Spin Cohomology

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Abstract

We explore differential and algebraic operations on the exterior product of spinor representations and their twists that give rise to cohomology, the spin cohomology. A linear differential operator $d$ is introduced which is associated to a connection $\nabla$ and a parallel spinor $\zeta$, $\nabla \zeta = 0$, and the algebraic operators $D(p)$ are constructed from skew-products of $p$ gamma matrices. We exhibit a large number of spin cohomology operators and we investigate the spin cohomologies associated with connections whose holonomy is a subgroup of $SU(m)$, $G_2$, $Spin(7)$ and $Sp(2)$. In the $SU(m)$ case, we find that the spin cohomology of complex spin and spin$^c$ manifolds is related to a twisted Dolbeault cohomology. On Calabi-Yau type of manifolds of dimension $8k + 6$, a spin cohomology can be defined on a twisted complex with operator $d + D$ which is the sum of a differential and algebraic one. We compute this cohomology on six-dimensional Calabi-Yau manifolds using a spectral sequence. In the $G_2$ and $Spin(7)$ cases, the spin cohomology is related to the de Rham cohomology.
1 Introduction

On spin manifolds apart from the exterior derivative $d$ and the associated de Rham complex $(\Lambda^*(M), d)$, one can define the Dirac operator $(\Delta(M), D)$, where $\Delta(M)$ is the spin bundle\(^1\). On complex manifolds the Dirac operator decomposes as $D = D + \bar{D}$ and the spin representation can be graded such that $(\Delta(M), \bar{D})$ can turn into a (graded) complex. The associated cohomology is called spinor cohomology \([1]\).

In even dimensions the complex Dirac spin representation is reducible and decomposes as $\Delta = \Delta^- \oplus \Delta^+$. Apart from the spin representation $\Delta$ both the exterior power, $\mathcal{C} = \Lambda^* (\Delta^*) (\mathcal{C}_\pm = \Lambda^*(\Delta_\pm))$, and the symmetric product, $\text{Sym}^*(\Delta^*) (\text{Sym}^*(\Delta_\pm))$, of the dual spinor representation $\Delta^*$ ($\Delta_\pm = (\Delta^\pm)^*$) have found applications in various problems in physics. The former has applications in supermanifolds. In particular all real supermanifolds that appear in the context of supersymmetry are isomorphic to $\mathcal{C}$ ($\mathcal{C}_\pm$) \([2]\). The latter is a model for the odd forms on supermanifolds and has appeared in the context of string theory \([3, 4]\) and the theory of deformations of the field equations of supersymmetric gauge theories and supergravity in superspace \([5, 6]\). It turns out that the theory of deforming the field equations of the supersymmetric gauge theories and supergravity can turn into a cohomological problem on $\Lambda^* \otimes \text{Sym}^*$ for the so called spinorial cohomology.

Motivated by these developments in physics, the aim of this paper is to investigate various cohomology operators that can be defined on $\mathcal{C}$, $\mathcal{C}_\pm$ and its various twistings. Let $(M, g)$ be a spin manifold equipped with a spin connection $\nabla$, which is not necessarily the Levi-Civita connection of the metric $g$. One can define a linear differential (spin) operator on $\mathcal{C}(M)$ or $\mathcal{C}_\pm(M)$ as

$$d\phi = \zeta \Gamma^\mu \bar{\Lambda} \nabla_\mu \phi \ , \quad \phi \in \mathcal{C}(M) \ ,$$

where $\bar{\Lambda}$ is the wedge product in $\mathcal{C}$, $\zeta$ is a cospinor and $\{\Gamma^\mu : \mu = 1, \ldots, \text{dim} M\}$ are the gamma matrices. In many applications $\zeta$ is taken to be a parallel cospinor with respect to $\nabla$, $\nabla \zeta = 0$. As we shall see there are various cohomology theories that can be defined depending on the choice of cospinor $\zeta$, the connection $\nabla$. The operator $d$ can always be defined on $\mathcal{C}$. However, the restriction of $d$ on $\mathcal{C}_\pm$ depends on the choice of $\zeta$ and the properties of the spinor inner products which in turn depend on the dimension of the manifold $M$. One of our aims is to investigate the conditions for $d$ to be nilpotent, $d^2 = 0$. These conditions can be expressed in terms of restrictions on the cospinor $\zeta$ and on the curvature $R$ of the connection $\nabla$, in addition to $\nabla \zeta = 0$.

In addition to differential operators, we shall present a large number of algebraic cohomology operators $D_{(p)}$ on some twisted complexes like for example $\Lambda^* (M) \otimes \mathcal{C}(M)$ and $\Lambda^* (M) \otimes \mathcal{C}_\pm(M)$. Some of these are constructed from skew-products of $p$ gamma matrices. We investigate the conditions for $D_{(p)}^2 = 0$ and relate them into the symmetry properties of gamma matrices. The latter again depend on the dimension of the manifold $M$. In addition we shall show that $D_{(p)}d + dD_{(p)} = 0$ and so the cohomology of $(\Lambda^* \otimes$  

\[^1\]We adopt the notation to denote a representation and its associated bundle with the same symbol, e.g. $\Delta = \Delta(\mathbb{R}^n)$ denotes the spin representation of $\text{Spin}(n)$ and $\Delta = \Delta(M)$ denotes also the spin bundle over $M$. In addition we shall denote the bundles and their sections with the same symbol. $\Lambda^*$ denotes the space of forms.
\(C, d + D(p)\) and \((\Lambda^* \otimes C_\pm, d + D(p))\) can be computed using a spectral sequence. We shall refer collectively to all of these cohomology theories with operators \(d, D(p)\) and \(d + D(p)\) as spin cohomologies.

We shall develop the general theory of spin cohomology. In particular, we shall compute the conditions on the curvature of the underlying manifold for \(d^2 = 0\). We shall also explain the relation to parallel spinors.

Next, we shall focus on a certain class of parallel spinors. In particular, we shall consider manifolds which admit a spin connection \(\nabla\) induced from the tangent bundle with holonomy contained in the groups \(SU(m)\) (\(n=2m\)), \(Sp(k)\) (\(n=4k\)), \(Spin(7)\) (\(n=8\)) and \(G_2\) (\(n=7\)), where in parenthesis is the dimension of the manifold. A special class of examples of manifolds with spin cohomology are those of which \(\nabla\) is the Levi-Civita connection.

We shall show that for spin complex manifolds which admit a holomorphic connection \(\nabla\) with \(\text{hol}\nabla \subseteq SU(m)\), there are two differential spin cohomologies with operators \(d_1\) and \(d_2\) related to two parallel spinors of the connection \(\nabla\). We refer to these spin cohomologies as complex spin cohomologies. We shall show that \(d_1\) and \(d_2\) restrict on \(C_\pm\). In particular one can construct complexes \((C_+, d_1)\) and \((C_+, d_2)\) for \(\dim M = 8k + 2, 8k + 6\) and complexes \((C_-, d_1)\) and \((C_-, d_2)\) for \(\dim M = 8k, 8k + 4\). We give the Laplace operators associated with \(d_1\) and \(d_2\) using a \(Spin(n)\)-invariant inner product. We show that the complex spin cohomology of \((C_-, d_2)\) in all dimensions is related to twisted Dolbeault cohomology. We extend this relation between this spin cohomology and Dolbeault cohomology to complex spin \(c\) manifolds as well. The complex spin cohomologies can be twisted with any holomorphic vector bundle. Apart from the differential complex spin cohomologies, there is an algebraic spin cohomology operator \(D = D(1)\) and the complex \(\Lambda^* \otimes C_-\) on all such manifolds of dimension \(n = 8k + 6\) and \(d_2 D + D d_2 = 0\). The cohomology of \((\Lambda^* \otimes C_-, d_2 + D)\) can be computed using a spectral sequence. As an example, we computed the cohomology of \((\Lambda^* \otimes C_-, d_2 + D)\) on six-dimensional Calabi-Yau manifolds.

On manifolds which admit a connection with holonomy \(Sp(k)\), there are \(k + 1\) differential spin operators associated to \(k + 1\) parallel spinors. Two of these are the same as those of the \(SU(2k)\) manifolds investigated above. We shall not present a full analysis in this case but we shall express a third spin differential operator on hyperKähler manifolds in terms of a Dolbeault operator.

On manifolds which admit a connection with holonomy \(Spin(7)\), there is one differential spin operator \(d\) associated to one parallel spinor and a real complex \((C_R, d)\). In addition, \(d^2 = 0\) provided the connection \(\nabla\) is the Levi-Civita connection of a \(Spin(7)\) metric. The spin cohomology is isomorphic to de Rham cohomology.

On manifolds which admit a connection with holonomy \(G_2\), there is again one differential spin operator \(d\) and a real complex \((C_R, d)\). In addition, \(d^2 = 0\) provided the connection \(\nabla\) is the Levi-Civita connection of a \(G_2\) metric. The spin cohomology of \((C_R, d)\) is isomorphic to two copies of the de Rham cohomology relatively shifted by one degree.

This paper has been organized as follows: In section two, we summarize the properties of Clifford algebras and spin representations which we use later. In section three, we explore the general properties of the linear differential operators \([\mathbf{1}1]\), define the twisted
complexes and present the algebraic cohomology operators. In section four, we investigate the properties of complex spin cohomology and derive the conditions for \(d^2 = 0\). In addition we compute the Laplace operators. In section five, we investigate various kinds of twisted complex cohomology. In section six, we relate the complex spin cohomology to the Dolbeault cohomology for spin and spin\(_c\) manifolds. In section seven, we compute the complex spin cohomology and a twisted spin cohomology on a six-dimensional Calabi-Yau manifold. In section eight, we investigate the spin cohomology of manifolds that admit a connection with holonomy contained in \(Sp(k)\). In section nine, we explore the properties of some real spin cohomologies. In sections ten and eleven, we investigate the spin cohomology of manifolds that admit a connection with holonomy \(Spin(7)\) and \(G_2\), respectively.

2 Preliminaries

The investigation of spin cohomology involves a detailed description of spinor representations. Because of this and to establish notation, we shall review some aspects of spinor representations in various dimensions [3, 4]. We shall focus on the manifolds with Euclidean signature but the analysis can be easily extended to other signatures.

Let \(V = \mathbb{R}^n\) be a real vector space equipped with the standard Euclidean inner product. If \(n = 2m\) even, the complex spin (Dirac) representation of \(Spin(2m)\), \(\Delta = \Delta(\mathbb{C})\), is reducible and decomposes to two irreducible representations, \(\Delta = \Delta^+ \oplus \Delta^-\).

To construct these spin representations let \(e_1, \ldots, e_n\) be an orthonormal basis in \(\mathbb{R}^n\), \(n = 2m\), and \(J\) be a complex structure in \(V\), \(J(e_i) = e_{i+m}\). We identify \(V\) and its dual using the Euclidean inner product. Next consider the subspace \(U = \mathbb{R}^m\) generated by \(e_1, \ldots, e_m\). Clearly \(V = U \oplus J(U)\). The Euclidean inner product on \(V\) can be extended to a hermitian inner product in \(V_\mathbb{C} = V \otimes \mathbb{C}\) denoted by \(\langle,\rangle\), ie

\[
\langle z^\mu e_\mu, w^\nu e_\nu \rangle = \sum_\mu \bar{z}^\mu w^\mu ,
\]

(2.1)

where \(\bar{z}\) is the standard complex conjugate of \(z\) in \(V_\mathbb{C}\). The space of spinors \(\Delta(V) = \Lambda^*(U_\mathbb{C})\), where \(U_\mathbb{C} = U \otimes \mathbb{C}\). In addition, \(\Delta^+ = \Lambda^{even}(U_\mathbb{C})\) and \(\Delta^- = \Lambda^{odd}U_\mathbb{C}\). The spinors in \(\Delta^+\) are called chiral while those in \(\Delta^-\) anti-chiral. The inner product (2.1) can be easily extended to \(\Delta\) and it is called the Dirac inner product on the space of spinors. The generators of the Clifford algebra \(e_\mu\) are represented on \(\Delta\) as

\[
\Gamma(e_i)\eta = e_i \cdot \eta \equiv e_i \wedge \eta + e_i \eta , \quad i \leq m
\]

\[
\Gamma(e_{m+i})\eta = e_{i+m} \cdot \eta = -ie_i \wedge \eta + ie_i \eta , \quad i \leq m ,
\]

