Superconvergent Recovery of Raviart–Thomas Mixed Finite Elements on Triangular Grids

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Received: 4 April 2019 / Revised: 22 August 2019 / Accepted: 10 October 2019 / Published online: 18 October 2019
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Abstract
For the second lowest order Raviart–Thomas mixed method, we prove that the canonical interpolant and finite element solution for the vector variable in elliptic problems are superclose in the $H(\text{div})$-norm on mildly structured meshes, where most pairs of adjacent triangles form approximate parallelograms. We then develop a family of postprocessing operators for Raviart–Thomas mixed elements on triangular grids by using the idea of local least squares fittings. Super-approximation property of the postprocessing operators for the lowest and second lowest order Raviart–Thomas elements is proved under mild conditions. Combining the supercloseness and super-approximation results, we prove that the postprocessed solution superconverges to the exact solution in the $L^2$-norm on mildly structured meshes.

Keywords Superconvergence · Mildly structured grids · Mixed methods · Raviart–Thomas elements · Second order elliptic equations

Mathematics Subject Classification 65N30 · 65N50

1 Introduction and Preliminaries
Gradient recovery methods for Lagrange elements have been studied extensively by many authors, see e.g., [3–5,26–30] and references therein. Let $u$ be the exact solution of Poisson’s equation and $u_h$ be the finite element solution from Lagrange elements. In general $\nabla u_h$ rather than $u_h$ is the main quantity of interest. Gradient recovery methods aim to get a new approximation $p_h$ to $\nabla u$ by postprocessing $u_h$ or $\nabla u_h$. Comparing to $\nabla u_h$, $p_h$ is often $H^1$-conforming and $p_h$ superconverges to $\nabla u$ in some situation. In addition, $p_h$ can be used to
develop a posteriori error estimators. The recovery-based a posteriori error estimators are popular for their simplicity and asymptotic exactness, see e.g., [4,27,29].

To derive recovery-type superconvergence, a common ingredient is the so-called supercloseness estimate showing that the canonical interpolant and finite element solution are superclose in some norm. In this paper, we consider the second lowest order Raviart–Thomas (denoted by $RT_1$) mixed method for the second order elliptic equation, namely, (1.5) with $r = 1$. We shall prove that the canonical interpolant $Π_h^1 p$ and the finite element solution $p_h$ are superclose in the $H(\text{div})$-norm under mildly structured grids, i.e., most pairs of adjacent triangles in grids form $O(h^{1+α})$-approximate parallelograms except for a region with measure $O(h^β)$, see Definitions 2.1 and 2.2. The supercloseness result in this paper generalizes a result for the $RT_0$ mixed method in [19]. For Poisson’s equation, Brandts [7] proved a supercloseness estimate for $RT_1$ on three-line grids, i.e., each edge in grids is parallel to one of three fixed lines.

To relax the restriction on mesh structures in supercloseness analysis, we give a constructive proof for Theorem 3.2 instead of using the odd-even argument and the Bramble–Hilbert lemma employed in [6,7]. For Lagrange elements over $(α, β)$-grids, the authors in [3] transferred the local error $∫_T \nabla (u - u_I) \cdot \nabla v_h$ on each element $T$ to line integrals using the divergence theorem, where $u_I$ is the linear Lagrange interpolant. Then line integrals are grouped in terms of tangential components of $\nabla v_h$ by delicate triangular integral identities. However, it’s not clear how to handle the local error $∫_T (p - Π_h^p) \cdot q_h$ for the $RT_r$ element in a similar fashion. Our key observation is that $RT_r$ elements satisfy the divergence-free property, i.e., $\text{div}(p_{r+1} - Π_h^p p_{r+1}) = 0$ on each triangle $T$ provided $p_{r+1} \in P_{r+1}(T)^2$. Hence $p_{r+1} - Π_h^p p_{r+1} = \nabla ⊥ w_{r+2}$ for some $w_{r+2} \in P_{r+2}(T)$ and it can be handled by Green’s theorem, see Sect. 5.

For mixed methods, the finite element solution $p_h$ approximating the vector variable $p \in H(\text{div}, Ω)$ is the main quantity of physical interest. As far as we know, existing post-processing/recovery techniques for $p$ and $p_h$ are restricted to strongly structured grids, e.g., three-line, translation invariant and rectangular grids, see e.g., [7,11,13,14]. As grids become increasingly unstructured, the rate of superconvergence of $\|p - K_h Π_h p\|_{0, Ω}$ deteriorates, where $Π_h$ is the canonical interpolation and $K_h$ is some postprocessing operator. In addition, most of the existing results of recovery methods focus on the lowest order case while the analysis of recovery operators for higher order elements is limited, especially on irregular grids. In this paper, we construct a new family of recovery operators $R_h^r$ for $RT_r$ ($r \geq 0$) elements by fitting the numerical solution $p_h$ with a vector polynomial of degree $r + 1$ in the least squares (LS) sense on each local patch surrounding each vertex in triangular grids. We shall show that $R_h^0$ and $R_h^1$ have nice super-approximation property under mild and easy-to-check conditions. The order of approximation of $R_h^r$ is almost independent of the mesh structure. Combining the supercloseness and $R_h^r$, we finally obtain the superconvergence of the postprocessed $RT_0$ and $RT_1$ solutions to the exact solution, see Theorem 4.4.

Recovery by local least squares fitting is not a new idea. The famous Zienkiewicz–Zhu (ZZ) superconvergent patch recovery $G_h$ is based on it, see e.g., [29,30]. For linear elements, $\|\nabla u - G_h \nabla u\| = O(h^2)$ under strongly regular grids (see [17]), that is, each pair of adjacent triangles form an $O(h^2)$ approximate parallelogram. Alternatively, Zhang and Naga [28] proposed a different LS-based patch recovery operator $G_h^r$ for Lagrange elements of degree $r$ by postprocessing the scalar function $u$ rather than $\nabla u$. Roughly speaking, $\|\nabla u - G_h^r u\| = O(h^{r+1})$ provided each LS problem has a unique solution on each local patch. $R_h^r$ can be viewed as a Raviart–Thomas version of $G_h^{r+1}$. In practice, the excellent superconvergence property of $G_h^r$ is attributed to the unique solvability of vertex-based LS problems, which is
difficult to prove on unstructured grids. For example, Naga and Zhang \cite{22} is mainly devoted to the analysis of the uniqueness of the LS solution for $G^1_h$ on unstructured grids. As far as we know, there is no similar analysis for $G^r_h$ with $r \geq 2$. We shall give a practical criterion of uniqueness for $G^2_h$ on unstructured grids, which also works for $R^1_h$, see Theorem 4.1.

In this paper, we consider the second order elliptic equation

$$\begin{align*}
-\text{div}(a_2(x)\nabla u + a_1(x)u) + a_0(x)u &= f(x), \quad x \in \Omega, \\
u &= g(x), \quad x \in \partial \Omega,
\end{align*}$$

(1.1a)

$$u = g(x), \quad x \in \partial \Omega,$$

(1.1b)

where $\text{div} = \nabla \cdot$ is the divergence operator, $a_2, a_0$ are scalar-valued and $a_1$ is vector-valued, $\Omega \subset \mathbb{R}^2$ is a bounded and simply-connected Lipschitz domain. Assume that $a_2, a_1, a_0$ are sufficiently smooth on $\Omega$ and $a_2 \geq \Lambda > 0$ for some constant $\Lambda$. Let

$$p = a_2\nabla u + a_1u, \quad a = a_2^{-1}, \quad b = a_2^{-1}a_1, \quad c = a_0.$$

Equation (1.1) is equivalent to the first order system

$$\begin{align*}
a p - bu - \nabla u &= 0, \quad x \in \Omega, \\
-\text{div} p + cu &= f, \quad x \in \Omega, \\
u &= g, \quad x \in \partial \Omega.
\end{align*}$$

(1.2a)

(1.2b)

(1.2c)

Let $Q = H(\text{div}, \Omega) := \{q \in L^2(\Omega)^2 : \text{div} q \in L^2(\Omega)\}$ and $\mathcal{V} = L^2(\Omega)$. The mixed formulation for (1.2) is to find the pair $(p, u) \in Q \times \mathcal{V}$, such that

$$\begin{align*}
(a p, q) - (q, bu) + (\text{div} q, u) &= \langle q \cdot n, g \rangle, \\
-(\text{div} p, v) + (cu, v) &= \langle f, v \rangle,
\end{align*}$$

(1.3a)

(1.3b)

for each pair $(q, v) \in Q \times \mathcal{V}$. Here $\langle \cdot, \cdot \rangle$ denotes the $L^2$-inner product on $\partial \Omega$.

Let $T_h$ be a collection of triangles that forms a triangulation of $\Omega$. Let $h_T = |T|^{\frac{1}{2}}$ be the diameter of $T$, where $|T|$ is the area of $T$. Let $h = \max_{T \in T_h} h_T < 1$ be the mesh-size. $T_h$ is assumed to quasi-uniform, namely, $\max_{T \in T_h} h_T \leq C_0(\min_{T \in T_h} h_T)$ for some generic constant $C_0$. The quasi-uniformity implies the minimum angle condition (MAC), namely, there exists a fixed constant $\Theta > 0$, such that $\theta \geq \Theta > 0$ for any angle $\theta$ of any triangle $T \in T_h$. Given a one-dimensional or two-dimensional subset $U \subset \mathbb{R}^2$, let

$$\mathcal{P}_r(U) = \{v : v \text{ is a polynomial on } U \text{ of degree } \leq r\}$$

denote the space of polynomials of degree $\leq r$. Let $E_h, E_h^0, E_h^3$ denote the set of edges, interior edges and boundary edges in $T_h$, respectively. Let $N_h$ denote the set of vertices in $T_h$. Several kinds of local patches are useful for finite element superconvergence analysis. For $z \in N_h$, let $\omega_z$ be the union of triangles in $T_h$ sharing $z$ as a vertex. For $e \in E_h$, let $\omega_e$ be the union of triangles in $T_h$ sharing $e$ as an edge. For $T \in T_h$, let $\omega_T$ be the union of $T$ and triangles in $T_h$ sharing at least one vertex with $T$. The local nodes, edges, and triangles in $U$ are $N_h(U) = \{z \in N_h : z \in U\}$, $E_h(U) = \{e \in E_h : e \subset U\}$, and $T_h(U) = \{T \in T_h : T \subset U\}$, respectively.

