Octonions, Exceptional Jordan Algebra and The Role of The Group $F_4$ in Particle Physics

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Abstract. Normed division rings are reviewed in the more general framework of composition algebras that include the split (indefinite metric) case. The Jordan–von Neumann–Wigner classification of finite euclidean Jordan algebras is outlined with special attention to the 27 dimensional exceptional Jordan algebra $J$. The automorphism group $F_4$ of $J$ and its maximal Borel-de Siebenthal subgroups $SU(3) \times SU(3)/\mathbb{Z}_3$ and $Spin(9)$ are studied in detail and applied to the classification of fundamental fermions and gauge bosons. Their intersection in $F_4$ is demonstrated to coincide with the gauge group $SU(2) \times U(3)$ of the Standard Model of particle physics.

Keywords. Automorphism groups, Clifford algebras, Exceptional Jordan algebra, octonions, Standard model.

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1. Introduction

Division and Clifford algebras were introduced in nineteenth century with an eye for applications in geometry and physics (for a historical survey see the last chapter of [35]). Pascual Jordan introduced and studied his algebras in the 1930s in order to describe observables in quantum mechanics (for “a taste of Jordan algebras” see, [36,37] along with the original paper [32]). Yet, the first serious applications of these somewhat exotic structures appeared (in mid-twentieth century) in pure mathematics: in the theory of exceptional Lie groups and symmetric spaces (cf. [24] as well as the later surveys [1,7,33,42]), in topology [2]. For an entertaining review on division algebras and quantum theory—see, [4,5]. Possible applications to particle physics were first advocated by Feza Gürsey and his students in the 1970s—see his lecture and his posthumous book (with C.-H.Tze) [30,31] and references therein). They continue in various guises to attract attention until these days, never becoming a mainstream activity. The present lectures are meant as a background for the ongoing work [21,47] centered on the exceptional 27 dimensional Jordan algebra \( J = \mathcal{H}_3(\mathbb{O}) \). In chapter 5 we elaborate on a possible application of the automorphism group \( F_4 \) of \( J \) as a novel “grand unified symmetry” of the Standard Model. Although such a proposal of an “exceptional finite quantum geometry” is still tentative, we feel that it is worth pursuing.\(^1\) In any case, the mathematical background which is the main subject of these notes is sound and beautiful—and deserves to be known by particle theorists.

2. Composition and Clifford Algebras

2.1. Normed Alternative Algebras ([42] Sect. 1)

A composition (or Hurwitz) algebra \( \mathcal{A} \) is a vector space over a field \( \mathbb{K} = (\mathbb{R}, \mathbb{C}, \ldots) \) equipped with a bilinear (not necessarily associative) product \( xy \)

\(^1\)For related attempts to provide an algebraic counterpart of the Standard Model of particle physics see [12,18–20,25,45] and references to earlier work cited there.
with a unit 1 (1x = x1 = x) and a nondegenerate quadratic form N(x), the norm, with values in \( \mathbb{K} \), satisfying
\[
N(xy) = N(x)N(y), \quad N(\lambda x) = \lambda^2 N(x) \quad \text{for} \quad x \in \mathcal{A}, \ \lambda \in \mathbb{K}.
\]  
(2.1)
The norm allows to define by polarization a symmetric bilinear form \(<x, y>\) setting:
\[
2 <x, y\rangle = N(x + y) - N(x) - N(y) = 2 <y, x\rangle.
\]  
(2.2)
(Nondegeneracy of \( N \) means that if \(<x, y\rangle = 0 \) for all \( y \in \mathcal{A} \) then \( x = 0 \).)

By repeated polarization of the identity \(<xy, xy\rangle = <x, x\rangle <y, y\rangle\) one obtains
\[
<x, y\rangle = <y, x\rangle.
\]  
(2.3)

Setting in (2.4) \( a = c = x, \ b = 1, \ d = y \) and using (2.3) we find:
\[
<x^2 + N(x)1 - t(x)x, y\rangle = 0,
\]  
where \( t(x) := 2 <x, 1\rangle \) is by definition the \textit{trace}. Using the non-degeneracy of the form \(<,\rangle\) we end up with a vector equation:
\[
x^2 - t(x)x + N(x)1 = 0, \quad t(x) = 2 <x, 1\rangle.
\]  
(2.5)
Thus every \( x \in \mathcal{A} \) satisfies a quadratic relation with coefficients the trace \( t(x) \) and the norm \( N(x) \), a linear and a quadratic scalar function.

The trace functional (2.5) allows to introduce \textit{Cayley conjugation},
\[
x \to x^* = t(x) - x, \quad (t(x) = t(x)1 \in \mathcal{A}),
\]  
(2.6)
an important tool in the study of composition algebras. It is an (orthogonal) reflection \( (<x^*, y^*> = <x, y>) \) that leaves the scalars \( \mathbb{K}1 \) invariant (in fact, \( t(\lambda 1) = 2\lambda \) implying \( (\lambda 1)^* = \lambda 1 \) for \( \lambda \in \mathbb{K} \)). It is also an involution and an antihomomorphism:
\[
(x^*)^* = x, \quad (xy)^* = y^*x^*.
\]  
(2.7)
Furthermore Eqs.(2.5) and (2.6) allow to express the trace and the norm as a sum and a product of \( x \) and \( x^*\):
\[
t(x) = x + x^*, \quad N(x) = xx^* = x^*x = N(x^*).
\]  
(2.8)
The relation (2.4) allows to deduce
\[
<x, y\rangle = <x, a^*y>, \quad <xa, y\rangle = <x, ya^*>.
\]  
From these identities it follows \(<ab, 1\rangle = <a, b^*> = <ba, 1\rangle \), hence, the trace is commutative:
\[
t(ab) = <b, a^*> = <a, b^* > = t(ba).
\]  
(2.9)
Similarly, one proves that \( t \) is associative and symmetric under cyclic permutations.
\[
t((ab)c) = t(a(bc)) = t(ab)c = t(abc) = t(bca).
\]  
(2.10)
Moreover, using the quadratic relation (2.5) and the above properties of the trace one proves the identities that define an alternative algebra:

\[ x^2y = x(xy), \quad yx^2 = (yx)x, \]  

(2.11)

(see, Sect. 1 of [42] for details). The conditions (2.11) guarantee that the associator

\[ [x, y, z] = (xy)z - x(yz), \]  

(2.12)

changes sign under odd permutations (and is hence preserved by even, cyclic, permutations). This implies, in particular, the flexibility conditions.

\[ (xy)x = x(yx). \]  

(2.13)

An unital alternative algebra with an involution \( x \rightarrow x^* \) satisfying (2.7) is a composition algebra if the norm \( N \) and the trace \( t \) defined by (2.9) are scalars (i.e., belong to \( K(=K1) \)) and the norm is non-degenerate.

Given a finite dimensional composition algebra \( A \) Cayley and Dickson have proposed a procedure to construct another composition algebra \( A' \) with twice the dimension of \( A \). Each element \( x \) of \( A' \) is written in the form

\[ x = a + eb, \quad a, b \in A, \]  

(2.14)

where \( e \) is a new “imaginary unit” such that

\[ e^2 = -\mu, \quad \mu \in \{1, -1\} \quad (\mu^2 = 1). \]  

(2.15)

Thus \( A \) appears as a subalgebra of \( A' \). The product of two elements \( x = a + eb, \ y = c + ed \) of \( A' \) is defined as

\[ xy = ac - \mu db + e(\bar{a}d + cb), \]  

(2.16)

where \( a \rightarrow \bar{a} \) is the Cayley conjugation in \( A \). (The order of the factors becomes important, when the product in \( A \) is noncommutative.) The Cayley conjugation \( x \rightarrow x^* \) and the norm \( N(x) \) in \( A' \) are defined by:

\[ x^* = (a + eb)^* = \bar{a} + \bar{b}e^* = \bar{a} - \bar{b}e = \bar{a} - eb \]

\[ N(x) = xx^* = a\bar{a} + \mu\bar{b}\bar{b} = x^*x. \]  

(2.17)

Let us illustrate the meaning of (2.16) and (2.17) in the first simplest cases.

For \( A = \mathbb{R} \), \( \bar{a} = a \), Eq. (2.16) coincides with the definition of complex numbers for \( \mu = 1 \) (\( e = i \)) and defines the split complex numbers for \( \mu = -1 \). Taking next \( A = \mathbb{C} \) and \( \mu = 1 \) we can identify \( A' \) with a \( 2 \times 2 \) matrix representations setting

\[ a = a_0 + e_1a_1 = a_0 + i\sigma_3a_1 = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \quad (a = a_0 + ia_1) \]

\[ x = a + eb, \quad e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} a & d \\ b & \bar{a} \end{pmatrix}, \quad b = b_0 + e_1b_1. \]  

(2.18)

Anticipating Baez Fano plane [3] notation for the octonion imaginary units (see, Appendix A) we shall set \( e = e_4, \ e_4e_1 = e_2 \ (= i\sigma_1) \).

It is then easily checked that the multiplication law (2.16) reproduces the standard matrix multiplication, the Cayley conjugation \( x \rightarrow x^* \) coincides...
with the hermitian conjugation of matrices, while the norm $N(x)$ in $\mathcal{A}'$ is given by the determinant:

$$H = \{ x \in \mathbb{C}[2]; \ xx^* = \det x(\geq 0) \}.$$ (2.19)

Similarly, starting with the split complex numbers, we can write

$$a_s = a_0 + \tilde{e}_1 a_1, \quad \tilde{e}_1 = \sigma_3 \Leftrightarrow a_s = \begin{pmatrix} a_s & 0 \\ 0 & \bar{a}_s \end{pmatrix} (a_s = a_0 + a_1, \ \bar{a}_s = a_0 - a_1).$$

and choosing the same $e$ as above we can identify the split quaternions $\mathbb{H}_s$ with real $2 \times 2$ matrices:

$$\mathbb{H}_s = \left\{ x = \begin{pmatrix} a_s & -\bar{b}_s \\ b_s & \bar{a}_s \end{pmatrix} \in \mathbb{R}[2], \ x^* = \begin{pmatrix} \bar{a}_s & \bar{b}_s \\ -b_s & a_s \end{pmatrix}, \ xx^* = \det x \right\},$$ (2.20)

its norm having signature $(2, 2)$.

The next step in Cayley–Dickson construction gives the octonions, which have a nonassociative (but alternative) multiplication and thus do not have matrix realization.

### 2.2. Relation to Clifford Algebras: Classification

Given a composition algebra $\mathcal{A}$ we define subspace $\mathcal{A}_0 \subset \mathcal{A}$ of pure imaginary elements with respect to the Cayley conjugation (2.6):

$$\mathcal{A}_0 = \{ y \in \mathcal{A}; \ y^* = -y \}. \quad (2.21)$$

It is a subspace of co-dimension one, orthogonal to the unit 1 of $\mathcal{A}$. For any $x \in \mathcal{A}$ we define its imaginary part as

$$x_0 = \frac{1}{2} (x - x^*) = x - <x, 1> \Rightarrow <x_0, 1> = 0. \quad (2.22)$$

From the expression $N(x) = xx^*$ (2.8) and from the defining property (2.21) of imaginary elements it follows that

$$x_0 \in \mathcal{A}_0 \Rightarrow x_0^2 = -N(x_0). \quad (2.23)$$

In other words, if the composition algebra $\mathcal{A}$ is $n$-dimensional then its $(n - 1)$-dimensional subalgebra $\mathcal{A}_0$ gives rise to a Clifford algebra. If the norm $N$ is positive definite then $\mathcal{A}_0 = Cl(0, n - 1) = Cl(1_{-n})$. In the case of split complex numbers, quaternions and octonions one encounters instead the algebras $Cl_1 \equiv Cl(1, 0)$, $Cl(2, 1)$ and $Cl(4, 3)$, respectively.

It turns out that the classification of the Clifford algebras $Cl(1_{-n})$ implies the classification of normed division rings of dimension $n$. So we recall it in Table 1.

Here we use the notation $\mathbb{A}[n]$ for the algebra of $n \times n$ matrices with entries in the (associative) algebra $\mathbb{A}$. As discovered by Élie Cartan in 1908 $Cl_{(-\nu - 8)} = Cl_{-\nu} \otimes \mathbb{R}[16]$ so that the above Table 1 suffices to reconstruct all Clifford algebras of type $Cl_{-\nu}$. We see that the (real) dimension of the irreducible representation of $Cl(1_{-n})$ coincides with $n$ for $n = 1, 2, 4, 8$ only, thus implying Hurwitz theorem (see, [3] Theorem 1 and the subsequent discussion).

