Canonical Heights on Shimura Varieties and the André-Oort conjecture

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1 Introduction

1.1 History and main results

The main purpose of this work is to prove the André-Oort conjecture in full generality. Recall the statement of the conjecture:

**Theorem 1.1.** Let $S$ be a Shimura variety. Let $V \subset S$ be a subvariety. Then there are only finitely many maximal special subvarieties contained in $V$.

The first unconditional result was obtained by André [1] for a product of two modular curves. In the past two decades there has been much work on the conjecture. First, spurred by an idea of Edixhoven [13], the conjecture was proven conditionally on GRH in a series of works by Klingler-Ullmo-Yafaev [27, 49].

Further unconditional results were obtained starting with [40], using a different strategy. This strategy was originally proposed by Zannier and had been implemented to reprove the Manin-Mumford conjecture in [42]. The approach has three main ingredients:

- Estimates [41] for counting rational points on a transcendental set
- Functional Transcendence theorems, specifically for the uniformization map of Shimura varieties
- Lower bounds for Galois orbits of special points.

The required functional transcendence results were generalized to arbitrary Shimura varieties in a series of works including [38, 25], and inspired stronger transcendence results in these and other contexts, including recently a proof of the so called Ax-Schanuel theorem in the context of arbitrary variations of mixed Hodge structures [37, 3, 9, 18]. Such generalizations are important for studying deeper unlikely intersection questions such as the Zilber-Pink conjecture.

The remaining missing ingredient was the lower bound for Galois orbits of special points. Such bounds were obtained for $\mathbb{A}_g$ in [17] by relying on the average Colmez conjecture (proved independently by Andreatta-Goren-Howard-Madapusi-Pera [2] and Yuan-Zhang [50]) to obtain bounds for the *height* of Galois orbits and the Masser-Wüstholz isogeny estimates. The isogeny estimates are not available in arbitrary Shimura varieties, but this ingredient was recently removed in a paper of Binyamini-Schmidt-Yafaev [5] based on a recent breakthrough of Binyamini [4] obtaining strong point-counting results in terms of both the height and degree of the points being counted. The idea for this strategy to obtain Galois orbit bounds first appeared in work of Schmidt [44] in the context of tori and elliptic curves.

In essence, this means that the conjecture is reduced to finding suitable upper bounds for heights of special points. Our main contribution is to establish this result:

**Theorem 1.2 (Theorem 9.11).** Fix a Shimura variety $S_K(G, X)$ with $G$ simple of adjoint type. Let $(T, r) \subset (G, X)$ be a (varying) 0-dimensional Shimura datum such that $K \cap T(A_f)$ is of index $M$ in the maximal compact $K_T$, and $T$ splits over a field $E$. Then the canonical height of $(T, r)$ w.r.t. an ample line bundle is $(M \text{ disc } E)^{(1)}$.

Gao [17] has shown how to deduce the André-Oort conjecture for a mixed Shimura variety from Galois bounds for the associated pure Shimura variety, and so the conjecture (which is as stated in Theorem 1.1 for $S$ a mixed Shimura variety) follows for all mixed Shimura varieties also.

1.2 Method of proof

The idea is to reduce the general case to the height bounds for CM points in $\mathbb{A}_g$. We thus need a way to compare heights for CM points which embed in different Shimura varieties. To facilitate this comparison, we require a ‘canonical’ height on arbitrary Shimura varieties, similar to the Faltings height on $\mathbb{A}_g$. At first glance this might feel minor, because all Weil heights on a line bundle are the same up to bounded functions, so why is it so important to get a canonical height function? The reason is that we have a different
comparison for each CM point, so knowing the result up to a bounded function for each CM point separately
tells us nothing! As such, we need to have better pointwise control. We explain how to do this in the next
subsection. Once this is done, then for every 0-dimensional Shimura variety and automorphic line bundle -i.e. a character of the split torus - we obtain a canonical height

We first explain in 3.1 how to associate a 0-dimensional Shimura datum \((E^x/F^x,r_Φ)\) to a partial CM
type \(\Phi\) associated to a CM field \(E/F\), and a canonical character \(χ_Φ\) on the associated torus. We are thus
reduced to bounding the corresponding intrinsic heights of \(χ_Φ\) on \(\{T_Φ, r_Φ\}\), and the case where \(\Phi\) is a full
CM-type is covered by the case of \(A_g\).

The idea is then to use Deligne’s construction for turning partial CM types into full CM types. Namely,
suppose that \(E_1, E_2\) are CM fields over the same real totally real field \(F\). Say \(Φ_1, Φ_2\) are partial CM types
for \(E_1, E_2\) respectively, such that their restrictions to \(F\) are complementary. Then one may form a complete
CM type \(Φ\) on \(E_{\text{tot}} := E_1E_2\) by taking the union of the pullbacks of the \(Φ_1\). Moreover, \((E_{\text{tot}}^x/F_{\text{tot}}^x, r_Φ)\) admits
maps (up to isogeny) to \((E_i^x/F_i^x, r_Φ)\) via the norm map on tori, and \(χ_Φ\) is the product of the pullbacks of the \(χ_Φ_i\). In this way, we are able to deduce from the case of full CM type, bounds for the sum of the intrinsic
heights of \(Φ_1\) and \(Φ_2\).

Armed with this technique, the key observation is that we may combine these partial CM types in many
different ways, allowing us to extract individual bounds by taking linear combinations. This is a simple
combinatorial argument, which we carry out in Theorem 9.12.

1.3 Constructing canonical heights

Given a \(\mathbb{Z}\)-local system \(V\) arising from an algebraic representation \(V\) of \(G_{\mathbb{Q}}\), we may form an associated
flat vector bundle \(\mathfrak{dr} V\) via the Riemann-Hilbert correspondence, which in this setting is equipped with a
filtration. Our line bundles will arise as determinants of sub-bundles of \(\mathfrak{dr} V\), and so we seek to give \(\mathfrak{Gr}^* V\)
the structure of a normed vector bundle. At the Archimedean places, we simply use the Hodge norm \(\|\cdot\|\).
Our construction of canonical heights is similar in spirit to previous constructions of heights for motives, in
particular, works of Kato [23] and Koshikawa [24].

At the finite places, work of Scholze [39] and Liu-Zhu [27] has shown how to interpret \(\mathfrak{dr} V\) in terms of a
\(p\)-adic Riemann-Hilbert correspondence, which specializes to the functor \(D_{\mathfrak{dr}}\) from \(p\)-adic Hodge theory
pointwise. This allows us to equip its grading with a canonical norm, and it is not hard to show that this
norm behaves well at each finite place.

A difficulty arises in showing that the above local norms piece together to give a well-behaved global
norm. We suspect that this is the case for all Shimura varieties (and perhaps much more generally), but
have been unable to establish this.

Here we use in an essential way the work of Esnault-Groechenig, which they generalize in the appendix
to accommodate non-proper Shimura varieties. Specifically, they establish that if the local system is rigid,
then for all but finitely many places it is in fact crystalline in the sense of Faltings. It is this extra data that
allows us to prove that the heights piece together well.

Here there is the good fortune that all irreducible Shimura varieties not of abelian type have real rank at
least 2, and thus satisfy Margulis super-rigidity. This dichotomy is used in many other places, for example in [22].

1.4 Organization of the paper

In §2 we recall the relevant integral \(p\)-adic Hodge theory results from Tsuji [18] and Liu-Zhu [27] and show a
compatibility between them. In §3 we recall some basics about Shimura varieties, introduce the concept of
partial CM-types and show that the associated 0-dimensional Shimura varieties are sufficient to understand
every 0-dimensional Shimura variety arising in a Shimura variety of adjoint type. We continue our study of

\[\text{1} \] In fact the details are a bit more complicated and require us to make some arbitrary and unaesthetic choices between what we dub the intrinsic and crystalline norms, but morally this is what is happening.

\[\text{2} \] This depends on the choice of polarization

\[\text{3} \] This would follow from the (conjectured) existence of suitable motives over exceptional Shimura Varieties
0-dimensional Shimura varieties in §4 by showing that in a given Shimura variety, all CM points are integral with respect to an appropriate integral model in the spirit of the Neron-Ogg-Shafarevich criterion.

In §5 we combine results of Esnault-Groechenig (14] and generalized to our setting in the appendix) and Margulis [31] to show that automorphic local systems on proper Shimura varieties are crystalline, so that we may apply the theory of relative Fontaine-Lafaille modules to them. In §6 we define notions of well-behaved norms on vector bundles that allows us to discuss global heights. In §7 we finally endow our Shimura varieties with canonical local heights. In §8 we show that our construction in §7 yields such ‘admissible’ families of norms, such that we may discuss global heights.

Finally, in section §9 we finally put the preceding theory together to define a ‘modified’ height on 0-dimensional Shimura varieties which behaves well with respect to Shimura embeddings. We then deduce the height bound for Partial-CM type 0-dimensional Shimura varieties from the full-CM case by using the known height bounds in [A] and Deligne’s idea, as described above. Finally, we deduce the André-Oort conjecture.

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2 p-adic Hodge Theory

2.1 Notation
We use the following notations from [45, 48, 30]:

- V - DVR of mixed characteristic p with residue field k and fraction field K. We assume p is a uniformizer of V, so that V ≅ W(k), the Witt-vectors of k.
- V is the integral closure of V in K.
- R - smooth V-algebra. Let R denote its p-adic completion.
- f : V[s1, s1−1, . . . , sn, s1−1] → R is an etale map.
- For a Zp-algebra S, Let Sφ be lim ←−x→x(S/pS).
- Define Ainf(S) to be W(Sφ), the ring of Witt-vectors of Sφ. This ring has a Frobenius ϕ by functoriality. For any element x ∈ Sφ, let [x] ∈ Ainf(S) denote its Teichmüller lift.
- For a ring S, Sφ is the p-adic completion of S.
- There is the following ring homomorphism θ : Ainf(S) → Sφ characterized by θ([a]) = limn→∞ anp, where a ∈ Sφ = (an) and an ∈ S is some lift of an.
- Assume now that S contains Zp. Fix a compatible system βn ∈ Zp of p nth roots of p with β0 = p. Define pn ∈ Sφ to be the element (βn mod p)n. Let [pn] denote the Teichmüller lift of pn.
- Define ξ ∈ Ainf(S) = p − [pn]. It generates the kernel of θ.
- Define ε to be an inverse limit of primitive p nth roots of unity, and π = [ε] − 1.
• Define \( \text{Fil}^i(A_{\inf}(S)) \) to be the ideal \( \ker(\theta)^i \) if \( r \geq 1 \), and to be \( A_{\inf}(S) \) if \( r \leq 0 \).

• Define \( A_{\text{cris},m}(S) \) to be divided power envelope of \( A_{\inf}(S)/p^m \) with respect to the ideal \( \ker(\theta) \mod p^m \).

• Define \( A_{\text{cris}}(S) \) to be the inverse limit \( \lim \limits_{\leftarrow} A_{\text{cris},m}(S) \) endowed with the inverse limit topology with respect to the discrete topology on \( A_{\text{cris},m}(S) \). This is the same as the following construction: consider the divided power envelope \( A_{0,\text{cris}}(S) \) of \( A_{\inf}(S) \) with respect to the ideal \( \ker(\theta) \). Then, \( A_{\text{cris}}(S) \) is the \( p \)-adic completion of \( A_{0,\text{cris}}(S) \).

• \( R_c := R[s_1^{1/p^m}, i \in \{1, \ldots, n\}, m \in \mathbb{N}] \otimes_V \mathcal{V} \)

• \( \overline{R} \) is the integral closure of \( R \) in the maximal etale extension of \( R[1/p] \), and define \( \overline{\mathcal{V}} \) analogously. Note that \( R_c \subset \overline{\mathcal{V}} \).

• Define \( A_{\text{cris},m}(\overline{R}) \) to be the divided power envelope of \( A_{\inf}(\overline{R}) \otimes_V R/p^m \) with respect to the kernel of the surjective homomorphism \( \theta \mod p^m \otimes \text{Id} \mod p^m : A_{\inf}(\overline{R}) \otimes R \mod p^m \to \overline{R} \mod p^m \). Define \( A_{\text{cris}}(\overline{R}) \) to be the inverse limit of \( A_{\text{cris},m}(\overline{R}) \). This ring equals \( A_{\text{cris}}(\overline{R})[v_1, \ldots, v_n]^{PD} \), which is the \( p \)-adic completion of the PD-algebra \( A_{\text{cris}}(\overline{R})[v_1, \ldots, v_n]^{PD} \). Here, the elements \( v_i \) map to \( [s_i] \otimes s_i^{-1} - 1 = (1 \otimes s_i^{-1} - 1)(s_i^1 \otimes 1) \).

• \( B_{\inf}(S) := A_{\inf}(S)[\frac{1}{p}] \). We use \( \theta \) to refer also to the natural extension \( B_{\inf}(S) \to \hat{S}[\frac{1}{p}] \).

• \( B_{\text{an}}^{\perp}(S) := \lim \limits_{\leftarrow} B_{\inf}(S)/(\ker(\theta))^n \) with its natural filtration \( \text{Fil}^i B_{\text{an}}^{\perp}(S) \) generated by \( (\ker(\theta))^i \).

• \( B_{\text{an}}(S) := B_{\text{an}}^{\perp}(S)[\frac{1}{p}] \).

• \( \mathcal{O}B_{\inf}(S) := S \otimes_V B_{\inf}(S) \). Note this still admits a map \( \theta \) to \( \hat{S}[\frac{1}{p}] \).

• \( \mathcal{O}B_{\text{an}}^{\perp}(S) \) is a suitable \( p \)-adic completion of \( \lim \limits_{\leftarrow} \mathcal{O}B_{\inf}(S)/(\ker(\theta))^n \) as in [40] with its natural filtration \( \text{Fil}^i \mathcal{O}B_{\text{an}}^{\perp}(S) \) generated by \( (\ker(\theta))^i \). It is naturally a power-series ring over \( B_{\text{an}}^{\perp}(S) \).

• \( \mathcal{O}B_{\text{an}}(S) := \mathcal{O}B_{\text{an}}^{\perp}(S)[\frac{1}{p}] \).

• Note that \( \mathcal{O}B_{\inf}(R), \mathcal{O}B_{\text{an}}^{\perp}(R), \mathcal{O}B_{\text{an}}(R) \) all admit integrable connections \( * \to * \otimes_R \Omega^1_{R/V} \) inherited from \( R \).

• Define \( t = \log([\epsilon]) := \sum_{i \geq 1}(-1)^{i+1} \frac{\pi^i}{1} \) as an element of \( B_{\text{an}}^{\perp}(S) \). Note that \( t - \pi \in \text{Fil}^2 B_{\text{an}}^{\perp}(S) \), and that the Galois action on \( B_{\text{an}}^{\perp}(\mathcal{O}_{C_p}) \) scales \( t \) via the Cyclotomic character.

2.2 The \( p \)-adic Riemann-Hilbert correspondence of Liu-Zhu

We summarize here (as a consequence of) one of the main results from [30], in the context of \( X = \text{Spec} \, R \). To set the stage, we let \( \mathcal{L} \) be a \( \mathbb{Z}_p \) - local system of finite rank \( r \) over \( \text{Spec} \, X_{\mathbb{Q}_p} \), which we can identify with a \( G = \text{Gal}(\overline{\mathcal{R}}/\mathcal{R}) \)-module \( \mathcal{L} \), free of rank \( r \) over \( \mathbb{Z}_p \). The following Theorem is [30] Thm 3.9].

Theorem 2.1. Define \( D^0_{\text{an}}(L) := (L \otimes \mathcal{O}B_{\text{an}}(\overline{R}))^G \). Then \( D^0_{\text{an}}(L) \) is a free \( R[\frac{1}{p}] \)-module with an integrable connection. It is of rank \( \leq r \) with equality iff \( L \) is a de Rham local system in the sense of Scholze.

Proof. The definition of \( D^0_{\text{an}}(L) \) is taken from [30] §3.2] once one makes the translations from sheaves to rings. The statement about ranks follows from the pullback compatibility [30] Thm 3.9(ii)] together with the corresponding statement for the classical case \( R = V \), as well as the statement [30] Thm 1.3] showing that a local system is de Rham iff its stalk is de Rham at a single classical point. \qed

6
2.3 Review of Tsuji’s constructions

Let $R, \mathcal{R}$ be as above. We fix a Frobenius lift $\varphi$ on $V[s_i]$, by setting $\varphi(s_i) = s_i^p$. There is a unique extension of $\varphi$ on $\mathcal{R}$ which we will also denote by the same $\varphi$. We recall the category $\text{MF}_{0p-2, \text{free}}^\nabla(\mathcal{R}, \varphi)$. An object of $\text{MF}_{0p-2, \text{free}}^\nabla(\mathcal{R}, \varphi)$ is given by the quadruple $(M, \text{Fil}, \nabla, \Phi)$ where (see [18 §4] for more details):

1. $M$ is a finitely-generated free $\mathcal{R}$-module.
2. A topologically nilpotent integrable connection $\nabla : M \to M \otimes \Omega^1$.
3. A decreasing filtration $\text{Fil}$ of $M$ by $\mathcal{R}$-submodules, satisfying:
   (a) $\text{Fil}^0 M = M$ and $\text{Fil}^{p-1} M = 0$.
   (b) $\text{Gr}_M$ is finitely generated free $\mathcal{R}$-module.
   (c) $\nabla(\text{Fil}^r M) \subset \text{Fil}^{r-1} M \otimes \Omega^1$ (Griffiths’ transversality).
4. An $\mathcal{R}$-linear homomorphism $\Phi : F^*(M) \to M$ where
   (a) $F^*(M)$ is a canonical-up-to-isomorphism defined $\mathcal{R}$-module with filtration and connection constructed out of $(M, \text{Fil}, \nabla)$
   (b) $\Phi$ is compatible with the connections
   (c) $\Phi$ satisfies appropriate integrality conditions.

Tsuji in [18 Section 5] associates the following objects to an object $M$ of $\text{MF}_{0p-2, \text{free}}^\nabla(\mathcal{R}, \varphi)$:

- An $A_{\text{cris}}(\mathcal{R})$-module $T_{\text{cris}}(M) := M \otimes_{\mathcal{R}} A_{\text{cris}}(\mathcal{R})$ [18 (37)]
- An $A_{\text{inf}}(\mathcal{R})$-module $T_{\text{inf}}(M)$, with a canonical isomorphism
  $$T_{\text{inf}}(M) \otimes_{A_{\text{inf}}(\mathcal{R})} A_{\text{cris}}(\mathcal{R}) \cong T_{\text{cris}}(M)$$

- A $G$-module $T_{\text{cris}} M$ of rank the same as $M$. This is defined as the dual of $T_{\text{cris}}^*(M)$ where
  $$T_{\text{cris}}^*(M) := \text{Hom}_{A, \text{Fil}, \varphi, \nabla}(M, A_{\text{cris}}(\mathcal{R})).$$

Any $G$-module obtained this way is defined to be a crystalline local system.

Remark 2.2. Let $\overline{X}/V$ denote a proper smooth scheme, let $D \subseteq X$ denote a relative normal crossing divisor, and let $X = \overline{X}\backslash D$. Faltings refers to the notion of a “logarithmic Fontaine-Laffaille module” on $\overline{X}$ (see [16 Theorem 2.6’, page 43, i]), associated to which is a local system on $X_{V[1/p]}$. Such local systems are also deemed crystalline, though we will not use this notion in this section. It will only be used in Section 5 and the appendix. A crystalline local system in this sense gives a crystalline local system in the usual sense when restricted to the generic fiber of the $p$-adic completion of $X$.

2.3.1 Recovering $M$ from $T_{\text{cris}} M$.

It is implicit though not formally stated in the text how to recover $M$ from $T_{\text{cris}}(M)$. We summarize it here in two steps:

- $T_{\text{inf}}(M)$ is the unique $G$-stable free $A_{\text{inf}}(\mathcal{R})$-submodule of $T_{\text{cris}} M \otimes A_{\text{inf}}(\mathcal{R})$ which generates $T_{\text{cris}} M \otimes A_{\text{inf}}(\mathcal{R})[\frac{1}{\pi}]$ and which is trivial modulo $\pi$. [18 Lemma 64(2) + (66) + Prop 76]
- $M \cong \left(T_{\text{inf}}(M) \otimes_{A_{\text{inf}}(\mathcal{R})} A_{\text{cris}}(\mathcal{R})\right)^G$. [18 Prop 61]

\[4\] Technically the proof of this proposition works with $T_{\text{cris}}(M)$ but the version written follows immediately from the definition of $T_{\text{inf}}(M)$ given above.
We denote the above procedure by $L \to S_{cris}(L)$. It follows from this description that if $L$ is a crystalline local system with $L = L_1 \oplus L_2$, then $L_1$ and $L_2$ are also crystalline local systems.

We shall require the following (almost certainly known) result for later:

**Lemma 2.3.** Let $L_1, L_2$ be crystalline local systems of weights in $[0, a]$ and $[0, b]$ such that $a + b \leq p - 2$. Then $L_1 \otimes L_2$ is crystalline and the natural map

$$S_{cris}(L_1) \otimes S_{cris}(L_2) \to S_{cris}(L_1 \otimes L_2)$$

is an isomorphism.

**Proof.** Define $M_i := S_{cris}(L_i)$ for $i = 1, 2$. Then $M := M_1 \otimes \mathbb{R} M_2$ is naturally an object of $MF_{[0, a+b]}^{\mathbb{Q}}(\mathcal{R}, \phi)$. Thus we may associate to $M$ the module $T^*_{cris}(M)$. Now it is clear that $T^*_{cris}(M) \cong T^*_{cris}(M_1) \otimes T^*_{cris}(M_2)$, and therefore there is an injective isogeny $f : T^*_{cris}(M) \to L_1 \otimes L_2$.

