The maximal regularity operator on tent spaces

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En l’honneur des 60 ans de Michel Pierre

Abstract

Recently, Auscher and Axelsson gave a new approach to non-smooth boundary value problems with $L^2$ data, that relies on some appropriate weighted maximal regularity estimates. As part of the development of the corresponding $L^p$ theory, we prove here the relevant weighted maximal estimates in tent spaces $T^{p,2}$ for $p$ in a certain open range. We also study the case $p = \infty$.

1 Introduction

Let $-L$ be a densely defined closed linear operator acting on $L^2(\mathbb{R}^n)$ and generating a bounded analytic semigroup $(e^{-tL})_{t \geq 0}$. We consider the maximal regularity operator defined by

$$M_L f(t, x) = \int_0^t L e^{-(t-s)L}f(s, .)(x)ds,$$

for functions $f \in C_c(\mathbb{R}_+ \times \mathbb{R}^n)$. The boundedness of this operator on $L^2(\mathbb{R}_+ \times \mathbb{R}^n)$ was established by de Simon in [16]. The $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$ case, for $1 < p < \infty$, turned out, however, to be much more difficult. In [10], Kalton and Lancien proved that $M_L$ could fail to be bounded on $L^p$ as soon as $p \neq 2$. The necessary and sufficient assumption for $L^p$ boundedness was then found by Weis [17] to be a vector-valued strengthening of analyticity, called R-analyticity. As many differential operators $L$ turn out to generate R-analytic semigroups, the $L^p$ boundedness of $M_L$ has subsequently been successfully used in a variety of PDE situations (see [14] for a survey).

Recently, maximal regularity was used in a different manner as an important tool in [2], where a new approach to boundary value problems with $L^2$ data for divergence form elliptic systems on Lipschitz domains, is developed. More precisely, in [2], the authors establish and use the boundedness of $M_L$ on weighted spaces $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^\beta dt dx)$, for certain values of $\beta \in \mathbb{R}$, under the additional assumption that $L$ has bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. This additional assumption was removed in [3, Theorem 1.3]. Here is the version when specializing the Hilbert space to be $L^2(\mathbb{R}^n)$.

**Theorem 1.1.** With $L$ as above, $M_L$ extends to a bounded operator on $L^2(\mathbb{R}_+ \times \mathbb{R}^n; t^\beta dt dx)$ for all $\beta \in (-\infty, 1)$.

The use of these weighted spaces is common in the study of boundary value problems, where they are seen as variants of the tent space $T^{2,2}$ which occurs for $\beta = -1$, introduced by Coifman, Meyer and Stein in [6]. For $p \neq 2$, the corresponding spaces are weighted versions of the tent spaces $T^{p,2}$, which are defined, for parameters $\beta \in \mathbb{R}$ and $m \in \mathbb{N}$, as the completion of $C_c(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to

$$\|g\|_{T^{p,2}m(t^\beta dt dy)} = \left( \int_0^\infty \left( \int_{\mathbb{R}^n} \frac{1}{t^m} \int_{\mathbb{R}^n} |g(t, y)|^2 t^\beta dy dt dx \right)^{\frac{2}{p}} dx \right)^{\frac{1}{2}},$$

the classical case corresponding to $\beta = -1, m = 1$, and being denoted simply by $T^{p,2}$. The parameter $m$ is used to allow various homogeneities, and thus to make these spaces relevant in the study of
differential operators $L$ of order $m$. To develop an analogue of \cite{2} for $L^p$ data, we need, among many other estimates yet to be proved, boundedness results for the maximal operator $\mathcal{M}_L$ on these tent spaces. This is the purpose of this note. Another motivation is well-posedness of non-autonomous Cauchy problems for operators with varying domains, which will be presented elsewhere. In the latter case, $\mathcal{M}_L$ can be seen as a model of the evolution operators involved. However, as $\mathcal{M}_L$ is an important operator on its own, we thought interesting to present this special case alone.

In Section 3 we state and prove the adequate boundedness results. The proof is based on recent results and methods developed in \cite{9}, building on ideas from \cite{5} and \cite{8}. In Section 2 we recall the relevant material from \cite{9}.

