Distributed adaptive control for optimal tracking of uncertain interconnected dynamical systems

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1. Introduction

Adaptive control is a methodology which can deal with uncertain systems because it can tolerate large parametric uncertainties to ensure the desired tracking performance. Thus it has been used in many applications [1]. The distinctive feature of adaptive control is that the knowledge of plant parameters is not required and can still achieve the boundedness of signals, system stability, and/or asymptotic output/state tracking. In particular, model reference adaptive control such that the closed loop system is matched with a reference model has been extensively studied and its theory and design techniques have been developed for decades [2].

Since several uncertainties should be treated in large-scale systems composed of several subsystems and interconnections, decentralized adaptive control was proposed in [3, 4], where bounded disturbances and nonlinear uncertainties are considered. After that, such a decentralized adaptive scheme was further investigated in [5, 6], where it is proved that the output of the system can asymptotically track the output of the reference model if all decentralized controllers share their prior information. On the other hand, the paper [7] provided a distributed control, where each controller which applies to each subsystem uses the information about her neighbours. Performance guarantee is also an issue even in decentralized adaptive control. In fact, the paper [7] provided an error bound of tracking based on a Lyapunov solution of a given reference model. This technique has further been investigated in the context of several types of uncertain systems in [8–10]. In this regard, when we introduce a performance index and consider an optimal tracking, it is natural to explore a performance guarantee with respect to the index. However, an available result does not exist. For example, the paper [11] considered an adaptive optimal control for large-scale systems, where an asymptotic optimality was investigated, while any error bound of tracking is not provided. It is important to evaluate the performance degradation caused by adaptation in terms of the performance index of the reference model which achieves optimal tracking. Then, an appropriate selection of control input which incorporates the evaluation of performance degradation is needed so that the tracking error is minimized for the nominal system and an explicit error bound of the optimal tracking is obtained.

In this paper, we consider a class of large-scale dynamical systems with uncertain interconnection between the subsystems. We investigate a distributed control, where each controller which applies to each subsystem uses the information about her neighbours. We first introduce a performance index and consider an optimal tracking problem for each nominal subsystem. We construct a reference model as optimal tracking for the nominal subsystem, where a Riccati solution of an LQ regulator determines the model. Then we develop a distributed adaptive tracking control law for the interconnected uncertain dynamical system, where the Riccati solution for optimal tracking is used in the update rule of the adaptive gain. We show that the proposed adaptive control law achieves the desirable optimal tracking asymptotically as well as the boundedness of all signals. We also establish an explicit error bound.
with respect to the nominal optimal tracking, where a role of the learning rate of the update rule is clarified. Numerical examples illustrate that the theoretical results developed in this paper are useful.

A preliminary version of this paper was presented at a conference [12], where an adaptive tracking of a single plant is considered for a step-type reference signal. On the other hand, the present paper deals with a distributed adaptive tracking of an interconnected system for a general reference signal, which clarifies a possible performance guarantee for a type of adaptive control law.

This paper is organized as follows. We state notations and definition in Section 2. We describe the statement of problem in Section 3 and select the desired reference model as optimal tracking in Section 4. We discuss the distributed adaptive control scheme for uncertain systems with interconnections in Section 5 and evaluate the performance degradation to the original optimal tracking in Section 6. Lastly, in Section 7, the effectiveness of the proposed scheme is demonstrated to show results of the theoretical findings and the concluding remarks are discussed in Section 8.

2. Notations and definitions

In this paper, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \)-dimensional real column vectors, and \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices. In addition, we write \( Q^T \) for the transpose of a real matrix \( Q \), \( R^{-1} \) for the inverse of a matrix \( R \), rank \( S \) for the rank of a matrix \( S \), and tr \( U \) for the trace of a square matrix \( U \).

3. Problem formulation

Let us consider an interconnected system consisting of \( N \) uncertain dynamical subsystems with uncertain interconnection. The topology of the interconnection is expressed by a graph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{1, 2, \ldots, N\} \) is the set of the nodes each of which corresponds to a subsystem, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the set of the edges which represents the interaction among the subsystems. The set of neighbour of the \( i \)th subsystem is denoted by \( \mathcal{N}_i = \{ j \in \mathcal{V} | (i, j) \in \mathcal{E} \} \), where the \( i \)th subsystem is affected by the \( j \)th subsystem through uncertain interconnection if \( j \in \mathcal{N}_i \). Here we assume that \( G \) is known and time invariant.

