Anchored expansion of Delaunay complexes in real hyperbolic space and stationary point processes

Itai Benjamini\textsuperscript{1} · Yoav Krauz\textsuperscript{2} · Elliot Paquette\textsuperscript{3}

This paper is dedicated to Harry Kesten, whom the first author holds in his loving memory and whom we all admire

Received: 16 July 2020 / Revised: 23 June 2021 / Accepted: 24 June 2021 / Published online: 23 July 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
We give sufficient conditions for a discrete set of points in any dimensional real hyperbolic space to have positive anchored expansion. The first condition is an anchored bounded density property, ensuring not too many points can accumulate in large regions. The second is an anchored bounded vacancy condition, effectively ensuring there is not too much space left vacant by the points over large regions. These properties give as an easy corollary that stationary Poisson–Delaunay graphs have positive anchored expansion, as well as Delaunay graphs built from stationary determinantal point processes.

Mathematics Subject Classification Primary 60D05; Secondary 52C20 · 60G55

1 Introduction

While still in graduate school, Harry Kesten initiated the study of random walk on groups and famously related nonamenability to strictly positive spectral gap in a truly pioneering work \cite{15}. Decades later he was leading the study of random walk in random
environment and of the geometry of randomly diluted media, and in particular the
theory of percolation. In the following note we study the geometry of random discrete
triangulations of hyperbolic spaces and of the speed of random walk on them. We
prove a type of relaxed spectral gap, anchored nonamenability, which is known to
have consequences for the speed of random walk and for return probability estimates,
a central topic in Kesten’s work.

Much of what we do is inspired by the construction of probabilistic relaxations
of lattices. Lattices in hyperbolic space (see [11] for an introduction, c.f. [10,12]) are
group theoretic constructions whose coarse geometric properties mirror the underlying
space. As a purely geometric consequence, any real hyperbolic space $\mathbb{H}^d$ contains
collections of points $S$ whose Delaunay graphs, defined below in Sect. 1.1, have a
linear isoperimetric inequality:

$$\inf_{V \subset S, |V| < \infty} \frac{|\partial V|}{|V|} > 0,$$

(1)

where $\partial V$ denotes the edges of this Delaunay graph with exactly one vertex in $V$. This
follows from the nonamenability of isometry group of $S$ which in turn is a relatively
simple consequence of the nonamenability of the ambient space ([11]).

However lattices are also quite special, for example there are not arbitrarily fine
lattices in $\mathbb{H}^d$ (in 2 dimensions this can be seen from applying Gauss–Bonnet—in
higher dimensions this is the Kazhdan–Margulis theorem). There is a larger class
of quasisymmetric analogues, such as aperiodic tilings and quasi-lattices, which we
define presently, of the hyperbolic plane which are good coarse approximations of $\mathbb{H}^d$
(see [9,17,20]).

A subset $S \subset \mathbb{H}^d$ is called coarsely dense if there is a real number $c > 0$ so that
every $y \in \mathbb{H}^d$ is less than distance $c$ from a point of $S$. On the other hand, a subset
$S \subset \mathbb{H}^d$ is called coarsely discrete if for every $r > 0$ there is a $K_r$ so that
$$|S \cap B_r(y)| \leq K_r$$

for every $y \in \mathbb{H}^d$. Define a quasi-lattice $S \subset \mathbb{H}^d$ as a set which is both coarsely dense
and coarsely discrete. As a corollary of [9, Theorem 3.1], any quasi-lattice $S \subset \mathbb{H}^d$ is
nonamenable in that (1) holds.

Another possible quasisymmetric generalization of a lattice is to replace $S$ by a
random collection of points whose law is invariant under the isometries of $\mathbb{H}^d$—a
stationary point process. As a principal motivating example, suppose that we consider a
Poisson point process $X$ with invariant intensity measure. Will this too be nonamenable
in the sense of (1)?

This is too much to ask. Because many Poisson points could arrive in any small ball,
and these points can be arranged in any way, there will be somewhere in the Delaunay
graph of $X$ a finite subgraph isomorphic to any neighborhood in a $d$-dimensional
Euclidean lattice.

However, there are relaxations of (1) that may still be satisfied by the Delaunay graph
of $X$. One natural such condition is anchored nonamenability. Say that a connected
graph $G$ has positive anchored expansion if for some fixed vertex $\rho$ in $G$,
Anchored expansion of Delaunay complexes...

\[
\inf_{V \ni \rho, V \text{ connected}} \frac{|\partial\text{out} V|}{|V|} > 0, \tag{2}
\]

and where \(\partial\text{out} V\) is the subset of \(V\) with a neighbor outside of \(V\). Say that a graph has anchored nonamenability if it has positive anchored expansion. Anchored nonamenability yields some attractive corollaries for random walks and other random processes (see [14,22,23]—note that in the weak form that we have formulated it, some standard corollaries may only hold under additional control on the degrees of the vertices). For unimodular random graphs embedded in \(\mathbb{H}^d\), almost sure anchored nonamenability implies that random walk has positive hyperbolic speed almost surely.\(^1\)

We will give a general criterion for any subset \(S \subset \mathbb{H}^d\) to be anchored nonamenable. Moreover, this criterion will be satisfied for any stationary Poisson process on \(\mathbb{H}^d\).

