A noncommutative-geometric interpretation of the resolution of equivariant instanton moduli spaces

C. I. Lazaroiu$^{1,a}$

$^1$Department of Physics
Columbia University
N.Y., N.Y. 10027

ABSTRACT

We generalize the recently proposed noncommutative ADHM construction to the case of $\Gamma$-equivariant instantons over $\mathbb{R}^4$, with $\Gamma$ a Kleinian group. We show that a certain form of the inhomogeneous ADHM equations describes instantons over a noncommutative deformation of the Kleinian orbifold $\mathbb{C}^2/\Gamma$ and we discuss the relation of this with Nakajima’s description of instantons over ALE spaces. In particular, we obtain a noncommutative interpretation of the minimal resolution of Kleinian singularities.

$^a$ lazaroiu@phys.columbia.edu
Introduction

In a recent paper [1], N. Nekrasov and A. Schwarz proposed a noncommutative version of the ADHM construction of instantons over $\mathbb{R}^4$. This can be obtained by deforming $\mathbb{R}^4$ to a noncommutative space and repeating the steps of the usual ADHM construction in the new, noncommutative set-up.

In the present paper we describe a generalization of these considerations to the equivariant case, i.e. to the case of instantons in $V$-bundles over an orbifold $\mathbb{R}^4/\Gamma$ with $\Gamma$ a discrete group. Since we are interested in making contact with the work of [7, 8, 9, 10, 11, 13] on the resolution of equivariant instanton moduli spaces, our discussion will be limited to Kleinian groups (i.e. to finite subgroups of $SU(2)$).

Our description is essentially a $\Gamma$-equivariant version of the construction of [1]. While the commutative limit of [1] is the usual ADHM construction over $\mathbb{R}^4$, the commutative limit in our case will be an equivariant version of the ADHM construction, which was studied (in the case $\Gamma = \mathbb{Z}_n$) in [4, 5, 6] and is implicit in the work of [7, 8, 9, 10, 11, 12, 13].

Our results provide a re-interpretation of the quiver varieties of [9] as moduli spaces of equivariant noncommutative instantons. Moreover, they give a different version of the resolution mechanism, which is natural from the point of view of matrix theory. In fact, the motivation of the present work lies in an effort of understanding certain aspects of the resolution of Kleinian singularities by D-branes, a subject which was investigated in [2, 3].

The plan of this paper is as follows. In section 1, we review equivariant instantons and the classical equivariant ADHM construction. For lack of a suitable reference, we give a rather detailed account of the steps involved. In section 2, we develop its noncommutative version. To display the equivariance properties in a way that is easy to manipulate, we follow an approach which is more geometric than that adopted in [1]. Besides avoiding complicated matrix arguments, this has the advantage of illuminating the simple geometric origin of the construction, while clearly displaying the mathematical structures and technical assumptions involved. In section 3, we discuss the relation between our results and the work of [7, 8, 9]. Finally, section 4 presents our conclusions and suggestions for further research.

1 The classical equivariant ADHM construction

The well-known ADHM construction of instantons over $\mathbb{R}^4$ has a natural equivariant version [1, 8, 8]. The latter describes equivariant instantons over $\mathbb{R}^4$, or (equivalently) over $S^4$.

For our purpose it is convenient to formulate this as follows. We start with a two dimensional complex hermitian vector space $Q$ and with a finite subgroup $\Gamma$ of $SU(Q)$. We let $Q_\mathbb{R}$ be the underlying real vector space of $Q$. The natural action of $\Gamma$ on $Q$ will be denoted by $\rho_Q$ (the fundamental representation of $\Gamma$). Choosing a real basis
of \( Q \) identifies it with \( \mathbb{R}^4 \), while the one-point compactification \( \overline{Q} = Q \cup \{\infty\} \) is identified with \( S^4 \). On the other hand, choosing a complex orthonormal basis of \( Q \) identifies \( Q \) with \( \mathbb{C}^2 \) and \( SU(Q) \) with \( SU(2) \). The action of \( \Gamma \) on \( Q \) extends to the Alexandrov compactification \( \overline{Q} \approx S^4 \). The resulting action on \( \overline{Q} \) has two fixed points, corresponding to \( 0 \in Q \) and to the point \( \infty \).

Our instantons can be thought of as connections in a \( \mathbb{V} \)-bundle (for background on \( \mathbb{V} \)-bundles, their topological invariants and equivariant connections, the reader is referred to [16]) over \( \overline{Q} \). The extension to \( \overline{Q} \) is essential for the topological classification of \( \mathbb{V} \)-bundles, but the analytic aspects are clearer if one works on \( Q/\Gamma \), and this paper we follow the latter approach. The only topological result we need is that \( SU(r) \)-\( \mathbb{V} \)-bundles \( E \) over \( S^4 \) are classified by their second Chern number \( c_2(E)[S^4] \) and the isotropy representations \( \rho_0, \rho_\infty \). The latter are defined as the actions of \( \Gamma \) on the fibers \( E_0, E_\infty \) over the fixed points.

1.1 V-bundles and equivariant connections over \( Q/\Gamma \)

In this section we briefly recall some basic facts about hermitian \( \mathbb{V} \)-bundles and their equivariant connections, as they apply to our situation.

Generalities

A hermitian \( \mathbb{V} \)-bundle on \( Q/\Gamma \) is a pair \((E, \phi)\), where \( E \) is a hermitian vector bundle over \( Q \) (of rank \( r \)) and \( \phi \) is a family of unitary isomorphisms:

\[
\phi_t(\gamma) : E_t \to E_{\rho_\gamma(Q)t}
\]

\((\gamma \in \Gamma, t \in Q)\) satisfying the compatibility conditions:

\[
\phi_t(\gamma_1 \gamma_2) = \phi_{\rho_\gamma(Q)\gamma_2}t(\gamma_1)\phi_{\gamma}(\gamma_2)
\]

A hermitian \( \mathbb{V} \)-bundle isomorphism is a unitary bundle isomorphism compatible with this structure.

The fundamental representation of \( \Gamma \) induces an action \( \nu \) by \(*\)-automorphisms of the algebra \( \mathcal{F}(Q) \) of smooth functions defined over \( Q \). \( \nu(\gamma)f := f^{\gamma} \) is given by:

\[
f^{\gamma}(t) = f(\rho_\gamma(\gamma^{-1})t)
\]

(1)

for all \( t \in Q \). On \( \mathcal{F}(Q) \) we consider the natural prehilbert module structure given by the hermitian form:

\[
< f, g > = \overline{fg},
\]

which is \( \nu \)-equivariant in the sense:

\[
< f^{\gamma}, g^{\gamma} > = < f, g >^{\gamma}.
\]
Let $E$ be the space of smooth sections of $E$. Then the $V$-structure of $E$ induces a representation $\nu_E$ by $\mathbb{C}$-linear automorphisms of $E$ via $\nu_E(s) := s^\gamma$, with:

$$s^\gamma(t) := \phi_{\rho_Q(\gamma^{-1})t}(\gamma)s(\rho_Q(\gamma^{-1})t) .$$

$E$ is an $\mathcal{F}(Q)$-bimodule, but $\nu_E$ is not an action by module automorphisms; rather, it is compatible with the action $\nu$ on $\mathcal{F}(Q)$:

$$(fs)^\gamma = f^\gamma s^\gamma .$$

The hermitian product of $E$ induces a hermitian metric $\langle , \rangle$ on $E$, which makes it into a hermitian $\mathcal{F}(Q)$-module (‘prehilbert module’). $\nu_E$ is also compatible with this structure (we say that $\nu_E$ is ‘quasiunitary’):

$$\langle s_1^\gamma, s_2^\gamma \rangle = \langle s_1, s_2 \rangle^\gamma . \quad (2)$$

We also consider the linear action of $\Gamma$ on the space $X(Q)$ of vector fields over $Q$:

$$X^\gamma := \rho_Q(\gamma)_* X$$

for all $X \in X(M)$. This action is again compatible with $\nu$:

$$(fX)^\gamma = f^\gamma X^\gamma .$$

If we pick a basis $\left(\epsilon_\alpha\right)_{\alpha=1.4}$ of $Q_\mathbb{R}$, then the vector fields $\partial_\alpha := \frac{\partial}{\partial \epsilon_\alpha}$ form a basis of the $\mathcal{F}(Q)$-module $X(Q)$. It is easy to see that:

$$\partial_\alpha^\gamma = \rho_Q(\gamma)_{\beta\alpha} \partial_\beta$$

where $\left(\rho_Q(\gamma)_{\alpha\beta}\right)_{\alpha,\beta=1.4}$ is the matrix of $\rho_Q(\gamma)$ in that basis.

For any vector $t \in Q_\mathbb{R}$, the usual directional derivative $(\partial_t f)(s) = (d_t f)(t)$ gives a vector field $\partial_t \in X(Q)$. If $t = t^\alpha \epsilon_\alpha$, then $\partial_t = t^\alpha \partial_\alpha$. The vector space $T$ of such fields is isomorphic to $Q_\mathbb{R}$; it forms an abelian Lie algebra, which can be naturally identified with the Lie algebra of $(Q_\mathbb{R}, +)$. Then $(\partial_t)^\gamma = \partial_t^\gamma$, with $t^\gamma = \rho_Q(\gamma)t$.

**Equivariant connections**

A connection $\nabla : X(Q) \times \mathcal{E} \rightarrow \mathcal{E}$ is *equivariant* if:

$$\nabla_{X^\gamma}(s^\gamma) = (\nabla_X (s))^\gamma \quad (3)$$

for all $X \in X(M), s \in \mathcal{E}$ and $\gamma \in \Gamma$. Such connections are called *invariant* in [4, 5, 6]. There one considered a more general set-up, in which equivariance cannot be realized in terms of a $V$-structure. In this sense, our situation (in which a $V$-bundle is given by
hypothesis) is only a particular one, but it is this case which is related to the work of [7].

We will always be interested in unitary connections, i.e. connections compatible with the hermitian structure of $E$:

$$<\nabla_X s_1, s_2> + <s_1, \nabla_X s_2> = X<s_1, s_2>, \forall s_1, s_2 \in \mathcal{E}, \forall X \in X(Q). \quad (4)$$

$\nabla$ is clearly determined by its restriction to $T$. Denoting $\nabla_\partial_t$ by $\nabla_t$, it therefore suffices to test (3,4) for the vector fields $\partial_t, t \in Q\mathbb{R}$:

$$\nabla_t(s_\gamma)(s_\gamma) = (\nabla_t(s))(s_\gamma) \quad (5)$$

$$<\nabla_t s_1, s_2> + <s_1, \nabla_t s_2> = \partial_t<s_1, s_2>. \quad (6)$$

Since $Q$ is retractible, any vector bundle $E$ over $Q$ is topologically trivial; in particular, the module $\mathcal{E}$ is free. Fixing a unitary trivialization $E \approx Q \times \mathbb{C}^r$, we can define the trivial connection $\nabla_0$ on $E$ by $\nabla_0 X(s) := ds(X)$. Changing the trivialization affects $\nabla_0$ by a gauge transformation, so $\nabla_0$ is essentially unique. It is immediate that $\nabla_0$ is equivariant.

A more explicit form of the equivariance condition (3) can be obtained by choosing an orthonormal frame $(s_i)_{i=1..r}$ of $E$. Such a frame defines a unitary trivialization $\psi : \mathbb{C}^r \to E$ via $\psi_t(u_i) := s_i(t) (t \in Q)$, where $\mathbb{C}^r = Q \times \mathbb{C}^r$. Here $(u_i)_{i=1..r}$ is the canonical basis of $\mathbb{C}^r$. The sections $(s_i)_{i=1..r}$ give a basis of the free $\mathcal{F}(Q)$-module $\mathcal{E}$. Since $\nu_E$ satisfies (2), $(s_i^\gamma)_{i=1..r}$ also form an orthonormal frame . Defining $\sigma_{\gamma,ij}$ by:

$$s_i^\gamma = \sum_{j=1..r} \sigma_{\gamma,ji} s_j,$$

we construct matrix-valued functions $\sigma_\gamma := (\sigma_{\gamma,ij})_{i,j=1..r}$. They satisfy:

$$\sigma_{\gamma_1 \gamma_2} = \sigma_{\gamma_2} (\sigma_{\gamma_1})^{\gamma_1}. \quad (7)$$

Clearly $\sigma_\gamma(t)$ is unitary for all $t \in Q$. The connection 1-form $A = (A_{ij})_{i,j=1..r}$ in our frame is given by:

$$\nabla s_i = s_j A_{ji}. \quad (8)$$

It is not hard to see that the equivariance conditions (3) are equivalent with:

$$\rho_Q(\gamma^{-1})^* A = \sigma^{-1}_\gamma (d + A)\sigma_\gamma \quad (9)$$

where $\rho_Q(\gamma^{-1})^*$ denotes the pull-back. This says that, under the action of $\Gamma$, $A$ transforms as a 1-form ‘up to gauge transformations’. Under a change of orthonormal frame (unitary gauge transformation) $s_i^\gamma = U_{ji}s_j$, $A$ and $\sigma_\gamma$ transform as:

$$A' = U^{-1}dU + U^{-1}AU \quad (10)$$

$$\sigma_\gamma' = U^{-1}\sigma_\gamma U^\gamma,$$
where $U := (U_{ij})_{i,j=1..r} \in U(r)$.

$\psi$ can be used to transport the $V$-structure of $E$ to a $V$-structure $\phi_e$ on $\mathbb{C}^r$ by requiring that the following diagrams commute:

\[
\begin{array}{ccc}
E_t & \xrightarrow{\phi_{E,t}(\gamma)} & E_{\rho Q}(\gamma)t \\
\psi_t \uparrow & & \uparrow \psi_{\rho Q}(\gamma)t \\
\mathbb{C}^r & \xrightarrow{\phi_{e,t}(\gamma)} & \mathbb{C}^r
\end{array}
\]

This makes $\psi$ into an isomorphism of $V$-bundles. Identifying $\mathbb{C}^r$ with the space of column vectors, $\phi_{e,t}(\gamma)$ acts by matrix multiplication:

$\phi_{e,t}(\gamma) = \hat{\phi}_{E,t}(\gamma) \cdot$

and $\hat{\phi}_{E,t}(\gamma)$ coincides with the matrix of $\phi_{E,t}(\gamma)$ taken in the bases $(s_i(t))_{i=1..r}$, $s_i(\rho Q(\gamma)t)_{i=1..r}$ of $E_t$, respectively $E_{\rho Q}(\gamma)t$. Moreover, we have:

$\sigma_\gamma(t) = \hat{\phi}_{E,\rho Q(\gamma^{-1})t}(\gamma)$ .