With the graph \( G \), we describe the dynamics of the \( i \)th subsystem as

\[
\begin{align*}
\dot{x}_i(t) &= A_i x_i(t) + B_i [u_i(t) + \sum_{j \in \mathcal{N}_i} \Delta_{ij}(x_j(t))] \\
&\quad + \Delta_{il}(x_l(t)), \quad x_i(0) = x_{i0}, \\
y_i(t) &= C_i x_i(t),
\end{align*}
\]

where \( x_i(t) \in \mathbb{R}^{n_i} \) is the state, \( u_i(t) \in \mathbb{R}^{m_i} \) is the control input restricted to the class of admissible controls consisting of measurable functions, \( y_i(t) \in \mathbb{R}^{m_i} \) is the controlled output. Throughout the paper, the subscripts \( i \) and \( j \) correspond to the \( i \)th and \( j \)th subsystems, respectively, i.e. \( i \in \mathcal{V} \) and \( j \in \mathcal{N}_i \subset \mathcal{V} \). The matrices \( A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}, \) and \( C_i \in \mathbb{R}^{m_i \times n_i} \) represent the nominal part of the subsystem, where the pair \((A_i, B_i)\) is controllable and the pair \((C_i, A_i)\) is observable. On the other hand, the matrix \( \Lambda_i \in \mathbb{R}^{m_i \times m_i} \) and the vector-valued function \( \Delta_{ij} : \mathbb{R}^{n_j} \to \mathbb{R}^{m_i} \) represent the uncertain part of the subsystem as well as the uncertain interaction among the subsystems. That is, \( \Delta_{ij}(x_j(t)) \) expresses the uncertain part of the \( i \)th subsystem itself, while \( \Delta_{ij}(x_j(t)) \ (j \neq i) \) expresses the uncertain influence from the \( j \)th subsystem to the \( i \)th subsystem.

In this regard, we introduce the following assumption for \( \Lambda_i \) and \( \Delta_{ij}(x_j(t)) \).

**Assumption 3.1**: The control effectiveness \( \Lambda_i \) of the \( i \)th subsystem is an unknown symmetric and positive definite matrix. The state-dependent uncertainty \( \Delta_{ij}(x_j) \) of the \( i \)th subsystem is linearly parameterized as

\[
\Delta_{ij}(x_j) = F_{ij} \alpha_{ij}(x_j),
\]

for all \( j \in \mathcal{N}_i \cup \{i\} \), where \( F_{ij} \in \mathbb{R}^{m_i \times n_j} \) is an unknown weight matrix and \( \alpha_{ij} : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i} \) is a given basis function.

For each subsystem (1), we define the corresponding nominal system as

\[
\begin{align*}
\dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \quad x_i(0) = x_{i0}, \\
y_i(t) &= C_i x_i(t).
\end{align*}
\]

That is, when \( \Lambda_i = I \) and \( \Delta_{ij}(x_j(t)) \equiv 0 \) for all \( j \in \mathcal{N}_i \cup \{i\} \), the \( i \)th subsystem (1) takes its nominal behaviour.

In this paper, we consider a reference signal \( r_i(t) \in \mathbb{R}^{m_i} \) generated by

\[
\begin{align*}
\dot{x}_{r_i}(t) &= A_{ri} x_{r_i}(t), \quad x_{r_i}(0) = x_{r_i0}, \\
r_i(t) &= C_{ri} x_{r_i}(t),
\end{align*}
\]

where \( x_{r_i}(t) \in \mathbb{R}^{n_i} \), the eigenvalues of \( A_{ri} \) are on the imaginary axis and all distinct from one another, and the pair \((C_{ri}, A_{ri})\) is observable. That is, \( x_{r_i}(t) \) and thus \( r_i(t) \) of (3) are bounded signals which are represented as a linear combination of a constant signal and sinusoidal signals having several frequencies and phases. The boundedness of \( x_{r_i}(t) \) will be used for establishing the zero steady-state tracking error. We define the initial time \( t = 0 \) at the time when the reference signal is applied. The initial state \( x_{r_i0} \) of (3) is arbitrary.

