ORTHOGONAL POLYNOMIALS ON THE SIERPINSKI GASKET

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ABSTRACT. The construction of a Laplacian on a class of fractals which includes
the Sierpinski gasket (SG) has given rise to an intensive research on analysis on
fractals. For instance, a complete theory of polynomials and power series on SG
has been developed by one of us and his coauthors. We build on this body of work
to construct certain analogs of classical orthogonal polynomials (OP) on SG. In
particular, we investigate key properties of these OP on SG, including a three-
term recursion formula and the asymptotics of the coefficients appearing in this
recursion. Moreover, we develop numerical tools that allow us to graph a number
of these OP. Finally, we use these numerical tools to investigate the structure of
the zero and the nodal sets of these polynomials.

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1. INTRODUCTION

The polynomials $\pi_0(t) = 1$, and $\pi_k(t) = t^k + \sum_{l=0}^{k-1} a_l t^l, k = 1, 2, \ldots$, defined on the
unit interval $I$ are called monic orthogonal polynomials with respect to a weight $w(t)$
if
\[ \langle \pi_k, \pi_j \rangle = \int_I \pi_k(t) \pi_j(t) w(t) dt = \| \pi_k \|_2^2 \delta_{k,j} \quad k = 0, 1, 2, \ldots, \]
where \( \delta_{k,j} \) denotes the Kronecker \( \delta \) sequence.

When renormalized, these monic orthogonal polynomials give rise to an orthonormal system of polynomials \( \{ \tilde{\pi}_k \}_{k=0}^\infty \), where \( \langle \tilde{\pi}_k, \tilde{\pi}_j \rangle = \delta_{j,k} \).

The monic OP are constructed by performing the Gram-Schmidt process on the monomials \( \{ t^k \}_{k=0}^\infty \), with respect to the inner product \( \langle a, b \rangle = \int_I a(t) b(t) w(t) dt \). Examples of such polynomials include the Legendre polynomials defined on \( I = [-1,1] \), and using the weight function \( w(t) = 1 \).

Numerical information about the OP can be obtained via a fundamental recursive equation called the three-term recursion formula. For example, the values of the polynomials at different points in \( I \), and their zeros can be obtained from the three-term recurrence relation. For comparison with our results, we state the general form of this recurrence relation and refer to [6, 13] for details on OP.

**Theorem A.** [6, Theorem 1.27] The orthogonal polynomials \( \{ \pi_k \}_{k \geq 0} \) satisfy a three-term recurrence relation. More specifically, \( \pi_{-1}(t) = 0, \pi_0(t) = 1 \), and for all \( k \geq 0 \) we have:
\[ \pi_{k+1}(t) = (t - \alpha_k) \pi_k(t) - \beta_k \pi_{k-1}(t), \]
where
\[ \begin{align*}
\alpha_k &= \frac{\langle t \pi_k, \pi_k \rangle}{\langle \pi_k, \pi_k \rangle} \quad \text{for } k \geq 0 \\
\beta_k &= \frac{\langle \pi_k, \pi_{k-1} \rangle}{\langle \pi_{k-1}, \pi_{k-1} \rangle} \quad \text{for } k \geq 1.
\end{align*} \]

Our goal in this paper is to construct and investigate the properties of the analogs of the above OP on \( SG \). But first what is a polynomial on a fractal set such as \( SG \)?

Let \( \Delta = \frac{d^2}{dt^2} \) denote the Laplacian on \( I = [0,1] \). If we let \( \pi_j(t) = t^j \), for \( t \in I = [0,1] \), it is immediate that \( \pi_j \) is a solution to a differential equation \( \Delta^\ell \pi_j(t) = 0 \), for some \( \ell \geq 1 \). This observation can be used to define polynomial on fractals in general and on the Sierpinski gasket in particular. In particular, on \( SG \), one defines a polynomial \( P \) to be any solution of \( \Delta^{j+1} P = 0 \) for some \( j \geq 0 \), where \( \Delta \) is the (fractal) Laplacian to be defined below. Using this analogy, monomials were introduced on the Sierpinski gasket \([4, 11, 12, 15, 16]\) based on the fractal Laplacian constructed by Kigami [9], see also [15, 14]. We also note that a theory of polynomials based on a different Laplacian, the so-called energy Laplacian, is developed in [17].

Consequently, our construction of OP on \( SG \) is based on these polynomials. In particular, we apply the Gram-Schmidt orthogonalization algorithm to produce the analogs to the Legendre polynomials on \( SG \). In the process we construct both a family of symmetric and antisymmetric OP on \( SG \), and the concatenation of the two families yields a tight frame for a proper closed subspace of \( L^2(SG) \). The reason that our construction does not yield a dense set of OP in \( L^2(SG) \) is due to the fact...
that the polynomials are not dense in this space [9, Theorem 4.3.6], [15, Section 5.1]. In addition, we investigate thoroughly a corresponding three-term recursion formula, paying a particular attention to the asymptotics of the coefficients involved in this recursion. Our three-term recurrence is fundamentally different from (2), and this is essentially due to the fact that the product of two polynomials on $SG$ is not a polynomial [1]! In addition, our analog of (2) is very unstable and thus cannot be used to generate the values of the OP on $SG$. However, we are able to exploit this three-term recurrence to recursively express the corresponding OP in terms of a well-known basis of polynomials on $SG$. This leads to the development of some numerical tools that allow us to plot graphs of many of these OP. In addition, we present preliminary results pertaining the zeros of these OP. In particular, we graph the zero sets of certain of these OP. These results are mainly experimental but seems to indicate that these zero sets have some structure. We have not been able to prove anything along these lines but hope that our experimental results lead to more investigations on the zero sets of the OP. Nonetheless, we have generated many graphs related to the OP on $SG$ that we did not include in the present paper due to space constraint. The interested reader is referred to the following web site which contains many figures and algorithms that were generated in the course of this research project [19].

Our paper is organized as follows: In Section 2 we review the basic facts of analysis on fractals needed to state and prove our results. We also prove some new results about the Green’s function that may be of independent interest. Section 3 is devoted to the construction of the OP on $SG$ and to the investigation of the three-term recursion and the asymptotic analysis of its coefficients. We also consider the Jacobi matrix associated with these coefficients. In Section 4 we carry out the numerical computations that enable us to plot many of the OP we constructed. In addition, we give some numerical evidence concerning the asymptotics of the coefficients involved in the three-term relation. Finally, we give numerical description of the zero sets of the OP and display some of these zero sets as well as some of their corresponding nodal sets on $SG$.

2. Polynomials on $SG$

2.1. Preliminaries. For more information on the theory of calculus on fractals see [9]; more specifically for calculus on the Sierpinski Gasket see [15, 14]. However, we collect below some key facts needed in the formulation of our results.

The Sierpinski Gasket ($SG$), shown in Figure 1, is the attractor of iterated function system (IFS) consisting of three contractions in the plane $F_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 0, 1, 2$, defined by $F_i = \frac{1}{2}(x - q_i) + q_i$ where $\{q_i\}_{i=0}^2$ are the vertices of an equilateral triangle. $SG$ satisfies the self-similar identity $SG = \bigcup_{i=0}^2 F_i(SG)$, and the $F_i(SG)$ are called cells of level 1. By iteration we can write $F_w = F_{w_1} \circ F_{w_2} \circ \ldots \circ F_{w_m}$ for $w = (w_1, w_2, \ldots, w_m)$, each $w_j = 0, 1, 2$. We call each $w$ a word of length $|w| = m$, and we have the self-similar identity $SG = \bigcup_{|w|=m} F_w(SG)$. Each $F_w(SG)$ is called a cell of level $m$. 

Figure 1. The Sierpinski Gasket

$SG$ can also be viewed as the limit of a sequence of graphs $\Gamma_m$ (with vertices $V_m$ and edge relation $x \sim_m y$ defined inductively as follows: $\Gamma_0$ is the complete graph on $V_0 = \{q_0, q_1, q_2\}$, and $V_m = \bigcup_{i=0}^{2^m} F_i V_{m-1}$ with $x \sim_m y$ if and only if $x$ and $y$ belong to the same cell of level $m$. Then $V_* = \bigcup_{m=1}^{\infty} V_m$, the set of all vertices is the analog of the dyadic points in $[0, 1]$ and is dense in $SG$. We consider $V_0$ the set of boundary points of $SG$, and $V_* \setminus V_0$ is the set of junction points. Note that every junction point in $V_m$ has four neighbors in the graph $\Gamma_m$.

The graph Laplacian $\Delta_m$ is defined by

$$
\Delta_m u(x) = \sum_{x \sim_m y} (u(y) - u(x)) \quad \text{for } x \in V_m \setminus V_0
$$

and the Laplacian $\Delta$ on $SG$ is defined as the renormalized limit

$$
\Delta u(x) = \lim_{m \to \infty} \frac{3}{2^m} \Delta_m u(x).
$$

At the three boundary points of $SG$, we have two different derivatives, the normal and tangential derivative, respectively defined by

$$
\begin{align*}
\partial_n u(q_i) &= \lim_{m \to \infty} \left( \frac{2}{3} \right)^m (2u(q_i) - u(F^m_i q_{i+1}) - u(F^m_i q_{i-1})) \\
\partial_T u(q_i) &= \lim_{m \to \infty} 5^m (u(F^m_0 q_{i+1}) - u(F^m_0 q_{i-1})),
\end{align*}
$$

identifying the boundary points $q_i$ and $q_j$ when $i = j \mod 3$.

