SHIFT-INvariant SPACES ON SI/Z LIE GROUPS

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Abstract. Given a simply connected nilpotent Lie group having unitary irreducible representations that are square-integrable modulo the center (SI/Z), we develop a notion of periodization on the group Fourier transform side, and use this notion to give a characterization of shift-invariant spaces in $L^2(N)$ in terms of range functions. We apply these results to study the structure of frame and Reisz families for shift-invariant spaces. We illustrate these results for the Heisenberg group as well as for other groups with SI/Z representations.

keywords: Shift-invariant spaces, translation invariant spaces, nilpotent Lie groups; range function; periodization operator; fibers; frame and Reisz bases;

1. INTRODUCTION

Let $G$ be a locally compact abelian (LCA) group, and let $H$ be a discrete subgroup of $G$ for which $G/H$ is compact. For a function $\phi : G \to \mathbb{C}$ write $L_g \phi = \phi(g^{-1} \cdot)$, $g \in G$. A shift invariant space (SIS) is a closed subspace $H$ of $L^2(G)$ that is invariant under the action of $H$ by the (unitary) operators $L_h$. The case where $G = \mathbb{R}^d$ has been studied extensively in the literature and plays a central role in the development of approximation theory and nonorthogonal expansions; see for example [8, 9, 1]. The main idea is due to Helson [12], whereby, via periodization, an SIS $H$ corresponds to a measurable range function. For general LCA groups, this idea is extended very successfully in [2] to give a characterization of SIS spaces exactly as in the Euclidean case; see also [13].

In this article we generalize the study of SIS in a different direction, to a natural class of non-abelian Lie groups called SI/Z groups. Let $N$ be a connected, simply connected nilpotent Lie group. Following [4], we say that $N$ is an SI/Z group if almost all of its irreducible representations are square-integrable modulo the center of the group. The effect of the SI/Z condition condition is that for this class of groups, the operator-valued Plancherel transform retains certain key features of the Euclidean case, and in particular, makes it possible to define a useful notion of periodization. Thus, techniques of representation theory, operator theory, spectral methods, Fourier analysis, and approximation theory, are related in a natural setting. We remark that the class of SI/Z groups is broad and in particular contains groups of an arbitrarily high degree of non-commutativity.

The outline of the paper is as follows. After the introduction, we introduce the class of SI/Z groups and some facts about these groups in Section 2. In Section 3 we study shift-invariant subspaces for our setting and their characterization in terms of range functions using group Fourier transform techniques. We apply these results to characterize bases.
for shift-invariant subspaces in Section \[4\]. We illustrate our results on some examples of such groups, including the Heisenberg group, in Section \[5\].

2. Notations and Preliminaries

Let \( N \) be a connected, simply connected nilpotent Lie group with Lie algebra \( \mathfrak{n} \). Recall that \( N \) acts naturally on \( \mathfrak{n} \) by the adjoint representation and on the linear dual \( \mathfrak{n}^* \) of \( \mathfrak{n} \) by the coadjoint representation. We denote these actions multiplicatively, and for \( l \in \mathfrak{n}^* \), denote by \( N(l) \) the stabilizer of \( l \) and \( \mathfrak{n}(l) \) its Lie algebra. The exponential mapping \( \exp : \mathfrak{n} \to N \) is a bijection, the center \( \mathfrak{z} \) of the Lie algebra of \( \mathfrak{n} \) is non-trivial, and \( Z = \exp \mathfrak{z} \) is the center of \( N \). We now recall certain classical results concerning the unitary representations of such groups originally due to J. Dixmier, A. Kirillov, and L. Pukanszky; citations for their work can be found in standard reference \[4\].

Let \( \pi \) be an irreducible unitary representation of \( N \). Then there is an analytic subgroup \( P \) of \( N \), and a unitary character \( \chi \) of \( P \), such that \( \pi \) is unitarily isomorphic with the representation induced by \( \chi \). Writing \( P = \exp \mathfrak{p} \) and \( \chi(\exp Y) = e^{2\pi il(Y)} \) where \( l \in \mathfrak{n}^* \), we have \( [\mathfrak{p}, \mathfrak{p}] \subseteq \ker l \). On the other hand, given any \( l \in \mathfrak{n}^* \), a subalgebra \( \mathfrak{p} \) of \( \mathfrak{n} \) is said to be subordinate to \( l \) if \( [\mathfrak{p}, \mathfrak{p}] \subseteq \ker l \), and in this case \( l \) defines a character \( \chi_l \) of \( P = \exp \mathfrak{p} \) as above. The representation induced from \( \chi_l \) is irreducible if and only if \( \mathfrak{p} \) is has maximal dimension among subordinate subalgebras, and if \( \mathfrak{p} \) and \( \mathfrak{p}' \) are both maximal subordinate subalgebras for \( l \), then the induced representations are isomorphic.

It follows that there is a canonical bijection between the space \( \hat{N} \) of equivalence classes of irreducible representations of \( N \), and the quotient space \( \mathfrak{n}^*/N \) of coadjoint orbits in \( \mathfrak{n}^* \). This bijection is a Borel isomorphism, and the Plancherel measure class for \( N \) is supported on the (dense, open) subset \( \hat{N}_{\text{max}} \) of \( \hat{N} \) corresponding to the coadjoint orbits of maximal dimension. For an irreducible representation \( \pi \) of \( N \), we denote the coadjoint orbit corresponding to the equivalence class of \( \pi \) by \( \mathcal{O}_\pi \).

An unitary irreducible representation of a unimodular Lie group \( N \) is said to be square integrable, or a discrete series representation, if \( s \mapsto \langle \pi(s)u, v \rangle \) belongs to \( L^2(N) \) for any vectors \( u \) and \( v \). If \( N \) is simply connected nilpotent, then the fact that the center is non-trivial implies that there are no discrete series representations. An irreducible representation \( \pi \) is square-integrable modulo the center if

\[
\int_{N/Z} |\langle \pi(n)u, v \rangle|^2 \, dn < \infty.
\]

We denote by \( SI/Z \) the subset of \( \hat{N} \) consisting of those (equivalence classes of) irreducible representations that are square-integrable modulo the center. For many unimodular groups the set \( SI/Z \) is empty. In the case where \( N \) is nilpotent simply connected, there is an orbital characterization of \( SI/Z \)-representations, first proved in \[15\] and also found in \[4\].

**Theorem 2.1.** Let \( \pi \) be an irreducible representation of a connected, simply connected nilpotent Lie group \( N \). Then the following are equivalent.

(i) \( \pi \) is square-integrable modulo the center.

(ii) For any \( l \in \mathcal{O}_\pi \), \( \mathfrak{n}(l) = \mathfrak{z} \) and \( \mathcal{O}_\pi = l + \mathfrak{z}^\perp \).
Moreover, if $SI/Z \neq \emptyset$, then $SI/Z = \hat{N}_{\text{max}}$.

We say that a connected simply connected nilpotent Lie group $N$ is an $SI/Z$ group if $SI/Z = \hat{N}_{\text{max}}$. We remark that for each positive integer $s$, there is a $SI/Z$ group $N$ of step $s$ (see [4]). The abelian case is just $\mathbb{R}^n$, and the simplest two-step example is the Heisenberg group, where $\hat{N}_{\text{max}}$ consists of the Schrödinger representations. From now on, we will assume that $N$ is a $SI/Z$ group.

The preceding shows that $\hat{N}_{\text{max}}$ is parametrized by a subset of $\mathfrak{z}^*$. Indeed, let $\pi \in \hat{N}_{\text{max}}$. Then $\dim \mathcal{O}_\pi = n - \dim \mathfrak{z}$, and since $\mathcal{O}_\pi$ is naturally a symplectic manifold, its dimension is even and we write $\dim \mathcal{O}_\pi = 2d$. Then Schur’s Lemma says that the restriction of $\pi$ to $Z$ is a character of $Z$, and hence $\pi$ determines a unique element $\lambda = \lambda_\pi \in \mathfrak{z}^*$ so that

$$\pi(z) = e^{2\pi i (\lambda, \log z) I},$$

where $I$ is the identity operator for the Hilbert space of a realization of $\pi$. It follows that $\mathcal{O}_\pi = \{ l \in \mathfrak{n}^*: l|_{\mathfrak{z}} = \lambda \}$. The description of orbits for $\hat{N}_{\text{max}}$ shows that $\pi \mapsto \lambda_\pi$ is injective.