2 Tools

When dealing with tent spaces, the key estimate needed is a change of aperture formula, i.e., a comparison between the $T_{p,2}$ norm and the norm

$$\|g\|_{T_{p,2}} := \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{\mathbb{R}^n} \frac{1}{t} B(x,\omega t)(y) \left| g(t,y) \right|^2 \frac{dydt}{t} \right)^\frac{1}{2} dx \right)^\frac{1}{2} ,$$

for some parameter $\alpha > 0$. Such a result was first established in \cite{6}, building on similar estimates in \cite{7}, and analogues have since been developed in various contexts. Here we use the following version given in \cite{9} Theorem 4.3].

**Theorem 2.1.** Let $1 < p < \infty$ and $\alpha \geq 1$. There exists a constant $C > 0$ such that, for all $f \in T_{p,2}$,

$$\|f\|_{T_{p,2}} \leq \|f\|_{T_{p,2}} \leq C(1 + \log \alpha)\alpha^{n/\tau} \|f\|_{T_{p,2}},$$

where $\tau = \min(p, 2)$ and $C$ depends only on $n$ and $p$. \footnote{1only on pour éviter les confusions}

Theorem 2.1 is actually a special case of the Banach space valued result obtained in \cite{9}. Note, however, that it improves the power of $\alpha$ appearing in the inequality from the $n$ given in \cite{6} to $\frac{n}{\tau}$.

This is crucial in what follows, and has been shown to be optimal in \cite{9}. Applying this to $(t,y) \mapsto \frac{1}{t^{\frac{m(d+1)}}{t^{\frac{m}{\tau}}}} f(t^m, y)$ instead of $f$, we also have the weighted result, where

$$\|g\|_{T_{p,2,m}(t^\beta dtdy)} = \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{\mathbb{R}^n} \frac{1}{t^{\frac{m}{\tau}}} \frac{1}{t^{m}} B(x,\omega t)(y) \left| g(t,y) \right|^2 t^{\beta} dydt \right)^\frac{1}{2} dx \right)^\frac{1}{2} .$$

**Corollary 2.2.** Let $1 < p < \infty$, $m \in \mathbb{N}$, $\alpha \geq 1$, and $\beta \in \mathbb{R}$. There exists a constant $C > 0$ such that, for all $f \in T_{p,2,m}(t^\beta dtdy)$,

$$\|f\|_{T_{p,2,m}(t^\beta dtdy)} \leq \|f\|_{T_{p,2,m}(t^\beta dtdy)} \leq C(1 + \log \alpha)\alpha^{n/\tau} \|f\|_{T_{p,2,m}(t^\beta dtdy)},$$

where $\tau = \min(p, 2)$ and $C$ depends only on $n$ and $p$.

To take advantage of this result, one needs to deal with families of operators, that behave nicely with respect to tent norms. As pointed out in \cite{9}, this does not mean considering $\mathcal{R}$-bounded families (which means $\mathcal{R}$-analytic semigroups when one considers $(tLe^{-tL})_{t \geq 0}$) as in the $L^p(\mathbb{R}_+ \times \mathbb{R}^n)$ case, but tent bounded ones, i.e. families of operators with the following $L^2$ off-diagonal decay, also known as Gaffney-Davies estimates.

**Definition 2.3.** A family of bounded linear operators $(T_t)_{t \geq 0} \subset B(L^2(\mathbb{R}^n))$ is said to satisfy off-diagonal estimates of order $M$, with homogeneity $m$, if, for all Borel sets $E, F \subset \mathbb{R}^n$, all $t > 0$, and all $f \in L^2(\mathbb{R}^n)$:

$$\|1_E T_t 1_F f\|_2 \lesssim \left( 1 + \frac{\text{dist}(E, F)^m}{t} \right)^{-M} \|1_F f\|_2 .$$

In what follows $\| \cdot \|_2$ denotes the norm in $L^2(\mathbb{R}^n)$.
As proven, for instance, in [4], many differential operators of order \( m \), such as (for \( m = 2 \)) divergence form elliptic operators with bounded measurable complex coefficients, are such that \((tLe^{-tL})_{t \geq 0}\) satisfies off-diagonal estimates of any order, with homogeneity \( m \). This condition can, in fact, be seen as a replacement for the classical gaussian kernel estimates satisfied in the case of more regular coefficients.

