EXISTENCE AND CONVERGENCE OF PUISEUX SERIES SOLUTIONS FOR FIRST ORDER AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. Given an algebraic first order autonomous ordinary differential equation $F(y, y') = 0$, we prove that every formal Puiseux series solution of $F(y, y') = 0$, expanded around any finite point or at infinity, is convergent. The proof is constructive and we provide an algorithm to describe all such Puiseux series solutions. Moreover, we show that for any point in the complex plane there exists a solution of the differential equation which defines an analytic curve passing through this point.

keywords Algebraic differential equation, algebraic curve, place, formal Puiseux series solution, convergent solution.

1. Introduction

We study local solutions of nonlinear autonomous first order ordinary differential equations of the form $F(y, y') = 0$, where $F(y, p)$ is a polynomial (or indeed a holomorphic function) in two variables. Rational and algebraic solutions of these equations have been studied in [11, 12] and [2]. In particular, they found degree bounds of the possible rational or algebraic solutions such that these global solutions can be computed algorithmically. In [10] it is proven that any formal power series solution of an autonomous first order ordinary differential equations is convergent. We extend this result to the case of fractional power series solutions and give an algorithm to compute all of them.

The problem of finding power series solutions of ordinary differential equations has been extensively studied in the literature. A method to compute generalized formal power series solutions, i.e. power series with real exponents, and describe their properties is the Newton
polygon method. A description of this method is given in [13, 14] and more recently in [15, 8, 1]. In [6], the first author, using the Newton polygon method, gives a theoretical description of all generalized formal power series solution of a non-autonomous first order ordinary differential equation as a finite set of one parameter families of generalized formal power series. This description of the solutions is in general not algorithmic by several reasons. One of them is that there is no bound on the number of terms which have to be computed in order to guarantee the existence of a generalized formal power series solution when extending a given truncation of a determined potential solution. Also the uniqueness of the extension can not be ensured a-priori. The direct application of the Newton polygon method to a first order autonomous differential equations does not provide any advantage with respect to the non-autonomous case, because during the computations the characteristic of being autonomous gets lost.

In [19] they derive an associated differential system to find rational general solutions of non-autonomous first order differential equations by considering rational parametrizations of the implicitly defined curve. We instead consider its places and obtain an associated differential equation of first order and first degree which can be treated by the Newton polygon method, described in [4]. Using the known bounds for computing places of algebraic curves (see e.g. [9]), existence and uniqueness of the solutions and the termination of our computations can be ensured.

The structure of the paper is as follows. Section 2 is devoted to recall the preliminary theory on formal Puiseux series and algebraic curves used throughout the paper. In Section 3 we show that every non-constant formal Puiseux series solution defines a place of the associated curve. We give a necessary condition on a place of the curve to contain in its equivalence class formal Puiseux series solutions of the original differential equation, and show the analyticity of them. In the case where the solutions are expanded around a finite point, the necessary condition turns out to be sufficient as well. As a byproduct, we obtain a new proof of the fact that there is an analytic solution curve of \( F(y, y') = 0 \) passing through any given point in the plane. This result is a consequence of section 6.10 in [3]. In Section 4 algorithms for computing all Puiseux series solutions are presented and illustrated by examples. Subsection 4.1 is devoted to solutions expanded around zero. For proving the correctness of the algorithm, we give a precise bound on the number of terms such that the solutions are in bijection with the corresponding truncations. In Subsection 4.2 we consider solutions expanded at infinity. Here we are able to compute for every solution a
2. Puiseux series solutions and places

In this section we introduce the notation, assumptions, and main notions that will be used throughout this paper.

Let us consider the differential equation

\[(2.1)\]
\[F(y, y') = 0,\]

where \(F \in \mathbb{C}[y, p]\) is non-constant in the variables \(y\) and \(p\).

We will study the existence and the convergence of formal Puiseux series solutions of (2.1). Formal Puiseux series can either be expanded around a finite point or at infinity. In the first case, since equation (2.1) is invariant under translations of the independent variable, without loss of generality we can assume that the formal Puiseux series is expanded around zero and it is of the form \(\varphi(x) = \sum_{j \geq j_0} a_j x^{j/n}\), where \(a_j \in \mathbb{C}\), \(n \in \mathbb{N} \setminus \{0\}\) and \(j_0 \in \mathbb{Z}\). In the case of infinity we can use the transformation \(x = 1/z\) obtaining the (non-autonomous) differential equation \(F(y(z), -z^2 y'(z)) = 0\). In order to deal with both cases in a unified way, we will study equations of the type

\[(2.2)\]
\[F(y(x), x^h y'(x)) = 0,\]

with \(h \in \mathbb{Z} \setminus \{1\}\) and its formal Puiseux series solutions expanded around zero. We note that for \(h = 0\) equation (2.2) is equal to (2.1) and for \(h = 2\) the case of formal Puiseux series solutions expanded at infinity is treated. In the sequel, we assume that \(h\) is fixed.

We use the notations \(\mathbb{C}[[x]]\) for the ring of formal power series, \(\mathbb{C}((x))\) for its fraction field and \(\mathbb{C}((x))^* = \bigcup_{n \geq 1} \mathbb{C}((x^{1/n}))\) for the field of formal Puiseux series expanded at zero. We call the minimal natural number \(n\) such that \(\varphi(x)\) belongs to \(\mathbb{C}((x^{1/n}))\) the ramification order of \(\varphi(x)\).

Associated to (2.2) there is an affine algebraic curve \(C(F) \subset \mathbb{C}^2\) defined by the zero set of \(F(y, p)\) in \(\mathbb{C}^2\). We denote by \(\mathcal{C}(F)\) the Zariski closure of \(C(F)\) in \(\mathbb{C}^2_\infty\), where \(\mathbb{C}^2_\infty = \mathbb{C} \cup \{\infty\}\). In addition we assume throughout the paper that \(F\) has no factor in \(\mathbb{C}[y]\) or \(\mathbb{C}[p]\).

Additionally, we may require that a formal Puiseux series solution \(y(x)\) of (2.2) fulfills the initial conditions \(y(0) = y_0, (x^h y'(x))(0) = p_0\) for some fixed \(p_0 = (y_0, p_0) \in \mathbb{C}^2_\infty\). In the case where \(y(0) = \infty\), \(\tilde{y}(x) = 1/y(x)\) is a Puiseux series solution of a new first order differential equation of the same type, namely the equation given by the numerator.
of the rational function $F(1/y, -x^h p/y^2)$, and $\tilde{y}(0) \in \mathbb{C}$. Therefore, in the sequel, we assume that $p_0 \in \mathbb{C} \times \mathbb{C}_\infty$.

Here we recall some classical terminology, see e.g. [20]. A formal parametrization centered at $p_0 \in \mathbb{C} \times \mathbb{C}_\infty$ is a pair of formal Puiseux series $A(t) \in \mathbb{C}((t)) \setminus \mathbb{C}$ such that $A(0) = p_0$ and $F(A(t)) = 0$. In the set of all formal parametrizations of $\mathbb{C}(F)$ we introduce the equivalence relation $\sim$ by defining $A(t) \sim B(t)$ if and only if there exists a formal power series $s(t) \in \mathbb{C}[[t]]$ of order one such that $A(s(t)) = B(t)$. A formal parametrization is said to be reducible if it is equivalent to another one in $\mathbb{C}((t^m))^2$ for some $m > 1$. Otherwise, it is called irreducible. An equivalence class of an irreducible formal parametrization $(a(t), b(t))$ is called a place of $\mathbb{C}(F)$ centered at the common center point $p_0$ and is denoted by $[(a(t), b(t))]$. In every place there is exactly one formal parametrization of the type $(a_0 + t^n b(t))$ and we refer to them as classical Puiseux parametrizations. We observe that $\text{ord}_{t}(a(t) - y_0)$ and $\text{ord}_{t}(b(t))$ are independent of the representative $(a(t), b(t))$ of a place of $\mathbb{C}(F)$ centered at $p_0$.

