D-particles, matrix integrals and KP hierarchy

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We study the regularized correlation functions of the light-like coordinate operators in the reduction to zero dimensions of the matrix model describing $D$-particles in four dimensions. We investigate in great detail the related matrix model originally proposed and solved in the planar limit by J. Hoppe. It also gives the solution of the problem of 3-coloring of planar graphs. We find interesting strong/weak ‘t Hooft coupling dependence. The partition function of the grand canonical ensemble turns out to be a tau-function of KP hierarchy. As an illustration of the method we present a new derivation of the large-$N$ and double-scaling limits of the one-matrix model with cubic potential.

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1. Introduction

Matrix models describing the behaviour of $Dp$-branes originate in the observation of Witten [1] that the massless modes propagating along the world volume of $N$ coincident $D$-branes are those of the supersymmetric Yang-Mills theory, obtained by the dimensional reductions of the $d = 10$ $\mathcal{N} = 1$ theory down to $p + 1$ space-time dimensions.

In various compactifications of string theory one encounters the nearly massless non-perturbative particles, obtained by wrapping the $Dp$-branes around vanishing $p$-cycles inside the internal Calabi-Yau manifold. Even in ten-dimensional Type IIA string theory there are solitonic particles [2], which are represented by certain black holes in the effective supergravity and are interpreted as Kaluza-Klein modes of the graviton multiplet in the compactification of $M$-theory on a circle [3,4]. Of course, these particles are no longer massless.

Despite the variety of mechanisms by which such objects appear, their internal description at low energies proves to be rather simple. In fact, if $N$ such particles in $d$ space-time dimensions are close to one another then their dynamics is described by the dimensional reduction of $\mathcal{N} = 1$ super-Yang-Mills theory from $d$ down to $0 + 1$ dimensions (first studied a long time ago for different reasons in [5]). The degrees of freedom in such quantum mechanics are represented by $U(N)$ matrices $X^i$, $i = 1, \ldots, d$, together with the gauge field $A_t$ and their fermionic partners.

Although the exact computations in quantum mechanics of interacting particles are rarely possible, the supersymmetry allows one to get some exact answers. In this paper we shall concentrate on the correlation functions of the light-like coordinate operator. To state more precisely what we mean by that, let us consider the quantum mechanics with periodic time $t \sim t + 2\pi \beta$ and with periodic boundary conditions on fermions. In this case one can show that the observable

$$\mathcal{O}_R = \text{Tr}_R P \exp \oint dt (A_t + X^3)$$

commutes with some of the supercharges (of course the choice of $X^3$ is arbitrary). In the limit $\beta \to 0$ (and after Wick rotation) the computations in the quantum mechanics reduce to the finite-dimensional integrals, where $A_t$ becomes the 0-th matrix $X_0 = -iX_4$. Then the observables $\mathcal{O}_R$ can be expanded in

$$\text{Tr} \left( X^+ \right)^l, \quad X^+ = X^3 + iX^4.$$
The paper is organized as follows. We are going to study the case $d = 4$ in great detail. We derive the determinant representation for the regularized generating function of the correlators of $\text{Tr} (X^+)/l$ and show that it obeys Hirota bilinear identities (when working with fixed chemical potential, e.g. in the grand canonical ensemble). Then we concentrate on the operators $\left(\text{Tr} (X^+)^2\right)/l$ and derive the asymptotics for the generating function in certain limits. We then briefly discuss $d = 6, 10$ cases. Then we proceed with the direct attack on the $d = 4$ integral for fixed but large $N$, using the saddle-point techniques, and derive interesting asymptotics both in the strong and in the weak coupling limit. In the weak coupling limit, we obtain agreement with the planar graph expansion. In the strong 't Hooft coupling limit we get the agreement with the predictions from KP hierarchy. In the first appendix we check the strong coupling asymptotics by the direct quasiclassical calculation of the matrix integral. In the second appendix we demonstrate our method based on the KP differential equation on the example of the usual one matrix model.

In the bulk of the paper we use the notation $\phi \equiv X^+, \bar{\phi} = X^-$. We also denote by $Z, F = \log Z$ the partition function and the free energy at fixed particle number $N$ and by $Z, \mathcal{F} = \log Z$ the corresponding quantities at the fixed chemical potential $\mu$.

2. Supersymmetric matrix integrals

2.1. Theory with four supercharges

The dimensional reduction of the $\mathcal{N} = 1$ SYM from $d = 4$ dimensions down to zero dimensions would produce a matrix model with 4 bosonic matrices $X_{\mu}$, $\mu = 1, 2, 3, 4$ and 2 complex fermionic matrices $\lambda_a$, $a = 1, 2$. All matrices are in the adjoint representation of the gauge group $G$, which we will take to be either $U(N)$ or $SU(N)/\mathbb{Z}_N$. The matrix integral has the form:

$$
\frac{1}{\text{Vol}(G)} \int dXd\lambda \exp \left( \frac{1}{2} \sum_{\mu<\nu} \text{Tr}[X_{\mu}, X_{\nu}]^2 + \text{Tr}\bar{\lambda}_a \sigma_{\mu}^{a\dot{a}} [X_{\mu}, \lambda_a] \right),
$$

where $\sigma_{\mu} = (1, \sigma_i), i = 1, 2, 3$, are the Pauli matrices. The two complex fermions $\lambda$ can be viewed as four real fermions, which we denote by $\chi, \eta, \psi_\alpha, \alpha = 1, 2$:

$$
\lambda_1 = \frac{1}{2} (\eta - i\chi), \quad \lambda_2 = \frac{1}{2} (\psi_1 + i\psi_2)
$$

and

$$
\bar{\lambda}_a = \sigma^{a\dot{a}}_2 \lambda^*_a.
$$
We also redefine the bosonic matrices as:
\[
\phi = \frac{1}{\sqrt{2}} (X_3 + iX_4), \quad \bar{\phi} = \frac{1}{\sqrt{2}} (X_3 - iX_4)
\] (2.2)
and introduce an auxiliary bosonic field \( H \) (also in the adjoint). Then the integral in (2.1) becomes:
\[
\int \frac{dX_\alpha d\psi_\alpha d\chi dH d\bar{\phi} d\eta d\phi}{\text{Vol}(G)} \exp(-S)
\]
where
\[
S = (i\text{Tr}Hs + \frac{1}{2}\text{Tr}H^2 + \text{Tr}[X_\alpha, \phi][X_\alpha, \bar{\phi}] + \frac{1}{2}\text{Tr}[\phi, \bar{\phi}]^2 + \ldots)
\] (2.3)
and \( s = [X_1, X_2] \) and \( \ldots \) represent the fermionic terms that are reconstructed using the following nilpotent symmetry of (2.3):
\[
\delta X_\alpha = \psi_\alpha, \quad \delta \psi_\alpha = [\phi, X_\alpha] \\
\delta \bar{\phi} = \eta, \quad \delta \eta = [\phi, \bar{\phi}] \\
\delta \chi = H, \quad \delta H = [\phi, \chi] \\
\delta \phi = 0.
\] (2.4)

The symmetry \( \delta \) squares to the gauge transformation generated by \( \phi \), hence it is nilpotent on the gauge-invariant quantities. This symmetry was formally studied in \[6\] in order to apply it to the model of \[3\]; it was powerfully exploited in \[8\] in the problem of computing the Witten index in certain quantum mechanical systems (first studied in the two-particle case in \[9-10\], see also \[11\]). The action of the matrix integral (2.3) is \( \delta \)-exact; in fact it may be written as
\[
S = \delta \left( -i\text{Tr}\chi s - \frac{1}{2}\text{Tr}\chi H - \sum_\alpha \text{Tr}\psi_\alpha[X_\alpha, \bar{\phi}] - \frac{1}{2}\text{Tr}\eta[\phi, \bar{\phi}] \right).
\] (2.5)

Now we proceed to reducing the integral (2.3) to an integral with respect to the single matrix variable \( \phi \). The strategy is known for some time \[12\], and it consists of two steps.

If the action is perturbed by the expression \( \delta(...) \), which has a nice behaviour at infinity, then the integral should not change, which can be shown by doing an integration by parts. Consider the modification of the action \( S \) by the term
\[
S \rightarrow S + i\delta R,
\]
with
\[
R = \frac{\kappa_1}{2} \epsilon^{\alpha\beta} \text{Tr}\psi_\alpha X_\beta + \kappa_2 \text{Tr}\chi \bar{\phi}.
\] (2.6)
This perturbation makes the integral (2.1) localized near the zeros of \( H, \bar{\phi}, \chi, \eta \) in the limit of large \( \kappa_2 \), which can be shown by the saddle-point approximation. It reduces the integral (2.1) to a simpler one

\[
\int \frac{d\phi dX_\alpha d\psi_\alpha}{\text{Vol}(G)} \exp(i\kappa_1 \text{Tr}[X_1, X_2] + \kappa_1 \psi_1 \psi_2). \tag{2.7}
\]

The behaviour of the integrand at large values of \( \phi \) is still not good enough. To make it better behaved we modify the transformation \( \delta \). The current \( \delta \) is designed to respect the ordinary gauge invariance. In particular \( \delta^2 \) is a gauge transformation generated by \( \phi \). We wish to invoke yet another symmetry of the integral in (2.1), which is the global group \( U(1) \) acting on the matrices \( X_\alpha, \psi_\alpha \) via the rotations:

\[
e^{i\theta} : X_1 + iX_2 \mapsto e^{i\theta} (X_1 + iX_2). \tag{2.8}
\]

