POSITIVE CURVATURE AND THE ELLIPTIC GENUS

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Abstract. We prove several results about the vanishing of the elliptic genus on positively curved Spin manifolds with logarithmic symmetry rank. The proofs are based on the rigidity of the elliptic genus and Kennard’s improvement of the Connectedness Lemma for transversely intersecting, totally geodesic submanifolds.

1. Introduction

Which manifolds are positively curved? This captivating question has intrigued geometers for more than a century and has been solved only in a few special cases. It turns out to be a difficult task to find or construct metrics of positive sectional curvature.

Indeed, all presently known, simply-connected, positively curved manifolds of dimension greater than 24 are the sphere, the complex projective space and the quaternionic projective space. This surprising lack of examples indicates that positive sectional curvature has a strong impact on the underlying topology and consequently, one aims to exhibit topological obstructions. In this note, we analyze the relation between positive curvature and cobordism invariants and provide new results about the vanishing of the elliptic genus on positively curved Spin manifolds.

This approach dates back to the sixties. Recall that a genus in the sense of Hirzebruch is a ring homomorphism from the oriented cobordism ring to some unital algebra. Classical examples are the signature \( \text{sign}(M) \) and the \( \hat{A} \)-genus \( \hat{A}(M) \). In analytic terms, both the signature and the \( \hat{A} \)-genus can be seen as the index of a first-order elliptic differential operator. Here, the signature is the index of the square root of the Laplacian, whereas for Spin manifolds the \( \hat{A} \)-genus equals the index of the Dirac-operator. These considerations culminated in the Atiyah-Singer Index Theorem.

Shortly after the Index Theorem was proven, Lichnerowicz provided in \[19\] a first obstruction to positive scalar curvature on Spin manifolds. By using a Bochner-type formula, he showed that the \( \hat{A} \)-genus vanishes on Spin manifolds carrying positive scalar curvature. This implies, for example, that the \( K3 \)-surface \( V^4 \) does not carry a metric of positive scalar curvature, since \( V^4 \) is Spin and \( \hat{A}(V^4) \neq 0 \).

In the years following Lichnerowicz’s result, an intensive study led to the classification of simply-connected manifolds of dimension greater than four admitting positive scalar curvature, a milestone in modern Riemannian Geometry. It was shown by Gromov-Lawson in \[10\] and by Stolz in \[22\] that the \( \hat{A} \)-genus (and more precisely, its \( KO \)-theoretic refinement, the \( \alpha \)-invariant) forms the only obstruction to positive scalar curvature on simply-connected Spin manifolds of dimension greater than four.
If one strengthens the curvature assumption to non-negative sectional curvature, then the signature also yields an obstruction. This follows from Gromov’s Betti number theorem. In [9], Gromov proved that the total Betti number of a non-negatively curved manifold is bounded by a constant only depending on the dimension. Therefore, the signature is also clearly bounded and so both classical genera are deeply related to curvature.

In the eighties, a new type of genus, the so-called elliptic genera, emerged from a discussion between topologists, number theorists and physicists (see the introductory article in [18]). The term elliptic originates from the fact that the logarithm of the genus corresponds to an elliptic integral. As example, we mention the universal elliptic genus \( \phi_0(M) \), which, according to Witten, can be thought of as the equivariant signature on the free loop space. The universal elliptic genus admits the expansion
\[
\phi_0(M^{4k}) = q^{-k/2} \cdot \hat{A}(M) \bigotimes_{\begin{subarray}{c} n \text{ odd} \\ n \geq 1 \end{subarray}} \Lambda_{-q^n} T\mathbb{C}M \bigotimes_{\begin{subarray}{c} n \text{ even} \\ n \geq 1 \end{subarray}} S_{q^n} T\mathbb{C}M,
\]
where each coefficient is a characteristic number.

In light of these new genera, one might again explore their connection to positive curvature. Indeed, it is fascinating to observe that this approach is still fruitful and has led to new exciting conjectures (see, for example, Stolz’s conjecture on the Witten genus in [23]).

In this context, Dessai raised the question in [7], whether the elliptic genus is constant on a positively curved Spin manifold and added some evidence to this conjecture. In particular, he showed that the coefficients of \( \phi_0(M) \) vanish linearly with the symmetry rank. If the conjecture happened to be true, this would give a new way to distinguish between positive sectional and positive Ricci curvature.

This paper is centered around this question and we hope that it provides a better understanding of the interplay between the elliptic genus and positive curvature. We prove here

**Theorem A.** Let \((M^n,g)\) be a closed, connected, positively curved Spin manifold. Suppose that a torus \(T^s\) acts isometrically and effectively on \(M\) with \(n \geq 2^s\) and \(s \geq 3\). Then, one of the following holds:

a) the first \( \min\{\lfloor \frac{n}{16} \rfloor + 1, 2^{s-3}\} \) coefficients of \( \phi_0(M) \) vanish.

b) the rational cohomology ring of \(M\) is 4–periodic.

After hearing about Theorem A, Amann and Kennard proved a slightly different version in their preprint [1].

Coming back to the statement, the notion of a 4–periodic cohomology ring will be explained in the next section. Nevertheless, we point out that, if the dimension of \(M\) is divisible by four, then the second statement implies that the rational cohomology ring of \(M\) is isomorphic to the one of \(S^n, \mathbb{C}P^n/2\) or \(\mathbb{H}P^n/4\).

We note that the case of a sphere is already covered by the first statement. Moreover, if \(n = 4 \pmod{8}\), a generalized version of Rokhlin’s theorem proven in [21] asserts that the signature must be divisible by 16. In this case, the second statement shows that \(M\) has the same rational cohomology ring as \(\mathbb{H}P^n/4\).

Finally, we observe that the \(\hat{A}\)–genus is the first term of \(\phi_0(M)\) and therefore, the first statement of the theorem remains true for \(s \leq 3\) by Lichnerowicz’s result regardless of any symmetry assumption.

In the introduction, we hinted at the fact that positively curved manifolds are scarce and that all presently known examples carry a lot of symmetry. This initiated the study of positively curved manifolds with large isometry groups, which shed considerable new light on this topic.
We mention, for example, Wilking’s achievement in [24] to classify positively curved manifolds with a linear symmetry rank up to homotopy equivalence. Theorem A should be considered in this context. However, we stress out that our symmetry assumption is of logarithmic type, i.e. the torus rank $s$ satisfies $s \sim \log_2(n)$.

The proof of Theorem A is based on the ideas developed by Dessai [6], [7], Wilking’s Connectedness Lemma [24] and its new improvement for transversely intersecting manifolds by Kennard [17]. The difficult task is to find high-dimensional, totally geodesic submanifolds, which intersect transversely. This can be achieved, since the non-vanishing of the elliptic genus guarantees the existence of high-dimensional submanifolds. In order to find transversely intersecting manifolds, we then use some elementary coding theory following an idea of Kennard [17]. This enables us to determine the cohomology ring.

Using the modular properties of the elliptic genus, it was shown that the entire power series $\phi_0(M^n)$ is determined by its first $\lfloor \frac{n}{12} \rfloor + 1$ coefficients. We can apply our methods to obtain

**Theorem B.** Let $(M^n, g)$ be a closed, connected, positively curved Spin manifold. Suppose that a torus $T^s$ acts isometrically and effectively on $M$ with $n \geq 3 \cdot 2^{s-2}$ and $s \geq 4$. Then, one of the following holds:

a) the first $\min\{\lfloor \frac{n}{12} \rfloor + 1, 2^{s-3} \}$ coefficients of $\phi_0(M)$ vanish.

b) there exists an element $x \in H^4(M; \mathbb{Q})$ such that $x^{n/4} \neq 0$.

It is interesting to remark that there is a trade-off between the vanishing of the elliptic genus and the computation of the cohomology ring. This is due to our proof strategy. Here, the non-vanishing of the coefficients of $\phi_0(M)$ yields totally geodesic submanifolds of relatively high codimension and therefore, we are not able to recover the entire cohomology ring.

