2N-WEAK MODULE AMENABILITY OF SEMIGROUP ALGEBRAS

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Abstract. Let $S$ be an inverse semigroup with the set of idempotents $E$. We prove that the semigroup algebra $\ell^1(S)$ is always $2n$-weakly module amenable as an $\ell^1(E)$-module, for any $n \in \mathbb{N}$, where $E$ acts on $S$ trivially from the left and by multiplication from the right.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A linear map $D : \mathcal{A} \to \mathcal{X}$ is called a derivation if $D(ab) = aD(b) + D(a)b$ for all $a, b \in \mathcal{A}$. Each map of the form $a \to ax - xa$, where $x \in \mathcal{X}$, is a continuous derivation which will be called an inner derivation.

For any Banach $\mathcal{A}$-module $\mathcal{X}$, its dual space $\mathcal{X}^*$ is naturally equipped with a Banach $\mathcal{A}$-module structure via

$\langle x, af \rangle = \langle xa, f \rangle$ \quad $\langle x, fa \rangle = \langle ax, f \rangle \quad (a \in \mathcal{A}, f \in \mathcal{X}^*, x \in \mathcal{X})$.

Note that the Banach algebra $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule under the algebra multiplication. So $\mathcal{A}^{(n)}$, the $n$-th dual space of $\mathcal{A}$, is naturally a Banach $\mathcal{A}$-bimodule in the above sense for each $n \in \mathbb{N}$. The Banach algebra $\mathcal{A}$ is called $n$-weakly amenable if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. If $\mathcal{A}$ is $n$-weakly amenable for each $n \in \mathbb{N}$ then it is called permanently weakly amenable.

The concept of $n$-weakly amenability was introduced by Dales, Ghahramani and Grønbæk in [6]. Johnson showed in [10] that for any locally compact group $G$, the group algebra $L^1(G)$ is always $1$-weakly amenable. It was shown further in [6] that $L^1(G)$ is in fact $n$-weakly amenable for all odd numbers $n$. Whether this is still true for even numbers $n$ was left open in [6]. Later in [11] Johnson proved that $\ell^1(G)$ is $2n$-weakly amenable for each $n \in \mathbb{N}$ whenever $G$ is a free group. The problem has been resolved affirmatively for general locally compact group $G$ in [5] and in [12] independently, using a theory established in [13]. In [19], as an application of a common fixed point property for semigroups, a short proof to $2m$-weak amenability of $L^1(G)$ was presented. Mewomo in [14] investigate the $n$-weak amenability of semigroup algebras and showed that for a Rees matrix semigroup $S$, $\ell^1(S)$ is $n$-weakly amenable when $n$ is odd. Also he obtained a similar result for a regular semigroup $S$ with finitely many idempotents.

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Let $\mathcal{A}$ and $\mathcal{U}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule with compatible actions, that is
\[ \alpha(ab) = (\alpha a)b, \quad (ab)\alpha = a(b\alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathcal{U}). \]

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathcal{U}$-bimodule with compatible actions, that is
\[ \alpha(ax) = (\alpha a)x, \quad a(\alpha x) = (\alpha a)x, \quad (\alpha x)a = \alpha(xa) \quad (a \in \mathcal{A}, \alpha \in \mathcal{U}, x \in \mathcal{X}), \]
and similarly for the right or two-sided actions. Then $\mathcal{X}$ is called a *Banach $\mathcal{A}$-$\mathcal{U}$-module*, and is called a *commutative* Banach $\mathcal{A}$-$\mathcal{U}$-module whenever $\alpha x = x\alpha$ for all $\alpha \in \mathcal{U}$ and $x \in \mathcal{X}$.

