CMB anisotropy power spectrum of 3-sphere universe for low $l$

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Abstract

We calculate the CMB anisotropy power spectrum of closed universe due to scale invariant fluctuation of primordial universe by considering spherical harmonics for 3-sphere. In particular, we show that this consideration affects CMB anisotropy power spectrum, contrary to the wide belief. We show that the best-fit for $\chi = \sin^{-1} r_L$, where $r_L$ is the radial distance of last scattering surface, is $\chi = 0.14^{+0.09}_{-0.03}$ whereas the previous analysis from WMAP+BAO+$H_0$ gives $\chi = 0.16^{+0.14}_{-0.16}$.

1 Introduction

It is well-known that the CMB anisotropy power spectrum gives us very valuable information about our universe. It is also well-known that the scale-invariance in primordial universe implies that $C_l(l + 1)$ is a constant for low $l$. However, it turns out that $C_l$ is severely suppressed for $l = 2$. This could be due to a big cosmic variance, but it could be due to another effect.

In this article, we will consider a closed universe. Under such a consideration, our universe is necessarily 3-sphere. In the analysis of CMB anisotropy, only the spherical harmonics of 2-sphere have been considered so far. However, as we will see in this article, considering the spherical harmonics of 3-sphere gives differences to the CMB anisotropy spectrum, even though only for low $l$. In particular, we succeeded in lowering $C_2$, even though not as low as the observed value, implying that the cosmic variance still plays a role, albeit to a less extent.

The organization of this article is as follows. In Section 2, we review the spherical harmonics for 3-sphere. In Section 3, we review how the traditional CMB anisotropy analysis is done by using the spherical harmonics for 2-sphere. The aim is to set a comparison with the case of 3-sphere in Section 4. In Section 4, we derive the CMB
anisotropy using the spherical harmonics of 3-sphere. This section is the main part of this paper. In particular, we show that for large \( l \), we recover the usual scaling law \( C_l(l+1) = \text{const.} \) In Section 5, we use data analysis to obtain the radial distance of the last scattering surface. In particular, we will see that it agrees with the one obtained earlier by another method.

2 Spherical harmonics for 3-sphere

The spherical harmonics on 3-sphere is given by

\[
\nabla^2 Y_{qlm}(\chi, \theta, \phi) = -q(q+2)Y_{qlm}(\chi, \theta, \phi) \tag{1}
\]

where \( q \) is a non-negative integer and \( l \) runs from 0 to \( q \) and \( m \) runs from \(-l\) to \( l \). Of course, we can write

\[
Y_{qlm}(\chi, \theta, \phi) = X_{ql}(\chi)Y_{lm}(\theta, \phi) \tag{2}
\]

for a suitable \( X_{ql} \). Given \( q \), there is a degeneracy of \((q+1)^2\), as

\[
\sum_{l=0}^{q} \sum_{m=-l}^{l} 1 = \sum_{l=0}^{q} (2l+1) = (q+1)^2 \tag{3}
\]

For our purpose, the following relation is important (see [1], for example)

\[
\sum_{lm} Y_{qlm}(\vec{u}) Y_{qlm}(\vec{v}) = \frac{q+1}{2\pi^2} U_q(\vec{u} \cdot \vec{v}) \tag{4}
\]

where \( U_q \) is Chebyshev polynomials of the second kind, i.e.,

\[
U_q(\cos \theta) = \frac{\sin(q+1)\theta}{\sin \theta} \tag{5}
\]

and \( \vec{u} \) are \( \vec{v} \) are 4-d unit vectors in unit 3-sphere.

