The dissipation distance for a 2D single crystal with two symmetric slip systems

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Abstract
We solve a model problem from single crystal plasticity. We consider 4 slip systems in the plane with orthogonal slip-directions and equal slip rates, forward as well as backwards. We compute the associated dissipation distance by solving an optimal control problem. It turns out that from a computational point of view computing the distance is inexpensive. We put special emphasis on visualization of the metric spheres and the associated length-minimizing curves. As a byproduct we also solve a related problem, optimal path planning for a car driving forwards and backwards with limited turning radius in the hyperbolic plane. This is a hyperbolic version of the Reeds-Shepp-Car-Problem first discussed in [17].

1 Introduction
In this paper we provide the solution to an optimization problem which has various interpretations. Although we will put special emphasis on the one mentioned in the title, the underlying mathematical problem does not require any knowledge from continuum mechanics and/or finite plasticity. Therefore we start with a naive formulation as an optimal factorization problem in the group $\text{SL}(2, \mathbb{R})$ of invertible 2 by 2-matrices with determinant 1.
A factorization problem

Let $\mathfrak{sl}(2)$ denote the set of 2 by 2-matrices with zero trace, and let

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

So $P$ and $Q$ are generators for shearings along the coordinate axes in $\mathbb{R}^2$. Then it is well-known that every $g \in \text{SL}(2, \mathbb{R})$ may be written as a product

$$g = \exp(t_1 A_1) \cdots \exp(t_k A_k) \quad \text{with} \quad k \in \mathbb{N}, \; t_k \in \mathbb{R}, \; \text{and} \; A_k \in \{P, Q\}.$$ 

We want to find factorization(s) of a given $g$ such that $\sum_i |t_i|$ is minimal. Therefore we define the factorization cost $\mathcal{T}(g)$ as

$$\mathcal{T}(g) = \inf \left\{ \sum_i |t_i| : g = \exp(t_1 A_1) \cdots \exp(t_k A_k), \; k \in \mathbb{N}, \; t_k \in \mathbb{R}, \; A_k \in \{P, Q\} \right\}.$$ 

The factorization cost $\mathcal{T}(g)$ can be interpreted as the distance of $g$ from the identity matrix. It can also be used to measure distances in the group $\text{SL}(2)$:

$$\hat{D}(g_0, g_1) \overset{\text{def}}{=} \mathcal{T}(g_0^{-1} g_1), \quad \text{or} \quad \check{D}(g_0, g_1) \overset{\text{def}}{=} \mathcal{T}(g_0 g_1^{-1}).$$ 

It turns out that $\hat{D}$, $\check{D}$ are metrics on $\text{SL}(2)$, by construction $\hat{D}$ is left-invariant while $\check{D}$ is right-invariant:

$$\hat{D}(g_0, g_1) = \hat{D}(gg_0, gg_1), \quad \check{D}(g_0, g_1) = \check{D}(g_0 g, g_1 g) \quad \text{for all} \quad g, g_0, g_1 \in \text{SL}(2).$$ 

In the sequel we will solve the problem of computing $\mathcal{T}(g)$ through an associated optimal control problem. Our technique is kind of standard in control theory in the sense that we use the Pontrjagin Maximum Principle (PMP) as a necessary condition for optimality plus some adhoc comparison arguments. We will also point out how the (PMP) relates to the yield surface and flow rule used in the plasticity literature.

Sneak preview

In order to give the reader an idea of the final outcome (and the computational complexity) we state a few consequences of our final results.

**Theorem 1.1** Every $g \in \text{SL}(2)$ has an optimal factorization of the form

$$g = \exp(t_1 A_1) \cdots \exp(t_6 A_6), \quad \text{with} \quad A_k \in \{P, Q, P + Q\}, \; t_k \in \mathbb{R}.$$ 

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So no more than 6 factors are needed, but it is necessary to allow factors of the form \( \exp(\pm t(P + Q)) \), too. Otherwise \( T(g) \) will be an infimum for some matrices \( g \in \text{SL}(2) \), in particular for \( g = \exp(t(P + Q)) \) with \( t \neq 0 \).

As a consequence, finding an optimal factorization is reduced to a finite problem. We will actually obtain the following, much more detailed information:

**Theorem 1.2** There exists a sufficient family consisting of 64 maps, i.e., there exist functions \( f_1, \ldots, f_{64} : \mathbb{R}^3 \rightarrow \text{SL}(2) \) with the following property: For every \( g \in \text{SL}(2) \) there exist \( k \in \{1, \ldots, 64\} \) and \( r, s, t \geq 0 \) such that \( f_k(r, s, t) \) provides an optimal factorization of \( g \).

If one is really interested in computing an optimal factorization explicitly one can exploit the symmetry of the problem and reduce the number of maps that have to be inverted to 13 (rather than 64). For efficient computation of the function \( T \) one can even reduce this to 12 maps.

To give a rough idea of the computational complexity we note that except for one map, nothing worse than solving quadratic equations is required. In this worst case the challenge consists of solving a cubic equation \( p(x) = y \), and this needs to be done only over an \( x \)-interval where the underlying cubic polynomial \( p \) is strictly increasing, convex, and, \( p'(x) \) is bounded away from 0.

Finally, we will show that for any other pair of rank-1 matrices \( S^1, S^2 \in \mathfrak{sl}(2) \) the solution of the associated factorization problem for \( A_k \in \{\pm S^1, \pm S^2\} \) can be obtained from \( T(g) \) in the following, very simple way:

**Theorem 1.3** Let \( S^1, S^2 \in \mathfrak{sl}(2) \) with \( \det(S^1) = \det(S^2) = 0 \), \( [S^1, S^2] \neq 0 \). Let \( T_S \) denote the factorization cost for \( \{\pm S^1, \pm S^2\} \). Then there exist \( \lambda > 0 \) and an automorphism \( \sigma : \text{SL}(2) \rightarrow \text{SL}(2) \) such that \( T_S(g) = \lambda T(\sigma(g)) \).

We will also show how \( \lambda > 0 \) and \( \sigma \in \text{Aut}(\text{SL}(2)) \) are obtained, given \( S^1, S^2 \). Thus we have determined the dissipation distance for any 2-slip system with symmetric dissipation functional.

**A reader’s guide**

This paper serves several purposes, therefore a few remarks concerning these seem to be in order.

The main purpose is to illustrate the application of optimal control techniques and Lie group methods to finite plasticity. So partly this paper is intended as a tutorial for non-specialists in optimal control on Lie groups. Therefore we will discuss everything in great detail and provide rigorous proofs. In this spirit this report is a successor of Sussmann’s and Tang’s
paper [18] on the Reeds-Shepp-Car-Problem. As our factorization problem is related to the Reeds-Shepp-Car-Problem in the hyperbolic plane, our arguments and results will bear some strong resemblance with those in [18]. Therefore we would like to stress that in this paper we put special emphasis on how to exploit the Lie group structure of SL(2). The latter is instrumental in reducing the complexity and streamlining the discussion. Moreover, it is indispensable if one’s aim is to treat similar problems in SL(2) and, eventually, in SL(3).

As the solution of the hyperbolic Reeds-Shepp-Car-Problem requires only little extra effort, we will provide it in an appendix. Although the result resembles that for the euclidean case, some aspects are different. For the geometer these are properties that distinguish hyperbolic from euclidean geometry. The interpretation as a path planning problem in the hyperbolic plane also provides a good visualization tool. It is noteworthy to mention that even if one does not care about hyperbolic geometry, one can benefit from it because some of the adhoc arguments suddenly have a simple interpretation—they might seem perfectly obscure and unmotivated, otherwise.

Visualization of the metric spheres (i.e., level sets of the factorization cost $T$) is another issue we deal with. Since the group SL(2) is three-dimensional, everything can be visualized in $\mathbb{R}^3$, but how? We will use a parametrization coming from a polar decomposition, first proposed by Hilgert and Hofmann in [4]. As a set, SL(2) is identified with $\mathbb{R}^2 \times [-\pi, \pi) \subseteq \mathbb{R}^3$, and $\mathbb{R}^3$ is identified with the simply connected Lie group with Lie algebra $\mathfrak{sl}(2)$. An advantage of this parametrization is that it immediately allows to recognize the symmetry inherent to the problem. A disadvantage is that the group operation is more complicated than matrix multiplication. The purpose of the first appendix is to collect information about this parametrization which is scattered around in the literature. This information is not necessary to understand and interpret the pictures of the metric spheres, but it is indispensable for generating them.

The remainder of the paper is organized as follows:

2. From finite plasticity to Lie groups
   Brief outline how plasticity leads to consider metrics on Lie groups.

3. The underlying optimal control problem

4. Symmetries and isometries

5. The structure of extremals
   Discussion of the (PMP), description of yield surface and flow rule.

6. A sufficient family for SL(2)
   Summary and short discussion of how to find best factorizations.
7. Comparison arguments
   Rigorous proofs for the sufficiency of the family described in Section 6.

8. Conclusion
   Brief outlook on future work and how to treat similar problems.

Appendix A: Parametrizing the simply connected group $\widetilde{SL}(2, \mathbb{R})$
   All information necessary to generate the graphics.

Appendix B: The hyperbolic Reeds-Shepp-Car
   Missing arguments and comparison with the results in [18].

Appendix C: More details for $\widetilde{SL}(2, \mathbb{R})$
   Additional information clarifying some of the arguments given in Section 7.

Notation. As we will have to write products of exponentials repeatedly, we need a shorthand notation. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and exponential function $\exp : \mathfrak{g} \to G$. Then we define

$$M(X_1, \ldots, X_k) := \exp(X_1) \cdots \exp(X_k), \quad k \in \mathbb{N}, \ X_1, \ldots, X_k \in \mathfrak{g}.$$ 

Thus $M(\cdot)$ is a map from $\bigcup_{k \in \mathbb{N}} \mathfrak{g}^k$ to $G$. The map $M(\cdot)$ depends, of course, on the group $G$, so one should write $M_G(X_1, \ldots)$. But except for a few situations in the appendix it will always be clear from the context in which group we are working, so we omit the subscript $G$ most of the time.

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2 From finite plasticity to dissipation distances on Lie groups

The idea to use left-invariant metrics on Lie groups in the modelling of elastoplastic material behavior is due to Mielke, cf. [9]. For a detailed overview of this approach we refer to [8]. Let us quickly outline some of the main ideas of this approach.

A global formulation of elastoplasticity

Consider a body $\Omega \subseteq \mathbb{R}^d$ that undergoes a deformation $\varphi : \Omega \to \mathbb{R}^d$. Let $F = D\varphi$ denote the deformation gradient. Inelastic material behavior is
described by an internal state $z$ from some set $Z$. The whole material model is based on two scalar constitutive functions, the elastic potential $\hat{\psi}$ and the dissipation potential $\hat{\Delta}$. These give rise to an elastic storage energy and a dissipation functional. Considering the evolution $(\varphi(t), z(t))$ under the influence of some time-varying external forces, the total elastic and potential energy (or Gibb’s energy) at time $t$ is

$$E(t, \varphi, z) = \int_{\Omega} \hat{\psi}(x, D\varphi, z) \, dx - \langle \ell(t), \varphi \rangle,$$

the second term corresponding to the work by external forces. The dissipation $\hat{\Delta}$ is supposed to depend only on the evolution of the internal state $z(t)$, i.e., $\hat{\Delta} = \hat{\Delta}(x, z, \dot{z})$. One defines the dissipation distance $\hat{D}$ as

$$\hat{D}(x, z_0, z_1) = \inf \left\{ \int_0^1 \hat{\Delta}(x, z(s), \dot{z}(s)) \, ds \mid z(\cdot) \in C^1([0, 1], Z), z(0) = z_0, z(1) = z_1 \right\}.$$

Integrating over $\Omega$ one defines $D(z_0, z_1) = \int_{\Omega} \hat{D}(x, z_0, z_1) \, dx$. Finally the total dissipation along a path $z(t)$ is defined as

$$\text{Diss}(z; [t_1, t_2]) = \sup \left\{ \sum_{j=1}^n D(z(\tau_j), z(\tau_{j-1})) \mid t_1 = \tau_0 < \cdots < \tau_n = t_2 \right\}.$$

With these functionals one obtains a notion of solution processes without making any differentiability assumptions. A process $(\varphi(t), z(t))$ is called a solution process over $[0, T]$ if it satisfies the following two conditions:

**Stability:** $E(t, \varphi, z) \leq E(t, \bar{\varphi}, \bar{z}) + D(z, \bar{z})$ for all $t \in [0, T]$ and all comparison states $(\bar{\varphi}, \bar{z})$.

**Energy inequality:**

$$E(t_1, \varphi(t_1), z(t_1)) + \text{Diss}(z; [t_1, t_2]) \leq E(t_2, \varphi(t_2), z(t_2)) - \int_{t_1}^{t_2} \langle \ell(s), \varphi(s) \rangle \, ds.$$

This formulation does not involve any derivatives, neither of $\hat{\psi}, \hat{\Delta}$ nor of $D\varphi, z$. As is shown in [8] this formulation is consistent with classical elastoplastic flow rules if the solution is sufficiently smooth. A particular advantage of this global formulation is that it allows to derive incremental time-stepping algorithms which are robust.
Multiplicative elastoplasticity: constitutive laws

So far we outlined the general approach without making any assumptions on the internal state space \( Z \). Multiplicative elastoplasticity uses the splitting \( D\varphi = F = F_{\text{el}}F_{\text{pl}} \) and considers \( F_{\text{pl}} \) as an internal variable while the elastic potential \( \hat{\psi} \) is supposed to depend only on \( F_{\text{el}} = FF_{\text{pl}}^{-1} \). Actually \( z = F_{\text{pl}}^{-1} \) is used as internal state, and the internal state space \( Z \) is a connected Lie subgroup, say \( G \), of \( \text{GL}(d) \). Typically, \( G = Z = \text{SL}(d) \), but other groups may be considered, too. The following constitutive laws are postulated:

1. **(Sy1) Objectivity:** (frame indifference) \( \hat{\psi}(x, RF, z) = \hat{\psi}(x, F, z) \) for all \( R \in \text{SO}(d) \);

2. **(Sy2) Plastic indifference:** \( \hat{\psi}(x, Fg^{-1}, gz) = \hat{\psi}(x, F, z) \), \( \hat{\Delta}(x, gz, g\dot{z}) = \hat{\Delta}(x, z, \dot{z}) \) for all \( g \in G \);

3. **(Sy3) Rate independence:** \( \hat{\Delta}(x, z, \alpha \dot{z}) = \alpha \hat{\Delta}(x, z, \dot{z}) \) for \( \alpha \geq 0 \);

Material symmetries may be captured, for example, in a group \( S \subseteq \text{O}(d) \cap G \). Following the notation in \( \text{[8]} \) we postulate this as constitutive law, too:

4. **(Sy4) Material symmetry:** \( \hat{\psi}(x, F, z\gamma) = \hat{\psi}(x, F, z) \), \( \hat{\Delta}(x, z\gamma, \dot{z}\gamma) = \hat{\Delta}(x, z, \dot{z}) \) for all \( \gamma \in S \).

Property (Sy2) implies that the dissipation distance \( \hat{D} \) defined in the previous subsection is invariant under left-multiplication with elements from \( G \), hence

\[
\hat{D}(z_0, z_1) = \hat{D}(1, z_0^{-1}z_1) =: \hat{D}(z_0^{-1}z_1).
\]

These metrics are the objects we want to study. Here we dropped the material point \( x \) for sake of simplicity. In the sequel we will never consider dependency on \( x \). This does not necessarily mean that our considerations are limited to homogeneous media. There are suitable formulations where all considerations are first limited to a fixed material point \( x \), and the final result is obtained by integration over \( \Omega \), cf. \( \text{[8]} \).

**Dissipation distances on Lie groups and time optimal control problems**

From now on we will assume that the internal state space is a connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \). Therefore we slightly change the notation. Henceforth, we write \( g \in G \) (instead of \( z \in Z \)). Our next goal is to discuss the consequences of the constitutive laws (Sy2) and (Sy3). By (Sy2) the
distance function $\tilde{D} : G \times G \to [0, \infty]$ is left-invariant. For the dissipation potential $\tilde{\Delta}$ this means that $\tilde{\Delta}(g, \dot{g}) = \tilde{\Delta}(g^{-1}\dot{g})$ for some $\tilde{\Delta} : \mathfrak{g} \to [0, \infty]$. Therefore the definition of $\tilde{D}$ still contains redundancy. We can reparametrize curves by their $\tilde{\Delta}$-length.

Indeed, assume (Sy3) and $\tilde{\Delta}(X) \geq 0$ for all $X \neq 0$. Now suppose that $g \in C^1([0, 1], G)$ is given. Let $L(t) = \int_0^t \tilde{\Delta}(g^{-1}(t)\dot{g}(t)) \, dt$ and set $L_1 = L(1)$. If $\dot{g}(t) \neq 0$ in $[0, 1]$, then $L'(t) > 0$, and $L$ has a differentiable inverse $L^{-1}$. An elementary computation shows that $g \circ L^{-1} : [0, L_1] \to G$ is parametrized by $\tilde{\Delta}$-length, hence for $\gamma(t) = g(L^{-1}(tL_1))$ we obtain $\gamma(0) = g(0)$, $\gamma(1) = g(1)$, and $\tilde{\Delta}(\gamma^{-1}
abla) \equiv L_1$.

In the general case ($\dot{g}(t) = 0$ is possible), $L(t)$ is only monotone increasing. In that case one uses $\tilde{L}(s) = \sup \{ t : L(t) \leq s \}$ and shows that $g \circ \tilde{L}$ is differentiable (although $\tilde{L}$ need not be differentiable). Hence $\tilde{D}$ can be characterized in the following ways

\[
\tilde{D}(g_0) = \inf \left\{ \int_0^1 \tilde{\Delta}(g^{-1}\dot{g}) \, dt : g \in C^1([0, 1]; G), \quad \tilde{\Delta}(g^{-1}\dot{g}) \equiv \text{const} \right\}.
\]

\[
= \inf \left\{ T : (\exists g \in C^1([0, T]; G)) \quad g(0) = 1, \quad g(T) = g_0, \quad \tilde{\Delta}(g^{-1}\dot{g}) \equiv 1 \right\}
\]

\[
= \inf \left\{ T : (\exists g \in C^1([0, T]; G)) \quad g(0) = 1, \quad g(T) = g_0, \quad \tilde{\Delta}(g^{-1}\dot{g}) \leq 1 \right\}.
\]

Now let $\mathcal{U} = \{ X \in \mathfrak{g} : \tilde{\Delta}(X) \leq 1 \}$. Then the last statement says we must look for solutions of the differential inclusion $g^{-1}\dot{g} \in \mathcal{U}$, with boundary data $g(0) = 1$, $g(T) = g_0$ such that $T$ is minimal. Thus computing the dissipation distance $\tilde{D}$ is equivalent to solving a time-optimal left-invariant control problem on the Lie group $G$. Such problems are well-studied and standard results are available, cf. [3, 6, 11].

**Theorem 2.1** Assume that $\mathcal{U}$ is compact and convex and $\tilde{D}(g_0) < \infty$. Then there exists an absolutely continuous $g : [0, \tilde{D}(g_0)] \to G$ such that $g^{-1}\dot{g} \in \mathcal{U}$ a.e., $g(0) = 1$, and $g(\tilde{D}(g_0)) = g_0$.  

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Thus length minimizing arcs always exist within the class of absolutely continuous paths, provided there is some path with finite length and the set $\mathcal{U} \subseteq \mathfrak{g}$ is compact and convex. If $\mathcal{U}$ is a 0-neighborhood then it is clear that $\bar{D}(g_0) < \infty$ for all $g_0 \in G$. But this is still true under much weaker hypotheses. For example, let $\langle \langle \mathcal{U} \rangle \rangle$ denote the smallest subalgebra of $\mathfrak{g}$ containing $\mathcal{U}$. Then we have:

**Theorem 2.2** Assume that $\mathcal{U} = -\mathcal{U}$ and $\langle \langle \mathcal{U} \rangle \rangle = \mathfrak{g}$. Then $\bar{D}(g_0) < \infty$ for all $g_0 \in G$.

In control theory language: every $g_0 \in G$ is reachable from the group identity $1$ along a trajectory of $g^{-1}\dot{g} \in \mathcal{U}$, see [1, Theorem 1] or [7, Thm. 5.1], for example. The condition $\langle \langle \mathcal{U} \rangle \rangle = \mathfrak{g}$ is also necessary because the set $\{g_0 \in G : \bar{D}(g_0) < \infty\}$ is nothing but the reachable set (from $1$) of the system $g^{-1}\dot{g} \in \mathcal{U}$. And if $\langle \langle \mathcal{U} \rangle \rangle \neq \mathfrak{g}$, this reachable set is contained in a proper subgroup of $G$.

Still $\bar{D}(g_0) < \infty$ for all $g_0 \in G$ may hold true under much weaker hypotheses. In fact, let $S(\mathcal{U}) := \langle \exp \mathbb{R}^+ \mathcal{U} \rangle$ denote the subsemigroup of $G$ generated by $\exp(\mathbb{R}^+ \mathcal{U})$. Then $\bar{D}(g_0) < \infty$ for all $g_0 \in S(\mathcal{U})$, and

$$\bar{D}(g_0) < \infty \text{ for all } g_0 \in G \iff S(\mathcal{U}) = G.$$ 

Actually, $S(\mathcal{U}) = G$ may hold under extremely weak assumptions. To give just one more example, consider $\mathfrak{g} = \mathfrak{sl}(2)$ and $\mathcal{U} = \text{conv}(0, P, -Q) = [0, 1] \text{conv}(P, -Q)$. Then $S(\mathcal{U}) = \text{SL}(2)$ holds true.

**Control systems on Lie groups and automorphisms**

Given a set $\mathcal{U} \subseteq \mathfrak{g}$ we now consider the left-invariant control system given by the differential inclusion $g^{-1}\dot{g} \in \mathcal{U}$ a.e., and analyze some of its properties. Left-invariance means that for a trajectory $g(t)$ and an arbitrary $g_0 \in G$ the path $\tilde{g}(t) := g_0 g(t)$ is a trajectory, too.

We now write $\tau_{\mu\mathcal{U}}(g_0)$ instead of $\bar{D}(g_0)$ because we want to consider various possibilities for $\mathcal{U}$. Our first observation is that

$$\tau_{\mu\mathcal{U}}(g_0) = \frac{1}{\mu} \tau_{\mathcal{U}}(g_0) \text{ for all } \mathcal{U} \subseteq \mathfrak{g}, \mu > 0, \text{ and } g_0 \in G. \tag{1}$$

In fact, this is obtained simply by reparametrization. Next we observe

**Proposition 2.3** If $\mathcal{U} = -\mathcal{U}$ then $\tau_{\mathcal{U}}(g_0) = \tau_{\mathcal{U}}(g_0^{-1})$ for all $g_0 \in G$. 

