then for a very general hypersurface \( H \in |L_1| \) we have
   \[
   \int_X c_2(\mathcal{E}) L_1 \cdots L_{n-2} = \int_H c_2(\mathcal{E}|_H) L_2|_H \cdots L_{n-2}|_H.
   \]

It is easy to see that this second Chern class is uniquely determined by equality

\[
\int_X c_2(F_X^*[\mathcal{E}]) L_1 \cdots L_{n-2} = p^2 \int_X c_2(\mathcal{E}) L_1 \cdots L_{n-2}
\]

for the Frobenius morphism \( F_X \), and by the Riemann–Roch type inequality analogous to that from Theorem 0.3 (see Remark 5.6). This suggests that one can define \( \int_X \text{ch}_2(\mathcal{E}) L_1 \cdots L_{n-2} \) as the limit

\[
\lim_{m \to \infty} \frac{\chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot F_X^m[\mathcal{E}])}{p^{2m}}.
\]

The main problem in the proof is to show that such a limit exists.
We give some applications of the obtained results. For example, we prove a
general Bogomolov type inequality (see Theorem 6.5), Bogomolov’s inequality
for strongly semistable reflexive sheaves on normal varieties (see Corollary 6.6)
and various boundedness results (see, e.g., Corollary 6.7). As a final application
we show a quick proof of certain boundedness theorem by Esnault and Srinivas on
F-divided sheaves on normal varieties in positive characteristic (see [ES, Theorem
2.1]).

Applications of the above theory to general restriction theorems and Bogo-
molov’s inequality (also for Higgs sheaves) can be found in [La4]. Other appli-
cations to non-abelian Hodge theory and Simpson’s correspondence for singular
varieties in positive characteristic are contained in [La5]. After the first version
of the paper was written, there appeared another application to the abundance
conjecture for threefolds in positive characteristic (see [Xu]).

The structure of the paper is as follows. In the first section we gather some
auxiliary results. In Section 2 we study intersection theory on normal varieties
and in particular we prove Theorem 0.1 In Section 3 we study relative Chern
classes for resolutions of surface singularities. Then in Section 4 we use them to
define Chern classes for reflexive sheaves on normal proper surfaces. In Section
5 we extend these results to higher dimensions proving Theorem 0.4 Finally in
Section 6 we give some applications of the obtained results.

Notation

Let $X$ be an integral normal locally Noetherian scheme. We say that an open
subset $U \subset X$ is big if every irreducible component of $X \setminus U$ has codimension $\geq 2$
in $X$. A vector bundle on $X$ is a finite locally free $O_X$-module. The category of
vector bundles on $X$ will be denoted by Vect $(X)$.

If $X$ is an algebraic variety defined over some algebraically closed field $k$ then
we write $Z^1(X)$ for the group of Weil divisors on $X$. We write $A_m(X)$ for the
group of $m$-dimensional cycles modulo rational equivalence on $X$ (see [Fu 1.3]).
By abuse of notation we also write $A^m(X)$ for $A_{dimX-m}(X)$. 

5
1 Preliminaries

1.1 Reflexive sheaves

In this subsection $X$ is an integral locally Noetherian scheme. Let $\text{Ref} (\mathcal{O}_X)$ be the category of coherent reflexive $\mathcal{O}_X$-modules. It is a full subcategory of the category $\text{Coh} (\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules. The inclusion functor $\text{Ref} (\mathcal{O}_X) \to \text{Coh} (\mathcal{O}_X)$ comes with a left adjoint $(-)^{**} : \text{Coh} (\mathcal{O}_X) \to \text{Ref} (\mathcal{O}_X)$ given by the reflexive hull. The category $\text{Ref} (\mathcal{O}_X)$ is an additive category with kernels and cokernels and it comes with an associative and symmetric tensor product $\hat{\otimes}$ defined by

$$E \hat{\otimes} F := (E \otimes_{\mathcal{O}_X} F)^{**}.$$ 

The following well-known lemma can be found in [SP, Lemma 0EBJ].

**Lemma 1.1.** Let $j : U \hookrightarrow X$ be an open subscheme with complement $Z$ such that the depth of $\mathcal{O}_{X,z}$ is $\geq 2$ for all $z \in Z$. Then $j_*$ and $j^*$ define adjoint equivalences of categories $\text{Ref} (\mathcal{O}_X)$ and $\text{Ref} (\mathcal{O}_U)$. In particular, the above assumptions are satisfied if $U$ is a big open subset of an integral locally Noetherian normal scheme $X$.

Let us also recall the following standard lemma (see, e.g., [SP, Tag 0EBF]).

**Lemma 1.2.** Let $f : X \to Y$ be a flat morphism of integral locally Noetherian schemes. If $\mathcal{F}$ is a coherent reflexive $\mathcal{O}_Y$-module then $f^* \mathcal{F}$ is reflexive on $X$.

In general, pull back of a reflexive sheaf need not be reflexive. If $f : X \to Y$ is a morphism between normal schemes and $\mathcal{E}$ is a coherent reflexive $\mathcal{O}_Y$-module then we set

$$f^{[n]} \mathcal{E} = (f^* \mathcal{E})^{**}.$$ 

If $\mathcal{F}$ is a coherent reflexive $\mathcal{O}_X$-module then we set

$$f_{[n]} \mathcal{F} = (f_* \mathcal{F})^{**}.$$ 

Note that in general these operations define functors on reflexive sheaves that are not functorial with respect to morphisms.

If $X$ is normal and $\mathcal{E} \in \text{Ref} (\mathcal{O}_X)$ then we also use the notation

$$\text{Sym}^{[m]} \mathcal{E} := (\text{Sym}^m \mathcal{E})^{**}.$$
for $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$.

If $X$ is a normal scheme of finite type over an algebraically closed field $k$ of characteristic $p > 0$ then $F_X : X \to X$ denotes the absolute Frobenius morphism. Let us recall that by Kunz’s theorem $F_X$ is flat only at regular points of $X$. If $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$ then for every $m$ we set

$$F_X^{[m]} \mathcal{E} := (F_X^m)[\mathcal{E}]^\ast.$$

The following well-known lemma is a corollary of Serre’s criterion on normality.

**Lemma 1.3.** Let us assume that $X$ is normal. If $\mathcal{E}$ is a coherent torsion free $\mathcal{O}_X$-module then the canonical map $\mathcal{E} \to \mathcal{E}^{**}$ is an isomorphism in codimension 1, i.e., it is injective and every irreducible component of the support of its cokernel has codimension $\geq 2$ in $X$.

Let $X$ be an excellent integral normal scheme (this is the situation in which we will use the remark below). Then the regular locus $U := X_{\text{reg}}$ is open and the above lemma implies that $j^* : \text{Ref}(\mathcal{O}_X) \to \text{Ref}(\mathcal{O}_U)$ and $j_* : \text{Ref}(\mathcal{O}_U) \to \text{Ref}(\mathcal{O}_X)$ are adjoint equivalences of categories, where $j : U \hookrightarrow X$ denotes the open embedding.

### 1.2 Mumford’s intersection theory on normal surfaces

Let $X$ be a normal proper surface defined over an algebraically closed field and let $f : \tilde{X} \to X$ be any resolution of singularities. For any Weil divisor $D$ on $X$ one can define its pull back $f^* D \in A^1(\tilde{X}) \otimes \mathbb{Q}$ as the only class of a $\mathbb{Q}$-divisor for which $f_* f^* D = D$ and $\int_{\tilde{X}} [f^* D] \cap [E_i] = 0$ for all exceptional curves $E_i$ (cf. [Fu, Example 8.3.11]). For any two Weil divisors $D_1$ and $D_2$ we set

$$[D_1] \cap [D_2] := f_*([f^* D_1] \cap [f^* D_2]).$$

This defines a $\mathbb{Z}$-bilinear symmetric form $A^1(X) \times A^1(X) \to A_0(X) \otimes \mathbb{Q}, (D_1, D_2) \mapsto [D_1] \cap [D_2]$. In the following we write $D_1 . D_2$ for the rational number $\int_X [D_1] \cap [D_2]$.

### 1.3 Bertini’s theorem

Here we recall some Bertini type theorems. In particular, we have Bertini’s theorem for smoothness, irreducibility and reducedness for unramified morphisms.
(see [Jo, Theoreme 6.3] ) and Seidenberg’s Bertini’s theorem for normality for embeddings into the projective space (see [Se, Theorem 7’]; see also [HL, Corollary 1.1.15] for a quick proof). However, we need these theorems in a slightly more general set-up and it is convenient to state them uniformly using [CGM, Theorem 1 and Remark below Corollary 2].

Let \( \mathcal{P} \) be one of the following properties of a locally noetherian scheme: being regular, normal, geometrically reduced or geometrically irreducible. Then we have the following result:

**THEOREM 1.4.** Let \( X \) be a scheme of finite type over an algebraically closed field and let \( \varphi : X \to \mathbb{P}^n_k \) be a morphism defined by a linear system \( \Lambda \). Let us assume that \( \varphi \) has separably generated residue field extensions. If \( X \) has property \( \mathcal{P} \) then there exists a nonempty Zariski open subset \( U \subset \Lambda \) such that every hypersurface \( H \in \Lambda \) also has property \( \mathcal{P} \).

As usual we will quote this theorem by saying that a general \( H \in \Lambda \) has property \( \mathcal{P} \).

### 1.4 Some results on Grothendieck’s group of a normal variety

Let \( X \) be a normal projective variety of dimension \( n \) defined over an algebraically closed field \( k \). Let \( K(X) \) denote the Grothendieck’s group of \( X \). For any \( 0 \leq m \leq n \) we denote by \( K_m(X) \) the subgroup of \( K(X) \) generated by classes of sheaves, whose support has dimension at most \( m \). For a line bundle \( L \) on \( X \) we have an additive endomorphism of \( K(X) \) defined by

\[
\mathcal{F} \to c_1(L) \cdot \mathcal{F} := \mathcal{F} - L^{-1} \otimes \mathcal{F}
\]

(see [Ko, Chapter VI, Definition 2.4]).

Let us recall the following result (see [Ko, Chapter VI, Proposition 2.5] and its proof):

**LEMMA 1.5.**

1. \( c_1(L) \cdot K_m(X) \subset K_{m-1}(X) \) for every \( m \).

2. \( c_1(L_1) \) and \( c_1(L_2) \) commute for any two line bundles \( L_1 \) and \( L_2 \).

3. For any two line bundles \( L_1 \) and \( L_2 \) we have equality of endomorphisms

\[
c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) - c_1(L_1) \cdot c_1(L_2).
\]
4. If \( Y \subset X \) is integral and \( L|_Y \simeq \mathcal{O}_Y(D) \) for some effective Cartier divisor \( D \) on \( Y \) then \( c_1(L) \cdot \mathcal{O}_Y = \mathcal{O}_D \).

We will also need the following lemmas:

**Lemma 1.6.** If \( D \) is a Weil divisor on \( X \) then \( [\mathcal{O}_D] \in K_{n-1}(X) \). Moreover, if we write \( D = \sum a_i D_i \) for some Weil divisors \( D_i \) and some integers \( a_i \) then the class \( [\mathcal{O}_D] - \sum a_i [\mathcal{O}_{D_i}] \) lies in \( K_{n-2}(X) \).

**Proof.** Since \( X \) is normal, Weil divisors are Cartier outside of a closed subset of codimension \( \geq 2 \). But for any closed subscheme \( Y \subset X \) we have a short exact sequence

\[
K(Y) \to K(X) \to K(X\setminus Y) \to 0.
\]

This together with Lemma 1.5(3) shows that \( [\mathcal{O}_D] - \sum a_i [\mathcal{O}_{D_i}] \) lies in \( K_{n-2}(X) \).

To prove the first part of the lemma let us write \( D = D_1 - D_2 \) for some effective Weil divisors \( D_1 \) and \( D_2 \). Since \( D_i, \ i=1,2 \) are effective, we have short exact sequences

\[
0 \to \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0
\]

showing that \( [\mathcal{O}_{D_i}] \in K_{n-1}(X) \). Now the first part of the proof implies that \( [\mathcal{O}_D] - [\mathcal{O}_{D_1}] + [\mathcal{O}_{D_2}] \in K_{n-2}(X) \), so \( [\mathcal{O}_D] \in K_{n-1}(X) \).\( \square \)

**Lemma 1.7.** Let \( \mathcal{F} \) be a coherent torsion free \( \mathcal{O}_X \)-module of rank \( r \). Then the class \( \alpha(\mathcal{F}) \) of \( \mathcal{F} - \mathcal{O}_X^{\oplus r} + [\mathcal{O}_{c_1(\mathcal{F})}] \) lies in \( K_{n-2}(X) \).

**Proof.** If \( r = 1 \) and \( \mathcal{F} \) is reflexive then by definition of \( c_1(\mathcal{F}) \) we have \( \alpha(\mathcal{F}) = 0 \) in \( K(X) \). In general, if \( r = 1 \) then

\[
\alpha(\mathcal{F}) = \alpha(\mathcal{F}) - \alpha(\mathcal{F}^{**}) = \mathcal{F} - \mathcal{F}^{**},
\]

\[
\mathcal{F} - \mathcal{O}_X^{\oplus r} + [\mathcal{O}_{c_1(\mathcal{F})}] = \mathcal{F} - \mathcal{O}_X^{\oplus r} + [\mathcal{O}_{c_1(\mathcal{F})}] - (\mathcal{F}^{**} - \mathcal{O}_X^{\oplus r} + [\mathcal{O}_{c_1(\mathcal{F})}]) = \mathcal{F} - \mathcal{F}^{**},
\]

so the assertion follows from the fact that the cokernel of \( \mathcal{F} \to \mathcal{F}^{**} \) is supported in codimension \( \geq 2 \).

If \( r > 1 \) then \( \mathcal{F} \) has a filtration \( \mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \supset \mathcal{F}^r = 0 \), whose quotients \( \mathcal{L}_i = \mathcal{F}^i / \mathcal{F}^{i+1} \) are rank 1 torsion free sheaves. Then we have

\[
\alpha(\mathcal{F}) = [\mathcal{O}_{c_1(\mathcal{F})}] - \sum [\mathcal{O}_{c_1(\mathcal{L}_i})] + \sum \alpha(\mathcal{L}_i).
\]

Since \( c_1(\mathcal{F}) = \sum c_1(\mathcal{L}_i) \) Lemma 1.6 shows that \( [\mathcal{O}_{c_1(\mathcal{F})}] - \sum [\mathcal{O}_{c_1(\mathcal{L}_i})] \) lies in \( K_{n-2}(X) \). Therefore the assertion follows from the rank 1 case. \( \square \)
2 Intersection theory on normal varieties

Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$. If $D$ is a Weil divisor on $X$ then we write $[\mathcal{O}_D]$ for the class of $\mathcal{O}_X - \mathcal{O}_X(-D)$ in $K(X)$. We also write $N^1(X)$ for the group of line bundles on $X$ modulo numerical equivalence. By the Néron–Severi theorem of the base, $N^1(X)$ is finitely generated free $\mathbb{Z}$-module.

2.1 Intersection of a Weil divisor with Cartier divisors

In this subsection we define a $K$-theoretic intersection of a Weil divisor with Cartier divisors and compare it with standard definition using intersections with Chern classes of line bundles.