3 Results

**Theorem 3.1.** Let \( m \in \mathbb{N}, \beta \in (-\infty, 1), p \in \left(\frac{2n}{n+m(1-\beta)}, \infty\right) \cap (1, \infty), \) and \( \tau = \min(p, 2) \). If \((tLe^{-tL})_{t \geq 0}\) satisfies off-diagonal estimates of order \( M > \frac{\beta}{m} \), with homogeneity \( m \), then \( \mathcal{M}_L \) extends to a bounded operator on \( T^{p,2,m}(t^3 dt dy) \).

**Proof.** The proof is very much inspired by similar estimates in [5] and [9]. Let \( f \in \mathcal{C}_c(\mathbb{R}_+ \times \mathbb{R}^n) \).

Given \( (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \), and \( j \in \mathbb{Z}_+ \), we consider:

\[
C_j(x,t) = \begin{cases} 
B(x,t) & \text{if } j = 0, \\
B(x,2^jt) \setminus B(x,2^{j-1}t) & \text{otherwise.}
\end{cases}
\]

We write \( \|\mathcal{M}_L f\|_{T^{p,2,m}} \leq \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} I_{k,j} + \sum_{j=0}^{\infty} J_{j} \) where

\[
I_{k,j} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{t^m} \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \right) ^{\frac{1}{p}}
\]

\[
J_{j} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{t^m} \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \right) ^{\frac{1}{p}}
\]

Fixing \( j \geq 0, k \geq 1 \) we first estimate \( I_{k,j} \) as follows. For fixed \( x \in \mathbb{R}^n \),

\[
\int_0^\infty \int_{B(x,t^\frac{1}{m})} \int_{2^{-k-1}t}^{2^{-k}t} \left( \frac{1}{t^m} \right) \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt
\]

\[
\leq \int_0^\infty \int_{B(x,t^\frac{1}{m})} \int_{2^{-k-1}t}^{2^{-k}t} \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt
\]

\[
\leq \int_0^\infty \int_{B(x,t^\frac{1}{m})} \int_{2^{-k-1}t}^{2^{-k}t} \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt \left( \int_0^1 \frac{1}{t^m} \right) \left( \int_0^t \left( \int_0^1 \frac{1}{t^m} \right) \right) dt
\]

In the second inequality, we use Cauchy-Schwarz inequality for the integral with respect to \( t \), the fact that \( t-s \sim t \) for \( s \in \cup_{k \geq 1} [2^{-k-1}t, 2^{-k}t] \subset [0, \frac{1}{2}] \) and Fubini’s theorem to exchange the integral in \( t \) and the integral in \( y \). The next inequality follows from the off-diagonal estimate verified by \((t-s)Le^{-(t-s)L}f\) and again the fact that \( t-s \sim t \). By Corollary 2.2 this gives

\[
I_{k,j} \lesssim (j+k)2^{-k}t^{\frac{t}{m}+\frac{1}{m}+\frac{1}{m}+\frac{1}{m}+\frac{1}{m}+\frac{1}{m}}2^{-j(mM-\frac{1}{m})} \|f\|_{T^{p,2,m}(t^3 dt dy)}.
\]
where \( \tau = \min(p, 2) \). It follows that \( \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} I_{k,j} \lesssim \|f\|_{T^{p,2,m}(t^\beta dt dy)} \) since \( M > \frac{n}{m\tau} \) and \( \frac{n}{m} + 1 - \beta > \frac{2n}{m\tau} \). (Note that for \( p \geq 2 \), this requires \( \beta < 1 \).

We now turn to \( J_0 \) and remark that \( J_0 \leq (\int_{\mathbb{R}^n} J_0(x) \frac{dx}{x})^{\frac{1}{n}} \), where

\[
J_0(x) = \int_0^\infty \int_{\mathbb{R}^n} Le^{-(t-s)L}(g(s,\cdot)(y)dy t^{3/2-m} dy dt
\]

with \( g(s, y) = 1_{B(x, 4s^{\frac{1}{2}})}(y)f(s, y) \). The inside integral can be rewritten as

\[
\mathcal{M}_L g(t, \cdot) = e^{-\frac{r}{2L}} \mathcal{M}_L g(t, \cdot).
\]

As \( \mathcal{M}_L \) is bounded on \( L^2(\mathbb{R} \times \mathbb{R}^n; t^{\beta - \frac{m}{2}} dy dt) \) by Theorem 1.1 and \( (e^{-tL})_{t \geq 0} \) is uniformly bounded on \( L^2(\mathbb{R}^n) \), we get

\[
J_0(x) \lesssim \int_0^\infty \left\| 1_{B(x, 4s^{\frac{1}{2}})} f(s, \cdot) \right\|_{2}^{2} s^{\beta - \frac{m}{2}} ds.
\]

