ISOMORPHISM CLASSES FOR HIGHER ORDER TANGENT BUNDLES

ALI SURI

Abstract. The tangent bundle $T^kM$ of order $k$, of a smooth Banach manifold $M$ consists of all equivalent classes of curves that agree up to their accelerations of order $k$. In the previous work of the author he proved that $T^kM$, $1 \leq k \leq \infty$, admits a vector bundle structure on $M$ if and only if $M$ is endowed with a linear connection or equivalently a connection map on $T^kM$ is defined. This bundle structure heavily depends on the choice of the connection. In this paper we ask about the extent to which this vector bundle structure remains isomorphic. To this end we define the notion of the $k$’th order differential $T^k g : T^k M \to T^k N$ for a given differentiable map $g$ between manifolds $M$ and $N$. As we shall see, $T^k g$ becomes a vector bundle morphism if the base manifolds are endowed with $g$-related connections. In particular, replacing a connection with a $g$-related one, where $g : M \to M$ diffeomorphism, follows invariant vector bundle structures. Finally, using immersions on Hilbert manifolds and manifold of $C^r$ maps, we offer two examples to support our theory and reveal its interaction with the known problems such as Sasaki lift of metrics.

Keywords: Banach manifold; Hilbert manifold; Linear connection; Connection map; Related connection, Higher order tangent bundle; Fréchet manifold; lifting of Riemannian metrics.

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1. Introduction

The tangent bundle of order $k$, $T^k M$, of a smooth manifold $M$ consists of all equivalent classes of curves that agree up to their accelerations of order
k. This bundle is a natural extension of the notion of the usual tangent bundle (see e.g. \cite{5, 17, 18, 27}) which, for example in classical mechanics, describes the Generalized Particle Mechanics in the autonomous sense \cite{4}.

A vector bundle structure for $T^k M$, $2 \leq k \leq \infty$, even for $k = 2$ is not as evident as in the case of tangent bundles and in fact it is not always possible \cite{5, 22, 23}.

The author in his previous work \cite{23} proved that at the presence of a linear connection on $M$ (or equivalently a connection map on $T^k M$), $T^k M$, $2 \leq k \leq \infty$, admits a vector bundle structure on $M$. Moreover we have shown that every linear connection on $M$, and hence a Riemannian metric on $M$, induces a connection map on $T^k M$.

As an immediate consequence, our suggested vector bundle structure allows us to solve an old problem of differential geometry formulated by Bianchi and Bompiani \cite{17}, namely the problem of prolongation of a Riemannian metric defined on the base manifold $M$ to $T^k M$, $k \geq 1$ even for infinite dimensional Hilbertable manifolds \cite{23}.

However, as one may have expected, these vector bundle structures depend crucially on the particular connection chosen \cite{5, 6, 22, 23}.

In this paper we ask about the extent of this vector bundle dependence. We will show that this dependence is closely related to the notion of related connection maps (or conjugate connections) which will be used for a classification of these vector bundle structures. More precisely we introduce the higher order differential $T^k g$ of a smooth map $g : M \to N$ between two manifolds $M$ and $N$ and we investigate under what conditions $T^k g$ is linear on fibres. Linearity of $T^k g$, $x \in M$, allows us to build a vector bundle morphism $T^k g : T^k M \to T^k N$ (see \cite{22} and \cite{6} for the special case $k = 2$). As a consequence, we show that the vector bundle structure on $T^k M$, defined by the aim of a connection map, remains invariant (isomorphic) if it is replaced by a $g$-related connection map, for any diffeomorphism $g : M \to M$.

If we take one step further by considering $T^\infty M$ and $T^\infty N$, as generalized Fréchet vector bundles over $M$ and $N$ respectively (\cite{23}), then proving $T^\infty g$ to be a generalized vector bundle morphism becomes much more complicated.

More precisely the set of linear maps between Fréchet spaces (the fibre types of $(T^\infty M, \pi^M_{\infty}, M)$ and $(T^\infty N, \pi^N_{\infty}, N)$) does not remain in the category of Fréchet spaces \cite{11, 19}. To get around this difficulty, we employ the projective limit methodology, as in \cite{11, 10, 23, 24} etc., to show that $(T^\infty g, g)$ becomes a generalized vector bundle morphism at the presence of $g$-related connections on $M$ and $N$.

Afterward, as an application, we settle our results to the special case of $f : M \to N$, where $f$ is an immersion and $N$ is a Riemannian Hilbert manifold. As a result, this special case tells us that higher order differential of an isometry remains isometry with respect to the induced Sasaki-type metrics.

We close this article by addressing, in part, the case of manifold of $C^r$ maps between manifolds $M$ and $N$ denoted by $C^r(N, M)$, as an advantage of working with Banach manifolds.
Trough this paper all the maps and manifolds are assumed to be smooth, but, except for section 4, less degrees of differentiability can be assumed.

The reader unfamiliar with infinite dimensional manifolds and spaces, can easily replace the model spaces with Euclidean spaces.

Most of the results of this paper are novel even for the case of finite dimensional manifolds.

2. Preliminaries

In this section we summarize the necessary preliminary material that we need for a self contained presentation of our paper.

At various points in this article, we will wish to have on hand explicit formulæ for the higher order differentials of compositions of smooth functions. Hence, we begin with a short description of the higher order chain rule. Let \( f : U \rightarrow V \) and \( g : V \rightarrow W \) be \( k \)-times Fréchet differentiable where \( U, V \) and \( W \) are open subsets of Banach spaces \( \mathbb{E}, \mathbb{E}' \) and \( \mathbb{E}'' \) respectively. Then it is known that \( g \circ f \) is \( k \)-times Fréchet differentiable and

\[
(1) \quad (g \circ f)^{(k)}(0) = \sum_{\mu} \frac{k!}{l_1! \cdots l_im_1! \cdots m_k!} d^i g(x)(f^{(l_1)}(0), \ldots , f^{(l_i)}(0))
\]

where the sum is over all ordered \( i \)-tuples \((l_1, \ldots , l_i)\) of integers \( l_1, \ldots , l_i\) such that \( l_1 + \cdots + l_i = k \) and \( 1 \leq l_1 \leq \cdots \leq l_i \leq k \) with \( i \) varying from 1 to \( k \). Moreover for any \( j \in \{1, \ldots , k\} \), \( m_j \) is the number of \( l_1, \ldots , l_i\) equal to \( j \) ([2], p. 234, [14], p. 359 or [20], p. 262). The coefficient \( \frac{k!}{l_1! \cdots l_im_1! \cdots m_k!} \) will henceforth be denoted by \( a_{l_1 \ldots l_i}^{k} \).

We proceed with a short description of infinite-dimensional manifolds and their tangent bundles. Let \( M \) be a manifold modeled on the Banach space \( \mathbb{E} \). For any \( x_0 \in M \) define

\[ C_{x_0} := \{ \gamma : (-\epsilon, \epsilon) \rightarrow M ; \gamma(0) = x_0 \text{ and } \gamma \text{ is smooth } \}. \]

As a natural extension of the tangent bundle \( TM \) define the following equivalence relation. For \( \gamma \in C_{x_0} \), set \( \gamma^{(1)}(t) = \gamma'(t) \) and \( \gamma^{(k)}(t) = \gamma^{(k-1)'}(t) \) where \( k \in \mathbb{N} \) and \( k \geq 2 \). Two curves \( \gamma_1, \gamma_2 \in C_{x_0} \) are said to be \( k \)-equivalent, denoted by \( \gamma_1 \approx_{x_0}^{k} \gamma_2 \), if and only if \( \gamma_1^{(j)}(0) = \gamma_2^{(j)}(0) \) for all \( 1 \leq j \leq k \). Define \( T_{x_0}^{k}M := C_{x_0} / \approx_{x_0}^{k} \) and the tangent bundle of order \( k \) or \( k \)-osculating bundle of \( M \) to be \( T^{k}M := \bigcup_{x \in M} T_{x}^{k}M \). Denote by \( \gamma, x_0 \) the representative of the equivalent class containing \( \gamma \) and define the canonical projections \( \pi_{x_0}^{k} : T^{k}M \rightarrow M \) which projects \( \gamma, x_0 \) onto \( x_0 \).

