The Existence of Coupling in the Category of Transitive Lie Algebroid

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Abstract
The coupling of the tangent bundle $TM$ with the Lie algebra bundle $L$ ([5], Definition 7.2.2) plays the crucial role in the classification of the transitive Lie algebroids for Lie algebra bundle $L$ with fixed finite dimensional Lie algebra $g$ as a fiber of $L$. Here we give a necessary and sufficient condition for the existence of such a coupling. Namely we define a new topology on the group $\text{Aut}(g)$ of all automorphisms of Lie algebra $g$ and show that tangent bundle $TM$ can be coupled with the Lie algebra bundle $L$ if and only if the Lie algebra bundle $L$ admits a local trivial structure with structural group endowed with such new topology.

1 Introduction and Preliminaries

For every transitive Lie algebroid, the adjoint bundle is a Lie algebra bundle and can be coupled with the tangent bundle of its base smooth manifold (See [5], Theorem 6.5.1 and Definition 7.3.4). Conversely, there is a problem whether there exists a transitive Lie algebroid with a given Lie algebra bundle as adjoint bundle. Mackenzie was concerned with this problem and formulated the definition of coupling and constructed the obstruction class that depends on the coupling in order to give a criterion of existence of a transition Lie algebroid(See [5], Section 7.2). The results from this problem are used for the description of the classifying space of transitive Lie algebroids (See [6]). Since in [5] the coupling was assumed to exist, it is natural to consider the problem of existence of a coupling. In this paper, we show a necessary and sufficient condition for existence of a coupling of the tangent bundle $TM$ with the Lie algebra bundle $L$.

In the beginning, we shall give some definitions and results about the Lie algebra and differential geometry.
Definition 1.1. (See [2]) Let \( \mathfrak{g} \) be a finite dimensional Lie algebra. Let \( \text{Aut} \mathfrak{g} \) denote the group of Lie algebra automorphisms of \( \mathfrak{g} \) and \( \text{Der} \mathfrak{g} \) denote the Lie algebra of derivations of \( \mathfrak{g} \). Define the exponential map \( \exp : \text{Der} \mathfrak{g} \to \text{Aut} \mathfrak{g} \) by the formula

\[
\exp : \psi \mapsto \sum_{i=1}^{\infty} \frac{\psi^i}{i!}
\]

where \( \psi \in \text{Der} \mathfrak{g} \). Usually we denote \( \exp(\psi) \) by \( e^\psi \).

Definition 1.2. Let \( \mathfrak{g} \) be a finite dimensional Lie algebra. An automorphism of the form \( \exp(\text{ad} u) \), where \( u \in \mathfrak{g} \), is called inner. More generally, the subgroup of \( \text{Aut} \mathfrak{g} \) generated by those is denoted by \( \text{Int} \mathfrak{g} \) and its elements are called inner automorphisms.

Proposition 1.3. (See [5], [2]) The subgroup \( \text{Int} \mathfrak{g} \) is a normal Lie subgroup of \( \text{Aut} \mathfrak{g} \). The Lie algebra of the group \( \text{Int} \mathfrak{g} \) is \( \text{ad} \mathfrak{g} \).

As the group \( \text{Int} \mathfrak{g} \) is a normal Lie subgroup of \( \text{Aut} \mathfrak{g} \), there is a quotient group denoted by \( \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \). Denote by \( q : \text{Aut} \mathfrak{g} \to \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) the corresponding quotient map. The Lie group structure of \( \text{Int} \mathfrak{g} \) is induced from the Lie group structure of the group \( \text{GL}(\mathfrak{g}) \) (See [3]), where \( \text{GL}(\mathfrak{g}) \) is the group of all linear isomorphisms from \( \mathfrak{g} \) to \( \mathfrak{g} \). Due to [4], the quotient group \( \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) has the topology induced from \( \text{Aut} \mathfrak{g} \) by the quotient map \( q \). It is well known that the topology of \( \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) is not always discrete. We can add more open subsets on \( \text{Aut} \mathfrak{g} \) such that the topology on \( \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) becomes the discrete topology. Let us denote by \( \text{Aut} \mathfrak{g}^d \) the space \( \text{Aut} \mathfrak{g} \) with a finer topology such that the topology of \( \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) becomes the discrete topology. In order to avoid confusion, we denote by \( \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) the space \( \text{Aut} \mathfrak{g}^d / \text{Int} \mathfrak{g} \) with discrete topology.