We finally turn to \( J_j \), for \( j \geq 1 \). For fixed \( x \in \mathbb{R}^n \),

\[
\int_0^\infty \int_{\mathbb{R}^n} 1_{B(x, t^\frac{1}{2}m)}(y) \left\| \int_{\frac{t}{2}}^t Le^{-(t-s)L}(1_{C_{J(x, 4s^{\frac{1}{2}})}}(s, \cdot))f(s, \cdot)(y) dy \right\|_{2}^{2} t^{3/2-m} dy dt
\]

\[
\leq \int_0^\infty \int_{\mathbb{R}^n} 1_{B(x, t^\frac{1}{2}m)}(y) \left\| \int_{\frac{t}{2}}^t (t-s)Le^{-(t-s)L}(1_{C_{J(x, 4s^{\frac{1}{2}})}}(s, \cdot))f(s, \cdot)(y) dy \right\|_{2}^{2} t^{3/2-m} dy dt
\]

\[
\leq \int_0^\infty \int_{\mathbb{R}^n} 1_{B(x, t^\frac{1}{2}m)}(y) \left\| \int_{\frac{t}{2}}^t (t-s)Le^{-(t-s)L}(1_{C_{J(x, 4s^{\frac{1}{2}})}}(s, \cdot))f(s, \cdot)(y) dy \right\|_{2}^{2} \frac{dy}{(t-s)^{2}} t^{3/2-m+1} dy dt
\]

\[
\leq \int_0^\infty \int_{\frac{t}{2}}^t (t-s)^{-2} \left( 1 + \frac{2jm}{t-s} \right)^{-2M} \left\| 1_{B(x, 2j+2s^{\frac{1}{2}}m)} f(s, \cdot) \right\|_{2}^{2} s^{\beta - \frac{m}{2}+1} ds dt
\]

\[
\leq \int_0^\infty \int_{\frac{t}{2}}^t (t-s)^{-2} \left( 1 + \frac{2jm}{t-s} \right)^{-2M} \left\| 1_{B(x, 2j+2s^{\frac{1}{2}}m)} f(s, \cdot) \right\|_{2}^{2} s^{\beta - \frac{m}{2}+1} ds dt
\]

\[
\leq 2^{-2jmM} \int_0^\infty \left\| 1_{B(x, 2j+2s^{\frac{1}{2}}m)} f(s, \cdot) \right\|_{2}^{2} s^{\beta - \frac{m}{2}+1} ds,
\]

where we have used Cauchy-Schwarz inequality in the second inequality, the off-diagonal estimates and the fact that \( s \leq t \) in the third, Fubini’s theorem and the fact that \( s \geq \frac{t}{2} \) in the fourth, and the change of variable \( \sigma = \frac{s}{t-s} \) in the last. An application of Corollary 2.2 then gives

\[
J_j \lesssim 2^{-jmM} \int t^{\beta} \|f\|_{T^{p,2,m}(t^\beta dt dy)} = j2^{-j(M-\frac{m}{2})} \|f\|_{T^{p,2,m}(t^\beta dt dy)},
\]

and the proof is concluded by summing the estimates.

An end-point result holds for \( p = \infty \). In this context the appropriate tent space consists of functions such that \( |g(t, x)|^2 \frac{dx}{dt} \) is a Carleson measure, and is defined as the completion of the
space $C_c(\mathbb{R}^+ \times \mathbb{R}^n)$ with respect to 

$$
\|g\|_{T^{\infty,2}}^2 = \sup_{(x,r) \in \mathbb{R}^n \times \mathbb{R}^+} r^{-n} \int_0^r \int_{B(x,r)} |g(t,x)|^2 \frac{dxdt}{t}.
$$

We also consider the weighted version defined by 

$$
\|g\|_{T^{\infty,2,m}(t^\beta dtdy)}^2 := \sup_{(x,r) \in \mathbb{R}^n \times \mathbb{R}^+} r^{-\frac{m}{2}} \int_0^r \int_{B(x,r^\frac{1}{m})} |g(t,x)|^2 t^\beta dxdt.
$$

**Theorem 3.2.** Let $m \in \mathbb{N}$, and $\beta \in (-\infty,1)$. If $(tLe^{-tL})_{t \geq 0}$ satisfies off-diagonal estimates of order $M > \frac{n}{2m}$, with homogeneity $m$, then $M_L$ extends to a bounded operator on $T^{\infty,2,m}(t^\beta dtdy)$.

