On exact solutions of a class of fractional Euler-Lagrange equations

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Abstract

In this paper, first a class of fractional differential equations are obtained by using the fractional variational principles. We find a fractional Lagrangian $L(x(t), \frac{\partial}{\partial a} D^\alpha t b(\frac{\partial}{\partial a} D^\alpha t x(t)))$ and $0 < \alpha < 1$, such that the following is the corresponding Euler-Lagrange

$$t D^\alpha_b (\frac{\partial}{\partial a} D^\alpha_t x(t)) + b(t, x(t))(\frac{\partial}{\partial a} D^\alpha_t x(t)) + f(t, x(t)) = 0. \quad (1)$$

At last, exact solutions for some Euler-Lagrange equations are presented. In particular, we consider the following equations

$$t D^\alpha_b (\frac{\partial}{\partial a} D^\alpha_t x(t)) = \lambda x(t), \quad (\lambda \in R) \quad (2)$$

$$t D^\alpha_b (\frac{\partial}{\partial a} D^\alpha_t x(t)) + g(t) \frac{\partial}{\partial a} D^\alpha_t x(t) = f(t), \quad (3)$$

where $g(t)$ and $f(t)$ are suitable functions.

Keywords: Fractional calculus, differential equations of fractional order, fractional variational calculus
1 Introduction

Fractional calculus is an emerging field and during the last decades it represents an alternative tool to solve several problems from various fields \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]\). During the last years the fractional variational principles \([12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]\) have developed and applied to fractional optimal control problems \([26, 27]\).

Despite of various efforts during the last years, the fractional Lagrangian and Hamiltonian formulation of both discrete and continuous systems is at the beginning of its development. Although the fractional variational principles were started to be investigated deeply the appropriate physical interpretation of the fractional derivatives creates problems in physical interpretation of the obtained equations. The existence of various fractional derivatives leads to several Hamiltonian formulations for a given dynamical system.

Very recently, based on finite difference \([28]\) it was proposed an alternative definition for the Riemann-Liouville (RL) derivatives. By using the approach presented in \([28]\) the troublesome effects of the initial conditions in the RL fractional derivative are removed.

By construction the fractional Lagrangian and fractional Hamiltonian contain as a particular case the classical counterparts. Due to the fractional integration by parts, the fractional Euler-Lagrange equations contains the left and the right Riemann-Liouville derivatives. Even if the fractional Lagrangian contains only Caputo derivatives the corresponding fractional Euler-Lagrange equations contains both RL and Caputo derivatives. From these reasons we expect to obtain new solutions of the fractional Euler-Lagrange equations. Another problem which presents interest is to find a fractional Lagrangian for a given fractional Euler-Lagrange equations and therefore we obtain a meaning for these equations. Until now quite a few exact solutions were reported for fractional Euler-Lagrange equations, therefore finding more general solutions having physical significance is an open issue in this area. This issue plays an important role in fractional quantisation models. Some type of functional involving the fractional derivatives are used in mathematical economy as well as utilized for describing the dissipative structures arising in nonlinear dynamical systems.

The plan of this manuscript is as follows:

Some basic definitions of fractional derivatives are shown in section two. Section three presents the fractional Lagrangian corresponding to a given second order fractional differential equations involving both RL and Caputo
derivatives. In section four an exact new solution for fractional oscillator as well as a generalization of it are obtained. Section five is dedicated to our conclusions.

2 Mathematical tools

In this section, we formulate the problem in terms of the left and the right RL fractional derivatives, which are defined as follows, the left RL fractional derivative

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (4) \]

and the right RL fractional derivative

\[ bD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau, \quad (5) \]

where the order \( \alpha \) fulfills \( n-1 \leq \alpha < n \) and \( \Gamma \) is the Euler’s gamma function. If \( \alpha \) becomes an integer, we recovered the usual definitions, namely,

\[ aD_t^\alpha f(t) = \left(\frac{df(t)}{dt}\right)^\alpha, \quad bD_t^\alpha f(t) = \left(-\frac{df(t)}{dt}\right)^\alpha; \quad (\alpha = 1, 2, ...). \quad (6) \]

Fractional RL derivatives have various interesting properties. For example the fractional derivative of a constant is not zero, namely

\[ aD_t^\alpha C = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. \quad (7) \]

The fractional derivative of a power of \( t \) has the following form

\[ aD_t^\alpha (t-a)^\beta = \frac{\Gamma(\alpha+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \quad (8) \]

for \( \beta > -1, \alpha \geq 0 \). The Caputo’s fractional derivatives are defined as follows, the left Caputo Fractional Derivative

\[ \epsilon D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \quad (9) \]
and the right Caputo Fractional Derivative

\[ \mathcal{D}_{t}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} (\tau - t)^{n-\alpha-1} \left( -\frac{d}{d\tau} \right)^{n} f(\tau) d\tau. \]  

(10)

Here \( \alpha \) represents the order of the derivative such that \( n - 1 < \alpha < n \).

