Group theoretical aspects of neutrino mixing
in Quantum Field Theory

M. Blasone\textsuperscript{1,2}, A. Capolupo\textsuperscript{1} and G. Vitiello\textsuperscript{1}

\textsuperscript{1} Dipartimento di Fisica and INFN, Università di Salerno, 84100 Salerno, Italy
\textsuperscript{2} Institute für Theoretische Physik, Freie Universität Berlin,
Arnimallee 14, D-14195 Berlin, Germany

Abstract

By resorting to recent results on the Quantum Field Theory of mixed particles, we discuss some aspects of three flavor neutrino mixing. Particular emphasis is given to the related algebraic structures and their deformation in the presence of CP violation. A novel geometric phase related to CP violation is introduced.

1 Introduction

Some progress has been done recently in the direction of finding a proper mathematical setting for the description of mixing in Quantum Field Theory (QFT). This is obviously a quite relevant task, in view of the importance of neutrino and meson mixing in the context of particle physics \cite{1, 2}.

It is worth to point out that \cite{3} mixing of states with different masses is not even allowed in non-relativistic Quantum Mechanics (QM). In spite of this fact, the quantum mechanical treatment \cite{4} is the one usually adopted for its simplicity and elegance. A review of the problems connected with the QM treatment of mixing and oscillations can be found in Ref.\cite{5}. Difficulties in the construction of the Hilbert space for mixed neutrinos were pointed out in Ref.\cite{6}.

Only recently \cite{7, 8, 9, 10, 11} a consistent treatment of mixing and oscillations in QFT has been achieved, based on the discovery of a rich non-perturbative structure associated to the vacuum for mixed particles. This vacuum appears to be a condensate of particle-antiparticle pairs, both for fermions and bosons. The structure of flavor vacuum reflects into observable quantities: exact oscillation formulas \cite{7} have been derived in QFT exhibiting corrections with respect to the usual QM ones.

In this paper, we first review some aspects of the quantization of neutrino mixing in the case of three flavors with CP violation and then discuss the group structure involved in the mixing and the related representations, both for two- and three-flavor mixing. The deformation of the associated algebra as well as the geometric phase due to CP violation are also discussed.
2 Three flavor fermion mixing

We discuss here some aspects of the QFT of three flavor fermion mixing [7].

Among the various possible parameterizations of the mixing matrix for three fields, we choose to work with the standard representation of the CKM matrix [1]:

\[
\Psi_f(x) = \mathcal{U} \Psi_m(x)
\]

\[
\mathcal{U} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

with \(c_{ij} = \cos \theta_{ij}\) and \(s_{ij} = \sin \theta_{ij}\), being \(\theta_{ij}\) the mixing angle between \(\nu_i, \nu_j\) and \(\Psi_m = (\nu_1, \nu_2, \nu_3)\), \(\Psi_m^T = (\nu_e, \nu_\mu, \nu_\tau)\). We work here with Dirac fields although similar conclusions are valid for Majorana neutrinos as well [7].

As shown in Ref.[7], the generator of the transformation (1) is\(^1\):

\[
\nu_\sigma^\alpha(x) \equiv G_\theta^{-1}(t) \nu_\alpha^\beta(x) G_\theta(t),
\]

with \((\sigma, i) = (e, 1), (\mu, 2), (\tau, 3)\), and

\[
G_\theta(t) = G_{23}(t)G_{13}(t)G_{12}(t),
\]

where \(G_{ij}(t) \equiv \exp[\theta_{ij}L_{ij}(t)]\) and

\[
L_{12}(t) = \int d^3x \left[ \nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x) \right],
\]

\[
L_{23}(t) = \int d^3x \left[ \nu_2^\dagger(x)\nu_3(x) - \nu_3^\dagger(x)\nu_2(x) \right],
\]

\[
L_{13}(\delta, t) = \int d^3x \left[ \nu_1^\dagger(x)\nu_3(x)e^{-i\delta} - \nu_3^\dagger(x)\nu_1(x)e^{i\delta} \right].
\]

It is clear that the phase \(\delta\) is unavoidable for three field mixing, while it can be incorporated in the definition of the fields in the two flavor case.