An explicit Plancherel transform is obtained by describing the set $\Sigma = \{ \lambda_\pi : \pi \in \hat{N}_{\text{max}} \}$, and explicit maximal subordinate subalgebras $\mathfrak{p}(\lambda), \lambda \in \Sigma$, that vary smoothly with $\lambda$. Fix a basis $\{ X_1, X_2, \ldots, X_n \}$ for $\mathfrak{n}$ for which $\mathfrak{n}_j := \text{span}\{ X_1, \ldots, X_j \}$ is an ideal in $\mathfrak{n}$, $1 \leq j \leq n$, and for which $\mathfrak{n}_j = \mathfrak{z}$ for some $r$. Having fixed such a basis, we identify $\mathfrak{z}$ with $\mathbb{R}^r$ and $\mathfrak{z}^*$ with the subspace $\{ l \in \mathfrak{n}^*: l(X_j) = 0, r < j \leq n \}$. Define $\text{Pf}(l), l \in \mathfrak{n}^*$ so that $|\text{Pf}(l)|^2 = \det M(l)$ where $M(l)$ is the skew-symmetric matrix

$$M(l) = [ [X_i, X_j]_{r \leq i, j \leq n}] .$$

The following can be gleaned from [16] and [15]; again a useful reference is [4].

**Proposition 2.2.** For an $SI/Z$ group, we have the following.

(i) $\text{Pf}$ is constant on each coadjoint orbit and $[\pi] \in SI/Z$ if and only if $\text{Pf}(l) \neq 0$ on $\mathcal{O}_\pi$.

(ii) $\Sigma := \{ \lambda \in \mathfrak{z}^*: \text{Pf}(\lambda) \neq 0 \}$ is a cross-section for coadjoint orbits of maximal dimension.

(iii) Fix $\lambda \in \Sigma$. Then

$$\mathfrak{p}(\lambda) := \sum_{j=1}^n \mathfrak{n}_j(\lambda|_{\mathfrak{n}_j})$$

is a maximal subordinate subalgebra for $\lambda$ and the corresponding induced representation $\pi_\lambda$ is realized naturally in $L^2(\mathbb{R}^d)$.

(iv) For $\phi \in L^1(N) \cap L^2(N)$, the Fourier transform

$$\hat{\phi}(\lambda) := \int_N \phi(x) \pi_\lambda(x) dx, \ \lambda \in \Sigma,$$

implements an isometric isomorphism – the Plancherel transform –

$$\mathcal{F} : L^2(N) \rightarrow L^2(\mathfrak{z}^*, \mathcal{H}_S(L^2(\mathbb{R}^d))), |	ext{Pf}(\lambda)||d\lambda|$$

where $\mathcal{H}_S(L^2(\mathbb{R}^d))$ is the Hilbert space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$, and $d\lambda$ is a suitably normalized Lebesgue measure on $\mathfrak{z}^*$. 
3. Shift invariant spaces

Let $N$ be an $SI/Z$ group. We retain the notations of the preceding section, and recall the basis $\{X_1, X_2, \ldots, X_n\}$ chosen above. Identify the center $\mathfrak{z}$ of $\mathfrak{n}$ with $\mathbb{R}^r$ via the ordered basis $\{X_1, X_2, \ldots, X_r\}$ ($r = n - 2d$), and identify the center $Z$ of $N$ with $\mathbb{R}^r$ by

$$z = \exp z_1X_1 \exp z_2X_2 \cdots \exp z_rX_r, \quad (z_1, z_2, \ldots, z_r) \in \mathbb{R}^r.$$ 

Write

$$x = \exp x_1X_{r+1} \exp x_2X_{r+2} \cdots \exp x_{2d}X_n, \quad (x_1, x_2, \ldots, x_r) \in \mathbb{R}^{2d}$$

so that the subset $\mathcal{X} = \exp \mathbb{R}X_{r+1} \exp \mathbb{R}X_{r+2} \cdots \exp \mathbb{R}X_n$ of $N$ is identified with $\mathbb{R}^{2d}$. For each $n \in N$, we have unique $x \in \mathcal{X}$ and $z \in Z$ such that $n = xz$ and $N$ is thus identified with $\mathbb{R}^{2d} \times \mathbb{R}^r$. The dual $\mathfrak{z}^*$ is identified with $\mathbb{R}^r$ via the dual basis $\{X_1^*, \ldots, X_r^*\}$.

Let $\phi \in \mathcal{L}^1 (N) \cap \mathcal{L}^2 (N)$. For $x \in \mathcal{X}$ and $z \in Z$, the Plancherel transform of the left translate $L_{xz}\phi$ is the operator-valued function on $\Sigma$ defined at each $\lambda \in \Sigma$ by

$$\mathcal{F} (L(xz) \phi) (\lambda) = e^{2\pi i (\lambda, z)} \hat{\pi}_\lambda (x) \hat{\phi}(\lambda).$$

We denote the $r$ dimensional torus by $\mathbb{T}^r$ and identify it with $[0, 1)^r$ in $\mathbb{R}^r$ as usual. For simplicity of notation we write $\mathcal{H} = \mathcal{H}S(\mathcal{L}^2 (\mathbb{R}^d)) = \mathcal{L}^2 (\mathbb{R}^d) \otimes \mathcal{L}^2 (\mathbb{R}^d)^*$, and let $\mathcal{L}$ denote the Hilbert space $\mathcal{L}^2 (\mathbb{Z}^r, \mathcal{H})$ with the norm

$$||h||^2_\mathcal{L} = \sum_{j \in \mathbb{Z}^r} ||h_j||^2_\mathcal{H}.$$

Define the periodized Plancherel transform map $T : \mathcal{L}^2 (N) \to \mathcal{L}^2 (\mathbb{T}^r, \mathcal{L})$ as follows. Denote by $\mathcal{C}$ the set of all $\mathcal{H}$-valued $\mathcal{L}^2$-functions on $\mathbb{R}^r$ that are continuous on each set $\mathbb{T}^r + j, j \in \mathbb{Z}^r$. Note that $\mathcal{C}$ determines a subspace of $\mathcal{L}^2 (\mathbb{R}^r, \mathcal{H})$, and we denote this subspace also by $\mathcal{C}$. Given $f \in \mathcal{C}$, define the sequence-valued function $a_f$ on $\mathbb{T}^r$ by

$$a_f (\sigma) = (f (\sigma + j))_{j \in \mathbb{Z}^r}, \quad \sigma \in \mathbb{T}^r.$$ 

Then it is easy to check that $a_f \in C (\mathbb{T}^r, \mathcal{L})$ and we have

$$\int_{\mathbb{R}^r} ||f (\lambda)||^2 d\lambda = \sum_{j \in \mathbb{Z}^r} \int_{\mathbb{T}^r} ||f (\lambda + j)||^2_\mathcal{H} d\sigma = \int_{\mathbb{T}^r} \sum_{j \in \mathbb{Z}^r} ||a_f (\sigma) ||^2_\mathcal{H} d\sigma.$$ 

Hence for a.e. $\sigma \in \mathbb{T}^r$, $a_f (\sigma) \in \mathcal{C}$, the function $a_f$ is an $\mathcal{L}^2, \mathcal{L}$-valued function on $\mathbb{T}^r$, and $||f|| = ||a_f||$. $\mathcal{C}$ is dense in $\mathcal{L}^2 (\mathbb{R}^r, \mathcal{H})$, and so we extend this mapping to an isometry $A : \mathcal{L}^2 (\mathbb{R}^r, \mathcal{H}) \to \mathcal{L}^2 (\mathbb{T}^r, \mathcal{L})$ given by $f \mapsto Af (\sigma) = a_f (\sigma)$. Recalling the Pfaffian $\text{Pf} (\lambda)$ and identifying $\mathfrak{z}^*$ with $\mathbb{R}^r$, let $M : \mathcal{L}^2 (\mathbb{R}^r, \mathcal{H}, |\text{Pf} (\lambda)| d\lambda) \to \mathcal{L}^2 (\mathbb{R}^r, \mathcal{H})$ be the natural isometric isomorphism given by

$$M f (\lambda) = f (\lambda)|\text{Pf} (\lambda)|^{1/2}$$

and put $T = A \circ M \circ \mathcal{F}$. The diagram below displays the composition of isometries as described.

