ON EXPOSED POINTS OF LIPSCHITZ FREE SPACES

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Abstract. In this note we prove that a molecule \(d(x, y)^{-1}(\delta(x) - \delta(y))\) is an exposed point of the unit ball of a Lipschitz free space \(\mathcal{F}(M)\) if and only if the metric segment \([x, y] = \{z \in M : d(x, y) = d(z, x) + d(z, y)\}\) is reduced to \([x, y]\). This is based on a recent result due to Aliaga and Pernecká which states that the class of Lipschitz free spaces over closed subsets of \(M\) is closed under arbitrary intersections when \(M\) has finite diameter.

1. Introduction

For a metric space \((M, d)\) with a distinguished point \(0 \in M\), we let \(\text{Lip}_0(M)\) be the real Banach space of Lipschitz maps from \(M\) to \(\mathbb{R}\) which vanish at 0. We recall that the norm of \(f \in \text{Lip}_0(M)\), denoted \(\|f\|_L\), is the best Lipschitz constant of \(f\), i.e.

\[
\|f\|_L = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)}.
\]

Next, for \(x \in M\), we let \(\delta(x) \in \text{Lip}_0(M)^*\) be Dirac measure, i.e. \(\langle \delta(x), f \rangle = f(x)\). We then define the Lipschitz free space over \(M\) to be the following closed subspace of \(\text{Lip}_0(M)^*\):

\[
\mathcal{F}(M) := \text{span}\{\delta(x) : x \in M\}.
\]

It follows from the fundamental linearisation property of Lipschitz free spaces that \(\mathcal{F}(M)\) is a canonical predual of \(\text{Lip}_0(M)\) (see [7] for more details).

In this note we are interested in extreme points and exposed points of the unit ball of Lipschitz free spaces. If \(B_X\) denotes the unit ball of a Banach space \(X\), we recall that \(x \in B_X\) is an extreme point of \(B_X\) whenever \(x \not\in \text{conv}(B_X \setminus \{x\})\). Next, \(x\) is an exposed point of \(B_X\) if there exists a linear functional \(f \in X^*\) such that \(f(x) > f(z)\) for every \(z \in B_X \setminus \{x\}\). In what follows, \(\text{ext}(B_X)\) denotes the set of extreme points of \(B_X\) while \(\text{exp}(B_X)\) denotes the set of exposed points of \(B_X\). Is is readily seen that \(\text{exp}(B_X) \subset \text{ext}(B_X)\).

The extremal structure of Lipschitz free spaces has already been investigated in a number of articles [1, 2, 5, 6, 9]. In any such study a special attention is dedicated to the elements of \(\mathcal{F}(M)\) of the form \(m_{xy} = \frac{\delta(x) - \delta(y)}{d(x, y)}\) which we call molecules (and which are called elementary molecules in [2]). It is simply a matter of writing down the corresponding convex combination to see that \(m_{xy} \in \text{ext}(B_{\mathcal{F}(M)})\) implies that \([x, y] = \{x, y\}\). However, it is only recently that Aliaga and Pernecká [2] managed to prove that, for a complete \(M\), the reverse implication is

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also valid. Here, using one of the ingredients of their proof, we show the following stronger result.

**Theorem 1.** Let $M$ be a complete metric space and $p \neq q \in M$ satisfy $[p, q] = \{p, q\}$. Then $m_{pq}$ is an exposed point of $B_{F(M)}$. It is exposed by the magic function

$$f_{pq}(t) := \frac{d(x, y)}{2} \left( \frac{d(t, q) - d(t, p)}{d(t, q) + d(t, p)} - \frac{d(0, q) - d(0, p)}{d(0, q) + d(0, p)} \right).$$

**2. Proof of the main result**

The authors of [2] had the following important insight which is likely to have many more applications in analysis of Lipschitz free spaces.

**Proposition 2** (Aliaga and Pernecká [2]). Let $M$ be a bounded complete metric space. Let $\{M_\alpha \subset M : \alpha \in A\}$ be a collection of closed subsets of $M$ containing 0. Then

$$\bigcap_{\alpha \in A} F(M_\alpha) = F \left( \bigcap_{\alpha \in A} M_\alpha \right).$$

For the proof of Theorem 1 we will need further some notation and few lemmas. Given a metric space $M$ we will set $\widetilde{M} := M \times M \setminus \{(x, x) : x \in M\}$ and $V = \left\{m_{xy} : (x, y) \in \widetilde{M} \right\}$ the set of molecules in $F(M)$. The following folklore fact is also stated in disguise as Lemma 2.1 in [2]. The proof here is different from the one in [2].

**Lemma 3.** Let $M$ be a metric space. Let us define $Q : \ell_1(\widetilde{M}) \to F(M)$ by $e_{(x,y)} \mapsto m_{xy}$ and linearly on span $\{e_{(x,y)}\}$. Then $Q$ extends to an onto norm-one mapping.

**Proof.** The fact that $\|Q\| = 1$ is clear so we can extend $Q$ to the whole space with the same norm. Let us call the extension $Q$ again. We will prove that $B_{F(M)}^Q \subset Q(B_{\ell_1}^0)$, where $B_{\ell_1}^0$ denotes the open unit ball of a Banach space $X$. For this it is enough to use Lemma 2.23 in [4], i.e. we need to check that $B_{F(M)}^Q \subset \text{conv}(V) \subset Q(B_{\ell_1}^0)$. But we have $B_{F(M)}^Q \subset \text{conv}(V) \subset Q(B_{\ell_1}^0)$. □

The next lemma is standard.

