Conserved matter superenergy currents for orthogonally transitive Abelian $G_2$ isometry groups

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Abstract

In a previous paper we showed that the electromagnetic superenergy tensor, the Chevreton tensor, gives rise to a conserved current when there is a hypersurface orthogonal Killing vector present. In addition, the current is proportional to the Killing vector. The aim of this paper is to extend this result to the case where we have a two-parameter Abelian isometry group that acts orthogonally transitive on non-null surfaces. It is shown that for four-dimensional Einstein–Maxwell theory with a source-free electromagnetic field, the corresponding superenergy currents lie in the orbits of the group and are conserved. A similar result is also shown to hold for the trace of the Chevreton tensor and for the Bach tensor, and also in Einstein–Klein–Gordon theory for the superenergy of the scalar field. This links up well with the fact that the Bel tensor has these properties and it gives further support to the possibility of constructing conserved mixed currents between the gravitational field and the matter fields.

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1. Introduction

In this paper we continue the investigation of conservation laws for the electromagnetic superenergy tensor, the Chevreton tensor, in Einstein–Maxwell spacetimes. In a previous paper [10], we showed that this tensor gives rise to a conserved current whenever there is a hypersurface orthogonal Killing vector present, i.e., if the Killing vector $\xi_a$ satisfies $\xi[a \nabla b \xi_c] = 0$, then

$$ H_{abcd} \xi^b \xi^c \xi^d = \omega \xi_a, \quad \nabla^a (H_{abcd} \xi^b \xi^c \xi^d) = 0, \quad (1) $$

where $H_{abcd}$ is the Chevreton tensor [9, 15],

$$ H_{abcd} = -\frac{1}{2} \left( \nabla_a F_{ce} \nabla_b F_d + \nabla_b F_{ce} \nabla_a F_d + \nabla_c F_{ae} \nabla_d F_b + \nabla_d F_{ae} \nabla_c F_b \right) $nabla_d F_{ae} \nabla_c F_b

$$ + \frac{1}{2} \left( g_{ab} \nabla_f F_{ce} \nabla^f F_d + g_{cd} \nabla_f F_{ae} \nabla^f F_b \right) $g_{ab} \nabla_f F_{ce} \nabla^f F_d

$$ + \frac{1}{2} \left( g_{ab} \nabla_e F_{cj} \nabla_d F_{bf} + g_{cd} \nabla_a F_{ej} \nabla_b F_{cf} \right) - \frac{1}{3} g_{ab} g_{cd} \nabla_g F_{ef} \nabla^g F_{de}. \quad (2) $$

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This result holds in four-dimensional Einstein–Maxwell spacetimes with a source-free electromagnetic field, \( F_{ab} \), that inherits the symmetry of the spacetime. A similar situation to when there is a hypersurface orthogonal Killing vector present occurs when there exists a two-parameter isometry group whose surfaces of transitivity are (locally) orthogonal to a family of two-surfaces. According to Frobenius’s theorem, the two Killing vectors generating the group, \( \xi_a \) and \( \eta_a \), then satisfy \( \xi_a \eta_b \nabla_c \xi_d = 0 = \xi_a \eta_b \nabla_c \eta_d \). We show in this paper, theorem 7, that when the two Killing vectors commute (i.e., the isometry group is Abelian) and form surfaces that are non-null, then the Chevreton tensor again gives rise to conserved currents,

\[
H_{abcd} \xi^a \xi^b \xi^c \xi^d = \omega_{1JK} \xi_a + \Omega_{1JK} \eta_a, \quad \nabla^a (H_{abcd} \xi^a \xi^b \xi^c \xi^d) = 0, \tag{3}
\]

where \( I, J, K = 1, 2 \) and \( \xi_i^1 = \xi_a \) and \( \xi_i^2 = \eta_a \). This result is interesting not only because it gives conserved quantities for the electromagnetic field, but also because it gives further support to the possibility of creating conserved currents between the electromagnetic field and the gravitational field at the superenergy level.

We also show that similar results hold for the trace of the Chevreton tensor and for the Bach tensor

\[
H_{ab} \xi^b = \omega_{1} \xi_a + \Omega_{1} \eta_a, \quad B_{ab} \xi^b = \omega' \xi_a + \Omega' \eta_a. \tag{4}
\]

The Bel–Robinson tensor [2, 3],

\[
T_{abcd} = C_{aefg} C_{b}^{e}C_{c}^{f} - \frac{1}{2} g_{ab} C_{efg} C_{c}^{e}C_{d}^{f} \nonumber
- \frac{1}{2} g_{cd} C_{aefg} C_{b}^{e}C_{c}^{f} + \frac{1}{8} g_{ab} g_{cd} C_{efg} C_{c}^{e}C_{d}^{f}, \tag{5}
\]

is a good candidate for representing gravitational energy since it satisfies the dominant property [4, 15] and is divergence-free in vacuum. When matter is present, however, neither the Bel–Robinson tensor, nor the Bel tensor,

\[
B_{abcd} = R_{aefg} R_{b}^{e}R_{c}^{f} - \frac{1}{2} g_{ab} R_{efg} R_{c}^{e}R_{d}^{f} \nonumber
- \frac{1}{2} g_{cd} R_{aefg} R_{b}^{e}R_{c}^{f} + \frac{1}{8} g_{ab} g_{cd} R_{efg} R_{c}^{e}R_{d}^{f}, \tag{6}
\]

are divergence-free in general. However, there are some cases when it is still possible to construct conserved currents for the gravitational field at the superenergy level. Lazkoz, Senovilla and Vera [11] have shown that the Bel tensor gives rise to independently conserved currents for general spacetimes when there is a hypersurface orthogonal Killing vector present or when two commuting Killing vectors that act orthogonally transitive on non-null surfaces are present. In the first case we have the current

\[
B_{abcd} \xi^b \xi^c \xi^d = \omega_{a}, \quad \nabla^a (B_{abcd} \xi^b \xi^c \xi^d) = 0, \tag{7}
\]

and in the second case the four currents

\[
B_{(abcd)} \xi^b \xi^c \xi^d = \omega_{ijk} \xi_a + \Omega_{ijk} \eta_a, \quad \nabla^a (B_{(abcd)} \xi^b \xi^c \xi^d) = 0. \tag{8}
\]

Also, Senovilla [15] has shown that for Einstein–Klein–Gordon theory, it is possible to construct a mixed conserved superenergy current between the gravitational field and the scalar field when there is a Killing vector present,

\[
\nabla^a ((B_{abcd} + S_{abcd}) \xi^b \xi^c \xi^d) = 0, \tag{9}
\]

where \( S_{abcd} \) is the superenergy of the scalar field. When the above isometries are present this breaks up into two separate conserved currents, and we show for completeness here and in [10] that the currents constructed from the superenergy of the scalar field also lie in the orbits of the isometry groups.
We hope that it will be possible to construct a similar conserved current between the
gravitational field and the electromagnetic field. Senovilla [15] has shown that this is possible
in the case of the propagation of discontinuities of the fields. For the general case it is not
known, but the results of this paper further support that this might be the case. Also, in the
spacetime we use as an example of our results, we do have mixed conserved currents.

