Abstract

We prove lower bounds on complexity measures, such as the approximate degree of a Boolean function and the approximate rank of a Boolean matrix, using quantum arguments. We prove these lower bounds using a quantum query algorithm for the combinatorial group testing problem.

We show that for any function $f$, the approximate degree of computing the OR of $n$ copies of $f$ is $\Omega(\sqrt{n})$ times the approximate degree of $f$, which is optimal. No such general result was known prior to our work, and even the lower bound for the OR of ANDs function was only resolved in 2013.

We then prove an analogous result in communication complexity, showing that the logarithm of the approximate rank (or more precisely, the approximate $\gamma_2$ norm) of $F : X \times Y \rightarrow \{0, 1\}$ grows by a factor of $\tilde{\Omega}(\sqrt{n})$ when we take the OR of $n$ copies of $F$, which is also essentially optimal. As a corollary, we give a new proof of Razborov’s celebrated $\Omega(\sqrt{n})$ lower bound on the quantum communication complexity of the disjointness problem.

Finally, we generalize both these results from composition with the OR function to composition with arbitrary symmetric functions, yielding nearly optimal lower bounds in this setting as well.
1 Introduction

Quantum computing promises to allow the efficient solution of certain problems believed to be intractable for classical computers, and is therefore of great practical interest. From a mathematical perspective, another important contribution of quantum computing is the rise of the “quantum method” as a proof technique. That is, often one can prove purely classical (i.e., not quantum) mathematical statements using techniques from quantum information for which no classical proof is known, or where the quantum proof is substantially simpler than its classical counterpart.¹ For example, the non-existence of efficient 2-locally-decodable codes was first proven using quantum arguments [KdW03]. The closure of the classical complexity class PP under intersection was first shown using classical techniques by Beigel, Reingold, and Spielman [BRS95], but Aaronson showed it could be reproven using quantum techniques in a simpler way [Aar05]. The survey by Drucker and de Wolf provides more examples of this proof technique [DW11].

**OR composition.** In this work, we apply the quantum method to resolve several composition questions for classical complexity measures in query complexity and communication complexity. A quintessential example of this type of question is the OR-composition question, which asks the following: Given a function \( f \), how hard is it to compute the function \( \text{OR}_n \circ f \), the OR of \( n \) copies of \( f \)? One particular strategy for computing \( \text{OR}_n \circ f \) is to compose the best algorithms for \( \text{OR}_n \) and \( f \) in the given model of computation. For many complexity measures (including all the measures studied in this paper), the product of the complexities of \( \text{OR}_n \) and \( f \) will yield an upper bound on the complexity of \( \text{OR}_n \circ f \). Typically, we conjecture that this upper bound is optimal, but it is not obvious that this must be the case, and hence establishing such a lower bound is usually difficult (or possibly even false for some complexity measures). For example, it is known that this upper bound is optimal for deterministic [Tal13, Mon14] and quantum query complexity [Rei11, LMR+11], but was only recently established for randomized query complexity [GJPW17].

In this paper we show an optimal OR-composition result for approximate degree, a complexity measure in query complexity first studied by Nisan and Szegedy [NS94], which lower bounds quantum query complexity [BBC+01], and a nearly optimal OR-composition theorem for approximate rank (or approximate \( \gamma_2 \)-norm or generalized discrepancy), a measure in communication complexity which lower bounds quantum communication complexity [BdW01, LS09a].

Our results significantly generalize previous OR-composition results for these measures. For instance, OR-composition for approximate degree was open for close to 20 years just for the special case that \( f \) is the AND function! After several incremental improvements (see Table 1) by Shi [Shi02], Ambainis [Amb05], and Sherstov [She13b], the problem was recently resolved by Sherstov [She13a] and Bun and Thaler [BT13] using a linear programming characterization of approximate degree.

In contrast, we show a tight OR-composition theorem for approximate degree for *arbitrary* functions \( f \), generalizing these works and newer results on constant-depth compositions of the AND and OR functions [BT15]. (In fact, we also provide an optimal lower bound on the approximate degree of the OR of possibly different functions \( f_i \).)

In communication complexity, to the best of our knowledge no OR-composition result was known for approximate rank. Indeed, such a result would directly imply Razborov’s celebrated \( \Omega(\sqrt{n}) \) lower bound on the quantum communication complexity of the disjointness function [Raz03]. To highlight the power of our techniques, we provide a short proof of the \( \Omega(\sqrt{n}) \) lower bound for disjointness. We

¹This is analogous to how it is sometimes easier to prove a statement about real numbers using complex numbers, as expressed in the following quote usually attributed to Jacques Hadamard [Kah91]: “The shortest path between two truths in the real domain passes through the complex domain”.

also provide a more direct proof of the recent lower bound on the quantum information complexity of disjointness [BGK+15].

Symmetric function composition. We then generalize our \( \text{OR} \)-composition results to hold for compositions with arbitrary symmetric functions, which are functions that only depend on the Hamming weight of the input. Other than \( \text{OR} \), compositions with symmetric functions like parity and majority have been studied in complexity theory. For instance, the question of how difficult it is to compute \( \text{XOR}_n \circ f \) was already studied in 1982 in Yao’s seminal paper on the XOR lemma [Yao82] (see [O’D04] for a general composition theorem for \( g \circ f \) in this setting.). Since the class of symmetric functions includes the \( \text{OR} \) function, proving composition theorems for arbitrary symmetric functions is even harder. Such composition theorems are known for deterministic [Tal13, Mon14] and quantum query complexity [Rei11, LMR+11]. But it remains open to show a similar theorem for randomized query complexity, where only partial results are known [GLS18, San18].

Techniques. Although the final results for approximate degree and approximate rank are purely classical, our proofs use quantum algorithms in a crucial way, and there is no known classical proof of these results. We therefore believe this to be a powerful example of the “quantum method” [DW11]. However, we only use quantum algorithms in a black-box manner and the reader is not required to be familiar with quantum query complexity. We only use its relationship with polynomials due to Beals et al. [BBC+01] and the existence of a quantum algorithm for the combinatorial group testing problem due to Belovs [Bel15].

Another salient feature of our proofs is that our lower bounds on various measures like approximate degree are proven using the existence of very fast quantum algorithms for related problems. This is part of a recent trend in complexity theory, sometimes called “ironic complexity theory” [Aar16], in which lower bounds are proven using upper bounds. For instance, Williams’ celebrated circuit lower bound for \( \text{ACC} \) uses this approach [Wil14].

Our approach of using a fast quantum algorithm (by Belovs [Bel15]) to prove lower bounds is inspired by the recent work of Hoza [Hoz17], who showed that fast quantum algorithms for certain query problems imply lower bounds in communication complexity. Hoza’s work was, in turn, inspired by work of Cleve, van Dam, Nielsen, and Tapp [CvDNT13], who used the Bernstein–Vazirani algorithm [BV97] to prove the first lower bound on the quantum communication complexity (with unlimited shared entanglement) of the inner product function. Similar proof techniques were also used by Buhrman and de Wolf [BdW98] to show a lower bound on the quantum query complexity of searching a sorted list by a reduction to the hardness of computing parity.

1.1 Our results

We now describe our results in more detail.

| Bound                      | Citation                      |
|---------------------------|-------------------------------|
| \( O(n) \)                | Høyer, Mosca and de Wolf [HMdW03] |
| \( \Omega(\sqrt{n}) \)    | Nisan and Szegedy [NS94]     |
| \( \Omega(\sqrt{n \log n}) \) | Shi [Shi02]            |
| \( \Omega(n^{0.66...}) \) | Ambainis [Amb05]            |
| \( \Omega(n^{0.75}) \)   | Sherstov [She13b]           |
| \( \Omega(n) \)          | Sherstov [She13a] and Bun and Thaler [BT13] |

Table 1: History of lower bounds on the approximate degree of \( \text{OR}_n \circ \text{AND}_n \) (from [She13a])
1.1.1 Approximate degree

For any Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), the approximate degree of \( f \), denoted \( \widetilde{\deg}(f) \), is the minimum degree of any real polynomial \( p \) over the variables \( x_1, \ldots, x_n \), such that \( |f(x) - p(x)| \leq 1/3 \) for all \( x \in \{0, 1\}^n \). Note that \( \widetilde{\deg}(f) \leq n \) for all Boolean functions since any Boolean function can be represented exactly with a polynomial of degree \( n \). Also note that negating the output of a function does not change its approximate degree, and neither does negating input bits. Hence \( \widetilde{\deg}(\text{OR}_n) = \widetilde{\deg}(\text{AND}_n) = \widetilde{\deg}(\text{NAND}_n) \) and results for one function carry over to the others.

Approximate degree was first studied by Nisan and Szegedy [NS94]. Since then, it has been used to prove oracle separations, design learning algorithms, and show lower bounds on quantum query complexity, formulas size, and communication complexity. (See [She13a, She13c, BT13] and the references therein for more information.) It can be used to prove lower bounds on quantum query complexity because for all (total or partial) functions \( f \), we have \( Q(f) \geq \frac{1}{2} \widetilde{\deg}(f) \) [BBC+01], where \( Q(f) \) denotes the bounded-error quantum query complexity of \( f \).

Although approximate degree has a simple definition in terms of polynomials, several simple questions about this measure remain open. Surprisingly, even the approximate degree of the depth-2 AND-OR tree \( \text{AND}_n \circ \text{OR}_m \) remained open for close to 20 years! In 2013, after several incremental improvements (described in Table 1), Sherstov [She13a] and Bun and Thaler [BT13] showed that

\[
\widetilde{\deg}(\text{AND}_n \circ \text{OR}_m) = \Omega(\sqrt{mn}),
\]

which is optimal [HMdW03]. These lower bounds were proved using a linear programming formulation of approximate degree, and exploited certain properties of the dual polynomial for the OR function. In contrast to these approaches using dual polynomials, our OR-composition result for approximate degree uses completely different techniques and is more general:

**Theorem 1.** For any Boolean function \( f : \{0, 1\}^m \rightarrow \{0, 1\} \), we have

\[
\widetilde{\deg}(\text{OR}_n \circ f) = \Omega(\sqrt{n} \widetilde{\deg}(f)).
\]

This lower bound is tight due to a matching upper bound of Sherstov [She13c]. This resolves the OR-composition question for approximate degree. As an example, this now allows us to show the optimal bound \( \widetilde{\deg}(\text{OR}_n \circ \text{MAJ}_n) = \Omega(n^{3/2}) \), where \( \text{MAJ} \) is the majority function. Prior to our work, the best lower bound that could be proved with known techniques was \( \widetilde{\deg}(\text{OR}_n \circ \text{MAJ}_n) = \Omega(n) \).

After characterizing the approximate degree of the depth-2 AND-OR tree, Bun and Thaler [BT15] also proved that the approximate degree of the depth-\( d \) AND-OR tree on \( n \) inputs is \( \Omega(\sqrt{n} / \log^{d/2-1} n) \). Theorem 1 straightforwardly implies the optimal bound of \( \Omega(\sqrt{n}) \).

We then generalize Theorem 1 to a composition theorem for arbitrary symmetric functions \( g \). Our OR-composition theorem plays a central role in the proof of our symmetric-function composition theorem, which we discuss in Section 1.2.

**Theorem 2.** For any symmetric Boolean function \( g : \{0, 1\}^n \rightarrow \{0, 1\} \) and any Boolean function \( f : \{0, 1\}^m \rightarrow \{0, 1\} \), we have

\[
\widetilde{\deg}(g \circ f) = \tilde{\Omega}(\widetilde{\deg}(g) \widetilde{\deg}(f)).
\]

This lower bound is also tight up to log factors due to a matching upper bound of Sherstov [She13c]. This resolves the symmetric-composition question for approximate degree.
1.1.2 Approximate rank or $\gamma_2$ norm

In communication complexity, we have two players Alice and Bob, who hold inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively. Their goal is to compute a function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$ on their inputs while minimizing the communication between them. One of the most studied functions in communication complexity is the set disjointness problem $\text{DISJ}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$, defined as $\text{DISJ}_n(x, y) = \bigvee_{i=1}^n (x_i \land y_i)$ for all $x, y \in \{0, 1\}^n$.

The quantum communication complexity of the disjointness problem was one of the early open problems in quantum communication complexity. Let $Q^*_cc(F)$ denote the bounded-error quantum communication complexity of a function $F$ with unlimited preshared entanglement. Then it follows from Grover’s algorithm [Gro96] and the query-to-communication simulation algorithm of Buhrman, Cleve, and Wigderson [BCW98] that $Q^*_cc(\text{DISJ}_n) = O(\sqrt{n}\log n)$, which was later improved to $Q^*_cc(\text{DISJ}_n) = O(\sqrt{n})$ [AA03]. However, the lower bound remained open until a breakthrough by Razborov [Raz03], who showed that $Q^*_cc(\text{DISJ}_n) = \Omega(\sqrt{n})$.

Razborov’s result actually lower bounds a smaller complexity measure. With any communication problem $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, we can associate a $\{-1, +1\}$ matrix, called the sign matrix of $F$, whose $(x, y)$ entry is $(-1)^{F(x,y)}$. Informally, the approximate rank of $F$, denoted $\tilde{\text{rank}}(F)$ is the least rank of any matrix that is entry-wise close to the sign matrix of $F$. (See Section 3 for a more precise definition.) Another measure that is essentially equivalent to approximate rank is the approximate $\gamma_2$-norm of the sign matrix of $F$, which we denote $\tilde{\gamma}_2(F)$, also defined in Section 3. For any function $F$, $\log \tilde{\gamma}_2(F)$ lower bounds its quantum communication complexity, and Razborov’s result proves the stronger statement that $\log \tilde{\gamma}_2(\text{DISJ}_n) = \Omega(\sqrt{n})$.

We first show that our techniques yield a new proof of Razborov’s celebrated $\Omega(\sqrt{n})$ lower bound for disjointness.

**Theorem 3.** Let $\text{DISJ}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ be the set disjointness function defined as $\text{DISJ}_n(x, y) = \bigvee_{i=1}^n (x_i \land y_i)$ for all $x, y \in \{0, 1\}^n$. Then

$$
\log \tilde{\text{rank}}(\text{DISJ}_n) = \Omega(\sqrt{n}) \text{ and } \log \tilde{\gamma}_2(\text{DISJ}_n) = \Omega(\sqrt{n}).
$$

(4)

Note that this lower bound is tight due to the matching quantum algorithm of Aaronson and Ambainis [AA03]. Building on this, we generalize our result to an OR-composition theorem.\(^2\)

**Theorem 4.** For any function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, we have $\log \tilde{\gamma}_2(\text{OR}_n \circ F) = \tilde{\Omega}(\sqrt{n}\log \tilde{\gamma}_2(F))$.

We then generalize this proof to show a nearly optimal composition theorem for an arbitrary symmetric function $g$ and an arbitrary communication problem $F$.

**Theorem 5.** For any Boolean function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, and any symmetric function $g : \{0, 1\}^n \to \{0, 1\}$, we have

$$
\log \tilde{\gamma}_2(g \circ F) \geq \tilde{\deg}(g)^{1-o(1)} \log \tilde{\gamma}_2(F).
$$

(5)

Note that these lower bounds are also essentially tight, as a matching upper bound of $\log \tilde{\gamma}_2(g \circ F) = \tilde{O}(\deg(g) \log \tilde{\gamma}_2(F))$ can be proved by composing a polynomial for $g$ with a matrix for $F$. (For example, this can be done using the construction in Lemma 30).

\(^2\)An astute reader may worry that an OR-composition theorem cannot possibly hold in communication complexity because some functions do not become harder as we take the OR of many copies of the function. For example, the function $\text{EQ}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$, defined as $\text{EQ}_n(x, y) = 0$ if and only if $x = y$, can be solved with $O(1)$ communication using a randomized or quantum protocol. Taking the OR of many copies of $\text{EQ}_n$ only yields a larger instance of $\text{EQ}$, which is no harder than before. However, Theorem 4 still holds because $\log \tilde{\gamma}_2(\text{EQ}_n) \leq 0$. 

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1.1.3 Further Extensions

We also prove two further extensions of our result. In Section 4.1, we generalize our tight OR-composition theorem for $\deg(\text{OR}_n \circ f)$ to the case of different functions $f_i$. We show a tight lower bound on the approximate degree of the OR of $n$ possibly different functions $f_i$ (which may possibly even have different input sizes). This completely characterizes the approximate degree of this function, and furthermore implies that the approximate degree of any constant-depth read-once formula is $\Omega(\sqrt{n})$. This lower bound is optimal, since an upper bound of $O(\sqrt{n})$ is known for the approximate degree of arbitrary read-once formulas (not just constant-depth) via the $O(\sqrt{n})$ upper bound on quantum query complexity [Rei11] and it is an interesting open question if this upper bound is tight for arbitrary read-once formulas.

In Section 4.2, we show a lower bound on the quantum information complexity of the disjointness function. Quantum information complexity [Tou15] is a information relaxation of quantum communication complexity, in the same sense that information complexity is a relaxation of communication complexity. Intuitively, instead of charging for the number of bits (or qubits) of communication if a protocol, information complexity only charges for the information transmitted by these bits (or qubits). We use our techniques to reprove the $\Omega(\sqrt{n})$ lower bound on the quantum information complexity of disjointness [BGK+15] up to log factors. This lower bound is already known, but the known proof uses an alternate characterization of quantum information complexity as amortized quantum communication complexity. In contrast, our proof is more direct and works with the information theoretic definition of quantum information complexity.

1.2 High-level overview of techniques

While we prove several different lower bounds against measures in query and communication complexity, our proofs share several common techniques. In particular, all our proofs use Belovs’ algorithm for the combinatorial group testing problem [Bel15], which we now describe. Combinatorial group testing has a long history originating in the testing of World War II draftees for Syphilis [DHH00], where the goal was to minimize the number of tests used to screen recruits. The basic idea was to pool multiple blood samples together before testing them; the blood test then reveals if anyone in the pool has the disease. In other words the test reveals the OR of the draftee’s disease statuses within the group. One can easily see that if only one person has the disease, then one can use binary search to use only $\log n$ tests to identify which of $n$ people has the disease; similarly one can show that if $k$ people have the disease then $k \log n$ tests suffice.

More formally, in this problem there is a hidden string $x \in \{0, 1\}^n$. One is allowed to query any subset $S \subseteq [n]$, and querying a subset $S$ returns the OR of the bits of $x$ in the subset, i.e., $\bigvee_{i \in S} x_i$. The goal is to use these subset queries to learn all of the bits of $x$. Clearly this can be achieved with $n$ queries in almost any reasonable measure of query complexity, by querying each bit of the input separately, i.e., by querying the subsets $\{1\}, \ldots, \{n\}$. And as previously mentioned, for sparse inputs one can use fewer than $n$ queries. But for worst-case inputs this trivial $O(n)$ query algorithm is optimal for classical (deterministic or randomized) query complexity. This is because if the string $x$ contains a single 0, then this problem reduces to search. Therefore even a quantum algorithm for this query problem would require $\Omega(\sqrt{n})$ queries by the lower bound for Grover search [BBBV97]. Surprisingly, Belovs [Bel15] showed that the quantum query complexity of this problem is at most $O(\sqrt{n})$ as well. This algorithm will play a key role in our proofs.

Approximate degree OR-composition. We first describe the ideas required to lower bound the approximate degree of functions of the form $\text{OR}_n \circ f$, making note of the parts of the proof that fail in communication complexity.
Suppose by way of contradiction that $\deg(\text{OR}_n \circ f) = T$, where $T$ is smaller than expected. This means we can compute the OR of $n$ copies of a function $f$ more easily than expected. But this also implies we can compute the OR of any subset $S \subseteq [n]$ of these $n$ copies of $f$, since we can apply this algorithm to any subset $S$ of our choice. (This argument already does not work in communication complexity when only one player knows the subset $S$, since that player would have to communicate $S$ to the other player.)

Now we view the $n$ outputs to the functions $f$ as the hidden string $x \in \{0,1\}^n$ in the combinatorial group testing problem. In the combinatorial group testing problem, we assume we have the ability to query the OR of any subset of the bits, which is exactly what the assumed polynomial for $\text{OR}_n \circ f$ gives us. From Belovs’ quantum algorithm, we can construct an approximating polynomial for combinatorial group testing using the results of Beals et al. [BBC+01]. More precisely, since combinatorial group testing has an $n$-bit output, which is the hidden string $x \in \{0,1\}^n$, we use a decision version of this problem that simply outputs the parity of all the bits. We would now like to compose this polynomial with the assumed polynomials that allow us to compute the OR of a subset of the functions $f$. However, since the polynomials we wish to compose are approximating polynomials, they do not straightforwardly compose as expected, and to make this work, we use Sherstov’s robust polynomial construction [She13c]. Finally, by composing these polynomials of degree $T$ and degree $O(\sqrt{n})$, we get a polynomial of degree $\tilde{O}(T\sqrt{n})$ for computing the parity of all the functions $f$, i.e., we have shown that $\deg(\text{XOR}_n \circ f) = O(T\sqrt{n})$.

