On super \((a, d)\)-edge-antimagic total labeling of Möbius ladder

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Abstract. A Möbius ladder is a simple graph obtained by introducing a twist in a prism graph. A Möbius ladder which has \(n\) pairs vertices is denoted by \(M_n\). A graph \(G\) is called \((a, d)\)-edge-antimagic total if there exists a bijection \(f\) from \(V(G) \cup E(G)\) to \([1, 2, ..., |V(G) + E(G)|]\) such that the edge-weights, \(w_f(uv) = f(u) + f(uv) + f(v), uv \in E(G)\), forms an arithmetic sequence with the first term \(a\) and common difference \(d\). Further, it called super if \(f(V) = \{1, 2, ..., |V(G)|\}\). In this paper we proved that \(M_n\) is super \((a, d)\)-edge-antimagic total with \(0 \leq d \leq 2\). Further, we showed that for every even \(n\), \(M_n\) is not \((a, 0)\)-edge-antimagic total.

1. Introduction
Let \(G(V, E)\) be a graph, where \(V(G)\) is the set of vertices and \(E(G)\) is the set of edges. A labeling on a graph \(G(V, E)\) is an assignment of \(V(G), E(G)\), or \(V(G) \cup E(G)\) to a set of non-negative integers. If the domain is \(V(G)\), then it is called the vertex labeling. Respectively, if the domain is \(E(G)\), then it is called the edge labeling. While, if the domain is \(V(G) \cup E(G)\), then it is called the total labeling.

An example of labeling is antimagic labeling which was introduced by Hartsfield and Ringel [1]. A graph \(G(V, E)\) is called antimagic if every \(uv \in E(G)\) can be labeled with distinct member of \([1, 2, ..., |E(G)|]\) such that the vertex-weights which is the sum of the edge labels’ incident to each vertex are distinct. Galian [2] summaries some antimagic graph, such as \(P_n\) for \(n \geq 3\), \(C_n\), \(K_n\) for \(n \geq 3\), etc. If the edge-weights form an arithmetic sequence, then the graph \(G(V, E)\) is called \((a, d)\)-antimagic. This concept was introduced by Bodendiek and Walther [3]. Bodendiek and Walther [3], [4] and [5] proved that parachutes \(P_{g,p}\) is \((a, d)\)-antimagic for some certain classes, while \(P_{2n}\), \(C_{2n}\), and stars are not \((a, d)\)-antimagic. Other concepts of this kind of labeling were proposed in [6], [7] and [8]. There are \((a, d)\)-vertex-antimagic total labeling, super \((a, d)\)-vertex-antimagic total labeling, \((a, d)\)-edge-antimagic total labeling, super \((a, d)\)-edge-antimagic total labeling and \((a, d)\)-edge-antimagic vertex labeling. The Definition of super \((a, d)\)-edge-antimagic total graph was given in [9], [6],[7] and [10].

Definition 1.1. A graph \(G(V, E)\) is called \((a, d)\)-edge-antimagic total if there exists a bijection \(f\) from \(V(G) \cup E(G)\) to \([1, 2, ..., |V(G) + E(G)|]\) such that the edge-weight, \(w_f(uv) = f(u) + f(uv) + f(v), uv \in E(G)\), forms an arithmetic sequence with the first term \(a\) and common difference \(d\). Further, it called super if \(f(V) = \{1, 2, ..., |V(G)|\}\).

There were some studies of graphs which are labeled by super \((a, d)\)-edge-antimagic total labeling. Bača, Kovář, Semaničová-Feňovčíková and Shafiq [11] proved that every even regular
graph and every odd regular graph with a 1-factor are super \((a, 1)\)-edge-antimagic total. While Dafik, Miller, Ryan, and Báca [9] proved that some \(mC_n\) are super \((a, 1)\)-edge-antimagic total and Báca and Barrientos [12] studied some properties of super \((a, d)\)-edge-antimagic total labeling on \(mK_n\).

Our focus in this paper is proving that Möbius ladder is a super \((a, d)\)-edge-antimagic total and discover its properties. In the next chapter, we introduced the definition and properties of Möbius ladder.

2. Möbius ladder
A Möbius ladder is a simple graph obtained by introducing a twist in a prism graph. A Möbius ladder which has \(n\) pairs vertices is denoted by \(M_n\). In the Figure 1 and Figure 2 below, we showed the comparison between a prism graph and a Möbius ladder which have 3 pairs vertices.

