The absence of cut–off effects for the fixed point action in 1–loop perturbation theory

Federico Farchioni
Dipartimento di Fisica dell’Università and I.N.F.N.
Piazza Torricelli 2, I–56126 Pisa, Italy.

Peter Hasenfratz, Ferenc Niedermayer and Alessandro Papa
Institute for Theoretical Physics
University of Bern
Sidlerstrasse 5, CH–3012 Bern, Switzerland

July 2018

Abstract

In order to support the formal renormalization group arguments that the fixed point action of an asymptotically free model gives cut–off independent physical predictions in 1–loop perturbation theory, we calculate the finite volume mass–gap \( m(L) \) in the non–linear \( \sigma \)–model. No cut–off effect of the type \( g^4 (a/L)^n \) is seen for any \( n \). The results are compared with those of the standard and tree level improved Symanzik actions.

\[ ^1 \text{Work supported in part by Schweizerischer Nationalfonds.} \]
\[ ^2 \text{On leave from the Institute of Theoretical Physics, Eötvös University, Budapest} \]
\[ ^3 \text{On leave from Dipartimento di Fisica, Università di Pisa and I.N.F.N., Pisa} \]
1 Introduction

Lattice actions lying on the renormalized trajectory of a renormalization group (RG) transformation are perfect in the sense that all the spectral quantities are free of lattice artefacts independently of the resolution. In asymptotically free theories the renormalized trajectory starts from a fixed point (FP) at \( g = 0 \), where \( g \) is the asymptotically free coupling of the continuum formulation. The FP action is the classically perfect lattice regularization of the field theory \( [1] \): its classical solutions (instantons) are scale invariant and in quadratic approximation in the fields the spectrum is exact. The FP action performs amazingly well when used in numerical simulations at small correlation lengths also \( [2] \).

Wilson remarked some time ago \( [3] \) that — according to formal RG arguments — the change of the FP action under a RG transformation in 1–loop perturbation theory is simple:

\[
\beta S_{FP} \quad \rightarrow \quad \beta' S_{FP},
\]

RG, 1–loop

where \( \beta' = \beta - \Delta \beta \) and \( \Delta \beta \) is fixed by the first coefficient of the \( \beta \)-function. Wilson did not elaborate this problem further. In \( [4] \) a set of formal RG arguments were presented to support the statement in eq. (1). The question is important since eq. (1) would imply that the FP action is 1–loop (quantum) perfect.

We are not able to make the formal arguments of \( [4] \) more rigorous\(^4\). We present here an explicit 1–loop calculation in the \( d = 2 \) non–linear \( \sigma \)-model to support the arguments in \( [4] \) further.

We calculate the mass gap \( m(L) \) in 1–loop perturbation theory using the FP action constructed and studied in \( [1] \). The mass gap \( m(L) \) has already been calculated up to 2 loops using the standard action \( [5] \). The 1–loop result in \( O(N) \) has the general form\(^5\)

\[
m(L) = \frac{N-1}{2} \left[ g^2 + g^4 \left( \frac{N-2}{2\pi} \ln \frac{L}{a} + R_1 + (N-1)R_2 \right) \right],
\]

(2)

where \( R_i, i = 1, 2 \) are independent of \( N \)

\[
R_i = A_i + \frac{a^2}{L^2} \left( c_{i1} + d_{i1} \ln \frac{L}{a} \right) + \frac{a^4}{L^4} \left( c_{i2} + d_{i2} \ln \frac{L}{a} \right) + \cdots .
\]

\(^4\)The statement in eq. (1) is, presumably, not even strictly correct due to possible redundant operators. They, however, would not change the physical content of eq. (1).

\(^5\)We denote the coupling constant in the action by \( g^2 \) deviating from the notation in \( [1] \).
The cut–off dependent terms are not universal, they depend on the explicit form of the lattice action. The constants $A_i$ determine the relation between the coupling constants (or the $\Lambda$–parameters) of the different lattice regularizations.