It is known that the controlled output \( y_i(t) \) of the nominal subsystem (2) can follow any reference signal
In this section, we select suitable reference models for our distributed adaptive control. To this end, we consider the nominal system (2) and revisit a standard optimal tracking control law for the nominal system (9), the resultant closed loop system is stable, and thus the optimal control law which minimizes $J_i$ with respect to (9) is given by

$$\tilde{u}_i(t) = K_i\tilde{x}_i(t),$$  \hspace{1cm} (11)

where

$$K_i = -R_i^{-1}B_i^TP_i$$  \hspace{1cm} (12)

and $P_i \in \mathbb{R}^{n_i \times n_i}$ is a symmetric and positive definite solution of the Riccati equation

$$A_i^TP_i + P_iA_i - P_iB_iR_i^{-1}B_i^TP_i + C_i^TQ_iC_i = 0.$$  \hspace{1cm} (13)

When the control law (11) is applied to the variation system (9), the resultant closed loop system is stable, and thus $\tilde{x}_i(t) \rightarrow 0$, $e_i(t) \rightarrow 0$ as $t \rightarrow \infty$, where the minimum value of $J_i$ is given by

$$\min_{\tilde{u}_i} J_i = \tilde{x}_{i0}^T \tilde{P}_{i0}\tilde{x}_{i0}.$$  \hspace{1cm} (14)

Using (6) and (8), we rewrite the control law (11) as

$$u_i(t) = K_i x_i(t) + H_i x_i(t),$$  \hspace{1cm} (15)

for the nominal system (2), where

$$H_i = -K_i L_{xi} + L_{ui}.$$  \hspace{1cm} (16)

That is, the optimal tracking control law for the nominal system (2) is composed of a feedback from the state $x_i(t)$ and a feedforward from the state $x_i(t)$. The resultant control system with (2) and (15) is described by

$$\dot{x}_i(t) = (A_i + B_i K_i) x_i(t) + B_i H_i x_i(t), \quad x_i(0) = x_{i0},$$  

$$y_i(t) = C_i x_i(t).$$  \hspace{1cm} (17)

When there is no uncertainty in the system (1), the optimal tracking (17) represents the best achievable behaviour of each subsystem. We therefore consider a distributed adaptive control framework for (1) which asymptotically realizes the optimal tracking (17). To this end, we employ

$$\dot{x}_{mi}(t) = (A_i + B_i K_i) x_{mi}(t) + B_i H_i x_i(t),$$

$$x_{mi}(0) = x_{mi0},$$

$$y_{mi}(t) = C_i x_{mi}(t).$$  \hspace{1cm} (18)

as the reference model for each subsystem, where $x_{mi}(t) \in \mathbb{R}^{n_i}$ is the state, $K_i$ is given by (12) based on the performance index $J_i$ of (10), and $H_i$ is given by (16) with this $K_i$.
5. Distributed adaptive control scheme

Let us go back to the uncertain interconnected system (1). According to the selected reference model (18), we rewrite each subsystem (1) as

\[ \dot{x}_i(t) = (A_i + B_i K_i)x_i(t) + B_i H_i x_{ri}(t) + B_i A_i [ u_i(t) + \delta_i(t) ] \]
\[ y_i(t) = C_i x_i(t), \]

where we define

\[ \delta_i(t) = -A_i^{-1} \left[ K_i x_i(t) + H_i x_{ri}(t) - \sum_{j \in \mathcal{N}_i} \Delta_{ij}(x_j(t)) - \Delta_{ii}(x_i(t)) \right] \]

\[ = -W_i \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \]

with \( x_{\mathcal{N}_i} \) used in (5). In fact, from Assumption 3.1, the signal \( \delta_i(t) \) must be linearly parameterized by using an unknown weight \( W_i \in \mathbb{R}^{m_i \times \tilde{q}_i} \) and the corresponding basis function \( \sigma_i : \mathbb{R}^{m_i + \tilde{n}_i} \rightarrow \mathbb{R}^{\tilde{n}_i} \) which contains \( x_i(t), x_{ri}(t), \) and \( \sigma_i(x_i(t)) (j \in \mathcal{N}_i \cup \{i\}) \), where \( \mathcal{N}_i = \{ j \in \mathcal{N}_i \cup \{i\} : s_{ij} \geq 0 \} \).