$$\begin{align*}
\mathcal{L}^2 (N) & \xrightarrow{\mathcal{F}} \mathcal{L}^2 (\mathbb{R}^r, \mathcal{H}, |\text{Pf} (\lambda)| d\lambda) \xrightarrow{M} \mathcal{L}^2 (\mathbb{R}^r, \mathcal{H}, d\lambda) \xrightarrow{A} \mathcal{L}^2 (\mathbb{T}^r, \mathcal{L}).
\end{align*}$$

Finally, for $n \in N$, we define the unitary operator $\tilde{\pi}_\sigma (n) : \mathcal{L} \to \mathcal{L}$ by

$$\left( \tilde{\pi}_\sigma (n) h \right)_j = \pi_{\sigma + j} (n) \circ h_j, \quad h \in \mathcal{L}$$
and define \( \tilde{\pi}(n) : L^2(\mathbb{T}^r, \mathcal{L}) \to L^2(\mathbb{T}^r, \mathcal{L}) \) by
\[
(\tilde{\pi}(n)a)(\sigma) = \tilde{\pi}_\sigma(n)a(\sigma), \ a \in L^2(\mathbb{T}^r, \mathcal{L}).
\]
Note that if \( z \in Z \), then \( (\tilde{\pi}(z)a)(\sigma) = e^{2\pi i (\sigma, z)}a(\sigma) \) for all \( a \in L^2(\mathbb{T}^r, \mathcal{L}) \). We have almost already proved the following.

**Lemma 3.1.** The mapping \( T \) is an isometric isomorphism, and for each \( n \in N \),
\[
(2) \quad T(L_n \phi)(\sigma) = (\tilde{\pi}(n)T \phi)(\sigma).
\]

**Proof.** We show that \( A \) is surjective. Take \( a \in C(\mathbb{T}^r, \mathcal{L}) \), and define \( f : \mathbb{R}^r \to \mathcal{H} \) by \( f(\lambda) = a(\sigma + j) = a(\sigma)_j \) where \( \sigma := \lambda - j \in \mathbb{T}^r \) and \( j \in \mathbb{Z}^r \). Then \( f \in C \) and the identity \( \{1\} \) shows that \( f \) is an \( L^2 \)-function on \( \mathbb{R}^r \) with \( Af = a \). By density of \( C \) and \( C(\mathbb{T}^r, \mathcal{H}) \), we have that \( A \) is surjective. It follows that \( T \) is unitary.

Next we turn to the definition of shift-invariant spaces and range functions. Recall that both \( \mathfrak{g} \) and \( \mathfrak{g}^* \) are identified with \( \mathbb{R}^r \) via the chosen basis \( X_1, \ldots, X_r \) for \( \mathfrak{g} \). Denote by \( \Gamma_0 \) the lattice of integral points in \( Z \). Then \( Z_{\mathbb{Z}^r} \) is identified by \( \mathbb{Z}^r \).

For any discrete subset \( \Gamma_1 \) of \( \mathcal{X} \), put
\[
\Gamma = \{xz \in N : x \in \Gamma_1, z \in \Gamma_0\}.
\]

**Remark:** The choice of basis for \( \mathfrak{g} \) is completely arbitrary, and given any lattice \( \Gamma_0 \) in \( Z \), there is a basis for \( \mathfrak{g} \) for which \( \Gamma_0 \) is the lattice of integral points.

We say a closed subspace \( S \) of \( L^2(N) \) is \( \Gamma \)-shift invariant if for any \( \gamma \in \Gamma \) and \( \phi \in S \)
\[
L_\gamma \phi \in S
\]

**Definition 3.2.** A range function is a mapping from \( \mathbb{T}^r \) into the set of closed subspaces of the Hilbert space \( \mathcal{L} \). For any \( \sigma \in \mathbb{T}^r \), we call \( J(\sigma) \) the fiber space associated to \( \sigma \). We say a range function is measurable if for any \( a \in L^2(\mathbb{T}^r, \mathcal{L}) \), the mapping \( \sigma \mapsto \langle P_\sigma u, v \rangle \), \( \forall u, v \in \mathcal{L} \) is measurable, where \( P_\sigma \) is the orthogonal projector of \( \mathcal{L} \) onto \( J(\sigma) \).

For a measurable range function \( J \), the condition “\( a(\sigma) \in J(\sigma) \) for a.e. \( \sigma \)” determines a subspace \( \mathcal{M}_J \) of \( L^2(\mathbb{T}^r, \mathcal{L}) \):
\[
\mathcal{M}_J = \{a \in L^2(\mathbb{T}^r, \mathcal{L}) : a(\sigma) \in J(\sigma) \text{ for a.e. } \sigma \}.
\]
It is easily seen that \( \mathcal{M}_J \) is closed in \( L^2(\mathbb{T}^r, \mathcal{L}) \). Observe that for range functions \( J, J' \), \( \mathcal{M}_J = \mathcal{M}_{J'} \) if and only if \( J(\sigma) = J'(\sigma) \) for a.e. \( \sigma \). In general we identify two range functions that are equal a.e.

**Lemma 3.3.** Let \( J \) be a range function and put \( S = T^{-1}(\mathcal{M}_J) \). If
\[
\tilde{\pi}_\sigma(\Gamma_1) (J(\sigma)) \subseteq J(\sigma)
\]
holds for a.e. \( \sigma \in \mathbb{T}^r \), then \( S \) is \( \Gamma \)-shift invariant.
Proof. For each \( \phi \in S \) and \( k \in \Gamma_1 \), we apply (2) to see that for a.e. \( \sigma \in \mathbb{T}^r \), \( k \in \Gamma_1 \), \( m \in \Gamma_0 \),

\[
T(L_{km}\phi)(\sigma) = e^{2\pi i \langle \sigma, m \rangle} \tilde{\pi}_\sigma(k)(T\phi(\sigma))
\]

and so by definition of \( \mathcal{M}_J \), \( T(L(\gamma)\phi) \in \mathcal{M}_J \) and \( L_\gamma \phi \in S \) for all \( \gamma \in \Gamma \).

\[ \square \]

The characterization of \( \Gamma \)-shift-invariant subspaces is given in

**Theorem 3.4.** Let \( S \subseteq L^2(N) \) be a closed subspace. Then the following are equivalent.

(i) \( S \) is \( \Gamma \)-shift invariant.

(ii) There is a unique range function \( J \) up to equivalency such that \( J(\sigma) \) is \( \tilde{\pi}_\sigma(\Gamma_1) \)-invariant for a.e. \( \sigma \), and \( T(S) = \mathcal{M}_J \).

Proof. To show (i) \( \implies \) (ii), we apply [12, Theorem 8]. Suppose that \( S \) is shift invariant and let \( \phi \in S \), \( a = T\phi \). For each \( m \in \Gamma_0 \) and \( \sigma \in \Sigma \), the relation (2) shows that

\[
e^{2\pi i m \cdot a(\sigma)} = T(L_m \phi)(\sigma)
\]

so that \( T(S) \) is doubly invariant in the sense of [12]. By [12, Theorem 8], there exists a range function \( J \) such that \( T(S) = \mathcal{M}_J \). It remains to show that \( J(\sigma) \) is \( \tilde{\pi}_\sigma(\Gamma_1) \)-invariant for a.e. \( \sigma \).

