Solutions from boundary condition changing operators in open superstring field theory

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Abstract

We construct analytic solutions of open superstring field theory in the Berkovits formulation using boundary condition changing operators under some regularity conditions, extending the previous construction in the bosonic string. We also consider the gauge-invariant observables corresponding to closed string one-point functions on the disk. We analytically calculate the gauge-invariant observables for the solutions both in the bosonic string and in the superstring and find the expected change of boundary conditions of the disk.
1 Introduction

String field theory can be thought of as a universal, effective theory when elementary excitations are string-like. To eliminate ghosts from such string-like excitations in unphysical directions, any covariant description in terms of a spacetime field theory would require gauge invariance. Then the gauge invariance seems to determine the interacting theory uniquely. This has been a guiding principle in constructing covariant string field theory [1,2].

On the other hand, we know from the perturbative world-sheet formulation of string theory that consistent backgrounds are described by conformal field theories in two dimensions. The classical equation of motion of string field theory determined by the spacetime gauge invariance should therefore reproduce this requirement of conformal invariance in the world-sheet perspective. We hope that deeper understanding of the relation between the world-sheet conformal invariance and the spacetime gauge invariance would reveal aspects of the non-perturbative theory behind the perturbative string theory.

In the case of the open string, a consistent background is given by a choice of boundary conformal field theory (BCFT), and different open string backgrounds correspond to different conformal boundary conditions. We would therefore like to have a systematic construction of solutions in open string field theory for a given BCFT. Since the change of boundary conditions can be described by insertions of boundary condition changing (bcc) operators in the original BCFT, one possible approach to such systematic constructions would be to use bcc operators.

Since the construction of an analytic solution for tachyon condensation by Schnabl [3], an impressive amount of analytic results for open string field theory have been obtained [4–67]. In [61], analytic solutions of open bosonic string field theory were constructed using bcc operators when they satisfy some regularity conditions. The starting point of the construction was the solutions in Schnabl gauge for marginal deformations when operator products of the marginal operators are regular [14,15]. The solutions take the form of a superposition of wedge-based states. When we write the wedge state $W_\alpha$ as $W_\alpha = e^{\alpha K}$ the solutions can be written in terms of the states $K$, $B$, $c$, and $V$, where these states are based on the wedge state $W_0$ of zero width with a line integral of the energy-momentum tensor and the $b$ ghost for $K$ and $B$, respectively, and with a local insertion of the $c$ ghost and the marginal operator for $c$ and $V$, respectively. The solutions in [14,15] were later generalized by Erler [16]. Just as the tachyon vacuum solution by Schnabl [3] written in terms of $K$, $B$, and $c$ was generalized by replacing $e^{K/2}$ with an arbitrary function $f(K)$ of $K$ in [4], the solutions in [14,15] were generalized by

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1 We denote wedge states [68,69] with operator insertions by wedge-based states.
2 Products of string fields in this paper are defined using Witten’s star product [1].
3 We follow the conventions of [4], but the states are rescaled as $K_{\text{here}} = (\pi/2) K_{\text{there}}$, $B_{\text{here}} = (\pi/2) B_{\text{there}}$, and $c_{\text{here}} = (2/\pi) c_{\text{there}}$. 

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replacing $e^{K/2}$ with an arbitrary function $f(K)$ of $K$. The resulting solutions depend explicitly on $V$ and thus cannot in general be written in terms of bcc operators. In [61] it was found that the solutions can be written in terms of bcc operators for a special choice of the function $f(K)$ given by $f(K) = 1/\sqrt{1-K}$. In this case the solutions depend on $V$ only through bcc operators, and one can show that the solutions satisfy the equation of motion when bcc operators obey some regularity conditions. Another important feature of the solutions using bcc operators is that one can construct the solutions only from the information on three-point functions on a disk of a pair of bcc operators and an arbitrary operator in the original BCFT, while the solutions in Schnabl gauge [14][15] depend in a complicated way on multi-point functions of the marginal operator.

The purpose of this paper is to extend the construction in [61] to open superstring field theory in the Berkovits formulation [70]. Namely, we would like to construct solutions of open superstring field theory written in terms of bcc operators. Actually, this can be immediately achieved by combining the following two observations. The first one is that solutions in [61] can be written only in terms of bcc operators, the energy-momentum tensor, the $b$ ghost, and the BRST operator $Q$ without explicitly using the $c$ ghost. We can prove that the solutions satisfy the equation of motion

$$Q\Psi + \Psi^2 = 0$$

(1.1)

of open bosonic string field theory, where $\Psi$ is the open string field of ghost number 1, only using the relations of these operators which we call $KB\sigma$ algebra. Since this $KB\sigma$ algebra also holds in open superstring field theory, we can construct a state $\Psi$ in open superstring field theory satisfying the bosonic equation of motion (1.1). It is important that the $c$ ghost does not appear explicitly in this form of the solutions because the BRST transformation of the $c$ ghost takes a different form in the superstring and requires an extension of the algebra to the superconformal ghost sector.

The second observation is that one can construct solutions of open superstring field theory from a state $\Psi$ in the small Hilbert space satisfying the bosonic string equation of motion (1.1) by inserting an operator $R(t)$ satisfying $Q \cdot R(t) = 1$ appropriately [38,50]. We can therefore construct solutions in open superstring field theory written in terms of bcc operators simply by inserting $R(t)$ appropriately to the state $\Psi$ in the $KB\sigma$ algebra. This is the first main result of this paper.

Since the solutions are written in terms of bcc operators, we expect that they describe the BCFT associated with the bcc operators we chose. As we mentioned earlier, the solutions in [61]

\begin{itemize}
\item Incidentally, this is the same choice of $f(K)$ in the phantomless solution for tachyon condensation [50] by Erler and Schnabl.
\end{itemize}
were originally constructed by rewriting solutions for marginal deformations with regular operator products, and thus these solutions in the bosonic string do describe the BCFT deformed by regular marginal deformations. We can also relate our solutions in the superstring to known solutions for regular marginal deformations. In this paper, we provide more direct evidence of the correspondence between the solutions using bcc operators and the associated BCFT. It was conjectured by Ellwood [32] that the gauge-invariant observables [71, 72] evaluated for a solution correspond to closed string one-point functions on the disk with the modified boundary conditions associated with the solution. We analytically calculate the gauge-invariant observables for the solutions using bcc operators both in the bosonic string and in the superstring and find the conjectured change of boundary conditions of the disk. This is the second main result of this paper.

The organization of this paper is as follows. In section 2 we review the solutions [61] in the bosonic string. We introduce the $KB\sigma$ algebra and demonstrate that the equation of motion is satisfied only using the $KB\sigma$ algebra. In section 3 we construct solutions in the superstring. After discussing the general prescription for the construction of solutions in the Berkovits formulation of open superstring field theory, we apply it to the solutions using bcc operators. In section 4 we analytically calculate the gauge-invariant observables for the solutions using bcc operators both in the bosonic string and in the superstring and find the conjectured change of boundary conditions of the disk. Section 5 is devoted to discussion.

2 Solutions to the bosonic string equation of motion

2.1 $KB\sigma$ algebra

The wedge state $W_\alpha$ with $\alpha \geq 0$ is defined by its BPZ inner product $\langle \varphi, W_\alpha \rangle$ as follows:

$$\langle \varphi, W_\alpha \rangle = \langle f \circ \varphi(0) \rangle_{C_{\alpha+1}}.$$ (2.1)

Here and in what follows we denote a generic state in the Fock space by $\varphi$ and its corresponding operator in the state-operator mapping by $\varphi(\xi)$. We denote the conformal transformation of $\varphi(\xi)$ under the map $f(\xi)$ by $f \circ \varphi(\xi)$, where

$$f(\xi) = \frac{2}{\pi} \arctan \xi.$$ (2.2)

The coordinate $z$ related through $z = f(\xi)$ to the coordinate $\xi$ on the upper half-plane used in the standard state-operator mapping is called the sliver frame. The correlation function is evaluated on the surface $C_{\alpha+1}$, which is the semi-infinite strip obtained from the upper half-plane of $z$ by the identification $z \sim z + \alpha + 1$. We usually use the region $-1/2 \leq \Re z \leq 1/2 + \alpha$ for $C_{\alpha+1}$. 

3
Just as the line integral $L_0$ of the energy-momentum tensor generates a surface $e^{-t L_0}$ in the standard open string strip coordinate, the wedge state $W_\alpha$ can be thought of as being generated by a line integral of the energy-momentum tensor in the sliver frame. We denote the wedge state $W_0$ of zero width with an insertion of the line integral by $K$ and write the wedge state $W_\alpha$ as

$$W_\alpha = e^{\alpha K}. \tag{2.3}$$

An explicit definition of the state $K$ is given by

$$\langle \varphi, K \rangle = \left\langle f \circ \varphi(0) \int_{\frac{i}{2} + i \infty}^{\frac{1}{2} - i \infty} \frac{dz}{2\pi i} T(z) \right\rangle_{C_1}, \tag{2.4}$$

where $T(z)$ is the energy-momentum tensor and we use the doubling trick. Note that the line integral is from a boundary to the open string mid-point before using the doubling trick, while the line integral $L_0$ is from a boundary to the other boundary.

Just as the line integral $L_0$ is the BRST transformation of the line integral $b_0$ of the $b$ ghost, the line integral that generates the wedge state is the BRST transformation of the same line integral with the energy-momentum tensor replaced by the $b$ ghost. Correspondingly, we define the state $B$ by

$$\langle \varphi, B \rangle = \left\langle f \circ \varphi(0) \int_{\frac{i}{2} + i \infty}^{\frac{1}{2} - i \infty} \frac{dz}{2\pi i} b(z) \right\rangle_{C_1}. \tag{2.5}$$

By construction, the state $K$ is the BRST transformation of $B$. Another important property of the state $B$ is that $B^2 = 0$.

Let us next consider wedge states with modified boundary conditions. The change of boundary conditions on a segment of the world-sheet boundary from a point $a$ to a point $b$ can be described by inserting a pair of bcc operators $\sigma_L(a)$ and $\sigma_R(b)$. We denote the wedge state $W_\alpha = e^{\alpha K}$ with modified boundary conditions by $\sigma_L e^{\alpha K} \sigma_R$:

$$\langle \varphi, \sigma_L e^{\alpha K} \sigma_R \rangle = \langle f \circ \varphi(0) \sigma_L \left( \frac{1}{2} \right) \sigma_R \left( \alpha + \frac{1}{2} \right) \rangle_{C_{\alpha+1}}. \tag{2.6}$$

In other words, we denote the state based on the wedge state $W_0$ of zero width with an insertion of $\sigma_L(t)$ by $\sigma_L$ and $W_0$ with an insertion of $\sigma_R(t)$ by $\sigma_R$. Since bcc operators are in the matter sector, they commute with any line integral of the $b$ ghost. We therefore have $[B, \sigma_L] = 0$ and $[B, \sigma_R] = 0$.

