UNIVALENCE CRITERIA AND QUASICONFORMAL EXTENSIONS

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Abstract. In the present paper, we obtain a more general conditions for univalence of analytic functions in the open unit disk \( \mathcal{U} \). Also, we obtain a refinement to a quasiconformal extension criterion of the main result.

1. Introduction

Let \( \mathcal{A} \) be the class of analytic functions \( f \) in the open unit disk \( \mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( f(0) = f'(0) - 1 = 0 \). We denote by \( \mathcal{U}_r \), the disk \( \{ z \in \mathbb{C} : |z| < r \} \), where \( 0 < r \leq 1 \), by \( \mathcal{U}_{1} \) the open unit disk of the complex plane and by \( I \) the interval \([0, \infty)\).

Most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Becker [4], Nehari [19] and Ozaki-Nunokawa [21]. Some extensions of these three criteria were given by (see [16], [18], [22], [26]-[29] and [31]). During the time, a lot of univalence criteria were obtained by different authors (see also [8], [10]-[12]).

In the present investigation we use the method of subordination chains to obtain some sufficient conditions for the univalence of an analytic function. Also, by using Becker’s method, we obtain a refinement to a quasiconformal extension criterion of the main result.

2. Preliminaries

Before proving our main theorem we need a brief summary of the method of Loewner chains and quasiconformal extensions.

A function \( L(z, t) : \mathcal{U} \times [0, \infty) \to \mathbb{C} \) is said to be subordination chain (or Loewner chain) if:

(i) \( L(z, t) \) is analytic and univalent in \( \mathcal{U} \) for all \( t \geq 0 \).
(ii) \( L(z, t) \prec L(z, s) \) for all \( 0 \leq t \leq s < \infty \), where the symbol ” \( \prec \) ” stands for subordination.

In proving our results, we will need the following theorem due to Ch. Pommerenke [25].

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Theorem 2.1. Let \( L(z, t) = a_1(t)z + a_2(t)z^2 + ... \), \( a_1(t) \neq 0 \) be analytic in \( U \), for all \( t \in I \), locally absolutely continuous in \( I \), and locally uniform with respect to \( U \). For almost all \( t \in I \), suppose that
\[
\frac{z}{\partial L(z, t)} = p(z, t) \frac{\partial L(z, t)}{\partial t}, \forall z \in U,
\]
where \( p(z, t) \) is analytic in \( U \) and satisfies the condition \( \Re p(z, t) > 0 \) for all \( z \in U \), \( t \in I \). If \( |a_1(t)| \to \infty \) for \( t \to \infty \) and \( \{L(z, t)/a_1(t)\} \) forms a normal family in \( U \), then for each \( t \in I \), the function \( L(z, t) \) has an analytic and univalent extension to the whole disk \( U \).

Let \( k \) be constant in \([0, 1)\). Then a homeomorphism \( f \) of \( G \subset \mathbb{C} \) is said to be \( k-\)quasiconformal, if \( \partial_z f \) and \( \partial_{\overline{z}} f \) in the distributional sense are locally integrable on \( G \) and fulfill the inequality
\[
|\partial_z f| \leq k|\partial_{\overline{z}} f|
\]
a almost everywhere in \( G \). If we do not need to specify \( k \), we will simply call that \( f \) is quasiconformal.

The method of constructing quasiconformal extension criteria is based on the following result due to Becker (see [4], [5] and also [6]).

Theorem 2.2. Suppose that \( L(z, t) \) is a Loewner chain. Consider
\[
w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}, \quad z \in U, \quad t \geq 0
\]
where \( p(z, t) \) is given in (2.1). If
\[
|w(z, t)| \leq k, \quad 0 \leq k < 1
\]
for all \( z \in U \) and \( t \geq 0 \), then \( L(z, t) \) admits a continuous extension to \( \overline{U} \) for each \( t \geq 0 \) and the function \( F(z, \overline{z}) \) defined by
\[
F(z, \overline{z}) = \begin{cases} 
L(z, 0), & \text{if } |z| < 1 \\
L(\frac{z}{|z|}, \log |z|), & \text{if } |z| \geq 1
\end{cases}
\]
is a \( k-\)quasiconformal extension of \( L(z, 0) \) to \( \mathbb{C} \).