(2.2)

where \(e_i \wedge\) is the adjoint of \(e_i\wedge\) with respect to \(\langle,\rangle\). It is convenient to denote the generators \(\Gamma(e_\mu) = \Gamma_\mu\) and they are often called gamma matrices. Clearly \(\Gamma_\mu : \Delta^\pm \to \Delta^\pm\). The linear maps \(\Gamma_\mu\) are hermitian with respect to the inner product \(\langle,\rangle\), \(\langle \eta, \Gamma_\mu \theta \rangle = \langle \eta, \Gamma_\mu \theta \rangle\), and satisfy the Clifford algebra relations \(e_\mu e_\nu + e_\nu e_\mu = \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 0\), for \(\mu \neq \nu\), \((e_\mu)^2 = (\Gamma_\mu)^2 = 1\).
Next define the maps \( A = \Gamma_1 \Gamma_2 \ldots \Gamma_m \) and \( B = \Gamma_{m+1} \ldots \Gamma_n \) and the inner products on \( \Delta \) as
\[
A(\eta, \theta) = \langle A(\bar{\eta}), \theta \rangle \\
B(\eta, \theta) = \langle B(\bar{\eta}), \theta \rangle ,
\]
which we denote with the same symbol, where \( \bar{\eta} \) is the standard complex conjugate of \( \eta \) in \( \Lambda^*(V_C) \). The inner products \( A, B \) are sometimes also called charge conjugation matrices. These have the following properties:
\[
A(\eta, \theta) = (-1)^{\frac{1}{2}m(m-1)} A(\theta, \eta) \\
B(\eta, \theta) = (-1)^{\frac{1}{2}m(m+1)} B(\theta, \eta) .
\]
Therefore \( A \) (\( B \)) is symmetric for \( m = 4k, 4k+1 \) (\( m = 4k, 4k+3 \)) and skew-symmetric for \( m = 4k + 2, 4k + 3 \) (\( m = 4k + 2, 4k + 1 \)). In addition, we have
\[
A(\Gamma_\mu \eta, \theta) = (-1)^{m-1} A(\eta, \Gamma_\mu \theta) , \quad 1 \leq \mu \leq n \\
B(\Gamma_\mu \eta, \theta) = (-1)^m B(\eta, \Gamma_\mu \theta) , \quad 1 \leq \mu \leq n
\]
and
\[
A(\Gamma_\mu \eta, \Gamma_\mu \theta) = (-1)^{m-1} A(\eta, \theta) , \quad 1 \leq \mu \leq n \\
B(\Gamma_\mu \eta, \Gamma_\mu \theta) = (-1)^m B(\eta, \theta) , \quad 1 \leq \mu \leq n .
\]
Therefore \( A \) is \( \text{Pin}(2m) \)-invariant for \( m = 4k+1, 4k+3 \) while \( B \) is \( \text{Pin}(2m) \)-invariant for \( m = 4k, 4k+2 \). Both \( A, B \) are \( \text{Spin}(n) \)-invariant. A consequence of the above relations is
\[
A(\eta, \Gamma_\mu \theta) = (-1)^{\frac{1}{2}(m-1)(m+2)} A(\theta, \Gamma_\mu \eta) \\
B(\eta, \Gamma_\mu \theta) = (-1)^{\frac{1}{2}m(m+3)} B(\theta, \Gamma_\mu \eta)
\]
Therefore the gamma-matrices are symmetric with respect to the inner product \( A(\cdot) \) for \( m = 4k+1, 4k+2 \) (\( m = 4k, 4k+1 \)) while they will be skew-symmetric for \( m = 4k, 4k+3 \) (\( m = 4k + 2, 4k + 3 \)).

Because of the existence of invariant non-degenerate inner products the dual of \( \Delta^* \) can be identified with \( \Delta \). To make this identification precise, let us denote with \( C \) either \( A \) or \( B \). Given a basis \( \{ \epsilon_A; A = 1, \ldots, \dim\Delta \} \) in \( \Delta \) let us denote with \( \{ \epsilon^A; A = 1, \ldots, \dim\Delta \} \) the dual basis in \( \Delta^* \), \( \epsilon^B(\epsilon_A) = \delta^B_A \). The inner product \( C^{-1} \) in \( \Delta^* \) induced by \( C \) is
\[
\sum_E C^{-1}(\epsilon^E, \epsilon^A)C(\epsilon_E, \epsilon_B) = \delta^A_B
\]
The co-spinor \( C(\eta) \) associated with the spinor \( \eta \) under the isomorphism \( C \) is defined as \( C(\eta)(\theta) = C(\theta, \eta), \eta, \theta \in \Delta \), ie in the above basis \( \eta_A = C_{AB} \eta^B \). The inverse transformation \( C^{-1} \) is defined as \( C^{-1}(\psi)(\chi) = C^{-1}(\psi, \chi), \psi, \chi \in \Delta^* \), ie \( \psi^A = \psi_B(C^{-1})^B_A \).

Notice that the maps \( A, B : \Delta^\pm \to \Delta^\pm \) for \( m = 4k, 4k+2 \) while \( A, B : \Delta^\pm \to \Delta^\pm \) for \( m = 4k + 1, 4k + 3 \). Therefore in the former case the dual \( \Delta^\pm \) of \( \Delta^\pm \) under \( A, B \) is
identified with $\Delta^\pm$, $\Delta_+ = \Delta^+\leq 1$ while in the latter the dual $\Delta_-$ of $\Delta^\pm$ is identified with $\Delta^\mp$, $\Delta_- = \Delta^\mp$.

There are two ways to construct the spin representation $\Delta$ in odd dimensions. One is to write $V' = \mathbb{R}^{2m+1} = V \oplus \mathbb{R} < e_{2m+1}>$ and extend the Euclidean inner product (2.1) from $V$ to $V'$, $< e_{2m+1}, e_{2m+1}> = 1$, $< V, e_{2m+1} > = 0$. The gamma matrices $\Gamma_\mu$, $1 \leq \mu \leq 2m$, are defined as in the even-dimensional case and

$$\Gamma_{2m+1} = i\tau^a \Gamma_1 \ldots \Gamma_{2m}.$$  \hspace{1cm} (2.9)

The $Spin(2m+1)$ spin representation, $\Delta$, is $\Delta = \Delta^+ \oplus \Delta^-$, where $\Delta^+, \Delta^-$ are the $Spin(2m)$ spin representations. (There are no chiral spinors in odd dimensions.) The invariant inner product on $Spin(2m+1)$ representation $\Delta$ is the $Pin(2m)$ invariant inner product on $\Delta^+ \oplus \Delta^-.$

Alternatively, we take $V = U \oplus J(U)$ as for $n = 2m$ and write $U = U_0 \oplus \mathbb{R} < e_{2m}>$. Then $V_0 = U \oplus J(U_0)$ has dimension $2m - 1$. The gamma matrices are $\tilde{\Gamma}_\mu = i\Gamma_\mu \Gamma_{2m}$, $1 \leq \mu \leq 2m - 1$, where $\Gamma_\mu$ are the gamma matrices of the $Spin(2m)$ spin representation. These induce a representation of $Pin(2m-1)$ onto the $\Delta^\pm$ representations of $Spin(2m)$.

For later convenience, we introduce the notation $(C\Gamma_\mu)(\eta, \theta) = C(\eta, \Gamma_\mu \theta)$ and similarly $(C\Gamma_{\mu_1 \ldots \mu_p})(\eta, \theta) = C(\eta, \Gamma_{\mu_1 \ldots \mu_p} \theta)$, where

$$\Gamma_{\mu_1 \ldots \mu_p} = \frac{1}{p!} \sum_\sigma (-1)^{|\sigma|} \Gamma_{\mu_{\sigma(1)}} \ldots \Gamma_{\mu_{\sigma(p)}}$$  \hspace{1cm} (2.10)

and $\sigma$ is a permutation. The symmetry of the inner product and that of the gamma matrices can be re-expressed as $C(\eta, \theta) = (-1)^{s_C} C(\theta, \eta)$, where $s_C = 0$ if $C$ is symmetric and $s_C = 1$ if $C$ is skew-symmetric, and similarly $C\Gamma_\mu(\eta, \theta) = (-1)^{s_{\Gamma}} C\Gamma_\mu(\theta, \eta)$, where $s_\Gamma = 0$ if $C\Gamma_\mu$ is symmetric and $s_\Gamma = -1$ if $C\Gamma_\mu$ is skew-symmetric. From these one can also find that

$$C\Gamma_{\mu_1 \ldots \mu_p}(\eta, \theta) = (-1)^{\frac{1}{2}p(p-1)}(-1)^{(p+1)s_C + ps_{\Gamma}} C\Gamma_{\mu_1 \ldots \mu_p}(\theta, \eta).$$  \hspace{1cm} (2.11)

Similarly we define $(\Gamma_{\mu_1 \ldots \mu_p}C^{-1})(\psi, \chi) = C^{-1}(\Gamma_{\mu_1 \ldots \mu_p}\psi, \chi)$, where $\chi, \psi \in \Delta^*.$

The product of two co-spinor representations can be decomposed in terms of forms as $\Delta^* \otimes \Delta^* = \sum_{p=1}^n \Lambda^p(V) \otimes \mathbb{C}$. In particular, one can write

$$(\psi \otimes \chi)(\eta \otimes \theta) = \frac{1}{\dim \Delta_n} \left( C^{-1}(\psi, \chi) C(\eta, \theta) \right. + \sum_{p=1}^n (-1)^{p(s_{\Gamma} + s_C)} \frac{1}{p!} \left( \Gamma_{\mu_1 \ldots \mu_p}C^{-1}(\psi, \chi) C\Gamma_{\mu_1 \ldots \mu_p}(\eta, \theta) \right),$$  \hspace{1cm} (2.12)

where $\eta, \theta \in \Delta$ and $\psi, \chi \in \Delta^*$. The above decomposition is valid after restricting to $\Delta_{\pm}$ co-spinor representations and to real co-spinor representations. We shall state the formulae later. The formula (2.12) is also known as Fierz identity.

In the above formalism, it is possible to explicitly present the spinors that are invariant under the action of certain subgroups of $Spin(n)$. We shall mainly focus on the subgroups $G \subset Spin(n)$ which arise as special holonomy groups in the Berger classification and the associated manifolds admit a parallel spinor. These spinors have been given
in \[3\]. Here we shall summarize the results and adjust the various formulae because of differences in the conventions.

(i) \(G = SU(m) \subset Spin(2m)\). The invariant spinors under the \(SU(m) \subset Spin(2m)\) are \(1, e_1 \wedge e_2 \wedge \ldots \wedge e_m\). This can be easily seen by decomposing the \(Spin(2m)\) representations \(\Delta^k\) under \(SU(m)\). If \(m = 4k, 4k + 2\) both invariant spinors are of the same chirality, i.e., they are elements of \(\Delta^+\) while if \(m = 4k + 1, 4k + 3\), they have opposite chiralities. In addition observe that \(\Gamma_j = i \Gamma_{m+j}(1) = 0\) and \(\Gamma_j + i \Gamma_{m+j}(e_1 \wedge \ldots e_m) = 0\), \(j = 1, \ldots, m\). Therefore, the invariant spinors are pure spinors with respect to the holomorphic and antiholomorphic parts of the decomposition of \(V \otimes \mathbb{C}\) with respect to the complex structure \(J\).

(ii) \(G = Sp(k) \subset Spin(4k)\). The invariant spinors are \(1, e_1 \wedge e_2 \wedge \ldots e_{2k}, \omega, \omega^2, \ldots, \omega^{k-1}\) where \(\omega = e_1 \wedge e_2 + \ldots + e_{2k-1} \wedge e_{2k}\) which is the symplectic form in \(U \subset \Delta^+\). Therefore there are \(k + 1\) parallel spinors.

(iii) \(G = Spin(7) \subset Spin(8)\). The invariant spinor is \(\frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)\).

(iv) \(G = G_2 \subset Spin(7)\). The invariant spinor is \(\frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)\).

\section{Differential and algebraic operations on spinors}

\subsection{First order differential operators}

Let \(M\) be a spin manifold equipped with a spin connection \(\nabla\) which admits a parallel spinor \(\zeta\), \(\nabla \zeta = 0\). We shall focus on even-dimensional manifolds. Some of the results can be easily extended to the odd dimensional case. We define \(C_\pm = \Lambda^* (\Delta_\pm)\) and \(C = \Lambda^* (\Delta^*)\) equipped with the wedge product \(\wedge\).

\textbf{Definition 1} The spin operator \(d\) is a linear differential operator \(d : C(M) \to C(M)\), and similarly \(d : C_\pm (M) \to C_\pm (M)\), such that

\[d\phi = \sum_{\mu=1}^{n} C^\mu_\zeta \wedge \nabla_\mu \phi,\]  

where \(C^\mu_\zeta (\eta) = C \Gamma^\mu (\zeta, \eta)\).

Clearly \(d : C^\ell \to C^{\ell+1}\) and \(d : C_\pm^\ell \to C_\pm^{\ell+1}\). Choosing a basis in the space of co-spinors \(\{e^A : A = 1, \ldots, 2^m\}\), the \(d\) operator can be written as

\[d\phi_{A_1 A_2 \ldots A_{\ell+1}} = \zeta^B C \Gamma^\mu_{BA_1} \nabla_\mu \phi_{A_2 \ldots A_{\ell+1}} + \text{cyclic}(A_1, A_2, \ldots, A_{\ell+1})\]

\[B, A_1, \ldots, A_{\ell+1} = 1, \ldots, \text{dim}\Delta.\]  

(3.2)

The operator \(d\) depends on the choice of parallel spinor \(\zeta\) and the connection \(\nabla\). Although subject to the data above, the operator \(d\) can always be defined on \(C(M)\), the restriction on \(d\) onto \(C_+\) or \(C_-\) depends on the choice of the parallel spinor \(\zeta\). Since this depends on the dimension of the manifold and the choice of the parallel spinor, we shall explain the general properties of the operator \(d\) acting on \(C\), and later we shall specialize into the various cases.
Evaluating $d^2$, we find
\[ d^2 \phi = \frac{1}{2} C^\mu_\zeta \wedge C^\nu_\zeta \wedge R_{\mu\nu} \phi, \] (3.3)
where $R$ is the curvature of the connection $\nabla$. Under certain conditions the operator $d$ can be nilpotent, $d^2 = 0$. This depend on the choice of the spinor $\zeta$ and the connection $\nabla$. There are two large classes of examples for which $d^2 = 0$.

