A new first order formalism for $\kappa$-supersymmetric Born Infeld actions: the $D3$-brane example†

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Abstract

We introduce a new first order formulation of world-volume actions for $p$-branes with $\kappa$-supersymmetry. In this language, which involves more auxiliary fields compensated by more local symmetries, the action is provided by a very compact, simple and elegant formula applicable to any supergravity background. The $\kappa$-supersymmetry variation against which it is invariant is obtained from the bulk supersymmetries by means of a projector that has a simple expression in terms of the auxiliary fields. The distinctive feature of our formalism is that all fermion fields are hidden into the definition of the curvatures and the action is formally the same, in terms of these differential forms as it would be in a purely bosonic theory. Typically our new formulation enables one to discuss the correct boundary actions for non trivial supergravity $Dp$–brane bulk solutions like the $D3$–brane solution on smooth ALE manifolds with flux recently constructed in the literature.

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1 Introduction

The $p$–branes and in particular $Dp$–branes are essential items in our new non perturbative understanding of string theory after the 2nd string revolution. There are three complementary aspects and as many complementary approaches to the study of these solitonic excitations of the string spectrum. Indeed they can be alternatively viewed as:

a classical solutions of the low energy supergravity field equations in the bulk,

b world–volume gauge theories described by suitable world–volume actions characterized by $\kappa$–supersymmetry

c boundary states in the superconformal field theory description (SCFT) of superstring vacua.

The three descriptions are closely intertwined. In particular the relation between [a] and [b] is the following. On one side the world–volume actions provide the source terms that specify the boundary conditions utilized in solving the supergravity field equations. On the other hand, for small fluctuations of the fields around a given classical solution, the world volume action provides the tool to explore the gauge/gravity correspondence, namely the pairing between certain composite operators in the world–volume gauge theory and certain corresponding bulk modes. Alternatively when we adopt the more abstract language of superconformal field theories, classical string backgrounds are identified with a specific SCFT and the brane is identified with a suitable boundary state for the same. The world–volume action encodes the interactions within the chosen boundary conformal field theory.

From this discussion it is evident that a full command on the world–volume actions of $p$–branes is a mostly essential weapon in the arsenal of the modern string theorist. As mentioned above the distinctive feature of these actions, which guides their construction, is the so called $\kappa$–supersymmetry [1]. This corresponds to the fermionic local symmetry that allows to halve the number of Fermi fields, originally equal to the number of $\theta$–coordinates for the relevant superspace, and obtain, on–shell, an equal number of bosonic and fermionic degrees of freedom as required by the general brane–scan [2]. As it is well understood in the literature since many years [3, 4, 5, 6] the $\kappa$–supersymmetries are nothing else but suitable chiral projections of the original supersymmetry transformation rules defined by supergravity. This was made particularly evident and handy by the construction of world–volume actions within the framework of the rheonomy approach to supergravity [3, 4, 5, 6]. In this geometric approach all Fermi fields are implicitly hidden in the definition of the geometric $p$–form potentials of supergravity and formally the action is the same as it would be in a purely bosonic theory. Yet it is supersymmetric and this supersymmetry, which fixes the relative coefficients of the kinetic terms with respect to the Wess–Zumino terms, can be shown through a simple calculation starting from the rheonomic parametrizations of the supergravity curvatures. In order to apply such a powerful method, the world–volume action must be presented in first–order rather than in second order formalism, namely à la Polyakov [4] rather than à la Nambu–Goto [4]. As a consequence the rheonomic method was successfully applied to those instances of $p$–brane actions where the Polyakov formulation (further generalized with the introduction of an additional auxiliary field representing the derivative of scalar fields) did exist: in particular the string or 1–brane [4], the M2–brane [4] and the particle or 0–brane [7]. More general $Dp$–branes were so far out of reach because of the following reason: their second order action is of the Born–Infeld type and a suitable first order formalism for the Born–Infeld lagrangian was not known.
In this paper we just fill this gap by constructing a **new first order formalism** that is able of generating second order actions of the Born–Infeld type. The new formulation which, in our opinion, is particularly compact and elegant is based on the introduction of an additional auxiliary field, besides the world volume vielbein, and on the enlargement of the local symmetry from the Lorentz group to the general linear group:

$$\text{SO}(1, d - 1) \xrightarrow{\text{enlarged}} \text{GL}(d, \mathbb{R})$$  \hspace{1cm} (1.1)

Within this new formalism we can easily apply the rheonomic method and as an example we construct the \(\kappa\)-supersymmetric action of a \(D3\)-brane. This choice is not random rather it is rooted in the main motivations to undertake this new construction. Indeed we need a suitable formulation of the \(D3\)-brane action holding true on a generic background in order to apply it to the newly constructed \(D3\)-brane bulk solutions of supergravity like that on smooth ALE manifolds [11] or the more recent and complicated ones of [12],[13].

Our paper is organized as follows. In section 2 we review the rheonomic formulation of \(\kappa\)-supersymmetry based on an essential use of the old 1st–order formalism. In section 3 we introduce the new first order formalism and we show how in this framework we can recover the Born–Infeld action by eliminating the auxiliary fields through their own equation of motion. In section 4 we apply our new machinery to the case of the \(D3\)-brane and we explicitly show \(\kappa\)-supersymmetry. Section 5 contains an outlook and our conclusions. In the appendices we have placed some important although more technical material. Particularly relevant is appendix A which is a short but comprehensive summary of type IIB supergravity in the rheonomic approach. It is entirely based on the original papers by Castellani and Pesando [26, 27] but it contains also some new useful results, in particular the transcription of curvatures from the complex SU(1,1) basis to the real SL(2, \(\mathbb{R}\)) basis and the comparison of the supergravity field equations as written in the rheonomy approach and as written in string text–books like Polchinsky’s [16].

2 From 2nd to 1st order and the rheonomy setup for \(\kappa\) supersymmetry

In this section we summarize the 1st order formulation of world–volume actions and we recall their essential role in setting up a simple, compact, rheonomic approach to \(\kappa\)-supersymmetry. Then we point out the problem arising with \(Dp\)-branes, related to the presence of the gauge–field \(A_\mu\). In this way we establish the need for the new first order formalism which is explained in the next section.

2.1 Nambu–Goto, Born–Infeld and Polyakov kinetic actions for \(p\)-branes

The 2nd order Nambu-Goto action of a bosonic string [10] that moves through a \(D\)-dimensional space–time endowed with a metric \(g_{\mu\nu}\), is simply given by the area of the world–sheet swept by the string. Namely we have:

$$A_{\text{string}}^{\text{Nambu-Goto}} = \int d^2 \xi \sqrt{-\det G_{\mu\nu}}$$ \hspace{1cm} (2.1)

where:

$$G_{\mu\nu} \equiv \partial_\mu X^\mu \partial_\nu X^\nu g_{\mu\nu}$$ \hspace{1cm} (2.2)
denotes the pull-back of the bulk metric $g_{\mu\nu}(X)$ onto the world-sheet. Such an action admits a straightforward generalization to the case of a $p$-brane, the area of the world-sheet being replaced by the value of the $d = p + 1$-dimensional world-volume:

$$A_{p-\text{brane}}^{\text{Nambu Goto}} = \int d^d\xi \sqrt{-\det G_{\mu\nu}} \quad (2.3)$$

As it is well known from the literature and thoroughly discussed in recent string theory textbooks \cite{16}, the kinetic part of $Dp$-brane actions is provided by a further generalization of the Nambu–Goto action (2.3) where the symmetric matrix $G_{\mu\nu}$ is modified by the addition of an antisymmetric part $F_{\mu\nu}$ that represents the field strength of a world volume gauge field $A_\mu$:

for $D$-branes $G_{\mu\nu} \mapsto G_{\mu\nu} + F_{\mu\nu}$

where $F_{\mu\nu} = \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu)$ \quad (2.4)

Seen from a different perspective the resulting second order action:

$$A_{D-\text{-brane}}^{\text{kinetic}} = \int d^d\xi \sqrt{-\det (G_{\mu\nu} + F_{\mu\nu})} \quad (2.5)$$

is a generalization of the Born-Infeld \cite{17} action of non linear electromagnetism. Indeed the latter was early shown to be the effective action for the zero mode gauge field of an open string theory \cite{18}.

In the context of superstrings and in the analysis of $Dp$–brane systems the important issue is to write world–volume actions that possess both reparametrization invariance and $\kappa$-supersymmetry \cite{1}. The former is needed to remove the unphysical degrees of freedom of the bosonic sector, while the latter removes the unphysical fermions. In this way we end up with an equal number of physical bosons and physical fermions as it is required by supersymmetry. As widely discussed in the literature \cite{3, 5, 6, 8} the appropriate $\kappa$-supersymmetry transformation rules are nothing else but the supersymmetry transformation rules of the bulk supergravity background fields with a special supersymmetry parameter $\epsilon$ that is projected onto the brane. For those $\kappa$-supersymmetric branes where the gauge field strength $F_{\mu\nu}$ is not required (for example the string itself or the M2–brane) such a projection is realized by imposing that the spinor $\epsilon$ satisfies the following condition:

$$\epsilon = \frac{1}{2} \left(1 + (i)^{d+1} \frac{1}{d!} \Gamma_{a_1\ldots a_d} V^a_{i_1} \ldots V^a_{i_d} \epsilon^{i_1\ldots i_d}\right) \epsilon \quad (2.6)$$

where $\Gamma_a$ are the gamma matrices in $D$–dimensions and $V^a_\mu$ are the component of the bulk vielbein $V_\mu$ onto a basis of world-volume vielbein $e^m$. Explicitly we write

$$V^a_\mu e^m = \varphi^* \left[V^a_\mu\right] \quad (2.7)$$

where $\varphi^* \left[V^a_\mu\right]$ denotes the pull–back of the bulk vielbein on the world volume,

$$\varphi : \mathcal{W}_d \rightarrow \mathcal{M}_D \quad (2.8)$$

being the injection map of the latter into the former. For all other branes with a full–fledged Born Infeld type of action the projection (2.6) becomes more complicated and involves also $F_{\mu\nu}$.

Certainly one can address the problem of $\kappa$–supersymmetrizing the 2nd–order action (2.5) and this programme was carried through in the literature to some extent \cite{19, 22}. Yet due to the highly non linear structure of such a bosonic action its supersymmetrization turns out to be quite involved.
Furthermore the geometric structure is not transparent and any modification is very difficult and obscure in such an approach.

On the contrary it was shown in [3] and recently illustrated with the case of the M2–brane in [3] and with the case of the 0–brane in four–dimensions in [3] that by using a first order formalism on the world volume the implementation of \( \kappa \)–supersymmetry is reduced to an almost trivial matter once the rheonomic parametrizations, consistent with superspace Bianchi identities, are given for all the the curvatures of the bulk background fields. It follows that an appropriate first order formulation of the Born–Infeld action (2.5) is an essential step for an easy and elegant approach to \( \kappa \)–supersymmetric \( Dp \)–brane world volume actions that are also sufficiently versatile to adapt to all type of bulk backgrounds.

The first order formulation of the Nambu–Goto action (2.3) is the Polyakov action for \( p \)–branes:

\[
\mathcal{L}^\text{Polyakov}_{p–brane} = \frac{1}{2(d-1)} \int d^d\xi \sqrt{-\det h_{\mu\nu}} \left\{ h^{\rho\sigma} \partial_\rho X_\mu \partial_\sigma X_\nu g_{\mu\nu} + (d-2) \right\} \tag{2.9}
\]

where the auxiliary field \( h_{\rho\sigma} \) denotes the world–volume metric. Varying the action (2.9) with respect to \( \delta h_{\rho\sigma} \) we obtain the equation:

\[
h_{\rho\sigma} = G_{\rho\sigma} \tag{2.10}
\]

and substituting (2.10) back into (2.9) we retrieve the second order action (2.3).

