On Skyrmion semiclassical quantization in the presence of an isospin chemical potential

Thomas D. Cohen
Department of Physics, University of Maryland, College Park, MD 20742-4111, USA

Juan A. Ponciano
CEFIMAS, Av. Santa Fe 1145, (1059) Buenos Aires, Argentina
CONICET, Rivadavia 1917, (1033) Buenos Aires, Argentina
Universidad Nacional de La Plata, C.C. 67, (1900) La Plata, Argentina

Norberto N. Scoccola
Physics Depart., Comisión Nacional de Energía Atómica, (1429) Buenos Aires, Argentina
CONICET, Rivadavia 1917, (1033) Buenos Aires, Argentina
Universidad Favaloro, Solís 453, (1078) Buenos Aires, Argentina

The semiclassical description of Skyrmions at small isospin chemical potential $\mu_I$ is carefully analyzed. We show that when the calculation of the energy of a nucleon is performed using the straightforward generalization of the vacuum sector techniques ($\mu_I = 0$), together with the "natural" assumption $\mu_I = \mathcal{O}(N_c^0)$, the proton and neutron masses are nonlinear in $\mu_I$ in the regime $|\mu_I| < m_\pi$. Although these nonlinearities turn out to be numerically quite small, such a result fails to strictly agree with the very robust prediction that for those values of $\mu_I$ the energy excitations above the vacuum are linear in $\mu_I$. The resolution of this paradox is achieved by studying the realization of the large $N_c$ limit of QCD in the Skyrme model at finite $\mu_I$. This is done in a simplified context devoid of the technical complications present in the Skyrme model but which fully displays the general scaling behavior with $N_c$. The analysis shows that the paradoxical result appears as a symptom of using the semi-classical approach beyond its regime of validity and that, at a formal level, the standard methods for dealing with the Skyrme model are only strictly justified for states of high isospin $I \sim N_c$.

PACS numbers: 12.39.Dc, 25.75.Nq

I. INTRODUCTION

This paper deals with a paradox associated with the description of Skyrmions[1, 2, 3] in systems with a non-zero isospin chemical potential $\mu_I \neq 0$—i.e., a chemical potential coupled to the third component of the isospin[4, 5]. Before discussing this in detail it is probably of some utility to discuss why one should care about the problem. One reason is simple: the problem of strong interactions with $\mu_I \neq 0$ is an interesting problem in theoretical physics in its own right, regardless of whether it has any immediate experimental implications. From the perspective of QCD there are certain aspects of the problem which are tractable at small isospin chemical potentials and others which are tractable at large isospin chemical potentials[6, 7]. Thus the study of this problem provides an additional avenue to gain insight into QCD dynamics. From the perspective of the Skyrme model—whose purpose is to mimic certain key aspects of QCD—the problem raises a distinct set of theoretical and mathematical issues which are interesting and illuminating in their own right. However, there is a more programmatic reason why this problem is of potential importance. One of the central problems in modern theoretical nuclear physics is the behavior of hadronic matter in extreme conditions of temperature or density. Ultimately, properties of such matter follows from the underlying theory of strong interactions, namely QCD. However, while ab initio calculations of QCD at high temperature and zero density are tractable via lattice simulations, the problem of dense matter is more problematic as they are afflicted by the notorious fermion sign problem. There has been considerable recent activity in attempting to develop viable lattice methods for systems at high temperature and low but nonzero chemical potential[8]. Unfortunately, these methods break down as the temperature decreases and are unsuitable for the study of cold dense matter. This is

*Electronic address: cohen@physics.umd.edu
†Electronic address: ponciano@fisica.unlp.edu.ar
‡Electronic address: scoccola@tandar.cnea.gov.ar
unfortunate since this regime is of potential importance in astrophysical applications.

Given this situation it is natural to consider models which (it is hoped) capture much of the essence of QCD, while being tractable in the regime of interest. One class of models which may be employed are Skyrme-type models. These have the virtue of encoding the scaling properties of QCD with $N_c$ (the number of colors). Moreover, the models become essentially classical at large $N_c$ and thus are far more tractable than fully quantum theories like QCD. In fact, over the years there has been a rather large literature dealing with cold dense Skyrmeon matter.\[12\]

Unfortunately, there is a fundamental difficulty in using the Skyrme model—or any other model—in place of QCD for cold dense matter. Even if a model is known to work well in reproducing the results of nature—and hence of QCD—in the vacuum sector, one is never sure whether the model is likely to continue to mimic QCD to good approximation in a quite different regime. Ideally one would calibrate the models against QCD in the regime of interest and test how they do. However, for cold dense matter this is not possible; the crux of the issue is that QCD is not tractable and observational data from astrophysics is, at best, both incomplete and quite indirect.

In these circumstances it becomes exceptionally important to test models against QCD anywhere where it is both computationally tractable and shares features in common with the regime of interest—cold dense matter. In this regard QCD with $\mu_I \neq 0$ can play a critical role. On the one hand, unlike the case of a non-vanishing baryon chemical potential, the functional determinant in a path integral can be shown to be real and non-negative for systems with a non-vanishing isospin chemical potential.\[9\] This means that the fermion sign problem is evaded and lattice simulations become practical. Preliminary simulations have already been done.\[10\] These were relatively small but increasing accurate lattice studies are possible.\[11\] On the other hand, this system has something in common with the problem of interest—namely a treatment of QCD with a baryon chemical potential. Thus, studies of models of QCD in the regime of $\mu_I \neq 0$ can play an important role in testing models against QCD.

Of course, if one intends ultimately to use a regime with $\mu_I \neq 0$ to test a model against QCD, it is essential that one has computed the $\mu_I \neq 0$ regime correctly in the model. It is in this context we wish to point out a paradoxical result that arises in the semiclassical treatment of the Skyrme model in the presence of a finite $\mu_I$.