1.1 Voronoi and Delaunay complexes

We recall here the definitions of the Voronoi tessellation and Delaunay complex for a discrete set of points \(S \subset \mathbb{H}^d\). The Voronoi tessellation is a partition of \(\mathbb{H}^d\) into cells, each of which is a hyperbolic polyhedron containing exactly one point of \(S\), its nucleus. For a point \(p \in S\), the Voronoi cell is given by

\[
\left\{ z \in \mathbb{H} : d_{\mathbb{H}}(z, p_0) = \min_{p \in S} d_{\mathbb{H}}(z, p) \right\},
\]

where \(d_{\mathbb{H}}\) denotes distance in the hyperbolic metric.

The Delaunay complex is a dual cell complex to the Voronoi tessellation: a collection of points \(U \subset S\) having \(|U| = d + 1\) is a \(d\)-dimensional simplex if and only if there exists an open ball \(V\) disjoint from \(S\) so that \(\partial V \supset U\). When the points \(S\) are in generic position, meaning there is no sphere containing more than \(d + 1\) points of \(S\), then the Delaunay complex canonically embeds into \(\mathbb{H}^d\) by representing each \(d\)-dimensional simplex by a hyperbolic simplex with vertices from \(S\). The Delaunay graph (the 1-skeleton of the Delaunay complex) can be succinctly represented by the edge set on \(S\) in which two vertices \(x\) and \(y\) are adjacent if and only if there is an open ball \(V\) disjoint from \(S\) with closure containing \(x\) and \(y\).

1.2 Main result

To formulate our result, we will need to make use of a fixed cocompact lattice \(\Gamma\) (see for example [18, Theorem 18.45]). These guarantee the existence of a Voronoi tessellation of \(\mathbb{H}^d\) into compact, convex polyhedra so that the isometries of \(\mathbb{H}^d\) act

---

\(^1\) For unimodular random graphs, anchored nonamenability implies the weaker invariant nonamenability. When the degree of root of such an invariantly nonamenable graph has finite expectation, random walk will escape from the root with linear rate in any (stationary) pseudometric on the graph in which balls grow subexponentially. In particular, random walk will have positive speed almost surely in the hyperbolic metric when run on a stationary point process (c.f. the proof of [19, Corollary 1.10] or [2]).
transitively on the set of polyhedra. As we will typically need to refer to the cells of the tessellation, we will use $\Gamma$ to denote the collection of these polyhedra.

We let $A \in \Gamma$ be any fixed cell. Say that two cells in $\Gamma$ are adjacent if their intersection has codimension-1. Let $L_A$ be the set of connected finite subsets of $\Gamma$ containing $A \in \Gamma$. We say that a set of points $S \subset \mathbb{H}^d$ has \textit{anchored bounded density} if

$$\sup_{J \in L_A} \frac{|S \cap (\bigcup_{B \in J} B)|}{|J|} < \infty,$$

Conversely, we say that a set of points $S \subset \mathbb{H}^d$ has \textit{anchored bounded vacancy} if there is an $M > 0$ so that

$$\inf_{J \in L_A} \frac{|\{B \in \Gamma : d_{\mathbb{H}}(B, J) \leq M, |B \cap S| > 0\}|}{|J|} > 0.$$

Note both of these definitions are independent of the choice of the base cell $A$ as well as the choice of cocompact lattice $\Gamma$. The lattice $\Gamma$ could just well be replaced by the Voronoi tessellation of any quasi-lattice as well. We further observe that any collection of points $S \subset \mathbb{H}^d$ that has both anchored bounded density and vacancy still have both of these properties after \textit{thinning}, i.e. randomly and independently removing each point of $S$ with any fixed probability $p > 0$.

Our main theorem is:

**Theorem 1.1** Suppose that $S \subset \mathbb{H}^d$ has both anchored bounded density and anchored bounded vacancy, then the Delaunay graph on $S$ has anchored nonamenability.

A Poisson point process satisfies both of these conditions easily (see Lemma 4.2), from which we derive:

**Corollary 1.2** Let $\Pi^\lambda$ be a Poisson point process on $\mathbb{H}^d$ with intensity measure which is a positive multiple $\lambda$ of hyperbolic volume measure. The Poisson–Delaunay graph with nuclei $\Pi^\lambda$ is almost surely anchored nonamenable.
It is also easy to verify that any **determinantal point process** with invariant intensity measure has anchored bounded density and anchored bounded vacancy (using negative association and standard tail properties of these properties, c.f. [16, Theorem 6.5]). We note that there is a single well-studied stationary determinantal point process on the hyperbolic plane, the hyperbolic GAF [21], but in fact there are many on the plane and many on higher dimensional real-hyperbolic space as well (see the extended version of this paper [5, Section 5]).