Lemma 2.1. The image of the map $Z^1(X) \times \text{Pic}X^{\times(n-1)} \to K(X)$ defined by

$$(D, L_1, \ldots, L_{n-1}) \mapsto c_1(L_1) \cdots c_1(L_{n-1}) \cdot [\mathcal{O}_D]$$

is contained in $K_0(X)$. Moreover, this map is $\mathbb{Z}$-linear with respect to each variable and symmetric with respect $(L_1, \ldots, L_{n-1})$ to for fixed Weil divisor $D$.

Proof. Let us write $D = \sum a_iD_i$ for some prime Weil divisors $D_i$ and some integers $a_i$. By Lemma 1.5 (1) we have $c_1(L_1) \cdots c_1(L_{n-1})K_{n-1}(X) \subset K_0(X)$ and $c_1(L_1) \cdots c_1(L_{n-1})K_{n-2}(X) = 0$. Therefore Lemma 1.6 implies the first assertion.

It also shows that

$$c_1(L_1) \cdots c_1(L_{n-1})[\mathcal{O}_D] = \sum a_i c_1(L_1) \cdots c_1(L_{n-1})[\mathcal{O}_D].$$

This proves that the map is $\mathbb{Z}$-linear with respect to $D$. By Lemma 1.5 (2) the map is symmetric for fixed $D$. Moreover, we have

$$c_1(L_1 \otimes M_1)c_1(L_2) \cdots c_1(L_{n-1}) \cdot [\mathcal{O}_D] - c_1(L_1) \cdots c_1(L_{n-1}) \cdot [\mathcal{O}_D] - c_1(M_1)c_1(L_2) \cdots c_1(L_{n-1}) \cdot [\mathcal{O}_D]$$

$$= -c_1(L_1)c_1(M_1)c_1(L_2) \cdots c_1(L_{n-1}) \cdot [\mathcal{O}_D] = 0,$$

which finishes the proof that the map is $\mathbb{Z}$-linear with respect to all variables.  

We also have a map $\text{Pic}X^{\times(n-1)} \to A_0(X)$ defined by

$$(L_1, \ldots, L_{n-1}) \mapsto c_1(L_1) \cap \cdots \cap c_1(L_{n-1}) \cap [D]$$
But any line bundle $L$ by sending $\mathcal{F}$ to $\sum_{x \in X(k)} l_x(\mathcal{F}) [x]$, where $l_x(\mathcal{F})$ is the length of $\mathcal{F}_x$ as an $\mathcal{O}_X$-module (cf. [Fu, Example 18.3.11], where an analogous map is defined also for cycles of higher dimension but it goes into the Chow group with rational coefficients). This map is an isomorphism with the inverse $\psi$ cycles of higher dimension but it goes into the Chow group with rational coefficients. This map is an isomorphism with the inverse $\phi : A_0(X) \to K_0(X)$ given by sending $[x]$ to $\mathcal{O}_x$.

The following lemma, generalizing the Riemann–Roch theorem on curves, compares the above maps.

**Lemma 2.2.** For any Weil divisor $D$ and any line bundles $L_1, \ldots, L_{n-1}$ we have

$$c_1(1) \cap \ldots \cap c_1(n) \cap [D] = \psi(c_1(1) \ldots c_1(n) \cdot [\mathcal{O}_D]).$$

**Proof.** By the above we know that both sides of our equality are $\mathbb{Z}$-linear in $L_i$. But any line bundle $L$ can be written as $A \otimes B^{-1}$ for some very ample line bundles $A$ and $B$. So it is sufficient to prove the above equality assuming that all $L_i$ are very ample.

For general divisors $H_i \in |L_i|$ the complete intersection $C = H_1 \cap \ldots \cap H_{n-1}$ is a smooth curve and $D \cap C$ is a 0-cycle representing $c_1(1) \cap \ldots \cap c_1(n) \cap [D]$. But by Lemma [L.5] (4) we have equality $c_1(1) \ldots c_1(n)[\mathcal{O}_D] = \mathcal{O}_{D \cap C}$ in $K(X)$, so the required assertion is clear. $\square$

Note that the above defined maps descend to the intersection product $A^1(X) \times (\text{Pic} X)^{(n-1)} \to A_0(X)$ given by $(D, L_1, \ldots, L_{n-1}) \to c_1(L_1) \cap \ldots \cap c_1(L_{n-1}) \cap [D]$. Since $\chi(X, \cdot) = \int_X \psi$, the above lemma shows that this descends to an intersection product $A^1(X) \times N^1(X)^{(n-1)} \to \mathbb{Z}$ given by

$$D \cdot L_1 \ldots L_{n-1} := \chi(X, c_1(L_1) \ldots c_1(L_{n-1}) \cdot [\mathcal{O}_D]) = \int_X c_1(L_1) \cap \ldots \cap c_1(L_{n-1}) \cap [D].$$

### 2.2 Intersection of two Weil divisors with Cartier divisors

In this subsection we define intersection number for two Weil divisors and a collection of Cartier divisors. We assume that $n \geq 2$.

**Proposition 2.3.** Let $D$ be a Weil divisor and let $L_1, \ldots, L_{n-2}$ be line bundles on $X$. Then the sequence

$$\left( \frac{\chi(X, c_1(L_1) \ldots c_1(L_{n-2}) \cdot [\mathcal{O}_mD])}{m^2} \right)_{m \in \mathbb{N}}$$
is convergent and its limit is a rational number.

Proof. By Lemma[1.5](3) we have

\[ \chi(X, c_1(L_1 \otimes M_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M]) = \chi(X, c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M]) + \chi(X, c_1(M_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M]) - \chi(X, c_1(M_1)c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M]). \]

Since

\[ \lim_{m \to \infty} \frac{\chi(X, c_1(M_1)c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2} = \lim_{m \to \infty} \frac{mDc_1(M_1)c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2} = 0, \]

we see that if \( \lim_{m \to \infty} \frac{\chi(X, c_1(M_1)c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2} \) and \( \lim_{m \to \infty} \frac{\chi(X, c_1(M_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2} \) exist then \( \lim_{m \to \infty} \frac{\chi(X, c_1(L_1 \otimes M_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2} \) exists and it is equal to the sum of these limits.

Since any line bundle \( L \) can be written as \( A \otimes B^{-1} \) for some very ample line bundles \( A \) and \( B \) and the formula in the sequence is symmetric in \( (L_1, \ldots, L_{n-2}) \), it is sufficient to prove existence of (rational) limit assuming that all line bundles \( L_i \) are very ample.

Let us write \( D = \sum d_i D_i \) as an integral combination of prime Weil divisors \( D_i \).

By Theorem[1.4] for a general sequence \( H_i \subseteq |L_i|, i = 1, \ldots, n-2 \), each intersection \( X_j := \bigcap_{i \leq j} H_i \) is normal and all \( D_i \cap X_j \) have codimension 1 in \( X_j \). So \( D_{X_j} = \sum d_i(D_i \cap X_j) \) is a well defined Weil divisor on \( X_j \) and for all \( m \in \mathbb{Z} \) and for all \( 1 \leq j \leq n-2 \) we have short exact sequences

\[ 0 \to \mathcal{O}_{X_{j-1}}(mD) \otimes L_j^{-1} \to \mathcal{O}_{X_{j-1}}(mD_{X_{j-1}}) \to \mathcal{O}_{X_j}(mD_{X_j}) \to 0. \]

These sequences show that \( \mathcal{O}_{X_j}(mD_{X_j}) = c_1(L_1) \cdot \mathcal{O}_{X_{j-1}}(mD_{X_{j-1}}) \) in \( K(X) \). So if we set \( S := X_{n-2} \) then \( \mathcal{O}_S(mD_S) = c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot \mathcal{O}_X(mD) \) in \( K(X) \). Now by the Riemann–Roch theorem on \( S \) (see Theorems[4.4] and[3.12]) we get

\[ \lim_{m \to \infty} \frac{\chi(X, c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2} = \lim_{m \to \infty} \frac{-\chi(X, c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_X(mD)])}{m^2} = \lim_{m \to \infty} \frac{-\chi(S, \mathcal{O}_S(−mD_S))}{m^2} = \frac{1}{2}D_S^2, \]

where \( D_S^2 \) is the self intersection of \( D_S \) in the sense of Mumford’s intersection pairing on \( S \).

For any Weil divisor \( D \) and any line bundles \( L_1, \ldots, L_{n-2} \) we set

\[ D^2.L_1\ldots L_{n-2} := 2 \lim_{m \to \infty} \frac{\chi(X, c_1(L_1)c_1(L_2)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_M])}{m^2}. \]
Theorem 2.4. Let us consider the map $Z^1(X) \times Z^1(X) \times \text{Pic}X \to \mathbb{Q}$ defined by sending $(D_1, D_2, L_1, \ldots, L_{n-2})$ to

$$D_1D_2L_1\ldots L_{n-2} := \frac{1}{2}((D_1+D_2)^2L_1\ldots L_{n-2} - D_1^2L_1\ldots L_{n-2} - D_2^2L_1\ldots L_{n-2}).$$

This map satisfies the following conditions:

1. It is $\mathbb{Z}$-linear in each variable.
2. It is symmetric in $D_1$ and $D_2$.
3. It is symmetric in $L_1, \ldots, L_{n-2}$.
4. If $D_2$ is a Cartier divisor then $D_1D_2L_1\ldots L_{n-2} = D_1\mathcal{O}_X(D_2)L_1\ldots L_{n-2}$.
5. If we fix $D_1, D_2 \in Z^1(X)$ and assume that $n > 2$ and $L_1$ is very ample then for a general hypersurface $H_1 \in |L_1|$ we have

$$D_1D_2L_1\ldots L_{n-2} = (D_1)_{H_1}(D_2)_{H_1}L_1\ldots L_{n-2}_{|H_1}.$$ 

Proof. The fact that the map is $\mathbb{Z}$-linear in $L_i$ follows from the first part of the proof of Proposition 2.3. If we fix $D_1, D_2 \in Z^1(X)$ and assume that all $L_1, \ldots, L_{n-2}$ are very ample then for a general complete intersection surface $S \in |L_1| \cap \ldots \cap |L_{n-2}|$ we have

$$D_1D_2L_1\ldots L_{n-2} = (D_1)_S(D_2)_S.$$ 

This follows from the proof of Proposition 2.3 and the fact that on a general complete intersection surface $S$ we have equality

$$(D_1)_S(D_2)_S = \frac{1}{2}((D_1+D_2)_S^2 - (D_1)_S^2 - (D_2)_S^2).$$

Now (5) follows from the above mentioned linearity, which allows us to assume that also $L_2, \ldots, L_{n-2}$ are very ample.

(2) is clear and (3) follows from Lemma 1.5 (2). To prove that the map is $\mathbb{Z}$-linear in $D_1$ and $D_2$ we can assume that $L_1, \ldots, L_{n-2}$ are very ample. In this case the assertion follows from (5). In the same way we can reduce (4) to the normal surface case, where the assertion is clear.
The above theorem implies that we have a well defined induced intersection form

\[ A^1(X) \times A^1(X) \times N^1(X)^{(n-2)} \to \mathbb{Q}, \]

generalizing Mumford’s intersection pairing on surfaces. Note that our approach reconstructs Mumford’s intersection pairing on normal surfaces without using any resolution of singularities.

**Corollary 2.5.** Let \( D_1, D_2 \) be Weil divisors and \( L_1, \ldots, L_{n-2} \) line bundles on \( X \). Let \( \pi : Y \to X \) is a finite morphism from a normal projective variety \( Y \). Let us assume that \( \pi \) has separably generated residue field extensions. Then we have

\[ \pi^* D_1 \cdot \pi^* D_2 \cdot \pi^* L_1 \cdots \pi^* L_{n-2} = \deg \pi \cdot D_1 \cdot D_2 \cdot L_1 \cdots L_{n-2}. \]

**Proof.** We can use similar arguments to that from the proof of Proposition 2.3 to reduce to the case, when all \( L_i \) are very ample. Then Theorem 1.4 allows us to reduce to the surface case. In this case the result follows from Remark 4.3.

Unfortunately, it is not clear how to prove the above corollary for arbitrary finite morphisms in positive characteristic. Clearly, it holds also for the Frobenius morphism as \( F^* D = pD \) for any Weil divisor \( D \).

Now let fix very ample line bundles \( L_1, \ldots, L_{n-2} \) and consider a \( \mathbb{Q} \)-valued intersection pairing \( \langle \cdot, \cdot \rangle : A^1(X) \times A^1(X) \to \mathbb{Q} \) by setting

\[ \langle D_1, D_2 \rangle := D_1 \cdot D_2 \cdot L_1 \cdots L_{n-2}. \]

Let us write \( N_L(X) \) for the quotient of \( A^1(X) \) by the radical of this intersection pairing. Then we have an induced non-degenerate intersection pairing

\[ \langle \cdot, \cdot \rangle : N_L(X) \times N_L(X) \to \mathbb{Q}. \]

If \( n = 2 \) then we write \( N(X) \) instead of \( N_L(X) \).

**Lemma 2.6.** \( N_L(X) \) is a free \( \mathbb{Z} \)-module of finite rank. If \( \text{rk} N_L(X) = s \) then the intersection pairing \( \langle \cdot, \cdot \rangle \) has signature \((1, s - 1)\). Moreover, there exists a positive integer \( N \) such that the intersection pairing \( \langle \cdot, \cdot \rangle \) takes values in \( \frac{1}{N} \mathbb{Z} \subset \mathbb{Q} \).

**Proof.** The lemma is well known in case of normal surfaces. Now let \( Y \) be the product of \((n - 2)\) projective spaces \(|L_i|\) and let us consider the incidence scheme \( \mathcal{S} \subset X \times_k Y \), whose points are of the form \((x; H_1, \ldots, H_{n-2}) \in X \times_k Y \) with \( x \in \)
\[ \bigcap_{j \leq n-2} H_j. \] Let \( \bar{\eta} : \text{Spec} K \to Y \) be a geometric generic point of \( Y \). Then the fiber \( \mathcal{S}_K \) of the projection \( \mathcal{S} \to Y \) over \( \bar{\eta} \) is a normal surface contained in \( X \otimes_k K \). Note that if \( D \) is a prime Weil divisor on \( X \) then \( \mathcal{S}_K \) is not contained in \( D_K \). Otherwise, \( \mathcal{S} \), which is the closure of the image of \( \mathcal{S}_K \) in \( X \times_k Y \), would be contained in \( D \times_k Y \), a contradiction. So \( D_{\mathcal{S}_K} = D_K \cap \mathcal{S}_K \) is a prime Weil divisor on \( \mathcal{S}_K \). Now Theorem 2.4 implies that for any two divisors \( D_1 \) and \( D_2 \) on \( X \) we have

\[ \langle D_1, D_2 \rangle = (D_1)_{\mathcal{S}_K} \cdot (D_2)_{\mathcal{S}_K}. \]

Note that if the class of a Weil divisor \( D_1 \) on \( X \) is 0 in \( N_L(X) \) then the class of \( (D_1)_{\mathcal{S}_K} \) is 0 in \( N(\mathcal{S}_K) \). This follows from the fact that the class of \( (D_1)_{\mathcal{S}_K} \) is 0 in \( N(\mathcal{S}_K) \) if and only if \((D_1)_{\mathcal{S}_K})^2 = 0 \) and \((D_1)_{\mathcal{S}_K} \cdot H_{\mathcal{S}_K} = 0 \) for some ample divisor \( H \) on \( X \) (cf. [La4, Lemma 1.15]). It follows that \( N_L(X) \) is a \( \mathbb{Z} \)-submodule of a free \( \mathbb{Z} \)-module \( N(\mathcal{S}_K) \) of finite rank \( t \). So \( N_L(X) \) is also a free \( \mathbb{Z} \)-module of finite rank. This implies that there exists \( N \) such that our intersection pairing takes values in \( \frac{1}{N}\mathbb{Z} \subset \mathbb{Q} \). To see the signature of \( \langle \cdot, \cdot \rangle \), we note that for any ample divisor \( H \) we have \( \langle H, H \rangle > 0 \) and Mumford’s intersection pairing on \( N(\mathcal{S}_K) \) has signature \((1, t - 1)\).