The Polyakov action (2.9) is not yet in a suitable form for a simple rheonomic implementation of \( \kappa \)–supersymmetry but can be easily converted to such a form. The required steps are:

1. replacing the world–volume metric \( h_{\mu\nu}(\xi) \) with a world–volume vielbein \( e^i = e^i_\rho d\xi^\rho \),
2. using a first order formalism also for the derivatives of target space coordinates \( X_\mu \) with respect to the world volume coordinates \( \xi^\rho \),
3. write everything only in terms of flat components both on the world volume and in the target space.

This programme is achieved by introducing an auxiliary 0–form field \( \Pi^a_i(\xi) \) with an index \( a \) running in the vector representation of \( SO(1,D-1) \) and a second index \( i \) running in the vector representation of \( SO(1,d-1) \) and writing the action [4]:

\[
\mathcal{A}^{kin}_{d} = \int_{\mathcal{W}_d} \left[ \Pi^a_i V^b_j \eta_i \eta_j e^i e^j \wedge \ldots \wedge e^i e^j \epsilon_{i_1 \ldots i_d} \right.
- \frac{1}{2d} \left( \Pi^a_i \Pi^b_j \eta_i \eta_j + d - 2 \right) e^i \wedge \ldots \wedge e^i e^j \epsilon_{i_1 \ldots i_d} \left. \right] \tag{2.11}
\]

The variation of (2.11) with respect to \( \delta \Pi^a_j \) yields an equation that admits the unique algebraic solution:

\[
V^a_i |_{\mathcal{W}_d} = \Pi^a_i e^i \tag{2.12}
\]

Hence the 0-form \( \Pi^a_i \) is identified with the intrinsic components along the world–volume vielbein \( e^i \) of the the bulk vielbein \( V^a_i \) pulled-back onto the world volume. In other words the field \( \Pi^a_i \) is identified by its own field equation with the field \( V^a_i \) defined in eq. (2.7). On the other hand with

\footnote{The need of a cosmological term for p-brane actions with \( p \neq 1 \) was first noted by Tucker and Howe in [4].}
the chosen numerical coefficients the variation of (2.11) with respect to the world–volume vielbein

$$\delta e^i$$

yields another equation with the unique algebraic solution:

$$\Pi^a_i \Pi^b_j \eta_{ab} = \eta_{ij}$$

(2.13)

which is the flat index transcription of eq. (2.10) identifying the world–volume metric with the

pull-back of the bulk metric. Hence eliminating all the auxiliary fields via their own equation of motion the first order action (2.11) becomes proportional to the 2nd order Nambu–Goto action (2.3). The first order form (2.11) of the kinetic action is the best suited one to discuss \( \kappa \)-supersymmetry. To illustrate this point we briefly consider the case of the M2–brane

2.2 \( \kappa \)-supersymmetry and the example of the M2–brane

In the case of the M2–brane in eleven dimensions the world–volume is three–dimensional and the complete action is simply given by the kinetic action (2.11) with \( d = 3 \) plus the Wess-Zumino term, namely the integral of the 3–form gauge field \( A^3 \). Explicitly we have \( W_3 \):

$$A_{M2} = A^{kin}[d = 3] - q \int_{W_3} A^3$$

(2.14)

where \( q = \pm 1 \) is the charge of the M2–brane. As explained in [3], the background fields, namely the bulk elfbein \( V^a \) an the bulk three–form \( A^3 \) are superspace differential forms which are assumed to satisfy the Bianchi consistent rheonomic parametrizations of \( D = 11 \) supergravity as originally given in [23, 24]. Hence, although implicitly, the action functional (2.14) depends both on 11 bosonic fields, namely the \( X^a(\xi) \) coordinates of bulk space–time, and on 32 fermionic fields \( \theta^a(\xi) \), forming an 11–dimensional Majorana spinor. A supersymmetry variation of the background fields is determined by the rheonomic parametrization of the curvatures and has the following explicit form:

$$\delta V^a = i\bar{\epsilon} \Gamma^a \Psi,$$

$$\delta \Psi = D\epsilon - \frac{i}{3} \left( \Gamma^a_{[ab]c} F_{abc} - \frac{1}{8} \Gamma^{ab_1...b_4} F_{ab_1...b_4} \right) V^a,$$

$$\delta A^3 = -i\bar{\epsilon} \Gamma^{ab} \Psi \wedge V^a \wedge V^b$$

(2.15)

(2.16)

where \( \Psi \) is the gravitino 1–form, \( F_{a_1...a_4} \) are the intrinsic components of the \( A^3 \) curvature and \( \epsilon \) is a 32–component spinor parameter. Essentially a supersymmetry transformation is a translation of the fermionic coordinates \( \theta \rightarrow \theta + \epsilon \), namely at lowest order in \( \theta \) it is just such a translation. With such an information the \( \kappa \)-supersymmetry invariance of the action (2.14) can be established through a two–line computation, using the so called 1.5–order formalism. Technically this consists of the following. In the action (2.14) we vary only the background fields \( V^a, A^3 \) with respect to the supersymmetry transformations (2.16) and, after variation, we use the first order field equations (2.12), (2.13). The action is supersymmetric if all the generated terms, proportional to the gravitino 1–form \( \Psi \) cancel against each other. This does not happen for a generic 32–component spinor \( \epsilon \) but it does if the latter is of the form:

$$\epsilon = \frac{1}{2} (1 - q i \bar{\Gamma}) \kappa,$$

$$\bar{\Gamma} \equiv \frac{\epsilon^{ijk}}{3!} \Gamma_{ijk} = \frac{\epsilon^{ijk}}{3!} \Pi^a_1 \Pi^b_2 \Pi^c_3 \Gamma_{abc}$$

(2.17)
where \( \kappa \) is another spinor. Eq. (2.17) corresponds to the anticipated projection (2.6) which halves the spinor components. It follows that of the 32 fermionic degrees of freedom 16 can be gauged away by \( \kappa \)-supersymmetry. The remaining 16 are further reduced to 8 by their field equation which is implicitly determined by the action (2.14). As one sees, once the M2-action is cast into the first order form (2.14), \( \kappa \)-supersymmetry invariance can be implemented in an extremely simple and elegant way that requires only a couple of algebraic manipulations with gamma matrices.

The example of the M2–brane is generalized to all other instances of \( p \)-branes where the world volume spectrum includes just the scalars (=target space coordinates) and their fermionic partners.

2.3 With \( Dp \)-branes we have a problem: the world–volume gauge field \( A^{[1]} \)

It is clear from what we explained above that to deal with \( \kappa \)-supersymmetry in an easy way we need a first order formulation of the action. Yet in the case of \( Dp \)-branes there is a new problem intrinsically related to the new structure of the Born–Infeld action (2.5) which, differently from the pure Nambu–Goto action (2.3) cannot be recast into a first order form of type (2.11). The solution of this problem is found through a procedure which is very frequent and traditional in Physics. Indeed, when a certain formulation of a theory cannot be generalized to a wider scenario including additional fields it usually means that there is a second formulation of the same theory which is equivalent to the former in the absence of the new fields, but which, differently from the former, can incorporate them in a natural way. Typical example is the relation of Cartan’s formulation of General Relativity in terms of vielbein and spin connection with respect to the standard metric formulation. Although they are fully equivalent in the absence of fermions, yet the former allows the coupling to spinors while the latter does not. The present case is similar. It turns out that there is a new, so far unknown, first order formulation of world-volume actions which, in the absence of world–volume gauge fields is fully equivalent to the formulation of eq. (2.11). Yet world–volume 1–forms can be naturally included in the new formalism while they have no place in the old. In full analogy with other examples of the same logical process the new formalism relies on the addition of a new auxiliary field and a new symmetry. The new field is a 0–form rank 2 tensor \( h_{ij} \) that is identified with the intrinsic components of the pulled-back bulk metric along a reference world–volume vielbein \( e^i \). The new symmetry is the independence of the action from the choice of the reference vielbein. Explicitly this means the following. Let \( K^i_j(x) \) be a generic \( d \times d \) matrix depending on the world–volume point. The new action we shall construct will be invariant against the local transformation:

\[
e^i \mapsto K^i_j e^j, \\
h^{ij} \mapsto (K^{-1})^i_{i'} (K^{-1})^j_{j'} h^{i'j'} (\det K)
\]

accompanied by suitable transformation of the other fields. The above symmetry generalizes the local Lorentz invariance of the previously known first order \( p \)-brane actions. Indeed, being generic, the matrix \( K \) can in particular be an element of the Lorentz group \( K \in SO(1,d-1) \). In this case there is no novelty. However \( K \) can also be a representative of a non trivial equivalence class of the coset \( GL(d,\mathbb{R})/SO(1,d-1) \). This latter is precisely parametrized by arbitrary symmetric

\[2\]Actually a partial first order formalism was already introduced in the literature for Dp-branes [20] in the context of the superembedding approach initiated by Kharkov group and extensively developed also in collaborations with the Padua group and other groups [13]. In particular in [21] an action with a partial first order formalism was introduced in the sense that there is an auxiliary \( F_{ij} \) field for the gauge degrees of freedom but the action is “second order” in the brane coordinates \( x \) and \( \theta \), which enter through the pullback of the target space supervielbine \( E^a \).
matrices. Hence the additional degrees of freedom introduced by the new auxiliary field $h^{ij}$ are taken away by the enlargement of the local symmetry from $SO(1,d-1)$ to $GL(d,\mathbb{R})$.

3 The new first order formalism

In the next subsection 3.1 we describe the new formalism as an alternative to the action (2.11). Then in subsection 3.2 we show how it allows the inclusion of world volume gauge fields and provides a first order formulation of the Born–infeld action (2.5).

3.1 An alternative to the Polyakov action for $p$–branes

To begin with we consider a world–volume Lagrangian of the following form:

$$L = \Pi_i^a V^b_m \eta^{i\ell_1} \wedge e^{\ell_2} \wedge \ldots \wedge e^{\ell_d} \epsilon_{\ell_1 \ldots \ell_d} + a_1 \Pi_i^a \Pi_j^b \eta^{i\ell_1} \wedge \ldots \wedge e^{\ell_d} \epsilon_{\ell_1 \ldots \ell_d} + a_2 (\det h)^{-\alpha} e^{\ell_1} \wedge \ldots \wedge e^{\ell_d} \epsilon_{\ell_1 \ldots \ell_d}$$

(3.1)

where $a_1$, $a_2$, $\alpha$ are real parameters to be determined and the other notations are recalled in eq.(C.1) of appendix C.

Performing the $\delta \Pi_i^a$ variation of the Lagrangian (3.1) we obtain:

$$\eta^{i\ell_1} V^b_m \eta^{j\ell_2} \ldots \epsilon_{\ell_1 \ldots \ell_d} + 2 (d!) a_1 \eta^{i\ell_1} \Pi_j^b h^{ij} = 0$$

(3.2)

If we choose:

$$a_1 = -\frac{1}{2d}$$

(3.3)

then equation (3.2) is solved by:

$$\Pi_i^b = V^b_m \eta^{ip} (h^{-1})_{pm}$$

(3.4)

Let us then introduce the following three $d \times d$ matrices:

$$\gamma_{ij} = \Pi_i^a \Pi_j^b \eta^{ab} ; \ G_{ij} = V^a_i V^b_j \eta^{ab} ; \ \tilde{G} = \eta G \eta$$

(3.5)

The solution (3.4) of the field equation (3.2) implies that:

$$\gamma = (h^{-1})^T \eta G \eta h^{-1} = (h^{-1})^T \tilde{G} h^{-1}$$

(3.6)

Next let us consider the variation of the action (3.1) with respect to the symmetric matrix $h^{ij}$. In matrix form such a variational equation reads as follows:

$$a_1 \gamma - a_2 \alpha h^{-1} (\det h)^{-\alpha} = 0$$

(3.7)

Setting:

$$a_2 = \frac{a_1}{\alpha} = -\frac{1}{2d \alpha}$$

(3.8)

eq(3.7) reduces to

$$\gamma = h^{-1} (\det h)^{-\alpha}$$

(3.9)

which can be solved by the ansatz:

$$h = \gamma^{-1} (\det \gamma)^\beta$$

(3.10)
provided:
\[ \beta = \frac{\alpha}{d\alpha + 1} \] (3.11)

On the other hand from eq.(3.6) we get:
\[ \det \gamma = \det (\det h)^{-2} \] (3.12)

so that:
\[ h = h \hat{G}^{-1} h (\det h)^{\beta} (\det h)^{-2\beta} \] (3.13)

Eq.(3.13) can be solved by the ansatz:
\[ h = \hat{G} (\det h)^p \] (3.14)

provided:
\[ p = -\frac{\alpha}{d\alpha - 1} \] (3.15)

Combining the last two results we have the final solution for the two auxiliary fields \( h \) and \( \gamma \):
\[ h = \hat{G} (\det G)^p ; \quad \gamma = \hat{G}^{-1} (\det G)^{-2p} \] (3.16)

in terms of \( G \) which is just the pull-back of the bulk metric onto the world volume, expressed in flat components with respect to an arbitrary reference vielbein \( e^\ell \) that lives on \( \mathcal{W} \).

