CALCULATING THERMAL COEFFICIENTS
USING A HYBRID METHOD

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Abstract: In this paper we study the problem of determining two thermal parameters of a cylindrical metal sample. This is an inverse problem in heat conduction where boundary conditions are determined on the basis of temperature measurements taken at the selected internal points in the sample. A hybrid method is used to find the parameters based on the experimental data of the temperature of a metallic sample. Both the direct and inverse problems are described and numerical results are given.

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1. Introduction

Parameters estimations of material samples and inverse problems are commonly used to derive physical models from experiments. To solve the inverse problem, one must first solve the direct problem, then solve the inverse problem for some coefficients and parameters. Solving such a problem therefore requires solving
an optimization (minimization) problem. These problems have many applications in scientific areas such as heat transfer, geophysics, electromagnetic, astronomy, electrocardiography, elastic waves and acoustics. Some important references on inverse problems can be found in [1, 2, 3, 4, 5]. Theory and application of ill-posed problems and their solutions can be found in the book edited by Bakushinsky and Goncharsky [6].

More mathematically oriented references on inverse problems include [7, 8, 9, 10, 11, 12, 13]. Tomography, particularly in medical imaging and seismology, is a very large field. Some general references on electrocardiography, are [14, 15, 16, 19, 20].

The recent development of theory, methods, and applications of one-dimensional inverse problems of dynamic elasticity can be found in [21, 22, 23, 24].

In [25] a good overview of many computational aspects of the subject, with applications and related areas that provide an entry point to some of the current research in this area. There is a wide research literature in the area of parameter estimation in [23].

Wave propagation problems in environmental applications such as seismic analysis, acoustic and electromagnetic scattering are described in [21] for both forward and inverse problems.

In our thermodynamics model we will study the problem of determining two thermal coefficients from a mixed set of data using a hybrid method. This is an inverse problem where the experimental data need to coincide the numerical solution of the model problem.

2. The model problem

Our model problem consists on a solid right circular cylinder of radius $R$ and height $L$ sitting on a table (see Fig. 1). The steady state temperature is denoted by $u(r, z)$ where we have introduced cylindrical coordinates. The bottom of the cylinder is at $z = 0$, and the top at $z = L$. In what follows, there will be no angular dependence. The origin for $r$ is at the center of the cylinder. A the top, a small circle of the cylinder of radius $\rho$ centred at $r = 0$ is indicated, the significance of which will soon be clear. The bottom is assumed to be insulated so there,

$$\frac{\partial u}{\partial z}(r, 0) = 0.$$

We take the ambient temperature to be zero. $k$ will denote the thermal
conductivity and $\sigma$ the heat transfer coefficient. We consider the model problem

$$\begin{align*}
\Delta u(r, z) &= \frac{\partial^2 u}{\partial r^2}(r, z) + \frac{1}{r} \frac{\partial u}{\partial r}(r, z) + \frac{\partial^2 u}{\partial z^2}(r, z) = 0, \\
0 < r < R, & \quad 0 < z < L, \\
\frac{\partial u}{\partial z}(r, 0) &= 0, \quad 0 \leq r \leq R, \\
\frac{\partial u}{\partial z}(R, z) &= -\gamma u(R, z), \quad \gamma = \frac{\sigma}{k}, \quad 0 \leq z \leq L, \\
u(r, L) &= f(r) = \begin{cases} S, & 0 \leq r < \rho \\ T, & \rho \leq r \leq R. \end{cases}
\end{align*}$$

The problem is solved by means of the standard technique of separation of variables:

$$u(r, z) = \varphi(r)\psi(z).$$

We find that $\varphi(r)$ satisfies the equation

$$\begin{align*}
\varphi''(r) + \frac{1}{r} \varphi'(r) + \lambda^2 \varphi(r) &= 0, \quad \lambda > 0, \\
\varphi'(R) &= -\gamma \varphi(R).
\end{align*}$$

