2-distance, injective, and exact square list-coloring of planar graphs with maximum degree 4

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Abstract

In the past various distance based colorings on planar graphs were introduced. We turn our focus to three of them, namely 2-distance coloring, injective coloring, and exact square coloring. A 2-distance coloring is a proper coloring of the vertices in which no two vertices at distance 2 receive the same color, an injective coloring is a coloring of the vertices in which no two vertices with a common neighbor receive the same color, and an exact square coloring is a coloring of the vertices in which no two vertices at distance exactly 2 receive the same color. We prove that planar graphs with maximum degree \(\Delta = 4\) and girth at least 4 are 2-distance list \((\Delta + 7)\)-colorable and injectively list \((\Delta + 5)\)-colorable. Additionally, we prove that planar graphs with \(\Delta = 4\) are injectively list \((\Delta + 7)\)-colorable and exact square list \((\Delta + 6)\)-colorable.

1 Introduction

A 2-distance coloring of a graph \(G\) is a proper coloring of the vertices of \(G\) such that no pair of vertices at distance at most 2 receive the same color. The 2-distance chromatic number of a graph \(G\), denoted by \(\chi_2(G)\), is the smallest integer \(k\) such that there exists a 2-distance coloring of \(G\) with \(k\) colors. An injective coloring of a graph \(G\) is a coloring of the vertices of \(G\) in which every pair of vertices with a common neighbor receive distinct colors. The injective chromatic number, denoted by \(\chi_i(G)\), is the smallest integer \(k\) such that there exists an injective coloring of \(G\) with \(k\) colors. An exact square coloring of a graph \(G\) is a coloring of the vertices of \(G\) in which every pair of vertices at distance exactly 2 receive distinct colors. The exact square chromatic number, denoted by \(\chi^{\#2}(G)\), is the smallest integer \(k\) such that there exists an exact square coloring of \(G\) with \(k\) colors. Given a list assignment \(L\) of \(G\), a 2-distance (injective, exact square) coloring \(\phi\) of \(G\) is called a 2-distance (injective, exact square) list-coloring if \(\phi(v) \in L(v)\) for every \(v \in V(G)\). A graph \(G\) is 2-distance (injective, exact square) \(k\)-choosable if \(G\) admits a 2-distance list-coloring for any list assignment \(L\) with \(|L(v)| \geq k\) for each \(v \in V(G)\). The 2-distance (injective, exact square) choosability number of \(G\), denoted by \(\chi_2^\ell(G)\) (\(\chi_i^\ell(G)\), \(\chi^{\#2\ell}(G)\)), is the smallest integer \(k\) such that \(G\) is 2-distance (injective, exact square) \(k\)-choosable.

Unlike the 2-distance coloring, both the injective coloring and the exact square coloring are not necessarily proper, i.e., adjacent vertices can receive the same color, provided that they satisfy certain conditions. For instance, in the exact square coloring two vertices can be colored with the same color if they are adjacent, and in the injective coloring two vertices can be colored with the same color if they are adjacent and do not share a common neighbor. See Figure 1 for a comparison of these colorings.

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A 2-distance coloring.

An injective coloring.

An exact square coloring.

Figure 1: A 2-distance coloring, injective coloring, and exact square coloring of the same graph.

It is therefore easy to observe that every 2-distance coloring is an injective coloring and every injective coloring is an exact square coloring. Thus, for every graph $G$ we have the following chain of inequalities:

\[ \chi^{\#2}(G) \leq \chi^i(G) \leq \chi^2(G). \]

Moreover, $\chi^{\#2}(G) = \chi^i(G)$ in the case of triangle-free graphs, i.e., graphs in which no pair of adjacent vertices share a common neighbor.

The notion of distance based colorings was first introduced in 1969 by Kramer and Kramer [29, 30] when they introduced the notion of a $p$-distance coloring. In this type of coloring, we require that vertices at distance at most $p$ receive distinct colors. When $p = 1$ we obtain the familiar proper coloring. Thus, a $p$-distance coloring is a generalization of the classical proper coloring.

Throughout the years, $p$-distance colorings, in particular the case when $p = 2$, became a focus for many researchers, see [2, 26, 27, 42]. Only in recent years, several results were investigated with regards to the 2-distance coloring of planar graphs, see [4, 13, 20]. For more of the recent results see also [31, 32, 33]. It is easy to observe that for every graph $G$, where $\Delta(G)$ (or simply $\Delta$ when $G$ is clear from the context) is the maximum degree of $G$, $\Delta(G) + 1 \leq \chi^2(G) \leq \Delta^2(G) + 1$. Moreover, the upper bound, which follows from a greedy algorithm, is known to be tight for the family of Moore graphs (see, e.g., [34]). A famous conjecture of Wegner from 1977 [43] states that for planar graphs $\chi^2(G)$ is linear in terms of $\Delta$.

**Conjecture 1** (Wegner [43]). Let $G$ be a planar graph with maximum degree $\Delta$. Then,

\[ \chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lceil \frac{3}{2} \Delta \right\rceil + 1, & \text{if } \Delta \geq 8. \end{cases} \]

If true, then these upper bounds are tight, as there exist graphs that attain them (see [43]). In 2018, the case when $\Delta \leq 3$ was proved independently by Thomassen [42] and by Hartke et al. [25]. Additionally, for $\Delta \geq 8$, Havet et al. [27] proved that the bound is $\frac{3}{2}(1 + o(1))$. Moreover, Conjecture 1 is known to be true for some subfamilies of planar graphs (e.g., $K_4$-minor free graphs [36]).

In [34], La and Montassier presented a summary of the latest known results regarding the 2-distance coloring of planar graphs for different girth values, where girth of a graph $G$, denoted by $g(G)$, is defined as the length of the shortest cycle. An additional more recent result is due to Bousquet et al. [4]. They improved a general result, in the case of the 2-distance coloring, stating that $2\Delta + 7$ colors are sufficient for all planar graphs with maximum degree between 6 and 31. Additionally, in [3], the same authors proved that 12 colors are sufficient when $\Delta = 4$, the case that we are particularly interested in.

The injective coloring was first introduced in 2002 by Hahn, Kratochvília, Širáň, and Sotteau [23]. The authors proved that for every graph $G$, $\Delta \leq \chi^i(G) \leq \Delta^2 - \Delta + 1$. They also characterized the regular graphs which achieve the lower bound, and the graphs which attain the upper bound. In 2005, Doyon, Hahn, and Raspaud [19] presented the first results on injective colorings of planar graphs and later

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1 The manuscript was presented in 2005 and the paper appeared in journal in 2010.
Chen et al. [12] proved that for every $K_4$-minor free graph $G$, $\chi^i(G) \leq \lceil \frac{3}{2} \Delta \rceil$. In the same paper they also posed the first conjecture which was proven to be incorrect by Lužar and Škrekovski [37] who provided an infinite family of planar graphs with small maximum degree (between 4 and 7), or of even maximum degree, for which the original conjecture is false. Although, the original conjecture was supported by several results proving that in the case of planar graphs with girth at least 5, $\Delta + C$ colors are sufficient, where $C$ is a small constant (see, e.g., [1, 7, 16, 17, 38]). Moreover, Lužar and Škrekovski [37] proposed a new Wegner type conjecture.

**Conjecture 2** (Lužar and Škrekovski [37]). Let $G$ be a planar graph with maximum degree $\Delta$. Then,

$$\chi^i(G) \leq \begin{cases} 5, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \left\lceil \frac{3}{2} \Delta \right\rceil + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Note that since injective coloring is a relaxation of the 2-distance coloring, proving Wegner’s conjecture would prove Conjecture 2, except in the case of subcubic graphs, i.e., the class of graphs with maximum degree 3. Brimkov et al. [6] proved that 5 colors suffice for subcubic planar graphs with girth at least 6, but in general that case is still open. If true, then the conjectured upper bound for subcubic graphs is also tight (see, e.g., [37]). For the sake of completeness we present a table summarizing the latest known results regarding the injective chromatic number of planar graphs for different girth values. A somewhat different table was presented in 2017 by Brimkov et al. [6].

| $\chi^i(G)$ | $\Delta$ | $\Delta + 1$ | $\Delta + 2$ | $\Delta + 3$ | $\Delta + 4$ | $\Delta + 5$ | $\Delta + 6$ | $\Delta + 7$ |
|------------|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 3          | $\Delta \geq 4$ [37] | $\Delta \geq 5$ [37] | $\Delta \geq 6$ [38] | $\Delta \geq 7$ [12] | $\Delta \geq 8$ [37] | $\Delta \geq 9$ [15] | $\Delta \geq 10$ [16] | $\Delta \geq 11$ [8] |
| 4          | $\Delta \geq 5$ [38] | $\Delta \geq 6$ [38] | $\Delta \geq 7$ [37] | $\Delta \geq 8$ [38] | $\Delta \geq 9$ [12] | $\Delta \geq 10$ [38] | $\Delta \geq 11$ [8] | $\Delta \geq 12$ [38] |
| 5          | $\Delta \geq 6$ [38] | $\Delta \geq 7$ [37] | $\Delta \geq 8$ [38] | $\Delta \geq 9$ [12] | $\Delta \geq 10$ [38] | $\Delta \geq 11$ [8] | $\Delta \geq 12$ [38] | $\Delta \geq 13$ [8] |
| 6          | $\Delta \geq 7$ [17] | $\Delta \geq 8$ [10] | $\Delta \geq 9$ [10] | $\Delta \geq 10$ [15] | $\Delta \geq 11$ [17] | $\Delta \geq 12$ [16] | $\Delta \geq 13$ [18] | $\Delta \geq 14$ [18] |
| 7          | $\Delta \geq 8$ [11] | $\Delta \geq 9$ [11] | $\Delta \geq 10$ [16] | $\Delta \geq 11$ [18] | $\Delta \geq 12$ [18] | $\Delta \geq 13$ [18] | $\Delta \geq 14$ [18] | $\Delta \geq 15$ [18] |
| 8          | $\Delta \geq 9$ [11] | $\Delta \geq 10$ [15] | $\Delta \geq 11$ [17] | $\Delta \geq 12$ [18] | $\Delta \geq 13$ [18] |
| 9          | $\Delta \geq 10$ [15] | $\Delta \geq 11$ [18] | $\Delta \geq 12$ [18] |
| 10         | $\Delta \geq 11$ [18] | $\Delta \geq 12$ [18] |
| 11         | $\Delta \geq 12$ [18] |

Table 1: Summary of the latest results with a coefficient 1 before $\Delta$ in the upper bound of $\chi^i$ for different girth values while for the crossed out cases there exist counterexamples.

Table 1 reads as follows. For example, the result from line “7” and column “$\Delta$” states that every planar graph $G$ of girth at least 7 and maximum degree $\Delta(G) \geq 16$ satisfies $\chi^i(G) \leq \Delta(G)$. In the first column (“$\Delta$”), the first four crossed out cases follow from the graph in Figure 2. In this same column, row “10” with a crossed out value of $\Delta = 3$ corresponds to a construction of a planar graph $G$ with $\Delta(G) = 3$, girth 10, and $\chi^i(G) \geq \Delta + 1$ [38]. Similar results in the same vein in columns “$\Delta + 1$” to “$\Delta + 4$” in row “3” are presented in [12] and [37]. There exists a planar graph $G$ with girth 3 and $\chi^i(G) \geq \lceil \frac{3}{2} \Delta \rceil + 1$ for all $\Delta \geq 8$ [37], which justifies the remaining crossed out cases in row “3”. Similarly, there also exists a planar graph $G$ with girth 4 and $\chi^i(G) \geq \lceil \frac{3}{2} \Delta \rceil$ for all $\Delta \geq 3$ [38], which justifies the crossed out cases in row “4”. Every highlighted result without a reference in Table 1 is a part of our contribution in this paper. Note that the highlighted result in row “5” follows from the result in row “4”,

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Finally, the study of the exact distance $p$-powers of graphs was started by Simić [41] and exact $p$-distance colorings have first been studied for graphs of bounded expansion [39], see also [28]. This parameter received an increasing attention in the last decade, see [5, 21, 28, 40]. In [21], Foucaud et al. began the first systematic study of the exact square coloring (i.e., exact 2-distance coloring) with respect to the maximum degree. In their paper, they considered the exact square coloring for some specific classes of subcubic graphs. Both in the case of subcubic $K_4$-minor free graphs and subcubic planar bipartite graphs, they prove that 4 colors suffice in any exact square coloring. Moreover, they provide examples attaining this bound. Furthermore, they prove that Conjecture 2 holds for fullerene graphs, i.e., cubic planar graphs in which every face has size 5 or 6. Since for every graph $G$ with girth at least 4 we have equality between $\chi^#_2(G)$ and $\chi^1(G)$, all the results in Table 1, except for the row corresponding to girth at least 3, hold also for the exact square coloring.

In this paper we present some results regarding the 2-distance, injective, and exact square coloring of planar graphs of small girth (3 or 4), thus filling some gaps from the literature. We focus on graphs which have maximum degree 4, also known in the literature as subquartic graphs.

With respect to the 2-distance coloring, we consider planar graphs with maximum degree 4 and girth at least 4 to get the following result for 2-distance choosability number.

**Theorem 3.** If $G$ is a planar graph with $g(G) \geq 4$ and $\Delta(G) = 4$, then $\chi^2_2(G) \leq \Delta(G) + 7$.

In the case of the injective coloring, we prove that the same bound as in Theorem 3 also holds, but without any restrictions on the girth.

**Theorem 4.** If $G$ is a planar graph with $\Delta(G) = 4$, then $\chi^1_i(G) \leq \Delta(G) + 7$.

Furthermore, in the case when girth is at least 4, we improve the bound implied by Theorem 3 to $\Delta + 5$ colors.

**Theorem 5.** If $G$ is a planar graph with $g(G) \geq 4$ and $\Delta(G) = 4$, then $\chi^1_i(G) \leq \Delta(G) + 5$.

Finally, in the case of the exact square coloring, we improve the bound implied by Theorem 4 to $\Delta + 6$ colors.

**Theorem 6.** If $G$ is a planar graph with $\Delta(G) = 4$, then $\chi^#_2(G) \leq \Delta(G) + 6$.

The structure of the paper is organized as follows. In Section 2, we present notations and auxiliary results. The proofs of Theorems 3 to 6 are then provided in Sections 3 to 6. We conclude the paper with some additional remarks in Section 7.

## 2 Preliminaries

We denote by $F(G)$ the set of faces of a planar graph $G$. Given two vertices $u$ and $v$ of a graph $G$, and a set $S \subseteq V(G)$, we denote by $d_G(u, v)$ the distance between $u$ and $v$ in $G$, and define the distance between the vertex $v$ and the set $S$ as $d_G(v, S) := \min\{d_G(v, w) \mid w \in S\}$. For a vertex $v$ of $G$, we define the neighborhood, 2-distance neighborhood, and exact 2-distance neighborhood respectively as follows:
Finally, to prove that Equation (1). 

\[ \sum_{u \in V(G)} (d(u) - 4) + \sum_{f \in F(G)} (d(f) - 4) < 0. \]  

(1)

Finally, to prove that \( G \) does not exist, we will redistribute the charges while preserving the total sum and prove that the final charge of each vertex and face is non-negative, which will be a contradiction to Equation (1).
3 2-distance list-coloring of triangle-free planar graphs

In this section, we provide the proof of Theorem 3. Let $G$ be a minimal counterexample to Theorem 3, namely $G$ has maximum degree 4, girth at least 4, and $\chi_2^f(G) \geq 12$.

