STABILITY OF TANGENT BUNDLE ON THE MODULI SPACE OF
STABLE BUNDLES ON A CURVE

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ABSTRACT. In this paper, we prove that the tangent bundle of the moduli space $SU_C(r, d)$
of stable bundles of rank $r$ and of fixed determinant of degree $d$ (such that $(r, d) = 1$), on
a smooth projective curve $C$ is always stable, in the sense of Mumford-Takemoto. This
proves a well-known conjecture and is related to a conjectural existence of a Kähler-
Einstein metric on Fano varieties with Picard number one.

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1. Introduction

Suppose $X$ is a compact Kähler manifold. The existence of a Kähler-Einstein metric
on $X$ has attracted wide interest since the conjecture of Calabi and work of Yau [Ya]
appeared, in the study of complex manifolds. Aubin [Au] and Yau show the existence
of a Kähler-Einstein metric whenever the canonical line bundle $K_X$ is ample or trivial.
The existence of a Kähler-Einstein metric when $-K_X$ is ample, i.e., when $X$ is a Fano
manifold, is an open problem. This has many interesting applications and are discussed
by Tian in [Ti2]. Kobayashi [Kb] and Lübke [Lu] show that the existence of a Kähler-
Einstein metric implies the stability of the tangent bundle, in the sense of Mumford and
Takemoto. In particular, the tangent bundle $T_X$ is stable when $X$ is of general type. Since
then the stability problem for Fano manifolds has brought a lot of attention. A very recent
announcement on $K$-stability and existence of Kähler-Einstein metrics is made in [CDS],
by Chen-Donaldson-Sun, and by G. Tian [Ti3].

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The significant works of Hwang \([Hw]\), Peternell-Wisniewski \([Pe-Wi]\), Steffens \([St]\), Subramanian \([Su]\), Tian \([Ti]\) (and the references therein), prove the stability result for certain Fano manifolds \(X\). In most of these cases, the Betti number \(b_2(X) = 1\). When \(b_2(X) > 1\), some examples are known when the stability fails, see \([Ti2, p.183]\). Since then it was speculated (for instance, by Peternell \([Pe2, p.14, Conjecture 5.2]\)) that the stability of the bundle \(T_X\) holds when \(X\) is a Fano manifold with \(b_2(X) = 1\). The list of examples where this is known to hold is rather small, and we investigate this problem for the following important class of varieties, namely the moduli spaces of stable bundles on a curve.

Suppose \(C\) is a smooth projective curve of genus \(g\). The moduli space \(SU_C(r,d)\) of stable vector bundles of rank \(r\) and of fixed determinant of degree \(d\), is a projective Fano manifold when \(r\) and \(d\) are coprime. Furthermore, the Picard number is one, generated by the determinant line bundle \(L\) and the canonical class is \(K = L^{-2}\). In other words, the moduli space is of index 2 \([Ra]\). When the rank \(r = 2\) and \(g \geq 2\), Hwang \([Hw2, Theorem 1]\), proved the stability of the tangent bundle on \(SU_C(2,1)\).

In this paper, we prove the stability of the tangent bundle, for higher rank smooth projective moduli spaces. More precisely,

**Theorem 1.1.** Suppose \(r \geq 3\) and \(d\) is an integer such that \((r,d) = 1\). Suppose \(C\) is a smooth projective curve of genus \(g(C) \geq 3\). Then the tangent bundle on the moduli space \(SU_C(r,d)\) is always stable, in the sense of Mumford-Takemoto.

We use the Hecke correspondence \([Na-Ra], [BLS, p.206]\), relating the moduli spaces \(SU_C(r,1)\) and \(SU_C(r,1-h)\) for any \(h, 0 < h < r\). In fact, it gives a correspondence given by a Grassmannian bundle, as shown in \([BLS]\). The key point is to use the structure of this correspondence and prove that the stability of the tangent bundle of the respective moduli spaces is preserved under this correspondence (see Corollary \(4.3\)). The proof is now by assuming to the contrary and consider destabilizing subsheaves of the cotangent sheaves. We consider the sum of the sheaves inside the direct sum of the relative cotangent sheaves on the Grassmannian bundle. The kernel subsheaf of the direct sum and the image are investigated, see \(14\). A careful analysis of induced sections in the Hodge cohomologies twisted by appropriate powers of ample generators of Picard groups, on the Grassmannian bundle is carried out. We then verify that these groups are zero.

Another approach using vanishing theorems on Hecke curves is indicated in \(2.3\) to rule out rank one destabilizing subsheaves. It is an interesting problem to see if the Hecke correspondence can be utilised to prove existence of a Kähler-Einstein metric on the moduli spaces.

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2. Determinant of destabilizing subsheaves of the cotangent bundle

We start with some preliminaries to fix notations and definitions we will use.

2.1. Preliminaries. Suppose $X$ is a projective manifold of dimension $n$ and $L$ is an ample line bundle on $X$. Suppose $E$ is any coherent sheaf on $X$ of rank $k$ and degree $d$ with respect to $L$. In other words, the determinant $\wedge^k E$ has the intersection number $d := (c_1(\wedge^k E).c_1(L)^n - 1)$. The slope of $E$ is defined to be $\mu(E) := \frac{d}{k}$. Mumford-Takemoto stability means that, for any coherent subsheaf $F \subset E$, $0 < rank(F) < k$, we have the inequality:

$$\mu(F) < \mu(E).$$

If the above strict inequality ($<$) is replaced by the inequality ($\leq$), then we say that $E$ is semistable.

In the proofs, we will need to look at sheaves on open smooth varieties but whose complementary locus in a compactification, has high codimension. We note the following lemma, which we will use.

**Lemma 2.1.** Suppose $X$ is a projective variety and $U \subset X$ be an open smooth subset. Let $S := X - U$ be the complementary closed subset and assume it has codimension at least two. Let $E$ be a coherent sheaf on $X$. Then $c_1(E)$ and $\mu(E)$ are well defined. In particular $c_1(E)$ and $\mu(E)$ are well-defined, when $X$ is a normal projective variety.