Computing the parity of $n$ copies of a function $f$ is usually $n$ times as hard as computing $f$ in most models of computation. Such a result is known for all the measures considered in this paper. The argument is now completed by combining the fact that $\deg(\text{XOR}_n \circ f) = \Omega(n \deg(f))$ [She12] and $\deg(\text{XOR}_n \circ f) = O(T\sqrt{n})$. Combining these gives us $T = \Omega(\sqrt{n} \deg(f))$, as desired.

Our results in communication complexity and the extension to arbitrary symmetric functions build on the ideas presented here. The flowchart in Figure 1 describes the flow of ideas as well as the dependencies between various sections.

**OR composition in communication complexity.** The general strategy outlined above also works in communication complexity for the measures approximate rank and approximate gamma 2 norm, but we need to make additional arguments to make some steps work.

First, as noted above if one player knows a subset $S$ of the shared input, but the other does not, it is not in general possible for them to run a communication protocol on that subset of their shared input. Thus our communication results have some overhead for dealing with this situation. Naively it would seem this overhead is too expensive, since Alice would need to communicate the entire subset $S$ to Bob, which might be more expensive than the rest of the protocol. However, a recursive argument based on self-reducibility of the OR function allows the conversion of the additive $O(n)$ loss into a multiplicative polylogarithmic loss.

The other technically challenging part of porting this argument to communication complexity is in composing approximating polynomials with approximating matrices. This composition does not work as cleanly as in query complexity, and in some cases leads to an additional log factor loss.

**Approximate degree PrOR-composition.** To lower bound the approximate degree of functions of the form $g \circ f$, where $g$ is a symmetric Boolean function, we first show an intermediate lower bound which will play a key role in our symmetric composition theorem. In particular we consider the Promise-OR function, denoted PrOR. The $\text{PrOR}_n : \{0,1\}^n \to \{0,1,\ast\}$ function is the same as the OR function with the additional promise that the input has Hamming weight either 0 or 1. We first extend our lower bound on the approximate degree of $\text{OR}_n \circ f$ to the partial function $\text{PrOR}_n \circ f$. 
Approximate degree OR composition (Section 2.2)

Approximate degree PrOR composition (Section 2.3)

Approximate rank of Disjointness (Section 3.2)

Approximate rank OR composition (Section 3.3)

Approximate rank PrOR composition (Section 3.4)

Approximate rank composition for symmetric functions (Section 3.5)

Quantum information complexity OR composition (Section 4.2)

Approximate degree unbalanced OR composition (Section 4.1)

Figure 1: Reading order for the results shown in this paper. An arrow from \( A \) to \( B \) indicates that \( A \) is a prerequisite for reading \( B \).

(For partial functions, we require that an approximating polynomial be close to the function on inputs in the domain, and be bounded in \([0, 1]\) on all inputs including those outside the domain.)

The main insight that allows us to extend our lower bounds from OR to PrOR is that Belovs’ algorithm actually solves a more general problem than combinatorial group testing, or more precisely, assumes a weaker access model to the input. In particular Belovs’ algorithm only requires that the queries that are supposed to return the OR of a subset \( S \), i.e., the value \( \bigvee_{i \in S} x_i \), return the correct answer when \( \sum_{i \in S} x_i \in \{0, 1\} \). The queries may return incorrect answers on those subsets \( S \) for which \( \sum_{i \in S} x_i > 1 \).

While this is the key conceptual step needed for the generalization, working with partial functions presents several technical challenges. One of the main challenges corresponds to the robustness of polynomials, for which we used Sherstov’s robust polynomial construction [She13c] previously.

Recall that our proof strategy is to compose the polynomial induced by Belovs’ algorithm with a too-good-to-be-true (approximating) polynomial for \( \PrOR_n \circ f \). This requires Belovs’ polynomial to be robust, i.e. handle noisy inputs, which it may not be. For OR composition, we applied Sherstov’s construction to obtain a robust version of Belovs’ polynomial, which tolerates \( 1/3 \) noise in the input bits. However, Sherstov’s robust polynomial construction has a downside—it constructs polynomials that are not multilinear and whose value may blow up when an input variable is not close to being Boolean. This is exactly what can happen when we plug in an approximating polynomial for a partial function.

For this reason we use a different strategy for composing polynomials without using Sherstov’s technique. In particular we prove that the polynomials constructed from quantum algorithms are already mildly robust, i.e., they can handle \( 1/poly(q) \) noise in the inputs, where \( q \) is the query complexity of the quantum algorithm. Since the polynomials induced by quantum algorithms are multilinear (and hence they are bounded whenever the inputs are in \([0, 1]\) ), this allows us to extend
our composition framework to the setting of partial functions (at the expense of losing a logarithmic factor).

We note we are not the first to prove intrinsic robustness of polynomials derived from quantum algorithms. For instance, Buhrman et al. [BNRdW07] show that the polynomials derived from a quantum algorithm computing a total function $f$ can tolerate $O(1/C(f))$ noise in the inputs, where $C(f)$ is the certificate complexity of $f$. However this result is insufficient for our application as we are applying it to a partial function. It is also not hard to show that all multilinear polynomials are $O(1/n)$ robust to noise, where $n$ is the number of input bits. However, this does not suffice for our application either because the number of input bits for CGT is exponentially larger than its quantum query complexity. To the best of our knowledge this particular robustness property of polynomials derived from quantum algorithms was not known before and might be of independent interest.

Approximate degree composition for symmetric functions. We now describe how an approximate degree lower bound for $\PrOR_n \circ f$ can be used to lower bound the approximate degree of $g \circ f$, where $g$ is a symmetric function. By a result of Paturi [Pat92], it is known that the approximate degree of a symmetric $n$-bit function is completely determined by the Hamming weight closest to $n/2$ where the function $g$ changes value. This implies that it suffices to prove the composition theorem for the case when the outer function $g$ is $\PrTH_k^n$, which is defined as follows:

$$\PrTH_k^n(x) = \begin{cases} 
0 & \text{if } |x| = k \\
1 & \text{if } |x| = k + 1 \\
\ast & \text{otherwise}
\end{cases}$$

Now the elementary, but crucial, observation is that $\PrTH_k^{2k} \circ \PrOR_{n/2k}$ is a sub-function of $\PrTH_k^n$. Hence we can obtain an approximate degree lower bound for $\PrTH_k^n \circ f$ using our composition theorem for $\PrOR$, and a prior composition theorem of Sherstov that works for $\PrTH_k^{2k}$. (Sherstov’s result yields optimal composition theorems whenever the outer function has linear approximate degree [She12]). This yields the composition theorem for arbitrary symmetric functions.

1.3 Open problems

We end with a discussion of main open problems left open by our work. The foremost open problem is whether the following conjecture is true.

**Conjecture 1.** For all Boolean functions $g : \{0,1\}^n \to \{0,1\}$ and $f : \{0,1\}^m \to \{0,1\}$, we have $\widetilde{\deg}(g \circ f) = \Omega(\widetilde{\deg}(g) \widetilde{\deg}(f))$.

Our result resolves this (up to log factors) when $g$ is symmetric and $f$ is arbitrary. A related question is whether any of our results can be reproved using the dual polynomials framework that has been used to show recent lower bounds for approximate degree [She13a, BT13, BT15]. In particular, is there a way to convert a dual witness for $\widetilde{\deg}(f)$ into a dual witness for $\widetilde{\deg}(\text{OR} \circ f)$?

A more open ended question is whether this technique can be generalized to functions other than OR by developing new quantum algorithms. Belovs’ algorithm used OR-queries to a hidden string $x$ to learn all of $x$. What other quantum algorithms of this form exist? Are there nontrivial quantum algorithms that use $g$-queries to learn $x$ for some function $g \notin \{\text{OR, XOR}\}$? Are there nontrivial quantum algorithms that use $g$-queries to compute some other function $h(x)$ of the input? This motivates the study of a whole class of quantum algorithms, which to the best of our knowledge has not been systematically studied other than in the work of Belovs [Bel15].
2 Approximate degree

In this section we prove our composition theorems for approximate degree. We start with some definitions and known results in Section 2.1. In Section 2.2 we prove the OR-composition theorem (Theorem 1), which is the starting point for the more general results proved in this paper. Section 2.3 generalizes the composition theorem to a partial function related to OR, and then Section 2.4 proves the final result with arbitrary symmetric functions.

2.1 Preliminaries

In this section we collect some basic definition and known results about partial functions, approximate degree, and quantum query complexity. Partial functions will play a key role in our proofs, even though the main results are about total Boolean functions. Hence it is necessary to formally define partial functions and extend the definitions of approximate degree and quantum query complexity to partial functions.

Definitions. A partial Boolean function on $m$ bits is a function that is only defined on a subset of $\{0,1\}^m$. There are two common ways to talk about partial functions. We can either view it as a function from $D$ to $\{0,1\}$, where $D \subseteq \{0,1\}^m$, or as a function $f : \{0,1\}^m \rightarrow \{0,1,*\}$, where the function evaluates to * outside $D$. We will mostly use the second definition and refer to the subset of $x \in \{0,1\}^m$ with $f(x) \neq *$ as the “promise” and denote it $\text{Dom}(f)$. We can now define the composition of two partial functions more formally.

Definition 6. Let $g : \{0,1\}^n \rightarrow \{0,1,*\}$ and $f : \{0,1\}^m \rightarrow \{0,1,*\}$ be partial Boolean functions. Then we define $g \circ f : \{0,1\}^{nm} : \{0,1,*\}$ to be the partial function $g \circ f(x_1, \ldots, x_n) = g(f(x_1), \ldots, f(x_n))$ on those inputs for which all $x_i \in \text{Dom}(f)$ and $(f(x_1), \ldots, f(x_n)) \in \text{Dom}(g)$. The function evaluates to * on all other inputs.

Most algorithmic models are easily generalized to partial functions. A (classical or quantum) algorithm for a partial function $f$ is only required to be correct on inputs in $\text{Dom}(f)$ and can have arbitrary behavior on inputs outside $\text{Dom}(f)$. Extending the definition of approximate degree to partial functions is more subtle, and we motivate it by using an example of a partial function.

Recall that the OR function on $n$ bits is defined as $\text{OR}_n(x) = 0$ if $|x| = 0$ and $\text{OR}_n(x) = 1$ if $|x| > 0$, where $|x|$ denotes the Hamming weight of $x$ or the number of 1s in $x$. Let us define a partial function related to OR, which we call PromiseOR, as follows. $\text{PrOR}_n : \{0,1\}^n \rightarrow \{0,1,*\}$ is the OR function with the additional promise that the input has Hamming weight 0 or 1. In other words,

$$\text{PrOR}_n(x) = \begin{cases} 0 & \text{if } |x| = 0 \\ 1 & \text{if } |x| = 1 \\ * & \text{otherwise} \end{cases} \quad (6)$$

Intuitively $\text{PrOR}$ contains the hardest instances of the OR function, and hence lower bounds for the OR function should hold against the PrOR function as well. For example, the quantum query complexity of $\text{PrOR}$ is still $\Theta(\sqrt{n})$, and the deterministic and randomized query complexities of $\text{PrOR}$ are $\Theta(n)$.

The approximate degree of $\text{PrOR}$ is also $\Theta(\sqrt{n})$ as one might expect, as long as we define approximate degree for partial functions appropriately. For a partial function we clearly want the polynomial to approximate the function value on inputs in the promise. But we additionally want the polynomial to be bounded. We say a polynomial $p$ on $m$ variables is bounded if for all $x \in \{0,1\}^m$, $p(x) \in [0,1]$.
We use the following standard generalization of approximate degree to partial functions that is sometimes called “bounded approximate degree” in the literature [BKT18].

**Definition 7** (Bounded approximate degree). For any partial Boolean function \(f : \{0, 1\}^m \rightarrow \{0, 1, \ast\}\), the bounded approximate degree of \(f\), denoted \(bdeg(f)\), is the minimum degree of any real polynomial \(p\) over the variables \(x_1, \ldots, x_m\), such that

1. \((p \text{ is bounded})\) for all \(x \in \{0, 1\}^m\), \(p(x) \in [0, 1]\), and
2. \((p \text{ approximates } f)\) for all \(x \in \text{Dom}(f)\), \(|f(x) - p(x)| \leq 1/3\).

With this generalization of approximate degree, it is indeed true that \(\widehat{\text{bdeg}}(\text{PrOR}) = \Theta(\sqrt{n})\), as expected. Note that if we did not require that the polynomial be bounded on all inputs in the domain, then there would be a degree-1 polynomial that exactly represents the \(\text{PrOR}\) function, which is the polynomial \(\sum_{i=1}^n x_i\).

Finally, we define what it means for a polynomial approximating a Boolean function to be \(\delta\)-robust to input noise. Informally it means the polynomial continues to approximate the Boolean function even if the input bits are \(\delta\)-far from being Boolean.

**Definition 8** (\(\delta\)-robustness to input noise). Let \(h : \{0, 1\}^n \rightarrow \{0, 1, \ast\}\) be a partial Boolean function, and let \(p : \{0, 1\}^n \rightarrow \mathbb{R}\) be a polynomial. We say that \(p\) approximately computes \(h\) with robustness \(\delta \in [0, 1/2]\) if for any \(x \in \text{Dom}(h)\) and any \(\Delta \in [-\delta, \delta]^n\), we have \(|h(x) - p(x + \Delta)| \leq 1/3\).

**Known results.** We now collect some facts about bounded polynomials and bounded approximate degree that we need to prove our results.

The first result we use is Sherstov’s result on making polynomials robust to noise [She13c, Theorem 1.1]. This result states that any polynomial \(p\) can be made \(1/3\)-robust to input noise by only increasing the degree of the polynomial by a constant factor.

**Theorem 9** (Sherstov). Let \(q : \{0, 1\}^n \rightarrow [0, 1]\) be a given polynomial. Then there exists a polynomial \(q' : \mathbb{R}^n \rightarrow \mathbb{R}\) of degree \(O(\deg(q) + \log(1/\epsilon))\) such that

\[
|q(x) - q'(x + \Delta)| < \epsilon, \tag{7}
\]

for all \(x \in \{0, 1\}^n\) and \(\Delta \in [-1/3, 1/3]^n\).

We will also need a result of Sherstov that establishes the hardness of computing the parity of \(n\) copies of a function \(f\), or more generally of \(n\) different functions \(f_1, f_2, \ldots, f_n\). We denote the parity of these \(n\) functions \(\text{XOR}_n \circ (f_1, f_2, \ldots, f_n)\). Sherstov shows that the approximate degree of the parity of \(n\) functions is at least the sum of their approximate degrees [She12, Theorem 5.9].

**Theorem 10** (Sherstov). For any partial Boolean functions \(f_1, f_2, \ldots, f_n\), we have

\[
\widehat{\text{bdeg}}(\text{XOR}_n \circ (f_1, f_2, \ldots, f_n)) = \Omega\left(\sum_i \widehat{\text{bdeg}}(f_i)\right). \tag{8}
\]

In particular, for any partial Boolean function \(f\), we have \(\widehat{\text{bdeg}}(\text{XOR}_n \circ f) = \Omega(n \widehat{\text{bdeg}}(f))\).

Finally, we also need the following result of Sherstov [She12, Theorem 6.6] that proves a composition theorem for bounded approximate degree when the outer function has high degree.

**Theorem 11** (Sherstov). Let \(g : \{0, 1\}^n \rightarrow \{0, 1, \ast\}\) and \(f : \{0, 1\}^m \rightarrow \{0, 1, \ast\}\) be partial Boolean functions. Then \(\text{bdeg} (g \circ f) = \Omega \left(\text{bdeg}(g)^2 \text{bdeg}(f)/n\right)\).
We now formally state the connection between quantum algorithms and approximating polynomials. Beals et al. [BBC+01] showed that the acceptance probability of a quantum query algorithm that makes few queries can be expressed as a low degree polynomial.

**Theorem 12** (Beals et al.). Let $A$ be a quantum query quantum algorithm that makes $T$ queries to an oracle string $x \in \{0,1\}^n$ and outputs 1 with probability $A(x)$. Then there exists a real polynomial $p$ of degree $2T$ over the variables $x_1, \ldots, x_n$ such that for all $x \in \{0,1\}^n$, $p(x) = A(x)$.

By choosing $A$ to be a quantum algorithm that computes a partial function $f$ to bounded error, we get the following corollary.

**Corollary 13.** For any partial Boolean function $f$, $Q(f) \geq \frac{1}{2} \tilde{\deg}(f)$.

### 2.2 OR composition

In this section we prove our first main result, Theorem 1. We start by formally defining the combinatorial group testing problem, whose quantum query complexity was first studied by Ambainis and Montanaro [AM14].

**Combinatorial group testing problem.** Let $\text{CGT}_{2^n}$ be the following problem. There is a hidden $n$-bit string $x$, which we have to determine using OR-queries to $x$. In an OR-query, we query the oracle with a subset $S \subseteq [n]$ and the oracle outputs 1 if there exists an $i \in S$ such that $x_i = 1$. In other words, the oracle’s output is the function $\bigvee_{i \in S} x_i$. Formally, combinatorial group testing is a partial function

$$\text{CGT}_{2^n} : \{0,1\}^{2^n} \rightarrow \{0,1\}^n \cup \{\ast\},$$

where the input is a $2^n$-bit string corresponding to the ORs of all possible subsets of $x$, and the promise is that all bits are indeed the OR of some string $x \in \{0,1\}^n$. When the promise is satisfied the desired output is the hidden string $x$. In other words, $y \in \{0,1\}^{2^n}$ is in Dom($\text{CGT}_{2^n}$) if there exists an $x \in \{0,1\}^n$ such that for all $S \subseteq \{0,1\}^n$, $y_S = \bigvee_{i \in S} x_i$. For such a $y$, $\text{CGT}_{2^n}(y) = x$. Note that for any $y \in \text{Dom}(\text{CGT}_{2^n})$, the string $x$ is uniquely defined by $x_i = y_{\{i\}}$.

Note that although the problem has an input size of $2^n$ bits, the problem is easily solved with $n$ queries as we can simply query all the singleton subsets $y_{\{i\}}$ for $i \in [n]$ to learn all the bits of $x$. Surprisingly, Belovs showed that the quantum query complexity of this problem is quadratically better than this [Bel15, Theorem 3.1].

**Theorem 14** (Belovs). The bounded-error quantum query complexity of $\text{CGT}_{2^n}$ is $\Theta(\sqrt{n})$.

**Decision problem associated with CGT.** Since we want to work with polynomials (and Boolean matrices in Section 3), it will be more convenient to consider a decision problem corresponding to combinatorial group testing. To do so, we define the problem

$$\text{XOR}_n \circ \text{CGT}_{2^n} : \{0,1\}^{2^n} \rightarrow \{0,1,\ast\},$$

which computes the parity of all the output bits of the CGT function.
In other words, $\text{XOR}_n \circ \text{CGT}_{2^n}(y) = \text{XOR}_n(\text{CGT}_{2^n}(y))$, which is the XOR of all the bits of $x$, the hidden string in the CGT problem. Of course, any quantum algorithm that solves CGT$_{2^n}$ and outputs $x$ can instead output the parity of all the bits of $x$.

We can now construct a polynomial that approximates this Boolean function. Using Theorem 12 and Theorem 14, we can get a polynomial of degree $O(\sqrt{n})$ that approximates $\text{XOR}_n \circ \text{CGT}_{2^n}$ on all inputs in the promise and is bounded in $[0,1]$ outside the promise.

For our application we need a more robust version of this polynomial. We need a polynomial that also works when the input variables are close to being Boolean. Combining this polynomial with Theorem 9, we get the following.

**Theorem 15.** There is a real polynomial $p$ of degree $O(\sqrt{n})$ acting on $2^n$ variables $\{y_S\}_{S \subseteq [n]}$ such that for any input $y \in \{0,1\}^{2^n}$ with $\text{XOR}_n \circ \text{CGT}_{2^n}(y) \neq \ast$, and any $\Delta \in [-1/3, 1/3]^{2^n}$,

$$|p(y + \Delta) - \text{XOR}_n \circ \text{CGT}_{2^n}(y)| \leq 1/3,$$

and for all $y \in \{0,1\}^{2^n}$, $p(y) \in [0,1]$.

