Local spacetime curvature effects on quantum orbital angular momentum

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Abstract
This paper claims that local spacetime curvature can nontrivially contribute to the properties of orbital angular momentum in quantum mechanics. Of key importance is the demonstration that an extended orbital angular momentum operator due to gravitation can identify the existence of orbital states with half-integer projection quantum numbers $m$ along the axis of quantization, while still preserving integer-valued orbital quantum numbers $l$ for a simply connected topology. The consequences of this possibility are explored in depth, noting that the half-integer $m$ states vanish as required when the locally curved spacetime reduces to a flat spacetime, fully recovering all established properties of orbital angular momentum in this limit. In particular, it is shown that a minimum orbital number of $l = 2$ is necessary for the gravitational interaction to appear within this context, in perfect correspondence with the spin-2 nature of linearized general relativity.

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1. Introduction

In the search for a self-consistent quantum theory of gravity over the past 70 years or more, there now exist many distinct avenues available to pursue. Arguably, the leading contenders in this pursuit are string theory [1, 2] and loop quantum gravity [3, 4]. This is followed—to varying degrees of interest—by the twistor theory [5], causal set theory [6], Regge calculus and causal dynamical triangulations [7, 8], noncommutative geometry [9], and so forth. In the absence of any discernable physical evidence to offer guidance toward achieving this ultimate goal, these and other competing theories are mathematically very sophisticated, requiring a great deal of ingenuity and intensive effort in order to make any headway along any one of these directions. In spite of their differences on a wide array of conceptual and computational details,
all these approaches effectively claim that some form of unification involving gravitation and quantum mechanics becomes relevant only at the Planck length scale of $10^{-33}$ cm, some 20 orders of magnitude smaller than the effective radius of a proton. Addressing this basic discrepancy of scale makes it extremely difficult to conceive of realistic tests for their efficacy, with the possible exception of identifying cosmologically driven observations to amplify any potential quantum gravity signatures to length scales large enough for detection [10].

Given this dilemma, it is worthwhile to ask if there are more indirect and modest means to seek out quantum gravity, but strongly driven by a simple desire to acquire some readily identifiable predictions involving established theories of gravitation and quantum mechanics when put under extreme conditions. In other words, is it possible to identify a suitable length scale many orders of magnitude higher than the Planck scale, in which general relativity and quantum mechanics may overlap and interact in unforeseen ways, such that a realistic possibility for experimental tests can be theorized? To this question, an affirmative answer exists, motivated largely by an exploration of known phenomena whereby certain critical assumptions are isolated and scrutinized in depth.

For example, in a large part to address the so-called hypothesis of locality [11] when applied to quantum mechanical particles while in noninertial motion or gravitationally accelerated [12], it is shown that Casimir invariance for spin, as described by the Poincaré group, no longer holds true [13–15], and that this breakdown of formalism results in readily identifiable and interesting physical predictions that may be potentially observable. Such a predicted breakdown can also be compared with other approaches [16] to the deformation of the Poincaré group, whose Casimir operators for mass and spin can be determined. A second example concerns the interaction of neutrinos in a curved spacetime background, in which it is shown that a single neutrino described as a wave packet is sensitive to variations of spacetime curvature [17], such that Dirac and Majorana neutrinos can be rendered distinguishable without recourse to hypothetical particle physics beyond the standard model. In addition, for this model as applied to a two-flavour oscillation, it is theoretically possible to infer the absolute neutrino masses due to gravitational interactions alone [17, 18]. For a third example, it is shown that spin-1/2 particle Zitterbewegung in the presence of a local gravitational background [19] while propagating through spacetime generates a quantum violation of the weak equivalence principle, due to the explicit coupling of the background Ricci curvature tensor to the particle’s mass-dependent Zitterbewegung frequency, which is inversely proportional to the particle’s Compton wavelength.

With the same underlying motivations as cited above, this paper is intended to present some predicted physical consequences for the interaction of a spinless quantum mechanical particle with nonzero orbital angular momentum within a locally curved spacetime setting. It is overwhelmingly evident that, for example, the presence of a terrestrial gravitational background has for all practical purposes no discernable impact on the local properties of orbital angular momentum for some valence electron surrounding an atom at rest with respect to its environment. At the same time, it should also be evident that, by the weak principle of equivalence and geodesic deviation in classical general relativity, this same electron must still interact with local tidal acceleration effects at some space-like separation away from a reference frame situated at the center of the atom’s nucleus. Since it is impossible to remove the perceived effects of a gravitational background beyond a mathematical point, in principle they must manifest some predictive consequences for the properties of orbital angular momentum, no matter how small they may be at a practical level. While it is reasonable to question whether it is realistic to envision sensing local gravitational effects for atomic systems, on a purely theoretical level the outcome of this exploration may have a potentially significant impact on how to sharpen the focus of inquiry as it pertains to quantum gravity research.
A major finding of this paper is the observation that the orbiting particle is sensitive to half-integer spacings along the axis of quantization when the expectation is to observe strictly integer-valued spacings only. Given this conventional wisdom for quantum mechanics, such an observation to the contrary is very surprising, whose impact is keenly felt throughout this paper. In retrospect, however, there may already exist some basis in the literature suggesting that a classical gravitational background can theoretically reveal the presence of half-integer spin angular momentum states within the context of either nontrivial topological spaces [20, 21] or specific metric configurations [22]. Since this paper implicitly assumes a simply connected topology for the spacetime, it follows from the relevant literature [23] that only the prediction of integer-valued spin for the gravitational field is allowed following a $2\pi$ rotation of an isolated spacetime patch with respect to its environment. Indeed, for this paper such a requirement is satisfied. However, a truly significant observation to follow is that the minimum allowable orbital quantum number for the spinless particle to be sensitive to the gravitational background is $l = 2$, which precisely matches the spin angular momentum of a graviton, the wider implications of which are worth exploring in detail.

This paper begins with a physical motivation found in section 2 to justify the study of orbital angular momentum in locally curved spacetime. An outline of the basic formalism of orbital angular momentum within standard quantum mechanics immediately follows in section 3, which lays the foundation for presenting in section 4 the main details of this paper as it concerns the role of local gravitation on observables and the consequences that follow. An in-depth discussion concerning the specifics of this paper, including its possible connections with the existing literature, is then found in section 5, with a brief conclusion given in section 6.

For this paper, the spacetime metric background is expressed in terms of +2 signature and the conventions adopted by Misner, Thorne, and Wheeler [24].

2. Physical motivation

It is worthwhile to consider the following physical motivations for this paper. Suppose that spacetime curvature is represented in terms of either Fermi or Riemann normal coordinates $x^\mu = (\tau, x(\tau))$, where $\tau$ is the proper time defined with respect to some reference worldline. An orthonormal tetrad $\hat{e}_\mu = \delta^\mu_\nu + \tilde{R}^\mu_\nu$ is then assumed, such that hatted indices describe a local Lorentz frame, and $\tilde{R}^\mu_\nu$ is a two-indexed spacetime curvature deviation away from the locally flat spacetime background, not to be confused with the Ricci tensor. Since the local curved spacetime metric is $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}_\alpha \hat{e}_\beta$, it follows naturally that [25]

$$F_{\tilde{R}^\mu_\nu} = \left[ \frac{1}{2} R^\mu_{\lambda\nu\beta}(\tau) \delta^\lambda_\nu + \frac{1}{6} R^\mu_{\lambda\beta\kappa}(\tau) \delta^\kappa_\nu \right] \delta x^\lambda \delta x^\mu$$  \hspace{1cm} (1)$$

in Fermi normal coordinates, and

$$R_{\tilde{R}^\mu_\nu} = \frac{1}{6} R^\mu_{\alpha\beta\nu}(\tau) \delta x^\alpha \delta x^\beta$$  \hspace{1cm} (2)$$

in Riemann normal coordinates, where $R^\mu_{\alpha\beta\nu}(\tau)$ describes the Riemann curvature tensor in the local Lorentz frame, and $\delta x^\mu$ is interpreted as a spacetime quantum fluctuation with $|\delta x^\mu| \ll |x^\mu|$ to satisfy $\tilde{R}^\mu_\nu \ll \delta^\mu_\nu$.