Choose an orthonormal basis \( \{E^{(n)}\}_{n \in \mathbb{N}} \) for \( \mathcal{L} \). For each \( n \in \mathbb{N} \) and \( p \in \mathbb{Z} \) put \( G_{p,n}(\sigma) = e^{2\pi i \langle \sigma, p \rangle} E^{(n)} \). Then \( \{G_{p,n} : n \in \mathbb{N}, p \in \mathbb{Z} \} \) is an orthonormal basis for \( L^2(\mathbb{T}^r, \mathcal{L}) \). Let \( P := P_T \) denote the orthogonal projector of \( L^2(\mathbb{T}^r, \mathcal{L}) \) onto \( T(S) \), and for each \( n, p \), choose a function \( F_{n,p} \) that belongs to the equivalence class of \( P(G_{n,p}) \). (From the proof of [12, Theorem 8] we see that \( J(\sigma) := \text{span}\{F_{n,p}(\sigma) : n \in \mathbb{N}, p \in \mathbb{Z} \} \) for all \( \sigma \).) Now by shift-invariance of \( S \), definition of \( \mathcal{M}_J = T(S) \) and (2), for each \( n \in \mathbb{N}, p \in \mathbb{Z} \), and \( k \in \gamma' \), we have a conull subset \( E_{n,p,k} \) of \( \mathbb{T}^r \) such that for all \( \sigma \in E_{n,p,k} \), the sequence

\[
\tilde{\pi}_\sigma(k)F_{n,p}(\sigma) = TL_kT^{-1}F_{n,p}(\sigma)
\]

belongs to \( J(\sigma) \). Put \( E = \cap \{E_{n,p,k} : n \in \mathbb{N}, p \in \mathbb{Z}, k \in \Gamma' \} \). Then \( E \) is conull, and for \( \sigma \in E \), we have \( \tilde{\pi}_\sigma(k)F_{n,p}(\sigma) \) belongs to \( J(\sigma) \) for all \( n, p, \) and \( k \). Since \( J(\sigma) \) is spanned by \( F_{n,p}(\sigma) \), then \( J(\sigma) \) is \( \tilde{\pi}_\sigma(\Gamma') \)-invariant. The proof of (ii) \( \implies \) (i) is obtained by Lemma 3.3.

\[ \square \]

4. Frames and Bases

4.1. **Frames.** Let \( X = \{\eta_\alpha\} \) be a countable family of vectors in a Hilbert space \( \mathcal{H} \). Recall that \( X \) is a Bessel family if there is a positive constant \( B \) such that

\[
\sum_\alpha |\langle h, \eta_\alpha \rangle|^2 \leq B \|h\|^2
\]

holds for all \( h \in \text{span}X \). If in addition there is \( 0 < A \leq B < \infty \) so that

\[
A \|h\|^2 \leq \sum_\alpha |\langle h, \eta_\alpha \rangle|^2 \leq B \|h\|^2
\]
holds for all \( h \in \text{span} X \), then we say that \( X \) is a frame (for its span). Finally, \( X \) is a Riesz family with positive finite constants \( A \) and \( B \) if

\[
A \sum_{\alpha} |a_{\alpha}|^2 \leq \left\| \sum_{\alpha} a_{\alpha} \eta_{\alpha} \right\|^2 \leq B \sum_{\alpha} |a_{\alpha}|^2
\]

holds for all finitely supported indexed sets \( (a_{\alpha})_\alpha \) of complex numbers. If a Riesz family is complete in \( \mathcal{H} \) we say that it is a Riesz basis. If \( A = B = 1 \) for a Riesz basis \( X \), then \( X \) is an orthogonal family and \( \| \eta_{\alpha} \| = 1 \) for all \( \alpha \).

Fix a discrete subset \( \Gamma \) of \( \mathbb{N} \) of the form \( \Gamma_1 \Gamma_0 \). Let \( A \subset L^2(\mathbb{N}) \) be a countable set. Define

\[
E(A) = \{ L_\gamma \phi : \gamma \in \Gamma, \phi \in A \}
\]

and put \( S = \text{span} E(A) \). Let \( J \) be the range function associated to \( S \). With these definitions we have

**Theorem 4.1.** The system \( E(A) \) is a frame with constants \( 0 < A \leq B < \infty \) (or a Bessel family with constant \( B \)) if and only if for almost \( \sigma \in \mathbb{T}^r \) the system \( T(E(A))(\sigma) := \{ T(L_k \phi)(\sigma) : \phi \in A, k \in \Gamma' \} \) constitutes a frame (Bessel) family for \( J(\sigma) \) with the unified constants.

**Proof.** Let \( f \in L^2(\mathbb{N}) \). Since \( \| Tf \| = \| f \| \) we have for each \( \gamma \in \Gamma \),

\[
\sum_{\phi \in A, \gamma \in \Gamma} |\langle f, L_\gamma \phi \rangle|^2 = \sum_{\phi \in A, \gamma \in \Gamma} |\langle Tf, T(L_\gamma \phi) \rangle|^2
= \sum_{\phi \in A, \gamma \in \Gamma} \left| \int_{\mathbb{T}^r} \langle Tf(\sigma), T(L_\gamma \phi(\sigma)) \rangle \, d\sigma \right|^2
= \sum_{\phi \in A, \gamma \in \Gamma} \left| \int_{\mathbb{T}^r} \langle Tf(\sigma), \tilde{\pi}_\gamma(\sigma)T\phi(\sigma) \rangle \, d\sigma \right|^2.
\]

Writing \( \gamma = km \), with \( k \in \Gamma_1, m \in \Gamma_0 \), we get

\[
\sum_{\phi \in A, \gamma \in \Gamma} \left| \int_{\mathbb{T}^r} \langle Tf(\sigma), \tilde{\pi}_\gamma(\sigma)T\phi(\sigma) \rangle \, d\sigma \right|^2
= \sum_{\phi \in A, (k,m) \in \Gamma} \left| \int_{\mathbb{T}^r} \langle Tf(\sigma), e^{2\pi i (\sigma, m)}\tilde{\pi}_\gamma(k)T\phi(\sigma) \rangle \, d\sigma \right|^2
= \sum_{\phi \in A, (k,m) \in \Gamma} \left| \int_{\mathbb{T}^r} \langle Tf(\sigma), \tilde{\pi}_\gamma(k)T\phi(\sigma)e^{-2\pi i (\sigma, m)} \rangle \, d\sigma \right|^2
\]

For each \( k \) put \( G_k(\sigma) := \langle Tf(\sigma), \tilde{\pi}_\gamma(k)T\phi(\sigma) \rangle \). Then \( G_k \) is integrable with square-summable Fourier coefficients, therefore \( G_k \) lies in \( L^2(\mathbb{T}^r) \). By Fourier inversion we then
continue the above as follows:

\[
\sum_{\phi \in \mathcal{A}(k,m) \in \Gamma} \left| \int_{\mathbb{T}} \langle Tf(\sigma), \tilde{\pi}_\sigma(k)T(\phi)(\sigma) \rangle e^{-2\pi i \langle \sigma, m \rangle} \, d\sigma \right|^2 = \sum_{\phi \in \mathcal{A}(k,m) \in \Gamma} |\hat{G}_k(m)|^2 = \sum_{\phi \in \mathcal{A}, k \in \Gamma'} \|G_k\|^2 = \sum_{\phi \in \mathcal{A}, k \in \Gamma'} \int_{\mathbb{T}} |G_k(\sigma)|^2 \, d\sigma
\]

By substituting back \( G_k(\sigma) := \langle Tf(\sigma), \tilde{\pi}_\sigma(k)T(\phi)(\sigma) \rangle \), we obtain

\[
\sum_{\phi \in \mathcal{A}, \gamma \in \Gamma} |\langle f, L_\gamma \phi \rangle|^2 = \sum_{\phi \in \mathcal{A}, k \in \Gamma'} \int_{\mathbb{T}} |G_k(\sigma)|^2 \, d\sigma = \sum_{\phi \in \mathcal{A}, k \in \Gamma'} \int_{\mathbb{T}} |\langle Tf(\sigma), \tilde{\pi}_\sigma(k)T(\phi)(\sigma) \rangle|^2 \, d\sigma = \int_{\mathbb{T}} \sum_{\phi \in \mathcal{A}, k \in \Gamma'} |\langle Tf(\sigma), T(L_k \phi)(\sigma) \rangle|^2 \, d\sigma.
\]

