TUTORIAL SESSION:
Synchronization in Coupled Oscillators:
Theory and Applications

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Exploring Synchronization in
Complex Oscillator Networks

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A Brief History of Sync
how it all began

Christiaan Huygens (1629 – 1695)
- physicist & mathematician
- engineer & horologist
observed "an odd kind of sympathy" between coupled & heterogeneous clocks
[Letter to Royal Society of London, 1665]

A Brief History of Sync
the odd kind of sympathy is still fascinating

watch movie online here:
http://www.youtube.com/watch?v=JWToUATLGzs&list=UUJIyXclKY8FQQwaKBaawl_A&index=3

Sync of 32 metronomes at Ikeguchi Laboratory, Saitama University, 2012

Recent reviews, experiments, & analysis
[M. Bennet et al. ’02, M. Kapitaniak et al. ’12]
A Brief History of Sync

- Sync in mathematical biology [A. Winfree '80, S.H. Strogatz '03, ...]
- Sync in physics and chemistry [Y. Kuramoto '83, M. Mézard et al. '87...]
- Sync in neural networks [F.C. Hoppensteadt and E.M. Izhikevich '00, ...]
- Sync in complex networks [C.W. Wu '07, S. Bocaletti '08, ...]
- ... and countless technological applications (reviewed later)

Phenomenology and Challenges in Synchronization

Synchronization is a trade-off: coupling vs. heterogeneity

- coupling small & $|\omega_i - \omega_j|$ large $\Rightarrow$ incoherence & no sync
- coupling large & $|\omega_i - \omega_j|$ small $\Rightarrow$ coherence & frequency sync

Some central questions: (still after 45 years of work)

- proper notion of sync & phase transition
- quantify “coupling” vs. “heterogeneity”
- interplay of network & dynamics

Applications of the Coupled Oscillator Model

Coupled oscillator model:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Some related applications:

- Sync in a population of fireflies [G.B. Ermentrout '90, Y. Zhou et al. '06, ...]
- Deep-brain stimulation and neuroscience [N. Kopell et al. '88, P.A. Tass '03, ...]
- Sync in coupled Josephson junctions [S. Watanabe et. al '97, K. Wiesenfeld et al. '98, ...]
- Countless other sync phenomena in physics, biology, chemistry, mechanics, social nets etc. [A. Winfree '67, S.H. Strogatz '00, J. Acebrón '01, ...]
Example 1: AC Power Transmission Network

- Power transfer on line $i \sim j$: $\frac{|V_i||V_j|}{Y_{ij}} \cdot \sin(\theta_i - \theta_j)$
- Power balance at node $i$: $P_i = \sum_j a_{ij} \sin(\theta_i - \theta_j)$
- Structure-Preserving Model [A. Bergen & D. Hill '81]:
  
  - Swing eq with $P_{m,i} > 0$: $M_i \dot{\theta}_i + D_i \dot{\theta}_i = P_{m,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$
  - $P_{i,i} < 0$ and $D_i \geq 0$: $D_i \dot{\theta}_i = P_{i,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

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Example 3: Flocking, Schooling, & Vehicle Coordination

- Network of Dubins' vehicles
  
  \[
  \dot{\bar{r}}_i = v e^{i\theta_i} \\
  \dot{\theta}_i = u_i(r, \theta)
  \]

  with speed $v$ and steering control $u_i(r, \theta)$

- Sensing/comm. graph $G = (V, E, A)$ for coordination of autonomous vehicles

- Relative sensing control $u_i = f_i(\theta_i - \theta_j)$

  For neighbors $\{i, j\} \in E$ yields closed-loop

  \[
  \dot{\theta}_i = \omega_0(t) - K \cdot \sum_j a_{ij} \sin(\theta_i - \theta_j)
  \]

[Example 4: Canonical Coupled Oscillator Model]

- Dynamical system with stable limit cycle $\gamma$ and weak perturb.
  
  \[
  \dot{x} = f(x) + \epsilon \cdot \delta(t)
  \]