**Proof.** We start with Theorem 14 which gives us a quantum algorithm that makes $O(\sqrt{n})$ queries and approximates $\text{XOR}_n \circ \text{CGT}_{2^n}$ to bounded error. Given a quantum algorithm computing a function with probability at least $2/3$, we can always boost the success probability to any constant in $(1/2, 1)$ by repeating the quantum algorithm and taking the majority vote of the outcomes. This only increases the quantum query complexity by a constant factor. Hence we can assume the quantum algorithm of Theorem 14 has error at most $1/6$ and apply Theorem 12 to get a polynomial $p'$ of degree $O(\sqrt{n})$ such that for all $y \in \{0,1\}^{2^n}$ with $\text{XOR}_n \circ \text{CGT}_{2^n}(y) \neq \ast$,

$$|p'(y) - \text{XOR}_n \circ \text{CGT}_{2^n}(y)| \leq 1/6.\tag{12}$$

Furthermore, because $p'$ arises from a quantum algorithm, we know that even on inputs outside the promise, i.e., inputs with $\text{XOR}_n \circ \text{CGT}_{2^n}(y) = \ast$, $p'(y) \in [0,1]$. Since $p'$ is bounded in $[0,1]$, we can apply Theorem 9 to it with $\delta = 1/6$ to obtain a new polynomial $p$ that is robust to input noise. Note that since $|p'(y) - p(y)| \leq 1/6$ on all inputs $y \in \{0,1\}^{2^n}$, including those outside the promise, $p'(y) \in [-1/6, 7/6]$ for all $y \in \{0,1\}^{2^n}$. By rescaling and shifting the polynomial, we can map the interval $[-1/6, 7/6]$ to the interval $[0,1]$. Explicitly, we map $p(y)$ to $\frac{3}{2}(p(y) + \frac{1}{2})$. Since the original polynomial was in $[0,1/6]$ for 0-inputs, this rescaled and shifted polynomial lies in $[0,1/4]$, and similarly for 1-inputs it lies in $[3/4, 1]$, satisfying the conditions of the theorem.

Using these results we can now prove the main result of this section.

**Theorem 1.** For any Boolean function $f : \{0,1\}^m \to \{0,1\}$, we have

$$\tilde{\deg}(\text{OR}_n \circ f) = \Omega(\sqrt{n} \tilde{\deg}(f)).\tag{2}$$

**Proof.** Assuming $\tilde{\deg}(f) \neq 0$ (otherwise the result is trivial), there is an input $w^*$ such that $f(w^*) = 0$. Let $w^*$ be any such input.
Let \( q \) be a polynomial of degree \( T := \widetilde{\deg}(\text{OR}_n \circ f) \) that approximates \( \text{OR}_n \circ f \). Let the input variables of the \( i \)th copy of \( f \), for \( i \in [n] \), be called \( w_{i1}, w_{i2}, \ldots, w_{im} \). Let us also define for all \( i \in [n] \), \( x_i := f(w_{i1}, w_{i2}, \ldots, w_{im}) \) to be the output of the \( i \)th function \( f \).

Thus \( q \) is a polynomial over the variables \( w_{i1} \) to \( w_{im} \) that approximately computes the Boolean function \( \bigvee_{i=1}^n x_i \). From \( q \), we can define for any \( S \subseteq [n] \), a new polynomial \( q_S \) over the same set of variables \( \{w_{ij} : i \in [n], j \in [m]\} \) that approximately computes the Boolean function \( \bigvee_{i \in S} x_i \). The polynomial \( q_S \) is obtained from \( q \) by setting all the inputs to \( f \) for which \( i \notin S \) equal to the special input \( w^* \) for which \( f(w^*) = 0 \). Thus the polynomial \( q_S \) is a polynomial over the same variables as \( q \) and has degree at most \( T \) and approximates the function \( \bigvee_{i \in S} x_i \).

Now from Theorem 15, we have a real polynomial \( p \) of degree \( O(\sqrt{n}) \) acting on \( 2^n \) variables \( \{y_S\}_{S \subseteq [n]} \) such that for any input \( y \in \{0, 1\}^{2^n} \) with \( \text{XOR}_n \circ \text{CGT}_{2^n}(y) \neq * \), and any \( \Delta \in [-1/3, 1/3]^{2^n} \),

\[
|p(y + \Delta) - \text{XOR}_n \circ \text{CGT}_{2^n}(y)| \leq 1/3. \tag{13}
\]

Now we define a polynomial \( r \) in the variables \( w_{i1} \) to \( w_{im} \) by taking the polynomial \( p \) over variables \( y_S \) and replacing each occurrence of the variable \( y_S \) with the polynomial \( q_S \).

Then because of equation (13) and the fact that the polynomial \( q_S \) approximates \( \bigvee_{i \in S} x_i \), the polynomial \( r \) approximates the parity of the bits \( x_i \) (recall \( x_i = f(w_{i1}, w_{i2}, \ldots, w_{im}) \)). Also note that \( r \) is of degree \( O(\sqrt{nT}) \). Thus we have

\[
\widetilde{\deg}(\text{XOR}_n \circ f) = O(\sqrt{nT}) = O(\sqrt{n \widetilde{\deg}(\text{OR}_n \circ f)}). \tag{14}
\]

Since \( \widetilde{\deg}(\text{XOR}_n \circ f)) = \Omega(n \widetilde{\deg}(f)) \) (Theorem 10), we get \( \widetilde{\deg}(\text{OR}_n \circ f) = \Omega(\sqrt{n \widetilde{\deg}(f)}) \). \( \square \)

Note that essentially the same proof yields a weak lower bound for the OR of \( n \) different functions \( f_1, f_2, \ldots, f_n \). The proof would follow similarly, except instead of (14), we would arrive at

\[
\widetilde{\deg}(\text{XOR}_n \circ (f_1, f_2, \ldots, f_n)) = O(\sqrt{n \widetilde{\deg}(\text{OR}_n \circ (f_1, f_2, \ldots, f_n))}). \tag{15}
\]

Now from Theorem 10, we have that \( \widetilde{\deg}(\text{XOR}_n \circ (f_1, f_2, \ldots, f_n)) = \Omega(n \min_i \widetilde{\deg}(f_i)) \), and hence

\[
\widetilde{\deg}(\text{OR}_n \circ (f_1, f_2, \ldots, f_n)) = \Omega(\sqrt{n \min_i \widetilde{\deg}(f_i)}). \tag{16}
\]

We will use this weak result in Section 4.1 to establish an optimal bound on the approximate degree of the OR of \( n \) different functions.

### 2.3 PrOR composition

We now extend the main result of the previous section to work with the partial function \( \text{PrOR}_n : \{0, 1\}^n \to \{0, 1, *\} \) introduced in Section 2.1. PrOR is just the OR function with the additional promise that the input has Hamming weight 0 or 1. As discussed in Section 2.1, \( \overline{\deg}(\text{PrOR}) = \Theta(\sqrt{n}) \).

We now generalize the result of the previous section to work for PrOR.

**Theorem 16.** For any Boolean function \( f : \{0, 1\}^n \to \{0, 1\} \), we have

\[
\overline{\deg}(\text{PrOR}_n \circ f) = \Omega(\sqrt{n \deg(f)} / \log n). \tag{17}
\]
Before proving this, we need to establish some properties of polynomials that arise from quantum algorithms using Theorem 12. Recall from Definition 8 that a polynomial approximating a Boolean function is $\delta$-robust to input noise if the polynomial correctly approximates the function even if the input bits are $\delta$-far from being Boolean.

We now prove that approximating polynomials for Boolean functions that are constructed via quantum algorithms (using Theorem 12) naturally possess some robustness to input noise. While it is not hard to show that all multilinear polynomials on $n$ bits are robust to input noise smaller than $O(1/n)$, this is not good enough for our applications, as the polynomials we are interested in act on exponentially many variables (see Section 1.2 for a detailed discussion). Instead, we show that polynomials that arise from quantum algorithms are much more robust to input noise if the function has small quantum query complexity.

**Theorem 17.** Let $h$ be a partial Boolean function. Then there is a bounded multilinear polynomial $q$ of degree $O(Q(h))$ that approximately computes $h$ with robustness $\Omega(1/Q(h)^3)$.

**Proof.** Consider the quantum query algorithm that computes $h$ to error $1/3$ using at most $Q(h)$ queries. By repeating the algorithm 3 times and taking a majority vote, we get an algorithm $A$ computing $h$ to error $7/27$ using at most $3Q(h)$ queries. Let $q$ be the polynomial associated with this algorithm; this polynomial has degree at most $6Q(h)$, and for each $x \in \{0,1\}^n$, $q(x)$ equals the acceptance probability of the quantum algorithm. Furthermore, make $q$ multilinear by replacing any squared variable $x_i^2$ with $x_i$ until no variables have power higher than 1; this does not change the behavior of $q$ on $\{0,1\}^n$, since $1^2 = 1$ and $0^2 = 0$.

The resulting polynomial $q$ is multilinear. In addition, since it evaluates to an acceptance probability on every input in $\{0,1\}^n$, it is bounded within $[0,1]$ on $\{0,1\}^n$. Together with multilinearity, we conclude $q$ is bounded in $[0,1]$ on all inputs in $[0,1]^n$. On inputs in $\text{Dom}(h)$, the polynomial $q$ evaluates to the acceptance probability of a quantum algorithm $A$ computing $h$, so $q$ computes $h$; it remains only to show that $q$ computes $h$ robustly.

Fix $x \in \text{Dom}(h)$, and fix $y \in [0,1]^n$ that is entry-wise within $\delta$ of $x$ for $\delta = 10^{-5}Q(h)^{-2}$. We must show $|p(y) - h(x)| \leq 1/3$.

Let $B$ be the probability distribution over $\{0,1\}^n$ given by sampling each bit $i$ independently from Bernoulli($y_i$), that is, bit $i$ is 0 with probability $1 - y_i$ and 1 with probability $y_i$. By the multilinearity of $q$, it is not hard to see that

$$q(y) = \mathbb{E}_{z \sim B}[q(z)].$$

Moreover, since $q(z) = A(z)$ for $z \in \{0,1\}^n$ (where $A(z)$ is the acceptance probability of $A$ when run on $z$), we get $q(y) = \mathbb{E}_{z \sim B} [A(z)].$

Split the strings $z \in \{0,1\}^n$ into two groups: call the strings with $|A(z) - A(x)| \leq 4/81$ “close” to $x$, and call the strings with $|A(z) - A(x)| > 4/81$ “far” from $x$. Let $C$ be the set of close strings. Then

$$q(y) = \Pr_{z \sim B} [z \in C] \cdot \mathbb{E}_{z \sim B | z \in C} [A(z)] + \Pr_{z \sim B} [z \notin C] \cdot \mathbb{E}_{z \sim B | z \notin C} [A(z)]$$

$$= \mathbb{E}_{z \sim B | z \in C} [A(z)] + \Pr_{z \sim B} [z \notin C] (\mathbb{E}_{z \sim B | z \notin C} [A(z)] - \mathbb{E}_{z \sim B | z \in C} [A(z)]).$$

The expectation $\mathbb{E}_{z \sim B | z \in C} [A(z)]$ is within $4/81$ of $A(x)$. The term $\mathbb{E}_{z \sim B | z \notin C} [A(z)] - \mathbb{E}_{z \sim B | z \in C} [A(z)]$ is has magnitude most 1, so if we upper bound $\Pr_{z \sim B} [z \notin C]$ by $2/81$, we will conclude that $q(y)$ is within $6/81 = 2/27$ of $A(x)$, and hence within $2/27 + 7/27 = 1/3$ of $h(x)$. For this reason, it suffices to show $\Pr_{z \sim B} [z \notin C] \leq 2/81$.

From the hybrid argument [BBBV97], let $m_{i,t}$ be the probability that if $A$ is run on $x$ for $t - 1$ queries and its query register is subsequently measured (right before the $t^{th}$ query), it is found to be
querying position \(i\). Let \(m_i = m_{i,1} + m_{i,2} + \cdots + m_{i,T}\) for all \(i \in [n]\), where \(T \leq 3Q(h)\) is the total number of queries made by \(A\). Then \(\sum_{i=1}^{n} m_i = T\). Moreover, the hybrid argument tells us that for any \(z \in \{0,1\}^n\), we have
\[
\sum_{i : x_i \neq z_i} m_i \geq \frac{|A(x) - A(z)|^2}{4T}.
\] (21)

For strings \(z \notin C\), this sum is at least \(4/3^8T\).

Note that \(\mathbb{E}_{z \sim B} \sum_{i : x_i \neq z_i} m_i \leq \delta \sum_{i=1}^{n} m_i = \delta T\), since each bit \(z_i\) differs from \(x_i\) with probability at most \(\delta\). By Markov’s inequality, the probability over \(z \sim B\) of the event \(\sum_{i : x_i \neq z_i} \geq 4/3^8T\) is less than \(3^8\delta^2/4\). This upper bounds the probability that \(z \notin C\) for \(z \sim B\). Since we have \(\delta \leq 10^{-3}Q(h)^{-2} \leq (8/3^12)^{-2}\), we conclude \(\Pr_{z \sim B}[z \notin C] \leq 2/81\), as desired. \(\square\)

We now generalize the combinatorial group testing problem defined in Section 2.2 to a problem we call “singleton combinatorial group testing.” In this problem instead of being able to query the \(\psi\) vector alone, and hence the solution for the SDP corresponding to an input \(z\) only

Definition 18 (Singleton CGT). Let \(\text{SCGT}_{2^n} : D \to \{0,1\}^n\) be a partial function with \(D \subseteq \{0,1\}^{2^n}\). Let \(D\) be the set of all \(x \in \{0,1\}^n\) with the property that for all \(S \subseteq [n]\) satisfying \(\sum_{i \in S} x_i \in \{0,1\}\), we have \(\sum_{i \in S} x_i = z_S\). Note that for all \(z \in D\), the string \(x\) is uniquely defined by \(x_i = z_{\{i\}}\), and we denote this string \(x(z)\). We define the partial Boolean function \(\text{SCGT}_{2^n} : D \to \{0,1\}^n\) by \(\text{SCGT}_{2^n}(z) := x(z)\).

Note that both \(\text{CGT}_{2^n}\) and \(\text{SCGT}_{2^n}\) are partial functions that agree on the inputs that are in both their domains, but \(\text{SCGT}_{2^n}\) is a more general problem in the sense that its promise strictly contains the promise of \(\text{CGT}_{2^n}\). In symbols, for all \(z \in \text{Dom}(\text{CGT}_{2^n})\), \(\text{CGT}_{2^n}(z) = \text{SCGT}_{2^n}(z)\), and \(\text{Dom}(\text{CGT}_{2^n}) \subseteq \text{Dom}(\text{SCGT}_{2^n})\).

Remarkably, Belovs’ algorithm for \(\text{CGT}\) also works for this more general problem.

Theorem 19. The bounded-error quantum query complexity of \(\text{SCGT}_{2^n}\) is \(\Theta(\sqrt{n})\).

This claim essentially follows from Belovs’ construction, but for completeness we include a full proof in Appendix B. We also provide a proof sketch here for readers familiar with Belovs’ proof.

Proof sketch. Belovs upper bounds the quantum query complexity of \(\text{CGT}\) by exhibiting a solution to the dual of the adversary SDP that is known to characterize quantum query complexity [Rei11, LMR+11]. Instead of reproducing his proof and observing that it works just as well for \(\text{SCGT}\), we explain why the proof goes through.

The SDP solution that Belovs constructs is fully described by the vectors constructed at the beginning of the proof, which in the notation of his paper are called \(\psi[A]\). His set \(A \subseteq [n]\) corresponds to the set of bits in the hidden input \(x\) that are equal to 1. For every input \(A\) (or \(x \in \{0,1\}^n\) in our notation), he constructs a vector \(\psi[A]\) indexed by \(S \subseteq [n]\), which corresponds to querying the subset \(S\).

However, the only nonzero entries of this vector are those for which \(|S \cap A| \in \{0,1\}\), i.e., where the subset queried has intersection 0 or 1 with the hidden input \(x \in \{0,1\}^n\). Hence this vector is easy to generalize to all inputs \(z \in \text{Dom}(\text{SCGT}_{2^n})\). Note that in \(\text{CGT}\), \(z\) is uniquely determined by the hidden string \(x \in \{0,1\}^n\), but in \(\text{SCGT}\) only some of the bits of \(z\) are fixed by \(x\). However, the vector \(\psi\) is fixed by \(x\) alone, and hence the solution for the SDP corresponding to an input \(z\) only
depends on the hidden input $x$ and not on the irrelevant variables of $z$ (i.e., the ones not fixed by the hidden input $x$).

Since the vectors are the same as in the original SDP, the value of the objective function remains the same. We only need to check that the constraint is satisfied. The constraint is on a pair of inputs $z$ and $z'$, and sums over bits on which they differ. But this sum will again not depend on the irrelevant bits of $z$ and $z'$, since the vectors only depend on the underlying hidden inputs, which completes the proof.

Before proving Theorem 16, we will prove an analogue of Theorem 15, showing the existence of a robust polynomial for $\text{XOR}_n \circ \text{SCGT}_{2^n}$. However, unlike in Theorem 15, we will require the polynomial to be multilinear. The reason for this extra requirement is that we will need plug in polynomials for $\text{PrOR}$ into the variables, and those might not approximate $\{0,1\}$ values on all inputs.

This multilinearity requirement means that we cannot use Sherstov’s robustification construction (Theorem 9). Instead, we use Theorem 17, which shows that polynomials coming from quantum algorithms are always slightly robust. The weaker robustness condition will later cause us to lose a logarithmic factor.

**Theorem 20.** There is a real polynomial $p$ of degree $O(\sqrt{n})$ acting on $2^n$ variables $\{y_S\}_{S \subseteq [n]}$ and a constant $c \geq 10^{-5}$ such that for any input $y \in \{0,1\}^{2^n}$ with $\text{XOR}_n \circ \text{SCGT}_{2^n}(y) \neq \ast$, and any $\Delta \in [-c/n,c/n]^{2^n}$,

$$|p(y + \Delta) - \text{XOR}_n \circ \text{SCGT}_{2^n}(y)| \leq 1/3,$$ \hfill (22)

and for all $y \in \{0,1\}^{2^n}$, $p(y) \in [0,1]$. In addition, $p$ is multilinear.

**Proof.** We know that Belovs’ quantum algorithm works for $\text{SCGT}_{2^n} : \{0,1\}^{2^n} \rightarrow \{0,1\}^n$ (Theorem 19). This algorithm can be modified to output the parity of the $n$ output bits, yielding a quantum algorithm for $\text{XOR}_n \circ \text{SCGT}_{2^n}$. Theorem 17 then gives us the desired polynomial $p$. \hfill \square

We are now ready to prove the main result of this section, Theorem 16. This proof is similar in structure to the proof of Theorem 1, with some crucial differences. We retain the proof structure and variable names of Theorem 1 to highlight the similarity in structure.

**Proof of Theorem 16.** Assuming $\overline{\deg}(f) \neq 0$ (otherwise the result is trivial), there is an input $w^*$ such that $f(w^*) = 0$. Let $w^*$ be any such input.

Let $q$ be a polynomial of degree $T := \overline{\deg}(\text{PrOR}_n \circ f)$ that approximates $\text{PrOR}_n \circ f$ and is bounded outside the promise. Let the input variables of the $i$th copy of $f$, for $i \in [n]$, be called $w_{i1}, w_{i2}, \ldots, w_{im}$. Let us also define for all $i \in [n]$, $x_i := f(w_{i1}, w_{i2}, \ldots, w_{im})$ to be the output of the $i$th function $f$. Thus $q$ is a polynomial over the variables $w_{11}$ to $w_{nm}$ that approximately computes the partial Boolean function $\text{PrOR}(x_1, x_2, \ldots, x_n)$.

We amplify the success probability of the polynomial $q$ until it agrees with $\text{PrOR}_n \circ f$ to within error $c/n$ on all inputs in the promise\textsuperscript{3}, where $c$ is the constant from Theorem 20. This amplified

\textsuperscript{3} This can be done by composing $q$ with the univariate “amplification polynomial” of Buhrman et al. (see proof of Lemma 1 in [BNRdW07]). This is a polynomial of degree $O(\log(1/\epsilon))$ that maps $[0,1/3]$ to $[0,\epsilon]$ and $[2/3, 1]$ to $[1-\epsilon, 1]$, and furthermore preserves the boundedness of the polynomial, i.e. $[1/3, 2/3]$ gets mapped to points in the interval $[0,1]$. 