Let $\mathcal{A}$ and $\mathcal{U}$ be as above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathcal{U}$-module. A bounded map $D : \mathcal{A} \to \mathcal{X}$ is called a *module derivation* if
\[
D(a \pm b) = D(a) \pm D(b), \quad D(ab) = aD(b) + D(a)b \quad (a, b \in \mathcal{A}),
\]
and
\[
D(\alpha a) = \alpha D(a), \quad D(a\alpha) = D(a).\alpha \quad (a \in \mathcal{A}, \alpha \in \mathcal{U}).
\]
Note that $D$ is not necessarily linear and if there exists a constant $M > 0$ such that $\|D(a)\| \leq M \|a\|$ for each $a \in \mathcal{A}$, then $D$ is bounded and its boundedness implies its norm continuity. When $\mathcal{X}$ is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module, each $x \in \mathcal{X}$ defines an $\mathcal{U}$-module derivation
\[ D_x(a) = xa - ax \quad (a \in \mathcal{A}), \]
these are called *inner* module derivations.

If $\mathcal{X}$ is a (commutative) Banach $\mathcal{A}$-$\mathcal{U}$-module, then so is $\mathcal{X}^*$, where the actions of $\mathcal{A}$ and $\mathcal{U}$ on $\mathcal{X}^*$ are naturally defined as above. So by letting $\mathcal{X}^{(0)} = \mathcal{X}$, if we define $\mathcal{X}^{(n)}$ ($n \in \mathbb{N}$) inductively by $\mathcal{X}^{(n)} = (\mathcal{X}^{(n-1)})^*$, then $\mathcal{X}^{(n)}$ is a (commutative) Banach $\mathcal{A}$-$\mathcal{U}$-module.

Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathcal{U}$-module, as we have not assumed the compatibility condition $a(\alpha b) = (a\alpha)b$ ($a, b \in \mathcal{A}, \alpha \in \mathcal{U}$). If we consider the closed ideal $J$ of $\mathcal{A}$ generated by elements of the form $(a\alpha)b - a(\alpha b)$ for $a, b \in \mathcal{A}, \alpha \in \mathcal{U}$, then $J$ is an $\mathcal{U}$-submodule of $\mathcal{A}$. So the quotient Banach algebra $\mathcal{A}/J$ is a Banach $\mathcal{U}$-module with compatible actions and hence from definition of $J$, when $\mathcal{A}/J$ acts on itself by algebra multiplication, it is a Banach $(\mathcal{A}/J)$-$\mathcal{U}$-module. Therefore, $(\mathcal{A}/J)^{(n)}$ ($n \in \mathbb{N}$) is a Banach $(\mathcal{A}/J)$-$\mathcal{U}$-module. In general $\mathcal{A}/J$ is not a commutative $\mathcal{U}$-module. If $\mathcal{A}/J$ is a commutative $\mathcal{U}$-module, then $(\mathcal{A}/J)^{(n)}$ ($n \geq 0$) is a commutative Banach $(\mathcal{A}/J)$-$\mathcal{U}$-module. Now it is clear when $\mathcal{A}$ is a commutative $\mathcal{U}$-module, then $J = \{0\}$ and hence by multiplication of $\mathcal{A}$ from both sides, $\mathcal{A}^{(n)}$ ($n \geq 0$) is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module.

Let the Banach algebra $\mathcal{A}$ be a Banach $\mathcal{U}$-module with compatible actions. From the above observations, $(\mathcal{A}/J)^{(n)}$ ($n \geq 0$) is a Banach $\mathcal{A}$-$\mathcal{U}$-module by the $\mathcal{A}$-module actions $a\Phi = (a + J)\Phi$ and $\Phi a = \Phi(a + J)$ for $a, b \in \mathcal{A}, \Phi \in (\mathcal{A}/J)^{(n)}$ (the $\mathcal{U}$-module actions are similar to actions on $(\mathcal{A}/J)^{(n)}$ as $\mathcal{U}$-module). Note that whenever $\mathcal{A}/J$ is a commutative $\mathcal{U}$-module, then $(\mathcal{A}/J)^{(n)}$ ($n \geq 0$) is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module by the above actions. Now we are ready to define the notion of $n$-weak module amenability. We say that $\mathcal{A}$ is *$n$-weakly module amenable* ($n \in \mathbb{N}$) if $(\mathcal{A}/J)^{(n)}$ is a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module, and each continuous module derivation $D : \mathcal{A} \to (\mathcal{A}/J)^{(n)}$ is inner; that is $D(a) = D_\Phi(a) = a\Phi - \Phi a$ for some
Now for any $s \in S$, give some properties of these module actions.

