SPECTRAL ESTIMATES FOR RUELLE TRANSFER OPERATORS WITH TWO PARAMETERS AND APPLICATIONS

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Abstract. For $C^2$ weak mixing Axiom A flow $\phi_t : M \to M$ on a Riemannian manifold $M$ and a basic set $\Lambda$ for $\phi_t$, we consider the Ruelle transfer operator $L_{f-g+s-t}$, where $f$ and $g$ are real-valued H"older functions on $\Lambda$, $\tau$ is the roof function and $s, z \in \mathbb{C}$ are complex parameters. Under some assumptions about $\phi_t$ we establish estimates for the iterations of this Ruelle operator in the spirit of the estimates for operators with one complex parameter (see [4], [20], [21]). Two cases are covered: (i) for arbitrary H"older $f, g$ when $|\text{Im} z| \leq B |\text{Im} s|^\mu$ for some constants $B > 0$, $0 < \mu < 1$ ($\mu = 1$ for Lipschitz $f, g$), (ii) for Lipschitz $f, g$ when $|\text{Im} s| \leq B_1 |\text{Im} z|$ for some constant $B > 0$.

Applying these estimates, we obtain a non zero analytic extension of the zeta function $\zeta(s, z)$ for $P_f - \epsilon < \text{Re}(s) < P_f$ and $|z|$ small enough with simple pole at $s = s(z)$. Two other applications are considered as well: the first concerns the Hannay-Ozorio de Almeida sum formula, while the second deals with the asymptotic of the counting function $\pi_p(T)$ for weighted primitive periods of the flow $\phi_t$.

1. Introduction

Let $M$ be a $C^2$ complete (not necessarily compact) Riemannian manifold, and let $\phi_t : M \to M$, $t \in \mathbb{R}$, be a $C^2$ weak mixing Axiom A flow (see [2], [11]). Let $\Lambda$ be a basic set for $\phi_t$, i.e. $\Lambda$ is a compact $\phi_t$-invariant subset of $M$, $\phi_t$ is hyperbolic and transitive on $\Lambda$ and $\Lambda$ is locally maximal, i.e. there exists an open neighborhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \cap_{t \in \mathbb{R}} \phi_t(V)$. The restriction of the flow $\phi_t$ on $\Lambda$ is a hyperbolic flow [11]. For any $x \in M$ let $W^s_\epsilon(x), W^u_\epsilon(x)$ be the local stable and unstable manifolds through $x$, respectively (see [2], [6], [11]).

When $M$ is compact and $M$ itself is a basic set, $\phi_t$ is called an Anosov flow. It follows from the hyperbolicity of $\Lambda$ that if $\epsilon_0 > 0$ is sufficiently small, there exists $\epsilon_1 > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \epsilon_1$, then $W^s_\epsilon(x)$ and $\phi_{[-\epsilon_0, \epsilon_0]}(W^u_\epsilon(y))$ intersect at exactly one point $[x, y] \in \Lambda$ (cf. [6]). This means that there exists a unique $t \in [-\epsilon_0, \epsilon_0]$ such that $\phi_t([x, y]) \in W^u_\epsilon(y)$. Setting $\Delta(x, y) = t$, defines the so called temporal distance function.

In the paper we will use the set-up and some arguments from [20]. First, as in [20], we fix a (pseudo-) Markov partition $\mathcal{R} = \{R_i\}_{i=1}^k$ of pseudo-rectangles $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$. Set $R = \cup_{i=1}^k R_i$, $U = \cup_{i=1}^k U_i$. Consider the Poincaré map $\mathcal{P} : R \to R$, defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in R$. The function $\tau$ is the so called first return time associated with $\mathcal{R}$. Let $\sigma : U \to \hat{U}$ be the shift map given by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \to U$ is the projection along stable leaves. Let $\hat{U}$ be the set of those points $x \in U$ such that $\mathcal{P}^m(x)$ is not a boundary point of a rectangle for any integer $m$. In a similar way define $\hat{R}$. Clearly in general $\tau$ is not continuous on $U$, however under the assumption that the 1991 Mathematics Subject Classification. Primary 37C30, Secondary 37D20, 37C35.

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holonomy maps are Lipschitz (see Sect. 3) \( \tau \) is essentially Lipschitz on \( U \) in the sense that there exists a constant \( L > 0 \) such that if \( x, y \in U_i \cap \sigma^{-1}(U_j) \) for some \( i, j \), then \( |\tau(x) - \tau(y)| \leq L d(x, y) \). The same applies to \( \sigma : U \rightarrow U \).

The hyperbolicity of the flow on \( \Lambda \) implies the existence of constants \( c_0 \in (0, 1) \) and \( \gamma_1 > \gamma_0 > 1 \) such that

\[
c_0 \gamma_0^m d(u_1, u_2) \leq d(\sigma^m(u_1), \sigma^m(u_2)) \leq \frac{\gamma_1^m}{c_0} d(u_1, u_2)
\]

whenever \( \sigma^j(u_1) \) and \( \sigma^j(u_2) \) belong to the same \( U_{ij} \) for all \( j = 0, 1, \ldots, m \).

Define a \( k \times k \) matrix \( A = \{A(i, j)\}_{i,j=1}^k \) by

\[
A(i, j) = \begin{cases} 
1 & \text{if } P(\text{Int } R_i) \cap \text{Int } R_j \neq \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

It is possible to construct a Markov partition \( \mathcal{R} \) so that \( A \) is irreducible and aperiodic (see [2]). Introduce \( R^r = \{(x, t) \in R \times R : 0 \leq t \leq \tau(x)\}/\sim \), where by \( \sim \) we identify the points \( (x, \tau(x)) \) and \( (\sigma x, 0) \). One defines the suspended flow \( \sigma^j_t(x, s) = (x, s + t) \) on \( R^r \) taking into account the identification \( \sim \). For a Hölder continuous function \( f \) on \( R \), the pressure \( \text{Pr}(f) \) with respect to \( \sigma \) is defined as

\[
\text{Pr}(f) = \sup_{m \in M_\sigma} \{ h(\sigma, m) + \int f \, dm \},
\]

where \( M_\sigma \) denotes the space of all \( \sigma \)-invariant Borel probability measures and \( h(\sigma, m) \) is the entropy of \( \sigma \) with respect to \( m \). We say that \( f \) and \( g \) are cohomologous and we denote this by \( f \sim g \) if there exists a continuous function \( w \) such that \( f = g + w \circ \sigma - w \). For a function \( v \) on \( R \) one defines

\[
v^n(x) := v(x) + v(\sigma(x)) + \ldots + v(\sigma^{n-1}(x)).
\]

Let \( \gamma \) denote a primitive periodic orbit of \( \phi_t \) and let \( \lambda(\gamma) \) denote its least period. Given a Hölder function \( F : \Lambda \rightarrow \mathbb{R} \), introduce the weighted period \( \lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x)) \, dt \), where \( x_\gamma \in \gamma \). Consider the weighted version of the dynamical zeta function (see Section 9 in [11])

\[
\zeta_\phi(s, F) := \prod_\gamma \left(1 - e^{\lambda_F(\gamma) - s \lambda(\gamma)}\right)^{-1}.
\]

Denote by \( \pi(x, t) : R^r \rightarrow \Lambda \) the semi-conjugacy projection which is one-to-one on a residual set and \( \pi(t, x) \circ \sigma^j_t = \phi_t \circ \pi(t, x) \) (see [2]). Then following the results in [2], [3], a closed \( \sigma \)-orbit \( \{x, \sigma x, \ldots, \sigma^{n-1}x\} \) is projected to a closed orbit \( \gamma \) in \( \Lambda \) with a least period

\[
\lambda(\gamma) = \tau^n(x) := \tau(x) + \tau(\sigma(x)) + \ldots + \tau(\sigma^{n-1}(x)).
\]

Passing to the symbolic model \( R \) (see [2], [11]), the analysis of \( \zeta_\phi(s, F) \) is reduced to that of the Dirichlet series

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{f(x) - s \tau^n(x)}.
\]

with a Hölder continuous function \( f(x) = \int_0^{\tau(x)} F(\pi(x, t)) \, dt : R \rightarrow \mathbb{R} \). On the other hand, to deal with certain problems (see Chapter 9 in [11] and [16]) it is necessary to study a more general series

\[
\eta_g(s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} g^n(x) e^{f^n(x) - s \tau^n(x)}
\]
with a Hölder continuous function \( G : \Lambda \to \mathbb{R} \) and \( g(x) = \int_0^1 G(\pi(x,t))dt : R \to \mathbb{R} \). For this purpose it is convenient to examine the zeta function

\[
\zeta(s,z) := \prod_\gamma \left( 1 - e^{\lambda_F(\gamma)-s\lambda(\gamma)+z\lambda_F(\gamma)} \right)^{-1} = \exp\left( \sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^nx=x} e^{n(x)-s\tau^n(x)+z g^n(x)} \right)
\]

(1.2)

depending on two complex variables \( s, z \in \mathbb{C} \). Formally, we get

\[
\eta_g(s) = \frac{\partial \log \zeta(s,z)}{\partial z} \bigg|_{z=0}.
\]

**Example 1.** If \( G = 0 \) we obtain the classical Ruelle dynamical zeta function

\[
\zeta_\phi(s) = \prod_\gamma \left( 1 - e^{-s\lambda(\gamma)} \right)^{-1}.
\]

Then \( Pr(0) = h \), where \( h > 0 \) is the topological entropy of \( \phi_t \) and \( \zeta_\phi(s) \) is absolutely convergent for \( \text{Re } s > h \) (see Chapter 6 in [11]).

**Example 2.** Consider the expansion function \( E : \Lambda \to \mathbb{R} \) defined by

\[
E(x) := \lim_{t \to 0} \frac{1}{t} \log |\text{Jac}(D\phi_t|E^u(x))|,
\]

where the tangent space \( T_x(M) \) is decomposed as \( T_x(M) = E_s(x) \oplus E^0(x) \oplus E^u(x) \) with \( E_s(x), E^u(x) \) tangent to stable and instable manifolds through \( x \), respectively. Introduce the function \( \lambda^u(\gamma) = \lambda_E(\gamma) \) and define \( f : R \to \mathbb{R} \) by

\[
f(x) = - \int_0^{\tau(x)} E(\pi(x,t))dt.
\]

Then we have \(-\lambda^u(\gamma) = f^u(x)\), \( f \) is Hölder continuous function and \( Pr(f) = 0 \) (see [3]). Consequently, the series

\[
\sum_{n=1}^\infty \frac{1}{n} \sum_{\sigma^nx=x} e^{f^n(x)-s\tau^n(x)}
\]

(1.3)
is absolutely convergent for \( \text{Re } s > 0 \) and nowhere zero and analytic for \( \text{Re } s \geq 0 \) except for a simple pole at \( \text{Re } s = 0 \) (see Theorem 9.2 in [11]). The roof functions \( \tau(x) \) is constant on stable leaves of rectangles \( R_i \) of the Markov family \( \mathcal{R} \), so we can assume that \( \tau(x) \) depends only on \( x \in U \). By a standard argument (see [11]) we can replace \( f \) in (1.3) by a Hölder function \( \tilde{f}(x) \) which depends only on \( x \in U \) so that \( f \sim \tilde{f} \). Thus the series (1.3) can be written by functions \( \tilde{f}, \tau \) depending only on \( x \in U \). We keep the notation \( f \) below assuming that \( f \) depends only on \( x \in U \). The analysis of the analytic continuation of (1.3) is based on spectral estimates for the iterations of the Ruelle operator

\[
L_f \sigma v(x) = \sum_{\sigma y = x} e^{f(y)-\sigma \tau(y)} v(y), \ v \in C^\alpha(U), \ s \in \mathbb{C}.
\]

(see for more details [1, 15, 20, 21, 23]).

**Example 3.** Let \( f, \tau \) be real-valued Hölder functions and let \( P_f > 0 \) be the unique real number such that \( Pr(f-P_f \tau) = 0 \). Let \( g(x) = \int_0^1 G(\pi(x,t))dt \), where \( G : \Lambda \to \mathbb{R} \) is a Hölder function. Then if the suspended flow \( \sigma^r_t \) is weak-mixing, the function (1.2) is nowhere zero analytic function.
for Re \( s > P_f \) and \( z \) in a neighborhood of 0 (depending on \( s \)) with nowhere zero analytic extension to Re \( s = P_f \) (\( s \neq P_f \)) for small \( |z| \). This statement is just Theorem 6.4 in [11]. To examine the analytic continuation of \( \zeta(s,z) \) for \( P_f - \eta_0 \leq \text{Re } s, \eta_0 > 0 \) and small \( |z| \), it is necessary to establish and to exploit some spectral estimates for the iterations of the Ruelle operator

\[
L_{f-st+zg}^n(x) = \sum_{\sigma y = x} e^{f(y)-s\tau(y)+zg(y)} v(y), \quad v \in C^\alpha(U), \quad s \in \mathbb{C}, z \in \mathbb{C}.
\]

The analytic continuation of \( \zeta(s,z) \) for small \( |z| \) and that of \( \eta_0(s) \) play a crucial role in the argument in [16] concerning the Hannay-Ozorio de Almeida sum formula for the geodesic flow on compact negatively curved surfaces. We deal with the same question for Axiom A flows on basic sets in Sect. 7.

**Example 4.** In the paper [17] the authors examine for Anosov flows the spectral properties of the Ruelle operator \( L_{f-(P_f+a+ib)\tau+iw}^n \) with \( f = 0 \) and \( z = iw, w \in \mathbb{R} \), as well as the analyticity of the corresponding \( L \)-function \( L(s,z) \). The properties of the Ruelle operator

\[
L_{f-(P_f+a+ib)\tau+iw}^n, \quad w \in \mathbb{R}, n \in \mathbb{N},
\]

are also rather important in the paper [22] dealing with the large deviations for Anosov flows. Here as above \( P_f \in \mathbb{R} \) is such that \( Pr(f-P_f\tau) = 0 \). However, it is important to note that in [7] and [22] the analysis of the Ruelle operators covers mainly the domain \( \text{Re } s \geq P_f \) and there are no results treating the spectral properties for \( P_f - \eta_0 \leq \text{Re } s < P_f \) and \( z = iw, w \in \mathbb{R} \). To our best knowledge the analytic continuation of the function \( \zeta(s,z) \) for these values of \( s \) and \( z \) has not been investigated in the literature so far which makes it quite difficult to obtain sharper results.

In this paper under some hypothesis on the flow \( \phi_t \) (see Sect. 3 for our standing assumptions) we prove spectral estimates for the iterations of the Ruelle operator \( L_{f-st+zg}^n \) with **two complex parameters** \( s, z \in \mathbb{C} \). These estimates are in the spirit of those obtained in [4], [19], [20], [21] for the Ruelle operators with **one complex parameter** \( s \in \mathbb{C} \). On the other hand, in this analysis some new difficulties appear when \( |\text{Im } s| \to \infty \) and \( |\text{Im } z| \to \infty \). First we prove in Theorem 5 spectral estimates in the case of arbitrary Hölder continuous functions \( f, g \), when there exist constants \( B > 0 \) and \( 0 < \mu < 1 \) such that \( |\text{Im } z| \leq B |\text{Im } s|^{\mu} \) and \( |\text{Im } s| \geq b_0 > 0 \). When \( f, g \) are Lipschitz one can take \( \mu = 1 \). This covers completely the case when \( |z| \) is bounded and the estimates have the same form as those for operators with one complex parameter. Moreover, these estimates are sufficient for the applications in [11] and [16] when \( |z| \) runs in a small neighborhood of 0 (see Sect. 6 and 7). In Sect. 5 we deal with the case when \( f, g \) are Lipschitz and there exists a constant \( B_1 > 0 \) such that \( |\text{Im } s| \leq B_1 |\text{Im } z| \) (see Theorem 6).

To study the analytic continuation of \( \zeta(s,z) \) for \( P_f - \eta_0 < \text{Re } s < P_f \), we need a generalization of the so called Ruelle’s lemma which yields a link between the convergence by packets of a Dirichlet series like (1.3) and the estimates of the iterations of the corresponding Ruelle operator. The reader may consult [23] for the precise result in this direction and the previous works (18], [15], [9]), treating this question. For our needs in this paper we prove in Sect. 2 an analogue of Ruelle’s lemma for Dirichlet series with two complex parameters following the approach in [23]. Combining Theorem 4 with the estimates in Theorem 5 (b), we obtain the following

**Theorem 1.** Assume the standing assumptions in Sect. 3 fulfilled for a basic set \( \Lambda \). Then for any Hölder continuous functions \( F, G : \Lambda \to \mathbb{R} \) there exists \( \eta_0 > 0 \) such that the function \( \zeta(s,z) \) admits
a non zero analytic continuation for 

\[(s,z) \in \{(s,z) \in \mathbb{C}^2 : P_f - \eta_0 \leq \text{Re} \, s, s \neq s(z), |z| \leq \eta_0\}\]

with a simple pole at \(s(z)\). The pole \(s(z)\) is determined as the root of the equation \(Pr(f-s\tau+tg) = 0\) with respect to \(s\) for \(|z| \leq \eta_0\).

Applying the results of Sects. 4, 5, we study also the analytic continuation of \(\zeta(s,\i w)\) for \(P_f - \eta_0 < \text{Re} \, s\) and \(w \in \mathbb{R}\), \(|w| \geq \eta_0\), in the case when \(F,G : \Lambda \rightarrow \mathbb{R}\) are Lipschitz functions (see Theorem 7). This analytic continuation combined with the arguments in [22] opens some new perspectives for the investigation of sharp large deviations for Anosov flows with exponentially shrinking intervals in the spirit of [12].

Our first application concerns the so called Hannay-Ozorio de Almeida sum formula (see [5], [10], [17]). Let \(\phi_t : M \rightarrow M\) be the geodesic flow on the unit-tangent bundle over a compact negatively curved surface \(M\). In [17] it was proved that there exists \(\epsilon > 0\) such that if \(\delta(T) = \mathcal{O}(e^{-\epsilon T})\), for every Hölder continuous function \(G : M \rightarrow \mathbb{R}\), we have

\[
\lim_{T \rightarrow +\infty} \frac{1}{\delta(T)} \sum_{T - \frac{\delta(T)}{2} \leq \lambda(\gamma) \leq T + \frac{\delta(T)}{2}} \lambda_G(\gamma) e^{-\lambda_G(\gamma)} = \int Gd\mu, \tag{1.5}
\]

where the notations \(\lambda(\gamma), \lambda_G(\gamma)\) and \(\lambda_u(\gamma)\) for a primitive periodic orbit \(\gamma\) are introduced above, while \(\mu\) is the unique \(\phi_t\)-invariant probability measure which is absolutely continuous with respect to the volume measure on \(M\). The measure \(\mu\) is called SRB (Sinai-Ruelle-Bowen) measure (see [3]). Notice that in the above case the Anosov flow \(\phi_t\) is weak mixing and \(M\) is an attractor. Applying Theorem 1 and the arguments in [17], we prove the following

**Theorem 2.** Let \(\Lambda\) be an attractor, that is there exists an open neighborhood \(V\) of \(\Lambda\) such that \(\Lambda = \cap_{t \geq 0} \phi_t(V)\). Assume the standing assumptions of Sect. 3 fulfilled for the basic set \(\Lambda\). Then there exists \(\epsilon > 0\) such that if \(\delta(T) = \mathcal{O}(e^{-\epsilon T})\), then for every Hölder function \(G : \Lambda \rightarrow \mathbb{R}\) the formula (1.5) holds with the SRB measure \(\mu\) for \(\phi_t\).