For $r \geq 0$ and $T \in T_h$, define the space of shape functions

$$\mathcal{R} T_r(T) := \left\{ \begin{pmatrix} v_1 \\ v_2 \\ x_1 \\ x_2 \end{pmatrix} : v_i \in \mathcal{P}_r(T), \quad i = 1, 2, 3 \right\}.$$

(1.4)
The $RT_r$ finite element spaces are
\[
Q_h^r := \{ q_h \in Q : q_h|_T \in RT_r(T), \quad \forall T \in \mathcal{T}_h \},
\]
\[
V_h^r := \{ v_h \in V : v_h|_T \in P_r(T), \quad \forall T \in \mathcal{T}_h \}.
\]
The mixed method for (1.3) is to find $(p_h^r, u_h^r) \in Q_h^r \times V_h^r$, such that
\[
(a \ p_h^r, q_h) - (q_h, bu_h^r) + (\text{div} \ q_h, u_h^r) = (q_h \cdot n, g), \quad q_h \in Q_h^r, \quad (1.5a)
\]
\[-(\text{div} \ p_h^r, v_h) + (cu_h, v_h) = (f, v_h), \quad v_h \in V_h^r. \quad (1.5b)
\]
Under mild assumptions, Douglas and Roberts \cite{12} proved the well-posedness and a priori error estimates for the method (1.5).

Given a positive integer $s$ and a sufficiently smooth function $v$, let
\[
|D^s v| := \sum_{\alpha_1+\alpha_2=s} \left| \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} v \right|.
\]
For a domain $U$, the Sobolev seminorms and norms are defined by
\[
|v|_{s,p,U} = \left( \int_U |D^s v|^p \right)^{1/p}, \quad \|v\|_{s,p,U} = \left( \sum_{m=0}^{s} |v|^p_{m,p,U} \right)^{1/p}.
\]
Sobolev norms with $\infty$-index and norms of vector-valued functions are generalized in usual ways.

Let $|v|_{h,m,U} := \left( \sum_{T \in \mathcal{T}_h} |v|^2_{m,T} \right)^{1/2}$ denote the mesh-dependent semi-norm w.r.t. $\mathcal{T}_h$. We say $A \lesssim B$ provided $A \leq CB$, where $C$ is a generic constant that may change from line to line, and depends only on the shape regularity of $\mathcal{T}_h$ measured by $C_0$ or $\Theta$. We say $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. The regularity condition will be indicated on right hand sides of estimates. In addition to $Q_h^r$ and $V_h^r$, we need the standard nodal finite element space
\[
W_h^r = \{ w \in C(\Omega) : w|_T \in P_r(T), \quad \forall T \in \mathcal{T}_h \},
\]
where $C(\Omega)$ is the space of continuous functions on $\Omega$. We present two well-known inequalities that will be used in the rest of this paper.

**Theorem 1.1** (Interpolation error) Let $I^r_h : C(\Omega) \to W_h^r$ denote the Lagrange interpolation of degree $r$. For $T \in \mathcal{T}_h$ and $r \geq 1$, it holds that
\[
\|v - I^r_h v\|_{0,r,T} \lesssim h^{r-\frac{1}{2}} |v|_{h,r+1,T}, \quad 1 \leq \gamma \leq \infty. \quad (1.6)
\]

**Theorem 1.2** (Trace inequality) For $T \in \mathcal{T}_h$ and $v \in H^1(T)$, it holds that
\[
\|v\|_{0,\partial T} \lesssim h^{-\frac{1}{2}} \|v\|_{0,T} + h^{\frac{1}{2}} \|\nabla v\|_{0,T}. \quad (1.7)
\]

### 2 Local Error Expansions

We begin with geometric identities on a local element $T$. It has three vertices $(z_k)_{k=1}^3$, oriented counterclockwise, and corresponding barycentric coordinates $(\lambda_k)_{k=1}^3$. Let $e_k$ denote the edge opposite to $z_k$, $\theta_k$ the angle opposite to $e_k$, $\ell_k$ the length of $e_k$, $d_k$ the distance from $z_k$ to $e_k$,
where \( \text{degrees of freedom of} \ R_T \text{r} \text{property about} \ \Pi \text{guaranteed. In addition,} \)

Fig. 1. Corresponding quantities on triangles \( \ell \text{. The subscripts are equivalent mod} \ 3, \text{e.g.,} \ \ell_4 = \ell_1, \theta_0 = \theta_3. \)

We have the rotational gradient \( \nabla^\perp v = ( -\partial_{x_2} v, \partial_{x_1} v )^\top \), and the adjoint \( \nabla \times q = \partial_{x_1} q_2 - \partial_{x_2} q_1. \nabla^\perp \) and \( \nabla \times \) are related by Green’s formula

\[
\int_T \nabla^\perp w \cdot q = \int_{\partial T} w q \cdot t - \int_T w \nabla \times q, \tag{2.1}
\]

where \( t \) is the unit tangent to \( \partial K \) oriented counterclockwise. For \( v \in \mathbb{R}^2 \), define \( v^\perp = (-v_2, v_1). \) Clearly, \( n_k^\perp = t_k, \ t_k^\perp = -n_k. \)

Now we introduce basic definitions for \( RT_r \text{ elements. For} \ e \in \mathcal{E}_h, \) let \( \{ (w_j, g_j) \}^{r+1}_{j=1} \) denote the Gaussian quadrature rule on \( e, \) where \( \{ g_j \} \) are quadrature points and \( \{ w_j \} \) are corresponding weights. \( \{ (w_j, g_j) \}^{r+1}_{j=1} \) is exact for \( P_{2r+1}(e), \) i.e.,

\[
\int_e v = |e| \sum_{j=1}^{r+1} w_j v(g_j) \quad \text{for all} \ v \in P_{2r+1}(e),
\tag{2.2}
\]

where \( |e| \) is the length of \( e. \) Let \( v_j \in P_r(e) \) be the polynomial that is \( w_j^{-1} \) at \( g_j \) and 0 at the rest of quadrature points. For \( T \in \mathcal{T}_h, \) let \( \{ \lambda_j \}^{(r+1)/2}_{j=1} \) be the nodal basis function of Lagrange elements of degree \( r - 1 \) on \( T \) (\( \{ \lambda_j \} = \emptyset \) if \( r = 0; \) \( \{ \lambda_j \} = \{ 1 \} \) if \( r = 1). \) We can specify degrees of freedom of \( RT_r \) elements as

\[
N_e^j(q) := \frac{1}{|e|} \int_e q \cdot n_e v_j, \quad N_T^m(q) := \frac{1}{|T|} \int_T q_m \lambda_l,
\]

where \( n_e \) is a unit normal to \( e, \ q = (q_1, q_2)^\top, \) and \( 1 \leq j \leq r + 1, 1 \leq l \leq r(r + 1)/2, m = 1, 2. \) By (2.2) and the definition of \( v_j, \) we have \( N_e^j(q) = q(g_j) \cdot n_e \) provided \( q \in P_{r+1}(e)^2. \) For \( q \in H^1(\Omega)^2, \) the \( RT_r \) interpolant \( \Pi_h^r q \in Q_h^r \) satisfies

\[
N_e^j(\Pi_h^r q) = N_e^j(q), \quad N_T^m(\Pi_h^r q) = N_T^m(q),
\]

for all indices \( j, l, m, \) and \( e \in \mathcal{E}_h, \ T \in \mathcal{T}_h. \) The existence and uniqueness of \( \Pi_h^r q \) is always guaranteed. In addition, \( \Pi_h^r \) is stable in the \( L^\infty \)-norm

\[
\| \Pi_h^r q \|_{0,\infty,T} \lesssim \| q \|_{0,\infty,T}, \quad T \in \mathcal{T}_h.
\tag{2.3}
\]

For \( v \in \mathcal{V}, \) the interpolant \( P_h^r v \) is the \( L^2 \)-projection of \( v \) onto \( V_h^r. \) There is a nice commuting property about \( P_h^r, \ \Pi_h^r \) and \( \text{div}, \) i.e.,

\[
\text{div}(\Pi_h^r q) = P_h^r(\text{div} q), \quad \forall q \in H^1(\Omega)^2.
\tag{2.4}
\]
The following interpolation error estimates hold, see e.g., [12].
\[
\|q - \Pi_h^r q\|_{0, \Omega} \lesssim h^{r+1}|q|_{h, r+1, \Omega},
\]
\[
\|\text{div}(q - \Pi_h^r q)\|_{0, \Omega} \lesssim h^{r+1}|\text{div} q|_{h, r+1, \Omega},
\]
\[
\|v - \Pi_h^r v\|_{0, \Omega} \lesssim h^{r+1}|v|_{h, r+1, \Omega}.
\]

In the rest of this section, we will present variational error expansions for the \textit{RT}ₙ element. Comparing to \textit{RT}₀, the theory of \textit{RT}ₙ is much more complicated. Let \(d\) be the diameter of the circumscribed circle of \(T\). For each edge \(e_k\), there are several associated geometric quantities
\[
\mu_{1,1,k} = \frac{1}{5760} (3e_k^4 - 3(e_{k-1}^2 - e_{k+1}^2)^2 - 4e_k^2(e_{k-1}^2 + e_{k+1}^2)),
\]
\[
\mu_{1,2,k} = \mu_{2,1,k} = \frac{1}{1440d} e_1 e_2 e_3 (e_{k-1}^2 - e_{k+1}^2),
\]
\[
\mu_{2,1,k} = \frac{1}{2880e_1 e_2 e_3} d_1 (e_{k-1}^2 - e_{k+1}^2)(4e_k^2 - (e_{k-1}^2 - e_{k+1}^2)^2 - 3e_k^2(e_{k-1}^2 + e_{k+1}^2)),
\]
\[
\mu_{2,2,k} = -\mu_{1,1,k}, \quad \mu_{2,2,k} = -\mu_{1,2,k},
\]

and second order differential operators \(\mathcal{D}_{i,j,k}^{ij} \mid i,j \leq 2\)
\[
\mathcal{D}_{1,1,k}^{11} = t_k \cdot \partial_{t_k}^2, \quad \mathcal{D}_{1,2,k}^{12} = \mathcal{D}_{1,2,k}^{21} = t_k \cdot \partial_{n_k}^2,
\]
\[
\mathcal{D}_{2,1,k}^{11} = n_k \cdot \partial_{n_k}^2, \quad \mathcal{D}_{2,2,k}^{12} = \mathcal{D}_{2,2,k}^{21} = n_k \cdot \partial_{n_k}^2.
\]

We define the second order differential operator \(\mathcal{B}_k(q) := \sum_{i,j,l} \mu_{i,j,k} \mathcal{D}_{i,j,k}^{ij}(q)\). The next lemma is our main tool for estimating the global variational error whose proof is left in Sect. 5.

