ON SHARP EMBEDDINGS OF BESOV AND
TRIEBEL-LIZORKIN SPACES IN THE SUBCRITICAL CASE

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ABSTRACT. We discuss the growth envelopes of Fourier-analytically defined Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in the limiting case $s = \sigma_p := n \max\left(\frac{1}{p} - 1, 0\right)$. These results may also be reformulated as optimal embeddings into the scale of Lorentz spaces $L^p_{p,q}(\mathbb{R}^n)$. We close several open problems outlined already in [H. Triebel, The structure of functions, Birkhäuser, Basel, 2001] and explicitly stated in [D. D. Haroske, Envelopes and sharp embeddings of function spaces, Chapman & Hall/CRC, Boca Raton, FL, 2007].

1. INTRODUCTION AND MAIN RESULTS

In this paper we prove sharp embedding theorems for Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in some limiting cases of the range guaranteeing that these spaces consist of locally integrable functions. As proven in [12, Theorem 3.3.2],

\begin{equation}
B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^n) \iff \begin{cases} 
\text{either } s > \sigma_p := n \max\left(\frac{1}{p} - 1, 0\right), \\
\text{or } s = \sigma_p, 1 < p \leq \infty, 0 < q \leq \min(p, 2), \\
\text{or } s = \sigma_p, 0 < p \leq 1, 0 < q \leq 1
\end{cases}
\end{equation}

and

\begin{equation}
F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^n) \iff \begin{cases} 
\text{either } s > \sigma_p, \\
\text{or } s = \sigma_p, 1 \leq p < \infty, 0 < q \leq 2, \\
\text{or } s = \sigma_p, 0 < p < 1, 0 < q \leq \infty.
\end{cases}
\end{equation}

The embeddings can be measured quantitatively by the growth envelope function of $X$ as defined by D. D. Haroske and H. Triebel (see [5], [6], [16] and the references given there) by

$$\mathcal{E}^X_G(t) := \sup_{\|f\|_X \leq 1} f^*(t), \quad 0 < t < 1,$$

where $f^*$ denotes the non-increasing rearrangement of $f$. 

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In the case where \( E_G^X(t) \approx t^{-\alpha} \) for \( 0 < t < 1 \) and some \( \alpha > 0 \) the growth envelope index \( u_X \) is given as the infimum of all numbers \( v \), \( 0 < v \leq \infty \), such that

\[
\left( \int_0^t \left[ \frac{f^*(t)}{E_G^X(t)} \right]^v \frac{dt}{t} \right)^{1/v} \leq c \| f \|_X
\]  

with the usual modification for \( v = \infty \) holds for some \( \epsilon > 0 \), \( c > 0 \) and all \( f \in X \).

The pair \( E_G(X) = (E_G^X, u_X) \) is called the growth envelope for the function space \( X \).

In the case \( \sigma_p < s \), the growth envelopes of \( A^s_{p,q}(\mathbb{R}^n) \) are known; cf. \[16\] Theorem 15.2 and \[8\] Theorem 8.1. If \( s = \sigma_p \) and \[11\] or \[24\] is fulfilled in the \( B \) or \( F \) case, respectively, then the growth function is given by \( t^{-\min(s,p)} \), but the known information about the growth index \( u \) is not complete; cf. \[16\] Remarks 12.5, 15.1 and \[8\] Props. 8.12, 8.14 and Remark 8.15.

The growth index of \( B^\sigma_{p,q}(\mathbb{R}^n) \) satisfies

\[
\begin{cases}
q \leq u \leq p & \text{if } 1 \leq p < \infty \text{ and } 0 < q \leq \min(p,2), \\
q \leq u \leq 1 & \text{if } 0 < p < 1 \text{ and } 0 < q \leq 1.
\end{cases}
\]

The growth index of \( F^\sigma_{p,q}(\mathbb{R}^n) \) satisfies \( p \leq u \leq 1 \) if \( 0 < p < 1 \) and \( 0 < q \leq \infty \) and is equal to \( p \) if \( 1 \leq p < \infty \) and \( 0 < q \leq 2 \).

The growth envelopes of \( B^\sigma_{p,q} \) defined on the torus \( \mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n \) with \( 1 \leq q \leq 2 \) were identified recently by Seeger and Trebels in \[10\] and are equivalent to \( | \log t |^{1/q'} \) for \( 0 < t \leq 1/2 \). We fill the remaining gaps for the range \( p < \infty \).