### 1.3 Related work and discussion

#### Poisson Voronoi tessellations

We give a criterion that shows that the Poisson–Delaunay graph almost surely is anchored nonamenable in any dimension of hyperbolic space. In [6], it is shown that in dimension 2, the Poisson–Delaunay graph has a slightly stronger version of anchored nonamenability almost surely, but the arguments were highly specific both to the plane and $S = \Pi^\lambda$. Corollary 1.2 answers Conjecture 1.8 in that paper. In [19], it is shown that the Delaunay graph of any stationary point process $S \subset \mathbb{H}^d$ has a type of nonamenability in law, and that argument extends to other nonpositively curved spaces; it is also shown there that simple random walk on the Delaunay graph has positive hyperbolic speed. For the case of the Poisson Delaunay graph, it is shown there that random walk has positive *graph* speed.

Poisson–Voronoi tessellations have been the setting for interesting percolation theory and other invariant probability, starting with [7]. Very recently, it was established that in the $\lambda \to \infty$ limit, the critical value for Bernoulli site percolation tends to $\frac{1}{2}$ [13], consistent with percolation on the Euclidean Poisson–Voronoi tessellation. Indeed one attractive feature of the Poisson–Voronoi tessellation on $\Pi^\lambda$ is that the parameter $\lambda$ is a continuously tunable (inverse) curvature parameter.

#### Controlling the degrees

The isoperimetric constant we use in (2) is not the one which is typically useful for finding corollaries of random walk (c.f. [6,23]). If we define for a finite set of vertices $V$ in graph $G$, $\text{Vol}_G(V)$ as the sum of the degrees of the vertices in $V$, then by the usual definition we say that a connected graph $G$ has positive anchored expansion if

$$\inf_{V \ni \rho \text{ connected}} \frac{\left| \partial_{\text{out}} V \right|}{\text{Vol}_G(V)} > 0.$$  \hspace{1cm} (5)

We expect that this holds for Poisson Voronoi complex at any intensity (indeed this is proven in [6] for the case of the hyperbolic plane) as well as more rigid point processes such as determinantal point processes. See the extended discussion in [5].
Other questions

We have given a sufficient condition for a collection of points \( S \) to have an anchored nonamenable Delaunay graph. Having too many points of \( S \) in a small region would seem to necessarily contradict that \( S \) is anchored nonamenable, as Euclidean geometry would then take over. On the other hand, having points at great distance would only seem to increase the boundary size of \( S \), suggesting that the anchored bounded vacancy is possibly unnecessary.

**Question 1.3** Suppose \( S \) is a collection of points whose Voronoi complex has all bounded cells and which is anchored nonamenable. Does it follow that \( S \) has anchored bounded density? Conversely, is there a \( S \subset \mathbb{H}^d \) whose Voronoi complex has all bounded cells but which does not have anchored bounded vacancy?

The Voronoi tessellation is only one way to construct a tiling from a collection of points. Particularly in 2-dimensions, circle packing is a very powerful tool for describing embeddings of abstract tilings [1,2,8].

**Question 1.4** Suppose that an anchored nonamenable, bounded degree graph \( G \) is the dual of a circle packing in \( \mathbb{H}^d \). Does it follow that the Delaunay graph on the vertices of \( G \) is anchored nonamenable?

While there are curvature constraints on the types on the fineness of lattices in hyperbolic space, perhaps there are less symmetric sets without this constraint which still look symmetric in a coarse sense.

**Question 1.5** Is there for any \( c > 0 \) a collection of points \( S \subset \mathbb{H}^d \) so that \( S \) is \( c \)-coarsely dense, and so that for each pair \( \{x, y\} \), there is a bilipschitz map \( \phi \) preserving \( S \) and interchanging \( x \) and \( y \)? If, conversely, we ask that the bilipschitz constant is sufficiently close to 1, is \( c \) bounded from below?

Organization

In Sect. 2 we give some basic structure of Voronoi complexes in hyperbolic space. In Sect. 3, we prove Theorem 1.1. In Sect. 4, we show Corollary 1.2.

2 Convexity and the structure of hyperbolic Voronoi complexes

A subset \( V \subset \mathbb{H}^d \) is convex if for any two points in \( V \), the hyperbolic geodesic connecting them is contained wholly in \( V \). For a set \( S \subset \mathbb{H}^d \), the convex hull \( \text{conv}_\mathbb{H}(S) \) is the intersection of all convex sets containing \( V \). Much as in Euclidean space, it is also the intersection of hyperbolic half-spaces containing \( S \). Indeed, in the Klein-model of \( \mathbb{H}^d \), in which a metric \( \rho \) is placed on the unit ball of \( \mathbb{R}^d \) so that \( \rho \)-geodesics are Euclidean geodesics, hyperbolically convex sets coincide with Euclidean convex sets. Hence, for example, a convex hull of a finite set of points is still the intersection of a family of finitely many half-spaces containing that set.
Hence for any discrete set of points \( S \subset \mathbb{H}^d \), the Voronoi complex with nuclei given by \( S \) is, as in Euclidean space, a partition of the space into convex cells which overlap on lower dimensional faces. Additionally, as with Euclidean space, a Voronoi cell is unbounded if and only if its nucleus is an extreme point of the convex hull of \( S \). However, in hyperbolic space, horoballs (limits of balls with radius tending to infinity that are tangent to a point) and half-spaces do not coincide, and hence we are led to some additional characterizations of hyperbolic convex hulls, as elaborated in the next two lemmas.