- Local phase dynamics near $\gamma$ with phase response curve $Q(\varphi)$

  \[
  \dot{\varphi} = \Omega + \epsilon \cdot Q(\varphi)\delta(t) + O(\epsilon^2)
  \]

  $\Rightarrow$ same phase reduction applied to interacting oscillators

  $\Rightarrow$ Coord. & time transf. + averaging

  \[
  \dot{\hat{\theta}} = \sum_j h_{ij}(\theta_i - \theta_j)
  \]

  $\Rightarrow$ 0th and 1st (odd) Fourier mode:

  \[
  \dot{\hat{\theta}} = \omega + \sum_j a_{ij} \sin(\theta_i - \theta_j)
  \]

[R. Sepulchre et al. '07, D. Klein et al. '09, L. Consolini et al. '10]

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Example 2: DC/AC Inverters in Microgrids

- (Islanded) microgrid = autonomously managed low-voltage network

- Inverter in microgrid = controllable AC source

- Physics: $P_{i,\ell} = a_{i\ell} \sin(\theta_i - \theta_\ell)$

Droop-control [M.C. Chandorkar et. al. '93]:

\[
D_i \dot{\theta}_i = P_i^* - P_{i,\ell}
\]

Closed-loop for inverters & load $\ell$ [J.W. Simpson-Porco et. al. '12]:

\[
D_i \dot{\theta}_i = P_i^* - a_{i\ell} \sin(\theta_i - \theta_\ell) \\
0 = P_\ell - \sum_j a_{ij} \sin(\theta_\ell - \theta_i)
\]
Example 5: Other technological applications

- Particle filtering to estimate limit cycles [A. Tilton & P. Mehta et al. '12]
- Clock synchronization over networks [Y. Hong & A. Scaglione '05, O. Simeone et al. '08, Y. Wang & F. Doyle et al. '12]
- Central pattern generators and robotic locomotion [J. Nakanishi et al. '04, S. Aoi et al. '05, L. Righetti et al. '06]
- Decentralized maximum likelihood estimation [S. Barbarossa et al. '07]
- Carrier sync without phase-locked loops [M. Rahman et al. '11]

Outline

1) Introduction and motivation
2) Synchronization notions, metrics, & basic insights
3) Phase synchronization and more basic insights
4) Synchronization in complete networks
5) Synchronization in sparse networks
6) Open problems and research directions

Order Parameter
(for homogenous coupling $a_{ij} = K/n$)

Define the order parameter (centroid) by $\hat{r}e^{i\psi} = \frac{1}{n} \sum_{j=1}^{n} e^{i\theta_j}$, then

\[ \dot{\hat{r}} = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j) \]

\[ \dot{\psi} = \omega_i - Kr \sin(\theta_i - \psi) \]

Intuition: synchronization = entrainment by mean field $\hat{r}e^{i\psi}$

K small & $|\omega_i - \omega_j|$ large

K large & $|\omega_i - \omega_j|$ small

⇒ analysis based on concepts from statistical mechanics & cont. limit:

[Y. Kuramoto '75, G.B. Ermentrout '85, J.D. Crawford '94, S.H. Strogatz '00, J.A. Acebrón et al. '05, E.A. Martens et al. '09, H. Yin et al. '12, ...]

Synchronization Notions & Metrics

1) frequency sync: $\dot{\theta}_i(t) = \dot{\theta}_j(t)$ $\forall i,j$
\[ \Leftrightarrow \dot{\theta}_i(t) = \omega_{\text{sync}} \quad \forall i \in \{1, \ldots, n\} \]

2) phase sync: $\theta_i(t) = \theta_j(t)$ $\forall i,j$
\[ \Leftrightarrow r = 1 \]

3) phase balancing: $r = 0$
(e.g., splay state = uniform spacing on $S^1$)

4) arc invariance: all angles in $\text{Arc}_n(\gamma)$
(closed arc of length $\gamma$) for $\gamma \in [0, 2\pi]$