The paradox concerns the behavior at small but non-zero values of $\mu_I$ and zero temperature. At zero temperature the behavior of QCD is known for small $\mu_I$. When $|\mu_I| < m_\pi$ the QCD vacuum state is unaltered by the presence of an isospin chemical potential. The only effect that adding a chemical potential has in that regime is to shift excitation energies above the vacuum (e.g., single particle energies for hadrons) by $\mu_I I_3$ where $I_3$ is the isospin of the excited state. Thus in this regime all energies are strictly linear in $\mu_I$ with a slope given by $I_3$. At $T = 0$, nonlinearities only appear for $|\mu_I| > m_\pi$. When $|\mu_I|$ reaches $m_\pi$, the energy for the lowest pionic excitation reaches zero. Beyond this point it is energetically favorable for pions to condense. This pion condensation alters the vacuum structure and thereby alters excitation energies beyond the linear shift due to the direct $\mu_I I_3$ contribution.

The paradox is that calculations of the energy of a proton or neutron in a Skyrme model using the natural generalization of the techniques used in the vacuum sector yield proton and neutron masses which are nonlinear in $\mu_I$ in the regime $\mu_I < m_\pi$. As we will show in Sec. III these nonlinearities are numerically quite small; however, the linear behavior in $\mu_I$ is a fundamental property of the quantum system the model is supposed to describe. Thus even small non-linearities appear to imply that there is something fundamentally wrong with the approach—or at the very least, something fundamental which is not fully understood.

This is quite problematic on two grounds. The first is what was alluded to above. In order to use the $\mu_I \neq 0$ regime as a test of Skyrme type models in regimes where chemical potentials play a role, it is essential to have reliable calculations. The second is more vexing: if the standard approach fails at some deep level in its straightforward generalization from the vacuum sector to $\mu_I \neq 0$, how certain can one be of its validity in the vacuum sector?

In this paper we resolve this paradox—at least at a formal level. The implication of this resolution for phenomenology, however, remains somewhat unclear.

The crux of the resolution involves the nature of the large $N_c$ limit of QCD. The standard methods for dealing with the Skyrme model—finding a classical static soliton solution and then requantizing the collective modes to restore symmetries and describe physical states—are only strictly justified at large $N_c$. Thus in using these methods one is implicitly studying a large $N_c$ world and then attempting to identify features which extrapolate back to the real world of $N_c = 3$. One feature of this large $N_c$ world is in an infinite tower of baryon states with $I = J$. For $N_c$ large but finite, the levels in this tower are split by effects of order $1/N_c$. However these effects grow with $I$ (or equivalently $J$). For sufficiently large $I$, the excitation energies become too large to neglect and the semiclassical methods based on the $1/N_c$ expansion ultimately break down. The usual interpretation of the $I = J$ tower of states for the physical world of $N_c = 3$ is that the $I = J = 1/2$ states correspond to the nucleon, the $I = J = 3/2$ states to the delta, while states with $I = J > 3/2$ correspond to the region beyond the scope of the approximation and are artifacts of the large $N_c$ limit.

In considering the formal large $N_c$ limit, it is important to recall that there are critical ways in which the large $N_c$ limit and the chiral limit do not commute.\[13\] The standard semiclassical treatment of the Skyrme model is based on a pure $1/N_c$ expansion. Thus implicitly when studying chiral properties in Skyrme models using standard quantization
methods one is implicitly taking the large $N_c$ limit before the chiral limit. Thus at least formally, $m_\pi \sim N_c^0$, even though $m_\pi$ is numerically small. This formal fact plays an essential role in understanding the origin of non-linearities for $|\mu_I| < m_\pi$.

The resolution to the paradox at the formal level is the following: when $\mu_I$ is treated as being of order $N_c^0$ (but less than $m_\pi$), the natural generalization of the methods used in the vacuum is formally correct at large $N_c$ for the state of the lowest free energy $E - \mu_I I$ and for a band of low-lying states above it. However, these states are formally states of high isospin—$I \sim N_c$—and as such do not include the nucleon. Thus in a formal sense (based on $1/N_c$ expansion), the nucleon is outside the domain of validity of the standard semiclassical methods. The non-linearities in $\mu_I$ are a symptom of using the semiclassical approach beyond its regime of validity.

The plan of this paper is as follows: The next section contains a brief discussion of why the hadrons’ energies are linear in $\mu_I$ for $|\mu_I| < m_\pi$. The following section contains a review of the semiclassical treatment of the nucleon in the Skyrme model; this yields non-linearities in $\mu_I$ for $|\mu_I| < m_\pi$, provided $\mu_I$ is treated as being formally of order $N_c^0$. This is in contrast to the general results of the preceding section and as such constitutes the heart of the apparent paradox. In the section following, a simple toy quantum mechanical model is introduced. This model has central features analogous to the soliton and its semiclassical treatment but in a far more transparent form; it illustrates explicitly the underlying issues. The model has the virtue of being tractable quantum mechanically as well as semiclassically and thus enables to explicitly verify one’s understanding. These model results illustrate the resolution of the paradox mentioned in this introduction. A final section discusses the implication of these results.

II. THE QUANTUM MECHANICS OF A SYSTEM IN THE PRESENCE OF AN ISOSPIN CHEMICAL POTENTIAL

In analyzing the Skyrme model at finite isospin chemical potential it is essential to recall that although the model is typically treated classically it is ultimately being used to model a quantum system—namely QCD. Accordingly it is essential to ensure that any treatments of the model correctly encode the underlying quantum mechanics. In this section, we review why elementary quantum considerations require that energy of the proton and the neutron are linear in $\mu_I$ below a critical value (provided that $T = 0$, as will be assumed throughout this paper).

To begin let us consider the problem a bit abstractly. Let us consider a general quantum mechanical system with Hamiltonian, $\hat{H}_0$. The system can be quite general with one important proviso—there must be a gap in the spectrum between the ground state and the lowest excited states. Suppose further that there is some operator, $\hat{Q}$ which corresponds to a conserved “charge”.

$$[\hat{H}_0, \hat{Q}] = 0.$$  \hfill (1)

For the present problem $\hat{Q}$ is the third component of the isospin (which is a conserved quantity in both QCD and in the Skyrme model). Because $\hat{H}_0$ and $\hat{Q}$ commute they have simultaneous eigenstates (if there are degeneracies in the spectrum of $\hat{H}_0$ we will work on a basis that where $\hat{H}_0$ and $\hat{Q}$ commute they have simultaneous eigenstates):

$$\hat{H}_0 |n, q\rangle = E_{n,q} |n, q\rangle, \quad \hat{Q} |n, q\rangle = q |n, q\rangle,$$  \hfill (2)

where $q$ labels the charge of the state and $n$ labels all other quantum numbers.