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Hence we denote left-multiplication with 

Proof. Take \( t \) close to \( g \) following way: right multiplication with 

For the metric \( \hat{D} \) this means that \( \hat{D}(g_0, g_1) \) is symmetric if \( \mathcal{U} = -\mathcal{U} \), for then 

Finally we observe that group automorphisms interact well with the ODE \( \dot{g}(t) = g(t)u(t) \). For \( g \in G \) we denote left-multiplication with \( g \) by \( \lambda_g : G \to G, \lambda_g(g_0) = gg_0 \). As \( \lambda_g \) is differentiable, we denote its differential by \( d\lambda_g \).

**Proposition 2.4** Let \( \sigma \in \text{Aut}(G) \) and let \( \sigma' = d\sigma(1) \). Then 

In particular, if \( g(t) \) is such that \( g^{-1}\dot{g} \in \mathcal{U} \), then \( \tilde{g}(t) := \sigma(g(t)) \) satisfies 

As an immediate consequence we obtain the following estimate:

**Proposition 2.5** Let \( \sigma \in \text{Aut}(G) \) and \( \mathcal{U}, \tilde{\mathcal{U}} \subseteq \mathfrak{g} \) such that \( \sigma'\mathcal{U} \subseteq \tilde{\mathcal{U}} \). Then 

\[ \mathcal{T}_{\mathcal{U}'}(g_0) \leq \mathcal{T}_{\mathcal{U}}(\sigma^{-1}(g_0)) \text{ for all } g_0 \in G. \]

Proof. Let \( g : [0, t^*] \to G \) be absolutely continuous with \( g^{-1}\dot{g} \in \mathcal{U} \) a.e., \( g(0) = 1, g(t^*) = \sigma^{-1}(g_0) \). Then \( \tilde{g}(t) := \sigma(g(t)) \) satisfies 

Hence \( \mathcal{T}_{\tilde{\mathcal{U}}}(g_0) \leq t^* \) follows. Since we may choose \( g \) such that \( t^* \) is arbitrarily close to \( \mathcal{T}_{\mathcal{U}}(\sigma^{-1}(g_0)) \), our claim follows.

The material symmetry axiom (Sy4) can be re-interpreted now in the following way: right multiplication with \( g \in G \) leaves \( \bar{D} \) invariant, iff the inner automorphism \( I_g = (g_0 \mapsto gg_0g^{-1}) : G \to G \) leaves \( \mathcal{U} \) invariant, i.e., \( \text{Ad}(g)\mathcal{U} = \mathcal{U} \). This implies that \( I_g \) leaves \( \mathcal{T}_{\mathcal{U}} \) invariant: \( \mathcal{T}_{\mathcal{U}} = \mathcal{T}_{\mathcal{U}'} \circ I_g \). In other words: \( I_g \) is an isometry for the distance \( \bar{D} \).
Single-crystal plasticity

In single-crystal plasticity the plastic flow occurs through plastic slips induced by movements of dislocations. These movements are generated by shearings or slip systems, say, $S^\alpha = x_\alpha y_\alpha^T$, $\alpha = 1 \ldots m$, where $x_\alpha, y_\alpha \in \mathbb{R}^d$, $\|x_\alpha\| = \|y_\alpha\| = 1$, and $x_\alpha \perp y_\alpha$. Geometrically, $x_\alpha$ is the slip direction and $y_\alpha$ is the unit normal of the slip plane. All plastic flow has the form

$$\dot{g} = g \sum_\alpha \nu_\alpha S^\alpha$$

with $\nu_\alpha \geq 0$. Formally one distinguishes between $S^\alpha$ and $-S^\alpha$ because mechanically the slip strains in these directions must be distinguished, cf. [3].

The associated Lie algebra is $\mathfrak{g} = \langle\langle\{S^\alpha : \alpha = 1, \ldots, m\}\rangle\rangle \subseteq \mathfrak{sl}(d)$ because $\text{trace}(S^\alpha) = 0$, $\alpha = 1, \ldots, m$. In this case the dissipation functional has the form

$$\tilde{\Delta}(X) = \min\left\{\sum_\alpha \kappa_\alpha \gamma_\alpha : \gamma_\alpha \geq 0, \ X = \sum_\alpha \gamma_\alpha S^\alpha\right\}$$

where $\kappa_\alpha > 0$ are threshold parameters. The set $U = \{X : \tilde{\Delta}(X) \leq 1\}$ is a convex polytope:

$$\{X : \tilde{\Delta}(X) \leq 1\} = \text{conv}\left(\{0\} \cup \{\kappa_\alpha^{-1} S^\alpha : \alpha = 1, \ldots, m\}\right).$$

Indeed, since $\tilde{\Delta}(S^\alpha) \leq \kappa_\alpha$, the inclusion $U \supseteq \text{conv}(\{0\} \cup \{\kappa_\alpha^{-1} S^\alpha\})$ obviously holds true. Conversely, if $\tilde{\Delta}(X) \leq 1$, we find $\gamma_1, \ldots, \gamma_m \geq 0$ such that $X = \sum_\alpha \gamma_\alpha S^\alpha$ and $\sum_\alpha \kappa_\alpha \gamma_\alpha \leq 1$. Hence $X = \sum_\alpha \lambda_\alpha (\kappa_\alpha^{-1} S^\alpha)$ with $\lambda_\alpha := \kappa_\alpha \gamma_\alpha \geq 0$, $\sum_\alpha \lambda_\alpha \leq 1$. Whence $X \in \text{conv}(\{0\} \cup \{\kappa_\alpha^{-1} S^\alpha\})$.

Thus the factorization problem described in the introduction can be interpreted as the problem of finding dissipation minimizing paths for a 2-dimensional single-crystal with four slip systems $S^1 = P$, $S^2 = Q$, $S^3 = -P$, $S^4 = -Q$ and equal slip rates $\kappa_\alpha \equiv 1$. Equivalently, we can say that we have two slip systems $S^1$, $S^2$ and a symmetric dissipation functional: $\tilde{\Delta}(-X) = \tilde{\Delta}(X)$ for all $X \in \mathfrak{sl}(2)$. The factorization cost $T(g)$ is nothing but the dissipation distance from the identity: $T(g) = \tilde{D}(g)$.

3 The associated optimal control problem

Before we state the control problem we fix some more notation. The Lie algebra $\mathfrak{sl}(2)$ is the set of $2 \times 2$-matrices of zero trace. The bracket is the commutator: $[X, Y] = XY - YX$. The following matrices form a basis of $\mathfrak{sl}(2)$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = P + Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad U = P - Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
We observe that $P = \frac{1}{2}(T + U)$ and $Q = \frac{1}{2}(T - U)$.

Figure 1: The set $\mathcal{U} = \text{conv}(\pm P, \pm Q) \subseteq \mathfrak{sl}(2)$ and the Lorentzian double cone $\{ X \in \mathfrak{sl}(2) : \det(X) = 0 \}$.

Let $\mathcal{U} = \text{conv}(\pm P, \pm Q) \subseteq \mathfrak{sl}(2)$. This set is simply a square in the plane $\mathbb{R}T + \mathbb{R}U \subseteq \mathfrak{sl}(2)$. Figure 1 shows how $\mathcal{U}$ is situated in $\mathfrak{sl}(2)$. The Lorentzian double cone consists of all matrices $X \in \mathfrak{sl}(2)$ such that $\det(X) = 0$. It is the set of all possible two-dimensional slip systems (plus the origin), cf. the discussion at the end of the previous section. The elements in the interior of the double cone all have purely imaginary spectrum, so they are generators for compact (circle) subgroups of SL(2). The plane $\mathbb{R}H + \mathbb{R}T \subseteq \mathfrak{sl}(2)$ is the set of symmetric matrices in $\mathfrak{sl}(2)$. All elements outside the double cone are diagonalizable (over $\mathbb{R}$), in fact they are conjugate to $\lambda H$ for some $\lambda > 0$.

Now we consider the left-invariant control system (ODE on SL(2)):

$$
\dot{g}(t) = g(t) \, u(t), \quad g(t) \in \text{SL}(2), \ u(\cdot) \in L^\infty(\mathbb{R}; \mathcal{U}).
$$

(LICS)

Admissible control functions are measurable, essentially bounded functions. The factorizations we are looking for are in one-to-one correspondence to the trajectories of (LICS) generated by piecewise constant controls. Given $t_1, \ldots, t_k > 0$ and $A_1, \ldots, A_k \in \mathcal{U}$, we let $\tau_j = \sum_{i=1}^j t_i$, $(j = 0, \ldots, k)$ and define the control $u \colon [0, \tau_k] \to \mathcal{U}$ by $u(t) = \sum_j A_j \chi(\tau_{j-1}, \tau_j)$.
Let \( g(t) \) denote the associated trajectory of (LICS) with \( g(0) = 1 \). Then

\[
g(t) = \begin{cases} 
\exp(tA_1), & t \in [0, \tau_1) \\
M((t_1A_1, (t - \tau_1)A_2), & t \in [\tau_1, \tau_2), \\
\vdots & \\
M((t_1A_1, t_2A_2, \ldots, t_{j-1}A_{j-1}, (t - \tau_{j-1})A_j), & t \in [\tau_{j-1}, \tau_j), \\
\end{cases}
\]

and \( g(\tau_k) = M(t_1A_1, \ldots, t_kA_k) \).

The system (LICS) is controllable because \( P, Q \) generate \( \mathfrak{sl}(2) \) as a Lie algebra: \([P, Q] = H, \mathfrak{sl}(2) = \mathbb{R}H + \mathbb{R}P + \mathbb{R}Q \). Therefore every \( g_0 \in \text{SL}(2) \) is reachable from the group identity. Our Optimal Control Problem (OCP) consists of finding a time-minimal trajectory from the group identity \( 1 \) to \( g_0 \):

\[
\int_0^{t^*} dt \longrightarrow \text{min, subj. to } \dot{g} = gu, \ u \in \mathcal{U}, \ g(0) = 1, \ g(t^*) = g_0. \quad \text{(OCP)}
\]

Since \( \mathcal{U} \) is compact and convex, a standard result in control theory states that time-optimal arcs (and controls) always exist. Using standard methods we will show that the corresponding controls are piecewise constant. This will tell us which factorizations have a chance to be optimal and in which sense the original factorization problem has to be modified in order to have solutions. Eventually our goal is to classify the optimal arcs of (OCP).

**Related problems**

There may be several non-isomorphic (connected) Lie groups having the same—i.e., isomorphic—Lie algebras. For example, the Lie algebras \( \mathfrak{so}(3) \) (real, skew-symmetric, \( 3 \times 3 \)) and \( \mathfrak{su}(2) \) (complex, skew-hermitian, \( 2 \times 2 \)) are isomorphic, but the groups \( \text{SO}(3) \) and \( \text{SU}(2) \) are not (\( \text{SO}(3) \) has trivial center while the center of \( \text{SU}(2) \) is \( \{\pm \text{id}_2\} \)).

The local structure of the control system (LICS) only depends on the Lie algebra structure. Therefore, if we consider (OCP) on any other group \( G \) with Lie algebra isomorphic to \( \mathfrak{sl}(2) \), the whole discussion of the (PMP) will apply equally to any such \( G \).

For example, if \( B = \text{diag}(1, 1, -1) \) denotes a bilinear form of Lorentzian signature on \( \mathbb{R}^3 \), then we can consider (OCP) on the group

\[
\text{SO}_0(2, 1) = \{g \in \text{mat}(3, \mathbb{R}) : g^TBg = B, \ \det(g) = 1, \ g_{33} > 0\}
\]

because the Lie algebras \( \mathfrak{so}(2, 1) = \{X : X^TB + BX = 0\} \) and \( \mathfrak{sl}(2) \) are indeed isomorphic. Note that \( \text{SO}_0(2, 1) \not\cong \text{SL}(2) \) (the centers consist of one, resp. two, elements).
Therefore Problem (OCP) is strongly related to the Hyperbolic Dubins’ Problem (HDP) and the Hyperbolic Reeds-Shepp-Car-Problem (HRSCP). We wish to emphasize this connection because it allows nice geometric interpretations of several of the arguments to come.

Explaining Dubins’ Problem (DP) is simple. Given two points \( x_0, x_1 \) and tangent directions \( v_0, v_1 \) in the plane \( \mathbb{R}^2 \), the goal is to find a \( C^1 \)-curve \( \gamma: [0, L] \to \mathbb{R}^2 \) such that

\[
(1) \quad \gamma(0) = x_0, \quad \dot{\gamma}(0) = v_0, \quad \gamma(L) = x_1, \quad \dot{\gamma}(L) = v_1,
\]

\[
(2) \quad \gamma(s) \text{ has curvature } \kappa(s) \text{ almost everywhere, and } |\kappa(s)| \leq 1,
\]

\[
(3) \quad \gamma \text{ is parametrized by arc-length, and } L \text{ is minimal.}
\]

One may interpret this as follows: Imagine driving a car in the plane. The car moves forward at constant speed 1 and its turning radius is limited. At time \( t = 0 \) the car is located at \( x_0 \), pointing into the \( v_0 \)-direction. The goal is to drive the car in minimal time to position \( x_1 \), pointing into the \( v_1 \)-direction. If we allow the car to move backward as well as forward, we obtain the so-called Reeds-Shepp-Car-Problem (RSCP).

The original sources for these problems are [2] and [17]. Variational problems with cost functional depending only on the curvature can be generalized directly to manifolds with curvature, in particular the sphere \( \mathbb{S}^2 \) and the hyperbolic plane \( \mathbb{H}^2 \) (constant curvature 1, resp. \(-1\)). In the latter case this leads to control systems on the Lie groups \( \text{SO}(3) \), resp. \( \text{SO}_0(2, 1) \), an observation first made by Jurdjevic in his paper [3] on non-euclidean elastica.

The hyperbolic Dubins’ Problem (HDP) has been investigated by Monroy in [13, 16]. We observed in [13] that (HDP) is equivalent to finding time-optimal paths for the control system:

\[
\dot{g} = d\lambda(g(1))u, \quad u \in \text{conv}(P, Q) \text{ a.e., } g \in \text{PSL}(2, \mathbb{R}),
\]

(HDP)

where \( \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm 1\} \cong \text{SO}_0(2, 1) \). The controls \( P, Q \) correspond to left, resp., right turns, and the control \( \frac{1}{2}(P + Q) = \frac{1}{2}T \) corresponds to a geodesic arc.

Thus a product of the form \( M(rP, sT, tQ, \ldots) \) corresponds to a path in the hyperbolic plane consisting of circular arcs and geodesic segments. Equality of two such products means that two (seemingly different) paths have the same initial and terminal positions and tangents.

In order to visualize such paths we use the so-called conformal disc model of \( \mathbb{H}^2 \). Here \( \mathbb{H}^2 \) is identified (as a set) with the open unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). The geodesics are either diameters of \( \mathbb{D} \) or circular arcs perpendicular to the unit circle. The circular arcs are parts of so-called
horocycles, i.e., ordinary (euclidean) circles touching the unit circle from inside.

4 Symmetries and isometries

Apparently the set $U = \text{conv}(\pm P, \pm Q) \subseteq \mathfrak{sl}(2)$ has some symmetries which we would like to exploit. Let $i$ denote inversion, i.e., $i(g) = g^{-1}$. Then we already observed that $T(i(g)) = T(g)$ for all $g \in \text{SL}(2)$ because of $U = -U$. The appropriate strategy for finding more symmetries is to look for group automorphisms preserving the factorization cost $T$.

Consider $\sigma_H, \sigma_T, \sigma_U : \text{SL}(2) \to \text{SL}(2)$,
$$
\sigma_H(g) = HgH, \quad \sigma_T(g) = TgT, \quad \sigma_U(g) = UgU^T = -UgU.
$$
We observe that $\sigma_H, \sigma_T, \sigma_U \in \text{Aut}(\text{SL}(2))$. For example, $\sigma_H(g_0g_1) = Hg_0g_1H = Hg_0Hg_1H = \sigma_H(g_0)\sigma_H(g_1)$. Also, $\sigma_H^2 = \sigma_T^2 = \text{id}_{\text{SL}(2)}$, and $\sigma_U = \sigma_H\sigma_T = \sigma_T\sigma_H$. Hence $\Gamma := \{\text{id}_{\text{SL}(2)}, \sigma_H, \sigma_T, \sigma_U\} \subseteq \text{Aut}(\text{SL}(2))$ is a group, actually $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

For any Lie group automorphism $\sigma$ its derivative at the identity $\sigma' := d\sigma(1)$ is a Lie algebra automorphism. In the present situation we have
$$
\sigma'_H, \sigma'_T, \sigma'_U : \mathfrak{sl}(2) \to \mathfrak{sl}(2), \quad \sigma'_H(X) = HXH, \quad \sigma'_T(X) = TXT, \quad \sigma'_U(X) = -UXU.
$$
Apparently there seems to be no difference between $\sigma_H$ and $\sigma'_H$, for example. Nevertheless the distinction makes sense because these maps have different domains and ranges. Next we observe that
$$
\sigma'_H(P) = -P, \quad \sigma'_T(P) = Q, \quad \sigma'_U(P) = -Q, \\
\sigma'_H(Q) = -Q, \quad \sigma'_T(Q) = P, \quad \sigma'_U(Q) = -P.
$$
Thus $\sigma'(U) = U$ for all $\sigma \in \Gamma$. These are the symmetries we have been looking for because the following proposition holds true for any Lie group $G$:

**Proposition 4.1** Assume that $\sigma \in \text{Aut}(G)$ and let $\sigma' = d\sigma(1) \in \text{Aut}(\mathfrak{g})$. If $\sigma'(U) = U$, then $T(\sigma(g)) = T(g)$ for all $g \in G$. Moreover, $D(g_0, g_1) = D(\sigma(g_0), \sigma(g_1))$ for all $g_0, g_1 \in G$, i.e., $\sigma$ is an isometry of the metric $D$.

**Remark 4.2** Although inversion $i$ preserves $T$, it is not an isometry of the metric $D$, in general. In fact, it is an easy exercise to show that for a left-invariant metric $D$ on $G$ we have:

$i$ is an isometry $\iff$ $D$ is right-invariant $\iff$ $D$ is bi-invariant.
The left-invariant metric defined by $U$ will be bi-invariant, for example, if $G$ is abelian or if $\text{Ad}(G)U = U$. But one cannot expect bi-invariance otherwise.

A general proof is easily obtained using Proposition 2.5. In the special problem that we consider here, an elementary computation allows to verify that every $\sigma \in \Gamma$ maps trajectories of (LICS) onto trajectories. Assume $\dot{\gamma}(t) = \gamma(t)u(t)$ and let, for example, $\eta(t) = \sigma_H(\gamma(t)) = H\gamma(t)H$. Then

$$\dot{\eta}(t) = H\dot{\gamma}(t)H = H\gamma(t)u(t)H = H\gamma(t)H H u(t)H = \eta(t) \sigma_H'(u(t)).$$

Hence $T(\sigma_H(g)) \leq T(g)$ for all $g$ follows. As the same is true for $\sigma_H^{-1}$, $T(\sigma_H(g)) = T(g)$ follows.

In terms of the basis $\{H, T, U\}$ of $\mathfrak{sl}(2)$ the maps $\sigma'_H, \sigma'_T, \sigma'_U$ are nothing but 180 degree rotations around the $H$-, $T$-, and $U$-axis. One can actually show that $\Gamma = \{\sigma \in \text{Aut}(SL(2)) : \sigma'(U) = U\}$. Finally, let $\tilde{\Gamma} \subseteq \text{Diffeo}(SL(2))$ denote the group generated by $\Gamma \cup \{I\}$, then

$$\tilde{\Gamma} = \{\text{id}_{SL(2)}, \sigma_H, \sigma_T, \sigma_U, I, \sigma_H I, \sigma_T I, \sigma_U I\}.$$ 

In particular, $\tilde{\Gamma}$ is abelian (isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$), and $T(\phi(g)) = T(g)$ for all $g \in SL(2), \phi \in \tilde{\Gamma}$.

We will use the group $\tilde{\Gamma}$ in various ways. First it will allow us to streamline the discussion of the (PMP) and the comparison arguments. Practically it will allow us to reduce the number of factorization maps that have to be inverted from 64 down to 13: instead of solving 64 (systems of three) equations for the same right hand side, say $g$, one may solve 13 systems for various righthand sides from $\{\phi(g) : \phi \in \tilde{\Gamma}\}$. Although this does not really affect the overall computational cost, it is a great help for programming, testing, and debugging.

It is also worthwhile to mention that the fixed point sets $\text{Fix}(\phi) := \{g : \phi(g) = g\}, (\phi \in \tilde{\Gamma})$ provide information about the cut-locus. In many cases geodesics lose their global optimality once they hit some $\text{Fix}(\phi)$.

5 The structure of extremals

We already observed that for $\mathcal{U} = \text{conv}(\pm P, \pm Q) \subseteq \mathfrak{sl}(2)$ the left-invariant control system

$$\dot{\gamma}(t) = \gamma(t)u(t), \quad u(t) \in \mathcal{U} \quad a.e.,$$

is controllable, and that every $g$ is reachable from the group identity in minimal time.

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Next we want to apply the Pontrjagin Maximum Principle (PMP) to obtain necessary conditions for optimality. For invariant systems on Lie groups the (PMP) takes a particularly simple form, cf. [3, 4]. A proper statement requires some extra terminology: the Lie algebra dual and the adjoint and coadjoint action.

**Adjoint and coadjoint action**

Let $G$ be an arbitrary Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ denote the vector space dual of $\mathfrak{g}$. Then there is a natural action of $G$ on $\mathfrak{g}$ and, by duality, also on $\mathfrak{g}^*$. These are the so-called **adjoint** and **coadjoint action**. Both actions come from conjugation on the group. For $g_0 \in G$ let $I_{g_0} = (g \mapsto g_0 g g_0^{-1}) : G \to G$. Then $I_{g_0} \in \text{Aut}(G)$. The adjoint action is obtained by differentiating $I_{g_0} : G \to G$ at the group identity $1$, $\text{Ad}(g_0) = dI_{g_0}(1)$. The coadjoint action is obtained via duality. Instead of a coordinate-free discussion we now give explicit representations (in coordinates) for $G = \text{SL}(2)$ because these will be needed in the subsequent discussion of the structure of extremals.

So consider $G = \text{SL}(2)$ and $\mathfrak{g} = \mathfrak{sl}(2)$. Fix the basis $\{H, T, U\}$ of $\mathfrak{g}$. So we can write $X \in \mathfrak{sl}(2)$ as $X = hH + tT + uU$ with $(h, t, u) \in \mathbb{R}^3$. Next we fix a dual basis in $\mathfrak{g}^*$, so we may write $p \in \mathfrak{g}^*$ as a row vector $p = (p_H, p_T, p_U)$, and $\langle p, hH + tT + uU \rangle = p_H h + p_T t + p_U u$.