**Theorem 1.1.** (i) Let \( 1 \leq p < \infty \) and \( 0 < q \leq \min(p,2) \). Then

\[
E_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, p).
\]

(ii) Let \( 0 < p < 1 \) and \( 0 < q \leq 1 \). Then

\[
E_G(B^\sigma_{p,q}) = (t^{-1}, q).
\]

(iii) Let \( 0 < p < 1 \) and \( 0 < q \leq \infty \). Then

\[
E_G(F^\sigma_{p,q}) = (t^{-1}, p).
\]

These results are closely related to optimal embeddings into the scale of Lorentz spaces. In this context, we prove the following.

**Theorem 1.2.** (i) Let \( 1 \leq p < \infty \) and \( 0 < q \leq \min(p,2) \). Then

\[
B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n).
\]

(ii) Let \( 0 < p < 1 \) and \( 0 < q \leq 1 \). Then

\[
B^\sigma_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n).
\]

(iii) Let \( 0 < p < 1 \) and \( 0 < q \leq \infty \). Then

\[
F^\sigma_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n)
\]

and all these embeddings are optimal with respect to the second fine parameter of the scale of the Lorentz spaces.

**Remark 1.3.** (i) Let us observe that \[5\] improves \[12\] Theorem 3.2.1 and \[11\] Theorem 2.2.3, where the embedding \( B^\sigma_{p,q}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n) \) is proved for all \( 0 < p < 1 \) and \( 0 < q \leq 1 \).
(ii) We also mention that growth envelopes for function spaces with minimal smoothness were recently studied in [2]. These authors worked with spaces defined by differences and their results differ from ours in logarithmic factors. This shows indirectly that the Fourier-analytical definition and the classical definition of Besov spaces do not coincide for \( s = 0 \), an effect observed in detail recently by Schneider [9].

We denote the Lebesgue and Lorentz spaces by \( L_p(\mathbb{R}^n) \) and \( L_{p,q}(\mathbb{R}^n) \), respectively. The reader may consult [13] Chapter 5, Section 3 or [1] Chapter 4, Section 4. We shall use the following well-known property of Lorentz spaces \( L_{1,q} \). Its proof follows immediately from Hardy’s lemma (cf. [1] Chapter 2, Proposition 3.6).

**Lemma 1.4.** Let \( 0 < q < 1 \). Then the \(|| \cdot ||_{L_{1,q}(\mathbb{R}^n)}|| \) is the \( q \)-norm; it means that
\[
||f_1 + f_2||_{L_{1,q}(\mathbb{R}^n)}||^q \leq ||f_1||_{L_{1,q}(\mathbb{R}^n)}||^q + ||f_2||_{L_{1,q}(\mathbb{R}^n)}||^q
\]
holds for all \( f_1, f_2 \in L_{1,q}(\mathbb{R}^n) \).

We work with Fourier analytically defined Besov and Triebel-Lizorkin spaces \( B_{p,q}^s(\mathbb{R}^n) \) and \( F_{p,q}^s(\mathbb{R}^n) \) as studied for example in [8], [14], [15] and [17]. We shall also use the sequence spaces \( b_{p,q}^s \) associated to \( B_{p,q}^s(\mathbb{R}^n) \) in a way described in [17] Chapters 2 and 3. This approach goes back to [3] and [4].

All the unimportant constants are denoted by the letter \( c \), whose meaning may differ from one occurrence to another. If \( \{a_n\}_{n=1}^\infty \) and \( \{b_n\}_{n=1}^\infty \) are two sequences of positive real numbers, we write \( a_n \lesssim b_n \) if, and only if, there is a positive real number \( c > 0 \) such that \( a_n \leq c b_n, n \in \mathbb{N} \). Furthermore, \( a_n \approx b_n \) means that \( a_n \lesssim b_n \) and simultaneously \( b_n \lesssim a_n \).

## 2. Proofs of the main results

### 2.1. Proof of Theorem 1.1 (i)

In view of (4), it is enough to prove that for \( 1 \leq p < \infty \) and \( 0 < q \leq \min(p,2) \) the index \( u \) associated to \( B_{p,q}^0(\mathbb{R}^n) \) is greater than or equal to \( p \).

