Generalized Network Growth: from Microscopic Strategies to the Real Internet Properties

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In this paper we present a generalized model for network growth that links the microscopical agent strategies with the large scale behavior. This model is intended to reproduce the largest number of features of the Internet network at the Autonomous System (AS) level. Our model of network grows by adding both new vertices and new edges between old vertices. In the latter case a “rewarding attachment” takes place mimicking the disassortative mixing between small routers to larger ones. We find a good agreement between experimental data and the model for the degree distribution, the betweenness distribution, the clustering coefficient and the correlation functions for the degrees.

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Networks or graphs are mathematical entities composed by sites (or vertices) connected by links (or edges).1 Due to the apparent simplicity of such definition, many attempt have been made in order to describe very different physical situations within this framework. More interestingly new unexpected properties (with respect to the traditional approach of Random Graphs due to P. Erdős and A. Rényi [2]), have been found in a variety of different systems: Internet [3,4], WWW [5,6], social structures [7,8,9,10] and even protein interactions [11] display a self-similar distribution $P(k) \propto k^{-\gamma}$ for the degree $k$ (i.e. the number of links per vertex). From a theoretical point of view some of the active ingredients that determine such self-similarity in the statistical properties of the degree have already been found [12,13,14,15]. In this Letter we want to present a simple statistical model able to reproduce many statistical properties of the Internet network at the Autonomous System (AS) level. Our model of network growth is achieved, as explained below, by relating the microscopic agents strategies and the macroscopic statistical properties.

A way to describe the complex phenomenology (even by restricting to the topology, ignoring the different weight of various links given by the traffic) can be made by considering the clustering correlation and centrality present in the graph.

Clustering measures the presence of part of the graph denser than the average. The most immediate measure of clustering is the clustering coefficient $c_i$ for every site $i$. This quantity gives the probability that two nearest neighbors of a vertex are also neighbors to each other. The clustering coefficient can then be averaged over the vertices in the structure, giving the total clustering $<C>$ or rather be decomposed by considering the function $c(k)$ giving the probability that a vertex has clustering coefficient $c(k)$ given its degree $k$.

Correlation is best represented by the conditional probability $P_C(k'|k)$ that a link belonging to a node with degree $k$ points to a node with degree $k'$. If this is independent on $k$, we have $P_C(k'|k) = P_C(k') \simeq k' P(k')$. If instead there is a dependence on $k$ we can establish the strength of correlation between vertices of different degree. The most immediate way to compute such a correlation is given by considering the quantity

$$<k_{nn}> = \sum_{k'} k' P(k'|k)$$

i.e. the nearest neighbors average degree of nodes with degree $k$.

Centrality of some vertices with respect to other zones is also a way to consider deviations from average behavior in the structure. In particular betweenness or closeness, are the measures of the centrality of a site with respect to the other vertices in the graph.20 The betweenness $b$ of a vertex $i$ gives the probability that the site $i$ is in the shortest path from vertex $j$ and vertex $k$ (for every $j$ and $k$). If the number of shortest paths between a pair of vertices $(j,k)$ is $D(j,k)$, we denote with $D_i(j,k)$ the number of such shortest paths running through $i$. The fraction $g_i(j,k) = D_i(j,k)/D(j,k)$ may be interpreted as the amount of the role played by the vertex $i$ in social relation between two persons $j$ and $k$. The betweenness of $i$ is defined as the sum of $g_i(j,k)$ over all the connected pairs.

