Stability Analysis of Sum Rules
for pion Compton Scattering

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ABSTRACT
Recently it has been suggested [1] that QCD sum rule analysis can be extended from their usual setting, which includes two and three-point functions [2, 3, 4], to processes of Compton type at moderate values of the Mandelstam invariants $s$, $t$ and $u$. Sum rules for a specific combination of the two invariant amplitudes in pion Compton scattering corresponding to different helicities, $H_1 + H_2$, have been given in ref. [1], while the evaluation of relevant power corrections has been discussed in [5]. Comparison of sum rule predictions - in the local duality limit - with predictions from a modified perturbative calculation which includes Sudakov suppression for soft gluon exchange [5] has been made in [7]. This new approach to Compton scattering, complementary to the usual perturbative one, is interesting in order to investigate the transition from non-perturbative to perturbative QCD in these processes. It also suggests that the non-perturbative information on this type of reactions can be parameterized by the lowest dimensional vacuum condensates through the operator product expansion of four interpolating currents.

For the simplest Compton reaction $\pi^+ \gamma \rightarrow \pi^+ \gamma$ two steps are still missing in completing the above programme. They are: 1) a stability analysis of the sum rule, and 2) the derivation of an individual sum rule for each of the two invariant amplitudes, $H_1$ and $H_2$. While here our attention is focused on the first point, we reserve the discussion of the second point to a future work. A stability analysis of the sum rule proposed in [5] is crucial in order to give physical justification to previous works and to allow further extension of the new formalism to many other similar reactions. As stated in [7], the
angular dependence of Compton scattering, absent in form factors, allows us to compare perturbative QCD and sum rule predictions in a nontrivial way. Our aim in this letter is first to characterize the stability region of sum rules for pion Compton scattering, and, second, to compare the sum rule predictions with the perturbative QCD approach based on the modified factorization formula.

Before proceeding to study the stability of the complete sum rule, we recall the conclusions of [1, 5], and quote the relevant results below. A sum rule for the sum of the two helicities of pion Compton scattering has been derived by studying the following correlation function of local currents [1]

$$\Gamma_{\sigma\mu\nu\lambda}(p_1^2, p_2^2, s, t) = i \int d^4x \; d^4y \; d^4z \; \exp(-ip_1 \cdot x + ip_2 \cdot y - iq_1 \cdot z) \times \langle 0| T \{ \eta_\sigma(y) J_\mu(z) J_\nu(0) \eta_\lambda^\dagger(x) \} | 0 \rangle,$$  \hspace{1cm} (1)

where

$$J_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d, \quad \eta_\sigma = \bar{u} \gamma_5 \gamma_\sigma d$$  \hspace{1cm} (2)

are the electromagnetic and axial currents respectively of up and down quarks. These currents interpolate with the two invariant amplitudes of the scattering process. The two photons carry on-shell momenta $q_1$ and $q_2$, and are physically polarized. The momenta of the two pions are denoted as $p_1$ and $p_2$, with $s_1 = p_1^2$ and $s_2 = p_2^2$ the virtualities. We also define $s = (p_1 + q_1)^2$, $t = (p_2 - p_1)^2$ and $u = (p_2 - q_1)^2$ for the Mandelstam invariants, which obey the relation $s + t + u = s_1 + s_2$. The invariant amplitudes are obtained selecting a specific time ordering in eq. (1) and projecting the two axial currents
onto single-pion states. $H_1$ and $H_2$ are isolated from the matrix element

$$M_{\mu\nu} = i \int d^4z \, e^{-iq_1\cdot z} \langle p_2 | T(J_\mu(z)J_\nu(0)) | p_1 \rangle$$

by the expansion

$$M_{\mu\nu} = H_1(s,t)e_{\mu}^{(1)}e_{\nu}^{(1)} + H_2(s,t)e_{\mu}^{(2)}e_{\nu}^{(2)},$$

where $e^{(1)}$ and $e^{(2)}$ are helicity vectors defined in [1, 5].