**Proof.** See [Ma, p.318-319]. The key point is that $U = X - S$ is smooth and the Chern class $c_1(E_{|U})$ of the restriction of $E$ on $U$ is well-defined, using a locally free resolution of $E_{|U}$. The Weil divisor $c_1(E_{|U})$ extends uniquely on $X$ since codim$(S) \geq 2$. Hence $c_1(E)$ and $\mu(E)$ are well-defined on $X$. □

2.2. Determinant of destabilizing subsheaf of $\Omega_X$ on the moduli space $SU_C(r,d)$. Suppose $C$ is a smooth projective curve of genus $g \geq 3$. Fix an integer $d$ and assume that $r$ is coprime to $d$. Let $X := SU_C(r,d)$ denote the moduli space of stable bundles of rank $r$ with fixed determinant $\eta$ of degree $d$ on $C$. Then $X$ is a projective manifold of dimension $N := (r^2 - 1)(g - 1)$.

We note that the Picard number of $X$ is one and let $L$ be the ample generator of Pic$X$. Also $X$ is of index two, i.e., the canonical line bundle $K_X = L^{-2}$ ([Ra, Theorem 1, p.69]).

Since the dual of a stable bundle is again stable, it suffices to prove that the cotangent bundle $\Omega_X^1$ of the Fano manifold $X$ is stable.

We remark that the stability of the cotangent bundle is implied by the vanishing of some Hodge cohomologies twisted by appropriate powers of the ample class $L$. This can be seen as follows. Suppose $S \subset \Omega_X^1$ is a coherent subsheaf of rank $s$ and $\wedge^s S = L^k$, for
some integer $k$. The inclusion of sheaves gives a non-trivial section of $\Omega_X^s \otimes L^{-k}$. The stability of the cotangent bundle will hold if we have the following vanishing:

$$H^0(X, \Omega_X^s \otimes L^{-k}) = 0,$$

for $0 < s < N$, and $k \geq s \cdot \frac{-2}{N}$.

Since $K_X = L^{-2}$, the condition on the slope is

$$\frac{k}{s} \geq \frac{-2}{N}, \text{ i.e. } -k \leq \frac{s}{N} \cdot 2 < 2.$$  

(1)

In this situation we note that the stability or semistability of $\Omega_X$ give the same inequality $-k < 2$.

**Lemma 2.2.** With notations as above, the only possibility for $k$ is equal to $-1$, i.e. $\det S = L^{-1}$, for a destabilizing subsheaf $S \subset \Omega_X$.

**Proof.** We exclude the other values of $k$ as follows:

**Case** $-k < 0$ : By Akizuki-Nakano vanishing theorem [Ak-Na], we have

$$H^0(X, \Omega_X^s \otimes L^{-k}) = 0 \text{ for any } 0 < s < N.$$

**Case** $k = 0$ : Since $X$ is Fano and hence rationally connected, the required Hodge cohomology vanish [Ko, p.202].

**Case** $-k > 0$ : The slope condition (1) gives the only possibility $-k = 1$.

$\square$

2.3. **Vanishing theorem on Hecke curves.** In this subsection, we prove a vanishing theorem of sheaves on a Hecke curve inside the moduli space $M$ or $M'$. This will exclude the case of rank one subsheaves inside $T_M$ with determinant equal to $L$.

**Lemma 2.3.** Suppose $R \subset M$ is a Hecke curve. Then the following vanishing holds on $R$:

$$H^0(R, \Omega_M^h \otimes L^{-k}) = 0$$

for $h > 0$ and $k > 0$. If $R \subset M$ is a very free rational curve, then the above vanishing holds for $h > 0$ and $k \geq 0$.

**Proof.** Let $R \subset M$ be a smooth Hecke curve passing through a general point of $M$. Since a Hecke curve is free of minimal degree, the tangent bundle $T_M$ of $M$, restricted to $R$, splits as follows ([Ko, p.195]):

$$T_{M|R} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \ldots \oplus \mathcal{O}.$$  

(2)

Here $\mathcal{O}(1)$ occurs $(d - 2)$ times in the above sum and $d := (-K_M, R)$. Hence

$$\Omega_{M|R} = \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \ldots \oplus \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \ldots \oplus \mathcal{O}.$$  

For $h > 0$, we get the expression:

$$\Omega_{M|R}^h = \mathcal{O}(-a_1) \oplus \mathcal{O}(-a_2) \oplus \ldots \oplus \mathcal{O}(-a_u) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \ldots \oplus \mathcal{O},$$  

for $h > 0$. 


where \( a_i \) are positive integers, for each \( i \).

By [Sn, Theorem 1, p.925], \( d = 2r \). Since \( K_M = L^{-2} \), the degree \((L.R) = r\). Hence the sheaf \( \Omega^h_M \otimes L^{-k} \), for \( k > 0 \), restricted to \( R \) looks like:

\[
(\Omega^h_M \otimes L^{-k})|_R = \mathcal{O}(-a_1-kr) \oplus \mathcal{O}(-a_2-kr) \oplus \cdots \mathcal{O}(-a_u-kr) \oplus \mathcal{O}(-kr) \oplus \cdots \mathcal{O}(-kr).
\]

Since \( R \) is a rational curve, this bundle on \( R \) has no global sections.

For the second assertion, since \( M \) is rationally connected, it contains very free rational curves and which cover \( M \). Note that if \( R \) is a very free rational curve then in (2), there are no trivial factors \( \mathcal{O}_R \). Hence in (3), all the factors are negative on \( R \) even when \( k = 0 \). Hence we again deduce the asserted vanishing. □

**Corollary 2.4.** There is no subsheaf \( F \subset T_M \) of rank one and \( \det(F) = L \).

**Proof.** Suppose \( R \subset M \) is a smooth Hecke curve. As in the proof of Lemma 2.3 we have the restriction:

\[
T_{M|R} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \cdots \mathcal{O}(1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}.
\]

Here \( \mathcal{O}(1) \) occurs \((d - 2)\) times in the above sum and \( d := (-K_M.R) \). By [Sn, Theorem 1, p.925], \( d = 2r \). Since \( K_M = L^{-2} \), the degree \((L.R) = r\). Now consider

\[
(T_M \otimes L^{-1})|_R = \mathcal{O}(2-r) \oplus \mathcal{O}(1-r) \oplus \cdots \oplus \mathcal{O}(1-r) \oplus \mathcal{O}(-r) \oplus \cdots \oplus \mathcal{O}(-r).
\]

If \( r \geq 3 \), then this bundle on \( R \) has no global sections. Since the Hecke curves cover \( M \), we get the vanishing \( H^0(M, T_M \otimes L^{-1}) = 0 \). This gives the assertion. □

### 3. Hecke correspondence between moduli spaces

Recall the Hecke correspondence introduced by Narasimhan and Ramanan in [Na-Ra]. It was investigated further by Beauville, Laszlo and Sorger in [BLS] and by others.