\textbf{Lemma 2.1} \textit{For} \(p_2 \in \mathcal{P}_2(T)\) \textit{and} \(w_2 \in \mathcal{P}_2(T)\),
\[
\int_T (p_2 - \Pi_h^1 p_2) \cdot \nabla^\perp w_2 = \sum_{k=1}^3 \int_{e_k} \mathcal{B}_k(p_2) \partial_{t_k}^2 w_2.
\]

Built upon Lemma 2.1, we derive the local error expansion for general \(p\).

\textbf{Theorem 2.1} \textit{For} \(w_2 \in \mathcal{P}_2(T)\),
\[
\int_T (p - \Pi_h^1 p) \cdot \nabla^\perp w_2 = \sum_{k=1}^3 \int_{e_k} \mathcal{B}_k(p) \partial_{t_k}^2 w_2 + O(h^3)\|p\|_{3,T} \|\nabla^\perp w_2\|_{0,T}.
\]

\textbf{Proof} \textit{Let} \(p_I\) \textit{be the quadratic interpolant of} \(p\). By Lemma 2.1, we have
\[
\int_T (p - \Pi_h^1 p) \cdot \nabla^\perp w_2 = \int_T (\text{id} - \Pi_h^1)(p - p_I) \cdot \nabla^\perp w_2
\]
\[
+ \sum_{k=1}^3 \int_{e_k} \mathcal{B}_k(p_I - p) \partial_{t_k}^2 w_2 + \sum_{k=1}^3 \int_{e_k} \mathcal{B}_k(p) \partial_{t_k}^2 w_2
\]
\[
:= I + II + III,
\]
where \(\text{id}\) is the identity operator. The inequalities (1.6) and (2.3) give the upper bound
\[
|I| \lesssim \|(\text{id} - \Pi_h^1)(p - p_I)\|_{0,T} \|\nabla^\perp w_2\|_{0,T}.
\]
\[
\begin{align*}
\lesssim h_T \| (\text{id} - \Pi_h^1) (p - p_I) \|_{0, \infty, T} \| \nabla^\perp w_2 \|_{0, T} \\
\lesssim h_T \| p - p_I \|_{0, \infty, T} \| \nabla^\perp w_2 \|_{0, T} \\
\lesssim h_T^2 |p|_{3, T} \| \nabla^\perp w_2 \|_{0, T} .
\end{align*}
\]

Using the trace inequality (1.7), inverse inequality, and \( \mu^e_{jkl} = O(h_T^4) \),

\[
|II| \lesssim \sum_{k=1}^3 \| B_k (p_I - p) \|_{0, e_k} \| \partial^2_{ek} w_2 \|_{0, e_k}
\]

\[
\lesssim \sum_{k=1}^3 (h_T^{-1} \| B_k (p_I - p) \|_{0, T} + h_T^2 |B_k (p_I - p)|_{1, T})
\]

\[
\times (h_T^{-1} \| D^2 w_2 \|_{0, T} + h_T^2 |D^2 w_2|_{1, T})
\]

\[
\lesssim \sum_{k=1}^3 (h_T^{-1} |B_k (p_I - p)|_{2, T} + h_T^4 |B_k (p_I - p)|_{3, T}) \times (h_T^{-3} \| \nabla^\perp w_2 \|_{0, T})
\]

\[
\lesssim h_T^3 |p|_{3, T} \| \nabla^\perp w_2 \|_{0, T} .
\]

(2.8)

Combining (2.6)–(2.8), we prove the theorem. \( \square \)

Our supercloseness estimates in this paper hold on mildly structured grids described as follows, see e.g., [3, 15, 16, 22, 27].

**Definition 2.1** For \( e \in \mathcal{E}_h^0 \), let \( T, T' \in \mathcal{T}_h \) be the two adjacent elements sharing \( e \). Define \( e_1 = e'_1 = e \). By going along \( \partial T \) and \( \partial T' \) counterclockwise, we obtain other two pairs of corresponding edges \( e_2, e'_2 \) and \( e_3, e'_3 \). We say \( \omega_e = T \cup T' \) is an \( O(h^{1+\alpha}) \)-approximate parallelogram provided \(|e_i| = |e'_i| + O(h^{1+\alpha})\) for \( i = 1, 2, 3 \).

**Definition 2.2** Assume \( \mathcal{E}_h^0 \) is the disjoint union of two subsets \( \mathcal{E}_{h,1}^0 \) and \( \mathcal{E}_{h,2}^0 \). We say the triangulation \( \mathcal{T}_h \) satisfies the \((\alpha, \beta)\)-condition provided for each \( e \in \mathcal{E}_{h,1}^0 \), \( \omega_e \) an \( O(h^{1+\alpha}) \)-approximate parallelogram, while \( \sum_{e \in \mathcal{E}_{h,2}^0} |\omega_e| = O(h^{\beta}) \).

Although the expression of \( B_k \) is complicated, it suffices to keep the following in mind.

1. \( \{B_k\}_{k=1}^3 \) are second order differential operators of magnitude \( h_T^4 \):

\[
B_k(q) = O(h_T^4) \sum_{i,j,l=1}^2 \partial_{x_i} \partial_{x_j} q_l .
\]

2. For \( e \in \mathcal{E}_h^0 \), we have \( \omega_e = T \cup T' \). Let \( t_e \) denote the unit tangent and and \( n_e \) the unit normal to \( e \) whose directions are induced by \( T \). Let \( \vec{a} = \frac{1}{|T|} \int_T a \) and \( \vec{a}' = \frac{1}{|T'|} \int_{T'} a \). Let \( B'_e \) be the operator based on \( T \) and \( B'_e \) based on \( T' \). If \( \omega_e \) is an \( O(h^{1+\alpha}) \)-approximate parallelogram, then on the edge \( e \), we have the cancellation

\[
\vec{a} B_e(q) - \vec{a}' B'_e(q) = O(h_e^{4+\min(1, \alpha)}) \sum_{i,j,m=1}^2 \partial_{x_i} \partial_{x_j} q_m .
\]

(2.9)
Indeed, $\omega_e$ is an approximate parallelogram implies that $\ell_k = \ell'_k + O(h^{1+\alpha})$, $t_k = t'_k + O(h^\alpha)$, $\sin \theta_k = \sin \theta'_k + O(h^\beta)$, $d = d' + O(h^{1+\alpha})$. Combining these estimates with $\tilde{a} = \tilde{a}' + O(h)$, (2.9) follows from the telescoping type inequality

$$\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right| \leq \sum_{i=1}^n |a_i - b_i| \prod_{j \neq i} \max(a_j, b_j).$$

### 3 Supercloseness Estimates

In this section, first we prove a superconvergence estimate for variational error which is a foundation of supercloseness estimates.

**Lemma 3.1** Let $T_h$ satisfy the $(\alpha, \beta)$-condition and $\tilde{a}$ be the piecewise constant with $\tilde{a}|_T = \frac{1}{|T|} \int_T a$ for each $T \in T_h$. For $w_h \in W_h^0$, it holds that

$$(\tilde{a}(p - \Pi_1^h p), \nabla \cdot w_h) \lesssim h^{2+\min(\frac{3}{2}, \alpha, \frac{1}{2})} (|p|_{2, \infty, \Omega} + |p|_{3, \Omega}) \| \nabla \cdot w_h \|_{0, \Omega}.$$  

**Proof** By Theorem 2.1 and the Cauchy–Schwarz inequality, the left hand side is

$$\left(\tilde{a}(p - \Pi_1^h p), \nabla \cdot w_h\right) = \sum_{T \in T_h} \sum_{k=1}^3 \int_{e_k} \tilde{a} B_k(p) \partial^2 T_k w_h + \sum_{T \in T_h} O(h^3) |p|_{3, T} \| \nabla \cdot w_h \|_{0, T}$$

$$= \left(\sum_{e \in E_{h, 1}} + \sum_{e \in E_{h, 2} \cup E_{h}^2}\right) \int_{e} (\tilde{a} B_e(p) - \tilde{a}' B'_e(p)) \partial^2 T_e w_h$$

$$+ O(h^3) |p|_{3, \Omega} \| \nabla \cdot w_h \|_{0, \Omega} : = I + II + O(h^3) |p|_{3, \Omega} \| \nabla \cdot w_h \|_{0, \Omega}. \quad (3.1)$$

Here the notations in (2.9) are adopted and $B'_e(p) = 0$ if $e \in E_{h}^2$. By the cancellation (2.9), the trace inequality (1.7), and the inverse inequality,

$$|I| \lesssim \sum_{e \in E_{h, 1}} h^{4+\min(1, \alpha)} \| D^2 p \|_{0, e} \| D^2 w_h \|_{0, e}$$

$$\lesssim \sum_{e \in E_{h, 1}} h^{4+\min(1, \alpha)} (h^{-\frac{1}{2}} \| D^2 p \|_{0, T} + h^\frac{1}{2} \| D^3 p \|_{0, T}) (h^{-\frac{1}{2}} \| D^2 w_h \|_{0, T})$$

$$\lesssim \sum_{e \in E_{h, 1}} h^{2+\min(1, \alpha)} \| p \|_{3, T} \| \nabla \cdot w_h \|_{0, T}$$

$$\lesssim h^{2+\min(1, \alpha)} \| p \|_{3, \Omega} \| \nabla \cdot w_h \|_{0, \Omega}. \quad (3.2)$$

For $e \in E_{h, 2}$, there is no cancellation. Let $\tilde{\Omega} = \cup_{e \in E_{h, 2} \cup E_{h}^2} \omega_e$. Using $|\tilde{\Omega}| = O(h^{\min(1, \beta)})$ and the inverse inequality, the sum over $E_{h, 2}$ is

$$|II| \lesssim \sum_{e \in E_{h, 2} \cup E_{h}^2} h^4 |D^2 p|_{0, \infty, e} \int_{e} |\partial^2 T_e w_h|.$$
Douglas and Roberts [12] have shown the standard a priori error estimates:

\[ \sum_{e \in E_h} \int_{e} |\nabla \cdot w_h| \lesssim h^2 |p|_{2,\infty,\Omega} \]

Combining (3.1)–(3.3) we prove the theorem.