A sum rule relates the timelike region of $s_1$ and $s_2$, where the resonant contribution to $M_{\mu\nu}$ is located, to the so called “deep Euclidean region”, where an operator product expansion (OPE) for the four-current correlator is made, through a dispersion relation. In the timelike region the spectral density appearing in the dispersion relation is commonly modeled by

$$\Delta_{\sigma\mu\nu\lambda} = f^2 \epsilon_{\lambda\mu\nu}(2\pi)^2 \delta(p_1^2)\delta(p_2^2)M_{\mu\nu} + \Delta_{\text{pert}} \left[ 1 - \theta(s_0 - p_1^2)\theta(s_0 - p_2^2) \right],$$

where the second term, the continuum contribution, which is nonvanishing only for $p_1^2, p_2^2 > s_0$, is chosen as the perturbative spectral density $\Delta_{\text{pert}}$, the leading term of OPE in the deep Euclidean region. Notice that the OPE of the spectral density includes, besides the perturbative part, non-perturbative power corrections proportional to the condensates of quarks and gluons, which are determined in close analogy with the canonical approach developed for the form factor case [3, 4].

A Borel transform then acts on both regions in order to enhance the resonant contribution respect to the continuum. Given the fact that the
dispersion relation for Compton scattering involves only a finite domain of the complex $p_i^2$ plane, a modified version of Borel transform, 

$$
\mathcal{B} = \int_C \frac{dp_1^2}{M_1^2} \int_C \frac{dp_2^2}{M_2^2} e^{-p_1^2/M_1^2} e^{-p_2^2/M_2^2} \left( 1 - e^{-\lambda^2 - p_1^2}/M_1^2 \right) \left( 1 - e^{-\lambda^2 - p_2^2}/M_2^2 \right)
$$

is introduced, with $C$ a contour of radius $\lambda^2$, which is kept finite in order to exclude the $u$-channel resonances from the phenomenological ansatz for the spectral density. In fact, $\lambda^2$ can vary from $s_0$ to $(s + t)/2$, though it was set approximately to the value $(s + t)/4$ in \cite{1, 5}. This modification introduces an extra unphysical parameter $\lambda^2$, in addition to the usual Borel mass $M^2$, and therefore, the stability of the sum rule has to be found in the two-dimensional $M^2-\lambda^2$ plane. The factor $1 - \exp(-\lambda^2 - p_i^2)/M^2)$ in (6), which is new compared to the standard Borel transform \cite{3, 4}, is to ensure that the transform is still regular when $C$ crosses the branch cut in the $p_i^2$ plane.

The resulting sum rule for $H = H_1 + H_2$ can be expanded asymptotically for the large invariant $Q^2$ \cite{3},

$$
Q^2 = \frac{1}{4} \left( s_1 + s_2 - t + \sqrt{(s_1 + s_2 - t)^2 - 4s_1s_2} \right),
$$

as

$$
f_{\pi}^2 H(s, t) \left( \frac{s(s + t)}{-t} \right) \left( 1 - e^{-\lambda/M^2} \right)^2 =
\left( \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho_{\text{pert}}^2 + \frac{\alpha_s}{\pi} \langle G^2 \rangle \int_0^{\lambda^2} ds_1 \int_0^{\lambda^2} ds_2 \rho_{\text{gluon}}^2 \right)
\times e^{-(s_1 + s_2)/M^2} \left( 1 - e^{-(\lambda^2 - s_1)/M^2} \right) \left( 1 - e^{-(\lambda^2 - s_2)/M^2} \right)
+C_{\text{quark}} \frac{\alpha_s}{\pi} \langle \bar{\psi} \psi \rangle^2 \left( 1 - e^{-\lambda^2/M^2} \right),
$$

(8)
where the perturbative, gluonic and quark contributions are given by, respectively,

\[
\begin{align*}
\rho_{\text{pert}} &= \frac{2560Q^{14}\tau_{\text{pert}}(s, Q^2, s_1, s_2)}{3\pi^2(s - 2Q^2)(4Q^4 - s_1s_2)^5(s_1s_2 - 2Q^2s)} , \\
\rho_{\text{gluon}} &= \frac{20480Q^{22}(s - 2Q^2)\tau_{\text{gluon}}(s, Q^2, s_1, s_2)}{27s(2Q^2 - s_1)^2(2Q^2 - s_2)^2(4Q^4 - s_1s_2)^5(2Q^2s - s_1s_2)(4Q^4 - 2Q^2s + s_1s_2)^2} , \\
C_{\text{quark}} &= \frac{16(8M^2s + 4s^2 + 2M^2t + 4st + t^2)}{9M^4t} ,
\end{align*}
\]

