Metrics of Horowitz-Myers type with the negative constant scalar curvature

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We construct a one-parameter family of complete metrics of Horowitz-Myers type with the negative constant scalar curvature. We also verify a positive energy conjecture of Horowitz-Myers for these metrics.

\section{I. INTRODUCTION}

In [8], Horowitz, Myers constructed so-called AdS solitons with toroidal topology

\begin{equation}
-r^2 dr^2 + \frac{1}{r^2} \left(1 - \frac{r_0^n}{r^n}\right) d\phi^2 + r^2 \left(1 - \frac{r_0^n}{r^n}\right) d\theta_i^2,
\end{equation}

for some $r_0 > 0$. These metrics are globally static vacuum $(n + 1)$-dimensional spacetime with cosmological constant

$\Lambda = -\frac{n(n-1)}{2\ell^2}$.

The induced Riemannian metrics $g_0$ on the constant time slice

\begin{equation}
g_0 = \frac{1}{r^2} \left(1 - \frac{r_0^n}{r^n}\right) dr^2 + r^2 \left(1 - \frac{r_0^n}{r^n}\right) d\phi^2 + r^2 \sum_{i=1}^{n-2} (d\theta_i)^2.
\end{equation}

are ALH and are referred as Horowitz-Myers metrics. Horowitz, Myers also verified that the Hawking-Horowitz mass of $g_0$ is negative and conjectured that, among all metrics which are asymptotic to $g_0$ and with scalar curvature

\begin{equation}
S \geq -\frac{n(n-1)}{\ell^2},
\end{equation}

$g_0$ has the lowest Hawking-Horowitz mass [8]. This conjecture was shown to be true for Horowitz-Myers metrics up to order $O(\frac{1}{\ell^2})$ on $R^2 \times T^{n-2}$ and with scalar curvature satisfying (1.3) [2].

On the other hand, ALE or ALH metrics of Eguchi-Hanson types attracted much attentions in physics and geometry and, in particular, these metrics with constant scalar curvature play important roles in constructing examples with the negative mass [3–6, 9, 10, 12, 13]. Therefore it is natural to study the analogous question for metrics of Horowitz-Myers type. And this is the purpose of this short paper.

The paper is organized as follows. In Section II, we construct complete metrics of Horowitz-Myers type with the negative constant scalar curvature. In Section III, we show that these metrics can not develop to certain vacuum spacetimes unless the Horowitz-Myers metrics. In Section IV, we show that there is a natural almost-complex structure, which is integrable and compatible with these metrics. But the metrics are not Kähler with respect to this almost-complex structure. In Section V, we compute the Hawking-Horowitz mass and the Hamiltonian energy of these metrics and verify a positive energy conjecture of Horowitz-Myers for these metrics.

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II. METRICS OF HOROWITZ-MYERS TYPE

In this section, we construct a one-parameter family complete metrics of asymptotically Horowitz-Myers type. Let \( r_+ \) be a positive number, \( V = V(r) \) be a positive smooth function for \( r > r_+ \). For \( n \geq 3 \), we consider the metric

\[
\dot{g} := \frac{1}{V} dr^2 + V d\phi^2 + r^2 \sum_{i=1}^{n-2} (d\theta^i)^2,
\]

where \( \theta^i \) and \( \phi \) are periodic. The periods of \( \theta^i \) are arbitrary, and the period of \( \phi \) is specified later. Let \( \nabla \) be the Levi-Civita connection of \( \dot{g} \). Denote \( V' := \frac{dV}{dr} \). By the curvature formulae for warped product (cf. [11, Chapter 7]), we have, for \( 1 \leq i, j \leq n-2 \), that

\[
\begin{align*}
\nabla_a \partial_r &= -\frac{1}{2} V^{-1} V' \partial_r, \\
\nabla_a \partial_r &= \nabla_a \partial_\phi = \frac{1}{2} V^{-1} V' \partial_\phi, \\
\nabla_a \partial_r &= \nabla_a \partial_{\theta^i} = -\frac{1}{r} \partial_{\theta^i}, \\
\nabla_a \partial_\phi &= -\frac{1}{2} V V' \partial_r, \\
\nabla_a \partial_\phi &= \nabla_a \partial_{\theta^i} = 0, \\
\nabla_a \partial_{\theta^i} = -r V \delta_{ij} \partial_j.
\end{align*}
\]

Moreover, the Ricci curvature tensor \( \text{Ric}_\dot{g} \) of \( \dot{g} \) is diagonal with respect to the coordinate frame

\[
\begin{align*}
\text{Ric}_\dot{g}(\partial_r, \partial_r) &= -\frac{1}{2} \left( V'' + \frac{n-2}{r} V' \right) V^{-1}, \\
\text{Ric}_\dot{g}(\partial_\phi, \partial_\phi) &= -\frac{1}{2} \left( V'' + \frac{n-2}{r} V' \right) V, \\
\text{Ric}_\dot{g}(\partial_{\theta^i}, \partial_{\theta^i}) &= -r V' - (n-3) V
\end{align*}
\]

for \( 1 \leq i \leq n-2 \), and the scalar curvature is

\[
\text{Scal}_\dot{g} = -V'' - \frac{2(n-2)}{r} V' - \frac{(n-2)(n-3)}{r^2} V.
\]