3. Puiseux solution places

Let us consider the sets $\text{Sol}(p_0)$ containing the non-constant formal Puiseux series solutions of equation (2.2) with initial values $p_0$, $\text{IFP}(p_0)$ containing all irreducible formal parametrizations of $\mathbb{C}(F)$ at $p_0$ and $\text{Places}(p_0)$ containing the places of $\mathbb{C}(F)$ centered at $p_0$. Let us define the mapping $\Delta : \text{Sol}(p_0) \longrightarrow \text{IFP}(p_0)$ as

$$\Delta(y(x)) = \left( y(t^n), t^{hn} \frac{d}{dx} (t^n) \right),$$

where $n$ is the ramification order of $y(x)$ and denote by $\delta : \text{Sol}(p_0) \longrightarrow \text{Places}(p_0)$ the map $\delta(y(x)) = [\Delta(y(x))]$. The map $\Delta$ is well defined because on the one hand, $\Delta(y(x))$ is a formal parametrization of $\mathbb{C}(F)$ centered at $p_0$ and on the other hand, by the definition of the ramification index, one deduces that $\Delta(y(x))$ is irreducible.

We remark that, since $\Delta$ is well defined, a necessary condition for $y(x) \in \text{Sol}(p_0)$ is that $p_0 \in \mathbb{C}(F)$.

**Definition 3.1.** A place $\mathcal{P} \in \text{Places}(p_0)$ is a (Puiseux) solution place of (2.2) if there exists $y(x) \in \text{Sol}(p_0)$ such that $\delta(y(x)) = \mathcal{P}$. Moreover, we say that $y(x)$ is a generating Puiseux (series) solution of the place $\mathcal{P}$. An irreducible formal parametrization $A(t) \in \text{IFP}(p_0)$ is called a solution parametrization if $A \in \text{Im}(\Delta)$.

Note that the above definition generalizes the notion of solution place in [10] for formal power series solutions to Puiseux series solutions.
Now we give a characterization for an irreducible formal parametrization to be a solution parametrization. Later we will show how to decide whether a given place contains a solution parametrization, i.e. whether it is a solution place.

Lemma 3.2. Let $y(x) \in \text{Sol}(p_0)$ be of ramification order $n$, and let $(a(t), b(t)) = \Delta(y(x))$. It holds that
\begin{align}
(3.1) \quad a'(t) &= n t^{n(1-h)-1} b(t), \\
(3.2) \quad n(1-h) &= \text{ord}_t(a(t) - y_0) - \text{ord}_t(b(t)).
\end{align}

Proof. Since $a(t) = y(t^n)$ and $b(t) = t^n y'(t^n)$, by the chain rule
\[ a'(t) = n t^{n-1} y'(t^n) = n t^{n(1-h)-1} b(t). \]
Equation (3.2) is obtained by taking the function order in $t$ on both sides of equation (3.1). \hfill \Box

Proposition 3.3. Let $(a(t), b(t)) \in \text{IFP}(p_0)$. Then $(a(t), b(t))$ is a solution parametrization if and only if there exists $n \in \mathbb{N}$ such that equation (3.1) holds. In this case, $n$ is the ramification order of $(a(t), b(t))$.

Proof. The first implication follows from Lemma 3.2. Let us now assume that (3.1) holds for an $n \in \mathbb{N}$ and write $a(t) = y_0 + \sum_{j=k}^\infty a_j t^j$ with $k > 0$, $a_k \neq 0$, and $b(t) = \sum_{j=k-n(1-h)}^\infty b_j t^j$. Let us consider $y(x) = y_0 + \sum_{j=k}^\infty a_j x^{j/n}$. By assumption, $y'(x) = x^{-h} b(x^{1/n})$ and
\[ F(y(x), x^h y'(x)) = F(a(x^{1/n}), b(x^{1/n})) = 0. \]
Thus, $y(x) \in \text{Sol}(p_0)$. It remains to show that $n$ is the ramification order of $y(x)$. Otherwise, there exists a natural number $m \geq 2$, such that $m$ divides $n$ and if $a_i \neq 0$ then $m$ divides $i$. By assumption, we have that $a_{j+n(1-h)} \neq 0$ if and only if $b_j \neq 0$. Hence, if $b_j \neq 0$, then $m$ divides $j$. This implies that $(a(t), b(t))$ is reducible in contradiction to our assumption. Therefore, $n$ is the ramification order of $y(x)$ and $\Delta(y(x)) = (a(t), b(t))$. \hfill \Box

Lemma 3.4. All Puiseux series solutions in $\text{Sol}(p_0)$, generating the same solution place in $\text{Places}(p_0)$, have the same ramification order. We call this number the ramification order of the solution place. As a consequence, the map $\Delta$ is injective.

Proof. Let $y_1, y_2 \in \text{Sol}(p_0)$ be such that $\delta(y_1) = \delta(y_2)$. Let $n$ and $m$ be the ramification orders of $y_1$ and $y_2$, respectively. Then there exists an order one formal power series $s(t)$ such that $\Delta(y_1)(s(t)) = \Delta(y_2)(t)$. Let us denote $\Delta(y_i)$ as $\Delta(y_i) = (a_i, b_i)$ with $i = 1, 2$. By equation (3.1)
\[ a'_2(t) = m t^{m(1-h)-1} b_2(t) = m t^{m(1-h)-1} b_1(s(t)) \]
and

$$a'_2(t) = (a_1(s(t)))' = a'_1(s(t)) s'(t) = n s(t)^{n(1-h)-1} b_1(s(t)) s'(t).$$

Since $y_1 \in \text{Sol}(p_0)$ is not constant, $b_1(s(t))$ is not zero. Therefore,

$$n s(t)^{n(1-h)-1} s'(t) = m t^{m(1-h)-1}.$$  

Finally, comparing orders, since $h \neq 1$ by assumption, we get that $n = m$.

Assume now that $\Delta(y_1) = \Delta(y_2)$. Then, $\delta(y_1) = \delta(y_2)$ and hence, $y_1(t^n) = y_2(t^n)$. Thus, $y_1(x) = y_2(x)$. \qed

**Definition 3.5.** The **ramification order** of a solution parametrization $A(t)$ is defined as the ramification order of $\Delta^{-1}(A(t))$.

In the following we analyze the number of solution parametrizations in a solution place. We start with a technical lemma.

**Lemma 3.6.** Let $a(t) \in \mathbb{C}((t))$ be non-constant and let $\alpha_1, \alpha_2 \in \mathbb{C}$ be two different $k$-th roots of unity. If $a(\alpha_1 t) = a(\alpha_2 t)$, then there exists $m \in \mathbb{N}$, with $1 < m \leq k$, such that $a(t)$ can be written as

$$a(t) = \sum_{j \geq j_0/m} a_{jm} t^{jm}.$$  

**Proof.** Let $a(t) = \sum_{j \geq j_0} a_j t^j$. Since $a(\alpha_1 t) = a(\alpha_2 t)$, then $a_j \alpha_1^j = a_j \alpha_2^j$. So, if $a_j \neq 0$ then $(\alpha_1/\alpha_2)^j = 1$. Let $m \in \mathbb{N}$ be such that $\alpha_1/\alpha_2$ is an $m$-th primitive root of unity. Then $(\alpha_1/\alpha_2)^j = 1$ if and only if $j$ is a multiple of $m$ and this implies that $a(t) = \sum_{j \geq j_0/m} a_{jm} t^{jm}$. \qed

**Lemma 3.7.** Let $[A]$ be solution place of ramification order $n$. It holds that

1. If $h \leq 0$, then there are exactly $n(1-h)$ solution parametrizations in $[A]$, $A(t)$ is a solution parametrization, and all solution parametrizations in the place are of the form $A(\alpha t)$ where $\alpha^{n(1-h)} = 1$.
2. If $h \geq 2$, then there are infinitely many solution parametrizations in $[A]$.