In the proofs we have opted for expanding the tensors in a basis where the two Killing
vectors are taken as two of the basis vectors. It is also possible to take exterior products with
the surface element \[\xi^{a}\eta^{b}\] and using expressions like
\[2\xi^{a}\eta^{b}\nabla_{c}\xi_{d} = -\eta_{d}\xi^{a}\nabla_{b}\xi_{c} + \xi_{d}\eta^{a}\nabla_{b}\xi_{c},\]
but this approach seems to require quite a lot of extra effort.

2. Conventions and some results

We assume that our spacetime is a four-dimensional manifold equipped with a metric of
signature \(-2\). We define the Riemann tensor by
\[
(\nabla_{a}\nabla_{b} - \nabla_{b}\nabla_{a})v_{c} = -R_{abcd}v^{d}.
\]
(10)
The Einstein equations are
\[
R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = -T_{ab}.
\]
(11)
We will keep the cosmological constant \(\Lambda\) throughout the calculations. If \(\xi_{a}\) is a Killing vector,
then \(\nabla_{a}\xi_{b} = -\nabla_{b}\xi_{a}\) and [18].
\[
\nabla_{a}\nabla_{b}\xi_{c} = R_{bcad}\xi^{d}.
\]
(12)
We also note that the Lie derivative along a Killing vector commutes with the covariant
derivative [19]
\[
[\xi_{f}, \nabla_{a}]T^{b_{1}...b_{l}c_{1}...c_{j}} = 0.
\]
(13)
We assume that we have two commuting Killing vectors \(\xi^{a}\) and \(\eta^{a}\) that act orthogonally
transitive on non-null surfaces [16].
\[
[\xi, \eta] = \xi^{b}\nabla_{b}\eta_{a} - \eta^{b}\nabla_{b}\xi_{a} = 0,
\]
(14)
\[
\xi_{a}\eta_{b}\nabla_{c}\xi_{d} = \xi_{a}\eta_{b}\nabla_{c}\eta_{d} = 0,
\]
(15)
\[
\xi_{a}\eta_{b}\xi^{a}\eta^{b} \neq 0.
\]
(16)
If we take a basis consisting of \(\xi_{a}, \eta_{a}, s_{a}\) and \(t_{a}\), where \(s_{a}\) and \(t_{a}\) are orthogonal to \(\xi_{a}\) and \(\eta_{a}\),
we can write
\[
\nabla_{a}\xi_{b} = C_{1}\xi_{a}\eta_{b} + C_{2}s_{b} + C_{3}t_{b} + \eta_{a}(C_{4}s_{b} + C_{5}t_{b}) + C_{6}s_{a}t_{b}.
\]
(17)
Taking an exterior product with \(\xi_{c}\eta_{d}\) gives \(C_{6} = 0\) and by contracting with \(\xi^{a}\eta^{b}\) we get
\[
\xi^{a}\eta^{b}\nabla_{a}\xi_{b} = -\xi^{a}\eta^{b}\nabla_{b}\xi_{a} = -\xi^{a}\eta^{b}\nabla_{b}\eta_{a} = 0 = C_{1}\xi_{a}\eta_{b}\xi^{a}\eta^{b},
\]
(18)
so \(C_{1} = 0\). The structure is the same for \(\nabla_{a}\eta_{b}\) and we can write
\[
\nabla_{a}\xi_{b} = 2\xi_{a}s_{b} + 2\eta_{a}y_{b},
\]
(19)
\[
\nabla_{a}\eta_{b} = 2\xi_{a}v_{b} + 2\eta_{a}w_{b},
\]
(20)
where \(x_{a}, y_{a}, v_{a}\) and \(w_{a}\) are orthogonal to \(\xi_{a}\) and \(\eta_{a}\). We will often write this as
\[
\nabla_{a}\xi^{I}_{a} = 2\xi_{a}s^{I}_{a} + 2\eta_{a}y^{I}_{a},
\]
(21)
where \( I = 1, 2 \) and \( x^1_a = x_a, x^2_a = v_a, y^1_a = y_a \) and \( y^2_a = w_a \). Via Einstein’s equations the energy–momentum tensor satisfies \([8, 11]\)
\[
\xi_{\alpha I} \eta_J T_{IJ} \xi^d = \xi_{\alpha I} \eta_J T_{IJ} \eta^d = 0, \tag{22}
\]
which implies
\[
\xi^{Ib} T_{ab} = \alpha^I \xi_a + \beta^I \eta_a, \tag{23}
\]
where \( \alpha^I = \alpha, \alpha^2 = \gamma, \beta^1 = \beta \) and \( \beta^2 = \delta \). By taking the Lie derivatives of this equation with respect to \( \xi_a \) and \( \eta_a \), we have that
\[
\xi^{\alpha} \nabla_a \alpha^I = 0, \quad \eta^{\alpha} \nabla_a \alpha^I = 0, \\
\xi^{\alpha} \nabla_a \beta^I = 0, \quad \eta^{\alpha} \nabla_a \beta^I = 0. \tag{24}
\]
When there is more than one matter field present this, in general, only applies to the total energy–momentum tensor. Here, as well as throughout the text, proportionality factors like \( \alpha \) in \( \alpha \xi_a \) are generally non-constant scalar functions.

3. Einstein–Klein–Gordon theory

In this section, we show that in Einstein–Klein–Gordon theory the superenergy tensor of the scalar (Klein–Gordon) field gives rise to conserved currents for Killing vectors that generate an Abelian two-parameter group of isometries that act orthogonally transitive on non-null surfaces. It has previously been shown that the Bel tensor in combination with the superenergy tensor of the scalar field gives rise to conserved currents for Killing vectors \([15]\) and that for this kind of symmetry, or for hypersurface orthogonal Killing vectors, the Bel tensor gives rise to independently conserved currents that lie in the orbits of the group \([11]\). Hence, the superenergy currents for the scalar field are also independently conserved, and we show here and in \([10]\) for completeness that these currents also lie in the orbits of the group.