Adding a chemical potential for $\hat{Q}$ to the Hamiltonian yields a new Hamiltonian

$$\hat{H}' = \hat{H}_0 - \mu_Q \hat{Q}.$$  \hfill (3)

This new Hamiltonian is a free energy operator; its ground state corresponds to the free energy of a system at fixed chemical potential and zero temperature. For field theories such as QCD (or the Skyrme model) it is the vacuum in the presence of the chemical potential. The commutativity of $\hat{H}_0$ with $\hat{Q}$ ensures that the eigenstates of $\hat{H}_0$ are also eigenstates of $\hat{H}'$. Equation (2) implies that its eigenvalues are given by

$$\hat{H}' |n, q\rangle = E'_{n,q} |n, q\rangle \quad \text{with} \quad E'_{n,q} = (E_{n,q} - \mu_Q).$$  \hfill (4)

Equation (4) immediately implies that the absolute energy of any state depends linearly on $\mu_Q$ for any value of $\mu_I$. It is important to note that this does not imply that excitation energies are also linear in $\mu_Q$ for any value of $\mu_I$. The reason for this is that excitation energies are measured relative to the ground state energy and ground state energies need not be linear in $\mu_Q$. Nonlinear behavior in the ground state energy occurs due to level crossing; the state which minimizes $\hat{H}'$ at one $\mu_Q$ need not minimize it another. In the context of quantum field theories this corresponds to a phase transition to a new “vacuum” state with a condensate carrying the charge $Q$. However, provided that in a
regime in which the ground state is unchanged from \( \mu_Q = 0 \), the excitation energies are linear in \( \mu_Q \). For theories with a gap in the spectrum one expects a finite region in \( \mu_Q \) over which the vacuum is unchanged and, accordingly, excitation energies are linear. Now let us return to the issue of QCD with an isospin chemical potential and zero temperature. The proton and neutron are excitations with baryon number of unity and a third component of isospin excitation energies are linear. Now let us return to the issue of QCD with an isospin chemical potential and zero \( \mu \) for which the phase transition occurs. There is very strong evidence that critical isospin chemical potential occurs at \( \mu_I = m_\pi \). Thus, as a very robust prediction one expects the energy of the proton and neutron excitation above the vacuum to be linear in \( \mu_I \) for \( |\mu_I| < m_\pi \).

One might worry that there is no rigorous demonstration that the critical chemical potential is at \( |\mu_I| = m_\pi \). In order for the critical chemical potential to be smaller than this, there must be a state in QCD with mass per unit isospin less than \( m_\pi \); this is highly implausible on its face. Moreover even if this highly implausible scenario were true, it would not alter the formal problem with the Skyrme model calculations. In those model calculations the vacuum does not undergo a phase transition—vacuum properties far from the soliton are unaltered by the chemical potential. Thus for the model the proton and neutron must be linear in \( \mu_I \) and yet, as we will see in the following section, a naive extension of the standard semiclassical treatment lead to deviations from linearity.

### III. Semiclassical Treatment of the Skyrme Model at \( \mu_I \neq 0 \)

In this section we analyze the semiclassical quantization of the skyrnymion paying special attention to the issues raised by the presence of a finite \( |\mu_I| \leq m_\pi \). For definiteness we consider here the lagrangian of the \( SU(2) \) Skyrme model with quartic term stabilization and finite pion mass. It is given by

\[
L = -\frac{f^2}{4} \text{Tr} \{ L_\alpha L^\alpha \} + \frac{1}{32e^2} \text{Tr} \{ [L_\alpha, L_\beta]^2 \} + \frac{m^2}{4} \frac{f^2}{\pi} \text{Tr} \{ U + U^\dagger - 2 \}. \tag{5}
\]

In Eq.\(^{(5)}\), as usual, \( U \) represents the \( SU(2) \) chiral field and the Maurier-Cartan operator \( L_\alpha \) is defined by \( L_\alpha = U^\dagger \partial_\alpha U \). The isospin chemical potential \( \mu_I \) is introduced by performing the replacement

\[
\partial_\alpha U \rightarrow \partial_\alpha U - i \frac{\mu_I}{2} [\tau_3, U] g_{\alpha 0}, \tag{6}
\]

where \( g_{\alpha \beta} \) is the metric tensor in Minkowski space and \( \tau_3 \) is the third Pauli matrix. In what follows we will assume that \( |\mu_I| < m_\pi \), namely that the perturbative vacuum \( U = 1 \) is stable against pion condensation.

We consider a spinning soliton configuration of the form

\[
U = A \, \hat{\bar{U}} (R^{-1} \hat{\vec{r}}) \, A^\dagger, \tag{7}
\]

where \( A \) and \( R \) are time-dependent isospin and spin rotations, respectively. \( \hat{\bar{U}} (\hat{\vec{r}}) \) is a static configuration. Inserting this form for \( U \) in Eq.\(^{(5)}\) we get

\[
L = -M_0 + \frac{1}{2} \Lambda_{ab}^I (\omega_a - \mu_I D_{3a}) (\omega_b - \mu_I D_{3b}) + \frac{1}{2} \Lambda_{ij}^I \Omega_i \Omega_j + \Lambda_{ai}^M (\omega_a - \mu_I D_{3a}) \Omega_i. \tag{8}
\]

Here, we have used

\[
A^{-1} \dot{A} = \frac{i}{2} \omega_a \tau_a,
\]

\[
\left( R^{-1} \dot{\bar{R}} \right)_{ij} = \epsilon_{ijk} \Omega_k, \tag{9}
\]

and

\[
D_{ab} = \frac{1}{2} \text{Tr} [\tau_a A \tau_b A^{-1}]. \tag{10}
\]