Examples of quasiconformal extension criteria can be found in [1], [3], [7], [17], [24] and more recently in [9], [13]-[15], [30].

3. Main Results

Making use of Theorem 2.1 we can prove now, our main results.

Theorem 3.1. Consider \( f \in A \) and \( g \) be an analytic function in \( U \), \( g(z) = 1 + b_1z + ... \). Let \( \alpha, \beta, A \) and \( B \) complex numbers such that \( \Re(\alpha) > \frac{1}{2} \), \( A + B \neq 0 \), \( |A - B| < 2 \), \( |A| \leq 1 \) and \( |B| \leq 1 \). If the inequalities
\[
\left| \frac{1}{\alpha} \left( \frac{f'(z)}{g(z)} - \beta \right) - 1 \right| < \frac{|A + B|}{2 - |A - B|}
\]
(3.1)
and
\[(3.2) \quad \left( \frac{f'(z)}{g(z) - \beta} - 1 \right) |z|^2 + \left( 1 - |z|^2 \right) \left( \frac{1 + \alpha}{\alpha} \frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z) - \beta} \right) - \frac{(A - B)(A + B)}{4 - |A - B|^2} \leq \frac{2|A + B|}{4 - |A - B|^2}\]
are satisfied for all $z \in \mathcal{U}$, then the function $f$ is univalent in $\mathcal{U}$.

Proof. We prove that there exists a real number $r \in (0, 1]$ such that the function $L : \mathcal{U}_r \times I \rightarrow \mathbb{C}$, defined formally by
\[(3.3) \quad L(z, t) = f^{1-\alpha}(e^{-t}z) \left[ f(e^{-t}z) + (e^t - e^{-t}) z (g(e^{-t}z) - \beta) \right] ^\alpha\]
is analytic in $\mathcal{U}_r$ for all $t \in I$.

Since $f(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$, the function
\[(3.4) \quad \varphi_1(z, t) = \frac{(e^t - e^{-t}) z (g(e^{-t}z) - \beta)}{f(e^{-t}z)}\]
is analytic in $\mathcal{U}$.

It follows from
\[(3.5) \quad \varphi_2(z, t) = 1 + \frac{(e^t - e^{-t}) z (g(e^{-t}z) - \beta)}{f(e^{-t}z)}\]
that there exist a $r_1, 0 < r_1 < r$ such that $\varphi_2$ is analytic in $\mathcal{U}_{r_1}$ and $\varphi_2(0, t) = (1 - \beta)e^{2t} + \beta, \varphi_2(z, t) \neq 0$ for all $z \in \mathcal{U}_{r_1}, t \in I$. Therefore, we choose an analytic branch in $\mathcal{U}_{r_1}$ of the function
\[(3.6) \quad \varphi_3(z, t) = [\varphi_2(z, t)]^\alpha.\]
From these considerations it follows that the function
\[L(z, t) = f^{1-\alpha}(e^{-t}z) \left[ f(e^{-t}z) + (e^t - e^{-t}) z (g(e^{-t}z) - \beta) \right] ^\alpha\]
\[= f(e^{-t}z)\varphi_3(z, t) = a_1(t)z + \ldots\]
is an analytic function in $\mathcal{U}_{r_1}$ for all $t \in I$.

After simple calculation we have
\[(3.7) \quad a_1(t) = e^{(2\alpha - 1)t} \left[ \beta e^{-2t} + 1 - \beta \right] ^\alpha\]
for which we consider the uniform branch equal to 1 at the origin. Because $\Re(\alpha) > \frac{1}{2}$, we have that
\[\lim_{t \to \infty} |a_1(t)| = \infty.\]
Moreover, $a_1(t) \neq 0$ for all $t \in I$.

