Large Deviation Principle for Some Measure-Valued Processes

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Abstract

We establish a large deviation principle for the solutions of a class of stochastic partial differential equations with non-Lipschitz continuous coefficients. As an application, the large deviation principle is derived for super-Brownian motion and Fleming-Viot process.

Key words: Large deviation principle, stochastic partial differential equation, Fleming-Viot process, super-Brownian motion.

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1 Introduction

Measure-Valued processes (MVP) arise from many fields of applications including population growth models and genetics. We refer the reader to the books of Dawson [2], Etheridge [8], Perkins [15], and Li [13] for an introduction to this topic. Two of the most studied measure-valued processes are super-Brownian motion (SBM) and Fleming-Viot process (FVP). An interesting problem concerns the limiting behavior of these processes when the branching rate (for SBM) or the mutation rate (for FVP) $\epsilon$, tends to zero. It is easy to see that the measure-valued processes, denoted by $\mu^\epsilon$, converge to a deterministic measure-valued process $\mu^0$, and it is desirable to study this rate of convergence.

Large deviation principle (LDP) is a very useful tool for the study of convergence rate. Roughly speaking, the goal of the LDP is to determine the rate $R(\delta) > 0$, for any $\delta > 0$ such that as $\epsilon \to 0$,

$$P\left(\rho\left(\mu^\epsilon, \mu^0\right) > \delta\right) \approx \exp\left(-\epsilon^{-1}R(\delta)\right),$$  

(1)

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for a suitable distance, $\rho$ in $C([0, 1]; \mathcal{M}_\beta(\mathbb{R}))$, the state space of the MVP, where $\mathcal{M}_\beta(\mathbb{R})$ is the set of $\sigma$-finite measures $\mu$ on $\mathbb{R}$ such that
\[
\int e^{-\beta|x|} d\mu(dx) < \infty.
\]
(2)

We refer the reader to the books of Dembo and Zeitouni [5], Deuschel and Stroock [6], and Dupuis and Ellis [7] for more background on this subject.

LDP for MVP has been studied by many authors. Fleischmann and Kaj [11] proved the LDP for SBM for a fixed time $t$. Later on, sample path LDP for SBM was derived independently by Fleischmann et al [10], and Schied [16] while the rate function was expressed by a variational form. To obtain an explicit expression for the rate function, [10] assumes a local blow-up condition which is not proven. On the other hand, [16] obtains the explicit expression of the rate function when the term representing the movements of the particles also tends to zero. The local blow-up condition of [10] was recently removed by Xiang for SBM with finite and infinite initial measure. [20], [19] respectively, and the same explicit expression was established. Fleischmann and Xiong [12] proved an LDP for catalytic SBM with a single point catalyst. The successes of the LDP for SBM depend on the branching property of this process. This property implies the weak LDP directly, and hence the problem diminishes to showing the exponential tightness of SBM, which yields the LDP, and identifying its rate function.

Since FVP does not possess the branching property, the derivation of LDP depends on new ideas. Dawson and Feng [4], [3], and Feng and Xiong [9] considered the LDP for FVP when the mutation is neutral. In [4], LDP was shown to hold when the process remains in the interior of the simplex, and in [3] the authors proved that if the process starts from the interior, it will not reach the boundary. On the other hand, authors in [9] focused on the singular case when the process starts from the boundary. For non-neutral case, Xiang and Zhang [21] derived an LDP for FVP when the mutation operator also tends to zero by projecting to the finite dimensional case.

The goal of this paper is to study LDP for MVP, with SBM and FVP as special cases. Comparing our LDP for SBM with that obtained in [10], [16] and [19], the rate function has the same explicit representation, but the approach is different. Our LDP for FVP contributes to the literature, by not requiring the neutrality and vanishing of mutation.
2 Notations and Main Results

Let \((\Omega, \mathcal{F}, P, \mathcal{F}_t)\) be a stochastic basis satisfying the usual conditions of right continuity and completeness. Suppose \(W\) is an \(\mathcal{F}_t\)-adapted space-time white noise random measure on \(\mathbb{R}_+ \times \mathcal{U}\) with intensity measure \(ds\lambda(da)\), where \((\mathcal{U}, \mathcal{U}, \lambda)\) is a measure space.

We consider the following stochastic partial differential equation (SPDE):

\[
\begin{align*}
\dot{u}_t^\epsilon(y) &= F(y) + \sqrt{\epsilon} \int_0^t \int_{\mathcal{U}} G(a, y, u_s^\epsilon(y)) \, W(da, ds) + \int_0^t \frac{1}{2} \Delta u_s^\epsilon(y) \, ds,
\end{align*}
\]

where \(F\) is a function on \(\mathbb{R}\) and \(G : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R}\) satisfies the following conditions: there exists a constant \(K > 0\) such that for any \(u_1, u_2, u, y \in \mathbb{R}\),

\[
\int_{\mathcal{U}} |G(a, y, u_1) - G(a, y, u_2)|^2 \lambda(da) \leq K |u_1 - u_2| \tag{4}
\]

and

\[
\int_{\mathcal{U}} |G(a, y, u)|^2 \lambda(da) \leq K (1 + |u|^2) \tag{5}
\]

This SPDE was studied by Xiong [22] in a Hilbert space denoted by him as \(\chi_0\). To study the LDP for the random field \(\{u_t^\epsilon(y)\}\), we need to consider the SPDE in a certain Hölder continuous space; that is, we study the regularity of the solution. For this purpose the spaces for the solution are introduced. Let \(\{\phi_j\}_{j \geq 1}\) be a complete orthonormal system (CONS) of \(L^2(\mathcal{U}, \mathcal{U}, \lambda)\) and define a system of stochastic processes as,

\[
B_j^t = \int_0^t \int_{\mathcal{U}} \phi_j(a) \, W(da, ds), \quad j = 1, 2, \cdots. \tag{6}
\]

which by Lévy’s characterization of Brownian motions, is a sequence of independent Brownian motions. Denote the measurable space,

\[
(S, \mathcal{S}) := (C([0, 1]; \mathbb{R}^\infty), \mathcal{B}(C([0, 1]; \mathbb{R}^\infty))).
\]