Now suppose that \( f \in \mathcal{S} \) and that for some \( 0 < A \leq B < \infty \), the system \( T(E(A))(\sigma) \) is an \((A, B)\)-frame for a.e. \( \sigma \in \mathbb{T} \). Then \( Tf(\sigma) \in J(\sigma) \) holds for a.e. \( \sigma \), so

\[
A \|Tf(\sigma)\|^2 \leq \int_{\mathbb{T}} \sum_{\phi \in \mathcal{A}, k \in \Gamma'} |\langle Tf(\sigma), T(L_k \phi)(\sigma) \rangle|^2 \, d\sigma \leq B \|Tf(\sigma)\|^2.
\]

holds for a.e. \( \sigma \). Integrating yields

\[
A \|f\|^2 = A \|Tf\|^2 = A \int_{\mathbb{T}} \|Tf(\sigma)\|^2 \, d\sigma \leq \int_{\mathbb{T}} \sum_{\phi \in \mathcal{A}, k \in \Gamma'} |\langle Tf(\sigma), T(L_k \phi)(\sigma) \rangle|^2 \, d\sigma \leq B \int_{\mathbb{T}} \|Tf(\sigma)\|^2 \, d\sigma \leq B \|f\|^2.
\]

By substituting (3) we obtain

\[
A \|f\|^2 \leq \sum_{\phi \in \mathcal{A}, \gamma \in \Gamma} \|\langle f, L_\gamma \phi \rangle\|^2 \leq B \|f\|^2.
\]

Now assume that \( E(A) \) is a frame family with constants \( A \) and \( B \). Let \( \mathcal{D} \) be a countable dense subset of \( \mathcal{L} \). To prove that the family \( T(E(A))(\sigma) \) constitutes a frame for \( J(\sigma) \) for almost every \( \sigma \), it is sufficient to show that for each \( h \in \mathcal{D} \),

\[
A \|P_\sigma h\|^2 \leq \sum_{\phi \in \mathcal{A}, k \in \Gamma} |\langle T(L_k \phi)(\sigma), h \rangle|^2 \leq B \|P_\sigma h\|^2
\]

holds for a.e. \( \sigma \in \mathbb{T} \), where \( P_\sigma \) is the orthogonal projection operator from \( \mathcal{L} \) onto \( J(\sigma) \). (For, if (4) holds for any \( h \in \mathcal{D} \), then we have \( N_h \) a measure zero subset in \( \mathbb{T} \) such that for any \( \sigma \in N_h^c \) the relation (1) holds. Put \( N = \bigcup_{h \in \mathcal{D}} N_h \). Then \( N \) has measure zero and
(4) holds for all \( h \in \mathcal{D} \) and all \( \sigma \in T/N \). Since \( P_\sigma(\mathcal{D}) \) is dense in \( J(\sigma) \) for all \( \sigma \in N_c \), the assertion holds, i.e., \( T(E(A))(\sigma) \) constitutes a frame for \( J(\sigma) \) with the identical constants for all \( \sigma \in N_c \). To complete the proof, we still need to show (4). For this, we assume that it fails for some \( h_0 \in \mathcal{D} \) and define \( G(\sigma) := \sum_{\phi \in A, \; k \in \Gamma} |\langle T(L_k \phi)(\sigma), h_0 \rangle|^2 \). Then one of the following sets must have positive measure.

\[
\{ \sigma \in T^r : G(\sigma) > B\|P_\sigma h_0\|^2 \}, \quad \{ \sigma \in T^r : G(\sigma) < A\|P_\sigma h_0\|^2 \}.
\]

Without lose of generality, we assume that the measure of the first set is positive. Therefore for some \( \epsilon > 0 \), the measure of \( C := \{ \sigma \in T^r : G(\sigma) > (B + \epsilon)\|P_\sigma h_0\|^2 \} \) is positive too. Put \( \tilde{h}_0(\sigma) = \chi_C(\sigma)P_\sigma h_0 \). Then \( \sigma \mapsto \tilde{h}_0(\sigma) \) is measurable and \( \tilde{h}_0(\sigma) \in J(\sigma) \) and \( \tilde{h}_0 \in T(S) = \int_T J(\sigma)d\sigma \). Therefore for some \( f_0 \in S \), \( T(f_0) = \tilde{h}_0 \). We show that the upper frame inequality does not hold for \( f_0 \) which leads us to a contradiction: Let \( \gamma = km \), with \( k \in \Gamma_1, m \in \Gamma_0 \). Then

\[
(5) \sum_{\phi \in A, \; \gamma \in \Gamma} |\langle f_0, L_\gamma \phi \rangle|^2 = \sum_{\phi \in A, \; \gamma \in \Gamma} |\langle T(f_0), T(L_\gamma \phi) \rangle|^2 = \sum_{\phi \in A, \; \gamma = (m,k) \in \Gamma} |\int_{T^r} \langle T(f_0)(\sigma), e^{2\pi i (\sigma,m)} \pi_\sigma(k)T\phi \rangle d\sigma|^2 = \sum_{\phi \in A, \; \gamma = (k,m) \in \Gamma} |\hat{G}_k(m)|^2
\]

where \( G_k(\sigma) = \langle T(f_0)(\sigma), \pi_\sigma(k)T\phi \rangle \). For any \( k \) we have \( \sum_m |\hat{G}_k(m)|^2 = \|G_k\|^2 \). By using this equality in our previous calculations and using the Plancherel theorem and substituting back the function \( G_k \), all together we arrive the following:

\[
(6) \sum_{\phi \in A, \; k \in \Gamma'} \sum_m |\hat{G}_k(m)|^2 = \sum_{\phi \in A, \; k \in \Gamma'} \|G_k\|^2 = \sum_{\phi \in A, \; k \in \Gamma'} \int_{T^r} |\langle T(f_0)(\sigma), \pi_\sigma(k)T\phi(\sigma) \rangle|^2 d\sigma = \sum_{\phi \in A, \; k \in \Gamma'} \int_{T^r} |\langle \tilde{h}_0(\sigma), \pi_\sigma(k)T\phi(\sigma) \rangle|^2 d\sigma = \int_{C} \sum_{\phi \in A, \; k \in \Gamma'} |\langle P_\sigma h_0, \pi_\sigma(k)T\phi(\sigma) \rangle|^2 d\sigma
\]
By substituting \( G(\sigma) = \sum_{\phi \in \mathcal{A}, k \in \Gamma'} \langle P_{\sigma} h_0, \tilde{\pi}_{\sigma}(k) T \phi(\sigma) \rangle^2 \) in the above, we have

\[
\begin{align*}
(6) &> (B + \epsilon) \int_C \| P_{\sigma} h_0 \|^2 d\sigma \\
&= (B + \epsilon) \int_{\Gamma'} \| \chi_C(\sigma) P_{\sigma} h_0 \|^2 d\sigma \\
&= (B + \epsilon) \int_{\Gamma'} \| \tilde{h}_0(\sigma) \|^2 d\sigma \\
&= (B + \epsilon) \| \tilde{h}_0 \|^2 \\
&= (B + \epsilon) \| T^{-1} \tilde{h}_0 \|^2 \\
&= (B + \epsilon) \| f_0 \|^2.
\end{align*}
\]

(7)

Now a combination of (5) and (7) contradicts the assumption and hence we are done. \( \square \)

4.2. Riesz Bases. Before we start with the characterization of Riesz bases obtained from \( \Gamma \) shifts of a countable sets in terms of range functions, we shall introduce the following notation: for any finite supported sequence \( a = \{a_m\}_{m \in \mathbb{Z}} \in l^2(\mathbb{Z}) \), denote by \( P_a \) the associated trigonometric polynomial \( P_a(\sigma) = \sum_m a_m e^{2\pi i \langle \sigma, m \rangle} \) for all \( \sigma \in \mathbb{T} \). Observe that

\[
\| P_a \|_2^2 = \sum_m |a_m|^2.
\]

We have the following.