In [61] solutions written in terms of $K$, $B$, $\sigma_L$, and $\sigma_R$ were constructed when operator products of bcc operators are regular, while in general they have singular operator products. The regularity conditions on bcc operators can be stated as

$$\lim_{\epsilon \to +0} \sigma_L(t) \sigma_R(t + \epsilon) = 1, \quad \lim_{\epsilon \to +0} \sigma_L(a) \sigma_R(b) \sigma_L(b + \epsilon) \sigma_R(c) = \sigma_L(a) \sigma_R(c), \tag{2.7}$$
and in the language of the states $\sigma_L$ and $\sigma_R$ the conditions are translated into the relations
\[ \sigma_L \sigma_R = \sigma_R \sigma_L = 1. \quad (2.8) \]

When we prove that the equation of motion is satisfied, we use states such as $\sigma_L K \sigma_R$, and it was implicitly assumed in [61] that these states are well defined. However, if the operator product of $\sigma_L(t)$ and $\sigma_R(t)$, for example, takes the form $\sigma_L(0) \sigma_R(t) = 1 + t^\alpha A(0) + \ldots$ with $0 < \alpha < 1$, the state $\sigma_L K \sigma_R$ is singular. It is also possible that operator products of more bcc operators develop similar singularities. In this paper we assume that there are no such singularities in the matter sector.

To summarize, we have defined the states $K$, $B$, $\sigma_L$, and $\sigma_R$ satisfying the relations
\[ B^2 = 0, \quad [B, \sigma_L] = 0, \quad [B, \sigma_R] = 0, \quad \sigma_L \sigma_R = 1, \quad \sigma_R \sigma_L = 1, \quad (2.9) \]
and $K = QB$. One can show that the solutions in [61] satisfy the equation of motion (1.1) only from these relations. While we have given an explicit realization of these states, one can construct a solution from any Grassmann-odd state $B$ and any Grassmann-even states $\sigma_L$ and $\sigma_R$ satisfying the relations (2.9), and $K = QB$ can be regarded as a definition of $K$.

By considering the BRST transformation of each of the relations in (2.9), we find
\[ [K, B] = 0, \quad \{B, Q \sigma_L\} = [K, \sigma_L] = -[1 - K, \sigma_L], \quad \{B, Q \sigma_R\} = [K, \sigma_R] = -[1 - K, \sigma_R], \quad (Q \sigma_L) \sigma_R + \sigma_L (Q \sigma_R) = 0, \quad (Q \sigma_R) \sigma_L + \sigma_R (Q \sigma_L) = 0. \quad (2.10) \]

We use these relations as well as (2.9).

### 2.2 Solutions using bcc operators in the bosonic string

The solution $\Psi$ written in terms of $K$, $B$, $\sigma_L$, and $\sigma_R$ is given by [61]
\[ \Psi = -\frac{1}{\sqrt{1 - K}} (Q \sigma_L) \sigma_R - \frac{1}{\sqrt{1 - K}} (Q \sigma_R) B \frac{1}{1 - K} (Q \sigma_L) \frac{1}{\sqrt{1 - K}}, \quad (2.11) \]
where
\[ \frac{1}{1 - K} = \int_0^\infty dt e^{-t} e^{tK}, \quad \frac{1}{\sqrt{1 - K}} = \int_0^\infty dt \sqrt{\pi t} e^{tK}. \quad (2.12) \]

It is convenient to introduce a non-real solution $\widetilde{\Psi}$ related to $\Psi$ as
\[ \widetilde{\Psi} = \frac{1}{\sqrt{1 - K}} \Psi \sqrt{1 - K} \]
\[ = -\frac{1}{1 - K} (Q \sigma_L) \sigma_R - \frac{1}{1 - K} (Q \sigma_R) B \frac{1}{1 - K} (Q \sigma_L), \quad (2.13) \]

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5 We would like to thank the referee of this paper for pointing this out.
where \( \sqrt{1-K} \) should be understood as a superposition of wedge-based states:

\[
\sqrt{1-K} = \frac{1-K}{\sqrt{1-K}} = \int_0^\infty dt \frac{e^{-t}}{\sqrt{\pi t}} e^{tK} (1-K).
\] (2.14)

Since any function of \( K \) is annihilated by the BRST operator, \( \Psi \) satisfies \( Q\bar{\Psi} + \bar{\Psi}^2 = 0 \) when \( \bar{\Psi} \) satisfies \( Q\bar{\Psi} + \bar{\Psi}^2 = 0 \). Although \( \bar{\Psi} \) is simpler than \( \Psi \), the solution \( \bar{\Psi} \) does not satisfy the reality condition on the string field \[73\]. Let us demonstrate that \( Q\bar{\Psi} + \bar{\Psi}^2 = 0 \) follows only from \( (2.9) \) and \( (2.10) \).

In calculating \( Q\bar{\Psi} \), we write \( \bar{\Psi} \) as follows:

\[
\bar{\Psi} = -\frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} \sigma_R - Q \left[ \frac{1}{1-K} (Q\sigma_L) \frac{B}{1-K} \sigma_R \right],
\] (2.15)

where we used

\[
Q \left[ \frac{B}{1-K} \right] = \frac{K}{1-K} = \frac{1}{1-K} - 1.
\] (2.16)

The second term in \( (2.15) \) is BRST-exact and does not contribute to \( Q\bar{\Psi} \). The BRST transformation of \( \bar{\Psi} \) is thus given by

\[
Q\bar{\Psi} = \frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} (Q\sigma_R).
\] (2.17)

Let us next calculate \( \bar{\Psi}^2 \). In this case we write \( \bar{\Psi} \) as

\[
\bar{\Psi} = \frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} (Q\sigma_R) B - \frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} \sigma_R (1-K)
\] (2.18)

or as

\[
\bar{\Psi} = \frac{B}{1-K} (Q\sigma_L) \frac{1}{1-K} (Q\sigma_R) + \sigma_L \frac{1}{1-K} (Q\sigma_R).
\] (2.19)

Multiplying \( \Psi \) in \( (2.18) \) and \( \Psi \) in \( (2.19) \), we find

\[
\bar{\Psi}^2 = \frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} (Q\sigma_R) B\sigma_L \frac{1}{1-K} (Q\sigma_R)
\]

\[
- \frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} \sigma_R B (Q\sigma_L) \frac{1}{1-K} (Q\sigma_R)
\]

\[
- \frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} \sigma_R (1-K) \sigma_L \frac{1}{1-K} (Q\sigma_R)
\]

\[
= -\frac{1}{1-K} (Q\sigma_L) \frac{1}{1-K} (Q\sigma_R),
\] (2.20)

where we used

\[
(Q\sigma_R)B\sigma_L - \sigma_R B(Q\sigma_L) = \{ Q\sigma_R, B \} \sigma_L - B[(Q\sigma_R)\sigma_L + \sigma_R (Q\sigma_L)] = -[1-K, \sigma_R] \sigma_L.
\] (2.21)
It follows from (2.17) and (2.20) that \( Q\tilde{\Psi} + \tilde{\Psi}^2 = 0 \) and therefore \( \Psi \) in (2.11) satisfies the equation of motion (1.1).

The solutions were originally constructed by rewriting solutions for marginal deformations with regular operator products \[61\]. While we do not necessarily assume that the modified boundary conditions are marginally connected to the original ones, the regularity conditions imply the existence of a marginal operator given by \( V(t) = \sigma_L \partial_t \sigma_R(t) \). We therefore need to relax the regularity conditions for the construction of more general solutions.

### 3 Solutions in open superstring field theory

In this section we construct solutions in open superstring field theory from bcc operators. We first discuss the general prescription for the construction of solutions in open superstring field theory. We then present solutions using bcc operators and discuss their regularity. In the last subsection, we consider marginal deformations with regular operator products.

#### 3.1 Open superstring field theory

The equation of motion for open superstring field theory in the NS sector \[70\] is

\[
\eta_0 \left( e^{-\Phi} Q e^\Phi \right) = 0 , \tag{3.1}
\]

where \( \Phi \) is the open superstring field. It is Grassmann-even and has ghost number 0 and picture number 0. The superconformal ghost sector is described by \( \eta, \xi \), and \( \phi \) \[74, 75\], and \( \eta_0 \) is the zero mode of \( \eta \). The BRST operator for the superstring is given by

\[
Q = \int \left[ \frac{dz}{2\pi i} j_B(z) - \frac{d\bar{z}}{2\pi i} \bar{j}_B(\bar{z}) \right], \tag{3.2}
\]

with

\[
j_B = c T_B^m + c T_B^\xi + c T_B^\phi + \eta e^\phi T_F^m + bc \partial c - b\eta \partial \eta e^{2\phi} , \quad T_B^\xi = - \eta \partial \xi , \quad T_B^\phi = - \frac{1}{2} \partial \phi \partial \phi - \partial^2 \phi , \tag{3.3}
\]

where \( T_B^m \) and \( T_F^m \) are the holomorphic components of the energy-momentum tensor and the supercurrent, respectively, in the matter sector and \( \bar{j}_B \) is the antiholomorphic counterpart of \( j_B \). Conformal normal ordering is implicit throughout this paper. The superconformal ghost sector can also be described by the free bosons \( \chi \) and \( \phi \) via the bosonization \[74,75\]:

\[
\xi(z) \cong e^\chi(z) , \quad \eta(z) \cong e^{-\chi(z)} . \tag{3.4}
\]

\[6\] We would like to thank the referee of this paper for explaining this to us.
The operator product expansions are given by
\[ \chi(z)\chi(0) \sim \ln z, \quad \phi(z)\phi(0) \sim -\ln z. \] (3.5)

We will use this description later. The operator \( \eta_0 \) is a derivation with respect to the star product. It is nilpotent and anticommutes with the BRST operator \( Q \):
\[ Q^2 = 0, \quad \eta_0^2 = 0, \quad \{Q, \eta_0\} = 0. \] (3.6)

The gauge transformation of the string field is given by
\[ e^\Phi \mapsto e^{\Phi'} = \Omega e^\Phi \Lambda \quad \text{with} \quad Q \Omega = 0, \quad \eta_0 \Lambda = 0. \] (3.7)

### 3.2 General prescription

For any solution \( \Phi \) in open superstring field theory, we can define a string field \( \Psi \) of ghost number 1 and picture number 0 by
\[ \Psi = e^{-\Phi} Q e^\Phi, \] (3.8)
which satisfies the following equations:
\[ Q\Psi + \Psi^2 = 0, \quad \eta_0 \Psi = 0. \] (3.9)