3 Traditional spherical harmonics analysis in CMB anisotropy

This section is important to set a comparison with our application of spherical harmonics for 3-sphere.

\[
\Delta T(\hat{n}) = \sum_{lm} a_{lm} Y_l^m(\hat{n}), \quad C_l \equiv \langle |a_{lm}|^2 \rangle \tag{6}
\]

\[
\langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle = \sum_{lm} C_l Y_l^m(\hat{n})Y_{l}^{-m}(\hat{n}') = \sum_{l} C_l \left( \frac{2l+1}{4\pi} \right) P_l(\hat{n} \cdot \hat{n}') \tag{7}
\]

Then, using

\[
\int d\Omega_k P_l(\hat{n} \cdot \hat{k}) P_l(\hat{n}' \cdot \hat{k}) = \frac{4\pi}{2l+1} P_l(\hat{n} \cdot \hat{n}') \delta_{ll'} \tag{8}
\]
$C_l$ can be obtained by

$$C_l = \frac{1}{4\pi} \int d^2\hat{n}d^2\hat{n}' P_l(\hat{n} \cdot \hat{n}') \langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle$$

(9)

In particular, when Sachs-Wolfe approximation is valid, we can write

$$\frac{\Delta T(\hat{n})}{T} = -\frac{1}{5} \mathcal{R}(\hat{n} r_L)$$

(10)

where $\mathcal{R}$ is the primordial curvature perturbation, and $r_L$ is the radial coordinate of the last scattering surface. When $\mathcal{R}$ satisfies approximate scale invariance, as widely believed, we have

$$\langle \mathcal{R}(\lambda \hat{x})\mathcal{R}(\lambda \hat{y}) \rangle = \langle \mathcal{R}(\hat{x})\mathcal{R}(\hat{y}) \rangle$$

(11)

in which case we have

$$C_l = \text{const} \frac{l(l+1)}{4\pi}$$

(12)

4 3d-spherical harmonics in CMB anisotropy

Let’s re-write (11) as

$$\sum_{lm} Y_{qlm}(\hat{n}) Y_{qlm}(\hat{v}) = \frac{(q+1)^2}{2\pi^2} \left( \frac{U_q(\cos \theta)}{q+1} \right) = \frac{(q+1)^2}{2\pi^2} \left( \frac{\sin(q+1)\theta}{(q+1)\sin \theta} \right)$$

(13)

where $\cos \theta \equiv \hat{n} \cdot \hat{v}$. Then, we can write

$$\langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle = \sum_q C_q \frac{(q+1)^2}{2\pi^2} \left( \frac{\sin(q+1)\theta}{(q+1)\sin \theta} \right), \quad C_q \equiv \langle |a_{qlm}|^2 \rangle$$

(14)

where $\cos \theta$ is the dot product between $\hat{n}$ and $\hat{n}'$ in 3-sphere. Here, by an abuse of notation, we denoted the average of $|a_{qlm}|^2$ as $C_q$; this is not the same one as $C_l$.

Now, let’s compare this with (11). For $\hat{n} \cdot \hat{n}' = \cos \theta_{nn'}$, and $\theta_{nn'}$ small, the right-hand side of (11) can be expanded as

$$C_l \frac{2l+1}{4\pi} \left( 1 - \frac{l(l+1)}{4} \theta_{nn'}^2 \right)$$

(15)

In case of (14) for $\theta$ small, we have

$$C_q \frac{(q+1)^2}{2\pi^2} \left( 1 - \frac{q(q+2)}{6} \theta^2 \right)$$

(16)

Thus, we see that they indeed have the similar structure. $C_l$ is replaced by $C_q$, the degeneracy $(2l+1)$ is replaced by $(q+1)^2$, the leading term in the parenthesis is both 1, and the coefficients for $\theta^2_{nn'}$ and $\theta^2$ are both proportional to the eigenvalues for the Laplacian. From this reason, we expressed (14) by pulling out the factor $(q+1)^2$ to the front, instead of the original expression $C_q(q+1)U_q(\cos \theta)/(2\pi^2)$.
Analogous to (8), we have (when \( \hat{n} = \hat{n}' \))

\[
\int_0^\pi \left( \frac{U_q(\cos \theta)}{q + 1} \right) \left( \frac{U'_q(\cos \theta)}{q' + 1} \right) \sin^2 \theta d\theta = \frac{2\pi^2}{(q + 1)^2} \delta_{qq'}
\]