with

\[
\begin{align*}
\tau_{\text{pert}} &= (s - Q^2)^2(2Q^4s s_1 - Q^2s^2 s_1 + 2Q^4s s_2 - Q^2s^2 s_2) \\
&\quad -2Q^4s_1s_2 - 6Q^2s_1s_2s_2 + 3s^2s_1s_2) , \\
\tau_{\text{gluon}} &= -8Q^{12}s - 8Q^{10}s^2 + 68Q^8s^3 - 64Q^6s^4 + 16Q^4s^5 \\
&+8Q^{10}s s_1 + 8Q^8s^2 s_1 - 108Q^6s^3 s_1 + 104Q^4s^4 s_1 - 26Q^2s^5 s_1 \\
&+8Q^{10}s s_2 + 8Q^8s^2 s_2 - 108Q^6s^3 s_2 + 104Q^4s^4 s_2 - 26Q^2s^5 s_2 .
\end{align*}
\]

Note that the perturbative contribution comes only from the interval \((0, s_0)\), since the contribution from \((s_0, \lambda^2)\) is cancelled by that from the phenomenological side of the sum rule. The gluon and quark condensates, \(\langle G^2 \rangle\) and \(\langle (\bar{\psi}\psi)^2 \rangle\), take the values

\[
\begin{align*}
\frac{\alpha_s}{\pi}\langle G^2 \rangle &= 1.2 \times 10^{-2}\text{GeV}^4 , \\
\alpha_s\langle (\bar{\psi}\psi)^2 \rangle &= 1.8 \times 10^{-4}\text{GeV}^6 .
\end{align*}
\]

The approximation made in the calculation of the gluonic power correction, with the gluonic coefficient obtained by integrating its double disconti-

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nuity along the real interval \((0, \lambda^2)\) for each of \(s_1\) and \(s_2\), has been discussed in [5]. A similar spectral representation of this coefficient has been used in [4] in the investigation of the form factor sum rule. In this latter case the integration interval of such spectral representation is \((0, \infty)\), since the spectral density for the triangle diagram is regular for all \(s_1, s_2 > 0\) and \(t < 0\) (see refs. [4] and [5] for details).

Based on the formulas (7)-(11), we perform the stability analysis of eq. (8). The \(\lambda^2\) dependence of \(H\) in a wide range of the Borel mass \(M^2 = 4-8\) GeV\(^2\) at \(s = 20, |t| = 4\) and \(s_0 = 0.7\) GeV\(^2\) is shown in fig. 1. Obviously, all the curves located in the region marked by the vertical bars, in which the power corrections do not exceed 50\% of the perturbative contribution, increase rapidly with \(\lambda^2\). The power corrections always dominate for \(M^2\) below 4 GeV\(^2\). This result indicates that there is not a stable region for \(H\) when the radius \(\lambda^2\) of the Borel transform is varied. A similar behavior is observed for other choices of \(s, t\) and \(s_0\). As already mentioned above, such dependence is not present in the form factor case mainly because the variables \(s_i\) in the spectral representation for the coefficient of the gluonic power correction runs up to infinity [4].

We do not expect the strong \(\lambda\) dependence in the sum rule for two reasons: 1) \(\lambda\) is an unphysical parameter, and a physical quantity like the invariant amplitude \(H\) should be insensitive to it; 2) the \(\lambda\) dependence, introduced by the Borel transform, should cancel from both sides of the sum rule. Therefore, the sensitivity of our results to the radius of the Borel transform just implies
that the phenomenological model in (3) is too simple to maintain a stability of the sum rule in the $M^2-\lambda^2$ plane.

We propose here a modified phenomenological model in order to remove this spurious dependence:

$$\Delta_{\sigma_{\mu\nu}\lambda} = f_\pi^2 p_{1\lambda} p_{2\sigma} (2\pi)^2 \delta(p_1^2) \delta(p_2^2) M_{\mu\nu}$$

$$+ \Delta^{\text{OPE}} \left[1 - \theta(s_0 - p_1^2) \theta(s_0 - p_2^2)\right],$$

(12)

with the continuum contribution replaced by $\Delta^{\text{OPE}}$, which is the same as the full spectral expression on the OPE side of the sum rule. This modification makes sense, because the region with large virtualities $p_1^2, p_2^2 > s_0$ can be regarded as perturbative, and an OPE is allowed. With this choice we are requiring that the hadronic spectral density for the continuum from $s_0$ to $\lambda^2$ truncates not only the perturbative part (the lowest order contribution), but also the power corrections, of the OPE side of the sum rule. Since the contributions from the region $(s_0, \lambda^2)$ have been removed by the above cancellation, $s_i$’s never reach the upper bound $\lambda^2$, and the remaining $\lambda$ dependent factors $1 - \exp[-(\lambda^2 - s_i)/M^2]$ from the transform (11) can be dropped. Therefore, the overall dependence on the radius of the Borel transform disappear completely from the sum rule.