The static mass \( M_0 \) is given by

\[
M_0 = -\int d^3 r \left[ \frac{f^2}{4} \text{Tr} \{ \dot{L}_i \dot{L}_i \} + \frac{1}{32e^2} \text{Tr} \{ [\dot{L}_i, \dot{L}_j]^2 \} + \frac{m^2}{4} \frac{f^2}{\pi} \text{Tr} \{ \dot{U} + \dot{U}^\dagger - 2 \} \right], \tag{11}
\]
while the tensors $\Lambda_{ab}^I$, $\Lambda_{ab}^J$, and $\Lambda_{ab}^M$ are the isospin, spin and mixed inertia tensors, respectively, given by

$$\Lambda_{ab}^I = \int d^3r \left[ \frac{f_2}{8} \text{Tr} \left\{ U^\dagger [\tau_a, U^\dagger \tau_b, U] \right\} + \frac{1}{32c^2} \text{Tr} \left\{ \left[ \hat{L}_i, U^\dagger [\tau_a, \hat{U}] \right] \left[ \hat{L}_i, U^\dagger [\tau_b, \hat{U}] \right] \right\} \right],$$

$$\Lambda_{ab}^J = \int d^3r \varepsilon_{ijk} \varepsilon_{mnl} r_j r_n \left[ -\frac{f_2}{2} \text{Tr} \left\{ \hat{L}_i \hat{L}_m \right\} - \frac{1}{8c^2} \text{Tr} \left\{ \left[ \hat{L}_i, \hat{L}_a \right] \left[ \hat{L}_m, \hat{L}_a \right] \right\} \right],$$

$$\Lambda_{ab}^M = \int d^3r \varepsilon_{ijk} r_j \left[ \frac{f_2}{4} \text{Tr} \left\{ \hat{U}^\dagger [\tau_a, \hat{U}] \hat{L}_i \right\} + \frac{1}{16c^2} \text{Tr} \left\{ \left[ \hat{U}^\dagger [\tau_a, \hat{U}], \hat{L}_m \right] \left[ \hat{L}_i, \hat{L}_m \right] \right\} \right].$$

The number of independent non-vanishing components of the inertia tensors is given by the explicit form of the static ansatz. All the configurations to be considered below will be symmetric under $\pi$-rotations along any of the three cartesian axes. In this case one can prove that all the inertia tensors are diagonal. Thus in what follows we will assume that

$$\Lambda_{ab}^I = \Lambda_{ab}^J = \Lambda_{ab}^M = 0 \quad \text{if } a \neq b. \quad (15)$$

Now let us pay attention to the $N_c$-order of the different quantities appearing in Eq. (15). As usual, $M_0$ and the inertia tensors $\Lambda$’s are of $O(N_c)$ while $\omega_a$ and $\Omega_i$ are taken to be of $O(N_c^{-1})$. In order to proceed we have to determine which order in $N_c$ has to be assigned to $\mu_I$. Since we are interested in values of $\mu_I \leq m_\pi$ it appears to be natural to take $\mu_I = O(N_c^2)$. In this case, to leading order in $N_c$ Eq. (15) reads

$$L^{(1)} = -M_0 + \frac{1}{2} \mu_3^2 \sum_{a=1,2,3} \Lambda_{aa}^I D_{3a} D_{3a}, \quad (16)$$

where Eq. (15) has been used. In principle the equations to determine the static soliton configuration should be obtained by minimizing this lagrangian. Some simplification can be performed by noting that since the chemical potential acts along the 3-axis in isospin space, the resulting configuration is expected to be axially symmetric along the third axis. In this case we have

$$\Lambda_{33}^I = \Lambda_{33}^J = \Lambda_{33}^M = \Lambda_3,$$

$$\Lambda_{11}^I = \Lambda_{22}^I = \Lambda_I,$$

$$\Lambda_{11}^J = \Lambda_{22}^J = \Lambda_J,$$

$$\Lambda_{11}^M = \Lambda_{22}^M = \Lambda_M. \quad (17)$$

Using $\sum_{a=1,2,3} D_{3a} D_{3a} = 1$, it is not difficult to find that Eq. (15) can be expressed as

$$L^{(1)} = -M_0 + \frac{1}{2} \Lambda_3 \mu_3^2 + \frac{1}{2} \left( \Lambda_3 - \Lambda_I \right) \mu_I^2 \sin^2 \beta. \quad (18)$$

Here, we have used the standard parametrization of the Wigner D-functions in terms of the Euler angles $\alpha, \beta, \gamma$. As we see, static mass $M_{st} = -L^{(1)}$ depends on the orientation of the soliton in flavor space. However, if we assume that the deformation induced by the chemical potential is not too large we have that $\Lambda_3 - \Lambda_I << \Lambda_3$. In this case the $\beta$-dependent term can be neglected and we have

$$M_{st} = M_0 - \frac{1}{2} \Lambda_3 \mu_3^2. \quad (19)$$

Namely, at the static level we have to minimize a $\mu_I$-dependent mass which, obviously, leads to a $\mu_I$-dependent soliton configuration. In what follows we will denote with an extra upper index the soliton quantities calculated with this $\mu_I$-dependent soliton configuration. Thus, the full lagrangian up to order $1/N_c$ reads

$$L = -M_0^{(\mu_I)} + \frac{1}{2} \Lambda_{I}^{(\mu_I)} [ (\omega_1 - \mu_I D_{31})^2 + (\omega_2 - \mu_I D_{32})^2 ] + \frac{1}{2} \Lambda_{J}^{(\mu_I)} \left[ \Omega_1^2 + \Omega_2^2 \right] + \frac{1}{2} \Lambda_{M}^{(\mu_I)} \left[ (\omega_3 - \mu_I D_{33}) + \Omega_3 \right]^2$$

$$+ \Lambda_{M}^{(\mu_I)} \left[ (\omega_1 - \mu_I D_{31}) \Omega_1 + (\omega_2 - \mu_I D_{32}) \Omega_2 \right]. \quad (20)$$

Defining the canonical conjugate momenta in the usual way, we have

$$p_a^{b} = \frac{\partial L}{\partial \omega_a} = \Lambda_{a}^{(\mu_I)} (\omega_a - \mu_I D_{3a}) + \Lambda_{M}^{(\mu_I)} \Omega_a; \quad \text{for } a = 1, 2 \quad (21)$$

$$j_a^{b} = \frac{\partial L}{\partial \Omega_a} = \Lambda_{a}^{(\mu_I)} (\omega_a - \mu_I D_{3a}) + \Lambda_{M}^{(\mu_I)} \Omega_a; \quad \text{for } a = 1, 2 \quad (22)$$

$$j_3^{b} = j_3^{b} = \frac{\partial L}{\partial \Omega_3} = \Lambda_{3}^{(\mu_I)} (\omega_3 - \mu_I D_{33} + \Omega_3). \quad (23)$$
Thus, following the same steps as before we get

\[ H = M_0^{(\mu_I)} + \mu_I \sum_{a=1,2,3} D_{3a} J_a^{bf} \]

\[ + \frac{1}{2} \left[ \frac{(J_3^{bf})^2}{\Lambda_3^{(\mu_I)}} + \frac{(J_1^{bf})^2 + (J_2^{bf})^2}{\Lambda_j^{(\mu_I)}} + \frac{(J_3^{bf})^2}{\Lambda_j^{(\mu_I)}} \right] - 2 \Lambda_M^{(\mu_I)} \left[ J_1^{bf} J_1^{bf} + J_2^{bf} J_2^{bf} \right]. \]  (24)