**Proof.** Pick a ball $B(z,r^\frac{1}{m})$. Let

$$
I^2 = \int_{B(z,r^\frac{1}{m})} \int_0^r \int_{B(x,r^\frac{1}{m})} |(M_{L}f)(t,x)|^2 t^\beta dxdt.
$$

We want to show that $I^2 \lesssim r^{\frac{n}{m}} \|f\|_{T^{\infty,2}(t^\beta dtdy)}^2$. We set

$$
I_j^2 = \int_{B(z,r^\frac{1}{m})} \int_0^r \int_{B(x,r^\frac{1}{m})} |(M_{L}f_j)(t,x)|^2 t^\beta dxdt
$$

where $f_j(s,x) = f(s,x)1_{C_j(z,4r^\frac{1}{m})}(x)1_{(0,r)}(s)$ for $j \geq 0$. Thus by Minkowsky inequality, $I \leq \sum I_j$. For $I_0$ we use again Theorem 1.1 which implies that $M_L$ is bounded on $L^2(\mathbb{R}^+ \times \mathbb{R}^n, t^\beta dxdt)$. Thus

$$
I_0^2 \lesssim \int_{B(z,4r^\frac{1}{m})} \int_0^r \int f(t,x)|^2 t^\beta dxdt \lesssim r^{\frac{n}{m}} \|f\|_{T^{\infty,2,m}(t^\beta dtdy)}^2.
$$

Next, for $j \neq 0$, we proceed as in the proof of Theorem 3.1 to obtain

$$
I_j^2 \lesssim \sum_{k=1}^{\infty} \int_0^r \int_{2^{-k-1}t}^{2^{-k}t} 2^{-kt} \Big(1 + \frac{2^{km}r}{t-s}\Big)^{-2M} \|f_j(s,.)\|_{L^2}^2 t^\beta dsdt
$$

$$
+ \int_0^r \int_{\frac{t}{2}}^{t} (t-s)^{-2} \Big(1 + \frac{2^{km}r}{t-s}\Big)^{-2M} \|f_j(s,.)\|_{L^2}^2 t^\beta dsdt.
$$

Exchanging the order of integration, and using the fact that $t \sim t-s$ in the first part and that $t \sim s$
in the second, we have the following.

\[ I_j^2 \lessapprox \sum_{k=1}^{\infty} 2^{-k} 2^{-2jM} r^{-2M} \int_0^r \int_{2^k s} \int_{t^\beta + 2M - 1} \|f_j(s, \cdot)\|^2_{L^2} dt ds \]

\[ + \int_0^r \int_{s} \int_{t-s} (1 + \frac{2jm}{l-s})^{-2M} \|f_j(s, \cdot)\|^2_{L^2} s^\beta ds dt \]

\[ \lessapprox \sum_{k=1}^{\infty} 2^{-k} 2^{-2jM} \int_0^r (2^k s)^\beta \|f_j(s, \cdot)\|^2_{L^2} ds + \int_0^r (1 + 2jm) r^{-2M} \|f_j(s, \cdot)\|^2_{L^2} s^\beta ds \]

\[ \lessapprox 2^{-2jM} \int_0^r \|f_j(s, \cdot)\|^2_{L^2} s^\beta ds, \]

where we used \( \beta < 1 \). We thus have

\[ I_j^2 \lessapprox 2^{-2jM} (2 j + \frac{m}{n})^n \|f\|^2_{T^{2n, m}(t, \cdot)_{L^2}} \]

and the condition \( M > \frac{n}{2m} \) allows us to sum these estimates.

**Remark 3.3.** Assuming off-diagonal estimates, instead of kernel estimates, allows to deal with differential operators \( L \) with rough coefficients. The harmonic analytic objects associated with \( L \) then fall outside the Calderón-Zygmund class, and it is common (see for instance [2]) for their boundedness range to be a proper subset of \((1, \infty)\). Here, our range \( (\frac{2n}{n+m(1-\beta)}, \infty) \) includes \([2, \infty]\) as \( \beta < 1 \), which is consistent with [2]. In the case of classical tent spaces, i.e., \( m = 1 \) and \( \beta = -1 \), it is the range \((2, \infty)\), where 2 denotes the Sobolev exponent \( \frac{2n}{n+2} \). We do not know, however, if this range is optimal.