Based on this fact, Erler divided the construction of a solution in open superstring field theory into the following two steps [16]: the first step is to construct a string field \( \Psi \) satisfying (3.9), and the second step is to solve the equation (3.8) for \( \Phi \). Note that different solutions for a given \( \Psi \) are all gauge-equivalent. For two solutions \( \Phi_1 \) and \( \Phi_2 \) we can show that \( \Omega \) given by

\[ e^{\Phi_1} = \Omega e^{\Phi_2} \] (3.10)

is annihilated by \( Q \):
\[ Q \Omega = Q(e^{\Phi_1} e^{-\Phi_2}) = (e^{\Phi_1} \Psi) e^{-\Phi_2} + e^{\Phi_1} (-\Psi e^{-\Phi_2}) = 0, \] (3.11)

where we used \( Q e^{\Phi_1} = e^{\Phi_1} \Psi \) following from \( e^{-\Phi_1} Q e^{\Phi_1} = \Psi \) and \( Q e^{-\Phi_2} = -\Psi e^{-\Phi_2} \) following from \( e^{-\Phi_2} Q e^{\Phi_2} = -Q(e^{-\Phi_2}) e^{\Phi_2} = \Psi. \)

The first equation in (3.9) takes the same form as the equation of motion in the modified cubic open superstring field theory [76][77].
3.2.1 The free theory

Let us consider the second step of the strategy by Erler for the free theory. Namely, for a string field $\Psi$ satisfying

$$Q\Psi = 0, \quad \eta_0 \Psi = 0,$$

we would like to solve

$$\Psi = Q \Phi,$$

for $\Phi$. This can be solved when $\Psi$ is in the Fock space because the cohomology of $Q$ is trivial in the large Hilbert space. This can be seen by the existence of an operator $R(t)$ satisfying

$$Q \cdot R(t) = 1,$$

and we choose

$$R(t) = -c\xi \partial \xi e^{-2\phi}(t)$$

in this paper. Consider the operator corresponding to the BRST-closed state $\Psi$ in the state-operator mapping and denote it by $\Psi(\xi)$. To solve (3.13), let us insert $R(t)$ to the left of $\Psi(0)$:

$$R(-\epsilon) \Psi(0)$$

with $0 < \epsilon < 1$. The BRST transformation of this pair of operators is $\Psi(0)$ because $Q \cdot R(-\epsilon) = 1$ and $Q \cdot \Psi(0) = 0$. Therefore, the state $\hat{\Psi}_L$ corresponding to $R(-\epsilon) \Psi(0)$ satisfies

$$Q \hat{\Psi}_L = \Psi.$$

We thus have a solution $\Phi = \hat{\Psi}_L$ to (3.13). Similarly, the BRST transformation of

$$\Psi(0) R(\epsilon)$$

with $0 < \epsilon < 1$ is $-\Psi(0)$, and the state $\hat{\Psi}_R$ corresponding to $\Psi(0) R(\epsilon)$ satisfies

$$Q \hat{\Psi}_R = -\Psi.$$

We have another solution $\Phi = -\hat{\Psi}_R$ to (3.13).

The solutions $\hat{\Psi}_L$ and $-\hat{\Psi}_R$ depend on the parameter $\epsilon$, but they are all gauge-equivalent. For example, $\hat{\Psi}_L$ with different values of $\epsilon$ are gauge-equivalent because

$$R(-\epsilon_1) \Psi(0) - R(-\epsilon_2) \Psi(0) = Q \cdot [R(-\epsilon_2) R(-\epsilon_1) \Psi(0)].$$

We can similarly show the equivalence of $-\hat{\Psi}_R$ with different values of $\epsilon$ and the equivalence between $\hat{\Psi}_L$ and $-\hat{\Psi}_R$. When the limit $\epsilon \to 0$ of $\hat{\Psi}_L$ or $-\hat{\Psi}_R$ is finite, the resulting state is a solution in the Fock space, while $\hat{\Psi}_L$ and $-\hat{\Psi}_R$ with $\epsilon > 0$ are not.
3.2.2 The interacting theory

- Non-real solutions  This construction can be generalized to the interacting theory and to the state Ψ outside the Fock space. Suppose that the state Ψ satisfying (3.9) is made of wedge-based states and we can insert \( R(t) \) to the left of all the other operator insertions. The resulting state \( \hat{Ψ}_L \) satisfies

\[
Q \hat{Ψ}_L = (1 + \hat{Ψ}_L) \Psi
\]

(3.21)

because when the BRST operator acts on \( R(t) \), we have \( \Psi \), and when the BRST operator acts on the other operator insertions, we have \( -Q \Psi = \Psi^2 \) with the insertion of \( R(t) \), which is \( \hat{Ψ}_L \Psi \).

Similarly, suppose we can insert \( R(t) \) to the right of all the other operator insertions. The resulting state \( \hat{Ψ}_R \) satisfies

\[
Q \hat{Ψ}_R = -\Psi (1 + \hat{Ψ}_R).
\]

(3.22)

From these equations we observe that

\[
(1 + \hat{Ψ}_L)^{-1} Q (1 + \hat{Ψ}_L) = (1 + \hat{Ψ}_L)^{-1} (1 + \hat{Ψ}_L) \Psi = \Psi,
\]

(3.23)

\[
(1 + \hat{Ψ}_R) Q (1 + \hat{Ψ}_R)^{-1} = -[Q(1 + \hat{Ψ}_R)] (1 + \hat{Ψ}_R)^{-1} = \Psi (1 + \hat{Ψ}_R) (1 + \hat{Ψ}_R)^{-1} = \Psi,
\]

and we can construct two solutions \( Φ_L \) and \( Φ_R \) to (3.8):

\[
e^{Φ_L} = 1 + \hat{Ψ}_L, \quad e^{−Φ_R} = 1 + \hat{Ψ}_R,
\]

(3.24)

or

\[
Φ_L = \ln (1 + \hat{Ψ}_L), \quad Φ_R = −\ln (1 + \hat{Ψ}_R),
\]

(3.25)

where \( \ln (1 + X) \) for a string field \( X \) is defined by

\[
\ln (1 + X) ≡ \sum_{n=1}^{∞} \frac{(-1)^{n-1}}{n} X^n.
\]

(3.26)

The solutions \( Φ_L \) and \( Φ_R \) depend on the insertion point of \( R(t) \) in \( \hat{Ψ}_L \) and \( \hat{Ψ}_R \), and we can also use \( R(\bar{z}) \) and \( \tilde{R}(\bar{z}) \) in the bulk instead of \( R(t) \) on the boundary. However, all these solutions are gauge-equivalent because by construction they give the same \( Ψ \).

- Real solutions  These solutions do not generally satisfy the reality condition [73]. For the open superstring field, the condition is stated as \( Φ^\dagger = −Φ \), where \( X^\dagger \) is the conjugate of \( X \) defined by the combination of the Hermitian conjugation (hc) and the inverse BPZ conjugation (bpz⁻¹):

\[
X^\dagger ≡ \text{bpz}^{-1} \circ \text{hc} (X).
\]

(3.27)
The conjugation satisfies
\begin{align}
(QX)\dagger &= -(-1)^{|X|} QX\dagger, \tag{3.28} \\
(XY)\dagger &= Y\dagger X\dagger, \tag{3.29}
\end{align}
where $|X|$ denotes the Grassmann property of the string field $X$: it is 0 mod 2 for a Grassmann-even state and 1 mod 2 for a Grassmann-odd state. See [20] for a detailed discussion on the reality condition relevant to our case, in particular, about the insertion of $R(t)$.

When $\Psi$ satisfies the condition $\Psi\dagger = \Psi$, which is the reality condition for the string field in bosonic string field theory, we can construct $\hat{\Psi}_L$ and $\hat{\Psi}_R$ such that they are conjugate to each other: $\hat{\Psi}_L\dagger = \hat{\Psi}_R$. This implies that
\begin{equation}
(e^{\Phi_L})\dagger = e^{-\Phi_R} \quad \text{or} \quad \Phi_L\dagger = -\Phi_R. \tag{3.30}
\end{equation}
Since $\Phi_L$ and $\Phi_R$ are solutions to (3.8) for the same $\Psi$, they are gauge-equivalent:
\begin{equation}
e^{\Phi_R} = \Omega e^{\Phi_L} \quad \text{with} \quad Q\Omega = 0. \tag{3.31}
\end{equation}
We also observe that the gauge transformation parameter $\Omega$ is real:
\begin{equation}
\Omega\dagger = (e^{\Phi_R} e^{-\Phi_L})\dagger = e^{\Phi_R} e^{-\Phi_L} = \Omega. \tag{3.32}
\end{equation}
Using these properties, we can construct a solution $\Phi_{\text{real}}$ satisfying the reality condition from $\Phi_L$ and $\Phi_R$ by the following gauge transformations [16]:
\begin{equation}
\e^{\Phi_{\text{real}}} = \Omega^{1/2} e^{\Phi_L} = \Omega^{-1/2} e^{\Phi_R}. \tag{3.33}
\end{equation}
Since $\Phi_{\text{real}}$ is gauge-equivalent to the solutions $\Phi_L$ and $\Phi_R$, it satisfies the equation of motion, and its reality follows from (3.30) and (3.32):
\begin{equation}
(e^{\Phi_{\text{real}}}\dagger) e^{\Phi_{\text{real}}} = (\Omega^{1/2} e^{\Phi_L})\dagger \Omega^{1/2} e^{\Phi_L} = e^{\Phi_R} e^{\Phi_L} = 1. \tag{3.34}
\end{equation}

The precise definition of $\Phi_{\text{real}}$ in terms of $\hat{\Psi}_L$ and $\hat{\Psi}_R$ is
\begin{equation}
e^{\Phi_{\text{real}}} = \Omega^{1/2} e^{\Phi_L} = \frac{1}{\sqrt{e^{\Phi_L} e^{-\Phi_R}}} e^{\Phi_L} = \frac{1}{\sqrt{(1 + \hat{\Psi}_L)(1 + \hat{\Psi}_R)}} (1 + \hat{\Psi}_L), \tag{3.35}
\end{equation}
where
\begin{equation}
\frac{1}{\sqrt{1 + X}} \equiv e^{-\frac{1}{2} \ln(1 + X)} \tag{3.36}
\end{equation}
with $\ln(1 + X)$ defined in [3.20]. More explicitly, we have
\begin{equation}
\frac{1}{\sqrt{(1 + \hat{\Psi}_L)(1 + \hat{\Psi}_R)}} = \sum_{n=0}^{\infty} \frac{\Gamma(1/2)}{\Gamma(n + 1) \Gamma(1/2 - n)} (\hat{\Psi}_L + \hat{\Psi}_R + \hat{\Psi}_L \hat{\Psi}_R)^n \tag{3.37}
\end{equation}
We can verify that $Q (\hat{\Psi}_L + \hat{\Psi}_R + \hat{\Psi}_L \hat{\Psi}_R) = 0$ and $(\hat{\Psi}_L + \hat{\Psi}_R + \hat{\Psi}_L \hat{\Psi}_R) = \hat{\Psi}_R \hat{\Psi}_L \hat{\Psi}_R$, consistent with the general discussion. The real solution $\Phi_{\text{real}}$ is expanded as follows:

$$
\Phi_{\text{real}} = \frac{1}{2} (\hat{\Psi}_L - \hat{\Psi}_R) - \frac{1}{4} (\hat{\Psi}_L^2 - \hat{\Psi}_R^2) + \frac{1}{6} \left( \hat{\Psi}_L^3 - \hat{\Psi}_R^3 \right) - \frac{1}{24} \left( \hat{\Psi}_L \hat{\Psi}_R \hat{\Psi}_L - \hat{\Psi}_L \hat{\Psi}_R \hat{\Psi}_L \right) + \ldots.
$$

(3.38)

We can confirm that $\Phi_{\text{real}}$ satisfies the reality condition $\Phi_{\text{real}}^\dagger = -\Phi_{\text{real}}$ for the first few terms in the expansion.