Then we introduce a distributed adaptive control law

\[ u_i(t) = \hat{W}_i(t) \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \]

where we define the update rule of the adaptive gain \( \hat{W}_i(t) \in \mathbb{R}^{m_i \times \tilde{q}_i} \) as

\[ \dot{\hat{W}}_i(t) = -\eta_i \bar{B}_i^T P_i (x_i(t) - x_{mi}(t)) \cdot \sigma_i^T(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \quad \hat{W}_i(0) = \hat{W}_{i0} \]  \hspace{1cm} \text{(21)}

The signal \( x_{mi}(t) \) of (21) is given by (18). The learning rate \( \eta_i \) is a positive real number and \( P_i \) is the symmetric and positive definite solution of the Riccati equation (13).

We see that the control law (20) with (21) is distributed indeed. In fact, the basis \( \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \) is distributed. Thus, if the interconnection of the overall system (1) is sparse, we enjoy a sparse structure of the control, where the control law (20) with (21) utilizes the knowledge of the \( i \)th subsystem itself and its surrounding neighbours. See also the numerical example in Section 7 for further details.

Now, let us define the errors from the ideal case as

\[ x_{ei}(t) = x_i(t) - x_{mi}(t), \quad x_{ei0} = x_{i0} - x_{mi0}, \]
\[ y_{ei}(t) = y_i(t) - y_{mi}(t), \]
\[ W_{ei}(t) = \hat{W}_i(t) - W_i, \quad W_{ei0} = \hat{W}_{i0} - W_i, \]

where \( x_{mi}(t) \) and \( y_{mi}(t) \) are defined in (18). With (18)–(21), we have

\[ \dot{x}_{ei}(t) = (A_i + B_i K_i)x_{ei}(t) + B_i A_i [ u_i(t) + \delta_i(t) ], \]
\[ y_{ei}(t) = C_i x_{ei}(t), \]
\[ W_{ei}(t) = -\eta_i \bar{B}_i^T P_i x_{ei}(t) \cdot \sigma_i^T(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)), \quad x_{ei0} = x_{ei0}, \]
\[ y_{ei}(t) = C_i x_{ei0}, \]
\[ W_{ei0} = W_{ei0}, \]

which describes the error dynamics from the reference model (18).

The next theorem presents the result of this section.

**Theorem 5.1:** Consider the uncertain interconnected dynamical system described by (1) subject to Assumption 3.1. Consider, in addition, the reference model given by (18) and the distributed adaptive control given by (20) and (21). Then, all of the solutions \( (x_{ei}(t), W_{ei}(t)) \) \( (i = 1, 2, \ldots, N) \) given by (22) and (23) are bounded. Furthermore, all of the tracking errors \( y_{ei}(t) \) \( (i = 1, 2, \ldots, N) \) satisfy

\[ \lim_{t \to \infty} y_{ei}(t) = 0 \]

for any \( (x_{ei}(0), W_{ei}(0)) \) \( (i = 1, 2, \ldots, N) \).

**Proof:** Consider a candidate of Lyapunov function

\[ V = \sum_{i=1}^{N} V_i(x_{ei}, W_{ei}), \]

where \( \eta_i \) and \( P_i \) are taken from (21) and \( A_i \) of (1) satisfies Assumption 3.1, which means that \( P_i = P_i^T > 0 \) and \( A_i = A_i^{-1} \) > 0. Thus the function \( V_i(x_{ei}, W_{ei}) \) is a continuously differentiable function such that \( V_i(0,0) = 0 \) and \( V_i(x_{ei}, W_{ei}) > 0 \) for all \( (x_{ei}, W_{ei}) \neq (0,0) \), which implies that \( V \) is also positive definite.

Differentiating each component \( V_i(x_{ei}, W_{ei}) \) of this candidate \( V \) along the trajectories of (22) and (23), we have

\[ \dot{V}_i(x_{ei}(t), W_{ei}(t)) \]
\[ = x_{ei}^T(t) P_i x_{ei}(t) + x_{ei}^T(t) P_i \dot{x}_{ei}(t) + \frac{2}{\eta_i} \text{tr} \hat{W}_i(t) A_i W_{ei}(t) \]
\[ = x_{ei}^T(t) \left( (A_i + B_i K_i) P_i + P_i (A_i + B_i K_i) \right) x_{ei}(t) \]
\[ + 2x_{ei}^T(t) P_i B_i A_i W_{ei}(t) \sigma_i(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) \]
\[ - 2\text{tr} \left( \sigma_i^T(x_i(t), x_{\mathcal{N}_i}(t), x_{ri}(t)) x_{ei}(t) \right) P_i B_i A_i W_{ei}(t) \]
\[ = x_{ei}^T(t) \left( (A_i + B_i K_i) P_i + P_i (A_i + B_i K_i) \right) x_{ei}(t) \]
\[ - x_{ei}^T(t) \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) x_{ei}(t), \]