The rest of the fields are invariant under this \( U(1) \) group action. Let us denote the generator of this group by \( \epsilon \). Then the new supercharge \( \delta \) acts as follows:

\[
\begin{align*}
\delta X_\alpha &= \psi_\alpha, & \delta \psi_\alpha &= [\phi, X_\alpha] + i\epsilon \cdot \epsilon^{\alpha\beta} X_\beta \\
\delta \bar{\phi} &= \eta, & \delta \eta &= [\phi, \bar{\phi}] \\
\delta \chi &= H, & \delta H &= [\phi, \chi] \\
\delta \phi &= 0.
\end{align*} \tag{2.9}
\]

The integral (2.1) has another \( U(1) \) symmetry (called the ghost number \( U(1)_{gh} \)) under which \( \delta \) has charge +1, the bosons \( X_\alpha, H \) have charge 0, the fermions \( \psi_\alpha \) have charge +1, the fermions \( \chi, \eta \) have charge -1, and the bosons \( \phi \) and \( \bar{\phi} \) have charges +2 and -2, respectively. The measure and the action have the over-all charge 0. The modification (2.9) is consistent with the ghost number symmetry iff the generator \( \epsilon \) is assigned the ghost number 2. If we compute the modification of the action (2.7), we obtain a better behaved integral

\[
\int \frac{d\phi dX_\alpha d\psi_\alpha}{\text{Vol}(G)} \exp \kappa_1 \text{Tr} \left( i\phi [X_1, X_2] - \frac{1}{2} \epsilon (X_1^2 + X_2^2) + \psi_1 \psi_2 \right). \tag{2.10}
\]

The factor \( \kappa_1 \) can be now reabsorbed into the \( X \)'s and \( \psi \)'s without affecting the measure, and the \( \psi \)'s can then be integrated out. Also the matrices \( X_1, X_2 \) can be integrated out, producing the determinant

\[
Z(N, \epsilon, V) = \int \frac{d\phi}{\text{Vol}(G)} \frac{1}{\det (ad(\phi) + \epsilon)}. \tag{2.11}
\]
The supersymmetry $\delta$ allows a modification of the action by the observables

$$S \rightarrow S + \sum_n T_n O_n$$

where $\delta O_n = 0$. In our case the operators $O_n$ are simply the gauge-invariant functions of $\phi$ as they are also $U(1)_c$-invariant. The simplest operators whose correlation functions may be evaluated are the gauge-invariant functionals of $\phi$, such as $\text{Tr}(\phi^l)$.

To summarize, we have shown that the computation of the (regularized) correlation functions of the observables $\text{Tr}(\phi^n)$ in the supersymmetric matrix integral (2.1) may be reduced to the computations of the integral over a single matrix $\phi$ of the form $V(z) = -\sum_n T_n z^n$:

$$\int d\phi \frac{e^{-\text{Tr}V(\phi)}}{\text{Vol}(G) \Det(ad(\phi) + \epsilon)}.$$

(2.12)

**Remarks.**

1. In ref.[8], the similar perturbation has been used in the computations of the Witten index, which can be reduced to the computation of the integral (2.1) for the group $SU(N)/\mathbb{Z}_N$ and without insertion of any observables. In that case the result of the computation was $\epsilon$-independent. Also, the integral over the eigenvalues of the matrix $\phi$ was to be understood as a contour integral, to avoid the contribution of the flat directions that corresponded to the unbound free particles. In our case, the flat directions contribute to the correlation functions as well, and the parameter $\epsilon$ serves as a regulator as in the computations of [13].

2. One may wonder about the physical meaning of the $\epsilon$-regularized integrals. Here it is:

$$Z(N, \epsilon, V) = \text{Tr}_H(-)^F e^{-\beta H} e^{-\epsilon J} e^{-\text{Tr}V(\phi)} ,$$

(2.13)

where the trace is taken over the Hilbert space $H$ of the quantum mechanical system, $H$ is the Hamiltonian, $F$ is the fermion number, $J$ is the generator of the global symmetry group (which we take to be $SO(d-2)$ for $d = 4, 6$ and $SO(6)$ for $d = 10$, see below). For example, in the case $d = 4$, $J = 2i\text{Tr}(\bar{\phi}[X_1, X_2])$. Just as in [10], this expression is related to the matrix integral in the $\beta \rightarrow 0$ limit. One can also consider directly the quantum mechanical path integral, i.e. the integral over the space of loops. In this case the rational functions in the formulas (2.12) and the similar formulas below are replaced by their trigonometric counterparts. Also one can consider a 1 + 1 model (Matrix strings) in which case the ratio of determinants leads to elliptic functions, just as in [14].
2.2. Theory with eight supercharges

This is the model obtained by the dimensional reduction of $\mathcal{N} = 1$, $d = 6$ theory. In this model the index $\alpha$ of the matrices $X_\alpha, \psi_\alpha$ runs from 1 to 4. The symmetry $U(1)_\epsilon$ is extended to $SO(4) \approx SU(2)_L \times SU(2)_R$. The matrices $X_\alpha, \psi_\alpha$ form two copies of the representation $\left(\frac{1}{2}, \frac{1}{2}\right)$ of this group. Also, the fermion $\chi$ is promoted to a triplet $\vec{\chi}$, which is in $(\frac{1}{2}, 0)$. The same metamorphosis is experienced by the auxiliary field $H \rightarrow \vec{H}$. The action is constructed by the same rules, the only difference being that

$$\chi(s - H) \rightarrow \vec{\chi} \cdot (\vec{s} - \vec{H}),$$

where

$$s_i = [X_4, X_i] + \frac{1}{2} \varepsilon_{ijk} [X_j, X_k].$$

The modification of the supercharge (2.9) is achieved by introduction of the generators $\epsilon_L \oplus \epsilon_R = \left(\frac{\epsilon_1 + \epsilon_2}{2}\right) \oplus \left(\frac{\epsilon_1 - \epsilon_2}{2}\right)$ of the Cartan subalgebra of $SO(4)$. The modified transformations are:

$$\begin{align*}
\delta \psi_1 &= [\phi, X_1] + i \epsilon_1 X_2, \\
\delta \psi_2 &= [\phi, X_2] - i \epsilon_1 X_1, \\
\delta \psi_3 &= [\phi, X_3] + i \epsilon_2 X_4, \\
\delta \psi_4 &= [\phi, X_4] - i \epsilon_2 X_3, \\
\delta \chi_i &= H_i, \\
\delta H_1 &= [\phi, \chi_1] + 2i \epsilon_L X_2, \\
\delta H_2 &= [\phi, \chi_2] - 2i \epsilon_L X_1.
\end{align*}$$  (2.14)

Now we get, instead of (2.12), the following one-matrix integral

$$\int d\phi e^{-\text{Tr} V(\phi)} \frac{\text{Det} (ad(\phi) + \epsilon_1 + \epsilon_2)}{\text{Vol}(G) \text{Det} (ad(\phi) + \epsilon_1) \text{Det} (ad(\phi) + \epsilon_2)}. $$  (2.15)

2.3. Theory with sixteen supercharges

It is of great interest to obtain the similar expression for the integrals occurring in the reductions of $d = 10$ SYM. In this theory the matrices $X_\alpha, \psi_\alpha$ have the index $\alpha$ transforming by the $8$ of the group $SO(8)$. The antighost $\bar{\chi}$ belongs to $1 \oplus 6$ of $SU(4) \subset SO(8)$. Introduce the notation

$$B_i = X_{2i-1} + i X_{2i}, i = 1, 2, 3, 4.$$  

The matrices $B_i$ are in $4$ of $SU(4)$ and $B_i^\dagger$ are in $\bar{4}$. The “gauge condition” $\vec{s}$ splits as:

$$\vec{s} = \mu \oplus \Phi \quad \mu = \sum_{i=1}^{4} [B_i, B_i^\dagger] \in 1$$

$$\Phi_{ij} = [B_i, B_j] + \frac{i}{2} \varepsilon_{ijkl} [B_k^\dagger, B_l^\dagger],$$

$$\Phi_{ij} = \varepsilon_{ijkl} \Phi_{kl}^\dagger, \quad \text{i.e.} \quad \Phi \in 6.$$  (2.16)
The action constructed by the standard rules coincides with that of the dimensional reduction of $d = 10$, $\mathcal{N} = 1$ SYM. The gauge field has ten components, which become $\phi, \bar{\phi}$ and $X_\alpha$. The sixteen-component fermion splits as $\psi_\alpha$, with eight components, $\bar{\chi}$ with seven components and $\eta$.

The global group $SU(4)$ (which is not to be confused with the $R$-symmetry group of $\mathcal{N} = 4$ SYM in four dimensions!) as before allows us to modify the supercharge $\delta$ in a manner analogous to (2.9)–(2.14). The Cartan generator of $SU(4)$ may be written as $\epsilon_1 \oplus \epsilon_2 \oplus \epsilon_3 \oplus (\epsilon_4 = -\epsilon_1 - \epsilon_2 - \epsilon_3)$. The integrals (2.12)–(2.15) generalize to:

$$
\int \frac{d\phi e^{-\text{Tr}V(\phi)}}{\text{Vol}(G)} \det (ad(\phi) + \epsilon_1) \det (ad(\phi) + \epsilon_2) \det (ad(\phi) + \epsilon_3) \det (ad(\phi) + \epsilon_4),
$$

(2.17)

3. Determinant representation of the correlation functions in the $d = 4$ case

In this section we study in detail the grand partition function

$$
\mathcal{Z}(\mu, \epsilon, V) = \sum_{N=0}^{\infty} e^{\mu N} Z(N, \epsilon, V).
$$

We show that $\mathcal{Z}(\mu, \epsilon, V)$ has a determinant representation, very much like the correlation functions in Sine-Gordon, and related models are expressed in terms of Fredholm determinants [13].