2.3 Comparison with the de Fernex–Hacon pullback and Enokizono’s construction

In this subsection we compare the obtained intersection theory with the one obtained by intersecting de Fernex–Hacon pullbacks of Weil divisors. This subsection is not needed in the following but we will need to use some results and notation introduced in further part of the paper.

Let us recall that if \( X \) is a normal variety and \( f : Y \to X \) is a birational morphism then for any Weil divisor \( D \) on \( X \) in [DFH, Definition 2.5] the authors define \( f^\sharp D \) so that \( \mathcal{O}_X(-f^\sharp D) = f^\ast \mathcal{O}_X(-D) \). If \( f : \tilde{X} \to X \) is a resolution of singularities and \( X \) is a proper normal surface then in the notation of Subsection 4.1 we have

\[ f^\ast (-mD) = -f^\sharp (mD) - \sum_{x \in f(E)} c_1(f_x, f^\ast \mathcal{O}_X(-mD)), \]

where on the left hand side we have Mumford’s pullback. But by Theorem 3.9 there are only finitely many possibilities for \( c_1(f_x, f^\ast \mathcal{O}_X(-mD)) \), so by \( \mathbb{Z} \)-linearity of Mumford’s pullback we have

\[ f^\ast (D) = \lim_{m \to \infty} \frac{f^\sharp (mD)}{m}. \]
This shows that in the surface case de Fernex–Hacon pullback of Weil divisors defined in [dFH, Definition 2.9] coincides with Mumford’s pullback.

In higher dimensions (in characteristic 0) the de Fernex–Hacon pullback satisfies $f^*(-D) \neq -f^*D$, so it is not useful for defining intersection form on $X$. If one wants to get pullback of Weil divisors similar to Mumford’s rational pullback, one needs to consider only very special morphisms (see [Sch]). One can also consider a partial (Mumford’s) pullback

$$f^*_M D := \sum_{\text{codim } f(E) \leq 2} a(E) \cdot E,$$

where $a(E)$ is the coefficient of $E$ in the de Fernex–Hacon pullback $f^*D$, and the sum is taken over all prime divisors $E$ on $\tilde{X}$ with centers of codimension 1 or 2. Cutting to the surface case and using the above surface case, we see that this coincides with Enokizono’s construction of Mumford’s pullback (see [En, Appendix]). Note that even if $X$ is smooth, the above pullback does not coincide with the usual pullback of Cartier divisors. However, this partial pullback can be used to define intersection of two Weil divisors with Cartier divisors.

3 Local relative Chern classes for resolutions of normal surfaces

In this section we revise the theory of local relative Chern classes for resolutions of normal surfaces. This theory was studied in [Wa] in the rank 2 case and in [La1] in general, but only in the complex analytic setup. Here we develop an algebraic approach in an arbitrary characteristic. Our approach is along similar lines as in [La1] but we give simplified and more detailed versions of several proofs.

Let $k$ be an algebraically closed field and let $A$ be an excellent normal 2-dimensional Henselian local $k$-algebra. The developed theory seems to work in more general situations as in [La1] but we will need it only for henselizations of local rings of algebraic surfaces defined over an algebraically closed field.

Let $X = \text{Spec} A$ and let $x \in X$ be the closed point of $X$. Let $f : \tilde{X} \to X$ be a desingularization of $X$, i.e., a proper birational morphism from a regular surface $\tilde{X}$. Here a surface is a reduced noetherian separated $k$-scheme of dimension 2 but we do not assume that it is of finite type over $k$. Let $E$ be the exceptional locus of $f$ considered with the reduced scheme structure. One can also assume that $f$ is good, i.e., $E$ is a simple normal crossing divisor, but this will not be used in the following.
3.1 Local relative Chern classes of vector bundles for resolutions of surfaces

Let $\mathcal{F}$ be a vector bundle on $\tilde{X}$. If $\{E_i\}$ denote the irreducible components of $E$ then the intersection matrix $[E_i,E_j]$ of the exceptional divisor is negative definite (see, e.g., [Li, Lemma 14.1]). So there exists a unique $\mathbb{Q}$-divisor $c_1(f, \mathcal{F})$ supported on $E$ such that for every irreducible component $E_i$ of $E$ we have

$$c_1(f, \mathcal{F}).E_i = \deg \mathcal{F}|_{E_i}.$$  

We call $c_1(f, \mathcal{F})$ the first relative Chern class of $\mathcal{F}$ with respect to $f$.

Let $\text{Div}_E(\tilde{X})$ denote the group of divisors on $\tilde{X}$ that are supported on $E$. Then $c_1(f, \mathcal{F})$ is an element of $\text{Div}_E(\tilde{X}) \otimes \mathbb{Q}$. Since the canonical map $\text{Div}_E(\tilde{X}) \rightarrow \text{Pic} \tilde{X}$ is injective, we can, without any loss of information, consider $c_1(f, \mathcal{F})$ as an element of $\text{Pic} \tilde{X} \otimes \mathbb{Q}$.

Let $\tilde{\pi} : \tilde{Y} \rightarrow \tilde{X}$ be a generically finite proper morphism from a regular surface $\tilde{Y}$. Let us consider the Stein factorization of $f \circ \tilde{\pi}$ into a proper birational morphism $g : \tilde{Y} \rightarrow Y$ and a finite morphism $\pi : Y \rightarrow X$. Clearly, $Y \rightarrow X$ corresponds to a finite extension $A \rightarrow B$ with $B$ a normal domain. Since $A$ is Henselian, $B$ is local and also Henselian. Since $A$ is excellent, $B$ is still excellent (of dimension 2). Possibly further blowing up $\tilde{Y}$ we can also assume that $g$ is good.

Let us fix a rank $r$ vector bundle $\mathcal{F}$ on $\tilde{X}$ and let us consider a filtration $\tilde{\pi}^* \mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \supset \mathcal{F}^r = 0$ such that all quotients $L_i = \mathcal{F}^i/\mathcal{F}^{i+1}$ are line bundles. Note that such a filtration always exists, possibly after further blowing up $\tilde{Y}$.

Let us note that

$$(r - 1)(\sum L_i)^2 - 2r \sum_{i<j} L_i L_j = \sum_{i<j} (L_i - L_j)^2 \leq 0,$$

where $L_i = c_1(g, L_i)$. But $(\sum L_i)^2 = c_1(g, \tilde{\pi}^* \mathcal{F})^2 = \deg \tilde{\pi} \cdot c_1(f, \mathcal{F})^2$, so we have

$$\frac{\sum_{i<j} L_i L_j}{\deg \tilde{\pi}} \geq \frac{r - 1}{2r} c_1(f, \mathcal{F})^2.$$  

Therefore the following definition makes sense:

**Definition 3.1.** The second relative Chern class $c_2(f, \mathcal{F})$ of $\mathcal{F}$ with respect to $f$ is defined as the real number

$$c_2(f, \mathcal{F}) = \inf \left( \frac{\sum_{i<j} L_i L_j}{\deg \tilde{\pi}} \right),$$

17
where the infimum is taken over all generically finite proper morphisms $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ from a regular surface and all filtrations $\tilde{\pi}^* \mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \supset \mathcal{F}^r = 0$, whose quotients $\mathcal{L}_i = \mathcal{F}^i / \mathcal{F}^{i+1}$ are line bundles.

In the rank 2 case in the complex analytic setting this definition was introduced by Wahl in [Wahl]. His definition was generalized to arbitrary rank in [La1] and studied there also in the complex analytic setting.

The following proposition summarizes some basic properties of the relative second Chern class.

**Proposition 3.2.**

1. For any line bundle $\mathcal{L}$ on $\tilde{X}$ we have $c_2(f, \mathcal{L}) = 0$.

2. (relative Bogomolov’s inequality) For any rank $r$ vector bundle $\mathcal{F}$ on $\tilde{X}$ we have

   $$\Delta(f, \mathcal{F}) := 2rc_2(f, \mathcal{F}) - (r-1)c_1(f, \mathcal{F})^2 \geq 0.$$  

3. For any rank $r$ vector bundle $\mathcal{F}$ and any line bundle $\mathcal{L}$ on $\tilde{X}$ we have

   $$\Delta(f, \mathcal{F} \otimes \mathcal{L}) = \Delta(f, \mathcal{F}).$$

4. If $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is a generically finite proper morphism from a regular surface $\tilde{Y}$ and $\tilde{Y} \xrightarrow{\tilde{\pi}} Y \xrightarrow{\pi} X$ is the Stein factorization of $f \circ \tilde{\pi}$ then

   $$c_2(g, \tilde{\pi}^* \mathcal{F}) = \deg \pi \cdot c_2(f, \mathcal{F}).$$

*Proof.* Properties (1)–(3) are obvious from the definition. To show (4) let us first remark that the category of finite extensions $A \to B$ with $B$ a normal domain, is filtered. Given two extensions $A \to B$ and $A \to C$ we can embed the quotient fields $K(B)$ and $K(C)$ into some fixed algebraic closure of $K(A)$. Then we can find a finite field extension $K(A) \subset L$ that contains both $K(B)$ and $K(C)$. Then the normalization $D$ of $A$ in $L$ gives a finite normal extension $A \to L$ dominating both $A \to B$ and $A \to C$.

This implies that the category of generically finite proper morphisms from a regular surface to $\tilde{X}$ is cofiltered. Indeed, if $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ and $\tilde{\tau} : \tilde{Z} \to \tilde{X}$ are generically finite proper morphisms from regular surfaces $\tilde{Y}$ and $\tilde{Z}$ then we consider the corresponding Stein factorizations $\tilde{Y} \xrightarrow{\tilde{g}} Y \xrightarrow{\tilde{\pi}} X$ and $\tilde{Z} \xrightarrow{\tilde{h}} Z \xrightarrow{\tilde{\tau}} X$. By the above we can find a finite morphism $T \to X$ from a normal surface $T$, that dominates both $\pi$ and $\tau$. Then we can find a good resolution of singularities $\tilde{T} \to T$ with generically finite proper morphisms to both $\tilde{Y}$ and $\tilde{Z}$ (it is sufficient to find a resolution dominating main irreducible components of both $\tilde{Y} \times_Y T$ and $\tilde{Z} \times_Z T$).
The above fact allows us to pull-back to \( \tilde{T} \) any filtration of the pullback of \( F \) to \( \tilde{Z} \). This gives a filtration of the pull back of \( \pi^*F \) implying (4). \qed

Remark 3.3. In the following we write \( \text{ch}_2(f, \mathcal{F}) \) for \( \frac{1}{4}c_1(f, \mathcal{F})^2 - c_2(f, \mathcal{F}) \). Note that definition of \( c_2(f, \cdot) \) implies that for any two vector bundles \( \mathcal{F}_1, \mathcal{F}_2 \) on \( \tilde{X} \) we have
\[
\text{ch}_2(f, \mathcal{F}_1 \oplus \mathcal{F}_2) \geq \text{ch}_2(f, \mathcal{F}_1) + \text{ch}_2(f, \mathcal{F}_2).
\]
Later we prove that if the base field \( k \) has positive characteristic then we have equality (see Corollary 3.17).

Remark 3.4. One of the main open problems related to local relative Chern classes is their behaviour under tensor operations (see \cite{La1}, Conjecture 8.1). For example, if we knew that one can compute \( c_2(f, \text{Sym}^m \mathcal{F}) \) using \( c_1(f, \mathcal{F}) \) and \( c_2(f, \mathcal{F}) \) using the same formulas as follow from the splitting principle for usual Chern classes, then Conjecture 3.18 holds.

### 3.2 Local relative Riemann–Roch theorem

**Definition 3.5.** For a vector bundle \( \mathcal{F} \) on \( \tilde{X} \) we define a *relative Euler characteristic* \( \chi(f, \mathcal{F}) \) by
\[
\chi(f, \mathcal{F}) := \text{dim} H^0(X, f\lceil_* \mathcal{F}/f_* \mathcal{F}) + \text{dim} H^0(X, R^1 f_* \mathcal{F}).
\]

**Definition 3.6.** For any rank \( r \) vector bundle \( \mathcal{F} \) on \( \tilde{X} \) we set
\[
a(f, \mathcal{F}) := \chi(f, \mathcal{F}) - r \chi(f, \mathcal{O}_X) + \frac{1}{2}c_1(f, \mathcal{F})(c_1(f, \mathcal{F}) - K_X) - c_2(f, \mathcal{F}).
\]

The proof of the following proposition is essentially the same as that of \cite{La1}, Proposition 2.9 so, since we cannot improve upon it, we skip it.

**Proposition 3.7.** Let \( f' : \tilde{X}' \to X \) be a desingularization of \( X \). Let \( \mathcal{F} \in \text{Vect}(\tilde{X}) \) and \( \mathcal{F}' \in \text{Vect}(\tilde{X}') \) be vector bundles such that \( f\lceil_* \mathcal{F} \) and \( f'\lceil_* \mathcal{F}' \) are isomorphic. Then we have \( a(f, \mathcal{F}) = a(f', \mathcal{F}') \).

If \( \mathcal{E} \) be a reflexive coherent \( \mathcal{O}_X \)-module then we set
\[
a(x, \mathcal{E}) := a(f, \mathcal{F}),
\]

19
where $f : \tilde{X} \to X$ is any desingularization of $X$ and $\mathcal{F}$ is any vector bundle on $\tilde{X}$ such that $f_\ast [\mathcal{F}] \simeq \mathcal{E}$. The above proposition implies that $a(x, \mathcal{E})$ is well defined. Therefore we get a function $a(x, \cdot) : \text{Ref}(\mathcal{O}_X) \to \mathbb{R}$. If $j : U := X \setminus \{x\} \hookrightarrow X$ denotes the open embedding then by Lemma 1.1 the functor $j_\ast : \text{Vect}(U) \to \text{Ref}(\mathcal{O}_X)$ is an equivalence of categories, so we can also treat $a(x, \cdot)$ as a function $\text{Vect}(U) \to \mathbb{R}$.

In the remaining part of the section we reprove the results of [La1, Section 4] giving more details and providing simpler proofs that avoid the use of reduction cycles. First we note the following lemma.

**Lemma 3.8.** Let $\mathcal{G}$ be a vector bundle of rank $r$ on $\tilde{X}$. If $\mathcal{G}$ is globally generated outside of a finite number of $k$-points $T$ of $E$ then for general $(r + 2)$ sections of $\mathcal{G}$ the cokernel of the induced map $\mathcal{O}_X^{(r+2)} \to \mathcal{G}$ is supported on $T$.

**Proof.** Let us choose a finite dimensional $k$-vector space $V \subset H^0(\tilde{X}, \mathcal{G})$ such that the evaluation map $V \otimes_k \mathcal{O}_{\tilde{X}} \to \mathcal{G}$ is surjective on $U := \tilde{X} \setminus T$. We can assume that $V$ has large dimension so that $s := \dim V - (r + 2) > 0$.