5) phase cohesiveness: all angles in
$\Delta G(\gamma) = \{ \theta \in \mathbb{T}^n : \max_{\{i,j\} \in \mathcal{E}} |\theta_i - \theta_j| \leq \gamma \}$
for some $\gamma \in [0, \pi/2]$
Geometric & Algebraic Insights I

Coupled oscillator model:
\[ \dot{\theta}_i = f(\theta) = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

- vector field \( f(\theta) \) possesses rotational symmetry: \( f(\theta^*) = f(\theta^* + \varphi 1_n) \)
- \( \sum_{i=1}^{n} \hat{\theta}_i(t) = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \omega_{\text{sync}} \Rightarrow \text{sync frequ.} \omega_{\text{sync}} = \omega_{\text{avg}} = \frac{1}{n} \sum_{i=1}^{n} \omega_i \)
  \( \Rightarrow \) transf. to rot. frame with freq. \( \omega_{\text{avg}} \Leftrightarrow \omega_{\text{sync}} = 0 \Leftrightarrow \omega_i \mapsto \omega_i - \omega_{\text{avg}} \)
- wlog: assume \( \omega_{\text{avg}} = 0 \) \( \Rightarrow \) frequency sync = equilibrium manifold
\[ \{ \theta^* \} = \{ \theta \in \mathbb{T}^n : \theta^* + \varphi 1_n, f(\theta^*) = 0, \varphi \in [0,2\pi] \} \]

Geometric & Algebraic Insights II

Jacobian is Laplacian

Coupled oscillator model:
\[ \dot{\theta}_i = f(\theta) = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

- negative Jacobian \(-\frac{\partial f}{\partial \theta}\) evaluated at \( \theta^* \in \mathbb{T}^n \) is given by
\[
L(\theta^*) = \begin{bmatrix}
\sum_{i=1}^{n} a_{1i} \cos(\theta^*_i - \theta^*_j) & -a_{i2} \cos(\theta^*_i - \theta^*_j) & \ldots & -a_{in} \cos(\theta^*_i - \theta^*_j) \\
-a_{i1} \cos(\theta^*_i - \theta^*_j) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
-a_{in} \cos(\theta^*_i - \theta^*_j) & \ldots & -a_{n,n-1} \cos(\theta^*_i - \theta^*_j) & \sum_{i=1}^{n} a_{in} \cos(\theta^*_i - \theta^*_j)
\end{bmatrix}
\]

- \( L(\theta^*) \) is p.s.d. and \( \ker(L(\theta^*)) = \text{span}(1_n) \)

Lemma [C. Tavora and O.J.M. Smith '72]

If there exists an equilibrium manifold \( \{ \theta^* \} \) in
\[ \Delta_G(\pi/2) = \{ \theta \in \mathbb{T}^n : \max_{\{i,j\} \in \mathcal{E}} |\theta_i - \theta_j| < \pi/2 \} , \]
then \( \{ \theta^* \} \) is
- locally exponentially stable (modulo symmetry), and
- unique in \( \tilde{\Delta}_G(\pi/2) \) (modulo symmetry).

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3. Phase synchronization and more basic insights
4. Synchronization in complete networks
5. Synchronization in sparse networks
6. Open problems and research directions
Phase Synchronization

a forced gradient system

\[ \dot{\theta}_i = \omega_i - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j) \]  \{\text{phase sync}\} = \{\theta \in \mathbb{T}^n: \theta_i = \theta_j \ \forall \ i, j\}

Classic intuition [P. Monzon et al. ’06, Sepulchre et al. ’07):

- Coupled oscillator model is forced gradient flow
  \[ \dot{\theta}_i = \omega_i - \nabla_i U(\theta) \]
  where \( U(\theta) = \sum_{(i,j) \in E} a_{ij} (1 - \cos(\theta_i - \theta_j)) \) (spring potential)
- Assume that \( \omega_i = 0 \ \forall \ i \in \{1,\ldots,n\} \Rightarrow \) gradient flow \( \dot{\theta} = -\nabla U(\theta) \)
  \Rightarrow global convergence to critical points \( \{\nabla U(\theta) = 0\} \supseteq \{\text{phase sync}\} \)
  \Rightarrow previous Jacobian arguments: \{\text{phase sync}\} is local minimum & stable