3.1 Structural properties of $G$

We start by proving that $G$ cannot be too sparse. More precisely, we have a lower bound on the minimum degree of $G$.

**Lemma 9.** The minimum degree of $G$ is at least 3.

**Proof.** If $G$ contains a 1-vertex $v$, then we can simply remove $v$ and color the resulting graph (with 11 colors), which is possible by minimality of $G$. Then, we add $v$ back and extend the coloring (at most 4 constraints and 11 colors).

If $G$ contains a vertex $u$ of degree 2 with neighbors $v$ and $w$ (see Figure 3), then let $H = G \setminus \{u\}$. We color $H + \{vw\}$ (resp. $H$) by minimality if $d_H(v, w) \geq 3$ (resp. $d_H(v, w) \leq 2$). Observe that in both cases, the girth (at least 4) and maximum degree (4) of the resulting graph are preserved. We extend such coloring to $G$ by coloring $u$ which sees only at most 8 different colors. \(\square\)

*Figure 3: Reducible configurations in Lemma 9.*

Along the same line, we prove that objects will a smaller neighborhood cannot be close together. Otherwise, $G$ would be colorable.

**Lemma 10.** Graph $G$ does not contain the following configurations:

(i) A 3-vertex incident to two 4-cycles.

(ii) A 3-vertex incident to a 4-cycle and adjacent to a 3-vertex.

(iii) A 3-vertex adjacent with two 3-vertices.

*Figure 4: Reducible configurations in Lemma 10.*

**Proof.** We reduce each configuration separately (see Figure 4) by precoloring a subgraph of $G$ and extending the coloring to $G$. 

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(i) Let $u$ be a 3-vertex incident to two 4-cycles. Let $e$ be the incident edge to $u$ that is also incident to both cycles. Color $G - \{e\}$ by minimality and uncolor $u$. Observe that $d^*_G(u) \leq 10$ so $u$ has at least one available color.

(ii) Let $u$ and $v$ be two adjacent 3-vertices and suppose that $u$ is incident to a 4-cycle. Color $G - \{uv\}$ and uncolor $u$ and $v$. Observe that $v$ has at least one and $u$ has at least two available colors.

(iii) Let $u$ be a 3-vertex with two 3-neighbors $v$ and $w$. Color $G - \{uv\}$ and uncolor $u$ and $v$. Observe that $v$ has at least one and $u$ has at least two available colors.

This concludes the proof. □

In a 2-distance coloring, 5-faces play an important role as they are right at the limit of being “big” enough objects that are not directly colorable, but “small” enough that a planar graph can contain only those. Thus, we turn our attention to configurations surrounding 5-faces.

**Lemma 11.** Let $f = v_1v_2v_3v_4v_5$ be a 5-face in $G$ such that $d(v_1) = d(v_3) = d(v_5) = 3$, and $v_2v_3$ is incident to a 4-face. Let $f' = v'_1v'_2v'_3v'_4v'_5$ be another 5-face in $G$ such that $d(v'_2) = d(v'_3) = d(v'_5) = 3$. Then, $v'_4 \neq v_4$ or $v'_5 \neq v_5$.

![Figure 5: Reducible configuration in Lemma 11.](image)

**Proof.** Suppose by contradiction that $v'_4 = v_4$ and $v'_5 = v_5$ (see Figure 5). First, observe that the vertices of $H = G[\{v_1, v_2, v_3, v_4, v_5, v'_1, v'_2, v'_3\}]$ are all distinct since $g(G) \geq 4$ and due to Lemma 10(ii). Color $G - \{v_4, v_5\}$ by minimality and uncolor the other vertices of $f$ and $f'$. Observe that the remaining list of colors for these vertices have size: $|L(v_1)| \geq 5$, $|L(v_2)| \geq 2$, $|L(v_3)| \geq 5$, $|L(v_4)| \geq 5$, $|L(v_5)| \geq 7$, $|L(v'_1)| \geq 2$, $|L(v'_2)| \geq 4$, and $|L(v'_3)| \geq 5$. Note that if for any pair of vertices $(u, v)$ in $V(H)$, if $d_G(u, v) \leq 2$ while $d_H(u, v) \geq 3$, then both $u$ and $v$ have at least one more remaining color.

We claim that $L(v_1) \cap L(v'_3) = \emptyset$ or $L(v_3) \cap L(v'_2) = \emptyset$. Suppose by contradiction that $L(v_1) \cap L(v'_3) \neq \emptyset$ and $L(v_3) \cap L(v'_2) \neq \emptyset$.

If $d_G(v_1, v'_3) \geq 3$. Then, we can color $v_1$ and $v'_3$ with the same color $c$, then we finish by coloring $v'_1$, $v_2$, $v'_3$, $v_3$, $v_4$, and $v_5$ in this order. This is a contradiction so $d_G(v_1, v'_3) \leq 2$. If $d_G(v_1, v'_3) = 1$, then $v_1$, $v'_3$, $v_4$, and $v_5$ form the configuration from Lemma 10(ii). So, $d_G(v_1, v'_3) = 2$.

If $d_G(v_3, v'_2) \geq 3$. Then, we can color $v_3$ and $v'_2$ with the same color $c$, then we finish by coloring $v'_1$, $v_2$, $v_1$, $v'_3$, $v_4$, and $v_5$ in this order. This is a contradiction so $d_G(v_3, v'_2) \leq 2$. If $d_G(v_3, v'_2) = 1$, then $v'_2$, $v'_3$, $v_4$, and $v_5$ form the configuration from Lemma 10(ii). So, $d_G(v_3, v'_2) = 2$.

As a result, by planarity, $v_1$, $v_3$, $v'_2$, and $v'_3$ must have a common neighbor $u$. However, $uv'_2v'_3$ is a triangle while $g(G) \geq 4$.

Consequently, we have $|L(v_1) \cup L(v'_3)| \geq 10$ or $|L(v'_2) \cup L(v_3)| \geq 9$. Moreover, note that $|L(v_1) \cup L(v'_3)| \geq 6$ and $|L(v_3) \cup L(v'_2)| \geq 6$. Indeed, these inequalities hold when $d_G(v_1, v'_3) \leq 2$ (resp. $d_G(v_3, v'_2) \leq 2$) as $|L(v_1)| \geq 6$ (resp. $|L(v_3)| \geq 6$). And when $d_G(v_1, v'_3) \geq 3$ (resp. $d_G(v_3, v'_2) \geq 3$), we also the same inequalities, otherwise $L(v_1)$ and $L(v'_3)$ (resp. $L(v_3)$ and $L(v'_2)$) will have a common color, in which case $G$ is colorable as seen previously.
Finally, $G$ is $L$-list-colorable by Corollary 8 as $|\bigcup_{u \in S} L(u)| \geq k$ for every subset $S$ of size $k$ in $V(H)$. \hfill \square

Lemma 12. Let $f = v_1v_2v_3v_4v_5$ be a 5-face in $G$ such that $d(v_1) = d(v_3) = d(v_5) = 3$, and $v_2v_3$ is incident to a 4-face. Let $f' = v'_1v'_2v'_3v'_4v'_5$ be another 5-face in $G$ such that $d(v'_1) = d(v'_3) = d(v'_5) = 3$, and $v'_2v'_3$ is incident to a 4-face. Then, $v'_1 \neq v_1$ or $v'_5 \neq v_5$.

![Figure 6: Reducible configuration in Lemma 12.](image)

Proof. Suppose by contradiction that $v'_1 = v_1$ and $v'_5 = v_5$ (see Figure 6). First, observe that the vertices of $H = G[\{v_1, v_2, v_3, v_4, v_5, v'_2, v'_3, v'_4\}]$ are all distinct since $g(G) \geq 4$ and due to Lemma 10(ii). Color $G - \{v_1, v_5\}$ by minimality and uncolor the other vertices of $f$ and $f'$. Observe that the remaining list of colors for these vertices have size: $|L(v_1)| \geq 7$, $|L(v_2)| \geq 3$, $|L(v_3)| \geq 4$, $|L(v_4)| \geq 2$, $|L(v_5)| \geq 7$, $|L(v'_2)| \geq 3$, $|L(v'_3)| \geq 4$, and $|L(v'_4)| \geq 2$. Note that if for any pair of vertices $(u, v)$ in $V(H)$, if $d_G(u, v) \leq 2$ while $d_H(u, v) \geq 3$, then both $u$ and $v$ have at least one more remaining color. We start with the following claims:

- $L(v_4) \cap L(v'_2) = \emptyset$ or $L(v'_4) \cap L(v_2) = \emptyset$. Suppose by contradiction that $L(v_4) \cap L(v'_2) \neq \emptyset$ and $L(v'_4) \cap L(v_2) \neq \emptyset$. Suppose w.l.o.g. that $d_G(v_4, v'_2) \geq 3$. Then, we can color $v_4$ and $v'_2$ with the same color $c$, then we finish by coloring $v'_4$, $v_2$, $v_3$, $v'_3$, $v_1$, and $v_5$ in this order. This is a contradiction so $d_G(v_4, v'_2) \leq 2$ and $d(v'_4, v_2) \leq 2$. If $d_G(v_4, v'_2) = 1$, then $v_1$, $v'_4$, $v_4$, and $v_5$ form the configuration from Lemma 10(ii). By symmetry, we get $d_G(v_4, v'_2) = d_G(v'_4, v_2) = 2$. As a result, by planarity, $v_2$, $v'_2$, $v_4$, and $v'_4$ must have a common neighbor $u$. However, $v_1, v_2, u,$ and $v'_2$ form the configuration from Lemma 10(ii).

As a consequence, we have $|L(v_4) \cup L(v'_2)| \geq 5$ or $|L(v'_4) \cup L(v_2)| \geq 5$.

- $L(v_4) \cap L(v'_3) = \emptyset$ or $L(v'_4) \cap L(v_3) = \emptyset$. Suppose by contradiction that $L(v_4) \cap L(v'_3) \neq \emptyset$ and $L(v'_4) \cap L(v_3) \neq \emptyset$. Suppose w.l.o.g. that $d_G(v_4, v'_3) \geq 3$. Then, we can color $v_4$ and $v'_3$ with the same color $c$, then we finish by coloring $v'_4$, $v_2$, $v_3$, $v'_3$, $v_1$, and $v_5$ in this order. This is a contradiction so $d_G(v_4, v'_3) \leq 2$ and $d_G(v'_4, v_3) \leq 2$. If $d_G(v_4, v'_3) = 1$, then $v_4$, $v_5$, $v'_4$, and $v'_3$ form the configuration from Lemma 10(ii). By symmetry, we get $d_G(v_4, v'_3) = d_G(v'_4, v_3) = 2$. As a result, by planarity, $v_3$, $v'_3$, $v_4$, and $v'_4$ must have a common neighbor $u$. However, $v_3v_4u$ is a triangle.

As a consequence, we have $|L(v_4) \cup L(v'_3)| \geq 5$ or $|L(v'_4) \cup L(v_3)| \geq 5$.

- $|L(v_2) \cup L(v_3) \cup L(v'_2) \cup L(v'_3)| \geq 8$. Suppose by contradiction that $|L(v_2) \cup L(v_3) \cup L(v'_2) \cup L(v'_3)| \leq 7$. Since $|L(v_3)| \geq 4$ and $|L(v'_3)| \geq 4$, we get $L(v_3) \cap L(v'_3) \neq \emptyset$. Suppose that $d_G(v_3, v'_3) \geq 3$. Then, we can color $v_3$ and $v'_3$ with the same color $c$. Since $L(v_4) \cap L(v'_3) = \emptyset$ or $L(v'_4) \cap L(v_3) = \emptyset$, w.l.o.g. we can color $v_4$ then $v'_4$. Since $L(v_4) \cap L(v'_3) = \emptyset$ or $L(v'_4) \cap L(v_3) = \emptyset$, w.l.o.g. we can color $v_2$ then $v'_2$. Finish by coloring $v_1$ and $v_5$. So, $d_G(v_3, v'_3) \leq 2$.

If $d_G(v_3, v'_3) = 1$, then let $w \notin \{v_2, v_4\}$ be the last neighbor of $v_3$, and we know that $v'_3 \in \{w, v_2, v_4\}$. We cannot have $v'_3 = v_2$ since $v_2v_2v'_2$ is a triangle. By symmetry, $v'_3 \neq v_4$. Finally, if $v'_3 = w$, then $v_3$ and $v'_3$ are adjacent 3-vertices lying on the same 4-cycle, which is impossible by Lemma 10(ii).
So, we have \( d(v_3, v'_3) = 2 \). In this case, observe that \(|L(v_3)| \geq 5\) and \(|L(v'_3)| \geq 5\). Recall that \(|L(v_2) \cup L(v_3) \cup L(v'_2) \cup L(v'_3)| \leq 7\). As a result, \( L(v_3) \cap L(v'_2) \neq \emptyset \) since \(|L(v_3)| \geq 5\) and \(|L(v'_2)| \geq 3\). The same holds for \( L(v'_3) \) and \( L(v_2) \). Suppose that \( d_G(v_3, v'_2) \geq 3\), then we can color \( v_3 \) and \( v'_2 \) with the same color \( c \). Since \( L(v_3) \cap L(v'_4) = \emptyset \) or \( L(v_4) \cap L(v'_2) = \emptyset \), w.l.o.g. we can color \( v_4 \) then \( v'_4 \). Finish by coloring \( v_2, v'_3, v_1 \) and \( v_5 \). So, \( d_G(v_3, v'_2) \leq 2\). By symmetry, the same holds for \( d_G(v'_3, v_2) \). By Lemma 10, \( d_G(v_3, v'_2) = d_G(v'_3, v_2) = 2\).

As a result, by planarity, \( v_2, v'_2, v_3, \) and \( v'_3 \) must have a common neighbor \( u \). However, \( uv_2v_3 \) is a triangle.

Due to the above claims, \( G \) is \( L \)-list-colorable by Corollary 8.

### 3.2 Discharging procedure

To prove that at least one of reducible configurations in \( G \) is unavoidable in a planar graph, we apply the following rules in the discharging procedure:

**R0** Every \( 5^+ \)-face \( f \) gives \( \frac{1}{3} \) to each incident 3-vertex that is not incident to a 4-face.

**R1** Every \( 5^+ \)-face \( f \) gives \( \frac{1}{2} \) to each incident 3-vertex that is incident to a 4-face.

**R2** Let \( f' = u_1u_2u_3u_4u_5 \) be a 5-face where \( d(u_1) = d(u_3) = d(u_4) = 3 \), \( u_1u_2 \) is incident to a 4-face and let \( f \) be incident to \( u_4u_5 \). If \( f \) is a \( 5^+ \)-face, then \( f \) gives \( \frac{1}{6} \) to \( f' \).

Figure 7: **R0**.

Figure 8: **R1**.

Figure 9: **R2**.

We are now ready to prove Theorem 3 using discharging procedure together with the structural properties of \( G \) proven in Section 3.1 and the discharging rules stated above.

**Proof of Theorem 3.** Let \( G \) be a minimal counterexample to the theorem. Let \( \mu(u) \) be the initial charge assignment for the vertices and faces of \( G \) with the charge \( \mu(u) = d(u) - 4 \) for each vertex \( u \in V(G) \), and \( \mu(f) = d(f) - 4 \) for each face \( f \in F(G) \). By Equation (1), we have that the total sum of the charges is negative.