The adjoint action is conjugation $\text{Ad}(g)X = gXg^{-1}$ for $g \in \text{SL}(2)$, $X \in \mathfrak{sl}(2)$. In terms of the basis $\{H, T, U\}$ $\text{Ad}(g)$ is a $3 \times 3$-matrix (actually $\text{Ad}(g) \in \text{SO}_0(2,1)$). Although we will not need the explicit expression, we state it just for sake of completeness:

$$
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(g) = 1 \quad \Rightarrow \quad \text{Ad}(g) = \begin{pmatrix} ad + bc & -ac + bd & -ac - bd \\ -ab + cd & \frac{a^2 - b^2 - c^2 - d^2}{2} & \frac{a^2 + b^2 - c^2 - d^2}{2} \\ -ab - cd & \frac{a^2 + b^2 + c^2 + d^2}{2} & \frac{a^2 - b^2 + c^2 - d^2}{2} \end{pmatrix}. \quad (2)
$$

For $X \in \mathfrak{sl}(2)$ let $\text{ad}(X) : \mathfrak{sl}(2) \to \mathfrak{sl}(2)$, $\text{ad}(X)Y = [X, Y]$. If $X = hH + tT + uU$, then in terms of the basis $\{H, T, U\}$ we obtain

$$
\text{ad}(X) = \begin{pmatrix} 0 & 2u & -2t \\ -2u & 0 & 2h \\ -2t & 2h & 0 \end{pmatrix}.
$$

We note that $\text{ad} : \mathfrak{sl}(2) \to \mathfrak{so}(2,1)$ is a Lie algebra isomorphism. For $P = \frac{1}{2}(T + U)$ and $Q = \frac{1}{2}(T - U)$ we obtain:

$$
\text{ad}(P) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{ad}(Q) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (3)
$$
Since \( \text{ad}(P)^3 = \text{ad}(Q)^3 = 0 \), we obtain

\[
e^{\tau \text{ad}(P)} = \begin{pmatrix} 1 & \frac{\tau}{2} & -\frac{\tau}{2} \\ -\frac{\tau}{2} & 1 - \frac{\tau^2}{2} & \frac{\tau^2}{2} \\ -\frac{\tau}{2} & -\frac{\tau^2}{2} & 1 + \frac{\tau^2}{2} \end{pmatrix}, \quad e^{\tau \text{ad}(Q)} = \begin{pmatrix} 1 & -\frac{\tau}{2} & -\frac{\tau}{2} \\ \frac{\tau}{2} & 1 - \frac{\tau^2}{2} & -\frac{\tau^2}{2} \\ -\frac{\tau}{2} & 1 + \frac{\tau^2}{2} & 1 + \frac{\tau^2}{2} \end{pmatrix}. \quad (4)
\]

Using the duality between \( \mathfrak{sl}(2) \) and \( \mathfrak{sl}(2)^* \) one obtains \( \text{Ad}(g)^*p = p \circ \text{Ad}(g) \), \( \text{ad}(X)^*p = p \circ \text{ad}(X) \), and with our choice of coordinates the latter is nothing but left-multiplication of the row vector \( p \) with the matrix \( \text{Ad}(g) \), resp. \( \text{ad}(X) \). The coadjoint action is defined as

\[
\text{CoAd}(g)p = \text{Ad}(g^{-1})^*p, \quad g \in G, \ p \in \mathfrak{g}^*.
\]

We are more in favor of this notation (rather than the frequently used \( \text{Ad}^*(g) \)) because it prevents confusion between \( \text{Ad}(g)^* \) and \( \text{Ad}^*(g) = \text{Ad}(g^{-1})^* \).

The Killing form is defined as \( \kappa(X,Y) = \text{trace}(\text{ad}(X) \text{ad}(Y)) \). For \( \mathfrak{sl}(2) \) it is a nondegenerate symmetric bilinear form of signature \(+, +, -\). The adjoint action leaves \( \kappa \) invariant, in particular the quadratic form \( q_\kappa(X) := \kappa(X,X) = \text{trace}(\text{ad}(X)^2) \) is \( \text{Ad} \)-invariant, \( q_\kappa(\text{Ad}(g)X) = q_\kappa(X) \) for all \( X \in \mathfrak{sl}(2) \), \( g \in \text{SL}(2) \). With our choice of coordinates \( q_\kappa(hH + tT + uU) = 8(h^2 + t^2 - u^2) \).

By duality, the quadratic form \( C(p) = p_H^2 + p_T^2 + p_U^2 \) is \( \text{Ad}^* \)-invariant, i.e., \( C(\text{Ad}(g)^*p) = C(p) \) for all \( p \in \mathfrak{sl}(2)^* \), \( g \in \text{SL}(2) \). As we will see soon, \( C(p) \) will appear as a first integral for any optimal control problem on any Lie group with Lie algebra isomorphic to \( \mathfrak{sl}(2) \).

For \( p \in \mathfrak{sl}(2)^* \) let \( \mathcal{O}_p = \{ \text{Ad}(g)^*p : g \in \text{SL}(2) \} \) denote its orbit under the coadjoint action. Roughly speaking, the coadjoint orbits are level sets of \( C \), more precisely we have four different types of orbits:

**Hyp1:** If \( C(p) > 0 \) the orbit \( \mathcal{O}_p \) is a one-sheeted hyperboloid.

**Hyp2:** If \( C(p) < 0 \) the orbit \( \mathcal{O}_p \) is the upper \((p_U > 0)\) or lower \((p_U < 0)\) part of a two-sheeted hyperboloid.

**Cone:** If \( C(p) = 0 \), then either \( p = 0 \) and \( \mathcal{O}_p = \{0\} \) is singleton, or \( p \neq 0 \) and \( \mathcal{O}_p \) is the upper \((p_U > 0)\) or lower \((p_U < 0)\) part of the (boundary of the) Lorentzian double cone, cf. Figure 1.

For \( p \in \mathfrak{sl}(2)^* \) we denote \( G_p = \{ g \in \text{SL}(2) : \text{Ad}(g)^*p = p \} \) the stabilizer group of \( p \), and \( \mathfrak{g}_p = \{ X \in \mathfrak{sl}(2) : \text{ad}(X)^*p = 0 \} \) the stabilizer algebra of \( p \). One quickly verifies that for \( p = (p_H, p_T, p_U) \neq 0 \) one has \( \mathfrak{g}_p = \mathbb{R}(p_H H + p_T T - p_U U) \).

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The Pontrjagin Maximum Principle on Lie groups
(for time-optimal problems)

Theorem 5.1 Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, $\mathcal{U} \subseteq \mathfrak{g}$, and consider the (LICS) \( \dot{\gamma}(t) = \gamma(t)u(t), \) $u(t) \in \mathcal{U}$. Assume that $g(t)$ is a trajectory which is time-optimal on some interval $I$. Let $u(t) = g(t)^{-1}\dot{g}(t) \in L^\infty(I; \mathcal{U})$. Then there exists an absolutely continuous covector function $p : I \to \mathfrak{g}^*$ such that

(0) $p(t) \neq 0$ for some (hence all) $t \in I$,
(1) $\text{Ad}(g(t)^{-1})^*p(t)$ is constant,
(2) $\langle p(t), u(t) \rangle = \min_{v \in \mathcal{U}} \langle p(t), v \rangle$, a.e. in $I$,
(3) $\langle p(t), u(t) \rangle$ is constant (a.e.), either $-1$, or $0$.

A triple $(g(t), p(t), u(t))$ consisting of a trajectory $g(t)$, a control $u(t)$ generating $g(t)$ and a covector $p(t)$ satisfying conditions (0)–(3) is called an extremal of the optimal control problem. In the present situation $u = g^{-1}\dot{g}$, so specifying $u$ is actually redundant.

Let $\mathcal{H}(p) = \min_{v \in \mathcal{U}} \langle p(t), v \rangle$ denote the optimal Hamiltonian. By (3) $\mathcal{H}$ is an integral of motion. An extremal for which $\mathcal{H}(p) \equiv 0$ is called an abnormal extremal. The other extremals are called normal or regular extremals. Geometrically an extremal being abnormal means it satisfies the first order necessary conditions for any cost functional, not only the one under consideration.

In view of our discussion of symmetries and isometries we observe:

Proposition 5.2 If $\sigma \in \text{Aut}(G)$ satisfies $\sigma'(\mathcal{U}) = \mathcal{U}$, then $\sigma$ maps extremals onto extremals. Similarly, if $\mathcal{U} = -\mathcal{U}$ and $(g, p, u)$ is an extremal (in $[0, t^*)$), then $(g(t^* - t), -u(t^* - t), -p(t^* - t))$ is an extremal, too.

The proof is an easy exercise. If $(g, p, u)$ is an extremal and $\sigma$ as above, the image extremal is $(\sigma(g(t)), ((\sigma')^{-1})^*p(t), \sigma'(u(t)))$. 

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Yield surface and flow rule for the model problem

Using the basis \( \{ H, T, U \} \) of \( g = \mathfrak{sl}(2) \) and fixing a dual basis (of \( \{ H, T, U \} \)) in \( g^* \), we write \( p = (p_H, p_T, p_U) \). With \( \mathcal{U} = \text{conv}(\pm P, \pm Q) \) the optimal Hamiltonian is

\[
\mathcal{H}(p) = -\frac{1}{2} (|p_T| + |p_U|).
\]

We already observed that there is another integral of motion:

\[
\mathcal{C}(p) = p_H^2 + p_T^2 - p_U^2
\]

because \( \mathcal{C} \) is constant along coadjoint orbits.

**Proposition 5.3** Abnormal extremals are not optimal.

**Proof.** Let \( (g(t), p(t), u(t)) \) be an abnormal extremal. Then \( \mathcal{H}(p) \equiv 0 \) implies \( p(t) = (p_H(t), 0, 0) \). Hence \( \mathcal{C}(p) = p_H(t)^2 \), whence \( p_H \) is constant.

As \( p \neq 0 \), \( p_H \neq 0 \) must hold. As \( p \) is constant, \( p \) is differentiable. Let \( u(t) = u_1(t)P + u_2(t)Q \). As \( 0 \equiv \dot{p} = p \circ \text{ad}(u) \), \( u(t) \in \mathcal{U} \cap g_p \) must hold. Since \( g_p = \mathbb{R}H, u(t) \equiv 0 \) follows. Or, elementary,

\[
0 = \dot{p} = p \text{ ad}(u_1P + u_2Q) = (p_H, 0, 0) \begin{pmatrix} 0 & u_1 - u_2 & -u_1 - u_2 \\ -u_1 + u_2 & 0 & 0 \\ -u_1 - u_2 & 0 & 0 \end{pmatrix},
\]

so \( u_1 - u_2 = 0 \) and \( u_1 + u_2 = 0 \), hence \( u_1 = u_2 = 0 \). Thus \( \dot{g} \equiv 0 \), i.e., \( g(t) \equiv g_0 \) is a constant path, hence not optimal. \( \square \)

For regular extremals the covector \( p(t) \) evolves on the level set (yield surface) \( \{ \mathcal{H} = -1 \} \). The latter is a cylinder over a square, cf. Figure 2 which also shows some flow lines. On each open face the minimizing control is uniquely determined. Switches (may) occur when \( p(t) \) hits one of the four edges, i.e., when \( p_U(t) = 0 \) or \( p_T(t) = 0 \). At such a point the minimizing condition \( \langle p(t), u(t) \rangle = \min_{v \in \mathcal{U}} \langle p(t), v \rangle \) does not suffice to characterize the control \( u(t) \) uniquely. But we also have the geometric information \( p(t) \in \{ \mathcal{H} = -1 \} \), and \( \dot{p}(t) = p(t) \circ \text{ad}(u(t)) \). Carefully exploiting all this information, one obtains that optimal controls are piecewise constant and that there are 4 types of extremals:

**(ALT)** The optimal control \( u(t) \) follows an alternating switching pattern.

\[
P \vdash (-Q) \vdash P \vdash (-Q) \ldots, \quad \text{resp.} \quad Q \vdash (-P) \vdash Q \vdash (-P) \ldots,
\]

and the time \( s \) between successive switches is a constant, \( 0 < s < 2\sqrt{2} \).

The corresponding path \( g(t) \) is a subarc of

\[
\mathbb{M}(sP, -sQ, sP, -sQ, \ldots) \quad \text{resp.,} \quad \mathbb{M}(sQ, -sP, sQ, -sP, \ldots).
\]
The yield surface \( \{ \mathcal{H} = -1 \} \).

(CSP) The circular switching pattern (CSP), here the control \( u(t) \) switches in either clockwise or counterclockwise order from vertex to vertex:

\[
P \vdash Q \vdash -P \vdash -Q \vdash P \ldots, \quad \text{resp.} \quad P \vdash -Q \vdash -P \vdash Q \vdash P \ldots,
\]

the time \( s \) between successive switches is constant, \( s \in (0, \sqrt{2}) \). The corresponding path \( g(t) \) is a subarc of

\[
\mathcal{M}(sP, -sQ, -sP, sQ, \ldots), \quad \text{resp.,} \quad \mathcal{M}(sP, -sQ, -sP, sQ, \ldots).
\]

(SSP) The singular switching pattern(s) (SSP). In this case singular arcs may occur, the corresponding control is \textit{not} bang-bang (i.e., in \( \{ \pm P, \pm Q \} \)). The singular controls are constant, \( \pm \frac{1}{2} T \), they may be applied on an interval of arbitrary length. The switching time for an intermediate bang arc is exactly \( \sqrt{2} \). Describing all possible switching sequences in general is complicated. All possibilities can be obtained as paths in the directed graph shown in Figure 3. If the \( S \)-arc between 2 \( B \)-arcs has zero length, the value of the control function \( u(t) \) need not change. We call these \textbf{virtual switches}. Example:

\[
\mathcal{M}\left( \sqrt{2} P, s_1 T, \sqrt{2} P, -\sqrt{2} Q, -\sqrt{2} Q, \sqrt{2} P, s_2 T, \sqrt{2} P, \ldots \right), \quad s_1, s_2 \geq 0.
\]

(U/2) The constant controls \( u_\pm \equiv \pm \frac{1}{2}(P - Q) = \pm \frac{1}{2} U \). The corresponding path is \( g(t) = \exp(\pm t \cdot \frac{1}{2} U), \ t \geq 0 \). These are singular controls, too.
From this characterization we can already deduce that every $g \in \text{SL}(2)$ has an optimal factorization with factors from the set $\exp(\mathbb{R}P) \cup \exp(\mathbb{R}Q) \cup \exp(\mathbb{R}T) \cup \exp(\mathbb{R}U)$.

**How to find the switching patterns and times**

In this subsection we prove that indeed, our list consists of extremals only, and we explain how this list is obtained. We also indicate why there are no other extremals, but a rigorous proof has to be carried out in a different way. Therefore it is provided separately in the subsequent subsection.

The yield surface has four faces, on each of the open faces we have $p_T p_U \neq 0$, and the minimizing condition (2) determines the control $u$ uniquely (the subgradient $\partial \mathcal{H}$ is singleton).

Due to the symmetries from $\Gamma = \{\text{id}, \sigma_H, \sigma_T, \sigma_U\}$ it suffices to consider only one of these faces, say,

$$\mathbb{F} = \{p : p_T + p_U = 2, \ p_T, p_U > 0\}, \quad \text{and} \quad \overline{\mathbb{F}}$$

because any other face is mapped onto $\mathbb{F}$ by $(\sigma')^{-*}$ for some $\sigma \in \Gamma$. The maps $(\sigma')^{-*}$ ($\sigma \in \Gamma$) are 180-degree rotations around the coordinate axes in $\mathfrak{sl}(2)^*$.

So let us consider $p \in \mathbb{F}$. Then $\partial \mathcal{H}(p) = \{-P\}$. It is crucial to analyze what happens when $p(t)$ hits the two boundary lines $(0,2,0) + \mathbb{R}(1,0,0)$, resp. $(0,0,2) + \mathbb{R}(1,0,0)$. Therefore our next step is to consider the flow

$$(p, \tau) \mapsto p e^{-\tau \text{ad}(P)}$$

for $p \in \overline{\mathbb{F}}$, $\tau \in \mathbb{R}$. We compute

$$e^{-\tau \text{ad}(P)} = \begin{pmatrix} 1 & -\frac{\tau^2}{2} & \frac{\tau^2}{2} \\ \tau & 1 - \frac{\tau^2}{2} & \frac{\tau^2}{2} \\ \tau & -\frac{\tau^2}{2} & 1 + \frac{\tau^2}{2} \end{pmatrix}, \quad \tau \in \mathbb{R}.$$
So \((p_H, 0, 2)e^{-\tau \text{ad}(P)} = (p_H + 2\tau, -\tau(p_H + \tau), \tau^2 + p_H \tau + 2)\). Hence we have

\[
\begin{array}{c}
p_H \\
(0,0,2) \\
(0,2,0)
\end{array}
\]

Figure 4: The flow on the face \(F = \{p_U + p_T = 2, p_U > 0, p_T > 0\}\).

the following cases:

1. \(p_H < -2\sqrt{2}\): \(p e^{-\tau \text{ad}(P)} \in F\) for \(\tau \in (0, \tau^*)\) where \(\tau^* = \frac{|p_H| - \sqrt{p_H^2 - 8}}{2}\), and \(p e^{-\tau^* \text{ad}(P)} = (-\sqrt{p_H^2 - 8}, 2, 0)\).

2. \(p_H = -2\sqrt{2}\): \(p e^{-\tau \text{ad}(P)} \in F\) for \(\tau \in (0, 2\sqrt{2}) \setminus \{\sqrt{2}\}\), and \(p e^{-\sqrt{2} \text{ad}(P)} = (0, 2, 0)\), \(p, e^{-2\sqrt{2} \text{ad}(P)} = (2\sqrt{2}, 0, 2)\).

3. \(p_H \in (-2\sqrt{2}, 0)\): \(p e^{-\tau \text{ad}(P)} \in F\) for \(\tau \in (0, \tau^*)\) where \(\tau^* = |p_H| > 0\), and \(p e^{-\tau^* \text{ad}(P)} = (|p_H|, 0, 2)\).

4. \(p_H \in (0, 2\sqrt{2})\): \(p e^{-\tau \text{ad}(P)} \in F\) for \(\tau \in (-\tau^*, 0)\) with \(\tau^* = p_H\), cf. the previous case.

5. \(p_H = 2\sqrt{2}\): \(p e^{-\tau \text{ad}(P)} \in F\) for \(\tau \in (-2\sqrt{2}, 0) \setminus \{-\sqrt{2}\}\), cf. the second case.

6. \(p_H > 2\sqrt{2}\): \(p e^{-\tau \text{ad}(P)} \in F\) for \(\tau \in (-\tau^*, 0)\) where \(\tau^* = \frac{p_H - \sqrt{p_H^2 - 8}}{2}\), \(p e^{-\tau^* \text{ad}(P)} = (\sqrt{p_H^2 - 8}, 2, 0)\).

Applying symmetries we obtain the flow lines on the other open faces, too. Figure 5 shows the yield surface “unfolded” and the various possibilities for \(p(t)\). Recalling that \(C(p) = p_H^2 + p_T^2 - p_U^2\) is constant along extremals we can state the following

**Proposition 5.4** Let \((g, p, u)\) be an extremal such that \(p(t_0) \in F\) for some \(t_0\). Let \(C = C(p)\) and

\[
\begin{align*}
\alpha &= \inf \{t < t_0 : (\forall \tau \in (t, t_0)) \ p(\tau) \in F\} \\
\beta &= \sup \{t > t_0 : (\forall \tau \in (t_0, t)) \ p(\tau) \in F\}
\end{align*}
\]

Then one of the following three cases occurs:
Figure 5: The Hamiltonian flow on \( \{ H = -1 \} \).

**CSP:** \( C > 4, p_H(t_0) \neq 0, \beta - \alpha = \frac{\sqrt{C+4} - \sqrt{C-4}}{2} \in (0, \sqrt{2}) \), and
\[
\begin{align*}
p_H(t_0) > 0 & \Rightarrow p(\alpha) = (\sqrt{C-4}, 2, 0), \quad p(\beta) = (\sqrt{C+4}, 0, 2), \\
p_H(t_0) < 0 & \Rightarrow p(\alpha) = (-\sqrt{C+4}, 0, 2), \quad p(\beta) = (-\sqrt{C-4}, 2, 0).
\end{align*}
\]

**SSP:** \( C = 4, p_H(t_0) \neq 0, \beta - \alpha = \sqrt{2} \), and
\[
\begin{align*}
p_H(t_0) > 0 & \Rightarrow p(\alpha) = (0, 2, 0), \quad p(\beta) = (2\sqrt{2}, 0, 2), \\
p_H(t_0) < 0 & \Rightarrow p(\alpha) = (-2\sqrt{2}, 0, 2), \quad p(\beta) = (0, 2, 0).
\end{align*}
\]

**ALT:** \( C \in (-4, 4), \beta - \alpha = \sqrt{C+4} \in (0, 2\sqrt{2}), \) and \( p(\alpha) = (-\sqrt{C+4}, 0, 2), p(\beta) = (\sqrt{C+4}, 0, 2) \).

It should be obvious why we used the labels (ALT),(SSP),(CSP). It is also obvious that the following triples \((g, p, u)\) are extremals \((t \in \mathbb{R})\):
\[
\left( \exp\left( \pm t \frac{1}{2} T \right), \ (0, \mp 2, 0), \ \pm \frac{1}{2} T \right), \quad \left( \exp\left( \pm t \frac{1}{2} U \right), \ (0, 0, \mp 2), \ \pm \frac{1}{2} U \right).
\]

We refer to these as the **singular arcs** because the control \( u(t) \) is not an extreme point of \( U \). The first one has \( \mathcal{C}(p) = 4 \) while the second one has \( \mathcal{C}(p) = -4 \). The geometric reason for the existence of these singular arcs is that for \( p \in \{(0, 0, \pm 2), (0, \pm 2, 0)\} \) the yield surface and the coadjoint orbit \( \mathcal{O}_p \) have first order contact: \( \partial H(p) \cap \mathbb{R} \partial \mathcal{C}(p) \neq \emptyset \).
We observe that an extremal with $C(p) = 4$ inevitably hits one of the points $(0, \pm 2, 0)$. There we may “glue” it together with a singular arc. Hence we obtain that our so-called list of extremals really consists of extremals.

In order to prove that our list of extremals is complete we must prove that switches must occur whenever $p(t)$ hits one of the edges at some $p \neq (0, \pm 2, 0)$. We first give a geometric explanation that would actually prove our claim if we knew that $p(t)$ is piecewise differentiable. Unfortunately the (PMP) only provides a $p(t)$ which is a.e. differentiable. Therefore a rigorous proof has to be provided. This will be done in the next subsection.

Let us consider $p = (p_H, 0, 2)$ first. The minimizing condition (2) implies $u \in \text{conv}(-P, Q)$. Next we compute

$$p \text{ ad}(-P) = (p_H, 0, 2) \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (2, -p_H, p_H)$$

$$p \text{ ad}(Q) = (p_H, 0, 2) \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = (-2, -p_H, -p_H).$$

Using the fact that $p(t)$ evolves on the yield surface, $p \text{ ad}(u)$ must be subtangent to the yield surface. A glance at Figure 6 convinces us that for $p_H \neq 0$ this determines $u$ uniquely. The same thing happens on the other edge for $p = (p_H, 2, 0)$ with $p_H \neq 0$.