By [48, (39)] it follows that

$$TA_{cris}(M) \cong TA_{cris}(M_1) \otimes_{A_{cris}(\mathcal{R})} TA_{cris}(M_2).$$

It follows by the characterization of $TA_{inf}(M)$ that

$$TA_{inf}(M) \cong TA_{inf}(M_1) \otimes_{A_{inf}(\mathcal{R})} TA_{inf}(M_2).$$

It follows by [48, Thm 70] that $f \otimes A_{inf}(\mathcal{R})[\frac{1}{\pi}]$ is an isomorphism. Since $A_{inf}(\mathcal{R})[\frac{1}{\pi}]$ is a domain it is flat over $\mathbb{Z}_p$. Moreover $A_{inf}(\mathcal{R})[\frac{1}{\pi}] \otimes \mathbb{F}_p \neq 0$ and therefore $f$ must be an isomorphism, which shows that $L_1 \otimes L_2$ is crystalline. Moreover, the proof shows it corresponds to $M_1 \otimes \mathbb{R} M_2$ which shows the second part of the claim as well.

\[\square\]

### 2.4 Compatibility of passage to fibers

We fix a map $i : \mathcal{R} \to V$. Our goal is to show that the constructions given in the previous section are compatible with restriction to $i$.

We fix an extension $i : \mathcal{R} \to V$ of a geometric point, inducing a map $G_V \to G_{\mathcal{R}}$. Let $L$ be a crystalline $G_{\mathcal{R}}$ representation, which also gives a $G_V$ representation $L_i$ via the map $G_V \to G_{\mathcal{R}}$ induced by $i$.

By following the description in [2.3.1] we first form $TA_{inf}(L)$ as the unique $G_{\mathcal{R}}$-stable free $A_{inf}(\mathcal{R})$-submodule of $L \otimes A_{inf}(\mathcal{R})$ which generates $T^*_{cris} M \otimes A_{inf}(\mathcal{R})[\frac{1}{\pi}]$ and which is trivial modulo $\pi$.

**Lemma 2.4.** There is a functorially induced isomorphism

$$TA_{inf}(L) \otimes_{A_{inf}(\mathcal{R})} A_{inf}(V) \to TA_{inf}(L_i).$$

**Proof.** Note that there is a natural map $A_{inf}(\mathcal{R}) \to A_{inf}(V)$ induced by functoriality, which is equivariant for the map of Galois groups. Now consider the map

$$i^# : TA_{inf}(L) \otimes_{A_{inf}(\mathcal{R})} A_{inf}(V) \to L \otimes A_{inf}(V).$$

Note that $i^#$ is surjective after tensoring with the fraction field $F$ of $A_{inf}(V)$ (in fact, even after inverting $\pi$). Since both sides have the same dimension as $L$, it follows that $i^#$ is an isomorphism after tensoring with $F$. It follows that $i^#$ is injective (here we are using that $A_{inf}(V)$ is torsion free).

Thus, the image of $i^#$ satisfies the universal property characterizing $TA_{inf}(L_i)$ and the claim follows.

\[\square\]

We come to our main theorem for this subsection.

**Theorem 2.5.** There is a functorially induced isomorphism $S_{cris}(L) \otimes_{\mathbb{R}} V \cong S_{cris}(L_i)$. 

Proof. Recall that $S_{\text{cris}}(L) \cong \left( TA_{\text{inf}}(L) \otimes_{A_{\text{inf}}(\mathcal{R})} A_{\text{cris}}(\mathcal{R}) \right)^{G_{\mathcal{R}}}$. Moreover, by [18] (39) it follows that

$$S_{\text{cris}}(L) \otimes_{\mathcal{R}} A_{\text{cris}}(\mathcal{R}) \cong TA_{\text{inf}}(L) \otimes_{A_{\text{inf}}(\mathcal{R})} A_{\text{cris}}(\mathcal{R}).$$

And so

$$(S_{\text{cris}}(L) \otimes_{\mathcal{R}} V) \otimes_V A_{\text{cris}}(V) \cong T A_{\text{inf}}(L) \otimes_{A_{\text{inf}}(\mathcal{R})} A_{\text{cris}}(V),$$

$$(S_{\text{cris}}(L) \otimes_{\mathcal{R}} V) \otimes_V A_{\text{cris}}(V) \cong (T A_{\text{inf}}(L) \otimes_{A_{\text{inf}}(\mathcal{R})} A_{\text{cris}}(V)) \otimes_{A_{\text{inf}}(V)} A_{\text{cris}}(V),$$

$$(S_{\text{cris}}(L) \otimes_{\mathcal{R}} V) \otimes_V A_{\text{cris}}(V) \cong T A_{\text{inf}}(L_0) \otimes_{A_{\text{inf}}(V)} A_{\text{cris}}(V)$$

where the last step follows by Lemma 2.4. Finally, it follows that

$$(S_{\text{cris}}(L) \otimes_{\mathcal{R}} V) \cong \left( T A_{\text{inf}}(L_0) \otimes_{A_{\text{inf}}(V)} A_{\text{cris}}(V) \right)^{G_{\text{V}}}$$

from which the Theorem follows.

\[\square\]

2.5 Compatibility of Morrow-Tsuji with Liu-Zhu

Proposition 2.6. Let $R$ be as above. There is a natural map $A_{\text{cris}}(\mathcal{R}) \to \mathcal{O}_{\text{dR}}(\mathcal{R})$, which induces the usual crystalline-de Rham isomorphism for closed points.

We first prove the following lemmas:

Lemma 2.7. The ring $A_{\text{inf}}(\mathcal{R})/\ker(\theta)^n$ is $p$-adically complete for all $n$.

Proof. The ring $A_{\text{inf}}(\mathcal{R})$ is clearly $p$-adically complete, as it is expressed as the ring of Witt-vectors of a perfect $\mathbb{F}_p$-algebra. The claim follows. \[\square\]

Proposition 2.8. There is a natural map $A_{\text{cris}}(\mathcal{R}) \to \mathcal{B}_{\text{dR}}(\mathcal{R})$.

Proof. It suffices to construct a series of compatible maps $A_{\text{cris}}(\mathcal{R}) \to B_{\text{inf}}(\mathcal{R})/\ker(\theta)^n$ for every integer $n$. We note that there is clearly a map $A_{0,\text{cris}}(\mathcal{R}) \to B_{\text{inf}}(\mathcal{R})/\ker(\theta)^n$ for every integer $n$. It suffices to show that such a map extends continuously to a map from $A_{\text{cris}}(\mathcal{R})$.

To that end, let $a_i \in A_{0,\text{cris}}(\mathcal{R})$ denote a sequence of elements, and consider the power-series $a = \sum a_ip^i$. Without loss of generality, we may assume that there exists an integer $n_0$ such that $p^{n_0}a_i \in A_{\text{inf}}(\mathcal{R})/\theta_i$. Indeed, suppose $a_i = \frac{b}{p^m} \in A_{\text{inf}}(\mathcal{R})$, with $m$ minimal. As $A_{0,\text{cris}}(\mathcal{R})$ is defined as the PD-envelope of $A_{\text{inf}}(\mathcal{R})$, it follows that $b_i \in \ker(\theta)^n$ if $m$ is a large enough integer relative to $n$, and hence $a_i$ maps to 0 in $B_{\text{inf}}(\mathcal{R})/\ker(\theta)^n$, and hence we may assume that $a_i = 0$ for such $i$.

We may replace the $a_i$ by $p^{n_0}a_i$, and hence assume that $a = \sum a_ip^i$ with $a_i \in A_{\text{inf}}(\mathcal{R})$, and hence the image of each $a_i$ is in the subring $A_{\text{inf}}(\mathcal{R})/\ker(\theta)^n$. This ring is complete for the $p$-adic topology, and hence the image of $\sum a_ip^i$ is indeed well defined, and therefore we obtain a natural map $A_{\text{cris}}(\mathcal{R}) \to \mathcal{B}_{\text{dR}}(\mathcal{R})$. \[\square\]

We are now ready to prove Proposition 2.6.

Proof. We have already constructed a map $A_{\text{cris}}(\mathcal{R}) \to \mathcal{B}_{\text{dR}}(\mathcal{R})$. To see that this extends to a natural map $A_{\text{cris}}(\mathcal{R}) \to \mathcal{O}_{\text{dR}}(\mathcal{R})$, we observe that there is an isomorphism $B_{\text{dR}}^+(\mathcal{R})[[X_1, \ldots, X_n]] \to \mathcal{O}B_{\text{dR}}^+(\mathcal{R})$ with $X_i \mapsto [u_i] \otimes 1 - 1 \otimes u_i$, and that $p$ is invertible in this ring. As $A_{\text{cris}}(\mathcal{R})$ is $p$-adically complete, we have that $A_{\text{cris}}(\mathcal{R})[[v_1, \ldots, v_n]]^{PD}$, the $p$-adic completion of $A_{\text{cris}}(\mathcal{R})[[v_1, \ldots, v_n]]^{PD}$, is a subring of $A_{\text{cris}}[\frac{1}{p}][[v_1, \ldots, v_n]]$. The result follows from the above observation, along with fact that $A_{\text{cris}}[\frac{1}{p}][[v_1, \ldots, v_n]]$ clearly maps naturally to $B_{\text{dR}}^+(\mathcal{R})[[X_1, \ldots, X_n]]$, with $v_i \mapsto (1 \otimes s_i^{-1})X_i$. \[\square\]
We now prove our main theorem of this subsection:

**Theorem 2.9.** Let $L$ be a crystalline $G_{\mathbb{R}}$-module. Then $L$ is de Rham in the sense of Scholze and there is a natural isomorphism of filtered vector bundles with connection $S_{\text{cris}}(L) \otimes \mathbb{Q}_p \to D_{\text{dR}}^0(L)$.

**Proof.** First off, we pick a $V$-point $i : \mathcal{R} \to V$. By specializing to $i$ and using Theorem 2.5 we see that $L_i$ is crystalline, and therefore also de Rham. Now [34, Thm 1.5(ii)] implies that $L_i$ is de Rham.

By Proposition 2.6 we have a natural map $\phi : S_{\text{cris}}(L) \otimes \mathbb{Q}_p \to D_{\text{dR}}^0(L)$. It is immediate by functoriality that this map specializes at Spec $V$-points to the usual crystalline-de Rham isomorphism. Now we use the fact that a map of flat vector bundles which is an isomorphism over a point is an isomorphism to conclude. 

We end this section with an elementary lemma.

**Lemma 2.10.** Let $\chi$ denote the cyclotomic character (thought of as a $\mathbb{Z}_p$-representation). Let $L$ be a crystalline $G_{\mathbb{R}}$-module with weights between $a$ and $b$, with $0 \leq a \leq b \leq p - 2$. Then, $L \otimes \chi^{-n}$ is also crystalline for $-a \leq n \leq p - 2 - b$.

**Proof.** Let $M, \text{Fil}, \nabla, \Phi$ denote the object in MF associated to $L$. For any $n \in \mathbb{Z}$, consider the data $M, \text{Fil}(n), \nabla, \Phi(n)$ where $\text{Fil}(n)^i = \text{Fil}^{-n}$, and $\Phi(n) = p^n : \Phi$. For $-a \leq n \leq p - 2 - b$, it’s clear that $\text{Fil}(n)^0 = M, \text{Fil}(n)^{p-1} = \{0\}$. The compatibility of $\Phi(n)$ with $\text{Fil}(n)$ follows directly from the compatibility of $\Phi$ and $\text{Fil}$.

## 3 Shimura Varieties

See [35] for further background on this section.

Let $G$ be a reductive group over $\mathbb{Q}$, $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ the Deligne torus, and $X$ be a conjugacy class of homomorphisms $f : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the Shimura axioms, and $K = \prod_{p} K_p \subset G(\mathbb{A}_f)$ be a neat compact subgroup. Let $E(G, X)$ be the reflex field, and $E$ be a field over which $G$ splits. Associated to this data we get

- A complex projective variety $\tilde{X}$ with a transitive action of $G$, such that $X$ injects into $\tilde{X}(\mathbb{C})$ and the actions of $G(\mathbb{R})$ are compatible.
- An algebraic variety $S_K(G, X)$ called a "Shimura Variety" whose complex points can be identified with $G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)) / K$.
- Canonical models of $S_K(G, X)$ and $\tilde{X}$ over $E(G, X)$ satisfying certain natural properties.

If $G$ is a simple adjoint group over $\mathbb{Q}$, then $G = \text{Res}_{F/\mathbb{Q}} G'$ for some totally real field $F$ [34, Thm 3.13]. Thus $G_{\mathbb{R}} := \prod_{\sigma \in I_{nc}} G'_\sigma$ where $G'_\sigma = G' \times_{F, \sigma} \mathbb{R}$. Let $I_{nc}$ be the set of real places of $\mathbb{R}$ such that $G'_\sigma$ is non-compact for each $\sigma \in I_{nc}$. We then have a corresponding splitting of hermitian symmetric domains $X \cong \prod_{\sigma \in I_{nc}} X_\sigma$.

### 3.1 Partial CM types

Let $K/F$ be a CM extension of a totally real field. We define a torus $R_K := \text{Res}_{K/\mathbb{Q}} \mathbb{G}_m / \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$. Note that the cocharacter lattice $X_\phi(R_K)$ is isomorphic to $\mathbb{Z} \langle e_\sigma, \sigma : K \to \mathbb{C}/(e_\sigma + e_\sigma) \rangle$ as a Galois module. Similarly, the character lattice $X^*(R_K) \subset X^*(\text{Res}_{K/\mathbb{Q}} \mathbb{G}_m)$ as a Galois module is isomorphic to $\mathbb{Z} \langle e_\sigma^* - e_\sigma^* \rangle$.

---

5We work with the convention that the cyclotomic character has weight -1
We define a partial CM type [20] 4.2 to be a subset $\Phi$ of $\text{Hom}(K, \mathbb{C})$ such that $\Phi \cap \overline{\Phi} = \emptyset$. We associate a homomorphism $r_\Phi : S \to R_{K,\mathbb{R}}$ as follows: We first note that

$$R_K(\mathbb{C}) \cong \prod_{\sigma \in \text{Hom}(K, \mathbb{C})} \mathbb{C}_\sigma^\chi / \sim$$

where $\sim$ identifies $\mathbb{C}_\sigma$ and $\mathbb{C}_\overline{\sigma}$ via inversion. Thus, identifying $S(\mathbb{C}) \cong \mathbb{C}^\chi \times \mathbb{C}^\chi$, the map from the first factor to $\mathbb{C}_\sigma$ is

$$
\begin{align*}
& z \to z \quad \sigma \in \Phi \\
& z \to 1 \quad \sigma \in \overline{\Phi} \\
& z \to 1 \quad \text{else}
\end{align*}
$$

The map from the second factor is determined by the conjugation action.

The data $(R_K, \rho_\Phi)$ defines a Shimura datum, which we shall show to be sufficient to discuss CM points in adjoint Shimura varieties. We finally define the (complex) character $\chi_\Phi$ of $R_{K,\mathbb{C}}$ to be

$$\chi_\Phi := \sum_{\sigma \in \Phi} e^\Phi_\sigma - e^\overline{\Phi}_\sigma$$

### 3.2 CM points in adjoint Shimura varieties

Let $(G, X)$ be an adjoint Shimura Datum with $G = \text{Res}_{F/\mathbb{Q}} G'$ for some totally real field $F$ [34] Thm 3.13]. As at the start of the section, we have $G := \prod_{\sigma : F \to \mathbb{R}} G'_\sigma$ and a corresponding splitting of hermitian symmetric domains $X \cong \prod_{\sigma : F \to \mathbb{R}} X_{\sigma}$. Now let $x \in X$ such that $f_x : S \to G$ factors through a $\mathbb{Q}$-torus $T_x$, and we denote by $i_x$ the map $S \to T_x$. Even though $T_x$ needn’t be a maximal torus of $G$, we will still use the term ‘root’ to refer to weights of $T_x$ that occur in the adjoint representation of $G$ restricted to $T_x$. For each compact factor $G'_\sigma$ the map $\pi_\sigma \circ f_x$ is trivial, whereas for each noncompact factor the corresponding co-root $\mu_\sigma := \pi_\sigma \circ f_x(z, 1)$ is special. Let $\mu = (\mu_\sigma)_\sigma$ be the corresponding coroot of $T_x$. The Galois group $\text{Gal}_F$ acts on coroots and the action on $\mu_{\tau_0}$ factors through a Galois CM field $K_x$ which contains $K$, with complex conjugation acting as negation [35] 12.4.e]. Let $F_i$ be the totally real subfield of $K_x$. Let $\tau_0$ be a complex place of $K_x$.

Let $\beta$ be a root of $T_x$ occurring in $\text{Lie} G$. Since $\beta$ is fixed by $\text{Gal}(\overline{\mathbb{Q}}/K_x)$ and acted on by complex conjugation as negation, there is a Galois-equivariant map $X^*(R_{K_x}) \to X^*(T_x)$ sending $\tau_0 - \overline{\tau_0}$ to $\beta$. Thus, there is a map $f_\beta : T_x \to R_{K_\mathbb{R}}$ such that $(e^\tau_0 - e^{\overline{\tau_0}}) \circ f_\beta = \beta$.

Moreover, consider the homomorphism $f_{g\beta} \circ i_x : S \to R_{K_x,\mathbb{R}}$. The root $e^\tau_0 - e^{\overline{\tau_0}}$ of $R_{K_x,\mathbb{C}}$ maps to $\beta$ via $f_{g\beta}^\ast$, for $g \in \text{Gal}_F$. Now note that $(\mu, g\beta) \in \{-1, 0, 1\}$ since $\mu_i$ is special for each $i$. Let $\Phi$ denote the corresponding embeddings $g_{\tau_0}$ of $K_x$ for which the inner product is 1. Then evidently $\Phi$ is a partial CM-type, and we see that $i_x \circ f_\beta = \rho_\Phi$.

**Theorem 3.1.** Let $(G, X)$ be a Shimura Datum with $G$ is a simple adjoint group over $\mathbb{Q}$, and let $(T_x, \rho)$ denote a 0-dimensional Shimura sub-datum which splits over a CM field $K$. Then there are $O(1)$ embeddings $a_i : (T_x, \rho) \to (R_{K}, \rho_\Phi)$ into a Shimura datum corresponding to a partial CM type, corresponding to the roots of $T_x$. Moreover, the dimension of $R_K$ and the cokernel of the co-character map $a^* : X^*(R_K) \to X^*(T_x)$ are bounded in terms of $G$.

Finally, $(\det F^1_{\text{ad}} \text{Lie} G | T_x)^N = \prod_{\beta} m_\beta \chi_\beta$, where $N$ is an integer bounded in terms of $G$.

**Proof.** Everything but the last claim follows from the discussion above. Note that by definition

$$\det F^1_{\text{ad}} \text{Lie} G | T_x = \sum_{\beta | (\mu, \beta) = 1} m_\beta \beta$$

where $m_\beta$ is the multiplicity with which $\beta$ occurs. The claim now follows by breaking up all the roots occurring in $\text{Lie} G$ into $\text{Gal}(K_x/\mathbb{Q})$-orbits. \[\square\]
3.3 Principle bundles

There is an algebraic variety $P = P_K(G, X)$ over $E(G, X)$ such that $P(C) \cong G(Q) \backslash (X \times G(C) \times G(A_f)) / K$ and maps

$$S_K(G, X) \xrightarrow{\tau} P \xrightarrow{\psi} \hat{X}$$

which over $\mathbb{C}$ are naturally identified with the maps $[(x, g, k)] \mapsto [(x, k)]$ and $[(x, g, k)] \mapsto g^{-1}x$. Moreover, there is a left action of $G$ on $P$ such that $g' \circ [(x, g, k)] = [(x, gg'^{-1}, k)]$ and $\psi([(x, g, k)]) = g'\psi([(x, g, k)])$.

3.4 Automorphic vector bundles

For an algebraic group $G$ defined over $\mathbb{Q}$, let $Z(G)$ denote its center, and $Z(G)_s < Z(G)$ denote the maximal connected subgroup which splits over $\mathbb{R}$. We define $G^c := G/Z(G)_s$.

A $G^c$-vector bundle $V$ on $\hat{X}$ is a vector bundle equipped with an equivariant $G^c$ action over $\hat{X}$. We may pull back $V$ to obtain a $G$-equivariant vector bundle $\psi^* V$ on $P$. This may then in turn be descended to obtain a vector bundle $W$ on $S_K(G, X)$. We call this the automorphic vector bundle. If $V$ is defined over a field $F \supset E(G, X)$ then $W$ has a canonical model over $F$ as well.

3.5 Rational representations

Let $\rho$ be a representation of $G^c$ acting on a vector space $V$ defined over $\mathbb{Q}$. This in turn induces a complex local system $\rho V_\mathbb{C}$ on $S_K(G, X)(\mathbb{C})$ as well as etale local systems $\rho V_\ell$ for each finite prime $\ell$. Fixing a $K$-stable lattice $V \subset V$ yields integral models $\rho V_\ell \subset {\rho} V_\ell$.

The Riemann-Hilbert correspondence yields a vector bundle with connection $\rho V$ on $S_K(G, X)$, which has a filtration $\Fil^*$, with each filtered piece being an automorphic vector bundle. Further, the data of $\rho V, \Fil^*$ descends to the reflex field. Moreover, by the work of Diao-Lan-Liu-Zhu [12, §4.1] we obtain algebraic vector bundles with connection $\rho_{dr}V_{E_\tau}$ equipped with a flat connection and filtration. By [12 Thm 5.3.1] these are naturally isomorphic in a way which is compatible with morphisms of Shimura data.

3.6 Canonical metrics at infinity

Let $S_K(G, X)$ be a Shimura Variety. Suppose that $\rho : G \to GL(V)$ is a rational representation of $G$. We pick a lattice $\mathbb{V}$ preserved by $K$. It follows by that $\rho \circ f_\mathbb{V}$ is a Hodge structure $V_\mathbb{C}$ on $V$ for any $x \in X$. By [32 Prop 3.2] it follows that $V$ admits a form $\psi : V \times \hat{V} \to \mathbb{Q}$ which becomes a polarization on $V_\mathbb{C}$ for all $x \in X$. Corresponding to such a polarization, we obtain a canonical metric on $\rho_{dr}V_{\mathbb{C}}$ on $S_K(G, X)$ by taking the Hodge norm (see [14, 6.6]), and this is compatible with pullbacks under maps of Shimura Varieties.

