TENSOR TOMOGRAPHY ON CARTAN-HADAMARD MANIFOLDS

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Abstract. We study the geodesic X-ray transform on Cartan-Hadamard manifolds, and prove solenoidal injectivity of this transform acting on functions and tensor fields of any order. The functions are assumed to be exponentially decaying if the sectional curvature is bounded, and polynomially decaying if the sectional curvature decays at infinity. This work extends the results of [Leh16] to dimensions $n \geq 3$ and to the case of tensor fields of any order.

1. Introduction

This article considers the geodesic X-ray transform on noncompact Riemannian manifolds. This transform encodes the integrals of a function $f$, where $f$ satisfies suitable decay conditions at infinity, over all geodesics. In the case of Euclidean space the geodesic X-ray transform is just the usual X-ray transform involving integrals over all lines, and in two dimensions it coincides with the Radon transform introduced in the seminal work of Radon in 1917 [Rad17]. For Euclidean or hyperbolic space in dimensions $n \geq 2$, one has the following basic theorems on the injectivity of this transform (see [Hel99], [Jen04], [Hel94]):

**Theorem A.** If $f$ is a continuous function in $\mathbb{R}^n$ satisfying $|f(x)| \leq C(1 + |x|)^{-\eta}$ for some $\eta > 1$, and if $f$ integrates to zero over all lines in $\mathbb{R}^n$, then $f \equiv 0$.

**Theorem B.** If $f$ is a continuous function in the hyperbolic space $\mathbb{H}^n$ satisfying $|f(x)| \leq Ce^{-d(x,o)}$, where $o \in \mathbb{H}^n$ is some fixed point, and if $f$ integrates to zero over all geodesics in $\mathbb{H}^n$, then $f \equiv 0$.

We remark that some decay conditions for the function $f$ are required, since there are examples of nontrivial functions in $\mathbb{R}^2$ which decay like $|x|^{-2}$ on every line and whose X-ray transform vanishes [Zal82], [Arm94]. Related results on the invertibility of Radon type transforms on constant curvature spaces or noncompact homogeneous spaces may be found in [Hel99], [Hel13].

The purpose of this article is to give analogues of the above theorems on more general, not necessarily symmetric Riemannian manifolds. We will work in the setting of Cartan-Hadamard manifolds, i.e. complete simply connected Riemannian manifolds with nonpositive sectional curvature. Euclidean and hyperbolic spaces are special cases of Cartan-Hadamard manifolds, and further explicit examples are recalled in Section 2. It is well known that any Cartan-Hadamard manifold is diffeomorphic to $\mathbb{R}^n$, the exponential map at any point is a diffeomorphism, and the map $x \mapsto d(x, p)^2$ is strictly convex for any $p \in M$ (see e.g. [Pet06]).

**Definition.** Let $(M, g)$ be a Cartan-Hadamard manifold, and fix a point $o \in M$. If $\eta > 0$, define the spaces of exponentially and polynomially decaying continuous functions by

$$E_\eta(M) = \{f \in C(M) ; |f(x)| \leq Ce^{-\eta d(x,o)} \text{ for some } C > 0\},$$

$$P_\eta(M) = \{f \in C(M) ; |f(x)| \leq C(1 + d(x,o))^{-\eta} \text{ for some } C > 0\}.$$
Also define the spaces
\[ E^1_\eta(M) = \{ f \in C^1(M) ; |f(x)| + |\nabla f(x)| \leq C e^{-\eta d(x,o)} \text{ for some } C > 0 \}, \]
\[ P^1_\eta(M) = \{ f \in C^1(M) ; |f(x)| \leq C (1 + d(x,o))^{-\eta} \text{ and } \]
\[ |\nabla f(x)| \leq C (1 + d(x,o))^{-\eta - 1} \text{ for some } C > 0 \} . \]

Here \( \nabla = \nabla_g \) is the total covariant derivative in \((M, g)\) and \(| \cdot | = | \cdot |_g\) is the \(g\)-norm on tensors.

It follows from Lemma 4.1 that if \( f \in P^1_\eta(M) \) for some \( \eta > 1 \), then the integral of \( f \) over any maximal geodesic in \( M \) is finite. For such functions \( f \) we may define the geodesic X-ray transform \( I_0 f \) of \( f \) by
\[ I_0 f(\gamma) = \int_{-\infty}^{\infty} f(\gamma(t)) \, dt, \quad \text{\( \gamma \) is a geodesic.} \]
The inverse problem for the geodesic X-ray transform is to determine \( f \) from the knowledge of \( I_0 f \). By linearity, uniqueness for this inverse problem reduces to showing that \( I_0 f = 0 \) implies \( f = 0 \).

More generally, suppose that \( f \) is a \( C^1\)-smooth symmetric covariant \( m \)-tensor field on \( M \), written in local coordinates (using the Einstein summation convention) as
\[ f = f_{j_1...j_m}(x) \, dx^{j_1} \otimes ... \otimes dx^{j_m} . \]

We say that \( f \in P^1_\eta(M) \) if \( |f|_g \in P^1_\eta(M) \), and \( f \in P^2_\eta(M) \) if \( |f|_g \in P^1_\eta(M) \) and \( |\nabla f|_g \in P^2_\eta(M) \), etc. Now if \( f \in P^1_\eta(M) \) for some \( \eta > 1 \), then the geodesic X-ray transform \( I_m f \) of \( f \) is well defined by the formula
\[ I_m f(\gamma) = \int_{-\infty}^{\infty} f_{\gamma(t)}(\dot{\gamma}(t), \ldots, \dot{\gamma}(t)) \, dt, \quad \text{\( \gamma \) is a geodesic.} \]
This transform always has a kernel when \( m \geq 1 \): if \( h \) is a symmetric \((m - 1)\)-tensor field satisfying \( h \in P^1_\eta(M) \) for some \( \eta > 0 \), then \( I_m(\sigma \nabla h) = 0 \) where \( \sigma \) denotes symmetrization of a tensor field (see Section 3.3). We say that \( I_m \) is solenoidal injective if \( I_m f = 0 \) implies \( f = \sigma \nabla h \) for some \((m - 1)\)-tensor field \( h \).

Our first theorem proves solenoidal injectivity of \( I_m \) for any \( m \geq 0 \) on Cartan-Hadamard manifolds with bounded sectional curvature, assuming exponential decay of the tensor field and its first derivatives. We will denote the sectional curvature of a two-plane \( \Pi \subset T_x M \) by \( K_x(\Pi) \), and we write \(-K_0 \leq K \leq 0 \) if \(-K_0 \leq K_x(\Pi) \leq 0 \) for all \( x \in M \) and for all two-planes \( \Pi \subset T_x M \).

**Theorem 1.1.** Let \((M, g)\) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \), and assume that
\[ -K_0 \leq K \leq 0, \quad \text{for some } K_0 > 0. \]
If \( f \) is a symmetric \( m \)-tensor field in \( E^1_\eta(M) \) for some \( \eta > \frac{n+1}{2} \sqrt{K_0} \), and if \( I_m f = 0 \), then \( f = \sigma \nabla h \) for some symmetric \((m - 1)\)-tensor field \( h \). (If \( m = 0 \), then \( f \equiv 0 \).)

The second theorem considers the case where the sectional curvature decays polynomially at infinity, and proves solenoidal injectivity if the tensor field and its first derivatives also decay polynomially.

**Theorem 1.2.** Let \((M, g)\) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \), and assume that the function
\[ K(x) = \sup \{|K_x(\Pi)| ; \Pi \subset T_x M \text{ is a two-plane}\} \]
satisfies \( K \in P^1_\kappa(M) \) for some \( \kappa > 2 \). If \( f \) is a symmetric \( m \)-tensor field in \( P^1_\eta(M) \) for some \( \eta > \frac{n+2}{2} \), and if \( I_m f = 0 \), then \( f = \sigma \nabla h \) for some symmetric \((m - 1)\)-tensor field \( h \). (If \( m = 0 \), then \( f \equiv 0 \).)
The second theorem is mostly of interest in two dimensions because of the following rigidity phenomenon: any manifold of dimension $\geq 3$ that satisfies the conditions of the theorem is isometric to Euclidean space $\mathbb{R}^3$. See Section 2 for a discussion. We will give the proof in any dimension since this may be useful in subsequent work.

We remark that Theorems 1.1–1.2 correspond to Theorems A and B above, but the manifolds considered in Theorems 1.1–1.2 can be much more general and include many examples with non-constant curvature (see Section 2). The results will be proved by using energy methods based on Pestov identities, which have been studied extensively in the case of compact manifolds with strictly convex boundary. We refer to [Muk77], [PS88], [Sha94], [Kni02], [PSU14] for some earlier results. In fact, Theorems 1.1–1.2 can be viewed as an extension of the tensor tomography results in [PS88] from the case of compact nonpositively curved manifolds with boundary to the case of certain noncompact manifolds. We remark that one of the main points in our theorems is that the functions and tensor fields are not compactly supported (indeed, the compactly supported case would reduce to known results on compact manifolds with boundary).

More recently, the work [PSU13] gave a particularly simple derivation of the basic Pestov identity for X-ray transforms and proved solenoidal injectivity of $I_m$ on simple two-dimensional manifolds. Some of these methods were extended to all dimensions in [PSU15] and to the case of attenuated X-ray transforms in [GPSU16]. Following some ideas in [PSU13], the work [Leh16] proved versions of Theorems 1.1–1.2 for the case of two-dimensional Cartan-Hadamard manifolds.

