Skew-orthogonal Laguerre polynomials for chiral real asymmetric random matrices

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Abstract

We apply the method of skew-orthogonal polynomials (SOP) in the complex plane to asymmetric random matrices with real elements, belonging to two different classes. Explicit integral representations valid for arbitrary weight functions are derived for the SOP and for their Cauchy transforms, given as the expectation values of traces and determinants or their inverses, respectively. Our proof uses the fact that the joint probability distribution function for all combinations of real eigenvalues and complex conjugate eigenvalue pairs can be written as a product. Examples for the SOP are given in terms of Laguerre polynomials for the chiral ensemble (also called the non-Hermitian real Wishart–Laguerre ensemble), both without and with the insertion of characteristic polynomials. Such characteristic polynomials play the role of mass terms in applications to complex Dirac spectra in field theory. In addition, for the elliptic real Ginibre ensemble, we recover the SOP of Forrester and Nagao in terms of Hermite polynomials.

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1. Introduction

Classical orthogonal polynomials (OP) are one of the principal standard tools used to solve problems in random matrix theory (RMT). The three classical Wigner–Dyson ensembles with Gaussian elements can be solved in terms of Hermite polynomials, whereas their chiral counterparts require the use of Laguerre polynomials. Whilst for the symmetry classes with unitary invariance ($\beta = 2$) the corresponding scalar product is symmetric with the standard weight, for the classes with orthogonal ($\beta = 1$) or symplectic ($\beta = 4$) symmetry the scalar product becomes skew-symmetric, with the details—including the weight—dependent on the symmetry class. The corresponding polynomials are then called skew-orthogonal polynomials.
(SOP). For details of all these cases we refer to [1], as well as to [2] and [3] for reviews on SOP. What all these ensembles have in common is that each of their solutions can be expressed using the kernel of the corresponding (S)OP as a building block.

The solution of RMT in terms of (S)OP is exact for finite matrix size $N$. Moreover, when taking one of the possible large-$N$ limits in the bulk of the spectrum, or at the soft or hard edge, the standard Plancherel–Rotach asymptotics can be used. Much work has been done on the question of universality, i.e. the extent to which the asymptotics also hold for non-Gaussian weights, see e.g. [4] for a review.

This setup of (S)OP has been generalized to the complex plane in order to solve non-Hermitian RMT. However, for the standard Ginibre ensembles the (S)OP are simply given in terms of monic powers (which holds for all weights that are rotationally invariant in $\mathbb{C}$). Only when considering so-called elliptic deformations of the Gaussian Ginibre weight do Hermite polynomials on $\mathbb{C}$ appear, as was first observed in [5]. The corresponding quaternionic ($\beta = 4$) elliptic Ginibre ensemble was solved in terms of some Hermite SOP in [6], and only very recently was the real elliptic Ginibre case ($\beta = 1$) solved in terms of another set of Hermite SOP [7].

The chiral counterparts of two of these ensembles were introduced in [8] ($\beta = 2$) and [9] ($\beta = 4$), where they were solved in terms of Laguerre OP on $\mathbb{C}$ and Laguerre SOP respectively. The obvious question then arises about the existence of Laguerre SOP for the chiral $\beta = 1$ ensemble [10, 11] which we answer affirmatively in this paper, thereby completing the set of classical Hermite and Laguerre (S)OP in the complex plane for these non-Hermitian RMT.

One complication arises for the ensembles with Laguerre (S)OP in the complex plane: due to the integration over angular variables the elliptic ensembles that are Gaussian in terms of the matrix elements lead to non-Gaussian weight functions for the eigenvalues (for $\beta = 1$ elliptic Ginibre, we also have a complementary error function). The Bessel function of the second kind appearing here for all three $\beta = 1, 2, 4$ makes the orthogonality question much more involved.

For this reason we first provide a new integral representation for the $\beta = 1$ SOP, valid in both the elliptic Ginibre and chiral symmetry classes for arbitrary weight functions. In view of earlier results for the SOP on $\mathbb{R}$ for both $\beta = 1, 4$ [12, 13], as well as for the complex SOP for $\beta = 4$ [6, 9], this representation comes very naturally. Moreover, it was shown very recently in [14] that the $\beta = 1$ and $\beta = 4$ Ginibre ensembles can be treated on an equal footing. We rederive this relation amongst these symmetry classes from a different angle. We also derive a new integral representation for the Cauchy transforms of the SOP on $\mathbb{C}$ valid for both $\beta = 1$ and 4. This extends the expression for Cauchy transforms on $\mathbb{C}$ for $\beta = 2$ in [15].

One important ingredient necessary in order to derive these results is the factorization of the joint probability distribution function (jpdf) for $\beta = 1$, which is originally given by a sum over all possible combinations of real and complex conjugate eigenvalues [16, 17]. Such a factorization, which uses the symmetrization over all eigenvalues, might have been expected from the fact that the partition function can be written as a single Pfaffian over double integrals [18].

Our integral representation allows us to derive the $\beta = 1$ Laguerre SOP on $\mathbb{C}$ in a straightforward fashion. The known Hermite SOP on $\mathbb{C}$ of [7] also follow easily. As a third and important example for our general formalism we explicitly compute the SOP for the chiral $\beta = 1$ ensemble with the insertion of mass terms. Such insertions play a crucial role in the application of RMT to the complex Dirac operator spectrum in quantum chromodynamics (QCD) and related field theories at a small quark chemical potential in the low-density phase, see e.g. [19] for reviews, as well as very recently in the high-density phase corresponding to maximal non-Hermiticity [20].
For this third example we exploit the fact that averages (and ratios) of the required characteristic polynomials have been computed very recently for the non-Hermitian $\beta = 1$ symmetry classes in [29]. Together with our interpretation of the building blocks there as the kernel, SOP and their Cauchy transforms this completes the analogy to earlier computations of such averages and ratios for $\beta = 2$ in [15, 22, 23] and $\beta = 4$ in [9, 24].

This paper is organized as follows. In section 2 we recall the definition of the two RMT with real asymmetric matrices including mass terms, their respective weight functions and eigenvalue representations. Section 3 is devoted to a factorization proof of the jpdf of real and complex eigenvalues, where we give two different arguments. The new integral representations for the SOP and their Cauchy transforms are then shown in subsections 4.1 and 4.2, respectively. In section 5 we provide three explicit examples for SOP including Hermite in subsection 5.1, Laguerre SOP in subsection 5.2 and the SOP including masses in subsection 5.3. In the appendices we collect together short proofs for some mathematical identities used in the text.

2. The matrix models

We show how to solve the following two-matrix models of real asymmetric matrices in terms of SOP in the complex plane.

The first model is given by the chiral extension of the elliptic real Ginibre ensemble

$$Z_{ch}^{(N_f)}(m) \sim \int_{\mathbb{R}^{2N(N+\nu)}} dAdB \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f \mathbb{I}_{N \times N} & A \\ B^T & m_f \mathbb{I}_{(N+\nu) \times (N+\nu)} \end{pmatrix},$$  \hspace{1cm} (2.1)

$$\times \exp \left[ -\frac{1}{2} \eta_+ \text{Tr}(AA^T + BB^T) + \eta_- \text{Tr}(AB^T) \right],$$

with

$$\eta_{\pm} \equiv \frac{1 \pm \mu^2}{4\mu^2}.$$  \hspace{1cm} (2.2)

Here $A$ and $B$ are real asymmetric matrices of size $N \times (N + \nu)$, and $\mu \in (0, 1]$ is a non-Hermiticity parameter. The integration runs over all independent real matrix elements of $A$ and $B$ with a flat measure. The product of determinants or characteristic polynomials is motivated by the addition of $N_f$ quark flavours in applications to QCD at finite density [19, 20]. The model can be written as a Gaussian two-matrix model with two independent real asymmetric matrices $P$ and $Q$, with $A = P + \mu Q$ and $B = P - \mu Q$ (see [11]). In the limit $\mu \to 0$, the model reduces to the chiral Gaussian orthogonal ensemble (chGOE).

The second matrix model is also a generalization of the elliptic real Ginibre ensemble and is given by

$$Z_{Gin}^{(N_f)}(m) \sim \int_{\mathbb{R}^{N^2}} dJ \prod_{f=1}^{N_f} \det[J - im_f] \exp \left[ -\frac{1}{1 - \tau^2} \text{Tr}(JJ^T - \tau J^2) \right].$$  \hspace{1cm} (2.3)

Here, $J$ is a real asymmetric matrix of size $N^2$, and $\tau \in [0, 1)$ is the non-Hermiticity parameter. We again integrate over all the independent matrix elements of $J$. The model can alternatively be written as a two-matrix model with symmetric and anti-symmetric matrices $S$ and $A$ with Gaussian elements, where $J = S + A \sqrt{(1 - \tau)/(1 + \tau)}$, see also [25] for $N_f = 0$. In the limit $\tau \to 1$, the model reduces to the GOE when $N_f = 0$. The extra determinants correspond to the imaginary mass terms coming in pairs of opposite sign. These are also motivated from applications to QCD with a chemical potential, but this time in three dimensions, see [26].
Once we switch to an eigenvalue basis for the Dirac matrix \( D \) for the first model, where
\[
D = \begin{pmatrix}
0_{N \times N} & A \\
B^T & 0_{(N+1) \times (N+1)}
\end{pmatrix},
\]
and to the eigenvalues of the matrix \( J \) for the second model, both models can be treated along the same lines. Because the characteristic equation for both \( D \) and \( J \) is real, its solutions are either real or come in complex conjugate pairs. However, because of the chirality of \( D \), there is a peculiarity here:
\[
0 = \det[A \mathbb{1}_{2N+\nu} - D] = \Lambda^\nu \det[A^2 \mathbb{1}_N - A B^T] = \Lambda^\nu \prod_{j=1}^{N} (\Lambda^2 - \Lambda_j^2).
\]
Whilst the \( \Lambda_j^2 \) are indeed either real or come in complex conjugate pairs, the Dirac eigenvalues \( \Lambda_j \) are consequently real (\( \Lambda_j^2 > 0 \)), purely imaginary (\( \Lambda_j^2 < 0 \)), or come in quadruplets (\( \pm \Lambda_j, \pm \Lambda_j' \)); there are also \( \nu \) generic zero-eigenvalues. For simplicity and to keep the presentation of the two models parallel, we mainly consider changed variables \( z_j = \Lambda_j^2 \) in the following.