Let \( \mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}_{\alpha \in I} \) be a \( C^\infty \) atlas for \( M \). For any \( \alpha \in I \) define

\[
\phi_\alpha^{k} : \pi_{x_0}^{k-1}(U_\alpha) \rightarrow \phi_\alpha(U_\alpha) \times \mathbb{E}^{k}
\]

\[ \gamma, x_0 \] \( \mapsto \) \( (\phi_\alpha \circ \gamma)(0), (\phi_\alpha \circ \gamma)'(0), \ldots, \frac{1}{k!}(\phi_\alpha \circ \gamma)^{(k)}(0) \)

**Theorem 2.1.** The family \( \mathcal{A}_k = \{ (\pi_{x_0}^{k-1}(U_\alpha), \phi_\alpha^{k}) \}_{\alpha \in I} \) declares a smooth fibre bundle (not generally a vector bundle) structure for \( T^{k}M \) over \( M \) [23].

Consider the \( C^\infty(T^{k}M) \)-linear map \( J : \mathcal{X}(T^{k}M) \rightarrow \mathcal{X}(T^{k}M) \) s.t. locally on a chart \( (\phi_\alpha^{k}, \pi_{x_0}^{k-1}(U_\alpha)) \) is given by
to any $u := \langle x, \xi_1, \ldots, \xi_k \rangle \in T^kM$ and every $(u; y, \eta_1, \ldots, \eta_k) \in T_u T^kM$.

**Definition 2.2.** A connection map on $T^kM$ is a vector bundle morphism

$$K = \left( K, K^1, \ldots, K^k \right) : T^kM \longrightarrow \left( \bigoplus_{i=1}^k TM, \bigoplus_{i=1}^k \pi_{M_i}, \bigoplus_{i=1}^k M \right)$$

such that for any $1 \leq a \leq k - 1$, $K \circ J^a = K^a$ and $K \circ J^k = \pi^k_M$ [3, 23].

In order to carry out the local structure of a connection map, we state the following lemma according to [23].

**Lemma 2.3.** Locally on a chart $(U_\alpha, \phi^\alpha_k)$ there are smooth maps $M_\alpha : U_\alpha \times E^k \longrightarrow L(E, E)$, $1 \leq i \leq k$, such that the connection map

$$K_\alpha := \bigoplus_{i=1}^k \phi^1 \circ K \circ \phi^k_{i-1} = (K_\alpha, \ldots, K_\alpha)$$

at $(u; y, \eta_1, \ldots, \eta_k) \in T_u T^kM$ is given by

\[
(2) \quad K|_{U_\alpha}(u; y, \eta_1, \ldots, \eta_k) = \bigoplus_{i=1}^{k} \left( x, \eta_i + \frac{1}{M_\alpha} (u) \eta_{i-1} + \frac{2}{M_\alpha} (u) \eta_{i-2} + \ldots + \frac{i}{M_\alpha} (u) y \right).
\]

Let $M$ and $N$ be two smooth manifolds modeled on the Banach spaces $E$ and $E'$. Motivated by [25], [6] and [22] we state the following two definitions.

**Definition 2.4.** Let $g : M \longrightarrow N$ be a smooth map. For $k \in \mathbb{N}$ define the $k$th order differential of $g$ by

$$T^k g : T^k M \longrightarrow T^k N$$

$$\left[ \gamma, x \right]^k \longmapsto \left[ g \circ \gamma, g(x) \right]^k$$

To show that the above definition is well defined consider another representative $[\delta, x]^k$ of the class $[\gamma, x]^k \in T^k_x M$. Then, for any integer $1 \leq i \leq k$, using (1) we have

\[
(g \circ \gamma)^{(i)}(0) = \sum_{\mu} a^{(i)}_{[\mu]} \delta \gamma^{[\mu]}(0) = \sum_{\mu} a^{(i)}_{[\mu]} \delta \gamma^{[\mu]}(0)
\]

which shows that $T^k g$ is well defined.

**Definition 2.5.** Let $K_M$ and $K_N$ be two connection maps on $M$ and $N$ respectively and $g : M \longrightarrow N$ be a smooth map. $K_M$ and $K_N$ are called $g$-related if they commute with the differentials of $f$ in the following manner

\[
(3) \quad K_N \circ TT^k g = \bigoplus_{i=1}^k Tg \circ K_M.
\]

**Remark 2.6.** If $k = 1$ then, the above definition agrees with that of [23], [6] and [22].
From now on fix the atlas \( \mathcal{B} = \{(V_\beta, \psi_\beta)\}_{\beta \in J} \) for \( N \) and construct the proposed atlas discussed in theorem 2.1 for \( T^kN \) which will be denoted by \( \mathcal{B}_k = \{(\pi_N^{-1}(V_\beta), \psi_\beta^k)\}_{\beta \in J} \). The model space of \( N \) is the Banach space \( \mathbb{E}' \).

For suitably chosen charts \( (\phi_\alpha^k, \pi_M^{-1}(U_\alpha)) \) and \( (\psi_\beta^k, \pi_N^{-1}(V_\beta)) \), of \( M \) and \( N \) respectively, we have

\[
(\oplus_{i=1}^k \psi_\beta^k) \circ K_N \circ TT^k g \circ T\phi_\alpha^{-1} = (\oplus_{i=1}^k \psi_\beta^k) \circ K_N \circ TT^k g \circ T\phi_\alpha^{-1} := K_{N,\beta} \circ TT^k g_{\beta\alpha}
\]

and

\[
(\oplus_{i=1}^k \psi_\beta^k) \circ (\oplus_{i=1}^k Tg) \circ K_M \circ T\phi_\alpha^{-1} := (\oplus_{i=1}^k Tg_{\beta\alpha}) \circ K_{M,\alpha}
\]

where \( g_{\beta\alpha} := \psi_\beta \circ g \circ \phi_\alpha^{-1} \) and \( K_{M,\alpha} \) and \( K_{N,\beta} \) stand for the local representations of the connection maps pointed out by lemma 2.3.

In order to reveal the local compatibility condition for \( g \)-related connections, for any \((x, \xi_1, \ldots, \xi_k, y, \eta_1, \ldots, \eta_k) \in U_\alpha \times \mathbb{E}^{2k+1} \), we define the following auxiliary curve.

\[
\tilde{c} : (-\epsilon, \epsilon)^2 \rightarrow \phi_\alpha(U_\alpha) \subseteq \mathbb{E}
\]

\[
(t, s) \rightarrow x + sy + \sum_{i=1}^k t^i(\xi_i + s\eta_i)
\]

Now, evaluating \( (\oplus_{i=1}^k Tg_{\beta\alpha}) \circ K_{M,\alpha} \) at \((u, y, \eta_1, \ldots, \eta_k) \) yields

\[
\bigoplus_{i=1}^k \left( g_{\beta\alpha}(x), d g_{\beta\alpha}(x)[\eta_i + \frac{1}{i!} M_\alpha(u) \eta_{i-1} + \frac{2}{i!} M_\alpha(u) \eta_{i-2} + \ldots + \frac{i}{i!} M_\alpha(u) y] \right).
\]

On the other hand

\[
K_{N,\beta} \circ TT^k g_{\beta\alpha}(u, y, \eta_1, \ldots, \eta_k)
\]

\[
= \bigoplus_{i=1}^k \left( g_{\beta\alpha}(x), \bar{\eta}_i + N_\beta(\bar{u}) \eta_{i-1} + \ldots + \frac{i}{i!} N_\beta(\bar{u}) \bar{y} \right)
\]

where \( \bar{\gamma}(t) := \tilde{c}(t, 0), \bar{u} = ((g_{\alpha \beta} \circ \bar{\gamma})(0), \ldots, \frac{1}{i!}(g_{\alpha \beta} \circ \bar{\gamma})(k)(0)), \bar{y} = \frac{\partial}{\partial s}(g_{\beta\alpha} \circ \bar{c})(0,0), \bar{\eta}_i = \frac{\partial^{i+1}}{\partial t^{i+1}}(g_{\beta\alpha} \circ \bar{c})(0,0) \) and \( N_\beta \) are local components of \( K_N \) for \( 1 \leq i \leq k \).

As a consequence we have the following important compatibility condition which locally declares \( g \)-related connection maps

\[
d g_{\beta\alpha}(x)[\eta_i + \frac{1}{i!} M_\alpha(u) \eta_{i-1} + \frac{2}{i!} M_\alpha(u) \eta_{i-2} + \ldots + \frac{i}{i!} M_\alpha(u) y] = \bar{\eta}_i + \frac{1}{i!} N_\beta(\bar{u}) \eta_{i-1} + \ldots + \frac{i}{i!} N_\beta(\bar{u}) \bar{y}; \ 1 \leq i \leq k.
\]

**Remark 2.7.** If \( k = 1 \), then the last equation coincides with the local compatibility condition for \( g \)-related connections \( K_M \) and \( K_N \) on \( M \) and \( N \) as stated in [25] p. 299.
Remark 2.8. Whenever \( M = N \), \( K_M = K_N := K \) and \( g = id_M \), the equation \( (4) \) reduces to the compatibility condition which locally the connection maps on common charts must satisfy \([23]\).