In this paper we combine the main ideas in [Leh16] with the methods of [PSU15] and prove solenoidal injectivity results on Cartan-Hadamard manifolds in any dimension $n \geq 2$. However, instead of using the Pestov identity in its standard form (which requires two derivatives of the functions involved), we will use a different argument from [PSU15] related to the $L^2$ contraction property of a Beurling transform on nonpositively curved manifolds. This argument dates back to [GK80a, GK80b], it only involves first order derivatives and immediately applies to tensor fields of arbitrary order. The $C^1$ assumption in Theorems 1.1–1.2 is due to this method of proof, and the decay assumptions are related to the growth of Jacobi fields. We mention that Theorems 1.1–1.2 also extend the two-dimensional results of [Leh16] by assuming slightly weaker conditions.

This article is organized as follows. Section 1 is the introduction, and Section 2 contains examples of Cartan-Hadamard manifolds. In Section 3 we review basic facts related to geodesics on Cartan-Hadamard manifolds, geometry of the sphere bundle and symmetric covariant tensors fields, following [Leh16], [PSU15], [DS10]. Section 4 collects some estimates concerning the growth of Jacobi fields and related decay properties for solutions of transport equations. Finally, Section 5 includes the proofs of the main theorems based on $L^2$ inequalities for Fourier coefficients.

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2. EXAMPLES OF CARTAN-HADAMARD MANIFOLDS

In this section we recall some facts and examples related to Cartan-Hadamard manifolds. Most of the details can be found in [BO69], [KW74], [GW79], [GW82], [Pet06]. We first discuss the case of two-dimensional manifolds, which is quite different compared to manifolds of higher dimensions.

2.1. Dimension two. Let $K \in C^\infty(\mathbb{R}^2)$. A theorem of Kazdan and Warner [KW74] states that a necessary and sufficient condition for existence of a complete Riemannian metric on $\mathbb{R}^2$ with Gaussian curvature $K$ is

$$\lim_{r \to \infty} \inf_{|x| \geq r} K(x) \leq 0.$$ (2.1)
This provides a wide class of Riemannian metrics satisfying the assumptions of Theorem 1.1 in dimension two. However, this does not directly give an example of a manifold satisfying the assumptions of Theorem 1.2 since the condition (2.1) is given with respect to the Euclidean metric of \(\mathbb{R}^2\).

Examples of manifolds satisfying the assumptions of Theorem 1.2 can be constructed using warped products. Let \((r, \theta)\) be the polar coordinates in \(\mathbb{R}^2\) and consider a warped product
\[
(2.2) \quad ds^2 = dr^2 + f^2(r)d\theta^2,
\]
where \(f\) is a smooth function that is positive for \(r > 0\) and satisfies \(f(0) = 0\) and \(f'(0) = 1\). This is a Riemannian metric on \(\mathbb{R}^2\) having Gaussian curvature
\[
(2.3) \quad K(x) = -\frac{f''(|x|)}{f(|x|)},
\]
which depends only on the Euclidean distance \(|x| := r(x)\) to the origin. We remark that distances to the origin in the Euclidean metric and in the warped metric coincide. It is shown in [GW79] Proposition 4.2] that for every \(k \in C^\infty((0, \infty))\) with \(k \leq 0\) there exists a unique warped metric of the form (2.2) such that \(k(|x|) = K(x)\). Hence warped products provide many examples of two-dimensional manifolds for which \(K(x) \leq C(1 + |x|)^{-\kappa}\) with \(\kappa > 0\), i.e. \(K \in P_\kappa(M)\).

2.2. Higher dimensions. Warped products can also be used to construct examples of higher dimensional Cartan-Hadamard manifolds satisfying the assumptions of Theorem 1.1 see e.g. [BO69].

In the case of Theorem 1.2 it turns out that the decay condition for curvature is very restrictive in higher dimensions: the only possible geometry is the Euclidean one. This follows directly from a theorem by Greene and Wu in [GWS82]. If \(M\) is a Cartan-Hadamard manifold with \(n = \text{dim}(M) \geq 3\), \(k(s) = \sup\{K(x); x \in M, d(x, o) = s\}\), where \(o\) is a fixed point, and one of the following holds:

1. \(n\) is odd and \(\lim \inf_{s \to \infty} s^2k(s) \to 0\) or
2. \(n\) is even and \(\int_0^\infty sk(s)\,ds\) is finite,

then \(M\) is isometric to \(\mathbb{R}^n\).

3. Geometric facts

Throughout this work we will assume \((M, g)\) to be an \(n\)-dimensional Cartan-Hadamard manifold with \(n \geq 2\) unless otherwise stated. We also assume unit speed parametrization for geodesics.

3.1. Behaviour of geodesics. By the Cartan-Hadamard theorem the exponential map \(\exp_o\) is defined on all of \(T_oM\) and is a diffeomorphism for every \(x \in M\). Hence every pair of points can be joined by a unique geodesic. Let \(SM = \{(x, v) \in TM; |v| = 1\}\) be the unit sphere bundle, and if \((x, v) \in SM\) denote by \(\gamma_{x, v}\) the unique geodesic with \(\gamma(0) = x\) and \(\dot{\gamma}(0) = v\). The triangle inequality implies that
\[
(3.1) \quad d_g(\gamma_{x, v}(t), o) \geq |t| - d_g(x, o)
\]
for all \(t \in \mathbb{R}, o \in M\).

We say that a geodesic \(\gamma\) is escaping with respect to the point \(o\) if the function \(t \mapsto d_g(\gamma(t), o)\) is strictly increasing on the interval \([0, \infty)\). The set of all such geodesics is denoted by \(\mathcal{E}_o\). For \(\gamma_{x, v} \in \mathcal{E}_o\) the triangle inequality gives
\[
(3.2) \quad d_g(\gamma_{x, v}(t), o) \geq \begin{cases} d_g(x, o), & \text{if } 0 \leq t \leq 2d_g(x, o), \\ t - d_g(x, o), & \text{if } 2d_g(x, o) < t. \end{cases}
\]
However, since $(M, g)$ is a Cartan-Hadamard manifold, Jacobi field estimates give a stronger bound. For $\gamma_{x,v} \in \mathcal{E}_o$ one has (see [Los08 Corollary 4.8.5] or [Pet06 Section 6.3])

\begin{equation}
(3.3) \quad d_g(\gamma_{x,v}(t), o) \geq \sqrt{d_g(x, o)^2 + t^2}, \quad t \geq 0.
\end{equation}

The following lemma is proved in [Leh16] in two dimensions. The proof in higher dimensions is identical, but we include a short argument for completeness.

**Lemma 3.1.** Suppose $o \in M$. At least one of the geodesics $\gamma_{x,v}$ and $\gamma_{x,-v}$ is in $\mathcal{E}_o$.

**Proof.** Since $(M, g)$ is a Cartan-Hadamard manifold, the function $h(t) = d_g(\gamma_{x,v}(t), o)^2$ is strictly convex, $h'' > 0$, on $\mathbb{R}$. If $h'(0) \geq 0$ then $\gamma_{x,v}$ is escaping, and if $h'(0) \leq 0$ then $\gamma_{x,-v}$ is escaping. □

### 3.2. On the geometry of the unit tangent bundle.

We first briefly explain the splitting of the tangent bundle into horizontal and vertical bundles. Then we give a short discussion on geodesics of $SM$. Finally, we include a proof that $SM$ is complete when $M$ is.

#### 3.2.1. The structure of the tangent bundle.

The following discussion is based on [Pat99], [PSU15], where these topics are considered in more detail. We denote by $\pi: TM \to M$ the usual base point map $\pi(x, v) = x$. The connection map $K_{TV}: T(TM) \to TM$ of the Levi-Civita connection $\nabla$ of $M$ is defined as follows. Let $\xi \in T_{x,v}TM$ and $c: (-\epsilon, \epsilon) \to TM$ be a curve such that $c(0) = \xi$. Write $c(t) = (\gamma(t), Z(t))$, where $Z(t)$ is a vector field along the curve $\gamma$, and define

$$K_{TV}(\xi) := D_t Z(0) \in T_x M.$$

The maps $K_{TV}$ and $d\pi$ yield a splitting

\begin{equation}
(3.4) \quad T_{x,v}TM = \tilde{\mathcal{H}}(x, v) \oplus \tilde{\mathcal{V}}(x, v)
\end{equation}

where $\tilde{\mathcal{H}}(x, v) = \ker K_{TV}$ is the horizontal bundle and $\tilde{\mathcal{V}}(x, v) = \ker d_{x,v}\pi$ is the vertical bundle. Both are $n$-dimensional subspaces of $T_{x,v}TM$.

On $TM$ we define the Sasaki metric $g_s$ by

$$\langle v, w \rangle_{g_s} = \langle K_{TV}(v), K_{TV}(w) \rangle_g + \langle d\pi(v), d\pi(w) \rangle_g,$$

which makes $(TM, g_s)$ a Riemannian manifold of dimension $2n$. The maps $K_{TV}: \tilde{\mathcal{V}}(x, v) \to T_x M$ and $d\pi: \tilde{\mathcal{H}}(x, v) \to T_x M$ are linear isomorphisms. Furthermore, the splitting (3.4) is orthogonal with respect to $g_s$. Using the maps $K_{TV}$ and $d\pi$, we will identify vectors in the horizontal and vertical bundles with corresponding vectors on $T_x M$.

The unit sphere bundle $SM$ was defined as

$$SM := \bigcup_{x \in M} S_x M, \quad S_x M := \{(x, v) \in T_x M : |v|_g = 1\}.$$

We will equip $SM$ with the metric induced by the Sasaki metric on $TM$. The geodesic flow $\phi_t(x, v): \mathbb{R} \times SM \to SM$ is defined as

$$\phi_t(x, v) := (\gamma_{x,v}(t), \tilde{\gamma}_{x,v}(t)).$$

The associated vector field is called the geodesic vector field and denoted by $X$.

