GLOBAL EXISTENCE AND BLOWUP ON THE ENERGY SPACE FOR THE INHOMOGENEOUS FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this article, we study the initial-value problem for inhomogeneous fractional nonlinear Schrödinger equation

\[ i \partial_t u = (-\Delta)^s u - |x|^{-b}|u|^{2\sigma} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

where \( \frac{1}{2} < s < 1 \), \( N \geq 2 \) and \( \frac{2s-b}{N} \leq \sigma < \frac{2s-b}{2s} \). We prove a Gagliardo-Nirenberg-type estimate and use it to establish sufficient conditions for global existence in \( H^s(\mathbb{R}^N) \). In addition, we derive a localized Virial estimate for inhomogeneous fractional nonlinear Schrödinger equation in \( \mathbb{R}^N \), which uses Balakrishnan’s formula for the fractional Laplacian \((-\Delta)^s\) from semigroup theory. By these estimates, we give the blowup criterion of radial solutions in \( \mathbb{R}^N \) for \( L^2 \)-critical, \( L^2 \)-supercritical and \( H^s \)-subcritical power.

1. Introduction. In this paper, we study the initial-value problem for the inhomogeneous fractional nonlinear Schrödinger equation

\[ i \partial_t u = (-\Delta)^s u - |x|^{-b}|u|^{2\sigma} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

where \( \sigma > 0 \), \( 0 < b < \min\{2s, N\} \) and the integer \( N \geq 2 \) denotes the space dimension. The fractional differential operator \((-\Delta)^s\) is defined by \((-\Delta)^s u = \mathcal{F}^{-1}(\xi^{2s}\mathcal{F}(u))\) where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are the Fourier transform and inverse Fourier transform, respectively. When \( s = 1, b = 0 \), \((1)\) is the well-know nonlinear Schrödinger equation which has been extensively studied. When \( 0 < s < 1 \) and \( b = 0 \), \((1)\) is a nonlocal model known as nonlinear fractional Schrödinger equation which has also attracted much attentions recently (see [1, 5–9, 12–15, 17, 18, 21, 22, 27, 30, 32–34]). The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics, which was derived by N. Laskin ([23, 24]) as a result of extending

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the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The purpose of this paper is to prove some analogue global well-posedness and blowup criterion for (1) in the radial case, and this work seems to be the first one dealing with blowup issues for the inhomogeneous fractional nonlinear Schrödinger equation with the pure power nonlinearity.

When $s = 1$, F. Merle [25] and P. Raphaël [28] investigated the problem of existence/nonexistence of minimal mass blow-up solution for the following inhomogeneous equation

$$\frac{i \partial_t u}{u} + \Delta u + k(x)|u|^{2\sigma} u = 0,$$

where $k(x)$ is a positive bounded function. When $k(x) = |x|^{-b}$, L.G. Farah in [11] established a Gagliardo-Nirenberg-type inequality and used it to derive sufficient conditions for global existence and blow-up in $H^1(\mathbb{R}^N)$. Further results about inhomogeneous nonlinear Schrödinger equation can be see in [3, 4, 16, 29] and reference therein.

In [29], T. Saanouni did the earliest researches about inhomogeneous fractional nonlinear Schrödinger equation. They studied the following equation

$$\frac{i \partial_t u}{u} = (-\Delta)^{s} u + \epsilon|x|^\gamma |u|^{2\sigma} u$$

and obtained the well-posedness issues by a sharp Gagliardo-Nirenberg-type inequality and potential well method under the condition $\gamma > 0$. But they didn’t consider the blowup results. Recently, T. Boulenger, D. Himmelsbach and E. Lenzmann [1] investigated problem (1) in the case of $b = 0$. They proved a general criterion for blowup of radial solution in $\mathbb{R}^N$ with $N \geq 2$ for $\frac{2s}{N} \leq \sigma \leq \frac{2s}{2s - 2s}$. The main results in their paper is the following theorem.

**Theorem 1.1.** Let $N \geq 2$, $s \in\left(\frac{1}{2}, 1]\right)$, $0 \leq s_c = \frac{N}{2} - \frac{s}{2} \leq s$ with $\sigma < 2s$. Assume that $u \in C([0, T]; H^{2s}(\mathbb{R}^N))$ is a radial solution of (1) with $b = 0$. Furthermore, we suppose that either

$$E(u_0) < 0$$

or, if $E(u_0) \geq 0$,

$$E(u_0)^{s_c} M(u_0)^{s - s_c} < E(Q)^{s_c} M(Q)^{s - s_c}$$

and

$$\|(-\Delta)^{\frac{s_c}{2}} u\|_{L^2}^{s_c} \|u_0\|_{L^2}^{s - s_c} > \|(-\Delta)^{\frac{s_c}{2}} Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{s - s_c},$$

where $Q$ is the ground states of (1). Then the following conclusions hold.

(i) $L^2$–supercritical case: If $0 < s_c \leq s$, then $u(t)$ blows up in finite time in the sense that $T < +\infty$ must hold.

(ii) $L^2$–critical case: If $s_c = 0$, then $u(t)$ either blows up in finite time in the sense that $T < +\infty$ must hold, or $u(t)$ blows up infinite time such that

$$\|(-\Delta)^{\frac{s_c}{2}} u(t)\|_{L^2} \geq C t^s$$

for all $t \geq t_*$, with some constants $C > 0$ and $t_* > 0$ that depend only on $u_0, s, N$.

The above conclusion is the generalization of classical blowup theory in (19, 20]) for classical nonlinear Schrödinger equation (i.e. when $s = 1$). The main difficult for fractional Schrödinger equation comes from the Variance-Virial Law will be invalid when we talk about the blowup problems. In [1], the authors employ a localized virial identity and Balakrishnan’s formula to overcome this difficult.
We recall the best Sobolev index where we can expect well-posedness for this model. First, note that if \( u(t, x) \) is a solution of (1) so is \( u_\lambda = \lambda^{\frac{2s-b}{2s}} u(\lambda^2 t, \lambda x) \), for all \( \lambda > 0 \). Computing the homogeneous Sobolev norm, we have

\[
\| u_\lambda (0, \cdot) \|_{H^s} = \lambda^{s + \frac{2s-b}{2s} - N} \| u_0 \|_{H^s}.
\]

Thus, the critical Sobolev index is given by \( s_\sigma = \frac{N}{2} - \frac{2s-b}{2s} \). When \( s_\sigma < 0 \), equation (1) is \( L^2 \)-subcritical and we will prove all \( H^s \) solutions are global by a Gagliardo-Nirenberg inequality. The smallest power for which blowup may occur is \( \sigma = \frac{2s-b}{N} \), which is referred to \( L^2 \)-critical case corresponding to \( s_\sigma = 0 \). When \( 0 < s_\sigma < 1 \), (1) is \( L^2 \)-supercritical and \( H^s \)-subcritical. When \( s_\sigma = 1 \), (1) is \( H^s \)-critical. In this paper, we are interested in the \( L^2 \)-critical, \( L^2 \)-supercritical and \( H^s \)-subcritical cases. Therefore, we restrict our attention to the case where \( 0 \leq s_\sigma < s \). Rewriting this last condition in terms of \( \sigma \), we obtain

\[
\frac{2s-b}{N} \leq \sigma < 2^*,
\]

where \( 2^* = \frac{2s-b}{N-2s} \) if \( N > 2s \) or \( 2^* = \infty \), if else. To avoid \( \sigma \) to be negative, we also assume the technical restriction \( 0 < b < \min\{2s, N\} \).

Motivated by the above discussion, in this paper, we study the global existence and blow up in finite time for equation (1). Our result is a generalization of Theorem 1.1 to inhomogeneous fractional nonlinear Schrödinger equation. Firstly, we give a new sharp Gagliardo-Nirenberg inequality and get the global existence for equation (1). Indeed, when \( 0 < \sigma < \frac{2s-b}{N} \), the solution exists globally and \( \sigma = \frac{2s-b}{N} \) is the critical exponent of the existence of finite time blowup solutions for Eq. (1) which is proved in Proposition 4.4. Secondly, we derive some localized radial virial estimate and obtain the blowup criterion in the case of \( L^2 \)-supercritical and \( H^s \)-subcritical. The key of proof for blowup solution is the estimate of the localized radial Virial equality. In [1], the authors provided a new method to get these estimates, but our proof is more elaborate due to the singular coefficient \( |x|^{-b} \). Finally, we get some results which imply that \( \| Q \|_{L^2} \) is the sharp threshold mass of the existence and finite-time blow-up for Eq. (1) with \( L^2 \)-critical exponent by a refined version of localized radial virial estimate.

Next, we give some remarks for our problem.

**Remark 1.** We exclude the half-wave case of \( s = \frac{1}{2} \), which is due to the lack of control for the pointwise decay of a radial function \( u \in H^2 (\mathbb{R}^N) \) with \( N \geq 2 \). See also that the Strauss’s inequality failed when \( s = \frac{1}{2} \), and it is an important result for our problem.

**Remark 2.** For classical nonlinear Schrödinger equation (i.e. when \( s = 1 \)) we have the Variance-Virial Law, which can be expressed as

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^N} |x|^2 |u(t)|^2 dx \right) = 2 \text{Im} \left( \int_{\mathbb{R}^N} \bar{u}(t) x \cdot \nabla u(t) dx \right),
\]

provided that \( \int_{\mathbb{R}^N} |x|^2 |u(t)|^2 dx < +\infty \). So, we obtain Glassey’s celebrated blowup result for classical nonlinear Schrödinger equation with negative energy \( E[u_0] < 0 \) and finite variance. However, this argument breaks down for \( s \neq 1 \). So, we use a localized version of Virial law to show blowup for our problem.

**Notation and Conventions.** We use \( C \) to denote various constants that may vary line by line. Given \( x, y \in \mathbb{R}^N \), \( x \cdot y \) denotes the inner product of \( x \) and \( y \) in \( \mathbb{R}^N \).
We use \( \| \cdot \|_{L^2}, \| \cdot \|_{L^\infty}, \| \cdot \|_{H^s}, \| \cdot \|_{H^s} \) to denote the \( L^2(\mathbb{R}^N), L^\infty(\mathbb{R}^N), H^s(\mathbb{R}^N), \) \( H^s(\mathbb{R}^N) \) norms. Moreover, we employ the notation \( X = OY \) by which we mean that \( |X| \leq CY \) holds.