The energy–momentum tensor in Einstein–Klein–Gordon theory is given by
\[
T_{ab} = -\nabla_a \phi \nabla_b \phi + \frac{1}{2} g_{ab} (\nabla_c \phi \nabla_d \phi + m^2 \phi^2), \tag{25}
\]
where the scalar field, \( \phi \), satisfies the Klein–Gordon equation, \( \nabla^c \nabla_c \phi = m^2 \phi \). The superenergy tensor of the scalar field is given by \([15]\)
\[
S_{abcd} = \nabla_a \nabla_c \phi \nabla_b \nabla_d \phi + \frac{1}{2} g_{ab} (\nabla_c \nabla^e \phi \nabla_d \phi + m^2 \nabla_c \phi \nabla_d \phi) \\
- g_{cd} (\nabla_a \nabla^e \phi \nabla_b \nabla_d \phi + m^2 \nabla_a \phi \nabla_b \phi) \\
+ \frac{1}{2} g_{ab} g_{cd} (\nabla_e \nabla_j \phi \nabla^e \nabla^j \phi + 2m^2 \nabla_e \phi \nabla^e \phi + m^4 \phi^2). \tag{26}
\]
It has the following symmetries, \( S_{abcd} = S_{(abc)d} = S_{cdab} \). We can construct the following currents:
\[
S_{abcd} \xi^{Ib} \xi^{Jc} \xi^{Kd} = (\nabla_a \nabla_c \phi \nabla_b \nabla_d \phi + \nabla_a \nabla_j \phi \nabla_b \nabla_d \phi) \xi^{Ib} \xi^{Jc} \xi^{Kd} \\
- \frac{1}{2} g_{cd} (\nabla_a \nabla^e \phi \nabla_b \nabla_d \phi + m^2 \nabla_a \phi \nabla_b \phi) \xi^{Ib} + \omega \xi_a + \Omega \eta_a. \tag{27}
\]
where the scalar functions \( \omega \) and \( \Omega \) are used to collect proportionality factors of \( \xi_a \) and \( \eta_a \). If the scalar field is massive, \( m \neq 0 \), it will have a vanishing Lie derivative \([15]\), \( \xi^{Ia} \nabla_a \phi = 0 \), so by using the Leibniz rule and expanding with \((21)\) we have that
\[
(\nabla_a \nabla^e \phi \nabla_b \nabla_d \phi + m^2 \nabla_a \phi \nabla_b \phi) \xi^{Ib} = \nabla_a \nabla^e \phi \nabla_b \nabla_d \phi \xi^{Ib} = \nabla_a \nabla^e \phi \nabla_d \phi \xi^{Ib} \\
= - \nabla_a \nabla^e \phi \nabla_b \phi \xi^{Ib} = - \xi^{Ib} \nabla_a \nabla_e \phi \phi \nabla_b \phi - \eta^I \nabla_a \nabla_e \phi y^{Ib} \nabla_b \phi \\
= \nabla_a \xi^I \nabla_e \phi y^{Ib} \nabla_b \phi + \eta^I \nabla_a \nabla_e \phi y^{Ib} \nabla_b \phi = \omega \xi_a + \Omega \eta_a. \tag{28}
\]
Conserved matter superenergy currents for orthogonally transitive Abelian

For the other type of term present in (27) we similarly have that

\[ \nabla_a \nabla_c \phi \nabla_b \nabla_d \phi \xi^{ib} \xi^{jc} \xi^{kd} = - \nabla_a \xi^{jc} \nabla_c \phi \nabla_b \nabla_d \phi \xi^{ib} \xi^{kd} = \omega \xi_a + \Omega \eta_a. \]  

Hence, for a massive scalar field the superenergy currents will lie in the orbits of the group,

\[ S_{abcd} \xi^{ib} \xi^{jc} \xi^{kd} = \omega \xi_a + \Omega \eta_a. \]

In the massless case, \( m = 0 \), the scalar field satisfies \( \xi^i \phi = \xi^i \nabla \phi = C \xi^i \), where \( C \) is a constant [15]. Here we also have from (23) and (25) that \( \nabla \phi = \alpha \xi_a + \beta \eta_a \). The calculations are similar to the massive case and the conclusion is the same.

To prove that the currents are divergence-free, we note that the Lie derivative commutes with covariant derivatives for Killing vectors, so the superenergy tensor has a vanishing Lie derivative. Since the Killing vectors commute, the Lie derivative of the currents therefore vanishes. We have proven

**Theorem 1.** For Einstein–Klein–Gordon spacetimes, possibly with a cosmological constant \( \Lambda \), which admit an Abelian two-parameter isometry group that acts orthogonally transitive on non-null surfaces, the superenergy tensor of the scalar field gives rise to conserved currents that lie in the orbits of the group, 

\[ S_{abcd} \xi^{ib} \xi^{jc} \xi^{kd} = \omega \xi_a + \Omega \eta_a, \]

\[ \nabla^a (S_{abcd} \xi^{ib} \xi^{jc} \xi^{kd}) = 0, \]  

where, in general, the proportionality factors \( \omega \) and \( \Omega \) will be non-constant.

Note, we have assumed a four-dimensional spacetime here, but the expansion of (21) is similar in the \( n \)-dimensional case and this result thus holds in \( n \) dimensions as well.

4. Einstein–Maxwell theory

In this section we will show that if a four-dimensional Einstein–Maxwell spacetime, possibly with a cosmological constant \( \Lambda \), admits an Abelian two-parameter group of isometries that act orthogonally transitive on non-null surfaces, then the corresponding Chevreton currents constructed from the Killing vectors of the group will lie in the orbits of the group and will be conserved. Furthermore, it is shown that this also holds for the trace of the Chevreton tensor and for the Bach tensor. The electromagnetic field is assumed to be source-free. For a null electromagnetic field we will assume that it inherits the symmetries of the spacetime.

The electromagnetic field is described by the Maxwell tensor, \( F_{ab} = -F_{ba} \), which in source-free regions satisfies

\[ \nabla^a F_{ab} = 0, \quad \nabla[a F_{bc]} = 0. \]  

The energy–momentum tensor is given by

\[ T_{ab} = -F_{ac} F_b^c + \frac{1}{4} g_{ab} F_{cd} F^{cd}. \]

The Ricci scalar, \( R \), satisfies \( R = 4 \Lambda \), where \( \Lambda \) is the cosmological constant. From (23), we have that

\[ \xi^{ib} F_{ac} F_{b^c} = \alpha \xi_a + \beta \eta_a. \]  

Generally, the Lie derivative of the electromagnetic field in four-dimensional Einstein–Maxwell theory satisfies for any Killing vector \( \xi^a \) [13, 17]

\[ \xi^a \xi_{ab} = \xi^c \nabla_c F_{ab} + F_{ca} \nabla_b \xi^c + F_{bc} \nabla_a \xi^c = \Psi F_{ab}, \]

where \( \Psi \) is a constant for non-null fields and satisfies \( l_a \nabla_b \Psi = 0 \) for null fields, where \( l_a \) is the repeated principal null direction of the field.  