Using eq.(3.16) we can rewrite the action (3.1) in second order formalism. The basic observation is that after implementation of the first order field equations the three terms appearing in (3.1) become all proportional to the same term, namely \((\det G)^{-p} \det e d^d \xi\), having named \( \xi \) the world volume coordinates. Indeed we have:

\[ \eta_{ab} \Pi^a_i \Pi^b_j h^{ij} e^{\ell_1} \wedge \ldots \wedge e^{\ell_d} e_{\ell_1 \ldots \ell_d} = d! (\det G)^{-p} \det e d^d \xi \]

\[ \Pi^a_i V^b_j \eta_{ab} e^{\ell_1} \wedge \ldots \wedge e^{\ell_d} e_{\ell_1 \ldots \ell_d} = d! (\det G)^{-p} \det e d^d \xi \] (3.17)

Hence the Lagrangian (3.1) becomes:
\[ \mathcal{L} = (d - 1)! (\det G)^{-p} \det e d^d \xi \]
\[ = (d - 1)! \frac{1}{2p} (\det G_{\mu\nu})^{-p} (\det e)^{2p+1} d^d \xi \] (3.18)

the second identity following from:
\[ G_{ij} = V^a_i V^b_j \eta_{ab} \partial_{\mu} X^a \partial_{\nu} X^b e^{\mu} e^{\nu} = g_{\mu\nu} \partial_{\mu} X^a \partial_{\nu} X^b e^{\mu} e^{\nu} \]
\[ \downarrow \]
\[ \det G_{ij} = (\det G_{\mu\nu}) (\det e)^{-2} \] (3.19)

where \( G_{\mu\nu} \) denotes the pull–back of the bulk space–time metric \( g_{\mu\nu} \) onto the world–volume of the brane.

If we choose:
\[ p = -\frac{1}{2} \Rightarrow \alpha = \frac{1}{d - 2} \] (3.20)
then the original world-volume lagrangian (3.1), already transformed to the second order form (3.18) becomes proportional to the Nambu–Goto lagrangian:

$$L = (d - 1)! \sqrt{\det G_{\mu \nu}} d^d \xi$$  \hspace{1cm} (3.21)

In this way the reference vielbein $e^{\mu}_{\mu}$ has disappeared from the lagrangian. This result is supported by the calculation of the variation in $\delta e^{k}$ of the first order action (3.1). After variation and substitution of the result for the first order equations $\delta \Pi^a_i$ and $\delta h_{ij}$ all terms are already Kronecker deltas proportional to $\det G$. With the choice $p = -1/2$ all terms in this stress energy tensor cancel identically.

Note also that if the transformation (2.18) is completed by setting:

$$\Pi^a_i \mapsto K^i_k \eta^{k\ell} \Pi^a_{\ell} (\det K)^{-1}$$  \hspace{1cm} (3.22)

it becomes an exact local symmetry of the action (3.1).

In this way we have shown how the standard first order formalism for the Nambu–Goto action can be replaced by a new first order formalism involving the additional field $h_{ij}$. So far the matrix $h$ was chosen to be symmetric. Including world–sheet vector fields corresponds to the generalization of the above construction to the case where $h$ has also an antisymmetric part.

### 3.2 Inclusion of a world–volume gauge field and the Born Infeld action in first order formalism

We consider a modification of the first order action (3.1) of the following form

$$L = \Pi^a_i V^2_{ab} \eta^{ij} \epsilon^i \land \epsilon^j \land \ldots \land \epsilon^d \epsilon^{\ell_1} \ldots \epsilon_{\ell_d} + a_1 \Pi^a_i \Pi^b_j \eta^{ab} h^{ij} \epsilon^i \land \ldots \land \epsilon^d \epsilon^{\ell_1} \ldots \epsilon_{\ell_d}$$

$$+ a_2 \left[ \det \left( h^{-1} + \mu F \right) \right]^\alpha \epsilon^i \land \ldots \land \epsilon^d \epsilon^{\ell_1} \ldots \epsilon_{\ell_d}$$

$$+ a_3 F^{ij} F^{[2]} \land \epsilon^{\ell_3} \land \ldots \land \epsilon^d \epsilon^{ij\ell_3} \ldots \epsilon_{\ell_d} + \text{WZT}$$

$$\hspace{2cm} (3.23)$$

where

$$F^{[2]} \equiv dA^{[1]}$$  \hspace{1cm} (3.24)

is the field strength of a world–volume 1–form gauge field, $F_{ij} = -F_{ji}$ is an antisymmetric 0–form auxiliary field and $a_3$ is a further numerical coefficient to be determined. Furthermore $WZT$ denotes the Wess–Zumino terms, i.e. the integrals on the world volume of various combinations of the Ramond–Ramond p–forms. These terms depend on the type of $Dp$–brane considered and will be discussed later in the case of the $D3$–brane.

Performing the $\delta \Pi^a_i$ variation we obtain :

$$(d - 1)! [\eta^{ab} V^2_{ab} \eta^{d} + 2a_1 d \eta^{ab} \Pi^a_i h^{ij}] = 0$$  \hspace{1cm} (3.25)

that is solved by :

$$\Pi^a_i = - \frac{1}{2 d a_1} V^2_{am} (h^{-1})^m j$$  \hspace{1cm} (3.26)

and :

$$\gamma = \frac{1}{(2 d a_1)^2} h^{-1} \hat{G} h^{-1}$$  \hspace{1cm} (3.27)
Varying in $\delta h_{ij}$ we also obtain a result similar to what we had before, namely:

$$a_1 \gamma - a_2 \alpha h^{-1} (h^{-1} + \mu F)^{-1} h^{-1} \left[ \det \left( h^{-1} + \mu F \right) \right]^\alpha = 0 \quad (3.28)$$

where the suffix $S$ denotes the symmetric part of the matrix to which it is applied.

From the variation in $\delta F_{ij}$ we obtain instead:

$$-d! a_2 \alpha \mu \left( h^{-1} + \mu F \right)^{-1} \left[ \det \left( h^{-1} + \mu F \right) \right]^\alpha + 2 (d-2)! a_3 F = 0 \quad (3.29)$$

where the suffix $A$ denotes the antisymmetric part of the matrix to which it is applied and where $F$ is the antisymmetric matrix $F_{ij}$ of flat components of the field strength 2–form:

$$F^{[2]} = F_{ij} e^i \wedge e^j \quad (3.30)$$

Hence from $\delta h_{ij}$ and $\delta F_{ij}$ we get:

$$2(d-2)! a_3 F_{ij} = \left( h^{-1} + \mu F \right)^{-1} \left[ \det \left( h^{-1} + \mu F \right) \right]^\alpha + 2(d-2)! a_3 F = 0 \quad (3.31)$$

Summing the two eq.s (3.31) together we obtain:

$$\frac{2(d-2)! a_3}{d! a_2 \alpha \mu} F_{ij} + \frac{1}{4 d^2 a_1 a_2 \alpha} \hat{G} = \left( h^{-1} + \mu F \right)^{-1} \left[ \det \left( h^{-1} + \mu F \right) \right]^\alpha \quad (3.32)$$

which can be uniquely solved by:

$$h^{-1} + \mu F = \left( a \hat{G} + b F \right)^{-1} \left[ \det \left( a \hat{G} + b F \right) \right]^\beta \quad ; \quad \beta = \frac{\alpha}{\alpha d - 1} \quad (3.33)$$

where:

$$a = \frac{1}{4 d^2 a_1 a_2 \alpha} \quad ; \quad b = \frac{2(d-2)! a_3}{d! a_2 \alpha \mu} \quad (3.34)$$

The coefficients $a_1$, $a_2$, $a_3$ are redundant since they can be reabsorbed into the definition of $\Pi^k_j$, $h$ and $F$; so we fix them by imposing:

$$a_1 = -\frac{1}{2d} \quad ; \quad a_1 = 1 \quad ; \quad b = -\frac{1}{\mu} \quad (3.35)$$

Hence using (3.34) and (3.35) we obtain:

$$a_2 = -\frac{1}{2d} \quad ; \quad a_3 = -\frac{d! a_2 \alpha}{2(d-2)!} = \frac{d-1}{4} \quad (3.36)$$

At this point everything proceeds just as in the previous case. Indeed inserting eq.s (3.27), (3.26) back into the action (3.23) we obtain:

$$\left[ \frac{(d-1)!}{2 d a_1} + a_1 d! \right] \frac{1}{(2 d a_1)^2} Tr(h^{-1} \hat{G}) + 2 a_3 (d-2)! F_{ij} F_{ij} +$$

$$+ a_2 d! \left[ \det \left( h^{-1} + \mu F \right) \right]^\alpha \det e^d \xi \quad (3.37)$$
Using (3.35) and (3.36) eq. (3.37) becomes:

\[
\left\{ \frac{(d-1)!}{2} [Tr(h^{-1} \hat{G}) - Tr(FF)] - \frac{(d-1)!}{2\alpha} \det (h^{-1} + \mu F) \right\} \det e d^d \xi
\]

(3.38)

Now we consider the variation \( \delta e \):

\[
\left[ -\frac{(d-1)!}{4da_1} Tr(G h^{-1}) - 2 (d-2)! a_3 Tr(FF) \right] \delta_p^i + 2 \left[ -\frac{(d-1)!}{4da_1} (G h^{-1}) p^t - 2 (d-2)! a_3 (F_{ti} F_{ip}) \right] + a_2 d! \det (h^{-1} + \mu F) \delta_p^i = 0
\]

(3.39)

the solution is:

\[
\left[ -\frac{(d-1)!}{4da_1} (G h^{-1}) p^t - 2 (d-2)! a_3 (F_{ti} F_{ip}) \right] = -\frac{a_2 d!}{d-2} \det (h^{-1} + \mu F) \delta_p^i
\]

(3.40)

Using (3.35) and (3.36) eq. (3.41) in matrix form becomes:

\[
h^{-1} \hat{G} - FF = \frac{1}{\alpha(d-2)} \det (h^{-1} + \mu F) \mathbb{I}
\]

(3.41)

Now using the result:

\[
[ \det (a \hat{G}_{ij} + b F_{ij}) ] = [ \det (a \hat{G}_{\mu\nu} + b F_{\mu\nu}) ] (\det e)^{-2}
\]

(3.42)

and implementing eq. (3.41) for \( \delta e \), we see that (3.38) becomes:

\[
(\det e) d^d \xi \frac{(d-1)!}{\alpha(d-2)} \det (h^{-1} + \mu F) = 0
\]

\[
= (\det e) d^d \xi \frac{(d-1)!}{\alpha(d-2)} [ \det (a \hat{G}_{ij} + b F_{ij}) ] = 0
\]

\[
= (\det e)^{1-2\beta} d^d \xi \frac{(d-1)!}{\alpha(d-2)} [ \det (a \hat{G}_{\mu\nu} + b F_{\mu\nu}) ] = 0
\]

(3.43)

Now we take \( \beta = 1/2 \) and so \( \alpha = 1/(d-2) \). The action becomes:

\[
S_{BI} = (d-1)! \int_{M_4} d^d \xi [ \det (\hat{G}_{\mu\nu} - \frac{1}{\mu} F_{\mu\nu}) ]^{1/2}
\]