Proof. For all $s \in S$, we have the followings

(i) $\delta_e \Phi = \Phi \delta_e$;
(ii) $\delta_e \Phi = \Phi \delta_e$.

Remark 2.1. With the above notation, for all $e \in E$ and $\Phi \in (\ell^1(S)/J)^{(n)} (n \geq 0)$ we have the followings

(i) $\delta_e \Phi = \Phi \delta_e$;
(ii) $\delta_e \Phi = \Phi \delta_e$.

Proof. For all $e, d \in E$, we have $\delta_e - \delta_d = \delta_e \Phi - \delta_d \Phi = \Phi \delta_e - \delta_d \Phi$ and also denote the $\ell^1(S)$ module actions of $f \in \ell^1(S)$ on $\Phi \in (\ell^1(S)/J)^{(n)}$ by $f \Phi$ and $f \Phi$. In the next remark we give some properties of these module actions.

2. Main result

A discrete semigroup $S$ is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^* s = s$ and $s^* ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of $S$ is denoted by $E$. There is a natural order on $E$, defined by

$$e \leq d \iff ed = e \quad (e, d \in E),$$

and $E$ is a commutative subsemigroup of $S$, which is also a semilattice [9, Theorem V.1.2]. Elements of the form $ss^*$ are idempotents of $S$ and in fact all elements of $E$ are in this form.

The algebra $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$. Hence $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions. In this article we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_{se} = \delta_{se} \delta_e = \delta_{se} \delta_e = \delta_{se} \delta_e \quad (s \in S, e \in E).$$

In this case, the ideal $J$ (see section 1) is the closed linear span of $\{\delta_{st} - \delta_{et} \mid s, t \in S, e \in E\}$. With the notations of the previous section $(\ell^1(S)/J)^{(n)} (n \geq 0)$ is a Banach $\ell^1(S)$-$\ell^1(E)$-module. Note that we show the $\ell^1(E)$-module actions of $f \in \ell^1(E)$ on $\Phi \in (\ell^1(S)/J)^{(n)}$ by $f \Phi$ and $f \Phi$, and also denote the $\ell^1(S)$ module actions of $f \in \ell^1(S)$ on $\Phi \in (\ell^1(S)/J)^{(n)}$ by $f \Phi$ and $f \Phi$. In the next remark we give some properties of these module actions.
Similarly, we get $\delta_e + J = \delta_e + J$ for $e \in E$ and $s \in S$. Hence

$$\delta_e.(\delta_s + J) = \delta_s + J = \delta_e + J = (\delta_e + J).\delta_e$$

and

$$\delta_e(\delta_s + J) = (\delta_e + J)(\delta_s + J) = \delta_e + J = \delta_e + J = (\delta_e + J)(\delta_e + J) = (\delta_e + J)\delta_e,$$

for all $e \in E$ and $s \in S$. Since $\text{lin}\{\delta_s \mid s \in S\}$ is dense in $\ell^1(S)$ and $J$ is closed in $\ell^1(S)$, it follows that

$$\delta_e.(f + J) = (f + J).\delta_e$$

and

$$\delta_e(f + J) = f + J = (f + J)\delta_e,$$

for all $e \in E$ and $f \in \ell^1(S)$. So by induction on $n$ we arrive at

$$\delta_e.\Phi = \Phi.\delta_e$$

and

$$\delta_e.\Phi = \Phi\delta_e = \Phi,$$

for all $e \in E$ and $\Phi \in (\ell^1(S)/J)^{(n)}$ $(n \geq 0)$. □

In view of this remark (i), we find that $(\ell^1(S)/J)^{(n)}$ $(n \geq 0)$ is a commutative $\ell^1(E)$-module.