Now suppose that the corresponding position ket vector for normal coordinates is described by $|x^\mu| = |(\tau, x)|$ that is subject to infinitesimal rotation of the spatial coordinates $x^i$ about the origin by the small angle $d\phi$ via a rotation operator $D_i^{(0)}(d\phi)$ [26]. Then, for a rotation about a locally defined $z$-direction as the axis of quantization, it follows that

$$D_z^{(0)}(d\phi)|x^\mu| = \left( 1 - \frac{i}{\hbar} d\phi L_z^{(0)} \right) |(\tau, x, y, z)|$$

$$= |(\tau, x - y d\phi, y + x d\phi, z)|,$$  \hspace{1cm} (3)$$
leading to an orbital angular momentum operator

\[
L_{\hat{i}}^{(0)} = \varepsilon_{ijk} x^j p^k
\]

defined for normal coordinates, where \(\varepsilon_{\mu\nu\rho\sigma}\) is the Levi-Civita tensor density [27] with \(\varepsilon_{0123} \equiv +1\). By following standard arguments involving the quantization of orbital angular momentum [26], it follows that eigenstates \(|l, m\rangle\) of \(L_{(0)}^2\) and \(L_{(0)}^z\) exist, denoted by integers \((l, m)\), such that

\[
L_{(0)}^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle, \tag{5a}
\]

\[
L_{(0)}^z |l, m\rangle = m\hbar |l, m\rangle, \quad -l \leq m \leq l. \tag{5b}
\]

Consider now the fact that a position vector in a local Lorentz frame is described by \(X_{\hat{\mu}}(\tau, x) = \tilde{e}_{\hat{\mu}} x^\nu X_{\hat{\nu}} = g_{\mu\nu} x^\mu x^\nu\). It is also possible to obtain a corresponding position ket vector \(|X_{\hat{\mu}}(\tau, x)\rangle_G\) that is unitarily equivalent to \(|x^\mu\rangle\), in the form \(|X_{\hat{\mu}}(\tau, x)\rangle_G = |x^\mu\rangle\), where

\[
U_{\text{Proj}}(\tau, x) = 1 + \tilde{\mathcal{R}}_{\alpha}^{\beta} x^\alpha \nabla_{\beta} \approx \varepsilon_{\alpha\mu\nu\sigma} \left[ \tilde{\mathcal{R}}_{\alpha}^{\hat{\nu}} x^\mu p^\rho \right]_S
\]

is a constructed unitary operator to project objects defined in a locally curved spacetime in terms of the local Lorentz frame for each instance of \(\tau\). Symmetrization ‘S’ of the position and momentum operators in accordance with Weyl ordering is imposed to ensure the unitarity of \(U_{\text{Proj}}(\tau, x)\). A visual representation of \(U_{\text{Proj}}(\tau, x)\) to illustrate its properties is given by figure 1.

If a unitary transformation of (4) is now applied using (6), it becomes evident that

\[
L_{\hat{i}}^{(0), \text{Proj}} = U_{\text{Proj}} L_{\hat{i}}^{(0)} U_{\text{Proj}}^{-1} = L_{\hat{i}}^{(0)} + L_{\hat{i}}^{(G), \text{Proj}}, \tag{7}
\]

where

\[
L_{\hat{i}}^{(G), \text{Proj}} \approx \varepsilon_{\hat{i}jk} \left[ \tilde{\mathcal{R}}_{\hat{i}}^{\mu} x^\rho p^\mu - x^j \tilde{\mathcal{R}}_{\hat{j}}^{\rho} p^\rho \right]. \tag{8}
\]
to leading order in $\tilde{R}^{\mu\nu}$, such that the spectra of eigenvalues for $|l, m\rangle$ and $\{U_{\text{Proj}} |l, m\rangle\}$ are identical [26]. This obvious but important property of unitarily equivalent kets in Hilbert space implies that the combination of terms that define (8) conspires to yield exactly no net changes in the observables of angular momentum under this restriction. As is shown below, however, a very different outcome occurs for orbital angular momentum operators defined explicitly for a local Lorentz frame, such that the gravitational contribution generates a nontrivial interaction that needs to be taken into account.

With this goal in mind, consider the rotation operator $D_{\hat{i}}(d\phi)$ suitably constructed to act directly on $|X^\hat{i}(\tau, x)\rangle_G$. For the specific case of rotation about the $z$-direction as defined by the local Lorentz frame,

$$D_{\hat{i}}(d\phi)|X^\hat{i}(\tau, x)\rangle_G = \left(1 - \frac{i}{\hbar}d\phi L_i\right)|(T, X, Y, Z)\rangle = \left|(T, X - Yd\phi, Y + Xd\phi, Z)\rangle\right.,$$

resulting in a frame-based orbital angular momentum operator

$$L_{\hat{i}} = \epsilon_{ijk}X^j P^k \approx L_{\hat{i}}^{(0)} + L_{\hat{i}}^{(G)} ,$$

where to leading order in $\tilde{R}^{\mu\nu}$,

$$L_{\hat{i}}^{(G)} \approx \epsilon_{ijk}\left[R^j_\alpha x^\alpha p^k + x^j \tilde{R}^k_\alpha p^\alpha\right] \neq L_{\hat{i}}^{(G),\text{Proj}} .$$

It becomes evident that, within the context of a local Lorentz frame, it is reasonable to expect that $|L, M\rangle$ denoted by integers $(L, M)$ are eigenstates of $L^2$ and $L^z$, such that

$$L^2 |L, M\rangle = L(L + 1)\hbar^2 |L, M\rangle, \quad L^z |L, M\rangle = M\hbar |L, M\rangle, \quad -L \leq M \leq L .$$

However, the fact that $L_{\hat{i}}^{(G)} \neq L_{\hat{i}}^{(G),\text{Proj}}$ via (11) indicates that $|L, M\rangle$ must be describable in general as

$$|L, M\rangle = aU_{\text{Proj}} |l, m\rangle + b|\lambda\rangle, \quad \langle\lambda|U_{\text{Proj}} |l, m\rangle = 0 ,$$

which is clearly not unitarily equivalent to $|l, m\rangle$ and corresponds to a physically distinct state from the one that is invisible to gravitational interactions due to local rotation. Therefore, the local Lorentz frame rotation operators are such that

$$D_{\hat{i}}(d\phi) \neq U_{\text{Proj}} \left[D_{\hat{i}}^{(0)}(d\phi)\right]U_{\text{Proj}}^{-1} ,$$

implying the existence of a nontrivial local gravitational interaction to distinguish between $|l, m\rangle$ and $|L, M\rangle$, with potential modifications in the observables of orbital angular momentum.