Following [10, 27] the partition functions in equations (2.1) and (2.3) above can be written as follows, where the normalization is to be determined later, see equation (4.16):
\[
Z_{2N+\chi} = N! \sum_{n=0}^{N} \prod_{k=1}^{2\nu+\chi} \left( \int dx_k \right) \frac{d^2 z_m}{\Delta_1}\mathcal{P}_{2\nu+\chi,N-n}(x, z, z^*)
\]
Here we sum over all the possible ways of splitting the total number \((2N + \chi)\) of eigenvalues into \( K \equiv 2n + \chi \) real eigenvalues \( \{x_k\} \) and \( M \equiv N - n \) complex conjugate eigenvalue pairs \( \{z_m, z_m^*\} \). A product with an upper limit less than its lower limit is defined as unity. Note that in this expression we have only one complex integration for each complex conjugate eigenvalue pair. The differentials of the complex eigenvalues are defined over the real and the imaginary parts, i.e. \( d^2 z_m = d\Re z_m d\Im z_m \). In the following we treat the cases of an even \( (\chi = 0) \) and odd \( (\chi = 1) \) total number of eigenvalues on the same footing. The jpdf for a fixed number \( K \) of real eigenvalues and \( M \) complex eigenvalue pairs is defined as
\[
P_{K,M}(x, z, z^*) = \prod_{k=1}^{K} b(x_k) \prod_{m=1}^{M} (g(z_m, z_m^*) 2i(\Im z_m) \Delta_K(\{x\}, \{z, z^*\}))
\]
\[
\times \prod_{k=2}^{K} \Theta(x_k - x_{k-1}) \prod_{m=2}^{M} \Theta(\Re z_m - \Re z_{m-1})
\]
with the weight specified in equations (2.9) and (2.11) below. Here \( \Theta \) is the Heaviside distribution, and the Vandermonde determinant is defined as
\[
\Delta_N(\{z\}) = \prod_{k=l}^{N} (z_k - z_l) = \det_{1 \leq a, b \leq N}^{N} [z_a^{b-1}].
\]
In equation (2.7) we explicitly specify the number \( K \) of real eigenvalues and \( M \) complex eigenvalue pairs, with the set of arguments labelled as \( x_1, \ldots, x_K, z_1, z_1^*, \ldots, z_M, z_M^* \) in \( \Delta_K(\{x\}, \{z, z^*\}) \). The factors \( 2i(\Im z_m) \) and the ordering of the real eigenvalues \( \Theta(x_k - x_{k-1}) \) in equation (2.7) allow us to omit the modulus sign around the Vandermonde determinant. The ordering of the real parts \( \Theta(\Re z_m - \Re z_{m-1}) \) is needed to make the
\footnote{In contrast to \[11\] we distinguish the weights for real and complex eigenvalues by different symbols \( (b \) and \( g \), respectively).}
transformation to upper triangular $2 \times 2$ block form of the matrices $A$ and $B^T$ (or $J$) unique when computing the Jacobian [11] (see also [28]). The latter and part of the former can be dropped later due to the symmetrizing integration as shown in the next section.

We also mention that the partition function can be written as a single Pfaffian [18, 27], equation (3.8) below, and we come back to the consequences for factorization in the next section.

We now give the weight functions for our two models. Looking at equation (2.5) for the chiral model we are only interested in the eigenvalues of the matrix $C = AB^T$. Because the $N_f$ extra mass terms compared with [11] depend only on $C$, their addition to the jpdf in [11] is trivial and so we only give the result. The corresponding weight functions in equation (2.7) for the real eigenvalues $x$ and complex eigenvalues $z = x + iy$ read

$$ h_{ch}(x) \equiv 2|x|^{v/2} K_{v/2}(\eta,|x|) \exp[\eta|x|] \prod_{f=1}^{N_f} \left(x + m_f^2\right), $$

$$ g_{ch}(z_1, z_2) \equiv 2|z_1 z_2|^{v/2} \exp[\eta_1(z_1 + z_2)] \prod_{f=1}^{N_f} (z_1 + m_f^2)(z_2 + m_f^2) $$

$$ \times \int_0^\infty \frac{dt}{t} \exp \left[-\eta_1^2 t(z_1^2 + z_2^2) - \frac{1}{4t} K_{v/2}(2\eta_1^2 t z_1 z_2) \text{erfc}(\eta_1 t(z_2 - z_1)) \right]. $$

(2.9)

In addition we have a trivial overall factor arising from the generic zero eigenvalues in the mass terms

$$ P_{K,M}^{ch}(x, z, z^*) = \prod_{f=1}^{N_f} m_f^2 P_{K,M}(x, z, z^*). $$

(2.10)

For the second model, equation (2.3), we have instead

$$ h_{Gia}(x) \equiv \exp[-x^2] \prod_{f=1}^{N_f} \left(x^2 + m_f^2\right), $$

(2.11)

$$ g_{Gia}(z_1, z_2) \equiv \exp[-z_1^2 - z_2^2] \text{erfc} \left(\frac{|z_1 - z_2|}{\sqrt{1 - \tau}}\right) \prod_{f=1}^{N_f} \left(z_1^2 + m_f^2\right) \left(z_2^2 + m_f^2\right). $$

For the above two Gaussian examples, the following relation is satisfied:

$$ \lim_{\tau \in \mathbb{C}, z \to 0} g(z, z^*) = h(\Re z)^2, $$

(2.12)

relating the two weights.

As a general remark here and in the following we can allow for more general weight functions $h(x)$ and $g(z_1, z_2)$ in equation (2.7) that do not necessarily follow from a matrix representation. For example, we could generalize the weights in equations (2.9) and (2.11) by multiplying by a factor $\exp[-V(z_1, z_2)]$, where $V$ is a polynomial in $z_1$ and $z_2$. Moreover, one can independently choose $h$ and $g$ instead fulfilling the relation (2.12).

### 3. Factorization of the joint probability distribution

In this section we prove that the jpdf inside the partition function can be written in a factorized form. For this to be possible, it is essential that we integrate over all the eigenvalues, leading

5 A so-called harmonic potential could be realized as a matrix model, by multiplying equations (2.1) and (2.5) with $\exp[-\text{Tr}(V(AB^T))]$ or $\exp[-\text{Tr}(V(J^2))]$. Although at finite $N$ these are formal expressions due to lack of convergence, this can be dealt with in the large-$N$ limit.
to a symmetrization. Hence this applies equally to the expectation value of any operator symmetric in all variables. Examples for this are the computation of the gap probability or expectation values leading to integral representations of the SOP in the next section. However, such a factorization can also be found for the k-point density correlation functions when summing over all possibilities of splitting k into real and complex eigenvalue pairs.

Let us first state the result for the partition function, equation (2.6), in terms of a single product for the weights

\[ Z_{2N+\chi} = \int_{\mathbb{R}} dy^x h^x(y) \prod_{k=1}^{2N} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{N} F(z_{2j-1}, z_{2j}) \Delta_{N+2N}(y, \{z\}), \]  

where we define the anti-symmetric function

\[ F(z_1, z_2) = ig(z_1, z_2)(\Theta(\Im z_1) - \Theta(\Im z_2)) \delta(z_2 - z_1^*) + \frac{1}{2} h(z_1) h(z_2) \delta(\Im z_1) \delta(\Im z_2) \operatorname{sgn}(\Re z_2 - \Re z_1). \]  

Note that for an even number of variables (\( \chi = 0 \) in equation (3.1)) the integration and weight for the real variable y have to be dropped, as well as the argument y inside the Vandermonde determinant. In equation (3.1) we integrate over 2N independent complex variables in contrast to the ordered integration in complex conjugated pairs. The standard two-dimensional delta function in equation (3.2) reads \( \delta^2(z) = \delta(x) \delta(y) \) for \( z = x + iy \).

We prove this factorization in two different ways. One is by explicitly summing up all the terms in equation (2.6) to make a single product. The results for expectation values of characteristic polynomials in [29] are built up on this idea, although a proof of this was not given. The second way is by starting from a single Pfaffian representation of the partition function derived in [18, 27] and using (in reverse) a proof of a slightly generalized version of the de Bruijn integral formula. Because both derivations are short and illustrate different aspects, we decided to present both.

The first derivation of the factorization of the jpdf goes as follows. From equations (2.7) and (2.6) we obtain

\[ Z_{2N+\chi} = N! \sum_{n=0}^{N} \frac{1}{(N-n)!} \prod_{l=1}^{2n+\chi} dx_l h(x_l) \prod_{j=2}^{2n+\chi} \Theta(x_j - x_{j-1}) \times \prod_{m=1}^{N-n} \int_{\mathbb{C}} d^2 z_m g(z_m, z_m^*) 2i \Theta(\Im z_m) \Delta_{2(n+\chi)+2(n-N)}(\{x\}, \{z, z^*\}), \]  

where we use that the integrand is totally symmetric under a permutation of two pairs of complex conjugated eigenvalues. Dropping the ordering of the real parts leads to a factor \( 1/(N - n)! \).

The ordering of the real variables can be simplified by applying the method of integration over alternating variables [1] twice. Pulling the integrations over odd variables in, dropping the symmetrization giving a factor \( 1/n! \) and then pulling them back out leads to the following result:

\[ Z_{2N+\chi} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} \prod_{l=1}^{2n+\chi} dx_l h(x_l) \prod_{j=1}^{n} \Theta(x_{2j} - x_{2j-1}) \times \prod_{m=1}^{N-n} \int_{\mathbb{C}} d^2 z_m g(z_m, z_m^*) 2i \Theta(\Im z_m) \Delta_{2(n+\chi)+2(n-N)}(\{x\}, \{z, z^*\}). \]
Isolating the integration $dx_x$, we can do the sum over multiple integrations, using the binomial formula and the permutation invariance of the integrand under exchanging pairs of complex numbers:

$$a = \int_{\mathbb{R}} dx_1 h(x_1) \int_{\mathbb{R}} dx_2 h(x_2) \Theta(x_2 - x_1)$$

$$= \int_{\mathbb{C}} d^2z_1 h(z_1) \int_{\mathbb{C}} d^2z_2 h(z_2) \Theta(\Re z_2 - \Re z_1) \delta(3m z_1) \delta(3m z_2)$$

$$b = \int_{\mathbb{C}} d^2z g(z, z^*) 2i\Theta(3m z) = \int_{\mathbb{C}} d^2z_1 \int_{\mathbb{C}} d^2z_2 g(z_1, z_2) 2i\Theta(3m z_1) \delta^2(z_2 - z_1^*),$$

(3.5)

with

$$\sum_{n=0}^{N} \frac{N!}{n!(N-n)!} a^n b^{N-n} = (a + b)^N.$$  

(3.7)

Pulling out all $2N$ independent complex integrations of the $N$-fold product this leads to equation (3.1) as claimed, after making the function $F(z_1, z_2)$ manifestly anti-symmetric.

