Kerr initial data

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Abstract
Exploiting a (3+1) analysis of the Mars–Simon tensor, conditions on a vacuum initial data set ensuring that its development is isometric to a subset of the Kerr spacetime are found. These conditions are expressed in terms of the vanishing of a positive scalar function defined on the initial data hypersurface. Applications of this result are discussed.

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1. Introduction

The Kerr spacetime is one of the most important exact solutions to the Einstein vacuum equations. Its relevance stems from the uniqueness theorems for black holes which state that under rather general conditions the Kerr spacetime is the only asymptotically flat, stationary, vacuum black hole solution—see e.g. the introduction of [19] for a critical review of the issue of black hole uniqueness and the involved assumptions. Although the Kerr spacetime is very well understood from a spacetime perspective, the same cannot be said if one adopts a (3+1) point of view—which would be the case if one tries to numerically calculate the spacetime from some Cauchy initial data.

As soon as one moves away from a (3+1) gauge which is adapted to the stationary and axial symmetries—which can occur in some applications, in particular in numerical ones—an analysis of the Kerr spacetime and initial data sets thereof becomes very complicated. The explicit nature of the Kerr solution makes it tempting to perform detailed calculations in order to, say, verify a particular property of the spacetime. This approach usually leads to very lengthy expressions which can be very hard to analyse. In a sense one could say that exact solutions contain too much information. In this case it can be more convenient to adopt more abstract approaches and, accordingly, it may prove useful to have at hand a characterization of Kerr initial data.
The question of providing an invariant characterization of initial data sets for the Schwarzschild spacetime has been addressed in [16, 28]. In particular, the analysis of [16] provides an algorithmic characterization of Schwarzschild data. That is, a procedure is provided to verify whether a given initial data set for the Einstein field equations will render a development which is isometric to a portion of the Schwarzschild spacetime.

One of the most important algebraic properties of the Kerr spacetime is that its Weyl tensor is of Petrov type D. The close relation between vacuum spacetimes with a Weyl tensor of Petrov type D and Killing spinors has been exploited in [17] to provide a characterization of initial data sets whose developments will be of Petrov type D. This characterization relies on one being able to decide whether a set of overdetermined partial differential equations has solutions for a given initial data set. Accordingly, such a characterization is not algorithmic. Although not explicitly stated in [17], from that analysis it should be possible to obtain a characterization of Kerr initial data by adding some global conditions.

The characterization of initial data sets discussed in [16, 17, 28] has followed the general strategy of starting from a given tensorial (respectively, spinorial) spacetime characterization of the class of spacetimes under consideration. Necessary conditions on the initial data set are obtained by performing a (3+1) decomposition of the spacetime characterization. Given a set of necessary conditions on the initial data, it is then natural to address the question of sufficiency. This is, usually, the most challenging part of the procedure as one has to discuss the evolution of complicated tensorial objects. The idea behind this is to show that if the necessary conditions are satisfied on some open subset of the initial hypersurface, then one can—possibly, under some additional assumptions—recover the spacetime characterization on the development of the open subset on the initial hypersurface from which one started.

In this paper a particular characterization of Kerr initial data is addressed. Our starting point is a certain spacetime characterization of the Kerr spacetime given in [21, 22]. This characterization was developed with the aim of providing an alternative way of proving the uniqueness of Kerr spacetime among the class of stationary, asymptotically flat black holes. This expectation has been recently fulfilled in [19], where a proof of the uniqueness of Kerr which does not make assumptions on the analyticity of the metric has been given. At the heart of the spacetime characterization given in [21, 22]—cf theorem 3—is a certain tensor, the Mars–Simon tensor, whose construction requires the existence of a timelike Killing vector. The Mars–Simon tensor is a spacetime version of the Simon tensor, a tensor, defined in the quotient manifold of a stationary spacetime, which characterizes the Kerr spacetime—see [27].

Following the general strategy for the construction of characterizations of initial data sets out of spacetime characterizations, necessary conditions for Kerr initial data are deduced from a (3+1) splitting of the Mars–Simon tensor. Accordingly, one assumes that the spacetime one is working with has a timelike Killing vector. This requirement can be encoded in the initial data by requiring that the data have a suitable Killing initial data (KID). The Mars–Simon tensor has the same symmetries as the Weyl tensor, and hence its (3+1) decomposition can be given in terms of its electric and magnetic parts. In order to discuss the propagation of the Mars–Simon tensor we make use of a framework for discussing the propagation tensorial fields using superenergy-type tensors—this framework has been discussed in e.g. [4]. It should be pointed out that the characterization discussed in this paper is not algorithmic. That is, like the one for type D initial data discussed in [17] it depends on being able to decide whether a certain overdetermined system of partial differential equations admits a solution.

The paper is structured as follows: in section 2 our main conventions are fixed and relevant aspects of the (3+1) formalism are discussed. Section 3 discusses the properties and causal propagation of Weyl candidates—i.e. tensors with the same symmetries of the Weyl tensor.
Section 4 is concerned with the properties of the Mars–Simon tensor. Section 5 discusses the causal propagation of the Mars–Simon tensor. Section 6 applies the previous discussion to the construction of a characterization of Kerr initial data. Our main result is provided in theorem 6. In section 6.1 we particularize to the case of the Schwarzschild spacetime, where, with the aim of the results of [16] it is possible to obtain an initial data characterization which is algorithmic. Some concluding remarks are provided in section 7. Finally, some technical details, too lengthy to be presented in the main text are presented in appendix A.

All the tensor computations of this paper have been performed with software xAct [23]. xAct is a suite of MATHEMATICA packages which has among its many features the capability to efficiently canonicalize tensor expressions by the use of powerful algorithms based on permutation group theory [24]. Currently no other software, either free or commercial, is capable to handle the tensor computations needed in this paper.

2. Preliminaries

Let \((M, g_{\mu\nu})\) denote a smooth orientable spacetime. The following conventions will be used: plain Greek letters \(\alpha, \beta, \gamma, \ldots\) denote abstract indices and boldface Latin characters \(a, b, c, \ldots\) will be used for component indices. The signature of the metric tensor \(g_{\mu\nu}\) will be taken to be \((- 1, + 3)\), while \(R_{\mu\nu\rho\si}, R_{\mu\nu} = R^\alpha_{\mu\alpha\nu},\) and \(W_{\alpha\beta\mu\nu}\) denote, respectively, the Riemann, Ricci and Weyl tensors of \(g_{\mu\nu}\). The tensor \(\eta_{\alpha\beta\si\nu}\) is the volume element which is used to define the Hodge dual of any antisymmetric tensor—denoted by attaching a star \(*\) to the tensor symbol. Sometimes we will need to work with complex tensors in which case the complex conjugation of a tensor is denoted by an overbar. The operator \(L_\vec{u}\) symbolizes the Lie derivative with respect to the vector field \(u^\mu\).

2.1. The orthogonal splitting

Let \(n^\mu\) be a unit timelike vector, \(n^\mu n_\mu = -1\) defined on \(M\). Then any tensor or tensorial expression can be decomposed with respect to \(n^\mu\) and the way to achieve it is the essence of the orthogonal splitting (also known as \((3+1)\) formalism) which is described in many places of the literature—see e.g. [11, 12]. We review the parts of this formalism needed in this work. The spatial metric is defined by \(h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu\) and it has the algebraic properties \(h^\mu_\mu = 3, h^\si_\mu h_\si\nu = h_{\mu\nu}\). We shall call a covariant tensor \(T_{\si_1 \cdots \si_a}\) spatial with respect to \(h_{\mu\nu}\) if it is invariant under \(h^{\mu}_{\nu}\) i.e. if

\[
h_{\si_1}^{\rho_1} \cdots h_{\si_a}^{\rho_a} T_{\rho_1 \cdots \rho_a \si_1 \cdots \si_a} = T_{\rho_1 \cdots \rho_a},
\]

with the obvious generalization for any mixed tensor. This property implies that the inner contraction of \(n^\mu\) with \(T_{\si_1 \cdots \si_a}\) (taken on any index) vanishes. The orthogonal splitting of a tensor expression consists of writing it as a sum of terms which are tensor products of the unit normal and spatial tensors of lesser degree—or the same degree in which case the unit normal is absent.

In order to find the orthogonal splitting of expressions containing covariant derivatives we need to introduce the spatial derivative \(D_\mu\) which is an operator whose action on any tensor field \(T^{\rho_1 \cdots \rho_p}_{\si_1 \cdots \si_q}\) is given by

\[
D_\mu T^{\rho_1 \cdots \rho_p}_{\si_1 \cdots \si_q} \equiv h^{\rho_1}_{\rho_1} \cdots h^{\rho_p}_{\rho_p} h^{\alpha}_{\beta_1} \cdots h^{\alpha_q}_{\beta_q} h^\si_\mu \nabla_\si T^{\rho_1 \cdots \rho_p}_{\si_1 \cdots \si_q}.
\]

From equation (2.2) it is clear that \(D_\mu T^{\rho_1 \cdots \rho_p}_{\si_1 \cdots \si_q}\) is spatial.

The results just described hold for an arbitrary unit timelike vector \(n^\mu\) but in our framework we only need to consider integrable timelike vectors which are characterized by the condition
\( n_{\mu} \nabla_{\nu} n_{\sigma} = 0 \) (the Frobenius condition). In this case there exists a foliation of \( \mathcal{M} \) such that the vector field \( n^\mu \) is orthogonal to the leaves of the foliation. We shall denote by \( \{ \Sigma_t \}, t \in I \subset \mathbb{R} \), the family of leaves of this foliation and \( \Sigma_0 \) is called the initial data hypersurface—it is assumed that \( 0 \in I \). The tensor \( h_{\mu\nu} \) plays the role of the first fundamental form for any of the leaves while the symmetric tensor \( K_{\mu\nu} \) defined by

\[
K_{\mu\nu} \equiv -\frac{1}{2} \mathcal{L}_h h_{\mu\nu},
\]

(2.3)
can be identified with the second fundamental form. Combining the previous definition with the Frobenius condition we easily derive

\[
\nabla_{\mu} n_{\nu} = -K_{\mu\nu} - n_{\mu} A_{\nu},
\]

(2.4)
where \( A^\mu \equiv n^\rho \nabla_{\rho} n^\mu \) is the acceleration of \( n^\mu \). By using these quantities one is in principle able to work out the orthogonal splitting of any tensorial expression. We supply below the explicit results of calculations which will be needed repeatedly in the following.

- **Orthogonal splitting of the volume element:**
  \[
  \eta_{\alpha\beta\gamma\delta} \equiv -n_\alpha \varepsilon_{\beta\gamma\delta} + n_\beta \varepsilon_{\gamma\delta\alpha} - n_\gamma \varepsilon_{\delta\alpha\beta} + n_\delta \varepsilon_{\alpha\beta\gamma}.
  \]
  (2.5)
  Here \( \varepsilon_{\alpha\beta\gamma\delta} \equiv n^\mu \eta_{\alpha\beta\gamma\delta} \) is the spatial volume element which is a fully antisymmetric spatial tensor.