Now we seek the function \( V \) such that \( \text{Scal}_\dot{g} = S < 0 \). It turns out that the general solutions are

\[
V = \frac{r^2}{\ell^2} \left( 1 + \frac{a}{r^{n-1}} + \frac{b}{r^n} \right), \quad \ell = \sqrt{-\frac{n(n-1)}{S}}
\]

where \( a, b \) are two arbitrary constants. We restrict to the subset of solutions with \( b = -r_0^a < 0 \), i.e.,

\[
V = \frac{r^2}{\ell^2} \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^a}{r^n} \right).
\]

When \( a = 0 \), they give rise to Horowitz-Myers metrics. Since

\[
\lim_{r \to +\infty} V(r) = +\infty, \quad \lim_{r \to 0^+} V(r) = -\infty,
\]

\( V(r) \) possesses the largest positive root \( r_+ > 0 \) for any \( a \in \mathbb{R} \). So \( V(r) > 0 \) for \( r > r_+ \), and \( V(r_+) = 0 \). The metric \( g \) has conical singularity at \( r = r_+ \). However, this singularity is removable, provided the period of \( \phi \) is specified appropriately.
Theorem II.1. Let \( g \) be the metric (2.7) with \( V \) given by (2.4) and \( a \in \mathbb{R} \). Assume that \( r \geq r_+ \), the periods of \( \{ \theta^i \}_{1 \leq i \leq n-2} \) are arbitrary and the period of \( \phi \) is

\[
\beta := \frac{4\pi \ell^2}{r_+ \left( n - 1 + \frac{2}{r_+} \right)}.
\] (2.5)

Then \( r = r_+ \) is a removable singularity, and \( g \) is a geodesically complete metric with negative constant scalar curvature \( S \) on \( \mathbb{R}^2 \times T^{n-2} \).

Proof: From \( V(r_+) = 0 \), we have

\[
a \frac{r_0^2}{r_+^{n-1}} = r_0^2 - 1.
\]

Hence

\[
V'(r_+) = \frac{r_+}{\ell^2} \left[ 2 - (n-3) a \frac{r_0^2}{r_+^{n-1}} - (n-2) b \frac{r_0^2}{r_+} \right] = \frac{4\pi}{\beta} > 0.
\]

By Implicit Function Theorem, \( V \) has smooth inverse \( \tilde{V} \) around \( r_+ \). Set \( \rho = \tilde{V}^{\frac{1}{2}}(r) \) for \( r > r_+ \). Then

\[
r = \frac{1}{2} V^{-\frac{1}{2}} V' dr.
\]

Replacing \( r \) with \( \rho \), we rewrite \( g \) as follows

\[
g = \frac{4}{V'(r_+)^2} \left[ h(\rho^2) \rho^2 d\rho^2 + (d\rho^2 + \rho^2 d\Phi^2) \right] + \tilde{V}^2(\rho^2) \sum_{i=1}^{n-2} (d\theta^i)^2,
\]

where \( \Phi = \frac{2\pi \phi}{\beta} \) and

\[
h(s) := s^{-1} \left[ \frac{V'(r_+)^2}{(V' \circ \tilde{V}(s))^2} - 1 \right].
\]

Clearly, \( \Phi \) is of period \( 2\pi \), and \( h(s) \) is smooth around \( s = 0 \). Now we apply coordinate transformation \( x = r \cos \Phi \) and \( y = r \sin \Phi \), and find that \( g \) becomes

\[
g = \frac{4}{V'(r_+)^2} \left[ h(x^2 + y^2)(dx + dy)^2 + (dx^2 + dy^2) \right] + \tilde{V}^2(x^2 + y^2) \sum_{i=1}^{n-2} (d\theta^i)^2.
\]

From this, we see that \( g \) is smooth at \( (x,y) = (0,0) \) and \( r = r_+ \) is a removable singularity. Thus \( g \) is a complete metric and the underlying manifold has the topology \( \mathbb{R}^2 \times T^{n-2} \). Moreover, \( g \) has constant scalar curvature \( S \). So the proof is complete.

Consequently, the initial data \((g,K \equiv 0)\) satisfying the constraint equations for the vacuum Einstein equations with negative cosmological constant \( \Lambda = \frac{1}{2}S \).

III. UNIQUENESS

In [7], Galloway, Surya and Woolgar established a uniqueness theorem of AdS solitons for static metrics which can be conformally compactified, see also [1] for certain extension. In their case some asymptotic behavior of the lapse \( N \) at spatial infinity is required. In the following we prove that the metric \( g \) constructed in Theorem II.1 cannot develop to certain vacuum spacetime unless \( a = 0 \), where the lapse \( N \) does not require to satisfy any asymptotic condition at spatial infinity.
**Theorem III.1.** Let $g$ be the metric constructed in Theorem [II.1]. Let $N = N(r, \phi, \theta^i)$ be a positive function where $r > r_+$. Suppose the metric

$$\tilde{g} = -N^2 dt^2 + \frac{1}{V} dr^2 + V d\phi^2 + r^2 \sum_{i=1}^{n-2} (d\theta^i)^2.$$ 

satisfies the vacuum Einstein field equations

$$\text{Ric}_{\tilde{g}} - \frac{1}{2} \text{Scal}_{\tilde{g}} \tilde{g} + \Lambda \tilde{g} = 0$$

(3.1)

for the negative cosmological constant $\Lambda < 0$. Then it holds that

$$N = cr, \quad \ell^2 = -\frac{n(n-1)}{2\Lambda}, \quad V = \frac{r^2}{\ell^2} \left(1 - \frac{r_0^2}{r^2}\right)$$

where $c$ is some positive constant which can be chosen as 1 by defining new time $\tilde{t} = ct$. Therefore $\tilde{g}$ is an AdS soliton.