It should be stressed that, as expected, using Eqs. (21)–(23) one can easily check that to leading order in \( N_c \) the minimization of \( H \) leads to the same result as the minimization of \( L^{(1)} \). Defining \( \vec{T} = \vec{I} - \vec{J} \) we obtain that \( H \) can be expressed as

\[ H = M_0^{(\mu_I)} - \mu_I I_3 + \frac{1}{2} \left[ \frac{(\Lambda_j^{(\mu_I)} - \Lambda_j^{(\mu_I)}) J^2}{\Lambda_j^{(\mu_I)} - \Lambda_j^{(\mu_I)}} + \frac{1}{\Lambda_j^{(\mu_I)}} \right] + \frac{1}{2} \left[ \frac{J_3^{bf}^2}{\Lambda_j^{(\mu_I)}} - \Lambda_j^{(\mu_I)} \right] T^2, \]  (25)

where we have used the well-known relation \( I_3 = -\sum_{a=1,2,3} D_{3a} J_a^{bf} \) between the lab-frame components of the isospin and those of the spin in the body-fixed frame. It should be noted that, of course, \( |I - J| \leq T \leq I + J \). This means that for the particular case of the nucleon \((J = I = 1/2)\) we have \( T = 0, 1 \). Since for not-too-deformed configurations \( \Lambda_j^{(\mu_I)} \approx \Lambda_j^{(\mu_I)} \approx \Lambda_j^{(\mu_I)} \), we expect that the mass of the \( T = 1 \) state will be much higher than that of the \( T = 0 \) state. Thus, for the physical nucleon we take \( T = 0 \). In this case, using \( J = I = 1/2, \left( J_3^{bf} \right)^2 = 1/4 \) we obtain that

\[ M_n^{(\mu_I)} = M_0^{(\mu_I)} + \mu_I + \frac{1}{2} \frac{1}{4} \left( \frac{1}{2} \Lambda_3^{(\mu_I)} \right) - \frac{1}{\Lambda_j^{(\mu_I)}} \Lambda_j^{(\mu_I)} \]  (26)

So far we have considered the axial symmetric soliton configuration which is expected to be the lowest energy configuration in this case. As an approximation to this, in Ref. [9] a spherical symmetric hedgehog configuration was used. For completeness, we provide here the corresponding formulae. In such case, we have

\[ \Lambda_j^{(\mu_I)} = \Lambda_j^{(\mu_I)} = \Lambda_j^{(\mu_I)} = \Lambda_0 \delta_{ij} \]  (27)

Thus, following the same steps as before we get

\[ M_n^{(\mu_I)} = M_0^{(\mu_I)} + \frac{\mu_I}{2} + \frac{3}{8 \Lambda_0^{(\mu_I)}}, \]  (28)

where \( M_0^{(\mu_I)} \) is related with \( M_H \) calculated in Ref. [9] by

\[ M_0^{(\mu_I)} = M_H(\mu_I) + \frac{1}{2} \Lambda_0^{(\mu_I)} \mu_I^2. \]  (29)

The \( \mu_I \) dependence of the proton and neutron shifts (\( \Delta_p \) and \( \Delta_n \), respectively) of the corresponding masses from their values at \( \mu_I = 0 \) is shown in Fig. 1. To perform the numerical calculations we have used the standard set of parameters of Ref. [14]: \( f_x = 54 \text{ MeV}; e = 4.84; m_\pi = 138 \text{ MeV} \). We can readily see that for both the axially symmetry exact configuration (dashed line) and for the spherical approximate configuration (dotted line) there is a nonlinear behavior. In fact, the results for these two configurations are almost identical, with small differences appearing only for \( \mu_I/m_\pi \) larger than 0.9. This provides an a posteriori justification for the use of the spherical approximate configuration made in Ref. [9]. While the deviation from linearity (full line) is numerically quite small, the very existence of such deviation implies an inconsistency with the quantum mechanics analysis discussed in the
FIG. 1: Shift of the proton and neutron masses with respect to their corresponding values at \( \mu_I = 0 \) as a function of the isospin chemical potential \( \mu_I \) for \( \mu_I \leq m_\pi \). Full line corresponds to the result obtained under the assumption \( \mu_I = O(N_c^{-1}) \) while the dashed (dotted) line corresponds to the use of axial (spherical) ansatz assuming \( \mu_I = O(N_c^0) \).

previous section. The fact that the “natural” choice \( \mu_I = O(N_c^0) \) leads to such an inconsistency represents the paradox which we would like to clarify in this article.

One might wonder whether there is a way to perform the semiclassical quantization which leads to a linear behavior. In fact, this is possible provided one makes the somewhat unnatural assignment \( \mu_I = O(N_c^{-1}) \). Then, to leading order in \( N_c \), the lagrangian is given by

\[
L^{(1)} = -M_0
\]  

(30)

Therefore, in this case to obtain the static soliton configuration one should only minimize \( M_0 \). In this way one obtains a static configuration \( U_{st} \) which is independent of \( \mu_I \) and, thus, of the spherically hedgehog type. The full expression for \( L \) reads

\[
L = -M_0 + \frac{1}{2} \Lambda_0 \left( \Omega_a + \omega_a - \mu_I D_{3a} \right) \left( \Omega_a + \omega_a - \mu_I D_{3a} \right)
\]

(31)

where we note that now \( M_0 \) and \( \Lambda_0 \) do not depend on \( \mu_I \). As before we define the momenta

\[
I_a = \frac{\partial L}{\partial \omega_a} = J_a = \frac{\partial L}{\partial \Omega_a} = \Lambda_0 \left( \Omega_a + \omega_a - \mu_I D_{3a} \right)
\]

