Linear stability of supersonic boundary layer over a cooled porous surface

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Abstract. The linear stability of the boundary layer at high supersonic Mach numbers on a flat plate with a cooled porous surface is studied. As is well known, at high supersonic speeds, both vortex disturbances of the first mode and acoustic disturbances of the second mode are unstable. The more unstable disturbances of the second mode can be stabilized using sound-absorbing porous coatings. At the same time, such coatings have a destabilizing effect on the disturbances of the first mode. Surface cooling is one of the factors potentially capable of preventing such a destabilization. Based on this assumption, in the present work, the linear theory of stability is used to consider the combined effect of a sound-absorbing coating and surface cooling.

1. Introduction
The instability of the supersonic boundary layer has been studied long enough. It is well known \cite{1, 2} that at moderate supersonic Mach numbers the instability arises due to vortical disturbances, Tollmien–Schlichting waves. In this case, the most unstable disturbances are those propagating at sufficiently large (50 \degree ÷ 60 \degree) angle to the direction of the basic flow. At higher speeds, along with unstable Tollmien–Schlichting waves (the first mode), multiple unstable acoustic-vortical high-frequency modes appear, known as Mack modes. The Mack mode with the lowest frequency is known as the second mode. For Mach numbers greater than approximately 4, the second-mode disturbances grow faster than the first-mode disturbances, and unlike the disturbances of the first mode, the fastest growing disturbances of the second mode are two-dimensional. These theoretical predictions were verified in a number of experiments with natural and artificial disturbances \cite{3, 4, 5, 6, 7}. In general, they confirmed the presence of rapidly growing high-frequency disturbances of the second mode in hypersonic boundary layers, although they did not play a dominant role in all experiments (see, in particular, \cite{6}).

Changes in the surface temperature and its acoustic and mechanical properties have significant effects on the stability of the flow. It is known that surface cooling stabilizes the disturbances of the first mode, while at the same time exerting a destabilizing effect on the second mode. Studies of the effect of surface permeability (porosity) on the stability of the compressible subsonic and supersonic boundary layers, started in \cite{8, 9}, showed that porosity destabilizes the first mode of disturbances. At the same time, theoretical analysis of the effect of ultrasound-absorbing porous coatings on the stability of the hypersonic boundary layer \cite{10} led to the conclusion that they are capable of exerting a strong stabilizing effect on disturbances of the second mode. This conclusion
was confirmed experimentally [10, 11, 12, 13, 14], and then numerically [15, 16, 17, 18, 19]. The calculations were carried out both with special boundary conditions simulating the effect of porosity [15, 18, 19], and with a real surface shape resolving individual pores [16, 17]. The calculations also confirmed the stabilization of disturbances of the second mode. Active studies of the suppression of instabilities using porous coatings are ongoing, in particular with the aim of choosing the optimal coating parameters [20].

As mentioned above, porous coatings have a destabilizing effect on the disturbances of the first mode. This circumstance coupled with the fact that the disturbances of the first mode remain unstable for a significantly greater spatial extent (along the surface of the body) than the disturbances of the second mode can make the first mode the most dangerous for the development of instability and transition to turbulence. It seems natural to apply surface cooling for first mode stabilization. In the works cited above one can find some examples of calculations of linear stability under the combined action of porosity and cooling, however this issue, as it seems, has not been studied in detail. This kind of study is the goal of the present work.

2. Linear stability problem formulation
We consider stability to small disturbances of the flow in the boundary layer on a semi-infinite flat plate at a zero angle of attack in a supersonic gas flow. The distributions of the mean velocity and temperature inside the boundary layer are found as self-similar solutions of the boundary layer equations.

It is assumed that the porous coating does not alter the mean flow; this assumption has been experimentally validated [12].

Small disturbances of the following form are added to the basic flow

\[ q'(x, y, z, t) = \hat{q}(y) e^{i(\alpha x + \beta y - \omega t)}. \]  

(1)

Here \( q \) is any of the gasdynamic quantities, \( x \) is the streamwise coordinate, \( y \) is the coordinate normal to the plate, \( z \) is the spanwise coordinate, \( t \) is time, \( \alpha \) and \( \beta \) are components of the wave vector of the disturbance, \( \omega \) is the disturbance frequency. We consider disturbances growing in the streamwise direction, so \( \omega \) and \( \beta \) are real numbers, and \( \alpha = \alpha_r + i\alpha_i \) is complex. The disturbance growth rate is \( -\alpha_i \), and the phase speed along \( x \) is given by the real part of the \( c = \omega/\alpha \) quantity.