This displays the way in which the $V$-structure of $E$ is encoded in the gauge transformations $\sigma_\gamma$.

**Product $V$-structures**

If $S$ is a hermitian $\Gamma$-module, construct an associated bundle $\mathcal{S} := Q \times S$ with the induced hermitian structure. The action $\rho_S$ of $\Gamma$ on $S$ induces a unitary $V$-structure on $\mathcal{S}$ via:

$\phi_{S,t}(\gamma) := \rho_S(\gamma), \forall t \in Q, \forall \gamma \in \Gamma$ .

Such hermitian $V$-bundles will be called *product* $V$-bundles. They are the natural analogues of trivial bundles. In this case, the action $\nu_S$ is simply:

$\nu_S := \rho_Q \otimes_C \nu$ . (11)

In fact, the free module $\mathcal{S}$ has a distinguished presentation:

$\mathcal{S} = S \otimes_C \mathcal{F}(Q)$ .

Note that, if $L$ is the trivial $\Gamma$-module (of dimension 1), then $\mathcal{L} = \mathcal{F}(Q)$ and $\nu_L = \nu$.

It is easy to see that a hermitian $V$-bundle $E$ is isomorphic with a product hermitian $V$-bundle iff $E$ admits an orthonormal frame $(s_i)_{i=1..r}$ in which all of the matrices $\sigma_\gamma$ are constant. In this case, $\sigma_\gamma$ form a unitary representation $\sigma$ of $\Gamma$, and $E$ is isomorphic to the product $V$-bundle $Q \times \Sigma$, where $\Sigma = (\mathbb{C}^r, \sigma)$ is the corresponding hermitian $\Gamma$-module.

6
1.2 The equivariant ADHM construction

The construction proceeds as follows. One starts with two hermitian \( \Gamma \)-modules \( V, W \), of dimensions \( v, w \) and with equivariant ADHM data \( B \in \text{Hom}_\Gamma(V,Q \otimes V), I \in \text{Hom}_\Gamma(W,\Lambda^2Q \otimes V), j \in \text{Hom}_\Gamma(V,W) \).

To display the equivariance properties, the more traditional data \( B_1, B_2 \in \text{Hom}(V,V) \) have been replaced by \( B \in \text{Hom}_\Gamma(V,Q \otimes V) \). \( B_1, B_2 \) can be recovered by choosing any orthonormal basis \( (e_1,e_2) \) of \( Q \) and decomposing \( B = e_1 \otimes B_1 + e_2 \otimes B_2 \). Then the equivariance of \( B \) is expressed by:

\[
\rho_Q(\gamma)e_1 \otimes \rho_V(\gamma)B_1 \rho_V(\gamma^{-1}) + \rho_Q(\gamma)e_2 \otimes \rho_V(\gamma)B_2 \rho_V(\gamma^{-1}) = e_1 \otimes B_1 + e_2 \otimes B_2
\]

(for all \( \gamma \in \Gamma \)). Writing \( \rho_Q(\gamma) = \begin{pmatrix} \gamma_1 & \gamma_2 \\ -\gamma_2 & \gamma_1 \end{pmatrix} \in SU(2) \) for the matrix of \( \gamma \) in our basis, this reads:

\[
\rho_V(\gamma^{-1})B_1 \rho_V(\gamma) = \gamma_1 B_1 + \gamma_2 B_2
\]
\[
\rho_V(\gamma^{-1})B_2 \rho_V(\gamma) = -\gamma_2 B_1 + \gamma_1 B_2
\].

The basis \( e_1, e_2 \) gives a nonzero vector \( e_1 \wedge e_2 \) of the 1-dimensional complex vector space \( \Lambda^2Q \); this allows us to identify \( \Lambda^2Q \) with \( \mathbb{C} \). Under this identification, \( I \) becomes an element \( i \) of \( \text{Hom}(W,V) \). Note that the representation of \( \Gamma \) induced on \( \Lambda^2Q \) is trivial. Therefore, as far as equivariance properties are concerned, we can always view \( I \) as the element \( i \) of \( \text{Hom}(W,V) \). Indeed, we will often do this below. With this identification, the equivariance conditions on \( i, j \) are:

\[
\rho_W(\gamma^{-1})i \rho_V(\gamma) = i
\]
\[
\rho_V(\gamma^{-1})j \rho_W(\gamma) = j
\].

Let \( P_\Gamma(V,W) = \text{Hom}_\Gamma(V,Q \otimes V) \oplus \text{Hom}_\Gamma(W,V) \oplus \text{Hom}_\Gamma(V,W) \) be the total space of ADHM data. This space admits a natural action of \( G_V := U_\Gamma(V) \) given by:

\[
(B_1, B_2, i, j) \rightarrow (UB_1U^{-1}, UB_2U^{-1}, Ui, jU^{-1})
\].

Here \( U_\Gamma(V) \) is the set of all unitary transformations of \( V \) which commute with the representation \( \rho_V \). \( P_\Gamma \) can be endowed with a natural structure of hermitian quaternionic vector space, given by the complex structures \( I, J, K \), where \( I \) is the original complex structure of \( P_\Gamma \), \( J \) is given by:

\[
J((e_1 \otimes B_1 + e_2 \otimes B_2) \oplus i \oplus j) := (-e_1 \otimes B_2^+ + e_2 \otimes B_1) \oplus (-j^+) \oplus (i^+)
\]

and \( K = IJ \).

With this quaternionic structure, the action of \( G_V \) is unitary symplectic. The associated hyperkahler moment map is given by the usual expressions:
Using the basis \((e_1, e_2)\) of \(Q\) to identify \(Q \otimes V \approx V \oplus V\) by

\[
u = v_1 \otimes e_1 + v_2 \otimes e_2 \in Q \otimes V \rightarrow (v_1, v_2) \in V \oplus V \text{ and } \Lambda^2 Q \approx \mathbb{C}
\]

gives the familiar expressions:

\[
\sigma_t := \begin{pmatrix} B_1 - t_1 \\ B_2 - t_2 \\ j \end{pmatrix} \in \text{Hom}(V, V \oplus V \oplus W) \\
\tau_t := \begin{pmatrix} -B_2 + t_2 & B_1 - t_1 & i \end{pmatrix} \in \text{Hom}(V \oplus V \oplus W, V)
\]
for all $t = t_1 e_1 + t_2 e_2 \in Q$.

For data satisfying $\mu_c(\mathcal{B}, i, j) = 0$ and $\mu_r(\mathcal{B}, i, j) = 0$, the trivial stabilizer requirement is equivalent with the pair of conditions:

(a) If $T$ is a $\Gamma$-invariant subspace of $V$ such that $B_1(T) \subset T$, $B_2(T) \subset T$ and $T \subset \ker j$, then we have $T = 0$

(b) If $T$ is a $\Gamma$-invariant subspace of $V$ such that $B_1(T) \subset T$, $B_2(T) \subset T$ and $\text{im} i \subset T$, then we have $T = V$,

and with the pair of (apparently stronger) conditions obtained from these by dropping the requirement of $\Gamma$-invariance of $T$.

On the other hand, if $\mu_c(\mathcal{B}, i, j) = 0$, then (a) is equivalent to injectivity of $\sigma_t$ for all $t \in Q$, while (b) is equivalent to surjectivity of $\tau_t$ for all $t \in Q$. Moreover, the complex ADHM equation implies that $\tau_t \sigma_t = 0$ for all $t$.

Thus for data satisfying both (12) and (13), we have monads

$0 \rightarrow V \rightarrow Q \otimes V \oplus W \rightarrow V \rightarrow 0$

for all $t \in Q$. These give a vector bundle monad over $Q$:

$0 \rightarrow V \rightarrow Q \otimes V \oplus W \rightarrow V \rightarrow 0$, \hspace{1cm} (16)

and the equivariance properties imply that $\sigma, \tau$ are $V$-bundle morphisms.

The instanton bundle is given by the cohomology of (16):

$E := \ker \tau / \text{im} \sigma$.

To induce an instanton connection on $E$, one proceeds as follows. First, define the operator:

$D_t := (\tau_t^+, \sigma_t) \in Hom(V \oplus V, Q \otimes V \oplus W)$ \hspace{1cm} (17)

Then $E_t$ can be identified with $\ker D_t^+$, which represents $E$ as a subbundle of the bundle $U := Q \otimes V \oplus W$.

Next, the trivial connection $\nabla^0$ on $U$ induces a connection on $E$ by:

$\nabla_X (s) := P \nabla_X^0 (s), \hspace{0.5cm} \forall X \in X(M), \hspace{0.5cm} \forall s \in E$

where we let $P$ be the orthogonal projector in the hermitian bundle $U$ on the fibers of the subbundle $E$ and $P$ the associated projector on sections. This connection turns out to be anti-self-dual and of square-integrable curvature; it is the desired instanton.

The equivariance properties (15) give:

$\rho_{V \oplus V}(\gamma) \circ D_t^+ = D_{\rho_Q(\gamma)t}^+ \circ \rho_{Q \otimes V \oplus W}(\gamma)$, \hspace{1cm} (18)

which says that $D^+$ is a $V$-bundle morphism. It follows that the map $\phi_{Q \otimes V \oplus W, t} = \rho_{Q \otimes V \oplus W}(\gamma)$ induces a unitary isomorphism $\phi_{E, t}(\gamma)$ between the fibers $E_t$ and $E_{\rho_Q(\gamma)t}$. These identifications define the $V$-bundle structure of $E$. For $t = 0$ we obtain the
isotropy representation $\rho_0 = \rho_{Q \otimes V \oplus W}(\gamma)|_{E_0}$. Since $\mathcal{D}_0^+$ is an isomorphism on the orthogonal complement of $E_0$, we have $E_0 = Q \otimes V \oplus W \ominus (V \oplus V)$ as an abstract $\Gamma$-module. On the other hand, taking $t \to \infty$, shows that the isotropy representation at $\infty$ is given by $E_\infty = W$.

$\mathcal{P}$ is an orthogonal projector of the hermitian $\mathcal{F}(Q)$-module $\mathcal{U}$, whose image coincides with the space of sections $\mathcal{E}$ of $E$. Property \cite{IS} can be rewritten:

$$\mathcal{P} \circ \nu_U = \nu_U \circ \mathcal{P}. \quad (19)$$

Since the trivial connection on $U$ is equivariant, (19) implies the equivariance of $\nabla$.

An explicit expression of $\nabla$ can be obtained by picking a unitary trivialization (‘a gauge’) of $E$, i.e. a map:

$$\psi : Q \to \text{Hom}(\mathbb{C}^w, Q \otimes V \oplus W)$$

such that $\psi_t^+ \psi_t = 1_{\mathbb{C}^w}$ and $\mathcal{D}_t^+ \psi_t = 0$. By noting that $\psi_t(u_i)$ gives an orthonormal basis of $E_t$ for all $t \in Q$, one sees that the connection 1-form $A$ of $\nabla$ coincides with the matrix of the (form-valued) linear operator:

$$\mathcal{A} := \psi^+ d\psi$$

in the canonical basis $(u_i)_{i=1..w}$ of $\mathbb{C}^w$. Equivariance of $\nabla$ can be formulated in terms of the connection 1-form $A$ as in (9), and a trivial computation shows that $\sigma_\gamma(t)$ coincides with the matrix of the linear operator:

$$\overline{\sigma}_\gamma(t) := \psi_t^+ \rho_{Q \otimes V \oplus W}(\gamma) \psi_t$$

in the canonical basis of $\mathbb{C}^w$.

### 1.3 Quiver description of the equivariant ADHM data

To make contact with the quiver description of \cite{HJ}, let $R_i$ ($i = 0..r$) denote the irreps of $\Gamma$ (with $R_0$ the trivial irrep.) and $R$ be its regular representation. Let $n_i := \text{dim}_\mathbb{C} R_i$ and $n := \text{dim}_\mathbb{C} R$. We have $R = \oplus_{i=0..r} \mathbb{C}^{n_i} \otimes R_i$. Let $V = \oplus_{i=0..r} V_i \otimes R_i$, $W = \oplus_{i=0..r} W_i \otimes R_i$ be the decompositions of $V,W$ in irreps of $\Gamma$. Define a symmetric matrix $A = (a_{ij})_{i,j=0..r}$ by:

$$Q \otimes R_i = \oplus_{j=0..r} \mathbb{C}^{a_{ij}} \otimes R_j,$$

and let $C = 2I - A$. The McKay correspondence asserts that $C$ is the extended Cartan matrix of a simply-laced Lie algebra $g\Gamma$. Then $A$ is the incidence matrix of the associated extended Dynkin diagram $\Delta$ (in particular, $a_{ij} = a_{ji} \in \{0,1\}$).

Schur’s lemma gives decompositions:

$$\text{Hom}_{\Gamma}(V, Q \otimes V) = \oplus_{i,j=0, a_{ij}=1} \text{Hom}(V_i, V_j)$$

$$\text{Hom}_{\Gamma}(W, V) = \oplus_{i=0..r} \text{Hom}(W_i, V_i)$$

$$\text{Hom}_{\Gamma}(V, W) = \oplus_{i=0..r} \text{Hom}(W_i, V_i).$$
Accordingly, we have decompositions:

\[ B = \bigoplus_{i,j=0..r, a_{ij}=1} B_{ij} \]
\[ i = \bigoplus_{k=0..r} i_k \]
\[ j = \bigoplus_{k=0..r} j_k \]

Viewing \( B_{ij}, B_{ji} \) (for \( a_{ij} = 1 \)) as associated to the links of \( \Delta \) and \( i_k, j_k \) as associated to its nodes recovers the desired quiver description. Define:

\[ \vec{v} := (v_0...v_r)^t \]
\[ \vec{w} := (w_0...w_r)^t \]

where \( v_i := \dim_{\mathbb{C}} V_i \) and \( w_i := \dim_{\mathbb{C}} W_i \). Then the isotropy representations at 0, \( \infty \) are given by:

\[ E_0 = Q \otimes V \oplus W - (V \oplus V) = \sum_{i=0..r} u_i R_i \]
\[ E_\infty = W = \bigoplus_{i=1..r} w_i R_i \]

with \( u_i = w_i - \sum_{j=0..r} c_{ij} v_j \). Since \( \ker C \) is one-dimensional (being spanned by the vector \( \vec{n} = (n_0...n_r) \) associated to the regular representation), knowledge of \( \rho_0, \rho_\infty \) determines \( \vec{w} \) completely, but fixes \( \vec{v} \) only up to a multiple of \( \vec{n} \).