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2 We adopt the sign convention of [30,31,35,46]; the opposite sign convention, $Cl_{(n-1)}$ for the positive definite $N(x)$, is used e.g. in [3].
Table 1. Irreducible spinors in the Clifford algebras $\mathcal{C}\ell_{(1-n)}$.

| $n$ | $\mathcal{C}\ell_{(1-n)}$ | Irreducible spinor | $n$ | $\mathcal{C}\ell_{(1-n)}$ | Irreducible spinor |
|-----|----------------|-------------------|-----|----------------|-------------------|
| 1   | $\mathbb{R}$  | $S_1 = \mathbb{R}$ | 5   | $\mathbb{H}[2]$ | $S_5 = \mathbb{H}^2$ |
| 2   | $\mathbb{C}$  | $S_2 = \mathbb{C}$ | 6   | $\mathbb{C}[4]$ | $S_6 = \mathbb{C}^4$ |
| 3   | $\mathbb{H}$  | $S_3 = \mathbb{H}$ | 7   | $\mathbb{R}[8]$ | $S_7 = \mathbb{R}^8$ |
| 4   | $\mathbb{H} \oplus \mathbb{H}$ | $S_4^+ = \mathbb{H}$, $S_4^- = \mathbb{H}$ | 8   | $\mathbb{R}[8] \oplus \mathbb{R}[8]$ | $S_8^+ = \mathbb{R}^8$, $S_8^- = \mathbb{R}^8$ |

Proceeding to the split alternative composition algebras we note that the type of $\mathcal{C}\ell(p, q)$ only depends on the signature $p - q$ which is 1 (similar to −7 modulo 8); we have: $\mathcal{C}\ell(1, 0) = \mathbb{R} \oplus \mathbb{R}$, $\mathcal{C}\ell(2, 1) = \mathbb{R}[2] \oplus \mathbb{R}[2]$, $\mathcal{C}\ell(4, 3) = \mathbb{R}[8] \oplus \mathbb{R}[8]$, for all above cases

$$\mathcal{C}\ell(p, p - 1) \cong \mathbb{R}[2^{(p-1)}] \oplus \mathbb{R}[2^{(p-1)}].$$ (2.24)

We note here the difference in the treatment of the representations of $\mathcal{C}\ell(p, p - 1)$ in the cases $p = 1, 2$, in which we are dealing with real associative composition algebras $\mathbb{C}_s$ and $\mathbb{H}_s$, and $p = 4$ of the split octonions. In the associative case we deal with the action of $\mathcal{C}\ell(p, p - 1)$ on the direct sum $\mathbb{R}^n \oplus \mathbb{R}^n$, $n = 2^{(p-1)}$ (for $p = 1, 2$) while in the non-associative case it acts on the irreducible subspace $\mathbb{R}^n$ ($n = 8$), thus again fitting the dimension of the corresponding alternative algebra.

Remark. The spinors $S_n$ are here understood as quantities transforming under the lowest order faithful irreducible representation of the (compact) group Spin(n) which consists of the norm one even elements of $\mathcal{C}\ell_{-n}$. In fact, the even part $\mathcal{C}\ell_0(p, q)$ of $\mathcal{C}\ell(p, q)$ is isomorphic, for $q > 0$ to $\mathcal{C}\ell(p, q - 1)$. Spin(n) is the double cover of the rotation group SO(n). The group of all norm one elements of $\mathcal{C}\ell_{(1-n)}$ is the double cover Pin(n − 1) of the full orthogonal group O(n − 1) and its irreducible representations are called “pinors”—see, [3], (Sect. 3.3).

In summary, the alternative algebras are classified as follows:

**Theorem 2.1.** ([42] Proposition 1.6) Let $(\mathcal{A}, N)$ be a composition algebra. For $\mu = \pm 1$, denote by $A(\mu)$ the algebra $A(\mu) = A \oplus eA$ with $e^2 = -\mu$ and product (2.16). Then

- $A(\mu)$ is commutative iff $A = \mathbb{K}$;
- $A(\mu)$ is associative iff $A$ is associative and commutative;
- $A(\mu)$ is alternative iff $A$ is associative.

**Theorem 2.2.** ([42] Theorems (2.7)–(2.10)). A composition algebra is, as a vector space, 1, 2, 4 or 8 dimensional. There are four composition algebras $A_j$ over $\mathbb{C}$ of dimension $2^j$, $j = 0, 1, 2, 3$. There are seven composition algebras over $\mathbb{R}$; the division algebras $A_j^+$, $(j = 0, 1, 2, 3)$ with $N(x) \geq 0$ and $x^{-1} = \frac{x^*}{N(x)}$ for $x \neq 0$, and the split algebras $A_j^s$, $j = 1, 2, 3$ of signature $(2j-1, 2j-1$).

All above algebras are unique up to isomorphism. The multiplication rule (2.16) varies in different expositions. Different conventions are related
by algebra automorphisms. (Our notation differs from Roos [42] only by the sign of $\mu$, as we set $e^2 = -\mu$.) The only nontrivial automorphism of the algebra of complex numbers is the complex conjugation. The automorphism group of the (real) quaternions is $SO(3)$ realized by

$$x \rightarrow u x u^*, \quad u \in SU(2) \ (u^* = u^{-1}).$$

Similarly, the automorphism group of the split quaternions is $SO(2,1)$:

$$\mathbb{H} \ni x \rightarrow g x g^{-1}, \quad g \in SL(2, \mathbb{R}).$$

(2.25)

We shall survey the octonions and their automorphisms in the next section.

2.3. Historical note

The simplest relation of type (2.1), the one applicable to the absolute value square of a product of complex numbers

$$(xu - yv)^2 + (xv + yu)^2 = (x^2 + y^2)(u^2 + v^2)$$

($x, y, u, v \in \mathbb{R}$), was found by Diophantus of Alexandria around 250. A more general relation of this type,

$$(xu + D yv)^2 - D(xv + yu)^2 = (x^2 - Dy^2)(u^2 - Dv^2)$$

occurs for special values of $D$ in Indian mathematics (Brahmegupta 598)—see, [49] Sect. 2. For $D$ positive it applies to the split complex numbers. The geometric interpretation by Gauss comes much later. (The fact that complex numbers are useful and should be taken seriously is sometimes attributed to Gerolamo Cardano (1501–1576), whose book Ars Magna (The Great Art) contains the solution of the cubic equation. In fact, it was his contemporary, Bologna’s mathematician Rafael Bombelli (1526–1572) who first thoroughly understood the complex numbers and described them in his L’Algebra, published in 1572.)

The multiplicativity of the norm of the quaternions was noted by Euler in 1748, a century before Hamilton discovered the algebra of quaternions in 1843 (when “in a famous act of a mathematical vandalism, he carved the equations $i^2 = j^2 = k^2 = ijk = -1$ into the stone of Brougham Bridge” [3] p. 145). The corresponding relation for the octonions was discovered by the Danish mathematician Degen in 1818—again before the discovery of the octonions (which took place in late 1843—in a letter to Hamilton by his college friend J. T. Graves). The first publication about octonions appears as an appendix to an otherwise erroneous paper of the English mathematician (at the time, lawyer) Arthur Cayley (1821–1895) in 1845 (see, Introduction and references 17, 18 of [3])

The American algebraist and author of a three-volume History of the Theory of Numbers, Leonard E. Dickson (1874–1954) contributed to the construction of composition algebras in 1919 [17]. The theorem that the only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ was proven by A. Hurwitz (1859–1919) in 1898. The extension of this result to alternative (including split) algebras belongs to M. Zorn (1906–1993) in 1930 and 1933. The fact that the only division algebras (without extra structure) have dimensions
1, 2, 4, 8 was established as late as in 1958 (independently by R. Bott and J. Milnor and by M. Kervaire).

3. Octonions. Isometries and Automorphisms

3.1. Eight Dimensional Alternative Algebras

The multiplication table of the imaginary octonions (see Appendix A) can be introduced by first selecting a quaternion subalgebra

\[ e_j e_k = \epsilon_{jkl} e_l - \delta_{jk}, \quad j, k, l = 1, 2, 4, \]
\[ \epsilon_{124} = 1 = \epsilon_{412} = \epsilon_{241} = 1 = -\epsilon_{214} = \ldots. \] (3.1)

The somewhat exotic labeling of the units (jumping over 3) is justified by the following memorable multiplication rules \( \text{mod} 7 \):

\[ e_i e_j = e_k \Rightarrow e_{i+1} e_{j+1} = e_{k+1}, \quad e_{2i} e_{2j} = e_{2k} \equiv e_{2k(\text{mod}7)} \]
\[ e_7 e_j = e_{3j(\text{mod}7)}, \quad \text{for } j = 1, 2, 4 (e_7 e_4 = e_5). \] (3.2)

A convenient complex isotropic basis for the vector representation of the isometry Lie algebra \( \mathfrak{so}(8) \) of \( O \) (which contains the automorphism algebra \( g_2 \) of the octonions) is given by:

\[ \rho^\epsilon = \frac{1}{2}(1 + i\epsilon e_7), \quad \zeta_j^\epsilon = \rho^\epsilon e_j = \frac{1}{2}(e_j + i\epsilon e_3j) \quad j = 1, 2, 4, \epsilon = \pm, \] (3.3)

(the imaginary unit \( i \) commutes with octonion units \( e_a \)). The multiplication table of the octonion units is summarized by the following relations:

\[ (\zeta_j^\epsilon)^2 = 0 = \rho^+ \rho^-, \quad (\rho^\epsilon)^2 = \rho^\epsilon, \quad \rho^+ + \rho^- = 1, \quad \zeta_j^\epsilon \zeta_k^\epsilon = \epsilon_{jkl} \zeta_l^{-\epsilon} \]
\[ \zeta_j^\epsilon \zeta_k^\epsilon = -\rho^- \epsilon \delta_{jk}, \quad [\zeta_j^+, \zeta_k^+] = \delta_{jk}, \quad j, k, l = 1, 2, 4. \] (3.4)

The idempotents \( \rho^\pm \) (which go back to Gürsey) are also exploited in [19]. The last Eq. (3.4) coincides with the canonical anticommutation relations for fermionic creation and annihilation operators (cf. [12]).

The split octonions \( x_s \in \mathbb{O}_s \) with units \( \tilde{e}_a \) can be embedded in the algebra \( \mathbb{C} \mathbb{O} \) of complexified octonions by setting \( \tilde{e}_\mu = e_\mu, \mu = 0, 1, 2, 4; \quad \tilde{e}_7 = i e_7, \tilde{e}_{3j} = i e_{3j(\text{mod}7)}, \) so that

\[ x_s = \sum_{a=0}^{7} x_s^a \tilde{e}_a \Rightarrow N(x_s) = x_s x_s^* = \sum_{\mu=0,1,2,4} (x_s^\mu)^2 - (x_s^7)^2 - (x_s^3)^2 - (x_s^6)^2 - (x_s^5)^2. \] (3.5)

The quark-lepton correspondence suggests the splitting of octonions into a direct sum,

\[ \mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3; \quad x = a + z e = a + z^1 e_1 + z^2 e_2 + z^4 e_4 (e_1 e_2 = e_4) \]
\[ a = x^0 + x^7 e_7, \quad z^j = x^j + x^{3j(\text{mod}7)} e_7 (x^{12} = x^5) \] (3.6)
thus $e_7$ playing the role of \textit{imaginary unit within the real octonions}. The Cayley–Dickson construction corresponds to the splitting of $\mathbb{O}$ into two quaternions:

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H} : x = u + e_7 v, \quad u = x^0 + x^j e_j, \quad v = x^7 + x^{3j} e_j. \tag{3.7}$$

One may speculate that upon complexification $u$ and $v$ could be identified with the “up-” and “down-”, leptons and quarks: $u = (\nu; u^j, \ j = 1, 2, 4)$. $v = (e^-, d_j)$, $j$ playing the role of a colour index, but we shall not pursue this line of thought. For $x_\alpha = u_\alpha + e_7 v_\alpha$, $\alpha = 1, 2$ the Cayley–Dickson formula (2.16) and its expression in terms of the complex variable $a_\alpha$, $z^\alpha_\alpha$ reads:

$$x_1 x_2 = u_1 u_2 - v_2 v_1^* + e_7 (u_1^* v_2 + u_2 v_1) = a_1 a_2 - z_1 z_2 + (a_1 z_2^* + a_2 z_1 + z_1 \times z_2) e, \tag{3.8}$$

where the star indicates quaternionic conjugation while the bar stands for a change of the sign of $e_7$ ($z^j = x^j - e_7 x^{3j}$). The two representations (3.8) display the covariance of the product under the action of two subgroups of maximal rank of the automorphism group of the octonions. If $p$ and $q$ are two unit quaternions

$$p = p^0 + p^j e_j, \quad q = q^0 + q^j e_j$$

$$pp^* = N(p) = (p^0)^2 + p^2 = 1 \Rightarrow (p, q) \in SU(2) \times SU(2), \ (3.9)$$

it is easy to verify, using the first Eq. (3.8), that the transformation

$$(p, q) : x = u + e_7 v, \quad \rightarrow pud^* + e_7 pqv^*, \quad (p, q) \in \frac{SU(2) \times SU(2)}{Z^2_{\text{diag}}}, \tag{3.10}$$

where $Z^2_{\text{diag}} = \{(p, q) = (1, 1), (-1, -1)\}$, is an automorphism of the octonion algebra. Similarly, if $U \in SU(3)$ acts on $x$ (3.6) as

$$U : x = a + z^j e_j \rightarrow a + U^j_z z^j e_i \text{ then } U(x_1) U(x_2) = U(x_1 x_2). \tag{3.11}$$

The subgroups (3.10), (3.11) are the two closed connected subgroups of maximal rank of the compact group $G_2$ corresponding to the Borel-de Siebenthal theory [8] that plays a central role in [47] as well as in Chapter 5 below.

3.2. Isometry Group of The (Split) Octonions. Triality

The norm $N(x)$ (2.17) and the associated scalar product of the (split) octonions,

$$N_{(s)}(x) = \sum_{a=0}^{7} \eta^a_a x_a^2, \quad \eta_a = 1 \quad \text{for all} \quad a, \quad \text{for} \quad x \in \mathbb{O}_s$$

$$\eta^0_s = \eta^1_s = \eta^2_s = \eta^4_s = 1 = -\eta^5_s = -\eta^6_s = -\eta^7_s, \quad \text{for} \quad x \in \mathbb{O}_s, \tag{3.12}$$

is invariant under the (compact) orthogonal group $O(8)$ (respectively, the split orthogonal group $O(4, 4)$). As stressed in [3], while invariant quadratic forms are common, trilinear forms are rare. It is, therefore, noteworthy, that the trilinear form $t(xyz)$ (2.10) is invariant under the 2-fold cover $\text{Spin}(8)$.
(respectively Spin(4, 4)) of the connected orthogonal group SO(8), (respectively, SO(4, 4)). The existence of such trilinear invariant is related to the exceptional symmetry of the Lie algebra $\mathfrak{D}_4 = so(8)$ which is visualized by the symmetry of its Dynkin diagram, Fig. 1.