In what follows, we determine a canonical connection map on \( T^k M \) depending only on a given linear connection (Riemannian metric) on the base manifold \( M \). Keeping the formalisms of \([21, 25, 26]\) we state the following proposition according to \([23]\).

Proposition 2.9. Let \( \nabla \) be a linear connection on \( M \) with the local components \( \{ \Gamma \}_{\alpha \in I} \). There exists an induced connection map on \( T^k M \) with the following local components.

\[
\begin{align*}
1 & \quad M_\alpha (x, \xi_1) y = \Gamma_\alpha (x, \xi_1) y \\
2 & \quad M_a (x, \xi_1, \xi_2) y = \frac{1}{2} \left( \sum_{i=1}^{2} \partial_i M_\alpha (x, \xi_1)(y, i\xi_i) + M_\alpha (x, \xi_1) [ M_\alpha (x, \xi_1)y] \right), \\
\vdots \\
\end{align*}
\]

\[
M (x, \xi_1, ..., \xi_k) y = \frac{1}{k} \left( \sum_{i=1}^{k} \partial_i M_\alpha (x, \xi_1, ..., \xi_{k-1})(y, i\xi_i) \\
+ M_\alpha (x, \xi_1) [ M_\alpha (x, \xi_1, ..., \xi_{k-1})y] \right).
\]

2.1. Lifting of related linear connections to higher order tangents bundles. In this section we show that lift of \( g \)-related linear connections remain \( g \)-related. More precisely let \( g : M \to N \) be a smooth map between differentiable manifolds \( M \) and \( N \) and \( \nabla_M \) and \( \nabla_N \) be two linear connections on \( M \) and \( N \) respectively. Moreover suppose that \( K_M \) and \( K_N \) be the lifted connection maps (as in proposition 2.9) generated by \( \nabla_M \) and \( \nabla_N \) on \( T^k M \) and \( T^k N \) respectively.

Proposition 2.10. If \( \nabla_M \) and \( \nabla_N \) are \( g \)-related, then \( K_M \) and \( K_N \) are \( g \)-related connection maps too.

Proof. We shall prove that, for any \( \alpha \in I, \beta \in J \) with \( g(U_\alpha) \subseteq V_\beta \), the local components \( \{ i_M \}_{i=1,...,k} \) and \( \{ i_N \}_{i=1,...,k} \) from proposition 2.9 satisfy condition \([4]\). The proof is by induction on \( i \).

For the base step of induction consider the Christoffel symbols \( \{ \Gamma_\alpha \}_{\alpha \in I} \) and \( \{ \Gamma_\beta \}_{\beta \in J} \) of the \( g \) related connections \( \nabla_M \) and \( \nabla_N \). The compatibility condition for \( \Gamma_\alpha \) and \( \Gamma_\beta \) is given by

\[
dg_\alpha (x)[\eta_1 + \Gamma_\alpha (x, \xi_1)y] = dg_\alpha (x)(\eta_1) + d^2 g_\alpha (x)(\xi_1, y) \\
+ \Gamma_\beta (g_\alpha (x), dg_\alpha (x)(\xi_1)) dg_\alpha (x)(y)
\]

(for more details see \([6, 22, 25]\)). Moreover we have \( 1_M = \Gamma_\alpha \) and \( 1_N = \Gamma_\beta \). These last three equalities show that \( \Gamma_\alpha \) and \( \Gamma_\beta \) satisfy \([4]\).
By induction, we assume that for $i = 1, \ldots, k - 1$, $M_\alpha$ and $N_\beta$; the rule \eqref{eq:4} is verified i.e.

\begin{equation}
\tag{5}
dg_{\beta a}(x)[\eta_i + \frac{1}{i} M_\alpha (u_1) \eta_{i-1} + \frac{2}{i} M_\alpha (u_2) \eta_{i-2} + \ldots + \frac{i}{i} M_\alpha (u_i) y] \\
= \bar{\eta}_i + N_{\beta} (\bar{u}_1) \bar{\eta}_{i-1} + \ldots + N_{\beta} (\bar{u}_i) \bar{y}
\end{equation}

where $u_j := (x, \xi_1, \ldots, \xi_j)$, $\bar{u}_j = ((g_{\beta a} \circ \bar{\gamma})(0), \ldots, \frac{1}{j!}(g_{\beta a} \circ \bar{\gamma})(j)(0))$, $\bar{y} = \frac{\partial}{\partial \tau}(g_{\beta a} \circ \bar{c})(0,0)$, $\bar{\eta}_j = \frac{\partial}{\partial \tau}(g_{\beta a} \circ \bar{c})(0,0)$ for $1 \leq j \leq i$ and $\bar{c}$ is as in \eqref{eq:3}. Then from the definition of $M_\alpha$ at $(u_k; y, 0, \ldots 0)$ and the induction hypothesis we get

\[kdg_{\beta a}(x) M_\alpha (u_k) y = dg_{\beta a}(x) \left\{ \sum_{i=1}^{k} \partial_i \frac{k-1}{k} M_\alpha (u_{k-1})(y, i \xi_i) + \frac{1}{i} M_\alpha (x, \xi_1) \frac{k-1}{k} M_\alpha (u_{k-1})y \right\}\]

It perhaps worth remarking that

\[dg_{\beta a}(x) \left\{ \partial_1 \frac{k-1}{k} M_\alpha (u_{k-1})(y, \xi_1) \right\} = -d^2 g_{\beta a}(x)(\xi_1, \frac{k-1}{k} M_\alpha (u_{k-1})y)\]

\[+ \frac{\partial}{\partial t} \left\{ \frac{\partial^k}{(k-1)! \partial s \partial t^{k-1}}(g_{\beta a} \circ \bar{c}_1)(t, s, h) \right\} \]

\[+ \frac{1}{N_{\beta}} [(g_{\beta a} \circ \bar{\gamma}_1)(t, h), \partial_t (g_{\beta a} \circ \bar{\gamma}_1)(t, h)] \frac{\partial^{k-1}}{(k-2)! \partial s \partial t^{k-2}}(g_{\beta a} \circ \bar{c}_1)(t, s, h)\]

\[+ \ldots + \frac{1}{N_{\beta}} [(g_{\beta a} \circ \bar{\gamma}_1)(t, h), \partial_t (g_{\beta a} \circ \bar{\gamma}_1)(t, h)] \frac{\partial^{k-1}}{(k-1)! \partial t^{k-1}}(g_{\beta a} \circ \bar{c}_1)(t, s, h) \bigg|_{t=s=h=0}\]

and for $2 \leq i \leq k$

\[dg_{\beta a}(x) \left\{ \partial_i \frac{k-1}{k} M_\alpha (u_{k-1})(y, i \xi_i) \right\} = \frac{\partial}{\partial t} \left\{ \frac{\partial^k}{(k-1)! \partial s \partial t^{k-1}}(g_{\beta a} \circ \bar{c}_i)(t, s, h) \right\} \]

\[+ \frac{1}{N_{\beta}} [(g_{\beta a} \circ \bar{\gamma}_i)(t, h), \partial_t (g_{\beta a} \circ \bar{\gamma}_i)(t, h)] \frac{\partial^{k-1}}{(k-2)! \partial s \partial t^{k-2}}(g_{\beta a} \circ \bar{c}_i)(t, s, h)\]

\[+ \ldots + \frac{1}{N_{\beta}} [(g_{\beta a} \circ \bar{\gamma}_i)(t, h), \partial_t (g_{\beta a} \circ \bar{\gamma}_i)(t, h)] \frac{\partial^{k-1}}{(k-1)! \partial t^{k-1}}(g_{\beta a} \circ \bar{c}_i)(t, s, h) \bigg|_{t=s=h=0}\]

where $\bar{c}_1(t, s, h) := x + h \xi_1 + sy + t \xi_1 + \ldots + t^k \xi_k$, $\bar{\gamma}_1(t, h) = \bar{c}_1(t, 0, h)$, $\ldots$, $\bar{c}_i(t, s, h) := x + sy + t \xi_1 + \ldots + t^{i-2} \xi_{i-2} + t^{i-1} (\xi_{i-1} + hi \xi_i) + t^i \xi_i + \ldots + t^k \xi_k$.
and $\tilde{\gamma}_i(t, h) = \tilde{c}_i(t, 0, h)$. As a result

$$kd\varphi_\alpha^k (x) M^k_\alpha (u_k) y$$

$$= -d^2 \varphi_\alpha(x) (\xi_1, M^k_\alpha (u_k - 1) y) + \sum_{i=1}^{k} \frac{\partial^{k}}{(k-1)! \partial \varphi s \partial h^{k-1}} \left( g_{\varphi_\alpha \circ \tilde{c}_i} (t, s, h) \right)$$

$$+ \frac{1}{(k-2)! \partial \varphi s \partial h^{k-2}} \left( g_{\varphi_\alpha \circ \tilde{c}_i} (t, s, h) \right)$$

$$+ \cdots + \frac{1}{N \beta (\tilde{u}_1) \left( \tilde{\eta}_{k-1} + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta}_{k-2} + \cdots + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta} \right) k_k + \frac{1}{(k-1)^2} \left[ \tilde{\eta}_{k-1} + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta}_{k-2} + \cdots + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta} \right]$$