The proof is similar to the proof of Theorem A but uses, in addition, the Lefschetz fixed point formula. We again point out that the symmetry assumption is of logarithmic type. As an immediate consequence, we get

**Corollary 1.** Let $(M^n, g)$ be a closed, connected, positively curved Spin manifold. Suppose that $b_4(M) = 0$ and assume the same symmetry condition as in Theorem B. Then, the first $\min\{\lfloor \frac{n}{12} \rfloor + 1, 2^{s-3} \}$ coefficients of $\phi_0(M)$ vanish.

In [24], Wilking listed all possible cohomology rings with field coefficients of positively curved manifolds $M^n$ with $n \geq 6000$ and a symmetry rank of at least $\frac{n}{6} + 1$. If we adopt a linear symmetry rank and use this classification, we obtain an extension of the preceding corollary.

**Theorem C.** Let $(M^n, g)$ be a closed, connected, positively curved Spin manifold with $n \geq 12000$. Suppose that $b_4(M) = 0$ and that a torus $T^s$ acts isometrically and effectively on $M$ with $s \geq \frac{n}{9} + 3$. Then, the elliptic genus $\phi(M)$ is constant. In particular, for $n = 4 \pmod{8}$, we have $\text{sign}(M) = 0$.

It would be interesting to know, whether a logarithmic symmetry rank such as in the corollary would suffice to conclude a similar statement.

Changing to a related subject, the starting point of the study of positively curved manifolds with symmetry goes back to the paper [11] by Grove-Searle (see also the paper by Hsiang-Kleiner [15]). In [11], the authors studied a positively curved manifold $M$ endowed with an isometric circle action fixing a component $N$ of codimension two. As a result, they obtained a diffeomorphism classification of positively curved manifolds with maximal symmetry rank.

The idea of the proof is to consider the orbit space $M/S^1$ and to identify $N$ as a boundary of the orbit space. By applying the soul theorem adapted to Alexandrov
spaces, one then estimates the number of remaining fixed points and uses equivariant bundle theory in order to obtain a diffeomorphism classification. Extending these ideas, Grove-Searle also classified in [12] fixed point homogeneous manifolds with positive curvature.

We consider now the case, where a circle action fixes a codimension four component. Of course, the technique sketched beforehand breaks down, since we cannot identify the boundary of the orbit space and apply the soul theorem. However, we are able to prove the following result.

**Theorem D.** Let \((M^n, g)\) be a closed, connected, positively curved Spin manifold. Suppose that a torus \(T^2\) acts isometrically and effectively on \(M\) with a circle \(S^1 \subset T^2\) fixing a codimension four component. Then, the elliptic genus \(\phi(M)\) is constant. In particular, for \(n \equiv 4 \pmod{8}\), we have \(\text{sign}(M) = 0\).

The proof involves the same geometric ingredients as before. Furthermore, we need to closely examine the integral cohomology ring.

This paper is structured as follows. The next section summarizes the geometric and topological methods needed for the proofs. In particular, we recall some useful properties of totally geodesic submanifolds and discuss the rigidity of the elliptic genus. In section 3, we develop some elementary coding theory. The proofs of the theorems then follow in the sections 4 and 5.

**Acknowledgments.** The results in this paper are part of the author’s doctoral thesis. It is a great pleasure for the author to thank Anand Dessai, his advisor, for introducing the subject to him and for many stimulating and helpful discussions.

2. Geometric and topological Background

Throughout these sections all manifolds are closed and smooth. Furthermore, all actions are smooth. We start with a short review on totally geodesic submanifolds and the elliptic genus.

2.1. **Totally geodesic submanifolds.** Let \((M^n, g)\) be a positively curved Riemannian manifold. A submanifold \(N \subset M\) is called totally geodesic, if any geodesic of \(N\) is also a geodesic of \(M\) with respect to the induced metric. It is well-known that in the presence of symmetry, totally geodesic submanifolds arise naturally as fixed point sets. The first theorem we would like to mention is Frankel’s Intersection Theorem.

**Theorem 2.1 (Intersection Theorem, [8]).** Let \((M^n, g)\) be a positively curved manifold and let \(N_1^{n_1}\) and \(N_2^{n_2}\) be two connected, totally geodesic submanifolds. If \(n_1 + n_2 \geq n\), then \(N_1^{n_1}\) and \(N_2^{n_2}\) intersect.

The topology of \((M^n, g)\) is strongly reflected in the topology of a totally geodesic submanifold as shown in Wilking’s Connectedness Lemma.

**Theorem 2.2 (Connectedness Lemma, [24]).** Let \((M^n, g)\) be a positively curved manifold. We recall that a map \(f : X \to Y\) is called \(k\)-connected, if the induced map on homotopy groups \(f_* : \pi_i(X) \to \pi_i(Y)\) is an isomorphism for \(i < k\) and an epimorphism for \(i = k\).
Using Poincaré duality and the Hurewicz theorem we note that a highly connected inclusion map \( N^{n-k} \hookrightarrow M^n \) implies a periodicity in the integral cohomology of \( M \). More precisely, we have

**Lemma 2.3** ([24]). Let \( M^n \) and \( N^{n-k} \) be two closed oriented manifolds. If the inclusion \( N^{n-k} \hookrightarrow M^n \) is \((n-k-l)\)-connected, then there exists \( e \in H^k(M;\mathbb{Z}) \) such that multiplication

\[
\cup e : H^i(M;\mathbb{Z}) \to H^{i+k}(M;\mathbb{Z})
\]

is surjective for \( l \leq i < n-k-l \) and injective for \( l < i \leq n-k-l \).

In this lemma the pullback of the class \( e \in H^k(M;\mathbb{Z}) \) via the inclusion map is the Euler class of the normal bundle of \( N \) in \( M \). Wilking applied this lemma in order to prove structure theorems for positively curved manifolds with large torus actions. As an example, we state the following result, which we will need for the proof of Theorem 4.

**Theorem 2.4** ([24]). Let \((M^n,g)\) be a simply-connected, positively curved manifold with \( n \geq 6000 \). Suppose that \( b_4(M) = 0 \) and that a torus \( T^s \) acts isometrically and effectively on \( M \) with \( s \geq \frac{n}{6} + 1 \). Then, \( M \) is a rational cohomology sphere.

The Connectedness Lemma is especially powerful, when two totally geodesic submanifolds intersect transversely. Let \( N_1^{n-k_1} \) and \( N_2^{n-k_2} \) be such as in the second part of the Connectedness Lemma and suppose that \( N_1 \) and \( N_2 \) intersect transversely. By Lemma 2.3, there exists \( e \in H^{k_1}(N_2;\mathbb{Z}) \) such that

\[
\cup e : H^i(N_2;\mathbb{Z}) \to H^{i+k_1}(N_2;\mathbb{Z})
\]

is surjective for \( 0 \leq i < n-k_1-k_2 \) and injective for \( 0 < i \leq n-k_1-k_2 \). Hence, \( e \in H^{k_1}(N_2;\mathbb{Z}) \) generates a periodicity on the entire ring of \( H^*(N_2;\mathbb{Z}) \). This leads to the following definition, which was introduced in [17].

**Definition.** Let \( R \) be a ring and let \( M^n \) be an oriented manifold. The cohomology ring \( H^*(M;R) \) is said to be \( k \)-periodic, if there exists \( e \in H^{k_1}(M;R) \) such that \( \cup e : H^i(M;R) \to H^{i+k}(M;R) \) is surjective for \( 0 \leq i < n-k \) and injective for \( 0 < i \leq n-k \).

By a small abuse of language, we will say that \( M \) is \( k \)-periodic with respect to \( R \), if \( H^*(M;R) \) is \( k \)-periodic.