Let \( \mu^* \) be the charge assignment after the discharging procedure. In what follows, we prove that:

\[
\forall x \in V(G) \cup F(G), \mu^*(x) \geq 0.
\]

First, we prove that the final charge on each vertex is non-negative. Let \( u \) be a vertex in \( V(G) \). Recall that \( \Delta(G) = 4 \) and \( \mu(u) = d(u) - 4 \). By Lemma 9, vertex \( u \) has degree at least 3, thus we consider the following two cases.

**Case 1:** \( d(u) = 4 \)

By **R0-R2**, \( u \) does not give any charge. So,

\[
\mu^*(u) = \mu(u) = d(u) - 4 = 0.
\]

**Case 2:** \( d(u) = 3 \)

Recall that \( \mu(u) = d(u) - 4 = -1 \) and we have the following cases:
• If $u$ is not incident to any 4-face, then it receives $\frac{1}{3}$ from each of the three incident $5^+$-faces (since $g(G) \geq 4$) by R0. So,

$$\mu^*(u) = -1 + 3 \cdot \frac{1}{3} = 0.$$

• If $u$ is incident to a 4-face, then it has exactly one incident 4-face due to Lemma 10(i). Therefore, $u$ is incident to two $5^+$-faces and receives $\frac{1}{2}$ from each by R1. So,

$$\mu^*(u) = -1 + 2 \cdot \frac{1}{2} = 0.$$

Secondly, we prove that the final charge on each face is non-negative. Let $f$ be a face in $F(G)$ and let $i_0$, $i_1$, and $i_2$ be respectively the number of times $f$ gives charge by R0, R1, and R2. Recall that $\mu(f) = d(f) - 4$. Moreover, $d(f) \geq 4$ since $g(G) \geq 4$, thus we consider the following three cases.

**Case 1: $d(f) \geq 6$**

By Lemma 10(ii), there are at most $\frac{2}{3}d(f)$ 3-vertices incident to $f$. As a result, we get $i_0 + i_1 \leq \frac{2}{3}d(f)$. Moreover, R2 can only be applied whenever R0 is applied since $u_4$ (from the statement of R2) is a 3-vertex that is not incident to a 4-face by Lemma 10(ii). Additionally, by Lemma 12, for each application of R0, R2 is applied at most once. In other words, $i_2 \leq i_0$. Finally,

$$\mu^*(f) \geq \mu(f) - \frac{1}{3}i_0 - \frac{1}{2}i_1 - \frac{1}{6}i_2$$

$$\geq \mu(f) - \frac{1}{3}i_0 - \frac{1}{2}i_1 - \frac{1}{6}i_0$$

$$\geq \mu(f) - \frac{1}{2}(i_0 + i_1)$$

$$\geq \mu(f) - \frac{1}{3}d(f)$$

$$\geq \frac{2}{3}d(f) - 4$$

$$\geq 0$$

since $d(f) \geq 6$.

**Case 2: $d(f) = 5$**

Recall that $\mu(f) = d(f) - 4 = 1$. Observe that we have the following inequalities:

• $i_0 + i_1 \leq 3$ and $i_2 \leq i_0$ (as in the previous case).

• $i_1 \leq 2$. Indeed, by Lemma 10(ii), the 3-vertex in R1 must be adjacent to only 4-vertices. As a result, if we have equality, then $f$ is incident to exactly two 3-vertices.

• $i_2 \leq 2$. Indeed, by Lemma 10(iii), the neighbors of $u_4$ (as in the statement of R2) that are incident to $f$ must be 4-vertices. As a result, if we have equality, then $f$ is incident to exactly two 3-vertices.

Recall that $f$ gives $\frac{1}{3}i_0 + \frac{1}{2}i_1 + \frac{1}{6}i_2$ by R0, R1, and R2. If $i_0 \leq 1$, $i_1 \leq 1$, and $i_2 \leq 1$, then

$$\mu^*(f) \geq 1 - \frac{1}{3} - \frac{1}{2} - \frac{1}{6} = 0.$$

If $i_2 \geq 2$, then $i_2 = 2$ since $i_2 \leq 2$ and $f$ is incident to exactly two 3-vertices. Thus, $i_0 + i_1 \leq 2$. Moreover, since $i_2 \leq i_0$, we also have $i_0 = 2$ and $i_1 = 0$. So,

$$\mu^*(f) \geq 1 - 2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} = 0.$$
If \( i_1 \geq 2 \), then \( i_1 = 2 \) since \( i_1 \leq 2 \) and \( f \) is incident to exactly two 3-vertices. Thus, \( i_0 + i_1 \leq 2 \). Therefore, \( i_0 = 0 \). Moreover, since \( i_2 \leq i_0 \), we also have \( i_2 = 0 \). So,

\[
\mu^*(f) \geq 1 - 2 \cdot \frac{1}{2} = 0.
\]

If \( i_0 \geq 3 \), then \( i_0 = 3 \) and \( i_1 = 0 \) since \( i_0 + i_1 \leq 3 \). Let \( f = v_1v_2v_3v_4v_5 \). W.l.o.g. we have \( d(v_1) = d(v_2) = d(v_3) = 3 \) and \( d(v_4) = d(v_5) = 4 \). Observe that \( f \) cannot give charge through \( v_5v_1, v_1v_2, \) or \( v_2v_3 \) by \textbf{R2} due to Lemma 10(i). If it gives through \( v_4v_5 \), then we have Lemma 11, which is a contradiction. The same holds for \( v_4v_3 \) by symmetry. As a result, \( i_2 = 0 \). So,

\[
\mu^*(f) \geq 1 - 3 \cdot \frac{1}{3} = 0.
\]

If \( i_0 = 2 \), then we have the following two cases since \( i_0 + i_1 \leq 3 \).

- If \( i_1 = 0 \), then we already know that \( i_2 \leq i_0 \leq 2 \). So,
  \[
  \mu^*(f) \geq 1 - 2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{6} = 0.
  \]

- If \( i_1 = 1 \), then let \( f = u_1u_2u_3u_4u_5 \) W.l.o.g. we have \( d(u_1) = d(u_3) = d(u_4) = 3 \) and \( d(u_2) = d(u_5) = 4 \). For \( 1 \leq i \leq 5 \), let \( f_i \neq f \) be the face incident to \( u_iu_{i+1(mod \ 5)} \). Observe that neither \( u_3 \) nor \( u_4 \) can be incident to a 4-face due to Lemma 10(ii) so \( f_2, f_3, \) and \( f_4 \) must be \( 5^+ \)-faces. Moreover, we can assume w.l.o.g. that \( f_1 \) is a 4-face. In such a case, \( f \) receives \( \frac{1}{5} \) from \( f_4 \) by \textbf{R2}. Additionally, \( f \) cannot give charge by \textbf{R2} to neither \( f_1 \) nor \( f_5 \) due to Lemma 10(ii). Furthermore, \( f \) cannot gives to \( f_2, f_3, \) nor \( f_4 \) by \textbf{R2} due to Lemma 10(iii). As a result, \( i_2 = 0 \). So,
  \[
  \mu^*(f) \geq 1 + \frac{1}{6} - 2 \cdot \frac{1}{3} - \frac{1}{2} = 0.
  \]

It follows that after the discharging procedure 5-faces have non-negative charge.

**Case 3:** \( d(f) = 4 \)

Recall that \( \mu(f) = d(f) - 4 = 0 \). Since \( f \) does not give any charge, we have

\[
\mu^*(f) = \mu(f) = 0.
\]

To conclude the proof, after the discharging procedure, which preserved the total sum, we end up with a non-negative total sum, a contradiction to Equation (1). \( \square \)

## 4 Injective list-coloring of planar graphs

In this section, we provide the proof to Theorem 4. Let \( G \) be a minimal counterexample to Theorem 4 with the fewest number of vertices plus edges. More precisely, \( G \) has maximum degree 4 and \( \chi'_L(G) \geq 12 \).

### 4.1 Structural properties of \( G \)

We follow the same ideas as in the previous proof, starting by bounding the minimum degree of \( G \) and proving that “smaller” objects are far away from each other. Recall that for injective coloring, two vertices see each other only when they share a neighbor.

**Lemma 13.** The minimum degree of \( G \) is at least 3.

**Proof.** Let \( u \) be a \( 2^- \)-vertex in \( G \). If \( u \) is a 1-vertex, then we color \( G - \{u\} \), and we extend the coloring to \( u \) by using one of the at least 8 remaining colors. On the other hand, if \( u \) is a 2-vertex and \( v \) one of its neighbors (see Figure 10), then we color \( G - \{u\} \) and uncolor \( v \). Observe that \( u \) has at least five and \( v \) has at least one available color. Thus, we first color \( v \) and then \( u \) to complete the coloring. \( \square \)
Lemma 14. Graph $G$ does not contain the following configurations:

(i) Two adjacent 3-vertices.

(ii) A 3-vertex incident to a 4-cycle.

(iii) A 3-vertex at distance 1 from two adjacent 3-cycles.

(iv) A 4-vertex incident to two adjacent 3-cycles and another 4-cycle.

Proof. We give a proof for each configuration separately (see Figure 11).

(i) Suppose by contradiction that there exist two adjacent 3-vertices $u$ and $v$. Color $G - \{uv\}$ by minimality and uncolor the vertices $u$ and $v$. Observe that each of them can see at most 8 colors. Thus, $u$ and $v$ are colorable.

(ii) Suppose by contradiction that there exists a 3-vertex $u$ that is incident to a 4-cycle $C$. Let $v$ be a vertex incident to $C$ and adjacent with $u$. Color $G - \{uv\}$ by minimality and uncolor the vertices $u$ and $v$. Observe that $|L(u)| \geq 2$ and $|L(v)| \geq 1$. Thus, $u$ and $v$ are colorable.

(iii) Let $uvw$ and $wxv$ be 3-cycles where $x \neq w$. First, suppose by contradiction that there exists a 3-vertex $y$ that is at distance 1 from $G[\{u,v,w,x\}]$. If $y$ is adjacent to $u$, color $G - \{uy\}$ by minimality and uncolor the vertices $u$ and $y$. Observe that $|L(u)| \geq 1$ and $|L(y)| \geq 2$. Thus, $u$ and $y$ are colorable. Now, suppose that $y$ is adjacent to $x$ instead. In this case, color $G - \{xy\}$ by minimality and uncolor the vertices $x$ and $y$. Observe that $|L(x)| \geq 1$ and $|L(y)| \geq 2$. Thus, $x$ and $y$ are colorable.
(iv) Let \( u \) be a 4-vertex that is incident to two adjacent 3-cycles \( uvw \) and \( uwx \) where \( x \neq w \). First, suppose by contradiction that \( u \) is incident to another 3-cycle. If there exists \( y \) such that \( y \notin \{x, w\} \) and \( wvy \) is a 3-cycle, then color \( G - \{uy\} \) by minimality and uncolor the vertices \( u \) and \( y \). Observe that \( |L(u)| \geq 2 \) and \( |L(y)| \geq 1 \). Thus, \( u \) and \( y \) are colorable. If \( wxy \) is a 3-cycle with \( y \notin \{v, w\} \), then the same exact arguments also hold in this case. Now, suppose that \( u \) is incident to a 4-cycle. Color \( G - \{uv\} \) by minimality and uncolor the vertices \( u \) and \( v \). Observe that \( |L(u)| \geq 2 \) and \( |L(v)| \geq 1 \). Thus, \( u \) and \( v \) are colorable.

This concludes the proof. \( \square \)

**Lemma 15.** A 3-cycle in \( G \) is not adjacent with a 4-cycle.

![Figure 12: Reducible configurations in the proof of Lemma 15.](image)

**Proof.** To prove Lemma 15, we will start by proving that the configurations in Figure 12 are reducible and observe that Lemma 15 corresponds to Figure 12(v). First, observe that the vertices in \( S = \{u_1, u_2, u_3, v_1, v_2, v_3, w, x\} \) are pairwise distinct due to Lemma 14(iv).

(i) Suppose first that \( G \) contains a configuration shown in Figure 12(i). Color \( G - \{u_2, u_3\} \) and uncolor the remaining vertices in \( S \). The remaining list of colors for the non-colored vertices have size: \( |L(u_1)| \geq 6 \), \( |L(u_2)| \geq 8 \), \( |L(u_3)| \geq 8 \), \( |L(v_1)| \geq 6 \), \( |L(v_2)| \geq 6 \), \( |L(v_3)| \geq 6 \), \( |L(x)| \geq 4 \), and \( |L(w)| \geq 4 \). Observe that \( u_1 \) cannot be adjacent to \( w \) as \( u_1, u_2, u_3, u_4, v_1, \) and \( w \) would form the configuration in Lemma 14(iv). The same holds for \( u_4 \) and \( w \). Moreover, \( v_1 \) and \( v_3 \) cannot both be adjacent to \( x \) as \( v_3, u_3, v_2, x, v_1, \) and \( w \) would form the configuration in Lemma 14(iv). Due to the previous observations and due to planarity, either \( u_1 \) or \( u_4 \) do not share a common neighbor with \( x \), say \( u_1 \) does not. In this case, we show that \( L(u_1) \cap L(x) = \emptyset \). Otherwise, we can color \( u_1 \) and \( x \) with the same color and finish the coloring by coloring \( w, v_1, v_3, u_3, u_2, \) and \( u_4, u_2, \) and \( u_3, \) in this order. Consequently, \( |L(u_1) \cup L(x)| \geq 9 \). But then, \( G \) is \( L \)-list-colorable by Corollary 8.

(ii) Suppose first that \( G \) contains a configuration shown in Figure 12(ii). Color \( G - \{u_2, u_3\} \) and uncolor the other vertices. The remaining list of colors for these vertices have size: \( |L(u_1)| \geq 6 \), \( |L(u_2)| \geq 7 \), \( |L(u_3)| \geq 7 \), \( |L(u_4)| \geq 6 \), \( |L(v_1)| \geq 5 \), \( |L(v_2)| \geq 5 \), \( |L(v_3)| \geq 5 \), and \( |L(w)| \geq 3 \). Due to (i), \( v_2 \) and \( w \) cannot share a neighbor. We claim that \( L(v_2) \cap L(w) = \emptyset \). Otherwise, we can
color $v_2$ and $w$ with the same color and finish the coloring in this order: $v_1$, $v_3$, $u_1$, $u_4$, $u_2$, and $u_3$. Consequently, $|L(v_2) \cup L(w)| \geq 8$. By Corollary 8, this configuration is reducible.

(iii) Suppose first that $G$ contains a configuration shown in Figure 12(iii). Color $G - \{u_2, u_3\}$ and uncolor the other vertices. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 5$, $|L(u_2)| \geq 6$, $|L(u_3)| \geq 6$, $|L(u_4)| \geq 5$, $|L(v_1)| \geq 4$, $|L(v_2)| \geq 5$, and $|L(v_3)| \geq 4$. Due to (ii), $v_1$ and $v_3$ cannot share a neighbor. We claim that $L(v_1) \cap L(v_3) = \emptyset$. Otherwise, we can color $v_1$ and $v_3$ with the same color and finish the coloring in this order: $u_1$, $u_4$, $v_2$, $u_2$, and $u_3$. Consequently, $|L(v_1) \cup L(v_3)| \geq 8$. By Corollary 8, this configuration is reducible.

(iv) Suppose first that $G$ contains a configuration shown in Figure 12(iv). Color $G - \{u_2, u_3\}$ and uncolor the other vertices. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 4$, $|L(u_2)| \geq 5$, $|L(u_3)| \geq 4$, $|L(u_4)| \geq 3$, $|L(v_1)| \geq 4$, and $|L(v_2)| \geq 4$. Due to (iii), $u_3$ and $u_4$ cannot share a neighbor. We claim that $L(u_3) \cap L(u_4) = \emptyset$. Otherwise, we can color $u_3$ and $u_4$ with the same color and finish the coloring in this order: $v_1$, $v_2$, $u_1$ and $u_2$. Consequently, $|L(u_3) \cup L(u_4)| \geq 7$. By Corollary 8, this configuration is reducible.