$$g_i = \sum_{j \neq k} g_i(j,k) = \sum_{j \neq k} \frac{D_i(j,k)}{D(j,k)}.$$  

All the above quantities have already been considered and analyzed in a series of papers [21,22]. The main result of these analysis is the detection of a complex inner structure resulting in a clustering larger than expected, a power law distribution of the $c(k)$ (i.e. $c(k) \propto k^{-\phi}$)
a power law distribution of the correlation \( k_{mn}(k) \) (i.e. \( k_{mn}(k) \propto k^{-\tau} \)) and finally a power law distribution of the values of the betweenness \( b \) (i.e. \( P(b) \propto b^{-\eta} \)). On the basis of these analysis the Internet at the level of Autonomous Systems (AS) shows a self-similar behavior in all these quantities signaling a possible presence of a critical state. Interestingly, all the statistical models introduced do not reproduce such features. The Barabási Albert model introduced the concept of preferential attachment such that the network grows by addition of new nodes that link with the older ones with a probability proportional to the degree of the latter ones. Even if this simple rule reproduces nicely the degree distribution it fails in reproducing the correlations, the centrality and the clustering of the AS systems. Some modifications of the above model give a better qualitative agreement with the real data. In particular in the stochastic growth model proposed in [17] a constant fraction of site are added at every timestep and a substantial rewiring takes place. Also the fitness model has a fairly nice agreement with the shape of the AS distributions. In this model [18] the preferential attachment is weighted through an individual site fitness.

The above models, anyway, do not give a precise quantitative prediction of all the properties measured in teh AS network. Here we want to present a new statistical model giving a better agreement with the data and linking the microscopic dynamic to the macroscopic evolution. The basic idea of the model is to allow both the addition of a vertex (with probability \( p \)) and the addition of a link (with probability \( 1-p \)). Typically such link relates two sites 1,2 whose degrees are \( k_1 \) and \( k_2 \). It is natural to define a "directionality" in the link. This is defined by deciding who pays the cost of the connections. This should mimic for example users that pay to get wired to Internet, flow of information etc. In this paper we will not consider directionality explicitly, leaving the extension to the oriented graphs to future work. In general, we can write the probability of addition of this link as \( \tilde{P}(k_1,k_2) \). The specific form can be directly linked to the microscopical agents strategies. There are two obvious limiting cases. The case \( k_1 = 0 \) corresponds to a new site which decides to join the network. The case \( k_2 = 0 \) corresponds to the creation of a link toward a site not connected to the network. The process is asymmetric due to the growth rules, that allows to write \( \tilde{P}(k_1,k_2) \) in terms of conditional probabilities. If site 1 pays for connection, then \( \tilde{P}(k_1,k_2) = \tilde{P}_2(k_2|k_1)\tilde{P}_1(k_1) \). A simple ansatz corresponding to the BA model, would be to assume \( \tilde{P}_1(k_1) = \delta_{k_1,0} \) and \( \tilde{P}_2(k_2|k_1) \propto k_2 \). The generalization presented in Ref. [13, 20] assumes instead \( \tilde{P}_2(k_2|k_1) = k_2 + \lambda \) (\( k_2 \) is the in-degree) and \( \tilde{P}_1(k_1) = k_1 + \mu \) (\( k_1 \) is the out-degree).

Here we propose to consider non oriented graphs (as it is the case of AS). Furthermore we assume \( \tilde{P}_1(k_1) \propto k_1 \) and we tune the form of the \( \tilde{P}_2(k_1|k_2) \) in order to obtain the different situations observed in the experimental data [21, 22]. For example a form of the type

\[
\tilde{P}_2(k_2|k_1) \propto \frac{1}{|k_1 - k_2| + 1}
\]

(3)

would produce the so-called assortative mixing [21] where vertices of the same degree tend to be connected to each other. We instead focus on the opposite limit where

\[
\tilde{P}_2(k_2|k_1) \propto |k_1 - k_2|
\]

(4)

in order to model the so-called disassortative mixing [24] with large hubs and poorly connected vertices. In this limit at any time step

1. Either a vertex is added and linked with vertex \( i \) with probability

\[
p \frac{k_i}{\sum_{j=1,N} k_j}.
\]

(5)

2. or an edge is added (if absent) between vertices \( i \) and \( j \) already present, with probability

\[
(1-p) \frac{k_i}{\sum_{k=1,N} k_k} \frac{|k_i - k_j|}{\sum_{k \neq i=1,N} |k_i - k_k|}.
\]

(6)

From the above rules, the case \( p = 1 \) (no edge creation) corresponds to a traditional AB model where only one edge is added for time step. Intuitively as the parameter \( p \) is tuned to 0 the edge growth becomes more and more important. This results in a larger connected core with respect to the AB model, as shown in Fig.1.