Using the above definitions, the analog of the Gauss-Green formula takes the following form:

$$
\int_{SG} (u\Delta v - v\Delta u) d\mu = \sum_{i=0}^{2} (u(q_i)\partial_n v(q_i) - v(q_i)\partial_n u(q_i)),
$$

where $\mu$ is the natural probability measure which assigns weight $3^{-m}$ to each cell of order $m$. 
These analytical tools can now be used to solve the boundary value problem
\[-\Delta u = f, \quad u|_{V_0} = 0,\]
whose solution is given by
\[u = \int_{SG} G(x, y) f(y) d\mu(y).\]

\(G\) is called the Green’s function, and is given by
\[G(x, y) = \lim_{M \to \infty} G_M(x, y), \quad G_M(x, y) = \sum_{m=0}^{M} \sum_{z' \in V_{m+1} \setminus V_m} g(z, z') \psi_{z'}^{(m+1)}(x) \psi_{z'}^{(m+1)}(y),\]
where \(g(z, z')\) is zero when \(z\) and \(z'\) are not in the same cell of level \(m + 1\),
\[g(z) = \frac{9}{50} \left(\frac{3}{5}\right)^m \quad \text{for} \quad z \in V_{m+1} \setminus V_m\]
and \(g(z, z') = \frac{3}{50} \left(\frac{3}{5}\right)^m \quad \text{for} \quad z \neq z', \quad z \text{ and } z' \text{ in the same cell of level } m + 1,\]
and \(\psi_{z'}^{(m+1)}\) is the piecewise harmonic function satisfying \(\psi_{z}^{(m+1)}(x) = \delta_{xz}\) and vanishing outside the \((m + 1)\)th cell to which \(z\) belongs.

We shall need an exact estimate of \(\|G\|_{L^2}\), the \(L^2\) norm of the Green function \(G\). It was conjectured in [10] and proved in [18] that
\[\|G\|_{L^\infty} = 178839/902500 \simeq 0.198.\]
This immediately implies that \(\|G\|_{L^2} < \|G\|_{L^\infty} < 1\). However, we obtain exact values for \(\|G\|_{L^2}\) and a related quantity in the next result. Though the result seems simple, to our knowledge, it has not appeared anywhere in the literature.

First, we briefly recall a description of the spectrum of \(\Delta\) on \(SG\), that was given in [5] using the method of spectral decimation introduced in [12]. In essence, the spectral decimation method completely determines the eigenvalues and the eigenfunctions of \(\Delta\) on \(SG\) from the eigenvalues and eigenfunctions of the graph Laplacians \(\Delta_m\). More specifically, for every Dirichlet \(\lambda_j\) of \(-\Delta\) on \(SG\), that is a solution of
\[-\Delta u = \lambda_j u, \quad u|_{V_0} = 0,\]
there exists an integer \(m \geq 1\), called the generation of birth, such that if \(u\) is a \(\lambda_j\)-eigenfunction and \(k \geq m\) then \(u|_{V_k}\) is an eigenfunction of \(\Delta_k\) with eigenvalue \(\lambda_j^{(k)}\). The only possible initial values \(\lambda_j^{(m)}\) are 2, 5 and 6, and subsequent values can be obtained from
\[\lambda_j^{(k+1)} = \frac{5 + \epsilon_k \sqrt{25 - 4\lambda_j^{(k)}}}{2} \quad \text{for} \quad k \geq m\]
where \(\epsilon_k\) can take the values \(\pm 1\). The sequence \(\lambda_j^{(k)}\) is related to \(\lambda_j\) by
\[\lambda_j = \frac{3}{2} \lim_{k \to \infty} 5^k \lambda_j^{(k)}.\]
Theorem 2.1. Let \( \{\lambda_j\}_{j=0}^{\infty} \) be the eigenvalues of the Laplacian on \( SG \). Then,

\[
\|G\|_{L^2}^2 = \int \int_{SG \times SG} G(x, y)^2 \, dx \, dy = \sum_{j=0}^{\infty} \frac{1}{\lambda_j} = \frac{45389}{3564000} \approx 0.0127
\]

and

\[
\int_{SG} G(x, x) \, dx = \sum_{j=0}^{\infty} \frac{1}{\lambda_j} = 1/6
\]

Proof. It is easily seen that \( G \) can be expressed as

\[
G(x, y) = \sum_{k=0}^{\infty} \lambda_k^{-1} u_k(x) u_k(y),
\]

where \( \lambda_k \) are the Laplacian eigenvalues corresponding to the eigenfunction \( u_k \) [13].

It then follows that

\[
\int_{SG} G(x, x) \, dx = \sum_{j} \frac{1}{\lambda_j} = \lim_{m \to \infty} \frac{2}{3} \sum_{j \in I_m} \frac{1}{5^m \lambda_j^{(m)}},
\]

where \( I_m \) is a finite set that depends on the multiplicity of the graph-Laplacian eigenvalues \( \lambda_j^{(m)} \). We shall now find explicit formula for the above sum.

Suppose that we know the graph-Laplacian eigenvalues \( \{\lambda_j^{(m)}\} \) whose multiplicity are \( \{d_j^{(m)}\} \): if \( \lambda_j^{(m)} = 6 \) it has multiplicity \( \frac{3^m - 3}{2} \), if \( \lambda_j^{(m)} = 5 \) it has multiplicity \( \frac{3^m - 1 + 3}{2} \) and if \( \lambda_j^{(m)} = 3 \) it has multiplicity \( \frac{3^m - 1 - 3}{2} \). Then, at the next level \( (m+1) \), the graph-Laplacian eigenvalues \( \{\lambda_j^{(m+1)}\} \) come from the spectral decimation method. Moreover, for every \( \lambda_j^{(m)} \neq 6 \) with multiplicity \( d_j^{(m)} \) there will be two eigenvalues \( \lambda_j^{(m+1)} \) with multiplicity \( d_j^{(m)} \) and given by

\[
\lambda_j^{(m+1)} = \frac{5 \pm \sqrt{25 - 4 \lambda_j^{(m)}}}{2}.
\]

Note that

\[
\frac{1}{5^{m+1} \lambda_j^{(m+1)}} + \frac{1}{5^{m+1} \lambda_{j+1}^{(m+1)}} = \frac{1}{5^m} \frac{\lambda_j^{(m+1)} + \lambda_{j+1}^{(m+1)}}{\lambda_j^{(m+1)} \lambda_{j+1}^{(m+1)}} = \frac{1}{5^m \lambda_j^{(m)}}.
\]

Now \( \sum_j \frac{1}{\lambda_j} = \lim_{m \to \infty} A_m \) where \( A_m = \sum_{\lambda_j \neq 6} \frac{1}{5^m \lambda_j^{(m)}} \), since the eigenvalue \( \lambda_j^{(m)} = 6 \) with multiplicity \( \frac{3^m - 3}{2} \) contributes \( O(\frac{3^m}{5^m}) \) to the sum. Therefore, by the spectral decimation, we have

\[
A_{m+1} = \frac{2}{5} \frac{3^m + 3}{5^m + 5} + \frac{2}{3} \frac{3^m - 3}{5^m - 3} + A_m,
\]

with \( A_1 = \frac{2}{5} (1/2 + 1/5 + 1/5) = 3/25 \). Consequently,

\[
\sum_j \frac{1}{\lambda_j} = A_1 + \sum_{m=1}^{\infty} \frac{2^{m-1} + 1}{5^{m+2}} + \frac{1}{3} \sum_{m=1}^{\infty} \frac{2^{m-1} - 1}{5^{m+1}} = 1/6.
\]
Similarly, if we let $B_m = \frac{4}{9} \sum_{\lambda \neq 0} \left( \frac{1}{5^m \lambda(m)} \right)^2$, then

$$\sum_{j} \frac{1}{\lambda_j} = \lim_{m \to \infty} B_m.$$  

One can check that

$$\left( \frac{1}{5^{m+1} \lambda_j} \right)^2 + \left( \frac{1}{5^{m+1} \lambda_j} \right)^2 = \frac{1}{25} \left( \frac{1}{\lambda_j} \right)^2 + \left( \frac{1}{\lambda_j} \right)^2 = \frac{1}{25} \frac{25 - 2 \lambda_j}{(5^m \lambda_j)^2} = \frac{1}{25} \left( \frac{1}{\lambda_j} \right)^2 - \frac{2}{25} \frac{1}{(5^m \lambda_j)^2}.$$  

Next, we can use the spectral decimation method to prove that

$$B_{m+1} = \frac{4}{9} \left[ \frac{3^m + 3}{2} \right] + \frac{4}{9} \left[ \frac{3^m - 3}{2} \right] + B_m - \frac{2}{25} C_m,$$

where $C_m = \sum_{\lambda \neq 0} \frac{1}{25^m A_m}$. Moreover,

$$C_{m+1} = \frac{3^m + 3}{2} \frac{1}{5^{m+2} \lambda_j} + \frac{3^m - 3}{2} \frac{1}{5^{m+2} \lambda_j} + \frac{1}{2} C_m,$$

with $C_1 = \frac{1}{25} (1/2 + 1/5 + 1/5) = 9/250$, and $B_1 = \frac{4}{9} \left[ \frac{1/4 + 1/25 + 1/25} \right] = \frac{41}{375}$. Consequently,

$$\sum_{j} \frac{1}{\lambda_j} = B_1 + \sum_{m=1}^{\infty} \frac{2}{5^{m+2} \lambda_j} + \sum_{m=1}^{\infty} \frac{2}{5^{m+2} \lambda_j} - \frac{2}{25} \sum_{m=1}^{\infty} C_m = \frac{1}{25} \left( \frac{3532}{27^2 \cdot 8111} - \frac{107}{1700} \right) = \frac{4539}{355400} \approx .0127.$$

2.2. Bases of polynomials on $SG$. For any integer $j \geq 0$, the set of polynomials of degree less than or equal to $j$ will be denoted $H^j$ and consists of the solutions of $\Delta^{j+1} u = 0$. $H^j$ is a space of dimension $3j + 3$ and it has a basis $\{ f_{ki}, 0 \leq k \leq j; i = 0, 1, 2 \}$ characterized by

$$\Delta^{\ell} f_{ki}(q_{\ell'}) = \delta_{\ell,k} \delta_{i,i'},$$

where $i' = 0, 1, 2$ and $0 \leq \ell \leq j$. In particular, $H^0$ is the space of harmonic functions, i.e., a function $h : SG \to \mathbb{R}$ belongs to $H^0$ if and only if it satisfies $\Delta h = 0$. We refer to [16] [11] [15] for details on polynomials on $SG$.