The hydrodynamic stability equations can be written [21] as the system of 8 linear ordinary differential equations

\[ \frac{d\hat{q}}{dy} = G\hat{q}, \quad \hat{q} = (\hat{u}, \hat{d}\hat{u}/dy, \hat{v}, \hat{p}, \hat{\theta}, d\hat{\theta}/dy, \hat{\theta}, d\hat{\theta}/dy), \]  

(2)

where \( u, v, w \) are the velocity components, \( p \) is the pressure, \( \theta \) is the temperature. The expressions for the entries of the matrix \( G \) dependent on the basic flow can be found, for example, in [21].

These equations should be supplemented by the conditions of boundedness of the disturbances at infinity and the boundary conditions on the plate surface. It is assumed that a large number of regularly arranged blind cylindrical holes (pores) of the radius \( r \) and the depth \( h \) are made in the plate material. The porosity \( n \) is defined as the total inlet area of all pores per unit area of the plate. For a solid surface \( n = 0 \), for a surface with the pores closely packed in a square lattice \( n = \pi/4 \). The boundary conditions on such a surface have the form

\[ \hat{u}(0) = 0, \quad \hat{v}(0) = A\hat{p}(0), \quad \hat{\theta}(0) = B\hat{\theta}(0), \]  

(3)

where coefficients \( A \) and \( B \) are proportional to \( n \), and depend on the geometrical sizes of the holes \( a, h \), the wave frequency \( \omega \) and the basic flow parameters near the plate surface.

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The boundary conditions can be derived and the coefficients $A$ and $B$ can be obtained considering the propagation of an acoustic wave in a cylindrical channel with a closed end [22].

The stability equations together with the boundary conditions form an eigenvalue problem with respect to the parameter $\alpha$. It is solved numerically by the shooting method, setting an initial guess for the eigenvalue and refining it with Newton iterations.

3. Results

3.1. Basic flow

In the course of calculations, it was assumed that the gas is perfect, with a ratio of specific heats $\gamma = 1.4$, the Prandtl number $Pr = 0.71$ and the power-law dependence of the viscosity on the temperature with an exponent of 0.75. All calculations were performed at the Mach number $M = 6$.

Figure 1 shows the velocity and temperature profiles of the basic flow at various ratios of the wall temperature $T_w$ and the temperature at the edge of the boundary layer $T_e$. Here, the Blasius thickness of the boundary layer $\delta$ is used as a length scale. It is also used below when determining the Reynolds number $Re$ and in the non-dimensionalization of the depth and radius of pores.

Wall cooling most apparently affects the temperature profile. With decreasing the wall temperature ratio from adiabatic conditions (corresponding to $T_w/T_e = 7.098$), the more and more prominent maximum appears in the temperature profile, and a strong temperature gradient emerges near the wall.

3.2. Effects of porosity parameters

The influence of the parameters characterizing the porous surface on the stability of the second mode was studied for a two-dimensional (propagation angle $\chi = \tan^{-1}(\beta/\alpha_r) = 0$) disturbance with a non-dimensional frequency $\omega = 0.14$ on nearly adiabatic ($T_w/T_e = 7$) surface at $Re = 800$. A disturbance with this frequency has a growth rate close to the maximum for the given Reynolds number.

The results of calculations with varying porosity and fixed values of the pore radius and their depth, which is assumed to be quite large ($h \to \infty$) in this case, indicate that the growth rate decreases with increasing porosity, and this effect is more pronounced for large values of the pore.
radius. So, at $r = 0.18$ the decrease in the growth rate upon reaching the maximum porosity is 24\%, while at $r = 0.8$ it decreases by 3.26 times (Fig. 2a). When the pore depth changes from zero, the growth rate first increases slightly, then rapidly decreases and reaches a minimum at a certain value of $h$. With a further increase in depth, it slightly increases again and then ceases to change, reaching a certain asymptotic value (Fig. 2b). The depth at which the growth rate ceases to change depends on the radius of the pores and increases with its growth.

![Figure 2](image)

**Figure 2.** The effects of the porous surface parameters on the growth rate of the second mode at $M = 6$, $T_w/T_e = 7$, $Re = 800$, $\chi = 0$. Dependence of the growth rate of the disturbance with $\omega = 0.14$ on porosity $n$ at $r = 0.18$, $0.8$ and $h \to \infty$ (a), on pore depth at $n = 0.5$ and $r = 0.18$ (b), and on pore radius $r$ at $n = 0.5$ and $h \to \infty$ (c).