- **Orthogonal splitting of an antisymmetric tensor** If \( H_{\mu\nu} \) is an arbitrary antisymmetric tensor then its orthogonal splitting takes the form

\[
H_{\mu\nu} = 2P_{\mu\nu} - Q_{\mu} P_{\nu},
\]

(2.6)
where \( P_{\mu\nu} = H_{\mu\nu} n^\nu, Q_{\mu} = H^*_{\mu\nu} n^\nu \) are clearly spatial tensors. It is straightforward from the previous equation to find an analogous formula for the orthogonal splitting of \( H^*_{\mu\nu} \). If we work with the self-dual part of \( H_{\mu\nu} \), denoted by \( \mathcal{H}_{\mu\nu} \) and defined by

\[
\mathcal{H}_{\mu\nu} = H_{\mu\nu} + i H^*_{\mu\nu},
\]

(2.7)
then its orthogonal splitting takes the form

\[
\mathcal{H}_{\mu\nu} = i \varepsilon_{\rho\mu\nu} P^\rho - 2P_{\nu} n_{\mu}, \quad P_\mu \equiv P_{\mu\nu} + i Q_{\mu}.
\]

(2.8)

- **Orthogonal splitting of the Weyl tensor:**
  \[
  W_{\mu\nu\lambda\rho} = 2(l_{\mu\nu} E_{\lambda\rho} - l_{\mu\lambda} E_{\nu\rho} - l_{\mu\rho} E_{\nu\lambda} + n_{\mu} B_{\nu\lambda} \varepsilon^{T}_{\nu\lambda\rho} + n_{\nu} B_{\mu\lambda} \varepsilon^{T}_{\mu\lambda\rho} + n_{\lambda} B_{\mu\nu} \varepsilon^{T}_{\mu\lambda\rho}),
  \]
  (2.9)
where

\[
E_{\mu\nu} \equiv W_{\tau\sigma\mu\nu} n^\tau n^\sigma, \quad B_{\mu\nu} \equiv W^*_{\tau\sigma\mu\nu} n^\tau n^\sigma.
\]

(10.10)
denote the Weyl tensor electric and magnetic parts respectively, and \( l_{\mu\nu} \equiv h_{\mu\nu} + n_\mu n_\nu \). The tensors \( E_{\mu\nu} \) and \( B_{\mu\nu} \) are symmetric, traceless and spatial. Equation (2.9) also holds for any rank-4 tensor possessing the same algebraic properties as the Weyl tensor. Such tensors will be designated collectively as Weyl candidates—see section 3. We can also choose to work with the self-dual Weyl tensor \( W_{\mu\nu\lambda\rho} \) which is given by

\[
W_{\mu\nu\lambda\rho} \equiv \frac{1}{2}(W_{\mu\nu\lambda\rho} + i W^*_{\mu\nu\lambda\rho}).
\]

(2.11)
Using (2.9) and the analogous formula for the orthogonal splitting of \( W^*_{\mu\nu\lambda\rho} \) one can obtain an expression for the orthogonal splitting of \( W_{\mu\nu\lambda\rho} \) as

\[
W_{\mu\nu\lambda\rho} = 2(l_{\mu\nu} E_{\lambda\rho} + l_{\mu\lambda} E_{\nu\rho} - l_{\mu\rho} E_{\nu\lambda} - n_{\mu} B_{\nu\lambda} \varepsilon^{T}_{\nu\lambda\rho} + n_{\nu} B_{\mu\lambda} \varepsilon^{T}_{\mu\lambda\rho} - n_{\lambda} B_{\mu\nu} \varepsilon^{T}_{\mu\lambda\rho}),
\]

(12.12)
where

\[
E_{\mu\nu} \equiv W_{\mu\nu\lambda\rho} n^\lambda n^\rho = \frac{1}{2}(E_{\mu\nu} - iB_{\mu\nu}), \quad E_{\mu\nu} = E_{\mu\nu}, \quad E^\rho_{\mu} = 0.
\]

(2.13)
Again a similar formula to (2.12) holds for any Weyl candidate.

- **Let \( T^\mu \) be any spatial vector.** Then the orthogonal splitting of \( \nabla_{\mu} T_{\nu} \) is

\[
\nabla_{\mu} T_{\nu} = D_{\mu} T_{\nu} - n_{\mu} (K_{\rho\nu} T^\rho + \mathcal{L}_h T_{\nu}) - T^\rho A_{\rho\nu} n_{\mu} n_{\nu} - n_{\nu} K_{\rho\mu} T^\rho.
\]

(2.14)
Note that since \( T_{\mu} \) is spatial then \( \mathcal{L}_h T_{\mu} \) is also spatial.
2.2. The Cauchy problem

We briefly review the standard formulation of the Cauchy problem for the vacuum Einstein equations. In this formulation one considers a three-dimensional connected Riemannian manifold \((\Sigma, h_{ij})\)—we use small plain Latin letters \(i, j, k, \ldots\) for the abstract indices of tensors on this manifold—and an isometric embedding \(\phi : \Sigma \rightarrow \mathcal{M}\). The map \(\phi\) is an isometric embedding if
\[
\partial_i \phi^\mu \partial_j \phi^\nu g_{\mu\nu} = h_{ij}
\]
where \(\partial_i \phi^\mu \partial_j \phi^\nu\) realizes the pullback, \(\phi^*\), of tensor fields from \(\mathcal{M}\) to \(\Sigma\). The metric \(h_{ij}\) defines a unique affine connection \(D_i\) (Levi-Civita connection) by means of the standard condition
\[
D_j h_{ik} = 0. \tag{2.15}
\]
The Riemann tensor of \(D_i\) is \(R^{ijkl}\) and from it we define its Ricci tensor by
\[
R^{ij} = R^{li} h_{lj},
\]
and its scalar curvature
\[
r = R^{ii}.
\]

**Theorem 1.** Let \((\Sigma, h_{ij})\) be a Riemannian manifold and suppose that there exists a symmetric tensor field \(K_{ij}\) on it which verifies the conditions (vacuum constraints)
\[
\begin{align*}
K &+ K^2 - K_{ij}K^{ij} = 0, \tag{2.16a} \\
D^i K_{ij} - D_j K^i = 0, \tag{2.16b}
\end{align*}
\]
where \(K \equiv K^i_i\). Provided that \(h_{ij}\) and \(K_{ij}\) are suitably smooth there exists an isometric embedding \(\phi\) of \(\Sigma\) into a globally hyperbolic, vacuum solution \((\mathcal{N}, g_{\mu\nu})\) of the Einstein field equations. The set \((\Sigma, h_{ij}, K_{ij})\) is then called a vacuum initial data set and the spacetime \((\mathcal{N}, g_{\mu\nu})\) is the data development. Furthermore the spacelike hypersurface \(\phi(\Sigma)\) is a Cauchy hypersurface in \(\mathcal{N}\).

A statement of this theorem containing precise regularity conditions on \(h_{ij}, K_{ij}\) is formulated in [20].

Under the conditions of theorem 1, we may construct a foliation of \(\mathcal{N}\) with \(n^\mu\) as the timelike unit vector which is orthogonal to the leaves and, in the above notation, set \(\Sigma_0 = \phi(\Sigma)\). It is then clear that
\[
\partial_i \phi^\mu \partial_j \phi^\nu h_{\mu\nu} = h_{ij}.
\]
Other key properties are
\[
\begin{align*}
\partial_i \phi^\mu \partial_j \phi^\nu (D_{\nu'} T_{\nu''...\nu'')} &= D_i (\partial_{\nu'} \phi^\nu' \cdots \partial_{\nu''} \phi^\nu'') \\
\partial_i \phi^\mu \partial_j \phi^\nu (D_{\nu} T_{\nu'...\nu''}) &= D_i \left( \partial_{\nu'} \phi^\nu' \cdots \partial_{\nu''} \phi^\nu'' \right) \\
\partial_i \phi^\mu \partial_j \phi^\nu \epsilon^{kl} (D_k K_{lj}) &= 0.
\end{align*}
\]

Another property which is needed later on is the following one: for any spatial tensor \(P_{\alpha_1...\alpha_p}, p \in \mathbb{N}\) with respect to \(n^\mu\) one can show that
\[
\begin{align*}
P_{\alpha_1...\alpha_p} |_{\phi(\Sigma)} = 0 &\iff \phi^* (P_{\alpha_1...\alpha_p}) = 0. \tag{2.19}
\end{align*}
\]

3. Weyl candidates

Fundamental for our discussion will be tensors of rank 4 having the same algebraic properties as the Weyl tensor. More precisely, one defines a Weyl candidate as any rank-4 tensor \(C_{\alpha\beta\gamma\delta}\) fulfilling the properties
\[
\begin{align*}
C_{[\alpha\beta]\gamma\delta} &= C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}, \\
C^\alpha_{\alpha\beta\gamma\delta} &= 0, \\
C_{[\alpha\beta\gamma\delta]} &= 0. \tag{3.1}
\end{align*}
\]
From (3.1) we easily deduce that \( *C_{a\beta\gamma\delta} = C^*_{a\beta\gamma\delta} = (1/2)\eta_{\gamma\delta\sigma\tau} C^\sigma_{a\beta} \) and indeed \( C^*_{a\beta\gamma\delta} \) is also a Weyl candidate. Given a Weyl candidate \( C_{a\beta\gamma\delta} \), its Weyl current \( J_{a\beta\gamma\delta} \) is defined by

\[
J_{a\beta\gamma\delta} \equiv \nabla_a C_{\beta\gamma\delta}.
\] (3.2)

The tensor \( J_{a\beta\gamma\delta} \) is trace-free, antisymmetric in the last pair of indices and it has the property

\[
J_{[\beta\gamma\delta]} = 0.
\]

From (3.2) we deduce the following identities:

\[
\nabla_{[a} C_{\beta\gamma\delta] \mu\nu} = -\frac{1}{3} \eta_{a\beta\gamma\delta} J^*_{\sigma\mu\nu}, \quad \nabla_{\beta} J^\beta_{a\gamma\delta} = R^\gamma_{\lambda\phi\rho\mu} C^\lambda_{a\delta\rho\mu},
\] (3.3)

where \( J^*_{\sigma\mu\nu} \equiv \eta_{\mu\nu} J^\sigma_{\rho\lambda} / 2. \) Equations (3.2) and (3.3) have counterparts involving \( C^*_{\mu\nu\rho\sigma} \) which we will not discuss. Also, if we combine (3.2) and (3.3) we get a wave equation for the Weyl candidate:

\[
\nabla_{\alpha} \nabla_{\alpha} C_{\beta\gamma\chi\rho} = -2 R^\gamma_{\delta\alpha \chi} C_{\beta\rho\delta\chi} + 2 R^\alpha_{\chi\rho} C_{\beta\gamma\delta\chi} + 2 R^\alpha_{\delta\rho\chi} C_{\beta\gamma\chi\delta} + \eta_{\gamma\rho\lambda\pi} \nabla_{\pi} J^\lambda_{\beta\gamma\delta} + 2 \nabla_{[\chi} J^\beta_{\delta\gamma]}.
\] (3.4)