If the relation (1) is valid then no cut–off effects in $R_i$ should be present to arbitrary order in $(a/L)$. We have calculated the terms $R_i$ in eq. (2) for the FP, standard and tree level improved Symanzik actions \cite{6}. For the FP action the coefficients $c_{ik}$ and $d_{ik}$ of the cut–off dependent terms in eq. (3) turned out to be zero within the numerical precision of the calculation. These coefficients are typically $O(1)$ for the standard and Symanzik actions.

Readers who are not interested in the technical details are advised to skip the next section and go directly to the results.

2 One–loop perturbation theory with the FP action

The FP action is a specific lattice regularization of the formal expression

$$\beta A^{\text{cont}}(\vec{S}) = \frac{\beta}{2} \int d^2 x \, \partial_\mu \vec{S} \partial_\mu \vec{S},$$

where $\beta = 1/g^2$ and the $N$–component vector $\vec{S}$ satisfies the constraint $\vec{S}^2(x) = 1$. It is convenient to parametrize the FP action as

$$A^{FP}(\vec{S}) = -\frac{1}{2} \sum_{n,r} \rho(r) \left( 1 - \vec{S}_n \vec{S}_{n+r} \right) + \sum_{n_1, n_2, n_3, n_4} c(n_1, n_2, n_3, n_4) \left( 1 - \vec{S}_{n_1} \vec{S}_{n_2} \right) \left( 1 - \vec{S}_{n_3} \vec{S}_{n_4} \right) + \cdots,$$

where the coupling constants $\rho, c, \cdots$ are determined by a classical saddle point equation \cite{1}. Writing

$$\vec{S}_n = \left( \sqrt{1 - g^2 \vec{\pi}_n^2} \right),$$

where the field $\vec{\pi}_n$ has $N-1$ components, a perturbation theory can be set up by considering $g\vec{\pi}$ as a small fluctuation. The higher order couplings, which are indicated only implicitly in eq. (5), do not enter in a 1–loop calculation.

There is a technical problem (which is independent of the action) when the mass gap is calculated in a finite periodic box. In order to obtain the mass
The zero (spatial) momentum two–point function is calculated at large time separations in a cylinder whose extension in time is much larger than $L$. In this finite euclidean space there are $N - 1$ zero modes which are, however easy to separate and handle. The real problem is the presence of quasi–zero modes related to the slow motion of the “magnetization” $\mathcal{M}(t) = \sum_x \vec{S}(t, x)$. The dynamics of $\mathcal{M}(t)$ is described in leading order by a rotator which has the spectrum
\[ E_l = \frac{g^2}{2L} (l + N - 2) , \quad l = 0, 1, \ldots . \] These are slow modes with energy much below the normal excitation energies $\sim \frac{2\pi}{L}$. A possible solution is to introduce collective coordinates for these quasi–zero modes and study their dynamics. A more elegant and technically simpler solution is to observe that free boundary conditions in time direction project on $O(N)$ singlet states and so these modes enter only as intermediate states.

We consider a cylinder of size $(2T + 1) \times L$ with free boundary conditions at $x_0 = \pm T$, periodic boundary conditions in $x_1$, $x_1 = 0, 1, \ldots, L - 1$, and calculate the correlation function
\[ C(\tau) = \frac{1}{L^2} \sum_{x_1, y_1} \langle \vec{S}(x) \vec{S}(y) \rangle_{x_0 = y_0 = \tau} . \] We shall stay close to the notations introduced in [4]. The form of the FP action eq. (5) and the need to use free boundary conditions suggest to work in configuration space.