The plan of this work is as follows: in the next section we introduce the sharp Gagliardo-Nirenberg inequality and global existence. In Section 3, we give the local Virial estimate and the blowup result for \( L^2 \)-supercritical and \( H^s \)-subcritical. Finally, Section 4, we give the refine local Virial estimate and the blowup result for \( L^2 \)-critical.

2. The sharp Gagliardo-Nirenberg inequality and global existence. Let us collect some classical results needed along this manuscript. We start with some properties of the free Schrödinger kernel.

**Proposition 1.** Denoting the free operator associated to the fractional Schrödinger equation

\[ \Gamma_s(t)u := e^{it(-\Delta)^{\frac{s}{2}}} u := \mathfrak{F}^{-1}(e^{-it|x|^{2\sigma}}) \ast u \]

yields the following:

1. \( \Gamma_s(t)u_0 - i \int_0^t \Gamma_s(t - \tau)(|x|^{-b}|u|^{2^*} u) d\tau \) is the solution to the problem (1),
2. \( (\Gamma_s(t))' = \Gamma_s(-t) \),
3. \( \Gamma_s \Gamma_t = \Gamma_{s+t} \),
4. \( \Gamma_s(t) \) is an isometry of \( \mathcal{L}^2(\mathbb{R}^N) \).

**Definition 2.1.** A couple of real numbers \((q, r)\) such that \( q, r \geq 2 \) are said to be admissible if

\[ \frac{4N + 2}{2N - 1} \leq q \leq +\infty, \quad \frac{2}{q} + \frac{2N - 1}{r} \leq N - \frac{1}{2}, \]

or

\[ 2 \leq q \leq \frac{4N + 2}{2N - 1}, \quad \frac{2}{q} + \frac{2N - 1}{r} < N - \frac{1}{2}. \]

**Proposition 2** ( [18]). Let \( N \geq 2, \mu \in \mathbb{R}, \frac{N}{2N - 1} < s < 1, \) and \( u_0 \in H^\mu(\mathbb{R}^N) \) is radial. Then

\[ \|u\|_{L^q_t(L^r)} \cap L^\sigma_t(H^\nu) \leq C(\|u_0\|_{\dot{H}^\mu} + \|iu_t - (-\Delta)^s u\|_{L^r_t(L^q)}), \]

if \((q, r)\) and \((\tilde{q}, \tilde{r})\) are admissible pairs such that \((\tilde{q}, \tilde{r}, N) \neq (2, \infty, 2)\) or \((q, r, N) \neq (2, \infty, 2)\) and satisfy the condition

\[ \frac{2\alpha}{q} + \frac{N}{r} = \frac{N}{2} - \mu, \quad \frac{2\alpha}{q} + \frac{N}{\tilde{r}} = \frac{N}{2} + \mu. \]

If we take \( \mu = 0 \) in the previous inequality, we obtain the classical Strichartz estimate. Using Theorem 2.2 which we will prove later, the local well-posedness of the Schrödinger equation (1) in the energy space holds.

**Proposition 3.** Let \( N \geq 2, 0 < b < \min\{2s, N\}, \frac{N}{2N - 1} < s < 1, 0 < \sigma \leq \frac{2s - b}{N - 2s} \) and \( u_0 \in H^\mu(\mathbb{R}^N) \) is radial. Then there exists \( T^* > 0 \) and a unique maximal radial solution to (1),

\[ u \in C([0, T^*), H^\nu(\mathbb{R}^N)). \]

Furthermore, the solution of equation Eq. (1) has the following conserved quantities

\[ M[u(t)] = \int_{\mathbb{R}^N} |u(t, x)|^2 dx \]
and
\[ E[u(t)] = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u(t, x)|^2 dx - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |x|^{-b}|u(t, x)|^{2\sigma + 2} dx. \]

**Remark 3.** The proof follows from Theorem 2.2 via Strichartz estimate in Proposition 2, the integral formula in Proposition 1, and a classical Picard fixed point method \([2, 18]\).

We begin with showing that the quantity \(E[u(t)]\) is well defined for functions in \(H^s(\mathbb{R}^N)\). This is guaranteed by the following sharp Gagliardo-Nirenberg inequality.

**Theorem 2.2.** Let \(0 < \sigma < \frac{2s-b}{N-2s}\) and \(0 < b < \min\{2s, N\}\), then the sharp Gagliardo-Nirenberg inequality
\[
\int_{\mathbb{R}^N} |x|^{-b}|u|^{2\sigma+2} dx \leq K_{opt}\|(-\Delta)\frac{1}{2} u\|_{L^2}^{\frac{\sigma N + b}{2s}} \|u\|_{L^2}^{(2\sigma+2) - \frac{\sigma N + b}{2s}}
\]
holds, and the sharp constant \(K_{opt}\) is explicitly given by
\[
K_{opt} = \frac{(\sigma N + b)}{2s(\sigma + 1) - (\sigma N + b)} \frac{2s(\sigma + 1)}{(N\sigma + b)\|Q\|_{L^2}^{2\sigma}}.
\]

where \(Q\) is the unique nonnegative, radially symmetric, decreasing solution of the equation
\[
-(-\Delta)^s Q - Q + |x|^{-b}|Q|^{2\sigma} Q = 0.
\]
Moreover, the solution \(Q\) satisfies the following relations
\[
\|(-\Delta)^{\frac{1}{2}} Q\|_{L^2} = \frac{\sigma N + b}{2s(\sigma + 1) - (\sigma N + b)} \|Q\|_{L^2}^{\frac{2s(\sigma + 1)}{2s(\sigma + 1) - (\sigma N + b)}}
\]
and
\[
\int_{\mathbb{R}^N} |x|^{-b}|Q|^{2\sigma+2} dx = \frac{2s(\sigma + 1)}{(N\sigma + b)\|Q\|_{L^2}^{2\sigma}}.
\]

**Proof.** We employ the ideas introduced by M.I. Weinstein \([31]\). First, define the Weinstein functional
\[
J(u) = \frac{\|(-\Delta)^{\frac{1}{2}} u\|_{L^2}^{\frac{\sigma N + b}{2s}} \|u\|_{L^2}^{(2\sigma+2) - \frac{\sigma N + b}{2s}}}{I(u)}
\]
where \(I(u) = \int_{\mathbb{R}^N} |x|^{-b}|u|^{2\sigma+2} dx\). We notice that \((2\sigma + 2) - \frac{\sigma N + b}{2s} > 0\) if \(0 < \sigma < \frac{2s-b}{N-2s}\).

By a similar method with Lemma 2.1 in F. Genoud \([16]\), we can prove that \(I \in C(H^s(\mathbb{R}^N); \mathbb{R})\) and is weakly sequentially continuous and \(J \in C(H^s(\mathbb{R}^N) \setminus \{0\}; \mathbb{R})\).

Now, since \(J(u) \geq 0\), there exists a minimizing sequence \(u_n \in H^s(\mathbb{R}^N)\) such that
\[
\lim_{n \to +\infty} J(u_n) = m = \min_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} J(u).
\]

By Schwarz symmetrization, we can assume that \(u_n\) is radial and radially non-increasing for all \(n\). Next, we rescale the sequence \(\{u_n\}_{n \in \mathbb{N}}\) by setting \(v_n(x) = \lambda_n u_n(\mu_n x)\) where
\[
\lambda_n = \frac{\|u_n\|_{L^2}^{\frac{N-2s}{2s}}}{{\|(-\Delta)^{\frac{1}{2}} u_n\|}_{L^2}^{\frac{\sigma N + b}{2s}}} \quad \text{and} \quad \mu_n = \frac{\|u_n\|_{L^2}^{\frac{1}{2}}}{{\|(-\Delta)^{\frac{1}{2}} u_n\|}_{L^2}^{\frac{1}{2}}},
\]
so that \(\|v_n\|_{L^2} = \|(-\Delta)^{\frac{1}{2}} v_n\|_{L^2} = 1\). Moreover, since \(J\) is invariant under this scaling, \(\{v_n\}_{n \in \mathbb{N}}\) is also a minimizing sequence which is bounded in \(H^s(\mathbb{R}^N)\). Therefore, there exists \(v^* \in H^s(\mathbb{R}^N)\) such that, up to a subsequence, \(v_n \rightharpoonup v^*\) weakly
in $H^s(\mathbb{R}^N)$. Furthermore, $v^*$ is nonnegative, spherically symmetric, radially nonincreasing, with

$$\|v^*\|_{L^2} \leq 1 \quad \text{and} \quad \|(-\Delta)^{\frac{s}{2}} v^*\|_{L^2} \leq 1.$$  

In this case,

$$m \leq J(v^*) \leq \frac{1}{I(v^*)} = \lim_{n \to +\infty} J(v_n) = m.$$  

Thus, $J(v^*) = \frac{1}{I(v^*)} = m$ and $\|v^*\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} v^*\|_{L^2} = 1$. In particular, $v^* \neq 0$ and $v_n \to v^*$ strongly in $H^s(\mathbb{R}^N)$.