Each of the solutions $\Phi_L$, $\Phi_R$, and $\Phi_{\text{real}}$ is defined by a formal infinite sum, and its convergence will depend on details of the solutions. While this issue of convergence is specific to our particular constructions, existence of $\Phi$ for a given $\Psi$ in general is an interesting question. As we noted earlier, $\Psi$ satisfying (3.9) is a solution to the modified cubic string field theory so that this question is related to existence of a solution in the Berkovits theory for a given solution in the modified cubic theory. See [59] for recent discussion.

Finally, let us consider the action for solutions constructed using this general prescription. The action of open superstring field theory in the Berkovits formulation is given by [70]

$$
S = -\frac{1}{2g^2} \int_0^1 dt \left[ \partial_t \text{Tr} (A_{\eta_0} A_Q) + \text{Tr} (A_t \{ A_Q, A_{\eta_0} \}) \right]
$$

with $A_{\eta_0} = e^{-\Phi(t)} \eta_0 e^{\Phi(t)}$, $A_Q = e^{-\Phi(t)} Q e^{\Phi(t)}$, $A_t = e^{-\Phi(t)} \partial_t e^{\Phi(t)}$, $\Phi(0) = 0$, $\Phi(1) = \Phi$, (3.39)

where $g$ is the open string coupling constant and $\text{Tr}(AB) = \langle A, B \rangle$. The value of the action is gauge invariant and the expression for the non-real solution $\Phi_L$ in terms of $\Psi$ and $\hat{\Psi}_L$ is

$$
S = \frac{1}{g^2} \sum_{n=3}^{\infty} \sum_{m=1}^{n-2} (-1)^{n-1} \frac{m}{n(n-1)} \text{Tr} \left( \hat{\Psi}_L^m (\eta_0 \hat{\Psi}_L) \hat{\Psi}_L^{n-m-2} \Psi \right).
$$

(3.40)

The derivation is given in appendix A.

### 3.3 Construction of solutions from bcc operators

Our goal is to construct solutions to the equation of motion (3.1) written in terms of bcc operators. We can construct such solutions if $\Psi$ in (3.8) is written in terms of bcc operators and if we can insert $R(t)$ appropriately to obtain $\hat{\Psi}_L$ and $\hat{\Psi}_R$ satisfying (3.21) and (3.22), respectively. Since the $KB\sigma$ algebra also holds in open superstring field theory, the solution

---

8 As we mentioned in the introduction, it is important that the $c$ ghost does not appear explicitly because the BRST transformation of the $c$ ghost in the superstring is different from that in the bosonic string.
can be seen as a string field in superstring field theory satisfying $Q \Psi + \Psi^2 = 0$. Furthermore, $\Psi$ is annihilated by $\eta_0$ because all elements of the $KB\sigma$ algebra are in the small Hilbert space. Therefore, $\Psi$ satisfies the two equations in (3.9). It also satisfies the reality condition $\Psi^\dagger = \Psi$.

Following the general prescription discussed in subsection 3.2, we construct solutions using $\Psi$ in (3.41) and the operator $R(t)$. We define $R$ by a state based on the wedge state $W_0$ of zero width with a local insertion of $R(t)$ on the boundary:

$$\langle \varphi, R \rangle = \langle f \circ \varphi(0) R(\frac{1}{2}) \rangle_{C_1}. \quad (3.42)$$

Using the state $R$, we can construct $\hat{\Psi}_L$ and $\hat{\Psi}_R$ as follows:

$$\hat{\Psi}_L = -\frac{1}{\sqrt{1-K}} R(Q\sigma_L) \sigma_R \frac{1}{\sqrt{1-K}} - \frac{1}{\sqrt{1-K}} R(Q\sigma_L) \frac{B}{1-K} (Q\sigma_R) \frac{1}{\sqrt{1-K}},$$
$$\hat{\Psi}_R = \frac{1}{\sqrt{1-K}} \sigma_L (Q\sigma_R) R \frac{1}{\sqrt{1-K}} - \frac{1}{\sqrt{1-K}} (Q\sigma_L) \frac{B}{1-K} (Q\sigma_R) R \frac{1}{\sqrt{1-K}}. \quad (3.43)$$

We therefore obtain two non-real solutions $\Phi_L$ and $\Phi_R$ given by

$$e^{\Phi_L} = 1 - \frac{1}{\sqrt{1-K}} R(Q\sigma_L) \sigma_R \frac{1}{\sqrt{1-K}} - \frac{1}{\sqrt{1-K}} R(Q\sigma_L) \frac{B}{1-K} (Q\sigma_R) \frac{1}{\sqrt{1-K}},$$
$$e^{-\Phi_R} = 1 + \frac{1}{\sqrt{1-K}} \sigma_L (Q\sigma_R) R \frac{1}{\sqrt{1-K}} - \frac{1}{\sqrt{1-K}} (Q\sigma_L) \frac{B}{1-K} (Q\sigma_R) R \frac{1}{\sqrt{1-K}}, \quad (3.44)$$

and a real solution $\Phi_{\text{real}}$ given by

$$e^{\Phi_{\text{real}}} = \frac{1}{\sqrt{(1 + \hat{\Psi}_L)(1 + \hat{\Psi}_R)}} (1 + \hat{\Psi}_L). \quad (3.45)$$

### 3.4 Regularity of the solutions

In our construction of the solutions, the operator $R(t)$ is inserted at the same point where the BRST transformation of the bcc operator $\sigma_L$ or $\sigma_R$ is inserted. Furthermore, in the expressions of $1/(1-K)$ and $1/\sqrt{1-K}$ in (2.12), the integral region of $t$ reaches $t = 0$, which corresponds to $W_0$ of zero width. Therefore, various operator insertions in the solutions collide and could make the solutions singular. In this subsection, we investigate the regularity of the solutions and demonstrate that they do not have any short-distance singularity of operator products.

---

9 The string field $\Psi$ can be regarded as a solution using bcc operators in the modified cubic theory.
Let us consider operator insertions in the superconformal ghost sector. There are two sources of such operator insertions: one is \( R(t) = -c_\xi \partial_\xi e^{-2\phi}(t) \) and the other is the BRST operator acting on the bcc operators \( \sigma_L(t) \) and \( \sigma_R(t) \). Since bcc operators behave as superconformal primary fields under superconformal transformations, their BRST transformations are determined by their weights. From the regularity condition

\[
\lim_{\epsilon \to +0} \sigma_L(t) \sigma_R(t + \epsilon) = 1,
\]

we assume that \( \sigma_L(t) \) and \( \sigma_R(t) \) behave as superconformal primary fields of weight 0. Then their BRST transformations are given by

\[
\begin{align*}
Q \cdot \sigma_L(t) &= c \partial_t \sigma_L(t) + \eta e^{\phi} G_{-1/2} \cdot \sigma_L(t), \\
Q \cdot \sigma_R(t) &= c \partial_t \sigma_R(t) + \eta e^{\phi} G_{-1/2} \cdot \sigma_R(t),
\end{align*}
\]

where \( G_{-1/2} \) generates the supersymmetry transformation:

\[
G_{-1/2} \cdot \varphi(t) \equiv \int_{C(t)} \left[ \frac{dz}{2\pi i} T_F(z) - \frac{d\bar{z}}{2\pi i} \bar{T}_F(\bar{z}) \right] \varphi(t).
\]

Here \( T_F(z) \) and \( \bar{T}_F(\bar{z}) \) are the holomorphic and antiholomorphic components, respectively, of the world-sheet supercurrent, and \( C(t) \) is a contour in the upper half-plane which runs from the point \( t + \epsilon \) on the real axis to the point \( t - \epsilon \) on the real axis in the limit \( \epsilon \to 0 \) with \( \epsilon > 0 \).

In the description using \( \phi \) and \( \chi \), the operator \( R(t) \) is written as

\[
R(t) = c e^{2\chi} e^{-2\phi}(t),
\]

and the BRST transformations of bcc operators are

\[
\begin{align*}
Q \cdot \sigma_L(t) &= c \partial_t \sigma_L(t) + e^{-\chi} e^{\phi} G_{-1/2} \cdot \sigma_L(t), \\
Q \cdot \sigma_R(t) &= c \partial_t \sigma_R(t) + e^{-\chi} e^{\phi} G_{-1/2} \cdot \sigma_R(t).
\end{align*}
\]

From these expressions, we notice that the superconformal ghost sector of (3.41) and (3.43) has a simple structure. Namely, the superconformal ghosts \( \chi \) and \( \phi \) always appear in the form \( e^{m\chi} e^{-m\phi} \). Because of the difference in the sign of the two operator product expansions in (3.5), the operator product of \( e^{m\chi} e^{-m\phi}(t) \) and \( e^{n\chi} e^{-n\phi}(t) \) is regular for arbitrary \( m \) and \( n \):

\[
\lim_{\epsilon \to 0} e^{m\chi} e^{-m\phi}(t) e^{n\chi} e^{-n\phi}(t + \epsilon) = (-1)^{mn} e^{(m+n)\chi} e^{-(m+n)\phi}(t).
\]

We therefore find that the superconformal ghost sector does not have any short-distance singularity of operator products. As the matter and \( bc \) ghost sectors do not suffer from any singularity, we conclude that the operator products in (3.41) and (3.43) are regular.