3.1. Eigenvalue integral

First we write the integral (2.12) in terms of the eigenvalues $i\phi_1, ..., i\phi_N$ of the anti-Hermitian matrix $\phi$:

$$
Z(N, \epsilon, V) = \int_{\mathbb{R}^N} \frac{d\phi_1 ... d\phi_N}{N!(2\pi \epsilon)^N} \prod_{i \neq j} \frac{\phi_i - \phi_j}{\phi_i - \phi_j + i\epsilon} \prod_i e^{-V(\phi_i)},
$$

(3.1)

where we changed $V(ix)$ to $V(x)$. The integral (3.1) can be rewritten, using the Cauchy formula, as

$$
Z(N, \epsilon, V) = \sum_{\sigma \in S_N} (-\epsilon)^\sigma \int_{\mathbb{R}^N} \frac{d\phi_1 ... d\phi_N}{N!(2\pi i)^N} \prod_k \frac{e^{-V(\phi_k)}}{\phi_k - \phi_{\sigma(k)} + i\epsilon}.
$$

(3.2)
It turns out that the grand partition function (that is, with fixed chemical potential) can be written as a Fredholm determinant of an integral operator. Let us introduce the notation

$$W_l(\epsilon, V) = \int_{\mathbb{R}} \prod_{k=1}^l \frac{dx_k}{2\pi} \frac{e^{-V(x_k)}}{\epsilon - i(x_k - x_{k+1})}, \quad x_{l+1} \equiv x_1. \quad (3.3)$$

We may rewrite the sum over all the elements of the permutation group in (3.2) as the sum over the conjugacy classes that are labelled by the partitions of $N$:

$$N = \sum_{l=1}^\infty l d_l, \quad d_l \geq 0.$$ 

Every permutation in the conjugacy class, labelled by $\vec{d} = (d_1, d_2, \ldots)$, is similar to the product of cycles of lengths $1, 2, \ldots$, the number of times the cycles with length $l$ appear being precisely $d_l$. The number of permutations in the given conjugacy class $\vec{d}$ is equal to

$$\frac{N!}{\prod_l l^{d_l} d_l!}$$

and the sign of any permutation in this class is $(-)^{\sum_l d_l} (-)^N$. Thus, (3.2) may be represented as

$$Z(N, \epsilon, V) = \sum_{\vec{d} : \sum_l l d_l = N} \prod_l \frac{1}{d_l!} \left( -\frac{(-)^l W_l}{l} \right)^{d_l} \quad (3.4)$$

and the grand partition function is equal to

$$Z(\mu, \epsilon, V) = \exp \sum_l (-)^{l-1} e^{l\mu} \frac{W_l(\epsilon, V)}{l}. \quad (3.5)$$

The quantity $W_l(\epsilon, V)$ may be represented as $W_l = \text{Tr}K^l$, where $K$ is a linear operator acting in the space of functions of one variable, as follows:

$$(Kf)(x) = e^{-V(x)} \int_{\mathbb{R}} \frac{dy}{2\pi} \frac{f(y)}{\epsilon - i(x - y)}. \quad (3.6)$$

Therefore, the grand partition function becomes

$$Z(\mu, \epsilon, V) = \exp \sum_l \frac{(-)^{l-1}}{l} \text{Tr}(e^{\mu K})^l = \text{Det} (I + e^{\mu K}). \quad (3.7)$$
3.2. Another representation for quadratic \( V \)

Let us consider the case of a Gaussian potential \( V(x) = \frac{1}{2} \left( \frac{\xi x}{\epsilon} \right)^2 \) and write the partition function (3.1) again as a matrix integral

\[
Z(N, \xi) = \int \frac{d\phi dX}{\text{Vol}(G)} \exp \left( -\frac{1}{2} \text{Tr}[\phi, X]^2 + \frac{\xi^2}{\epsilon^2} \text{Tr}\phi^2 - \frac{1}{2}\epsilon^2 \text{Tr}X^2 \right). \tag{3.8}
\]

Considering the matrices \( X \) and \( \phi \) as the Hermitian and anti-Hermitian part of the same complex matrix \( Z = (\epsilon/\sqrt{\xi})X + (\sqrt{\xi}/\epsilon)\phi \), we rewrite the matrix integral as

\[
Z(N, \xi) = \frac{1}{\text{Vol}(G)} \int dZdZ^\dagger e^{-S}, \quad S = \frac{1}{2} \text{Tr}[Z, Z^\dagger]^2 + \frac{\xi}{2} \text{Tr}ZZ^\dagger. \tag{3.9}
\]

In polar coordinates

\[
Z = UH^{1/2}
\]

where \( U \) is unitary and \( H \) is a Hermitian matrix with positive eigenvalues \( y_1, \ldots, y_N \), the measure and the action read

\[
dZdZ^\dagger = dUdH, \quad S = \text{Tr}H^2 - \text{Tr}U^{-1}HU + \frac{1}{2}\xi \text{Tr}H.
\]

Using the Harish-Chandra-Itzykson-Zuber formula [16] we perform the \( U \)-integration and find

\[
Z(N, \xi) = \frac{1}{N!} \int_{\mathbb{R}^N_+} d^N y \ e^{-\sum_i \left( y_i^2 + \frac{1}{2} \xi y_i \right)} \text{Det}_{ij} (e^{y_i y_j}). \tag{3.10}
\]

The grand partition function will be expressed in terms of the quantities

\[
W_l = \int_{\mathbb{R}^+} \prod_{i=1}^l dy_i \ e^{-\frac{1}{2} \left[ \xi y_i + (y_i - y_{i+1})^2 \right]}, \quad (y_{l+1} \equiv y_1), \tag{3.11}
\]

as

\[
Z(\mu, \xi) = \exp \sum_{l=1}^\infty (-1)^{l-1} e^{\mu l} \frac{W_l}{l}. \tag{3.12}
\]

Clearly \( W_l = \text{Tr}K^l \), where \( K \) now acts on the functions on the positive semi-axis as

\[
Kf(y) = e^{-\frac{\xi}{2} y} \int_0^\infty e^{-\frac{1}{2} (y - y')^2} f(y') dy'. \tag{3.13}
\]

Therefore we arrive at the same determinant representation (3.7) where the kernel \( K \) is replaced by its Fourier transform \( \hat{K} \).

For small \( \xi \),

\[
(-)^{l-1} W_l = \frac{1}{l\xi} g_l(\xi), \tag{3.14}
\]

where \( g_l \) is an analytic function.
4. The grand partition function as a tau-function

In this section we show that the grand partition function
\[ Z(\mu, \epsilon, V) = \sum_{N=0}^{\infty} e^{\mu N} \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \ e^{-V(x_i)} \prod_{i<j} (x_i - x_j)^2 \prod_{i,j} (x_i - x_j - i\epsilon) \] (4.1)
can be represented as a tau-function of the KP hierarchy.

4.1. Vertex operator construction – bosonic representation

Introduce the bosonic field \( \varphi(z) \) with mode expansion
\[ \varphi(z) = \hat{q} + \hat{p} \ln z + \sum_{n \neq 0} \frac{J_n}{n} z^{-n}, \] (4.2)
\[ [J_n, J_m] = n\delta_{m,n,0}; \ [\hat{p}, \hat{q}] = 1 \] (4.3)
and the vacuum state \( |l\rangle \) defined by
\[ J_n |l\rangle = 0, \ (n > 0); \ \hat{p}|l\rangle = l|l\rangle. \] (4.4)
The associated normal ordering is defined by putting \( J_n \) with \( n > 0 \) to the right. Define the vertex operator
\[ \mathcal{V}_\epsilon(z) = : e^{\varphi(z+i\epsilon/2)} : e^{-\varphi(z-i\epsilon/2)} : , \] (4.5)
which satisfies the OPE
\[ \mathcal{V}_\epsilon(z) \mathcal{V}_\epsilon(z') = \frac{(z - z')^2}{(z - z')^2 + \epsilon^2} : \mathcal{V}_\epsilon(z) \mathcal{V}_\epsilon(z') : , \] (4.6)
the Hamiltonian
\[ H[t] = \sum_{n>0} t_n J_n, \] (4.7)
and the operator
\[ \Omega_\mu = \exp \left( e^{\mu} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} \mathcal{V}_\epsilon(z) \right). \] (4.8)
Then the vacuum expectation value
\[ \tau_0[t] = \langle 0 | e^{H[t]} \Omega_\mu | 0 \rangle \] (4.9)
is equal to the canonical partition function \((4.1)\), with chemical potential \(\mu\) and potential

\[
V(z) = U\left(z + \frac{i}{2}\epsilon\right) - U\left(z - \frac{i}{2}\epsilon\right) = -\sum_{n=0}^{\infty} T_n z^n, \\
U(z) = -\sum_{n=1}^{\infty} t_n z^n,
\]

where \(\mu\) can be reabsorbed into the definition of \(V\) and where the coefficients \(T_n\) and \(t_n\) are related by

\[
T_{n-1} = i \sum_{k=0}^{\infty} (-)^k \binom{n+2k}{n-1} \frac{\epsilon^{2k+1}}{4^k} t_{n+2k}.
\]

### 4.2. Fermionic representation

The fermionic representation of the partition function is constructed using the bosonization formulas

\[
\psi(z) = e^{-\varphi(z)} ; \quad \psi^*(z) = e^{\varphi(z)} ; \quad \partial \varphi(z) = \psi^*(z) \psi(z) ;
\]

where the fermion operators

\[
\psi(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r z^{-r - \frac{1}{2}}, \quad \psi^*(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^*_{-r} z^{-r - \frac{1}{2}}
\]

satisfy the anticommutation relations

\[
[\psi_r, \psi^*_{s}]_+ = \delta_{rs}.
\]