(v) Finally, suppose first that $G$ contains a configuration shown in Figure 12(v). Color $G - \{u_2, u_3\}$ and uncolor the other vertices. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 2$, $|L(u_2)| \geq 3$, $|L(u_3)| \geq 3$, $|L(u_4)| \geq 2$, and $|L(v_2)| \geq 3$. Due to (iv), $u_1$ and $u_2$ cannot share a neighbor. The same holds for $u_3$ and $u_4$. We claim that $L(u_1) \cap L(u_2) = \emptyset$. Otherwise, we can color $u_1$ and $u_2$ with the same color and finish the coloring in this order: $u_4$, $v_2$, and $u_3$. Consequently, $|L(u_1) \cup L(u_2)| \geq 5$. By symmetry, the same holds for $L(u_3)$ and $L(u_4)$. Thus, a 3-cycle adjacent to a 4-cycle in $G$ is reducible by Corollary 8.

This concludes the proof.

We now look at adjacent 3-cycles.

**Lemma 16.** If $uvw$ and $uvx$ are two adjacent 3-cycles in $G$ with $x \neq w$, then $x$ is not incident to another 3-cycle $xyz$ with $y, z \notin \{u, v, w\}$.

![Figure 13: Reducible configurations in the proof of Lemma 16.](image)

**Proof.** To prove Lemma 16, we will start by proving that the configurations in Figure 13 are reducible and observe that Lemma 16 corresponds to Figure 13(iii). First, observe that the vertices in $\{t, u, v, w, x, y, z\}$ are pairwise distinct due to Lemma 14(iv).

(i) Color $G - \{ux\}$ and uncolor $u$ and $x$. Observe that $|L(u)| \geq 1$ and $|L(x)| \geq 2$. Thus, $u$ and $x$ are colorable.

(ii) First, observe that $s$ is distinct from all other vertices due to (i) and Lemma 14(iv). Moreover, $v$ cannot be adjacent to $s$ due to Lemma 14(iv). The same holds for $u$ and $s$. Also, $s$ cannot be adjacent to $t$ since $z, y, t, s$, and $x$ would form the configuration in Lemma 15. Now, color $G - \{x\}$ and uncolor the other vertices. The remaining list of colors for these vertices have
size: $|L(u)| \geq 7$, $|L(v)| \geq 7$, $|L(w)| \geq 5$, $|L(x)| \geq 7$, $|L(y)| \geq 5$, $|L(z)| \geq 5$, $|L(s)| \geq 4$, and $|L(t)| \geq 4$. Due to the previous observations, $w$ and $s$ cannot share a neighbor. We claim that $L(w) \cap L(s) = \emptyset$. Otherwise, we can color $w$ and $s$ with the same color and finish the coloring in this order: $t$, $y$, $z$, $u$, $v$, and $x$. Consequently, $|L(w) \cup L(s)| \geq 9$. By Corollary 8, this configuration is reducible.

(iii) Color $G - \{x\}$ and uncolor the other vertices. The remaining list of colors for these vertices have size: $|L(u)| \geq 5$, $|L(v)| \geq 5$, $|L(w)| \geq 3$, $|L(x)| \geq 5$, $|L(y)| \geq 3$, and $|L(z)| \geq 3$. Due to (ii), either $w$ and $y$, or $w$ and $z$ cannot share a neighbor, say $w$ and $y$. We claim that $L(w) \cap L(y) = \emptyset$. Otherwise, we can color $w$ and $y$ with the same color and finish the coloring in this order: $z$, $u$, $v$, and $x$. Consequently, $|L(w) \cup L(y)| \geq 6$. By Corollary 8, this configuration is reducible.

This concludes the proof.\qed

Similar to 2-distance coloring, the constraint at distance 2 in an injective coloring also naturally pushes us to look at configurations around 5-faces.

**Lemma 17.** Let $f = u_1u_2u_3u_4u_5$ be a 5-face in $G$ and let $u_4u_5v_4$ and $u_5v_4w$ be two distinct 3-faces. Then, $f$ is adjacent to at most one other 3-face.

![Figure 14: Reducible configuration in Lemma 17.](image)

**Proof.** Suppose by contradiction that $f$ is adjacent to at least two other 3-faces. In such a case, the only possible configuration must be the one in Figure 14 due to Lemma 14(iv) and Lemma 16. First, observe that the vertices in $S = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_4, w\}$ are pairwise distinct due to Lemma 14(ii, iv), Lemma 15, and Lemma 16. Color $G - \{u_1, u_2, u_3, u_4, u_5\}$ and uncolor $\{v_1, v_2, v_4, w\}$. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 6$, $|L(u_2)| \geq 5$, $|L(u_3)| \geq 5$, $|L(u_4)| \geq 6$, $|L(u_5)| \geq 6$, $|L(v_1)| \geq 4$, $|L(v_2)| \geq 4$, $|L(v_3)| \geq 5$, and $|L(w)| \geq 4$.

Observe that $v_4$ cannot be adjacent to $v_2$ nor $v_1$ due to Lemma 15. Thus, $u_2$ and $v_4$ cannot share a neighbor. If $L(u_2) \cap L(v_4) = \emptyset$, then we color $u_2$ and $v_4$ with the same color. Now, we color $w$, $v_1$, and $v_2$. Observe that each of the remaining list of colors for $u_1, u_3, u_4,$ and $u_5$ has size at least 2 as they each see at most 4 different colors and if $u_3$ sees $w$ then $|L(u_3)| \geq 6$. Moreover, $u_1$ (resp. $u_3$) cannot see $u_5$ (resp. $u_4$) by Lemma 14(iv) (resp. Lemma 16). Since each of the remaining four vertices sees exactly two others, we can finish the coloring by the 2-choosability of even cycles. Thus, we must have $L(u_2) \cap L(v_4) = \emptyset$.

We distinguish the following cases.

**Case 1:** $v_1$ shares a neighbor with $u_4$

Let $u$ be their common neighbor. We get $u \neq v_4$ (resp. $u_3$) by Lemma 15 (resp. Lemma 14(iv)). We also have $u \neq u_5$ since vertices of $S$ are all distinct. As a result, $v_2$ cannot share a neighbor with $u_5$ by planarity. Moreover, if $w$ sees $u_3$, then their common neighbor must be $u$ by planarity. However, $w$, $v_4$, $u_4$, $u$, and $u_5$ form the configuration in Lemma 15. Thus, $w$ cannot share a neighbor with $u_3$.

- If $L(u_3) \cap L(w) \neq \emptyset$, color $u_3$ and $w$ with the same color $c$. Let $L'(x)$ be the remaining list of colors for $x \in S \setminus \{u_3, w\}$.
Lemma 18. If $L'(u_5) \cap L'(v_2) \neq \emptyset$, then color $u_5$ and $v_2$ with the same color $c'$ and complete the coloring by Corollary 8 as $L(u_2) \cap L(v_4) = \emptyset$ (as proven above).

If $L'(u_5) \cap L'(v_2) = \emptyset$, then we complete the coloring of $S \setminus \{u_3, w\}$ by Corollary 8, which is possible since $|L'(u_2) \cup L'(v_4)| \geq 8$ and $|L'(u_5) \cup L'(v_2)| \geq 8$.

- If $L(u_4) \cap L(w) = \emptyset$, then $|L(u_3) \cup L(w)| \geq 9$.
  - If $L(u_3) \cap L(v_2) \neq \emptyset$, then color $u_5$ and $v_2$ with the same color $c$ and complete the coloring by Corollary 8 as $L(u_2) \cap L(v_4) = \emptyset$ and $L(u_3) \cap L(w) = \emptyset$.
  - If $L(u_3) \cap L(v_2) = \emptyset$, then we complete the coloring of $S$ by Corollary 8, which is possible since $|L(u_5) \cup L(v_2)| \geq 10$, $|L(u_3) \cup L(w)| \geq 9$, and $|L(u_2) \cup L(v_4)| \geq 10$.

Case 2: $v_2$ shares a neighbor with $u_5$.

Let $u$ be their common neighbor. We get $u \neq u_1$ (resp. $u_4$, $v_4$) by Lemma 14(iv) (resp. Lemma 16, Lemma 15). So, $u = w$. Observe that in this case, $|L(u_1)| \geq 6$, $|L(u_2)| \geq 6$, $|L(u_3)| \geq 6$, $|L(u_4)| \geq 6$, $|L(u_5)| \geq 7$, $|L(v_1)| \geq 4$, $|L(v_2)| \geq 6$, $|L(v_4)| \geq 6$, and $|L(w)| \geq 6$.

- If $L(u_4) \cap L(v_1) \neq \emptyset$, then color $u_4$ and $v_1$ with the same color $c$ and complete the coloring by Corollary 8 as $L(u_2) \cap L(v_4) = \emptyset$.
- If $L(u_4) \cap L(v_1) = \emptyset$, then we complete the coloring of $S$ by Corollary 8, which is possible since $|L(u_4) \cup L(v_1)| \geq 10$ and $|L(u_2) \cup L(v_4)| \geq 12$.

Case 3: $v_1$ does not share a neighbor with $u_4$ and $v_2$ does not share a neighbor with $u_5$.

- If $L(u_4) \cap L(v_1) \neq \emptyset$, then color $u_4$ and $v_1$ with the same color $c$. Let $L'(x)$ be the remaining list of colors for $x \in S \setminus \{u_4, v_1\}$.
  - If $L'(u_5) \cap L'(v_2) \neq \emptyset$, then color $u_5$ and $v_2$ with the same color $c'$ and complete the coloring by Corollary 8 as $L(u_2) \cap L(v_4) = \emptyset$.
  - If $L'(u_5) \cap L'(v_2) = \emptyset$, then we complete the coloring by Corollary 8, which is possible since $|L'(u_2) \cup L'(v_4)| \geq 8$ and $|L'(u_5) \cup L'(v_2)| \geq 8$.
- If $L(u_4) \cap L(v_1) = \emptyset$, then we complete the coloring of $S$ by Corollary 8, which is possible since $|L(u_4) \cup L(v_1)| \geq 10$ and $|L(u_2) \cup L(v_4)| \geq 12$.
  - If $L(u_3) \cap L(v_2) \neq \emptyset$, then color $u_5$ and $v_2$ with the same color $c$ and complete the coloring by Corollary 8 as $L(u_2) \cap L(v_4) = \emptyset$ and $L(u_4) \cap L(v_1) = \emptyset$.
  - If $L(u_3) \cap L(v_2) = \emptyset$, then we complete the coloring by Corollary 8, which is possible since $|L(u_2) \cup L(v_4)| \geq 10$, $|L(u_4) \cup L(v_1)| \geq 10$, and $|L(u_5) \cup L(v_2)| \geq 10$.

This concludes the proof.

Lemma 18. A 5-face in $G$ is not incident to a 3-vertex and adjacent with three 3-faces.

Figure 15: Reducible configuration in Lemma 18.
Proof. Suppose to the contrary that such a configuration exists in $G$. In which case, it must be the one in Figure 15 due to Lemma 14(ii). First, observe that the vertices in $S = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3\}$ are pairwise distinct due to Lemma 14(iv) and Lemma 16. Color $G - \{u_1, u_2, u_3, u_4, u_5\}$ and uncolor $\{v_1, v_2, v_3\}$. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 5$, $|L(u_2)| \geq 6$, $|L(u_3)| \geq 6$, $|L(u_4)| \geq 6$, $|L(v_1)| \geq 6$, $|L(v_2)| \geq 4$, $|L(v_3)| \geq 5$, and $|L(v_3)| \geq 4$. Observe that $u_1$ and $u_5$ cannot share a neighbor due to Lemma 14(ii). The same holds for $u_4$ and $u_5$.

Moreover, we claim that either $v_1$ and $u_4$, or $v_3$ and $u_1$ do not share a neighbor. Indeed, say $v_1$ and $u_4$ share a neighbor $w$, and $v_3$ and $u_1$ share a neighbor $w'$. Note that $w$ is distinct from $u_1$ by Lemma 14(ii), from $u_2$ and $u_3$ since vertices of $S$ are pairwise distinct, from $v_2$ by Lemma 14(iv), and from $v_3$ by Lemma 15. By symmetry and planarity, $w = w'$. However, this is impossible since $w$, $u_1$, $u_5$, $u_4$, and $v_1$ form the configuration in Lemma 15.

Now, we can assume w.l.o.g. that $v_1$ and $u_4$ do not share a neighbor. In the following cases, we argue that $G$ is always colorable.

- If $L(v_1) \cap L(u_4) \neq \emptyset$, then color $v_1$ and $u_4$ with the same color $c$. Now, let $L'(x)$ be the remaining list of colors for $x \in S \setminus \{v_1, u_4\}$.
  
  - If $L'(u_1) \cap L'(u_5) \neq \emptyset$, then color $u_1$ and $u_5$ with the same color (recall that they do not share a common neighbor) and finish by coloring $v_2$, $v_3$, $u_2$, and $u_3$ in this order.
  
  - If $L'(u_1) \cap L'(u_5) = \emptyset$, then $|L'(u_1) \cup L'(u_5)| \geq 9$. In this case, we conclude by coloring $S \setminus \{v_1, u_4\}$ due to Corollary 8.

- If $L(v_1) \cap L(u_4) = \emptyset$, then $|L(v_1) \cup L(u_4)| \geq 9$. Now, we distinguish the two following cases.
  
  - If $L(u_1) \cap L(u_5) \neq \emptyset$, then color $u_1$ and $u_5$ with the same color $c$. Now, let $L'(x)$ be the remaining list of colors for $x \in S \setminus \{u_1, u_5\}$. Since $|L(v_1) \cup L(u_4)| \geq 9$, we have $|L'(v_1) \cup L'(u_4)| \geq 7$. Thus, we conclude by coloring $S \setminus \{u_1, u_5\}$ due to Corollary 8.
  
  - If $L(u_1) \cap L(u_5) = \emptyset$, then $|L(u_1) \cup L(u_5)| \geq 11$. Since we also have $|L(v_1) \cup L(u_4)| \geq 9$, we can conclude by coloring $S$ due to Corollary 8.

This concludes the proof.

\[\square\]

**Lemma 19.** A 5-face in $G$ is not adjacent with five 3-faces.

![Figure 16: Reducible configuration in Lemma 19.](image)

\[\begin{proof}
Suppose by contradiction that there exists a 5-face $f = u_1u_2u_3u_4u_5$ that is adjacent to five 3-faces like in Figure 16. First, observe that the vertices in $S = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$ are pairwise distinct due to Lemma 14(iv) and Lemma 16. Color $G - \{u_1, u_2, u_3, u_4, u_5\}$ and uncolor $\{v_1, v_2, v_3, v_4, v_5\}$. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 7$, $|L(u_2)| \geq 7$, $|L(u_3)| \geq 7$, $|L(u_4)| \geq 7$, $|L(v_1)| \geq 6$, $|L(v_2)| \geq 5$, $|L(v_3)| \geq 5$, $|L(v_4)| \geq 5$, and
\end{proof}\]
\(|L(v_5)| \geq 5\). Observe that, for \(1 \leq i \leq 5\), and \(v_i\) and \(v_{i+2(\text{mod}\ 5)}\) cannot be adjacent due to Lemma 15. By symmetry, the same holds for \(v_i\) and \(v_{i+3(\text{mod}\ 5)}\). As a result, \(v_i\) and \(u_{i+3(\text{mod}\ 5)}\) cannot share a neighbor for \(1 \leq i \leq 5\). We have the following cases.