Proposition 5.3 gives a flavor of how to attack a rigorous proof. The
remainder of this section is devoted to providing one, it is not mandatory for understanding the rest of this paper.

Rigorous proofs

In the previous discussion we already observed that the cases to distinguish are $C(p) = -4$, $C(p) \in (-4, 4)$, $C(p) = 4$, and $C(p) > 4$. They correspond (in this order) to $(U/2)$-, $(ALT)$-, $(SSP)$-, and $(CSP)$-extremals, respectively. W.l.o.g. we only consider extremals $(g, p, u)$ starting at $g(0) = 1$.

**Proposition 5.5** Let $(g, p, u)$ be an extremal such that $p_U = 0$ on an open interval $I$. Then $C(p) = 4$, $p(t)$ is constant in $I$, $p \equiv (0, \pm 2, 0)$, and $u(t) \equiv \mp \frac{1}{2} T$ in $I$.

**Proof.** Suppose $p_U \equiv 0 \in I$, then $h(p) \equiv -1$ implies $|p_T| \equiv 2$ in $I$. Hence $C(p) = p_H^2 + p_T^2 - p_U^2 = 4$ holds, whence $p(t)$ is constant in $I$. Thus $0 = \dot{p} = p \operatorname{ad}(u)$, so $u \in C_p \cap -\operatorname{sign}(p_T) \operatorname{conv}(P, Q)$. Since $g_p = \mathbb{R}(p_H H + p_T T)$, this intersection is nonempty iff $p_H = 0$. Thus $p \equiv (0, \pm 2, 0)$ and $u \equiv \mp \frac{1}{2} T$ in $I$. So $C(p) = 0 + 4 - 0$ in $I$ and hence for all $t$ as it is a first integral. □

**Proposition 5.6** Let $(g, p, u)$ be an extremal such that $p_T = 0$ on an open interval $I$. Then $C(p) = -4$, $p(t)$ is constant for all $t$, either $(0, 0, \pm 2)$, $u(t)$ is constant (either $\mp \frac{1}{2} U$), and $g(t) = \exp(\mp \frac{1}{2} U)$.

**Proof.** Suppose $p_T \equiv 0$ in some interval $I$. Then $|p_U| \equiv 2$ in $I$, and as in the previous proof we deduce that $p_H$ is constant in $I$, hence $p(t) \equiv (p_H, 0, p_U)$ is constant in $I$. Thus $0 = \dot{p} = p \operatorname{ad}(u)$ yields $u(t) \in g_p \cap -\operatorname{sign}(p_U) \operatorname{conv}(P, Q)$. As $g_p = \mathbb{R}(p_H H - p_U U)$, the intersection is nonempty iff $p_H = 0$. Hence $p(t) \equiv (0, 0, \pm 2)$ and $u(t) \equiv \mp \frac{1}{2} U$ in $I$. Also $C(p) = 0^2 + 0^2 - 4 = -4$.

Since $|p_U| \leq 2$ and $p_H^2 + p_T^2 - p_U^2 = -4$ hold for all $t$ (not just in $I$), we deduce $p_H^2 + p_T^2 = 0$ and $|p_U| = 2$ for all $t$. So $p \equiv (0, 0, \pm 2)$ and $u \equiv \mp \frac{1}{2} U$ follows for all $t$. Hence $g(t) = \exp(\mp \frac{1}{2} U)$ follows, too. □

The switching surface for our problem is $\Sigma := \{p : p_U = 0\} \cup \{p : p_T = 0\}$. Up to symmetry ($\Gamma$) we have to consider only two cases: $p_H < 0, p_T = 0, p_U = 2$ and $p_H < 0, p_T = 2, p_U = 0$.

**Proposition 5.7** Let $(g, p, u)$ be an extremal on some interval $I$ and $t_0 \in I$ such that $p(t_0) = (p_H^0, 0, 2)$ with $p_H^0 < 0$. Then there exists an $\epsilon > 0$ such that $p_T(t) < 0$ in $(t_0 - \epsilon, t_0)$ and $p_T(t) > 0$ in $(t_0, t_0 + \epsilon)$. In particular, $u(t) = Q$ a.e. in $(t_0 - \epsilon, t_0)$ and $u(t) = -P$ a.e. in $(t_0, t_0 + \epsilon)$. Hence a switch occurs.
Proof. Since \( p_U(t_0) = 2 \), take \( \epsilon \) such that \( p_U(t) > 0 \) for \( t \in I_\epsilon := I \cap (t_0 - \epsilon, t_0 + \epsilon) \). Then \( u(t) \in \text{conv}(-P, Q) \), so \( u(t) = -\frac{1}{2} U + \lambda(t) \cdot \frac{1}{2} T \) for some measurable function \( \lambda : I_\epsilon \to [-1, 1] \). Since \( p(t) \) is absolutely continuous, we have \( \dot{p} = p \text{ ad}(u) \) a.e. in \( I_\epsilon \). Hence

\[
\dot{p} = p \text{ ad}\left(-\frac{1}{2} U + \lambda \frac{1}{2} T\right) = (p_H, p_T, p_U) \begin{pmatrix}
0 & -1 & -\lambda \\
1 & 0 & 0 \\
-\lambda & 0 & 0
\end{pmatrix} = (*, -p_H, *).
\]

Thus \( \dot{p}_U(t) = -p_H(t) \) a.e. in \( I_\epsilon \). As \( p_H \) is continuous, \( p_T \) is differentiable in \( I_\epsilon \), and since \( p_T(t_0) = 0 \), \( \dot{p}_T(t_0) = -p_H(t_0) > 0 \) we find \( \epsilon \) such that \( p_U(t) < 0 \) in \((t_0 - \epsilon, t_0)\) and \( p_T(t) > 0 \) in \((t_0, t_0 + \epsilon)\). Hence \( u(t) = Q \) a.e. in \((t_0 - \epsilon, t_0)\), \( u(t) = -P \) a.e. in \((t_0, t_0 + \epsilon)\).

The other case is treated similarly.

**Proposition 5.8** Let \((g, p, u)\) be an extremal on some interval \( I \) and \( t_0 \in I \) such that \( p(t_0) = (p^0_H, 2, 0) \) with \( p^0_H < 0 \). Then there exists an \( \epsilon > 0 \) such that \( p_U(t) > 0 \) in \((t_0 - \epsilon, t_0)\) and \( p_T(t) < 0 \) in \((t_0, t_0 + \epsilon)\). In particular, \( u(t) = -P \) a.e. in \((t_0 - \epsilon, t_0)\) and \( u(t) = -Q \) a.e. in \((t_0, t_0 + \epsilon)\). Hence a switch occurs.

**Proof.** Since \( p_T(t_0) = 2 \), we find \( \epsilon > 0 \) such that \( p_T(t) > 0 \) for \( t \in I_\epsilon := I \cap (t_0 - \epsilon, t_0 + \epsilon) \). Then \( u(t) \in \text{conv}(-P, -Q) \), so \( u(t) = -\frac{1}{2} T + \lambda(t) \cdot \frac{1}{2} U \) for some measurable function \( \lambda : I_\epsilon \to [-1, 1] \). Since \( p(t) \) is absolutely continuous, we have \( \dot{p} = p \text{ ad}(u) \) a.e. in \( I_\epsilon \). Hence

\[
\dot{p} = p \text{ ad}\left(-\frac{1}{2} T + \lambda \frac{1}{2} U\right) = (p_H, p_T, p_U) \begin{pmatrix}
0 & \lambda & 1 \\
-\lambda & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} = (*, *, p_H).
\]

Thus \( \dot{p}_U(t) = p_H \) a.e., so \( p_U \) is differentiable in \( I_\epsilon \) because \( p_H \) is continuous. Since \( p_U(t_0) = 0 \) and \( \dot{p}_U(t_0) = p_H(t_0) < 0 \), we find \( \epsilon_1 > 0 \) such that \( p_U(t) > 0 \) in \((t_0 - \epsilon_1, t_0)\) and \( p_U(t) < 0 \) in \((t_0, t_0 + \epsilon_1)\). Hence \( u(t) = -P \) a.e. in \((t_0 - \epsilon_1, t_0)\) and \( u(t) = -Q \) a.e. in \((t_0, t_0 + \epsilon_1)\).

This finishes the proof that our list of extremals is complete. It is noteworthy to mention that the whole discussion involves only the Lie algebra, its dual, and the adjoint, resp. coadjoint action. In particular our results are valid for **any** group with Lie algebra isomorphic to \( sl(2) \).
6 A sufficient family for $\text{SL}(2)$

Although the (PMP) provides very detailed information, it is only a first-order necessary condition. We need two types of additional arguments: global arguments (no local condition could replace them) and higher order variations. Our goal is to provide a sufficient family of paths, i.e., a finite set of maps $f_k : \mathbb{R}^3 \to \text{SL}(2)$ such that for every $g \in \text{SL}(2)$ some map $f_k$ provides an optimal path $1 \sim g$.

Our classification of extremals provides us with candidates. Looking at alternating extremals we come up with

$$A_3(r, s, t) = M(rP, -sQ, tP), \quad 0 \leq r, t \leq s < 2\sqrt{2},$$

$$A_4(r, s, t) = M(rP, -sQ, sP, -tQ),$$

$$A_5(r, s, t) = M(rP, -sQ, sP, -sQ, tP),$$

and $A_6, A_7, \ldots$ are defined similarly. Due to symmetry it suffices to consider only (ALT)-extremals starting with an $\exp(rP)$-arc because the group $\Gamma$ acts transitively on $\{\pm P, \pm Q\}$.

Similarly, when considering (CSP)-extremals, there are up to 8 possibilities (4 for the first vertex, 2 for the orientation: clockwise/anticlockwise). The group $\tilde{\Gamma}$ has 8 elements, and it turns out that for an odd number of factors it suffices to consider 1 case, whereas for an even number of factors there are 2 distinguished cases. We may always assume that the first factor is $\exp(rP)$, but we must distinguish between clockwise and anticlockwise switching patterns if the total number of factors is even. We obtain a sequence of maps starting with:

$$C_3(r, s, t) = M(rP, sQ, -tP), \quad 0 \leq r, t \leq s < \sqrt{2},$$

$$C_4a(r, s, t) = M(rP, -sQ, -sP, tQ),$$

$$C_4c(r, s, t) = M(rP, sQ, -sP, -tQ).$$

For (SSP)-extremals we can always assume that (up to symmetry) the first singular ($S$-) arc is $\exp(sT)$, and that the preceding bang ($B$-) arc is $\exp(rP)$. For example, for 3 or 4 factors with one singular arc we obtain the maps:

$$S_3P(r, s, t) = M(rP, \frac{s}{2}T, tP), \quad s \geq 0, \quad r, t \in [0, \sqrt{2}],$$

$$S_3Q(r, s, t) = M(rP, \frac{s}{2}T, tQ),$$

$$S_4P(r, s, t) = M(rP, \frac{s}{2}T, \sqrt{2}P, -tQ),$$

$$S_4Q(r, s, t) = M(rP, \frac{s}{2}T, \sqrt{2}Q, -tP).$$

Enumerating all possible patterns for a large number, say $n$, of factors is an unpractical task since the number of possibilities grows exponentially in
After a singular \((S-)\) arc we always have two choices for the next bang \((B-)\) arc. But fortunately enough, it turns out that it suffices to consider (SSP)-extremals with one \(S\)-arc only. The proof is basically a verification of the identity:

\[
M(sT, \sqrt{2} P, -\sqrt{2} Q) = M(\sqrt{2} P, -\sqrt{2} Q, -sT). \tag{*}
\]

Let us abbreviate \(w := \sqrt{2}\). Then \((*)\) implies

\[
M(rP, s_1 T, wP, -wQ, -s_2 T) = M(rP, (s_1 + s_2)T, wP, -wQ). \tag{8}
\]

Occurrence of this identity is not a miracle but it is kind of locally detected by the (PMP):

\[e^{r\text{ad}Q}e^{-r\text{ad}P}T = -T \iff r^2 = 2.\]

So the singular switching time is in a certain sense geometrically distinguished. The hyperbolic Reeds-Shepp-Car problem allows to visualize this neatly, (cf. Fig. 9, p. 57). This is completely analogous to the euclidean case, cf. [18, Fig. 18, p.59].

Eventually it turns out that one only needs three more maps:

\[
S5P(r, s, t) = M(-rQ, wP, \frac{1}{2} T, wP, -tQ),
S5Q(r, s, t) = M(-rQ, wP, \frac{1}{2} T, wQ, -tP),
S7b(r, s, t) = M(-rQ, wP, \frac{1}{2} T, wP, -wQ, -wQ, tP) \tag{8}
\]

In addition, it is convenient to consider also the following maps which are derived from the previous ones:

\[
S5a(r, s, t) = M(rP, \frac{1}{2} T, wP, -wQ, -tQ) = S4P(r, s, t + w)
S6(r, s, t) = M(-rQ, wP, \frac{1}{2} T, wP, -wQ, -tQ) = S5P(r, s, t + w)
S7a(r, s, t) = M(rP, \frac{1}{2} T, wP, -wQ, -wQ, wP, tP) = \sigma_U(S5P(r + w, s, t + w)). \tag{9}
\]

Table 1 specifies a family \(\mathcal{F}\) consisting entirely of extremals. We obtain a list of \(|\mathcal{F}| = 76\) maps, but since \(\mathcal{F}\) is \(\tilde{\Gamma}\)-invariant, it suffices to specify one map from each \(\tilde{\Gamma}\)-orbit. There are 16 such orbits. We list a representative, a subgroup generating the orbit and, if nontrivial, “the” stabilizer in columns 2–4.

Now we can state the main result for SL(2).

**Theorem 6.1** Let \(\mathcal{F}\) denote the family of maps specified in Table 1. Then \(\mathcal{F}\) is a sufficient family for (OCP) in SL(2), i.e., for every \(g \in \text{SL}(2)\) there exist \(f \in \mathcal{F}\) and \((r, s, t) \in \text{dom}(f)\) such that \(f(r, s, t) = g\), and the associated path from \(1\) to \(g\) has minimal length \(T(g)\).

The proof will be given in the next section. Observing \(S7b(r, s, t) = S5P(r, s, 2\sqrt{2}) \exp(tP)\), we immediately obtain
| Type | Map | Symmetry | Domain | Cost |
|------|-----|----------|--------|------|
| ALT  | A3  | Γ 4 $\sigma_H$ | $0 \le r, t \le s \le 2\sqrt{2}$ | $r + s + t \in [0, 6\sqrt{2}]$ |
|      | A4  | Γ 4 $\sigma_T$ | $s \ge 1$ and $s \in [\sqrt{2}, \sqrt{3}]$ | $r + 2s + t \in [2, 8\sqrt{2}]$ |
|      | A5  | Γ 4 $\sigma_H$ | $s \in [\sqrt{2}, \sqrt{3}]$ and $s \le 2\sqrt{2}$ | $r + 3s + t \in [3\sqrt{2}, 5\sqrt{3}]$ |
| CSP  | C3  | $\tilde{\Gamma}$ 8 | $0 \le r, t \le s \le 2\sqrt{2}$ | $r + s + t \in [0, 3\sqrt{2}]$ |
|      | C4a | Γ 4 $\sigma_U$ | and $s \le 1$ | $r + 2s + t \in [0, 4]$ |
|      | C4c | Γ 4 $\sigma_T$ | | $r + 2s + t \in [0, 4]$ |
| SSP  | S3P | Γ 4 $\sigma_H$ | $s \ge 0$, $r, t \in [0, \sqrt{2}]$ | $r + s + t \ge 0$ |
|      | S3Q | Γ 4 $\sigma_U$ | | |
|      | S4P | $\tilde{\Gamma}$ 8 | $r + s + t + \sqrt{2} \ge \sqrt{2}$ | |
|      | S4Q | $\tilde{\Gamma}$ 8 | | |
|      | S5P | Γ 4 $\sigma_H$ | | $r + s + t + 2\sqrt{2} \ge 2\sqrt{2}$ |
|      | S5Q | Γ 4 $\sigma_U$ | | |
|      | S5a | Γ 4 $\sigma_T$ | | |
|      | S6  | $\tilde{\Gamma}$ 8 | | $r + s + t + 3\sqrt{2} \ge 3\sqrt{2}$ |
|      | S7b | Γ 4 $\sigma_T$ | | |
| $\mathcal{F}$ | 16 | 76 | |

Use bigger domains and drop some maps

| drop | S5a | Γ 4 | $S4P : [0, \sqrt{2}] \times \mathbb{R}_+ \times [0, 2\sqrt{2}] \to \text{SL}(2)$ |
| drop | S6  | $\tilde{\Gamma}$ 8 | $S5P : [0, 2\sqrt{2}] \times \mathbb{R}_+ \times [0, 2\sqrt{2}] \to \text{SL}(2)$ |
| $\mathcal{F}_1$ | 13 | 64 | |
| drop | A3  | Γ 4 | |
| drop | C3  | $\tilde{\Gamma}$ 8 | |
| add  | B3  | $\sigma_T$ 2 | $\mathbb{M}(rP, sQ, tP), \ |r|, |s|, |t| \le 2\sqrt{2}$ |
| $\mathcal{F}_2$ | 12 | 54 | |

Table 1: A sufficient family of extremals for $\text{SL}(2)$ and smaller families that suffice for computing optimal factorizations, resp., $\mathcal{T}(g)$
Corollary 6.2 Every $g \in \text{SL}(2)$ has an optimal factorization with at most 6 factors from $\{\pm P, \pm Q, \pm T\}$. There always exists a geodesic from 1 to $g$ with at most 5 switches.

As is indicated in Table 1, there exist smaller families (fewer maps but with bigger domains) which are sufficient, too. For example, we find a family $F_1$ consisting of 64 maps, resp., 13 $\tilde{\Gamma}$-orbits, and another family $F_2$ consisting of 54 maps, resp., 12 orbits.

Corollary 6.3 The families $F_1$, $F_2$ specified in Table 1 are sufficient for (OCP), too.

The proof is an immediate consequence of the preceding theorem. The family $F$ is the appropriate one for visualizing the metric spheres. That’s the reason why we wanted to keep the domains as small as possible. The family $F_2$ is appropriate for computing $T(g)$. In that case we want to keep the number of maps that have to be inverted as small as possible. Finally, $F_1$ is appropriate if we want to find optimal factorizations because in that case we have to keep track of the type of factorization (ALT or CSP). Thus it makes sense not to merge $A_3, C_3$ into $B_3$.

Since computation of $T(g)$ as well as finding optimal factorizations requires inverting the maps from $F_2$, resp., $F_1$, it is important to realize that this is an easy task which can be carried out efficiently at low computational costs. Therefore we observe:

Remark 6.4 We have $\exp(tP) = \text{id} + tP$, and $\exp(tQ) = \text{id} + tQ$. In particular $f(r, s, t)$ is linear in $r$ and $t$ for every $f \in F$. The dependence on $s$ is as follows:

| $A_3, C_3, B_3$ | $A_4, C_4a, C_4c$ | $A_5$ | SSP |
|-----------------|-------------------|-------|-----|
| linear in $s$   | quadratic in $s$  | cubic in $s$ | linear in $\xi, \xi^{-1}$ where $\xi = e^s > 0$.

$\sim$ quadratic equation in $\xi$.

Hence inverting the maps in $F$ is nothing but a (time consuming) exercise in college algebra, except for $A_5$ where we have $p(s) = e_2^T A_5(r, s, t)e_1 = s^3 - 2s$.

It suffices to invert one map from each $\tilde{\Gamma}$-orbit because $\sigma(f(r, s, t)) = g$ iff $f(r, s, t) = \sigma^{-1}(g)$. So instead of inverting, say, all 54 maps in $F_2$, it suffices to invert 12 representatives and apply them to several right hand sides, according to Table 1. An efficient implementation can also use the information on the domains to “discard” solutions of $f(r, s, t) = g$ as soon as one of the parameters $r, s, t$ is not in the appropriate range, i.e., dom($f$). A smart implementation also uses the fact that some extremals inevitably generate large cost. For example, an optimal $A_5$-extremal has at least cost $3\sqrt{2}$. Hence it is not necessary to check $A_5$-type factorizations if some other factorization has already given cost less than $3\sqrt{2}$.
7 Comparison arguments

In this section we provide all the arguments necessary to prove Theorem 6.1. The proof is based on comparison arguments. We will show that certain factorizations are not optimal. Hence they can never appear as a subarc of an optimal arc.

We must treat the four classes of extremals separately. The singular \((U/2)\)-extremals and the (CSP)-extremals are most pleasant in the sense that the proofs are purely Lie algebraic. Therefore they apply to any group with Lie algebra \(sl(2)\). The arguments for (ALT)- and (SSP)-extremals (partly) make very explicit use of the underlying group. We start with the elimination of \((U/2)\)-extremals.

**Proposition 7.1** For all \(\alpha \in \mathbb{R}\) the “factorization” \(\exp(2\alpha \frac{1}{2}(P - Q))\) is not optimal. In particular, optimal controls take values only in \(\{\pm P, \pm Q, \pm \frac{1}{2}T\}\), and optimal factorizations only use factors from \(\exp(\mathbb{R}P \cup \mathbb{R}Q \cup \mathbb{R}T)\).

**Proof.** W.l.o.g. let us consider \(\alpha > 0\) small. With \(r = \tan(\alpha/2)\) and \(s = \sin(\alpha)\) we obtain

\[
\exp(2\alpha \frac{1}{2}(P - Q)) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} = M\begin{pmatrix} rP, -sQ, rP \end{pmatrix} = A3(r, s, r).
\]

The cost of the lefthand side is \(2\alpha\), on the righthand side we have cost \(\sin(\alpha) + 2\tan(\alpha/2)\). Consider the difference \(f(\alpha) = 2\alpha - \sin(\alpha) - 2\tan(\alpha/2)\). Then a simple Taylor expansion yields

\[
0 = f(0) = f'(0) = f''(0), \quad \text{and} \quad f'''(0) = \frac{1}{2}.
\]

Hence for small \(\alpha > 0\) the factorization on the righthand side is better than the factorization on the lefthand side.

**Remark 7.2** For the Hyperbolic Reeds-Shepp-Car Problem the previous result has an interesting consequence. It says that a “rotation on the spot” is never optimal. So if one wants to drive and return to the starting point but heading into a different direction, it is better to move forward (or backward)! This is in contrast to the euclidean case where it doesn’t matter if one moves forward or turns on the spot: both paths have equal cost. Of course this is important, because a rotation on the spot is not feasible for the Reeds-Shepp-Problem, it is only feasible for the convexified problem!

This property reflects the fact that in a euclidean triangle the sum of the angles equals \(\pi\) whereas in the hyperbolic plane this sum is strictly less than \(\pi\).
7.1 Circular (CSP)-extremals

Our next goal is to prove:

**Proposition 7.3** Optimal (CSP)-extremals have at most 4 factors. This is actually true for any Lie group \( G \) with Lie algebra \( \mathfrak{sl}(2) \).