Our second application concerns the counting function

\[
\pi_F(T) = \sum_{\lambda(\gamma) \leq T} e^{\lambda_F(\gamma)},
\]

where \(\gamma\) is a primitive period orbit for \(\phi_t : \Lambda \rightarrow \Lambda\), \(\lambda(\gamma)\) is the least period and \(\lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))dt, x_\gamma \in \gamma\). For \(F = 0\) we obtain the counting function \(\pi_0(T) = \#\{\gamma : \lambda(\gamma) \leq T\}\). These counting functions have been studied in many works (see [15] for references concerning \(\pi_0(T)\) and [11], [13] for the function \(\pi_F(T)\)). The study of \(\pi_F(T)\) is based on the analytic continuation of the function

\[
\zeta_F(s) = \prod_\gamma \left(1 - e^{\lambda_F(\gamma) - s\lambda(\gamma)}\right)^{-1}, s \in \mathbb{C}
\]

which is just the function \(\zeta(s,0)\) defined above. We prove the following

**Theorem 3.** Let \(\Lambda\) be a basic set and let \(F : \Lambda \rightarrow \mathbb{R}\) be a Hölder function. Assume the standing assumptions of Sect. 3 fulfilled for \(\Lambda\). Then there exists \(\epsilon > 0\) such that

\[
\pi_F(T) = li(e^{Pr(F)T})(1 + \mathcal{O}(e^{-\epsilon T})), T \rightarrow \infty,
\]

where \(li(x) := \int_2^x \frac{1}{\log y} dy \sim \frac{x}{\log x}, x \rightarrow +\infty\).
In the case when $\phi_t : T^1(M) \to T^1(M)$ is the geodesic flow on the unit tangent bundle $T^1(M)$ of a compact $C^2$ manifold $M$ with negative section curvatures which are $\frac{1}{4}$-pinching the above result has been established in [14]. Following [20], [21], one deduces that the special case of a geodesic flow in [14] is covered by Theorem 3.

2. Ruelle lemma with two complex parameters

Let $B(\hat{U})$ be the space of bounded functions $q : \hat{U} \to \mathbb{C}$ with its standard norm $\|q\|_0 = \sup_{x \in \hat{U}} |g(x)|$. Given a function $q \in B(\hat{U})$, the Ruelle transfer operator $L_q : B(\hat{U}) \to B(\hat{U})$ is defined by $(L_q h)(u) = \sum_{\sigma(v)=u} e^{q(v)} h(v)$. If $q \in B(\hat{U})$ is Lipschitz on $\hat{U}$ with respect to the Riemann metric, then $L_q$ preserves the space $C^{\text{Lip}}(\hat{U})$ of Lipschitz functions $q : \hat{U} \to \mathbb{C}$. Similarly, if $q$ is $\nu$-Hölder for some $\nu > 0$, the operator $L_q$ preserves the space $C^{\nu}(\hat{U})$ of $\nu$-Hölder continuous functions on $\hat{U}$. In this section we assume that, $g, \tau$ and $f$ are real-valued $\nu$-Hölder continuous functions on $\hat{U}$. Then we can extend these functions as Hölder continuous on $U$.

We define the Ruelle operator $L_{f-s\tau+zg} : C^\nu(\hat{U}) \to C^\nu(\hat{U})$ by

$$L_{f-s\tau+zg} v(x) = \sum_{\sigma(y)=x} e^{f(y)-s\tau(y)+zg(y)} v(y), \quad s,z \in \mathbb{C}.$$ 

Next, for $\nu > 0$ define the $\nu$-norm on a set $B \subset U$ by

$$|w|_\nu = \sup \left\{ \frac{|w(x) - w(y)|}{d(x,y)^\nu} : x, y \in B \cap U_i, \, i = 1, \ldots, k, \, x \neq y \right\}.$$ 

Let

$$\|w\|_\nu = \|w\|_\infty + |w|_\nu,$$

and denote by $\|\cdot\|_\nu$ be the corresponding norm for operators. Let $\chi_i(x)$ be the characteristic function of $U_i$.

Introduce the sum

$$Z_n(f-s\tau+zg) := \sum_{\sigma^n(x)=x} e^{f^n(x)-s\tau^n(x)+zg^n(x)}.$$

Our purpose is to prove the following statement which can be considered as Ruelle’s lemma with two complex parameters.

**Theorem 4.** For every Markov leaf $U_i$ fix an arbitrary point $x_i \in U_i$. Then for every $\epsilon > 0$ and sufficiently small $a_0 > 0, c_0 > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\left| Z_n(f-s\tau+zg) - \sum_{i=1}^k L^n_{f-s\tau+zg} \chi_i(x_i) \right| \leq C_\epsilon (1 + |s|)(1 + |z|) \sum_{m=2}^n \|L^{n-m}_{f-s\tau+zg} \|_{\nu}^{-m} e^{m(\epsilon + \nu (s+c)g)}, \quad \forall n \in \mathbb{N} \quad (2.1)$$

for $s = a + ib, z = c + iw, \quad |a| \leq a_0, |c| \leq c_0$.

The proof of this theorem follows that of Theorem 3.1 in [23] with some modifications. We have to take into account the presence of a second complex parameter $z$. Given a string $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$
of symbols $\alpha_j$ taking the values in $\{1, ..., k\}$, we say that $\alpha$ is an admissible word if $A(\alpha_j, \alpha_{j+1}) = 1$ for all $0 \leq j \leq n - 1$. Set $|\alpha| = n$ and define the cylinder of length $n$ in the leaf $U_{\alpha_0}$ by

$$U_{\alpha} = U_{\alpha_0} \cap \sigma^{-1}U_{\alpha_1} \cap ... \cap \sigma^{-(n-1)}U_{\alpha_{n-1}}.$$  

Each $U_i$ is a cylinder of length 1. Next we introduce some other words (see Section in [23]). Given a word $\alpha = (\alpha_0, ..., \alpha_{n-1})$ and $i = 1, ..., k$, if $A(\alpha_{n-1}, i) = 1$ and $A(i, \alpha_0) = 1$, we define

$$\alpha_i = (\alpha_0, ..., \alpha_{n-1}, i), \text{ } i\alpha = (i, \alpha_0, ..., \alpha_{n-1}), \text{ } \bar{\alpha} = (\alpha_0, ..., \alpha_{n-2}).$$

We have the following

**Lemma 1.** Let $w$ be a $\nu$-Hölder real-valued function on $U$. Let $x$ and $y$ be on the same cylinder $U_{\alpha}$ with $|\alpha| = m$. Then there exists a constant $B > 0$ depending only on $w, \nu$ and the constants $c_0$ and $\gamma_0$ in [11] such that

$$|w^m(x) - w^m(y)| \leq B(d(x, y)\nu)^\nu.$$  

The proof is a repetition of that of Lemma 2.5 in [23] and we leave the details to the reader.

**Proposition 1.** Let $m \geq 1$ and let $w$ be a function which is $\nu$-Hölder continuous on all cylinder of length $m + 1$. Then for the transfer operator $L_{f^{s\tau+z\nu}}$ we have

$$L_{f^{s\tau+z\nu}} := \oplus_{|\alpha| = m+1} C^\nu(U_{\alpha}) \ni w \rightarrow L_{f^{s\tau+z\nu}}w \in \oplus_{|\alpha| = m} C^\nu(U_{\alpha}).$$

**Proof.** Let $w$ be $\nu$-Hölder on $U_{i\alpha}$ for all $i$ such that $A(i, \alpha_0) = 1$. Let $x, y \in \text{Int } U_{\alpha}$ and let $|U| = \max_{i=1, ..., k} \text{diam}(U_i)$. Then

$$|L_{f^{s\tau+z\nu}}w(x) - L_{f^{s\tau+z\nu}}w(y)| = \left| \sum_{A(i, \alpha_0) = 1} e^f(ix) - e^f(iy)w(ix) - \sum_{A(i, \alpha_0) = 1} e^f(iy)w(iy) \right|$$

$$\leq \sum_{A(i, \alpha_0) = 1} |e^{-s\tau(iy)}\left(|e^{s\tau(iy)-s\tau(ix)} - 1||e^f(iy)+zg(iy)w(ix)| + |e^f(iy)+zg(iy)w(iy) - e^f(ix)+zg(ix)w(ix)|\right)|$$

$$\leq e^{c_0|\tau|}\sum_{A(i, \alpha_0) = 1} \left(|s||g|_\nu e^{c_0|g|_\nu U}|f|_\infty + |e^f(iy)w(iy) - e^f(ix)w(ix)|\right).$$

Repeating this argument, we get

$$\sum_{A(i, \alpha_0) = 1} |e^f(iy)+zg(iy)w(iy) - e^f(ix)+zg(ix)w(ix)|$$

$$\leq e^{c_0|g|}\sum_{A(i, \alpha_0) = 1} \left(|s||g|_\nu e^{c_0|g|_\nu U}|f|_\infty + |e^f(iy)w(iy) - e^f(ix)w(ix)|\right)$$

and we conclude that

$$|L_{f^{s\tau+z\nu}}w(x) - L_{f^{s\tau+z\nu}}w(y)| \leq C|w|_\nu d(x, y)^\nu.$$  

Now, as in [23], we will choose in every cylinder $U_{\alpha}$ a point $x_{\alpha} \in U_{\alpha}$. For the reader’s convenience we recall the choice of $x_{\alpha}$.

1. If $U_{\alpha}$ has an $n$-periodic point, then we take $x_{\alpha} \in U_{\alpha}$ so that $\sigma^n x_{\alpha} = x_{\alpha}$.
2. If $U_{\alpha}$ has no $n$-periodic point and $n > 1$ we choose $x_{\alpha} \in U_{\alpha}$ arbitrary so that $x_{\alpha} \notin \sigma(U_{\alpha_{n-1}})$.
3. if $|\alpha| = n = 1$, then we take $x_{\alpha} = x_i$, where $i = \alpha_0$ and $x_i \in U_i$ is one of the points fixed in
Theorem 4.

Let \( \chi_\alpha \) be the characteristic function of \( U_\alpha \). Then Lemma 3.4 and Lemma 3.5 in [23] are applied without any change and we get

\[
Z_n(f - s\tau + zg) = \sum_{|\alpha| = n} (L^n_{f-s\tau+ zg}(x_\alpha)).
\]

Proposition 2. We have

\[
Z_n(f - s\tau + zg) - \sum_{i=1}^k L^n_{f-s\tau+ zg}(x_i)
\]

\[
= \sum_{m=2}^n \left( \sum_{|\alpha| = m} L^n_{f-s\tau+ zg}(x_\alpha) - \sum_{|\beta| = m-1} L^n_{f-s\tau+ zg}(x_\beta) \right).
\]

The proof is elementary by using the fact that

\[
\sum_{i=1}^k (L^n_{f-s\tau+ zg}(U_i))(x_i) = \sum_{|\alpha| = 1} (L^n_{f-s\tau+ zg}(x_\alpha)).
\]

Now we repeat the argument in [23] and conclude that

\[
\sum_{|\beta| = m-1} L^n_{f-s\tau+ zg}(x_\beta) = \sum_{|\alpha| = m} L^n_{f-s\tau+ zg}(x_\alpha).
\]

Thus the proof of (2.1) is reduced to an estimate of the difference

\[
L^n_{f-s\tau+ zg}(x_\alpha) - L^n_{f-s\tau+ zg}(x_\bar{\alpha}).
\]

Observe that \( x_\alpha \) and \( x_\bar{\alpha} \) are on the same cylinder \( U_\bar{\alpha} \). According to Proposition 1, the function \( L^n_{f-s\tau+ zg} \) is \( \nu \)-Hölder continuous on \( U_\bar{\alpha} \). Consequently, for every \( n \geq 2 \) we obtain

\[
|L^n_{f-s\tau+ zg}(x_\alpha) - L^n_{f-s\tau+ zg}(x_\bar{\alpha})| \leq \|L^n_{f-s\tau+ zg} \|_{\nu} d(x_\alpha, x_\bar{\alpha})^\nu,
\]

where \( \| \|_{\nu} \) denotes the operator norm derived from the \( \nu \)-Hölder norm. Going back to (2.2), we deduce

\[
\left| Z_n(f - s\tau + zg) - \sum_{i=1}^k L^n_{f-s\tau+ zg}(x_i) \right|
\]

\[
\leq \sum_{m=2}^n \sum_{|\alpha| = m} \|L^{n-m}_{f-s\tau+ zg} \|_{\nu} \|L^m_{f-s\tau+ zg} \|_{\nu} d(x_\alpha, x_\bar{\alpha}).
\]

(2.3)

This it makes possible to apply (1.1) and to conclude that

\[
d(x_\alpha, x_\bar{\alpha}) \leq C^{n-\nu(m-2)} d(\sigma^{m-2} x_\alpha, \sigma^{m-2} x_\bar{\alpha})^\nu \leq C_2 \gamma_0^{m \nu}.
\]

To finish the proof we have to estimate the term \( \|L^m_{g-s\tau+ zf} \chi_\beta \|_{\nu} \). Given a word \( \alpha \) of length \( n > 1 \) and \( x \in \sigma(U_{\alpha_{n-1}}) \cap \text{Int} U_i \), for any \( i \) with \( A(\alpha_{n-1}, i) = 1 \), we define \( \sigma_i^{-1}(x) \) to be the unique point \( y \) such that \( \sigma^n(y) = x \) and \( y \in U_\alpha \). For a symbol \( i \) we define \( ix = \sigma_i^{-1}(x) \).

First we have
Lemma 2.

\[ (L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x))(y) = \begin{cases} e^{(f^{-\mathbf{st}}+\mathbf{zg})^m\sigma^{-1}y}, & \text{if } x \in \sigma(U_{\beta_{m-1}}), \\ 0, & \text{otherwise.} \end{cases} \]

The proof is a repetition of that of Lemma 3.7 in [23] and it is based on the definition of \( \sigma^{-1} \) above and the fact that

\[ (L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x))(y) = \sum_{\sigma^m y = x} e^{(f^{-\mathbf{st}}+\mathbf{zg})^m y}(y)\chi_y(y). \]

For every admissible word \( \beta \) with \( |\beta| = m \), we fix a point \( y, \beta \in \sigma(U_{\beta_{m-1}}) \) which will be chosen as in [23]. Define \( z_\beta = \sigma^{-1}(y, \beta) \).

Lemma 3. There exist constants \( B_0 > 0, B_1 > 0, B_2 > 0 \) such that we have the estimate

\[ \|L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x)\|_\nu \leq B_0 \left( e^{a_0\|U\|B_1 + B_1\|s\|e^{a_0\|U\|r(1+\gamma_c^{-1})B_1)} \right) \times \left( e^{c_0\|U\|B_2 + B_2\|c\|e^{c_0\|U\|r(1+\gamma_c^{-1})B_2)} \right) e^{(f^{-\mathbf{st}}+\mathbf{zg})^m(z, \beta)}. \]

**Proof.** We will follow the proof of Lemma 3.8 in [23]. Let \( x \) and \( y \) be in the same Markov leaf. If \( y \notin \sigma(U_{\beta_{m-1}}) \), then \( |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x)| = |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x) - L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(y)| = 0 \). In the case when \( x \notin \sigma(U_{\beta_{m-1}}) \), we repeat the same argument. So we will consider the case when both \( x \) and \( y \) are in \( \sigma(U_{\beta_{m-1}}) \).

We have

\[ |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x)| = |e^{(f^{-\mathbf{st}}+\mathbf{zg})^m\sigma^{-1}x}| \leq \exp\left( (f^{-\mathbf{st}}+\mathbf{zg})^m(\sigma^{-1}x) - (f^{-\mathbf{st}}+\mathbf{zg})^m(\sigma^{-1}y) \right) e^{(f^{-\mathbf{st}}+\mathbf{zg})^m(z, \beta)}. \]

On the other hand, applying Lemma 1 with \( w = \tau \), we get

\[ |\tau^m(\sigma^{-1}x) - \tau^m(\sigma^{-1}y)| \leq B_1(d(\sigma^{-1}x, \sigma^{-1}y)) \leq B_1\|U\|\nu. \]

The same argument works for the terms involving \( f^m \) and \( g^m \), applying Lemma 1 with \( w = f, g \), respectively. Thus we obtain

\[ |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x)| \leq e^{(C_0+a_0B_1+c_0B_2)}\|U\|\nu e^{(f^{-\mathbf{st}}+\mathbf{zg})^m(z, \beta)}. \]

and this implies an estimate for \( |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x)| \). Next,

\[ |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x) - L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(y)| \leq |e^{m(\sigma^{-1}(x)) - e^{m(\sigma^{-1}(y))}|} - 1||e^{m(\sigma^{-1}(x)) + \sigma^{-1}(y))} - 1||e^{-m(\sigma^{-1}(y))} | \times |e^{g^m(\sigma^{-1}(x)) - g^m(\sigma^{-1}(y))} - 1||e^{g^m(\sigma^{-1}(y))} |. \]

As in [23], we have

\[ |e^{m(\sigma^{-1}(x)) + \sigma^{-1}(y))} - 1||e^{-m(\sigma^{-1}(y))} | \leq B_1\|U\|\nu e^{a_0B_1(1+\gamma_c^{-1})}\|U\|e^{-a_0(\gamma_c^{-1})}d(x, y)\nu. \]

For the product involving \( zg^m \) we have the same estimate with \( B_2, |z|, c_0 \) and \( c \) in the place of \( B_1, |s|, a_0 \) and \( a \). A similar estimate holds for the term containing \( f^m \) with a constant \( B_3 \) in the place of \( B_1 \). Taking the product of these estimates, we obtain a bound for \( |L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x) - L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(y)| \), this implies the desired estimate for the \( \nu \)-Hölder norm of \( L^m_{f^{-\mathbf{st}}+\mathbf{zg}}(x) \). This completes the proof. \( \square \)
Now the proof of (2.1) is reduced to the estimate of
\[ \sum_{|\beta|=m} e^{(f^m_a - \alpha + cg^m)(z_\beta)}. \]

Introduce the real-valued function \( h = f - \alpha + cg \). Then we must estimate
\[ \sum_{|\beta|=m} e^{h^m(z_\beta)}. \]

For this purpose we repeat the argument on pages 232-234 in [23] and deduce with some constant \( d_0 > 0 \) depending only on the matrix \( A \) and every \( \epsilon > 0 \) the bound
\[ \sum_{|\beta|=m} e^{h^m(z_\beta)} \leq e^{d_0 |h|_{\infty}} B_\epsilon e^{(m+d_0)(\epsilon+\Pr(h))}. \]

Combing this with the previous estimates, we get (2.1) and the proof of Theorem 4 is complete. \( \square \)

3. Ruelle operators – definitions and assumptions

For a contact Anosov flows \( \phi_t \) with Lipschitz local stable holonomy maps it is proved in Sect. 6 in [20] that the following local non-integrability condition holds:

(LNIC): There exist \( z_0 \in \Lambda, \epsilon_0 > 0 \) and \( \theta_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0] \), any \( \hat{z} \in \Lambda \cap W^u_\epsilon(z_0) \) and any tangent vector \( \eta \in E^u(\hat{z}) \) to \( \Lambda \) at \( \hat{z} \) with \( \|\eta\| = 1 \) there exist \( \tilde{z} \in \Lambda \cap W^u_\epsilon(\hat{z}) \), \( \tilde{y}_1, \tilde{y}_2 \in \Lambda \cap W^s_\epsilon(\tilde{z}) \) with \( \tilde{y}_1 \neq \tilde{y}_2 \), \( \delta = \delta(\hat{z}, \tilde{y}_1, \tilde{y}_2) > 0 \) and \( \epsilon' = \epsilon'(\hat{z}, \tilde{y}_1, \tilde{y}_2) \in (0, \epsilon] \) such that
\[ |\Delta(\exp_x^\epsilon(v), \pi_{\tilde{y}_1}(\tilde{z})) - \Delta(\exp_x^\epsilon(v), \pi_{\tilde{y}_2}(\tilde{z}))| \geq \delta \|v\| \]
for all \( z \in W^u_\epsilon(\hat{z}) \cap \Lambda \) and \( v \in E^u(\hat{z}; \epsilon') \) with \( \exp_x^\epsilon(v) \in \Lambda \) and \( \frac{\langle \frac{v}{\|v\|}, \eta_\epsilon \rangle}{\|v\|} \geq \theta_0 \), where \( \eta_\epsilon \) is the parallel translate of \( \eta \) along the geodesic in \( W^u_\epsilon(z_0) \) from \( \hat{z} \) to \( z \).