Therefore, $v^*$ is a minimizer for the Weinstein functions $J$. Moreover, $v^*$ is a solution of the Euler-Lagrange equation

$$\frac{d}{dc} J(v^* + c\eta)|_{c=0} = 0 \quad \text{for all} \quad \eta \in C_0^\infty(\mathbb{R}^N),$$

and so we obtain that $v^*$ satisfies the equation

$$\frac{N\sigma + b}{2s}(-\Delta)^{\frac{s}{2}} v^* - (\sigma + 1 - \frac{N\sigma + b}{2s})v^* + m(\sigma + 1)|x|^{-b}|v^*|^{2\sigma} v^* = 0. \quad (7)$$

Next, we rescale $v^*$ to a solution of Eq. (4). First, we take $\psi^* = (m(\sigma + 1))^\frac{2}{\sigma} v^*$. It is easy to see that $\psi^*$ is a solution of

$$\frac{N\sigma + b}{2s}(-\Delta)^{\frac{s}{2}} \psi^* - (\sigma + 1 - \frac{N\sigma + b}{2s})\psi^* + |x|^{-b}|\psi^*|^{2\sigma} \psi^* = 0. \quad (8)$$

Furthermore, since $\|v^*\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} v^*\|_{L^2} = 1$, we have

$$m = \frac{\|\psi^*\|^2_{L^2}}{\sigma + 1} = \frac{\|(-\Delta)^{\frac{s}{2}} \psi^*\|^2_{L^2}}{\sigma + 1}$$

and

$$\|\psi^*\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} \psi^*\|_{L^2}.$$  

Now, set $Q(x) = \lambda\psi^*(\mu x)$, where

$$\left\{ \begin{array}{l}
\lambda = \left(\frac{\sigma N + b}{2s(\sigma + 1) - (\sigma N + b)}\right)^\frac{2\sigma - b}{2s}\left(\frac{2s}{\sigma N + b}\right)^\frac{1}{2s}, \\
\mu = \left(\frac{\sigma N + b}{2s(\sigma + 1) - (\sigma N + b)}\right)^\frac{1}{2s},
\end{array} \right.$$  

so $Q$ is a solution of (4) and

$$\|Q\|_{L^2} = \frac{\lambda}{\mu^\frac{s}{2}} \|\psi^*\|_{L^2}. \quad (9)$$  

By the definition of $m$ and relation (9), we have

$$K_{opt} = \frac{1}{m} = \frac{\sigma + 1}{\|\psi^*\|^2_{L^2}} = \frac{\sigma N + b}{2s(\sigma + 1) - (\sigma N + b)} \left(\frac{2s(\sigma + 1)}{\sigma N + b}\right)^\frac{2s - (b + N\sigma)}{2s} \frac{2s(\sigma + 1)}{b + N\sigma} \|Q\|^2_{L^2},$$

which implies (3).

To finish the proof, we need to show the relation (5) and (6). Indeed, the definition of $Q$ yields

$$\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2} = \lambda \mu^{s - \frac{2s}{\sigma}} \|(-\Delta)^{\frac{s}{2}} \psi^*\|_{L^2}.$$  

Moreover, since $\|\psi^*\|_{L^2} = \|(-\Delta)^{\frac{s}{2}} \psi^*\|_{L^2}$, we obtain

$$\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2} = \mu^{s - \frac{2s}{\sigma}} \|Q\|_{L^2} = \left(\frac{\sigma N + b}{2s(\sigma + 1) - (\sigma N + b)}\right)^\frac{1}{2s} \|Q\|_{L^2}. \quad (10)$$

On the other hand, by multiplying (4) $Q$ and integrating by parts we have

$$\int_{\mathbb{R}^N} |x|^{-b}|Q|^{2\sigma + 2} dx = \|(-\Delta)^{\frac{s}{2}} Q\|^2_{L^2} + \|Q\|^2_{L^2}.$$
So:
\[
\int_{\mathbb{R}^N} |x|^{-b}|Q|^{2\sigma+2} \, dx = \frac{2s(\sigma + 1)}{2s(\sigma + 1) - (\sigma N + b)} \|Q\|_{L^2}^2,
\]
which we use (10). This completes the proof. \(\square\)

**Remark 4.** In the case of \(L^2\)-Critical, the sharp Gagliardo-Nirenberg inequality will be
\[
\int_{\mathbb{R}^N} |x|^{-b}|u|^{\frac{4s - 2b + 2N}{N}} \, dx \leq K_{\text{opt}} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 \|u\|_{L^2}^{4s - 2b},
\]
where \(K_{\text{opt}} = \frac{N}{2s - b + N}\|Q\|_{L^2}^{2s - 2b}.

Thanks to the above sharp Gagliardo-Nirenberg inequality, we can prove the following global well-posedness result.

**Theorem 2.3.** Let \(\frac{2s - b}{N} < \sigma < \frac{2s - b}{N - 2}\), \(0 < b < \min\{2s, N\}\) and set \(s_\sigma = \frac{N}{2} - \frac{2s - b}{\sigma}\).
Suppose that \(u(t)\) is the solution of Eq. (1) with initial data \(u_0 \in H^s(\mathbb{R}^N)\) satisfying
\[
E[u_0]^{s_\sigma} M[u_0]^{s - s_\sigma} < E[Q]^{s_\sigma} M[Q]^{s - s_\sigma},
\]
and
\[
\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 \|u\|_{L^2}^{s - s_\sigma} < \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 \|Q\|_{L^2}^{s - s_\sigma},
\]
then \(u(t)\) is a global solution in \(H^s(\mathbb{R}^N)\).
Moreover, for any \(t \in \mathbb{R}\) we have
\[
\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \|u(t)\|_{L^2}^{s - s_\sigma} < \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 \|Q\|_{L^2}^{s - s_\sigma},
\]
where \(Q\) is unique positive symmetric solution of the elliptic Eq. (4).

**Proof.** By the local theory, we just need to control \(H^s(\mathbb{R}^N)\) norm of \(u(t)\) for all \(t \in \mathbb{R}\). Using the quantities \(M[u(t)], E[u(t)]\) and the sharp Gagliardo-Nirenberg inequality, we have
\[
2E[u_0] = \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \frac{1}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b}|u(t)|^{2\sigma + 2} \, dx
\]
\[
\geq \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \frac{K_{\text{opt}}}{\sigma + 1} \|u_0\|_{L^2}^{2\sigma + 2 - \frac{N\sigma + b}{\sigma}} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{N\sigma + b}{\sigma}}.
\]
Let
\[
X(t) = \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2, \quad A = 2E[u_0], \quad B = \frac{K_{\text{opt}}}{\sigma + 1} \|u_0\|_{L^2}^{2\sigma + 2 - \frac{N\sigma + b}{\sigma}}
\]
then we have
\[
X(t) - BX(t)^{\frac{N\sigma + b}{\sigma}} \leq A \quad \text{for} \quad t \in (0, T),
\]
where \(T\) is the maximum time of existence given by the local theory.
Define the function \(f(x) = x - x^{\frac{N\sigma + b}{\sigma}}\) for \(x \geq 0\), since \(\sigma > \frac{2s - b}{N}\), we have \(\text{deg}(f) > 1\). Moreover, a simple computation shows that \(f\) has a local maximum at:
\[
x_0 = \left(\frac{2s}{B(N\sigma + b)}\right)^{\frac{2s}{\sigma N + b}}
\]
with maximum value
\[
f(x_0) = \left(\frac{2s}{B(N\sigma + b)}\right)^{\frac{2s}{\sigma N + b}} \frac{N\sigma + b - 2s}{N\sigma + b}.
\]
Using the relation
\[ E[Q] = \frac{N\sigma + b - 2s}{2(N\sigma + b)} \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 = \frac{\sigma N + b - 2s}{2(2s(\sigma + 1) - (\sigma N + b))} \|Q\|_{L^2}^2, \] (14)
the condition (11) implies that \(2E[u_0] < f(x_0)\), so
\[ f(||(-\Delta)^{\frac{s}{2}} u||_{L^2}^2) \leq 2E[u_0] < f(x_0). \] (15)
Moreover, note that condition (12) is equivalent to \(||(-\Delta)^{\frac{s}{2}} u(t)||_{L^2}^2 < x_0\). If initially it holds, then the continuity of \(||(-\Delta)^{\frac{s}{2}} u(t)||_{L^2}^2\) and (15) imply that
\[ ||(-\Delta)^{\frac{s}{2}} u(t)||_{L^2}^2 < x_0 \]
for any \(t\) as long as the solution exists, which gives (13). By mass conservation, we thus proved that \(H^s(\mathbb{R}^N)\) norm of the solution \(u(t)\) is bounded, which completes the proof of Theorem 2.3.

**Remark 5.** Global existence for \(\sigma < \frac{2s}{N} - \frac{1}{2}\) will be proved in Proposition 4.4 which illustrating that \(\sigma = \frac{2s}{N} - \frac{1}{2}\) is critical exponent of the existence of finite time blowup solutions for Eq (1).

3. The blowup for \(L^2\)-supercritical and \(H^s\)-subcritical. In this section, according to the methods in [1], we investigate the blowup criterion for the case of \(L^2\)-supercritical and \(H^s\)-subcritical. Firstly, let us assume that \(\varphi : \mathbb{R}^N \rightarrow \mathbb{R}\) is a real-valued function with \(\nabla \varphi \in W^{3,\infty}(\mathbb{R}^N)\). We define the localized virial identity of \(u = u(t, x)\) to be the quantity given by
\[
\mathcal{W}_\varphi[u(t)] := 2\Im \int_{\mathbb{R}^N} \bar{u}(t) \nabla \varphi \cdot \nabla u(t) dx = 2\Im \int_{\mathbb{R}^N} \bar{u}(t) \partial_k \varphi \partial_k u(t) dx. \] (16)
Recall that we use the convention by summing over repeated indices form 1 to \(N\).

By applying Lemma 3.1, we derive the bound
\[
\mathcal{W}_\varphi[u(t)] \leq C(\|\nabla \varphi\|_{L^\infty} \|\Delta \varphi\|_{L^\infty}) \|u(t)\|_{H^{s,\frac{3}{2}}}^2.
\]

Hence the quantity \(\mathcal{W}_\varphi[u(t)]\) is well-defined due to \(u(t) \in H^s(\mathbb{R}^N)\) with some \(s \geq \frac{1}{2}\) by assumption.