Now, following [32 II.4.2] for any element $\tau \in Gal(\mathbb{Q}/E(G, X))$ one may associate to $\tau S_K(G, X)$ the structure of a shimura variety $S_{K}(\tau G, \tau X)$ in a manner unique up to a canonical isomorphism. Following [32 III.6.2] one obtains canonical representation $V_{\tau}$ of $\tau G$ corresponding to $\tau V$ in a way compatible with the passage to the De-Rham vector bundles or the $\ell$-adic local systems. Moreover, conjugating $\psi$ gives us a polarization $\psi_{\tau}$. Thus we obtain canonical Hodge metrics on $\rho_{dr}V_{\mathbb{C}, \tau}$ as well.

3.7 Automorphic line bundles on tori

We shall require the following lemma, which tells us that for tori, there is essentially only one way given an automorphic line bundle to find it as a sub-bundle of the associated vector bundle corresponding to a local system. To see the relation recall that for tori, automorphic bundles are in bijection with complex representations of the torus, whereas local systems are in bijection with rational representations.

Lemma 3.2. The irreducible representations of a Torus defined over $k$ are in bijection with $G_k$-orbits on its character lattice $X^*(k^{\text{et}})$.

Proof. This is [36 Theorem 14.22]
3.8 Motives associated to Special points

Let \((T, h)\) denote a Shimura datum with \(T = T^c\) a torus. Let \(K = \prod_p K_p \subset T(\mathbb{A}_f)\) denote a neat compact open subgroup. Let \(V\) denote a \(\mathbb{Q}\)-representation of \(T\) with non-negative Hodge weights, and let \(\mathcal{V} \subset V\) denote a lattice stabilized by \(K\). We make crucial use of the following fact.

**Theorem 3.3.** The rational hodge structure associated to \(V\) is in the Tannakian category generated by complex CM abelian varieties.

**Proof.** This is the content of [33, Proposition 4.6].

Let \(x \in S_K(T, h)\) denote some point, and let \(E\) denote the field of definition of \(x\). The fiber of the local system \(e_t V_p\) at \(x\) yields a Galois representation \(\rho_p: \text{Gal}_E \to \text{GL}(\mathcal{V}_p)\). The following result asserts that the representations \(\rho_p\) are potentially crystalline for large enough primes \(p\).

**Proposition 3.4.** Let the setup be as above. Then, there exists a finite extension \(E_1 \supseteq E\) such that for \(p \gg 1\), \(\rho_p|_{\text{Gal}_{E_1}}\) is crystalline at all places of \(E_1\) dividing \(p\).

**Proof.** Without loss of generality, we assume that there is no subtorus \(T'' \subset T\) defined over \(\mathbb{Q}\) which contains the image of \(h\) (otherwise, we may just replace \(T\) by \(T''\)). We already have that the rational hodge structure associated to \(V\) is in the Tannakian category generated by some CM complex abelian variety \(A\). Let \(E'/E\) denote a number field over which \(A\) is defined, and has CM (and therefore has good reduction everywhere). It follows that \((T, h)\) is a quotient of the Shimura variety \((T', r')\) corresponding to \(H^1(A)\). We can therefore reduce to the case of \((T', r')\), and we work over the field \(E'\). The rational representation \(V\) of \(T'\) is a direct summand of some tensor power of \(H^1(A)\), and therefore the \(\mathbb{Z}_p\)-valued Galois representations \(e_t V_p\) is a direct summand of the same tensor power of \(H^1_{\text{et}}(A, \mathbb{Z}_p)\) for large enough primes \(p\). The crystallinity now follows from the fact that \(A\) has good reduction modulo primes above \(p\) and that we may choose \(p\) such that \(p - 2\) is larger than the hodge-weights of \(V\) (which we have also assumed to be non-negative).

4 Good Reduction of CM Points

The purpose of this section is to show that all CM points in a Shimura variety are integral with respect to some fixed model. Ideally, we would show this for all primes, even the finitely many ‘bad ones’. However, we have been unable to show this, settling instead for handling almost all primes. This is sufficient for our purposes, but somewhat unsatisfying, so we leave this open as a question, to which we strongly expect the answer ‘yes’:

**Question:** Given a Shimura Variety, is there an integral model with respect to which all CM points are integral?

As such, our main theorem is as follows:

**Theorem 4.1.** Let \(S = S_K(G, X)\) be a Shimura variety defined over a number field \(F\). For some positive integer \(N\), there exists an integral model \(\mathcal{S}\) over \(\mathcal{O}_F[N^{-1}]\) such that every CM point of \(S(\overline{\mathbb{Q}})\) extends to a point \(\mathcal{S}(\overline{\mathbb{Z}}[N^{-1}])\).

4.1 Punctured formal neighborhoods

**Lemma 4.2.** Let \(R\) be a regular local ring of mixed characteristic \(p\), and consider \(S = R[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \left[\frac{1}{y_1}, \ldots, \frac{1}{y_n}\right]\). Let \(R_0, S_0\) denote the maximal prime-to-\(p\) Galois etale extension of \(R, S\) respectively. Then \(S_0\) is generated by \(R_0\) and the prime-to-\(p\) roots of the \(y_i\).
Proof. Without loss of generality we may assume \( R = R_0 \) by base-changing. Let \( W \) be a Galois etale extension of \( S \) of degree prime-to-\( p \). Let \( T \) denote the normal closure of \( R[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \) in \( W \). Now for each \( j \in \{1, \ldots, m\} \) we let \( R_j \) and \( T_j \) denote the localization of \( R[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \) and \( T \) at the prime ideals \((y_j)\) and \( \mathfrak{n}_j \), respectively, where \( \mathfrak{n}_j \) is some prime ideal of \( T_j \) sitting above \((y_j)\). Now \( T_j, R_j \) are discrete valuation rings. Let \( e_j \) denote the ramification degree and let \( e = \prod_j e_j \). Now let \( W' \) denote the compositum of \( W \) and the \( e \)'th roots of all the \( y_i \), and let \( T', T_j' \) be as before. Then by Abhyankar’s lemma \([\text{Tag 0BRM}]\) it follows that \( T' \) is unramified over \( R[[x_1, \ldots, x_n, y_1^\frac{1}{e}, \ldots, y_m^\frac{1}{e}]] \). Finally, etale covers of \( R \) correspond bijectively to etale covers of \( R[[x_1, \ldots, x_n, y_1^\frac{1}{e}, \ldots, y_m^\frac{1}{e}]] \) and thus \( T' = R[[x_1, \ldots, x_n, y_1^\frac{1}{e}, \ldots, y_m^\frac{1}{e}]] \). The claim is thus proven. \( \square \)

### 4.2 Specialization homomorphisms and setup

Let \( S = S_K(G, X) \) be a Shimura variety, and let \( \overline{\mathcal{S}} \) be a log-smooth compactification of \( S \) with \( D = \overline{\mathcal{S}} - S \). By blowing up further, we may and do ensure that the irreducible components of \( D \) are smooth, i.e. they have no self-intersections. By spreading out, we may form a model \((\overline{\mathcal{S}}, S, D) \) over \( O_E[\frac{1}{D}] \) for some large even integer \( N \) such that \( \overline{\mathcal{S}} \) is proper smooth and \( D \) is a normal crossings divisor of \( \overline{\mathcal{S}} \) over \( O_E[\frac{1}{E}] \).

We fix a faithful \( G_O \)-representation \( V \) with a lattice \( \mathcal{V} \) invariant under \( K \). By shrinking \( K \) we may assume that the action of \( K_2, K_3 \) on \( \mathcal{V}_2 = \mathcal{V} \otimes \mathbb{Z}_2 \) is trivial mod 4. We fix a CM point \( x \in S(\mathcal{Q}) \) and by possibly enlarging \( N \) assume that \( x \) is induced by an integral point of \( \mathcal{S} \), unramified outside of primes dividing \( N \).

The following lemma is essentially \([15] \text{ Prop. 3.1}]\):

**Lemma 4.3.** For all primes \( p \nmid N \), the local system \( e_t \mathcal{V}_2 \) extends to \( S_W(\mathcal{F}_p) \).

**Proof.** For a profinite group \( H \), denote by \( H' \) denote the prime-to-\( p \) quotient of \( H \). By \([29] \text{ Thm A.7}]\) we have that \( \pi'_1(S_{W}(\mathcal{F}_p)) \cong \pi'_1(S_{\mathcal{F}_p}) \).

It follows that we have a canonical direct product decomposition

\[
\pi'_1(S_{\mathcal{Q}_p}^{ur}) \cong \pi'_1(S_{\mathcal{Q}_p}) \times \pi'_1(\mathcal{Q}_p^{ur}). \tag{4.2.1}
\]

Indeed, we always have a corresponding exact sequence, and here we have a natural section \( \pi'_1(\mathcal{Q}_p^{ur}) \to \pi'_1(S_{\mathcal{Q}_p}^{ur}) \) given by the generic fiber of the maximal prime-to-\( p \) etale extension of \( S_W(\mathcal{F}_p) \). Moreover, the projections corresponding to equation \([12,21] \) can be realized via the maps from \( S_{\mathcal{Q}_p}^{ur} \) to \( S_{W(\mathcal{F}_p)}, \text{Spec } \mathcal{Q}_p^{ur} \) respectively.

Finally, via this decomposition, we see that a local system on \( S_{\mathcal{Q}_p}^{ur} \) extends to \( S_{W(\mathcal{F}_p)} \) if and only if the induced action of \( \pi'_1(\mathcal{Q}_p^{ur}) \) is trivial, which can be checked on any \( \mathcal{Q}_p^{ur} \) point of \( S \) which extends to a \( \mathcal{Z}_p^{ur} \) point of \( \mathcal{S} \). Indeed, this follows from \( \mathcal{Z}_p^{ur} \) having no etale extensions.

Now, by assumption, \( x \) is a point of \( \mathcal{S} \) which extends to \( \mathcal{S} \), so it is enough to check that \( e_t \mathcal{V}_{2, x} \) is unramified at \( p \). However, since \( x \) is a CM point and \( 2 \neq p \) we see that the image of inertia on \( e_t \mathcal{V}_{2, \mathcal{F}_p} \) is finite, and therefore is trivial since the monodromy of \( e_t \mathcal{V}_2 \) is contained in \( K_2 \) which is torsion-free by construction. This completes the proof. \( \square \)

The following lemma completes the proof of Theorem \([6]\)

**Lemma 4.4.** With the above notation, for every place \( v \nmid N \), every CM point of \( S_v \) extends to a point of \( S_v \).

---

6Note that Theorem \([11]\) is invariant under finite covers. 

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Proof. Let $y$ be such a CM point with field of definition $E' \supset E$ and consider its extension $y_0$ to $\mathbb{S}(E_v) = \mathbb{S}(\mathbb{O}_E)$. It corresponds to some place $w$ of $E'$. Let $M = E_w^\ur$.

Suppose for the sake of contradiction that $y_0|_p \in D$. Then we may pick a regular system of parameters such that $\hat{O}_{S,y_0|_p} \cong \mathcal{O}_{E,v}[[t_1, \ldots, t_n]][[s_1, \ldots, s_m]]$ with $D$ cut out by the $t_i$. Since $y_0$ reduces to $D$ we see that the $t_i$ pull back to elements of the maximal ideal of $\mathcal{O}_M$.

This induces a map

$$f : \mathcal{O}_M[[t_1, \ldots, t_n, s_1, \ldots, s_m]] \left[ \frac{1}{t_i}, \ldots, \frac{1}{t_n} \right] \to M.$$ 

Now by lemma 12, $f$ induces a map of prime-to-$p$ Galois Groups $G^{(p)}_M \to \prod_{\ell \neq p} \mathbb{Z}_\ell^*(1)$. Moreover, by the same lemma the image of this map is completely determined by the (positive) valuations of $f(t_i)$.

Now, by choosing an embedding $\iota : \mathcal{O}_M \to \mathbb{C}$ we obtain an induced map

$$\mathcal{O}_M[[t_1, \ldots, t_n, s_1, \ldots, s_m]] \left[ \frac{1}{t_i}, \ldots, \frac{1}{t_n} \right] \to \mathbb{C}[[t_1, \ldots, t_n, s_1, \ldots, s_m]] \left[ \frac{1}{t_i}, \ldots, \frac{1}{t_n} \right]$$

inducing an isomorphism of prime-to-$p$ Galois groups. Moreover, we may consider a map

$$F : \mathbb{C}[[t_1, \ldots, t_n, s_1, \ldots, s_m]] \left[ \frac{1}{t_i}, \ldots, \frac{1}{t_n} \right] \to \mathbb{C}[[t]] \left[ \frac{1}{t} \right]$$

defined by $F(s_i) = 0$, $F(t_i) = t^{M}(f(t_i))$ which thus has the same fundamental group image as $f$.

Since $\mathcal{V}_{G,M}$ is trivial, it follows that $F^*_{et} \mathcal{V}_2$ is also trivial. Now, since the $t_i$, $s_j$ are regular parameters, the map $F$ is the completion of an holomorphic map $F' : \Delta^* \to S(\mathbb{C})$ where $t$ is identified with a coordinate of $\Delta$ vanishing at the origin, and since the (profinite completion of the) topological fundamental group of $\Delta^*$ is naturally identified with the etale fundamental group of Spec $\mathbb{C}[[t]] \left[ \frac{1}{t} \right]$, it follows that $F^*_{et,B} \mathcal{V}_C$ is also trivial. However, $B_\mathcal{V}_C$ underlies a variation of Hodge structures with maximal domain of definition $S_\mathbb{C}$, and thus $F^*_{et,B} \mathcal{V}_C$ must have infinite monodromy around $0$ ([3]). This is our desired contradiction.

5 $p$-adic Local Systems on Shimura Varieties are crystalline

In this section, we use work of Esnault-Groechenig to prove results pertaining to the crystallinity of $p$-adic local systems on Shimura varieties of non-abelian type. We first set up required notation. Let $(G,X)$ be an adjoint $\mathbb{Q}$-simple Shimura datum, with reflex field $E(G,X)$. We also fix a neat compact open subgroup $K \subset G(\mathbb{A}_f)$. It is well known that for any integer $n$, there are only finitely many rigid vector bundles of rank $n$ with flat connection on $S_K(G,X)_\mathbb{C}$, and these must all be defined over some number field $E'$ which we may assume contains $E(G,X)$. We fix a log-smooth compactification of $S_K(G,X)$, which spreads out to such a compactification at almost all places and allows us to discuss crystallinity on almost all rigid fibers:

**Theorem 5.1.** Let $(p,V)$ denote a degree $n$ rational representation of $G$, and let $V$ be a lattice stable under $K$. Assume that the Hodge weights corresponding to $V$ are non-negative. Recall that $\mathcal{V}_p$ denotes the corresponding $\mathbb{Z}_p$-local system on $S_K(G,X)$. There exists a positive integer $N'$ such that the following holds: Let $p \nmid N'$ denote a prime, and let $v$ be a place of $E'$ dividing $p$. Then $\mathcal{V}_p|_{S_K(G,X)_{E'_v}}$ is a crystalline local system.

5.1 Rigidity of local systems on Shimura varieties of exceptional type

**Theorem 5.2** (Margulis). Every local system on an irreducible adjoint Shimura variety of non-abelian type is strongly cohomologically rigid, in the sense of Definition [4,1] Moreover, the category of local systems is semisimple.
Proof. Without loss of generality, we assume that the Shimura variety in question is associated to an (adjoint) \( \mathbb{Q} \)-simple group \( G \).

Firstly, we claim that \( G_{\mathbb{R}} \) has rank at least 2. There is a unique real Shimura datum for the group \( E_7 \), and the real form of \( E_7 \) corresponding to this Shimura datum has rank 3 (for instance, see [28, p. 28]). There are two real Shimura data for the group \( E_6 \), and the real forms of \( E_6 \) corresponding to these data both have rank 2 (see [28, p.32]). These are the only real Shimura data associated to exceptional groups \( G \). Finally, the only other non-abelian adjoint Shimura datum occurs when \( G \) is an orthogonal group, and \( G_{\mathbb{R}} \) has at least one non-compact factor isomorphic to \( \text{SO}(2, n) \) (see [11, §2.3] [28, p. 61]), whence \( G_{\mathbb{R}} \) clearly has rank \( \geq 2 \).

The fundamental group of the Shimura variety is \( \Gamma \), some arithmetic subgroup of \( G \), whence a rank \( m \) local system on the Shimura variety corresponds to an \( m \)-dimensional complex representation of \( \Gamma \). As the rank of \( G_{\mathbb{R}} \) is \( \geq 2 \), we may apply Margulis super-rigidity ([31, Thm.IX.15.ii]) to see that this representation is cohomologically rigid.

By combining Theorem A.22 and Theorem 5.2 we obtain the following result: Let \( (\rho, V) \) be as in Theorem 5.1 and denote the vector bundle with connection associated to \( \rho \).

**Corollary 5.3.** For every place \( v \) of \( K \) not dividing a sufficiently large integer \( N' \), there exists a positive integer \( f \) such that \( \mathrm{et} V_{p} \otimes W(F_{q}^{1/p}) |_{S_{K}(G,X)c_{v}} \) and therefore \( \mathrm{et} V_{p}^{1/p} |_{S_{K}(G,X)c_{v}} \) admits a descent to \( S_{K}(G,X)W(F_{q}^{1/p}) \) that is crystalline.

### 5.2 Crystallinity of \( p \)-adic local systems

We are now ready to prove the main result of this section.

**Proof of Theorem 5.4.** We first assume that \( p \) is unramified in \( E' \). Corollary 5.3 yields a crystalline local system \( W/S_{K}(G,X)W(F_{q}^{1/p}) \) with dimension at most \( p-2 \). The base-change to \( \mathbb{C}_{p} \) of \( W \) is isomorphic to \( V_{p}^{1/p} W(F_{q}) \). Without loss of generality, we assume that \( q \) is such that \( G_{W(F_{q})} \) is a split reductive group, and that \( F_{q} \) contains the residue fields of \( E' \) at all places dividing \( p \). We observe the following facts about algebraic representations of the reductive group \( G_{W(F_{q})}^{1/p} \):

1. Consider an irreducible representation of \( G_{W(F_{q})}^{1/p} \) with dimension at most \( p-2 \). The mod \( p \) reduction of any \( G_{W(F_{q})}^{1/p} \)-stable lattice yields an irreducible representation of \( G_{W(F_{q})} \) ([22]), and therefore there is a unique isomorphism class of \( G_{W(\mathbb{F}_{q})}^{1/p} \)-stable lattices.

2. Non-isomorphic irreducible representations of \( G_{W(F_{q})}^{1/p} \) with dimension at most \( p-2 \) have non-isomorphic reductions mod \( p \) ([22]).

3. Every algebraic representation of \( G_{W(F_{q})} \) of dimension at most \( p-2 \) is completely reducible ([19]).

In light of the above, we may decompose \( \rho \) into a direct sum \( \bigoplus_{i} V_{i} \) of \( G_{W(F_{q})}^{-}\)stable \( W(F_{q}) \)-local systems whose stalks are rationally irreducible as \( G_{W(F_{q})}^{-}\)-representations. Since \( G \) is adjoint, geometric monodromy is Zariski dense in \( G_{W(F_{q})}^{1/p} \) ([34, Lemma 5.13]). It follows that the \( V_{i} \otimes \mathbb{Q}_{p} \) are geometrically irreducible \( \mathbb{Q}_{p} \)-local systems. We shall show that all of the \( V_{i} \) are crystalline, and therefore that \( \rho \) is as well. Without loss of generality, we focus on \( V_{1} \).

Let \( V_{1}^{\prime} \) be the corresponding isotypic component of \( \rho \). Let \( W_{1}^{\prime} \) denote the image of \( V_{1}^{\prime} \) in \( W_{q}^{1/p} |_{S_{K}(G,X)c_{v}} \). Since the geometric fundamental group is normal in the entire fundamental group it follows that \( W_{1}^{\prime} \) descends to an (arithmetic) sub-local-system of \( W \). Here, we use that the Zariski closures of arithmetic and geometric monodromy of \( V^{1/p} \) is the same, and equals \( G_{W_{q}^{1/p}} \). We then apply Schur's lemma to obtain \( W_{1}^{\prime} \cong V_{1}^{\prime} \otimes \mathbb{L} \) where \( \mathbb{L} \) is a representation of absolute Galois group of \( E' \).

We claim first that \( \mathbb{L} \) is crystalline up to cyclotomic twist. To see this, we fix a special point \( x \) in the Shimura variety. After replacing \( N \) by a bigger integer and replacing \( q \) by a larger power of \( p \) if need be,
Proposition 5.4 implies that the fiber of $e_t V_p$ at $x$ is crystalline. Since $V_{1,x}$ is a direct summand it must also be crystalline. Now as $V^\vee_1(m)$ (with $m$ chosen such that the smallest Hodge-Tate weight is 0) is also crystalline by Theorem 2.6 & Lemma 2.10, we see that $L \otimes V_{1,x} \otimes V^\vee_1(m)$ is crystalline by Lemma 2.3 as long as $p$ is large enough relative to the Hodge-Tate weights and dimension of $V$. Now for $p > \dim V_1$ we have that $Z_p(m)$ is a direct summand of $V_{1,x} \otimes V^\vee_1(m)$, and thus $L(m)$ is a direct summand of $L \otimes V_{1,x} \otimes V^\vee_1(m)$, and hence is crystalline, as desired.

Similarly, it now follows that $V_1 \otimes L \otimes L^\vee$ is crystalline up to twist, and hence so is $V_1$ by Lemma 2.10.

Since $e_t V_q$ is a direct sum of its irreducible components, each of which is crystalline, the result follows.