The proof of this result requires two different types of arguments. The first one is elementary and involves only the adjoint action. It is a matter of elementary computations to verify that

\[
e^{r \text{ad} Q}(-P) = e^{1/r \text{ad} P}(r^2 Q), \quad \text{and} \quad e^{r \text{ad} P}(-Q) = e^{1/r \text{ad} Q}(r^2 P), \quad r \neq 0.
\]

**Proposition 7.4** The factorization \( \mathbb{M}(rP, -sQ, -rP) \) is not optimal for \( s \in (1, \sqrt{2}) \) and \( r > 1 \). The factorization \( \mathbb{M}(P, -Q, -P, tQ) \) is not optimal for \( t > 0 \). In particular, optimal (CSP)-extremals with switching time \( s \geq 1 \) have at most 4 factors.

**Proof.** Let \( s \in (1, \sqrt{2}) \) and \( r > 1 \), assume w.l.o.g. \( r < s \). Then we obtain \( \mathbb{M}(rQ, -sP, -rQ) = \mathbb{M}(\frac{1}{r}P, r^2 s Q, -\frac{1}{r}P) \) from the above equation. Comparing the factorization costs on both sides we obtain

\[
(s + 2r) - \left( \frac{2}{r} - r^2 s \right) = \frac{1}{r} (r^2 - 1)(2 - rs) > 0 \quad \text{if } s \in (1, \sqrt{2}), \ r \in (1, s).
\]

Thus \( \mathbb{M}(rP, -sQ, -rP) \) is not optimal for \( s \in (1, \sqrt{2}) \) and \( r > 1 \).

To prove the second claim we consider \( r = s = 1 \) and obtain

\[
\mathbb{M}(P, -Q, -P, tQ) = \mathbb{M}(Q, -P, -Q, tQ) = \mathbb{M}(Q, -P, -(1 - t)Q).
\]

The lefthand side has cost \( 3s + t \) as opposed to \( 3s - t \) on the righthand side. Thus our claim is proved. \( \square \)

While this was quite elementary, proving that optimal (CSP)-extremals with switching time \( s \in (0, 1) \) have at most 4 factors requires some more sophisticated arguments. Therefore we first supply another, relatively simple argument, that allows to reduce the number of factors to 5:

**Proposition 7.5** For all \( s > 0 \) the factorization \( \mathbb{M}(sP, -sQ, -sP, sQ) \) is not optimal.
Proof. For $s > 0$ let $r = \frac{s}{1+s^2} \in (0, s)$. Then an elementary computation yields:

$$\mathbb{M}(rP, -sQ, -sP, rQ) = \mathbb{M}(-rP, sQ, sP, -rQ).$$

Thus

$$\mathbb{M}(sP, -sQ, -sP, sQ) = \mathbb{M}((s-r)P, rP, -sQ, -sP, rQ, (s-r)Q) = \mathbb{M}((s-2r)P, -sQ, sP, (s-2r)Q).$$

The cost of LHS and RHS are $4s$, resp., $2s + 2|s-2r|$. As $0 < r < s$, we deduce $|s-2r| < s$, i.e., RHS is better.

Remark 7.6 The preceding argument has a nice geometric interpretation. Both factorizations (or arcs) actually generate a parallel translation along the geodesic perpendicular to the initial tangent. So initial and terminal tangent are both perpendicular to the geodesic joining the initial and terminal point.

A kind of second order variational argument allows us to prove the following stronger result:

Proposition 7.7 Let $G$ be an arbitrary Lie group with Lie algebra $\mathfrak{sl}(2)$. Then for $s \in (0, 1)$ and $t > 0$ the factorization $\mathbb{M}(sQ, -sP, -sQ, tP)$ is not optimal.

Proof. Fix $s \in (0, 1)$ and consider $\gamma(t) = \mathbb{M}(sQ, -sP, -sQ, tP)$. Let $F : \mathbb{R}^3 \to G$, $F(x) = \mathbb{M}(x_1P, x_2Q, -x_2P, -x_3Q)$. Then $\gamma(0) = F(0, s, s)$. We claim that the following statements hold true:

1. The differential $dF(0, s, s)$ is invertible. Therefore there exist $t_0 > 0$ and a smooth (actually analytic) curve $x : (-t_0, t_0) \to \mathbb{R}^3$ such that $F(x(t)) = \gamma(t)$ for all $|t| < t_0$. 

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(2) We have \( x_k(t) > 0 \) for \( t \in (0, t_0) \), \( k = 1, 2, 3 \).

(3) Let \( \delta(t) = 3s + t - (x_1(t) + 2x_2(t) + x_3(t)) \).

Then \( 0 = \delta(0) = \delta'(0) \), and \( \delta''(0) > 0 \).

Once we have proved (1)–(3) it is clear that \( M(sQ, -sP, -sQ, tP) \) cannot be optimal because for small \( t > 0 \) the map \( F(x(t)) \) provides a better factorization. So let us prove (1)–(3). First we must compute the differential \( dF(x) \). We observe:

\[
\begin{align*}
\frac{dF(x)e_1}{x} &= F(x)e^{x_3 \text{ad}Q}x_2 \text{ad}P e^{-x_2 \text{ad}Q}P, \\
\frac{dF(x)e_2}{x} &= F(x)e^{x_3 \text{ad}Q}x_2 \text{ad}P (Q - P), \\
\frac{dF(x)e_3}{x} &= F(x)(-Q) \\
&= d\lambda_{F(x)}e^{x_3 \text{ad}Q}x_2 \text{ad}P e^{-x_2 \text{ad}P}(-Q).
\end{align*}
\]

Thus \( dF(x) = F(x)e^{x_3 \text{ad}Q}x_2 \text{ad}P M(x_2) \) with

\[
M(x_2) = \left[ e^{-x_2 \text{ad}P}Q, Q - P, e^{-x_2 \text{ad}P}(-Q) \right].
\]

In terms of the basis \( \{H, P, Q\} \) we compute

\[
M(s) = \begin{pmatrix}
  s & 0 & s \\
  1 & -1 & s^2 \\
-s^2 & 1 & -1
\end{pmatrix}, \quad \det M(s) = -2s(s^2 - 1).
\]

Since \( 0 < s < 1 \), Claim (1) is proved. Next we use the Implicit Function Theorem (IFT) to compute the derivate \( \dot{x}(0) \). Differentiating \( F(x(t)) = \gamma(t) \) we obtain \( dF(x(t)) \dot{x}(t) = \dot{\gamma}(t) = d\lambda_{\gamma(t)}(1)P \). Thus

\[
e^{x_3(t) \text{ad}Q}x_2(t) \text{ad}P M(x_2(t)) \dot{x}(t) \equiv P.
\]

Since \( x(0) = (0, s, s) \), we deduce \( \dot{x}(0) = M(s)^{-1}e^{-s \text{ad}P}e^{-s \text{ad}Q}P \). Simplification with MATHEMATICA yields:

\[
\dot{x}(0) = \begin{pmatrix}
  s^2 + 2 \\
  2(1 - s^2) \\
-2s^2 \\
2(1 + s^2)
\end{pmatrix}.
\]

Since \( x_2(0) = x_3(0) = s > 0 \), continuity of \( x(t) \) yields \( x_2(t), x_3(t) > 0 \) for sufficiently small \( t > 0 \). Now \( x_1(0) = 0 \) and \( \dot{x}_1(0) > 0 \) because of \( 1 - s^2 > 0 \), hence \( x_1(t) > 0 \) for small positive \( t \) follows, too. Thus we may assume w.l.o.g. that \( x_k(t) > 0 \) for \( t \in (0, t_0) \), \( k = 1, 2, 3 \) which proves Claim (2).

Verifying \( \delta(0) = 0 = \delta'(0) \) is straight forward. Finally, we must use the (IFT) again to compute \( \delta''(0) = -(\dot{x}_1(0) + 2\dot{x}_2(0) + \dot{x}_3(0)) \). Differentiating the identity

\[
M(x_2(t)) \dot{x}(t) = e^{-x_2(t) \text{ad}P}e^{-x_3(t) \text{ad}Q}P.
\]

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we obtain
\[ M(x_2)\ddot{x} = -\dot{x}_2 M'(x_2)\dot{x} - \dot{x}_2 e^{-x_2(t)} ad_P ad(P) e^{-x_3(t)} ad_Q P \]
\[ - \dot{x}_3 e^{-x_2(t)} ad_P e^{-x_3(t)} ad_Q ad(Q) P. \]
Simplification with Mathematica yields
\[ \ddot{x}(0) = \begin{pmatrix} \frac{-3s^3(3+s^2)}{4(-1+s^2)^3} \\ \frac{-s^3}{4} \\ \frac{s^3(-1+5s^2+8s^4)}{4(-1+s^2)^2} \end{pmatrix}, \text{ and } \delta''(0) = \frac{3s^3(2+s^2)}{2(1-s^2)}. \]
For \(0 < s < 1\) the denominator is positive, hence \(\delta''(0) > 0\) follows. This proves Claim (3) and finishes the proof of the proposition.

Now we are ready to prove Proposition 7.3.

Proof. Consider a (CSP)-extremal with 5 factors. Up to symmetry (\(\Gamma\)) we may assume that the first factor is \(\exp(rP)\). The switching pattern is clockwise or counterclockwise:
\[ M(rP, sQ, -sP, -sQ, tP), \text{ or } M(rP, -sQ, -sP, sQ, tP), \]
with \(0 < r, t < s < \sqrt{2}\). Recalling that \(\iota \sigma_H\) simply reverts the factors, we see that the second pattern is transformed into the first pattern.

For \(s \geq 1\) Proposition 7.4 applies, showing that this arc contains a non-optimal subarc. For \(s \in (0, 1)\) we apply the previous proposition and obtain that \(M(sQ, -sP, -sQ, tP)\) is not optimal. This finishes the proof.

We conclude our discussion of (CSP)-extremals with one more observation:

Proposition 7.8 Let \(r > 0\) and \(s \in (1, \sqrt{2})\). Then \(M(rP, sQ, -sP)\) is not optimal. In particular, an optimal (CSP)-extremal with switching time \(> 1\) has at most 3 factors. Conversely, if C4a\((r, s, t)\) or C4c\((r, s, t)\) is optimal, then \(s \leq 1\) and \(r + 2s + t \leq 4\).

Proof. We compute \(M(rP, sQ, -sP) = M(x_1Q, x_2P, x_3Q)\) with
\[ x_2 = -(s + r(s^2 - 1)), \quad x_1 = \frac{s^2}{x_2}, \text{ and } x_3 = \frac{rs}{x_2}. \]
For \(s \geq 1\) and \(r \geq 0\) we have \(x_2 \leq -1, x_1 \geq 0, \text{ and } x_3 \leq 0\). Hence \(|x_1| + |x_2| + |x_3| = x_1 - x_2 - x_3, \text{ and }\)
\[ (2s + r) - (x_1 - x_2 - x_3) = \frac{r^2(s^2 - 1)(2 - s^2)}{s + r(s^2 - 1)}. \]
For \(r > 0\) and \(s \in (1, \sqrt{2})\) the righthand side is positive, hence \(M(rP, sQ, -sP)\) is not optimal.
7.2 Alternating (ALT)-extremals

While the results for optimal (CSP)-extremals are valid in any group $G$ with Lie algebra $\mathfrak{sl}(2)$, some of the results for (ALT)-extremals depend explicitly on $G$. The crucial parameter is actually $|Z(G)| =: n$, the cardinality of the center of $G$. We start with results that hold for any $G$. Whenever we make explicit use of $G$ (resp. $n$) we will indicate this clearly.

**Proposition 7.9** For $s \in (0, 1)$ and $r > 0$ the factorization $M(rP, -sQ, sP)$ is not optimal. In particular, an optimal (ALT)-extremal with switching time $s \in (0, 1)$ has at most 3 factors. Conversely, if $A4(r, s, t)$ is optimal, then $s \geq 1$.

**Proof.** We compute $M(rP, -sQ, sP) = M(s_1Q, x_2P, s_3Q)$ with

$$x_1 = -\frac{s^2}{s + r(1 - s^2)}, \quad x_2 = s + r(1 - s^2), \quad x_3 = -\frac{rs}{s + r(1 - s^2)}.$$  

For $s \in (0, 1)$ and $r > 0$ we deduce $x_2 > 0$ and $x_1, x_3 < 0$, so the second factorization has cost $-x_1 + x_2 - x_3$. The cost difference is

$$r + 2s - (-x_1 + x_2 - x_3) = \frac{r^2 s^2 (1 - s^2)}{s + r (1 - s^2)}.$$  

For $s \in (0, 1)$ and $r > 0$ numerator and denominator are both positive, so $r + 2s - (-x_1 + x_2 - x_3) > 0$. Hence $M(rP, -sQ, sP)$ is not optimal. $\Box$

**Proposition 7.10** The factorizations

$$M(P, -Q, P, -tQ) \quad \text{and} \quad M(\sqrt{2}P, -\sqrt{2}Q, \sqrt{2}P, -\sqrt{2}Q, tP)$$

are not optimal for $t > 0$. In particular, an optimal (ALT)-extremal with $s = 1$ has at most 4 factors, and with $s = \sqrt{2}$ it has at most 5 factors.

**Proof.** Let $w = \sqrt{2}$. An elementary computation yields

$$M(P, -Q, P) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M(wP, -wQ, wP, -wQ) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

Since $\sigma_U(U) = U$ and $\sigma_U(-1) = -1$, we get $M(-Q, P, -Q) = M(P, -Q, P)$ and $M(wP, -wQ, wP, -wQ) = M(-wQ, wP, -wQ, wP)$. Hence

$$M(P, -Q, P, -tQ) = M(-Q, P, -(t + 1)Q),$$

$$M(wP, -wQ, wP, -wQ, tP) = M(-wQ, wP, -wQ, (t + w)P).$$
In each case LHS and RHS have equal cost, but for \( t > 0 \) the righthand side is not an extremal, hence it cannot be optimal. We also observe that for all \( r \in \mathbb{R} \) the identities

\[
\mathbb{M}((1-r)P, -Q, P, -rQ) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
\mathbb{M}((w-r)P, -wQ, wP, -wQ, rP) \equiv -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

hold true. Therefore an optimal (ALT)-extremal with \( s = 1 \) and 4 factors may be replaced (at equal cost) by another extremal with 3 factors. Similarly, for switching time \( s = \sqrt{2} \) it suffices to consider at most 4 factors. \( \square \)

The same technique as in the proof of Proposition 7.7 can be used to obtain:

**Proposition 7.11** For \( 1 < s < \sqrt{2} \) the factorization \( \mathbb{M}(-sQ, sP, -sQ, tP) \) is not optimal. In particular, optimal (ALT)-extremals with switching time \( s \in (1, \sqrt{2}) \) have at most 4 factors. Conversely, if \( A5(r, s, t) \) is optimal, then \( s \geq \sqrt{2} \).

**Proof.** Let \( \gamma(t) = \mathbb{M}(-sQ, sP, -sQ, tP) \) and consider \( F: \mathbb{R}^3 \to G, F(x) = \mathbb{M}(x_1 P, -x_2 Q, x_2 P, -x_3 Q) \). Then \( \gamma(0) = F(0, s, s) \). We claim that \( dF(0, s, s) \) is invertible and that there exists a \( t_0 > 0 \) and a smooth curve \( x: (-t_0, t_0) \to \mathbb{R}^3 \) such that \( x_k(t) > 0 \) for \( t \in (0, t_0) \), \( k = 1, 2, 3 \), and for \( \delta(t) = 3s + t - (x_1(t) + 2x_2(t) + x_3(t)) \) we have \( 0 = \delta(0) = \delta'(0), \delta''(0) > 0 \).

Computing the differential of \( F \) yields:

\[
dF(x) = F(x) e^{x_3 \text{ad}_Q} e^{-x_2 \text{ad}_P} M(x_2)
\]

with

\[
M(x_2) = \begin{bmatrix} e^{x_2 \text{ad}_Q} P, & -Q + P, & e^{x_2 \text{ad}_P} (-Q) \end{bmatrix}.
\]

In terms of the basis \( \{H, P, Q\} \) we obtain

\[
M(s) = \begin{pmatrix} -s & 0 & -s \\ 1 & 1 & s^2 \\ -s^2 & -1 & -1 \end{pmatrix}, \quad \text{det } M(s) = -2s(s^2 - 1).
\]

Differentiating \( F(x(t)) = \gamma(t) \) yields

\[
e^{x_3 \text{ad}_Q} e^{-x_2 \text{ad}_P} M(x_2) \dot{x} = P, \quad M(x_2) \dot{x} = e^{x_2 \text{ad}_P} e^{-x_3 \text{ad}_Q} P.
\]
For $x(0) = (0, s, s)$ we obtain
\[ \dot{x}(0) = \left( \frac{s^2}{-2 + 2s^2}, 1 - \frac{s^2}{2}, \frac{2 - 5s^2 + 2s^4}{-2 + 2s^2} \right). \]

As $s > 1$, $\dot{x}_1(0) > 0$. Therefore $x_k(t) > 0$ for $t > 0$ small follows. With $\delta(t) = 3s + t - (x_1 + 2x_2 + x_3)$ we have $\delta(0) = 0 = \delta'(0)$. Differentiating once more we deduce
\[ M(x_2)\ddot{x} = -\dot{x}_2 M'(x_2)\dot{x}_2 e^{x_2 \text{ad} P} \text{ad}(P)e^{-x_3 \text{ad} Q} P - \dot{x}_3 e^{x_2 \text{ad} P} e^{-x_3 \text{ad} Q} \text{ad}(Q) P. \]

MATHEMATICA yields:
\[ \ddot{x}(0) = \left( \frac{4 - 8s^2 - 3s^4 + 3s^6}{4s(-1 + s^2)^2}, \frac{-(-2 + s^2)^2}{4s}, \frac{(-2 + s^2)^2 (1 - 5s^2 + 8s^4)}{4s(-1 + s^2)^2} \right), \]
and
\[ \delta''(0) = \frac{s(4 - 8s^2 + 3s^4)}{2 - 2s^2} = \frac{(s(2 - s^2)(-2 + 3s^2))}{2(-1 + s)(1 + s)}. \]

For $1 < s < \sqrt{2}$ the numerator is positive and so is the denominator, hence $\delta''(0) > 0$ as we claimed. This finishes the proof.

Up to this point all statements hold true in any group with Lie algebra $\mathfrak{sl}(2)$. The last proposition in this subsection makes very explicit use of $\text{SL}(2)$.

**Proposition 7.12** Let $s > \sqrt{2}$ and $r, t \geq 0$. If $r + t > \frac{2s}{s^2 - 1}$, then the factorization $M(rP, -sQ, sP, -tQ)$ is not optimal in $\text{SL}(2)$. In particular,
\[
\begin{align*}
& s > \frac{\sqrt{3}}{3}, \quad r, t \geq 0 \quad \Rightarrow \quad M(-sQ, sP, -sQ) \quad \text{not optimal in } \text{SL}(2), \\
& s = \frac{\sqrt{3}}{3}, \quad t > 0 \quad \Rightarrow \quad M(-sQ, sP, -sQ, tP) \quad \text{not optimal in } \text{SL}(2), \\
& s \in (\sqrt{2}, \sqrt{3}) \quad \Rightarrow \quad M(-sQ, sP, -sQ, sP) \quad \text{not optimal in } \text{SL}(2).
\end{align*}
\]

In particular, optimal (ALT)-extremals in $\text{SL}(2)$ have at most 5 factors, and if $A_5(r, s, t)$ is optimal, then $s \in [\sqrt{2}, \sqrt{3}]$.

**Proof.** For $s \neq \pm 1$ let $\mu = \frac{2s}{s^2 - 1}$, then
\[ M(\mu P, -sQ, sP) = M(sQ, -sP, \mu Q). \]

Now assume w.l.o.g. that $r, t \leq \mu$. Then
\[
\begin{align*}
M(rP, -sQ, sP, -tQ) &= M(-(\mu - r)P, \mu P, -sQ, sP, -\mu Q, (\mu - t)Q) \\
&= M(-(\mu - r)P, sQ, -sP, (\mu - t)Q).
\end{align*}
\]
The LHS has cost $2s + r + t$, the RHS has cost $2s + 2\mu - (r + t)$. As $r + t > \mu$, $2s + r + t > 2s + \mu > 2s + 2\mu - (r + t)$, so LHS is not optimal.

For $s > \sqrt{3}$ we can apply the previous result with $r = 0$ and $t = s > \mu$. For $s = \sqrt{3}$ we observe $\mu = s = \sqrt{3}$, and

$$M(sP, -sQ, sP, -tQ) = M(sQ, -sP, sQ, -tQ) = M(sQ, -sP, -(s - t)Q).$$

Comparing costs we obtain $3s + t$ (LHS) and $3s - t$ (RHS), so LHS is not optimal for $t > 0$.

Finally, for $s \in (\sqrt{2}, \sqrt{3})$ we have $s^2 - 1 > 1$, hence $2s > \frac{2s}{s^2 - 1}$. Taking $r = t = s$ we therefore have $r + t = 2s > 2\mu$. Hence $M(sP, -sQ, sP, -sQ)$ is not optimal in SL(2), whence optimal (ALT)-extremals cannot have $4 + 2 = 6$ factors.

The following picture in the hyperbolic plane shows geometrically where the mysterious identity comes from.

\[
M(\mu P, -sQ, sP) = M(sQ, -sP, \mu Q).
\]

It is noteworthy to mention that the bound on the number of factors of optimal (ALT)-extremals always depends on the group. In particular we would like to stress that for the simply connected group $\widetilde{\text{SL}}(2, \mathbb{R})$ there is no a priori bound on the number of factors.

### 7.3 Singular (SSP)-extremals

There are two reasons for non-optimality of (SSP)-extremals. One of them is purely local, the other one is global in nature. Looking at the graph from Figure 3 we can say this as follows: For (SSP)-extremals that switch between top and bottom line (of the graph) we get a bound on the number of factors using purely local arguments. An (SSP)-extremal that does not switch between top and bottom basically looks like an (ALT)-extremal with switching
time \(2\sqrt{2}\) plus an interspersed \(S\)-arc. A global argument (depending explicitly on the group) is indispensable to bound the number of factors for these (SSP)-extremals.

To save some typing throughout this subsection we let \(w = \sqrt{2}\). We recall that this is the ”switching” time of the bang-arcs of an (SSP)-extremal. The quotation marks indicate that a switch might be virtual because the singular arc between two identical bang-arcs may have zero length.