**Lemma 4.2.** Let \( \mathcal{A} \) be a countable set in \( L^2(N) \). Let \( a = \{a_{\phi,k,m}\}_{\phi \in \mathcal{A}, km \in \Gamma} \) be a finitely supported sequence in \( l^2(\mathbb{A} \times \Gamma) \). For each \( \phi \in \mathcal{A} \) and \( k \in \Gamma_1 \) put \( P_{\phi,k}(\sigma) = \sum_m a_{\phi,k,m} e^{2\pi i \langle \sigma, m \rangle} \).

Then

\[
\left\| \sum_{\phi \in \mathcal{A}, km \in \Gamma} a_{\phi,k,m} L_{km} \phi \right\|_2^2 = \int_{\mathbb{T}'} \left\| \sum_{\phi,k} P_{\phi,k}(\sigma) \tilde{\pi}_{\sigma}(k) T(\phi)(\sigma) \right\|_\mathcal{L}^2 d\sigma
\]

**Proof.** The proof is based on some elementary calculations as follows: Since \( T \) is isometric, we have
By applying Lemma 4.2 to the above middle sum and substituting (8) we arrive at
\[ \sum_{\phi \in A, k, m} a_{\phi, k, m} L_{km} \phi \leq \sum_{\phi \in A, k, m} a_{\phi, k, m} T(L_{km} \phi) \leq \int_{T^r} \sum_{\phi \in A, k, m} a_{\phi, k, m} T(L_{km} \phi) d\sigma \]
which yields
\[ \sigma \text{ as desired.} \]

**Proposition 4.3.** Let \( A \) be a countable subset of \( L^2(N) \) and \( E(A) \) be the set of \( \Gamma \)-translates of elements in \( A \) with associated range function \( J \). Assume that for some \( 0 < A \leq B < \infty \), \( \{ \tilde{\pi}_\sigma(k) T(\phi)(\sigma) : k \in \Gamma_1, \phi \in A \} \) is a Riesz basis for \( J(\sigma) \) with constants \( A \) and \( B \), for almost every \( \sigma \in T \). Then \( E(A) \) is a Riesz basis for its span with the same constants.

**Proof.** Let a finitely supported sequence \( \{a_{\phi, k, m}\}_{\phi \in A, k, m} \) be given. For each \( \phi, k \), let \( P_{\phi, k} \) be the trigonometric polynomial defined in Lemma 4.2. Then with our assumptions, for almost every \( \sigma \in T^r \) and the finite supported sequence \( \{ P_{\phi, k}(\sigma) \}_{\phi, k} =: \{ b_{\phi, k} \} \)
\[ A \sum_{\phi, k} |P_{\phi, k}(\sigma)|^2 \leq \sum_{\phi, k} |P_{\phi, k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma)|^2 \leq B \sum_{\phi, k} |P_{\phi, k}(\sigma)|^2 \]
where all the sums in above run on a finite index set of \( \phi, k \). Integrating (11) over \( T^r \) yields
\[ A \sum_{\phi, k} \|P_{\phi, k}\|^2 \leq \int_{T^r} \sum_{\phi, k} |P_{\phi, k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma)|^2 d\sigma \leq B \sum_{\phi, k} \|P_{\phi, k}\|^2 \]
By applying Lemma 4.2 to the above middle sum and substituting (8) we arrive at
\[ A \sum_{\phi, k, m} |a_{\phi, k, m}|^2 \leq \sum_{\phi, k, m} |a_{\phi, k, m} L_{km} \phi|^2 \leq B \sum_{\phi, k, m} |a_{\phi, k, m}|^2 \]
as desired.

**Theorem 4.4.** The set \( E(A) \) is a Riesz family with constants \( A \) and \( B \) if and only if \( \{ \tilde{\pi}_\sigma(k) T(\phi)(\sigma) : k \in \Gamma_1, \phi \in A \} \) is a Riesz family with constants \( A \) and \( B \), for a.e. \( \sigma \in T \).
Proof. The proof is similar to those of [1, Theorem 2.3, part (ii)]; see also [2, Theorem 4.3].

Choose any \( a = \{a_{\phi, (k, m)}\}_{\phi \in \mathcal{A}, (k, m) \in \Gamma} \), having finite support. For each \( \phi, k \), let \( P_{\phi, k} \) be the trigonometric polynomial defined in Lemma 1.2.

For sufficiency, suppose that \( \{\tilde{\pi}_\sigma(k) T(\phi)(\sigma) : k \in \Gamma, \phi \in \mathcal{A}\} \) is a Riesz family with constants \( A \) and \( B \), for a.e. \( \sigma \in \mathbb{T} \). Then for almost every \( \sigma \in \mathbb{T} \)

\[
(11) \quad A \sum_{\phi, k} |P_{\phi, k}(\sigma)|^2 \leq \left\| \sum_{\phi, k} P_{\phi, k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|^2 \leq B \sum_{\phi, k} |P_{\phi, k}(\sigma)|^2
\]

where all the sums in above run on a finite index set of \( \phi, k \). Integrating (11) over \( \mathbb{T} \) yields

\[
A \sum_{\phi, k} \|P_{\phi, k}\|^2 \leq \int_{\mathbb{T}} \left\| \sum_{\phi, k} P_{\phi, k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|^2 d\sigma \leq B \sum_{\phi, k} \|P_{\phi, k}\|^2
\]

By applying Lemma 1.2 to the above middle sum and substituting (8) we arrive at

\[
(12) \quad A \sum_{\phi, k, m} |a_{\phi, k, m}|^2 \leq \left\| \sum_{\phi, k, m} a_{\phi, k, m} L_{km} \phi \right\|^2 \leq B \sum_{\phi, k, m} |a_{\phi, k, m}|^2
\]

as desired.

On the other hand, suppose that \( E(\mathcal{A}) \) is a Riesz family in \( L^2(\mathbb{N}) \) with constants \( A \) and \( B \). Then for any finitely supported \( \{a_{\phi, k, m}\}_{\phi \in \mathcal{A}, (k, m) \in \Gamma} \)

\[
(13) \quad A \sum_{\phi, k, m} |a_{\phi, k, m}|^2 \leq \left\| \sum_{\phi, k, m} a_{\phi, k, m} L_{km} \phi \right\|^2 \leq B \sum_{\phi, k, m} |a_{\phi, k, m}|^2.
\]

Using Lemma 1.2 and (8) this becomes

\[
(14) \quad A \int_{\mathbb{T}^r} \sum_{\phi, k} |P_{\phi, k}(\sigma)|^2 d\sigma \leq \int_{\mathbb{T}^r} \left\| \sum_{\phi, k} P_{\phi, k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|^2 d\sigma \leq B \int_{\mathbb{T}^r} \sum_{\phi, k} |P_{\phi, k}(\sigma)|^2 d\sigma.
\]

Observe that (14) holds for any finite collection \( \{P_{\phi, k}\} \) of trigonometric polynomials.

(Not this that this is correct since we can take \( \{a_{\phi, k, m}\} \) in the definition of \( P_{\phi, k} \) on \( \mathbb{T}^r \) and plug the sequence back in (13) and then get (14).)

Moreover, we can strengthen (14) as follows. Let \( F \) be any finite subset of \( \mathcal{A} \times \Gamma_1 \) and for each \( (\phi, k) \in F \) let \( m_{\phi, k} \) be a bounded and measurable function on \( \mathbb{T}^r \). Then by Lusin’s Theorem and density of trigonometric polynomials, for each \( (\phi, k) \in F \), we have a sequence of trigonometric polynomials \( (P_{\phi, k}^i)_{i \in \mathbb{N}} \) such that

\[
\|P_{\phi, k}^i\|_{\infty} \leq \|m_{\phi, k}\|_{\infty}, \quad \text{for every } i \in \mathbb{N},
\]
and $P_{\phi,k}^i \to m_{\phi,k}$ a.e.. Note that (14) holds for $P_{\phi,k}^i$ for all $i \in \mathbb{N}$. Hence by the Lebesgue Dominated Convergence Theorem we have that (15)

$$A \int_{\mathbb{T}^r} \sum_{\phi,k} |m_{\phi,k}(\sigma)|^2 d\sigma \leq \int_{\mathbb{T}^r} \left\| \sum_{\phi,k} m_{\phi,k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|_2^2 d\sigma \leq B \int_{\mathbb{T}^r} \sum_{\phi,k} |m_{\phi,k}(\sigma)|^2 d\sigma$$

holds.