**Remark 3.4.** Theorem 3.2 is a maximal regularity result for parabolic Carleson measure norms. This is quite natural from the point of view of non-linear parabolic PDE (where maximal regularity is often used), and such norm have, actually, already been used in the context of Navier-Stokes equations in [11], and, subsequently, for some geometric non-linear PDE in [12]. Theorem 5.1 is also reminiscent of Krylov’s Littlewood-Paley estimates [13], and of their recent far-reaching generalization in [15]. In fact, the methods and results from [11], on which this paper relies, use the same circle of ideas (R-boundedness, Kalton-Weis \( \gamma \) multiplier theorem...) as [15]. The combination of these ideas into a “conical square function” approach to stochastic maximal regularity will be the subject of a forthcoming paper.

**References**

[1] P. Auscher, On necessary and sufficient conditions for \( L^p \) estimates of Riesz transforms associated to elliptic operators on \( \mathbb{R}^n \) and related estimates. *Mem. Amer. Math. Soc.* 871 (2007).

[2] P. Auscher, A. Axelsson, Weighted maximal regularity estimates and solvability of elliptic systems I. To appear in *Inventiones Math.* [arXiv:0911.4344].

[3] P. Auscher, A. Axelsson, Remarks on maximal regularity estimates. To appear in *Parabolic Problems: Herbert Amann Festschrift*, Birkhäuser. [arXiv:0912.4482].

[4] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, P. Tchamitchian, The solution of the Kato square root problem for second order elliptic operators on \( \mathbb{R}^n \). *Ann. of Math.* 156(2) (2002) 633–654.
[5] P. Auscher, A. McIntosh, E. Russ, Hardy spaces of differential forms and Riesz transforms on Riemannian manifolds. *J. Geom. Anal.* 18 (2008) 192–248.

[6] R. Coifman, Y. Meyer, E.M. Stein, Some new function spaces and their applications to harmonic analysis. *J. Funct. Anal.* 62 (1985) 304–335.

[7] C. Fefferman; E.M. Stein, $H^p$ spaces of several variables. *Acta Math.* 129 (1972) 137–193.

[8] T. Hytönen, A. McIntosh, P. Portal, Kato’s square root problem in Banach spaces. *J. Funct. Anal.* 254 (2008) no. 3, 675–726.

[9] T. Hytönen, J. van Neerven, P. Portal, Conical square function estimates in UMD Banach spaces and applications to $H^\infty$-functional calculi. *J. Analyse Math.* 106 (2008) 317–351.

[10] N. Kalton, G. Lancien, A solution to the problem of $L_p$ maximal-regularity. *Math. Z.* 235 (2000), 559-568.

[11] H. Koch, D. Tataru, Well-posedness for the Navier-Stokes equations. *Adv. Math.* 157 (2001), 22–35.

[12] H. Koch, T. Lamm, Geometric flows with rough initial data. preprint, [arXiv:0902.1488v1](https://arxiv.org/abs/0902.1488v1).

[13] N.V. Krylov, A parabolic Littlewood-Paley inequality with applications to parabolic equations. *Topol. Methods Nonlinear Anal.* 4 (1994), no. 2, 355–364.

[14] P.C. Kunstmann, L. Weis, Maximal $L^p$ regularity for parabolic problems, Fourier multiplier theorems and $H^\infty$-functional calculus, in *Functional Analytic Methods for Evolution Equations* (Editors: M. Iannelli, R. Nagel, S. Piazzera). Lect. Notes in Math. 1855, Springer-Verlag (2004).

[15] J. van Neerven, M. Veraar, L. Weis, Stochastic maximal $L^p$ regularity. submitted, [arXiv:1004.1309v2](https://arxiv.org/abs/1004.1309v2).

[16] L. de Simon. Un’applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineare astratte del primo ordine. *Rend. Sem. Mat., Univ. Padova* (1964) 205-223.

[17] L. Weis, Operator-valued Fourier multiplier theorems and maximal $L_p$-regularity. *Math. Ann.* 319 (2001), 735–758.

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