Accordingly, the symmetric group $S_3$ which permutes the three external nodes of $\mathfrak{D}_4$ is the group of outer automorphisms of the Lie algebra $so(8)$. This is the *triality group* of $\mathfrak{D}_4$.

We proceed to describing the Lie algebra $\mathfrak{D}_4$ and to displaying the action of the outer automorphisms on it.

To begin with $\mathfrak{D}_4$ is generated by left multiplications $L_\alpha$ (as well as by right multiplications $R_\alpha$) by pure imaginary elements $\alpha \in \text{Im}\mathbb{O}$ acting on the algebra of octonions. If we set (following [40,51])

\[ L_\alpha x = \alpha x, \quad R_\alpha x = x\alpha, \quad T_\alpha = L_\alpha + R_\alpha \]

(i.e. $T_\alpha x = \alpha x + x\alpha$) for $\text{Re} \alpha = \langle \alpha, 1 \rangle = 0$), (3.13)

then each of the three actions annihilates the inner product:

\[ \langle L_\alpha x, y \rangle + \langle x, L_\alpha y \rangle = 0 \quad \text{etc. since} \quad \alpha^* = -\alpha. \]

Note that the product of operators $L_a L_b$, $a, b \in \mathbb{O}$ cannot be written in general as $L_{ab}$ (or as left multiplication by any element of $\mathbb{O}$). But, as emphasized in the thesis [25] one can work with composition of maps $L_a L_b$ which is associative while the product of octonions is not. In fact, the Lie algebra $so(8)$

---

**Figure 1.** The Dynkin diagram of the Lie algebra $\mathfrak{D}_4$. The three outer nodes correspond to the two 8-dimensional chiral spinor representations $S^\pm_8$ of Spin(8) and to the vector representation of SO(8), the central node corresponds to the 28-dimensional adjoint representation. The *triality* automorphisms implementing the symmetries of the diagram were discovered in 1925 by Élie Cartan [11].
is spanned by $L_{e_i}$ and $[L_{e_j}, L_{e_k}]$ for $1 \leq i, j, k \leq 7$. The action of the group $S_3$ of outer automorphism of $D_4$ is generated by an automorphism of order three $\nu$ and an involution $\pi$ defined on the triple $(L_\alpha, R_\alpha, T_\alpha)$ by

$$
\nu L_\alpha = R_\alpha, \quad \nu R_\alpha = \nu^2 L_\alpha = -T_\alpha \Rightarrow \nu T_\alpha = -L_\alpha, \quad \nu^3 = id
$$

$$
\pi L_\alpha = T_\alpha, \quad \pi R_\alpha = -R_\alpha \Rightarrow \pi T_\alpha = L_\alpha, \quad \pi^2 = id. \quad (3.14)
$$

We leave it as an exercise to the reader to verify that the product $\kappa = \pi \nu$ is an involution ($\kappa^2 = id$) satisfying

$$
\kappa L_\alpha = -R_\alpha, \quad \kappa R_\alpha = -L_\alpha, \quad \kappa T_\alpha = -T_\alpha, \quad \nu \kappa = \pi, \quad \nu \pi = \nu \pi^2. \quad (3.15)
$$

The involutive automorphism $\kappa$ can be extended to an arbitrary element $D$ of the Lie algebra $D_4$ with the formula (51 Sect. 1.3).

$$
F_{ab} x = \frac{1}{2} e_a (e_b x) = -F_{ba} x \quad (F_{aa} := 0, \quad e_0^* = e_0), \quad (3.17)
$$

which satisfy the same commutation relations:

$$
[X_{ab}, X_{cd}] = \delta_{bc} X_{ad} - \delta_{bd} X_{ac} + \delta_{ad} X_{bc} - \delta_{ac} X_{bd}, \quad X = G, F. \quad (3.18)
$$

Using the identity $L_{e_k} = 2F_{e_k0}$ (for $k = 1, \ldots, 7$) and Eq. (3.14) one verifies that $\pi(F_{k0}) = G_{k0}$. More generally, we have

$$
\pi(G_{ab}) = F_{ab}, \quad \pi(F_{ab}) = G_{ab}, \quad (3.19)
$$

(cf. Lemma 1.3.3 of [51]). We shall demonstrate in Appendix B that the involution $\pi$ splits into seven 4-dimensional involutive transformations. Here, we display one of them which involves our choice of the Cartan subalgebra of $so(8)$:

$$
F_7 = X_7 G_7, \quad G_7 = \begin{pmatrix}
G_{07} & \\
G_{13} & \\
G_{26} & \\
G_{45} & 
\end{pmatrix}, \quad F_7 = \begin{pmatrix}
F_{07} & \\
F_{13} & \\
F_{26} & \\
F_{45} & 
\end{pmatrix},
$$

$$
X_7 = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 
\end{pmatrix}, \quad (3.20)
$$

a straightforward calculation gives $X_7^2 = \mathbb{I}$, $\det X_7 = -1$.

**Theorem 3.1.** (Infinitesimal triality) For any $D \in D_4$ and $x, y \in \mathbb{O}$ the following identity holds:

$$
(Dx)y + x\nu(D)y = \pi(D)(xy). \quad (3.21)
$$

\footnote{We note that the convention $e_1 e_2 = e_3$ (rather than our $e_1 e_2 = e_4$ which follows [3,15]) is used in [40,51] so that our formulas relating $G$ and $F$ (Appendix B) differ from theirs.}
For a given $D \neq 0$ the automorpisms $\nu$ and $\pi$, given by (3.14), (3.19) are uniquely determined from (3.21).

For $D = L_\alpha$ (Re$\alpha = 0$) the Theorem 3.1 follows from the definition (3.14). For a general $D \in D_4$ one uses the fact that it can be written as a linear combination of $L_\alpha$ and their commutators (see Theorem 1.3.6 of [51]).

There exists a group theoretic counterpart of the principle of infinitesimal triality which we shall only sketch. (For a pedagogical exposition based on the concepts of Moufang loops and their isotopies—see Chapters 7 and 8 of [4, 15].)

The notion of left and right multiplication are well defined and preserve the norm for unit octonions; for instance $N(L_\alpha x) = N(x)$ for $aa^* = 1$. In order to obtain a general $SO(8)$ action on the octonions one needs seven left (or seven right) multiplications ([4, 15] Sect. 8.4). The group theoretic counterpart of $T_\alpha$ is the bimultiplicaton $B_a x = axa$; for $a_t = \exp(ta), t \in \mathbb{R}$, $\alpha \in \text{Im}$\mathbb{O}$

$$
\lim_{t \to 0} \frac{d}{dt} B_{a_t}x = T_\alpha x = ax + x\alpha. \quad (3.22)
$$

The automorphisms (3.14), (3.15) can be lifted to group automorphisms:

$$
\nu(L_\alpha, R_\alpha, B_\alpha) = (R_\alpha, B_\alpha^*, L_\alpha^*), \quad \pi(L_\alpha, R_\alpha, B_\alpha) = (B_\alpha, R_\alpha^*, L_\alpha) \quad \kappa(L_\alpha, R_\alpha, B_\alpha) = (R_\alpha^*, B_\alpha^*, L_\alpha^*), \quad \kappa^2 = \pi^2 = \nu^3 = 1. \quad (3.23)
$$

**Theorem 3.2.** For each rotation $g \in SO(8)$ there are elements $g^\pm$ of the double cover $\text{Spin}(8)$ of $SO(8)$ such that the normed trilinear form

$$
t_8(x, y, z) = \frac{1}{2} t(x, y, z) = <xyz, 1>, \quad x, y, z \in \mathbb{O} \quad (3.24)
$$

satisfy the invariance condition

$$
t_8(gx, g^+ y, g^- z) = t_8(x, y, z). \quad (3.25)
$$

The elements $g^\pm$ are determined from (3.25) up to a common sign: for each $g$ there are exactly two pairs $(g^+, g^-)$ and $(-g^+, -g^-)$ obeying (3.25).

For $g = L_\alpha$ $(aa^* = 1)$ we can set

$$
g^+ = \nu(L_\alpha) = R_\alpha, \quad g^- = \nu^2 L_\alpha = B_\alpha^*. \quad (3.26)
$$

(The factor $\frac{1}{2}$ in (3.24) follows the definitionn of normed triality of [3] which demands $|t_8(x, y, z)|^2 \leq N(x)N(y)N(z)$.)

### 3.3. Automorphism Group $G_2$ of The Octonions

As proven by Élie Cartan in 1914, the automorphism group of the octonions is the rank 2 exceptional Lie group $G_2$ which can thus be defined by

$$
G_2 = \{g \in \text{L}(\mathbb{O}, \mathbb{R})|(gx)(gy) = g(xy), \quad x, y \in \mathbb{O}\}, \quad (3.27)
$$

where $\text{L}(\mathbb{O}, \mathbb{R})$ is the group of non-singular linear transformations of the 8-dimensional real vector space $\mathbb{O}$. It follows from $e_0^2 = e_0 = 1$ and $e_k^* = -e_k$ for $k > 0$ that $G_2$ preserves the octonion unit and the Cayley conjugation and hence the norm $N(x)$:

$$
g1 = 1, \quad (gx)^* = g(x^*), \quad N(gx) = N(x). \quad (3.28)
$$
Thus $G_2$ is a subgroup of the isometry (orthogonal) group $O(O) = O(8)$ of the 8-dimensional euclidean space of the octonions. In fact, it is a subgroup of the connected orthogonal group $SO(7)$ of the 7-dimensional space $\Im O$ of imaginary octonions, the Lie algebra $\mathfrak{so}(7)$ splitting as a vector space into a direct sum of the Lie algebra $\mathfrak{g}_2$ and the vector representation $\mathfrak{r}$ of $\mathfrak{so}(7)$. Thus the dimension of $G_2$ is $(\binom{7}{2}) - 7 = 14:

\mathfrak{so}(7) \cong \mathfrak{g}_2 \oplus \mathfrak{r} \ (\cong \mathfrak{g}_2 \oplus \mathfrak{r}^7).

Moreover, the group $G_2$ acts transitively on the unit sphere $S^6$ in $\mathbb{R}^7$; every point of $S^6$ can be transformed, say, into $e_7$ by an authomorphism of $O$. The stabilizer of $e_7$ is the subgroup $SU(3)$ of $G_2$, defined by (3.11):

$$(G_2)_{e_7} = SU(3) \Rightarrow \frac{G_2}{SU(3)} \cong S^6. \quad (3.29)$$

It follows that the group $G_2$ is connected. The maximal subgroups of $G_2$, whose action was defined by (3.10) and (3.11) (and which correspond to the Borel–de Siebenthal theory) can be characterized as follows. The complex conjugation $\gamma$ (in the notation of [51]): $e_7 \rightarrow -e_7$ belongs to the automorphism group $G_2$ of $O$ (corresponding, in fact, to the reflection of four imaginary units $e_7$, $e_7 e_1 = e_3$, $e_7 e_2 = e_6$ and $e_7 e_4 = e_5$) and has square one:

$$\gamma x = \gamma(u + e_7 v) = u - e_7 v, \quad \gamma^2 = 1. \quad (3.30)$$

The rank two (semisimple) subgroup (3.9), (3.10) of $G_2$ can be characterized as the commutant of $\gamma$ in $G_2$:

$$G_2^\gamma = \{ g \in G_2 | \gamma g = g \gamma \} = \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}. \quad (3.31)$$

Denote, on the other hand, by $\omega$ the generator of the center of $SU(3)$ acting on $z$ by (3.11)

$$\omega x = a + \omega_7 z^j e_j, \quad \omega_7 = -\frac{1}{2} + \frac{\sqrt{3}}{2} e_7 \ (\omega_7^3 = 1 = \omega^3). \quad (3.32)$$

Then the subgroup (3.11) of $G_2$ is characterized by

$$G_2^\omega := \{ \omega \in G_2 | \omega g = g \omega \} = SU(3) = (G_2)_{e_7}. \quad (3.33)$$

(One uses, in particular, the relation $\omega_7 ze_7 \omega_7 = \omega_7 z e_7 ze = ze.$)

### 3.4. Roots and Weights of $g_2 \subset \mathfrak{b}_3 \subset \mathfrak{d}_4$

It is convenient (in particular, for the study of the Jordan algebra $JSpin_9$ in Sect. 5.1 below) to view the Lie algebra $\mathfrak{d}_4$ as embedded into the Clifford algebra $\mathfrak{cl}_8$. Indeed, its $16 \times 16$ matrix-generators $\Gamma_a$, $a = 0, 1, \ldots, 7$, satisfy the same anticommutation relations as the $2 \times 2$ hermitian octonionic matrices

$$\hat{e}_a = \begin{pmatrix} 0 & e_a \\ e_a^* & 0 \end{pmatrix}, \quad i.e. \quad \hat{e}_0 = \begin{pmatrix} 0 & e_0 \\ e_0 & 0 \end{pmatrix}, \ (e_0 = 1), \quad \hat{e}_k = \begin{pmatrix} 0 & e_k \\ -e_k & 0 \end{pmatrix}, \quad (3.34)$$