From the analyticity of $L(z, t)$ in $\mathcal{U}_{r_1}$, it follows that there exists a number $r_2, 0 < r_2 < r_1$, and a constant $K = K(r_2)$ such that
\[\left| \frac{L(z, t)}{a_1(t)} \right| < K, \forall z \in \mathcal{U}_{r_2}, t \in I.\]
Then, by Montel’s Theorem, \( \left\{ \frac{L(z,t)}{\sigma_1(t)} \right\}_{t \in I} \) is a normal family in \( U \). From the analyticity of \( \frac{\partial L(z,t)}{\partial t} \), we obtain that for all fixed numbers \( T > 0 \) and \( r_3, 0 < r_3 < r_2 \), there exists a constant \( K_1 > 0 \) (that depends on \( T \) and \( r_3 \)) such that
\[
\left| \frac{\partial L(z,t)}{\partial t} \right| < K_1, \quad \forall z \in U_{r_3}, \ t \in [0,T].
\]
Therefore, the function \( L(z,t) \) is locally absolutely continuous in \( I \), locally uniform with respect to \( U_{r_3} \).

Let \( p : U_r \times I \to \mathbb{C} \) be the analytic function in \( U_r \), \( 0 < r < r_3 \), for all \( t \in I \), defined by
\[
p(z,t) = \frac{\partial L(z,t)}{\partial t} / z \frac{\partial L(z,t)}{\partial z}.
\]
If the function
\[
w(z,t) = \frac{p(z,t) - 1}{A + Bp(z,t)} = \frac{\frac{\partial L(z,t)}{\partial t} - z\frac{\partial L(z,t)}{\partial z}}{A\frac{\partial L(z,t)}{\partial t} + B\frac{\partial L(z,t)}{\partial z}}
\]
is analytic in \( U \times I \) and \( |w(z,t)| < 1 \), for all \( z \in U \) and \( t \in I \), then \( p(z,t) \) has an analytic extension with positive real part in \( U \), for all \( t \in I \). From equality (3.8) we have
\[
w(z,t) = \frac{-2\phi(z,t)}{(A - B)\phi(z,t) + A + B}
\]
for \( z \in U \) and \( t \in I \), where
\[
\phi(z,t) = \left( \frac{f'(e^{-t}z)}{\alpha g(e^{-t}z) - \beta} - 1 \right) e^{-2t} + (1 - e^{-2t}) \left[ \left( \frac{1 - \alpha}{\alpha} \right) \frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} + \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z) - \beta} \right]
\]
From (3.1), (3.9), (3.10) and \( \Re(\alpha) > \frac{1}{2} \), we have
\[
|w(z,0)| = \frac{1}{\alpha} \left( \frac{f'(z)}{g(z) - \beta} - 1 \right) < \frac{|A + B|}{2 - |A - B|}
\]
and
\[
|w(0,t)| = \left( \frac{1}{\alpha (1 - \beta)} - 1 \right) e^{-2t} < \frac{|A + B|}{2 - |A - B|}
\]
where \( A + B \neq 0 \), \( |A - B| < 2 \), \( |A| \leq 1 \) and \( |B| \leq 1 \).

Since \( |e^{-t}z| = |e^{-t}| = e^{-t} < 1 \) for all \( z \in \overline{U} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and \( t > 0 \), we find that \( w(z,t) \) is an analytic function in \( U \). Using the maximum modulus principle it follows that for all \( z \in U - \{ 0 \} \) and each \( t > 0 \) arbitrarily fixed there exists \( \theta = \theta(t) \in \mathbb{R} \) such that
\[
|w(z,t)| < \max_{|z|=1} |w(z,t)| = |w(e^{i\theta},t)|,
\]
for all \( z \in U \) and \( t \in I \).