To investigate the time evolution of \(\mathcal{W}_\varphi[u(t)]\), we shall need the following auxiliary function
\[
u_m := c_s \frac{1}{-\Delta + m} u(t) = c_s^{-1}(\frac{\bar{u}(t, \xi)}{\xi^2 + m}), \quad \text{with} \quad m > 0, \quad (17)
\]
where the constant
\[
c_s := \sqrt{\frac{\sin \pi s}{\pi}} \quad (18)
\]
turns out to be a convenient normalization factor. By the smoothing properties of \((\Delta + m)^{-1}\), we clearly obtain that \(u_m(t) \in H^{\alpha+2}(\mathbb{R}^N)\) holds for any \(t \in [0, T]\) whenever \(u(t) \in H^\alpha(\mathbb{R}^N)\).

Before the proof of blowup result, we give the following lemmas.

**Lemma 3.1 ([1]).** Let \(N \geq 1\) and suppose \(\varphi : \mathbb{R}^N \rightarrow \mathbb{R}\) is such that \(\nabla \varphi \in W^{1,\infty}(\mathbb{R}^N)\). Then, for all \(u \in H^{s,\frac{3}{2}}(\mathbb{R}^N)\), it holds that
\[
\int_{\mathbb{R}^N} \varphi(x) \nabla \varphi(x) \cdot \nabla u(x) dx \leq C(||\nabla^{\frac{s}{2}} u||_{L^2} + ||u||_{L^2} ||\nabla^{\frac{s}{2}} u||_{L^2}),
\]
with some constant \(C > 0\) that depends only on ||\(\nabla \varphi||_{W^{1, \infty}}\) and \(N\).
Lemma 3.2 ([1]). Let $N \geq 1$, $s \in (0,1)$ and suppose $\varphi : \mathbb{R}^N \to \mathbb{R}$ with $\Delta \varphi \in W^{2,\infty}(\mathbb{R}^N)$. Then, for all $u \in L^2(\mathbb{R}^N)$, we have

$$|\int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi)| u_m|^2 dx dm| \leq C \|\Delta^2 \varphi\|_{L^\infty} \|\Delta \varphi\|_{L^\infty}^{-s} \|u\|_{L^2}^2.$$ 

The following lemma give a formula for $\mathcal{M}_\varphi[u(t)]$ by Balakrishnan’s representation formula (21) for $(-\Delta)^s$.

Lemma 3.3. For any $t \in [0, T)$, we have the identity

$$\frac{d}{dt} \mathcal{M}_\varphi[u(t)] = \int_0^\infty m^s \int_{\mathbb{R}^N} \left\{ 4 \partial_k u_m (\partial_k \varphi) \partial_l u_m - (\Delta^2 \varphi)| u_m |^2 \right\} dx dm$$

$$- \frac{2\sigma}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma + 2} \Delta \varphi dx - \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} x \cdot \nabla \varphi |x|^{-b - 2} |u|^{2\sigma + 2} dx$$

where $u_m = u_m(t,x)$ is defined in (17) above.

Remark 6. From the definition of $u_m$, we have

$$\int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 dx dm = \int_{\mathbb{R}^N} \left( \frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s dm}{(|\xi|^2 + m)^2} \right) |\xi|^2 |\hat{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^N} (s|\xi|^{2s - 2}) |\xi|^2 |\hat{u}(\xi)|^2 d\xi = s \|(-\Delta)^{\frac{2}{2}} u(t)\|^2_{L^2}$$

denotes the commutator of $X$ and $Y$.

Proof. Define the (formally) self-adjoint differential operator

$$\Gamma_\varphi := -i(\nabla \varphi \cdot \nabla + \nabla \cdot \nabla \varphi)$$

which acts on functions according to

$$\Gamma_\varphi f = -i(\nabla \varphi \cdot \nabla f + \nabla \cdot ((\nabla \varphi) f)) = -i(2\nabla \varphi \cdot \nabla f + f \Delta \varphi).$$

Using the Leibniz rule, we readily check that

$$[-\Delta, i\Gamma_\varphi] = -4 \partial_k (\partial_k \varphi) \partial_l - \Delta^2 \varphi, \tag{19}$$

where we recall that $[X,Y] \equiv XY - YX$ denotes the commutator of $X$ and $Y$. Also, we can get

$$\mathcal{M}_\varphi[u(t)] = \langle u(t), \Gamma_\varphi u(t) \rangle.$$ 

By taking the time derivative and using the equation satisfied by $u(t)$, we get

$$\frac{d}{dt} \mathcal{M}_\varphi[u(t)] = \langle u(t), (-\Delta)^s, i\Gamma_\varphi u(t) \rangle + \langle u(t), [-|x|^{-b}|u|^{2s}, i\Gamma_\varphi u(t) \rangle. \tag{20}$$

Since $u(t) \in H^{2s}(\mathbb{R}^N)$, we have $(-\Delta)^s u(t) \in L^2(\mathbb{R}^N)$ and $\Gamma_\varphi u(t) \in H^{2s-1}(\mathbb{R}^N)$ for $s \geq \frac{1}{2}$. In particular, the terms above are well-defined a-priori. Next, we discuss the terms on the right side of (20) separately as follows.

Step1 (Dispersive Term). For $s \in (0,1)$, we have the formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s - 1} \frac{-\Delta}{-\Delta + m} dm, \tag{21}$$

which follows from spectral calculus applied to the self-adjoint operator $-\Delta$ and the formula $x^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s - 1} \frac{x}{x + m} dm$ hold for any real number $x > 0$ and $s \in (0,1)$.

In semigroup theory, the formula (21) usually goes by the name Balakrishnan’s formula. Therefore, we have

$$[(-\Delta)^s, i\Gamma_\varphi] = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s - 1} \frac{1}{-\Delta + m} [-\Delta, i\Gamma_\varphi] \frac{1}{-\Delta + m} dm. \tag{22}$$
which we use the formula (21). Let us now apply the formal identities above to the situation at hand. Indeed, let us first assume that $u \in C_c^\infty(\mathbb{R}^N)$ holds. We claim that

$$
(u(t), ((-\Delta)^s, i\Gamma_{\varphi})u(t)) = \int_0^\infty m^s \int_{\mathbb{R}^N} \{4\partial_k u_m(\partial^2_{k\ell}\varphi)\partial_l u_m - (\Delta^2 \varphi)|u_m|^2\} dx dm,
$$

where we use the definition of $u_m$ in (18). Now, for $u \in C_c^\infty(\mathbb{R}^N)$, we can apply formula (21) (where the $m-$integral is a convergent Bochner integral) to express $(-\Delta)^s u$. Furthermore, it is legitimate to use (22) with (19) and. Using Fubini’s theorem, we arrive at (23) provided that $u \in C_c^\infty(\mathbb{R}^N)$. As a next step, we extend the identity (23) to any $u \in H^{2s}(\mathbb{R}^N)$ by the approximation argument similar with Lemma 2.1 in [1]. So, we omit it.

**Step 2 (Nonlinear Term).**

$$
\langle u, [-|x|^{-b}|u|^{2\sigma}, i\Gamma_{\varphi}]u \rangle = -(u, [x|^{-b}|u|^{2\sigma}, \nabla \varphi \cdot \nabla + \nabla \cdot \nabla \varphi]u) \\
= 2\int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma} \nabla \varphi \nabla |u|^{2\sigma} dx + 2\int_{\mathbb{R}^N} |x|^{-b} \nabla \varphi |u|^{2\sigma + 2} dx \\
= -\frac{2\sigma}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma + 2} \Delta \varphi dx - \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b - \frac{2b}{\sigma}} \cdot \nabla \varphi |u|^{2\sigma + 2} dx
$$

where we made use of the identity

$$
\nabla(|x|^{-b}) = -b|x|^{-b - \frac{2b}{\sigma}} \text{ and } \nabla(|u|^{2\sigma + 2}) = \frac{\sigma + 1}{\sigma} \nabla(|u|^{2\sigma})|u|^2.
$$

This completes the proof. 

For the time evolution of the localized virial $\mathfrak{M}_\varphi[u(t)]$ with $\varphi_R$ as above, we have the following estimate.

**Lemma 3.4. (Localized Radial Virial Estimate)** Let $N \geq 2, s \in (\frac{1}{2}, 1)$ and assume in addition that $u(t, x)$ is a radial solution of (1). We then have

$$
\frac{d}{dt} \mathfrak{M}_\varphi[u(t)] \leq (4\sigma N + 4b)E[u_0] - 2(\sigma N + b - 2s)\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 \\
+ C(R^{-2s} + R^{-b - \sigma(N - 1) + \epsilon})(\|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^{\frac{s}{2} + \epsilon})^2
$$

for any $0 < \epsilon < \frac{\sigma(2s - 1)}{s}$. Here $C = C(\|u_0\|_{L^2}, \epsilon, s, \sigma, b)$ is some positive constant.

**Proof.** The proof is similar to the one in [1]. We provide it for the sake of completeness. Let $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ be as above. In addition, we assume that $\varphi = \varphi(r)$ is radial and satisfies

$$
\varphi(r) = \begin{cases} 
\frac{r^2}{4} & \text{for } r \leq 1, \\
\text{const.} & \text{for } r \geq 10,
\end{cases}
$$

and $\varphi''(r) \leq 1$ for $r \geq 0$. For $R > 0$ given, we define the rescaled function $\varphi_R : \mathbb{R}^N \rightarrow \mathbb{R}$ by setting

$$
\varphi_R := R^2 \varphi\left(\frac{r}{R}\right).
$$

We readily verify the inequalities

$$
1 - \varphi''_R(r) \geq 0, \ 1 - \frac{\varphi'_R(r)}{r} \geq 0, \ N - \Delta \varphi_R(r) \geq 0, \ 1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} \geq 0
$$

for all $r \geq 0$. Indeed, this first inequality follows from $\varphi''_R(r) = \varphi''(\frac{r}{R}) \leq 1$. We obtain the second inequality by integrating the first inequality on $[0, r]$ and using $\varphi'_R(0) = 0$. For the third inequality, we find that $N - \Delta \varphi_R(r) = 1 + \varphi''_R(r) + (N - 1)(1 - \frac{1}{r}\varphi'_R(r)) \geq 0$ holds thanks to the first two inequalities. Finally, we noticed that

$$
\frac{x \cdot \nabla \varphi_R(r)}{|x|^2} = \frac{R\varphi'_R(\frac{r}{R})}{|x|^2} = \frac{R}{r} \varphi'_R(\frac{r}{R}).
$$

So, $1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} \geq 0$ due to $\varphi'_R(r) \leq r$.