The previous considerations hold regardless of whether the Weyl candidate is real or complex. Indeed, from any real Weyl candidate \( C_{\mu\nu\rho\sigma} \) we may construct a complex Weyl candidate \( C^*_{\mu\nu\rho\sigma} \) by means of

\[
C^*_{\mu\nu\rho\sigma} = \frac{1}{2} (C_{\mu\nu\rho\sigma} + i C^\sigma_{\mu\nu\rho\sigma}).
\] (3.5)

A Weyl candidate constructed in such a way is self-dual.\(^3\)

\[
C^*_{\mu\nu\rho\sigma} = i C^\sigma_{\mu\nu\rho\sigma}.
\] (3.6)

Inversely, if a complex Weyl candidate \( C^*_{\mu\nu\rho\sigma} \) fulfils (3.6) then its real part \( \text{Re}(C_{\mu\nu\rho\sigma}) \) and its imaginary part \( \text{Im}(C_{\mu\nu\rho\sigma}) \) are also Weyl candidates related via

\[
\text{Im}(C_{\mu\nu\rho\sigma}) = \frac{1}{2} \eta_{\rho\lambda\mu\nu} \text{Re}(C_{\lambda\sigma\mu\nu}), \quad \text{Re}(C_{\mu\nu\rho\sigma}) = -\frac{1}{2} \eta_{\rho\lambda\mu\nu} \text{Im}(C_{\lambda\sigma\mu\nu}).
\] (3.7)

### 3.1. The causal propagation of a Weyl candidate

Besides the basic algebraic properties of a Weyl candidate just explained we need to introduce a further concept which will play a crucial role in the sequel.

**Definition 1.** Let \((\mathcal{M}, g_{\mu\nu})\) be a spacetime and consider a Weyl candidate \( C_{a\mu\nu\rho} \) defined in the whole of \( \mathcal{M} \). We say that the Weyl candidate \( C_{a\mu\nu\rho} \) propagates causally on \( \mathcal{M} \) if for any embedded spacelike hypersurface \( B \subset \mathcal{M} \) the condition \( C_{a\mu\nu\rho}|_B = 0 \) implies \( C_{a\mu\nu\rho} = 0 \) on \( D(B) \).

Given a set \( B \), a point \( p \in \mathcal{M} \) belongs to the future Cauchy development \( D^+(B) \) if any past-inextendible causal curve containing \( p \) intersects \( B \). There is a similar notion of \( D^-(B) \) and the set \( D(B) \equiv D^+(B) \cup D^-(B) \) is called the Cauchy development of \( B \). It is well known that the interior of \( D(B) \) is globally hyperbolic and since \( B \) is an embedded spacelike hypersurface then it will be a Cauchy hypersurface for \( D(B) \)—see e.g. [29] for an elementary introduction to these concepts of causality theory.

The simplest example of a Weyl candidate which propagates causally is the Weyl tensor \( W_{a\mu\nu\rho} \) of a vacuum spacetime—see e.g. [5]. It is possible to address the causal propagation of a Weyl candidate by following the general techniques given in [4]—see also [26]—in which the causal propagation of any tensor is analysed. These ideas are reviewed next.

\(^3\) Note that sometimes the opposite convention is followed in the literature whereby the complex conjugate of \( C_{\mu\nu\rho\sigma} \) is given the name of ‘self-dual’.
An essential object to study the causal propagation of a Weyl candidate is its Bel–Robinson tensor. Given a real Weyl candidate \(C_{\alpha\beta\mu\nu}\), we define its Bel–Robinson tensor by

\[
T_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + C_{\alpha\beta\gamma\delta} \epsilon^\gamma_{\mu} \epsilon^\delta_{\nu}. 
\]

The previous definition has been described in a more general framework in [25] as the superenergy tensor of the Weyl candidate \(C_{\alpha\beta\mu\nu}\). In our particular context we prefer to call this tensor the Bel–Robinson tensor of \(C_{\alpha\beta\mu\nu}\) in analogy with the Bel–Robinson tensor constructed out of the Weyl tensor [3]. The Bel–Robinson tensor of a Weyl candidate has the following properties—see [25] for detailed proofs of points (i)–(iii); point (iv) follows by taking the covariant divergence of (3.8) and then using repeatedly (3.2) and (3.3).

**Theorem 2.** If \(T_{\alpha\beta\mu\nu}\) is the Bel–Robinson tensor of the Weyl candidate \(C_{\alpha\beta\mu\nu}\), then

(i) \(T_{\alpha\beta\mu\nu} = T_{\alpha\beta\mu\nu}, T^a_{\alpha\beta\mu\nu} = 0\).

(ii) Generalized dominant property: if \(u_1^\alpha, u_2^\alpha, u_3^\alpha, u_4^\alpha\) are causal future-directed vectors then \(T_{\alpha\beta\gamma\delta} u_1^\alpha u_2^\alpha u_3^\alpha u_4^\alpha \geq 0\). This property admits an alternative formulation: let \(E = \{e_0^\alpha, e_1^\alpha, e_2^\alpha, e_3^\alpha\}\) be any orthonormal frame with \(e_0^\alpha\) the timelike vector. Then the following inequality holds:

\[
T_{\alpha\beta\gamma\delta} = \left|T_{\alpha\beta\gamma\delta}\right|, 
\]

where the component indices refer to the frame \(E\).

(iii) \(T_{\alpha\beta\mu\nu} = 0 \iff C_{\alpha\beta\mu\nu} = 0 \iff \exists \ a \ timelike \ vector \ u^\alpha \ such \ that \ T_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu = 0\).

(iv) The covariant divergence of \(T_{\alpha\beta\mu\nu}\) is given by

\[
\nabla_\alpha T_{\alpha\beta\gamma\delta} = 4 J^{\alpha}_{\ (\gamma} C_{\beta)\delta\alpha\beta} - 2 J^{\alpha}_{\ (\gamma} C_{\beta)\delta\beta\sigma} - g_{\beta\gamma} J^{\alpha\beta\rho\sigma} C_{\delta\rho\sigma}. \]

Next, we explain how the properties summarized in theorem 2 are useful in the study of the causal propagation of \(C_{\alpha\beta\mu\nu}\). We follow closely [4] in our exposition. Let \(D^+(B)\) be the future Cauchy development of a closed achronal hypersurface \(B\) and pick up any event \(q \in D^+(B)\). Furthermore, let \(J^-(q)\) denote the causal past of the event \(q\). The set \(K = J^-(q) \cap D^+(B)\) is compact and it contains the set \(\hat{B} \equiv K \cap B\). The boundary of \(K\) is \(\partial K = \hat{B} \cup H^+(B)\) where \(H^+(B)\) is the future Cauchy horizon of \(B\). Also, since the interior of \(D^+(B)\) is globally hyperbolic we deduce that it can be foliated by a family of spacelike hypersurfaces \(\{\Sigma_t\}, t \in I\), where \(t_0 < t < t_1, t_0 < 0 < t_1\) and \(\Sigma_0 = B\). Figure 1 shows a schematic view of this geometric construction.

Let \(n_\mu\) be the timelike normal 1-form to the leaves of \(\{\Sigma_t\}_{t \in I}\) and define the quantity

\[
W(t) \equiv \int_{-J^-(\Sigma_t) \cap K} T_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma \ d\mathcal{M} = \int_0^t \left( \int_{\Sigma_t \cap K} T_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma \ d\Sigma_t \right) dt. \]

Here, \(d\mathcal{M}\) is the positive measure on the spacetime \(\mathcal{M}\) constructed from the volume form \(\eta_{\mu\nu\rho\sigma}\) (any measure defined from a volume form will be understood as positive) and \(d\Sigma_t\) is the measure on the hypersurface \(\Sigma_t\) obtained from its volume form \(n^\mu \eta_{\mu\nu\rho\sigma} |_{\Sigma_t}\). The scalar \(T_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma\) is called the superenergy density and is everywhere non-negative—see point (ii) of theorem 2—which entails \(W(t) \geq 0\). Also, from the definition of \(W(t)\) we get

\[
W'(t) = \int_{\Sigma_t \cap K} T_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma \ d\Sigma_t = \int_{\Sigma_t \cap K} P_\mu n^\mu \ d\Sigma_t, \]

where \(P_\mu\) is defined by

\[
P_\mu \equiv T_{\mu\nu\rho\sigma} n^\nu n^\rho n^\sigma. \]
Figure 1. Schematic depiction of the sets $B$, $\tilde{B}$, $D^*(B)$ and $H^*(\tilde{B})$—see the main text for further details.