### 1.4 Ideal instantons

A partial compactification of \( \mathcal{M}_{0}^{reg}(V, W) \) is given by the moduli space of ideal equivariant instantons:

\[ \mathcal{M}_0(V, W) := \{ (B, i, j) \in P_1(V, W) | \mu_e(B, i, j) = \mu_e(B, i, j) = 0 \} / G_V \]

which is obtained by dropping the trivial stabilizer condition. Since the action of \( G_V \) is no longer free, the ideal instanton moduli space is, in general, singular.

As explained for example in [4], ideal instantons are pairs formed by a usual instanton (of lower second Chern number) and a set of points (which may be void). The latter are sometimes called ‘small instantons’ in the physics literature and are obtained as follows. If \((B, i, j)\) satisfies the homogeneous ADHM equations but has nontrivial stabilizer in \( G_V \), then one of the conditions (a), (b) above fails, so that there will exist a (finite) set of points \( t_1..t_s \in Q \) such that \( D_{t_i} \) is not surjective (i.e. either \( \sigma_{t_i} \) is not injective, either \( \tau_{t_i} \) is not surjective). At such points, the dimension of \( E_t \) jumps, so that the bundle \( E \) is replaced by a sheaf. If one attempts to apply the ADHM construction nonetheless, one will obtain a connection having singularities at these points. Intuitively, a part of the instanton has collapsed to zero size. In general, the dimension of \( E_t \) may jump by more than one, and in this case the point \( t \) is taken with multiplicity equal to this jump. Such points correspond to ‘coalescing
small instantons’. Since they are directly related to ADHM data with a nontrivial stabilizer, small instantons are responsible for the singularities of $\mathcal{M}_0(V,W)$.

In our case, the equivariance properties of $\sigma, \tau$ show that:

$$\rho_V(\gamma)\ker_t \sigma = \ker_{\rho_Q(\gamma)t} \sigma$$
$$\rho_V(\gamma)\text{im}_t \tau = \text{im}_{\rho_Q(\gamma)t} \tau$$

so that the sets $S_\sigma, S_\tau$ of points where $\sigma, \tau$ fail to have maximal rank are $\Gamma$-invariant (including their multiplicities). Hence the set $S_\sigma \cup S_\tau$ of ‘small instantons’ (with multiplicity) also has this property. Clearly the situation is different according to whether $t = 0$ (the fixed point of the action of $\Gamma$ on $Q$) or $t$ belongs to the set $Q - \{0\}$, on which $\Gamma$ acts freely. In the latter case, we have $|\Gamma|$ different ‘images’ and one can deduce a constraint on $V$.

Therefore, if $V$ does not contain $R$, then small instantons can only occur at the fixed point (in this case, $\ker_0 \sigma$ and $\ker_0 \tau$ must be invariant subspaces of $V$, which constrains the allowed multiplicities). Hence for such choices of $V$, the singularities of $\mathcal{M}_0(V,W)$ are a direct consequence of the presence of the fixed point.

## 2 Noncommutative version

### 2.1 Intuitive considerations

Fix $\zeta < 0$. In this section, we wish to repeat the ADHM construction by starting from equivariant data $(\mathcal{B}, i, j) \in P_1(V,W)$ which obey the following inhomogeneous form of the ADHM equations:

$$\mu_r(B_1, B_2, i, j) = -\frac{i}{2} \zeta \otimes \text{Id}_V$$
$$\mu_c(B_1, B_2, i, j) = 0 .$$

We do not impose any extra-conditions on $(\mathcal{B}, i, j)$.

In the classical construction, anti-self-duality of the ADHM connection is a consequence of the relations $\tau \sigma = 0, \sigma^+ \sigma = \tau \tau^+$, which assure that $\mathcal{D}_t^+ \mathcal{D}_t = 1_2 \otimes \Delta$, with $\Delta$ a positive operator. These relations are induced by the homogeneous equations (12), (13). The beautiful observation of [1] is that these key properties can also be satisfied for data obeying (21) if one promotes $t_1, t_2$ to non-commuting variables $z_1, z_2$ such that:

$$[z_1, z_1^+] + [z_2, z_2^+] = \zeta; \ [z_1, z_2] = 0 .$$

Then the steps of the usual ADHM construction can be repeated in this noncommutative set-up.
The conditions (22) are solved by taking \([z_i, z_j^+] = 2\eta \delta_{ij}\), with \(\eta = \frac{\zeta}{4} < 0\). Since the fundamental action of \(\Gamma\) preserves these relations, it follows that \(\rho_Q\) extends to a morphism of the \(*\)-algebra generated by \(z_i\), allowing us to define a noncommutative deformation of the orbifold \(Q/\Gamma\) and to carry over the classical equivariant arguments to the noncommutative case.

We now proceed to give a more rigorous form to these intuitive ideas. For this, we must give a clear description of the noncommutative deformation of the base space and develop the noncommutative analogues of the building blocks which entered in the equivariant form of the classical construction.

### 2.2 The noncommutative version of the base space

The desired noncommutative deformation of \(Q\) can be achieved in two different but equivalent ways. The first method (which we will follow here) is inspired by the ideas of deformation quantization while the second approach proceeds via Weyl systems and is explained in Appendix 1.

**Approach via deformation quantization**

By virtue of the Gelfand-Naimark theorem, the commutative space \(Q_\mathbb{R}\) can be described by its (non-unital) \(C^*\)-algebra \(C_0(Q_\mathbb{R}) := A_0\) of continuous functions vanishing at infinity. For our purpose it is more convenient to start with the algebra \(C_0(Q_\mathbb{R})\) of bounded uniformly continuous functions on \(Q_\mathbb{R}\), which has a unit.

To define a noncommutative deformation controlled by the parameter \(\eta\), view \(Q_\mathbb{R}\) as a symplectic space with the symplectic form \(\omega := \text{Im} \langle,\rangle\). Consider the natural (strongly-continuous) action \(\alpha\) of \((Q_\mathbb{R}, +)\) by \(*\)-automorphisms of \(C_0(Q_\mathbb{R})\) given by translations \(\alpha_u : f \mapsto f_u\), where:

\[
f_u(t) := f(t + u)
\]

for all \(t \in Q_\mathbb{R}\). The set of smooth (i.e. \(C^\infty\)) vectors for this action is the subalgebra \(B_0^\infty\) of infinitely differentiable functions over \(Q_\mathbb{R}\), whose partial derivatives of any order are bounded on \(Q_\mathbb{R}\). The deformation quantization of \(B_0^\infty\) along the symplectic form \(\eta \omega\) can be achieved [18] by replacing the original product of \(B_0^\infty\) by the Moyal product, which is formally given by:

\[
(f \times_\eta g)(t) := [e^{-\frac{i}{2\eta} \pi_{\mu\nu} \partial_\mu \partial_\nu} f(u)g(v)]_{u = v = t}
\]

(24)

(here \(\pi^{\mu\nu}\) is the inverse of the matrix \(\omega_{\mu\nu}\)). This gives a new involutive algebra structure on the set \(B_0^\infty\), which we denote by \(B_\eta^\infty\). Note that the sets \(B_0^\infty\) and \(B_\eta^\infty\) coincide, and the involution of \(B_\eta^\infty\) is still given by the usual complex conjugation. As explained in [18], one can introduce a norm \(||\cdot||_\eta\) on \(B_\eta^\infty\) compatible with the deformed product \(\times_\eta\). Taking the completion of \(B_\eta^\infty\) with respect to this norm gives a (unital) \(C^*\)-algebra \(B_\eta^\infty\). It turns out [18] that \(\alpha\) extends to an action by \(*\)-automorphisms of the
new algebra structure \((B_\eta^\infty, \times_\eta, +, ||||\eta)\), and the subalgebra of smooth vectors for this action coincides with \(B_\eta^\infty\).

For our purpose, the completion \(B_\eta^\infty\) will be of little import, since we will concentrate mainly on the geometric aspects of the problem. For this, it will suffice to consider the restricted formalism of gauge connections developed in [19, 20], which involves only the subalgebra \(B_\eta^\infty\). To summarize, then, \(B_\eta^\infty\) is the set \(B_0^\infty\), with the usual addition and complex conjugation of functions, but with the new product \(\times_\eta\) given by (24). The action \(\alpha\) of \((Q_R, +)\) on \(B_\eta^\infty\) is the usual action by translations (23), which is an algebra morphism of \(B_\eta^\infty\):

\[
\alpha_u(f \times_\eta g) = \alpha_u(f) \times_\eta \alpha_u(g).
\]

Since \(\alpha\) acts by \(*\)-automorphisms of \(B_\eta^\infty\), one obtains an action \(\text{ad}_\alpha\) of \((Q_R, +)\) by derivations, compatible with the involution. Clearly \(\text{ad}_\alpha(t)(f)\) is the usual directional derivative \(\partial_t f\) and the derivation property reads:

\[
\partial_t(f \times_\eta g) = (\partial_t f) \times_\eta g + f \times_\eta (\partial_t g).
\]

For later use, define the differentiation operator \(d : B_\eta^\infty \rightarrow (Q_R)^* \otimes B_\eta^\infty\) by:

\[
(df)(t) := \partial_t f.
\]

Here \((Q_R)^* := \text{Hom}_R(Q_R, R)\).

The action \([1]\) of \(F(Q_R)\) restricts to an action by \(*\)-automorphisms of \(B_0^\infty\). It turns out that \(\nu(\gamma)\) are also \(*\)-automorphisms of the deformed algebra structure \(B_\eta^\infty\) for any \(\eta > 0\), so that we have:

\[
(f \times_\eta g)^\gamma = f^\gamma \times_\eta g^\gamma
\]

(this is shown in Appendix 1). The pair \((B_\eta^\infty, \nu)\) is the natural noncommutative generalization of the orbifold \(Q/\Gamma\) and will henceforth be called a ‘noncommutative orbifold’.

For part of the following, we will further restrict to the \(*\)-subalgebra \(A_\eta\) of \(B_\eta^\infty\) whose underlying set is the Schwarz space \(S(Q_R)\). As discussed in [18], \(A_\eta\) is not only a subalgebra, but also a bilateral ideal of \(B_\eta^\infty\). The reason for considering \(A_\eta\) is that we wish the operators:

\[
\begin{align*}
        z_i \times_\eta &= (z_i + \eta \partial_{z_i}) \\
\overline{z}_i \times_\eta &= (\overline{z}_i - \eta \partial_{z_i})
\end{align*}
\]

(25)

to preserve our modules. Indeed, these operators preserve \(A_\eta\), but not \(B_\eta^\infty\). Note that \(A_\eta\) does not have a unit. As we explain below, if one restricting to \(A_\eta\) modules only leads to some unpleasant features of the formalism. Since \(B_\eta^\infty\) and \(A_\eta^\infty\) contain only smooth vectors for the action (23), the formalism of [19] is applicable to both of them. However, nonunitality of \(A_\eta\) obstructs a standard formulation of connection matrices and matrix unitary gauge transformations, due to lack of a good analogue of orthonormal bases of the associated hermitian modules. We believe that a better way
of treating this problem is to work with the unital algebra $\mathcal{B}_\eta^\infty$ from the outset, in which case a more natural formalism of connections can be recovered; this is the approach we will propose in this paper. In Appendix 1 we sketch how this can be technically achieved, provided that a certain ‘regularity’ condition holds. Note, however, that one can always construct a noncommutative connection operator even if one uses $\mathcal{A}_\eta$-modules only (provided that the assumptions needed for the argument of [1] hold). The regularity condition of Appendix 2 is needed only if one wishes to recover a standard matrix formalism.

2.3 Noncommutative hermitian V-bundles

Generalities

The considerations of section 1 suggest the following definition: A ‘noncommutative hermitian V-bundle’ over $(\mathcal{B}_\eta^\infty, \nu)$ is a pair $(\mathcal{E}, \nu_E)$ where $\mathcal{E}$ is a finite projective right hermitian $\mathcal{B}_\eta^\infty$-module and $\nu_E$ an action of $\Gamma$ on $\mathcal{E}$ by $\mathbb{C}$-linear automorphisms, satisfying the compatibility conditions:

$$\nu_E(sf) = \nu_E(s)\nu(f)$$
$$<\nu_E(\gamma)(s_1), \nu_E(\gamma)(s_2)> = \nu(\gamma)(<s_1, s_2>).$$

We will denote $\nu_E(\gamma)(s)$ by $s^\gamma$. An isomorphism of noncommutative hermitian V-bundles is a unitary module isomorphism compatible with this action.

We will be interested in unitary connections as defined in [19], i.e. in maps:

$$\nabla : \mathcal{E} \to (\mathcal{Q}_\mathbb{R})^* \otimes \mathcal{E}$$

satisfying the ‘Leibniz’ property:

$$\nabla_t(sf) = \nabla_t(s)f + s\partial_t(f), \forall s \in \mathcal{E}, \forall f \in \mathcal{B}_\eta^\infty, \forall t \in \mathcal{Q}_\mathbb{R}$$

where $\nabla_t := (\nabla s)(t)$ as usual, and the ‘Ricci property’ (6). Such a connection will be called equivariant if it also satisfies condition (5). Clearly the map $\nabla_t$ obtained by restricting a usual equivariant connection $\nabla_X$ to the space of vector fields $T$ defined in section 1 is a noncommutative equivariant connection in the above sense. Therefore, we have a generalization of the usual notion of equivariant connections.

Given a right $\mathcal{B}_\eta^\infty$-module $\mathcal{E}$ and a connection $\nabla$, an associated $\mathcal{A}_\eta$-module with a connection is obtained by restriction of scalars. We use the same notations $\mathcal{E}$, $\nabla$ for the latter.