We now come to our second argument, starting from the result derived in [18, 27]. This states that, including normalization,

$$Z_{2N} = N! \text{Pf}_{1 \leq a,b \leq 2N} \left[ \int_{\mathbb{C}^2} d^2z_1 d^2z_2 F(z_1, z_2) \left[z_1^{a-1} z_2^{b-1} - z_2^{a-1} z_1^{b-1}\right]\right]$$

(3.8)

where for simplicity we only state the even case, i.e. with $2N$ eigenvalues. The sign of the Pfaffian is defined as in [1] such that the Pfaffian of the matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) is unity. Here $F(z_1, z_2)$ is the function from equation (3.2). By using a slight generalization of the proof of the de Bruijn integral formula, in reverse,

$$\prod_{k=1}^{2N} \int_{\mathbb{C}} d^2z_k \prod_{j=1}^{N} F(z_{2j-1}, z_{2j}) \det_{1 \leq a,b \leq N} \left[ \{ f_a(z_{2b-1}), g_a(z_{2b}) \} \right]$$

$$= N! \text{Pf}_{1 \leq a,b \leq 2N} \left[ \int_{\mathbb{C}^2} d^2u d^2v F(u, v) \left[ f_a(u) g_a(v) - f_a(v) g_a(u) \right] \right],$$

(3.9)

where we refer to appendix A for a derivation (cf appendix C.2 in [21]), we obtain

$$Z_{2N} = \prod_{k=1}^{2N} \int_{\mathbb{C}} d^2z_k \prod_{j=1}^{N} F(z_{2j-1}, z_{2j}) \det_{1 \leq a,b \leq 2N; 1 \leq b \leq N} \left[ z_{2b-1}^{a-1} z_{2b}^{b-1} \right].$$

(3.10)

Here the last determinant is simply the Vandermonde determinant, and thus we have arrived again at equation (3.1).

All of the above arguments can be generalized, by including an arbitrary observable that is symmetric under the exchange of all the eigenvalues. Perhaps the simplest example, which we will also encounter in the next section, is a factorizing operator

$$f(x, z, z^*) = f^x(x) \prod_{j=1}^{2N} f(z_j).$$

(3.11)

The individual factors do not affect the symmetry arguments above and we obtain

$$\langle f(x, z, z^*) \rangle_{2N+X} = \frac{1}{Z_{2N+X} N!} \int_{\mathbb{R}} dy (h(y) f(y))^X \prod_{k=1}^{2N} \int_{\mathbb{C}} d^2z_k f(z_k) \prod_{j=1}^{N} F(z_{2j-1}, z_{2j})$$

$$\times \Delta_X \prod_{k=1}^{2N+X} (y, [z])$$

(3.12)
for general expectation values. An explicit example for such an operator is the characteristic polynomial. This result can be generalized to non-factorizing observables symmetric in the eigenvalues. Since the monomials in the traces of a matrix can be traced back to products over characteristic polynomials, equation (3.12) is true for all symmetric polynomials in the eigenvalues. With Weierstraß’ approximation theorem all symmetric functions consist of polynomials in the traces, and as a limiting case one can also construct distributions like the Dirac distribution. This means that all observables symmetric in the eigenvalues fulfill a similar equation to (3.12) in a weak sense.

As a further remark the symplectic non-Hermitian ensembles with $\beta = 4$ \cite{6,9} are already of the factorized form equation (3.1) from the onset, with

$$F(\beta = 4)(z_1, z_2) = i(z_2 - z_1)u(z_1, z_2)\delta^2(z_2 - z_1^*)$$

(3.13)

and we always have $\chi = 0$ due to symmetry. The symmetric real weight $u(z_1, z_2)$ can be found in \cite{6} and \cite{9} for the Gaussian Ginibre and chiral classes, respectively. Therefore, all statements we derive from the form of equation (3.1) automatically hold true for these symmetry classes as well. The factorization thus unifies the non-Hermitian ensembles for $\beta = 1$ and 4; in fact, this was already pointed out in \cite{14,29}. In \cite{14}, this was found in a different way without using factorization.

4. Integral representation of skew-orthogonal polynomials and their Cauchy transforms

In this section we derive integral representations for the SOP for general weight functions, using the results from the previous section. For this we only need the result for an even total number of eigenvalues (i.e. $\chi = 0$).

All the matrix or complex eigenvalue models introduced previously can be solved for all eigenvalue density correlation functions in terms of the following skew-symmetric kernel:

$$K_{2N}(z_1, z_2) = \sum_{k,l=0}^{2N-1} A_{kl}^{-1} p_k(z_1) p_l(z_2),$$

(4.1)

where

$$A_{kl} = 2 \int_{\mathbb{C}^2} d^2z_1 d^2z_2 F(z_1, z_2) p_k(z_1) p_l(z_2).$$

(4.2)

In fact, the kernel is only a property of the measure $F(z_{2j-1}, z_{2j})$ and not of the particular choice of the polynomials $\{p_k(z)\}$; in \cite{27} these were chosen to be monic. For an odd total number of eigenvalues a similar representation to equation (4.1) holds, but with a modification to the last row and column of the matrix $A$; see \cite{28} and \cite{30}. Here we choose the polynomials $p_k(z)$ to be skew-orthogonal with respect to the following anti-symmetric scalar product:

$$\langle f | g \rangle = -\langle g | f \rangle = \int_{\mathbb{C}^2} d^2z_1 d^2z_2 F(z_1, z_2) \det \begin{bmatrix} f(z_1) & g(z_1) \\ f(z_2) & g(z_2) \end{bmatrix},$$

(4.3)

defined for the two functions $f(z)$ and $g(z)$ that are integrable with respect to the weight functions contained in $F(z_1, z_2)$. This includes the particular function $g(z_1, z_2)$ from equation (2.7).

Our SOP $q_k(z)$ are defined to satisfy

$$\langle q_k | q_{l+1} \rangle = h_k \delta_{kl},$$
$$\langle q_k | q_l \rangle = 0 = \langle q_{k+1} | q_{l+1} \rangle \quad \forall \, k, l \geq 0,$$

(4.4)

where the $h_k > 0$ are their positive (squared skew) norm, see equation (4.16). This leads to a block diagonal matrix $A = \text{diag}(h_0 \epsilon, \ldots, h_{N-1} \epsilon)$ that can be easily inverted, where $\epsilon$ is the
anti-symmetric $2 \times 2$ matrix with elements $\epsilon_{12} = 1 = -\epsilon_{21}$. The kernel can be written as a single sum in terms of the SOP:

$$K_{2N}(z_1, z_2) = \sum_{k=0}^{N-1} \frac{1}{R_k} (q_{2k+1}(z_1)q_{2k}(z_2) - q_{2k+1}(z_2)q_{2k}(z_1)).$$

(4.5)

The kernel for $2N + 1$ contains the same SOP plus a correction term, see also [30] for the Ginibre ensemble and [1] for the GOE.

### 4.1. Skew-orthogonal polynomials

After all this preparation we come to our second result, an explicit representation of the SOP. They are given in terms of the following expectation values:

$$q_{2n}(z) = \langle \det(z - J) \rangle_{2n} = \left( \prod_{j=1}^{2n} (z - z_j) \right)_{2n},$$

(4.6)

for the even polynomials, and

$$q_{2n+1}(z) = \langle \det(z - J) \rangle_{2n+1} = \langle \det(z - J) \rangle_{2n+1} + (z + c)q_{2n}(z)$$

$$= \left( \prod_{j=1}^{2n} (z - z_j) \left( \sum_{i=1}^{2n} z_i + c \right) \right)_{2n},$$

(4.7)

for the odd polynomials, which are both expectation values over an even number of eigenvalues $2n, n \geq 1$. For $n = 0$, we simply have $q_0(z) = 1$ and $q_1(z) = z + c$, by definition. It is easy to see by taking large arguments that these representations are in monic normalization, namely $q_n(z) = z^n + O(z^{n-1})$. Similar expressions hold in terms of the matrix $D$ from equation (2.4) for the chiral model (see subsection 5.2 for more details), whilst the representations given in terms of squared eigenvalues $\lambda_j^2 = z_j$ are identical. Equations (4.6) and (4.7) were also shown for particular non-Hermitian ensembles in [12, 13].

The set of odd polynomials is not unique because of the anti-symmetry of the skew product (4.3), as the arbitrary constant $c$ times the even polynomial drops out. In most of the following, we will keep the constant $c \neq 0$ though.

The proof of the first integral representation, equation (4.6), goes as follows. We can write the product times the Vandermonde determinant of dimension $2n$ as a Vandermonde determinant of dimension $2n + 1$, and so

$$q_{2n}(z) = \frac{1}{Z_{2n}} \prod_{k=1}^{2n} \int d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \det_{1 \leq a \leq 2n+1; 1 \leq b \leq 2n} \left[ \left[ z_{a}^{n-1} \right] \left[ z_{b}^{n-1} \right] \right].$$

(4.8)

We can now apply a slight modification of the generalization of the de Bruijn integral formula proved in appendix C.2 of [21]:

$$\prod_{k=1}^{2n} \int d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \det_{1 \leq a \leq 2n+m; 1 \leq j \leq n; 1 \leq i \leq m} \left[ \left[ f_a(z_{2j-1}), g_a(z_{2j}) \right] \alpha_{ai} \right]$$

$$= (-)^{m(2m-1)/2} n! \prod_{1 \leq a, b \leq 2n+m; 1 \leq i \leq m} \left( \int \mathbb{C} d^2 u \left[ f_a(u)g_b(u) - f_b(u)g_a(u) \right] \alpha_{ai} \right) \alpha_{ib}.$$  

(4.9)

The overall sign can be seen by choosing $|\alpha_{ai}| = \left( \mathbb{0}_{m, 2n+m} \right)^T$. Here $\alpha$ is a constant matrix, which in our case, equation (4.8), is a simple vector with $m = 1$. In contrast to the usual
de Bruijn formula we integrate over 2n variables here instead of n, as is shown to hold in appendix A (see also appendix C.2 in [21]).