**Proof:** Let $g$ be the spatial part of $\tilde{g}$. By (3.1), we have

$$\nabla^2 N = N \left(\text{Ric}_{\tilde{g}} - \frac{2\Lambda}{n-1} g\right).$$

(3.2)

**Claim 1:** There exist smooth functions

$$k = k(r), \quad h = h(\phi), \quad f_i = f_i(\theta^i)$$

where $h$ and $f_i$ are periodic, $1 \leq i \leq n-2$, such that

$$N = k(r) + V^\frac{1}{2} (r) h(\phi) + r \sum_{i=1}^{n-2} f_i(\theta^i).$$

(3.3)

**Proof of the Claim 1:** Recall that $\text{Ric}_{\tilde{g}}$ is diagonal. It follows from (3.2) that $\nabla^2 N$ is also diagonal. From this and (2.2), we have

$$\nabla^2 N(\partial_r, \partial_\phi) = \nabla^2 N(\partial_r, \partial_\theta^i) = \nabla^2 N(\partial_\phi, \partial_\theta^i) = \nabla^2 N(\partial_\phi, \partial_\theta^i) = 0$$

for $1 \leq i, j \leq n-2$ ($i \neq j$), we obtain

$$\partial_r \partial_\phi (V^{-\frac{1}{2}} N) = \partial_r \partial_\theta^i (r^{-1} N) = \partial_\phi \partial_\theta^i N = \partial_\phi \partial_\theta^i N = 0.$$

Therefore $N$ must take the form (3.3). Since $\phi$ and $\theta^i$ are periodic, $h$ and $f_i$ are periodic.

**Claim 2:** $N$ depends only on $r$.

**Proof of the Claim 2:** By (3.2), we have

$$\partial_\phi \partial_\phi N - \nabla^2 \partial_\phi (\partial_\phi N) = \nabla^2 N(\partial_\phi, \partial_\phi) = N \left(\text{Ric}_{\tilde{g}}(\partial_\phi, \partial_\phi) - \frac{2\Lambda}{n-1} g(\partial_\phi, \partial_\phi)\right).$$

By (2.2) and (2.3), we obtain

$$\partial_\phi \partial_\phi N + \frac{1}{2} V V' \partial_r N = -N \left[\frac{1}{2} V'' + \frac{n-2}{2r} V' + \frac{2\Lambda}{n-1}\right] V.$$ 

(3.4)

Taking the derivative $\partial_\phi$ on both sides, and noting that $\partial_\phi N = V^\frac{1}{2} \frac{dh}{d\phi}$ by (3.3), we then obtain,

$$-4 \frac{d^3 h}{d\phi^3} = \frac{dh}{d\phi} \left[2 \left(V'' + \frac{n-2}{r} V'\right) V + (V')^2 + \frac{8\Lambda}{n-1} V\right].$$

(3.5)
Note the term in the bracket is equal to

\[
\frac{r^2}{\ell^2} \left[ 4 \left( n + \frac{2\Lambda \ell^2}{n-1} \right) + (n-3)^2 \frac{a^2}{r^{2(n-1)}} + n(n-2) \frac{b^2}{r^{2n-1}} + 2(n-2)^2 \frac{ab}{r^{2n-3}} + 8 \left( \frac{\Lambda \ell^2}{n-1} + 1 \right) \frac{a}{r^{n-1}} + \left( 2n + \frac{8\Lambda \ell^2}{n-1} \right) \frac{b}{r^n} \right]
\] \hspace{1cm} (3.6)

where \( b = -r_0^\alpha \). Now we assert that \( \frac{dh}{d\phi} \equiv 0 \). If \( \frac{dh}{d\phi}(\phi_0) \neq 0 \) for some \( \phi_0 \). As \( h \) depends only on \( \phi \), we conclude that (3.6) must be constant for \( r > r_+ \). This implies that

\[
n + \frac{2\Lambda \ell^2}{n-1} = b = 0.
\]

But \( b = -r_0^\alpha \neq 0 \), this yields a contradiction. Consequently, \( \frac{dh}{d\phi} \equiv 0 \), and \( \partial_\phi N = 0 \).

Similarly, we can show that \( \partial_\theta \varphi N = 0 \). Indeed, by (3.2), we obtain

\[
\partial_\theta \partial_\phi N - \nabla_\theta \partial_\phi (N) = \nabla^2 N(\partial_\theta, \partial_\phi) = N \left( \text{Ric}_{\varphi}(\partial_\theta, \partial_\phi) - \frac{2\Lambda}{n-1} \delta(\partial_\theta, \partial_\phi) \right)
\]

Using (2.2) and (2.3), we obtain

\[
\partial_\theta \partial_\phi N + rV \partial_r N = -N r^2 \left( \frac{1}{r} V' + \frac{n-3}{r^2} V + \frac{2\Lambda}{n-1} \right).
\] \hspace{1cm} (3.7)