For any \(\alpha \in (0, 1)\) and \(0 < \beta \in \mathbb{R}\), let the space, \(\mathcal{B}_{\alpha, \beta}\) be the collection of all functions \(f : \mathbb{R} \rightarrow \mathbb{R}\) such that for all \(m \in \mathbb{N}\),

\[
|f(y_1) - f(y_2)| \leq Ke^{\beta m} |y_1 - y_2|^\alpha, \quad \forall |y_1|, |y_2| \leq m, \tag{7}
\]

and

\[
|f(y)| \leq Ke^{\beta |y|}, \quad \forall y \in \mathbb{R}. \tag{8}
\]
We define the metric on $B_{\alpha,\beta}$ as follows:

$$d_{\alpha,\beta}(u, v) = \sum_{m=1}^{\infty} 2^{-m} (\|u - v\|_{m,\alpha,\beta} \wedge 1), \quad u, v \in B_{\alpha,\beta}$$

where

$$\|u\|_{m,\alpha,\beta} = \sup_{x \in \mathbb{R}} e^{-\beta|\alpha x|} |u(x)| + \sup_{y_1 \neq y_2} \left| \frac{u(y_1) - u(y_2)}{|y_1 - y_2|^\alpha} \right| e^{-\beta m}.$$

Notice that SPDE (3) can be rewritten as an SPDE driven by Brownian motions, $\{B_t^i\}$ as follows:

$$u_{\varepsilon}^t(y) = F(y) + \sqrt{\varepsilon} \sum_j \int_0^t G_j(y, u_{\varepsilon}^s(y))dB^j_s + \int_0^t \frac{1}{2} \Delta u_{\varepsilon}^s(y)ds, \quad (9)$$

where

$$G_j(y, u) = \int_U G(a, y, u)\phi_j(a)\lambda(da), \quad j = 1, 2, \ldots. \quad (10)$$

In this paper, we let $\beta_0 \in (0, \beta)$.

**Theorem 1.** For any $\alpha \in (0, \frac{1}{2})$, there exists a measurable map, $g^\varepsilon : B_{\alpha,\beta_0} \times \mathbb{S} \rightarrow C([0,1]; B_{\alpha,\beta})$ such that for $F \in B_{\alpha,\beta_0}$, $u^\varepsilon = g^\varepsilon(F, \sqrt{\varepsilon}B)$ is the unique mild solution of (3).

In order to study the LDP of the process $u_{\varepsilon}^t$, one needs to consider the controlled version of (3) with the noise replaced by the control. For any $h \in L^2([0,1] \times U, ds\lambda(da))$, this version has the following deterministic form,

$$u_t(y) = F(y) + \int_0^t \int_U G(a, y, u_s(y))h_s(a)\lambda(da)ds + \int_0^t \frac{1}{2} \Delta u_s(y)ds \quad (11)$$

Because of the non-Lipschitz continuity of the coefficient, the topology of the state space, $C([0,1]; B_{\alpha,\beta})$, needs to be modified.

**Definition 1.** We say that $u, v \in C([0,1]; B_{\alpha,\beta})$ are equivalent, denoted by $u \sim v$, if there exists an $h \in L^2([0,1] \times U, ds\lambda(da))$ such that both $u, v$ are solutions to (11). If $u$ is not a solution to equation (11) for a suitable $h$, then $u$ belongs to the equivalent class consisting of itself only.

From this point on, we establish the LDP of $u^\varepsilon$ in the quotient space of $C([0,1]; B_{\alpha,\beta})$ under the equivalence relation $\sim$ given above. We abuse the notation a bit by using the same notation for this quotient space. Note that
when \( h = 0 \), equation (11) has a unique solution, \( u^0_t(y) \). Therefore, this modification of topology does not affect the exponential rate of the form (11) derived from the LDP at a neighborhood of \( u^0 \).

Let \( \gamma \) be a map from \( \mathbb{B}_{\alpha,\beta_0} \times L^2([0,1] \times U, \lambda(da)) \) to \( C([0,1]; \mathbb{B}_{\alpha,\beta}) \) whose domain consists of \( (F,h) \) such that (11) has a solution, and denote the equivalence class of the solution as \( u = \gamma(F,h) \).

**Theorem 2.** Suppose \( F \in \mathbb{B}_{\alpha,\beta_0} \), then the family \( \{u^\epsilon\} \) satisfies the LDP in \( C([0,1]; \mathbb{B}_{\alpha,\beta}) \) with rate function,

\[
I(u) = \begin{cases} 
\frac{1}{2} \inf \left\{ \int_0^1 \int_U |h_s(a)|^2 \lambda(da) ds : u = \gamma(F,h) \right\} & \exists h \text{ s.t. } u = \gamma(F,h) \\
\infty & \text{otherwise.} 
\end{cases}
\] (12)

We now apply Theorem 2 to SBM and FVP. Suppose \( \{\mu^\epsilon\} \) is an SBM with branching rate \( \epsilon \). As indicated by Xiong [22], for all \( y \in \mathbb{R} \),

\[
u^\epsilon_t(y) = \int_0^y \mu^\epsilon_t(dx)
\] (13)

is the unique solution to SPDE (3) with

\[
F(y) = \int_0^y \mu_0(dx), \quad U = \mathbb{R}, \quad \lambda(da) = da \quad \text{and} \quad G(a,y,u) = 1_{a<u}.
\] (14)

Assume \( \mathcal{D} \) is the Schwartz space of test functions with compact support in \( \mathbb{R} \) and continuous derivatives of all orders. Denote the dual space of real distributions on \( \mathbb{R} \) by \( \mathcal{D}^\ast \). Similar to [10], for a fixed \( \nu \in M_{\beta}(\mathbb{R}) \), let the Cameron-Martin space, \( H_\nu \), be the set of measures \( \mu \in C([0,1]; M_{\beta}(\mathbb{R})) \) satisfying the conditions below.