We refer to the Hecke correspondence and the properties we use from [BLS, p.206]. Assume \( r \geq 3 \) in the rest of the paper and consider the correspondence:

\[
\mathcal{P} \xrightarrow{q} M' \xrightarrow{q} M
\]

Here \( M := SU_C(r, \eta) \) and \( M' := SU_C(r, \eta') \), such that \( \deg \eta = 1 \) and \( \deg \eta' = 1 - h \), for \( 0 < h < r \). Hence \( 1 - r < 1 - h < 1 \). We can choose \( h \) so that \( 1 - h \) and \( r \) are coprime. So the degree \( d \) in statement of main Theorem [1.1] is \( d := 1, 1 - h \).

Hence \( M \) and \( M' \) are smooth projective moduli spaces of dimension \( N := (r^2 - 1)(g - 1) \). Moreover they are Fano varieties of index 2. The family \( \mathcal{P} \to M \) is a family of
Grassmannian varieties $G(h, r)$ and $q' : \mathcal{P} \rightarrow M'$ is a rational map. The general fibre of $q'$ is a Grassmannian $G(r - h, r)$. See \cite{BLS} Proof of Lemma 10.3. Let $N' := \dim \mathcal{P}$.

As in the previous section, denote the ample generator of Pic$M$ by $L$ and of Pic$M'$ by $L'$. Then the canonical class $K_{\mathcal{P}}$ of $\mathcal{P}$ is (\cite{BLS} Lemma 10.3, p.207):

\begin{equation}
K_{\mathcal{P}} = q^*L^{-1} \otimes q'^*L'^{-1}.
\end{equation}

**Lemma 3.1.** There is a subset $U \subset \mathcal{P}$ such that codim$(\mathcal{P} - U) \geq 2$. Furthermore, $U$ contains the generic fibre of $q$ and $q'$.

**Proof.** The map $q'$ is a morphism outside a codimension two subset of $\mathcal{P}$, call this subset $U \subset \mathcal{P}$ on which $q'$ is defined. Furthermore, $q'$ is defined on a generic fibre of $q$ (see proof of \cite{BLS} Lemma 10.3). In other words, the subset $U$ contains the generic fibre of $q$. We now show that it contains the generic fibre of $q'$, and codim$(\mathcal{P} - U) \geq 2$.

We note that $Z := M' - \text{image}(q')$ is a subset whose closure in $M'$, is of codimension at least two. Otherwise the closure $\bar{Z}$ is an effective divisor linearly equivalent to a positive multiple of $L'$ on $M'$. But $L'$ restricts on a generic fibre $G$ of $q$, as $\mathcal{O}_G(r)$. Hence $\bar{Z}$ is of codimension at least two. Denote $U'' := q'^{-1}(M' - \bar{Z})$. Then $U'' \subset U$. Since codim$(\mathcal{P} - U'') \geq 2$, we deduce that codim$(\mathcal{P} - U) \geq 2$ and $U$ contains a generic fibre of $q'$. \hfill \Box

In the rest of the paper, we will consider sheaves on $U \subset \mathcal{P}$, and in particular we will use that Pic$(q'(U)) = \text{Pic}(M') = \mathbb{Z}.L'$ and codim$(\mathcal{P} - U) \geq 2$.

We also note the following, on the structure of the Grassmannian bundles, with notations as in the previous lemma.

**Lemma 3.2.** The Grassmannian bundles $q : \mathcal{P} \rightarrow M$ and $q' : U'' \rightarrow (M' - \bar{Z}) \subset M'$ are Zariski locally trivial.

**Proof.** Since the moduli space $M$ parametrises stable bundles of rank coprime to their degree, there is a Poincaré bundle on $C \times M$. The Grassmannian bundle $\mathcal{P} \rightarrow M$ is associated to (restriction of) the Poincaré bundle, see \cite{BLS} p.206. Hence the fibration $q$ is Zariski locally trivial. Similar statement holds for $q'$ on $U''$. \hfill \Box

In the later sections, we will use the following identification of the cohomology groups.

**Lemma 3.3.** Suppose $\mathcal{E}$ is a coherent sheaf on $\mathcal{P}$. Then

$$H^0(U, \mathcal{E}) = H^0(M, q_* \mathcal{E}).$$

In particular when $\mathcal{E} = q^* \mathcal{H}$ for some coherent sheaf $\mathcal{H}$ on $M$, we have the equality:

$$H^0(U, \mathcal{E}) = H^0(M, \mathcal{H}).$$

Here $U$ can be replaced by any open subset of $\mathcal{P}$, of codimension at least two. This also holds for the map $q'$, where it is proper.
Proof. Since codimension of $\mathcal{P} - \mathcal{U}$ is at least two, by Hartog’s theorem, we have the equality:

$$H^0(\mathcal{U}, \mathcal{E}) = H^0(\mathcal{P}, \mathcal{E}).$$

This gives us,

$$H^0(\mathcal{P}, \mathcal{E}) = H^0(M, q_*\mathcal{E}).$$

Since $q$ is a proper morphism with connected fibres, when $\mathcal{E} = q^*\mathcal{H}$, by projection formula, we have $q_*q^*\mathcal{H} = \mathcal{H}$. This gives the claim.

□

4. Stability of tangent bundle is preserved under Hecke correspondence: $r \geq 3$

In this subsection, we prove that the stability of the tangent bundle is preserved under Hecke correspondence. This will help us to conclude the stability of the tangent bundle for any $SU_C(r, d)$, when $(r, d) = 1$, in the final section.

Recall the Hecke correspondence from the previous section, and the choice of $\mathcal{U} \subset \mathcal{P}$, such that $q' : \mathcal{U} \rightarrow M'$ is a morphism, containing a generic fibre of $q$ and $q'$, and $\text{codim}(\mathcal{P} - \mathcal{U}) \geq 2$.

Recall from (1) that stability of $\Omega_X$ or $T_X$ (here $(X, \mathcal{L}) := (M, L)$ or $(M', L')$) will hold if there does not exist a coherent subsheaf

$$\mathcal{F} \subset T_X$$

of rank $p \leq \frac{N}{2}$ and $\text{det}(\mathcal{F}) = L$.

Consider the exact sequence of tangent sheaves on $\mathcal{U}$:

$$(5) \quad 0 \rightarrow T_{\mathcal{P}/M} \rightarrow T_{\mathcal{P}} \rightarrow q^*T_M \rightarrow 0.$$ A similar exact sequence corresponds to the fibration $q'$.