3. Formalism for orbital angular momentum

3.1. Commutation relations

Having established the physical motivation to justify examination of curvature effects on orbital angular momentum, it is necessary to now present the computational details to explicitly demonstrate their existence. Referring now exclusively to objects defined with respect to a local Lorentz frame, for notational simplicity all indices are now expressed in unhatted subscript form, and that $\epsilon_{0ijk} \equiv \epsilon_{ijk}$.
It is well known that the properties for $L_i (i = x, y, z)$ and the ladder operators $L_{\pm} \equiv L_i \pm i L_j$ acting on states described by $\{|L, M\rangle\}$ must satisfy [26]

\[
[L_i, L_j] = i \hbar \epsilon_{ijk} L_k, \quad (15a)
\]
\[
[L_z, L_{\pm}] = \pm \hbar L_{\pm}, \quad (15b)
\]
\[
[L_z, L_{\mp}] = 2\hbar L_z, \quad (15c)
\]
\[
[L_z^2, L_z] = 0, \quad (15d)
\]
\[
L_{\pm} |L, M\rangle = C_{L,M}^\pm |L, M \pm 1\rangle, \quad (15e)
\]

with

\[
C_{L,M}^\pm = \sqrt{(L \mp M)(L \pm M + 1)}.
\]

Simultaneously, the orbital angular momentum expressed in terms of $L_i^{(0)} (i = x, y, z)$ and its corresponding ladder operators $L_{\pm}^{(0)} \equiv L_i^{(0)} \pm i L_j^{(0)}$ acting on $|l, m\rangle$ must also satisfy

\[
[L_i^{(0)}, L_j^{(0)}] = i \hbar \epsilon_{ijk} L_k^{(0)}, \quad (17a)
\]
\[
[L_i^{(0)}, L_{\pm}^{(0)}] = \pm \hbar L_{\pm}^{(0)}, \quad (17b)
\]
\[
[L_i^{(0)}, L_{\mp}^{(0)}] = 2\hbar L_i^{(0)}, \quad (17c)
\]
\[
[L_{\pm}^{(0)}, L_k^{(0)}] = 0, \quad (17d)
\]
\[
L_{\pm}^{(0)} |l, m\rangle = c_{l,m}^{(0)\pm} |l, m \pm 1\rangle, \quad (17e)
\]

with

\[
c_{l,m}^{(0)\pm} = \sqrt{(l \mp m)(l \pm m + 1)}. \quad (18)
\]

It should be noted that, while $L$ must equal $l$ to conserve the orbital quantum number, it proves useful to use these separate labels to notationally distinguish between $|l, m\rangle$ and $|L, M\rangle$ for reasons that become clear later in this paper. Following from (10), substitutions of $L_i = L_i^{(0)} + L_i^{(G)}$ into (15a)–(15c) reveal that, to first-order in $L_i^{(G)}$,

\[
[L_i^{(G)}, L_j^{(0)}] + [L_i^{(0)}, L_j^{(G)}] = i \hbar \epsilon_{ijk} L_k^{(G)}, \quad (19a)
\]
\[
[L_i^{(G)}, L_{\pm}^{(0)}] + [L_i^{(0)}, L_{\pm}^{(G)}] = \pm \hbar L_{\pm}^{(G)}, \quad (19b)
\]
\[
[L_i^{(G)}, L_{\mp}^{(0)}] + [L_i^{(0)}, L_{\mp}^{(G)}] = 2\hbar L_i^{(G)}. \quad (19c)
\]

A solution to (19a) exists, in the form

\[
[L_i^{(G)}, L_j^{(0)}] = \frac{i \hbar}{2} \epsilon_{ijk} L_k^{(G)} \quad (20)
\]

subject to the conditions that

\[
[L_i^{(G)}, L_{\pm}^{(0)}] = 0, \quad (21a)
\]
\[
[L_i^{(G)}, L_{\mp}^{(0)}] = 0. \quad (21b)
\]

which need to be imposed. For the conditions to ensure that (21a) and (21b) are satisfied—the details of which are investigated later in this paper—it not only follows from (20) that (15d)

is satisfied to first order in $L_i^{(G)}$, explicit computations involving (20), (21a), and (21b) reveal independently that

\[
[L_i^{(0)}, L_{\pm}^{(0)}] = \frac{\pm \hbar}{2} L_i^{(G)}, \quad (22a)
\]
\[
[L_{\pm}^{(0)}, L_{\mp}^{(0)}] = \pm \hbar L_{\pm}^{(G)}, \quad (22b)
\]

automatically satisfying (19b) and (19c), respectively.
3.2. Physical consequences

Some significant implications follow from (20) to (22) as presented above. To begin, consider the first commutation relation of (22a) acting on $|l, m\rangle$. It is shown that

$$L^{(G)}_z [L^{(G)}_\pm |l, m\rangle] = \left( L^{(G)}_+ L^{(G)}_z + [L^{(G)}_z, L^{(G)}_\pm] \right) |l, m\rangle = (m \pm \frac{1}{2}) \hbar [L^{(G)}_\pm |l, m\rangle],$$

revealing that $L^{(G)}_\pm$ is indeed a ladder operator like $L^{(0)}_\pm$, but one that raises and lowers $m$ by half-integer steps, such that

$$L^{(G)}_\pm |l, m\rangle = c^{(G)\pm}_{l,m} |l, m \pm 1/2\rangle,$$

with the coefficients $c^{(G)\pm}_{l,m}$ to be determined. This is very interesting because (24) suggests that, in principle, $L^{(G)}_z$ is sensitive to half-integer $m$ projections of $|l, m\rangle$ that are otherwise considered either nonexistent or irrelevant to orbital angular momentum. In particular, this observation is the basis for suggesting a possible connection with the existence of a spin-1/2 internal structure within spacetime [20–22]. Despite this unexpected feature, there is no a priori reason to discount its validity, given that any prediction of a gravitational interaction in orbital angular momentum due to $L^{(G)}_i$ must be very small in comparison with a purely flat spacetime computation involving $L^{(0)}_i$ alone. A more quantitative statement to this effect follows later in this paper.

With (24) in hand, it is also possible to determine $L^{(G)}_z$ acting on $|l, m\rangle$, in terms of (19c) and (22b), with the result that

$$L^{(G)}_z |l, m\rangle = (L^{(G)}_+ + L^{(G)}_-) |l, m\rangle = (\mathcal{M}^{(G)+}_{l,m} \hbar) |l, m + 1/2\rangle + (\mathcal{M}^{(G)-}_{l,m} \hbar) |l, m - 1/2\rangle,$$

where

$$\mathcal{M}^{(G)+}_{l,m} = \pm \frac{1}{2\hbar} (c^{(G)+}_{l,m,\pm} c^{(G)-}_{l,m,\pm} - c^{(G)-}_{l,m,\pm} c^{(G)+}_{l,m,\mp}).$$

Clearly, $L^{(G)}_z$ does not generate $m$-projection eigenvalues associated with $|l, m\rangle$, but rather shifts the $|l, m\rangle$ state by a half-integer above and/or below $m$. Following from (25), a necessary constraint for (26) is that

$$\mathcal{M}^{(G)+}_{l,\pm l} = 0,$$

(27)

to ensure that $L^{(G)}_z |l, m\rangle$ remains confined to $-l \leq m \leq l$, such that

$$L^{(G)}_z |l, \pm l\rangle = (\mathcal{M}^{(G)+}_{l,\pm l} \hbar) |l, \pm (l - 1/2)\rangle.$$

(28)

Given that $c^{(G)+}_{l,\pm l} = 0$ from (18), this implies that

$$c^{(G)+}_{l,\pm l} = 0$$

(29)

is required from (26). While it is not obvious at present whether this constraint holds true, it is shown later in this paper that (29) is indeed satisfied.

4. Orbital angular momentum in the presence of local spacetime curvature

It should be clear that, while the operators $L^{(G)}_\pm$ and $L^{(G)}_z$ are describable in terms of $|l, m\rangle$, they also need to be expressed within the context of $|L, M\rangle$, since they contribute to defining the total orbital angular momentum $L_i$ that act on $|L, M\rangle$, as outlined in (12a)–(12b) and (15a)–(15e). This is necessary in order to determine the coefficients $c^{(G)\pm}_{l,m}$ with respect
to known quantities and also establish the conditions in which (21a)–(21b) are satisfied. It is also necessary to determine the scalar products \( \langle l, m | L, M \rangle \) relating \([l, m]\) with \([|L,M]\), now allowing for \(m\) to take on half-integer values, such that they can become parameters to fit with known measurement bounds.