For any \( x \in \Lambda, T > 0 \) and \( \delta \in (0, \epsilon] \) set
\[ B_T^\delta(x, \delta) = \{ y \in W^u_\epsilon(x) : d(\phi_t(x), \phi_t(y)) \leq \delta, \ 0 \leq t \leq T \}. \]

We will say that \( \phi_t \) has a regular distortion along unstable manifolds over the basic set \( \Lambda \) if there exists a constant \( \epsilon_0 > 0 \) with the following properties:

(a) For any \( 0 < \delta \leq \epsilon \leq \epsilon_0 \) there exists a constant \( R = R(\delta, \epsilon) > 0 \) such that
\[ \text{diam}(\Lambda \cap B_T^\delta(z, \epsilon)) \leq R \text{diam}(\Lambda \cap B_T^\epsilon(z, \delta)) \]
for any \( z \in \Lambda \) and any \( T > 0 \).

(b) For any \( \epsilon \in (0, \epsilon_0] \) and any \( \rho \in (0, 1) \) there exists \( \delta \in (0, \epsilon] \) such that for any \( z \in \Lambda \) and any \( T > 0 \) we have
\[ \text{diam}(\Lambda \cap B_T^\epsilon(z, \delta)) \leq \rho \text{diam}(\Lambda \cap B_T^\epsilon(z, \delta)). \]

A large class of flows on basic sets having regular distortion along unstable manifolds is described in [21].

In this paper we work under the following Standing Assumptions:

(A) \( \phi_t \) has Lipschitz local holonomy maps over \( \Lambda \),
(B) the local non-integrability condition (LNIC) holds for \( \phi_t \) on \( \Lambda \),
(C) \( \phi_t \) has a regular distortion along unstable manifolds over the basic set \( \Lambda \).
A rather large class of examples satisfying the above conditions is provided by imposing the following pinching condition:

\((P)\): There exist constants \(C > 0\) and \(\beta \geq \alpha > 0\) such that for every \(x \in M\) we have

\[
\frac{1}{C} e^{\alpha t} \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C e^{\beta t} \|u\|, \quad u \in E_u(x), t > 0
\]

for some constants \(\alpha_x, \beta_x > 0\) with \(\alpha \leq \alpha_x \leq \beta \leq \beta_x\) and \(2\alpha_x - \beta_x \geq \alpha\) for all \(x \in M\).

We should note that \(P\) holds for geodesic flows on manifolds of strictly negative sectional curvature satisfying the so-called \(\frac{1}{4}\)-pinching condition. \(P\) always holds when \(\dim(M) = 3\).

**Simplifying Assumptions:** \(\phi_t\) is a \(C^2\) contact Anosov flow satisfying the condition \(P\).

As shown in [21] the pinching condition \((P)\) implies that \(\phi_t\) has Lipschitz local holonomy maps and regular distortion along unstable manifolds. Combining this with Proposition 6.1 in [20], shows that the Simplifying Assumptions imply the Standing Assumptions.

As in Sect. 1 consider a fixed Markov family \(\mathcal{R} = \{R_i\}_{i=1}^k\) for the flow \(\phi_t\) on \(\Lambda\) consisting of rectangles \(R_i = [U_i, S_i]\) and let \(U = \cup_{i=1}^k U_i\). The Standing Assumptions imply the existence of constants \(c_0 \in (0, 1]\) and \(\gamma_1 > \gamma_0 > 1\) such that (1.1) hold.

In what follows we will assume that \(f\) and \(g\) are fixed real-valued functions in \(C^\alpha(\widehat{U})\) for some fixed \(\alpha > 0\). Let \(P = P_f\) be the unique real number so that \(\Pr(f - P\tau) = 0\), where \(\Pr(h)\) is the topological pressure of \(h\) with respect to the shift map \(\sigma\) defined in Section 2. Given \(t \in \mathbb{R}\) with \(t \geq 1\), following [3], denote by \(f_t\) the average of \(f\) over balls in \(U\) of radius \(1/t\). To be more precise, first one has to fix an arbitrary extension \(f \in C^\alpha(V)\) (with the same Hölder constant), where \(V\) is an open neighborhood of \(U\) in \(M\), and then take the averages in question. Then \(f_t \in C^\infty(V)\), so its restriction to \(U\) is Lipschitz (with respect to the Riemann metric) and:

(a) \(\|f - f_t\|_\infty \leq |f|_\alpha/t^\alpha\);
(b) \(\text{Lip}(f_t) \leq \text{Const} \|f\|_\infty t\);
(c) For any \(\beta \in (0, \alpha)\) we have \(|f - f_t|_\beta \leq 2 |f|_\alpha/t^{\alpha - \beta}\).

In the special case \(f \in C^{\text{Lip}}(U)\) we set \(f_t = f\) for all \(t \geq 1\). Similarly for \(g\). Let \(\lambda_0 > 0\) be the largest eigenvalue of \(L_{f - P\tau}\), and let \(\nu_0\) be the (unique) probability measure on \(U\) with \(L_{f - P\tau}^* \nu_0 = \nu_0\). Fix a corresponding (positive) eigenfunction \(h_0 \in C^\alpha(U)\) such that \(\int_U h_0 \, d\nu_0 = 1\). Then \(dv_0 = h_0 \, d\nu_0\) defines a \(\sigma\)-invariant probability measure \(v_0\) on \(U\). Setting

\[
f_0 = f - P\tau + \ln h_0(u) - \ln h_0(\sigma(u)),
\]

we have \(L_{f_0}^* \nu_0 = \nu_0\), i.e.

\[
\int_U L_{f_0} H \, dv_0 = \int_U H \, dv_0 \text{ for any } H \in C(U), \text{ and } L_{f_0} 1 = 1.
\]

Given real numbers \(a\) and \(t\) (with \(|a| + \frac{1}{|t|} \text{ small}\)), denote by \(\lambda_{at}\) the largest eigenvalue of \(L_{f_1 - (P+a)\tau}\) on \(C^{\text{Lip}}(U)\) and by \(h_{at}\) the corresponding (positive) eigenfunction such that \(\int_U h_{at} \, dv_{at} = 1\), where \(\nu_{at}\) is the unique probability measure on \(U\) with \(L_{f_1 - (P+a)\tau}^* \nu_{at} = \nu_{at}\).

As is well-known the shift map \(\sigma : \widehat{U} \rightarrow \widehat{U}\) is naturally isomorphic to an one-sided subshift of finite type. Given \(\theta \in (0, 1)\), a natural metric associated by this isomorphism is defined (for \(x \neq y\)) by \(d_\theta(x, y) = 0^m\), where \(m\) is the largest integer such that \(x, y\) belong to the same cylinder of length \(m\). There exist \(\theta = \theta(\alpha) \in (0, 1)\) and \(\beta \in (0, \alpha)\) such that \((d(x, y))^{\alpha} \leq \text{Const} \, d_\theta(x, y)\) and \(d_\theta(x, y) \leq \text{Const} \, (d(x, y))^{\beta}\) for all \(x, y \in \widehat{U}\). One can then apply the Ruelle-Perron-Frobenius
fixed real-valued functions \( f, g \). However this is not enough for our purposes – in Lemma 4 below we get a bit more.

Consider an arbitrary \( \beta \in (0, \alpha) \). It follows from properties (a) and (c) above that there exists a constant \( C_0 > 0 \), depending on \( f \) and \( \alpha \) but independent of \( \beta \), such that

\[
\| [f_t - (P + a)\tau] - (f - P\tau) \|_{\beta} \leq C_0 \| a \| + 1/t^{\alpha - \beta} \tag{3.1}
\]

for all \( |a| \leq 1 \) and \( t \geq 1 \). Since \( \text{Pr}(f - P\tau) = 0 \), it follows from the analyticity of pressure and the eigenfunction projection corresponding to the maximal eigenvalue \( \lambda_{at} = e^{\text{Pr}(f_t - (P + a)\tau)} \) of the Ruelle operator \( L_{f_t - (P + a)\tau} \) on \( C^\beta(U) \) (cf. e.g. Ch. 3 in [11]) that there exists a constant \( a_0 > 0 \) such that, taking \( C_0 > 0 \) sufficiently large, we have

\[
|\text{Pr}(f_t - (P + a)\tau)| \leq C_0 \left( |a| + \frac{1}{t^{\alpha - \beta}} \right), \quad \| h_{at} - h_0 \|_{\beta} \leq C_0 \left( |a| + \frac{1}{t^{\alpha - \beta}} \right) \tag{3.2}
\]

for \( |a| \leq a_0 \) and \( 1/t \leq a_0 \). We may assume \( C_0 > 0 \) and \( a_0 > 0 \) are taken so that \( 1/C_0 \leq \lambda_{at} \leq C_0 \), \( \| f_t \|_{\infty} \leq C_0 \) and \( 1/C_0 \leq h_{at}(u) \leq C_0 \) for all \( u \in U \) and all \( |a|, 1/t \leq a_0 \).

Given real numbers \( a \) and \( t \) with \( |a|, 1/t \leq a_0 \) consider the functions

\[
f_{at} = f_t - (P + a)\tau + \ln h_{at} - \ln(h_{at} \circ \sigma) - \ln \lambda_{at}
\]

and the operators

\[
L_{abt} = L_{f_{at} - P\tau} : C(U) \to C(U) \quad \text{and} \quad M_{at} = L_{f_{at}} : C(U) \to C(U).
\]

One checks that \( M_{at} 1 = 1 \).

Taking the constant \( C_0 > 0 \) sufficiently large, we may assume that

\[
\| f_{at} - f_0 \|_{\beta} \leq C_0 \left[ |a| + \frac{1}{t^{\alpha - \beta}} \right], \quad |a|, 1/t \leq a_0. \tag{3.3}
\]

We will now prove a simple uniform estimate for \( \text{Lip}(h_{at}) \). With respect to the usual metrics on symbol spaces this a consequence of general facts (see e.g. Sect. 1.7 in [1] or Ch. 3 in [11]), however here we need it with respect to the Riemann metric.

The proof of the following lemma is given in the Appendix.

**Lemma 4.** Taking the constant \( a_0 > 0 \) sufficiently small, there exists a constant \( T' > 0 \) such that for all \( a, t \in \mathbb{R} \) with \( |a| \leq a_0 \) and \( t \geq 1/a_0 \) we have \( h_{at} \in C^{\text{Lip}(\hat{U})} \) and \( \text{Lip}(h_{at}) \leq T't \).

It follows from the above that, assuming \( a_0 > 0 \) is chosen sufficiently small, there exists a constant \( T > 0 \) (depending on \( |f|_\alpha \) and \( a_0 \)) such that

\[
\| f_{at} \|_{\infty} \leq T, \quad \| g_t \|_{\infty} \leq T, \quad \text{Lip}(h_{at}) \leq Tt, \quad \text{Lip}(f_{at}) \leq Tt \tag{3.4}
\]

for \( |a|, 1/t \leq a_0 \). We will also assume that \( T \geq \max\{ \| \tau \|_0, \text{Lip}(\tau|_\beta) \} \). From now on we will assume that \( a_0, C_0, T, 1 < \gamma_0 < \gamma_1 \) are fixed constants with (1.1) and (3.1) – (3.4).

4. **Ruelle operators depending on two parameters – the case when \( b \) is the leading parameter**

Throughout this section we work under the Standing Assumptions made in Sect. 3 and with fixed real-valued functions \( f, g \in C^\alpha(\hat{U}) \) as in Sect. 3. Throughout \( 0 < \beta < \alpha \) are fixed numbers.
We will study Ruelle operators of the form $L_{f-(P_x+a+ib)t+sz}$, where $z = c + iw$, $a, b, c, w \in \mathbb{R}$, and $|a|, |c| \leq a_0$ for some constant $a_0 > 0$. Such operators will be approximates by operators of the form

$$L_{ab} = L_{f-at^b + zgt} : C^0(\hat{U}) \rightarrow C^0(\hat{U}).$$

In fact, since $f_at - ibt + zgt$ is Lipschitz, the operators $L_{ab}$ preserves each of the spaces $C^{0'}(\hat{U})$ for $0 < a' \leq 1$ including the space $C^{Lip}(\hat{U})$ of Lipschitz functions $h : \hat{U} \rightarrow \mathbb{C}$. For such $h$ we will denote by $\text{Lip}(h)$ the Lipschitz constant of $h$. Let $\|h\|_0$ denote the standard sup norm of $h$ on $\hat{U}$. For $|b| \geq 1$, as in [4], consider the norm $\|h\|_{\text{Lip},b}$ on $C^{Lip}(\hat{U})$ defined by $\|h\|_{\text{Lip},b} = \|h\|_0 + \frac{\text{Lip}(h)}{|b|}$ and also the norm $\|h\|_{b,\beta} = \|h\|_{\infty} + \frac{|h|_c}{|b|}$ on $C^\beta(U)$.

Our aim in this section is to prove the following

**Theorem 5.** Let $\phi_t : M \rightarrow M$ satisfy the Standing Assumptions over the basic set $\Lambda$, and let $0 < \beta < \alpha$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for $\phi_t$ over $\Lambda$ as in Sect. 1. Then for any real-valued functions $f, g \in C^\alpha(\hat{U})$ we have:

(a) For any constants $\epsilon > 0$, $B > 0$ and $\nu \in (0, 1)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$, $A_0 > 0$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then

$$\|L^m_{f-at^b + (c+ib)t}\|_{\text{Lip},b} \leq C \rho^m |b|^{\epsilon} \|h\|_{\text{Lip},b}$$

for all $h \in C^{Lip}(\hat{U})$, all integers $m \geq 1$ and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$, $1 < t \leq \frac{1}{A_0} \log |b|^{\nu}$ and $|w| \leq B |b|^{\nu}$.

(b) For any constants $\epsilon > 0$, $B > 0$, $\nu \in (0, 1)$ and $\beta \in (0, \alpha)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then

$$\|L^m_{f-(P_x+a+ib)t^b + (c+ib)t}\|_{b,\beta} \leq C \rho^m |b|^{\epsilon} \|h\|_{b,\beta}$$

for all $h \in C^\beta(\hat{U})$, all integers $m \geq 1$ and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B |b|^{\nu}$.

(c) If $f, g \in C^{Lip}(\hat{U})$, then for any constants $\epsilon > 0$, $B > 0$ and $\beta \in (0, \alpha)$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C = C(B, \epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then

$$\|L^m_{f-(P_x+a+ib)t^b + (c+ib)t}\|_{\text{Lip},b} \leq C \rho^m |b|^{\epsilon} \|h\|_{\text{Lip},b}$$

for all $h \in C^\beta(\hat{U})$, all integers $m \geq 1$ and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq B |b|$. 

We will first prove part (a) of the above theorem and then derive part (b) by a simple approximation procedure. To prove part (a) we will use the main steps in Section 5 in [20] with necessary modifications. The proof of part (c) is just a much simpler version of the proof of (b).

Define a new metric $D$ on $\hat{U}$ by

$$D(x, y) = \min\{\text{diam}(C) : x, y \in C, C \text{ a cylinder contained in } U_i\}$$

if $x, y \in U_i$ for some $i = 1, \ldots, k$, and $D(x, y) = 1$ otherwise. Rescaling the metric on $M$ if necessary, we will assume that $\text{diam}(U_i) < 1$ for all $i$. As shown in [19], $D$ is a metric on $\hat{U}$ with $d(x, y) \leq D(x, y)$ for $x, y \in \hat{U}_i$ for some $i$, and for any cylinder $C$ in $U$ the characteristic function $\chi_C$ of $C$ on $\hat{U}$ is Lipschitz with respect to $D$ and $\text{Lip}_D(\chi_C) \leq 1/\text{diam}(C)$.

We will denote by $C^{Lip}_D(\hat{U})$ the space of all Lipschitz functions $h : \hat{U} \rightarrow \mathbb{C}$ with respect to the metric $D$ on $\hat{U}$ and by $\text{Lip}_D(h)$ the Lipschitz constant of $h$ with respect to $D$. 

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Given $A > 0$, denote by $K_A(\hat{U})$ the set of all functions $h \in C^I_D(\hat{U})$ such that $h > 0$ and $|h(u) - h(u')| \leq A D(u, u')$ for all $u, u' \in \hat{U}$ that belong to the same $\hat{U}_i$ for some $i = 1, \ldots, k$. Notice that $h \in K_A(\hat{U})$ implies $|\ln h(u) - \ln h(v)| \leq A D(u, v)$ and therefore $e^{-A D(u, v)} \leq \frac{h(u)}{h(v)} \leq e^A D(u, v)$ for all $u, v \in \hat{U}_i$, $i = 1, \ldots, k$.

We begin with a lemma of Lasota-Yorke type, which necessarily has a more complicated form due to the more complex situation considered. It involves the operators $L_{abtz}$, and also operators of the form

$$M_{ate} = L_{fat+ct} : C^0(\hat{U}) \to C^0(\hat{U}).$$

**Fix arbitrary constants** $\nu \in (0, 1)$ and $\hat{\gamma}$ with $1 < \hat{\gamma} < \gamma_0$.

**Lemma 5.** Assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $A_0 > 0$ such that for all $a, c, t \in \mathbb{R}$ with $|a|, |c| \leq a_0$ and $t \geq 1$ the following hold:

(a) If $H \in K_E(\hat{U})$ for some $E > 0$, then

$$\frac{|(M_{ate}^m H)(u) - (M_{ate}^m H)(u')|}{(M_{ate}^m H)(u')} \leq A_0 \left[ \frac{E}{\hat{\gamma}^m} + e^{A_0 t} \right] D(u, u')$$

for all $m \geq 1$ and all $u, u' \in U_i$, $i = 1, \ldots, k$.

(b) If the functions $h$ and $H$ on $\hat{U}$ and $E > 0$ are such that $H > 0$ on $\hat{U}$ and $|h(v) - h(v')| \leq E H(v') D(v, v')$ for any $v, v' \in \hat{U}_i$, $i = 1, \ldots, k$, then for any integer $m \geq 1$ and any $b, w, t \in \mathbb{R}$ with $|b|, t, |w| \geq 1$, for $z = c + i w$ we have

$$|L_{abtz}^m h(u) - L_{abtz}^m h(u')| \leq A_0 \left[ \frac{E}{\hat{\gamma}^m} (M_{ate}^m H)(u') + (|b| + e^{A_0 t} t + |w|)(M_{ate}^m H)(u') \right] D(u, u')$$

whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$. In particular, if

$$t \leq \frac{\log |b|^\nu}{A_0}, \quad t \leq B |b|^{1-\nu}, \quad |w| \leq B |b|^\nu$$

(4.1)

for some constant $B > 0$, then

$$|L_{abtz}^m h(u) - L_{abtz}^m h(u')| \leq A_1 \left[ \frac{E}{\hat{\gamma}^m} (M_{ate}^m H)(u') + |b|(M_{ate}^m H)(u') \right] D(u, u').$$

for some constant $A_1 > 0$.