Denoting the basis functions of monic powers by $e_a(z) \equiv z^a$ and using the fact that we have equal functions $f_a(z) = g_a(z) = e_{a-1}(z)$ above, we arrive at

$$q_{2n}(z) = \frac{n!}{Z_{2n}} \text{Pf}_{1 \leq a, b \leq 2n+1} \left[ \begin{array}{cc} \{e_{a-1}|e_{b-1}\} & e_{a-1}(z) \\ -e_{b-1}(z) & 0 \end{array} \right].$$

(4.10)

It is easy to see that using the definition of the skew product in equation (4.3) and performing one more integration, we have

$$\langle q_{2n}|e_c \rangle = 0 \quad \forall c = 0, \ldots, 2n,$$

(4.11)

because the corresponding Pfaffian vanishes. Using the linearity of the skew product we can deduce that the even polynomials $q_{2n}(z)$ in equation (4.6) are skew-orthogonal to all polynomials of lower and equal degree.

To prove the second integral representation, equation (4.7), we need a further identity for manipulating Vandermonde determinants

$$\sum_{a=1}^{N} z_a \Delta_N(\{z\}) = \text{det} \begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_N \\ \vdots & \cdots & \vdots \\ z_{N-1} & \cdots & z_{N} \end{bmatrix} = \tilde{\Delta}_N(\{z\}),$$

(4.12)

which is proved in appendix B. We can now proceed as in equation (4.8), by first incorporating the product in equation (4.7) into a larger Vandermonde determinant, and then applying the identity (4.12) for 2n + 1:

$$q_{2n+1}(z) = \frac{1}{Z_{2n}} \prod_{k=1}^{2n} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \text{det}_{1 \leq a, b \leq 2n} \left[ \begin{array}{cc} z_a^{-1} & z_{b-1}^{-1} \\ e_{2n+1} z_a^{a-1} & e_{2n+1} z_b^{b-1} \end{array} \right].$$

(4.13)

For simplicity we have set $c = 0$ here as it does not affect the proof. Again applying the integral formula, equation (4.9), with a slightly modified range of indices compared with the even polynomial case, we obtain

$$q_{2n+1}(z) = \frac{n!}{Z_{2n}} \text{Pf}_{1 \leq a, b \leq 2n+1} \left[ \begin{array}{ccc} \{e_{a-1}|e_{b-1}\} & \{e_{a-1|e_{2n+1}\}} & e_{a-1}(z) \\ \langle q_{2n+1}|e_{b-1} \rangle & 0 & e_{2n+1}(z) \\ -e_{b-1}(z) & -e_{2n+1}(z) & 0 \end{array} \right].$$

(4.14)

On taking the skew product (4.3) of this result, it obviously follows that

$$\langle q_{2n+1}|e_c \rangle = 0 \quad \forall c = 0, \ldots, 2n - 1,$$

(4.15)

this being the skew-orthogonality of the odd polynomials $q_{2n+1}(z)$ given by equation (4.7) to all polynomials of degree less than or equal to 2n − 1.

As a last step we verify the coefficient of the only non-vanishing skew product which due to linearity and equation (4.11) equals $\langle q_{2n}|e_{2n+1} \rangle = \langle q_{2n}|q_{2n+1} \rangle$. To do so we first determine the partition function in terms of the norms. It follows along the lines of equation (4.8). Inside the Vandermonde determinant there we could choose any set of polynomials in monic normalization, after applying the invariance properties of the determinant. We thus have

$$Z_{2n} = \prod_{k=1}^{2n} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n} F(z_{2j-1}, z_{2j}) \text{det}_{1 \leq a, b \leq 2n} \left[ q_{a-1}(z_b) \right] = n! \text{Pf}_{1 \leq a, b \leq 2n} \left[ \{q_{a-1}|q_{b-1}\} \right].$$

(4.16)
In the second step we applied once more the integral identity (4.9), with \( m = 0 \) and the matrix \( \alpha \) absent. Due to the skew-orthogonality, the matrix inside the Pfaffian becomes block diagonal, with the norms \( h_k \) times \( \epsilon \) down the diagonal, which finally leads to the product of the norms.

Using equation (4.10) after replacing the monic powers with the polynomials \( q_k \), we have

\[
\langle q_{2n} | q_{2n+1} \rangle = \frac{n!}{Z_{2n}} \left[ \prod_{a=0}^{n} h_a = h_n, \right] (4.17)
\]

and thus the consistency of the normalization of our integral representations (4.6) and (4.7) with respect to equation (4.4). This concludes our proof of the integral representations of the SOP satisfying equation (4.4). An entirely different derivation of the same results can be made by a mapping to the \( \beta = 4 \) symplectic case. While mapping our \( F(z_1, z_2) \) as in equation (3.13) we could in principle copy the orthogonality proof from [6] where the representations, equations (4.6) and (4.7), were derived for \( \beta = 4 \).

It is worth mentioning that the same representation for the SOP holds for Hermitian RMT at \( \beta = 1 \) and 4 with real eigenvalues as was shown earlier in [12, 13]. However, for all four cases—two Hermitian and two non-Hermitian—the jpdf and corresponding skew products are different. In contrast, for \( \beta = 2 \) all OP are obtained from a single relation as in equation (4.6), in both Hermitian and non-Hermitian RMT [22]. The fact that the same integral representation for the SOP holds both in non-chiral [12, 13] and chiral ensembles is straightforward in the Hermitian case. However, for non-Hermitian ensembles this becomes nontrivial comparing [6] versus [9] for \( \beta = 4 \), and [10] for \( \beta = 1 \). This is due to the two-matrix model structure of the chiral ensembles, where the change to an eigenvalue basis requires detailed calculations.

Let us finish this subsection with some remarks. For an even number of eigenvalues \( \chi = 0 \) the anti-symmetric kernel, equation (4.1) (or (4.5)), can itself be expressed as the expectation value of two characteristic polynomials for \( \beta = 1 \) [10] (and \( \beta = 4 \) [24])

\[
\langle \det(\lambda - J) \det(\gamma - J) \rangle_{2N} = h_N \frac{\mathcal{K}_{2N+2}(\lambda, \gamma)}{\lambda - \gamma} \quad \text{with} \quad \lambda \neq \gamma, \quad (4.18)
\]

and similarly for the chiral ensemble. This equation is valid for arbitrary weight functions. In fact we partly use this relation to determine the set of odd polynomials, equation (4.7), in section 5 below. So why are equations (4.6) and (4.7) interesting if the kernel itself can be independently determined as a building block? It is because the integral representations that we just derived, and the explicit determination of the SOP in some examples in the next section, complete the list of classical polynomials in the complex plane for the three elliptic Ginibre ensembles and their chiral extensions.

The determination of the SOP through an ansatz, and subsequently the direct verification of relations (4.4) for skew-orthogonal Hermite polynomials, was already a formidable task for the elliptic real Ginibre ensemble as can be seen from [7]. Because of the non-Gaussian form of the chiral weight, equation (2.9), this is even more so true for skew-orthogonal Laguerre polynomials. The integral representations derived here provide an alternative, constructive approach, leading to a new result for skew-orthogonal Laguerre polynomials.

\[ \text{Note that the overall constant in front of the kernel has been chosen here to be consistent with the standard choice in equation (4.5).} \]
4.2. Cauchy transforms

We now come to the definition and integral representation of the Cauchy transforms $t_κ(z)$. It is very natural to define them with respect to the scalar product, equation (4.3), as follows:

$$ t_κ(z) ≡ \int_{\mathbb{C}^2} d^2 z_1 d^2 z_2 F(z_1, z_2) \det \begin{pmatrix} q_κ(z_1) & 1 \\ q_κ(z_2) & 1 \end{pmatrix} = \langle q_κ \mid \frac{1}{κ - z} \rangle. $$

(4.19)

We now show that the following integral representations hold:

$$ t_{2κ}(z) = \frac{1}{\det(κ - J)} \int_{2κ+2} d^2 z \frac{1}{κ - z}. $$

(4.20)

for the Cauchy transforms of the even polynomials, and

$$ t_{2κ+1}(z) = \frac{\text{Tr}(J - (κ + c))}{\det(κ - J)} \int_{2κ+2} d^2 z \frac{1}{κ - z} - (κ + c)t_{2κ}(z) $$

(4.21)

for the odd polynomials. Note that the averages for $t_{2κ}(z)$ and $t_{2κ+1}(z)$ run over 2n + 2 variables, instead of 2n as for the polynomials $q_{2κ}(z)$ and $q_{2κ+1}(z)$. This implies in particular that $t_κ(z) \neq \text{const}$, see also equation (4.22) below.

The correct overall prefactors can also easily be seen. From expanding the geometric series in the definition, equation (4.19), for large arguments, and using equations (4.11) and (4.15) as well as the anti-symmetry of the first non-vanishing skew product, it follows that the Cauchy transforms are indeed Laurent series with the following coefficients:

$$ t_{2κ}(z) = + \frac{h_n}{κ^{2n+2}} + O\left(\frac{1}{κ^{2n+3}}\right), $$

$$ t_{2κ+1}(z) = - \frac{h_n}{κ^{2n+1}} + O\left(\frac{1}{κ^{2n+2}}\right). $$

(4.22)

The form given in equations (4.20) and (4.21) is completely analogous to equations (4.6) and (4.7), as well as to the corresponding result for $β = 2$. Let us also remark that such a representation was not known before in the non-Hermitian $β = 4$ symmetry class, and that both translate into new representations for $β = 1, 4$ in the Hermitian limit.