In the same way, we can show that star products made of \( \hat{\Psi}_L \) and \( \hat{\Psi}_R \) are also regular. This guarantees the regularity of operator products in the solutions \( \Phi_L, \Phi_R, \) and \( \Phi_{\text{real}} \), although there still remains the issue of convergence in the infinite sum mentioned in subsection 3.2.
3.5 Marginal deformations with regular operator products

We conclude this section by discussing the relation between our solutions constructed from bcc operators and previous solutions for marginal deformations with regular operator products \[16, 17, 19\], where the regularity conditions of bcc operators are satisfied. The marginal operator $V_1$ in the superstring is the supersymmetry transformation of a superconformal primary field $\hat{\chi}_{1/2}$ in the matter sector of weight $1/2$:

$$V_1(t) = G_{-1/2} \cdot \hat{\chi}_{1/2}(t).$$  \hspace{1cm} (3.52)

An integrated vertex operator in the 0 picture is an integral of $V_1$ on the boundary:

$$\int_a^b dt V_1(t) = \int_a^b dt G_{-1/2} \cdot \hat{\chi}_{1/2}(t).$$  \hspace{1cm} (3.53)

It is invariant under the BRST transformation up to nonvanishing terms from the endpoints of the integral region:

$$Q \cdot \int_a^b dt V_1(t) = \int_a^b dt \partial_t [cV_1(t) + \eta e^\phi \hat{\chi}_{1/2}(t)]$$
$$= [cV_1(b) + \eta e^\phi \hat{\chi}_{1/2}(b)] - [cV_1(a) + \eta e^\phi \hat{\chi}_{1/2}(a)].$$  \hspace{1cm} (3.54)

The operator $cV_1(t) + \eta e^\phi \hat{\chi}_{1/2}(t)$ is annihilated by the BRST operator, so it can be written as a BRST transformation of an operator. Since

$$\lim_{\epsilon \to 0} R(t - \epsilon) [cV_1(t) + \eta e^\phi \hat{\chi}_{1/2}(t)] = c\xi e^{-\phi} \hat{\chi}_{1/2}(t),$$  \hspace{1cm} (3.55)

we have

$$cV_1(t) + \eta e^\phi \hat{\chi}_{1/2}(t) = Q \cdot \chi(t)$$  \hspace{1cm} (3.56)

with

$$\chi(t) = c\xi e^{-\phi} \hat{\chi}_{1/2}(t).$$  \hspace{1cm} (3.57)

When operator products of $V_1$ are regular, \[10\] bcc operators are given by

$$\sigma_L(a) \sigma_R(b) = \exp \left[ \lambda \int_a^b dt V_1(t) \right] = \exp \left[ \lambda \int_a^b dt G_{-1/2} \cdot \hat{\chi}_{1/2}(t) \right],$$  \hspace{1cm} (3.58)

\[10\] In the following, we also need regularity conditions involving $\hat{\chi}_{1/2}$. A set of conditions, which is sufficient for the solutions to be well defined, can be stated in the following way: the operators $V_1(t_1)V_1(t_2)^n$ and $\hat{\chi}_{1/2}(t_1)\hat{\chi}_{1/2}(t_2)^n$ are finite in the limit $t_1 \to t_2$ for any positive integer $n$ and the operator $\hat{\chi}_{1/2}(t_1)\hat{\chi}_{1/2}(t_2)V_1(t_2)^n$ vanishes in the limit $t_1 \to t_2$ for any positive integer $n$. 

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where $\lambda$ is a deformation parameter. We can confirm that the conditions described in subsection 2.1 are satisfied. The BRST transformation of (3.58) is given by

$$Q \cdot \exp \left[ \lambda \int_a^b dt V_1(t) \right] = \exp \left[ \lambda \int_a^b dt V_1(t) \right] \lambda (Q \cdot V) (b) - \lambda (Q \cdot V) (a) \exp \left[ \lambda \int_a^b dt V_1(t) \right].$$

(3.59)

We can see that the effect of the BRST transformation is localized at the endpoints $a$ and $b$, which is consistent with the local property of bcc operators. Furthermore, the BRST transformations of the two terms on the right-hand side of (3.59) are

$$Q \cdot \left[ \lambda (Q \cdot V) (a) \exp \left[ \lambda \int_a^b dt V_1(t) \right] \right] = -\lambda (Q \cdot V) (a) \exp \left[ \lambda \int_a^b dt V_1(t) \right] \lambda (Q \cdot V) (b),$$

$$Q \cdot \exp \left[ \lambda \int_a^b dt V_1(t) \right] \lambda (Q \cdot V) (b) = -\lambda (Q \cdot V) (a) \exp \left[ \lambda \int_a^b dt V_1(t) \right] \lambda (Q \cdot V) (b),$$

(3.60)

where we used

$$\lim_{t_1 \to t_2} Q \cdot V(t_1) Q \cdot V(t_2) = 0.$$  

(3.61)

From (3.59) and (3.60), we identify the BRST transformations of bcc operators as

$$\sigma_L(a) (Q \cdot \sigma_R) (b) = \exp \left[ \lambda \int_a^b dt V_1(t) \right] \lambda (Q \cdot V) (b),$$

$$ (Q \cdot \sigma_L) (a) \sigma_R(b) = -\lambda (Q \cdot V) (a) \exp \left[ \lambda \int_a^b dt V_1(t) \right],$$

$$ (Q \cdot \sigma_L) (a) (Q \cdot \sigma_R) (b) = -\lambda (Q \cdot V) (a) \exp \left[ \lambda \int_a^b dt V_1(t) \right] \lambda (Q \cdot V) (b).$$

(3.62)

When the operators are inserted on wedge states, these relations can be translated as follows:

$$\sigma_L e^{\alpha K} (Q \sigma_R) = e^{\alpha (K + \lambda V_1)} \lambda (Q V),$$

$$(Q \sigma_L) e^{\alpha K} \sigma_R = -\lambda (Q V) e^{\alpha (K + \lambda V_1)},$$

$$(Q \sigma_L) e^{\alpha K} (Q \sigma_R) = -\lambda (Q V) e^{\alpha (K + \lambda V_1)} \lambda (Q V),$$

(3.63)

where the states $V_1$ and $V$ are defined by the states based on the wedge state $W_0$ of zero width with a local insertion of $V_1(t)$ and $V(t)$, respectively, on the boundary. Note that $e^{\alpha (K + \lambda V_1)}$ is a wedge state with the modified boundary conditions [61].

We can now write the solutions (3.44) in terms of $K$, $B$, $V_1$, and $V$ as follows:

$$e^{\Phi_L} = 1 + \frac{1}{\sqrt{1 - K}} \lambda V \lambda \frac{1}{\sqrt{1 - K}} + \frac{1}{\sqrt{1 - K}} \lambda V \lambda \frac{B}{1 - K - \lambda V} \lambda (Q V) \frac{1}{\sqrt{1 - K}},$$

$$e^{-\Phi_R} = 1 - \frac{1}{\sqrt{1 - K}} \lambda V \lambda \frac{1}{\sqrt{1 - K}} - \frac{1}{\sqrt{1 - K}} \lambda (Q V) \lambda \frac{B}{1 - K - \lambda V} \lambda V \frac{1}{\sqrt{1 - K}}.$$  

(3.64)
Here we used the relations

\[ R(QV) = V, \quad (QV)R = -V, \]  

which follow from (3.55). It is straightforward to express a real solution (3.45) in terms of \( K, B, V_1, \) and \( V \). As in the bosonic theory [61], our solutions correspond to the special choice \( f(K) = 1/\sqrt{1-K} \) in the class of solutions by Erler [16].

4 Gauge-invariant observables

A classical solution in open string field theory is expected to correspond to a BCFT. In particular, the solutions constructed from bcc operators should correspond to the BCFT associated with the bcc operators. In this section, we provide evidence by calculating the gauge-invariant observables introduced in [71, 72].

The gauge-invariant observable \( W(\Psi, O) \) is defined for a solution \( \Psi \) in open bosonic string field theory and an on-shell closed string vertex operator \( O \) of weight \((0, 0)\) [71, 72]. In [32], Ellwood conjectured that it is related to the closed string one-point function on the disk with the modified boundary conditions associated with the solution. The relation is given by

\[ W(\Psi, O) = A_0(O) - A_0^{\ast}(O), \]  

where \( A_0(O) \) and \( A_0^{\ast}(O) \) are the closed string one-point functions on the unit disk

\[ A_0(O) = \frac{1}{2\pi i} \left\langle O(0)c(1) \right\rangle_{\text{disk}}, \quad A_0^{\ast}(O) = \frac{1}{2\pi i} \left\langle O(0)c(1) \right\rangle_{\text{disk}, \text{BCFT}^*}, \]  

in the original BCFT for \( A_0(O) \) and in the BCFT\(_{\ast}\) associated with the solution \( \Psi \) for \( A_0^{\ast}(O) \), as indicated by the subscript.

Based on this conjecture by Ellwood, it is expected that the gauge-invariant observables for the solutions constructed from bcc operators reproduce the one-point functions in the BCFT associated with the bcc operators. After recalling some basic properties of the gauge-invariant observables, we evaluate \( W(\Psi, O) \) for the solution (2.11) in the bosonic string constructed in [61], and then we extend the calculation to the solutions (3.44) and (3.45) in the superstring.

4.1 Properties of \( W(\varphi, O) \)

For a state \( \varphi \) in the Fock space, \( W(\varphi, O) \) is defined by the following correlator on the upper half-plane:

\[ W(\varphi, O) = \langle O(i)f_I \circ \varphi(0) \rangle_{\text{UHP}} \quad \text{with} \quad f_I(\xi) = \frac{2\xi}{1-\xi^2}. \]  

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The conformal transformation \( f_I(\xi) \) is associated with the wedge state \( W_0 \) of zero width, which is also called the identity state:

\[
\langle \varphi, W_0 \rangle = \langle f \circ \varphi(0) \rangle_{C_1} = \langle f_I \circ \varphi(0) \rangle_{UHP} .
\] (4.4)

Therefore, \( W(\varphi, \mathcal{O}) \) can be thought of as a BPZ inner product of \( \varphi \) and the identity state \( W_0 \) with an insertion of \( \mathcal{O} \) at the conical singularity, and we can formally express \( W(\varphi, \mathcal{O}) \) in the sliver frame as

\[
W(\varphi, \mathcal{O}) = \langle \mathcal{O}(i\infty) f \circ \varphi(0) \rangle_{C_1} .
\] (4.5)

This expression is formal because the point \( i\infty \) where \( \mathcal{O} \) is inserted is outside the coordinate patch, and we need another patch for a more rigorous treatment.