The solution of (3) is well known as

$$\varphi(r) = J_0(\lambda r),$$
and $\lambda$ is obtained by solving for $\mu$

$$\mu J_1(\mu) = \gamma R J_0(\mu).$$

Here $J_0$ and $J_1$ are Bessel functions. We find

$$0 < \mu_1 < \mu_2 < \ldots < \mu_n < \ldots, \quad \mu_n \to \infty \text{ as } n \to \infty.$$

Now, $\lambda_n = \mu_n/R$. Then

$$\varphi_n(r) = J_0(\lambda_n r)$$

and

$$\psi_n(z) = \cosh(\lambda_n z).$$

The solution is

$$u(r, z) = \sum_{n=1}^{\infty} \alpha_n J_0(\lambda_n r) \cosh(\lambda_n z). \tag{4}$$

The $\alpha_n$ are determined from the boundary conditions

$$u(r, L) = f(r)$$

and for that we make use of the orthogonality relations

$$\int_0^R \varphi_m(r) \varphi_n(r) r dr = \begin{cases} 0, & m \neq n \\ \frac{R^2}{2} J_1^2(\lambda_n R) \left[ \frac{\gamma^2}{\lambda_n^2} + 1 \right], & m = n. \end{cases}$$

Let

$$\nu_n = \frac{R^2}{2} J_1^2(\lambda_n R) \left[ \frac{\gamma^2}{\lambda_n^2} + 1 \right].$$

Then

$$u(r, L) = f(r) = \sum_{n=1}^{\infty} \alpha_n \varphi_n(r) \cosh(\lambda_n L),$$

and

$$\alpha_n \nu_n \cosh(\lambda_n L) = \int_0^R f(r) \varphi_n(r) r dr.$$ 

So

$$\alpha_n = \frac{1}{\nu_n \cosh(\lambda_n L)} \left[ S \int_0^\rho J_0(\lambda_n r) r dr + T \int_\rho^R J_0(\lambda_n r) r dr \right]. \tag{5}$$
We now construct a problem that will illustrate the hybrid method. Choose values for \(k, \sigma, S, T\), all constants, positive, and take \(S > T\). This allows us to construct the solution \(u(r, z)\). Let
\[
g(r) = ku_z(r, L), \quad 0 \leq r \leq \rho.
\]
We assume at this point that \(u(r, z)\) is not known but that we have measured values, say \(h_1 = u(0, L/3)\) and \(h_2 = u(0, 2L/3)\).

The values \(T, g(r), h_1\) and \(h_2\) are assumed known and we wish to determine \(k\) and \(\sigma\). The ambient temperature is zero.

The problem as it stands is not amenable to an eigenvalue and eigenfunction approach because \(u(r, L)\) is not given for all \(r\) but instead a mixed set of data is given at \(z = L\). The first step is to solve for arbitrary \(k, \sigma\) Eqns. (1)-(2) to get values of \(u(r, z)\) along \(z = L\), that is \(u(r, L)\). We then use these values to define \(g(r)\) and solve numerically the model problem
\[
\begin{align*}
\Delta u(r, z) &= 0 \quad \text{in} \quad 0 \leq r < R, \quad 0 < z < L, \\
\frac{\partial u}{\partial r}(R, z) &= -\gamma u(R, z), \quad \gamma = \sigma/k, \quad 0 \leq z \leq L, \\
u_z(r, 0) &= 0, \quad 0 \leq r \leq R, \\
k u_z(r, L) &= g(r), \quad 0 \leq r < \rho, \\
u(r, L) &= T, \quad \rho \leq r \leq R,
\end{align*}
\]
(6)

at the grid points. Let us call these values \(v_{ij}\) at \(z = L, v_{i,j}\). This allows us to define a function \(f(r)\) on \(z = L\) and compare \(u(0, L/3)\) with \(h_1\) and \(u(0, 2L/3)\) with \(h_2\). The solution is done in an iteration process. We adjust the values for \(k\) and \(\sigma\) and repeat the process until that the computed values \(u(0, L/3)\) and \(u(0, 2L/3)\) coincide with the experimental values \(h_1\) and \(h_2\).