To simplify the above terms, we use the induction hypothesis for the last line and we get

$$kd\varphi_\alpha^k (x) M^k_\alpha (u_k) y$$

$$= \frac{\partial^{k+1}}{(k-1)! \partial \varphi s \partial h^{k-1}} \left( g_{\varphi_\alpha \circ \tilde{c}_i} (t, s, h) \right)_{t=s=h=0}$$

$$+ \sum_{j=1}^{k} \partial_{j}^{1} \frac{\partial^{j}}{(j-1)! \partial \varphi s \partial h^{j-1}} \left( g_{\varphi_\alpha \circ \tilde{c}_i} (t, s, h) \right)_{t=s=h=0}$$

$$+ \cdots + \frac{1}{(k-2)! \partial \varphi s \partial h^{k-2}} \left( g_{\varphi_\alpha \circ \tilde{c}_i} (t, s, h) \right)_{t=s=h=0}$$

$$+ \cdots + \frac{1}{N \beta (\tilde{u}_1) \left( \tilde{\eta}_{k-1} + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta}_{k-2} + \cdots + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta} \right) k_k + \frac{1}{(k-1)^2} \left[ \tilde{\eta}_{k-1} + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta}_{k-2} + \cdots + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta} \right]$$

Now, using lemma 6.2 it follows that $kd\varphi_\alpha^k (x) M^k_\alpha (x, \xi_1, \ldots, \xi_k) y$ is equal to

$$k \tilde{\eta}_k + \sum_{j=1}^{2} \partial_{j}^{1} \frac{\partial^{j}}{(j-1)! \partial \varphi s \partial h^{j-1}} \left( g_{\varphi_\alpha \circ \tilde{c}_i} (t, s, h) \right)_{t=s=h=0}$$

$$+ \cdots + \frac{1}{(k-1)^2} \left[ \tilde{\eta}_{k-1} + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta}_{k-2} + \cdots + \frac{1}{N \beta (\tilde{u}_1)} \tilde{\eta} \right]$$
\[ = k\eta_k + 2N_\beta(u_2)\eta_{k-2} + \cdots + kN_\beta(u_k)\eta_0 + (k - 1)N_\beta(\bar{u}_1)\eta_{k-1} \]
\[ + (k - 2)N_\beta(u_2)\eta_{k-2} + \cdots + k\eta_{k-1} = k\eta_k + (1 + (k - 1))N_\beta(\bar{u}_1)\eta_{k-1} + \cdots + ((k - 1) + 1)N_\beta(\bar{u}_1)\eta_{k-1} \]
\[ + kN_\beta(\bar{u}_k)\eta_0 \]
where \( \xi_j = \text{Proj}_{j+1}(\bar{u}_k) \) and \( \text{Proj}_{j+1}, 0 \leq j \leq k - 1 \), is the projection map to \((j + 1)\)th factor. As a consequence we obtain
\[ dg_{\alpha}(y)^k M_\alpha(x, \xi_1, \ldots, \xi_k)y = \eta_k + N_\beta(\bar{u}_1)\eta_{k-1} + \cdots + N_\beta(\bar{u}_k)\eta_0. \]

However, as we are about to see, the desired compatibility condition is finally handled. More precisely we have shown that for vector bundle morphisms \( K_M \) and \( K_N \) locally
\[ \oplus_{i=1}^k TG_{\beta\alpha}(x)K_{M,\alpha}(u; y, 0, \ldots, 0) = K_{N,\beta} \circ TT^k g_{\beta\alpha}(u; y, 0, \ldots, 0). \]
Moreover by the induction hypothesis we have
\[ \oplus_{i=1}^k TG_{\beta\alpha}(x)K_{M,\alpha}(u; 0, \eta_1, 0, \ldots, 0) = K_{N,\beta} \circ TT^k g_{\beta\alpha}(u; 0, \eta_1, 0, \ldots, 0). \]
Summing these last two equalities conclude the verification of our claim that is, \( K_M \) and \( K_N \) are \( g \)-related connection maps on the \( k \)th order tangent bundles of \( M \) and \( N \) respectively, constructed only with the help of the connections \( \nabla_M \) and \( \nabla_N \). \( \Box \)

2.2. \( T^k M \) as a vector bundle. For \( k \geq 2 \), the bundle structure defined in theorem [23] is quite far from being a vector bundle due to the complicated nonlinear transition functions. However, according to [23] we have the following main theorem.

**Theorem 2.11.** Let \( \nabla \) be a linear connection on \( M \) and \( K \) the induced connection map introduced in proposition [2.4]. The following trivializations define a vector bundle structure on \( \pi^k M : T^k M \rightarrow M \) with the structure group \( GL(\mathbb{E}^k) \).

\( \Phi^k_{\alpha} : \pi^k M^{-1}(U_\alpha) \rightarrow \psi_\alpha(U_\alpha) \times \mathbb{E}^k \)
\[ [\gamma, x]_k \mapsto (\gamma_\alpha(0), \gamma^{(1)}(0), z^2([\gamma, x]_k), \ldots, z^k([\gamma, x]_k)) \]
where \( \gamma_\alpha = \phi_\alpha \circ \gamma \) and
\[ z^0_{\alpha}([\gamma, x]_k) = \frac{1}{2} \left\{ \frac{\gamma^{(2)}(0)}{2!} + M_\alpha [\gamma_\alpha(0), \gamma^{(1)}(0)] \gamma^{(1)}(0) \right\}, \]
\[ \vdots \]
\[ z^k_{\alpha}([\gamma, x]_k) = \frac{1}{k} \left\{ \frac{\gamma^{(k)}(0)}{k!} + M_\alpha [\gamma_\alpha(0), \gamma^{(1)}(0)] \gamma^{(k-1)}(0) \right\} + \frac{k}{k-1} M_\alpha [\gamma_\alpha(0), \gamma^{(1)}(0)] \gamma^{(k-1)}(0) \gamma^{(k)}(0)/(k-1)! \gamma^{(1)}(0). \]}
Moreover setting $\Phi^{k}_{\alpha\beta} = \Phi^{k}_{\alpha} \circ \Phi^{-1}_{\beta}$, $\phi_{\alpha\beta} = \phi_{\alpha} \circ \phi^{-1}_{\beta}$ and $U_{\alpha\beta} = U_{\alpha} \cap U_{\beta}$ the transition map

$$\Phi^{k}_{\alpha\beta} : U_{\alpha\beta} \rightarrow GL(E^{k})$$

is given by $\Phi^{k}_{\alpha\beta}(x)(\xi_{1}, \xi_{2}, \ldots, \xi_{k}) = \left(\phi_{\alpha\beta}(x), d\phi_{\alpha\beta}(x)\xi_{1}, \ldots, d\phi_{\alpha\beta}(x)\xi_{k}\right)$ that is $T^{k}M$, as a vector bundle, is isomorphic to $\oplus_{i=1}^{k}TM$.

The converse of the above theorem is also true i.e. if $\pi^{k}_{M} : T^{k}M \rightarrow M$, $k \geq 2$, admits a vector bundle structure isomorphic to $\oplus_{i=1}^{k}TM$, then a linear connection on $M$ can be defined $[23]$.

**Remark 2.12.** One can replace the induced connection map in theorem 2.11 with a general connection map (in the sense of definition 2.2) and prove the theorem in a similar fashion. One reason for using this rather elaborate model (induced connection maps) is that it permits a concrete way of constructing.