It follows from the definition that \( W(\varphi, \mathcal{O}) \) vanishes for any BRST-exact state and has a cyclic property:

\[
W(Q\Lambda, \mathcal{O}) = 0 ,
\] (4.6)

\[
W(\varphi_1 \varphi_2, \mathcal{O}) = (-1)^{|\varphi_1||\varphi_2|} W(\varphi_2 \varphi_1, \mathcal{O}) .
\] (4.7)

It is easy to see the cyclic property (4.7) from the expression (4.5). The observable \( W(\Psi, \mathcal{O}) \) is thus invariant under gauge transformations in the bosonic string:

\[
W(\delta\Psi, \mathcal{O}) = W(Q\Lambda + [\Psi, \Lambda], \mathcal{O}) = 0 .
\] (4.8)

We can also represent \( W(\varphi, \mathcal{O}) \) as a correlator on the unit disk by the conformal transformation \( f_{\text{disk}}(\xi) \) from the upper half-plane to the disk:

\[
W(\varphi, \mathcal{O}) = \langle \mathcal{O}(0) f_{\text{disk}} \circ f_I \circ \varphi(0) \rangle_{\text{disk}} \quad \text{with} \quad f_{\text{disk}}(\xi) = \frac{1 + i\xi}{1 - i\xi} ,
\] (4.9)

where the explicit form of \( f_{\text{disk}}(f_I(\xi)) \) is

\[
f_{\text{disk}}(f_I(\xi)) = \left( \frac{1 + i\xi}{1 - i\xi} \right)^2 .
\] (4.10)

Consider a state \( \Sigma \) based on \( W_s \) defined by

\[
\langle \varphi, \Sigma \rangle = \left\langle f \circ \varphi(0) \prod_{i=1}^n O_i(z_i) \right\rangle_{C_{s+1}} \quad \text{with} \quad \frac{1}{2} \leq \Re z_i \leq s + \frac{1}{2} ,
\] (4.11)

where \( O_i(z_i) \) are some local operators. A formal expression for \( W(\Sigma, \mathcal{O}) \) in the sliver frame can be written as

\[
W(\Sigma, \mathcal{O}) = \left\langle \mathcal{O}(i\infty) \prod_{i=1}^n O_i(z_i) \right\rangle_{C_s} ,
\] (4.12)
where the surface $C_s$ is represented in the region $\frac{1}{2} \leq \Re z_i \leq s + \frac{1}{2}$ of the upper half-plane. A precise expression for $W(\Sigma, \mathcal{O})$ can be obtained by a conformal transformation from the surface $C_s$ to the unit disk:

$$W(\Sigma, \mathcal{O}) = \left\{ \mathcal{O}(0) \prod_{i=1}^{n} h_s \circ O_i(z_i) \right\}_{\text{disk}},$$

where the map $h_s(z)$ from $C_s$ to the unit disk is given by

$$h_s(z) = f_{\text{disk}} \left( f^{-1} \left( \frac{2z_s}{s} \right) \right) = \exp \frac{2\pi i z}{s}.$$  

In the following, we use the expression of the gauge-invariant observables on the unit disk.

### 4.2 Gauge-invariant observables in the bosonic string

We start with the evaluation of the gauge-invariant observable for the solution (2.11) in the bosonic string

$$W(\Psi, \mathcal{O}) \quad \text{with} \quad \Psi = -\frac{1}{\sqrt{1-K}}(Q\sigma_L)\sigma_R \frac{1}{\sqrt{1-K}} - \frac{1}{\sqrt{1-K}}(Q\sigma_L) \frac{B}{1-K}(Q\sigma_R) \frac{1}{\sqrt{1-K}},$$

where $\mathcal{O} = c\tilde{c}O_m$ with $O_m$ being a matter conformal primary field of weight $(1,1)$. Using the cyclic property (4.7), $W(\Psi, \mathcal{O})$ reduces to the form

$$W(\Psi, \mathcal{O}) = W(\tilde{\Psi}, \mathcal{O}) \quad \text{with} \quad \tilde{\Psi} = -\frac{1}{1-K}(Q\sigma_L) \frac{1}{1-K} \sigma_R - Q \left[ \frac{1}{1-K}(Q\sigma_L) \frac{B}{1-K} \sigma_R \right],$$

where $\tilde{\Psi}$ is defined in (2.13) and we wrote its expression given in (2.15). Since BRST-exact terms do not contribute to the gauge-invariant observables, $W(\tilde{\Psi}, \mathcal{O})$ further reduces to the following form:

$$W(\tilde{\Psi}, \mathcal{O}) = W(\tilde{\Psi}_*, \mathcal{O}) \quad \text{with} \quad \tilde{\Psi}_* = -\frac{1}{1-K}(Q\sigma_L) \frac{1}{1-K} \sigma_R.$$

When $\sigma_L(t)$ and $\sigma_R(t)$ are matter primary fields of weight 0, their BRST transformations are

$$Q \cdot \sigma_L(t) = c\partial_t \sigma_L(t), \quad Q \cdot \sigma_R(t) = c\partial_t \sigma_R(t),$$

and $\tilde{\Psi}_*$ is written as

$$\tilde{\Psi}_* = -\frac{1}{1-K}c[\sigma_L, \sigma_R] \frac{1}{1-K} \sigma_R,$$

where $c$ is a state based on the wedge state $W_0$ of zero width with a local insertion of $c(t)$ on the boundary. Note that an insertion of $\partial_t \sigma_L(t)$ is translated into the commutator $[\sigma_L, \sigma_R]$. We
can further rewrite $\tilde{\Psi}_\ast$ as
\[
\tilde{\Psi}_\ast = \frac{1}{1 - K}[1 - K, c\sigma_L] - \frac{1}{1 - K}\sigma_R + \frac{\sigma_L}{1 - K}\sigma_L - \frac{1}{1 - K}c + \frac{1}{1 - K}[K, c]\sigma_L = \frac{1}{1 - K}\sigma_R.
\]

The commutator $[K, c]$ corresponds to an insertion of $\partial_t c(t)$ on the boundary, and the operator $\partial_t c(t)$ is transformed under the conformal map $h_s(z)$ in (4.14) as
\[
h_s \circ \partial_t c(t) = \partial_t \left( \frac{s}{2\pi i} e^{-2\pi i t/s} c(e^{2\pi i t/s}) \right) = -i \partial_\theta \left( e^{-i\theta} c(e^{i\theta}) \right),
\]
where $\theta = \frac{2\pi t}{s}$. Using the expression of the gauge-invariant observables on the unit disk, we find
\[
W(\tilde{\Psi}_\ast, \mathcal{O}) = \int_0^\infty ds e^{-s} \lim_{\epsilon \to +0} \langle \mathcal{O}(0)c\sigma_L(1)\sigma_R(e^{(2\pi - \epsilon)i}) \rangle_{\text{disk}} - \int_0^\infty ds e^{-s} \langle \mathcal{O}(0)c(1) \rangle_{\text{disk}}
\]
\[
- i \int_0^\infty ds \int_0^1 d\tau se^{-\gamma} \partial_\theta \left[ e^{-i\theta} \langle \mathcal{O}(0)c(e^{i\theta})\sigma_L(1)\sigma_R(e^{2\pi i\tau}) \rangle_{\text{disk}} \right] \bigg|_{\theta = 0}.
\]

We can factorize the matter and ghost sectors of the correlator in the last term as
\[
e^{-i\theta} \langle \mathcal{O}(0)c(e^{i\theta})\sigma_L(1)\sigma_R(e^{2\pi i\tau}) \rangle_{\text{disk}} = e^{-i\theta} \langle c\bar{c}(0)c(e^{i\theta}) \rangle_{\text{disk}}^{\text{bc}} \langle \mathcal{O}_m(0)\sigma_L(1)\sigma_R(e^{2\pi i\tau}) \rangle_{\text{disk}}^{\text{matter}}.
\]

From the rotation invariance of the disk amplitude, we find
\[
\partial_\theta \left[ e^{-i\theta} \langle c\bar{c}(0)c(e^{i\theta}) \rangle_{\text{disk}}^{\text{bc}} \right] = 0.
\]

Therefore, the last term in (4.22) vanishes, and the final expression of the gauge-invariant observable is
\[
W(\Psi, \mathcal{O}) = W(\tilde{\Psi}_\ast, \mathcal{O}) = \frac{1}{2\pi i} \lim_{\epsilon \to +0} \langle \mathcal{O}(0)c\sigma_L(1)\sigma_R(e^{(2\pi - \epsilon)i}) \rangle_{\text{disk}} - \frac{1}{2\pi i} \langle \mathcal{O}(0)c(1) \rangle_{\text{disk}}.
\]

The second term on the right-hand side is $A_0(\mathcal{O})$. The first term is $A_\ast(\mathcal{O})$ because of the insertions of bcc operators:
\[
\frac{1}{2\pi i} \lim_{\epsilon \to +0} \langle \mathcal{O}(0)c\sigma_L(1)\sigma_R(e^{(2\pi - \epsilon)i}) \rangle_{\text{disk}} = \frac{1}{2\pi i} \langle \mathcal{O}(0)c(1) \rangle_{\text{disk}}^{\text{BCFT}_\ast} = A_\ast(\mathcal{O}).
\]

We have thus obtained the relation (4.1) conjectured by Ellwood.
4.3 Gauge-invariant observables in the superstring

Let us extend the calculation to the superstring. For a solution $\Phi$ in superstring field theory, the gauge-invariant observable is defined by $W(e^{-\Phi}Qe^{\Phi},\mathcal{O})$ \[32\], where $\mathcal{O}$ is an on-shell closed string vertex operator of weight $(0,0)$. In the NS-NS sector we can take the operator $\mathcal{O}$ to be

$$\mathcal{O} = (\xi + \tilde{\xi})c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}\tilde{O}_m,$$

(4.27)

where $\tilde{O}_m$ is a matter superconformal primary field of weight $(\frac{1}{2}, \frac{1}{2})$. The conjecture by Ellwood in the superstring can be stated as \[32\]

$$W(e^{-\Phi}Qe^{\Phi},\mathcal{O}) = A^*\mathcal{O} - A_0\mathcal{O}.$$   \hspace{1cm} (4.28)

For the solutions (3.44) and (3.45), the gauge-invariant observables are

$$W(e^{-\Phi}Qe^{\Phi},\mathcal{O}) = W(\Psi,\mathcal{O}),$$

(4.29)

where $\Psi$ is given by (3.41). As in the case of the bosonic string, the calculation of the observables reduces to the following form:

$$W(e^{-\Phi}Qe^{\Phi},\mathcal{O}) = W(\tilde{\Psi},\mathcal{O}) \text{ with } \tilde{\Psi} = -\frac{1}{1-K}(Q\sigma_L)\frac{1}{1-K}\sigma_R.$$ \hspace{1cm} (4.30)

The explicit form of $Q \cdot \sigma_L(t)$ in the superstring is given by

$$Q \cdot \sigma_L(t) = c\partial_t\sigma_L(t) + \eta e^\phi(G_{-1/2} \cdot \sigma_L)(t)$$ \hspace{1cm} (4.31)

when $\sigma_L(t)$ and $\sigma_R(t)$ are matter superconformal primary fields of weight 0. Since the bc ghost sector of $\mathcal{O}$ is $c\tilde{c}$, the second term on the right-hand side of (4.31) does not contribute to the observables because it does not contain any $c$ ghost. In the case of the R-R sector, the explicit form of $\mathcal{O}$ is not simple \[32\]. However, the bc ghost sector is $c\tilde{c}$ so that the second term on the right-hand side of (4.31) does not contribute to the observables either. The calculations in both sectors therefore reduce to that in the bosonic string, and we exactly obtain the relation (4.28) conjectured by Ellwood.