Subtracting (1.5) from (1.3) gives the error equation

\[
(a(p - p_h^r), q_h) - (q_h, b(u - u_h^r)) + (\text{div } q_h, u - u_h^r) = 0, \quad q_h \in Q_h, \\
-(\text{div}(p - p_h^r), v_h) + (c(u - u_h^r), v_h) = 0, \quad v_h \in V_h^r.
\]

Douglas and Roberts [12] have shown the standard a priori error estimates:

\[
\|p - p_h^r\|_{0,\Omega} \lesssim h^{r+1} \|u\|_{r+2,\Omega}, \\
\|\text{div}(p - p_h^r)\|_{0,\Omega} \lesssim h^{r+1} \|u\|_{r+3,\Omega}, \\
\|u - u_h^r\|_{0,\Omega} \lesssim h^{r+1} \|u\|_{r+1+\delta_{r_0},\Omega}.
\]

(3.5)

where \(\delta_{r_0} = 1\) if \(r = 0\) and \(\delta_{r_0} = 0\) if \(r \neq 0\). In addition, Douglas and Roberts [12] gives the well-known supercloseness result for the scalar unknown \(u\)

\[ \|P_h^r u - u_h^r\|_{0,\Omega} \lesssim h^{r+2} \|u\|_{2+r+\delta_{r_0},\Omega}.
\]

(3.6)

(3.6) holds on unstructured meshes and implies that \(\|\text{div}(P_h^r p - p_h^r)\|_{0,\Omega}\) is supersmall. For convenience, let \(\xi_h := P_h^r p - p_h^r\).

**Theorem 3.1** For general shape regular \(T_h\) and \(r \geq 0\),

\[ \|\text{div}(P_h^r p - p_h^r)\|_{0,\Omega} \lesssim h^{r+2} \|u\|_{2+r+\delta_{r_0},\Omega}.
\]

**Proof** Let

\[ v_h := \frac{\text{div } \xi_h}{\|\text{div } \xi_h\|_{0,\Omega}} \in V_h^r.
\]

By (2.4) and (3.4), we have

\[
\|\text{div } \xi_h\|_{0,\Omega} = (\text{div } \xi_h, v_h) = (P_h^r \text{div } p - \text{div } p_h^r, v_h) \\
= (\text{div}(p - p_h^r), v_h) = (u - P_h^r u, cv_h) + (P_h^r u - u_h^r, c v_h).
\]

It then follows from (2.5), (3.5), (3.6), and \(\|v_h\|_{0,\Omega} = 1\) that

\[
\|\text{div } \xi_h\|_{0,\Omega} = (u - P_h^r u, cv_h - P_h^r (cv_h)) + O(h^{r+2}) \|u\|_{2+r+\delta_{r_0},\Omega} \\
= O(h^{2r+2}) \|u\|_{r+1,\Omega} |cv_h|_{h,r+1,\Omega} + O(h^{r+2}) \|u\|_{2+r+\delta_{r_0},\Omega} \\
= O(h^{r+2}) \|u\|_{2+r+\delta_{r_0},\Omega}.
\]

In the last step, we use \(v_h | T \in \mathcal{P}_r(T)\) and the inverse inequality.

Before proving the superconvergence estimate of \(\|P_h^r p - p_h^r\|_{0,\Omega}\), it is necessary to discuss the \(L^2\) de Rham complex in \(\mathbb{R}^2\):

\[ H^1(\Omega) \xrightarrow{\nabla^\perp} Q \xrightarrow{\text{div}} \mathcal{V} \rightarrow 0. \]
Here $\mathcal{V} = L^2(\Omega)$ is equipped with the standard $(\cdot, \cdot)$ inner product. Since we are dealing with variable coefficients, $Q$ is equipped with the weighted $L^2$ inner product $(\cdot, \cdot)_a$ given by

$$(q_1, q_2)_a := (aq_1, q_2), \quad q_1, q_2 \in L^2(\Omega)^2.$$  

The weighted $L^2$-norm is $\|q\|_a = (aq, q)^{\frac{1}{2}}$. Clearly, $\|q\|_{0,\Omega} \approx \|q\|_a$ for all $q \in L^2(\Omega)^2$.

Similarly, we have the discrete subcomplex

$$\mathcal{W}_h^{r+1} \rightarrow \mathcal{Q}_h^r \rightarrow \mathcal{V}_h^r \rightarrow 0.$$  

(3.7)

Let $\oplus$ denote the direct sum w.r.t. $(\cdot, \cdot)_a$. Since $\Omega$ is simply connected, (3.7) is exact and the discrete Helmholtz/Hodge decomposition (see e.g., [1,2,9,18]) holds:

$$\mathcal{Q}_h^r = \nabla^\perp \mathcal{W}_h^{r+1} \oplus \text{grad}_h \mathcal{V}_h^r,$$  

(3.8)

where $\text{grad}_h : \mathcal{V}_h^r \rightarrow \mathcal{Q}_h^r$ is the adjoint of $-\text{div} : \mathcal{Q}_h^r \rightarrow \mathcal{V}_h^r$ w.r.t. the weighted inner product $(\cdot, \cdot)_a$, namely, $(a \text{grad}_h v_h, q_h) = -(v_h, \text{div} q_h)$ for all $q_h \in \mathcal{Q}_h^r$.

The last ingredient for our supercloseness analysis is a discrete Poincaré inequality.

**Lemma 3.2**

$$\|v_h\|_{0,\Omega} \lesssim \|\text{grad}_h v_h\|_a, \quad v_h \in \mathcal{V}_h^r.$$  

**Proof**

$\text{div} : \mathcal{Q}_h^r \rightarrow \mathcal{V}_h^r$ is surjective and there exists $q_h \in \mathcal{Q}_h^r$ and $\text{div} q_h = v_h$. In addition, $q_h$ can be chosen (see [23]) such that $\|q_h\|_a \approx \|q_h\|_{0,\Omega} \lesssim \|v_h\|_{0,\Omega}$. It then follows

$$\|v_h\|^2_{0,\Omega} = -(a \text{grad}_h v_h, q_h) \lesssim \|\text{grad}_h v_h\|_a \|v_h\|_{0,\Omega},$$

which completes the proof. \hfill \square

With the above preparations, we are able to prove supercloseness estimates for the $RT_1$ mixed methods.

**Theorem 3.2** Assume that $\mathcal{T}_h$ satisfies the $(\alpha, \beta)$-condition. Then

$$\|\Pi_h^1 p - p_h^1\| \lesssim h^{2+\min\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)} \left(\|p\|_{2,\infty,\Omega} + \|p\|_{3,\Omega}\right).$$  

**Proof** For simplicity, the super-index $r = 1$ is suppressed in the proof. Consider the discrete Helmholtz decomposition

$$\xi_h := \Pi_h p - p_h = \nabla^\perp w_h \oplus \text{grad}_h v_h,$$  

(3.9)

for some $\{v_h, w_h\} \in \mathcal{V}_h^1 \times \mathcal{W}_h^2$. Let $q_h = \text{grad}_h v_h / \|\text{grad}_h v_h\|_a$. By Lemmas 3.2 and 3.1,

$$\|\text{grad}_h v_h\|_a = (\text{grad}_h v_h, q_h)_a = -(v_h, \text{div} q_h)$$

$$= -(v_h, \frac{\text{div} \xi_h}{\|\text{grad}_h v_h\|_a}) \lesssim \|\text{div} \xi_h\|_{0,\Omega} \lesssim h^{r+2} \|u\|_{r+2,\delta,\Omega}.$$  

(3.10)

It remains to bound $\nabla^\perp w_h$. Let $q_h = \nabla^\perp w_h / \|\nabla^\perp w_h\|_a$. The orthogonality implies

$$\|\nabla^\perp w_h\|_a = -(a(p - \Pi_h p), q_h) + (a(p - p_h), q_h) := I + II.$$  

(3.11)

$I$ is split as

$I = ((\tilde{a} - a)(p - \Pi_h p), q_h) - (\tilde{a}(p - \Pi_h p), q_h)$.
By \( \| \tilde{a} - a \|_{0,\infty,\Omega} = O(h) \), (2.5) and Lemma 3.1,
\[
| I | \lesssim h^3 | p |_{2,\Omega} + h^{2+\min\left( \frac{1}{2}, \frac{\beta}{2} \right)} (| p |_{2,\infty,\Omega} + \| p \|_{3,\Omega}).
\]  
(3.12)

By \( \text{div} \ q_h = 0 \), \( \| q_h \|_{0,\Omega} \approx 1 \), (3.4) and (3.6),
\[
II = (q_h, b(u - u_h))
\]
\[
= (b \cdot q_h, u - P_h u + P_h u - u_h)
\]
\[
= (b \cdot q_h - P_h (b \cdot q_h), u - P_h u) + O(h^3)\|u\|_{3,\Omega}
\]
\[
= O(h^3) \| b \cdot q_h \|_{h,2,\Omega} | u |_{2,\Omega} + O(h^3)\|u\|_{3,\Omega}.
\]  
(3.13)

Since \( q_h |_T \in \mathcal{P}_1(T)^2 \), the inverse estimate implies
\[
| b \cdot q_h |_{2,T} \lesssim \| q_h \|_{0,T} + \| D^1 q_h \|_{0,T} \lesssim h^{-1} | q_h |_{0,T}.
\]  
(3.13) then reduces to
\[
II = O(h^3)\|u\|_{3,\Omega}.
\]  
(3.14)

Then the theorem follows from (3.10)–(3.12), and (3.14). \( \square \)

## 4 Superconvergent Recovery

In this section, we introduce a new recovery operator \( R^r_h : Q_h^r \rightarrow \mathcal{W}_h^{r+1} \times \mathcal{W}_h^{r+1} \). For \( q_h \in Q_h^r \), it suffices to specify nodal values of \( R^r_h q_h \). Here a node is the location of the degree of freedom of Lagrange elements, which can be a vertex of a triangle or an interior point of an edge/ triangle. For vertices \( z_1, z_2, z_3 \in \mathcal{N}_h \), let \( \overline{z_1z_2} \) denote the edge with endpoints \( z_1, z_2 \) and \( z_1z_2z_3 \) the triangle with vertices \( z_1, z_2, z_3 \). \( R^r_h \) is defined in three steps.

**Step 1.** For each vertex \( z \in \mathcal{N}_h \), let \( R^r_h q_h(z) := q_z(z) \), where \( q_z \in \mathcal{P}_{r+1}(\omega_z)^2 \) minimizes the quadratic functional
\[
\mathcal{F}(q) = \sum_{e \in \mathcal{E}_h(\omega_z)} \sum_{j=1}^{r+1} \left( N^j_e(q) - N^j_e(\tilde{q}_h) \right)^2
\]
\[
+ \sum_{T \in \mathcal{T}_h(\omega_z)} \sum_{l=1}^{r(r+1)/2} \sum_{m=1}^2 \left( N^{2l}_T(q) - N^{2l}_T(\tilde{q}_h) \right)^2,
\]
subject to \( q \in \mathcal{P}_{r+1}(\omega_z)^2 \).