**Proof.** Let, for $i = 1, 2$, $(a_i, b_i) \in [A]$ be two different solution parametrizations. As a consequence of equation (3.3), we get that the order one formal power series $s(t)$ relating $(a_1, b_1)$ and $(a_2, b_2)$ satisfies

$$s(t)^n(1-h)-1 s'(t) = t^{n(1-h)-1},$$

where $n$ is the ramification order of the place. Conversely, let $s(t)$ be a solution of (3.4) with $\text{ord}_t(s(t)) = 1$ and $(a_3(t), b_3(t)) = (a_1(s(t)), b_1(s(t)))$. Then

$$a'_3(t) = (a_1(s(t)))' = a'_1(s(t)) s'(t) = n s(t)^{n(1-h)-1} b_1(s(t)) s'(t),$$

$$\text{ord}_t(s(t)) = 1.$$
and by using equation (3.4),

\[ a_3'(t) = n t^{n(1-h)-1} b_3(t). \]

Then, by Proposition 3.3 \((a_3(t), b_3(t)) = (a_1(s(t)), b_1(s(t)))\) is a solution parametrization.

Let us compute the solutions of (3.4) by separation of variables. If \( h \leq 0 \), then \( s(t) = \alpha t \), where \( \alpha n(1-h) = 1 \). Therefore, the set of all solution parametrizations in \([A]\) is

\[ A := \{ (a_1(\alpha t), b_1(\alpha t)) \mid \alpha n(1-h) = 1 \}. \]

Let us verify that \(#(A) = n(1-h)\). If \( n(1-h) = 1 \), the result is trivial. Let \( n(1-h) > 1 \), and let us assume that \(#(A) < n(1-h)\).

Then, there exist two different \( n(1-h)\)-th roots of unity, \( \alpha_1, \alpha_2 \), such that \((a_1(\alpha_1 t), b_1(\alpha_1 t)) = (a_1(\alpha_2 t), b_1(\alpha_2 t))\). By Lemma 3.6 there exists \( m \in \mathbb{Z} \) with \( 1 < m \leq n(1-h) \), such that \( a_1(t), b_1(t) \) can be written as \( a_1(t) = \sum_{j \geq k_0/m} c_{jm} t^{jm} \) and \( b_1(t) = \sum_{j \geq k_0/m} d_{jm} t^{jm} \). This implies that \((a_1, b_1)\) is reducible, which is a contradiction.

If \( h \geq 2 \), the solutions of (3.4) are of the form

\[ s(t) = \frac{\alpha t}{n^{(h-1)}} \sqrt{1 + t^{n(h-1)}} c, \]

where \( c \) is an arbitrary constant and \( \alpha n(h-1) = 1 \). Note that \( s(t) \) can indeed be written as a formal power series of first order and for every choice \( c \in \mathbb{C} \) the solution parametrization is distinct.

Now, in the case of non-positive \( h \), we are in the position to decide whether a given place \( \mathcal{P} \in \text{Places}(p_0) \) is a solution place by a simple order comparison.

**Theorem 3.8.** Let \( \mathcal{P} = [(a(t), b(t))] \in \text{Places}(p_0) \) and \( h \leq 0 \). Then \( \mathcal{P} \) is a solution place if and only if equation (3.2) holds for an \( n \in \mathbb{N}^* \). In the affirmative case the ramification order of \( \mathcal{P} \) is equal to \( n \).

**Proof.** The first direction is Lemma 3.2. For the other direction let \((a(t), b(t)) \) and \( n \in \mathbb{N}^* \) be such that equation (3.2) holds. For every \( s(t) \in \mathbb{C}[[t]] \) with \( \text{ord}_t(s(t)) = 1 \) we claim that \((a(s(t)), b(s(t)))\) is a solution parametrization if and only if \( s(t) \) satisfies the following associated differential equation

\[ a'(s(t)) \cdot s'(t) = n t^{n(1-h)-1} b(s(t)). \]

For showing that such a solution exists, we need the technical Lemma 3.9. Now let \( s(t) \in \mathbb{C}[[t]] \) with \( \text{ord}_t(s(t)) = 1 \) be a solution of (3.5).
Then \((a(t), \tilde{b}(t)) = (a(s(t)), b(s(t)))\) fulfills equation (3.1) and by Proposition 3.3 \((a(t), \tilde{b}(t))\) is a solution parametrization with ramification order equal to \(n\).

The following lemma analyzes the solvability and properties of solutions of the associated differential equation.

**Lemma 3.9.** Let \(\mathcal{P} = [(a(t), b(t))] \in \text{Places}(\mathfrak{p}_0)\) be such that equation (3.2) holds for an \(n \in \mathbb{N}^*\). If \(h \leq 0\), there exist exactly \(n(1-h)\) distinct formal power series \(s(t) \in \mathbb{C}[[t]]\), with \(\text{ord}_t(s(t)) = 1\), satisfying the associated differential equation (3.5). If \(h \geq 2\), then (3.5) has either no solution or a family of solutions involving one free parameter. Moreover, the following statement holds:

1. If \(a(t)\) and \(b(t)\) are convergent as Puiseux series, then \(s(t)\) is convergent.
2. If the coefficients of \(a(t)\) and \(b(t)\) belong to a subfield \(\mathbb{L}\) of \(\mathbb{C}\), then the coefficients of \(s(t)\) belong to the extension field \(\mathbb{L}(\sigma_1, c)\), where \(\sigma_1\) is the first coefficient of \(s(t)\) and \(\sigma_1^{n(1-h)} \in \mathbb{L}\) and \(c\) is an arbitrary constant. If \(h \leq 0\), the constant \(c\) does not appear and the field extension is simple radical.
3. For any \(m \in \mathbb{N}\), the first \(m\) coefficients of \(s(t)\) depend only on the first \(m\) coefficients of \(a(t)\) and \(b(t)\), on \(\sigma_1\) and, if \(h \geq 2\), on \(c\).

**Proof.** Let us denote \(k = \text{ord}_t(a(t) - y_0)\) and \(r = \text{ord}_t(b(t))\). By hypothesis, \(n = \frac{k-r}{1-h} \geq 1\). First, let \(h \leq 0\). Multiplying both sides of (3.5) by \(s(t)^{-r}\), we obtain \(c(s(t)) \cdot s'(t) = nt^{n(1-h)-1}d(s(t))\), or equivalently

\[
G(t, s, s') = c(s) s' - nt^{n(1-h)-1}d(s) = 0,
\]

where \(c(s) = s^{-r} a'(s) = \sum_{i=n(1-h)-1}^{\infty} c_i s^i\) and \(d(s) = s^{-r} b(s) = \sum_{i=0}^{\infty} d_i s^i\), with \(c_{n(1-h)} \neq 0\) and \(d_0 \neq 0\). We observe that \(G \in \mathbb{L}[[s]][t, s']\) is convergent in \(s\) provided that \(c(t)\) and \(d(t)\) are convergent as power series in \(t\).

Let us apply the Newton polygon method for differential equations (see section 1 of [4]) to describe all possible solutions \(s(t)\) of \(G = 0\). The Newton polygon \(\mathcal{N}(G)\) of \(G\) is sketched in the left picture of figure 1. Let us write \(s(t) = \sigma_1 t + \bar{s}(t)\), where \(\bar{s}(t)\) is a formal power series of order greater than 1. By Lemma 1 of [4], the constant \(\sigma_1\) is a root of the polynomial \(\Phi_{(G,1)}(C) = c_{n(1-h)-1} C^{n(1-h)} - nd_0\) associated to the slope \(-1\). Since \(d_0, c_{n(1-h)} \neq 0\), there are exactly \(n(1-h)\) possibilities to choose \(\sigma_1\) such that \(\sigma_1^{n(1-h)} = nd_0/c_{n(1-h)} - 1\).