- Group manifolds equipped with the left or the right invariant connections.
- Manifolds that admit a pure parallel spinor.

In the case of group manifolds $R = 0$. Therefore $d$ operators associated with the left- or right-invariant connections are all nilpotent. The different $d$ operators that can be constructed on group manifolds are determined by the orbits of $Spin(n)$ in $\Delta$.

A spinor is pure if the subspace
\[ W(\zeta) = \{ v \in V_C, v_\mu \Gamma^\mu \zeta = 0 \} \] (3.4)
of $V_C$ has dimension $\frac{1}{2} \dim_C(V_C)$. We can use the inner product to decompose $V_C = W(\zeta) \oplus Z$.

**Proposition 1** The operator $d$ is nilpotent if, in addition to $\nabla \zeta = 0$, the curvature $R$ vanishes along the subspace $\Lambda^2(Z) \subset \Lambda^2(V_C)$, ie
\[ R|_{\Lambda^2(Z)} = 0. \] (3.5)

**Proof:** The curvature $R$ can be viewed as a map from $R : \Lambda^2(M) \to \Lambda^2(M)$. Therefore
\[ d^2 \phi = \frac{1}{2} C^\mu_\zeta \wedge C^\nu_\zeta \wedge R_{\mu\nu} \phi = \frac{1}{2} [C^\mu_\zeta \wedge C^\nu_\zeta \wedge R_{\mu\nu}]|_{\Lambda^2(Z)} \phi = 0. \] (3.6)
q.e.d.

Clearly, this condition can be generalized to spinors $\zeta$ which are not pure but $W(\zeta) \neq \emptyset$.

The conditions on the curvature required for $d^2 = 0$ can also be determined using (2.12). In particular we have

**Proposition 2** The conditions on the curvature $R$ for $d^2 = 0$ can be expressed in terms of the forms associated with the parallel spinor $\zeta$.

**Proof:** We compute $d^2$ using (2.12) to find
\[ d^2 \phi = \frac{1}{2 \dim \Delta_n} \left( \sum_{p=0}^n (-1)^p (s_1 + s_c) \frac{p!}{p!} (C^\rho_1 \cdots C^\rho_p \Gamma^{-1})(\zeta, C^\mu_\zeta) C \Gamma_{\rho_1 \cdots \rho_p} \wedge R_{\mu\nu} \phi \right) \]
\[ = \frac{1}{2 \dim \Delta_n} \left( \sum_{p=0}^n (-1)^{(p+1)(s_1 + s_c)} \frac{p!}{p!} C(\zeta, \Gamma^\rho_1 \cdots \Gamma^\rho_p \zeta) C \Gamma_{\rho_1 \cdots \rho_p} \wedge R_{\mu\nu} \phi \right) \]
\[
\frac{1}{2 \dim \Delta_n} \left( \sum_{p=0}^{n} \frac{(-1)^{(p+1)(s_p+s_C)}}{p!} \right) \left[ C(\zeta, \Gamma^{\mu \rho_1 \cdots \rho_p \nu} \zeta) + p(p-1)g^{\mu \rho_1}C(\zeta, \Gamma^{\rho_2 \cdots \rho_{p-1} \nu} \zeta) g^{\rho \rho_p} \right]
\]
where \( g \) is the metric on the manifold. In the above sum over \( p \) only the terms with \( C_{\rho_1 \cdots \rho_p} \) skew-symmetric contribute. Sufficient conditions on the curvature for \( d^2 = 0 \) are

\[
[C(\zeta, \Gamma^{\mu \rho_1 \cdots \rho_p \nu} \zeta) + p(p-1)C(\zeta, \Gamma^{\rho_2 \cdots \rho_{p-1} \nu} \zeta) g^{\mu \rho_1} g^{\nu \rho_p}] R_{\mu \nu} = 0 \tag{3.8}
\]
for \( \frac{1}{2} p(p-1) + (p+1)s_C + ps_T \in 2\mathbb{Z} + 1 \). In some cases these conditions are also necessary. q. e. d.

Provided that the condition for \( d^2 = 0 \) are met, we can define a cohomology theory associated with the linear differential operator \( d \).

**Definition 2** The spin cohomology, \( H_d(C) \), is that of the graded complex \((C, d)\), \( d^2 = 0 \), where \( d \) is as in (3.1). Similarly, the spin cohomology, \( H_d(C_{\pm}) \), is that of the graded complex \((C_{\pm}, d)\).

### 3.2 Twisted Complexes

There are several ways to twist the complexes \( C \) and \( C_{\pm} \). Here we shall consider two cases which we shall describe below.

#### 3.2.1 The complexes \( C \otimes E \) and \( C_{\pm} \otimes E \)

Let \( E \) be a vector bundle \( E \) over the spin manifold \( M \) equipped with a connection \( \nabla^E \). One way to twist the complexes \( C \) and \( C_{\pm} \) is to consider \( C \otimes E \) and \( C_{\pm} \otimes E \). Let \( \zeta \) be a parallel spinor with respect to a spin connection \( \nabla^M \) on the manifold \( M \) induced from the tangent bundle, \( \nabla^M \zeta = 0 \). The spin differential operator \( d \) is

\[
d\phi = C^{\mu}_{\zeta} \bar{\zeta} \gamma_\mu \phi , \tag{3.9}
\]
where \( \nabla = \nabla^M \otimes 1 + 1 \otimes \nabla^E \) on \( C \otimes E \) or on \( C_{\pm} \otimes E \) and \( \phi \in C \otimes E \) or \( C_{\pm} \otimes E \), respectively.

The condition \( d^2 = 0 \) implies conditions on both the curvature \( R \) of \( M \) and the curvature \( F \) of the connection \( \nabla^E \) of the bundle \( E \).

**Theorem 1** The operator \( d \) is nilpotent providing that both the curvature \( R \) of the manifold \( M \) and the curvature \( R \) of \( E \) satisfy either (3.5) or (3.7).

**Proof:** This is similar to the proof given in the previous section. q. e. d.

There is a particular twisted complex of this type that we shall consider by taking \( E = \Lambda^*(M) \) or \( E = \Lambda^*(M) \otimes \mathbb{C} \). We shall see that in this case one can define certain algebraic operators with are nilpotent. The spin cohomology, \( H_d(C \otimes E) \), of the linear operator \( d \) for the twisted complex \((C \otimes E, d)\), can be defined in analogy with the spin cohomology of the untwisted case in the previous section. This definition can be extended for \( H_d(C_{\pm} \otimes E) \).
3.2.2 The complexes \( C(E) = \Lambda^*(\Delta^* \otimes E) \) and \( C(E)_\pm = \Lambda^*(\Delta^\pm \otimes E) \)

These complexes allow the definition of the spin operator \( d \) operator on manifolds that do not admit a spin structure but admit a \( Spin_c \) or in general a \( Spin_G \) structure. Another use of twisted complexes \( C(E) \) is that they allow the imposition of a reality condition. It is known that there are not real (Majorana) spin representations for \( Spin \) do not admit a spin structure but admit a \( d \). These complexes allow the definition of the spin operator \( q \) if \( p \geq 1 \) as it can been seen from (2.11).

Let \( \zeta \) be a parallel section of \( \Delta \otimes E \) with respect to a connection \( D = \nabla^M \otimes 1 + 1 \otimes \nabla^E \), where \( \nabla^M \) is a spin connection on the manifold \( M \) induced from the tangent bundle and \( \nabla^E \) is a connection on the vector bundle \( E \). The operator \( d \) on \( C(E) \) or \( C(E)_\pm \) is defined as

\[
d\phi = C^\mu_\zeta \wedge D_\mu \phi, \quad (3.10)
\]

where \( \wedge \) is the wedge operation in \( \Lambda^*(\Delta^* \otimes E) \). One consequence of this definition is that \( \zeta \) is not necessary a parallel section of the spinor bundle but of \( \Delta \otimes E \).

The case that it is of most interest to us is that for which it is a line bundle. In this case, the conditions for \( d^2 = 0 \) can be expressed as conditions on the curvature \( R \) and \( F \) of the manifold and of the line bundle, respectively. The formulae are similar to those in (3.9) and (3.7).

The construction can be further generalized in the case for which there is no a spin structure but there is a \( Spin_c \) structure. In this case although the spin bundle \( \Delta \) is not well-defined \( \Delta \otimes E \) is and so is \( C(E) \).

Provided that the conditions for \( d^2 = 0 \) are met, we can define the twisted spin cohomology, \( H_d(C(E)) \), of the graded complex \((C(E), d)\). Similarly we can define the twisted spin cohomology, \( H_d(C(\pm(E))) \), of the graded complex \((C(\pm(E), d)\).

3.3 Algebraic operations

3.3.1 The algebraic operator \( D_{(p)} \)

There are several algebraic cohomology operations that can be defined on the twisted complexes \( \Lambda^* \otimes C, \Lambda^* \otimes C_\pm, Sym^* \otimes C \) and \( Sym^* \otimes C_\pm \), where \( Sym^* = \bigoplus_{p=0}^\infty Sym^p \) and \( Sym^p \) is the symmetrized product of \( p \) copies of \( \Lambda^1 \).

The maps \( CT_{(p)} : \Delta \otimes \Delta \to \Lambda^p \)

\[
CT_{(p)}(\eta, \theta) = \frac{1}{p!} \left( CT_{\mu_1 \ldots \mu_p}(\eta, \theta) e^{\mu_1} \wedge \ldots \wedge e^{\mu_p} \right) \quad (3.11)
\]

are skew-symmetric, i.e \( CT_{(p)}(\eta, \theta) = -CT_{(p)}(\theta, \eta) \), provided that \( \frac{1}{2} p(p-1) + (p+1) s_C + ps_F \in 2\mathbb{Z} + 1 \) as it can been seen from (2.11).

**Definition 3** The algebraic spin operator \( D_{(p)} : \Lambda^q(M) \otimes C^\ell(M) \to \Lambda^{q-p}(M) \otimes C^{\ell+2}(M) \) is

\[
D_{(p)} \phi = \left( -1 \right)^{\frac{p(p-1)+\ell}{2}} (CT_{\mu_1 \ldots \mu_p})_{A_1 A_2 \ldots} \phi_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_{q-p} \nu A_3 \ldots A_{\ell+2}} e^{\nu_1} \wedge \ldots \wedge e^{\nu_{q-p}} \otimes e^{A_1} \bar{\nu} \ldots \bar{\nu} e^{A_{\ell+2}}, \quad (3.12)
\]

if \( q \geq p \) and \( D_{(p)} = 0 \) for \( p > q \), where \( CT_{(p)} \) is skew-symmetric.
It is straightforward to show that

**Proposition 3** $D_{(p)}$ is nilpotent, $D_{(p)}^2 = 0$, provided that $p \in 2\mathbb{Z} + 1$.

**Proposition 4** The algebraic spin operator $D_{(p)}$ can be restricted on $\Lambda^* \otimes C_\pm$, iff $\dim M = 8k + 2, 8k + 6$.

**Proof:** It can be seen from the properties of spinor inner product $C$ summarized in section two that for $\dim M = 8k + 2, 8k + 6$, $C \Gamma^{(p)} : \Delta_\pm \otimes \Delta_\pm \to \Lambda^p, p \in 2\mathbb{Z} + 1$. q. e. d.

In what follows, when we refer to the algebraic spin operator on $\Lambda^* \otimes C_\pm$ complexes we shall assume the condition of the above proposition applies and $\dim M = 8k + 2, 8k + 6$.

The algebraic spin operator $D_{(p)}$ can be extended to twisted complexes $\Lambda^* \otimes C \otimes E$ and $\Lambda^* \otimes C_\pm \otimes E$ in a straightforward way. There is also an extension to the twisted complexes $\Lambda^* \otimes C(E)$ and $\Lambda^* \otimes C_\pm(E)$ provided that $E$ is equipped with and inner product $h$. In particular we define

$$D_{(p)} : \Lambda^q(M) \otimes C^\ell(E) \to \Lambda^{q-p}(M) \otimes C^{\ell+2}(E)$$

where

$$D_{(p)} \phi = \frac{(-1)^{\frac{1}{2} p(p-1) + \ell}}{2 (q-p)! p! \ell!} \left( C \Gamma^\mu \cdots \Gamma^\nu \otimes h \right)_{I_1 A_1, I_2 A_2} \phi_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_q} \cdots I_{p+2} A_{p+2}$$

$$e^{\nu_1} \wedge \cdots \wedge e^{\nu_{q-p}} \otimes \epsilon_{I_1} \Lambda_{A_1} \cdots \Lambda_{I_{q-p+2}} A_{q-p+2},$$

(3.13)

if $q \geq p$ and $D_{(p)} = 0$ for $p > q$. This operation is well defined provided that $C \Gamma^{(p)} \otimes h$ is skew-symmetric. This is the case when either $C \Gamma^{(p)}$ is symmetric and $h$ is skew-symmetric or $C \Gamma^{(p)}$ is skew-symmetric and $h$ is symmetric. In all the above cases $D_{(p)}^2 = 0$ provided $p \in 2\mathbb{Z} + 1$.