A proof of this lemma is given in the Appendix.

From now on we will assume that $a_0$, $\eta_0$ and $A_0$ are fixed with the properties in Lemma 5 above and $a, b, c, w, t \in \mathbb{R}$ are such that $|a| \leq a_0$, $c \leq \eta_0$, $|b|, t, |w| \geq 1$ and (4.1) hold. As before, set $z = c + id$.

We will use the entire set-up and notation from Section 5 in [20]. In what follows we recall the main part of it.

Following Sect. 4 in [20], fix an arbitrary point $z_0 \in \Lambda$ and constants $\epsilon_0 > 0$ and $\theta_0 \in (0, 1)$ with the properties described in (LNIC). Assume that $z_0 \in \text{Int}_\Lambda(U_1)$, $U_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$ and $S_1 \subset \Lambda \cap W^u_{\epsilon_0}(z_0)$. Fix an arbitrary constant $\theta_1$ such that

$$0 < \theta_0 < \theta_1 < 1.$$

Next, fix an arbitrary orthonormal basis $e_1, \ldots, e_n$ in $E^u(z_0)$ and a $C^1$ parametrization $r(s) = \exp^u_{z_0}(s)$, $s \in V'_0$, of a small neighborhood $W_0$ of $z_0$ in $W^u_{\epsilon_0}(z_0)$ such that $V'_0$ is a convex compact
neighbourhood of 0 in $\mathbb{R}^n \approx \text{span}(e_1, \ldots, e_n) = E^n(z_0)$. Then $r(0) = z_0$ and $\frac{\partial}{\partial s} r(s)|_{s=0} = e_i$ for all $i = 1, \ldots, n$. Set $U_0' = W_0 \cap \Lambda$. Shrinking $W_0$ (and therefore $V_0'$ as well) if necessary, we may assume that $\overline{U_0'} \subset \text{Int}_\Lambda(U_1)$ and $\left| \left\langle \frac{\partial r}{\partial x_i}(s), \frac{\partial r}{\partial x_j}(s) \right\rangle - \delta_{ij} \right|$ is uniformly small for all $i, j = 1, \ldots, n$ and $s \in V_0'$, so that

$$\frac{1}{2} \langle \xi, \eta \rangle \leq \langle dr(s) \cdot \xi, dr(s) \cdot \eta \rangle \leq 2 \langle \xi, \eta \rangle, \quad \xi, \eta \in E^n(z_0), \ s \in V_0',$$

and $\frac{1}{2} \|s - s'\| \leq d(r(s), r(s')) \leq 2 \|s - s'\|, \ s, s' \in V_0'$.

**Definitions** ([20]): (a) For a cylinder $C \subset U_0'$ and a unit vector $\xi \in E^n(z_0)$ we will say that a separation by a $\xi$-plane occurs in $C$ if there exist $u, v \in C$ with $d(u, v) \geq \frac{1}{2} \text{diam}(C)$ such that $\left\langle \frac{r^{-1}(u) - r^{-1}(v)}{\|r^{-1}(u) - r^{-1}(v)\|}, \xi \right\rangle \geq \theta_1$.

Let $S_\xi$ be the family of all cylinders $C$ contained in $U_0'$ such that a separation by an $\xi$-plane occurs in $C$.

(b) Given an open subset $V$ of $U_0'$ which is a finite union of open cylinders and $\delta > 0$, let $C_1, \ldots, C_p$ ($p = p(\delta) \geq 1$) be the family of maximal closed cylinders in $\overline{V}$ with $\text{diam}(C_j) \leq \delta$. For any unit vector $\xi \in E^n(z_0)$ set $M^{(\delta)}_\xi(V) = \cup \{C_j : C_j \in S_\xi, 1 \leq j \leq p \}$.

In what follows we will construct, amongst other things, a sequence of unit vectors $\xi_1, \xi_2, \ldots, \xi_{j_0} \in E^n(z_0)$. For each $\ell = 1, \ldots, j_0$ set $B_{\ell} = \{\eta \in S^{n-1} : \langle \eta, \xi_\ell \rangle \geq \theta_0\}$. For $t \in \mathbb{R}$ and $s \in E^n(z_0)$ set $I_{t,s}g(s) = \frac{g(s+t) - g(s)}{t}$, $t \neq 0$ (increment of $g$ in the direction of $\eta$).

**Lemma 6.** ([20]) There exist integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, a sequence of unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^n(z_0)$ and a non-empty open subset $U_0$ of $U_0'$ which is a finite union of open cylinders of length $n_1$ such that setting $\mathcal{U} = \sigma^{n_1}(U_0)$ we have:

(a) For any integer $N \geq N_0$ there exist Lipschitz maps $v_1^{(\ell)}, v_2^{(\ell)} : U \rightarrow U$ ($\ell = 1, \ldots, \ell_0$) such that $\sigma^N(v_1^{(\ell)}(x)) = x$ for all $x \in U$ and $v_1^{(\ell)}(U)$ is a finite union of open cylinders of length $N$ ($i = 1, 2; \ell = 1, 2, \ldots, \ell_0$).

(b) There exists a constant $\hat{\delta} > 0$ such that for all $\ell = 1, \ldots, \ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ with $s + h \eta \in r^{-1}(U_0 \cap \Lambda)$ we have

$$\left[ I_{t,s} \left( \tau^N(v_2^{(\ell)}(\tilde{r}(\cdot))) - \tau^N(v_1^{(\ell)}(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{\hat{\delta}}{2}.$$  

(c) We have $v_1^{(i,\ell)}(U) \cap v_2^{(i',\ell')}(U) = \emptyset$ whenever $(i, \ell) \neq (i', \ell')$.

(d) For any open cylinder $V$ in $U_0$ there exists a constant $\delta' = \delta'(V) > 0$ such that

$$V \subset M_{\eta_1}^{(\delta)}(V) \cup M_{\eta_2}^{(\delta)}(V) \cup \ldots \cup M_{\eta_{\ell_0}}^{(\delta)}(V)$$

for all $\delta \in (0, \delta']$.

Fix $U_0$ and $\mathcal{U}$ with the properties described in Lemma 1; then $\overline{U} = U$.

Set $\tilde{\delta} = \min_{1 \leq \ell \leq \ell_0} \delta_j$, $n_0 = \max_{1 \leq \ell \leq \ell_0} m_\ell$, and fix an arbitrary point $\tilde{z}_0 \in U_0^{(\ell_0)} \cap \tilde{U}$.

Fix integers $1 \leq n_1 \leq N_0$ and $\ell_0 \geq 1$, unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0} \in E^n(z_0)$ and a non-empty open subset $U_0$ of $W_0$ with the properties described in Lemma 6. By the choice of $U_0$, $\sigma^{n_1} : U_0 \rightarrow \mathcal{U}$ is one-to-one and has an inverse map $\psi : \mathcal{U} \rightarrow U_0$, which is Lipschitz.
Set $E = \max \left\{ 4A_0, \frac{2A_0}{\gamma_1^N} \right\}$, where $A_0 \geq 1$ is the constant from Lemma 5.4, and fix an integer $N \geq N_0$ such that

$$\gamma^N \geq \max \left\{ 6A_0, \frac{200\gamma_1^{n_1}A_0}{c_0}, \frac{512\gamma_1^{n_1}E}{c_0\delta \rho} \right\}.$$ 

Then fix maps $v_i^{(\ell)} : U \rightarrow U$ ($\ell = 1, \ldots, \ell_0$, $i = 1, 2$) with the properties (a), (b), (c) and (d) in Lemma 6. In particular, (c) gives

$$v_i^{(\ell)}(U) \cap v_i^{(\ell')}(U) = \emptyset, \quad (i, \ell) \neq (i', \ell').$$ 

Since $U_0$ is a finite union of open cylinders, it follows from Lemma 6(d) that there exist a constant $\delta' = \delta'(U_0) > 0$ such that

$$M_{\eta_1}^{(\delta)}(U_0) \cup \ldots \cup M_{\eta_0}^{(\delta)}(U_0) \supset U_0, \quad \delta \in (0, \delta'].$$

**Fix $\delta'$ with this property.**

Set

$$\epsilon_1 = \min \left\{ \frac{1}{32C_0}, \frac{1}{4E}, \frac{1}{\delta \rho^{n_0 + 2}}, \frac{c_0\rho^0}{\gamma_1^n}, \frac{c_0^2(\gamma - 1)}{16E\gamma_1^n} \right\},$$

and let $b \in \mathbb{R}$ be such that $|b| \geq 1$ and

$$\frac{\epsilon_1}{|b|} \leq \delta'.$$

Let $C_m$ ($1 \leq m \leq p$) be the family of maximal closed cylinders contained in $\overline{U_0}$ with diam$(C_m) \leq \frac{\epsilon_1}{|b|}$ such that $U_0 \subset \bigcup_{j=m}^p C_m$ and $\overline{U_0} = \bigcup_{m=1}^p C_m$. As in [20],

$$\rho \frac{\epsilon_1}{|b|} \leq \text{diam}(C_m) \leq \frac{\epsilon_1}{|b|}, \quad 1 \leq m \leq p.$$  \hspace{1cm} (4.2)

Fix an integer $q_0 \geq 1$ such that

$$\theta_0 < \theta_1 - 32\rho^{q_0 - 1}.$$ 

Next, let $D_1, \ldots, D_q$ be the list of all closed cylinders contained in $\overline{U_0}$ that are subcylinders of co-length $p_0q_0$ of some $C_m$ ($1 \leq m \leq p$). Then $\overline{U_0} = C_1 \cup \ldots \cup C_p = D_1 \cup \ldots \cup D_q$. Moreover,

$$\rho^{p_0q_0 + 1} \cdot \frac{\epsilon_1}{|b|} \leq \text{diam}(D_j) \leq \rho^{q_0} \cdot \frac{\epsilon_1}{|b|}, \quad 1 \leq j \leq q.$$ 

Given $j = 1, \ldots, q$, $\ell = 1, \ldots, \ell_0$ and $i = 1, 2$, set $\tilde{D}_j = D_j \cap \tilde{U}$, $Z_j = \sigma^{n_1}(\tilde{D}_j)$, $\tilde{Z}_j = Z_j \cap \tilde{U}$, $X_{i,j}^{(\ell)} = v_i^{(\ell)}(\tilde{Z}_j)$, and $\hat{X}_{i,j}^{(\ell)} = X_{i,j}^{(\ell)} \cap \tilde{U}$. It then follows that $\tilde{D}_j = \psi(Z_j)$, and $\tilde{U} = \bigcup_{j=1}^q Z_j$. Moreover, $\sigma^{n_1}(v_i^{(\ell)}(x)) = \psi(x)$ for all $x \in \mathcal{U}$, and all $X_{i,j}^{(\ell)}$ are cylinders such that $X_{i,j}^{(\ell)} \cap X_{i',j'}^{(\ell')} = \emptyset$ whenever $(i, j, \ell) \neq (i', j', \ell')$ and

$$\text{diam}(X_{i,j}^{(\ell)}) \geq \frac{c_0\rho^{p_0q_0 + 1}}{\gamma_1^n} \cdot \frac{\epsilon_1}{|b|}$$

for all $i = 1, 2$, $j = 1, \ldots, q$ and $\ell = 1, \ldots, \ell_0$. The characteristic function $\omega_{i,j}^{(\ell)} = \chi_{\hat{X}_{i,j}^{(\ell)}} : \tilde{U} \rightarrow [0, 1]$ of $\hat{X}_{i,j}^{(\ell)}$ belongs to $C_{D}^{1}(\tilde{U})$ and $\text{Lip}_D(X_{i,j}^{(\ell)}) \leq 1/\text{diam}(X_{i,j}^{(\ell)})$.

Let $J$ be a subset of the set $\Xi = \{ (i, j, \ell) : 1 \leq i \leq 2, 1 \leq j \leq q, 1 \leq \ell \leq \ell_0 \}$. Set

$$\mu_0 = \mu_0(N) = \min \left\{ \frac{1}{4}, \frac{c_0\rho^{p_0q_0 + 2}\epsilon_1}{4\gamma_1^n}, \frac{1}{4e^{2TN}} \sin^2 \left( \frac{\delta \rho \epsilon_1}{256} \right) \right\},$$
and define the function $\omega = \omega_J : \widehat{U} \to [0,1]$ by $\omega = 1 - \mu_0 \sum_{(i,j,\ell) \in J} \omega^{(\ell)}_{i,j}$. Clearly $\omega \in C_D^{\text{Lip}}(\widehat{U})$ and $1 - \mu \leq \omega(u) \leq 1$ for any $u \in \widehat{U}$. Moreover, 
\[
\text{Lip}_D(\omega) \leq \Gamma = \frac{2\mu \gamma_N}{C_0 \rho^{\rho_0 \theta_0 + 2}} \cdot \frac{|b|}{\epsilon_1}.
\]

Next, define the contraction operator $\mathcal{N} = \mathcal{N}_J(a,b,t,c) : C_D^{\text{Lip}}(\widehat{U}) \to C_D^{\text{Lip}}(\widehat{U})$ by
\[
(\mathcal{N}h)(\omega) = \mathcal{M}_{a_0,c}(\omega_J \cdot h).
\]

Using Lemma 5 above, the proof of the following lemma is the same as that of Lemma 5.6 in [20].

**Lemma 7.** Under the above conditions for $N$ and $\mu$ the following hold:

(a) $\mathcal{N}h \in K_{E|b|}(\widehat{U})$ for any $h \in K_{E|b|}(\widehat{U})$;

(b) If $h \in C_D^{\text{Lip}}(\widehat{U})$ and $H \in K_{E|b|}(\widehat{U})$ are such that $|h| \leq H$ in $\widehat{U}$ and $|h(v) - h(v')| \leq E|b|H(v') D(v,v')$ for any $v,v' \in U_j$, $j = 1, \ldots, k$, then for any $i = 1, \ldots, k$ and any $u,u' \in \widehat{U}_i$ we have
\[
|\langle \mathcal{L}^{N}_{abz \cdot h}(u) - \mathcal{L}^{N}_{abz \cdot h}(u') \rangle| \leq E|b|(N H(u')) D(u,u').
\]

**Definition.** A subset $J$ of $\Xi$ will be called dense if for any $m = 1, \ldots, p$ there exists $(i,j,\ell) \in J$ such that $\mathcal{D}_J \subset C_m$.

Denote by $J = J(a,b)$ the set of all dense subsets $J$ of $\Xi$.

Although the operator $\mathcal{N}$ here is different, the proof of the following lemma is very similar to that of Lemma 5.8 in [20].

**Lemma 8.** Given the number $N$, there exist $\rho_2 = \rho_2(N) \in (0,1)$ and $a_0 = a_0(N) > 0$ such that
\[
\int_{\widehat{U}} (\mathcal{N}H)^2 dv \leq \rho_2 \int_{\widehat{U}} H^2 dv \quad \text{whenever } |a|,|c| \leq a_0, \ t \geq 1/a_0, \ J \text{ is dense and } H \in K_{E|b|}(\widehat{U}).
\]

In what follows we assume that $h,H \in C_D^{\text{Lip}}(\widehat{U})$ are such that
\[
H \in K_{E|b|}(\widehat{U}) , \quad |h(u)| \leq H(u) , \quad u \in \widehat{U} , \quad (4.3)
\]
and
\[
|h(u) - h(u')| \leq E|b|H(u') D(u,u') \quad \text{whenever } u,u' \in \widehat{U}_i , \ i = 1, \ldots, k . \quad (4.4)
\]

Let again $z = c + i\omega$. Define the functions $\chi^{(1)}_\ell : \widehat{U} \to \mathbb{C} (\ell = 1, \ldots, j_0, i = 1,2)$ by
\[
\chi^{(1)}_\ell(u) = \left[ e^{(f_{d}^{N} - 4\nu N + zg)^{v_1^{(\ell)}(u)}} h(v_1^{(\ell)}(u)) + e^{(f_{d}^{N} - 4\nu N + zg)^{v_2^{(\ell)}(u)}} h(v_2^{(\ell)}(u)) \right] / \left( (1 - \mu)e^{f_{d}^{N}(v_1^{(\ell)}(u)) + cg^{v_1^{(\ell)}(u)}} H(v_1^{(\ell)}(u)) + \mu e^{f_{d}^{N}(v_2^{(\ell)}(u)) + cg^{v_2^{(\ell)}(u)}} H(v_2^{(\ell)}(u)) \right),
\]
\[
\chi^{(2)}_\ell(u) = \frac{e^{(f_{d}^{N} - 4\nu N + zg)^{v_1^{(\ell)}(u)}} h(v_1^{(\ell)}(u)) + e^{(f_{d}^{N} - 4\nu N + zg)^{v_2^{(\ell)}(u)}} h(v_2^{(\ell)}(u))}{e^{f_{d}^{N}(v_1^{(\ell)}(u)) + cg^{v_1^{(\ell)}(u)}} H(v_1^{(\ell)}(u)) + (1 - \mu)e^{f_{d}^{N}(v_2^{(\ell)}(u)) + cg^{v_2^{(\ell)}(u)}} H(v_2^{(\ell)}(u))},
\]
and set $\gamma_\ell(u) = b[\tau^{N}(v_2^{(\ell)}(u)) - \tau^{N}(v_1^{(\ell)}(u))]$, $u \in \widehat{U}$.

**Definitions.** We will say that the cylinders $\mathcal{D}_J$ and $\mathcal{D}_{J'}$ are adjacent if they are subcylinders of the same $C_m$ for some $m$. If $\mathcal{D}_J$ and $\mathcal{D}_{J'}$ are contained in $C_m$ for some $m$ and for some $\ell = 1, \ldots, \ell_0$
there exist \( u \in \mathcal{D}_j \) and \( v \in \mathcal{D}_{j'} \) such that \( d(u, v) \geq \frac{1}{4} \text{diam}(\mathcal{C}_m) \) and \( \left\langle \frac{r^{-1}(v)-r^{-1}(u)}{\|r^{-1}(v)-r^{-1}(u)\|}, \xi \right\rangle \geq \theta_1 \), we will say that \( \mathcal{D}_j \) and \( \mathcal{D}_{j'} \) are \( \eta \)-separable in \( \mathcal{C}_m \).

As a consequence of Lemma 6(b) one gets the following.

**Lemma 9.** (Lemma 5.9 in [20]) Let \( j, j' \in \{1, 2, \ldots, q\} \) be such that \( \mathcal{D}_j \) and \( \mathcal{D}_{j'} \) are contained in \( \mathcal{C}_m \) and are \( \eta \)-separable in \( \mathcal{C}_m \) for some \( m = 1, \ldots, p \) and \( l = 1, \ldots, \ell_0 \) . Then \( |\gamma_l(u) - \gamma_l(u')| \geq c_2 \varepsilon_1 \) for all \( u \in \tilde{Z}_j \) and \( u' \in \tilde{Z}_{j'} \), where \( c_2 = \frac{\delta_{\rho}}{16} \).