**Lemma 4.** Let $a \in S_{\ell_1}$ and $b \in B_{\ell_\infty}$. Assume that $1 - \alpha \varepsilon \leq \langle a, b \rangle$ for some $0 < \alpha, \varepsilon < 1$. Denote $B = \{n \in \mathbb{N} : |b_n| \leq (1 - \alpha)\}$. Then $\sum_{n \in B} |a_n| \leq \varepsilon$.

**Proof.** We denote $G := \mathbb{N} \setminus B$. We have

$$1 - \varepsilon \alpha \leq \sum_{n=1}^{\infty} a_nb_n \leq \sum_{n \in G} |a_nb_n| + \sum_{n \in B} |a_nb_n|$$

$$\leq \sum_{n \in G} |a_n| + (1 - \alpha) \sum_{n \in B} |a_n|$$

$$\leq \sum_{n \in \mathbb{N}} |a_n| - \alpha \sum_{n \in B} |a_n| = 1 - \alpha \sum_{n \in B} |a_n|. $$

It follows that $\sum_{n \in B} |a_n| \leq \varepsilon$. □
For a metric space $M$, points $p, q \in M$ and $\varepsilon > 0$ we will denote

$$[p, q]_\varepsilon := \left\{ x \in M : d(p, x) + d(x, q) \leq \frac{1}{1-\varepsilon}d(p, q) \right\}.$$  

The properties of the magic function collected in the following lemma have been proved already in [8].

**Lemma 5.** Let $(p, q) \in \widetilde{M}$. We have

1. $f_{pq}$ is Lipschitz and $\|f_{pq}\|_L \leq 1$.
2. Let $u \neq v \in M$ and $\varepsilon > 0$ be such that $\frac{d_{pq}(u) - d_{pq}(v)}{d(u, v)} > 1 - \varepsilon$. Then both $u, v \in [p, q]_\varepsilon$.
3. If $(u, v) \in \widetilde{M}$ and $\frac{d_{pq}(u) - d_{pq}(v)}{d(u, v)} = 1$, then both $u, v \in [x, y]$.

Let us remark at this point that if $[p, q] = \{p, q\}$, then $f_{pq}$ exposes $m_{pq}$ among molecules (immediate from Lemma 5 (3)) and also among those $\mu \in M$ which have finite support (or more generally such that $\|\mu\| = \|a\|_1$ in the representation coming from Lemma 3). The next lemma prepares the ground for the remaining cases.

**Lemma 6.** Let $M$ be a metric space with the base point $0 = q$ and let $p \neq q \in M$ be such that $[p, q] = \{p, q\}$. Assume that $\mu \in M$ satisfies $\langle \mu, f_{pq} \rangle = 1$. Then for every $\varepsilon, \alpha \in (0, \frac{1}{2})$ we have $\mu \in F_1([p, q]_\alpha) + 2\varepsilon B_{\|\mu\|}$.

**Proof.** Let us observe right away that by the hypothesis $\|\mu\| = 1$. Let $\varepsilon, \alpha \in (0, \frac{1}{2})$ be fixed. By Lemma 3 there exist $a_i = (a_i, q_i) \in \ell_1$ and $(p_i, q_i) \subset M$ such that $\mu = \sum_{i=1}^\infty a_i m_{p_i/q_i}$ and $\|a\|_1 \leq \|\mu\| + \frac{\alpha}{1-\alpha}$. We have

$$1 - \varepsilon \alpha \leq \frac{1}{\|a\|_1} = \left\langle \frac{\mu}{\|a\|_1}, f_{pq} \right\rangle = \sum_{i=1}^\infty \frac{a_i}{\|a\|_1} \langle m_{p_i/q_i}, f_{pq} \rangle.$$  

Now if we denote $B = \{i \in \mathbb{N} : \langle m_{p_i/q_i}, f_{pq} \rangle \leq (1 - \alpha)\}$, then Lemma 4 yields that $\sum_{i \in B} \left| \frac{a_i}{\|a\|_1} \right| \leq \varepsilon$ and so $\sum_{i \notin B} |a_i| \leq 2\varepsilon$. It follows from Lemma 5 (2) that for every $i \in \mathbb{N} \setminus B$ we have $p_i, q_i \in [p, q]_\alpha$. The conclusion is now immediate. \hfill $\Box$

**Proof of Theorem 1.** We can assume without loss of generality that $0 = q$. Indeed, a change of the base point in $M$ induces a linear isometry between the corresponding Lipschitz free spaces which preserves the molecules. Lemma 6 shows that if $\mu \in M$ satisfies $\langle \mu, f_{pq} \rangle = 1$ then $\mu \in F([p, q]_\alpha)$. Since $[p, q]_1$ is bounded, Proposition 2 yields that $\mu \in F([p, q]) = F([p, q])$. This is a 1-dimensional vector space so $\mu = \pm m_{pq}$ but only the choice of the plus sign is reasonable. \hfill $\Box$

**Remark 7.** Apart from the obvious fact that Theorem 1 strengthens and generalizes some of the results in [5] let us also point out that one of the proofs of the main result in [8] (i.e. the characterization of $M$ such that $F(M) = \ell_1(\Gamma)$) becomes now much simpler.

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