$k = 1, \ldots, 7$; we have

$$[\hat{e}_a, \hat{e}_b]_+ = 2\delta_{ab} \mathbb{I}_2 \quad \leftrightarrow \quad [\Gamma_a, \Gamma_b]_+ = 2\delta_{ab} \mathbb{I}_{16}, \quad a, b = 0, 1, \ldots, 7. \quad (3.35)$$
(The symbol \( I_n \), the \( n \times n \) unit matrix, will henceforth be omitted wherever this would not give rise to ambiguity.) The Clifford algebra gives also room for the symmetry generators. If we set (recalling (3.16) and (3.17))

\[
\hat{G}_{ab} = \begin{pmatrix} G_{ab} & 0 \\ 0 & \kappa(G_{ab}) \end{pmatrix}, \quad \kappa(G_{ab}) = \begin{cases} -G_{ab} & \text{for } ab = 0 \\ G_{ab} & \text{for } a, b > 0, \end{cases}
\]

(3.36)

so that \( \kappa(G_{ab})e^*_c = \delta_{bc}e^*_a - \delta_{ac}e^*_b \), we shall have

\[
\hat{G}_{ab} \hat{e}_c = \delta_{bc} \hat{e}_a - \delta_{ac} \hat{e}_b, \quad a, b, c = 0, 1, \ldots, 7.
\]

(3.37)

Writing the \( \Gamma_a \), in analogy with (3.34), as

\[
\Gamma_0 = \sigma_1 \otimes P_0 = \begin{pmatrix} 0 & P_0 \\ P_0 & 0 \end{pmatrix}, \quad \Gamma_k = c \otimes P_k = \begin{pmatrix} 0 & P_k \\ -P_k & 0 \end{pmatrix},
\]

\( k = 1, \ldots, 7 \), \( P_0 = I_8 \), \( c = i\sigma_2 \), \( [P_j, P_k]_+ = -2\delta_{jk} \), \( P_1 P_2 \ldots P_7 = I_8 \),

(3.38)

we see that the generators \( G_{ab} \) of the Lie algebra \( D_4 \) are represented by

\[
\frac{1}{2} \Gamma_{ab} = \frac{1}{4} [\Gamma_a, \Gamma_b], \quad \left[ \frac{1}{2} \Gamma_{ab}, \Gamma_c \right] = \delta_{bc} \Gamma_a - \delta_{ac} \Gamma_b, \quad a, b, c = 0, 1, \ldots, 7.
\]

(3.39)

We stress that the map \( e_a \to P_a \) (and hence the map \( \hat{e}_a \to \Gamma_a \))—unlike the representation \( q_j \to -i\sigma_j \) of the imaginary quaternion units - only respects the anticommutators and the \( \text{SO}(O) \) symmetry properties of the octonions, not their commutators.

The four operators \( \Gamma_7 \) of (3.20) are diagonalized in the counterpart of the isotropic octonionic basis (3.3), (3.4) given by

\[
\Gamma_0^\varepsilon = \frac{1}{2}(\Gamma_0 + i\epsilon \Gamma_7), \quad \Gamma_j^\varepsilon = \frac{1}{2}(\Gamma_j + i\epsilon \Gamma_{3j}), \quad j = 1, 2, 4, \quad (3 \times 4 = 5(\text{mod} 7))
\]

\[
\epsilon = \pm, \quad \text{s.t. } (\Gamma_\mu^\varepsilon)^2 = 0, \quad \Gamma_\mu - \Gamma_\mu^\varepsilon =: \rho_\mu^- \text{, } (\rho_\mu^\varepsilon)^2 = \rho_\mu^\varepsilon \text{, } \rho_\mu^+ + \rho_\mu^- = I_{16}.
\]

(3.40)

Their commutators define an orthogonal weight basis in \( D_4 \) (as well as in \( \mathfrak{B}_4 = \mathfrak{so}(9) \)—see Sect. 5.1):

\[
\Lambda_\mu = \frac{1}{2} [\Gamma_\mu^-, \Gamma_\mu^+] = \frac{1}{2}(\rho_\mu^+ - \rho_\mu^-), \quad \Lambda_0 = \frac{i}{2} \Gamma_{07}, \quad \Lambda_j = \frac{i}{2} \Gamma_{j3j}, \quad j = 1, 2, 4,
\]

(3.41)

which can be represented by real diagonal matrices (belonging actually to \( \mathfrak{so}(4, 4) \subset Cl(4, 4) \)). The operators \( \Lambda_\mu \) are normalized to have eigenvalues \pm 1 (and 0) under commutation:

\[
[\Lambda_\mu, \Gamma_\nu^\varepsilon] = -\epsilon \delta_{\mu\nu} \Gamma_\nu^\varepsilon, \quad \mu, \nu = 0, 1, 2, 4, \quad \epsilon = \pm.
\]

(3.42)

They also satisfy the orthogonality relation \( \langle \Lambda_\mu, \Lambda_\nu \rangle := \frac{1}{4} \text{tr} (\Lambda_\mu, \Lambda_\nu) = \delta_{\mu\nu} \). A Chevalley-Cartan basis corresponding to the simple roots \( \alpha_\mu \) of \( D_4 \) is given by
\[
\begin{align*}
\alpha_0 &\leftrightarrow H_0 = \Lambda_0 - \Lambda_1 = \rho_0^+ - \rho_1^+ , \quad E_0 = \Gamma_0^+ \Gamma_1^+ , \quad F_0 = \Gamma_1^+ \Gamma_0^+ \\
\alpha_1 &\leftrightarrow H_1 = \Lambda_1 - \Lambda_2 = \rho_1^+ - \rho_2^+ , \quad E_1 = \Gamma_1^+ \Gamma_2^+ , \quad F_1 = \Gamma_2^+ \Gamma_1^+ \\
\alpha_2 &\leftrightarrow H_2 = \Lambda_2 - \Lambda_4 = \rho_2^+ - \rho_4^+ , \quad E_2 = \Gamma_2^+ \Gamma_4^+ , \quad F_2 = \Gamma_4^+ \Gamma_2^+ \\
(H_4 = )H_{\alpha_4} & = \Lambda_2 + \Lambda_4 = \pm \rho_2^+ \mp \rho_4^+ , \quad E_{\alpha_4} = \Gamma_4^+ \Gamma_2^+ , \quad F_{\alpha_4} = \Gamma_2^+ \Gamma_4^+ 
\end{align*}
\]
and satisfy the standard commutation relations
\[
[H_\mu, E_\nu] = c_{\mu\nu}^D E_\nu , \quad [H_\mu, F_\nu] = -c_{\mu\nu}^D F_\nu , \quad [E_\mu, F_\nu] = \delta_{\mu\nu} H_\nu
\]
where \( c_{\mu\nu}^D \) is the \( \mathfrak{D}_4 \) Cartan matrix
\[
(c_{\mu\nu}^D) = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix}.
\]

As it follows from our discussion in Sect. 3.3 of the automorphism group of the octonions, the derivations of \( \mathcal{O} \) span, in fact, a subalgebra of the 21-dimensional Lie algebra
\[
\mathfrak{B}_3 = \mathfrak{so}_7 = \{ \mathfrak{D} \in \mathfrak{D}_4 : \quad De_0 = 0 \} = \{ D \in \mathfrak{D}_4 : \quad \kappa(D) = D \}
= \text{Span}\{G_{kl}; k, l = 1, \ldots, 7\}.
\]
(3.45)
The first two simple roots \( \alpha_1 \) and \( \alpha_2 \) of \( \mathfrak{B}_3 \) coincide with those of \( \mathfrak{D}_4 \), while the third, \( \alpha_s \), is a short root:
\[
\{\text{Simple roots of } \mathfrak{B}_3\} = \{\alpha_1 = \Lambda_1 - \Lambda_2, \quad \alpha_2 = \Lambda_2 - \Lambda_4, \quad \alpha_s = \Lambda_4\}, \quad (3.46)
\]
the coroot corresponding to \( \alpha_s \) being
\[
\alpha_s^\vee = \frac{2\alpha_s}{(\alpha_s, \alpha_s)} = 2\Lambda_4
\]
(so that \( c_{2s}^D = (\alpha_2, \alpha_s^\vee) = -2 \)).

Comparing the defining relation for a derivation \( D \in \mathfrak{g}_2 \),
\[
(Dx)y + xDy = D(xy)
\]
(3.47)
with the infinitesimal triality relation (3.21) we conclude that \( D \in \mathfrak{g}_2 \) only if \( \nu(D) = \pi(D) = D \). Taking into account the fact that \( \kappa(D) = D \) in \( \mathfrak{B} \) and that \( \nu = \pi \kappa \) (according to (3.15)) we arrive at the following

**Proposition 3.3.** If \( D \in \mathfrak{B}_3 \) then each of the condition \( \nu(D) = D \) and \( \pi(D) = D \) implies the other and the resulting triality invariance is necessary and sufficient for \( D \) to belong to \( \mathfrak{g}_2 \).

Combining this result with the last Eq. (3.45) we deduce that a linear combination of \( G_{kl} \) belongs to \( \mathfrak{g}_2 \) iff it is invariant under the involution \( \pi \) (3.19). In particular, taking (3.20) into account, we deduce that \( \lambda G_{13} + \mu G_{26} + \nu G_{45} \in \mathfrak{g}_2 \) iff \( \lambda + \mu + \nu = 0 \). In other words, the Cartan subalgebra \( \mathfrak{g}_2 \) is spanned by \( G_{12} - G_{26} \) and \( G_{26} - G_{45} \) whose representatives within \( \Gamma_{ab} \) are...
proportional to \( H_1 \) and \( H_2 \) of Eq. (3.43). More generally, using Appendix B, we find that the following seven linear combinations of \( G_{lk} \) span \( g_2 \)

\[
\lambda G_{24} + \mu G_{37} + \nu G_{56}, \quad \lambda G_{14} - \mu G_{35} + \nu G_{76},
\lambda G_{17} + \mu G_{25} - \nu G_{46}, \quad -\lambda G_{12} + \mu G_{36} + \nu G_{75},
\lambda G_{16} - \mu G_{23} + \nu G_{47}, \quad -\lambda G_{15} + \mu G_{27} + \nu G_{43},
\lambda G_{13} + \mu G_{26} + \nu G_{45}, \text{ with } \lambda + \mu + \nu = 0.
\]

(3.48)

4. Jordan Algebras and Related Groups

4.1. Classification of Finite Dimensional Jordan Algebras

Pascual Jordan (1902–1980) the “unsung hero among the creators of quantum theory” (in the words of Schweber 1994) asked himself in 1932 a question you would expect of an idle mathematician: can one construct an algebra of (hermitian) observables without introducing an auxiliary associative product? He arrived, after some experimenting with the special Jordan product

\[
A \circ B = \frac{1}{2} (AB + BA) = B \circ A,
\]

at two axioms (Jordan 1933)

\[
(i) : \quad A \circ B = B \circ A; \quad (ii) : \quad A^2 \circ (B \circ A) = (A^2 \circ B) \circ A,
\]

where \( A^2 := (A \circ A) \). They imply, in particular, power associativity and

\[
A^m \circ A^n = A^{m+n}, \quad m, n = 0, 1, 2, \ldots
\]

(4.3)

(Jordan algebras are assumed to contain a unit and \( A^0 = 1 \).) Being interested in extracting the properties of the algebra of hermitian matrices (or selfadjoint operators) for which \( A^2 \geq 0 \), Jordan adopted Artin’s formal reality condition

\[
A_1^2 + \cdots + A_n^2 = 0 \Rightarrow A_1 = 0 = \cdots = A_n.
\]

(4.4)

(It is enough to demand (3.4) for \( n=2 \).) Algebras over the real numbers satisfying both (4.2) and (4.4) are now called euclidean Jordan algebras. In a fundamental paper of 1934 Jordan, von Neumann and Wigner [32] classified all finite dimensional euclidean Jordan algebras. They split into a direct sum of simple algebras, which belong to four infinite families,

\[
\mathcal{H}_n(\mathbb{R}), \quad \mathcal{H}_n(\mathbb{C}), \quad \mathcal{H}_n(\mathbb{H}), \quad J\text{Spin}(n),
\]

(4.5)

and a single exceptional one

\[
\mathcal{J} (= J_3^8) = \mathcal{H}_3(\mathbb{O}).
\]

(4.6)

Here \( \mathcal{H}_n(A) \) stands for the set of \( n \times n \) hermitian matrices with entries in the division ring \( A (= \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}) \), equipped with the commutative product (4.1). (One uses the same notation when \( A \) is replaced by one of the alternative split composition rings, \( \mathbb{C}_s, \mathbb{H}_s \) or \( \mathbb{O}_s \) albeit the resulting algebra is not euclidean in that case.) \( J\text{Spin}(n) \) is an algebra of elements \((\xi, x; \xi \in \mathbb{R}, x \in \mathbb{R}^n)\) where \( \mathbb{R}^n \) is equipped with the (real) euclidean scalar product \( < x, y > \) and the product in \( J\text{Spin}(n) \) is given by

\[
(\xi, x)(\eta, y) = (\xi\eta + < x, y >, \xi y + \eta x).
\]

(4.7)
The first three algebras $\mathcal{H}_n(\mathbb{A})$ (4.5) are special the matrix product $AB$ in (4.1) is associative. The algebra $JSpin(n)$ is also special as a Jordan subalgebra of the $2^n$ dimensional (associative) Clifford algebra $C\ell_n$.