Using eq. (12), (8) is modified to

$$f_\pi^2 H(s, t) \left(\frac{s(s + t)}{-t}\right) =$$

$$\left(\int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho^{\text{pert}} + \frac{\alpha_s}{\pi} (G^2) \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho^{\text{gluon}} \right) e^{-(s_1 + s_2)/M^2}$$

$$+ \hat{C}^{\text{quark}} \frac{\pi \alpha_s}{\pi} \langle \bar{\psi} \psi \rangle^2,$$

(13)
with the modified quark contribution

\[
\hat{C}_{\text{quark}} = -\frac{16}{9} \left( 8M^4s^2 + 8M^4st + 8M^2s^2t + 8M^2st^2 + 4s^2t^2 + 4st^3 + t^4 \right) M^4t^3.
\] (14)

Note the change of the upper bound from \( \lambda^2 \) to \( s_0 \) in the integral for the gluonic power correction, which is due to the cancellation from the phenomenological side.

Before studying the new sum rule (13), we shall examine how the modified parametrization for the continuum affects the sum rule calculation of the pion form factor. This is a significant check for the process we aim to discuss, since it has been shown [1] that Compton scattering has a strong similarity to the corresponding form factor case at moderate \( s \) and \( t \) and at a fixed angle. The modified sum rule for pion form factor \( F_\pi \) can be derived easily based on [4]:

\[
\frac{f_\pi^2}{4} F_\pi(Q^2) = \left( \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho_{\pi}^{\text{pert}} + \frac{\alpha_s}{\pi} \langle G^2 \rangle \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho_{\pi}^{\text{gluon}} \right) e^{-(s_1+s_2)/M^2} + C_{\pi} \alpha_s \langle (\bar{\psi}\psi)^2 \rangle,
\] (15)

with

\[
\rho_{\pi}^{\text{pert}} = \frac{3Q^4}{16\pi^2 \delta^{3/2}} \left[ 3\delta(s_1 + s_2 + Q^2)(s_1 + s_2 + 2Q^2) - \delta^2 - 5Q^2(s_1 + s_2 + Q^2) \right],
\]

\[
\rho_{\pi}^{\text{gluon}} = \frac{1}{48M^4} \left[ -Q^2(s_1 + s_2) + (s_1 - s_2)^2 - 2\delta - 2Q^4 \delta^{3/2} \\
+ 4Q^2(s_1 + s_2 + Q^2) \frac{s_1 + s_2 + Q^2}{\delta} + \frac{s_1 + s_2 + Q^2}{M^2\delta^{1/2}} \right],
\]

\[
C_{\pi} = \frac{52}{81M^4} \left( 1 + 2Q^2 \frac{2}{13M^2} \right),
\]

\[
\delta = (s_1 + s_2 - t)^2 - 4s_1s_2.
\] (16)
Note that the expression for the quark power correction $C_{\pi}^{\text{quark}}$ based on the modified phenomenological model is the same as in [3, 4]. A Borel transform picks up only the residue of the pole $1/p_1^2 p_2^2$ in the quark power correction, and there is not such a pole term in the integral from $s_0$ to $\lambda^2$ on the phenomenological side of the sum rule. Therefore, the quark part in $\Delta^{\text{OPE}}$ does not contribute, when the Borel transform is applied. There is not cancellation from the phenomenological side for the quark power correction. The difference of the modified expression (14) in the case of pion Compton scattering from the corresponding one in (3) is due to the neglect of the suppressing factors mentioned above, which results in the change of the spectral density.

Results for $F_\pi$ obtained from the stability analysis of (15) are shown in fig. 2 with the best values of $s_0 = 0.7$ and $M^2 = 2$ GeV$^2$ substituted into the sum rule. It is evident that the behavior of $F_\pi$ in $Q^2$ is indeed in good agreement with experimental data [8] even after these amendments, and consistent with those derived in [3, 4], which are based on the standard phenomenological model (5).