1 It was erroneously stated in [10] that \( \Psi \) is always constant, though this did not interfere with the calculations since the electromagnetic field was assumed to inherit the symmetry of the spacetime.
\( \Psi \) is zero, then the electromagnetic field is said to inherit the symmetry of the spacetime. It has been shown that for an Abelian two-parameter group of isometries that act orthogonally transitive on non-null surfaces, a non-null electromagnetic field inherits those symmetries of the spacetime \([13]\). In the case of a null electromagnetic field we will assume that it inherits the symmetries. Hence, \( \xi_{\ell} F_{ab} = 0 \), or
\[
\xi_{\ell} \nabla_{\ell} F_{ab} = - F_{eb} \nabla_{\ell} \xi_{a} - F_{ae} \nabla_{\ell} \xi_{b}.
\] 
(35)
The basic superenergy tensor of the electromagnetic field is given by \([15]\)
\[
E_{abcd} = -\nabla_{a} F_{ce} \nabla_{b} F_{de} - \nabla_{b} F_{ce} \nabla_{a} F_{de} + \frac{1}{2} g_{cd} \nabla_{a} F_{ef} \nabla_{b} F^{ef} - \frac{1}{4} g_{ab} g_{cd} \nabla_{e} F_{fg} \nabla^{e} F^{fg}.
\] 
(36)
The Chevreton tensor is defined as
\[
H_{abcd} = \frac{1}{2} (E_{abcd} + E_{cdab}),
\] or
\[
H_{abcd} = -\frac{1}{2} \left( \nabla_{a} F_{ce} \nabla_{b} F_{de} + \nabla_{b} F_{ce} \nabla_{a} F_{de} + \nabla_{c} F_{ae} \nabla_{d} F_{b} \varepsilon + \nabla_{d} F_{ae} \nabla_{c} F_{b} \varepsilon \right)
+ \frac{1}{2} \left( g_{ab} \nabla_{j} F_{ce} \nabla^{j} F_{de} + g_{cd} \nabla_{j} F_{ae} \nabla^{j} F_{b} \varepsilon \right)
+ \frac{1}{4} \left( g_{ab} g_{cd} \nabla_{e} F_{fg} \nabla^{e} F^{fg} \right) + \omega \xi_{\alpha} + \Omega_{\ell} \eta_{\alpha}.
\] 
(37)
This tensor is more interesting physically than the basic superenergy tensor, because it gives unique currents and a unique divergence and because it shares the symmetries of the Bel tensor. We will now examine the currents that arise when this tensor is contracted with the Killing vectors, \( \xi_{a} \) and \( \eta_{a} \), of our two-parameter group. Since the Chevreton tensor is symmetric there are only four different currents, and by interchange of \( \xi_{a} \) and \( \eta_{a} \), we only need to consider currents of the form
\[
H_{abcd} \xi_{b} \xi_{c} \xi_{d} = -\nabla_{a} F_{ce} \nabla_{b} F_{de} \xi_{b} \xi_{c} \xi_{d} - \nabla_{c} F_{ae} \nabla_{d} F_{b} \varepsilon \xi_{b} \xi_{c} \xi_{d}
+ \frac{1}{2} \xi_{d} \xi_{c} \xi_{b} \nabla_{a} F_{ef} \nabla_{b} F_{ef} \xi_{c} \xi_{d}
+ \frac{1}{4} g_{ab} g_{cd} \nabla_{e} F_{fg} \nabla^{e} F^{fg}.
\] 
(38)
Here, and later, \( \omega \) and \( \Omega \) are again used to collect the proportionality factors of \( \xi_{a} \) and \( \eta_{a} \). We want to show that the remaining terms also lie in the orbits of the group. The proof is divided into three lemmas. We treat the second and third terms separately and then the first and fourth together. The proofs involve some quite lengthy calculations.

**Lemma 2.** Under the assumptions of theorem 7 (see below),
\[
\xi_{\alpha} \eta_{\ell} \nabla^{\ell} F_{a\ell} \nabla_{d} F_{b} \varepsilon \xi_{lb} = 0.
\] 
(39)
The proof is rather long and given in the appendix.

Before we present the next two lemmas, we note that, as in our previous paper \([10]\), this lemma can be applied to the trace of the Chevreton tensor, which is given by \([6]\)
\[
H_{ab} = \frac{1}{4} (E_{abcd} + E_{cdab}),
\] or
\[
H_{ab} = -\frac{1}{2} \left( \nabla_{a} F_{ce} \nabla_{b} F_{de} + \nabla_{b} F_{ce} \nabla_{a} F_{de} + \nabla_{c} F_{ae} \nabla_{d} F_{b} \varepsilon + \nabla_{d} F_{ae} \nabla_{c} F_{b} \varepsilon \right)
+ \frac{1}{2} \left( g_{ab} \nabla_{j} F_{ce} \nabla^{j} F_{de} + g_{cd} \nabla_{j} F_{ae} \nabla^{j} F_{b} \varepsilon \right)
+ \frac{1}{4} \left( g_{ab} g_{cd} \nabla_{e} F_{fg} \nabla^{e} F^{fg} \right) + \omega \xi_{\alpha} + \Omega_{\ell} \eta_{\alpha}.
\] 
(40)
Hence

**Theorem 3.** Assume that we have four-dimensional Einstein–Maxwell theory, possibly with a cosmological constant \( \Lambda \), with a source-free electromagnetic field that inherits the symmetry of the spacetime. If \( \xi_{a} \) and \( \eta_{a} \) are two commuting Killing vectors that act orthogonally transitive on non-null surfaces, then the currents \( H_{ab} \xi_{b} \) and \( H_{ab} \eta_{b} \), where \( H_{ab} \) is the trace of the Chevreton tensor, lie in the orbits of the group,
\[
H_{ab} \xi_{b} = \omega_{1} \xi_{a} + \Omega_{1} \eta_{a}, \quad H_{ab} \eta_{b} = \omega_{2} \xi_{a} + \Omega_{2} \eta_{a},
\] 
(41)
where the proportionality factors \( \omega_{i} \) and \( \Omega_{i} \), in general are non-constant.
These currents are trivially conserved, since the trace of the Chevreton tensor is divergence-free [6]. Note that for a non-null electromagnetic field we automatically have inherited symmetry. It was shown in [5] that the trace of the Chevreton tensor is related to the Bach tensor

$$B_{ab} = \nabla^c \nabla^d C_{acbd} - \frac{1}{2} R^{cd} C_{acbd}$$

by

$$B_{ab} = 2H_{ab} + \frac{1}{2} \Lambda T_{ab}.$$  

Hence, the Bach currents constructed from the Killing vectors $\xi_a$ and $\eta_a$ will also lie in the orbits of the group. This also applies to the case with a hypersurface orthogonal Killing vector [10].