We begin by proving the representation for the Cauchy transforms of the even polynomials. Inserting the result, equation (4.6), into definition (4.19), we have

$$ t_{2κ}(z) = \int_{\mathbb{C}^2} d^2 z_1 d^2 z_2 F(z_1, z_2) \left[ \frac{(\det(κ - J))_{2κ}}{κ - z_2} - \frac{(\det(κ - J))_{2κ}}{κ - z_1} \right] $$

$$ = \frac{1}{Z_{2κ}} \int_{\mathbb{C}^2} d^2 z_{2κ+1} d^2 z_{2κ+2} F(z_{2κ+1}, z_{2κ+2}) \prod_{j=1}^{2κ} \int_{\mathbb{C}^2} d^2 z_j \prod_{j=1}^{n} F(z_{2κ+1 - j}, z_{2κ+2 - j}) Δ_{2n}(\{z\}) $$

$$ \times \left[ \prod_{j=1}^{2κ} \frac{(z_{2κ+1 - j} - z_j)}{κ - z_{2κ+1}} - \frac{2κ+1}{κ - z_{2κ+1}} \right] $$

$$ = \frac{1}{(n + 1)Z_{2κ}} \int_{\mathbb{C}^2} d^2 z_{2κ+1} \prod_{j=1}^{n+1} F(z_{2κ+1 - j}, z_{2κ+2 - j}) $$

12
In the first step we have simply written out the expectation value and renamed the additional two integration variables. The products in the numerator can be incorporated into a larger Vandermonde determinant. Next we can symmetrize the integrand with respect to an exchange two integration variables. The products in the numerator can be incorporated into a larger Vandermonde determinant as before, as well as the modified Vandermonde determinant $\tilde{\Delta}$ defined on the right-hand

\begin{align}
\times \sum_{k=1}^{n+1} \left[ \frac{\Delta_{2n+1}(\{z\}_{\neq 2k})}{\kappa - z_{2k}} - \frac{\Delta_{2n+1}(\{z\}_{\neq 2k-1})}{\kappa - z_{2k-1}} \right]

= \frac{1}{(n+1)Z_{2n+2}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^{2}z_{k} \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j})

= \frac{1}{(n+1)Z_{2n}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^{2}z_{k} \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j})

= h_{n} \left( \frac{1}{\det(\kappa - J)} \right)^{2n+2}.

(4.23)

In the first step we have simply written out the expectation value and renamed the additional two integration variables. The products in the numerator can be incorporated into a larger Vandermonde determinant. Next we can symmetrize the integrand with respect to an exchange two integration variables. The products in the numerator can be incorporated into a larger Vandermonde determinant as before, as well as the modified Vandermonde determinant $\tilde{\Delta}$ defined on the right-hand

\begin{align}
\det_{1 \leq a < 2n; 1 \leq b \leq 2n-1} \left[ \{z_{a}^{-1}\} \right] \frac{1}{\kappa - z_{a}} = \prod_{a > b}(z_{a} - z_{b}) \prod_{i=1}^{2n}(\kappa - z_{i}).

(4.24)

This can be used to express the Cauchy transform as an expectation value, after providing the correct normalization factor from equation (4.16) in the last step.

The proof for the odd Cauchy transforms follows along the same lines. For simplicity we set $c = 0$ here, which can easily be reinstated at the end:

\begin{align}
t_{2n+1}(\kappa) = & \frac{1}{Z_{2n}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^{2}z_{k} \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j})

= \frac{1}{(n+1)Z_{2n}} \prod_{k=1}^{2n+2} \int_{\mathbb{C}} d^{2}z_{k} \prod_{j=1}^{n+1} F(z_{2j-1}, z_{2j})

= h_{n} \left( \frac{\text{Tr} J - \kappa}{\det(\kappa - J)} \right)^{2n+2}.

(4.25)

Here we included the product into the Vandermonde determinant as before, as well as the additional sum leading to the modified Vandermonde determinant $\tilde{\Delta}$ defined on the right-hand

\[\text{Note that we order products here so that there is no sign in equation (2.8) for the Vandermonde determinant.}\]
side of equation (4.12). In the last step we simply need a slightly modified version of the identity, equation (4.24), which is derived in appendix C. The inclusion of the arbitrary constant \( c \neq 0 \) follows simply by shifting \( \kappa \to \kappa + c \) in the numerator but not in the denominator. This concludes the derivation of all the integral representations of the SOP and their Cauchy transforms. In principle the simple expectation values in equations (4.6) and (4.7) as well as in equations (4.20) and (4.21) could be computed explicitly using supersymmetric vectors depending on ordinary variables and Grassmannians. In the explicit examples given in the next section we give the resulting SOP only.

5. Examples for skew-orthogonal polynomials

In this section we give three examples of SOP in the complex plane: Hermite, Laguerre and Laguerre with mass terms. Although the first of these were already known, our derivation is new. The second and third are new examples.

In principle there are two different methods. In the first of these we directly use the integral representations; for the even polynomials these are the expectations of a single determinant, equation (4.6), and for the odd polynomials the expectations of a determinant multiplied by a trace, equation (4.7). Both can be calculated in one step by computing the expectation of the product of two determinants (which is proportional to the kernel) and either taking limits or differentiating, and using the fact that the determinant is the generating functional of all independent matrix invariants. The expectations can then be computed using Grassmannians, and because this was already explicitly done in [10], we can be very brief here.

The second method follows the general setup outlined at the start of section 4. Given the kernel in terms of general polynomials, equation (4.1), and choosing them to be skew-orthogonal, equation (4.3), the individual polynomials can be ‘read off’ from the kernel in equation (4.5) by differentiation (or taking limits):

\[
q_{2n}(z) = h_n \left( \frac{1}{(2n + 1)!} \frac{\partial^{2n+1}}{\partial u^{2n+1}} K_{2n+2}(u, z) \right) = h_n \lim_{u \to \infty} \frac{K_{2n+2}(u, z)}{u^{2n+1}}
\]

This is possible whenever the kernel has already been independently determined, e.g., by the above procedure detailed in [10] (see also [27] for another method). In addition the norms \( h_k \) can be read off from the kernel as the leading coefficients.

Of course both methods lead to the same answer. In the third example the kernel including the masses as well as the partition function itself have not previously been computed explicitly and so this also constitutes a new result.

5.1. Example I: skew-orthogonal Hermite polynomials

In general the calculation of the expectation of a single determinant (or the product of two determinants) is straightforward, even without switching to an eigenvalue basis: we express the determinant as an integral over anti-commuting (Grassmann) variables, and then the Gaussian random matrices can be integrated out. After a Hubbard–Stratonovich transformation the anti-commuting variables can also be integrated out. Because the procedure was carried out and explained in detail for two determinants with \( N_f = 0 \) in this model in [10], we
only quote here the result for our first example, the expectation with respect to the model, equation (2.3)³
det(z − J) \det(u − J))_N = N! \sum_{l=0}^{N} \sum_{k=0}^{l} \frac{1}{k!2^k} H_k \left( \frac{z}{\sqrt{2\tau}} \right) \frac{u}{\sqrt{2\tau}} \cdot \quad (5.2)

where \(\tau\) is the non-Hermiticity parameter, and the \(H_k(z)\) are the standard Hermite polynomials.

Hence, for the even polynomials we can simply project out the second determinant to give

\[
q_{2k}(z) = \lim_{u \to \infty} \frac{\det(z - J) \det(u - J)}{u^{2k}} \cdot \quad (5.3)
\]

Here we used the following result to calculate the single term in the double sum that survives the limiting process

\[
\lim_{u \to \infty} \frac{1}{u^{N}} H_N \left( \frac{u}{\alpha} \right) = \left( \frac{2}{\alpha} \right)^N. \quad (5.4)
\]

This equation also implies that the even polynomials, equation (5.3), are in monic normalization as they should be, starting with \(q_0(z) = 1\). Alternatively we could of course have differentiated equation (5.2) \(N\) times with respect to \(u\).

For the odd polynomials, we use the fact that the determinant is the generating functional for symmetric functions, and in particular for the trace

\[
\frac{1}{(N-1)!} \frac{\partial^{N-1}}{\partial u^{N-1}} \det(u \mathbb{1}_N - J) \bigg|_{u=0} = -\text{Tr} J, \quad (5.5)
\]

where \(J\) is an \(N \times N\) matrix, and \(N \geq 1\). Applying this to equation (5.2) on the left-hand side allows us to obtain \(q_{2k+1}(z)\) from equation (4.7), for \(k \geq 1\),

\[
q_{2k+1}(z) = -\frac{1}{(2k-1)!} \frac{\partial^{2k-1}}{\partial u^{2k-1}} \langle \det(z - J) \det(u - J) \rangle_{2k}|_{u=0} + (z + c)q_{2k}(z)
\]

\[
= -(2k) \left( \sqrt{\frac{\tau}{2}} \right)^{2k-1} (z + c) \frac{\tau^{k} H_{2k-1} \left( \frac{z}{\sqrt{2\tau}} \right)}{2^{2k-1} 2^{k-1}} + c \frac{\tau^{k} H_{2k} \left( \frac{z}{\sqrt{2\tau}} \right)}{2^{k}} \cdot \quad (5.6)
\]

In the first step only two terms survive the differentiation after setting \(u = 0\); we used the following properties of Hermite polynomials in addition to equation (5.4):

\[
\frac{d^{n-1}}{dz^{n-1}} H_n(z) = 2^n n! z \quad (5.7)
\]

\[
H_{n+1}(z) = 2z H_n(z) - 2n H_{n-1}(z), \quad \text{for } n \geq 1. \quad (5.8)
\]

as well as the recurrence relation to simplify equation (5.6) in the last line. This form makes it more transparent that the arbitrary addition of \(cq_{2k}(z)\) is the only even Hermite polynomial appearing in this example. From equation (5.4) it also follows that \(q_{2k+1}(z)\) is in monic normalization, and for \(k = 0\), we have \(q_1(z) = z + c\) by definition. Defining

\[
C_k(z) \equiv \left( \frac{\tau}{2} \right)^{k/2} H_k \left( \frac{z}{\sqrt{2\tau}} \right) \quad (5.9)
\]

³ The double sum can be simplified by using the Christoffel–Darboux identity [10].
we reobtain the following final simple result from [7]:

\[ q_{2k}(z) = C_{2k}(z), \]
\[ q_{2k+1}(z) = C_{2k+1}(z) - 2kC_{2k-1}(z) - C_{2k}(z). \]  

The norms \( h_k = 2(\tau + 1)\sqrt{2\pi}(2k)! \) can be determined either by direct calculation of the scalar product, equation (4.4), of the SOP which we just obtained, as was done in [7], or by computing the partition function\(^9\).