For later use, we record the following properties of $\varphi_R$, which can be easily checked:

$$
\varphi(r) = \begin{cases} 
\nabla \varphi(r) = R\varphi'_R(\frac{r}{R}) \frac{x}{|x|} & \text{for } r \leq R, \\
\|\nabla^j \varphi\|_{L^\infty} \leq CR^{2-j} & \text{for } 0 \leq j \leq 4, \\
\text{supp}(\nabla^j \varphi) \subset \{|x| \leq 10R\} & \text{for } j = 1, 2, \\
\{|R \leq |x| \leq 10R\} & \text{for } j = 3, 4.
\end{cases}
$$

Noted that the Hessian of a radial function $f : \mathbb{R}^N \to \mathbb{C}$ can be written as

$$
\partial^2_{kl}f = (\delta_{kl} - \frac{x_k x_l}{r^2}) \frac{\partial f}{r} + \frac{x_k x_l}{r^2} \partial^2_{kl}f.
$$

Thus, we can rewrite the first term on the right-hand side in Lemma 3.3 as follows.

$$
4 \int_0^\infty m^s \int_{\mathbb{R}^N} \partial_k u_m(\partial^2_{kl} \varphi) \partial_l u_m dx dm = 4 \int_0^\infty m^s \int_{\mathbb{R}^N} \partial^2_r \varphi R |\nabla u_m|^2 dx dm.
$$

Recalling Remark 6, we deduce that

$$
4 \int_0^\infty m^s \int_{\mathbb{R}^N} \partial_k u_m(\partial^2_{kl} \varphi) \partial_l u_m dx dm
$$

$$
= 4s\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - 4 \int_0^\infty m^s \int_{\mathbb{R}^N} (1 - \partial^2 R |\nabla u_m|^2) dx dm
$$

$$
\leq 4s\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2.
$$

Moreover, from Lemma 3.2 we have the bound

$$
| \int_{\mathbb{R}^N} (\Delta^2 \varphi_R) |u_m|^2 dx dm | \leq C\|\Delta^2 \varphi_R\|_{L^\infty}^s \|\Delta \varphi_R\|_{L^\infty}^{1-s} \|u\|_{L^2}^2,
$$

where we also used the properties of $\varphi_R$ and the conservation of $L^2$-mass of $u(t)$.

Due to $N - \Delta \varphi_R(r) \equiv 0$ and $1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} \equiv 0$ on $\{r \leq R\}$, we obtain that

$$
- \frac{2\sigma}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} \Delta \varphi_R dx
$$

$$
= - \frac{2\sigma N}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} dx - \frac{2\sigma}{\sigma + 1} \int_{|x| \geq R} |x|^{-b} |u|^{2\sigma+2} (\Delta \varphi_R - N) dx
$$

and

$$
- \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} (x \cdot \nabla \varphi_R) |x|^{-b-2} |u|^{2\sigma+2} dx
$$

$$
= - \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} dx - \frac{2b}{\sigma + 1} \int_{|x| \geq R} \left( \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} - 1 \right) |x|^{-b} |u|^{2\sigma+2} dx.
$$
From [10], we have the following fractional Sobolev (generalized Strauss) inequality 
\[ \sup_{x \neq 0} |x|^\frac{\alpha}{s} \|u(x)\| \leq C(N, \alpha)(\|N\|^2 u \|L^2 \)
\]
for all radial functions \( u \in \mathcal{H}^\alpha(\mathbb{R}^N) \) provided that \( \frac{1}{2} < \alpha < \frac{N}{2} \). Now, let \( 0 < \epsilon < \frac{\alpha(2s-1)}{2} \) and set \( \alpha = \frac{1}{2} + \epsilon \cdot \frac{N}{2s} \), which implies \( \frac{1}{2} < \alpha < s < \frac{N}{2} \). From the interpolation inequality \( \|(-\Delta)^{\frac{s}{2}} u \|_{L^2} \leq \|u\|_{L^2}^{1-\frac{s}{2}} \|(-\Delta)^{\frac{s}{2}} u \|_{L^2}^{\frac{s}{2}} \) and Strauss’s inequality, we deduce
\[
| - \frac{2\sigma}{\sigma+1} \int_{|x| \geq R} |x|^{-b} |u|^{2\sigma+2} (\Delta \varphi_R - N) dx | 
\leq C \int_{|x| \geq R} |x|^{-b} |u|^{2\sigma+2} dx 
\leq CR^-b \int_{|x| \geq R} |u|^{2\sigma+2} dx 
\leq C(N, \alpha, \epsilon) R^{-b-2\alpha(\frac{1}{2} - \epsilon)} \|(-\Delta)^{\frac{s}{2}} u \|_{L^2}^{2\alpha} 
\leq C(N, \alpha, \epsilon) R^{-b-\alpha(N-1)+\epsilon s} \|(-\Delta)^{\frac{s}{2}} u \|_{L^2}^{\frac{s}{2} + \epsilon}.
\]
By the similar method, we deduce
\[
| - \frac{2b}{\sigma+1} \int_{|x| \geq R} \left( \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} - 1 \right) |x|^{-b} |u|^{2\sigma+2} dx | 
\leq C(N, b, \alpha, \epsilon) R^{-b-\alpha(N-1)+\epsilon s} \|(-\Delta)^{\frac{s}{2}} u \|_{L^2}^{\frac{s}{2} + \epsilon}.
\]
In summary, we have shown that
\[
\frac{d}{dt} \mathcal{M}_u[u(t)] 
\leq 4s \|(-\Delta)^{\frac{s}{2}} u(t) \|_{L^2}^2 - \frac{2\sigma N}{\sigma+1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} dx - \frac{2b}{\sigma+1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} dx 
+ C(R^{-2s} + R^{-b-\alpha(N-1)+\epsilon s}) \|(-\Delta)^{\frac{s}{2}} u \|_{L^2}^{\frac{s}{2} + \epsilon} 
= (4\sigma N + 4b) E[u_0] - 2(\sigma N + b - 2s) \|(-\Delta)^{\frac{s}{2}} u(t) \|_{L^2}^2 
+ C(R^{-2s} + R^{-b-\alpha(N-1)+\epsilon s}) \|(-\Delta)^{\frac{s}{2}} u \|_{L^2}^{\frac{s}{2} + \epsilon}
\]
for any \( 0 < \epsilon < \frac{\alpha(2s-1)}{2s} \) with some constant \( C = C(\|u_0\|_{L^2}, N, s, \alpha, \sigma, b) > 0 \). Note that we use the conservation of energy \( E[u(t)] \) in the last step. The proof of Lemma 3.4 is complete.

By the above localized radial Virial estimate, we can prove the following blowup result.

Theorem 3.5. (\( L^2 \)-Supercritical and \( H^s \)-Subcritical) Let \( N \geq 2, s \in (\frac{1}{2}, 1), \frac{2s-1}{N-2s} < \sigma < \frac{2s-b}{N-2s}, 0 < b < \min \{ 2s, N \} \) with \( \sigma < 2s \) and set \( s_\sigma = \frac{N}{2} - \frac{2s-b}{2} \sigma \). Suppose that \( u(t) \in C([0,T); H^s(\mathbb{R}^N)) \) is a radial solution of Eq.(1). Furthermore, we assume that \( E[u_0] < 0 \),
or, if \( E[u_0] \geq 0 \), we assume that
\[
\begin{align*}
E[u_0]^{s_\sigma} M[u_0]^{s_\sigma} &< E[Q]^{s_\sigma} M[Q]^{s_\sigma}, \\
\|(-\Delta)^{\frac{s}{2}} u_0 \|_{L^2}^2 u_0^{s_\sigma} &> \|(-\Delta)^{\frac{s}{2}} Q \|_{L^2}^2 Q^{s_\sigma},
\end{align*}
\]
then \( u(t) \) blows up in finite time in the sense that \( T < +\infty \) must hold.
Thus we deduce that (25) holds. So, we find

\[ \text{proven by M.I. Weinstein [31].} \]

**Remark 7.** This theorem can be viewed as an unified blowup theory result for both inhomogeneous nonlinear Schrödinger equation and fractional nonlinear Schrödinger equation. Indeed, if \( s = 1 \) we deduce L.G. Farah’s result [11], if \( b = 0 \) our result is consistent with [1] and, finally, if \( s = 1 \) and \( b = 0 \) we obtain the classical result proved by M.I. Weinstein [31].

**Proof. Case 1:** \( E[u_0] < 0 \). Let us define \( \delta := \sigma N + b - 2s > 0 \). From Lemma 3.4 with \( \epsilon > 0 \) sufficiently small and fixed, we deduce the inequality (with \( \sigma_R(1) \to 0 \) as \( R \to +\infty \) uniformly in \( t \))

\[
\frac{d}{dt} M_\varphi[u(t)] \leq (4\sigma N + 4b)E[u_0] - 2\delta\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 + \sigma_R(1)(1 + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}^{\frac{2}{s} - \epsilon}) \leq 2(\sigma N + b)E[u_0] - \delta\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^2 \quad \text{for all } t \in [0, T),
\]

provided that \( R \gg 1 \) is taken sufficiently large. In the last step, we used that \( E[u_0] < 0 \), and that \( s > \epsilon < 2 \) when \( \epsilon > 0 \) is sufficiently small and \( \sigma < 2s \).