### 3 Fractional Variational Principles

Let us consider the following fractional second order differential equation:

\[ t D^\alpha_{b} (c_a D^\alpha_{t} x(t)) + b(t, x(t))(c_a D^\alpha_{t} x(t)) + f(t, x(t)) = 0, \]  

(11)

where \( 0 < \alpha \leq 1 \). Our aim is to find a fractional Lagrangian

\[ L(x(t), c_a D^\alpha_{t} x(t)), \quad 0 < \alpha < 1, \]  

(12)

such that

\[ \frac{\partial L}{\partial x} + t D^\alpha_{b} \left( \frac{\partial L}{\partial (c_a D^\alpha_{t} x)} \right) = t D^\alpha_{b} (c_a D^\alpha_{t} x) + b(t, x)(c_a D^\alpha_{t} x) + f(t, x). \]  

(13)

We assume a solution of this problem as follows

\[ L(x, c_a D^\alpha_{t} x) = \frac{1}{2} (c_a D^\alpha_{t} x)^2 + h(t, x)c_a D^\alpha_{t} x + g(t, x). \]  

(14)

Then, we evaluate the left hand side of (13) and we obtain

\[ (t D^\alpha_{b} (c_a D^\alpha_{t} x)) + (t D^\alpha_{b} h(t, x)) + \frac{\partial h}{\partial x}(c_a D^\alpha_{t} x) + \frac{\partial g(t, x)}{\partial x}. \]  

(15)

Therefore, by using (14) we obtain

\[ \frac{\partial h(t, x)}{\partial x} = b(t, x), \]  

(16)

and

\[ t D^\alpha_{b} h(t, x) + \frac{\partial g(t, x)}{\partial x} = f(t, x). \]  

(17)

By using (16) and (17) we obtain the functions \( g(t, x) \) and \( h(t, x) \) respectively.
4 Exact solutions of fractional Euler-Lagrange equations

The scope of this section is to present the exact solutions for a class of problems arising from a fractional variational principles.

4.1 One-dimensional fractional oscillator

The first fractional Euler-Lagrange is given below

\[ tD^\alpha b(\alpha D^\alpha t x(t)) = \lambda x(t), \]  
(18)

where \( \lambda \in R \). Our purpose is to solve the equation (18). For this reason we assume the solution in the following form

\[ x(t) = \sum_{n=0}^{\infty} a_n (t - a)^{n\alpha + 1}, \]  
(19)

where \( a_n \) is to be determined. The first step is to calculate \( c_a D^\alpha t x(t) \) taking into account (18). Therefore, we obtain the following

\[ c_a D^\alpha t x(t) = \sum_{n=1}^{\infty} a_n \frac{\Gamma((n + 1)\alpha)}{\Gamma(n\alpha)} (t - a)^{n\alpha - 1}, \]  
(20)

Then

\[ tD^\alpha b(\alpha D^\alpha t x(t)) = \sum_{n=0}^{\infty} a_{n+2} \frac{\Gamma((n + 3)\alpha)}{\Gamma((n + 1)\alpha)} e^{i\pi\alpha} (t - a)^{n\alpha + 1}, \]  
(21)

with \( a < x < 2b - a \).

Now we find the following relation that permits us to find the coefficients \( a_n \), by using (18) and (21)

\[ a_{n+2} = \frac{\Gamma((n + 1)\alpha)}{\Gamma((n + 3)\alpha)} a_n, \]  
(22)

that is the coefficients of the solution of (18) are given by

\[ a_{2(n+1)} = (\lambda e^{-i\pi\alpha})^{n+1} \frac{\Gamma(\alpha)}{\Gamma((2n + 3)\alpha)} a_0, n \geq 0, \]  
(23)

\[ a_{2(n+1)+1} = (\lambda e^{-i\pi\alpha})^{n+1} \frac{\Gamma(2\alpha)}{\Gamma((2n + 2)\alpha)} a_1, n \geq 0. \]  
(24)
So we have

\[ x(t) = a_0(t - a)^{\alpha - 1} \left[ 1 + \sum_{n=0}^{\infty} \frac{(\lambda e^{-i\pi\alpha})^{n+1} \Gamma(\alpha)(t - a)^{2(n+1)\alpha}}{\Gamma((2n+3)\alpha)} \right] + a_1(t - a)^{2\alpha - 1} \left[ 1 + \sum_{n=0}^{\infty} \frac{(\lambda e^{-i\pi\alpha})^{n+1} \Gamma(2\alpha)(t - a)^{2(n+1)\alpha}}{\Gamma(2(n+1)\alpha)} \right]. \] \hspace{1cm} (25)