Kennard studied integrally \( k \)-periodic manifolds and showed by means of the Steenrod power operations that \( k \)-periodic manifolds are rationally \( 4 \)-periodic. Combining this fact with the Connectedness Lemma one retrieves

**Theorem 2.5** ([17]). Let \((M^n,g)\) be a simply-connected, positively curved manifold. Let \( N_1^{n-k_1}, N_2^{n-k_2} \subset M^n \) be two totally geodesic submanifolds that intersect transversely. If \( 2k_1 + 2k_2 \leq n \), then \( M \) is rationally \( 4 \)-periodic.

If the dimension of \( M \) is divisible by four, then it follows at once that the cohomology ring of a simply-connected \( 4 \)-periodic manifold is generated by a single cohomology class. So, \( H^*(M;\mathbb{Q}) \) is isomorphic to the rational cohomology ring of a sphere, a complex projective space or a quaternionic projective space.

The next lemma is a direct consequence of the Connectedness Lemma. A similar statement can be found in [17] and therefore, we omit the proof.

**Lemma 2.6.** Let \((M^n,g)\) be a positively curved manifold of even dimension and \( n \geq 8 \). Let \( N^{n-k} \subset M^n \) be a totally geodesic submanifold of even codimension with \( k \leq \frac{n}{4} \). If \( N \) is rationally \( 4 \)-periodic, then so is \( M \).
2.2. Elliptic genus. We now move on to the topological part of this paper. We give a brief description of the elliptic genus and mention its different expansions as well as its rigidity properties with regard to compact Lie group actions. For an introduction to the subject, the reader is referred to [13], [14] and [18].

A genus in the sense of Hirzebruch is a ring homomorphism from the oriented cobordism ring $\Omega^*_{SO} \otimes \mathbb{Q}$ to a commutative unital $\mathbb{Q}$-algebra $R$. As examples, we mention the signature $\text{sign}(M)$ and the $\hat{A}$-genus $\hat{A}(M)$. As these examples suggest, genera often arise in the context of Index Theory. The Hirzebruch formalism describes a correspondence between genera and power series $Q(x)$ with coefficients in $R$. The elliptic genus $\phi(M)$ is the genus associated to the power series $Q(x) = x/f(x)$ with

$$f(x) = \frac{1 - e^{-x}}{1 + e^{-x}} \prod_{n=1}^{\infty} \frac{1 - q^n e^{-x}}{1 + q^n e^{-x}}.$$ 

Since the function $f(x)$ is attached to a certain lattice, this yields a close relation between the elliptic genus $\phi(M)$ and the theory of modular forms. For instance, we note that for $n \equiv 0 \pmod{8}$ the elliptic genus $\phi(M^n)$ is a modular function for the subgroup $\Gamma_0(2) \subset SL_2(\mathbb{Z})$.

According to Witten [25], the elliptic genus $\phi(M)$ admits a remarkable interpretation. It can be thought of as the equivariant signature of the free loop space $\mathcal{L}M$ with respect to the natural circle action on $\mathcal{L}M$. We recall that $S^1$ acts on the loop space by simply reparametrizing the loops. This approach yields the following power series for the elliptic genus

(1) $$\phi(M) = \text{sign}(M, \bigotimes_{n=1}^{\infty} S_q^n T_C M \otimes \bigotimes_{n=1}^{\infty} \Lambda_q^n T_C M)$$

$$= \text{sign}(M) + 2 \text{sign}(M, T_C M) \cdot q + \ldots,$$

where $T_C M$ denotes the complexified tangent bundle and

$$S_i T_C M = \sum_{i=0}^{\infty} S^{i} T_C M \cdot t^i \quad \text{and} \quad \Lambda_i T_C M = \sum_{i=0}^{\infty} \Lambda^{i} T_C M \cdot t^i.$$

It follows that in the cusp given by $q = 0$ the elliptic genus equals the signature. We say that the elliptic genus is strongly rigid, if it is constant as a power series. In the other cusp, $\phi(M)$ can be described in terms of twisted $\hat{A}$-genera. More precisely, $\phi(M)$ has the following $q$-development

(2) $$\phi_0(M^{4k}) = q^{-k/2} \cdot \hat{A}(M, \bigotimes_{n \geq 1 \text{ add}}^{\infty} \Lambda_{-q^n} T_C M \otimes \bigotimes_{n \geq 1 \text{ even}}^{\infty} S_q^n T_C M)$$

$$= q^{-k/2} \cdot (\hat{A}(M) - \hat{A}(M, T_C M) \cdot q \pm \ldots).$$

If $M$ is Spin, then $\hat{A}(M, W)$ can be geometrically seen as the index of the Dirac operator twisted with some complex vector bundle $W$. In this case, the coefficients of the power series $[2]$ are integers. We conclude that (1) and (2) reveal a wonderful connection between Spin and signature geometry.

2.3. Rigidity property. We turn our attention to the equivariant setting. Let $G$ be a compact, connected Lie group acting on a Spin manifold $M$. Then, the associated vector bundles in (1) become $G$-bundles and the elliptic genus $\phi(M)$ refines to an equivariant genus $\phi(M)_g$ depending on $g \in G$. However, Bott-Taubes proved in [2] that the elliptic genus is rigid, i.e.

$$\phi(M)_g = \phi(M), \quad \forall g \in G.$$
Suppose now that $S^1$ acts on a Spin manifold $M$ and let $\sigma \in S^1$ be the non-trivial involution. Using the rigidity property and the Lefschetz fixed point formula of Atiyah-Segal-Singer and Bott, it was shown in [13] that

$$\phi(M) = \phi(M) = \phi(M^\sigma \circ M^\sigma),$$

where $M^\sigma \circ M^\sigma$ denotes the transverse self-intersection. By studying the order of the pole in expression (2), one then deduces the following result, which will play a crucial role in our arguments.

**Theorem 2.7 ([13]).** Let $S^1$ act on a Spin manifold $M^{4k}$. If the action is odd, then $\phi(M) = 0$. If the action is even and $\text{codim } M^\sigma > 4r$, then the first $(r + 1)$ coefficients of $\phi_0(M)$ vanish.

We recall that a circle action is called *even*, if the circle action lifts to the Spin structure. If this is not the case, then the action is called *odd*. Hirzebruch and Slodowy made use of this theorem to show that any Spin homogeneous space has constant elliptic genus. In particular, $\phi(\mathbb{HP}^k) = \text{sign}(\mathbb{HP}^k)$.

We finish this section by describing the Lefschetz fixed point formula of Atiyah-Segal-Singer and Bott [2] for the elliptic genus. This is needed in the proof of Theorem B. The fixed point formula gives a way to compute the equivariant elliptic genus $S(M)_{\sigma}$ in terms of the fixed point set $M^\sigma$ and the normal bundle $M^\sigma \hookrightarrow M$.

Let $\sigma \in S^1$ be an involution. Every coefficient occurring in (1) is a twisted signature $\text{sign}(M, W)$ and therefore, refines to an equivariant twisted signature $S(M, W)_{\sigma}$. The latter can be expressed as a sum

$$S(M, W)_{\sigma} = \sum_{F \subset M^\sigma} a_{F, W}.$$

Thus every fixed point component $F \subset M^\sigma$ contributes to $S(M, W)_{\sigma}$. The local datum $a_{F, W}$ at the component $F$ is given by the rational cohomology class

$$u \cdot \text{ch}(W|_F)_{\sigma} \cdot e(\nu_F) \in H^*(F; \mathbb{Q})$$

evaluated on the fundamental cycle of $F$. Here, $e(\nu_F)$ is the Euler class of the normal bundle $F \hookrightarrow M$, $\text{ch}(W|_F)_{\sigma}$ is the equivariant Chern character of $W$ restricted to $F$ and $u$ is a specific cohomology class related to the signature. We conclude that the equivariant twisted signature $S(M, W)_{\sigma}$ vanishes, if the Euler class $e(\nu_F)$ vanishes for every component $F$ of $M^\sigma$.