(This figure is in colour only in the electronic version)

Again point (ii) of theorem 2 implies that $P_\mu n_\mu$ is also a non-negative quantity and hence $W'(t) \geq 0$. Now, according to the Gauss theorem we have

$$\int_{J^- (\Sigma_t) \cap K} \nabla_\mu P^\mu \, d\mathcal{M} = \int_{\Sigma_t \cap K} P_\mu n_\mu \, d\Sigma_t - \int_B P_\mu n_\mu \, d\tilde{B} + \int_{H^* (\tilde{B})} P_\mu k_\mu \, d(H^* (\tilde{B})), \quad (3.13)$$

where $d\tilde{B}$ and $d(H^* (\tilde{B}))$ denote the measures on the hypersurfaces $\tilde{B}$ and $H^* (\tilde{B})$ respectively induced by the volume forms $n_\mu \eta_{\mu \nu \rho \sigma} |_{\tilde{B}}$ and $k_\mu \eta_{\mu \nu \rho \sigma} |_{H^* (\tilde{B})}$. In this last expression $k_\mu$ is a causal vector which is orthogonal to the null hypersurface $H^* (\tilde{B})$, points outward to $K$ (therefore it is future directed) and is such that $k_\mu \eta_{\mu \nu \rho \sigma} |_{H^* (\tilde{B})} \neq 0$. Note that these three conditions do not fix the vector $k_\mu$ univocally. By combining (3.11) and (3.13) we deduce

$$W'(t) = \int_{J^- (\Sigma_t) \cap K} \nabla_\mu P^\mu \, d\mathcal{M} + \int_B P_\mu n_\mu \, d\tilde{B} - \int_{H^* (\tilde{B}) \cap J^- (\Sigma_t)} P_\mu k_\mu \, d(H^* (\tilde{B})). \quad (3.14)$$

Now, we need to estimate the right-hand side of (3.14). According to point (ii) of theorem 2, the quantity $P_\mu k_\mu$ is positive and therefore

$$0 \leq \int_{H^* (\tilde{B}) \cap J^- (\Sigma_t)} P_\mu k_\mu \, d(H^* (\tilde{B})). \quad (3.15)$$

On the other hand, if $C_{\alpha \beta \mu \nu}$ vanishes on $B$ then $W(0) = 0$ and

$$\int_B P_\mu n_\mu \, d\tilde{B} = 0. \quad (3.16)$$

Next, we set up an orthonormal frame $E = \{ e_0^\mu, e_1^\mu, e_2^\mu, e_3^\mu \}$ with $e_0^\mu = n^\mu$ and we consider the set of components $\nabla_a R_{b c}$ in such a frame. Since these are continuous functions we deduce that in the compact set $K$ the estimate $\nabla_a R_{b c} \leq m_0, \forall a, b, c, d = 0, 1, 2, 3$ holds for some constant $m_0$. Hence

$$\nabla_a P^a = 3n_a n_b T^{c d a b} \nabla_a R_{b c} + n_a n_b n_c \nabla_d T^{d a b c} \leq 3m_0 \sum_{c,d=0}^3 T^{00 c d} + n_a n_b n_c \nabla_d T^{d a b c} \leq 3m_0 T^{a b c d} n_a n_b n_c n_d + n_a n_b n_c \nabla_d T^{d a b c}, \quad (3.17)$$

where in the last step, point (ii) of theorem 2 was used. From here, we deduce

$$\int_{J^- (\Sigma_t) \cap K} \nabla_\mu P^\mu \, d\mathcal{M} \leq \int_{J^- (\Sigma_t) \cap K} (\nabla_\mu T^{\mu \nu \rho \sigma}) n_\mu n_\nu n_\rho n_\sigma \, d\mathcal{M} + 3m_0 \int_{J^- (\Sigma_t) \cap K} T^{\mu \nu \rho \sigma} n_\mu n_\nu n_\rho n_\sigma \, d\mathcal{M}. \quad (3.18)$$
Using the above estimates in (3.14) we get

\[
0 \leq W(t) \leq \int_{\mathcal{J}^{-}(\Sigma \cap K)} (\nabla_{\mu} T^{\mu\nu\rho\sigma}) n_{\nu} n_{\rho} n_{\sigma} \, dM + 3m_{0}W(t).
\]  

(3.19)

Now it only remains to estimate \( n_{\nu} n_{\rho} n_{\sigma} \nabla_{\mu} T^{\mu\nu\rho\sigma} \) and to that end (3.9) is used. The actual estimate will depend on the particular expression of the Weyl current. For example, the simplest case arises when \( J_{\mu\nu} = 0 \) where no further estimates are needed and Grönwall’s lemma entails

\[
W(t) = 0, \quad t \in [0, t_{1}),
\]

(3.20)

which in turn implies that the scalar \( T_{\mu\rho\sigma} n^{\mu} n^{\rho} n^{\sigma} \) vanishes in \( K \). From the arbitrariness of \( q \) we deduce that the superenergy density is zero in \( D^{+}(B) \) and by a similar argument as the above one can show that this is also the case for \( D^{-}(B) \). Hence point (iii) of theorem 2 implies that \( C_{\alpha\beta\mu\nu} = 0 \) on \( D^{+}(B) \). Consequently the tensor \( C_{\alpha\beta\mu\nu} \) propagates causally whenever the Weyl current is zero.

**Remark 1.** Point (iii) of theorem 2 entails \( C_{\alpha\beta\mu\nu}|_{B} = 0 \) if and only if \( (T_{\alpha\beta\mu\nu} n^{\alpha} n^{\beta} n^{\mu})|_{B} = 0 \). Accordingly, the causal propagation of a Weyl candidate can also be formulated in terms of the vanishing of the superenergy density at the hypersurface \( B \).

4. The Mars–Simon tensor

We start this section by recalling some basic background about vacuum spacetimes possessing a Killing vector—see e.g. [18] for a complete account. Let \((\mathcal{M}, g_{\mu\nu})\) be a vacuum solution of Einstein field equations and let \( \xi_{\mu} \) be a Killing vector,

\[
\nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} = 0.
\]

(4.1)

This condition enables us to define a 2-form, the *Killing form*, by

\[
F_{\mu\nu} \equiv \nabla_{(\mu} \xi_{\nu)} = \nabla_{\mu} \xi_{\nu}.
\]

(4.2)

Elementary manipulations lead to

\[
\nabla_{\mu} F_{\nu\kappa} = W_{\nu\kappa\mu\rho} \xi_{\rho}.
\]

(4.3)

The **self-dual Killing form** \( F_{\mu\nu} \) is then

\[
F_{\mu\nu} = F_{\mu\nu} + iF^{*}_{\mu\nu},
\]

(4.4)

with \( F^{*}_{\mu\nu} = (1/2)\eta_{\mu\nu\rho\sigma} F^{\rho\sigma} \) the Hodge dual of \( F_{\mu\nu} \). From the previous definition it is straightforward to obtain the property (self-duality)

\[
F_{\mu\nu} = iF^{*}_{\mu\nu}.
\]

(4.5)

We shall write \( \mathcal{F} = F_{\mu\nu} F^{\mu\nu} \). The Killing form is said to be degenerate if \( \mathcal{F} = 0 \). Other basic algebraic properties of \( F_{\mu\nu} \) are

\[
F_{\mu\rho} F^{\rho}_{\nu} = \frac{1}{2} g_{\nu\mu} \mathcal{F}, \quad F_{(\mu} F_{\nu)} = 0, \quad F_{\mu\nu} F^{\alpha\nu} = 0.
\]

(4.6)

From a Killing vector \( \xi^{\mu} \) we may define its associated *Ernst 1-form* by

\[
\sigma_{\nu} \equiv 2\xi^{\mu} F_{\mu\nu} = -\nabla_{\nu} \Lambda - i\Omega_{\nu},
\]

(4.7)

where \( \Lambda \equiv \xi^{\mu} \xi_{\mu} \) is the norm of the Killing vector and \( \Omega_{\mu} \equiv \eta_{\mu\nu\lambda\sigma} \xi^{\nu} \nabla^{\lambda} \xi^{\sigma} \) denotes its *twist form*. It is well known that in a vacuum spacetime \( \sigma_{\mu} \) is closed—i.e. \( \nabla_{\nu} \sigma_{\mu} = 0 \). Thus locally—if we are in a simply connected region—there exists a scalar field \( \sigma \), the *Ernst potential*, such that \( \sigma_{\mu} = \nabla_{\mu} \sigma \). The self-dual Weyl tensor and the self-dual Killing form are related by

\[
\nabla_{\mu} F_{\nu\rho} = 2\nabla_{(\nu\rho} \xi_{\mu)}.
\]

(4.8)
a property which is easily obtained from (4.2). Finally, we introduce the tensor $I_{\mu\nu\lambda\rho}$ defined by

$$I_{\mu\nu\lambda\rho} \equiv \frac{1}{2} (g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda} + i\eta_{\mu\nu\lambda\rho}).$$

(4.8)

The tensor $I_{\mu\nu\lambda\rho}$ has the obvious symmetries $I_{\mu\nu\lambda\rho} = I_{\nu\lambda\mu\rho} = I_{\lambda\rho\mu\nu}$. This tensor can be regarded as a metric in the space of self-dual 2-forms.

We have now all the ingredients to define the Mars–Simon tensor. This is a 4-rank tensor $S_{\mu\nu\lambda\rho}$ whose expression is given by

$$S_{\mu\nu\lambda\rho} \equiv 2\mathcal{W}_{\mu\nu\lambda\rho} + \frac{6}{1 - \sigma} \left( \mathcal{F}_{\mu\nu} - \frac{1}{6} \mathcal{F}^2 I_{\mu\nu\lambda\rho} \right).$$

(4.9)

From the previous expression we deduce that in a vacuum spacetime admitting a Killing vector, the Mars–Simon tensor is determined up to a choice for the Ernst potential. We make a choice such that $\sigma \neq 1$ on $M$. Our interest in the Mars–Simon tensor lies in the following result—see [21, 22] for a proof.

**Theorem 3.** Let $(M, g_{\mu\nu})$ be a smooth, vacuum spacetime with a Killing vector $\xi^\mu$. Let $\mathcal{F}_{\mu\nu}$ denote the associated self-dual Killing form. If there is a non-vanishing real constant $M$ such that the conditions

$$\mathcal{F}^2 = -M^2 (1 - \sigma)^4,$$

(4.10a)

$$S_{\mu\nu\lambda\rho} = 0$$

(4.10b)

hold on a non-empty $\mathcal{N} \subset M$ then $(\mathcal{N}, g_{\mu\nu})$ is locally isometric to the Kerr spacetime.

Condition (4.10a) of theorem 3 can be reformulated in the form

$$\Xi(M, \tilde{\xi}) = 0,$$

(4.11)

where the scalar $\Xi(M, \tilde{\xi})$ is defined for any Killing vector $\tilde{\xi}$ and any real constant $M$ by the expression

$$\Xi(M, \tilde{\xi}) \equiv \mathcal{F}^2 + M^2 (1 - \sigma)^4.$$ 

(4.12)

A very important property of the scalar $\Xi(M, \tilde{\xi})$ is that it fulfills the differential equation

$$\nabla_\mu \Xi = -2\mathcal{F}^{\alpha\beta} S_{\mu\alpha\beta\xi} + 4\Xi \sigma_{\mu \sigma \xi}.$$ 

(4.13)

To see it, differentiate both sides of (4.12) and use (4.7) to replace the covariant derivatives of $\mathcal{F}_{\mu\nu}$. The result is

$$\nabla_\mu \Xi = -4(\mathcal{F}^{\alpha\beta} \mathcal{W}_{\mu\alpha\beta\xi}^\rho + M^2 \sigma_{\mu}(1 - \sigma)^3).$$

(4.14)

Next use (4.9) to write $\mathcal{W}_{\mu\alpha\beta\xi}$ in terms of $S_{\mu\alpha\beta\xi}$ and replace all the occurrences of $I_{\mu\nu\lambda\rho}$ by means of (4.8) to obtain

$$\nabla_\mu \Xi = 2(2M^2 \sigma_{\mu}(\sigma - 1)^3 - \mathcal{F}^{\alpha\beta} S_{\mu\alpha\beta\xi}) + \frac{\mathcal{F}^2 (-10\mathcal{F}_{\mu\rho} + 2i\mathcal{F}^\ast_{\mu\rho}) \xi^\rho}{\sigma - 1}.$$ 

(4.15)

We use here (4.4) and (4.6) with the result

$$\nabla_\mu \Xi = -2\mathcal{F}^{\alpha\beta} S_{\mu\alpha\beta\xi} + \frac{4\sigma_{\mu}(\mathcal{F}^2 + M^2 (1 - \sigma)^4)}{\sigma - 1},$$

(4.16)

from which (4.13) is deduced.