**Definition 4** The algebraic spin cohomology $H_{D_{(p)}}(\Lambda^* \otimes C)$ is defined as the cohomology of the double graded complex $(\Lambda^* \otimes C, D_{(p)})$. This definition can be extended to the rest of the twisted and untwisted spin complexes.

A particular case of this operation is for $p = 1$. In this case, we have $D = D_{(1)} : \Lambda^* \otimes C \to \Lambda^* \otimes C$, where

$$D \phi = \frac{(-1)^{\ell}}{2 (q-1)! \ell!} \left( C \Gamma^\mu \right)_{A_1 A_2} \phi_{\mu_1 \nu_1 \cdots \nu_q} A_3 \cdots A_{q+2} e^{\nu_1} \wedge \cdots \wedge e^{\nu_{q-1}} \otimes \epsilon_{A_1} \cdots \Lambda_{A_{q+2}},$$

(3.15)

if $q \geq 1$ and $D_{(p)} = 0$ for $q = 0$. In particular $D$ is defined on $\Lambda^* \otimes C$ and $\Lambda^* \otimes C \otimes E$ for $m = 4k, 4k + 2$ if $C = A$ and for $m = 4k + 3$ if $C = B$. It is also defined on $\Lambda^* \otimes C_\pm$ and $\Lambda^* \otimes C_\pm \otimes E$ for $m = 4k + 2$ if $C = A$ and for $m = 4k + 3$ if $C = B$, $\dim M = 2m$. In the twisted case $\Lambda^* \otimes C(E)$, the operator $D$ can be defined in all the cases for which $E$ admits a fibre inner product such that $C \Gamma^{(1)} \otimes h$ is skew-symmetric.
Proposition 5. $D_{(p)}$ anti-commutes with the differential operator $d$, ie

$$dD_{(p)} + D_{(p)}d = 0.$$  \hspace{1cm} (3.16)

Proof: We can show this after a direct computation using the property of the connection $\nabla^M$ of the manifold to be a spin connection induced from the tangent bundle. In the twisted case (3.14), this also the case provided that $\nabla^E h = 0$. q. e. d.

The differential and algebraic spin cohomology operators on the various untwisted and twisted complexes above can be combined into a new spin cohomology operator $d + D$. The new cohomology operator $d + D$ defines a new cohomology, $H_{d+D}$, which can be computed using spectral sequences. We shall describe such computation on Calabi-Yau manifolds of dimension six.

### 3.3.2 The algebraic operator $\hat{D}$

Apart from the $D_{(p)}$ algebraic operator, there is another algebraic operation $\hat{D}$.

**Definition 5** The algebraic operator $\hat{D}$ on the complex $\text{Sym}^* \otimes C$ is

$$\hat{D} : \text{Sym}^q(M) \otimes C^\ell \rightarrow \text{Sym}^{q-1}(M) \otimes C^{\ell+1},$$ \hspace{1cm} (3.17)

where

$$\hat{D}\phi = (-1)^{\ell} \frac{1}{(q-1)!\ell!} (C_\zeta)^{\mu}_{A_1 \ldots A_{q-1} \nu_1 \ldots \nu_{\ell+1}} \phi^{\nu_1 \ldots \nu_{q-1} A_2 \ldots A_\ell} e^{\nu_1} \wedge \ldots \wedge e^{\nu_{q-1}} \otimes e^{A_1} \bar{\wedge} \ldots \bar{\wedge} e^{A_{\ell+1}}$$ \hspace{1cm} (3.18)

if $q \geq 1$ and $\hat{D} = 0$ if $q = 0$.

It is straightforward to extend this definition to the other untwisted and twisted complexes. Moreover one can show that

**Proposition 6** $\hat{D}$ is nilpotent, $\hat{D}^2 = 0$.

As in the case of $D_{(p)}$ algebraic spin operator

**Proposition 7** $\hat{D}$ anti-commutes with the differential operators $d$, ie

$$d\hat{D} + \hat{D}d = 0.$$ \hspace{1cm} (3.19)

A consequence of this is that one can define a new cohomology operator $d + \hat{D}$ and an associated cohomology $H_{d+\hat{D}}$ which can be computed using a spectral sequence.

**Proposition 8** Let $\hat{D}$ be the spin algebraic operator on $\text{Sym}^* \otimes C$ defined as in (3.18). If $C_\zeta$ is an isomorphism, then $H^*_D = C$. 

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Proof: Since $C_\zeta$ is an isomorphism, then $\text{Sym}^* \otimes C = \text{Sym}^* \otimes \Lambda^*$. $\text{Sym}^p \otimes \Lambda^q$ can be decomposed under $GL(n, \mathbb{C})$ into two irreducible representations. These have dimensions

$$
\delta_1 = \frac{n(n+1)\ldots(n+p-1)(n-1)\ldots(n-q)}{(p+q)(p-1)!q!},
\delta_2 = \frac{n(n+1)\ldots(n+p)(n-1)\ldots(n-q+1)}{(p+q)p!(q-1)!}.
$$

(3.20)

Clearly $\ker \hat{D}|_{\text{Sym}^p \times \Lambda^q} = \Lambda^q$. In addition

$$
\ker \hat{D}|_{\text{Sym}^p \times \Lambda^q} = \hat{D}(\text{Sym}^{p+1} \times \Lambda^{q-1}), \quad p > 0,
$$

with $\dim \ker \hat{D}|_{\text{Sym}^p \times \Lambda^q} = \delta_2$ Therefore all cohomology $H^p,q = 0$ for $p > 0$ and $H^0,q = \Lambda^q = \mathbb{C}^q$. q. e. d.

This theorem can be thought as a consequence of the Spencer cohomology [3]. Clearly the above result can be generalized to $\text{Sym}^* \otimes C_{\pm}$ and twisted complexes.

4 Manifolds with connections of holonomy $SU(m)$ and spin cohomology

As we have mentioned on even-dimensional Riemannian manifolds, there are two complex spin representations $\Delta_{\pm}$. In addition there are real spin representations provided that $m = 4k, 4k + 2$. In what follows, we shall focus on the spin cohomology associated with the complex representations. The spin cohomology associated with real representations will be investigated later.

4.1 Complex spinor representations

Let $M$ be a Riemannian manifold equipped with a spin connection $\nabla$ with $\text{hol}(\nabla) \subseteq SU(m)$. We take that the metric on $M$ to be compatible with the parallel almost complex structure $J$. There are two distinct $\nabla$-parallel complex spinors. These are given by $\zeta_1 = 1, \zeta_2 = e_1 \wedge \ldots \wedge e_m$. These spinors are of different chirality if $m = 4k+1, 4k+3$ and of the same chirality if $m = 4k, 4k + 2$. Therefore there are two first order differential spin operators $d_1$ and $d_2$ associated with the spinors $\zeta_1$ and $\zeta_2$, respectively. If $m$ is odd, $d_1 : \mathcal{C}_+ \to \mathcal{C}_+$ and $d_2 : \mathcal{C}_- \to \mathcal{C}_-$ while if $m$ is even, $d_1, d_2 : \mathcal{C}_- \to \mathcal{C}_-$. We shall treat the two cases separately.

4.1.1 The $m = 4k + 1, 4k + 3$ case

Theorem 2 The operator $d_1 : \mathcal{C}_+ \to \mathcal{C}_+$ is nilpotent, $d_1^2 = 0$, provided that the $(2,0)$ part of the curvature $R$ of the connection $\nabla$ with respect to the almost complex structure $J$ vanishes.
Proof: For this we compute \( d_1^2 \) on \( C_+^\ell \) to find

\[
d_1^2 \phi = \frac{1}{2} C_{\mu}^\nu \bar{C}^\rho_\sigma R_{\mu\rho\nu\sigma} \phi ,
\]

where \( R_{\mu\nu} = [\nabla_{\mu}, \nabla_{\nu}] \) is the curvature of the connection \( \nabla \). In a spinor basis \( \{ \epsilon^a : a = 1, \ldots, \dim \Delta_+ \} \), the above expression can be written as

\[
d_1^2 \phi = \frac{1}{\ell!} (C_1^\mu a_1^\nu (C_1^\rho a_2^\tau R_{\mu\rho\tau\nu} \phi) a_3^\ldots a_{\ell+2} \epsilon^{a_1} \wedge \epsilon^{a_2} \wedge \epsilon^{a_3} \wedge \ldots \wedge \epsilon^{\ell+2} .
\]

Observe that the product representation \( \Delta_+ \otimes \Delta_+ \) can be decomposed as

\[
\Delta_+ \otimes \Delta_+ = \sum_{p=1}^{m-3} \Lambda^{2p+1}(V_C) \oplus \Lambda^{m+}(V_C) .
\]

In particular using (4.1), we have

\[
\chi \otimes \psi(\eta \otimes \theta) = \frac{1}{\dim C_+ \Delta_+} \sum_{p=1}^{m-3} \left( (-1)^{(2p+1)(s_1+s_C)} / (2p+1)! \right) (\Gamma_{\mu_1 \ldots \mu_{2p+1}} C^{-1}) (\chi, \psi) \Gamma_{\mu_1 \ldots \mu_{2p+1}}(\eta, \theta) .
\]

The only non-vanishing form associated with \( \zeta_1 \) spinor is the \( m \)-form given by

\[
\epsilon_1 = \frac{1}{m!} C(\zeta_1, \Gamma_{\rho_1 \ldots \rho_m} \zeta_1) \epsilon_{\rho_1} \wedge \ldots \wedge \epsilon_{\rho_m} .
\]

Moreover, from section two, we have that \( \Gamma_{\rho_1 \ldots \rho_m} \) is symmetric and \( \Gamma_{\rho_1 \ldots \rho_{m-2}} \) is skew-symmetric. Applying the formula (4.4) for \( \chi = C_1^\mu \) and \( \psi = C_1^\nu \), we find that the only non-vanishing term is

\[
d_1^2 \phi = \frac{1}{2} \frac{(-1)^{(m-2)(s_1+s_C)}}{(m-2)! \dim C_+ \Delta_+} (\Gamma_{\rho_1 \ldots \rho_{m-2}} C^{-1}) (C_1^\mu, C_1^\nu) \Gamma_{\rho_1 \ldots \rho_{m-2}} \bar{\Lambda} R_{\mu\rho\nu\sigma} \phi
\]

\[
= \frac{1}{2} \frac{(-1)^{(m-1)(s_1+s_C)}}{(m-2)! \dim C_+ \Delta_+} C(1, \Gamma_{\mu_1 \ldots \mu_{m-2}} \nu) \Gamma_{\rho_1 \ldots \rho_{m-2}} \bar{\Lambda} R_{\mu\rho\nu\sigma} \phi .
\]

Therefore \( d_1^2 = 0 \), if the \( (2,0) \) component of the curvature \( R \) vanishes since \( (\Gamma_j - i\Gamma_{j+m}) 1 = 0 \) and so \( \epsilon_1 \) is an \( (0,m) \) form. q. e. d.

**Theorem 3** The operator \( d_2 : \mathcal{C}_- \rightarrow \mathcal{C}_- \) is nilpotent, \( d_2^2 = 0 \), provided that the \( (0,2) \) part of the curvature \( R \) of the connection \( \nabla \) with respect to the almost complex structure \( J \) vanishes.

**Proof:** The proof of this is similar to the one presented above for the case of the \( \zeta_1 \) parallel spinor. One difference is the decomposition

\[
\Delta_- \otimes \Delta_- = \sum_{p=1}^{m-3} \Lambda^{2p+1}(V_C) \oplus \Lambda^{m+}(V_C) .
\]
A direct computation reveals that

\[
d^2 \phi = \frac{1}{2(m-2)\dim C_+} \Gamma^{\rho_1 \ldots \rho_{m-2}} C^{-1}(C_2^\mu, C_2^\nu) C \Gamma_{\rho_1 \ldots \rho_{m-2}} \bar{R}_{\mu \nu} \phi
\]

Using (4.8)

\[
d^2 = 0 \quad \text{if} \quad \Gamma_{1} + \Gamma_{j+m}(e_1 \wedge \ldots \wedge e_m, \Gamma^{\rho_1 \ldots \rho_{m-2}} e_1 \wedge \ldots \wedge e_m)\]

\[
C \text{ symmetric with respect to both inner products } J\text{ to the almost complex structure } A\text{.}
\]

**Corollary 1** The operator \( d = d_1 \oplus d_2 : C_+ \oplus C_- \rightarrow C_+ \oplus C_- \) is nilpotent provided that the curvature \( R \) of the connection \( \nabla \) is (1,1) with respect to the almost complex structure \( J \).

**Proof:** It follows immediately from the two theorems above. q. e. d.

We therefore conclude that there are three kinds of untwisted differential spin cohomology associated with a manifold that admits a connection with holonomy contained in \( SU(m) \), \( m = 4k + 1, 4k + 3 \). The complexes are \((C_+, d_1), (C_-, d_2)\) and \((C_+ \oplus C_-, d_1 \oplus d_2)\) and the associated spin cohomologies are denoted as \( H_{d_1}(C_+), H_{d_2}(C_-) \) and \( H_d(C_+ \oplus C_-) \), respectively.