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polynomial, which we call \( q' \), has degree \( O(T \log n) \). Furthermore, \( q' \) remains bounded in \([0, 1]\) at all Boolean points outside the promise as well.

From \( q' \), we can define for any \( S \subseteq [n] \), a new polynomial \( q'_S \) over the same set of variables \( \{w_{ij} : i \in [n], j \in [m]\} \) that approximately computes the \( \text{PrOR} \) function only on those \( x_i \) with \( i \in S \). The polynomial \( q'_S \) is obtained from \( q' \) by setting all the inputs to \( f \) for which \( i \notin S \) equal to the special input \( w^* \) for which \( f(w^*) = 0 \). Thus the polynomial \( q'_S \) is a polynomial over the same variables as \( q' \) and has degree at most \( \deg(q') \) and approximates the function \( \sum_{i \in S} x_i \) when \( \sum_{i \in S} x_i \in \{0, 1\} \).

As before, we define a polynomial \( r \) in the variables \( w_{11} \) to \( w_{nm} \) by taking the polynomial \( p \) (from Theorem 20) over variables \( y_S \) and replacing each occurrence of the variable \( y_S \) with the polynomial \( q'_S \). Then because of the robust approximation condition of \( p \) in Theorem 20 and the fact that the polynomial \( q'_S \) approximates \( \sum_{i \in S} x_i \) when \( \sum_{i \in S} x_i \in \{0, 1\} \), we would like to argue that the polynomial \( r \) approximates the parity of the bits \( x_i \). We know that the polynomials \( q'_S \) approximates (to error \( c/n \)) the function \( \sum_{i \in S} x_i \) when \( \sum_{i \in S} x_i \in \{0, 1\} \). Now if \( q'_S \) was additionally also \( c/n \)-close to having Boolean output on all other inputs, then we would be done. This is because this perfectly fits the input of SCGT, which expects the correct answer on subsets \( S \) with \( \sum_{i \in S} x_i \in \{0, 1\} \), but still expects a Boolean answer on the remaining inputs with \( \sum_{i \in S} x_i > 1 \). However, the fact that the polynomial \( q'_S \) may output any value in \([0, 1]\) when \( \sum_{i \in S} x_i > 1 \) is not a problem because of the multilinearity of the polynomial \( p \). We can simply view \( q'_S \) as the convex combination of polynomials that always output a value \( c/n \)-close to being Boolean, and since \( p \) is multilinear, the value of \( p \) on a non-Boolean input is the same as its expected value over some Boolean inputs. Hence if \( p \) works correctly when all inputs are in \([0, 1]\), it must also work correctly on inputs in \([0, 1]\).

In more detail, fix an input \( w \in \{0, 1\}^n \) to \( \text{PrOR}_n \circ f \), and consider the vector \( v_w \) (of length \( 2^n \)) of all the real numbers \( q'_S(w) \) for \( S \subseteq [n] \). We have \( r(w) = p(v_w) \). For sets \( S \) for which \( \sum_{i \in S} x_i \in \{0, 1\} \), the entry of \( v_w \) at \( S \) is \( c/n \)-close to \( \sum_{i \in S} x_i \). Call those entries of \( v_w \) the “good” entries. For sets \( S \) without this property, the entry of \( v_w \) at \( S \) is a number in \([0, 1]\). We can write \( v_w \) as a convex combination of vectors that agree with \( v_w \) on the good entries and have \( \{0, 1\} \) values on the bad entries. Since \( p \) is multilinear, its value on a convex combination of vectors is a convex combination of its values on each vector. Since \( p \) robustly computes \( \text{SCGT} \), and since all vectors in the convex combination are extremely close to points in \( \{0, 1\}^{2^n} \) that satisfy the \( \text{SCGT} \) promise with hidden string \( x_1 x_2 \ldots x_n \), the value of \( p(v_w) \) is within \( 1/3 \) of \( \text{XOR}_n \circ f(w) = \bigoplus_{i=1}^n x_i \).

Since \( \deg(r) \leq \deg(p) \deg(q') \), \( r \) is of degree \( O(\sqrt{n} T \log n) \). Thus we have

\[
\tilde{\deg}(\text{XOR}_n \circ f) = O(\sqrt{n} T \log n) = O(\sqrt{n} \log n \ \tilde{bdeg}(\text{PrOR}_n \circ f)).
\]

Combining this with \( \tilde{\deg}(\text{XOR}_n \circ f) = \Omega(n \ \tilde{\deg}(f)) \) (Theorem 10), we get \( \tilde{\deg}(\text{PrOR}_n \circ f) = \Omega(\sqrt{n} \ \tilde{\deg}(f)/\log n) \).

### 2.4 Symmetric function composition

We can now prove the final result of this section, which generalizes the results of Section 2.2 from OR to arbitrary symmetric functions.

**Theorem 2.** For any symmetric Boolean function \( g : \{0, 1\}^n \to \{0, 1\} \) and any Boolean function \( f : \{0, 1\}^m \to \{0, 1\} \), we have

\[
\tilde{\deg}(g \circ f) = \tilde{\Omega}(\tilde{\deg}(g) \ \tilde{\deg}(f)).
\]
Proof. By the result of Paturi [Pat92], we know that the approximate degree of a symmetric function $g : \{0, 1\}^n \rightarrow \{0, 1\}$ is completely determined by the Hamming weight closest to $n/2$ where the function changes value. More precisely, let $k$ be the closest number to $n/2$ such that $g$ gives different values to strings of Hamming weight $k$ and $k + 1$. Let us assume that $k \leq n/2$ (otherwise let $k$ be defined as $n - k$). Then Paturi showed that $\tilde{\deg}(g) = \Theta(\sqrt{n}k)$.

So to prove our result it suffices to show $\deg(g \circ f) = \Omega(\deg(f)\sqrt{n}k / \log n)$. Let us now define a partial function that captures the hard inputs of the function $g$. Let $PrTH^k_n : \{0, 1\}^n \rightarrow \{0, 1, *\}$ be the partial function

$$
PrTH^k_n(x) = \begin{cases} 
0 & \text{if } |x| = k \\
1 & \text{if } |x| = k + 1 \\
* & \text{otherwise} 
\end{cases}.
$$

Now we have

$$
\tilde{\deg}(g \circ f) \geq \tilde{\deg}(PrTH^k_n \circ f),
$$

since $g$ contains either $PrTH^k_n$ or its negation as a sub-function, which implies $g \circ f$ contains either $PrTH^k_n \circ f$ or its negation as a sub-function.

Let $PrOR_{n/2k} : \{0, 1\}^{n/2k} \rightarrow \{0, 1, *\}$ be the partial function that maps the all-zeros string to 0, the Hamming weight 1 strings to 1, and the remaining strings to *. Observe that $PrTH^k_{2k} \circ PrOR_{n/2k}$ is a sub-function of $PrTH^k_n$, so we have

$$
\bdeg(PrTH^k_n \circ f) \geq \bdeg(PrTH^k_{2k} \circ PrOR_{n/2k} \circ f).
$$

By Theorem 11, we know that

$$
\bdeg(PrTH^k_{2k} \circ PrOR_{n/2k} \circ f) = \Omega(\bdeg(PrTH^k_{2k})^2 \bdeg(PrOR_{n/2k} \circ f)/2k)
$$

$$
= \Omega(k \bdeg(PrOR_{n/2k} \circ f)),
$$

since $\bdeg(PrTH^k_{2k}) = \Theta(k)$ [Pat92].

Combining equations (25), (26), and (27), we get $\tilde{\deg}(g \circ f) = \Omega(k \bdeg(PrOR_{n/2k} \circ f))$. Finally, using Theorem 16, we have

$$
\tilde{\deg}(g \circ f) = \Omega(k \sqrt{n}/2k \bdeg(f)/\log n) = \Omega(\sqrt{n}k \bdeg(f)/\log n) = \Omega(\bdeg(g) \bdeg(f)/\log n),
$$

which proves the claim.

We note that all the composition theorems in this section also extend to partial functions $f$.

3 Approximate rank or $\gamma_2$ norm

In this section we prove our composition theorems for approximate rank and approximate $\gamma_2$. Section 3.1 starts with some definitions and known results in communication complexity. In Section 3.2 we prove the $\Omega(\sqrt{n})$ lower bound on the approximate rank and approximate $\gamma_2$ of the disjointness problem (Theorem 3), first proved by Razborov [Raz03]. We also show that the same proof technique yields an OR-composition theorem in the special case that the function $F$ contains an all-zeros row or column (Corollary 29). In Section 3.3 we prove a general OR-composition theorem for approximate $\gamma_2$ (Theorem 4). We then extend this composition result to the PrOR function in Section 3.4, and finally to arbitrary symmetric functions in Section 3.5.
3.1 Preliminaries

In this section we describe the setting of communication complexity and define the complexity measures we are interested in, approximate rank and approximate $\gamma_2$ norm, their properties, and the relationships between these measures. We then end with some results related to quantum communication and query complexity that we use.

Rank and $\gamma_2$. In communication complexity, we have a known function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$ for some finite sets $\mathcal{X}$ and $\mathcal{Y}$. Two players, typically called Alice and Bob, receive inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively, and their goal is to compute $F(x, y)$ using as little communication as possible.

We identify a communication function $F$ with a sign matrix whose rows are indexed by $\mathcal{X}$ and whose columns are indexed by $\mathcal{Y}$, and where the entry corresponding to $(x, y)$ is $(-1)^{1-F(x,y)} \in \{-1, 1\}$. Note that this matrix completely specifies the communication problem.

We are interested in two complexity measures of matrices, the rank of a matrix, and the $\gamma_2$ norm of a matrix. The latter is defined by

$$\gamma_2(A) := \min_{B,C : BC=A} \|B\|_{\text{row}}\|C\|_{\text{col}},$$

where $\|B\|_{\text{row}}$ and $\|C\|_{\text{col}}$ denote the largest $\ell_2$ norm of a row of $B$ and the largest $\ell_2$ norm of a column of $C$, respectively.

The $\gamma_2$ norm is in many ways similar to rank and has several useful properties [She12, LSS08], which we list here and compare with rank. The following hold for all matrices $A$ and $B$:

1. $\gamma_2(A + B) \leq \gamma_2(A) + \gamma_2(B)$ and $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.
2. For any scalar $\lambda \neq 0$, $\gamma_2(\lambda A) = |\lambda|\gamma_2(A)$ and $\text{rank}(\lambda A) = \text{rank}(A)$.
3. If $B$ is a submatrix of $A$, $\gamma_2(B) \leq \gamma_2(A)$ and $\text{rank}(B) \leq \text{rank}(A)$.
4. $\gamma_2(A)$ and $\text{rank}(A)$ are invariant under duplicating, rearranging, or negating rows or columns of $A$.
5. $\gamma_2(A \otimes B) = \gamma_2(A)\gamma_2(B)$ and $\text{rank}(A \otimes B) = \text{rank}(A)\text{rank}(B)$.
6. $\gamma_2(A \circ B) \leq \gamma_2(A)\gamma_2(B)$ and $\text{rank}(A \circ B) \leq \text{rank}(A)\text{rank}(B)$.
7. $\gamma_2(J) = 1$ and $\text{rank}(J) = 1$.
8. $\|A\|_{\infty} \leq \gamma_2(A) \leq \|A\|_{\infty} \sqrt{\text{rank}(A)}$.

In the above, $A \otimes B$ denotes the Kronecker (tensor) product, $A \circ B$ denotes the Hadamard product, $J$ denotes the all-ones matrix, and $\|A\|_{\infty}$ denotes the maximum absolute value of an entry of $A$. Since we will only deal with matrices whose entries are bounded in $[-1, 1]$, item 8 will give us $\gamma_2(A) \leq \sqrt{\text{rank}(A)}$.

Approximate rank and $\gamma_2$. We are also interested in matrices that approximate the sign matrix of a function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, where we say that a matrix $A$ of real numbers approximates (the sign matrix of) $F$ to error $\epsilon$ if $|A_{xy} - (-1)^{1-F(x,y)}| \leq \epsilon$ for all $(x, y)$ and $|A_{xy}| \leq 1$ for all $(x, y)$. If $A$ approximates $F$ to error $\epsilon$, we denote this by $A \approx_{\epsilon} F$.

If $M : \mathbb{R}^{\mathcal{X}\times\mathcal{Y}} \to \mathbb{R}^+$ is a function on matrices, we define the $\epsilon$-approximate version of $M$ as

$$M_\epsilon(F) := \inf_{A : A \approx_{\epsilon} F} M(A).$$

(30)
We are particularly interested in the case where \( M(A) = \text{rank}(A) \) and \( M(A) = \gamma_2(A) \). For both \( \gamma_2 \) and rank, the infimum above is achieved and can be replaced by a minimum.

We now list some useful properties of approximate rank and \( \gamma_2 \) shown in the literature. The first result shows that the exact value of \( \epsilon \) chosen is not really too important.

**Lemma 21 (Amplification).** For any \( F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\} \) and any \( 0 < \epsilon' < \epsilon < 1 \), there exist constants \( c, d > 0 \) that only depend on \( \epsilon \) and \( \epsilon' \), such that

\[
\log \text{rank}_\epsilon(F) \leq \log \text{rank}_{\epsilon'}(F) \leq c (\log \text{rank}_\epsilon(F)), \quad \text{and}
\]

\[
\log \gamma_{2,\epsilon}(F) \leq \log \gamma_{2,\epsilon'}(F) \leq d (\log \gamma_{2,\epsilon}(F) + 1).
\]

**Proof.** The left two inequalities follow from the definition of \( \epsilon \)-approximation. For the right inequalities, the idea is to find a univariate polynomial \( p \) that maps the range \([-1, -1 + \epsilon]\) to \([-1, -1 + \epsilon']\) and maps the range \([1 - \epsilon, 1]\) to \([1 - \epsilon', 1]\). (See footnote 3 for more information.) Then take a matrix \( A \) that approximates \( F \) to error \( \epsilon \) and apply \( p \) to \( A \), using the Hadamard product for the matrix product. The result is that \( p \) gets applied to each entry of \( A \), producing a matrix \( A' \) that approximates \( F \) to error \( \epsilon' \). Finally, the rank or gamma 2 norm of \( A' \) can be upper bounded by appealing to sub-additivity (item 1 above) and to sub-multiplicativity under the Hadamard product (item 6 above). The rank and \( \gamma_2 \) will increase polynomially, and taking logarithms on both sides will yield a result of the form \( \log \text{rank}_{\epsilon'}(F) \leq c (\log \text{rank}_\epsilon(F) + 1) \).

To remove the additive constant, we need to worry about what happens when \( \text{rank}_\epsilon(F) \leq 1 \) or \( \gamma_{2,\epsilon}(F) \leq 1 \). For rank, this can only happen if all rows of the approximating matrix are constant multiples of each other; but in that case, even the original sign matrix of \( F \) is rank 1. It follows that \( \text{rank}_{\epsilon'}(F) = 1 \) as well, so a multiplicative constant is sufficient. For gamma 2, however, this is not the case; we therefore have to lose an additive constant as well as a multiplicative constant. \(\square\)

In light of Lemma 21, we do not need to worry about the exact error \( \epsilon \) so long as it is bounded between 0 and 1. We will pick \( \epsilon = 2/3 \) to be the default, and use \( \tilde{\gamma}_2(F) \) and \( \text{rank}(F) \) to denote the approximate gamma 2 and rank of \( F \) to error 2/3. (Note that since we are approximating the values \([-1, 1]\), an error of 2/3 is analogous to the usual 1/3 error for approximating \( \{0, 1\} \) values).

While these two measures seem somewhat different, Lee and Shraibman \[LS09a\] showed that \( \log \tilde{\gamma}_2(F) \) and \( \text{log rank}(F) \) are actually closely related.

**Theorem 22 (Lee and Shraibman).** Let \( F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\} \) be a communication problem and define \( |F| := |\mathcal{X}||\mathcal{Y}| \). Then

\[
2 \log \tilde{\gamma}_2(F) \leq \log \text{rank}(F) \leq 6 \log \tilde{\gamma}_2(F) + O(\log \log |F|).
\]

Finally, we will need the following result of Sherstov \[She12, Theorem 4.18\] that says that taking the XOR of \( n \) copies of a function \( F \) increases its approximate \( \gamma_2 \) by a factor of \( n \) as long as \( \tilde{\gamma}_2(F) \) is larger than some universal constant.

**Theorem 23 (Sherstov).** There is a universal constant \( c \) such that for all \( F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\} \),

\[
\log \tilde{\gamma}_2(\text{XOR}_n \circ F) = \Omega(n(\log \tilde{\gamma}_2(F) - c)).
\]

**Quantum results.** Next, we note the relevance of \( \tilde{\gamma}_2 \) for lower bounding quantum communication complexity. The relation between these measures follows from the following theorem of Linial and Shraibman \[LS09b\].
**Theorem 24** (Linial and Shraibman). Let $\Pi$ be a $T$-qubit quantum communication protocol with arbitrary preshared entanglement on inputs from $X$ and $Y$. Let $P \in \mathbb{R}^{X \times Y}$ be the matrix of acceptance probabilities of $\Pi$; i.e., $P_{xy}$ is the probability that $\Pi$ accepts when Alice and Bob start with input $(x, y) \in X \times Y$. Then $\gamma_2(P) \leq 2^T$.

This theorem naturally yields the following corollary.

**Corollary 25** (Linial and Shraibman). Let $F : X \times Y \to \{0, 1\}$ be a communication problem. Then $Q^*_{cc}(F) \geq \log \gamma_2(F) - 1$.

**Proof.** Let $\Pi$ be a communication protocol computing $F$ to error 1/3 using $Q^*_{cc}(F)$ qubits of communication. Let $P$ be the matrix of its acceptance probabilities. Note that by negating the answers at the end of $\Pi$, we get a new protocol $\Pi'$ with acceptance probability matrix $J - P$ that has the same communication cost. Then $\gamma_2(2P - J) \leq \gamma_2(P) + \gamma_2(J - P) \leq 2 \cdot 2^{Q^*_{cc}(F)}$ so $Q^*_{cc}(F) \geq \log 2^{2P - J} - 1$. Finally, observe that the entries of $2P - J$ lie in $[-1, 1]$ and approximate the sign matrix of $F$ to error 2/3. Thus $\gamma_2(2P - J)$ upper bounds $\gamma_2(F)$, and the result follows. \hfill \Box

We will additionally need the existence of a low-degree robust polynomial for combinatorial group testing described in Section 2, which we restate here in the $\{-1, 1\}$ basis. Note that in the theorem below, we abuse notation by assuming that the functions $\text{XOR}_n$ and $\text{CGT}_{2^n}$ take input bits in $\{-1, 1\}$ and give output bits in $\{-1, 1\}$.

**Theorem 26.** There is a real polynomial $p$ of degree $O(\sqrt{n})$ acting on $2^n$ variables $\{y_S\}_{S \subseteq [n]}$ such that for any input $y \in \{-1, 1\}^{2^n}$ with $\text{XOR}_n \circ \text{CGT}_{2^n}(y) \neq *$ and any $\Delta \in [-2/3, 2/3]^{2^n}$,

$$|p(y + \Delta) - \text{XOR}_n \circ \text{CGT}_{2^n}(y)| \leq 2/3,$$

and for all $y \in \{-1, 1\}^{2^n}$, $p(y) \in [-1, 1]$. Moreover, the sum of absolute values of coefficients of $p$ is at most $(2^n)^{O(\sqrt{n})} \leq 2^{O(n^{1.5})}$.

**Proof.** This follows immediately from Theorem 15 by a variable substitution of $2y_S - 1$ for each variable $y_S$, mapping $\{0, 1\}$ inputs to $\{-1, 1\}$ inputs. We also apply this linear mapping to the output of the polynomial. The condition on the sum of absolute values of the coefficients follows from the fact that a robustified multilinear polynomial of degree $d$ on $N$ variables has sum of absolute coefficients at most $N^{O(d)}$, which we prove in Appendix A (see Theorem 46). \hfill \Box

### 3.2 Disjointness lower bound

We start by reproving the disjointness lower bound for approximate log rank, first proven by Razborov [Raz03]. We note that the techniques used in our proof are quite different from the previous proofs by Razborov [Raz03] and Sherstov [She11]. Our result for disjointness, Theorem 3, is restated below:

**Theorem 3.** Let $\text{DISJ}_n : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ be the set disjointness function defined as $\text{DISJ}_n(x, y) = \bigvee_{i=1}^n (x_i \wedge y_i)$ for all $x, y \in \{0, 1\}^n$. Then

$$\log \text{rank}(\text{DISJ}_n) = \Omega(\sqrt{n}) \text{ and } \log \widetilde{\gamma}_2(\text{DISJ}_n) = \Omega(\sqrt{n}).$$

**Proof.** Let $G = \text{DISJ}_n = \text{OR}_n \circ \text{AND}_{cc}$, where $\text{AND}_{cc}$ is the communication function where Alice and Bob each get one bit, and they must compute the $\text{AND}$ of their bits. In later theorems, we will replace $\text{AND}_{cc}$ by an arbitrary communication function $F$. 