Let $G$ be the Grassmannian of $s$-dimensional quotient spaces of $V$. Then $U \times_k G \to U$ is the Grassmann scheme representing rank $s$ quotient vector bundles of the trivial bundle $V \otimes_k \mathcal{O}_U$. Let $\mathcal{H}$ be the kernel of the evaluation map $V \otimes_k \mathcal{O}_{\tilde{X}} \to \mathcal{G}$ and let us consider the subfunctor of the appropriate Grassmann functor, such that $S$-points consist of those quotients $V \otimes_k \mathcal{O}_S \to \mathcal{F}$ for which the induced map $\bigwedge^s \mathcal{H} \to \bigwedge^s \mathcal{F}$ vanishes. By [Kl, Proposition 2.2] this functor is represented by a closed subscheme $Z$ of $U \times_k G$. For $x \in U(k)$, $k$-points of $Z_x \subset G$ correspond to $(r + 2)$-dimensional vector subspaces $W \subset V$ such that $\dim(W \cap k(x) \cap \mathcal{H} \otimes k(x)) \geq 3$ (which is why $Z$ is also called the 3-rd special Schubert cycle defined by $\mathcal{H}$). Naively speaking, $Z$ parametrizes pairs $(x, [W \subset V])$ such that the induced map $W \otimes k(x) \to \mathcal{G} \otimes k(x)$ is not surjective.

By [Kl, Corollary 2.9] the scheme $Z$ has relative dimension $\dim G - 3$ over $U$ (this also follows from the standard dimension computation of the Schubert cell defined by condition $\dim(W \cap \mathcal{H} \otimes k(x)) \geq 3$ in $G$). It follows that $\dim Z = \dim U + \dim G - 3 < \dim G$. So there exists $W \subset V$ of dimension $(r + 2)$ (corresponding to some point of $G \setminus p_2(Z)$, where $p_2 : Z \to G$ comes from the projection $U \times_k G \to G$) such that the evaluation map $W \otimes k(x) \to \mathcal{G}$ is surjective over $U$. □

**Proposition 3.9.** Let $\mathcal{E}$ be a reflexive coherent $\mathcal{O}_X$-module of rank $r$ and let $\mathcal{F} = f_\ast[\mathcal{E}]$. Then the following conditions are satisfied:

1. $f_\ast \mathcal{F} = \mathcal{E}$,
2. \( \dim H^0(X, R^1 f_* \mathcal{F}) \leq (r + 2) \dim H^0(X, R^1 f_* \mathcal{O}_\bar{X}) \).

3. There exists some \( C > 0 \) depending only on \( f : \bar{X} \to X \) such that for every irreducible component \( E_j \) of \( E \) we have
   \[
   0 \leq c_1(f, \mathcal{F}) \cdot E_j \leq C \cdot r. 
   \]

**Proof.** To see (1) note that we have the canonical map
   \[
   \mathcal{E} \to f_* f^* \mathcal{E} \to f_* f^{[s]} \mathcal{E},
   \]
which is an isomorphism on \( X \setminus f(E) \). Since \( \mathcal{E} \) is reflexive and \( f_* f^{[s]} \mathcal{E} \) is torsion free, it is an isomorphism on the whole \( X \).

To see (2) let us note that \( \mathcal{E} \) is globally generated as \( X \) is affine. So \( \mathcal{F} \) is globally generated outside a finite number of \( k \)-points (lying on \( E \)) and by Lemma 3.8 for general \( (r + 2) \) sections of \( \mathcal{F} \) the cokernel of the corresponding map \( \mathcal{O}_{\bar{X}}^{(r+2)} \to \mathcal{F} \) is supported on a finite set of points. This shows that we have a surjective map \( (R^1 f_* \mathcal{O}_X)^{(r+2)} \to R^1 f_* \mathcal{F} \), which gives (2). Note also that for every \( j \) the restriction of the above map to \( E_j \) gives \( \mathcal{O}_{E_j}^{(r+2)} \to \mathcal{F}|_{E_j} \) with cokernel supported on a finite set of points. Therefore \( c_1(f, \mathcal{F}) \cdot E_j = \deg \mathcal{F}|_{E_j} \geq 0. \)

Now let us consider a short exact sequence
   \[
   0 \to \mathcal{F} \to \mathcal{F}(E_j) \to \mathcal{F}(E_j)|_{E_j} \to 0.
   \]
(1) implies that \( f_* \mathcal{F} = \mathcal{E} = f_* (\mathcal{F}(E_j)) \), so the map \( f_* (\mathcal{F}(E_j)|_{E_j}) \to R^1 f_* \mathcal{F} \) is injective. Therefore we have
   \[
   \chi(\mathcal{F}(E_j)|_{E_j}) \leq \dim H^0(\bar{X}, \mathcal{F}(E_j)|_{E_j}) \leq \dim H^0(X, R^1 f_* \mathcal{F}) \leq (r + 2) \dim H^0(X, R^1 f_* \mathcal{O}_\bar{X}).
   \]
But by the Riemann–Roch theorem on \( E_j \) we have
   \[
   \chi(\mathcal{F}(E_j)|_{E_j}) = r \chi(\mathcal{O}_{E_j}) + rE_j^2 + \deg \mathcal{F}|_{E_j},
   \]
so we get
   \[
   c_1(f, \mathcal{F}) \cdot E_j \leq (r + 2) \dim H^0(X, R^1 f_* \mathcal{O}_\bar{X}) - r \chi(\mathcal{O}_{E_j}) - rE_j^2,
   \]
which gives the second inequality in (3).

**COROLLARY 3.10.** In the notation of Proposition 3.9 the \( \mathbb{Q} \)-divisor \( -c_1(f, \mathcal{F}) \) is effective and there exists a constant \( \tilde{C} > 0 \) depending only on \( f : \bar{X} \to X \) such that for every \( \mathcal{E} \) we have
   \[
   -c_1(f, \mathcal{F})^2 \leq \tilde{C} \cdot r^2. 
   \]
Proof. The first assertion follows from the inequalities $0 \leq c_1(f, \mathcal{F}) \cdot E_j$ (see [Gi (7)]). To see the second assertion let us write $c_1(f, \mathcal{F}) = -\sum \alpha_i E_i$ for some non-negative rational numbers $\alpha_i$. Then by Cramer’s rule the $\alpha_i$ depend linearly on the numbers $c_1(f, \mathcal{F}) \cdot E_j$. So the assertion follows from the last part of Proposition 3.9. □

Lemma 3.11. If $\mathcal{G}$ is a globally generated rank $r$ vector bundle on $\tilde{X}$ then $c_2(f, \mathcal{G}) \leq 0$.

Proof. Let us choose a finite dimensional $k$-vector subspace $V \subset H^0(\tilde{X}, \mathcal{G})$ such that the evaluation map $V \otimes_k \mathcal{O}_{\tilde{X}} \to \mathcal{G}$ is surjective. Let $G$ be the Grassmannian of $(r - 1)$-dimensional $k$-subspaces of $V$ and let $Z \subset \tilde{X} \times G$ be the subscheme parametrizing pairs $(x, [W \subset V])$ such that the induced map $W \otimes k(x) \to \mathcal{G} \otimes k(x)$ is not injective. Strictly speaking, we should again use the functorial approach of [Kl] (as in the proof of Lemma 3.8) but we sketch a naive approach leaving formalization of proof to the reader.

Let us consider projections $p_1 : Z \to \tilde{X}$ and $p_2 : Z \to G$. For any $k$-point of $\tilde{X}$ the map $V \otimes k(x) \to \mathcal{G} \otimes k(x)$ is surjective. So if $K$ denotes its kernel (which is of dimension $\dim V - r$) then we have

$$p_1^{-1}(x) \simeq \{[W \subset V] \in G : \dim((W \otimes k(x)) \cap K) \geq 1\}.$$ 

A standard computation shows that this Schubert cell has codimension 2 in $G$. It follows that $\dim p_1^{-1}(E) = \dim E + \dim G - 2 < \dim G$. So there exists $W \subset V$ of dimension $(r - 1)$ (corresponding to some point of $G \setminus p_2(p_1^{-1}(E))$) such that the evaluation map $W \otimes k \mathcal{O}_{\tilde{X}} \to \mathcal{G}$ is injective and its cokernel is locally free along $E$. This implies that we have a short exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}^{\oplus(r-1)} \to \mathcal{G} \to \mathcal{L} \to 0,$$

where $\mathcal{L}$ is a line bundle. This immediately implies the required inequality. □

The following theorem is an important improvement upon [La1 Theorem 4.12 and Corollary 4.13]. In particular, this theorem immediately implies [La1 Conjecture 8.2]. Since this conjecture is equivalent to the more natural formula in [La1 Corollary 8.3] that is the content of theorem below, we do not repeat the original formulation.
Lemma 3.11 we have $c$ determined by $(\mathcal{O}_T \text{twisting it by } 0$. This implies that for all $n \geq n_0$ such that for all irreducible components $E_j$ of $E$ we have $R^1 f_*(\mathcal{O}_X(-E_j) \otimes \mathcal{O}_X(n)) = 0$.

Let us recall that by Lemma 3.8 the cokernel of the map $\mathcal{O}_X \oplus (r+2) \to \mathcal{F}$ determined by $(r+2)$ general sections of $\mathcal{F}$ is supported on a finite set of points. Twisting it by $\mathcal{O}_X(-E_j) \otimes \mathcal{O}_X(n)$ we see that $R^1 f_*(\mathcal{F}(n) \otimes \mathcal{O}_X(-E_j)) = 0$

for all $n \geq n_0$. So we have surjective maps $H^0(\mathcal{F}(n)) \to H^0(E_j, \mathcal{F}(n)|_{E_j})$.

This implies that for all $n \geq n_0$ the bundle $\mathcal{F}(n)$ is globally generated. So by Lemma 3.11 we have $c_2(f, \mathcal{F}(n_0)) \leq 0$ and hence $0 \leq \Delta(f, \mathcal{F}) = \Delta(f, \mathcal{F}(n_0)) \leq -(r-1)c_1(f, \mathcal{F}(n_0))^2$ $= -(r-1) (c_1(f, \mathcal{F})^2 + n_0 c_1(f, \mathcal{F}) c_1(f, \mathcal{O}_X(1)) + n_0^2 c_1(f, \mathcal{O}_X(1))^2)$.

By Proposition 3.9 we also know that $0 \leq \chi(f, \mathcal{F}) \leq (r+2) \chi(f, \mathcal{O}_X)$. Since $a(x, \mathcal{E}) = \chi(f, \mathcal{F}) - r \chi(f, \mathcal{O}_X) + \frac{1}{2r} c_1(f, \mathcal{F})(c_1(f, \mathcal{F}) - rK_\mathcal{X}) - \frac{\Delta(f, \mathcal{F})}{2r}$, the required inequalities follow from Proposition 3.9 and Corollary 3.10.

**Remark 3.13.** The proof shows that in the above theorem one can find $A$ and $B$ that depend only on numerical invariants of $X$ and its fixed resolution $f : \tilde{X} \to X$. More precisely, these constants can be determined by $\dim R^1 f_* \mathcal{O}_X$ (called the geometric genus of the singularity $X$), discrepancies of the exceptional divisor and the intersection matrix of exceptional curves of $f$. This fact is useful when studying how Chern classes change in families of reflexive sheaves on normal surfaces.

**Remark 3.14.** It is natural to expect that in notation of the proof of Theorem 3.12 there exists a constant $\hat{A}$ such that $\Delta(f, \mathcal{F}) \leq \hat{A}r^2$. This would imply that one can find $A$ such that $Ar^2 \leq a(x, \mathcal{E})$. This conjecture is equivalent to [La1, Conjecture 8.2].
3.3 Characterization of the relative second Chern class in positive characteristic

In this subsection we assume that the base field $k$ has characteristic $p > 0$.

**Theorem 3.15.** There exists a uniquely determined function $c_2(f, \cdot) : \text{Vect}(\tilde{X}) \to \mathbb{R}, F \mapsto c_2(f, F)$ such that the following conditions are satisfied:

1. For every $F \in \text{Vect}(\tilde{X})$ we have $c_2(f, F^* \tilde{X}) = p^2 c_2(f, F)$.

2. There exists a function $\varphi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that for every rank $r$ vector bundle $F \in \text{Vect}(\tilde{X})$ we have

$$\left| \chi(f, F) + \frac{1}{2} c_1(f, F)(c_1(f, F) - K_{\tilde{X}}) - c_2(f, F) - r \chi(f, O_{\tilde{X}}) \right| \leq \varphi(r).$$

**Proof.** Existence of the function $c_2(f, \cdot)$ was proven in previous sections. More precisely, (1) follows from the last part of Proposition 3.2 applied to the Frobenius morphism. Condition (2) follows from Theorem 3.12.

To prove uniqueness note that $c_1(f, (F_X^m)^* \tilde{X}) = p^m c_1(f, \tilde{X})$. So (1) and (2) imply that

$$\left| \chi(f, (F_X^m)^* \tilde{X}) + \frac{1}{2} p^m c_1(f, \tilde{X})(p^m c_1(f, \tilde{X}) - K_{\tilde{X}}) - p^{2m} c_2(f, F) - r \chi(f, O_{\tilde{X}}) \right| \leq \varphi(r)$$

Dividing the above inequality by $p^{2m}$ and passing to the limit we get

$$c_2(f, F) = \frac{1}{2} c_1(f, F)^2 + \lim_{m \to \infty} \frac{\chi(f, (F_X^m)^* \tilde{X})}{p^{2m}}.$$

\[ \square \]

**Remark 3.16.** Theorem 3.12 shows that in fact we can take $\varphi$ to be quadratic and independent of $f$. This is not needed for the proof of uniqueness of $c_2(f, \cdot)$.

The last formula in the above proof and additivity of the relative Euler characteristic imply the following corollary.

**Corollary 3.17.** For any $\mathcal{F}_1, \mathcal{F}_2 \in \text{Vect}(\tilde{X})$ we have

$$\text{ch}_2(f, \mathcal{F}_1 \oplus \mathcal{F}_2) = \text{ch}_2(f, \mathcal{F}_1) + \text{ch}_2(f, \mathcal{F}_2).$$

Moreover, for any coherent reflexive $O_X$-modules $\mathcal{E}_1$ and $\mathcal{E}_2$ we have

$$\text{a}(x, \mathcal{E}_1 \oplus \mathcal{E}_2) = \text{a}(x, \mathcal{E}_1) + \text{a}(x, \mathcal{E}_2).$$
3.4 Conjectural characterization of the relative second Chern class in characteristic zero

Assume that the base field $k$ has characteristic zero. In this case one expects that the following asymptotic Riemann–Roch formula works (see the conjecture in [Wa, Introduction] for the rank 2 case; we need this conjecture in an arbitrary rank as hinted in [La1, Section 8]).

**Conjecture 3.18.** If $\mathcal{F}$ is a rank $r$ vector bundle on $\tilde{X}$ then

$$\chi(f, \text{Sym}^{m} \mathcal{F}) = -\frac{m^{r+1}}{r!} \left( c_1(f, \mathcal{F})^2 - c_2(f, \mathcal{F}) \right) + O(m^r).$$

This conjecture would allow us to characterize $c_2(f, \mathcal{F})$ as

$$c_2(f, \mathcal{F}) = c_1(f, \mathcal{F})^2 + r! \lim_{m \to \infty} \frac{\chi(f, \text{Sym}^{m} \mathcal{F})}{m^{r+1}}.$$

Using Theorem 3.12 it is easy to prove inequality $\leq$ in the above conjecture (cf. [La1, Theorem 4.15] for the rank 2 case). This allows us to consider

$$\liminf_{m \to \infty} \frac{\chi(f, \text{Sym}^{m} \mathcal{F})}{m^{r+1}}$$

(or similar limits) to define local relative Chern classes (see, e.g., [La2] for one example of use of such definition).