**Lemma 4.1.** Suppose $\mathcal{F}' \subset T_{M'}$ is a coherent subsheaf of rank $p$ and $\text{det}(\mathcal{F}') = L'$. Then it corresponds to a non-zero section $s$ in the cohomology group $H^0(\mathcal{U}, \bigwedge^p T_{\mathcal{P}} \otimes q'^*L'^{-1})$, for the subset $\mathcal{U} \subset \mathcal{P}$, as in Lemma 3.1. Furthermore, $\text{codim}(M - q(\mathcal{U})) \geq 2$. Similar statement holds, replacing $M$ by $M'$, $M'$ by $M$ and $L'$ by $L$.

**Proof.** Suppose $\mathcal{F}' \subset T_{M'}$ is a coherent subsheaf of rank $p$ and $\text{det}(\mathcal{F}') = L'$. Then taking determinants we have a nonzero morphism on $\mathcal{U}$,

$$\text{det}(\mathcal{F}') \rightarrow \bigwedge^p T_{M'}.$$ This corresponds to a non-zero section $s$ in $H^0(M', \bigwedge^p T_{M'} \otimes L'^{-1})$.

Consider the exact sequence of tangent sheaves on $\mathcal{U}$:

$$(5) \quad 0 \rightarrow T_{\mathcal{P}/M'} \rightarrow T_{\mathcal{P}} \rightarrow q'^*T_{M'} \rightarrow 0.$$
Then there is a finite filtration of each exterior power of $T_P$ ([Ha Ex 5.16 d), Chap.II]):

$$
\bigwedge^p T_P = G^0 \supset G^1 \supset G^2 \supset \ldots \supset G^{(N)+1} = 0
$$

such that the $i$-th graded piece is

$$
\frac{G^i}{G^{i+1}} = (\bigwedge q^* T_{M'}) \otimes (\bigwedge T_{P/M'}).
$$

Tensor above graded pieces with $q^* L'^{-1}$.

We want to prove the surjectivity of the map:

$$
H^0(\mathcal{U}, \bigwedge^p T_P \otimes q^* q'^* L' - 1) \to H^0(\mathcal{U}, G^0/I \otimes q^* L' - 1).
$$

Using the above filtration, the surjectivity will hold if $H^1(\mathcal{U}, G^1/I \otimes q^* L' - 1) = 0$. This vanishing will hold if

$$
H^1(\mathcal{U}, G^i/I \otimes q^* L' - 1) = 0
$$

for each $i > 0$.

In other words, we want to show the vanishing:

$$
H^1(\mathcal{U}, \bigwedge^i q^* T_{M'} \otimes \bigwedge T_{P/M'} \otimes q^* L' - 1) = 0
$$

for $i \geq 1$.

When $i = m \leq p$, using $K_{P/M'}^{-1} = q^* L' - 1 \otimes q^* L$, the cohomology group in (7) is

$$
H^1(\mathcal{U}, \bigwedge^m q^* T_{M'} \otimes \bigwedge q^* L' - 1) = 0
$$

Restrict a nonzero class of this group on the generic fibre $G'$ of $q'$ to give a non zero class in $H^1(G', L) = H^1(G', K_{G'}^{-1})$. But this vanishes by Kodaira vanishing theorem. Hence (7) holds.

When $0 \leq i < m$, a non zero cohomology class in (7) restricted to $G'$ gives a non-zero class in $H^1(G', \bigwedge^i T_{G'})$. By [Le Corollaire 3], these groups are zero.

Hence (7) holds for $i \geq 1$, as required. This gives the required surjectivity in (6).

Hence the non-zero section $s$ associated to the rank $p$ subsheaf $\mathcal{F}' \subset T_{M'}$, lifts to a nonzero section

$$
s' \in H^0(\mathcal{U}, \bigwedge^p q^* T_P \otimes q^* L' - 1).
$$

Note that since $q, q'$ are fibre bundle maps (more generally they are flat maps), they are open maps [Du]. Hence $q(\mathcal{U})$ is a Zariski open subset of $M$. Suppose $\text{codim}(M - q(\mathcal{U})) = 1$. The closed subset $D := M - q(\mathcal{U})$ is an effective divisor and $q^* D$ does not intersect $\mathcal{U}$. Since $Pic(M) = \mathbb{Z}.L$, $D$ is linearly equivalent to a positive power $L'$. By [BLS Lemma 10.3], $q^* L$ restricted on $G'$ is $O_{G'}(r)$. Hence $q^* D$ intersects $\mathcal{U}$. This contradicts our assumption on $\text{codim}(M - q(\mathcal{U})) = 1$. This shows that $\text{codim}(M - q(\mathcal{U})) \geq 2$.
Above arguments also hold replacing $M$ by $M'$ and $M'$ by $M$, since we only need to note that $q'$ is defined on a generic fibre of $q$ (see proof of [BLS Lemma 10.3]). Since $Pic(M') = Pic(q'(U))$, we can conclude by similar arguments as above.

\[ \square \]

**Proposition 4.2.** Assume that $0 < h < r$, $r \geq 3$ and $\text{gcd}(r, 1-h) = 1$. Let $U \subset P$, be as in Lemma 3.1. Then

\[ H^0(U, \bigwedge^p T_P \otimes q^* L'^{-1}) = H^0(M, \bigwedge^{p-m} T_M \otimes L^{-1}). \]

Here $m := \dim G(h, r) < p$.

Similarly, the statement also holds if we replace $M$ by $M'$, $M'$ by $M$ and $L'$ by $L$.

**Proof.** Suppose there is a nonzero section $s' \in H^0(U, \bigwedge^p T_P \otimes q^* L'^{-1})$.

On $U$, consider the exact sequence (5), associated to the fibration $q$. Then there is a finite filtration of each exterior power of $T_P$ ([Ha, Ex 5.16 d), Chap.II]):

\[ \bigwedge^p T_P = G^0 \supset G^1 \supset G^2 \supset \ldots \supset G^{(N)} + 1 = 0 \]

such that the $i$-th graded piece is

\[ \frac{G^i}{G^{i+1}} = (\bigwedge^i q^* T_M) \otimes (\bigwedge^i T_{P/M}). \]

Tensor above graded pieces with $q^* L'^{-1}$.