Proof of "\( \Rightarrow \)": wlog in rot. frame: \( \omega_i = \omega_j = 0 \Rightarrow \) follow previous args

Proof of "\( \Leftarrow \)": phase sync’d solutions satisfy \( \theta_i = \theta_j \) & \( \dot{\theta}_i = \dot{\theta}_j \Rightarrow \omega_i = \omega_j \)

Remark: “almost global phase sync” for certain topologies (trees, cmplt., short cycles) [P. Monzon, E.A. Canale et al. ’06–’10, A. Sarlette ’09]

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Synchronization in a Complete & Homogeneous Graph

Classic Kuramoto model of coupled oscillators:
\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)
\]

One appropriate sync notion:
1. **arc invariance**: \( \theta \in \overline{\text{Arc}}_n(\gamma) \) for small \( \gamma \in [0, \pi/2] \)
2. **frequency sync**: \( \dot{\theta}_i = \omega_{\text{avg}} \) with \( \omega_{\text{avg}} = \frac{1}{n} \sum_{j=1}^{n} \omega_j \)

Numerous results on sync conditions & bifurcations
[A. Jadbabaie et al. '04, P. Monzon et al. '07, F. de Smet et al. '07, N. Chopra et al. '09, A. Franci et al. '10, S.Y. Ha et al. '10, D. Aeyels et al. '04, J.L. van Hemmen et al. '93, R.E. Mirollo et al. '05, M. Verwoerd et al. '08, ...]

Synchronization in a Complete & Homogeneous Graph

main result
\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)
\]

The following statements are equivalent:
1. Coupling dominates heterogeneity, i.e., \( K > K_{\text{critical}} \equiv \omega_{\text{max}} - \omega_{\text{min}} \);
2. \( \exists \gamma_{\text{max}} \in [\pi/2, \pi] \) s.t. all Kuramoto models with \( \omega_i \in [\omega_{\text{min}}, \omega_{\text{max}}] \) and \( \theta(0) \in \text{Arc}_n(\gamma_{\text{max}}) \) achieve exponential frequency sync; and
3. \( \exists \gamma_{\text{min}} \in [0, \pi/2] \) s.t. all Kuramoto models with \( \omega_i \in [\omega_{\text{min}}, \omega_{\text{max}}] \) feature a locally exp. stable equilibrium manifold in \( \text{Arc}_n(\gamma_{\text{min}}) \).

Moreover, we have \( K_{\text{critical}} / K = \sin(\gamma_{\text{min}}) = \sin(\gamma_{\text{max}}) \) and **practical phase synchronization**: from \( \gamma_{\text{max}} \) arc \( \to \gamma_{\text{min}} \) arc

Synchronization in a Complete & Homogeneous Graph

brief review

\[
\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)
\]

- **Implicit equations** for existence of sync’d fixed points
  [D. Aeyels et al. '04, R.E. Mirollo et al. '05, M. Verwoerd et al. '08]
- **Necessary conditions**: \( \dot{\theta}_i = \dot{\theta}_j \ \forall i, j \Rightarrow K > \frac{\omega_{\text{max}} - \omega_{\text{min}}}{2} \cdot \frac{n}{n-1} \)
  [N. Chopra et al. '09, A. Jadbabaie et al. '04, J.L. van Hemmen et al. '93]
- **Sufficient conditions**, e.g., \( K > \| (\omega_1 - \omega_2, \ldots) \|_2 \cdot f(n, \gamma) \)
  [J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, F. Dörfler and F. Bullo '09, S.J. Chung and J.J. Slotine '10, A. Franci et al. '10, S.Y. Ha et al. '10, ...]