For an inverse semigroup $S$, the quotient $S/\approx$ is a discrete group, where $\approx$ is an equivalence relation on $S$ as follows:

$$s \approx t \iff \delta_s - \delta_t \in J \quad (s, t \in S).$$

Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image $G_S$ [15] of $S$ (see [3], [16] and [17]). As in [18, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(G_S)$. Also see [8].

Since for proof of the main result we use a common fixed point property for semigroups, now we recall some notions related to common fixed point theorem. Let $S$ be a (discrete) semigroup. The space of all bounded complex valued functions on $S$ is denoted by $\ell^\infty(S)$. It is a Banach space with the uniform supremum norm. In fact $\ell^\infty(S) = (\ell^1(S))^\ast$. For each $s \in S$ and each $f \in \ell^\infty(S)$ let $\ell_s f$ be the left translate of $f$ by $s$, that is $\ell_s f(t) = f(st)$ $(t \in S)$ (the right translate $r_s f$ is defined similarly). We recall that $f \in \ell^\infty(S)$ is weakly almost periodic if its left orbit $\mathcal{L}_0(f) = \{\ell_s f \mid s \in S\}$ is relatively compact in the weak topology of $\ell^\infty(S)$. We denote by $WAP(S)$ the space of all weakly almost periodic functions on $S$, which is a closed subspace of $\ell^\infty(S)$ containing the constant function and invariant under the left and right translations. A linear functional $m \in WAP(S)^\ast$ is a mean on $WAP(S)$ if $\|m\| = m(1) = 1$. A mean $m$ on $WAP(S)$ is a left invariant mean (abbreviated $LIM$) if $m(\ell_s f) = m(f)$ for all $s \in S$ and all $f \in WAP(S)$. If $S$ is an inverse semigroup, it is well known that $WAP(S)$ always has a $LIM$ [7, Proposition 2]. Let $C$ be a subset of a Banach space $X$. We say that $\Gamma = \{T_s \mid s \in S\}$ is a representation of $S$ on $C$ if for each $s \in S$, $T_s$ is a mapping from $C$ into $C$ and $T_{st}(x) = T_s(T_t(x))$ $(s, t \in S, x \in C)$. We say that $x \in C$ is a common fixed point for (the representation of) $S$ if $T_s(x) = x$ for all $s \in S$.

Let $X$ be a Banach space and $C$ a nonempty subset of $X$. A mapping $T : C \rightarrow C$ is called nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. The mapping $T$ is called affine if $C$ is convex and $T(\gamma x + \eta y) = \gamma T(x) + \eta T(y)$ for all constants
\( \gamma, \eta \geq 0 \) with \( \gamma + \eta = 1 \) and \( x, y \in C \). A representation \( \Gamma \) of a semigroup \( S \) on \( C \) acts as nonexpansive affine mappings, if each \( T_s \) \( (s \in S) \) is nonexpansive and affine.

A Banach space \( X \) is called \( L\)-embedded if there is a closed subspace \( X_0 \subseteq X^{**} \) such that \( X^{**} = X \oplus_{\ell_1} X_0 \). The class of \( L \)-embedded Banach spaces includes all \( L^1(\Sigma, \mu) \) (the space of of all absolutely integrable functions on a measure space \( (\Sigma, \mu) \)), preduals of von Neumann algebras, dual spaces of \( M \)-embedded Banach spaces and the Hardy space \( H_1 \). In particular, given a locally compact group \( G \), the space \( L^1(G) \) is \( L \)-embedded. So are its even duals \( L^1(G)^{(2n)} \) \((n \geq 0)\). For more details we refer the reader to [19] and the references therein.

The next lemma is the common fixed point theorem for semigroups in [19, Theorem 2], which will be used in our proof to the main result.

**Lemma 2.2.** Let \( S \) be a discrete semigroup and \( \Gamma \) a representation of \( S \) on an \( L \)-embedded Banach space \( X \) as nonexpansive affine mappings. Suppose that \( WAP(S) \) has a \( LIM \) and suppose that there is a nonempty bounded set \( B \subseteq X \) such that \( B \subseteq \overline{T_s(B)} \) for all \( s \in S \), then \( X \) contains a common fixed point for \( S \).