**Remark 2.13.** For $i < k$

$$\sigma_{M}^{k,i} : T^{k}M \rightarrow T^{i}M \quad [\gamma, x]_{k} \mapsto [\gamma, x]_{i}$$

also admits a vector bundle structure $[23]$.

As we have shown, the vector bundle structure on $T^{k}M$ depends on the chosen linear connection only. In the next section we ask about the extent to which this vector bundle structure remains isomorphic.

### 3. $T^{k}g$ as a Vector Bundle Morphism

For the differentiable map $g : M \rightarrow N$, in contrast to $T^{i}g = Tg : TM \rightarrow TN$, the tangent map $T^{k}g$, even for $k = 2$, is not necessarily a vector bundle morphism $[6, 22]$.

Let $[\gamma, x]_{k}$ be an arbitrary $k$’th order tangent vector to $M$ and $(\Phi^{k}_{\alpha}, \pi^{-1}_{k}(U_{\alpha}))$ and $(\Psi^{k}_{\beta}, \tau^{-1}_{k}(V_{\beta}))$, as in theorem 2.11 be two vector bundle trivializations around $x \in M$ and $g(x) \in N$ respectively. For $(x, \xi_{1}, \ldots, \xi_{n}) \in U_{\alpha} \times E^{k}$, define the curve $\bar{\mu}_{k}$ inductively as follows; $\bar{\mu}_{1}(t) = x + t\xi_{1}$, $\bar{\mu}_{2}(t) = \bar{\mu}_{1}(t) + \frac{t^{2}}{2}(2\xi_{2} - M_{\alpha}(x, \xi_{1})\xi_{1})$ and for $i \geq 2$,

$$\bar{\mu}_{i}(t) = \bar{\mu}_{i-1}(t) + \frac{t}{i!}\left[i\xi_{i} - M_{\alpha}(x, \xi_{1})(\bar{\mu}^{(i-1)}_{(i-1)}(0))\xi_{1}\right] - \ldots$$

$$i^{-1}\left[M_{\alpha}(x, \xi_{1})(\bar{\mu}^{(2)}_{i-1}(0)), \ldots, \frac{1}{(i-1)!}(\bar{\mu}^{(i-1)}_{i-1}(0))\xi_{1}\right].$$

In order to reduce the intricate computations as much as possible, we will use the following lemma.

**Lemma 3.1.** Fix the positive integer $k \geq 2$ and suppose that $\bar{\mu} := \bar{\mu}_{k}$ be the curve defined above, $\mathcal{O} \subseteq E$ be open and $f : \mathcal{O} \rightarrow E$ be any smooth map. Then

$$\frac{\partial^{k}}{\partial s\partial t^{k-1}}(f \circ \bar{d}_{k})(t, s)|_{t=s=0} = (f \circ \bar{\mu})^{(k)}(t)|_{t=0}$$

(7)
where
\[ \bar{d}_k : (-\epsilon, \epsilon)^2 \to \mathbb{E} : (t, s) \mapsto \sum_{i=1}^{k-1} \frac{i^i}{i!} (\bar{\mu}^{(i)}(0) + s\bar{\mu}^{(i+1)}(0)). \]

**Proof.** See appendix. \(\square\)

Now suppose that \(\mu := \phi^{-1}_\alpha \circ \bar{\mu}\). Then the local representation of \(T^k g\) is
\[ \Psi^k_\beta \circ T^k g \circ \Phi^{-1}_\alpha(x, \xi_1, \ldots, \xi_k) = \Psi^k_\beta \circ T^k g([\mu, x]_k) = \Psi^k_\beta([\mu \circ g, x]_k). \]

With the fact \(\psi_\beta \circ g \circ \mu = \psi_\beta \circ g \circ \phi^{-1}_\alpha \circ \psi_\alpha \circ \mu := g_{\beta \alpha} \circ \bar{\mu}\) in mind, we apply the vector bundle trivialization of theorem 2.11 to \([g \circ \mu, g(x)]_k\) and we get
\[ \Psi^k_\beta([g \circ \mu, g(x)]_k) = \left( g_{\beta \alpha}(x), dg_{\beta \alpha}(x)\xi_1, z^j_{\beta}(\mu \circ g, g(x))_k, \ldots, z^{k}_{\beta}(\mu \circ g, g(x))_k \right) \]

However, for \(2 \leq i \leq k\) we have
\[ 2_{\beta}(g \circ \mu, g(x))_k = d g_{\beta \alpha}(x) \left( i \xi_i - \frac{1}{1 \alpha} M_\alpha (v_1) (\bar{\mu})^{(i-1)}(0) \right. \]
\[ \left. + \frac{1}{(i-1)!} \left( \sum_{j=2}^{i-1} a_{i,j}^j d^2 g_{\beta \alpha}(x) \left( \bar{\mu}^{(i-j)}(0), \bar{\mu}^{(j)}(0) \right) + \right. \right. \]
\[ \left. \left. + d^2 g_{\beta \alpha}(x) \left( \bar{\mu}^{(i)}(0), \bar{\mu}^{(i)}(0) \right) \right\} \right) \]
\[ + 1 \left( v_1 \right) \left( g_{\beta \alpha} \circ \bar{\mu}^{(i-1)}(0) \right) \]
\[ + \cdots + \left( v_{i-1} \right) \left( g_{\beta \alpha} \circ \bar{\mu}^{(1)}(0) \right) \]

where \(v = (\bar{\mu}(0), \bar{\mu}^{(1)}(0), \ldots, \bar{\mu}^{(i)}(0))\), \(v_j = \text{Proj}_{i+1} (v)\),
\[ \bar{v} = \left( (g_{\beta \alpha} \circ \bar{d})(0,0), \frac{\partial}{\partial t} (g_{\beta \alpha} \circ \bar{d})(0,0), \ldots, \frac{\partial^k}{\partial t^k} (g_{\beta \alpha} \circ \bar{d})(0,0) \right) \]
\[ \left. \frac{\partial}{\partial t} (g_{\beta \alpha} \circ \bar{\mu})(0,0), \left( g_{\beta \alpha} \circ \bar{\mu}(1)(0) \right), \ldots, \left( g_{\beta \alpha} \circ \bar{\mu}^{(i)}(0) \right) \right) \]

and \(\bar{v}_j = \text{Proj}_{i+1} (\bar{v})\).

Alas, as we can see now, due to the presence of higher order derivatives, as \(k\) increases, generally it becomes increasingly difficult for \(T^k g\) to be a vector bundle morphism.

To get around this difficulty, we assume that \(K_M\) and \(K_N\) are \(g\) related.

**Theorem 3.2.** Let \(\nabla_M\) and \(\nabla_N\) be two linear connections \(M\) and \(N\) respectively. If \(\nabla_M \sim_g \nabla_N\) then, \(T^k g : T^k M \to T^k N\) becomes a vector bundle morphism.

**Proof.** If \([\mu, x]_k \in T^k x M\) then, \(\pi_N \circ T^k g([\mu, x]_k) = \pi_N([g \circ \mu, g(x)]_k)\) meaning that \(T^k g\) is fibre preserving. Now consider the trivializations \(\Phi^{-1}_\alpha\) and \(\Phi^{-1}_\beta\), as introduced in theorem 2.11 around \(x\) and \(g(x)\) respectively. Our main task is to show that \(\Psi^k_\beta \circ T^k g \circ \Phi^{-1}_\alpha\) is linear on fibres.
Step 1. Setting \( \eta_i := \frac{1}{(i-1)!} \bar{\mu}^{(i)}(0) \), \( y := \bar{\mu}^{(1)}(0) \) and \( \xi_i := \frac{1}{i!} \bar{\mu}^{(i)}(0) \) in compatibility condition \([4]\), and using lemma \([3,2]\) with \( f := g_{\beta \alpha} \) we get

\[
dg_{\beta \alpha}(x) \left( \frac{\bar{\mu}^{(0)}}{(i-1)!} + \frac{\bar{\mu}^{(i-1)}}{(i-2)!} + \cdots + \frac{\bar{\mu}^{(1)}}{(i-1)!} \right) + dg_{\beta \alpha}(x) \left( v_1 \frac{\bar{\mu}^{(0)}}{(i-1)!} + \cdots + \frac{\bar{\mu}^{(1)}}{(i-1)!} \right) = \frac{1}{(i-1)!} N_{\beta} \left( \bar{v}_1 \right) \bar{\mu}(0) + \cdots + \frac{1}{(i-1)!} N_{\beta} \left( \bar{v}_{i-1} \right) \bar{\mu}(0).
\]

