Hamilton-Jacobi quantization of the finite dimensional systems with constraints

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Abstract

The Hamiltonian treatment of constrained systems in Güler’s formalism leads us to the total differential equations in many variables. These equations are integrable if the corresponding system of partial differential equations is a Jacobi system. The main aim of this paper is to investigate the quantization of the finite dimensional systems with constraints using the canonical formalism introduced by Güler. This approach is applied for two systems with constraints and the results are in agreement with those obtained by Dirac’s canonical quantization method and path integral quantization method.

1 Introduction

The canonical formulation\cite{1,2,3,4,5,6} gives the set of Hamilton-Jacobi partial-differential equation as

\begin{equation}
H'_{\alpha}(t_\beta, q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial t_\alpha}) = 0, \alpha, \beta = 0, n - r + 1, \cdots, n, a = 1, \cdots, n - r, \tag{1}
\end{equation}

where

\begin{equation}
H'_{\alpha} = H_\alpha(t_\beta, q_\alpha, p_\alpha) + p_\alpha \tag{2}
\end{equation}

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The equations of motion are obtained as total differential equations in many variables as follows

\[
\begin{align*}
dq_\alpha &= \frac{\partial H_\alpha'}{\partial p_\alpha} dt_\alpha, \\
dp_\alpha &= -\frac{\partial H_\alpha'}{\partial q_\alpha} dt_\alpha, \\
dp_\mu &= -\frac{\partial H_\mu'}{\partial t_\mu} dt_\mu, \mu = 1, \ldots, r
\end{align*}
\]

where \( z = S(t_\alpha, q_\alpha) \). The set of equations (4,5) is integrable if

\[
dH_\alpha' = 0, \quad dH_\mu' = 0, \quad \mu = 1, \ldots, r
\]

If conditions (6) are not satisfied identically, one should consider them as new constraints and again should test the consistency conditions. Thus repeating this procedure one may obtain a new set of conditions.

The main aim of this paper is to investigate the quantization of finite dimensional systems using Güler’s formalism. We test our formalism on two systems with first and second class constraints respectively.

The plan of this paper is the following:

A brief information of the Güler’s formalism is given in Sect. 1. In Sec. 2 quantization of systems with constraints is investigated. The examples are worked out in Sec. 3 In Sec. 4 conclusions are presented.

2 Quantization of the systems with constraints in Güler’s formalism

Let us suppose that for a finite dimensional system with constraints we found all independent hamiltonians \( H_\mu' \) using the calculus of variations. Because all the hamiltonians \( H_\mu' \) are constraints we will use Dirac’s procedure of quantization. We have

\[
H_\mu' \Psi = 0, \mu = 1, \ldots, r
\]

where \( \Psi \) is the wave function. The consistency conditions are

\[
[H_\mu', H_\nu'] \Psi = 0, \mu, \nu = 1, \ldots, r
\]

If for a finite dimensional system the hamiltonians \( H_\mu' \) satisfy

\[
[H_\mu', H_\nu'] = C_{\mu\nu} H_\alpha'
\]
the system has first class constraints in the Dirac’s classification.

On the other hand if

\[ [H'_\mu, H'_\nu] = C_{\mu\nu} \]  

(10)

where \( C_{\mu\nu} \) do not depend of \( q_i \) and \( p_i \) then from(9) we will be lead naturally to Dirac’s brackets and the canonical quantization will be performed taking Dirac’s brackets into commutators.

\textbf{Güler's formalism give us and the action when all hamiltonians} \( H'_\mu \) \textit{are in involution. Because in this formalism we work from the beginning in the extended space we suppose that variables} \( t_\alpha \) \textit{depend of} \( \tau \). \textit{Here} \( \tau \) \textit{is canonical conjugate to} \( p_0 \). \textit{We propose the following expression for the action}

\[ z = \int (-H_a + p_0 \frac{\partial H'_a}{\partial p_a}) \dot{t}_a d\tau \]  

(11)

where \( \dot{t}_a = \frac{dt_a}{d\tau} \).