With estimate (24) at hand, we can now adapt the strategy of T. Ogawa-Y. Tsutsumi [1] to our problem. Suppose \( u(t) \) exists for all time \( t \geq 0 \), i.e., we can take \( T = +\infty \). From (24) it follows that \( \frac{d}{dt} M_\varphi[u(t)] \leq -c \) with some constant \( c > 0 \). By integrating this bound, we conclude that \( M_\varphi[u(t)] < 0 \) for all \( t \geq t_1 \) with some time sufficiently large time \( t_1 \gg 1 \). Thus, if we integrate (24) on \([t_1, t]\), we obtain

\[
M_\varphi[u(t)] \leq -\delta \int_{t_1}^t \|(-\Delta)^{\frac{s}{2}}u(\tau)\|_{L^2}^2 d\tau \quad \text{for all } t \geq t_1.
\]

On the other hand, we use Lemma 3.1 and \( L^2 \)-mass conservation to find that

\[
\| M_\varphi[u(t)] \| \leq C(\varphi_R)(\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{1}{2}} + \|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{1}{2}}),
\]

where we used the interpolation estimate \( \|\nabla U\|_{L^2} \leq \|U\|_{L^2}^{1-s/2} \|(-\Delta)^{\frac{s}{2}}U\|_{L^2}^{s/2} \) for \( s > \frac{1}{2} \). Next, we claim the lower bound

\[
\|(-\Delta)^{\frac{s}{2}}u\|_{L^2} \geq C \quad \text{for } t \geq 0.
\]

Indeed, suppose this bound is not true, we have that \( \|(-\Delta)^{\frac{s}{2}}u(t_k)\|_{L^2} \to 0 \) for some sequence of time \( t_k \in [1, \infty) \). However, by \( L^2 \)-mass conservation and the sharp Gagliardo-Nirenberg inequality, this implies that \( \int_{\mathbb{R}^N} |x|^{-b}|u|^{2s+2} dx \to 0 \) as well. Hence we get \( E[u(t_k)] \to 0 \), which is a contradiction to \( H^1 \)-energy conservation. Thus we deduce that (25) holds. So, we find

\[
\| M_\varphi[u(t)] \| \leq C(\varphi_R)(\|(-\Delta)^{\frac{s}{2}}u(t)\|_{L^2}^{\frac{1}{2}}).
\]

Thus we conclude that

\[
M_\varphi[u(t)] \leq -C(\varphi_R) \int_{t_1}^t \| M_\varphi[u(\tau)] \|_{L^2}^{\frac{2s}{s}} d\tau \quad \text{for } t \geq t_1.
\]

So, we have \( M_\varphi[u(t)] \leq -C(\varphi_R)|t-t_*|^{1-2s} \) for \( s > \frac{1}{2} \) with some \( t_* < +\infty \). Therefore we have \( M_\varphi[u(t)] \to -\infty \) as \( t \to t_* \). Hence the solution \( u(t) \) cannot exist for all time \( t \geq 0 \) and consequently we must have that \( T < +\infty \) holds.
Case 2: $E[u_0] \geq 0$. From the conservation of energy and $L^2$-mass combined with Gagliardo-Nirenberg inequality we get

$$E[u_0] = \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u(t)\|^2_{L^2} - \frac{1}{2(\sigma + 1)} \int_{\mathbb{R}^N} |x|^{-b} |u(t)|^{2\sigma + 2} dx$$

$$\geq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u(t)\|^2_{L^2} - \frac{K_{opt}}{2(\sigma + 1)} \|u_0\|^{2\sigma + 2-N_{\sigma+b}} \|(-\Delta)^{\frac{s}{2}} u(t)\|^{N_{\sigma+b}}_{L^2}$$

$$= F(\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}).$$

Where the function $F : [0, +\infty) \to \mathbb{R}$ is defined as

$$F(y) = \frac{1}{2} y^2 - \frac{K_{opt}}{2(\sigma + 1)} (M[u_0])^{\sigma + 1-N_{\sigma+b}} y^{\frac{N_{\sigma+b}}{\sigma}}$$

$$= \frac{1}{2} y^2 - \frac{K_{opt}}{2(\sigma + 1)} (M[u_0])^{\frac{s}{2}(s-s)} y^{2+2\sigma - s}. $$

We readily verify that $F(y)$ has a unique global maximus

$$F(y_{max}) = \frac{N\sigma + b - 2s}{2(N\sigma + b)} y_{max}^2,$$

which is attained at

$$y_{max} = \left(\frac{2s(\sigma + 1)}{(N\sigma + b)K_{opt}}\right)^{\frac{s}{2}} \|u_0\|^{\frac{2}{\sigma}}_{L^2}.$$ 

Thus, the condition of Case 2 tell us that

$$E[u_0] < F(y_{max}) \text{ and } \|(-\Delta)^{\frac{s}{2}} u_0\|_{L^2} > y_{max},$$

where we used the relations

$$E[Q] = \frac{N\sigma + b - 2s}{2(N\sigma + b)} \|(-\Delta)^{\frac{s}{2}} Q\|^2_{L^2} = \frac{N\sigma + b - 2s}{2s(\sigma + 1) - (N\sigma + b)} \|Q\|^2_{L^2}. \quad (28)$$

By continuity in time, we deduce that

$$\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} > y_{max} \text{ for all } t \in [0, T). \quad (29)$$

Indeed, suppose this bound was not true, there is some time $t_\ast \in (0, T)$ such that $\|(-\Delta)^{\frac{s}{2}} u(t_\ast)\|_{L^2} = y_{max}$ by continuity. But this is contradicts

$$E[u_0] \geq F(\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}).$$

Therefore the bound (29) holds.

Picking $\eta > 0$ sufficiently small, we obtain

$$E[u_0] \leq (1 - \eta) E[Q] M[Q]^{\frac{s}{2}} M[u_0]^{-\frac{s}{2}}$$

$$= (1 - \eta) \frac{N\sigma + b - 2s}{2s(\sigma + 1) - (N\sigma + b)} \|Q\|^2_{L^2} \|u_0\|^{-\frac{s}{\sigma}}_{L^2}$$

$$= (1 - \eta) \frac{N\sigma + b - 2s}{2(N\sigma + b)} y_{max}^2.$$

So, we have

$$\delta(1 - \eta) \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \geq 2(N\sigma + b)E[u_0] \text{ for all } t \in [0, T), \quad (30)$$
due to the relations (28) and the definition of $K_{opt}$. By inserting this bound into the differential inequality from Lemma 3.3, we get
\[
\frac{d}{dt} \mathcal{M}_\varphi[u(t)] \leq 4(\sigma N + b)E[u_0] - 2\delta \|(-\Delta)^{\kappa} u(t)\|_{L^2}^2 + \sigma_R(1)(1 + \|(-\Delta)^{\kappa} u\|_{L^2}^{2+\epsilon}) \\
\leq -(\delta + \sigma_R(1))\|(-\Delta)^{\kappa} u(t)\|_{L^2}^2 + \sigma_R(1),
\]
provided that $R \gg 1$ is taken sufficiently large. In the last step, we used that $E[u_0] < 0$, and that $\frac{\sigma}{r} + \epsilon < 2$ when $\epsilon > 0$ is sufficiently small and $\sigma < 2s$. By following exactly the steps after (24) above, we deduce that $u(t)$ cannot exist for all time $t \geq 0$.

The proof of Theorem 3.5 is complete.

4. The blowup for $L^2$-Critical. In this section, we give the blowup result in the case of $L^2$-Critical i.e., $\sigma = \frac{2s-4}{N}$. Firstly, we shall need the following refined of Lemma 3.3 involving the nonnegative radial functions
\[
\psi_1 = 1 - \varphi''_R(r) \geq 0, \quad \psi_2 = N - \Delta \varphi_R(r) \geq 0, \quad \text{and} \quad \psi_3 = 1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} \geq 0.
\]

Lemma 4.1. (A Refined Version of Lemma 3.4) Let $N \geq 2, s \in (\frac{1}{2}, 1)$ and assume in addition that $u(t, x)$ is a radial solution of (1) for any $t \in [0, T)$ and $\sigma = \frac{2s-4}{N}$. We then have
\[
\frac{d}{dt} \mathcal{M}_\varphi[u(t)] \leq 8sE[u_0] - 4\int_0^\infty m^s \int_{\mathbb{R}^N} (\psi_1 - C_2(\eta)\psi_2^{\frac{N}{\sigma}} - C_3(\eta)\psi_3^{\frac{N}{\sigma}})\|
abla u_m\|^2 dx dm \\
+ O((1 + \eta^{-\beta})R^{-2s} + \eta(1 + R^{-2} + R^{-4})),
\]
for any $\eta > 0$ and $R > 0$, where $C_2(\eta) = \frac{(2s-b)\eta}{2s(N+2s-b)}$, $C_3(\eta) = \frac{N\eta(N-1)}{2s(N+2s-b)}$ and
\[
\beta = \frac{2s-b}{N-2s+b}.
\]

Remark 8. By Plancherel’s identity, we have following estimates
\[
\|\frac{\Delta}{-\Delta + m}\|_{L^2 \to L^2} \leq C, \quad \|\nabla \frac{\Delta}{-\Delta + m}\|_{L^2 \to L^2} \leq Cm^{-\frac{1}{2}}, \quad \|\frac{1}{-\Delta + m}\|_{L^2 \to L^2} \leq Cm^{-1}.
\]

Proof. The proof is similar to the one in [1]. We provide it for the sake of completeness. Inspecting the proof of Lemma 3.4, we immediately get
\[
\frac{d}{dt} \mathcal{M}_\varphi[u(t)] \\
= 4\int_0^\infty m^s \int_{\mathbb{R}^N} (\partial_s^2 \varphi_R)|\nabla u_m|^2 dx dm - \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi_R)|u_m|^2 dx dm \\
- \frac{2s}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} \Delta \varphi_R dx - \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} |x|^{-b} |u|^{2\sigma+2} dx \\
= 4s\|(-\Delta)^{\kappa} u(t)\|_{L^2}^{2} - 4\int_0^\infty m^s \int_{\mathbb{R}^N} (1 - \partial_s^2 \varphi_R)|\nabla u_m|^2 dx dm \\
- \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi_R)|u_m|^2 dx dm + \frac{2\sigma}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} (N - \Delta \varphi_R) dx \\
+ \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} (1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2}) |x|^{-b} |u|^{2\sigma+2} dx \\
- \frac{2N\sigma}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} dx - \frac{2b}{\sigma + 1} \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} dx.
\[ = 8sE[u_0] - 4 \int_0^\infty m^s \int_{\mathbb{R}^N} \psi_1 |\nabla u_m|^2 \, dx \, dm \]

\[ - \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi_R)|u_m|^2 \, dx \, dm + \frac{2(2s - b)}{N + 2s - b} \int_{\mathbb{R}^N} \psi_2 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx \]

\[ + \frac{2Nb}{N + 2s - b} \int_{\mathbb{R}^N} \psi_3 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx \]

\[ = 8sE[u_0] - 4 \int_0^\infty m^s \int_{\mathbb{R}^N} \psi_1 |\nabla u_m|^2 \, dx \, dm \]

\[ - \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2 \varphi_R)|u_m|^2 \, dx \, dm + \frac{2(2s - b)}{N + 2s - b} \int_{\mathbb{R}^N} \psi_2 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx \]

\[ + \frac{2Nb}{N + 2s - b} \int_{\mathbb{R}^N} \psi_3 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx + \mathcal{O}(R^{-2s}). \]