**Step 2.** For each node \( z \) in the interior of an edge \( e = \overline{z_1z_2} \in \mathcal{E}_h \), let
\[
R^r_h q_h(z) := (1 - \alpha) q_{z_1}(z) + \alpha q_{z_2}(z), \quad \alpha = \| z - z_1 \| / |e|.
\]

**Step 3.** For each node \( z \) in the interior of the triangle \( T = \overline{z_1z_2z_3} \in \mathcal{T}_h \), let
\[
R^r_h q_h(z) := \alpha_1 q_{z_1}(z) + \alpha_2 q_{z_2}(z) + \alpha_3 q_{z_3}(z),
\]
where \( \alpha_1, \alpha_2, \alpha_3 \) are barycentric coordinates of \( z \) w.r.t. \( z_1, z_2, \) and \( z_3 \).

In some cases, \( \omega_z \) needs be enlarged to ensure that the above LS problem has a unique solution. Since \( R^r_h \) depends only on the degrees of freedom of the \( RT_r \) element, \( R^r_h q \) is well-defined for all \( q \in \mathcal{Q} \) and \( R^r_h \Pi_h q = R^r_h q \). Recall that \( N^j_e(q) = q(g_j) \cdot n_e \) if \( q \in \mathcal{P}_{r+1}(T)^2 \) and \( e \in \mathcal{E}_h(T) \).
To clarify the recovery procedure, we give details to two important cases: \(RT_0\) and \(RT_1\) elements.

**Example 1** \(RT_0\) elements on triangular meshes. In this case, \(R_0^0 q_h\) is a continuous piecewise linear function. At step 1, let \(\{e_j\}_{j=1}^J = \mathcal{E}_h(\omega_z)\). Let \(m_j = (m_{j1}, m_{j2})^T\) be the midpoint of \(e_j\) and \(n_j = (n_{j1}, n_{j2})^T\) be a unit normal to \(e_j\). Then \(q_z = (c_1 + c_2 x_1 + c_3 x_2, c_4 + c_5 x_1 + c_6 x_2)^T \in \mathcal{P}_1(\omega_z)^2\) is the minimizer of

\[
\mathcal{F}(q) = \sum_{j=1}^J \left( \left( q(m_j) \cdot n_j - q_h(m_j) \cdot n_j \right)^2 \right),
\]

subject to \(q \in \mathcal{P}_1(\omega_z)^2\).

Equivalently, \(c_z = (c_1, \ldots, c_6)^T\) satisfies the normal equation \(A_z^T A_z c_z = A_z^T d_z\), where \(d_z = (q_h(m_1) \cdot n_1, \ldots, q_h(m_J) \cdot n_J)^T\), \(A_z = (a_1^T, \ldots, a_J^T)^T\) is an \(N \times 6\) matrix, \(a_j = (n_{j1}, m_{j1} n_{j1}, m_{j1} n_{j2}, m_{j2} n_{j1}, m_{j2} n_{j2})\). Then \(R_0^0 q_h(z) = q_z(z)\) for \(z \in \mathcal{N}_h\).

To avoid ill-conditioned \(A_z\) on grained meshes, we calculate \(q_z\) by scaling it properly. Let \(h_z = |\omega_z|^2\) and \(\hat{q}_z(\hat{x}) = q_z(z + h_z \hat{x}) = (\hat{c}_1 + \hat{c}_2 \hat{x}_1 + \hat{c}_3 \hat{x}_2, \hat{c}_4 + \hat{c}_5 \hat{x}_1 + \hat{c}_6 \hat{x}_2)^T\). Then \(\hat{c}_z = (\hat{c}_1, \ldots, \hat{c}_6)^T\) solves \(A_z^T \hat{A}_z \hat{c}_z = \hat{A}_z^T d_z\), where \(\hat{A}_z = (\hat{a}_1^T, \ldots, \hat{a}_J^T)^T\), \(\hat{a}_j = (n_{j1}, \hat{m}_{j1} n_{j1}, \hat{m}_{j1} n_{j2}, n_{j2}, \hat{m}_{j2} n_{j1}, \hat{m}_{j2} n_{j2})\). Then \(R_0^0 q_h(z) = (\hat{c}_1, \hat{c}_4)^T\).

**Example 2** \(RT_1\) elements on triangular meshes. In this case, \(R_1^1 q_h\) is a continuous piecewise quadratic function. At step 1, let \(\{e_j\}_{j=1}^J = \mathcal{E}_h(\omega_z)\) and \(\{T_l\}_{l=1}^L = \mathcal{T}_h(\omega_z)\). Let

\[
q_z = \left( \begin{array}{c} c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1^2 + c_5 x_1 x_2 + c_6 x_2^2 \\ c_7 + c_8 x_1 + c_9 x_2 + c_{10} x_1^2 + c_{11} x_1 x_2 + c_{12} x_2^2 \end{array} \right) \in \mathcal{P}_2(\omega_z)^2
\]

minimize

\[
\mathcal{F}(q) = \sum_{j=1}^J \left( q(x_j) \cdot n_j - q_h(x_j) \cdot n_j \right)^2 + \left( q(y_j) \cdot n_j - q_h(y_j) \cdot n_j \right)^2
\]

\[
+ \sum_{l=1}^L \sum_{m=1}^M \left( \frac{1}{|T_l|} \int_{T_l} q_m - \frac{1}{|T_l|} \int_{T_l} q_{h,m} \right)^2, \quad q \in \mathcal{P}_2(\omega_z)^2,
\]

where \(q = (q_1, q_2)^T\), \(q_h = (q_{h,1}, q_{h,2})^T\), \(x_j = \frac{3+\sqrt{3}}{6} a_j + \frac{3-\sqrt{3}}{6} b_j\), \(y_j = \frac{3-\sqrt{3}}{6} a_j + \frac{3+\sqrt{3}}{6} b_j\), and \(e_j = \frac{a_j b_j}{a_j^2 + b_j^2}\). Equivalently, \(c_z = (c_1, \ldots, c_{12})^T\) solves the normal equation \(A_z^T A_z c_z = A_z^T d_z\), where

\[
d_z = (q_h(x_1) \cdot n_1, q_h(y_1) \cdot n_1, q_h(x_2) \cdot n_2, q_h(y_2) \cdot n_2, \ldots,
\]

\[
q_h(y_j) \cdot n_j, \quad \frac{1}{|T_l|} \int_{T_l} q_{h,1}, \quad \frac{1}{|T_l|} \int_{T_l} q_{h,2}, \ldots, \quad \frac{1}{|T_L|} \int_{T_L} q_{h,2}
\]

\(\mathcal{T}\), and \(A_z = (a_1^T, \ldots, a_{2J+2L}^T)^T\) is a \((2J + 2L) \times 12\) matrix,

\[
a_{2j-1} = (n_{j1} \xi_j, n_{j2} \xi_j), \quad a_{2j} = (n_{j1} \eta_j, n_{j2} \eta_j),
\]

\[
\xi_j = (1, x_{j1}, x_{j2}, x_{j1}^2, x_{j1} x_{j2}, x_{j2}^2),
\]

\[
\eta_j = (1, y_{j1}, y_{j2}, y_{j1}^2, y_{j1} y_{j2}, y_{j2}^2), \quad 1 \leq j \leq J.
\]
The proof is complete.

Hence it suffices to study the unisolvence of Lemma 4.1 to the case of Lagrange interpolation.

The following statements are equivalent:

1. There exists a unique \( q_z \) at \( z \).
2. \( \Pi h^r q_z = 0 \) implies \( c = 0 \).
3. \( \Pi h^r q_z = 0 \) on \( \omega_z \) implies \( q_z \equiv 0 \).

Hence it suffices to study the unisolvence of \( \Pi h^r \) on \( P_{r-1}(\omega_z) \). \( \Pi h^r \) is moment-based interpolation while nodal interpolation is often easier to analyze. The next lemma reduces Statement 3 in Lemma 4.1 to the case of Lagrange interpolation.

Assume that the solution of each local LS problem at each vertex \( z \) is unique. By definition \( R_h^r \) preserves \((r+1)\)-degree polynomials, namely, \( R_h^r q = q \) on \( T \) for \( q \in P_{r+1}(\omega_T) \), which leads to the super-approximation property \( \| q - R_h^r q \|_{0,\Omega} = O(h^{r+2}) \). However, it’s not obvious that these local LS problems are uniquely solvable. The next obvious lemma gives several statements equivalent to uniqueness.

**Lemma 4.1** The following statements are equivalent:

1. There exists a unique \( q_z \) at \( z \).
2. \( \Pi h^r q_z = 0 \) implies \( c = 0 \).
3. \( \Pi h^r q_z = 0 \) on \( \omega_z \) implies \( q_z \equiv 0 \).

Hence it suffices to study the unisolvence of \( \Pi h^r \) on \( P_{r+1}(\omega_z) \). \( \Pi h^r \) is moment-based interpolation while nodal interpolation is often easier to analyze. The next lemma reduces Statement 3 in Lemma 4.1 to the case of Lagrange interpolation.

**Lemma 4.2** Assume \( \Pi h^r q_z = 0 \) on \( \omega_z \). Then \( q_z = \nabla^\perp w \) for some \( w \in P_{r+2}(\omega_z) \). In addition, for \( e \in E_h(\omega_z) \), \( w(l) = 0 \) at any Lobatto quadrature point \( l \) on \( e \).

**Proof** \( \Pi h^r q_z = 0 \) and (2.4) imply

\[
\text{div} \ q_z = \text{div} (q_z - \Pi h^r q_z) = \text{div} \ q_z - P_h^r \text{div} \ q_z = 0.
\]

Hence \( q_z = \nabla^\perp w \) for some \( w \in P_{r+2}(\omega_z) \). Given \( e = \overline{ab} \in E_h(\omega_z) \),

\[
w(b) - w(a) = \int_e \partial_\mathbf{e} w = \int_e q_z \cdot n_e = \int_e \Pi h^r q_z \cdot n_e = 0.
\]

Hence \( w(z) \equiv c \) for all vertices \( z \) in \( \omega_z \). By subtracting \( c \) from \( w \), we can assume that \( w \) vanishes at all vertices. For \( v \in P_r(e) \),

\[
\int_e w \partial_\mathbf{e} v = - \int_e v \partial_\mathbf{e} w = - \int_e q_z \cdot n_e v = - \int_e \Pi h^r q_z \cdot n_e v = 0,
\]

and thus

\[
\int_e w \tilde{v} = 0 \quad \text{for all } \tilde{v} \in P_{r-1}(e). \tag{4.1}
\]

Note that on \( e = \overline{ab} \), the Lobatto quadrature \( \int_e f = \sum_{j=1}^{r+2} \mu_j f(l_j) \) is exact for \( f \in P_{2r+1}(e) \), where \( l_j = a + (b - a) \hat{l}_j \), \( \hat{l}_j \) are zeros of the polynomial \( \frac{d^r}{ds} (s^{r+1}(1-s)^{r+1}) \) and \( \{\mu_j\}_{j=1}^{r+2} \) are corresponding weights. Let \( \tilde{v} \) be the polynomial which is \( \mu_j^{-1} \) at \( l_j \) and 0 at rest of the \((r - 1)\) interior quadrature points \( \{l_j\}_{i=2,i\neq j}^{r+1} \) in (4.1). Then \( w(l_j) = \int_e w \tilde{v} = 0 \). The proof is complete.