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Let $A$ be the optimal approximating matrix for $G$, so $A$ approximates $G$ and $\text{rank}(A) = \widetilde{\text{rank}}(G)$. For each set $S \subseteq [n]$, let $G_S$ be the disjointness problem restricted to that set of bits. That is, in $G_S$, Alice and Bob still get $n$-bit strings as input, but now they ignore the bits whose position is outside of $S$, and must compute disjointness of the bits whose position is in $S$. We use $A$ to construct an approximating matrix for $G_S$ for every $S \subseteq [n]$. For a fixed $S$, we define $(A_S)_{xy}$ to be $A_{xy}^S$, where $y^S$ is the string $y$ will all the bits outside of $S$ replaced with 0; that is, $y^S_i = y_i$ for $i \in S$, and $y^S_i = 0$ for $i \notin S$. Since $A$ approximates $\text{OR}_n \circ \text{AND}^cc$ entrywise, each entry $(A_S)_{xy}$ approximates the disjointness function on $x$ and $y$ restricted to the subset $S$; thus $A_S$ approximates $G_S$. Moreover, for each $S \subseteq [n]$, the matrix $A_S$ is simply the matrix $A$ with some columns replaced by others; that is, you can get from $A$ to $A_S$ by deleting and duplicating columns. Hence $\text{rank}(A_S) \leq \text{rank}(A) = \widetilde{\text{rank}}(G)$.

Now, from Theorem 26, there is a polynomial $p$ of degree $d = O(\sqrt{n})$ acting on $2^n$ variables $\{z_S\}_{S \subseteq [n]}$ that is bounded between $-1$ and 1 on all inputs in $\{-1, 1\}^{2^n}$. Moreover, this polynomial $p$ has the property that when the variables $z_S$ approximate the OR of the subset $S$ of $n$ fixed bits $w_1, w_2, \ldots, w_n$, the polynomial outputs an approximation of the parity of those bits.

We apply $p$ to the matrices $A_S$, plugging in $z_S = A_S$ for all $S \subseteq [n]$. We use the Hadamard product as the matrix product in this polynomial evaluation. This gives a matrix $B$ with the property that $B_{xy}$ is the result of evaluating $p$ on $z_S = (A_S)_{xy}$. Since $(A_S)_{xy}$ approximates $G_S(x, y)$, which is the OR of the bitwise AND$^{cc}$ of the positions in $S$, we conclude that $B_{xy}$ approximates the parity of the bitwise AND$^{cc}$ of the strings $x$ and $y$. In other words, $B$ is a matrix approximating the function $\text{IP}_n$, which means $\text{rank}(B) \geq \widetilde{\text{rank}}(\text{IP}_n)$.

We now upper bound the rank of $B$ using the rank of $A$. We can write

$$B = \sum_m \alpha_m \prod_{S \in m} A_S,$$

where $m$ are the monomials of $p$ (represented as sets of size at most $d$ of subsets $S \subseteq [n]$), $\alpha_m$ are the coefficients, and the product refers to the Hadamard product of the matrices $A_S$. Consider a single term of the sum, and let $T$ be the resulting matrix product in that term. Then each column of $T$ is the Hadamard product of at most $d$ columns of the $A_S$ matrices. But recall that all the columns of the $A_S$ matrices are columns of $A$; therefore, each column of $T$ is the Hadamard product of at most $d$ columns of $A$. If $A'$ denotes the matrix $A$ with an added all-ones column, then each column of $T$ is the Hadamard product of exactly $d$ columns of $A'$.

Let $C$ be the matrix consisting of all $d$-wise Hadamard products of columns of $A'$ (in any order). Then $C$ is a submatrix of $(A')^{\otimes d}$, the $d$-fold tensor product of $A'$. We saw that each term $T$ in the sum has columns from $C$. The columns of the final sum $B$ are therefore linear combinations of the columns of $C$, meaning the column span of $B$ is a subspace of the column span of $C$. We conclude

$$\text{rank}(\text{IP}_n) \leq \text{rank}(B) \leq \text{rank}(C) \leq \text{rank}((A')^{\otimes d}) = \text{rank}(A')^d \leq (1 + \text{rank}(A))^d = (1 + \widetilde{\text{rank}}(G))^d.$$  

(37)

Taking logarithms and using $d = O(\sqrt{n})$, we get

$$\log \text{rank}(\text{DISJ}_n) = \Omega(\log \text{rank}(\text{IP}_n)/\sqrt{n}),$$  

(38)

where we used the easy-to-check fact that $\text{rank}(\text{DISJ}_n) > 1$ for all $n \geq 1$ to replace the additive 1 term with a multiplicative factor. Finally, it is known that $\log \text{rank}(\text{IP}_n) = \Omega(n)$ (e.g., it follows from Forster’s lower bound on the sign rank of inner product [For02]). A more elementary way of showing this is to lower bound approximate rank using discrepancy [LS09b] and then show that the discrepancy of inner product is $\Omega(n)$ [KN06]. The result for $\gamma_2(\text{DISJ}_n)$ follows from Theorem 22. □
The disjointness lower bound lets us conclude that for any communication problem $F$ containing $\text{AND}^\text{cc}$ as a sub-problem, the approximate gamma 2 norm of $\text{OR}_n \circ F$ is at least $\Omega(\sqrt{n})$ (so in particular, it goes to infinity as $n \to \infty$). We also observe that $\overline{\text{EQ}}$, the negation of the equality problem, is essentially the only communication problem which does not contain $\text{AND}$ as a sub-problem; it is therefore the only communication task that does not grow like $\Omega(\sqrt{n})$ when we take the $\text{OR}$ of $n$ copies of it (indeed, the $\text{OR}$ of $\overline{\text{EQ}}$ is a larger $\overline{\text{EQ}}$ instance, which can be solved with bounded error using constant communication complexity).

**Corollary 27.** Let $F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be a communication problem that is not equivalent (i.e. identical up to permuting rows/columns and deleting repeated rows/columns of the communication matrix) to $\overline{\text{EQ}}$. Then $\log \tilde{\gamma}_2(\text{OR}_n \circ F) = \Omega(\sqrt{n})$.

**Proof.** Remove rows and columns from $F$ until all rows and columns are distinct; this doesn’t change the communication problem. Suppose $F$ has a row or column with at least two zeros. Without loss of generality, let that be a row, and name it $x_1 \in \mathcal{X}$. Then there are $y_1, y_2 \in \mathcal{Y}$ with $F(x_1, y_1) = F(x_1, y_2) = 0$. Since the columns $y_1$ and $y_2$ are distinct, there is some row $x_2$ where they disagree. By exchanging $y_1$ and $y_2$ if necessary, we get $F(x_2, y_1) = 0$ and $F(x_2, y_2) = 1$, which means that on the inputs $\{x_1, x_2\} \times \{y_1, y_2\}$, the function $F$ acts like $\text{AND}^\text{cc}$. In particular, if we restrict the sign matrix of $F$ to the rows and columns corresponding to these inputs, we get the sign matrix of $\text{AND}^\text{cc}$. It follows that if $A$ is an approximating matrix for $\text{OR}_n \circ F$, a submatrix of $A$ is an approximating matrix for $\text{OR}_n \circ \text{AND}^\text{cc} = \text{DISJ}_n$. Since gamma 2 is nonincreasing under submatrices,

$$\log \tilde{\gamma}_2(\text{OR}_n \circ F) \geq \log \tilde{\gamma}_2(\text{DISJ}_n) = \Omega(\sqrt{n}),$$

using Theorem 3.

If $F$ does not have a row or column with at least two zeros, sort its rows by the position of the zero in that row. The resulting matrix is simply $J - I$, with the possible addition of an all-ones row or column. But $J - I$ is the $\overline{\text{EQ}}$ function. An addition of an all ones row is equivalent to adding an element to $\mathcal{X}$ that’s not in $\mathcal{Y}$ and solving $\overline{\text{EQ}}$ on $(\mathcal{X}, \mathcal{Y})$; an addition of an all ones column is similarly equivalent to adding a new element to $\mathcal{Y}$. In all cases, we get that $F$ is equivalent to a version of $\overline{\text{EQ}}$. \hfill \Box

Observe that since the sign matrix of $\overline{\text{EQ}}$ is $J - 2I$ (or a submatrix of $J - 2I$ in the case that the sets $\mathcal{X}$ and $\mathcal{Y}$ are not identical), we have $\gamma_2(\overline{\text{EQ}}) \leq \gamma_2(J) + 2\gamma_2(I) = 3$. One approximating matrix for any sign matrix $A$ (to error 2/3) is simply $(1/3)A$; hence $\tilde{\gamma}_2(\overline{\text{EQ}}) \leq (1/3)\gamma_2(\overline{\text{EQ}}) \leq 1$, and $\log \tilde{\gamma}_2(\overline{\text{EQ}}) \leq 0$. This conveniently allows us to hope that $\log \tilde{\gamma}_2(\text{OR}_n \circ F) = \Omega(\sqrt{n}\log \tilde{\gamma}_2(F))$ for all functions $F$. We don’t quite manage to prove this, but we get quite close in Corollary 29 and Theorem 4.

Indeed, we note that the proof of Theorem 3 actually proved the following generalization, which lets us reduce $\text{OR}$-composition lower bounds to $\text{XOR}$-composition lower bounds if $F$ has an all-zero row or column.

**Theorem 28.** Let $F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be a communication function. Suppose $F$ has an all-zero row or all-zero column (that is, an input $a \in \mathcal{X}$ such that $F(a, y) = 0$ for all $y \in \mathcal{Y}$, or an input $b \in \mathcal{Y}$ such that $F(x, b) = 0$ for all $x \in \mathcal{X}$). Then

$$\log \text{rank}(\text{OR}_n \circ F) = \Omega(\log \text{rank}(\text{XOR}_n \circ F) / \sqrt{n}).$$

**Proof.** The proof is identical to the proof of Theorem 3. We use $F$ in place of $\text{AND}^\text{cc}$, and set $G = \text{OR}_n \circ F$. For $S \subseteq [n]$, we define $G_S$ to be the OR of the $F$-inputs in the set $S$ (out of the $n$
given $F$-inputs). We need an all-zero column in order to define $A_S$ approximating $G_S$ by deleting and duplicating columns of $A$; we do this by setting $(A_S)_{xy} := A_{xy}^S$, as before, but we must be careful when defining $y^S$. Note that $y$ is a Bob-input to $n$ copies of $F$, say $y_1, y_2, \ldots, y_n$, and we wish to “zero out” the copies that are outside of $S$. We can do this by setting $y_i^S = b$ for all $i \notin S$, and $y_i^S = y_i$ for all $i \in S$. The rest of the argument proceeds as before. \hfill \square

We can then use Theorem 23 to immediately get the following OR-composition theorem.

**Corollary 29.** Let $F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be a communication problem with an all-zero row or column and let $|F| = |\mathcal{X}||\mathcal{Y}|$. Then we have

$$
\log \tilde{\operatorname{rank}}(\operatorname{OR}_n \circ F) = \Omega(\sqrt{n} \log \tilde{\gamma}_2(F)), \quad \text{and}
\log \tilde{\gamma}_2(\operatorname{OR}_n \circ F) = \Omega(\sqrt{n} \log \tilde{\gamma}_2(F)) - O(\log \log |F|).
$$

**Proof.** By Theorem 28, we have

$$
\log \tilde{\operatorname{rank}}(\operatorname{OR}_n \circ F) = \Omega(\log \tilde{\operatorname{rank}}(\operatorname{XOR}_n \circ F)/\sqrt{n}) = \Omega(\log \tilde{\gamma}_2(\operatorname{XOR}_n \circ F)/\sqrt{n}).
$$

Applying Theorem 23 to this equation, we get

$$
\log \tilde{\operatorname{rank}}(\operatorname{OR}_n \circ F) = \Omega(\sqrt{n}(\log \tilde{\gamma}_2(F) - c)).
$$

We now get rid of the $c$ term. If $\log \tilde{\gamma}_2(F) > 2c$, we can of course remove the $c$ term and collapse the loss into the $\Omega$-notation. So suppose $\log \tilde{\gamma}_2(F) \leq 2c$. Without loss of generality, suppose $F$ has an all-zero row, so $F(a, y) = 0$ for all $y \in \mathcal{Y}$. If $F$ has any other row that has both a 1 and a 0, then $F$ has $\operatorname{AND}^c$ as a sub-problem. If not, then each row of $F$ is either all-ones or all-zeros, meaning the sign matrix of $F$ has rank 1. In other words, either $\gamma_2(F) \leq 1$ (which implies $\log \tilde{\gamma}_2(F) \leq 0$), or else $F$ contains $\operatorname{AND}^c$ as a sub-problem. In the former case, the statement $\log \tilde{\operatorname{rank}}(\operatorname{OR}_n \circ F) = \Omega(\sqrt{n} \log \tilde{\gamma}_2(F))$ is trivial. In the latter case, Corollary 27 applies, and we get

$$
\log \tilde{\operatorname{rank}}(\operatorname{OR}_n \circ F) = \Omega(\sqrt{n}) = \Omega(c \sqrt{n}) = \Omega(\sqrt{n} \log \tilde{\gamma}_2(F)).
$$

Finally, we appeal to Theorem 22 to get the desired result (noting that $\log \log |\operatorname{OR}_n \circ F| = \log n + \log \log |F|$). \hfill \square

Although we have a general OR composition result for functions $F$ with an all-zero row or column, this does not cover all functions. We modify the proof of Theorem 3 in the next section to cover all functions $F$.

### 3.3 OR composition

We now prove our OR composition theorems for approximate rank and approximate gamma 2 norm for general functions. We start with the following useful lemma on the composition of a polynomial with matrices.

**Lemma 30.** Let $p$ be a real polynomial on $N$ variables. Let $A_1, A_2, \ldots, A_N$ be real matrices of the same size. Let the matrix $B = p(A_1, A_2, \ldots, A_N)$ be the result of plugging in the matrices into $p$ and using the Hadamard product for the matrix product. Then

$$
\log \gamma_2(B) = O \left( \deg(p) \max_{i \in [N]} \gamma_2(A_i) + \log C \right),
$$

where $C$ is the sum of the absolute values of coefficients of $p$. If $p$ is bounded inside $[-1,1]$ for all inputs in $\{-1,1\}^N$, we have $\log C = O(\deg(p) \log N)$. 26
Proof. Write \( p = \sum_m \alpha_m \prod_{i \in m} x_i \) where \( m \) ranges over monomials, each \( \alpha_m \) is a real coefficient, and \( x_i \) are the variables. We have
\[
B = \sum_m \alpha_m \prod_{i \in m} A_i,
\]
where the product is the Hadamard product. Recall that \( \gamma_2 \) satisfies \( \gamma_2(X + Y) \leq \gamma_2(X) + \gamma_2(Y) \), \( \gamma_2(X \circ Y) \leq \gamma_2(X) \gamma_2(Y) \), and \( \gamma_2(\alpha X) = |\alpha| \gamma_2(X) \). Setting \( d = \deg(p) \) and \( M = \max_{i \in [N]} \gamma_2(A_i) \), we immediately get \( \gamma_2(B) \leq M^d \sum_m |\alpha_m| \). Taking logarithms on both sides yields the desired upper bound on \( \log \gamma_2(B) \). The upper bound on the sum of absolute values of coefficients of a bounded polynomial is proved in Theorem 46.

The above lemma, combined with the polynomial we get from Belovs’s algorithm, will turn any approximating matrix for \( \text{OR} \circ F \) into an approximating matrix for \( \text{XOR} \circ F \) whose gamma 2 norm is not too much larger. We can then use Sherstov’s XOR lemma to get a lower bound on the gamma 2 norm of \( \text{OR} \circ F \). This gives us the following result.

**Lemma 31.** Let \( F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\} \) be a communication problem. Then
\[
\log \tilde{\gamma}_2(\text{OR}_n \circ F) \geq \Omega(\sqrt{n} \log \tilde{\gamma}_2(F)) - O(n). \tag{48}
\]

**Proof.** Let \( c \) be the constant in Theorem 23. If \( \log \tilde{\gamma}_2(F) < 2 + c \), the statement is trivial: we just need an \( \Omega(\sqrt{n}) \) lower bound on \( \log \tilde{\gamma}_2(\text{OR}_n \circ F) \), which follows from Corollary 27 unless \( F \) is equivalent to \( \overline{\text{EQ}} \) (and in the latter case, \( \log \tilde{\gamma}_2(F) \leq 0 \) anyway, so the theorem holds trivially as the right hand side is less than 0). Therefore, suppose \( \log \tilde{\gamma}_2(F) \geq 2 + c \). In particular, \( F \) is not constant, so let \( (a, b) \) be an input to \( F \) such that \( F(a, b) = 0 \).

Let \( G = \text{OR}_n \circ F \). For each \( S \subseteq [n] \), let \( G_S \) be the OR of the inputs in the set \( S \), as in the proof of Theorem 3. Let \( A \) be the best approximating matrix for \( G \), so \( \gamma_2(A) = \tilde{\gamma}_2(G) \), \( A_{xy} \) is within 2/3 of \((-1)^{G(x, y)}\) if \((x, y)\) is in the promise of \( G \), and \( A_{xy} \in [-1, 1] \). For each \( S \subseteq [n] \), we construct the matrix \( A_S \) as follows. First, consider the submatrix \( A'_S \) of \( A \) we get by restricting \( A \) to rows \( x \) and columns \( y \) satisfying \( x_i = a \) and \( y_i = b \) for each \( i \notin S \). Next, let \( J \) denote the all-ones matrix with the same dimensions as \( F \), and tensor product \( A'_S \) with \( J \) \( n - |S| \) times (once for each \( i \notin S \)). Intuitively, this adds \( n - |S| \) inputs that are always ignored. Finally, rearrange the rows and columns of the resulting matrix so that the ignored inputs are in the positions \( i \notin S \). This final matrix is \( A_S \).

It is not hard to see that \( A_S \) approximates \( G_S \) to the usual error 2/3 for all \( S \). Moreover, since \( \gamma_2 \) is invariant under rearrangements of rows and columns and under tensor products with \( J \), and since it is nonincreasing under restriction to a submatrix, we have \( \gamma_2(A_S) \leq \gamma_2(A) \) for all \( S \).

From Theorem 26, we have a polynomial \( p \) of degree \( d = O(\sqrt{n}) \) on \( 2^n \) variables \( \{z_S\}_{S \subseteq [n]} \) such that if there are \( n \) bits \( w_1, w_2, \ldots, w_n \) and if each \( z_S \) is instantiated to be within 1/3 of \( \overline{\bigvee_{i \in S} w_i} \), then \( p(z) \) evaluates to within 1/3 of \( \overline{\bigoplus_{i \in [n]} w_i} \). We represent \( p \) in the \( \{-1, 1\} \) basis, so that the inputs and outputs of \( p \) approximate \( -1 \) or \( 1 \) instead of \( 0 \) or \( 1 \). Each monomial \( m \) of \( p \) is a set of size at most \( d \) of subsets \( S \subseteq [n] \).

We use Lemma 30 to plug the matrices \( A_S \) into the polynomial \( p \). This gives us a matrix \( B \) such that \( B_{xy} \) is the result of applying \( p \) to \( \{(A_S)_{xy}\}_S \). As we saw in the proof of Theorem 3, this means the matrix \( B \) approximates \( \text{XOR} \circ F \) to error 2/3. Theorem 26 also gives us an upper bound on the sum of absolute coefficients. We therefore get
\[
\log \tilde{\gamma}_2(\text{XOR} \circ F) \leq \log \gamma_2(B) = O(\sqrt{n} \log \tilde{\gamma}_2(\text{OR} \circ F) + n^{1.5}). \tag{49}
\]
By Theorem 23, we have \( \log \tilde{\gamma}_2(\text{OR}_n \circ F) = \Omega(n \log \tilde{\gamma}_2(F)) \), from which we get \( \log \tilde{\gamma}_2(G) \geq \Omega(\sqrt{n} \log \tilde{\gamma}_2(F)) - O(n) \).
Finally, we apply Lemma 31 recursively to turn the additive $O(n)$ loss into a multiplicative \(poly\log n\) loss. The idea is as follows: for functions \(F\) such that \(\log \tilde{\gamma}_2(F)\) is sufficiently large (larger than \(c\cdot\sqrt{n}\) for some constant \(c\)), we can get rid of the additive \(O(n)\) factor simply because \(\sqrt{n}\log \tilde{\gamma}_2(F) - O(n) = \Omega(\sqrt{n}\log \tilde{\gamma}_2(F))\). So the only trouble is with functions \(F\) that whose approximate gamma 2 norm is small compared to \(n\) (the size of the desired OR). The key insight is to use the fact that \(\OR_n\) is the same function as \(\OR_n\circ\OR_n\circ\OR_n\), and therefore \(\OR_n\circ F = \OR_n\circ(\OR_n\circ F)\). The inner function \(\OR_n\circ F\) will then intuitively have large approximate gamma 2 norm compared with the outer OR of size \(\sqrt{n}\), so we can use Lemma 31 on the outer composition. Finally, to show that the approximate gamma 2 norm of the inner function \(\OR_n\circ F\) is as large as suspected, we recurse the argument.