(3.44)

For \( d = 4 \), which is the interesting case of the D3–brane we obtain:

\[
a_1 = -\frac{1}{8} \quad a_2 = -\frac{1}{4} \quad a_3 = \frac{3}{4} \quad ; \quad \alpha = \beta = \frac{1}{2}
\]

(3.45)

In this way we have shown how the kinetic part of a \( Dp \)-brane action, namely the Born-Infeld type of Lagrangian can be written in first order formalism. The new formalism can be applied to all cases except \( d = 2 \) where the formulae become singular. This is just welcome since for \( d = 2 \) we have ordinary strings for which the Polyakov formalism is sufficient and no world–volume cosmological term is necessary. For \( d = 3 \), we are instead in the case of the M2 brane or of its descendant, the D2 brane, for which no Born Infeld action is necessary either.
3.3 Explicit solution of the equations for the auxiliary fields for $F$ and $h^{-1}$

In the transition to second order formalism and in the discussion of $\kappa$-supersymmetry through the use of 1.5 order formalism we need the explicit solution of the first order equations and the expression of the auxiliary fields $F$, $h^{-1}$ in terms of the physical degrees of freedom. This is what we can do most conveniently by fixing the gauge related to the local symmetry (2.18) and (3.22). Our gauge choice is provided by setting:

$$\hat{G} = \eta$$

(3.46)

which is identical with the yield (2.13) of the $\delta e^i$ variation in the old first order formalism. This gauge can certainly be reached by using the degrees of freedom of $\text{GL}(d, \mathbb{R})/\text{SO}(1, d - 1)$. Taking (3.46) into account let us rewrite our constraint equations into matrix form. Eq. (3.41) for the $\delta e$ variation is:

$$h^{-1} \hat{G} - FF = [\det (h^{-1} + \mu F)]^\alpha \mathbb{I}$$

(3.47)

and the other equation that we must solve is (3.33):

$$(h^{-1} + \mu F) \left( \hat{G} - \frac{1}{\mu} F \right) = \left[ \det \left( \hat{G} - \frac{1}{\mu} F \right) \right]^{1/2} \mathbb{I}$$

(3.48)

Using our previous result for $[\det (h^{-1} + \mu F)]^\alpha$ we conclude that we have the following linear system of matrix equations:

$$\begin{cases}
(h^{-1} + \mu F) \left( \hat{G} - \frac{1}{\mu} F \right) = \left[ \det \left( \hat{G} - \frac{1}{\mu} F \right) \right]^{1/2} \mathbb{I} \\
(\mathbb{I} - \frac{1}{\mu^2} F \eta F \eta) \left[ \det \left( \eta - \frac{1}{\mu} F \right) \right]^{-1/2}
\end{cases}$$

(3.49)

the solution in the gauge (3.46) is:

$$\begin{cases}
\hat{G} = \eta \\
F = \frac{1}{\mu^2} h^{-1} F \eta \\
h \eta = \left( \mathbb{I} - \frac{1}{\mu^2} F \eta F \eta \right) \left[ \det \left( \eta - \frac{1}{\mu} F \right) \right]^{-1/2}
\end{cases}$$

(3.50)

Since the $\eta$ metric just raises and lowers the indices we can just ignore it and write, in more compact form:

$$h = (\eta - \frac{1}{\mu^2} F^2) \left[ \det \left( \eta - \frac{1}{\mu} F \right) \right]^{-1/2}$$

(3.51)

4 The $D3$–brane example and $\kappa$-supersymmetry

In this section we focus on the case $d = 4$ and we apply our new first order formalism to the description of the $\kappa$-supersymmetric action of a $D3$–brane. As claimed in the introduction, $\kappa$–supersymmetry just follows, via a suitable projection, from the bulk supersymmetries as derived from supergravity, the type II B theory, in this case. The latter has a duality symmetry with respect to an $\text{SL}(2, \mathbb{R})$ group of transformations that acts non linearly on the two scalars of massless spectrum, the dilaton $\phi$ and the Ramond scalar $C_0$. Indeed these two parametrize the coset manifold $\text{SL}(2, \mathbb{R})/\text{O}(2)$ and actually correspond to its solvable parametrization (see eq. (4.13) of the appendix). Hence the $D3$–brane action we want to write, not only should be cast into first order
formalism, but should also display manifest covariance with respect to SL(2, \mathbb{R}). This covariance relies on introducing a two component charge vector \( q_\alpha \) that transforms in the fundamental representation of SU(1, 1) and expresses the charges carried by the D3 brane with respect to the 2–forms \( A^{\alpha}_{[2]} \) of bulk supergravity (both the Neveu Schwarz \( B_{[2]} \) and Ramond–Ramond \( C_{[2]} \)). According to the geometrical formulation of type IIB supergravity which is summarized in the appendix we set:

\[
\begin{align*}
A^\Lambda &= (B_{[2]}, C_{[2]}); \quad A^\alpha &= C^\alpha \wedge A^\Lambda \\
A^{\alpha=1} &= \frac{1}{\sqrt{2}} (B_{[2]} - i C_{[2]}); \quad A^{\alpha=2} &= \frac{1}{\sqrt{2}} (B_{[2]} + i C_{[2]})
\end{align*}
\] (4.1)

and by definition we call \( \epsilon_{\alpha\beta} q^\beta \) the orthogonal complement of \( q^\alpha \):

\[
q_\alpha q^\alpha = 1 \quad ; \quad q_\alpha q_\beta \epsilon_{\alpha\beta} = 0
\] (4.2)

In terms of these objects we write down the complete action of the D3–brane as follows:

\[
\mathcal{L} = \Pi^2 \eta_{\mu\nu} \epsilon_1 e_1 \wedge e_2 \wedge \ldots \wedge e_4 \epsilon_4 + a_1 \Pi^2 \Pi^2 \eta_{\mu\nu} h^{ij} e^i \epsilon_1 \wedge \ldots \wedge e^4 \epsilon_4 + a_2 \left( \det (h^{-1} + \mu F) \right)^{\alpha} e_1 \wedge \ldots \wedge e^4 \epsilon_4
\]

\[
+ a_3 F^{ij} F_{ij} \wedge e_3 \wedge e_4 \epsilon_3 \epsilon_4 + \nu F \wedge F + a_5 q^\alpha \epsilon_{\alpha\beta} A^\beta \wedge F + a_6 C_{[4]}
\] (4.3)

where \( C_{[4]} \) is the 4–form potential, the coefficients

\[
\alpha = \frac{1}{2} \quad a_1 = -\frac{1}{8} \quad a_2 = -\frac{1}{4} \quad a_3 = \frac{3}{4} \]

(4.4)

have already been determined, while \( a_5, a_6, \nu \) are new coefficients to be fixed by \( \kappa \)–supersymmetry. The first two are numerical, while \( \nu \) will also depend on the bulk scalars. In the action (4.3)

\[
F^{[2]} \equiv dA_{[1]} + q_\alpha A^\alpha
\] (4.5)

is the field strength of the world–volume gauge field and depends on the charge vector \( q^\alpha \). The physical interpretation of \( F^{[2]} \) is as follows. By definition a \( Dp \)–brane is a locus in space–time where open strings can end or, in the dual picture, boundaries for closed string world–volumes can be located. The type IIB theory contains two kind of strings, the fundamental strings and the \( D \)-strings which are rotated one into the other by the SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R}) group. Correspondingly a D3 brane can be a boundary either for fundamental or for \( D \)-strings or for a mixture of the two. The charge vector \( q^\alpha \) just expresses this fact and characterizes the D3–brane as a boundary for strings of \( q \)–type. Furthermore the definition (4.3) of \( F^{[2]} \) encodes the following idea: the world–volume gauge 1–form \( A_{[1]} \) is just the parameter of a gauge transformation for the 2–form \( q_\alpha A^\alpha \), which in a space–time with boundaries can be reabsorbed everywhere except on the boundary itself. Note that if we take \( q_\alpha = \frac{1}{\sqrt{2}} (1, 1) \) we obtain :

\[
q_\alpha A^\alpha = B_{[2]} \quad ; \quad -i q^\alpha \epsilon_{\alpha\beta} A^\beta = C_{[2]}
\] (4.6)

### 4.1 \( \kappa \)–supersymmetry

Next we want to prove that with an appropriate choice of \( \nu, a_5 \) and \( a_6 \) the action (4.3) is invariant against bulk supersymmetries characterized by a projected spinor parameter. For simplicity we do
this in the case of the choice $q_0 = \frac{1}{\sqrt{2}}(1, 1)$. For other choices of the charge type the modifications needed in the prove will be obvious from its details.

To accomplish our goal we begin by writing the supersymmetry transformations of the bulk differential forms $V^a$, $B_{[2]}$, $C_{[2]}$ and $C_{[4]}$ which appear in the action. From the rheonomic parametrizations (A.13, A.14, A.15, A.16) we immediately obtain:

$$
\delta V^a = i \frac{1}{2} (\bar{\epsilon} \Gamma^a \psi + \bar{\epsilon}^* \Gamma^a \psi^*)
$$

$$
\delta B_{[2]} = -2i \left[ (\Lambda^1_+ + \Lambda^2_+) \bar{\epsilon} \Gamma^a \psi^* V^a + (\Lambda^1_+ + \Lambda^2_-) \bar{\epsilon}^* \Gamma^a \psi V^a \right]
$$

$$
\delta C_{[2]} = 2 \left[ (\Lambda^1_+ - \Lambda^2_-) \bar{\epsilon} \Gamma^a \psi^* V^a + (\Lambda^1_- - \Lambda^2_-) \bar{\epsilon}^* \Gamma^a \psi V^a \right]
$$

$$
\delta C_{[4]} = -\frac{1}{6} \left[ (\bar{\epsilon} \Gamma_{abc} \psi - \bar{\epsilon}^* \Gamma_{abc} \psi^*) V^{abc} \right] + \frac{1}{8} \left[ B_{[2]} \delta C_{[2]} - C_{[2]} \delta B_{[2]} \right]
$$

Note that in writing the above transformations we have neglected all terms involving the dilatino field. This is appropriate since the background value of all fermion fields is zero. The gravitino 1-form $\psi$ is instead what we need to keep track of. Proving $\kappa$-supersymmetry is identical with showing that all $\psi$ terms cancel against each other in the variation of the action. Relying on (4.7) the variation of the W.Z.T term is as follows:

$$
\delta (\nu F \wedge F + a_5 C_{[2]} \wedge F + C_{[4]} =)
$$

$$
= 2 \nu B \delta B + a_5 B \delta C_{[2]} + \frac{1}{8} a_6 B \delta C + a_5 C \delta B - \frac{1}{8} a_6 C \delta B + a_6 \delta C_{[4]}
$$

if we set $a_6 = 8a_5$ the variation (4.8) simplifies to:

$$
\delta (W.Z.T) = 2B (\nu \delta B + a_5 \delta C) + 8a_5 \delta C_{[4]}
$$

and with such a choice the complete variation of the Lagrangian under a supersymmetry transformation of arbitrary parameter is:

$$
\delta \mathcal{L} = \delta \mathcal{L}_\psi + \delta \mathcal{L}_{\psi^*}
$$

$$
\delta \mathcal{L}_\psi = [-3i \bar{\epsilon} P^{\alpha \beta} (\bar{\epsilon} \Gamma^a \psi) \eta_{ab} + (\mu_1 F^{ij} + \mu_2 \bar{F}^{ij}) V^a_i (\bar{\epsilon}^* \Gamma^a \psi) +
$$

$$
-\frac{4}{3} a_5 (\bar{\epsilon} \Gamma_{abc} \psi) V^a_i V^b_j V^c_k \epsilon^{ijkp} \Omega_p^3 \right]
$$

$$
\delta \mathcal{L}_{\psi^*} = [-3i \bar{\epsilon} P^{\alpha \beta} (\bar{\epsilon}^* \Gamma^a \psi^*) \eta_{ab} + (\mu_3 F^{ij} + \mu_4 \bar{F}^{ij}) V^a_i (\bar{\epsilon} \Gamma^a \psi^*) +
$$

$$
+\frac{4}{3} a_5 (\bar{\epsilon}^* \Gamma_{abc} \psi^*) V^a_i V^b_j V^c_k \epsilon^{ijkp} \Omega_p^3 \right]
$$

where:

$$
\mu_1 = -8ia_3 (\Lambda^1_+ + \Lambda^2_+) ; \quad \mu_2 = 8[-i\nu (\Lambda^1_+ + \Lambda^2_-) + a_5 (\Lambda^1_- - \Lambda^2_-)]
$$

$$
\mu_3 = -8ia_3 (\Lambda^1_- + \Lambda^2_+) ; \quad \mu_4 = 8[-i\nu (\Lambda^1_- + \Lambda^2_-) + a_5 (\Lambda^1_+ - \Lambda^2_+)]
$$