A Möbius ladder which has \(n\) pairs of vertices has \(3n\) edges and every vertex has degree 3. As we can see in Figure 3, graph \(C_k\) is subgraph of \(M_n\) for \(4 \leq k \leq 2n\). These properties are important to study some properties of the super \((a, d)\)-edge-antimagic total labeling on Möbius ladder.

The application of Möbius ladder can be found in chemistry and chemical stereography. Graph \(M_3\) had been used by Walba to describe synthesis molecular structures [13]. While
Flapan [14] used Möbius ladder to describe molecular chirality. Other application also can be found in integer programming [15]. Möbius ladder is used in describing a linear programming relaxation of the problem.

Some labelings on Möbius ladder have been studied, for instance Jayasekaran and Little Flower [16] proved that $M_n$ is edge trimagic total. Pasotti [17] proved that $M_n$ is graceful. While, Chunling, Xiaohui, Yuansheng and Liping [18] showed that when $M_n$ is labeled with irregular labeling, the total edge irregularity strength of $M_n$ is $n + 1$, while the total vertex irregularity strength of $M_n$ is $\left\lceil \frac{n}{2} \right\rceil + 1$.

3. On super $(a, d)$-edge-antimagic total labeling on Möbius ladder

In order to define a labeling on $M_n$ so that it satisfies super $(a, d)$-edge-antimagic total labeling, we are looking for a pattern by examining some examples.

Example 3.1. The Graph $M_3$ and $M_4$ are super $(a, 1)$-edge-antimagic total.

Let the labeling pattern be as in the Figure 4 and Figure 6. By calculating the edge-weight for each edges, we got arithmetic sequences. In the Figure 5, the edge-weights forms a sequence \{14, 15, 16, ..., 22\}, an arithmetic sequence with the first term 14 and common difference 1.
While in the Figure 7, the edge-weights forms a sequence \{18,19,20,...,29\}, an arithmetic sequence with the first term 18 and common difference 1. The set of vertex-labels of \(M_3\) is \(f(V(M_3)) = \{1,2,...,6\}\) and the set of vertex-labels of \(M_4\) is \(f(V(M_4)) = \{1,2,...,8\}\). So, we can conclude that both \(M_3\) and \(M_4\) are super \((a, 1)\)-edge-antimagic total.

**Theorem 3.1.** Graph \(M_n\) is super \((a, d)\)-edge-antimagic total for every \(n \geq 3\).

**Proof.** Define a label \(f\) on \(V(M_n) \cup E(M_n)\).

For \(i = 1,2,\ldots,n\),

\[
\begin{align*}
    f(v_i) &= i \\
    f(u_i) &= n + i \\
    f(u_iv_i) &= 3n - (i - 1)
\end{align*}
\]

For \(i = 1,2,\ldots,n-1\)

\[
\begin{align*}
    f(v_iv_{i+1}) &= 5n - i \\
    f(u_iu_{i+1}) &= 4n - i
\end{align*}
\]

and

\[
\begin{align*}
    f(u_nv_1) &= 5n \\
    f(u_1v_n) &= 4n
\end{align*}
\]

Considering that on graph \(M_n\), \(E(G) = \{u_iu_{i+1}|i = 1,2,...,n-1\} \cup \{v_iv_{i+1}|i = 1,2,...,n-1\} \cup \{u_iv_i|i = 1,2,...,n\} \cup \{u_nv_1\} \cup \{u_nv_1\}\), then we calculated the edge-weights for each classes of edges.

For \(\{u_iu_{i+1}\}\), \(i = 1,2,...,n-1\)

\[
\begin{align*}
    w_f(u_iu_{i+1}) &= f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) \\
                   &= (n + 1) + (n + 1 + i) + (4n - i) \\
                   &= 6n + i + 1
\end{align*}
\]

For \(\{v_iv_{i+1}\}\), \(i = 1,2,...,n-1\)

\[
\begin{align*}
    w_f(v_iv_{i+1}) &= f(v_i) + f(v_{i+1}) + f(v_iv_{i+1}) \\
                   &= i + (i + 1) + (5n - i) \\
                   &= 5n + i + 1
\end{align*}
\]

For \(\{u_iv_i\}\), \(i = 1,2,...,n\)

\[
\begin{align*}
    w_f(u_iv_i) &= f(u_i) + f(v_i) + f(u_iv_i) \\
               &= (n + i) + i + (3n - (i - 1)) \\
               &= 4n + i + 1
\end{align*}
\]

For \(\{u_1v_n\}\),

\[
\begin{align*}
    w_f(u_1v_n) &= f(u_1) + f(v_n) + f(u_1v_n) \\
               &= (n + 1) + n + 4n \\
               &= 6n + 1
\end{align*}
\]
For \( \{u_n v_1\} \),
\[
    w_f(u_n v_1) = f(u_n) + f(v_1) + f(u_n v_1)
    = 2n + 1 + 5n
    = 7n + 1
\]