Let us emphasize that in our derivation the skew-orthogonality of the polynomials is automatically satisfied due to their general integral representations, equations (4.6) and (4.7). This is in contrast to [7], where the skew-orthogonality was explicitly verified for the weights, equation (2.11), in the complex plane.

### 5.2. Example II: skew-orthogonal Laguerre polynomials

In this subsection we turn to entirely new expressions for skew-orthogonal Laguerre polynomials for our chiral model, equation (2.1). We start with the so-called quenched case \((N_f = 0)\) and then add mass terms in the next subsection.

Expressed in terms of the eigenvalues \( z_j \) of \( ABT \) these are the same as before; however, the matrix \( D \) in equation (2.4) also has \( \nu \) generic zero eigenvalues, and so we repeat the integral representations, equations (4.6) and (4.7), for completeness, and also to make contact with [10]. For the even polynomials, we have

\[ q_{2n}(z) = \frac{1}{\sqrt{2\kappa}} \langle \det(z I_{2n} - ABT) \rangle_{2n} \]

and for the odd polynomials,

\[ q_{2n+1}(z) = \frac{1}{\sqrt{2\kappa}} \left( \langle \det(z I_{2n+1} - D) \left( \frac{1}{2} TrD^2 + z + c \right) \rangle_{2n+1} \right) \]

\[ = \langle \det(z I_{2n} - AB^T) TrAB^T \rangle_{2n} + (z + c)q_{2n}(z). \]  

The starting point for what we need for our calculations, namely the expectation of two determinants, was again given in detail in [10] and thus we merely state the result

\[ \langle \det(z I_{2n} - ABT) \det(u I_{2n} - AB^T) \rangle_{2n} = \]

\[ (2n)! \langle 2n + v \rangle ! (4\mu^2\eta_+) \sum_{l=0}^{2n} \left( \frac{\eta_-}{\eta_+} \right) ^{2l} \sum_{k=0}^{l} k! L_k^\nu \left( \frac{z}{4\mu^2\eta_-} \right) L_k^\nu \left( \frac{u}{4\mu^2\eta_-} \right). \]  

where we recall notation (2.2).  

We thus obtain the even polynomials by simply projecting out the second determinant

\[ q_{2k}(z) = \lim_{u \to \infty} \frac{1}{\mu^{2k}} \langle \det(z I_{2k} - AB^T) \det(u I_{2k} - ABT) \rangle_{2k} \]

\[ = (4\mu^2\eta_-)^{2k} \langle 2k \rangle ! L_{2k}^\nu \left( \frac{z}{4\mu^2\eta_-} \right). \]  

Here we have used the following relation for the Laguerre polynomials:

\[ \lim_{u \to \infty} \frac{1}{\mu^N} L_N^\nu \left( \frac{u}{\mu} \right) = \frac{(-1)^N}{N!\alpha^N}, \]

which also confirms that the even polynomial is properly normalized.

\(^9\) The lower order terms from combining equations (4.5) and (4.18) will provide the ratios of norms \( h_N / h_k \) and thus the \( k \)-dependence only.
For the odd polynomials we again need to take derivatives as in equation (5.5) to obtain for \( k \geq 1, \)

\[
q_{2k+1}(z) = -\frac{1}{(2k - 1)!} \frac{d^{2k-1}}{dz^{2k-1}} \left( \det(zI_{2k} - AB^T) \det(uI_{2k} - AB^T) \right)_{u=0} + (z + c)q_{2k}(z)
\]

\[
= (4\mu^2 \eta_-)^{2k+1} (2k)! (2k + v) \left( 2kL^v_{2k} \left( \frac{z}{4\mu^2 \eta_-} \right) + \left( 1 + \frac{\eta^2}{\eta_-^2} \right) L^v_{2k-1} \left( \frac{z}{4\mu^2 \eta_-} \right) \right)
\]

\[
+ (z + c)(4\mu^2 \eta_-)^{2k} (2k)! L^v_{2k} \left( \frac{z}{4\mu^2 \eta_-} \right)
\]

\[
= -(4\mu^2 \eta_-)^{2k+1} (2k + 1)! L^v_{2k+1} \left( \frac{z}{4\mu^2 \eta_-} \right) + c'(4\mu^2 \eta_-)^{2k} (2k)! L^v_{2k} \left( \frac{z}{4\mu^2 \eta_-} \right)
\]

\[
+ (2k + v)(4\mu^2 \eta_-)^2 (4\mu^2 \eta_-)^{2k-1} (2k)! L^v_{2k-1} \left( \frac{z}{4\mu^2 \eta_-} \right), \tag{5.16}
\]

where the new arbitrary constant

\[
c' \equiv c + (4\mu^2 \eta_-)(4k^2 + 4k + 1 + (2k + 1)v) \tag{5.17}
\]

now depends on \( k, v \) and \( \mu \). The above result was obtained after using the corresponding relations for Laguerre polynomials

\[
\frac{d^{n-1}}{dz^{n-1}} L^v_n(z) = (-1)^n(z - (n + v)) \tag{5.18}
\]

\[
(n + 1)L^v_{n+1}(z) = (2n + v + 1 - z) L^v_n(z) - (n + v)L^v_{n-1}(z), \quad \text{for } n \geq 1. \tag{5.19}
\]

It is easy to see that the polynomials are again monic, due to equation (5.15). This once more fixes \( q_1(z) = z + c \). We can now define

\[
C^v_{2k}(z) \equiv (4\mu^2 \eta_-)^k L^v_k \left( \frac{z}{4\mu^2 \eta_-} \right) \tag{5.20}
\]

allowing us to write

\[
q_{2k}(z) = +C^v_{2k}(z),
\]

\[
q_{2k+1}(z) = -C^v_{2k+1}(z) + (1 + \mu^2)^2 (2k)(2k + v)C^v_{2k-1}(z) + c'C^v_{2k}(z), \tag{5.21}
\]

giving the new skew-orthogonal Laguerre polynomials up to an arbitrary constant. The final result compares with the similar form of equation (5.10).

For the norms we find \( h_k = 8\pi (4\mu^2)(2k)! (2k + v)! (4\mu^2 \eta_-)^{4k+4v+1} \) where the \( k \)-dependence again follows from the ratio of the norms \( h_{k+1}/h_k \), see equations (4.18) and (4.5), whereas the overall constant factor can be deduced from the partition function, see equation (3.46) in [11], and taking the ratio for consecutive values of \( N \).

5.3. **Example III: inclusion of mass terms in the chiral model**

Our third example gives the SOP again for weights including \( N_f > 0 \) mass terms, which is also called the unquenched case. We exemplify this using the chiral model, equation (2.1), where such terms are more common due to applications in QCD. However, the same insertion of mass terms can be done in the non-chiral model (2.3) following the same lines.

Our main point here will be to express the SOP for \( N_f > 0 \) in terms of the SOP for \( N_f = 0 \) (and the corresponding kernel), which we have already calculated. To indicate which polynomials we are referring to we will use a superscript, as in \( q_k^{(N_f)}(z) \), and correspondingly for the kernel and expectations.
Our derivation relies heavily on [29] where all the expectation values of products and ratios of characteristic polynomials (or determinants) have been expressed in terms of Pfaffian expressions of matrices containing a small number of building blocks; in our case these building blocks will be the quenched \((N_f = 0)\) SOP and kernel.

To begin we first express the unquenched integral representations (4.6) and (4.7) in terms of the ratios of quenched expectations. For the even polynomials we have

\[
q^{(N_f)}_{2n}(z) = \frac{\langle \det(z I_{2n} - AB^T) \rangle^{(N_f)}_{2n}}{\langle \prod_{j=1}^{N_f} \det (m_j^2 I_{2n} - AB^T) \rangle^{(0)}_{2n}},
\]

and similarly for the odd polynomials we have

\[
q^{(N_f)}_{2n+1}(z) = \frac{\langle \det(z I_{2n} - AB^T) \text{Tr} A B^T \prod_{j=1}^{N_f} \det (m_j^2 I_{2n} - AB^T) \rangle^{(0)}_{2n}}{\langle \prod_{j=1}^{N_f} \det (m_j^2 I_{2n} - AB^T) \rangle^{(0)}_{2n}} + (z + c)q^{(N_f)}_{2n}(z).
\]

We now give all the building blocks for these expressions. The first building block in the denominator, the expectation value of the mass term, simultaneously provides us with the massive partition function itself:

\[
\frac{Z_{ch2N}^{(N_f)}(m)}{Z_{ch2N}^{(0)}} = \prod_{j=1}^{N_f} m_j^{N_f} \left( \prod_{j=1}^{N_f} \det (m_j^2 I_{2N} - AB^T) \right)^{(0)}_{2N}.
\]

The masses to the power of \(v\), the number of generic zero eigenvalues, of course cancel in the ratios for \(d_k^{(N_f)}(z)\) above. Using the results from [29] and expressing the expectation values there in terms of our quenched kernel and even SOP, we obtain

\[
\begin{align*}
\frac{Z_{ch2N}^{(N_f)}(m)}{Z_{ch2N}^{(0)}} &= \frac{\prod_{j=1}^{N_f} m_j^{N_f} \Delta_{N_f}^{N_f} (m^2)}{\Delta_{N_f}^{(m^2)}} \left( \prod_{j=1}^{N_f} h_j^{(0)} \right)^{(N_f)}_{\Delta_{N_f}^{N_f}} \left[ \mathcal{K}^{(0)}_{\Delta_{N_f}^{N_f}} (m_j^2, m_g^2) \right], & N_f \text{ even} \\
\frac{Z_{ch2N}^{(N_f)}(m)}{Z_{ch2N}^{(0)}} &= \frac{\prod_{j=1}^{N_f} m_j^{N_f} \Delta_{N_f}^{N_f} (m^2)}{\Delta_{N_f}^{(m^2)}} \left( \prod_{j=1}^{N_f} h_j^{(0)} \right)^{(N_f)}_{\Delta_{N_f}^{N_f}} \left[ -q_{2N+N_f-1}^{(0)} (m_j^2) \mathcal{K}^{(0)}_{\Delta_{N_f}^{N_f}} (m_j^2, m_g^2) \right], & N_f \text{ odd}
\end{align*}
\]

where we have to distinguish even and odd numbers of flavours \(N_f\). The product over the norms in the prefactor is equal to unity when the upper limit is \(N - 1\). Compared to [29] we have the following identity:

\[
\begin{align*}
\mathcal{Pf} & \left[ \begin{array}{cc}
0 & q_{2M}^{(0)} (m_g^2) \\
-q_{2M}^{(0)} (m_f^2) & k_{2M}^{(0)} (m_f^2, m_g^2)
\end{array} \right] = \mathcal{Pf} & \left[ \begin{array}{cc}
0 & q_{2M}^{(0)} (m_f^2) \\
-q_{2M}^{(0)} (m_g^2) & k_{2M}^{(0)} (m_f^2, m_g^2)
\end{array} \right]
\end{align*}
\]

which can be easily seen by adding multiples of the first row and column to the remaining rows and columns, in order to eliminate the leading SOP in the kernels and hence shifting their index down by 2. This result for the partition function (or expectation values of characteristic polynomials) precisely equals the corresponding result for \(\beta = 4\) in [24].