\section{Adelically Metrized Bundles}

Throughout we work with the norm on $\bar{\mathbb{Q}}_p$, and all of its subfields such that $|p| = \frac{1}{p}$. Given a $\bar{\mathbb{Q}}_p$ vector space $V$, we work with norms $| \cdot |$ on $V$ such that $|\alpha v| = |\alpha||v|$ for all $\alpha \in \bar{\mathbb{Q}}_p$.

**Definition 6.1.** Let $F$ be a non-Archimedean local field, and let $X/F$ denote a rigid-analytic variety with a vector bundle $V$ on it. Following \cite{[6] 2.7.1}, we define a norm on $V$ to be a norm $| \cdot |_x$ on every fiber $V_x, x \in X(F)$. We say such a norm is *acceptable* if it is Galois-invariant, and for every affinoid $U \subset X$ on which $V$ is trivial, there exist sections $s_1, \ldots, s_n \in V(U)$ trivializing $V$ on $U$, and an integer $N$ such that for every $u \in E$ we have

1. $\log |s_i(u)|_u \leq N$
2. $p^N \cdot \{ v \in V_u : |v|_u \leq 1 \} \subset \bigoplus_{i=1}^n \mathcal{O}_F \cdot s_i(u)$

**Lemma 6.2.** If $U$ is an affinoid, $V, s_i$ is a vector bundle with a set of trivializing section and $| \cdot |$ is a norm such that the $s_i$ satisfy the conditions above, then any trivializing sections $t_1, \ldots, t_n$ satisfy the two conditions above.

**Proof.** Let $g \in \Gamma(U, GL(\mathcal{O}_U))$ be the matrix such that $gs = \tilde{t}$. Note that $g, g^{-1}$ have entries that are elements of $\mathcal{O}_U(U)$ and therefore are globally bounded. It follows that the norms of $\tilde{t}$ are uniformly bounded by those of $s$, and also that the two lattices $\mathcal{O}_F \cdot s_i(u)$ and $\mathcal{O}_F \cdot t_i(u)$ are comparable uniformly for $u \in U$. The claim follows.

**Lemma 6.3.** Let $X/F, V, | \cdot |$ be as in Definition 6.1. In order to check that the norm is acceptable, it suffices to check this on some finite affinoid cover.

**Proof.** Let $X = \bigcup_{j=1}^J U_j$ denote an affinoid cover, such that there are trivializing sections $s_{i,j}$ of $V|_{U_j}$ which satisfy the conditions of Definition 6.1. Let $U'$ denote any affinoid on which $V$ is trivial, and let $t_1 \ldots t_n$ denote a set of trivializing sections of $V|_{U'}$. Define $U'_j = U' \cap U_j$. By Lemma 6.2 that the $\{ t_i \}|_{U'_j}$ satisfy the conditions of Definition 6.1 follows from the fact that the $\{ s_{i,j} \}|_{U'_j}$ do. The lemma follows.

Given an integral model $\mathcal{Y}/\mathcal{O}_F$ and vector bundle $V$ on $Y$ there is a natural norm on the fiber of analytification $\mathcal{Y}_v$, which is easily seen to be acceptable. We call this the $\mathcal{V}$-norm on $V$. We now come to our central definition:

**Definition 6.4.** Let $X$ be a proper variety over a number field $F$, $V$ a vector bundle on $X$, and $Y$ an open subscheme of $X$. We define an *admissible collection of norms* on $V$ to consist of the following data:

1. An integral model $\mathcal{Y}$ of $Y$ over $\mathcal{O}_F[N^{-1}]$
2. A vector bundle $\mathcal{V}$ on $\mathcal{Y}$ extending $V$.
3. For each infinite place $v$, a continuous metric $h_v$ on $V_v | Y$. 

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4. For each finite place \( v \mid N \), a norm \( \cdot \mid_v \) on \( V_v \mid Y_v \) which extends to an acceptable norm on \( X \).

5. For each finite place \( v \mid N \), an acceptable norm \( \cdot \mid_v \) on \( V \mid Y_v \) such that for almost all finite places it is the \( V_v \)-norm.

Abusing notation somewhat, we refer to \((V, (\cdot \mid_v)_v)\) as an \textit{admissible normed vector bundle} on the triple \((X,Y,V)\).

\textbf{Lemma 6.5.} For a vector bundle \( V \) on a proper variety \( X/F \) with open subscheme \( Y \), any two admissible collections of norms on \((X,Y,V)\) agree at almost all finite places, and differ by \( O(1) \) at every finite place.

\textit{Proof.} Any two integral models agree at almost all places, which implies the corresponding norms agree. As for the second claim, it suffices to prove that two acceptable norms differ by a uniform \( O(1) \).

It is sufficient to work over affinoids since every proper variety is covered by finitely many such, on which the vector bundle is trivial.

So let \( U \) be an affinoid and \( \mathcal{S} \) be any set of trivializing the vector bundle \( V \). The claim immediately follows by property 2 of definition 6.1. \( \square \)

The following lemma is straightforward:

\textbf{Lemma 6.6.} The restriction of an acceptable (resp. admissible) norm to a sub-vector bundle is admissible (resp. admissible). Likewise for the induced norm on a quotient bundle.

Note that if the vector bundle is a line bundle, then an acceptable collection of norms is - when \( X = Y \) - nothing other than an \( \mathbb{M} \)-metric \( [\mathbb{M}] \) on our line bundle. Analogously to that context, we have an associated height function for points \( P \in \mathcal{Y}(\mathcal{O}_F[N^{-1}]) \) given by taking a non-vanishing section \( s \) of \( V \) not vanishing at \( P \) and defining

\[ h_{\mathcal{Y},(\cdot \mid_v),v\in \mathcal{M}_F}(P) := -\sum_v \frac{[F_v : \mathbb{Q}_p]}{[F : \mathbb{Q}]} \log |s(P)|_v. \]

It follows from Lemma 6.6 that up to an additive \( O(1) \) the height depends only on the triple \((X,Y,V)\).

\section{Canonical Heights and Admissible Metrics}

\subsection{Metrizing \( D_{HT} \)}

Let \( K \) be a finite extension of \( \mathbb{Q}_p \). We let \( G_K \) be the absolute Galois group of \( K \). We let \( \mathbb{C}_p \) be the completion of \( K \), and we let \( B_{HT} := \oplus_i \mathbb{C}_p(i) \) where we think of \( B_{HT} \) as a \( \mathbb{C}_p \)-vector space with a semi-linear \( G_K \)-action. We metrize \( \mathbb{C}_p(i) \) by identifying \( \mathbb{C}_p(i) \) with \( \mathbb{C}_p \) with the galois action on 1 being through the cyclotomic character, and then pulling back the metric on \( \mathbb{C}_p \) under this identification. See [7] for background on \( p \)-adic Hodge theory.

We let \( V/\mathbb{Q}_p \) be a finite dimensional \( G_K \)-representation, and define \( D_{HT}(V) := (V \otimes B_{HT})^{G_K} \). We assume that \( V \) is Hodge-Tate, which means that \( \dim D_{HT}(V) = \dim V \).

In integral \( p \)-adic Hodge theory, one typically fixes lattices in \( V \) and attempts to define a lattice in \( D_{HT}(V) \). For our purposes it will be more convenient to work with norms, which record more information since they may not be \( p \)-valued. Thus, let assume that \( (\cdot \mid \cdot) \) is a \( p \)-adic norm on \( V \) which is invariant under \( G_K \). We call such a \( (V, (\cdot \mid \cdot)) \) a \textit{metrized \( G_K \)-representation}, and say it is Hodge-Tate if \( V \) is Hodge-Tate (likewise for de Rham, crystalline,...)

We shall construct a norm on \( D_{HT}(V) \) (in fact on all graded pieces of it) in a way which will works well in families:

\textbf{Definition 7.1.} Let \((V, (\cdot \mid \cdot))\) be a metrized \( G_K \)-representation which is Hodge-Tate. For each integer \( n \), let \( V_n = (V \otimes \mathbb{C}_p(n))^G_k \otimes \mathbb{C}_p \). There is a natural map \( V_n \mid (\cdot \mid \cdot) \to V \otimes \mathbb{C}_p \) and we let \( V_n \) be the image of this.
map. Finally let \( V_{<n} := \bigoplus_{m<n} V_m \). Now \( V_{<n} \) has an induced norm, and thus we may equip \( V_{<n} \otimes \mathbb{C}_p \) with the quotient norm. This then yields a norm on

\[
V_{<n} \otimes \mathbb{C}_p \otimes \mathbb{C}_p(n).
\]

Note that this latter space is isomorphic to \( V^n \) and we endow it with the corresponding norm. We call it the intrinsic metric on \( V\mathbf{n} \) and likewise on \( D_{HT}(V) \). We call the corresponding norm \( \leq 1 \) set on \( V^n \) the intrinsic lattice.

Recall that \( \text{Gr} B_{\text{an}} \cong B_{HT} \) as rings with Galois-actions. This isomorphism is unique up to an element of \( \mathbb{Q}_p^* \). We choose an isomorphism where the image of \( t \pmod{\text{Fil}^2 B_{\text{an}}} \) is sent to the element \( 1 \in \mathbb{C}_p(1) \). Note that this means \( p \pmod{\text{Fil}^2 B_{\text{an}}} \) also maps to \( 1 \) in \( \mathbb{C}_p(1) \).

Now let \( \mathcal{V} \) be a crystalline representation. Then by Theorem 7.2 we obtain a map \( \text{Gr} S_{\text{cris}}(\mathcal{V}) \to \text{Gr} D_{\text{an}}(\mathcal{V}) \cong D_{HT}(\mathcal{V}) \). The image of the crystalline lattice defines another norm, which we call the crystalline norm.

In the next subsection, we will define the intrinsic norm in the case of families, and show that the intrinsic and crystalline norms are comparable for crystalline local systems. One advantage to carrying out the argument in families is it allows us to deal with points defined over ramified fields by specializing, without having to directly confront the Faltings-Fontaine-Laffaille theory in that context (for example, Tsuji\[48\] insists the argument in families is it allows us to deal with points defined over ramified fields by specializing, without

\[ D_{HT}(V) \]

\[
\hat{\chi} = \hat{\chi}(\mathbf{n}) \mod \text{Fil}^2 B_{\text{an}}.
\]

Therefore, the crystalline lattice is the intrinsic lattice.

Proof of Theorem 7.2. Let \( K \) denote an unramified extension of \( \mathbb{Q}_p \) and suppose that \( \mathcal{V} \) is a crystalline \( \mathbb{Z}_p \)-representation of \( \text{Gal}_K \) (in the sense of Fontaine-Laffaille) with non-negative Hodge Tate weights \( \leq a \), such that the sum of all the weights is \( \leq p - 2 \). Then the intrinsic norm on \( \text{Gr} D_{\text{an}}(\mathcal{V}) \) agrees with the crystalline norm (induced by the image of \( \text{Gr} S_{\text{cris}}(\mathcal{V}) \)) upt to a multiple of \( p^{\frac{1}{1-p}} \).

We first prove this result for 1-dimensional representations, in which case the conclusion is that the two norms exactly agree.

Lemma 7.3. Let the the setting be as above, with the further condition that \( \mathcal{V} \) is a one-dimensional representation. Then, the two norms agree.

Proof. After replacing \( K \) with \( W(\mathbf{F}_p)[1/p] \), the Galois representation being crystalline must be of the form \( \chi^{-a} \) with \( 0 \leq a \leq p - 2 \). Recall the element \( t \in \text{Fil}^2 B_{\text{an}} \), on which the galois action is through the cyclotomic character (\( t \) is unique up to scaling by \( \mathbb{Z}_p^* \)). For brevity, we will use the term intrinsic lattice to denote the norm \( \leq 1 \) elements of \( D_{HT}(\mathcal{V}) \). It suffices to prove that this equals the crystalline lattice.

We first compute the intrinsic lattice. Let \( t^n \) also denote the image of \( t^n \pmod{\text{Fil}^n B_{\text{an}}} \in \mathbb{C}_p(n) \). We have that \( D_{HT}(\mathcal{V}) = \{ e \otimes t^n z : z \in \mathbb{C}_p \} \subset L \otimes \mathbb{C}_p(n) \) where \( e \) is a generator of \( L \). Therefore, the intrinsic lattice consists of elements \( \{ e \otimes t^n z : z \in W(\mathbf{F}_p)[1/p] \} \) with \( t^n \cdot z \in O_{\mathbb{C}_p} \), where we identify \( t^n \cdot z \) with \( \theta(t^n z) \in \mathbb{C}_p \). As \( \frac{1}{p} \to 1 \) under \( \theta \), we see that \( z \in W(\mathbf{F}_p) \).

We now compute the crystalline lattice. Note that the crystalline lattice is defined by considering the image of \( S_{\text{cris}}(L) \subset D_{HT}(L) \). To compute \( S_{\text{cris}}(L) \), observe that \( T A_{\text{inf}} \) (as in Section 2) is \( L \otimes \pi^n A_{\text{inf}} \) (this follows from \[48\] Lemma 7.5) and the defining property of \( T A_{\text{inf}} \). Therefore, the crystalline lattice is the image of \( (L \otimes \pi^n A_{\text{cris}})^{\text{Gal} = 1} \) in \( (L \otimes \pi^n B_{\text{an}})^{\text{Gal} = 1} \). This shows that the crystalline lattice consists of elements of the form \( \{ e \otimes z t^n : z \in W(\mathbf{F}_p)[1/p] \} \) with \( z t^n \in \pi^n A_{\text{cris}} \). As \( \frac{1}{p} \) is a unit in \( A_{\text{cris}} \), we see that \( z \in W(\mathbf{F}_p) \), and hence the two lattices are the same.

We now prove the higher rank case.

Proof of Theorem 7.2. We have that \( S_{\text{cris}}(\mathcal{V}) \subset \mathcal{V} \otimes A_{\text{cris}} \). Note that for \( 0 \leq i \leq p - 2 \), \( \text{Gr}^i(A_{\text{cris}}) \cong \xi^i O_{\mathbb{C}_p} \).

Therefore, the image of \( \text{Gr}^i S_{\text{cris}}(\mathcal{V}) \) in \( \text{Gr}^i D_{\text{an}}(\mathcal{V}) \) is contained in the Galois invariants of \( \mathcal{V} \otimes \xi^i O_{\mathbb{C}_p} \). As we have metrized \( \text{Gr}^i B_{\text{an}} \) with the convention that \( t^i \) has norm 1, it follows that \( \text{Gr}^i S_{\text{cris}}(\mathcal{V}) \) is contained in

\[
\mathcal{V} \otimes \xi^i O_{\mathbb{C}_p} \].
\]
Theorem 7.4. (see [16] Page 29), the crystalline lattice is contained in \( p^{-\infty} \) times the intrinsic lattice.

To obtain the other inclusion, note that the intrinsic norm and crystalline norm are both compatible with tensor and wedge powers, the former by construction as a subquotient and the latter by Lemma 2.3.

We thus get two \( \mathcal{O}_C_p \) lattices \( A, B \) such that \( A \subset p^{-\infty} B \) and \( \det A = \det B \).

Let \( b_1 \in B \) be primitive and such that \( p\nu b_1 \in A \) is primitive, and complete it to a basis \( p\nu b_1, a_2, \ldots, a_{\dim V} \) of \( A \). Then since \( \det A = \det B \) we have

\[
1 \leq |p\nu b_1|_B \prod_{i=2}^{\dim V} |a_i|_B \leq p^{-c} \frac{p^{(\dim V - 1)}}{p-1}
\]

from which the desired inequality \( c \leq \frac{p^{(\dim V - 1)}}{p-1} \) follows.

\[\square\]

7.2 Metrizing Hodge-Tate metrics in families

We will borrow heavily from [30], [12], and [45]. Let \( X \) be a smooth geometrically connected rigid-analytic variety over \( k \). In the case where one exists, we let \( \hat{X} \) denote a partial compactification of \( X \) such that \( D := X - \hat{X} \) is a smooth normal crossings divisor. In this setting, We equip \( \hat{X} \) with the structure of a smooth log-adic space in the natural way. We will assume that \( L \) is a \( \mathbb{Z}_p \)-local system on \( X_{et} \) which is de Rham in the sense of [15], and such that the local geometric monodromy around the components of \( D \) is unipotent.

We shall work with the Pro-etale site \( X_{proet} \) and the Pro-Kummer-etale site \( \hat{X}_{proket} \). We shall denote by \( L_{ket} \) the extension of \( L \) to \( \hat{X}_{ket} \), which is also a \( \mathbb{Z}_p \)-local system.

Let \( \hat{O}_X \) and \( \hat{O}_{\hat{X}} \) be the completed structure sheaf on \( X_{proet}, \hat{X}_{proket} \) respectively. Let \( \hat{L} \) be the induced sheaf of local systems on \( X_{proet} \) and likewise let \( \hat{L}_{ket} \) be the sheaf on \( \hat{X}_{proket} \). Finally we set \( \hat{L} := \hat{L} \otimes \mathbb{Z}_p \hat{O}_X \), and \( \hat{L}_{ket} := \hat{L}_{ket} \otimes \mathbb{Z}_p \hat{O}_{\hat{X}} \). These are locally free sheaves on \( X_{proet}, \hat{X}_{proket} \) (over their respective structure sheaves). Let \( \nu : X_{proet} \to X_{et}, \nu_{ket} : \hat{X}_{proket} \to \hat{X}_{et} \) be the natural maps. Note that \( \hat{L}, \hat{L}_{ket} \) have natural norms at every classical point of \( X, \hat{X} \) which agree on \( X \).

We will now define an ascending filtration on \( \hat{L}, \hat{L}_{ket} \), and will use this filtration to metrize the graded pieces of \( D^0_{DR}(L) \). We work only in the Kummer case as it is more general, though it is unnecessary (and we will drop it) when we work in settings without log-structures (or, equivalently, trivial log-structures).

First, recall that by combining the second displayed equation in the proof of [12] Lemma 3.3.17, the first equation in the proof of [12] Cor 3.4.22, and the isomorphism at the bottom of page 36 of [12], we obtain

\[
\bigoplus_r \nu_{k,et}^* (\hat{L}_{ket} \otimes \mathcal{O}_C \log (r)) \otimes_{\mathcal{O}_C} \mathcal{O}_X \otimes \mathcal{O}_C \log (-r) \cong \hat{L}_{ket} \otimes \mathcal{O}_C \log.
\]

Locally on \( X \), we may pick a smooth toric chart with co-ordinates \( x_1, \ldots, x_m \). Recall also that we have [12] 2.3.17]

\[
\mathcal{O}_C \log \cong \hat{O}_X \left[ \frac{y_1}{l}, \ldots, \frac{y_m}{l} \right]
\]

where the \( y_i \) are as in [12] 2.3.6].

Assume the weights for \( L \) are in \([0, m]\). We have the following important structural theorem(see also [21] §5):

**Theorem 7.4.** Let \( L \) be as above. Then:

8To those unfamiliar, the point is that \( L \) is an inverse limit of local systems with finite fibers. On each of those, monodromy is finite and can therefore be unwound by taking a sufficiently large Kummer extension.
1. For $r \geq 0$ the elements of $\nu_{k\text{et},*}(\hat{L}_{k\text{et}} \otimes \mathcal{O}_{\log}(r))$ have degree (in the variables $y_i$) $\leq r$ under the isomorphism (7.2.2).

2. For $r \geq 1$, the positive degree monomials of $\nu_{k\text{et},*}(\hat{L}_{k\text{et}} \otimes \mathcal{O}_{\log}(r))$, lie in the $\mathcal{O}_{\log}$-span of $\nu_{k\text{et},*}(\hat{L}_{k\text{et}} \otimes \mathcal{O}_{\log}(s))(r - s)$ for $0 \leq s < r$, when pulled back via $\nu_{k\text{et}}^*$ to $\hat{L}_{k\text{et}} \otimes \mathcal{O}_{\log}(r)$ under the identification (7.2.2).

Proof. 1. We proceed by strong induction on $r$. Consider the induced map
\[
\nabla_r : \nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log}(r)\right) \to \nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log} \otimes \mathcal{O}_{\tilde{X}^\nu} \Omega_{X}^{\log}(r - 1)\right)
\]
given by [12, 2.4.2(4)]. Now $\Omega_{X}^{\log}$ is locally free and so
\[
\nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log} \otimes \mathcal{O}_{\tilde{X}^\nu} \Omega_{X}^{\log}(r - 1)\right) \cong \nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log}(r - 1)\right) \otimes \mathcal{O}_{\tilde{X}^\nu} \Omega_{X}^{\log}
\]

Therefore by induction, the image consists of elements of degree $\leq r - 1$. We now finish with the claim that if $\alpha \in \mathcal{O}_{\log}$ and $\nabla(\alpha)$ has degree $\leq r - 1$ then $\alpha$ has degree $\leq r$. Indeed, writing
\[
\alpha = \sum_{h \in \mathbb{Z}_{\geq 0}} a_h \left(\frac{y}{t}\right)^h
\]
we compute that
\[
\nabla \alpha = \sum_{h \in \mathbb{Z}_{\geq 0}} a_h \sum_{1 \leq i \leq m} \left(\frac{y_i}{t}\right)^{h - e_i} \delta(y_i)
\]
where we have used the notation in [12, §2.4]. Now since $\Omega_{X}^{\log}$ is locally free over $\mathcal{O}_{\tilde{X}^\nu}$ with generators $\delta(y_i)$ (see proof of [12, 2.4.2]) the claim follows.

2. We again proceed by induction. As above, applying $\nabla_r$ to an element $f$ of $\nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log}(r)\right)$ gives an element $\nabla_r(f)$ of $\nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log}(r - 1)\right) \otimes \mathcal{O}_{\tilde{X}^\nu} \Omega_{X}^{\log}$. By induction, all the positive degree monomials of this element lie in the $\mathcal{O}_{\log}$-span of $\nu_{k\text{et},*}(\hat{L}_{k\text{et}} \otimes \mathcal{O}_{\log}(s))(r - 1 - s) \otimes \mathcal{O}_{\tilde{X}^\nu} \Omega_{X}^{\log}$ for $0 \leq s < r - 1$, and therefore the constant term is in the $\mathcal{O}_{\log}$-span of $\nu_{k\text{et},*}(\hat{L}_{k\text{et}} \otimes \mathcal{O}_{\log}(s))(r - 1 - s) \otimes \mathcal{O}_{\tilde{X}^\nu} \Omega_{X}^{\log}$ for $0 \leq s \leq r - 1$.