Synchronization in a Complete & Homogeneous Graph

main proof ideas

1. **Arc invariance**: \( \theta(t) \) in \( \gamma \) arc \( \Leftrightarrow \) arc-length \( V(\theta(t)) \) is non-increasing

   \[
   \left\{ \begin{array}{l}
   V(\theta(t)) = \max_{i,j \in \{1,...,n\}} |\theta_i(t) - \theta_j(t)| \\
   D^+ V(\theta(t)) \leq 0
   \end{array} \right.
   \]

   true if \( K \sin(\gamma) \geq K_{\text{critical}} \)

2. **Frequency synchronization** \( \Leftrightarrow \) consensus protocol in \( \mathbb{R}^n \)

   \[
   \frac{d}{dt} \dot{\theta}_i = -\sum_{j=1}^{n} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j),
   \]

   where \( a_{ij}(t) = \frac{K}{n} \cos(\theta_i(t) - \theta_j(t)) > 0 \) for all \( t \geq T \)

3. **Necessity**: all results exact for bipolar distribution \( \omega_i \in \{\omega_{\text{min}}, \omega_{\text{max}}\} \)

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**Synchronization in a Complete & Homogeneous Graph**

**recall definitions**

[Florian Dörfler (UCSB)] Power Networks Synchronization Advancement to Candidacy 31 / 36
Synchronization in a Complete & Homogeneous Graph

robustness and extensions

1 Switching natural frequencies: dwell-time assumption ✓

Kuramoto model with $\omega_i \in [-1, 1]$:

\[ \dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j) \]

Cont. limit predicts largest $K_{\text{critical}} = 2$ for bipolar distribution & smallest $K_{\text{critical}} = 4/\pi$ for uniform distribution [Y. Kuramoto ’75, G.B. Ermentrout ’85]

2 Slowly time-varying: $\|\dot{\omega}(t) - \dot{\omega}_{\text{avg}}(t)\|_\infty$ sufficiently small ✓

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Primer on Algebraic Graph Theory

Undirected graph $G = (V,E,A)$ with weight $a_{ij} > 0$ on edge $\{i,j\}$

- adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ (induces the graph)
- degree matrix $D \in \mathbb{R}^{n \times n}$ is diagonal with $d_{ii} = \sum_{j=1}^{n} a_{ij}$
- Laplacian matrix $L = D - A \in \mathbb{R}^{n \times n}$, $L = L^T \geq 0$

Notions of connectivity

- topological: connectivity, path lengths, degree, etc.
- spectral: 2nd smallest eigenvalue of $L$ is “algebraic connectivity” $\lambda_2(L)$

Notions of heterogeneity

$\|\omega\|_{E,\infty} = \max_{\{i,j\} \in E} |\omega_i - \omega_j|$, $\|\omega\|_{E,2} = \left( \sum_{\{i,j\} \in E} |\omega_i - \omega_j|^2 \right)^{1/2}$
Synchronization in Sparse Networks

a brief review I

\[ \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

Assume connectivity &
\[ \omega_{avg} = \frac{1}{n} \sum_{i=1}^{n} \omega_i = 0 \]

\[ \sum_{j=1}^{n} a_{ij} \geq |\omega_j| \iff \text{sync} \]

[C. Tavara and O.J.M. Smith ’72]

Proof idea: \( \dot{\theta}_i = 0 \) has no solution if condition is not true

\[ \lambda_2(L) > \lambda_{\text{critical}} \triangleq \|\omega\|_{\text{cplmt},2} \Rightarrow \text{sync} \]

[F. Dörfler and F. Bullo ’09]

Proof idea: analogous Lyapunov proof with \( V(\theta) = \sum_{i<j} |\theta_i - \theta_j|^2 \);
condition also implies \( \theta^* \in \text{Arc}_n(\lambda_{\text{critical}}/\lambda_2(L)) \Rightarrow \text{evtl. too strong!} \)

Synchronization in Sparse Networks
problems ...