10 Note a typo in [24] in equation (2.8) compared to the correct theorem 1 in equation (3.1) there.
The even polynomials now easily follow from equation (5.25), by choosing one of the masses to be the argument. We obtain

$$q_{2N}^{(N_f)}(z) = \prod_{m=1}^{N_f} (z - m_f^2) \frac{K_{2N+N_f}(z, m_f^2)}{\prod_{m=1}^{N_f} (z - m_m^2) \text{Pf}[K_{2N+N_f}(m_m^2, m_m^2)]}$$

(5.27)

for \( N_f \) even. Here and in the following we suppress the indices of the Pfaffian which run from 1 to \( N_f \) in both the even and odd cases. For \( N_f \) odd, we obtain

$$q_{2N}^{(N_f)}(z) = -\frac{h_{N_f(N_f-1)/2}^{(0)}}{h_{N_f-1}^{(N_f)}} \frac{\det(z \mathbb{I}_{2N} - AB^T) \text{det}(u \mathbb{I}_{2N} - AB^T)}{\prod_{m=1}^{N_f} (z - m_m^2)} \frac{q_{2N+n+1}(z, m_m^2)}{q_{2N+n-1}(m_m^2)}$$

Next we determine the massive kernel using equation (4.18):

$$K_{2N}^{(N_f)}(z, u) = \frac{(z - u) \text{det}(z \mathbb{I}_{2N} - AB^T) \text{det}(u \mathbb{I}_{2N} - AB^T)}{h_{N_f-1}^{(N_f)}} \frac{\prod_{m=1}^{N_f} (z - m_m^2) (u - m_m^2)}{\prod_{m=1}^{N_f} \text{Pf}[K_{2N+n}^{(0)}(m_m^2, m_m^2)]}$$

Using equation (5.25) with two extra masses we obtain for \( N_f \) even

$$K_{2N}^{(N_f)}(z, u) = -\frac{h_{N_f-1}^{(N_f)}}{h_{N_f}^{(0)}} \frac{\prod_{m=1}^{N_f} (z - m_m^2)}{\prod_{m=1}^{N_f} (z - m_m^2)} \frac{q_{2N+n+1}(z, m_m^2)}{q_{2N+n-1}(m_m^2)} \text{Pf}[K_{2N+n}^{(0)}(m_m^2, m_m^2)]$$

(5.30)

Here the mass-dependent inverse norm \( 1/h_{N_f-1}^{(N_f)} \) has been eliminated using the following identity, leading to a shift in the index of the kernels in the denominator by +2. Following equation (4.16) we can write

$$h_{N_f-1}^{(N_f)} = \frac{Z_{ch2N}^{(0)}}{Z_{ch2N}^{(0)}} \frac{Z_{ch2N-2}^{(0)}}{Z_{ch2N-2}^{(0)}}$$

$$h_{N_f}^{(0)} = \frac{h_{N_f-1}^{(0)}}{h_{N_f-2}^{(0)}} \frac{\text{Pf}[K_{2N+n}^{(0)}(m_m^2, m_m^2)]}{\text{Pf}[K_{2N+n}^{(0)}(m_m^2, m_m^2)]}$$

(5.31)
Likewise for \( N_f \) odd we have

\[
\mathcal{K}_{2N}^{(N_f)}(z, u) = \left[ \prod_{f=1}^{N_f} (z - m_f^2) \right] \left( u - m_f^2 \right) Pf \begin{bmatrix}
-1 \\
0 \quad q^{(0)}_{2N+N_f-1}(m_f^2) \\
\cdots \\
-q^{(0)}_{2N+N_f-1}(z) \quad q^{(0)}_{2N+N_f-1}(u) \quad k^{(0)}_{2N+N_f-1}(z, u) \quad k^{(0)}_{2N+N_f-1}(z, m_f^2) \\
-q^{(0)}_{2N+N_f-1}(m_f^2) \\
-k^{(0)}_{2N+N_f-1}(m_f^2, z) \\
-k^{(0)}_{2N+N_f-1}(m_f^2, u) \\
-k^{(0)}_{2N+N_f-1}(m_f^2, m_f^2)
\end{bmatrix}
\]

\( (5.32) \)

Let us pause with a few remarks. Following [11] these expressions for the massive kernel determine all massive eigenvalue correlation functions for \( 2N_f \), in terms of the known quenched kernel and the quenched even SOP that were given in the second example above. In particular it is transparent that even for finite \( N \) the unquenched kernel (when properly normalized by the massive weight) is given by the quenched kernel plus some correction terms. The same structure thus prevails for all unquenched eigenvalue correlation functions.

For \( 2N + 1 \), correction terms to the massive kernel will also include the massive SOP \( q^{(N_f)}_{2N+1}(z) \), when following e.g. [30]. Thus all massive eigenvalue correlation functions follow in this case as well.

As the final step we give the massive odd SOP \( q^{(N_f)}_{2N+1}(z) \). Here we will follow equation (5.1) and determine them from the kernel, rather than equation (5.5). As an aside, above we could have alternatively determined the kernel first and then the even SOP from equation (5.1) as well. A slight generalization of equation (5.1) reads

\[
q^{(N_f)}_{2n+1}(z) = -h^{(N_f)}_n \left( \frac{1}{2n+k} \right) \frac{\partial^{2n}}{\partial u^{2n}} \left( \prod_{j=1}^{k} (u - a_j) \mathcal{K}_{2n+2}^{(N_f)}(u, z) \right) \bigg|_{u=0} + c^{(N_f)} q^{(N_f)}_{2n}(z), \quad (5.33)
\]

where \( k \geq 0 \), and the \( a_j \) are some arbitrary constants. It is easy to see that we only get a non-vanishing result when \( 2n \) or \( 2n+1 \) of the derivatives act on the kernel and not the prefactor. This is true because the function in the brackets is a polynomial of order \( 2n + k + 1 \) in the variable \( u \). Hence, the differentiation yields the coefficients of the monomials of order \( 2n \) and \( 2n+1 \) in the variable \( u \) of the kernel \( \mathcal{K}_{2n+2}^{(N_f)} \).

In choosing \( k = N_f \) and \( a_j = m_f^2 \) we can use this relation to cancel the factor \( \prod_{j=1}^{N_f} (u - m_f^2) \) in the denominator of equation (5.30) that would otherwise have to be differentiated as well. We thus obtain from equations (5.33) and (5.30) that

\[
q^{(N_f)}_{2n+1}(z) = -h^{(N_f)}_n \left( \frac{1}{2n+N_f} \right) \frac{\partial^{2n}}{\partial u^{2n}} \left( \prod_{j=1}^{N_f} (u - m_f^2) \mathcal{K}_{2n+2}^{(N_f)}(u, z) \right) \bigg|_{u=0} + c^{(N_f)} q^{(N_f)}_{2n}(z)
\]

\( = \frac{0}{\prod_{j=1}^{N_f} (z - m_f^2)} Pf \begin{bmatrix}
0 \quad q^{(0)}_{2N+N_f+1}(z) \\
\cdots \\
-q^{(0)}_{2N+N_f+1}(m_f^2) \\
-k^{(0)}_{2N+N_f}(m_f^2, z) \\
-k^{(0)}_{2N+N_f}(m_f^2, m_f^2)
\end{bmatrix}
\]

for \( N_f \) even.

\( (5.34) \)
Here we have pulled the derivatives inside the Pfaffian leading to the quenched SOP of the shifted odd index, and used equation (5.31) and the identity corresponding to equation (5.26) for the odd polynomials. For odd $N_f$ we obtain

\[ q_{2N+1}^{(N_f)}(z) = \prod_{j=1}^{N_f} (z - m_j^2) \text{Pf} \begin{bmatrix} 0 & q_{2N+1}^{(0)}(m_j^2) \\ -q_{2N+1}^{(0)}(z) & -q_{2N+1}^{(0)}(m_j^2) \end{bmatrix} \]

\[ \times \text{Pf} \begin{bmatrix} 0 & q_{2N+1}^{(0)}(z) & \tilde{c} & q_{2N+1}^{(0)}(m_j^2) \\ -q_{2N+1}^{(0)}(z) & 0 & q_{2N+1}^{(0)}(z) & q_{2N+1}^{(0)}(m_j^2) \\ -\tilde{c} & -q_{2N+1}^{(0)}(z) & 0 & q_{2N+1}^{(0)}(z) \\ -q_{2N+1}^{(0)}(m_j^2) & q_{2N+1}^{(0)}(m_j^2, z) & -q_{2N+1}^{(0)}(m_j^2) & q_{2N+1}^{(0)}(m_j^2, z) \end{bmatrix} \]

\[ + c' d_{2N}^{(N_f)}(z), \quad \text{for } N_f \text{ odd.} \] (5.35)

The constants $\tilde{c}$ in the Pfaffian in the numerator can be absorbed into the even polynomial $c' d_{2N}^{(N_f)}(z)$, which can be seen as follows. Just as the determinants of two matrices that only differ by a single row (or column) can be added, a similar statement holds for Pfaffians: due to linearity the Pfaffians of two anti-symmetric matrices that only differ by a single row and its transposed column can be added. We can thus split off the $\tilde{c}$-part from the Pfaffian above to obtain

\[ \text{Pf} \begin{bmatrix} 0 & q_{2N+1}^{(0)}(z) & \tilde{c} & q_{2N+1}^{(0)}(m_j^2) \\ -q_{2N+1}^{(0)}(z) & 0 & q_{2N+1}^{(0)}(z) & q_{2N+1}^{(0)}(m_j^2) \\ -\tilde{c} & -q_{2N+1}^{(0)}(z) & 0 & q_{2N+1}^{(0)}(z) \\ -q_{2N+1}^{(0)}(m_j^2) & q_{2N+1}^{(0)}(m_j^2, z) & -q_{2N+1}^{(0)}(m_j^2) & q_{2N+1}^{(0)}(m_j^2, z) \end{bmatrix} \]

\[ = - \text{Pf} \begin{bmatrix} 0 & \tilde{c} & 0 & 0 \\ -\tilde{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{2N+1}^{(0)}(z, m_j^2) & k_{2N+1}^{(0)}(m_j^2, z) \end{bmatrix} \] (5.36)

which is proportional to the numerator of the even polynomials with odd $N_f$ in equation (5.28).