Free noncommutative V-bundles

If the $\mathcal{B}_\eta^\infty$-module $\mathcal{E}$ is free (of rank $r$), then we can give a more concrete description of our objects, as in the commutative case. For this, choose an orthonormal basis
(s_\nu = 1, r) \in \mathcal{E}. Quasiunitarity (3) of \nu_E allows us to define unitary \mathcal{B}_\eta^\infty - valued matrices 
\sigma_\gamma := (\sigma_{\gamma, ij})_{i, j = 1, r} \in Mat(r, \mathcal{B}_\eta^\infty) by:
\nu_E(\gamma)(s_i) = \sum_{j=1}^{r} s_j \sigma_{\gamma, ji}

These satisfy (\ref{eq:noncommutative-structure}) and encode the noncommutative \textit{V}-structure of \mathcal{E}. The connection 1-form is again defined by (\ref{eq:connection-form}); it is an element of \((Q_\mathbb{R}^r) \otimes Mat(r, \mathcal{B}_\eta^\infty)\) or, equivalently, an \mathbb{R}\text{-linear map} \ A : Q_\mathbb{R} \to Mat(r, \mathcal{B}_\eta^\infty). Its values are ‘anti-hermitian’ matrices. A simple computation shows that the equivariance condition is again equivalent to (\ref{eq:equivariance-condition}), if we define \(\rho_Q(\gamma^{-1})^* A\) by:

\((\rho_Q(\gamma^{-1})^* A)(t) := A(\rho_Q(\gamma^{-1})t)^\gamma, \forall t \in Q_\mathbb{R}\).

This reduces to the usual pull-back in the commutative case. Under a change of orthonormal basis \(s'_j := s_j U_{ji}\) of \mathcal{E}, we again have the transformations (\ref{eq:transformations}), where now \(U := (U_{ji})_{i, j = 1, r} \in U(r, \mathcal{B}_\eta^\infty).\) The curvature \(F\) of our connection is an element of \(\Lambda^2(Q_\mathbb{R}^r)^* \otimes Hom_{\mathbb{R}_\eta^\infty}(\mathcal{E}, \mathcal{E})\), i.e. a skew-symmetric map \(F : Q_\mathbb{R} \times Q_\mathbb{R} \to Hom_{\mathbb{R}_\eta^\infty}(\mathcal{E}, \mathcal{E}),\) defined by:

\(F(s, t)\sigma := (\nabla_s \nabla_t - \nabla_t \nabla_s)\sigma, \forall \sigma \in \mathcal{E},\)

The matrix of \(F(s, t)\) in the basis \(s_i\) is:

\(F(s, t) = \partial_s A(t) - \partial_t A(s) + [A(s), A(t)]\).

Choosing an orthonormal basis \((\epsilon_\alpha)_{\alpha = 1, 4}\) of \(Q_\mathbb{R}\) and defining \(F_{\alpha\beta} := F(\epsilon_\alpha, \epsilon_\beta)\) recovers the standard formula:

\(F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]_\times\)

(where the commutator \([A_\alpha, A_\beta]_\times\) is computed with the Moyal product). Finally, the frame \((s_i)_{i=1, r}\) gives a right-linear module isomorphism \(\psi : \mathbb{C}^r \otimes \mathcal{B}_\eta^\infty \to \mathcal{E},\) defined by \(\psi(u_i) := s_i,\) where \((u_i)_{i=1, r}\) is the canonical basis of \(\mathbb{C}^r,\) viewed as a basis of \(\mathbb{C}^r \otimes \mathcal{B}_\eta^\infty\) via \(u_i \equiv u_i \otimes 1_{\mathcal{B}_\eta^\infty}.\) Then it is easy to repeat the considerations of section 1 , regarding \(\psi,\) in our noncommutative context.

The existence of a unit of \(\mathcal{B}_\eta^\infty\) is crucial for being able to define orthonormal bases \((s_i)_{i=1, r}\) of \(\mathcal{B}_\eta^\infty\)-modules. Indeed, the orthonormality condition \(< s_i, s_j > = \delta_{ij} 1_{\mathcal{B}_\eta^\infty}\) presupposes a unit in the base algebra. Therefore, if one considers only \(\mathcal{A}_\eta\text{-modules}\) \(\mathcal{E},\) one is not able to define such bases. In this case, it is only possible to define the connection form in a more general basis of \(\mathcal{E} ;\) as a result, its values will not be hermitian matrices. Moreover, unitary automorphisms of \(\mathcal{E}\) will no longer be represented by ‘unitary’ matrices. Therefore, in this situation one is not, in general, able to recover a nice noncommutative analogue of the usual matrix formalism.

**Noncommutative product \textit{V}-bundles**
The bundles appearing in (17) are product V-bundles. This allows for a straightforward definition of their noncommutative deformation. In the commutative case, the space of sections $S$ of $\mathcal{S}$ has a distinguished presentation as the space of smooth maps from $Q$ into the vector space $S$, that is, we have a module isomorphism $S = S \otimes_C \mathcal{B}_0^\infty$ (from now on, in the commutative case, we restrict scalars from $F(Q) \otimes_C \mathcal{B}_0^\infty$). The noncommutative deformation of this is simply: $S_\eta^\infty := S \otimes_C \mathcal{B}_\eta^\infty$, which has a natural bimodule structure over $\mathcal{B}_\eta^\infty$. As a $\mathbb{C}$-vector space, $S_\eta^\infty$ coincides with $S$, but the multiplications $sf, fs$ for $f \in \mathcal{B}_\eta^\infty$ and $s \in S$ are now computed on components by replacing the usual point-wise product with the Moyal product.

In the following, we will denote such submodules $s$ imply by $\sigma, \tau$, we view to reinterpret the ADHM construction in the language of noncommutative geometry, $D_{2.4}$

Finally, given a $\mathcal{B}_\eta^\infty$-module $S_\eta = S \otimes \mathcal{B}_\eta^\infty$, we can consider the $\mathcal{B}_\eta^\infty$-submodule $S_\eta := S \otimes \mathcal{A}_\eta = S_\eta^\infty \otimes_{\mathcal{B}_\eta^\infty} \mathcal{A}_\eta$, where the bilateral ideal $\mathcal{A}_\eta$ of $\mathcal{B}_\eta^\infty$ is viewed as a $\mathcal{B}_\eta^\infty$-bimodule. In the following, we will denote such submodules simply by $S \otimes \mathcal{A}_\eta$, but it is understood that they are always to be viewed as $\mathcal{B}_\eta^\infty$-modules. For any two sections $s_1, s_2$ of $S_\eta$, we have $<s_1, s_2> \in \mathcal{A}_\eta$. In particular, no $\mathcal{B}_\eta^\infty$-submodule of such a module admits orthonormal bases. Note that the modules $S \otimes \mathcal{A}_\eta$ are not free over $\mathcal{B}_\eta^\infty$.

### 2.4 $\mathcal{D}, \mathcal{D}^+$ as operators on sections

To reinterpret the ADHM construction in the language of noncommutative geometry, we view $\sigma, \tau, \mathcal{D}, \mathcal{D}^+$ as operators on sections. To achieve this, we define a ‘tautological’ section $z$ of $\mathcal{Q}$, whose value at any point $t \in Q$ is given by $t$. Then the section form of our operators is obtained by replacing $t \otimes, t \wedge$ in the definition of $\sigma, \tau$ with the point-wise
tensor/wedge products of sections $z \otimes_0, z \wedge_0$:

$$\sigma := (B - z \otimes_0) \oplus j$$
$$\tau := [B \wedge_Q - id_V \otimes (z \wedge_0)] + i . \quad (26)$$

We obtain $B_0^\infty$-right linear operators:

$$\sigma : V \otimes A_0 \to (Q \otimes V \oplus W) \otimes A_0$$
$$\tau : (Q \otimes V \oplus W) \otimes A_0 \to V \otimes A_0$$
$$D : (V \oplus V) \otimes A_0 \to (Q \otimes V \oplus W) \otimes A_0$$
$$D^+ : (Q \otimes V \oplus W) \otimes A_0 \to (V \oplus V) \otimes A_0$$

Note that $z \otimes_0, z \wedge_0$ preserve $B_0^\infty$-modules of the form $S \otimes A_0$, but not modules of the form $S \otimes B_0^\infty$. This is the reason for restricting to submodules of the form $S \otimes A_0$.

If $\nu_Q$ is the action of $\Gamma$ on the space $Q = Q \otimes A_0$ of sections of $Q$, then $z$ obeys the trivial equivariance law $\nu_Q(\gamma)(z) = z$. This implies:

$$(z \otimes_0) \circ \nu_V(\gamma) = \nu_Q \otimes V(\gamma) \circ (z \otimes_0)$$
$$(z \wedge_0) \circ \nu_Q \otimes V(\gamma) = \nu_V(\gamma) \circ (z \wedge_0) \quad (27)$$

which gives the following reformulation of the equivariance properties:

$$\sigma \circ \nu_V(\gamma) = \nu_Q \otimes V \oplus W(\gamma) \circ \sigma$$
$$\tau \circ \nu_Q \otimes V \oplus W(\gamma) = \nu_V(\gamma) \circ \tau$$
$$D \circ \nu_Q \otimes V \oplus W(\gamma) = \nu_Q \otimes V \oplus W(\gamma) \circ D$$
$$D^+ \circ \nu_Q \otimes V \oplus W(\gamma) = \nu_V(\gamma) \circ D^+ . \quad (28)$$

### 2.4.1 The noncommutative equivariant ADHM construction

Let $\eta := \zeta/2$. Then we define $B_0^\infty$-right linear operators:

$$\sigma : V \otimes A_\eta \to (Q \otimes V \oplus W) \otimes A_\eta$$
$$\tau : (Q \otimes V \oplus W) \otimes A_\eta \to V \otimes A_\eta$$
$$D : (V \oplus V) \otimes A_\eta \to (Q \otimes V \oplus W) \otimes A_\eta$$
$$D^+ : (Q \otimes V \oplus W) \otimes A_\eta \to (V \oplus V) \otimes A_\eta$$

by replacing $z \otimes, z \wedge$ in (26) with $z \otimes_\eta, z \wedge_\eta$:

$$\sigma := (B - z \otimes_\eta) \oplus j$$
$$\tau := [B \wedge_Q - id_V \otimes (z \wedge_\eta)] + i .$$

Here $z \otimes_\eta, z \wedge_\eta$ are defined by using the usual tensor product on the vector space parts and the Moyal product on the scalar components. These operators are right $B_0^\infty$-linear.
Note that, since $z$ does not belong to $Q \otimes B^\infty_\eta$, this is not quite the module tensor product consider above (despite the similar notation). However, it clearly has the same equivariance properties.

As remarked in [1], the commutation relations obeyed by $z_i, \overline{z}_i$ assure that the validity of the key properties $\tau\sigma = 0, \sigma^+\sigma = \tau\tau^+$. Thus we have:

$$D^+D = 1_2 \otimes \Delta = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$

with $\Delta : V \otimes A_\eta \rightarrow V \otimes A_\eta$.

We define a right $B^\infty_\eta$-module by $E_\eta := \ker D^+$; this is a submodule of $U_\eta := (Q \otimes V \oplus W) \otimes A_\eta$. In the commutative limit $\eta = 0$, $E_\eta$ reduces to the module of sections $E$ of the bundle $E$ constructed before. Note that $E_\eta$ does not, in general, coincide with $E$ as a set. $E_\eta$ carries an $A_\eta$-valued hermitian product induced from $U_\eta$.

The equivariance properties of $z, D, D^+$ are again given by (27), (with $\otimes_0$ replaced by $\otimes_\eta$) and (28). It follows that $E_\eta$ is $\Gamma$-invariant. In particular, $E_\eta$ carries a natural action $\nu_{E_\eta}$ of $\Gamma$ by $C$-linear quasiunitary automorphisms, given by the restriction of $\nu_U$. Therefore, $(E_\eta, \nu_{E_\eta})$ is a hermitian noncommutative $V$-bundle over $(A_\eta, \nu)$.

By assuming that $\Delta$ is invertible on $V \otimes A_\eta$, it follows that the operator:

$$P_\eta := 1 - D(1_2 \otimes \Delta^{-1})D^+$$

(30)

gives an orthogonal projector of the hermitian module $U_\eta$ onto $E_\eta$. Then the null connection $\nabla^0 = d$ of $U_\eta$ induces a connection in $E_\eta$:

$$\nabla_t^0(s) := P_\eta \nabla^0_t(s), \forall t \in Q_\mathbb{R}, \forall s \in E_\eta.$$  

It is easy to see that the usual argument [7] for anti-self-duality of the induced connection carries over unaffected to the noncommutative case. Therefore, the connection $\nabla_t^\ast$ has anti-self-dual curvature. This is our noncommutative instanton. Indeed, the curvature $F$ of $\nabla$ is given by the ‘Gauss equation’:

$$< F(v, w)s_1, s_2 > = < \Pi_v s_1, \Pi_w s_2 > - < \Pi_v s_1, \Pi_w s_2 > ,$$

where $\Pi = (1 - P_\eta)d$ and $\Pi_v = (1 - P_\eta)\partial_v$. Since $D^+s = 0$ for all $s \in E_\eta$, we have $D^+d = (dD^+)(d\xi)^+\pi_{Q\otimes V}$ (on $E_\eta$), where $\pi_{Q\otimes V} : (Q \otimes V \oplus W) \otimes A_\eta \rightarrow (Q \otimes V) \otimes A_\eta$ is the orthogonal projection and :

$$\xi := \begin{pmatrix} -\overline{z}_2 & z_1 \\ \overline{z}_1 & z_2 \end{pmatrix}$$

with $z_i : Q \rightarrow \mathbb{C}$ the coordinate functions in an orthonormal basis of $Q$. This gives

$$\Pi |_{c_\eta} = D(1_2 \otimes \Delta^{-1})(d\xi)^+\pi_{Q\otimes V} |_{c_\eta},$$

which implies:

$$F(v, w) = P_\eta [\pi_{Q\otimes V} d\xi(v)(1_2 \otimes \Delta)d\xi(w)^+\pi_{Q\otimes V} - (v < -- > w)]$$
i.e.
\[
F = \mathcal{P}_\eta \pi_{Q \otimes W} \left( \begin{array}{cc}
\frac{1}{\Delta} d \bar{z}_2 dz_2 + d z_1 \frac{1}{\Delta} d \bar{z}_1 & -d \bar{z}_2 \frac{1}{\Delta} d z_1 + d z_1 \frac{1}{\Delta} d \bar{z}_2 \\
-d z_1 \frac{1}{\Delta} d \bar{z}_2 + d \bar{z}_2 \frac{1}{\Delta} d z_1 & d \bar{z}_1 \frac{1}{\Delta} d z_1 + d z_2 \frac{1}{\Delta} d \bar{z}_2
\end{array} \right) \pi_{Q \otimes W}.
\]

(31)

\[\Gamma\text{-invariance of } \mathcal{E}_\eta \text{ reads:} \]
\[\nu_U \mathcal{P}_\eta = \mathcal{P}_\eta \nu_U,\]

which (together with equivariance of \(\nabla^0\)) immediately yields equivariance of \(\nabla\).