---

\(^1\)Let us consider for example the generation of the first row of the mixing matrix \(\mathcal{U}\). We have \(\partial \nu_e / \partial \theta_{23} = 0\); and \(\partial \nu_e / \partial \theta_{13} = G_{12}^{-1}G_{13}^{-1}\nu_1, L_{13}|G_{13}G_{12} = G_{12}^{-1}e^{-i\delta}\nu_3G_{13}G_{12}\), thus:

\[
\partial^2 \nu_e / \partial \theta_{13}^2 = -\nu_e \Rightarrow \nu_e = f(\theta_{12}) \cos \theta_{13} + g(\theta_{12}) \sin \theta_{13};
\]

with initial conditions: \(f(\theta_{12}) = \nu_e\mid_{\theta_{13}=0} = 0\) and \(g(\theta_{12}) = \partial \nu_e / \partial \theta_{13}\mid_{\theta_{13}=0} = e^{-i\delta}\nu_3\). We also have

\[
\partial^2 f(\theta_{12}) / \partial \theta_{13}^2 = -f(\theta_{12}) \Rightarrow f(\theta_{12}) = A \cos \theta_{12} + B \sin \theta_{12}
\]

with the initial conditions \(A = \nu_e\mid_{\theta_{13}=0} = \nu_1\) and \(B = \partial f(\theta_{12}) / \partial \theta_{12}\mid_{\theta_{13}=0} = \nu_2\), and \(\theta = (\theta_{12}, \theta_{13}, \theta_{23})\).
The free fields $\nu_i$ can be quantized in the usual way (we use $t \equiv x_0$):

$$\nu_i(x) = \sum_r \int d^3k \left[ u_{k,i}^r(t) \alpha_{k,i}^r + v_{-k,i}^r(t) \beta_{-k,i}^r \right] e^{ik \cdot x}, \quad i = 1, 2, 3,$$

(7)

with $u_{k,i}^r(t) = e^{-i\omega_{k,i} t} u_{k,i}^r$, $v_{k,i}^r(t) = e^{i\omega_{k,i} t} v_{k,i}^r$, and $\omega_{k,i} = \sqrt{k^2 + m_i^2}$. The vacuum for the mass eigenstates is denoted by $|0\rangle_m$. The flavor annihilation operators defined as $G_{k,i}$ are those of Ref.[7]. The relations are the usual ones; the wave function orthonormality and completeness relations are those of Ref.[7].

There it was also shown that the above generator of mixing transformations has a non-trivial action on $|0\rangle_m$. The vacuum for the flavor fields can be then defined as:

$$|0(t)\rangle_f \equiv G^{-1}_0(t)|0\rangle_m. \quad (8)$$

The flavor annihilation operators defined as $\alpha_{k,\sigma}^r(t) \equiv G^{-1}_0(t) \alpha_{k,\sigma}^r G_0(t)$ and $\beta_{k,\sigma}^r(t) \equiv G^{-1}_0(t) \beta_{k,\sigma}^r G_0(t)$ were studied in Ref.[7] and shown to exhibit a non-standard Bogoliubov like term. For example, the annihilation operator for electron neutrino is (in the reference frame $k = (0, 0, |k|)$):

$$\alpha_{k,e}^r(t) = e^{-i\theta_{12}}(U_{12}^k(t) \alpha_{k,1}^r + e^{i\theta_{13}}U_{13}^k(t) \alpha_{k,2}^r) + e^{-i\delta_{13}}U_{13}^k(t) \alpha_{k,3}^r + e^{i\theta_{12}}U_{12}^k(t) \beta_{k,2}^r + e^{-i\delta_{13}}U_{13}^k(t) \beta_{k,3}^r, \quad (9)$$

with Bogoliubov coefficients defined as:

$$V_{ij}^k(t) = |V_{ij}^k| e^{i(\omega_{k,j} + \omega_{k,i}) t}, \quad U_{ij}^k(t) = |U_{ij}^k| e^{i(\omega_{k,j} - \omega_{k,i}) t} \quad (10)$$

$$|U_{ij}^k| = \left( \frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,j} + m_j}{2\omega_{k,j}} \right)^{\frac{1}{2}} \left( 1 + \frac{|k|^2}{(\omega_{k,i} + m_i)(\omega_{k,j} + m_j)} \right) \quad (11)$$

$$|V_{ij}^k| = \left( \frac{\omega_{k,i} + m_i}{2\omega_{k,i}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,j} + m_j}{2\omega_{k,j}} \right)^{\frac{1}{2}} \left( \frac{|k|}{(\omega_{k,j} + m_j)} - \frac{|k|}{(\omega_{k,i} + m_i)} \right) \quad (12)$$

where $i, j = 1, 2, 3$ and $j > i$. We also have $|U_{ij}^k|^2 + |V_{ij}^k|^2 = 1$.