- If there exists \(1 \leq i \leq 5\) such that \(L(v_i) \cap L(u_{i+3(\text{mod}\ 5)}) \neq \emptyset\), say \(i = 1\), then color \(v_1\) and \(u_4\) with the same color \(c\). Now, let \(L'(x)\) be the remaining list of colors for \(x \in S \setminus \{v_1, u_4\}\).
  
  - If there exists \(2 \leq i \leq 5\) such that \(L'(v_i) \cap L'(u_{i+3(\text{mod}\ 5)}) \neq \emptyset\), say \(i = 2\), then color \(v_2\) and \(u_5\) with the same color \(c'\) and let \(L''(x)\) be the remaining list of colors for \(x \in S \setminus \{v_1, v_2, u_4, u_5\}\).
    
    * If \(L''(v_3) \cap L''(u_1) \neq \emptyset\), then color \(v_3\) and \(u_1\) with the same color \(c''\) and finish by coloring \(v_4, v_5, u_2,\) and \(u_3\).
    
    * If \(L''(v_3) \cap L''(u_1) = \emptyset\), then \(|L''(v_3) \cup L''(u_1)| \geq 8\). We complete the coloring of \(S \setminus \{v_1, v_2, u_4, u_5\}\) by Corollary 8.

- If, for every \(2 \leq i \leq 5\), \(L'(v_i) \cap L'(u_{i+3(\text{mod}\ 5)}) = \emptyset\), then \(|L'(v_i) \cup L'(u_{i+3(\text{mod}\ 5)})| \geq 10\). In such a case, we can conclude by coloring \(S \setminus \{v_1, u_4\}\) due to Corollary 8.

- If, for every \(1 \leq i \leq 5\), \(L(v_i) \cap L(u_{i+3(\text{mod}\ 5)}) = \emptyset\), then \(|L(v_i) \cup L(u_{i+3(\text{mod}\ 5)})| \geq 12\). In such a case, we can conclude by coloring \(S\) due to Corollary 8.

It follows that a 5-face in \(G\) is adjacent with at most four 3-faces, thus concluding the proof.

For conciseness, we introduce a useful definition when considering 5-faces.

**Bad faces:** We call a 5-face bad if it is adjacent to four 3-faces (see Figure 17).

Observe that bad 5-faces are adjacent to exactly four 3-faces due to Lemma 19.

![Figure 17: A bad 5-face.](image)

We now turn our attention to bad 5-faces.

**Lemma 20.** A bad 5-face in \(G\) is not adjacent with a 4-face.

![Figure 18: Reducible configuration in Lemma 20.](image)
Proof. Suppose by contradiction that there exists a 5-face \( f = u_1u_2u_3u_4u_5 \) that is adjacent to four 3-faces and one 4-face like in Figure 18. First, observe that the vertices in \( \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, w_2, v_3, v_4, v_5\} \) are pairwise distinct due to Lemma 14(iv), Lemma 15, and Lemma 16. Color \( G - \{u_1, u_2, u_3, u_4\} \) and uncolor \( \{u_5, v_1, v_2, w_2, v_3\} \). Let \( S = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, w_2, v_3\} \). The remaining list of colors for these vertices have size: \(|L(u_1)| \geq 5, |L(u_2)| \geq 6, |L(u_3)| \geq 6, |L(u_4)| \geq 5, |L(u_5)| \geq 5, |L(v_1)| \geq 4, |L(v_2)| \geq 3, |L(v_3)| \geq 3, \) and \(|L(v_5)| \geq 4.\)

Observe that \( v_1 \) and \( v_3 \) (resp. \( v_4 \)) cannot be adjacent due to Lemma 15. The same holds for \( v_3 \) and \( v_5 \). As a result, \( v_1 \) and \( u_4 \) (resp. \( v_3 \) and \( u_1 \)) cannot share a neighbor.

Similarly, \( v_2 \) (resp. \( w_2 \)) cannot be adjacent to \( v_5 \) (resp. \( v_4 \)) by Lemma 15. As a result, if \( v_2 \) (resp. \( w_2 \)) share a neighbor with \( u_5 \), then \( v_2 \) must be adjacent to \( v_4 \) (resp. \( v_3 \)). By planarity and by symmetry, we can assume w.l.o.g. that \( u_5 \) cannot share a neighbor with \( v_2 \).

We distinguish the following cases.

- If \( L(v_1) \cap L(u_4) \neq \emptyset \), then color \( v_1 \) and \( u_4 \) with the same color \( c \). Now, let \( L'(x) \) be the remaining list of colors for \( x \in S - \{v_1, u_4\} \).
  
  - If \( L'(v_3) \cap L'(u_1) \neq \emptyset \), then color \( v_3 \) and \( u_1 \) with the same color \( c' \). Due to Lemma 15, \( v_2 \) and \( w_2 \) (resp. \( u_2 \) and \( v_2, u_3 \) and \( w_2 \)) cannot share a neighbor. So, we can finish by coloring \( v_2, w_2, u_5, u_2 \) and \( u_3 \).
  
  - If \( L'(v_3) \cap L'(u_1) = \emptyset \), then \( |L'(v_3) \cup L'(u_1)| \geq 7 \).
    * If \( L'(v_2) \cap L'(u_5) \neq \emptyset \), then color \( v_2 \) and \( u_5 \) with the same color \( c' \) and finish by coloring \( v_2, v_3, u_1, u_2 \) and \( u_3 \).
    * If \( L'(v_2) \cap L'(u_5) = \emptyset \), then \( |L'(v_2) \cup L'(u_5)| \geq 6 \). We complete the coloring of \( S - \{v_1, u_4\} \) by Corollary 8.

- If \( L(v_1) \cap L(u_4) = \emptyset \), then by symmetry, \( L(v_3) \cap L(u_1) = \emptyset \) due to the previous case. Thus, we have \(|L(v_1) \cup L(u_4)| \geq 9\) and \(|L(v_3) \cup L(u_1)| \geq 9\).
  
  - If \( L(v_2) \cap L(u_5) \neq \emptyset \), then color \( v_2 \) and \( u_5 \) with the same color \( c \). Let \( L'(x) \) be the remaining list of colors for \( x \in S - \{v_2, u_5\} \). Since \(|L(v_1) \cup L(u_4)| \geq 9\) and \(|L(v_3) \cup L(u_1)| \geq 9\), we have \(|L(v_1) \cup L(u_4)| \geq 7\) and \(|L(v_3) \cup L(u_1)| \geq 7\). Thus, we can the coloring of \( S - \{v_2, u_5\} \) by Corollary 8.
  
  - If \( L(v_2) \cap L(u_5) = \emptyset \), then \( |L(v_2) \cup L(u_5)| \geq 8 \). We can color \( S \) by Corollary 8.

This concludes the proof.

\[\square\]

**Lemma 21.** A bad 5-face in \( G \) is not adjacent with a 5-face.

![Figure 19: Reducible configuration in Lemma 21.](image-url)
Proof. Suppose be contradiction that there exists a bad 5-face $f = u_1 u_2 u_3 u_4 u_5$ that is adjacent to a 5-face as shown in Figure 19. First, observe that the vertices in $S = \{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$ are pairwise distinct due to Lemma 14(iv), Lemma 15, and Lemma 16. Color $G - \{u_1, u_2, u_3, u_4, u_5\}$ and uncolor $\{v_1, v_2, v_3, v_4, v_5\}$. The remaining list of colors for these vertices have size: $|L(u_1)| \geq 7, |L(u_2)| \geq 6, |L(u_3)| \geq 6, |L(u_4)| \geq 7, |L(u_5)| \geq 7, |L(v_1)| \geq 5, |L(v_2)| \geq 3, |L(v_3)| \geq 3, |L(v_4)| \geq 5, |L(v_5)| \geq 5, |L(v_6)| \geq 5$. Observe that the following pairs of vertices cannot share a neighbor:

- $(u_1, v_3)$ and $(u_4, v_1)$. Otherwise, by symmetry, say $v_3$ sees $u_1$, then it would need to be adjacent to $v_1$ or $v_5$, which is impossible due to Lemma 15.
- $(u_2, v_2)$ and $(u_3, v_2)$ due to Lemma 16.
- $(u_2, u_3)$ since they neighbors are distinct vertices.

We distinguish the following cases:

**Case 1:** $L(u_2) \cap L(v_2) \neq \emptyset$.

Color $u_2$ and $v_2$ with the same color $c$. Now, let $L'(x)$ be the remaining list of colors for $x \in S \setminus \{u_2, v_2\}$.

- If $L'(u_2) \cap L'(u_3) \neq \emptyset$, then color $u_2$ and $u_3$ with the same color $c'$. Let $L'(x)$ be the remaining list of colors for $x \in S \setminus \{u_2, u_3, v_2, v_2\}$. We have the following: $|L''(u_1)| \geq 5, |L''(u_4)| \geq 5, |L''(u_5)| \geq 5, |L''(v_1)| \geq 3, and |L''(v_3)| \geq 3$. Observe that $|L''(v_4)| \geq 4$ because if $v_4$ sees the color $c$, then $v_4$ sees either $v_2$ or $u_2$, in which case we had $|L(v_4)| \geq 6$. The same holds for $v_5$ and $c'$.

- If $L''(u_1) \cap L''(u_4) \neq \emptyset$ or $L''(u_3) \cap L''(u_4) \neq \emptyset$, say it holds for $L''(u_1)$ and $L''(u_4)$, then we color $v_1$ and $u_4$ with the same color $c''$ and finish by coloring $v_3, v_4, v_5, u_1$, and $u_5$.

- If $L''(u_1) \cap L''(u_4) = \emptyset$ and $L''(u_3) \cap L''(u_4) = \emptyset$, then we complete the coloring by Corollary 8.

- If $L'(u_2) \cap L'(u_3) = \emptyset$, then we have following cases.

- If $L'(v_1) \cap L'(u_4) \neq \emptyset$, then we color $v_1$ and $u_4$ with the same color $c'$. Let $L''(x)$ be the remaining list of colors for $x \in S \setminus \{u_2, u_4, v_2, v_1\}$. We have the following: $|L''(u_1)| \geq 5, |L''(u_3)| \geq 4, |L''(u_5)| \geq 5, |L''(v_3)| \geq 3, and |L''(v_5)| \geq 3$. Similar to the previous case, even if $v_4$ sees $c$, we get $|L''(v_4)| \geq 4$. Moreover, we have $|L''(u_3) \cup L''(u_2)| = |L''(u_3)| + |L''(u_2)| \geq 7$ since $L''(u_3) \cap L''(u_2) = \emptyset$.

- If $|L''(u_2)| = 1$, then $|L''(v_3)| \geq 6$. Thus, we can coloring the remaining vertices in this order: $w_2, v_3, v_5, v_4, u_1, u_5$, and $u_3$.

- If $|L''(u_2)| \geq 2$, then we can always color $u_2$ last. Now, we either complete the coloring by Corollary 8, or $L''(u_1) \cap L''(v_3) = \emptyset$, in which case, we color $u_1$ and $v_3$ with the same color and finish by coloring $v_5, v_4, u_3, u_5$, and $u_2$ in this order.

- If $L'(v_3) \cap L'(u_1) \neq \emptyset$, then we color $v_3$ and $u_1$ with the same color $c'$. Let $L''(x)$ be the remaining list of colors for $x \in S \setminus \{u_1, u_2, v_2, v_3\}$. We have the following: $|L''(u_3)| \geq 4, |L''(u_4)| \geq 5, |L''(u_5)| \geq 5, |L''(v_1)| \geq 3, |L''(v_4)| \geq 4, |L''(v_5)| \geq 3, and |L''(v_5)| \geq 3$. Moreover, we have $|L''(u_3) \cup L''(u_2)| = |L''(u_3)| + |L''(u_2)| \geq 7$ since $L''(u_3) \cap L''(u_2) = \emptyset$.

- If $|L''(u_2)| = 1$, then $|L''(v_3)| \geq 6$. Thus, we can coloring the remaining vertices in this order: $w_2, v_1, v_5, v_4, u_3, u_5$, and $u_3$.

- If $|L''(u_2)| \geq 2$, then we can always color $u_2$ last. Now, we either complete the coloring by Corollary 8, or $L''(u_4) \cap L''(v_1) = \emptyset$, in which case, we color $u_4$ and $v_1$ with the same color and finish by coloring $v_5, v_3, u_3, u_5$, and $u_2$ in this order.

- If $L'(v_3) \cap L'(u_4) = \emptyset$ and $L'(u_4) \cap L'(u_1) = \emptyset$, then we have $|L'(u_1) \cup L'(u_4)| \geq 10$, $|L'(v_3) \cup L'(v_4)| \geq 10$, and $|L'(w_2) \cup L'(u_3)| \geq 8$. Thus, we can conclude by Corollary 8.

**Case 2:** $L(u_2) \cap L(v_2) = \emptyset$.

By symmetry, we also have $L(u_2) \cap L(u_3) = \emptyset$ or we would be back in Case 1.
If \( L(v_1) \cap L(u_4) \neq 0 \), then we color \( v_1 \) and \( u_4 \) with the same color \( c \). Now, color \( v_2, w_2, v_3, \) and \( v_4 \) in this order. Note that \( v_5 \) has at least one color left, otherwise \( |L(v_5)| = 5 \) and it would need to see every colored vertex in \( S \), in which case \( |L(v_5)| \geq 6 \). So, we can also color \( v_5 \). Let \( L'(x) \) be the discharging rules stated above.

Every \( R_0 \) and the discharging rules above.

If \( L(v_1) \cap L(u_4) = \emptyset \), then by symmetry, we also have \( L(v_3) \cap L(u_1) = \emptyset \) or we would be back to the previous case. In this case, we start by coloring \( v_2 \) and \( w_2 \). Let \( L'(x) \) be the remaining list of colors for \( x \in S \setminus \{v_2, w_2\} \).

- If \( L'(u_2) \cap L'(u_3) \neq \emptyset \), then we color \( u_2 \) and \( u_3 \) with the same color \( c' \). Let \( L''(x) \) be the remaining list of colors for \( x \in S \setminus \{u_2, u_3, v_2, w_2\} \). Observe that, by similar arguments as above, recounting the colors for \( L''(x) \) gives us \( |L''(u_1)| \geq 5, |L''(u_4)| \geq 5, |L''(u_5)| \geq 6, |L''(v_1)| \geq 3, |L''(v_3)| \geq 3, |L''(v_4)| \geq 4, \) and \( |L''(v_5)| \geq 4 \). We can conclude by Corollary 8 since \( |L''(u_1) \cup L''(u_3)| \geq 8 \) and \( |L''(v_3) \cup L''(u_1)| \geq 8 \) as \( L(v_1) \cap L(u_4) = \emptyset \) and \( L(v_3) \cap L(u_1) = \emptyset \).

- If \( L'(u_2) \cap L'(u_3) = \emptyset \), then we have \( |L'(u_2) \cup L'(u_3)| \geq 10, |L'(v_1) \cup L'(u_4)| \geq 10, \) and \( |L'(v_3) \cup L'(u_1)| \geq 10 \) as \( L(v_1) \cap L(u_4) = \emptyset \) and \( L(v_3) \cap L(u_1) = \emptyset \). We complete the coloring by Corollary 8.

