Calculation of the bound states of power-law and logarithmic potentials through a generalized pseudospectral method

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Abstract

Bound states of the power-law and logarithmic potentials are calculated using a generalized pseudospectral method. The solution of the single-particle Schrödinger equation in a nonuniform and optimal spatial discretization offers accurate eigenvalues, densities and expectation values. The calculations are carried out for states with arbitrary $n$ and $\ell$ quantum numbers. Comparisons are made with the available literature data and excellent agreement is observed. In all the cases, the present method yields considerably improved results over the other existing calculations. Some new states are reported.

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I. INTRODUCTION

Quantum mechanical treatment of a large number of important physical processes in various branches of physics and chemistry often requires solving the time-independent Schrödinger equation involving a central potential. For example, the Hellmann potential [1] has been used to study the electron-core or electron-ion interactions or the atomic inner-shell ionization problems, the short-range Hulthén potential [2] is relevant in the nuclear and particle physics, the exponential cosine screened Coulomb potential and the celebrated Morse potential [3] found applications in solid-state physics, etc. Unfortunately, the exact solutions of such systems are obtainable only in an exceptionally few instances and for almost all practical purposes, one has to take resort to the approximation methods or direct numerical techniques. Therefore, almost ever since the inception of quantum mechanics, a large number of attractive formalisms have been proposed by many workers to calculate the eigenvalues and eigenfunctions of the spherically symmetric potentials, varying in terms of complexity, efficiency and elegance. Significant strides have been made over the years and it is still an active area of research. Possibly the most widely used approximation schemes are the Rayleigh-Schrödinger perturbation theory and the Rayleigh-Ritz variational method. Both of these methods have witnessed outstanding successes to deliver physically meaningful results for many systems of interest. The other popular schemes are the $1/N$ expansion or many of its variants, the WKB method, etc. However the important shortcomings in some of these methods are that these often involve extensive and elaborate algebraic manipulations and the limited nature of the eigenvalues and eigenfunctions to be expressible in compact analytical forms.

This work is devoted to a detailed calculation of the power-law and the logarithmic potentials which have relevant applications in the field of particle physics [4-13]. These potentials have been studied from various perspectives by several researchers employing a number of approximations; e.g., the WKB treatment [14], the shifted $1/N$ expansion method [15-17], the variational technique [18], through an interpolation formula [19] and also by the direct numerical integration methods [14,19]. Although several formally attractive and elegant formalisms exist in the literature, there is lack of accurate eigenvalues, especially those states characterized by higher quantum numbers and the eigenfunctions. Here we employ a generalized pseudospectral (GPS) method for this purpose which has shown considerable promise
in the field of static and dynamic processes in atomic and molecular systems recently (see, e.g., [20-23]) involving mainly the Coulomb potentials. However, to our knowledge, no attempts have been made so far to apply this scheme to other potentials of physical interest including the ones of current study. Therefore the purpose of this article is two-fold, (a) To assess the validity and performance of the GPS method for the power-law and logarithmic potentials, and (b) To determine accurately the bound state spectra of these systems. As is demonstrated in Section 3, it appears that the current method is capable of producing excellent results of these systems; for the low as well as higher states. The layout of the article is as follows. Section 2 presents a brief overview of the method of calculation. In Section 3 we first make some test calculations on some simple quantum mechanical systems and subsequently present the results for the power-law and the logarithmic potentials. Finally we make a few concluding remarks in Section 4.

II. METHODOLOGY

In this section, we present an overview of the generalized pseudospectral method (GPS) employed to solve the radial eigenvalue problem with the power-law and the logarithmic potentials. A more detailed account can be found in the refs. [24,25,20-23].

The desired radial Schrödinger equation to be solved, can be written in the following operator form,

\[ \hat{H}(r) \phi(r) = \varepsilon \phi(r), \]  

where the Hamiltonian includes the usual kinetic and potential energy operators,

\[ \hat{H}(r) = -\frac{1}{2} \frac{d^2}{dr^2} + v(r), \]  

with

\[ v(r) = V(r) + \frac{\ell(\ell + 1)}{2r^2} \]  

and \( V(r) = \text{sgn}(\nu)Ar^{\nu}(\nu > -2) \) or \( V(r) = \ln r \). The symbols have their usual significances.