The flavor fields can be expanded in terms of the flavor ladder operators as:

$$\nu_\sigma(x) = \sum_r \int d^3k \left[ u_{k,i}^r(t) \alpha_{k,\sigma}^r(t) + v_{-k,i}^r(t) \beta_{-k,\sigma}^r(t) \right] e^{ik \cdot x}, \quad (13)$$

with $(\sigma, i) = (e, 1), (\mu, 2), (\tau, 3)$.

Let us now investigate the algebraic structures associated with the mixing generator Eq.(3). To this end we introduce the following Lagrangian:

$$\mathcal{L}(x) = \bar{\Psi}_m(x) (i \not\partial - M_d) \Psi_m(x) = \bar{\Psi}_f(x) (i \not\partial - M) \Psi_f(x), \quad (14)$$

3
where $M_d = \text{diag}(m_1, m_2, m_3)$ and the matrix $M$ is non-diagonal, being fixed by the mixing relations Eq.(1).

The above Lagrangian is invariant under global $U(1)$ phase transformations, leading to a conserved (total) charge $Q = \int d^3x \Psi_m^\dagger(x) \Psi_m(x) = \int d^3x \Psi_f^\dagger(x) \Psi_f(x)$.

We then study the invariance of $\mathcal{L}$ under global phase transformations of the kind:

$$\Psi_m'(x) = e^{i\alpha_j} \tilde{F}_j \Psi_m(x), \quad j = 1, 2, ..., 8. \quad (15)$$

where $\tilde{F}_j \equiv \frac{1}{2} \tilde{\lambda}_j$ and the $\tilde{\lambda}_j$ are a generalization of the usual Gell-Mann matrices $\lambda_j$:

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & e^{i\delta_2} & 0 \\ e^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\lambda}_2 = \begin{pmatrix} 0 & -i e^{i\delta_2} & 0 \\ i e^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & e^{-i\delta_5} \\ 0 & 0 & 0 \\ e^{i\delta_5} & 0 & 0 \end{pmatrix}, \quad \tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i e^{-i\delta_5} \\ 0 & 0 & 0 \\ i e^{i\delta_5} & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\delta_7} \\ 0 & e^{-i\delta_7} & 0 \end{pmatrix}, \quad \tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i e^{i\delta_7} \\ 0 & i e^{-i\delta_7} & 0 \end{pmatrix}, \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (16)$$

The normalization is $tr(\tilde{\lambda}_j \tilde{\lambda}_k) = 2\delta_{jk}$. One then obtains the following set of charges [7]:

$$\tilde{Q}_{m,j}(t) = \int d^3x \Psi_m^\dagger(x) \tilde{F}_j \Psi_m(x), \quad j = 1, 2, ..., 8. \quad (17)$$

Thus the matrix Eq.(1) is generated by $\tilde{Q}_{m,2}(t), \tilde{Q}_{m,5}(t)$ and $\tilde{Q}_{m,7}(t)$, with $\{\delta_2, \delta_5, \delta_7\} \to \{0, \delta, 0\}$. An interesting point is that the algebra generated by the matrices Eq.(16) is not $su(3)$ unless the condition $\Delta \equiv \delta_2 + \delta_5 + \delta_7 = 0$ is imposed: such a condition is however incompatible with the presence of a CP violating phase. When CP violation is allowed, then $\Delta \neq 0$ and the $su(3)$ algebra is deformed. To see this, let us introduce the raising and lowering operators, defined as [1]:

$$\tilde{T}_+ \equiv \tilde{F}_1 \pm i \tilde{F}_2 , \quad \tilde{U}_\pm \equiv \tilde{F}_6 \pm i \tilde{F}_7 , \quad \tilde{V}_\pm \equiv \tilde{F}_4 \pm i \tilde{F}_5 \quad (18)$$