Taking the derivative \( \partial_\theta \) on both sides, and noting that (3.3) gives \( \partial_\theta N = r \frac{df_i}{d(\theta^i)} \), we get,

\[
\frac{d^3 f_i}{d(\theta^i)^3} = -\frac{df_i}{d(\theta^i)} \cdot r^2 \left( \frac{1}{r} V' + \frac{n-2}{r^2} V + \frac{2\Lambda}{n-1} \right) = -\frac{df_i}{d(\theta^i)} \cdot r^2 \cdot \left( \frac{2\Lambda \ell^2}{n-1} + \frac{a}{r^{n-1}} \right).
\] \hspace{1cm} (3.8)

Now we assert that \( \frac{df_i}{d(\theta^i)} \equiv 0 \). If it is not true, same as before, we can conclude that

\[
\frac{r^2}{\ell^2} \left( n + \frac{2\Lambda \ell^2}{n-1} + \frac{a}{r^{n-1}} \right) = C, \quad r > r_+.
\] \hspace{1cm} (3.9)

for certain constant \( C \).

Case 1: \( n > 3 \). Note that (3.9) gives that

\[
n + \frac{2\Lambda \ell^2}{n-1} = a = 0.
\]

Then (3.8) becomes \( \frac{df_i}{d(\theta^i)} \equiv 0 \). Since \( f_i \) is periodic, this implies \( f_i \) must be a constant, which yields a contradiction as \( \frac{df_i}{d(\theta^i)}(\theta^i_0) \neq 0 \). Consequently, \( \frac{df_i}{d(\theta^i)} \equiv 0 \), and \( \partial_\theta N = 0 \).

Case 2: \( n = 3 \). In this case, \( \frac{df_i}{d(\theta^i)}(\theta^i_0) \neq 0 \), and (3.9) only gives that

\[
n + \frac{2\Lambda \ell^2}{n-1} = 0 \quad \Rightarrow \quad \Lambda = -\frac{3}{\ell^2}.
\]

Since \( \partial_\theta N = 0 \), (3.4) gives \( \partial_\theta N = \frac{1}{\ell^2} N \). When \( n = 3 \), (3.3) implies \( \frac{dh}{d\phi} = 0 \), we obtain \( N = \kappa r + rf_1(\theta^1) \) for some constant \( \kappa \). Putting this into (3.7) with \( i = 1 \), we obtain

\[
\frac{d^2 f_1}{d(\theta^1)^2} + \frac{a}{\ell^2} f_1 = -\frac{a\kappa}{\ell^2}.
\] \hspace{1cm} (3.10)

If \( a = 0 \), then (3.10) becomes \( \frac{d^2 f_1}{d(\theta^1)^2} \equiv 0 \). Since \( f_1 \) is periodic, this implies \( f_1 \) must be a constant, which yields a contradiction as \( \frac{df_i}{d(\theta^i)}(\theta^i_0) \neq 0 \).
If \( a < 0 \), then the solutions of (3.10) are

\[ f_1(\theta^1) = -\kappa + A_1 e^{\frac{-a}{\ell} \theta^1} + A_2 e^{\frac{-a}{\ell} \theta^1}, \]

with \( A_1, A_2 \in \mathbb{R} \). Since \( f_1 \) is periodic, this implies \( A_1 = A_2 = 0 \) and so \( f_1 = -\kappa \). Again, this is impossible as \( \frac{df_1}{d\theta^1}(\theta^1_0) \neq 0 \).

If \( a > 0 \), then the solutions of (3.10) are

\[ f_1(\theta^1) = -\kappa + A_1 \sin \left( \frac{\sqrt{-a}}{\ell} \theta^1 + A_2 \right) \]

with \( A_1, A_2 \in \mathbb{R} \). It follows

\[ N = A_1 r \sin \left( \frac{\sqrt{-a}}{\ell} \theta^1 + A_2 \right), \]

But this is impossible since \( N > 0 \) for all \( \theta^1 \).

As a result, we conclude that \( \frac{df_1}{d\theta^1} \equiv 0 \), so \( \partial_{\theta^1} N = 0 \). This proves the claim.

Now we obtain

\[
\begin{align*}
\text{Ric}_\tilde{g}(\partial_\theta, \partial_\theta) &= NN''V + NN'(V' + \frac{n-2}{r}V), \\
\text{Ric}_\tilde{g}(\partial_\tau, \partial_\tau) &= -N^{-1}N'' - \frac{1}{2}N^{-1}N'V^{-1}V' - \frac{1}{2}V^{-1} \left( V'' + \frac{n-2}{r}V' \right), \\
\text{Ric}_\tilde{g}(\partial_\phi, \partial_\phi) &= -\frac{1}{2}N^{-1}N'V' - \frac{1}{2}V \left( V'' + \frac{n-2}{r}V' \right), \\
\text{Ric}_\tilde{g}(\partial_{\theta^1}, \partial_{\theta^1}) &= -rV'' - (n-3)V - N^{-1}N'rV,
\end{align*}
\]

for \( 1 \leq i \leq n - 2 \). Since Einstein equations (3.11) gives that

\[ \text{Ric}_\tilde{g}(V^{\frac{1}{2}} \partial_\theta, V^{\frac{1}{2}} \partial_\tau) - \text{Ric}_\tilde{g}(V^{-\frac{1}{2}} \partial_\phi, V^{-\frac{1}{2}} \partial_\phi) = 0. \]