We perform the change of variable \(s(t) = \sigma_1 t + \bar{s}(t)\) in the differential equation \(G = 0\), obtaining a new differential equation \(G_1(t, \bar{s}, s') = 0\).
Note that the coefficients of \( G_1 \) are in \( \mathbb{L}(\sigma_1) \). We are looking for a formal power series solution of \( G_1 = 0 \) of the form \( \bar{s}(t) = \sum_{i=2}^{\infty} \sigma_i t^i \). The Newton polygon \( \mathcal{N}(G_1) \) of \( G_1 \) is sketched in the right picture of figure 1. We can guarantee the existence of the vertices \( v_0 = (-1, n(1-h)) \) and \( v_1 = (n(1-h) - 2, 1) \) in \( \mathcal{N}(G_1) \). Moreover, the monomials of \( G_1 \) corresponding to the vertex \( v_1 \) are \( A t^{n(1-h)-2} \bar{s}(t) + B t^{n(1-h)-1} \bar{s}'(t) \), where

\[
A = c_{n(1-h)-1} \sigma_1^{n(1-h)-1}(n(1-h) - 1) \quad \text{and} \quad B = c_{n(1-h)-1} \sigma_1^{n(1-h)-1}.
\]

We have that \( -A/B = -(n(1-h) - 1) \leq 0 \) and in particular \( -A/B \notin \mathbb{Q}_{\geq 1} \).

In order to prove the first part of the Lemma, we first show the existence and convergence and then the uniqueness of a formal power series solution \( \bar{s} \) of \( G_1 = 0 \) with order greater than one.

If \( \bar{s} = 0 \) is a solution of \( G_1 = 0 \) we are already done. Otherwise \( \mathcal{N}(G_1) \) has a vertex \( v_2 \) lying on the abscissa axis. By construction of \( G_1 \), the point \( (n(1-h) - 1, 0) \) does not appear in \( \mathcal{N}(G_1) \), and the abscissa of \( v_2 \) is an integer greater than \( n(1-h) - 1 \). Let \( L \) the side of \( \mathcal{N}(G_1) \) containing \( v_1 \) and \( v_2 \). The slope of \( L \) is \( -1/\mu_1 \), where \( \mu_1 \) is an integer greater than 1. Since \( -A/B \notin \mathbb{Q}_{\geq 1} \), we can apply Theorem 1 of [4] and \( L \) is the “principal side” of \( G_1 \). Thus, there exists a formal Puiseux series solution \( \bar{s} \) of \( G_1 = 0 \) of order \( \mu_1 > 1 \). Moreover, since the “pivot point” (see the proof of Theorem 1 in [4]) is reached at the initial step, the Puiseux series solution is in fact a formal power series solution. Theorem 2 in the same reference guarantees that the obtained solution is convergent provided \( c(\bar{s}) \) and \( d(\bar{s}) \) are convergent.

In order to prove the uniqueness of the solution \( \bar{s} \) of \( G_1 = 0 \) and the remaining points (2) and (3), we explicitly describe how to compute the coefficients of \( \bar{s}(t) \). The coefficient \( \sigma_2 \) is a root of the polynomial \( \Phi_{(G_1,2)}(C) \) of \( G_1 = 0 \) associated to the slope \(-1/2\). We have that \( \Phi_{(G_1,2)}(C) = (A + 2B) C + C_2 \), where \( C_2 \) is the coefficient of \( t^{n(1-h)} \) in \( G_1(t, \bar{s}, \bar{s}') \), namely \( c_{n(1-h)} \sigma_1^{n(1-h)+1} - d_1 \). Since \( A + 2B \neq 0 \), then \( \sigma_2 = -C_2/(A + 2B) \) is uniquely determined. Note that \( \sigma_2 \in \mathbb{L}(\sigma_1) \).

Let us recursively define \( G_{m+1} = G_m(t, \sigma_m t^m + \bar{s}, m \sigma_m t^{m-1} + \bar{s}') \) for \( m \geq 1 \), where \( \bar{s}(t) \) is a new variable. Then \( v_0 \) and \( v_1 \) are vertices of \( \mathcal{N}(G_{m-1}) \) and the monomials corresponding to \( v_1 \) in \( G_{m-1} \) are the same as in \( G_1 \). Hence, \( \Phi_{(G_{m-1},1)}(C) = (A + mB) C + C_m \), where \( C_m \) is the coefficient of \( t^{n(1-h)-2+m} \) in \( G_{m-1} \). Because \( A + mB \neq 0 \), then

\[
\sigma_m = -C_m/(A + mB)
\]

is uniquely determined and it is an element of \( \mathbb{L}(\sigma_1) \).
It remains to prove item (3). Since $A$ and $B$ only depends on $\sigma_1$, by induction, it is enough to show that $G_m$ depends only on $\sigma_1, \ldots, \sigma_{m-1}$, $c_{n(1-h)-1}, \ldots, c_{n(1-h)+m-2}$ and $d_0, \ldots, d_{m-1}$. By definition of $G_m$, $C_m$ is the coefficient of $t^{n(1-h)-2+m}$ in

$$G(t, \psi(t), \psi'(t)) = \left( \sum_{i \geq n(1-h)-1} c_i \psi(t)^i \right) \psi(t)' - n \sum_{i \geq 0} d_i \psi(t)^i,$$

where $\psi(t) = \sigma_1 t + \cdots + \sigma_{m-1} t^{m-1}$. Item (3) is a consequence of the equation above and the fact that $\text{ord}_i(\psi(t)^i) = i$.

Second, let $h \geq 2$. Then we obtain similarly as above

$$G(t, s, s') = t^{n(h-1)+1} c(s) s' - n d(s) = 0,$$

where $c(s) = s^{-k+1} a'(s) = \sum_{i=0}^{\infty} c_i s^i$ with $c_0 \neq 0$ and $d(s) = s^{-k+1} b(s) = \sum_{i=n(h-1)+1}^{\infty} d_i s^i$ with $d_{n(h-1)+1} \neq 0$. Hence, $\sigma_1$ is a root of $\Phi_{(G,1)}(C) = c_0 - n d_{n(h-1)+1} C^{n(h-1)}$. After the change of variables $s(t) = \sigma_1 t + \bar{s}(t)$, the Newton polygon $\mathcal{N}(G_1)$ of the differential equation $G_1(t, \bar{s}, \bar{s}') = 0$ has a vertex $v_1 = (n(h-1), 1)$ with monomials $A t^{n(h-1)} \bar{s}(t) + B t^{n(h-1)+1} \bar{s}'(t)$, where

$$A = -(n(h-1) - 1) c_0 \quad \text{and} \quad B = c_0.$$

So the critical value for the slope is $-A/B = n(h-1) + 1$. For $1 < \mu < n(h-1) + 1$ the characteristic polynomial $\Phi_\mu(C)$ is uniquely solvable. For $\mu = n(h-1) + 1$ the characteristic polynomial is a constant. If it is non-zero, then $s(t)$ cannot be continued to a solution of (3.3). If it is zero, then $\sigma_\mu$ can be chosen arbitrary and for $\mu > n(h-1) + 1$ the coefficients $\sigma_\mu$ are again uniquely determined as the roots of $\Phi_\mu(C)$.

In the case that a solution $s(t)$ exists, since the above coefficient $B \neq 0$, the linearized operator along $s(t)$ has a regular singularity and by the main result from [17], $s(t)$ is convergent. Alternatively one could use directly Theorem 2 in [5], because the pivot point of $s(t)$ with respect to $G$ is $v_1$ and the coefficient of the highest derivative $B$ is non zero. The remaining items in the case of $h \geq 2$ follow as above. \qed

**Theorem 3.10.** Any formal Puiseux series solution of (2.1), expanded around a finite point or at infinity, is convergent.

**Proof.** In order to prove the statement we show that every formal Puiseux series solution of equation (2.2), in particular for $h \in \{0, 2\}$, expanded around zero is convergent. Let $y(x) \in \text{Sol}(p_0)$. Performing the change of variable $\bar{y}(x) = 1/y(x)$ if necessary, we can assume that $y_0 \in \mathbb{C}$. Let $n \geq 1$ be the ramification order of $y(x)$ and $\Delta(y(x)) = (a(t), b(t))$. By Lemma 3.2, equations (3.1) and (3.2) hold.
Let $k = \text{ord}_t(a(t) - y_0) \geq 1$. By Section 2 of Chapter IV in [20], there exists a formal power series $s(t) \in \mathbb{C}[[t]]$, with $\text{ord}_t(s(t)) = 1$, such that

$$a(s(t)) - y_0 = t^k.$$ 

Let $\bar{a}(t) = a(s(t))$ and $\bar{b}(t) = b(s(t))$. Then $(\bar{a}(t) - y_0, \bar{b}(t)) = (t^k, \bar{b}(t))$ is a local parametrization of the non-trivial algebraic curve defined by $F(y - y_0, p)$. Hence, by Puiseux’s theorem, $\bar{b}(t)$ is convergent.