where we use the fact that Riccati equation (13) can be rewritten as

\[ (A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) \]
Note also that \( v_i(t) \) is represented by the signals \( v_i(t) = \sum_{j=1}^{N} v_{ij}(t) \) given by (22) and (23) are bounded. We therefore see that
\[
\sum_{i=1}^{N} v_i(t) < \infty
\]
holds true for all \( t \geq 0 \). Hence all of the solutions \( (x_i(t), W_i(t)) \) given by (22) and (23) are bounded.

Now, let us recall a standard fact
\[
\int_{0}^{t} v_i(x_i(t), W_i(t)) \, dt = V_i(x_i(t), W_i(t)) - V_i(x_{i0}, W_{i0}),
\]
which holds for all \( t \geq 0 \). With (24), we have
\[
\int_{0}^{t} v_i(x_i(t), W_i(t)) \, dt \geq V_i(x_{i0}, W_{i0})
\]
for all \( t \geq 0 \). We therefore see that
\[
\int_{0}^{\infty} \left( \sum_{i=1}^{N} v_i(t) \right) \, dt = \sum_{i=1}^{N} V_i(x_{i0}, W_{i0}).
\]
Furthermore, since \( y_i(t) = C_i x_i(t) + \dot{x}_i(t) \) of (22) is represented by the signals \( x_i(t), W_i(t), x_i(t), x_{i'}(t), \) and \( x_{i'}(t) \), we can see that
\[
\sup_{t \geq 0} \left( \sum_{i=1}^{N} y_i(t) \right) \leq \sum_{i=1}^{N} \sup_{t \geq 0} y_i(t) = \infty.
\]
In fact, \( x_i(t) \) and \( W_i(t) \) are bounded as we have proved above, while \( x_{i'}(t) \) of (3) is bounded as we have assumed. The boundedness of \( x_i(t) \) and \( x_{i'}(t) \) follows the boundedness of \( x_i(t) \) and \( W_{i'}(t) \) of (18). Using (26), (27), and [15] with \( Q_i = Q_i^T > 0 \), we conclude that all of the tracking errors \( y_i(t) \) satisfy
\[
\lim_{t \to \infty} y_i(t) = 0
\]
for any \( (x_i(t), W_i(t)) \) of (22) and (23) are bounded. We therefore see that
\[
\int_{0}^{\infty} V_i(x_i(t), W_i(t)) \, dt = \sum_{i=1}^{N} V_i(x_{i0}, W_{i0})
\]
holds true for all \( t \geq 0 \). Hence all of the solutions \( (x_i(t), W_i(t)) \) given by (22) and (23) are bounded.

Now, let us recall a standard fact
\[
\int_{0}^{t} y_i(t) Q y_i(t) \, dt \leq -\int_{0}^{t} \dot{V}_i(x_i(t), W_i(t)) \, dt
\]
which holds for all \( t \geq 0 \). With (24), we have
\[
\int_{0}^{t} y_i(t) Q y_i(t) \, dt \leq \int_{0}^{t} \dot{V}_i(x_i(t), W_i(t)) \, dt + V_i(x_{i0}, W_{i0})
\]
for all \( t \geq 0 \). We therefore see that
\[
\int_{0}^{\infty} \left( \sum_{i=1}^{N} y_i(t) \right) Q y_i(t) \, dt = \sum_{i=1}^{N} V_i(x_{i0}, W_{i0}).
\]
Furthermore, since \( y_i(t) = C_i x_i(t) + \dot{x}_i(t) \) of (22) is represented by the signals \( x_i(t), W_i(t), x_i(t), x_{i'}(t), \) and \( x_{i'}(t) \), we can see that
\[
\sup_{t \geq 0} \left( \sum_{i=1}^{N} y_i(t) \right) \leq \sum_{i=1}^{N} \sup_{t \geq 0} y_i(t) = \infty.
\]
In fact, \( x_i(t) \) and \( W_i(t) \) are bounded as we have proved above, while \( x_{i'}(t) \) of (3) is bounded as we have assumed. The boundedness of \( x_i(t) \) and \( x_{i'}(t) \) follows

6. Performance evaluation
Since the distributed adaptive control given by (20) and (21) employs the optimal tracking system (18) as the reference model, one of our interests should be to evaluate the performance degradation from the optimal response.