The majority of the finite-difference discretization schemes for the solution of the radial Schrödinger equation arising in these situations often require a large number of spatial grid points predominantly because of the uniform distributional nature of the spatial grid. The GPS method, on the other hand allows the nonuniform and optimal spatial discretization, and as is mentioned below, maintains the same accuracy at both the small and large \( r \) regions.
Therefore one has the advantage of working with a much lesser grid points compared to many other methods in the literature, and can have a denser mesh at the smaller \( r \) while a coarser mesh at the large \( r \). The principal feature of this scheme is that a function \( f(x) \) defined in the interval \( x \in [-1, 1] \) can be approximated by the polynomial \( f_N(x) \) of order \( N \) so that,

\[
f(x) \approx f_N(x) = \sum_{j=0}^{N} f(x_j) g_j(x),
\]

and the approximation is exact at the collocation points \( x_j \), i.e.,

\[
f_N(x_j) = f(x_j).
\]

In the Legendre pseudospectral method that we use in this study, \( x_0 = -1, x_N = 1 \), and the \( x_j(j = 1, \ldots, N - 1) \) are obtained from the roots of the first derivative of the Legendre polynomial \( P_N(x) \) with respect to \( x \), i.e.,

\[
P_N'(x_j) = 0.
\]

The \( g_j(x) \) in Eq. (2) are called the cardinal functions and given by the following expression,

\[
g_j(x) = -\frac{1}{N(N + 1)P_N(x_j)} \frac{(1 - x^2) P_N'(x)}{x - x_j}, \tag{7}
\]

satisfying the unique property \( g_j(x_j') = \delta_{j,j'} \). Now the semi-infinite domain \( r \in [0, \infty] \) is mapped into the finite domain \( x \in [-1, 1] \) by the transformation \( r = r(x) \). Now the following algebraic nonlinear mapping is introduced

\[
r = r(x) = L \frac{1 + x}{1 - x + \alpha}, \tag{8}
\]

where \( L \) and \( \alpha = 2L/r_{\text{max}} \) may be termed as the mapping parameters. At this stage introduction of the following relation,

\[
\psi(r(x)) = \sqrt{r'(x)}f(x)
\]

coupled with the symmetrization procedure [31,32] leads to the transformed Hamiltonian as below,

\[
\hat{H}(x) = -\frac{1}{2} \frac{1}{r'(x)} \frac{d^2}{dx^2} \frac{1}{r'(x)} + v(r(x)) + v_m(x), \tag{10}
\]

where \( v_m(x) \) is given by the following relation,

\[
v_m(x) = \frac{3(r'')^2 - 2r''r'}{8(r')^4}. \tag{11}
\]
The advantage is that this leads to a symmetric matrix eigenvalue problem which can be readily solved to give accurate eigenvalues and eigenfunctions and at the same time. For the particular transformation used above, \( v_m(x) = 0 \). This discretization then finally leads to the following set of coupled equations,

\[
\sum_{j=0}^{N} \left[ -\frac{1}{2} D^{(2)}_{j'j} + \delta_{j'j} v(r(x_j)) + \delta_{j'j} v_m(r(x_j)) \right] A_j = E A_j', \quad j = 1, \ldots, N - 1, \tag{12}
\]

where

\[
A_j = [r'(x_j)]^{1/2} \psi(r(x_j)) \left[ P_N(x_j) \right]^{-1}. \tag{13}
\]

and the symmetrized second derivative of the cardinal function, \( D^{(2)}_{j'j} \) is given by,

\[
D^{(2)}_{j'j} = [r'(x_{j'})]^{-1} d^{(2)}_{j'j} [r'(x_j)]^{-1}, \tag{14}
\]

with

\[
d^{(2)}_{j'j} = \frac{1}{r'(x_j)} \frac{(N + 1)(N + 2)}{6(1 - x_j)^2} \frac{1}{r'(x_{j'})}, \quad j = j',
\]

\[
d^{(2)}_{j'j} = \frac{1}{r'(x_{j'})} \frac{1}{(x_j - x_{j'})^2} \frac{1}{r'(x_j)}, \quad j \neq j'. \tag{15}
\]