The next theorem gives practical criteria of checking the well-posedness of \( R_h^0 \) and \( R_h^1 \).
Theorem 4.1 Let \( z \) be a vertex in \( T_h \). If \( \#T(\omega_z) \geq 5 \) and the sum of each pair of adjacent angles in \( \omega_z \) is \( \leq \pi \), then there exists a unique \( q_z \) at \( z \) for \( R_h^0 \). If \( \#T(\omega_z) \geq 4 \), then there exists a unique \( q_z \) at \( z \) for \( R_h^1 \).

Proof Assume \( \Pi_h q_z = 0 \) on \( \omega_z \). By Lemma 4.2, \( q_z = \nabla^\perp w \) for some \( w \in P_{r+2}(\omega_z) \). If \( r = 0 \), then \( w \in P_2(\omega_z) \) vanishes at all vertices in \( \omega_z \) and thus \( w = 0 \) by Theorem 2.3 in [22]. Hence \( q_z = 0 \).

If \( r = 1 \), \( w \in P_3(\omega_z) \) vanishes at all vertices and midpoints of edges in \( \omega_z \). Without loss of generality, we can assume that \( z = (0,0) \) and the reference triangle \( \hat{T} \) spanned by \( (0,0), (0,1), (1,0) \) is in \( T_h(\omega_z) \).

If \( w \) is reducible, then the zero set \( w^{-1}(0) \) is the union of three straight lines (counting multiplicity) or the union of a straight line and a conic. Clearly three lines cannot pass all vertices and midpoints of edges in \( \omega_z \) provided \( \#T_h(\omega_z) \geq 4 \). If \( w^{-1}(0) \) contains a conic branch \( C \), then \( C \) must contain at least two vertices \( a, b \) in \( \omega_z \) because \( \#T_h(\omega_z) \geq 4 \). However, \( C \) cannot pass through \((a + b)/2\) by elementary geometry.

Hence reducible \( w \) cannot vanish at all nodes in \( \omega_z \) and we can assume

\[
w = c_1 x_1^3 + c_2 x_1^2 x_2 + c_3 x_1 x_2^2 + c_4 x_2^3 + c_5 x_1^2 + c_6 x_1 x_2 + c_7 x_2^2 + c_8 x_1 + c_9 x_2
\]

is irreducible. Furthermore, we can assume one of the coefficients of highest order terms is 1, say \( c_1 = 1 \) (similar argument for \( c_2, c_3 \) or \( c_4 = 1 \)). Let \((\alpha, \beta)\) be the vertex outside \( \hat{T} \) next to \((0,1)\), see Fig. 2.

Solving the linear system of equations

\[
\begin{align*}
w(1,0) &= w(0,1) = w(1/2,0) = w(0,1/2) = w(1/2,1/2) \\
&= w(\alpha, \beta) = w\left(\frac{\alpha}{2}, \frac{\beta + 1}{2}\right) = w\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) = 0,
\end{align*}
\]

we have

\[
c_1 = \frac{3 - 3\alpha}{1 + \beta}, \quad c_2 = \frac{3\alpha(\alpha - 1)}{\beta(1 + \beta)}.
\] (4.2)

Note that \( \beta \neq 0, \beta \neq -1 \) in (4.2), otherwise the irreducible cubic curve \( w^{-1}(0) \) intersects with a line at five distinct points, which is impossible by Bézout’s theorem (see [25]). Also \( \alpha \neq 1 \) otherwise it violates the topology of the patch \( \omega_z \). Hence \( \alpha/\beta = -c_2/c_1 \). Let \((\alpha', \beta')\) be the vertex outside \( \hat{T} \) next to \((1,0)\). Similarly we have \( \alpha'/\beta' = -c_2/c_1 \). Then it forces \((\alpha, \beta) = (\alpha', \beta')\), which contradicts \( \#T_h(\omega_z) \geq 4 \). Hence \( w = 0 \) and \( q_z = 0 \).

Therefore by Lemma 4.1, there exists a unique \( q_z \) for \( r = 0, 1 \). \( \square \)

We say a vertex \( z \) is good if the condition in Theorem 4.1 holds at \( z \), otherwise it is a bad vertex. In practice, \( T_h \) typically has a few bad vertices, e.g., boundary vertices. There are several ways of dealing with a bad vertex \( z \). If \( z \) is directly connected to a good vertex \( z' \),
one can define \( \omega_z := \omega^z \) and thus \( A_z \) is of full column rank. A more convenient way is to empirically add some extra elements to the patch \( \omega_z \) in practice, e.g., enlarge \( \omega_z \) by one layer. Alternatively, one can solve a rank-deficient local least squares problem, which might reduce the rate of superconvergence of \( R^r_h \).

In the rest of this paper, we assume that

At each vertex \( z \), there exists a unique \( q_z \).

Using the uniqueness of the LS solution, we obtain the boundedness of \( R^r_h \).

**Theorem 4.2** For \( q_h \in Q^r_h \) and \( T \in T_h \),

\[
\| R^r_h q \|_{0,T} \lesssim \| q \|_{0,\omega_T}, \quad r = 0, 1.
\]

**Proof** For \( z \in N_h \), Let \( \sigma_{\min} \) and \( \sigma_{\max} \) be the minimum and maximum singular values of \( A_z \) respectively. The goal is to show that \( \sigma_{\min} \) is uniformly bounded away from 0. MAC implies \( \#T_h(\omega_z) \leq N_{\max} = 2\pi/\Theta \). Hence it suffices to consider the case \( \#T_h(\omega_z) = N \) for some fixed \( N \leq N_{\max} \). In this case, \( \#E_h(\omega_z) = 2N \). Let \( N_1 = 2N \), \( N_2 = 6 \) provided \( r = 0 \) and \( N_1 = 6N \), \( N_2 = 12 \) provided \( r = 1 \). Let \( M_{N_1 \times N_2} \) and \( S_{N_1 \times N_2} \) be the set of \( N_1 \times N_2 \) matrices and \( N_1 \times N_2 \) rank-deficient matrices, respectively. It is well known that \( \sigma_{\min} = \text{dist}(\hat{A}_z, S_{N_1 \times N_2}) \), the distance (measured by matrix 2-norm) from \( \hat{A}_z \) to rank-deficient matrices. dist(\( \cdot \), \( S_{N_1 \times N_2} \)) is continuous on \( M_{N_1 \times N_2} \). Recall that \( \hat{A}_z \) is the scaled LS coefficient matrix determined by \( \omega_z \). Consider all possible \( \omega_z \) and define

\[
A_z = \{ \hat{A}_z \in M_{N_1 \times N_2} : \#T_h(\omega_z) = N, \omega_z \text{ satisfies MAC} \}.
\]

Clearly \( A_z \) is a compact set in \( M_{N_1 \times N_2} \) and any \( \hat{A}_z \in A_z \) is of full rank by the uniqueness assumption. Hence \( \sigma_{\min} = \text{dist}(\hat{A}_z, S_{N_1 \times N_2}) \geq C_1 > 0 \), where \( C_1 \) depends only on the minimum angle \( \Theta \). The maximum singular value \( \sigma_{\max} \leq C_2 \), where \( C_2 \) only depends on \( \Omega \). For \( q_h \in Q^r_h \),

\[
|\hat{c}_z| \leq \| (\hat{A}^T \hat{A}_z)^{-1} \|_2 |\hat{A}^T d_z| \leq \sigma_{\min}^2 \sigma_{\max} |d_z| \\
\leq C_1^{-2} C_2 \| q_h \|_{0,\omega_T} \lesssim h^{-r}_{\omega_T} \| q_h \|_{0,\omega_T},
\]

where \( |\cdot| \) is the Euclidian norm. Finally by (4.3), we have

\[
\| R^r_h q_h \|_{0,T} \lesssim h \| R^r_h q_h \|_{0,\omega_T} \lesssim h |\hat{c}_z| \lesssim \| q_h \|_{0,\omega_T},
\]

which completes the proof.

The superapproximation property of \( R_h \) follows from the uniqueness and boundedness results.

**Theorem 4.3** For \( q \in H^{r+2}(\Omega) \),

\[
\| q - R^r_h q \|_{0,\Omega} \lesssim h^{r+2} |q|_{r+2,\Omega}, \quad r = 0, 1.
\]

**Proof** Let \( T = \overline{z_1 z_2 z_3} \in T_h \) and \( T_1 \subset \overline{\Omega} \) be a smallest local triangle containing \( \omega_T \). Let \( q_{r+1} \in P_{T_1} \) be the degree-(\( r+1 \)) local Lagrange interpolant of \( q \) using based on \( T_1 \). By the uniqueness assumption, \( R^r_h q_{r+1} = q_{r+1} \) on \( T \). It then follows from \( R^r_h \Pi^r_h = R^r_h \) that

\[
\| q - R^r_h q \|_{0,T} \leq \| q - q_{r+1} \|_{0,T} + \| R^r_h \Pi^r_h (q_{r+1} - q) \|_{0,T}.
\]

Using the boundedness from Theorem 4.2, the stability in (2.3), and (1.6),

\[
\| R^r_h \Pi^r_h (q_{r+1} - q) \|_{0,T} \lesssim \| \Pi^r_h (q_{r+1} - q) \|_{0,\omega_T}
\]
By counting the dimension, it is clear that
\[\|p - R_h^r p\|_{0, \Omega} \lesssim h^{r+1} \min(1, \beta, \frac{\ell}{2}) (\|p\|_{r+1, \infty, \Omega} + \|p\|_{r+2, \Omega}), \quad r = 0, 1.\]

\[4.5\]

Combining (4.4), (4.5) and the shape regularity \(T_h\) completes the proof. \(\square\)

In the end, we present the superconvergent recovery estimate.