For $SM$ we obtain an orthogonal splitting

\begin{equation}
(3.5) \quad T_{x,v}SM = \mathbb{R}X(x, v) \oplus \mathcal{H}(x, v) \oplus \mathcal{V}(x, v)
\end{equation}

where $\mathbb{R}X \oplus \mathcal{H}(x, v) = \tilde{\mathcal{H}}(x, v)$ and $\mathcal{V}(x, v) = \ker d_{x,v}(\pi|_{SM})$. Both $\mathcal{H}(x, v)$ and $\mathcal{V}(x, v)$ have dimension $n - 1$ and can be canonically identified with elements in the codimension one subspace $\{v\}^\perp \subset T_x M$ via $d\pi$ and $K_{TV}$, respectively. We will freely use this identification.
Following [PSU15], if \( u \in C^1(SM) \), then the gradient \( \nabla_{SM} u \) has the decomposition

\[
\nabla_{SM} u = (X u) X + h u + v u,
\]

according to (3.5). The quantities \( h u \) and \( v u \) are called the horizontal and the vertical gradients, respectively. It holds that \( \langle \nabla u(x, v), v \rangle = 0 \) and \( \langle \nabla u(x, v), v \rangle = 0 \) for all \( (x, v) \in SM \).

As discussed in [PSU15], on two-dimensional manifolds the horizontal and vertical gradients reduce to the horizontal and vertical vector fields \( X_\perp \) and \( V \) via

\[
\nabla h u(x, v) = -(X_\perp u(x, v)) v^\perp \quad \text{and} \quad \nabla v u(x, v) = (V u(x, v)) v^\perp
\]

where \( v^\perp \) is such that \( \{v, v^\perp\} \) is a positive orthonormal basis of \( T_x M \). In [Leh16] the flows associated with \( X_\perp \) and \( V \) were used to derive estimates for \( X_\perp u \) and \( V u \). We will proceed in a similar manner in the higher dimensional case.

Let \( (x, v) \in SM \) and \( w \in S_x M \), \( w \perp v \). We define \( \phi^h_{w,t} : \mathbb{R} \to SM \) by \( \phi^h_{w,t}(x, v) = (\gamma_{x,w}(t), V(t)) \), where \( V(t) \) is the parallel transport of \( v \) along \( \gamma_{x,w} \). It holds that

\[
K \nabla \left( \frac{d}{dt} \phi^h_{w,t}(x, v) \right)_{t=0} = 0 \quad \text{and} \quad d\pi \left( \frac{d}{dt} \phi^h_{w,t}(x, v) \right)_{t=0} = w.
\]

We define \( \phi^v_{w,t} : \mathbb{R} \to SM \) by \( \phi^v_{w,t}(x, v) = (x, (\cos t)v + (\sin t)w) \). It holds that

\[
K \nabla \left( \frac{d}{dt} \phi^v_{w,t}(x, v) \right)_{t=0} = v \quad \text{and} \quad d\pi \left( \frac{d}{dt} \phi^v_{w,t}(x, v) \right)_{t=0} = 0.
\]

The following lemma states the relation between \( \phi^h_{w,t} \) and \( \phi^v_{w,t} \) and the horizontal and the vertical gradients of a function.

**Lemma 3.2.** Suppose \( u \) is differentiable at \( (x, v) \in SM \). Fix \( w \in S_x M \), \( w \perp v \). Then it holds that

\[
\langle \nabla h u(x, v), w \rangle = \frac{d}{dt} u(\phi^h_{w,t}(x, v))_{t=0}
\]

and

\[
\langle \nabla v u(x, v), w \rangle = \frac{d}{dt} u(\phi^v_{w,t}(x, v))_{t=0}.
\]

**Proof.** Using the chain rule and the equations (3.6) we get

\[
\frac{d}{dt} u(\phi^h_{w,t}(x, v))_{t=0} = \langle \nabla_{SM} u(\phi^h_{w,t}(x, v)), d\pi \phi^h_{w,t}(x, v) \rangle_{g^*} = \langle \nabla u(x, v), w \rangle.
\]

For \( \nabla v \) we use the equations (3.7) in a similar fashion. \( \square \)

The maps \( \phi^h_{w,t} \) and \( \phi^v_{w,t} \) are related to normal Jacobi fields along geodesics. We can define

\[
J^h_w(t) := \frac{d}{ds} \pi \left( \phi^h_{w,s}(x, v) \right)_{s=0} = d_{\phi_t(x,v)} \pi \left( \frac{d}{ds} \phi^h_{w,s}(x, v) \right)_{s=0}.
\]

Since \( \Gamma(s, t) = \pi \left( \phi^h_{w,s}(x, v) \right) \) is a variation of \( \gamma_{x,w} \) along geodesics, \( J^h_w(t) \) is a Jacobi field along \( \gamma_{x,w} \). It has the initial conditions \( J^h_w(0) = w \) and \( D_t J^h_w(0) = 0 \) by the symmetry lemma (see e.g. [Lee97]).

Replacing \( \phi^h_{w,s} \) with \( \phi^v_{w,s} \) gives a Jacobi field \( J^v_w(t) \) with the initial conditions \( J^v_w(0) = 0 \) and \( D_t J^v_w(t)(0) = w \). In the both cases the Jacobi field is normal because \( \langle v, w \rangle_g = 0 \).

By the symmetry lemma

\[
K \nabla \left( \frac{d}{ds} \phi^h_{w,s}(x, v) \right)_{s=0} = D_s \partial_s \gamma^h_{w,s}(x, v)(t)_{s=0} = D_t \partial_s \gamma^v_{w,s}(x, v)(t)_{s=0} = D_t J^v_w(t).
\]
From the definition of the Sasaki metric we then see that
\[
\langle \nabla_{SM} u(x,v), \frac{d}{ds} \phi_{t}(\phi^h_{w,t}(x,v)) \rangle_{g} = \langle \nabla u(x,v), J^h_{w,t}(t) \rangle_g + \langle \nabla u(x,v), D_t J^h_{w,t}(t) \rangle_g.
\]
and
\[
\langle \nabla_{SM} u(x,v), \frac{d}{ds} \phi_{t}(\phi^v_{w,t}(x,v)) \rangle_{g} = \langle \nabla u(x,v), J^v_{w,t}(t) \rangle_g + \langle \nabla u(x,v), D_t J^v_{w,t}(t) \rangle_g.
\]

**Remark.** The constructions in this subsection remain valid at a.e. \((x,v) \in SM\) if one assumes that \(u\) is in the space \(W^{1,\infty}_{loc}(SM)\). Functions in \(W^{1,\infty}_{loc}(SM)\) are characterized as locally Lipschitz functions, and further by Rademacher’s theorem, differentiable almost everywhere and weak gradients equal to gradients almost everywhere (see e.g. [Eva98] Chapters 5.8.2–5.8.3).

### 3.2.2. Geodesics on the unit tangent bundle

Next we describe some facts related to geodesics on \(SM\) (see e.g. [BBNV03] and references therein). Let \(R(U,V)\) denote the Riemannian curvature tensor. A curve \(\Gamma(t) = (x(t), V(t))\) on \(SM\) is a geodesic if and only if

\[
\begin{align}
\nabla_{\dot{x}} \dot{x} &= -R(V, \nabla_{\dot{x}} V) \dot{x} \\
\nabla_{\dot{V}} \dot{V} &= -|\nabla_{\dot{x}} V|_g^2 V, \quad |\nabla_{\dot{x}} V|_g^2 \text{ is a constant along } t(t)
\end{align}
\]

holds for every \(t\) in the domain of \(\Gamma\) (see [Sas62] Equations 5.2)). Given \((x,v) \in SM\), the horizontal lift of \(w \in T_xM\) is denoted by \(w^h\), i.e. the unique vector \(w^h \in T_{x,v}(SM)\) such that \(d(\pi|_{SM})(w^h) = w\) and \(K_{\pi}(w^h) = 0\), and the vertical lift \(w^v\) is defined similarly. Initial conditions for \(x, \dot{x}, V, \nabla_{\dot{x}} V\) at \(t = 0\) with \(g(V(0), \nabla_{\dot{x}(0)} V(0)) = 0\) and \(|V(0)|_g = 1\) determine a unique geodesic \(\Gamma = (x,v)\), by (3.8), which satisfies the initial conditions \(\Gamma(0) = (x(0), V(0))\) and \(\dot{\Gamma}(0) = (\dot{x}(0)^h + (\nabla_{\dot{x}(0)} V(0))v\)

where the lifts are done with respect to \((x(0), V(0)) \in SM\). The geodesics of \(SM\) are of the following three types:

1. If \(\nabla_{\dot{x}(0)} V(0) = 0\), then \(\Gamma\) is a parallel transport of \(V(0)\) along the geodesic \(x\) on \(M\) (horizontal geodesics).
2. If \(\dot{x}(0) = 0\), then \(\Gamma\) is a great circle on the fibre \(\pi^{-1}(x(0))\) and \(x(t) = x(0)\) (vertical geodesics, in this case one interprets the system (3.8) via \(\nabla_{\dot{x}} = D_t\)).
3. All the rest, i.e. solutions of (3.8) with initial conditions \(\dot{x}(0) \neq 0\) and \(\nabla_{\dot{x}(0)} V(0) \neq 0\) (oblique geodesics).

We state the following lemma for the sake of clarity.

**Lemma 3.3.** Fix \((x,v) \in SM\) and \(w \in S_xM\), \(w \perp v\). Then \(\phi_t(x,v)\) and \(\phi^h_{w,t}(x,v)\) are horizontal unit speed geodesics and \(\phi^v_{w,t}(x,v)\) is a vertical unit speed geodesic with respect to \(t\).