In order to make a judicious choice of the mapping parameters, a large number of tests have been made to check the performance of this scheme for a broad range of parameter sets in these potentials available in the literature. The results are reported only up to the precision that were found to maintain stability with respect to these variations. In this way, a consistent set of the numerical parameters (\( r_{\text{max}} = 200 \), \( \alpha = 25 \) and \( N = 300 \)) has been chosen which seemed to be appropriate and satisfactory for the current problem.

**III. RESULTS AND DISCUSSION**

First we present results for two simple test cases where accurate results are available for comparison. The first one is the well-known Morse potential [26], having the following form,

\[
V(r) = 25(e^{-4(r-3)} - 2e^{-2(r-3)}) \tag{16}
\]

The above potential supports four bound states and the corresponding energies are given by,

\[
E_n = -[5 - \sqrt{2}(n + \frac{1}{2})]^2 \tag{17}
\]
TABLE I: Comparison of the eigenvalues E (in a. u.) of the Morse potential with the literature data.

| n | Energy This work | B-splines (Ref. [27]) | Exact (Ref. [26]) |
|---|------------------|-----------------------|------------------|
| 0 | −18.428932188134 | −18.428932188135 | −18.428932188134 |
| 1 | −8.2867965644036 | −8.2867965644044 | −8.2867965644035 |
| 2 | −2.144669409726 | −2.14466940973 | −2.144669409726 |
| 3 | −0.0025253169419 | −0.0025253169424 | −0.0025253169416 |

where \( n = 0, 1, 2, 3 \). Table I presents the calculated eigenvalues along with the exact analytical values [26] and the B-spline results [27]. While for the ground and second excited states, the current results match the exact values completely, slight overestimation is noticed for the first and third excited states. It may be noted that the first three states reached the precision of Table I with an \( r_{\text{max}} \) of 50 a. u. and with 200 grid points. However, the fourth state required a grid of size 200 a. u. having 300 radial points, presumably due to its weakly bound nature and therefore having a long tail. The B-spline basis set calculations were done using a box of radius 250 a. u. and 263 B-splines.

As a second case we take the common and frequently studied anharmonic oscillator potential having a quartic perturbation given as,

\[
V(r) = \frac{1}{2} m \omega^2 r^2 + \frac{1}{4} \lambda r^4, \quad \lambda > 0
\]

This potential has been studied quite extensively in the literature as a testing ground for many theoretical methodologies. Table II shows the agreement of the current results with the numerically calculated “exact” values of [28] for selected values of the quantum numbers. A very large segment of the quantum numbers are chosen covering ground to the very high excited states (having high values for both the radial as well as the angular quantum numbers), and for all these states the GPS results reproduce the reference values nicely. A careful glance through the table shows that within a given \( \ell \), the eigenvalues show better accuracy as the quantum number \( n \) increases. Such features have been observed in other works also (e.g., the phase-integral calculations [28]).

After these tests we now in Table III present the ground state energies (in units of \( \hbar = 2m = 1 \)) for the various power-law potentials having several values of \( \nu \). The first row shows the results for \( \nu = -1 \) and 2 corresponding to the well-known Coulomb and the harmonic oscillator potentials for which exact solutions are known. In both these cases, our
TABLE II: The calculated eigenvalues $E$ (in a. u.) of the anharmonic oscillator with $m = 1$, $\omega = 1$, $\lambda = 2$ for the various combinations of the quantum numbers $n$ and $\ell$. The numerically calculated “exact” reference values are quoted from [28].