The following lemma is the analogue of Lemma 5.10 in [20] and represents the main step in proving Theorem 1.

**Lemma 10.** Assume \( |b| \geq b_0 \) for some sufficiently large \( b_0 > 0 \), \( |a|, |c| \leq a_0 \), and let (4.1) hold. Then for any \( j = 1, \ldots, q \) there exist \( i \in \{1, 2\} \), \( j' \in \{1, 2\} \) and \( \ell \in \{1, \ldots, \ell_0\} \) such that \( \mathcal{D}_j \) and \( \mathcal{D}_{j'} \) are adjacent and \( \chi_{\ell}^{(i)}(u) \leq 1 \) for all \( u \in \tilde{Z}_j, \mathcal{Z}_{j'} \).

To prove this we need the following lemma which coincides with Lemma 14 in [4] and its proof is almost the same.

**Lemma 11.** If \( h \) and \( H \) satisfy (4.3)-(4.4), then for any \( j = 1, \ldots, q \), \( i = 1, 2 \) and \( \ell = 1, \ldots, \ell_0 \) we have:

(a) \( \frac{1}{2} \leq \frac{H(v_{j}^{(\ell)}(u'))}{H(v_{j}^{(\ell)}(u''))} \leq 2 \) for all \( u', u'' \in \tilde{Z}_j \);

(b) Either for all \( u \in \tilde{Z}_j \) we have \( |h(v_{j}^{(\ell)}(u))| \leq \frac{3}{4} H(v_{j}^{(\ell)}(u)) \), or \( |h(v_{j}^{(\ell)}(u))| \geq \frac{1}{4} H(v_{j}^{(\ell)}(u)) \) for all \( u \in \tilde{Z}_j \).

**Sketch of proof of Lemma 10.** We use a modification of the proof of Lemma 5.10 in [20].

Given \( j = 1, \ldots, q \), let \( m = 1, \ldots, p \) be such that \( \mathcal{D}_j \subset \mathcal{C}_m \). As in [20] we find \( j', j'' \in 1, q \) such that \( \mathcal{D}_{j'} \subset \mathcal{C}_m \) and \( \mathcal{D}_{j''} \) and \( \mathcal{D}_{j''} \) are \( \eta \)-separable in \( \mathcal{C}_m \).

Fix \( \ell, j' \) and \( j'' \) with the above properties, and set \( \tilde{Z} = \tilde{Z}_j \cup \tilde{Z}_j' \cup \tilde{Z}_j'' \). If there exist \( t \in \{ j, j', j'' \} \) and \( i = 1, 2 \) such that the first alternative in Lemma 11(b) holds for \( \tilde{Z}_t, \ell \) and \( i \), then \( \mu \leq 1/4 \) implies \( \chi_{\ell}^{(i)}(u) \leq 1 \) for any \( u \in \tilde{Z}_t \).

Assume that for every \( t \in \{ j, j', j'' \} \) and every \( i = 1, 2 \) the second alternative in Lemma 11(b) holds for \( \tilde{Z}_t, \ell \) and \( i \), i.e. \( |h(v_{j}^{(\ell)}(u))| \geq \frac{1}{4} H(v_{j}^{(\ell)}(u)), u \in \tilde{Z}_t \).

Since \( \psi(\tilde{Z}) = \tilde{D}_j \cup \tilde{D}_j' \cup \tilde{D}_j'' \subset \mathcal{C}_m \), given \( u, u' \in \tilde{Z} \) we have \( \sigma^{N-n_1} v_{j}^{(\ell)}(u), \sigma^{N-n_1} v_{j}^{(\ell)}(u') \subset \mathcal{C}_m \). Moreover, \( C' = v_{j}^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m)) \) is a cylinder with \( \text{diam}(C') \leq \frac{\delta_{\rho}}{c_0 \gamma_n} \). Thus, the estimate (8.3) in the Appendix below implies

\[
|g_{\ell}^{N}(v_{j}^{(\ell)}(u)) - g_{\ell}^{N}(v_{j}^{(\ell)}(u'))| \leq \frac{C_1 t \varepsilon_1}{c_0 \gamma_n} \frac{1}{|b|}.
\]

Using the above assumption, (4.1), (4.2) and (3.5), and assuming e.g.

\[
e^c g_{\ell}^{N}(v_{j}^{(\ell)}(u))|h(v_{j}^{(\ell)}(u))| \geq e^c g_{\ell}^{N}(v_{j}^{(\ell)}(u))|h(v_{j}^{(\ell)}(u'))|,
\]
we get

\[
\frac{|e^{2gN(\theta_i(t))} h(\theta_i(t)) - e^{2gN(\theta_i(t'))} h(\theta_i(t'))|}{\min\{|e^{2gN(\theta_i(t))} h(\theta_i(t))|, |e^{2gN(\theta_i(t'))} h(\theta_i(t'))|\}} \\
= \frac{|e^{2gN(\theta_i(t))} h(\theta_i(t)) - e^{2gN(\theta_i(t'))} h(\theta_i(t'))|}{e^{2gN(\theta_i(t))} |h(\theta_i(t))|} \\
\leq \frac{|e^{2gN(\theta_i(t))} - e^{2gN(\theta_i(t'))}|}{e^{2gN(\theta_i(t))}} + \frac{e^{2gN(\theta_i(t))} |h(\theta_i(t)) - h(\theta_i(t'))|}{e^{2gN(\theta_i(t))} |h(\theta_i(t'))|} \\
\leq \frac{|e^{2gN(\theta_i(t))} - e^{2gN(\theta_i(t'))}|}{e^{2gN(\theta_i(t))}} + e^{2gN(\theta_i(t))} |h(\theta_i(t)) - h(\theta_i(t'))| E|b(H(\theta_i(t)) - H(\theta_i(t')))|D(\theta_i(t), \theta_i(t')) \\
\leq (C_1_t + |w| C_1_t) D(\theta_i(t), \theta_i(t')) + 4E|b|e^{2N\alpha T} \gamma^{N} e^N \epsilon \frac{\pi}{12}
\]

assuming \(\alpha_0 > 0\) is is chosen sufficiently small and \(N\) sufficiently large. So, the angle between the complex numbers

\[e^{2gN(\theta_i(t))} h(\theta_i(t))\quad \text{and}\quad e^{2gN(\theta_i(t'))} h(\theta_i(t'))\]

( regarded as vectors in \(\mathbb{R}^2\) ) is \(\leq \pi/6\). In particular, for each \(i = 1, 2\) we can choose a real continuous function \(\theta_i(u), u \in \mathbb{Z}\), with values in \([0, \pi/6]\) and a constant \(\lambda_i\) such that

\[e^{2gN(\theta_i(t))} h(\theta_i(t)) = e^{i(\lambda_i + \theta_i(u))} e^{2gN(\theta_i(t))} |h(\theta_i(t))|\]

for all \(u \in \mathbb{Z}\). Fix an arbitrary \(u_0 \in \mathbb{Z}\) and set \(\lambda = \gamma(\theta_0)\). Replacing e.g. \(\lambda_2\) by \(\lambda_2 + 2m\pi\) for some integer \(m\), we may assume that \(122 - \lambda + \lambda \leq \pi\). Using the above, \(\theta \leq 2\sin \theta\) for \(\theta \in [0, \pi/6]\), and some elementary geometry yields \(|\theta_i(u) - \theta_i(u')| \leq 2 \sin \theta_i(u) - \theta_i(u')| < \frac{\pi}{12}\).

The difference between the arguments of the complex numbers

\[e^{iB(\tau N(\theta_i(t)) - e^{2gN(\theta_i(t))} h(\theta_i(t))\quad \text{and}\quad e^{iB(\tau N(\theta_i(t)) - e^{2gN(\theta_i(t))} h(\theta_i(t))}\]

is given by the function

\[\Gamma(t) = [b \tau N(\theta_i(t)) + \theta_2(u) + \lambda_2] - [b \tau N(\theta_i(t)) + \theta_1(u) + \lambda_1] = (\lambda_2 - \lambda_1) + \gamma(\theta_2(u) - \theta_1(u))\]

Given \(u' \in \mathbb{Z}\) and \(u'' \in \mathbb{Z}\), since \(\mathbb{D}\) and \(\mathbb{D}'\) are contained in \(\mathbb{C}\) and are \(\eta\)-separable in \(\mathbb{C}\), it follows from Lemma 9 and the above that

\[|\Gamma(t)(u') - \Gamma(t)(u'')| \geq |\gamma(\theta_2(u') - \gamma(\theta_2(u'')) - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{2c_1 e_1}{2}\]

Thus, \(|\Gamma(t)(u') - \Gamma(t)(u'')| \geq \frac{c_1 e_1}{2}\) for all \(u' \in \mathbb{Z}\) and \(u'' \in \mathbb{Z}\). Hence either \(|\Gamma(t)(u')| \geq \frac{c_1 e_1}{4}\) for all \(u' \in \mathbb{Z}\) or \(|\Gamma(t)(u')| \geq \frac{c_1 e_1}{4}\) for all \(u'' \in \mathbb{Z}\).

\(^1\)Using some estimates as in the proof of Lemma 5(b) in the Appendix below and \(|c_0| \leq a_0 NT\) by (3.5).
Assume for example that $|\Gamma^{(l)}(u)| \geq \frac{3\pi}{4}\epsilon_1$ for all $u \in \hat{Z}_j'$. Since $\hat{Z} \subset \sigma^{n_1}(\mathcal{L}_m)$, as in [20] we have for any $u \in \hat{Z}$ we get $|\Gamma^{(l)}(u)| < \frac{3\pi}{2}$. Thus, $\frac{3\pi}{4}\epsilon_1 \leq |\Gamma^{(l)}(u)| < \frac{3\pi}{2}$ for all $u \in \hat{Z}_j'$. Now as in [4] (see also [20]) one shows that $\chi_t^{(1)}(u) \leq 1$ and $\chi_t^{(2)}(u) \leq 1$ for all $u \in \hat{Z}_j'$.

Parts (a) and (b) of the following lemma can be proved in the same way as the corresponding parts of Lemma 5.3 in [20], while part (c) follows from Lemma 5(b).

Lemma 12. There exist a positive integer $N$ and constants $\rho = \rho(N) \in (0, 1)$, $a_0 = a_0(N) > 0$, $b_0 = b_0(N) > 0$ and $E \geq 1$ such that for every $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ such that (4.1) hold, there exists a finite family $\{N_J\}_{J \in \mathbb{N}}$ of operators

$$N_J = N_J(a, b, t, c) : C^{\lip}_D(\hat{U}) \rightarrow C^{\lip}_D(\hat{U}),$$

where $J = J(a, b, t, c)$, with the following properties:

(a) The operators $N_J$ preserve the cone $K_{E|b|}(\hat{U})$;

(b) For all $H \in K_{E|b|}(\hat{U})$ and $J \in \mathbb{N}$ we have

$$\int_{\hat{U}} (N_J H)^2 \, dv_0 \leq \rho \int_{\hat{U}} H^2 \, dv_0.$$

(c) If $h, H \in C^{\lip}_D(\hat{U})$ are such that $H \in K_{E|b|}(\hat{U})$, $|h(u)| \leq H(u)$ for all $u \in \hat{U}$ and $|h(u) - h(u')| \leq E|b| H(u') D(u, u')$ whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$, then there exists $J \in \mathbb{N}$ such that $|L_{abw}^N h(u)| \leq (N_J H)(u)$ for all $u \in \hat{U}$ and for $z = c + iw$ we have

$$|(L_N^{abw}h')(u) - (L_N^{abw}h')(u')| \leq E|b|(N_J H)(u') D(u, u')$$

whenever $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$.

Proof of Theorem 5(a). Using an argument from [4] one derives from Lemma 12 that there exist a positive integer $N$ and constants $\rho \in (0, 1)$ and $a_0 > 0$, $b_0 \geq 1$, $A_0 = 0$ such that for any $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c| \leq a_0$, $|b| \geq b_0$ for which (4.1) hold, and for any $h \in C^{\lip}(\hat{U})$ with $\|h\|_{\lip,b} \leq 1$ we have

$$\int_{\hat{U}} |L_{abw}^N h|^2 \, dv_0 \leq \rho^m , \quad m \geq 0. \tag{4.5}$$

Then the estimate claimed in Theorem 5(a) follows as in [4] (see also the proof of Corollary 3.3(a) in [19]). \hfill \Box

The proof of Theorem 5(b) can be derived using an approximation procedure as in [4] – see the Appendix below for some details.

5. Spectral estimates when $w$ is the leading parameter

Here we try to repeat the arguments from the previous section however changing the roles of the parameters $b$ and $w$. We continue to use the assumptions made at the beginning of Sect. 4, however now we suppose that $f \in C^{\lip}(\hat{U})$. We will consider the case

$$|b| \leq B|w| \tag{5.1}$$

for an arbitrarily large (but fixed) constant $B > 0$.

Assume that $G : \Lambda \rightarrow \mathbb{R}$ is a Lipschitz functions which is constant on stable leaves of $B_t = \{\phi_t(x) : x \in \tau_t, 0 \leq t \leq \tau(x)\}$ for each rectangle $\tau_t$ of the Markov family and $A = \min_{x \in \Lambda} G(x) > 0$. Set

$$L = \lip(G) , \quad D = \text{diam}(\Lambda),$$
where without loss of generality we may assume that \( D \geq 1 \). We will also assume that
\[
L \leq \hat{\mu} A, \quad \text{where} \quad \hat{\mu} = \frac{c_0 \delta}{128 C_0 C_1 D}.
\]
(5.2)
The function
\[
g(x) = \int_0^{\tau(x)} G(\phi_t(x)) \, dt, \quad x \in R,
\]
is constant on stable leaves of \( R \), so it can be regarded as a function on \( U \). Clearly \( g \in C^{\text{Lip}}(\hat{U}) \).

**Remark.** Notice that if we replace \( G \) by \( G + d \) for some constant \( d > 0 \), then
\[
g'(x) = \int_0^{\tau(x)} (G(\phi_t(x)) + d) \, dt = g(x) + d \tau(x),
\]
so
\[
\mathcal{L}_{f_a - 1br + 1wg} = \mathcal{L}_{f_a - 1br + 1w(g' - d\tau)} = \mathcal{L}_{f_a - 1(b + dw)\tau - 1wg'}.
\]
Choose and fix \( d > 0 \) so that \( \frac{\text{Lip}(G)}{G_0 + d} \leq \hat{\mu} \). Then for \( G' = G + d \) and \( g' = g + d\tau \) we have \( \frac{\text{Lip}(G')}{\min G'} \leq \hat{\mu} \), and the operator \( \mathcal{L}_{f_a - 1br + 1wg} = \mathcal{L}_{f_a - 1b\tau + 1wg'} \), where \( b' = b + dw \). Thus, without loss of generality we may assume that \( \frac{\text{Lip}(G)}{\min G} \leq \hat{\mu} \), which is equivalent to (5.2). As in \cite{12}, this will imply a non-integrability property for \( g \) (see Lemma 10 below). In other words, dealing with an initial function \( G \) one has to first change it to arrange (5.2), and then with the new parameters \( b \) and \( w \) that appear in front of \( \tau \) and \( ig \) consider the cases \( |w| \leq B|b| \) (as in Theorem 5(c)) and \( |b| \leq B|w| \), which is considered in this section.

As in Sect. 4, we will use the set-up and some arguments from \cite{20}. Let \( \mathcal{R} = \{R_i\}_{i=1}^k \) be a Markov family for \( \phi_t \) over \( \Lambda \) as in Sect. 1.

Here we prove the following analogue of Theorem 5(c).

**Theorem 6.** Let \( \phi_t : M \rightarrow M \) be a \( C^2 \) flow satisfying the Standing Assumptions over the basic set \( \Lambda. \) Assume in addition that (5.2) holds. Then for any real-valued functions \( f, g \in C^{\text{Lip}}(\hat{U}) \), any constants \( \epsilon > 0 \) and \( B > 0 \) there exist constants \( 0 < \rho < 1 \), \( a_0 > 0 \), \( w_0 \geq 1 \) and \( C = C(B, \epsilon) > 0 \) such that if \( a, c \in \mathbb{R} \) satisfy \( |a|, |c| \leq a_0 \), then
\[
\|L_{f - (P + a)\tau + (c + iw)a}^m h\|_{\text{Lip}, b} \leq C \rho^m \|b\|^\epsilon \|h\|_{\text{Lip}, b}
\]
(5.3)
for all integers \( m \geq 1 \) and all \( b, w \in \mathbb{R} \) with \( |w| \geq w_0 \) and \( |b| \leq B|w| \).

Recall the definitions of \( \lambda_0 > 0 \), \( \nu_0 \), \( h_0 \), \( f_0 \) from Sect. 3; now we have \( h_0, f_0 \in C^{\text{Lip}}(\hat{U}) \). Fix a small \( a_0 > 0 \). Given a real number \( a \) with \( |a| \leq a_0 \), denote by \( \lambda_a \) the largest eigenvalue of \( L_{f - (P + a)\tau} \) on \( C^{\text{Lip}}(U) \) and by \( h_a \) the corresponding (positive) eigenfunction such that \( \int_U h_a \, d\nu_a = 1 \), where \( \nu_a \) is the unique probability measure on \( U \) with \( L_{f - (P + a)\tau}^* \nu_a = \nu_a \). Given real numbers \( a, b, c, w \) with \( |a|, |c| \leq a_0 \) consider the function
\[
\tilde{f}_a = f - (P + a)\tau + \ln h_a - \ln(h_a \circ \sigma) - \ln \lambda_a
\]
and the operators
\[
\mathcal{L}_{abz} = L_{f_a - 1b\tau + zg} : C(U) \rightarrow C(U), \quad \tilde{\mathcal{M}}_{ac} = L_{f_a + cg} : C(U) \rightarrow C(U),
\]
where \( z = c + iw \). Notice that \( L_{f_a^1} = 1 \).
Taking the constant $C_0 > 0$ sufficiently large, we may assume that
\[ \text{Lip}(\tilde{f}_a - f_0) \leq C_0 |a|, \quad ||\tilde{f}_a - f_0||_0 \leq C_0 |a|, \quad |a| \leq a_0. \quad (5.4) \]
Thus, assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $T > 0$ (depending on $f$ and $a_0$) such that
\[ ||\tilde{f}_a||_\infty \leq T, \quad \text{Lip}(h_a) \leq T, \quad \text{Lip}(\tilde{f}_a) \leq T \quad (5.5) \]
for $|a| \leq a_0$. As before, we will assume that $T \geq \max\{||\tau||_0, \text{Lip}(\tilde{\tau}|_E)\}$, and also that Lip($g$) $\leq T$ and $||g||_0 \leq T$.

Essentially in what follows we will repeat (a simplified version of) the proof of Theorem $5$, so we will use the set-up in Sect. $4$ – see the text after Lemma $6$, up to and including the definition of $\epsilon_1$.

Let $a, b, c, w \in \mathbb{R}$ be so that $|a|, |c| \leq a_0$, $|w| \geq w_0$, where $w_0$ is a sufficiently large constant defined as $b_0$ in Sect. 4, and $|b| \leq B|w|$. Set $z = c + iw$.