To this end, let us rewrite the minimum value (14) of the performance index (10) for the nominal system as
\[
\min_{\tilde{u}} J_i = \tilde{x}_{i0}^T P \tilde{x}_{i0} = \int_{0}^{\infty} \tilde{x}_i(t) \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) \tilde{x}_i(t) \, dt,
\]
where \( K_i \) is the optimal gain (12). For the reference model (18), this means that
\[
J_{mi} = \int_{0}^{\infty} (x_{mi}(t) - x_i(t))^T \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) (x_{mi}(t) - x_i(t)) \, dt
\]
\[
= (x_{mi0} - x_{i0})^T P (x_{mi0} - x_{i0}).
\]
Referring the above, we define a performance index for the adaptive control as
\[
J_{ci} = \int_{0}^{\infty} (x_i(t) - x_{mi}(t))^T \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) (x_i(t) - x_{mi}(t)) \, dt
\]
This index (29) is reasonable for evaluating the degradation caused by adaptation since its weight \( (C_i^T Q_i C_i + K_i^T R_i K_i) \) coincides with that of (28). In fact, when we evaluate the tracking error from the preferable state \( x_{i0}(t) \) which achieves \( y_{i0}(t) = r_i(t) \) rather than the tracking error from the reference model state \( x_{mi}(t) \), if we introduce the performance index
\[
J_{ci} = \int_{0}^{\infty} (x_i(t) - x_{i0}(t))^T \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) (x_i(t) - x_{i0}(t)) \, dt,
\]
we immediately obtain its evaluation as

\[ J_{ei} \leq \left( \frac{J_{mi}}{2} + J_{ei}^{1/2} \right)^2 \]

by employing the triangle inequality. Notice also that the value \( J_{ei} \) becomes 0 (i.e. the value \( J_{mi} \) becomes \( J_{mi} \)) if the perfect tracking \( x_i(t) = x_{mi}(t) \) is achieved.

For this index (29), we have the following result.

**Theorem 6.1:** Consider the uncertain interconnected dynamical system described by (1) subject to Assumption 3.1. Consider, in addition, the reference model given by (18) and the distributed adaptive control given by (20) and (21). Then, all of the indices (29) \((i = 1, 2, \ldots, N)\) are bounded as

\[ J_{ei} \leq x^T_{e0} P_i x_{e0} + \frac{1}{\eta_i} \text{tr} W^T_{e0} \Lambda_i W_{e0} \tag{30} \]

**Proof:** In the proof of Theorem 5.1, we have established (24) and (25), which implies that

\[ \int_0^t x^T_{ei}(\tau) \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) x_{ei}(\tau) \, d\tau \leq V(x_{e0}, W_{e0}) \]

holds true for all \( t \geq 0 \). Thus it turns out that

\[ J_{ei} = \int_0^{t_i} x^T_{ei}(t) \left( C_i^T Q_i C_i + K_i^T R_i K_i \right) x_{ei}(t) \, dt \]

\[ \leq V(x_{e0}, W_{e0}) = x^T_{e0} P_i x_{e0} + \frac{1}{\eta_i} \text{tr} W^T_{e0} \Lambda_i W_{e0} \]

which establishes the bound (30). \( \blacksquare \)

The upper bound given by this theorem guarantees the transient performance of the proposed adaptive control. It shows that the performance of the distributed adaptive control applied to the uncertain interconnected dynamical system captured by \( x_{ei}(t) \) cannot
be more than the right hand side of (30) at each subsystem. In addition, if we make the learning rate $\eta_i$ large, the transient performance will be better, which will be confirmed in the numerical example in the following section.