**Lemma 2.1** Let \( S \subset \mathbb{H}^d \) a discrete set of points, and let \( p \in S \). Then, in the Voronoi complex corresponding to \( S \), the cell with nucleus \( p \) is unbounded if and only if there exists a horoball \( V \) so that \( V \cap S = \emptyset \) and so that \( p \in \partial V \).

**Proof** The cell with nucleus \( p \) is unbounded if and only if there is a sequence of balls \( \{ B_\mathbb{H}(q_n, r_n) \}_{n=1}^{\infty} \subset \mathbb{H}^d \) with \( r_n \to \infty \) so that \( p \in \partial B_\mathbb{H}(q_n, r_n) \) but \( B_\mathbb{H}(q_n, r_n) \cap S = \emptyset \). By compactness, we may extract a subsequence \( n_k \) of balls so that the normal vectors at \( p \) converge to some \( b \). It then follows that the union \( \bigcup_{k=1}^{\infty} B_\mathbb{H}(q_{n_k}, r_{n_k}) \) is disjoint from \( S \). As this union of balls contains the horoball \( V \) with \( p \in \partial V \) and with normal vector \( b \) at \( p \), the proof is complete. \( \square \)

As a corollary, we see that:

**Proposition 2.2** For a discrete point set \( S \) which has anchored bounded vacancy, every Voronoi cell in the complex with nuclei \( S \) is finite.

**Proof** Suppose that \( S \) is a discrete point set whose Voronoi complex has an unbounded cell with nucleus \( p \). Then there is a horoball \( V \) tangent to the ideal boundary at some point \( \omega \). Let \( \gamma \) be a geodesic from \( p \) to \( \omega \). Then any fixed size tubular neighborhood \( N \) of \( \gamma \) is eventually contained in \( V \) in that \( V^c \cap N \) is compact.

We may take \( A \in \Gamma \) in the definition of anchored bounded vacancy to be the cell containing \( p \). Let \( J_k \) be an increasing sequence in \( \mathcal{L}_A \) with \( |J_k| = k \) and every cell of \( J_k \) intersecting \( \gamma \). Then for any \( M > 0 \), the collection of \( \Gamma \) cells near to \( J_k \) by \( d_A \) distance \( M \) is contained in some tubular neighborhood \( N \) of \( \gamma \). Hence all but finitely many cells are contained in \( V \), and consequently

\[
\lim_{k \to \infty} \frac{|\{ B \in \Gamma : d_\mathbb{H}(B, J_k) \leq M, |B \cap S| > 0 \}|}{|J_k|} = 0.
\]

\( \square \)

We can also use Lemma 2.1 to give two alternative representations of a convex hull of a set of points in hyperbolic space. We define the Voronoi boundary of a discrete set of points \( S \) to be those points in the Voronoi complex with nuclei \( S \) whose cells are unbounded.

**Lemma 2.3** For each point \( w \) on the ideal boundary of \( \mathbb{H}^d \), let \( H_w \) be the maximal horoball that passes through \( w \) that is disjoint from \( S \). Let \( Q \) be the compact set...
$\cap_{\mu} H^c_{\nu}$, with the intersection over the entire boundary of $\mathbb{H}^d$. Let $T \subset S$ be the Voronoi boundary of $S$. Then

$$\text{conv}_H(Q) = \text{conv}_H(S) = \text{conv}_H(T).$$

**Proof** The inclusions $\text{conv}_H(Q) \supset \text{conv}_H(S) \supset \text{conv}_H(T)$ follow trivially from the inclusions $Q \supset S \supset T$. Hence to prove the lemma, it is enough to show that $\text{conv}_H(Q) \subset \text{conv}_H(T)$. Let $p$ be an extreme point of $Q$, so that there exists a hyperplane $L$ such that $L \cap Q = \{p\}$ and so that one of the closed half-spaces bounded by $L$ contains $Q$. Suppose on way to a contradiction that $p \notin S$. Let $V$ be the horoball that is tangent to $L$ at $p$ and that is disjoint from $Q$. As $p$ is separated from $S$, it is possible to slightly enlarge $V$ to $V'$, another horoball tangent to the ideal boundary at the same point that still is disjoint from $S$. As $V' \supset \overline{V}$, we have that $p \in Q^c$, a contradiction. Hence $p \in S$. By Lemma 2.1, we have that $Q \cap S \subset T$, so that we have shown that all the extreme points of $Q$ are in $T$. Hence $\text{conv}_H(Q) \subset \text{conv}_H(T)$. \hfill \quad \Box

A major difference between Euclidean Voronoi complexes and hyperbolic Voronoi complexes is that in a hyperbolic Voronoi complex with nuclei $S$, points that are close to the boundary of $\text{conv}_H(S)$ must actually be close to the Voronoi boundary of $S$.

**Proposition 2.4** For any $D > 0$ there is an $R > 0$ so that for all $S \subset \mathbb{H}^d$ finite,

$$\bigcup_{p \in S} B_{\mathbb{H}}(p, D) \subset \text{conv}_H(T) \cup \bigcup_{p \in T} B_{\mathbb{H}}(p, R),$$

where $T \subset S$ is the Voronoi boundary of $S$.