Numerical simulations of this model are presented below for different cases. The quantities we decide to monitor are the distribution of the degree \( P(k) \), the distribution of both betweenness \( b \) and closeness \( c \), the clustering coefficient \( c(k) \) of vertices whose degree is \( k \) and finally the average degree \( \langle k \rangle \) of neighbours of a vertex whose degree is \( k \). All these quantities together with experimental data for the AS system are reported in Table 1.

As reported in Fig.2, the probability distribution of the degree, \( P(k) \), follows a power law behavior for every value of the parameter \( p \).

In particular the exponent \( \gamma \) diminishes as loops start to form in the system when \( p < 1 \). Using such numerical evidence about the shape of the \( P(k) \) we can present an analytical estimate of the exponent \( \gamma \) through simple arguments. We firstly notice that the number of edges in the system increases by one unity at every time step. Consequently the total degree over the network increases by two

\[
\sum_{k=1}^{k_{max}} kN_k(t) = 2t.
\]

(7)

Here \( N_k(t) \) is the number of vertices with degree \( k \) at time \( t \).
The total number of vertices, instead, increases at a rate $p$. Therefore the total number of vertices is

$$\sum_{k=1}^{k_{\text{max}}} N_k(t) = pt$$

(8)

We assume that there is a stationary state, so that the number of vertices grows linearly in time, $N_k(t) = n_k t$. As stated above, we also assume that the degree distribution is a power law, $n_k = a k^{-\gamma}$, as seen from simulations in agreement with the preferential attachment rule. Then we can write

$$\sum_{k=1}^{k_{\text{max}}} n_k \simeq a \int_1^{\infty} k^{-\gamma} dk = \frac{a}{\gamma - 1} = p$$

(9)

and

$$\sum_{k=1}^{k_{\text{max}}} kn_k \simeq a \int_1^{\infty} k^{-\gamma+1} dk = \frac{a}{\gamma - 2} = t$$

(10)

Using Eqs(8) and (10) we obtain $\gamma(p) = 2 + \frac{p}{2 - p}$, which provides good estimates for the results of the simulations, and correctly recovers the limiting cases, $\gamma(1) = 3$. A striking feature of the model is that as soon as $p$ is different from 0 one still deals with a scale free network. The limit value for the distribution is $\gamma(0^+) = 2$. The case $p = 0$ is degenerate giving rise to a complete graph whose degree distribution is a delta function peaked around the size $n$ of the system. One can argue that this limit is peculiar. Indeed a complete graph of $N$ nodes is characterized by a number of edges of order $N^2$.

In our model, for arbitrary small but strictly positive $p$, both the number of nodes and the number of edges grow linearly in time, with a fixed ratio, so that the graph will never be complete. As regards the distance distribution, we find the small world effect, that is a peak around a characteristic value.

It has recently been shown that the probability distribution $P(b)$ for the betweenness $b$ follows a power law,

$$P_B(g) \sim g^{-\eta},$$

(11)

where $\eta = 2$ is about 2. For our model we find, in agreement with Ref. [27], that the exponent $\eta$ is equal to 2.0 if $p = 1$. From the data for $p \neq 1.0$ we can conclude that the exponent changes to $\eta = 2.2$, as it happens for the BA model when $m > 1$ and loops start to appear in the network (see inset of Fig.2). Another important measure of centrality is given by the closeness $c$. Closeness of a site $i$ is simply the inverse of the sum of the distances from $i$ toward all the other vertices. Not surprisingly since the distance distribution has a small-world effect this quantity has a frequency distribution decreasing exponentially.