Now we must show that for a.e. $\sigma \in \mathbb{T}^r$, the following holds for any $b = \{b_{\phi,k}\} \in l^2(\mathcal{A} \times \Gamma_1)$.

$$A \sum_{\phi,k} |b_{\phi,k}|^2 \leq \left\| \sum_{\phi,k} b_{\phi,k} \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|_2^2 \leq B \sum_{\phi,k} |b_{\phi,k}|^2.$$

Now let $D$ be a countable dense subset of $l^2(\mathcal{A} \times \Gamma_1)$ consisting of sequences of finite support. It is enough to show that for each $d \in D$, there is a conull measurable subset $E$ of $\mathbb{T}^r$ such that $\mathbb{T}^r \setminus E$ is a c.e. subset of $\mathbb{T}^r$. Suppose that this is false. Then there is a measurable subset $G$ of $\mathbb{T}^r$ with $|G| > 0$ and $\epsilon > 0$ such that at least one of the following holds:

(16) $$\left\| \sum_{\phi,k} b_{\phi,k} \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|_2^2 > (B + \epsilon) \sum_{\phi,k} |b_{\phi,k}|^2$$

(17) $$\left\| \sum_{\phi,k} b_{\phi,k} \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|_2^2 < (A - \epsilon) \sum_{\phi,k} |b_{\phi,k}|^2.$$

Suppose that (16) is the case, and consider the functions $m_{\phi,k} = b_{\phi,k} 1_G$. Then

$$\int_{\mathbb{T}^r} \left\| \sum_{\phi,k} m_{\phi,k}(\sigma) \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|_2^2 d\sigma = \int_G \left\| \sum_{\phi,k} b_{\phi,k} \tilde{\pi}_\sigma(k) T(\phi)(\sigma) \right\|_2^2 d\sigma$$

$$\geq (B + \epsilon)|G| \sum_{\phi,k} |b_{\phi,k}|^2$$

$$= (B + \epsilon) \int_{\mathbb{T}^r} \sum_{\phi,k} |m_{\phi,k}(\sigma)|^2 d\sigma$$

which contradicts (15). If (17) holds, we get a contradiction with using an analogous argument.

Since a Parseval frame which is also a Riesz basis must be orthonormal, we have the following.
Corollary 4.5. Let $\phi \in L^2(N)$ where $N$ is $SI/Z$ group. The system $\{L_{\gamma}\phi : \gamma = km \in \Gamma\}$ is orthonormal iff for a.e. $\sigma$ the system $\{\widetilde{\pi}_\sigma(k)T\phi(\sigma) : k \in \Gamma_1\}$ is orthogonal and $\|T\phi(\sigma)\|_L = 1$

5. Examples and Applications

Example 1. The Heisenberg group. Let $N$ denote the 3 dimensional Heisenberg group. We choose a basis $\{X_1, X_2, X_3\}$ for its Lie algebra $\mathfrak{n}$ where $[X_3, X_2] = X_1$ and other Lie brackets are zero. Thus the center of $N$ is $Z = \exp \mathbb{R}X_1$ and $X = \exp \mathbb{R}X_2 \exp \mathbb{R}X_3$. With the coordinates $(x, y, z) = \exp(yX_2)\exp(xX_3)\exp(zX_1)$, we have $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$.

As is well-known, $N$ is an $SI/Z$ group and when $g^*$ is identified with $\mathbb{R}$ as above, then $\Sigma = \mathbb{R}^*$. and $|\mathbf{Pf}(\lambda)| = |\lambda|, \lambda \in \mathbb{R}^*$. It is also well-known that the Schrödinger representations act by a translation followed by a modulation. Specifically, for each $\lambda \in \mathbb{R}^*$, take the maximal subordinate subalgebra $p(\lambda) = \mathbb{R}$-span$\{X_1, X_2\}$; then the corresponding irreducible induced representation acts via natural isomorphisms on functions $f$ in $L^2(\mathbb{R})$ by

$$\pi_\lambda(x, y, z)f(t) = e^{2\pi i\lambda z}e^{-2\pi i\lambda y t}f(t-x), \ (x, y, z) \in N.$$ 

Thus $\pi_\lambda(x, y, 0) = M_{\lambda y}L_x$, where $L_x f(t) = f(t-x)$ and $M_y f(t) = e^{-2\pi i\lambda t}f(t)$.

Thus for $\phi \in L^1(N) \cap L^2(N)$, we have

$$\mathcal{F}(L(x, y, 0)\phi)(\lambda) = M_{\lambda y}L_x\hat{\phi}(\lambda), \ \lambda \in \mathbb{R}^*.$$ 

With above notations, we have $T(\phi)(\sigma)_j = |\sigma + j|^{1/2}\hat{\phi}(\sigma + j)$ for all $\phi \in L^1(N) \cap L^2(N)$, $\sigma \in \mathbb{T}$, and $j \in \mathbb{Z}$. For any $a \in L^2(\mathbb{T}, \mathcal{L})$ and $\lambda$, let $\tilde{\pi}_\lambda(k, l, m) a(\lambda) := (\pi_{\lambda + j}(k, l, m) o a(\lambda))_j$.

Therefore

\begin{equation}
T(L(k, l, m)\phi)(\lambda) = \tilde{\pi}_\lambda(k, l, m)T(\phi)(\lambda).
\end{equation}

Take $\Gamma_1 = \exp \mathbb{Z}X_2 \exp \mathbb{Z}X_3$, so that, when $N$ is identified with $\mathbb{R}^2 \times \mathbb{R}$ by the coordinates above, $\Gamma$ is identified with the integer lattice. It is shown in [10] that if $H$ is a left-invariant subspace of $L^2(N)$ for which there is a Parseval frame of the form $L_{\gamma}\psi, \gamma \in \Gamma$, then for all $\phi \in L^1(N) \cap L^2(N)$, $\mathcal{F}\phi$ is supported in $[-1, 1]$. Thus in this case $T\phi(\sigma)_j = 0$ for all $j \leq -2$, and $j \geq 1$. See [7] for an explicit example of a left-invariant subspace $H$ and an orthonormal basis for $H$ of the form $L_{\gamma}\psi, \gamma \in \Gamma$.

Example 2. A six-dimensional two step case. Let $N$ be an $SI/Z$ with the following matrix realization.

$$N = \begin{bmatrix}
1 & 0 & x_2 & x_1 & -y_2 & -y_1 & 2z_2 - x_1y_1 - x_2y_2 \\
0 & 1 & x_1 & x_2 & -y_1 & -y_2 & 2z_1 - x_1y_2 - x_2y_1 \\
0 & 0 & 1 & 0 & 0 & 0 & y_2 \\
0 & 0 & 0 & 1 & 0 & 0 & y_1 \\
0 & 0 & 0 & 0 & 1 & 0 & x_2 \\
0 & 0 & 0 & 0 & 0 & 1 & x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{pmatrix}
z_2 \\
z_1 \\
y_2 \\
y_1 \\
x_2 \\
x_1
\end{pmatrix} \in \mathbb{R}^6,$$
with Lie algebra \( \mathfrak{n} \) spanned by the vectors \( Z_1, Z_2, Y_1, Y_2, X_1, X_2 \) with the following non-trivial Lie brackets,

\[
[X_1, Y_1] = [X_2, Y_2] = Z_1
\]

\[
[X_1, Y_2] = [X_2, Y_1] = Z_2.
\]

Here, for \( f \in L^2(N) \cap L^1(N) \) we have the group Fourier transform defined as follows.

\[
\hat{f}(\lambda) = \int_N f(n) \pi_\lambda(n) \, dn \quad \text{for all } \lambda \in \Sigma,
\]

where \( \Sigma \) is identified with an open dense subset of \( \mathbb{R}^2 \), and the Plancherel measure is given by

\[
d\mu(\lambda) = d\mu(\lambda_1, \lambda_2) = |\lambda_1^2 - \lambda_2^2| \, d\lambda_1d\lambda_2.
\]

For each \( \lambda \in \Sigma \), \( \pi_\lambda \) is a corresponding irreducible representation acting in \( L^2(\mathbb{R}^2) \) such that,

\[
(19) \quad \pi_\lambda (z_1, z_2, y_1, y_2, x_1, x_2) f(t_1, t_2) = e^{2\pi i (z_1\lambda_1 + z_2\lambda_2)} e^{-2\pi i (t_1 M_{12} + t_2 M_{21})} f(t_1 - x_1, t_2 - x_2),
\]

where

\[
t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad M_{\lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}.
\]

Let

\[
\Gamma = \exp(ZZ_1 + ZZ_2) \exp(ZY_1 + ZY_2) \exp(ZX_1 + ZX_2), \quad \text{and}
\]

\[
\Gamma_1 = \exp(ZY_1 + ZY_2) \exp(ZX_1 + ZX_2).
\]

A range function here is a mapping from \( \mathbb{T}^2 \) into \( l^2(\mathbb{Z}^2, L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R}^2)) \).