Let $\mathcal{C}_m (1 \leq m \leq p)$ be the family of maximal closed cylinders contained in $\overline{U}_0$ with $\text{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|w|}$ such that $U_0 \subset \bigcup_{j=1}^p \mathcal{C}_m$ and $\overline{U}_0 = \bigcup_{m=1}^p \mathcal{C}_m$. As before we have
\[ \rho \frac{\epsilon_1}{|w|} \leq \text{diam}(\mathcal{C}_m) \leq \frac{\epsilon_1}{|w|}, \quad 1 \leq m \leq p. \]

Fix an integer $q_0 \geq 1$ as in Sect. 4, and let $\mathcal{D}_1, \ldots, \mathcal{D}_q$ be the list of all closed cylinders contained in $\overline{U}_0$ that are subcylinders of co-length $p_0 q_0$ of some $\mathcal{C}_m (1 \leq m \leq p)$. Then $\overline{U}_0 = \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_p = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_q$ and
\[ \rho^{p_0 q_0 + 1} \frac{\epsilon_1}{|w|} \leq \text{diam}(\mathcal{D}_j) \leq \rho^{p_0} \frac{\epsilon_1}{|w|}, \quad 1 \leq j \leq q. \]

Next, define the cylinders $Z_j = \sigma^{n_1}(\mathcal{D}_j)$ and $X^{(\ell)}_{i,j} = v_{i,\ell}(Z_j)$ as in Sect. 4, and consider the characteristic functions $\omega_{i,j}^{(\ell)} = \chi_{X^{(\ell)}_{i,j}} : \tilde{U} \longrightarrow [0,1]$. Let $J$ be a subset of the set $\Xi = \Xi(a, w) = \{ (i,j,\ell) : 1 \leq i \leq 2, 1 \leq j \leq q, 1 \leq \ell \leq \ell_0 \}$. Define $\mu_0 > 0$ as in Sect. 4 and $\omega = \omega_J : \tilde{U} \longrightarrow [0,1]$ by $\omega = 1 - \mu_0 \sum_{(i,j,\ell) \in J} \omega_{i,j}^{(\ell)}$. Finally define $\mathcal{N} = \mathcal{N}_{J}(a, b, c) : C_{D}^{\text{Lip}}(\tilde{U}) \longrightarrow C_{D}^{\text{Lip}}(\tilde{U})$ by
\[ (\mathcal{N} h)(z) = \mathcal{N}_{J}(\omega_J \cdot h). \]

Then we have the following analogue of Lemma 5.

**Lemma 13.** Assuming $a_0 > 0$ is chosen sufficiently small, there exists a constant $A_0 > 0$ such that for all $a, c \in \mathbb{R}$ with $|a|, |c| \leq a_0$ the following hold:

(a) If $h \in K_E(\tilde{U})$ for some $E > 0$, then
\[ \frac{|(\tilde{\mathcal{N}}_{ac} H)(u) - (\tilde{\mathcal{N}}_{ac} H)(u')|}{(\tilde{\mathcal{N}}_{ac} H)(u')} \leq A_0 \left[ \frac{E}{\gamma_0^m} + 1 \right] D(u, u') \]
for all $m \geq 1$ and all $u, u' \in \tilde{U}_i, i = 1, \ldots, k$.

(b) If the functions $h$ and $H$ on $\tilde{U}$ and $E > 0$ are such that $H > 0$ on $\tilde{U}$ and $|h(v) - h(v')| \leq E H(v') D(v, v')$ for any $v, v' \in \tilde{U}_i, i = 1, \ldots, k$, then for any integer $m \geq 1$ and any $b, w \in \mathbb{R}$ with $|b|, |w| \geq 1$, for $z = c + iw$ we have
\[ |(\mathcal{L}_{ac}^n h)(u) - (\mathcal{L}_{ac}^n h)(u')| \leq E|w|(\mathcal{N} H)(u') D(u, u'). \]
whenever $u, u' \in \tilde{U}_i$ for some $i = 1, \ldots, k$. 

The proof is a simplified version of that of Lemma 5 and we omit it.

Next, changing appropriately the definition of a dense subset $J$ of $\Xi$, Lemma 8 holds again replacing $K_{E|_B}(\tilde{U})$ by $K_{E|_w}(\tilde{U})$.

Assume that $h,H \in C^L_D(\tilde{U})$. If are such that
\[
H \in K_{E|_w}(\tilde{U}), \quad |h(u)| \leq H(u), \quad u \in \tilde{U},
\]
and
\[
|h(u) - h(u')| \leq E|w|H(u')D(u,u') \quad \text{whenever } u,u' \in \tilde{U}, i = 1, \ldots, k.
\]
Define the functions $\chi^i_u : \tilde{U} \to \mathbb{C}$ by
\[
\chi^1_u(u) = \frac{e^{(f_a^N - \lambda r^N + g^N)(v_1^u(\ell) - h(v_1^u(\ell)))} e^{(f_a^N - \lambda r^N + g^N)h(v_2^u(\ell))}}{(1 - \mu)e^{f_a^N(\ell)(u) + c\gamma^N(v_1^u(\ell))} e^{f_a^N(\ell)(u)} + e^{f_a^N(\ell)(u) + c\gamma^N(v_2^u(\ell))} e^{f_a^N(\ell)(u)}},
\]
and set $\gamma_u(u) = w [\tau_N(v_2^u(\ell)) - \tau_N(v_1^u(\ell))], u \in \tilde{U}$. The crucial step in this section is to prove the following analogue of Lemma 9:

**Lemma 14.** Let $j,j' \in \{1,2,\ldots,q\}$ be such that $D_j$ and $D_{j'}$ are contained in $C_m$ and are $\eta$-separable in $C_m$ for some $m = 1,\ldots,p \ell \ell_0$. Then $|\gamma_u(u) - \gamma_u(u')| \geq c_3 \epsilon$ for all $u \in \tilde{U}$ and $u' \in \tilde{U}'$, where $c_3 = \frac{A\delta}{32}$.

To prove the above we need the following.

**Lemma 15.** (Lemma 6 in [12]) Assume that (5.2) holds. Under the assumptions and notation in Lemma 1, for all $\ell = 1,\ldots,\ell_0$, $s \in r^{-1}(U_0)$, $0 < |h| \leq \hat{\delta}$ and $\eta \in B_\ell$ so that $s + h \eta \in r^{-1}(U_0 \cap \Lambda)$ we have
\[
\left[ I_{\eta,h} \left( g^N(v_2^u(\ell)(\tilde{r}(\cdot))) - g^N(v_1^u(\ell)(\tilde{r}(\cdot))) \right) \right](s) \geq \frac{A\hat{\delta}}{4}.
\]

**Proof of Lemma 14.** This just a repetition of the proof of Lemma 5.9 in [20], where instead of using Lemma 6(b) we use the above Lemma 14. We omit the details. \hfill \Box

Next, we need to prove the analogue of Lemma 10.

**Lemma 16.** Assume $|w| \geq w_0$ for some sufficiently large $w_0 > 0$ and let $|b| \leq B|w|$. Then for any $j = 1,\ldots,q$ there exist $i \in \{1,2\}$, $j' \in \{1,\ldots,q\}$ and $\ell \in \{1,\ldots,\ell_0\}$ such that $D_j$ and $D_{j'}$ are adjacent and $\chi^i_u(u) \leq 1$ for all $u \in \tilde{U}'$.

**Sketch of proof of Lemma 16.** We will use Lemma 11 which holds again with (4.3)-(4.4) replaced by (5.6)-(5.7).

Given $j = 1,\ldots,q$, let $m = 1,\ldots,p$ be such that $D_j \subset C_m$. As in [20] we find $j',j'' = 1,\ldots,q$ such that $D_{j'}, D_{j''} \subset C_m$ and $D_j'$ and $D_{j''}$ are $\eta$-separable in $C_m$. \hfill \Box

The proof is a simplified version of that of Lemma 5 and we omit it.
Fix $\ell$, $j'$ and $j''$ with the above properties, and set $\tilde{Z} = \tilde{Z}_j \cup \tilde{Z}_j' \cup \tilde{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 11(b) holds for $\tilde{Z}_t$, $\ell$ and $i$, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \tilde{Z}$.

Assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 11(b) holds for $\tilde{Z}_t$, $\ell$ and $i$, i.e. $|h(v_i^{(\ell)}(u))| \geq \frac{1}{2} H(v_i^{(\ell)}(u))$, $u \in \tilde{Z}$.

Again we have $\psi(\tilde{Z}) = \hat{D}_{j} \cup \hat{D}_{j'} \cup \hat{D}_{j''} \subset \mathcal{C}_m$, and $\mathcal{C}' = v_i^{(\ell)}(\sigma_m^{n_1}(\mathcal{C}_m))$ is a cylinder with diam($\mathcal{C}'$) $\leq \frac{\sigma_m^{n_1}|w|}{\gamma_1^{N-n_1}}$. Thus, assuming e.g. $|h(v_i^{(\ell)}(u))| \geq |h(v_i^{(\ell)}(u'))|$, we get

$$\frac{|e^{ib\tau_N(v_i^{(\ell)}(u)) h(v_i^{(\ell)}(u))} - e^{ib\tau_N(v_i^{(\ell)}(u')) h(v_i^{(\ell)}(u'))}|}{\min\{|h(v_i^{(\ell)}(u))|, |h(v_i^{(\ell)}(u'))|\}} \leq \frac{|e^{ib\tau_N(v_i^{(\ell)}(u))} - e^{ib\tau_N(v_i^{(\ell)}(u'))}| + E|w| H(v_i^{(\ell)}(u')) D(v_i^{(\ell)}(u), v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|} \leq \frac{|b| C_1 D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + 4 E|w| D(v_i^{(\ell)}(u), v_i^{(\ell)}(u'))}{|b| C_1 + 4 E|w|} \text{ diam}(\mathcal{C}') \leq \frac{(BC_1 + 4E)\epsilon_1}{\gamma_1^{N-n_1}} \leq \frac{\pi}{12}$$

assuming $N$ is chosen sufficiently large. So, the angle between the complex numbers

$$e^{ib\tau_N(v_i^{(\ell)}(u)) h(v_i^{(\ell)}(u))} \quad \text{and} \quad e^{ib\tau_N(v_i^{(\ell)}(u')) h(v_i^{(\ell)}(u'))}$$

(regarded as vectors in $\mathbb{R}^2$) is $< \pi/6$. In particular, for each $i = 1, 2$ we can choose a real continuous function $\theta_i(u)$, $u \in \tilde{Z}$, with values in $[0, \pi/6]$ and a constant $\lambda_i$ such that $h(v_i^{(\ell)}(u)) = e^{i(\lambda_i + \theta_i(u))} |h(v_i^{(\ell)}(u))|$ for all $u \in \tilde{Z}$.

Fix an arbitrary $u_0 \in \tilde{Z}$ and set $\lambda = \gamma_\ell(u_0)$. Replacing e.g $\lambda_2$ by $\lambda_2 + 2m\pi$ for some integer $m$, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \leq \pi$. Using the above, $\theta \leq 2\sin\theta$ for $\theta \in [0, \pi/6]$, and some elementary geometry yields $|\theta_i(u) - \theta_i(u')| \leq 2\sin|\theta_i(u) - \theta_i(u')| < \frac{\pi}{8}$.

The difference between the arguments of the complex numbers

$$e^{ib\tau_N(v_i^{(\ell)}(u)) e^{ig_N(v_i^{(\ell)}(u)) h(v_i^{(\ell)}(u))}} \quad \text{and} \quad e^{ib\tau_N(v_i^{(\ell)}(u')) e^{ig_N(v_i^{(\ell)}(u)) h(v_i^{(\ell)}(u))}}$$

is given by the function

$$\Gamma^{(\ell)}(u) = |w g_N(v_i^{(\ell)}(u)) + \theta_2(u) + \lambda_2| - |w g_N(v_i^{(\ell)}(u)) + \theta_1(u) + \lambda_1| = (\lambda_2 - \lambda_1 + \gamma_\ell(u) + (\theta_2(u) - \theta_1(u))).$$

Given $u' \in \tilde{Z}_j'$ and $u'' \in \tilde{Z}_j''$, since $\hat{D}_{j'}$ and $\hat{D}_{j''}$ are contained in $\mathcal{C}_m$ and are $\eta_\ell$-separable in $\mathcal{C}_m$, it follows from Lemma 9 and the above that

$$|\Gamma^{(\ell)}(u') - \Gamma^{(\ell)}(u'')| \geq |\gamma_\ell(u') - \gamma_\ell(u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \geq \frac{C_1 \epsilon_1}{2}.$$
6. Analytic continuation of the function $\zeta(s, z)$

Consider the function $\zeta(s, z)$ introduced in Section 1. Recall that $s = a + ib, z = c + iw$ with real $a, b, c, w \in \mathbb{R}$. First, we assume that $f$ and $g$ are functions in $C^\alpha(\Lambda)$ with some $0 < \alpha < 1$. Passing to the symbolic model of the Markov family $\mathcal{R}$ we obtain function 3 in $C^\alpha(R)$ which we denote again by $f$ and $g$. We assume that $Pr(f - P_f \tau) = 0$ and we set $s = P_f + a + ib$. The functions $f, g$ depend on $x \in \mathbf{R}$. A second reduction is to replace $f$ and $g$ by functions $\hat{f}, \hat{g} \in C^{\alpha/2}(U)$ depending only on $x \in U$ so that $f = \hat{f} + h_1 - h_1 \circ \sigma, g = \hat{g} + h_2 - h_2 \circ \sigma$ (see Proposition 1.2 in [11]). Since for periodic points with $\sigma^n x = x$ we have $f^n(x) = \hat{f}(x), g^n(x) = \hat{g}(x)$, we obtain the representation

$$\zeta(s, z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{a^n x = x} e^{f^n(x)-(P_f+a+ib)\tau^n(x)+(c+iw)\hat{g}(x)}\right).$$

In this section we will prove under the standing assumptions that there exists $\epsilon > 0$ and $\epsilon_0 > 0$ such that the function $\zeta(s, z)$ has a non non zero analytic continuation for $-\epsilon \leq a \leq 0$ and $|z| \leq \epsilon_0$ with a simple pole at $s = s(z), s(0) = P_f$. Here $s(z)$ is determined from the equation $Pr(f - s \tau + zg) = 0$. For simplicity of the notation we denote below $\hat{f}$ and $\hat{g}$ again by $f, g$.

First consider the case $0 < \delta \leq |b| \leq b_0$. Since our standing assumptions imply that the flow of $\phi_t$ is weak mixing, Theorem 6.4 in [11] says that for every fixed $b$ lying in the compact interval $[\delta, b_0]$ there exists $\epsilon(b) > 0$ so that the function $\zeta(s, z)$ is analytic for $|s - P_f + ib| \leq \epsilon(b), |z| \leq \epsilon(b)$. This implies that there exists $\eta_0 = \eta_0(\delta, b_0) > 0$ such that $\zeta(s, z)$ is analytic for $P_f - \eta_0 \leq \Re s \leq P_f + \eta_0, \delta \leq \Im s \leq b_0, |z| \leq \eta_0$. Decreasing $\delta > 0$ and $\eta_0$, if it is necessary, we apply once more Theorem 6.4 in [11], to conclude that $\zeta(s, z)(1 - e^{Pr(f - s \tau + zg)})$ is analytic for

$$s \in \{s \in \mathbb{C} : |\Re s - P_f| \leq \eta_0, |\Im s| \leq \delta\}$$

and $|z| \leq \eta_0$. Consequently, the singularities of $\zeta(s, z)$ are given by $(s, z)$ for which we have $Pr(f - s \tau + zg) = 0$ and, solving this equation, we get $s = s(z)$ with $s(0) = P_f$. It is clear that we have a simple pole at $s(z)$ since $\frac{d}{ds}Pr(f - s \tau + zg) \neq 0$ for $|z|$ small enough.

Now we pass to the case when $|\Im s| = |b| \geq b_0 > 0, |z| \leq \eta_0$. Then we fix a $\beta \in (0, \alpha/2)$ and we get with $0 < \mu < 1$ the inequality $|\Im b| \geq B_0 |z|^\mu$ with $B_0 = \frac{b_0}{\eta_0}$. Thus we are in position to apply the estimates of Theorem 5(b) saying that for every $\epsilon > 0$ there exist $0 < \rho < 1$ and $C_\epsilon > 0$ so that

$$\|L_{f-(P_f+a+ib)\tau+zg}\|_{\beta, b} \leq C_\epsilon \rho^m |b|^{\epsilon^d}, \forall m \in \mathbb{N} \quad (6.1)$$

for $|a| \leq a_0, |b| \geq b_0, |z| \leq \eta_0$. Next we apply Theorem 4 with functions $f, g \in C^\beta(U)$. For

$$|\Re s - P_f| \leq \eta_0, \Im s \geq b_0 \text{ and } |z| \leq \eta_0 \text{ we deduce}$$

$$|Z_n(f - (P_f + a + ib)\tau + zg)| \leq \sum_{i=1}^{k} |L_{f-(P_f+a+ib)\tau+zg}(\chi_i)(x_i)|$$

$$+ C(1 + |b|) \sum_{m=2}^{n} \|L_{f-(P_f+a+ib)\tau+zg}\|_{\beta, b}^{-m\beta} e^{mPr(f-(P_f+a)\tau+(Re)g)}$$

\footnote{In fact, one has to define first $f$ and $g$ as functions in $C^\alpha(\hat{R})$ and then extend them as $\alpha$-Hölder functions on $R$. In the same way one should proceed with Hölder functions on $U$.}
\[
\leq k\|L^n_{f-(P_f+a+ib)+zg}\|_\beta + C_\epsilon(1+|b|)\rho^n \sum_{m=2}^{n} \rho^{n-m} \gamma_0^{-m}\epsilon^m e^{\epsilon + Pr(f-(P_f+a)+cg)}.
\]

Taking \(\eta_0\) and \(\epsilon\) small, we arrange
\[
\gamma_0^{-\beta} e^{\epsilon + Pr(f-(P_f+a)+cg)} \leq \gamma_2 < 1
\]
for \(|a| \leq \eta_0, |c| \leq \eta_0\), since \(Pr(f-P_f\tau) = 0\) and \(\gamma_0^{-\nu} < 1\). Next increasing \(0 < \rho < 1\), if it is necessary, we get \(\frac{2\rho}{\rho} < 1\). Thus the sum above will be bounded by
\[
C_\epsilon(1+|b|)|b|^\epsilon \rho^n \sum_{m=2}^{\infty} \left(\frac{\gamma_2}{\rho}\right)^m \leq C_\epsilon |b|^{1+\epsilon} \rho^n
\]
for \(|a| \leq \eta_0, |z| \leq \eta_0\). The analysis of the term \(\|L^n_{f-(P_f+a+ib)+zg}\|_\beta\) follows the same argument and it is simpler. Finally, we get
\[
|Z_n(f-(P_f+a+ib)\tau + zg)| \leq B_\epsilon |b|^{1+\epsilon} \rho^n, \forall n \in \mathbb{N}
\]
and the series
\[
\sum_{n=1}^{\infty} \frac{1}{n}Z_n(f-(P_f+a+ib)\tau + zg)
\]
is absolutely convergent for \(|a| \leq \eta_0, |b| \geq b_0, |z| \leq \eta_0\). This implies the analytic continuation of \(\zeta(s,z)\) for \(\text{Re } s - P_f| \leq \eta_0, |\text{Im } s| \geq b_0, |z| \leq \eta_0\), thus completing the proof of Theorem 1.