**Theorem 4.4** Assume that \(T_h\) satisfies the \((\alpha, \beta)\)-condition. Then
\[\|p - R_h^r p\|_{0, \Omega} \lesssim h^{r+1} \min(1, \alpha, \frac{\beta}{2}) (\|p\|_{r+1, \infty, \Omega} + \|p\|_{r+2, \Omega}), \quad r = 0, 1.\]

**Proof** The theorem follows from
\[\|p - R_h^r p\|_{0, \Omega} \leq \|p - R_h^r p\|_{0, \Omega} + \|R_h^r (\Pi_h^r p - p)\|_{0, \Omega}.\]

Theorems 4.2 and 4.3, Theorem 3.2\((r = 1)\) or Theorem 4.5\((r = 0)\) in [19]. \(\square\)

### 5 Proof of Lemma 2.1

The following elementary triangular identities hold:
\[
\begin{align*}
\cos \theta_k &= (\ell_{k-1}^2 + \ell_{k+1}^2 - \ell_k^2)/(2\ell_{k-1}\ell_{k+1}), \quad \sin \theta_k = \ell_k/d, \quad d_k = \ell_{k-1}\ell_{k+1}/d, \\
n_{k-1} &= -\sin \theta_{k+1} t_k - \cos \theta_{k+1} n_k, \quad n_{k+1} = \sin \theta_{k-1} t_k - \cos \theta_{k-1} n_k, \\
\ell_{k-1}^2 &= \cos^2 \theta_{k+1} t_k - 2\cos \theta_{k+1} \sin \theta_{k+1} \ell_k n_k + \sin^2 \theta_{k+1} n_k^2, \\
\ell_{k+1}^2 &= \cos^2 \theta_{k-1} t_k + 2\cos \theta_{k-1} \sin \theta_{k-1} \ell_k n_k + \sin^2 \theta_{k-1} n_k^2.
\end{align*}
\]

(5.1)

For each edge \(e_k\), we define several associated geometric quantities \(\{\alpha_{ij,k}\}_{1 \leq i, j, l \leq 2}\)
\[
\begin{align*}
\alpha_{11,k} &= \frac{1}{24d^2 \ell_k^2} \ell_{k-1} \ell_{k+1} (3\ell_k^4 - (\ell_{k-1}^2 - \ell_{k+1}^2)^2), \\
\alpha_{12,k} &= \alpha_{21,k} = \frac{1}{12d^2 \ell_k^2} \ell_{k-1}^2 \ell_{k+1}^2 (\ell_{k-1}^2 - \ell_{k+1}^2), \\
\alpha_{12,k} &= \alpha_{22,k} = \frac{1}{48d^3 \ell_k^3} (\ell_{k-1}^2 - \ell_{k+1}^2) (9\ell_k^4 - (\ell_{k-1}^2 - \ell_{k+1}^2)^2), \\
\alpha_{22,k} &= \alpha_{21,k} = -\alpha_{11,k}, \quad \alpha_{22,k} = -\alpha_{12,k}.
\end{align*}
\]

To prove Lemma 2.1, we introduce cubic bubble functions
\[
\psi_0 = \lambda_1 \lambda_2 \lambda_3, \quad \psi_k = \lambda_{k-1} \lambda_k \lambda_{k+1}(\lambda_{k-1} - \lambda_{k+1}), \quad 1 \leq k \leq 3.
\]

By counting the dimension, it is clear that \(\{\psi_k\}_{k=0}^3\) can span polynomials in \(P_3(T)\) that vanish at \(\{z_k\}_{k=1}^3\) and midpoints of \(\{\ell_k\}_{k=1}^3\). In fact, \(\{\psi_k\}_{k=0}^3\) has been used to derive superconvergence of quadratic Lagrange elements (cf. [15]) and a posteriori error estimators (cf. [5]).

**Lemma 5.1** For \(p_2 \in P_2(T)^2\),
\[
p_2 - \Pi_h^1 p_2 = \nabla^\perp w,
\]

where
\[
w = \alpha_{ij,k} D_{ij,k}^1 p_2 \psi_0 + \sum_{k=1}^3 \frac{\ell_k^3}{12} D_{2,k}^1 p_2 \psi_k, \quad 1 \leq \beta \leq 3.
\]
Proof By $\Pi_h (p_2 - \Pi_h p_2) = 0$ and using Lemma 4.2, we have
\[
p_2 - \Pi_h p_2 = \nabla \left( \sum_{k=0}^{3} c_k \psi_k \right).
\tag{5.2}
\]
For a unit vector $d$ and the directional derivative $\partial_d$, the definition of $\mathcal{RT}_1(T)$ implies that $\partial_d^2 \Pi_h p_2$ is proportional to $d$. Then applying $d^\perp \cdot \partial_d^2$ to (5.2) gives
\[
d^\perp \cdot \partial_d^2 p_2 = \sum_{k=0}^{3} c_k \partial_d^2 \psi_k. \tag{5.3}
\]
By direct calculation,
\[
\partial_d^3 \psi_0 = 6 \partial_d \lambda_1 \partial_d \lambda_2 \partial_d \lambda_3, \tag{5.4a}
\]
\[
\partial_d^3 \psi_k = 6 \partial_d \lambda_{k-1} \partial_d \lambda_{k+1} (\partial_d \lambda_{k-1} - \partial_d \lambda_{k+1}), \quad 1 \leq k \leq 3. \tag{5.4b}
\]
In particular, $\partial_d^3 \psi_0 = 0$ and $\partial_d^3 \psi_j = -12 \delta_{jk} / \ell_k^3$. By (5.4) and (5.3) with $d = t_k$, we have
\[
c_k = \frac{\ell_k^3}{12} n_k \cdot \partial_d^2 p_2 = \frac{\ell_k^3}{12} D_{2,k}^{11} (p_2), \quad 1 \leq k \leq 3. \tag{5.5}
\]
It remains to determine $c_0$. (5.3) with $d = n_k$ implies that
\[
D_{1,k}^{22} (p_2) = c_0 \partial_{n_k}^3 \psi_0 + c_k \partial_{n_k}^3 \psi_k + c_{k-1} \partial_{n_k}^3 \psi_{k-1} + c_{k+1} \partial_{n_k}^3 \psi_{k+1}, \tag{5.6}
\]
By $\partial_{n_k} \lambda_k = -1/d_k, \partial_{n_k} \lambda_{k+1} = \cos \theta_{k-1}/d_{k+1}, \partial_{n_k} \lambda_{k-1} = \cos \theta_{k+1}/d_{k-1}$, (5.4) with $d = n_k$, (5.5), and (5.6), we obtain
\[
c_0 = -\frac{d_{k-1} d_k d_{k+1}}{6 \cos \theta_{k-1} \cos \theta_{k+1}} D_{1,k}^{22} (p_2) + \frac{\ell_k^3}{12} d_k \left( \frac{\cos \theta_{k+1}}{d_{k-1}} - \frac{\cos \theta_{k-1}}{d_{k+1}} \right) D_{2,k}^{11} (p_2) - \frac{\ell_k^3}{12} \frac{d_{k-1}}{\cos \theta_{k+1}} \left( \frac{1}{d_k} + \frac{\cos \theta_{k-1}}{d_{k+1}} \right) D_{2,k-1}^{11} (p_2) + \frac{\ell_k^3}{12} \frac{d_{k+1}}{\cos \theta_{k-1}} \left( \frac{1}{d_k} + \frac{\cos \theta_{k+1}}{d_{k-1}} \right) D_{2,k+1}^{11} (p_2). \tag{5.7}
\]
Then using (5.1) and (5.7), we obtain $c_0 = \alpha_{j_l,k}^{j_l} D_{1,k}^{jl} (p_2), 1 \leq k \leq 3$. \qed

Now we can prove Lemma 2.1. In the proof, we shall use the integral formula
\[
\int_T \lambda_1^{m_1} \lambda_2^{m_2} \lambda_3^{m_3} = \frac{2|m_1!m_2!m_3!}{(m_1 + m_2 + m_3 + 2)!}, \quad \int_e \lambda_1^{m_1} \lambda_2^{m_2} = \frac{|e|m_1!m_2!}{(m_1 + m_2 + 1)!}, \tag{5.8}
\]
where $\lambda_1, \lambda_2$ are barycentric coordinates w.r.t. the edge $e$.

Proof Using (2.1) and Lemma 5.1, we have
\[
\int_T (p_2 - \Pi_h p_2) \cdot \nabla w_2 = \sum_{k=1}^{3} \int_{e_k} w \nabla w_2 \cdot t_k - \int_T w \Delta w_2 := I + II. \tag{5.9}
\]
Recall that $\phi_k = \lambda_{k-1}\lambda_{k+1}$ and let $I_h$ be the linear interpolation. Then using the hierarchical representation

$$w_2 - I_h w_2 = -\frac{1}{2} \sum_{k=1}^{3} \ell_k^3 \phi_k\partial_{t_k}^2 w_2.$$  \hspace{1cm} (5.10)

and $\Delta \phi_k = 2\nabla \lambda_{k-1} \cdot \nabla \lambda_{k+1} = -2\cos \theta_k/(d_{k-1}d_{k+1})$, we obtain

$$\Delta w_2 = \frac{1}{4|T|^2} \sum_{k=1}^{3} \ell_k^3 \ell_{k-1} \ell_{k+1} \cos \theta_k \partial_{t_k}^2 w_2.$$  \hspace{1cm} (5.11)

It then follows from Lemma 5.1, (5.11), and $\int_T \psi_0 = |T|/60$, $\int_T \psi_k = 0$, $1 \leq k \leq 3$, that

$$II = -\frac{|T|}{60} c_0 \Delta w_2 = -\frac{1}{240|T|} \sum_{k=1}^{3} c_0 \ell_k^3 \ell_{k-1} \ell_{k+1} \cos \theta_k \partial_{t_k}^2 w_2$$

$$= -\frac{1}{120} \sum_{k=1}^{3} \int_{e_k} \alpha_{j,l,k}^2 D_{j,l,k}^3(p_2) \ell_k \cot \theta_k \partial_{t_k}^2 w_2.$$  \hspace{1cm} (5.12)