**Proof** Let $x \in S$ be arbitrary. We will take $R > D$, and so if $x \in T$ there is nothing to show. Likewise if $B_{\mathbb{H}}(x, D) \subset \text{conv}_H(T)$, there is nothing to show, and so there is a supporting hyperbolic half-space $L \supset \text{conv}_H(S)$ not containing $B_{\mathbb{H}}(x, D)$. Let $w$ be the point on the ideal boundary of $\mathbb{H}^d$ which is the endpoint of the geodesic ray from $x$ that is normal to $L$, and let $p$ the point of intersection between this ray and $L$. Let $H_w(y)$ be the open horoball defined by the ray from $y$ to $w$. Note that $H_w(p) \subset L^c$ and therefore contains no points of $S$. However, $H_w(x)$ is a horoball whose closure contains $x$ and by the assumption that $x \notin T$, it must be that $H_w(x)$ intersects $S$. Hence, there is a point $y$ on the geodesic $[p, x]$ that is the closet point to $p$ at which the closure of $H_w(y)$ intersects $S$. Let $t \in S$ be such a point, and note that $H_w(y)$ is an open horoball whose closure contains $t$ and is disjoint from $S$, and hence $t \in T$. This implies that $T$ intersects $H_w(x) \cap L$. The diameter of $H_w(x) \cap L$ can be controlled solely as a function of $D$, which completes the proof. \hfill \quad \Box

Finally, we show that to estimate the number of unbounded cells in a hyperbolic Voronoi complex, then it suffices to estimate, up to constants, the volume of a neighborhood of those points.

**Lemma 2.5** There is a constant For every $D > 0$ there is an $\beta$ so that for any finite set $S \subset \mathbb{H}^d$ the Voronoi boundary $T$ of $S$ satisfies

$$\text{Vol}_H(\cup_{p \in S} B_{\mathbb{H}}(p, D)) \leq \beta |T|.$$
This lemma is effectively an immediate corollary of Proposition 2.4 and the following proposition, proved in [4]:

**Proposition 2.6** For any finite set \( S \subset \mathbb{H}^d \),

\[
\text{Vol}_{\mathbb{H}}(\text{conv}_{\mathbb{H}}(S)) \leq \alpha_d |S|,
\]

where \( \text{conv}_{\mathbb{H}}(S) \) denotes the hyperbolic convex hull \( S \), and \( \alpha_d \) is a constant depending only on \( d \).

**Proof of Lemma 2.5** Using Proposition 2.4 and Proposition 2.6, letting \( T \) be the Voronoi boundary of \( S \) and letting \( o \in \mathbb{H}^d \) be any point,

\[
\text{Vol}_{\mathbb{H}}(\bigcup_{p \in S} B_{\mathbb{H}}(p, D)) \leq \text{Vol}_{\mathbb{H}}(\text{conv}_{\mathbb{H}}(T)) + |T|\text{Vol}_{\mathbb{H}}(B_{\mathbb{H}}(o, R)) \\
\leq (\alpha_d + \text{Vol}_{\mathbb{H}}(B_{\mathbb{H}}(o, R)))|T|.
\]

Thus the bound follows with \( \beta = (\alpha_d + \text{Vol}_{\mathbb{H}}(B_{\mathbb{H}}(o, R))) \).

\( \square \)

### 3 Lattice discretizations of Voronoi complexes

We suppose that \( S \) is a countable subset of \( \mathbb{H}^d \) so that the Voronoi complex on \( S \) has only finite cells. We will now formulate conditions under which we can show the Delaunay graph of \( S \) has anchored nonamenability.

We again let \( \Gamma \) be any cocompact lattice, which we once more identify with the cells in a Voronoi tessellation with nuclei given by the orbit of \( \Gamma \) of a point. By transitivity, we have that the volumes of the cells are equal, and we let \( V \) denote that volume. We also let \( D \) be the diameter of one (hence all) of these cells.

We would like to say that connected sets of points in \( S \) resemble lattice animals in \( \Gamma \) in some way. We let \( A \) be the graph with vertex set \( \Gamma \) and edges between any \( A, B \) having \( d_{\mathbb{H}}(A, B) \leq 2D \). For any set \( S \) of points in \( \mathbb{H}^d \), let

\[
\mathcal{I}(S) := \{ A \in \Gamma : |A \cap S| > 0 \}, \\
\mathcal{E}(S) := \{ A \in \Gamma : |A \cap S| = 0, A \text{ intersects an } S-\text{Voronoi cell with nucleus in } S \}.
\]

**Lemma 3.1** Suppose that \( S \) is a discrete set of points in \( \mathbb{H}^d \) for which all Voronoi cells are bounded. There is a \( C > 0 \) so that for any finite set \( S \subset \mathbb{S} \), we have

\[
|\partial_{\text{out}} S| \geq C|\mathcal{I}(S)|
\]

**Proof of Lemma 3.1** By assumption, there are no unbounded cells in the Delaunay complex on \( S \). Therefore, any cell in the Voronoi boundary of \( S \) is in \( \partial_{\text{out}} S \).