7. Numerical example

Let us consider a mass-spring-damper system having $N$ masses in one line shown in Figure 1. Each mass $m_i$ $(1 < i < N)$ is possibly connected with its neighbours $m_{i-1}$ and $m_{i+1}$ by springs $k_{i-1,i}$, $k_{i,i+1}$ and dampers $c_{i-1,i}$, $c_{i,i+1}$, that is,

$$m_i \ddot{q}_i(t) = -k_{i-1,i} \left( \dot{q}_i(t) - \dot{q}_{i-1}(t) \right) - c_{i-1,i} \left( \dot{q}_i(t) - \dot{q}_{i-1}(t) \right) - k_{i,i+1} \left( \dot{q}_i(t) - \dot{q}_{i+1}(t) \right) - c_{i,i+1} \left( \dot{q}_i(t) - \dot{q}_{i+1}(t) \right) + u_i(t),$$

where $q_i(t) \in \mathbb{R}$ is the position of the mass $m_i$ to be controlled, and $\dot{q}_i(t)$ and $\ddot{q}_i(t)$ are its velocity and acceleration, respectively. The force $u_i(t) \in \mathbb{R}$ is applied to the mass $m_i$ as the control input, while we assume that all physical parameters $m_i > 0$, $k_{i-1,i} \geq 0$, $k_{i,i+1} \geq 0$, $c_{i-1,i} \geq 0$, and $c_{i,i+1} \geq 0$ are unknown. The cases $i = 1$ and $i = N$ having one neighbour are not explicitly stated in this section, though it is clear that they can be described in a similar way.

The mass-spring-damper system (31) stated above is consistent with the system description (1). In fact, we can rewrite (31) as

$$\dot{x}_i(t) = A_i x_i(t) + B_i [\Lambda_i u_i(t) + \sum_{j=i-1}^{i+1} \Delta_{ij}(x_j(t))],$$

$$y_i(t) = C_i x_i(t),$$

where we define

$$x_i(t) = \begin{bmatrix} q_i(t) \\ \dot{q}_i(t) \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \Lambda_i = \frac{1}{m_i},$$

$$\Delta_{i,i-1}(x_{i-1}(t)) = \begin{bmatrix} k_{i-1,i} \/ m_i \\ c_{i-1,i} / m_i \end{bmatrix} x_{i-1}(t),$$

$$\Delta_{i,i}(x_i(t)) = \begin{bmatrix} -k_{i-1,i} / m_i + k_{i,i+1} / m_i \\ -c_{i-1,i} / m_i + c_{i,i+1} / m_i \end{bmatrix} x_i(t),$$

$$\Delta_{i,i+1}(x_{i+1}(t)) = \begin{bmatrix} k_{i,i+1} / m_i + c_{i,i+1} / m_i \end{bmatrix} x_{i+1}(t).$$

That is, all unknown parameters are included in $\Lambda_i$ and $\Delta_{ij}(x_j(t))$. Apparently, $(A_i, B_i)$ is controllable, $(C_i, A_i)$ is observable, and the uncertainties $\Lambda_i$ and $\Delta_{ij}(x_j(t))$ satisfy Assumption 3.1. That is, the mass-spring-damper system (31) can be represented as (1), where we define $N_i = \{i-1, i+1\}$.

For this system, we consider a sinusoidal reference signal such as $\sin \omega_i t$, i.e. we define the coefficient matrices of (3) as

$$A_{ri} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix}, \quad C_{ri} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where $\omega_i > 0$. We see that $(C_{ri}, A_{ri})$ is observable. Also, the rank condition (4) is satisfied for the eigenvalues of $A_{ri}$. Actually, we have the solutions of (7) as

$$L_{ri} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_{ui} = [-\omega_i^2 \ 0].$$

Regarding the performance index $J_i$ of (10) with for $Q_i = 1$ and $R_i = 1$, we obtain the positive definite solution of the Riccati equation (13) as

$$P_i = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}.$$

Then the optimal tracking gains (12) and (16) are

$$K_i = \begin{bmatrix} -1 & -\sqrt{2} \end{bmatrix}, \quad H_i = \begin{bmatrix} 1 - \omega_i^2 & \sqrt{2} \end{bmatrix}.$$

In this way, we can construct the reference model (18) which achieves the optimal tracking for the sinusoidal reference signal.
According to this reference model, we can rewrite the system (32) as (19), i.e.

\[
\dot{x}_i(t) = (A_i + B_i K_i)x_i(t) + B_i H_i x_{ri}(t) + B_i \Lambda_i[u_i(t) + \delta_i(t)],
\]

\[
y_i(t) = C_i x_i(t).
\]