1. \( \mu_0 = \nu \),

2. the \( \mathcal{D}^\ast \)-valued map \( t \mapsto \mu_t \) defined on \([0,1]\) is absolutely continuous with respect to time. Let \( \dot{\mu} \) and \( \Delta^\ast \mu \) be its generalized derivative and Laplacian respectively,

3. for every \( t \in [0,1] \), \( \dot{\mu}_t - \frac{1}{2} \Delta^\ast \mu_t \in \mathcal{D}^\ast \) is absolutely continuous with respect to \( \mu_t \) with \( \frac{d(\mu_t - \frac{1}{2} \Delta^\ast \mu_t)}{d\mu_t} \) being the (generalized) Radon Nikodym derivative,

4. \( \frac{d(\mu_t - \frac{1}{2} \Delta^\ast \mu_t)}{d\mu_t} \) is in \( L^2([0,1] \times \mathbb{R}, ds\mu(dy)) \).
The topology of $M_\beta(\mathbb{R})$ is defined by the following modified weak convergence topology. We say that $\mu^n \rightarrow \mu$ in $M_\beta(\mathbb{R})$ if for any $f \in C_b(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)e^{\beta|x|}\mu^n(dx) \rightarrow \int_{\mathbb{R}} f(x)e^{\beta|x|}\mu(dx).$$

**Theorem 3.** If $\mu_0 \in M_\beta(\mathbb{R})$ such that $F \in \mathcal{B}_{\alpha,\beta}$, then $\{\mu^\epsilon\}$ satisfies the LDP on $C([0,1];M_\beta(\mathbb{R}))$ with rate function,

$$I(\mu) = \begin{cases} 
\frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left( \frac{\dot{\mu}_t - \frac{1}{2}\Delta^*\mu_t}{\mu_t(dy)} \right)^2 \mu_t(dy)dt & \text{if } \mu \in H_{\mu_0} \\
\infty & \text{otherwise.} 
\end{cases}$$

(15)

As for FVP, if $\{\mu^\epsilon\}$ is an FVP, then $u^\epsilon$ is defined as

$$u^\epsilon_t(y) = \mu^\epsilon_t((-\infty,y])$$

for all $y \in \mathbb{R}$, and by this definition, $u^\epsilon_t$ is the solution of SPDE (3) with

$$F(y) = \mu_0((-\infty,y]), \quad U = [0,1], \quad \lambda(da) = da \quad \text{and } G(a,y,u) = 1_{a<u} - u. \quad \text{(16)}$$

In this case, let $\tilde{H}_{\nu}$ be the space for which conditions for $H_{\nu}$ hold with $M_\beta(\mathbb{R})$ replaced by $\mathcal{P}_\beta(\mathbb{R}) := M_\beta(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, the collection of Borel probability measures on $\mathbb{R}$, and with the additional assumption,

$$\left\langle \mu_t, \left( \frac{\dot{\mu}_t - \frac{1}{2}\Delta^*\mu_t}{\mu_t(dy)} \right) \right\rangle = 0.$$

**Theorem 4.** Suppose $\mu_0 \in \mathcal{P}_\beta(\mathbb{R})$ such that $F \in \mathcal{B}_{\alpha,\beta}$. Then, $\{\mu^\epsilon\}$ satisfies the LDP on $C([0,1];\mathcal{P}_\beta(\mathbb{R}))$ with rate function,

$$I(\mu) = \begin{cases} 
\frac{1}{2} \int_0^1 \int_{\mathbb{R}} \left( \frac{\dot{\mu}_t - \frac{1}{2}\Delta^*\mu_t}{\mu_t(dy)} \right)^2 \mu_t(dy)dt & \text{if } \mu \in \tilde{H}_{\mu_0} \\
\infty & \text{otherwise.} 
\end{cases}$$

(17)

Proofs of Theorems 1-4 will be given in Sections 3-6. Throughout the rest of this paper, $K$ will denote a constant whose value can be changed from place to place.
3 Regularity of SPDE

This section is devoted to the proof of Theorem 1. For the simplicity of notation, we take $\epsilon = 1$ and denote $u_t^{\epsilon}(y)$ by $u_t(y)$. The solution to SPDE (3) can then be written in the following mild form,

$$u_t(y) = \int_{\mathbb{R}} p_t(y - x) F(x) dx + \int_0^t \int_{\mathbb{R}} p_{t-s}(y - x) G(a, x, u_s(x)) W(dsda) dx$$

(18)

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp \left(-\frac{x^2}{2t}\right)$ is the heat kernel. Here we refer to the first term on the RHS of (18) by $u_0^\epsilon t(y)$, and the second term by $v_t(y)$.

The following lemma offers an estimate that is the starting point for other more refined estimates of the solution. The proof is identical to that of Lemma 2.3 in [22] so we omit it.

**Lemma 1.** For any $n \geq 2$ and $\beta_1 \in (\beta_0, \beta)$, we have

$$M := \sup_{0 \leq s \leq 1} \mathbb{E} \left( \int_{\mathbb{R}} |u_s(x)|^2 e^{-2\beta_1|x|} dx \right)^n < \infty. \quad (19)$$

Inspired by Shiga [17], to obtain the regularity of the solution to SPDE (18) and for its tightness to be used in a later section, the following refined version of Kolmogorov’s criterion is proved and applied.

**Lemma 2.** Let $\{u_t^\epsilon(y)\}$ be a sequence of random fields and suppose $\beta_1 \in (\beta_0, \beta)$. If there exist constants $n, q, K > 0$ such that

$$\mathbb{E} \left| u_{t_1}^\epsilon(y_1) - u_{t_2}^\epsilon(y_2) \right|^n \leq Ke^{n\beta_1(|y_1| + |y_2|)} (|y_1 - y_2| + |t_1 - t_2|)^{2+q}, \quad (20)$$

then,

$$\sup_{\epsilon > 0} \mathbb{E} \left| \sup_{m} \sup_{t_i \in [0,1], |y| \leq m, i=1,2} \frac{|u_{t_1}^\epsilon(y_1) - u_{t_2}^\epsilon(y_2)|}{(|y_1 - y_2| + |t_1 - t_2|)^{n\beta}} e^{-\beta m} \right|^n < \infty. \quad (21)$$

Furthermore, if $\sup_{\epsilon > 0} \mathbb{E} \left| u_{t_0}^\epsilon(y_0) \right|^n < \infty$ for some $(t_0, y_0) \in [0,1] \times \mathbb{R}$, then

$$\sup_{\epsilon > 0} \mathbb{E} \left| \sup_{(t,y) \in [0,1] \times \mathbb{R}} e^{-\beta |y|} |u_t^\epsilon(y)| \right|^n < \infty. \quad (22)$$

With the above additional assumption, the sequence $\{u^\epsilon\}$ is tight in $C([0,1]; \mathbb{B}_{\alpha,\beta})$. 