The operators \((4.7)\) and \((4.8)\) are represented by

\[
H\left[\cdot\right] = \sum_{n > 0} t_n \sum_{r} : \psi^*_r \psi_r : \quad (n \in \mathbb{Z})
\]

\[
\Omega_\mu = \exp \left[ e^\mu \int_{-\infty}^{\infty} \frac{dx}{2\pi i} \psi(x + i\frac{\epsilon}{2}) \psi^*(x - i\frac{\epsilon}{2}) \right].
\]

and the vacuum states with given electric charge \(l\) satisfy

\[
\langle l | \psi_{-r} = \langle l | \psi_r^* = 0 \quad (r > l) \\
\psi_r | l \rangle = \psi_{-r}^* | l \rangle = 0 \quad (r > l).
\]

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The original expression (4.1) is obtained from the expectation value (4.9) by first commuting the operator \( e^H \) to the right until it hits the right vacuum by using the formulas

\[
\begin{align*}
 e^H[t] \psi(z) e^{-H[t]} &= e^{\sum_{n=1}^{\infty} t_n z^n} \psi(z) \\
 e^H[t] \psi^*(z) e^{-H[t]} &= e^{-\sum_{n=1}^{\infty} t_n z^n} \psi^*(z)
\end{align*}
\] (4.18)

and then applying the Wick theorem to calculate the expectation value

\[
\langle l | \prod_i \psi(z_i) \psi^*(w_i) | l \rangle = \prod_i \left( \frac{z_i}{w_i} \right)^l \prod_{i<j} \frac{(z_i-z_j)(w_i-w_j)}{(z_i-w_j)(w_i-z_j)}. \] (4.19)

4.3. The KP hierarchy

The partition function (4.9) is a particular case of the “general solution” of the KP hierarchy obtained as the limit \( N \to \infty \) of a general \( N \)-soliton solution [18]

\[
\tau_l[t] = \langle l | e^{\sum_{n>0} t_n J^n} \Omega_a | l \rangle, \] (4.20)

where the \( GL(\infty) \) rotation

\[
\Omega_a = \exp \left( \int dx dy \ a(x,y) \psi(x) \psi^*(y) \right) \] (4.21)

is parametrized by an arbitrary integrable function \( a(x,y) \).

The tau-functions \( \tau_l, l \in \mathbb{Z} \), are Fredholm determinants

\[
\tau_l = \det(1 + e^H K_l) \] (4.22)

of the kernels \( K_l \)

\[
K_l(x,y) = \frac{E_l(x+i\epsilon)}{E_l(x-i\epsilon)} \frac{1}{x-y-2i\epsilon}, \quad E_l(x) = x^l \exp \left( \sum_n t_n x^n \right). \] (4.23)

For \( l = 0 \) we get precisely the operator \( K \) (3.6).

The KP hierarchy of differential equations is generated by the Hirota bilinear equations [19]:

\[
\oint \frac{dz}{2\pi i} z^{l-l'} \exp \left( \sum_{n>0} (t_n-t'_n) z^n \right) \tau_l \left( t_n - \frac{1}{n} z^{-n} \right) \tau_{l'} \left( t'_n + \frac{1}{n} z^{-n} \right) = 0 \quad (l' \leq l). \] (4.24)

1 In the soliton solutions it is a sum of delta-functions, \( a(x,y) = \sum_{k=1}^{N} a_i \delta(x-p_i)\delta(y-q_i) \), so that the operator \( \Omega_a \) is a finite sum, \( \Omega_a = \sum_{k=1}^{N} a_i \psi(p_i) \psi^*(q_i) \).
Let us sketch the proof of (4.24). First we remark that each element \( \Omega \in GL(\infty) \) is represented by an infinite c-number matrix \( a = \{ a_{rs} \} \)

\[
\Omega \psi_r \Omega^{-1} = \sum_s \psi_s a_{sr}, \quad \Omega^{-1} \psi_r^* \Omega = \sum_s a_{rs} \psi_s^*.
\] (4.25)

As a consequence, there exists a tensor Casimir operator

\[
S_{12} = \sum_r \psi_r \otimes \psi_r^* = \oint \frac{dz}{2\pi i} \psi(z) \otimes \psi^*(z)
\] (4.26)

which satisfies, for any \( \Omega \in GL(\infty) \),

\[
S_{12} \otimes \Omega = \Omega \otimes \Omega S_{12}.
\] (4.27)

On the other hand \( S_{12} |l\rangle \otimes |l\rangle = 0 \) because, according to (4.17), for each \( r \) either \( \psi_r \) or \( \psi_r^* \) is annihilated by the right vacuum \( |l\rangle \). Therefore (4.27) implies that \( S_{12} \Omega |l\rangle \otimes \Omega |l\rangle = 0 \).

Taking the scalar product with \( \langle l + 1 | e^{H[t]} \otimes (l - 1 | e^{H[t']} \) we find

\[
\oint \frac{dz}{2\pi i} (l + 1 | e^{H[t]} \psi(z) \Omega_{\mu} |l\rangle \langle l - 1 | e^{H[t']} \psi^*(z) \Omega_{\mu} |l\rangle = 0,
\] (4.28)

where the integration contour surrounds the origin. Equation (4.28) simply reflects the fact that the tensor Casimir (4.26) is constant on the orbits of \( GL(\infty) \). Finally we use the bosonization formulas (4.12) to represent the fermions as vertex operators, \( \psi(z) \rightarrow V_-(z) \), \( \psi^*(z) \rightarrow V_+(z) \), where \( V_{\pm} \) act in the space of the coupling constants as

\[
V_{\pm}(z) = \exp \left( \pm \sum_{n=0}^{\infty} t_n z^n \right) \exp \left( \mp \ln \frac{1}{z} \frac{\partial}{\partial \mu} \mp \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n} \right).
\] (4.29)

The general case \( l \neq l' \) is treated similarly, and one obtains the following identity

\[
\oint \frac{dz}{z} \left( V_+(z) \cdot \tau_l [t] \right) \left( V_-(z) \cdot \tau_{l'} [t'] \right) = 0 \quad (l' \leq l),
\] (4.30)

which is identical to the Hirota equation (4.24).

The differential equations of the KP hierarchy are obtained by expanding (4.24) in the differences \( y_n = \frac{1}{2} (t_n - t'_n) \). In the case \( l' = l \), the first non-trivial equation (the KP equation) is obtained by requiring that \( y_3 \) term vanishes:

\[
\left( \frac{\partial^4}{\partial y_1^4} + 3 \frac{\partial^2}{\partial y_2^2} - 4 \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_3} \right) \tau_l [t + y] \tau_l [t - y] \bigg|_{y=0} = 0.
\] (4.31)
In terms of the “specific heat”
\[ u[t] = 2 \frac{\partial^2}{\partial t_1^2} \log \tau, \]  
(4.32)
the KP equation reads
\[ 3 \frac{\partial^2 u}{\partial t_2^2} + \frac{\partial}{\partial t_1} \left[ -4 \frac{\partial u}{\partial t_3} + 6u \frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3} \right] = 0. \]  
(4.33)

5. The \( d = 4 \) integral with quadratic potential: KP equation, weak coupling and double scaling

In this section we study the case \( t_n = 0, \ n > 3 \).

5.1. Reduction to a single equation

A potential of the form \( V(x) = -\mu + \lambda x^2 \), \( \lambda = \xi^2 \) is related to the three couplings \( t_1, t_2, t_3 \) by
\[ \mu = i\epsilon \left( t_1 - \frac{1}{4} \epsilon^2 t_3 - \frac{t_2^2}{2t_3} \right), \quad \xi^2 = -3i\epsilon t_3. \]  
(5.1)
Rescale \( \epsilon \to 1 \). Then \( u = -2\partial^2 \log \kappa \) and it is easy to show that (4.33) implies the following partial differential equation for the function \( \psi = \xi u(\mu, \xi) \):
\[ \psi_\xi + \psi \psi_\mu + \frac{\xi}{6} (\psi - \psi_\mu)_\mu = a(\xi). \]  
(5.2)
By comparing it with the expansion in (3.12)–(3.14), we see that \( a(\xi) \equiv 0 \). If we expand
\[ \psi(\mu, \xi) = \sum_{l=1}^{\infty} e^{\mu l} e^{-\frac{\xi^2}{12}(t^3 - l)} \eta_l(\xi), \]  
(5.3)
then eq.(5.2) is equivalent to the infinite system of recursive first-order differential equations:
\[ \eta_l' = \frac{l}{2} \sum_{p+q=l} \eta_p \eta_q e^{-\frac{\xi^2}{12}l p q}. \]  
(5.4)

The form (5.3) is dictated by the semi-classical approximation to the integral (2.10). Indeed, the expression \( \frac{\xi^2}{12}(t^3 - l) \) is nothing but the classical action evaluated on the solution to the equations of motion:
\[ [X_1, X_2] = -2i\lambda \phi, \]
\[ [\phi, X_2] = -iX_1, \]  
(5.5)
\[ [\phi, X_1] = +iX_2. \]
The solutions to (5.5) are classified by the decompositions of $N$-dimensional representation into irreducibles of $SU(2)$. The logarithm of the grand partition function takes into account only irreducible $l$-dimensional representations, the rest is generated by the exponentiation. The functions $\eta_l$ therefore describe the quantum fluctuations around the saddle points.