As a consequence we obtain

\[
\frac{1}{(i-1)!} N_{\beta} \left( \bar{v}_1 \right) \left( g_{\beta \alpha} \circ \bar{\mu} \right)^{(i-1)}(0) + \cdots + \frac{1}{(i-1)!} N_{\beta} \left( \bar{v}_{i-1} \right) \left( g_{\beta \alpha} \circ \bar{\mu} \right)^{(1)}(0) = dg_{\beta \alpha}(x) \left( \frac{\bar{\mu}^{(0)}}{(i-1)!} + \cdots + \frac{\bar{\mu}^{(1)}}{(i-1)!} \right) + \frac{1}{(i-1)!} g_{\beta \alpha}(x) \left( v_1 \frac{\bar{\mu}^{(0)}}{(i-1)!} + \cdots + \frac{\bar{\mu}^{(1)}}{(i-1)!} \right) - \frac{g_{\beta \alpha} \circ \bar{\mu}^{(i)}(0)}{(i-1)!}.
\]

Step 2. We now apply the previous observation to \([5]\) and we obtain

\[
iz_{\beta}^k \left( [g \circ \mu, g(x)]_k \right) = idg_{\beta \alpha}(x) \xi_1 + \frac{1}{(i-1)!} \left\{ \sum_{l_1 + l_2 = i} a_{l_1, l_2} \partial g_{\beta \alpha}(x) \left( (\bar{\mu})^{(l_1)}(0), (\bar{\mu})^{(l_2)}(0) \right) + \cdots + \partial g_{\beta \alpha}(x) \left( (\bar{\mu})^{(1)}(0), (\bar{\mu})^{(0)}(0) \right) \right\} + \frac{1}{(i-1)!} \left( g_{\beta \alpha} \circ \bar{\mu}^{(0)} \right) \partial g_{\beta \alpha}(x) \xi_1.
\]

As a consequence we have

\[
\Psi_{\beta}^k \circ T^k g \circ \Phi_{\alpha}^{-1} \left( x, \xi_1, \ldots, \xi_k \right) = \left( g_{\beta \alpha}(x), dg_{\beta \alpha}(x)(\xi_1), \ldots, dg_{\beta \alpha}(x)(\xi_k) \right).
\]

This last means that \( T^k g \) is fibre linear and

\[
T^k g_{\beta \alpha} : U_\alpha \to L(E^k, E^k)
\]

\[
x \mapsto \Psi_{\beta, g(x)}^k \circ T^k g \circ \Phi_{\alpha, x}^{-1}
\]

is a smooth morphism which completes the proof.

**Remark 3.3.** If \( M = N \) and \( g = id_M \) then theorem \([3,2]\) yields

\[
\Phi_{\beta}^k \circ \Phi_{\alpha}^{-1} : U_\alpha \to L(E^k, E^k)
\]

\[
x \mapsto (d\phi_{\beta} \circ \phi_{\alpha}^{-1}(x)(.), \ldots, d\phi_{\beta} \circ \phi_{\alpha}^{-1}(x)(.))
\]
as was noted by theorem 3.3 of [23].

The next corollary is a direct consequence of theorems 3.2 and 3.4.

**Corollary 3.4.** If \( g \) is a diffeomorphism, then \( T^kg \) becomes a vector bundle isomorphism.

**Remark 3.5.** Again, one can replace the induced connection maps with general connection maps and prove theorem 3.2 in an exactly similar way.

**Remark 3.6.** Let \( \nabla_M \) and \( \nabla_N \) be two \( g \)-related connections on \( M \) and \( N \) respectively. Consider the vector bundle structures on \((T^kM, \pi_M^k, T^iM)\) and \((T^kN, \pi_N^k, T^iN)\) proposed by theorem 2.11. Then the argument of theorem 3.2 can easily be modified to yield a proof to show that

\[(T^kg, T^ig) : (T^kM, \pi_M^k, T^iM) \rightarrow (T^kN, \pi_N^k, T^iN)\]

is also a vector bundle morphism for \( i < k \).

4. \( T^\infty g \) AS A VECTOR BUNDLE MORPHISM

As we have seen in the previous sections, at the presence of \( g \)-related connections on \( M \) and \( N \) for any \( k \in \mathbb{N} \), \( (T^kg, g) \) becomes a vector bundle morphism. If we take one step further by considering \( T^\infty M \) and \( T^\infty N \), as generalized Fréchet vector bundles over \( M \) and \( N \) respectively [23], then proving \( T^\infty g \) to be a generalized vector bundle morphism becomes much more complicated.

More precisely the set of linear maps between \( F := \lim \mathbb{E}^k \) and \( F' := \lim \mathbb{E}^k \) (the fibre types of \((T^\infty M, \pi_M^\infty, M)\) and \((T^\infty N, \pi_N^\infty, N)\) respectively) does not remain in the category of Fréchet spaces [11, 14].

In this section employing the projective limit methodology, as in [1, 10, 23, 24] etc., we show that \((T^\infty g, g)\) becomes a vector bundle morphism at the presence of \( g \)-related connections on \( M \) and \( N \).

Of course one can consider a projective system of \( g \)-related connection maps on \( T^kM \) and \( T^kN \), \( k \in \mathbb{N} \) and prove the same results.

Let the notation be as in the preceding sections and for the natural numbers \( j \geq i \) consider the projections \( \pi_M^{j,i} : T^jM \rightarrow T^iM \) and \( \pi_N^{j,i} : T^jN \rightarrow T^iN \) mapping \([\gamma, x]_j\) onto \([\gamma, x]_i\) as connecting morphisms of the projective families \( \{T^kM\}_{k \in \mathbb{N}} \) and \( \{T^kN\}_{k \in \mathbb{N}} \) (for more details see [23, 24]). The family \( \{T^kg\}_{k \in \mathbb{N}} \) form a projective system of maps since \( \pi_N^{j,i} \circ T^jg = T^ig \circ \pi_M^{j,i} \). More precisely

\[\pi_N^{j,i} \circ T^jg([\gamma, x]_j) = \pi_N^{j,i}([g \circ \gamma, g(x)]_j) = [g \circ \gamma, g(x)]_i,\]

and

\[T^ig \circ \pi_M^{j,i}([\gamma, x]_j) = T^ig([\gamma, x]_i) = [g \circ \gamma, g(x)]_i.\]

As a consequence the limit map \( T^\infty g := \lim T^kg \) exists and maps the thread \( ([\gamma, x]_k)_{k \in \mathbb{N}} \in T^\infty M := \lim T^kM \) to \( ([g \circ \gamma, x]_k)_{k \in \mathbb{N}} \in T^\infty N = \lim T^kN \). It is easily checked that, for any \( x \in M \), the families \( \{\Phi^k_{\alpha, x}\}_{k \in \mathbb{N}} \) and \( \{\Psi^k_{\beta, g(x)}\}_{k \in \mathbb{N}} \), as in theorem 2.11, form projective systems of trivializations with the limits \( \Phi^{\infty}_{\alpha, x} \) and \( \Psi^{\infty}_{\beta, g(x)} \) respectively (see also [10]).
But $T^\infty g$ seems to be far from being called a vector bundle morphism due to the difficulties emerged in $\mathcal{L}(\mathbb{F}, \mathbb{F}')$ and therefor the problematic map

$$T^\infty g_{\beta \alpha} : U_\alpha \rightarrow \mathcal{L}(\mathbb{F}, \mathbb{F}')$$

and $x \mapsto \Psi^\infty_{\beta, g(x)} \circ T^\infty \circ g \circ \Phi^\infty_{\alpha, x}^{-1}$.