Recalling eqs (A.7) and (A.3) of the appendix the above eqs (4.11) become:

$$
\mu_1 = -6i e^{\phi/2} ; \quad \mu_2 = 8a_5 e^{-\phi/2}
$$

$$
\mu_3 = -6i e^{\phi/2} ; \quad \mu_4 = -8a_5 e^{-\phi/2}
$$
where we have chosen:

$$\nu = -a_5 C_0 = a_5 \text{Re} \mathcal{N}$$  (4.13)

In the above equation we have introduced the complex kinetic matrix which would appear in a
gauge theory with scalars sitting in SU(1, 1)/U(1) and determined by the classical Gaillard–Zumino
general formula applied to the specific coset:

$$\mathcal{N} = i \frac{\Lambda^1 - \Lambda^2}{\Lambda^+ + \Lambda^-} \Rightarrow \begin{cases} \text{Re} \mathcal{N} = -C_0 \\ \text{Im} \mathcal{N} = e^{-\phi} \end{cases}$$  (4.14)

It is convenient to rewrite the full variation (4.10) of the Lagrangian in matrix form in the 2–
dimensional space spanned by the fermion parameters ($\psi, \psi^*$):

$$\delta \mathcal{L} = \delta \mathcal{L}_\psi + \delta \mathcal{L}_{\psi^*} = (\bar{\psi}, \bar{\psi}^*) A \left( \begin{array}{c} \psi \\ \psi^* \end{array} \right)$$  (4.15)

$$A_k = \left( \begin{array}{c} -6i \gamma_k + \frac{4}{3} a_5 \epsilon_{ijkl} \gamma^{ijlm} h_{mk} \\ (\mu_1 F_{lm} + \mu_2 \tilde{F}_{lm}) h_{mk} \gamma_l \end{array} \right)$$  (4.16)

where $A = A_k \Omega_k^3$, and $\Omega_k^3 \equiv \eta^{k\ell} (\epsilon_{ijl} \epsilon^j \wedge \epsilon^l \wedge \epsilon^k)$ denotes the quadruplet of three–volume forms The matrix $A_k$ is a tensor product of a matrices in spinor space and $2 \times 2$ matrices in the space spanned by ($\epsilon, \epsilon^*$). It is convenient to spell out this tensor product structure which is achieved by the following rewriting:

$$A_k = f_1 \gamma_k \otimes \mathbb{I} + f_2 \tilde{\gamma}^m h_{mk} \otimes \sigma_3 + f_3 \Pi^m h_{mk} \otimes \sigma_1 + f_4 \Pi^m h_{mk} \otimes \sigma_2$$  (4.17)

where:

$$f_1 = -6i \quad f_2 = -6i \quad f_3 = -\frac{4}{3} a_5 \quad f_4 = -8ia_5$$  (4.18)

and:

$$\tilde{\gamma}^m \equiv \gamma_{ijl} \epsilon^{ijlm} \quad \Pi^m_1 \equiv e^{\phi/2} F_{jm} \gamma_l \quad \Pi^m_2 \equiv e^{-\phi/2} \tilde{F}_{jm} \gamma_l$$  (4.19)

now using (3.50), (3.51) we set

$$\frac{1}{\mu} = e^{-\phi/2} = \sqrt{\text{Im} \mathcal{N}}$$

$$\tilde{F} \equiv \sqrt{\text{Im} \mathcal{N}} F$$  (4.20)

and we obtain:

$$\Pi_1^m h_{mk} = e^{\phi/2} F_{jm} \gamma_l h_{mk} = e^{\phi/2} e^{-\phi(F^{-1})jm} h_{mk} \gamma_l \equiv \tilde{F}_{lk} \gamma^l \equiv \Pi_k$$

$$\Pi_2^m = e^{-\phi/2} \tilde{F}_{jm} \gamma_l \equiv \tilde{F}_{jm} \gamma_l \equiv \Pi^m$$  (4.21)

This observation further simplifies the expression of $A_k$ which can be rewritten as:

$$A_k = f_1 \gamma_k \otimes \mathbb{I} + f_2 \tilde{\gamma}^m h_{mk} \otimes \sigma_3 + f_3 \Pi^k \otimes \sigma_1 + f_4 \Pi^m h_{mk} \otimes \sigma_2$$  (4.22)
The proof of \(\kappa\)-supersymmetry can now be reduced to the following simple computation. Assume we have a matrix operator \(\Gamma\) with the following properties:

\[
\begin{align*}
[a] \quad \Gamma^2 &= \mathbb{I} \\
[b] \quad \Gamma A_k &= A_k
\end{align*}
\]

(4.23)

It follows that

\[
P = \frac{1}{2}(\mathbb{I} - \Gamma)
\]

(4.24)

is a projector since \(P^2 = \mathbb{I}\) and that

\[
PA_k = \frac{1}{N}(\mathbb{I} - \Gamma)A_k = 0
\]

(4.25)

Therefore if we use supersymmetry parameters \((\xi, \xi^\ast) = (\epsilon, \epsilon^\ast)\) projected with this \(P\), then the action is invariant and this is just the proof of \(\kappa\)-supersymmetry.

The appropriate \(\Gamma\) is the following\[28\]

\[
\Gamma = \frac{1}{N}[\omega[4] + \omega[0] \otimes \sigma_3 + \omega[2] \otimes \sigma_2]
\]

(4.26)

where:

\[
\begin{align*}
\omega[4] &= \alpha_4 \epsilon^{ijkl} \gamma_{ijkl} \\
\omega[0] &= \alpha_0 \epsilon^{ijkl} \hat{F}_{ij} \hat{F}_{kl} \\
\omega[2] &= \alpha_2 \epsilon^{ijkl} \hat{F}_{ij} \gamma_{kl} \\
N &= \left[\det \left(\eta \pm \hat{F}\right)\right]^{1/2}
\end{align*}
\]

and the coefficients are fixed to:

\[
\begin{align*}
\alpha_4 &= \frac{1}{24} ; \quad \alpha_0 = \frac{1}{8} ; \quad \alpha_2 = \frac{i}{4}
\end{align*}
\]

(4.28)

This choice suffices to guarantee property [a] in the above list. Property [b] is also verified if one chooses:

\[
a_5 = \frac{3}{4}i
\]

(4.29)

The proof of the two properties is given in the appendix\[3\]. Essential ingredients in that proof are the following identities holding true for any antisymmetric tensor \(\hat{F}\):

\[
\det \left(\eta \pm \hat{F}\right) = -1 + \frac{1}{2} Tr(\hat{F}^2) + \left(\frac{1}{8} \epsilon^{ijkl} \hat{F}_{ij} \hat{F}_{kl}\right)^2
\]

(4.30)

and

\[
\hat{F} \hat{F} = -\frac{1}{8} \left(F_{ij} F_{kl} \epsilon^{ijkl}\right) \mathbb{I} = -\omega[0] \mathbb{I}
\]

\[
\hat{F}^2 + \hat{F} = \frac{1}{2} Tr(F^2) \mathbb{I}
\]

(4.31)

4In the paper quoted above the \(\kappa\)-supersymmetry projector presented here was originally introduced within a 2nd order formulation of the theory. It is particularly significant and rewarding that the same projector is valid also in first order formulation. As shown in the appendix the mechanism by means of which it works are very subtle and take advantage of the explicit solutions for the auxiliary fields in terms of the physical ones. In this way one finds an overall non trivial check of all the algebraic machinery of our new first order formalism.
5 Outlook and conclusions

In this paper we have introduced a new first order formalism for $p$–brane world volume actions that allows to reproduce the Born–Infeld second order action via the elimination of a set composed by three auxiliary fields:

- $\Pi^a$
- $h^{ij}$ (symmetric)
- $\mathcal{F}^{ij}$ (antisymmetric)

Distinctive properties of our new formulation are:

1. All fermion fields are implicitly hidden inside the definition of the $p$–form potentials of supergravity
2. $\kappa$–supersymmetry is easily proven from supergravity rheonomic parametrization
3. The action is manifestly covariant with respect to the duality group $\text{SL}(2, \mathbb{R})$ of type IIB supergravity.
4. The action functional can be computed on any background which is an exact solution of the supergravity bulk equations.

Of specific interest in applications are precisely the last two properties. Putting together our result we can summarize the $D_3$ brane action we have found as follows:

$$
\mathcal{L} = \Pi^a_i V^b \eta^{ab} h^{i\ell_1} \wedge e^{\ell_2} \wedge \ldots \wedge e^{\ell_4} \epsilon_{\ell_1 \ldots \ell_4} - \frac{1}{8} \Pi^a_i \Pi^b_j \eta^{ab} h^{ij} e^{\ell_1} \wedge \ldots \wedge e^{\ell_4} \epsilon_{\ell_1 \ldots \ell_4}
$$

$$
- \frac{1}{4} \left[ \det \left( h^{-1} + \sqrt{\text{Im} N} \mathcal{F} \right) \right]^{1/2} e^{\ell_1} \wedge \ldots \wedge e^{\ell_4} \epsilon_{\ell_1 \ldots \ell_4}
$$

$$
+ \frac{3}{4} \mathcal{F}^{ij} F \wedge e^{\ell_5} \wedge e^{\ell_4} \epsilon_{ij\ell_5\ell_4}
$$

$$
+ \frac{3}{4} i \text{Re} N F \wedge F + \frac{3}{4} q^{a} \epsilon_{a\beta} A^\beta \wedge F + 6i C^{[4]}
$$

$$
F = dA^{[1]} + q_{a} A^a
$$

$$
\mathcal{N} = i \frac{\Lambda_1^2 - \Lambda_2^2}{\Lambda_2 - \Lambda_2^2}
$$

(5.1)

Evaluating for instance the above action on the background provided by the bulk solution found in [11] which describes a $D3$–brane with an $\mathbb{R}^2 \times ALE$ transverse manifold we can finally write the appropriate source term of that exact solution which was so far missing. Alternatively by expanding (5.1) for small fluctuations around the same background we can use it as a token to explore the gauge/gravity correspondence.

In relation with the solution [11] which is one of the main motivations for undertaking the present investigation we note that the $D3$–brane action (5.1) is not sufficient to work out all the sources for such a solution. Indeed (5.1) gives account of the $C^{[4]}$ charge but not of the $B^{[2]}$ or $C^{[2]}$ charges. The latter are however essential in the solution [11] since there we also have nontrivial 2–forms. It follows that to complete our task we need to adjoin to (5.1) also a source action for the 2–forms which cannot be anything else but a 5–brane, since the 2–forms are magnetically coupled. A distinctive feature of the needed 5–brane is that it should not couple to the dilaton field since the latter is constant in the solution [11]. For a Neveu-Schwarz 5–brane or for a $D5$–brane this is
impossible yet it can be possible for a mixture of the two. This singles out an obvious research
direction which we are presently pursuing. That is applying our new first order formalism to the
case of a $q$–type 5–brane.

Several other applications of our formalism are possible since it provides a general way to deal
with Born–Infeld, $\kappa$–supersymmetric actions. In particular it can give new insight on the open
problem of writing supersymmetric, non abelian, Born Infeld actions.