In 2-sphere case, we used (8) to obtain \( C_l \). However, in 3-sphere case, we shall not use (17), because what we want to obtain is \( C_l \) not \( C_q \). Moreover, the integration range of (17) is not the subdomain 2-sphere, but the whole 3-sphere, as we can see from the measure \( \sin^2 \theta d\theta d\Omega \). In other words, we still need to use (17), but only if the integration range is properly considered. As we have

\[
\cos \theta = \cos^2 \chi + \sin^2 \chi \hat{n} \cdot \hat{n}'
\]

the integration range is

\[
\cos 2\chi \leq \cos \theta \leq 1 \quad \rightarrow \quad 0 \leq \theta \leq 2\chi
\]

Thus to obtain \( C_l \), we have

\[
C_l(2l + 1) = \sum_{q=l}^{\infty} C_q \frac{(q + 1)^2}{2\pi^2} \int_{\cos \theta = \cos 2\chi}^{\cos \theta = 1} \left( \frac{U_q(\cos \theta)}{q + 1} \right)^2 \sin^2 \theta d\theta
\]

\[
= \sum_{q=l}^{\infty} C_q \frac{1}{2\pi^2} \int_0^{2\chi} d\theta \sin^2(q + 1)\theta
\]

The range for the infinite sum comes from the fact that, for a given \( l \), the possible \( q \) runs from \( l \) to \( \infty \). The \( (2l + 1) \) term in the left-hand side comes from the fact that there is a degeneracy of \( (2l + 1) \) for a given \( l \). In other words, we have \( (2l + 1) \) factor on the left-hand side and \( (q + 1)^2 \) factor on the right-hand side as expected from (3).

Now, we need to find \( C_q \). Recall that the Lagrangian for \( \mathcal{R} \) in inflation is given by

\[
S_2 = \frac{1}{2} \int d^3x dt 2a^3 \varepsilon \left( \left( \partial_i \mathcal{R} \right)^2 - \frac{\left( \partial_i \mathcal{R} \right)^2}{a^2} \right), \quad \varepsilon \equiv -\frac{\dot{H}}{H^2}
\]

Considering that \( \mathcal{R} \) is conserved, \( \partial_i \mathcal{R} \) is zero. Thus,

\[
\nabla^2_3(\mathcal{R}(x)\mathcal{R}(y)) = \frac{1}{2a^3} \delta^3(x - y)
\]

Therefore, the two-point function of Fourier mode is given by the inverse of the Laplacian. As the eigenvalues of Laplacian is proportional to \( q(q + 2) \), we conclude \( C_q \) is proportional to \( 1/(q(q + 2)) \). Thus, we have

\[
C_l(2l + 1) = 2C \sum_{q=l}^{\infty} \frac{1}{q(q + 2)} \left( \chi - \frac{\sin(4(q + 1)\chi)}{4(q + 1)} \right)
\]

\[
= C\chi \left( \frac{(2l + 1)}{l(l + 1)} - \frac{1}{2\chi} \sum_{q=l}^{\infty} \frac{\sin(4(q + 1)\chi)}{q(q + 1)(q + 2)} \right)
\]
for some constant $C$.

Here, we see that $C_l$ is roughly proportional to $1/\ell(\ell + 1)$ from the first term. The second term rapidly converges to zero for higher $\ell$, not only because there are fewer terms to add (even though there are infinite terms to do so), but also because the sine function is oscillating.

5 Data analysis

Let’s set the notation. We have

$$\chi = \sqrt{\Omega_{\text{tot}} - 1} \int_{1/(1+z_L)}^1 \frac{da}{a^2 \sqrt{\Omega_\Lambda - (\Omega_{\text{tot}} - 1)a^{-2} + \Omega_M a^{-3}}}, \quad r_L = \sin \chi$$  \hspace{1cm} (26)

where we ignored the contribution from the radiation. We use $\Omega_M = \Omega_{\text{tot}} - \Omega_\Lambda$.

