Some stumbling first steps towards linear homology in a nutshell

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Abstract

In 1985 Bayer and Billera defined a flag vector $f(X)$ for every convex polytope $X$, and proved some fundamental properties. The flag vectors $f(X)$ span a graded ring $\mathcal{R} = \bigoplus_{d \geq 0} \mathcal{R}_d$. Here $\mathcal{R}_d$ is the span of the $f(X)$ with $\dim X = d$. It has dimension the Fibonacci number $F_{d+1}$.

This paper introduces and explores the conjecture, that $\mathcal{R}$ has a counting basis $\{e_i\}$. If true then the equation $f(X) = \sum g_i(X)e_i$ conjecturally provides a formula for the Betti numbers $g_i(X)$ of a new homology theory. As the $g_i(X)$ are linear functions of $f(X)$, we call the new theory linear homology.

Further, assuming the conjecture each $g_i$ will have a rank $r \geq 0$. The rank zero part of linear homology will be (middle perversity) intersection homology. The higher rank $g_i$ measure successively more complicated singularities. In dimension $d$ we will have $\dim \mathcal{R}_d$ linearly independent Betti numbers.

This paper produces a basis $\{e_i\}$ for $\mathcal{R}$, that is conjecturally a counting basis.

Warning. Conjecture withdrawn in version 2.

Contents

1 Advice to the reader 2
2 Background and overview 3
3 The cone product formula 6
4 Simplices 7
5 Rank one examples 9
6 The rank basis 10
7 The rank basis product formula 12
8 Rank and cone 14
9 Product shadows 18
10 Cone shadows 21
1 Advice to the reader

0. There is a serious error in this paper which prevents it reaching its goal. In the ‘rank basis’ and is introduced. It gives an decomposition of the flag vector ring. I assumed that the counting basis is consistent with (is a refinement of) this ‘rank decomposition’. I’ve since done calculations which strongly suggest (or perhaps prove) that this assumption is false.

Thus, a different way of proceeding is needed to obtain the counting basis. It is no longer possible (or necessary) to rely on the ‘rank basis’. In particular, much of the latter part of this paper is a failed attempt to ‘square the circle’.

The only changes made in the second version are additions to the title and abstract, and the addition of this subsection (and of course changed page breaks). All numbering and cross-references remain the same, as the first version started at 1.

1. This paper is the first exposition of a conjectural homology theory. It is intended for experts, and others who already have a strong interest in the area. It is longer than the author would like. This is because it has dependencies, which are in the process of being written up [4, 5]. Once done, this paper can be simplified and shortened.

2. The statement [Exercise] has been added to paragraphs that omit a proof, that a determined reader could provide, based on what is in the present paper. The exercises are, for example, algebraic and combinatorial calculations based already stated definitions and results.

3. Terms being defined are emphasized. Sometimes definitions are made incrementally, perhaps first by use or purpose and then implementation. So the same term might be emphasized several times. Sometimes emphasis is used just for emphasis.

4. This paper is a journey. In §2 we describe the starting point, the direction of travel, and the goal. In §9 and §10 we construct the candidate counting basis. This concludes the paper.

5. This paper involves four rings, all of which are isomorphic. Equivalently, we have four different presentations of the same ring. The rings have a cone operator $C$, and are generated by cone and product. Thus, the isomorphisms are unique.

6. The first ring is the convex polytope flag vector ring. This is largely due to Bayer and Billera [1]. It provides a link to combinatorial and counting arguments in geometry, particularly convex polytopes.

7. The second ring is the cone product ring. This has a concise algebraic description, via equations (1), (3) and (5). It allows the use of algebraic methods.

8. The third ring, which is only implicitly defined, is the rank basis ring. It has basis the $\langle I \rangle$ for $I \in \mathcal{I}$, where $\mathcal{I}$ is defined in [88]. Each $I$ is a non-empty sequence of pairs of counting numbers. It has product as defined in (15), and cone as in (56) and (57).

9. The fourth ring, which is not even implicitly defined, is the candidate counting ring. It has basis $[I]$ for $I \in \mathcal{I}$, and cone as in [174] and §9. Product is defined only by reference to the rank basis ring. The author conjectures that this product can be explicitly defined, in such a way that the coefficients are clearly counting numbers.
10. A word about proofs is helpful here. If a system of constraints has a single solution, then any two solutions are isomorphic. This follows from satisfaction of constraints, the isomorphism need not be explicitly constructed. This can be used to prove, for example, combinatorial identities.

11. We apply this to our four rings. The author has proved that the first and third rings satisfy the cone product formulas (see [5] and [4] respectively). From this it is easy to prove that they are isomorphic to the cone product ring.

12. The author intends to use a similar approach to the fourth ring. Once the structure coefficients (see [26]) are correctly known for the counting ring, all that is left to do is verify that they satisfy the cone product formula (4). This might be hard.

13. Thus, any gaps, obscurities or even errors in the present paper are of less importance, once these structure coefficients have be obtained. The author suggest that the reader focus on progress towards the discovery of the counting structure coefficients, and put to one side any difficulty in the details.

14. This paper takes the form of a journey from one ring to the next. The journey from first to second is done in a single step, namely by referencing [5] (which builds on the ideas in [2]). This is done because no details from this step are used in the rest of the paper. Similarly, there is only passing mention of intersection homology.

15. Finally, there is a conjectural fifth ring, which might interest and help some readers. It is the representation ring of an unknown algebraic object \( G \) (see [37] to [40]). This ring should again provide a connection to geometry, but this time via the construction of linear homology.

2 Background and overview

16. By homology we mean, in this paper, any systematic way to construct useful finite dimensional vector spaces. The dimension \( n \) of such a vector space is of course a non-negative integer, or in other words a member of the set \( \{0, 1, 2, 3, \ldots \} = \mathbb{N} \) of counting numbers.

17. Put another way, homology is any systematic way to construct useful integers, that are non-negative because they are the dimension of a vector space. And sometimes being non-negative is already a useful property. These integers, until proved non-negative, we will call candidate Betti numbers.

18. Suppose for each \( X \in \mathcal{X} \) we have a linear flag vector \( f(X) \). In this situation we will say that a homology theory is linear if the associated Betti numbers are linear functions of \( f(X) \).

19. In this paper \( \mathcal{X} \) will be the set \( \mathcal{P} \) of all convex polytopes. Every \( X \in \mathcal{X} \) has a dimension \( \dim X \). Let \( \mathcal{P}_d \) be all \( X \in \mathcal{P} \), with \( \dim X = d \). In 1985 Bayer and Billera [1] defined the (linear) flag vector \( f(X) \), for \( X \in \mathcal{P} \). Let \( \mathcal{R}_d \) denote the span of \( f(X) \), where \( X \) ranges over \( \mathcal{P}_d \).
20. Bayer and Billera also proved that the sequence \( \dim \mathcal{R}_d \) is the sequence 1, 1, 2, 3, 5, 8, \ldots of Fibonacci numbers. Along the way they proved that the cone (or pyramid) operator \( C : \mathcal{P}_d \to \mathcal{P}_{d+1} \) induces, via the equation \( C(f(X)) = f(C(X)) \), a linear map \( C : \mathcal{R}_d \to \mathcal{R}_{d+1} \), which we also call cone and denote by \( C \).

21. Finally, consider the Cartesian product \( X = X \times Y \) of two convex polytopes. Bayer and Billera also proved, mostly, that the equation \( f(X)f(Y) = f(X \times Y) \) induces a bilinear product and thus a graded ring structure on \( \mathcal{R} = \bigoplus_{d \geq 0} \mathcal{R}_d \). (They needed a special case of this result. The argument they used easily extends to give the general result.)