**Corollary 4.** Assume that we have four-dimensional Einstein–Maxwell theory, possibly with a cosmological constant $\Lambda$, with a source-free electromagnetic field that inherits the symmetry of the spacetime. If $\xi_a$ is a hypersurface orthogonal Killing vector, then the Bach current $B_{ab}\xi^b$ is proportional to $\xi_a$,

$$B_{ab}\xi^b = \omega \xi_a.$$  

If $\xi_a$ and $\eta_a$ are two commuting Killing vectors that act orthogonally transitive on non-null surfaces, then the Bach currents $B_{ab}\xi^b$ and $B_{ab}\eta^b$ lie in the orbits of the group

$$B_{ab}\xi^b = \omega_3 \xi_a + \Omega_3 \eta_a, \quad B_{ab}\eta^b = \omega_4 \xi_a + \Omega_4 \eta_a.$$  

In general, the proportionality factors $\omega$, $\omega_i$ and $\Omega_i$ are non-constant.

Again, in the second case, for a non-null electromagnetic field, we automatically have inherited symmetry for those two Killing vectors generating the group.

We return now to the Chevreton currents (38).

**Lemma 5.** Under the assumptions of theorem 7,

$$\xi_{\lbrack f \eta_{g]} \nabla_{c]} F_{a e} \nabla_{d} F_{b f} \xi^b \xi^c \xi^d = 0.$$  

The proof is given in the appendix.

**Lemma 6.** Under the assumptions of theorem 7,

$$-\xi_{\lbrack g \eta_{h]} \nabla_{a]} F_{e b} \nabla_{a} F_{c f} \xi^h \xi^c \xi^f + \frac{1}{2} \xi_{\lbrack e \eta_{f]} \nabla_{a]} F_{c f} \nabla_{a} F_{e b} \xi^e \xi^f = 0.$$  

The proof is given in the appendix.

From (38) together with lemmas 2, 5 and 6 we have that $H_{abcd} \xi^b \xi^c \xi^d = \omega_{ijk} \xi_a + \Omega_{ijk} \eta_a$. The Lie derivative commutes with the covariant derivative for Killing vectors (13), so $\xi_{\xi \xi} \nabla_{d} F_{bc} = 0$ and we have that $\xi_{\xi \xi} H_{abcd} = 0$. Since the Killing vectors commute, we have that

$$\xi_{\xi \xi}(H_{abcd} \xi^b \xi^c \xi^d) = 0.$$  

Hence, the proportionality factors $\omega$ and $\Omega$ satisfy $\xi^a \nabla_a \omega = 0 = \xi^a \nabla_a \Omega$. We have proven

**Theorem 7.** Assume that we have four-dimensional Einstein–Maxwell theory, possibly with a cosmological constant, $\Lambda$, with a source-free electromagnetic field that inherits the symmetry of the spacetime. If $\xi^a$ and $\eta^a$ generate a two-parameter Abelian isometry group that acts
orthogonally transitive on non-null surfaces, then the Chevreton currents constructed from these vectors lie in the orbits of the group and are divergence-free,

\[ H_{abcd} ξ^b ξ^c ξ^d = \omega^5 ξ^a + \Omega_1^5 η^a, \]

\[ \nabla^a (H_{abcd} ξ^b ξ^c ξ^d) = 0, \]

\[ H_{abcd} ξ^b ξ^c η^d = \omega^6 ξ^a + \Omega_1^6 η^a, \]

\[ \nabla^a (H_{abcd} ξ^b ξ^c η^d) = 0, \]

\[ H_{abcd} η^b η^c η^d = \omega^8 \xi^a + \Omega_1^8 \eta^a, \]

\[ \nabla^a (H_{abcd} η^b η^c η^d) = 0. \]

(49)

In general, the proportionality factors \( \omega_i \) and \( \Omega_1^i \) will be non-constant.

Note that for a non-null electromagnetic field, the symmetry is automatically inherited.

We would like to stress here that, even though one can construct plenty of conserved currents by taking combinations of Killing vectors with functions that are constant on the orbits of the Killing vectors, the currents that appear here are physically interesting because they are constructed from a contraction of the Killing vectors and the Chevreton tensor; the fact that they happen to be of the mentioned form is a result valid only in particular situations, and do not deprive them of their physical meaning. In contrast, this implies that the particular functions \( \omega_I \) and \( \Omega_1^I \) contain information at the superenergy level of the electromagnetic field. The result that the Chevreton currents lie in the orbits of the group may also be interesting from a geometric point of view.

In four-dimensional Einstein–Maxwell theory, the Bel tensor (6) can be decomposed as [7, 10]

\[ B_{abcd} = T_{abcd} + T_{ab} T_{cd} + \frac{1}{4} R^2 g_{ab} g_{cd}, \]

(50)

where \( T_{abcd} \) is the Bel–Robinson tensor (5) and \( T_{ab} \) is the electromagnetic energy–momentum tensor (32). As was shown in [11], the Bel currents \( B_{abcd} ξ^b ξ^c ξ^d \) lie in the orbits of the group. We note that when we contract with the Killing vectors, both the second and the third terms above will lie in the orbits of the group. Both terms also have vanishing Lie derivative, so both terms will give rise to independently conserved currents. From this we see that this also applies to the Bel–Robinson tensor, i.e., it will also give rise to conserved currents that lie in the orbits of the group.

5. Example

Theorem 7 applies to a wide class of stationary axisymmetric spacetimes, notably the Kerr–Newman solution. The expressions for the Chevreton currents in the Kerr–Newman solution are quite large, so we will not treat it here, but one could follow a procedure similar to the treatment of the Reissner–Nordström solution in section 3 of [11]. Instead, we give another interesting example—the algebraically general Einstein–Maxwell spacetime found by Barnes [1]. The metric is given by

\[ ds^2 = r \sin(\sqrt{3} \theta) dx^2 - r \sin(\sqrt{3} \theta) dy^2 + r 2 \cos(\sqrt{3} \theta) dx \, dy - r^2 d\theta^2 - dr^2, \]

(51)

and it admits a three-parameter group of isometries generated by the Killing vectors

\[ \xi_{1a} = \delta_x^a, \quad \xi_{2a} = \delta_y^a, \quad \xi_{3a} = y \sqrt{2} \delta_{xa} - x \sqrt{2} \delta_{ya} + \delta_{ha}. \]

(52)

Here \( \xi_1 \) and \( \xi_2 \) form an orthogonally transitive Abelian \( G_2 \) subgroup. None of the three Killing vectors are hypersurface orthogonal. The electromagnetic field is given by

\[ F_{ab} = 2 \cos(p) \delta_{xa} \wedge \delta_{yb} + 2 \sin(p) \delta_{ra} \wedge \delta_{rb}, \]

(53)

where \( p \) is an arbitrary constant determining the complexion of the field.
The four Chevreton and Bel currents constructed from $\xi_1$ and $\xi_2$ are of course independently conserved. The Chevreton currents are very simple here,

\[
H_{abcd} \xi_b^1 \xi_c^1 \xi_d^1 = \frac{3}{2r^2} \xi_a^1, \\
H_{abcd} \xi_b^2 \xi_c^2 \xi_d^2 = \frac{1}{2r^2} \xi_a^2.
\]  

(54)

None of the six Chevreton currents involving the third Killing vector $\xi_3$ are divergence-free and the same holds for the corresponding Bel currents. However, a combination of the symmetrized Bel tensor with the Chevreton tensor gives conserved currents for all possible combinations of the Killing vectors

\[
\nabla^a \left( (B_{abcd}) + \frac{1}{3} H_{abcd} \xi_b^1 \xi_c^2 \xi_d^1 \right) = 0.
\]  

(55)

This combination is the one expected for the construction of mixed superenergy currents, though the factor $\frac{1}{3}$ is in disagreement with the calculations of the propagation of discontinuities between the gravitational and electromagnetic field [15]. It would be interesting to find other examples where the Bel and Chevreton currents are not independently conserved to see if this factor is to be generally expected.