Let \( M \) be a smooth manifold and \( \varphi : M \to \text{Aut} \mathfrak{g} \) be a smooth map. Let \( q : \text{Aut} \mathfrak{g} \to \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) be the quotient map defined above.

Theorem 1.4. The composition \( q \circ \varphi : M \to \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) is locally constant if and only if \( \frac{\partial \varphi}{\partial x} \varphi(x)^{-1} \in \text{ad} \mathfrak{g} \) for arbitrary \( x \in M \) and \( X \in T_xM \).

Proof. Fix \( x \in M \) and \( X \in T_xM \). Let \( \gamma : (-\varepsilon, \varepsilon) \to M \) be a curve such that \( \gamma(0) = x \) and \( \dot{\gamma}(0) = X \). Since \( q \circ \varphi : M \to \text{Aut} \mathfrak{g} / \text{Int} \mathfrak{g} \) is locally constant, the map \( \varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \text{Aut} \mathfrak{g} \) has the form

\[
\varphi \circ \gamma(t) = C \cdot \varphi(\gamma(t)),
\]

where \( C \in \text{Aut} \mathfrak{g} \) is constant and \( \varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \text{Int} \mathfrak{g} \). Then by Proposition 1.3 one has

\[
\left. \frac{d\varphi}{dt} \right|_{t=0} \varphi(\gamma(0))^{-1} = C \cdot \left. \frac{d\varphi}{dt} \right|_{t=0} \cdot \varphi(\gamma(0))^{-1} \cdot C^{-1} \in \text{ad} \mathfrak{g}.
\]

Thus \( \frac{\partial \varphi}{\partial x} \varphi(x)^{-1} \in \text{ad} \mathfrak{g} \).
Consider the composition
\[
T_x M \xrightarrow{T_x \varphi(x)} (\text{Aut} \; g) \xrightarrow{T_x R_{\varphi(x)}(\cdot)} \text{Aut} \; g \xrightarrow{T_x q} (\text{Aut} \; g/\text{Int} \; g)
\]
where \( x \in M \) and \( R_{\varphi(x)}^{-1} : \text{Aut} \; g \rightarrow \text{Aut} \; g, R_{\varphi(x)}^{-1}(\theta) = \varphi(x)^{-1}\theta, \) for \( \theta \in \text{Aut} \; g. \)

We identify \( T_e (\text{Aut} \; g) \) with \( \text{Der} \; g \) and \( T[e] (\text{Aut} \; g/\text{Int} \; g) \) with \( \text{Der} \; g/\text{ad} \; g. \)

For arbitrary \( X \in T_x M, \)
\[
\partial \varphi \over \partial X \varphi(x) - 1 \in \text{ad} \; g
\]
is equivalent to
\[
T_x R_{\varphi(x)}^{-1} \circ T_x \varphi(X) \in \text{ad} \; g.
\]
Then \( T_x q \circ T_x R_{\varphi(x)}^{-1} \circ T_x \varphi(X) = 0. \)

Since \( q \circ R_{\varphi(x)}^{-1} \circ \varphi = R_{q \circ \varphi(x)}^{-1} \circ (q \circ \varphi), \) it follows that \( T_x q \circ T_x \varphi(X) \equiv 0. \)
Consequently, \( q \circ \varphi : M \rightarrow \text{Aut} \; g/\text{Int} \; g \) is locally constant.