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A Summary of type IIB supergravity in a geometrical set up

The formulation of type IIB supergravity as it appears in string theory textbooks [25, 16] is tailored for the comparison with superstring amplitudes and is quite appropriate to this goal. Yet, from the viewpoint of the general geometrical set up of supergravity theories this formulation is somewhat unwieldy. Specifically it neither makes the $SU(1,1)/U(1)$ coset structure of the theory manifest, nor it relates the supersymmetry transformation rules to the underlying algebraic structure which, as in all other instances of supergravities, is a simple and well defined Free Differential algebra.

The Free Differential Algebra of type IIB supergravity was singled out many years ago by Castellani in [26] and the geometric manifestly $SU(1,1)$ covariant formulation of the theory was constructed by Castellani and Pesando in [27]. In this appendix we summarize their formulae giving also their transcription from a complex $SU(1,1)$ basis to a real $SL(2, \mathbb{R})$ basis. Furthermore we provide the translation vocabulary between these intrinsic notations and those of Polchinski’s textbook [16] frequently used in current superstring literature.

### A.1 The $SU(1,1)/U(1) \sim SL(2, \mathbb{R})/O(2)$ coset

The basic ingredient in all supergravity constructions is the parametrization of the scalar manifold geometry that, with few exceptions, corresponds to a homogeneous scalar manifold [29]. In all these cases the essential building block appearing in the Lagrangian and supersymmetry transformation rules is the coset representative $L(\phi)$ that provides a parametrization of the coset manifold $G/H$ in terms of some chosen patch of coordinates. A very useful choice is given by the so called solvable Lie algebra parametrization [5]. This is true also in the present case where the solvable parametrization of the coset $SU(1,1)/U(1) \sim SL(2, \mathbb{R})/O(2)$ is precisely that which allows for the identification of the massless superstring fields inside the covariant formulation of supergravity.

Our notations are as follows.

**SL(2, $\mathbb{R}$) Lie algebra**

\[
[L_0, L_\pm] = \pm L_\pm \quad ; \quad [L_+, L_-] = 2 L_0 \tag{A.1}
\]

with explicit 2-dimensional representation:

\[
L_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ; \quad L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.2}
\]

**Coset representative of SL(2, $\mathbb{R}$)/O(2) in the solvable parametrization**

\[
L(\varphi, C_{[0]}) = \exp [\varphi L_0] \exp [C_{[0]} e^{\varphi} L_-] = \begin{pmatrix} \exp[\varphi/2] & 0 \\ C_{[0]} e^{\varphi/2} \exp[-\varphi/2] \end{pmatrix} \tag{A.3}
\]

where $\varphi(x)$ and $C_{[0]}$ are respectively identified with the dilaton and with the Ramond-Ramond 0-form of the superstring massless spectrum. The isomorphism of SL(2, $\mathbb{R}$) with SU(1,1) is realized by conjugation with the Cayley matrix:

\[
{\mathcal{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \tag{A.4}
\]

---

5 For a review see either [30] or [29] and all references therein.
Table 1: **Field content of type IIB supergravity.** The early Greek indices $\alpha, \beta, \ldots = 1, 2$ run in the fundamental representation of SU(1, 1), while the early capital Latin indices $A, B, \ldots = 1, 2$ run in the fundamental representation of SL(2, \mathbb{R}). The $p$-gauge forms of the Ramond Ramond sector are denoted by $C_{[p]}$.

| Field in SU(1, 1) basis | SU(1,1) repres. | U(1) charge | superstring zero modes |
|------------------------|----------------|-------------|------------------------|
| $V^\alpha_\mu$         | $J = 0$       | 0           | graviton $h_{\mu\nu}$  |
| $\psi_\mu$             | $J = 0$       | $\frac{1}{2}$ | gravitinos $\psi_{A\mu}$ |
| $A^\alpha_{\mu\nu}$    | $J = \frac{1}{2}$ | 0           | $B_{[2]}$, $C_{[2]}$   |
| $C_{\mu\nu\rho\sigma}$| $J = 0$       | 0           | $C_{[4]}$              |
| $\lambda$              | $J = 0$       | $\frac{3}{2}$ | dilatinos $\lambda_A$  |
| $L^\alpha_{\beta}$     | $J = \frac{1}{2}$ | $\pm 1$     | $\varphi, C_{[0]}$     |

Introducing the SU(1, 1) coset representative

$$SU(1, 1) \ni \Lambda = CLC^{-1}$$  \hspace{1cm} (A.5)

from the left invariant 1-form $\Lambda^{-1} d\Lambda$ we can extract the 1-forms corresponding to the scalar vielbein $P$ and the U(1) connection $Q$

**The SU(1,1)/U(1) vielbein and connection**

$$\Lambda^{-1} d\Lambda = \begin{pmatrix} -iQ & P \\ P^* & iQ \end{pmatrix}$$  \hspace{1cm} (A.6)

Explicitly

$$P = \frac{1}{2} \left( d\varphi - i e^{\varphi} dC_{[0]} \right) \text{ scalar vielbein}$$

$$Q = \frac{1}{2} \exp[\varphi] dC_{[0]} \text{ U(1)-connection}$$  \hspace{1cm} (A.7)

**A.2 The free differential algebra, the supergravity fields and the curvatures**

Following Castellani and Pesando the field content of type IIB supergravity is organized into representations of SU(1, 1) as displayed in table 1. In order to write down the free differential algebra the critical issue is the correct identification of the fermionic terms contributing to the curvature of the complex 2-form doublet $A^\alpha_{\mu\nu}$. These latter transform in the 2-dimensional representation of SU(1, 1) and are related by the Cayley matrix of eq.(A.4) to a doublet of real 2-forms $A^\Lambda_{\mu\nu}$ that transform in the 2-dimensional representation of SL(2, \mathbb{R}):

$$\begin{pmatrix} A^1_{\mu\nu} \\ A^2_{\mu\nu} \end{pmatrix} = C \begin{pmatrix} A^1_{\mu\nu} \\ A^2_{\mu\nu} \end{pmatrix}$$  \hspace{1cm} (A.8)
We introduce a doublet of Majorana-Weyl spinor 1-forms (the gravitinos) having the same chirality:

\[ \Gamma_{11} \psi_A = -\psi_A \quad ; \quad C \bar{\psi}_A = \psi_A \quad , \quad A = 1, 2 \]  

(A.9)

In terms of these we define the complex doublet of gravitinos:

\[
\begin{pmatrix}
\psi^* \\
\psi
\end{pmatrix} = \mathcal{C}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\]  

(A.10)

and we introduce the following doublet made by a complex 3-form current and its complex conjugate:

\[
\mathbb{J}^x_{SU} = \begin{pmatrix}
\bar{\psi}^* \Gamma_a \psi & V^a \\
\bar{\psi} \Gamma_a \psi^* & V^a
\end{pmatrix} (x = \pm) \tag{A.11}
\]

By means of an inverse Cayley transformation we get a doublet of real currents:

\[
\mathbb{J}^A_{SL} = \mathcal{C}^{-1} A^x \mathbb{J}^x_{SU} = \begin{pmatrix}
i(\bar{\psi}_1 \Gamma_a \psi_1 - \bar{\psi}_2 \Gamma_a \psi_2) & V^a \\
-2i\bar{\psi}_1 \Gamma_a \psi_2 & V^a
\end{pmatrix} = d^{A[BC} i \bar{\psi}_B \Gamma_a \psi_C \wedge V^a \tag{A.12}
\]

The formula (A.12) is understood as follows. Recall that the fermions transform only with respect to the isotropy subgroup \( H = U(1) \sim O(2) \) of the scalar coset (are neutral under \( G \)) and that all irreducible representations of \( O(2) \) are 2-dimensional. The coefficients \( d^{A[BC} \) defined by equation (A.12) are the Clebsch Gordon coefficients that extract the doublet of helicity \( s = 2 \) from the tensor product of two representations of helicity \( s = 1 \). Relying on these notations we can write the type IIB curvature definitions in two equivalent bases related by a Cayley transformation:

1. the complex \( SU(1, 1) \) basis originally used by Castellani and Pesando [27]
2. the real \( SL(2, \mathbb{R}) \), introduced here and best suited for comparison with string theory massless modes.

The curvatures of the free differential algebra in the complex basis:

\[
\begin{align*}
R^a &= D V^a - i \bar{\psi} \wedge \Gamma_a \psi \\
R^{ab} &= d \omega^{ab} - \omega^{ac} \wedge \omega^{bd} \eta_{cd} \\
\rho &= D \psi \equiv d \psi - \frac{1}{4} \omega^{ab} \wedge \Gamma_{ab} \psi - \frac{1}{2} i Q \wedge \psi \\
\mathcal{H}^a_{[3]} &= \sqrt{2} d A_{[2]}^a + 2 i \Lambda^a_{\pm} \bar{\psi} \wedge \Gamma_a \psi^* \wedge V^a + 2 i \Lambda^a_{\pm} \bar{\psi} \wedge \Gamma_a \bar{\psi} \wedge V^a \\
\mathcal{F}^{[5]} &= d C_{[4]} + \frac{1}{16} i \epsilon_{\alpha \beta} \sqrt{2} A_{[2]}^\alpha \wedge \mathcal{H}^\beta_{[3]} + \frac{1}{8} \bar{\psi} \wedge \Gamma_{abc} \psi \wedge V^a \wedge V^b \wedge V^c \\
& \quad + \frac{1}{8} i \epsilon_{\alpha \beta} \sqrt{2} A_{[2]}^\alpha \wedge \left( \Lambda_+ \bar{\psi} \Gamma_a \psi^* + \Lambda_- \bar{\psi} \Gamma_a \bar{\psi} \right) \wedge V^a \\
D \lambda &= d \lambda + \frac{1}{4} \omega^{ab} \Gamma_{ab} \lambda - \frac{i}{2} Q \lambda \\
D \Lambda^\alpha_{\pm} &= d \Lambda^\alpha_{\pm} \mp i Q \Lambda^\alpha_{\pm}.
\end{align*}
\]

\[ ^6 \text{Comparing with the original paper by Castellani and Pesando, note that we have changed the normalization:} \\
A_\alpha \rightarrow \sqrt{2} A_\alpha \text{ and } B^\alpha_{\mu \nu \rho} \rightarrow 6 C^\alpha_{\mu \nu \rho} \text{ so that eventually the 4–form } C_{[4]} \text{ will be identified with that used in Polchinski's book [13].} \]
alternatively using the real \( SL(2, \mathbb{R}) \) basis we can write:

The curvatures of the free differential algebra in the real basis

\[
R^a = D\bar{\psi}_a - i\bar{\psi}_a \wedge \Gamma^a \psi_A \\
R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{bd} + \bar{\psi}_A \wedge \Gamma^{ab} \psi_A + \frac{1}{2} Q \wedge \epsilon_{AB} \psi_B \\
\rho_A = D\psi_A \equiv d\psi_A - \frac{1}{2} \bar{\psi} \wedge \Gamma^{ab} \psi_A + \frac{1}{2} Q \wedge \epsilon_{AB} \psi_B \\
H^A_{[3]} = dA^A_{[3]} + i L^A \omega^{[BC]} \bar{\psi}_B \wedge \Gamma^a \psi_C \wedge V^a \\
F[A] = dC[A] - \frac{1}{16} \epsilon_{\Lambda \Sigma} A^A_{[2]} \wedge H^\Lambda_{[3]} + i \frac{1}{2} \bar{\psi}_A \wedge \Gamma^{abc} \psi_B \epsilon^{AB} V^a \wedge V^b \wedge V^c + \\
+ i \frac{1}{4} \epsilon_{\Lambda \Sigma} A^A_{[2]} \wedge \Lambda^\Lambda_{[3]} \wedge V^2 \\
D\lambda = d\lambda - \frac{1}{4} \omega^{ab} \Gamma_{ab} \lambda - \frac{3}{2} i Q \lambda \\
D\Lambda_{\pm} = d\Lambda_{\pm} + \epsilon_{AB} Q L^A_{\pm}.
\]

In the above formulae, eq.s (A.18) and (A.24) define the covariant derivative of the coset representative of the scalar coset in the \( SU(1,1) \) and \( SL(2, \mathbb{R}) \) basis respectively. They follow from the Maurer Cartan equations of \( G/H \).