We also define:

$$\tilde{Y} = \frac{2}{\sqrt{3}} \tilde{F}_8 , \quad \tilde{T}_3 \equiv \tilde{F}_3 , \quad \tilde{U}_3 \equiv \frac{1}{2} (\sqrt{3} \tilde{F}_8 - \tilde{F}_3) , \quad \tilde{V}_3 \equiv \frac{1}{2} (\sqrt{3} \tilde{F}_8 + \tilde{F}_3) \quad (19)$$

The commutation relations are

$$[\tilde{T}_3, \tilde{T}_\pm] = \pm \tilde{T}_\pm , \quad [\tilde{T}_3, \tilde{U}_\pm] = \pm \frac{1}{2} \tilde{U}_\pm , \quad [\tilde{T}_3, \tilde{V}_\pm] = \pm \frac{1}{2} \tilde{V}_\pm , \quad [\tilde{T}_3, \tilde{Y}] = 0 , \quad (20)$$

$$[\tilde{Y}, \tilde{T}_\pm] = 0 , \quad [\tilde{Y}, \tilde{U}_\pm] = \pm \tilde{U}_\pm , \quad [\tilde{Y}, \tilde{V}_\pm] = \pm \tilde{V}_\pm , \quad [\tilde{T}_+, \tilde{T}_-] = 2\tilde{T}_3 , \quad (21)$$

$$[\tilde{U}_+, \tilde{U}_-] = 2\tilde{U}_3 , \quad [\tilde{V}_+, \tilde{V}_-] = 2\tilde{V}_3 , \quad [\tilde{T}_+, \tilde{V}_-] = [\tilde{T}_+, \tilde{U}_+] = [\tilde{U}_+, \tilde{V}_+] = 0. \quad (22)$$
that are similar to the standard $SU(3)$ commutation relations. However, the following commutators are deformed:
\[
[\tilde{T}_+, \tilde{V}_-] = -\mathcal{U}_- e^{2i\Delta \tilde{U}_3}, \quad [\tilde{T}_+, \tilde{U}_+] = \tilde{V}_+ e^{-2i\Delta \tilde{U}_3}, \quad [\tilde{U}_+, \tilde{V}_-] = \tilde{U}_- e^{2i\Delta \tilde{V}_3}
\] (23)

Let us define the operators:
\[
Q_1 \equiv \frac{1}{3}Q + Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8}, \quad Q_2 \equiv \frac{1}{3}Q - Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8}, \quad Q_3 \equiv \frac{1}{3}Q - \frac{2}{\sqrt{3}}Q_{m,8}
\] (24) (25) (26)

\[
Q_i = \sum_r \int d^3k \left( \alpha_{k,i}^r \alpha_{k,i}^r - \beta_{-k,i}^r \beta_{-k,i}^r \right), \quad i = 1, 2, 3.
\] (27)

These are nothing but the Noether charges associated with the non-interacting fields $\nu_1$, $\nu_2$ and $\nu_3$: in the absence of mixing, they are the flavor charges, separately conserved for each generation.

In a similar way with the above derivation, we can study the invariance properties of the Lagrangian Eq.(14) under the transformations:
\[
\Psi'_f(x) = e^{i\alpha_j \tilde{F}_j} \Psi_f(x), \quad j = 1, 2, ..., 8.
\] (28)

Then the following charges are obtained
\[
\tilde{Q}_{f,j}(t) = \int d^3x \Psi_f^\dagger(x) \tilde{F}_j \Psi_f(x), \quad j = 1, 2, ..., 8.
\] (29)