### 3. Elementary Coding Theory

In this section, we introduce linear codes and develop some elementary properties of linear codes with a given maximum distance. We give a simple variant of the Griesmer step. Although these results have already been used in the setting of positively curved manifolds, we include them here in order to have a more transparent presentation at hand. Furthermore, we explain, how linear codes appear naturally in the context of group actions on manifolds. For an introduction to coding theory see [16].

#### 3.1. Linear codes

We consider the finite dimensional vector space $\mathbb{Z}_2^n$. A *binary linear code* $C$ of length $n$ and rank $k$ is a $k$-dimensional subspace of $\mathbb{Z}_2^n$. A linear code $C$ may be presented by a *generating matrix* $G$, where the rows of $G$ form a basis of the code $C$. Moreover, the code $C$ comes with the *Hamming distance* function defined by

$$d(x, y) := \text{ord}(\{i \mid x_i \neq y_i\}) \quad \text{for } x, y \in C.$$
We define the weight \( wt(x) \) of a codeword \( x \in C \) to be the number of coordinates that are non-zero, i.e. \( wt(x) = d(x,0) \). Finally, we define the maximum distance \( d_{\max}(C) \) of a code \( C \) to be

\[
d_{\max}(C) := \max_{x \in C} wt(x).
\]

The minimal distance \( d_{\min}(C) \) of a code \( C \) is defined in a similar way. We now derive some properties of the maximal distance \( d_{\max}(C) \) by using the construction of the residual code.

Let \( C \) be a binary linear code of length \( n \) and of maximum distance \( d_{\max}(C) \). Let \( x \in C \) be a codeword such that \( wt(x) = d_{\max}(C) \). We construct the residual code \( C_{res}(x) \) with respect to \( x \) in the following manner. We permute the ones of \( x \) to the front and write down a generating matrix

\[
G = \begin{pmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
G^1 & \phantom{1} & & & & & & \\
G^2 & \phantom{1} & & & & & & 
\end{pmatrix}
\]

for \( C \). We then define the residual code \( C_{res}(x) \) to be the linear code generated by the matrix \( G^2 \). This new code is of length \( n - d_{\max}(C) \) and there is a simple estimate on the maximal distance.

Lemma 3.1. The maximal distance of \( C_{res}(x) \) satisfies

\[
d_{\max}(C_{\text{res}}(x)) \leq \left\lfloor \frac{d_{\max}(C)}{2} \right\rfloor.
\]

Proof. Let \( c \in C \) be a codeword, which lies in the span of \((G^1|G^2)\). We decompose \( c = (c_1|c_2) \) with respect to \( G^1 \) and \( G^2 \). We have to show that

\[
wt(c_2) \leq \left\lfloor \frac{d_{\max}(C)}{2} \right\rfloor.
\]

Since \( c \in C \), we first observe that

\[
wt(c) = wt(c_1) + wt(c_2) \leq d_{\max}(C).
\]

Using now the first row of \( G \) we also have

\[
(d_{\max}(C) - wt(c_1)) + wt(c_2) \leq d_{\max}(C).
\]

Adding up the two inequalities gives the claim. \( \square \)

There is a natural way to iterate the construction of the residual code. Let \( C_{\text{res}}(x_1,\ldots,x_{k-1}) \) be the residual code with respect to \( x_i \in C_{\text{res}}(x_1,\ldots,x_{i-1}) \) for all \( i \leq k - 1 \). Choose now \( x_k \in C_{\text{res}}(x_1,\ldots,x_{k-1}) \) with maximal weight and construct the code \( C_{\text{res}}(x_1,\ldots,x_k) \) as illustrated above. The new code is of length \( n - \sum_{i=1}^k wt(x_i) \) and by the previous lemma we obtain

\[
d_{\max}(C_{\text{res}}(x_1,\ldots,x_k)) \leq \left\lfloor \frac{d_{\max}(C)}{2^k} \right\rfloor.
\]

Thus we obtain

Corollary 3.2. Let \( C \) be a binary linear code of length \( n \) and of maximum distance \( d_{\max}(C) < 2^k \). Then, \( d_{\max}(C_{\text{res}}(x_1,\ldots,x_k)) = 0 \). In other words, \( C_{\text{res}}(x_1,\ldots,x_k) \) is a zero code of length \( n - \sum_{i=1}^k wt(x_i) \).

The corollary describes a particular situation that we will encounter in the proof of Theorem A.
3.2. Group actions on manifolds. We now point out the relation between linear codes and group actions on manifolds. This has been often used in recent papers in the area of positively curved manifolds with symmetry such as in [17] and [24].

Let \((M^n, g)\) be an oriented, positively curved manifold of even dimension with an isometric and effective torus \(T^s\)-action and let \(pt \in M\) be a torus fixed point. So, the torus acts linearly on the tangent space \(T_{pt}M\) and we obtain the isotropy representation

\[
\hat{\rho} : T^s \to SO(T_{pt}M).
\]

Passing to the subgroup of involutions

\[
\rho : \mathbb{Z}_2^s \to \mathbb{Z}_2^s
\]

induces a linear code \(C := \mathrm{im}(\rho)\) of length \(\frac{s}{2}\). Since the torus action is effective, \(\rho\) is a monomorphism and therefore, \(C\) has rank \(s\).

Let \(F(\sigma) \subset M^s\) be the fixed point component of the involution \(\sigma \in T^s\) containing \(pt\) and let \(\tilde{\sigma}\) be the associated codeword in \(C\). Then, the weight \(\text{wt}(\tilde{\sigma})\) measures exactly one half of the codimension of \(F(\sigma)\) in \(M\).

The residual code also admits a geometric description. Let \(\sigma \in T^s\) be an involution such that \(F(\sigma)\) is of maximal codimension and let \(\tilde{\sigma} \in C\) be the associated codeword. Then, \(C^{\text{res}}(\tilde{\sigma})\) is the associated code to the induced torus action on \(F(\sigma)\). These facts will be used to find totally geodesic submanifolds, which intersect transversally.

4. Proofs of Theorem A, B and C

4.1. Proofs of Theorem A and B. This subsection deals with the proofs of Theorem A and B. In both proofs, we examine closely the fixed point components of the involutions in \(T^s\). The main idea is that, if these totally geodesic submanifolds have high codimension, then the coefficients of \(\bar{\phi}_0(M)\) vanish, whereas, if the codimensions are low, we are able to compute the rational cohomology ring of \(M\).

Before we begin the proof, we fix the following notations.

Notation. Let \(N \subset M\) be a submanifold. Then, \(\text{cod}_M N\) denotes the codimension of \(N\) in \(M\).

Let \(\sigma_1, \sigma_2 \in T^s\) be two involutions and let \(F(\sigma_1)\) resp. \(F(\sigma_2)\) denote the corresponding fixed point components at a fixed point \(pt \in M^T\). We consider the induced action of \(\sigma_2\) on \(F(\sigma_1)\).

Notation. We write \(F(\langle \sigma_1, \sigma_2 \rangle)\) for the corresponding fixed point set at \(pt\). We notice that \(F(\langle \sigma_1, \sigma_2 \rangle)\) is the connected component of the intersection \(F(\sigma_1) \cap F(\sigma_2)\) at \(pt\).

Proof of Theorem A. The first coefficient of \(\phi_0(M)\) is the \(\hat{A}\)-genus. Therefore, the first statement remains true for \(n \leq 12\) by [10]. Let \(n \geq 16\). First, we set up the proof and then we shall divide it in several steps.