Note that \( \mathcal{I}(S) \) is contained in the \( D \)-neighborhood of \( S \), and \( \text{Vol}_{\mathbb{H}}(\bigcup_{A \in \mathcal{I}(S)} A) = V|\mathcal{I}(S)| \). Hence by Lemma 2.5,

\[
|\partial_{\text{out}} S| \geq |\mathcal{I}(S)|V/R.
\]

\( \square \)
We also show that Voronoi complexes can be discretized to form connected subsets of \( \mathcal{A} \) in the sense of the following lemma.

**Lemma 3.2** For any \( S \subset \mathcal{S} \) that is connected in the Delaunay graph on \( \mathcal{S} \), then \( \mathcal{E}(S) \cup \mathcal{I}(S) \) is connected in \( \mathcal{A} \).

**Proof of Lemma 3.2** Let \( p \in S \) be arbitrary, and let \( X \) be the \( S \)-Voronoi cell \( X \) with nucleus given by \( p \). Let \( h \) be any point in \( X \), and let \( L_h \) be any cell of \( \Gamma \) in \( X \). It suffices to show that \( L_h \) is connected to \( L_p \in \mathcal{I}(\{p\}) \) in \( \mathcal{A} \).

Let \( \gamma \) be a geodesic connecting \( p \) to \( h \). As \( h \) is in \( X \), the \( S \)-Voronoi cell with nucleus \( p \), the ball \( B_{\mathbb{H}}(h, r) \) with \( r = d_{\mathbb{H}}(h, p) \) is disjoint from \( S \). Hence any \( A \in \Gamma \) intersecting \( B_{\mathbb{H}}(h, r - D) \) is empty. If \( r \leq D \), then \( d_{\mathbb{H}}(L_h, L_p) \leq D \) and so \( L_h \) and \( L_p \) are connected in \( \mathcal{A} \). Hence, the only portion of \( \gamma \) not necessarily covered by cells from \( \mathcal{E}(S) \) is the initial segment of length \( D \). In particular, the distance from \( L_p \) to \( B_{\mathbb{H}}(h, r - D) \) is at most \( D \), and hence \( L_h \) and \( L_p \) are connected in \( \mathcal{A} \) through \( \mathcal{E}(S) \).

Furthermore, if we enlarge these discretizations of some Voronoi cells, we do not cover many more cells of \( \mathcal{I}(S) \). Let \( d_\mathcal{A} \) be the graph distance in \( \mathcal{A} \).

**Lemma 3.3** For any \( M \in \mathbb{N} \), there is a \( C = C(M) < \infty \) so that for all \( S \subset \mathcal{S} \) the following holds. Let \( W = \{A \in \Gamma : d_\mathcal{A}(A, \mathcal{E}(S) \cup \mathcal{I}(S)) \leq M\} \) then

\[
|\{A \in W : |A \cap S| > 0\}| \leq C|\mathcal{I}(S)|.
\]

**Proof** Let \( p \in S \) be arbitrary, and let \( X \) be the \( S \)-Voronoi cell with nucleus \( p \). Suppose \( A \in \mathcal{E}(S) \) is an empty cell intersecting \( X \) at a point \( q \). As \( q \) is contained in the \( S \)-Voronoi cell with nucleus \( p \), with \( r = d_{\mathbb{H}}(q, \mathcal{I}(p)) \) we have that \( B_{\mathbb{H}}(q, r) \cap S = \emptyset \). Furthermore,

\[
B_{\mathbb{H}}(q, r) \supset \left( \bigcup \{B \in \Gamma : d_\mathcal{A}(B, A) < (r - 2D)/(4D)\} \right).
\]

Hence, any \( A \in \mathcal{E}(S) \) intersecting \( X \) for which \( d_{\mathbb{H}}(A, \mathcal{I}(p)) > 4DM + 3D \) has the property that every \( B \in \Gamma \) within \( d_\mathcal{A} \)-distance \( M \) contains no points of \( S \). Thus, there is a constant \( C = C(M) \) so that

\[
|\{A \in W : A \cap X \neq \emptyset, |A \cap S| > 0\}| \leq C,
\]

and summing over all \( \mathcal{I}(S) \) proves the Lemma.

These tools combine to give a proof Theorem 1.1 for showing when the Delaunay graph of \( \mathcal{S} \) has anchored nonamenability. Before launching into the proof, we remark that in (3) or (4), we may replace \( \mathcal{L}_A \) by the connected subsets of \( \mathcal{A} \) which contain \( A \), as for any connected set \( J \) in \( \mathcal{A} \), we may find a connected set \( J' \supset J \) in \( \mathcal{L}_A \) for which \( |J'| < C J \) for some constant \( C \) depending only on \( \Gamma \).