Our construction of OP on $SG$ will be based on another basis for the space $H^j$ introduced in [11]. Functions in this basis are called monomials and are essentially the fractal analogs of $\partial_j^{n}$.  

**Definition 2.1.** Fix a boundary point $q_n$ for $n = 0, 1, 2$. The monomials $P_{j}^{(n)}$ for $i = 1, 2, 3$ and $j \geq 0$ are defined to be the functions in $H^j$ satisfying

$$\Delta^{m} P_{j}^{(n)}(q_n) = \delta_{m,j} \delta_{i,1},$$

$$\partial_n \Delta^{m} P_{j}^{(n)}(q_n) = \delta_{m,j} \delta_{i,2},$$

$$\partial_T \Delta^{m} P_{j}^{(n)}(q_n) = \delta_{m,j} \delta_{i,3},$$

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \lim_{m \to \infty} B_m.$$
with $0 \leq m \leq j$. When $n = 0$ we will sometimes delete the upper index and just write $P_{ji}$ for $P_{ji}^{(0)}$.

Notice that for a fixed $n = 0, 1, 2$, the set of monomials $\{P_{ki}, i = 1, 2, 3\}_{k=0}^{j}$ forms a basis for $\mathcal{H}^j$. Moreover, the monomials in this basis satisfy $\Delta P_{ji}^{(n)} = P_{(j-1)k}^{(n)}$, and possess some symmetry properties. When $i = 1$ or 2, $P_{ji}^{(n)}$ is symmetric with respect to the line passing through $q_n$ and the midpoint of the side opposing $q_n$. In fact, these symmetric polynomials can be viewed as analogs of the even polynomials $t^{2k}$ on the unit interval. When $i = 3$, $P_{j3}^{(n)}$ is antisymmetric with respect to the line passing through $q_n$ and the midpoint of the side opposing $q_n$. In this case, $P_{j3}^{(n)}$ can be viewed as analogs of the odd polynomials $t^{2k+1}$.

In our construction of OP on $SG$, we will need the following result which was proved in [11] and which gives recursively values of the above monomials and their derivatives at boundary points.

**Theorem B** [11] Theorem 2.3] For $j \geq 0$, let

$$
\alpha_j = P_{j1}(q_1), \quad \beta_j = P_{j2}(q_1), \quad \gamma_j = P_{j3}(q_1), \quad \eta_j = \partial_n P_{j1}(q_1)
$$

The following recursion relations hold:

\begin{equation}
\begin{cases}
\alpha_j &= \frac{4}{5j-6} \sum_{\ell=1}^{j-1} \alpha_{j-\ell} \alpha_{\ell} \\
\beta_j &= \frac{2}{5(5j-1)} \sum_{\ell=0}^{j-1} (3 \cdot 5^{j-\ell} - 5^{\ell+1} + 6) \alpha_{j-\ell} \beta_{\ell} \\
\gamma_j &= 3\alpha_{j+1} \\
\eta_j &= \frac{5^{j+1}}{2} \alpha_j + 2 \sum_{\ell=0}^{j-1} \eta_{\ell} \beta_{j-\ell}
\end{cases}
\end{equation}

where the initial values are: $\alpha_0 = 1$, $\alpha_1 = 1/6$, $\beta_0 = -1/2$, $\eta_0 = 0$, and $\partial_n P_{02}(q_1) = -1/2$. Moreover,

$$
\partial_n P_{j2}(q_1) = -\alpha_j \quad \text{for} \quad j \geq 1 \quad \text{and} \quad \partial_n P_{j3}(q_1) = 3\eta_{j+1} \quad \text{for} \quad j \geq 0.
$$

We can use Theorem B along with the Green-Gauss formula to compute the inner product among the monomials $P_{kj}^{(n)}$.

**Lemma 2.1.** Consider the monomials $\{P_{ji}\}_{j \geq 0}$ for $i = 1, 2, 3$. Then the following formulas hold:

\begin{equation}
\begin{cases}
\langle P_{j1}, P_{k1} \rangle &= 2 \sum_{\ell=j-m}^{\ell=j-m} \alpha_{j-\ell} \eta_{k+\ell+1} - \alpha_{k+\ell+1} \eta_{j-\ell} \\
\langle P_{j2}, P_{k2} \rangle &= -2 \sum_{\ell=j-m}^{\ell=j-m} \beta_{j-\ell} \alpha_{k+\ell+1} - \beta_{k+\ell+1} \alpha_{j-\ell} \\
\langle P_{j3}, P_{k3} \rangle &= 18 \sum_{\ell=j-m}^{\ell=j-m} \alpha_{j-\ell} \eta_{k+\ell+2} - \alpha_{k+\ell+2} \eta_{j-\ell+1} \\
\langle P_{j1}, P_{k2} \rangle &= -2 \sum_{\ell=0}^{\ell=0} \alpha_{j-\ell} \alpha'_{k+\ell+1} - \beta_{k+\ell+2} \eta_{j-\ell+1}
\end{cases}
\end{equation}

where $\alpha'_k = \alpha_k$ for $k \neq 1$ and $\alpha'_1 = -1/2$, and where $m_s = \min(j, k)$.

Moreover,

\begin{equation}
\langle P_{j1}, P_{k3} \rangle = \langle P_{j2}, P_{k3} \rangle = 0.
\end{equation}
In addition, for \( i = 3 \) the inner products among the different monomials associated with each of the boundary points \( q_n, n = 0, 1, 2 \) are given by

\[
\langle P^{(n)}_{j\bar{3}}, P^{(n)}_{k\bar{3}} \rangle = \langle P^{(0)}_{j\bar{3}}, P^{(0)}_{k\bar{3}} \rangle \\
\langle P^{(n)}_{j\bar{3}}, P^{(n')}_{k\bar{3}} \rangle = -\frac{1}{2} \langle P^{(0)}_{j\bar{3}}, P^{(0)}_{k\bar{3}} \rangle
\]

for \( n \neq n' \).

**Proof.** It is enough to prove that for each \( k, j \) and any \( i, i' \) we have:

\[
\langle P_{ji}, P_{ki'} \rangle = \sum_{\ell=0}^{j} \sum_{n=0}^{2} (P_{(j-\ell)i}(q_n) \partial_n P_{(k+1+\ell)i'}(q_n) - P_{(k+1+\ell)i'}(q_n) \partial_n P_{(j-\ell)i}(q_n))
\]

Notice that by a symmetry argument we can assume that \( k \geq j \) and proved the result by induction on \( j \). Fix \( i, i' \in \{1, 2, 3\} \) and notice that for all \( k \geq 0 \) Green-Gauss’ formula gives:

\[
\langle P_{0i}, P_{ki'} \rangle = \int_{SG} P_{0i} P_{ki'} d\mu = \int_{SG} P_{0i} \Delta P_{(k+1)i'} d\mu \\
= \int_{SG} \Delta P_{0i} P_{ki'} d\mu + \sum_{n=0}^{2} P_{0i}(q_n) \partial_n P_{ki'}(q_n) - P_{ki'}(q_n) \partial_n P_{0i}(q_n) \\
= \sum_{n=0}^{2} P_{0i}(q_n) \partial_n P_{(k+1)i'}(q_n) - P_{(k+1)i'}(q_n) \partial_n P_{0i}(q_n),
\]

where we have used the fact that \( \Delta P_{0i} = 0 \). This establishes (19) for \( j = 0 \), and all \( k \geq 0 \).

Now assume that for some \( j \) and for \( k \geq j \) we have

\[
\langle P_{ji}, P_{ki'} \rangle = \sum_{\ell=0}^{j} \sum_{n=0}^{2} (P_{(j-\ell)i}(q_n) \partial_n P_{(k+1+\ell)i'}(q_n) - P_{(k+1+\ell)i'}(q_n) \partial_n P_{(j-\ell)i}(q_n))
\]
we just have to show that this last relation holds for \( j + 1 \) and \( k \geq j + 1 \). But, by the Gauss-Green formula we have

\[
\langle P_{(j+1)i}, P_{kr} \rangle = \int_{SG} P_{(j+1)i} P_{kr} \, d\mu = \int_{SG} P_{(j+1)i} \Delta P_{(j+1)i} \, d\mu
\]

\[
= \langle P_{ji}, P_{(k+1)i'} \rangle + \sum_{n=0}^{2} (P_{(j+1)i}(q_n) \partial_n P_{(k+1)i'}(q_n) - P_{(k+1)i'}(q_n) \partial_n P_{(j+1)i}(q_n))
\]

\[
= \sum_{\ell=0}^{j} \sum_{n=0}^{2} (P_{(j-\ell)i}(q_n) \partial_n P_{(k+2+\ell)i'}(q_n) - P_{(k+2+\ell)i'}(q_n) \partial_n P_{(j-\ell)i}(q_n))
\]

\[
+ \sum_{n=0}^{2} (P_{(j+1)i}(q_n) \partial_n P_{(k+1)i'}(q_n) - P_{(k+1)i'}(q_n) \partial_n P_{(j+1)i}(q_n))
\]

\[
= \sum_{\ell=1}^{j+1} \sum_{n=0}^{2} (P_{(j+1-\ell)i}(q_n) \partial_n P_{(k+1+\ell)i'}(q_n) - P_{(k+1+\ell)i'}(q_n) \partial_n P_{(j+1-\ell)i}(q_n))
\]

\[
+ \sum_{n=0}^{2} (P_{(j+1)i}(q_n) \partial_n P_{(k+1)i'}(q_n) - P_{(k+1)i'}(q_n) \partial_n P_{(j+1)i}(q_n))
\]

\[
= \sum_{\ell=0}^{j+1} \sum_{n=0}^{2} (P_{(j+1-\ell)i}(q_n) \partial_n P_{(k+1+\ell)i'}(q_n) - P_{(k+1+\ell)i'}(q_n) \partial_n P_{(j+1-\ell)i}(q_n))
\]

This shows that (19) holds for all \( j \geq 0 \) and \( k \geq j \).