Thus, the set of edge-weights \( \{w_f\} = \{4n + i + 1|i = 1, 2, ..., n\} \cup \{5n + i + 1|i = 1, 2, ..., n - 1\} \cup \{6n + 1\} \cup \{6n + i + 1|i = 1, 2, ..., n - 1\} \cup \{7n + 1\} \) forms an arithmetic sequence which \( a = 4n + 2, \ d = 1, \) and \( f(V) = \{1, 2, ..., 2n\} \) where \( 2n \) is the order of \( M_n \). So, we can conclude that \( M_n \) is a super \((a,1)\)-edge-antimagic total.

The Theorem 3.1 showed that \( M_n \) is a super \((a, d)\)-edge-antimagic total with \( d \) equals 1, then we investigated the possibility of other values of \( d \). We proved the range of \( d \) in the next following theorem.

**Theorem 3.2.** Graph \( M_n \) is super \((a, d)\)-edge-antimagic total with \( 0 \leq d \leq 2 \).

**Proof.** Suppose that \( M_n \) is labelled so that \( M_n \) is a super \((a, d)\)-edge-antimagic total. We observed the minimum and the maximum possible edge weights of \( M_n \).

The minimum possible edge weight : \( 1 + 2 + (2n + 1) = 2n + 4 \)

The maximum possible edge weight : \( (2n - 1) + 2n + 5n = 9n - 1 \)

Notice that the number of edges is \( 3n \), thus
\[
    d \leq \frac{9n - 1}{2n + 1} = \frac{7n - 5}{3n - 1}
\]

It is clear that \( \frac{7(n-1)-5}{3(n-1)-1} < \frac{7n-5}{3n-1} \) for all \( n \). Notice that \( \lim_{n \to \infty} \frac{7n-5}{3n-1} = \frac{7}{3} \). Thus \( \left\{\frac{7n-5}{3n-1}\right\} \) is an increasing sequence which its supremum is \( \frac{7}{3} \). So, the maximum \( d \) is 2 and the possible interval of \( d \) so that \( M_n \) is a super \((a, d)\)-edge-antimagic total is \( 0 \leq d \leq 2 \).

Regarding to the Theorem 3.1 and Theorem 3.2, if we want to prove the existence of super \((a, d)\)-edge-antimagic total labeling for the other \( d \), we should prove for \( d = 0 \) or \( d = 2 \). Respectively, if we want to prove that there is no super \((a, d)\)-edge-antimagic total labeling except \( d = 1 \), we have to prove that it is not possible to have a super \((a, d)\)-edge-antimagic total labeling for \( d = 0 \) or \( d = 2 \). By considering the properties of Möbius ladder, in the next following theorem, we proved that \( M_3 \) is a super \((a, d)\)-edge-antimagic total with a unique \( d = 1 \).

**Theorem 3.3.** Graph \( M_3 \) is not a super \((a, d)\)-edge-antimagic total for \( d = 0 \) or \( d = 2 \).

**Proof.** Suppose that \( M_3 \) is a super \((a, d)\)-edge-antimagic total, we will proved that \( d \neq 0 \) and \( d \neq 2 \). The idea of our proof is checking that there is no possible labeling that satisfies \( M_3 \) to be a super \((a, d)\)-edge-antimagic total with \( d = 0 \) or \( d = 2 \). The total sum of the edge-weights of \( M_3 \) is 162. Suppose that \( M_3 \) is a super \((a, 0)\)-edge-antimagic total, then the edge-weights of each edge is 18. So, the combinations of edge-labels and vertex-labels are : \( (7, 5, 6), (8, 4, 6), (9, 3, 6) \) and these are unique, but we can not find combination for edge-label equals 10, because \( (10, 4, 4) \) is not possible since the vertex-labels have to be distinct. Notice that graph \( C_k \) is a subgraph of \( M_n \) for \( 4 \leq k \leq 2n \), so triplet \( (10, 5, 3) \) is not possible since it will make a \( C_3 \) because we have fixed the triplet \( (7, 5, 6) \) and \( (9, 3, 6) \) before. While, \( (10, 2, 6) \) is not possible, because the degree of each vertex is 3 and we have fixed triplets \( (7, 5, 6), (8, 4, 6), (9, 3, 6) \) before. Thus, \( M_3 \) is not a super \((a, 0)\)-edge-antimagic total.