Setup: If the first statement does not hold, then by Theorem [24], there exists for every involution \(\sigma \in T^s\) a connected component \(F \subset M^s\) with codimension \(\text{cod}_M F \leq \frac{n}{16}\) and \(\text{cod}_M F < 2^{s-1}\). In order to see this, we distinguish two cases. If \(2^s \leq n < 2^{s+1}\), then

\[
\text{cod}_M F \leq 4 \cdot \left\lfloor \frac{n}{16} \right\rfloor \leq \frac{n}{4} < 2^{s-1}.
\]

On the other hand, if \(n \geq 2^{s+1}\), we have

\[
\text{cod}_M F \leq 4 \cdot (2^{s-3} - 1) < 2^{s-1} \leq \frac{n}{4}.
\]

Furthermore, we may assume that the action is even. So, \(\text{cod}_M F\) is divisible by four for every connected component \(F \subset M^s\). Since the torus action is isometric,
the connected components of $M^s$ are totally geodesic submanifolds. Therefore, the Intersection Theorem implies that every component $F$ of $M^s$ has either
\begin{equation}
\text{cod}_M F \leq \frac{n}{4} \quad \text{or} \quad \text{cod}_M F \geq \frac{3}{4}n + 4.
\end{equation}

The assumption that $\theta_0(M)$ is non-zero, ensures the existence of a torus fixed point $pt \in M^T$. As described in the previous section, we can associate a linear code $C$ of length $\frac{n}{2}$ and rank $s$ to the torus action. Let $\sigma \in T^s$ be an involution and let $\bar{\sigma}$ be the associated codeword. Following the fixed point configuration (3), we either have
\begin{equation}
wt(\bar{\sigma}) \leq \frac{n}{8} \quad \text{or} \quad wt(\bar{\sigma}) \geq \frac{3}{8}n + 2.
\end{equation}

We define the set $C_1 = \{ \bar{\sigma} \in C \mid wt(\bar{\sigma}) \leq \frac{n}{8} \}$. $C_1$ is obviously closed under addition and therefore, is a linear subspace of $C$. We will see that the rank of $C_1$ is at least $s - 1$.

In order to get an estimate on $rk(C_1)$, we choose a linear code $C_2$ such that $C$ is the direct sum of $C_1$ and $C_2$. Therefore, we have $rk(C) = rk(C_1) + rk(C_2)$. By definition of $C_1$, we observe that $d_{\text{min}}(C_2) \geq \frac{3}{8}n + 2$. Let $c_1, c_2 \in C_2$ be two non-trivial codewords. An easy computation shows that
\begin{equation}
wt(c_1 + c_2) < \frac{3}{8}n + 2.
\end{equation}
So, $c_1 + c_2 \notin C_2$ and we conclude that $rk(C_2) \leq 1$ and subsequently we get
\begin{equation}
rk(C_1) \geq s - 1.
\end{equation}

To summarize the setup we note that the torus action at $pt$ induces a linear code $C_1$ of length $\frac{n}{2}$ with rank $s - 1$ and maximum distance $d_{\text{max}}(C_1) < 2^{n-2}$.

**Step 1:** In this step, we are going to construct a family $(N_k)_{k=0, \ldots, s-2}$ of totally geodesic submanifolds
\[ N_{s-2} \subseteq N_{s-3} \subseteq \ldots \subseteq N_1 \subseteq N_0 = M \]
of small codimensions.

We consider the associated code $C_1$ and use the construction of the residual code to obtain codewords $\bar{\sigma}_k \in C_1^{\text{res}}(\bar{\sigma}_1, \ldots, \bar{\sigma}_{k-1})$. We define the submanifolds $N_k$ inductively in correspondence to the codewords $\bar{\sigma}_k$. Let $\sigma_1 \in T^s$ be the involution associated to $\bar{\sigma}_1 \in C_1$ and put
\[ N_1 := F(\sigma_1). \]

Consider the induced torus action on $N_1$. We choose the involution $\sigma_2 \in T^s$ such that the $\sigma_2$–action restricted to $N_1$ corresponds to $\bar{\sigma}_2 \in C_1^{\text{res}}(\bar{\sigma}_1)$ and set
\[ N_2 := F(\langle \sigma_1, \sigma_2 \rangle). \]

From here we proceed inductively. Let $N_k = F(\langle \sigma_1, \ldots, \sigma_k \rangle)$ be given. Choose the involution $\sigma_{k+1} \in T^s$ such that the $\sigma_{k+1}$–action restricted to $N_k$ corresponds to $\bar{\sigma}_{k+1} \in C_1^{\text{res}}(\bar{\sigma}_1, \ldots, \bar{\sigma}_k)$ and put
\[ N_{k+1} := F(\langle \sigma_1, \ldots, \sigma_{k+1} \rangle). \]

We therefore obtain a family of totally geodesic submanifolds with
\[ wt(\bar{\sigma}_k) = \frac{1}{2} \text{cod}_{N_{k-1}} N_k. \]

By Lemma 4.1, the codimensions decrease by a factor of at least $\frac{1}{2}$, i.e.
\[ \text{cod}_{N_{k-1}} N_{k+1} \leq \frac{1}{2} \text{cod}_{N_{k-1}} N_k. \]
By iterating this last inequality, it follows that
\[ \text{cod}_{N_k} N_{k+1} \leq \frac{1}{2^k} \text{cod}_{M} N_1 \leq \frac{1}{2^{k+2}} n. \]
In particular, \( \dim N_k \geq \frac{n}{2} \) for each \( k = 1, \ldots, s - 2 \).

**Step 2:** We claim that there exists \( k \in \{0, \ldots, s - 2\} \) such that \( N_k \) is rationally 4–periodic.

We proceed via contradiction and suppose that \( N_k \) is not rationally 4–periodic for any \( k \in \{0, \ldots, s - 2\} \). Let \( \mathbb{Z}_2^{s - 1} \subset T^{s - 1} \) be the subgroup of involutions corresponding to \( C_1 \).

We show that there exists a group \( \mathbb{Z}_2^{s - 2} \) acting effectively on \( N_1 = F(\sigma_1) \). For this we consider the induced \( \mathbb{Z}_2^{s - 1} \)-action on \( N_1 \). If the kernel of this action has rank greater than one, then there is an involution \( \rho \in \mathbb{Z}_2^{s - 1} \) such that \( F(\sigma_1 \cdot \rho) \) and \( F(\rho) \) intersect transversely in \( M \), the intersection being \( N_1 \). Since
\[ 2 \cdot \text{cod}_{M} N_1 \leq \frac{n}{2}, \]
we have that \( M \) is rationally 4–periodic by Theorem \( \text{2.3} \). However, this would be a contradiction. Therefore, there exists \( \mathbb{Z}_2^{s - 2} \) acting effectively on \( N_1 \).

In the same way, we show by an inductive argument that there is a group \( \mathbb{Z}_2^k \) acting effectively on \( N_{s - k - 1} \) for \( k = 1, \ldots, s - 2 \). This is true, since the codimensions are small enough
\[ 2 \cdot \text{cod}_{N_k} N_{k+1} < \frac{n}{2} \leq \dim N_k, \]
so that we can use Theorem \( \text{2.5} \).

It follows that there is \( \mathbb{Z}_2 \) acting effectively on \( N_{s - 2} \), where the \( \mathbb{Z}_2 \)-action comes from the induced \( \mathbb{Z}_2^{s - 1} \)-action on \( N_{s - 2} \). However, since \( d_{\text{max}}(C_1) < 2^{s - 2} \), we note that \( C_1^{\text{res}}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{s - 2}) \) is the zero code by Corollary \( \text{5.2} \). Since \( C_1^{\text{res}}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{s - 2}) \) is the associated code to the \( \mathbb{Z}_2 \)-action on \( N_{s - 2} \), we conclude that \( \mathbb{Z}_2 \) acts trivially. Hence, we established a contradiction. This implies that there is \( k \in \{0, \ldots, s - 2\} \) such that \( N_k \) is rationally 4–periodic.

**Step 3:** We finish the proof by applying successively Lemma \( \text{2.6} \) to our family of totally geodesic submanifolds.