At this point, we have constructed a noncommutative instanton connection in the \(\mathcal{B}_\eta^\infty\)-module \(\mathcal{E}_\eta\). Since this module cannot admit orthonormal bases, we do not yet have a nice analogue of the usual matrix formalism. Assuming as in [1] that \(\mathcal{E}_\eta\) is free as an \(\mathcal{A}_\eta\)-module, one can consider orthogonal bases of the \(\mathcal{A}_\eta\)-module obtained from \(\mathcal{E}_\eta\) by restriction of scalars and define connection matrices in such bases. Thus we can always define matrices \(A, \sigma\) at this level, one runs into various problems of normalization which make the formalism rather cumbersome. Instead of attempting to develop a formalism along these lines, we take a different point of view, arguing for the existence of a nonextension of the rather cumbersome. Instead of attempting to develop a formalism along these lines, we take a different point of view, arguing for the existence of an extension of the 

Extension to \(\mathcal{E}_\eta^\infty\)

Although the operators \(D, D^+\) do not preserve the \(\mathcal{B}_\eta^\infty\)-modules \((V \oplus V) \otimes \mathcal{B}_\eta^\infty\) and \((Q \otimes V \oplus W) \otimes \mathcal{B}_\eta^\infty\), they are always defined on these modules if we allow them to take values in a more general space (essentially a space of distributions). Thus we can always define \(\mathcal{E}_\eta^\infty\) to be the kernel of \(D^+\) on \((Q \otimes V \oplus W) \otimes \mathcal{B}_\eta^\infty\), which is a right \(\mathcal{B}_\eta^\infty\)-module. In particular, relation (24) still holds for these generalized operators. If we make the (nontrivial) assumption that the operator \(D(1_2 \otimes \Delta^{-1})D^+\) extends to an operator which preserves the space \(\mathcal{U}_\eta^\infty := (Q \otimes V \oplus W) \otimes \mathcal{B}_\eta^\infty\), then (24) gives an orthogonal projector \(\mathcal{P}_\eta^\infty\) of \(\mathcal{U}_\eta^\infty\), which extends the projector \(\mathcal{P}_\eta\) and maps \(\mathcal{U}_\eta^\infty\) onto \(\mathcal{E}_\eta^\infty\). Using \(\mathcal{P}_\eta^\infty\) instead of \(\mathcal{P}_\eta\) and starting with the null connection on \(\mathcal{U}_\eta^\infty\), we have an induced connection on \(\mathcal{E}_\eta^\infty\) which extends the connection \(\nabla\). The self-duality argument proceeds unaffected and the curvature is still given by (24).

By further assuming that there exists a unitary isomorphism \(\psi : \mathbb{C}^w \otimes \mathcal{B}_\eta^\infty \rightarrow \mathcal{E}_\eta^\infty\) (which amounts to saying that \(\mathcal{E}_\eta^\infty\) is free as a \(\mathcal{B}_\eta^\infty\)-module), we can construct \(\mathcal{B}_\eta^\infty\)-valued matrices \(A, \sigma\) as before. Again a trivial computation shows that:

\[A_{ji} = \langle u_j, (\psi^+ \circ d \circ \psi)(u_i) \rangle\]
\[\sigma_{\gamma, ji} = \langle u_j, (\psi^+ \circ \nu_U(\gamma) \circ \psi)(u_i) \rangle.\]
where \( u_i \equiv u_i \otimes 1_{B_\infty} \). The construction of \( \psi \) is further discussed in Appendix 1.

To give at least a rough justification for the extendibility of \( P_\eta \), note that, in the commutative case (and for ADHM data of trivial stabilizer), the projector \( P \) of section 2 preserves the module \( U_0^\infty \). Indeed, \( P_t \in \text{Hom}(U,U) \) is a smooth function of \( t \in Q \) and has a limit at infinity given by the orthogonal projector \( P_\infty \) on the fiber \( E_\infty = W \). Since \( P_\eta^\infty \) should ‘continuously’ reduce (in a suitable sense) to \( P \) as \( \eta \) tends to zero, this indicates that the desired condition may be satisfied. Further discussion of how the extension of \( P_\eta \) could be achieved is given in Appendix 1.

3 Discussion of the moduli space

Our noncommutative instanton moduli space:

\[
\mathcal{M}_{(\zeta,0)}(V,W) := \{ (B,i,j) \in P_T(V,W)|\mu_r(B,i,j) = -\frac{i}{2}\zeta Id_V; \mu_c(B,i,j) = 0 \} / G_V
\]  

(32)

is a particular example of a quiver variety. Such varieties were studied in detail in [9], to which we refer the reader for background. For the convenience of the non-expert reader, the main results of relevance for us are summarized in Appendix 2.

As expected, the limit \( \zeta = 0 \) gives the moduli space \( \mathcal{M}_0(V,W) \) of \( \Gamma \)-equivariant ideal instantons on \( Q \), which was discussed in section 1. In the case \( \zeta < 0 \), the results of [8] imply that \( \mathcal{M}_{(\zeta,0)}(V,W) \) is always smooth and that it provides a resolution of singularities of \( \mathcal{M}_0(V,W) \). The differential and complex structures of \( \mathcal{M}_{(\zeta,0)}(V,W) \) are independent of the parameter \( \zeta \), which controls only its Kahler structure. Our construction gives a noncommutative-geometric interpretation of this resolution: as in [1], deforming the orbifold to its noncommutative version resolves the singularities of the ideal instanton moduli space. However, as we discuss below, the geometry of the orbifold situation is richer. The reason is that the noncommutative deformation effectively eliminates the orbifold point, thus implicitly achieving the resolution of the orbifold.

As reviewed in Appendix 2, a general quiver variety is obtained by considering arbitrary central levels \( \bar{\zeta} \) of both the real and the complex moment maps in (32). For generic values of \( \bar{\zeta} \), such a variety is a smooth hyperkahler manifold. For fixed \( V \) and \( W \), the differential structure of a generic (in the sense defined in Appendix 2) quiver variety is independent of the value of \( \bar{\zeta} \), which controls only its hyperkahler structure. Since diagonal levels as in (32) are always generic, it follows that \( \mathcal{M}_{(\bar{\zeta},0)}(V,W) \) gives a differentiable model for all generic quiver varieties of the same type \( (V,W) \). Therefore, working with our particular deformation of the ADHM equations suffices to capture all of the differential-geometric aspects of the resolution. The situation is more complicated if one takes into account the complex structure, or the full hyperkahler structure of \( \mathcal{M}_{(\bar{\zeta},0)}(V,W) \), as we discuss below.
3.1 Instanton interpretation

In certain situations, the *hyperkahler* manifold $\mathcal{M}(\zeta,0)(V,W)$ admits an interpretation as a moduli space of instantons over a smooth ALE space. Such moduli spaces were constructed in \[7\]. They are also given by quiver varieties, but not any quiver variety admits such an interpretation. The obstruction to this is a ‘tracelessness’ condition, discussed in detail in Appendix 2, which can be satisfied by our moment map level only if the $\Gamma$-module $V$ is such that there exists an $i \in \{0, \ldots, r\}$ for which $V_i = 0$.

If this condition is satisfied, one can in general find a smooth ALE space $X(\xi,0)$ such that $\mathcal{M}(\zeta,0)(V,W)$ coincides (as a *hyperkahler* manifold) with the moduli space of instantons over $X(\xi,0)$ (the parameter $\xi$ is related to $\zeta$ as explained in Appendix 2). The underlying complex manifold of $X(\xi,0)$ is the minimal resolution of the Kleinian orbifold $Q/\Gamma$, while $\xi$ controls its Kahler structure. Therefore, in this case, the resolution mechanism admits a purely classical geometric description, namely as a blow-up of the orbifold $Q/\Gamma$ to $X(\xi,0)$, which takes equivariant ideal instantons over $Q/\Gamma$ into instantons over the resolved space.

The reason why the blow-up of $Q/\Gamma$ suffices (in this case) to resolve the equivariant ideal instanton moduli space can be understood as follows. When $V_i = 0$ for some $i$, the $\Gamma$-module $V$ does not contain the regular representation. Therefore, as we discussed in section 1, point-like equivariant instantons can only occur at the orbifold point. The resolution of $Q/\Gamma$ removes this point, and, since $V$ does not contain $R$, Proposition 9.2. of \[7\] assures us that no point-like instantons can form on the resolved space $X(\xi,0)$.

A particularly pleasant outcome of this situation is that we can obtain the ALE space $X(\xi,0)$ itself via our noncommutative construction. Indeed, it was shown in \[7, 9\] that $X(\xi,0) = \mathcal{M}(\zeta,0)(R \oplus R_0, Q)$, i.e. $X(\xi,0)$ coincides with a particular moduli space of SU(2) instantons over itself. By our construction, $X(\xi,0)$ also coincides with the moduli space of noncommutative equivariant instantons over the deformation of the orbifold $Q/\Gamma$. The limit $\xi \to 0$ ($\zeta \to 0$) gives the moduli space of equivariant ideal instantons over $Q/\Gamma$, of $c_2(E)[S^4] = |\Gamma| - 1$ and isotropy representations $E_0 = 2R_0$ (trivial 2-dimensional representation) and $E_\infty = Q$, which indeed coincides \[9\] with $X_0 = Q/\Gamma$.

3.2 Sheaf-theoretic interpretation

The underlying complex manifold of $\mathcal{M}(\zeta,0)(V,W)$ also admits an interpretation as a moduli space of equivariant torsion-free sheaves (framed over the line at infinity) over $\mathbb{C}\mathbb{P}^2$. We give a sketch of the relevant arguments in Appendix 2.

3.3 Methodological considerations

The instanton interpretation of the quiver varieties is obtained \[7, 9\] essentially by replacing the bundles $\mathcal{V}$, $Q \otimes \mathcal{V} \oplus W$ appearing in \[10\] with bundles $(V \otimes R)^\Gamma$, $(Q \otimes V \otimes R)^\Gamma \oplus (W \otimes R)^\Gamma$, where $R$ is a rank $|\Gamma|$ ‘tautological’ bundle over $X_\xi$ related to

22
the description of $X_\xi$ as a hyperkahler quotient. Then one lifts the operator $z \otimes$ to a section $\lambda$ of $Q \otimes \text{End}(\mathcal{R})$. This lifts the monad (16) to the ALE space, allowing one to perform an ADHM construction over $X_\xi$.

Our procedure above was entirely different in spirit, insamuch as we did not resolve the orbifold $Q/\Gamma$ ‘by hand’; rather, we deformed its algebra of smooth functions, which effectively resolved it via the uncertainty principle: since individual points cease to have a well-defined meaning, the orbifold singularity is lost in the noncommutative ‘fuzz’. The remarkable fact is that this apparently purely algebraic procedure gives a moduli space which has a classical geometric interpretation: for $V_0 = 0$, at least, instantons over the noncommutative deformation of the orbifold correspond to instantons over the resolution of the orbifold.

In [1] it was shown that Nakajima’s resolution [15] of the ideal instanton moduli space over $\mathbb{R}^4$ admits a noncommutative interpretation. The singularities of that space are due to instantons collapsing to zero size, and the heuristic reason for their smoothing out in the noncommutative set-up is that such a collapse is not possible once the fuzziness of the base space is taken into account. The moduli space of equivariant ideal instantons has a qualitatively new singularity due to instantons collapsing to zero size at the orbifold point. The noncommutative deformation of the orbifold effectively eliminates this point, thereby smoothing out this type of singularity as well. Therefore, the resolution mechanism is heuristically identical to that of [1].

4 Conclusions and further directions

By studying the noncommutative version of the equivariant ADHM construction, we showed that the quiver varieties of [9] admit an interpretation as moduli spaces of instantons over a noncommutative deformation of the Kleinian orbifold. In particular, the ALE spaces of [24] can be described as a moduli space of $|\Gamma|-1$ $SU(2)$ noncommutative equivariant instantons. These spaces also admit a description as moduli spaces of equivariant torsion-free sheaves over $\mathbb{C}^2$, which lends further credence to the conjectural correspondence [1] between noncommutative instantons and torsion-free sheaves.

There are two main lessons that we can learn from this, one of a physical and one of a mathematical nature. The physical lesson is that it is possible to describe moduli spaces of objects over the resolution of a singular space as moduli spaces of similar objects over a noncommutative deformation of that singular space. We believe that this approach can play an important role in clarifying the nature of space-time resolution processes in matrix theory. The mathematical lesson is that rather nontrivial manifolds can be constructed as moduli spaces of noncommutative objects, and that classical geometric processes such as the minimal resolution of orbifold singularities admit a noncommutative geometric realization.

One point which we believe to be worth further study is the precise nature of the relation between torsion-free sheaves and noncommutative instantons. Both in
and in the present paper, such a correspondence was constructed at the level of moduli spaces only, but it is important to understand it at a more fundamental level. Namely, one would hope to have a relation between torsion-free sheaves and anti-self-dual noncommutative connections which is similar to the Hitchin-Kobayashi correspondence between stable holomorphic bundles and instantons. The best approach to this problem may be to view the noncommutative deformation of the base space as a certain type of regularization. Indeed, a generalized Hitchin-Kobayashi correspondence was constructed in [32]. This associates a singular Einstein-Hermitian connection (of square-integrable curvature) to a polystable reflexive sheaf, and one may suppose that similar results could be derived for more general sheaves, if one replaces square-integrability with a weaker condition. We believe that the noncommutative deformation acts as a regulator of such connection singularities, which probably renders the curvature ‘square-integrable’ in the noncommutative sense, similar to what happens in the examples considered in [1]. Such a ‘noncommutative’ Hitchin-Kobayashi correspondence may shed considerable light on the precise meaning of the relation between D-brane and instanton moduli spaces in string theory.