**Remark 1.5.** In the theorem above, the condition that \( q \circ \varphi : M \rightarrow \text{Aut} \; g/\text{Int} \; g \) is locally constant is equivalent to the condition that \( \varphi : M \rightarrow \text{Aut} \; g^\delta \) is continuous.

It is well known that given a vector bundle endowed with a connection, parallel transport along paths can be defined.

**Definition 1.6.** (See [1] Definition 3.1.1 and Lemma 3.1.3) Let \( E \) be a vector bundle on a smooth manifold \( M \) and \( \nabla \) be a connection on \( E. \) Let \( \gamma : [0,1] \rightarrow M \) be a smooth path. Then for each \( e \in E_{\gamma(0)}, \) there exists a unique section \( \sigma \) along \( \gamma \) which satisfies \( \nabla_\gamma \sigma \equiv 0 \) and \( \sigma(0) = e. \) Define \( P_\gamma(e) = e. \) Then \( P_\gamma : E_{\gamma(0)} \rightarrow E_{\gamma(1)} \) is a well defined linear map called the parallel transport map.

**Lemma 1.7.** (See [1], Lemma 3.1.8) The parallel translation along path \( \gamma \) does not depend on the parameterization of \( \gamma. \)

More generally, Definition 1.6 and Lemma 1.7 extend to the piecewise smooth situation (See [1], page 22). Let \( \gamma, \gamma' : [0,1] \rightarrow M \) be two piecewise paths with \( \gamma(1) = \gamma'(0). \) Define the inverse path \( \gamma^{-1} \) and composition \( \gamma' \gamma \) by
\[
\gamma^{-1}(t) = \gamma(1 - t) \quad 0 \leq t \leq 1 ; \quad \gamma' \gamma = \begin{cases} 
\gamma(2t) & 0 \leq t \leq \frac{1}{2}; \\
\gamma'(2t - 1) & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Since the composition of smooth paths is piecewise smooth, we have

**Lemma 1.8.** (See [1], 3.1.18) Given two piecewise smooth paths \( \gamma, \gamma' : [0,1] \rightarrow M \) with \( \gamma(1) = \gamma'(0). \) Then
\[
P_{\gamma^{-1}} = P_{\gamma}^{-1} \quad \text{and} \quad P_{\gamma' \gamma} = P_{\gamma'} \circ P_{\gamma}.
\]
Definition 1.9. (See [1], page 23) Let $M$ be a smooth map. Consider a continuous map
\[ H : [0, 1] \times [0, 1] \to M, \quad h_s = H(s, \cdot) : [0, 1] \to M \]
where $h_s$ is a family of piecewise smooth maps, and $H(s, t)$ is smooth in $s$. The map $H$ is called a piecewise smooth homotopy.

Consider a vector bundle $E \to M$ endowed with a connection $\nabla$ with curvature $R$. Through piecewise smooth homotopy, we can give some important relation between parallel transport and curvature. We denoted by $P_{s,t}$ the parallel transport from $E_{h_s(t)}$ to $E_{h_s(1)}$ through path $h_s$ and set
\[ R_{s,t} = P_{s,t} \circ R(\partial_t h(s, t), \partial_s h(s, t)) \circ P_{s,t}^{-1} : E_{h_s(1)} \to E_{h_s(1)} \]

Lemma 1.10. (See [1], Lemma 3.1.11) Let $\sigma$ be a piecewise smooth section along a piecewise smooth homotopy $H$, defined as above, such that $\nabla_{\partial_t} \sigma(s, t) \equiv 0$ and $\nabla_{\partial_s} \sigma(s, 0) \equiv 0$ for all $s$. Then
\[ \nabla_{\partial_s} \sigma(s, 1) = \left( \int_0^1 R_{s,t} dt \right) \sigma(s, 1). \]

In the case $H(s, 0)$ and $H(s, 1)$ are constant, then $P_{s,0} = P_{h_s}$ and satisfies
\[ \partial_s P_{s,0} = \left( \int_0^1 R_{s,t} dt \right) P_{s,0}. \]