This concludes the proof.

\[ \square \]

4.2 Discharging

Knowing the reducible structures in \( G \), we apply the following rules in the discharging procedure to get a contradiction with Equation (1):

- **R0** Every 5+face \( f \) gives \( \frac{1}{3} \) to each incident 3-vertex.
- **R1** Every 5+face \( f \) gives \( \frac{1}{2} \) to each adjacent 3-face that is not adjacent with any 3-faces.
- **R2** Every 5+face \( f \) gives \( \frac{1}{3} \) to each adjacent 3-face that is adjacent with another 3-face.
- **R3** Every 6+face \( f \) gives \( \frac{1}{3} \) to each adjacent bad 5-face.

Figure 20: **R0.**

Figure 21: **R1.**

Figure 22: **R2.**

Figure 23: **R3.**

Now we prove Theorem 4 using both discharging procedure and structural properties of \( G \) proven in Section 4.1 and the discharging rules stated above.

**Proof of Theorem 4.** Let \( G \) be a minimal counterexample to the theorem and let \( \mu(u) \) be the initial charge assignment for the vertices and faces of \( G \) with the charge \( \mu(u) = d(u) - 4 \) for each vertex.
\( u \in V(G) \), and \( \mu(f) = d(f) - 4 \) for each face \( f \in F(G) \). By Equation (1), we have that the total sum of the charges is negative.

Let \( \mu^* \) be the assigned charges after the discharging procedure. In what follows, we will prove that:

\[
\forall x \in V(G) \cup F(G), \mu^*(x) \geq 0.
\]

We first prove that the final charge on each vertex is non-negative. Let \( u \) be a vertex in \( V(G) \). Note that \( u \) has degree at least 3 by Lemma 13. Recall also that \( \Delta(G) = 4 \) and \( \mu(u) = d(u) - 4 \), thus we now consider the following two cases.

**Case 1:** \( d(u) = 4 \)

Then, \( u \) does not give any charge. So,

\[
\mu^*(u) = \mu(u) = d(u) - 4 = 0.
\]

**Case 2:** \( d(u) = 3 \)

Then, \( \mu(u) = d(u) - 4 = -1 \). By Lemma 14(ii), \( u \) is incident to only 5\(^+\)-faces. Thus, \( u \) receives three times \( \frac{1}{3} \) by \( R0 \). So,

\[
\mu^*(u) = -1 + 3 \cdot \frac{1}{3} = 0.
\]

Secondly, we prove that the final charge on each face is also non-negative. Let \( f \) be a face in \( F(G) \) and recall that \( \mu(f) = d(f) - 4 \). We distinguish the following cases.

**Case 1:** \( d(f) \geq 6 \)

We claim that \( f \) never gives more than \( \frac{1}{3}d(f) \) by \( R0-R3 \). Indeed, we argue that \( f \) sends at most \( \frac{1}{3} \) per incident edge. Observe that we can also view \( R0 \) and \( R2 \) as \( f \) giving charge along an incident edge as shown in Figure 24. Consider an edge \( uv \) incident to \( f \), if \( f \) gives \( \frac{1}{6} \) along \( uv \) by \( R0 \), then it cannot give additional charge along \( uv \) by any other rule since a 3-vertex cannot be incident to a 3-cycle by Lemma 14(ii), and it cannot be at distance 1 from two adjacent 3-cycles by Lemma 14(iii). If \( f \) gives \( \frac{1}{3} \) along \( uv \) by \( R1 \), then it cannot give along \( uv \) by \( \text{R2} \) due to Lemma 14(iv), nor by \( \text{R3} \) by definition. If \( f \) gives (at most \( \frac{1}{3} \)) along \( uv \) by \( \text{R2} \), then it cannot also give along \( uv \) by \( \text{R3} \) due to Lemma 16. To conclude, we have

\[
\mu^*(u) \geq \mu(u) - \frac{1}{3}d(f) = d(f) - 4 - \frac{1}{3}d(f) = \frac{2}{3}d(f) - 4 \geq 0
\]

since \( d(f) \geq 6 \).

![Figure 24: A 6\(^+\)-face f sending charges through incident edges.](image)

**Case 2:** \( d(f) = 5 \)

Recall that \( \mu(f) = d(f) - 4 = 1 \). Let \( i_0, i_1, \) and \( i_2 \) be respectively the number of times \( f \) gives charge by \( R0, R1, \) and \( R2 \). Note that \( \text{R3} \) does not apply to 5-faces. Observe now that we have the following inequalities:
• \( i_1 \leq 4 \) by Lemma 19.
• \( i_0 + i_2 \leq 2 \) by Lemma 14 and lemma 16.
• If \( i_0 \geq 1 \), then \( i_0 + i_1 \leq 3 \) by Lemma 14(i, ii) and Lemma 18.

Recall that \( f \) gives \( \frac{1}{3}i_0 + \frac{1}{3}i_1 + \frac{1}{2}i_2 \) by \textbf{R0}, \textbf{R1}, and \textbf{R2}.

If \( i_2 = 2 \), then \( i_0 = 0 \) since \( i_0 + i_2 \leq 2 \). We have \( i_1 = 0 \) due to Lemma 14(iv) and Lemma 16. Thus,
\[
\mu^*(f) \geq 1 - 2 \cdot \frac{1}{2} = 0.
\]

If \( i_2 = 1 \), then we have the two following cases since \( i_0 + i_2 \leq 2 \):
• If \( i_0 = 1 \), then \( i_1 = 0 \) due to Lemma 14(ii, iii, iv) and Lemma 16. Thus,
\[
\mu^*(f) \geq 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}.
\]
• If \( i_0 = 0 \), then \( i_1 \leq 1 \) due to Lemma 17. Thus,
\[
\mu^*(f) \geq 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}.
\]

If \( i_2 = 0 \), then we distinguish two cases:
• If \( i_0 \geq 1 \), then recall \( i_0 + i_1 \leq 3 \). Thus,
\[
\mu^*(f) \geq 1 - 3 \cdot \frac{1}{3} = 0.
\]
• If \( i_0 = 0 \) and \( i_1 \leq 3 \), then
\[
\mu^*(f) \geq 1 - 3 \cdot \frac{1}{3} = 0.
\]
• If \( i_0 = 0 \) and \( i_1 = 4 \) (recall \( i_1 \leq 4 \)), then \( f \) is adjacent to a 6-\(^+\)-face due to Lemma 20 and Lemma 21. So \( f \) receives \( \frac{1}{3} \) by \textbf{R3}. Thus,
\[
\mu^*(f) \geq 1 - 4 \cdot \frac{1}{3} + \frac{1}{3} = 0.
\]

Thus, we have that after the discharging procedure 5-faces have non-negative charge.

Case 3: \( d(f) = 4 \)
Since 4-faces do not give any charge, we have
\[
\mu^*(f) = \mu(f) = d(f) - 4 = 0.
\]

Case 4: \( d(f) = 3 \)
Recall that \( \mu(f) = d(f) - 4 = -1 \). If \( f \) is adjacent only with 5\(^+\)-faces, then \( f \) receives three times \( \frac{1}{3} \) by \textbf{R1}. So,
\[
\mu^*(f) \geq -1 + 3 \cdot \frac{1}{3} = 0.
\]

On the other hand, if \( f \) is adjacent to a 4\(^-\)-face, then by Lemma 15 and Lemma 14(iv) \( f \) is adjacent with exactly one 3-face. In such a case, it receives \( \frac{1}{2} \) from each of the two adjacent 5\(^+\)-faces by \textbf{R2}. So,
\[
\mu^*(f) \geq -1 + 2 \cdot \frac{1}{2} = 0.
\]

Finally, it follows that after the discharging procedure the charge of every vertex and every face is non-negative and thus the final total sum is non-negative, a contradiction to Equation (1). \( \square \)
5 Injective list-coloring of triangle-free planar graphs

Let $G$ be a minimal counterexample to Theorem 5. More precisely, $G$ has maximum degree 4, girth at least 4, and $\chi^i(G) \geq 10$.

5.1 Structural properties of $G$

Observe that since $g(G) \geq 4$, whenever two vertices are adjacent, they do not see each other (they do not share a common neighbor). Otherwise, $G$ would contain a 3-cycle. As a result, an injective coloring of $G$ is also an exact square coloring as only vertices at distance exactly 2 see each other.

Lemma 22. The minimum degree of $G$ is at least 2.

Proof. If $G$ contains a 1-vertex $v$, then we can simply remove $v$ and color the resulting graph, which is possible by minimality of $G$. Then, we add $v$ back and extend the coloring, since $v$ shares a neighbor with at most 3 other vertices and we have 9 colors in total.

Unlike the previous cases, we do not have enough colors to reduce a 2-vertex directly. However, the presence of such a “small” vertex guarantees that its neighbors must have a larger neighborhood.

Lemma 23. If a 4-vertex $u$ in $G$ is adjacent to a 2-vertex, then $d^{\#2}(u) \geq 9$.

Proof. Suppose by contradiction that $u$ is a 4-vertex that is adjacent to a 2-vertex $v$ and $d^{\#2}(u) \leq 8$. Then, color $G - \{v\}$ by minimality and uncolor $u$. Vertex $u$ sees as many colors as $d^{\#2}(u) \leq 8$, so $u$ is colorable. Finish by coloring $v$ which sees only $d^{\#2}(v) \leq 6$ colors.

Lemma 24. Graph $G$ cannot contain the following configurations:

(i) Two adjacent 3-vertices.
(ii) A 4-vertex adjacent to two 2-vertices.
(iii) A 4-vertex adjacent to a 2-vertex and two 3-vertices.
(iv) A 2-vertex incident to a 4-cycle.
(v) A 3-vertex incident to two 4-cycles.
(vi) A 4-vertex $u$ adjacent to a 2-vertex and a 3-vertex $v$, and $uv$ is incident to a 4-cycle.

Figure 25: Reducible configurations in Lemma 24.
Proof. We separate the proof into four parts based on the configurations.

(i) Suppose by contradiction that there exist two adjacent $3^-$-vertices $u$ and $v$. Color $G - \{uv\}$ by minimality. Uncolor $u$ and $v$. Observe that $d#2(u) \leq 8$. The same holds for $v$. Thus, $u$ and $v$ are colorable.

(iv) Suppose by contradiction that there exists a 2-vertex $u$ incident to a 4-cycle. Color $G - \{u\}$ by minimality. Observe that the two neighbors of $u$ will also have different colors in $G$ since they are at distance 2 in $G - \{u\}$. Thus, we only need to color $u$ which sees only $d#2(u) \leq 5$ colors.

(v) Suppose by contradiction that there exists a 3-vertex $u$ incident to two 4-cycles. Let $e$ be the edge incident to $u$ that is incident to both cycles. Color $G - \{e\}$ by minimality and uncolor $u$. Observe that every pair of neighbors of $u$ are still at distance 2 in $G - \{e\}$. Thus, we only need to color $u$ which sees only $d#2(u) \leq 7$ colors.

(ii), (iii), and (vi) Observe that the 4-vertex $u$ with the 2-neighbor in these configurations always verifies $d#2(u) \leq 8$, which is impossible due to Lemma 23.

Thus, if $G$ contains any of the above configurations, then $\chi^e_i(G) \leq 9$, a contradiction. \qed

Before continuing with proving some more structural results, we first give some additional useful definitions and observations.

**Good and bad faces:** We call a 5-face **bad** if it is incident to a 2-vertex and a 3-vertex. Additionally, we call a 5$^+$-face **good**, if it is not a bad 5-face.

The following observation is a direct consequence of Lemma 24(i).

**Observation 25.** A 2-vertex and a 3-vertex on a bad 5-face $f$ in $G$ must be at distance 2 and they are the only $3^-$-vertices on $f$.

![Figure 26: A bad face.](image)

To further help us with the proofs, we now divide $3^-$-vertices into three different types.

**Small, medium, and large $3^-$-vertices:** We call a $3^-$-vertex **small**, if it is either a 2-vertex or a 3-vertex incident to a bad 5-face and a 4-face. A 3-vertex is called **medium**, if it is incident to either a bad 5-face or a 4-face. Finally, a 3-vertex is called **large**, if it is neither medium nor small.

Due to Lemma 24(vi) we have the following observation.

**Observation 26.** A 4-face in $G$, adjacent with a bad 5-face $f$ and incident to a small 3-vertex $v$, is not incident to the common neighbor of a vertex $v$ and the 2-vertex on $f$. 

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Let $f = v_1v_2v_3v_4v_5$ be a bad 5-face in $G$ where $v_1$ is the 3-vertex and $v_3$ is the 2-vertex. Let $f' = v'_1v'_2v'_3v'_4v'_5 \neq f$ be another 5-face incident to $v_1 = v'_1$. Then, we have the following:

- If $f'$ is incident to $v_1v_2$, then $f'$ does not contain any other $3^-$-vertices (distinct from $v_1$).
- If $f'$ is incident to $v_1v_5$, then $f'$ does not contain any other small vertices (distinct from $v_1$).

We are now ready to prove some structural properties regarding bad 5-faces.

**Lemma 27.** Let $f = v_1v_2v_3v_4v_5$ be a bad 5-face in $G$ where $v_1$ is the 3-vertex and $v_3$ is the 2-vertex. Let $f' = v'_1v'_2v'_3v'_4v'_5 \neq f$ be another 5-face incident to $v_1 = v'_1$. Then, we have the following:

- If $f'$ is incident to $v_1v_2$, then $f'$ does not contain any other $3^-$-vertices (distinct from $v_1$).
- If $f'$ is incident to $v_1v_5$, then $f'$ does not contain any other small vertices (distinct from $v_1$).

**Proof.** We assume w.l.o.g. that $v'_1 = v_1$. Since $g(G) \geq 4$, every vertex of $f$ and $f'$ (except for the two common vertices that is $v_1$ and one of its neighbor) is distinct.

Suppose by contradiction that $f'$ contains (another) small vertex different from $v_1$.

**Case 1:** If $f'$ is incident to $v_1v_2$, say $v'_5 = v_2$. First, observe that $d(v'_5) = 4$ due to Lemma 24(iii) and $d(v'_3) = 4$ due to Lemma 24(i). Thus, $v'_4$ must be a $3^-$-vertex. Color $G - \{v_3\}$ and uncolor $v_1$, $v_2$, and $v'_4$. Observe that $|L(v_1)| \geq 2$, $|L(v_2)| \geq 1$, $|L(v_3)| \geq 4$, $|L(v'_4)| \geq 2$. Therefore, we can color $v_2$, $v'_4$, $v_1$, and $v_3$ in this order.

**Case 2:** If $f'$ is incident to $v_1v_5$, say $v'_5 = v_5$. By Lemma 24(i), $v'_2$ cannot be a small vertex, and at most one of $v'_3$ and $v'_4$ can be. Thus, we have the following two cases:

- If $v'_4$ is a $3^-$-vertex, then color $G - \{v_3\}$ and uncolor $v_1$, $v_4$, and $v'_4$. Observe that $|L(v_1)| \geq 3$, $|L(v_4)| \geq 1$, $|L(v_3)| \geq 4$, $|L(v'_4)| \geq 2$. Therefore, we can color $v_4$, $v'_4$, $v_1$, and $v_3$ in this order.
- If $v'_3$ is a $3^-$-vertex, then recall that $v'_3$ is a small vertex.
  - If $v'_3$ is a small 3-vertex, then it is incident to a bad 5-face $f'' \neq f'$ (since $f'$ is a good face) and a 4-face. If $f''$ is incident to $v'_2v'_3$, then the 4-face must be incident to $v'_3v'_4$. By Lemma 24(iii, vi), $f''$ cannot be incident to a 2-vertex, which is a contradiction. Thus, $f''$ must be incident to $v'_2v'_4$. By Lemma 24(vi), the 2-vertex incident to $f''$ must be adjacent to $v'_4$. However, in this case, we can use the same proof as in **Case 1** from the point of view of $v'_3$, $f''$, and $f'$ instead.
Thus, we can conclude that

Finally, we show that small vertices cannot be close to each other from the perspective of a face of Lemma 29.