We conclude this section by listing the two equivalent forms of the equation obeyed by $u$:

$$
\psi\xi + \psi\psi_\mu + \frac{\xi}{6}(\psi - \psi_\mu)_\mu = 0
$$

$$
2u_\lambda + \frac{1}{\lambda}u + uu_\mu + \frac{1}{6}(u - u_\mu)_\mu = 0.
$$

5.2. Weak coupling limit

For low values of $l$, eqs. (5.4) can be solved explicitly. It is interesting to look at the large $\lambda = \xi^2$ asymptotics of the solutions. We expect that as $\lambda \to \infty$ the partition function at fixed $N$ has the following scaling behaviour of $Z(N, \lambda) = e^{F(N, \lambda)}$:

$$
Z(N, \lambda) = \frac{1}{\lambda^{N^2/2}} (1 + \ldots) = \frac{1}{\xi^{N^2}} (1 + \ldots).
$$

On the other hand it follows from (5.4) that

$$
u = -\frac{e^\mu}{\xi} + \frac{e^{2\mu}}{\xi^2} \sum_{n=0}^{\infty} \frac{(2n - 1)!!}{\xi^{2n}} + \ldots
$$

and therefore indeed

$$
Z(1, \lambda) = \frac{1}{2\xi}, \quad Z(2, \lambda) = -\frac{1}{\xi^4} \sum_{n=1}^{\infty} \frac{(2n - 1)!!}{\xi^{2n}},
$$

in accordance with the scaling (5.7).

It turns out that equation (5.2) has another interesting property. Suppose we are studying the 't Hooft limit, where the free energy has an expansion of the form

$$
F(N, \lambda) = \sum_{g=0}^{\infty} N^{2-2g} F_g(N/\lambda).
$$

Given $Z(\mu, \lambda) = e^{\mathcal{F}(\mu, \lambda)}$ we extract the fixed-$N$ partition function via the Fourier transform

$$
Z(N, \lambda) = \int \frac{d\mu}{2\pi i} e^{-i\mu N + \mathcal{F}(i\mu, \lambda)},
$$
which can be taken, in the large-$N$ limit, using the saddle-point approximation. In the planar limit ($N = \infty$, with $\lambda/N$ finite) the functions $F(\mu, \lambda)$ and $F(N, \lambda)$ are Legendre transforms of each other. If we keep all the $1/N$ corrections, then the corresponding expansion of $F(\mu, \lambda)$ is:

$$F(\mu, \lambda) = \sum_{g=0}^{\infty} \lambda^{2-2g} F_g(\mu/\lambda)$$

(5.12)

and

$$u(\mu, \lambda) = \sum_{g=0}^{\infty} u_g(\mu/\lambda) \lambda^{-2g}, \quad u_g(x) = -2 F''_g(x).$$

(5.13)

Let us introduce the variables

$$x = \mu/\lambda, \quad y = \lambda^{-2}, \quad \chi(x, y) = u(\mu, \lambda),$$

(5.14)

which are relevant for the 't Hooft limit. Then eq. (5.2) may be rewritten as:

$$\chi - 2 \left(x - \frac{1}{12}\right) \chi_x - 4 y \chi_y + \chi \chi_x - \frac{y}{6} \chi_{xxx} = 0,$$

(5.15)

which after the expansion

$$\chi(x, y) = \sum_{g=0}^{\infty} y^g \chi_g(x)$$

(5.16)

reduces to the infinite system of recursive equations for $\chi_g$’s:

$$\left(\frac{1}{6} - 2x + \chi_0\right) \chi'_g + (1 - 4g - \chi'_0) = \frac{\chi''_g - 1}{6} - \sum_{a+b=g, a,b \neq 0} \chi_a \chi'_b, \quad g > 0$$

$$\chi_0 - 2 \left(x - \frac{1}{12}\right) \chi'_0 + \chi_0 \chi'_0 = 0.$$  

(5.17)

The equation for $\chi_0$ is the only non-linear one. Its solution is:

$$x = \frac{\alpha_0}{2} \chi_0^2 + \chi_0 + \frac{1}{12}.$$

(5.18)

Of course eq. (5.2) has more general solutions, in particular those for which the expansion (5.16) is not bounded as $g \geq 0$. It turns out that the solution corresponding to the matrix integral in question does have the form (5.16). In Appendix A we show that $\alpha_0 = -\pi^2$. 

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It follows that, for large $N$ and $\lambda$, the free energy has 't Hooft-like behaviour:\(^2\)

$$F(N, \lambda) = -\frac{1}{10} \left(\frac{243\pi^2}{4}\right)^{1/3} N^2 \left(\frac{\lambda}{N}\right)^{1/3} + \ldots$$  \tag{5.19}$$

5.3. Double scaling limit near the quadratic singularity in $\chi(x)$

So far we investigated (3.1) only in the large-$N$ ('t Hooft) limit in the canonical ensemble or, equivalently, in the large-$\mu$ limit for the grand canonical ensemble.

As we know from [20] and Appendix B, the universal scaling behaviour of higher $1/N$ corrections sometimes can be summed up to some functions obeying non-linear differential equations, such as Painlevé II for the pure 2$d$ gravity.

It is reasonable to ask whether we can do the same with the $1/\mu$ expansion for our model starting from the general KP equation (5.15) and what the physical or geometrical meaning of this expansion is (we recall that in the pure gravity described by the one-matrix model of Appendix B the corresponding $1/N$ expansion has the meaning of the expansion over the genera of the topologies of the two-dimensional manifold).

Let us concentrate on the square-root singularity of (5.18) at $\chi_c = \frac{1}{\alpha_0}$, $x_c = -\frac{1}{2\alpha_0} + \frac{1}{12}$. We try the following ansatz:

$$\chi = \chi_c + y^a v(z), \quad z = y^b (x - x_c).$$  \tag{5.20}$$

As in the case of the one-matrix model, the presence of a quadratic singularity implies that $b = -2a$. In full analogy with Appendix B (the only difference being that $y$ takes the place of $1/N^2$), inserting this ansatz into (5.15), and neglecting the subleading corrections, we obtain $b = -\frac{2}{5}$, $a = \frac{1}{5}$ and that function $v(z)$ satisfies the Painlevé II equation:

$$v'' - 3v^2 + \frac{12}{\pi^2} z = 0.$$  \tag{5.21}$$

From this equation we find the following coefficients of the $1/\lambda$ expansion in (5.12) (which is the same as the $1/\mu$ expansion in this approximation) for the singular part of the free energy near the critical point:

$$\mathcal{F}^{\text{sing}}_0 = -\Delta^{\frac{3}{2}}, \quad \mathcal{F}^{\text{sing}}_1 = -\frac{1}{24} \log \Delta, \quad \cdots,$$  \tag{5.22}$$

---

\(^2\) Recall that $F = \mathcal{F} - \mu N$ in our conventions.
where $\Delta = \text{const}(x - x_c)$ (we choose the constant in such a way that the coefficient in front of $\Delta^{5/2}$ be $-1$, then the next coefficients, $-\frac{1}{24}, \cdots$, are universal constants).

So everything goes just as in the pure 2d quantum gravity. The $1/\mu$ expansion looks like the topological $1/N$ expansion, the coefficients giving the leading scaling behaviour of the partition functions of successive topologies (see the details in [20]). It is tempting to speculate that the quadratic singularity in $\chi(x)$ corresponds to the pure gravity. It may be related to the large planar graph expansion with respect to $g$ in the model of dense self-avoiding random paths (we will argue at the end of section 6 that our matrix integral describes such a model in the large-$N$ limit). It would be interesting to demonstrate it by passing from the grand canonical to the canonical ensemble for the free energy.

### 6. Saddle-point approach

#### 6.1. $d = 4$ integral

So far we managed to calculate the grand canonical version of the integral (3.5) in the large-$\mu$ limit by the use of the KP equations. It is not clear whether we can derive from this asymptotics the large $N$ limit of canonical partition function. In fact, we shall show that it is possible by comparing the results with a more direct approach, originally proposed in this context by Hoppe [21]. Namely, in the case of Gaussian potential $V(x) = \lambda x^2$, it is possible to solve the integral saddle-point equation for the distribution of the eigenvalues of a matrix in (3.1). We work out the details of the solution (correcting some minor mistakes in [21] and actually deriving the result) and extract interesting critical behaviours of our system.

It is natural to scale the coupling $\lambda$ as $N$, and rescale $\epsilon$ to 1, i.e. to set:

$$\lambda = \frac{N}{g^2}, \quad \epsilon \to 1. \quad (6.1)$$

Indeed by rewriting the integral (3.1) as:

$$Z(N, \lambda) = \frac{1}{\text{vol}(G)\lambda^\frac{N^2}{4}} \int dX d\phi \exp \left( -\text{Tr} \left( \phi^2 + X^2 \right) + \frac{1}{\lambda} \text{Tr}[X, \phi]^2 \right) \quad (6.2)$$

we see that $\frac{1}{\sqrt{\lambda}}$ plays the role of the coupling constant while $\frac{N}{\lambda}$ is the ’t Hooft coupling. In the large-$N$ limit the integral (3.1) localizes onto the critical point of the effective potential:

$$V_{\text{eff}}(\phi) = \sum_i V(\phi_i) + \sum_{i<j} \log \left( 1 + \frac{1}{(\phi_i - \phi_j)^2} \right). \quad (6.3)$$
Its critical point is found from the equation:

\[
\frac{\phi_k}{g^2} = \frac{1}{N} \sum_{j \neq k} \frac{1}{(\phi_k - \phi_j)(1 + (\phi_k - \phi_j)^2)}.
\] (6.4)

In the usual fashion we assume that the eigenvalues in the large-\(N\) limit form a continuous medium of density

\[
\rho(x) = \frac{1}{N} \sum_i \delta(x - \phi_i).
\] (6.5)