For this spacetime, all Bel–Robinson currents are independently conserved, and it is only the $T_{ab} T_{cd}$ part of the Bel tensor that contributes to the mixed current. For example, the current $v_a = (B_{abcd}) + \frac{1}{3} H_{abcd} \xi_b^1 \xi_c^2 \xi_d^1$, which equals

\[
v_x = -\frac{\sqrt{3}(3x \cos(2\sqrt{3}\theta) - 3y \sin(2\sqrt{3}\theta) + 11x)}{48r^2}, \\
v_y = \frac{\sqrt{3}(3y \cos(2\sqrt{3}\theta) + 3x \sin(2\sqrt{3}\theta) + 11y)}{48r^2}, \\
v_r = \frac{\sqrt{3} \sin(\sqrt{3}\theta)}{6r^2}, \\
v_\theta = \frac{\cos(\sqrt{3}\theta)}{3r^2},
\]

(56)

is conserved. It is also interesting to note that this current is not proportional to a combination of the Killing vectors. This actually holds for all the currents here that involve the third Killing vector $\xi_3$.

6. Conclusion

We have shown that if a four-dimensional Einstein–Maxwell spacetime admits an Abelian two-parameter isometry group that acts orthogonally transitive on non-null surfaces and the electromagnetic field is source-free and inherits the symmetries of the spacetime, then the Chevreton currents generated from the isometry group lie in the orbits of the group and are conserved. Hence, by Gauss’s theorem, these currents give rise to conserved quantities.

Since the Bel currents have similar properties under this isometry group, this gives further support to the possibility of constructing mixed conserved currents that could govern the interchange of superenergy between the electromagnetic field and the gravitational field.

In the proof of lemma 2 we needed to make use of an identity which holds only in four dimensions, so our result seems to be restricted to this dimension. The results for the Bel currents are $n$ dimensional, so possible mixed conservation laws may be restricted to four dimensions.
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Appendix A. Proofs of lemmas 2, 5 and 6

Here we give the proofs of the lemmas in section 4. We assume that we have a four-dimensional source-free Einstein–Maxwell spacetime admitting two commutative Killing vectors that act orthogonally transitive on non-null surfaces. The electromagnetic field is assumed to inherit the symmetries of the spacetime for these two Killing vectors. This is a restriction only for a null electromagnetic field.

Proof of lemma 2

Proof. We rewrite \( \nabla^d F_{ac} \nabla_d F_{b}^{\ c} \xi^{1b} \) by applying the d’Alembertian \( \square = \nabla^d \nabla_d \) to the energy–momentum tensor (32)

\[
\nabla^d F_{ac} \nabla_d F_{b}^{\ c} \xi^{1b} = -\frac{1}{2} \xi^{1b} \left( \square T_{ab} + F_{b}^{\ c} \square F_{ac} + F_{a}^{\ c} \square F_{bc} \right) + \omega \xi^{1b}. \quad (A.1)
\]

By using the four-dimensional Maxwell wave equation, \( \square F_{ab} = 2 C_{ab}^{\ cd} F_{dc} - \frac{1}{3} R F_{ab} \) [14], we get

\[
-\frac{1}{2} \xi^{1b} \square T_{ab} - \xi^{1b} \left( C_{ac}^{\ e} F_{b}^{\ e} F_{ed} + C_{bc}^{\ e} F_{a}^{\ e} F_{ed} \right) + \frac{1}{3} R \xi^{1b} F_{ac} F_{b}^{\ c} + \omega \xi^{1b}. \quad (A.2)
\]

By (33), the next to last term equals \( \omega \xi_a + \Omega \eta_a \). The two terms involving the Weyl tensor are rewritten using the four-dimensional identity \[10, 12\]

\[
C_{[ab|cd]}^{\ [ef]} = 0. \quad (A.3)
\]

We are then left with

\[
-\frac{1}{2} \xi^{1b} \square T_{ab} - C_{ebad} F_{ac}^{\ d} \xi^{1b} + \omega \xi_a + \Omega \eta_a. \quad (A.4)
\]

Substituting the Weyl tensor for the Riemann tensor and simplifying with (23) and Einstein’s equations (11) yields

\[
-\frac{1}{2} \xi^{1b} \square T_{ab} - R_{ebad} F_{ac}^{\ d} \xi^{1b} + \omega \xi_a + \Omega \eta_a. \quad (A.5)
\]

By (12) we then have that

\[
\nabla^d F_{ac} \nabla_d F_{b}^{\ c} \xi^{1b} = -\frac{1}{2} \xi^{1b} \left( \square T_{ab} - F_{ac}^{\ d} \nabla_d \nabla_a \xi^{1b} \right) + \omega \xi_a + \Omega \eta_a. \quad (A.6)
\]

We can rewrite the first term, \( A \), with the Leibniz rule as

\[
\xi^{1b} \square T_{ab} = \square (T_{ab} \xi^{1b}) - T_{ab} \square \xi^{1b} - 2 \nabla_f T_{ab} \nabla^f \xi^{1b}. \quad (A.7)
\]

For the first term on the right-hand side, use (23), expand, and use (21) and (24). For the second term, use (12) and (23). We are then left with

\[
\xi^{1b} \square T_{ab} = -2 \nabla_f T_{ab} \nabla^f \xi^{1b} + \omega \xi_a + \Omega \eta_a. \quad (A.8)
\]

Expanding this with (21) and rewriting with the Lie derivative of the energy–momentum tensor, \( \xi^l \nabla_l T_{ab} = \xi^l \nabla_l T_{ab} + T_{ac} \nabla_c \xi^l + T_{ac} \nabla_c \xi^l = 0 \), and the Leibniz rule, we get

\[
2 x^{1b} T_{fb} \nabla_b \xi^f + 2 y^{1b} T_{fb} \nabla_a \eta^f + 2 x^{1f} \nabla_f (T_{ab} \xi^b) + 2 y^{1f} \nabla_f (T_{ab} \eta^b) + \omega \xi_a + \Omega \eta_a. \quad (A.9)
\]