Let us recall that a quotient surface singularity is a quotient of the spectrum of a regular 2-dimensional ring by a linear action of a finite group. The following result follows from [La1, Theorem 5.1].

**Theorem 3.19.** Assume that $X = \text{Spec} A$, where $A$ is the henselization of a local ring of a quotient surface singularity. Then Conjecture 3.18 holds for any desingularization of $X$. Moreover, for any vector bundle $\mathcal{F}$ on $\tilde{X}$ the number $c_2(f, \mathcal{F})$ is rational.

See [La1] for further discussion of Conjecture 3.18 and its proof in some other cases.
4 Chern classes of reflexive sheaves on normal surfaces

Let $X$ be a proper normal variety defined over an algebraically closed field $k$. For any coherent $\mathcal{O}_X$-module $\mathcal{E}$ of rank $r$ the sheaf $\det \mathcal{E} = (\bigwedge^r \mathcal{E})^{**}$ is a reflexive coherent $\mathcal{O}_X$-module of rank 1. So we can define $c_1(\mathcal{E}) \in A^1(X)$ as the class of a divisor $D$ such that $\det \mathcal{E} = \mathcal{O}_X(D)$.

In this section we assume that $\dim X = 2$.

4.1 Second Chern class

Let $f : \tilde{X} \to X$ be any resolution of singularities and let $E$ be its exceptional locus. For every $x \in f(E)$ we consider the map $\nu_x : \text{Spec} \mathcal{O}_{\tilde{X}, x} \to \tilde{X}$ from the spectrum of the henselization of the local ring of $X$ at $x$ and the base change $f_x = \nu_x^* f : \tilde{X} \to \text{Spec} \mathcal{O}_{\tilde{X}, x}$ of $f$ via $\nu_x$. Note that the group of divisors on $\tilde{X}_x$ that are supported on the exceptional locus $E_x$ of $f_x$ embeds into $A^1(\tilde{X})$. For any vector bundle $\mathcal{F}$ on $\tilde{X}$ we write $c_1(f_x, \mathcal{F}) \in A^1(\tilde{X}) \otimes \mathbb{Q}$ for the image of $c_1(f_x, \mathcal{F}|_{\tilde{X}_x})$. Then for any $\mathcal{F} \in \text{Vect}(\tilde{X})$ we have

$$c_1(\mathcal{F}) = f^* c_1(f_* \mathcal{F}) + \sum_{x \in f(E)} c_1(f_x, \mathcal{F})$$

in $A^1(\tilde{X}) \otimes \mathbb{Q}$. So if $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$ and we choose some vector bundle $\mathcal{F}$ on $\tilde{X}$ such that $f_* \mathcal{F} \simeq \mathcal{E}$ then

$$f^* c_1(\mathcal{E}) = c_1(\mathcal{F}) - \sum_{x \in f(E)} c_1(f_x, \mathcal{F})$$

treated as an element of $A^1(\tilde{X}) \otimes \mathbb{Q}$.

**Definition 4.1.** If $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$ then we choose some vector bundle $\mathcal{F}$ on $\tilde{X}$ such that $f_* \mathcal{F} \simeq \mathcal{E}$. Then we use the homomorphism $f_* : A_0(\tilde{X}) \to A_0(X)$ to define the second Chern class of $\mathcal{E}$ as

$$c_2(\mathcal{E}) := f_* c_2(\mathcal{F}) - \sum_{x \in f(E)} c_2(f_x, \mathcal{F}) [x]$$

treated as an element of $A_0(X) \otimes \mathbb{R}$.
Note that the function \( c_2 : \text{Ref}(\mathcal{O}_X) \to A_0(X) \otimes \mathbb{R} \) is well defined, i.e., the class \( f_* c_2(\mathcal{F}) - \sum_{x \in f(E)} c_2(f_x, \mathcal{F})[x] \) does not depend on the choice of \( \mathcal{F} \) and \( f \). Namely, if we choose another resolution of singularities \( f' : \tilde{X}' \to X \) and \( \mathcal{F}' \in \text{Vect}(\tilde{X}') \) such that \( f'_* f'^{-1} \mathcal{F}' \cong \mathcal{E} \) then \( f_* c_2(\mathcal{F}) - f_* c_2(\mathcal{F}') \) is a well-defined 0-cycle supported on \( S = f(E) \cup f'(E') \) and Proposition 3.7 implies equality of the corresponding degrees locally at each point of \( S \).

Using the above definition we can also define for \( \mathcal{E} \in \text{Ref}(\mathcal{O}_X) \) the discriminant \( \Delta(\mathcal{E}) \) as

\[
\Delta(\mathcal{E}) := 2rc_2(\mathcal{E}) - (r - 1)c_1(\mathcal{E})^2 \in A_0(X) \otimes \mathbb{R},
\]

where \( r \) is the rank of \( \mathcal{E} \). Let us also recall that we have the degree map \( \int_X : A_0(X) \otimes \mathbb{R} \to \mathbb{R} \).

The following proposition summarizes some of the basic properties of Chern classes on normal surfaces:

**Proposition 4.2.** 1. For any rank 1 reflexive coherent \( \mathcal{O}_X \)-module \( \mathcal{L} \) on \( X \) we have \( c_2(\mathcal{L}) = 0 \).

2. For any \( \mathcal{E} \in \text{Ref}(\mathcal{O}_X) \) and any rank 1 reflexive coherent \( \mathcal{O}_X \)-module \( \mathcal{L} \) on \( X \) we have \( \Delta(\mathcal{E} \otimes \mathcal{L}) = \Delta(\mathcal{E}) \).

3. For any vector bundle \( \mathcal{F} \) on \( \tilde{X} \) we have \( \int_{\tilde{X}} \Delta(\mathcal{F}) \geq \int_X \Delta(f_* \mathcal{F}) \).

4. If \( \pi : Y \to X \) is a finite morphism from a normal surface \( Y \) then for any \( \mathcal{E} \in \text{Ref}(\mathcal{O}_X) \) we have \( \int_Y c_2(\pi^* \mathcal{E}) = \deg \pi \cdot \int_X c_2(\mathcal{E}) \).

**Proof.** All the above properties follow easily from the corresponding properties of relative local Chern classes listed in Proposition 3.2 and from properties of Chern classes of vector bundles on smooth surfaces.

**Remark 4.3.** Note that for any two Weil divisors \( D_1 \) and \( D_2 \) on \( X \) we have

\[
\int_X c_2(\mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2)) = D_1 \cdot D_2.
\]

In the local case this follows from [Wa, Proposition 2.5]. The global assertion follows from this fact and definition of Mumford’s intersection numbers on normal surfaces. Note that this equality implies that Mumford’s intersection numbers behaves well under finite coverings (see Proposition 4.2 (4)).
4.2 Riemann–Roch theorem on normal surfaces

For \( E \in \text{Ref}(\mathcal{O}_X) \) we write \( a(x, E) \) for \( a(x, \nu_*^x E) \) (see the previous subsection for the notation). Then we have the following Riemann–Roch type theorem:

**Theorem 4.4.** Let \( X \) be a normal proper algebraic surface defined over an algebraically closed field \( k \). Then for any \( E \in \text{Ref}(\mathcal{O}_X) \) we have

\[
\chi(X, E) = \frac{1}{2} c_1(E). (c_1(E) - K_X) - \int_X c_2(E) + \text{rk} E \cdot \chi(X, \mathcal{O}_X) + \sum_{x \in \text{Sing} X} a(x, E).
\]

**Proof.** Let \( \mathcal{F} \) be a vector bundle on \( \tilde{X} \). By the Leray spectral sequence we have

\[
\chi(X, f_* \mathcal{F}) = \chi(\tilde{X}, \mathcal{F}) + h^0(X, R^1 f_* \mathcal{F}).
\]

This shows that

\[
\chi(X, f_*[\mathcal{E}]) = \chi(\tilde{X}, \mathcal{F}) + \sum_{x \in f(E)} \chi(f_x, \mathcal{F}).
\]

For any \( E \in \text{Ref}(\mathcal{O}_X) \) we can find \( \mathcal{F} \) such that \( E \simeq f_*[\mathcal{E}] \) (e.g., one can take \( \mathcal{F} = f^*[\mathcal{E}] \)). Then the required formula follows from the Riemann–Roch theorem for \( \mathcal{F} \) on \( \tilde{X} \), Definition 3.6 and definitions of \( c_1(E) \) and \( c_2(E) \).

4.3 Second Chern class in positive characteristic

The following theorem shows that in positive characteristic the second Chern class is uniquely determined by two very simple properties.

**Theorem 4.5.** Let \( X \) be a normal proper algebraic surface defined over an algebraically closed field of characteristic \( p > 0 \). Then there exists a uniquely determined function \( \int_X c_2 : \text{Ref}(\mathcal{O}_X) \to \mathbb{R}, E \mapsto \int_X c_2(E) \), such that the following conditions are satisfied:

1. There exists a function \( \varphi : \mathbb{N} \to \mathbb{R}_{\geq 0} \) such that for every \( E \in \text{Ref}(\mathcal{O}_X) \) of rank \( r \) we have

\[
\left| \chi(X, E) - \left( \frac{1}{2} c_1(E). (c_1(E) - K_X) - \int_X c_2(E) + r \chi(X, \mathcal{O}_X) \right) \right| \leq \varphi(r).
\]

2. For every \( E \in \text{Ref}(\mathcal{O}_X) \) we have \( \int_X c_2(F^m_X E) = p^{2m} \int_X c_2(E) \).
Proof. Existence of $\int_X c_2$ satisfying the above conditions was already proven. Namely, by Theorem 4.4 we have

$$\chi(X, E) - \left( \frac{1}{2} c_1(E) \cdot c_2(E) - K_X \right) = \int_X c_2(E) + r\chi(X, O_X) = a(E),$$

where $a(E) := \sum_{x \in \text{Sing} X} a(x, E)$. Since $X$ has finitely many singular points, (1) follows from Theorem 3.12. The second condition follows from Proposition 4.2 (4).

Now let us prove that $\int_X c_2$ is uniquely determined by the conditions (1) and (2). Note that $c_1(F_X^m, E) = p^m c_1(E)$. So using (1) we get

$$\left| \chi(X, F_X^m, E) - \left( \frac{1}{2} p^m c_1(E) \cdot (p^m c_1(E) - K_X) - p^{2m} \int_X c_2(E) + r\chi(X, O_X) \right) \right| \leq C_r.$$

Dividing the above inequality by $p^{2m}$ and passing to the limit we get

$$\int_X c_2(E) = \frac{1}{2} c_1(E)^2 - \lim_{m \to \infty} \frac{\chi(X, F_X^m, E)}{p^{2m}}.$$

Remark 4.6. We can also talk about the second Chern character $\text{ch}_2(E)$, which is defined as usual by setting $\text{ch}_2(E) = \frac{1}{2} c_1(E)^2 - c_2(E)$. By the above, we see that

$$\int_X \text{ch}_2(E) = \lim_{m \to \infty} \frac{\chi(X, F_X^m, E)}{p^{2m}}.$$

Existence of this limit is the main reason why we can define the second Chern character for higher dimensional varieties in positive characteristic (see Theorem 5.2).

4.4 Second Chern class in characteristic zero

Let us assume that the base field $k$ has characteristic 0. The following result follows from the Riemann–Roch theorem on resolution of singularities of $X$ and from Theorem 3.19.

Theorem 4.7. Let $X$ be a normal proper algebraic surface with at most quotient singularities. Then for any $E \in \text{Ref}(O_X)$ of rank $r$ we have $\int_X c_2(E) \in \mathbb{Q}$ and

$$\chi(X, \text{Sym}^m E) = \frac{m^{r+1}}{r!} \left( c_1(E)^2 - \int_X c_2(E) \right) + O(m^r).$$
5 Chern classes in higher dimensions

In this section we develop the theory of the second Chern classes and characters for reflexive sheaves on higher dimensional normal varieties. This works well in case of positive characteristic. In characteristic zero in dimensions \( > 2 \) the theory depends heavily on Conjecture 3.18, so we can do it only for varieties with at most quotient singularities in codimension 2.

In this section we assume that \( n \geq 2 \).

5.1 The second Chern character in positive characteristic

Let us assume that the base field \( k \) has positive characteristic \( p \).

**Proposition 5.1.** Let \( \mathcal{E} \) be a reflexive coherent \( \mathcal{O}_X \)-module and let \( L_1, \ldots, L_{n-2} \) be line bundles on \( X \). Then the sequence

\[
\left( \frac{\chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E})}{p^{2m}} \right)_{m \in \mathbb{N}}
\]

is convergent to some real number.

**Proof.** First we use similar arguments to that from the proof of Proposition 2.3 to reduce to the case, when all \( L_i \) are very ample. For any line bundle \( M_1 \) on \( X \), Lemma 1.5, (1) and Lemma 1.7 imply that

\[
c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E} = c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot (\mathcal{O}_X^\oplus r - [\mathcal{O}_{-c_1(F_X^{[m]} \mathcal{E})}]),
\]

where \( r \) denotes the rank of \( \mathcal{E} \). Since \( c_1(F_X^{[m]} \mathcal{E}) = p^m c_1(\mathcal{E}) \), Lemma 1.6 shows that \([\mathcal{O}_{-c_1(F_X^{[m]} \mathcal{E})}] + p^m [\mathcal{O}_{c_1(\mathcal{E})}] \in K_{n-2}(X)\). So Lemma 1.5 (1) gives

\[
c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot [\mathcal{O}_{-c_1(F_X^{[m]} \mathcal{E})}] = p^m c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot [\mathcal{O}_{c_1(\mathcal{E})}]
\]

and we get

\[
\lim_{m \to \infty} \frac{\chi(X, c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E})}{p^{2m}} = 0.
\]
But by Lemma \[1.5(3)\] we have

\[
\chi(X, c_1(L_1 \otimes M_1) c_1(L_2) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E}) = \chi(X, c_1(L_1) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E}) + \chi(X, c_1(M_1) c_1(L_2) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E}) - \chi(X, c_1(M_1) c_1(L_1) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E}).
\]

So if the limits

\[
\lim_{m \to \infty} \frac{\chi(X, c_1(L_1) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E})}{p^{2m}}
\]

and

\[
\lim_{m \to \infty} \frac{\chi(X, c_1(M_1) c_1(L_2) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E})}{p^{2m}}
\]

exist, then

\[
\lim_{m \to \infty} \frac{\chi(X, c_1(L_1 \otimes M_1) c_1(L_2) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E})}{p^{2m}}
\]

also exists and it is equal to their sum.

Since any line bundle \(L\) can be written as \(A \otimes B^{-1}\) for some very ample line bundles \(A\) and \(B\) and the formula in the sequence is symmetric in \((L_1, \ldots, L_{n-2})\), it is sufficient to prove convergence of the sequence assuming that all line bundles \(L_i\) are very ample.

Let \(K\) be any uncountable algebraically closed field containing \(k\) and let \(X_K \to X\) be the base change. Since

\[
\chi(X_K, c_1((L_1)_K) \ldots c_1((L_{n-2})_K) \cdot F_{X_K}^{[m]} \mathcal{E}_K) = \chi(X, c_1(L_1) \ldots c_1(L_{n-2}) \cdot F_X^{[m]} \mathcal{E}),
\]

it is sufficient to prove convergence of the considered sequence after base change to \(K\). So in the following we can assume that our base field \(k\) is uncountable.