22. All this will be explained in more detail, from the point of view of linear homology, in the author’s preprint [5]. In particular, a geometric construction will explain why the Bayer-Billera flag vector should rightly be regarded as the linear flag vector (for convex polytopes).

23. To summarize, the (convex polytope linear) flag vector ring \( \mathcal{R} = \bigoplus_{d \geq 0} \mathcal{R}_d \) is a graded ring with, of course, a bilinear product, and a linear cone operator \( C : \mathcal{R}_d \to \mathcal{R}_{d+1} \). Further, \( \dim \mathcal{R}_d \) is the sequence of Fibonacci numbers.

24. Let \( e_0 \) denote the flag vector of a point. We have, of course, that \( e_0 = 1 \in \mathcal{R}_0 \). If \( U = C(B) \) for \( B, U \in \mathcal{R} \), we say that \( U \) is a cone, or the cone with base \( B \). We write \( e_{n+1} = C(e_n) \). This defines the simplices \( e_n \), for \( n \geq 0 \). Every simplex \( e_n \), except \( e_0 = 1 \), is also a cone.

25. Now suppose that \( \{e_i\} \) is a basis for \( \mathcal{R} \), with each \( e_i \) lying in some \( \mathcal{R}_d \). The equation

\[
f(X) = \sum g_i(X)e_i
\]

now defines linear functions \( g_i(X) \) of the flag vector \( f(X) \). If \( d = \dim X \) then \( g_1(X) = 0 \), unless \( e_i \in \mathcal{R}_d \). Thus, on \( \mathcal{P}_d \) we get a Fibonacci number of linearly independent linear functions.

26. Again using the basis \( \{e_i\} \), the equations

\[
e_i e_j = \sum \lambda_{ijk} e_k
\]

\[
C(e_i) = \sum \mu_{ij} e_j
\]

define the structure coefficients \( \lambda_{ijk} \) and \( \mu_{ij} \) for \( \mathcal{R} \) (in the \( \{e_i\} \) basis).

27. The goal of this paper is to define a candidate Betti number formula for the at present only conjectural linear homology theory (for convex polytopes). Doing this is what we mean by describing linear homology in a nutshell. Hence the title of this paper.

28. To do this we use the structure coefficients. We say that \( \{e_i\} \) is a counting basis for \( \mathcal{R} \) if (i) each \( e_i \in \mathcal{R}_d \) is an integer linear combination of \( f(X) \) for \( X \in \mathcal{P}_d \) and vice versa. And (ii) the \( \lambda_{ijk} \) are counting numbers (lie in \( \mathbb{N} \)). And (iii) similarly the \( \mu_{ij} \) are counting numbers.

29. This paper constructs a basis \( \{e_i\} \) for \( \mathcal{R} \) that is, conjecturally, a counting basis. If so, then the author conjectures that the associated \( g_i(X) \) are the Betti numbers for a new way of constructing useful vector spaces, to be called linear homology.
30. The author hopes that showing \( \{e_i\} \) is a counting basis will be a straightforward calculation, once we have a correct guess for the structure coefficients. This paper defines some of the candidate structure coefficients. Product and cone on \( \mathcal{R} \) will determine the remainder.

31. This paper constructs the candidate counting basis incrementally. The steps are: (i) The set of \( \langle W \rangle \) is the \( CD \)-basis. (ii) The \( \langle I \rangle \) give the same basis. (iii) Here, \( I \) is a non-empty finite sequence of pairs of counting numbers. (iv) The \( \{I\} \) give the rank basis. (v) The \([I]\) give the candidate counting basis.

32. We use \( e_i \) to denote an element of a general basis. We use \( \langle W \rangle, \langle I \rangle, \{I\}, [I] \) to respectively denote an element of the \( CD, CD, rank \) and candidate counting bases respectively. However, we also use \( e_a \) (for \( a \in \mathbb{N} \)) to denote the simplices in \( \mathcal{R} \). We have \( e_a = \langle C^a \rangle = \langle a^0 \rangle = \{a^0\} = [a^0] \).

33. We can think of the product structure coefficients \( \lambda_{ijk} \) as providing a list of ingredients for constructing the homology of \( Z = X \times Y \) from that of the factors \( X \) and \( Y \). The \( \{e_i\} \) being a counting basis is a strong constraint on this list. Künneth, in his PhD thesis [7], stated and proved correct the list of ingredients for the product of compact manifolds.

34. Similarly, the cone structure coefficients \( \mu_{ij} \) should provide a list of ingredients for the homology of \( X = C(B) \) in terms of that of the base \( B \).

35. The concept of a ring with a counting basis is not new. Suppose \( G \) is a finite group. Representations of \( G \) can be added using direct sum of vector spaces, and multiplied by using tensor product.

36. When Schur’s Lemma applies, every representation decomposes uniquely into a direct sum of irreducibles. Thus, from \( G \) the representation ring \( \mathcal{R}_G \) is formed, and the irreducibles provide a counting basis for this ring. Each structure coefficient is a number, that counts how many times an irreducible appears in a product.

37. The author conjectures that the flag vector ring \( \mathcal{R} \) is isomorphic to the representation ring \( \mathcal{R}_G \) for some algebraic object \( G \). For this to work, \( G \) and its representations must have a cone \( C \) operator, and satisfy Schur’s Lemma.

38. Even though \( G \) is at present unknown, its conjectured existence provides a way of thinking about \( \mathcal{R} \), which the author finds helpful. This way of thinking is used in explicitly in [73, 80] and implicitly throughout [6].

39. For non-singular projective algebraic varieties, linear homology should reduce to the usual homology theory. And in this setting, the representation ring of the Lie algebra \( \mathfrak{sl}_2 \) is crucial to the proof of the hard Lefschetz theorem.

40. Thus, \( G \) might be some sort of amplification of \( \mathfrak{sl}_2 \), with a cone operator \( C \). Successful construction of \( \mathcal{R}_G \cong \mathcal{R} \) should then provide the desired counting basis \( \{e_i\} \) for \( \mathcal{R} \). However, it may be that it is the construction of \( \{e_i\} \) that provides the insight needed to construct \( G \).
In any case, the goal of this paper is to construct a basis \( \{ e_i \} \) for \( R \), that is conjecturally a counting basis. From this follows conjecturally a formula for linear homology Betti numbers.

3 The cone product formula

In the previous section we saw how linear homology is related to the problem, of finding a counting basis for the flag vector ring \( R \). In this section we provide a purely algebraic description of \( R \).

Recall that \( R = \bigoplus_{d \geq 0} R_d \) is a graded ring with a linear operator \( C : R_d \to R_{d+1} \). Recall also that \( e_0 = 1 \in R_0 \), and that \( e_{n+1} = C(e_n) \) defines the simplices \( e_n \), for \( n \geq 0 \).

We also need the following. Bayer and Billera [1] showed that \( \dim R_d \) gives the Fibonacci sequence 1, 1, 2, 3, 5, 8 and so on. In proving this, they showed that the operators \( C \) and product together generate \( R \).

Suppose \( U, V \in R \). In [4] the author will prove the cone product formula

\[ C(U)C(V) = C(J(U,V)) + DUV \]  \hspace{1cm} (4)

where

\[ J(U,V) = UC(V) + C(U)V - e_1UV \]  \hspace{1cm} (5)

is the join formula.