Now by the computation in part 1, each positive-degree monomial in $\alpha$ can be extracted from the monomials of $\nabla_r(\alpha)$ by taking the projection operators for $\Omega_{X}^{\log}$ with respect to the basis $\delta(y_i)$, and the claim follows.

We now define $\mathcal{T}_{< r} \subset \mathcal{T}_{k\text{et}}$ to be the generated by all coefficients of the constant terms of $\nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log}(s)\right)(-s)$ for $s < r$.

It follows from Theorem 7.4 that
\[
\mathcal{T}_{< r} \otimes \mathcal{O}_{\log} \cong \bigoplus_{s < r} \nu_{k\text{et}}^* \nu_{k\text{et},*}\left(\hat{L} \otimes \mathcal{O}_{\log}(s)\right)(-s) \otimes \mathcal{O}_{\tilde{X}^\nu} \mathcal{O}_{\log}
\]
and consequently by (7.2.2) that $\mathcal{T}_{< r}$ is a locally split filtration of $\mathcal{T}_{k\text{et}}$. Finally, we define $\mathcal{T}_n := \mathcal{T}_{< n+1}/\mathcal{T}_{< n}$. 21
Proposition 7.5. We have a natural isomorphism
\[ \nu_{\text{kct}}^*(\tilde{L} \otimes \mathcal{O}_{\log}(n)) \otimes \nu_{\text{kct}}^* \hat{\mathcal{O}}_{Y}(-n) \cong \hat{L}_n. \]

Proof. First, note there is a natural map
\[ \psi: \nu_{\text{kct}}^*(\tilde{L} \otimes \mathcal{O}_{\log}(n)) \otimes \nu_{\text{kct}}^* \hat{\mathcal{O}}_{X}(-n) \to \tilde{L} \otimes \mathcal{O}_{\log}. \]

The image of \( \psi \) is actually in \( \tilde{L}_{<n+1} \otimes \mathcal{O}_{\log} \) by (7.2.3) and therefore there is (etale locally on \( X \)) a map
\[ \phi: \nu_{\text{kct}}^*(\tilde{L} \otimes \mathcal{O}_{\log}(n)) \otimes \nu_{\text{kct}}^* \hat{\mathcal{O}}_{X}(-n) \to \tilde{L}_n \otimes \mathcal{O}_{\log} \]
given by composing \( \psi \) with the quotient map. Using the isomorphism (7.2.2) and part 2 of Theorem 7.4 we conclude that the image of \( \phi \) is in fact simply \( \tilde{L}_n \), which gives us our desired map.

It remains to show this map is an isomorphism. But it becomes one when tensored with \( \mathcal{O}_{\log} \) by (7.2.3) and so the claim follows since the \( \mathcal{O}_{\log} \) is faithfully flat (in fact free) over \( \mathcal{O}_{\mathcal{X}} \).

Given the theorem, we make the following definition.

Definition 7.6. We endow \( \text{Gr}^n \mathcal{D}_\text{dr} (L) = \nu_{\text{kct}}^*(\tilde{L}_{\text{kct}} \otimes \mathcal{O}_{\log}(n)) \) with the norm coming from \( \tilde{L}_n \), induced from the quotient norm on \( \tilde{L} \), and we call this the intrinsic norm.

Lemma 7.7. The intrinsic norms on \( \nu_{\text{kct}}(\tilde{L} \otimes \mathcal{O}_{\mathcal{C}}(n)) \) and \( \nu_{\text{kct}}^*(\tilde{L}_{\text{kct}} \otimes \mathcal{O}_{\log}(n)) \) are acceptable, (according to definition 6.1).

Proof. We deal with the \( \overline{X} \) case, the other case being identical. By Lemma 6.3, it suffices to check the conditions of acceptability with respect to any one finite affinoid cover. Therefore, we may assume that all the graded pieces \( \text{Gr}^i \mathcal{D}_\text{dr} (L) \) are trivial vector bundles.

We work with a subaffinoid \( U \subset \overline{X} \) and a perfectoid cover \( \tilde{U} = \lim_{\leftarrow i} U_i \) on which \( L_{\text{kct}} \) trivializes. We may then pick a basis \( e_j \) for \( L_{\text{kct}} \) and a basis \( l_i \) for \( \tilde{L}_{\text{kct}} \) compatible with the filtration \( \tilde{L}_{<n,\text{kct}} \). Let \( t_{i,j} \in \tilde{\mathcal{O}}_{\overline{X}}(\tilde{U}) \) be the coefficients of the \( l_i \) w.r.t the \( e_j \). That the norms of the \( t_{i,j} \) are bounded above is trivial, hence condition 1 of definition 6.1 is satisfied.

For condition 2, note that the norm 1 lattice in the quotient is the image of the norm 1 lattice in the source which is generated by the \( e_j \). Since the \( l_i \) are a basis, that means that they generate the \( e_j \) over \( \tilde{\mathcal{O}}_{\overline{X}}(\tilde{U}) \). The denominators of the coefficients are bounded, which immediately yields condition 2.

The intrinsic norm is completely functorial, as per the following result:

Proposition 7.8. Let \( X \) be a smooth rigid analytic variety, and let \( L \) be a \( \mathbb{Z}_p \)-local system on \( X_{\text{et}} \) which is de Rham. Let \( \phi: Y \rightarrow X \) be a map of smooth rigid analytic varieties. Then the intrinsic norm on
\[ \nu_{\phi}^*(\tilde{L} \otimes \mathcal{O}_{X,\log}(n)) \otimes \nu_{\phi}^* \hat{\mathcal{O}}_{Y}(-n) \]
pulls back to the intrinsic norm on
\[ \nu_{\phi}^*(\tilde{L} \otimes \mathcal{O}_{Y,\log}(n)) \otimes \nu_{\phi}^* \hat{\mathcal{O}}_{Y}(-n). \]

Proof. Note that the natural map \( \phi^*(\nu_{\phi}^*(\tilde{L} \otimes \mathcal{O}_{X,\log}(n))) \rightarrow (\tilde{L} \otimes \mathcal{O}_{Y,\log}(n)) \) is an isomorphism by [33 Thm 2.1(3)]. It follows from comparing the direct sum decompositions (7.2.2) for \( X \) and \( Y \) that \( \phi^*\mathcal{T}_{n} \cong \phi^{-1}\mathcal{T}_{n} \) and thus that \( \phi^*\mathcal{T}_{n} \cong \phi^{-1}\mathcal{T}_{n} \). The claim follows.
7.3 Comparison of the two metrics

Now, suppose that $X$ has a smooth integral model $X'$ over $\mathcal{O}_k$ and let $L$ be a crystalline local system on the generic fiber of the formal completion $\hat{X}_{\text{gen}}$.

Then by Theorem 2.3 we get a natural map $S_{\text{cris}}(L) \to D^0_{\text{dR}}(L)$ which induces a map

$$\text{Gr } S_{\text{cris}}(L) \to \text{Gr } D^0_{\text{dR}}(L) \cong D_{HT}(L) = \bigoplus_n \nu_*(\hat{L} \otimes \mathcal{O}(n)).$$

We thus obtain a canonical lattice in $L^\ast_{\text{gen}}$, called the crystalline lattice. We define the corresponding norm the crystalline norm.

Our goal is to show that the crystalline and intrinsic norms are comparable.

**Theorem 7.9.** Let $X$ be a smooth geometrically connected scheme over $\mathcal{O}_k$, and let $L$ a $\mathbb{Z}_p$-Crystalline local system on $\hat{X}_{\text{gen}}$. Assume further that the weights of $L$ are contained in $[0, a]$ where $a \dim L \leq p - 2$. Then the crystalline norm and the intrinsic norm associated to $L$ are comparable up to $p^{\frac{a}{p-1}}$.

**Lemma 7.10.** Let $L$ denote a crystalline local system with all Hodge-Tate weights $0$. Then, the crystalline and intrinsic norms agree.

**Proof.** We will first show that the crystalline lattice is contained in the norm $\leq 1$ subset of the intrinsic norm. For brevity, call the latter subset the intrinsic lattice.

As $0$ is the only Hodge-Tate weight, we have that $D^0_{\text{dR}}(L) = \nu_*(\hat{L} \otimes \hat{\mathcal{O}}_X)$. Further, we have $S_{\text{cris}}(L) \subset (L \otimes \mathcal{G}^0_{\text{cris}})^{\text{Gal}=1} = (L \otimes \overline{\mathcal{R}})^{\text{Gal}=1}$. It follows that the image of $S_{\text{cris}}(L)$ in $D^0_{\text{dR}}(L)$ is contained in $L \otimes \overline{\mathcal{R}}$, and therefore the crystalline lattice is contained in the intrinsic lattice.

To finish the lemma, it suffices to prove the other containment. In particular, it suffices to prove that the crystalline and intrinsic lattices agree for the determinant of $L$ (as both norms and therefore lattices are compatible with tensor products, see Lemma 2.3). Therefore, we assume that $L$ is a crystalline rank-1 local system with Hodge-Tate weight $0$. By [16, Theorem 2.6, h)], the dual local system $L^\ast$ of $L$ is also crystalline. As $L^\ast$ (and $L \otimes L^\ast$, which is the trivial local system) also has Hodge-Tate weight $0$, we have that the crystalline lattices for $L, L^\ast$ and $L \otimes L^\ast$ are each contained in the intrinsic lattices for each local system.

As $L \otimes L^\ast$ is the trivial local system, the two lattices clearly agree for $L \otimes L^\ast$. It follows from lemma 2.3 that the two lattices therefore must agree for both $L$ and $L^\ast$. The lemma follows.

**Lemma 7.11.** Let $L$ denote a rank 1 crystalline local system with Hodge-Tate weight $0 \leq n \leq p - 2$. Then the intrinsic norm is the same as the crystalline norm.

**Proof.** Consider the local system $L^\prime = L \otimes \chi^n$. This is crystalline with Hodge-Tate weight $0$, and so the intrinsic and crystalline norms agree for $L^\prime$.

We identify $L, L^\prime$ as rank-1 $\mathbb{Z}_p$-modules (with the Galois action differing by a twist), and therefore also identify $L \otimes A$ with $L^\prime \otimes A$ with $A = \mathcal{A}_{\text{inf}}(\overline{\mathcal{R}})$ or $\mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$. We choose a generator $e$ of $L$. By [16, Lemma 7.5], we have that $T\mathcal{A}_{\text{inf}}(L) = \pi^n T\mathcal{A}_{\text{inf}}(L^\prime)$. We view $T\mathcal{A}_{\text{inf}}(L^\prime), T\mathcal{A}_{\text{inf}}(L)$ as subsets of $L^\prime \otimes \mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$. We claim that $e \otimes f$ (with $f \in \mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$) is an element of $S^\ast_{\text{cris}}(L^\prime)$ if and only if $e \otimes t^n f$ is an element of $S^\ast_{\text{cris}}(L)$. In other words, we claim that $e \otimes f$ is in the $\mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$-span of $T\mathcal{A}_{\text{inf}}(L^\prime)$ if and only if $e \otimes t^n f$ is in the $\mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$-span of $T\mathcal{A}_{\text{inf}}(L)$. This follows from the fact that $\frac{1}{\pi} \in \mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$ is a unit. The claim follows from the easy observation that $e \otimes f \in L \otimes \mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$ is Galois invariant if and only if $e \otimes t^n f \in L \otimes \mathcal{A}_{\text{cris}}(\overline{\mathcal{R}})$ is.

Therefore, the crystalline norm of $e \otimes f \in S^\ast_{\text{cris}}(L^\prime)$ is the same as the crystalline norm of $e \otimes t^n f \in S^\ast_{\text{cris}}(L)$.

We now calculate the intrinsic norms. By Lemma 7.10, we have that $S^\ast_{\text{cris}}(L^\prime)$ generates the intrinsic lattice in $D^0_{\text{dR}}(L^\prime)$. Again, by identifying $L, L^\prime$ as $\mathbb{Z}_p$-modules (with Galois actions differing by $\chi^n$), we may identify $\hat{L} \otimes \mathcal{O}_{B_{\text{dR}}}$ and $\hat{L}^\prime \otimes \mathcal{O}_{B_{\text{dR}}}$ (with the Galois action again differing by $\chi^n$). As in the crystalline case, $e \otimes f \in \hat{L} \otimes \mathcal{O}_{B_{\text{dR}}}$ is Galois invariant if and only if $e \otimes t^n f \in \hat{L} \otimes \mathcal{O}_{B_{\text{dR}}}$ is. We normalized the intrinsic norm so that the element $t$ has norm $1$, and therefore the intrinsic norm of $e \otimes f \in \hat{L} \otimes \mathcal{O}_{B_{\text{dR}}}$ is the same as the intrinsic norm of $e \otimes t^n f \in \hat{L} \otimes \mathcal{O}_{B_{\text{dR}}}$. The result follows.

\[\square\]
Proof. of Theorem 7.9
Without loss of generality, we may and do assume \( X = \text{Spec}(R) \) is affine with an etale map
\[
\mathcal{O}_k[s_1, s_1^{-1}, \ldots, s_n, s_n^{-1}] \to R.
\]
Recall that \( v_i = [s_i] \otimes s_i^{-1} - 1 \) are elements of \( \mathcal{A}_{cris}(\mathcal{R}) \) such that \( \mathcal{A}_{cris}(\mathcal{R}) = \mathcal{A}_{cris}(\mathcal{R})^{PD} \). Moreover,
\[
S_{cris}(L) \subset \mathcal{A}_{cris}(\mathcal{R}) \otimes L.
\]
It follows that the image \( S_i \) of \( \text{Fil}^i S_{cris}(L) \) in \( \text{Gr}^i D^0_{dR}(L) \) is contained in the image of \( \text{Fil}^i A_{cris}(\mathcal{R}) \otimes L \) which is \( \mathcal{L} \otimes \xi^{\mathcal{L}}[\frac{a_1}{2}, \ldots, \frac{a_n}{2}] \leq i \) by [18] (5).

Moreover, it follows by Theorem 7.4 that the positive degree components of \( S_i \) are irrelevant when computing the intrinsic norm. Since we embed \( D_{HT}(L) \) inside \( \mathcal{L} \otimes \mathcal{O}_X \) by scaling by \( \frac{1}{\mathcal{R}} \), it follows that the crystalline lattice is contained inside \( \theta(\frac{a}{2})^i \) times the intrinsic lattice. As \( v_p(\theta(\frac{a}{2})) = -\frac{1}{p-1} \) (see for example [16], page 29), the crystalline lattice is contained within \( p^{-\frac{i}{p-1}} \) times the intrinsic lattice.

To get the other inclusion, note first of all that by Lemma 7.11 that \( S_{cris}(L \otimes L') = S_{cris}(L) \otimes S_{cris}(L') \), and by taking direct summands that \( \lambda^{\dim L} S_{cris}(L) = S_{cris}(\lambda^{\dim L} L) \) - note that the Hodge-Tate weights are \( \leq a \dim L \leq p-2 \) by assumption. The same compatibility for the intrinsic height is immediate.

It follows by Lemma 7.11 that the top wedge powers of the crystalline lattice and of the intrinsic lattices are the same. An argument identical to one used to conclude the proof of Theorem 7.2 proves this result.

\[ \square \]

8 Canonical norm on local systems of Shimura varieties

Let \( S = S_K(G, X) \) be a Shimura variety, and let \( \mathcal{S} \) be a smooth toroidal compactification. We assume as always that \( K \) is neat. Let \( \rho \) be a rational representation of \( G \) on a polarized vector space \( (V, \psi) \), and let \( \forall \subset V \) denote a lattice. The associated filtered vector bundle \( (\mathcal{F}, \mathcal{V}, \psi) \) extends via the Deligne extension to \( \mathcal{S} \). Let \( (\mathcal{S}, \mathcal{V}) \) be a smooth integral model of \( (S, \mathcal{V}) \) over \( \mathcal{O}_E(N^{-1}) \) as in Theorem 4.1. We assign an admissible norm to \( (\mathcal{G}, \mathcal{F}, \mathcal{V}, \mathcal{S}) \) as follows:

- For each Archimedean place, we simply use the norm on the graded pieces induced by the Hodge norm as described in section 3.6.
- For each non-Archimedean place \( v \) (dividing \( p \in \mathbb{Z} \)) at which \( \text{et} V_p \) is crystalline and which doesn’t divide \( N \) from Theorem 4.1 we identify \( \mathcal{F}_{V_p} \) with \( \text{et} V_p \) via [12] Thm 5.3.1, and use the crystalline norm on the graded pieces of \( \text{et} V_p \) - note that the Hodge-Tate weights are \( \leq a \dim L \leq p-2 \) by assumption.
- For the (finitely many) other non-Archimedean placed \( v \), we identify \( \mathcal{F}_{V_{et}} \) with \( \text{et} V_{et} \) via [12] Thm 5.3.1, and use the intrinsic norm on the graded pieces of \( \text{et} V_{et} \).

Theorem 8.1. With the notation above, the normed vector bundle \( \text{Gr}_{et} V \) is admissible.

Proof. For every finite place \( v \), the intrinsic norm on the graded pieces of \( D_{HT}(\text{et} V_{et}) \) extends to an acceptable norm on \( \mathcal{S}_v \).

The fact that the norms at infinity are continuous is immediate, and the acceptability of the norms at each finite place follows from the discussion above Theorem 7.9 and Lemma 7.7. Let \( N \) be a positive integer such that \( S_K(G, X) \) and the filtered bundle \( (\mathcal{F}, \mathcal{V}, \psi) \) extends to a proper smooth model \( S_{G}^{\mathcal{F}} \) above \( \mathcal{O}_{E[G, X]}(N) \), and the result of Theorem 4.1 holds for primes larger than \( N \).

The theorem is then reduced to the proof of the following proposition:

Proposition 8.2. For almost all places \( v \), the image of the integral model \( S_{cris}(\text{et} V_{et}) \) in \( \mathcal{F}_{V_{et}} \) under the canonical isomorphism agrees with \( \mathcal{V}_v \).
Proof. We first reduce to the case where \( G = G^c \). To do this, note that by our local system by definition comes form a rational representation that factors through \( G^c \), therefore all our associated data is canonically pulled back along the map \( S(G, X) \to S(G^c, X^c) \). Hence we assume \( G = G^c \) in what follows.

To prove this proposition, we first note that both \( S_{\text{et}}(e_1 V_{E_1}) \) and \( V_r \) are equipped with flat connections. By Theorem \ref{thm:comparison} these are abstractly isomorphic since they both correspond under the Fontaine-Lafaille-Faltings correspondence to isomorphic local systems. Since the categories of Relative Fontaine-Laffaille modules and crystalline local systems are equivalent by \cite[Thm 77]{flf}, they are the same up to a global automorphism of the generic fiber \( _{\text{an}} V_{E_1} \otimes \mathbb{Q}_p \), which corresponds to a global automorphism of the associated \( \mathbb{Q}_p \)-local system. It follows that they agree everywhere if they agree at a single point.

We may thus check that these two structures agree at a single point, and so we can restrict to a 0-dimensional Shimura variety. By the crystalline compatibility on passage to fibers in Theorem \ref{thm:crystalline} we are thus reduced to checking the agreement for a 0-dimensional Shimura datum \((T, r)\) that is a sub of \((G, X)\). We pick \( T \) so that it contains no proper \( \mathbb{Q}_p \)-subtorus which contains the image of \( r \).

Since the \( \mathbb{R} \)-split center of \( G = G^c \) is \( \mathbb{Q} \)-split, it follows that the weight character of \((G, X)\) is rational, and hence the same is true for \((T, r)\). Thus, by \cite[Prop 4.1]{flf} it follows that \((T, r)\) arises from a rational Hodge structure \( V \) of CM type. Moreover, by \cite[Prop 4.7]{flf} there is a CM abelian variety \( A \) such that \( V \) is in the Tannakian category generated by the rational Hodge structure \( H^1(A) \). It follows that \((T, r)\) is a quotient of the Shimura variety \((T', r')\) corresponding to \( H^1(A) \). It is therefore sufficient to prove the proposition for the Shimura variety \((T', r')\).

Now \((T', r')\) admits a faithful representation on \( V_0 := H^1(A) \). For \( V_0 \), and therefore any representation of \( T' \) (as the Tannakian category generated by \( V_0 \) gives yields all representations of \( T' \)), the comparison isomorphism is induced by the usual de Rham to Betti comparison theorem as in \cite[5.5.3]{flf}, and the statement here follows from the integral version of this theorem \cite[Thm 5.3]{flf}.

Moreover, as \( V \otimes \mathbb{Q} \) is in the Tannakian category generated by \( V_0 \otimes \mathbb{Q} \), this remains true integrally for almost all places. The claim now follows.

\[ \square \]

**Corollary 8.3.** For any integer \( a \), the height \( h \) corresponding to the line bundle \( \mathcal{L} = \det \text{Gr}^a_{\text{an}} V \) differs from the corresponding Weil height \( h_{\mathcal{L}} \) induced from the Deligne extension to \( \mathcal{S} \) by at most \( O(\max(1, \log h_A)) \) where \( A \) is an ample line bundle on \( \mathcal{S} \). In particular, if \( \mathcal{L} \) is ample then \( h \) is comparable to \( h_{\mathcal{L}} \).

**Proof.** The key is that the Hodge metric has logarithmic singularities by \cite[Thm 6.6]{flf}.

By Theorem \ref{thm:comparison} the difference \( h - h_{\mathcal{L}} \) is \( O(1) \) plus the difference at the Archimedean places. By Schmid’s work this is bounded by the logarithm of a power of the absolute value of logarithm of the distance to \( D \). Since the absolute value of the logarithm of the distance to the boundary is bounded above by \( h_A \), the result follows.

\[ \square \]

**Definition 8.4.** We keep the setup of this section, and suppose that the weights of \( _{\text{an}} V \) are in \([0, a]\), such that \( \text{Gr}^a_{\text{an}} V \) is 1-dimensional. Call the height constructed using the above norms the **canonical height**.