Next, (16) follows from (19) by symmetry consideration and by observing that the values and normal derivatives at \( q_0 \) vanish. In addition, Theorem B, especially (15) can be used to further simplify (19) and establish (16).

Note that (17) is trivial because of the symmetry of \( P_{j1} \) and \( P_{j2} \) and the anti-symmetry of \( P_{j3} \).

To prove (18), it is enough to show that

\[
\langle P_{j3}^{(0)}, P_{k3}^{(1)} \rangle = -\frac{1}{2} \langle P_{j3}^{(0)}, P_{k3}^{(0)} \rangle
\]

since the other inner products can be computed using similar arguments.

Using Equation (19), and canceling out zero terms yields

\[
\langle P_{j3}^{(0)}, P_{k3}^{(1)} \rangle = \sum_{\ell=0}^{j} P_{(j-\ell)3}^{(0)}(q_2) \partial_n P_{(k+\ell+1)3}^{(1)}(q_2) - P_{(k+\ell+1)3}^{(1)}(q_2) \partial_n P_{(j-\ell)3}^{(0)}(q_2)
\]

\[
= -9 \sum_{\ell=0}^{j} \alpha_{j-\ell+1} \eta_{k+\ell+2} - \alpha_{k+\ell+1} \eta_{k-\ell+1}
\]

\[
= -\frac{1}{2} \langle P_{j3}^{(0)}, P_{k3}^{(0)} \rangle
\]

□
3. Orthogonal polynomials on SG

We are now ready to construct the families of orthogonal polynomials on SG. These OP will be obtained by applying the Gram-Schmidt algorithm on the polynomials \( \{P_{jk}^{(n)}\}_{j \geq 0} \), where \( k = 1, 2, 3 \), and \( n = 0, 1, \text{ or } 2 \). In Subsections 4.1, and 4.2, we plot the graphs of various OP obtained from \( \{P_{jk}\}_{j \geq 0} \), and \( k = 1, 2, 3 \). As mentioned in the introduction, more graphs of these OP can be found at [19].

But we first derive a three-term recursion formula for our OP and give estimates on the size of the coefficients appearing in this recursion formula. These coefficients are extremely small. This creates major instability problems when plotting the corresponding OP. To circumvent this problem, we carry out our numerical simulation to arbitrary precision. A similar phenomenon was already observed in [11], and rational arithmetic was used. We also considered the use of rational arithmetic, but because it involves a prohibitive time cost, we settled for arbitrary precision instead.

3.1. General theory of orthogonal polynomials on SG.

**Definition 3.1.** Fix \( k = 1, 2 \text{ or } 3 \) and denote by \( \{p_j\}_{j=0}^\infty := \{p_{jk}\}_{j=0}^\infty \) the orthogonal polynomials obtained from \( \{P_{jk}\}_{j=0}^\infty \) by the Gram-Schmidt process, i.e.,

\[
p_j(x) = P_{jk} - \sum_{l=0}^{j-1} d_l^2 \langle P_{jk}, p_l \rangle p_l \text{ for each } j \geq 1.
\]

Consequently, there exists a set of coefficients \( \{\omega_{j,l}\}_{l=0}^j \), with \( \omega_{j,j} = 1 \), and such that

\[
p_j(x) = P_{jk} - \sum_{l=0}^{j-1} d_l^2 \langle P_{jk}, p_l \rangle p_l = P_{jk}(x) + \sum_{l=0}^{j-1} \omega_{j,l} P_{lk}(x), j \geq 1.
\]

Moreover,

\[
\langle p_j, p_\ell \rangle = d_j^{-2} \delta_{j,\ell} \text{ where } \|p_j\|_{L^2}^2 = d_j^{-2}.
\]

By normalizing the orthogonal polynomials \( \{p_j\}_{j=0}^\infty \) we obtain the family of orthonormal polynomials \( \{Q_j\}_{j=0}^\infty \) characterized by

\[
\begin{align*}
\langle Q_j, Q_k \rangle &= \delta_{j,k} \\
Q_j &= d_j p_j = d_j P_{jk} + d_j \sum_{l=0}^{j-1} \omega_{j,l} P_{lk}, j \geq 1.
\end{align*}
\]

**Theorem 3.1.** Given \( k = 1, 2 \text{ or } 3 \), and for each \( j \geq 0 \) the following holds:

\[
\|p_j\|_{L^2} = d_j^{-1} \leq \|P_{j,k}\|_{L^2}.
\]

Moreover, for any \( 0 < r < \infty \), there exist constants \( c_1, c_r > 0 \) such that for all \( j \geq 0 \)

\[
\|p_j\|_{L^2} = d_j^{-1} \leq c_1 (j!)^{-\log 5/\log 2} + c_r r^{-j}.
\]

In particular,

\[
\lim_{j \to \infty} \|p_j\|_{L^2} = \lim_{j \to \infty} d_j^{-1} = 0.
\]
Proof. Recall that by the definition of $p_j$ we can write that for $j \geq 1$, the Gram-Schmidt process yields $p_j = P_{jk} - \sum_{l=0}^{j-1} d_j^l \langle P_{jk}, p_l \rangle p_l$ for each $j \geq 1$. Therefore,

$$\|P_{jk}\|_{L^2}^2 = \|p_j\|_{L^2}^2 + \sum_{l=0}^{j-1} \langle P_{jk}, p_l \rangle^2 = d_j^{-2} + \sum_{l=0}^{j-1} \langle P_{jk}, p_l \rangle^2 \geq d_j^{-2}$$

which establishes the first estimate.

Now recall from \[11, Theorem 2.7\] and \[11, Theorem 2.13\], that for each $r \in (0, \infty)$, there are $c > 0$ and $c_r > 0$ such that

$$0 < \alpha_j < c(j!)^{-\log 5/\log 2} \quad 0 < |\eta_j| \leq c_r r^{-j} \quad \text{for all} \quad j \geq 0.$$

When $k = 1$, we get from Lemma 2.1 that

$$\|P_{j1}\|_{L^2}^2 = 2 \sum_{l=0}^{j-1} \alpha_{j-l} \eta_{j-l+1} - \alpha_{j+l+1} \eta_{j-l}$$

$$= 2 \sum_{l=0}^{j} \alpha_l \eta_{2j-l+1} - \sum_{l=0}^{j} \alpha_{2j-l+1} \eta_l$$

$$\leq 2 c r^{-j-1} \sum_{l=0}^{j} (l!)^{-\log 5/\log 2} + 2 c c_r (j+1)!^{-\log 5/\log 2} \sum_{l=0}^{j} r^{-\log 5/\log 2}$$

from which (21) follows. When $k = 2$ or $k = 3$ similar arguments are used to obtain the same estimate. It then follows that $\lim_{j \to \infty} d_j^{-1} = 0$ with at least an exponential rate of decay. \[\square\]

Our first main result is the following theorem that establishes the three-term recursion formula for OP on $SG$.

**Theorem 3.2.** Let $\{p_k\}_{k=0}^\infty$ be the orthogonal polynomials defined above. Let $f_0(x) = 0$ and for $k \geq 0$, let $f_{k+1}$ be the polynomial defined by

$$f_{k+1}(x) := -\int G(x, y) p_k(y) \, d\mu(y).$$

Set $p_{-1}(x) := 0$, and $p_0(x) = P_{03}(x)$. Then, for each $k \geq 0$

$$p_{k+1}(x) = f_{k+1}(x) - b_k p_k(x) - c_k p_{k-1}(x),$$

where

$$\begin{align*}
  b_k &= \frac{d_{k}^2}{d_{k}^1} \langle f_{k+1}, p_k \rangle, \\
  c_k &= \frac{d_{k+1}^2}{d_{k}^1} = \frac{\|p_k\|_{L^2}^2}{\|p_{k-1}\|_{L^2}^2}.
\end{align*}$$

Consequently,

$$d_k^{-2} = \|p_k\|_{L^2}^2 = d_0^{-2} c_1 c_2 c_3 \ldots c_{k-1} c_k.$$
Proof. Using the definition of $f_{k+1}$ we can write
\[
\langle f_{k+1}, p_j \rangle = -\int \int G(x, y) p_k(y) p_j(x) \, d\mu(y) \, d\mu(x) = \langle f_j, p_k \rangle
\]

Notice that $f_{j+1}$ is a polynomial of degree $j + 1$. Thus, $f_{j+1}$ is orthogonal to all $p_k$ with $j + 1 < k$. Therefore, $f_{k+1} = a_k p_{k+1} + b_k p_k + c_k p_{k-1}$ for some coefficients $a_k, b_k,$ and $c_k$. It is easy to see that $a_k = 1$ this follows from the fact $f_{k+1} - p_{k+1}$ is orthogonal to $p_{k+1}$. Thus, $f_{k+1} = b_k p_k + c_k p_{k-1}$. Taking inner product with $p_k$ yields the first equation in (24). Taking inner product with $p_{k-1}$ yields

\[
c_k \langle p_{k-1}, p_{k-1} \rangle = c_k d_{k-1}^2 = \langle f_{k+1}, p_{k-1} \rangle
\]
\[
= -\int_{SG} \int_{SG} G(x, y) p_k(y) p_{k-1}(x) \, d\mu(x) \, d\mu(y)
\]
\[
= \langle f_k, p_k \rangle = \langle p_k + b_{k-1} p_{k-1} + c_{k-1} p_{k-2}, p_k \rangle
\]
\[
= d_{k-1}^2
\]

which is exactly the second equation in (24).