We prove this recursive argument formally in the following technical lemma, from which the desired result will directly follow. We phrase this lemma in a very general setting (making no direct reference to \(\OR\) or to communication complexity or gamma 2 norm) because in Section 3.4 we will need to apply this lemma to the \(\Pr\OR\) function instead of the \(\OR\) function (and it will have to compose with partial functions instead of total functions).

**Lemma 32.** Let \(\{\alpha_n\}_{n\in\mathbb{N}}\) be a family of functions \(\alpha_n: A\to A\) on some domain \(A\), with the property that \(\alpha_n\circ\alpha_m = \alpha_{nm}\) for all \(n, m\in\mathbb{N}\). Let \(M: A\to \mathbb{R}\) be a function satisfying \(M(\alpha_n(G)) \geq M(\alpha_n(G))\) for all \(G\in A\) whenever \(n\geq m\).

Fix a positive integer \(n\) and a domain element \(F\in A\). Let \(S := \{\alpha_k(F): k \leq n\}\). Let \(a, b\) and \(c\) be positive integers such that for all \(k \leq n\) and \(G\in S\), we have

\[
M(\alpha_k(G)) \geq \frac{\sqrt{k}}{c} - d \quad \text{and} \quad M(\alpha_k(G)) \geq \frac{\sqrt{k}M(G)}{a} - bk. \tag{50}
\]

Then we also have

\[
M(\alpha_n \circ F) \geq \frac{\sqrt{n}M(F)}{16cba^4 + 5a(\log n)^2\log2a} - d. \tag{51}
\]

Before proving this lemma, we will show how it implies Theorem 4, which we restate here for convenience.

**Theorem 4.** For any function \(F: \mathcal{X} \times \mathcal{Y} \to \{0, 1\}\), we have \(\log \tilde{\gamma}_2(\OR_n \circ F) = \Omega(\sqrt{n}\log \tilde{\gamma}_2(F))\).

**Proof.** First, we observe that when \(F\) is equivalent to \(\overline{E}Q\), we have \(\log \tilde{\gamma}_2(F) \leq 0\) so the theorem follows trivially. We focus on the case where \(F\) is not equivalent to \(\overline{E}Q\).

We use Lemma 32 with \(A = \{\OR_k \circ F: k \in \mathbb{N}\}\), \(\alpha_k\) being the composition-with-OR\(_k\) operator (so \(\alpha_k(G) = \OR_k \circ G\) for all \(G\in A\)), and \(M = \log \tilde{\gamma}_2(\cdot)\). It is clear that \(\alpha_n \circ \alpha_m = \alpha_{nm}\) (from the associativity of composition). The property that \(M(\alpha_n(G)) \geq M(\alpha_n(G))\) when \(n \geq m\), that is, the property that \(\log \tilde{\gamma}_2(\OR_n \circ G) \geq \log \tilde{\gamma}_2(\OR_m \circ G)\), follows from the fact that the sign matrix of \(\OR_m \circ G\) is a submatrix of the sign matrix of \(\OR_n \circ G\), and \(\gamma_2\) is non-increasing under submatrices (and hence so is \(\tilde{\gamma}_2\)).

From Corollary 27, since \(F\) is not equivalent to \(\overline{E}Q\), we have \(\log \tilde{\gamma}_2(\OR_n \circ F) = \Omega(\sqrt{n})\). Hence we can pick \(c\) and \(d\) in Lemma 32 to be universal constants independent of \(F\) and \(n\). Moreover, from Lemma 31, we can also pick \(a\) and \(b\) to be universal constants independent of \(F\) and \(n\). Since \(a, b, c,\) and \(d\) are all constants, Lemma 32 gives us \(\log \tilde{\gamma}_2(\OR_n \circ F) = \Omega(\sqrt{n}\log \tilde{\gamma}_2(F))\), as desired. \(\square\)

We now prove the lemma.

**Proof of Lemma 32.** We compare \(M(F)\) to \(b(2a)^4\). If \(M(F) < b(2a)^4\), then

\[
M(\alpha_n(F)) \geq \frac{\sqrt{n}}{c} - d \geq \frac{\sqrt{n}M(F)}{cb(2a)^4} - d \tag{52}
\]
from which the desired result follows.

For the rest of the proof, we assume $M(F) \geq b(2a)^4$. We prove by induction on $t$ that $M(\alpha_{ct}(F)) \geq \sqrt{c}M(F)/(2a)^t$, where $c_t$ is defined by $c_{t+1} = c_t^2/(2a)^{2t}$ and $c_1 = (2a)^6$. The base case follows from

$$M(\alpha_{ct}(F)) \geq (1/a)(2a)^3M(F) - b(2a)^6 \geq (1/2a)(2a)^3M(F),$$

where we used $M(F) \geq b(2a)^4$.

For the induction step, we have

$$M(\alpha_{ct+1}(F)) = M(\alpha_{ct/(2a)^t}(\alpha_{ct}(F))) \geq (1/a)\sqrt{c_t/(2a)^{2t}}M(\alpha_{ct}(F)) - bc_t/(2a)^{2t}$$

where we used the property in the lemma of the constants $a$ and $b$ to remove the outer function $\alpha_{ct/(2a)^t}$. Next, apply the induction hypothesis for $c_t$ to get $M(\alpha_{ct}(F)) \geq \sqrt{c}M(F)/(2a)^t$. This gives

$$M(\alpha_{ct+1}(F)) \geq (1/a)c_tM(F)/(2a)^{2t} - bc_t/(2a)^t = (1/a)\sqrt{c_{t+1}}M(F)/(2a)^t - (b/(2a)^t)\sqrt{c_{t+1}}.$$  

Since $M(F) \geq 2ab$, the subtracted term is at most half the first term, so we get

$$M(\alpha_{ct+1}(F)) \geq (1/2a)\sqrt{c_{t+1}}M(F)/(2a)^t = \sqrt{c_{t+1}}M(F)/(2a)^{t+1},$$

finishing the induction step.

Now, the recursion for $c_t$ is $c_{t+1} = c_t^2/(2a)^{2t}$. Using the identity $1.5^t \geq t$ for all positive integers $t$, it is not hard to prove by induction that $c_t \geq (2a)^{4^t-1}$. It is also clear that $c_t \leq (2a)^{4^t}$.

Consider the largest $t$ such that $c_t \leq n$. Then $t \leq \log_{1.5} \log(2a)^2 \leq \log_{1.5} \log n \leq 2 \log \log n$, and $n < c_{t+1} = c_t^2/(2a)^{2t}$, so $(2a)^t \sqrt{n} < c_t \leq n$. If $c_t \geq n/2$, we have

$$M(\alpha_n(F)) \geq M(\alpha_{ct}(F)) \geq \sqrt{c}M(F)/(2a)^t = (1/\sqrt{2})(2a)^{-t}\sqrt{n}M(F).$$

Otherwise, $[n/c_t] \geq n/2c_t$, and we have

$$M(\alpha_n(F)) \geq M(\alpha_{[n/c_t]}(\alpha_{ct}(F))) \geq (1/a)\sqrt{n/2c_t}M(\alpha_{ct}(F)) - bn/c_t \geq (1/a)\sqrt{n/2c_t}(2a)^{-t}\sqrt{c}M(F) - bn/c_t = (1/\sqrt{2})(2a)^{-t}\sqrt{n}M(F) - bn/c_t.$$ 

Since $c_t \geq \sqrt{n}(2a)^t$, we have $n/c_t \leq \sqrt{n}(2a)^{-t}$. Also, since $M(F) \geq 2ab$, we have $b \leq (1/2a)M(F)$. Hence the subtracted term above is at least a factor of $\sqrt{2}$ smaller than the first term, which means that subtracting it off decreases the first term by a factor of at most $\sqrt{2} - 1$. Using $(\sqrt{2} - 1)/\sqrt{2} \geq 1/5$, we get

$$M(\alpha_n(F)) \geq (1/5a)/(2a)^{-t}\sqrt{n}M(F).$$

Note that since $\sqrt{2} < 5$, the above inequality is also satisfied in the first case, where $c_t \geq n/2$.

Finally, using $t \leq 2 \log \log n$, we get

$$M(\alpha_n(F)) \geq \frac{\sqrt{n}M(F)}{5a(2a)^{2\log \log n}} = \frac{\sqrt{n}M(F)}{5a(\log n)^{2\log 2a}},$$

from which the desired result follows.
### 3.4 PrOR Composition

In this section, we extend the composition result to PrOR. To do so, we first have to define the notion of partial functions in communication complexity. As in the query setting, a partial communication problem will be a function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1, *\}$. We use $\text{Dom}(F)$ to denote the set of pairs $(x, y)$ with $F(x, y) \neq *$. We associate a communication matrix with a sign matrix, which will have $\{-1, 1, *\}$ entries. We say that a real matrix $A$ approximates (the sign matrix of) $F$ to error $\epsilon$ if $A_{xy}$ is within $\epsilon$ of $(-1)^{1-F(x,y)}$ for all $(x, y) \in \text{Dom}(F)$, and in addition, $|A_{xy}| \leq 1$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ (even those outside the promise of $F$). The measures $\gamma_2(F)$ and $\tilde{\gamma}_2(F)$ are then defined as they were previously, minimizing over all matrices approximating $F$ under this new definition of approximation.

Finally, we define composition for partial functions. A partial Boolean function $f : \{0, 1\}^n \to \{0, 1, *\}$ composed with a partial communication function $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1, *\}$ is a partial communication function $f \circ F : \mathcal{X}^n \times \mathcal{Y}^n \to \{0, 1, *\}$. On inputs where $(x_1, y_1), \ldots, (x_n, y_n)$ are all inside $\text{Dom}(F)$, the output of $f \circ F$ is given by the normal function composition – that is, it is equal to $f(F(x_1, y_1), \ldots, F(x_n, y_n))$. On inputs where some $(x_i, y_i)$ is not in the domain of $F$, the output of $f \circ F$ is simply *. Another way to view it is to extend $f$ to take inputs from $\{0, 1, *\}^n$, where $f(x)$ for any $x \notin \{0, 1\}^n$ is defined to be *, and then let $f \circ F$ be the usual function composition.

We are now ready to prove a composition theorem for PrOR. To do so, we need the a version of Theorem 20 showing the existence of a slightly robust polynomial for the SCGT function that is also multilinear. This polynomial will be in the $\{-1, 1\}$ basis, and we abuse notation by assuming XOR and SCGT have input and output bits in $\{-1, 1\}$ instead of $\{0, 1\}$.

**Lemma 33.** There is a real polynomial $p$ of degree $O(\sqrt{n})$ acting on $2^n$ variables $\{z_S\}_{S \subseteq [n]}$ and a constant $c \geq 10^{-5}$ such that for any input $z \in \{-1, 1\}^{2^n}$ with $\text{XOR}_n \circ \text{SCGT}_{2^n}(z) \neq *$ and any $\Delta \in [-c/n, c/n]^{2^n}$,

$$|p(z + \Delta) - \text{XOR}_n \circ \text{SCGT}_{2^n}(z)| \leq 2/3,$$

and for all $z \in \{0, 1\}^{2^n}$, we have $p(z) \in [-1, 1]$. In addition, $p$ is multilinear.

**Proof.** The existence of this polynomial follows directly from Theorem 20. All we need to do is change bases from $\{0, 1\}$ to $\{-1, 1\}$, which we can do by applying the variable substitution $x \to 2x - 1$ to all input variables and to the output. \qed

The proof of the following composition lemma for PrOR will closely follow the proofs of Lemma 31 (the analogous composition theorem for OR) and of Theorem 16 (the composition theorem for PrOR in query complexity).

**Lemma 34.** Let $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1, *\}$ be a partial communication problem. Then

$$\log \gamma_2(\text{PrOR}_n \circ F) \geq \Omega(\sqrt{n \log \gamma_2(F) / \log n}) - O(n / \log n).$$

**Proof.** Let $C$ be the constant from Theorem 23. If $\log \gamma_2(F) \leq C + 1$, the theorem holds trivially by picking the constant in the $O(n)$ term to be larger than the constant in the $\Omega(\sqrt{n \log \gamma_2(F) / \log n})$ term by a factor of at least $C + 1$. Therefore, it suffices to prove the result for functions $F$ satisfying $\log \gamma_2(F) > C + 1$.

We follow the proof of Lemma 31 fairly closely. Fix $F$ with $\log \gamma_2(F) > C + 1$. $F$ cannot be constant, so there is an input pair $(a, b)$ such that $F(a, b) = 0$. Let $G = \text{PrOR}_n \circ F$. For each $S \subseteq [n]$, let $G_S$ be the function PrOR applied to the copies of $F$ in the set $S$. Let $A$ be a matrix approximating $G$ to error $c/n$ with minimum $\gamma_2$ norm, where $c$ is the constant from Lemma 33. That is, $A_{xy} \in [-1, 1]$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $A_{xy}$ is within $c/n$ of $(-1)^{1-G(x,y)}$ for $(x, y) \in \text{Dom}(G)$. 

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We take a moment to upper bound $\gamma_2(A)$. Note that $\log \overline{\gamma}_2(\cdot)$ can be amplified by plugging the matrix into a univariate amplification polynomial (see Lemma 21). A univariate polynomial can amplify constant error to error $\epsilon$ using degree $O(\log 1/\epsilon)$ (see footnote 3). Moreover, if we plug in a matrix with gamma 2 norm $\ell$ into a polynomial of degree $d$ (using the Hadamard product), the gamma 2 norm of the result will be at most $\ell^d$ times the total sum of the absolute values of the coefficients of the polynomial. Since a bounded univariate polynomial of degree $d$ has coefficients that are at most $4^d$ (see, for example, [She13c]), we conclude the gamma 2 norm of the result is at most $d(4\ell)^d$. In our case, $d = O(\log n)$, so we conclude that $\log \overline{\gamma}_2(A) = O(\log \overline{\gamma}_2(G) \cdot \log n)$ (since one way to construct a good approximation matrix like $A$ is to start with a 2/3-approximation matrix with logrank equal to $\log \overline{\gamma}_2(G)$ and then amplify it at $O(\log n)$ cost).

As before, we construct the matrices $A_S$ for each $S \subseteq [n]$ as follows. First, let $A_S^{'}$ be the submatrix of $A$ we get by restricting $A$ to rows $x$ and columns $y$ satisfying $x_i = a$ and $y_i = b$ for all $i \notin S$. This effectively fixes the values on the copies of $F$ outside the set $S$ to always be 0. Next, let $J$ denote the all-ones matrix with the same dimensions as $F$, and tensor product $A_S^{'}$ with $J$ $n - |S|$ times (once for each $i \notin S$). This effectively adds $n - |S|$ inputs to $F$ that are always ignored. Finally, rearrange the rows and columns of the resulting matrix so that the ignored inputs are in the positions $i \notin S$. The final matrix is $A_S$.

It is not hard to see that $A_S$ approximates $G_S$ to the usual $2/3$ error for all $S$. Moreover, since $\gamma_2$ is invariant under rearrangements of rows and columns and under tensor products with $J$, and since it is nonincreasing under restrictions to a submatrix, we have $\gamma_2(A_S) \leq \gamma_2(A)$ for all $S$.

From Lemma 33, we have a polynomial $p$ of degree $d = O(\sqrt{n})$ that (approximately) computes $\text{XOR}_n \circ \text{SCGT}_{2^n}$ with robustness $c/n$. We plug in the matrices $A_S$ into the variables $z_S$ of $p$ using the Hadamard product as the matrix product, to get a matrix $B$, and apply Lemma 30. This gives $\log \overline{\gamma}_2(B) = O(\deg(p) \log \overline{\gamma}_2(A) + \deg(p) \log(2^n)) = O(\sqrt{n} \log n \log \overline{\gamma}_2(G) + n^{1.5})$.

We now wish to show that $B$ approximates the sign matrix of $\text{XOR}_n \circ F$. That is, we need to show each entry of $B$ is within $2/3$ of the corresponding entry of the sign matrix. Fix an entry $(x, y)$ of $B$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then $B_{xy}$ is the result of applying $p$ to the variables $z_S = (A_S)_{xy}$. We know that $(A_S)_{xy}$ approximates $G_S(x, y)$ for $S$ such that $(x, y) \in \text{Dom}(G_S)$. For $S$ such that $(x, y) \notin \text{Dom}(G_S)$, we have $(A_S)_{xy} \in [-1, 1]$. Call the $S$ such that $(x, y) \in \text{Dom}(G_S)$ good and the rest bad. As in the proof of Theorem 16, we can the vector $(v_S)_{S\subseteq[n]}$ with $v_S = (A_S)_{xy}$ as a convex combination of vectors that agree with $z_S$ on all good $S$ and take $\{-1, 1\}$ values on all bad $S$. The evaluation of $p$ on each of these support vectors will then be within $2/3$ of the sign matrix of $\text{XOR}_n \circ F$ at the entry $(x, y)$, since these support vectors are within $c/n$ of being integer points that satisfy the SCGT promise. Since $p$ is multilinear, it follows that it evaluates to within $2/3$ of the right value on $v$ itself as well. Hence $B_{xy}$ approximates the sign matrix of $\text{XOR}_n \circ F$, as we hoped.

Theorem 23 tells us that $\log \overline{\gamma}_2(B) = \Omega(n \log \overline{\gamma}_2(F))$. But we also know that $\log \overline{\gamma}_2(B) = O(\sqrt{n} \log n \log \overline{\gamma}_2(G) + n^{1.5})$, from which the desired result follows.

As before, we can replace the additive $O(n)$ with a multiplicative term; this time, we lose a quasi-polylogarithmic factor in $n$ instead of merely a polylogarithmic factor.

Theorem 35. For all $F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$, \[
\log \overline{\gamma}_2(\text{PrOR}_n \circ F) \geq n^{1/2 - o(1)} \log \overline{\gamma}_2(F). \tag{64}
\]

We note that we do not prove this theorem for partial functions due to a technicality: we cannot handle partial functions which do not contain $\text{AND}^c$ as a subproblem yet are not equivalent to $\text{EQ}$. It is possible that some such function exists that has $\overline{\gamma}_2(F) > 1$ and yet behaves like
\[ \overline{\text{EQ}} \] in that \( \log \tilde{\gamma}_2(\text{OR}_n \circ F) = O(1) \). However, we note that our proof does apply to partial functions \( F \) that contain the two-bit \( \text{AND}^{cc} \) as a subproblem or can otherwise be shown to obey \( \tilde{\gamma}_2(\text{PrOR}_n \circ F) = \Omega(\sqrt{n}) \).

**Proof.** We use Lemma 32. The key to the proof is the observation that

\[
\text{PrOR}_s \circ \text{PrOR}_t = \text{PrOR}_{st}. 
\]

This equality holds exactly: the function on the left has the same domain as the function on the right, and the two give the same values on that domain.

This means we can apply Lemma 32 in the same way as we did for Theorem 4; the only difference is that the term \( a \) is no longer constant, but instead as large as \( O(\log n) \) due to the loss in Lemma 34. Since the final loss in Theorem 4 was \( (\log n)^{O(\log a)} \), this means we lose a quasi-polylogarithmic factor in \( n \).

An additional requirement for applying Lemma 32 is that \( \log \tilde{\gamma}_2(\text{PrOR}_n \circ F) = \tilde{\Omega}(\sqrt{n}) \). For this, note that if \( F \) contains the two-bit \( \text{AND}^{cc} \) function as a subproblem, then \( \text{PrOR}_n \circ F \) contains unique disjointness as a subproblem, which satisfies the desired lower bound (one can show this directly by a modification of Theorem 3, losing a log factor, or using the known lower bound for unique set disjointness [Raz03, She11]). If \( F \) does not contain the two-bit \( \text{AND}^{cc} \) function as a subproblem, and if \( F \) is a total function, then \( F \) is equivalent to \( \overline{\text{EQ}} \). In this case, \( \log \tilde{\gamma}_2(F) \leq 0 \) and the theorem follows trivially.