Let $r(t)$ be the compositional inverse of $s(t)$, i.e. $r(s(t)) = t = s(r(t))$. Then $r(t)$ is a formal power series of order one and $a(t) = \bar{a}(r(t)), b(t) = \bar{b}(r(t))$. Since equation (3.5) holds for $(\bar{a}(t), \bar{b}(t))$ and $r(t)$, by Lemma 3.9, $r(t)$ is convergent. This implies that $a(t)$ is convergent and therefore, $y(x) = a(x^{1/n})$ is convergent as a Puiseux series.

**Theorem 3.11.** Let $F(y, p)$ be a non-constant polynomial with no factor in $\mathbb{C}[y]$ or $\mathbb{C}[p]$. For any point in the plane $(x_0, y_0) \in \mathbb{C}^2$, there exists an analytic solution $y(x)$ of $F(y, y') = 0$ such that $y(x_0) = y_0$.

**Proof.** It is sufficient to prove the existence of a convergent formal Puiseux series solution $y(x) = y_0 + \sum_{i=1}^{\infty} c_i (x - x_0)^{i/n}$. Performing the change of variable $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$, we may assume that $x_0 = 0$ and $y_0 = 0$.

Let us write $F(y, p) = \sum F_{i,j} y^i p^j$. If $F(0, 0) = F_{0,0} = 0$, then we have that $y(x) = 0$ is a solution of $F(y, y') = 0$ and $\Delta(y)$ passes through $(0, 0)$. We may assume that $F_{0,0} \neq 0$. Consider $\mathcal{N}(F)$ the Newton polygon of the algebraic curve $F(y, p) = 0$ in the variables $y$ and $p$. The point $(0, 0)$ is a vertex of $\mathcal{N}(F)$, because $F_{0,0} \neq 0$. This implies that all the sides of $\mathcal{N}(F)$ have slope greater or equal to zero (see figure 2). Since the degree of $F(y, p)$ with respect to $p$ is positive, $\mathcal{N}(F)$ has at least one side. Therefore, by Puiseux’s theorem,
there exists a convergent Puiseux series solution $p(y)$ of the algebraic equation $F(y, p(y)) = 0$ of the form $p(y) = \sum_{i=k}^{\infty} c_i y^{i/n}$, where $c_k \neq 0$ and $k \leq 0$. Let us define $a(t) = t^n$ and $b(t) = \sum_{i=k}^{\infty} c_i t^i$. Then $(a(t), b(t))$ is a convergent parametrization of $\mathcal{C}(F)$ satisfying

$$m = \text{ord}_x(a(t) - a(0)) - \text{ord}_x(b(t)) = n - k \geq n \geq 1.$$ 

By Theorem 3.8 there exists a formal Puiseux series solution $y(x)$ of the differential equation $F(y, y') = 0$ and $\text{ord}_x(y(x)) > 0$ which proves the theorem. □

**Figure 2.** The Newton polygon of the algebraic curve $F(y, p) = 0$. All its sides have non-negative slope, because the point $(0, 0) \in \mathcal{N}(F)$.

Notice that in Theorem 3.11 we can give a lower and an upper bound for the number of solution parametrizations passing through a given point $(x_0, y_0) \in \mathbb{C}^2$. First, every side with slope greater or equal to zero defines a different solution parametrization. Thus, a lower bound can easily be derived after computing the Newton polygon $\mathcal{N}(F)$.

Second, let $\Sigma_{(x_0,y_0)}$ denote the set of solution parametrizations passing through $(x_0, y_0)$. The set of corresponding solution places are denoted by $\mathcal{P}(y_0) = \{[(a(t), b(t))] \mid (a(t), b(t)) \in \Sigma_{(x_0,y_0)}\}$. Since every solution parametrization passing through $(x_0, y_0)$ is a solution parametrization centered at $(y_0, p_0)$ for some $p_0 \in \mathbb{C}_\infty$, by Lemma 3.7,

$$\#\Sigma_{(x_0,y_0)} = \sum_{P \in \mathcal{P}(y_0)} \text{ramification index of } P \leq \text{deg}_p(F).$$

The last inequality is a well known result for algebraic curves and can be found for example in [9][Theorem 1].

As a consequence for example the family of functions

$$y(x) = x + cx^2,$$
where \( c \) is an arbitrary constant, cannot be a solution of any first order autonomous ordinary differential equation. Otherwise, there are infinitely many distinct formal parametrizations \((y(x), y'(x))\) with \( y_0 = 0 \) as initial value and the sum of the ramification indexes of \( P \in \mathcal{P}(y_0) \) is infinite in contradiction to the bound above.

We note that there might be families of formal Puiseux series solutions at infinity for a first order autonomous ordinary differential equation as we will see in example 4.5.

4. Algorithms and Examples

In this section we outline an algorithm that is derived from the results in Section 3, in particular, for \( h \in \{0, 2\} \). We can describe algorithmically all formal Puiseux series solutions of the differential equation (2.1). For each formal Puiseux series solution we will provide what we call a determined solution truncation. A determined solution truncation is an element of \( \mathbb{C}[x^{1/n}][x^{-1}] \), for some \( n \in \mathbb{N} \), that can be extended uniquely to a formal Puiseux series solution.

If \( F \) is reducible, one could factor it and consider its irreducible components and the solutions of the corresponding differential equations. However, from a computational point of view, this is not optimal, and we compute the square-free part of \( F \) instead. So let us assume \( F \in \mathbb{C}[y, p] \) to be square-free and have no factor in \( \mathbb{C}[y] \) or \( \mathbb{C}[p] \) in the remaining of the paper. Since each formal Puiseux series solution \( y(x) \) gives rise to an initial tuple \( p_0 = (y(0), (x^h y'(x))(0)) \) in \( \mathcal{C}(F) \), we will describe for each point \( p_0 \in \mathcal{C}(F) \) the set \( \text{Sol}(p_0) \). We note that if \( \text{ord}_x(y) \geq 0 \) and \( h \geq 2 \), then \( p_0 \) will necessarily be of the type \((y_0, 0)\) for some \( y_0 \in \mathbb{C} \).

4.1. Solutions expanded around zero. In this subsection we consider formal Puiseux series solutions of (2.1), or equivalently, solutions of (2.2) with \( h = 0 \) expanded around zero. A point \( p_0 = (y_0, p_0) \in \mathcal{C}(F) \) is called a critical curve point if either \( p_0 = \infty \) or \( \frac{\partial F}{\partial p}(p_0) = 0 \) (compare [10]). Under our assumptions, the set of critical curve points, denoted by \( \mathcal{B}(F) \), is finite.

If \( p_0 = (y_0, p_0) \in \mathcal{C}(F) \setminus \mathcal{B}(F) \), we can apply the method of limits (see Chapter XII in [16]). The only formal Puiseux series solution with \( p_0 \) as initial tuple is a formal power series and its determined solution truncation is given by \( y_0 + p_0 x \).

The points \( p_0 = (y_0, \infty) \in \mathcal{C}(F) \) with \( y_0 \in \mathbb{C} \), can be computed by considering \( F \in \mathbb{C}[y][p] \) and determining the zeros of the leading coefficient in \( y \). As already remarked in Section 2, the possible curve
point \((\infty, \infty)\) can be handled by a suitable change of variables. Note that there cannot be a solution with an initial tuple of the form \((\infty, p_0)\) with \(p_0 \in \mathbb{C}\), because if \(\text{ord}_x(y(x)) < 0\) then \(\text{ord}_x(y'(x)) < 0\) as well.