Observe that the last equation in (24) is equivalent to $c_k d_k^2 = d_{k-1}^2$, which implies (25).

\[\square\]

The above results deserve some discussions. While (23) resembles (2), these two relations are fundamentally different. In fact, (2) is essentially the statement that $t \pi_k$ is a monomial of degree $k + 1$, and thus can be expressed in terms of $\pi_l$, $l \leq k + 1$. Because the product of polynomials is not a polynomial on $SG$, $t \pi_k$ is replaced by $f_{k+1}(x) = -\int G(x, y) p_k(y) \, d\mu(y)$. Therefore, unlike in the classical case in which all information on $\pi_{k+1}$ can be gathered from the coefficients in (2), on $SG$ we must also evaluate the auxiliary polynomial $f_{k+1}$. This presents an additional difficulty in carrying out any numerical simulations with our OP.

We now prove a version of Theorem 3.2 dealing with the three-term recurrence for the orthonormal polynomial $\{Q_k\}_{k=0}^\infty$.

**Theorem 3.3.** Let $\{Q_k\}_{k=0}^\infty$ be the orthogonal polynomials defined above. Let $\tilde{f}_0 = 0$ and for $k \geq 0$

\[
(26) \quad \tilde{f}_{k+1}(x) := -\int G(x, y) Q_k(y) \, d\mu(y).
\]

Then, for each $k \geq 0$

\[
(27) \quad \sqrt{k+1} Q_{k+1}(x) = \tilde{f}_{k+1}(x) - b_k Q_k(x) - \sqrt{k} Q_{k-1}(x),
\]

where $Q_{-1}(x) := 0$, $Q_0(x) = d_0 P_{03}(x)$, and $b_k$ and $c_k$ were defined in Theorem 3.2.

Proof. The proof follows from the fact that $p_k = \|p_k\|_{L^2} Q_k$ and (??). \[\square\]
We prove below certain properties about the coefficients $b_k$ and $c_k$ appearing in the three-term recursion formula.

**Theorem 3.4.** For each $k \geq 0$, we have

\begin{align*}
    &- \|G\|_{L^2} \leq b_k < 0, \quad (28) \\
    &0 < c_k \leq \|G\|_{L^2}^2, \quad (29) \\
    &\text{and} \\
    &d_{k+1}^{-1} \leq \|G\|_{L^2} d_k^{-1}. \quad (30)
\end{align*}

In particular,

\[ d_k^{-1} \leq d_0^{-1} \|G\|_{L^2}^k, \]

where $\|G\|_{L^2} = \left( \int_S G(x,y)^2 \, dx \, dy \right)^{1/2}$.

**Proof.**

\[ |b_k| = d_k^2 |\langle f_{k+1}, p_k \rangle| \leq d_k^2 \|f_{k+1}\|_{L^2} \|p_k\|_{L^2} \leq d_k^2 \|G\|_{L^2} \|p_k\|_{L^2} \|p_{k-1}\|_{L^2} = \|G\|_{L^2}. \]

Using the expression of $G(x,y)$ given in the proof of Theorem 2.1, we can write

\[ \langle f_{k+1}, p_k \rangle = - \sum_{l=1}^\infty \lambda_l^{-1} \langle p_k, u_l \rangle^2 < 0. \]

Hence, $b_k < 0$. Similarly,

\[ 0 < c_k d_{k-1}^{-2} = \langle f_{k+1}, p_{k-1} \rangle \leq \|f_{k+1}\|_{L^2} \|p_{k-1}\|_{L^2} \leq \|G\|_{L^2} \|p_k\|_{L^2} \|p_{k-1}\|_{L^2} = \|G\|_{L^2} d_k^{-1} d_{k-1}^{-1}. \]

Using the fact that $c_k = d_k^{-2} d_{k-1}^2$, we have

\[ 0 < c_k \leq \|G\|_{L^2} d_{k-1}^{-1} d_k^{-1} = \|G\|_{L^2} \sqrt{c_k}, \]

from which the second estimate follows.

The last estimate follows by observing that

\[ 0 < d_{k+1}^{-2} < \|f_{k+1}\|_{L^2}^2 = d_{k+1}^{-2} + b_k^2 d_k^{-2} + c_k^2 d_{k-1}^{-2} \leq \|G\|_{L^2}^2 d_k^{-2}. \]

Consequently,

\[ d_k^{-1} \leq d_0^{-1} \|G\|_{L^2}^k. \]

\[ \square \]

**Remark 3.1.** The last estimate in Theorem 3.4 along with the estimate on the $\|G\|_{L^2}$ given in Theorem 2.1 gives another proof that the sequence $d_k^{-1}$ decays exponentially fast.
3.2. **Recurrence relations for orthogonal polynomials on SG.** We have now proved via two different arguments that the coefficients $d_{k-1}$ are very small. This, together with the fact that we must also evaluate the auxiliary polynomial $f_{k+1}$ prevents us to use Theorem 3.2 to directly plot the OP we construct. Rather we use (23) together with the representation of $p_k$ in Definition 3.1 to plot these OP. More specifically, we identify each $p_j$ with a vector $\Omega_j \in \mathbb{R}_{j+1}^j$ where $\Omega_j(l) = \omega_{j,l}$ for $l = 0, 1, \ldots, j - 1$ and $\Omega_j(j) = 1$. This can be simply written as

$$p_j = \sum_{\ell=0}^{j} \omega_{j,\ell} P_{\ell,k} \leftrightarrow \Omega_j = \begin{pmatrix} \omega_{j,0} \\ \omega_{j,1} \\ \vdots \\ \omega_{j,j} \end{pmatrix} = \begin{pmatrix} \omega_{j,0} \\ \omega_{j,1} \\ \vdots \\ 1 \end{pmatrix}$$

We now derive a recursion relation for the coefficients $\Omega_j$ appearing in the above representation of $p_j$. We shall later, implement this recurrence using arbitrary precision to give accurate graphs of $Q_j = d_{j-1}^{-1} p_j$.

**Theorem 3.5.** The recurrence relation for the orthogonal polynomials $p_k$ is accomplished by evaluating the following equations for each $k \geq 0$.

$$\Omega_{k+1} = \begin{pmatrix} \zeta_k \\ \Omega_k \end{pmatrix} - b_k \begin{pmatrix} \Omega_k \\ 0 \end{pmatrix} - c_k \begin{pmatrix} \Omega_{k-1} \\ 0 \end{pmatrix},$$

or more specifically,

$$\begin{pmatrix} \omega_{k+1,0} \\ \omega_{k+1,1} \\ \vdots \\ \omega_{k+1,k-1} \\ \omega_{k+1,k} \\ 1 \end{pmatrix} = \begin{pmatrix} \zeta_k \\ \omega_{k,0} \\ \vdots \\ \omega_{k,k-2} \\ \omega_{k,k-1} \\ 1 \end{pmatrix} - b_k \begin{pmatrix} \omega_{k,0} \\ \omega_{k,1} \\ \vdots \\ \omega_{k,k-1} \\ 1 \\ 0 \end{pmatrix} - c_k \begin{pmatrix} \omega_{k-1,0} \\ \omega_{k-1,1} \\ \vdots \\ \omega_{k-1,k-2} \\ 1 \\ 0 \end{pmatrix},$$

where $b_k, c_k$ where defined in Theorem 3.2 and $\zeta_k = -\frac{1}{\gamma_0} \sum_{\ell=0}^{k} \omega_{k,\ell} \gamma_{\ell+1}$, with $\gamma_{\ell}$ given in (15).
Proof. Using (22) we can write

\[
f_{j+1} = - \int G(x, y)p_j(x)\,d\mu(y) = - \int G(x, y)j \sum_{\ell=0}^j \omega_{j,\ell}P_{\ell,k}\,d\mu(y)
\]

\[
= \sum_{\ell=0}^j \omega_{j,\ell} \left( - \int G(x, y)P_{\ell,k}\,d\mu(y) \right) = \sum_{\ell=0}^j \omega_{j,\ell} \left( P_{\ell+1,j} - \frac{\gamma_{\ell+1}}{\gamma_0}P_{0,k} \right)
\]

\[
= \zeta_j P_{0,k} + \sum_{\ell=0}^j \omega_{j,\ell}P_{\ell+1,k}
\]

\[
= \begin{pmatrix}
\zeta_j \\
\omega_{j,0} \\
\vdots \\
\omega_{j,j-2} \\
\omega_{j,j-1} \\
1
\end{pmatrix}
\]

\]

We now derive a version of the Christoffel-Darboux formulas for the OP \(\{Q_k\}\). As an application, we use these formulas to find the expansion of \(\Delta Q_k\) in terms of \(\{Q_l\}_{l=0}^{k-1}\).