We assume to the contrary that (3) is fulfilled for some \( 0 < v < p, \epsilon > 0, c > 0 \) and all \( f \in B_{p,q}^0(\mathbb{R}^n) \). Let \( \psi \) be a non-vanishing \( C^\infty \) function in \( \mathbb{R}^n \) supported in \( [0,1]^n \) with \( \int_{\mathbb{R}^n} \psi(x) dx = 0 \).

Let \( J \in \mathbb{N} \) be such that \( 2^{-Jn} < \epsilon \) and consider the function
\[
f_j = \sum_{m=1}^{2^{(j-J)n}} \lambda_{jm} \psi(2^j(x - (m,0,\ldots,0))), \quad j > J,
\]
where
\[
\lambda_{jm} = \frac{1}{m^\frac{p}{p-1} \log^\frac{1}{p} (m+1)}, \quad m = 1, \ldots, 2^{(j-J)n}.
\]
Then (3) represents an atomic decomposition of \( f \) in the space \( B_{p,q}^0(\mathbb{R}^n) \) according to [17] Chapter 1.5, and we obtain (recall that \( v < p \))
\[
||f_j||_{B_{p,q}^0(\mathbb{R}^n)}|| \lesssim 2^{-J\phi}\left(\sum_{m=1}^{2^{(j-J)n}} \lambda_{jm}^p\right)^{1/p} \leq 2^{-J\phi}\left(\sum_{m=1}^{\infty} m^{-1} (\log(m+1))^{-\frac{\phi}{v}}\right)^{1/p} \lesssim 2^{-J\phi}.
\]
On the other hand,
\[
\left( \int_0^t \left[ f_j^*(t) t^{\frac{p}{2}} \right]^v \frac{dt}{t} \right)^{1/v} \geq \left( \int_0^{\frac{2^{-jn}}{f_j^*(t) t^{v/p-1}}} dt \right)^{1/v} \\
\geq \left( \sum_{m=1}^{2^{(j-j)n}} \lambda_{jm}^v \int_{c}^{2^{2-jn}m} t^{v/p-1} dt \right)^{1/v} \geq \left( \sum_{m=1}^{2^{(j-j)n}} \lambda_{jm}^v 2^{-jv/v/p-1} \right)^{1/v} \\
= 2^{-j\frac{v}{p}} \left( \sum_{m=1}^{\frac{1}{m\log(m+1)}} \right)^{1/v}.
\]

As the last series is divergent for \( j \to \infty \), this is in contradiction with (7), and (3) cannot hold for all \( f_j, j > J \).

**Remark 2.1.** Observe that Theorem 1.2 (i) is a direct consequence of Theorem 1.1 (i). The embeddings \( B_{1,q}^0(\mathbb{R}^n) \hookrightarrow B_{1,1}^0(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n) \) if \( p = 1 \) and \( B_{p,q}^0(\mathbb{R}^n) \hookrightarrow F_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n) \) if \( 1 < p < \infty \) show that \( B_{p,q}^0(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \). Theorem 1.1 (i) implies that if \( B_{p,q}^0(\mathbb{R}^n) \hookrightarrow L_{p,v}(\mathbb{R}^n) \) for some \( 0 < v < \infty \), then \( p \leq v \). This proves the optimality of Theorem 1.2 (i) in the frame of the scale of Lorentz spaces.

### 2.2. Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii).

Let \( 0 < p < 1, 0 < q \leq 1 \) and \( s = \sigma_p = n \left( \frac{1}{p} - 1 \right) \). We first prove Theorem 1.2 (ii); i.e. we show that
\[
B_{p,q}^{\frac{s}{p}-n}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n),
\]

or, equivalently, that
\[
\left( \int_0^\infty |tf^*(t)|^q \frac{dt}{t} \right)^{1/q} \leq c \|f|B_{p,q}^{\frac{s}{p}-n}(\mathbb{R}^n)||, \quad f \in B_{p,q}^{\frac{s}{p}-n}(\mathbb{R}^n).
\]

Let
\[
f = \sum_{j=0}^\infty f_j = \sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda_{jm} a_{jm}
\]
be the optimal atomic decomposition of an \( f \in B_{p,q}^{\frac{s}{p}-n}(\mathbb{R}^n) \), again in the sense of [17, Chapter 1.5]. Then
\[
\|f|B_{p,q}^{\frac{s}{p}-n}(\mathbb{R}^n)|| \approx \left( \sum_{j=0}^\infty 2^{-jn} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{jm}|^p \right)^{q/p} \right)^{1/q}
\]

and by Lemma 1.4
\[
\|f|L_{1,q}(\mathbb{R}^n)|| = \| \sum_{j=0}^\infty f_j |L_{1,q}(\mathbb{R}^n)|| \leq \left( \sum_{j=0}^\infty \|f_j|L_{1,q}(\mathbb{R}^n)|| \right)^{1/q}.
\]