When \(g\) is real, it is natural to expect \(\rho\) to vanish outside of the interval \([-a, +a]\), and to be an even function \(\rho(x) = \rho(-x)\). We introduce the resolvent:

\[
W(z) = \int_{-a}^{+a} dx \frac{\rho(x)}{z - x},
\] (6.6)

and rewrite the eq. (6.4) as:

\[
\frac{2x}{g^2} = W(x + i0) + W(x - i0) - W(x + i) - W(x - i),
\] (6.7)

and where it is assumed that \(x \in [-a, a]\). The spectral density is given by the discontinuity of \(W(z)\) along the cut at \((-a, +a)\):

\[
W(x + i0) - W(x - i0) = -2\pi i \rho(x).
\] (6.8)

Let us introduce the holomorphic function

\[
G(z) = \frac{z^2}{g^2} + i \left[ W \left( z + \frac{i}{2} \right) - W \left( z - \frac{i}{2} \right) \right]
\] (6.9)

which is related to the expectation value of the fermionic current \(\langle \psi^*(z)\psi(z) \rangle\) in the grand canonical ensemble (4.1). The saddle-point equation (6.7) can now be rewritten as

\[
G \left( x + \frac{i}{2} \right) = G \left( x - \frac{i}{2} \right), \quad x \in (-a, +a).
\] (6.10)

The definitions (6.6) and (6.9) imply that, in case of positive coupling \(\lambda = N/g^2\)

\[
W(z) = W(\bar{z}), \quad W(z) = -W(-z),
\]

\[
G(z) = G(\bar{z}), \quad G(z) = G(-z),
\] (6.11)

and also that the function \(G(z)\) has the cuts at \((\pm \frac{i}{2} - a, \pm \frac{i}{2} + a)\). It is also clear from (6.10) and (6.11) that \(G(z)\) is real when \(z \in \mathbb{R}, i\mathbb{R}, (\pm \frac{i}{2} - a, \pm \frac{i}{2} + a)\). Hence \(G(z)\) defines
a holomorphic map of the region \( \mathcal{U} \) bounded by \( \mathbb{R}_+, i\mathbb{R}_+ \) and by the sides of the interval \((\frac{i}{2}, \frac{i}{2} + a)\) onto the upper half-plane \( H \). The inverse map is given by the following integral formula: \( G(z) = \zeta \), where:

\[
z = A \int_{x_1}^{\zeta} \frac{dt(t-x_3)}{\sqrt{(t-x_1)(t-x_2)(t-x_4)}}. \tag{6.12}
\]

The map acts on the special points \( x_1 > x_2 > x_3 > x_4 \) and \( \infty \) as follows:

| \( \zeta \) | \( z \) |
|---|---|
| \( +\infty \) | \( +\infty \) |
| \( x_1 \) | 0 |
| \( x_2 \) | \( \frac{i}{2} \) |
| \( x_3 \) | \( \frac{i}{2} + a \) |
| \( x_4 \) | \( \frac{i}{2} \) |
| \( -\infty \) | \( +i\infty \) |

These conditions imply the following equations on \( x_i, A, a \):

\[
\frac{1}{2} = A \int_{x_2}^{x_1} dt \frac{t-x_3}{\sqrt{(t-x_2)(x_1-t)(t-x_4)}} \tag{6.13}
\]

\[
a = A \int_{x_3}^{x_2} dt \frac{x_3-t}{\sqrt{(x_2-t)(x_1-t)(t-x_4)}}
\]

\[
a = A \int_{x_4}^{x_3} dt \frac{x_3-t}{\sqrt{(x_2-t)(x_1-t)(t-x_4)}}.
\]

From (6.9) we know the large-\( z \) asymptotics of \( \zeta \):

\[
\zeta = \frac{1}{g^2} z^2 + z^{-2} + \delta z^{-4} + \ldots, \tag{6.14}
\]

\[
z = g \zeta^\frac{1}{2} - \frac{1}{2g} \zeta^{-\frac{1}{2}} - \frac{\delta}{2g^3} \zeta^{-\frac{3}{2}} + \ldots,
\]

where

\[
\delta = -\frac{1}{4} + 3\nu, \quad \nu = \int_{-a}^{+a} \rho(y)y^2dy. \tag{6.15}
\]

At large \( t \) the function \( z(\zeta) \), as given by (6.12), has the following expansion:

\[
z = 2A \left( \zeta^\frac{1}{2} + a_0 + a_1 \zeta^{-\frac{1}{2}} + a_2 \zeta^{-\frac{3}{2}} + a_3 \zeta^{-\frac{5}{2}} + \ldots \right), \tag{6.16}
\]
where
\[ a_0 = \int_{x_1}^{\infty} dt \left( \frac{(t - x_3)}{\sqrt{(t - x_1)(t - x_2)(t - x_4)}} - \frac{1}{\sqrt{t - x_1}} \right) \]
\[ a_k = \frac{(-)^{k-1}}{2k-1} (\gamma_k + x_3 \gamma_{k-1}), \quad k > 0 \] (6.17)
\[ \gamma_k = \sum_{p+q+r=k} \left( -\frac{1}{2} \right)^p \left( -\frac{1}{2} \right)^q \left( -\frac{1}{2} \right)^r x_1^p x_2^q x_4^r. \]

By comparing (6.14) and (6.17), we get the following equations on \( x_i, A \):
\[ A = \frac{g}{2} \]
\[ a_0(x_i) = 0 \]
\[ x_1 + x_2 + x_4 = 2x_3 \] (6.18)
\[ x_1^2 + x_2^2 + x_4^2 - 2x_3^2 = \frac{6}{g^2}, \]
which, together with eqs. (6.13), fixes everything completely. As shown in [21], the equation \( a_0 = 0 \) follows from (6.13) by a contour-deformation argument. We introduce more notation:
\[ y_i = gx_i, \quad \lambda_i = \frac{y_i}{y_1 - y_4} \]
\[ m = \frac{y_2 - y_4}{y_1 - y_4}, \quad 1 > m > 0, \quad m' = 1 - \frac{1}{m}, \] (6.19)
with which eqs. (6.13) assume the following form:
\[ (y_2 + y_4 - y_1)K(m) + 2(y_1 - y_4)E(m) = 0 \]
\[ -(y_1 + y_2 - y_4)K(m') + 2(y_2 - y_4)E(m') = \sqrt{\frac{y_2 - y_4}{g}}, \] (6.20)
where we use the standard elliptic functions
\[ K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - msin^2\theta}}, \quad E(m) = \int_0^{\frac{\pi}{2}} d\theta \sqrt{1 - msin^2\theta}, \] (6.21)
which have the following crucial properties:
\[ K(m') = \sqrt{m}K(1 - m), \quad E(m') = \frac{1}{\sqrt{m}}E(1 - m) \]
\[ E(m)K(1 - m) + E(1 - m)K(m) - K(m)K(1 - m) = \frac{\pi}{2}. \] (6.22)
In the sequel we use the short-hand notations $E = E(m), K = K(m), \vartheta = E/K$. The first equation in (6.20) allows us to express $\lambda_2$ in terms of $m$, while the second, together with (6.22), gives $g(y_1 - y_4)$:

$$\lambda_2 = 1 - 2\vartheta, \quad g(y_1 - y_4) = \frac{1}{\pi^2} K^2. \quad (6.23)$$

From (6.18) and the equations $\lambda_4 = \lambda_2 - m$ and $\lambda_1 = \lambda_4 + 1$ we get an expression for $y_1 - y_4$ and consequently for $g$:

$$g^2(m) = \frac{K^4}{3\pi^4} \left( 4m\lambda_2 + 1 - 3\lambda_2^2 - 2\lambda_2 \right)$$

$$= \frac{K^4}{3\pi^4} (-3\vartheta^2 + 2(2 - m)\vartheta - (1 - m)). \quad (6.24)$$

We can now compute $\frac{1}{N}\langle \text{Tr} \phi^2 \rangle = \nu$ and $F(N, g)$:

$$\nu(m) = \frac{g^4}{N^2} \frac{\partial F(N, g)}{\partial g^2}$$

$$= \frac{1}{12} + \frac{K^2}{5\pi^2} \left( \frac{4m\lambda_2(1 - m) + (5\lambda_2^2 - 1)(2m - 1 - \lambda_2)}{4m\lambda_2 + 1 - 3\lambda_2^2 - 2\lambda_2} \right)$$

$$= \frac{1}{12} - \frac{K^2}{5\pi^2} \frac{10\vartheta^2(\vartheta + m - 2) + 2\vartheta(6 - 6m + m^2) + (1 - m)(m - 2)}{3\vartheta^2 + 2(m - 2)\vartheta + 1 - m}. \quad (6.25)$$

The formulae (6.24) and (6.25) give the exact analytic solution of the large-$N$ model in the parametric form.