2 The Existence of Coupling

Definition 2.1. (See [2], Definition 3.1.1) A Lie algebroid $A$ over a smooth manifold $M$ is a vector bundle $p : A \to M$ together with a Lie algebra structure $[ , ]$ on the space $\Gamma(A; M)$ of sections and a bundle map $a : A \to TM$ called the anchor such that

(i) the induced map $a : \Gamma(A; M) \to \Gamma(TM; M)$ is a Lie algebra homomorphism

(ii) for any sections $\sigma, \tau \in \Gamma(A; M)$ and smooth function $f \in C^\infty(M)$ we have the Leibniz identity
\[ [\sigma, f\tau] = f[\sigma, \tau] + a(\sigma)(f)\tau. \]

We call $A$ a transitive Lie algebroid if $a$ is surjective. We often use
\[ 0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0 \]
to denote a transitive Lie algebroid. Here $L = \text{Ker} a$ is called the adjoint bundle. Sometimes we use $(A, M, [ , ], a)$ to denote a Lie algebroid in order to highlight the bracket.
Let \((A', M, [\cdot, \cdot], a')\) be a second Lie algebroid on the same base \(M\). Then a morphism of Lie algebroids \(g : A \to A'\) over \(M\), or a Lie algebroid homorphism, is a vector bundle morphism such that \(a' \circ g = a\) and \(f([\sigma, \tau]) = [f(\sigma), f(\tau)]\), for all \(\sigma, \tau \in \Gamma(A; M)\).

**Notation 2.2.** Given a Lie algebra bundle \(L\), the transitive Lie algebroid of co-variant derivations on \(\Gamma(L; M)\) (See \[5\], Corollary 3.6.11) is denoted by \(D_{\text{Der}}(L)\). The bundle \(D_{\text{Der}}(L)\) is included in the Atiah exact sequence

\[
0 \to \text{Der} (L) \to D_{\text{Der}}(L) \to TM \to 0
\]

where \(\text{Der} (L)\) is the bundle of fiberwise derivative of \(L\). The elements of \(\Gamma(D_{\text{Der}}(L); M)\) are called bracket derivations on \(L\).

We denote by \(\text{ad} L\) the image of the adjoint representation \(\text{ad} : L \to D_{\text{Der}}(L)\). One can prove that \(\text{ad} L\) is an ideal of \(D_{\text{Der}}(L)\)(See \[5\]), page 271).

**Definition 2.3.** (See \[5\], Definition 7.2.1) Let \(L\) be a Lie algebra bundle on a smooth manifold \(M\). Then the quotient Lie algebroid

\[
0 \to \text{Der} (L)/\text{ad} L \to D_{\text{Der}}(L)/\text{ad} L \to TM \to 0,
\]

is denoted by

\[
0 \to \text{Out Der} (L) \to \text{Out} D_{\text{Der}}(L) \to TM \to 0
\]

and elements of \(\Gamma(\text{Out Der} D_{\text{Der}}L; M)\) are called outer bracket derivations on \(L\). Two quotient maps are denoted by \(\natural^+ : \text{Der} (L) \to \text{Out Der} (L)\) and \(\natural : D_{\text{Der}}(L) \to \text{Out} D_{\text{Der}}(L)\).

Since the tangent bundle of a smooth manifold is a Lie algebroid (\[5\], Section 3.3), we have following definition.

**Definition 2.4.** (See \[5\], Definition 7.2.2) Let \(L\) be a Lie algebra bundle on a smooth manifold \(M\). A coupling of \(TM\) with \(L\) is a Lie algebroid homomorphism \(\Xi : TM \to \text{Out} D_{\text{Der}}(L)\). We also say that \(TM\) and \(L\) are coupled by \(\Xi\).