We claim that, for $i \leq p$ and except when $i = m < p$,

\[ H^0(U, \bigwedge^{p-i} q^* T_M \otimes \bigwedge^i T_{P/M} \otimes q^* L'^{-1}) = 0. \]

**case 1:** $i = m \leq p$.

The cohomology group in this case is

\[ H^0(U, \bigwedge^{p-i} q^* T_M \otimes K_{P/M}^{-1} \otimes q^* L'^{-1}) = H^0(U, \bigwedge^{p-i} q^* T_M \otimes q^* L^{-1}). \]

Indeed, using (4), we have

\[ \bigwedge^{m} T_{P/M} \otimes q^* L'^{-1} = K_{P/M}^{-1} \otimes q^* K_M \otimes q^* L'^{-1} = q^* L \otimes q^* L' \otimes q^* L^{-2} \otimes q^* L'^{-1} = q^* L^{-1}. \]

If $i = p$ then the cohomology group is $H^0(U, q^* L^{-1}) = 0$, since $L$ is ample and $q^* L$ restricted on a generic fibre of $q'$ is ample.

If $i = m < p$, then we note that $\text{codim}(P - U) \geq 2$ and $\text{codim}(M - q(U)) \geq 2$ (Lemma 4.1). Hence, by Hartog’s theorem and by projection formula, (see Lemma 3.3), we have
the equality:

\[ H^0(\mathcal{U}, \bigwedge^{p-i} q^* T_M \otimes K_{P/M}^{-1} \otimes q^* L'^{-1}) = H^0(M, \bigwedge^{p-m} T_M \otimes L^{-1}) \]

**case 2):** \( i < m \).

Let \( s \in H^0(\mathcal{U}, \bigwedge^{p-i} q^* T_M \otimes K_{P/M}^{-1} \otimes q^* L'^{-1}) \). Then the restriction of \( s \) to the generic fibre \( G \) of \( q \) is zero. In fact, \( \bigwedge^{p-i} q^* T_M \) restricted to \( G \) is trivial and \( q^* L' \) restricted to \( G \) is \( K_G^{-1} = \mathcal{O}_G(r) \) (see the proof of [BLS, Lemma 10.3]). It is therefore sufficient to prove that \( H^0(G, \bigwedge^i T_G \otimes \mathcal{O}_G(-r)) = 0 \).

Now notice that

\[ H^0(G, \bigwedge^i T_G \otimes K_G) = H^0(G, \Omega_G^{m-i} \otimes \mathcal{O}_G). \]

But this group is zero since \( G \) is rationally connected, and we deduce the claim.

Hence \( H^0(\mathcal{U}, \bigwedge^{p-i} q^* T_M \otimes \bigwedge^i T_{P/M} \otimes q^* L'^{-1}) = 0 \) as required. This completes the proof of the first assertion. The second assertion follows similarly.

\[ \square \]

**Corollary 4.3.** If the tangent bundle \( T_M \) of \( M \) is stable then the tangent bundle \( T_{M'} \) of \( M' \) is also stable. Similar statement holds, replacing \( M \) by \( M' \) and \( M' \) by \( M \).

**Proof.** Suppose \( T_{M'} \) is not stable. Then, by (1), there exists a subsheaf \( \mathcal{F}' \subset T_{M'} \) such that \( \text{det}(\mathcal{F}') = L' \) and rank of \( \mathcal{F}' \) is \( p \leq \frac{N}{2} \).

By Lemma 4.1, \( \mathcal{F}' \) corresponds to a nonzero section

\[ s' \in H^0(\mathcal{U}, \bigwedge^p T_P \otimes q^* L'^{-1}). \]

By Proposition 4.2, it corresponds to a nonzero section on \( M \):

\[ s : L \to \bigwedge^{p-m} T_M. \]

Here \( m = \dim G(h, r) \).

We compute the slope of these sheaves. The slope of \( L \) is \( \mu(L) = L.N^{-1} \). The slope \( \mu(\bigwedge^{p-m} T_M) \) of \( \bigwedge^{p-m} T_M \) is

\[ \frac{\text{det}(\bigwedge^{p-m} T_M) \cdot L^{N-1}}{\binom{N}{p-m} \cdot L^N} = \frac{2 \cdot \binom{N-1}{p-m-1}}{\binom{N}{p-m}} \cdot L^N. \]

Claim: \( \mu(L) > \mu(\bigwedge^{p-m} T_M) \).
We need to show that
\[ 1 > \frac{2 \binom{N-1}{p-m-1}}{\binom{N}{p-m}} = \frac{2 \big( \binom{N}{p-m} - \binom{N-1}{p-m} \big)}{\binom{N}{p-m}} = 2 \big( 1 - \frac{\binom{N-1}{p-m}}{\binom{N}{p-m}} \big) = 2 \left( \frac{p - m}{N} \right). \]

Since \( p - m < p \leq \frac{N}{2} \), this inequality always holds.

Since \( T_M \) is assumed to be stable, the exterior powers \( \bigwedge^{p-i} T_M, i \geq 0 \), are semistable \([Ma]\). Hence the above slope inequality gives a contradiction to the assumption.

When we replace \( M \) by \( M' \) and \( M' \) by \( M \), Lemma 4.1 and Proposition 4.2 also hold. Hence above arguments also hold true, to get the second assertion. \( \square \)

5. Stability of the tangent bundle \( T_M \): Main Theorem

Recall the Hecke correspondence and notations, from [33]

\[ \mathcal{P} \xrightarrow{q'} M' \]
\[ \downarrow q \]
\[ M. \]

Here \( q' \) is a rational map and there is a subset \( \mathcal{U} \subset \mathcal{P} \) where \( q' \) is defined and such that \( \text{codim}(\mathcal{P} - \mathcal{U}) \geq 2 \). (The choice of \( \mathcal{U} \) is made in Lemma 3.1 with \( \text{codim}(\mathcal{P} - \mathcal{U}) \geq 2 \) and \( \mathcal{U} \) contains a generic fibre of \( q \) and \( q' \)).

In the rest of the proofs, we will consider sheaves on \( \mathcal{U} \subset \mathcal{P} \).

Consider the product variety \( M \times M' \) with the projections \( l : M \times M' \to M, l' : M \times M' \to M' \). The inclusion in the following lemma was pointed out by C. Simpson.

**Lemma 5.1.** There is a map:
\[ \gamma : \mathcal{P} \to M \times M' \]
defined on \( \mathcal{U} \), and compatible with the projections \( l, l', q, q' \). The map \( \gamma \) is injective on \( \mathcal{U} \).

The natural sheaf map:
\[ \gamma^* \big( \Omega_{M \times M'} = l^* \Omega_M \oplus l'^* \Omega_{M'} \big) \to \Omega_\mathcal{P} \]
is generically surjective.