| $n$ | $\ell$ | Energy       | $n$ | $\ell$ | Energy       |
|-----|--------|--------------|-----|--------|--------------|
|     |        | $\text{This work}$ |     |        | $\text{Exact (Ref. [28])}$ |
| 0   | 0      | 2.32440635210 | 5   | 3      | 13.94920888000 |
| 2   | 0      | 6.57840194902 | 10  | 4      | 29.510012607268 |
| 10  | 0      | 30.06476147967 | 5   | 5      | 13.26445879906 |
| 50  | 0      | 213.9909056603 | 10  | 10     | 27.092492305227 |
| 1   | 1      | 4.19017126505 | 50  | 20     | 209.482265121044 |
| 5   | 1      | 14.33886698777 | 50  | 50     | 187.5297080403 |
| 2   | 2      | 6.24277802550 |     |        |              |

TABLE III: The calculated ground states ($n = 0$, $\ell = 0$) along with the literature results for various power law potentials in units of $\hbar = 2m = 1$.

| $V(r)$  | Energy       | $V(r)$  | Energy       |
|---------|--------------|---------|--------------|
| $-r^{-1}$ | $-0.25^a, -0.25^b$ | $r^2$  | 3.000000000 |
| $r^{0.15}$ | 1.327945844 | $r^{2.5}$ | 3.242232312 |
| $r^{0.5}$ | 1.83393609 | $r^{3.0}$ | 3.45052689 |
| $r^{0.75}$ | 2.108136909 | $r^{3.5}$ | 3.634394905 |
| $r^{1.0}$ | 2.338107410 | $r^{4.0}$ | 3.799673029 |
| $r^{1.5}$ | 2.708092416 | $2.70816^c, 2.70809^b$ | 3.799673030^d, 3.799673029^e |

$^a$Ref. [18]
$^b$Numerical results, ref. [16]
$^c$Ref. [29]
$^d$Ref. [17]
$^e$Exact value, as quoted in ref. [17]

calculation reproduces the exact results very well. The variational as well as the numerical results exist for all of these states excepting $\nu = 2.5$ and 3.5 For the $\nu = 4$ case, several accurate calculations are available in the literature including the large-order shifted 1/N expansion [17] and the present result is in complete agreement with the most accurate “exact” result of [17]. However to the best of our knowledge, such definitive and accurate results are not available for the other members of this series and the present values show considerable improvement over the existing results. The GPS results are more in conformity with the numerical results of [16] but noticeably deviate from the values of [18].
TABLE IV: The first six eigenvalues \( \ell \) (in a. u.) of the \( r^{0.5} \) potential along with the literature data for \( \ell = 0, 1, \cdots, 5 \).

| \( \ell \) | Energy | \( \ell \) | Energy |
|-----------|--------|-----------|--------|
|           | This work | Literature | This work | Literature |
| 0         | 1.83339360 | 1.83352\textsuperscript{a}, 1.83339\textsuperscript{b} | 2.55064749 | 3.35759\textsuperscript{a}, 3.35759\textsuperscript{b} |
|           | 2.55152\textsuperscript{a}, 2.55065\textsuperscript{b} | 3.05177\textsuperscript{a}, 3.05111\textsuperscript{b} | 3.70270499 | 3.70299\textsuperscript{a}, 3.70270\textsuperscript{b} |
|           | 3.45197\textsuperscript{a}, 3.45213\textsuperscript{b} | 3.79233\textsuperscript{a}, 3.79336\textsuperscript{b} | 4.00736733 | 4.00796\textsuperscript{a}, 4.00737\textsuperscript{b} |
|           | 3.79336044 | 4.28195944 | 3.35759\textsuperscript{a}, 3.35759\textsuperscript{b} |
|           | 4.009392584 | 4.53316865 | 3.79233\textsuperscript{a}, 3.79336\textsuperscript{b} |
| 1         | 2.20049623 | 2.30056\textsuperscript{a}, 2.30050\textsuperscript{b} | 2.85433592 | 3.35759\textsuperscript{a}, 3.35759\textsuperscript{b} |
|           | 2.85473\textsuperscript{a}, 2.85434\textsuperscript{b} | 3.28666\textsuperscript{a}, 3.28583\textsuperscript{b} | 3.8897564 | 3.88913\textsuperscript{a}, 3.88898\textsuperscript{b} |
|           | 3.64738542 | 3.6483\textsuperscript{a}, 3.64739\textsuperscript{b} | 4.17268190 | 4.17308\textsuperscript{a}, 4.17208\textsuperscript{b} |
|           | 3.96267650 | 3.96361\textsuperscript{a}, 3.96268\textsuperscript{b} | 4.43130627 | 4.43196\textsuperscript{a}, 4.43131\textsuperscript{b} |
|           | 4.24465838 | 4.66989741 | 3.96361\textsuperscript{a}, 3.96268\textsuperscript{b} |
| 2         | 2.65756336 | 2.65760\textsuperscript{a}, 2.65756\textsuperscript{b} | 3.12032849 | 3.77041929 |
|           | 3.12048\textsuperscript{a}, 3.12033\textsuperscript{b} | 3.50296\textsuperscript{a}, 3.50245\textsuperscript{b} | 4.0636036 | 4.32947933 |
|           | 3.83254391 | 3.83338\textsuperscript{a}, 3.83254\textsuperscript{b} | 4.12580907 | 4.57430430 |
|           | 4.12580907 | 4.12581\textsuperscript{a} | 4.39138573 | 4.80174799 |
|           | 4.39138573 | 4.80174799 |