Applying (5) with \( U = V = e_0 \) we obtain \( J(U,V) = e_0e_1 + e_1e_0 - e_1e_0e_0 = e_1 \). Thus \( C(e_0)C(e_0) = C(e_1) + De_0e_0 \), and so \( e_1e_1 = e_2 + D \). Thus

\[ D = e_1e_1 - e_2 \]  \hspace{1cm} (6)

is forced by (4) and (5).

We can now define the CD-basis for \( R \). Let \( W \) be any finite word in symbols \( C \) and \( D \). Use \( \emptyset \) to denote the empty word. The rules (i) \( \langle \emptyset \rangle = e_0 \), and (ii) \( \langle CW \rangle = C(\langle W \rangle) \), and (iii) \( \langle DW \rangle = D \langle W \rangle \) now define \( \langle W \rangle \) for any finite CD-word \( W \). This is the CD-basis for \( R \).

We will now show that this is indeed a basis. Writing \( \deg C = 1 \) and \( \deg D = 2 \) gives a degree \( \deg W \) for every CD-word. Clearly, \( \langle W \rangle \in R_d \) where \( d = \deg W \). Further, the number of \( W \) with \( \deg W = d \) is equal to the Fibonacci number that is \( \dim R_d \). So the candidate basis has the correct number of elements.

Recall that \( C \) and product generate \( R \). Every element of the candidate basis, other than \( \langle \emptyset \rangle = e_0 = 1 \), is either a cone on, or \( D \) times, a candidate basis element. After removing leading powers of \( D \), any product of candidate basis elements either has \( e_0 \) has a factor (and so is trivial), or is a product of cones.

The cone product formula allows us to write a product of candidate basis cones as the cone on something plus \( D \) times something else. In both cases, the something is a linear combination of lower degree candidate basis products. (For \( e_1UV \) we use that \( e_1 = C(e_0) \) is a cone.)
51. Thus, via the cone product formula, any product of candidate basis elements is a linear combination of candidate basis elements. Trivially, the same is true for the cone on a candidate basis element. As cone and product generate $R$, the span of the candidate basis elements is the whole of $R$. Counting, as in 48, now shows that the candidate CD-basis is indeed a basis.

52. We now have our algebraic description of $R$. We can define $R_d$ to be the vector space with basis $\langle W \rangle$, where $\deg W = d$. Now as usual write $R = \bigoplus_{d \geq 0} R_d$.

53. The rule $D \langle W \rangle = \langle DW \rangle$, together with the cone product formula, define the ring structure on $R$. (We also need the consequence $D = e_1 e_1 - e_2$ of the cone product formula.) Finally, we use the rule $C(\langle W \rangle) = \langle CW \rangle$ to define the linear operator $C : R_d \to R_{d+1}$.

54. The arguments in this section, and the existence and properties of the flag vector ring, show that the rules in 52 and 53 define a ring $R$ (with operator $C$) that is isomorphic to the flag vector ring.

55. The ring defined by the rules in 52 and 53 we will call the (universal) cone product ring, and we will also denote it by $R$. It is isomorphic (via $C \mapsto C$) to the flag vector ring of the previous section, which we will henceforth denote by $R_f$.

56. Thus, the cone product ring $R$ is defined algebraically. The flag vector ring $R_f$ is defined via convex polytopes. By definition, there is a flag vector map $X \mapsto f(X) \in R_{f, \dim X}$. The arguments in this section, and in particular the cone product formula, establish an isomorphism $R \cong R_f$. Thus, we can think of the flag vector $f(X)$ as lying in the cone product ring $R$.

4 Simplices

57. Most of the rest of this paper is a study of the cone product ring $R$, or in other words the algebraic consequences of the cone product formula (4). In this section we focus on simplices.

58. Recall that $e_0 = 1 \in R_0$ and $e_{n+1} = C(e_n) \in R_{n+1}$ together define the simplices $e_n$ for $n \geq 0$. Every simplex, except $e_0$, is a cone.

59. If $e_a e_b$ is a non-trivial product of simplices then $a, b \geq 1$ and so the cone product formula (4) gives

$$e_a e_b = C(J(e_{a-1}, e_{b-1})) + D e_{a-1} e_{b-1}$$

where $J$ is given by the join formula (5).

60. For $a, b \geq 0$ we have

$$J(e_a, e_b) = e_a e_{b+1} + e_{a+1} e_b - e_1 e_a e_b$$

and so the formulas (7) and (8), for the product and join of two simplices, are intertwined.
61. Here is a statement of results. For \(a, b \geq 0\) we obtain

\[
J(e_a, e_b) = e_{a+b+1}
\]  

(9)

while for \(a, b \geq 1\) we obtain

\[
e_a e_b = e_{a+b} + De_{a-1} e_{b-1}
\]  

(10)

which is easily seen to be a consequence of (9).

62. From (8) we have \(J(e_0, e_b) = e_{b+1}\), which starts the inductive proof of (9). Note that for \(a = 0\), the \(e_1 UV\) term in (8) gives a cancellation, of \(e_1 e_a e_b\) with \(e_{a+1} e_b\).

63. Now consider \(e_1 e_a\) for \(a \geq 1\). We get

\[
e_1 e_a e_b = (e_1 e_a)e_b = e_{a+1} e_b + De_{a-1} e_b
\]  

(11)

and so

\[
J(e_a, e_b) = e_a e_{b+1} - De_{a-1} e_b
\]  

(12)

where here \(e_1 e_a e_b\) gives a partial cancellation with \(e_{a+1} e_b\).

64. Assuming (10) for \(e_a e_{b+1}\) we obtain as required \(J(e_a, e_b) = e_{a+b+1}\). Finally, do we have a well founded induction? In 62 we established (9) for \(a = 0\), and hence (10) for \(a = 1\). More generally, given (9) for \(a = n\) we get (10) for \(a = n+1\). We have just shown that (10) for \(a = n\), gives (9), also for \(a = n\). Thus, the intertwined induction for both (7) and (8) is established.

65. Because this proof involved induction only on \(a\), it also proves, for \(a \geq 1\)

\[
e_a C(U) = C^{a+1}(U) + De_{a-1} U
\]  

(13)

which is a generalisation of (10), via \(U = e_{b-1}\). This result is important in [49].

66. Here’s the proof. As before we have, for \(a \geq 1\)

\[
e_a C(U) = C(J(e_{a-1}, U)) + De_{a-1} U
\]  

(14)

and as \(J(e_0, U) = e_0 C(U) + e_1 U - e_1 e_0 U = C(U)\) we get (13) for \(a = 1\).

67. More generally, for \(a \geq 2\) we have

\[
J(e_{a-1}, U) = e_{a-1} C(U) + e_a U - e_1 e_{a-1} U
\]  

(15)

and by induction on \(a\) we have

\[
e_{a-1} C(U) = C^a(U) + De_{a-1} U
\]  

(16)

while from (10) we have \(e_1 e_{a-1} = e_a + De_{a-1}\).

68. Substituting these results in (15) we obtain, for \(a \geq 1\)

\[
J(e_{a-1}, U) = C^a(U)
\]  

(17)

and hence the required formula (13) for \(e_a C(U)\). (If \(U = e_b\) then \(C^a(U) = e_{a+b}\), as in [49].)
69. We now introduce some notation. We will write

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \langle D^b C^a \rangle = D^b e_a
\]

and now the more general formula

\[
\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} a-1 \\ c-1 \end{bmatrix}
\]

is a consequence of (10), provided \( a, c \geq 1 \).