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Proof. For \( i = 1, 2 \), let \( y'_i := \frac{1}{m} y_i \) and \( \tilde{u}^r_i(y'_i) := u^r_i(y_i) \). By the hypothesis,

\[
\mathbb{E} |\tilde{u}^r_1(y_1') - \tilde{u}^r_2(y_2')|^n = \mathbb{E} |u^r_1(my_1') - u^r_2(my_2')|^n \\
\leq Ke^{n\beta_1(\|y_1'\|+|y_2'|)} (m|y_1' - y_2'| + |t_1 - t_2|)^{2+q} \\
\leq Km^{2+q}e^{n\beta_1m} (|y_1' - y_2'| + |t_1 - t_2|)^{2+q}. \tag{23}
\]

By Kolmogorov’s criterion (cf. Corollary 1.2 in Walsh [18]), there exists a random variable \( Y_m \) such that \( \mathbb{E} Y_m^n \leq Km^{2+q}e^{n\beta_1m} \) and

\[
|\tilde{u}^r_1(y_1') - \tilde{u}^r_2(y_2')| \leq Y_m (|y_1' - y_2'| + |t_1 - t_2|)^{q/n}
\]

therefore,

\[
|u^r_1(y_1) - u^r_2(y_2)| \leq Y_m (|y_1 - y_2| + |t_1 - t_2|)^{q/n}. \tag{24}
\]

Let \( Y := \sup_m \{Y_m e^{-\beta m}\} \). Then,

\[
\mathbb{E} Y^n \leq \sum_m Y^n_m e^{-\beta mn} \tag{25}
\]

\[
= \sum_m \mathbb{E} Y^n_m e^{-\beta mn} \\
\leq \sum_m Km^{2+q}e^{-(\beta-\beta_1)mn} < \infty.
\]

Thus, \( Y \) is a finite random variable, and (24) implies

\[
|u^r_1(y_1) - u^r_2(y_2)| \leq Ye^{\beta m} (|y_1 - y_2| + |t_1 - t_2|)^{q/n}. \tag{26}
\]

Now, we suppose there exists \( (t_0, y_0) \in [0, 1] \times \mathbb{R} \) such that

\[
\sup_{t>0} \mathbb{E}|u^r_{t_0}(y_0)|^n < \infty.
\]

Note that (26) remains true with \( \beta \) replaced by \( \beta_2 \in (\beta_1, \beta) \). For the simplicity of notation, we choose \( t_0 = y_0 = 0 \). Taking \( t_1 = t, y_1 = y \) and \( t_2 = y_2 = 0 \) in (26), gives

\[
|u^r_t(y)| \leq |u^r_0(0)| + Ye^{\beta m} (|y| + |t|)^{q/n}.
\]

Suppose that \( |y| \leq m \). Then,

\[
e^{-\beta|y|}|u^r_t(y)| \leq e^{-\beta|y|} |u^r_0(0)| + Ye^{-(\beta-\beta_2)|y|} e^{\beta_2} (|y| + |t|)^{q/n} \\
\leq K \left( e^{-\beta_0|y|} |u^r_0(0)| + Y \right)
\]
for a suitable constant \( K \) (independent of \( m \)). Inequality \((22)\) then follows easily.

Uniform boundedness and equicontinuity are implied by \((22)\) and \((21)\), respectively. Therefore, tightness of the sequence follows from Arzelà-Ascoli and Prohorov theorems.

The following lemmas illustrate \( u_0^t \) and \( v_t \) are included in \( \mathbb{B}_{\alpha,\beta} \) space. These lemmas along with the result in \([22]\) on existence and uniqueness of a mild solution to SPDE \((3)\), prove Theorem 1.

**Lemma 3.** \( u_0^t \) is an element of \( C([0,1];\mathbb{B}_{\alpha,\beta}) \).

**Proof.** Suppose \( t \in [0,1] \) and \( y_1, y_2 \) are any real numbers such that \(|y_i| \leq m\) for \( i = 1,2 \). Let \( B_t \) be a Brownian motion. Then,

\[
u_0^t(y) = \mathbb{E} F(y - B_t).
\]

Choosing \( \gamma > 0 \) such that \((1 + \gamma) \beta_0 \leq \beta \) gives,

\[
|u_0^t(y_1) - u_0^t(y_2)| \leq \mathbb{E} |F(y_1 - B_t) - F(y_2 - B_t)|
\]

\[
= \sum_{j=0}^{\infty} \mathbb{E} |F(y_1 - B_t) - F(y_2 - B_t)| 1_{j m \gamma \leq |B_t| \leq (j + 1) \gamma m}
\]

\[
\leq \sum_{j=0}^{\infty} Ke^{((j+1) \gamma + 1) \beta_0 m} |y_1 - y_2|^\alpha P(|B_t| \geq j \gamma m)
\]

\[
\leq K \sum_{j=0}^{\infty} e^{(j+1) \gamma \beta_0 m - \frac{1}{2} j^2 m^2 \gamma^2 + \beta_0 m} |y_1 - y_2|^\alpha
\]

\[
= Ke^{\beta_0 m} \sum_{j=0}^{\infty} e^{(j+1) \beta_0 m - \frac{1}{2} j^2 m^2 \gamma^2} |y_1 - y_2|^\alpha
\]

\[
\leq Ke^{(1+\gamma) \beta_0 m \gamma} |y_1 - y_2|^\alpha
\]

\[
\leq Ke^{m \gamma} |y_1 - y_2|^\alpha
\] (27)

On the other hand, let \( y \) in \( \mathbb{R} \) be fixed such that \(|y| \leq m\), then for any
0 < t_1 \leq t_2 < 1,