Proof. By Lemma 24(vi), this 5-face cannot be incident to any 2-vertex. As a result, \(v_2v_3\) must be incident to a 4-face. By symmetry, \(v_1v_2\) is also incident to a 4-face. Additionally, \(v_3v_4\) must be incident to a bad 5-face. By Lemma 24(vi), the 2-vertex \(u\) on this bad 5-face must be adjacent to \(v_4\). Now, color \(G - \{u\}\) and uncolor \(v_4\) and \(v_2\). Observe that \(|L(v_4)| \geq 1\), \(|L(v_2)| \geq 2\), and \(|L(u)| \geq 3\). Thus, we can finish by coloring \(v_1\), \(v_3\), and \(v_3\) in this order.

Thus, we can conclude that \(\chi^e(G) \leq 9\), a contradiction. □

Finally, we show that small vertices cannot be close to each other from the perspective of a face of size at least 6.

**Definition 28** (Facial-distance). Let \(f = u_1u_2 \ldots u_{d(f)}\) be a face in \(F(G)\), and let \(u_i\) and \(u_j\) be vertices incident to \(f\). The facial-distance on \(f\) between \(u_i\) and \(u_j\) is their distance on the cycle \(u_1u_2 \ldots u_{d(f)}\) (which is \(\min(i - j \mod d(f)), j - i \mod d(f))\)).

**Lemma 29.** Two small vertices incident to a same 6+-face \(f\) in \(G\) are at facial-distance at least 3 on \(f\).

Proof. By Lemma 24(i), small vertices cannot be adjacent. By Lemma 24(ii), two 2-vertices must be at distance at least 3. We only need to check if a small 3-vertex and a 2-vertex, or two small 3-vertices can be at facial-distance 2 on \(f\). Let \(f = v_1v_2v_3v_4 \ldots\) is a 6+-face.

Suppose that \(v_1\) is a 2-vertex and \(v_3\) a small 3-vertex. Observe that \(v_2v_3\) cannot be incident to a 4-face by Lemma 24(vi), so \(v_2v_4\) must be incident to a 4-face and \(v_2v_3\) is incident to a bad 5-face (different from \(f\) since \(f\) is a 6+-face). However, due to Lemma 24(ii, vi), the bad 5-face incident to \(v_2v_3\) cannot be incident to a 2-vertex, which is a contradiction.

Now, suppose that \(v_1\) and \(v_3\) are small 3-vertices. They must both be incident to some 4-faces and bad 5-faces. If \(v_3v_4\) is incident to a 4-face, then \(v_2v_3\) must be incident to a bad 5-face. However, by Lemma 24(i, iii, vi), this 5-face cannot be incident to any 2-vertex. As a result, \(v_2v_3\) must be incident to a 4-face. By symmetry, \(v_1v_2\) is also incident to a 4-face. Additionally, \(v_3v_4\) must be incident to a bad 5-face. By Lemma 24(vi), the 2-vertex \(u\) on this bad 5-face must be adjacent to \(v_4\). Now, color \(G - \{u\}\) and uncolor \(v_4\) and \(v_2\). Observe that \(|L(v_4)| \geq 1\), \(|L(v_2)| \geq 2\), and \(|L(u)| \geq 3\). Thus, we can finish by coloring \(v_1\), \(v_2\), and \(v_3\) in this order. □
5.2 Discharging procedure

To get a contradiction with Equation (1) we apply the following rules in the discharging procedure:

**R0** Every $5^+$-face $f$ gives 1 to each 2-vertex.

**R1** Every good $5^+$-face $f$ gives 1 to each small 3-vertex.

**R2** Every good $5^+$-face $f$ gives $\frac{1}{2}$ to each medium 3-vertex.

**R3** Every $5^+$-face $f$ gives $\frac{1}{3}$ to each large 3-vertex.

Now we can proceed by proving Theorem 5 using the discharging procedure together with the structural properties of $G$ proven in Section 5.1 and the discharging rules stated above.

**Proof of Theorem 5.** Let $G$ be a minimal counterexample to the theorem and let $\mu(u)$ be the initial charge assignment for the vertices and faces of $G$ with the charge $\mu(u) = d(u) - 4$ for each vertex $u \in V(G)$, and $\mu(f) = d(f) - 4$ for each face $f \in F(G)$. By Equation (1), we have that the total sum of the charges is negative.

Let $\mu^*$ be the assigned charges after the discharging procedure. In what follows, we will prove that:

$$\forall x \in V(G) \cup F(G), \mu^*(x) \geq 0.$$

Let $u$ be a vertex in $V(G)$. Vertex $u$ has degree at least 2 by Lemma 22. Recall that $\Delta(G) = 4$ and $\mu(u) = d(u) - 4$. 

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Figure 29: Reducible configurations in Lemma 29.

Figure 30: **R0**.

Figure 31: **R1**.

Figure 32: **R2**.

Figure 33: **R3**.
Case 1: If \( d(u) = 4 \), then \( u \) does not give any charge. So,

\[
\mu^*(u) = \mu(u) = d(u) - 4 = 0.
\]

Case 2: If \( d(u) = 3 \), then \( \mu(u) = d(u) - 4 = -1 \) and we have the following cases:

- If \( u \) is a small 3-vertex, then \( u \) is incident to one good \( 5^+ \)-face due to Lemma 24(v) and Lemma 27. By \( R_1 \), we have
  
  \[
  \mu^*(u) \geq -1 + 1 = 0.
  \]

- If \( u \) is a medium 3-vertex, then it is incident to two good \( 5^+ \)-faces due to Lemma 24(v) and Lemma 27. By \( R_2 \), we have
  
  \[
  \mu^*(u) \geq -1 + 2 \cdot \frac{1}{2} = 0.
  \]

- If \( u \) is a large 3-vertex, then by definition, \( u \) is incident to only \( 5^+ \)-face. Thus, by \( R_3 \), we have
  
  \[
  \mu^*(u) \geq -1 + 3 \cdot \frac{1}{3} = 0.
  \]

Case 3: If \( d(u) = 2 \), then \( u \) has to be incident to only \( 5^+ \)-faces due to Lemma 24(iv). By \( R_0 \), we have

\[
\mu^*(u) \geq d(u) - 4 + 2 \cdot 1 = 0.
\]

Let \( f \) be a face in \( F(G) \). Recall that \( \mu(f) = d(f) - 4 \) and \( d(f) \geq 4 \) since \( g(G) \geq 4 \). Let \( i_0, i_1, i_2, \) and \( i_3 \) be respectively the number of times \( f \) gives charge by \( R_0, R_1, R_2, \) and \( R_3 \). We distinguish the following cases.

Case 1: \( d(f) \geq 7 \)

Let \( u \) and \( v \) be two small vertices on \( f \). By Lemma 29, \( u \) and \( v \) must be at facial-distance at least 3 on \( f \). As a result, the neighbors of \( u \) and \( v \) on \( f \) are distinct. Moreover, due to Lemma 24(i), those neighbors are 4-vertices. Thus, we also have \( i_2 + i_3 \leq d(f) - 3(i_0 + i_1) \). Due to Lemma 24(i), we also have \( i_2 + i_3 \leq \frac{1}{2}d(f) \). Consequently, \( i_2 + i_3 \leq \min(d(f) - 3(i_0 + i_1), \frac{1}{2}d(f)) \).

We claim that \( f \) gives at most \( \frac{5}{12}d(f) \) charge away. Indeed, recall that \( f \) gives \( i_0 + i_1 + \frac{1}{2}i_2 + \frac{1}{3}i_3 \) by \( R_0, R_1, R_2, \) and \( R_3 \). By the above inequalities,

- if \( d(f) - 3(i_0 + i_1) \leq \frac{1}{2}d(f) \), then \( i_0 + i_1 \geq \frac{1}{6}d(f) \). Moreover, we get

  \[
  i_0 + i_1 + \frac{1}{2}i_2 + \frac{1}{3}i_3 \leq i_0 + i_1 + \frac{1}{2}(i_2 + i_3) \\
  \leq i_0 + i_1 + \frac{1}{2}(d(f) - 3(i_0 + i_1)) \\
  \geq \frac{1}{2}(d(f) - (i_0 + i_1)) \\
  \geq \frac{1}{2}(d(f) - \frac{1}{6}d(f)) \\
  \geq \frac{5}{12}d(f)
  \]

- if \( d(f) - 3(i_0 + i_1) > \frac{1}{2}d(f) \), then \( i_0 + i_1 < \frac{1}{6}d(f) \). Moreover, we get

  \[
  i_0 + i_1 + \frac{1}{2}i_2 + \frac{1}{3}i_3 \leq i_0 + i_1 + \frac{1}{2}(i_2 + i_3) \\
  \leq \frac{1}{6}d(f) + \frac{1}{2} \cdot \frac{1}{2}d(f) \\
  \geq \frac{5}{12}d(f)
  \]
To conclude, we have
\[ \mu^*(f) \geq d(f) - 4 - \frac{5}{12} d(f) \geq \frac{7}{12} d(f) - 4 \geq 0 \]
since \( d(f) \geq 7 \).

**Case 2: \( d(f) = 6 \)**

Similar to the previous case, two small vertices cannot be at facial-distance 2 on \( f \) by Lemma 29. As a result, we get \( i_0 + i_1 \leq 2 \). Moreover, by Lemma 24(i), two 3\(^-\)vertices cannot be adjacent, so we get \( i_0 + i_1 + i_2 + i_3 \leq 3 \iff i_2 + i_3 \leq 3 - (i_0 + i_1) \). Now, we distinguish the following cases.

- If \( i_0 + i_1 = 2 \), then observe that, since small vertices cannot share neighbors on \( f \) and that their neighbors are all 4-vertices, we have exactly two 3\(^-\)vertices on \( f \). In other words, \( i_2 + i_3 = 0 \).

  Recall that \( \mu(f) = d(f) - 4 = 2 \) and that \( f \) gives \( i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3 \) by \( R0, R1, R2, \) and \( R3 \).

  Thus,
  \[ \mu^*(f) \geq 2 - (i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3) = 2 - 2 + 0 = 0. \]

- If \( i_0 + i_1 \leq 1 \), then we get
  \[ \mu^*(f) \geq 2 - (i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3) \geq 2 - (i_0 + i_1 + \frac{1}{2} (3 - (i_0 + i_1))) \geq \frac{1}{2} - \frac{1}{2} (i_0 + i_1) \geq 0. \]

**Case 3: \( d(f) = 5 \)**

Recall that \( \mu(f) = d(f) - 4 = 1 \). Observe that we have the following inequalities.

- \( i_0 + i_1 + i_2 + i_3 \leq 2 \) since there are no adjacent 3\(^-\)vertices by Lemma 24(i).

- \( i_0 \leq 1 \) due to Lemma 24(ii).

- \( i_1 \leq 1 \) due to Lemma 27.

Recall that \( f \) gives \( i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3 \).

- If \( i_0 = 1 \), then either \( f \) is incident to a 3-vertex, in which case, it is a bad 5-face and \( R1, R2, R3 \) do not apply (by definition of a bad face), or it is not incident to any 3-vertex. In both cases, \( i_1 + i_2 + i_3 = 0 \). So,
  \[ \mu^*(f) \geq 1 - (i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3) \geq 1 - (1 + 0) = 0. \]

- If \( i_0 = 0 \) and \( i_1 = 1 \), then \( f \) cannot be incident to any other (than the small 3-vertex) 3\(^-\)vertices due to Lemma 27. As a result, \( i_2 + i_3 = 0 \). So,
  \[ \mu^*(f) \geq 1 - (i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3) \geq 1 - (1 + 0) = 0. \]

- If \( i_0 = i_1 = 0 \), then
  \[ \mu^*(f) \geq 1 - (i_0 + i_1 + \frac{1}{2} i_2 + \frac{1}{3} i_3) \geq 1 - (0 + \frac{1}{2} \cdot 2) = 0. \]

**Case 4: \( d(f) = 4 \)**

Recall that \( \mu(f) = d(f) - 4 = 0 \). Since \( f \) does not give any charge, we have
\[ \mu^*(f) = \mu(f) = 0. \]

We started with a negative total charge, but after the discharging procedure, which preserved the total sum, we end up with a non-negative total sum, a contradiction with Equation (1). In other words, there exist no counter-examples to Theorem 5. \( \square \)
6 Exact square list-coloring of planar graphs

Let $G$ be a minimal counterexample to Theorem 6. More precisely, $G$ has maximum degree 4 and $\chi_{#2}^2(G) \geq 11$.

6.1 Structural properties of $G$

Unlike injective coloring where vertices incident to the same triangle need different colors, in exact square coloring, two vertices see each if and only if the distance between them is exactly 2.

**Lemma 30.** The minimum degree of $G$ is at least 2.

**Proof.** Suppose that $v$ is a 1-vertex in $G$. We now color $G - \{v\}$ by minimality and then we can color $v$, since $v$ sees at most three vertices and we have 10 colors.

Similarly as in the previous section, a 2-vertex is not reducible by itself, but it provides a nice counting argument to prove that “smaller” objects must be further away.

**Lemma 31.** If a 4-vertex $u$ is adjacent with a 3-vertex $v$, then $d_{#2}(u) \geq 10$. Moreover, if $v$ is a 2-vertex, then every vertex in $N_{#2}(u)$ is a 4-vertex.

**Proof.** Suppose to the contrary and let $u$ be a 4-vertex with $d_{#2}(u) < 10$ adjacent with a 3-vertex $v$. Note that if $d(u) \leq 3$, then the condition $d_{#2}(u) < 10$ is always satisfied. Thus, we have $d_{#2}(v) < 10$.

Now we color $G - \{uv\}$ by minimality and uncolor the vertices $u$ and $v$. Next, observe that since $d_{#2}(u) < 10$ and $d_{#2}(v) < 10$, we have that $|L(u)| \geq 1$ and $|L(v)| \geq 1$. We can thus finish the coloring by first coloring $u$ and then $v$. Assume now that $v$ is a 2-vertex. From above we have that $u$ must be a 4-vertex adjacent with three other 4-vertices. Thus, we have that $d_{#2}(u) = 10$. Suppose that there exists a 3-vertex $w \in N_{#2}(u)$. In that case we color $G - uv$ by minimality and uncolor the vertices $u$, $v$, and $w$. Since $w \in N_{#2}(u)$, $d(v) = 2$, and $d(w) \leq 3$, we have that the remaining list of colors for these vertices have size: $|L(u)| \geq 1$, $|L(v)| \geq 4$, and $|L(w)| \geq 2$. We can then finish by coloring $u$, $w$, and $v$ in this order. 

As a consequence of Lemma 31, we obtain the configurations in Lemma 32 directly.

**Lemma 32.** Graph $G$ cannot contain the following configurations:

(i) Two adjacent 3-vertices.

(ii) A 4-vertex adjacent with a 2-vertex and a 3-vertex.

(iii) A 4-vertex adjacent with three 3-vertices.

(iv) A 3-vertex incident to a 3-cycle.