To obtain a representation of the function \(\eta_g(s) = \frac{\partial \log \zeta(s,z)}{\partial z}|_{z=0}\) for \(s\) sufficiently close to \(P_f\), notice that for such values of \(s\) we have
\[
\eta_g(s) = -\frac{\partial \log (1-e^{Pr(f-s+tg)})}{\partial z}|_{z=0} + A_0(s)
\]
\[
= \frac{1}{s-P_f} \int \frac{gdm}{\tau dm} + A_1(s) = \frac{\int Gd\mu_F}{s-P_f} + A_1(s),
\]
where \(m\) is the equilibrium state of \(f-P_f\tau\), \(\mu_F\) is the equilibrium state of \(F\) and \(A_0(s)\) and \(A_1(s)\) are analytic in a neighborhood of \(P_f\) (see Chapter 6 in [11]). More precisely, \(\mu_F\) is a \(\sigma_1^T\) invariant probability measure on \(R^T\) such that
\[
Pr(F) = h(\sigma_1^T, \mu_F) + \int F(\pi(x,t))d\mu_F,
\]
where \(h(\sigma_1^T, \mu_F)\) is the metric entropy of \(\sigma_1^T\) with respect to \(\mu_F\) (see Chapter 6 in [11]).

Taking \(\eta_0\) small enough, for \(|z| \leq \eta_0, |\text{Re } s - P_f| \leq \eta_0\) and \(|\text{Im } s| \geq \eta_0\) from the estimates for \(Z_n(f-(P_f+a+ib)\tau + zg)\) above, we deduce
\[
|\log \zeta(s,z)| \leq C_\epsilon \max \left(1, |\text{Im } s|^{1+\epsilon}\right).
\]

To estimate \(\eta_g(s)\), as in [16], we apply the Cauchy theorem for the derivative
\[
\frac{\partial}{\partial z} \log \zeta(s,z)|_{z=0} = \frac{1}{2\pi i\delta} \int_{|\xi| = \delta} \frac{\log \zeta(s,\xi)}{\xi^2} d\xi = O(|\text{Im } s|^{1+\epsilon}), |\text{Im } s| \geq 1.
\]
with $\delta > 0$ sufficiently small. Thus we obtain a $O(\max(1, |\text{Im} s|^{1+\epsilon}))$ bound for the function

$$A(s) = \eta_0(s) - \frac{1}{s - P_f} \int Gd\mu_F$$

which is analytic for $|\text{Re} s - P_f| \leq \eta_0$. Decreasing $\eta_0$ and applying Phragmén-Lindelöf theorem, by a standard argument we obtain a bound $O(\max(1, |\text{Im} s|^\alpha))$ with $0 < \alpha < 1$. Consequently, we have the following

**Proposition 3.** Under the assumptions of Theorem 1 there exist $\eta_0 > 0$ and $0 < \alpha < 1$ such that for $|\text{Re} s - P_f| < \eta_0$ we have

$$\eta_0(s) = \frac{1}{s - P_f} \int Gd\mu_F + A(s)$$

(6.2)

with an analytic function $A(s)$ satisfying the estimate

$$|A(s)| \leq C \max\left(1, |\text{Im} s|^\alpha\right).$$

(6.3)

Next define $F^r(\mathbb{C}) := \{F : R^r \to \mathbb{C}\}$ and $F^r(\mathbb{R}) := \{F : R^r \to \mathbb{R}\}$ the spaces of complex-valued (real-valued) functions which are continuous. If $G \in F^r(\mathbb{C})$ is Lipschitz continuous and if the standing assumptions for $\Lambda$ are satisfied, the function

$$g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$$

is Lipschitz continuous on $R$. Moreover, if the representative of $G$ in the suspension space $R^r$ is constant on stable leaves, the function $g(x)$ depends only on $x \in U$. Now we introduce two definitions of independence.

**Definition 1.** Two functions $f_1, f_2 : U \to \mathbb{R}$ are called $\sigma-$ independent if whenever there are constants $t_1, t_2 \in R$ such that $t_1 f_1 + t_2 f_2$ is co homologous to a function in $C(U : 2\pi \mathbb{Z})$, we have $t_1 = t_2 = 0$.

For a function $G \in F^r(\mathbb{R})$ consider the skew product flow $S^G_t$ on $S^1 \times R^r$ by

$$S^G_t(e^{2\pi i\alpha}, y) = (e^{2\pi i(\alpha + G^t(y))}, \sigma_t^\tau(y)).$$

**Definition 2.** Let $G \in F^r(\mathbb{R})$. Then $G$ and $\sigma_t^\tau$ are flow independent if the following condition is satisfied. If $t_0, t_1 \in \mathbb{R}$ are constants such that the skew product flow $S^H_t$ with $H = t_0 + t_1 G$ is not topologically mixing, then $t_0 = t_1 = 0$.

Notice that if $G$ and $\sigma_t^\tau$ are flow independent, then the flow $\sigma_t^\tau$ is topologically weak mixing and the function $G$ is not co homologous to a constant function. On the other hand, if $G$ and $\sigma_t^\tau$ are flow independent, then $g(x) = \int_0^{\tau(x)} G(\pi(x,t))dt$ and $\tau$ are $\sigma-$ independent.

Below we assume that $g$ and $\tau$ are $\sigma-$ independent and we suppose that $F, G$ is a Lipschitz functions $\Lambda$ having representative in $R^r$ which are constant on stable leaves. Thus we obtain functions $f, g$ which are in $C^{\text{Lip}}(\hat{U})$. We will now obtain an analytic continuation of $\zeta(s, z)$ for $P_f - \eta_0 < \text{Re} s < P_f$ and $z = iw$. Set $r(s, w) = f - (P_f + a + ib)\tau + iw g$. We choose $M > 0$ large enough so that we can apply Theorem 6 for $|w| \geq M$. We consider two cases.
**Case 1.** \( \eta_0 \leq |w| \leq M \). We consider two sub cases: 1a) \( \text{Im } s \leq M_1 \), 1b) \( \text{Im } s \geq M_1 \). Here \( M_1 > 0 \) is chosen large enough so that Theorem 5 (b) holds with \( |\text{Im } s| \geq M_1 \).

Let \( |\text{Im } s| \leq M_1 \). Assume first that \( \text{Im } r(s_0, w_0) \) is cohomologous to \( c + 2\pi Q \) with an integer-valued function \( Q \in C(U; \mathbb{Z}) \) and a constant \( c \in [0, 2\pi) \). If \( c = 0 \), since \( g \) and \( \tau \) are \( \sigma \)-independent, from the fact that \( b \tau + wg \) is cohomologous to a function in \( C(U; 2\pi \mathbb{Z}) \), we deduce \( b = w = 0 \) which is impossible because \( b = \text{Im } s \neq 0 \). Thus we have \( c \neq 0 \). Consequently, the operator \( L_{f - s_0 \tau + iw_0} \) has an eigenvalue \( e^{ic} \). Then there exists a neighborhood \( U_1 \subset \mathbb{C} \times \mathbb{R} \) of \((s_0, w_0)\) such that for \((s, w) \in U_1\) we have \( Pr(r(s, w)) \neq 0 \) and for \((s, w) \in U_2\) we have an analytic extension of \( \log \zeta(s, w) \) given by

\[
\log \zeta(s, w) = \frac{K_1(s, w)}{1 - e^{Pr(r(s, w))}} + J_1(s, w)
\]

with functions \( K_1(s, w), J_1(s, w) \) analytic with respect to \( s \) for \((s, w) \in U_1\). Second, let \( \text{Im } r(s_0, w_0) \) be not cohomologous to \( c + 2\pi Q \). Then the spectral radius of \( L_{f - s_0 \tau + iw_0} \) is strictly less than 1 and this will be the case for \((s, w)\) is a small neighborhood \( U_2 \subset \mathbb{C} \times \mathbb{R} \) of \((s_0, w_0)\). Applying Theorem 4, this implies easily that \( \log \zeta(s, iw) \) has an analytic continuation with respect to \( s \).

Passing to the case 1b), we observe that \( |\text{Im } s| \geq \frac{M_1}{\eta_0} |w| \). Then, we apply Theorem 5, (c) combined with Theorem 4 to obtain an analytic continuation of \( \log \zeta(s, iw) \). Moreover, our argument works for \( z = c + iw \) with \(|c| \leq \eta_0 \) and \( \eta_0 \leq |w| \leq M \) and we obtain an analytic continuation of \( \log \zeta(s, z) \) for \( P_f - \eta_0 \leq \text{Re } s < P_f, |c| \leq \eta_0, \eta_0 \leq |w| \leq M \).

**Case 2.** \(|w| \geq M \). We consider two sub cases: 2a) \( |\text{Im } s| \geq B|w| \), 2b) \( |\text{Im } s| \leq B|w|, B = \frac{M_1}{M} \). If we have 2a), we apply Theorem 5 (c). In the case 2b) we use the argument of Section 5 replacing \( g(x) \) by \( g'(x) = g(x) + d\tau(x) \), where the constant \( d > 0 \) is chosen so that for the function \( G' = G + d \) we have

\[
\frac{\text{Lip } G'}{\min G'} \leq \hat{\mu},
\]

where \( \hat{\mu} > 0 \) is the constant introduced in Section 5. Next we write

\[
L_{f - (P_f - \eta_0)\tau + iw} = L_{f - (P_f + a + b dw)\tau + iw g'}.
\]

For the Ruelle operator involving \( g' \) we can apply Theorem 6 since \(|b + dw| \leq (B + d)|w|, |w| \geq M \) and \( g \) is a Lipschitz function. An application of Theorem 4 implies the analytic continuation of \( \log \zeta(s, iw) \) for \( P_f - \eta_0 \leq \text{Re } s < P_f \) and \(|w| \geq M \). From the above analysis we deduce the following

**Theorem 7.** Assume the standing assumptions fulfilled for the basic set \( \Lambda \). Let \( F, G : \Lambda \to \mathbb{R} \) be Lipschitz functions having representatives in \( R^\tau \) which are constant on stable leaves. Assume that \( g \) and \( \tau \) are \( \sigma \)-independent. Then there exists \( \eta_0 > 0 \) such that \( \zeta(s, iw) \) admits a non zero analytic continuation with respect to \( s \) for \( P_f - \eta_0 \leq \text{Re } s, w \in \mathbb{R} \) and \(|w| \geq \eta_0 \).

### 7. Applications

7.1. **Hannay-Ozorio de Almeida sum formula.** The proof of (1.5) in [17] is based on the analytic continuation of the Dirichlet series

\[
\eta(s) = \sum_\gamma \sum_{m=1}^\infty \lambda_G(\gamma) e^{m(-\lambda^u(\gamma)-(s-1)\lambda(\gamma))}, s \in \mathbb{C}
\]
TWO PARAMETERS

for $1 - \eta_0 \leq \text{Re } s < 1$. For this purpose the authors examine the analytic continuation of the symbolic function $\eta_0(s)$ with $g(x) = \int_0^T G(x, t)dt$ defined in Section 1 and they use the fact that the difference $\eta(s) - \eta_0(s)$ is analytic in a region $\text{Re } s > 1 - \epsilon'$, $\epsilon' > 0$. Next for the geodesic flow on surfaces with negative curvature they establish Proposition 3 with $P_f = 1$. Since $M$ is an attractor, the equilibrium state of the function $-E(x)$ is just the SRB measure $\mu$ of $\phi_t$ (see [3]) and the residuum in (6.2) becomes $\int Gd\mu$.

For the proof of Proposition 3 in [17] the authors exploit the link between the analytic continuation of $\zeta(s, z)$ and the spectral estimates of the Ruelle operator obtained by Dolgopyat [4]. However, in [17] Ruelle’s lemma in [15] was used whose proof is rather sketchy and contains some steps which are not done in detail (see [23] for more information and comments concerning these steps and the gaps in their proofs). On the other hand, the estimates of Dolgopyat [4] are established only for Ruelle operators with one complex parameter, and to take into account the second parameter $z$ some complementary analysis is necessary.

We would like to mention that [23] contains a correct and complete proof of Ruelle’s lemma in the case of one complex parameter and H"older function $\tau(x)$. A version of this lemma with two complex parameters is given in Section 2 above. Next, in Theorem 5 the spectral estimates for the Ruelle operator with two complex parameters are established for Axiom A flows on a basic set $\Lambda$ of arbitrary dimension under the standing assumptions. If $\Lambda$ is an attractor, according to [3], the equilibrium state of $-E(x)$ coincides with the SRB measure $\mu$ of $\phi_t$. Thus we can apply Proposition 3 to obtain a representation of $\eta_0(s)$ with residue $\int Gd\mu$. Using (6.2) and repeating the argument of Section 4 in [17], we obtain Theorem 2.

7.2. Asymptotic of the counting function for period orbits. As we mentioned in Sect. 1, the analysis of $\pi_F(T)$ is based on the analytic continuation of the function $\zeta(s, 0)$ defined in Section 1. From the arguments in Section 6 with $z = 0$ and the proof of Proposition 3 we get the following

**Proposition 4.** Under the standing assumptions in Sect. 3 there exists $\eta_0 > 0$ such that $\frac{\zeta'(F)}{\zeta(F)}$ admits an analytic continuation for $\text{Pr}(F) - \eta_0 \leq \text{Re } s$ with a simple pole at $s = \text{Pr}(F)$ with residue 1. Moreover, there exists $0 < \alpha < 1$ such that for $|\text{Im } s| \geq 1$ we have

$$\left| \frac{\zeta'(F)}{\zeta(F)} \right| \leq C|\text{Im } s|^\alpha. \quad (7.1)$$

To obtain an asymptotic of $\pi_F(T)$, we examine the functions

$$\Psi(T) = \sum_{e^{n\text{Pr}(F)\lambda(\gamma)\leq T}} \lambda(\gamma)e^{\text{Pr}(F)\lambda(\gamma)}, \quad \Psi_1(T) = \int_0^T \Psi(y)dy.$$

By a standard argument (see [15] and [14]) we obtain the representation

$$\psi_1(T) = \frac{T^2}{2} + \int_{\text{Re } s = (1-\eta_0)\text{Pr}(F)} \frac{T^s}{s(s + 1)} ds = \frac{T^2}{2} + O(T^{1+\alpha}),$$

where in the second equality the estimate (7.1) is used. This implies an asymptotic for $\Psi(T)$ and repeating the argument in [15], [14], one obtains Theorem 3.
8. Appendix: Proofs of some lemmas

Proof of Lemma 4. Denote by $F_\theta(\hat{U})$ the space of all functions $h : \hat{U} \to \mathbb{R}$ that are Lipschitz with respect to $d_a$. Let $g \in C^{\text{Lip}}(\hat{U})$, and let $\theta = \theta_\alpha \in (0, 1)$ be as in Sect. 3. Then $g \in F_\theta(\hat{U})$. Let $\lambda > 0$ be the maximal positive eigenvalue of $L_g$ on $F_\theta(\hat{U})$ and let $h > 0$ be a corresponding normalized eigenfunction. By the Ruelle-Perron-Frobenius theorem, we have that \( \frac{1}{\lambda^m} L_g^m 1 \) converges uniformly to $h$. We will show that there exists a constant $C > 0$ such that \( \frac{1}{\lambda^m} \text{Lip}(L_g^m 1) \leq C \) for all $m$; this would then imply immediately that $h \in C^{\text{Lip}}(\hat{U})$ and Lip($h$) $\leq C$.

Take an arbitrary constant $K > 0$ such that $1/K \leq h(x) \leq K$ for all $x \in \hat{U}$. Given $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k$ and an integer $m \geq 1$ for any $v \in \hat{U}$ with $\sigma^m(v) = u$, denote by $v' = v'(v)$ the unique $v' \in \hat{U}$ in the cylinder of length $m$ containing $v$ such that $\sigma^m(v') = u'$. By (1.1) we have

\[
|g_m(v) - g_m(v')| \leq \sum_{j=0}^{m-1} |g(\sigma^j(v)) - g(\sigma^j(v'))| \leq \text{Lip}(g) \sum_{j=0}^{m-1} \frac{d(u, u')}{c_0 \gamma^m} \leq C' \text{Lip}(g) d(u, u')
\]

for some constant $C' > 0$. Thus,

\[
|(L_g^m 1)(u) - (L_g^m 1)(u')| \leq \sum_{\sigma^m(v) = u} |e^{g_m(v)} - e^{g_m(v')}| = \sum_{\sigma^m(v) = u} e^{g_m(v)} \left| e^{g_m(v)} - e^{g_m(v')} \right| - 1 \leq e^{C' \text{Lip}(g)} \sum_{\sigma^m(v) = u} e^{g_m(v)} \left| g_m(v) - g_m(v') \right| \leq e^{C' \text{Lip}(g)} \text{Lip}(g) d(u, u') \sum_{\sigma^m(v) = u} e^{g_m(v)} \leq e^{C' \text{Lip}(g)} \text{Lip}(g) d(u, u') \sum_{\sigma^m(v) = u} e^{g_m(v)} K h(v) = e^{C' \text{Lip}(g)} \text{Lip}(g) d(u, u') (L_g^m h)(u) = e^{C' \text{Lip}(g)} \text{Lip}(g) d(u, u') \lambda^m h(u) \leq \lambda^m C' K^2 e^{C' \text{Lip}(g)} \text{Lip}(g) d(u, u').
\]

Thus, for every integer $m$ the function \( \frac{1}{\lambda^m} L_g 1 \in C^{\text{Lip}}(\hat{U}) \) and \( \frac{1}{\lambda^m} \text{Lip}(L_g^m 1) \leq C' K^2 e^{C' \text{Lip}(g)} \text{Lip}(g) \). As mentioned above this proves that the eigenfunction $h \in C^{\text{Lip}}(\hat{U})$.

Using this with $g = f_i$ proves that $h_{at} \in C^{\text{Lip}}(\hat{U})$ for all $|a| \leq a_0$ and $t \geq 1/a_0$. However the above estimate for Lip($h_{at}$) would be of the form $\leq C' e^{C \gamma} t$ for some constant $C > 0$, which is not good enough.

We will now show that, taking $a_0 > 0$ sufficiently small, we have Lip($h_{at}$) $\leq C t$ for some constant $C > 0$ independent of $a$ and $t$.

Using (3.2) and choosing $a_0 > 0$ sufficiently small, we have $\lambda_{at} \gamma > \hat{\gamma}$ for all $|a| \leq a_0$ and $t > 1/a_0$. Fix an integer $m_0 \geq 1$ so large that \( \frac{\gamma^2}{\alpha_\gamma^m} < \frac{1}{4} \) for $m \geq m_0$. There exists a constant $d_0 > 0$ depending on $m_0$ such that for any $u, u'$ belonging to the same $U_i$ but not to the same cylinder of length $m_0$ we have $d(u, u') \geq d_0$. For such $u, u'$ we have

\[
\frac{|h_{at}(u) - h_{at}(u')|}{d(u, u')} \leq 2 \frac{\|h_{at}\|_0}{d_0} \leq 2C_0/d_0.
\]
So, to estimate \( \text{Lip}(\hat{h}_t) \) it is enough to consider pairs \( u, u' \) that belong to the same cylinder of length \( m_0 \).