It is obvious to prove that the above series is convergent. Thus, we obtain the following two general solutions as follows

\[ x_1(t) = a_0(t - a)^{\alpha - 1} \left[ 1 + \sum_{n=0}^{\infty} \frac{(\cos(n\pi \alpha)\lambda)^{n+1} \Gamma(\alpha)}{\Gamma(2(n+1)\alpha + \alpha)}(t - a)^{2(n+1)\alpha} \right] \]

\[ + a_1(t - a)^{2\alpha - 1} \left[ 1 + \sum_{n=0}^{\infty} \frac{(\cos(n\pi \alpha)\lambda)^{n+1} \Gamma(2\alpha)}{\Gamma(2(n+2)\alpha)}(t - a)^{2(n+1)\alpha} \right], \] \hspace{1cm} (26)

and

\[ x_2(t) = a_0(t - a)^{\alpha - 1} \left[ \sum_{n=0}^{\infty} \frac{(\sin(n\pi \alpha)\lambda)^{n+1} \Gamma(\alpha)}{\Gamma(2(n+1)\alpha + \alpha)}(t - a)^{2(n+1)\alpha} \right] \]

\[ + a_1(t - a)^{2\alpha - 1} \left[ \sum_{n=0}^{\infty} \frac{(\sin(n\pi \alpha)\lambda)^{n+1} \Gamma(2\alpha)}{\Gamma(2(n+2)\alpha)}(t - a)^{2(n+1)\alpha} \right]. \] \hspace{1cm} (27)

We observe that for \( \alpha = 1 \), \( x_1(t) = a_0\cos(t) + a_1\sin(t) \) and \( x_2(t) = 0 \), therefore the classical result is obtained.

### 4.2 A more general case

In the following we consider the fractional differential equation

\[ \frac{d^\alpha}{dt^\alpha} \left( \zeta \frac{d^\alpha}{dt^\alpha} x(t) \right) + g(t) \frac{d^\alpha}{dt^\alpha} x(t) = f(t), \] \hspace{1cm} (28)

where \( g(t) \) and \( f(t) \) are suitable functions. We denote \( \zeta \frac{d^\alpha}{dt^\alpha} x(t) = z(t) \) and we rewrite the equation (28) as

\[ \frac{d^\alpha}{dt^\alpha} z(t) + g(t)z(t) = f(t). \] \hspace{1cm} (29)

The equation (29) can be written as follows

\[ L(z(t)) = \frac{f(t)}{g(t)}, \] \hspace{1cm} (30)
where $L = g(t)^{-1} \frac{D^\alpha}{D_t^\alpha} + 1$. The solution of (30) can be written in the following form
\[
z(t) = L^{-1}\left\{\frac{f(t)}{g(t)}\right\},
\] (31)
where, we will consider
\[
L^{-1} = \sum_{i=0}^{\infty} (-1)^i [g(t)^{-1} \frac{D^\alpha}{D_t^\alpha}]^i.
\] (32)
The second step is to solve the following equation
\[
c_\alpha D^\alpha t x(t) = z(t).
\] (33)
The solution of (33) is as follows
\[
x(t) = a I^\alpha_t z(t) + c_1 (t - a)^{\alpha-1} + c_2,
\] (34)
that is
\[
x(t) = \sum_{i=0}^{\infty} (-1)^i a^i I^\alpha_t [g(t)^{-1} \frac{D^\alpha}{D_t^\alpha}]^i \left\{\frac{f(t)}{g(t)}\right\} + c_1 (t - a)^{\alpha-1} + c_2.
\] (35)
It is very easy to check directly that the above function $x(t)$ is a solution of equation (28), and it is a convergent series if $f(x)$ and $g(x)$ are suitable functions.

5 Conclusions

The solutions of the complex fractional Euler-Lagrange equations were obtained by using the numerical techniques for most of the cases. In this paper we found a new and more general solution, as a series solution, of the fractional oscillator within Caputo derivatives. The classical solution is recovered but a new solution was also reported. A fractional Lagrangian that produces a given class of second order ordinary fractional differential equations was found. By using the operational approach an exact solution of a particular Euler-Lagrange equation was obtained.

6 Acknowledgements

The research reported in this paper have been partially supported by the Scientific and Technical Research Council of Turkey and by MEC of Spain (MTM2004-0317).
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