**Proof.** The existence of the map \( \gamma \) and compatibility with projections, follows from universality of products over \( \text{Spec}(\mathbb{C}) \). More concretely, on \( \mathcal{U} \), \( \gamma \) is given as \( u \mapsto (q(u), q'(u)) \).
To show that the natural sheaf map $\gamma^*(l^*\Omega_M \oplus l'^*\Omega_{M'}) \to \Omega_P$ is generically surjective, it suffices to show that the dual map on tangent spaces:

$$T_P \to T_M \oplus T_{M'},$$

is injective.

This map restricted to $G$ is injective, since $q' : \mathcal{U} \to M'$ restricted to $\{u\} \times G$ is injective, for any $u \in M$, whenever $q'$ is defined (see proof of [BLS, Lemma 10.3]). In particular,

$$\gamma : \mathcal{U} \to M \times q'(\mathcal{U})$$

is injective. □

A dimension count shows that the image of $\mathcal{U}$ under the map $\gamma$ has dimension strictly smaller than the product $M \times M'$. We also note the following consequence of the inclusion $\mathcal{U} \hookrightarrow M \times M'$.

**Lemma 5.2.** There is a (generically) surjective map of sheaves on $\mathcal{U}$:

$$\Omega_{\mathcal{U}} \to \Omega_{\mathcal{P}/M'} \oplus \Omega_{\mathcal{P}/M}.$$

In particular, there is a (generically) surjective map of sheaves, compatible with direct sums:

$$q^*\Omega_M \oplus q'^*\Omega_{M'} \to \Omega_{\mathcal{P}/M'} \oplus \Omega_{\mathcal{P}/M}.$$

**Proof.** Using Lemma 5.1, we have the following inclusion of sheaves on $\mathcal{U}$:

$$(q^*T_M \cap T_P) \oplus (q'^*T_{M'} \cap T_P) \hookrightarrow T_P \hookrightarrow q^*T_M \oplus q'^*T_{M'}.$$

Consider the exact sequence of tangent sheaves on $\mathcal{U}$ associated to $\mathcal{P} \to M$ (and a similar one associated to $\mathcal{P} \to M'$):

$$0 \to T_{\mathcal{P}/M} \to T_P \to q^*T_M \to 0.$$

Now consider the exact sequence:

$$0 \to q'^*T_{M'} \to q^*T_M \oplus q'^*T_{M'} \to q^*T_M \to 0$$

$$\cup \quad \cup \quad =$$

$$0 \to q'^*T_{M'} \cap T_P \quad \to \quad T_P \quad \to \quad q^*T_M \to 0.$$

We deduce that (inside $q^*T_M \oplus q'^*T_{M'}$):

(8) $$q'^*T_{M'} \cap T_P = T_{\mathcal{P}/M}$$

and similarly

(9) $$q^*T_M \cap T_P = T_{\mathcal{P}/M'}.$$

This gives the inclusion of sheaves on $\mathcal{U}$:

$$T_{\mathcal{P}/M'} \oplus T_{\mathcal{P}/M} \hookrightarrow T_P.$$
Since the sheaves are locally free, dualizing we get a (generically) surjective map of sheaves on $U$:

$$\Omega_P \to \Omega_{P/M'} \oplus \Omega_{P/M}.$$ 

The second assertion in the lemma follows from the above arguments. \qed

5.1. **Main theorem.** Now we proceed to show:

**Theorem 5.3.** The cotangent bundle $\Omega_M$ of the moduli space $M = SU_C(r, d)$ where $(r, d) = 1$ and $r \geq 3$ is always stable.

We start by assuming to the contrary, and gather the consequences in this subsection. The next subsection will analyse the possible cases which will be ruled out, using rational connectedness and Kodaira-Akizuki-Nakano theorem.

By Corollary 4.3, we assume that both $\Omega_M$ and $\Omega_M'$ are not stable.

Suppose $F \subset \Omega_M$ (resp. $G \subset \Omega_M'$) is a coherent subsheaf destabilizing $\Omega_M$ (resp. $\Omega_M'$). Then we have seen in §2.2 that we have the following only possibilities

(10) $$\det(F) = L^{-1}, \quad \det(G) = L'^{-1}$$

and

(11) $$\text{rank}(F) = p \geq \frac{N}{2}, \quad \text{rank}(G) = p' \geq \frac{N}{2}.$$ 

Consider the product variety $M \times M'$ with the projections $l : M \times M' \to M$, $l' : M \times M' \to M'$, and compatible with $q, q'$ as in Lemma 5.1, together with the injective map

$$\gamma : U \to M \times q'(U).$$

The choice of $U$ is made in Lemma 3.1.

Consider the map of sheaves, on $M \times q'(U)$ (use Lemma 5.2):

$$\eta : l^* F \oplus l'^* G \hookrightarrow l^* \Omega_M \oplus l'^* \Omega_M' \to \gamma_*(\Omega_U) \to \gamma_* (\Omega_{P/M'} \oplus \Omega_{P/M}).$$

The composed map is taking sums inside $\gamma_*(\Omega_{P/M'} \oplus \Omega_{P/M}).$

This gives a left exact sequence on $M \times q'(U)$ :

(12) $$0 \to \tilde{\mathcal{K}} \to l^* F \oplus l'^* G \to \gamma_* (\Omega_{P/M'} \oplus \Omega_{P/M}).$$

Here $\tilde{\mathcal{K}} := \text{kernel } (\eta)$. Note that the image of $\eta$ is supported on $\gamma(U)$, so the image has rank zero, and hence

(13) $$\text{rank}(\tilde{\mathcal{K}}) = \text{rank}(l^* F \oplus l'^* G).$$

We note the following lemma, which we will use.
Lemma 5.4. There is a commutative diagram on $M \times q'(U) \subset M \times M'$:

$$
\begin{array}{ccccccc}
0 & 0 & & & & & \\
\downarrow & & & & & & \\
0 \to T & \to B & \to & \\
\downarrow & & & & & & \\
0 \to \tilde{K} & \to l^*F \oplus l'^*G & \to & \gamma_*(\Omega_{P/M'} \oplus \Omega_{P/M}) & \\
\downarrow f & & & \downarrow & & & \\
\to \tilde{K} \otimes \gamma_*O_U & \to (l^*F \oplus l'^*F) \otimes \gamma_*O_U & \to & \gamma_*(\Omega_{P/M'} \oplus \Omega_{P/M}) \otimes \gamma_*O_U & \\
\downarrow & & & \downarrow & & & \\
0 & 0 & & & \to & 0.
\end{array}
$$