We divide the rest of the proof into following steps:

**Step 1 (Control of Nonlinearity).**

Recall that \( \text{supp} \psi_2 \in \{|x| \geq R\} \). We apply Strauss’s inequality to the radial function \( \psi_2^{\frac{2s-b}{N}} u \in H^s(\mathbb{R}^N) \) and use that \( ||u||_{L^2} \leq C \). Which together yields

\[ \int_{\mathbb{R}^N} \psi_2 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx \]

\[ \leq R^{-b} ||\psi_2^{\frac{2s-b}{N}} u||_{L^\infty \{|x| \geq R\}} ||u||_{L^2}^2 \]

\[ \leq CR^{-b} ||\psi_2^{\frac{2s-b}{N}} (\psi_2^{\frac{2s-b}{N}} u)||_{L^{\frac{2s-b}{N}}}^\infty \]

\[ \leq \eta ||(\Delta)^{\frac{b}{2}} (\psi_2^{\frac{2s-b}{N}} u)||_{L^2}^2 + \mathcal{O}(\eta^{-\beta} R^{-2s}), \]

where \( \beta = \frac{2s-b}{N-2s+b} \) and we used Young’s inequality \( ab \leq C(\eta a^q + \eta^{-\frac{q}{p}} b^p) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) such that \( q = \frac{N}{2s-b}, p = \frac{N}{N-2s+b} \), \( \beta = \frac{p}{q} = \frac{2s-b}{N-2s+b} \) and \( \eta > 0 \) is an arbitrary number.

By the same method, we have

\[ \int_{\mathbb{R}^N} \psi_3 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx \]

\[ = \int_{\mathbb{R}^N} \psi_3 |x|^{-b} |u|^{ \frac{2(2s-b)}{N} } + 2 \, dx \]

\[ \leq CR^{-b} ||\psi_3^{\frac{2s-b}{N}} (\psi_3^{\frac{2s-b}{N}} u)||_{L^{\frac{2s-b}{N}}}^\infty \]

\[ \leq \eta ||(\Delta)^{\frac{b}{2}} (\psi_2^{\frac{2s-b}{N}} u)||_{L^2}^2 + \mathcal{O}(\eta^{-\beta} R^{-2s}), \]

where \( \beta = \frac{2s-b}{N-2s+b} \).

For notational convenience, let us define \( \Psi_i = \psi_i^{\frac{N}{2s-b}} \), and we have

\[ ||(\Delta)^{\frac{b}{2}} (\Psi_i u)||_{L^2}^2 = \int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla (\Psi_i u)_m|^2 \, dx \, dm, \]

where \( i = 2, 3 \) and we denote

\[ (\Psi_i u)_m = c_s \frac{1}{\Delta + m} (\Psi_i u) \]
as in (17) above. To estimate the right-hand side of (32), we split the \( m \)-integral in the regions \( \{ 0 < m \leq 1 \} \) (low frequencies) and \( \{ m \geq 1 \} \) (high frequencies).

To estimate the contribution in the low-frequency region, we notice that

\[
| \int_0^1 m^s \int_{\mathbb{R}^N} \frac{\nabla}{-\Delta + m} (\Psi, u)^2 \, dx \, dm | \leq \int_0^1 m^{s-1} \| \Psi, u \|^2_{L^2} \, dm \leq C,
\]

where we make use of the Remark 8 and \( \| \Psi_i \|_{L^\infty} \leq C (i = 2, 3) \). To control the right-hand side of (32) in the high frequency region \( m \geq 1 \), we need a more elaborate argument worked out in the next step,

**Step 2 (Control of Nonlinearity).**

By using the commutator identity \( [\frac{1}{-\Delta + m}, \Psi_i] = \frac{1}{-\Delta + m} [\Delta, \Psi_i] - \frac{1}{-\Delta + m} \), we conclude

\[
\nabla (\Psi, u)_m = \Psi_i \nabla u_m + \nabla \Psi_i u_m + \frac{\nabla}{-\Delta + m} [\Delta, \Psi_i] u_m.
\]

Thus, we get

\[
| \int_1^\infty m^s \int_{\mathbb{R}^N} \frac{\nabla}{-\Delta + m} [\Delta, \Psi_i] u_m \, dx \, dm | \leq C \int_1^\infty m^{s-1} \| \Psi_i \|^2_{L^2} \, dm
\]

\[
\leq C \int_1^\infty m^{s-1} (\| \nabla \Psi_i \|^2_{L^\infty} + \| \Delta \Psi_i u_m \|^2_{L^2}) \, dm
\]

\[
\leq C \int_1^\infty (m^{s-2} \| \nabla \Psi_i \|^2_{L^\infty} + m^{s-3} \| \Delta \Psi_i \|^2_{L^\infty}) \, dm
\]

\[
\leq C \left( \frac{\| \nabla \Psi_i \|^2_{L^\infty}}{1 - s} + \frac{\| \Delta \Psi_i \|^2_{L^\infty}}{2 - s} \right),
\]

where we used that \( [\Delta, \Psi_i] = 2(\nabla \Psi_i) \cdot \nabla + \Delta \Psi_i \) as well as Remark 8 and conservation of mass in the last line. Similarly, we get

\[
| \int_1^\infty m^s \int_{\mathbb{R}^N} |\nabla \Psi_i u_m|^2 \, dx \, dm | \leq C \frac{\| \nabla \Psi_i \|^2_{L^\infty}}{1 - s}.
\]

Recalling that

\[
\Psi_2 = \psi_2 (x^{-b}), \quad \Psi_2 = N - \Delta \varphi_R, \quad \Psi_3 = \psi_3 (x^{-b}), \quad \psi_3 = 1 - \frac{x \cdot \nabla \varphi_R}{|x|^2}.
\]

By the properties of \( \varphi_R \), we have

\[
\| \nabla \Psi_2 \|_{L^\infty} \leq CR^{-1}, \quad \| \Delta \Psi_2 \|_{L^\infty} \leq CR^{-2},
\]

and

\[
\nabla \Psi_3 = -\frac{N}{2(2s - b)} (1 - \frac{x \cdot \nabla \varphi_R}{|x|^2}) \nabla (\frac{x \cdot \nabla \varphi_R}{|x|^2})
\]

\[
= -\frac{N}{2(2s - b)} (1 - \frac{x \cdot \nabla \varphi_R}{|x|^2}) \nabla (\frac{x \cdot \nabla \varphi_R}{|x|^2}) + \frac{x \cdot (\partial_{xx} x \cdot \varphi_R)}{|x|^2} - \frac{2(x \cdot \nabla \varphi_R) x}{|x|^4}.
\]

So, we have

\[
\| \nabla \Psi_3 \|_{L^\infty} \leq CR^{-1}, \quad \| \Delta \Psi_3 \|_{L^\infty} \leq CR^{-2}
\]
Step 3 (Conclusion).

We define the radial function

\[ \psi \leq \frac{\eta}{s} \int_0^\infty m^s \int_{\mathbb{R}^N} |\psi_2|^2 |\nabla u_m|^2 dx \, dm + \mathcal{O}(\eta^{-\beta} R^{-2s} + \eta(1 + R^{-2} + R^{-4})) \]

and

\[ \psi \leq \frac{\eta}{s} \int_0^\infty m^s \int_{\mathbb{R}^N} |\psi_3|^2 |\nabla u_m|^2 dx \, dm + \mathcal{O}(\eta^{-\beta} R^{-2s} + \eta(1 + R^{-2} + R^{-4})) \]

By inserting this back into (32) and setting \( C_2(\eta) = \frac{(2s-b)\eta}{2s(N+2s-b)} \), \( C_3(\eta) = \frac{Nbn}{2s(N+2s-b)} \), we complete the proof.

Next, we prove the blowup result by refined version of Lemma 3.4.

**Theorem 4.2.** (\( L^2 \)-Critical) Let \( N \geq 2 \), \( s \in (\frac{1}{2}, 1) \), \( \sigma = \frac{2s-b}{N} \), \( 0 < b < \min \{2s, N\} \) and \( \sigma < 2s \). Suppose that \( u(t) \in C([0,T); H^{2s}(\mathbb{R}^N)) \) is a radial solution of Eq. (1). Furthermore, we suppose that

\[ E[u_0] < 0, \]

then \( u(t) \) blows up in finite time in the sense that \( T < +\infty \) must hold, or \( u(t) \) blows up infinite time such that

\[ \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2} \geq Ct \quad \text{for all} \ t \geq t_*, \]

with some constants \( C > 0 \) and \( t_* > 0 \) that depend only on \( u_0 \), \( s \), \( N \).