**Proposition 7.13** Let \(w = \sqrt{2}\). Then the following identities hold true in \(SL(2)\):

\[
\begin{align*}
\mathcal{M}(sP, -\frac{2}{s}Q) &= -\mathcal{M}\left(\frac{2}{s}Q, -sP\right), \quad \text{if } s \neq 0, \\
\mathcal{M}(wP, -wQ) &= -\mathcal{M}(wQ, -wP), \\
\mathcal{M}(sT, wP, -wQ) &= \mathcal{M}(wP, -wQ, -sT), \quad s \in \mathbb{R}, \\
\mathcal{M}(rP, wQ, -wP) &= \mathcal{M}\left(\frac{2}{r+w}Q, -(r+w)P, -\frac{r}{r+w}Q\right), r \in \mathbb{R}, \\
\mathcal{M}(wP, wQ, -wP) &= \mathcal{M}\left(\frac{w}{2}Q, -2wP, -\frac{w}{2}Q\right).
\end{align*}
\]

The proof consists of nothing but elementary computations. The next result is valid for any group. Basically it means that the crucial (SSP)-extremals are those that look like (ALT)-extremals with switching time \(2\sqrt{2}\) with an interspersed \(S\)-arc.

**Proposition 7.14** The factorization \(\mathcal{M}(wP, wQ, -wP)\) is not optimal in any group. In particular, if an optimal (SSP)-extremal contains 5 \(B\)-arcs, then for each \(S\)-arc the two adjacent \(B\)-arcs must be equal.

**Proof.** In view of the last item of the previous proposition we observe that

\[
\mathcal{M}(wP, wQ, -wP) = \mathcal{M}\left(\frac{w}{2}Q, -2wP, -\frac{w}{2}Q\right).
\]

Both factorizations have equal cost \(3w\), but the righthand side cannot be optimal because it does not come from an extremal! Indeed, (ALT)-extremals are impossible because the switching pattern is circular. (CSP)-extremals are impossible because they have switching time \(s \in (0, \sqrt{2})\) whereas here the switching time is \(2\sqrt{2}\). Finally, (SSP)-extremals are impossible, too, because for an (SSP)-extremal with middle arc \(\exp(-2wP)\) the third arc would have to be \(\exp(tQ)\) with \(t \geq 0\).

A similar argument allows to prove the following stronger result:
Proposition 7.15 For $r, s > 0$ the factorization $M(rP, sT, wQ, -wP)$ is not optimal. In particular, an optimal (SSP)-extremal that switches between top and bottom has at most 5 factors.

Proof. Let $r, s > 0$. Using Proposition 7.13(12) and (13) we obtain

$$M(rP, sT, wQ, -wP) = M\left(\frac{2}{r + w}Q, -(r + w)P, -\frac{r}{r + w}Q, -sT\right).$$

Both sides have equal cost $r + s + 2w$, but the righthand side is not an extremal if $r > 0$ and $s > 0$, hence it cannot be optimal. \(\square\)

One more argument is needed.

Proposition 7.16 In SL(2) $M(rP, sT, wP, -wQ, -wQ, wP)$ is not optimal. In particular, optimal (SSP)-extremals in SL(2) have at most 7 factors.

Proof. Due to Eqn. (18) we may shift the $S$-arc to the right end, and it suffices to show that $M(rP, wP, -wQ, -wQ, wP)$ is not optimal for $r > 0$.

Next we use the seemingly obscure Eqn. (10) whose geometric meaning becomes clear in the discussion of (HRSCP) (cf. proof of Proposition B.6), namely

$$M\left(sP, -\frac{2}{s}Q\right) = -M\left(\frac{2}{s}Q, -sP\right),$$

which is valid for all $s \neq 0$. We consider $s = r + w > w$. Then $\frac{2}{s} < \frac{2}{w} = w$. Let $\epsilon = w - \frac{2}{w} + r > 0$. Then we obtain:

$$M\left((r + w)P, -wQ, -wQ, wP\right) = M\left(\frac{2}{r + w}Q, -(r + w)P, -\epsilon Q, -wP, wQ\right).$$

Both factorizations have equal cost. But the second factorization cannot be optimal because it does not come from an extremal! Indeed, the switching pattern is neither alternating nor circular. Besides the switching times $r + w, \epsilon, w$ are not equal. Hence $M((r + w)P, -wQ, -wQ, wP)$ cannot be optimal in SL(2). An (SSP)-extremal with 8 factors will always contain a subarc of the above (non-optimal) form. \(\square\)

Our last task in this subsection is to show that the previous propositions imply that the singular extremals listed in Table 1 are sufficient. Up to five factors we obtain the factorization maps $S3P, S4Q, S4P, S4Q, S5P, S5Q$, and $S5a$. Due to Proposition 7.13 we know that optimal (SSP)-extremals with more than five factors look like (ALT)-extremals with switching time $2\sqrt{2}$.
and an interspersed $S$-arc. In view of Eqn. (12) the $S$-arc may be the second or third arc in the product. But if the number of factors is even, we may revert the order of the factors (i.e., apply $\iota \sigma_H$), if necessary, so that we can also assume that the $S$-arc is the second arc. Therefore it suffices to consider a single factorization map (say, $S6, S8, \ldots$) for an even number of factors, but it is necessary to distinguish between, say, $S7a, S8, \ldots$ and $S7b = \mathbb{M} (-rQ, wP, \frac{\pi}{2} T, \ldots)$, resp. $S9a, S9b, \ldots$, for an odd number of factors. As $S7a$-extremals are not optimal in $\text{SL}(2)$, this naming convention may look surprising, but it perfectly makes sense if one also wants to consider the simply connected group $\widetilde{\text{SL}}(2, \mathbb{R})$. Since $S8$-extremals cannot be optimal, we see that the family $F$ given in Table 1 exhausts all possibilities. Thus the proof of Theorem 6.1 is finished.

8 Conclusion

Having proved that the families $F, F_1, F_2$ are sufficient, it is natural to ask if they are minimal with this property. The answer is affirmative in the sense that for every $f \in F$ there exist $(r, s, t) \in \text{dom}(f)$ such that $f(r, s, t)$ is optimal, and no $\tilde{f} \in F \setminus f$ allows us to reach the same endpoint at equal cost. Rather than providing a list of such cases, we include pictures of metric spheres $S(c) = \{g : T(g) = c\}$ for some values of $c$, cf. Fig. 4. To generate these pictures we used the parametrization of $\text{SL}(2)$, resp., its simply connected covering group $\widetilde{\text{SL}}(2, \mathbb{R})$ described in Appendix A. To understand the pictures it suffices to know that $\text{SL}(2)$ is identified as a set with $\mathbb{R}H + \mathbb{R}T + (-\pi, \pi]U \subseteq \mathfrak{s}(2) \cong \mathbb{R}^3$. In this parametrization the symmetries $\sigma_H, \sigma_T, \sigma_U$ are 180-degree rotations around the $H$-, resp., $T, U$-axes. Inversion is simply, $\iota(X) = -X$, and $\iota \sigma_H$ is reflection in the $TU$-plane. The horizontal plane $\mathbb{R}H + \mathbb{R}T$ corresponds to symmetric, positive definite elements of $\text{SL}(2)$ while the vertical segment $(-\pi, \pi]U$ corresponds to the circle group $\exp(\mathbb{R}U) = \exp([-\pi, \pi]U) \cong \text{SO}(2)$.

For $c = 1$ the relevant maps are $A3, C3, C4a, C4c, S3P, \text{and } S3Q$. The four $A3$-patches make up the top and bottom part. The thin sides with the figure eight curves consist of $S3P$- and $S3Q$-patches, the $S3P$-patch is inside the figure eight. The “flat” sides consist of eight patches: four $C3$- and two $C4a$- resp., $C4c$-patches. The $C4a$-patches connect (horizontally) to an $S3Q$-patch (because $C4a(0, s, 0) = \mathbb{M} (-sQ, -sP)$) whereas the $C4c$-patches connect (vertically) to $A3$-patches (because of $C4c(0, s, 0) = \mathbb{M} (sQ, -sP)$). One can see that the $C3$- and $S3$-patches fit in perfectly while the $A3$- and $C4$-patches have nontrivial intersections.

The shape of the sphere $S(1)$ still reminds us of the fact that the gener-
S(c) for $c = 3\sqrt{3} \approx 5.196$

S(c) for $c = \frac{32}{7}\sqrt{2} \approx 6.465$

Figure 7: Metric spheres $S(c)$ for some values of $c$. 
ating set \( \mathcal{U} = \text{conv}(\pm P, \pm Q) \) is a flat square.

For \( c = 2 \) we also have \( S4P \)- and \( S4Q \)-patches emanating from the \( S3 \)-patches, and connecting them to the \( A3 \)- and \( C3 \)-patches, respectively. One can also see how larger portions of \( C4a \)- and \( C4c \)-patches intersect.

For \( c = 3 \) tiny triangular patches appear, the \( S5P \)- and \( S5Q \)-patches. One also sees how \( A4 \)-extremals arise on the top surface. The \( S5a \)-patch is also a tiny triangle adjacent to the \( S4P \)-patches.

For \( c = 4 \) the \( C4 \)-patches have completely disappeared (cf. Proposition 7.8). The only change compared to \( S3(3) \) is, of course, that the \( S5 \)- and \( A4 \)-patches are much larger, hence better visible.

The last two pictures show \( S(c) \) for especially critical values of \( c \). For \( c = 3 \) \( \sqrt{3} \) the sphere touches the plane \( RH + RT + \pi U \) for the first time, \( A3 \)-extremals make up only a very thin portion of the top while \( A4 \)- and \( A5 \)-patches are clearly visible. Nevertheless the \( A5 \)-patches are quite small which means that the set of matrices for which \( A5(r, s, t) \) is the optimal factorization is quite small and close to \(-1 \in SL(2)\).

For \( c = \frac{32}{7} \sqrt{2} \) (ALT)-extremals have disappeared (which means they are above \( \pi U \)) and all patches come from (SSP)-extremals. Except for the \( S4Q \)- and \( S5Q \)-patches there are no further intersections. So in particular, \( S3P, S4P, S5P, S5a, S6, S7b \)-patches perfectly fit together, only the portions where the vertical coordinate exceeds \( \pi \) are chopped off in \( SL(2) \). They would still be present in \( \tilde{SL}(2, \mathbb{R}) \), though.

Finally these pictures also make clear that the set \( \text{Fix}(\iota \sigma_T) \subseteq SL(2) \) is a cut locus for the problem. In the pictures \( \text{Fix}(\iota \sigma_T) \) is the plane \( RH + RU \). An extremal that hits this plane transversally will lose its global optimality. A glance at Table \[ \] reveals that this applies to \( A3, A5, C4a, S4Q, S5Q \)-extremals; it does not apply to \( S5a, S7b, C4c \)-extremals because the latter are \( \iota \sigma_T \)-invariant.

Before we give an outlook on generalizations and future work we now give a proof for Theorem 1.3.

**Proposition 8.1** Let \( S1, S2 \in \mathfrak{sl}(2) \) with \( \det(S1) = \det(S2) = 0 \), and assume that \( [S1, S2] \neq 0 \). Let \( \tilde{U} = \text{conv}(\pm S1, \pm S2) \). Then there exist \( \mu > 0 \) and a \( g_0 \in SL(2) \) such that \( \tilde{U} = \mu \text{Ad}(g_0)U \). In particular, we have \( \mathcal{T}_{g}(g) = \mu^{-1} \mathcal{T}_{g_0^{-1}gg_0} \) for all \( g \in SL(2) \).

**Proof.** We recall that \( \{ X \in \mathfrak{sl}(2) : \det(X) = 0, X \neq 0 \} \) is the boundary of a Lorentzian double cone \( C \cup -C \) (minus the vertex), cf. Figure \[ \]. W.l.o.g. we may assume that \( S1, S2 \) lie on the upper part, say \( C \). Finding \( \mu \) and \( g_0 \) is a 3-step procedure:
1. Rotation: There exist $u \in [0, \pi]$ and $\rho_1 > 0$ such that $e^{u \text{ad} U} S^1 = \rho_1 P$. Let $g_1 = \exp(uU)$.

2. Shearing: We first observe that (in $HTU$-coordinates:)

$$e^{\tau \text{ad}(P)}(-Q) = \left(-\tau, \frac{\tau^2 - 1}{2}, \frac{\tau^2 + 1}{2} \right) = \frac{\tau^2 + 1}{2} \left(-\frac{2\tau}{1 + \tau^2}, \frac{\tau^2 - 1}{\tau^2 + 1}, 1 \right).$$

Since $(1 + \tau^2)^{-1}(2\tau, 1 - \tau^2)$ parametrizes the unit circle (except for the point $(0, 1)$), we see that $(0, \infty) e^{R \text{ad}(P)}(-Q) = C \setminus (0, \infty)P$. Hence for $X = \text{Ad}(g_1)S^2$ we can find $\tau^* \in \mathbb{R}$ and $\rho_2 > 0$ such that $e^{\tau^* \text{ad}(P)}X = -\rho_2 Q$. We let $g_2 = \exp(\tau^*P)$ and observe that $\text{Ad}(g_2)P = P$.

3. Hyperbolic rotation: Let $h = \frac{1}{4} \log(\rho_2/\rho_1) \in \mathbb{R}$, so $e^{2h}\rho_1 = e^{-2h}\rho_2 = \sqrt{\rho_1\rho_2} := \mu > 0$. Then $e^{h \text{ad}(H)}\rho_1P = \mu P$, and $e^{h \text{ad}(H)}(-\rho_2Q) = -\mu Q$. Let $g_3 = \exp(hH)$. Then $\text{Ad}(g_3g_2g_1)S^1 = \mu P$ and $\text{Ad}(g_3g_2g_1)S^2 = -\mu Q$.

Finally let $g_0 = (g_3g_2g_1)^{-1}$. Then $\tilde{U} = \mu \text{Ad}(g_0)U$. Hence

$$\mathcal{T}_{\tilde{U}}(g) = \mathcal{T}_{\mu \text{Ad}(g_0)U}(g) = \frac{1}{\mu} \mathcal{T}_{\text{Ad}(g_0)U} = \frac{1}{\mu} \mathcal{T}_U(\text{Id}_{g_0}^{-1}(g)) = \frac{1}{\mu} \mathcal{T}_U(g_0^{-1}g g_0).$$

So the claim of Theorem 1.3 follows with $\sigma = \text{Id}_{g_0}^{-1}$ and $\lambda = \mu^{-1}$.

Thus all claims made in the introduction have been proved by now. For an arbitrary 2-slip system with symmetric slip rates we now know how to find the dissipation distance and geodesics.

**Generalizations and future work**

It is clear that two different types of generalizations are of interest, namely more slip systems and passage to dimension $d = 3$. In a forthcoming paper we will treat the 2D-hexagonal lattice, i.e., we consider slips along the sides of an equilateral triangle. This is of practical interest because such systems arise in reality.

Observing that in the problem analyzed in this paper the optimal controls are piecewise constant, it is natural to ask whether this is true for general slip systems, or, for general polytopes, i.e., $U = \text{conv}(X_1, \ldots, X_m)$ for $X_1, \ldots, X_m \in \mathfrak{sl}(d)$. The following example shows that the answer is negative for arbitrary polytopes.

**Example 8.2** In $\mathfrak{sl}(2)$ let $X_1 = H + 2Q$, $X_2 = H - 2Q$, $X_3 = -H + 2P$, $X_4 = -H - 2P$, and $U = \text{conv}(X_1, \ldots, X_4)$. Then $U$ is a simplex and $0 \in \text{int}\, U$. For the polar $Q = \{ p : H(p) \geq -1 \}$ we obtain $Q = \text{conv}(u, v, w^1, w^2, w^3, w^4)$.
with \( p^1 = (1, 1, -1), p^2 = (1, -1, 1), p^3 = (-1, 1, 1), p^4 = (-1, -1, -1) \). Now consider \( p(t) \in \text{conv}(p^1, p^2) \), then (PMP:2) implies \( u(t) \in \text{conv}(X_3, X_4) = -H + [-2, 2]P \). Let \( \mathfrak{h} = \mathbb{R}H + \mathbb{R}P \) denote the subalgebra generated by \( X_3, X_4 \). The associated subgroup is \( \langle \exp \mathfrak{h} \rangle = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\} \).

Now \( \langle (e^{ad \mathfrak{h}})^t \rangle \) leaves the line \( \text{aff}(p^1, p^2) = (1, 0, 0) + \mathbb{R}(0, 1, -1) \) invariant. Take \( u(t) = -H + v(t)P \) with measurable \( v: [0, \infty) \to [-2, 2] \), \( g(t) \) the corresponding trajectory with \( g(0) = 1 \), and \( p(t) = \text{Ad}(g(t))^{1/2}(p^1 + p^2) \).

Then \( g(t) \in \langle \exp \mathfrak{h} \rangle \) and \( p(t) \in \text{aff}(p^1, p^2) \) for all \( t \). By continuity, there exists \( t^* > 0 \) such that \( p(t) \in \text{conv}(p^1, p^2) \) for \( t \in [0, t^*] \). Hence \((g, p, u)\) is an extremal in \([0, t^*] \).

The geometric reason for the occurrence of this degeneracy in the previous example is that the edge \( \text{conv}(p^1, p^2) \) lies in a single coadjoint orbit—the one-sheeted hyperboloid \( \{ (e^{ad \mathfrak{h}})^t \} \). Considering the dual picture, we already observed that the opposed edge \( \text{conv}(X_3, X_4) \) lies in a 2-dimensional subalgebra of \( \mathfrak{sl}(2) \). This is a very special situation that will never occur for slip systems because the \( 2D \)-subalgebras in \( \mathfrak{sl}(2) \) are precisely the hyperplanes tangent to the double cone. Therefore we state:

**Conjecture 8.3** Let \( \mathcal{U} = \text{conv}(X_1, \ldots, X_m) \subseteq \mathfrak{sl}(2) \) with \( \det(X_j) = 0 \). Then optimal controls are piecewise constant.

A careful look at our analysis of the flow rule actually suggests a much stronger conjecture. For \( X \in \mathfrak{sl}(2) \setminus \{0\} \) with \( \det(X) = 0 \), the induced flow \( (p, t) \mapsto p e^{t \text{ad}(X)} \) on the opposed face \( \text{op}(X) = \{ p : \langle p, X \rangle = -1 \} \) is (up to symmetry) the same flow as the one we analyzed on \( p \mathcal{U} \). All flow lines are parabolas, like in Figure 4. Consequently, an affine line, say \( \ell \subseteq \text{op}(X) \) will be tangent to at most one of these. This suggests that we get at most one singular control for each edge of \( \mathcal{Q} \). At a vertex, say \( p, \) of \( \mathcal{Q} \) the stabilizer \( \mathfrak{g}_p \) is one-dimensional. Hence \( \mathfrak{g}_p \cap \mathcal{U} \cap \{ X : \langle p, X \rangle = -1 \} \) is singleton, or empty. Since the polytopes \( \mathcal{U} \) and \( \mathcal{Q} \) are (combinatorial) duals of each other, we can formulate the following.

**Conjecture 8.4** Consider 2-dimensional slip systems, so \( \mathcal{U} \subseteq \mathfrak{sl}(2) \) is a convex polytope, \( \mathcal{U} = \text{conv}(S^1, \ldots, S^m) \) with \( \det(S^\alpha) = 0 \). Assume that \( 0 \in \text{int}(\mathcal{U}) \). Let \( f_0, f_1, f_2 \) denote the number of vertices, edges, and faces of the polytope \( \mathcal{U} \). Then optimal controls are piecewise constant, and the number of possible values is at most \( f_0 + f_1 + f_2 \).

In other words, for each vertex of \( \mathcal{Q} \) (resp., face of \( \mathcal{U} \)) and each edge of \( \mathcal{U} \), we get at most one singular control. In order to obtain rigorous proofs for
these conjectures, Proposition 8.1 and our discussion of the flow rule for the square lattice are the appropriate tools (cf. propositions 5.7 and 5.8). For if we consider an arbitrary (but fixed) edge of $Q$, we see that up to automorphism, we may assume that the two active slip-systems are either $-P, Q$, or $-P, -Q$, as in Proposition 5.7 and 5.8. Hence the arguments given there apply, whence switches must occur except for some isolated points.

Formulating reasonable conjectures for 3D-slip systems seems to be more difficult. Of course one can hope that optimal controls must be piecewise constant, but we do not have such strong evidence as in the 2D-case. In particular it is not yet clear, what types of singular controls one has to expect.

Computational tools

Programming and debugging are usually time consuming tasks, therefore we will make some tools available on the web. In particular, we will provide Mathematica notebooks with procedures for finding the factorization cost and optimal factorizations in SL(2). The pictures of the metric spheres were generated with Maple. But since the numerical computations are too slow, a more efficient C-program produces a file defining the necessary plot data structures. An interested reader should follow the links starting at URL http://www.mathematik.uni-stuttgart.de/mathA/lst1/mittenhuber/

Appendix A: The group $\tilde{\text{SL}}(2, \mathbb{R})$

In this appendix we provide a summary of facts about the simply connected group $\tilde{\text{SL}}(2, \mathbb{R})$. It is well-known that $\tilde{\text{SL}}(2, \mathbb{R})$ cannot be represented (faithfully) as a matrix group (subgroup of GL($n$) for some $n \in \mathbb{N}$). When analyzing SL(2) and related groups the following functions are ubiquitous:

$$C(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2n)!} = \begin{cases} \cosh(\sqrt{z}), & \text{if } z \geq 0, \\ \cos(\sqrt{-z}), & \text{if } z < 0, \end{cases}$$

$$S(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2n+1)!} = \begin{cases} \sinh(\sqrt{z}), & \text{if } z > 0, \\ 1, & \text{if } z = 0, \\ \sin(\sqrt{-z}), & \text{if } z < 0. \end{cases}$$
The ubiquity of these functions is due to the fact that $C$ and $S$ describe the (matrix) exponential function $\exp_{\text{SL}(2)} : \mathfrak{sl}(2) \to \text{SL}(2)$:

$$\sum_{k=0}^{\infty} \frac{1}{k!} X^k = C(-\det(X)) \text{id} + S(-\det(X)) X \quad \text{for every } X \in \mathfrak{sl}(2).$$

Due to a lack of letters we let $T(z) = \frac{S(z)}{C(z)}$, assuming that the function $T(\cdot)$ will not be confused with the matrix $T = P + Q$.

The covering map and a local inverse

As a set we identify $\tilde{\text{SL}}(2, \mathbb{R})$ with the Lie algebra $\mathfrak{sl}(2)$. Using the basis $\{H, T, U\}$ and writing $X \in \mathfrak{sl}(2)$ as $X = hH + tT + uU$ with $h, t, u \in \mathbb{R}$, the covering map $f : \mathfrak{sl}(2) \to \text{SL}(2)$ is defined as

$$f(hH + tT + uU) = C(h^2 + t^2) \begin{pmatrix} \cos(u) & \sin(u) \\ -\sin(u) & \cos(u) \end{pmatrix} + S(h^2 + t^2) \begin{pmatrix} h & t \\ t & -h \end{pmatrix}. \quad (15)$$

Hilgert and Hofmann introduced this map in [4] to parametrize the group $\tilde{\text{SL}}(2, \mathbb{R})$. Since $f$ is an analytic covering, it allows to define a group operation $\circ : \mathfrak{sl}(2) \times \mathfrak{sl}(2) \to \mathfrak{sl}(2)$ such that

$$f(X \circ Y) = f(X) \ f(Y) \quad \text{for all } X, Y \in \mathfrak{sl}(2).$$

From now on we will always identify the group $\tilde{\text{SL}}(2, \mathbb{R})$ with $(\mathfrak{sl}(2), \circ)$.