We now can prove the main result of the paper.

**Theorem 2.3.** Let \( S \) be an inverse semigroup with the set of idempotents \( E \). Consider \( \ell^1(S) \) as a Banach module over \( \ell^1(E) \) with the trivial left action and natural right action. Then the semigroup algebra \( \ell^1(S) \) is 2n-weakly module amenable as an \( \ell^1(E) \)-module for each \( n \in \mathbb{N} \).

**Proof.** Let \( D : \ell^1(S) \to (\ell^1(S)/J)^{(2n)} \) be a continuous module derivation. Since \( ss^* \in E \) for all \( s \in S \), from Remark 2.1(ii), we have

\[
D(\delta_{ss^*}) = D(\delta_{ss^*} \ast \delta_{ss^*}) = D(\delta_{ss^*} \ast \delta_{ss^*}) \\
= \delta_{ss^*} D(\delta_{ss^*}) + D(\delta_{ss^*}) \delta_{ss^*} \\
= 2D(\delta_{ss^*}).
\]

Hence \( D(\delta_{ss^*}) = 0 \) for all \( s \in S \). Define \( \phi : S \to (\ell^1(S)/J)^{(2n)} \) by

\[
\phi(s) = D(\delta_s) \delta_s \quad (s \in S).
\]

We see that

\[
\phi(st) = D(\delta_s \ast \delta_t) \delta_{st} \\
= (\delta_s D(\delta_t)) \delta_s \ast \delta_t + (D(\delta_s) \delta_t) \delta_s \ast \delta_t \\
= \delta_s (D(\delta_t) \delta_t) \delta_s \ast \delta_t + (D(\delta_s) \delta_t) \delta_s \ast \delta_t \\
= \delta_s (\delta(\delta_t) \delta_t) \delta_s \ast \delta_t + D(\delta_s) \delta_s \ast \delta_t \\
= \delta_s \phi(t) \delta_s \ast \phi(s),
\]

for all \( s, t \in S \). Let \( B = \phi(S) \). Then \( B \) is a nonempty bounded subset of \( (\ell^1(S)/J)^{(2n)} \). For any \( s \in S \) define the mapping \( T_s : (\ell^1(S)/J)^{(2n)} \to (\ell^1(S)/J)^{(2n)} \) by

\[
T_s(\Phi) = \delta_s \Phi \delta_s \ast \phi(s) \quad (\Phi \in (\ell^1(S)/J)^{(2n)}).
\]

Clearly each \( T_s \) \( (s \in S) \) is an affine mapping and for every \( \Phi, \Psi \in (\ell^1(S)/J)^{(2n)} \) and \( s \in S \) we have

\[
\| T_s(\Phi) - T_s(\Psi) \| = \| \delta_s \Phi \delta_s \ast \phi(s) - \delta_s \Psi \delta_s \ast \phi(s) \| \leq \| \Phi - \Psi \|.
\]
So each $T_s$ ($s \in S$) is nonexpansive. Now by using (1) for any $s, t \in S$ and $\Phi, \Psi \in (\ell^1(S)/J)^{(2n)}$ we find
\[
T_{st}(\Phi) = \delta_{st} \Phi \delta_{(st)^*} + \phi(st)
\]
\[
= \delta_s (\delta_t \Phi \delta_{(st)^*}) \delta_{s^*} + \delta_t \phi(t) \delta_{s^*} + \phi(s)
\]
\[
= \delta_s T_t(\Phi) \delta_{s^*} + \phi(s)
\]
\[
= T_s(T_t(\Phi)).
\]
So $\Gamma = \{T_s \mid s \in S\}$ defines a representation of $S$ on $(\ell^1(S)/J)^{(2n)}$ which is nonexpansive and affine. From definition of $T_s$ and (1), for any $s, t \in S$ it follows that $T_s(\Phi(t)) = \delta_s \phi(t) \delta_{s^*} + \phi(s) = \phi(st)$. Therefore $T_s(B) \subseteq B$ ($s \in S$). Let $\Phi \in B$. Now by Remark 2.1(ii) and the fact that $D(\delta_{s^*}) = 0$ ($s \in S$), we have
\[
T_s(T_{s^*}(\Phi)) = T_{ss^*}(\Phi) = \delta_{s^*} \Phi \delta_{ss^*} + \phi(ss^*) = \Phi \quad (s \in S).
\]
Since $T_{ss^*}(\Phi) \in B$, it follows that $T_s(B) = B$ for each $s \in S$. Here $S$ is regarded as a discrete semigroup.