**Small-$g$ expansion.** In this case we expect to get a regular planar graph expansion of the matrix integral (3.8) with respect to the quartic term in the action. The careful analysis shows that the $g \to 0$ limit corresponds to $m \to 0$. In this limit we can expand:

$$g^2 = \frac{1}{128} m^2 + \frac{1}{128} m^3 + \frac{119}{16384} m^4 + \ldots$$

$$\nu = \frac{1}{256} m^2 + \frac{1}{256} m^3 + \frac{133}{4096\sqrt{2}} m^4 + \ldots, \quad (6.26)$$

which implies

$$\nu = \frac{1}{2} g^2 - \frac{1}{2} g^4 + \ldots$$

$$F(N, g) = N^2 \left( \frac{1}{2} \log g^2 - \frac{1}{2} g^2 + \ldots \right) \quad (6.27)$$

in perfect agreement with the planar graph expansion and formula (5.7).
Large-$g$ limit. This limit corresponds to the situation where we are far beyond the convergence radius of the $g$-series. The integral (3.8) is dominated by the commutator term in the action and hence the fluctuations of the matrices are very large due to the presence of the zero modes. It follows from the careful study of (6.23) and (6.24) that the $g \to \infty$ limit corresponds to $m \to 1$. In this limit ($m = 1 - \varepsilon$):

$$g^2 = \frac{1}{12\pi^4} \log^3 \left( \frac{16}{\varepsilon} \right) - \frac{1}{4\pi^4} \log^2 \left( \frac{16}{\varepsilon} \right) \ldots$$

$$\nu = + \frac{1}{20\pi^2} \log^2 \left( \frac{16}{\varepsilon} \right) - \frac{7}{20\pi^2} \log \left( \frac{16}{\varepsilon} \right) + \ldots$$

(6.28)

Hence

$$\nu = \frac{(12\pi)^\frac{2}{3}}{20} g^{\frac{4}{3}} - \frac{3}{(12\pi)^\frac{2}{3}} g^{\frac{2}{3}} \ldots$$

$$F(N, g) = - N^2 \left( \frac{3(12\pi)^\frac{2}{3}}{40} g^{-\frac{2}{3}} - \frac{9}{5(12\pi)^\frac{2}{3}} g^{-\frac{2}{3}} + \ldots \right)$$

(6.29)

in perfect agreement with (5.19). In fact, the strong coupling expansion can be greatly simplified if we choose $L = \frac{1}{\log \left( \frac{16}{\varepsilon} \right)}$ as an expansion parameter and systematically neglect all non-perturbative (in $L$) corrections, i.e. we consider the leading logarithmic approximation. We then get the very simple formulas:

$$g^2 = \frac{1}{12\pi^4 L^3} (1 - 3L)$$

$$\nu = \frac{1}{20\pi^2 L^2} \left( \frac{1 - 10L + 20L^2}{1 - 3L} \right) + \frac{1}{12}.$$

(6.30)

The surprise is that formulas (6.30) are exactly equivalent to (5.18).

Proof. Equation (5.18) with $\alpha_0 = \frac{\pi^2}{2}$ gives:

$$\mathcal{F}_0' = \frac{\pi^2}{12} \chi_0^3 - \frac{1}{4} \chi_0^2, \quad \mathcal{F}_0 = \frac{\pi^4}{120} \chi_0^5 - \frac{5\pi^4}{96} \chi_0^4 + \frac{1}{12} \chi_0^3.$$

(6.31)

The Legendre transform leading from $\mathcal{F}_0$ to $F_0$ yields: $\nu = x - 2 \frac{\mathcal{F}_0}{\mathcal{F}_0'}$, while $g^2 = -\mathcal{F}_0'$. Clearly, it leads to (6.30) if we substitute

$$\chi_0 = \frac{\log \left( \frac{16}{\varepsilon} \right)}{\pi^2}.$$

(6.32)

\[\frac{1}{10} \left( \frac{243\pi^2}{4} \right)^\frac{1}{3} = \frac{3(12\pi)^\frac{2}{3}}{40}.\]
This confirms once again that indeed $\alpha_0 = \frac{\pi^2}{2}$.

Remarks.

1. It is very tempting to speculate that the relation to supersymmetric gauge theories, which was one of the original motivations of this work, is somehow revealed by the appearance of a family of elliptic curves, parametrized by the value of coupling $g$ just as in [22]. Notice that the solution that we studied here has something to do with mirror symmetry. Indeed, the naive coordinates in the space of our Lagrangians (which is just the coupling $g$ for a quadratic potential) have been replaced by the period of a certain differential on the elliptic curve. It is conceivable that a similar construction takes place for more general potentials.

2. The partition function does not change, in the large-$N$ limit, if we substitute the commutator by the anticommutator in the action in (3.8). The latter model generates the statistical ensemble of $\phi^4$-type random graphs covered by dense non-oriented self-avoiding random loops and has been studied in [23]. It describes the dense phase of the $O(n)$ loop-gas model [24] with $n = 1$. It also can be viewed as a matrix model counting 3-colored planar graphs of the $\phi^3$ type: each propagator has one of 3 colors and all three propagators meeting at any vertex have different colors. This 3-coloring problem can be formulated as the following matrix model of three hermitean matrices $B$(lue), $W$(hite) and $R$(ed) with the action:

$$S = N \text{Tr}(gBWR + gBRW + B^2 + W^2 + R^2) \quad (6.33)$$

To see that it coincides with the original Hoppe’s model it is enough to change the sign of one of 2 cubic terms (which is equivalent to the change of anticommutator of $W, R$ to the commutator) and integrate over one of the matrices, say, $B$. The critical behaviour (thermodynamical limit) is due to the dominance of graphs of infinite size, which renders the $g$-expansion of the partition function divergent and is therefore determined by the singularity in $g^2$ closest to the origin. The latter is a solution of the equation $g'(m) = 0$. It should appear for negative $g^2$ and corresponds to the situation where all three cuts are located on the real axis, symmetrically with respect to the origin. When $g$ increases, the end-points of the cuts get closer and a singularity occurs when they touch one another. One can show, using the symmetry (6.11) of $W(z)$, that when $g^2$ is real and negative, the saddle-point equation (6.7) becomes identical to eq. (3.3) of ref. [23]. It is known [23] that the critical behaviour of the
dense $O(1)$ model is in the universality class of the pure 2$d$ quantum gravity. For example, the one-point function behaves as

$$\nu \sim (g_c - g)^{\frac{3}{2}}.$$ 

Here is the explicit formula for $(g^2)'$:

$$\frac{dg^2}{dm} = -9\pi^4 K^4 \frac{\partial(\partial - 1)(\partial - 1 + m)}{m(1-m)}. \quad (6.34)$$

3. The last assertion can be partially confirmed by the study of the specific heat at the fixed chemical potential. From the eq. (5.18) we get the critical point $x_c = -(\frac{1}{\pi^2} + \frac{1}{12})$, which can be inserted into eqs. (6.30)–(6.32) to yield $g_c^2 = -\frac{1}{3\pi^4}$, which is negative indeed. Note, however, that the corresponding value of $\varepsilon = 2.16$ is way larger than the leading log approximation allows us to see.

7. On the correlation functions in the $d = 6, 10$ cases and directions for the future

This section is devoted to a work in progress. We sketch the possible similar saddle-point approach to the $d = 6$ integral. We also attempt a fermionic representation for the $d = 10$ integral.

7.1. Saddle-point approach to $d = 6$ integral

We keep the same notations for the resolvent $F$ and density $\rho$. We set $\varepsilon_1 + \varepsilon_2 = 1$, $\varepsilon_1 = \beta$, $\varepsilon_2 = \gamma$. Equation (6.7) is replaced by:

$$\frac{x}{g^2} = \hat{F}(x) - \frac{1}{2} (F(x + i\beta) + F(x - i\beta))$$

$$- \frac{1}{2} (F(x + i\gamma) + F(x - i\gamma)) + \frac{1}{2} (F(x + i) + F(x - i)). \quad (7.1)$$

Let us introduce the derived functions:

$$f(x) = \frac{ix^2}{2g^2\gamma} + F(x + \frac{i\beta}{2}) - F(x - \frac{i\beta}{2}) \quad (7.2)$$

$$g(x) = \frac{1}{2} \left( f(x + \frac{i\gamma}{2}) - f(x - \frac{i\gamma}{2}) \right).$$

The saddle-point equation (7.1) is equivalent to

$$g \left( x + \frac{i}{2} \right) + g \left( x - \frac{i}{2} \right) = 0. \quad (7.3)$$

The function $g(x)$ has four cuts: at $x \in \pm \frac{1}{2}, \pm \frac{1}{2}(\beta - \gamma) + (-a, +a)$, it is real: $g(\bar{z}) = \overline{g(z)}$ and it is purely imaginary for $z \in i\mathbb{R}$. It would be nice to guess the correct function from the stated properties. We plan to return to this problem in the future.
7.2. Fermionic representation for the $d = 10$ integral

We now proceed with the fermionic representation of the $d = 10$ integral (2.17). Unfortunately we were not able to find such a representation for all values of $\epsilon_1, \epsilon_2, \epsilon_3$. However, let us consider the limit $\epsilon_3 \to -\epsilon_1, \epsilon_4 \to -\epsilon_2$. At the same time we keep $e^{i(\epsilon_1 + \epsilon_3)} = e^{\bar{\mu}}$ finite. We claim that, in this limit,

$$ Z^{d=10}_{\mu, \epsilon} = \langle 0 | e^{H[t]} e^{\bar{\Omega}_\mu} | 0 \rangle, \quad (7.4) $$

where

$$ \bar{\Omega}_\mu = e^{i(\epsilon_1 + \epsilon_3)} \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} : \psi(z - a) \psi(z + a) \psi^*(z - b) \psi^*(z + b) : $$

with $a = \frac{1}{2}(\epsilon_1 + \epsilon_2), b = \frac{1}{2}(\epsilon_1 - \epsilon_2)$. Now the relation between $V$ and $U$ is modified into:

$$ V(z) = U(z + a) + U(z - a) - U(z + b) - U(z - b). \quad (7.6) $$

As in the index computation it is possible that, by making the appropriate mass perturbation, we reduce the $d = 10$ integral for the gauge group $U(N)$ to the products of $d = 4$ integrals for gauge groups $U(n_1) \times \ldots \times U(n_k)$ with $N = \sum_{l=1}^{\infty} l n_l$. Under this assumption:

$$ Z^{d=10}(\mu, \epsilon, V) = \prod_{l=1}^{\infty} Z^{d=4}(l \mu, \epsilon, V) = \prod_{l=1}^{\infty} \text{Det}(I + e^{i \mu} K) = \Theta(\mu, K). \quad (7.7) $$

The latter expression is very interesting, since it possesses certain modular properties and allows one to deduce the large-$N$ asymptotics using very little information about the operator $K$ itself:

$$ \Theta(\mu, K) \sim \exp 2 \sqrt{N |\text{Li}_2(-K)|}, \quad (7.8) $$

where

$$ \text{Li}_2(-K) = \sum_{l=1}^{\infty} \frac{1}{l^2} \text{Tr}(-K)^l. \quad (7.9) $$

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8. Conclusions

Here we summarize the results of our computations. The integral (3.8), which is a cousin of (3.1), is studied in two regimes: at fixed $N$ and at fixed $\mu$. In the first case we got the large-$N$ asymptotics in the 't Hooft limit, see (6.24) and (6.25). In the second case we showed that the grand partition function is a particular tau-function of the KP hierarchy. In particular, we obtained eq. (5.2) for the specific heat $u = -2\partial^2_\mu F$ in the case of quadratic potential $V \sim \lambda z^2$:

$$2u_\lambda + \frac{1}{\lambda} u + uu_\mu + \frac{1}{6}(u - u_{\mu\mu})_\mu = 0.$$  

In the large-$\mu, \lambda$ limit, we obtained the simple explicit formula for the specific heat as a function of $x = \mu/\lambda$:

$$-\frac{\pi^2}{4} u^2 + u + \frac{1}{12} = x. \quad (8.1)$$

We observed various similarities to the properties of supersymmetric gauge theories in four dimensions and we hope that our results will find their place in the study of dynamics of D-particles in various dimensions.