**Proof.** The fact that \(\phi_t(x,v)\) and \(\phi^h_{w,t}(x,v)\) are horizontal geodesics and \(\phi^v_{w,t}(x,v)\) is a vertical geodesic follows immediately from their definitions and the above discussion based on the system of differential equations (3.8). The fact that \(\phi_t(x,v), \phi^h_{w,t}(x,v)\) and \(\phi^v_{w,t}(x,v)\) are unit speed follows from the equations (3.6) and (3.7) and the definition of the Sasaki metric.

**Corollary 3.4.** Let \((x,v) \in SM\). Assume that \(T_{x,v}(SM)\) has the decomposition
\[
Y = aX|_{x,v} + H + V, \quad H \in \mathcal{H}(x,v), V \in \mathcal{V}(x,v), a \in \mathbb{R}.
\]
Then
\[
\begin{align}
(D \phi_t)_{x,v}(aX|_{x,v}) &= aX|_{\phi_t(x,v)}, \\
(D \phi_t)_{x,v}(H) &= |H|_{g_s} \left[ (J^h_{w_t}(t))^h + (D_t J^h_{w_t}(t))^v \right], \\
(D \phi_t)_{x,v}(V) &= |V|_{g_s} \left[ (J^v_{w_t}(t))^h + (D_t J^v_{w_t}(t))^v \right].
\end{align}
\]
Lemma 3.2, gives

Moreover, using the unit speed geodesic \( \Gamma(s) \) such that

\[ N = \frac{d\pi}{ds} \]

which is also orthogonal to \( V \).

Choose an open set \( U \) where \( \Gamma : [0, \tau] \rightarrow SM \)

which converges, say to \( p \). Following Lemma 3.2 gives

\[ (D\phi_t)_{x,v}(X|_{x,v}) = D\phi_t(\hat{\Gamma}(0)) = (\phi_t \circ \Gamma)'(0) = X|_{\phi_t(x,v)}. \]

Moreover, using the unit speed geodesic \( \Gamma(s) = \phi^u_{w,v,s}(x,v) \) on \( SM \), and using the formulas after Lemma 3.2 gives

\[ (D\phi_t)_{x,v}(H) = D\phi_t(|H|_{gs}(\hat{\Gamma}(0))) = |H|_{gs}(\phi_t \circ \Gamma)'(0) \]

which is orthogonal to \( X|_{\phi_t(x,v)} \). Finally, the unit speed geodesic \( \Gamma(s) = \phi^w_{w,v,s}(x,v) \) on \( SM \) gives

\[ (D\phi_t)_{x,v}(V) = D\phi_t(|V|_{gs}(\hat{\Gamma}(0))) = |V|_{gs}(\phi_t \circ \Gamma)'(0) \]

which is also orthogonal to \( X|_{\phi_t(x,v)} \).

\[ \Box \]

3.2.3. Completeness of the unit tangent bundle. We will need the fact that \( SM \) is complete when \( M \) is complete. This need arises from theory of Sobolev spaces on manifolds (see Section 5). We could not find a reference so a proof is included.

**Lemma 3.5.** Let \( M \) be a complete Riemannian manifold with or without boundary. Then \( SM \) is complete.

**Proof.** Let \( (y^{(j)}) \) be a Cauchy sequence in \( (SM, d_{gs}) \). We show that it converges in the topology induced by \( g_s \). The definition of the Sasaki metric implies that

\[ L_{gs}(\Gamma) \geq \int_0^{t} \left| d\pi_{\Gamma}(t) \right|_{g} dt = L_g(\pi \circ \Gamma) \geq d_g(\pi(\Gamma(0)), \pi(\Gamma(t))) \]

where \( \Gamma : [0, \tau] \rightarrow SM \) is any piecewise \( C^1 \)-smooth curve. Hence

\[ (3.9) \quad d_{gs}(a,b) \geq d_g(\pi(a), \pi(b)) \]

for all \( a, b \in SM \). The above inequality implies that \( (\pi(y^{(j)})) \) is a Cauchy sequence in \( (M, g) \) and converges, say to \( p \in M \), by completeness of \( M \).

Consider a coordinate neighborhood \( U \) of \( p \) in \( M \), so that \( \pi^{-1}(U) \) is diffeomorphic to \( U \times S^{n-1} \). Choose an open set \( V \) and a compact set \( K \) so that \( p \in V \subset K \subset U \). Now \( \pi^{-1}(K) \) is homeomorphic to \( K \times S^{n-1} \) which is compact as a product of two compact sets. Since \( \pi(y^{(j)}) \rightarrow p \), there exists \( N \) such that \( \pi(y^{(j)}) \in V \) for all \( j \geq N \), and this implies \( y^{(j)} \in \pi^{-1}(K) \) for all \( j \geq N \). Hence \( (y^{(j)}) \) has a limit in \( \pi^{-1}(K), d_{gs}|_{\pi^{-1}(K)} \) since it is a Cauchy sequence, and thus \( (y^{(j)}) \) converges also in \( (SM, d_{gs}) \). \[ \Box \]

3.3. Symmetric covariant tensors fields. We denote by \( S^m(M) \) the set of \( C^1 \)-smooth symmetric covariant \( m \)-tensor fields and by \( S^m_0(M) \) the symmetric covariant \( m \)-tensors at point \( x \). Following [DS10] (where more details are also given), we define the map \( \lambda_x : S^m_0(M) \rightarrow C^\infty(S_x M) \),

\[ \lambda_x(f)(v) = f_v (v, \ldots, v) \]

which is given in local coordinates by

\[ \lambda_x(f_{i_1 \ldots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m})(v) = f_{i_1 \ldots i_m}(x)v^{i_1} \ldots v^{i_m}. \]
If $S^m_x(M)$ and $C^\infty(S_xM)$ are endowed with their usual $L^2$-inner products, then $\lambda_x$ is an isomorphism and even isometry up to a factor. It smoothly depends on $x$ and hence we get an embedding $\lambda: S^m_x(M) \to C^1_x(SM)$. The mapping $\lambda$ identifies symmetric covariant $m$-tensor fields with homogeneous polynomials (with respect to $v$) of degree $m$ on $SM$. We will use this identification and do not always write $\lambda$ explicitly.

The symmetrization of a tensor is defined by

$$\sigma(\omega_1 \otimes \cdots \otimes \omega_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} \omega_{\pi(1)} \otimes \cdots \otimes \omega_{\pi(m)},$$

where $\Pi_m$ is the permutation group of $\{1, \ldots, m\}$. From the above expression we see that if a covariant $m$-tensor field $f$ is in $E^1_\eta(M)$ or $P^1_\eta(M)$ for some $\eta > 0$, then so is $\sigma f$ too. Furthermore, for $f \in S^m(M)$ one has

$$\lambda(\sigma \nabla f) = X\lambda(f). \tag{3.10}$$

It follows from the last identity and the fundamental theorem of calculus that if $f \in P^1_\eta(M)$ for some $\eta > 0$, then $I_m(\sigma \nabla f) = 0$. This shows that $I_m$ always has a nontrivial kernel for $m \geq 1$, as described in the introduction.

The next lemma states how the decay properties of a tensor field carry over to functions on $SM$.

**Lemma 3.6.** Suppose $f \in S^m(SM)$ and $\eta > 0$.

(a) If $f \in E^1_\eta(M)$, then

$$\sup_{v \in S^*_xM} |Xf(x,v)| \in E^1_\eta(M), \quad \sup_{v \in S^*_xM} |\nabla f(x,v)| \in E^1_\eta(M) \quad \text{and} \quad \sup_{v \in S^*_xM} |\nabla f(x,v)| \in E^1_\eta(M).$$

(b) If $f \in P^1_\eta(M)$, then

$$\sup_{v \in S^*_xM} |Xf(x,v)| \in P^1_{\eta+1}(M), \quad \sup_{v \in S^*_xM} |\nabla f(x,v)| \in P^1_{\eta+1}(M) \quad \text{and} \quad \sup_{v \in S^*_xM} |\nabla f(x,v)| \in P^1_\eta(M).$$

**Proof.** (a) The result for $Xf$ follows from (3.10). To prove the other statements we take $x \in M$ and use local normal coordinates $(x^1, \ldots, x^n)$ centered at $x$ and the associated coordinates $(v^1, \ldots, v^n)$ for $T_xM$. In these coordinates $f(x) = f_{i_1 \cdots i_m}(x) \, dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ and $\nabla f(x) = \partial_{x_j} f_{i_1 \cdots i_m}(x) \, dx^j \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_m}$. We see that

$$|f(x)|_g = \left( \sum_{i_1, \ldots, i_m} |f_{i_1 \cdots i_m}(x)|^2 \right)^{1/2} \quad \text{and} \quad |\nabla f(x)|_g = \left( \sum_{j,i_1, \ldots, i_m} |\partial_{x_j} f_{i_1 \cdots i_m}(x)|^2 \right)^{1/2}.$$
and, using the orthogonality of $Xf(x,v)X(x,v)$ and $\nabla f(x,v)$ and the Cauchy-Schwarz inequality,

$$\sup_{v \in S_x M} |\nabla f(x,v)| = \left( \sum_{j,i_1,\ldots,i_m} |\partial_{x_j} f_{i_1\ldots i_m}(x)|^2 \right)^{1/2} = |\nabla f(x)|_g.$$ 

This implies that $\sup_{v \in S_x M} |\nabla f(x,v)| \in E_\eta(M)$.