**Remark 2.5.** The definition of coupling in \[5\] is more general, as Mackenzie defined coupling between an arbitrary Lie algebroid and Lie algebra bundle on the same base. Here we only consider the coupling between a tangent bundle and a Lie algebra bundle. Nevertheless, it is still worth considering the special coupling in Definition 2.4 since it is useful in describing the classifying space of a transitive Lie algebroid in \[5\].

Given a Lie algebra bundle, the coupling need not always exist. An example follows.

**Example 2.6.** Let \(S^2\) be two dimensional sphere. The tangent bundle of \(S^2\) is denoted by \(TS^2 \to S^2\) and is not trivial(See \[3\], Page 5). Then there is no coupling \(\Xi : TS^2 \to \text{Out} D_{\text{Der}}(TS^2)\).
To see this, we first note that $\text{ad}(TS^2) = 0$ since $TS^2$ is an abelian Lie algebra bundle. Then $\text{Out} \mathcal{D}_{Der}(TS^2) \equiv \mathcal{D}_{Der}(TS^2)$. Thus a coupling $\Xi$ is also a Lie algebroid homomorphism $\Xi : TS^2 \to \mathcal{D}_{Der}(TS^2)$. It can be regard as a flat connection on $TS^2$ (See [5], Page 186). But $TS^2$ admits no flat connection, since $S^2$ is simply connected, a flat connection would trivialize the bundle.

In the remainder, we will be concerned with the existence of a coupling.

**Lemma 2.7.** (See [4], Section 7.2) Let $L$ be a Lie algebra bundle on a smooth manifold $M$ with coupling $\Xi : TM \to \text{Out} \mathcal{D}_{Der}(L)$. There is a connection $\nabla$ on $L$ such that

i. $\nabla_X[u,v] = [\nabla_Xu, v] + [u, \nabla_Xv]$;

ii. $R^\nabla(X,Y) = \nabla_{[X,Y]} - \{\nabla_X, \nabla_Y\} = \text{ad} \circ \Omega(X,Y)$.

Here $u,v \in \Gamma(L;M), X,Y \in \Gamma(TM;M)$ and $\Omega : TM \to L$ is a vector bundle morphism. The connection $\nabla$ satisfying property (i) is called a Lie connection.

**Lemma 2.8.** Let $\gamma : [0,1] \to M$ be a piecewise smooth path on $M$ and $L \to M$ a Lie algebra bundle with Lie connection $\nabla$. Then the parallel transport map $P_\gamma : L_{\gamma(0)} \to L_{\gamma(1)}$, as defined in Definition 1.8, is a Lie algebra isomorphism.

**Proof.** Without lose of generality, we suppose that $\gamma : [0,1] \to M$ is smooth path. Let $u,v \in L_{\gamma(0)}$. There are sections $\sigma, \tau$ along path $\gamma$ such that $\nabla_\gamma \sigma = 0$, $\nabla_\gamma \tau = 0$ and $\sigma(\gamma(0)) = u$, $\tau(\gamma(0)) = v$.

By Definition 1.8, $P_\gamma(u) = \sigma(\gamma(1))$ and $P_\gamma(v) = \tau(\gamma(1))$. Since $\nabla$ is a Lie connection, $\nabla_\gamma([\sigma, \tau]) = [\nabla_\gamma \sigma, \tau] + [\sigma, \nabla_\gamma \tau] = 0$. Thus $[\sigma, \tau]$ is also a parallel section along $\gamma$. As $\sigma, \tau(\gamma(0)) = [u,v]$,

$$P_\gamma([u,v]) = [\sigma(\gamma(1)), \tau(\gamma(1))]$$

$$= [\sigma(\gamma(1)), \tau(\gamma(1))],$$

$$= [P_\gamma(u), P_\gamma(v)].$$

Due to the above formula and Lemma 1.8, $P_\gamma$ is a Lie algebra isomorphism.