Denote \( u = e^{-t}e^{i\theta} \). Then \( |u| = e^{-t} \) and from (3.8) we have
\[
|w(e^{i\theta},t)| = \left| \frac{2\phi(e^{i\theta},t)}{(A - B)\phi(e^{i\theta},t) + A + B} \right|
\]
where
\[ \phi(e^{i\theta}, t) = \left( \frac{1}{\alpha} \frac{f'(u)}{g(u) - \beta} - 1 \right) |u|^2 + \left( 1 - |u|^2 \right) \left[ \frac{1 - \alpha}{\alpha} \frac{uf'(u)}{f(u)} + \frac{ug'(u)}{g(u) - \beta} \right] \]

Because \( u \in U \), the inequality (3.2) implies that
\[ |w(e^{i\theta}, t)| \leq 1, \]
for all \( z \in U \) and \( t \in I \). Therefore \( |w(z, t)| < 1 \) for all \( z \in U \) and \( t \in I \).

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function \( L(z, t) \) has an analytic and univalent extension to the whole unit disk \( U \), for all \( t \in I \). For \( t = 0 \) we have \( L(z, 0) = f(z) \), for \( z \in U \) and therefore the function \( f \) is analytic and univalent in \( U \). \( \square \)

If we take \( A = B \) in Theorem 3.1 we get the following univalence criterion.

**Corollary 3.2.** Consider \( f \in A \) and \( g \) be an analytic function in \( U \), \( g(z) = 1 + b_1 z + ... \). Let \( \alpha, \beta \) and \( A \) are complex numbers such that \( \Re(\alpha) > \frac{1}{2} \), \( A \neq 0 \), \( |A| \leq 1 \). If the inequalities
\begin{equation}
(3.14) \left| \frac{f'(z)}{g(z) - \beta} - 1 \right| < |A|
\end{equation}
and
\begin{equation}
(3.15) \left| \frac{f'(z)}{g(z) - \beta} - 1 \right| |z|^2 + \left( 1 - |z|^2 \right) \left[ \frac{1 - \alpha}{\alpha} \frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z) - \beta} \right] \leq |A|
\end{equation}
are satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \).

If we choose \( \alpha = 1 \) in Theorem 3.1 we obtain the following univalence criterion.

**Corollary 3.3.** Consider \( f \in A \) and \( g \) be an analytic function in \( U \), \( g(z) = 1 + b_1 z + ... \). Let \( \beta, A \) and \( B \) complex numbers such that \( A + B \neq 0 \), \( |A - B| < 2 \), \( |A| \leq 1 \) and \( |B| \leq 1 \). If the inequalities
\begin{equation}
(3.16) \left| \frac{f'(z)}{g(z) - \beta} - 1 \right| < \frac{|A + B|}{2 - |A - B|}
\end{equation}
and
\begin{equation}
(3.17) \left| \frac{f'(z)}{g(z) - \beta} - 1 \right| |z|^2 + \left( 1 - |z|^2 \right) \left[ \frac{zg'(z)}{g(z) - \beta} - \frac{(\overline{A - B})(A + B)}{4 - |A - B|^2} \right] \leq \frac{2|A + B|}{4 - |A - B|^2}
\end{equation}
are satisfied for all \( z \in U \), then the function \( f \) is univalent in \( U \).

**Corollary 3.4.** Consider \( f \in A \) and \( g \) be an analytic function in \( U \), \( g(z) = 1 + b_1 z + ... \). Let \( A \) and \( B \) complex numbers such that \( A + B \neq 0 \), \( |A - B| < 2 \), \( |A| \leq 1 \) and \( |B| \leq 1 \). If the inequalities
\begin{equation}
(3.18) \left| \frac{f'(z)}{g(z) - 1} \right| < \frac{|A + B|}{2 - |A - B|}
\end{equation}
and
\begin{equation}
(3.19) \left| \frac{f'(z)}{g(z) - 1} \right| |z|^2 + \left( 1 - |z|^2 \right) \left[ \frac{zf'(z)}{g(z)} - \frac{(\overline{A - B})(A + B)}{4 - |A - B|^2} \right] \leq \frac{2|A + B|}{4 - |A - B|^2}
\end{equation}
are satisfied for all \( z \in \mathcal{U} \), then the function \( f \) is univalent in \( \mathcal{U} \).