For $\nabla f$, the identity $\partial_{x_j} v^k = \delta_j^k - v_j v^k$ (see [PSU15]) implies that

$$\nabla f(x,v) = \sum_{j=1}^n (f_{ji_1\ldots i_m} v^{i_2} \ldots v^{i_m} - f(x,v) v_j) \partial_{x_j} + \ldots + \sum_{j=1}^n (f_{i_1\ldots i_{m-1} j} v^{i_1} \ldots v^{i_{m-1}} - f(x,v) v_j) \partial_{x_j}$$

$$= m \sum_{j=1}^n (f_{ji_1\ldots i_m} v^{i_2} \ldots v^{i_m} - f(x,v) v_j) \partial_{x_j},$$

Thus orthogonality and expanding the squares gives

$$\left| \nabla f(x,v) \right|^2 = m^2 \sum_{j=1}^n \sum_{i_1,\ldots,i_m} (f_{ji_1\ldots i_m} v^{i_2} \ldots v^{i_m})^2 \leq m^2 \sum_{i_1,\ldots,i_m} \left| f_{i_1\ldots i_m}(x) \right|^2 = m^2 |f(x)|^2_g$$

which in turn implies that $\sup_{v \in S_x M} |\nabla f(x,v)| \in E_\eta(M)$. The proof for (b) is the same. \qed

4. Growth estimates

Throughout this section we assume that $f$ is a symmetric covariant $m$-tensor field in $P_\eta(M)$ for some $\eta > 1$. We begin by observing that the geodesic X-ray transform is well defined for such $f$.

**Lemma 4.1.** Let $f \in P_\eta(M)$ for some $\eta > 1$. For any $(x,v) \in SM$ one has

$$\int_{-\infty}^{\infty} |f_{\gamma_x,v(t)}(\hat{\gamma}_x,v(t),\ldots,\hat{\gamma}_x,v(t))| \, dt < \infty.$$ 

**Proof.** The assumption implies that $|f_{\gamma_x,v(t)}(\hat{\gamma}_x,v(t),\ldots,\hat{\gamma}_x,v(t))| \leq C(1 + d(\gamma_x,v(t),o))^{-\eta}$. One can then change variables so that $t = 0$ corresponds to the point on the geodesic that is closest to $o$, split the integral over $t \geq 0$ and $t \leq 0$, and use the fact that the integrands are $\leq C(1 + |t|)^{-\eta}$ by the estimate (3.3). \qed

If $f \in P_\eta(M)$ for some $\eta > 1$, we may now define

$$u^f(x,v) := \int_0^\infty f_{\gamma_x,v(t)}(\hat{\gamma}_x,v(t),\ldots,\hat{\gamma}_x,v(t)) \, dt.$$ 

It easy to see that

$$u^f(x,v) + (-1)^m u^f(x,-v) = If(x,v)$$

for all $(x,v) \in SM$.

We have the usual reduction to the transport equation.

**Lemma 4.2.** Let $f \in P_\eta(M)$ for some $\eta > 1$. Then $Xu^f = -f$.

**Proof.** By definition

$$Xu^f(x,v) = \lim_{s \to 0} -\frac{1}{s} \int_0^s f_{\gamma_x,v(t)}(\hat{\gamma}_x,v(t),\ldots,\hat{\gamma}_x,v(t)) \, dt = -f(x,v,\ldots,v).$$ \qed

Next we derive decay estimates for $u^f$ under the assumption that $If = 0$. 

Lemma 4.3. Suppose that $I f = 0$.
(a) If $f \in E_\eta(M)$ for $\eta > 0$, then
\[
|u^f(x,v)| \leq C(1 + d_g(x,o))e^{-\eta d_g(x,o)}
\]
for all $(x,v) \in SM$.
(b) If $f \in P_\eta(M)$ for $\eta > 1$, then
\[
|u^f(x,v)| \leq \frac{C}{(1 + d_g(x,o))^{\eta-1}}
\]
for all $(x,v) \in SM$.

Proof. Since $I f = 0$, one has $|u^f(x,v)| = |u^f(x,-v)|$. By Lemma 3.1 possibly after replacing $(x,v)$ by $(x,-v)$, we may assume that $\gamma_{x,v}$ is escaping. We have
\[
|u^f(x,v)| = \left| \int_0^\infty f(\gamma(t))\langle \dot{\gamma}(t), \ldots, \dot{\gamma}(t) \rangle dt \right| \leq \int_0^\infty |f(\gamma(t))|_g dt.
\]
The rest of the proof is as in [Leh16 Lemma 3.2].

Lemma 4.4. Let $f \in P_\eta(M)$ for some $\eta > 1$. If $I f = 0$ and $u^f$ is differentiable at $(x,v) \in SM$, then
\[
\begin{align*}
\hat{h} \nabla u^f(x,-v) &= (-1)^{m-1} \hat{h} \nabla u^f(x,v) \quad \text{and} \quad \hat{v} \nabla u^f(x,-v) = (-1)^m \hat{v} \nabla u^f(x,v).
\end{align*}
\]

Proof. From $I f = 0$ it follows that
\[
u^f(x,v) + (-1)^m u^f(x,-v) = 0.
\]
Fix $w \in S_xM$, $w \perp v$. We note that
\[
u^f(\phi^h_{w,s}(x,-v)) + (-1)^m u^f(\phi^h_{-w,-s}(x,v)) = 0
\]
and hence
\[
\left. \frac{d}{ds} u^f(\phi^h_{w,s}(x,-v)) \right|_{s=0} = -(-1)^m \left. \frac{d}{ds} (u^f(\phi^h_{-w,-s}(x,v))) \right|_{s=0} = (-1)^m \left. \frac{d}{ds} (u^f(\phi^h_{w,s}(x,v))) \right|_{s=0}.
\]
By Lemma 3.2
\[
\langle \hat{h} \nabla u^f(x,-v), w \rangle = (-1)^m \langle \hat{h} \nabla u^f(x,v), -w \rangle = -(-1)^m \langle \hat{v} \nabla u^f(x,v), w \rangle.
\]
For $\hat{v} \nabla u^f$ we use that
\[
u^f(\phi^v_{w,s}(x,-v)) + (-1)^m u^f(\phi^v_{-w,s}(x,v)) = 0
\]
and by Lemma 3.2 we get that
\[
\langle \hat{v} \nabla u^f(x,-v), w \rangle = (-1)^{m-1} \langle \hat{v} \nabla u^f(x,v), -w \rangle = (-1)^m \langle \hat{v} \nabla u^f(x,v), w \rangle.
\]

We move on to prove growth estimates for Jacobi fields. These estimates will be used to derive estimates for $\hat{h} \nabla u^f$ and $\hat{v} \nabla u^f$.

Lemma 4.5. Suppose $J(t)$ is a normal Jacobi field along a geodesic $\gamma$.
(a) If all sectional curvatures along $\gamma([0,\tau])$ are $\geq -K_0$ for some constant $K_0 > 0$, and if $J(0) = 0$ or $D_tJ(0) = 0$, then
\[
|J(t)| \leq |J(0)| \cosh(\sqrt{K_0}t) + |D_tJ(0)| \frac{\sinh(\sqrt{K_0}t)}{\sqrt{K_0}}
\]
for $t \in [0,\tau]$. 


(b) If $t_0 \in (0, \tau)$, then

$$|D_t J(t)| + \left| \frac{J(t)}{t} - D_t J(t) \right| \leq \left[ |D_t J(t_0)| + \left| \frac{J(t_0)}{t_0} - D_t J(t_0) \right| \right] e^{2 \int_{t_0}^t \kappa(s) \, ds}$$

for $t \in [t_0, \tau]$.

**Proof.** (a) follows from the Rauch comparison theorem [Jos85] Theorem 4.5.2. For (b), we follow the argument in [Leh10]. Consider an orthonormal frame $\{\hat{\gamma}(t), E_1(t), \ldots, E_{n-1}(t)\}$ obtained by parallel transporting an orthonormal basis of $T_{\gamma(0)}M$ along $\gamma$. Write $J(t) = u^j(t)E_j(t)$, so that the Jacobi equation becomes

(4.1) \[ \ddot{u}(t) + R(t)u(t) = 0 \]

where $u(t) = (u^1(t), \ldots, u^{n-1}(t))$ and $R_{jk} = R(E_j, \hat{\gamma}, \hat{\gamma}, E_k)$. We wish to estimate $v(t) = \frac{u(t)}{t}$, and we do this by writing $v(t) = A(t) + \frac{B(t)}{t}$ where

$$A(t) = \dot{u}(t), \quad B(t) = u(t) - t\dot{u}(t).$$

By using the equation, we see that

$$A(t) - A(t_0) = -\int_{t_0}^t sR(s)v(s) \, ds,$$

$$B(t) - B(t_0) = \int_{t_0}^t s^2 R(s)v(s) \, ds.$$

Write $g(t) = |A(t)| + \left| \frac{B(t)}{t} \right|$. If $t \geq t_0$ one has

$$g(t) = \left| A(t_0) - \int_{t_0}^t sR(s)v(s) \, ds \right| + \frac{1}{t} \left| B(t_0) + \int_{t_0}^t s^2 R(s)v(s) \, ds \right| \leq g(t_0) + 2 \int_{t_0}^t s\|R(s)\|g(s) \, ds.$$

The Gronwall inequality implies that

$$g(t) \leq g(t_0)e^{2 \int_{t_0}^t \|R(s)\| \, ds}.$$

The result follows from this, since $\|R(s)\| = \sup_{|\xi|=1} R(s)\xi \cdot \xi = \sup_{\gamma(s) \in \Pi} K(\Pi) \leq K(\gamma(s))$. \hfill \Box

**Corollary 4.6.** Suppose that $(M, g)$ is a Cartan-Hadamard manifold. Let $\gamma$ be a geodesic and $J$ a normal Jacobi field along it, satisfying either $J(0) = 0$ and $|D_t J(0)| \leq 1$ or $|J(0)| \leq 1$ and $D_t J(0) = 0$.