Substituting this into (3.11), we get \( N^{-1}N''V = 0 \). This implies \( N'' = 0 \), so \( N = cr + d \) for some \( c, d \in \mathbb{R} \). Putting this into the following equation

\[ \text{Ric}_\tilde{g}(\partial_\theta, \partial_\theta) = \frac{2\Lambda}{n-1} \delta(\partial_\theta, \partial_\theta) = -\frac{2\Lambda}{n-1} N^2, \]

and applying the explicit expression for \( \text{Ric}_\tilde{g}(\partial_\theta, \partial_\theta) \) in (3.11), we have

\[
-\frac{c}{cr + d} \left( V' + \frac{n-2}{r}V \right) = \frac{2\Lambda}{n-1}, \quad V' + \frac{n-2}{r}V = \ell^2 \left( n + \frac{a}{r^{n-1}} \right).
\]

Therefore

\[
-\frac{cr}{cr + d} \left( n + \frac{a}{r^{n-1}} \right) = \frac{2\Lambda \ell^2}{n-1},
\]

for all \( r > r_+ \). It follows

\[ a = d = 0, \quad \Lambda = -\frac{n(n-1)}{2\ell^2}, \quad N = cr, \quad c > 0. \]

Thus the proof is completed.
IV. COMPLEX STRUCTURE

The natural complex structures for metrics of Eguchi-Hanson types were studied in [3, 10, 12]. In this section, we assume dimension $n = 2 + 2k$ for some $k \geq 1$. We show that, on the metric $g$ constructed in Theorem II.1, there exists a natural almost-complex structure $J$, which is integrable and $g$-compatible. However $g$ is not Kähler with respect to this $J$.

For $r > r_+$, we define the almost-complex structure $J$ by

$$
\frac{1}{V}dr \mapsto d\phi, \quad d\theta^i \mapsto d\theta^{k+j}, \quad 1 \leq j \leq k.
$$

(4.1)

**Proposition IV.1.** It holds that $J$ extends smoothly to $r = r_+$. So $J$ is an almost-complex structure defined on the entire initial data.

**Proof:** Recall the coordinates $(\rho, \Phi)$ and $(x, y)$ which are introduced in the proof of Theorem II.1. We rewrite

$$
\frac{1}{V}dr = 2V' \circ V^{-1}(\rho^2) \frac{1}{\rho}d\rho = 2V' \circ V^{-1}(\rho^2) \frac{1}{\rho^2}(xdx + ydy)
$$

and

$$
d\phi = \frac{\beta}{2\pi}d\Phi = \frac{\beta}{2\pi} \frac{1}{\rho^2}(-ydx + xdy).
$$

Note that $\beta$ is defined by (2.5). So we have

$$
J(xdx + ydy) = \frac{\beta}{2\pi} \frac{V' \circ V^{-1}(\rho^2)}{2}(-ydx + xdy).
$$

Letting

$$
u(s) = \frac{\beta}{2\pi} \frac{V' \circ V^{-1}(s)}{2},
$$

we rewrite

$$
J(xdx + ydy) = u(\rho^2)(-ydx + xdy).
$$

By the proof of Theorem II.1 it is easy to see that $u(s)$ is smooth positive function around $s = 0$ and it satisfies $u(0) = 1$. From the above equation and from the fact that $J^2 = -\text{Id}$, we see that $J$ maps $\text{span}\{dx, dy\}$ onto $\text{span}\{dx, dy\}$. Denoting by $A$ the transformation matrix so that

$$
J(dx, dy) = (dx, dy)A,
$$

we find

$$
A = \frac{1}{\rho^2} \left( \begin{array}{cc}
\frac{xy(\frac{1}{u(\rho^2)} - u(\rho^2))}{(u(\rho^2) - 1)x^2 + (\frac{1}{u(\rho^2)} - 1)y^2} & (1 - \frac{1}{u(\rho^2)})x^2 + (1 - u(\rho^2))y^2 \\
\frac{1}{xy(\rho^2)} - \frac{1}{u(\rho^2)} & 1
\end{array} \right) + \left( \begin{array}{c}
1 \\
-1
\end{array} \right).
$$

Since $u(s)$ is smooth around $s = 0$ and $u(0) = 1$, it is easy to see that $A$ extends smoothly to $\rho = 0$ and it holds $A = \left( \begin{array}{c}
1 \\
1
\end{array} \right)$ at $\rho = 0$. So the proof is complete.

**Proposition IV.2.** The almost-complex structure $J$ is integrable and $g$-compatible.

**Proof:** With respect to $J$ given by (4.1), the $(1, 0)$-forms are spanned by

$$
\frac{1}{V}dr + id\phi, \quad d\theta^i + id\theta^{k+j}, \quad 1 \leq j \leq k.
$$
It is easy to see that all of these are closed forms. So $J$ is integrable. Now we compute

$$g(J, J) = \frac{1}{V} J(dr) \otimes J(dr) + V J(d\phi) \otimes J(d\phi) + r^2 \sum_{j=1}^{2k} J(d\theta^j) \otimes J(d\theta^j)$$

$$= V d\phi \otimes d\phi + \frac{1}{V} dr \otimes dr + r^2 \sum_{j=1}^{2k} d\theta^j \otimes d\theta^j = g.$$ 

Therefore $J$ is $g$-compatible and the proof is complete.