Fix for a moment \( a, t \) with \(|a| \leq a_0 \) and \( t \geq 1/a_0 \). Set

\[
L = \sup_{u \neq u'} \frac{|\hat{h}_t(u) - \hat{h}_t(u')|}{d(u, u')},
\]

where the supremum is taken over all pairs \( u \neq u' \) that belong to the same cylinder of length \( m_0 \). If \( L < \text{Lip}(\hat{h}_t) \), then the above implies

\[
\text{Lip}(\hat{h}_t) \leq \frac{2C_0}{d_0} \leq \frac{2C_0}{d_0} t.
\]

Assume that \( L = \text{Lip}(\hat{h}_t) \). Then there exist \( u, u' \) belonging to the same cylinder of length \( m_0 \) such that

\[
\frac{3L}{4} < \frac{|\hat{h}_t(u) - \hat{h}_t(u')|}{d(u, u')}.
\]

(8.1)

Fix such a pair \( u, u' \). Let \( m \geq m_0 \) be an integer. For any \( v \in \hat{U} \) with \( \sigma^m(v) = u \), denote by \( v' = v'(v) \) the unique \( v' \in \hat{U} \) in the cylinder of length \( m \) containing \( v \) such that \( \sigma^m(v') = u' \). By (1.1),

\[
d(\sigma^j(v), \sigma^j(v')) \leq \frac{1}{c_0 \gamma^{m-j}} d(u, u') \quad , \quad j = 0, 1, \ldots, m - 1
\]

so

\[
|f_t^m(v) - f_t^m(v')| \leq \sum_{j=0}^{m-1} |f_t(\sigma^j(v)) - f_t(\sigma^j(v'))| \leq \text{ConstLip}(f_t) d(u, u') \leq \text{Const} t d(u, u').
\]

At the same time, by property (i), \( \|f_t\|_0 \leq T'' \) for some constant \( T'' > 0 \), so

\[
|f_t^m(v) - f_t^m(v'(v))| \leq 2m \|f_t\|_0 \leq 2m T''.
\]

Similarly,

\[
|(P + a)\tau^m(v) - (P + a)\tau^m(v'(v))| \leq \text{Const} d(u, u') \leq T'',
\]

assuming \( T'' > 0 \) is chosen sufficiently large. Thus,

\[
\left| e^{(f_t - (P+a)\tau)^m(v')} - (f_t - (P+a)\tau)^m(v) - 1 \right| \\
\leq e^{3mT''} \left| (f_t - (P+a)\tau)^m(v) - (f_t - (P+a)\tau)^m(v') \right| \leq e^{3mT''} \text{Const} t d(u, u').
\]
Using \( L_{f_t-(P+\alpha)}^{m} h_{at} = \lambda_{at}^{m} h_{at} \), we obtain

\[
\lambda_{at}^{m} |h_{at}(u) - h_{at}(u')| = \left| \sum_{\sigma_{m}^{u}=u} e^{(f_{t}-(P+\alpha)\tau)^{m}(v)} h_{at}(v) - \sum_{\sigma_{m}^{u}=u} e^{(f_{t}-(P+\alpha)\tau)^{m}(v')} h_{at}(v') \right|
\]

\[
\leq \sum_{\sigma_{m}^{u}=u} |e^{(f_{t}-(P+\alpha)\tau)^{m}(v)} h_{at}(v) - h_{at}(v')| + \|h_{at}\|_{0} \sum_{\sigma_{m}^{u}=u} \left| e^{(f_{t}-(P+\alpha)\tau)^{m}(v)} - e^{(f_{t}-(P+\alpha)\tau)^{m}(v')} \right|
\]

\[
\leq \text{Lip}(h_{at}) \frac{d(u, u')}{c_{0}\gamma_{m}} \sum_{\sigma_{m}^{u}=u} e^{(f_{t}-(P+\alpha)\tau)^{m}(v)} + C_{0} e^{3mT''} \text{Const } t \frac{d(u, u')}{c_{0}\gamma_{m}} \sum_{\sigma_{m}^{u}=u} e^{(f_{t}-(P+\alpha)\tau)^{m}(v)} \lambda_{at}^{m}(v)
\]

\[
\leq \left( \frac{L}{c_{0}\gamma_{m}} + C_{0} e^{3mT''} \text{Const } t \right) d(u, u') \sum_{\sigma_{m}^{u}=u} e^{(f_{t}-(P+\alpha)\tau)^{m}(v)} \lambda_{at}^{m}(v)
\]

\[
= \left( \frac{L}{c_{0}\gamma_{m}} + C_{0} e^{3mT''} \text{Const } t \right) d(u, u') C_{0} \lambda_{at}^{m}(u) \leq \left( \frac{L}{c_{0}\gamma_{m}} + C_{0} e^{3mT''} \text{Const } t \right) d(u, u') C_{0}^{2} \lambda_{at}^{m}
\]

This, (8.1) and the choice of \( m_{0} \) imply

\[
\frac{3L}{4} < \frac{L C_{0} \gamma_{m}^{2}}{C_{0} \gamma_{m}} + C_{0}^{3} e^{3mT''} \text{Const } t \leq \frac{L}{2} + C_{0}^{3} e^{3mT''} \text{Const } t.
\]

This is true for all \( m \geq m_{0} \). In particular for \( m = m_{0} \) we get

\[
\frac{L}{4} < C_{0}^{2} e^{3m_{0}T''} \text{Const } t,
\]

and so \( \text{Lip}(h_{at}) = L \leq \text{Const } t \). □

**Proof of Lemma 5.** (a) Let \( u, u' \in \hat{U}_{i} \) for some \( i = 1, \ldots, k \) and let \( m \geq 1 \) be an integer. For any \( v \in \hat{U} \) with \( \sigma_{m}^{m}(v) = u \), denote by \( v' = v'(v) \) the unique \( v' \in \hat{U} \) in the cylinder of length \( m \) containing \( v \) such that \( \sigma_{m}^{m}(v') = u' \). Then

\[
|f_{at}^{m}(v) - f_{at}^{m}(v')| \leq \sum_{j=0}^{m-1} |f_{at}(\sigma_{j}^{m}(v)) - f_{at}(\sigma_{j}^{m}(v'))| \leq \frac{Tt}{c_{0}(\gamma - 1)} d(u, u') \leq C_{1} t D(u, u')
\]

(8.2) for some constant \( C_{1} > 0 \). Similarly,

\[
|g_{at}^{m}(v) - g_{at}^{m}(v')| \leq C_{1} t D(u, u').
\]

(8.3)

Also notice that if \( D(u, u') = \text{diam}(C') \) for some cylinder \( C' = C[i_{m+1}, \ldots, i_{p}] \), then \( v, v'(v) \in C'' = C[i_{0}, i_{1}, \ldots, i_{p}] \) for some cylinder \( C'' \) with \( \sigma_{m}(C'') = C' \), so

\[
D(v, v') \leq \text{diam}(C'') \leq \frac{1}{c_{0} \gamma_{m}} \text{diam}(C') = \frac{D(u, u')}{c_{0} \gamma_{m}}.
\]
Using the above, $\text{diam}(U_i) \leq 1$, the definition of $M_{atc}$, we get

$$\frac{|(M_{atc}^m H)(u) - (M_{atc}^m H)(u')|}{M_{atc}^m H(u')} = \frac{\left| \sum_{\sigma^m v = u} e^{f^m_{at}(v) + c_1^m(v)} H(v) - \sum_{\sigma^m v = u} e^{f^m_{at}(v') + c_1^m(v')} H(v') \right|}{M_{atc}^m H(u')} \leq \frac{\left| \sum_{\sigma^m v = u} e^{f^m_{at}(v)}(H(v) - H(v')) \right|}{M_{atc}^m H(u')} + \frac{\left| \sum_{\sigma^m v = u} \left| e^{f^m_{at}(v) + c_1^m(v)} - e^{f^m_{at}(v') + c_1^m(v')} \right| H(v') \right|}{M_{atc}^m H(u')} \leq \frac{\sum_{\sigma^m v = u} e^{f^m_{at}(v)} + c_1^m(v) E H(v') D(v, v')}{M_{atc}^m H(u')} \leq \frac{\sum_{\sigma^m v = u} \left| e^{f^m_{at}(v)} + c_1^m(v) \right|[f^m_{at}(v) + c_1^m(v)] - 1)}{M_{atc}^m H(u')} \leq e^{2C_1 t} D(u, u') \leq 2C_1 t,$$

and therefore

$$e^{[f^m_{at}(v) + c_1^m(v)] - [f^m_{at}(v') + c_1^m(v')] - 1} \leq e^{2C_1 t} 2C_1 t D(u, u').$$

However (8.4) is not good enough to estimate the first term in the right-hand-side above. Instead we use (3.3) and (3.4) to get

$$\left| f^m_{at}(v) + c_1^m(v) - [f^m_{at}(v') + c_1^m(v')] \right| \leq 2m \| f_t - f_0(0) + [f^m_{at}(v) - f^m_{at}(v')] + \text{Const} D(u, u') + 4C_0 + 2ma_0 ||g_t - g||_0 \leq \text{Const} D(u, u') + C_2 ma_0 \leq C_2 + C_2 m a_0$$

for some constant $C_2 > 0$. We will now assume that $a_0 > 0$ is chosen so small that

$$e^{C_2 a_0} < \gamma/\gamma.$$

Hence

$$\frac{|(M_{atc}^m H)(u) - (M_{atc}^m H)(u')|}{M_{atc}^m H(u')} \leq \frac{E D(u, u')} c_0 \gamma^m \frac{\sum_{\sigma^m v = u} e^{f^m_{at}(v) + c_1^m(v)} - [f^m_{at}(v') + c_1^m(v')] e^{f^m_{at}(v') + c_1^m(v')} H(v')}{M_{atc}^m H(u')} + e^{2C_1 t} \frac{\sum_{\sigma^m v = u} 2C_1 t e^{f^m_{at}(v)} H(v')}{M_{atc}^m H(u')} \leq e^{C_2} e^{C_2 ma_0} \frac{E D(u, u')} c_0 \gamma^m + 2C_1 t e^{2C_1 t} D(u, u') \leq A_0 \left[ \frac{E}{\gamma^m} + e^{A_0 t t} \right] D(u, u'),$$
for some constant $A_0 > 0$ independent of $a$, $c$, $t$, $m$ and $E$.

(b) Let $m \geq 1$ be an integer and $u, u' \in \tilde{U}_i$ for some $i = 1, \ldots, k$. Using the notation $v' = v'(v)$ and the constant $C_2 > 0$ from part (a) above, where $\sigma^m v = u$ and $\sigma^m v' = u'$, and some of the estimates from the proof of part (a), we get

\[
|\mathcal{L}^m_{atc} h(u) - \mathcal{L}^m_{atc} h(u')| \leq C_2 e^{C_2 m} E D(u, u') \sum_{m=1}^{\infty} e^{f_{at}^m(v') + c_{at}^m(v')} H(v')
\]

Using the constants $C_1, C_2 > 0$ from the proof of part (a), (8.5) and (8.6) we get

\[
\sum_{m=1}^{\infty} e^{f_{at}^m(v') + c_{at}^m(v')} |h(v) - h(v')| \leq C_2 e^{C_2 m} E D(u, u') \sum_{m=1}^{\infty} e^{f_{at}^m(v') + c_{at}^m(v')} H(v')
\]

This, (8.3) and (8.5) imply

\[
|\mathcal{L}^m_{atc} h(u) - \mathcal{L}^m_{atc} h(u')| \leq C_2 e^{C_2 m} (\mathcal{M}_{atc}^m)(u') D(u, u') + e^{2C_1 t} 2C_1 t D(u, u') (\mathcal{M}_{atc}^m |h|)(u') + \text{Const} \ |b| + |w| C_1 t D(u, u')
\]

Thus, taking the constant $A_0 > 0$ sufficiently large we get

\[
|(\mathcal{L}^N_{atc} h)(u) - (\mathcal{L}^N_{atc} h)(u')| \leq A_0 \left( E_{\tilde{c} \gamma m} (\mathcal{M}_{atc}^m)(u') + \left( |b| + \epsilon A_0 t + t|w| (\mathcal{M}_{atc}^m |h|)(u') \right) \right) D(u, u')
\]

which proves the assertion.

As in [4] and [20] we need the following lemma whose proof is omitted here, since it is very similar to the proof of Lemma 5 given above.
Lemma 17. Let $\beta \in (0, \alpha)$. There exists a constants $A_0' > 0$ such that for all $a, b, c, t, w \in \mathbb{R}$ with $|a|, |c|, 1/|b|, 1/t \leq a_0$ such that (4.1) hold, and all positive integers $m$ and all $h \in C^\beta(U)$ we have

$$|\mathcal{L}_{abtz}^m h(u) - \mathcal{L}_{abtz}^m h(u')| \leq A_0' \left[ |h|_{\mathcal{M}_{abc}^m} + \frac{|b|}{t^n} \right] (d(u, u'))^\beta$$

for all $u, u' \in U_i$.

We will derive Theorem 5(b) from Theorem 5(a), proved in Sect. 4, and Lemma 17 above.

Proof of Theorem 5(b). We essentially repeat the proofs of Corollaries 2 and 3 in [4] (cf. also Sect. 3 in [19]).

Let $\epsilon > 0$, $B > 0$ and $\beta \in (0, \alpha)$. Take $\hat{\rho} \in (0, 1)$, $a_0 > 0$, $b_0 > 0$, $A_0 > 0$ and $N$ as in Theorem 2(a). We will assume that $\hat{\rho} \geq \frac{1}{a_0}$. Let $a, b, c, w \in \mathbb{R}$ be such that $|a|, |c| \leq a_0$ and $|b| \geq b_0$. Let $t > 0$ be such that $1/t^{\alpha-\beta} \leq a_0$. Assume that (4.1) hold and set $z = c + iw$.

First, as in [4] (see also Sect. 3 in [19]) one derives from Theorem 5(a) and Lemma 17 (approximating functions $h \in C^\beta(\hat{U})$ by Lipschitz functions as in Sect. 3) that there exist constants $C_3 > 0$ and $\rho_1 \in (0, 1)$ such that

$$\|F_{abtz}^n h\|_{\beta, B} \leq C_3 |b|^2 \rho_1^n, \quad n \geq 0,$$

(8.7)

for all $h \in C^\beta(\hat{U})$.

Next, given $h \in C^\beta(\hat{U})$, we have

$$F_{abtz}^n(h/h_{at}) = \frac{1}{\lambda_{at}^n} L_{ft-(P+a+b)t+zg}^n h,$$

so by (8.7) we get

$$\|L_{ft-(P+a+b)t+zg}^n h\|_{\beta, B} \leq \lambda_{at}^n \|h_{at} F_{abtz}^n(h/h_{at})\|_{\beta, B} \leq \text{Const}(\lambda_{at}\rho_1^n) |b|^\epsilon \|h/h_{at}\|_{\beta, B} \leq \text{Const} \rho_2^n |b|^\epsilon \|h\|_{\beta, B},$$

where $\lambda_{at}\rho_1 \leq e^{C_3a_0} \rho_2 = \rho_2 < 1$, provided $a_0 > 0$ is small enough.

We will now approximate $L_{ft-(P+a+b)t+zg}$ by $L_{ft-(P+a+b)t+zg}$ in two steps. First, using the above it follows that

$$\|L_{ft-(P+a+b)t+zg}^n h\|_{\beta, B} = \|L_{ft-(P+a+b)t+zg}^n (F_{ft}^{(f^n-f_t^n)+c(g^n-g_t^n)}h)\|_{\beta, B} \leq \text{Const} \rho_2^n |b|^\epsilon \|e^{(f^n-f_t^n)+c(g^n-g_t^n)}h\|_{\beta, B},$$

Choosing the constant $C_4 > 0$ appropriately, $\|f - f_t\|_0 \leq C_4 a_0$ and $\|f - f_t\|_\beta \leq C_4/t^{\alpha-\beta} \leq C_4 a_0$, so $\|f^n - f_t^n\|_0 \leq n \|f - f_t\| \leq C_4 na_0$, and similarly $\|f^n - f_t^n\|_\beta \leq C_4 na_0$. Similar estimates hold for $g^n - g_t^n$. Thus,

$$\|e^{(f^n-f_t^n)+c(g^n-g_t^n)}h\|_0 \leq e^{C_4a_0t} \|h\|_0$$

and

$$\|e^{(f^n-f_t^n)+c(g^n-g_t^n)}h\|_{\beta, B} \leq \|e^{(f^n-f_t^n)+c(g^n-g_t^n)}\|_0 \|h\|_\beta + \|e^{(f^n-f_t^n)+c(g^n-g_t^n)}\|_\beta \|h\|_\infty \leq e^{C_4a_0t} \|h\|_\beta + e^{C_4a_0t} \|f^n - f_t^n\|_\beta \|h\|_\infty \leq C_5 n e^{C_4a_0t} \|h\|_\beta.$$
Combining this with the previous estimate gives
\[ \|e^{(P_n - f^n) + c(g^n - g^n)} h\|_{\beta, b} \leq C_5 n e^{C_4 a_0} \|h\|_{\beta}, \]
so
\[ \|L^n f_{(P+a+ib)\tau + cg + iw_g} h\|_{\beta, b} \leq C_5 \rho_2^n |b|^\epsilon n e^{C_4 a_0} \|h\|_{\beta, b}. \]
Taking \( a_0 > 0 \) sufficiently small, we may assume that \( \rho_2 e^{C_4 a_0} < 1 \). Now take an arbitrary \( \rho_3 \) with \( \rho_2 e^{C_4 a_0} < \rho_3 < 1 \). Then we can take the constant \( C_6 > 0 \) so large that \( n \rho_3^n e^{C_4 a_0} \leq C_6 \rho_3^n \) for all integers \( n \geq 1 \). This gives
\[ \|L^n f_{(P+a+ib)\tau + cg + iw_g} h\|_{\beta, b} \leq C_6 \rho_3^n |b|^\epsilon \|h\|_{\beta, b}, \quad n \geq 0. \tag{8.8} \]

Using the latter we can write
\[ \|L^n f_{(P+a+ib)\tau + zg} h\|_{\beta, b} = \left\| L^n f_{(P+a+ib)\tau + cg + iw_g} \left( e^{iw(g^n - g^n)} h \right) \right\|_{\beta, b} \leq C_6 \rho_3^n \|b|^\epsilon \left\| e^{iw(g^n - g^n)} h \right\|_{\beta, b}. \]

However, \( \left\| e^{iw(g^n - g^n)} h \right\|_0 = \|h\|_0, \ |g - g^n|_\beta \leq C_4/\epsilon^{\alpha - \beta} \leq C_4 a_0 \leq 1 \) (assuming \( a_0 > 0 \) is sufficiently small), and by (4.1), \( |w| \leq B |b|^\mu \leq B |b| \), so
\[ \left\| e^{iw(g^n - g^n)} h \right\|_{\beta} \leq \left\| e^{iw(g^n - g^n)} \right\|_0 \|h\|_\beta + \left\| e^{iw(g^n - g^n)} \right\|_{\beta} \|h\|_{\infty} \leq \|h\|_\beta + Bn |b| \|h\|_{\infty}. \]

Thus,
\[ \left\| e^{iw(g^n - g^n)} h \right\|_{\beta, b} = \left\| e^{iw(g^n - g^n)} h \right\|_0 + \frac{1}{|b|} \left\| e^{iw(g^n - g^n)} h \right\|_{\beta} \leq 2Bn \|h\|_{\beta, b}, \]
and therefore
\[ \left\| L^n f_{(P+a+ib)\tau + zg} h \right\|_{\beta, b} \leq C_7 \rho_3^n |b|^\epsilon n \|h\|_{\beta, b}. \]

Now taking an arbitrary \( \rho \) with \( \rho_3 < \rho < 1 \) and taking the constant \( C_8 > C_7 \) sufficiently large, we get
\[ \left\| L^n f_{(P+a+ib)\tau + zg} h \right\|_{\beta, b} \leq C_8 \rho^n |b|^\epsilon \|h\|_{\beta, b} \]
for all integers \( n \geq 0. \]

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