In contrast with the previous case, note that the diagonal elements $\tilde{Q}_{f,3}$ and $\tilde{Q}_{f,8}$ are now time-dependent. We define the flavor charges for mixed fields as
\[
Q_e(t) \equiv \frac{1}{3}Q + Q_{f,3}(t) + \frac{1}{\sqrt{3}}Q_{f,8}(t),
\] (30)
\[
Q_\mu(t) \equiv \frac{1}{3}Q - Q_{f,3}(t) + \frac{1}{\sqrt{3}}Q_{f,8}(t),
\] (31)
\[
Q_\tau(t) \equiv \frac{1}{3}Q - \frac{2}{\sqrt{3}}Q_{f,8}(t).
\] (32)

with $Q_e(t) + Q_\mu(t) + Q_\tau(t) = Q$. These charges have a simple expression in terms of the flavor ladder operators:
\[
Q_\sigma(t) = \sum_r \int d^3k \left( \alpha_{k,\sigma}^r(t) \alpha_{k,\sigma}^r(t) - \beta_{-k,\sigma}^r(t) \beta_{-k,\sigma}^r(t) \right), \quad \sigma = e, \mu, \tau,
\] (33)

because of the connection with the Noether charges of Eq.(27) via the mixing generator:
\[
Q_\sigma(t) = G_\sigma^{-1}(t)Q_i G_\sigma(t).
\]

In Ref.[8] the above flavor charges were used to derive oscillation formulas which generalize the usual ones obtained in Quantum Mechanics.
3 Group representations and the oscillation formula

We now study the group representations. Let us first consider the simple case of two generations and then discuss the three flavor case.

3.1 Two flavors

In this case [7], the group is $SU(2)$ and the charges in the mass basis read [7]:

$$Q_{m,j}(t) = \frac{1}{2} \int d^3x \, \bar{\Psi}^m(x) \tau_j \Psi_m(x), \quad j = 1, 2, 3,$$

(34)

where $\Psi^T_m = (\nu_1, \nu_2)$ and $\tau_j = \sigma_j/2$ with $\sigma_j$ being the Pauli matrices.

The states with definite masses can then be defined as eigenstates of $Q_{m,3}$:

$$Q_{m,3}|\nu_1\rangle = \frac{1}{2}|\nu_1\rangle \quad ; \quad Q_{m,3}|\nu_2\rangle = -\frac{1}{2}|\nu_2\rangle$$

(35)

and similar ones for antiparticles. We have $|\nu_i\rangle = \alpha^+_i(r)|0\rangle_m$, $i = 1, 2$. Eq.(35) expresses the obvious fact that the mass eigenstates, treated as free particle states, are eigenstates of the conserved $U(1)$ charges associated to $\nu_1$ and $\nu_2$:

$$Q_1 \equiv \frac{1}{2}Q + Q_{m,3} \quad ; \quad Q_2 \equiv \frac{1}{2}Q - Q_{m,3}.$$  

(36)

The next step is to define flavor states using a similar procedure. We need to be careful here since the diagonal $SU(2)$ generator $Q_{f,3}$ is time-dependent in the flavor basis. Thus we define states (Hilbert space) at a reference time $t = 0$ from:

$$Q_{f,3}(0)|\nu_e\rangle = \frac{1}{2}|\nu_e\rangle \quad ; \quad Q_{f,3}(0)|\nu_\mu\rangle = -\frac{1}{2}|\nu_\mu\rangle.$$  

(37)

with $|\nu_\sigma\rangle = \alpha^+_\sigma(r)|0\rangle_f$, $\sigma = e, \mu$ and similar ones for antiparticles.

The flavor states so defined are eigenstates of the flavor charges at time $t = 0$:

$$Q_e(t) = \frac{1}{2}Q + Q_{f,3}(t) \quad ; \quad Q_\mu(t) = \frac{1}{2}Q - Q_{f,3}(t),$$  

(38)

$$Q_e(0)|\nu_e\rangle = |\nu_e\rangle \quad ; \quad Q_\mu(0)|\nu_\mu\rangle = |\nu_\mu\rangle.$$  

(39)

and $Q_e(0)|\nu_\mu\rangle = Q_\mu(0)|\nu_e\rangle = 0$.

This result is far from being trivial since the usual Pontecorvo states [4]:

$$|\nu_e\rangle_P = \cos \theta \, |\nu_1\rangle + \sin \theta \, |\nu_2\rangle$$  

(40)

$$|\nu_\mu\rangle_P = -\sin \theta \, |\nu_1\rangle + \cos \theta \, |\nu_2\rangle,$$  

(41)
are *not* eigenstates of the flavor charges. The Lorentz invariance properties of the flavor states Eq.(39) have been discussed in Ref.[12].