---

4 We follow the convention set in [19] of calling this tensor the Mars–Simon tensor. Note however, that our conventions are slightly different.
4.1. Mars–Simon tensor as a Weyl candidate

In [19] it is proved that the Mars–Simon tensor $\mathcal{S}_{\mu
u\alpha\beta}$ is a Weyl candidate. Furthermore, it was shown that its Weyl current is a linear function of $\mathcal{S}_{\mu
u\alpha\beta}$ in which case the wave equation (3.4) is homogeneous in $\mathcal{S}_{\mu
u\alpha\beta}$. The derivation of these results are discussed as some of the intermediate calculations will be needed in section 5.

**Proposition 1.** The Mars–Simon tensor is a Weyl candidate.

**Proof.** From (4.9) it is clear that $\mathcal{S}_{[\mu
u\alpha\lambda]} = \mathcal{S}_{[\mu
u\alpha\lambda]} = \mathcal{S}_{0}$. Using (4.5) one can show that $\mathcal{S}_{\mu
u\lambda\alpha} = 0$ and therefore the Mars–Simon tensor is traceless. To finish our proof we need to show that the Mars–Simon tensor fulfills the cyclic identity. To that end we compute $\mathcal{S}_{\mu[\nu\rho\lambda]}$ getting

$$\mathcal{S}_{\mu[\nu\rho\lambda]} = \frac{6}{1 - \sigma} \mathcal{F}_{\mu[\nu\rho\lambda]} - \frac{i \mathcal{F}^{2}}{2(1 - \sigma)} \eta_{\mu\nu\rho\lambda}. \quad (4.17)$$

The right-hand side of this expression vanishes by virtue of the identity

$$\mathcal{F}_{\mu[\nu\rho\lambda]} = \frac{1}{12} \mathcal{F}^{2} \eta_{\mu\nu\rho\lambda}, \quad (4.18)$$

which in turn is a consequence of the algebraic property $\mathcal{F}_{\mu[\nu\rho\lambda]} = \mathcal{F}_{[\mu\nu\rho\lambda]}$ and the fact that in four dimensions $\mathcal{F}_{[\mu\nu\rho\lambda]}$ is proportional to $\eta_{\mu\nu\rho\lambda}$. \qed

A direct computation shows that the Mars–Simon tensor is self-dual

$$\mathcal{S}_{\mu
u\rho\lambda} = i \mathcal{S}^{*}_{\mu
u\rho\lambda}. \quad (4.19)$$

Therefore $\text{Re}(\mathcal{S}_{\mu
u\rho\lambda})$ and $\text{Im}(\mathcal{S}_{\mu
u\rho\lambda})$ are then related by duality as explained in section 3:

$$\text{Im}(\mathcal{S}_{\mu
u\rho\lambda}) = \frac{1}{2} \eta_{\rho\lambda\mu\nu} \text{Re}(\mathcal{S}^{\rho\lambda}_{\mu\nu}), \quad \text{Re}(\mathcal{S}_{\mu
u\rho\lambda}) = -\frac{1}{2} \eta_{\rho\lambda\mu\nu} \text{Im}(\mathcal{S}^{\rho\lambda}_{\mu\nu}). \quad (4.20)$$

Any property of $\mathcal{S}_{\mu
u\rho\lambda}$ will admit an equivalent counterpart formulated in terms of either its real or imaginary part. In particular, this shows that both $\text{Re}(\mathcal{S}_{\mu
u\rho\lambda})$ and $\text{Im}(\mathcal{S}_{\mu
u\rho\lambda})$ are Weyl candidates and any of them contains the same information as the original Mars–Simon tensor.

**Proposition 2.** The covariant divergence of the Mars–Simon tensor is given by

$$\nabla_{a} \mathcal{S}^{\mu}_{\beta\gamma\delta} = \frac{1}{\sigma - 1} \left(4(\mathcal{F}_{[\alpha\beta\gamma\delta]} + \mathcal{F}^{\alpha\beta}_{\gamma\delta}) + 2 g_{\beta\gamma} \mathcal{S}^{\alpha}_{\mu\rho\lambda} \mathcal{F}^{\rho\lambda} \right) \xi^{a}. \quad (4.21)$$

**Proof.** To calculate $\nabla_{a} \mathcal{S}^{\mu}_{\beta\gamma\delta}$ we start from (4.9) and take the covariant divergence of $\mathcal{S}_{\mu
u\rho\lambda}$. Next we apply relations (4.7), $\nabla_{\mu} \sigma = \sigma_{\mu}$, $\nabla_{a} \mathcal{W}^{\mu} \sigma_{\nu} = 0$ and $\nabla_{a} \mathcal{I}_{\mu\nu\rho\sigma} = 0$. Finally we expand all the occurrences of $\mathcal{I}_{\mu\nu\rho\sigma}$ using (4.8) with the result

$$\nabla_{a} \mathcal{S}^{\mu}_{\beta\gamma\delta} = \frac{1}{2(1 - \sigma)^{2}} \left(-12 \mathcal{F}_{\mu\nu} \mathcal{F}^{\sigma\nu} + \eta_{\beta\gamma\delta\mu} \mathcal{S}^{\mu}_{\alpha\rho\lambda} \mathcal{F}^{\alpha\rho\lambda} + 2 g_{\beta\gamma} \mathcal{S}^{\alpha}_{\mu\rho\lambda} \mathcal{F}^{\rho\lambda} \right)$$

$$+ \frac{1}{1 - \sigma} \left((2i \eta_{\beta\gamma\delta\mu} \mathcal{W}^{\mu}_{\rho\sigma} \xi^{\rho} + 4 g_{\beta\gamma} \mathcal{W}^{\mu}_{\rho\sigma} \sigma^{\rho}) \mathcal{F}^{\rho\sigma} - 12 \mathcal{F}_{\mu} \mathcal{W}^{\rho\sigma} \xi^{\rho} \right). \quad (4.22)$$

The dependence on the Ernst 1-form can be removed from this expression if we write $\sigma_{\mu}$ in terms of $\mathcal{F}_{\mu\nu}$ and $\xi^{\mu}$ by means of (4.6). The final step is to rearrange the terms in the resulting expression in order to render it in the form of (4.21). This is accomplished by using (4.9) to write $\mathcal{W}^{\mu}_{\rho\sigma}$ in terms of the Mars–Simon tensor, expanding again all the occurrences of $\mathcal{I}_{\mu\nu\rho\sigma}$ by means of (4.8) and using (4.5), (4.4) where necessary.

Equation (4.21) enables us to define the Weyl current of the Mars–Simon tensor as

$$\mathcal{J}_{\beta\gamma\delta} = \frac{1}{\sigma - 1} \left(4(\mathcal{F}_{[\alpha\beta\gamma\delta]} + \mathcal{F}^{\alpha\beta}_{\gamma\delta}) + 2 g_{\beta\gamma} \mathcal{S}^{\alpha}_{\mu\rho\lambda} \mathcal{F}^{\rho\lambda} \right) \xi^{a}. \quad (4.23)$$
Since $J_{\mu\nu\beta}$ is linear in the Mars–Simon tensor, we conclude—in view of (3.4)—that $S_{\mu\nu\rho\sigma}$ fulfils a homogeneous wave equation. This interesting property was used in [19] to show that the domain of outer communication of an asymptotically flat smooth stationary spacetime is, under certain additional conditions, isometric to the Kerr domain of outer communication—i.e. the uniqueness of Kerr.

Using $\nabla_\alpha S_{\alpha\mu\nu\rho} = J_{\mu\nu\rho}$ together with the self-duality of the Mars–Simon tensor one deduces

$$ J_{\mu\nu\rho} = i J^*_{\mu\nu\rho} = \frac{i}{2} \eta_{\nu\rho\sigma} J_{\mu}^{\sigma}, $$

that is to say, $J_{\mu\nu\rho}$ is also self-dual with respect to the block of indices $\nu\rho$.

### 4.2. Orthogonal splitting of the Mars–Simon tensor

In our forthcoming calculations it is necessary to use the orthogonal splitting of $S_{\mu\nu\rho\lambda}$ with respect to a unit timelike vector field $n^\mu$. Since $S_{\mu\nu\rho\lambda}$ is a Weyl candidate we can use (2.12) to find such an orthogonal splitting. The resulting split is given by

$$ S_{\mu\nu\rho\lambda} = 2(t_{\nu\lambda} T_{\rho\mu} + t_{\rho\mu} T_{\nu\lambda}) + \eta_{\rho\lambda\eta} n_{(\mu} T_{\nu)\mu} + i \eta_{\rho\lambda\mu} n_{\nu} T_{\eta\mu}, $$

where

$$ T_{\mu\nu} = S_{\mu\nu\rho\lambda} n^\rho n^\lambda. $$

The tensor $T_{\mu\nu}$ is symmetric, spatial and traceless. Alternatively, the orthogonal splitting of the Mars–Simon tensor can be calculated directly from (4.9). To that end one needs to find the orthogonal splitting of the different quantities which appear in (4.9). The orthogonal splitting of $W_{\mu\nu\rho\lambda}$ is given in (2.12) and the orthogonal splitting of $F_{\mu\nu}$ is calculated by means of (2.8) and is given by

$$ F_{\mu\nu} = i \varepsilon_{\mu\nu\rho} E^\rho - 2 \varepsilon_{[\mu} n_{\nu]}, $$

where

$$ E_\mu \equiv F_{\mu\rho} n^\rho. $$

One also needs the orthogonal splitting of $I_{\mu\nu\rho\lambda}$ which is easily calculated from (4.8) and (2.5) so that

$$ I_{\mu\nu\rho\lambda} = h_{\mu\nu} h_{\rho\lambda} + 2 n_{[\mu} h_{\nu]\rho\lambda} + i (n_{[\mu} \varepsilon_{\nu]\rho\lambda} + n_{\mu} \varepsilon_{\rho\lambda} n_{\nu}). $$

We insert relations (2.12), (4.27) and (4.29) into formula (4.9) and then equal the resulting expression to (4.25). The result of that is the relation

$$ T_{\mu\nu} = 2 \bar{E}_{\mu\nu} + \frac{2}{1 - \sigma} (3 \bar{E}_\mu E_\nu - h_{\mu\nu} E_\rho E^\rho), $$

which shall be used later on. Also needed later on is the relation

$$ \mathcal{F}^2 = -4 E_\mu E^\mu \quad \Rightarrow \quad \mathcal{F}(M, \tilde{\xi}) = -4 E_\mu E^\mu + M^2 (1 - \sigma)^4, $$

which is derived from equation (4.27).