\begin{align*}
&\left| u_0^0(y) - u_0^0(y) \right| \\
&\leq \mathbb{E} \left| F(y - B_{t_1}) - F(y - B_{t_2}) \right| \\
&= \sum_{j_1, j_2} \mathbb{E} \left| F(y - B_{t_1}) - F(y - B_{t_2}) \right| \mathbf{1}_{j_1 m \vee j_2 m \leq |B_{t_1}| \leq (j_1 + 1)m \gamma} \cdot \mathbf{1}_{j_2 m \gamma \leq |B_{t_2}| \leq (j_2 + 1)m \gamma} \\
&\leq \sum_{j_1, j_2} K e^{(m \vee (j_1 + 1)m \gamma) \beta_0} \mathbb{E} \left| B_{t_1} - B_{t_2} \right|^\alpha \mathbf{1}_{|B_{t_1}| \geq j_1 m \gamma} \mathbf{1}_{|B_{t_2}| \geq j_2 m \gamma} \\
&\leq \sum_{j_1, j_2} K e^{(j_1 \vee j_2 + 1)m \gamma + m \beta} (\mathbb{E} |B_{t_1} - B_{t_2}|^{2\alpha})^{\frac{1}{2}} \mathbb{P} \left( |B_{t_1} - B_{t_2}| \geq j_1 m \gamma | B_{t_2} | \geq j_2 m \gamma \right)^{\frac{1}{2}} \\
&\leq \sum_{j_1, j_2} K e^{(j_1 \vee j_2 + 1) \gamma + \beta_0 m} |t_1 - t_2|^{\alpha/2} e^{-\frac{1}{2} m \gamma^2 (j_1^2 + j_2^2)} \\
&\leq K (m \gamma)^2 e^{\beta_0 (1 + \gamma)m} |t_1 - t_2|^{\alpha/2} \\
&\leq K e^{\beta_0 |t_1 - t_2|^{\alpha/2}}. \tag{28}
\end{align*}

Estimates (27) and (28) imply that $u_0^0 \in C([0, 1]; \mathbb{B}_{\alpha, \beta})$. \hfill \Box

**Lemma 4.** v. takes values in $C([0, 1]; \mathbb{B}_{\alpha, \beta})$, a.s.

**Proof.** Similar to the proof of Lemma 3, two cases are demonstrated for this lemma. Considering the first case, denote

\[ G := G(a, x, u_s(x)) \text{ and } P_1 := p_{t-s}(y_1 - x) - p_{t-s}(y_2 - x) \]

and let $t \in [0, 1]$ be fixed, while $y_1, y_2 \in \mathbb{R}$ are arbitrary numbers such that $|y_i| \leq m$ for $i = 1, 2$. Using Burkholder-Davis-Gundy and Hölder’s inequalities, we obtain,

\begin{align*}
\mathbb{E} \left| v_1(y_1) - v_1(y_2) \right|^n &= \mathbb{E} \left| \int_0^t \int_U P_1 G dx W(ds, da) \right|^n \\
&\leq K \mathbb{E} \left| \int_0^t \int_U \left| \int_\mathbb{R} P_1 G dx \right|^2 \lambda(da) ds \right|^{n/2} \\
&\leq K \mathbb{E} \left| \int_0^t \int_U \int_\mathbb{R} |P_1|^2 e^{2\beta_1 |x|} dx \int_\mathbb{R} G^2 e^{-2\beta_1 |x|} dx \lambda(da) ds \right|^{n/2} \\
&\leq K \mathbb{E} \left| \int_0^t \int_{F_{t-s}(y_1, y_2)} (1 + |u_s(x)|^2) e^{-2\beta_1 |x|} dx ds \right|^{n/2}
\end{align*}

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where
\[ J_s(y_1, y_2) = \int_{\mathbb{R}} |p_s(y_1 - x) - p_s(y_2 - x)|^2 e^{2\beta_1|x|} \, dx \]
is estimated below using the simplified notation,
\[ P_2 := p_s(y_1 - x) - p_s(y_2 - x). \]

\[ J_s(y_1, y_2) = \int_{\mathbb{R}} |P_2|^\alpha |P_2|^{2-\alpha} e^{2\beta_1|x|} \, dx \]

\[ \leq \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi s}} \right|^\alpha \frac{(y_1 - x)^2 - (y_2 - x)^2}{2s} \left| P_2 \right|^{2-\alpha} e^{2\beta_1|x|} \, dx \]

\[ \leq K \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi s}} \right|^\alpha \frac{|y_1 - y_2|^\alpha |y_1 + y_2 - 2x|^\alpha}{(2s)^\alpha(2\pi s)^{(2-\alpha)/2}} e^{-\frac{(2-\alpha)(y_1 - x)^2}{2s}} e^{2\beta_1|x|} \, dx \]

\[ + K \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2\pi s}} \right|^\alpha \frac{|y_1 - y_2|^\alpha |y_1 + y_2 - 2x|^\alpha}{(2s)^\alpha(2\pi s)^{(2-\alpha)/2}} e^{-\frac{(2-\alpha)(y_2 - x)^2}{2s}} e^{2\beta_1|x|} \, dx \]

\[ \leq K |y_1 - y_2|^\alpha \frac{s^{-(1+\alpha)}}{s^{-\frac{1}{2}(1+\alpha)}} \int_{\mathbb{R}} |y_1 + y_2 - 2x|^\alpha e^{-\frac{(2-\alpha)(y_1 - x)^2}{2s}} e^{2\beta_1|x|} \, dx \]

\[ + K |y_1 - y_2|^\alpha \frac{s^{-(1+\alpha)}}{s^{-\frac{1}{2}(1+\alpha)}} \int_{\mathbb{R}} |y_1 + y_2 - 2x|^\alpha e^{-\frac{(2-\alpha)(y_2 - x)^2}{2s}} e^{2\beta_1|x|} \, dx \]

\[ \leq K e^{2\beta_1(|y_1| + |y_2|)} s^{-\frac{1}{2}(1+\alpha)} |y_1 - y_2|^\alpha. \]

Note that, we may choose \( p > 1 \) such that \( \left( \frac{1}{2} + \alpha \right) p < 1 \) and let \( q \) be the conjugate index. Then,

\[ \mathbb{E} \left( \int_0^t (t-s)^{-\left(\frac{1}{2} + \alpha\right)} \int_{\mathbb{R}} (1 + |u_1(x)|^2)^{2p} e^{-2\beta_1|x|} \, dx \, ds \right)^{n/2} \]

\[ \leq \left( \int_0^t (t-s)^{-\left(\frac{1}{2} + \alpha\right)p} ds \right)^{n/(2p)} \mathbb{E} \left( \int_0^t \left( \int_{\mathbb{R}} (1 + |u_1(x)|^2)^{2p} e^{-2\beta_1|x|} \, dx \right)^q ds \right)^{n/(2q)} \]

\[ \leq K \mathbb{E} \int_0^t \left( \int_{\mathbb{R}} (1 + |u_1(x)|^2)^{2p} e^{-2\beta_1|x|} \, dx \right)^{n/2} ds \]

\[ \leq K. \] (31)

Plugging (30) back into (29) and noting (31) to obtain,

\[ \mathbb{E}|v_t(y_1) - v_t(y_2)|^n \leq K e^{n\beta_1(|y_1| + |y_2|)} |y_1 - y_2|^{\frac{n p}{2}}. \]