**4.1.2 The \( m = 4k, 4k + 2 \) case**

In this case both parallel spinors \( \zeta_1, \zeta_2 \in \Delta^+ \). Therefore \( d_1, d_2 : C_- \rightarrow C_- \).

**Theorem 4** The operators \( d_1, d_2 : C_- \rightarrow C_- \) are nilpotent, \( d_1^2 = 0 \) and \( d_2^2 = 0 \), provided that the either (2,0) or the (0,2) part of the curvature \( R \) of the connection \( \nabla \) with respect to the almost complex structure \( J \) vanishes, respectively.

**Proof:** The proof of this statement is similar to that for the cases \( m = 4k + 1, 4k + 3 \) described in the previous section. In particular, we have

\[
\Delta_- \otimes \Delta_- = \sum_{p=0}^{m-2} \Lambda^{2p}(V_C) \oplus \Lambda^{m-2}(V_C). \quad (4.9)
\]

From the results of section two, the map \( C \Gamma^{\mu_1 \ldots \mu_m} \) is symmetric and \( C \Gamma^{\mu_1 \ldots \mu_{m-2}} \) is skew-symmetric with respect to both inner products \( C = A, B \). The expressions for \( d_1^2 \) and \( d_2^2 \) are given by [1,3] and [1,5], respectively. From these, it is straightforward to see that \( d_1^2 = 0 \) (\( d_2^2 = 0 \)) if the (0,2) ((2,0)) part of the curvature \( R \) of \( \nabla \) vanishes. q. e. d.

Since both operators \( d_1, d_2 \) act on the same complex, one can define the operator \( d = d_1 + d_2 : C_- \rightarrow C_- \). If \( d_1^2 = d_2^2 = 0, d^2 = d_1 d_2 + d_2 d_1 \). Therefore \( d \) is nilpotent iff the operators \( d_1, d_2 \) anti-commute.
**Theorem 5** The operator $d_1d_2 + d_2d_1 = 0$ iff the curvature of $\nabla$ vanishes $R = 0$.

**Proof:** Applying the definition of $d_1$ and $d_2$, one can find that

$$ (d_1d_2 + d_2d_1)\phi = C_1^{\mu} \tilde{\nabla} \tilde{\nabla}_\mu \nabla_\nu \phi - C_2^{\nu} \tilde{\nabla} \tilde{\nabla}_\nu \nabla_\mu \phi = C_2^{\mu} \tilde{\nabla} C_1^{\nu} \tilde{\nabla}_\nu R_{\mu\nu} \phi \ . \quad (4.10) $$

The Kähler form associated with the parallel spinors is

$$ \Omega = \frac{-i}{2C(\zeta_2, \zeta_1)} C(\zeta_2, \Gamma_{\mu\nu} \zeta_1)e^\mu \wedge e^\nu \ . \quad (4.11) $$

It can then be seen that

$$ C\Gamma_{(2p)}(\zeta_2, \zeta_1) = \frac{1}{(2p)!} C(\zeta_2, \Gamma_{\rho_1...\rho_2p} \zeta_1) e^{\rho_1} \wedge \ldots \wedge e^{\rho_{2p}} = \frac{(-i)^p C(\zeta_2, \zeta_1)}{p!} \wedge^p \Omega \ . \quad (4.12) $$

Applying (4.9), we find

$$ (d_1d_2 + d_2d_1)\phi = \frac{1}{\dim C(\Delta_+)} \sum_{p=0}^{m} \frac{(-1)^{s_c + s_\Gamma}}{(2p)!} C(\zeta_2, \Gamma^\mu \Gamma_{\rho_1...\rho_2p} \Gamma^\nu \zeta_1) C\Gamma_{\rho_1...\rho_2p} \tilde{\nabla} R_{\mu\nu} \phi $$

$$ \quad = \frac{1}{\dim C(\Delta_+)} \sum_{p=0}^{m} \frac{(-1)^{s_c + s_\Gamma}}{(2p)!} \left[ C(\zeta_2, \Gamma^\mu \Gamma_{\rho_1...\rho_2p^\nu} \zeta_1) \right. $$

$$ \quad \quad + \left. (2p)(2p-1)g^{\rho_1\rho_2} C(\zeta_2, \Gamma_{\rho_2...\rho_2p-1} \zeta_1) g^{\rho_2\rho_2p^\nu} \right] C\Gamma_{\rho_1...\rho_2p} \tilde{\nabla} R_{\mu\nu} \phi \ , \quad (4.13) $$

where $g$ is the metric on the manifold $M$. This can be rewritten as

$$ (d_1d_2 + d_2d_1)\phi = \frac{C(\zeta_2, \zeta_1)}{\dim C(\Delta_+)} \sum_{p=0}^{m} \frac{(-1)^{s_c + s_\Gamma}}{(2p)!} $$

$$ \left[ (-i)^{p+1}(2p+2)! \frac{\Omega^{\mu\nu} \Omega^{\rho_1\rho_2} \ldots \Omega^{\rho_{2p-2}\rho_{2p-1}} C\Gamma_{\rho_1...\rho_{2p}} \tilde{\nabla} R_{\mu\nu} \phi}{2p+1(2p+1)(p+1)!} \right. $$

$$ \quad \quad + \left. (-i)^{p-1}(2p)! \frac{\Omega^{\rho_1\rho_3} \ldots \Omega^{\rho_{2p-2}\rho_{2p-1}} \right] C\Gamma_{\rho_1...\rho_{2p}} \tilde{\nabla}(R_{\mu\nu} J_{\rho_1...\rho_2p}^\nu \phi + R_{\rho_1...\rho_{2p}}^\nu \phi) \right] \quad (4.14) $$

For $m = 4k$, $C\Gamma_{(2p)}$ is symmetric for $p = 2q$ while they are skew-symmetric for $p = 2q + 1$. Therefore in this case only the latter terms contribute in the sum. Similarly for $m = 4k + 2$, $C\Gamma_{(2p)}$ is skew-symmetric for $p = 2q$ while they are symmetric for $p = 2q + 1$. Therefore only the former terms contribute is the sum.

It is clear that if the $(1,1)$ part of the curvature vanishes, then the proposition is satisfied. However the $(2,0)$ and the $(0,2)$ parts of the curvature vanish as well. Thus $R = 0$. q.e.d.

We therefore conclude that there are three kinds of untwisted differential spin cohomology associated with a manifold that admits a connection with holonomy contained in $SU(m)$, $m = 4k, 4k + 2$. The complexes are $(C_-, d_1)$, $(C_-, d_2)$ and $(C_-, d = d_1 + d_2)$ and the associated spin cohomologies are denoted as $H_{d_1}(C_-)$, $H_{d_2}(C_-)$ and $H_d(C_-)$, respectively. Unlike the case case where $m = 4k + 1, 4k + 3$, all three cohomologies are cohomologies of the complex $C_-$.
4.2 Adjoint operators and Laplacians

As we have mentioned in section two, \( \Delta_{\pm} \) are equipped with a \( \text{Spin}(n) \)-invariant inner product. Because of this, one can find the adjoints of the spin cohomology operators \( d_1, d_2 \) and their associated Laplacians. As in the previous section, we shall distinguish between the \( m = 4k + 1, 4k + 3 \) and \( m = 4k, 4k + 2 \) cases. This is because of the properties of the inner product are different-see section two.

4.3 The \( m = 4k + 1, 4k + 3 \) case

We extend the inner product \( C^{-1} \) from \( \Delta_{+} \oplus \Delta_{-} \) to the space of sections of \( C_{+} \oplus C_{-} \) and denote it with the same symbol. The inner product \( C^{-1} \) vanishes if it is restricted on either \( C_{+} \) or \( C_{-} \).

**Definition 6** The adjoint operator \( \delta_1 : C_- \to C_- \) of \( d_1 : C_+ \to C_+ \) is

\[
C^{-1}(\phi, d_1 \psi) = C^{-1}(\delta_1 \phi, \psi) .
\]

Similarly, the adjoint operator \( \delta_2 : C_+ \to C_+ \) of \( d_2 : C_- \to C_- \) is

\[
C^{-1}(\psi, d_2 \phi) = C^{-1}(\delta_2 \phi, \phi) .
\]

Using these adjoints, one can define two Laplace operators \( \Delta_1 = \delta_2 d_1 + d_1 \delta_2 \) and \( \Delta_2 = \delta_1 d_2 + d_2 \delta_1 \). The Laplace operator of \( d : C_+ \oplus C_- \to C_+ \oplus C_- \) is \( \Delta = \Delta_1 \oplus \Delta_2 \).

To compute the Laplace operator \( \Delta_1 \), we use the above definitions to find

\[
\Delta_1 \phi = -(1)^{s_c + s_f} \left( C(\zeta_1, \Gamma^\mu \zeta_2) \nabla_\mu \nabla_\nu \phi + (1)^{s_c + s_f} C_1^{\nu} \overline{\Lambda}(C_2^{\mu} R_{\nu \mu} \phi) \right) , \tag{4.17}
\]

where \( \eta \overline{\phi} \) denotes inner derivation with respect to spinor \( \eta \), ie

\[
\eta \overline{\phi} = \frac{(1)^{s_c}}{\ell!} \eta^B \phi_{BA_1 ... A_\ell} \epsilon^{A_1} \overline{\Lambda} \ldots \overline{\Lambda} \epsilon^{A_\ell} = \frac{(1)^{s_c}}{\ell!} (C^{-1})^{BE} \eta_B \phi_{EA_1 ... A_\ell} \epsilon^{A_1} \overline{\Lambda} \ldots \overline{\Lambda} \epsilon^{A_\ell} ,
\]

which is equivalent to

\[
(\eta \overline{\phi})_{A_1 ... A_\ell} = (\ell + 1)(1)^{s_c} \eta^B \phi_{BA_1 ... A_\ell} . \tag{4.19}
\]

More generally, we have

\[
(\eta \overline{\phi})_{E_1 ... E_q A_1 ... A_\ell} = (\ell + 1)(1)^{s_c} \eta_{E_1 ... E_q} \eta^B \phi_{BA_1 ... A_\ell} \tag{4.20}
\]

The product of the co-spinor representations \( \Delta_{\pm} \) can be decomposed as

\[
\Delta_+ \otimes \Delta_- = \sum_{p=0}^{[\frac{n}{2}]} \Lambda^{2p} .
\]  

The formula that relates the product of two co-spinors to forms is given by \( 2.12 \) after the appropriate restrictions. Applying this to the second term in the Laplace operator, we find

\[
\Delta_1 \phi = -(1)^{s_c + s_f} \left( C(\zeta_1, \Gamma^\mu \zeta_2) \nabla_\mu \nabla_\nu \phi
\]

\[= -(1)^{s_c + s_f} \left( C(\zeta_1, \Gamma^\mu \zeta_2) \nabla_\mu \nabla_\nu \phi
\]

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In this case we have that the expressions of these operators are the same as those in the previous section. In particular we find that the Laplace operators $\Delta_1$ and $\Delta_2$ of the even and odd cases, respectively, are related by

$$\Delta_2 = -\frac{1}{2} \left[ \sum_{p=0}^{\frac{d}{2}} \frac{1}{(2p)!} \left( \sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \sum_{\rho_1 \cdots \rho_{2p}} \sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \right) \right] \Delta_1$$


In turn this can be written as

$$\Delta_1 \phi = -(-1)^{s_C + s_f} C(\zeta_1, \zeta_2) \left((g^\mu{}_{\nu} + i\Omega^{\mu\nu})\nabla_\mu \nabla_\nu \phi \right)$$

$$+ \frac{1}{\dim \mathbb{C}(\nabla^+)} \sum_{p=0}^{\frac{d}{2}} \frac{1}{(2p)!} \left[ \sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \right] \right] \Delta_1 \phi.$$

\[ (4.22) \]

is the $(1,1)$ part of the curvature with respect to the almost complex structure $J$.

The Laplace operators $\Delta_2$ is given as in (4.17) but with the parallel spinors $\zeta_1$ and $\zeta_2$ interchanged. The effect that this can be easily computed from (4.17) using the symmetry properties of $CT_{\rho_1 \cdots \rho_q}$. In particular we find that

**Corollary 2** The $\Delta_2$ Laplace operator is

$$\Delta_2 \phi = -(-1)^{s_C + s_f} C(\zeta_1, \zeta_2) \left((g^\mu{}_{\nu} + i\Omega^{\mu\nu})\nabla_\mu \nabla_\nu \phi \right)$$

$$+ \frac{1}{\dim \mathbb{C}(\nabla^+)} \sum_{p=0}^{\frac{d}{2}} \frac{1}{(2p)!} \left[ (\sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \right] \right] \Delta_2 \phi.$$

\[ (4.23) \]

\[ (4.24) \]

**4.4 The $m = 4k, 4k + 2$ case**

In this case $d_1, d_2 : \mathcal{C}_- \to \mathcal{C}_-$ and the inner product $C^{-1}$ when restricted on $\mathcal{C}_-$ is non-degenerate. We define the adjoints $\delta_1, \delta_2 : \mathcal{C}_- \to \mathcal{C}_-$ of the $d_1, d_2$ operators, respectively. Again there are two Laplace operators $\Delta_1 = \delta_2 d_1 + d_1 \delta_2$ and $\Delta_2 = \delta_1 d_2 + d_2 \delta_1$. The expressions of these operators are the same as those in the previous section. In particular we have that

**Corollary 3** The $\Delta_1$ Laplace operator is

$$\Delta_1 \phi = -(-1)^{s_C + s_f} C(\zeta_1, \zeta_2) \left((g^\mu{}_{\nu} + i\Omega^{\mu\nu})\nabla_\mu \nabla_\nu \phi \right)$$

$$+ \frac{1}{\dim \mathbb{C}(\nabla^+)} \sum_{p=0}^{\frac{d}{2}} \frac{1}{(2p)!} \left[ (\sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \sum_{\rho_1 \cdots \rho_{2p}} \frac{(2p)!}{(2p+1)(2p+1)(p+1)!} \right] \right] \Delta_1 \phi.$$

\[ (4.25) \]
more, one can use the connection \( \nabla \) bundle \( m \) part of the curvature, \( R \) the curvature, \( d \) and associated curvature \( F \).