**Remark.** The Jordan algebras $\mathcal{H}_2(\mathbb{A})$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are isomorphic to $JSpin(n)$ for $n = 2, 3, 5, 9$, respectively. In fact, more generally, let $x = x^0 + x$ where $x = \sum_{k=1}^{n-2} x^k e_k \in C\ell_{2-n}$; then the two-by-two (Clifford valued) matrix:

$$X = \begin{pmatrix} \xi + x_{n-1} & x \\ x^* & \xi - x_{n-1} \end{pmatrix}$$

(4.8)

satisfies Eq. (2.5) with $N(X) = \det X$, $t(X) = \text{tr}(X) = 2\xi$ where the determinant of $X$ has a (time-like) Minkowski space signature:

$$\det X = \xi^2 - x^2_{n-1} - xx^*, \quad xx^* = x^*x = (x^0)^2 + (x^1)^2 + \ldots + (x^{n-2})^2$$

(4.9)

and is thus invariant under the Lorentz group $\text{SO}(n,1)$. We leave it to the reader to verify that the multiplication law (4.7) can be obtained from Eq. (2.5) for the matrix (4.8) by polarization. (Cf. Sect. 2.2.)

On the other hand, the algebras $\mathcal{H}_n(\mathbb{O})$ for $n > 3$ are not Jordan since they violate condition $(ii)$ of (4.2). The exceptional Jordan algebra $\mathcal{J} = \mathcal{H}_3(\mathbb{O})$ did not seem to be special but the authors of [32] assigned the proof that the product $A \circ B$ of two elements of $\mathcal{J}$ cannot be represented in the form (4.1) with an associative product $AB$ to A. Adrian Albert (1905–1972)—a PhD student of L. Dickson. As a result, many authors, including [36,37] call $\mathcal{J}$ an Albert algebra.

With the realization (4.8) of the elements of $JSpin(n)$ we see that each (simple) euclidean Jordan algebra is a matrix algebra of some kind and so has a well defined (matrix) trace. The trace $\rho$ of the unit element of a Jordan algebra defines its rank; in particular, the rank of the algebra $JSpin(n)$ for any $n$ is $\rho(JSpin(n)) = 2$. The real dimension $\delta$ of the non-diagonal elements of the matrix representation of a Jordan algebra is called its degree. (For a concise survey of euclidean Jordan algebras and their two numerical characteristics, the rank and the degree—see, [39] (Sect. 2); note that Meng denotes the $JSpin(n)$ by $\Gamma(n)$ and calls them Dirac type.) The algebras listed in (4.5) have degrees $\delta = 1, 2, 4, n-1$, respectively, while the Albert algebra (4.6) has degree 8. The dimension of an euclidean Jordan algebra $V = J^\delta_\rho$ of rank $\rho$ and degree $\delta$ is $\text{dim} J^\delta_\rho = \binom{\rho}{2}\delta + \rho$. The rank and the degree completely characterize the simple euclidean Jordan algebras.

We introduce the 1-dimensional projectors $E_i$ and the hermitian octonionic matrices $F_i(x_i)$ writing down a general element of $\mathcal{H}_3(\mathbb{O})$ as

$$X = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix}$$

$$= \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + F_1(x_1) + F_2(x_2) + F_3(x_3).$$

(4.10)
We can then write the Jordan multiplication $X \circ Y$ setting
\[
E_i \circ E_j = \delta_{ij} E_j, \quad E_i \circ F_j(x) = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{2} F_j(x), & \text{if } i \neq j. \end{cases}
\]
\[
F_i(x) \circ F_i(y) = < x, y > (E_{i+1} + E_{i+2}),
\]
\[
F_i(x) F_{i+1}(y) = \frac{1}{2} F_{i+2}(y^* x^*), \quad (4.11)
\]
where the indices are counted mod 3: $E_4 \equiv E_1$, $F_5 \equiv F_2$, \ldots. We define the trace, a symmetric bilinear inner product and a trilinear scalar product in $J$ by
\[
\text{tr}X = \xi_1 + \xi_2 + \xi_3, \\
< X, Y > = \text{tr}(X \circ Y), \quad \text{tr}(X, Y, Z) = < X, Y \circ Z >. \quad (4.12)
\]
The exceptional algebra $J$ also admits a (symmetric) Freudenthal product:
\[
X \times Y = \frac{1}{2} [2X \circ Y - X \text{tr}Y - Y \text{tr}X + (\text{tr}X \text{tr}Y - < X, Y >) E] \quad (4.13)
\]
where $E$ is the $3 \times 3$ unit matrix, $E = E_1 + E_2 + E_3$. Finally, we define a 3-linear form $(X, Y, Z)$ and the determinant $\det X$ by
\[
(X, Y, Z) = < X, Y \times Z > = < X \times Y, Z >, \\
\det X = \frac{1}{3} (X, X, X) = \\
\xi_1 \xi_2 \xi_3 + 2 \text{Re}(x_1 x_2 x_3) - \xi_1 x_1 x_1^* - \xi_2 x_2 x_2^* - \xi_3 x_3 x_3^*. \quad (4.14)
\]
The following identities hold:
\[
X \times X \circ X = (\det X)E \quad \text{(Hamilton-Cayley)} \\
(X \times X) \times (X \times X) = (\det X)X. \quad (4.15)
\]

4.2. The Tits–Kantor–Koecher (TKK) Construction

The symmetrized product $u \circ x$ (4.1) is not the only way to construct a hermitian operator out of two such operators $u$ and $x$. The quadratic in $u$ binary operation $P_u(x) = u x u$ also gives a hermitian result whenever $u$ and $x$ are hermitian. Its expression in terms of the Jordan product looks somewhat clumsy:
\[
P_u = 2L_u^2 - L_u^2, \quad \text{i.e.} \quad P_u x = u \circ (u \circ x) - u^2 \circ x, \quad (4.16)
\]
but, as emphasized by McCrimmon [36,37], it can be advantageously taken as a basic notion defined axiomatically. Introduce first the polarized form of $P$:
\[
S_{uv}w = \frac{1}{2} (P_{u+w} - P_u - P_w) v = \{ uvw \} := \\
u \circ (v \circ w) + w \circ (v \circ u) - (u \circ w) \circ v = \{ wvu \}. \quad (4.17)
\]
A unital quadratic Jordan algebra is a space together with a distinguished element 1 and a product $P_u(x)$ linear in $x$ and quadratic in $u$, which is unital and satisfies the Commuting Formula and the Fundamental Formula:
\[
P_1 = 1, \quad P_u S_{vu} = S_{uv} P_u, \quad P_{P_u(v)} = P_u P_v P_u. \quad (4.18)
\]
The triple product \( \{uvw\} \) is symmetric (according to the last equation (4.17)) and obeys the 5-linear relation
\[
\{x y \{uvw\}\} = \{\{x y u\} v w\} - \{u \{y x v\} w\} + \{u v \{x y w\}\}. \tag{4.19}
\]
This identity can be read as a Lie algebra relation:
\[
[S_{xy}, S_{uw}] = S_{\{xyu\}v} - S_{\{yuv\}x} \tag{4.20}
\]
applied to an arbitrary element \( w \in V \). It defines the structure Lie algebra \( \text{str} \) of \( V \). The generators \( S_{uw} \) can be expressed in terms of the (left) multiplication operators \( L_x \) as follows:
\[
S_u = -S_{u1} = L_u, \quad S_{uv} = L_{uv} + [L_u, L_v]. \tag{4.21}
\]
The derivation algebra \( \text{der}(V) \) of \( V \), spanned by the commutators \( [L_u, L_v] \), appears as the maximal compact Lie subalgebra of \( \text{str}(V) \).

The conformal algebra \( \text{co}(v) \) is an extension of \( \text{str}(V) \) defined, as a vector space, as
\[
\text{co}(v) = V + \text{str}(V) + V^* \tag{4.22}
\]
with the natural TKK commutation relations (under the action of the structure algebra, \( u \) and \( v \) in \( S_{uv} \) transforming as a vector and a covector, respectively—see Theorem 3.1 of [39]). Here there are three relevant examples (the reader can find a complete list in [39]):
\[
\begin{align*}
\text{der}(JSpin_n) &= \mathfrak{so}(n), \quad \text{str}(JSpin_n) = \mathfrak{so}(n,1) \oplus \mathbb{R}, \quad \text{co}(JSpin_n) = \mathfrak{so}(n+1,2) \oplus \mathbb{R} \\
\text{der}(\mathcal{H}_n(\mathbb{C})) &= \mathfrak{su}(n), \quad \text{str}(\mathcal{H}_n(\mathbb{C})) = \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}, \quad \text{co}(\mathcal{H}_n(\mathbb{C})) = \mathfrak{su}(n,n) \\
\text{der}(\mathcal{H}_3(\mathbb{O})) &= \mathfrak{so}(1), \quad \text{str}(\mathcal{H}_3(\mathbb{O})) = \mathfrak{e}_6(-26) \oplus \mathbb{R}, \quad \text{co}(\mathcal{H}_3(\mathbb{O})) = \mathfrak{e}_7(-25).
\end{align*}
\tag{4.23}
\]

A Jordan triple system \( V \) with a triple product \( V^3 \rightarrow V, \{uvw\} \), satisfying (i) \( \{uvw\} = \{wvu\} \) and (4.19) is a generalization of a Jordan algebra (as every Jordan algebra generates a Jordan triple system). The same structure arises\(^4\) in any 3-graded Lie algebra \( \mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \) with an involution \( \tau \) exchanging \( \mathfrak{g}_{\pm 1} \) (see, [41]):
\[
\{uvw\} = \{[u, \tau(v)], w\}, \quad u, v, w \in \mathfrak{g}_1, \quad \tau(v) \in \mathfrak{g}_{-1}, \quad [u, \tau(v)] \in \mathfrak{g}_0. \tag{4.24}
\]

Another important discovery, due to Koecher and his school, is the existence of a one-to-one correspondence between (simple) euclidean Jordan algebras \( V \) and (irreducible) symmetric cones \( \Omega(V) \) (see, [23,26,33]). To the four matrix algebras correspond the cones of positive definite matrices \( \Omega_3(\mathbb{R}), \Omega_n(\mathbb{C}), \Omega_n(\mathbb{H}), \Omega_3(\mathbb{O}) \) of rank \( \rho = tr(v) = 1 \) equal to \( n \) or \( 3 \), respectively. The positive cone of \( JSpin_n \) coincides with the forward light cone:
\[
\Omega_n(JSpin_n) = \{(\xi, x) \in JSpin_n | \xi > \sqrt{x_1^2 + \cdots + x_n^2}\}. \tag{4.25}
\]
In all cases \( \Omega(V) \) is spanned by (convex) linear combinations of squares of elements \( x \in V \) with positive coefficients; equivalently, \( \Omega(V) \) is the connected component of the unit element \( \mathbb{I} \in V \) of the invertible elements of \( V \).

\(^4\)This fact has been discovered by Isaiah Kantor (1964)—see the emotional essay [52] by Efim Zelmanov.
cones $\Omega(V)$ are all selfdual and invariant under the structure group $\text{Str}(V)$. The conformal group $\text{Co}(V)$ can be defined as the automorphism group of the tube domain:

$$\text{Co}(V) = \text{Aut}\{V + i\Omega(V)\}.$$  \hfill (4.26)

### 4.3. Automorphism Groups of The Exceptional Jordan Algebras $H_3(O_s)$ and Their Maximal Subgroups

Classical Lie groups appear as symmetries of classical symmetric spaces. For quite some time there was no such interpretation for the exceptional Lie groups. The situation only changed with the discovery of the exceptional Jordan algebra $H_3(O)$ and its split octonions’ cousin $H_3(O_s)$.

The automorphism group of $H_3(O)$ is the rank four compact simple Lie group $F_4$. It leaves the unit element $E$ invariant and is proven to preserve the trace \hfill (4.12) (see Lemma 2.2.1 in [51]). The stabilizer of $E_1$ in $F_4$ is the double cover $\text{Spin}(9)$ of the rotation group in nine dimensions (which preserves $X_0^2$ \hfill (4.8)). Moreover, we have

$$F_4/\text{Spin}(9) \simeq \mathbb{OP}^2 \implies \dim F_4 = \dim \text{Spin}(9) + \dim \mathbb{OP}^2 = 36 + 16 = 52. \quad (4.27)$$

Building on our treatment of $D_4 \supset g_2$ of Sect.3.4 we shall first construct the Cartan subalgebra of the Lie algebra $f_4$ of derivation (infinitesimal automorphisms) of $H_3(O_s)$. It is again spanned by the orthonormal weight basis $\Lambda_\mu$, $\mu = 0, 1, 2, 4$ (that actually belongs to the real form $f_{4(4)} \supset \text{so}(4,4)$), their restriction to $D_4$ being given by (3.41). The simple roots $\alpha$ and the corresponding coroots $\alpha^\vee$ of $f_4$ are given by:

$$\alpha_1 = \Lambda_1 - \Lambda_2 = \alpha_1^\vee = H_1, \quad \alpha_2 = \Lambda_2 - \Lambda_4 = \alpha_2^\vee = H_2$$

$$\alpha_4(s) = s_4 = \Lambda_4, \quad \alpha_4^\vee = 2\Lambda_4 = H_4^s, \quad s_0 = \frac{1}{2}(\Lambda_0 - \Lambda_1 - \Lambda_2 - \Lambda_4) = \alpha_0(s)$$

$$s_0^\vee = \Lambda_0 - \Lambda_1 - \Lambda_2 - \Lambda_4 = H_0^s. \quad (4.28)$$

The corresponding Cartan matrix reads:

$$\left( c^f_{ij} = <\alpha_i, \alpha_j^\vee> = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix} \right). \quad (4.29)$$

The highest root $\theta$ coincides with that of the rank four simple subalgebra $\mathfrak{B}_4 \supset D_4$ (respectively, of the real forms $\text{so}(5,4) \supset \text{so}(4,4)$):

$$\theta = \Lambda_0 + \Lambda_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_4 + 2\alpha_0. \quad (4.30)$$

---

\[5\]This was proven by Claude Chevalley and Richard Schafer in [43,44]. The result was prepared by Ruth Moufang’s study in 1933 of the octonionic projective plane, then Jordan’s construction in 1949 of $\mathbb{OP}^2$ in terms of 1-dimensional projections in $H_3(O)$ and Armand Borel’s observation that $F_4$ is the isometry group of $\mathbb{OP}^2$; for a review and references - see [3] (Sect. 4.2). Octonionic quantum mechanics in the Moufang plane was considered in [29].
The elements $D$ of $so(8)$ act on $X$ of $\mathcal{J}$ (4.10) through their action on the octonions.

$$DX = F_1(Dx_1) + F_2(\kappa(D)x_2) + F_3(\pi(D)x_3),$$  \hfill (4.31)

where $D =: D_1$, $\kappa(D) =: D_2$, $\pi(D) =: D_3$, obey the principle of infinitesimal triality:

$$(D_1x)y + x(D_2y) = (D_3((xy)^*))^*.$$

For $D \in G_2$ we have $D_1 = D_2 = D_3 = D$.