(a) If $-K_0 \leq K \leq 0$ and $K_0 > 0$, then

$$|J(t)| \leq C e^{\sqrt{K_0} t} \quad \text{and} \quad |D_t J(t)| \leq C e^{\sqrt{K_0} t}$$

for $t \geq 0$ where the constants do not depend on the geodesic $\gamma$.

(b) If $K \in P_\kappa(M)$ for some $\kappa > 2$, then

$$|J(t)| \leq C(t + 1) \quad \text{and} \quad |D_t J(t)| \leq C$$

for $t \geq 0$. If in addition $\gamma \in E_0$, then the constants do not depend on the geodesic $\gamma$.

**Proof.** (a) The estimate for $|J(t)|$ follows directly from Lemma 4.3. Using the same notations as in the proof of that Lemma we have $|D_t J(t)| = |\dot{u}(t)|$ and by integrating (4.1) from 0 to $t$ we get

$$|\dot{u}(t)| \leq |\dot{u}(0)| + \int_0^t \|R(s)\|u(s)\| \, ds$$

$$\leq |D_t J(0)| + \int_0^t K_0|J(s)| \, ds$$

$$\leq C e^{\sqrt{K_0} t}.$$
Lemma 4.7. Suppose that $t$ for $s$ and suppose that $\Gamma(t)$ for $u$ for $t$ $u$

Proof of (4.2): Let us fix $t_0 = 1$ and suppose that $J$ is a Jacobi field along a geodesic in $E_o$ whose initial values satisfy the given assumptions. From Lemma 4.5 and (a) we then get that

$$|J(t)| \leq e^{2A} (2 |D_t J(1)| + |J(1)|) t$$

$$\leq e^{2A} C e^{\sqrt{K_o} t}$$

for $t \geq 1$, where $K_0 = \sup_{x \in M} |\mathcal{K}(x)|$.

For $t \in [0, 1]$ we can estimate $|J(t)| \leq C e^{\sqrt{K_o}}$. By combining these two estimates we get

$$|J(t)| \leq C (1 + e^{2A}) t$$

for $t \geq 0$, and the constants do not depend on $\gamma \in E_o$.

For $|D_t J(t)|$, Lemma 4.5 gives the estimate

$$|D_t J(t)| \leq e^{2A} (2 |D_t J(1)| + |J(1)|)$$

for $t \geq 1$, and for $t \in [0, 1]$ we get a bound from (a). Neither of these bounds depends on $\gamma \in E_o$. □

Lemma 4.7. Suppose that $f = 0$.

(a) If $-K_0 \leq K \leq 0$, $K_0 > 0$ and $f \in E_{1\eta}^1(M)$ for some $\eta > \sqrt{K_o}$, then $u^f$ is differentiable along every geodesic on $SM$, $u^f \in W^{1, \infty}(SM)$ and

$$\left| \frac{h}{\nabla} u^f(x,v) \right| \leq C e^{-\eta - \sqrt{K_o} d_g(x,o)}$$

for a.e. $x \in SM$.

(b) If $\mathcal{K} \in P_{\kappa}(M)$ for some $\kappa > 2$ and $f \in P_{1\eta}^1(M)$ for some $\eta > 1$, then $u^f$ is differentiable along every geodesic on $SM$, $u^f \in W^{1, \infty}(SM)$ and

$$\left| \frac{h}{\nabla} u^f(x,v) \right| \leq \frac{C}{(1 + d_g(x,o))^\eta - 1}$$

for a.e. $(x,v) \in SM$.

The same estimates hold for $\nabla u^f$ with the same assumptions.

Proof of $u^f \in W^{1, \infty}_{loc}(SM)$. We show that $u^f$ is locally Lipschitz continuous. Fix $(x_0, v_0) \in SM$, and suppose that $\Gamma(s)$ is a unit speed geodesic on $SM$ through $(x_0, v_0)$. We have

$$\frac{u^f(\Gamma(r)) - u^f(\Gamma(0))}{r} = \int_0^r f(\phi_t(\Gamma(s))) - f(\phi_t(\Gamma(0))) \frac{dr}{r}$$

$$= \int_0^r \frac{1}{r} \int_0^r \frac{\partial}{\partial s} [f(\phi_t(\Gamma(s)))] ds dt$$

$$= \int_0^r \frac{1}{r} \int_0^r \langle \nabla_{SM} f(\phi_t(\Gamma(s))), D\phi_t(\Gamma(s))\tilde{v}(s) \rangle ds dt.$$
When we apply Corollary 3.4 to the right hand side of (4.2) (and omit the identifications), we find that

\[
\frac{u^I(\Gamma(t)) - u^I(\Gamma(0))}{t} = \int_0^\infty \frac{1}{r} \int_0^r \left[ Xf(\phi_t(\Gamma(s)))\langle \hat{\Gamma}(s), X \rangle \\
+ \langle \nabla f(\phi_t(\Gamma(s)))\rangle, |\hat{\Gamma}(s)|^h J_{h_{w_h(s)}}(t) + |\hat{\Gamma}(s)|^v J_{v_{w_v(s)}}(t) \rangle \\
+ \langle \nabla f(\phi_t(\Gamma(s)))\rangle, |\hat{\Gamma}(s)|^h D_t J_{h_{w_h(s)}}(t) + |\hat{\Gamma}(s)|^v D_t J_{v_{w_v(s)}}(t) \rangle \right] \, ds \, dt
\]

(4.3)

where \( w_h(s) = \hat{\Gamma}(s)^h / |\hat{\Gamma}(s)^h| \) and \( w_v(s) = \hat{\Gamma}(s)^v / |\hat{\Gamma}(s)^v| \). Here the Jacobi fields are along the geodesic \( \gamma_{\Gamma(s)}(t) := \pi(\phi_t(\Gamma(s))) \). By definition their initial values fulfill the assumptions of Corollary 4.6.

From this point on we will work under assumptions of (b). The proof under assumptions of (a) is similar but simpler. We fix a small \( \varepsilon > 0 \). We show that the integral (4.3) has a uniform upper bound for every \( r \in (0, 1] \) and every geodesic \( \Gamma \) through a point in \( B(x_0,v_0)(\varepsilon) \subset SM \). For \( (x,v) \in SM \) we denote by \( G(x,v) \) the set of unit speed geodesics on \( SM \) through \( (x,v) \), and define

\[
J(x_0,v_0,\varepsilon) := \{ \Gamma \in G(x,v) \mid (x,v) \in B(x_0,v_0)(\varepsilon) \}.
\]

For all \( \Gamma \in J(x_0,v_0,\varepsilon), \Gamma(0) = (x,v) \), and \( s \in (0,r] \) the estimate (3.9) gives that \( d_g(x,x_0) \leq \varepsilon \) and

\[
d_g(\gamma_{\Gamma(s)}(0), x) = d_g(\pi(\Gamma(s)), x) \leq d_g(\Gamma(s), (x,v)) \leq s.
\]

The estimate (3.1) implies that

\[
d_g(\pi(\phi_t(\Gamma(s))), o) = d_g(\gamma_{\Gamma(s)}(t), o) \geq t - d_g(\gamma_{\Gamma(s)}(0), x_0) \\
\geq t - \sup_{s \in (0,r]} d_g(\gamma_{\Gamma(s)}(0), o) \geq t - d_g(x,o) - r \\
\geq t - d_g(x_0, o) - \varepsilon - r
\]

for all \( t \geq t_0 \) where \( t_0 := d_g(x_0, o) + r + \varepsilon \). We can use a trivial estimate \( d_g(\pi(\phi_t(\Gamma(s))), o) \geq 0 \) on the interval \([0,t_0]\). Further, the estimate (4.4) gives

\[
K(\gamma_{\Gamma(s)}(t)) \leq \frac{C}{(1 + d_g(\gamma_{\Gamma(s)}(t), o))^\eta} \leq \frac{C}{(1 + t - d_g(x_0, o) - \varepsilon - r)^\eta}
\]

(4.5)

for all \( t \geq t_0 \) where the constant \( C \) does not depend on \( s \in (0,r] \) or the geodesic \( \Gamma \in J(x_0,v_0,\varepsilon) \), and hence

\[
\sup_{\Gamma \in J(x_0,v_0,\varepsilon)} \int_0^\infty tK(\gamma_{\Gamma(s)}(t)) \, dt < \infty.
\]

Using the proof of Corollary 4.6 together with (4.6), we can find a constant \( C \) which does not depend on \( s \in (0,r] \) so that one has

\[
|J_{h_{w_h(s)}}(t)| \leq Ct, \quad |D_t J_{h_{w_h(s)}}(t)| \leq C
\]

(4.4)

for all \( t \geq 0 \) and \( \Gamma \in J(x_0,v_0,\varepsilon) \). Similar estimates hold also uniformly for \( J_{v_{w_v(s)}}(t) \) and \( D_t J_{v_{w_v(s)}}(t) \).

Recall that \( |\hat{\Gamma}(s)^h|, |\hat{\Gamma}(s)^v| \leq |\hat{\Gamma}(s)| = 1 \), and that \( w_h(s), w_v(s) \) depend on \( \Gamma \). By combining the above estimates for Jacobi fields with estimate (4.4) and Lemma 3.6 we get for the integrand
in (4.3) that

\[
|Xf(\hat{\phi}_t(\Gamma(s)))| \leq |Xf(\hat{\phi}_t(\Gamma(s)))| + |\hat{\phi}_t(\Gamma(s))|J^h_{\Gamma(s)} (t) + |\hat{\phi}_t(\Gamma(s))|J^v_{\Gamma(s)} (t)
\]

for all \( t \in [0, \infty) \), \( s \in (0, r] \) and \( \Gamma \in J(x_0, v_0, \varepsilon) \). On the interval \([0, t_0]\) we also get a uniform upper bound since \( f \), its covariant derivative and sectional curvatures are all bounded.