\textbf{If we are able, for a given finite system with constraints, to find the independent hamiltonians} \( H'_\mu \) \textit{in involution then we can perform the quantization of this system using path integral quantization method with the action gives as in (11).}

\section{Examples}

\subsection{A system with first class constraints}

Consider the following lagrangian(for more details see[2])

\[ 2L = a_{ij} \dot{q}_i \dot{q}_j + 2b \dot{q}_2 - 2c, i, j = 1, 2, 3 \]  

(12)

The generalized momenta read as

\[ p_1 = a_1 \dot{q}_1, p_2 = a_2 (\dot{q}_3 - \dot{q}_2) + b, p_3 = a_2 (\dot{q}_3 - \dot{q}_2) \]  

(13)

Then we have two hamiltonians in the \textbf{Güler’s formalism}

\[ H'_0 = p_0 + \frac{1}{2}(\frac{p_1^2}{a_1} - \frac{p_3^2}{a_2}) + c, H'_2 = p_2 + p_3 - b \]  

(14)

Here \( a, b \) and \( c \) are constants.

At this stage we have two ways for quantization of the system presented above.

Because we have two constraints \( H'_0 \) and \( H'_2 \) Dirac’s canonical formalism for the systems with constraints will be applied[7].

Then we have

\[ H'_2 \Psi = 0, H'_0 \Psi = 0 \]  

(15)

where \( \Psi \) is the wave function.
The consistency condition gives the following commutation relation

\[ [H'_2, H'_0] \Psi = 0 \]  

(16)

and it is automatically satisfied because the Hamiltonians \( H'_2 \) and \( H'_0 \) commute.

We found that a solution of eq. (15) has the following form

\[ \Psi = (-b^2 + 2c)e^{bq_1}[\sin q_1 + \cos q_1] \]  

(17)

For the path integral quantization we need the action \( z \). Using G"uler's formalism we found

\[ dz = (-c + \frac{p_1^2}{2a_1} - \frac{p_2^2}{2a_2} + b\dot{q}_2)d\tau \]  

(18)

or

\[ z = \int (-c + \frac{p_1^2}{2a_1} - \frac{p_2^2}{2a_2} + b\dot{q}_2)d\tau \]  

(19)

We know that for a system with \( n \) degrees of freedom with \( r \) first class constraints \( \phi^a \) the path integral representation is given as [for more details see Ref. [8]]

\[ <q' \mid \exp[-i(t' - t)\hat{H}_0] \mid q> = \int \prod_t d\mu(q_j, p_j)e^{i\{\int_{-\infty}^{+\infty} dt(p_j\dot{q}_j - H_0)\}}, \]

\[ j = 1, \ldots, n \]  

(20)

where the measure of integration is given as

\[ d\mu(q,p) = det \{ \psi^a, \psi^b \} | \prod_{a=1}^{r} \delta(\chi^a)\delta(\phi^a) \prod_{j=1}^{n} dq^j dp_j \]  

(21)

and \( \chi^a \) are \( r \) gauge constraints.

If we perform now the usual path integral quantization using (21) for the system (12), after imposing the gauge condition and integrate over \( q_2 \), we get the action (19) when \( \tau \) is replaced by \( t \).

### 3.2 A system with second class constraints

Let us consider the Lagrangian [4]

\[ L = \frac{1}{2}a_1\dot{q}_1^2 - \frac{1}{2}a_2(\dot{q}_2^2 - 2\dot{q}_2\dot{q}_3 + q_3^2) + b\dot{q}_2 - c \]  

(22)

Here \( a_1, a_2, b, c \) are functions of \( q_1, q_2, q_3, t \). Let us specify the functions \( a_1, a_2, b, c \) as

\[ a_1 = 1, a_2 = \frac{1}{2}, b = q_1 + q_3, c = q_1 + q_2 + q_3^2. \]

From (22) we found two Hamiltonians \( H'_0 \) and \( H'_2 \) as

\[ H'_0 = p_0 + \frac{1}{2}(p_1^2 - 2p_3^3) + q_1 + q_2 + q_3^2 = 0 \]  

(23)
\[ H'_2 = p_2 + p_3 - q_1 - q_3 = 0 \] (24)

If we impose the variations of (23) and (24) to be zero we get a new independent hamiltonian
\[ H'_1 = -p_1 + 2p_3 - 2q_3 - 1 = 0 \] (25)

Then we have three independent hamiltonians \( H'_0, H'_1 \) and \( H'_2 \).