Since the form

$$g(\cdot, J \cdot) = \frac{1}{V} dr \otimes J(dr) + V d\phi \otimes J(d\phi) + r^2 \sum_{j=1}^{2k} d\theta^j \otimes J(d\theta^j)$$

$$= dr \otimes d\phi - d\phi \otimes dr + r^2 \sum_{j=1}^{k} \left( d\theta^j \otimes d\theta^{k+j} - d\theta^{k+j} \otimes d\theta^j \right)$$

$$= dr \wedge d\phi + r^2 \sum_{j=1}^{k} d\theta^j \wedge d\theta^{k+j}$$

is not closed, $g$ is not Kähler with respect to $J$.

V. ENERGY

There are several definitions of the total energy for asymptotically Horowitz-Myers metrics, e.g., the Hawking-Horowitz mass used in [8], and the Ashtekar-Magnon mass used in [7]. Two masses are the same for Horowitz-Myers metrics [7]. In this section, we compute the total energy of the metric $g$ constructed in Theorem II.1. We show that the Hamiltonian energy defined in [2] is equal to the Hawking-Horowitz mass up to certain constant. We also verify a positive energy conjecture of Horowitz-Mayers for metric $g$.

Firstly, we compute the Hawking-Horowitz mass of $g$. Let $T^{n-1}_r$ be the constant $r$ slice in metric $g$. Recall that the period of $\phi$ is $\beta$. The period of $\theta^i$ is denoted as $\lambda_i$, $1 \leq i \leq n-2$. Let $H$ be the mean curvature of $T^{n-1}_r$ with respect to the unit normal $V^\frac{1}{2} \partial_r$. We obtain

$$H = \text{div}_{T^{n-1}_r} (V^\frac{1}{2} \partial_r)$$

$$= \langle V^\frac{1}{2} \partial_\phi, V^\frac{1}{2} \partial_\phi \rangle + \sum_{i=1}^{n-2} \langle V^{\frac{1}{2}} \partial_r, V^{\frac{1}{2}} \partial_{\theta^i} \rangle$$

$$= \frac{1}{2} V^{\frac{1}{2}} \partial_r g_{\phi\phi} + \frac{1}{2} \sum_{i=1}^{n-2} r^{-2} V^{\frac{1}{2}} \partial_r g_{\theta^i \theta^i}$$

$$= V^{\frac{1}{2}} \left[ \frac{1}{2} V' + (n-2) r^{-1} V \right].$$

Since $V$ is given by (2.4), we further obtain

$$H = \frac{n-1}{\ell} + \frac{r_0^2}{2 \ell^2 \rho^2} + O(r^{-2(n-1)}). \quad (5.1)$$

The reference spacetime metric is chosen as

$$-r^2 dt^2 + \frac{r^2}{r^2} dr^2 + \frac{r^2}{r^2} d\phi^2 + r^2 \sum_{i=1}^{n-2} (d\theta^i)^2 \quad (5.2)$$
by taking \( r_0 = 0 \) in (1.1), where the period of \( \theta^i \) is \( \lambda_i \) and the period of \( \phi \) is \( 2\pi \ell^2 \). Denote

\[
\lambda := \lambda_1 \cdots \lambda_{n-2}
\]  

(5.3)

the volume of the torus \( T_{\theta}^{n-2} \). The perimeter of \( T_\phi \) with respect to the metric \( \frac{1}{\ell^2} d\phi^2 \) is \( 2\pi \ell \). Let

\[
\bar{g} = \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} d\phi^2 + r^2 \sum_{i=1}^{n-2} (d\theta^i)^2
\]  

(5.4)

be the induced Riemannian metric of constant time slice in (5.2). Note that \( \bar{g} \) is of constant sectional curvature \( -\frac{1}{\ell^2} \). Let \( H_0 \) be the mean curvature of \( T_{r}^{n-1} \) with respect to \( \bar{g} \). Letting \( a = 0 \) and \( r_0 = 0 \) in (5.1), we obtain

\[
H_0 = \frac{n - 1}{\ell}.
\]  

(5.5)

Recall the lapse of reference metric (5.2)

\[
N = r.
\]  

(5.6)

The Hakwing-Horowitz mass of \( g \) is then defined as [8]

\[
E_{HH}(g) = -\frac{1}{8\pi G} \lim_{r \to \infty} \int_{T_{r}^{n-1}} N(H - H_0) dv_{g_{(r^n-1)}}.
\]  

(5.7)

where \( G \) is the \((n+1)\)-dimensional Newton’s constant. Note that the area of \( T_{r}^{n-1} \) is \( \lambda \beta r^n \). Inserting this and equations (5.1) and (5.5) into (5.7), we obtain

\[
E_{HH}(g) = -\frac{1}{8\pi G} \lim_{r \to \infty} \lambda \beta V^\frac{1}{2} r^n \left( \frac{r_0^n}{2\ell r^n} + O(r^{-2(n-1)}) \right) = -\frac{\lambda \beta r_0^n}{16\pi Gr^2}.
\]  

(5.8)

Since \( \beta \) is given by (2.5), we obtain

\[
E_{HH}(g) = -\frac{\lambda r_0^n}{4Gr_+ \left( n - 1 + \frac{2\pi}{\ell} \right)}.
\]  

(5.9)

From (5.8) or (5.9), one sees immediately that

\[
E_{HH}(g) < 0
\]

for all \( a \in \mathbb{R} \).