70. The \( \begin{bmatrix} a \\ b \end{bmatrix} \) are part of the candidate counting basis. In §9 and §10 we define candidate counting basis elements

\[
\begin{bmatrix} a_0 a_1 \ldots a_r \\ b_0 b_1 \ldots b_r \end{bmatrix}
\]

for \( r \geq 1 \). But first we must define the rank basis.

5 Rank one examples

71. The \( CD \)-basis is not a counting basis. The term \(-e_1UV\) in the join formula (5) is the cause of difficulty. In the previous examples, a cancellation took place. In general, it does not. To resolve this, we have to modify the \( CD \)-basis.

72. The simplest example is \( \langle CD \rangle \langle CD \rangle \). We have

\[
J(D, D) = DC(D) + C(D)D - e_1DD = 2\langle DCD \rangle - \langle DDC \rangle
\]

and so

\[
\langle CD \rangle \langle CD \rangle = 2\langle DCD \rangle - \langle CDDC \rangle + \langle DDD \rangle
\]

which contains totally uncancelled the negative \( C(e_1UV) \) term \( \langle CDDC \rangle \).

73. Recall from [37] that conjecturally each element of the counting basis is an irreducible representation of \( \mathcal{G} \). We will now use this, but for guidance only. We don’t rely on the conjecture. The defining equation (see [46])

\[
e_1 e_1 = e_2 + D
\]

for \( D \) we thus will think of as a decomposition of \( e_1 e_1 \) into irreducibles.

74. Now consider (22). If \( D \) is a representation, then so are \( \langle CD \rangle \) and \( \langle CD \rangle \langle CD \rangle \). But the cone product formula (22) is not yet writing \( \langle CD \rangle \langle CD \rangle \) as a sum of representations. However, if \( \langle CDCD \rangle \) or \( \langle DDD \rangle \) were reducible, perhaps something could be done.

75. Suppose

\[
\langle CDCD \rangle = (\langle CDCD \rangle - \langle CDDD \rangle) + \langle CDDC \rangle
\]

were a representation decomposition of \( \langle CDCD \rangle \). In that case we would have

\[
\langle CD \rangle \langle CD \rangle = 2(\langle CDCD \rangle - \langle CDDD \rangle) + \langle CDDC \rangle + \langle DDD \rangle
\]

as a representation decomposition of the cone product \( \langle CD \rangle \langle CD \rangle \).
76. This is an example of the route we will follow, to cancel the \(-e_1UV\) term in \(J(U, V)\). Notice that in this example we get positive cancelling terms from both \(UC(V)\) and \(C(U)V\), and so the total effect is to change the sign of \(-e_1UV\). In practice, the justification for following this route is that the subsequent calculations successfully remove the negative structure coefficients.

77. Consider now \(\langle CD \rangle\). Suppose, following (24) we also adopt
\[
\langle CD \rangle = (\langle CD \rangle - \langle DC \rangle) + \langle DC \rangle
\]  
(26)
as a representation decomposition. (This decomposition also arises from intersection homology, see \([5]\).)

78. In that case, we will also want a representation decomposition of:
\[
(\langle CD \rangle - \langle DC \rangle)^2 = 2 \langle CD\langle CD \rangle - \langle CDDC \rangle + \langle DDD \rangle \\
- 2 \langle DCCD \rangle - 2 \langle DDD \rangle \\
+ \langle DDCC \rangle + \langle DDD \rangle
\]
(The details of the expansion are left to the reader.)

79. Anticipating a notation from the next section, we write:
\[
\begin{align*}
\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} &= \langle CD \rangle - \langle DC \rangle \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 \end{bmatrix} &= \langle CDDC \rangle - \langle DDCC \rangle \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= \langle CD\langle CD \rangle - \langle DCCD \rangle - \langle CDDC \rangle + \langle DDCC \rangle
\end{align*}
\]
(27)(28)(29)

80. Using this notation, we obtain
\[
\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} = 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]
(30)
as the representation decomposition of \((\langle CD \rangle - \langle DC \rangle)^2\). (The details are left to the reader.)

81. Further examples, using
\[
\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \langle CDD^bC^a \rangle - \langle D^{b+1}C^{a+2} \rangle
\]
(31)
can be done in the same way.

82. This is an example in the rank basis. See the next section of the full definition. See \([7]\) for the product formula in the rank basis.

6 The rank basis

83. This section introduces the concept of rank, and uses it to define the rank decomposition \(\mathcal{R} = \bigoplus_{r \geq 0} \mathcal{R}_{[r]}\) of the cone product ring. We do this by explicitly giving the decomposition
\[
\langle W \rangle = \sum_{r \geq 0} \langle W \rangle_{[r]}
\]  
(32)
of each element \(\langle W \rangle\) of the CD-basis, where \(U_{[r]}\) is the rank \(r\) part of \(U \in \mathcal{R}\).
84. For each \( W \) the sum (32) is finite, and so has a top rank component. These components, which can be indexed by \( CD \)-words \( W \), form the rank basis of \( \mathcal{R} \). This will become clear in 89.

85. In the next section we state the product formula in the rank basis. (This formula will be proved in [4].) The product structure coefficients will clearly be counting numbers.

86. First, we introduce a new way of writing words \( W \) in \( C \) and \( D \). The basic idea is to use each occurrence of \( CD \) in \( W \) to split \( W \) into pieces. Each piece (less the \( CD \)) is then of the form \( DbCa \). The number of occurrences of \( CD \) in \( W \) is called the rank of \( W \), written as \( \text{rank } W \).

87. If \( W \) has rank zero, it is of the form \( DbCa \). We already write \( [a\ b] \) for \( \langle DbCa \rangle \), and the \( [a\ b] \) has a nice product formula, given by (19). It is the absence of \( CD \) in \( W \) that gives this nice formula. If \( W \) has rank one, it is of the form \( Db0Ca0CD Db1Ca1 \). Similarly, each rank \( r \) word \( W \) is given by a sequence of \( (r+1) \) pairs of counting numbers, and vice versa.

88. We define the (linear homology) index set \( \mathcal{I} \) as follows. Let \( I \) be a sequence of one or more pairs of counting numbers. We define \( I \) to be the set of all such sequences. Each \( I \in \mathcal{I} \) is a sequence of \( (r+1) \) pairs (of counting numbers), for some \( r \geq 0 \). We write \( r = \text{rank } I \). We write \( \mathcal{I} = \bigsqcup_{r \geq 0} \mathcal{I}_r \) for the resulting disjoint union.

89. From the previous discussion, it follows that each \( I \in \mathcal{I}_r \) gives a rank \( r \) word \( W \) in \( C \) and \( D \), and vice versa. We use this bijection to define \( \langle I \rangle \) for each index \( I \) in \( \mathcal{I} \). We define \( \text{deg } I \) to be \( \text{deg } W \), or in other words \( \langle I \rangle \in \mathcal{R}_d \) where \( d = \text{deg } I \).

90. The following implicitly defines the rank basis, up to rank 2,

\[
\begin{align*}
\left\langle a_0 \right\rangle &= \left\{ a_0 \right\} \\
\left\langle a_0 a_1 \right\rangle &= \left\{ a_0 a_1 \right\} + \left\{ a_0 + a_1 + 1 \right\} \\
\left\langle a_0 a_1 a_2 \right\rangle &= \left\{ a_0 a_1 a_2 \right\} + \left\{ a_0 + a_1 + a_2 + 1 \right\} \\
&\quad + \left\{ a_0 + a_1 + a_2 + 2 \right\} \\
\end{align*}
\]

and gives the rank decomposition promised in 84.