The Killing vector $\xi^\mu$ can be decomposed in the form $\xi^\mu = -Y n^\mu + Y^\mu$ where $Y, Y^\mu$ are called the Killing lapse and the Killing shift respectively. We need to find a formula relating $E_\mu$ to $Y, Y^\mu$. To that end, we start by taking the covariant derivative of both sides of $\xi_\mu = -Y n_\mu + Y^\mu$ and replace $\nabla_\mu n_\nu$ by means of (2.4). The result is

$$ \nabla_\mu \xi_\nu = Y (K_{\mu\nu} + A_{\nu} n_\mu) + \nabla_\mu Y_\nu - n_\nu \nabla_\mu Y. $$
Next we use in this expression the property
\[ \nabla_{\mu} Y = D_{\mu} Y - n_{\mu} \mathcal{L}_{\vec{n}} Y \]
and equation (2.14) to replace \( \nabla_{\mu} Y \) and \( \nabla_{\mu} Y_{\nu} \) respectively yielding
\[
\nabla_{\mu} Y_{\nu} = Y K_{\mu \nu} + D_{\mu} Y_{\nu} + n_{\mu} n_{\nu} (\mathcal{L}_{\vec{n}} Y - A^\rho Y_{\rho})
\]
\[ + n_{\mu} (Y A_{\nu} - K_{\nu \rho} Y_{\rho} - \mathcal{L}_{\vec{n}} Y_{\nu}) - n_{\nu} (K_{\mu \rho} Y_{\rho} + D_{\mu} Y). \tag{4.32a}
\]
\[ \text{The antisymmetric part of this expression gives the orthogonal splitting of the 2-form } F_{\mu \nu}, \text{ which is}
\]
\[ F_{\mu \nu} = -Y A_{[\mu} n_{\nu]} + n_{[\mu} D_{\nu]} Y + D_{[\mu} Y_{\nu]} - n_{[\mu} \mathcal{L}_{\vec{n}} Y_{\nu]}. \tag{4.33a}
\]
Also the symmetric part of (4.32a) is the Killing condition \( \nabla_{(\mu} \xi_{\nu)} = 0 \). Explicitly one has
\[
Y K_{\mu \nu} + D_{(\mu} Y_{\nu]} + n_{\mu} n_{\nu} (\mathcal{L}_{\vec{n}} Y - A^\rho Y_{\rho}) + n_{(\mu} (A_{\nu]} Y - 2K_{\nu \rho} Y^\rho - D_{\nu]} Y - \mathcal{L}_{\vec{n}} Y_{\nu]) = 0,
\]
from which we deduce
\[
\mathcal{L}_{\vec{n}} Y_{\mu} = A_{\mu} Y - D_{\mu} Y - 2K_{\mu \rho} Y^\rho, \tag{4.34a}
\]
which renders relation (4.33a) in the form
\[
F_{\mu \nu} = 2n_{[\mu} K_{\nu] \rho} Y^\rho + D_{[\mu} Y_{\nu]} + 2n_{[\mu} D_{\nu]} Y. \tag{4.35a}
\]
From this expression we may compute the orthogonal splitting of \( F_{\mu \nu}^* \) which is
\[
F_{\mu \nu}^* = \epsilon_{\mu \nu \beta \gamma} (K_{\beta} \gamma Y^\alpha + D_{\beta} Y) + n_{[\mu} \epsilon_{\nu] \alpha \beta \gamma} D^\alpha Y^\beta, \tag{4.36a}
\]
where equation (2.5) was used along the way. Inserting equations (4.35a) and (4.36a) into (4.3) and applying (4.28) on the resulting expression we deduce
\[
E_{\mu} = K_{\mu \rho} Y^\rho + D_{\mu} Y - \frac{i}{2} \epsilon_{\mu \nu \rho \sigma} D^\nu Y^\rho, \tag{4.37a}
\]
which is the required relation.

5. The causal propagation of the Mars–Simon tensor

In this section a proof of the following result is provided.

**Theorem 4.** The Mars–Simon tensor \( S_{\mu \nu \rho \lambda} \) propagates causally.

**Proof.** According to (4.20) it is enough to show that either \( \text{Re}(S_{\mu \nu \rho \lambda}) \) or \( \text{Im}(S_{\mu \nu \rho \lambda}) \) propagates causally. We choose to work with the former, so let us set \( S_{\mu \nu \rho \lambda} \equiv \text{Re}(S_{\mu \nu \rho \lambda}) \) and define \( L_{\mu \nu \lambda} \equiv \text{Re}(J_{\mu \nu \lambda}) \). Equation (4.21) entails
\[
\nabla_{\mu} S^a_{\rho \lambda} = L_{\rho \lambda}. \tag{5.1a}
\]
Hence, \( L_{\rho \lambda} \) is the Weyl current of the Weyl candidate \( S_{\mu \nu \rho \lambda} \). To study the causal propagation of \( S_{\mu \nu \rho \lambda} \) we follow the general procedure explained in section 3.1. We denote by \( B_{\mu \nu \rho \lambda} \), the Bel–Robinson tensor constructed from the Weyl candidate \( S_{\mu \nu \rho \lambda} \). To prove our result, we need to find a good estimate for the quantity \( n^\nu n^\rho n^\lambda \nabla_{\mu} B_{\nu \rho \lambda}^a \)—see equation (3.19). One can write
\[
n^\nu n^\rho n^\lambda \nabla_{\mu} B_{\nu \rho \lambda}^a = L_{\rho \lambda} n^\mu S_{\mu \nu \rho \lambda} + 4L_{\rho \nu}^\mu n^\alpha n^\rho n^\lambda S_{\mu \nu \rho \lambda}^a, \tag{5.2a}
\]
where (3.9) was used. In order to work out the right-hand side of equation (5.2a) we calculate the orthogonal splitting of \( S_{\mu \nu \rho \lambda} \) and \( L_{\mu \nu \rho} \) with respect to \( n^\mu \). The tensors \( S_{\mu \nu \rho \lambda} \) and \( L_{\mu \nu \rho} \) are related to \( S_{\mu \nu \rho \lambda} \) and \( J_{\mu \nu \rho} \) by means of the relations
\[
S_{\mu \nu \rho \sigma} = \frac{1}{2} (S_{\mu \nu \rho \sigma} + S_{\mu \rho \nu \sigma}), \quad L_{\mu \nu \rho} \equiv \frac{1}{2} (J_{\mu \nu \rho} + J_{\mu \rho \nu}), \tag{5.3a}
\]
and therefore their orthogonal splittings can be calculated once those of \( S_{\mu \nu \rho \sigma} \) and \( J_{\mu \nu \rho} \) are known. The orthogonal splitting of \( S_{\mu \nu \rho \sigma} \) is given in equation (4.25) and the orthogonal
splitting of $J_{\mu\nu\rho}$ can be calculated from equation (4.23) given that we know the orthogonal splitting of all the quantities which appear in the definition of $J_{\mu\nu\rho}$—see (A.1) and (A.2)-(A.5) in appendix A.1 for the precise formulae. Inserting these orthogonal splittings in (5.2) we get

$$n^\nu n^\rho n^\sigma \nabla_\mu B^\mu_{\nu\rho\sigma} = 12 \text{ Re} \left[ \frac{1}{\sigma - 1} \left( \mathcal{E}^\mu Y_\mu T^{\rho\lambda} T_{\rho\lambda} - 2 \mathcal{E}^\mu Y^\nu T^{(\mu}_\lambda T^{\nu)}_\rho \right) \right]$$

$$- 6i \varepsilon_{\mu\nu\rho} T^{\lambda\nu} \mathcal{F}^{\rho\sigma} Y \left( \frac{\mathcal{E}^\mu}{\sigma - 1} - \frac{\mathcal{E}^\nu}{\sigma - 1} \right).$$

(5.4)

Now, using equations (A.13) and (A.14) of appendix A.2 we can deduce the property

$$B_{\mu\nu\rho}(E^\alpha n^\nu n^\rho Y_\mu - E^\alpha n^\mu n^\nu Y_\rho) = -i \varepsilon_{\mu\nu\rho} E^\alpha T^{\mu\rho} \mathcal{F}^{\nu\sigma} Y + \mathcal{E}^\mu Y_\mu T^{\alpha\beta} T_{\alpha\beta} - 2 \mathcal{E}^\mu Y^\nu T^{(\nu}_\mu \mathcal{F}^{\mu)}_\rho,$$

which when combined with (5.4) yields

$$n^\nu n^\rho n^\sigma \nabla_\mu B^\mu_{\nu\rho\sigma} = \text{ Re} \left[ \frac{12}{1 - \sigma} B_{\mu\nu\rho\sigma}(E^\mu n^\nu n^\sigma Y - E^\mu n^\sigma n^\rho Y^\nu) \right].$$

(5.5)

Equation (5.5) is the key to get the required estimate. To obtain it, let us introduce an orthonormal frame $E = \{e^0_0, e^1_0, e^2_0, e^3_0\}$ with $n^\mu = e^0_0$. Since $\mathcal{E}^\mu, Y^\mu$ are spatial vectors they can be written as a linear combination of the vectors $e^1_0, e^2_0, e^3_0$. Therefore, point (ii) of theorem 2 entails the estimates

$$\text{Re} \left[ \frac{12}{1 - \sigma} B_{\mu\nu\rho\sigma} E^\mu n^\nu n^\sigma Y \right] \leq m_1 B_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma,$$

(5.6a)

$$\text{Re} \left[ \frac{12}{1 - \sigma} B_{\mu\nu\rho\sigma} E^\mu n^\sigma n^\rho Y^\nu \right] \leq m_2 B_{\mu\nu\rho\sigma} n^\mu n^\rho n^\sigma n^\rho,$$

(5.6b)

for some scalar continuous functions $m_1, m_2$ defined on $K$. Using these estimates in equation (5.5) we obtain our final estimate

$$n^\nu n^\rho n^\sigma \nabla_\mu B^\mu_{\nu\rho\sigma} \leq m_3 B_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma,$$

(5.7)

where $m_3$ is the upper bound of $m_1 + m_2$ in the compact set $K$ and thus (3.19) can be written in the form

$$0 \leq W'(t) \leq (m_3 + 3m_0) W(t), \quad \forall t \in [0, t_1).$$

(5.7)

Since $W(0) = 0$, then an application of Grönwall’s lemma enables us to conclude that $W(t) = 0, \forall t \in [0, t_1)$ which, according to the considerations made at the end of section 3.1 proves the causal propagation.