(32)

Thus, the resulting Hamiltonian \( H = \Omega \cdot I - L \) is

\[
H = M_{sol} + \mu_I D_{3a} J_a + \frac{J_a^2}{2 \Lambda_0}
\]

(33)

which leads to

\[
M_{p,n} = M_0 \mp \frac{\mu_I}{2} + \frac{3}{8 \Lambda_0}
\]

(34)

Comparing with the situation described above we see that here there is not intrinsic dependence of \( M_0 \) and \( \Lambda_0 \) on \( \mu_I \). In this case we obtain the energy shift linear in \( \mu_I \) represented by a full line in Fig[1]. Although this might be a resolution of the problem one is still left with the question of why the “natural” assignment \( \mu_I = O(N_c^0) \) is not fully compatible with general expectation from general quantum mechanical principles while the “unnatural” assignment \( \mu_I = O(N_c^{-1}) \) does. In the following section we will address this issue in detail.
IV. A TOY MODEL

As seen in the previous section the results from the semiclassical quantization of the Skyrme model with the “natural” assignment $\mu_I = \mathcal{O}(N_c^0)$ are inconsistent with the formal requirement of the analysis in Sect. III that the proton and neutron energies are strictly linear in $\mu_I$ below the critical value of $m_\pi$. However, the calculation in the Skyrme model is technically complicated and these complications may tend to obscure the origin of the discrepancy. These complications are of two sorts. The first is that the system is a field theory with an infinite number of degrees of freedom. A few of these are collective modes which are requantized, but the remainder are internal degrees of freedom associated with the shape of the soliton. These internal degrees of freedom come into play when one alters the profile in response to the inclusion of $\mu_I$. The second complication is that collective degrees of freedom—those which are requantized—do not commute with one another so the structure of the quantum mechanical system is nontrivial.

However, the general scaling behavior of the system with $N_c$—which we believe to be the core of the problem—can be illustrated in a much simpler context: a quantum system with two degrees of freedom; one which plays the role of the internal degrees of freedom and the other the collective ones. This system is supposed to be analogous to the soliton excitations acting above the vacuum and hence the analogy only holds in the regime where the isospin chemical potential has not induced a phase transition of the vacuum—which is the regime where our paradox has been identified. The absolute energies in this model thus correspond to the excitation energy of the baryons above vacuum. The collective degree of freedom in the model is associated with a conserved quantity (which we will take to be the analog of the third component of isospin); we can add a chemical potential for this quantity. The system can be treated semiclassically. Such a “toy model” is a useful place to test approximation schemes since it is solvable at the quantum level and, thus, one can directly compare approximate with exact results.

The model, in units with $\hbar = 1$, is given by the following Hamiltonian which corresponds to a particle of unit charge moving in two dimensions:

$$\hat{H} = \frac{1}{2m} \left( \vec{\beta} - \vec{A} \right)^2 + \lambda V \left( \lambda^{-1} \vec{\beta}^2 \right)$$

(35)

where $\lambda$ plays the role of $N_c$ and $\vec{A}$ is an infinitely thin magnetic flux localized at the origin and with a strength of precisely one half of a flux quantum: $\vec{A}(\vec{x}) = -\frac{\hbar \lambda}{2e} \vec{\beta} \vec{1}$ where $\hat{\beta}$ and $\vec{1}$ indicate unit vectors (as opposed to quantum operators). The purpose for including this vector potential will become clear presently. Note that this model is axially symmetric and hence commutes with $\hat{L}$, the two-dimensional angular momentum; the Hamiltonian can be written as $\hat{H} = \hat{H}_r + (\vec{L} - 1/2)^2$, where $\hat{H}_r$ acts entirely on the radial degrees of freedom and the $\vec{L} - 1/2$ structure reflects the presence of the vector potential. It is convenient to introduce a new operator $\hat{L} \lambda = \vec{L} - 1/2$. Note that a well-defined Hilbert space (i.e., a single-valued wave function) requires that the eigenvalues of $\hat{L}$ be integers; accordingly the eigenvalues of $\hat{I}$ are $\pm 1/2, \pm 3/2, \pm 5/2 \cdots$. Thus $\hat{I}$ plays the role of $I_3$ in the Skyrme model and the vector potential was included to ensure half-integer values for $I$.

Note that the model has a magnetic flux localized at the origin. Accordingly, the vector potential is undefined at the origin. Thus, to be sensible our model must exclude the particle from hitting the origin. This can be achieved by choosing a form for $V$ which is repulsive at the origin and sufficiently singular. Having a singular repulsive potential at the origin has another virtue: if there is attraction elsewhere in the potential, there will be a minimum of the potential. This model is technically complicated and the two-dimensional angular momentum; the Hamiltonian can be written as $\hat{H} = \hat{H}_r + (\vec{L} - 1/2)^2$, where $\hat{H}_r$ acts entirely on the radial degrees of freedom and the $\vec{L} - 1/2$ structure reflects the presence of the vector potential. It is convenient to introduce a new operator $\hat{L} \lambda = \vec{L} - 1/2$. Note that a well-defined Hilbert space (i.e., a single-valued wave function) requires that the eigenvalues of $\hat{L}$ be integers; accordingly the eigenvalues of $\hat{I}$ are $\pm 1/2, \pm 3/2, \pm 5/2 \cdots$. Thus $\hat{I}$ plays the role of $I_3$ in the Skyrme model and the vector potential was included to ensure half-integer values for $I$.

A classical solution localized (in $r$) at this minimum plays the role of the soliton; angular motion is then the collective motion.