The remaining 24 generators of $\mathfrak{f}_4$ (outside $so(8)$) can be identified with the skew-hermitian matrices $A_i(e_a)$, $i = 1, 2, 3, a = 0, 1, \ldots, 7,$ where

$$A_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x^* \\ 0 & x^* & 0 \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} 0 & 0 & -x^* \\ 0 & 0 & 0 \\ x^* & 0 & 0 \end{pmatrix},$$

$$A_3(x) = \begin{pmatrix} 0 & x & 0 \\ -x^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.33)$$

They act on $\mathcal{J}$ through the commutators

$$\tilde{A}_i(e_a)X = \frac{1}{2}[A_i(e_a), X], \quad i = 1, 2, 3, a = 0, 1, \ldots, 7. \quad (4.34)$$

The Borel–de Siebenthal theory [8] describes the maximal rank closed connected subgroups of a compact Lie group. In order to reveal the physical meaning of the symmetry of $\mathcal{H}_3(\mathbb{O})$ we shall consider, elaborating on [47], those maximal rank subgroups of $F_4$ which contain the (unbroken) colour symmetry group $SU(3)_c \subset G_2 \subset F_4$. There are two such subgroups:

$$\frac{SU(3) \times SU(3)}{ \mathbb{Z}_3} \quad \text{and} \quad Spin(9). \quad (4.35)$$

Postponing the study of the maximal subgroup $Spin(9)$ to Sect. 5.1 we shall display here the action of

$$F_4^{\omega} = \frac{SU(3) \times SU(3)}{ \mathbb{Z}_3}, \quad \omega(a + z^j e_j) = a + \omega_7 z^j e_j \quad (\omega^3 = 1 = \omega_7^2)$$

$$(a = a^0 + a^7 e_7, \quad z^j e_j = z^1 e_1 + z^2 e_2 + z^4 e_4) \quad (4.36)$$

(cf. (3.32)) on the exceptional Jordan algebra. To do that we shall first extend the splitting of the octonions $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ to a splitting of the exceptional Jordan algebra, $\mathcal{H}_3(\mathbb{O}) = \mathcal{H}_3(\mathbb{C}) \oplus \mathbb{C}[3]$:

$$(\mathcal{H}_3(\mathbb{O}) \ni) X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} = X(\xi, a) + X(0, ze), \quad (4.37)$$
where

\[
X(\xi, a) = \begin{pmatrix}
\xi_1 & a_3 & \bar{a}_2 \\
\bar{a}_3 & \xi_2 & a_1 \\
a_2 & \bar{a}_1 & \xi_3
\end{pmatrix},
\]

\[a_r = x_r^0 + x_r^7 e_7, \quad \bar{a}_r = x_r^0 - x_r^7 e_7, \quad r = 1, 2, 3;\]

\[
X(0, ze) = \begin{pmatrix}
0 & z_3 e & -z_2 e \\
-z_3 e & 0 & z_1 e \\
z_2 e & -z_1 e & 0
\end{pmatrix},
\]

\[
z_r e = z_r^1 e_1 + z_r^2 e_2 + z_r^4 e_4, \quad z_r^j = x_r^j + x_r^{3j mod 7} e_7,
\]

(we have used the conjugation property \((ze)^* = -ze\) of imaginary octonions). Multiplication mixes the two terms in the right hand side of (4.37). The Freudenthal product \(X(\xi, x) \times Y(\eta, y)\) can be expressed in a nice compact way if we substitute the skew symmetric octonionic matrices \(X(0, ze), X(0, we)\) by 3 \times 3 complex matrices \(Z, W\):

\[X(0, ze) \leftrightarrow Z = (z_r^j, \quad r = 1, 2, 3, \quad j = 1, 2, 4) \in \mathbb{C}[3],\]

which transform naturally under the subgroup (4.36).

Indeed, using the fact that the matrices \(X(0, ze)\) and \(X(0, we)\) are traceless, their Freudenthal product (4.13) simplifies and we find:

\[X(\xi, a) \times X(0, we) = X(\xi, a) \circ X(0, we) - \frac{\xi_1 + \xi_2 + \xi_3}{2} X(0, we)\]

\[\implies X(\xi, a) \times W = -\frac{1}{2} WX(\xi, a), \quad \text{for } W = (w_r^j);\]

\[X(0, ze) \times X(0, we) = X(0, ze) \circ X(0, we) - \frac{1}{2} tr(X(0, ze) X(0, we)) E\]

\[X(0, ze) \times X(0, we) \leftrightarrow -\frac{1}{2} (W^* Z + Z^* W + Z \times W)\]

where \(Z \times W = (\epsilon_{rst}(z_s \times w_t)^j)\), so that

\[(X(\xi, a) + Z) \times (X(\eta, b) + W) = X(\zeta, c) + V\]

\[X(\zeta, c) = X(\xi, a) \times X(\eta, b) - \frac{1}{2} (Z^* W - W^* Z)\]

\[V = -\frac{1}{2} \left(W X(\xi, a) + Z X(\eta, b) + \bar{Z} \times W\right).\]

(4.42)

Thus, if we set \(V = (v_r^j)\) we would have

\[2v_1 = -\xi_1 w_1 - \bar{a}_3 w_2 - a_2 w_3 - \bar{z}_2 \times \bar{w}_3\]
\[2v_2 = -a_3 w_1 - \xi_2 w_2 - \bar{a}_1 w_3 - \bar{z}_3 \times \bar{w}_1\]
\[2v_3 = -\bar{a}_2 w_1 - a_1 w_2 - \xi_3 w_3 - \bar{z}_1 \times \bar{w}_2.\]
The inner product in $\mathcal{J}$ is expressed in terms of the components $X(\xi,a)$ and $Z$ as:

$$(X,Y)(=trX \circ Y) = (X(\xi,a),X(\eta,b)) + 2(Z,W)$$

where $2(Z,W) = Tr(Z^*W + W^*Z) = 2\sum_{r=1}^{3}\sum_{j=1,2,4}(\bar{z}_r^1w_r^j + \bar{w}_r^1z_r^j)$.

(4.43)

In the applications to the standard model of particle physics the (upper) index $j$ of $z_j$ ($j = 1, 2, 4$) labels quark’s colour while $r \in \{1, 2, 3\}$ is a flavour index acted upon by the electroweak gauge group. The $SU(3)$ subgroup of $G_2$, displayed in Sect. 3, acting on individual (imaginary) octonions is the colour group.

The subgroup $F_4^\omega$ (4.36) is defined as the commutant of the automorphism $\omega$ of order three in $F_4$ (see, (3.32)):

$$\omega X(\xi,x) = \begin{pmatrix} \xi_1 & \omega x_3 & (\omega x_2)^* \\ (\omega x_3)^* & \xi_2 & \omega x_1 \\ \omega x_2 & (\omega x_1)^* & \xi_3 \end{pmatrix},$$

$$\omega(X(\xi,a) + Z) = X(\xi,a) + \omega_7 Z.$$  

(4.44)

The automorphisms $g \in F_4^\omega$ that commute with $\omega$ (4.44) are given by pairs $g = (A,U) \in SU(3) \times SU(3)$ acting on $\mathcal{H}_3(\emptyset)$ by

$$(A,U) (X(\xi,a) + Z) = AX(\xi,a)A^* + UZA^*.$$  

(4.45)

The central subgroup

$$\mathbb{Z}_3 = \{(1,1), (\omega_7,\omega_7), (\omega_7^2,\omega_7^2)\} \in SU(3) \times SU(3)$$

acts trivially on $\mathcal{H}_3(\emptyset)$. We see that the unitary matrix $U$ acts in (4.44) on the colour index $j$ and hence belongs to the (unbroken) colour group $SU_c(3)$, while the action of $A$ on the flavour indices will be made clear in Sect. 5.2 below.

5. $F_4$ as A Grand Unified Symmetry of The Standard Model

This chapter provides a tentative application of the exceptional Jordan algebra to the standard model (SM) of particle physics. We first study, in Sect. 5.1, the special Jordan subalgebra $J_{Spin9}$ of $\mathcal{J}$ and its automorphism group $Spin(9) \subset F_4$ singling out the 16-dimensiion spinor representation of $Spin(9)$ and interpret it in terms of the fundamental fermionic doublets of the SM. Then, in Sect. 5.2, we examine the full 26-dimensional representation 26 of $F_4$, considering its restriction to both its maximal rank subgroups that contain the colour $SU(3)_c$ as subgroup. We demonstrate that 26 gives also room, to the electroweak gauge bosons.
5.1. The Jordan Subalgebra $J \text{Spin}_9$ of $\mathcal{H}_3(\mathbb{O})$: $\text{Spin}(9)$ as A Symmetry of One Generation

The ten dimensional Jordan algebra $J \text{Spin}_9$ can be identified with the algebra of $2 \times 2$ hermitian octonionic matrices $\mathcal{H}_2(\mathbb{O})$ equipped with the Jordan matrix product. It is generated by the 9-dimensional vector subspace $s\mathcal{H}_2(\mathbb{O})$ of traceless matrices of $\mathcal{H}_2(\mathbb{O})$ whose square is, in fact, a positive real scalar:

$$X = \begin{pmatrix} \xi & x \\ x^* & -\xi \end{pmatrix} \Rightarrow X^2 = (\xi^2 + x^*x)I,$$

$x \in \mathbb{O}, \xi \in \mathbb{R}$.

($J \text{Spin}_9$ is a (special) Jordan subalgebra of the (associative) matrix algebra $\mathbb{R}[2^4]$ that provides an irreducible representation of $\text{Cl}_9$. The algebra $\mathcal{H}_2(\mathbb{O})$ can be identified as a subalgebra $\mathcal{J}_i$ of $\mathcal{J} = \mathcal{H}_3(\mathbb{O})$—which commutes with and is orthogonal to either of the three projectors $E_i, i = 1, 2, 3$. Each $\mathcal{J}_i$ involves a single octonion $x_i$ (and its conjugate). The three $x_i$—and thus the $\mathcal{J}_i$’s—are related by triality. It is tempting to speculate that the three $\mathcal{J}_i$ correspond to the three generations of fundamental fermions in the SM$^6$ [10].

We shall now restrict our attention to the subalgebra $\mathcal{J}_1$ consisting of $3 \times 3$ matrices of $\mathcal{J}$ with vanishing first row and first column and will interpret it as the observable algebra of the first generation of the SM. Its automorphism group is the subgroup $\text{Spin}(9) \subset \text{F}_4$ which stabilizes the projector $E_1$: $\text{Spin}(9) = (\text{F}_4)E_1 \subset \text{F}_4$.

We shall see that the spinor representation of $\text{Spin}_9$ can be interpreted as displaying the first generation of (left chiral) doublets of quarks and leptons

$$\begin{pmatrix} \nu_L & \omega^j_L \\ \nu^*_L & \omega^j_L \end{pmatrix} (j \text{ is the colour index})$$

and their antiparticles. In fact, the irreducible representation 26 of $\text{F}_4$ appears to accommodate only particles of a single generation of the SM.