Next, using the results of Castellani and Pesando [27], we can write the rheonomic parametrizations of the curvatures (A.13-A.18) (alternatively (A.19-A.24)) that determine the supersymmetry transformation rules of all the fields. Prior to that, in order to make contact with superstring massless modes as normalized in Polchinski’s book, it is convenient to introduce the following identifications:

\[
A^1_{[2]} = 2 \sqrt{2} B_{[2]} ; \quad A^2_{[2]} = 2 \sqrt{2} C_{[2]} \\
\]

where \( B_{[2]} \) is the 2–form gauge field of the Neveu-Schwarz sector that couples to ordinary fundamental strings, while \( C_{[2]} \) is the 2–form of the Ramond-Ramond sector that couples to \( D1 \)–branes. For simplicity we write the rheonomic parametrizations only in the complex basis and we disregard the bilinear fermionic terms calculated by Castellani and Pesando. We have:

\[
R^a = 0 \\
\rho = \rho_{ab} V^a \wedge V^b + \frac{5}{16} i \Gamma^{a_1-a_4} \psi V^{a_5} \left( F_{a_1-a_5} + \frac{1}{5!} \epsilon_{a_1-a_5} F_{a_5-a_10} \right) \\
+ \frac{1}{32} \left( -\Gamma^{a_1-a_4} \psi V_{\dot{a}_4} + 9 \Gamma^{a_3} a_4 \psi^{a_5} V^{a_5} \right) \Lambda^\alpha_{\dot{a}_2-\dot{a}_4} \epsilon_{a \beta} \Lambda^\alpha_{\dot{a}_2-\dot{a}_4} \epsilon_{a \beta} \wedge \text{fermion bilinears} \\
\]

\[
H^\alpha_{[3]} = H^\alpha_{abc} V^a \wedge V^b \wedge V^c + \Lambda^\alpha_\psi \bar{\psi} \Gamma_{abc} \lambda V^a \wedge V^b \wedge V^c + \Lambda^\alpha_\bar{\psi} \Gamma_{abc} \lambda V^a \wedge V^b \\
F[A] = F_{a_1-a_5} V^{a_1} \wedge \ldots \wedge V^{a_5} \\
D\lambda = D\lambda V^a + iP_{a} \Gamma^{a} \psi^* - \frac{1}{8} i \Gamma^{a_1-a_4} \epsilon_{a \beta} \Lambda^\alpha_\psi \bar{\psi} \Gamma_{abc} \lambda V^a \wedge V^b
\]
\[ \mathcal{D} \Lambda^a_+ = \Lambda^a_+ P_a^\omega V^2 + \Lambda^a_+ \bar{\psi}^a \lambda \] (A.31)

\[ \mathcal{D} \Lambda^a = \Lambda^a P^a_\omega V^2 + \Lambda^a \bar{\psi} \lambda^a \] (A.32)

\[ R^{ab} = R^{ab}_{\varphi} V^2 \wedge V^2 + \text{fermionic terms} \] (A.33)

A.3 The bosonic field equations and the standard form of the bosonic action

Following Castellani and Pesando we write next the general form of the bosonic field equations and using the identifications of eqs. (A.25), (A.3), (A.7) we reduce them to those following from a standard supergravity action for \( p \)-branes. As discussed in the literature \([\S, Z, \mathcal{W}]\), the standard form of a supergravity action truncated to the graviton, the dilaton and the \( n_i = p_i + 2 \) field strengths that can couple to the world–volume actions of \( p_i \)-branes is as follows:

\[
\mathcal{A}_{\text{standard}} = \int d^D x \det V \left[ -2 R [\omega(V)] - \frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi \right] - \int \sum_i \frac{1}{2} \exp [-a_i \varphi] F_{[n_i]} \wedge \ast F_{[n_i]} + \text{Chern Simons couplings} \] (A.34)

where \( R = R^{ab}_{\varphi} \) is the scalar curvature in the geometric normalizations always adopted in the rheonomic framework \([\S]\), and \( a_i \) are characteristic exponents dictated by the structure of supergravity and playing an essential role in dictating the properties of \( p \)-brane solutions. Furthermore in eq. (A.34) we have defined:

\[
| F_{[n]} |^2 = g_{\mu_1 \nu_1} \ldots g_{\mu_n \nu_n} F_{\mu_1 \ldots \mu_n} F_{\nu_1 \ldots \nu_n} \] (A.35)

\[
F_{[n]} = F_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} \] (A.36)

and we have not made explicit the Chern Simons couplings between field strengths that are on the other hand essential in the derivation of the exact field equations.

Introducing the definition of the dressed 3–form field strengths:

\[
\hat{H}_{[a_1 a_2 a_3]} = \epsilon_{\alpha \beta} \Lambda^\alpha_+ H_{a_1 a_2 a_3} \ ; \ \hat{H}_{A[a_1 a_2 a_3]} = \epsilon_{A \Sigma} \Lambda^\Sigma A H_{a_1 a_2 a_3} \] (A.37)

it was shown by Castellani and Pesando \([27]\) that the exact bosonic field equations implied by the closure of the supersymmetry algebra have the following form:

\[
R^{ab}_{\varphi} - \frac{1}{2} \delta^2_R R^{ab}_{\varphi} = -75 \left( F_{[a_1 a_2 a_3]} F^{[a_1 a_2 a_3]} - \frac{1}{10} \delta^2 P_{a_1 a_2 a_3} \right) - \frac{9}{16} \left( \tilde{H}_{a_1 a_2 a_3} - \tilde{H}_{a_1 a_2 a_3} + \tilde{H}_{a_1 a_2 a_3} - \tilde{H}_{a_1 a_2 a_3} \right) - \frac{1}{2} \left( P^a_2 P^a_2 + P^a_2 P^a_2 - \delta^2 P^a_2 P^a_2 \right) \] (A.38)

\[
\mathcal{D}^2 P^a_2 = -\frac{3}{8} \hat{H}_{a_1 a_2 a_3} \hat{H}_{a_1 a_2 a_3} \] (A.39)
At the purely bosonic level (i.e. disregarding all fermionic contributions), using the solvable parametrization \((A.3)\) of the \(\text{SL}(2, \mathbb{R})/\text{O}(2)\) coset and inserting the identifications \((A.25)\) we obtain the following expression for the dressed \(3\)-forms in terms of string massless fields denoted \(\text{NS}\) or \(\text{RR}\) according to their origin in the Neveu Schwarz or Ramond Ramond sector:

\[
\hat{\mathcal{H}}_\pm = \pm 2 e^{-\varphi/2} F_{[3]}^{\text{NS}} + i 2 e^{\varphi/2} F_{[3]}^{\text{RR}}
\]

\[
P = \frac{1}{2} d\varphi - i \frac{1}{2} e^{\varphi} F_{[1]}^{\text{RR}}
\]

\[
F_{[3]}^{\text{NS}} = dB_{[2]}
\]

\[
F_{[1]}^{\text{RR}} = dC_{[0]}
\]

\[
F_{[3]}^{\text{RR}} = (dC_{[2]} - C_{[0]} dB_{[2]})
\]

\[
F_{[5]}^{\text{RR}} = F_{[5]} = dC_{[4]} - \frac{1}{2} (B_{[2]} \wedge C_{[2]} - C_{[2]} \wedge dB_{[2]})
\]

\[(A.43)\]

Using the convention \((C.6)\) for the Hodge dual of \(\ell\)-forms in space–time dimensions \(D\), the field equations \((A.39-A.42)\) can be written in more compact form. Let us begin with the scalar equation \((A.39)\), it becomes:

\[
d(*)P - 2 i Q \wedge P + \frac{1}{16} \hat{\mathcal{H}}_+ \wedge * \hat{\mathcal{H}}_+ = 0
\]

\[(A.44)\]

and separating its real from imaginary part we obtain the two equations:

\[
d \star d\varphi - e^{2\varphi} F_{[1]}^{\text{RR}} \wedge F_{[1]}^{\text{RR}} = -\frac{1}{2} \left( e^{-\varphi} F_{[3]}^{\text{NS}} \wedge * F_{[3]}^{\text{NS}} - e^{\varphi} F_{[3]}^{\text{RR}} \wedge * F_{[3]}^{\text{RR}} \right)
\]

\[
d \left( e^{2\varphi} * F_{[1]}^{\text{RR}} \right) = -e^{\varphi} F_{[3]}^{\text{NS}} \wedge * F_{[3]}^{\text{RR}}
\]

\[(A.45)\]

\[(A.46)\]

Considering next the \(3\)-form eq. \((A.40)\) it can be rewritten as:

\[
d \star \hat{\mathcal{H}}_+ - i Q \wedge * \hat{\mathcal{H}}_+ = i F_{[5]} \wedge \hat{\mathcal{H}}_+ - P \wedge * \hat{\mathcal{H}}_-
\]

\[(A.47)\]

Separating the real and imaginary parts of eq.(A.47) we obtain:

\[
d \left( e^{-\varphi} \star F_{[3]}^{\text{NS}} \right) + e^{\varphi} F_{[1]}^{\text{RR}} \wedge * F_{[3]}^{\text{RR}} = -F_{[3]}^{\text{RR}} \wedge F_{[5]}^{\text{RR}}
\]

\[
d \left( e^{\varphi} \star F_{[3]}^{\text{RR}} \right) = -F_{[5]}^{\text{RR}} \wedge F_{[5]}^{\text{NS}}
\]

\[(A.48)\]

Finally the equation for the Ramond-Ramond \(5\)-form, namely equation \((A.42)\) is rewritten as follows:

\[
d \star F_{[5]}^{\text{RR}} = i \frac{1}{8} \hat{\mathcal{H}}_+ \wedge \hat{\mathcal{H}}_- = -F_{[3]}^{\text{NS}} \wedge F_{[5]}^{\text{RR}}
\]

\[(A.49)\]
B The $\kappa$-supersymmetry projector

Let us begin with property [a] and consider the ansatz in eq. (4.26). By direct calculation we find:

$$\omega^2_{[4]} = \alpha^2 (4!)^2$$
$$\omega^2_{[2]} = \frac{(\alpha^2)^2 \omega_0 \omega_{[4]}}{3! \alpha_0 \alpha_4} + 8(\alpha^2)^2 Tr(\hat{F}^2)$$

(B.1)

so that we get:

$$\Gamma^2 = \frac{1}{N^2} \left[ (4! \alpha_4)^2 + \omega_0 \omega_{[4]} \left( \frac{(\alpha^2)^2}{3! \alpha_0 \alpha_4} + 2 \right) + 8(\alpha^2)^2 Tr(\hat{F}^2) \right]$$

(B.2)

so we obtain $\Gamma^2 = 1$ if the normalization factor $N$ is chosen as in eq. (4.27) and if the coefficients are chosen as in eq. (4.28). This conclusion is easily reached using the identity (4.30) of the main text.

Let us now turn to property [b], namely to the condition

$$\Gamma A_k = A_k$$

(B.3)

To implement it we need to calculate some $\gamma$ matrix products:

$$\omega_{[4]} \gamma_k = \frac{1}{6} \tilde{\gamma}_k$$
$$\omega_{[4]} \tilde{\gamma}_k = 6 \gamma_k$$
$$\omega_{[4]} \Pi_k = -\frac{1}{2} \tilde{F}^{ij} \gamma_{ijk} \equiv -\frac{1}{2} \tilde{\Delta}_k$$
$$\omega_{[4]} \Pi_k = -\frac{1}{2} \tilde{F}^{ij} \gamma_{ijk} \equiv -\frac{1}{2} \Delta_k$$

(B.4)

$$\omega_{[2]} \gamma_k = \frac{i}{2} \Delta_k + i \Pi_k$$
$$\omega_{[2]} \tilde{\gamma}_k = -3i \Delta_k - 6i \Pi_k$$
$$\omega_{[2]} \Pi_k = -\frac{i}{6} (\tilde{F}^2)_{kl} \tilde{\gamma}^l - i \omega_0 \gamma_k$$
$$\omega_{[2]} \Pi_k = i (\tilde{F}^2)_{kl} \gamma^l + \frac{i}{6} \omega_0 \tilde{\gamma}_k$$

(B.5)

Now we impose equation (B.3) and we obtain the following equations.