**Proof.** It results from Corollary 3.3 with \( \alpha = 1 \) and \( \beta = 0 \). \( \square \)

For \( g(z) = f'(z) \) in Corollary 3.4 we have the following univalence criterion.

**Corollary 3.5.** Consider \( f \in \mathcal{A} \). Let \( A \) and \( B \) complex numbers such that \( A + B \neq 0 \), \( |A - B| < 2 \), \( |A| \leq 1 \) and \( |B| \leq 1 \). If the inequality

\[
\left| \left( 1 - |z|^2 \right) \frac{zf''(z)}{f'(z)} - \frac{(\overline{A - B})(A + B)}{4 - |A - B|^2} \right| \leq \frac{2|A + B|}{4 - |A - B|^2}
\]

is satisfied for all \( z \in \mathcal{U} \), then the function \( f \) is univalent in \( \mathcal{U} \).

**Corollary 3.6.** Consider \( f \in \mathcal{A} \). Let \( \beta, A \) and \( B \) complex numbers such that \( A + B \neq 0 \), \( |A - B| < 2 \), \( |A| \leq 1 \) and \( |B| \leq 1 \). If the inequalities

\[
\left| \frac{\beta}{f'(z) - \beta} \right| < \frac{|A + B|}{2 - |A - B|}
\]

and

\[
\left| \frac{\beta |z|^2 + \left( 1 - |z|^2 \right) zf''(z)}{f'(z) - \beta} - \frac{(\overline{A - B})(A + B)}{4 - |A - B|^2} \right| \leq \frac{2|A + B|}{4 - |A - B|^2}
\]

are satisfied for all \( z \in \mathcal{U} \), then the function \( f \) is univalent in \( \mathcal{U} \).

**Proof.** It results from Corollary 3.3 with \( g(z) = f'(z) \). \( \square \)

**Corollary 3.7.** Consider \( \beta < 0 \) in Corollary 3.6. By elementary calculation we obtain that the inequality (3.2) for \( A = B = 1 \) is equivalent to

\[
\mathbb{R}f'(z) > \frac{1}{2\beta} |f'(z)|^2, \ z \in \mathcal{U}.
\]

If in the last inequality we let \( \beta \to -\infty \) we obtain that

\[
\mathbb{R}f'(z) > 0.
\]

Since (3.22) for \( A = B = 1 \) and \( \beta \to -\infty \) it follows from Corollary 3.6 that the function \( f \) is univalent in \( \mathcal{U} \). Therefore, we can conclude that the univalence criterion due to Alexander-Noshiro-Warshawski [2, 20, 32] is a limit case of Corollary 3.6.

**Remark 3.8.** Some particular cases of Theorem 3.1 are the following:

(i) When \( \alpha = 1, \beta = 0, A = B = 1 \) and \( g(z) = f'(z) \) inequality (3.3) becomes

\[
\left( 1 - |z|^2 \right) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \ z \in \mathcal{U}
\]

which is Becker’s condition of univalence [4].
(ii) A result due to N. N. Pascu [23] is obtained when \( \alpha = 1, A = B = 1 \) and \( g(z) = f'(z) \).

**Remark 3.9.** It is worth to notice that the condition \((3.2)\) assures the univalence of an analytic function in more general case than that of condition \((3.3)\).

**Remark 3.10.** If we put \( g(z) = \frac{f(z)}{z} \) into \((3.12)\), we have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{|A + B|}{2 - |A - B|}, \quad z \in \mathbb{U}
\]
the class of functions starlike with respect to origin.