Suppose that \( M_3 \) is a super \((a, 2)\)-edge-antimagic total. Then the edge-weights are 10, 12, 14, ...., 26. There are unique combinations for edge-weights 10, 12, 14, namely \((1, 2, 7), (1, 3, 8), \) and \((1, 4, 9)\). In the opposite way, there is no suitable combination for edge-weight 16, because
triplets (1, 5, 10), (2, 4, 19), (3, 3, 10), (1, 4, 11), (2, 3, 11), (1, 3, 12), (2, 2, 12), (1, 2, 13) are not satisfying some properties of super (a, 2)-edge-antimagic total of Möbius ladder. Thus, $M_3$ is not a super (a, 2)-edge-antimagic total. Therefore we can conclude that there is a unique $d = 1$, so that $M_3$ is a super (a, d)-edge-antimagic total.

In the next theorem, we proved that for all even $n$, graph $M_n$ is not super (a, 0)-edge-antimagic total.

**Theorem 3.4.** For all even $n$, graph $M_n$ is not super (a, 0)-edge-antimagic total.

**Proof.** Suppose that $M_n$ is super (a, 0)-edge-antimagic total, thus $f(V(M_n)) = \{1, 2, ..., 2n\}$. Consider that the degree of each vertex is 3, so the total sum of all edge weights is $3(1 + 2 + ... + 2n) + (2n + 1) + (2n + 2) + ... + 5n = \frac{1}{2}3n(11n + 3)$. Since, the number of edges is $3n$ and we assumed $d = 0$, then for each edge, the edge weight is $\frac{1}{2}(11n + 3)$. Notice that $n$ is even, then $(11n + 3)$ is odd, so $\frac{1}{2}(11n + 3)$ is not an integer. Since the edge-weight is always an integer, we can conclude that our supposition is false.

4. Conclusion and future work

In this paper, we proved that $M_n$ is super (a, d)-edge-antimagic total with $a = 4n + 2$ and $d = 1$. By considering the maximum and minimum edge weights, we showed that $0 \leq d \leq 2$. We proved that $M_3$ is super (a, d)-edge-antimagic total with a unique $d = 1$, and for every even $n$, $M_n$ is not super (a, 0)-edge-antimagic total. Our future work is investigating whether for every even $n$, $M_n$ is super (a, 2)-edge-antimagic total and the existence of super (a, d)-edge-antimagic total labelings for $d = 0$ or $d = 2$ on $M_n$, for every odd $n \geq 5$.

References

[1] Hartsfield N and Gerhard R 1990 *Pearls in Graph Theory* (San Diego: Academic Press)
[2] Galian J A 2016 *The Electronic Journal of Combinatorics* #DS6
[3] Bodendiek R and Walther G 1993 * antimagische Graphen Graphentheorie III*
[4] Bodendiek R and Walther G 1996 *Ars Combin* 42 129–149
[5] Bodendiek R and Walther G 1998 *Mitt. Math. Ges. Hamburg* 17 85–99
[6] Bača M and Miller M 2007 *Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solutions* (Boca Raton: BrownWalker Press)
[7] Bača M, Baskoro E T, Miller M, Ryan J, Simanjuntak R and Sugeng K A 2006 *J. Indonesian Math. Soc.(MIHMI)* 12 113–130
[8] Sugeng K A 2005 *Magic and Antimagic Labeling of Graphs* (Victoria: University of Ballarat)
[9] Daňik, Miller M, Ryan J and Bača M. 2009 *Discrete Mathematics* 309 4909–4915
[10] Bača M, Lin Y, Miller M and Youssf M Z 2007 *Discrete Mathematics* 307 1232–1244
[11] Bača M, Ková P, Semaníčková-Fešová A and Shafiq M K 2010 *Discrete Mathematics* 310 1408–1412
[12] Bača M and Barrientos C 2008 *Discrete Mathematics* 308 5032–5037
[13] Kawauchi A 1996 *Survey on Knot Theory* (Basel, Boston, Berlin, Birkäuser: Springer Science & Business Media)
[14] Flapan E 2000 *When Topology Meets Chemistry : A Topological Look at Molecular Chirality* (Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi: Cambridge University Press)
[15] Newman A and Vempala S 2001 *IPCO Conference Proceeding* pp 333–347
[16] Jayasekaran C and Little Flower J 2017 *Annals of Pure and Applied Mathematics* 13 151–163
[17] Pasotti A 2010 *Discrete Mathematics* 310 3080–3087
[18] Chunling T, Xiaohui L, Yuansheng Y and Liping W 2009 *Indian J. Pure Appl Math* 40 155–181