Next to prove case two, let \( y \in \mathbb{R} \) and choose any \( 0 \leq t_1 < t_2 \leq 1 \). Note
that

$$\mathbb{E} \left| v_{t_1}(y) - v_{t_2}(y) \right|^n \leq K \mathbb{E} \left| \int_0^{t_1} I_s(t_1, t_2) \int_{\mathbb{R}} (1 + |u_s(y)|^2) e^{-2\beta|x|} dx ds \right|^{n/2} + K \mathbb{E} \left| \int_{t_1}^{t_2} \int_{\mathbb{R}} p_{t_2-s}(y-x)e^{2\beta|x|} dx \int_{\mathbb{R}} (1 + |u_s(x)|^2) e^{-2\beta|x|} dx ds \right|^{n/2}$$

where

$$I_s(t_1, t_2) := I_s^1(t_1, t_2) + I_s^2(t_1, t_2),$$

$$I_s^1(t_1, t_2) := \int_{\mathbb{R}} |p_{t_1-s}(y-x) - p_{t_2-s}(y-x)|^\alpha p_{t_1-s}(y-x)^{2-\alpha} e^{2\beta|x|} dx$$

for \(i = 1, 2\). We estimate \(I_s^1(t_1, t_2)\) by \(K (I_s^{11}(t_1, t_2) + I_s^{12}(t_1, t_2))\) where

$$I_s^{11}(t_1, t_2) := \int_{\mathbb{R}} \left| \frac{1}{\sqrt{t_1-s}} - \frac{1}{\sqrt{t_2-s}} \right|^\alpha p_{t_1-s}(y-x)^{2-\alpha} e^{2\beta|x|} dx$$

and

$$I_s^{12}(t_1, t_2) := \int_{\mathbb{R}} \left| \frac{1}{\sqrt{t_1-s}} - \frac{1}{t_1-s} - \frac{1}{t_2-s} \right| (y-x)^2 \left| p_{t_1-s}(y-x)^{2-\alpha} e^{2\beta|x|} dx \right|^\alpha$$

Now we continue with

$$I_s^{11}(t_1, t_2) \leq K \int_{\mathbb{R}} \left| \frac{t_2-t_1}{\sqrt{t_1-s(t_2-s)}} \right|^\alpha \frac{e^{2\beta|x|}}{t_1-s} p_{(t_1-s)/(2-\alpha)}(y-x) dx \leq K \frac{|t_1-t_2|^\alpha}{\sqrt{t_1-s(t_2-s)}} e^{2\beta_1|y|}$$

and

$$I_s^{12}(t_1, t_2) \leq K \int_{\mathbb{R}} \left| \frac{|t_2-t_1|^\alpha (y-x)^{2\alpha}}{(t_2-s)^{\frac{3\alpha}{2}}(t_1-s)^{\frac{1-\alpha}{2}}} p_{(t_1-s)/(2-\alpha)}(y-x) e^{2\beta_1|x|} dx \right| \leq K \frac{(t_2-t_1)^\alpha}{(t_2-s)^{3\alpha} (t_1-s)^{1-\alpha}} e^{2\beta_1|y|}$$

Recall \(0 \leq t_1 < t_2 \leq 1\) so for \(\alpha \in (0, \frac{1}{2})\),

$$\int_0^{t_1} I_s^{11}(t_1, t_2) ds \leq K e^{2\beta_1|y|} |t_1-t_2|^\alpha \int_0^{t_1} (t_1-s)^{-\left(\frac{1}{2}+\alpha\right)} ds \leq K e^{2\beta_1|y|} |t_1-t_2|^\alpha$$
and
\[ \int_0^{t_1} I_s^{12}(t_1, t_2) ds \leq Ke^{2\beta_1|y||t_1 - t_2|^{\alpha}} \int_0^{t_1} (t_1 - s)^{-(\frac{1}{2} + \alpha)} ds \leq Ke^{2\beta_1|y||t_1 - t_2|^{\alpha}}, \]
where we used the fact that \( t_2 - s > t_1 - s \). Making use of (19), we see that the first term of (32) can be estimated above by
\[ K \left( \int_0^{t_1} (I_s^{11}(t_1, t_2) + I_s^{12}(t_1, t_2) + I_s^{21}(t_1, t_2) + I_s^{22}(t_1, t_2)) ds \right)^{n/2} \leq Ke^{n\beta_1|y||t_1 - t_2|^{\frac{\alpha}{2}}}, \]
where \( I_s^{21} \) and \( I_s^{22} \) are defined and estimated similarly as those for \( I_s^{11} \) and \( I_s^{12} \).

Finally, we consider the second term of (32). Notice that,
\[ \int_1^{t_2} \int_\mathbb{R} p_{t_2-s}(y-x) e^{2\beta_1|x|} dx ds \leq K \int_1^{t_2} \int_\mathbb{R} \frac{e^{2\beta_1|x|}}{\sqrt{t_2-s}} p_{\frac{1}{2}(t_2-s)}(y-x) dx ds \leq Ke^{2\beta_1|y|} \int_1^{t_2} ds \leq K|t_1 - t_2|^{\alpha/2} e^{2\beta_1|y|} \]
Thus, we see that the second term of (32) is bounded by
\[ Ke^{n\beta_1|y||t_1 - t_2|^{\frac{\alpha}{2}}} \]

\[ \Box \]

4 LDP for SPDE

LDP describes the asymptotic behavior of the sequence \( \{u^\epsilon\} \) of the above SPDE as \( \epsilon \to 0 \). This principle gives the following two bounds.

LDP Lower bound: For all open sets, \( U \subset C([0, 1]; \mathbb{B}_{\alpha, \beta}) \),
\[ \liminf_{\epsilon \to 0} \epsilon \log P(u^\epsilon \in U) \geq - \inf_{x \in U} I(x) \]

LDP Upper bound: For every closed set \( C \subset C([0, 1]; \mathbb{B}_{\alpha, \beta}) \),
\[ \limsup_{\epsilon \to 0} \epsilon \log P(u^\epsilon \in C) \leq - \inf_{x \in C} I(x) \]
where \( I : \mathcal{C}([0,1];\mathbb{B}_{\alpha,\beta}) \rightarrow [0,\infty] \) is a lower semicontinuous map called a rate function. For more introduction to the theory of large deviations, we refer the reader to [5], [6] and [7]. In this section, we derive the LDP for SPDE (3) by using the powerful technique developed by Budhiraja et al [1]. More specifically, we apply Theorem 6 of that paper with \( \mathcal{E}_0 := \mathbb{B}_{\alpha,\beta}^0 \) and \( \mathcal{E} := \mathcal{C}([0,1];\mathbb{B}_{\alpha,\beta}) \).