Let a Riemannian manifold equipped with a connection \( \nabla^M \) such that \( \text{hol}(\nabla^M) \subseteq SU(m) \). In addition let \( E \) be a vector bundle over \( M \) equipped with a connection \( \nabla^E \) and associated curvature \( F \).

As in the previous section one can construct two first order differential spin operators \( d_1 \) and \( d_2 \) associated with the two parallel spinors of \( \nabla^M \). Then one can use the vector bundle \( E \) to twist the complexes \( C_+ \) and \( C_- \) as \( C_+ \otimes E \) and \( C_- \otimes E \), respectively. Furthermore, one can use the connection \( \nabla^E \) to extend \( d_1 \) and \( d_2 \) to the twisted complexes as it has been described in (3.9). We denote the extended operators with the same symbols. In particular we have

**Corollary 4** The operators \( d_1 : C_+ \otimes E \to C_+ \otimes E \) for \( m = 4k + 1, 4k + 3 \) and \( d_1 : C_- \otimes E \to C_- \otimes E \) for \( m = 4k, 4k + 2 \) are nilpotent, \( d_1^2 = 0 \), if the (2,0) part of the curvature, \( R \) and \( F \), of both the connections \( \nabla \) and \( \nabla^E \) vanishes. Similarly, The operator \( d_2 : C_- \otimes E \to C_- \otimes E \), \( m = 4k, 4k + 1, 4k + 2, 4k + 3 \), is nilpotent, if the (0,2) part of the curvature, \( R \) and \( F \), of both the connections \( \nabla \) and \( \nabla^E \) vanishes. For \( m = 4k, 4k + 2 \), \( d_1 d_2 + d_2 d_1 = 0 \), if \( F = R = 0 \).
The Laplace operators can be easily computed. In particular we find that

\[ \Delta_1 \phi = -(-1)^{s_c+s_f} C(\zeta_1, \zeta_2) \left( (g^{\mu\nu} + i\Omega^{\mu\nu}) \nabla_\mu \nabla_\nu \phi \right) \]

\[ + \frac{1}{\dim_{\mathbb{C}}(\Delta_+)} \sum_{p=0}^{\dim_{\mathbb{C}}(\Delta_+)} \frac{1}{(2p)!} \left[ \frac{-i^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!} \Omega^{\rho_1\rho_2} \cdots \Omega^{\rho_{2p-1}\rho_{2p}} \Gamma_{\rho_1 \cdots \rho_{2p}} \bar{\gamma}(R_{\mu\nu} + F_{\mu\nu}) \phi \right] + \frac{-i^{p-1}(2p)!}{2^{p-2}(p-1)!} \Omega^{\rho_{2p-3} \cdots \rho_{2p-2}\rho_{2p-1}} g^{\rho_1\mu} g^{\rho_2\nu} \Gamma_{\rho_1 \cdots \rho_{2p-2}} \bar{\gamma}(R_{\mu\nu}^{1,1} + F_{\mu\nu}^{1,1}) \phi \right] . \] (5.1)

Similarly, the \( \Delta_2 \) Laplace operator is

\[ \Delta_2 \phi = -(-1)^{s_c+s_f} C(\zeta_1, \zeta_2) \left( (1) \left( g^{\mu\nu} - i\Omega^{\mu\nu} \right) \nabla_\mu \nabla_\nu \phi \right) \]

\[ + \frac{(-1)^{s_c}}{\dim_{\mathbb{C}}(\Delta_+)} \sum_{p=0}^{\dim_{\mathbb{C}}(\Delta_+)} \frac{(-1)^{p+1}(2p+2)!}{2^{p+1}(2p+1)(p+1)!} \Omega^{\rho_1\rho_2} \cdots \Omega^{\rho_{2p-1}\rho_{2p}} \Gamma_{\rho_1 \cdots \rho_{2p}} \bar{\gamma}(R_{\mu\nu} + F_{\mu\nu}) \phi \]

\[ + \frac{(-1)^{p-1}(2p)!}{2^{p-2}(p-1)!} \Omega^{\rho_{2p-3} \cdots \rho_{2p-2}\rho_{2p-1}} g^{\rho_1\mu} g^{\rho_2\nu} \Gamma_{\rho_1 \cdots \rho_{2p-2}} \bar{\gamma}(R_{\mu\nu}^{1,1} + F_{\mu\nu}^{1,1}) \phi \right] . \] (5.2)

### 5.2 Twisted \( C_{\pm}^1(E) \) complexes

Suppose that the parallel spinors with respect to the \( \mathcal{D} \) connection on \( \Delta_\pm \otimes E \) are in the direction of either \( 1 \otimes 1 \) or \( e^1 \wedge \ldots \wedge e^m \otimes 1 \) and \( E \) is a vector bundle with a fibre inner product \( h, \nabla^E h = 0 \). One can construct an invariant inner product on \( \Delta_\pm \otimes E \) as \( C^{-1} \otimes h \) and extended to the twisted complexes \( C_{\pm}(E) \). We can again define operators \( d_1 \) and \( d_2 \). In particular we have

**Proposition 9** The operators \( d_1 : C_{\pm}(E) \to C_{\pm}(E) \) for \( m = 4k + 1, 4k + 3 \) and \( d_1 : C_{\pm}(E) \to C_{\pm}(E) \) for \( m = 4k, 4k+2 \) are nilpotent, \( d_1^2 = 0 \), if the \((2,0)\) part of the curvature of \( \mathcal{D} \) vanishes. Similarly, The operator \( d_2 : C_{\pm}(E) \to C_{\pm}(E) \), \( m = 4k, 4k+1, 4k+2, 4k+3 \), is nilpotent, \( d_2^2 = 0 \), if the \((0,2)\) part of the curvature of \( \mathcal{D} \) vanishes. For \( m = 4k, 4k+2 \), \( d_1 d_2 + d_2 d_1 = 0 \), if the curvature of \( \mathcal{D} \) vanishes.

**Proof:** The proof is similar to that we have already investigated in the previous sections for the untwisted \( d_1 \) and \( d_2 \) operators. However, there is one difference. If \( E \) is not a line bundle, then in the expression for \( d_1^2 \) and \( d_2^2 \) both symmetric and skew-symmetric \( CT(p) \) contribute. This is unlike the untwisted case where only the skew-symmetric \( CT(p) \) contribute. However, there is no additional restriction on the curvature of \( \mathcal{D} \). q. e. d.

The Laplace operators \( \Delta_1 \) and \( \Delta_2 \) can be easily computed in this case. The expressions are as in (5.1) and (5.2) with the curvature \( R \) and \( F \) replaced by the curvature of \( \mathcal{D} \).
5.3 Algebraic Cohomologies

We have seen that the operator $D$ on the complex $\Lambda^* \otimes C$ is defined provided that $C \Gamma$ is skew-symmetric which is the case for $C = A$ if $m = 4k, 4k + 3$ and for $C = B$ if $m = 4k + 2$. Moreover $D$ restricts on $C_\pm$, $D : C_\pm \to C_\pm$ if $m = 4k + 3$. Therefore we conclude that the $D$ operator can be defined on the complexes $\Lambda^* \otimes C_\pm$ and $\Lambda^* \otimes C_\pm \otimes E$ only for $m = 4k + 3$. The operator $D$ can also defined for twisted complexes $\Lambda^* \otimes C_\pm(E)$ but we shall not investigate this further here.

**Corollary 5** The algebraic operator $D$ anticommutes with both $d_1$ and $d_2$ differential operators

$$d_1 D + D d_1 = d_2 D + D d_2 = 0 .$$

(5.3)

Therefore one can define the operators $d_1 + D$ and $d_2 + D$ which are nilpotent provided $d_1^2 = d_2^2 = 0$. The cohomology of $d_1 + D$ and $d_2 + D$ can be computed using spectral sequences. We shall not do a general computation. Instead, we shall give the cohomology of the operator $d_2 + D$ in the special case where $M$ is a six-dimensional Calabi-Yau manifold.

6 Complex manifolds with holonomy $SU(m)$ and spin cohomology

It is clear from the results of the previous section that complex spin cohomology is related to the Dolbeault cohomology. Here we shall establish the precise relation and we shall give the classes of the spin cohomology in terms of those of the Dolbeault cohomology, see e.g. [10, 11].

6.1 Spin and Dolbeault cohomologies

Let $M$ be a complex manifold equipped with a connection $\nabla$, $\text{hol}(\nabla) \subseteq SU(m)$. On $M$, it is known that

$$\Delta = \oplus_q \Lambda^{0,q} = \Lambda^{0,*} .$$

(6.1)

This can be easily seen from $\Delta = \Lambda^{0,q}(1)$, where $\Lambda^{0,q}$ acts on 1 with Clifford multiplication. In particular, we have $\Delta_+ = \Lambda^{0,\text{even}}$ and $\Delta_- = \Lambda^{0,\text{odd}}$. Thus

$$C_+ = \Lambda^*(\Lambda^{0,\text{even}})$$

$$C_- = \Lambda^*(\Lambda^{0,\text{odd}})$$

(6.2)

Write $\Delta_- = \Lambda^{0,1} \oplus Z$, where $Z = \oplus_{p \geq 1} \Lambda^{0,2p+1}$. The complex $C_-$ can now be decomposed as

$$C'_- = \oplus_{p+q = \ell} \Lambda^{0,p} \otimes \Lambda^q(Z) .$$

(6.3)

**Proposition 10**

$$d_2 = \bar{\partial} : \Lambda^{0,p} \otimes \Lambda^q(Z) \to \Lambda^{0,p+1} \otimes \Lambda^q(Z) .$$

(6.4)
Proof: To show this, we first evaluate the action of $d_2$ on $\Lambda^{0,1} \subset C_-$. Indeed let $\eta_i e^i \in \Lambda^{0,1}$, then we have
\[
d_2(\eta_i e^i) = \lambda(-1)^{\frac{1}{2}m(m-1)}(\nabla_i + i\nabla_{i+m})\eta_j e^i \wedge e^j = \lambda(-1)^{\frac{1}{2}m(m-1)}\bar{\partial}\eta_j e^i \wedge e^j, \tag{6.5}
\]
where $\lambda = 1$ for the $A$ inner product and $\lambda = i^m$ for $B$ inner product. After suppressing the numerical coefficient which is inconsequential for the computation of cohomology, we have $d_2 = \bar{\partial}$ on $\Lambda^{0,1}$. Using the definition of the $\Lambda$ product, it is straightforward to extend the proof to the rest of the complex $C_-$. q. e. d.

Corollary 6 Let $M$ be a complex manifold as described in the beginning of the section. Then the spin cohomology
\[
H_{d_2}^0(C_-) = \oplus_{p+q=\ell} H^{0,p}_\partial(\Lambda^q(Z)) \tag{6.6}
\]
Therefore the spin cohomology of the $d_2$ operator can be computed in terms of Dolbeault cohomology of a twisted complex by the bundle $\Lambda^*(Z)$, where $Z = \oplus_{p \geq 1} \Lambda^{0,2p+1}$.