91. We can simplify this by first writing

\[ I = I_0 I_1 \ldots I_r \]

as a sequence \( {a\ b}_i \) of pairs, and then introducing a combining operator. We write

\[
{a\ b}_i \# {c\ d} = {a+c+1\ b+d+1}
\]

and call \# the mix operator. (In \( {a\ b}_i \# {c\ d} \), the two pairs lose their separate identity.)
For example, we can rewrite (35) as
\[
\langle L_0 L_1 L_2 \rangle = \{ L_0 L_1 L_2 \} + \{ L_0 \# L_1 L_2 \} + \{ L_0 L_1 \# L_2 \} + \{ L_0 \# L_1 \# L_2 \}
\] (38)
by using the mix operator. Like addition, \# is both associative and commutative.

This completes the definition of the rank basis, whose elements are \{ I \} for \( I \) in the index set \( \mathcal{I} \). Clearly, each \( \langle I \rangle \) is the sum of \( 2^r \) distinct elements of the rank basis, where \( r = \text{rank } I \).

For clarity, the top row of \( I \) gives the exponents of \( C \), and the second row exponents for \( D \). Further, the degree \( \deg I \) of \( I \) is defined by \( \langle I \rangle \in \mathcal{R}_d \). It is the sum of the top row, plus twice the sum of the second row, plus three times the rank.

### 7 The rank basis product formula

Products are as nice as possible, in the rank basis. For example
\[
\{ a \} \{ b \} = \{ L \}
\] (39)
holds for any sequence \( L \) of \( r \geq 0 \) pairs of counting numbers. (This result, and others in this section stated without proof, will be proved in [4].)

This is a powerful result. It greatly simplifies product calculations in the rank basis. It provides a form of orthogonality. To help explain this, we now introduce some notation.

Every rank \( r \) index \( I \) in \( \mathcal{I} \) can be written uniquely as \( I = AL \), where \( A \in \mathcal{I}_{[0]} \) is a pair (of counting numbers) and \( L \) is a sequence of \( r \) pairs (of counting numbers). We call \( A \) the head and \( L \) the body of the index \( I \). We write \( \mathcal{B} = \bigsqcup_{r \geq 0} \mathcal{B}_r \) for the set of all possible body indexes.

Now let \( AL \) and \( BM \) be two indexes, in head-body form. From (39) we have
\[
\{ AL \} \{ BM \} = \{ A \} \{ B \} \{ 0 \# L \} \{ 0 \# M \}
\] (40)
which reduces the general product to computing \( \{ 0 \# L \} \{ 0 \# M \} \). (Because \( \langle a \rangle = \{ a \} = [a] \), we can use (19) to compute \( \{ A \} \{ B \} \).

Recall equation (30)
\[
\{ 0 \} \{ 0 \} = 2 \{ 0 \} + \{ 1 \}
\]
from §4. This is a special case of a general property.

We define the point-like elements of \( \mathcal{R} \) as follows. First, every \( \{ 0 \# L \} \) is point-like. Second, any linear combination of point-like is also point-like. Finally, no other elements are point-like. In other words, the point-like elements of \( \mathcal{R} \) is the subspace spanned by the \( \{ 0 \# L \} \), for \( \mathcal{I} \in \mathcal{B} \).

Also proved in [4] is that any product of point-like elements is also point-like. This is also a powerful result. Equation (30), quoted above, is an example of this.
102. We will now introduce further notation to help us. By using formal sums the equation
\[
\{^0_0 L\} \{^0_0 M\} = \{^0_0 L * M\}
\] (41)
both (i) states that the product of point-like is point-like, and (ii) defines merge product \(L * M\), which is a formal sum over \(B\).

103. Here’s how. Suppose \(F : S \to V\) is any function, that takes values in a vector space \(V\). We assume no structure on \(S\). The equation
\[
F(\lambda_1 s_1 + \ldots + \lambda_n s_n) = \lambda_1 F(s_1) + \ldots + \lambda_n F(s_n)
\] (42)
defines a map, also denoted by \(F\), from formal sums (of elements of \(S\)) to the vector space \(V\).

104. Note that if \(S\) has structure such as addition, care must be taken to distinguish formal addition from actual addition. For example, if \(F(n) = n^2\) for \(n \in \mathbb{N}\), then is \(F(2 + 3)\) equal to \(2^2 + 3^2\), or is it \((2 + 3)^2\)? Sometime we use \((x) + (y)\) to denote a formal sum.

105. We will use this with the vector-valued function \(\{\} : T \to R\). We return to (41). Recall that the product of point-likes is also point-like. Thus we can write
\[
\{^0_0 L\} \{^0_0 M\} = \lambda_1 \{^0_0 P_1\} + \ldots + \lambda_n \{^0_0 P_n\}
\] (43)
for suitable \(\lambda_i, P_i\).

106. Now use these same values to define
\[
P = \lambda_1 P_1 + \ldots + \lambda_n P_n
\] (44)
as a formal sum. By definition, \(\{^0_0 L\} \{^0_0 M\} = \{^0_0 P\}\).

107. As promised in 102, the equation (41)
\[
\{^0_0 L\} \{^0_0 M\} = \{^0_0 L * M\}
\]
now makes two statements. The first is that \(\{^0_0 L\} \{^0_0 M\}\) is point-like. The second is that \(L * M\) is defined to be the formal sum, which we previously denoted by \(P\) (for product).

108. We can now state the rank basis point-like product formula. First, the trivial cases. If \(L \in B_{[0]}\) then \(\{^0_0 L\} = \{^0_0 0\}\), and so \(\{^0_0 L\} \{^0_0 M\} = \{^0_0 0\}\). Similarly for \(M \in B_{[0]}\).

109. Any element of \(B_{[r]}\), for \(r \geq 1\), can be written uniquely as \(AL\), with \(A \in B_{[1]}\) and \(L \in B_{[r-1]}\). In [4] the equation
\[
\{^0_0 AL\} \{^0_0 BM\} = \{^0_0 A(L * BM)\} + \{^0_0 B(AL * M)\} + \{^0_0 (A#B)(L * M)\}
\] (45)
is proved. This deals with the general case.

110. We call \(L * M\) the merge product of the body indexes \(L, M \in B\). The general case equation (45) writes the merge product \((AL) * (BM)\) as the sum of the left shuffle \(A(L * MB)\), the right shuffle \(B(AL * M)\) and the mix combine \((A#B)(L * M)\).
111. For clarity, by definition, the equations

\[
\begin{align*}
\{0 \cdot L\} \{0 \cdot BM\} &= \{0 \cdot L \ast BM\} \\
\{0 \cdot AL\} \{0 \cdot M\} &= \{0 \cdot AL \ast M\} \\
\{0 \cdot L\} \{0 \cdot M\} &= \{0 \cdot L \ast M\}
\end{align*}
\]

define the quantities \((L \ast BM)\), \((AL \ast M)\) and \((L \ast M)\) in (45), which thus implicitly contains a recursion.

112. For example, if \(A, B \in \mathcal{B}_{[1]}\) then

\[
\{0 \cdot A\} \{0 \cdot B\} = \{0 \cdot AB\} + \{0 \cdot BA\} + \{0 \cdot A\#B\}
\]

follows from (45), together with 108. This is a generalisation of (30). [Exercise].

113. Suppose \(L \in \mathcal{B}_{[r]}\) and \(M \in \mathcal{B}_{[s]}\), with \(r \geq s\). From (45), say treated as a definition, it follows that the rank \(t\) part \((L \ast M)_{[t]}\) is zero unless \(r \leq t \leq (r + s)\). Further, within that range the coefficients are counting numbers, whose sum is non-negative. In fact, the sum depends only on the pair \(r, s\). [Exercise], or see [4].