3. Numerical solution

To set up the finite difference method, we subdivide the interval \([0, L]\) into \(m\) intervals each of width \(l\) such that
\[
z_j = jl, \quad j = 1, 2, ..., m - 1, \quad l = L/m.
\]

Let \(h\) be the radius-step size with \(r_i = ih, \quad i = 1, 2, ..., n, \quad h = R/n\).

The differential equation (1) is discretized at \((r_i, z_j)\) using the central difference in the \(z\) and \(r\)-directions, giving
\[
\begin{align*}
\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{1}{ih} u_{i+1,j} - u_{i-1,j} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{2h} &= 0, \\
i &= 1, 2, ..., n, \quad j = 0, 1, ..., m.
\end{align*}
\]
Rewrite to get
\[ u_{i,j-1} + \lambda^2 u_{i-1,j} (1 - \frac{1}{2^2}) - 2u_{i,j} (\lambda^2 + 1) + \lambda^2 u_{i+1,j} (1 + \frac{1}{2^2}) + u_{i,j+1} = 0, \]
\[ i = 1, 2, ..., n, \quad j = 0, 1, ..., m, \]
where \( u_{i,j} = u(ih, jl) \) and \( \lambda = l/h. \) This is a five-point difference formula.

For the boundary conditions at \( r = R, \) and \( z = 0, \) we have
\[ z = 0, \quad u_{i,1} = u_{i,-1}, \]
\[ u_{i,m+1} = u_{i,m} + \frac{1}{k} g(ih), \quad ih < \rho, \]
\[ u_{i,m} = T, \quad ih > \rho, \]
\[ u_{n+1,j} = u_{n,j} (1 - \gamma h), \]
\[ u_{1,j} = u_{0,j}, \quad r = 0. \]

4. Numerical results

In this section we will show some numerical results that determine the values of \( k \) and \( \sigma \) and therefore \( u(x,t). \) For the infinite series in equations (4) we took 50 terms to guarantee the convergence of the series.

In this example we consider an unknown cylindrical metal material with height \( L = 5 \) cm, radius \( R = 4 \) cm. We choose \( S = 100, \) and \( T = 80. \) The results of the experiment are shown in Fig. 2 with the measured values of \( h_1 = u(0, L/3) = 49.013 \) and \( h_2 = u(0, 2L/3) = 62.122 \) which was done at Zayed University. We want to determine the values of \( k \) and \( \sigma \) of the sample. First we choose arbitrary values for \( k = 4.0 \) and \( \sigma = 2.0 \) that is \( \gamma = 0.5 \) and use 4 to get the values of the temperature at \( u(r, L) \) which will be used to define \( g(z). \) We then solve the model problem (6) numerically to obtain the values of the temperature at \( u(0, L/3) \) and \( u(0, 2L/3). \)

We compare these values with the experimental values \( h_1 \) and \( h_2. \) We continue the iteration process by using the new value for \( \sigma \) obtained from the condition
\[ \frac{\partial u}{\partial r}(R, z) = -\gamma u(R, z), \quad \gamma = \sigma/k, \quad 0 \leq z \leq L, \]
until \( h_1 \) and \( h_2 \) coincide with \( u(0, L/3) \) and \( u(0, 2L/3). \) Table 1 shows the values of \( \gamma \) obtained during the iteration process.
After 3 iterations we get convergence with $\gamma = 0.2529$, that gives $k = 0.6$ and $\sigma = 2.370$. These values are in good agreement with the actual parameters of the sample. Fig. 3 shows the numerical and experimental results along $r = 0$.

**Nomenclature**

$\sigma =$ Heat transfer coefficient (W-s/Kg-K).

$k =$ Thermal conductivity (W/cm-K).

**5. Conclusion**

This paper deals with the determination of two thermal coefficients of a metallic sample. Using a hybrid method and a set of experimental data, the parameters
were obtained by solving an inverse problem. The present study shows that the values of the thermal coefficients of the sample obtained from the numerical results were in good agreement with experimental data. The model problem was presented and numerical results were given.

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