(v) A 3-vertex at distance 1 from a 3-cycle.

(vi) A 2-vertex incident to a 4-cycle.

(vii) A 3-vertex incident to two adjacent 4-cycles.

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Figure 34: Reducible configurations in Lemma 32.

We finish the study of $G$ structural properties by looking at small cycles.

Figure 35: Reducible configurations in Lemma 33. Figure 36: Reducible configuration in Lemma 34.

Figure 37: Reducible configuration in Lemma 35.

**Lemma 33.** A 3-cycle in $G$ is not adjacent with a 4$^-$-cycle.

**Proof.** Suppose to the contrary and let $C = uwv$ be a 3-cycle in $G$. Let $C' = uwx$ be a 4$^-$-cycle in $G$. Note that possibly $x = y$, in which case $C'$ is a 3-cycle. We now color $G - uv$ by minimality and uncolor the vertices $u$ and $v$. Observe that whether $x = y$, or $x \neq y$, we have that $|L(u)| \geq 1$ and $|L(v)| \geq 1$. Therefore, we can finish the coloring by first coloring $u$ and then $v$.

**Lemma 34.** A 3-cycle in $G$ is not incident to another 3-cycle.

**Proof.** Suppose to the contrary and let $C = uwv$ be a 3-cycle in $G$. Let $C' = uxy$ be another 3-cycle in $G$. Note that possibly the vertices $v, w, x, y$ are all distinct, otherwise we are done by Lemma 33. We now color $G - u$ by minimality and uncolor the vertices $v, w, x, y$. Observe that the remaining list of colors for vertices $u, v, w, x, y$ have size: $|L(u)| \geq 2$, $|L(v)| \geq 2$, $|L(w)| \geq 2$, $|L(x)| \geq 2$, and $|L(y)| \geq 2$. Since each of the vertices $v, w, x, y$ sees at most two others, we can color these vertices by the 2-choosability of even cycles and finally coloring $u$ with one of the two remaining colors.

**Lemma 35.** A 5-cycle in $G$ incident to a 2-vertex is not adjacent to a 3-cycle.

**Proof.** Suppose to the contrary and let $C = u_1u_2u_3u_4u_5$ be a 5-cycle with $d(u_1) = 2$. Due to Lemma 32(iv, v) we have that the adjacent 3-cycle must be incident with the edge $u_3u_4$. We now color
G – u3u4 by minimality and uncolor all the vertices of C. Note that the remaining list of colors for these vertices have size: |L(u_1)| ≥ 6, |L(u_2)| ≥ 2, |L(u_3)| ≥ 2, |L(u_4)| ≥ 2, and |L(u_5)| ≥ 2. Observe that u_2 does not see u_3, u_3 does not see u_4 and u_4 does not see u_5. Thus, we can color the vertices u_2, u_3, u_4, and u_5 by the 2-choosability of paths and finally coloring u_1 with one of the four remaining colors.

6.2 Discharging procedure

To get a contradiction with Equation (1) we apply the following rules in the discharging procedure:

- **R0** Every 5+ -face f gives 1 to each incident 2-vertex.
- **R1** Every 5+ -face f gives $\frac{1}{3}$ to each incident 3-vertex that is not incident to a 4-face.
- **R2** Every 5+ -face f gives $\frac{1}{2}$ to each incident 3-vertex that is incident to a 4-face.
- **R3** Every 5+ -face f gives $\frac{1}{3}$ to each adjacent 3-face.

Now we can proceed by proving Theorem 6 using the discharging procedure together with the structural properties of G proven in Section 6.1 and the discharging rules stated above.

**Proof of Theorem 6.** Let G be a minimal counterexample to the theorem and let $\mu(u)$ be the initial charge assignment for the vertices and faces of G with the charge $\mu(u) = d(u) - 4$ for each vertex $u \in V(G)$, and $\mu(f) = d(f) - 4$ for each face $f \in F(G)$. By Equation (1), we have that the total sum of the charges is negative.

Let $\mu^*$ be the assigned charges after the discharging procedure. In what follows, we will prove that:

$$\forall x \in V(G) \cup F(G), \mu^*(x) \geq 0.$$  

Let u be a vertex in $V(G)$. Vertex u has degree at least 2 by Lemma 30. Recall that $\Delta(G) = 4$ and $\mu(u) = d(u) - 4$.

**Case 1:** If $d(u) = 4$, then u does not give any charge. So,

$$\mu^*(u) = \mu(u) = d(u) - 4 = 0.$$

**Case 2:** If $d(u) = 3$, then $\mu(u) = d(u) - 4 = -1$ and due to Lemma 32(iv, vii) we have the following two cases:

- If u is not incident to any 4-face, then it receives $\frac{1}{3}$ from each of the three incident 5+ -faces by R1. So,
  $$\mu^*(u) = -1 + 3 \cdot \frac{1}{3} = 0.$$

- If u is incident to a 4-face, then it has exactly one incident 4-face and two incident 5+ -faces due to Lemma 32(vii). Therefore, u receives $\frac{1}{2}$ from each incident 5+ -face by R2. So,
  $$\mu^*(u) = -1 + 2 \cdot \frac{1}{2} = 0.$$
Case 3: If \(d(u) = 2\), then \(\mu(u) = d(u) - 4 = -2\) and due to Lemma 32(iv, vi) \(u\) is incident with two \(5^-\)-faces. Thus, \(u\) receives 1 from each incident face by \(R0\). So,

\[\mu^*(u) = -2 + 2 \cdot 1 = 0.\]

Let \(f\) be a face in \(F(G)\). Recall that \(\mu(f) = d(f) - 4\). Let \(i_0, i_1, i_2,\) and \(i_3\) be respectively the number of times \(f\) gives charge by \(R0, R1, R2,\) and \(R3\). We distinguish the following cases.

Case 1: \(d(f) \geq 6\)

We claim that \(f\) never gives more than \(\frac{1}{3}d(f)\) by \(R0-R3\). Indeed, we argue that \(f\) sends at most \(\frac{1}{3}\) per incident edge. Observe that we can also view \(R0, R1,\) and \(R2\) as \(f\) giving charge along incident edges of \(f\) as shown in Figure 42. Consider now an edge \(uv\) incident to \(f\). If \(f\) gives \(\frac{1}{4}\) by \(R0\), then it cannot give any additional charge along \(uv\) since a \(2^-\)-vertex is at distance at least 2 from any \(3^-\)-face by Lemma 32(iv) and is also at distance at least 4 from any other \(3^-\)-vertex by Lemma 31. If \(f\) gives either \(\frac{1}{6}\) (respectively \(\frac{1}{4}\)) along \(uv\) by \(R1\) (respectively \(R2\)), then \(f\) cannot give any additional charge along \(uv\). Indeed, as argued above, \(f\) cannot give charge along \(uv\) by \(R0\) and also \(f\) cannot give additional charge along \(uv\) by \(R1-R3\) due to Lemma 32(i, iv, v). Finally, if \(f\) gives \(\frac{1}{3}\) along \(uv\) by \(R3\), then, due to Lemma 32(iv, v) and Lemma 33, \(f\) cannot give any additional charge along \(uv\). To conclude, we have

\[\mu^*(u) \geq \mu(u) - \frac{1}{3}d(f) = d(f) - 4 - \frac{1}{3}d(f) = \frac{2}{3}d(f) - 4 \geq 0\]

since \(d(f) \geq 6\).

Case 2: \(d(f) = 5\)

Recall that \(\mu(f) = d(f) - 4 = 1\). Observe that we have the following inequalities regarding the values \(i_0, i_1, i_2,\) and \(i_3\):

- \(i_0 \leq 1\), due to Lemma 32(i, ii).
- \(i_1 + i_2 + i_3 \leq 2\). Indeed, by Lemma 32(i) we have that \(f\) is incident to at most two \(3^-\)-vertices. Moreover, by Lemma 34 we have that \(f\) is adjacent with at most two \(3^-\)-faces. Next, if \(f\) is incident to exactly one \(3^-\)-vertex, then by Lemma 32(iv, v) we have that \(f\) is adjacent with at most one \(3^-\)-face. Finally, if \(f\) is incident to exactly two \(3^-\)-vertices, then again by Lemma 32(iv, v) we have that \(f\) is not adjacent to any \(3^-\)-faces.

Recall that \(f\) gives \(i_0 + \frac{1}{3}i_1 + \frac{1}{2}i_2 + \frac{1}{3}i_3\) by \(R0, R1, R2,\) and \(R3\).

If \(i_0 = 1\), then, by Lemma 32(i, ii, iv, v) and Lemma 35, we have that \(i_1 + i_2 + i_3 = 0\). So,

\[\mu^*(f) \geq 1 - 1 = 0.\]
If \( i_0 = 0 \), then since \( i_1 + i_2 + i_3 \leq 2 \),
\[
\mu^*(f) \geq \mu(f) - \frac{1}{3}i_1 - \frac{1}{2}i_2 - \frac{1}{3}i_3
\geq 1 - \frac{1}{2}(i_1 + i_2 + i_3)
\geq 1 - 2 \cdot \frac{1}{2} = 0.
\]

**Case 3:** \( d(f) = 4 \)
Recall that \( \mu(f) = d(f) - 4 = 0 \). Since \( f \) does not give any charge, we have
\[
\mu^*(f) = \mu(f) = 0.
\]

**Case 4:** \( d(f) = 3 \)
Recall that \( \mu(f) = d(f) - 4 = -1 \). Due to Lemma 33 we have that \( f \) receives \( \frac{1}{3} \) from each adjacent face by \( R3 \). So,
\[
\mu^*(f) = -1 + 3 \cdot \frac{1}{3} = 0.
\]

To conclude, we started with a charge assignment with a negative total sum, but after the discharging procedure, which preserved that sum, we end up with a non-negative one, which is a contradiction. In other words, there exists no counter-example to Theorem 6.

\[\square\]

### 7 Conclusion

Almost all results in 2-distance (and also injective) coloring of planar graphs (of high girth) are proved using the discharging method. Many of these proofs use planarity only to bound the number of edges with respect to the number of vertices in any induced subgraph, which can also be achieved by bounding the maximum average degree. The maximum average degree, denoted by \( \text{mad}(G) \), is taken as the maximum over all subgraphs \( H \) of \( G \) of average degree of \( H \). Note that for planar graphs of girth \( g \) we have the inequality \( \text{mad}(G) < \frac{2g}{g^2-2} \). This allows one to easily translate the condition on maximum average degree to the condition on girth in the case of planar graphs. Thus, we can ask if our results can be extended to non-planar graphs \( G \) which have maximum average degree \( \text{mad}(G) < \frac{24}{24-2} = 4 \).

To answer this question we need to look at finite projective planes. For a prime power \( q \), a projective plane \( PG(2,q) \), of order \( q \), consists of \( q^2 + q + 1 \) points and \( q^2 + q + 1 \) lines such that any two points belong to exactly one line and any two lines intersect in exactly one point. Moreover, each point belongs to exactly \( q + 1 \) lines and each line contains exactly \( q + 1 \) points. Let \( IG(q) \) be the incidence graph of a projective plane \( PG(2,q) \) (i.e., a bipartite graph whose vertices are the points and lines of \( PG(2,q) \) and an edge \( uv \in E(IG(q)) \) if and only if a line \( u \) contains a point \( v \)). In [23], Hahn et al. proved the following result.

**Theorem 36** (Hahn et al. [23]). Let \( G \) be a connected graph of maximum degree \( \Delta \geq 3 \). Then, \( \chi^i(G) = \Delta^2 - \Delta + 1 \) if and only if there exists a projective plane of order \( q = \Delta - 1 \) and \( G \) is isomorphic to \( IG(q) \).

A similar result, but for the exact square coloring was shown by Focaud et al. in [22].

**Theorem 37** (Focaud et al. [22]). Let \( G \) be a connected graph of maximum degree \( \Delta \geq 3 \). Then, \( \chi^{#2}(G) = \Delta^2 - \Delta + 1 \) if and only if there exists a projective plane of order \( q = \Delta - 1 \) and \( G \) is isomorphic to \( IG(q) \).
None of our results on injective and exact square coloring can be extended to non-planar graphs $G$ with maximum average degree $\text{mad}(G) < 4$ even in their non-list version. Indeed, note that for the graph $G = IG(3)$ as seen in Figure 43 we have $\chi_i(G) = \chi^2(G) = 13$ by Theorems 36 and 37. Moreover, removing any vertex from $G$ yields a graph $G'$ with $\text{mad}(G') < 4$ and $\chi_i(G') = \chi^2(G') = 12$, which still proves that our results are “optimal” in the sense that planarity is needed not only for sparseness.

It is easy to see that $\chi^2(\ell(G)) \leq \chi_i(\ell(G)) \leq 12$ for every graph $G$ with $\Delta(G) = 4$ and $\text{mad}(G) < 4$ and this bound is tight for the graph $G'$ as explained above.

**Theorem 38.** For every graph $G$ with $\Delta(G) = 4$ and $\text{mad}(G) < 4$, $\chi_i(G) \leq 12$.

Indeed, if a counter-example $G$ to Theorem 38 exists, then it would have minimum degree 4 (the proof is similar to Lemma 13), which is a contradiction since $G$ would be a 4-regular graph but $\text{mad}(G) < 4$.

For 2-distance coloring, our result is also not extendable to graphs with $\text{mad}(G) < 4$ due to the graph in Figure 44 for which $\chi^2(G) = 13$. In 2016, Cranston and Rabern [14] showed the following result.

**Theorem 39 (Cranston and Rabern [14]).** If $G$ is a connected graph with maximum degree $\Delta \geq 3$ and $G$ is not the Peterson graph, the Hoffman-Singleton graph, or a Moore graph with $\Delta = 57$, then $\chi^2(G) \leq \Delta^2 - 1$.

Thus, when $\Delta(G) = 4$, we immediately get that $\chi^2(G) \leq 15$. We believe that the following conjecture is true.

**Conjecture 40.** For every graph $G$ with $\Delta(G) = 4$ and $\text{mad}(G) < 4$, $\chi^2(G) \leq 13$.

We are also interested in planar graphs with girth 4. Due to constructions in [35] and [43] we have some lower bounds for the 2-distance coloring of planar graphs with girth 4. We believe these lower bounds to be tight and thus we conjecture the following.

**Conjecture 41.** For every planar graph $G$ with girth at least 4 and maximum degree $\Delta$, $\chi^2(G) \leq \Delta + 3$ for $3 \leq \Delta \leq 5$ and $\chi^2(G) \leq \lfloor \frac{3}{2}\Delta \rfloor$ for $\Delta \geq 6$.

Similarly, due to constructions in [38], we conjecture the following for injective and exact square coloring which is the same in the case of planar graphs with girth 4.

**Conjecture 42.** For every planar graph $G$ with girth at least 4 and maximum degree $\Delta$, $\chi_i(G) \leq 4$ for $\Delta = 3$, $\chi_i(G) = \chi^2(G) \leq \Delta + 2$ for $4 \leq \Delta \leq 5$ and $\chi_i(G) = \chi^2(G) \leq \lfloor \frac{3}{2}\Delta \rfloor$ for $\Delta \geq 6$. 
Finally, unlike in the injective coloring, in the exact square coloring we are only concerned by conflicts between vertices at distance exactly 2, hence the presence of triangles does not create any conflicts. Thus, we believe that a similar conjecture holds for exact square coloring.

**Conjecture 43.** For every planar graph $G$ with maximum degree $\Delta$, $\chi^\#_2(G) = \left\lfloor \frac{3}{2} \Delta \right\rfloor$.

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