**Theorem 2.9.** Let $L$ be a Lie algebra bundle on a smooth manifold $M$ with fibre $\mathfrak{g}$. Then a coupling $\Xi : TM \to \text{Out} \mathcal{D}_{Der}(L)$ exists if and only if $L$ admits a locally trivial structure with structural group $\text{Aut} \mathfrak{g}$.

**Proof.** By Lemma 2.7, there is a connection $\nabla$ on $L$ with curvature $R^\nabla = \text{ad} \circ \Omega$, where $\Omega : TM \to L$. Let $\{U_\alpha, f_\alpha \cong R^n\}_{\alpha \in \Delta}$ be a class of charts of $M$.

For each $\alpha \in \Delta$, define $x_\alpha \in U_\alpha$ by $x_\alpha = f_\alpha^{-1}(0)$. Let $x \in U_\alpha$ be arbitrary and define a smooth path

$$\gamma_{\alpha, x} : [0,1] \to U_\alpha \quad \text{by} \quad \gamma_{\alpha, x}(t) = f_\alpha^{-1}(tf_\alpha(x)).$$
By Lemma 2.8 and the path defined above, we can define a new class of charts for \( L \rightarrow M \),

\[
\varphi_\alpha : U_\alpha \times L_{x_\alpha} \rightarrow L_{U_\alpha}, \quad \varphi_\alpha(x, u) = P_{\gamma_{x,\alpha}}(u)
\]

for \( x \in U_\alpha, u \in L_{x_\alpha} \).

If \( U_\alpha \cap U_\beta \neq \emptyset \), then \( \varphi_\beta^{-1} \circ \varphi_\alpha : U_\alpha \cap U_\beta \times L_{x_\alpha} \rightarrow U_\alpha \cap U_\beta \times L_{x_\beta} \) is

\[
\varphi_\beta^{-1} \circ \varphi_\alpha(x, u) = (x, P_{\gamma_{x,\beta}} \circ P_{\gamma_{x,\alpha}}(u))
\]

\[
= (x, P_{\gamma_{x,\beta} \circ \gamma_{x,\alpha}}(u))
\]

Let us define

\[
\gamma_x = \gamma_{x,\beta} \circ \gamma_{x,\alpha}^{-1}.
\]

By (2), (3) and the definition of new charts (1), the transition function

\[
\varphi_{\alpha \beta} : U_\alpha \cap U_\beta \rightarrow \text{Aut } g
\]

for the new charts is

\[
\varphi_{\alpha \beta}(x) = P_{\gamma_x}.
\]

Given arbitrary \( x_0 \in U_\alpha \cap U_\beta \) and \( x \in T_{x_0} U_\alpha \cap U_\beta \). Let \( c : [-\varepsilon, \varepsilon] \rightarrow U_\alpha \cap U_\beta \) be a smooth path with \( c(0) = x_0 \) and \( \dot{c}(0) = X \).

Define

\[
H : [-\varepsilon, \varepsilon] \times [0, 1] \rightarrow U_\alpha \cup U_\beta
\]

by

\[
H(s, t) = \begin{cases} 
  f_{\alpha}^{-1}(2tf_\alpha(c(s))) & 0 \leq t \leq \frac{1}{2}, \\
  f_{\beta}^{-1}((2 - 2t)f_\beta(c(s))) & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

The map \( H \) is a piecewise homotopy and \( H(s, \cdot) = h_s = \gamma_{c(s)}, \) as defined in (3), for each \( s \in [-\varepsilon, \varepsilon] \).