We can conclude that integral on the right hand side of (4.3) converges absolutely with some uniform bound \( C < \infty \) over \( t \in (0, 1] \) and the set \( J(x_0, v_0, \varepsilon) \). This shows that \( u^f \) is locally Lipschitz, i.e. \( u^f \in W^{1,\infty} (SM) \) (cf. Remark 1). Moreover, the uniform estimate together with the dominated convergence theorem guarantees that the limit \( r \to 0 \) of (4.2) exists for all geodesics \( \Gamma \) on \( SM \). This finishes the first part of the proof.

**Proof of the gradient estimates.** By Rademacher’s theorem \( u^f \) is differentiable almost everywhere, and thus we can assume that \( u^f \) is differentiable at \((x, v) \in SM\). By Lemmas 3.1 and 3.2 we can assume that \((x, v) \) satisfies \( \gamma = \gamma_{x, v} \in \mathcal{E}_o \). We may also assume that \( \nabla u^f(x, v) \neq 0 \). Since \( \langle \nabla u^f(x, v), v \rangle = 0 \), we can take \( w = \nabla u^f(x, v) / |\nabla u^f(x, v)| \) in Lemma 3.2, and get that

\[
\left| \frac{h}{ds} u^f(x, v) \right| = \frac{d}{ds} u^f(\phi^{h}_{s,0}(x, v)) \bigg|_{s=0}
\]

(4.8)

where \( J^h \) is again a Jacobi field along \( \gamma \) fulfilling the assumptions of Corollary 4.6.

Under the conditions in part (a), the estimate (4.3) implies

\[
\left| \frac{h}{ds} u^f(x, v) \right| \leq Ce \int_0^\infty e^{-\eta d_g(\gamma(t), o)} e^{\sqrt{K_0} t} dt \leq \int_0^\infty e^{-\eta \sqrt{d_g(\gamma(t), o)^2 + r^2}} e^{\sqrt{K_0} t} dt.
\]

Writing \( r = d_g(x, o) \) and splitting the integral over \([0, r] \) and \([r, \infty) \) gives

\[
\left| \frac{h}{ds} u^f(x, v) \right| \leq Ce \int_0^r e^{-\eta r \sqrt{K_0}} dt + \int_0^\infty e^{-\eta t \sqrt{K_0}} dt \leq Ce^{-(\eta - \sqrt{K_0}) r d_g(x, o)}.
\]
The above estimate also shows that $|\nabla u^f|$ is bounded. Similarly, under the conditions in part (b), Lemma 3.6, Corollary 4.6 and (3.3) imply
\[
\left|\frac{h}{\nabla u^f(x,v)}\right| \leq C \int_0^\infty \frac{1 + t}{(1 + d_g(\gamma(t), o))^{\eta+1}} \, dt + C \int_0^\infty \frac{C}{(1 + d_g(\gamma(t), o))^\eta} \, dt
\]
\[
\leq C \left[ \int_0^r \frac{1}{(1 + r)^{\eta+1}} \, dt + \int_r^\infty \frac{1 + t}{(1 + t)^{\eta+1}} \, dt \right] \leq C(1 + r)^{-(\eta-1)}
\]
where $r = d_g(x, o)$. The same arguments apply to $\nabla u^f$. Hence $u^f \in W^{1,\infty}(SM)$ in the both cases, (a) and (b).

\[\square\]

**Lemma 4.8.**
(a) If $-K_0 \leq K \leq 0$ and $K_0 > 0$, then
\[
\text{Vol } S_o(r) \leq C e^{(n-1)\sqrt{Ko}r}
\]
for all $r \geq 0$.
(b) If $K \in P_\kappa(M)$ for $\kappa > 2$, then
\[
\text{Vol } S_o(r) \leq C r^{n-1}
\]
for all $r \geq 0$.

\[\text{Proof.}\] We define the mapping $f : S_oM \to S_o(r)$,
\[
f(v) = (\pi \circ \phi_r)(o, v) = \exp_o(rv).
\]
We denote by $d\Sigma$ the volume form on $S_o(r)$ and have that
\[
\text{Vol } S_o(r) = \int_{S_o(r)} d\Sigma = \int_{S_o(r)} f^*(d\Sigma) = \int_{S_oM} \mu \, dS,
\]
where $dS$ denotes the volume form on $S_oM$ (induced by Sasaki metric) and $\mu : S_oM \to \mathbb{R}$.

Let $v \in S_oM$ and $\{w_i\}_{i=1}^{n-1}$ be an orthonormal basis for $T_vS_oM$ with respect to Sasaki metric. By the Gauss lemma $\{d_vf(w_i)\}_{i=1}^{n-1}$ is an orthonormal basis for $T_{f(v)}S_o(r)$ and
\[
f^*(d\Sigma)_v(w_1, \ldots, w_{n-1}) = d\Sigma_{f(v)}(d_vf(w_1), \ldots, d_vf(w_{n-1})).
\]
It holds that $d_vf(w_i) = J_i(r)$ where $J_i$ is a Jacobi field along the geodesic $\gamma_o,v$ with initial values $J(0) = d_v\pi(w_i)$ and $D_tJ_i(0) = K\nabla(w_i)$. We get that
\[
|\mu(v)| \leq \prod_{i=1}^{n-1} |d_vf(w_i)| = \prod_{i=1}^{n-1} |J_i(r)|.
\]

Since the tangent vectors $w_i$ lie in $\mathcal{V}(o,v)$ we have $|J_i(0)|_{g_o} = 0$ and $|D_tJ_i(0)|_{g_o} = |w_i|_{g_o} = 1$, and the estimates for the volume of $S_o(r)$ then follow from Corollary 4.6.

\[\square\]

5. **Proof of the main theorems**

We begin by introducing some useful notation related to operators on the sphere bundle and spherical harmonics. One can find more details in [GK80b], [DS10] and [PSU15]. We prove the main theorems of this work in the end of this section.

The norm $\| \cdot \|$ in this section will always be the $L^2(SM)$-norm. We define the Sobolev space $H^1(SM)$ as the set of all $u \in L^2(SM)$ for which $\|u\|_{H^1(SM)} < \infty$, where
\[
\|u\|_{H^1(SM)} = \left( \|u\|^2 + \|\nabla_{SM} u\|^2 \right)^{1/2}
\]
\[
= \left( \|u\|^2 + \|Xu\|^2 + \|\nabla u\|^2 + \|\nabla u\|^2 \right)^{1/2}.
\]
Let $C^\infty_c(SM)$ denote the smooth compactly supported functions on $SM$. It is well known that if $N$ is complete Riemannian manifold, then $C^\infty_c(N)$ is dense in $H^1(N)$ (see [Eic88, Satz 2.3]). By Lemma 3.5 $SM$ is complete when $M$ is complete. Hence $C^\infty_c(SM)$ is dense in $H^1(SM)$.

For the following facts see [PSU15]. The vertical Laplacian $\Delta : C^\infty(SM) \to C^\infty(SM)$ is defined as the operator

$$\Delta := -\text{div}\nabla.$$ The Laplacian $\Delta$ has eigenvalues $\lambda_k = k(k + n - 2)$ for $k = 0, 1, 2, \ldots$, and its eigenfunctions are homogeneous polynomials in $v$. One has an orthogonal eigenspace decomposition

$$L^2(SM) = \bigoplus_{k \geq 0} H_k(SM),$$

where $H_k(SM) := \{f \in L^2(SM) : \Delta f = \lambda_k f\}$. We define $\Omega_k = H_k(SM) \cap H^1(SM)$. In particular, by Lemma 5.1 below any $u \in H^1(SM)$ can be written as

$$u = \sum_{k=0}^{\infty} u_k, \quad u_k \in \Omega_k,$$

where the series converges in $L^2(SM)$.

One can split the geodesic vector field in two parts, $X = X_+ + X_-$, so that (by Lemma 5.1) $X_+ : \Omega_k \to H_{k+1}(SM)$ and $X_- : \Omega_k \to H_{k-1}(SM)$. The next lemma gives an estimate for $X_\pm u$ in terms of $Xu$ and $\nabla^h u$.

**Lemma 5.1.** Suppose $u \in H^1(SM)$. Then $X_\pm u \in L^2(SM)$ and

$$\|X_+ u\|^2 + \|X_- u\|^2 \leq \|Xu\|^2 + \|\nabla^h u\|^2.$$

Moreover, for each $k \geq 0$ one has $u_k \in H^1(SM)$, and there is a sequence $(u_k^{(j)})_{j=1}^\infty \subset C^\infty_c(SM) \cap H_k(SM)$ with $u_k^{(j)} \to u_k$ in $H^1(SM)$ as $j \to \infty$.