Using (23) and expanding again with (21) we are left with only \( \omega \xi_a + \Omega \eta_a \).
The second term of (A.6), \( B \), is expanded with (21) and using (33) we get
\[
F^e \Phi^d \nabla_e \nabla_d \xi^f = -(\alpha^\prime \xi^e + \beta^\prime \eta^e) \nabla_e x^f - (\gamma^\prime \xi^e + \delta^\prime \eta^e) \nabla_e y^f + \omega \xi_a + \Omega \eta_a.
\] (A.10)

Expanding \( \xi^e \nabla_e \xi^f = 0 \) gives \( \xi^e x^f = 0 = \xi^e y^f \), or \( \xi^f \nabla_a x^f = \alpha \xi_a + \Omega \eta_a \) and \( \xi^f \nabla_a y^f = \omega \xi_a \). Hence,
\[
F^e \Phi^d \nabla_e \nabla_d \xi^f = \alpha \xi_a + \Omega \eta_a.
\] (A.11)

So, taken together,
\[
\xi^e \eta^f \nabla_d F_{ae} \nabla_d F_{bf} \xi^f = 0.
\] (A.12)

□

For the proofs of lemma 5 and 6 we will need to divide into two different cases depending on whether the electromagnetic field is invertible or skew invertible. For a non-null electromagnetic field we can write [13]
\[
F_{ab} = \tau_{ab} \cos \alpha + \tau^*_{ab} \sin \alpha,
\] (A.13)
where \( \tau_{ab} \) is the extremal field and \( \alpha \) is the complexion scalar. The extremal field here satisfies one of the following three sets of conditions [13]:

1. \( \tau_{ab} \xi^a \eta^b = 0 \), \( \tau^*_{ab} \xi^a \eta^b = 0 \), \( \tau_{ab} \xi^a \eta^b = 0 \), \( \tau^*_{ab} \xi^a \eta^b = 0 \).
2. \( \tau_{ab} \xi^a \eta^b \neq 0 \), \( \tau^*_{ab} \xi^a \eta^b = 0 \), \( \tau_{ab} \eta^a = 0 \), \( \tau^*_{ab} \eta^a = 0 \).
3. \( \tau^*_{ab} \xi^a \eta^b \neq 0 \), \( \tau_{ab} \xi^a \eta^b = 0 \), \( \tau_{ab} \eta^a = 0 \), \( \tau^*_{ab} \eta^a = 0 \).

In the first case the electromagnetic field satisfies \( F_{ab} \xi^a \eta^b = 0 = \tau_{ab} \xi^a \eta^b \) and is said to be skew invertible. It can then be written as
\[
F_{ab} = 2 \xi_{[a} s_{b]} + 2 \eta_{[a} t_{b]},
\] (A.17)
where \( s_a \) and \( t_a \) are orthogonal to \( \xi_a \) and \( \eta_a \). Carter [8] showed that the two scalars \( F_{ab} \xi^a \eta^b \) and \( \tau_{ab} \xi^a \eta^b \) are constants and if we, for example, have a spacetime with a symmetry axis where one of the Killing vectors vanishes, the constants vanish everywhere and the electromagnetic field will be skew invertible. In the two other cases, the electromagnetic field is invertible and can be written as
\[
F_{ab} = 2 \lambda \xi_{[a} \eta_{b]} + 2 \eta_{[a} \xi_{b]},
\] (A.18)

where again \( s_a \) and \( t_a \) are orthogonal to \( \xi_a \) and \( \eta_a \).

For a null electromagnetic field with the principal null direction \( l_a \) we can write
\[
F_{ab} = 2 l_{[a} \xi_{b]}, \quad \tau_{ab} = 2 l_{[a} \eta_{b]},
\] (A.19)
where \( \xi_a \) and \( \eta_a \) are space-like vectors satisfying \( A^a l_a = B^a l_a = A^a \eta_a = 0 \). By expanding (33) we see that either \( l_a = \omega \xi_a + \Omega \eta_a \) or \( \xi^a l_a = 0 = \eta^a l_a \), in which either case implies \( F_{ab} \xi^a \eta^b = 0 = \tau_{ab} \xi^a \eta^b \) and the electromagnetic field is therefore skew invertible.

In the following two proofs we will only show the calculations for the skew invertible case. The invertible case works similarly, noting that \( F_{ab} \xi^b = \lambda \xi_a + \mu \eta_a \), where \( \xi^a \nabla_a \lambda^f = 0 \) and \( \xi^a \nabla_a \mu^f = 0 \).
Proof of lemma 5

Proof. We start by rewriting with the Lie derivative (35),
\[ \nabla_c F_{ab} \nabla_d F_{ef} \xi^b \xi^c \xi^d = \xi^b \nabla_a \xi^c \left( F_{ce} F_{bd} \nabla \xi^d \xi^d + F_{ce} F_{bd} \nabla \xi^e \right) + \xi^b F_{ac} \left( F_e \nabla \xi^c \xi^c \nabla \xi^d + F_{ed} \nabla \xi^e \right). \tag{A.20} \]
Expanding the first term of the right-hand side with (21) and using (33), we have
\[ \xi^b \nabla_a \xi^c F_{ce} F_{bd} \nabla \xi^d \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.21} \]
The second term of (A.20) is expanded by (21), and if the electromagnetic field is skew invertible, all terms like \( F_{ab} \xi^a \xi^b \) vanish. Hence,
\[ \xi^b \nabla_a \xi^c F_{ce} F_{bd} \nabla \xi^d \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.22} \]
For the third term, using (A.17) and (18) we have
\[ \xi^b F_{ac} F_{bd} \nabla \xi^c \xi^c \nabla \xi^d \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.23} \]
For the last term of (A.20) we expand using (21) and (A.17) to get
\[ \xi^b F_{ac} F_{bd} \nabla \xi^c \xi^c \nabla \xi^d \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.24} \]
Hence, taken together, we have that
\[ \xi^b \nabla \xi^c F_{ce} F_{bd} \nabla \xi^d \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.25} \]
The proof is similar for the invertible electromagnetic field. \( \square \)