Recall from Section 2, the definition of the map \( \gamma \) and let \( g^\epsilon \) be the map given in Theorem 1. Denote \( S^N(\ell_2) := \left\{ k \in L^2([0,1]:\ell_2) : \int_0^1 \|k_s\|_{\ell_2}^2 ds \leq N \right\} \) and define a map \( \zeta \) from \( k \in S^N(\ell_2) \) to \( h = \zeta(k) \in L^2([0,1] \times U) \) as follows:

\[
    h_s(a) = \sum_j k^j_s \phi_j(a).
\]

Let \( g^0 : \mathbb{B}_{\alpha,\beta}^0 \times S \rightarrow \mathcal{C}([0,1]:\mathbb{B}_{\alpha,\beta}) \) given by,

\[
    g^0 \left( F, \int_0^1 k_s ds \right) = \gamma \left( F, \zeta(k) \right). \tag{33}
\]

To obtain the LDP, it is sufficient to verify Assumption 2 imposed by [1]. Suppose \( \{k^\epsilon\} \) is a family of random variables taking values in \( S^N(\ell_2) \) such that \( k^\epsilon \rightarrow k \) in distribution and \( F^\epsilon \rightarrow F \) as \( \epsilon \rightarrow 0 \). Denote the solution to

\[
    u_t(y) = \int_\mathbb{R} p_t(y-x)F^\epsilon(x)dx + \theta \sum_j \int_0^t \int_\mathbb{R} p_{t-s}(y-x)G_j(x,u_s(x))dB^j_sdx \\
    + \sum_j \int_0^t \int_\mathbb{R} p_{t-s}(y-x)G_j(x,u_s(x))k^\epsilon_{s,j}dxds \tag{34}
\]

as \( u^\theta,\epsilon_t(y) \), where \( G_j(y,u) \) is defined in Section 2.

**Lemma 5.** \( \{u^\theta,\epsilon\} \) is tight in \( \mathcal{C}([0,1];\mathbb{B}_{\alpha,\beta}) \). In particular, Assumption 2 of [1] holds under the current setup.

**Proof.** To prove the tightness of \( \{u^\theta,\epsilon\} \), we need to determine estimates for \( u^\theta,\epsilon \) similar to those obtained in Section 3. Since the main difference is in the last term, we restrict our attention to

\[
    w_t(y) := \sum_j \int_0^t \int_\mathbb{R} p_{t-s}(y-x)G_j(x,u_s(x))k^\epsilon_{s,j}dxds.
\]

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Using $P_1 := p_{t-s}(y_1 - x) - p_{t-s}(y_2 - x)$,

$$
\mathbb{E} |w_t(y_1) - w_t(y_2)|^n = \mathbb{E} \left| \int_0^t \int_\mathbb{R} P_1 \sum_j G_j(x, u_s(x)) k^j_s dxds \right|^n
\leq \mathbb{E} \left| \int_0^t \int_\mathbb{R} |P_1| \left( \sum_j G_j(x, u_s(x))^2 \right)^{1/2} \|k^j_s\|_{L^2} dxds \right|^n
\leq \mathbb{E} \left| \int_0^t \left( \int_\mathbb{R} |P_1| \sqrt{K (1 + |u_s(x)|^2)} dx \right)^2 ds \right|^{n/2} N^{n/2}
\leq MN^{n/2} \left( \int_0^t \int_\mathbb{R} |P_1|^2 e^{2\beta_1 |x|} dxds \right)^{n/2}
\leq Ke^{n\beta_1 (|y_1|\vee|y_2|)} |y_1 - y_2|^\alpha
$$

where the last step follows from an analogous argument as in the proof of Lemma 4 and $M$ is given by (19). The estimate for fixed $y$ and $t_1, t_2$ arbitrary can be derived similarly. Now the first condition in Assumption 2 follows from the above argument by taking $\theta = 0$, while the second condition is verified by taking $\theta = \sqrt{\epsilon}$.

Suppose $u^{0,0}$ is a limit point of $\{u^{0,\epsilon}\}$ as $\theta, \epsilon \to 0$. By taking limits on both sides of (34), $u^{0,0}$ becomes a solution to the following equation,

$$
u_t^{0,0}(y) = \int_\mathbb{R} p_t(y - x) F(x) dx
+ \sum_j \int_0^t \int_\mathbb{R} p_{t-s}(y - x) G_j(x, u^{0,0}_s(x)) k^j_s dxds
= \int_\mathbb{R} p_t(y - x) F(x) dx
+ \int_0^t \int_\mathbb{R} \int_\mathbb{R} p_{t-s}(y - x) G(a, x, u^{0,0}_s(x)) h_s(a) \lambda(da) dxds
$$

which is the mild form of (11), where $h = \zeta(k)$. The definition of $\gamma$ implies $u^{0,0} = \gamma(F, h)$. Thus, using the above lemma to apply Theorem 6 in [1], the rate function for SPDE (3) is given as,

$$\tilde{I}(u) = \begin{cases} \frac{1}{2} \inf_{k_s} \left\{ \int_0^t \|k_s\|_{L^2}^2 ds : u = \gamma(F, \zeta(k)) \right\} & \exists k \text{ s.t. } u = \gamma(F, \zeta(k)) \\ \infty & \text{otherwise.} \end{cases}$$
By the relationship between \( k \) and \( h \), it is easy to see that \( \tilde{I} \) coincides with rate function \( I \) defined by (12). This concludes the proof of Theorem 2.

Function \( G(a, x, u) \) for SBM and FVP satisfies conditions (1) and (5); hence, by the results attained in this section, to establish the LDP for SBM and FVP, one needs only to determine the corresponding rate functions. The identification of these rate functions is given in Sections 5 and 6 respectively.