Problems: the sharpest general nec. & suff. conditions known to date

\[ \sum_{j=1}^{n} a_{ij} < |\omega_j|, \quad \lambda_2(L) > \|\omega\|_{\text{cplmt},2}, \quad \text{and} \quad \lambda_2(L) > \|\omega\|_{\text{c},2} \]

have a large gap and are conservative!

Why?

\[ \sum_{j=1}^{n} a_{ij} < |\omega_j|, \quad \lambda_2(L) > \|\omega\|_{\text{cplmt},2}, \quad \text{and} \quad \lambda_2(L) > \|\omega\|_{\text{c},2} \]

\[ \sum_{j=1}^{n} a_{ij} < |\omega_j| \]

\[ \lambda_2(L) > \lambda_{\text{critical}} \triangleq \|\omega\|_{\text{cplmt},2} \]

\[ \lambda_2(L) > \|\omega\|_{\text{c},2} \]

\[ \lambda_2(L) > \|\omega\|_{\text{c},2} \]

A Nearly Exact Synchronization Condition

a “back of the envelope calculation”

Recall: if \( \exists \) equilibrium \( [\theta^*] \in \tilde{\Delta}_G(\gamma) \), then it is unique and stable

\[ \omega_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

(\(*\))

Consider linear “small-angle” approximation of (\(*\)):

\[ \omega_i = \sum_{j=1}^{n} a_{ij} (\delta_i - \delta_j) \iff \omega = L \delta \]

(\(**\))

Unique solution (modulo symmetry) of (\(**\)) is \( \delta^* = L^\dagger \omega \)

\[ \omega_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) = \sum_{j=1}^{n} a_{ij} \sin(\arcsin(\delta_i^* - \delta_j^*)) = \omega_i \]

(\(\checkmark\))

\[ \Rightarrow \text{Solution ansatz for (\(*\))}: \quad \theta_i^* - \theta_j^* = \arcsin(\delta_i^* - \delta_j^*) \quad \text{(for a tree)} \]

\[ \Rightarrow \text{Theorem: (for a tree)} \quad \exists [\theta^*] \in \tilde{\Delta}_G(\gamma) \iff \|L^\dagger \omega\|_{\text{c},\infty} \leq \sin(\gamma) \]
Theorem [F. Dörfler, M. Chertkov, and F. Bullo '12]

Under one of following assumptions:

1) graph is either tree, homogeneous, or short cycle \( n \in \{3, 4\} \)
2) natural frequencies: \( L^T \omega \) is bipolar, small, or symmetric (for cycles)
3) arbitrary one-connected combinations of 1) and 2)

If \( \| L^T \omega \|_{\mathcal{E}, \infty} \leq \sin(\gamma) \) where \( \gamma < \pi/2 \)

\[ \Rightarrow \exists \text{ a unique & locally exponentially stable equilibrium manifold in} \]

\[ \bar{\Delta}_G(\gamma) = \{ \theta \in \mathbb{T}^n | \max_{(i,j) \in \mathcal{E}} |\theta_i - \theta_j| \leq \gamma \} . \]