### 4.1. Constraint equations

To begin, recall that \(L_\pm = \sum_{M=-L}^{L} |L, M \pm 1 \rangle \langle L, M|\), \(L_\pm^{(0)} = \sum_{m=-l}^{l} |l, m \pm 1 \rangle c_{l,m}^{(0)\pm} \langle l, m|\), \(L_z = \sum_{M=-L}^{L} |L, M \rangle m \hbar \langle L, M|\), \(L_z^{(0)} = \sum_{m=-l}^{l} |l, m \rangle m \hbar \langle l, m|\), keeping in mind that summations of \(m\) from \(-l\) to \(l\) now go by half-integer steps, amounting to \(4l + 1\) terms in the sum, while \(M\) remains integer valued with \(2L + 1\) terms in the sum, and that \(L = l\) to conserve the orbital quantum number. As well, since it is true that \(L = l\), the combination of (30a), (30b), and (31) to form \(\langle l, m_1 | L_\pm | l, m_2 \rangle\) leads to

\[
c_{l,m_2}^{(G)\pm} \delta_{m_1,m_2\pm 1} = \sum_{M=-L}^{L} C_{L,M}^{\pm} \langle l, m_2 | L, M \rangle^* \langle l, m_1 | L, M \pm 1 \rangle - c_{l,m_2}^{(0)\pm} \delta_{m_1,m_2\pm 1} = 0
\]

which for \(m_1 \neq m_2 \pm \frac{1}{2}\) results in a constraint equation

\[
\sum_{M=-L}^{L} C_{L,M}^{\pm} \langle l, m_2 | L, M \rangle^* \langle l, m_1 | L, M \pm 1 \rangle - c_{l,m_2}^{(0)\pm} \delta_{m_1,m_2\pm 1} = 0
\]  

for \(\langle l, m | L, M \rangle\), while it follows that

\[
c_{l,m}^{(G)\pm} = \sum_{M=-L}^{L} C_{L,M}^{\pm} \langle l, m | L, M \rangle^* \langle l, m \pm 1/2 | L, M \pm 1 \rangle
\]  

for \(m_1 = m_2 \pm \frac{1}{2}\). As expected, (34) requires nonzero scalar products with half-integer values of \(m\), in order for \(c_{l,m}^{(G)\pm} \neq 0\). In addition, the magnitude of \(c_{l,m}^{(G)\pm}\) can be determined according to \(|c_{l,m}^{(G)\pm}|^2 = \langle l, m | L_\pm^{(G)} | l, m \rangle\), leading to

\[
|c_{l,m}^{(G)\pm}|^2 = |c_{l,m}^{(0)\pm}|^2 + \sum_{M=-L}^{L} \left| C_{L,M}^{\pm} \langle l, m | L, M \rangle \right|^2 - 2 \text{Re} \left[ C_{L,M}^{\pm} c_{l,m}^{(0)\pm} \langle l, m \pm 1 | L, M \pm 1 \rangle^* \langle l, m | L, M \rangle \right].
\]
Considering now the commutators (21a)–(21b) that are assumed true when used in conjunction with (20), it is a straightforward matter to evaluate them directly using (30a)–(30d). Starting with (21a), it follows that

\[
[L_{\pm}^{(G)}, L_{\pm}^{(0)}] = \sum_{m=-l}^{l} \left[ c_{l,m}^{(0)\pm} - c_{l,m+1}^{(0)\pm} - c_{l,m+1}^{(G)\pm} + c_{l,m}^{(G)\pm} \right]\langle l, m \pm 3/2 | l, m \rangle = 0,
\]

which is always true if the prefactor for each \(m\) is set to zero. Therefore, (36) is satisfied at the operator level, provided that the constraint equation

\[
\sum_{M=-L}^{L} C_{L,M} \langle l, m \pm 1 | L, M \rangle \langle l, m \pm 3/2 | L, M \pm 1 \rangle = 0
\]

for \(\langle l, m | L, M \rangle\) is satisfied, using (34) in place of \(c_{l,m}^{(G)\pm}\).

As for (21b), a similar approach with the incorporation of

\[
|L, M\rangle = \sum_{m=-l}^{l} \langle l, m | L, M \rangle |l, m\rangle
\]

results in

\[
[L_{z}^{(G)}, L_{z}^{(0)}] = \sum_{m=-l}^{l} \sum_{m'=-l}^{l} \left\{ \hbar^{2} \sum_{M=-L}^{L} M(m' - m) \langle l, m' | L, M \rangle \langle l, m | L, M \rangle \right\} \times |l, m \rangle \langle l, m' | \]

\[
\langle l, m | = 0.
\]

4.2. Evaluation of the constraint equations

Unlike (36), it is not obvious at present whether (39) holds true. In part, to address this issue, it is very useful to introduce a physically motivated simplification by letting (38) approximate to

\[
|L, M\rangle \approx \alpha_{M} |l, M\rangle + \beta_{M}^{(+)} |l, M + 1/2\rangle + \beta_{M}^{(-)} |l, M - 1/2\rangle,
\]

or equivalently

\[
|l, m\rangle \approx \alpha_{m}^{*} |L, m\rangle + \beta_{m+\frac{1}{2}}^{(+)} |L, m - 1/2\rangle + \beta_{m+\frac{1}{2}}^{(-)} |L, m + 1/2\rangle,
\]

with

\[
\langle L, M | L, M \rangle = |\alpha_{M}|^{2} + |\beta_{M}^{(+)}|^{2} + |\beta_{M}^{(-)}|^{2} \approx 1
\]

such that \(|\beta_{M}^{(\pm)}|^{2} \ll |\alpha_{M}|^{2}\) for each integer-valued \(M\), and \(\beta_{\pm L}^{(\pm)} = 0\).

With (40), it is possible to conveniently express (30d) in the form

\[
L_{z}^{(0)} = \hbar \sum_{M=-L}^{L} |L, M\rangle \left[ M |\alpha_{M}|^{2} + \left( M + \frac{1}{2} \right) |\beta_{M}^{(+)}|^{2} + \left( M - \frac{1}{2} \right) |\beta_{M}^{(-)}|^{2} \right] \langle L, M | L, M \rangle
\]

\[
+ \hbar \sum_{M=-L}^{L} |L, M\rangle \left( M + \frac{1}{2} \right) \beta_{M}^{(+)} \beta_{M+1}^{(-)} \langle L, M + 1 | L, M \rangle
\]

\[
+ \hbar \sum_{M=-L}^{L} |L, M\rangle \left( M - \frac{1}{2} \right) \beta_{M}^{(-)} \beta_{M-1}^{(+)} \langle L, M - 1 | L, M \rangle.
\]
Therefore, it becomes straightforward to show that

\[
[L^{(G)}_\pm, L^{(0)}_\pm] = \hbar^2 \sum_{M=-L}^L |L, M \rangle \left( M - \frac{1}{2} \right) \beta^{(-)}_M \beta^{(+)}_{M+1} |L, M - 1\rangle 
- \left( M + \frac{1}{2} \right) \beta^{(+)}_M \beta^{(-)}_{M+1} |L, M + 1\rangle .
\]  

(44)

Clearly, (44) is nonzero, with a structure that is somewhat analogous to angular momentum orthogonal to the z-direction [28]. However, (44) is only a second-order operator in \( \beta^{(\pm)}_M \), while the right-hand side of (20) is necessarily a first-order operator. Therefore, (44) satisfies (21b) within this approximation structure and still satisfies (19a) outright. In retrospect, observing this discrepancy as it concerns (21b) is not surprising, considering that spacetime curvature has the effect of introducing other breakdowns in quantum mechanical concepts, such as the non-Hermiticity of the Dirac Hamiltonian for spin-1/2 particles in a gravitational background [19, 29].