A direct consequence of this is that
\[
H_{d_2}^0(C_- \otimes E) = \oplus_{p+q=\ell} H^{0,p}_\partial(\Lambda^q(Z) \otimes E) \tag{6.7}
\]
Using the corollary, we can also compute the index of the spin complex $(C_-, d_2)$ in terms of the index of the twisted $\bar{\partial}$ complex. In particular, we have
\[
\text{Index}_{d_2}(C_-) = \sum_{q \geq 0} (-1)^q \text{Index}_{\bar{\partial}}(\Lambda^q(Z)) \tag{6.8}
\]
or more generally
\[
\text{Index}_{d_2}(C_- \otimes E) = \sum_{q \geq 0} (-1)^q \text{Index}_{\bar{\partial}}(\Lambda^q(Z) \otimes E) \tag{6.9}
\]

It remains to investigate the cohomology of $d_1$. We shall consider the cases $m = 4k + 1, 4k + 3$ and $m = 4k, 4k + 2$ separately. In the former case $d_1 : C_+ \to C_+$. Writing $\Delta_+ = \Lambda^{0,m-1} \oplus W$, where $W = \oplus_{p < \frac{m-1}{2}} \Lambda^{0,2p}$, we have
\[
C_\ell^+ = \oplus_{p+q=\ell} \Lambda^p(\Lambda^{0,m-1}) \otimes \Lambda^q(W), \tag{6.10}
\]
and
\[
d_1 : \Lambda^p(\Lambda^{0,m-1}) \otimes \Lambda^q(W) \to \Lambda^{p+1}(\Lambda^{0,m-1}) \otimes \Lambda^q(W). \tag{6.11}
\]
Since there is a $SU(m)$ structure, we can identify $\Lambda^{0,m-1} = \Lambda^{1,0}$, $\Lambda^p(\Lambda^{0,m-1}) = \Lambda^{p,0}$ and
\[
d_1 = \partial : \Lambda^{p,0} \otimes \Lambda^q(W) \to \Lambda^{p+1,0} \otimes \Lambda^q(W). \tag{6.12}
\]
Therefore, we conclude that
\[
H_{d_1}^\ell(C_+) = \oplus_{p+q=\ell} H^{p,0}_\partial(\Lambda^q(W)) \tag{6.13}
\]
Thus $H_{d_1}^\ell(C_+) = H_{d_2}^\ell(C_-)$. The same applies for $m = 4k, 4k + 2$.  

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6.2 Algebraic operations on twisted complexes

On complex manifolds with $\text{hol}(\nabla) \subseteq SU(m)$ apart from the differential spin operators $d_1, d_2$, there is also an algebraic spin operator $D$ provided $m = 4k + 3$. We shall focus on the twisted spin cohomology associated with the operators $d_2$ and $D$. In the context of complex geometry, there are several versions that one can consider. In particular one can defined a twisted spin cohomology on $\Lambda^* \otimes \mathcal{C}_-$ as we have already mentioned in section 5.3. However it is also possible to twist $\mathcal{C}_-$ with either $\Lambda^* \otimes \mathcal{C}_0$ or $\Lambda^0 \otimes \mathcal{C}_\ast$. In each of these cases the twisted spin cohomology of the operator $d_2 + D$, or equivalently $\bar{\partial} + D$, can be computed using a spectral sequence.

The twisted complex $C = \Lambda^* \otimes \mathcal{C}_-$ is a double complex, $C^{p,\ell}$, with grading induced from the space of forms $\Lambda^*$ and that of $\mathcal{C}_-$. However in this grading, $d_2 = \bar{\partial}$ and $D$ do not act with horizontal and vertical operations. In particular, $d_2 : C^{p,\ell} \rightarrow C^{p,\ell+1}$ and $D : C^{p,\ell} \rightarrow C^{q-1,\ell+2}$. It is therefore convenient to introduce a new grading as

$$ C^{[-p,\ell+2]} = C^{p,\ell} = \Lambda^p \otimes \mathcal{C}_\ell. \quad (6.14) $$

Note know that $d_1 : C^{[-p,\ell+2]} \rightarrow C^{[-p,\ell+2p+1]}$ and $D : C^{[-p,\ell+2]} \rightarrow C^{[-p+1,\ell+2p]}$ as expected. The twisted complexes $\Lambda^* \otimes \mathcal{C}_-$ and $\Lambda^0 \otimes \mathcal{C}_-$ can be treated in a similar way. The machinery of spectral sequences can now be used to do the computation, see e.g [9] and references within. Instead of developing the general theory of computing the cohomology of $\bar{\partial} + D$ for the various complexes above, we shall give the cohomology of $(\Lambda^* \otimes \mathcal{C}_-, \bar{\partial} + D)$ for six-dimensional Calabi-Yau manifolds in an example below.

6.3 $\text{Spin}_c$ structures and spin cohomology

Let $M$ be a complex manifold equipped with a $\text{Spin}_c$ structure and compatible connection $\nabla$, $\text{hol}(\nabla) \subseteq SU(m)$. Suppose that $L$ is a (locally defined) complex line bundle associated with the $\text{Spin}_c$ structure. On $M$, it is known that

$$ \Delta^* \otimes L = \oplus_q \Lambda^{0,q} = \Lambda^0. \quad (6.15) $$

This is similar to the standard complex case, we have investigated. In particular, we have $\Delta_+ \otimes L = \Lambda^{0,\text{even}}$ and $\Delta_- = \Lambda^{0,\text{odd}}$. Thus

$$ \mathcal{C}_+ (L) = \Lambda^*(\Lambda^{0,\text{even}}) \quad \mathcal{C}_- (L) = \Lambda^*(\Lambda^{0,\text{odd}}) \quad (6.16) $$

Write $\Delta_- \otimes L = \Lambda^{0,1} \otimes L \oplus Z \otimes L$, where $Z = \oplus_{p \geq 1} \Lambda^{0,2p+1}$. The complex $\mathcal{C}_- (L)$ can now be decomposed as

$$ \mathcal{C}_-^\ell (L) = \oplus_{p+q=\ell} \Lambda^p(\Lambda^{0,1}) \otimes \Lambda^q(Z). \quad (6.17) $$

Proposition 11

$$ d_2 = \bar{\partial} : \Lambda^p(\Lambda^{0,1}) \otimes \Lambda^q(Z \otimes L) \rightarrow \Lambda^{p+1}(\Lambda^{0,1}) \otimes \Lambda^q(Z). \quad (6.18) $$
Proof: To show this, we first evaluate the action of \( d_2 \) on \( \Lambda^{0,1} \subset C_-(L) \). Indeed let \( \eta \in \Lambda^{0,1} \), then we have
\[
d_2(\eta e^i) = \lambda(-1)^{\frac{1}{2}m(m-1)}(\nabla_i + i\nabla_{i+m})\eta_j e^i \Lambda e^j = \lambda(-1)^{\frac{1}{2}m(m-1)}\partial_j \eta_j e^i \wedge e^j , \tag{6.19}
\]
where \( \lambda = 1 \) for the \( A \) inner product and \( \lambda = i^m \) for \( B \) inner product. After suppressing the numerical coefficient which is inconsequential for the computation of cohomology, we have \( d_2 = \bar{\partial} \) on \( \Lambda^{0,1} \). Using the definition of the \( \bar{\Lambda} \) product, it is straightforward to extend the proof to the rest of the complex \( C_-(L) \). q. e. d.

**Corollary 7** Let \( M \) be a complex \( \text{Spin}_c \) manifold as described in the beginning of the section. Then the spin cohomology
\[
H^\ell_{d_2}(C_-(L)) = \oplus_{p+q=\ell}H^0_\bar{\partial}(\Lambda^q(Z)) . \tag{6.20}
\]
Therefore the spin cohomology of the \( d_2 \) operator can be computed in terms of Dolbeault cohomology of a twisted complex by the bundle \( L^p \otimes \Lambda^q(Z) \), where \( Z = \oplus_{p \geq 1} \Lambda^{0,2p+1} \).

### 7 Spin cohomology and six-dimensional Calabi-Yau manifolds

#### 7.1 Differential spin cohomology

Applying the general theory of the previous section to this case, we have \( \Delta_- = \Lambda^{0,\text{odd}} = \Lambda^{0,1} \oplus \Lambda^{0,3} \). In addition for six-dimensional Calabi-Yau manifolds \( \Lambda^{0,3} \) is trivial line bundle. Using these, we find that
\[
C_\ell_\text{m} = \Lambda^{0,\ell} \oplus \Lambda^{0,\ell-1} \tag{7.1}
\]
and
\[
d_2 = \bar{\partial} : \Lambda^{0,\ell} \oplus \Lambda^{0,\ell-1} \to \Lambda^{0,\ell+1} \oplus \Lambda^{0,\ell} . \tag{7.2}
\]
Therefore
\[
H^\ell_{d_2}(C_-) = H^0_\bar{\partial} \oplus H^0_{\bar{\partial}} . \tag{7.3}
\]
In particular for an irreducible six-dimensional Calabi-Yau manifold, we have that
\[
H^0_{d_2} = \mathbb{C} , \quad H^1_{d_2} = \mathbb{C} , \quad H^2_{d_2} = 0 , \quad H^3_{d_2} = \mathbb{C} , \quad H^4_{d_2} = \mathbb{C} . \tag{7.4}
\]
It is also straightforward to compute the cohomology of the twisted complex \( (\Lambda^{*,0} \otimes C_-, d_2) \). In particular, we find that
\[
H^{p,\ell}_{d_2}(\Lambda^{*,0} \otimes C_-) = H^p_\bar{\partial} \oplus H^p_{\bar{\partial}} . \tag{7.5}
\]
7.2 Twisted complexes and algebraic spin cohomology

To compute the cohomology of the complex \((\Lambda^*, C_-, d_2 + D)\), we first investigate the complex \((\Lambda^{*,0} \otimes C_-, D)\). Using (7.1), we find that the operator \(D : \Lambda^{p,0} \otimes C^\ell_- \to \Lambda^{p-1,0} \otimes C_{\ell+2}^\ell\) acts as

\[
D : \Lambda^{p,0} \otimes [\Lambda^{0,\ell} \oplus \Lambda^{0,\ell-1}] \to \Lambda^{p-1,0} \otimes [\Lambda^{0,\ell+2} \oplus \Lambda^{0,\ell+1}]
\]

or equivalently \(D : \Lambda^{p,\ell} \oplus \Lambda^{p,\ell-1} \to \Lambda^{p-1,\ell+2} \oplus \Lambda^{p-1,\ell+1}\). Since it acts on the two parts in the sum separately, it is enough to consider only its action in the first part. After some computation, one finds that

\[
D\psi = -(-1)^q \frac{1}{(p-1)!q!2} \psi_{\gamma\alpha_1...\alpha_{p-1},\beta_3...\beta_{q+2}} e^{\gamma} \beta_1 \beta_2 e^{\alpha_1} \Lambda ... e^{\alpha_{p-1}} \Lambda e^{\beta_1} \Lambda e^{\beta_2} \Lambda ... e^{\beta_{q+2}},
\]

where \(\psi \in \Lambda^{p,q}\).

Next we shall compute the cohomology of the double complex \((\Lambda^{*,*}, \bar{\partial} + D)\) using a spectral sequence, see e.g. [9] and references within. The most convenient filtration is that for which \(E_1 = H^{p,q}_{\bar{\partial}}\). Then from the general theory of spectral sequences for double complexes \(E_2 = H_2 H_{\bar{\partial}}\) and \(E_2\) is graded as the double complex. It is known that for six-dimensional irreducible Calabi-Yau manifolds the non-vanishing Dolbeault groups are \(H^{0,0}_{\bar{\partial}}, H^{1,1}_{\bar{\partial}}, H^{2,1}_{\bar{\partial}}, H^{1,2}_{\bar{\partial}}, H^{2,2}_{\bar{\partial}}\) and \(H^{3,3}_{\bar{\partial}}\). Moreover \(H^{3,0}_{\bar{\partial}} = H^{0,3}_{\bar{\partial}} = \mathbb{C}\) which are generated by the parallel \((3,0)\)- and \((0,3)\)-forms, respectively. To compute \(E_2\) observe that

\[
0 \to H^{3,0}_{\bar{\partial}} \to H^{2,2}_{\bar{\partial}} \to 0 \quad (7.8)
\]

and

\[
0 \to H^{1,1}_{\bar{\partial}} \to H^{0,3}_{\bar{\partial}} \to 0 \quad (7.9)
\]

and the rest of the cohomology groups of \(E_1 = H^{p,q}_{\bar{\partial}}\) live in \(E_2\). It is easy to see that \(\ker D|_{H^{3,0}_{\bar{\partial}}} = \{0\}\). Therefore \(E_2^{3,0}\) vanishes. In addition \(D(H^{3,0}_{\bar{\partial}}) = \mathbb{C} < \Omega \wedge \Omega\), where \(\Omega\) is the Kähler form. Since \(DH^{2,2}_{\bar{\partial}} = 0\), we conclude that \(E_2^{2,2} = H^{2,2}_{\bar{\partial}} / \mathbb{C} < \Omega \wedge \Omega\), i.e. \(E_2^{2,2} = PH^{2,2}_{\bar{\partial}}\) is generated by the primitive \((2,2)\) harmonic forms.

Next observe that

\[
\ker D|_{H^{1,1}_{\bar{\partial}}} = \{\alpha \in H^{1,1}_{\bar{\partial}} \text{ such that } \Omega \cdot \alpha = 0\}.
\]

Therefore \(E_2^{1,1} = PH^{1,1}_{\bar{\partial}}\) is generated by the primitive \((1,1)\) harmonic forms. In addition we have that \(DH^{1,1}_{\bar{\partial}} = H^{0,3}_{\bar{\partial}}\), therefore \(E_2^{0,3} = 0\). The only non-vanishing groups are \(E_2^{0,0} = \mathbb{C}, E_2^{2,1} = H^{2,1}_{\bar{\partial}}, E_2^{1,2} = H^{1,2}_{\bar{\partial}}, E_2^{1,1} = PH^{1,1}_{\bar{\partial}}, E_2^{2,2} = PH^{2,2}_{\bar{\partial}}\) and \(E_2^{3,3} = \mathbb{C}\).