A convenient choice for the functional form of the potential is $V(r^2) = \frac{1}{2m \lambda} + \frac{m \omega^2 r^2}{2\lambda}$, since this system is exactly solvable. Including our scaling factor $\lambda$ (playing the role of $N_c$) the effective potential for radial motion associated with a state of fixed $I$ becomes

$$V_{\text{eff}}^I(r^2) = \lambda \left( \frac{\lambda + \frac{\omega^2}{2m} r^2}{\frac{\lambda}{2m} r^2} \right).$$

(36)

and the radial Schrödinger equation becomes:

$$\left[ -\frac{1}{2m} \frac{1}{r} \partial_r r \partial_r + \lambda \left( \frac{\lambda + \frac{\omega^2}{2m} r^2}{\frac{\lambda}{2m} r^2} \right) \right] \psi_I(r) = E_I \psi(r)$$

(37)
This can be solved exactly. The energies of the lowest-lying state for any $I$ are given by

$$E_I = \omega \left( 1 + \sqrt{\lambda^2 + I^2} \right)$$

$$= \omega \left( \lambda + 1 + \frac{I^2}{2\lambda} + \mathcal{O}(\lambda^{-2}) \right).$$

(38)

Note that these energies at large $\lambda$ accord with our large $N_c$ expectations: the overall energy of the states go as $\lambda \sim N_c$ as do the mass of baryons while the energy splitting between states with different $I^2$ (of order 1) scales as $\lambda^{-1} \sim N_c^{-1}$ as does, for example, the N-Δ mass splitting.

Before putting a chemical potential on this system, it is useful to show that a semiclassical approach analogous to that used from Skyrmions reproduces the leading large $\lambda$ (large $N_c$ results) for both the overall energy and the splittings. As a first step one needs to find the analog of the soliton: a static solution of the classical equations of motion. This amounts to finding $r_0$, the value of $r$ which minimizes the potential. It is straightforward to find $r_0$ and the value of the potential at its minimum, $E_0$

$$r_0 = \sqrt{\frac{\lambda}{m\omega}},$$

$$E_0 = \lambda \omega$$

(39)

Note that $E_0$ is the analog of the “static soliton mass” $M_0$ appearing in e.g. Eq. (38); a direct comparison with Eq. (38) shows that it accurately predicts the leading contribution to the energy. There is collective angular motion around the minimum. The corresponding moment of inertia is $\Lambda_0 = \Lambda(r_0)$ with $\Lambda(r) = mr^2$. It is easy to see that the quantization of the collective motion—in analogy to the standard treatment of the Skyrme model—yields a collective motion contribution to the energy of

$$E_I^{\text{coll}} = \frac{I^2}{2m r_0^2} = \frac{\omega I^2}{2\lambda}$$

(40)

which directly reproduces the leading $I$-dependent contribution of Eq. (38). Thus the analog of the standard Skyrmion semiclassical approach correctly describes the dynamics of this toy model up to the accuracy for which it is supposed to work.

In doing this comparison to Eq. (38) the classical “soliton” correctly gives the order $\lambda \sim N_c$ part of the exact energy. However, there is also a term of order unity. In the Skyrme model terms of order $N_c^0$ correspond to the zero point fluctuations of the pion fields about the soliton background. In Skyrme models they are difficult to compute and generally neglected. By analogy, in this model the $\lambda^0$ piece of the energy should correspond to the zero point motion of the internal degree of freedom. This zero-point energy is given by $1/2$ the small amplitude vibration frequency about the minimum:

$$E_0^{\text{zero-point}} = \frac{1}{2} \sqrt{\frac{dV}{dr}} |_{r_0} = \frac{\omega}{m}.$$ 

(41)

It is worth noting that this exactly reproduces the term of order $\lambda^0 \sim N_c^0$ in Eq. (38).

Now consider what happens if we add a chemical potential for $I$ to the Hamiltonian: $\hat{H}' = \hat{H} - \mu_I \hat{I}$. The exact eigenstates of $\hat{H}'$ corresponding to the lightest state with fixed $I$ is given by $E_I'(\mu_I) = E_I - \mu_I I$. Since we are interested in the effect of the chemical potential, it is useful to focus on the difference of the energy from the ground state “baryon” at $\mu_I = 0$

$$\Delta E_I(\mu_I) \equiv E_I'(\mu_I) - E_{I/2} = \omega \left( \sqrt{\lambda^2 + I^2} - \sqrt{\lambda^2 + \frac{1}{4}} \right) - \mu_I I,$$

(42)

which is plotted in Fig. (2) for a number of different values of $I$ for the case $\lambda = 15$.

We can now proceed to treat the problem semiclassically using an algorithm analogous to the one in the previous section. If one implements this assuming $\mu_I \sim \mathcal{O}(1)$, one has a “soliton” whose static energy, in analogy to Eq. (19), is obtained by minimizing

$$V_{st}(r, \mu_I) = V_{eff}^{(0)}(r) - \frac{1}{2} \Lambda(r) \mu_I^2 = \lambda \left( \frac{\lambda}{2mr^2} + \frac{m\omega^2 r^2}{2\lambda} \right) - \frac{1}{2} m^2 \mu_I^2$$

(43)
FIG. 2: Energy shifts as defined in Eq. (42) (in units of $\omega$) versus the chemical potential $\mu_I$ for $\lambda = 15$. The solid lines represent exact solutions. At $\mu_I = 0$ the lowest line corresponds to $I = 1/2$, the line above it to $I = 3/2$ with $I$ increasing by one unit each line up. The dashed line corresponds to the semiclassical solution of Eq. (44). Figure (a) goes out to $\mu_I = .6$ and illustrates how the semiclassical result follows the minimum of the exact solution. Figure (b) shows the low $\mu_I$ region and illustrates the breakdown of the semiclassical treatment at very small $\mu_I$.

Minimizing this with respect to $r$ to obtain the analog of the soliton hedgehog mass $M_H(\mu_I)$ (see Eq. (29)) one gets

$$
\begin{align*}
&i_0^{(\mu_I)} = \frac{\lambda^{1/2}}{m^{1/2}(\omega^2 - \mu_I^2)^{1/4}} \\
&E_H(\mu_I) = V_{st}(i_0^{(\mu_I)}, \mu_I) = \lambda \sqrt{\omega^2 - \mu_I^2} \\
&\Delta E_H(\mu_I) = E_H(\mu_I) - E_H(0)
\end{align*}
$$

where $\Delta E_H(\mu_I)$ is the shift in the classical “hedgehog mass” due to the chemical potential. It is also plotted in Fig. (2) for the case $\lambda = 15$.