The 16-dimensional (real) spinor representation $S_9$ of $\text{Spin}(9)$ splits into a direct sum of the two 8-dimensional chiral representations $S^+_8$ and $S^-_8$ of $\text{Spin}(8)$ that appear as eigenvectors of the Coxeter element $\omega_8$ of $\text{Cl}_8$:

$$S_9 = S^+_8 \oplus S^-_8, \quad \omega_8 S^+_8 = \pm S^+_8.$$

We shall use the relation (3.38) $(\Gamma_0 = \sigma_1 \otimes P_0, \Gamma_a = c \otimes P_a, \ a = 1, \ldots, 7)$ with real $P_j (j = 1, 2, 4)$ and $iP_7, iP_3, iP_6, iP_5$.

$$P_1 = I \otimes \sigma_1 \otimes c, \quad P_2 = \sigma_1 \otimes \sigma_3 \otimes c^*, \quad P_4 = c \otimes \sigma_1 \otimes \sigma_1$$

$$iP_7 = -I \otimes I \otimes \sigma_3, \quad iP_3 = \sigma_3 \otimes \sigma_1 \otimes \sigma_1, \quad iP_6 = -\sigma_1 \otimes I \otimes \sigma_1,$$

$$iP_5 = \sigma_1 \otimes c \otimes c, \quad \Rightarrow \quad \omega^-_7 := P_1P_2P_3P_4P_5P_6P_7 = I \otimes I \otimes I,$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = i\sigma_2, \quad c^* = -c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

($\mathcal{H}_8$) of One Generation

The Coxeter element $\omega_8$ of $\text{Cl}_8$, appearing in (5.2) is given by:

$$\Gamma_8 = \omega_8 = \Gamma_0 \Gamma_1 \ldots \Gamma_7 = \sigma_3 \otimes \omega^-_7 = \sigma_3 \otimes I \otimes I \otimes I \otimes I.$$

---

$^6$This idea was advanced by M. Dubois-Violette and is being explored in work in progress.
Inserting the matrices in the tensor products from right to left, so that
\[
\mathbb{I} \otimes \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad \sigma_3 \otimes \mathbb{I} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \sigma_3 \otimes \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}
\]
we can write the weight matrices \(\Lambda_\mu\) (3.41) as:
\[
\Lambda_0 = \frac{i}{2} \Gamma_{07} = \frac{1}{2} \sigma_3 \otimes \mathbb{I} \otimes \mathbb{I} \otimes \sigma_3 = \frac{1}{2} \text{diag} (\sigma_3, \sigma_3, \sigma_3, -\sigma_3, -\sigma_3, -\sigma_3, -\sigma_3)
\]
\[
\Lambda_1 = \frac{i}{2} \Gamma_{13} = -\frac{1}{2} \mathbb{I} \otimes \sigma_3 \otimes \mathbb{I} \otimes \sigma_3 = -\frac{1}{2} \mathbb{I} \otimes \text{diag} (\sigma_3, \sigma_3, -\sigma_3, -\sigma_3)
\]
\[
\Lambda_2 = \frac{i}{2} \Gamma_{26} = -\frac{1}{2} \mathbb{I} \otimes \mathbb{I} \otimes \sigma_3 \otimes \sigma_3 = -\frac{1}{2} \mathbb{I} \otimes \text{diag} (\sigma_3, -\sigma_3, -\sigma_3, -\sigma_3)
\]
\[
\Lambda_4 = \frac{i}{2} \Gamma_{45} = -\frac{1}{2} \mathbb{I} \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 = -\frac{1}{2} \mathbb{I} \otimes \text{diag} (\sigma_3, -\sigma_3, -\sigma_3, \sigma_3).
\]
They form a (commuting) orthonormal basis in the weight space:
\[
\Lambda_\mu^2 = \frac{1}{4} \mathbb{I}_{16}, \quad [\Lambda_\mu, \Lambda_\nu] = 0, \quad <\Lambda_\mu, \Lambda_\nu> := \frac{1}{4} tr (\Lambda_\mu \Lambda_\nu) = \delta_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 4.
\]
The Lie algebra \(\mathfrak{B}_4 (= \mathfrak{so}(9))\) of \(\text{Spin}(9)\) is spanned by the commutators
\[
\Gamma_{ab} = \frac{1}{2} [\Gamma_a, \Gamma_b], \quad a, b = 0, 1, \ldots, 8.
\]
The physical meaning of \(\mathfrak{B}_4\) is best revealed by identifying its maximal (rank 4) subalgebra
\[
\mathfrak{so}(2) \oplus \mathfrak{su}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(6) \subset \mathfrak{so}(9), \quad (5.6)
\]
that is part of the Pati–Salam grand unified Lie algebra
\[
\mathfrak{g}_{PS} := \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{su}(4). \quad (5.7)
\]
(See, for a nice review [6].) The SU(4) of Pati–Salam is designed to unify the quark colour group SU(3)\(_c\) with the lepton number. The colour Lie algebra \(\mathfrak{su}(3)_c\) is identified with the commutant in \(\mathfrak{su}(4)\) of the hypercharge \(Y(4)\). In the defining 4-dimensional representation \(4\) of \(\mathfrak{su}(4)\) it is given by the traceless diagonal matrix
\[
Y(4) = \begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \quad (5.8)
\]
With our splitting of the octonions \(\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3\) and the corresponding embedding of \(\text{SU}(3)_c\) into \(G_2\) [see, (3.11)] the subalgebra \(\mathfrak{so}(6)\) is spanned by \(\Gamma_{jk}\) with \(1 \leq j < k \leq 6\). The Lie subalgebra \(\mathfrak{su}(3)_c\) appears as the intersection of \(\mathfrak{g}_{PS}\) and \(\mathfrak{so}(6)\) in \(\mathfrak{so}(7)\). As a consequence of (3.43) it is spanned by the following combinations of \(\Gamma_{ab}\):
\[
\mathfrak{su}(3)_c = \text{Span}\{ \Gamma_{13} - \Gamma_{26} (= -iH_1), \quad \Gamma_{26} - \Gamma_{45} (= -iH_2), \quad \Gamma_{12} + \Gamma_{36}, \quad \Gamma_{24} + \Gamma_{45}, \quad \Gamma_{14} + \Gamma_{35} \}
\]
\[
\Gamma_{16} + \Gamma_{23}, \quad \Gamma_{25} + \Gamma_{46}, \quad \Gamma_{15} + \Gamma_{43}. \quad (5.9)
\]
The first two span the Cartan subalgebra of \(\mathfrak{su}(3)_c\)—cf. (4.28). The next three span the maximal compact subalgebra \(\mathfrak{so}(3)\) of the real form \(\mathfrak{sl}(3, \mathbb{R}) \subset\)
so(4, 4). The generator \( Y \) of the commutant of \( su(3) \) in (the complexification of) \( su(4) \) is given by

\[
Y = \frac{2}{3} (\Lambda_1 + \Lambda_2 + \Lambda_4) = \frac{i}{3} (\Gamma_{13} + \Gamma_{26} + \Gamma_{45}) \quad (\in so(3, 3)),
\]

\[
3Y = \mathbb{I} \otimes \left( \begin{array}{ccc}
-3\sigma_3 & 0 & 0 \\
0 & \sigma_3 & 0 \\
0 & 0 & \sigma_3 \\
0 & 0 & \sigma_3
\end{array} \right) = \mathbb{I} \otimes \text{diag}(-3\sigma_3, \sigma_3, \sigma_3, \sigma_3).
\]

It (commutes with and) is orthogonal to the Cartan \( \{H_1, H_2\} \) of \( su(3) \). The algebra \( su(2) \) in the left hand side of (5.6) is then spanned by

\[
I_3 := \Lambda_0 = \frac{i}{2} \Gamma_{07}, \quad I_+ = \Gamma_0 \Gamma_8, \quad I_- = \Gamma_8 \Gamma_0^+-
\]

\[
([I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3).
\]

The spinor representation \( 16 \) of \( Spin(9) \) can now be associated with the doublet representation \( (4^*, 2, 1) \oplus (4, 2, 1) \in g_{PS}' \).

It is natural to identify \( \Lambda_0 \) which commutes with \( su(4) \) and has eigenvalues \( \pm \frac{1}{2} \) with the third component \( I_3 \) of the weak isospin, while \( Y \) that commutes with \( su(3) \), and is orthogonal to \( I_3 \) should coincide with the weak hypercharge (see [50]). The spinor representation consists of two octets of the fermion (lepton-quark) left chiral doublets and antifermion right chiral doublets

\[
\left( \begin{array}{c}
\nu_L \\
e_L
\end{array} \right), \quad Y = -1, \quad \left( \begin{array}{c}
u^+_L \\
e^+_L
\end{array} \right), \quad Y = 1, \quad \left( \begin{array}{c} u_L \\
d_L
\end{array} \right), \quad Y = \frac{1}{3}, \quad \left( \begin{array}{c} u_R \\
d_R
\end{array} \right), \quad Y = -\frac{1}{3}
\]

\[
I_3 = \left( \begin{array}{c}
\frac{1}{2}
\end{array} \right), \quad I_3 = \left( \begin{array}{c}
\frac{1}{2}
\end{array} \right).
\]

These are the fundamental (anti) fermions that participate in the weak interactions. The missing fundamental fermion representations are the 16 singlets with respect to the weak isospin, \( I = 0(= I_3) \).

5.2. The Representation 26 of \( F_4 \)

As the automorphism group \( F_4 \) of \( J = \mathcal{H}_3(\mathbb{O}) \) preserves the unit element and the trace of \( J \) it acts faithfully and irreducibly on the 26-dimensional subspace \( J_0 = s\mathcal{H}_3(\mathbb{O}) \subset J \) of traceless \( 3 \times 3 \) hermitian octonionic matrices. In fact, 26 is the lowest nontrivial (fundamental) representation (of highest weight \( \Lambda_0 \)) of \( F_4 \). Restricted to the maximal subgroup \( Spin(9) \) of \( F_4 \) it splits into three irreducible components:

\[
26 = 16 + 9 + 1.
\]

We have identified in the preceding Sect. 5.1 the spinor representation as four isospin chiral doublets of quarks and leptons and their antiparicles. The 9-vector representation of \( so(9) \) is spanned by the generators \( \Gamma_a \) of \( C\ell_9 \). The
matrices $\Gamma^\epsilon_\mu$ (3.40) and $\Gamma_8$ diagonalize the adjoint action of the physical Cartan elements $Y$ (5.10) and $I_3$ (5.11); we find:

$$[I_3, \Gamma^{\pm}_0] = \pm \Gamma^{\pm}_0, \quad [I_3, \Gamma_8] = 0 = [I_3, \Gamma^j_3], \quad (j = 1, 2, 4)$$

$$[Y, \Gamma^{\pm}_0] = 0 = [Y, \Gamma_8], \quad [Y, \Gamma^j_3] = \pm \frac{2}{3} \Gamma^j_3. \quad (5.15)$$

This allows to consider $\Gamma^{\pm}_0$ as representatives of the $W^\pm$ bosons while $\Gamma_8$ appears as the neutral component, $W^0$, of the isotopic triplet. Then the $Z$ boson and the photon can be associated with appropriate mixtures of $W^0$ and the trivial representation of $\mathfrak{so}(9)$. The isotropic matrices $\Gamma^{\pm}_j$, on the other hand, carry the quantum numbers $(I_3 = 0, Y = \pm \frac{2}{3})$ of the right handed $d$-quarks $d_R$ and the left handed $d$-antiquarks, $\bar{d}_L$, respectively ($j = 1, 2, 4$ being the colour index). Let us note that all eigenvalues of the pair $(I_3, Y)$ appearing in the representation 26 of $F_4$ satisfy the SM constraint

$$I_3 + \frac{3}{2} Y \in \mathbb{Z}.$$  

We recall that according to the identification of the $2^5$-dimensional fermion Fock space with the exterior algebra $\Lambda \mathbb{C}^5$ (see, [6]) all fundamental fermions (of the first generation) can be expressed as exterior products of $d_R, e^+_R$, and $\bar{\nu}_R$ (with $\bar{\nu}_L$ identified with the Fock vacuum). In particular, all fundamental fermions can be written as wedge products of $d_R, \bar{d}_L$ and the chiral spinors in (5.13):

$$e^+_L(I_3 = 0, Y = 2) = e^+_R \wedge \bar{\nu}_R$$

$$u_R(I_3 = 0, Y = \frac{4}{3}) = e^+_L \wedge d_R$$

$$u^j_L(I_3 = 0, Y = -\frac{4}{3}) = \epsilon^{jkl} d^k_R \wedge d^l_R$$

$$e^-_R(I_3 = 0, Y = -2) = \sum_{j=1,2,4} \bar{u}^j_L \wedge d^j_R$$

$$\nu_R(I_3 = 0 = Y) = \sum_j \bar{d}^j_L \wedge d^j_R, \quad (j, k, l = 1, 2, 4). \quad (5.16)$$

(Here $\epsilon^{ijk}$ is the fully antisymmetric unit tensor with $\epsilon^{124} = 1$.) The isosinglets $\bar{\nu}_L, d_R, \bar{d}_L$ and the wedge products (5.16) together with the isotopic doublets (5.13) exhaust the 32 first generation fundamental fermions of the SM. We assume that right handed fermions have negative chirality so that wedge product with $d_R$ or $\bar{\nu}_R$ changes the sign of chirality (i.e., transforms right to left and vice versa). If we associate the Higgs boson with the trace (or the unit element) of the exceptional Jordan algebra $\mathcal{J}$ we see that all fundamental particles of the first family of the SM are generated either directly or as exterior products of elements of $\mathcal{J}$.

The same assignment of quantum numbers to fundamental particles is obtained, as it should, if we consider the restriction of $F_4$ to the other admissible maximal subgroup $\text{F}_4^\omega$ (4.36). On the way of demonstrating this we shall express the weak isospin $I^{(1)}_k, k = 1, 2, 3$ and the hypercharge $Y^{(1)}$ of
the first generation in terms of the Cartan elements of the extended electroweak symmetry group SU(3)e. Here we shall use the commutativity of the isospin $I_k^{(1)}$ with the projector $E_1$. In the defining representation 3 of SU(3)e (implemented by $A$ in (4.45)); we set

$$2I_k^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad k = 1, 2, 3,$$

$$3Y^{(1)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These operators act on the part $X(\xi, a)$ (4.38) of $J$ by commutation and on the matrix $Z$ (4.39) by right multiplication with $(-I_3, -Y)$. Their eigenvectors are given by

$$\hat{a}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{a}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{a}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z_1 = (z_1^j, 0, 0), \quad Z_2 = (0, z_2^j, 0), \quad Z_3 = (0, 0, z_3^j)$$

and their conjugates and by the real diagonal matrices. We have, in particular,

$$[I_3, \hat{a}_1] = \hat{a}_1, \quad [I_3, \hat{a}_2] = -\frac{1}{2} \hat{a}_2, \quad [I_3, \hat{a}_3^*] = \frac{1}{2} \hat{a}_3^*,$$

$$[Y, \hat{a}_1] = 0, \quad [Y, \hat{a}_2] = -\hat{a}_2, \quad [Y, \hat{a}_3^*] = -\hat{a}_3^*;$$

$$Z_1(-I_3) = 0, \quad Z_2(-I_3) = -\frac{1}{2} Z_2, \quad Z_3(-I_3) = \frac{1}{2} Z_3,$$

$$Z_1(-Y) = -\frac{2}{3} Z_1, \quad Z_{2,3}(-Y) = \frac{1}{3} Z_{2,3},$$

while their conjugates have the same eigenvalues with opposite sign and the diagonal matrices $X(\xi, 0)$ correspond to $I_3 = Y = 0$ ($I = 0, 1$). Comparing with the quantum numbers of (5.13) we end up with the following correspondence

$$\hat{a}_1 \rightarrow W^+, \quad \hat{a}_2 \rightarrow e_L^-, \quad \hat{a}_3^* \rightarrow \nu_L, \quad Z_1 \rightarrow d_R, \quad Z_2 \rightarrow d_L, \quad Z_3 \rightarrow u_L$$

$$\hat{a}_1^* \rightarrow W^-, \quad \hat{a}_2^* \rightarrow e_R^+, \quad \hat{a}_3 \rightarrow \bar{\nu}_R, \quad Z_1^* \rightarrow d_L, \quad Z_2^* \rightarrow \bar{d}_R, \quad Z_3^* \rightarrow \bar{u}_R.$$ (5.21)

Thus the basic representation 26 of $F_4$ splits when restricted to SU(3)e × SU(3)c into the following irreducible components:

$$26 = \bar{3} \otimes 3 + \bar{3} \otimes \bar{3} + 8 \otimes 1$$

(5.22)

where the adjoint representation 8 of the extended electroweak SU(3)e consists of two doublets (of leptons and antileptons) and of the three massive gauge bosons and the photon.