It is interesting to study the structure of such quantity with respect to the degree distribution. In particular we checked the behavior of $c(k)$ defined as the average clustering coefficient for a site whose degree is $k$. Also this quantity could be fitted with a power law $c(k) \simeq k^{-\phi}$ as shown in Fig.3. The model for $p = 1.0$ is a BA tree and therefore by definition (since no loops are present) the clustering coefficient is always zero. Instead in the BA model where $m$ is larger than 1, loops are present and the distribution of $c(k)$ with respect to $k$ is flat. A very similar behavior can be found for the $< k_{nn}(k) >$ (inset of Fig.4). Again we have a power law of the form $< k_{nn}(k) > \simeq k^{-7}$ for large probability of rewiring ($p << 1$) while this structuring disappears when $p = 1$.

In conclusion we presented a model whose topological features depend on the parameter $p$ tuning the ratio of vertices to edges creation. Interestingly, we find that for $p = 0.5(1)$ the model nicely reproduces most of the properties measured in the real case. It would be then very tempting to assume that substantial rewiring in existing routers is the key ingredient that makes the statistical properties of Internet networks so different from other growing networks.

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FIG. 1: Plot of graphs obtained for different values of $p$. Above a graph with $p = 1$ corresponding to AB tree; below the rewiring produced by a $p = 0.5$ simulation, gives rise to a more connected structure. Pictures made with Pajek

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FIG. 2: Plot of the degree distribution for various values of $p$. In the inset the integrated betweenness distribution. For the latter one only the symbols for $p = 0.1$, $p = 0.7$, $p = 0.9$, $p = 1.0$ have been explicitly plotted.

FIG. 3: Plot of the average clustering coefficient for vertices whose degree is $k$. In the inset the average degree of the nearest neighbours of a vertex whose degree is $k$.

TABLE I: Data from Numerical simulation of the model. The last row refers to the Internet (AS) network.

| $p$ | $< k >$ | $\gamma$ | $2 + \frac{d}{k}$ | $< d >$ | $\eta$ | $\phi$ | $\tau$ |
|-----|---------|----------|------------------|--------|-------|-------|-------|
| 0.1 | 19.9(5) | 2.15(5)  | 2.05             | 2.8(3) | 2.1(1) | 0.8(1) | 0.5(1) |
| 0.2 | 10.0(3) | 2.2(1)   | 2.11             | 2.9(3) | 2.1(1) | 0.8(2) | 0.5(1) |
| 0.3 | 6.6(3)  | 2.3(2)   | 2.18             | 3.0(2) | 2.1(2) | 0.7(2) | 0.5(1) |
| 0.4 | 5.0(2)  | 2.3(3)   | 2.25             | 3.1(2) | 2.2(1) | 0.7(2) | 0.5(2) |
| 0.5 | 4.0(2)  | 2.5(2)   | 2.33             | 3.4(2) | 2.2(1) | 0.6(3) | 0.5(2) |
| 0.6 | 3.3(2)  | 2.5(2)   | 2.43             | 3.9(3) | 2.1(1) | 0.5(4) | 0.5(3) |
| 0.7 | 2.8(1)  | 2.6(1)   | 2.54             | 4.4(3) | 2.3(2) |       |       |
| 0.8 | 2.5(1)  | 2.7(1)   | 2.67             | 5.5(1) | 2.1(1) |       |       |
| 0.9 | 2.2(1)  | 2.9(1)   | 2.82             | 6.5(4) | 2.2(2) |       |       |
| 1.0 | 2.0(1)  | 3.0(1)   | 3.00             | 8.7(3) | 2.0(1) |       |       |
| AS  | 3.8(1)  | 2.22(1)  |       | 4.16(1) | 2.2(1) | 0.75   | 0.55   |