114. Formal sums can also be used for the head part of a product. Assume \(A, B \in \mathcal{I}_{[0]}\). The equation

\[
[A] [B] = [\pi(A, B)]
\]

defines the formal sum \(\pi(A, B)\) (which by \(\S\) is over \(\mathcal{I}_{[0]}\)). As \(\{A\} = [A]\), this is equivalent to:

\[
\{A\} \{B\} = \{\pi(A, B)\}
\]

115. Thus

\[
\{AL\} \{BM\} = \{\pi(A, B) \ast L \ast M\}
\]

states the rank basis product formula. In it, the product rule decomposes orthogonally into a head-product and a body-product, given respectively by \(\pi(A, B)\) and \(L \ast M\). For clarity, here \(A, B \in \mathcal{I}_{[0]}\) and \(L, M \in \mathcal{B}\). Further, \(\pi(A, B)\) is defined by (51), and \(L \ast M\) by (45).

8 Rank and cone

116. Although product is counting in the subset basis, cone is not. In the subset basis, cones often produce negative coefficients, for a simple reason. For example:

\[
C \{0 \cdot 0\} = C (\langle CD \rangle - \langle DC \rangle) = \langle CCD \rangle - \langle CDC \rangle \\
= (\langle CCD \rangle - \langle DCC \rangle) - (\langle CDC \rangle - \langle DCC \rangle) \\
= \{1 \cdot 0\} - \{0 \cdot 1\} = \{0 \cdot 1\} \{0 \cdot 0\} - \{0 \cdot 0\} \{0 \cdot 1\}
\]

(53)
To resolve this particular problem first rewrite (53) as
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = C \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
which then shows, assuming \( G \), that \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is a reducible representation. Thus, \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is not part of the counting basis.

Put another way, the negative coefficient arises because \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is too large, to be part of a counting basis. We could fix it by using \( C \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) instead of \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) as part of the basis.

In the \( CD \) basis, the structure coefficients for \( C \) are always counting numbers, but those for product are not. To solve this we introduced the rank basis. Now product is always counting, and \( C \) sometimes is not. Despite this, we have made progress.

First, the failure of counting for product in the \( CD \) basis is deep problem, embedded in the \( -e_1UV \) term in the join formula. At root, it is a geometric problem. The failure of cone in the rank basis is a relatively shallow problem. So going from \( CD \) to rank basis solves a hard problem, at the expense of introducing an easier problem.

Second, the rank basis introduces the rank decomposition \( R = \bigoplus_{r \geq 0} R_{[r]} \). The \( CD \) basis has no similar feature. (The degree grading \( R = \bigoplus_{d \geq 0} R_d \), which makes \( R \) a graded ring, is compatible with both bases. In other words, each basis element \( \langle I \rangle \) or \( \{ I \} \) lies in some \( R_d \), where \( d = \text{deg} I \).

Third, the rank decomposition introduces a useful, and perhaps novel, structure on \( R \). For example, in [113] we saw that rank \( r \) times rank \( s \) is a sum of terms, whose rank lies between \( \max(r, s) \) and \( (r + s) \).

In particular, if \( U \) and \( V \) are both known up to and including rank \( r \), then the same is true of the product \( UV \) (and trivially also the sum \( U + V \)). Thus we can form the \textit{rank} \( r \) \textit{quotient ring} \( R/R_{[>r]} \cong R_{[0]} + \ldots + R_{[r]} \). The author hasn’t seen such a structure before.

Fourth, consider the conjectural \( G \). If it exists, its irreducible representations induce a decomposition of \( R \) into one-dimensional subspaces. For the rank decomposition to be directly useful, this \( G \) decomposition should refine the rank decomposition.

In other words, we expect (i) each irreducible representation \( \rho \) to have a rank \( r \), and (ii) \( R_{[r]} \) to have as basis the rank \( r \) representations, and (iii) when a product of irreducibles is fully decomposed, the resulting ranks are as in \( R \), and (iv) there is a cone operator \( C \) on representations, and (v) the representation \( C \) operator satisfies the cone product formula.

Thus, we see that the rank decomposition is conjecturally a step towards the construction of \( G \), and it introduces a novel and interesting mathematical structure. In the rest of this section, we explore the interaction between \( C \) and the rank. First we look at the rank decomposition of the cone \( C(\{ 0 \}_{0L} \)\), with point-like base \( \{ 0 \}_{0L} \).

Consider \( C(\{ 0 \}_{0L}) \). If rank \( L = 0 \), then \( \{ 0 \}_{0L} = \{ 0 \}_{0} \) and so \( C(\{ 0 \}_{0L}) = \{ 1 \}_{0} = \{ 1 \}_{0} \).
This takes care of the trivial case. For the general case of rank \( L \geq 0 \), we have \( L = L_0L' \), where \( L_1 = a \) is a pair of counting numbers.
128. Now consider $C(\{0_0L_1L'\})$. To evaluate this, we can unpack into the \(CD\) basis (giving an alternating sum), apply \(C\), and then repack. When unpacked the leading rank \(CD\) term is \(\langle 0_0L_1L' \rangle = C(D \{L_1L'\})\). After applying \(C\) we get \(C^2(D \{L_1L'\}) = \langle 1_0L_1L' \rangle\). Thus, repacking, \(\{0_0L_1L'\}\) is one of the terms in \(C(\{0_0L_1L'\})\).

129. Terms in \(\{0_0L_1L'\}\) also arise by applying the mix operator \(\#\) at least once (and alternating the sign). Thus, \(-\langle 0_0\#L_1L' \rangle\) appears. But \(0_0\#a = a+b+1\) and so \(C(\langle 0_0\#L_1L' \rangle)\) is the rank \(r\) element \(\langle 0_0L_{1}^{(1)}L' \rangle\), where \(L_1^{(1)} = a+b\). Thus, \(\{0_0L_1^{(0)}\}\) is one of the terms in \(C(\{0_0L_1L'\})\).

130. The equation

\[
C(\{0_0L_1L'\}) = \{1_0L_1L'\} - \{0_1L_1^{(1)}L'\}
\]  

(55)

now follows, once it is checked that the remaining lower rank terms cancel. [Exercise]

131. The same method of proof shows the more general result

\[
C(\{n_0L_1L'\}) = \{n+1_0L_1L'\} - \{0_1L_1^{(n)}L'\}
\]  

(56)

where \(L_1^{(n)} = a+b\), when \(L_1 = a\). [Exercise]

132. Similarly, for \(a, b \geq 0\) the same style of argument shows

\[
C(\{b+1_0L\}) = \{0_aL\} + \{a+1_0L\}
\]  

(57)

which completes the calculation of \(C\) in the rank basis. [Exercise]

133. We now use (56) and (57) to understand better the interaction between \(C\) and the rank decomposition. First we will give a rank decomposition of the \(C\) operator. We then use this decomposition to redefine point-like, and to introduce the related concept simplex-like.

134. First, suppose \(U \in R_{[r]}\). From inspecting (56) and (57) the equation

\[
C(U) \in R_{[r]} \oplus R_{[r+1]}
\]

(58)

now follows. Thus, we can write

\[
C = C_{[0]} + C_{[1]}
\]

(59)

where the operation \(C_{[i]}\) has rank \(i\) (or in other words is a map \(R_{[r]} \rightarrow R_{[r+1]}\)).