In the first step, we built a family of totally geodesic submanifolds
\[ N_{s - 2} \subseteq N_{s - 3} \subseteq \ldots \subseteq N_1 \subset N_0 = M \]
and showed in a second step that \( N_k \) is rationally 4–periodic. We have
\[ 4 \cdot \text{cod}_{N_{k-1}} N_k \leq \frac{n}{2^{k-1}} \leq \frac{n}{2} \leq \dim N_{k-1} \quad \text{for } k \geq 2. \]
Using Lemma \( \text{2.6} \) we conclude that \( N_{k-1} \) is also rationally 4–periodic. Proceeding inductively, we eventually see that \( N_1 \) is rationally 4–periodic. Moreover, we have \( \text{cod}_{M} N_1 \leq \frac{n}{4} \) and so Lemma \( \text{2.6} \) implies that \( M \) is rationally 4–periodic. This ends the proof.

**Proof of Theorem \( \text{A} \).** The proof is centered around the same ideas as the proof of Theorem \( \text{A} \) and therefore, we leave the preliminaries and the setup to the reader. First and foremost, one has to keep track of the codimensions in order to apply Theorem \( \text{2.5} \) and Lemma \( \text{2.6} \). Finally, we use the fixed point formula and the Connectedness Lemma. We divide the proof in several steps.

**Step 1:** We suppose that the first statement does not hold. Let \( n \geq 12 \) and without loss of generality assume that the action is even. By Theorem \( \text{2.7} \) any involution \( \sigma \in T^q \) has a fixed point component \( F \subset M^q \) such that \( \text{cod}_{M} F \leq \frac{n}{4} \) and
cod_{M} F < 2^{s-1}. In order to see this, we consider two cases. If \( 3 \cdot 2^{s-2} \leq n < 3 \cdot 2^{s-1} \), then

\[
\text{cod}_{M} F \leq 4 \cdot \left\lfloor \frac{n}{12} \right\rfloor \leq \frac{n}{3} < 2^{s-1}.
\]

Moreover, if \( n \geq 3 \cdot 2^{s-1} \), then

\[
\text{cod}_{M} F \leq 4 \cdot (2^{s-3} - 1) < 2^{s-1} \leq \frac{n}{3}.
\]

Using the symmetry condition as in the proof of Theorem A, we construct a family of totally geodesic submanifolds

\[
N_{s-2} \subseteq N_{s-3} \subseteq \ldots \subseteq N_{1} \subset N_{0} = M
\]

with the property that

\[
\text{cod}_{N_{k}} N_{k+1} \leq \frac{1}{2} \text{cod}_{N_{k-1}} N_{k} \quad \text{for } k \geq 1.
\]

Combining this with the fact that \( \text{cod}_{M} N_{1} \leq \frac{n}{3} \), we obtain

(4) \[
\text{cod}_{N_{1}} N_{k+1} \leq \frac{n}{3} \cdot 2^{k}.\]

In particular, \( \text{dim } N_{k} \geq \frac{n}{3} \). It follows that

\[
4 \cdot \text{cod}_{N_{1}} N_{2} \leq \frac{2}{3} n \leq \text{dim } N_{1}
\]

and using (4)

\[
4 \cdot \text{cod}_{N_{1}} N_{k+1} \leq \frac{n}{3} \cdot 2^{k-2} \leq \frac{n}{3} \leq \text{dim } N_{k} \quad \text{for } k \geq 2.
\]

So, the codimensions are small enough to use Theorem 2.6 and Lemma 2.3. Applying the same methods as in the proof before, we deduce that \( N_{1} \) is rationally 4-periodic. Since the action is even, the dimension of \( N^{\nu}_{1} \) is divisible by four and hence, \( N^{\nu}_{1} \) is either a rational cohomology sphere \( S^{n_{1}}, \mathbb{C}P^{n_{1}/2} \) or \( \mathbb{H}P^{n_{1}/4} \).

**Step 2:** By means of the Lefschetz fixed point formula, we rule out the case of \( N_{1} \) being a rational cohomology sphere.

Let \( N_{1} \) be a rational cohomology sphere. By the Intersection Theorem, \( M^{\sigma_{1}} \) consists of the component \( N_{1} \) and components of dimension less than \( \frac{n}{3} \). Therefore, the Euler class of the corresponding normal bundles vanish and the fixed point formula implies the vanishing of the elliptic genus.

We remark that geometrically the vanishing of the Euler class of the normal bundle \( e(\nu_{N_{1}}) \) yields a nowhere vanishing section over the normal bundle. So, we can move the submanifold \( N_{1} \) in normal direction along this section such that the transverse self-intersection \( N_{1} \cap N_{1} \) is empty. So, we conclude

\[
\phi_{0}(M) = \phi_{0}(M^{\sigma_{1}} \circ M^{\sigma_{1}}) = \phi_{0}(N_{1} \cap N_{1}) = 0.
\]

Therefore, we may assume that \( N_{1} \) is not a rational cohomology sphere.

**Step 3:** We use the Connectedness Lemma and its implications to wrap up the proof.

Since \( \text{cod}_{M} N_{1} \leq \frac{n}{3} \), the Connectedness Lemma implies that the inclusion map \( N_{1} \hookrightarrow M \) is at least \( (\frac{n}{4} + 1) \)-connected. Let \( x \in H^{k_{1}}(M; \mathbb{Q}) \) be a generator and let \( k_{1} := n - n_{1} \) be the codimension of \( N_{1} \) in \( M \), which is divisible by four. By Lemma 2.3 there exists a class \( e \in H^{k_{1}}(M; \mathbb{Q}) \) such that multiplication

(5) \[
\cup e : H^{i}(M; \mathbb{Q}) \rightarrow H^{i+k_{1}}(M; \mathbb{Q})
\]

is an isomorphism for \( k_{1} \leq i \leq n - 2k_{1} \).

Since \( N_{1} \) is rationally 4-periodic and the inclusion map is highly connected, we observe that \( e = x^{k_{1}/4} \) up to some constant, possibly zero. Since \( H^{k_{1}}(N_{1}; \mathbb{Q}) \cong \mathbb{Q} \).
$H^{k_1}(M; \mathbb{Q})$ is non-zero and multiplication with $e$ is an isomorphism in this degree, it follows that the class $e$ is non-zero. Hence, we may assume $e = x^{k_1/4}$.

Finally, by (5) there exists an integer $m \in \mathbb{N}$ such that $e^m \neq 0 \in H^{m \cdot k_1}(M; \mathbb{Q})$ with $m \cdot k_1 \geq 2n$. So, $x^{m \cdot k_1/4}$ generates $H^{m \cdot k_1}(M; \mathbb{Q})$. Thus, by the cup product version of Poincaré duality, we have that $x^{n/4}$ is non-zero.

### 4.2. Proof of Theorem C

The proof of Theorem C is a combination of the Structure Theorem 2.4 and the ideas involved in the proof of Theorem B.

**Proof of Theorem C**

We argue by contradiction. Let $n = 4k$ and suppose that the elliptic genus is not constant. So, we may assume that the action is even. Moreover, according to Theorem 2.7, there exists for every involution $\sigma \in T^s$ a connected component $F^\sigma \subset M^\sigma$ with dimension $\dim F^\sigma > 2k$. Choose $l \in \mathbb{N}$ maximal such that $l \leq \frac{k_3}{4}$. We consider now two cases depending on the dimension of $F^\sigma$.

**Case 1:** Let $2k < \dim F^\sigma \leq n - 4l - 4$. We consider the induced $T^s$–action on $F^\sigma$. If this induced action contains a two-dimensional kernel, then there is an involution $\rho \in T^s$ such that $F^\rho$ and $F^\rho \cdot \sigma$ intersect transversely. We note that the intersection is exactly $F^\sigma$ and therefore, $M$ is a rational cohomology sphere by Theorem 2.5. This implies that the elliptic genus is constant. On the other hand, if the induced action has just a one-dimensional kernel, then

$$\frac{\dim F^\sigma}{6} + 1 \leq \frac{n - 4l - 4}{6} + 1 \leq \frac{n}{9} + 1 < s - 1.$$  

So, we observe that the symmetry rank of $F^\sigma$ is big enough to use the Structure Theorem 2.4. It follows that $F^\sigma$ is a rational cohomology sphere. Hence, the Lefschetz fixed point formula implies that the elliptic genus vanishes.