Thus, the final result for the odd polynomial with odd $N_f$ is equation (5.35) with $\tilde{c} = 0$ and $c' \to c''$. This ends our third example for the SOP and kernel including masses.

Similar expressions could be given for the non-chiral model (2.3), as well as for the Cauchy transforms of the unquenched SOP.

6. Conclusions

In this paper we have completed the analysis of the set of (S)OP in the complex plane that apply to the chiral extensions of the three elliptic Ginibre ensembles. By constructing an explicit integral representation, we found a new set of skew-orthogonal Laguerre polynomials in the complex plane which provide an alternative method of solving the chiral ensemble with real asymmetric elements. Our integral representation is also valid for the real elliptic Ginibre ensemble; in fact, it holds for arbitrary weight functions $g$ and $h$ in these two classes. Furthermore we also gave a new integral representation of the Cauchy transforms of these
polynomials which holds not only for the two symmetry classes with real matrix elements ($\beta = 1$) but also for quaternion real matrix elements ($\beta = 4$).

An important ingredient for our results was a proof that the probability distribution in the partition function factorizes for $\beta = 1$. This offers another unifying view of the non-Hermitian $\beta = 1$ and $\beta = 4$ symmetry classes, both chiral and non-chiral.

There are many more non-Hermitian ensembles, in addition to the three Ginibre classes and their chiral counterparts, all six of which have now been solved. It thus remains an open question whether corresponding sets of OP or SOP exist for the other ensembles. It is possible that, just as in the real case, the known polynomials also apply to some of these other non-Hermitian symmetry classes, once a complex eigenvalue representation has been found for them.

As an application of our results we have shown how to construct the SOP and kernel when including $N_f$ characteristic polynomials or mass terms in our models. These constitute the building blocks for the massive partition function and eigenvalue density correlation functions. Consequently, this will allow us to study the complex Dirac operator spectrum for QCD with two colours and a non-vanishing quark chemical potential, both in the low- and high-density phases. The study of the large-$N$ limit needed for this partly follows from the known quenched $N_f = 0$ results and is left for future investigations.

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Appendix A. Generalization of the de Bruijn integral formula

In this appendix we slightly generalize the standard de Bruijn integral formula that reads

$$
\prod_{j=1}^{n} \int_{\mathbb{C}} d^2 z_j w(z_j) \det_{1 \leq a, b \leq 2n, 1 \leq j \leq n} [\{f_{a}(z_j), g_{a}(z_j)\}] = n! P_{a \leq b \leq 2n} \left[ \int_{\mathbb{C}} d^2 u d^2 v \sum_{\sigma} (-1)^{\sigma} \prod_{j=1}^{n} \det \left[ \begin{array}{cc}
 f_{\sigma(j-1)}(z_{j-1}) & f_{\sigma(j)}(z_j) \\
 g_{\sigma(j-1)}(z_{j-1}) & g_{\sigma(j)}(z_j)
 \end{array} \right] \right]. \quad (A.1)
$$

Here $w(z)$ is a weight function in the complex plane, and $f$ and $g$ are functions such that the integrals exist. The proof is usually done by a Laplace expansion into $2 \times 2$ blocks that each depend on a single variable $z_l$.

If we start with $2n$ instead of $n$ integrations over a product of an anti-symmetric weight $F(u, v)$ and let $f$ and $g$ depend on different variables, we have on the left-hand side

$$
\prod_{k=1}^{2n} \int_{\mathbb{C}} d^2 z_k \prod_{j=1}^{n} \int_{\mathbb{C}} d^2 z_j F(z_{2j-1}, z_{2j}) \det_{1 \leq a, b \leq 2n, 1 \leq j \leq n} [\{f_{a}(z_{2b-1}), g_{a}(z_{2b})\}] = \prod_{j=1}^{n} \int_{\mathbb{C}} d^2 z_{2j-1} d^2 z_{2j} F(z_{2j-1}, z_{2j}) \sum_{\sigma} (-1)^{\sigma} \prod_{j=1}^{n} \det \left[ \begin{array}{cc}
 f_{\sigma(2j-1)}(z_{2j-1}) & f_{\sigma(2j)}(z_{2j}) \\
 g_{\sigma(2j-1)}(z_{2j-1}) & g_{\sigma(2j)}(z_{2j})
 \end{array} \right] = n! P_{a \leq b \leq 2n} \left[ \int_{\mathbb{C}} d^2 u d^2 v F(u, v) [f_{a}(u) g_{b}(v) - f_{b}(u) g_{a}(v)] \right]. \quad (A.2)
$$

Here $w(z)$ is a weight function in the complex plane, and $f$ and $g$ are functions such that the integrals exist. The proof is usually done by a Laplace expansion into $2 \times 2$ blocks that each depend on a single variable $z_l$. 
Here \((-)^n\) is the sign of the permutation of the \(2n\) variables, and the sum is over all \((2n)!\) permutations which satisfy the restriction \(\sigma(1) < \sigma(2) < \cdots < \sigma(2n)\). Note that each pair \(\{z_{2j-1}, z_{2j}\}\) only appears in one subdeterminant. This gives the Pfaffian as a result (see e.g. [1] for a definition).

**Appendix B. Relation to a modified Vandermonde determinant**

In this appendix we prove an identity related to Vandermonde determinants, which is needed to derive the integral representation equation (4.7) for the odd polynomials \(q_{2n+1}(z)\). For completeness we repeat the relation (4.12) which is to be shown here:

\[
\sum_{i=1}^{N} z_i \, \Delta_N([z]) = \det \begin{bmatrix}
1 & \ldots & 1 \\
z_1 & \ldots & z_N \\
\vdots & & \vdots \\
z_1^{N-2} & \ldots & z_N^{N-2} \\
z_1^N & \ldots & z_N^N
\end{bmatrix} = \widetilde{\Delta}_N([z]). \tag{B.1}
\]

Using the second representation from equation (2.8), \(\Delta_N([z]) = \det_{a,b \leq N} \left[z_a^{b-1}\right]\), one can see that the modified Vandermonde determinant in equation (B.1) has a mismatch of 1 in the powers in the last row compared to the Vandermonde determinant.

Our proof uses that \(\widetilde{\Delta}_N([z])\) is simply the coefficient of power \(u^{N-1}\) in a Leibniz expansion of the Vandermonde determinant \(\Delta_{N+1}([z], u)\) of size \(N+1\) with respect to the last column in the extra variable \(u\). This term can be singled out by a differentiation

\[
\frac{1}{(N - 1)!} \frac{\partial^{N-1}}{\partial u^{N-1}} \bigg|_{u=0} \Delta_{N+1}([z], u) = -\tilde{\Delta}_N([z]). \tag{B.2}
\]

On the other hand, using again equation (2.8) that \(\Delta_N([z]) = \prod_{j>k}^N (z_j - z_k)\), one can write

\[
\Delta_{N+1}([z], u) = \prod_{a=1}^N (u - z_a) \Delta_N([z]). \tag{B.3}
\]

Combining equations (B.2) and (B.3), we obtain result (B.1).

**Appendix C. Cauchy-type identity for the modified Vandermonde determinant**

In this appendix we prove the following identity:

\[
\det_{1 \leq a \leq 2n+2, 1 \leq b \leq 2n} \left[ \begin{array}{c}
1 \\
\vdots \\
1 \\
\end{array} \right] \left[ \begin{array}{c}
z_a^{b-1} \\
\vdots \\
z_a^{2n+1} \\
\end{array} \right] \frac{1}{z_a - z_b} = \frac{\sum_{k=1}^{2n+2} \prod_{j=1}^{2n+2} (z_j - k)}{\prod_{i=1}^{2n+2} (k - z_i)} \tag{C.1}
\]

which is a slight modification of identity (4.24) with a mismatch by 1 in the power in the last but one column. In fact we use identity (4.24) to prove the above. Expanding the left-hand side with respect to the last but one column, we have

\[
\sum_{k=1}^{2n+2} (-)^{2n+2-k} \frac{\prod_{j>i, j \neq k}^{2n+2} (z_j - z_i) \prod_{j=k}^{2n+2} (k - z_j)}{\prod_{i=1}^{2n+2} (k - z_i)}
\]

\[
= (-) \sum_{k=1}^{2n+2} (-)^{2n+2-k} \frac{\prod_{j>i, j \neq k}^{2n+2} (z_j - z_i) \prod_{j=k}^{2n+2} (k - z_j)}{\prod_{i=1}^{2n+2} (k - z_i)}
\]

\[
= (-) \sum_{k=1}^{2n+2} (-)^{2n+2-k} \frac{\prod_{j>i, j \neq k}^{2n+2} (z_j - z_i) \prod_{j=k}^{2n+2} (k - z_j)}{\prod_{i=1}^{2n+2} (k - z_i)}
\]

23
\[
\begin{align*}
&= (-)^n \Delta_{2n+2}(\{z\}) - \tilde{\Delta}_{2n+2}(\{z\}) \\
&= \left( \sum_{k=1}^{2n+2} z_k - \kappa \right) \prod_{i<j} (z_i - z_j) \\
&\quad \prod_{l=1}^{2n+2} (\kappa - z_l).
\end{align*}
\]

(C.2)

In the second step we used identity (4.24) and multiplied by unity to complete the product in the denominator. For the numerator we obtain the modified Vandermonde determinant \(\tilde{\Delta}_{2n+2}\) from equation (4.12) and a proper Vandermonde determinant, both of size \(2n + 2\), after resumming the expansion. Writing out explicitly \(\tilde{\Delta}_{2n+2}\) from the left-hand side of equation (B.1) yields the right-hand side of our identity (C.1).

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