$^7$The element $X(\xi, 0) = \xi_1 A_3$ carries total isospin $I = 1$ while $\xi_2 A_8$ corresponds to $I = 0$, both having $I_3 = 0$. 
Remark. Had we started, say, with the algebra $J_3$ orthogonal to the projector $E_3$ we would have had, instead of (5.17), to set:

$$2I_k^{(3)} = \begin{pmatrix} \sigma_k & 0 \\ 0 & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad 3Y^{(3)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In this case we would have ended up with the \textit{third generation} assignment:

$$\hat{a}_1 \rightarrow \tau_L^-, \quad \hat{a}_2 \rightarrow \nu_{\tau L}, \quad Z_1 \rightarrow b_L, \quad Z_2 \rightarrow t_L, \quad Z_3 \rightarrow b_R, \quad \hat{a}_3 \rightarrow W^+_3.$$  \hfill (5.23)

(Here $\nu_{\tau L}$ stands for the left chiral tau neutrino and $t, b$ for the top and bottom quark, respectively.) In general, for the family $i$ we would have (with indices counted mod 3):

$$[I_3^{(i)}, \hat{a}_i] = \hat{a}_i, \quad [I_3^{(i)}, \hat{a}_{i+1}] = -\frac{1}{2}\hat{a}_{i+1}, \quad [I_3^{(i)}, \hat{a}_{i+2}] = \frac{1}{2}\hat{a}_{i+2} \quad (5.24)$$

(5.19)) Remarkably, the action of the colour group $SU(3)_c$ does not depend on the choice of generation.

5.3. The Symmetry Group of The Standard Model

It has been observed by Baez and Huerta [6] that the gauge group of the SM,

$$G_{ST} = SU(2) \times U(1) \cong SU(3)_c \times SU(3) \times U(1) \cong Spin(9) \cong SU(3)_c \times SU(3) \times SU(3)/Z_3 \cong Spin(10).$$

(5.25)

can be obtained as the intersection of the Georgi-Glashow and Pati–Salam \textit{grand unified theory groups} $SU(5)$ and $(SU(4) \times SU(2) \times SU(2))/Z_2$ viewed as subgroups of Spin(10).

Here we elaborate on the suggestion of [47] that one can deduce the symmetry of the SM by applying the Borel–de Siebenthal theory to admissible maximal rank subgroups of $F_4$—those that contain the colour symmetry $SU(3)_c$.

Our analysis in this chapter demonstrates that the intersection of the two admissible maximal subgroups $Spin(9)$ and $SU(3) \times SU(3)$ of $F_4$ is precisely the symmetry group $G_{ST}$ (5.25) of the SM. The constraints on the $U(1)_Y$ term coming from factoring the 6-element central subgroup $Z_6$ imply

$$I_3 + \frac{3}{2}Y (= Q + Y) \in \mathbb{Z},$$

(5.26)

(where $Q = I_3 + \frac{1}{2}Y$ is the electric charge); furthermore, if the central element of $SU(3)_c$ is represented by a nontrivial phase factor $\omega$, i.e., if

$$\omega^2 + \omega + 1 = 0 \quad \text{then} \quad U_Y^2 + U_Y + 1 = 0 \quad \text{for} \quad U_Y = e^{2\pi i Y}. \quad (5.27)$$

The “colourless” Cartan elements $I_3$, $Y$ complemented with the total isospin $I_1$, $(I_1^2 + I_2^2 + I_3^2 = I(I + 1))$ and the chirality $\gamma$ (which has eigenvalue 1 for left and $-1$ for right chiral fermions) completely characterize all 32 first generation fermions as well as the four electroweak gauge bosons. The colour index $j$ can be related to the eigenvalues of the $SU(3)_c$ Cartan matrices, but
different “colours” are physically indistinguishable. We only need the chirality $\gamma$ to separate $\nu_R(\gamma = -1)$ from $\bar{\nu}_L(\gamma = 1)$.

6. Concluding Remarks

The idea that exceptional structures in mathematics should characterize the fundamental constituents of matter has been with us since the Ancient Greeks first contemplated the Platonic solids\textsuperscript{8}. The octonions, the elements of the ultimate division algebra, have been linked to the Standard Model of particle physics starting with the paper of Günaydin and Gürsey [28] which related them to the coloured quarks. Vigorous attempts to implement them in superstring theory [16, 27] remained inconclusive. When the idea of a finite quantum geometry emerged [14, 22] (see also the recent contributions [9, 13, 34] and references therein) it became natural to look for a role of special algebraic structures in such a context. The application of the exceptional Jordan algebra $\mathcal{J}$ to the SM, put forward in Chapter 5, a continuation of [47], was triggered by the paper [21] of Michel Dubois-Violette (see also [10] where differential calculus and the theory of connections on Jordan algebras and Jordan modules is developed).

We find it remarkable that our version of the Borel–de Siebenthal theory [8] applied to the automorphism group $F_4$ of $\mathcal{J}$, yields unambiguously the gauge group $G_{ST}$ of the SM and that the relevant irreducible representation 26 of $F_4$ combines in a single multiplet all quarks’ and leptons’ doublets and the d-quarks’ isotopic singlets of a given generation with the gauge bosons of the electroweak interactions.

In fact, the exceptional Jordan algebra is intimately related to all exceptional Lie groups (Sect. 4.2—see also, [7, 36]). It will be interesting to reveal the role of the structure group $E_{6(-26)}$ and the conformal group $E_{7(-25)}$ of $\mathcal{J}$ in the physics of the SM. We intend to return to this problem in future work.

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\textsuperscript{8}Theaetetus, a contemporary of Plato, gave the first mathematical description of the five regular solids. Plato in the dialog Timaeus (c. 360 B.C.) related the four classical elements, earth, air, water and fire to the four regular solids, cube, octahedron, icosahedron and tetrahedron, respectively, while, according to him “God used the dodecahedron for arranging the constellations on the whole heaven”. In the Twentieth Century special structures became part of the excitement with the A–D–E classification that includes the exceptional Lie groups—see [38] and references therein.
Appendix A. The Fano Plane of Imaginary Octonions ([3])

The multiplication rules for the seven imaginary quaternionic units can be summarized by

\[ e_a e_b = -\delta_{ab} + f_{abc} e_c \]  \hspace{1cm} (A.1)

where \( f_{abc} \) are fully antisymmetric and

\[ f_{124} = f_{235} = f_{346} = f_{457} = f_{561} = f_{672} = f_{713} = 1. \]  \hspace{1cm} (A.2)

The relation (A.2) obey the rules

\[ f_{ijk} = 1 \Rightarrow f_{i+1j+1k+1} = 1 = f_{2i2j2k} \]  \hspace{1cm} (A.3)

where indices are counted \( \mod 7 \). Equation (A.2) can be recovered from anyone of them and the first relation Eq.(A.3).

We have displayed on Fig. 2. the points \( e_i \) as non-zero triples of homogeneous coordinates taking values 0 and 1 such that the product \( e_i e_j \) (in clockwise order) is obtained by adding the coordinates \( (a, b, c) \), \( a, b, c \in \{0, 1\} \), modulo two.

Appendix B. Two bases of \( so(8) \) related by the outer automorphism \( \pi \).

The generators \( G_{ab} \) of \( so(8) \) are given directly by their action on the octonion units (2.17):

\[ G_{ab} e_b = e_a, \quad G_{ab} e_a = -e_b, \quad G_{ab} e_c = 0 \text{ for } a \neq c \neq b. \]  \hspace{1cm} (B.1)

\[ e_1 = (0, 0, 1), e_2 = (0, 1, 0) \Rightarrow e_1 e_2 = e_4 = (0, 1, 1) \]
[diagram]

\[ e_3 = (1, 0, 0) \Rightarrow e_2 e_3 = e_5 = (1, 1, 0) \]
\[ e_1 e_5 = e_6 = (1, 1, 1) \]
\[ e_4 e_5 = e_7 = (1, 0, 1). \]

Figure 2. Projective plane in \( \mathbb{Z}_2^3 \) with seven points and seven lines.
The action of $F_{ab}$ can also be deduced from definition (2.17) and multiplication rules:

$$F_{ab}e_b = \frac{1}{2}e_a, \quad F_{ab}e_a = -\frac{1}{2}e_b, \quad (a \neq b) \quad F_{ab} = -F_{ba}.$$ 

$$F_{j0}e_{2j} = \frac{1}{2}e_{4j} \mod 7 (= -F_{0j}e_{2j}) \text{ for } j = 1, 2, 4,$$

$$F_{j0}e_{3j} = \frac{1}{2}e_7, \quad F_{70}e_7 = \frac{1}{2}e_{3j}, \quad F_{07} = \frac{1}{2}e_j,$$

$$F_{0j}e_{6j} = \frac{1}{2}e_{5j}, \quad F_{j0}e_{5j} = \frac{1}{2}e_{6j},$$

$$[F_{j0}, F_{0k}] = F_{jk}. \quad (B.2)$$

(All indices are counted mod 7.) From (B.1) and (B.2) we find

$$2F_{0j} = G_{0j} + G_{2j4j} + G_{3j7} + G_{5j6j}$$

$$2F_{03j} = G_{03j} - G_{j17} - G_{2j5j} + G_{4j6j}, \quad j = 1, 2, 4$$

$$2F_{07} = G_{07} + G_{13} + G_{26} + G_{45}. \quad (B.3)$$

In particular, taking the skew symmetry of $G_{ab}$ and the counting mod 7 into account we can write

$$2F_{02} = G_{02} - G_{14} + G_{35} - G_{76},$$

$$2F_{04} = G_{04} + G_{12} - G_{36} - G_{75},$$

$$2F_{03} = G_{03} - G_{17} - G_{25} + G_{46},$$

$$2F_{06} = G_{06} + G_{15} - G_{27} - G_{43},$$

$$2F_{05} = G_{05} - G_{16} + G_{23} - G_{47}. \quad (B.4)$$

Note that with the $abc$ are ordering $(1, 2, 4, 3, 7, 5, 6)$ The first (positive) indices of $G (2, 3, 5; 1, 3, 7; 1, 2, 4)$ correspond to quaternionic triples: $e_2 e_3 = e_5$, $e_1 e_3 = e_7$, $e_1 e_2 = e_4$. Setting

$$G_1 = \begin{pmatrix} G_{01} \\ G_{24} \\ G_{37} \\ G_{56} \end{pmatrix}, \quad G_2 = \begin{pmatrix} G_{02} \\ G_{14} \\ G_{35} \\ G_{76} \end{pmatrix}, \quad G_4 = \begin{pmatrix} G_{04} \\ G_{12} \\ G_{36} \\ G_{75} \end{pmatrix}, \quad G_3 = \begin{pmatrix} G_{03} \\ G_{17} \\ G_{25} \\ G_{46} \end{pmatrix},$$

$$G_7 = \begin{pmatrix} G_{07} \\ G_{13} \\ G_{26} \\ G_{45} \end{pmatrix}, \quad G_5 = \begin{pmatrix} G_{05} \\ G_{16} \\ G_{23} \\ G_{47} \end{pmatrix}, \quad G_6 = \begin{pmatrix} G_{06} \\ G_{15} \\ G_{27} \\ G_{43} \end{pmatrix},$$

and similarly for $F_1, \ldots F_6$ we find
\[ F_a = X_a G_a, \quad a = 1, 2, 4, 3, 7, 5, 6, \text{with} \]

\[ X_1 = X_7 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad X_2 = X_5 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \]

\[ X_4 = X_6 = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \quad X_3 = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

They all define involutive transformations:

\[ X_k^2 = I, \quad \det X_k = -1, \quad k = 1, 2, 3, 4. \quad (B.5) \]

and satisfy

\[ X_1 X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -X_3 X_4, \quad X_1 X_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -X_2 X_4 \]

\[ X_1 X_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -X_2 X_3 \quad \Rightarrow \quad X_1 X_2 X_3 X_4 = -I. \]

\[ (X_i X_j = X_j X_i), \quad X_1 X_2 + X_3 X_4 = 0, \quad X_1 X_3 + X_1 X_4 = 0. \] \quad (B.6)

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