By Theorem \[1.4\] a general divisor \(H_1 \in |\mathcal{L}_1|\) is normal and irreducible. Since \(F_X^{[m]} \mathcal{E}\) is reflexive, \(H_1\) is \(F_X^{[m]} \mathcal{E}\)-regular and the restriction \((F_X^{[m]} \mathcal{E})|_{H_1}\) is torsion free as \(\mathcal{O}_{H_1}\)-module (see, e.g., \[HL\], Lemma 1.1.13). By \[HL\] Corollary 1.1.14, for fixed \(m\) and general \(H_1\) the restriction \((F_X^{[m]} \mathcal{E})|_{H_1}\) is also reflexive. Therefore, since \(k\) is uncountable, there exists a divisor \(H_1 \in |\mathcal{L}_1|\) such that \(H_1\) is normal, irreducible and the restriction \((F_X^{[m]} \mathcal{E})|_{H_1}\) is reflexive for all non-negative integers \(m\). Since \(\mathcal{E}\) is locally free outside of a closed subset of codimension \(\geq 2\) in \(X\), there exists a point \(x \in H_1\) such that \(\mathcal{E}_x\) is a free \(\mathcal{O}_{X,x}\)-module of some rank \(r\). Therefore \(\mathcal{E}|_{H_1}\) has the same rank \(r\) as \(\mathcal{E}\) and we have \((F_X^{[m]} \mathcal{E})|_{H_1} = F_{H_1}^{[m]} (\mathcal{E}|_{H_1})\). In particular, we have \(c_1(L_1) \cdot F_X^{[m]} \mathcal{E} = F_{H_1}^{[m]} (\mathcal{E}|_{H_1})\) in \(K(X)\).
Proceeding in the same way, we can construct a sequence of divisors $H_i \in |\mathcal{L}_i|$, $i = 1, \ldots, n - 2$, such that for all non-negative integers $m$ we have

1. the intersection $X_i := \bigcap_{j \leq i} H_j$ is normal and irreducible,
2. the restriction $(F^m_X(\mathcal{E}|_{X_i}))|_{X_i}$ is reflexive of rank $r$,
3. we have $F^m_X(\mathcal{E}|_{X_i}) = c_1(L_1) \cdots c_1(L_i) \cdot F^m_X(\mathcal{E})$.

In particular, $S = X_{n - 2}$ is a normal surface and by the Riemann–Roch theorem on $S$ (see Remark 4.6) we obtain

$$\lim_{m \to \infty} \frac{\chi(X, c_1(L_1) \cdots c_1(L_{n - 2}) \cdot F^m_X(\mathcal{E}))}{p^{2m}} = \lim_{m \to \infty} \frac{\chi(S, F^m_S(\mathcal{E}|_{S}))}{p^{2m}} = \int_S \text{ch}_2(\mathcal{E}|_{S}).$$

\[ \square \]

**Theorem 5.2.** Let us fix $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$ and consider the map $\int_X \text{ch}_2 : N^1(X)^{n(n - 2)} \to \mathbb{R}$ sending $(L_1, \ldots, L_{n - 2})$ to

$$\int_X \text{ch}_2(\mathcal{E})L_1 \cdots L_{n - 2} := \lim_{m \to \infty} \frac{\chi(X, c_1(L_1) \cdots c_1(L_{n - 2}) \cdot F^m_X(\mathcal{E}))}{p^{2m}}.$$

This map satisfies the following properties:

1. It is $\mathbb{Z}$-linear in all $L_i$.
2. It is symmetric in $L_1, \ldots, L_{n - 2}$.
3. If $\mathcal{E}$ is a vector bundle on $X$ then

$$\int_X \text{ch}_2(\mathcal{E})L_1 \cdots L_{n - 2} = \int_X \text{ch}_2(\mathcal{E}) \cap c_1(L_1) \cap \cdots \cap c_1(L_{n - 2}) \cap [X].$$

4. If $k \subseteq K$ is an algebraically closed field extension then

$$\int_X \text{ch}_2(\mathcal{E}_K)(L_1)_K \cdots (L_{n - 2})_K = \int_X \text{ch}_2(\mathcal{E})L_1 \cdots L_{n - 2}.$$

5. If $n > 2$ and $L_1$ is very ample then for a very general hypersurface $H \in |L_1|$ we have

$$\int_X \text{ch}_2(\mathcal{E})L_1 \cdots L_{n - 2} = \int_H \text{ch}_2(\mathcal{E}|_H)L_2|_H \cdots L_{n - 2}|_H.$$
**Proof.** The map is well defined by Proposition 5.1. (1) follows from the first part of the proof of this proposition. (2) follows from the definition and Lemma 1.5. (3) follows from the definition and the fact that Euler characteristic does not change under base field extension. To prove (5) let us first note that by Bertini’s theorem general $H \in |L_1|$ is normal and irreducible. So both sides of the equality are well defined. Moreover, by [HL, Corollary 1.1.14] for fixed $m$ and general $H \in |L_1|$, the restriction $(F_X^{[m]}E)|_H$ is reflexive and hence isomorphic to $F_H^{[m]}(E|_H)$ (if $E|_H$ is also reflexive). So for very general $H \in |L_1|$ we have

$$\chi(X, c_1(L_1)\ldots c_1(L_{n-2}) \cdot F_X^{[m]}E) = \chi(H, c_1(L_2|_H)\ldots c_1(L_{n-2}|_H) \cdot F_H^{[m]}(E|_H))$$

for all $m$ at the same time. Dividing the above equality by $p^{2m}$ and passing to the limit gives (5). Finally, (3) follows from (4), (5) and the analogous fact in the surface case. 

**Definition 5.3.** For any reflexive coherent $\mathcal{O}_X$-module $E$ of rank $r$ and any line bundles $L_1, \ldots, L_{n-2}$ we set:

$$\int_X c_2^2(E) L_1 \ldots L_{n-2} := c_1(E)^2 L_1 \ldots L_{n-2},$$

$$\int_X c_2(E) L_1 \ldots L_{n-2} := \frac{1}{2} \int_X c_1^2(E) L_1 \ldots L_{n-2} - \int_X c_2(E) L_1 \ldots L_{n-2},$$

$$\int_X \Delta(E) H L_1 \ldots L_{n-2} := 2r \int_X c_2(E) L_1 \ldots L_{n-2} - (r-1) \int_X c_1^2(E) L_1 \ldots L_{n-2}.$$

If $D$ is a Weil divisor on $X$ then $\mathcal{O}_X(D)$ is a reflexive $\mathcal{O}_X$-module of rank 1. In this case our definitions agree and since $F_X^{[m]}\mathcal{O}_X(D) = \mathcal{O}_X(p^mD)$, we have

$$\int_X \text{ch}_2(\mathcal{O}_X(D)) L_1 \ldots L_{n-2} = \lim_{m \to \infty} \frac{\chi(X, c_1(L_1)\ldots c_1(L_{n-2}) \cdot \mathcal{O}_X(p^mD))}{p^{2m}} = \lim_{m \to \infty} \frac{\chi(X, c_1(L_1)\ldots c_1(L_{n-2}) \cdot [\mathcal{O}_X(p^mD)])}{(p^m)^2} = \frac{1}{2} D^2 L_1 \ldots L_{n-2} = \frac{1}{2} \int_X c_1^2(\mathcal{O}_X(D)) L_1 \ldots L_{n-2}.$$

In particular, $\int_X c_2(\mathcal{O}_X(D)) L_1 \ldots L_{n-2} = 0$ and $\int_X \Delta(\mathcal{O}_X(D)) L_1 \ldots L_{n-2} = 0.$
5.2 Properties of Chern classes in positive characteristic

In this subsection we assume that $k$ has positive characteristic. Apart from Theorem 5.2, the second Chern classes have the following properties analogous to that from Proposition 4.2. We can assume in this proposition that $n > 2$.

**Proposition 5.4.** For any line bundles $L_1, \ldots, L_{n-2}$ on $X$ the following conditions are satisfied:

1. For any rank 1 reflexive coherent $\mathcal{O}_X$-module $\mathcal{L}$ on $X$ we have
   \[
   \int_X c_2(\mathcal{L})L_1 \ldots L_{n-2} = 0.
   \]

2. For any $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$ and any rank 1 reflexive coherent $\mathcal{O}_X$-module $\mathcal{L}$ on $X$ we have
   \[
   \int_X \Delta(\mathcal{E} \hat{\otimes} \mathcal{L})L_1 \ldots L_{n-2} = \int_X \Delta(\mathcal{E})L_1 \ldots L_{n-2}.
   \]

3. If $f : \tilde{X} \to X$ is a resolution of singularities and it has separably generated residue field extensions then for any vector bundle $\mathcal{F}$ on $\tilde{X}$ we have
   \[
   \int_{\tilde{X}} \Delta(\mathcal{F})f^*L_1 \ldots f^*L_{n-2} \geq \int_X \Delta(f_\ast \mathcal{F})L_1 \ldots L_{n-2}.
   \]

4. Let $\pi : Y \to X$ be a finite morphism from a normal projective variety $Y$. Assume that either $\pi$ has separably generated residue field extensions or it is the Frobenius morphism (or composition of such maps). Then for any $\mathcal{E} \in \text{Ref}(\mathcal{O}_X)$ we have
   \[
   \int_Y c_2(\pi_\ast \mathcal{E})\pi^*L_1 \ldots \pi^*L_{n-2} = \deg \pi \cdot \int_X c_2(\mathcal{E})L_1 \ldots L_{n-2}.
   \]

**Proof.** Let us first assume that $k$ has positive characteristic. Passing to the base change we can assume that the base field is uncountable. By Theorem 5.2 (4) it is sufficient to check properties (1) and (2) after restricting to a very general complete intersection surface. In this case these properties follow from Proposition 4.2 (1) and (2). If $\pi$ is the Frobenius morphism then (4) follows from the definition of $\int_X c_2(\mathcal{E})L_1 \ldots L_{n-2}$. Apart from that case, we use Theorem 1.4 and Theorem 5.2 (4) to reduce (3) and (4) to the surface case where these properties follow from Proposition 4.2 (3) and (4). For example in (4), Theorem 1.4 says that for general $H \in |-L_1|$ both $H$ and $\pi^{-1}(H) \in |\pi^*L_1|$ are normal and irreducible, so we can reduce the statement to lower dimension.

34
One of the most important results that follow from our theory is the following Riemann–Roch type inequality:

**THEOREM 5.5.** Let $L_1, \ldots, L_{n-2}$ be very ample line bundles on $X$. Then there exists a constant $C$ such that for all $E \in \text{Ref} (\mathcal{O}_X)$ we have

$$\left| \chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot E) - r \chi(SK, \mathcal{O}_{SK}) + \frac{1}{2} c_2(E, L_1 + \ldots + L_{n-2}) \right| \leq C \cdot r^2,$$

where $r$ is the rank of $E$.

**Proof.** Here we use notation from the proof of Lemma 2.6. Note that for any $E \in \text{Ref} (\mathcal{O}_X)$ we have by Theorem 4.4

$$\chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot E) = \chi(SK, E) = \int_{SK} \text{ch}_2(E) - \frac{1}{2} c_1(E) \cdot (K_{SK} + L_1 + \ldots + L_{n-2}) \cdot L_1 \ldots L_{n-2}.$$

By adjunction we have $K_{SK} = K_{SK} + (L_1 + \ldots + L_{n-2})$. Therefore

$$c_1(E) \cdot K_{SK} = c_1(E) \cdot (K_{SK} + L_1 + \ldots + L_{n-2}) \cdot L_1 \ldots L_{n-2}.$$

We have $\chi(SK, E) = \chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot \mathcal{O}_X)$. By Theorem 5.2 (4) we also have

$$\int_{SK} \text{ch}_2(E) = \int_X \text{ch}_2(E) L_1 \ldots L_{n-2}.$$

Now the required inequality follows from Theorem 3.12 applied to singularities of $SK$. 

**Remark 5.6.** By definition of $\int_X c_2$ we immediately have

$$\int_Y c_2(F_X^{|s|} \mathcal{O}) L_1 \ldots L_{n-2} = p^2 \cdot \int_X c_2(E) L_1 \ldots L_{n-2}$$

for all $E \in \text{Ref} (\mathcal{O}_X)$. As in Theorem 4.5 $\int_X c_2 : \text{Ref} (\mathcal{O}_X) \times N^1(X) \times (n-2) \rightarrow \mathbb{R}$ is uniquely determined by this property and inequality from Theorem 5.5.
5.3 The second Chern class in characteristic zero

Here we assume that $k$ is an algebraically closed field of characteristic 0. In this subsection we assume that $X$ is a normal projective variety with at most quotient singularities in codimension 2. This means that any general complete intersection surface in $X$ has at most quotient singularities.

**Proposition 5.7.** Let $\mathcal{E}$ be a reflexive coherent $\mathcal{O}_X$-module of rank $r$ and let $L_1, \ldots, L_{n-2}$ be line bundles on $X$. Then the sequence

$$
\left( \frac{\chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot \text{Sym}^m[\mathcal{E}])}{m^{r+1}} \right)_{m \in \mathbb{N}}
$$

is convergent to some rational number.

**Proof.** Proof is similar to the proof of Proposition 5.1. Namely, for any line bundle $M_1$ on $X$, Lemma 1.5, (1) and Lemma 1.7 imply that

$$
c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot \text{Sym}^m[\mathcal{E}] = c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot (\mathcal{O}_X(\mathcal{O}_X^{m+r-1})[\text{Sym}^m[\mathcal{E}]]).
$$

Since $c_1(\text{Sym}^m[\mathcal{E}]) = \binom{m+r-1}{r} c_1(\mathcal{E})$, Lemma 1.5 (1) and Lemma 1.6 give

$$
c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot [\mathcal{O}_{c_1(\text{Sym}^m[\mathcal{E}])}] = - \binom{m+r-1}{r} c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot [\mathcal{O}_{c_1(\mathcal{E})}].
$$

Therefore

$$
\lim_{m \to \infty} \frac{\chi(X, c_1(M_1)c_1(L_1) \cdots c_1(L_{n-2}) \cdot \text{Sym}^m[\mathcal{E}])}{m^{r+1}} = 0.
$$

Now the same arguments at that in the proof of Proposition 5.1 reduce the assertion to the case when all $L_i$ are very ample. Similarly as before we reduce to the case when the base field $k$ is uncountable and then restrict to a very general complete intersection surface $S \in |L_1| \cap \ldots \cap |L_{n-2}|$. Then we get

$$
\lim_{m \to \infty} \frac{\chi(X, c_1(L_1) \cdots c_1(L_{n-2}) \cdot \text{Sym}^m[\mathcal{E}])}{m^{r+1}} = \lim_{m \to \infty} \frac{\chi(S, \text{Sym}^m[\mathcal{E}]_{|S|})}{m^{r+1}},
$$

which by Theorem 4.7 exists and is a rational number. \qed
The method of proof of Proposition 5.7 works for any normal projective variety $X$ in characteristic 0 for which we know Conjecture 3.18 for any general complete intersection surface in $X$. As in the previous subsections, the above proposition allows us to define
\[ \int_X c_2(\mathcal{E}) L_1 \ldots L_{n-2} \]
by
\[ \int_X c_2(\mathcal{E}) L_1 \ldots L_{n-2} := c_1(\mathcal{E})^2 \cdot L_1 \ldots L_{n-2} - r! \lim_{m \to \infty} \frac{\chi(X, c_1(L_1) \ldots c_1(L_{n-2}) \cdot \text{Sym}^m \mathcal{E})}{m^r + 1}. \]

We can use this to define $\int_X \text{ch}_2(\mathcal{E}) L_1 \ldots L_{n-2}$ and $\int_X \Delta(\mathcal{E}) L_1 \ldots L_{n-2}$.