135. Recall from (100) that \(U \in R\) is point-like if it is the span of the \(\{0_0L\}\). We can free this concept from its dependence on the rank basis, and also introduce the related simplex-like. (Key here is that we defined point-like as a vector subspace of \(R\), rather than as a property of basis vectors.)

136. First, let's solve the equation \(U = C_{[1]}(V)\), for \(U\). We start with \(V = \{AL\}\) in the rank basis. If \(A = a_0\), then (55) applies, and \(C(V) = C_{[0]}(V)\), and so \(C_{[1]}(V) = 0\). If \(A = a_{b+1}\) then \(C_{[1]}(V) = \{0_aL\}\).
This shows that the point-like elements $U$ of $R$, as defined in [100], are precisely the ones for which the equation $U = C_{[1]}(V)$ can be solved. In other words, the point-like elements are precisely the range of the operator $C_{[1]}$.

Now let’s solve $C_{[1]}(V) = 0$, for $V$. Again, we start with $V = \{AL\}$ in the rank basis. If $A = \begin{pmatrix} a \\ 0 \end{pmatrix}$ then (56) shows that $C_{[1]}(V) = 0$. Further, if $A = \begin{pmatrix} a \\ b+1 \end{pmatrix}$ then by (57) we have $C_{[1]}(V) \neq 0$. This is almost enough. Some non-trivial linear combination of not-simplex-like basis vectors might be zero under $C_{[1]}$.

To deal with this, note that, if $A = \begin{pmatrix} a \\ b+1 \end{pmatrix}$ then $C_{[1]}(V) = \begin{pmatrix} 0 \\ 0 \\ a \\ b \end{pmatrix}$. Thus, $C_{[1]}$ takes distinct not-simplex-like elements of the rank basis to distinct elements of the rank basis. This is enough. It shows that if a linear combination of not-simplex-like rank basis elements goes to zero under $C_{[1]}$, then the linear combination is trivial.

We define the simplex-like elements $V$ of $R$ to be those such that $C_{[1]}(V) = 0$. In other words, the simplex-like elements are the kernel of the operator $C_{[1]}$. We have shown that the $\{aL\}$ provide a basis of the simplex-like elements.

In the next section we use equation (13) $e_a C(U) = C^{a+1}(U) + De_a - 1 U$ and properties of rank to resume the definition of the counting basis. In particular, we resolve systematically the problem first considered in [116].

This requires the following result. Recall $C = C_{[0]} + C_{[1]}$. Consider $C^n = (C_{[0]} + C_{[1]})^n$. Then the equations

\begin{align*}
(C^n)_{[0]} &= (C_{[0]})^n \\
(C^n)_{[1]} &= \sum_{i=0}^n (C_{[0]})^i C_{[1]} (C_{[0]})^{n-i}
\end{align*}

(60) (61)

give the rank zero and rank one parts of $C^n$. Further, the other components are zero.

The proof goes as follows. First, suppose that in the binomial expansion of $(C_{[0]} + C_{[1]})^n$ we choose $C_{[1]}$ exactly $r$ times. If so, then the resulting product has rank $r$. This proves (60) and (61).

Now suppose we choose $C_{[1]}$ at least twice. If so, then the resulting product contains

\[ C_{[1]}(C_{[0]})^a C_{[1]} \]

(62)
as a subexpression, for some $a \geq 0$. Now, $V = C_{[1]}(U)$ is always point-like, so $V' = (C_{[0]})^a(V)$ is simplex-like, so $C_{[1]}(V')$ is zero. This proves that the other rank components $(C^n)_{[r]}$ of $C^n$ are zero.
9 Product shadows

145. The equations

\[ C\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]  
(63)

\[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]  
(64)

are suggested by [116] and [117]. They are a special case of equations and definitions in this section. In (64) we call \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) a shadow term. We call \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) the leading term.

146. Assume \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is known. Equation (63) now determines \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). And now (64) determines \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). In this section we do something similar first for \( \begin{bmatrix} a & L \\ 0 & 0 \end{bmatrix} \), and then for \( \begin{bmatrix} a & L \end{bmatrix} \). The next section determines \( \begin{bmatrix} 0 & L \end{bmatrix} \), for rank \( L \geq 0 \). (If rank \( L = 0 \), \( \begin{bmatrix} 0 & L \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \), determined by \( \begin{bmatrix} 0 & 0 \end{bmatrix} = 1 \in \mathcal{R} \).)

147. The previous section showed that \( C = C_{[0]} + C_{[1]} \) is the rank decomposition of \( C \). It defined the point-wise subspace of \( \mathcal{R} \) to be the image of \( C_{[1]} \), and the simplex-like subspace to the kernel (or nullspace) of \( C_{[1]} \). In the candidate counting basis, the \( \begin{bmatrix} 0 & L \end{bmatrix} \) will be a basis for the point-like subspace. Similarly, \( \begin{bmatrix} a & L \end{bmatrix} \) a basis for the simplex-like.

148. Suppose now we are give \( \begin{bmatrix} 0 & L \end{bmatrix} \), for some \( L \in \mathcal{B} \) of rank \( r + 1 \geq 1 \). By assumption (or convention) it is point-like. So we can solve \( \begin{bmatrix} 0 & L \end{bmatrix} = C_{[1]}(U) \) for \( U \). This solution is unique, up to the addition of a simplex-like \( V \). Thus, we can assume rank \( U = r \geq 0 \). (For rank \( \begin{bmatrix} 0 & L \end{bmatrix} = 0 \), see [146].)

149. Now consider \( e_a \begin{bmatrix} 0 & L \end{bmatrix} \), for \( a \geq 0 \). By the rank product property, this also has rank \( r + 1 \). But \( \begin{bmatrix} 0 & L \end{bmatrix} = C_{[1]}(U) \) and \( C = C_{[0]} + C_{[1]} \). So we can apply formula (13) to \( e_a C(U) \), to expand \( e_a \begin{bmatrix} 0 & L \end{bmatrix} \).

150. We will do this for a typical case, namely \( a = 2 \). We have

\[ e_2 \begin{bmatrix} 0 & L \end{bmatrix} = e_2 C(U) - e_2 C_{[0]}(U) \]
(65)

\[ = C^3(U) + D e_1 U - e_2 C_{[0]}(U) \]
(66)

and now look at the rank \((r+1)\) part.

151. Using (61) and the rank product property we obtain

\[ e_2 \begin{bmatrix} 0 & L \end{bmatrix} = C_{[0]} C_{[0]} C_{[1]}(U) + C_{[0]} C_{[1]} C_{[0]}(U) + C_{[1]} C_{[0]} C_{[0]}(U) \]
(67)

which, we will soon see, determines the shadow terms for \( \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \).

152. An aside. We obtained (67) by using properties of rank. This amounts to a massive cancellation of terms in (65). Perhaps some other massive cancellations also arise from rank properties, perhaps for an at present unknown rank.

153. To return to (67). First note that \( C_{[1]} \) is point-like, and so \( C_{[0]} C_{[0]} C_{[1]} = C^2 C_{[1]} \). In the rank basis, point-like elements have the form \( \begin{bmatrix} 0 & L \end{bmatrix} \). This leads to the general rule

\[ C \begin{bmatrix} a & L \end{bmatrix} = \begin{bmatrix} a+1 & L \end{bmatrix} \]
(68)

which determines the simplex-like elements, given the point-like.
154. Recall that \( [0]_0^L = C_{[1]}(U) \) determines \( U \) only up to a simplex-like element \( V \). But if \( V \) is simplex-like, then so are \( C_{[0]}(V), C_{[0]}C_{[0]}(V) \) and so on. Thus, the individual terms in (67) don’t depend on the choice of \( U \). They depend only on \( [0]_0^L \).