Dirac’s method of quantization give us the following relations [7]
\[ H'_0 \Psi = 0, H'_1 \Psi = 0, H'_2 \Psi = 0 \] (26)

and the consistency conditions
\[
[H'_0, H'_1] \Psi = 0 \tag{27}
\]
\[
[H'_0, H'_2] \Psi = 0 \tag{28}
\]
\[
[H'_1, H'_2] \Psi = 0 \tag{29}
\]

Here \( \Psi \) is the wave function.

Because
\[
[H'_1, H'_2] = 1 \tag{30}
\]

and the consistency condition (29) is not satisfied. In the Dirac’s classification of constraints \( H'_1 \) and \( H'_2 \) are second class constraints. At this stage we will introduce the Dirac’s brackets for our system.

Some calculations gives the following form
\[
\{F, G\}_{D.B.} = \{F, G\} + \{F, H'_2\}\{H'_1, G\} - \{F, H'_1\}\{H'_2, G\} \tag{31}
\]

and we can perform the canonical quantization taking Dirac’s brackets into commutators.

Now we would like to find the action using Güler’s formalism. Because the hamiltonians \( H'_0, H'_1, H'_2 \) are not in involution we extend the phase-space with another pairs of conjugate variables \((\lambda, p_\lambda)\). The new hamiltonians \( \tau'_1, \tau'_2 \) and \( \tau'_3 \) are in involution and have the following expressions
\[
\tau'_1 = p_0 + \frac{1}{2}(p_1^2 - 2p_3^3) + q_1 + q_2 + q_3 + \lambda(-1 - 4p_3 + 4q_3)
+ p_\lambda(p_1 - 2p_3 - 1 + 2q_3) - \frac{1}{2}p_\lambda^2 = p_0 + \tau_1 \tag{32}
\]
\[
\tau'_2 = p_2 + p_3 - q_1 - q_3 + \lambda = p_2 + \tau_2 \tag{33}
\]
\[
\tau'_3 = -p_1 - 2q_3 + 2p_3 - 1 - p_\lambda = -p_\lambda + \tau_3 \tag{34}
\]

From (33) we get the following expression for the action \( z \)
\[
dz = (-\tau_1 + p_1q_1 + 2p_2q_2 + p_3q_3 + p_\lambda\dot{\lambda} + \dot{q}_2p_3 + \dot{\lambda}(-p_1 + 2p_3))d\tau \tag{35}
\]
or
\[ z = \int \left( -\tau_1 + p_1 \dot{q}_1 + p_2 \dot{q}_2 + p_3 \dot{q}_3 + p_\lambda \dot{\lambda} + \dot{q}_2 p_3 + \dot{\lambda} (-p_1 + 2 p_3) \right) d\tau \] (36)

In this case the extended system has first class constraints in Dirac’s classification. The action (36) gives the same result as the expression for the effective action from (20) for the system with first class constraints (32), (33), (34) when we consider \( t = \tau \). The only difference is that the gauge conditions have different expressions in Güler's formalism.

4  Concluding remarks

In this paper the quantization of the finite dimensional systems with constraints using canonical formalism introduced by Güler was investigated.

In this formalism we have no first or second class constraints as in the Dirac’s classification at the classical level but the constraints arises naturally in the set of consistency conditions (3) at the quantum level. The secondary constraints are obtained in Güler’s formalism using calculus of variations. When a system has second class constraints the Dirac’s brackets in Güler’s formalism are defined on the extended space \((q_i, \tau)\). In this case we enlarge the system and convert the second class constraints into first class constraints in order to obtain the hamiltonians in involution. When the hamiltonians \( H'_\mu \) are in involution we can construct the action in Güler’s formalism. In this formalism all dependent variables \( t_\alpha \) are gauge variables and we suppose that these variables have a dependence on \( \tau \) (a canonical conjugate variable with \( p_0 \)). In the case when hamiltonians \( H'_\mu \) commute the action (11) has the same form as obtained by path integral quantization method after performing all the calculations. In this case the results are in perfect agreement with those obtained by usual path integral quantization methods of a system with constraints.

5  Acknowledgements

One of the authors (D.B.) would like to thank TUBITAK for financial support and METU for the hospitality during his working stage at Department of Physics.

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