Now we provide the Hamiltonian energy of an asymptotically Horowitz-Myers metric \( g \) given by Barzegar, Chruściel, Hörzinger, Maliborski and Nguyen [2]. Let

\[
\tilde{e}_1 = \frac{r}{\ell} \partial_r, \quad \tilde{e}_2 = \frac{\ell}{r} \partial_\phi, \quad \tilde{e}_i = \frac{1}{r} \partial_{\theta^i}, \quad i = 3, \ldots, n.
\]

Then \( \{ \tilde{e}_i \}_{i=1}^n \) is an orthonormal frame of \( \bar{g} \). Let \( \{ \tilde{e}^i \}_{i=1}^n \) be its dual frame. Let \( \tilde{\nabla} \) be the Levi-Civita connection of \( \bar{g} \). It is easy to compute

\[
\tilde{\nabla}_{\tilde{e}_1} \tilde{e}_1 = 0, \quad \tilde{\nabla}_{\tilde{e}_2} \tilde{e}_1 = \frac{1}{r} \partial_\phi, \quad \tilde{\nabla}_{\tilde{e}_2} \tilde{e}_2 = -\frac{r}{\ell^2} \partial_r
\]

and, for \( 3 \leq i \leq n \),

\[
\tilde{\nabla}_{\tilde{e}_i} \tilde{e}_1 = \frac{1}{r\ell} \partial_{\theta^{i-2}}, \quad \tilde{\nabla}_{\tilde{e}_i} \tilde{e}_i = -\frac{r}{\ell^2} \partial_r.
\]
where $V_0$ and the scalar curvature $\text{Scal}_g$ satisfy

$$
\int_M \left( \text{Scal}_g + \frac{n(n-1)}{\ell^2} \right) \nabla_1^2 \wedge \cdots \wedge \nabla_n < \infty.
$$

(5.11)

For this metric, the Hamiltonian energy up to certain constant is [2]

$$
E(g) = \frac{1}{4 \text{Vol}(\mathbb{T}^{n-1})} \lim_{\ell \to \infty} \int_{\mathbb{T}^{n-1}} \mathcal{E} \nabla_2^2 \wedge \cdots \wedge \nabla_n,
$$

(5.12)

where $\text{Vol}(\mathbb{T}^{n-1}) = 2 \pi \lambda$, and

$$
\mathcal{E} = \nabla_1^2 g_{11} - \nabla_1 \text{tr}_g(g) - \frac{1}{\ell} (a_{11} - g_{11} \text{tr}_g(a)).
$$

Under the frame $\{e_i\}_{i=1}^n$, the metric $g$ constructed in Theorem II.1 satisfies

$$
(g_{ij}) = \text{diag} \left( \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} \right)^{-1}, \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} \right), 1, \ldots, 1 \right),
$$

$$
(a_{ij}) = \text{diag} \left( -\frac{a}{r^{n-1}} + \frac{r_0^n}{r^n} + o(r^{n+1}), \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n}, 0, \ldots, 0 \right)
$$

(5.13)

Therefore, $g$ does not satisfy the condition (5.10). But its Hamiltonian energy is still finite. Indeed, we can prove

**Proposition V.1.** For metrics of Horowitz-Myers type constructed in Theorem II.1

$$
E_{\text{hm}}(g) = \frac{\lambda \ell}{2G} E(g).
$$

**Proof:** By a direct computation, we have

$$
\nabla_1^2 g_{11} = (n + 1) \frac{r^2}{\ell^3} V^{-1} - \frac{r^3}{\ell^2} V^{-2} \nabla - \frac{n-2}{r} - \frac{a}{\ell r^{n-1}} + O(r^{-(n+1)}),
$$

$$
\nabla_1 \text{tr}_g(g) = \frac{r}{\ell} \partial_r \left( \frac{r^2}{\ell^2} V^{-1} + \frac{r^2}{\ell^3} V^2 + n - 2 \right) = O(r^{-(n+1)}),
$$

and

$$
\frac{1}{\ell} (a_{11} - g_{11} \text{tr}_g(a)) = \frac{3r^2}{\ell^3} V^{-1} - \frac{r^4}{\ell^5} V^{-2} - 2 = -\frac{a}{\ell r^{n-1}} + \frac{r_0^n}{\ell r^n} + O(r^{-(n+1)}).
$$

Putting these into $\mathcal{E}$, we find that

$$
\mathcal{E} = -\frac{r_0^n}{\ell r^n} + O(r^{-2(n-1)}).
$$
Therefore
\[ E(g) = \frac{1}{4 \text{Vol}(T^{n-1})} \lim_{r \to \infty} \int_{r^{-1}}^{r} e \alpha^2 \wedge \cdots \wedge \alpha^n = -\frac{\beta r_0^n}{8 \pi \ell^3} = \frac{2G}{\lambda \ell} E_{\text{in}}(g). \]

In [2], Barzegar, Chruściel, Hörzinger, Maliborski and Nguyen proved the positive energy conjecture of Horowitz-Myers for the following metrics
\[ e^{2u} dr^2 + e^{2v} d\phi^2 + e^{2w} \sum_{i=1}^{n-2} (d\theta_i)^2 \]
and \( u, v \) and \( w \) satisfy the following asymptotic conditions (Note \( \ell = 1 \) in [2], here we choose arbitrary \( \ell \))
\[ u = -\ln \frac{r}{\ell} - \frac{1}{2} \ln \left( 1 - \frac{r_0^n}{r^n} \right) + \frac{u_n}{r^n} + o(r^{-n}), \quad v = \ln \frac{r}{\ell} + \frac{1}{2} \ln \left( 1 - \frac{r_0^n}{r^n} \right) + \frac{v_n}{r^n} + o(r^{-n}), \quad w = \ln r + \frac{w_n}{r^n} + o(r^{-n}). \]