A Nearly Exact Synchronization Condition

statistical analysis for power networks

Randomized power network test cases

with 50% randomized loads and 33% randomized generation

| Randomized test case (1000 instances) | Correctness of condition: \( \| L^T \omega \|_{\mathcal{E}, \infty} \leq \sin(\gamma) \) | Accuracy of condition: \( \| L^T \omega \|_{\mathcal{E}, \infty} \leq \sin(\gamma) \) | Phase cohesiveness: \( \max_{(i,j) \in \mathcal{E}} |\theta_i^* - \theta_j^*| \) |
|-------------------------------------|---------------------------------|-------------------------------|---------------------------------|
| 9 bus system                        | always true                     | 4.1218 \times 10^{-3} \text{ rad} | 0.128689 \text{ rad}            |
| IEEE 14 bus system                  | always true                     | 2.7995 \times 10^{-4} \text{ rad} | 0.166224 \text{ rad}            |
| IEEE RTS 24                         | always true                     | 1.7089 \times 10^{-3} \text{ rad} | 0.223094 \text{ rad}            |
| IEEE 30 bus system                  | always true                     | 2.6140 \times 10^{-4} \text{ rad} | 0.104432 \text{ rad}            |
| New England 39                      | always true                     | 0.6355 \times 10^{-3} \text{ rad} | 0.168214 \text{ rad}            |
| IEEE 57 bus system                  | always true                     | 2.0630 \times 10^{-3} \text{ rad} | 0.202959 \text{ rad}            |
| IEEE RTS 96                         | always true                     | 2.6076 \times 10^{-3} \text{ rad} | 0.245931 \text{ rad}            |
| IEEE 118 bus system                 | always true                     | 5.9959 \times 10^{-3} \text{ rad} | 0.235264 \text{ rad}            |
| IEEE 300 bus system                 | always true                     | 5.2618 \times 10^{-4} \text{ rad} | 0.432904 \text{ rad}            |
| Polish 2383 bus system (winter peak 1999/2000) | always true | 4.2183 \times 10^{-3} \text{ rad} | 0.251444 \text{ rad}            |

\[ \Rightarrow \text{ condition } \| L^T \omega \|_{\mathcal{E}, \infty} \leq \sin(\gamma) \text{ is extremely accurate for } \gamma \leq 25^\circ \]

A Nearly Exact Synchronization Condition

statistical analysis for complex networks

Comparison with exact \( K_{\text{critical}} \) for

\[ \dot{\theta}_i = \omega_i - K \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]

\[ \Rightarrow \text{ condition } \| L^T \omega \|_{\mathcal{E}, \infty} \leq \sin(\gamma) \text{ is extremely accurate for } \gamma = \pi/2 \]
Exciting Open Problems and Research Directions

1. Q: What about networks of **second-order oscillators**?
   
   \[ M_i \ddot{\theta}_i + D_i \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]
   
   **Apps:** mechanics, synchronous generators, Josephson junctions, …
   
   **Problems:** kinetic energy is a mixed blessing for transient dynamics

2. Q: What about **asymmetric interactions**?
   
   e.g., directed graphs: \( a_{ij} \neq a_{ji} \) or phase shifts: \( a_{ij} \sin(\theta_i - \theta_j - \varphi_{ij}) \)
   
   **Apps:** sync protocols, lossy circuits, phase/time-delays, flocking, …
   
   **Problems:** algebraic & geometric symmetries are broken

3. Q: How to derive **sharper results** for heterogeneous networks?

4. Q: What about the **transient dynamics** beyond \( \text{Arc}_n(\pi) \), **general equilibria** beyond \( \Delta_C(\pi/2) \), or the **basin of attraction**?
   
   **Apps:** phase balancing, volatile power networks, flocking, …
   
   **Problems:** lack of analysis tools (only for simple cases), chaos, …

5. Q: Beyond **continuous, sinusoidal, and diffusive coupling**?
   
   \[ \dot{\theta}_i \in \omega_i - \sum_{\{i,j\} \in E} f_{ij}(\theta_i, \theta_j), \quad \theta \in C \subset \mathbb{T}^n \]
   
   \[ \theta_i^+ \in \theta_i + \sum_{\{i,j\} \in E} g_{ij}(\theta_i, \theta_j), \quad \theta \in D \subset \mathbb{T}^n \]
   
   **Apps:** impulsive coupling, relaxation oscillators, neuroscience, …
   
   **Problems:** lack of analysis tools, coping with heterogeneity, …

6. Q: Does anything extend from phase to **state space oscillators**?

Conclusions

- **Coupled oscillator model:**
  
  \[ \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \]
  
  **history:** from Huygens’ clocks to power grids
  
  **applications in sciences, biology, & technology**
  
  **synchronization phenomenology**
  
  **network aspects & heterogeneity**
  
  **available analysis tools & results**