It remains to show that \(E_2 = E_\infty\). This is easily verified by computing the action of the differential \(d_2\) of \(E_2\). For this, we need to convert to the grading of the double complex \(\Lambda^{*,*}\) for which \(\bar{\partial} : \Lambda^{[m,n]} \to \Lambda^{[m,n+1]}\) acts vertically and \(D : \Lambda^{[m,n]} \to \Lambda^{[m+1,n]}\) acts horizontally. As we have explained \((m,n) = (-p,2p+q)\), i.e. \(E_2^{[-p,2p+q]} = H^p_{\bar{\partial}} H^{q}_{\bar{\partial}}\).

The differential \(d_2 : E_2^{[m,n]} \to E_2^{[m-2,n-1]}\). It is easy then to see that the \(d_2\) differential is the zero map and \(E_2 = E_\infty\). Therefore the cohomology of the operator \(d + D\) is given by \(E_2\). Thus we have shown the proposition.
Proposition 12 Let $M$ be an irreducible six-dimensional Calabi-Yau manifold. The non-vanishing cohomology groups of the complex $(\Lambda^p, \bar{\partial} + D)$ are $H^0_{\bar{\partial}+D} = \mathbb{C}, H^2_{\bar{\partial}+D} = PH^1_{\bar{\partial}}$, $H^3_{\bar{\partial}+D} = H^1_{\bar{\partial}} \oplus H^2_{\bar{\partial}}$, $H^4_{\bar{\partial}+D} = PH^2_{\bar{\partial}}$ and $H^6_{\bar{\partial}+D} = H^3_{\bar{\partial}} = \mathbb{C}$. Finally, we have $H^\ell_{\bar{\partial}+D}(\Lambda^* \otimes \mathbb{C}^-, \bar{\partial} + D) = H^\ell_{\bar{\partial}+D} \oplus H^{\ell-1}_{\bar{\partial}+D}$.

8 Manifolds with connections with holonomy $Sp(k)$ and spin cohomology

Let $M$ be a Riemannian manifold which admits a connection $\nabla$ with holonomy $Sp(k)$. In this case, there are $k+1$ parallel spinors and so $k+1$ spin differential operators that one can define. The spin differential operators associated with the parallel spinors $e_1 \wedge \ldots \wedge e_{2k}$ are the same as the $d_1$ and $d_2$ spin differential operators that we have investigated for complex manifolds. There are another $k-1$ spin differential operators $d_{\omega^k}$ associated with the $\omega^k$, $1 \leq k \leq k-1$, parallel spinors of section two. Since $\omega \in \Lambda^2$ and $m = 2k$ even, $d_{\omega^k} : \mathcal{C}^- \to \mathcal{C}^-$. We shall not present a complete analysis of all cases. Instead we shall focus on the spin differential operator $d_0 = d_{\omega^{k-1}}$ associated with the parallel spinor $\omega^{k-1}$.

As we have explained for complex manifolds, $\mathcal{C}^\ell = \bigoplus_{p+q=\ell} \Lambda^{0,p} \otimes \Lambda^q(Z)$.

Proposition 13 Let $M$ be a hyperKähler manifold, then

$$d_0 = K_{\omega} : \Lambda^{0,p} \otimes \Lambda^q(Z) \to \Lambda^{0,p+1} \otimes \Lambda^q(Z),$$

where $K$ is the second complex structure on $M$.

Proof: To show this, we first evaluate the action of $d_0$ on $\Lambda^{0,1} \subset \mathcal{C}^-$. Indeed let $\eta e^i \in \Lambda^{0,1}$, then we have

$$d_0(\eta e^i) = \lambda < \omega, \Gamma^i e_k > (\nabla_i - i\nabla_{i+m}) \eta_j e^k \Lambda^e^j = \lambda K^i_k \partial_i \eta_j e^k \wedge e^j,$$

where $\lambda$ is inconsequential numerical coefficient that depends on the inner product and the normalization of the parallel spinor and $< v, K w > = \omega(v, w)$. Thus we have that $d_2 = K_{\omega} \partial$ on $\Lambda^{0,1}$. Using the definition of the $\Lambda$ product, it is straightforward to extend the proof to the rest of the complex $\mathcal{C}^-$. q. e. d.

9 Real Spin Cohomologies

So far we have investigated the complex spinor cohomologies. Now we shall turn to investigate the real ones. The real spinor representations for $m = 4k, 4k+1, 4k+3$ can
be constructed by imposing a reality condition on the complex representations. These reality conditions are

\[
\begin{align*}
\eta &= \pm A(\bar{\eta}), \quad \eta \in \Delta^\pm, \quad m = 4k \\
\eta &= A(\bar{\eta}), \quad \eta \in \Delta^+ \oplus \Delta^-, \quad = 4k + 1 \\
\eta &= B(\bar{\eta}), \quad \eta \in \Delta^+ \oplus \Delta^-, \quad = 4k + 3
\end{align*}
\] (9.1)

and \( \Delta^\pm_R = \{ \eta \in \Delta^\pm, \text{s.t. } \eta = \pm A(\bar{\eta}) \} \), \( \Delta^+_R = \{ \eta \in \Delta^+ \oplus \Delta^-, \text{s.t. } \eta = A(\bar{\eta}) \} \) and \( \Delta^-_R = \{ \eta \in \Delta^+ \oplus \Delta^-, \text{s.t. } \eta = B(\bar{\eta}) \} \), respectively.

9.1 \( SU(m) \) invariant spinors
We begin with a summary of the properties of real spin representation in various cases.

9.1.1 \( m=4k \)
The real parallel spinors in the \( SU(4k) \) case are

\[
\begin{align*}
\tau_1 &= \frac{1}{\sqrt{2}}(1 + e_1 \wedge \ldots \wedge e_m), \\
\tau_2 &= \frac{i}{\sqrt{2}}(1 - e_1 \wedge \ldots \wedge e_m).
\end{align*}
\] (9.2)

Both parallel spinors \( \tau_1, \tau_2 \in \Delta^+_R \). We can again define spin cohomologies \( d_1, d_2 \) associated with \( \tau_1, \tau_2 \) for \( C = A \) on the real complex \( C_R \). Thus \( s_c = 0, s_\Gamma = 1 \). The decomposition of the real spinor representations is

\[
\Delta^+_R \otimes \Delta^+_R = \sum_{p=0}^{2k-1} \Lambda^{2p}_R \oplus \Lambda^{m\pm}_R.
\] (9.3)

9.2 \( m=4k+1 \)
The real parallel spinors are

\[
\begin{align*}
\tau_1 &= \frac{1}{\sqrt{2}}(1 + e_1 \wedge \ldots \wedge e_m), \\
\tau_2 &= \frac{i}{\sqrt{2}}(1 - e_1 \wedge \ldots \wedge e_m).
\end{align*}
\] (9.4)

The parallel spinors \( \tau_1, \tau_2 \in \Delta_R \). We can again define spin cohomologies \( d_1, d_2 \) associated with \( \tau_1, \tau_2 \) for \( C = A \) on the real complex \( C_R \). Thus \( s_c = 0, s_\Gamma = 0 \). The decomposition of the real spinor representations is

\[
\Delta_R \otimes \Delta_R = \sum_{p=0}^{2m} \Lambda^{2p}_R.
\] (9.5)
9.3 \( m=4k+3 \)

The real parallel spinors are

\[
\begin{align*}
\tau_1 &= \frac{1}{\sqrt{2}}(1 + i e_1 \wedge \ldots \wedge e_m) \\
\tau_2 &= \frac{1}{\sqrt{2}}(i1 + e_1 \wedge \ldots \wedge e_m).
\end{align*}
\] (9.6)

The parallel spinors \( \tau_1, \tau_2 \in \Delta_{\mathbb{R}} \). We can again define spin cohomologies \( d_1, d_2 \) associated with \( \tau_1, \tau_2 \) for \( C = B \) on the real complex \( C^\mathbb{R} \). Thus \( s_c = 0, s_r = 1 \). The decomposition of the real spinor representations is

\[
\Delta_{\mathbb{R}} \otimes \Delta_{\mathbb{R}} = \sum_{p=0}^{2m} \Lambda^{2p}_{\mathbb{R}}.
\] (9.7)

In all the above cases we find the following:

**Theorem 6** The operators \( d_1, d_2 \) are nilpotent iff the curvature of the connection \( \nabla \) vanishes.

**Proof:** This is a consequence of the results we have already demonstrated in sections four and six. q. e. d.

We also have that

**Theorem 7** The Laplacians \( \Delta_1, \Delta_2 \) of the operators \( d_1, d_2 \) are

\[
\Delta_2 \phi = \Delta_1 \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi
\] (9.8)

**Proof:** This is a consequence of the results we have already demonstrated in section four. q. e. d.

**Corollary 8** The real spin cohomologies \( H^*(C^\mathbb{R}) \) defined above are generated by the parallel elements in \( C^\mathbb{R} \) with respect to \( \nabla \).

**Proof:** This follows from a partial integration formula and the fact that the inner product \( C = A, B \) restricted in \( \Delta_{\mathbb{R}} \) is definite. q. e. d.

A class of manifolds which we can define a real spin cohomology are group manifolds equipped with the left or right invariant connections. One can also defined twisted real spin cohomology but we shall not pursue this further here.
10 Spin(7) spin cohomology

The $Spin(7)$ invariant spinor is $\zeta = \frac{1}{\sqrt{2}}(e_1 - e_2 \wedge e_3 \wedge e_4)$. Therefore $\zeta \in \Delta^-$. So the spin cohomology operator is $d : C_+ \to C_+$. 

**Theorem 8** The spin operator is nilpotent, $d^2 = 0$, if the connection $\nabla$ is the Levi-Civita connection of a metric on the manifold $M$ with holonomy $Spin(7)$.

**Proof:** The representations $\Delta^\pm$ are real. The map $\tau : \Delta^+ \to \Lambda^1(\mathbb{R}^8)$ given by $\tau(\eta) = CT_\mu(\zeta, \eta) e^\mu = C^\mu_{\zeta}(\eta) e_\mu$ induces an isomorphism between the $\Delta^+$ and the vector representations, $C = A$. This can be easily seen by observing that there is a (real) basis in $\Delta^+$ such that $\tau$ is diagonal. This basis is

$$
1 + e_1 \wedge \ldots \wedge e_4, \quad i(1 - e_1 \wedge \ldots \wedge e_4), \\
(e_1 \wedge e_2 - e_3 \wedge e_4), \quad e_1 \wedge e_3 + e_2 \wedge e_4, \\
i(e_2 \wedge e_3 + e_1 \wedge e_4), \quad (e_2 \wedge e_3 - e_1 \wedge e_4). 
$$

(10.1)

In addition we have that

$$
d^2 \phi = \frac{1}{2} C^\mu_\zeta \wedge C_\nu_\zeta R_{\mu\nu} \phi.
$$

(10.2)

The right-hand-side will vanish if the curvature $R$ of the connection $\nabla$ is that of a Levi-Civita connection for a metric with holonomy $Spin(7)$ by virtue of the Bianchi identity. 

q. e. d.

**Corollary 9** Let $M$ be a manifold equipped with a metric with holonomy $Spin(7)$. Then $H^*(C_+) = H^*_{dR}(M)$.

**Proof:** The map $\tau$ is an isomorphism between the spin cohomology complex $(C_-, d)$ and the de Rham complex $(\Lambda^*(M), d)$. Therefore it induces an isomorphism in cohomology. 

q. e. d.

11 $G_2$ Spin Cohomology

The spinor $Spin(7)$ representation $\Delta$ decomposes under $G_2$ as $\Delta = \mathbb{R} \oplus \Lambda^1(\mathbb{R}^7)$. The $G_2$ invariant spinor is $\zeta = \frac{1}{\sqrt{2}}(e_1 - e_{234})$. As in the previous cases, one can define a linear operator $d$ on $C$ using the spinor $\zeta$.

**Theorem 9** Let $M$ a manifold equipped with a metric $g$ with holonomy contained in $G_2$. The operator $d$ associated to the Levi-Civita connection is nilpotent.

**Proof:** Since $\Delta = \mathbb{R} \oplus \Lambda^1(\mathbb{R}^7)$, we have that $C^\ell = \Lambda^\ell(M) \oplus \Lambda^{\ell-1}(M)$. Moreover, $\tau : \Delta \to \Lambda^1(\mathbb{R}^7)$ such that $\tau(\eta) = CT_\mu(\zeta, \eta) e_\mu$ is onto and has kernel $\mathbb{R} \times \zeta$. Next

$$
d^2 \phi = \frac{1}{2} C^\mu \wedge C^\nu \wedge R_{\mu\nu} \phi
$$

(11.1)

which vanishes because of the Bianchi identity where $R$ is the curvature of the $G_2$ metric. 

q. e. d.
Corollary 10  Let $M$ a manifold equipped with a metric with holonomy $G_2$. Then $H^\ell(C) = H^\ell_{dR}(M) \oplus H^{\ell-1}_{dR}(M)$.

Proof: The map $\tau$ induces an isomorphism between the two complexes. This establishes the isomorphism between the cohomologies. q. e. d.

Acknowledgements

I would like to thank P.S. Howe for many helpful discussions and for suggesting the term Spin Cohomology. I would like to also thank Stefan Ivanov who took part in the initial stage of this project.

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