It should be noted that its uncertainty is described as

\[
\delta_i(t) = -\Lambda_i^{-1} \left[ K_i x_i(t) + H_i x_{ri}(t) - \sum_{j=i-1}^{i+1} \Lambda_{ij}(x_j(t)) \right]
\]

\[
= -W_i \sigma_i(x_i(t), x_{\mathcal{N}i}(t), x_{ri}(t)),
\]

where

\[
W_i = \begin{bmatrix}
-k_{i-1,i} & -c_{i-1,i} & k_{i-1,i} + k_{i,i+1} - m_i \\
-c_{i-1,i} + c_{i,i+1} - \sqrt{2} m_i & k_{i,i+1} - c_{i+1} & \\
(1 - \omega_i^2) m_i & \sqrt{2} m_i & -c_{i+1}
\end{bmatrix}
\]

\[
\sigma_i(x_i(t), x_{\mathcal{N}i}(t), x_{ri}(t)) = \begin{bmatrix} x_i^T(t) \\ \hat{x}_{i+1}(t) \\ x_{ri}(t) \end{bmatrix}.
\]

That is, due to the sparse structure of the system (31) of Figure 1, the basis \(\sigma_i(x_i(t), x_{\mathcal{N}i}(t), x_{ri}(t))\) contains only the states of its neighbours, i.e. \(x_{i-1}(t)\) and \(x_{i+1}(t)\). Since the adaptive control law (20) and the update rule of the adaptive gain (21) have the form

\[
u_i(t) = \hat{W}_i(t) \sigma_i(x_i(t), x_{i+1}(t), x_{ri}(t)),
\]

\[
\dot{\hat{W}}_i(t) = -\eta_i B_i^T P_i (x_i(t) - x_{mi}(t)) \cdot \sigma_i^T(x_i(t), x_{i+1}(t), x_{ri}(t)),
\]

these become in fact a distributed control law thanks to the sparse structure of the basis \(\sigma_i(x_i(t), x_{\mathcal{N}i}(t), x_{ri}(t))\).

Now, let us consider a numerical simulation. We investigate the case \(N = 3\), where we set the unknown and uncertain parameters as \(m_1 = m_2 = m_3 = 3\), \(k_{1,2} = k_{2,3} = 2\), and \(c_{1,2} = c_{2,3} = 1\). We chose all initial states are zero except for the reference signal generators, where we used \(x_{r1} = x_{r2} = x_{r3} = [0 \ 1]^T\). We set \(\omega_1^2 = \omega_2^2 = \omega_3^2 = 0.1\).

Figures 2–7 show the tracking responses and the gain behaviours of the proposed distributed adaptive control for subsystems 1, 2 and 3, respectively, where we set the learning rates as \(\eta_1 = \eta_2 = \eta_3 = 50\).

On the other hand, Figures 8–13 show the tracking responses and the gain behaviours of the proposed adaptive control, where we use \(\eta_1 = \eta_2 = \eta_3 = 15\). Note that \(y_1(t), y_2(t), y_3(t)\) are indicated as solid lines and \(y_{mi}(t), y_{m2}(t), y_{m3}(t)\) are indicated as dashed lines in Figures 2, 4, 6, 8, 10, and 12. The elements of the adaptive gains \(\hat{W}_1(t), \hat{W}_2(t)\) and \(\hat{W}_3(t)\) for subsystems 1, 2 and 3 are indicated as solid lines in Figures 3, 5, 7, 9, 11, and 13.

In Figures 2–7, all signals are bounded and \(y_i(t)\) tends to \(y_{mi}(t)\) (\(i = 1, 2, 3\)) as \(t\) tends to infinity, which
is consistent with Theorem 5.1. Furthermore, comparing Figures 2, 4, and 6 with Figures 8, 10, and 12, we see that a larger learning rate $\eta_i$ gives a better performance, which is consistent with Theorem 6.1.

8. Conclusion

In this paper, we have investigated a distributed adaptive control such that the output of an uncertain interconnected dynamical system asymptotically tracks the output of a reference model in the presence of system/interconnection uncertainties. We have constructed the reference model as optimal tracking for the nominal system, where a Riccati solution of an LQ regulator determines the model. We have employed this Riccati solution in the update rule of the adaptive gain and have shown that the proposed adaptive control law actually achieves the desirable tracking as well as the boundedness of all signals. We have also established an explicit error bound regarding optimal tracking. The numerical examples have shown that the theoretical results developed in this paper are useful. Although the state feedback case under no disturbance has been investigated in this paper, the extension to the output feedback case in the presence of disturbance is an important future work. In this regard, a preliminary result [16] has been obtained, where $L_2$ disturbance is considered for a single plant in an $H_\infty$ tracking setting.

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