Assume that \(p_0\) is a critical curve point. Let \(R\text{Trunc}_N(p_0) \subseteq \mathbb{C}[t][t^{-1}]\) denote the set of truncations of non-equivalent classical Puiseux parametrizations \((y_0 + t^k, b(t)) \in \text{Places}(p_0)\), where the first \(N\) terms of \(b(t)\) are computed. In [9] is presented an algorithm to compute \(R\text{Trunc}_N(p_0)\) and with \(N = 2(\deg_p(F) - 1) \deg_y(F) + 1\) or the Milnor number (see [18]) is given a bound for the truncation such that \(R\text{Trunc}_N(p_0)\) is in one-to-one correspondence to \(\text{Places}(p_0)\). Moreover, the ramification indexes of the approximated places are determined then such that we can check whether equation (3.2) holds and follow the proof of Lemma 3.9 to describe all formal Puiseux series solutions with \(p_0\) as initial tuple as Algorithm PuiseuxSolve shows. By choosing the bound \(N = 2(\deg_p(F) - 1) \deg_y(F) + 1\) no further extensions of the ground field for computing the coefficients are required.

Algorithm 1 PuiseuxSolve

**Input:** A first-order AODE \(F(y, y') = 0\), where \(F \in \mathbb{C}[y, p]\) is square-free with no factor in \(\mathbb{C}[y]\) or \(\mathbb{C}[p]\).

**Output:** A set consisting of all determined solution truncations of \(F(y, y') = 0\) expanded around zero.

1. If \((\infty, \infty) \in \mathcal{C}(F)\), then perform the transformation \(\bar{y} = 1/y\) and apply the algorithm to the numerator of \(F(1/y, -p/y^2)\) and \(p_0 = (0, 0)\).
2. Compute the set of critical curve points \(\mathcal{B}(F)\).
3. For every point \((y_0, p_0) \in \mathcal{C}(F) \setminus \mathcal{B}(F)\) a determined solution truncation is \(y_0 + p_0x\).
4. For every critical curve point \(p_0 = (y_0, p_0) \in \mathcal{B}(F)\) with \(y_0 \in \mathbb{C}\) we compute the finite set \(R\text{Trunc}_N(p_0)\), where \(N = 2(\deg_p(F) - 1) \deg_y(F) + 1\).
5. If \(p_0 = 0\), then add to the output the constant solution \(y(x) = y_0\).
6. For every truncation \((\hat{a}(t), \hat{b}(t)) \in R\text{Trunc}_N(p_0)\) corresponding to \([\hat{a}(t), \hat{b}(t)] \in \text{Places}(p_0)\), equation (3.2) can be checked.
7. In the negative case, \([\hat{a}(t), \hat{b}(t)]\) is not a solution place.
8. In the affirmative case compute by the Newton polygon method for differential equations the first \(N\) terms of the solutions \(s_1(t), \ldots, s_n(t)\) of (3.5), denoted by \(\hat{s}_1(t), \ldots, \hat{s}_n(t)\).
9. Then the first \(N\) terms of \(\hat{a}(\hat{s}_i(x^{1/n}))\) are the determined solution truncations with \(p_0\) as initial values.
In the following Lemma we show that the output truncations are indeed determined solution truncations, which also proves correctness of Algorithm PuiseuxSolve.

**Theorem 4.1.** Let $F \in \mathbb{C}[y,p]$ be square-free with no factor in $\mathbb{C}[y]$ or $\mathbb{C}[p]$. Then the set of truncated solutions obtained by the Algorithm PuiseuxSolve with $p_0$ as initial tuple, denoted by $\text{STrunc}_N(p_0)$, and $\text{Sol}(p_0)$ are in one-to-one correspondence.

**Proof.** Let $p_0 \in \mathcal{C}(F)$. From [9] it follows that $\# \text{RTrunc}_N(p_0) = \# \text{Places}(p_0)$. By Proposition 3.3, $\# \text{Sol}(p_0) = \sum \# \text{Places}(p_0) n_i$, where every summand $n_i$ is equal to the ramification index of the corresponding place (or 0, if (3.2) is not fulfilled). It remains to prove that $\# \text{STrunc}_N(p_0) = \sum i \in \mathbb{N} \# \text{RTrunc}_N(p_0) n_i$, or in other words, that all output elements of PuiseuxSolve are distinct.

Let $\hat{A}_1 = (\hat{a}_1(t), \hat{b}_1(t)) \in \text{RTrunc}_N(p_0)$ with $\hat{a}_1(t) = y_0 + t^r$, $\hat{b}_1(t) = \sum_{j \geq h} b_{i,j} t^j$ and $n_i = r_i - h_i > 0$ for $i = 1, 2$. If $r_1 \neq r_2$ or $h_1 \neq h_2$, the statement holds. So let us assume that $r = r_1 = r_2$ and $h = h_1 = h_2$.

Let $\hat{A}_1 \neq \hat{A}_2$. Then, by the definition of $\text{RTrunc}_N(p_0)$, $\hat{b}_1(t) \neq \hat{b}_2(\lambda t)$ for every $\lambda \in \mathbb{C}$ with $\lambda^r = 1$. Let $m \in \mathbb{N}$ be the first index such that $b_{1,h+m} \neq b_{2,h+m}$. If the quotient $b_{1,h+m}/b_{2,h+m}$ is equal to a $\lambda \in \mathbb{C}$ with $\lambda^r = 1$, then consider $\hat{A}_2(\lambda t)$ instead of $\hat{A}_2(t)$ and set $m$ to the first index where the coefficient of $t^{h+m}$ in $\hat{b}_1(t)$ and $\hat{b}_2(\lambda t)$ are distinct.

Let $\hat{s}_i(t) = \sum_{j=1}^{H_i} c_{i,j} t^j$ be the truncated solutions of (3.5) corresponding to $\hat{A}_i(t)$. First, assume that $m = 0$. Then

$$\sigma^r_{1,1} = \left( \sqrt[n]{\frac{b_{1,h}}{r}} \right)^r \neq \sigma^r_{2,1} = \left( \sqrt[n]{\frac{b_{2,h}}{r}} \right)^r.$$  

The coefficient of $t^r$ in $\hat{a}_1(\hat{s}_i(t))$ is equal to $\sigma^r_{i,1}$ and thus, the outputs $\hat{a}_1(\hat{s}_i(x^{1/n}))$ are distinct already in the first coefficient.

Now let us consider $m > 0$. Then $\sigma^m_{1,1} = \sigma^m_{2,1}$ and without loss of generality we can choose $\sigma_{1,1} = \sigma_{2,1}$. By Lemma 3.9, item (3), and the fact that $\hat{a}_1(t) = \hat{a}_2(t)$, the coefficients $\sigma_{1,1}, \ldots, \sigma_{1,m}$ coincide with $\sigma_{2,1}, \ldots, \sigma_{2,m}$. As we have seen in the proof of the same Lemma, $C_{m+1}$ in formula (3.6) is equal to the coefficient of $t^{n-1+m}$ in

$$G(t, \hat{s}_i, \hat{s}_i') = (r \hat{s}_i(t)^{n-1}) \hat{s}_i'(t) - n t^{n-1} \frac{\hat{b}_i(\hat{s}_i(t))}{\hat{s}_i(t)^h},$$

namely $-n b_{i,h+m} \sigma^m_{i,1} \sigma^r_{i,1}$ plus terms involving $b_{i,h}, \ldots, b_{i,h+m-1}$ and $\sigma_{i,1}, \ldots, \sigma_{i,m}$. Since $b_{1,h+m} \neq b_{2,h+m}$, it follows that $-n b_{1,h+m} \sigma^m_{1,1} \neq -n b_{2,h+m} \sigma^m_{2,1}$ and by formula (3.6), $\sigma_{1,m+1} \neq \sigma_{2,m+1}$. The coefficient of $t^{r+m}$ in
\( \hat{a}_i(\hat{s}_i(t)) \) is equal to \( r \sigma_{i,1}^{-1} \sigma_{i,m+1} \) plus terms involving \( \sigma_{i,1}, \ldots, \sigma_{i,m} \).