### 3.5 Symmetric function composition

We have reached the final result of this section: a composition theorem for approximate gamma 2 norm with arbitrary symmetric functions on the outside.

**Theorem 36.** For all \( F : X \times Y \to \{0, 1\} \) and all symmetric functions \( g : \{0, 1\}^n \to \{0, 1\} \),

\[
\log \tilde{\gamma}_2(g \circ F) \geq \tilde{\deg}(g)^{1-o(1)} \log \tilde{\gamma}_2(F). 
\]

We note that as for Theorem 35, this theorem generalizes to partial functions \( F \) so long as \( F \) contains the two-bit \( \text{AND}^{cc} \) and two-bit \( \text{OR}^{cc} \) as subproblems or can otherwise be shown to obey \( \tilde{\gamma}_2(\text{PrOR}_n \circ F) = \Omega(\sqrt{n}) \) and \( \tilde{\gamma}_2(\text{PrAND}_n \circ F) = \Omega(\sqrt{n}) \). We do not analyze partial functions \( g \) at all, and leave this for future work.

**Proof.** The proof mirrors that of Theorem 2. Summarizing, we use a theorem of Paturi to characterize \( \tilde{\deg}(g) \) in terms of the most central Hamming layer \( k \) such that \( g \) behaves differently on Hamming layers \( k \) and \( k+1 \). We then restrict \( g \) to those Hamming layers; this reduces the problem to showing a composition theorem with \( \text{PrTH}_k^n \) the outside. By negating the function if necessary, we assume without loss of generality that \( k \leq n/2 \). A further restriction to a promise reduces the problem to showing a composition theorem with \( \text{PrTH}_k^{2k} \circ \text{PrOR}_{n/2k} \) on the outside. The composition theorem for \( \text{PrOR}_{n/2k} \) follows from Theorem 35. Finally, the composition theorem for \( \text{PrTH}_k^{2k} \) follows from the fact that \( \tilde{\deg}(\text{PrTH}_k^{2k}) = \Omega(k) \) (as shown by Paturi) together with Sherstov’s composition theorem [She12] that applies when the approximate degree of the outer function is linear, a communication analogue of Theorem 11.

### 4 Extensions of our results

In this section we show two extensions of our main results.
4.1 Extension to OR of different functions

In Section 2, we showed a lower bound on the approximate degree of the OR of \( n \) copies of a function \( f \) (Theorem 1). In this section, we will extend this result to a non-uniform version of this theorem for the case when the functions \( f_i \) in the OR are possibly different.

For \( n \) Boolean functions \( f_1, \ldots, f_n \), let \( \text{OR}_n \circ (f_1, f_2, \ldots, f_n) \) denote the function \( \bigvee_{i=1}^n f_i(x_i) \), where \( x_i \) is an input to \( f_i \) and \( x_i \)'s have disjoint variables for different \( i \). These functions may be on different input sizes, hence the \( x_i \) may be of different sizes. We completely characterize the approximate degree of this function.

**Theorem 37.** For any Boolean functions \( f_1, f_2, \ldots, f_n \), we have

\[
\tilde{\deg}\left(\text{OR}_n \circ (f_1, f_2, \ldots, f_n)\right) = \Theta\left(\sqrt{\sum_i \tilde{\deg}(f_i)^2}\right).
\]

**Proof.** Our proof will only the following two facts:

\[
\begin{align*}
\tilde{\deg}(\text{OR}_n \circ (f_1, f_2, \ldots, f_n)) &= \Omega(\sqrt{n} \min_i \tilde{\deg}(f_i)), \text{ and} \\
\tilde{\deg}(\text{OR}_n \circ (f_1, f_2, \ldots, f_n)) &= O(\sqrt{n} \max_i \tilde{\deg}(f_i)).
\end{align*}
\]

The first equation is the same as (16), proved at the end of Section 2. The second equation follows from Sherstov’s robust polynomial construction (Theorem 9), since we can take a robust polynomial for \( \text{OR}_n \) of degree \( O(\sqrt{n}) \) and plug in for the \( i^{th} \) variable the approximating polynomial for \( f_i \).

To see how these imply the claim, consider the function \( F = \text{OR}_n \circ (f_1, f_2, \ldots, f_n) \). Let

\[
d_i = \tilde{\deg}(f_i)^2.
\]

Now let \( k \) be the least common multiple of the numbers \( d_i \), which is well defined as the each \( d_i \) is a positive integer. Consider the function \( G = \text{OR}_k \circ F \). By (68) and (69), we have that

\[
\tilde{\deg}(G) = \Theta(\sqrt{k} \deg(F)).
\]

Now since OR is associative, we can also write \( G \) as

\[
G = \text{OR}_k \circ F = \text{OR}_{nk} \circ (f_1, f_1, f_1, f_2, \ldots, f_2, \ldots, f_n, \ldots, f_n),
\]

where \( k \) times each \( f_i \). So the \( k \) copies of \( f_i \) is regrouped into \( d_i \) groups, each containing \( k/d_i \) copies of \( f_i \). Let \( \ell = \sum_{i=1}^{n} d_i \). We can rewrite \( G \) as

\[
G = \text{OR}_{\ell} \circ \left( \text{OR}_{k/d_1} \circ (f_1, \ldots, f_1), \ldots, \text{OR}_{k/d_i} \circ (f_1, \ldots, f_1), \ldots, \text{OR}_{k/d_n} \circ (f_n, \ldots, f_n), \ldots, \text{OR}_{k/d_n} \circ (f_n, \ldots, f_n) \right).
\]

For example, if \( d_1 = 1, d_2 = 2, \) and \( d_3 = 3 \), then \( k = 6 \) and this would be

\[
G = \text{OR}_{18} \circ (f_1, f_1, f_1, f_1, f_1, f_2, f_2, f_2, f_2, f_2, f_3, f_3, f_3, f_3, f_3, f_3)
\]

\[
= \text{OR}_6 \circ (\text{OR}_6(f_1, \ldots, f_1), \text{OR}_3(f_2, f_2, f_2), \text{OR}_3(f_2, f_2, f_2), \text{OR}_2(f_3, f_3), \text{OR}_2(f_3, f_3), \text{OR}_2(f_3, f_3))
\].
Now by (68) and (69), we can compute the approximate degree of each of these groups. We have for all $i \in [n]$, 
\[ \tilde{\deg}(\text{OR}_{k/d_i} \circ f_i) = \Theta\left(\sqrt{k/d_i \tilde{\deg}(f_i)}\right) = \Theta(\sqrt{k}). \] (74)

Using this, we can apply (68) and (69) to (73), and get 
\[ \tilde{\deg}(G) = \Theta(\sqrt{k}). \] (75)

Combining equations (71) and (75), we get 
\[ \tilde{\deg}(F) = \Theta\left(\sqrt{\sum_i \tilde{\deg}(f_i)^2}\right). \] (76)

### 4.2 Extension to quantum information complexity

In this section we prove an OR-composition theorem for quantum information complexity and in particular, we establish Theorem 38, which we state below:

**Theorem 38.** Let $\text{DISJ}_n$ be the set disjointness function and let $QIC(F, \epsilon)$ denote the $\epsilon$-error distribution-free quantum information complexity of $F$. Then 
\[ QIC(\text{DISJ}_n, 1/3) = \Omega(\sqrt{n}/\log n). \] (77)

#### 4.2.1 Preliminaries

In this section we assume the reader is familiar with quantum information complexity. The reader is referred to [Tou15] for more details.

Let us start by recalling the definition of quantum information complexity. Let $\Pi$ be a quantum communication protocol and let $\mu$ be a probability distribution over inputs $(x, y)$. At the end of round $i$ of the protocol, we assume there are three registers $A_i$, $B_i$, and $C_i$. $A_i$ is with Alice and $B_i$ is with Bob and $C_i$ was sent as a message in the $i$th round by either Alice or Bob depending on whether $i$ is odd or even, respectively. Let $|\psi_{i,x,y}^j\rangle_{A_i,B_i,C_i}$ be the joint state on the three registers on input $(x, y)$. The total purified state at the end of round $i$ is given by 
\[ |\psi_{i,x,y}^j\rangle_X,Y,R,A_i,B_i,C_i = \sum_{x,y} \sqrt{\mu(x,y)} |x\rangle_X |y\rangle_Y |x,y\rangle_R |\psi_{i,x,y}^j\rangle_{A_i,B_i,C_i}. \] (78)

Then the quantum information cost of $\Pi$ on the distribution $\mu$, $QIC(\Pi, \mu)$, is given by 
\[ QIC(\Pi, \mu) = \sum_{i \text{ odd}} I(R; C_i|Y, B_i)_{\psi^i} + \sum_{i \text{ even}} I(R; C_i|X, A_i)_{\psi^i}. \] (79)

Note that our definition of quantum information cost is off by a factor of 2 from the definition in [Tou15]. However, for simplicity we ignore this factor, since our lower bounds ignore constant factors anyway.

Now the quantum information complexity of a function $F$ on distribution $\mu$ with error $\epsilon$, denoted $QIC(F, \mu, \epsilon)$, is defined as 
\[ QIC(F, \mu, \epsilon) = \inf_{\Pi: \Pi \text{ computes } F} \text{ with error at most } \epsilon QIC(\Pi, \mu), \] (80)
the infimum of quantum information costs of protocols \( \Pi \) (with respect to \( \mu \)) which compute \( F \) with error at most \( \epsilon \). Finally, the distribution-free quantum information complexity of \( F \), \( \text{QIC}(F, \epsilon) \), is defined as

\[
\text{QIC}(F, \epsilon) = \max_{\mu} \text{QIC}(F, \mu, \epsilon),
\]

(81)

the maximum over distributions \( \mu \) of \( \text{QIC}(F, \mu, \epsilon) \). If \( \epsilon \) is unspecified, it is taken to be \( 1/3 \), and we use \( \text{QIC}(F) \) to mean \( \text{QIC}(F, 1/3) \).

We will need the following basic lemmas about quantum information cost and quantum information complexity. The first lemma is about switching the quantifiers in the definition of quantum information complexity.

**Lemma 39** ([BGK+15]). Consider the following alternate definition of quantum information complexity:

\[
\tilde{\text{QIC}}(F, \epsilon) = \inf_{\Pi: \Pi \text{ computes } F \text{ with error at most } \epsilon} \max_{\mu} \text{QIC}(\Pi, \mu).
\]

(82)

Then

\[
\tilde{\text{QIC}}(F, 2\epsilon) \leq 2 \cdot \text{QIC}(F, \epsilon).
\]

(83)

The next lemma is about cleaning up a protocol, using the standard “uncomputation” trick, so that all the registers except an output bit are essentially returned to their original states. The proofs in [CvDNT13, Hoz17] are for quantum communication cost but the same proofs work for quantum information cost as well [LT17].

**Lemma 40.** Suppose \( \Pi \) is a quantum protocol such that \( \text{QIC}(\Pi, \mu) \leq I \) for all \( \mu \) and that computes a Boolean function \( f \) with error at most \( \epsilon \). Then there is a cleaned up version of \( \Pi \), \( \Pi' \) which satisfies \( \text{QIC}(\Pi', \mu) \leq 2I \) and the following property. Suppose \( |\psi\rangle_{AB} \) is the entanglement shared at the start of \( \Pi' \) and \( |\phi^{x,y}\rangle_{ABB_{out}} \) be the final state of the protocol \( \Pi' \) on inputs \( x, y \) with register \( A \) held by Alice and registers \( B, B_{out} \) held by Bob. Also let \( |\tilde{\phi}^{x,y}\rangle_{ABB_{out}} \) be the state \( |\psi\rangle_{AB} \otimes |f(x, y)\rangle_{B_{out}} \). Then it holds that

\[
\|\phi^{x,y} - \tilde{\phi}^{x,y}\|_{\text{tr}} \leq 16\sqrt{\epsilon}
\]

(84)

for all inputs \( x, y \).

The next lemma is about reducing the error probability and its effect on quantum information complexity [BGK+15].

**Lemma 41.** For any problem \( F \), \( \text{QIC}(F, \epsilon) = O(\text{QIC}(F, 1/3) \cdot \log(1/\epsilon)) \).

The next proposition states an elementary fact about quantum conditional mutual information which can proven by combining Uhlmann’s theorem and unitary equivalence of quantum conditional mutual information (e.g. see [BGK+15]).

**Proposition 42.** Let \( |\psi\rangle_{R,A,B,C} \) and \( |\phi\rangle_{R,A,B,C} \) be two pure states s.t. the marginals of \( \psi \) and \( \phi \) on the registers \( A, B, C \) are equal i.e. \( \psi_{A,B,C} = \phi_{A,B,C} \). Then

\[
I(R; C| B)_\psi = I(\tilde{R}; C| B)_\phi
\]

(85)

The next lemma is a direct sum theorem for quantum information complexity.

**Lemma 43** ([Tou15]). Let \( F \) be a (possibly partial) Boolean function and \( \mu \) be a distribution. Then

\[
\text{QIC}(F^n, \epsilon) \geq n \cdot \text{QIC}(F, \epsilon).
\]

(86)
The direct sum usually gives an equality but there are two differences between the above lemma and the usual direct sum statement: absence of a prior and different error model (where we demand that all the copies are solved except w.p. \( \leq \epsilon \)). But it is easy to see that the usual direct sum statement (in [Tou15]) implies the above inequality.

### 4.2.2 OR composition

Now we are ready to prove our general result, an OR-composition theorem for quantum information complexity.

**Theorem 44.** Let \( F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\} \) be a Boolean function that has an all-zeroes column. Then

\[
\text{QIC}(\text{OR}_n \circ F, 1/3) \geq \Omega \left( \frac{\sqrt{n}}{\log(n)} \cdot \text{QIC}(F, 1/3) \right).
\]  

(87)

The above theorem implies the following OR lemma for general functions as a corollary. The proof (of the reduction from general functions to functions with an all-zeroes column) is similar to the self-reducibility arguments in the derivation of Theorem 4 and Lemma 31 from Corollary 29. So we skip it to avoid repetition of the same arguments.

**Corollary 45.** Let \( F : \mathcal{X} \times \mathcal{Y} \to \{0, 1\} \) be a Boolean function. Then

\[
\text{QIC}(\text{OR}_n \circ F, 1/3) \geq n^{1/2-\omega(1)} \cdot \left( \text{QIC}(F, 1/3) - n^{\omega(1)} \right).
\]  

(88)

Theorem 44 also implies Theorem 38, which lower bounds the quantum information complexity of disjointness. As mentioned before, this was already known [BGK+15] (and in fact without the log factor) but our proof is simpler and more intuitive.

**Theorem 38.** Let \( \text{DISJ}_n \) be the set disjointness function and let \( \text{QIC}(F, \epsilon) \) denote the \( \epsilon \)-error distribution-free quantum information complexity of \( F \). Then

\[
\text{QIC}(\text{DISJ}_n, 1/3) = \Omega(\sqrt{n}/\log n).
\]  

(77)

This follows from Theorem 44 by plugging in \( F = \text{AND} \), which does have an all-zeroes column, and observing that \( \text{QIC}(\text{AND}, 1/3) \geq \Omega(1) \) because the protocol needs to learn the value of AND (with some accuracy). Note that here quantum information complexity is measured w.r.t. an arbitrary distribution in contrast with the quantum information complexity w.r.t. distributions having tiny mass on inputs evaluating to 1 which arise in the study of disjointness (see [BGK+15] and the references therein) where quantum information complexity can approach zero (for unbounded round protocols).

Now we are ready to prove the general result of this section, Theorem 44.

**Proof of Theorem 44.** Suppose \( \text{QIC}(\text{OR}_n \circ F, 1/3) = I \). By Lemma 41, \( \text{QIC}(\text{OR}_n \circ F, 1/n^4) \leq O(I \cdot \log(n)) \). By the combination of Lemma 39 and Lemma 40, there is an almost clean protocol \( \Pi \) s.t. \( \text{QIC}(\Pi, \mu, 2/n^4) \leq O(I \cdot \log(n)) \) for all distributions \( \mu \).

We will use this protocol \( \Pi \) as a black box along with Belovs’ algorithm (with error 1/5) for combinatorial group testing (Theorem 14) to design a protocol \( \tau \) for solving \( n \) copies of \( F \), with quantum information cost \( \leq O(I \cdot \sqrt{n} \log(n)) \), which will imply the lower bound we need using Lemma 43.

Suppose Alice and Bob want to solve \( n \) copies of \( F \) on inputs \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) jointly distributed according to some distribution \( \mu \) (with bounded probability of error). Bob will run
Belovs’ algorithm (with the goal of learning the string $F(X_1, Y_1), \ldots, F(X_n, Y_n)$) with a query to $\vee_{i\in S} F(X_i, Y_i)$ simulated by running $\Pi$ with Alice where Alice’s input is $X_1, \ldots, X_n$ and Bob’s input is $(Y_i)_{i\in S}$ and fixed to $y^*$ outside $S$, where $y^*$ is the input corresponding to the all-zeroes column in $F$, i.e. $F(x, y^*) = 0$ for all $x$. It is not hard to see that this protocol allows Bob to predict $F(X_1, Y_1), \ldots, F(X_n, Y_n)$ except with probability $O(1/n) + 1/5$ (using the fact that $\Pi$ is almost clean). Also one can prove that the quantum information cost of $\tau$ is $\leq O(\sqrt{n} \cdot I \log(n))$. To see this, suppose the (unnormalized) state at the end of the $i^{\text{th}}$ round of the $j^{\text{th}}$ simulation of $\Pi$ (while running $\tau$) be

$$|\psi^{j,i}_{X,Y,R,A_j,B_j,A'_j,C'_j,B'_j} = \sum_{x,y} |x\rangle_X |y\rangle_Y |\overline{y}\rangle_Y |x,y\rangle_R |\psi^{j,x,y}_{A_j,B_j} |q^{j,i,x,y}_{A'_j,C'_j,B'_j} \rangle$$

(89)

Here $X, Y$ registers contain the actual inputs to protocol $\tau$ while the registers $X, \overline{Y}$ contain the inputs to $\Pi$ in the $j^{\text{th}}$ simulation. The registers $\overline{A}_j, \overline{B}_j$ contain the garbage left from previous simulations of $\Pi$ including the previous answers, Bob’s query register $S$ for the current round etc. Note that in the $j^{\text{th}}$ simulation of $\Pi$, Alice and Bob apply a sequence of unitaries on registers $A'_j, C'_j, B'_j$ controlled on the registers $X, \overline{Y}$ without touching the registers $Y, \overline{A}_j, \overline{B}_j$. This will be crucial. Now the quantum information cost of $\tau$ is the following:

$$\sum_{j=1}^{q} \left( \sum_{i \text{ odd}} I(R; C'_j | Y, \overline{Y}, B'_j, \overline{B}_j)_{\psi^{j,i}} + \sum_{i \text{ even}} I(R; C'_i | X, A'_i, \overline{A}_i)_{\psi^{j,i}} \right)$$

(90)

where $q$ is the query cost of Belovs’ algorithm ($q \leq O(\sqrt{n})$). We will prove that for every $j$,

$$\sum_{i \text{ odd}} I(R; C'_i | Y, \overline{Y}, B'_i, \overline{B}_j)_{\psi^{j,i}} + \sum_{i \text{ even}} I(R; C'_i | X, A'_i, \overline{A}_i)_{\psi^{j,i}} \leq O(I \cdot \log(n))$$

(91)

which will complete the proof. We will in fact prove a stronger statement:

$$\sum_{i \text{ odd}} I(R, Y, \overline{A}_i, \overline{B}_i; C'_i | \overline{Y}, B'_i)_{\psi^{j,i}} + \sum_{i \text{ even}} I(R, Y, \overline{A}_i, \overline{B}_i; C'_i | X, A'_i)_{\psi^{j,i}} \leq O(I \cdot \log(n))$$

(92)

from which (91) follows by applying the chain rule and positivity of quantum conditional mutual information. Now note that the marginal state on registers $X, \overline{Y}$ in $\psi^{j,i}$ is a classical distribution since there is a copy of $x$ in $R$ and $\overline{y}$ is a deterministic function of $y$ and the value in query register $S$ inside $\overline{B}_j$. Denote this distribution by $\nu$. Consider the following alternate states:

$$|\phi^{j,i}_{X,Y,R,A_j,C'_j,B'_j} = \sum_{x,y} \sqrt{\nu(x,y)} |x\rangle_X |\overline{y}\rangle_Y |x,y\rangle_R |\phi^{j,i,x,y}_{A_j,C'_j,B'_j} \rangle$$

(93)

Because QIC($\Pi, \nu$) $\leq O(I \cdot \log(n))$, we get that

$$\sum_{i \text{ odd}} I(\overline{R}; C'_i | \overline{Y}, B'_i)_{\phi^{j,i}} + \sum_{i \text{ even}} I(\overline{R}; C'_i | X, A'_i)_{\phi^{j,i}} = \text{QIC}(\Pi, \nu) \leq O(I \cdot \log(n))$$

(94)

Also note that the marginal states on registers $X, \overline{Y}, A'_j, C'_j, B'_j$ are the same in the two states $\phi^{j,i}$ and $\psi^{j,i}$. This along with Proposition 42 implies that

$$\sum_{i \text{ odd}} I(\overline{R}; C'_i | \overline{Y}, B'_i)_{\phi^{j,i}} + \sum_{i \text{ even}} I(\overline{R}; C'_i | X, A'_i)_{\phi^{j,i}}$$

$$= \sum_{i \text{ odd}} I(R, Y, \overline{A}_i, \overline{B}_i; C'_i | \overline{Y}, B'_i)_{\phi^{j,i}} + \sum_{i \text{ even}} I(R, Y, \overline{A}_i, \overline{B}_i; C'_i | X, A'_i)_{\phi^{j,i}}$$

(95)

Combining (94) and (95) gives us (92) which completes the proof. \qed
Acknowledgements

We would like to thank Mark Bun and Justin Thaler for helpful discussions and feedback on an early draft of this work. We would also like to thank Harry Buhrman for bringing reference [BdW98] to our attention. R.K. would like to thank Jeongwan Haah for helpful discussions regarding the proof of Theorem 46.