Let us use \( P_{s,t} \) to denote parallel transport over the path \( h_s \) from \( L_{h_s(t)} \) to \( L_{h_s(0)} \). Since \( R^X = \text{ad } \Omega \),

\[
R_{s,t} = P_{s,t} \circ R(\partial_1 H(s, t), \partial_1 H(s, t)) \circ P_{s,t}^{-1}
\]

\[
= P_{s,t} \circ (\text{ad } \Omega(\partial_1 H(s, t), \partial_1 H(s, t))) \circ P_{s,t}^{-1}
\]

\[
= \text{ad } P_{s,t}(\Omega(\partial_1 H(s, t), \partial_1 H(s, t)))
\]

By Lemma 1.10,

\[
\partial_s P_{s,0} = (\int_0^1 R_{s,t}(dt) \cdot P_{s,0}
\]

\[
= \text{ad}(\int_0^1 P_{s,t}(\Omega(\partial_1 H(s, t), \partial_1 H(s, t))) dt) \cdot P_{s,0}.
\]

Since \( P_{s,0} = P_{h_s} \) and \( h_s = \gamma_{c(s)} \), formula (4) implies we have \( P_{s,0} = \varphi_{\alpha \beta}(c(s)) \).

Thus

\[
\frac{d\varphi_{\alpha \beta}(c(s))}{ds}_{|s=0} \cdot \varphi_{\alpha \beta}^{-1}(c(0)) = \partial_s P_{s,0} \big|_{s=0} \cdot P_{s,0}^{-1}
\]

\[
= \text{ad}(\int_0^1 P_{s,t}(\Omega(\partial_1 H(s, t), \partial_1 H(s, t))) dt) \big|_{s=0} \in \text{ad } g.
\]
That is $\frac{\partial (\phi^-_\alpha)}{\partial X} : \varphi^-_\alpha(x_0)^{-1} \in \text{ad}g$.

Then by Theorem 1.4, we have $\varphi^-_\alpha : U_\alpha \cap U_\beta \to \textbf{Aut} g^\delta$. So the Lie algebra bundle $L \to M$ admits structural group $\textbf{Aut} g^\delta$.

Now let us prove the other direction. As $L \to M$ admits structural group $\textbf{Aut} g^\delta$, there is a class of charts $\{U_\alpha, \varphi^-_\alpha : U_\alpha \times g \to L_{U_\alpha}\}_{\alpha \in \Delta}$ with transition functions $\{\psi^-_{\alpha\beta} : U_\alpha \cap U_\beta \to \textbf{Aut} g^\delta\}_{\alpha,\beta \in \Delta}$.

For each $\alpha \in \Delta$, we define a connection $\nabla^\alpha$ on $L_{U_\alpha}$ that is a bundle morphism $\nabla^\alpha : T_{U_\alpha} \to D(\text{Der}(L)|_{U_\alpha})$ by

$$\nabla^\alpha_X s_\alpha = \varphi^-_\alpha \left( \frac{\partial \varphi^-_\alpha^{-1}(s_\alpha)}{\partial X} \right)$$

where $X \in \Gamma(T_{U_\alpha})$ and $s_\alpha \in \Gamma(L_{U_\alpha})$.

Let $\{h_\alpha\}_{\alpha \in \Delta}$ be a partition of unity corresponding to $\{U_\alpha\}_{\alpha \in \Delta}$. Then define a connection $\nabla$ on $L$, that is $\nabla : TM \to D(\text{Der}(L))$, by the formula

$$\nabla_X s(m) = \sum_{\alpha \in \Delta} h_\alpha(m) \cdot \nabla^\alpha_X s|_{U_\alpha}(m)$$

where $X \in \Gamma(TM)$, $s \in \Gamma(L; M)$, $m \in M$ and the corresponding items in the sum are considered to be zero when $m \notin U_\alpha$.

Let us define $\Xi^\alpha : T_{U_\alpha} \to \textbf{Out} D(\text{Der}(L)|_{U_\alpha})$ by

$$\Xi^\alpha = \natural \circ \nabla^\alpha.$$

By straightforward calculation, for $X_\alpha, Y_\alpha \in \Gamma(T_{U_\alpha})$,

$$R^{\Xi^\alpha}(X_\alpha, Y_\alpha) = \Xi^\alpha([X_\alpha, Y_\alpha]) - \{\Xi^\alpha(X_\alpha), \Xi^\alpha(Y_\alpha)\}$$