The identity $f(X + 2k\pi U) = f(X)$ for $X \in \mathfrak{sl}(2)$, $k \in \mathbb{Z}$ is obvious. Conversely, for $g \in \text{SL}(2)$ the preimage $f^{-1}(g)$ has the form $X + 2\pi ZU$ for some suitable $X$. Next let $\mathcal{E} = \mathbb{R}H + \mathbb{R}T$ denote the set of symmetric matrices in $\mathfrak{sl}(2)$. Then the restriction $f : \mathcal{E} + (-\pi, \pi] U \to \text{SL}(2)$ is injective. An inverse of this restriction is easily obtained. For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(X) \implies X = hH + tT + uU \quad \text{with}$$

$$u = \arg((a + d) + i(b - c)) \in (-\pi, \pi],$$

$$\begin{pmatrix} h \\ t \end{pmatrix} = \frac{1}{2S(\rho)} \begin{pmatrix} a - d \\ b + c \end{pmatrix}, \quad \text{where } \rho = \text{arcosh} \left( \frac{\sqrt{(a + d)^2 + (b - c)^2}}{4} \right).$$

We denote this local inverse map simply $f^{-1} : \text{SL}(2) \to \mathcal{E} + (-\pi, \pi] U$. Practically this means that we obtain the group $\text{SL}(2)$ from $\tilde{\text{SL}}(2, \mathbb{R})$ simply by taking the $U$-coordinate mod $2\pi$. 

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Advantages and disadvantages

This parametrization of \( \widetilde{SL}(2, \mathbb{R}) \) has advantages and disadvantages. The main disadvantage is that the group operation is much more complicated than ordinary matrix multiplication. We will give an explicit expression for \( X \circ Y \), but this expression is practical mainly for numerical computations.

The main advantages, on the other hand, are the rotational symmetry and the fact that every one-parameter subgroup \( \exp(\mathbb{R}X) \subseteq \widetilde{SL}(2, \mathbb{R}) \) lies in \( \mathbb{R}X + \mathbb{R}U \subseteq \widetilde{SL}(2, \mathbb{R}) \).

A particular advantage for the problem (OCP) is that in the above parametrization all maps from the symmetry group \( \widetilde{\Gamma} \) are simply linear maps on the vector space \( \mathfrak{sl}(2) \). For \( X = hH + tT + uU \) one quickly verifies that:

\[
\begin{align*}
    f(-X) &= f(X)^{-1}, \\
    f(hH - tT - uU) &= \sigma_H(f(X)), \\
    f(-hH + tT - uU) &= \sigma_T(f(X)), \\
    f(-hH - tT + uU) &= \sigma_U(f(X)).
\end{align*}
\]

The first equation shows that inversion in \( \widetilde{SL}(2, \mathbb{R}) \) is simply \( \widetilde{\iota}(X) = -X \). Similarly, the second equation implies that \( \widetilde{\sigma}_H := (X \mapsto HXH) : \widetilde{SL}(2, \mathbb{R}) \rightarrow \widetilde{SL}(2, \mathbb{R}) \) is an automorphism of \( \widetilde{SL}(2, \mathbb{R}) \). So the 180-degree rotations around the \( H, T, U \)-axes, respectively, are all group automorphisms.

**Group multiplication**

Multiplication and conjugation with elements from \( \mathbb{R}U \) is easily obtained—this is the rotational symmetry we already mentioned.

\[
\begin{align*}
    (uU) \circ X &= e^{\frac{u}{2} \text{ad} U} X + uU, \\
    X \circ (uU) &= e^{-\frac{u}{2} \text{ad} U} X + uU, \\
    (uU) \circ X \circ (-uU) &= e^{u \text{ad} U} X,
\end{align*}
\]

for all \( X \in \mathfrak{sl}(2), u \in \mathbb{R} \). In particular,

\[
(k\pi U) \circ X = X \circ (k\pi U), \quad (2k\pi U) \circ X = X + 2k\pi U, \quad \text{for all } k \in \mathbb{Z}.
\]

In order to find \( X_1 \circ X_2 \) for arbitrary \( X_1, X_2 \in \mathfrak{sl}(2) \), we use the observation that \( \mathcal{E} \circ \mathcal{E} \subseteq \mathcal{E} \times (-\pi/2, \pi/2) \), cf. [12, Lemma 1]. Hence

\[
X_1, X_2 \in \mathcal{E} \implies X_1 \circ X_2 = f^{-1}(f(X_1) f(X_2))
\]
Writing $X_i = h_i H + t_i T + u_i U$, $i = 1, 2$, and observing $(-u_1 U) \circ X_1, X_2 \circ (-u_2 U) \in \mathcal{E}$, we obtain

$$X_1 \circ X_2 = (u_1 U) \circ (-u_1 U) \circ X_1 \circ X_2 \circ (-u_2 U) \circ (u_2 U)$$

$$= (u_1 U) \circ ( ((-u_1 U) \circ X_1) \circ (X_2 \circ (-u_2 U)) ) \circ (u_2 U)$$

$$= (u_1 U) \circ f^{-1} \left( f(-u_1 U) f(X_1) f(X_2) f(-u_2 U) \right) \circ (u_2 U).$$

The last expression is as explicit as can be, but of course its usefulness is mainly restricted to numerical computations.

**One-parameter groups**

Let $\text{exp} : \mathfrak{sl}(2) \to \tilde{\text{SL}}(2, \mathbb{R})$ denote the exponential function (of the group $\tilde{\text{SL}}(2, \mathbb{R})$). Our first observation is that the covering map $f(X)$ coincides with the matrix exponential function if $X$ is either symmetric or skew-symmetric. Hence

$$\text{exp}(X) = X, \quad \text{and} \quad \text{exp}(RX) = RX \quad \text{for all} \quad X \in \mathcal{E} \cup \mathbb{R}U.$$ 

It turns out that $\text{exp}(RX) \subseteq RX + RU$ holds true for all $X \in \mathfrak{sl}(2)$. More precisely, if $X_0 = h_0 H + u_0 U$, $X_0 \neq 0$, then (cf. [14])

$$\text{exp}(RX_0) = \{ hH + uU : u_0 \tanh(h) = h_0 \sin(u) \}.$$ 

Qualitatively there are two different cases (for $X_0 \neq 0$):

$$\text{exp}(RX_0) = \left\{ uU + \text{artanh} \left( \frac{h_0}{u_0} \sin(u) \right) H : u \in \mathbb{R} \right\}, \quad \text{if} \quad |u_0| > |h_0|,$$

$$\text{exp}(RX_0) = \left\{ hH + \text{arcsin} \left( \frac{u_0}{h_0} \tanh(h) \right) U : h \in \mathbb{R} \right\}, \quad \text{if} \quad |u_0| \leq |h_0|.$$ 

The other one-parameter groups are obtained via rotation around the $U$-axis. Figure 8 shows one-parameter groups in the $HU$-plane. The group SL(2) is obtained by reading the picture modulo $2\pi U$. The dashed curves indicate the boundary of the complement of the image of the exponential function. An explicit expression for $\text{exp} : \mathfrak{sl}(2) \to \tilde{\text{SL}}(2, \mathbb{R})$ is available, too, cf. [10, 12].

Recalling that we defined

$$T(z) = \frac{S(z)}{C(z)} = \begin{cases} \frac{\tanh(\sqrt{z})}{\sqrt{-z}}, & z > 0, \\ \frac{\tan(\sqrt{-z})}{\sqrt{-z}}, & z < 0, \\ 1, & z = 0, \end{cases}$$

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we observe that
\[
\text{dom}(T) = \mathbb{R} \setminus \left\{ -\frac{n^2 \pi^2}{4} : n \in \mathbb{N}, n \text{ odd} \right\}.
\]

Now let \( X = \rho_0 E + u_0 U \in \mathfrak{sl}(2) \) with \( E = \cos \alpha H + \sin \alpha T \in \mathcal{E} \), and let \( k(X) = \rho_0^2 - u_0^2 \). Then \( \exp(X) = \rho E + u U \) with
\[
\rho = \text{arsinh}(\rho_0 S(k(X))), \\
u = \begin{cases} 
\arctan(u_0 T(k(X))), & k(X) \geq 0, \\
\text{sign}(u_0) \sqrt{-k(X)}, & k(X) \notin \text{dom}(T), \\
\arctan(u_0 T(k(X))) + \text{sign}(u_0) \left\lfloor \frac{1}{2} + \frac{\sqrt{-k(X)}}{\pi} \right\rfloor, & \text{otherwise}.
\end{cases}
\]

Special cases of particular interest are
\[
\exp(tP) = \exp \left( \frac{t}{2} (T + U) \right) = \text{arsinh} \left( \frac{t}{2} \right) T + \arctan \left( \frac{t}{2} \right) U. \\
\exp(tQ) = \exp \left( \frac{t}{2} (T - U) \right) = \text{arsinh} \left( \frac{t}{2} \right) T - \arctan \left( \frac{t}{2} \right) U.
\]
Appendix B: The hyperbolic Reeds-Shepp-Car

As we already mentioned (cf. p.14) the Hyperbolic Dubins Problem (HDP) can be considered as an optimal control problem on $\text{PSL}(2, \mathbb{R})$. Hence solving (HRSCP) is equivalent to finding time-optimal paths of

$$\dot{\gamma}(t) = \gamma(t)u(t), \quad u \in \pm \text{conv}(P, Q) \text{ a.e., } \gamma \in \text{PSL}(2).$$

(HRSCP)

At this point it is not clear, if optimal arcs exist for (HRSCP). We must convexify the set of admissible control values, i.e., we must pass to $\text{conv}(\pm P, \pm Q)$. The only difference between this Convexified Hyperbolic Reeds-Shepp-Car Problem (CHRSCP) and the problem (OCP) is that the first one evolves on PSL(2) and the second one on SL(2). From the characterization of extremals and the elimination of (U/2)-extremals we deduce that (HRSCP) is indeed solvable.

By definition, $\text{PSL}(2) = \text{SL}(2)/\{\pm 1\}$. Since the adjoint representation $\text{Ad}$ has kernel $\ker \text{Ad} = \{\pm 1\}$, we can write $\text{Ad}: \text{SL}(2) \to \text{PSL}(2)$ to denote the quotient map.

Proposition B.5 Let $T_{\text{PSL}(2)}$ denote the factorization cost in $\text{PSL}(2)$. Then

$$T_{\text{PSL}(2)}(\text{Ad}(g)) = \min \{T(g), T(-g)\} \quad \text{for all } g \in \text{SL}(2).$$

This is due to the fact that for a trajectory $\gamma(t)$ in SL(2) its projection $\text{Ad}(\gamma(t))$ is a trajectory (for the same control $u(t)$) in PSL(2). So “≤” follows immediately. Conversely, if $\eta(t) \in \text{PSL}(2)$ is an optimal path from $\eta(0) = \text{Ad}(1)$ to $\eta(t^*) = \text{Ad}(g)$, then $\eta(t)$ lifts to a trajectory $\gamma(t) \in \text{SL}(2)$. Since $\eta(t^*) = \text{Ad}(g)$, $\gamma(t^*) \in \{\pm g\}$, hence “≥” follows, too.

A practical consequence of this observation is that the sufficient families listed in Table [I] are also sufficient for PSL(2). In particular, there always exists an optimal path in $\mathbb{H}^2$ with at most 6 factors, resp. 5 switches. Of course our goal is to show that we can drop some maps from Table [I] so we find smaller sufficient families for PSL(2). Briefly, we will show that one may drop $A_5, S_5a, S_6, S_7b$ from $\mathcal{F}$. So optimal paths for (HRSCP) need at most 5 pieces, resp. 4 switches.

Another important observation is that inversion of the factorization maps is much nicer in SL(2) because $\exp(tP) = \text{id} + tP$ is linear in $t$ whereas $\text{Ad}(\exp tP) = e^{t\text{ad}P}$ is quadratic in $t$. Thus finding optimal paths for given boundary data is preferably done working in SL(2).
The missing comparison arguments

We need two new arguments, one for (ALT)- and one for (SSP)-extremals. The first argument also shows where the, perhaps obscure, identity used in the proof of Proposition 7.16 came from:

Proposition B.6 Let $s > 1$ and $r, t > 0$ such that $r + t > \frac{2}{s}$. Then the factorization $M(rP, -sQ, tP)$ is not optimal in $\text{PSL}(2)$. In particular,

- $s > 1 \implies M(sP, -sQ, sP)$ not optimal in $\text{PSL}(2)$,
- $s > \sqrt{2} \implies M(sP, -sQ)$ not optimal in $\text{PSL}(2)$,
- $r > 0 \implies M(rP, -\sqrt{2}Q, \sqrt{2}P)$ not optimal in $\text{PSL}(2)$.

Furthermore, optimal (ALT)-extremals in $\text{PSL}(2)$ have at most 4 factors, and if $A4(r, s, t)$ is optimal in $\text{PSL}(2)$, then $s \in [1, \sqrt{2}]$.

Proof. Let $\mu(s) = 2/s$. We already observed that

$$M_{\text{SL}(2)}(\mu P, -sQ) = -M_{\text{SL}(2)}(sQ, -\mu P),$$

hence $M(\mu P, -sQ) = M(sQ, -\mu P)$ in $\text{PSL}(2)$. Thus $M(rP, -sQ, tP) = M(-(\mu - r)P, sQ, -(\mu - t)P)$. Comparing costs we find $r + s + t$ for LHS and $s + 2\mu - (r + t)$ for RHS. If $r + t > \mu$, then $s + r + t > s + \mu > s + 2\mu - (r + t)$. So RHS is better than LHS.

For $s > 1$ we choose $r = t = s$ and obtain $r + t = 2s > 2 > \frac{2}{s}$. For $s > \sqrt{2}$ we choose $r = 0$, $t = s$, and obtain $r + t = s > \frac{2}{s}$. Finally, $s = \sqrt{2} = t$ and $r > 0$ also implies $r + t > \frac{2}{s}$. $\square$

The second argument is similar, it eliminates $S5a$-extremals.

Proposition B.7 For $r > 0$ the factorization $M(rP, \sqrt{2}P, -\sqrt{2}Q)$ is not optimal in $\text{PSL}(2)$. In particular, $S5a(r, s, t)$ is not optimal for $r > 0$, and optimal (SSP)-extremals in $\text{PSL}(2)$ have at most 5 factors.
Proof. Let \( w = \sqrt{2} \) and \( s = r + w > w \). Then \( \frac{2}{s} < \frac{2}{w} = w \). Let \( \varepsilon = w - \frac{2}{r+w} > 0 \). In the previous proof we already observed that \( \mathbb{M} \left( sP, -\frac{2}{s}Q \right) = \mathbb{M} \left( \frac{2}{s}Q, -sP \right) \) holds true in PSL(2). Hence
\[
\mathbb{M} \left( rP, wP, -wQ \right) = \mathbb{M} \left( sP, -wQ \right) = \mathbb{M} \left( sP, -\frac{2}{s}Q, -\varepsilon Q \right) = \mathbb{M} \left( \frac{2}{s}Q, -sP, -\varepsilon Q \right).
\]
All these factorizations have equal cost because of \( s + \frac{2}{s} + \varepsilon = r + 2w \). But the last one cannot be optimal because it does not come from an extremal. Indeed, the switching pattern \( Q \vdash -P \vdash -Q \) is **circular**, but the switching time is \( s = r + \sqrt{2} > \sqrt{2} \). And we know that (CSP)-extremals have switching time less than \( \sqrt{2} \). The second claim is obvious because of \( S5a(r, s, t) = \mathbb{M} \left( rP, wP, -wQ, -sT, -tQ \right) \). Finally, if an (SSP)-extremal has six factors, then it contains at most two singular arcs. Therefore one can always find a subarc like \( \mathbb{M} \left( rP, wP, -wQ \right) \), or \( \mathbb{M} \left( rP, sT, wQ, -wP \right) \). \( \square \)

**Comparison with the euclidean case**

The hyperbolic and the euclidean problem bear similarities, but they also differ in some respect. The euclidean case has a degeneracy which reflects the geometric fact that the sum of the angles in a triangle equals \( \pi \) in euclidean geometry. In the hyperbolic case the sum is strictly less than \( \pi \), this basically accounts for the fact that we could eliminate the \((U/2)\)-extremals by hand. Consequently, an optimal path for (CHRSCP) is also admissible for (HRSCP), and (HRSCP) always has a solution (for all possible boundary data).

Table 2 shows a sufficient family for (HRSCP), the symmetry groups are the same as in Table 1, so we do not list the groups here. Instead, the last column contains a reference to [18], namely where the corresponding paths appear, resp. where they are eliminated.

It is interesting to observe that the hyperbolic case in some sense swaps the role of the two types of paths corresponding to \( S5P \)- and \( S5a \)-factorizations. In \( \mathbb{H}^2 \) \( S5a \) is not optimal and \( S5P \) yields some optimal arcs whereas in the euclidean setting the paths corresponding to \( S5P \) are not optimal (cf. [18, Lemma 12, Fig. 9]) but the paths corresponding to \( S5a \) belong to the sufficient family ([18, Theorem 8, Item 7]).
Visualization

The problem (HRSCP) is also useful for visualization purposes. Every trajectory \( \gamma(t) \in \text{PSL}(2) \) (or SL(2)) yields a path \( \zeta(t) \) in the open unit disc \( \mathbb{D} \). One may assume w.l.o.g. that the controls \( P \) and \( Q \) correspond to forward left- and right-turns, while \( \frac{1}{2}(P + Q) \) corresponds to a geodesic arc.

Some obscure identities and constants suddenly get a clear geometric interpretation. For example, consider the singular switching time \( w = \sqrt{2} \) and the identities

\[
\mathbb{M}(s_1 T, wP, -wQ, -s_2 T) = \mathbb{M}((s_1 + s_2)T, wP, -wQ) = \mathbb{M}(wP, -wQ, -(s_1 + s_2)T).
\]

Computational verification is a tedious exercise that gives no insight at all why these identities hold true. Now consider the path, say \( \zeta(t) \in \mathbb{D} \), corresponding to \( \mathbb{M}(wP, -wQ) \). Let \( \zeta_0, v_0, \zeta_1, v_1 \) denote the boundary data (positions and tangents). Assume w.l.o.g. that \( \zeta_0 = 0 \) and \( v_0 = 1 \). Then \( \zeta_1 \in (0, 1) \) (actually: \( \zeta_1 = \frac{1}{2} \sqrt{2} \)), and \( v_1 = -1 \). So \( \zeta(t) \) is tangent to the geodesic through \( \zeta_0, \zeta_1 \), cf. Figure 9. So geometrically it is clear why these identities hold. We start and finish on the same geodesic, but with reversed orientation (forward/backward motion). The \( \mathbb{M}(wP, -wQ) \)-maneuver accomplishes the U-turn part, and it does not matter, if we perform this turn at the beginning, in the end, or somewhere inbetween. If we reflect Figure 9 along the horizontal axis, then it becomes obvious that we could have performed the

| Type | Map | Domain | Remark | \( \mathbb{R}^2 \) (Sussmann/Tang) |
|------|-----|--------|--------|----------------------------------|
| ALT  | A3  |        |        |                                  |
|      | A4  | \( s \in [1\sqrt{2}] \) | 2 cusps |                                 |
| CSP  | C3  |        |        | [8.3]                            |
| SSP  | C4a |        | 2 cusps| [8.4]                            |
|      | C4c |        | 1 cusp |                                  |
|      | S3P |        |        | [8.2]                            |
|      | S3Q |        |        |                                  |
|      | S4P |        |        | [8.6]                            |
|      | S4Q |        |        |                                  |
|      | S5P | new    |        | not optimal in \( \mathbb{R}^2 \) |
|      | S5Q |        |        | [8.5]                            |
|      | S5a |        | not optimal in \( \mathbb{H}^2 \) | [8.7]                            |

Table 2: A sufficient family for PSL(2)
Figure 9: The geometric meaning of the singular switching time

*turning maneuver* starting with a right turn (instead of a left turn). Hence we see that $M(wP, -wQ) = M(wQ, -wP)$ holds true in $\text{PSL}(2)$. These observations are actually in full analogy to the euclidean case, cf. [18, Fig.18, p.59]. It also becomes evident that for (SSP)-extremals the path $\zeta(t)$ will never intersect itself, so $\zeta(t_1) = \zeta(t_2)$ iff $t_1 = t_2$ because $\zeta(t)$ will consist of subarcs of a fixed geodesic and interspersed turning maneuvers. W.l.o.g. one may assume that this geodesic is the diameter $(-1, 1)$ as in Fig. 9, and then it becomes obvious that $\zeta(t)$ is doublepoint free.

**Proposition B.8** For (SSP)-extremals and (CSP)-extremals with $\leq 4$ factors the path $\zeta(t)$ has no self-intersections (i.e. $\zeta(t_1) = \zeta(t_2)$ implies $t_1 = t_2$).

Instead of a rigorous proof (which would not be elucidating at all) we simply provide pictures showing why neither $C4a$-extremals (2 cusps) nor $C4c$-extremals (1 cusp) have self-intersections.

Figure 10: $C4a$- and $C4c$-extremals have no self-intersections

The reason why we look at self-intersections is that the corresponding subpath is nothing but a turning maneuver (same initial and terminal point)
by some angle $\alpha$. Since the group $\exp(\mathbb{R}U)$ is the stabilizer of the point $0 \in \mathbb{D}$, we see that it is impossible to reach a nontrivial rotation $\exp(\alpha U)$ from the identity $1$ along an extremal of type (SSP), (C4a), or (C4c).

**Proposition B.9** For $\alpha \in (0, \pi/2)$ let $s(\alpha) = \sin(\alpha)$ and $r(\alpha) = \tan(\alpha/2)$. Then $\mathbb{M}(rP, -sQ, rP) = \exp(\alpha U)$ is optimal in PSL(2) as well as in any other group.

**Proof.** We know that $g := \exp(\alpha U)$ has an optimal factorization. Since we have a sufficient family, the previous proposition implies that the best factorization is of type A3 or A4. In view of Table [1] we must solve $A3(r, s, t) = \sigma(g)$ and $A4(r, s, t) = \sigma(g)$ for all $\sigma \in \Gamma$. Since $\sigma_U(g) = g$, we only need to consider two different right-hand sides, namely $g$ and $\sigma_T(g) = \exp(-\alpha U)$. We already observed $A3(r(\alpha), s(\alpha), r(\alpha)) = g$, whereas $\sigma_T(g) = A3(r, s, t)$ implies $s = -\sin \alpha < 0$, so this solution is not feasible (cf. Table [1]). In SL(2) the equation $A4(r, s, t) = \exp(\pm \alpha U)$ leads to:

$$
\begin{pmatrix}
* & r + s - rs^2 \\
-s + s^2t - t & 1 - s^2
\end{pmatrix} = 
\begin{pmatrix}
\cos(\alpha) & \pm \sin(\alpha) \\
\mp \sin(\alpha) & \cos(\alpha)
\end{pmatrix},
$$

so $s = \sqrt{1 - \cos(\alpha)}$. Since $\alpha \in (0, \pi/2)$ we deduce $s < 1$. But optimal A4-extremals have switching time $\geq 1$. Hence none of the A4-factorizations is optimal, whence $g = \mathbb{M}(rP, -sQ, rP) = \mathbb{M}(-rQ, sP, -rQ)$ are the optimal factorizations. 