Since $\ell^1(S)/J \cong \ell^1(G_S)$, where $G_S$ is the maximal group homomorphic image, it follows that $(\ell^1(S)/J)^{(2n)}$ is $L$-embedded. Also WAP$(S)$ has a LIM. So by Lemma 2.2, there is $\Upsilon \in (\ell^1(S)/J)^{(2n)}$ such that $T_s(\Upsilon) = \Upsilon$ for all $s \in S$, or
\[
\delta_s \Upsilon \delta_{s^*} + \phi(s) = \Upsilon,
\]
for all $s \in S$. So $\delta_s \Upsilon \delta_{s^*} + D(\delta_s) \delta_{s^*} = \Upsilon$ ($s \in S$). Hence
\[
D(\delta_s) = \Upsilon \delta_s - \delta_s \Upsilon,
\]
for all $s \in S$. By definition of left module action of $\ell^1(E)$ on $\ell^1(S)$, we have $\delta_e \delta_s = \delta_s$ ($e \in E, s \in S$). Since $\text{lin}\{\delta_s \mid s \in S\}$ is dense in $\ell^1(S)$, we find $\delta_\varepsilon f = f$ for all $e \in E$ and $f \in \ell^1(S)$. Hence $\delta_e (f + J) = f + J$ ($e \in E, f \in \ell^1(S)$). Furthermore a routine inductive argument shows that for each $e \in E$ and $\Phi \in (\ell^1(S)/J)^{(2n)}$ ($n \geq 0$), we have $\delta_e \Phi = \Phi$. From this result and the fact that $D$ is a module mapping, for any $s \in S$ and $\lambda \in \mathbb{C}$ we have
\[
D(\lambda \delta_s) = D(\lambda \delta_{ss^*} \delta_s)
\]
\[
= \lambda \delta_{ss^*} \lambda D(\delta_s)
\]
\[
= \lambda \delta_{ss^*} \Upsilon \delta_s - \delta_s \Upsilon
\]
\[
= \lambda \Upsilon \delta_s - \delta_s \Upsilon.
\]
Since $D$ is additive, we get $D(f) = \Upsilon f - f \Upsilon$ for any $f \in \ell^1(S)$ of finite support. But $D$ is continuous and functions of finite support are dense in $\ell^1(S)$, hence
\[
D(f) = \Upsilon f - f \Upsilon = D(\Upsilon^{-1}) f \quad (f \in \ell^1(S)),
\]
therefore $D$ is inner. The proof is complete.

In [4], it has been proved that $\ell^1(S)$ is $(2n + 1)$-weakly module amenable as an $\ell^1(E)$-module, for each $n \in \mathbb{N}$, where $S$ is an inverse semigroup with the set of idempotents $E$. From this result and above theorem we get the next corollary.

**Corollary 2.4.** Let $S$ be an inverse semigroup with the set of idempotents $E$. Consider $\ell^1(S)$ as a Banach module over $\ell^1(E)$ with the trivial left action and natural right action. Then the semigroup algebra $\ell^1(S)$ is permanently weakly module amenable as an $\ell^1(E)$-module.
With the notations in previous corollary, we have the next result.

**Corollary 2.5.** Each continuous module derivation $D : \ell^1(S) \to (\ell^1(G_S))^{(n)}$ ($n \in \mathbb{N}$) is inner.

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