**Remark 2.** We stress that the fact that $S_{\mu\nu\rho\lambda}$ satisfies a wave equation and the property (4.21) do not, by themselves, suffice to show the causal propagation of this tensor. If we impose that $S_{\mu\nu\rho\lambda}$ and some directional derivative of this quantity vanish on a spacelike hypersurface $B$ then using the wave equation and standard results on hyperbolic partial differential equations allow us to conclude that $S_{\mu\nu\rho\lambda}$ vanishes on a neighbourhood of $B$. In doing so, extra conditions needed to be imposed so that $S_{\mu\nu\rho\lambda}$ is indeed zero in $D(B)$. This approach leads, in principle, to a weaker result than that given in theorem 4.

Consider now any vacuum spacetime admitting a Killing vector $\xi^\mu$ and use $\xi^\mu$ to construct the Mars–Simon tensor $S_{\mu\nu\rho\lambda}$ and as above, define its real part $S_{\mu\nu\rho\lambda}$ and the Bel–Robinson tensor $B_{\mu\nu\rho\lambda}$ of $S_{\mu\nu\rho\lambda}$. For any unit timelike vector $n^\mu$, define the positive scalar

$$\Phi(\bar{\mu}, \xi) \equiv B_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma.$$
One can find an explicit expression for $\Phi(\vec{n}, \vec{\xi})$ if we use the property—see equation (A.12) in appendix A.2—

$$B_{\mu\nu\rho\alpha}n^\mu n^\nu n^\rho n^\alpha = T_{\mu\nu}T_{\mu\nu} = 4E_{\mu\nu}E_{\mu\nu} - 24 \text{Re} \left[ \frac{\epsilon^{\mu\nu\rho\sigma}E_{\mu\nu}E_{\rho\sigma}}{\sigma - 1} \right] - 36 \left| \frac{\epsilon^{\mu\nu\rho\sigma}E_{\mu\nu}}{\sigma - 1} \right|^2,$$

(5.9)

which shows that $\Phi(\vec{n}, \vec{\xi})$ can be expressed in terms of $E_{\mu\nu}, E_{\mu}$ and $\sigma$ exclusively. When $\Phi(\vec{n}, \vec{\xi})$ vanishes on a Cauchy hypersurface, the causal propagation of $S_{\mu\nu\rho\lambda}$ entails the following result.

**Theorem 5.** Let $(\mathcal{M}, g_{\mu\nu})$ be a globally hyperbolic vacuum spacetime and suppose that $\mathcal{M}$ admits a Killing vector $\xi^\mu$. If there are a Cauchy hypersurface $\mathcal{B} \subset \mathcal{M}$ and a real constant $M$ such that $\Phi(\vec{N}, \vec{\xi})|_\mathcal{B} = 0$ and $\Xi(\mathcal{M}, \vec{\xi})|_\mathcal{B} = 0$ where $\vec{N}$ is the unit timelike normal to $\mathcal{B}$, then $\mathcal{M}$ is locally isometric to the Kerr spacetime.

**Proof.** If $\Phi(\vec{N}, \vec{\xi})$ is zero on $\mathcal{B}$ then by (5.8) $B_{\mu\nu\rho\alpha}N^\mu N^\nu N^\rho N^\alpha|_\mathcal{B} = 0$. Therefore, remark 1 leads us to

$$S_{\mu\nu\rho\alpha}|_\mathcal{B} = 0.$$

Using that the Mars–Simon tensor propagates causally, one deduces that $S_{\mu\nu\rho\alpha} = 0$ on $\mathcal{M}$. In particular, this implies that (4.13) is rendered in the form

$$\nabla_\mu \Xi(\mathcal{M}, \vec{\xi}) = \frac{4\sigma_\mu \Xi(\mathcal{M}, \vec{\xi})}{\sigma - 1},$$

(5.10)

which can be integrated to yield $\Xi = A(1 - \sigma)^4$ for some complex constant $A$. The condition $\Xi(\mathcal{M}, \vec{\xi})|_\mathcal{B} = 0$ implies that $A = 0$ and hence $\Xi(\mathcal{M}, \vec{\xi})$ vanishes on $\mathcal{M}$. Theorem 3 can now be applied and we conclude that $\mathcal{M}$ is locally isometric to the Kerr spacetime. □

**6. Application: characterization of Kerr initial data**

We explain in this section how to use previous results to find conditions which ensure that the data development of a vacuum initial data set is isometric to a subset of the Kerr spacetime. The basic idea behind is to express the result of theorem 5 in terms of conditions on a vacuum initial data set $(\Sigma, h_{ij}, K_{ij})$.

**Theorem 6 (Kerr initial data).** Let $(\Sigma, h_{ij}, K_{ij})$ be a vacuum initial data set and assume that there exist two scalar fields $\tilde{Y}, \tilde{\sigma}$, a vector field $\tilde{Y}_j$, and a real constant $M$, all defined on $\Sigma$, fulfilling the following conditions:

$$\epsilon^{ij}E^j = \frac{1}{4} M^2 (1 - \tilde{\sigma})^4,$$

(6.1a)

$$4\epsilon^{ij}E^j - 24 \text{Re} \left[ \frac{\epsilon^{ij}E^j}{\sigma - 1} \right] - 3 M^4 |1 - \tilde{\sigma}|^6 + 36 \left| \frac{\epsilon^{ij}E^j}{\sigma - 1} \right|^2 = 0,$$

(6.1b)

$$D_\mu \tilde{\sigma} = 2\tilde{Y} K_{\mu j} \tilde{Y}^j - 2i \epsilon_{jml} (K_{\mu}^{\rho} \tilde{Y}^\rho \tilde{Y}^m \tilde{Y}^j) + 2\tilde{Y} D_\mu \tilde{Y} + 2\tilde{Y}^j D_{[\mu} \tilde{Y}_{j]} - 2\tilde{Y} D_{[\mu} K_{j]} - 2K_{[\mu},$$

(6.1c)

$$D_\mu \tilde{Y}_j + \tilde{Y} K_{\mu j} = 0,$$

(6.1d)

$$D_\mu \tilde{Y} + L_{[\mu} K_{j]} = \tilde{Y} (r_{ij} + K K_{ij} - 2K_{[\mu} K_{j]}),$$

(6.1e)
where
\[ E_j \equiv K_{jk} \tilde{Y}^k + D_j \tilde{Y} - \frac{1}{2}i\varepsilon_{jkl} D_k \tilde{Y}^l, \tag{6.2a} \]
\[ \bar{E}_{jk} \equiv \frac{1}{i}(E_{jk} - iB_{jk}). \tag{6.2b} \]

Then the data development \((M, g_{\mu\nu})\) of \((/\Sigma_1, h_{ij}, K_{ij})\) is locally isometric to an open subset of the Kerr spacetime.

**Remark 3.** Conditions (6.1d) and (6.1e) are the Killing initial data (KID) equations. It is well known that if both a solution \((\tilde{Y}, \tilde{Y}_i)\) and the initial data set \((/\Sigma_1, h_{ij}, K_{ij})\) are suitably smooth, then the development of the initial data \((/\Sigma_1, h_{ij}, K_{ij})\) possesses a Killing vector such that the pull-back of its restriction to \(\Sigma\) coincides with the given Killing data—see for example, [9, 13], and in particular [8] for an explicit set of smoothness conditions on the Killing data. Alternatively, if the Killing data is transversal, that is \(\tilde{Y} \neq 0\), then one can make use of the concept of Killing development to ensure the existence of a spacetime with a Killing vector associated with the Killing data under question—see [1]. Again, under suitable regularity conditions it can be shown that the Killing development always includes the Cauchy development of \((/\Sigma_1, h_{ij}, K_{ij})\)—see again [9]: the Killing development is unique and maximal among the class of spacetimes containing the relevant Cauchy data and a Killing vector with non-closed orbits; hence, if the Cauchy development has a timelike Killing vector, then it must be a subset of the Killing development.

**Proof.** Let \(M\) be the data development of the vacuum initial data set \((/\Sigma_1, h_{ij}, K_{ij})\) and let \(\xi^\mu\) be the Killing vector mentioned in remark 3. As usual consider a foliation \(\{/\Sigma_t\}, t \in I\), of \(M\) with \(\Sigma_0 = \phi(\Sigma)\) and let \(n^\mu\) be the unit timelike normal to the leaves of this foliation. Let us adopt the formalism and notation explained in section 4 with \(\xi^\mu\) being the Killing vector. Then the orthogonal splitting of \(\xi^\mu\) is \(\xi^\mu = -Yn^\mu + Y^\mu\). As explained in remark 3 we have the property
\[ \phi^*Y = \tilde{Y}, \quad \partial_j \phi^\mu Y_\mu = \tilde{Y}_j, \tag{6.3} \]
which combined with equations (6.2a) and (4.37) entails \(\partial_j \phi^\mu (E_\mu) = \bar{E}_j\). The orthogonal splitting of \(\sigma_\mu\) is calculated on the one hand from equation (4.6) using expressions (4.27), (4.37) and the property \(\xi^\mu = -Yn^\mu + Y^\mu\). The result is
\[ \sigma_\mu = s_\mu + sn_\mu, \tag{6.4} \]
where
\[ s \equiv -2K_{\mu\nu}Y^\mu Y^\nu - 2Y^\mu D_\mu Y + i\varepsilon_{\mu\nu\rho\sigma} Y^\mu D^\nu Y, \]
\[ s_\mu \equiv 2YK_{\mu\rho}Y^\rho - 2i\varepsilon_{\mu
u\rho\sigma} (K_{\alpha}^\rho Y^\sigma Y^\nu + Y^\nu D^\sigma \tilde{Y}) + 2YD_\rho Y + 2Y^\nu D_\nu Y_\mu. \tag{6.5} \]
On the other hand, the orthogonal splitting of \(\sigma_\mu\) can be calculated from the relation \(\sigma_\mu = \nabla_\mu \sigma\) yielding
\[ s = -L_\mu \sigma, \quad s_\mu = D_\mu \sigma. \tag{6.6} \]
Combining the previous equation with the pull-back of equation (6.5) under \(\phi\) and (6.1c) we deduce
\[ D_j (\tilde{\sigma}) = D_j (\phi^*(\sigma)). \tag{6.7} \]
Upon a suitable choice of the Ernst potential \(\sigma\), this equation can be integrated on \(\Sigma\) to give \(\phi^*(\sigma) = \tilde{\sigma}\). Next consider the Mars–Simon tensor \(S_{\mu\nu\rho\sigma}\) constructed with the Killing vector
ξμ and calculate its orthogonal splitting with respect to nμ—equation (4.25). Using (4.30) and our previous results, we get

\[ \phi^*(\Phi(\vec{\eta}, \vec{\xi})) = \phi^*(\bar{T}_{\mu\nu}) \]

\[ = \left( 2E_{jk} + \frac{2}{1 - \sigma} (3E_{j}E_{k} - h_{jk}E_{l}E_{l}) \right) \left( 2E^{jk} + \frac{2}{1 - \sigma} (3E^{j}E^{k} - h^{jk}E_{m}E^{m}) \right). \]

(6.8)

Expanding this product and using conditions (6.1a) and (6.1b) we conclude

\[ \Phi(\vec{\eta}, \vec{\xi})|_{\phi(\Sigma)} = 0. \]

(6.9)

Moreover, equation (6.1a) implies \[ \Xi(M, \vec{\xi})|_{\phi(\Sigma)} = 0 \] —see (4.31). Thus, theorem 5 applies and therefore we conclude that \( (\mathcal{M}, g_{\mu\nu}) \) must be locally isometric to an open subset of the Kerr spacetime.

\[ \square \]

6.1. Schwarzschild initial data

Theorem 6 asserts that under certain conditions the development of a vacuum initial data set is isometric to an open subset of the Kerr spacetime but nothing is said as to the resulting development being one of the specializations of the Kerr spacetime: the Schwarzschild solution. In this section we show the form which theorem 6 takes in that case. This complements previous work of the present authors about the same issue [16].