Now we proceed to the stability analysis of (13) following a method similar to [3, 4]. Since the $\lambda$ dependence has been removed completely, we concentrate simply on the variation of $H$ with respect to $M^2$. The $M^2$ dependence of $H$ for $s_0 = 0.5 - 0.7$ GeV$^2$ at $s = 20$ and $|t| = 4$ GeV$^2$ is displayed in fig. 3. The region on the right-hand side of vertical bars is the one where the power corrections do not exceed 50% of the perturbative contribution. It is obvious that as $s_0 = 0.6$ GeV$^2$ there is the largest $M^2$ interval, in which
$H$ is approximately constant. Therefore, $s_0 = 0.6 \text{ GeV}^2$ is the best choice which makes both sides of the sum rule most coincident. This value of the duality interval is close to that given in the form factor case, and consistent with its conjectured value $0.7 \text{ GeV}^2$ in [7]. Different sets of $s$ and $t$ have been investigated. The best value of $s_0$ does not vary significantly, and $H$ is almost constant within the range $2 < M^2 < 6 \text{ GeV}^2$.

Results for $H$ at different scattering angles of the photon, $\theta$, $\sin(\theta/2) = -t/s$, with $s_0 = 0.6$ and $M^2 = 4 \text{ GeV}^2$ are exhibited in fig. 4, where $|H|$ denotes the magnitude of $H$. Basically, they show a similar dependence on angles and momentum transfers $|t|$ to those derived using local duality approximation [7], but with the magnitude lower by 20% only at small $|t|$. These predictions are compared to the perturbative predictions obtained from the modified factorization formula [7]. The transition to perturbative QCD at about $|t| = 4 \text{ GeV}^2$ and $-t/s = 0.5$ ($\theta = 40^\circ$), where the perturbative contributions begin to dominate, is observed. Sum rule results are always smaller for $-t/s = 0.6$ ($\theta = 50^\circ$), and always larger for $-t/s = 0.2$ ($\theta = 15^\circ$), than the perturbative results. This is also consistent with the conclusion in [7].

We have analyzed in more detail the sum rule which describes the behaviour of the sum of the two helicities of pion Compton scattering close to the resonant region, and its dependence on the two parameters, $M^2$ and $\lambda^2$, which characterize the modified Borel transform. We have seen that the strong dependence of the sum rule on the radius of the transform can be re-
moved consistently under the assumption that resonances from mass higher than the duality interval $s_0$ contribute equally both to the phenomenological side and to the OPE side. The results for the two helicities are found to be stable under the new phenomenological parametrization for the continuum in a wide variation of the Borel mass, and is characterized by a local duality interval which takes a value similar to the form factor case. A more detailed discussion of these issues on individual sum rules for the two helicities will be considered elsewhere [9].

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**Figure Captions**

**Fig. 1** Dependence of $H$ on $\lambda^2$ at $s = 20$, $|t| = 4$ and $s_0 = 0.7$ GeV$^2$ for $M^2 = 4$ (solid line), 6 (dashed line), and 8 GeV$^2$ (dotted line).

**Fig. 2** Dependence of $F_\pi$ on $Q^2$ derived from the modified phenomenological model (12) with $s_0 = 0.7$ and $M^2 = 2$ GeV$^2$ (solid line). Results from ref. [4] (dashed line) and from ref. [3] (dotted line), and experimental data (dots) are also shown.

**Fig. 3** Dependence of $H$ on $M^2$ at $s = 20$ and $|t| = 4$ GeV$^2$ for (a) $s_0 = 0.7$ GeV$^2$, (b) $s_0 = 0.6$ GeV$^2$, and (c) $s_0 = 0.5$ GeV$^2$.

**Fig. 4** Dependence of $|t||H|$ on $|t|$ derived from the full analysis of QCD sum rules with $s_0 = 0.6$ and $M^2 = 4$ GeV$^2$ (solid lines) for (a) $-t/s = 0.6$ ($\theta = 50^\circ$), (b) $-t/s = 0.5$ ($\theta = 40^\circ$), and (c) $-t/s = 0.2$ ($\theta = 15^\circ$). Corresponding results from the modified perturbative QCD calculation (dashed lines) are also shown.