At a time $t \neq 0$, oscillation formulas can be derived \[7\] for the flavor charges from the following relation

$$
\langle \nu_e | Q_{f,3}(t) | \nu_e \rangle = \frac{1}{2} - |U_{12}^k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_{12}^k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right)
$$

where the non-standard oscillation term do appear.

### 3.2 Three flavors

Having discussed above the procedure for the definition of flavor states in the case of two flavors, we can directly write, for three flavors,

$$
Q_{\sigma}(0) | \nu_\sigma \rangle = | \nu_\sigma \rangle , \quad Q_{\sigma}(0) | \bar{\nu}_\sigma \rangle = - | \bar{\nu}_\sigma \rangle , \quad \sigma = e, \mu, \tau,
$$

leading to

$$
| \nu_\sigma \rangle \equiv \alpha_{k,\sigma}^{\dagger}(0) | 0(0) \rangle_f , \quad | \bar{\nu}_\sigma \rangle \equiv \beta_{k,\sigma}^{\dagger}(0) | 0(0) \rangle_f , \quad \sigma = e, \mu, \tau.
$$

These neutrino and antineutrino states can be related to the fundamental representation $3$ and $3^*$ of the (deformed) $SU(3)$ mixing group above introduced, as shown in Fig.1 for neutrinos. Note that the position of the points in the $\tilde{Y} - \tilde{T}_3$ is the same as for the ordinary $SU(3)$, since the diagonal matrices $\tilde{\lambda}_3, \tilde{\lambda}_8$ do not contain phases. However,
a closed loop around the triangle gives a non-zero phase which is of geometrical origin [13] and only depends on the CP phase. A similar situation is valid for antineutrinos.

To see this more in detail, let us consider the octet representation as in Fig.2 and define the normalized state $|A\rangle$: $\langle A|A\rangle = 1$. Then all the other states are also normalized, except for $|G\rangle$: $|G\rangle = \frac{1}{\sqrt{2}} \tilde{T}_-|A\rangle$. We obtain the following paths

\begin{align*}
(AGBA) & : \quad \tilde{V}_+\tilde{U}_-\tilde{T}_-|A\rangle = \tilde{V}_+([\tilde{U}_-, \tilde{T}_-] + \tilde{T}_-\tilde{U}_-)|A\rangle = \tilde{V}_+\tilde{V}_e^{2i\Delta\tilde{V}_3}|A\rangle = e^{i\Delta}|A\rangle \\
(ABGA) & : \quad \tilde{T}_+\tilde{U}_+\tilde{V}_-|A\rangle = e^{-i\Delta}|A\rangle \\
(AFGA) & : \quad \tilde{T}_+\tilde{V}_-\tilde{U}_+|A\rangle = -e^{-i\Delta}|A\rangle \\
(AGFA) & : \quad \tilde{U}_-\tilde{V}_+\tilde{T}_-|A\rangle = -e^{i\Delta}|A\rangle \\
(AFGBA) & : \quad \tilde{V}_+\tilde{U}_-\tilde{V}_-\tilde{T}_-\tilde{U}_+|A\rangle = |A\rangle \\
(AFEDCBA) & : \quad \tilde{V}_+\tilde{T}_+\tilde{U}_-\tilde{V}_-\tilde{T}_-\tilde{U}_+|A\rangle = |A\rangle
\end{align*}

where we have used

\begin{align*}
\tilde{U}_-|A\rangle = \tilde{T}_+|A\rangle = \tilde{V}_+|A\rangle = 0, \quad \tilde{T}_3|A\rangle = |A\rangle, \quad \tilde{V}_3|A\rangle = \frac{1}{2}|A\rangle, \quad \tilde{U}_3|A\rangle = -\frac{1}{2}|A\rangle
\end{align*}

and the commutation relations.

We thus see that the phase sign change if we change the versus of the path on the triangles; the paths on two opposite triangles and around the hexagon bring no phase.

\[8\]
4 Conclusions

In this report, we have discussed some aspects of the quantization of mixed fermions (neutrinos) in the context of Quantum Field Theory.

In particular, we have analyzed the algebraic structures arising in connection with field mixing and their deformation due to the presence of CP violating phase, in the case of neutrino mixing among three generations.