**Theorem 3.6.** Let \(N \geq 0\), \(x, y \in SG\), and define \(K_N(x, y) = \sum_{k=0}^N Q_k(x)Q_k(y)\). Using the notations of the last theorem we have

\[
K_N(x, y) = \sqrt{c_{N+1}}[Q_N(x)\Delta Q_{N+1}(y) - Q_{N+1}(x)\Delta Q_N(y)] + \sum_{k=0}^N \tilde{f}_{k+1}(x)\Delta Q_k(y).
\]

In particular, for each \(x \in SG\) and \(N \geq 0\),

\[
K_N(x, x) = \sqrt{c_{N+1}}[Q_N(x)\Delta Q_{N+1}(x) - Q_{N+1}(x)\Delta Q_N(x)] + \sum_{k=0}^N \tilde{f}_{k+1}(x)\Delta Q_k(x).
\]

**Proof.** Let \(k \in \{0, 1, \ldots, N\}\), and \(x, y \in SG\). By (27) we have

\[
\tilde{f}_{k+1}(x)Q_k(y) - \tilde{f}_{k+1}(y)Q_k(x) = \sqrt{c_{k+1}}[Q_{k+1}(x)Q_k(y) - Q_{k+1}(y)Q_k(x)] - \sqrt{c_k}[Q_k(x)Q_{k-1}(y) - Q_k(y)Q_{k-1}(x)].
\]

Therefore, summing both sides we obtain

\[
\sum_{k=0}^N [\tilde{f}_{k+1}(x)Q_k(y) - \tilde{f}_{k+1}(y)Q_k(x)] = \sqrt{c_{N+1}}[Q_{N+1}(x)Q_N(y) - Q_{N+1}(y)Q_N(x)].
\]
Now, observe that
\[
\sum_{k=0}^{N} [\tilde{f}_{k+1}(x)Q_k(y) - \tilde{f}_{k+1}(y)Q_k(x)] = \sum_{k=0}^{N} \tilde{f}_{k+1}(x)Q_k(y) + \int_{SG} \sum_{k=0}^{N} G(y, z)Q_k(z)Q_k(x)dz
\]
\[
= \sum_{k=0}^{N} \tilde{f}_{k+1}(x)Q_k(y) + \int_{SG} G(y, z)K_N(z, x)dz
\]

Using this last equation and taking the Laplacian with respect to \(y\) of both sides of (34) yields,
\[
\Delta_y \left( \sum_{k=0}^{N} [\tilde{f}_{k+1}(x)Q_k(y) - \tilde{f}_{k+1}(y)Q_k(x)] \right) = \sqrt{c_{N+1}}[Q_{N+1}(x)\Delta Q_N(y) - Q_N(x)\Delta Q_{N+1}(y)]
\]
\[
= \sum_{k=0}^{N} \tilde{f}_{k+1}(x)\Delta Q_k(y) - K_N(x, y)
\]

Consequently,
\[
K_N(x, y) = \sqrt{c_{N+1}}[Q_{N+1}(x)\Delta Q_N(y) - Q_N(x)\Delta Q_{N+1}(y)] + \sum_{k=0}^{N} \tilde{f}_{k+1}(x)\Delta Q_k(y).
\]

We now use this theorem to find the coordinates of \(\Delta Q_k\) in terms of lower order polynomials. That is since, \(\Delta Q_k \in \text{span}\{Q_l\}_{l=0}^{k-1}\), we have
\[
\Delta Q_k(x) = \sum_{l=0}^{k-1} A_l^{(k)} Q_l(x),
\]
where \(A_l^{(k)} = \langle \Delta Q_k, Q_l \rangle\). These coefficients can be computed recursively using Theorem 3.6. More specifically, we have

**Corollary 3.1.** Let \(k \geq 1\), and assume \(\Delta Q_k(x) = \sum_{\ell=0}^{k-1} A_\ell^{(k)} Q_\ell(x)\). Then, \(A_{k-1}^{(k)} = \frac{1}{\sqrt{c_k}}\), and for \(\ell = 0, 1, \ldots, k-2\) we have
\[
A_\ell^{(k)} = -A_{\ell-1}^{(k-1)} \frac{b_{k-1}}{\sqrt{c_k}},
\]
with the initial condition \(\Delta Q_1 = A_0^{(1)} Q_0\) where \(A_0^{(1)} = \frac{1}{\sqrt{c_1}}\).

Furthermore, for each \(k \geq 1\) and \(\ell = 0, 1, \ldots, k-1\), \(A_\ell^{(k)} > 0\).

**Proof.** Use \(N = 0\) in (33), gives \(K_0(x, x) = Q_0^2(x) = \sqrt{c_1}Q_0(x)\Delta Q_1(x)\), and integrating this equality leads to \(1 = \sqrt{c_1}\langle \Delta Q_1, Q_0 \rangle\). Hence, \(\Delta Q_1(x) = A_0^{(1)} Q_0(x) = \frac{1}{\sqrt{c_1}}Q_0(x)\).

Next, write \(\Delta Q_2(x) = A_1^{(2)} Q_1(x) + A_0^{(2)} Q_0(x)\). Using \(N = 1\) in (33) gives
\[
K_1(x, x) = \sqrt{c_2}[Q_1(x)\Delta Q_2(x) - Q_2(x)\Delta Q_1(x)] + \tilde{f}_2(x)\Delta Q_1(x)
\]
where we use the fact that $\Delta Q_0 = 0$. Integrating this last expression leads to

$$2 = \sqrt{c_2} \langle \Delta Q_2, Q_1 \rangle - \frac{1}{\sqrt{c_1}} \langle Q_2, Q_0 \rangle + \frac{1}{\sqrt{c_1}} \langle \tilde{f}_2, Q_0 \rangle = \sqrt{c_2} A_1^{(2)} + \frac{1}{\sqrt{c_1}} \sqrt{c_1}.$$

Therefore, $A_1^{(2)} = \frac{1}{\sqrt{c_2}}$.

To compute $A_0^{(2)}$ we utilize (32) with $N = 1$. In particular, we multiply $K_1(x, y)$ by $Q_1(x)Q_0(y)$ and integrate the resulting equation with respect to both $x$ and $y$. This will give

$$0 = \sqrt{c_2} \langle \Delta Q_2, Q_0 \rangle + \frac{1}{\sqrt{c_1}} \langle \tilde{f}_2, Q_1 \rangle \langle Q_0, Q_0 \rangle = \sqrt{c_2} A_0^{(2)} + \frac{b_1}{\sqrt{c_1}}.$$

Consequently, $A_0^{(2)} = -\frac{1}{\sqrt{c_2}} \frac{b_1}{\sqrt{c_1}} = -A_0^{(1)} \frac{b_1}{\sqrt{c_1}}$. Observe that $\tilde{f}_2(x) = \sqrt{c_2} Q_2(x) + b_1 Q_1(x) + \sqrt{c_1} Q_0(x)$ was used in the above arguments.

Assume now that we have determined the coefficients $\{A_\ell^{(k)}\}_{\ell=0}^{k-1}$ of $\Delta Q_k$ with respect to the orthonormal set $\{Q_\ell\}_{\ell=0}^{k-1}$. We can use an induction argument to find the coefficients of $\Delta Q_{k+1}$ in the orthonormal set $\{Q_\ell\}_{\ell=0}^{k}$. Set $N = k$ in (33), and integrate the resulting equation to obtain $A_k^{(k+1)} = \frac{1}{\sqrt{c_{k+1}}}$.

To obtain the remaining coefficients, $A_\ell^{(k+1)}$ for $l = 0, 1, \ldots, k - 1$, set $N = k$ in (32), multiply the resulting equation by $Q_k(x)Q_l(y)$ and integrate with respect to both $x$ and $y$.

To obtain the positivity of the coefficients $A_\ell^{(k)}$, we proceed by induction on $k$. Clearly, $A_0^{(1)} = \frac{1}{\sqrt{c_1}} > 0$, and $A_0^{(2)} = -A_0^{(1)} \frac{b_1}{\sqrt{c_2}} > 0$ since $b_k < 0$ for all $k$. In addition, $A_1^{(2)} = \frac{1}{\sqrt{c_2}} > 0$. For assume, that for $k \geq 2$, and each $\ell = 0, 1, \ldots, k - 1$, we have $A_\ell^{(k)} > 0$. Then, $A_\ell^{(k+1)} = \frac{1}{\sqrt{c_{k+1}}}$ $> 0$ and for each $\ell = 0, 1, \ldots, k - 1$ we have $A_\ell^{(k+1)} = -A_\ell^{(k)} \frac{b_k}{\sqrt{c_{k+1}}} > 0$ since $b_k < 0$ and $A_\ell^{(k)} > 0$ by the induction hypothesis. This concludes the proof. \qed

\textbf{Remark 3.2}. Note that Theorem 3.6 and Corollary 3.1 hold for the symmetric OP $S_k$ defined in Subsection 4.2.

We end this section with an investigation of the tri-diagonal symmetric Jacobi matrix corresponding to the OP $\{Q_k\}$. As in the classical case of OP defined on the real line, each of the families of orthogonal polynomials we constructed above is related to a tri-diagonal, symmetric Jacobi matrix that we denote $J_n$. In particular, we can write (27) of Theorem 3.3 in a matrix form that will involve this Jacobi polynomial. To do this we use the following notations:

$$\tilde{F}_n = [\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{n-1}]^T \quad Q^{(n)} = [Q_0, Q_1, \ldots, Q_{n-1}]^T,$$
and
\[
J_n = \begin{pmatrix}
    b_0 & \sqrt{c_1} & 0 & 0 & \cdots & 0 & 0 \\
    \sqrt{c_1} & b_1 & \sqrt{c_2} & 0 & \cdots & 0 & 0 \\
    0 & \sqrt{c_2} & b_2 & \sqrt{c_3} & \cdots & 0 & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & 0 & 0 & \cdots & \sqrt{c_{n-1}} & b_{n-1}
\end{pmatrix}
\]

By writing (27) in a matrix-vector form we have for each \(n \geq 2\)
\[
\tilde{F}_n(x) = J_n Q^{(n)}(x) + \sqrt{c_n} Q_n(x)e_n
\]
where \(e_n = [0, 0, \ldots, 0, 1]^T\).

For the OP we consider in this paper, though \(J_n\) is invertible for all \(n\), its determinant is extremely small. Recall [8, Theorem 0.9.10] that the determinant of a tri-diagonal symmetric Jacobi matrix obeys the following recursion formula. Let \(D_n = \text{Det}(J_n)\), then
\[
D_{n+1} = b_{n-1} D_n - c_{n-1} D_{n-1}
\]
with initial conditions \(D_0 = b_0\) and \(D_{-1} = 1\).

To illustrate the fact that the Jacobi matrices associated to the OP considered here have small determinant, we show below a graph of \(\log |D_n|\) as a function of \(n\). This graph is generated using the antisymmetric OP that will be constructed explicitly in the next section.