**Theorem 5.8.** Let us fix $\mathcal{E} \in \text{Ref} (\mathcal{O}_X)$. The map $\int_X \text{ch}_2(\mathcal{E}) : N^1(X)^\times (n-2) \to \mathbb{Q}$, sending $(L_1, \ldots, L_{n-2})$ to $\int_X \text{ch}_2(\mathcal{E}) L_1 \ldots L_{n-2}$, satisfies the following properties:

1. It is $\mathbb{Z}$-linear and symmetric.
2. If $\mathcal{E}$ is a vector bundle on $X$ then
\[ \int_X \text{ch}_2(\mathcal{E}) L_1 \ldots L_{n-2} = \int_X \text{ch}_2(\mathcal{E}) \cap c_1(L_1) \cap \ldots \cap c_1(L_{n-2}) [X]. \]
3. If $n > 2$ and $L_1$ is very ample then for a very general hypersurface $H \in |L_1|$ we have
\[ \int_X \text{ch}_2(\mathcal{E}) L_1 \ldots L_{n-2} = \int_H \text{ch}_2(\mathcal{E}|_H) L_2|_H \ldots L_{n-2}|_H. \]
4. There exists some $N \in \mathbb{N}$ such that the image of $\int_X \text{ch}_2$ is contained in $\frac{1}{N} \mathbb{Z} \subset \mathbb{Q}$.

**Proof.** Properties (1)–(3) can be proven in the same way as Theorem 5.2. To prove (4) it is sufficient to note that this holds on normal surfaces with only quotient singularities and use the proof of Theorem 5.5 to reduce to this case. \( \square \)

**Remark 5.9.** Let $(X, D)$ be a projective klt pair defined over an algebraically closed field of characteristic zero. In that case $X$ is well known to have quotient singularities in codimension 2. Then for fixed $\mathcal{E} \in \text{Ref} (\mathcal{O}_X)$, the multilinear forms $\text{Pic} X^{\times (n-2)} \to \mathbb{Q}$ given by sending $(L_1, \ldots, L_{n-2})$ to $c_1(\mathcal{E})^2 \cdot L_1 \ldots L_{n-2}$, $\int_X c_2(\mathcal{E}) L_1 \ldots L_{n-2}$ or $\int_X \text{ch}_2(\mathcal{E}) L_1 \ldots L_{n-2}$ coincide (modulo passing to numerical equivalence classes) with analogous forms considered in [GKPT, Theorem 3.13]. This follows, e.g., from [La1, Theorem 5.1] and [GKPT, (3.13.2)]. The construction of these forms in [GKPT] uses Mumford’s Chern classes for $\mathbb{Q}$-bundles on $\mathbb{Q}$-varieties and it does not generalize to varieties that do not have quotient singularities in codimension 2. Moreover, this construction works well only in the characteristic zero case.
5.4 Properties of Chern classes in characteristic zero

In this subsection we keep the notation from previous subsection. Then we have the following proposition analogous to Proposition 4.2.

**Proposition 5.10.** For any line bundles $L_1, \ldots, L_{n-2}$ on $X$ the following conditions are satisfied:

1. For any rank 1 reflexive coherent $\mathcal{O}_X$-module $\mathcal{L}$ on $X$ we have
   \[ \int_X c_2(\mathcal{L})L_1 \ldots L_{n-2} = 0. \]

2. For any $E \in \text{Ref}(\mathcal{O}_X)$ and any rank 1 reflexive coherent $\mathcal{O}_X$-module $\mathcal{L}$ on $X$ we have
   \[ \int_X \Delta(E \otimes \mathcal{L})L_1 \ldots L_{n-2} = \int_X \Delta(E)L_1 \ldots L_{n-2}. \]

3. If $f : \tilde{X} \to X$ is a resolution of singularities then for any vector bundle $\mathcal{F}$ on $\tilde{X}$ we have
   \[ \int_{\tilde{X}} \Delta(\mathcal{F})f^*L_1 \ldots f^*L_{n-2} \geq \int_X \Delta(f_*[\mathcal{F}])L_1 \ldots L_{n-2}. \]

4. If $\pi : Y \to X$ is a finite morphism from a normal projective variety $Y$ then for any $E \in \text{Ref}(\mathcal{O}_X)$ we have
   \[ \int_Y c_2(\pi^*[E])\pi^*L_1 \ldots \pi^*L_{n-2} = \deg \pi \cdot \int_X c_2(E)L_1 \ldots L_{n-2}. \]

**Proof.** The proof is the same as that of Proposition 5.4 except that our assumption on morphisms to have separably generated residue field extensions is automatically satisfied in characteristic zero. \qed

**Remark 5.11.** In case $k$ has characteristic 0 and $f : Y \to X$ is a quasi-étale morphism of klt pairs the property (4) was proven in [GKPT, Lemma 3.16]. Note that our assertion is much stronger as it does not require $f$ to be quasi-étale.

In characteristic 0 the Riemann–Roch type inequality analogous to that from Theorem 5.5 is also satisfied but we will not use it in the following.
5.5 K-theoretic formulation

Let $X$ be a normal projective variety defined over an algebraically closed field $k$. We define the Grothendieck group $K^{\text{ref}}(X)$ of reflexive sheaves on $X$ as the free abelian group on the isomorphism classes $[\mathcal{E}]$ of coherent reflexive $\mathcal{O}_X$-modules modulo the relations $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$ for each locally split short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

of coherent reflexive $\mathcal{O}_X$-modules. Note that $K^{\text{ref}}(X)$ is a ring with unity $[\mathcal{O}_X]$ and multiplication given by

$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] = [\mathcal{E}_1 \otimes \mathcal{E}_2].$$

We have well defined $\mathbb{Z}$-linear maps $\text{ch}_1 : K^{\text{ref}}(X) \to A^1(X)$, $[\mathcal{E}] \to c_1(\mathcal{E})$ and $\chi : K^{\text{ref}}(X) \to \mathbb{Z}$, $[\mathcal{E}] \to \chi(X, \mathcal{E})$.

Let us assume that $k$ has positive characteristic $p$. Since the functor $F^*[\mathcal{X}]$ is exact on locally split short exact sequences, we have a well defined homomorphism of rings

$$F^*[\mathcal{X}] : K^{\text{ref}}(X) \to K^{\text{ref}}(X)$$

sending $[\mathcal{E}]$ to $[F^*[\mathcal{X}] \mathcal{E}]$ (here we use that in the definition of $K^{\text{ref}}(X)$ we consider only locally split short exact sequences). If $n \geq 2$ and we fix some line bundles $L_1, ..., L_{n-2}$ then $\int_X \text{ch}_2(\cdot)L_1...L_{n-2}$ defines a $\mathbb{Z}$-linear map $K^{\text{ref}}(X) \to \mathbb{R}$ (this can be proven as Corollary 3.17; see [La4, Lemma 2.1]).

If $X$ is a surface then we define the Chern character

$$\text{ch} : K^{\text{ref}}(X) \to A^*(X) \otimes \mathbb{R}$$

by setting $\text{ch}([\mathcal{E}]) := \text{rk } \mathcal{E} + c_1(\mathcal{E}) + \text{ch}_2(\mathcal{E})$. Corollary 3.17 and our definitions imply that this extends to a homomorphism of abelian groups. Note that $A^*(X) \otimes \mathbb{R}$ is a ring and conjecturally, $\text{ch}$ is also a homomorphism of rings (see Remark 3.4).

6 Applications

Let $X$ be a normal projective variety of dimension $n$ defined over an algebraically closed field $k$. Let $\mathcal{O}_X(1)$ be an ample line bundle on $X$. All the results below hold when we replace $\mathcal{O}_X(1)$ by a collection of ample line bundles but proofs become much more complicated and we deal with these results in [La4].
6.1 Boundedness on normal varieties

We will often write $H$ for a Cartier divisor such that $\mathcal{O}_X(1) = \mathcal{O}_X(H)$. Let $\mathcal{E}$ be a torsion free coherent $\mathcal{O}_X$-module of rank $r$. We will write $H^i \cdot \mathcal{E}$ for the class $c_1(\mathcal{O}_X(1))^i \cdot \mathcal{E}$ in $K(X)$. By [Ko, Chapter VI, Theorem 2.13] we have

$$\chi(X, \mathcal{E}(m)) = \sum_{i=0}^{n} \chi(X, H^i \cdot \mathcal{E}) \left(\frac{m+i-1}{i}\right).$$

There are also uniquely determined integers $a_{H0}^H(\mathcal{E}), \ldots, a_{Hn}^H(\mathcal{E})$ such that

$$\chi(X, \mathcal{E}(m)) = \sum_{i=0}^{n} a_{Hi}^H(\mathcal{E}) \left(\frac{m+n-i}{n-i}\right).$$

**Lemma 6.1.** We have $a_{H0}^H(\mathcal{E}) = rH^n$,

$$a_{H1}^H(\mathcal{E}) = c_1(\mathcal{E})H^{n-1} - \frac{r}{2}(K_X + (n-1)H).H^{n-1} - rH^n$$

and

$$a_{H2}^H(\mathcal{E}) = \chi(X, H^{n-2} \cdot \mathcal{E}) - c_1(\mathcal{E})H^{n-1} + \frac{r}{2}(K_X + (n-1)H).H^{n-1}.$$

**Proof.** Using Theorem 1.4 and [HL, Lemma 1.1.12 and Corollary 1.1.14]) we can construct a sequence of divisors $H_1, \ldots, H_n \in |\mathcal{O}_X(1)|$ such that for all $i = 1, \ldots, n$ the following conditions are satisfied:

1. the intersection $\bigcap_{j \leq i} H_j$ is normal,
2. the restriction $\mathcal{E}|_{\bigcap_{j \leq i} H_j}$ is torsion free of rank $r$.

From the definition we see that $a_{H0}^H(\mathcal{E}) = \chi(\mathcal{E}|_{\bigcap_{j \leq n} H_j})$ and

$$a_{Hi}^H(\mathcal{E}) = \chi(\mathcal{E}|_{\bigcap_{j \leq n-i} H_j}) - \chi(\mathcal{E}|_{\bigcap_{j \leq n-i+1} H_j})$$

for $i > 0$. So the equality $a_{H0}^H(\mathcal{E}) = rH^n$ is clear. By the above the curve $C := \bigcap_{j \leq n-1} H_j$ is smooth and by the adjunction formula we have

$$-2\chi(\mathcal{O}_C) = \text{deg } \omega_C = \text{deg } \mathcal{O}_C(K_X + H_1 + \ldots + H_{n-1}) = (K_X + (n-1)H).H^{n-1}.$$

This, together with the Riemann–Roch theorem for $\mathcal{E}|_C$, gives the formula for $a_{H1}^H(\mathcal{E})$. The last formula follows easily from the previous two formulas and equality $a_{H2}^H(\mathcal{E}) = \chi(\mathcal{E}|_{\bigcap_{j \leq n-2} H_j}) - (a_{H0}^H(\mathcal{E}) + a_{H1}^H(\mathcal{E})).$
We define a slope of \( \mathcal{E} \) with respect to \( H \) as
\[
\mu_H(\mathcal{E}) := \frac{c_1(\mathcal{E}).H^{n-1}}{r}.
\]
This allows us to define slope \( H \)-semistability and the maximal \( H \)-destabilizing slope \( \mu_{\max,H}(\mathcal{E}) \) for any coherent torsion free \( \mathcal{O}_X \)-module \( \mathcal{E} \).

Let us recall the following special case of [La3, Theorem 4.4].

**Theorem 6.2.** Let \((X,H)\) be as above and let us fix some integers \(a_0, a_1, a_2\) and a rational number \(\mu_{\max}\). Then the set of coherent reflexive \( \mathcal{O}_X \)-modules \( \mathcal{E} \) with \( a_0 H(\mathcal{E}) = a_0, a_1 H(\mathcal{E}) = a_1, a_2 H(\mathcal{E}) \geq a_2 \) and \( \mu_{\max,H}(\mathcal{E}) \leq \mu_{\max} \) is bounded.

Using Lemma 6.1 we can rewrite the above theorem in the following way:

**Corollary 6.3.** Let us fix some positive integer \( r \), integers \( c_1 \) and \( \chi \) and some rational number \( \mu_{\max} \). Then the set \( \mathcal{A} \) of coherent reflexive \( \mathcal{O}_X \)-modules \( \mathcal{E} \) of rank \( r \) with \( [c_1(\mathcal{E}).H^{n-1} = c_1, \chi(X,H^{n-2} \cdot \mathcal{E}) \geq \chi \) and \( \mu_{\max,H}(\mathcal{E}) \leq \mu_{\max} \) is bounded. In particular, the set of slope \( H \)-semistable coherent reflexive \( \mathcal{O}_X \)-modules \( \mathcal{E} \) of rank \( r \) with \( c_1(\mathcal{E}).H^{n-1} = c_1 \) and \( \chi(X,H^{n-2} \cdot \mathcal{E}) \geq \chi \) is bounded.

### 6.2 Boundedness and Bogomolov’s inequality on normal varieties in positive characteristic

In this subsection we assume that the base field \( k \) has positive characteristic. In the following, we write \( N_H(X) \) for the group \( N_L(X) \) introduced in Subsection 2.2 in the case of \((L_1,...,L_{n-2}) = (\mathcal{O}_X(1),...,\mathcal{O}_X(1))\).

**Theorem 6.4.** Let us fix some positive integer \( r \) and some real numbers \( c_2 \) and \( \mu_{\max} \). Let us also fix some class \( c_1 \in N_H(X) \). Then the set \( \mathcal{A} \) of coherent reflexive \( \mathcal{O}_X \)-modules \( \mathcal{E} \) of rank \( r \) with \( [c_1(\mathcal{E})] = c_1 \in N_H(X), \int_X c_2(\mathcal{E}) H^{n-2} \leq c_2 \) and \( \mu_{\max,H}(\mathcal{E}) \leq \mu_{\max} \) is bounded.

**Proof.** By Theorem 5.5 we have
\[
\chi(X,H^{n-2} \cdot \mathcal{E}) \leq \frac{1}{2} c_1 . (c_1 - (K_X + (n - 2)H)).H^{n-2} - \int_X c_2(\mathcal{E}) H^{n-2} + r\chi(X,H^{n-2} \cdot \mathcal{O}_X) + C \cdot r.
\]
So our assertion follows from Corollary 6.3. \( \square \)
The proof of the following theorem was motivated by a similar proof by Maruyama in the smooth case (see the proof of [Ma, Corollary 2.10]).