9 Controlling Heights of CM Points via \( A_g \)

9.1 Notation

Given two functions \( f, g \) we write \( f <_x g \) \((g >_x f)\) if there exist functions \( A, B > 0 \) depending only on \( x \) such that \( f \leq A g^B \).

For a torus \( T \) over a number field \( E \) with splitting field \( F \) and an open compact subgroup \( K \subset T(A^1_q) \) and maximal compact \( K_T \), we define \( d(K) := [K_T : K] \text{disc}(F) \).
9.2 Asymptotics of tori representations

9.2.1 Complexity of representations

Given a character $\chi = \prod_{i=1}^{d} \chi_i$ of $G_{m}^d$, we say that the complexity of $\chi$ is the maximum degree of $\chi_i$. Note that this depends on a choice of co-ordinates for $G_{m}^d$. Given a representation $\rho$ of $G_{m}^d$, we say that the complexity of $\rho$ is the maximum complexity of any character occurring in $\rho$.

For any torus $T$ over a field $k$ and variety $V$, we say that the complexity of $V$ is the minimum complexity of the representation of $G_{m}^{\dim T}$ under an isomorphism $G_{m}^{\dim T}$ with $T_{F}$.

9.2.2 Endomorphism ring of representations

Let $T$ be a torus over a number field $E$, $(\rho, V)$ an algebraic representation of $T$, $\mathcal{V} \subseteq V$ an $\mathcal{O}_E$-sublattice, and $K$ the stabilizer of $\mathcal{V}$. We define the $E$-algebra $E_{\rho} \subseteq \text{End}(V)$ associated to $\rho$ to be the algebra generated by $\rho(T(E))$. We define $E_{\rho, \mathcal{V}} \subset E_{\rho}$ to be the stabilizer of $\mathcal{V}$. We shall require the following proposition, which is [13] Theorem 4.1. Technically they prove something slightly more restrictive by insisting that $V$ comes from a fixed representation from an ambient Shimura variety - which is still enough for us - but their proof works verbatim in this more general context.

**Proposition 9.1.** Let $T$ be a torus over a number field $E$ of dimension $O(1)$, and $(\rho, V)$ be a representation of dimension and complexity $O(1)$. Let $K$ be the stabilizer of $\mathcal{V} \otimes \mathbb{Z}$. Then $\text{disc } E_{\rho, \mathcal{V}} < d(K)$.

9.3 Heights on Siegel modular varieties

Consider the Shimura datum $(GSp_{2g}, X)$ and the associated Siegel modular variety $A_g$, which has a smooth model over $\text{Spec } \mathbb{Z}$. Let $\mathscr{A} \to A_g$ denote the universal $g$-dimensional principally polarized abelian scheme with identity section given by $e$.

Let $(\mathcal{V}, q)$ be the canonical symplectic representation, with $\mathcal{V}_{\mathbb{Z}}$ the integral self-dual sublattice. As in Section $3$, $\mathcal{V}_{\mathbb{Z}}$ yields $p$-adic etale local systems $\mathcal{V}_p$ for every prime $p$, as well as a complex vector bundle with connection $\mathcal{V} \to A_g$. It is well known that the local systems $\mathcal{V}_p$ are crystalline on the generic fiber of the formal completion of $A_g$, for large enough $p$.

The $p$-adic local systems are canonically isomorphic to the relative etale cohomology of $\mathscr{A}$, $\mathcal{V}$ is canonically isomorphic to the relative de Rham cohomology of $\mathscr{A}$ and $\mathcal{V}$ has a canonical integral model given by integral de Rham cohomology. The vector bundle also admits a metric at infinity, namely the Hodge metric. Finally we note that $\mathcal{V}$ has a two-step descending filtration with $\text{Fil}^1_{\mathcal{V}}\mathcal{V}$ canonically being identified with $e^n\mathcal{V}$.

The Faltings height of a principally polarized abelian scheme over $\mathcal{O}_K$ (where $K$ is some number field) is precisely the height of the associated Spec $\mathcal{O}_K$-valued point of $A_g$ with respect to the line bundle $\text{Fil}^1_{\mathcal{V}}\mathcal{V}$ equipped with the Hodge metric at infinity and the integral structure described above. We will now describe the (well known) fact that the integral structure on $\text{Fil}^1_{\mathcal{V}}\mathcal{V}$ obtained using the relative Fontaine-Laffaille correspondence and the results of [12] also yields the Faltings height.

In order to establish this, it suffices to show that the $(p$-adic) integral structures on $\mathcal{V}$ endowed by (the vector bundle with connection underlying) $S_{\text{cris}}(\mathcal{V}_p)$ is the same as the the integral structure endowed by integral de Rham cohomology, for large enough $p$. This follows as below:

1. The vector bundle with connection $p_{\text{an}}\mathcal{V}$ on $A_g|_{\mathbb{Q}_p}$ is canonically isomorphic to $\mathcal{V}$ via comparison isomorphisms for abelian varieties. Therefore, it suffices to prove that (the vector bundle with connection underlying) $S_{\text{cris}}(\mathcal{V}_p)/A_g \times \mathbb{Q}_p$, which is naturally a lattice inside $p_{\text{an}}\mathcal{V}$ and hence $H^1_{\text{an}}(\mathcal{V} \times \mathbb{Q}_p)$, coincides with integral de Rham cohomology.

2. It suffices to check that the two lattices coincide at $W = W(\mathbb{F}_p)$-valued points of $A_g$. To that end, pick any such point $x$, and let $A/W$ denote the corresponding abelian scheme. By [16], the lattice $S_{\text{cris}}(\mathcal{V}_p|_x) = S_{\text{cris}}(H^1_{\text{et}}(A, \mathbb{Z}_p)) \subseteq H^1_{\text{an}}(A_{W[1/p]})$ is just the crystalline cohomology of $A_{\mathbb{F}_p}$, thought of as a subset of $H^1_{\text{an}}(A_{W[1/p]})$ (where the latter inclusion arises from the canonical isomorphism
\[H^1_{cr} (A_W/W)[1/p] \cong H^1_{dR} (A_{W[1/p]}).\] On the other hand, it is well known that the lattice inside \(H^1_{dR} (A_{W[1/p]})\) induced by integral de Rham cohomology of \(A/W\) is the same as the above lattice.

Moreover, at the finitely many exceptional primes \(p\), the intrinsic height is acceptable by Lemma \(\text{[7.9]}\).

It now follows from Lemma \(\text{[6.5]}\) that the height on \(\mathcal{O}_K\)-valued points of \(A_p\) induced by the metrics on \(\nabla\) arising from \(S_{\text{reg}}(\nabla_p)\) differs from the Faltings height of the corresponding \(\mathcal{O}_K\)-abelian scheme by a uniform \(O(1)\).

### 9.4 Modified heights on 0-dimensional Shimura varieties

In this section we associate a height on automorphic line bundles on 0-dimensional Shimura Varieties in a way which is natural, to allow for easy comparisons between distinct Shimura embeddings.

**Definition 9.2.** Let \(S_K(T, r)\) be a 0-dimensional Shimura variety, and let \((V, q)\) be a polarized rational representation of \(T\) with lattice \(V\) and weights in \([0, a]\) with \(\text{Gr}_F V_Q\) having dimension 1, and set \(K\) to be the stabilizer of \(V \otimes \mathbb{Z}\). We equip \(\text{Gr}_F V\) with a set of norms as follows:

- For each Archimedean place, we simply use the norm induced by the Hodge norm as described in section 3.0.
- For each non-Archimedean place \(v\) such that
  - \(T\) is unramified at \(p\),
  - \(K_p\) is maximal, and
  - \(p \geq a \dim V + 2\)
  we identify \(\text{Gr}_F V_{E_v}\) with \(\text{Gr}_F V_{E_v}\) via [12 Thm 5.3.1], and use the crystalline norm.
- For the (finitely many) other non-Archimedean placed \(v\), we identify \(\text{Gr}_F V_{E_v}\) with \(\text{Gr}_F V_{E_v}\) via [12 Thm 5.3.1], and use the intrinsic norm.

We call this the **modified norm**. Note that it is admissible by \(\text{[8.1]}\) and we call the corresponding height the **modified height**\(^{10}\).

**Lemma 9.3.** Let \(S_K(T, r)\) be a 0-dimensional Shimura Variety. Let \((V, \nabla, q)\) be as above with \(\text{Gr}_F \nabla_Q\) having dimension 1. Then \(\text{Gr}_F \nabla\) acquires the structure of a normed line bundle. Then the modified height of every point on \(S_K(T, r)\) with respect to \(\text{Gr}_F \nabla\) is the same.

**Proof.** This proof is morally similar to the proof that CM abelian varieties of the same type all have the same height.

Recall that the complex points of \(S_K(T, r)\) can be described as \(T(\mathbb{Q}) \backslash T(A_f)/K\). Let \(x_1, x_2\) be two distinct points. Then there are many elements \(t\) in \(T(A_f)\) such that \(tx_1 = x_2\). Each such element defines a map \(f_t : S_K(T, r) \rightarrow S_K(T, r)\). Moreover, the pullback of the automorphic local system \(\nabla\) is naturally isomorphic to the local system \(t^{-1} \nabla\) where

\[t^{-1} \nabla := t^{-1} (\nabla \otimes \mathbb{Z}) \cap \nabla_Q.\]

It follows that

\[h_{\text{Gr}_F \nabla} (x_2) = h_{\text{Gr}_F \nabla} (t^{-1} \nabla) (x_1).\]

The two lattices \(\nabla, t^{-1} p \nabla\) differ only at the set \(S(t)\) of primes \(p\) where \(t_p \neq K_p\). It follows that the Archimedean contributions to the heights differ only by rational multiples of \(\log p\), \(p \in S(T)\), and the non-Archimedean contributions only differ at primes in \(S(t)\). Thus

\[h_{\text{Gr}_F \nabla} (x_2) - h_{\text{Gr}_F \nabla} (x_1)\]

\(^9\)This condition allows us to compare this norm with the intrinsic norm, by Theorem \(\text{[7.9]}\).

\(^{10}\)It would be much cleaner to simply use the intrinsic norm at all finite primes. Unfortunately, we are not able to prove this height is admissible, though we suspect it to be the case.
are sums of rational multiples of $\log p, p \in S(t)$.

However, by weak approximation we may pick $t$ so as to make $S(t)$ exclude any given prime. Since logs of primes are $\mathbb{Q}$-linearly independent, the claim follows. \hfill $\blacksquare$

**Lemma 9.4.** Let $S_K(G,X)$ be a fixed Shimura variety, and take a faithful polarized representation $V$ with a sublattice $\mathbb{V}$ fixed by $K$. Let $(T,r) \subset (G,X)$ be a 0-dimensional Shimura subdatum. Let $\text{Fil}^q V$ be the smallest piece of the Hodge filtration, and assume its one-dimensional. Then the modified heights of $(\text{any point of}) S_K(T,r)$ with respect to $\text{Fil}^q V$ on $S_{K \cap T \langle \mathbb{A}_f \rangle}(T,r)$ differs from the the canonical height on $S_K(G,X)$ by $O_{p,\dim V}(\log d(K \cap T \langle \mathbb{A}_f \rangle))$.

**Proof.** At every prime we are using either the crystalline norm or the intrinsic norm, and the intrinsic norm only comes up for primes $\ll_{p,\dim V,S} 1$ or at which $K \cap T \langle \mathbb{A}_f \rangle$ is non-maximal or at which $T$ is ramified. The claim now follows immediately by Theorem [7.3]. \hfill $\blacksquare$

### 9.5 Comparing realizations of a rational representation

**Theorem 9.5.** Let $T$ be a Torus of dimension $O(1)$ Let $V_1,V_2$ be isomorphic rational representations of $T$ of dimension and complexity $O(1)$ with sublattices $\mathbb{V}_1,\mathbb{V}_2$ preserved by $K$. Then there is an isomorphism $\phi : V_1 \rightarrow V_2$ sending $\mathbb{V}_1$ to $\mathbb{V}_2$ such that $[\mathbb{V}_2 : \phi(\mathbb{V}_1)] < d(K)$.

**Proof.** First consider $\mathbb{V}' := \mathcal{O}_{\mathbb{V}_1/\mathbb{V}_1}$, which in particular is fixed by $K_T$. Then by Proposition [9.1] it follows that $[\mathbb{V}' : \mathbb{V}_1] < d(K)$. We thus assume that $\mathbb{V}_1,\mathbb{V}_2$ are stabilized $\mathcal{O}_{\mathbb{V}_1/\mathbb{V}_1}$. The claim follows from the following well known facts:

- $\mathcal{O}_T$ modules are sums of ideal classes
- Every ideal class has a representative of norm $O(\text{disc}(F)^{1/2})$.

**Theorem 9.6.** Let $T$ be a Torus of dimension $O(1)$, and $(T,r)$ be a Shimura variety. Let $V$ be a rational representation of $T$ of dimension and complexity $O(1)$ with a sublattice $\mathbb{V}$ preserved by $K$. Assume the weights of $V$ are in $[0,a]$ with $\dim \text{Gr}^a V = 1$. Let $q_1,q_2$ be distinct polarizations of $V$ of discriminant $O(N)$. Then the modified heights of $\text{Gr}^a V$ with respect to $q_1$ and $q_2$ differ by $O(\log N)$.

**Proof.** Note that the heights differ only at the archimedean place and so it is sufficient to understand how they differ there. Modifying the lattice $\mathbb{V}$ does not affect the ratio of the archimedean heights. Thus, we may replace $\mathbb{V}$ by $\mathbb{V}' := \mathcal{O}_{\mathbb{V}_1/\mathbb{V}_1} \mathbb{V}$, which in particular is fixed by $K_T$. We therefore reduce to the setting where $K = K_T$ and $\mathbb{V}$ is stabilized by $\mathcal{O}_L$ where $L := \mathbb{Q}_p \mathbb{V}$.

Now, $\mathbb{V}$ breaks up into a direct sum or irreducible representations of $\mathcal{O}_L$, compatible with the direct sum decomposition of $V$ into irreducible representations of $L$, which must underlie Hodge substructures since $L$ consists of Hodge endomorphisms. Since $F^a V$ is one-dimensional it is contained in a single such summand $V_0$ which therefore occurs in $V$ with multiplicity one. Let $L_0$ be the summand of $L$ which acts non-trivially on $V_0$. It follows that $V_0$ is a 1-dimensional representation of $L_0$. Moreover, $F^a V$ must be preserved by $L_0$ and therefore cannot be defined over a smaller field than $L_0$. Thus $V_0$ is an irreducible Hodge structure, and therefore has no non-zero maps as a hodge structure to the other summands of $V$. It follows that $q_i$ pairs $V_0$ trivially with every other summand of $V$. Finally, by restricting to $V_0$ we may reduce to the case where $L$ is a field, $\mathbb{V}$ is an ideal class of $\mathcal{O}_L$, and $q_i$ must respect that $\mathcal{O}_L$ structure for $i = 1,2$.

Finally, it follows that $q_1 = q_2$ for some element $\alpha \in L$ whose norm is the ratio between the discriminants of $q_1$ and $q_2$ and is therefore $< N$. Now on each archimedean place $v$ of $L$ it follows that the hodge norms differ by $\log |\alpha|_v$, and so the claim follows from the product formula. \hfill $\blacksquare$
**Definition 9.7.** Let \((T, r)\) be a 0-dimensional Shimura variety. Given a character \(\chi\) of \(T\), let \(V\), be the smallest rational representation whose complexification contains \(\chi\). Assume moreover that \(\chi = \text{Fil}^p V\) is the highest weight piece of \(V\).

Let \(V\) be a maximal lattice, i.e. stabilized by \(K_T\). Let \(q\) be a polarization of \(V\) of discriminant \(< d(K)\). The modified norm then induces a height of any point with respect to the normed bundle \(\text{Fil}^p V\). By the previous two theorems, this is well defined up to \(O(\log d(K_T))\). We call it \(h_r(\chi)\).

**Lemma 9.8.** Given two characters \(\chi_1, \chi_2\) of \(T\) of bounded complexity which satisfy the assumption of the definition above, we have \(h_r(\chi_1) + h_r(\chi_2) = h_r(\chi_1(\chi_2))\).

**Proof.** Consider \(W = \bigoplus_{\chi} V\). This is a maximal lattice, such that \(W\) surjects onto \(\otimes_{\chi} V\), inducing an isomorphism on highest graded pieces. It follows that the latter is a direct summand of the former up to an isogeny of index \(< d(K_T)\). The claim follows.

**Lemma 9.9.** Let \(T\) be a Torus of dimension \(O(1)\), and \((T, r)\) be a Shimura variety. Let \(\chi\) be a character of \(T\) which occurs as the highest weight piece of \(V\). Let \(m\) be a positive integer of size \(O(1)\). Then \(h_r(\chi) = mh(\chi)\).

**Proof.** Let \(V\) be a maximal maximal lattice in \(V\). Notice that the \(m\)'th power map \(f_m : (T, r) \rightarrow (T, r^m)\) is a map of Shimura data. Note that \(f_m^* V = V^m\) and \(f_m^* V\) is also a maximal lattice. Now, the norms on \(G\) and \(G\) are the same and thus we conclude that \(h_r(\chi^m) = h_r(\chi)\). The lemma now follows since \(h_r(\chi^m) = mh_r(\chi)\).

More generally, we have the following functorial relation, proved in the same way as above:

**Lemma 9.10.** Let \(T_1, T_2\) be a Tori of dimension \(O(1)\), and \((T_1, r_1), (T_2, r_2)\) be Shimura varieties. Let \(\chi\) be a character of \(T_2\) which occurs as the highest weight piece of \(V\), and let \(\phi : (T_1, r_1) \rightarrow (T_2, r_2)\) be a morphism of Shimura varieties. Then \(h_r(\chi \circ \phi) = h_{r_1}(\chi)\).

### 9.6 Bounding heights of special points

We let \(S_K(G, X)\) be a Shimura datum, with \(G\) an adjoint simple \(Q\)-group. Let \(V\) denote the adjoint representation of \(G\) which we twist (by \(1\)) so that \(\text{Fil}^0 V = V\) and \(\text{Fil}^1 V = 0\), and a lattice \(V\) stabilized by \(K\). Then \(\text{Fil}^0 V\) is naturally equipped with an admissible collection of norms via the canonical norm. This equips \(\text{det} \text{Fil}^0 V\) with a metric with respect to which we can take heights. Our goal is to prove the following:

**Theorem 9.11.** Fix a Shimura variety \(S_K(G, X)\) with \(G\) simple of adjoint type. Let \((T, r) \subset (G, X)\) be a (varying) 0-dimensional Shimura datum such that \(K \cap T(A_f)\) is of index \(M\) in \(K\), and \(T\) splits over a field \(E\). Then the canonical height of \((T, r)\) with respect to \(\text{det} \text{Fil}^0 V\) is admissible, and is \((M \text{ disc } E)^{o(1)}\).

By Theorem 5.1 and 9.4 and it is sufficient to show the following:

**Theorem 9.12.** Let \((R_E, r_E)\) be a 0-dimensional Shimura datum of partial CM type, where \(\text{dim } E = O(1)\). Then \(h_{r_E}(\chi_E) = O(\text{disc } E^{o(1)})\).

The proof of this will heavily use an idea of Deligne [11] Prop. 2.3.10 for combining partial CM types to get a complete CM type for a larger CM fields, which he used to classify Shimura Varieties of abelian type.

**Proof.** First, note that if \(\Psi\) is a full CM type, then by embedding into a Siegel modular variety, the claim follows from the corresponding bound on Faltings heights of CM abelian varieties. [11][11]

**Step 1: Reduction to the Galois case**

[11] Ultimately the bound here relies on the averaged form of the Colmez conjecture, see [50] and [2].
We may reduce to the case where \( E \) is Galois as follows: Let \( E' \) denotes the Galois closure of \( E \), and \( m = [E' : E] \), and \( \Psi' \) the pullback of \( \Psi \). Then there is a norm map \( R_{E'} \rightarrow R_E \) which gives a map of Shimura Data \( (R_{E'},r_{\Psi'}) \rightarrow (R_E,r_{\Psi}^d) \), and \( \chi_{\Psi} \) pulls back to \( \chi_{\Psi'} \). The reduction now follows from lemma 9.10.

**Step 2:** \(|\Phi| = 1\)

We next handle the case where \( \Phi \) consists of a single place. Suppose that \( F \) is a Galois totally real field of degree \( d \), and \( E_1,\ldots,E_d \) be disjoint Galois CM extensions of \( F \). Now suppose that \( \Phi_1,\ldots,\Phi_d \) are partial CM types of \( E_1,\ldots,E_d \) consisting of a single place, and we assume all of these places are above a distinct place of \( F \). Let \( E = E_1 \cdots E_d \), and consider \( \Phi'_i \) the pullback of \( \Phi_i \) to \( E \). Then \( \Phi := \bigcup_{i=1}^{d} \Phi'_i \) is a CM type. The norm map \( \text{Nm}_{E_i/E} \) induces a map of Shimura data \( (R_K,\rho_{\Phi}) \rightarrow (R_K,\rho_{\Phi_i}^{d-1}) \). Moreover, \( \chi_{\Phi} = \prod_{i=1}^{d} \chi_{\Phi_i} \circ \text{Nm}_{E_i/E} \). It follows from lemma 9.10 that

\[
h_{r_{\Phi}}(\chi_{\Phi}) = 2^{d-1} \sum_{i=1}^{d} h_{r_{\Phi_i}}(\chi_{\Phi_i}). \tag{9.6.1}\]

Now let \( E_0 \) be an arbitrary Galois extension, with totally real subfield \( F \) of degree \( d \). Let \( E_1,\ldots,E_d \) be disjoint Galois CM extensions of \( F \) s.t. \( E_0,E_1,\ldots,E_d \) are disjoint, and the discriminants of \( E_1,\ldots,E_d \) are bounded by a power of the discriminant of \( E_0 \). One may construct such fields easily by setting \( E_i = F(\sqrt{m_i}) \) for appropriate primes \( m_i \).