\( ^{a}\text{Ref. [18]} \)

\( ^{b}\text{Numerical results, ref. [19]} \)

Now we show the efficacy of the current method for the excited states by presenting the first six eigenvalues corresponding to \( \ell = 0, 1, \cdots, 5 \) in Table IV for the \( r^{0.5} \) potential. The variational results are from [16], while the numerical reference values are taken from [19] for the \( n, \ell \leq 4 \) states. As in the previous table, there is good agreement between the current and the numerical values. However, the results of [18] are consistently overestimated in all cases except the third and fourth states belonging to \( \ell = 0 \). No results could be found for \( \ell > 4 \). As in the last table, here also we have produced superior results for these systems than the previous works in the literature. Some new states are reported. Additionally Table VI displays the energies (in a. u.) for the two potentials \( V(r) = -2^{1.7}r^{-0.2} \) and \( V(r) = 2^{7/2}r \) which have been examined earlier by some workers. All the states with \( n \leq 4 \) and \( \ell \leq 3 \) are calculated and compared wherever possible. In the former case, accurate results are not available for comparison and the present results match closely to all of them, while for the latter case, the GPS results are in excellent agreement with the accurate calculations [17].
TABLE V: The calculated eigenvalues $E$ (in a.u.) of the power-law potentials for several values of $n$ and $\ell$. The left and the right hand sides correspond to the potentials $V(r) = -2^{1.7} r^{-0.2}$ and $V(r) = 2^{7/2} r$ respectively.