![Graph of \(\log |D_k|\) of the determinants of the Jacobi matrix, \(D_k\)](image)

**Figure 2.** Graph of \(\log |D_k|\) of the determinants of the Jacobi matrix, \(D_k\)

### 4. Numerical results

We shall now generate graphs of the OP we constructed above starting from two families of monomials on SG. We also present graphs of the sequences appearing in the three-terms recursion formula. We start by considering a family of antisymmetric monomials.
4.1. **Antisymmetric orthonormal polynomials.** Recall the monomials \( \{P_j\} \) in Definition 3.1 are centered around the point \( q_0 \) and antisymmetric with respect to that point. We now use these monomials to generate the family of antisymmetric orthogonal polynomials \( \{P_j\}_{j=0}^{\infty} \) and their corresponding normalized version \( \{Q_j\}_{j=0}^{\infty} \). We do this inductively using Theorems 3.2 and 3.5. In the process, we compute the sequence \( d_k, c_k \) and \( b_k \).

More specifically, it is easily seen that \( \|P_0\|_{L^2} = \frac{3}{10} \). Next we use Theorem 3.5 to find the auxiliary polynomial \( f_1(x) = -\int_{SG} G(x, y) P_0(y) dy = -\int_{SG} G(x, y) P_{0,3}(y) dy \).

Notice that we impose the condition \( f_1|_{V_0} = 0 \). Thus, \( b_0 \) can now be computed, which, together with the polynomial \( f_1 \), is used to find the polynomial \( p_1 \). Next, \( f_2 \) is computed again using Theorem 3.5 from which one gets \( d_1 \). The next step is to compute \( c_1 \). Using this, \( b_1 \) and the polynomial \( p_2 \) are calculated yielding \( c_2 \). Continuing these recursions, we construct \( p_k, k \geq 2 \) and all the related sequences.

To better display the results of these computations, we use a logarithmic scale to plot the numerical sequences obtained above. In particular, we plot \( \log(\|P_k\|_{L^2}) = \log(d_k), \log(-b_k), \) and \( \log c_k \) versus \( k \). These graphs are displayed in Figure 3. In particular, these graphs illustrate Theorems 3.1 and 3.4. That is \( \|P_k\|_{L^2} \) tends to infinity exponentially fast. In addition the graphs of \( \log(-b_k) \) and \( \log c_k \) suggest that \( c_k < -b_k \) for \( k > k_0 \) for some \( k_0 \). This seems to be the case for \( k < 200 \), but we do not have data beyond this range of \( k \) to confirm or infirm the observation.

In Figure 4 we plot the orthonormal polynomials \( Q_j = d_j P_j \) for \( j = 0, 3, 4, 7 \), on \( \Gamma_7 \) the level 7 approximation to \( SG \). To do this, we use the recursion formula given by Theorem 3.5 along with the second half of the representation of \( P_j \) given in (20), as well as the graphs of the monomials \( P_{j,3} \) which were given in (11). Moreover, most of the computations have been carried out to arbitrary precision since the coefficients \( \omega_{j,l} \) are very small. We refer to Table 1 for approximate values of these coefficients for \( p_j, j = 0, 1, \ldots, 6 \).

The fact that \( Q_k \) are antisymmetric is apparent from Figure 4. In addition, we observe that for \( i = 1, 2, |Q_k(q_i)| \) is larger compare to \( |Q_k(x)| \) for any \( x \in SG \setminus \{q_1, q_2\} \). This is consistent with the behavior of classical OP such as the Legendre polynomial on \([0, 1]\); see (7).

4.2. **Symmetric orthogonal polynomials.** We next carried out the constructions of subsection 4.1 to the fully symmetric polynomials which we first define. By fully symmetric we mean symmetric under all rotations and flips in \( D_3 \). Then we will apply the results of section 3 to this family of fully symmetric polynomials to obtain the symmetric OP \( \{s_k\}_{k=0}^{\infty} \) and their normalized counterpart \( \{S_k\}_{k=0}^{\infty} \).
Figure 3. Plots of the log $d_k^2$, log($-b_k$), and log $c_k$ in log scale for the antisymmetric OP $p_k$.

Table 1. $p_k$ and their coefficients $\omega_{j\ell}$

| $p_k$  | $P_{0,3}$ | $P_{1,3}$ | $P_{2,3}$ | $P_{3,3}$ | $P_{4,3}$ | $P_{5,3}$ | $P_{6,3}$ |
|--------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $p_0$  | 1         | 0         | 0         | 0         | 0         | 0         | 0         |
| $p_1$  | -2.04E-02 | 1         | 0         | 0         | 0         | 0         | 0         |
| $p_2$  | 1.08E-04  | -1.30E-02 | 1         | 0         | 0         | 0         | 0         |
| $p_3$  | -3.23E-07 | 6.03E-05  | -9.74E-03 | 1         | 0         | 0         | 0         |
| $p_4$  | 1.99E-10  | -6.41E-08 | 2.14E-05  | -6.01E-03 | 1         | 0         | 0         |
| $p_5$  | -8.20E-14 | 5.16E-11  | -2.94E-08 | 1.41E-05  | -5.03E-03 | 1         | 0         |
| $p_6$  | 4.43E-17  | -3.53E-14 | 2.63E-11  | -1.77E-08 | 9.99E-06  | -4.23E-03 | 1         |

Definition 4.1. Define the fully symmetric monomial as follows for each $j \geq 0$:

$$\rho_j = P_{j1}^{(0)} + P_{j1}^{(1)} + P_{j1}^{(2)}$$

The symmetric OP denoted $\{s_j\}_{j=0}^\infty$ are now obtained from these symmetric polynomials by applying the Gram-Schmidt process and satisfies

$$\langle s_j, s_k \rangle = d_j^{-2}\delta_{jk},$$

where $d_j^{-2} = \|s_j\|^2$. Moreover, $s_j = \rho_j + \sum_{l=0}^{j-1} \mu_{j,l} \rho_l$ for a set of coefficients $\{\mu_{j,l}\}_{l=0}^{j-1}$.

By normalizing the orthogonal polynomials $s_j$ we obtain the orthonormal OP denoted $\{S_j\}_{j=0}^\infty = \{d_j s_j\}_{j=0}^\infty$. 
Remark 4.1. Note that the above definitions will remain unchanged if in defining the symmetric polynomials $\rho_j$ we used $P_j^2$ instead of $P_j^1$; see [11] for details about this.

Observe that for each $j, k \geq 0$ we have $\langle \rho_j, \rho_k \rangle = 6\langle P_j^1, P_k^1 \rangle$, which implies that for $j \geq 0$, $\| \rho_j \|_{L^2}^2 = d_j^2 = 6\| P_j^1 \|_{L^2}^2$. We remark that the size of $\| s_j \|_{L^2}$ was already estimated in Theorem 3.1. Furthermore, one checks easily that $d_0^2 = \| \rho_0 \|_{L^2}^2 = 6\| P_0^1 \|_{L^2}^2 = 6$

As in subsection 4.1, we now plot the symmetric orthonormal polynomials as well as some of the related sequence. In particular, Figure 5 displays plots of the sequences $\log(\| s_k \|_{L^2}^{-2}) = \log(d_k^2)$, $\log(-b_k)$, and $\log c_k$ for these symmetric OP. The behavior of the coefficients $c_k, b_k$, and $d_k$ in the case of the symmetric OP $S_k$ are very similar to those we observed for the antisymmetric OP.

In Table 2 we show the coefficients $\mu_{j, \ell}$ for $s_j$, when $j = 0, 1, \ldots, 7$

Four of the symmetric orthogonal polynomials $S_j = d_j s_j$, corresponding to $j = 0, 3, 5, \text{ and } 6$ are shown in Figures 6. As observed for the antisymmetric OP, $|S_k(q_i)|$, which is constant for $i = 1, 2, 3$ (due to symmetry) is very large compare to the values of $S_k$ at non-boundary points in $SG$.

4.3. Orthonormal system of polynomials. By combining both the antisymmetric and symmetric OP, we can form an orthogonal system of polynomials in $L^2(SG)$.

![Figure 4](image-url) 4 antisymmetric orthonormal polynomials.
Orthogonal Polynomials on the Sierpinski Gasket

Figure 5. Plots of the log $d_k^2$, log($-b_k$), and log $c_k$ for the symmetric OP $s_k$ in log scale.

Table 2. $s_j$ and their coefficients $\mu_{j,\ell}$

| $s_j$ | $\rho_0$ | $\rho_1$ | $\rho_2$ | $\rho_3$ | $\rho_4$ | $\rho_5$ | $\rho_6$ |
|-------|----------|----------|----------|----------|----------|----------|----------|
| $s_0$ | 1        | 0        | 0        | 0        | 0        | 0        | 0        |
| $s_1$ | -5.56E-02| 0        | 0        | 0        | 0        | 0        | 0        |
| $s_2$ | 5.34E-04 | 1        | 0        | 0        | 0        | 0        | 0        |
| $s_3$ | -6.98E-07| 9.72E-05| -1.23E-02| 1        | 0        | 0        | 0        |
| $s_4$ | 1.56E-09 | -3.20E-07| 6.15E-05| -9.93E-03| 1        | 0        | 0        |
| $s_5$ | -1.46E-12| 3.54E-10| -9.41E-08| 2.62E-05| -6.48E-03| 1        | 0        |
| $s_6$ | 1.44E-16 | -8.51E-14| 5.21E-11| -2.94E-08| 1.41E-05| -5.02E-03| 1        |

Notice that this system will not span the whole space $L^2$, since the space of polynomials is not dense in $L^2(SG)$ either.