Now note that since our fields are Galois, for any \( \Phi \) on \( E_i \) consisting of a single place the heights of \( h_{r_{\Phi}}(\chi_{\Phi}) \) is the same. We simply call this \( h_1(E_i) \). Now by Equation \( 9.6.1 \) the sum of the heights of any \( d \) of \( h_1(E_0),\ldots,h_1(E_d) \) equal the height corresponding to a complete CM type, and therefore is of size \( \text{disc}(E)_{\text{ad}}^{(1)} \). It follows by taking linear combinations that the same is true for the \( h_1(E_i) \), and thus in particular for \( h_1(E_0) \).

**Step 3:** The general case

Finally, we handle the case of general partial CM type \( \Phi_0 \). If \( E_0,F \) are as above, we set \( e = d - |\Phi_0| \). We pick singleton sets \( \Phi_i \) on \( E_i \) for \( i = 1,\ldots,e \) such that each place of \( F \) is below an element of \( \Phi_i \) for exactly one \( i \). Now we set \( E_{\text{tot}} := E_0E_1\ldots E_e \) and \( \Phi \) the union of the pullbacks of the \( \Phi_i \) to \( E_{\text{tot}} \), which is a complete CM type. Then as above, we obtain:

\[
h_{r_{\Phi}}(\chi_{\Phi}) = 2^e \sum_{i=0}^{e} h_{r_{\Phi_i}}(\chi_{\Phi_i}). \tag{9.6.2}\]

Given the theorem has already been proven for \( h_{r_{\Phi}}(\chi_{\Phi_i}) \) for \( i > 0 \) we obtain the result for \( i = 0 \) as well, as desired.

\[\square\]

**9.7 Using the height bound to prove André-Oort**

We finally obtain André-Oort conjecture for Shimura Varieties:

**Theorem 9.13.** Let \( S_K(G,X) \) be a Shimura Variety. Then the André-Oort conjecture holds.

*Proof.* We have a finite map \( S_K(G,X) \rightarrow S_{K^\text{ad}}(G^{\text{ad}},X^{\text{ad}}) \) which sends CM points to CM points, and thus the André-Oort conjecture for \( S_K(G,X) \) follows from the same for \( S_{K^\text{ad}}(G^{\text{ad}},X^{\text{ad}}) \). We thus assume from the start that \( G \) is adjoint. For the same reason we may also pick \( K \) without affecting the validity of our conjecture. From [5] Theorem 1 it is sufficient to provide a sub-polynomial height bound for special points as in Theorem 9.11.
We thus have a splitting $G = \prod G_i$ where the $G_i$ are all $\mathbb{Q}$-simple and adjoint. If we choose $K$ to split up as $K = \prod K_i$, we get a corresponding decomposition of Shimura varieties. It is therefore sufficient to establish the height bound in the case that $G$ is simple.

Now if $G$ is abelian type, the height bound follows from [39]. If $G$ is non-abelian type, it follows from Theorem [9.11]. This completes the proof.

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A Frobenius structures and unipotent monodromy at infinity by Hélène Esnault and Michael Groechenig

We fix an irreducible affine base scheme S which is of finite type over a universally Japanese ring. For the purpose of this appendix, S will either be Spec C, Spec F_q or Spec R, where R is a finite type algebra. Let us denote by X_S a smooth and projective S-scheme with a relatively very ample line bundle O_X_S(1). Let X_S ⊂ X_S be an open subscheme such that X_S\X_S is a strict normal crossings divisor (snc) D_S = ∪_{μ=1}^c D^μ_S. The sheaf of degree α Kähler differentials with log-poles along D will be denoted by Ω^n_X_S/S(D). For μ = 1, ..., c we write res_μ : Ω^n_X_S/S(D) → O_{D^μ_S/S} for the residue map.

Definition A.1. (a) A log-dR local system on X_S is a pair (E_S, ∇_S) where E_S is a vector bundle of rank r on X_S and

∇ : E_S → E_S ⊗ Ω^1_X_S/S(D)

is a flat logarithmic connection such that res_μ(∇) ∈ H^0(D^μ_S, End(E_S|_{D^μ_S})) is nilpotent for all μ = 1, ..., c.
(b) We say that \((E_S, \nabla_S)\) is **strongly cohomologically rigid**, if

\[
\mathbb{H}^1(\bar{X}_S, [\text{End}(E_S) \xrightarrow{\text{End}(\nabla_S)} \text{End}(E_S) \otimes \Omega^1_D \xrightarrow{\text{End}(\nabla_S)} \cdots]) = 0. \tag{A.0.1}
\]

(c) A **log-Higgs bundle** on \(\bar{X}_S\) is a pair \((V_S, \theta_S)\), where \(V_S\) is a vector bundle of rank \(r\) on \(\bar{X}_S\) and \(\theta_S\) is an \(\mathcal{O}\)-linear morphism \(V \to V \otimes \Omega^1_{\bar{X}}(D)\) satisfying \(\theta_S \wedge \theta_S = 0\).

(d) A log-Higgs bundle \((V_S, \theta_S)\) is called **strongly cohomologically rigid**, if

\[
\mathbb{H}^1(\bar{X}_S, [\text{End}(V_S) \xrightarrow{\text{End}(\theta_S)} \text{End}(V_S) \otimes \Omega^1_D \xrightarrow{\text{End}(\theta_S)} \cdots]) = 0. \tag{A.0.2}
\]

**Remark A.2.** (a) If \(S = \text{Spec } \mathbb{C}\), the underlying vector bundle \(E\) of a log-dR local system has vanishing Chern classes. This follows from the formula for the Atiyah class of \(E\) given in [EV86, Proposition B.1]. In addition, the left-hand side of (A.0.1) computes \(H^1(X, \text{End}(E) \otimes \Omega^1_{\bar{X}})\). Indeed, as \(\text{res}_\mu(\nabla_C)\) is nilpotent for \(\mu = 1, \ldots, c\), so is \(\text{res}_\mu(\text{End}(\nabla_C))\), thus \(\text{End}(E)\) is Deligne’s extension the cohomology of which computes analytically \(Rf_*\) where \(f : X_{\text{can}} \to X_{\text{can}}\), see [Del70, II, Proposition 3.13, Corollaire 3.14].

(b) The notion of **strong cohomological rigidity** is more restrictive than the one of **cohomological rigidity**, used in [Kat96, EGIS]. A cohomological rigid local system, in the traditional sense, does not have any non-trivial infinitesimal deformations which leave the monodromies at infinity invariant. A strongly cohomologically rigid log-dR local system does not have any non-trivial infinitesimal deformations, independently of any constraints at the boundary.

**Definition A.3** (Arithmetic models). Let \((\bar{X}_C, D_C, \mathcal{O}_{\bar{X}}(1))\) be a triple consisting of a smooth projective complex variety \(\bar{X}_C\), an snc divisor \(D_C\), and a very ample line bundle \(\mathcal{O}_{\bar{X}_C}(1)\).

(a) An **arithmetic model** for \((\bar{X}_C, D_C, \mathcal{O}_{\bar{X}}(1))\) is given by an affine scheme \(S\) where \(\Gamma(S, \mathcal{O}_S)\) is a finite type subring \(R \subset \mathbb{C}\), a smooth projective \(S\)-scheme \(\bar{X}_S\) together with an snc divisor \(D_S\) such that

\[\bar{X}_C = \bar{X}_S \times S \text{Spec } \mathbb{C}\] 

and a relatively very ample line bundle \(\mathcal{O}_{\bar{X}_S}(1)\) pulling back to \(\mathcal{O}_{\bar{X}_C}(1)\).

(b) Let \(\{(E^i_C, \nabla^i_C)\}_{i \in I}\) be a family of log-dR local systems on \(X_C\). An arithmetic model for \((\bar{X}_C, D_C, \mathcal{O}_{\bar{X}_C}(1))\) is given by an arithmetic model for \((\bar{X}_C, D_C, \mathcal{O}_{\bar{X}_C}(1))\) as in (a), and log-dR local systems \(\{(E^i_C, \nabla^i_C)\}_{i \in I}\) on \(X/S\) satisfying

\[(E^i_C, \nabla^i_C) = (E^i_S, \nabla^i_S)|_{\bar{X}_C}\text{ for all } i \in I.\]

**Theorem A.4.** Suppose that every stable log-dR local system \((E_C, \nabla_C)\) of rank \(r\) on \((\bar{X}_C, D_C)\) is strongly cohomologically rigid. Then, there exists a finite type subalgebra \(R \subset \mathbb{C}\) and a model of \((\bar{X}_C, X_C, D_C)\) over \(S = \text{Spec } R\) such that every stable log-dR local system of rank \(r\) on \((\bar{X}_C, D_C)\) has an \(S\)-model \((E_S, \nabla_S)\) such that for every finite field \(k\) and every morphism \(R \to W(k)\) the formal flat connection

\[(\hat{E}_W, \nabla_W)\]

is endowed with the structure of a torsionfree Fontaine-Lafaille module on \(X_W = \hat{X}_W \setminus D_W\).

**Remark A.5.** In [EG20] we prove a stronger result for the case where \(D_C = \emptyset\). The assumptions of loc. cit. are less stringent, as they apply more generally to arbitrary rigid dR local system, i.e. isolated points of the moduli space \(\mathcal{M}_{dR}\). The additional assumptions above allow one to simplify the argument significantly.
A.1 Construction of a suitable arithmetic model

Moduli spaces of logarithmic flat connections on complex varieties were constructed by Nitsure in [Nit93]. Using Langer’s boundedness (see [Lan14]), this construction was extended to more general base schemes ([Lan14] Theorem 1.1):

**Theorem A.6** (Langer). For a fixed polynomial \( P \) there exists a quasi-projective \( S \)-scheme \( \mathcal{M}_{dR}(\tilde{X}_S, D_S) \) of stable flat logarithmic connections on \( \tilde{X}_S \) with Hilbert polynomial \( P \).

More generally, Langer constructs moduli spaces for semistable \( \Lambda \)-modules, where \( \Lambda \) is a ring of operators in the sense of [Sim94]. It is explained on p. 87 of loc. cit. that flat logarithmic connections are special case of the general theory of \( \Lambda \)-modules. We are interested in moduli spaces of flat logarithmic connections with vanishing Chern classes (see Remark A.2). The corresponding Hilbert polynomial satisfies

\[
P_R(n) = \int r \cdot t \chi_c(O_{\bar{X}_c}(n)) \quad \text{for all } n \in \mathbb{N}.
\]

**Corollary A.7.** There exists a closed subscheme \( \mathcal{M}_{\log-dR}(\tilde{X}_S, D_S) \subset \mathcal{M}_{dR}(\tilde{X}_S, D_S) \), which is the moduli space of stable \( \log-dR \) local systems with Hilbert polynomial \( P_R \).

**Proof.** There is an étale covering \( (U_i \rightarrow \mathcal{M}_{dR}(\tilde{X}_S, D_S))_{i \in I} \) such that we have a universal family \( (\mathcal{E}_{U_i}, \nabla_{U_i}) \) on \( U_i \times \tilde{X}_S \). By stability, such a universal \( \log-dR \) \( U_i \)-family is well-defined up to tensoring by a line bundle on \( U_i \).

By construction, the characteristic polynomial \( \chi_{i, \mu}(T) \) of \( \text{res}_{U_i}(\nabla_{U_i}) \) is a section of a locally free sheaf on \( U_i \). We let \( Z_i \rightarrow U_i \) be the closed immersion corresponding to the vanishing locus of \( (\chi_{i, \mu}(T) - T)^r, \mu = 1, \ldots, c \). This closed immersion is independent of the choice of a \( U_i \)-universal family, since tensoring by a line bundle on \( U_i \) leaves \( \chi_{i, \mu} \) invariant. We may thus apply faithfully flat descent theory to glue those closed immersions to a closed embedding

\[
Z \hookrightarrow \mathcal{M}_{dR}(\tilde{X}_S, D_S).
\]

The scheme \( Z \) is the sought-for moduli space \( \mathcal{M}_{\log-dR}(\tilde{X}_S, D_S) \).

We record the following consequence of non-abelian Hodge theory for later reference.

**Theorem A.8.** For every strongly cohomologically rigid \( \log-dR \) local system \( (E_C, \nabla_C) \) on \( \tilde{X}_C \) there exists an \( F \)-filtration \( \cdots \subset F^i \subset F^{i-1} \subset \cdots \subset F^0 = E \) satisfying Griffiths transversality \( \nabla: F^i \rightarrow F^{i-1} \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_X^1(D) \) and with the associated graded sheaves \( \text{gr}_F^i E = F^i/F^{i+1} \) being locally free. The associated Higgs bundle is denoted by

\[
(\text{gr}_F^i E, \text{KS}),
\]

where KS stands for Kodaira-Spencer and is defined by the linear maps

\[
\text{gr}_F^i \nabla: \text{gr}_F^i E \rightarrow \text{gr}_F^{i-1} E \otimes_{\mathcal{O}_{\tilde{X}}} \Omega_X^1(D).
\]

**Proof.** Mochizuki proved in [Mo06] Theorem 10.5 that every \( \log-dR \) local system on \( \tilde{X}_C \) can be complex analytically deformed to a polarised variation of Hodge structures, which implies the existence of the requisite \( F \)-filtration on the rigid \( (E_C, \nabla_C) \). In loc. cit., this is stated in terms of Betti local systems on \( X_C = \tilde{X}_C \backslash D_C \).

This is an equivalent perspective, by virtue of the Riemann-Hilbert correspondence which is complex analytic. Due to strong cohomological rigidity, \( (E_C, \nabla_C) \) cannot be deformed in a non-trivial manner. We conclude that \( (E_C, \nabla_C) \) underlies a polarised variation of Hodge structures.

**Remark A.9.** Stability of the log-Higgs bundle \( \left( \bigoplus_j \text{gr}_F^j E_C^{i, j}/\text{KS}(\nabla_C^{i, j}) \right) \) is implied by Mochizuki’s parabolic Simpson correspondence [Mo06]. We remark that the monodromies around the divisor at infinity are unipotent and therefore in this case, parabolic stability amounts to stability in the usual sense of log-Higgs bundles. See [Sim90] p. 722 where the triviality of the parabolic structure is justified for the curve case. The argument given there generalises directly to higher dimensional varieties.
Consider the set $p$ of stable and locally free models of finitely many homogeneous equations, there exists $r$ such that $X_{\overline{S}}$, $D_{\overline{S}}$, $O_{X_{\overline{S}}(1)}$ are jointly surjective. This implies (e). By inverting $(2r+2)!$ we can achieve (f). And, property (g) can be arranged by passing to the maximal open subset of $S$ which is smooth over $\text{Spec } \mathbb{Z}$.

**Proposition A.10.** We keep the assumptions of Theorem [A.4](#). There exists an arithmetic model $(S, \overline{X}_S, D_S, O_{X_S}(1))$ of $(X_{\overline{C}}, D_{\overline{C}}, O_{X_{\overline{C}}}(1))$ such that

(a) all rank $r$ log-dR local systems $(E^i_{\overline{C}}, \nabla^i_{\overline{C}})_{i \in I}$ have a locally free model $(E^i_S, \nabla^i_S)_{i \in I}$ over $S$,

(b) the models $(E^i_S, \nabla^i_S)_{i \in I}$ are also strongly cohomologically rigid,

(c) the filtrations $(F^{ij}_S \subset E^i_S)$ are defined over $S$ such that the $S$-relative filtrations $F_S^{ij} \subset E^i_S$ satisfy the Griffiths-transversality condition,

(d) for every $i \in I$ the associated graded

$$\left( \bigoplus_{j} \text{gr}^{ij}_E E^i_S, \text{KS}(\nabla^i_S) \right)$$

is a stable logarithmic Higgs bundle which is also locally free.

Furthermore, if $s : \text{Spec } \overline{k} \to S$ is a geometric point of $S$, then

(e) $(E_s, \nabla_s)$ is a log-dR local system on $\overline{X}_s = \overline{X}_S \times_S \text{Spec } \overline{k}$, then there exists $i \in I$ such that $(E_s, \nabla_s) = (E^i_S, \nabla^i_S)|_{X_s}$,

(f) $p = \text{char}(\overline{k}) > 2r + 2$, and

(g) $S \to \text{Spec } \mathbb{Z}$ is smooth.

**Proof.** The proof is analogous to the one of [EG20](#) Proposition 3.3] and will therefore only be sketched. Consider the set $\mathcal{R}$ of all finite type subrings $R \subset \mathbb{C}$. Since $\mathbb{C} = \bigcup_{R \in \mathcal{R}} R$ and $(\overline{X}_S, D_S)$ are defined in terms of finitely many homogeneous equations, there exists $\overline{R} \in \mathcal{R}$ such that $(\overline{X}_S, D_S)$ are obtained by base change from a pair of projective schemes $(\overline{X}_{\overline{S}}, D_{\overline{S}}) \subset \mathbb{P}^N_{\overline{S}}$, where we write $\overline{S}$ for $\text{Spec } \overline{R}$. We may assume that $D_{\overline{S}}$ is an snc divisor and that $\overline{X}_{\overline{S}}$ is smooth.

We now consider the moduli space $\mathcal{M}_{\log-dR}(\overline{X}_{\overline{S}}/\overline{S})$. Since $\mathcal{M}_{\log-dR}(\overline{X}_{\overline{S}}/\overline{S}) \times \overline{S} \text{Spec } \mathbb{C}$ is finite and flat over $\text{Spec } \mathbb{C}$, there exists a finite type algebra $\overline{R} \subset R$, such that the base change (we denote $\text{Spec } \overline{R}$ by $\overline{S}$)

$$\mathcal{M}_{\log-dR}(\overline{X}_{\overline{S}}/\overline{S}) = \mathcal{M}_{\log-dR}(\overline{X}_{\overline{S}}/\overline{S}) \times \overline{S} S$$

is finite and flat over $S$.

Since there are only finitely many log-dR local systems $(E^i_{\overline{C}}, \nabla^i_{\overline{C}})_{i \in I}$ over $\mathbb{C}$, we may assume that they have stable and locally free models $(E^i_S, \nabla^i_S)_{i \in I}$ over $S$. This amounts to property (a) above. By further enlarging $R$ we obtain strong cohomological rigidity (property (b)), and properties (c,d) about the $F$-filtrations and the associated graded log-Higgs bundles.

The $S$-models above give rise to sections

$$[(E^i_S, \nabla^i_S)]_{i \in I} : S \to \mathcal{M}_{\log-dR}(\overline{X}_{\overline{S}}/\overline{S}).$$

(A.1.1)

Since the structural morphism $\mathcal{M}_{\log-dR}(\overline{X}_{\overline{S}}/\overline{S}) \to S$ is finite and flat, we infer that the sections of (A.1.1) are jointly surjective. This implies (e). By inverting $(2r+2)!$ we can achieve (f). And, property (g) can be arranged by passing to the maximal open subset of $S$ which is smooth over $\text{Spec } \mathbb{Z}$. \qed
A.2 Applications of the Higgs-de Rham flow

In this subsection, we apply the logarithmic Higgs-de Rham flow from [LSYZ19] (the smooth and proper case is due to [LSZ13]).

We fix an arithmetic model as in Proposition A.10. Let $\bar{k}$ be an algebraic closure of a finite field and let $s \colon \text{Spec} \bar{k} \to S$ be a geometric point of $S$.

**Definition A.11 ([LSZ13] [LSYZ19]).** An $f$-periodic Higgs-de Rham flow on $X_s$ is a tuple

$$(E_0, \nabla_0, F_0, \phi_0, E_1, \nabla_1, F_1, \ldots, E_{f-1}, \nabla_{f-1}, F_{f-1}, \phi_{f-1}),$$

where for all $i \in \mathbb{Z}/f\mathbb{Z}$ we have a log-dR local system $(E_i, \nabla_i, F_i)$ with nilpotent $p$-curvature of level $\leq p - 1$, a Griffiths-transversal filtration $F_i$, and an isomorphism $\phi_i : C_{i-1}^{-1}(\text{gr}_p E_i, \text{KS}_i) \cong (E_{i+1}, \nabla_{i+1})$.

We denote the set of isomorphism classes of stable rank $r$ logarithmic Higgs bundles on $X_s$ with Hilbert polynomial $P_0$ by $M_{\text{Dol}}(s)$. Likewise, we write $M_{\text{dR}}(s)$ for the set of isomorphism classes of stable rank $r$ log-dR local systems on $X_s$ with Hilbert polynomial $P_0$. For the purpose of this subsection, it will not matter that those sets are $\bar{k}$-rational points of moduli spaces, which could be constructed with Langer’s methods (see Theorem A.6).

We informally refer to the following diagram as the **Higgs-de Rham flow**:

$$M_{\text{Dol}}(s) \xrightarrow{C^{-1}} \text{gr} \xrightarrow{\phi} M_{\text{dR}}(s).$$

The dashed arrows represent merely correspondences, rather than actual maps. The reason is that $\text{gr}(E, \nabla)$ could be not stable, and $C^{-1}$ can only be defined if the $p$-curvature is nilpotent of level $\leq p - 1$ and the residues at infinity are nilpotent.

Using this viewpoint, one calls an element $[(E, \nabla)]$ of $M_{\text{dR}}(s)$ periodic, if there exists $f \in \mathbb{N}$ with

$$[(E, \nabla)] = (C^{-1} \circ \text{gr})^f([(E, \nabla)]).$$

We let $R_{\text{Dol}}(s) \subseteq M_{\text{Dol}}(s)$ denote the subset of stable rank $r$ log Higgs bundles with nilpotent Higgs field $\theta$ and nilpotent $\text{res}_\mu \theta$ for all $\mu = 1, \ldots, c$ of level $\leq p - 1$. We denote by $R_{\text{dR}}(s) \subseteq M_{\text{dR}}(s)$ the subset of stable log-dR local systems with nilpotent residues or level $\leq p - 1$. Restricting the Higgs-de Rham flow to these subsets has the added advantage of turning the correspondences above into maps of sets:

$$R_{\text{Dol}}(s) \xrightarrow{C^{-1}} \text{gr} \xrightarrow{\phi} R_{\text{dR}}(s). \quad (A.2.1)$$

It is not immediately obvious that the above maps are well-defined, since one has to justify that strong cohomological rigidity and stability is preserved by $\text{gr}$ and $C^{-1}$.