**Case 2:** Let $\dim F^\sigma \geq n - 4l \geq \frac{2n}{9}$. According to case 1, we may assume that this case is valid for every involution $\sigma \in T^s$. However, this is exactly the situation that we encountered in the proof of Theorem B. We conclude that there is an involution $\rho \in T^s$ such that $F^\rho$ is rationally 4–periodic and consequently, a rational cohomology sphere. The theorem follows now from the fixed point formula.

For $n = 4 \pmod{8}$, the pole of the expansion (2) is of half-order, in other words

$$\phi_0(M) \in q^{-k-1/2}\mathbb{Z}[q].$$

If the elliptic genus is constant, this implies that the constant has to be zero and so $\text{sign}(M) = 0$.

**Remark.** These theorems are also true for non-Spin manifolds as long as the equivariant elliptic genus remains rigid. This is, for example, the case for $\text{Spin}^c$ manifolds with $c_1(M)$ a torsion class (see [5]).

### 5. Proof of Theorem D

We come to the proof of Theorem D. The difficulty of Theorem A and B lies in finding totally geodesic submanifolds which intersect transversely. One has to find a clever way to achieve this, for example via coding theory, whereas in Theorem D the symmetry assumption is more explicit and provides a clear picture of the various fixed point sets.

Nonetheless, in order to show that the elliptic genus is strongly rigid, we need to compute the integral cohomology ring. We will see that $M$ is integrally 4–periodic and Theorem D will then follow by a recognition theorem for the quaternionic projective space given in [24].
5.1. Symmetry assumption. First, we fix the symmetry assumption. Throughout this section let \((M^n, g)\) be a Riemannian manifold endowed with an effective and isometric \(T^2\)-action. Let \(\sigma_1, \sigma_2\) and \(\sigma_1 \cdot \sigma_2\) denote the non-trivial involutions in the torus and let \(M^{\sigma_1}\) have a component of codimension four.

5.2. Proof of Theorem \[\text{[13]}\]. The proof of Theorem \[\text{[13]}\] relies on the following proposition, which we will show in the next subsection.

**Proposition 5.1.** Let \((M^n, g)\) be a positively curved Spin manifold with the above symmetry assumption. Then, \(M\) is integrally 4–periodic or the elliptic genus \(\phi(M)\) is strongly rigid.

In light of this proposition, we characterize integrally 4–periodic manifolds. Most considerations can also be found in [24, Ch. VII].

**Lemma 5.2.** Let \(M^{4k}\) be an oriented, simply-connected and integrally 4–periodic manifold with \(k \geq 2\). Then, \(M\) is an integral cohomology sphere, complex projective space or quaternionic projective space.

**Proof.** First, we claim that \(M\) is torsion-free. Since \(M\) is simply-connected, we observe that \(H^1(M; \mathbb{Z})\) and \(H^{4k-1}(M; \mathbb{Z})\) are trivial. Therefore, the odd cohomology groups vanish by 4–periodicity. Moreover, the Universal Coefficient Theorem yields

\[
\text{Tor}(H^1(M; \mathbb{Z})) \cong \text{Tor}(H_{4k-1}(M; \mathbb{Z})).
\]

Thus, the even homology groups are torsion-free and so are the even cohomology groups by Poincaré duality. This gives the claim.

We determine now the multiplicative structure of \(H^*(M; \mathbb{Z})\). Let \(x \in H^4(M; \mathbb{Z})\) be the element inducing 4–periodicity. Unless \(M\) is an integral cohomology sphere, complex projective space or quaternionic projective space, there exists \(y \in H^{4k-4}(M; \mathbb{Z})\) such that \(x \cdot y\) generates \(H^{4k}(M; \mathbb{Z})\). By 4–periodicity, it is clear that \(y = m \cdot x^{k-1}\) for some integer \(m \in \mathbb{Z}\). Therefore, \(m \cdot x^k\) generates \(H^{4k}(M; \mathbb{Z})\). This is only the case, if \(m = \pm 1\) and so, \(x^k\) is a generator of the top cohomology group.

We turn our focus to \(H^2(M; \mathbb{Z})\). Let \(a \in H^2(M; \mathbb{Z})\) be a generator of a \(\mathbb{Z}\)-summand. Again by Poincaré duality and 4–periodicity, there is \(b \in H^2(M; \mathbb{Z})\) such that \(a \cdot b \cdot x^{k-1} = x^k\). Since \(a \cdot b\) is a multiple of \(x\), it follows that \(a \cdot b = x\). We recall that multiplication with \(x\) is an isomorphism in any degree. By considering the compositions \(a \cdot b\) resp. \(b \cdot a\), we conclude that multiplication with \(a\) is also an isomorphism in degrees \(2 \leq i \leq 4k - 4\). In degrees \(i = 0\) and \(i = 4k - 2\), one considers the compositions \(a \cdot x\) resp. \(x \cdot a\) to get that multiplication with \(a\) is an isomorphism. As a consequence, \(M\) is an integral cohomology complex projective space.

If \(H^2(M; \mathbb{Z})\) is trivial, then we obviously get that \(M\) is an integral cohomology quaternionic space. This finishes the proof. \(\square\)

**Proof of Theorem \[\text{[14]}\]** We assume for the moment that Proposition \[\text{[5.3]}\] is true. Furthermore, \(S^1 \subset T^2\) fixes a component of codimension four. The statement holds in dimensions 4 and 8 by [19] and so, let \(n \geq 12\).

Without loss of generality we may assume that the action is of even type. Let \(\sigma_1 \in S^1\) be the involution contained in the relevant circle group. Since every isolated \(\sigma_1\)-fixed point is a torus fixed point, we conclude that \(M^{\sigma_1} = M^{\sigma_1}\). In particular, \(\sigma_1\) contains a connected component of codimension four.

According to Proposition \[\text{[5.3]}\], \(\phi(M)\) is either strongly rigid or \(M\) is an integral cohomology sphere, complex projective space or quaternionic projective space. We investigate these three cases.
Case 1: $M$ is an integral cohomology $S^n$. It is clear that all the Pontryagin classes vanish except possibly the top class. However, the signature is zero and therefore, the $L$-genus vanishes. We conclude that the top Pontryagin class also vanishes and so does the elliptic genus.

Case 2: $M$ is an integral cohomology $CP^n/2$. We note that $M$ is, in particular, a $\mathbb{Z}_2$-cohomology $CP^n/2$. Therefore, the Stiefel-Whitney classes of $M$ are standard by the Wu formula. Since the dimension of $M$ is divisible by four, we obtain that $M$ is not Spin (see [20]), which yields a contradiction.

Case 3: $M$ is an integral cohomology $\mathbb{H}P^n/4$. Let $N_1 \subset M^{\sigma_1}$ denote the codimension four component. Since $n \geq 12$, we see that $N_1$ is a $\mathbb{Z}_2$-cohomology $\mathbb{H}P^{n/4-1}$ (see [4]). By [24, Lemma 10.2.], we conclude that $M$ is homeomorphic to $\mathbb{H}P^n/4$. Since the rational Pontryagin classes are invariant under homeomorphisms by the work of Novikov, it follows that $\phi(M) = \phi(\mathbb{H}P^n/4)$ is constant. □

5.3. Proof of Proposition 5.1 We start this subsection by proving a rational version of Proposition 5.1. Parts of the proof are similar to [7, Theorem 17] and [24, Proposition 7.3].

Lemma 5.3. Let $(M^n, g)$ be a positively curved Spin manifold with the above symmetry assumption. Then, $M$ is rationally 4–periodic or the elliptic genus $\phi(M)$ is strongly rigid.

Proof. The specific symmetry assumption enables us to reduce the fixed point configurations to three cases, which we will study independently.

The lemma is true in dimensions 4 and 8 by [19], so let $n = 4k \geq 12$. The proof goes by contradiction. We suppose that the elliptic genus is not constant. Therefore, the action has to be of even type and according to Theorem 2.5, the fixed point set of any involution contains a component of codimension less than 2k.