We have defined flavor states in terms of the representations of the group associated with field mixing. A new geometric phase arising from CP violation was discovered. Other geometric phases related to fermion mixing have been discussed in Refs.[14].

Acknowledgements

M.B. and G.V. are grateful to the organizers of the XII-th International Baksan School “Particles and Cosmology” for the invitation. The ESF Program COSLAB, EPSRC, INFN and INFM are also acknowledged for partial financial support.

References

[1] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford, 1989.

[2] J. Davis, D. S. Harmer and K. C. Hoffmann, Phys. Rev. Lett. 20 (1968) 1205. M. Koshiba, in “Erice 1998, From the Planck length to the Hubble radius”, 170; S. Fukuda et al. [Super-Kamiokande collaboration], Phys. Rev. Lett. 86 (2001) 5656. Q. R. Ahmad et al. [SNO collaboration] Phys. Rev. Lett. 87 (2001) 071301; Phys. Rev. Lett. 89 (2002) 011301. K. Eguchi et al. [KamLAND Collaboration], Phys. Rev. Lett. 90 (2003) 021802 M. H. Ahn et al. [K2K Collaboration], Phys. Rev. Lett. 90 (2003) 041801

[3] V. Bargmann, Annals Math. 59 (1954) 1; A. Galindo and P. Pascual, *Quantum Mechanics*, (Springer Verlag, 1990). See also: D. M. Greenberger, Phys. Rev. Lett. 87 (2001) 100405.

[4] S. M. Bilenky and B. Pontecorvo, Phys. Rep. 41 (1978) 225.

[5] M. Zralek, Acta Phys. Polon. B 29 (1998) 3925.

[6] C. Giunti, C. W. Kim and U. W. Lee, Phys. Rev. D 45 (1992) 2414; C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, Harwood Academic Press, Chur, Switzerland,1993.
[7] M. Blasone and G. Vitiello, Annals Phys. 244 (1995) 283 [Erratum-ibid. 249 (1995) 363]. M. Blasone, P. A. Henning and G. Vitiello, Phys. Lett. B 451 (1999) 140; M. Blasone, in “Erice 1998, From the Planck length to the Hubble radius” 584, [hep-ph/9810329]. M. Blasone and G. Vitiello, Phys. Rev. D60 (1999) 111302. M. Blasone, P. Jizba and G. Vitiello, Phys. Lett. B 517 (2001) 471. M. Blasone, A. Capolupo, O. Romei and G. Vitiello, Phys. Rev. D 63 (2001) 125015. M. Blasone and J. S. Palmer, [hep-ph/0305257] M. Blasone, P. P. Pacheco and H. W. Tseung, Phys. Rev. D 67 (2003) 073011. M. Blasone, P. Jizba and G. Vitiello, [hep-ph/0308009].

[8] M. Blasone, A. Capolupo and G. Vitiello, Phys. Rev. D 66 (2002) 025033;

[9] M. Binger and C. R. Ji, Phys. Rev. D60 (1999) 056005. C. R. Ji and Y. Mishchenko, Phys. Rev. D 64 (2001) 076004; Phys. Rev. D 65 (2002) 096015.

[10] K. Fujii, C. Habe and T. Yabuki, Phys. Rev. D 59 (1999) 113003 [Erratum-ibid. D 60 (1999) 099903]; Phys. Rev. D 64 (2001) 013011.

[11] K. C. Hannabuss and D. C. Latimer, J. Phys. A36 (2003) L69; A33 (2000) 1369.

[12] M. Blasone, J. Magueijo and P. Pires-Pacheco, [hep-ph/0307205].

[13] Y. Aharonov and J. Anandan Phys. Rev. Lett. 58 (1987) 1593; 65 (1990) 1697.

[14] M. Blasone, P. A. Henning and G. Vitiello, Phys. Lett. B 466 (1999) 262; X. B. Wang, L. C. Kwek, Y. Liu and C. H. Oh, Phys. Rev. D 63 (2001) 053003. Y. Liu, Phys. Rev. D 61 (2000) 033010. S. Capozziello and G. Lambiase, Europhys. Lett. 52 (2000) 15.