The columns are exact and rows are left exact. Here $T := \tilde{K}\mathcal{I}_{\gamma(U)}$, where $\mathcal{I}_{\gamma(U)}$ denotes the ideal sheaf of the subset $\gamma(U)$ in $M \times q'(U)$. In particular we have the equalities:

$$
\text{rank}(T) = \text{rank}(\tilde{K}) = \text{rank}(l^*F \oplus l'^*G).
$$

Proof. This commutative diagram follows from restriction of (12) on the image $\gamma(U)$ in $M \times q'(U)$. The rank equality is clear, using (13), since $\tilde{K} \otimes \gamma_*O_U$ is supported on $\gamma(U)$, whose dimension is strictly smaller than $M \times M'$. □

Since $q = l \circ \gamma$ and $q' = l' \circ \gamma$, pullback of the left exact sequence (12), on $U$, via $\gamma$ gives the exact sequence of sheaves:

(14) $$
0 \to T \to \gamma^*\tilde{K} \to q^*F \oplus q'^*G \to q^*F + q'^*G \to 0.
$$

The right most term is just projecting the direct sum into $\gamma^*\gamma_*(\Omega_{P/M'} \oplus \Omega_{P/M})$. Via the generically injective map $\gamma^*\gamma_*(\Omega_{P/M'} \oplus \Omega_{P/M}) \to \Omega_{P/M'} \oplus \Omega_{P/M}$, we obtain a generically injective map

(15) $$
q^*F + q'^*G \to \Omega_{P/M'} \oplus \Omega_{P/M}.
$$

The ranks of these sheaves will be crucial in the rank estimates in the final section.

Here $T := \ker(\alpha)$.

Lemma 5.5. There is a subsheaf $\tilde{T} \subset \tilde{K}$ on $M \times q'(U)$, such that

$$
\gamma^*(\tilde{T}) = T
$$

and

$$
\text{rank}(T) = \text{rank}(\tilde{T}) = \text{rank}(l^*F \oplus l'^*G).
$$

Proof. Pushforward of (13) on $M \times q'(U)$ gives the exact sequence:

$$
0 \to \gamma_*(T) \to \tilde{K} \otimes \gamma_*(\mathcal{O}_U) \to (l^*F \oplus l'^*G) \otimes \gamma_*(\mathcal{O}_U) \to .
$$
In the commutative diagram in Lemma 5.4, we take the inverse image \( \tilde{T} := f^{-1}(\gamma_\ast T) \) of \( \gamma_\ast T \) under the map \( f \), which gives a short exact sequence, compatible with the maps in Lemma 5.4:

\[
0 \to T \to \tilde{T} \to \gamma_\ast(T) \to 0.
\]

Since \( \gamma_\ast(T) \) is supported on \( \gamma(U) \), we get the rank equality:

\[
\text{rank}(\tilde{T}) = \text{rank}(T).
\]

Pullback via \( \gamma \) on \( U \) gives the exact sequence:

\[
\gamma^\ast T \to \gamma^\ast(\tilde{T}) \to T \to 0.
\]

However the left hand map is zero. Hence we get \( \gamma^\ast(\tilde{T}) = T \).

□

Here \( K := \text{image}(\gamma^\ast \tilde{K} \to q^*F \oplus q^*G) \). Then we get the short exact sequences on \( U \) (by breaking up (14)):

(16) \[
0 \to T \to \gamma^\ast \tilde{K} \to K \to 0
\]

and

(17) \[
0 \to K \to q^*F \oplus q^*G \to q^*F + q^*G \to 0.
\]

**Remark 5.6.** We note that it is relevant to look at the exact sequence (12) on the product \( M \times q'(U) \) and then restrict on \( U \), instead of taking sums on \( U \) as in (17). This is essential to deduce the triviality of \( \det(K) \), in Corollary 5.3.

In the following lemma we note the determinants, (see [Kd-Mu], for a definition and functorial properties of the functor \( \det \)).

**Lemma 5.7.** We have

a) \( \det(T) = \det(\tilde{K}) \),

b) \( \det(\mathcal{T}) = \det(\tilde{T}) \),

c) \( \det(T) \otimes \det(K) = q^*L^{-1} \otimes q'^*L^{-1} \)

and

d) \( \gamma^\ast \det(\tilde{T}) = \det(T) \).

**Proof.** Using (12), we have on \( M \times q'(U) \):

\[
\det(\tilde{K}) \otimes \det(l^*F + l'^*G) = \det(l^*F \oplus l'^*G) = l^*L^{-1} \otimes l'^*L'^{-1}.
\]

Since \( \text{Image}(\eta) \), \( \gamma_\ast(T) \) and \( \tilde{K} \otimes \gamma_\ast(O_U) \) are supported on \( \gamma(U) \) whose codimension is at least two, their determinants are trivial, i.e equal to \( O_{M \times q(U)} \) [Hu-Le, p.10]. Similarly, we note that

\[
\det(T) = \det(\tilde{K})
\]

and

\[
\det(T) = \det(\tilde{T}).
\]
Putting them together, using (16) and \( \det(\gamma^*\tilde{K}) = \gamma^*(\det(\tilde{K})) \), we get

\[
\det(T) \otimes \det(K) = q^*L^{-1} \otimes q'^*L^{-1}.
\]

Since, by Lemma 5.5 \( \gamma^*\tilde{T} = T \) we obtain \( \gamma^*\det(\tilde{T}) = \det(T) \). \( \square \)

Hence, we deduce:

**Corollary 5.8.**

\[
\det(K) = \mathcal{O}.
\]

**Proof.** Using (16) and, a) and d) of Lemma 5.7 we deduce that \( \det(K) = \mathcal{O} \). \( \square \)

5.2. **Proof of main theorem.** Recall the short exact sequence (17) on \( \mathcal{U} \):

\[
0 \to K \to q^*\mathcal{F} \oplus q'^*\mathcal{G} \to q^*\mathcal{F} + q'^*\mathcal{G} \to 0.
\]

Let \( s := \text{rank}(K) \) and \( s' := \text{rank}(q^*\mathcal{F} + q'^*\mathcal{G}) \) on \( \mathcal{U} \). (Note that these ranks may be different from the ranks on \( M \times M' \)).