Turning attention now to (37), the constraint equation for \([L^{(G)}_\pm, L^{(0)}_\pm] = 0\), substitution of (40) results in

\[
c^{(0)}_{l,m} \left[ C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(\pm)}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(\pm)*}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(-)}_{m;3} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(-)*}_{m;3} \right] 
- c^{(0)}_{l,m;1} \left[ C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(\pm)}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(\pm)*}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(-)}_{m;3} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(-)*}_{m;3} \right] = 0 .
\]  

(45)

When evaluated for the upper and lower signs separately, (45) can be expressed in a single constraint equation as

\[
c^{(0)}_{l,m} \left[ C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(\pm)}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(\pm)*}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(-)}_{m;3} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(-)*}_{m;3} \right] 
- c^{(0)}_{l,m;1} \left[ C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(\pm)}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(\pm)*}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(-)}_{m;3} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(-)*}_{m;3} \right] = 0 .
\]  

(46)

The structure of (46) demands that, for \( m \) equal to an integer,

\[
c^{(0)}_{l,m;1} C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(\pm)}_{m;2} + c^{(0)}_{l,m;1} C_{L,m;1}^\pm \alpha_{m;1} \beta^{(\pm)*}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(-)}_{m;3} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(-)*}_{m;3} = 0 ,
\]  

(47)

while for \( m \) equal to a half-integer,

\[
c^{(0)}_{l,m;1} C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(\pm)}_{m;2} + c^{(0)}_{l,m;1} C_{L,m;1}^\pm \alpha_{m;1} \beta^{(\pm)*}_{m;2} + C_{L,m;1}^\pm \alpha_{m;1}^* \beta^{(-)}_{m;3} + C_{L,m;1}^\pm \alpha_{m;1} \beta^{(-)*}_{m;3} = 0 .
\]  

(48)

By letting \( m \rightarrow m \mp \frac{1}{2} \) in (48) for direct comparison with (47) and recognizing that \( C_{L,m}^\pm = c^{(0)}_{l,m;1} \) for \( L = l \) and \( M = m \), it follows that

\[
\left( \frac{\alpha_{M;1}}{\alpha_M} \right)^* = \left( \frac{C_{L,M;1}}{C_{L,M;1}} \right)^* \left( \frac{\beta^{(\pm)}_{M;1}}{\beta^{(\pm)}_{M;2}} \right) = \left( \frac{C_{L,M;1}}{C_{L,M;1}} \right)^* \left( \frac{\alpha_{M;1}}{\alpha_M} \right)^* \left( \frac{\beta^{(\pm)}_{M;1}}{\beta^{(\pm)}_{M;2}} \right) ,
\]  

(49)

which leads to a recursion relation for \( \beta^{(\pm)}_{M;2} \) of the form

\[
\beta^{(\pm)}_{M;2} = \left( \frac{C_{L,M;1}^*}{C_{L,M;1}^*} \right)^* \left( \frac{C_{L,M;1}^*}{C_{L,M;1}^*} \right)^* \left( \frac{\alpha_{M;1}}{\alpha_M} \right)^* \beta^{(\pm)}_{M;1} \right) , \quad (\beta^{(\pm)}_{M;1} = 0 .
\]  

(50)

Without knowing anything further about this recursion relation, it is immediately obvious that (50) is subject to the restriction \((-L - 2) \leq M \leq L - 2\), which implies a minimum
allowable value of $L = 2$ in order for any gravitational corrections of orbital angular momentum to occur within this framework. This is also the basis for suggesting a deep connection with the properties of the graviton for linearized general relativity, since it is anticipated to propagate with spin angular momentum $\tilde{S} = 2$, matching perfectly with the constraint conditions of (50).

For the final constraint equation (33) to consider, substitution of (40) leads to

$$\left[ C_{L,m_1 \mp 1}^{\pm} \alpha_m \alpha_m^{* \mp 1} - c_{l,m_1 \mp 1}^{(0) \pm} \right] \delta_{m_1, m_{1, \pm 1}}$$

$$+ \left[ C_{L,m_1 \mp 1}^{\pm} \alpha_m^{* \pm 1} \beta_{m_1 \mp 1}^{\pm} + C_{L,m_1 \mp 1}^{\mp} \alpha_m \beta_{m_1 \mp 1}^{(\mp \pm)} \right] \delta_{m_1, m_{1, \pm 1}}^2 = 0,$$

subject to $m_1 \neq m_2 \pm \frac{1}{2}$. It is evident from (51) that the only physically relevant constraint to be satisfied is the recurrence relation corresponding to $m_1 = m_2 \pm 1$, since while the one corresponding to $m_1 = m_2 \pm \frac{3}{2}$ demands the existence of $\alpha_M$ and $\beta_M^{(\pm \pm)}$ for half-integer $M$ to ensure that nontrivial integer-valued $M$ terms can appear, such quantities are not accessed within the framework of this problem and can be ignored. Therefore, the recursion relations for $\alpha_M$ are

$$\alpha_{M \pm 1}^{(\pm)} = \frac{\alpha_M^{(\pm)} |\alpha_M|}{|\alpha_M|^2},$$

and from substituting (52a) into (49) and (50), the recursion relations for $\beta_M^{(\pm \pm)}$ are

$$\beta_{M \pm 1}^{(\pm \pm)} = \left( \frac{C_{L,M}^{\pm \pm}}{C_{L,M}^{\pm \pm}} \right) \beta_M^{(\pm \pm)} - (L - 2) \leq M \leq L - 2,$$

$$\beta_{M \pm 2}^{(\pm \pm)} = \left( \frac{C_{L,M}^{\pm \pm}}{C_{L,M}^{\pm \pm}} \right) |\alpha_M| \beta_M^{(\pm \pm)},$$

$$\beta_{M \pm 2}^{(\pm \pm)} = \left( \frac{C_{L,M}^{\pm \pm}}{C_{L,M}^{\pm \pm}} \right) |\alpha_M| \beta_M^{(\pm \pm)},$$

with $|\alpha_M|^2 \approx 1 - |\beta_M^{(\pm \pm)}|^2$ from (42). With $\beta_{M \mp 2}^{(\pm \pm)} = 0$ as a boundary condition, only $\beta_{M \mp 1}^{(\pm \pm)}$ is left to specify separately. This is accomplished by first letting $M \rightarrow M \pm 1$ in (53b), such that

$$\beta_{M \pm 1}^{(\pm \pm)} = \left( \frac{C_{L,M}^{\pm \pm}}{C_{L,M}^{\pm \pm}} \right) |\alpha_M| \beta_M^{(\pm \pm)}.$$

Therefore, by substituting $M = \mp (L - 1)$ into (54) and re-arranging, it follows that

$$\beta_{M \pm (L - 1)}^{(\pm \pm)} = \left( \frac{C_{L,M}^{\pm \pm}}{C_{L,M}^{\pm \pm}} \right) |\alpha_M| \beta_M^{(\pm \pm)}.$$

With (52a)–(55), it becomes clear that all of the $\alpha_M$ and $\beta_M^{(\pm \pm)}$ can be computed in relation to $\alpha_0$ and $\tilde{\beta}_0^{(\pm \pm)}$ as input parameters to be determined from experimental data.