## 4. Quasiconformal Extension

In this section we will obtain the univalence condition given in Theorem 3.1 to a quasiconformal extension criterion.

**Theorem 4.1.** Consider \( f \in \mathcal{A} \), \( g \) be an analytic function in \( \mathbb{U} \), \( g(z) = 1 + b_1 z + \ldots \) and \( k \in [0, 1) \). Let \( \alpha, \beta, A \) and \( B \) complex numbers such that \( \Re(\alpha) > \frac{1}{2}, A + B \neq 0, k |A - B| < 2, |A| \leq 1 \) and \( |B| \leq 1 \). If the inequalities
\[
(4.1) \quad \left| \frac{1}{\alpha} \left( \frac{f'(z)}{g(z)} - \beta \right) - 1 \right| < \frac{k |A + B|}{2 - k |A - B|}
\]
and
\[
(4.2) \quad \left( \frac{f'(z)}{g(z) - \beta} - 1 \right) |z|^2 + \left( 1 - |z|^2 \right) \left( \frac{1 - \alpha}{\alpha} \frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z) - \beta} \right) - \frac{k^2 (A - B) (A + B)}{4 - k^2 |A - B|^2} \leq \frac{2k |A + B|}{4 - k^2 |A - B|^2}
\]
are satisfied for all \( z \in \mathbb{U} \), then the function \( f \) has a \( k \)-quasiconformal extension to \( \mathbb{C} \).

**Proof.** In the proof of Theorem 3.1 has been proved that the function \( L(z, t) \) given by \((3.8)\) is a subordination chain in \( \mathbb{U} \). Applying Theorem 2.2 to the function \( w(z, t) \) given by \((3.3)\), we obtain that the assumption
\[
(4.3) \quad |w(z, t)| = \left| \frac{-2\phi(z, t)}{(A - B) \phi(z, t) + A + B} \right| \leq k, \; z \in \mathbb{U}, \; t \geq 0, \; k \in [0, 1)
\]
where \( \phi(z, t) \) is defined by \((3.10)\).

Lengthy but elementary calculation shows that the last inequality \((4.3)\) is equivalent to
\[
\left( \frac{1}{\alpha} \frac{f'(e^{-t}z)}{g(e^{-t}z) - \beta} - 1 \right) e^{-2t} + (1 - e^{-2t}) \left( \frac{1 - \alpha}{\alpha} \frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)} + \frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z) - \beta} \right) - \frac{k^2 (A - B) (A + B)}{4 - k^2 |A - B|^2} \leq \frac{2k |A + B|}{4 - k^2 |A - B|^2}.
\]
The inequality \((4.4)\) implies \( k \)-quasiconformal extensibility of \( f \).

The proof is complete. \( \square \)
If we choose $\alpha = 1, \beta = 0, g = f'$ in Theorem 4.1, we obtain following corollary.

**Corollary 4.2.** Consider $f \in A$ and $k \in [0,1)$. Let $A$ and $B$ complex numbers such that $A + B \neq 0, k |A - B| < 2, |A| \leq 1$ and $|B| \leq 1$. If the inequality

$$\left| \left(1 - |z|^2\right) \left(\frac{zf''(z)}{f'(z)}\right) - \frac{k^2 (\overline{A} - B) (A + B)}{4 - k^2 |A - B|^2} \right| \leq \frac{2k |A + B|}{4 - k^2 |A - B|^2}$$

is satisfied for all $z \in U$, then the function $f$ has a $k$–quasiconformal extension to $\mathbb{C}$.

For $A = B = 1$ in Corollary 4.2 we have result of Becker [4].

**Corollary 4.3.** Consider $f \in A$ and $k \in [0,1)$. If the inequality

$$\left(1 - |z|^2\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq k$$

is satisfied for all $z \in U$, then the function $f$ has a $k$–quasiconformal extension to $\mathbb{C}$.

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