5 Discussion

– Universal coefficients   In this paper, we constructed a class of analytic solutions of open superstring field theory in the Berkovits formulation from boundary condition changing operators satisfying the regularity conditions described in subsection 2.1. In the case of the solutions in the bosonic string, the dependence on the matter sector is simple, and we only need the information of three-point functions with a pair of bcc operators and an arbitrary operator in
the original BCFT [61]. In the case of the solutions in the superstring, the dependence on the matter sector is again simple. All the solutions \( \Phi_L, \Phi_R, \) and \( \Phi_{\text{real}} \) are constructed from star products of \( \hat{\Psi}_L \) and \( \hat{\Psi}_R \), which are specified by giving \( \langle \varphi, \hat{\Psi}_L \rangle \) and \( \langle \varphi, \hat{\Psi}_R \rangle \) for an arbitrary state \( \varphi \) in the Fock space, and the dependence of \( \langle \varphi, \hat{\Psi}_L \rangle \) and \( \langle \varphi, \hat{\Psi}_R \rangle \) on the matter sector reduces to the three-point functions

\[
\begin{align*}
\langle \varphi_m(t_1)(G_{-1/2} \cdot \sigma_L)(t_2)\sigma_R(t_3) \rangle^\text{matter}_{\text{UHP}},
\langle \varphi_m(t_1)\sigma_L(t_2)(G_{-1/2} \cdot \sigma_R)(t_3) \rangle^\text{matter}_{\text{UHP}},
\langle \varphi_m(t_1)(G_{-1/2} \cdot \sigma_L)(t_2)(G_{-1/2} \cdot \sigma_R)(t_3) \rangle^\text{matter}_{\text{UHP}},
\end{align*}
\]

where \( \varphi_m(t) \) is an arbitrary matter conformal primary field. For example, \( \langle \varphi, \hat{\Psi}_L \rangle \) for \( \varphi(t) = -c\partial c e^{-\varphi} \varphi_m(t) \) is given by

\[
\langle \varphi, \hat{\Psi}_L \rangle = C_{\varphi} \cdot g(h) \quad \text{with} \quad C_{\varphi} = \langle \varphi_m(0)(G_{-1/2} \cdot \sigma_L)(1)\sigma_R(\infty) \rangle^\text{matter}_{\text{UHP}},
\]

and \( g(h) \) is a universal function of the weight \( h \) of \( \varphi_m \), which does not depend on the particular choice of \( \varphi_m \) or the bcc operators \( \sigma_L \) and \( \sigma_R \). The explicit form of \( g(h) \) is

\[
g(h) = \left( h - \frac{1}{2} \right) \int_0^\infty dx \int_0^\infty ds \int_0^\infty dy \frac{e^{1-x-s-y}}{2\pi \sqrt{(x-\frac{1}{2})(y-\frac{1}{2})}} \frac{2\sin\theta_s}{L\sin\theta_s\sin\theta_{x+s}} \frac{\theta_y \sin^2\theta_x + \theta_x \sin^2\theta_y - \sin\theta_x \sin\theta_y \sin\theta_{x+s}}{\sin^2\theta_s} \Bigg|^{h+\frac{1}{2}}_{h+\frac{1}{4}}
\]

for \( h \neq \frac{1}{2} \) and \( g(\frac{1}{2}) = 1 \), where \( L = x + s + y \) and \( \theta = \frac{\ell}{2}\pi \). The details of the calculation are presented in appendix [13]. It is straightforward to calculate \( \langle \varphi, \hat{\Psi}_L \rangle \) and \( \langle \varphi, \hat{\Psi}_R \rangle \) for a different \( \varphi \).

— **Towards the generalization to singular bcc operators** For a systematic construction of solutions in open string field theory for a given BCFT, we would like to generalize the construction of solutions to the case when operator products of the bcc operators are singular [14]. We expect that the structure we observed in the calculation of the gauge-invariant observables in section 4 gives us some insight into this generalization. The solution takes the form

\[
\tilde{\Psi} = c\sigma_L \frac{1}{1-K}\sigma_R - \frac{1}{1-K}c + (\text{irrelevant terms}), \quad (5.4)
\]

where the irrelevant terms are the terms which do not contribute to the gauge-invariant observables such as BRST-exact terms or those containing \( \partial c \). From this structure, it is obvious that the gauge-invariant observables for the solution reproduce the expected change of boundary conditions of the disk.

11 See [46][60][65] for other interesting approaches to this problem.
We also observe this structure in the phantomless solution for tachyon condensation by Erler and Schnabl [50]. It is given by

\[ \Psi = -\frac{1}{\sqrt{1-K}} (c - cKBc) \frac{1}{\sqrt{1-K}}, \]  

(5.5)

and the associated \( \tilde{\Psi} \) has the same structure:

\[ \tilde{\Psi} = \frac{1}{\sqrt{1-K}} \Psi \sqrt{1-K} = -\frac{1}{1-K} c + Q \left( \frac{1}{1-K} Bc \right). \]  

(5.6)

In [50] a class of gauge conditions called dressed \( B_0 \) gauges were introduced. In fact the solutions (5.5) and (2.11) satisfy the same gauge condition in the class of dressed \( B_0 \) gauges. In the generalization to bcc operators with singular operator products, the same structure may continue to hold, and this gauge condition may play an important role.

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A Evaluating the action

We discuss the structure of the action for solutions constructed by the general prescription explained in subsection 3.2. As shown in [78], the action (3.39) of open superstring field theory in the Berkovits formulation can be written as

\[ S = -\frac{1}{g^2} \int_0^1 dt \, \text{Tr} \left( (\eta_0 A_t) A_Q \right). \]  

(A.1)

Since the solutions \( \Phi_L, \Phi_R, \) and \( \Phi_{\text{real}} \) in subsection 3.2 are gauge-equivalent, the values of the action for them are the same. Here we evaluate the action for \( \Phi_L = \ln(1 + \tilde{\Psi}_L) \). Choosing the
\[ t \text{-dependence of } \Phi(t) \text{ to be} \]

\[ e^{\Phi(t)} = 1 + t \widehat{\Psi}_L , \]  

we have

\begin{align*}
A_t &= (1 + t \widehat{\Psi}_L)^{-1} \widehat{\Psi}_L , \\
A_Q &= (1 + t \widehat{\Psi}_L)^{-1}(1 + \widehat{\Psi}_L) t \Psi ,
\end{align*}

and

\begin{align*}
\eta_0 A_t &= -(1 + t \widehat{\Psi}_L)^{-1}(t \eta_0 \widehat{\Psi}_L) (1 + t \widehat{\Psi}_L)^{-1} \widehat{\Psi}_L + (1 + t \widehat{\Psi}_L)^{-1}(\eta_0 \widehat{\Psi}_L) \\
&= (1 + t \widehat{\Psi}_L)^{-1}(\eta_0 \widehat{\Psi}_L) \left[ 1 + (1 + t \widehat{\Psi}_L)^{-1}(-t \widehat{\Psi}_L) \right] , \quad (A.4)
\end{align*}

Therefore, the action for the solution \( \Phi = \Phi_L \) is

\begin{align*}
S &= -\frac{1}{g^2} \int_0^1 dt \, \text{Tr} \left[ (1 + t \widehat{\Psi}_L)^{-1}(\eta_0 \widehat{\Psi}_L)(1 + t \widehat{\Psi}_L)^{-1}(1 + \widehat{\Psi}_L) t \Psi \right] . \quad (A.5)
\end{align*}

The expansion of the action in \( \Psi \) and \( \widehat{\Psi}_L \) is given by

\begin{align*}
S &= -\frac{1}{g^2} \sum_{a,b=0}^\infty \int_0^1 dt \, t^{a+b+1} \text{Tr} \left[ (-\widehat{\Psi}_L)^a(\eta_0 \widehat{\Psi}_L)(b+1)(-\widehat{\Psi}_L)^b(1 + \widehat{\Psi}_L) \Psi \right] \\
&= -\frac{1}{g^2} \sum_{a,b=0}^\infty \frac{b+1}{a+b+2} \text{Tr} \left[ (-\widehat{\Psi}_L)^a(\eta_0 \widehat{\Psi}_L)(-\widehat{\Psi}_L)^b \Psi - (-\widehat{\Psi}_L)^a(\eta_0 \widehat{\Psi}_L)(-\widehat{\Psi}_L)^{b+1} \Psi \right] \\
&= -\frac{1}{g^2} \sum_{a,b=0}^\infty \frac{a+1}{(a+b+2)(a+b+1)} \text{Tr} \left[ (-\widehat{\Psi}_L)^a(\eta_0 \widehat{\Psi}_L)(-\widehat{\Psi}_L)^b \Psi \right] \\
&= \frac{1}{g^2} \sum_{n=2}^{n-2} \sum_{m=0}^{n-2} (-1)^{n-1} \frac{m+1}{n(n-1)} \text{Tr} \left[ \widehat{\Psi}_L^m(\eta_0 \widehat{\Psi}_L) \widehat{\Psi}_L^{n-m-2} \Psi \right] . \quad (A.6)
\end{align*}

Since \( \text{Tr} \left[ \eta_0 (AB) \right] = 0 \), which follows from the BPZ property \( \langle \eta_0 A, B \rangle = -(1)^{|A|} \langle A, \eta_0 B \rangle \), we find that

\begin{align*}
\sum_{m=0}^{n-2} \text{Tr} \left[ \widehat{\Psi}_L^m(\eta_0 \widehat{\Psi}_L) \widehat{\Psi}_L^{n-m-2} \Psi \right] &= \text{Tr} \left[ \eta_0 (\widehat{\Psi}_L^{n-1} \Psi) \right] = 0 , \quad (A.7)
\end{align*}

where we also used \( \eta_0 \Psi = 0 \). We therefore obtain

\begin{align*}
S &= \frac{1}{g^2} \sum_{n=3}^{\infty} \sum_{m=1}^{n-2} (-1)^{n-1} \frac{m}{n(n-1)} \text{Tr} \left[ \widehat{\Psi}_L^m(\eta_0 \widehat{\Psi}_L) \widehat{\Psi}_L^{n-m-2} \Psi \right] . \quad (A.8)
\end{align*}

The first few terms in the expansion are

\begin{align*}
S &= \frac{1}{g^2} \text{Tr} \left[ \frac{1}{6} \widehat{\Psi}_L(\eta_0 \widehat{\Psi}_L) \Psi - \frac{1}{12} \left\{ \widehat{\Psi}_L(\eta_0 \widehat{\Psi}_L) \widehat{\Psi}_L \Psi + 2 \widehat{\Psi}_L^2(\eta_0 \widehat{\Psi}_L) \Psi \right\} + \ldots \right] . \quad (A.9)
\end{align*}
The state $\hat{\Psi}_L$ is constructed by inserting $R(t)$ to $\Psi$, and the operator $\eta_0$ annihilates $\Psi$ and changes $R(t)$ to the picture-lowering operator $Y(t) = c\partial_\xi e^{-2\phi}(t)$. Schematically, each term in (A.8) consists of $n$ states of $\Psi$, $n - 2$ insertions of $R(t)$, and one $Y(t)$.