4.3. Evaluation of the gravitational ladder coefficients

It remains to present the computation of the gravitational ladder coefficients $c_{l,m}^{(G) \pm}$ and their magnitudes according to (34) and (35), respectively, based on the approximation employed for
follows from a straightforward evaluation of (34) that

| L, M ⟩ via (40). Not surprisingly, the coefficients are defined in accordance with the choice of m as either integer- or half-integer valued. Therefore, with the use of (52a) and (53a), it follows from a straightforward evaluation of (34) that

\[
c_{l,m}^{(G)±} = c_{l,m}^{(0)±} \alpha_m^* \beta_m^{(±)} = \left( \frac{c_{l,m}^{(0)±}}{\alpha_m^*} \right) \left( \frac{c_{l,m}^{(0)±}}{\beta_m^{(±)}} \right) \alpha_m^* \beta_m^{(±)}
\]

(56)

for m an integer, while

\[
c_{l,m}^{(G)±} = c_{l,m}^{(0)±} \alpha_m^* \beta_m^{(±)*} = \left( \frac{c_{l,m}^{(0)±}}{\alpha_m^*} \right) \left( \frac{c_{l,m}^{(0)±}}{\beta_m^{(±)*}} \right) \alpha_m^* \beta_m^{(±)*}
\]

(57)

for m a half-integer. Clearly, both (56) and (57) vanish in the limit as \(|\beta_M^{(±)}| \to 0\), as expected. As well, (56) satisfies the condition that \(c_{l,m}^{(G)±} = 0\) as claimed in (29), such that (27) is also satisfied, ensuring that \(L_z^{(G)} |l, m\rangle\) is confined to \(-l \leq m \leq l\).

It is interesting to note from (52a) and (53a) that each iteration of \(c_M^*_m\) preserves the same phase angle of its predecessor, as denoted by tan \(\gamma_M = \text{Im}(\alpha_M)/\text{Re}(\alpha_M)\) and tan \(\delta_M^{(±)} = \text{Im}(\beta_M^{(±)}/\text{Re}(\beta_M^{(±)})\), though it is still possible for \(\alpha_M\) and \(\beta_M^{(±)}\) to have a relative phase difference. As a result, both (56) and (57) demonstrate the existence of a generally nonzero phase angle associated with \(c_{l,m}^{(G)±}\), given by

\[
\tan \varphi_m = -\frac{\tan \gamma_m - \tan \delta_M^{(±)}}{1 + \tan \gamma_m \tan \delta_M^{(±)}} = -\tan \left( \gamma_m - \delta_M^{(±)} \right)
\]

(58)

for integer-valued \(m\), while

\[
\tan \varphi_m = \frac{\tan \gamma_M^{±} + \tan \delta_M^{(±)}}{1 + \tan \gamma_M^{±} \tan \delta_M^{(±)}} = \tan \left( \gamma_M^{±} - \delta_M^{(±)} \right)
\]

(59)

for half-integer-valued \(m\), with tan \(\varphi_m^{±}\) = tan \(\varphi_m\) for both cases.

Regarding the magnitude of the gravitational ladder coefficients in terms of (40) and (41), recall that \(|c_{l,m}^{(G)±}|^2 = (l, m | L_z^{(G)} L_z^{(G)} | l, m\rangle\), where

\[
L_z^{(G)} L_z^{(G)} = L_z^{(0)} L_z^{(0)} + \sum_{M=-L}^{L} |L, M\rangle |c_{L,M}^{±}|^2 \langle L, M|
\]

\[
- \sum_{M=-L}^{L} |C_{L,M}^{±}| |c_{l,M}^{(0)±}|^2 \left[ |L, M\rangle \alpha_M^{±} \langle L, M| + |L, M\rangle \alpha_M^{±} \langle L, M| \right]
\]

\[
- \sum_{M=-L}^{L} |C_{L,M}^{±}| |c_{l,M}^{(0)±}|^2 \left[ |L, M+1/2\rangle \beta_M^{(±)} \langle L, M| + |L, M-1/2\rangle \beta_M^{(±)} \langle L, M| \right]
\]

\[
+ |L, M\rangle \beta_M^{(±)} \langle L, M+1/2\rangle
\]

\[
- \sum_{M=-L}^{L} |C_{L,M}^{±}| |c_{l,M}^{(0)±}|^2 \left[ |L, M-1/2\rangle \beta_M^{(±)} \langle L, M| + |L, M\rangle \beta_M^{(±)} \langle L, M-1/2\rangle \right]
\]

(60)

and
\[ L_{z}^{(0)} L_{+}^{(0)} = \sum_{m=-1}^{L} |L, m\rangle |c_{l,m}^{(0)\pm}\rangle^{2} |L, m\rangle \]
\[ \quad = \sum_{M=-L}^{L} \left[ \langle L, M \rangle \left( |\alpha_{M}|^{2} |c_{l,M}^{(0)\pm}\rangle^{2} + |\beta_{M}^{(\pm)}|^{2} |c_{l,M+\frac{1}{2}}^{(0)\pm}\rangle^{2} + |\beta_{M}^{(-)}|^{2} |c_{l,M-\frac{1}{2}}^{(0)\pm}\rangle^{2} \right) \right. \]
\[ \quad \times \langle L, M \rangle + |L, M\rangle \, \beta_{M}^{(\pm)} \beta_{M+1}^{(-)} |c_{l,M+\frac{1}{2}}^{(0)\pm}\rangle^{2} \langle L, M + 1 \rangle \]
\[ \quad + |L, M\rangle \, \beta_{M}^{(-)} \beta_{M-1}^{(\pm)} |c_{l,M-\frac{1}{2}}^{(0)\pm}\rangle^{2} \langle L, M - 1 \rangle \right]. \]

It happens that \(|c_{l,m}^{(G)\pm}\rangle^{2} = 0\) for both (60) and (61) in the limit as \(|\beta_{M}^{(\pm)}| \rightarrow 0\) and \(|\alpha_{M}| \rightarrow 1\), again ensuring that all gravitationally induced expressions involving orbital angular momentum vanish smoothly in the flat spacetime limit.

\[ (61) \]

\[ |c_{l,m}^{(G)\pm}\rangle^{2} = \left\{ (1 + |\alpha_{m}|^{2}) |c_{l,m}^{(0)\pm}\rangle^{2} + |\beta_{m}^{(\pm)}|^{2} |c_{l,m+\frac{1}{2}}^{(0)\pm}\rangle^{2} + |\beta_{m}^{(-)}|^{2} |c_{l,m-\frac{1}{2}}^{(0)\pm}\rangle^{2} \right\} |\alpha_{m}|^{2} \]

for \(m\) integer, while

\[ |c_{l,m}^{(G)\pm}\rangle^{2} = \left\{ (1 + |\alpha_{m+\frac{1}{2}}|^{2}) |c_{l,m+\frac{1}{2}}^{(0)\pm}\rangle^{2} + |\beta_{m+\frac{1}{2}}^{(\pm)}|^{2} |c_{l,m+1}^{(0)\pm}\rangle^{2} + |\beta_{m+\frac{1}{2}}^{(-)}|^{2} |c_{l,m+1}^{(0)\pm}\rangle^{2} \right\} |\beta_{m+\frac{1}{2}}^{(\pm)}|^{2} \]

for half-integer \(m\). It is self-evident that \(|c_{l,m}^{(G)\pm}\rangle^{2} \rightarrow 0\) for both (62) and (63) in the limit as \(|\beta_{M}^{(\pm)}| \rightarrow 0\) and \(|\alpha_{M}| \rightarrow 1\), again ensuring that all gravitationally induced expressions involving orbital angular momentum vanish smoothly in the flat spacetime limit.

\[ (62) \]

\[ (63) \]

4.4. Physical consequences for observables

It is important to understand the physical consequences for the observables of orbital angular momentum when incorporating local spacetime curvature in this proposed fashion. This entails performing the direct computation of \(L^{2} \mid L, M \rangle \rangle \) and \(L_{z} \mid L, M \rangle \rangle \) to show the dependence of \(\alpha_{M}\) and \(\beta_{M}^{(\pm)}\) on the expressed quantities. As shown explicitly below, it follows that while both \(|L^{2} - L(L + 1)\beta_{M}^{2}\rangle \langle L, M \rangle \rangle \) and \(|L_{z} - M \hbar L, M \rangle \rangle \) equal zero to first order in \(\beta_{M}^{(\pm)}\), the second-order terms which survive are not proportional to \(|L, M\rangle \rangle \), indicating that \(|L, M\rangle \rangle \) can no longer be classified as the eigenstates of \(L^{2}\) and \(L_{z}\). Given (44) for the evaluation of \([L_{z}^{(G)}, L_{i}^{(G)}]\), this should not come as a surprise. Nonetheless, it is necessary to fully understand all the consequences that result from introducing local curvature effects into the current description of orbital angular momentum.