**Appendix C: More details for $\widetilde{\text{SL}}(2, \mathbb{R})$**

It is quite natural to ask for a characterization of optimal arcs in other groups (than SL(2) and PSL(2)), and in particular in $\widetilde{\text{SL}}(2, \mathbb{R})$. First it is important to notice that there is no apriori bound for the number of factors of an optimal extremal in $\widetilde{\text{SL}}(2, \mathbb{R})$. The reason is very simple: we could never reach all points if we considered only (ALT)-extremals and (SSP)-extremals with a fixed number, say, $N$, of factors.

**Proposition C.10** Let $\mathcal{M} = \exp(\mathbb{R} \text{conv}(P, Q)) \subseteq \widetilde{\text{SL}}(2, \mathbb{R})$. then

$$
\mathcal{M}^n = \underbrace{\mathcal{M} \cdots \mathcal{M}}_n \subseteq \mathcal{E} + (2n - 1) \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) U \quad \text{for all } n \in \mathbb{N}.
$$

In particular $\mathcal{M}^n \neq \widetilde{\text{SL}}(2, \mathbb{R})$ for all $n \in \mathbb{N}$, and there is no apriori bound on the number of factors of an optimal extremal.
Proof. For $n = 1$ the claim holds true, cf. the description of one-parameter groups. Now assume the claim holds true for some $n \in \mathbb{N}$. Let $g_0 \in \mathcal{E} + (2n - 1) \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) U$ and $g_1 \in \mathcal{E} + \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) U$. We recall that $\mathcal{E} \circ \mathcal{E} \subseteq \mathcal{E} + (-\frac{\pi}{2}, \frac{\pi}{2}) U$. Writing $g_j = X_j + u_j U$ with $X_j \in \mathcal{E}$ and $u_j \in \mathbb{R}$ our multiplication formula for $\tilde{\text{SL}}(2, \mathbb{R})$ yields:

$$g_0 \circ g_1 = (u_0 U) \circ g' \circ (u_1 U)$$

with $g' = X' + u' U \in \mathcal{E} \circ \mathcal{E} \subseteq \mathcal{E} + (-\frac{\pi}{2}, \frac{\pi}{2}) U$. Hence $g_0 \circ g_1 = X'' + u'' U$ where $u'' = u_0 + u_1 + u'$. Since $|u_0| \leq (2n - 1)\frac{\pi}{2}$ and $|u_1|, |\bar{u}| \leq \frac{\pi}{2}$, we obtain $|u''| \leq (2n + 1)\frac{\pi}{2}$, hence $M^{n+1} \subseteq \mathcal{E} + (2n + 1) \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) U$ follows. This proves the first claim from which we immediately deduce the second claim. \qed

Nevertheless, for finite coverings of PSL(2) we get an a priori bound on the numbers of factors of optimal (SSP)-extremals. We will also conjecture a bound for (ALT)-extremals and present some evidence why this conjecture should be true. In any case the crucial parameter to consider is the cardinality of the center $|Z(G)|$ where $G$ denotes an arbitrary covering of PSL(2).

Singular extremals

Reconsidering our results for PSL(2) and SL(2) we notice that we obtained the bounds 5, resp., 7 if we count all virtual switches. This is actually the appropriate thing to do. Our goal is to show that for $|Z(G)| = N$ we obtain the bound $2N + 3$. The first step is to see how Proposition 7.13 generalizes to $\tilde{\text{SL}}(2, \mathbb{R})$. Except for Equation (10) nothing changes.

Proposition C.11 Let $w = \sqrt{2}$. Then the following equalities hold true:

\begin{align*}
\mathbb{M} \left( sP, -\frac{2}{s} Q \right) &= \mathbb{M} \left( \frac{2}{s} Q, -sP, \pi U \right), \quad (\forall s > 0). \\
\mathbb{M} (wP, -wQ) &= \mathbb{M} (wQ, -wP, \pi U). \\
\mathbb{M} (sT, wP, -wQ) &= \mathbb{M} (wP, -wQ, -sT) \quad s \in \mathbb{R}. \\
\mathbb{M} (wP, -wQ, -wP, wQ) &= \exp(\text{arsinh}(2wT)). \\
\mathbb{M} (rP, wQ, -wP) &= \mathbb{M} \left( \frac{2}{r + w} Q, -(r + w)P, -r \frac{w}{r + w} Q \right) \\
\mathbb{M} (wP, -wQ, -wP) &= \mathbb{M} \left( \frac{w}{2} Q, -2wP, -\frac{w}{2} Q \right).
\end{align*}

Proof. From Eqn. (10) we deduce $\mathbb{M} \left( sP, -\frac{2}{s} Q \right) = \mathbb{M} \left( \frac{2}{s} Q, -sP, \pi U \right) \circ (2\pi k) U$ for some $k \in \mathbb{Z}$. Since $s > 0$, we deduce $\exp(sP), \exp(-2s^{-1} Q) \in$
\[ \mathcal{E} + (0, \pi/2)U. \] Therefore

\[ \mathbb{M}(sP, -2s^{-1}Q) \in \mathcal{E} + \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right) U, \quad \mathbb{M}(2s^{-1}Q, -sP) \in \mathcal{E} + \left( -\frac{3\pi}{2}, \frac{\pi}{2} \right) U. \]

Hence \( \mathbb{M}(2s^{-1}Q, -sP, \pi U) \in \mathcal{E} + \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right) U, \) whence \( k = 0, \) and Eqn. (16) follows for all \( s > 0. \)

Eqn. (18) is equivalent to \( e^{-\text{ad}(P) \text{ad}(Q)}T = -T. \) As it involves only the adjoint action, it holds true regardless of the group \( G. \) To prove Eqn. (19) we observe \( \mathbb{M}(wP, -wQ, -wP) = \exp(-we^{\text{ad}(P)}Q) \in \mathcal{E} + (0, \pi/2)U \) and \( \exp(wQ) \in \mathcal{E} + (-\pi/2, 0)U, \) hence \( \mathbb{M}(wP, -wQ, -wP, wQ) \in \mathcal{E} + (-\pi, \pi)U. \)

As an elementary computation in \( \text{SL}(2) \) yields

\[ f(\mathbb{M}(wP, -wQ, -wP, wQ)) = f(\exp(\text{arsinh}(2w)T)), \]

we deduce \( \mathbb{M}(wP, -wQ, -wP, wQ) \in \mathcal{E} \circ 2\pi \mathbb{Z} U, \) hence (19) follows.

Eqn. (20) trivially holds true for \( r = 0. \) As both sides are continuous in \( r, \) it must hold for all \( r \in (-\sqrt{2}, \infty). \) The special choice \( r = w \) yields Eqn. (21).

\[ \blacksquare \]

Note that Eqn. (16) is the only equation that has changed (in comparison to Proposition 7.13). In particular, propositions 7.14 and 7.15 hold true in \( \widetilde{\text{SL}}(2, \mathbb{R}) \) because their proofs only required Eqn. (21), resp., eqns. (18, 20). The only instance where we used the result corresponding to Eqn. (16) was in the proof of Proposition 7.16.

**Proposition C.12** Let \( |Z(G)| = N \in \mathbb{N}. \) Then the factorization

\[ \mathbb{M}(rP, wP, -wQ, -wQ, wP, \ldots) \]

is not optimal in \( G \) for \( r > 0. \) In particular, optimal (SSP)-extremals in \( G \) have at most \( 2N + 3 \) factors.

**Proof.** Considering \( \mathbb{M}(rP, wP, -wQ, -wQ, wP, \ldots) \) we let \( \mu = r + w > 0 \) and observe that

\[ \mathbb{M}(\mu P, -wQ) = \mathbb{M} \left( \frac{2}{\mu} Q, -\mu P, -\varepsilon Q \right) \circ (\pi U). \quad \text{with } \varepsilon = w - \frac{2}{\mu} > 0. \]
Next we observe $M(wP, wQ) = M(wQ, wP) \circ (\pi U)$ and $M(wQ, wP) = M(-wP, wQ) \circ (\pi U)$. As $|Z(G)| = N$ and $\pi U \in Z(G)$, $(\pi U)^N = 1$. Hence

$$M(\mu P, wQ) \circ M(-wQ, wP) \circ M(wP, -wQ) \cdots$$

Both factorizations have equal cost $r + 2Nw$. But the second factorization cannot be optimal because it does not come from an extremal. The switching pattern $Q \vdash -P \vdash -Q \vdash -P \ldots$ is neither alternating nor circular. For $r > 0$ small the switching time for the second arc is $\mu = r + w \in (w, 2w)$, so it cannot be an (SSP)-extremal either. Hence the LHS factorization $M(\mu P, -wQ, -wQ, wP, \ldots)$ cannot be optimal. An (SSP)-extremal with $2N + 4$ factors always contains a subarc of the above form. Note that $2N + 3$ arcs are possible, just consider $M(-rQ, wP, sT, wP, -wQ, \ldots)$. $\blacksquare$

### Alternating extremals

One may look for a general pattern behind the arguments we used to obtain bounds on the number of factors of optimal (ALT)-extremals in SL(2) and PSL(2). And of course, there is one. It is very natural to ask the following questions:

How can we reach the central elements of $\tilde{\text{SL}}(2, \mathbb{R})$, i.e., $k\pi U$ for $k \in \mathbb{N}$? And what is the fastest way to do so?

An answer to the first question (how?) is relatively easy to find. Let

$$g(s) = M(sP, -sQ) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} = \begin{pmatrix} 1 - s^2 & s \\ -s & 1 \end{pmatrix} \in \text{SL}(2).$$

Then $\text{trace}(g(s)) = 2 - s^2$. Thus for $s \in (0, 2)$ $\text{trace}(g(s)) \in (-2, 2)$. Hence

$$\text{spec}(g(s)) = \{\cos \alpha \pm i \sin \alpha\}, \quad \text{with} \quad \cos \alpha = 1 - \frac{s^2}{2}.$$  

Let $\alpha(s) = \arccos \left(1 - \frac{s^2}{2}\right) \in (0, \pi)$. Then $g(s)$ is actually conjugate to $\exp(\alpha(s)U)$, i.e., there exists a matrix $V \in \text{SL}(2)$ such that

$$V^{-1}g(s)V = \begin{pmatrix} \cos(\alpha(s)) & \sin(\alpha(s)) \\ -\sin(\alpha(s)) & \cos(\alpha(s)) \end{pmatrix}.$$  

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For $s = 0$ this is true, and for $s \in (0, 2)$ it follows by continuity. In $\tilde{S}L(2, \mathbb{R})$ we therefore have

$$\gamma(s) := \mathbb{M}(sP, -sQ) = \mathbb{M}(X, \alpha(s)U, -X)$$

for some $X \in \mathfrak{sl}(2)$.

Next we look for solutions of $\gamma(s)^m = k\pi U$, $m, k \in \mathbb{N}$. As $k\pi U$ is central, this is equivalent to $m \alpha(s) = k\pi$. Since $\alpha(s) \in (0, \pi)$, solutions exist only for $m > k$:

$$\alpha_{k,j}(s) = \frac{k}{k+j} \pi, \quad j \in \mathbb{N}.$$

Since $\cos \alpha = 1 - \frac{s^2}{2}$, we find

$$s^2 = 2(1 - \cos \alpha) = 2 \cdot 2 \sin^2 \left(\frac{\alpha}{2}\right), \quad \text{hence} \quad s(\alpha) = 2 \sin \left(\frac{\alpha}{2}\right).$$

Thus for $\alpha_{k,j}$ the appropriate switching time is $s_{k,j} = 2 \sin(\alpha_{k,j}/2)$. The cost of this factorization is

$$2(k+j) s_{k,j} = 2(k+j) \sin \left(\frac{k\pi}{2(k+j)}\right) = k\pi \frac{\sin \left(\frac{k\pi}{2(k+j)}\right)}{\left(\frac{k\pi}{2(k+j)}\right)}.$$

Since $x \mapsto \frac{\sin x}{x}$ is decreasing and nonnegative in $[0, \pi]$ and $\frac{k\pi}{2(k+j)} \in [0, \frac{\pi}{2}]$ for $k, j \in \mathbb{N}$, we deduce that $j = 1$ provides the best factorization of the form $\gamma(s)^{k+j}$. Let us write

$$A(n; r, s, t) = \underbrace{\mathbb{M}(rP, -sQ, \ldots)}_{\text{n factors}}$$

$$= \begin{cases} 
\exp(rP) \mathbb{M}(-sQ, sP)^{n-1} \exp(-tQ), & n \text{ even}, \\
\exp(rP) \mathbb{M}(-sQ, sP)^{n-1} \mathbb{M}(sP, -tQ), & n \text{ odd},
\end{cases}$$

and let $A_n(s) = A(n; s, s, s)$. Then we can prove:

**Proposition C.13** For $n \in \mathbb{N}$ let $s_n = 2 \cos \left(\frac{\pi}{n}\right)$. Then

$$A_n(s_n) = (n - 2) \frac{\pi}{2} U \quad \text{for all } n \geq 3.$$

Moreover, $A(n+1; t, s_n, s_n)$ is not optimal for $t > 0$.

**Proof.** First we observe that $1 - \frac{s_n^2}{2} = -\cos(\frac{2\pi}{n}) = \cos \left(\frac{n-2}{n} \pi\right)$, hence

$$\gamma(s_n) = \mathbb{M} \left( X, \frac{n-2}{n} \pi U, -X \right)$$

for some $X \in \mathfrak{sl}(2)$. 62
Now we distinguish two cases. First, if $n = 2k$ is even then we obtain

\[
\gamma(s_n)^k = M \left( X, \frac{n-2}{n} \pi U, -X \right)^k = M \left( X, \frac{k(2k-2)}{2k} \pi U, -X \right) = (k-1)\pi U = \frac{n-2}{2} \pi U.
\]

On the other, if $n = 2k + 1$ is odd, we let $\tilde{\gamma} = A_n(s_n)$. We observe that $u\sigma_H(\tilde{\gamma}) = \tilde{\gamma}$, so $\tilde{\gamma} = \tau T + uU \in \mathbb{R}T + \mathbb{R}U$. Moreover,

\[
\tilde{\gamma} \sigma_U(\tilde{\gamma}) = \gamma(s_{2k+1})^{2k+1} = M \left( X, (2k+1) \frac{2k-1}{2k+1} \pi U, -X \right) = (2k-1)\pi U = (n-2)\pi U.
\]

As $\sigma_U(\tau T + uU) = -\tau T + uU$, we compute

\[
f(\tau T + uU) f(-\tau T + uU) = \left( \cosh^2(\tau) \cos(2u) + \sinh^2(\tau) \right) \text{id} + \cosh^2(\tau) \sin(2u) U - 2 \sinh(\tau) \sin(u) H.
\]

Since $f((2k-1)\pi U) = -\text{id}$, we deduce $\cos(2u) = -1$, hence $2u \in \pi + 2\pi \mathbb{Z}$, whence $u \in \frac{\pi}{2} + \pi \mathbb{Z}$. Thus $\sin(u) \neq 0$, so $\tau = 0$ and $\tilde{\gamma} = uU$ for some $u \in \frac{\pi}{2} + \pi \mathbb{Z}$. Thus $\sigma_U(\tilde{\gamma}) = \tilde{\gamma}$, and therefore $\tilde{\gamma} \sigma_U(\tilde{\gamma}) = (n-2)\pi U$ implies $u = \frac{n-2}{2} \pi$.

In order to prove that $A(n+1; t, s_n, s_n)$ is not optimal, we observe that $A_n(s_n) \in \mathbb{R}U$ and $\text{Fix}(\sigma_U) = \mathbb{R}U$. Therefore $A_n(s_n) = \sigma_U(A_n(s_n)) = M((-s_n Q, s_n P, \ldots)$, and we obtain

\[
A(n+1; t, s_n, s_n) = \exp(tP) M(-s_n Q, s_n P, \ldots) = \exp(tP) \sigma_U(A_n(s_n)) = \exp(tP) A_n(s_n) = M((t + s_n) P, -s_n Q, s_n P, -s_n Q, \ldots),
\]

and both factorizations have equal cost $t + n s_n$. But since $t > 0$, the RHS does not come from an extremal, hence it cannot be optimal. \qed

**Remark C.14** It is clear that $s_n$ is algebraic as it is twice the real part of a $2n$-th root of $1 \in \mathbb{C}$. For small $n$ we obtain:

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|
| $s_n$ | 1 | $\sqrt{2}$ | $\frac{1 + \sqrt{5}}{2}$ | $\sqrt{3}$ | $2 \cos(\pi/7)$ | $\sqrt{2 + \sqrt{2}}$ |
|     | 1.41421 | 1.61803 | 1.73205 | 1.80194 | 1.87939 |
Algebraically, for \( n = 2k \) even let
\[
p_n(\xi) = \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k}{2j} \xi^{k-2j}(\xi^2 - 1)^j
\]
and for \( n = 2k + 1 \) odd let
\[
p_n(\xi) = \sum_{j=0}^{2k} (-1)^j \xi^j + \sum_{j=1}^{k} \binom{n}{2j} \xi^{n-2j}(\xi - 1)(\xi^2 - 1)^{j-1}.
\]

Then \( p_n(\frac{1}{2}s_n) = 0 \) for all \( n \in \mathbb{N} \). In fact for \( n = 2k \) even, \( p_n \) is derived from \( \Im(z^k - i) = 0 \) while for \( n = 2k + 1 \) odd, \( p_n \) is derived from \( \Re\frac{1+s_n}{1+z} = \sum_{j=0}^{2k} (-z)^j = 0 \), for \( z = \xi + i\sqrt{1-\xi^2} \). For small \( n \) we obtain
\[
\begin{align*}
p_3(s/2) &= (s-1)^2, & p_4(s/2) &= \frac{1}{2}(s^2 - 2), \\
p_5(s/2) &= (s^2 - s - 1)^2, & p_6(s/2) &= \frac{1}{2} s(s^2 - 3), \\
p_7(s/2) &= (s^3 - s^2 - 2s + 1)^2, & p_8(s/2) &= \frac{1}{2}(s^4 - 4s^2 + 2), \\
p_9(s/2) &= (s-1)^2(s^3 - 3s - 1)^2, & p_{10}(s/2) &= \frac{1}{2} s(s^4 - 5s^2 + 5).
\end{align*}
\]

Now we can formulate our conjecture concerning optimal (ALT)-extremals

**Conjecture C.15** Let \( n \in \mathbb{N}, n \geq 3 \). Then \( A_n(s_n) \) is optimal in \( \widetilde{\text{SL}}(2,\mathbb{R}) \) and in any group with \( |Z(G)| \geq n - 2 \).

If \( A(n; r, s, t) \) is optimal, then \( s \geq s_{n-1} \). This is true for any group with Lie algebra \( \mathfrak{sl}(2) \).

If \( |Z(G)| = N \), then optimal (ALT)-extremals have at most \( N + 3 \) factors. If \( A(N + 3; r, s, t) \) is optimal, then \( s \in [s_{N+2}, s_{2N+2}] \).

After our discussion on how to reach the central elements it should be clear why we expect the first statement to hold true. The second part of the conjecture is trivial for \( n = 3 \) (\( s \geq s_2 = 0 \)), and it has already been proved for \( n = 4 \) and \( n = 5 \), cf. Propositions 7.9, 7.11. Finally, the last statement generalizes the arguments given in Propositions 7.12 and B.6. We conjecture that in general there exists a function \( \mu_N(s) \) with the following properties:

- For \( g := A(N + 1; \mu_N(s), s, s) \) we have
  \[
g \circ g = N\pi U \quad \text{if } N \text{ is odd,} \\
g \circ \sigma_U(-g) = N\pi U \quad \text{if } N \text{ is even.}
\]
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(N\) & \(\mu_N(s)\) & \(\mu_N(s) = 2s\) & \(\mu_N(s) = s\) \\
\hline
1 & \(\frac{2}{s}\) & 1 = \(s_3\) & \(\sqrt{2} = s_4\) \\
2 & \(\frac{2s}{s^2 - 1}\) & \(\sqrt{2} = s_4\) & \(\sqrt{3} = s_6\) \\
3 & \(\frac{2(s^2 - 1)}{s(s^2 - 2)}\) & \(\frac{1 + \sqrt{5}}{2} = s_5\) & \(\sqrt{2} + \sqrt{2} = s_8\) \\
4 & \(\frac{2s(s^2 - 2)}{s^4 - 3s^2 + 1}\) & \(\sqrt{3} = s_6\) & \(\sqrt{\frac{5 + \sqrt{5}}{2}} = s_{10}\) \\
5 & \(\frac{2(s^2 - 3s^2 + 1)}{s(s^4 - 4s^2 + 3)}\) & \(s_7\) & \(\frac{1 + \sqrt{7}}{\sqrt{2}} = s_{12}\) \\
\hline
\end{tabular}
\caption{The function \(\mu_N(s)\) for \(N = 1, \ldots, 5\).}
\end{table}

- \(\mu_N(s)\) is well-defined for \(s > s_{N+1}, s < \mu_N(s) < 2s\) for \(s \in (s_{N+2}, s_{2N+2})\), and \(\mu_N(s) = s\) for \(s = s_{2N+2}, \mu_N(s) = 2s\) for \(s = s_{N+2}\).

For small \(N\) one can verify this explicitly, cf. Table 3. Provided we have such a function \(\mu_N\) we obtain that \(A(N + 2; r, s, t)\) is not optimal if \(s > s_{N+2}\) and \(r + t > \mu_N(s)\). Hence

\[
\begin{align*}
\text{if } s &> s_{2N+2}, \quad r = s, t = 0 \implies A_{N+1}(s) \text{ not optimal.} \\
\text{if } s = s_{2N+2}, \quad r = s, t > 0 \implies A(N + 2; s, s, t) \text{ not optimal.} \\
\text{if } s \in (s_{N+2}, s_{2N+2}), \quad r = t = s \implies A_{N+2}(s) \text{ not optimal.}
\end{align*}
\]

Recalling our conjecture that optimal \(A(N+4; \cdot)\)-extremals must have switching time \(s \geq s_{N+3}\), it is clear why we are convinced that \(A(N + 4; r, s, t)\) cannot be optimal if \(r > 0\) or \(t > 0\).

We conclude with one more

**Conjecture C.16** The factorization \(A_n(s)\) is optimal in \(\widetilde{SL}(2, \mathbb{R})\) for all \(s \in [2, 2\sqrt{2}], n \geq 3\).

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