Thus, the outputs \( \hat{a}_i(\hat{s}_i(x^{1/n})) \) are distinct. \( \square \)

**Example 4.2** (Example 2 in [10]). Let us consider

\[
F = ((y' - 1)^2 + y^2)^3 - 4(y' - 1)^2 y^2 = 0.
\]

The critical set is \( B = \{(0, 1), (\alpha, 0), \left(\frac{4\beta}{9}, \gamma\right), (\infty, \infty)\} \) where \( \alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0, \beta^2 = 3, \) and \( 27\gamma^2 - 54\gamma + 19 = 0. \) Observe that, since the leading coefficient of \( F \) w.r.t. \( y \) is 1, there is no curve point of the form \( (y_0, \infty) \) with \( y_0 \in \mathbb{C}. \)

We now analyze the critical curve points. Let \( c_\alpha = (\alpha, 0) \) where \( \alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0. \) We get the place

\[
\left( \alpha + t, \left( \frac{11}{19} \alpha^5 + \frac{36}{19} \alpha^3 + \frac{4}{19} \alpha \right) t + \mathcal{O}(t^2) \right),
\]

which does not provide any solution (see equation (3.2)). Thus, the constant \( \alpha \) is the only solution with the initial tuple \( c_\alpha. \)

Let \( c_1 = (0, 1). \) The truncated classical Puiseux parametrizations at \( c_1 \) are

\[
\mathcal{P}_1 = (t^2, 1 + \sqrt{2}t - \frac{3t^2}{4\sqrt{2}} - \frac{15t^5}{64\sqrt{2}} + \mathcal{O}(t^6)) \quad \mathcal{P}_3 = (t, 1 + \frac{t^2}{2} + \frac{3t^4}{16} + \mathcal{O}(t^6))
\]

\[
\mathcal{P}_2 = (t^2, 1 - \sqrt{2}t - \frac{3t^2}{4\sqrt{2}} + \frac{15t^5}{64\sqrt{2}} + \mathcal{O}(t^6)) \quad \mathcal{P}_4 = (t, 1 - \frac{t^2}{2} - \frac{3t^4}{16} + \mathcal{O}(t^6)).
\]

So we have \( n = 2 \) for \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) and \( n = 1 \) for \( \mathcal{P}_3 \) and \( \mathcal{P}_4. \) Then equation (3.5) corresponding to \( \mathcal{P}_1 \) is

\[
s(t) s'(t) = t \left( 1 + \sqrt{2}s(t) - \frac{3s(t)^2}{4\sqrt{2}} - \frac{15s(t)^3}{64\sqrt{2}} \right).
\]

We obtain the solutions

\[
s_1(t) = t + \frac{\sqrt{2}t^2}{3} - \frac{t^3}{18} - \frac{89\sqrt{2}t^4}{1080} + \mathcal{O}(t^5),
\]

\[
s_2(t) = -t + \frac{\sqrt{2}t^2}{3} + \frac{t^3}{18} + \frac{89\sqrt{2}t^4}{1080} + \mathcal{O}(t^5).
\]

Therefore, \( \mathcal{P}_1(s_1(x^{1/2})) \) and \( \mathcal{P}_1(s_2(x^{1/2})) \) are determined solution truncations of \( F(y, y') = 0. \)

Similarly we can find two determined solution truncations coming from \( \mathcal{P}_2 \) and one for each \( \mathcal{P}_3 \) and \( \mathcal{P}_4. \) We note that the solutions corresponding to \( \mathcal{P}_3 \) and \( \mathcal{P}_4 \) are formal power series and already detected
in [10]. Thus,

\[
\mathcal{U}_{c_1} = \begin{cases} 
\mathcal{P}_1(s_1(x^{1/2})) = x + \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\
\mathcal{P}_1(s_2(x^{1/2})) = x - \frac{2\sqrt{2}x^{3/2}}{3} + \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\
\mathcal{P}_2(s_1(x^{1/2})) = x + \frac{2\sqrt{2}x^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\
\mathcal{P}_2(s_2(x^{1/2})) = x - \frac{2\sqrt{2}x^{3/2}}{3} - \frac{x^2}{3} + \mathcal{O}(x^{5/2}), \\
\mathcal{P}_3(s(x)) = x + \frac{x^3}{6} + \frac{17x^2}{28} + \mathcal{O}(x^6), \\
\mathcal{P}_4(s(x)) = x - \frac{x^3}{6} + \frac{240x}{17} + \mathcal{O}(x^6) \end{cases}
\]

is the set of all determined solution truncations with \(c_1\) as initial tuple.

Let \(c_{\beta,\gamma} = \left(\frac{43}{9}, \gamma\right)\), where \(\beta^2 = 3\), and \(27\gamma^2 - 54\gamma + 19 = 0\). We get the place

\[
\left(\frac{4\beta}{9} + t^2, \gamma + \frac{\beta i}{\sqrt{3}}t + \mathcal{O}(t^2)\right).
\]

Thus, (3.2) is fulfilled with \(n = 2\). Similarly as before, we obtain at \(c_{\beta,\gamma}\) the set of solutions

\[
\mathcal{U}_{c_{\beta,\gamma}} = \begin{cases} 
\frac{4\beta}{9} + \gamma x + \frac{43\cdot 807 - 113/2 + 2\cdot 17\cdot 27}{19^2} \cdot 2 \cdot 3^3 i x^{3/2} \\
+ \left(\frac{57}{52} - \frac{143}{864}\right) \beta x^2 + \mathcal{O}(x^{5/2}), \\
\frac{4\beta}{9} + \gamma x + \frac{43\cdot 807 - 113/2 + 2\cdot 17\cdot 27}{19^2} \cdot 2 \cdot 3^3 i x^{3/2} \\
+ \left(\frac{57}{52} - \frac{143}{864}\right) \beta x^2 + \mathcal{O}(x^{5/2}) \end{cases}
\]

Let us analyze \(c_{\infty} = (\infty, \infty)\). The numerator of \(F(1/y, -y'/y^2)\) is equal to

\[
G = (6y' - 1)y^{10} + (15y^2 + 4y' + 3)y^8 + (20y^3 + 14y^2 + 6y' + 1)y^6 +
(15y^4 + 12y^3 + 3y^2)y^4 + (6y^5 + 3y^4)y^2 + y^6 + y^{12}.
\]

The places at the origin of \(\mathcal{C}(G)\) are given by

\[
(\pm it^3, t^3 + \mathcal{O}(t^4)),
\]

which do not define a solution place.

Now the set \(\{y(x; y_0)\} \cup \{\alpha\} \cup \mathcal{U}_{c_1} \cup \mathcal{U}_{c_{\beta,\gamma}}\) describes all formal Puiseux series solutions of \(F = 0\).

4.2. Solutions expanded at infinity. In this subsection we describe the formal Puiseux series solutions of (2.1) expanded around infinity, or equivalently, formal Puiseux series solutions of (2.2) with \(h = 2\) expanded around zero.

That (3.2) is only a necessary and not a sufficient condition in this is shown in the following example.
Example 4.3. Let us consider
\[ F(y(x), -x^2 y'(x)) = -x^2 yy' + y^2 - x^2 y' = 0 \]
coming from \( F(y, p) = yp + y^2 + p \). By the Newton polygon method for
differential equations we directly see that there is no formal Puiseux
series solution with \( y(0) = 0 \) except the constant zero. On the other
hand, \( (t, -t^2 + O(t^3)) \) is a formal parametrization of \( C(F) \) fulfilling
\( (3.2) \) with \( n = 1 \).

Nevertheless, it can still be checked whether there exists a solutions
fulfilling the necessary condition \( (3.2) \) or not and therefore, we can
algorithmically compute all solutions as in the previous subsection.