| Energy | Literature | $n$ | $\ell$ | Energy | Literature |
|--------|------------|-----|--------|--------|------------|
| 9.352429641 | 9.352429643 | $0$ | $0$ | $2.68602822$ | $-2.68601^a$, $-2.6859^b$, $-2.686^c$ |
| 16.35179777 | 16.35179777 d | $1$ | $0$ | $-2.25351412$ | $-2.25483^a$, $-2.2530^b$, $-2.253^c$ |
| 22.08223931 | 22.08223931 d | $2$ | $0$ | $-2.04431800$ | $-2.04658^a$, $-2.0446^b$, $-2.044^c$ |
| 27.14683236 | $0$ | $1$ | $0$ | $-1.91063527$ | $3$ | $0$ |
| 29.62266174 | $0$ | $1$ | $0$ | $-1.8144352$ | $4$ | $0$ |
| 31.77653434 | $0$ | $1$ | $0$ | $-1.76427587$ | $3$ | $1$ |
| 32.03881169 | $0$ | $1$ | $0$ | $-1.78852162$ | $3$ | $2$ |
| 34.0693721 | $0$ | $1$ | $0$ | $-1.7193045$ | $4$ | $2$ |
| 32.02906490 | $0$ | $1$ | $0$ | $-1.71073849$ | $-2.10103^a$, $-2.1006^b$, $-2.101^c$ |
| 31.71993045 | $0$ | $1$ | $0$ | $-1.95072177$ | $-1.95147^a$, $-1.9504^b$, $-1.951^c$ |
| 32.01858256 | $0$ | $1$ | $0$ | $-1.84490090$ | $2$ | $1$ |
| 36.3081220 | $0$ | $1$ | $0$ | $-1.8750325$ | $1$ | $1$ |
| 34.0693721 | $0$ | $1$ | $0$ | $-1.78852162$ | $3$ | $2$ |
| 36.3081220 | $0$ | $1$ | $0$ | $-1.7193045$ | $4$ | $2$ |
| 20.20370253 | $0$ | $1$ | $0$ | $-2.15626090$ | $-2.15626^a$, $-2.1562^b$, $-2.156^c$ |
| 22.51883350 | $0$ | $1$ | $0$ | $-1.99005560$ | $-1.99015^a$, $-1.9900^b$, $-1.990^c$ |
| 24.83049317 | $0$ | $1$ | $0$ | $-1.8753225$ | $-1.87535^a$, $-1.8749^b$, $-1.875^c$ |
| 27.4755075 | $0$ | $1$ | $0$ | $-1.78852162$ | $3$ | $2$ |
| 30.01858256 | $0$ | $1$ | $0$ | $-1.7193045$ | $4$ | $2$ |
| 34.38846804 | $0$ | $1$ | $0$ | $-1.7398751$ | $3$ | $3$ |
| 38.50906805 | $0$ | $1$ | $0$ | $-1.68053739$ | $4$ | $3$ |

| aRef. [16] | bRef. [18] | cNumerical results, from [16] | dRef. [17] | eExact values, as quoted in [17] |

Table VI displays the computed eigenvalues for selected $n$ and $\ell$ quantum numbers of the logarithmic potential. The numerical results [4] exist for all the states with $n, \ell \leq 4$, while for states with $n \leq 4, \ell \leq 3$ and $\ell = 4; n = 3, 4$, the shifted $1/N$ expansion [16] as well as the variational results [16] are also available. Results are presented for $n$ and $\ell$ ranging up to a maximum of 7 and 10. There is a scarcity of accurate reference values for the comparison of these states and the present results show general agreement with them. Some of the states are reported here for the first time and may constitute a useful reference for future purposes.

The usefulness of the method is further illustrated by calculating the wave functions and
TABLE VI: The calculated eigenvalues $E$ (in a.u.) of the logarithmic potential for several values of $n$ and $\ell$.

| $n$ | $\ell$ | This work | Literature |
|-----|-----|-------|----------|
| 0 | 0 | 1.04433226 | 1.04436<sup>a</sup>| 1.04433226 |
| 1 | 0 | 1.84744258 | 1.8485<sup>b</sup> |
| 2 | 0 | 2.28961571 | 2.2897831 |
| 3 | 0 | 2.59570686 | 2.59570686 |
| 4 | 0 | 2.82992843 | 2.82992843 |
| 5 | 0 | 3.01965502 | 3.01965502 |
| 6 | 0 | 3.17910756 | 3.17910756 |
| 7 | 0 | 3.31662376 | 3.31662376 |
| 8 | 0 | 3.46364190 | 3.46364190 |
| 9 | 0 | 3.60863856 | 3.60863856 |
| 10 | 0 | 3.75168528 | 3.75168528 |

| $n$ | $\ell$ | This work | Literature |
|-----|-----|-------|----------|
| 0 | 1 | 1.6414133 | 1.6414133 |
| 1 | 1 | 2.15094678 | 2.15094678 |
| 2 | 1 | 2.49094221 | 2.49094221 |
| 3 | 1 | 2.74559643 | 2.74559643 |
| 4 | 1 | 2.94900787 | 2.94900787 |
| 5 | 1 | 3.11827840 | 3.11827840 |
| 6 | 1 | 3.26318814 | 3.26318814 |
| 7 | 1 | 3.38984841 | 3.38984841 |
| 8 | 1 | 3.51570496 | 3.51570496 |
| 9 | 1 | 3.63978506 | 3.63978506 |
| 10 | 1 | 3.76206557 | 3.76206557 |