To illustrate this more fully, recall that to first order in \(L_{i}^{(G)}\)

\[ L^{2} = L_{0}^{(0)} + \{ L_{i}^{(G)}, L_{j}^{(0)} \}, \]

where it is shown that

\[ \{ L_{i}^{(G)}, L_{j}^{(0)} \} = L_{i}^{(0)} L_{j}^{(G)} + L_{j}^{(0)} L_{i}^{(G)} + 2L_{i}^{(0)} L_{j}^{(G)} + \left[ L_{i}^{(G)}, L_{i}^{(0)} \right] \]

(65)
and

\[
L_z^{(G)} = \sum_{m=-L}^{L} \left( (|l, m + 1/2) \mathcal{M}_{l,m}^{(G)^+} \hbar (|l, m + 1/2) \mathcal{M}_{l,m}^{(G)^-} \hbar (|l, m) \right)
\]

(66)

from (25) and (26). Knowing (30b), (30d), and (31) in terms of \(|l, m\)|, their equivalent expressions in terms of \(|L, M\)| in combination with (44) result in

\[
\{ L_z^{(G)}, L_i^{(0)} \} = \sum_{M=-L}^{L} |L, M \rangle \left( \alpha_M \left( \beta_M^{(+) \star} \mathcal{F}_{L,M}^{(G)+} + \beta_M^{(-) \star} \mathcal{F}_{L,M}^{(G)-} \right) \hbar^2 \right) \langle L, M |
\]

\[
+ \sum_{M=-L}^{L} |L, M \rangle \left( \beta_M^{(-)} \left[ \mathcal{F}_{L,M-1}^{(G)+} + \left( M - \frac{1}{2} \right) \beta_M^{(-)} \hbar^2 \right] \right) \langle L, M - 1 |
\]

\[
+ \sum_{M=-L}^{L} |L, M \rangle \left( \beta_M^{(+)} \left[ \mathcal{F}_{L,M+1}^{(G)-} - \left( M + \frac{1}{2} \beta_M^{(+)} \hbar^2 \right] \right) \langle L, M + 1 |
\]

(67)

where

\[
\mathcal{F}_{L,m}^{(G)\pm} = (2m + 1) \mathcal{M}_{l,m}^{(G)\pm} + \frac{\hbar^2}{2} \mathcal{Q}_{l,m}^{(G)\pm} + \frac{1}{\hbar^2} \mathcal{Q}_{l,m}^{(G)\pm}
\]

(68)

is first order in \(\beta_M^{(\pm)}\). Therefore, it follows that

\[
L^2 |L, M\rangle = \left[ L (L + 1) + \alpha_M \left( \beta_M^{(+) \star} \mathcal{F}_{L,M}^{(G)+} + \beta_M^{(-) \star} \mathcal{F}_{L,M}^{(G)-} \right) \right] \hbar^2 |L, M\rangle
\]

\[
+ \beta_M^{(-)\star} \left[ \alpha_M \mathcal{F}_{L,M-1}^{(G)+} + \frac{1}{2} \left( 2M + 1 \right) \beta_M^{(-)} \hbar^2 \right] |L, M + 1\rangle
\]

\[
+ \beta_M^{(+)\star} \left[ \alpha_M \mathcal{F}_{L,M+1}^{(G)-} - \frac{1}{2} \left( 2M - 1 \right) \beta_M^{(+)} \hbar^2 \right] |L, M - 1\rangle,
\]

(69)

which is strictly no longer an eigenvalue equation, though only at second order in \(\beta_M^{(\pm)}\). By a similar procedure, it can be shown for \(L_z\) that

\[
L_z |L, M\rangle = \left[ M + \frac{1}{2} \left( \beta_M^{(+) \star} - \beta_M^{(-) \star} \right) \alpha_M \left( \beta_M^{(+)} \mathcal{M}_{L,M}^{(G)+} + \beta_M^{(-)} \mathcal{M}_{L,M}^{(G)-} \right) \right] \hbar |L, M\rangle
\]

\[
+ \alpha_M \left( \beta_M^{(-)} \mathcal{M}_{L,M+1}^{(G)+} + \beta_M^{(+) \star} \mathcal{M}_{L,M-1}^{(G)-} \right) \hbar |L, M + 1\rangle
\]

\[
+ \left[ \left( M + \frac{1}{2} \right) \beta_M^{(+) \star} \beta_M^{(+) \star} + \beta_M^{(-)} \beta_M^{(+)} \alpha_M \mathcal{M}_{L,M}^{(G)+} + \beta_M^{(+) \star} \mathcal{M}_{L,M}^{(G)-} \hbar \right] |L, M + 1\rangle
\]

\[
+ \left[ \left( M - \frac{1}{2} \right) \beta_M^{(+) \star} \beta_M^{(-)} + \beta_M^{(-) \star} \beta_M^{(+)} \alpha_M \mathcal{M}_{L,M}^{(G)-} + \beta_M^{(-)} \mathcal{M}_{L,M}^{(G)+} \hbar \right] |L, M - 1\rangle.
\]

(70)

also no longer an eigenvalue equation at second order in \(\beta_M^{(\pm)}\).

An interesting observation results from considering the diagonal matrix elements for \(L^2\) and \(L_z\). By supposing that

\[
\langle L, M | L^2 | L, M \rangle \approx L (L + 1) \hbar^2 \text{eff}.
\]

(71)

\[
\langle L, M | L_z | L, M \rangle \approx M \hbar \text{eff},
\]

(72)

where \(\hbar \text{eff}\) becomes a predicted spacetime curvature-dependent Planck’s constant defined with respect to the observed Planck’s constant of \(\hbar = 1.05457195(07) \times 10^{-34}\) J s [30] in a flat spacetime background, it is possible to express the gravitational contributions to (69) and (70) in terms of

\[
\hbar \text{eff} = \left( 1 + \frac{\Delta \alpha(G(L, M))}{\hbar} \right) \hbar,
\]

(73)
with $\Delta h_{G}(L, M)$ a second-order function of $\hat{p}_M^{(\pm)}$. By relating $\Delta h_{G}(L, M)$ to the relative uncertainty in the measurement of Planck’s constant, in the form

$$\frac{\Delta h_{G}(L, M)}{\hbar} < \frac{\Delta h}{\hbar}_{\text{expt}} \approx 6.63776 \times 10^{-8},$$

(74)

it is possible to establish an upper bound for $\hat{p}_M^{(\pm)}$, such that $|\hat{p}_M^{(\pm)}|^2 < 10^{-8} \ll 1$. This observation provides a strong confirmation that, for quantum phenomena measured in the presence of the Earth’s gravitational field, the local spacetime curvature has a negligible impact on the observables of orbital angular momentum, as expected, and provides further justification for adopting the approximations used to motivate this investigation.