Similar to section 4.1 let \( \text{RTrunc}_N(p_0) \subseteq \mathbb{C}[t][t^{-1}] \) denote the set of
truncations of non-equivalent classical Puiseux parametrizations where
the first \( N \) terms are computed.

**Algorithm 2 PuiseuxSolveInfinity**

**Input:** A first-order AODE \( F(y, y') = 0 \), where \( F \in \mathbb{C}[y, p] \) is square-
free with no factor in \( \mathbb{C}[y] \) or \( \mathbb{C}[p] \).

**Output:** A set consisting of all solution truncations of \( F(y, y') = 0 \)
expanded around infinity.

Compute the algebraic set \( \mathbb{V}(F(y, 0)) \).

For every \( y_0 \in \mathbb{V}(F(y, 0)) \) compute the finite set \( \text{RTrunc}_N(p_0) \), where
\( N = \max(2(\deg_p(F) - 1) \deg_y(F) + 1, \deg_y(F) + 1) \).

Add to the output the constant solutions \( y(x) = y_0 \).

For every truncation \( (\hat{a}(t), b(t)) \in \text{RTrunc}_N(p_0) \) corresponding to
\( [(a(t), b(t)) \in \text{Places}(p_0), \) equation \( (3.2) \) can be checked.

In the negative case, \( [(a(t), b(t)) \) is not a solution place.

In the affirmative case check by the Newton-polygon method for
differential equations whether \( (3.5) \) is solvable. Note that in \( (3.5) \)
the critical term with slope \( \mu = n + 1 \) is already covered by the first
\( N \) terms, since \( n \leq \deg_y(F) + 1 \leq N \).

In the affirmative case compute the first \( N \) terms of the solutions
\( s_1(t), \ldots, s_n(t) \) denoted by \( \hat{s}_1(t), \ldots, \hat{s}_n(t) \), which contain a transcendental
element.

The first \( N \) terms of \( \hat{a}(\hat{s}_i(x^{-1/n})) \) are solution truncations with \( y_0 \) as
initial value.

Let \( y_0 \in \mathbb{C} \) be such that \( p_0 = (y_0, 0) \in \mathcal{C}(F) \) and let us again denote
the formal Puiseux series solutions expanded around infinity with \( p_0 \)
as initial tuple by \( \text{Sol}(p_0) \) and the output of Algorithm PuiseuxSolve-
Infinity by \( \text{STrunc}_N(p_0) \).
Since $\mathbb{V}(F(y,0))$ is a finite set and termination of the Newton-polygon method for computing formal parametrizations and of the Newton-polygon method for computing the reparametrizations is ensured, also termination of Algorithm 2 follows. Correctness of Algorithm 2 follows from section 3 and the following Corollary.

**Corollary 4.4.** Let $F \in \mathbb{C}[y,p]$ be square-free with no factor in $\mathbb{C}[y]$ or $\mathbb{C}[p]$. Then every solution truncation $\tilde{y} \in \text{STrunc}_N(p_0)$ can be extended to $y \in \text{Sol}(p_0)$ and vice versa, for every $y \in \text{Sol}(p_0)$ there exists a truncation $\tilde{y} \in \text{STrunc}_N(p_0)$.

**Proof.** Since in Algorithm 2 all places fulfilling the necessary conditions (3.2) and all solutions of (3.5) are treated, the statement holds. $\square$

In Theorem 4.1 we were able to additionally show that the corresponding output truncations $\tilde{y}_1, \tilde{y}_2 \in \text{RTrunc}_N(p_0)$ coming from different places or different reparametrizations do not coincide. However, in Algorithm PuiseuxSolveInfinity we cannot guarantee this. The problematic cases are those where the order of the first distinct (up to multiplication with roots of unity) coefficient of some formal parametrizations $(y_0 + t^r b_1(t)) \neq (y_0 + t^r b_2(t))$, let us say $m \in \mathbb{N}$, and the ramification order $n$ of the solutions coincide.

**Example 4.5.** Let us consider

$$F(y, y') = y' + y^2 = 0$$

and its formal Puiseux series solutions expanded around infinity. We obtain $\mathbb{V}(F(y,0)) = \{0\}$. For $p_0 = (0,0)$ compute the formal parametrization $(a,b) = (t,-t^2)$, which fulfills (3.2) with $n = 1$. Equation (3.5) simplifies to $s'(t) = -t^{-2} s(t)^2$ having the solutions

$$s(t) = \frac{-t}{1 - ct} = -t - \frac{ct^2}{2} - \frac{3c^2 t^3}{8} + \mathcal{O}(t^4)$$

for an arbitrary constant $c$. Hence,

$$a(s(x^{-1})) = \frac{1}{x} + \frac{c}{x^2} + \frac{c^2}{x^3} + \mathcal{O}(x^{-4})$$

describes all formal Puiseux series solutions expanded around infinity.

**REFERENCES**

[1] J.M. Aroca. Puiseux Solutions of Singular Differential Equations. In *Resolution of Singularities*, 129–145. Birkhäuser Basel, 2000.

[2] J.M. Aroca, J. Cano, R. Feng, and X.-S. Gao. Algebraic general solutions of algebraic ordinary differential equations. In *Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation*, 29–36. ACM Press, 2005.
[3] F. Aroca Bisquert. Métodos algebraicos en ecuaciones diferenciales ordinarias en el campo complejo. PhD Thesis, Universidad de Valladolid, 2000.

[4] J. Cano. An extension of the Newton-Puiseux polygon construction to give solutions of Pfaffian forms. *Ann. Inst. Fourier (Grenoble)*, 43:125–142, 1993.

[5] J. Cano. On the series defined by differential equations, with an extension of the puiseux polygon construction to these equations. *Analysis*, 13:125–142, 1993.

[6] J. Cano. The Newton Polygon Method for Differential Equations. In *Proceedings of the 6th International Conference on Computer Algebra and Geometric Algebra with Applications*, 18–30. Springer-Verlag, 2005.

[7] J. Cano and P. Fortuny. The Space of Generalized Formal Power Series Solution of an Ordinary Differential Equations. *Astérisque*, 323:61–82, 2009.

[8] J. Della Dora and F. Richard-Jung. About the Newton Algorithm for Non-linear Ordinary Differential Equations. In *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, 298–304, ACM Press, 1997.

[9] D. Duval. Rational puiseux expansion. *Compositio Mathematica*, 70:119–154, 1989.

[10] S. Falkensteiner and J.R. Sendra. Formal Power Series Solutions of First Order Autonomous Algebraic Ordinary Differential Equations. *Mathematics in Computer Science*, 2019 (to appear).

[11] R. Feng and X.-S. Gao. Rational general solutions of algebraic ordinary differential equations. In *Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation*, 155–162. ACM Press, 2004.

[12] R. Feng and X.-S. Gao. A polynomial time algorithm for finding rational general solutions of first order autonomous odes. *J. Symb. Comput.*, 41(7):739–762, 2006.

[13] H. Fine. On the functions defined by differential equations, with an extension of the Puiseux Polygon construction to these equations. *Amer. Jour. of Math.*, 11:317–328, 1889.

[14] H. Fine. Singular solutions of ordinary differential equations. *Amer. Jour. of Math.*, 12:295–322, 1890.

[15] D. Grigor’ev and M.F. Singer. Solving Ordinary Differential Equations in Terms of Series with Real Exponents. *Transactions of the American Mathematical Society*, 327(1):329–351, 1991.

[16] E. Ince. *Ordinary Differential Equations*. Dover Books on Mathematics, 1926.

[17] B. Malgrange. Sur le théorème de Maillet. *Asymptotic Analysis*, 2:1–4, 1989.

[18] P. Stadelmeyer. *On the Computational Complexity of Resolving Curve Singularities and Related Problems*. PhD Thesis, Johannes Kepler University Linz, 2000.

[19] N.T. Vo, G. Grasegger, and F. Winkler. Deciding the Existence of Rational General Solutions for First-Order Algebraic ODEs. *J. Symb. Comput.*, 87:127–139, 2018.

[20] R.J. Walker. *Algebraic curves*. Princeton University Press, 1950.
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