The propagator has the form
\[ D(x_0, x_0'; x_1 - x_1') = \frac{1}{L} \sum_q e^{iq(x_1 - x_1')} R^{-1}(x_0, x_0'; q) , \] where $q = \frac{2\pi}{L} \cdot k$, $k = 0, \ldots, L - 1$ and the $(2T + 1) \times (2T + 1)$ dimensional matrix $R(x_0, x_0'; q)$ ($q$ fixed) is defined as
\[ R(x_0, x_0'; q) = \rho(x_0 - x_0'; q) - \delta_{x_0, x_0} f(x_0) , \quad q \neq 0 , \] with
\[ f(x_0) = \sum_{x_0' = -T}^{T} \rho(x_0 - x_0'; q = 0) . \]
In Eqs. (10), (11) \( \rho(x_0; q) \) are the quadratic couplings Fourier transformed in space:

\[
\rho(x_0; q) = \sum_{x_1=0}^{L-1} e^{-iqx_1} \rho(x_0, x_1).
\] (12)

For \( q = 0 \) \( R \) has an extra term (related to the constraint \( \sum_x \vec{\pi}(x) = 0 \) which enters the path integral when eliminating the global zero mode)

\[
R(x_0, x'_0; q = 0) = \rho(x_0 - x'_0; q = 0) - \delta_{x_0, x'_0} f(x_0) + \lambda,
\] (13)

where \( \lambda \) is an arbitrary positive parameter. The limit \( \lambda \to \infty \) corresponds to the constraint \( \delta(\sum_x \vec{\pi}(x)) \), but it is easy to see that the final results are independent of \( \lambda \). Due to the free boundary conditions the propagator is not translation invariant in time.

Using the explicit representation of the quadratic couplings \( \rho \) in eq. (17) of Ref. [1] (with the optimized parameter \( \kappa = 2 \)) one can obtain the propagator and those vertices which are proportional to \( \rho \) to high precision (close to machine precision). On the other hand, in solving the FP equations for \( c(n_1, n_2, n_3, n_4) \) we had to introduce cuts. The numerical errors in our results are dominated by the errors in the couplings \( c \).

3 Results

In order to simplify the discussion and save space we present the results for \( N = 3 \). We introduce the notations \( A = A_1 + 2A_2, c_1 = c_{11} + 2c_{21}, \) etc.

The constant \( A \) can be calculated simply by using the 1-loop results on \( m(L) \) in continuum perturbation theory in the \( \overline{\text{MS}} \) scheme [1], and the ratios between \( \Lambda_{\overline{\text{MS}}} \) and the \( \Lambda \)-parameter of the lattice action under consideration. For the standard and the Symanzik actions this ratio is known [12] [13]. Using the general expression in the Appendix of [1] we obtained for the FP action

\[
\Lambda^{(N=3)}_{FP} = 9.424754598 \Lambda_{st},
\] (14)

where \( \Lambda_{FP} \) is the \( \Lambda \)-parameter of the action defined by the couplings \( \rho \) and \( c \) of eq. [1], which are used in the following mass gap calculation. The number in eq. (14) is somewhat different from that corresponding to a parametrized form of the FP action which was used in numerical simulations earlier [1]. The corresponding constants \( A \) are given in Table 1 for the standard, Symanzik and FP actions.
The constant $A = A_1 + (N - 1)A_2$ (see eq. (3)) is given for $N = 3$ for the different actions considered.

|                   | standard          | Symanzik          | FP               |
|-------------------|-------------------|-------------------|------------------|
| $A$               | 0.214836206       | 0.087964307       | −0.142202395     |

We calculated the two point function $C(\tau)$ of eq. (8) on a cylinder $(2T+1) \times L$, where $\tau \cdot 4\pi/L \gg 1$ and $(T - \tau) \cdot 4\pi/L \gg 1$, where $4\pi/L$ is the energy of the first excited state in the singlet channel [5]. Depending on $L$ we used $T$ and $\tau$ in the range $40 - 90$ and $9 - 45$, respectively. The consistency conditions assuring correct exponentialization [5] were satisfied up to 9 digits, or better. The cut–off dependent part of the $O(g^4)$ result $(R - A)$ is given in Table 2 for the different actions and $L = 2, 3, \ldots, 10$.