**Proof of Theorem 1.1** We wish to show there is a \( \delta > 0 \) so that if \( S \ni x \) is any finite set in \( \mathcal{S} \) which is connected in the Delaunay graph on \( \mathcal{S} \) to an anchor point \( p \in S \) then

\[
|\partial_{\text{out}} S| > \delta|S|.
\]
Let $A$ be an anchor cell of $\Gamma$ that contains $\rho$. Let $J$ be $E(S) \cup I(S)$, which contains $A$. By Lemma 3.2, $J$ is connected in $A$. By the hypothesis that $S$ has anchored bounded density (recall (3)), we have that there is a $\delta_0 > 0$ so that

$$|J| > \delta_0 |S|.$$  

By Lemma 3.2, $J$ is connected in $A$. By the hypothesis that $S$ has anchored bounded density, we have that there is a $\delta_0 > 0$ so that

$$|J| > \delta_0 |S|.$$  

By the hypothesis that $S$ has anchored bounded vacancy, we have there is an $M$ and a $\delta_1 > 0$ so that

$$|\{B \in \Gamma : d_{\mathcal{A}}(B, J) \leq M, |B \cap S| > 0\}| > \delta_1 |J|.$$  

By Lemma 3.3, there is a $\delta_2 > 0$ so that

$$|I(S)| > \delta_2 |\{B \in \Gamma : d_{\mathcal{A}}(B, J) \leq M, |B \cap S| > 0\}|.$$  

By Lemma 3.1, there is a $\delta_3 > 0$ so that

$$|\partial_{\text{out}} S| > \delta_3 |I(S)|.$$  

Hence combining this with (8), (7) and (6), we have the desired conclusion. 

\[\square\]

### 4 Application to Poisson point processes

We make some simple observations which allow Theorem 1.1 to be useful, for example when applied to Poisson points. We begin by observing that on account of $A$ having bounded degree (in fact being regular), there is a constant $\Delta_1$ so that

$$|J \in \mathcal{L}_A : |J| = n| \leq \Delta_1^{n-1} \forall n \in \mathbb{N}.$$  

We also observe that enlarging a set $J \in \mathcal{L}_A$ enlarges it by a multiple that can be made arbitrarily large depending on $M$.

**Lemma 4.1** For any $R > 0$, there is an $M$ so that

$$\inf_{J \in \mathcal{L}_A} \frac{|\{B \in \Gamma : d_{\mathcal{A}}(B, J) \leq M\}|}{|J|} > R,$$

**Proof** This follows as $\mathcal{L}_A$ is an expander, i.e.

$$\inf_{V \subseteq \mathcal{L}_A} \frac{|\partial V|}{|V|} > 0,$$

which can be deduced from the isoperimetric inequality for $\mathbb{R}^d$. 

We now check that Corollary 1.2 follows from Theorem 1.1, which is to say that $\Pi^\Lambda$ has anchored bounded density and anchored bounded vacancy.
Lemma 4.2 A stationary Poisson point process has anchored bounded density and anchored bounded vacancy.

Proof We check each criterion of Theorem 1.1 separately. Let $\lambda \cdot dV$ be the intensity of $\Pi^{\lambda}$, with $dV$ the volume measure on $\mathbb{H}^d$. By the properties of the Poisson process, for $J \in \mathcal{L}_A$ of cardinality $n$,

$$|\Pi^{\lambda} \cap (\bigcup_{B \in J} B)| \overset{\mathcal{L}}{=} \text{Poisson}(\lambda \cdot V \cdot n).$$

In particular, using Markov’s inequality and the moment generating function of the Poisson distribution, for all $x > 0$, all $t \geq 0$, all $n \in \mathbb{N}$,

$$\mathbb{P}\left[|\Pi^{\lambda} \cap (\bigcup_{B \in J} B)| \geq (1 + t)\lambda V n\right] \leq e^{\lambda V n(e^x - 1 - (1 + t)x)}.
$$

Thus fixing any $x > 0$ and picking $t$ large enough that $\lambda V (e^x - 1 - (1 + t)x) + \log \Delta < 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\exists J \in \mathcal{L}_A : |J| = n, |\Pi^{\lambda} \cap (\bigcup_{B \in J} B)| \geq (1 + t)\lambda V n\right]
\leq \sum_{n=1}^{\infty} \Delta^n e^{\lambda V n(e^x - 1 - (1 + t)x)} < \infty.
$$

So, by Borel–Cantelli,

$$\sup_{J \in \mathcal{L}_A} \frac{|\Pi^{\lambda} \cap (\bigcup_{B \in J} B)|}{|J|} < \infty,$$

almost surely.

Conversely, for $J \in \mathcal{L}_A$ of cardinality $n$,

$$|\{B \in \Gamma : d_A(B, J) \leq M, |B \cap \Pi^{\lambda}| > 0\}| \overset{\mathcal{L}}{=} \text{Binom}(X, 1 - \exp(-\lambda V)),$$

where $X = |\{B \in \Gamma : d_A(B, J) \leq M\}|$. Hence, there is an absolute constant $\delta > 0$ (see for example [3, Corollary A.1.14]—this also follows from the “multiplicative form” of the Chernoff bound) so that

$$\mathbb{P}\left[|\{B \in \Gamma : d_A(B, J) \leq M, |B \cap \Pi^{\lambda}| > 0\}| \leq X(1 - \exp(-\lambda V))/2\right]
\leq e^{-\delta X(1 - \exp(-\lambda V))}.
$$