Given a \( \Gamma \)-shift-invariant subspace of \( L^2(N) \). Then \( \{L_\gamma \phi : \gamma \in \Gamma \} \) is a frame with frame constant \( A, B \) for the closure of its \( \mathbb{C} \)-span if and only if \( \{\pi_\sigma(\gamma)T\phi : \gamma \in \Gamma \} \) forms a frame with the same constants for a closed subspace \( \{T(L_k \phi)(\sigma) : k \in \Gamma_1 \} \) of \( \mathcal{L} = l^2(\mathbb{Z}^2, L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R}^2)) \) for a.e \( \sigma \in \mathbb{T} \).

**Application.** We will give an example of a \( \Gamma \)-shift-invariant space of \( L^2(N) \) which is not left-invariant. We start by defining a specific range function

\[
J : \mathbb{T}^2 \to \{ \text{closed subspaces of } l^2(\mathbb{Z}^2, L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R}^2)) \},
\]

such that for a.e \( \sigma \in \mathbb{T}^2 \), \( J(\sigma) = l^2(I, \mathcal{H}_\sigma) \) where \( I = \{(n, n) : n \in \mathbb{Z}, |n| \leq 4 \} \) and,

\[
\mathcal{H}_\sigma = \overline{\mathbb{C} \text{-span } \left\{ \pi_\sigma(k) \circ (f \otimes g) : f = 1_{[0,1/2]^2}, g = 1_{[0,1)^2}, \text{and } k \in \Gamma_1 \right\} }.
\]

We consider \( M_J \) as defined earlier. It follows that \( S = T^{-1}(M_J) \) is \( \Gamma \)-shift-invariant but not left-invariant simply because the span of \( \{\pi_\sigma(k) f : k \in \Gamma_1 \} \) is a proper subspace of \( L^2(\mathbb{R}^2) \). In fact, given \( \phi \in S = T^{-1}(M_J) \), and referring to the action of the representation defined in \( (19) \),

\[
L_{\exp(\frac{\phi}{2} + \frac{\phi}{2})} \phi = L_{(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)} \phi \notin S.
\]
Example 4. A 3-step case. Let us consider a nilpotent Lie group \( N \) with its Lie algebra \( \mathfrak{n} \) spanned by the vectors \( \{ Z, Y, X_1, X_2, X_3 \} \) so that we have the following non-trivial Lie brackets: \([X_1, Y] = Y, \; [X_1, Z] = [X_2, Y] = Z\). Observe that the center of \( \mathfrak{n} \) is the 1-dimensional vector space spanned by \( Z \). It can be shown that \( N \) is square integrable modulo the center and that its unitary dual is a subset of the dual of \( \mathbb{R}Z \) which can be identified with \( \mathbb{R}^* \). The Plancherel measure on \( \mathbb{R}^* \) is up to multiplication by a constant equal to \( \lambda^2 d\lambda \). For each \( \lambda \in \mathbb{R}^* \), the corresponding unitary representation \( \pi_\lambda \) acts in \( L^2 (\mathbb{R}^2) \) in the following ways.

\[
\pi_\lambda (\exp tZ) F (t_1, t_2) = e^{2\pi i t \lambda} F (t_1, t_2)
\]
\[
\pi_\lambda (\exp (y_2 Y + y_1 Y_1)) F (t_1, t_2) = e^{\pi i t (t_2 y_1 - 2t_1 y_2)} e^{-2\pi i t \lambda y_1} F (t_1, t_2)
\]
\[
\pi_\lambda (\exp (x_2 X_2 + x_1 X_1)) F (t_1, t_2) = F (t_1 - x_1, t_2 - x_2)
\]

We use the following exponential coordinates \( (x_1, x_2, y_1, y_2, z) = \exp (x_1 X_1) \exp (x_2 X_2) \exp (y_1 Y_1) \exp (y_2 Y_2) \exp (z Z) \). We define

\[
T : L^2 (N) \rightarrow L^2 (\mathbb{T}, \mathcal{L})
\]

with \( \mathcal{L} = l^2 (\mathbb{Z}, \mathcal{HS} (L^2 (\mathbb{R}^2))) \) such that for almost every \( \sigma \in \mathbb{T} \) we have \( T \phi (\sigma) \) which is a sequence of Hilbert Schmidt operators in \( \mathcal{HS} (L^2 (\mathbb{R}^2)) \). More precisely, \( T \phi (\sigma, j) = \sigma + j |(\mathcal{F} \phi)(\sigma + j)|, \; j \in \mathbb{Z}, \) and

\[
T (L_{(x_1, x_2, y_1, y_2, z)} \phi) (\sigma) = \tilde{\pi}_{\sigma + j} (x_1, x_2, y_1, y_2, z) T \phi (\sigma + j)
\]

with \( T \phi (\sigma + j) \in \mathcal{HS} (L^2 (\mathbb{R}^2)) \). We define the following discrete subsets of \( N \).

\[
\Gamma = \{(k_1, k_2, k_3, k_4, m) \in N : k_i \in \mathbb{Z}, m \in \mathbb{Z}\}, \; \text{and}
\]
\[
\Gamma_1 = \{(k_1, k_2, k_3, k_4, 0) \in N : k_i \in \mathbb{Z}\}.
\]

Also, we define the range function \( J \) such that for almost every \( \sigma \in \mathbb{T} \), \( J (\sigma) \) is a closed subspace of \( L^2 (\mathbb{T}, l^2 (\mathbb{Z}, \mathcal{HS} (L^2 (\mathbb{R}^2)))) \). The following must hold.

1. \( S \) is a \( \Gamma \)-shift-invariant subspace of \( L^2 (N) \) if and only if for almost every \( \sigma \in \mathbb{T} \), \( J (\sigma) \) is invariant under the action of \( \tilde{\pi}_\sigma (k_1, k_2, k_3, k_4, 0) \) for all \( (k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 \). Furthermore

\[
T (S) = \{ a \in L^2 (\mathbb{T}, l^2 (\mathbb{Z}, \mathcal{HS} (L^2 (\mathbb{R}^2)))) : a (\sigma) \in J (\sigma) \}.
\]

2. Let \( \mathcal{A} \) be a countable set in \( L^2 (N) \). Define \( S \) such that

\[
S = \overline{C \text{-span} \{ L_\gamma \phi : \gamma \in \Gamma, \phi \in \mathcal{A} \}}.
\]

The system \( \{ L_\gamma \phi : \gamma \in \Gamma, \phi \in \mathcal{A} \} \) constitutes a frame (or a Riesz basis) in \( S \) with constants \( 0 < A \leq B < \infty \) if and only if for almost every \( \sigma \in \mathbb{T} \), the system

\[
\{ \tilde{\pi}_\sigma (k_1, k_2, k_3, k_4, 0) T \phi (\sigma) : (k_1, k_2, k_3, k_4) \in \mathbb{Z}^4, \phi \in \mathcal{A} \}
\]

constitutes a frame (or a Riesz basis) in \( J (\sigma) \) with the same frame constants.
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