To overcome this obstacle define

$$\mathcal{H}(\mathbb{F}, \mathbb{F}') = \{(l_k)_{k \in \mathbb{N}} \in \prod_{k=1}^{\infty} \mathcal{L}(\mathbb{E}^k, \mathbb{E}^k) : \rho'_{ji} \circ l_j = l_i \circ \rho_{ji}, \forall j \geq i \}$$

where $\rho_{ji} : \mathbb{E}^j \rightarrow \mathbb{E}^i$ and $\rho'_{ji} : \mathbb{E}'^j \rightarrow \mathbb{E}'^i$ are the projection maps to the first $i$ factors. $\mathcal{H}(\mathbb{F}, \mathbb{F}')$ is a Fréchet space $\mathbb{H}^{\infty}([10])$ isomorphic to the projective limit of the projective system of Banach spaces $\{\mathcal{H}^k(\mathbb{E}^k, \mathbb{E}^k)\}_{k \in \mathbb{N}}$ where

$$\mathcal{H}^k(\mathbb{E}^k, \mathbb{E}^k) = \{(l_i)_{1 \leq i \leq k} \in \prod_{i=1}^{k} \mathcal{L}(\mathbb{E}^i, \mathbb{E}^i) : \rho'_{ji} \circ l_j = l_i \circ \rho_{ji}, \forall 1 \leq i \leq j \leq k \}.$$

However, for any $\xi_1, \ldots, \xi_j \in \mathbb{E}$ and $j \geq i$ we have

$$\rho'_{ji} \circ (\Psi^j_{\beta, g(x)} \circ T^j g \circ \Phi^j_{\alpha, x}^{-1})(\xi_1, \ldots, \xi_j) = \rho'_{ji}(dg_{\beta \alpha}(x)\xi_1, \ldots, dg_{\beta \alpha}(x)\xi_j)$$

$$= (dg_{\beta \alpha}(x)\xi_1, \ldots, dg_{\beta \alpha}(x)\xi_j) = (\Psi^j_{\beta, g(x)} \circ T^j g \circ \Phi^j_{\alpha, x}^{-1}) \circ \rho_{ji}(\xi_1, \ldots, \xi_j)$$

meaning that $\{(\Psi^k_{\beta, g(x)} \circ T^k g \circ \Phi^k_{\alpha, x}^{-1})\}_{k \in \mathbb{N}}$, with the limit $\Psi^\infty_{\beta, g(x)} \circ T^\infty g \circ \Phi^\infty_{\alpha, x}^{-1}$, belongs to $\mathcal{H}(\mathbb{F}, \mathbb{F}')$. As a consequence $T^\infty g_{\beta \alpha} : U_\alpha \rightarrow \mathcal{L}(\mathbb{F}, \mathbb{F}'); x \mapsto \varepsilon \circ \Psi^\infty_{\beta, g(x)} \circ T^\infty g \circ \Phi^\infty_{\alpha, x}^{-1}$ is smooth in the sense of Leslie and Galanis $\mathbb{H}^{\infty}([15][16][9][10])$ where

$$\varepsilon : \mathcal{H}(\mathbb{F}, \mathbb{F}') \rightarrow \mathcal{L}(\mathbb{F}, \mathbb{F}')$$

$$(l_k)_{k \in \mathbb{N}} \mapsto \lim_{k \rightarrow \infty} l_k.$$

As a consequence, $T^\infty g$ is smooth.

Finally, following the argument in section $\mathbb{B}$ we get:

**Theorem 4.1.** The pair $(T^\infty g, g) : (T^\infty M, \pi^\infty_M, M) \rightarrow (T^\infty N, \pi^\infty_N, N)$ is a generalized vector bundle morphism. Moreover, the bundle morphism $(T^\infty g, g)$ is a vector bundle isomorphism if $g$ is a diffeomorphism.

**Remark 4.2.** It is easy to check that $(T^\infty g, T^k g) : (T^\infty M, \pi^\infty_M, T^k M) \rightarrow (T^\infty N, \pi^\infty_N, T^k N), 1 \leq k < \infty$, is also a vector bundle morphism where $\pi^\infty_M$ maps the class $[\gamma, x]_\infty$ to $[\gamma, x]_k$.

5. Applications and examples

In this section we settle our results to the special case of $f : M \rightarrow N$, where $f$ is an immersion and $N$ is a Riemannian Hilbert manifold and to the case of manifold of $C^r$ maps between manifolds $M$ and $N$ denoted by $C^r(N, M)$.

**Example 5.1.** Let $M$ and $N$ be two smooth manifolds modeled on the Hilbert spaces $\mathbb{E}$ and $\mathbb{F}$ respectively and $f : M \rightarrow N$ be a smooth immersion. Moreover suppose that $h$ be a Riemannian metric on $N$ with the
Levi-Civita connection $\nabla_N$. It is known that the immersion $f$ induces a Riemannian metric $g$ on $M$ defined by
\[ g(p)(u, v) := h(f(p))(d(f(p))u, d(f(p))v); \ \forall p \in M \ 	ext{and} \ \forall u, v \in T_pM. \]
Denote by $\nabla_N$ the associated Levi-Civita connection of $g$. In what follows we will show that $\nabla_N$ and $\nabla_N$ are $f$-related.

For $p \in M$ consider the charts $(U_\alpha, \phi_\alpha)$ and $(V_\alpha, \psi_\beta)$ around $p$ and $f(p) := q$ respectively. Setting
\[ g_\alpha(p_0) := g(p) \circ (d_{p_0}^{-1} \circ d_{p_0}^{1}) : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \]
and
\[ h_\alpha(q_0) := h(q) \circ (d_{q_0}^{1} \circ d_{q_0}^{-1}) : \mathbb{F} \times \mathbb{F} \to \mathbb{R} \]
we observe that
\[ g_\alpha(p_0)(u, v) = g(p)(d_{p_0}^{-1}u, d_{p_0}^{1}v) \]
\[ = h(f(p))(d_{p_0}f \circ \phi_{-1}^{-1}u, d_{p_0}f \circ \phi_{1}^{-1}v) \]
\[ = h_\beta(q_0)(d_{p_0}f_{\beta_0}u, d_{p_0}f_{\beta_0}v) \]
where $p_0 := \phi(p)$, $q_0 := f_{\beta_0}(p_0)$ and $f_{\beta_0} := \psi_\beta \circ f \circ \phi_{-1}^{-1}$. As a consequence of (10) and Leibniz’s rule we have
\[
dg_\alpha(p_0).u(u, v) = dh_\beta(q_0).d_{p_0}f_{\beta_0}u(d_{p_0}f_{\beta_0}u, d_{p_0}f_{\beta_0}v) \\
+ h_\beta(q_0)(d_{p_0}^2f_{\beta_0}(u, v), d_{p_0}f_{\beta_0}(u, v)) \]
Now, using the Koszul formula (e.g. [8, 12]) we get
\[ g_\alpha(p_0)(\Gamma^M_\alpha(p_0)(u, v), w) = \frac{1}{2}\left\{ dg_\alpha(p_0).u(v, w) + dg_\alpha(p_0).v(u, w) \\
- dg_\alpha(p_0).w(u, v) \right\} \]
\[ = \frac{1}{2}\left\{ dh_\beta(q_0).d_{p_0}f_{\beta_0}u(d_{p_0}f_{\beta_0}u, d_{p_0}f_{\beta_0}w) + h_\beta(q_0)(d_{p_0}^2f_{\beta_0}(u, v), d_{p_0}f_{\beta_0}w) \\
+ h_\beta(q_0)(d_{p_0}f_{\beta_0}v, d_{p_0}^2f_{\beta_0}(u, v)) \right\} \]
\[ + dh_\beta(q_0).d_{p_0}f_{\beta_0}v(d_{p_0}f_{\beta_0}u, d_{p_0}f_{\beta_0}w) + h_\beta(q_0)(d_{p_0}^2f_{\beta_0}(u, v), d_{p_0}f_{\beta_0}w) \\
+ h_\beta(q_0)(d_{p_0}f_{\beta_0}u, d_{p_0}^2f_{\beta_0}v, d_{p_0}f_{\beta_0}w) \right\} \]
\[ = h_\beta(q_0)(\Gamma^N_\beta(q_0)(d_{p_0}f_{\beta_0}u, d_{p_0}f_{\beta_0}v), d_{p_0}f_{\beta_0}w) + h_\beta(q_0)(d_{p_0}^2f_{\beta_0}(u, v), d_{p_0}f_{\beta_0}w) \]
for any $u, v, w \in \mathbb{E}$. On the other hand equation (10) now reads
\[ g_\alpha(p_0)(\Gamma^M_\alpha(p_0)(u, v), w) = h_\beta(q_0)(d_{p_0}f_{\beta_0}\Gamma^M_\alpha(p_0)(u, v), d_{p_0}f_{\beta_0}w) \]
Since $g$ is non-degenerate we deduce that
\[ d_{p_0}f_{\beta_0}\Gamma^M_\alpha(p_0)(u, v) = \Gamma^N_\beta(q_0)(d_{p_0}f_{\beta_0}u, d_{p_0}f_{\beta_0}v) + d_{p_0}^2f_{\beta_0}(u, v) \]
that is $\nabla^M_\beta$ and $\nabla^N_\beta$ are $f$-related linear connections.
As a consequence of theorems 2.11, 3.2 and 4.1 for \( k \in \mathbb{N} \cup \{ \infty \} \), \( T^k M \) and \( T^k N \) admit vector bundle structures and in this case \((T^k f, f) : (T^k M, \pi^k_M, M) \rightarrow (T^k N, \pi^k_N, N)\) becomes a vector bundle morphism. Moreover if \( f \) is a diffeomorphism (isometry) then, \( T^k f \) is a vector bundle isomorphism. In this case \( T^k g \) is also an isometry for \( k \in \mathbb{N} \) and the induced metrics introduced in section 3.3 of [23].