**B Universal coefficients**

In this appendix we evaluate $\langle \varphi, \hat{\Psi}_L \rangle$ for $\varphi(t) = -c\partial_\xi e^{-\phi}\varphi_m(t)$, where $\varphi_m(t)$ is a matter conformal primary field of weight $h$. When the bcc operators are superconformal primary fields of weight 0, $\hat{\Psi}_L$ is given by

$$\hat{\Psi}_L = \hat{\Psi}_{L1} + \hat{\Psi}_{L2} + \hat{\Psi}_{L3}$$

with

$$\hat{\Psi}_{L1} = -\frac{1}{\sqrt{1 - K}}ce^x e^{-\phi}(G_{-1/2} \sigma_L) \frac{1}{\sqrt{1 - K}}$$,

$$\hat{\Psi}_{L2} = -\frac{1}{\sqrt{1 - K}}ce^x e^{-\phi}(G_{-1/2} \sigma_L) \frac{B}{1 - K} e^{[K, \sigma_R]} \frac{1}{\sqrt{1 - K}}$$,

$$\hat{\Psi}_{L3} = -\frac{1}{\sqrt{1 - K}}ce^x e^{-\phi}(G_{-1/2} \sigma_L) \frac{B}{1 - K} e^{-x} e^{\phi}(G_{-1/2} \sigma_R) \frac{1}{\sqrt{1 - K}}$$,

where the states $e^m\chi$ and $e^n\phi$ are defined by the states based on the wedge state $W_0$ of zero width with a local insertion of $e^m\chi(t)$ and $e^n\phi(t)$, respectively, on the boundary. For $\varphi(t) = -c\partial_\xi e^{-\phi}\varphi_m(t)$, the inner product $\langle \varphi, \hat{\Psi}_{L3} \rangle$ vanishes because of the constraint from the bc ghost number:

$$\langle \varphi, \hat{\Psi}_L \rangle = \langle \varphi, \hat{\Psi}_{L1} \rangle + \langle \varphi, \hat{\Psi}_{L2} \rangle$$.

We first evaluate the second term $\langle \varphi, \hat{\Psi}_{L2} \rangle$ given by

$$\langle \varphi, \hat{\Psi}_{L2} \rangle = -\int_{\frac{1}{2}}^{\infty} dx \int_{0}^{\infty} ds \int_{\frac{1}{2}}^{\infty} dy \frac{e^{1 - x - s - y}}{\pi \sqrt{(x - \frac{1}{2})(y - \frac{1}{2})}}$$

$$\times \langle f \circ \varphi(0) ce^x e^{-\phi}(G_{-1/2} \sigma_L)(x) B c\partial_\xi \sigma_R(x + s) \rangle_{CL}$$,

where $L = x + s + y$ and the operator $B$ is a line integral of the $b$ ghost defined by

$$B = \int_{-\infty}^{-i\infty} \frac{dz}{2\pi i} b(z)$$.
The correlator in (B.4) can be factorized as follows:

\[
\langle f \circ \varphi(0) e^\chi e^{-\phi} (G_{-1/2} \cdot \sigma_L)(x) \mathcal{B} \partial_s \sigma_R(x + s) \rangle_{C_L} = - \left( \frac{\pi}{2} \right)^{\frac{1}{2}} \langle c \partial c(0) c(x) \mathcal{B} c(x + s) \rangle_{C_L} \left( e^{-\phi(0)} e^\chi e^{-\phi(x)} \right)_{C_L}\ 
\times \partial_b \langle f \circ \varphi_m(0) (G_{-1/2} \cdot \sigma_L)(x) \sigma_R(b) \rangle_{C_L} \bigg|_{b = x + s}.
\]

(B.6)

Since \( \varphi_m, G_{-1/2} \cdot \sigma_L, \) and \( \sigma_R \) are primary fields of weight \( h, \frac{1}{2}, \) and 0, respectively, we find

\[
\langle f \circ \varphi_m(0) (G_{-1/2} \cdot \sigma_L)(a) \sigma_R(b) \rangle_{C_L} = C_\varphi \left( \frac{2}{L} \right)^h \left( \frac{\pi}{L \cos^2 \theta_a} \right)^{\frac{1}{2}} \langle \varphi_m(0) (G_{-1/2} \cdot \sigma_L)(\tan \theta_a) \sigma_R(\tan \theta_b) \rangle_{UHP}^{\text{matter}}.
\]

(B.7)

where \( 0 < a < b < L, \theta_\ell = \frac{\ell}{L} \pi, \) and \( C_\varphi \) is a constant independent of \( a, b, \) and \( L. \) It is related to the coefficient of the matter three-point function of \( \varphi_m, G_{-1/2} \cdot \sigma_L, \) and \( \sigma_R \) as follows:

\[
\langle \varphi_m(t_1) (G_{-1/2} \cdot \sigma_L)(t_2) \sigma_R(t_3) \rangle_{UHP}^{\text{matter}} = C_\varphi \frac{|t_3 - t_2|^{\frac{h - \frac{1}{2}}{2}}}{|t_2 - t_1|^2 |t_3 - t_1|^{\frac{h + \frac{3}{2}}{2}}},
\]

(B.8)

In other words, \( C_\varphi \) is the matter three-point function with operators \( \varphi_m, G_{-1/2} \cdot \sigma_L, \) and \( \sigma_R \) inserted at 0, 1, and \( \infty, \) respectively.

\[
C_\varphi = \langle \varphi_m(0) (G_{-1/2} \cdot \sigma_L)(1) \sigma_R(\infty) \rangle_{UHP}^{\text{matter}}.
\]

(B.9)

Therefore, the matter correlator in (B.6) is

\[
\partial_b \langle f \circ \varphi_m(0) (G_{-1/2} \cdot \sigma_L)(x) \sigma_R(b) \rangle_{C_L} \bigg|_{b = x + s} = \left( h - \frac{1}{2} \right) C_\varphi \left( \frac{2}{L} \right)^h \left( \frac{\pi}{L} \right)^{\frac{3}{2}} \left( \frac{1}{\sin \theta_{x+s} \sin \theta_s} \right) \left( \frac{\sin \theta_s}{\sin \theta_x \sin \theta_{x+s}} \right)^{\frac{h - \frac{1}{2}}{2}}.
\]

(B.10)

The ghost sector correlators take the form

\[
\langle c \partial c(0) c(x) \mathcal{B} c(x + s) \rangle_{C_L} = \frac{L^2}{\pi^3} \left( -\theta_y \sin^2 \theta_x - \theta_x \sin^2 \theta_y + \sin \theta_x \sin \theta_s \sin \theta_y \right),
\]

\[
\langle e^{-\phi(0)} e^\chi e^{-\phi(x)} \rangle_{C_L} = - \frac{\pi}{L \sin \theta_x}.
\]

(B.11)

\textsuperscript{12} The normalization of the correlator on the upper half-plane is \( \langle e^\chi c \partial c \partial^2 c e^{-\phi(z)} \rangle_{UHP} = -2 \) up to a factor of the spacetime volume. We also use the following convention of factorization: \( \langle e^\chi c \partial c \partial^2 c e^{-\phi(z)} \rangle_{UHP} = -\langle c \partial c \partial^2 c(z) \rangle_{UHP} \langle e^\chi e^{-\phi(z)} \rangle_{UHP} \langle 1 \rangle_{UHP}^{\text{matter}} \) with \( \langle c \partial c \partial^2 c(z) \rangle_{UHP} = -2 \) and \( \langle e^\chi e^{-\phi(z)} \rangle_{UHP} = -1. \)

\textsuperscript{13} Since the weight of \( \sigma_R \) vanishes, we can simply send the position of \( \sigma_R \) to infinity without considering the conformal transformation \( I(\xi) = -1/\xi. \)
Combining all the sectors, we have
\[
\langle \varphi, \hat{\Psi}_{L2} \rangle = C_\varphi \left( h - \frac{1}{2} \right) \int_0^\infty dz \int_0^\infty ds \int_0^\infty dy \frac{e^{1-x-y}}{2\pi \sqrt{(x-\frac{1}{2})(y-\frac{1}{2})}} \left( \frac{2 \sin \theta_s}{L \sin \theta_x \sin \theta_{x+s}} \right)^{h+\frac{1}{2}} \times \frac{\theta_y \sin^2 \theta_x + \theta_x \sin^2 \theta_y - \sin \theta_x \sin \theta_{s} \sin \theta_y}{\sin^2 \theta_s}.
\]
(B.12)

Let us next consider the first term \( \langle \varphi, \hat{\Psi}_{L1} \rangle \) given by
\[
\langle \varphi, \hat{\Psi}_{L1} \rangle = -\int_\frac{1}{2}^\infty dx \int_\frac{1}{2}^\infty dy \frac{e^{1-x-y}}{\pi \sqrt{(x-\frac{1}{2})(y-\frac{1}{2})}} \langle f \circ \varphi(0) ce^{-\phi(G_{-1/2} \cdot \sigma_L)} \sigma_R(x) \rangle_{C_s+y}.
\]
(B.13)

It vanishes for \( h \neq \frac{1}{2} \). For \( h = \frac{1}{2} \), it is evaluated as
\[
\langle \varphi, \hat{\Psi}_{L1} \rangle = C_\varphi.
\]
(B.14)

We conclude that the inner product \( \langle \varphi, \hat{\Psi}_{L} \rangle \) can be written as a product of the constant \( C_\varphi \) given by \( \text{(B.9)} \) and the universal function \( g(h) \) independent of the particular choice of \( \varphi_m \) or the bcc operators:
\[
\langle \varphi, \hat{\Psi}_{L} \rangle = C_\varphi g(h)
\]
(B.15)

with
\[
g(h) = \left( h - \frac{1}{2} \right) \int_\frac{1}{2}^\infty dx \int_0^\infty ds \int_\frac{1}{2}^\infty dy \frac{e^{1-x-y}}{2\pi \sqrt{(x-\frac{1}{2})(y-\frac{1}{2})}} \left( \frac{2 \sin \theta_s}{L \sin \theta_x \sin \theta_{x+s}} \right)^{h+\frac{1}{2}} \times \frac{\theta_y \sin^2 \theta_x + \theta_x \sin^2 \theta_y - \sin \theta_x \sin \theta_{s} \sin \theta_y}{\sin^2 \theta_s}.
\]
(B.16)

for \( h \neq \frac{1}{2} \) and \( g(\frac{1}{2}) = 1 \).

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