Then we obtain
\[ e^{2u} = \frac{\ell^2}{r^2} \left( 1 + \frac{r_0^n}{r^n} + o(r^{-n}) \right), \quad e^{2v} = \frac{r^2}{\ell^2} \left( 1 - \frac{r_0^n}{r^n} - 2v_n + o(r^{-n}) \right), \quad e^{2w} = r^2 \left( 1 + \frac{2w_n}{r^n} + o(r^{-n}) \right). \quad (5.14) \]

Compare (5.13) with (5.14), we know that, if \( a \neq 0 \), the fall-offs of \( g_{11} \) and \( g_{22} \) of are slower than that of (5.14), thus we can not apply the main theorem for the positive energy conjecture of Horowitz-Myers in [2] to \( g \). In order to verify it for \( g \), we require that \( \phi \) has the same period in \( g \) as the period \( 4\pi \ell^2 \) in the following Horowitz-Myers metric
\[ g_{\text{adm}} = \frac{1}{r^2} \left( 1 - \frac{r_0^n}{r^n} \right) dr^2 + \frac{\ell^2}{r^2} \left( 1 - \frac{r_0^n}{r^n} \right) d\phi^2 + \frac{2}{n} \sum_{i=1}^{n-2} (d\theta_i)^2. \]

This implies
\[ r_0 = n \left( n - 1 + \frac{r_0^n}{r_+^n} \right). \quad (5.15) \]

**Theorem V.1.** For metric \( g \) of Horowitz-Myers type constructed in Theorem [II.7] with fixed \( r_0 \), it holds that
\[ E(g) \geq E(g_{\text{adm}}), \]
with equality if and only if \( a = 0 \), and \( g \) is the Horowitz-Myers metric \( g_{\text{adm}} \).

**Proof:** For the Horowitz-Myers metric \( g_{\text{adm}} \), we have
\[ E_{\text{in}}(g_{\text{adm}}) = \frac{\lambda r_0^{n-1}}{4Gr_+} < 0. \]
Combining this with (5.15), it follows from (5.9) that
\[ E_{\text{in}}(g) = \frac{\lambda r_0^n}{4Gr_+} \left( n - 1 + \frac{r_0^n}{r_+^n} \right)^n. \]

Since it is easy to see that \( n - 1 + s^n \geq ns \) for \( s \geq 0 \) and the equality occurs if and only if \( s = 1 \), applying this with \( s = \frac{r_0}{r_+} \) to the above equation, we get
\[ E_{\text{in}}(g) \geq E_{\text{in}}(g_{\text{adm}}), \]
with equality if and only if \( r_+ = r_0 \). In this case, we have
\[ 0 = V(r_+) = V(r_0) = a \frac{r_0^n}{\ell^2}, \]
therefore \( a = 0 \). By Proposition [V.1], we complete the proof of the theorem.
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[1] M. Anderson, P. Chruściel, E. Delay, Non-trivial, static, geodesically complete, vacuum space-times with a negative cosmological constant. J. High Energy Phys. 10 (2002) 063.
[2] H. Barzegar, P. Chruściel, M. Hörzinger, M. Maliborski, L. Nguyen, Remarks on the energy of asymptotically Horowitz-Myers metrics. Phys. Rev. D 101, (2020) 024007.
[3] J. Chen, X. Zhang, Metrics of Eguchi-Hanson types with the negative constant scalar curvature, arXiv:2007.15964 (2020).
[4] D. Dold, Global dynamics of asymptotically locally AdS spacetimes with negative mass, Class. Quantum Grav. 35 (2018) 095012, 26 pp.
[5] T. Eguchi, A.J. Hanson, Asymptotically flat self-dual solutions to Euclidean gravity, Phys. Lett. 74B (1978) 249-251.
[6] T. Eguchi, A.J. Hanson, Self-dual solutions to Euclidean gravity, Ann. Phys. 120 (1979) 82-106.
[7] G. Galloway, S. Surya, and E. Woolgar, On the Geometry and Mass of Static, Asymptotically AdS Spacetimes, and the Uniqueness of the AdS Soliton, Commun. Math. Phys. 241 (2003) 1-25.
[8] G. Horowitz, R. Myers, AdS-CFT correspondence and a new positive energy conjecture for general relativity, Phys. Rev. D 59, no. 2 (1998): 026005.
[9] C. LeBrun, Counter-examples to the generalized positive action conjecture, Commun. Math. Phys. 118 (1988) 591-596.
[10] C. LeBrun, The Einstein-Maxwell Equations and Conformally Kahler Geometry, Commun. Math. Phys. 344 (2016) 621-653.
[11] B. O’Neill, Semi-Riemannian Geometry with Application to Relativity, vol. 103, Pure and Applied Mathematics. Academic Press, New York (1983).
[12] H. Pedersen, Eguchi-Hanson metrics with cosmological constant, Class. Quantum Grav. 2 (1985) 579-587.
[13] X. Zhang, Scalar flat metrics of Eguchi-Hanson type, Commun. Theor. Phys. (Beijing, China) 42 (2004) 235-237.