Our aim is now to show that the short exact sequence (17) does not exist on \( \mathcal{U} \). To show this, we first show that rank(\( K \)) > 0. Then we consider the non-zero section induced by taking appropriate determinants. We then check that the section lies in a Hodge cohomology twisted by powers of \( L \) and \( L' \). Then we will note that these groups are zero, using rational connectedness and Kodair-Akizuki-Nakano theorem. The relevant cohomology groups are detailed below, which depend on the powers of \( L \) and \( L' \). When rank(\( K \)) = 0, we make a rank estimate of the direct sum with respect to the target sheaf, to obtain a contradiction.

Before analysing the induced sections, we note the following lemma, which we will need.

**Lemma 5.9.** Consider the subsheaf \( K \subset q^*\mathcal{F} \oplus q'^*\mathcal{G} \), from (17). Then there is a short exact sequence of sheaves:

\[
0 \to K \to \mathcal{K} \to K' \to 0.
\]

such that the sheaves \( K, K' \) are subsheaves of \( q^*\Omega_M \cap q'^*\Omega_{M'} \subset \Omega_{\mathcal{U}} \).
Proof. Consider the commutative diagram:

\[
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
0 \rightarrow K \rightarrow \mathcal{K} \rightarrow (q^*\mathcal{F} + q'^*\mathcal{G})_U \cap (q^*\Omega_M \cap q'^*\Omega_{M'}) & \subset q^*\Omega_M \cap q'^*\Omega_{M'} \\
\| & \downarrow & \downarrow \\
0 \rightarrow K \rightarrow q^*\mathcal{F} \oplus q'^*\mathcal{G} \rightarrow (q^*\mathcal{F} + q'^*\mathcal{G})_U & \subset \Omega_M \\
\downarrow & \| & \downarrow \\
0 \rightarrow \mathcal{K} \rightarrow q^*\mathcal{F} \oplus q'^*\mathcal{G} \rightarrow q^*\mathcal{F} + q'^*\mathcal{G} & \subset \Omega_{P/M'} \oplus \Omega_{P/M} \\
\downarrow & & \downarrow \\
0 & 0 \\
\end{array}
\]

The columns on the right are exact sequences, and the bottom row is the same as (17).

Since the sheaf $K$ is generically $q^*\mathcal{F} \cap q'^*\mathcal{G}$ (inside $q^*\Omega_M \cap q'^*\Omega_{M'}$), there is an inclusion of sheaves

\[ K \hookrightarrow q^*\Omega_M \cap q'^*\Omega_{M'} . \]

Denote

\[ K' := (q^*\mathcal{F} + q'^*\mathcal{G})_U \cap (q^*\Omega_M \cap q'^*\Omega_{M'}) \subset q^*\Omega_M \cap q'^*\Omega_{M'} . \]

This gives the exact sequence

\[ 0 \rightarrow K \rightarrow \mathcal{K} \rightarrow K' \rightarrow 0 \]

as claimed. \qed

We now proceed as follows:

Using above lemma, we have $\text{rank}(\mathcal{K}) \neq 0$, and by Corollary 5.8 we know $\det(\mathcal{K}) = \mathcal{O}$. \qed
To compute the determinants of above sheaves, we use the fact that the Picard group of \( \mathcal{P} \) is generated by \( q^* \text{Pic} \mathcal{M} \) and \( \mathcal{O}_{\mathcal{P}}(1) \). Note that this also holds outside a codimension two subset of \( \mathcal{P} \), in particular on \( \mathcal{U} \).

We can write
\[
\det(K) = \mathcal{O}.
\]
The above exact sequence (17), and (10) give
\[
\det(q^* F + q'^* G) = q^* L^{-1} \otimes q'^* L'^{-1}.
\]
Using the exact sequence in Lemma 5.9, we can write
\[
\det(K) = \det(K) \cdot \det(K').
\]
Let \( \det(K') = L^{a_1} \otimes L'^{b_1} \), then \( \det(K) = L^{-a_1} \otimes L'^{-b_1} \).

Case I: Suppose that \( \text{rank}(K) > 0 \) and \( \text{rank}(K') > 0 \).

Hence taking determinants in (18), we get nonzero morphisms
\[
q^* L^{-a_1} \otimes q'^* L'^{-b_1} \to \bigwedge^s q^* \Omega_M, \quad q^* L^{a_1} \otimes q'^* L'^{b_1} \to \bigwedge^s q^* \Omega_M.
\]
This give nonzero sections in
\[
H^0(\mathcal{U}, \bigwedge^s q^* \Omega_M \otimes q^* L^{-a'} \otimes q'^* L'^{-b'})
\]
where \( (a', b') := (-a_1, -b_1) \) and when \( (a', b') := (a_1, b_1) \).

Restricting on a generic fibre \( G \) of \( q \), we get a nonzero section in
\[
(20) \quad H^0(G, q^* L^{-b'}).\]
If \( b' > 0 \), then the cohomology \( H^0(G, q^* L^{-b'}) \) is zero, since \( q^* L' \) on \( G \) is \( \mathcal{O}_G(r) \) (see proof of [BLS Lemma 10.3]). Hence the group in (20) is zero.

Hence \( b' \leq 0 \).

Similarly, using (19), if we consider the fibration \( q' : \mathcal{P} \to M' \) we deduce that \( a' \leq 0 \).

Hence
\[
a', b' \leq 0.
\]
Substituting the values of \( (a', b') \), we deduce that
\[
-a_1 \leq 0, -b_1 \leq 0, a_1 \leq 0, b_1 \leq 0.
\]
This implies that \( a_1 = b_1 = 0. \)

**Case II:** Suppose \( \text{rank}(K) = 0 \) or \( \text{rank}(K') = 0. \) Then since \( \mathcal{F} \) and \( \mathcal{G} \) are torsion free sheaves on \( \mathcal{U} \), the sheaf \( K = 0 \) (resp \( K' = 0 \) since it is a subsheaf of \( \Omega_{\mathcal{U}} \)) and hence

\[
(21) \quad a_1 = b_1 = 0.
\]

Since \( \text{rank}(\mathcal{K}) \neq 0 \), either \( K \) or \( K' \) has non zero rank.

Hence we get a non zero section in \( H^0(\mathcal{U}, q^*\Omega^*_M) \), for some \( s > 0. \) Since \( \mathcal{P} - \mathcal{U} \subset \mathcal{P} \) is of codimension at least two, \( M - q(\mathcal{U}) \subset M \) is of codimension at least two (see proof of Lemma [4.1]), by Lemma [3.3] the non zero section extends to give a non-zero section in \( H^0(M, \Omega^*_M) \). But this group is zero using rational connectedness of \( M \), since \( s > 0. \)

This completes the proof.

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