By the elementary identity $t_k = \cos \theta_k/n_{k+1} - \cos \theta_{k-1}/n_{k+1}$, Lemma 5.1, and $\psi_k = -\ell_k \partial_{t_k}(\phi_k^2)/2$, we have

$$I = -\sum_{k=1}^{3} \frac{1}{12} \int_{e_k} \ell_k^3 D_{j,l,k}^{11}(p_2) \psi_k \nabla \cdot w_2 \cdot \left( \frac{\cos \theta_{k-1}}{\sin \theta_k} n_{k-1} - \frac{\cos \theta_{k+1}}{\sin \theta_k} n_{k+1} \right)$$

$$= \sum_{k=1}^{3} \frac{1}{24} \int_{e_k} \ell_k^4 D_{j,l,k}^{11}(p_2) \phi_k^2 \left( \frac{\cos \theta_{k-1} \partial_{t_{k-1}}^2 w_2}{\sin \theta_k} - \frac{\cos \theta_{k+1} \partial_{t_{k+1}}^2 w_2}{\sin \theta_k} \right).$$  \hspace{1cm} (5.13)

Then using the quadrature rule (5.8),

$$I = \frac{1}{720} \sum_{k=1}^{3} \ell_k^5 D_{j,l,k}^{11}(p_2) \left( \frac{\cos \theta_{k-1} \partial_{t_{k-1}}^2 w_2}{\sin \theta_k} - \frac{\cos \theta_{k+1} \partial_{t_{k+1}}^2 w_2}{\sin \theta_k} \right).$$

In addition, (5.10) gives

$$\partial_{t_{k-1}}^2 w_2 = -\frac{\ell_k}{2\ell_{k-1}} \partial_{t_k}^2 w_2 + \frac{\ell_k^2}{2\ell_{k-1}\ell_k} \partial_{t_{k-1}}^2 w_2 - \frac{\ell_{k-1}}{2\ell_k} \partial_{t_{k-1}}^2 w_2,$$

$$\partial_{t_{k+1}}^2 w_2 = -\frac{\ell_k}{2\ell_{k+1}} \partial_{t_k}^2 w_2 + \frac{\ell_{k+1}}{2\ell_k} \partial_{t_{k+1}}^2 w_2 + \frac{\ell_k^2}{2\ell_k \ell_{k+1}} \partial_{t_{k-1}}^2 w_2.$$  

Therefore,

$$I = \frac{1}{1440} \sum_{k=1}^{3} \int_{e_k} \left\{ \ell_k^5 D_{j,l,k}^{11}(p_2) \left( \frac{\cos \theta_{k+1}}{\ell_{k+1}} - \frac{\cos \theta_{k-1}}{\ell_{k-1}} \right) \right\} \partial_{t_k}^2 w_2$$

$$+ \frac{\ell_k^4}{\sin \theta_{k-1}} D_{j,l,k-1}^{11}(p_2) \left( \cos \theta_{k+1} + \frac{\ell_k}{\ell_{k+1}} \cos \theta_{k+1} \right)$$

$$- \frac{\ell_k^4}{\sin \theta_{k+1}} D_{j,l,k+1}^{11}(p_2) \left( \frac{\ell_k}{\ell_{k-1}} \cos \theta_{k-1} + \cos \theta_{k} \right) \partial_{t_k}^2 w_2.$$  \hspace{1cm} (5.14)
Combining (5.9), (5.12), (5.14) and using (5.1), we obtain Lemma 2.1.

6 Numerical Experiments

We test our recovery operators $R^r_h$ with $r = 1, 2, 3$ by the Poisson equation

$$-\Delta u = f \text{ in } \Omega,$$

where $\Omega$ and $u$ will be given in the next three experiments. Readers are referred to [19] for numerical results on recovery superconvergence of the $RT_0$ element. The experiments are implemented using the iFEM package [8] in Matlab 2018b. In tables, $\| \cdot \|$ is the $L^2$-norm $\| \cdot \|_{0, \Omega}$, ‘nt’ denotes the number of triangles. The order of convergence is $p$ such that error $\approx \text{ndof}^{-\frac{p}{2}}$, where ndof is the number of degrees of freedom. The value of $p$ is computed by least squares.
**Problem 1** In the first experiment, let $\Omega$ be the unit square $[0, 1]^2$ and

$$u = \exp(x_1 + x_2) \sin(2\pi x_1) \sin(\pi x_2)$$

be the exact solution. We test the performance of $R_h^1$. Due to Theorem 4.1, we do not enlarge the patch $\omega_z$ when $z$ is an interior vertex. If $z$ is a boundary vertex, extra neighboring elements are added to $\omega_z$ such that $\#\omega_z \geq 8$. It turns out that all local least squares problems are uniquely solvable. We start with the Delaunay triangulation in Fig. 3, and computed a sequence of meshes by regular refinement, i.e., dividing an element into four similar subelements by connecting the midpoints of each edge, see Table 1. We also computed a sequence of meshes by newest vertex bisection (cf. [8, 20]), see Fig. 4 and Table 2.

For regular refinement, the sequence of grids satisfies $(\alpha, \beta)$-condition with $(\alpha, \beta) = (\infty, 1)$. For $RT_1$ elements, Theorem 3.2 predicts that $\|P_h p - p_h^1\| = O(h^{2.5})$, which is confirmed by Table 1. In view of the high order recovery superconvergence $\|p - R_h^1 p_h^1\| = O(h^{3.4})$, our supercloseness estimate $\|p - R_h^1 p_h^1\| = O(h^{2.5})$ in Theorem 4.4 may be sub-optimal.

The sequence of grids created by newest vertex bisection is far from uniformly parallel, i.e., almost no pair of adjacent triangles forms an $O(h^{1+\alpha})$ approximate parallelogram with some positive $\alpha$. Hence there is no supercloseness in Table 2. Surprisingly, we still observe apparent superconvergence for $\|p - R_h^1 p_h^1\|$.

**Problem 2** Although our supercloseness estimates only work for $RT_0$ and $RT_1$ elements, we perform numerical experiments on the recovery operators $R_h^2$ and $R_h^3$ for $RT_2$ and $RT_3$ elements. We use the same $\Omega$, $u$, and initial mesh with regular refinement in Problem 1. Local patches $\omega_z$ is chosen in the same way as in Problem 1. The numerical results are presented in Tables 3 and 4.

As mentioned in Problem 1, the sequence of grids satisfies $(\alpha, \beta)$-condition with $(\alpha, \beta) = (\infty, 1)$. Unlike $RT_0$ and $RT_1$ elements, there is no supercloseness phenomenon for $RT_2$ and $RT_3$ even on regularly refined meshes. However, it can be observed that the rate of recovery superconvergence is at least $\|p - R_h^r p_h^r\| = O(h^{r+2})$ with $r = 2, 3$. Therefore, the supercloseness estimate is not a necessary ingredient of superconvergence analysis. We conjecture that the superconvergence is due to a large number of locally symmetric patches, see [24] for the theory of Lagrange elements.

**Problem 3** Postprocessing superconvergence is often used to develop recovery-type a posteriori error estimator and adaptive FEMs. In the end, we test the adaptivity performance of $R_h^1$ on the domain $\Omega = [-1, 1]^2 \setminus \Omega_0$, where $\Omega_0$ is a right triangle whose smallest angle is $\omega = \pi/24$, see Fig. 5(left). Let

$$u(r, \theta) = r^{\pi/\omega} \sin \left( \frac{\pi}{2\pi - \omega} \theta \right) - \frac{r^2}{4},$$

where $(r, \theta)$ is the polar coordinate. The corresponding source $f = -\Delta u = 1$. We use the classical adaptive feedback loop (cf. [10, 21])

SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE.

It will return a sequence of meshes $\{T_{h\ell}\}_{\ell \geq 0}$ and numerical solutions $\{p_{h\ell}\}_{\ell \geq 0}$. The algorithm starts from the initial grid $T_{h0}$ in Fig. 5(left). In the procedure ESTIMATE, $\eta_{\ell, T} = \|R_{h\ell}^1 p_{h\ell} - p_{h\ell}^1\|_{0, T}$ serves as a posteriori error estimator on each triangle $T \in T_{h\ell}$. The procedure MARK
selects a collection of triangles $\mathcal{M}_\ell \subset \mathcal{T}_h^\ell$ such that

$$\sum_{T \in \mathcal{M}_\ell} \eta_{\ell,T}^2 \geq 0.3 \sum_{T \in \mathcal{T}_h^\ell} \eta_{\ell,T}^2.$$ 

Then the elements in $\mathcal{M}_\ell$ and necessary neighboring elements are refined by local mesh refinement strategy to yield a conforming subtriangulation $\mathcal{T}_{h+1}^\ell$ of $\mathcal{T}_h^\ell$. In particular, we use regular refinement with bisection closure in the procedure REFINE, see Fig. 5(right) for an adaptively refined triangulation. The numerical results are presented in Fig. 6.

It can be observed that the adaptive algorithm yields optimal rate of convergence and apparent recovery superconvergence. A distinct feature of the a posteriori error estimator $\eta_{h\ell} := \| R_h^1 p_h^1 - p_h^1 \|_{0,\Omega} = (\sum_{T \in \mathcal{T}_h^\ell} \eta_{\ell,T}^2)^{1/2}$ is the well-known asymptotic exactness:

$$\lim_{\ell \to \infty} \frac{\eta_{h\ell}}{\| p - p_h^1 \|_{0,\Omega}} = 1,$$

which can be numerically confirmed using the rates of superconvergence in Fig. 6 with a triangle inequality, see e.g., [4,27] for details.
Table 4  $RT_3$ with regular refinement

| nt   | $\|p - p_h^3\|$ | $\|\Pi_h^3 p - p_h^3\|$ | $\|p - R_h^3 p_h^3\|$ |
|------|------------------|--------------------------|---------------------|
| 86   | 3.733e−3         | 3.394e−3                 | 4.022e−2            |
| 344  | 2.359e−4         | 2.140e−4                 | 1.180e−3            |
| 1376 | 1.478e−5         | 1.338e−5                 | 2.668e−5            |
| 5504 | 9.242e−7         | 8.354e−7                 | 6.377e−7            |
| 22106| 5.777e−8         | 5.217e−8                 | 2.116e−8            |
| Order| 3.995            | 3.998                    | 5.257               |

Fig. 5  (left) Initial grid for the adaptive algorithm. (right) Adaptive grid, 2026 elements

Fig. 6  Error curves for $RT_1$

7 Concluding Remarks

In this paper, we develop supercloseness estimate for the second lowest order RT element and a family of postprocessing operators $R_h^r$ for higher order RT elements applied to second order elliptic equations. Since both the analysis of supercloseness and postprocessing operators are local, our superconvergence results can be adapted to Neumann and mixed boundary
conditions. In practice, $R^e_h$ can be extended to 3-dimensional RT elements in a straightforward way although the theoretical analysis in this paper may need significant modifications, e.g., the supercloseness estimate and well-posedness of the local least squares problem would be more complicated. Readers are also referred to [13] for numerical experiments on a different postprocessing operator for the lowest order RT elements in $\mathbb{R}^3$.

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