Some of this work was performed while the first two authors were students at the Massachusetts Institute of Technology and the last author was a postdoctoral associate at the Massachusetts Institute of Technology. This work was partially supported by ARO grant W911NF-12-1-0541, NSF grant CCF-1410022, NSF grant CCF-1629809, and a Vannevar Bush faculty fellowship.

A Coefficients of bounded polynomials

In this section, we prove an elementary result about the maximum absolute value of coefficients of (not necessarily multilinear) multivariate polynomials that are bounded inside the unit cube. Our main result is the following theorem.

**Theorem 46.** Let \( p \) be a polynomial with real coefficients on \( n \) variables with degree \( d \) such that for all \( x \in [0,1]^n \), \( |p(x)| \leq 1 \). Then the magnitude of any coefficient of \( p \) is at most \((2d)^3d\), and the sum of magnitudes of all coefficients of \( p \) is at most \((2(n+d))^{3d}\).

For univariate polynomials a similar theorem is known; see [She13c] for an elementary proof of this fact. Our Theorem 46 follows from the following lemma, which at first looks like a weaker result.

**Lemma 47.** Let \( p \) be a polynomial with real coefficients on \( n \) variables with degree \( d \) such that for all \( x \in [0,1]^n \), \(|p(x)| \leq 1 \). Then the magnitude of any coefficient of \( p \) is at most \((2nd(n+d))^d\).

Let us first prove the main result (Theorem 46) from this lemma.

**Proof of Theorem 46.** Suppose we want to bound the coefficient of some monomial \( M \). Since \( M \) has degree at most \( d \), at most \( d \) variables appear in \( M \). We can set all remaining variables (that do not appear in \( M \)) to 0 and this reduces the number of variables to at most \( d \). This new polynomial is also bounded on the cube and hence we can apply Lemma 47 with \( n = d \) to get the desired bound.

To bound the total magnitude of all coefficients, we first need to count the number of coefficients. The number of monomials of degree \( d \) on \( n \) variables is \( \binom{n+d-1}{d} \). This is smaller than \( \binom{n+d}{d} \). We upper bound the latter by \( \left( \frac{e(n+d)}{d} \right)^d \), which is a standard inequality for binomial coefficients. Multiplying by the bound from Lemma 47, we get an upper bound of \((2en(n+d)^2)^d\) on the magnitude of all coefficients of monomials of degree \( d \). Some monomials may have degree smaller than \( d \); we therefore need to sum the above over degrees from \( d \) down to 0. But this series decreases faster than a geometric series – indeed, assuming \( n, d \geq 1 \), it decreases strictly faster than a geometric series that decreases by a factor of \( 8e \geq 21 \) each term. Such a geometric series has a sum at most \( 21/20 \) times the largest term. Replacing \( 2e \) by 8 in the equation is an increase by a factor larger than \( 21/20 \), so we can safely upper bound the total magnitude of all coefficients by \((8n(n+d)^2)^d \leq (2(n+d))^{3d}\).

The remainder of this section is devoted to proving Lemma 47. The main idea is to use multivariate Lagrange interpolation similar to the proof for univariate case in [She13c]. We first develop some Lagrange interpolators and prove some basic properties about them.
Consider the set of points
\[ S = \{ \alpha \in \mathbb{R}^n : d\alpha \in \{0, 1, \ldots, d\}^n \text{ and } \sum_i \alpha_i \leq 1 \}. \] (96)

Note that the elements of \( S \) are in one-to-one correspondence with degree \( \leq d \) monomials in \( n \) variables and \( |S| = \binom{n+d}{d} \). We first define Lagrange interpolators with respect to the points in \( S \).

**Proposition 48.** For every \( \alpha \in S \), there is a degree \( d \) polynomial \( p_\alpha \) s.t. \( p_\alpha(\alpha) = 1 \) and \( p_\alpha(\beta) = 0 \) for all \( \beta \in S \) with \( \beta \neq \alpha \).

**Proof.** We first define polynomials \( q_\alpha \) that satisfy all the properties of \( p_\alpha \), except that \( q_\alpha(\alpha) \neq 0 \) instead of \( q_\alpha(\alpha) = 1 \). We then define \( p_\alpha(x) \) to be \( q_\alpha(x)/q_\alpha(\alpha) \). Let us first construct \( q_\alpha \) for all \( \alpha \) such that \( \sum_i \alpha_i = 1 \). We define \( q_\alpha \) as
\[ q_\alpha(x) = \prod_{i=1}^{n} \prod_{j_i=0}^{d\alpha_i-1} (x_i - j_i/d). \] (97)

Clearly \( q_\alpha \) has degree \( d \) and \( q_\alpha(\alpha) \neq 0 \) because all the terms of the form \( (\alpha_i - j_i/d) \) are nonzero by construction. But why is \( q_\alpha(\beta) = 0 \) for all \( \beta \in S \) with \( \beta \neq \alpha \)? Since \( \sum_i \alpha_i = 1 \) and \( \sum_i \beta_i \leq 1 \) as well as \( \beta \neq \alpha \), it follows that there is some \( i \in [n] \) s.t. \( \beta_i < \alpha_i \) and hence \( \beta_i = j_i/d \) for some \( j_i \leq d\alpha_i - 1 \) (due to the integrality of \( d\beta_i \) and \( d\alpha_i \)). For that choice of \( i \), the term \((\beta_i - j_i/d)\) in the product above will be zero.

Now let us construct the polynomial corresponding to \( \alpha = 0^n \), \( q_{0^n} \). Here the polynomial is quite simple as well.
\[ q_{0^n}(x) = \prod_{j=1}^{d} \left( \sum_{i=1}^{n} x_i - j/d \right). \] (98)

It is clear that \( q_{0^n}(\beta) = 0 \) for all \( \beta \in S \) with \( \beta \neq 0 \). Also \( q_{0^n} \) has degree \( d \) and \( q_{0^n}(0) \neq 0 \).

Now let us construct the polynomial in the general case, which will be a combination of the two cases discussed above. Suppose \( \sum_i \alpha_i = k/d \). Then the polynomial is as follows.
\[ q_\alpha(x) = \prod_{j=k+1}^{d} \left( \sum_{i=1}^{n} x_i - j/d \right) \prod_{i=1}^{n} \prod_{j_i=0}^{d\alpha_i-1} (x_i - j_i/d) \] (99)

It is clear that the degree of the polynomial is \( d \) and that \( q_\alpha(\alpha) \neq 0 \). The property that \( q_\alpha(\beta) = 0 \) for \( \beta \in S \), \( \beta \neq \alpha \) follows from a combination of arguments given above. \( \square \)

We now list some properties of the polynomials \( p_\alpha \) which will be useful for us.

**Proposition 49.** The following statements hold for the polynomials constructed in **Proposition 48**.

1. The collection of polynomials \( \{p_\alpha\}_{\alpha \in S} \) forms a basis for the space of \( n \)-variate degree-\( d \) polynomials.

2. For each \( \alpha \in S \), the magnitude of coefficients of \( p_\alpha \) is at most \( d^d(2n)^d \).

**Proof.** For item 1, it suffices to prove that the polynomials \( p_\alpha \) are linearly independent since the number of polynomials is the same as the dimension of the space. Suppose we have a linear combination of these polynomials which is 0: \( \sum_{\alpha \in S} c_\alpha p_\alpha = 0 \). Evaluating this linear combination at all \( \beta \in S \), we get
\[ c_\beta = c_\beta p_\beta(\beta) = 0, \] (100)
which proves their linear independence.

To bound the coefficients of \( p_\alpha(x) = q_\alpha(x)/q_\alpha(\alpha) \), first note that we can instead scale \( q_\alpha \) up by a factor of \( d^d \). This scaled up version of \( q_\alpha \) is the following:

\[
d^d q_\alpha(x) = \prod_{j=k+1}^{d} \left( d \sum_{i=1}^{n} x_i - j \right) \prod_{i=1}^{n} \prod_{j_i=0}^{d-1} (dx_i - j_i).
\]

(101)

It’s easy to see that \( |d^d q_\alpha(\alpha)| \geq 1 \), since each term in the product is a nonzero integer. Hence to upper bound the coefficients of \( p_\alpha \), it is sufficient to upper bound the coefficients of \( d^d q_\alpha \).

A crude upper bound on the magnitude of each coefficient in the above polynomial is \( d^d(n+1)^d \leq d^d(2n)^d \). One way to see this is that the polynomial is a product of degree-1 polynomials, which are of the form \( (d \sum_{i=1}^{n} x_i - j) \) or \( (dx_i - j_i) \). For each of these, the sum of magnitudes of all coefficients is at most \( d(n+1) \). When taking the product of several polynomials, the sum of magnitudes of all coefficients is submultiplicative, hence the sum of magnitudes of all coefficients in the above polynomial is \( d^d(n+1)^d \leq d^d(2n)^d \).

Now we are ready to prove Lemma 47.

Proof of Lemma 47. Since \( \{p_\alpha\}_{\alpha \in S} \) is a basis for the space of degree-\( d \) polynomials on \( n \) variables (by Proposition 49), we know that the given bounded polynomial \( p \) can be written as a linear combination of these polynomials

\[
p = \sum_{\alpha \in S} c_\alpha p_\alpha,
\]

(102)

for some real numbers \( c_\alpha \). By evaluating the above expression at \( \beta \in S \), we can compute the coefficient \( c_\beta \) and we get

\[
|c_\beta| = |p(\beta)| \leq 1.
\]

(103)

Since \( p \) is a linear combination of the \( \binom{n+d}{d} \) polynomials \( p_\alpha \) with coefficients \( |c_\alpha| \leq 1 \), we get that the magnitude of the largest coefficient of \( p \) is at most \( \binom{n+d}{d} d^d(2n)^d \leq (2nd(n+d))^d \), where we used the fact that the largest coefficient of any \( p_\alpha \) is at most \( d^d(2n)^d \) (using Proposition 49).

\[ \square \]

B Quantum query complexity of SCGT

In this appendix we formally prove Theorem 19, which follows from [Bel15].

Theorem 19. The bounded-error quantum query complexity of \( \text{SCGT}_{2^n} \) is \( \Theta(\sqrt{n}) \).

Before proving this, let us recall the definition of \( \text{SCGT} \) (Definition 18):

Definition 18 (Singleton CGT). Let \( \text{SCGT}_{2^n} : D \rightarrow \{0, 1\}^n \) be a partial function with \( D \subseteq \{0, 1\}^{2^n} \). Let \( D \) be the set of all \( z \in \{0, 1\}^{2^n} \) for which there exists an \( x \in \{0, 1\}^n \) with the property that for all \( S \subseteq [n] \) satisfying \( \sum_{i \in S} x_i \in \{0, 1\} \), we have \( \sum_{i \in S} x_i = z_S \). Note that for all \( z \in D \), the string \( x \) is uniquely defined by \( x_i = z_{\{i\}} \), and we denote this string \( x(z) \). We define the partial Boolean function \( \text{SCGT}_{2^n} : D \rightarrow \{0, 1\}^n \) by \( \text{SCGT}_{2^n}(z) := x(z) \).

Proof. The lower bound on the complexity of \( \text{SCGT} \) follows from Theorem 14, since CGT is a special case of SCGT. To show the upper bound, we use the dual of the adversary bound [Bel15, Rei11, LMR+11], which we now restate using notation convenient for this problem.

40
For each \( S \subseteq [n] \), let \( X_S \) be a \( D \times D \) matrix, where \( D \) is the domain of \( \text{SCGT} \) as defined above. Then the quantum query complexity of \( \text{SCGT}_2 \) is given by the following semidefinite program (SDP):

\[
\begin{align*}
\text{minimize} & \quad \max_{z \in D} \sum_{S \subseteq [n]} X_S[z, z] \\
\text{s.t.} & \quad \sum_{S: z \neq z'} X_S[z, z'] = 1 \quad \forall z, z' \in D \text{ with } x(z) \neq x(z') \\
& \quad X_S \succeq 0 \quad \forall S \subseteq [n]
\end{align*}
\]

Following Belovs, we used the notation \( X_S[z, z'] \) for \( z, z' \in D \) to denote the \((z, z')\) entry of the matrix \( X_S \). We will also talk about matrices where each entry is a function of some parameter \( p \in \mathbb{R} \). For this usage, we will use the notation \( X_S(p) \) to talk about the matrix, and \( X_S(p)[z, z'] \) to talk about a particular entry of the matrix.

So to prove an upper bound on \( Q(\text{SCGT}) \) it suffices to exhibit \( X_S \) obeying the above constraints whose objective value is \( O(\sqrt{n}) \). We construct these matrices in steps. We start by constructing vectors \( \psi_S(p) \) of length \( D \) as follows. For all \( S \subseteq [n] \), and \( p \in [0, 1] \), define the vector \( \psi_S(p) \) as

\[
\psi_S(p)[z] = \frac{1}{(1 - p)^{|x(z)|} / 2} \times \begin{cases} \sqrt{np/(1 - p)} & \sum_{i \in S} x(z)_i = 0 \\ (1 - p) / np & \sum_{i \in S} x(z)_i = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Now consider the following \( D \times D \) rank-one matrices parameterized by \( p \in [0, 1] \):

\[
Y_S(p) = \frac{p^{|S|}(1 - p)^n - |S|}{2p} \psi_S(p) \psi_S^*(p).
\]

Finally, we define \( X_S \) as

\[
X_S = \int_0^1 Y_S(p) dp.
\]

We now claim these \( X_S \) are a solution to the above SDP with objective value \( O(\sqrt{n}) \).

To see this, first note that since the \( Y_S(p) \) are all rank 1 and hence positive semidefinite (PSD), and the PSD matrices form a convex cone, the \( X_S \) matrices are PSD as well. This satisfies one of the constraints of the SDP.

Next we show the objective value is \( O(\sqrt{n}) \). To see this, note that for any \( z \in D \), we have that

\[
Y_S(p)[z, z] = \frac{p^{|S|}(1 - p)^n - |S|}{2p(1 - p)^{|x(z)|}} \times \begin{cases} \sqrt{np/(1 - p)} & \sum_{i \in S} x(z)_i = 0 \\ (1 - p) / np & \sum_{i \in S} x(z)_i = 1 \\ 0 & \text{otherwise} \end{cases}
\]

Consider the Bernoulli distribution \( P \) on \([n]\), and note that the probability of obtaining \( S \) under distribution, denoted \( P(S) \), is \( p^{|S|}(1 - p)^{n - |S|} \). One can see that summing the above over \( S \subseteq [n] \)
gives
\[
\sum_{S \subseteq [n]} Y_S(p)[z, z] = \frac{1}{2p(1-p)^{|x(z)|}} \left( \Pr_{S \sim P} \left[ \sum_{i \in S} x(z)_i = 0 \right] \sqrt{\frac{np}{(1-p)}} + \Pr_{S \sim P} \left[ \sum_{i \in S} x(z)_i = 1 \right] \sqrt{\frac{1-p}{np}} \right) \tag{110}
\]
\[
= \frac{1}{2p(1-p)^{|x(z)|}} \left( (1-p)^{|x(z)|} \sqrt{\frac{np}{(1-p)}} + |x(z)|p(1-p)^{|x(z)|-1} \sqrt{\frac{1-p}{np}} \right) \tag{111}
\]
\[
= \frac{1}{2} \left( \sqrt{\frac{n}{p(1-p)}} + |x(z)| \frac{1}{n p(1-p)} \right) \leq \sqrt{\frac{n}{p(1-p)}} \tag{112}
\]

Hence for all \(z \in D\), we have that
\[
\sum_{S \subseteq [n]} X_S[z, z] \leq \int_{0}^{1} dp \sqrt{\frac{n}{p(1-p)}} = \pi \sqrt{n}, \tag{113}
\]
as desired.

Finally we show these \(X_S\) satisfy the remaining constraint. Suppose \(z, z' \in D\) such that \(x(z) \neq x(z')\), and consider \(Y_S(p)[z, z']\). Since the \(\psi_S\) vectors only have mass on points for which \(\sum_{i \in S} x(z)_i \in \{0, 1\}\), the only sets \(S\) for which the value of \(Y_S(p)[z, z']\) is nonzero are those for which either \(\sum_{i \in S} x(z)_i = \sum_{i \in S} x(z')_i = 0\), \(\sum_{i \in S} x(z)_i = \sum_{i \in S} x(z')_i = 1\), or \(\sum_{i \in S} x(z)_i + \sum_{i \in S} x(z')_i = 1\).

Now by the definition of SCGT, if \(\sum_{i \in S} x(z)_i \in \{0, 1\}\), then \(z_S = \sum_{i \in S} x(z)_i\). Hence if \(\sum_{i \in S} x(z)_i = \sum_{i \in S} x(z')_i = 0\) or \(\sum_{i \in S} x(z)_i = \sum_{i \in S} x(z')_i = 1\), then we have that \(z_S = z'_S\). Therefore if one considers the sum
\[
\sum_{S:z_S \neq z'_S} Y_S(p)[z, z'] \tag{114}
\]
Then the only nonzero terms are those for which \(\sum_{i \in S} x(z)_i + \sum_{i \in S} x(z')_i = 1\). Hence we have that
\[
\sum_{S:z_S \neq z'_S} Y_S(p)[z, z'] = \frac{\Pr_{S \sim P}[\sum_{i \in S} x(z)_i + \sum_{i \in S} x(z')_i = 1]}{2p(1-p)^{|x(z)|+|x(z')|/2}} \tag{115}
\]
\[
= \frac{|x(z) \oplus x(z')|(1-p)^{|x(z)|+|x(z')|/2}}{2p(1-p)^{|x(z)|+|x(z')|/2}} \tag{116}
\]
\[
= \frac{|x(z) \oplus x(z')|}{2} (1-p)^{|x(z)|+|x(z')|/2}(1-p)^{-|x(z)|/2-|x(z')|/2} \tag{117}
\]
\[
= \frac{|x(z) \oplus x(z')|}{2} (1-p)^{-|x(z)|/2-|x(z')|/2} \tag{118}
\]
Where \(x(z) \oplus x(z')\) denotes the bitwise XOR of the strings and \(x(z) \lor x(z')\) denotes their bitwise OR. Hence for all \(z, z'\) such that \(x(z) \neq x(z')\),
\[
\sum_{S:z_S \neq z'_S} X_S[z, z'] = \int_{0}^{1} dp \frac{|x(z) \oplus x(z')|}{2} (1-p)^{-|x(z)|/2-|x(z')|/2} - 1 = 1 \tag{119}
\]
Where we used that fact that \(|x(z) \oplus x(z')| \geq 1\) as \(x(z) \neq x(z')\), and that for any positive real \(a > 0\), we have \(\int_{0}^{1} a(1-p)^{a-1} dp = 1\). □
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