- The contributions from $\Delta_k$ and $\tilde{\Delta}_k$ are:

$$\Delta^m h_{mk} \left( \frac{i}{2} f_4 + 3 f_2 \right) \otimes \sigma_1 = 0$$
$$\tilde{\Delta}_k \left( -\frac{i}{2} f_3 + \frac{i}{2} f_1 \right) \otimes \sigma_2 = 0$$

(B.6)

- For the contributions with $\gamma_k$ we have two equations, one proportional to $\sigma_3$ and one proportional to $\Pi$, namely:

$$\gamma_k \omega_0 (f_1 - f_3) \otimes \sigma_3 = 0$$

(B.7)
and
\[
\frac{1}{N} \left( 6f_2\mathbb{I}_{4\times 4} + i f_4 \tilde{\mathcal{F}}^2 \right) h \otimes \mathbb{I}_{2\times 2} = f_1 \mathbb{I}_{4\times 4} \otimes \mathbb{I}_{2\times 2} \tag{B.8}
\]

For:
\[
6f_2 = f_1 \\
f_1 = -i f_4 \tag{B.9}
\]

and using the property (4.31) we obtain:
\[
N^{-1} \left( \mathbb{I} - \tilde{\mathcal{F}}^2 \right) h = \mathbb{I} \\
N^{-1} \left( \mathbb{I} - \tilde{\mathcal{F}}^2 \right) N^{-1} \left( \mathbb{I} - \hat{\mathcal{F}}^2 \right) = \mathbb{I} \\
\left( \mathbb{I} - \hat{\mathcal{F}}^2 - \tilde{\mathcal{F}}^2 + \tilde{\mathcal{F}}^2 \hat{\mathcal{F}}^2 \right) = N^2 \\
\left[ \mathbb{I} - \frac{1}{2} Tr(\hat{\mathcal{F}}^2) + \left( \frac{1}{8} F_{ij} F_{kl} \epsilon^{ijkl} \right) \mathbb{I} \right] = N^2 \tag{B.10}
\]

• For the contributions with \( \tilde{\gamma}_k \) we get the following equations:
\[
\tilde{\gamma}_m h^m_k \omega[0] \left( f_2 + \frac{i}{6} f_4 \right) \otimes \mathbb{I}_{2\times 2} = 0 \\
N^{-1} \tilde{\gamma}_m \left[ \frac{1}{6} f_1 \delta^m_k - \frac{1}{6} f_3 (\hat{\mathcal{F}}^2)^m_k \right] \otimes \sigma_3 = f_2 \tilde{\gamma}_m h^m_k \otimes \sigma_3 \tag{B.11}
\]

Then if:
\[
f_2 = -\frac{i}{6} f_4 \\
f_1 = f_3 \\
f_1 = 6f_2 \tag{B.12}
\]
we obtain that the second of equations (B.11) as a matrix equation becomes:
\[
N^{-1} [\mathbb{I} - (\hat{\mathcal{F}}^2)] = h \tag{B.13}
\]
and just coincides with the solution (3.51) for the auxiliary field \( h \) in terms of the physical ones.

• Now we consider \( \Pi \) and \( \Pi \).

The equation proportional to \( \sigma_1 \) is:
\[
\alpha \omega[0] \Pi^m h_{mk} + \beta \Pi^m h_{mk} = N \gamma \Pi^k \\
\alpha \omega[0] \tilde{\mathcal{F}} \gamma_\ell \ h_{mk} \gamma_\ell + \beta \hat{\mathcal{F}}^l h_{mk} \gamma_\ell = \gamma N \hat{\mathcal{F}}^l k \gamma_\ell \tag{B.14}
\]

in matrix form we have:
\[
\alpha \omega[0] (\tilde{\mathcal{F}} h) + \beta (\hat{\mathcal{F}} h) = \gamma N \hat{\mathcal{F}}
\]
\[
\alpha \omega[0] \hat{F}^2 \left[ \mathbb{I} - (\hat{F}^2) \right] \mathbb{N}^{-1} + \beta \hat{F}^2 \left[ \mathbb{I} - (\hat{F}^2) \right] \mathbb{N}^{-1} = \gamma \mathbb{N} \hat{F}
\]

\[
\alpha \omega[0] \hat{F}^2 \left[ \mathbb{I} - (\hat{F}^2) \right] + \beta \hat{F}^2 \left[ \mathbb{I} - (\hat{F}^2) \right] = \gamma \mathbb{N}^2 \hat{F}
\]

\[
\alpha \omega[0] \hat{F}^2 \left[ \mathbb{I} - (\hat{F}^2) \right] + \beta \hat{F}^2 \left[ \mathbb{I} - (\hat{F}^2) \right] = \gamma [1 - \frac{1}{2} \text{Tr}(\hat{F}^2) + \omega^2] \hat{F}
\]

\[
\alpha \omega[0] \hat{F} - \alpha \omega[0] \left( \hat{F} \hat{F} - \beta \hat{F}^2 \hat{F} \right) + \beta \hat{F}^2 \hat{F} = \gamma \hat{F} - \frac{\gamma}{2} \text{Tr}(\hat{F}^2) \hat{F} + \gamma \omega[0] \hat{F}
\]  \hspace{1cm} (B.15)

if:

\[
\beta = \gamma \hspace{1cm} (B.16)
\]

and using (4.31), than (B.15) become:

\[
\alpha \omega[0] \hat{F} + \alpha \omega[0] \hat{F} - \beta \hat{F}^2 \hat{F} = -\frac{\gamma}{2} \text{Tr}(\hat{F}^2) \hat{F} + \gamma \omega[0] \hat{F}
\]  \hspace{1cm} (B.17)

if:

\[
\alpha = \gamma \hspace{1cm} (B.18)
\]

\[
\alpha \omega[0] \hat{F} - \beta \hat{F}^2 \hat{F} = -\frac{\gamma}{2} \text{Tr}(\hat{F}^2) \hat{F}
\]

\[
\alpha \omega[0] \hat{F} - \alpha \hat{F}^2 \hat{F} = -\alpha (\hat{F}^2 + \hat{F}^2 \hat{F})
\]

\[
\alpha \omega[0] \hat{F} = -\alpha \hat{F}^2 \hat{F}
\]  \hspace{1cm} (B.19)

and it is correct by (4.31).

The equation proportional to \(\sigma_2\) is:

\[
\mu \omega[0] \Pi_k + \nu \Pi_k = N \rho \Pi^m h_{mk}
\]

\[
\mu \omega[0] \hat{F}_{lk} \gamma^l + \nu \hat{F}_{lk} \gamma^l = N \rho \hat{F}_{lm}^m h_{mk} \gamma^l
\]

\[
\mu \omega[0] \hat{F}_{lk} + \nu \hat{F}_{lk} = N \rho \hat{F}_{lm} \left( \delta^m_k - [\hat{F}^2]_m^k \right) N^{-1}
\]  \hspace{1cm} (B.20)

for:

\[
\nu = \rho
\]

\[
\mu = \rho
\]  \hspace{1cm} (B.21)

we obtain the first of the relations (4.31).

Where:

\[
\alpha = -if_4 \hspace{1cm} \beta = 6if_2 \hspace{1cm} \gamma = f_3
\]

\[
\mu = if_3 \hspace{1cm} \nu = if_1 \hspace{1cm} \rho = f_4
\]  \hspace{1cm} (B.22)

Using the fact that \(a_5 = \frac{3}{4}i\) and (4.18) we have that (B.6), (B.7), (B.9), (B.12), and (B.16), (B.18), (B.21) are automatically satisfied. This concludes the proof of property [b] and hence of \(\kappa\) supersymmetry.
C Notations and Conventions

General adopted notations for first order actions are the following ones:

\[
\begin{align*}
    d & = \text{dimension of the world-volume } \mathcal{W}_d \\
    D & = \text{dimension of the bulk space–time } \mathcal{M}_D \\
    V^a & = \text{vielbein 1–form of bulk space–time} \\
    \Pi^a_i & = D \times d \text{ matrix. } 0–\text{form auxiliary field} \\
    h^{ij} & = d \times d \text{ symmetric matrix. } 0–\text{form auxiliary field} \\
    e^\ell & = \text{vielbein 1–form of the world-volume} \\
    \eta_{ab} & = \text{flat metric on the bulk} \\
    \eta^{ij} & = \text{flat metric on the world–volume} \\
\end{align*}
\]

The supersymmetric formulation of type IIB supergravity we rely on is that of Castellani and Pesando [27] that uses the rheonomy approach [5]. Hence, as it is customary in all the rheonomy constructions, the adopted signature of space–time is the mostly minus signature:

\[
\eta_{ab} = \text{diag}\{+,−,\ldots,−\} = \text{flat metric on the bulk} \\
\eta^{ij} = \text{flat metric on the world–volume}
\]

(C.1)

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\eta_{ab} = \text{diag}\{+,−,\ldots,−\} = \text{flat metric on the bulk} \\
\eta^{ij} = \text{flat metric on the world–volume}
\]

(C.2)

The supersymmetric formulation of type IIB supergravity we rely on is that of Castellani and Pesando [27] that uses the rheonomy approach [5]. Hence, as it is customary in all the rheonomy constructions, the adopted signature of space–time is the mostly minus signature:

\[
\eta_{ab} = \text{diag}\{+,−,\ldots,−\} = \text{flat metric on the bulk} \\
\eta^{ij} = \text{flat metric on the world–volume}
\]

(C.2)

The index conventions are the following ones:

\[
\begin{align*}
    a, b, c, \ldots & = 0, 1, 2, \ldots, 9 \quad \text{Lorentz flat indices in } D = 10 \\
    i, j, k, \ldots & = 0, \ldots, d \quad \text{Lorentz flat indices on the world-volume} \\
    \alpha, \beta, \ldots & = 1, 2 \quad \text{SU}(1,1) \text{ doublet indices} \\
    A, B, C, \ldots & = 1, 2 \quad \text{O}(2) \text{ indices for the scalar coset} \\
    \Lambda, \Sigma, \Gamma, \ldots & = 1, 2 \quad \text{SL}(2, \mathbb{R}) \text{ doublet indices}
\end{align*}
\]

(C.3)

(C.4)

For the gamma matrices our conventions are as follows:

\[
\left\{ \Gamma^a, \Gamma^b \right\} = 2 \eta^{ab}
\]

(C.5)

The convention for constructing the dual of an \( \ell \)-form \( \omega \) in \( D \)-dimensions is the following:

\[
\omega = \omega_{i_1 \ldots i_{\ell}} \ V^{i_1} \wedge \ldots \wedge V^{i_{\ell}} \quad \Leftrightarrow \quad * \omega = \frac{1}{(D - \ell)!} \epsilon_{a_1 \ldots a_D \ b_1 \ldots b_\ell} \omega^{b_1 \ldots b_\ell} \ V^{a_1} \wedge \ldots \wedge V^{a_{D-\ell}}
\]

(C.6)

Note that we also use \( \ell \)-form components with strength one: \( \omega = \omega_{i_1 \ldots i_{\ell}} \ V^{i_1} \wedge \ldots \wedge V^{i_{\ell}} \) and not with strength \( \ell! \) as it would be the case if we were to write \( \omega = \frac{1}{\ell!} \omega_{i_1 \ldots i_{\ell}} \ V^{i_1} \wedge \ldots \wedge V^{i_{\ell}} \). When it is more appropriate to use curved rather than flat indices then the convention for Hodge duality is summarized by the formula:

\[
* (dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}) = \frac{\sqrt{-\det(g)}}{(10 - n)!} G^{\mu_1 \nu_1} \ldots G^{\mu_n \nu_n} \epsilon_{\rho_1 \ldots \rho_{10-n} \nu_1 \ldots \nu_n} \ dx^{\rho_1} \wedge \ldots \wedge dx^{\rho_{10-n}}
\]

(C.7)
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