$$= \natural \circ R^{\nabla^\alpha}(X_\alpha, Y_\alpha) = 0$$

Define a bundle homorphism $\Xi : TM \to \textbf{Out} D(\text{Der}(L))$ by

$$\Xi = \natural \circ \nabla.$$  

Actually, due to the definition of $\Xi^\alpha$,

$$\Xi(X_m) = \sum_{\alpha \in \Delta} h_\alpha(m) \cdot \Xi^\alpha(X_m)$$

for $X_m \in T_{m}M$ and the corresponding items in the sum are considered to be zero if $m \notin U_\alpha$.

In the case $U_\alpha \cap U_\beta \neq \emptyset$, choose arbitrary $X \in \Gamma(T_{U_\alpha} \cap U_\beta : U_\alpha \cap U_\beta)$ and...
section \( s \in \Gamma(L_{U_{\alpha}} \cap U_{\beta}; U_{\alpha} \cap U_{\beta}) \). Then 

\[
\nabla_X^\beta s = \varphi_\beta \left( \frac{\partial \varphi^{-1}_\beta(s)}{\partial X} \right)
\]

\[
= \varphi_\beta \left( \frac{\partial \varphi^{-1}_\alpha \circ \varphi_\alpha^{-1}(s)}{\partial X} \right) = \varphi_\beta \left( \frac{\partial (\varphi_\alpha \circ \varphi^{-1}_\alpha(s))}{\partial X} \right)
\]

\[
= \varphi_\beta \circ \varphi_\alpha \circ \left( \frac{\partial \varphi^{-1}_\alpha(s)}{\partial X} \right) + \varphi_\beta \circ \left( \frac{\partial \varphi_\alpha}{\partial X} \right) \circ \varphi^{-1}_\alpha(s)
\]

\[
= \varphi_\beta \circ \varphi^{-1}_\alpha \circ \varphi_\alpha \circ \left( \frac{\partial \varphi^{-1}_\alpha(s)}{\partial X} \right) + \varphi_\beta \circ \left( \frac{\partial \varphi_\alpha}{\partial X} \right) \circ \varphi^{-1}_\alpha(s)
\]

\[
= \nabla_X^\alpha s + \varphi_\alpha \circ \varphi^{-1}_\alpha \circ \varphi_\beta \circ \left( \frac{\partial \varphi_\alpha}{\partial X} \right) \circ \varphi^{-1}_\alpha(s)
\]

\[
= \nabla_X^\alpha s + \varphi_\alpha \circ (\varphi^{-1}_\alpha \circ \left( \frac{\partial \varphi_\alpha}{\partial X} \right)) \circ \varphi^{-1}_\alpha(s)
\]

\[
= \nabla_X^\alpha s + [\varphi_\alpha(h_{\alpha \beta}(X)), s]
\]

here \( h_{\alpha \beta} : TU_{\alpha} \cap U_{\beta} \to U_{\alpha} \cap U_{\beta} \times \mathfrak{g} \) such that \( \varphi^{-1}_{\alpha \beta} \cdot \left( \frac{\partial \varphi_\alpha}{\partial X} \right) = \text{ad} \circ h_{\alpha \beta}(X) \) is from the fact that \( \varphi_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \to \text{Aut} \mathfrak{g}^\delta \) and Theorem 1.4.

Thus \( \nabla_X^\alpha = \nabla_X^\beta \in \text{ad}L \). So \( \Xi^\alpha = \Xi^\beta \) on \( TU_{\alpha} \cap U_{\beta} \). Hence \( \Xi|_{TU_{\alpha}} = \Xi^\alpha \). The bundle morphism \( \Xi : TM \to \text{Out} \mathcal{D}_{\text{Der}}(L) \) defined in (6) is a coupling since \( R^\Xi|_{TU_{\alpha}} = R^{\Xi^\alpha} = 0 \) for arbitrary \( \alpha \in \Delta \).

\[\square\]

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