Proof of lemma 6

Proof. Taking two covariant derivatives of the energy–momentum tensor (32) yields
\[ - \nabla_a F_{ce} \nabla_b F_{ef} \xi^b \xi^e \xi^d + \frac{1}{4} \xi^b \xi^e \xi^d F_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^e \xi^d F_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^e \xi^d F_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^e \xi^d F_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^e \xi^d F_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d \right) \] (A.26)
We rewrite term \( A \) in terms of a covariant derivative of the Lie derivative of the energy–
momentum tensor, \( \nabla_a \xi^b \nabla \xi^c T_{cd} = 0 \),
\[ \nabla_a \xi^b \nabla \xi^c T_{cd} = - \nabla_a \xi^b \nabla \xi^c T_{cd} \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^c \xi^d T_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^c \xi^d T_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d - \frac{1}{4} \xi^b \xi^c \xi^d T_{ce} \nabla \xi^c \xi^c \nabla \xi^d \xi^d . \tag{A.27} \]
By expanding with (21), the Lie derivative of energy–momentum tensor, (23), and (18),
\[ \nabla_a \xi^b \nabla \xi^c T_{cd} \xi^d \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.28} \]
Similarly, by using the Leibniz rule, expanding with (21) and using (23) and (18),
\[ \nabla_a T_{bc} \nabla \xi^b \xi^c \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.29} \]
Finally, by expanding with (21) twice and using (23), the Leibniz rule, and (18),
\[ T_{bc} \nabla \xi^b \xi^c \xi^d = \omega \xi^b + \Omega \eta_a. \tag{A.30} \]
Term \( B \), rewritten by taking a covariant derivative of the Lie derivative of the electromagnetic field (35) equals
\[ F_{ce} \nabla_a \nabla \xi^c \xi^b \xi^e \xi^d = - F_c \nabla_a F_{be} \nabla \xi^a \xi^b \xi^e \xi^d - F_c \nabla_a F_{bd} \nabla \xi^a \xi^b \xi^e \xi^d - F_c \nabla_a F_{de} \nabla \xi^a \xi^b \xi^e \xi^d - F_c F_{be} \nabla \xi^a \xi^b \xi^e \xi^d . \tag{A.31} \]
Conserved matter superenergy currents for orthogonally transitive Abelian 4967

The first term is expanded with (21) and if the electromagnetic field is skew invertible, we use (A.17) to yield

$$-F_{a}^{\epsilon} \nabla_{d} F_{\beta d} \nabla_{d} \xi^{b} \xi^{c} \xi^{J} \xi^{J} = \omega_{\xi a} + \Omega_{\eta a}.$$  \hspace{1cm} (A.32)

The second term is similarly expanded with (21) and (A.17) and then the Leibniz rule is used to give us

$$-F_{a}^{\epsilon} \nabla_{d} F_{\beta d} \nabla_{d} \xi^{b} \xi^{c} \xi^{J} \xi^{J} = \omega_{\xi a} + \Omega_{\eta a}.$$  \hspace{1cm} (A.33)

The third term is expanded with (21) to yield

$$-F_{a}^{\epsilon} \nabla_{d} F_{d e} \nabla_{d} \xi^{b} \xi^{c} \xi^{J} \xi^{J} = x_{a}^{c} F_{e}^{\epsilon} \nabla_{d} F_{d e} \xi^{c} \xi^{J} \xi^{J} + y_{a}^{c} F_{e}^{\epsilon} \nabla_{d} F_{d e} \xi^{c} \xi^{J} \xi^{J} + \omega_{\xi a} + \Omega_{\eta a}$$

$$= \frac{1}{2} x_{a}^{c} \nabla_{d} \left(F_{e}^{\epsilon} F_{d e} \right) \xi^{c} \xi^{J} \xi^{J} + \frac{1}{2} y_{a}^{c} \nabla_{d} \left(F_{e}^{\epsilon} F_{d e} \right) \xi^{c} \xi^{J} \xi^{J} + \omega_{\xi a} + \Omega_{\eta a},$$  \hspace{1cm} (A.34)

where, by (33), (24) and (18) we have

$$\xi^{b} \nabla_{b} \left(F_{e}^{\epsilon} F_{d e} \right) \xi^{c} \xi^{J} \xi^{J} = \xi^{b} \nabla_{b} \left(F_{e}^{\epsilon} F_{d e} \xi^{c} \right) \xi^{J} \xi^{J} - F_{e}^{\epsilon} F_{d e} \xi^{b} \nabla_{b} \xi^{c} \xi^{J} \xi^{J}$$

$$= \xi^{b} \xi^{J} \xi^{J} \nabla_{b} \left( \alpha^{J} \xi_{d} + \beta^{J} \eta_{d} \right) - \left( \alpha^{J} \xi_{d} + \beta^{J} \eta_{d} \right) \xi^{b} \nabla_{b} \xi^{J} \xi^{J} = 0,$$  \hspace{1cm} (A.35)

and likewise for the other term. Hence,

$$-F_{a}^{\epsilon} \nabla_{d} \xi^{b} \nabla_{d} \xi^{c} \xi^{J} \xi^{J} = \omega_{\xi a} + \Omega_{\eta a}.$$  \hspace{1cm} (A.36)

For the last term we use (33), expand with (21) and the Leibniz rule to get

$$-F_{a}^{\epsilon} \nabla_{d} \xi^{b} \nabla_{d} \xi^{c} \xi^{J} \xi^{J} = \omega_{\xi a} + \Omega_{\eta a}.$$  \hspace{1cm} (A.37)

Term C, rewritten by taking a covariant derivative of Lie derivative of the electromagnetic field, equals

$$F_{e f} \nabla_{a} \nabla_{b} F_{e f} \xi^{b} \xi^{J} \xi^{J} = 2 F_{e f} \nabla_{a} F_{f b} \nabla_{b} \xi^{b} \xi^{J} \xi^{J} - F_{e f} \nabla_{a} \xi^{b} \nabla_{b} F_{e f}.$$  \hspace{1cm} (A.38)

The first term is expanded with (21) and (A.17) to yield

$$F_{e f} \nabla_{a} \nabla_{b} F_{e f} \xi^{b} \xi^{J} \xi^{J} = \omega_{\xi a} + \Omega_{\eta a}.$$  \hspace{1cm} (A.39)

The second term is expanded with (21) and rewritten with the Lie derivative (35),

$$F_{e f} \nabla_{a} \xi^{b} \nabla_{b} F_{e f} = -x_{a}^{J} F_{e f} \xi^{b} \nabla_{b} F_{e f} - y_{a}^{J} F_{e f} \eta^{b} \nabla_{b} F_{e f} + \omega_{\xi a} + \Omega_{\eta a} = \omega_{\xi a} + \Omega_{\eta a}.$$  \hspace{1cm} (A.40)

Hence, taken together, we have that

$$-\xi_{a}^{J} \eta_{b} \nabla_{a} F_{e f} \xi^{b} \xi^{J} \xi^{J} \xi^{J} + \frac{1}{2} \xi_{a}^{J} \xi^{J} \xi^{J} \xi^{J} \nabla_{b} \eta_{a} F_{e f} \xi^{b} \xi^{J} \xi^{J} = 0.$$  \hspace{1cm} (A.41)

The case with an invertible electromagnetic field works similarly.

□

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