By Lemma 4.1, by choosing $M$ sufficiently large, we can guarantee that

$$\delta X(1 - \exp(-\lambda V)) \geq n(\log \Delta + 1),$$

so that once more we can sum over all $J \in \mathcal{L}_A$ of cardinality $n$ to conclude that

$$\mathbb{P}\left[\exists J \in \mathcal{L}_A : |J| = n, |\{B \in \Gamma : d_A(B, J) \leq M, |B \cap \Pi^{\lambda}| > 0\}| \leq e^{-n}\right].$$

Once more applying Borel–Cantelli, the desired conclusion holds. \qed

 Springer
References

1. Angel, O., Hutchcroft, T., Nachmias, A., Ray, G.: Unimodular hyperbolic triangulations: circle packing and random walk. Invent. Math. **206**(1), 229–268 (2016). https://doi.org/10.1007/s00222-016-0653-9

2. Angel, O., Hutchcroft, T., Nachmias, A., Ray, G.: Hyperbolic and parabolic unimodular random maps. Geom. Funct. Anal. **28**(4), 879–942 (2018). https://doi.org/10.1007/s00039-018-0446-y

3. Alon, N., Spencer, J.H.: The Probabilistic Method. Fourth. Wiley Series in Discrete Mathematics and Optimization. p. xiv+375. Wiley, Hoboken, NJ (2016)

4. Benjamini, I., Eldan, R.: Convex hulls in the hyperbolic space. Geom. Dedicata **160**, 365–371 (2012). https://doi.org/10.1007/s10711-011-9687-8

5. Benjamini, I., Kraus, Y., Paquette, E.: Anchored expansion of Delaunay complexes in real hyperbolic space and stationary point processes (2020). arXiv:2008.01063 [math.PR]

6. Benjamini, I., Paquette, E., Pfeffer, J.: Anchored expansion, speed and the Poisson–Voronoi tessellation in symmetric spaces. Ann. Probab. **46**(4), 1917–1956 (2018). https://doi.org/10.1214/17-AOP1216

7. Benjamini, I., Schramm, O.: Percolation in the hyperbolic plane. J. Am. Math. Soc. **14**(2), 487–507 (2001)

8. Benjamini, I., Timar, A.: Invariant embeddings of unimodular random planar graphs. In: arXiv e-prints, arXiv:1910.01614 (Oct 2019). arXiv:1910.01614 [math.PR]

9. Block, J., Weinberger, S.: Aperiodic tilings, positive scalar curvature and amenability of spaces. J. Am. Math. Soc. **5**(4), 907–918 (1992). https://doi.org/10.2307/2152713

10. Cornulier, Y., de la Harpe, P.: Metric Geometry of Locally Compact Groups, Vol. 25. EMS Tracts in Mathematics. Winner of the 2016 EMS Monograph Award, pp. viii+235. European Mathematical Society (EMS), Zürich (2016). https://doi.org/10.4171/166

11. Gelander, T.: Lectures on lattices and locally symmetric spaces. In: Geometric Group Theory, Vol. 21, pp. 249–282. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI (2014)

12. Gromov, M.: Infinite groups as geometric objects. In: Proceedings of the International Congress of Mathematicians, Vol. 1, 2, pp. 385–392 (Warsaw, 1983). PWN, Warsaw (1984)

13. Hansen, B.T., Müller, T.: The critical probability for Voronoi percolation in the hyperbolic plane tends to 1/2. In: arXiv e-prints, arXiv:2004.01464 (Apr 2020). arXiv: 2004.01464 [math.PR]

14. Häggström, O., Schonmann, R.H., Steif, J.E.: The Ising model on diluted graphs and strong amenability. Ann. Probab. **28**(3), 1111–1137 (2000). https://doi.org/10.1214/aop/1019160327

15. Kesten, H.: Symmetric random walks on groups. Trans. Am. Math. Soc. **92**, 336–354 (1959). https://doi.org/10.2307/1993160

16. Lyons, R.: Determinantal probability measures. Publ. Math. Inst. Hautes Études Sci. **98**, 167–212 (2003). https://doi.org/10.1007/s10240-003-0016-0

17. Margulis, G., Mozes, S.: Aperiodic tilings of the hyperbolic plane by convex polygons. Israel J. Math. **107**, 319–325 (1998). https://doi.org/10.1007/BF02764015

18. Morris, D.W.: Introduction to Arithmetic Groups, pp. xii+475. Deductive Press (2015)

19. Paquette, E.: Distributional lattices on Riemannian symmetric spaces. Unimod Randoml Generated Graphs **719**, 63 (2018). arXiv:1707.00308 [math.PR]

20. Penrose, R.: Pentaplexity: a class of nonperiodic tilings of the plane. Math. Intell. **2**(1), 32–37 (1979/80). https://doi.org/10.1007/BF03024384

21. Peres, Y., Virág, B.: Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process. Acta Math. **194**(1), 1–35 (2005). https://doi.org/10.1007/BF02392515

22. Thomassen, C.: Isoperimetric inequalities and transient random walks on graphs. Ann. Probab. **20**(3), 1592–1600 (1992)

23. Virág, B.: Anchored expansion and random walk. Geom. Funct. Anal. **10**(6), 1588–1605 (2000). https://doi.org/10.1007/PL00001663

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.