Another interesting issue is the generalization of the previous results to the case of a compact base space, in particular to the case of orbifolds of noncommutative tori. This may help solve the problem of understanding the resolution of such singularities by D-branes. A direct approach to this problem leads to rather singular connections, which could be regularized by noncommutative-geometric methods.

Finally, let us note that the procedure we used consists essentially of deforming the standard moment map to a noncommutative version, thereby absorbing nontrivial values of its level into the non-commutativity of the base space. Since symplectic quotients appear in the description of moduli spaces of various types of objects (as well as in the theory of Hamiltonian reduction of mechanical systems), it would very interesting to see whether a systematic deformation procedure exists in a more general set-up. Such a theory may contribute to a better understanding of the relation between classical and ‘quantum’ geometry.

### Acknowledgements

The author wishes to thank N. Nekrasov and A. Schwarz for correspondence, I. Krichever for enlightening discussions and B. R. Greene for constant support and encouragement. This work was supported by a C.U. Pfister Fellowship and by the DOE grant DE-FG02-92ER40699B.

### A Weyl quantization

We start with the hermitian vector space \((Q, \langle, \rangle)\) (\(\langle, \rangle\) is anti-linear in the first variable). Let \((, ) := \text{Re} \langle, \rangle\) and \(\omega(, ) := \text{Im} \langle, \rangle\) be the associated Euclidean scalar
product and symplectic form on the underlying real vector space \( Q_{\mathbb{R}} \). Fix \( \eta \neq 0 \). Quantization of the symplectic space \((Q_{\mathbb{R}}, \eta \omega)\) is achieved by considering a (strongly continuous) Weyl system \( W_\eta : Q_{\mathbb{R}} \to \mathcal{U}(\mathcal{H}) \), with \( \mathcal{H} \) a (separable) Hilbert space, satisfying the property:

\[ W_\eta(s)W_\eta(t) = W_\eta(s + t)e^{-i\frac{\eta}{2} \omega(s,t)} \]

\((s, t \in Q)\). By von-Neumann’s theorem, any such system is unitarily equivalent with the standard one, realized on the symmetric Fock space \( \mathcal{H} \) over \((Q, <.,.>)\) by the operators we use the formalism of [30], with the trivial deformation obtained by rescaling the creation/annihilation operators given there by \( \sqrt{\eta} \):

\[ W_\eta(s) := e^{iQ_\eta(s)} \].

Here \( Q_\eta(s) := \frac{1}{\sqrt{2}}[a_\eta^+(s), a_\eta(s)] \) is the Segal operator with \( a_\eta^+(s), a_\eta(s) \) the generation and annihilation operators of \( [30] \). The latter are \( \mathbb{C} \)-linear, respectively anti-linear in \( s \) and satisfy:

\[ [a_\eta(s), a_\eta^+(t)] = |\eta| < s, t > , \quad [a_\eta(s), a_\eta(t)] = 0 , \quad [a_\eta^+(s), a_\eta^+(t)] = 0 \]

for all \( s, t \in Q \). The usual creation and annihilation operators \( a_\eta^+, a_\eta^\eta (i = 1, 2) \) are obtained by choosing an orthonormal basis \( E := (e_i)_{i=1,2} \) of \((Q, <.,.>)\) and considering the associated real basis \( (\epsilon_i := e_i, \epsilon_{i+2} := i\epsilon_i)_{i=1,2} \) of \( Q_{\mathbb{R}} \). The latter is an orthonormal basis of \((Q_{\mathbb{R}}, (,))\) and a canonical (i.e. Liouville) basis of \((Q_{\mathbb{R}}, \omega)\). Then:

\[ a_\eta^\eta := a_\eta(e_i) , \quad a_\eta^\eta^+ := a_\eta^+(e_i) \]

while the usual momentum and position operators are:

\[ Q_i^\eta := Q_\eta(e_i) , \quad P_i^\eta := Q_\eta(e_{i+2}) \].

These have the nontrivial commutators:

\[ [a_i^\eta, a_j^\eta^+] = |\eta| \delta_{ij} , \quad [Q_i^\eta, P_j^\eta] = i|\eta| \delta_{ij} \].

The subspace \( \mathcal{H}_0 \) of states with a finite particle number is a dense subspace of the symmetric Fock space \( \mathcal{H} \). In the position representation, we have \( \mathcal{H} \approx L^2(\mathbb{R}^2) \) and we define a dense subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) by \( \mathcal{H}_0 \approx S(\mathbb{R}^2) \). All operators we consider are densely defined on subspaces of \( \mathcal{H} \) containing \( \mathcal{H}_0 \) and preserve \( \mathcal{H}_0 \).

For any Schwarz function \( f \in S(Q_{\mathbb{R}}) \), define its Fourier transform by:

\[ \hat{f}(s) := \frac{1}{(2\pi)^2} \int_{Q_{\mathbb{R}}} dt \ f(t)e^{i(s,t)} \]

where \( dt \) is the natural Lesbegue measure on \((Q_{\mathbb{R}}, (,))\). Then Weyl quantization of the phase space distribution \( f \) is achieved by the operator:

\[ W_f^\eta := \frac{1}{(2\pi)^2} \int_{Q_{\mathbb{R}}} dt \ \hat{f}(t)W_\eta(t) \] \quad (33)
where

\[ f_c := \begin{cases} f, & \eta > 0, \\ \bar{f}, & \eta < 0 \end{cases} \]

This has the conjugation property:

\[ W^\eta f = (W^\eta f)^+ . \]

The Weyl correspondence \[ (33) \] can be extended to various spaces of functions and distributions on \( Q_\mathbb{R} \) as explained in \[ [23, 29, 26] \].

If \( f, g \) are such that \( W^\eta f, W^\eta g \) are well-defined operators from \( \mathcal{H}_0 \) to \( \mathcal{H}_0 \), then the Weyl correspondence maps the operator product \( W^\eta f W^\eta g \) into the Moyal product \( f \times_\eta g \) via the relation:

\[ W^\eta f \times_\eta g := W^\eta f W^\eta g \]

In particular, this product makes \( \mathcal{S}(Q_\mathbb{R}) \) into an involutive algebra, which we denoted above by \( \mathcal{A}_\eta \). As a set, \( \mathcal{A}_\eta \) is independent of \( \eta \). The \( \eta \)-dependence enters only through the product \( \times_\eta \). For \( f, g \in \mathcal{S}(Q_\mathbb{R}) \), the Moyal product is given by:

\[ f \times_\eta g = [e^{-\frac{i}{2} \eta \pi(d_u, d_v)} f(u)g(v)]_{u=v=s} \]

where \( \pi \) is the symplectic form induced by \( \omega \) on the dual space \( (Q_\mathbb{R})^* = \text{Hom}_\mathbb{R}(Q_\mathbb{R}, \mathbb{R}) \) and \( d_u, d_v \) denote the total differentials \( d_u(f) := d_u f \in \text{Hom}_\mathbb{R}(Q_\mathbb{R}, \mathbb{R}) \). This relation is valid for any sign of \( \eta \). Note that taking \( \eta \) into \( -\eta \) amounts to replacing \( f \times_\eta g \) by \( f \times_{-\eta} g = g \times_\eta f \), i.e. ‘transposing’ the algebra structure on \( \mathcal{A}_\eta \).

Choosing a basis \( \mathcal{E} \) as before, one defines the real and complex coordinate functions:

\[ x_i(s) := s^i, \quad x_{i+2}(s) := p_i(s) := s^{2+i}, \]

\[ z_i := x_i + ix_{i+2}, \quad \bar{z}_i := x_i - ix_{i+2} \]

where \( s = \sum_{\alpha=1,4} s^\alpha e_\alpha = z^i e_i \in Q_\mathbb{R} \). Then \( \pi_{\alpha,\beta} := \pi(e_\alpha^*, e_\beta^*) = -\omega_{\alpha,\beta} := -\omega(\epsilon_\alpha, \epsilon_\beta) \) and \( df = \sum_{i=1,2} (\frac{\partial f}{\partial s^i} + \frac{\partial f}{\partial p_i}) \), where \( (\epsilon_\alpha)^* = 1,4 \) is the basis dual to \( (\epsilon_\alpha)_{1,4} \). This gives the standard form of the Moyal product:

\[ f \times_\eta g = f \epsilon^{\frac{i}{\eta}} \sum_{i=1,2} (\partial_{s^i} \partial_{p_i} - \partial_{s^i} \partial_{x_i}) g . \]

The Weyl quantizations of \( x_i, p_i \) are the operators \( Q_i^\eta, P_i^\eta \), respectively. The Weyl quantizations of \( (z_i, \bar{z}_i) \) are (in this order) \( (\sqrt{2}a_i^\eta, \sqrt{2}a_i^\eta^+) \) for \( \eta > 0 \) and \( (\sqrt{2}a_i^\eta^+, \sqrt{2}a_i^\eta) \) for \( \eta < 0 \). These give the Moyal products \( (27) \) for all \( \eta \neq 0 \).

**Action of phase-space translations and smooth vectors**

The Weyl group has an ‘adjoint’ action by \( * \)-automorphisms of the \( \mathbb{C}^* \)-algebra \( \mathcal{B}(\mathcal{H}) \) of bounded operators in \( \mathcal{H} \):

\[ B \rightarrow W(t)BW(t)^{-1}, \forall B \in \mathcal{B}(\mathcal{H}) . \]
Let $B_c(\mathcal{H})$, $B^\infty(\mathcal{H})$ be the subalgebras of continuous, respectively smooth vectors for this action. It was shown in [21, 22] that $B_c(\mathcal{H})$ is a $C^*$- subalgebra of $B(\mathcal{H})$, $B^\infty(\mathcal{H})$ is a dense *-subalgebra of $B_c(\mathcal{H})$ and that the Weyl transform is a *-algebra isomorphism from $B^\infty_\eta$ to $B^\infty(\mathcal{H})$. The adjoint action of the Weyl group is mapped into the action (23) of $(Q, +)$ on $B^\infty_\eta$: 

$$W^\eta_{\alpha^\eta}(f) = W^\eta(i\eta|\eta|^{-1}u)W^\eta_f W^\eta(i|\eta|^{-1}u)^{-1}.$$ 

**The action of the orbifold group**

For any $\gamma \in \Gamma$, let $U_\gamma$ be the ‘second-quantized’ form of the unitary operator $\rho_Q(\gamma) \in U(Q, <, >)$ in our symmetric Fock space. $U_\gamma$ is independent of $\eta$ and satisfies:

$$a^\eta(\rho_Q(\gamma)s) = U_\gamma a_\eta(s)U^{-1}_\gamma$$

$$a^\eta(\rho_Q(\gamma)s) = U_\gamma a^\eta_\gamma(s)U^{-1}_\gamma,$$

which give:

$$W_\eta(\rho_Q(\gamma)s) = U_\gamma W_\eta(s)U^{-1}_\gamma.$$

Then it is easy to see that:

$$W^\eta_f = U_\gamma W^\eta_f U^{-1}_\gamma,$$

which implies that the transformations $\nu_\gamma : f \to f_{\gamma}$ are *-automorphisms of $B^\infty_\eta$.

**A.1 The operator version of the ADHM construction**

For what follows we take $\eta < 0$. The Weyl transform maps the algebraic ADHM construction of section 2 into an operatorial construction, which we explain below.

**The operator $z$**

The Weyl quantization rule (33) has a trivial generalization to vector-valued functions $s$ if we replace $f$ by $s$ in both sides. In this way, we can ‘quantize’ any section of a trivial vector bundle $S$. In particular, we can define the Weyl transform $\hat{z}_\eta := W^\eta_z$ of the tautological section $z$ of $Q$. In an orthonormal basis, this is given by:

$$\hat{z}_\eta = \sqrt{2}a^\eta_i + e_i = z^\eta_i e_i$$

and is a (densely-defined) closed operator from $\mathcal{H}$ to $Q \otimes \mathcal{H}$.

**Operator form of $D$, $D^+$**

The Weyl quantizations $\hat{D}_z, \hat{D}^+_z$ of $D_z, D^+_z$ (viewed as sections of the associated bundles of homomorphisms) are given by replacing $t$ with $\hat{z}_\eta$ in (17).
If $S$ is a (finite dimensional) hermitian vector space, let $\mathcal{H}_S$ be the Hilbert space $S \otimes \mathcal{H}$ and $\mathcal{H}_S^0 := S \otimes \mathcal{H}^0$ the associated dense subspace. Then $\mathcal{D}_z, \mathcal{D}_z^+$ are densely defined operators between $\mathcal{H}_{V \oplus V}$ and $\mathcal{H}_{Q \otimes V \oplus W}$, whose domains contain $\mathcal{H}_{V \oplus V}$, respectively $\mathcal{H}_{Q \otimes V \oplus W}$. They are closable on these subspaces, since $\dot{z}_i^+, \dot{z}_i^\eta$ are closable. Denote their closures by the same letters. For any operator $A$, denote its domain by $d(A)$.

The construction

Consider the operator $\mathcal{D}_z^+ \mathcal{D}_z$, defined on the set $\mathcal{M} = \{ x \in d(\mathcal{D}_z^+) | \mathcal{D}_z x \in d(\mathcal{D}_z^+) \}$ (which includes $\mathcal{H}_{Q \otimes V \oplus W}$). It is well-known (see, for example, [23]) that $\mathcal{D}_z^+ \mathcal{D}_z$ is self-adjoint on $\mathcal{M}$. By the ADHM equations, this operator has the form:

$$\hat{\mathcal{D}}_z^+ \hat{\mathcal{D}}_z = 1_2 \otimes \hat{\Delta} = \begin{pmatrix} \hat{\Delta} & 0 \\ 0 & \hat{\Delta} \end{pmatrix}$$

with $\hat{\Delta} = \hat{\sigma}_z^+ \hat{\sigma}_z = \hat{\tau}_z^+ \hat{\tau}_z$ a self-adjoint operator on $M = d(\hat{\tau}_z^+ \hat{\tau}_z^\eta) = d(\hat{\sigma}_z^+ \hat{\sigma}_z) \subset \mathcal{H}_V$ (containing $\mathcal{H}_V^0$) with values in $\mathcal{H}_V$. The set $\mathcal{M}$ equals $M \oplus M$. Assuming as in [1] that $\Delta$ is (strictly) positive, its inverse $\Delta^{-1}$ is a bijective bounded operator from $\mathcal{H}_V$ to $M$. We further assume that $\Delta^{-1}(\mathcal{H}_V^0) \subset \mathcal{H}_V^0$.