Following the Intersection Theorem this component is unique. Thus, we get the following possible fixed point configurations

$$M^{\sigma_1} = N_1^{4k-4} \cup \text{pts},$$

$$M^{\sigma_2} = N_2^{4k-4l} \cup X^{4l-4} \cup X^{4l-8} \cup \ldots \cup \text{pts} \quad \text{for} \quad l < \frac{k}{2}.$$

We also fix $N_{12} \subset M^{\sigma_1 \sigma_2}$ to be the component of $M^{\sigma_1 \sigma_2}$ of largest dimension.

Since $N_1$ and $N_2$ are two high-dimensional, totally geodesic submanifolds, they intersect. Take a point $p \in (N_1 \cap N_2)^T$. By considering the $\mathbb{Z}_2 \times \mathbb{Z}_2$–isotropy representation at $p$, there occur exactly three possibilities for the intersection.

Case 1: $N_1$ and $N_2$ intersect transversely. We need to verify two cases here. First, let $\dim N_2 \geq 2k + 4$, then

$$2 \cdot \text{cod}_M N_1 + 2 \cdot \text{cod}_M N_2 \leq n$$

and we use Theorem 2.5 to get that $M$ is rationally 4–periodic.

If the special case $\dim N_2 = 2k + 2$ occurs, then we proceed by a different argument. We recall that $N_1 \cap N_2 \subset M^{\sigma_1 \sigma_2}$ and that $\dim N_1 \cap N_2 = 2k - 2$, since $N_1$ and $N_2$ intersect transversely. Since the action is even and using the Intersection Theorem, we observe that $\dim M^{\sigma_1 \sigma_2} = 2k - 2$. Thus, the elliptic genus vanishes, since the codimension is greater than 2k.

Case 2: $N_2$ is a subset of $N_1$. Since $N_2$ is contained in $N_1$, we have that $\sigma_1$ and $\sigma_1 \cdot \sigma_2$ act trivially on $N_2$. Therefore, $N_2$ is the transverse intersection of $N_1$ with $N_{12}$. Hence, $M$ is rationally 4–periodic.
Case 3: $N_1 \cap N_2$ has codimension two in $N_1$. If $N_1 \cap N_2$ has codimension two in $N_2$, then $\dim N_1 = \dim N_2$ and so the possible fixed point configuration for $M^{\sigma_1 \sigma_2}$ resembles the one of $M^{\sigma_2}$.

We claim that $M^{\sigma_1}$ is connected. Indeed, $M^{\sigma_1}$ consists of $N_1$ and a possibly empty set of isolated fixed points. Let $q$ be such an isolated $\sigma_1$-fixed point. In fact, $q$ is fixed by the torus and we consider the induced linear code at $q \in M^T$. The weights are given by the fixed point configurations

\[
\begin{align*}
wt(\tilde{\sigma}_1) &= 2k, \\
wt(\tilde{\sigma}_2) &\in \{2l, 2k - 2l + 2, 2k - 2l + 4, \ldots, 2k\}, \\
wt(\tilde{\sigma}_1 + \tilde{\sigma}_2) &\in \{2l, 2k - 2l + 2, 2k - 2l + 4, \ldots, 2k\}.
\end{align*}
\]

On the other hand, we obtain

\[
wt(\tilde{\sigma}_1 + \tilde{\sigma}_2) = 2k - wt(\tilde{\sigma}_2) \in \{2k - 2l, 2l - 2, 2l - 4, \ldots, 0\}.
\]

This yields $2k - 2l = 2l$ and so $\text{codim}_M N_{12} = 2k$, which is a contradiction. We conclude that $M^{\sigma_1}$ is connected.

We remark that the odd Betti numbers of $N_2$ vanish. This follows directly from the Connectedness Lemma for the inclusion $N_1 \cap N_2 \hookrightarrow N_2$ and the fact that $N_2$ is simply-connected.

We now use the fixed point formula for the Euler characteristic. Since $M^{\sigma_1}$ is connected and $\sigma_1$ comes from a torus action, we have that

\[
\chi(M) = \chi(M^{\sigma_1}).
\]

Using the Connectedness Lemma for $M^{\sigma_1} \hookrightarrow M$, we obtain that $b_i(M) = b_i(M^{\sigma_1})$ for $i \leq 4k - 7$. A comparison of the Euler characteristics of $M$ and $M^{\sigma_1}$ gives

\[
-b_3(M) + b_4(M) - b_5(M) + b_6(M) \leq 0.
\]

Moreover, the inclusion $N_2 \hookrightarrow M$ is at least 5–connected and so, the induced pullback map $H^*(M; \mathbb{Z}) \to H^*(N_2; \mathbb{Z})$ is a monomorphism for $* \leq 5$. Since $b_{\text{odd}}(N_2) = 0$, we conclude that $b_3(M) = 0$ and $b_5(M) = 0$ as well as $b_4(M) = 0$ and $b_6(M) = 0$.

Finally, the Connectedness Lemma for $M^{\sigma_1} \hookrightarrow M$ induces an isomorphism

\[
H^i(M; \mathbb{Z}) \to H^{i+4}(M; \mathbb{Z})
\]

for $4 \leq i \leq 4k - 8$. Hence, all the Pontryagin classes of $M$ vanish except possibly the top class $p_k(M)$. However, the $A$–genus vanishes on positively curved Spin manifolds and therefore, $p_k(M) \cdot [M] = 0$. So, all the Pontryagin numbers are zero and $\phi(M) = 0$. We are done with all cases and finished the proof.

As a consequence of this lemma, we can compute the signature in the case of $n = 4 \pmod{8}$. We state the following observation, which might be of independent interest.

**Corollary 5.4.** Let $(M^n, g)$ be a positively curved Spin manifold with the above symmetry assumption. If $n = 4 \pmod{8}$, then $\text{sign}(M) = 0$.

**Proof.** According to Lemma 5.3 we have that $M$ is rationally 4–periodic or $\phi(M)$ is constant. Let $n = 8k + 4$ and let us consider the two cases.

**Case 1:** $M$ is rationally 4–periodic. If $M$ is rationally 4–periodic, then we have $b_{4k+2}(M) \leq 1$ and so $|\text{sign}(M)| \leq 1$. However, a generalized version of Rohlin’s theorem proven in [21] asserts that any Spin manifold of dimension $8k + 4$ has signature divisible by 16. The statement follows immediately.

**Case 2:** $\phi(M)$ is strongly rigid. We note that the pole of the expansion [2] is of half-order. Since the elliptic genus is constant, we deduce that $\text{sign}(M) = 0$. \hfill $\square$
We continue our discussion within the setting of the proof of Lemma 5.3 and analyze the integral cohomology ring. In general, one cannot detect the integral cohomology from the rational cohomology. Yet the Connectedness Lemma enables us to compute the integral cohomology ring.

**Lemma 5.5.** Let $M^{4k}$ be rationally 4-periodic as in the proof of Lemma 5.3. Then, in fact, $M$ is integrally 4-periodic.

**Proof.** We use the same notations as before. By considering the proof of Lemma 5.3, we may assume that $N_1$ and $N_2$ intersect transversely and $\dim N_2 = 4k - 4l \geq 2k + 4$.

The Connectedness Lemma implies that $N_2$ is integrally 4-periodic. Recall that the inclusion $N_1 \hookrightarrow M$ induces an isomorphism

$$\cup e : H^i(M; \mathbb{Z}) \to H^{i+4}(M; \mathbb{Z})$$

for $4 \leq i \leq 4k - 8$.

Moreover, the inclusion map $N_2 \hookrightarrow M$ is at least 9-connected. Using similar methods as in the proof of Lemma 5.2, it is not difficult to show that $e \in H^4(M; \mathbb{Z})$ induces 4-periodicity on $H^*(M; \mathbb{Z})$.

□

This completes the proofs of Proposition 5.1 and Theorem 10.

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