Using Lemma 2.1 we can prove that the set $\{P_{j3}^{(0)}$, $P_{j3}^{(1)}$, $P_{j3}^{(2)}\}$ forms a tight frame for its span $F_j = \text{span}\{P_{j3}^{(0)}$, $P_{j3}^{(1)}$, $P_{j3}^{(2)}\}$. That is, there exists a constant $A_j > 0$ such that for each $P \in F_j = \text{span}\{P_{j3}^{(0)}$, $P_{j3}^{(1)}$, $P_{j3}^{(2)}\}$ we have

$$P(x) = A_j \sum_{n=0}^{2} \langle P, P_{j3}^{(n)} \rangle P_{j3}^{(n)}(x), \forall x \in SG.$$  

For more on frame theory we refer to [3].
Theorem 4.1. For each $j \geq 0$, the set $\{P_{j3}^{(0)}, P_{j3}^{(1)}, P_{j3}^{(2)}\}$ forms a tight frame for its two-dimensional span $\mathcal{F}_j = \text{span}\{P_{j3}^{(0)}, P_{j3}^{(1)}, P_{j3}^{(2)}\}$ with frame bound $A_j = \frac{3}{2} \|P_{j3}^{(0)}\|_{L^2}^2$.

Proof. The frame bounds of $\{P_{j3}^{(0)}, P_{j3}^{(1)}, P_{j3}^{(2)}\}$ are determined by the eigenvalues of its Gram matrix $G$ given by

$$G = \begin{pmatrix} P_{j3}^{(0)} & P_{j3}^{(1)} & P_{j3}^{(2)} \\ P_{j3}^{(1)} & P_{j3}^{(0)} & P_{j3}^{(2)} \\ P_{j3}^{(2)} & P_{j3}^{(1)} & P_{j3}^{(0)} \end{pmatrix} = \frac{1}{2} \|P_{j3}^{(0)}\|_{L^2}^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$G$ has eigenvalues $\{0, \frac{3}{2} \|P_{j3}^{(0)}\|_{L^2}^2, \frac{3}{2} \|P_{j3}^{(0)}\|_{L^2}^2\}$. This automatically shows that the frame operator

$$S = \begin{pmatrix} P_{j3}^{(0)} & P_{j3}^{(1)} & P_{j3}^{(2)} \\ P_{j3}^{(1)} & P_{j3}^{(0)} & P_{j3}^{(2)} \\ P_{j3}^{(2)} & P_{j3}^{(1)} & P_{j3}^{(0)} \end{pmatrix}$$

is the $2 \times 2$ diagonal matrix with $\frac{3}{2} \|P_{j3}^{(0)}\|_{L^2}^2$ on the diagonal. Thus, $\{P_{j3}^{(0)}, P_{j3}^{(1)}, P_{j3}^{(2)}\}$ is a tight frame with frame bound $A_j = \frac{3}{2} \|P_{j3}^{(0)}\|_{L^2}^2$. 

\[\square\]

Figure 6. 4 symmetric orthonormal polynomials.
More generally, the same argument can be used to prove that, 

\[ \{ \phi_j^{(i)} , i = 0, 1, 2 \} \] 

is an orthonormal basis for a subspace \( \mathcal{P} \) of \( L^2(SG) \).

**Proof.** For \( j \geq 0 \) and \( i \in \{0, 1, 2\} \), let an element in the above system be defined by 

\[ \phi_j^{(i)} = \sqrt{\frac{2}{3}} Q_j^{(i)} + \sqrt{\frac{1}{3}} S_j. \]

Then, the inner product of two orthonormal system elements is:

\[
\langle \phi_j^{(i)}, \phi_k^{(\ell)} \rangle = \langle \sqrt{\frac{2}{3}} Q_j^{(i)} + \sqrt{\frac{1}{3}} S_j, \sqrt{\frac{2}{3}} Q_k^{(\ell)} + \sqrt{\frac{1}{3}} S_k \rangle
= \frac{2}{3} \langle Q_j^{(i)}, Q_k^{(\ell)} \rangle + \frac{\sqrt{2}}{3} \langle S_j, Q_k^{(\ell)} \rangle + \frac{\sqrt{2}}{3} \langle Q_j^{(i)}, S_k \rangle + \frac{1}{3} \langle S_j, S_k \rangle.
\]

Recall that \( S_j \) is fully symmetric and \( Q_j^{(n)} \) is antisymmetric, so their inner product is zero for all \( j \) and \( n \). Moreover, if \( n \neq n' \), by the Gauss Green formula, we can rewrite

\[
\langle Q_k^{(n)}, Q_k^{(n')} \rangle = \sum_{j=0}^{k} \sum_{i=0}^{2} Q_j^{(n)}(q_i) \partial_n Q_j^{(n')}(q_i) - Q_j^{(n')}(q_i) \partial_n Q_j^{(n)}(q_i).
\]

We now evaluate the right-hand side of the last equation. When \( i = n \) or \( i = n' \), \( Q_j^{(n)}(q_i) = 0 \), and this eliminates two terms for each \( j \). Let \( a_j = Q_j^{(n)}(q_{n+1}) \), then \( -a_j = Q_j^{(n)}(q_{n-1}) \) (by symmetry). If we let \( b_j = \partial_n Q_j^{(n)}(q_{n+1}) \), then \( -b_j = \partial_n Q_j^{(n)}(q_{n-1}) \). If we let \( c_j = \partial_n Q_j^{(n)}(q_n) \), then \( c_j = 0 \) since \( \partial_n P_{j+3}^{(n)}(q_n) = 0 \) for all \( j \). Then we have

\[
\langle Q_k^{(n)}, Q_k^{(n')} \rangle = \sum_{j=0}^{k} \left[ 0 - (-a_j)(c_j) + (a_j)(-b_j) - (-a_j)(b_j) + (a_j)(c_j) - 0 \right] = 2a_j c_j = 0.
\]

More generally, the same argument can be used to prove that, \( \langle Q_k^{(n)}, Q_{k'}^{(n')} \rangle = \delta_{k,k'} \delta_{n,n'} \).

Consequently,

\[
\langle \phi_j^{(i)}, \phi_k^{(\ell)} \rangle = \frac{2}{3} \delta_{j,k} \delta_{i,\ell} + \frac{1}{3} \delta_{j,k} = \delta_{j,k} \delta_{i,\ell}.
\]

**Remark 4.2.** Notice that using a different normalization we can show that the set

\[
\left\{ \sqrt{\frac{2}{3}} \frac{d_j}{d_j} p_j^{(i)} + \sqrt{\frac{1}{3}} s_j, i = 0, 1, 2 \right\}_{j=0}^{\infty}
\]

is an orthonormal basis for the same subspace \( \mathcal{P} \).
is also an orthogonal system for $L^2(SG)$.

Four of these orthonormal polynomials are plotted in Figure 7 and as might be expected, these OP do not seem to possess any obvious symmetry.

![Plot of orthonormal polynomials](image)

**Figure 7.** 4 orthonormal polynomials.

### 4.4. Zero sets of orthogonal polynomials on $SG$

Classical orthogonal polynomial theory suggests that zeros of higher order polynomials should interweave between the zeros of the next higher order polynomial. Specifically, if $x_0$ is a zero of $p_n$, then there exist $x_a$ and $x_b$ zeros of $P_{n+1}$ such that $x_a < x_0 < x_b$.

On $SG$, we have not been able to establish a similar result. In fact, it is not clear what ‘interweaving’ will mean in the fractal setting. More generally, on $SG$, the zero sets of the OP seem very difficult to fully describe. However, using (35) we see that an exact characterization of the zero set of $Q_n$ is the set of all points $x_0 \in SG$ such that

$$\tilde{F}_n(x_0) = J_n Q^{(n)}(x_0).$$

Though this looks similar to the characterization of the zeros of classical orthogonal polynomials in terms of the eigenvectors of a Jacobi matrix, there is a main difference in that $\tilde{F}_n(x_0)$ is a function of the auxiliary polynomials $\tilde{f}_k, k-0,1,\ldots,n-1$. Despite this difficulty, we have some numerical data, suggesting some structures in these zero sets. In particular, we shall display below nodal domains corresponding to some of these OP.
We first consider the zeros of the OP on the edges of $SG$. Notice that there are two types of edges, which we call bottom edge (the edge across from $q_0$) and side edges (edges which meet $q_0$). We present some numerical data in figures 8, 9, and 10, which seem to indicate that the zeros on these edges behave like zeros of classical orthogonal polynomials. Because $S_k$ is fully symmetric its behavior on side edges is the same as their behavior on the bottom edge. The restriction of $Q_k$ to the side edge of $SG$ as illustrated in Figure 10.

More generally, we have been able to numerically generate the nodal domains of both the first 20 antisymmetric and first 20 symmetric OP on $SG$. For example, the nodal domains of a polynomial $f$ on $SG$ are defined as follows: Let

$$Z(f) = f^{-1}\{0\} = \{x \in SG : f(x) = 0\}$$

be the zero set (or the nodal set) of $f$. Then, $SG \setminus Z(f)$ can be partitioned into finitely many connected domains $D_1, D_2, \ldots, D_{\nu_k}$, where $\nu_k$ depends on the degree of the polynomial $f$. These domains $\{D_\ell\}_{\ell=1}^{\nu_k}$ are the nodal domains of $f$. We refer to [19] for color graphs that support the above bound on $\nu_k$. 
Remark 4.3. As mentioned in the Introduction, we have generated many data for all the OP constructed here. These data as well as the codes used to generate them are available at [19]. Certain data that are not included in the present article, deal with the dynamics of the OP at certain points of SG. That is, a choice of OP type and a fixed point $x \in SG$ yield a set of points $\{p_k(x), p_{k+1}(x)\}$ in the plane. The passage from $k$ to $k+1$ is thought of as a dynamical system in the plane with some ”attractor” toward which the points tend as $k \to \infty$. With our numerics, we had hoped to get a ”snapshot” of this attractor. The computational complexity associated with our construction, limited us to generate data for these dynamics only for $1 \leq k \leq 200$. Thus, we could not make some conclusive statements about the long term behavior of these dynamics. However, we refer the interested reader to [19] for graphs of some of these dynamics. Our investigation in this context is parallel to recent results for the dynamics of certain OP related to self-similar measures on the unit interval [7].

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