The projector $\mathcal{P}$

Let $K := \ker \hat{\mathcal{D}}_z^+$ and $\mathcal{P}$ be the orthogonal projector on $K$ in $\mathcal{H}_{Q \otimes V \oplus W}$. Then it is not hard to see that (with the assumptions above):

$$P|_{d(\mathcal{D}_z^+)} = 1 - \mathcal{D}_z (1_2 \otimes \Delta^{-1}) \mathcal{D}_z^+.$$ 

For any (finite -dimensional) hermitian vector space $S$, consider the space $\mathcal{B}_S := \mathcal{B}(\mathcal{H}, \mathcal{H}_S) = S \otimes \mathcal{B}(\mathcal{H})$ of bounded operators from $\mathcal{H}$ to $\mathcal{H}_S$. This has a natural hermitian $\mathcal{B}(\mathcal{H})$- module structure given by $< A, B > := A^\dagger B$. We also let $\mathcal{A}_S, \mathcal{B}_S^\infty$ be the subsets of operators in $\mathcal{B}_S$ whose Weyl symbols belong to $S \otimes \mathcal{S}(\mathcal{Q}_\mathbb{R})$, $S \otimes \mathcal{B}_0^\infty$, both viewed as $\mathcal{B}^\infty(\mathcal{H})$-modules. $\mathcal{A}_S$ is a $\mathcal{B}(\mathcal{H})$-submodule of $\mathcal{B}_S$. The Weyl transform (acting on both the module and the base algebra) gives a unitary isomorphism between the modules $S \otimes \mathcal{B}_\eta^\infty, S \otimes \mathcal{A}_\eta$ and $\mathcal{B}_\eta^\infty, \mathcal{A}_S$ respectively.

$\mathcal{P}$ induces an orthogonal projector $\mathcal{P}$ of the hermitian module $\mathcal{B}_{Q \otimes V \oplus W}$ via:

$$\mathcal{P}(A) := PA, \forall A \in \mathcal{B}_{Q \otimes V \oplus W}.$$ 

$\mathcal{P}$ projects onto the subspace $\mathcal{E}_b$ of all $A \in \mathcal{B}_{Q \otimes V \oplus W}$ whose range is a subset of $K$. If $A \in \mathcal{B}_{Q \otimes V \oplus W}$ preserves $\mathcal{H}_0$, then clearly $A \in \mathcal{E}_b$ iff $\mathcal{D}_z^+ A|_{\mathcal{H}_0} = 0$, which is equivalent to the fact that $\mathcal{D}^+$ annihilates the Weyl symbol of $A$.

Now further assume that $\mathcal{P}$ preserves $\mathcal{A}_{Q \otimes V \oplus W}$. In this case, let $\mathcal{E} := \mathcal{P}(\mathcal{A}_{Q \otimes V \oplus W})$. Then the Weyl transform maps $\mathcal{E}$ to the $\mathcal{B}_\eta^\infty$-module $\mathcal{E}_\eta$ and $\mathcal{P}$ to the projector $\mathcal{P}_\eta$ used in section 2.

28
The map $\psi$

Choosing an orthonormal basis of $Q$ for simplicity, we can write:

$$\hat{D}^+_z = (\hat{D}^+_z, V),$$

where $\hat{D}^+_z$ is the closed densely-defined operator from $H_{Q \otimes V}$ to $H_{V \oplus V}$ given by:

$$\hat{D}^+_z := \begin{pmatrix} -B_2 + z_2^2 & B_1 - z_1^1 \\ B_1 - z_1^1 & B_2^+ - z_2^2 + \end{pmatrix},$$

while $V$ is the bounded operator from $H_W$ to $H_{V \oplus V}$ given by:

$$V := \begin{pmatrix} i \\ j^+ \end{pmatrix}.$$

The domain of $\hat{D}^+_z$ is the direct sum of the domain of $\hat{D}^+_z$ with the entire space $W \otimes \mathcal{H}$. Assuming as in [1] that 0 does not belong to the spectrum of $\hat{D}^+_z$ we have a bounded inverse $R_0 := (\hat{D}^+_z)^{-1}$ defined on $H_{V \oplus V}$. Defining a bounded operator $S := R_0 V : H_W \rightarrow H_{Q \otimes \mathcal{V}}$, the kernel $K$ of $\hat{D}^+_z$ coincides with the graph of $-S$. Define $\Psi_0 := (-S) \oplus id_{H_W} : H_W \rightarrow H_{Q \otimes V \oplus W}$ and let $\Psi := (\Psi_0^+ \Psi_0)^{-1/2} \Psi_0$ be its unitary part. Then $\Psi$ induces a map of hermitian modules $\psi : \mathcal{B}_W \rightarrow \mathcal{B}_{Q \otimes V \oplus W}$ by

$$\psi(A) := \Psi A, \forall A \in \mathcal{B}_W.$$

Since $im \Psi = K$, this gives a unitary isomorphism between the hermitian modules $\mathcal{B}_W$ and $\mathcal{E}_b$. Assuming that $\psi(A_W) = \mathcal{E}$, the inverse Weyl transform gives the map used in section 2.

**Extension to $\mathcal{E}^{\infty}$**

It is clear from the above that we can extend the whole discussion of the ADHM construction to the modules $V \otimes \mathcal{B}^{\infty}$, $(Q \otimes V \otimes B^{\infty})$ etc. provided that $P$ satisfies a slightly different condition, namely that left multiplication by $P$ maps $\mathcal{B}_{Q \otimes V \oplus W}^{\infty}$ into itself. In this case, we can define $\mathcal{E}^{\infty} = \mathcal{P}(\mathcal{E}_b) = \{ A \in \mathcal{B}_{Q \otimes V \oplus W}^{\infty} | im A \in K \}$. Then the inverse Weyl transform of $P$ gives the projector $\mathcal{P}_n^{\infty}$ of section 3, while the inverse Weyl transform of $\mathcal{E}^{\infty}$ gives the module $\mathcal{E}_{n}^{\infty}$. Despite its apparent simplicity, this ‘regularity’ condition on $P$ is nontrivial. It could be tested, in principle, by computing the anti-Wick symbol of $P$ and using the results of [31].
B Quiver varieties, instantons over ALE spaces and the resolution of the moduli space of orbifold instantons

Definitions and basic results

One starts with equivariant ADHM data as in section 2. Let $g_V$ be the Lie algebra of $G_V$ and $Z_V$ be its center. By the decomposition $V = \oplus_{i=0..r} V_i \otimes R_i$, we have $G_V = \Pi_{i=0..r} U(V_i)$, $g_V = \Pi_{i=0..r} u(V_i)$ and $Z_V$ is the subset of skew-hermitian operators which are diagonal on each $V_i$. We identify $Z_V$ with the set:

$$Z_V := \{ \zeta = (\zeta_0...\zeta_r) \in \mathbb{R}^{r+1} | \zeta_i = 0 \text{ if } V_i = 0 \} \subset \mathbb{R}^{r+1}$$

via the map $\zeta \in Z_V \rightarrow \hat{\zeta} := \frac{1}{2} \oplus_{k=0..r} \zeta_k 1_{V_k}$. As before, we have a hyperkahler moment map $\vec{\mu} : P \rightarrow \mathbb{R}^{3} \otimes g_V$ or, equivalently, a real and a complex moment map $\mu_r = \mu_1$, $\mu_c = \mu_2 + i\mu_3$.

The resolution of the equivariant instanton moduli space can be achieved by replacing the usual ADHM equations with their inhomogeneous version. For this, let $\vec{\zeta} \in \mathbb{R}^{3} \otimes Z_V$. Define the following quiver varieties:\n
$$\mathcal{M}_{\vec{\zeta}}(V,W) := \{ (B,i,j) \in P(V,W) | \vec{\mu}(B,i,j) = -\hat{\zeta} \}/G_V$$

$$\mathcal{M}_{\vec{\zeta}}^{reg}(V,W) := \{ (B,i,j) \in P^{reg}(V,W) | \vec{\mu}(B,i,j) = -\hat{\zeta} \}/G_V \subset \mathcal{M}_{\vec{\zeta}}(V,W)$$

In general, $\mathcal{M}_{\vec{\zeta}}(V,W)$ has complicated singularities, but $\mathcal{M}_{\vec{\zeta}}^{reg}(V,W)$ is a smooth hyperkahler manifold, of dimension $d(V,W) = 4v^2 - 2\vec{v}C\vec{v}$.

To give a sufficient criterion of smoothness of $\mathcal{M}_{\vec{\zeta}}(V,W)$, one defines:

$$R_+ := \{ \theta = (\theta_0...\theta_r) \in (\mathbb{Z}_{\geq 0})^{r+1} | \theta^tC\theta \leq 2 \} - \{0\}$$

$$R_+(\vec{v}) := \{ \theta \in R_+ | \theta_k \leq v_k, \forall k = 0..r \} \subset R_+$$

and, for all $\theta \in R_+$:

$$D_\theta := \{ x = (x_0...x_r) \in \mathbb{R}^{r+1} | \sum_{k=0..r} x_k \theta_k = 0 \}.$$

We say that $\vec{\zeta} \in \mathbb{R}^{3} \otimes Z_V$ is generic if $\vec{\zeta}$ does not belong to $\cup_{\theta \in R_+(\vec{v})} \mathbb{R}^{3} \otimes D_\theta$. If $\vec{\zeta}$ is generic, then it is shown in \[4\] that:

(1) $\mathcal{M}_{\vec{\zeta}}(V,W) = \mathcal{M}_{\vec{\zeta}}^{reg}(V,W)$. Thus $\mathcal{M}_{\vec{\zeta}}(V,W)$ is smooth; moreover, its hyperkahler metric is complete.

(2) Assuming further that $\mathcal{M}_{\vec{\zeta}}^{reg}(0,\vec{\zeta})$ is nonempty, there exists a map $\mathcal{M}_{\vec{\zeta}}(V,W) \rightarrow \mathcal{M}_{(0,\vec{\zeta})}(V,W)$ which is a resolution of singularities. In particular, given $\zeta_r \in Z_V$ such that $(\zeta_r,0)$ is generic, we have a resolution $\mathcal{M}_{(\zeta_r,0)}(V,W) \rightarrow \mathcal{M}_0(V,W)$. 

30
For any $\zeta$, $\tilde{\zeta}$ which are generic, the manifolds $\mathcal{M}_\zeta(V, W)$ and $\mathcal{M}'_\zeta(V, W)$ are diffeomorphic. Hence given a generic $(\zeta, 0)$, the manifold $\mathcal{M}_{(\zeta, 0)}$ provides a differential model for all $\mathcal{M}_\zeta(V, W)$ with a generic $\tilde{\zeta}$.

**Instanton interpretation**

If the parameters $\zeta \in \mathbb{R}^3 \otimes Z_V$ have a particular form, then $\mathcal{M}^{reg}_\zeta(V, W)$ admits an interpretation as a moduli space of instantons over an ALE space, while $\mathcal{M}_\zeta(V, W)$ coincides with the associated moduli space of *ideal* instantons. To explain this, we first describe Kronheimer’s construction of ALE spaces as hyperkahler quotients.

**Kronheimer’s construction of ALE spaces**

According to [24], all 4-dimensional ALE spaces can be obtained as hyperkahler quotients. For this, consider $V = R$ (the regular representation of $\Gamma$) and $W = 0$. In this rather degenerate case, the central $U(1)$ subgroup of $G_R = U_1(R)$ acts trivially on $P^{reg}_\Gamma(V, W)$. To overcome this pathology, one replaces $G_R$ by the quotient group $G'_R := G_R/U(1)$, which acts freely. The Lie algebra $\mathfrak{g}'_R$ of $G'_R$ is the traceless part of the Lie algebra of $G_R$. Its center $Z'_R$ is formed of $r \times r$ traceless diagonal matrices, with eigenvalues occurring in blocks of dimensions $n_j$ ($j = 0..r$). This can be identified with:

$$Z'_R := \{(\xi_0..\xi_r)\mid \sum_{i=0..r} n_i \xi_i = 0\} \subset \mathbb{R}^{r+1}$$

via the map $\xi = (\xi_0..\xi_r) \mapsto \hat{\xi} := \frac{1}{r} \oplus_{k=0..r} \xi_k 1_{V_k}$. One chooses $\hat{\zeta} \in \mathbb{R}^3 \otimes Z'_R$ and defines:

$$X_{\hat{\zeta}} := \{(B, i, j) \in P_1(R, 0)|\tilde{\mu}(B, i, j) = +\hat{\zeta}\}/G'_R$$

$$X^{reg}_{\hat{\zeta}} := \{(B, i, j) \in P^{reg}_1(R, 0)|\tilde{\mu}(B, i, j) = +\hat{\zeta}\}/G'_R.$$  

The parameter $\hat{\xi}$ is called *non-degenerate* if $\hat{\xi} \in \mathbb{R}^3 \otimes Z'_R - \cup_{\theta \in R_+ - \{q\tilde{\mu}|q \in \mathbb{Z}\}} \mathbb{R}^3 \otimes D_\theta$.

If $\hat{\zeta}$ is non-degenerate, then one has $\tilde{X}_\zeta = X^{reg}_{\hat{\zeta}}$ and $X_{\tilde{\zeta}}$ is a smooth ALE hyperkahler manifold of real dimension 4, which is diffeomorphic with the minimal resolution of the Kleinian singularity $Q/\Gamma$. For $\hat{\zeta} = 0$, one has $X_0 = Q/\Gamma$.