**Theorem 6.5.** Let us fix some positive integer r and some non-negative rational number α. There exists some constant \( \tilde{C} = \tilde{C}(X,H,r,\alpha) \) depending only on \( X, H, r \) and \( \alpha \) such that for every coherent reflexive \( \mathcal{O}_X \)-module \( \mathcal{E} \) of rank \( r \) with \( \mu_{\max,H}(\mathcal{E}) - \mu_H(\mathcal{E}) \leq \alpha \) we have

\[
\int_X \Delta(\mathcal{E}) H^{n-2} \geq \tilde{C}.
\]

**Proof.** Let us choose a basis \( L_1, \ldots, L_a \) of \( N_H(X) \) as a \( \mathbb{Z} \)-module (see Lemma 2.6) and let us write \( [c_1(\mathcal{E})] = \sum a_iL_i \) for some \( a_i \in \mathbb{Z} \). There exist uniquely determined integers \( q_i \) and \( r_i \) such that \( a_i = q_ir + r_i \) and \( 0 \leq r_i < r \). Since

\[
\int_X \Delta(\mathcal{E}) H^{n-2} = \int_X \Delta(\mathcal{E}(-\sum q_iL_i)) H^{n-2}
\]

and there are only finitely many possibilities for \( [c_1(\mathcal{E}(-\sum q_iL_i))] = \sum r_iL_i \), it is sufficient to prove existence of the above constant assuming that \( c_1 = [c_1(\mathcal{E})] \in N_H(X) \) is fixed.

If \( \int_X c_2(\mathcal{E}) H^{n-2} \geq 0 \) then \( \int_X \Delta(\mathcal{E}) H^{n-2} \geq -(r-1)c_1^2 H^{n-2} \).

On the other hand, by Theorem 6.4 the set \( \mathcal{A} \) of reflexive coherent \( \mathcal{O}_X \)-modules \( \mathcal{E} \) of rank \( r \) with \( [c_1(\mathcal{E})] = c_1 \in N_H(X) \), \( \int_X c_2(\mathcal{E}) H^{n-2} \leq 0 \) and

\[
\mu_{\max,H}(\mathcal{E}) \leq \mu_{\max} = \alpha + \frac{1}{r}c_1 H^{n-1}
\]

is bounded. So there exists some constant \( D \) such that for every \( \mathcal{E} \in \mathcal{A} \) and any \( \mathcal{E} \)-regular sequence \( H_1, \ldots, H_{n-2} \in |\mathcal{O}_X(1)| \) we have

\[
\chi(X,H^{n-2} \cdot \mathcal{E}) = \chi(X,\mathcal{E} \mid_{\cap_{j=1}^{n-2} H_j}) \leq D.
\]

But Theorem 5.5 gives

\[
\chi(X,H^{n-2} \cdot \mathcal{E}) \geq \frac{1}{2r}c_1(c_1 - r(K_X + (n-2)H)).H^{n-2} - \frac{1}{2r} \int_X \Delta(\mathcal{E}) H^{n-2} + r\chi(X,H^{n-2} \cdot \mathcal{O}_X) - C \cdot r^2.
\]

Therefore for any \( \mathcal{E} \in \mathcal{A} \) we have

\[
\int_X \Delta(\mathcal{E}) H^{n-2} \geq \max\{-(r-1)c_1^2,H^{n-2},c_1(c_1 - r(K_X + (n-2)H)).H^{n-2} + 2r^2\chi(X,H^{n-2} \cdot \mathcal{O}_X) - 2C \cdot r^3 + 2D \cdot r\}.
\]

\( \square \)
The above theorem easily implies Bogomolov’s inequality for strongly semistable reflexive sheaves:

**Corollary 6.6.** Let \( \mathcal{E} \) be a coherent reflexive \( \mathcal{O}_X \)-module. If \( \mathcal{E} \) is strongly slope \( H \)-semistable then

\[
\int_X \Delta(\mathcal{E}) H^{n-2} \geq 0.
\]

**Proof.** Let us consider the set \( \{ F_X^{[m]} \mathcal{E} \}_{m \geq 0} \). Note that by assumption each sheaf in this set is reflexive of rank \( r \) and slope \( H \)-semistable, i.e., \( \mu_{\max,H}(F_X^{[m]} \mathcal{E}) - \mu_H(F_X^{[m]} \mathcal{E}) = 0 \). So by Theorem 6.5 there exists a constant \( \tilde{C} \) such that

\[
\int_X \Delta(F_X^{[m]} \mathcal{E}) H^{n-2} = p^{2m} \int_X \Delta(\mathcal{E}) H^{n-2} \geq \tilde{C}
\]

for all \( m \geq 0 \). Dividing by \( p^{2m} \) and passing with \( m \) to infinity, we get the required inequality. \( \square \)

**Corollary 6.7.** Let us fix some positive integer \( r \), integer \( ch_1 \) and some real numbers \( ch_2 \) and \( \mu_{\max} \). Then the set \( \mathcal{B} \) of coherent reflexive \( \mathcal{O}_X \)-modules \( \mathcal{E} \) of rank \( r \) with \( \int_X ch_1(\mathcal{E}) H^{n-1} = ch_1, \int_X ch_2(\mathcal{E}) H^{n-2} \geq ch_2 \) and \( \mu_{\max,H}(\mathcal{E}) \leq \mu_{\max} \) is bounded.

**Proof.** By Theorem 6.5 there exists a constant \( \tilde{C} \) such that for all \( \mathcal{E} \in \mathcal{B} \) we have

\[
\tilde{C} \leq \int_X \Delta(\mathcal{E}) H^{n-2} = c_1(\mathcal{E})^2 H^{n-2} - 2r \int_X ch_2(\mathcal{E}) H^{n-2}.
\]

Therefore \( c_1(\mathcal{E})^2 H^{n-2} \geq \tilde{C} + 2r ch_2 \). Let us write \( [c_1(\mathcal{E})] = \alpha H \in N_H(X) \), where \( \alpha = \frac{ch_1}{mp} \). Then \( D.H^{n-1} = 0 \) and \( c_1(\mathcal{E})^2 H^{n-2} = \alpha^2 H^n + D^2 H^{n-2} \) and we have

\[
D^2 H^{n-2} \geq \tilde{C} + 2r ch_2 - \alpha^2 H^n.
\]

But by the Hodge index theorem (see Lemma 2.6) the intersection form is negative definite on \( H^\perp \subset N_H(X) \), so there are only finitely many possibilities for \( D \) and hence there are also finitely many possibilities for the classes \( [c_1(\mathcal{E})] \in N_H(X) \). Now the assertion follows from Theorem 6.4 \( \square \)

43
6.3 Application to F-divided sheaves on normal varieties

In this section we use the above developed theory to reprove [ES, Theorem 2.1] in the style well-known from the smooth varieties.

Let \( X \) be a normal projective variety of dimension \( \geq 1 \) defined over an algebraically closed field \( k \). Let \( j : U \hookrightarrow X \) be an open embedding of a big open subset contained in the regular locus of \( X \). Let \( E = (E_m, \sigma_m)_{m \geq 0} \) be an F-divided sheaf on \( U \) and let us set \( \mathcal{E}_m := j_* E_m \).

**Lemma 6.8.** For every ample divisor \( H \) we have \([c_1(\mathcal{E}_m)] = 0 \in N_H(X)\) for all \( m \) and \( \lim_{m \to \infty} \int_X c_2(\mathcal{E}_m)H^{n-2} = 0 \).

**Proof.** The isomorphisms \( \sigma_m : E_m \to F_X^m E_{m+1} \) extend uniquely to isomorphisms \( \mathcal{E}_m \to F_X^m \mathcal{E}_{m+1} \). In particular, we see that \( c_1(\mathcal{E}_0) = p^m c_1(\mathcal{E}_m) \). By Lemma 2.6 there exists some \( N \) such that for every Weil divisor \( D \) on \( X \) we have

\[
N \cdot c_1(\mathcal{E}_0) \cdot D \cdot H^{n-2} = p^m N \cdot c_1(\mathcal{E}_m) \cdot D \cdot H^{n-2} \in p^m \mathbb{Z}.
\]

Since \( N_H(X) \) is a free \( \mathbb{Z} \)-module, the class of \( c_1(\mathcal{E}_0) \) in \( N_H(X) \) must vanish. This also implies vanishing of all classes \( [c_1(\mathcal{E}_m)] \in N_H(X) \). The second assertion follows from equalities \( \int_X c_2(\mathcal{E}_0)H^{n-2} = p^m \int_X c_2(\mathcal{E}_m)H^{n-2} \).

The following corollary is due to Esnault and Srinivas (see [ES, Theorem 2.1] for a slightly weaker statement).

**Corollary 6.9.** There exists a bounded set of (slope \( H \)-semistable) sheaves \( \mathcal{S} \) such that for every \( r > 0 \) and every rank \( r \) F-divided sheaf \( E = (E_m, \sigma_m)_{m \geq 0} \) on \( U \) there exists \( m_0(E) \in \mathbb{Z}_{\geq 0} \) such that for all \( m \geq m_0(E) \) the sheaves \( E_m \) belong to the set \( \mathcal{S} \).

**Proof.** Let us fix a positive real number \( c_2 \) and let \( \mathcal{S} \) denote the set of slope \( H \)-semistable sheaves \( \mathcal{F} \) on \( X \) such that \([c_1(\mathcal{F})] = 0 \in N_H(X) \) and \( \int_X c_2(\mathcal{F})H^{n-2} \leq c_2 \). By Theorem 6.4 this set is bounded.

Now fix \( E = (E_m, \sigma_m)_{m \geq 0} \) on \( U \) and as before set \( \mathcal{E}_m := j_* E_m \). Since

\[
\mu_{\max, H}(\mathcal{E}_m) \geq p \mu_{\max, H}(\mathcal{E}_m),
\]

we have \( \mu_{\max, H}(\mathcal{E}_m) \geq p^m \mu_{\max, H}(\mathcal{E}_m) \). Since \( r! \cdot \mu_{\max, H}(\mathcal{E}_m) \in \mathbb{Z} \), there exists \( m_1(E) \in \mathbb{Z}_{\geq 0} \) such that for all \( m \geq m_1(E) \) we have \( \mu_{\max, H}(\mathcal{E}_m) \leq 0 \). Since \( \mu_H(\mathcal{E}_m) = 0 \) we see that the sheaves \( \mathcal{E}_m = j_* E_m \) are slope \( H \)-semistable for all \( m \geq m_1(E) \).

Now the above lemma implies that for any \( E \) there exists \( m_0(E) \in \mathbb{Z}_{\geq 0} \) such that \( \mathcal{E}_m \in \mathcal{S} \) for all \( m \geq m_0(E) \).

44
Remark 6.10. The above proof shows that one can choose $m_0$ that does not depend on $E$ but only on the rank $r$ if and only if $\int_X c_2(\mathcal{E}_m)H^{n-2} = 0$ for all $m \geq 0$. In general, this would follow if we knew that the second relative Chern classes of vector bundles defined in Subsection 3.1 are rational with bounded denominators but this seems very unlikely. However, this happens, e.g., if $X$ has only quotient singularities in codimension 2 (cf. Theorem 5.8 (5)).

Acknowledgements

The author would like to thank M. Enokizono for pointing out \[\text{En}\].

A large part of the paper was written while the author was an External Senior Fellow at Freiburg Institute for Advanced Studies (FRIAS), University of Freiburg, Germany. The author would like to thank Stefan Kebekus for his hospitality during the author’s stay in FRIAS.

The author was partially supported by Polish National Centre (NCN) contract numbers 2018/29/B/ST1/01232 and 2021/41/B/ST1/03741. The research leading to these results has received funding from the European Union’s Horizon 2020 research and innovation programme under the Maria Skłodowska-Curie grant agreement No 754340.

References

[BFM] Baum, Paul; Fulton, William; MacPherson, Robert Riemann-Roch for singular varieties. Inst. Hautes Études Sci. Publ. Math. No. 45 (1975), 101–145.

[CGM] Cumino, Caterina; Greco, Silvio; Manaresi, Mirella An axiomatic approach to the second theorem of Bertini. J. Algebra 98 (1986), 171–182.

[dFH] de Fernex, Tommaso; Hacon, Christopher D. Singularities on normal varieties. Compos. Math. 145 (2009), 393–414.

[dJ] de Jong, Aise Johan Smoothness, semi-stability and alterations. Publ. Math. Inst. Hautes Étud. Sci. 83 (1996), 51–93.
[En] Enokizono, Makoto Vanishing theorems and adjoint linear systems on normal surfaces in positive characteristic. *Pacific J. Math.* **324** (2023), 71–110.

[ES] Esnault, Hélène; Srinivas, Vasudevan, Simply connected varieties in characteristic $p > 0$. with an appendix by Bost, Jean-Benoit. *Compos. Math.* **152** (2016), 255–287.

[Fu] Fulton, William Intersection theory. 2nd ed. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* 2. Berlin: Springer, 1998. xiii+470 pp.

[Gi] Giraud, Jean Improvement of Grauert–Riemenschneider’s theorem for a normal surface. *Ann. Inst. Fourier* **32** (1982), 13–23.

[GKPT] Greb, Daniel; Kebekus, Stefan; Peternell, Thomas; Taji, Behrouz The Miyaoka-Yau inequality and uniformisation of canonical models. *Ann. Sci. Éc. Norm. Supér. (4)* **52** (2019), 1487–1535.

[HL] Huybrechts, Daniel; Lehn, Manfred, The geometry of moduli spaces of sheaves. Second edition. *Cambridge Mathematical Library*. Cambridge University Press, Cambridge, 2010. xviii+325 pp.

[Jo] Jouanolou, Jean-Pierre Théoremes de Bertini et applications. *Progress in Mathematics* **42**. Boston-Basel-Stuttgart: Birkhäuser, 1983. 127 pp.

[Kl] Kleiman, Steven L. Geometry on Grassmannians and applications to splitting bundles and smoothing cycles. *Publ. Math. Inst. Hautes Étud. Sci.* **36** (1969), 281–297.

[Ko] Kollár, János, Rational curves on algebraic varieties. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.* **32**. Berlin: Springer–Verlag, 1995. viii+320 pp.

[La1] Langer, Adrian, Chern classes of reflexive sheaves on normal surfaces. *Math. Z.* **235** (2000), 591–614.

[La2] Langer, Adrian, The Bogomolov-Miyaoka-Yau inequality for log canonical surfaces. *J. London Math. Soc.* **64** (2001), 327–343.

[La3] Langer, Adrian, Semistable sheaves in positive characteristic. *Ann. of Math.* **159** (2004), 251–276.
[La4] Langer, Adrian, Bogomolov’s inequality and Higgs sheaves on normal varieties in positive characteristic. [arXiv:2210.08767] preprint.

[La5] Langer, Adrian, Simpson’s correspondence on singular varieties in positive characteristic. [arXiv:2302.05668] preprint.

[Li] Lipman, Joseph, Rational singularities, with applications to algebraic surfaces and unique factorization. *Publ. Math., Inst. Hautes Étud. Sci.* 36 (1969), 195–279.

[Ma] Maruyama, Masaki, On boundedness of families of torsion free sheaves. *J. Math. Kyoto Univ.* 21 (1981), 673–701.

[Sch] Schröer, Stefan A higher-dimensional generalization of Mumford’s rational pullback for Weil divisors. *J. Singul.* 19 (2019), 53–60.

[Se] Seidenberg, Abraham, The hyperplane sections of normal varieties. *Trans. Am. Math. Soc.* 69 (1950), 357–386.

[SP] The Stacks Project Authors, Stacks Project. [https://stacks.math.columbia.edu](https://stacks.math.columbia.edu)

[Wa] Wahl, Jonathan, Second Chern class and Riemann–Roch for vector bundles on resolutions of surface singularities. *Math. Ann.* 295 (1993), 81–110.

[Xu] Xu, Zheng, Abundance for threefolds in positive characteristic when $\nu = 2$. [arXiv:2307.03938] preprint.