| $n$ | $\ell$ | This work | Literature |
|-----|-----|-------|----------|
| 0 | 2 | 2.01330864 | 2.01330864 |
| 1 | 2 | 2.38743285 | 2.38743285 |
| 2 | 2 | 2.66249204 | 2.66249204 |
| 3 | 2 | 2.87749935 | 2.87749935 |
| 4 | 2 | 3.05848949 | 3.05848949 |
| 5 | 2 | 3.21070014 | 3.21070014 |

<sup>a</sup>Ref. [16]  
<sup>b</sup>Ref. [18]  
<sup>c</sup>Numerical results, from [4]

The expectation values $\langle r^{-1} \rangle$ and $\langle r^1 \rangle$ for both the potentials. Table VII shows a cross-section of the expectation values, for which no literature results could be found. The first three states belonging to $\ell = 0, 1, 2$ are given. Finally Figs. 1 and 2 depict the radial densities for the first four states of $\ell = 0, 1, 2$ of the logarithmic and the $r^{0.5}$ case, along with the respective potentials. As expected they show the desired numbers of peaks.

IV. CONCLUSION

The generalized pseudospectral method is shown to deliver arbitrary bound states of the power-law and logarithmic potentials. The method is simple, computationally efficient,
FIG. 1: The radial probability distribution function, $|r R_{n\ell}|^2$ for the first four states corresponding to $\ell = 0, 1, 2$ for the logarithmic potential. (a) the potential (b) ground state (c) first excited state (d) second excited state and (e) third excited state.
FIG. 2: The radial probability distribution function, $|r R_{n\ell}|^2$ for the first four states corresponding to $\ell = 0, 1, 2$ for the power law potential $r^{0.5}$. (a) the potential (b) ground state (c) first excited state (d) second excited state and (e) third excited state.
TABLE VII: The calculated expectation values (in a.u.) of the power-law and logarithmic potentials for the first three states belonging to $\ell = 0, 1, 2$.

| $V(r)$ | $\ell$ | $\langle r^{-1} \rangle$ | $\langle r^1 \rangle$ | $V(r)$ | $\ell$ | $\langle r^{-1} \rangle$ | $\langle r^1 \rangle$ |
|-------|-------|-----------------|-----------------|-------|-------|-----------------|-----------------|
| $\ln r$ | 0     | 0.975829609     | 1.39052517      | $r^{0.5}$ | 0     | 0.767168993     | 1.72566470      |
|       | 0     | 0.497961528     | 3.15106068      | 0     | 0.469136231     | 3.36898957      |
|       | 0     | 0.339365144     | 4.91871111      | 0     | 0.354202831     | 4.82937427      |
|       | 1     | 0.493205837     | 2.38769029      | 1     | 0.437206279     | 2.65352685      |
|       | 1     | 0.327683049     | 4.14985349      | 1     | 0.323564598     | 4.16612795      |
|       | 1     | 0.247106879     | 5.91466141      | 1     | 0.263022498     | 5.55753612      |
|       | 2     | 0.330196264     | 3.38659900      | 2     | 0.315487133     | 3.50669405      |
|       | 2     | 0.245961257     | 5.14983438      | 2     | 0.253812312     | 4.93344373      |
|       | 2     | 0.196769673     | 6.91391694      | 2     | 0.215382759     | 6.27078423      |

reliable and accurate. Application of this method on these systems show that the current scheme offers results which are considerably better than the existing results available in the literature. It has the capability to handle the low as well as the very high excited states with equal ease and accuracy which often pose problems for the variational methods. As a test of the quality of the wave functions, we have also calculated the expectation values and the densities.

Acknowledgments

I gratefully acknowledge the University of New Brunswick, Fredericton, New Brunswick, Canada, for providing warm hospitality.

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