We shall need only one property of the atoms \( a_{jm} \), namely, that their support is contained in the cube \( Q_{jm} \), a cube centred at the point \( 2^{-j}m \) with sides parallel to the coordinate axes and side length \( \alpha 2^{-j} \), where \( \alpha > 1 \) is fixed and independent.
of \( f \). We denote by \( \tilde{\chi}_{jm}(x) \) the characteristic functions of \( \tilde{Q}_{jm} \) and by \( \chi_{jl} \) the characteristic function of the interval \((12^{-jn}, (l+1)2^{-jn})\). Hence

\[
f_j(x) \leq c \sum_{m \in \mathbb{Z}^n} |\lambda_{jm}| \tilde{\chi}_{jm}(x), \quad x \in \mathbb{R}^n
\]

and

\[
||f_j||_{L^\infty(\mathbb{R}^n)} \lesssim \left( \int_0^\infty \sum_{l=0}^\infty \left| (\lambda_j l^\alpha) \tilde{\chi}_{jl}(t) \right|^q t^{q-1} dt \right)^{1/q}
\]

\[
\leq \left( \sum_{l=0}^\infty \left| (\lambda_j l^\alpha) \tilde{\chi}_{jl} \right|^q \int_{2^{-jn}}^{2^{-jn}(l+1)} t^{q-1} dt \right)^{1/q}
\]

\[
\lesssim 2^{-jn} \left( \sum_{l=0}^\infty \left| (\lambda_j l^\alpha) \right|^q (l+1)^{q-1} \right)^{1/q} \lesssim 2^{-jn} ||\lambda_j||_{\ell_p}.
\]

The last inequality follows by \((l+1)^{q-1} \leq 1 \text{ and } \ell_p \hookrightarrow \ell_q \text{ if } p \leq q\). If \( p > q \), the same follows by Hölder’s inequality with respect to the indices \( \alpha = \frac{q}{p} \) and \( \alpha' = \frac{p}{p-q} \):

\[
\left( \sum_{l=0}^\infty \left| (\lambda_j l^\alpha) \right|^q (l+1)^{q-1} \right)^{1/q} \leq \left( \sum_{l=0}^\infty \left| (\lambda_j l^\alpha) \right|^q \right)^{\frac{q}{p}} \prod_{l=0}^\infty (l+1)^{\frac{(q-1)(p-1)}{pq}} \leq c ||\lambda_j||_{\ell_p}.
\]

Here, we used that for \( 0 < q < p < 1 \) the exponent \( \frac{(q-1)p}{p-q} = -1 + \frac{(p-1)q}{p-q} \) is strictly smaller than \(-1\).

The proof now follows by (8), (9) and (10):

\[
||f||_{L^1(\mathbb{R}^n)} \leq \left( \sum_{j=0}^\infty ||f_j||_{L^{1,q}(\mathbb{R}^n)}^q \right)^{1/q} \leq c \left( \sum_{j=0}^\infty 2^{-jnq} ||\lambda_j||_{\ell_p}^q \right)^{1/q}
\]

\[
\leq c ||f||_{B_{p,q}^\alpha(\mathbb{R}^n)}.
\]

**Remark 2.2.** We actually proved that \( 8 \) holds for \( X = B_{p,q}^{\frac{\alpha}{p}}(\mathbb{R}^n) \), \( v = q \) and \( \epsilon = \infty \). This, together with (4), implies immediately Theorem 1.1 (ii).

**2.3. Proof of Theorem 1.1 (iii) and Theorem 1.2 (iii).** Let \( 0 < p < 1 \) and \( 0 < q \leq \infty \). By the Jawerth embedding (cf. [7] or [18]) and Theorem 1.1 (ii) we get for any \( 0 < p < \tilde{p} < 1 \),

\[
F_{p,q}^\alpha(\mathbb{R}^n) \hookrightarrow B_{\tilde{p},p}^{\alpha}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)
\]

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