**Proof.** Let $u \in C^\infty_c(SM)$. One has the decomposition

$$\nabla^h u = \nabla \left[ \sum_{l=1}^{\infty} \left( \frac{l}{l+n-2} X_+ u_{l-1} - \frac{1}{l+n-2} X_- u_{l+1} \right) \right] + Z(u)$$

where $Z(u)$ is such that $\text{div} Z(u) = 0$ (see [PSU15, Lemma 4.4]). Hence

$$\|\nabla^h u\|^2 = \sum_{l=1}^{\infty} \left( \frac{l(l+n-2)}{l} \|X_+ u_{l-1}\|^2 + \frac{1}{l+n-2} \|X_- u_{l+1}\|^2 \right) + \|Z(u)\|^2$$

$$= \sum_{l=1}^{\infty} \left( \frac{l+n-2}{l} \|X_+ u_{l-1}\|^2 - 2 \langle X_+ u_{l-1}, X_- u_{l+1} \rangle + \frac{l}{l+n-2} \|X_- u_{l+1}\|^2 \right) + \|Z(u)\|^2.$$ 

We also have

$$\|X u\|^2 = \|X_- u_1\|^2 + \sum_{l=1}^{\infty} \left( \|X_+ u_{l-1} + X_- u_{l+1}\|^2 \right)$$

$$= \|X_- u_1\|^2 + \sum_{l=1}^{\infty} \left( \|X_+ u_{l-1}\|^2 + 2 \langle X_+ u_{l-1}, X_- u_{l+1} \rangle + \|X_- u_{l+1}\|^2 \right)$$

$$= \sum_{l=1}^{\infty} \left( \frac{l+n-2}{l} \|X_+ u_{l-1}\|^2 - 2 \langle X_+ u_{l-1}, X_- u_{l+1} \rangle + \frac{l}{l+n-2} \|X_- u_{l+1}\|^2 \right) + \|Z(u)\|^2.$$
by the definition of $X_+$ and $X_-$. Adding up these estimates gives that
\[
\|Xu\|^2 + \|\nabla u\|^2 = \|Z(u)\|^2 + \|X_u u_1\|^2 + \sum_{l=1}^{\infty} (A(n,l)\|X_{u_{l-1}}\|^2 + B(n,l)\|X_{u_{l+1}}\|^2)
\]
where $A(n,l) = 2 + \frac{n-2}{2}$ and $B(n,l) = 1 + \frac{l}{n+2}$. Since $A(n,l) \geq 1$ and $B(n,l) \geq 1$ for all $l = 1, 2, \ldots$ and $n \geq 2$, the estimate for $\|X_u u\|^2 + \|X_u u\|^2$ follows when $u \in C_c^{\infty}(SM)$, and it extends to $H^1(SM)$ by density and completeness.

Moreover, if $u \in C_c^{\infty}(SM)$ and if $k \geq 0$, then the triangle inequality $\|X u_k\| \leq \|X u_k\| + \|X_{-u_k}\|$ and orthogonality imply that
\[
\|u_k\| + \|X u_k\| + \|\nabla u_k\| \leq \|u\| + \|X u\| + \|\nabla u\|.
\]

We may also estimate $\|\nabla u_k\|$ by [PSU15, Proposition 3.4] and orthogonality to obtain
\[
\|\nabla u_k\|^2 \leq (2k + n - 1)\|X u_k\|^2 + (\sup_{M} K)\|\nabla u_k\|^2 \leq C_k(\|X u\|^2 + \|\nabla u\|^2).
\]

It follows from the first part of this lemma that
\[
\|u_k\|_{H^1(SM)} \leq C_k\|u\|_{H^1(SM)}, \quad u \in C_c^{\infty}(SM).
\]

This extends to $u \in H^1(SM)$ by density and completeness. Finally, if $u \in H^1(SM)$ and the sequence $(u^{(j)}) \subset C_c^{\infty}(SM)$ satisfies $u^{(j)} \to u$ in $H^1(SM)$, then also $u_k^{(j)} \to u_k$ in $H^1(SM)$ by the above inequality.

**Corollary 5.2.** Suppose $u \in H^1(SM)$. Then
\[
\lim_{k \to \infty} \|X u_k\|_{L^2(SM)} = 0.
\]

**Proof.** By Lemma 5.1 one has
\[
\|X u\|^2 = \sum_{k=0}^{\infty} \|X u_k\|^2 < \infty
\]
which implies the claim. \(\square\)

**Lemma 5.3.** Let $u \in H^1(SM)$ and $k \geq 1$. Then one has that
\[
\|X_{-u_k}\| \leq D_n(k)\|X u_k\|
\]
where
\[
D_2(k) = \begin{cases}
\sqrt{2}, & k = 1 \\
1, & k \geq 2,
\end{cases}
\]
\[
D_3(k) = \left[1 + \frac{1}{(k+1)^2(2k-1)}\right]^{1/2}
\]
\[
D_n(k) \leq 1 \quad \text{for} \ n \geq 4.
\]

**Proof.** This result was shown for smooth compactly supported functions in [PSU15, Lemma 5.1]. The result follows for $u \in H^1(SM)$ by an approximation argument using Lemma 5.1. \(\square\)

The estimates from Section 4 allow us to prove the following result:

**Lemma 5.4.** Suppose that $f$ is a symmetric $m$-tensor field and either of the following holds:

(a) $-K_0 \leq K \leq 0$, $K_0 > 0$ and $f \in E_1^\eta(M)$ for $\eta > \frac{(n+1)\sqrt{K_0}}{2}$

(b) $K \in P_\kappa(M)$ for $\kappa > 2$ and $f \in P_1^\eta(M)$ for $\eta > \frac{n+2}{2}$.
Then \( u^f \in H^1(SM) \).

**Proof.** We prove only (a), the proof for (b) is similar. By Lemma \ref{lem:4.7} we have that \( u^f \in W^{1,\infty}(SM) \).

Lemma \ref{lem:4.3} gives that
\[
|u^f(x,v)| \leq C(1 + d_g(x,o))e^{-\eta d_g(x,o)}
on SM. \]

By using the coarea formula with Lemma \ref{lem:4.8} we get
\[
\int_{SM} |u^f(x,v)|^2 \, dV_g \leq C \int_M (1 + d_g(x,o))^2 e^{-2\eta d_g(x,o)} \, dV_g
= C \int_0^\infty (1 + r)^2 e^{-2\eta r} \left( \int_{S_o(r)} dS \right) \, dr
\leq C \int_0^\infty (1 + r)^2 e^{-2\eta r} e^{(n-1)\sqrt{K_0}} r \, dr.
\]

The last integral above is finite and hence \( u^f \in L^2(SM) \). Similar calculations using Lemmas \ref{lem:4.2} and \ref{lem:4.7} show that \( Xu^f, \nabla u^f \) and \( \nabla u^f \) all have finite \( L^2 \)-norms under the assumption \( \eta > \frac{(n+1)\sqrt{K_0}}{2} \), and therefore the \( H^1 \)-norm of \( u^f \) is finite. \( \square \)

We are ready to prove our main theorems.

**Proof of Theorems \ref{thm:1.1} and \ref{thm:1.2}** Suppose that the \( m \)-tensor field \( f \) and the sectional curvature \( K \) satisfy the assumptions of Theorem \ref{thm:1.1} or \ref{thm:1.2} Recall that we identify \( f \) with a function on \( SM \) as described in Section \ref{sec:3.3}. Then \( u = u^f \) is in \( H^1(SM) \) by Lemma \ref{lem:5.4} and Lemma \ref{lem:4.2} states that \( Xu = -f \) on \( SM \). Note also that \( f \in H^1(SM) \), which follows as in the proof of Lemma \ref{lem:5.4}.

Since \( f \) is of degree \( m \) it has a decomposition
\[
f = \sum_{k=0}^m f_k, \quad f_k \in \Omega_k,
\]
and \( u \) has a decomposition
\[
u = \sum_{k=0}^\infty u_k, \quad u_k \in \Omega_k.
\]

We first show that \( u_k = 0 \) for \( k \geq m \). From \( Xu = -f \) it follows that for \( k \geq m \) we have
\[
X_+ u_k + X_- u_{k+2} = 0.
\]

This implies that
\[
\|X_+ u_k\| \leq ||X_- u_{k+2}||, \quad k \geq m.
\]

Fix \( k \geq m \). We apply Lemma \ref{lem:5.3} and the inequality \ref{eq:5.1} iteratively to get
\[
\|X_- u_k\| \leq D_n(k)\|X_+ u_k\|
\leq D_n(k)\|X_- u_{k+2}\|
\leq D_n(k)D_n(k + 2)\|X_+ u_{k+2}\|
\leq \left[ \prod_{l=0}^N D_n(k + 2l) \right] \|X_+ u_{k+2N}\|.
\]

By Corollary \ref{cor:5.2}
\[
\lim_{l \to \infty} \|X_+ u_{k+2l}\| = 0.
\]
Moreover, as stated in [PSU15, Theorem 1.1], one has
\[ \prod_{l=0}^{\infty} D_n(k + 2l) < \infty. \]
Thus we obtain that
\[ \| X_k \| = \| X_k \| = 0. \]
This gives \( Xu_k = 0 \), which implies that \( t \mapsto u_k(\varphi_t(x, v)) \) is a constant function on \( \mathbb{R} \) for any \((x, v) \in SM\). Since \( u \) decays to zero along any geodesic we must have \( u_k = 0 \), and this holds for all \( k \geq m \).

It remains to verify that the equation \( Xu = -f \) on \( SM \) together with the fact \( u = \sum_{k=0}^{m-1} u_k \) imply the conclusions of Theorems 1.1 and 1.2. This is done as in [PSU13, end of Section 2]. Suppose that \( m \) is odd (the case where \( m \) is even is similar). The function \( f \) is a homogeneous polynomial of order \( m \) in \( v \) and hence its Fourier decomposition has only odd terms, i.e.
\[ f = f_m + f_{m-2} + \cdots + f_1. \]
It follows that the decomposition of \( u \) has only even terms,
\[ u = u_{m-1} + u_{m-3} + \cdots + u_0. \]

By taking tensor products with the metric \( g \) and symmetrizing it is possible to raise the degree of a symmetric tensor: if \( F \in S^m(M) \), then \( \alpha F := \sigma(F \otimes g) \in S^{m+2}(M) \). One has \( \lambda(\alpha F) = \lambda(F) \), since \( \lambda(g) \) has a constant value 1 on \( SM \).

We define \( h \in S^{m-1}(M) \) by
\[ h := -\sum_{j=0}^{(m-1)/2} \alpha^j(\lambda^{-1}(u_{m-1-2j})). \]
Then \( \lambda(h) = -u \), so equation (3.10) gives \( \lambda(\sigma \nabla h) = \lambda(f) \), which implies \( f = \sigma \nabla h \).

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