**Lemma A.12.** The maps in (A.2.1) are well-defined.

**Proof.** Proposition A.10(c) allows us to fix for every $(E_s, \nabla_s) \in R_{\text{dR}}(s)$ an $F$-filtration. It follows from Proposition A.10(d) that $\text{gr}(E_s, \nabla_s) = (\text{gr}_p E_s, \text{KS})$ is stable. This shows that $\text{gr} : R_{\text{dR}}(s) \to R_{\text{Dol}}(s)$ is a well-defined map, which a priori depends on the chosen filtration (but see the end of the proof of Lemma A.15). Arguing as in [Lan14] Corollary 5.10 one shows that $C^{-1}$ preserves stability. \qed

**Lemma A.13.** Every element of $R_{\text{Dol}}(s)$ is strongly cohomologically rigid.

**Proof.** There is an equivalence of categories (see [LSYZ19] Theorem 6.1])

$$C^{-1} : \text{Higgs}_{p-1}(\bar{X}_s, D_s) \cong \text{MIC}_{p-1}(\bar{X}_s, D_s),$$

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where the left-hand side denotes a subcategory of logarithmic Higgs bundles $(V, \theta)$ satisfying several technical assumptions, and similarly, the right-hand side denotes a subcategory of log-dR local systems with nilpotent $p$-Higgs bundles which are required to satisfy various assumptions. We refer the reader to [LSYZ19 Section 6] for more details. This is an equivalence of categories, and therefore

$$\text{Ext}(C^{-1}(V_s, \theta_s), C^{-1}(V_s, \theta_s)) = \text{Ext}((V_s, \theta_s), (V_s, \theta_s)) = 0.$$ 

Here, we implicitly use Proposition A.10(e) to guarantee that all self-extensions of $(V_s, \theta_s)$ belong to $\text{Higgs}_{p-1}(X_s, D_s)$ (respectively $\text{MIC}_{p-1}(X_s, D_s)$). Indeed, since $p > 2r + 2$ by Proposition A.10(f), the Higgs field of such a self-extension is automatically nilpotent of level $\leq p - 1$. Thus, $C^{-1}$ preserves strong cohomological rigidity. The same assertion holds for its inverse functor $C$.

We conclude the proof of the lemma by applying assertion (e) of Proposition A.10, according to which every log-dR local system with Hilbert polynomial $P$ preserves strong cohomological rigidity. The same assertion holds for its inverse functor $C$. Therefore, for every $[(V, \theta)] \in R_{\text{dR}}(s)$ we have that $C^{-1}(V, \theta)$ is strongly cohomologically rigid. This implies that $(V, \theta) = \text{gr} \circ C^{-1}(V, \theta)$ is strongly cohomologically rigid. □

**Lemma A.14.** The set $R_{\text{dR}}(s)$ is finite.

**Proof.** Let $(E, \nabla)$ be a strongly cohomologically rigid log-dR local system which is stable and has Hilbert polynomial $P_0$. The hypercohomology group A.0.1 computes the tangent space of $\mathcal{M}_{\text{dR}}(X, D)$ in $\mathcal{M}(E, \nabla)$. By the vanishing assumption, the point $[(E, \nabla)]$ is isolated. We conclude the proof by recalling that the number of isolated points of a Noetherian scheme is finite. □

**Lemma A.15.** The maps $\text{gr}$ and $C^{-1}$ are bijections.

**Proof.** It suffices to prove that $\text{gr}$ and $C^{-1}$ are injective. Indeed, it then follows from Lemma A.14 they must be of equal cardinality if both maps are injective. The pigeonhole principle is used to conclude that $\text{gr}$ and $C^{-1}$ are bijections.

Since $C^{-1}$ is defined using an equivalence of categories (see [LSYZ19 Theorem 6.1]), it is clear that $C^{-1}: R_{\text{dR}}(s) \to R_{\text{dR}}(s)$ is injective.

The associated graded $\text{gr}$ is injective for different reasons. In particular, we will use strong cohomological rigidity to prove this. The Artin-Rees construction applied to the $F$-filtration on $(E, \nabla)$ yields a $\mathbb{G}_m$-equivariant $\mathbb{A}^1$-family of vector bundles $(V, \nabla_t)$, endowed with a log-$t$-connection $\nabla_t$, where $t: \mathbb{A}^1 \to \mathbb{A}^1_s$ denotes the identity map. Furthermore, we have $(V, \nabla_t)|_{t=0} \simeq (\text{gr}_F E, \text{KS})$. Recall from Proposition A.10(d) that the right-hand side is a strongly cohomologically rigid log-Higgs bundle. There is therefore a unique way to lift it to a $t$-connection over $\text{Spec} \bar{k}[t]/(t^2)$, and likewise for $\text{Spec} \bar{k}[t]/(t^n)$. We infer from the Grothendieck existence theorem that there is a unique way to lift it to a $t$-connection on $\text{Spec} \bar{k}[[t]]$. This implies that there cannot be a pair of distinct elements

$$(E^1_s, \nabla^1_s), (E^2_s, \nabla^2_s) \in R_{\text{dR}}(s)$$

such that $(\text{gr}_F E^1_s, \text{KS}) \simeq (\text{gr}_F E^2_s, \text{KS})$.

Otherwise, we would have

$$(E^1_s, \nabla^1_s) \otimes \bar{k}((t)) \simeq (E^2_s, \nabla^2_s) \otimes \bar{k}((t)),$$

which implies the existence of an isomorphism over $\bar{k}$ (by stability). This concludes the proof of injectivity, and furthermore proves that the map $\text{gr}$ doesn’t depend on the chosen $F$-filtration. □

**Proposition A.16.** The $p$-curvature of $[(E, \nabla)] \in R_{\text{dR}}(s)$ is nilpotent.

**Proof.** By virtue of definition of $C^{-1}$, every log-dR local system in the image of $C^{-1}$ has nilpotent $p$-curvature. According to Lemma A.15 the map $C^{-1}$ is bijective. This concludes the proof. □

**Proposition A.17.** Every $[(E, \nabla)] \in R_{\text{dR}}(s)$ is periodic.

**Proof.** Let $\sigma = C^{-1} \circ \text{gr}$. By definition, it is a permutation of the finite set $R_{\text{dR}}(s)$. Let $f'$ be the order of $\sigma$. We then have that $\sigma^{f'}([(E, \nabla)]) = [(E, \nabla)]$, and thus $[(E, \nabla)]$ is $f$-periodic for some $f | f'$. □
A.3 Higgs-de Rham flow over truncated Witt rings

As before, we denote by \( \bar{k} \) the algebraic closure of a finite field of characteristic \( p \), and let \( s : \text{Spec} \bar{k} \to S \) be a \( k \)-point of \( S \). Furthermore, we write \( W = W(\bar{k}) \) for the associated Witt ring, and \( K \) for its fraction field. Hensel’s lemma and Proposition A.10(g) implies that \( s \) can be extended to a morphism
\[
s_W : \text{Spec} W \to S.
\]

For \( n \in \mathbb{N} \) we denote by \( W_n \) the ring of \( n \)-th Witt vectors and by \( \bar{X}_n \) the base change \( \bar{X}_S \times_S W_n \).

We define \( \mathcal{H}(\bar{X}_n/W_n) \) to be the category of tuples \( (V, \theta, \bar{E}, \nabla, F, \phi) \), where \( (V, \theta) \) is a graded log-Higgs bundle on \( \bar{X}_n \) of level \( \leq p - 1 \), \( (\bar{E}, \nabla, F) \) is a log-dR local system on \( \bar{X}_{n-1} \) with a Griffiths-transversal filtration \( F \) of level \( \leq p - 2 \), and \( \phi : gr_F(\bar{E}, \nabla) \cong (V, \theta) \times_{W_n} W_{n-1} \) is an isomorphism of graded log-Higgs bundles.

Similarly, we denote by \( \text{MIC}(\bar{X}_n/W_n) \) the category of quasi-coherent sheaves with \( W_n \)-linear flat connections on \( \bar{X}_n \). There is a functor
\[
C_n^{-1} : \mathcal{H}(\bar{X}_n/W_n) \to \text{MIC}(\bar{X}_n/W_n)
\]
which extends the logarithmic inverse Cartier transform. In the proper non-logarithmic case this is due to \cite[Theorem 4.1]{LSZ13}. Closely related results were obtained by Xu in \cite[X.19]{Xu19}. The logarithmic version is covered in \cite[Section 5]{LSYZ19} immediately before the proof of Proposition 5.2.

Let \( (E_{W_n}, \nabla_{W_n}, F_n) \) be an \( W_n \)-linear log-dR local system endowed with an \( F \)-filtration. We denote by \( gr(E, \nabla, F) \) the tuple \( (gr_F(E), KS, (E, \nabla, F)_{W_{n-1}}, id) \).

**Definition A.18** (Lan–Sheng–Zuo & Lan–Shen–Yang–Zuo). An \( f \)-periodic log-dR local system on \( \bar{X}_n/W_n \) is a tuple
\[
(E_{W_n}^0, \nabla_{W_n}^0, F_{W_n}, \phi_0, E_{W_n}^1, \nabla_{W_n}^1, F_{W_n}, \ldots, E_{W_n}^{f-1}, \nabla_{W_n}^{f-1}, F_{W_n}, \phi_{f-1}),
\]
where for all \( i \) we have that \( (E_i, \nabla_i, F_i) \) is a log-dR local system on \( \bar{X}_{W_n} \) (nilpotent of level \( \leq p - 2 \) on the special fibre) with a Griffiths-transversal filtration \( F_{W_n} \), such that for all integers \( n \) we have that \( gr_F(E_{W_n}^i, \nabla_{W_n}^i) \) belongs to \( \mathcal{H}(\bar{X}_n/W_n) \) and \( \phi_i : C_i^{-1}(gr_F(E_{W_n}^i, KS_i) \cong (E_{W_n}^{i+1}, \nabla_{W_n}^{i+1}) \).

By taking the inverse limit with respect to \( n \), we obtain a notion of periodicity relative to \( W \). Using \cite[p.3, Theorem 3.2, Variant 2]{LSZ13}, \cite[Theorem 1.1]{LSYZ19} together with \cite[Theorem 2.6*,p.43 i)]{Fal88} one obtains:

**Theorem A.19** (Lan–Sheng–Zuo & Lan–Shen–Yang–Zuo). A 1-periodic log-dR local system on \( \bar{X}_W/W \) gives rise to a torsion-free Fontaine–Lafaille module on \( X_W = \bar{X}_W \setminus D_W \). Furthermore, we can associate to an \( f \)-periodic log-dR local system on \( \bar{X}_W/W \) a crystalline étale local system of free \( W(\mathbb{F}_p) \)-modules on \( X_K \). This is a fully faithful functor.

We remark that Faltings only treats the case \( f = 1 \), in which he constructs a fully faithful functor from Fontaine–Lafaille modules to étale local systems of \( \mathbb{Z}_p \)-modules. The general case can be reduced to this one using a categorical construction, as explained in \cite[Variant 2]{LSZ13}. It is clear that this formal procedure preserves fully faithfulness of the functor. In combination with the above, the following result concludes the proof of Theorem A.4.

**Theorem A.20.** Every element \( [(E_s, \nabla_s)] \in R_{dR}(s) \) can be lifted to a periodic Higgs-de Rham flow over \( W_n \) on \( \bar{X}_n \).

**Proof.** Recall from Lemma A.14 that there is a finite number of non-isomorphic log-dR local systems \( (E^s_\tau, \nabla^s_\tau)_{\tau \in I} \) in \( R_{dR}(s) \). For every \( \tau \in I \) there exists an extension to an \( S \)-family of log-dR local systems \( (E^s_\tau, \nabla^s_\tau) \) (see Proposition A.10(a,b)). By pulling back along \( s_W : \text{Spec} W \to S \) we therefore obtain a lift to a \( W \)-family \( (E^s_W, \nabla^s_W) \), and hence also a \( W_n \)-lift \( (E^s_{W_n}, \nabla^s_{W_n}) \).

Since \( (E^s_\tau, \nabla^s_\tau) \) is strongly cohomologically rigid, deformation theory implies that such a \( W \)-lift is unique up to isomorphism.
We have seen in Proposition [A.17] that every \((E_s^i, \nabla_s^i)\) is periodic over \(s\) (this corresponds to the case \(n = 0\)) since the map
\[
\sigma = C^{-1} \circ \text{gr}: R_{dR}(s) \to R_{dR}(s)
\]
is a permutation (Lemma [A.13]). The \(W_n\)-relative log-dR local system \((E_{W_n}^i, \nabla_{W_n}^i)\) is endowed with an \(F\)-filtration by Proposition [A.10] c). We can therefore evaluate \(\mathfrak{p}(E_{W_n}^i, \nabla_{W_n}^i)\).

Since the functor \(C_n^{-1}\) extends \(C^{-1}\) on the special fibre, we see that we have
\[
(C_n^{-1} \circ \text{gr})(E_{W_n}^i, \nabla_{W_n}^i) \simeq (E_{W_n}^\sigma(i), \nabla_{W_n}^\sigma(i)).
\]
In particular, for \((E_s^i, \nabla_s^i)\) being \(f\)-periodic, we have \(\sigma^f(i) = i\), and thus
\[
(C_n^{-1} \circ \text{gr})(E_{W_n}^i, \nabla_{W_n}^i) \simeq (E_{W_n}^i, \nabla_{W_n}^i).
\]
This equation establishes periodicity of \((E_{W_n}^i, \nabla_{W_n}^i)\) relative to \(W_n\).

We will now state a Betti version of Theorem [A.4] For this purpose, let us recall that the Betti moduli space \(\mathcal{M}_B(X)\) of irreducible rank \(r\) complex local systems of is zero-dimensional, since every such local system is assumed to be strongly cohomologically rigid. Furthermore, \(\mathcal{M}_B(X)\) is defined over \(\mathbb{Q}\), and therefore the irreducible rank \(r\) local systems \(\rho_1, \ldots, \rho_n\) are defined over a number field \(F\). By finite generation of \(\pi_1^{\text{top}}(X)\) we have that the representations \(\rho_1, \ldots, \rho_n\) can be defined over \(\mathcal{O}_F[M^{-1}]\), for a sufficiently big positive integer \(M\).

**Remark A.21.** As by Remark [A.2] a), \(H^1(X(\mathbb{C}), \text{End}(E_{C,n})^{\text{End}(\nabla_C)})\) computes the left-hand side of [A.01], so is equal to zero, we can apply [EG18, Theorem 1.1] to conclude that Simpson’s integrality conjecture holds. Therefore, \(M\) can be chosen to be \(1\). In fact, under the assumption that all log-dR bundles in a given rank are rigid, the proof of loc. cit. applies without verifying this vanishing assumption. We do not use this remark in the sequel.

For every prime \(p \nmid M\), and every choice of an embedding \(\mathcal{O}_F \to W(\overline{\mathbb{F}}_p)\) we can therefore consider the induced \(W(\overline{\mathbb{F}}_p)\)-representations
\[
\rho_1^{W(\overline{\mathbb{F}}_p)}, \ldots, \rho_n^{W(\overline{\mathbb{F}}_p)}: \pi_1^{\text{top}}(X_{\mathbb{C}}) \to \text{GL}_r(W(\overline{\mathbb{F}}_p)).
\]

The étale fundamental group \(\pi_1(X_{\mathbb{C}})\) is the profinite completion of \(\pi_1^{\text{top}}(X_{\mathbb{C}})\). Thus, we obtain continuous representations
\[
\rho_1^{W(\overline{\mathbb{F}}_p)}, \ldots, \rho_n^{W(\overline{\mathbb{F}}_p)}: \pi_1(X_{\mathbb{C}}) \to \text{GL}_r(W(\overline{\mathbb{F}}_p)).
\]

**Theorem A.22.** Let \((\tilde{X}_C, X_C, D_C)\) and \((\tilde{X}_S, X_S, D_S)\) be as in Theorem [A.4] Suppose that \(p\) is a prime, which belongs to the image of \(S \to \text{Spec } \mathbb{Z}\). Let \(k\) be a finite field of characteristic \(p\) and fix a morphism \(\text{Spec } \mathbb{C}(k) \to S\). Then, the representations \(\{\rho_1^{W(\overline{\mathbb{F}}_p)}\}_{i=1, \ldots, n_r}\) descend to crystalline representations \(\{\rho_1^{\text{cris}}\}_{i=1, \ldots, n_r}: \pi_1(X_{\mathbb{C}}) \to \text{GL}_r(W(\overline{\mathbb{F}}_p)), \text{ where } K = \text{Frac}(W(k))\).

**Proof.** By combining Theorem [A.19] and Theorem [A.4] we obtain crystalline representations
\[
\{\pi_i\}_{i=1, \ldots, n_r}: \pi_1(X_{\mathbb{C}}) \to \text{GL}_r(W(\overline{\mathbb{F}}_p))
\]
associated to the corresponding log-dR systems \((E_i^i, \nabla_i^i)_{i=1, \ldots, r}\) on \(\tilde{X}_K\) Restricting these representations further to the geometric fundamental group \(\pi_1(X_{\overline{\mathbb{F}}_p})\), we obtain
\[
(\pi_i^{\text{geom}})_{i=1, \ldots, n_r}: \pi_1(X_{\overline{\mathbb{F}}_p}) \to \text{GL}_r(W(\overline{\mathbb{F}}_p)).
\]

**Claim.** The geometric representations \(\{\pi_i^{\text{geom}}\}_{i=1, \ldots, n_r}\) are irreducible.
Proof of the claim. Assume by contradiction that there exists \( \pi_i^{\text{geom}} \) which is reducible. Then, the residual representation \( \pi_i^{\text{geom}} \otimes \overline{\mathbb{F}}_p \colon \pi_1(X_K) \to \text{GL}_r(\overline{\mathbb{F}}_p) \) is reducible as well. The continuous representation \( \pi_i \otimes \overline{\mathbb{F}}_p : \pi_1(X_K) \to \text{GL}_r(\overline{\mathbb{F}}_p) \) factors through the finite group \( \text{GL}_r(\mathbb{F}_q) \) for a \( p \)-power \( q \). By Proposition A.10 (g) we may assume that \( X_K \) has a rational point, which yields a section \( \text{Gal}(\overline{K}/K) \to \pi_1(X_K) \). The kernel of the restriction \( \pi_i|_{\text{Gal}(\overline{K}/K)} \) yields a finite extension \( K'/K \) such that \( \pi_i \otimes \overline{\mathbb{F}}_p|_{\pi_1(X_{K'})} \) is reducible.

Let \( \alpha \) be a subrepresentation of \( \pi_i \otimes \overline{\mathbb{F}}_p|_{\pi_1(X_{K'})} \). We will now use work by Sun–Yang–Zuo. It develops a version of the Higgs-de Rham flow over ramified extensions of \( W \). Theorem 5.15 in [SYZ22] implies that the subrepresentation \( \alpha \) gives rise to a sub-log-dR local system of \((E_i, \nabla)\), which is furthermore periodic and thus of slope 0. This contradicts stability. Note that loc. cit. deals with the more general setting of twisted Higgs-de Rham flows and projective representations. When applying their result we may therefore assume that the twisting line bundle \( \mathcal{L} \) is trivial, since our representations are not projective. \( \square \)

Claim. The geometric representations \( \pi_i^{\text{geom}} \) are pairwise non-isomorphic.

Proof of the claim. As before, it suffices to show that the residual \( \overline{\mathbb{F}}_p \)-representations are pairwise non-isomorphic. We will use the same strategy as before. An isomorphism between \( \pi_i^{\text{geom}} \otimes \overline{\mathbb{F}}_p \) and \( \pi_j^{\text{geom}} \otimes \overline{\mathbb{F}}_p \) therefore implies the existence of a finite extension \( K'/K \) such that there exists an isomorphism

\[
\pi_i^{\text{geom}} \otimes \overline{\mathbb{F}}_p \cong \pi_j^{\text{geom}} \otimes \overline{\mathbb{F}}_p : \pi_1(X_{K'}) \to \text{GL}_r(\overline{\mathbb{F}}_p).
\]

According to [SYZ22] Theorem 5.15 (which relies on [SYZ22] Theorem 5.8(ii)), the functor from periodic Higgs-de Rham flows to \( \overline{\mathbb{F}}_p \)-linear representations of \( \pi_1(X_{K'}) \) is fully faithful. Thus, we obtain an isomorphism of the associated Higgs-de Rham flows, and in particular we have that \( (E_i, \nabla_i)_{K'} \cong (E_j, \nabla_j)_{K'} \). This implies \( i = j \) since the associated complex log-dR local systems \((E_i, \nabla_i)_{\mathbb{C}} \) and \((E_j, \nabla_j)_{\mathbb{C}} \) are non-isomorphic, and hence concludes the proof. \( \square \)

Applying these two claims we see that the geometric representations

\[
\pi_i|_{\pi_1(X_K)} : \pi_1(X_K) \to \text{GL}_r(W(\overline{\mathbb{F}}_p))
\]

for \( i = 1, \ldots, n_r \) remain irreducible since the set \( \{\pi_i|_{\pi_1(X_K)}\}_{i=1, \ldots, n_r} \) defines \( n_r \) pairwise non-isomorphic \( W(\overline{\mathbb{F}}_p) \)-local systems on \( X_K \), which by the pigeonhole principle has to be the set of \( W(\overline{\mathbb{F}}_p) \)-local systems defined by \( \{\rho^i_{W(\overline{\mathbb{F}}_p)}\}_{i=1, \ldots, n_r} \), and thus each single one of them descends to a crystalline representation. \( \square \)

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