An increase in the pore radius at fixed porosity and depth leads first to a rapid, then to an ever slower decrease in the growth rate (Fig. 2c).

### 3.3. Stability on adiabatic surface

Figure 3 shows the neutral stability curves $\alpha_i(Re, \omega) = 0$ and $\alpha_i(Re, F) = 0$ and the growth rates $(-\alpha_i)$ for two-dimensional disturbances in the boundary layer over the solid wall at $T_w/T_e = 7$. Here $F = \omega/Re$ is the so-called frequency parameter which is convenient since, in contrast to the non-dimensional frequency $\omega$, it remains constant for a disturbance of a given (dimensional) frequency as the boundary layer thickness grows when moving downstream.

![Figure 3](image)

**Figure 3.** Neutral stability curves in the planes $Re, \omega$ (a) and $Re, F$ (b) and growth rates of the disturbances at $Re = 800$ (c) for a supersonic boundary layer at $M = 6$, $T_w/T_e = 7$, $\chi = 0$.

Two parts corresponding to low-frequency disturbances of the first mode and higher-frequency disturbances of the second mode can be distinguished on the neutral curves. In a certain range of Reynolds numbers (e.g., at $Re = 800$), the region of unstable frequencies is divided into two separate intervals corresponding to disturbances of these two modes (Fig. 3c). The maximum
growth rate of the higher-frequency disturbance of the second mode is many times, more than an order of magnitude, higher than the growth rates of the first mode. The porous surface 
\[(n = 0.5, r = 0.8, h \rightarrow \infty)\] causes merging of the two instability regions into one. In this case, the low-frequency disturbances of the first mode are destabilized: the range of unstable frequencies widens, the maximum growth rate noticeably increases. However, the porosity affects the disturbances of the second mode in the opposite way: it leads to a significant decrease in their growth rates. As a result, the maximum growth rate for all frequencies decreases almost twofold.

3.4. Stability on cooled surface

The calculations of the linear stability characteristics were also performed for a different ratio of the wall and free-stream temperatures \(T_w/T_e = 1.4\), i.e., under conditions of a strongly cooled wall. The main feature of the boundary layer under such conditions is the complete stabilization of the two-dimensional disturbances of the first mode. On the neutral curves (Fig. 4), there is no portion corresponding to the first mode, the low-frequency disturbances are stable. The calculations show that with a decrease in \(T_w/T_e\), the critical Reynolds number for the low-frequency disturbances grows rapidly and they soon become completely stable. As a result, the axis \(\omega = 0\) ceases to be the asymptote of the lower branch of the neutral curve.

![Figure 4](image)

**Figure 4.** Neutral stability curves in the planes \(\text{Re}, \omega\) (a) and \(\text{Re}, F\) (b) and growth rates of the disturbances at \(\text{Re} = 800\) (c) for a supersonic boundary layer at \(M = 6, T_w/T_e = 1.4, \chi = 0\).

It should also be noted that the region of Reynolds numbers, at which the disturbance of the given frequency is unstable, on the cooled surface becomes very narrow (Fig. 4b). Those disturbances that become unstable close enough to the leading edge, when moving downstream will quickly leave the instability region because of the growth of the boundary layer thickness and will not have time to significantly increase their amplitude.

The disturbances of the second mode are still strongly stabilized by the porous surface. The calculations show that at \(\text{Re} = 800\) the maximum disturbance growth rate on a porous \((n = 0.5, r = 0.18, h \rightarrow \infty)\) surface decreases by more than 8 times (Fig. 4).

4. Conclusion

The calculations of the linear stability of the supersonic boundary layer performed at the Mach number \(M = 6\) show that the simultaneous use of sound-absorbing porous coatings and surface cooling can be efficient for boundary layer stabilization. The sound-absorbing porous coatings suppress the growth of the disturbances of the second mode, although they destabilize the first mode to some extent. Surface cooling is a very efficient way of suppressing the disturbances of the first mode. In the future, we plan to continue this work by performing calculations for three-dimensional disturbances propagating at an angle with respect to the free stream direction.
Acknowledgments
The current investigation was supported by the Russian Foundation for Basic Research (Project No. 18-01-00866). This support is greatly appreciated.

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