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Appendix A. Determination of $\alpha_0$

We now present a trick that allows us to find the value of the unknown coefficient $\alpha_0$ in (5.18). If $\alpha_0 \neq 0$ then for large $x$ one has $u \sim \pm \sqrt{\frac{2x}{\alpha_0}}$ and

$$\partial_\mu F \sim \pm \frac{\sqrt{2} \mu^{\frac{3}{2}}}{3 \xi \sqrt{\alpha_0}}. \quad (A.1)$$
Below, we rescale $\lambda \rightarrow \lambda/2$ to be in agreement with the notations of (3.13).

Let $e^{-E_k}$ be the eigenvalues of the integral operator $\mathcal{K}$. From the determinant representation (3.7) of the partition function, it follows that $e^{\mathcal{F}(\mu,\xi)} = \prod_k \left(1 + e^{\mu-E_k}\right)$. Correspondingly, the mean value of the number of particles is given by

$$\langle N \rangle \equiv \partial_\mu \mathcal{F} = \sum_k \frac{1}{1 + e^{E_k-\mu}}. \quad \text{(A.2)}$$

We are interested in the limit where both $\mu$ and $\xi$ (and therefore $E_k$, see below) are very large, i.e. a kind of low-temperature limit for a Fermi-gas with energy levels given by the spectrum of the operator $\log \mathcal{K}$. In the low-temperature limit we simply need to count the number of energy levels below the Fermi level $\mu$.

The eigenvalue problem for the operator $\mathcal{K}$ is similar (although far from being equivalent in general) to the eigenvalue problem of the particle of unit mass, which is confined to move at the positive semi-axis $y > 0$ and subject to the spike-like potential

$$U(z,t) = \frac{\xi^2}{2} \sum_{n\in\mathbb{Z}} \delta(t-n), \quad \xi > 0.$$  

The operator $\mathcal{K}$ is to be compared with the operator $\mathcal{U}_1$ of the evolution during the unit imaginary time. The latter can be easily diagonalized:

$$\mathcal{U}_1 f_E = e^{-E} f_E, \quad \text{with} \quad f_E(y) = A(y - \frac{2\tilde{E}}{\xi}), \quad \tilde{E} = E + \log \sqrt{2\pi} + \frac{\xi^2}{48},$$

and $A(y)$ is the modified Airy function

$$A(y) = \int_{\gamma} \frac{dp}{2\pi} e^{ipy + \frac{p^2}{4} + i\frac{p^3}{3\xi}} \quad \text{(A.4)}$$

where the contour $\gamma$ is such that $\Im p^3 > 0$ as $p \to \infty$ along $\gamma$. The spectrum is determined from the condition that $f_E(0) = 0$, i.e. $A(-\frac{2\tilde{E}}{\xi}) = 0$. For large values of $E$ this equation can be solved using quasi-classics. It gives

$$E_k \sim \frac{1}{2} \left(\frac{3\pi \xi k}{2}\right)^{\frac{2}{3}}, \quad k \to \infty. \quad \text{(A.5)}$$

Assuming that for our problem we may use the same asymptotics we conclude that

$$\langle N \rangle \sim \frac{2^{\frac{2}{3}}}{3\pi \xi} \mu^{\frac{4}{3}}, \quad \text{(A.6)}$$

which means that

$$\alpha_0 = \frac{\pi^2}{2}. \quad \text{(A.7)}$$

Remark. It is interesting to note that a similar Schrödinger problem arises in the Born-Oppenheimer approximation to the quantum mechanics of a particle in two dimensions confined by the potential $x^2y^2$, which is a good model for the matrix potential $\text{Tr}[X, Y]^2$, see [21].
Appendix B. Solution of the one-matrix model from the KP equation and double scaling limit.

As an example illustrating the application of the KP hierarchy to matrix integrals, we will derive the critical singularity and the double scaling limit of the one-matrix integral:

\[ Z_N(t) = \int D M \exp N \text{Tr} \sum_{q=1}^{3} t_q M^q \]  

in the case of a cubic potential. The fact that the matrix integral (B.1) is a tau-function of the KP hierarchy has been established in [26]. A direct derivation of the Hirota equations (4.24) from the matrix integral can be found in [27]. The free energy

\[ F_N = \frac{1}{N^2} \log Z_N \]  

and the specific heat

\[ u(t) = 2 \partial_t^2 F_N \]  

can be obtained from the KP equation (4.33), if we take it into account that the specific heat is actually a function of a single parameter:

\[ u = t_3 \frac{2}{3} \psi \left( \frac{t_1}{t_3^{\frac{2}{3}}} - \frac{t_2^{\frac{3}{4}}}{3t_3^{\frac{3}{4}}} \right) \]  

and is given in the Gaussian limit by

\[ t_3 = 0 : \quad u = -\frac{1}{t_2}. \]

By inserting (B.4) into (4.33) we derive that the function \( f \) obeys the following ordinary differential equation

\[ \psi + 2x \psi' + 9 \psi' + \frac{3}{2N^2} \psi''' = \psi_0 \]  

where \( \psi_0 \) is a constant,

\[ x = \frac{t_1}{t_3^{\frac{1}{3}}} - \frac{t_2^{\frac{2}{3}}}{3t_3^{\frac{5}{3}}} , \quad \psi = t_3^{\frac{2}{3}} u. \]

In the large-\( N \) limit, we have an algebraic equation for \( x(\psi) \):

\[ (\psi - \psi_0) \frac{dx}{d\psi} + 2x + 9 \psi = 0, \]  

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whose solution depends on two constants \((\psi_0, \beta)\):

\[
x = \frac{\beta}{(\psi - \psi_0)^2} - 3 \left( \psi - \frac{1}{2} \psi_0 \right) .
\]  

(B.9)

By comparing this with the Gaussian limit, we get: \(\psi_0 = 0, \beta = -\frac{1}{3}\). Hence the large-\(N\) result is:

\[
t_1 t_3 - \frac{1}{3} t_2^2 = -3t_3^2 u - \frac{1}{3u^2}.
\]  

(B.10)

The critical point is

\[
x_c = -\left( \frac{9}{2} \right)^\frac{4}{9}, \quad \psi_c = \left( \frac{2}{9} \right)^\frac{1}{9}.
\]  

(B.11)

To compare with the results of Brezin et al. [28] we set:

\[
t_1 = 0, \quad t_2 = \frac{1}{2N}, \quad t_3 = \frac{g}{N^\frac{3}{2}}
\]

and get \(x = -\frac{1}{12g^\frac{4}{3}}\). Hence

\[
g_c^2 = \frac{1}{108\sqrt{3}},
\]

which is exactly the result of the tedious computations [28] using the distribution of the eigenvalues in the large-\(N\) limit. Equation (4.33) also allows a trivial derivation of the double scaling limit for the pure 2d quantum gravity [20]. This limit consists in sending \(x-x_c\) to zero and \(N\) to infinity in such a way that the double scaling variable \(z = N^b(x-x_c)\) remains finite. We try the ansatz: \(\psi = \psi_c + N^a v(z)\). Since \(\psi \simeq \psi_c + \text{const} \cdot \sqrt{x-x_c}\) we have: \(b = -2a\). We then plug this ansatz into (4.33) and obtain

\[
v'''' + 6N^{2+5a} uv' + 2N^{2+5a} + \left( \frac{4}{3} zv' + \frac{2}{3} v \right) N^{2+6a} = 0.
\]  

(B.12)

To keep here the non-linear term (the source of all higher-genus corrections \(\sim 1/N^{2g}\)) we impose the condition: \(2 + 5a = 0\), which gives

\[
a = -\frac{2}{5}, \quad b = \frac{4}{5}.
\]

One immediately sees that the last term in (B.12) vanishes in the double scaling limit \((N^{2+6a} = N^\frac{-6}{5} \to 0\). Integrating (B.12) once with respect to \(z\), we finally obtain the Painlevé II equation:

\[
v'' + 3v^2 = cz,
\]  

(B.13)
where $c = -\frac{2}{3} \psi_c = -\frac{2^4}{3^3}$.

Remark. We obtained this equation without making use at all of the method of orthogonal polynomials – the only technique known until now for these purposes. This raises the hope that the method of Hirota equations proposed here will allow us to obtain the non-perturbative description (beyond the loop expansion) of some interesting non-critical string theories, such as $O(2)$ model [24] whose grand partition is known to satisfy the KdV hierarchy [29].
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