To state our result we need a preliminary lemma whose proof can be found in [6].

**Lemma 1.** Let \( (\Sigma, h_{ij}, K_{ij}) \) be a vacuum initial data set and assume that there exist a scalar \( \tilde{Y} \) and a vector field \( \tilde{Y}_{j} \), all defined on \( \Sigma \), fulfilling the following conditions:

\[ \tilde{Y}_{D_{i}} \tilde{Y}_{j} + 2\tilde{Y}_{[i}D_{j]} \tilde{Y} + 2\tilde{Y}_{i}K_{ji} \tilde{Y}^{i} = 0, \]

(6.10a)

\[ \tilde{Y}_{i}D_{j} \tilde{Y}_{k} = 0. \]

(6.10b)

Then there exists an integrable Killing vector \( \xi^{\mu} \) in the data development of \( (\Sigma, h_{ij}, K_{ij}) \).

With the aid of this lemma we can now derive conditions guaranteeing that the development of a vacuum initial data set is isometric to an open subset of the Schwarzschild spacetime.

**Theorem 7** (Schwarzschild initial data). Let \( (\Sigma, h_{ij}, K_{ij}) \) be a vacuum initial data set and assume that there exist two scalar fields \( \tilde{Y} \) and \( \tilde{\sigma} \) and a vector field \( \tilde{Y}_{j} \), all defined on \( \Sigma \), fulfilling the conditions of theorem 6 with (6.1d) and (6.1e) replaced by (6.10a) and (6.10b). Then the data development \( \mathcal{M} \) is locally isometric to an open subset of the Schwarzschild spacetime.

**Proof.** The first step in the proof of this theorem is to use lemma 1 to show that the data development \( \mathcal{M} \) admits an integrable Killing vector \( \xi^{\mu} \). The remaining part of the proof follows the same pattern as that of theorem 6 and therefore we reach the conclusion that \( \mathcal{M} \) is isometric to an open subset of a specialization of the Kerr spacetime possessing an integrable Killing vector which, as is well known, is the Schwarzschild spacetime.

\[ \square \]

**Remark 4.** It is worth recalling that the analysis of [16] has provided an explicit formula for a timelike KID candidate in terms of concomitants of the initial data \( (h_{ij}, K_{ij}) \). Hence, a combination of theorem 7 with the such timelike KID candidate renders an algorithmic characterization of Schwarzschild initial data alternative to the one given in [16].
7. Conclusions

The main result obtained in this paper, theorem 6 provides a characterization of initial data for the Kerr spacetime by first requiring that a set of overdetermined elliptic differential equations—the KID equations—admit a solution; and then in turn requiring that certain objects constructed out of the solutions to the KID equation satisfy very particular relations. This characterization fails to be algorithmic in two ways. First, it is in practise a non-trivial problem to verify whether a certain initial data set admits a KID or not. A pointwise procedure for doing this has been discussed in [2]. Alternatively, one could resort to a non-time symmetric generalization of the geometric invariant characterizing static time symmetric initial data sets constructed in [10]—which requires solving a fourth-order elliptic partial differential equation. In any case—and this leads to the second way the result fails to be algorithmic—even if one could ascertain the existence of a KID on the initial data, one requires to know the KID in an explicit manner in order to be able to verify conditions (6.1a)–(6.1c). It is plausible that from a hypothetical characterization of the Kerr spacetime in terms of, say, concomitants of the Weyl tensor—analogous to that for the Schwarzschild spacetime given in [15]—one would be able to construct, following the ideas of [16], KID candidates for the initial data set \((\Sigma, h_{ij}, K_{ij})\). If such a KID candidate were available, then theorem 6 would constitute a powerful tool to analyse the behaviour of dynamical black hole spacetimes, both from an analytic and a numerical point of view.

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Appendix A. Technical details about the calculations

A.1. The orthogonal splitting of \(J_{\mu
u\rho}\)

Given the symmetries of the Weyl current \(J_{\mu
u\rho}\) it is immediate to realize that its orthogonal splitting must take the form

\[
J_{\mu
u\rho} = n_{\mu}Z_{\nu\rho} + Z_{\mu}X_{\nu\rho} + V_{\mu}P_{\nu\rho}. \tag{A.1}
\]

It takes a greater effort, however, to find the explicit expression of the spatial tensors \(Z_{\mu\nu}, X_{\mu\nu}, V_{\mu}, P_{\mu\rho}\). These expressions are found by inserting into equation (4.23) the orthogonal splitting of \(S_{\mu
u\rho\lambda}\) shown in (4.25), the orthogonal splitting of \(F_{\mu\nu}\) shown in (4.27) and the relation \(\xi^\mu = -Y n^\mu + Y^\mu\). A computer calculation yields the results

\[
Z_{\mu\nu} \equiv \frac{4}{1 - \sigma} \{i Y E^\alpha T_{\mu\nu \rho} e_{\nu\rho\mu} + 3 E_{\mu} T_{\nu\rho} Y^\rho + Y_{\nu} T_{\mu\rho} e_{\mu \rho} - i e_{\mu \rho} e^\alpha T_{\alpha \rho} Y^\mu\}, \tag{A.2}
\]

\[
X_{\mu\nu} \equiv \frac{1}{1 - \sigma} \{E^\alpha (4i Y (T_{\mu\nu \rho} e_{\nu\rho\mu} - e_{\nu\rho\mu} T_{\mu\rho}) + 6 T_{\mu\rho} Y_{\nu} - 2 T_{\nu\rho} Y_{\mu})
- 4 T_{\nu\alpha} Y_{\mu} + 4 h_{\mu\rho} T_{\nu\rho} Y^\rho + 6 e_{\mu\rho} T_{\nu\rho} Y^\rho), \tag{A.3}
\]

\[
V_{\mu} \equiv \frac{E^\alpha}{1 - \sigma} (4i Y (e_{\mu\lambda\rho} T_{\lambda\rho} e_{\gamma\mu\lambda} e_{\alpha \mu}) - 2 Y T_{\mu\alpha}), \tag{A.4}
\]
\[
\mathcal{P}_{\mu\nu\rho} \equiv \frac{2}{\Lambda^2}\left(2i\varepsilon_{\nu\lambda\rho}e_\mu T_\alpha^\lambda Y^\alpha + 2\varepsilon_{\rho\lambda} \left(3Y T_\nu T_\mu + i\varepsilon_{\nu\lambda\mu} T_\alpha^\lambda Y^\alpha\right) + Y^\lambda \left(\varepsilon_{\nu\lambda\rho} T_\mu T_\alpha^\lambda + 2\varepsilon_{\mu\lambda\rho} T_\alpha^\lambda Y^\alpha\right) + 2\varepsilon_{\mu\nu\lambda\rho} T_\alpha^\lambda\eta^\delta \left(\varepsilon_{\rho\lambda\delta} T_\alpha^\lambda + T_\rho^\lambda e_\alpha\eta^\delta + e_\rho\eta T_\alpha^\delta\right)\right).
\]

(A.5)

A.2. The orthogonal splitting of \(B_{\mu\nu\rho\sigma}\)

In [14] the orthogonal splitting of the Bel–Robinson tensor was calculated and the different parts resulting from such splitting were studied in detail. The calculations are similar for the Bel–Robinson tensor of a Weyl candidate and therefore we only need to adapt these results for the particular case of \(S_{\mu\nu\rho\lambda}\). In this case the orthogonal splitting of \(B_{\mu\nu\rho\lambda}\) reads—see section 5.1 of [14],

\[
B_{\mu\nu\rho\lambda} = W_{\mu\nu\rho\lambda} + 4P_{\mu\nu\rho\lambda} + 6t_{\mu\nu\rho\lambda} + 4Q_{\mu\nu\rho\lambda} + t_{\mu\nu\rho\lambda},
\]

(A.6)

where the different parts of this splitting are given by

\[
W \equiv e_{\alpha\beta}e_{\alpha\beta} + b_{\alpha\beta}b_{\alpha\beta},
\]

(A.7)

\[
P_{\mu\nu} \equiv 2b_{\lambda}^\alpha e_{\rho\lambda}^\mu e_{\rho\beta}^\nu,
\]

(A.8)

\[
t_{\mu\nu} \equiv Wh_{\mu\nu} - 2(b_{\mu}^\rho b_{\nu}\rho + e_{\mu}^\rho e_{\nu}\rho),
\]

(A.9)

\[
Q_{\mu\nu\rho\lambda} \equiv h_{\mu\nu}P_{\rho\lambda} - 2(\varepsilon_{\mu\alpha\nu\beta} + \varepsilon_{\nu\alpha\mu\beta}) e_{\rho\beta}^\mu,
\]

(A.10)

\[
t_{\mu\nu\rho\lambda} \equiv 4(b_{\mu\lambda}b_{\rho\nu} + e_{\mu\lambda}e_{\rho\nu}) - h_{\mu\nu}t_{\rho\lambda} + 2(h_{\mu\lambda\rho}t_{\nu\nu} + 2h_{\mu\lambda\nu}t_{\rho\nu} - h_{\mu\lambda\nu}t_{\rho\nu} + W(h_{\mu\nu}h_{\rho\lambda} - 2h_{\mu\nu}h_{\rho\lambda}),
\]

(A.11)

where the relations

\[
e_{\mu\nu} = T_{\mu\nu} + \nabla_{\mu\nu}, b_{\mu\nu} = i(T_{\mu\nu} - \nabla_{\mu\nu}),
\]

need to be used.

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