155. Put another way, if \( [0]_0^L = C_{[1]}(U) \) then the equation

\[
[0]_0^L^{(1)} = C_{[1]}C_{[0]}(U) \tag{69}
\]

determines an element \( [0]_0^L^{(1)} \) of \( \mathcal{R} \), which does not depend on the choice of \( U \). (This can be though of as a form of orthogonality.)

156. Thus

\[
[0]_0^L \rightarrow [0]_0^L^{(1)} \tag{70}
\]
defines the shadow operator on the point-like subspace of \( \mathcal{R} \). It preserves rank, and increase degree by one. We use \( L^{(n)} \) to denote the \( n \)-fold iteration of this operator.

157. We call \( [a]_0^L^{(n)} \) the \( n \)-th shadow of \( [0]_0^L \). Note that \( L^{(1)} \) is a formal sum over \( \mathcal{B} \). This definition agrees with the definition in [129].

158. We can now rewrite (67) as

\[
[2]_0^L = [2]_0^L + [1]_0^L^{(1)} + [0]_0^L^{(2)} \tag{71}
\]

which, as promised in [151] defines the leading and shadow terms for \( [2]_0^L \). Note that in (71), all terms have the same rank and degree. The general case of \( [a]_0^L \) is similar. [Exercise]

159. We can rewrite the general case of (71) as

\[
[a]_0^L = [a]_0^L - [a-1]_0^L^{(1)} \tag{72}
\]

for \( a \geq 1 \). This writes \( [a]_0^L \) as a linear combination, with simplex coefficients, of point-like basis elements. (For \( a = 0 \) we have of course \( [a]_0^L = [0]_0^L = [a]_0^L \).)

160. The remainder of this section provides a process to determine the general \( [a]_0^L \), from the point-like elements of the form \( [0]_0^L^{(i)} \). We have already done \( b = 0 \). For \( b \geq 1 \) we need a new method. We look at products.

161. We start at \( b = 1 \). Consider the product \( [a]_0^L [a]_0^M \), with \( a \geq 1 \). Using (72) this product unpacks into:

\[
([a]_0^L - [a-1]_0^L^{(1)})([0]_0^M) = ([a]_0^L - [a-1]_0^L^{(1)})([a]_0^M - [a-1]_0^M^{(1)}) \tag{73}
\]

162. In (73) expanded, the coefficient of \( [0]_0^L [0]_0^M \) is \( [1]_1^L [a]_0^M = [a+1]_0^M + [a-1]_0^M \). All other rank-zero coefficients are of the form \( [a]_0^M \). Thus, we get an equation with one ‘unknown’. To simplify the calculations, we first study the coefficients, in isolation.

163. Taking a slightly more general starting point, we have

\[
([a]_0^M - [a-1]_0^M)([b]_0^L - [b-1]_0^L) = [a+b]_0^M - 2[a+b-1]_0^M[a+b-2]_0^L \\
+ [0]_1^M ([a-1]_0^M - [a-2]_0^M)([b-1]_0^L - [b-2]_0^L) \tag{74}
\]

assuming \( a, b \geq 0 \).
Now write
\[ [a_\sim^0] = [a_0^0] - [a_{-1}^0] \] (75)
for \( a \geq 1 \). This is similar to (72). We call \([a_\sim^0]\) a quasi-simplex. It is part of the shadow basis for \( R_0 \). Note that it has mixed degree.

Equation (74) now becomes
\[ [a_\sim^0] [b_\sim^0] = [a+b_\sim^0] - [a+b_{-1}^0] + D [a_{-1}^0] [b_{-1}^0] \] (76)
which holds for \( a, b \geq 1 \). This is the quasi-simplex product formula. We use \( D \) rather than \([0]^1\) to emphasise the similarity with the cone product formula. [Exercise]

Now consider the special case \( a = b = 1 \). Our new product formula reduces to:
\[ [1_\sim^0] [1_\sim^0] = [2_\sim^0] - [0_\sim^0] + D \] (77)
Part of our process is to extend the shadow basis to a counting basis for \( R_0 \). Bearing (77) in mind, how should we express \( D \) in the shadow basis?

How to write \( D \) in the shadow basis is the unknown. Because \( D \) is not a linear combination of simplices, nor is it of quasi-simplices. So it must use a new basis vector, which we will call \([1_\sim^0]\). In addition, (77) contains a negative. This must be cancelled, and only \( D \) is available to do this.

This leads to the equation
\[ D = [0] = [1_\sim^0] + [0_\sim^0] \] (78)
which defines \([1_\sim^0]\) in the shadow basis. It also leads to
\[ [0]_1^0 [L] = [0_1^0] [L] + [0_1^0 (1)] \] (79)
which determines \([0_1^0 L]\) in the candidate counting basis (given the point-like elements).

When applied to \([1_\sim^0] [n_{-1}^0_0] \) this process works in exactly the same way, to give
\[ [1]_n^0 [n_0^0] = [n_0^0] + [n_{-1}^0_0] \] (80)
\[ [1]_n^0 [L] = [1_0^0] + [n_{-1}^0_0 L (1)] \] (81)
which respectively define \([n_0^0]\) and \([n_0^0 L]\). [Exercise]

Now consider:
\[ [2_\sim^0] [2_\sim^0] = [4_0^0] - [3_0^0] + D [1_0^0] [1_0^0] \]
\[ = [4_0^0] - [3_0^0] + D [2_0^0] - D [1_0^0] + D^2 \] (82)
The big unknown is the shadow basis. The unknown here is $D^2$ (in the shadow basis). The first negative $[3_{0\sim}^\sim]$ is $-\[1_{1\sim}^\sim\]$. However, the next term is $D[1_{1\sim}^\sim]$, which is equal to $[2_{0\sim}^\sim] + [3_{0\sim}^\sim]$. Thus, the first negative is cancelled. The remaining negative is $-D[1_{0\sim}^\sim] = -[1_{1\sim}^\sim] - [2_{0\sim}^\sim]$.

Thus, using the previous logic, we have

$$D^2 = [0_2^\sim] + [1_0^\sim] + [0_0^\sim]$$

$[0_2^\sim][0_0^\sim] = [0_2^L] + [1_1^L(1)] + [0_0^L(2)]$ (83)

$[0_2^\sim][n_0^L] = [n_2^L] + [n_1^L(1)] + [0_0^L(2)]$ (84)

where the last equation comes from $[2_{0\sim}^\sim] [2+n_{0\sim}^\sim]$. This process can be continued to provide a definition for all $[a_b^L]$, by using recursive on $b$. [Exercise].

For clarity, this construction of the $[a_b^L]$ depends on knowing the point-like elements. This we do in the next section.

## 10 Cone shadows

Here is the construction for the point-like $[0_0^L]$, for $r = \text{rank} L \geq 1$. Assume $L = L_1 L'$, and $L = a$ $\frac{b}{b}$. The formula is

$$[0_a^L] = C_{[1]}([a_{b+1}^L]) - C_{[1]}([a^+2_b^L])$$

where the extra term $C_{[1]}([a^+2_b^L])$ can give rise to shadow terms in $C_{[1]}([L])$.

The extra term is present to ensure consistency with the calculations in §3 Prop. 19. If $b = 0$ then $[a^+2_b^L]$ is simplex-like, and so the extra term is zero.

The understanding and explanation of this is outside the scope of the present paper. The author expects this feature to be necessary for the basis we have just constructed to be a counting basis. See §1 for some ideas as how that might be proved.

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