To calculate $\chi$, we use the data in [2]. For WMAP+BAO+$H_0$, they obtained

$$\Omega_{\text{tot}} = 1.0023^{+0.0056}_{-0.0054}, \quad \Omega_\Lambda = 0.728^{+0.015}_{-0.016}, \quad z_L = 1090.89^{+0.68}_{-0.69}$$  \hspace{1cm} (27)

which yields

$$\chi = 0.16_{-0.16}^{+0.14}$$  \hspace{1cm} (28)

For CMB anisotropy data, we used [3]. As the present author does not know well about statistics and data processing, we tried to find the best fit $\chi$ by trial and error. First, we defined $D_\ell \equiv C_\ell \cdot \ell(\ell + 1)$ and obtained the value for

$$\lim_{\ell \to \infty} D_\ell = D$$  \hspace{1cm} (29)

by averaging from $\ell = 2$ to 29, which we got 851. (If we use the notation of (25), we have $D = C \chi$.) Then, we tried to minimize

$$\frac{(a_{\text{obs}} - a_{\text{th}})^2}{\sigma_a^2} + \frac{(b_{\text{obs}} - b_{\text{th}})^2}{\sigma_b^2} + \frac{(c_{\text{obs}} - c_{\text{th}})^2}{\sigma_c^2}$$  \hspace{1cm} (30)

where “obs” denotes observed value, and “th” denotes theoretical value, and

$$a = C_2 \cdot 2 \cdot 3, \quad b = C_3 \cdot 3 \cdot 4, \quad c = \frac{(C_4 \cdot 4 \cdot 5) + (C_5 \cdot 5 \cdot 6)}{2}$$  \hspace{1cm} (31)

$\sigma$s are also in the Table 1.

We found that $\chi = 0.14$ minimizes (30). Then, as you can see from [3], $\ell = 2$ is suppressed while $\ell = 5$ is augmented. This is true for $\chi = 0.11$ to 0.23. Therefore, our conclusion is $\chi = 0.14_{-0.03}^{+0.09}$ which agrees with (28).
Table 1: $D_l(\chi)$

| $l$ | $D_{\text{obs}}$ | $D_{\text{th}}(0.11)$ | $D_{\text{th}}(0.14)$ | $D_{\text{th}}(0.16)$ | $\sigma_{D_l}$ |
|-----|-----------------|----------------------|----------------------|----------------------|----------------|
| 2   | 150             | 562                  | 664                  | 721                  | 708            |
| 3   | 902             | 705                  | 800                  | 844                  | 565            |
| 4.5 | 1099            | 833                  | 887                  | 899                  | 312            |

6 Discussions and Conclusions

In this article, we examined the CMB anisotropy power spectrum by a novel approach, and found an agreement with the earlier analysis. Future work should consider other effects, going beyond the scale invariant primordial universe.

Acknowledgments

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References

[1] “Wavelets on the Three-Dimensional Sphere $S^3$,” Svend Ebert, Diploma Thesis
   [http://www.mathe.tu-freiberg.de/inst/amm1/Mitarbeiter/Ebert/Wavelets.pdf](http://www.mathe.tu-freiberg.de/inst/amm1/Mitarbeiter/Ebert/Wavelets.pdf)

[2] N. Jarosik et al., “Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Sky Maps, Systematic Errors, and Basic Results,” Astrophys. J. Suppl. 192, 14 (2011) doi:10.1088/0067-0049/192/2/14 [arXiv:1001.4744 [astro-ph.CO]].

[3] https://lambda.gsfc.nasa.gov/data/map/dr5/dcp/spectra/wmap_tt_spectrum_9yr_v5.txt
    https://lambda.gsfc.nasa.gov/data/map/dr5/dcp/spectra/wmap_binned_tt_spectrum_9yr_v5.txt