**Proof.** Let \( N \geq 2 \), \( s \in (\frac{1}{2}, 1) \) and assume in addition that \( u(t, x) \) is a radial solution of (1). To construct a suitable Virial function \( \varphi(r) \) for the \( L^2 \)-critical case, we can adapt the choice made in [26] used for classical nonlinear Schrödinger equation. Let \( g \in W^{3,\infty}(\mathbb{R}^N) \) be a radial function such that

\[ g(r) = \begin{cases} 
  r & \text{for } 0 \leq r \leq 1, \\
  r - (r - 1)^3 & \text{for } 1 < r \leq 1 + \frac{1}{\sqrt{3}}, \\
  g(r) \text{ smooth and } g'(r) < 0 & \text{for } 1 + \frac{1}{\sqrt{3}} < r \leq 10, \\
  0 & \text{for } r > 10.
\end{cases} \]

We define the radial function \( \varphi(r) \) by setting

\[ \varphi(r) = \int_0^r g(s) \, ds. \]

Recall that we set \( \varphi_R := R^2 \varphi(R) \) for \( R > 0 \) given. Furthermore, recall the definitions of the nonnegative functions \( \psi_1 = 1 - \varphi_R'(r) \geq 0, \psi_2 = N - \Delta \varphi_R(r) \geq 0 \) and
ψ₃ = 1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} ≥ 0. Let C₂(η) = \frac{(2s-b)η}{2s(N+2s-b)} and C₃(η) = \frac{Nb⁴}{2s(N+2s-b)} for η > 0. We claim that if η > 0 sufficiently small and any R > 0, we have

\psi₁ - C₂(η)ψ₃^{\frac{2s-b}{N}} - C₃(η)ψ₃^{\frac{4}{N}} ≥ 0 for all r ≥ 0. \hspace{1cm} (38)

To prove (38), we argue as follows. First, by scaling, we can assume R = 1 without loss of generality. So, we have:

- In the case of 0 ≤ r ≤ 1, ψ₁ = ψ₂ = ψ₃ = 0.
- In the case of 1 < r ≤ 1 + \frac{1}{\sqrt{3}}, ψ₁ = 3(r-1)^2, |ψ₂|^{\frac{2s}{N}} = |N - \Delta \varphi₁(r)|^{\frac{2s}{N}} ≤ C(r-1)^{\frac{2s}{N}}, |ψ₃|^{\frac{2s}{N}} = 1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2} = (r-1)^3.
- In the case of 1 + \frac{1}{\sqrt{3}} < r ≤ 10, ψ₁ ≥ 1, |ψ₂| = |N - \Delta \varphi₁(r)| ≤ C, |ψ₃| = |1 - \frac{x \cdot \nabla \varphi_R(r)}{|x|^2}| ≤ C.
- In the case of r > 10, ψ₁ = ψ₂ = ψ₃ = 0.

So, (38) holds for η > 0 sufficiently small.

Thus if we choose η ≪ 1 sufficiently small and R ≫ 1 sufficiently large, we can apply Lemma 4.1 to deduce that

\frac{d}{dt} \mathcal{M}_μ[u(t)] ≤ 4sE[u₀] for all t > 0. \hspace{1cm} (39)

Next, we suppose that u(t) exists for all time t ≥ 0, i.e., we can take T = +∞. From (39) we infer that

\mathcal{M}_μ[u(t)] ≤ -ct for t > t₀, \hspace{1cm} (40)

with some sufficiently large time t₀ > 0 and some constant c > 0 depending only on s and initial energy. On the other hand, if we invoke Lemma 3.1, we see that

\begin{align*}
|\mathcal{M}_μ[u(t)]| &≤ C(\varphi_R)(\|\nabla \frac{1}{2} u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}[\|\nabla \frac{1}{2} u(t)\|_{L^2}]) \\
&≤ C(\varphi_R)[\|\nabla \frac{1}{2} u(t)\|_{L^2}^2 + 1] ≤ C(\varphi_R)[\|(-\Delta)^{\frac{1}{2}} u(t)\|_{L^2}^2 + 1],
\end{align*} \hspace{1cm} (41)

where we also used the conservation of L²-mass of u(t) together with the interpolation estimate [\|\nabla \frac{1}{2} u(t)\|_{L^2} ≤ \|u\|_{L^2}^{\frac{1}{2}} \|(-\Delta)^{\frac{1}{2}} u(t)\|_{L^2}^{\frac{1}{2}}] for s > \frac{1}{2}. By combining (40) and (41), we finally get

\|(-\Delta)^{\frac{1}{2}} u(t)\|_{L^2} ≥ Ct for all t ≥ t*,

with some constants C > 0 and t* > 0 that depend only on u₀, s, N. The proof of Theorem 4.2 is now complete.

Next, we give two propositions which imply that \|Q\|_{L^2} is the sharp threshold mass of the existence of finite-time blow-up for Eq.(1) with L²-critical exponent.

Proposition 4. Let N ≥ 2, \(\frac{1}{2} < s < 1\) and Q be the ground state solution of (4) with \(σ = \frac{2s-1}{N}\). Then the following results hold.

(i) If the initial data \(u₀ \in H^s(\mathbb{R}^N)\) and \(\|u₀\|_{L²} < \|Q\|_{L²}\), then the corresponding solution of Eq.(1) exists globally in \(H^s(\mathbb{R}^N)\).

(ii) If the initial data \(u₀ \in H^{2s} \) is radial symmetric in the form

\(u₀(0, x) = u₀ = cρ^{\frac{2s}{2}} Q(ρx), \hspace{1cm} (42)\)

where the complex number c satisfies |c| > 1 and the real number ρ > 0, then \(\|u₀\|_{L²} > \|Q\|_{L²}\), and corresponding solution \(u(t, x)\) of Eq.(1) must blow up in finite time \(0 < T < +∞\), or \(u(t)\) blows up infinite time such that

\(\|(-\Delta)^{\frac{1}{2}} u(t)\|_{L²} ≥ Ct\) for all \(t ≥ t*\).
with some constants $C > 0$ and $t_* > 0$ that depend only on $u_0, s, N$.

Proof. It follows from the conservation laws and the sharp Gagliardo-Nirenberg inequality in Remark 2.6 that for all $t \in I$ (the maximal time interval)

$$
E(u(t)) = \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2 - \frac{N}{4s - 2b + 2N} \int_{\mathbb{R}^N} |x|^{-b} |u|^{\frac{4s - 2b + 2N}{N}} dx
$$

$$
\geq \frac{1}{2} (1 - \frac{\|u_0\|_{L^2}^{\frac{4s - 2b}{N - 2s}}}{\|Q\|_{L^2}^{\frac{4s - 2b}{N - 2s}}}) \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2.
$$

(43)

From the hypothesis $\|u_0\|_{L^2} < \|Q\|_{L^2}$, there exists a constant $C > 0$ such that $E(u_0) = E(u(t)) \leq C \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}^2$ for all $t \in I$. Then, $u(t, x)$ is bounded in $H^s(\mathbb{R}^N)$ for all $t \in I$ by the conservation of mass, and $u(t, x)$ exists globally in $H^s(\mathbb{R}^N)$. This completes the proof of (i).

(ii) From Remark 2.1, we see that $Q$ satisfies

$$
\int_{\mathbb{R}^N} |x|^{-b} |Q|^{\frac{2(4s - 2b + 2N)}{N - 2s}} dx = \frac{N + 2s - b}{2s - b} \|Q\|_{L^2}^2, \quad (44)
$$

$$
\|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 = \frac{N}{2s - b} \|Q\|_{L^2}^2. \quad (45)
$$

Inject these identities and initial data (42) into $E(u_0)$. There exists $C_0 > 0$ such that

$$
E(u_0) = \frac{|c|^2 \rho^{2s}}{2} \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2 - \frac{N \rho^{2s} |c|^{\frac{4s - 2b + 2N}{N}}}{4s - 2b + 2N} \int_{\mathbb{R}^N} |x|^{-b} |Q|^{\frac{4s - 2b + 2N}{N}} dx
$$

$$
= \rho^{2s} \left( |c|^2 - \frac{|c|^{\frac{4s - 2b + 2N}{N}}}{2} \right) \|(-\Delta)^{\frac{s}{2}} Q\|_{L^2}^2
$$

$$
\leq -c_0 < 0, \quad (46)
$$

where we make use of the fact $\frac{4s - 2b + 2N}{N} > 2$. So, we complete the proof of (ii) by Theorem 4.2.

\[\square\]

**Proposition 5.** $\sigma = \frac{2s - b}{N}$ is the critical exponent of the existence of finite time blowup solutions for Eq.(1).

Proof. When $0 < \sigma < \frac{2s - b}{N}$, inject the sharp Gagliardo-Nirenberg inequality into $E(u(t))$. We deduce that if $0 < \sigma < \frac{2s - b}{N}$, then for all $t \in I$ (the maximal time interval), there exists a constant $C(\epsilon, \sigma, s, b, \|u_0\|_{L^2}) > 0$ such that for all $0 < \epsilon < \frac{1}{2}$,

$$
E(u(t)) = \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |x|^{-b} |u(t)|^{2\sigma + 2} dx
$$

$$
\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \frac{K_{opt}}{2\sigma + 2} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^{\frac{2\sigma + 2}{2\sigma}} \|u(t)\|_{L^2}^{\frac{2\sigma}{2\sigma}}
$$

$$
\geq \frac{1}{2} \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - \epsilon \|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 - C(\epsilon, \sigma, s, b, \|u_0\|_{L^2}), \quad (47)
$$

where $K_{opt} = \left( \frac{\sigma N + b}{2\sigma (\sigma + 1) - (\sigma N + b)} \right)^{2\sigma (\sigma + 1) - (\sigma N + b)} \|u_0\|_{L^2}^2$. Then, we can choose a constant $K \geq \frac{E(u_0) + C(\epsilon, \sigma, s, b, \|u_0\|_{L^2})}{\frac{1}{2} - \frac{\epsilon}{2}} + \|u_0\|_{L^2}^2 > 0$ such that $\|(-\Delta)^{\frac{s}{2}} u(t)\|_{L^2}^2 \leq K$ for all $t \in I$. That is, the existence interval of $u(t, x)$ is $I = [0, +\infty)$.

By applying Theorem 3.5, Theorem 4.2 and Proposition 4, we can see that there exist finite-time blowup solutions for Eq.(1) when $\frac{2s - b}{N} \leq \sigma < \frac{2s - b}{N - 2s}$. So, we complete the proof of Proposition 5.

\[\square\]
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