A more useful form of the non-degeneracy condition can be obtained as follows. Consider the set $\Phi := \{\tilde{\theta} \in \mathbb{Z}^r\mid \sum_{i,j=1..r} C_{ij} \tilde{\theta}_i \tilde{\theta}_j = 2\}$, which can be identified with the set of simple roots of $\mathfrak{g}_R$. Then $(\theta_0..\theta_r) \mapsto (\theta_1 - n_1 \theta_0,..,\theta_r - n_r \theta_0)$ maps the set $R^0_+ := R_+ - \{q\tilde{\mu}|q \in \mathbb{Z}\}$ onto $\Phi$, while the map $(\xi_1..\xi_r) \mapsto (-\sum_{k=1..r} n_k \xi_k, \xi_1..\xi_r)$ is a bijection from $\mathbb{R}^r$ to $Z'_R$. Using this, is is easy to see that $\hat{\zeta} \in \mathbb{R}^3 \otimes Z'_R$ is non-degenerate iff $\sum_{i=1..r} \tilde{\theta}_i \xi_i \neq 0, \forall \tilde{\theta} \in \Phi$.

$X_{\hat{\zeta}}$ can be presented as a quiver variety as follows. Choose $V = R \oplus R_0$ and $W = Q$. Then $R_+(R \oplus R_0) \subset R^0_+$ can be identified with $\Phi$ in the obvious way and it is clear
that $\vec{\zeta} \in Z'_R$ is non-degenerate iff $\phi(\vec{\zeta}) = (\vec{\zeta}_1..\vec{\zeta}_r)$ is generic. In this case, it is easy to see that
\[
X_{\vec{\zeta}} = \mathcal{M}_{\phi(\vec{\zeta})}(R \oplus R_0, Q).
\]
Moreover, by the results we discuss below, $\mathcal{M}_{\phi(\vec{\zeta})}(R \oplus R_0, Q)$ coincides with the moduli space of instantons over $X_{\vec{\zeta}}$ (framed at infinity) of topological invariants $c_1 = 0, c_2 |X_{\vec{\zeta}}| = \frac{\dim V}{3}$ and isotropy representation at infinity given by $Q$.

**Instanton description**

Define a map $\phi : Z'_R \to Z_V$ by:
\[
\phi(\xi_0..\xi_r) := (s_0\xi_0...s_r\xi_r)
\]
where $s_i := 0$ or $1$ according to whether $\dim V_i = 0$ or $\neq 0$, and let $Z^\phi_V$ be the image of this map in $Z_V$. In general, $Z^\phi_V$ is strictly smaller than $Z_V$. The desired interpretation is possible only if $\vec{\zeta} \in \mathbb{R}^3 \otimes Z^\phi_V$. More precisely, let $\vec{\zeta} \in \mathbb{R}^3 \otimes Z'_R$ be non-degenerate and let $\vec{\zeta} := \phi(\vec{\xi})$. Then we have the following statements:

(a) $\mathcal{M}^\text{reg}_{\vec{\zeta}}(V, W)$ coincides with the moduli space of instantons (framed at infinity) over $X_{\vec{\zeta}}$ (of Chern classes and isotropy representation at infinity determined by $V, W$ as explained in [7])

(b) $\mathcal{M}_{\vec{\zeta}}(V, W)$ coincides with the associated moduli space of ideal instantons.

Note that:

(a) If $V_i \neq 0$ for at least one $i \in \{0..r\}$, then $Z^\phi_V = Z_V$, i.e. $\phi$ is surjective; in this case, $Z^\phi_V = Z_V$ and we have an instanton interpretation for all $\vec{\zeta}$ associated via $\phi$ with a non-degenerate $\vec{\xi} \in Z'_R$.

(b) If $V_i \neq 0$ for all $i = 0..r$, then $Z^\phi_V = \{\zeta = (\zeta_1..\zeta_r) \in \mathbb{R}^{r+1} | \sum_{i=0..r} n_i\zeta_i = 0\}$, while $Z_V = \mathbb{R}^{r+1}$. In particular, $\phi$ is not surjective and $Z^\phi_V$ is strictly smaller than $Z_V$.

**Diagonal levels of the real moment map**

Given a real parameter $\zeta$, consider the moment map level $\mu_r = -\frac{1}{2} \zeta_1 V_1, \mu_c = 0$, which is represented by the parameters $\zeta_r = (s_0..s_r)\zeta, \zeta_c = 0$. It is immediate that $(\zeta_r, 0)$ is always generic, so that $\mathcal{M}_{(\zeta, 0)}(V, W) := \mathcal{M}_{(\zeta_r, 0)}(V, W)$ is always smooth. Now consider the ALE instanton interpretation for this space:

(A) If $V_i \neq 0, \forall i = 0..r$, then all $s_i$ are strictly positive and $\zeta_r$ never belongs to $Z^\phi_V$. Therefore, an ALE instanton interpretation is not possible.

(B) On the other hand, if $V_i = 0$ for at least one $i$ (in which case $Z^\phi_V = Z_V$), and if $(\zeta_r, 0) = \phi(\vec{\eta}_r)$, with a non-degenerate $\vec{\eta}_r$, then the hyperkahler manifold $\mathcal{M}_{(\zeta, 0)}(V, W)$ coincides with the moduli space of instantons over the smooth ALE space $X_{\vec{\eta}_r}$.

To see whether $(\zeta_r, 0)$ can be represented by a non-degenerate $\vec{\eta}_r$, consider the most restrictive case, namely when $V_m = 0$ for some $m \in \{0..r\}$ and $V_k \neq 0, \forall k \neq m$. In
this case, the equation \( \phi(\vec{\xi}) = (\zeta_r, 0) \) has exactly one solution, namely \( \xi^c = 0 \) and 
\[
\xi^m = -\sum_{k=0}^{n_m} \zeta, \quad \xi^r = \zeta, \quad \forall i \neq m,
\]
where \( N := \sum_{k=0}^{n_m} n_k \). It is easy to see that such a \((\xi_r, 0)\) is in general non-degenerate, but we do not have a uniform proof that this holds for any group \( \Gamma \). The case \( V_0 = 0, V_i \neq 0, \forall i = 1..r \) is particularly simple, since in this situation we have 
\[
\sum_{i=1}^{r} \xi^r_i \tilde{\theta}_i = \zeta \sum_{i=1}^{r} \tilde{\theta}_i,
\]
which is nonzero for any \( \tilde{\theta} \in \Phi \).

Thus \((\xi_r, 0)\) is non-degenerate for any \( \Gamma \) and any \( V \) such that \( V_0 = 0 \), and in this case, at least, one can represent the hyperkahler manifold \( M(\zeta_r, 0) \) as a moduli space of instantons over the smooth ALE space \( X(\xi_r, 0) \).

**Sheaf-theoretic interpretation of \( M(\zeta, 0) \)\((V,W)\)**

The sheaf-theoretic interpretation of \( M(\zeta, 0) \)\((V,W)\) can be obtained essentially as follows. Consider the (‘commutative’) sequence (16) for data satisfying the homogeneous complex ADHM equation. By virtue of this equation, (16) is still a complex: \( \tau \sigma = 0 \). Replacing the bundles by their sheaves of holomorphic sections, we can pull this back to a sheaf complex over \( \mathbb{C}P^2 \) via the usual projection \( p : \mathbb{C}P^2 \rightarrow Q \). \( p \) takes the line \( l_\infty \) at infinity in \( \mathbb{C}P^2 \) into the point \( \infty \in Q \), so the resulting sheaves over \( \mathbb{C}P^2 \) are trivial over \( l_\infty \). Moreover, \( p^* \) takes any framing over \( \infty \) into a framing over \( l_\infty \). An equivariant version of the argument in [14] shows that the first map of the resulting complex is always injective (as a sheaf map), while the second map is surjective iff condition (b) is satisfied, which is in turn equivalent (using the ‘transpose’ form of Proposition 3.5. of [9]) with the requirement that the \( G^C_V \)-orbit of \((B, i, j)\) intersects the level \( \mu_r = -\frac{i}{2} \xi Id_V \) (the fact that \( \zeta < 0 \) is crucial at this point). Thus, we have a sheaf monad over \( \mathbb{C}P^2 \) iff there exists a \( g \in G^C_V \) such that
\[
\mu_r(gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}) = -\frac{i}{2} \xi Id_V.
\]
Moreover, such a \( g \) is determined up to multiplication on the left by an element of \( G_V \). The cohomology of the \( \mathbb{C}P^2 \) monad gives a sheaf, which carries an induced action of \( \Gamma \). Repeating the arguments of [33, 14] in this context, the quotient:
\[
\{(B, i, j) | \mu_c(B, i, j) = 0 \text{ and } G^C_V(B, i, j) \cap \mu_c^{-1}(-\frac{i}{2} \xi Id_V) \neq \Phi \} / (G_V)^C
\]
gives a moduli space of \( \Gamma \)-equivariant torsion free sheaves (framed over \( l_\infty \)) over \( \mathbb{C}P^2 \), which can be identified with the complex manifold \( M(\zeta, 0) \)\((V,W)\).

**References**

[1] N. Nekrasov and A. Schwarz, *Instantons on noncommutative \( \mathbb{R}^4 \), and (2, 0) superconformal six dimensional theory*, Commun.Math.Phys. 198 (1998) 689-703, [hep-th/ 9802068](http://arxiv.org/abs/hep-th/9802068).

[2] M. Douglas and G. Moore, *D-branes, Quivers, and ALE Instantons*, [hep-th/9603167](http://arxiv.org/abs/hep-th/9603167).
[3] C. V. Johnson and R. C. Myers, *Aspects of type IIB theory on ALE spaces*, Phys.Rev. D55 (1997) 6382–6393.

[4] M. Furuta and Y. Hashimoto, *Invariant instantons on $S^4$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 37 (1990), no. 3, 585–600.

[5] M. Furuta, *$Z_a$-invariant SU(2) instantons over the four sphere*, Geometry of low-dimensional manifolds, 1 (Durham, 1989), 161–174, London Math. Soc. Lecture Note Ser., 150, Cambridge Univ. Press, Cambridge, 1990.

[6] D. M. Austin, *SO(3)-instantons on $L(p,q) \times \mathbb{R}$*, J. Differential Geom. 32 (1990), no. 2, 383–413.

[7] P. B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. 288(1990)263-307

[8] H. Nakajima, *Moduli spaces of anti-self-dual connections on ALE gravitational instantons*, Invent. Math. 102(1990), 267–303

[9] ——, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. 76(1994), 365–416

[10] ——, *Homology of moduli spaces of instantons on ALE spaces I.*, J. Diff. Geom. 40(1994), 105–127

[11] ——, *Gauge theory on resolutions of simple singularities and simple Lie algebras*, Internat. Math. Res. Notices 1994, no. 2, 61–74.

[12] ——, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) 145 (1997), no. 2, 379–388, [alg-geom/9507012](http://arxiv.org/abs/alg-geom/9507012).

[13] ——, *Instantons and affine Lie algebras, S-duality and mirror symmetry*, Nuclear Phys. B Proc. Suppl. 46 (1996), 154–161, [alg-geom/9510003](http://arxiv.org/abs/alg-geom/9510003).

[14] ——, *Lectures on Hilbert schemes of points on surfaces*, Hiraku Nakajima’s homepage

[15] ——, *Resolutions of moduli spaces of ideal instantons on $\mathbb{R}^4$*, Topology, geometry and field theory, 129–136, World Sci. Publishing, River Edge, NJ, 1994.

[16] I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan 9 (1957), 464-492, T. Kawasaki, *The index of elliptic operators over V-manifolds*, Nagoya Math. J. 84 (1981), 135-157.

[17] S. K. Donaldson, P. B. Kronheimer, *The geometry of four-manifolds*, Oxford mathematical monographs, Clarendon Press, 1990.

[18] M. Rieffel, *Deformation quantisation for actions of $\mathbb{R}^d$*, Memoirs of AMS, No. 506 (1993).

[19] A. Connes and M. A. Rieffel, *Yang-Mills for noncommutative two-tori*, Contemp. Math., vol 62, 1987, 237–267.

34
[20] A. Connes, $C^*$-algebres et geometrie differentielle, C. R. Acad. Sc. Paris, \textbf{290} (1980), 559–604.

[21] H. O. Cordes, On pseudodifferential operators and smoothness of special Lie group representations, Manuscripta Math., \textbf{28} (1979), 51–69.

[22] K. R. Payne, Smooth tame Frechet algebra and Lie groups of pseudodifferential operators, Commun. Pure and Appl. Math., — (1991), 309 –337.

[23] G. B. Folland, Harmonic analysis in phase space, Annals of Mathematics Studies, Princeton U. P., Princeton, N. J., 1989.

[24] P. B. Kronheimer, The construction of ALE spaces as hyperkahler quotients, J. Diff. Geom \textbf{28} (1989), 665–683, A Torelli-type theorem for gravitational instantons, J. Diff. Geom. \textbf{29} (1989), 685–697.

[25] M. Reed and B. Simon, Methods of Modern Mathematical Physics, vol I, Academic Press, 1973.

[26] R. F. Anderson, The Weyl functional calculus, J. Func. Anal. \textbf{4} (1963), 240–267, On the Weyl functional calculus, J. Func. Anal. \textbf{6} (1970), 110–115.

[27] I. E. Segal, Transforms for operators and symplectic automorphisms over a locally compact abelian group, Math. Scand. \textbf{13} (1963), 31–43

[28] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, D. Sternheimer: Deformation theory and quantisation.I. Deformations of symplectic structure, Ann. of Phys. \textbf{111} (1978), 61–110, II. Physical applications, Ann. of Phys. \textbf{110} (1978), 111–151

[29] J. C. Varilly and J. M. Gracia-Bondia, Algebras of distributions suitable for phase-space quantum mechanics. II. Topologies on the Moyal algebra, J. Math. Phys \textbf{1990} (1989), 107–148

[30] J. M. Cook, The mathematics of second quantization, Trans. Amer. Math. Soc. \textbf{74} (1953), 222–245

[31] F. A. Berezin, The method of second quantisation, Academic Press, 1966

[32] S. Bando, T. Siu, Stable sheaves and Einstein-Hermitian metrics, Geometry and analysis on complex manifolds, 39–50, World Sci. Publishing, River Edge, NJ, 1994.

[33] C. Okonek, M. Schneider, H. Spindler, Vector bundles on complex projective spaces, Progress in mathematics, v. \textbf{3}, Birkhauser, 1988.