It is apparent from these plots that the classical treatment of the “soliton” works in the sense that it describes the shift in the minimum energy solution as a function of $\mu_I$. Moreover, it is equally clear how it works. Although each quantum level is linear in $\mu_I$, as $\mu_I$ increases the quantum levels cross and the value of $I$ of the lowest state increases. The classical calculation tracks this level crossing—in a continuous way. Thus it is able to accurately reflect the behavior of the minimum energy state when $\mu_I$ is large enough so that $I \gg 1$. Parametrically, this regime occurs for $\mu_I/\omega \gg 1/\lambda \sim 1/N_c$ where, therefore, rotational corrections can be neglected. Conversely at small $\mu_I$ one expects the semiclassical analysis to fail to reproduce the shifts—and, as it can be seen in Fig. 2b, this is indeed the case.

Let us now look at the analog of the semiclassical calculation of the proton’s energy. Following the same steps as in previous section we get for the present toy model

$$
E_{sc}^p = E_0^{(\mu_I)} - \frac{\mu_I}{2} + \frac{1}{8\Lambda(\mu_I)}
$$

FIG. 3: Energy shifts for the “proton” \((I = 1/2)\) state as a function of \(\mu_I\) for the case of \(\lambda = 15\). All energies are given in units of \(\omega\). The solid line is the exact result; the dashed line is the semiclassical result.

where

\[
E^{(\mu_I)}_0 = V_{cI}^{(\mu_I)(\mu_I)} = E_H^{(\mu_I)} + \frac{1}{2} \Lambda^{(\mu_I)} \mu_I^2
\]

\[
\Lambda^{(\mu_I)} = m \left[ r_0^{(\mu_I)} \right]^2
\]

(46)

The explicit form of “proton” energy resulting from this calculation is given by

\[
E_{\text{sc}}^{(\mu_I)}(\mu_I) = \omega \left[ \frac{1 - \mu_I^2}{2\omega^2} + \frac{1}{8\lambda} \sqrt{1 - \frac{\mu_I^2}{\omega^2}} \right] - \frac{\mu_I}{2\omega}
\]

(47)

For ease of comparison we introduce the shift in the semiclassically calculated energy from \(\mu_I = 0\) defined by \(\Delta_{\text{sc}}^{\text{p}} = E_{\text{sc}}^{\text{p}}(\mu_I) - E_{\text{sc}}^{\text{p}}(0)\). The corresponding exact result from Eq. (42) is \(\Delta_{\text{exact}}^{\text{p}} = -\mu_I/2\). Both the exact and semiclassical result are plotted in Fig. 3. It is apparent that the semiclassical procedure analogous to that in the preceding section does not accurately reproduce the exact result except at very small \(\mu_I\) where nonlinearities are negligible. The disease it suffers from is precisely the one identified for the analogous Skyrmion calculation. Namely, that in contrast to the exact result, the calculated energy is nonlinear in \(\mu_I\) with a nonzero curvature all the way down to \(\mu_I = 0\). It is also clear why it fails. The “soliton” used in the calculation was the minimum classical solution with fixed \(\mu_I\). Now it is clear from Fig. 2 that for \(\mu_I\) of order one \((\lambda^0)\) this classical configuration corresponds to quantum states with relatively high \(I\), i.e., states which parametrically have \(I \sim \lambda\). In situations where a classical solution breaks a symmetry, and the underlying quantum system is in the semiclassical limit, a classical configuration is associated with a band of states which are all qualitatively similar and all have energies which are near this minimum. Thus, when \(\mu_I\) is parametrically of order unity, the classical solution is valid only for describing states whose energy differs from the minimum by energies of order \(1/\lambda\). The “proton” state is not in this class.

To summarize the results of a semiclassical treatment of this toy model for \(\mu_I \neq 0\): in the large \(\lambda\) limit (analogous to the large \(N_c\) limit) and \(\mu_I\) of order unity, the analog of the classical soliton gives an accurate description of the lowest-lying “baryon” states in terms of free energy—states which have \(I \sim \lambda\)—but does not accurately describe the “nucleon” state.

V. DISCUSSION

The toy model of the previous section illustrates the resolution of the paradox of the existence of small nonlinearities in the nucleon’s energy as a function of \(\mu_I\) in Skyrme model calculations. The key point is that at large \(N_c\) and zero temperature with \(\mu_I \sim N_c^0\) but below the phase transition (i.e., \(|\mu_I| \leq m_\pi\)), semiclassical calculations are formally valid for the lowest-lying baryons states (which have \(I \sim N_c\)) but not for the nucleon.

This formal resolution of the paradox is useful for understanding what is happening from a mathematical perspective. However, important phenomenological problems remain. The crux of the issue concerns the fact that in nature \(N_c = 3\) and the parameters of the model are fit from the \(N_c = 3\) world. The high isospin states critical to the formal resolution of this problem do not exist as baryon resonances in the physical world. Thus, there appears to be no useful regime for
which the semiclassical method for describing how baryon free energies vary with $\mu_I$. The issue is further complicated by the fact that $m_\pi$, while formally of order $N_c^0$, is, in fact, very small due to approximate chiral symmetry. Indeed, $m_\pi$ is smaller than $M_\Delta - M_N$ which is formally of order $1/N_c$. Thus if one were to increase $\mu_I$ starting from zero, even before the first level crossing occurs in the baryons (the $J_3 = 3/2$ state of the $\Delta$ crossing the proton), a phase transition occurs and the semiclassical methods discussed here need to be modified. It is not immediately obvious how to do this.

Apart from phenomenology, an interesting formal question remains: If the semiclassical method fails, how does one accurately describe the nucleon (and other low isospin states, such as the $\Delta$) for $\mu_I \neq 0$ in the large $N_c$ limit of the Skyrme model? In the regime $|\mu_I| < m_\pi$, where the paradox discussed here is manifest, this is easily accomplished. One first does semiclassical quantization at $\mu_I = 0$ and identifies the quantum states. Subsequently one imposes the chemical potential at the quantum level on these states. Such a prescription is guaranteed to give the correct linear behavior. However, the problem gets more interesting when $|\mu_I| > m_\pi$. In this regime pions condense. It is clear that new methods need to be developed to describe baryons in this regime.

One of the authors (TDC) gratefully acknowledges the support of the U. S. Department of Energy under grant no. DEFG02-93ER-40762. NNS acknowledges the support of CONICET (Argentina) grant PIP 6084, and ANPCyT (Argentina) grant PICT04 03-25374 (NNS)

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