5. Discussion

Having established the details of orbital angular momentum for a spinless particle in a locally curved spacetime background, it is useful to examine some of its relevant implications. An immediate example comes from the recursion relation (53c), showing that $l = 2$ is the minimum orbital quantum number that accommodates the presence of local curvature. A transition to $l = 0$ most likely occurs as a two-step process with the emission of a single photon at each step, with the understanding that the first transition to $l = 1$ destroys the boundary conditions necessary to incorporate the curvature contributions. As stated earlier, the minimum boundary condition at $l = 2$ coincides with the spin of a graviton and suggests that a one-step transition to $l = 0$ is also theoretically possible, with the simultaneous emission of a single graviton to conserve angular momentum. For $l > 2$, it is possible to envision a combination of photon and graviton emissions that also satisfy known selection rules for each transition.

If graviton emission is reasonable to expect, then conversely such a suggestion also implies that graviton absorption is possible, in like fashion to photon absorption under similar conditions. At the macroscopic level, it is commonly understood that gravitational wave radiation due to astrophysical sources occurs as freely propagating ripples in spacetime, described quantum mechanically as coherent graviton emissions. In particular, it is also well understood that gravitational waves can propagate to very distant observers without experiencing any noticeable dispersion while passing through matter. Therefore, at the quantum mechanical level there is an expectation that coherent emissions must also occur for a many-body quantum system to ultimately preserve the macroscopic properties of the emitted waveform. This means that any coherent graviton absorptions for such a system must also translate into a virtually instantaneous and coherent emission to account for the anticipated lack of any dispersive effects.

It is no surprise that, given the small local curvature deviation away from an otherwise flat spacetime background, there is no realistic possibility of observing gravitational contributions to orbital angular momentum under current laboratory conditions. That is, $|\hat{p}_M^{(\pm)}|^2 \sim 0$ and $|\alpha_M|^2 \sim 1$ for likely all relevant experiments performed on the Earth. However, this condition does not necessarily apply when dealing with much stronger gravitational fields, in which a sufficiently small radius of curvature in the spacetime background exists compared to the particle’s orbital radius. It is conceivable, therefore, that such a situation arises near the event horizon of a microscopic black hole [14]. In fact, it seems possible that if such a black hole undergoes Hawking radiation, the backreaction due to time-dependent induced curvature variations may have a significant impact on the orbital angular momentum of an orbiting quantum particle nearby, such that corresponding signatures may appear in photon and graviton emissions corresponding to transitions from higher to lower levels of $l$. 

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As a matter of principle, the concept of a curvature-dependent Planck’s constant $\hbar_{\text{eff}}$ should have an impact on all known physical observables in the quantum domain, such as decay rates and scattering cross-sections, and any other measurements involving quantized energy states. In addition, given the interest in trying to observe time variations in the fine structure constant over cosmological time scales [31], it may be conceivable to also envision local curvature variations impacting on spectral line emissions due to matter-energy fluctuations in the early Universe. Another possibility for identifying the presence of $\hbar_{\text{eff}}$ may occur with ultra-high precision measurements of the gravitational redshift effect, as originally demonstrated by the Pound–Rebka experiment and its variants [32–34]. In particular, it is conceivable that different theoretical predictions may appear in the gravitational redshift effect in response to differences in the metric gravitation theories under consideration. Nevertheless, it remains unrealistic to expect any clearly observable signatures to appear in near-future measurements of the gravitational redshift effect.

It is important to recall that this paper assumes a simply connected topology for the spacetime background. This necessitates the restriction of an integer-valued spin to describe the gravitational field [23], if rotated by $2\pi$ with respect to an environment described by a larger spacetime background in which the field in question is treated as an isolated system. The claim by Friedman and Sorkin [20] that it is possible to identify the existence of a spin-1/2 gravitational background within a nontrivial topological space is a very interesting observation suggesting that spacetime is fundamentally spinorial in structure. It is, therefore, worthwhile to consider if a similar analysis of orbital angular momentum within a nontrivial topological setting leads to the same conclusions as first drawn by Friedman and Sorkin, but based upon a completely different set of motivations. Given this paper’s suggestion that the gravitational part of the orbital angular momentum operator is sensitive to half-integer spacings along the axis of quantization, it seems reasonable to suggest that a pre-existent spin-1/2 structure is somehow embedded within the restrictions imposed by a simply connected topology, due to the boundary conditions that define $\alpha_M$ and $\beta_M^{(\pm)}$. Within the context of quantum gravity, this is an important issue to consider if Wheeler’s ‘spacetime foam’ concept [35] is well posed, such that topologically dynamical structures spontaneously emerge at a sufficiently small length scale. In principle, such effects may be reflected within $\alpha_M$ and $\beta_M^{(\pm)}$ as time-dependent signatures with potentially large variations in their relative magnitudes, though the likelihood of envisioning an experiment to test this hypothesis seems remote.

Finally, a different type of consideration to follow from this paper involves the coupling of rotation $\Omega$ to the orbital and spin angular momentum of a quantum particle. Recently, it is shown by Shen [36] that Mashhoon’s proposed spin-rotation coupling interaction $\Omega \cdot \mathbf{S}$ [37], introduced as an extension of the hypothesis of locality for a quantum system, can be generalized to describe the coupling of graviton spin $\mathbf{S}^{(G)}$ to a gravitomagnetic field identified with $\Omega$. This results in a gravitational self-interaction contained within the spacetime background. A subsequent computation by Ramos and Mashhoon [38] shows how this generalization is applicable for the propagation of gravitational waves, while Papini [40] develops the approach further to accommodate massive spin-2 particles and explore its application to wave optics phenomena, such as gravitational lenses. In light of Shen’s generalization of the Mashhoon effect, it may be possible to interpret the gravitational orbital angular momentum operator $\mathbf{L}^{(G)}$ as an analogous modification of the orbital-rotation coupling term to accompany the spin-rotation coupling generalization. In like fashion, this interaction results in a generalization of the Sagnac effect $\Omega \cdot \mathbf{L}$, such that for a spinless particle,

$$
\mathbf{\Omega} \cdot \mathbf{J} = \mathbf{\Omega} \cdot \left( \mathbf{J}^{(0)} + \mathbf{J}^{(G)} \right)
$$

(75)
gives rise to an overall phase shift in the wavefunction generated by the Sagnac and Mashhoon effects [39], where $J^{(0)} = L^{(0)}$ and $J^{(G)} = L^{(G)} + S^{(G)}$. It remains to be seen whether these types of quantum mechanical predictions involving classical gravitation are observable with current or near-future experimental means available.

6. Conclusion

This paper demonstrates the impact of local spacetime curvature on the orbital angular momentum of a spinless particle in quantum mechanics. It suggests the existence of half-integer spacings along the axis of quantization that are otherwise not accessed in the absence of gravitational interactions with quantum matter. In addition, the constraints required to preserve consistency of the formalism demonstrates that $l = 2$ is the minimum allowable value for the orbital quantum number to incorporate curvature contributions, which precisely coincides with the spin of a graviton in linearized general relativity. The consequences of these details are compared with previous research that suggests the existence of a spin-1/2 internal structure embedded in spacetime, with interesting consequences for potentially advancing quantum gravity research into the future.

Possible future research may involve studying the addition of both orbital and spin angular momentum in the presence of a local gravitational background, to determine if curvature-dependent quantities make a contribution to the Clebsch–Gordan coefficients. Other possibilities may include the incorporation of nontrivial topological backgrounds to see if a spin-1/2 gravitational background alters the constraint equations required to specify the gravitational ladder coefficients. It is also worthwhile to determine if these observations of orbital angular momentum get adequately reflected within a quantum field theory framework. These and other possibilities may be investigated in due course.

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