**Example 5.2.** Let \( N \) be a \( C^\infty \) compact manifold and \( M \) be a \( C^\infty \) Banach (possibly infinite dimensional) manifold with a linear connection \( \nabla_M \). According to [7], \( C^r(N, M) \), the space of all \( C^r \) maps \( 0 \leq r < \infty \) from \( N \) to \( M \), form a Banach manifold with the following charts. Let \( exp : O \subset TM \rightarrow M \) be the exponential map corresponding to the linear connection \( \nabla_M \). Moreover suppose that \( D \) be an open neighborhood of zero section in \( TM \) such that \((\pi_M, exp)]_D\) form \( D \) to \((\pi_M, exp)(D) \subset M \times M \) is a diffeomorphism. For the \( C^r \) map \( h : N \rightarrow M \) the chart \((\phi_h, U_h)\) defined by

\[
C^r(exp) : C^r(h^*D) \rightarrow C^r(N, M) : \xi \mapsto exp \circ \xi.
\]

In the notation above \( C^r(h^*D) \) is the set of all sections \( \xi : N \rightarrow TM \) with the property \( \pi_M \circ \xi = h \). In this case \( C^r(h^*D) \) becomes a Banach space with the norm

\[
\|\xi\|_{C^r} = \sum_{j=0}^{r} \|\nabla^j \xi\|_{C^0} := \sum_{j=0}^{r} \sup_{p \in N} \|\nabla^j \xi(p)\|.
\]

which serves as the model space of \( C^r(N, M) \).

Moreover the connection \( \nabla_M \) induces a connection on \( C^r(N, M) \) with the connection map \( C^r(\nabla_M) : TT C^r(N, M) \simeq C^r(N, TT M) \rightarrow C^r(N, TM) \simeq TC^r(N, M) \) which maps \( A \in C^r(N, TT M) \) to \( C^r(\nabla_M)(A) = \nabla_M \circ A \). Since \( \nabla_M \) is a linear connection, so \( C^r(\nabla_M) \) also is ([7] Theorem 5.4).

According to theorem 2.11, \( T^k M \) and \( T^k C^r(N, M) \) admits vector bundle structures on \( M \) and \( C^r(N, M) \) respectively. Moreover \( T^k(\nabla_M) \simeq \oplus_{j=0}^{k} TT C^r(N, M) \) and \( C^r(N, T^k M) \simeq C^r(N, \oplus_{j=1}^{k} TM) \) are isomorphic vector bundles over \( C^r(N, M) \).

For the Banach manifolds \( M \) and \( M' \) and the smooth map \( g : M \rightarrow M' \) the map \( C^r(g) : C^r(N, M) \rightarrow C^r(N, M') \) defined by \( f \mapsto g \circ f \) is differentiable and \( TC^r(g) = C^r(Tg) \) [7].

Now, suppose that \( \nabla_M \) and \( \nabla_{M'} \) be two \( g \)-related connections on \( M \) and \( M' \) respectively. Then, \( C^r(\nabla_{M'}) \circ TT C^r(g) = C^r(\nabla_{M'}) \circ C^r(TT g) = C^r(\nabla_{M'} \circ TT g) = C^r(T g \circ \nabla_M) = TC^r(g) \circ C^r(\nabla_M) \) that is \( C^r(\nabla_M) \) and \( C^r(\nabla_{M'}) \) are \( C^r(g) \)-related. As a consequence of theorem 5.2, \( T^k g : T^k M \rightarrow T^k M' \) and \( T^k C^r(g) : T^k C^r(N, M) \rightarrow T^k C^r(N, M') \) are vector bundle morphisms.

6. Appendix

In this section, using the chain rule formula (1) we prove lemma 3.1. For \((x, \xi_1, \ldots, \xi_n) \in U_\alpha \times E^k\), define the curve \( \bar{\mu}_h \) inductively as in section 3 by
Lemma 6.2. Let \( \bar{\mu} \) be the map defined above, \( O \subseteq \mathbb{E} \) be open and \( f : O \rightarrow \mathbb{E} \) be any smooth map. Then
\[
\begin{align*}
\frac{\partial^k}{\partial s \partial t^{k-1}}(f \circ \bar{d}_k)(t, s)|_{t=s=0} &= (f \circ \bar{\mu})^{(k)}(t)|_{t=0}
\end{align*}
\]
\( \bar{d}_k := (\epsilon, e)^2 \rightarrow \mathbb{E} ; (t, s) \mapsto \sum_{i=1}^{k-1} \frac{t^i}{i!}(\bar{\mu}^{(i)}(0) + s\bar{\mu}^{(i+1)}(0)). \)

Proof. Using the chain rule formula (1) we observe that
\[
\begin{align*}
\frac{\partial^k}{\partial s \partial t^{k-1}}(f \circ \bar{d}_k)(t, s)|_{t=s=0} &=
\sum_{l_1, l_2 = 0}^{k-1} \frac{d^{l_1} f(x + s\xi_1)}{l_1!} \bigg|_{l_1 = 0} \cdot \frac{d^{l_2} f(x + s\xi_2)}{l_2!} \bigg|_{l_2 = 0} + \sum_{l_1 + l_2 = k} \frac{d^{l_1} f(x + s\xi_1)}{l_1!} \cdot \frac{d^{l_2} f(x + s\xi_2)}{l_2!} \\
&= df(x)[\bar{\mu}^{(k)}(0)] + d^2 f(x)[\bar{\mu}^{(1)}(0), \bar{\mu}^{(k-1)}(0)] + \cdots + d^{k-1} f(x)[\bar{\mu}^{(1)}(0), \bar{\mu}^{(k-1)}(0)]
\end{align*}
\]

as claimed. \( \square \)

Finally, we leave it to the reader to verify that

Lemma 6.1. Let \( \bar{\mu} \) be the map defined above, \( O \subseteq \mathbb{E} \) be open and \( f : O \rightarrow \mathbb{E} \) be any smooth map. Then
\[
\begin{align*}
\frac{\partial^k}{\partial s \partial t^{k-1}}(f \circ \bar{d}_k)(t, s)|_{t=s=0} &= (f \circ \bar{\mu})^{(k)}(t)|_{t=0}
\end{align*}
\]
\( \bar{d}_k := (\epsilon, e)^2 \rightarrow \mathbb{E} ; (t, s) \mapsto \sum_{i=1}^{k-1} \frac{t^i}{i!}(\bar{\mu}^{(i)}(0) + s\bar{\mu}^{(i+1)}(0)). \)

Proof. Using the chain rule formula (1) we observe that
\[
\begin{align*}
\frac{\partial^k}{\partial s \partial t^{k-1}}(f \circ \bar{d}_k)(t, s)|_{t=s=0} &=
\sum_{l_1, l_2 = 0}^{k-1} \frac{d^{l_1} f(x + s\xi_1)}{l_1!} \bigg|_{l_1 = 0} \cdot \frac{d^{l_2} f(x + s\xi_2)}{l_2!} \bigg|_{l_2 = 0} + \sum_{l_1 + l_2 = k} \frac{d^{l_1} f(x + s\xi_1)}{l_1!} \cdot \frac{d^{l_2} f(x + s\xi_2)}{l_2!} \\
&= df(x)[\bar{\mu}^{(k)}(0)] + d^2 f(x)[\bar{\mu}^{(1)}(0), \bar{\mu}^{(k-1)}(0)] + \cdots + d^{k-1} f(x)[\bar{\mu}^{(1)}(0), \bar{\mu}^{(k-1)}(0)]
\end{align*}
\]

as claimed. \( \square \)
\[
\frac{\partial^j}{\partial h \partial t^j} \sum_{i=1}^{k} (f \circ \tilde{c}_i)(t, s, h)|_{t=s=h=0} = \frac{\partial^j}{\partial y^j}(f \circ \tilde{c})(t, s)|_{t=s=0} \\
\]
\(j = 1, \ldots, k.\)

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Department of Mathematics, Faculty of sciences, Bu-Ali Sina University, Hamedan 65178, Iran.

E-mail address: ali.suri@yahoo.com & a.suri@math.iut.ac.ir & a.suri@basu.ac.ir