| $L$ | $R_{st} - A_{st}$ | $R_{Sym} - A_{Sym}$ | $R_{FP} - A_{FP}$ |
|-----|-------------------|--------------------|-------------------|
| 2   | 0.086457646       | 0.020052439        | −0.000320928      |
| 3   | 0.037574920       | 0.004628052        | −0.00011974       |
| 4   | 0.020334020       | 0.001463304        | −0.000003526      |
| 5   | 0.012697662       | 0.000607401        | −0.000001045      |
| 6   | 0.008699585       | 0.000298780        | −0.000000314      |
| 7   | 0.006343465       | 0.000163770        | −0.000000103      |
| 8   | 0.004834753       | 0.000097027        | −0.000000036      |
| 9   | 0.003808864       | 0.000061034        | −0.000000008      |
| 10  | 0.003078960       | 0.000040271        | −0.000000005      |

Both the standard and the Symanzik actions give power decaying cut–off corrections. In the latter case the $\sim a^2/L^2$ leading term seems to be missing, or very small. For the FP action the power–like cut–off effects are tiny, in the range of $L = 5, \ldots, 10$ are about 5 orders of magnitude smaller than those of the standard action. The numerical errors in the results of the FP action are dominated by the errors in the quartic couplings $c$. They are obtained by solving the FP equation by iteration where unavoidably cuts have to be introduced [1]. For the quartic couplings one can derive different sum rules which are satisfied by our couplings up to 6–digits accuracy. We did not attempt to translate this error into a quantitative error estimate on the mass gap $m(L)$, but it seems to us plausible that it can produce the tiny power–like cut–off effects seen.
Figure 1: The value of \( R = R_1 + 2R_2 \) of eq. (3) vs. \((a/L)^2\) for the FP action for \( L/a = 2, 3, \ldots, 10 \). The fit is \(-0.1422022 - 0.0000176 (a/L)^2\). Note that the exact limiting value is \(-0.1422024\).

The formal RG arguments which lead to eq. (1) are valid in an infinite system. In a box whose size is comparable to the range of the interaction cut–off effects are generated which should go however to zero exponentially as the size of the system is increased. Similar cut–off effects were observed in the correlation function of FP operators \([4]\). The fit in fig. 1 shows this additional cut–off effect at \( L = 2 \). This is a real effect which decays rapidly and becomes part of the numerical error for \( L > 3 \) and it is related to the finite extension of the FP action.

References

[1] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B414 (1994) 785; P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 34 (1994) 3; F. Niedermayer, ibid 513.

[2] T. DeGrand, A. Hasenfratz, P. Hasenfratz and F. Niedermayer, preprint COLO–HEP–362, BUTP–95/15, (1995).

[3] K.G. Wilson, in Recent developments of gauge theories, ed. G. ’t Hooft et al. (Plenum, New York, 1980).
[4] T. DeGrand, A. Hasenfratz, P. Hasenfratz and F. Niedermayer, preprint COLO–HEP–361, BUTP–95/14, (1995).

[5] M. Lüscher, P. Weisz, U. Wolff, Nucl. Phys. B359 (1991) 221.

[6] K. Symanzik, Nucl. Phys. B226 (1983) 187, 205.

[7] P. Hasenfratz, Phys. Lett. B141 (1984) 385.

[8] E. Brézin and J. Zinn–Justin, Nucl. Phys. B257 [FS14] (1985) 867.

[9] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B343 (1990) 241.

[10] P. Hasenfratz and F. Niedermayer, Z. Phys. B92 (1993) 91.

[11] M. Lüscher, Phys. Lett. B118 (1982) 391.

[12] G. Parisi, Phys. Lett. B92 (1980) 133; J. Shigemitsu and J.B. Kogut, Nucl. Phys. B190 [FS3] (1981) 365.

[13] Y. Iwasaki and T. Yoshié, Phys. Lett. B125 (1983) 201; A. Hasenfratz and A. Margaritis, Phys. Lett. B133 (1983) 211; ibid B148 (1984) 129.