5 LDP for super-Brownian Motion

SBM is one of the main models in studying the evolution of populations. It assumes that each individual moves randomly according to a Brownian motion and she leaves a random number of offsprings upon her death. Therefore, SBM is a measure-valued process with an associated branching rate, \( \epsilon \). Formally speaking, this measure-valued process, also referred to as a superprocess, is defined as the unique solution, \( \mu^\epsilon_t \), to the following martingale problem: for all \( f \in C^2_b(\mathbb{R}) \),

\[
M_t(f) := \langle \mu^\epsilon_t, f \rangle - \langle \mu_0, f \rangle - \int_0^t \left\langle \mu^\epsilon_s, \frac{1}{2} \Delta f \right\rangle ds,
\]

is a square-integrable martingale with quadratic variation,

\[
\langle M_t(f) \rangle = \epsilon \int_0^t \langle \mu^\epsilon_s, f^2 \rangle ds.
\]

For more information on this superprocess see [8] and [13]. Our aim in this section is to prove the LDP for SBM as the branching rate \( \epsilon \) is set to converge to zero. We define

\[
J_\beta(x) = \int_{\mathbb{R}} e^{-\beta|y|} \rho(x - y)dy
\]  (35)

where \( \rho \) is the mollifier given by

\[
\rho(x) = K \exp \left(\frac{-1}{1 - x^2}\right) 1_{|x|<1}
\]

and \( K \) is a constant such that \( \int_{\mathbb{R}} \rho(x)dx = 1 \). Then for all \( m \in \mathbb{Z}_+ \), there are constants \( c_m, C_m \) such that

\[
c_m e^{-\beta|x|} \leq J_\beta^{(m)}(x) \leq C_m e^{-\beta|x|} \quad \forall x \in \mathbb{R}
\]

(cf. Mitoma [14], (2.1)). Therefore, we may and will replace \( e^{-\beta|x|} \) by \( J_\beta(x) \) in the definition of \( \mathcal{M}_\beta(\mathbb{R}) \) given by (2).
Lemma 6. Let $\mathcal{A}$ be the set of all nondecreasing functions, then the map
$\xi : \mathcal{B}_{\alpha,\beta} \cap \mathcal{A} \to \mathcal{M}_{\beta}(\mathbb{R})$ defined as $\xi(u)(A) = \int 1_A(y) du(y)$ for all $A \in \mathcal{B}(\mathbb{R})$, is continuous.

Proof. Suppose $u_n \to u$ in the space $\mathcal{B}_{\alpha,\beta} \cap \mathcal{A}$. Then for every $f \in C_0(\mathbb{R})$,
\[
\int f(x) J_\beta(x) \xi(u_n) dx = \int f(x) J_\beta(x) du_n(x) = - \int (f J_\beta)'(x) u_n(x) dx
\]
\[
\to - \int (f J_\beta)'(x) u(x) dx = \int f J_\beta(x) \xi(u)(dx)
\]
verifying the continuity of $\xi$ map.

Proof of Theorem 3 Recall the definition of $u^\epsilon$ given by (13). By Theorem 2, $u^\epsilon$ satisfies the LDP on $C([0,1]; \mathcal{B}_{\alpha,\beta})$ and because $u^\epsilon_t \in \mathcal{A}$ a.s. for all $t$, we see that $u^\epsilon$ obeys LDP on $C([0,1]; \mathcal{B}_{\alpha,\beta} \cap \mathcal{A})$, as well. Since for SBM, $\mu^\epsilon_t = \xi(u^\epsilon_t)$, then by Lemma 6 and the contraction principle, LDP holds for $\mu^\epsilon$ on $C([0,1]; \mathcal{M}_{\beta}(\mathbb{R}))$ with the rate function determined below.

If $I(\mu) < \infty$, then there exists $h \in L^2([0,1] \times \mathbb{R}_+, dsda)$ such that (11) holds. Let $C_c(\mathbb{R})$ be the collection of functions with compact support on $\mathbb{R}$, then for $f \in C_c^1(\mathbb{R})$,
\[
\langle \mu_t, f \rangle = - \langle u_t, f' \rangle_{L^2(\mathbb{R})}.
\]
Using the controlled version SPDE (11), for every $f \in C_c^1(\mathbb{R})$,
\[
\langle \mu_t, f \rangle = - \langle F, f' \rangle - \int_0^t \left\langle u_s, \frac{1}{2} \Delta f \right\rangle ds
\]
\[
- \int_0^t \int_{\mathbb{R}} \int_{-\infty}^{u_s(y)} h_s(a) da f'(y) dy ds
\]
\[
= \langle \mu_0, f \rangle + \int_0^t \left\langle \mu_s, \frac{1}{2} \Delta f \right\rangle ds + \int_0^t \int_{\mathbb{R}} f (u_s^{-1}(a)) h_s(a) da ds
\]
\[
= \langle \mu_0, f \rangle + \int_0^t \left\langle \mu_s, \frac{1}{2} \Delta u_s, f \right\rangle ds + \int_0^t \int_{\mathbb{R}} f(y) h_s(u_s(y)) du_s(y) dy ds
\]
\[
= \langle \mu_0, f \rangle + \int_0^t \left\langle \frac{1}{2} \Delta^* \mu_s, f \right\rangle ds + \int_0^t \langle \mu_s, fh^s(u_s) \rangle ds
\]
which implies $\mu \in H_{\mu_0}$ and
\[
\frac{(\dot{\mu}_t - \frac{1}{2} \Delta^* \mu_t)(dy)}{\mu_t(dy)} = h_t(u_t(y)).
\]
Moreover,
\[
\int \left| h_t(u_t(y)) \right|^2 \mu_t(dy) = \int \left| h_t(u_t(y)) \right|^2 du_t(y) = \int |h_t(a)|^2 da.
\]
Denote the right hand side of (15) by $I_0(\mu)$ and observe that in this case, $I_0(\mu) = I(\mu)$.

If $I_0(\mu) < \infty$, we may reverse the above calculation to obtain the finiteness of $I(\mu)$. This completes the proof of Theorem 3.

6 LDP for Fleming-Viot

Besides SBM, FVP is another important model used in population evolution. In this model population size stays fixed throughout time and the gene mutation and selection of individuals are observed. A rigorous definition for Fleming-Viot process is a probability measure-valued process $\mu^\epsilon_t$ solving the following martingale problem: for all $f \in C^2_c(\mathbb{R})$,
\[
N_t(f) := \langle \mu^\epsilon_t, f \rangle - \langle \mu_0, f \rangle - \int_0^t \left\langle \mu^\epsilon_s, \frac{1}{2} \Delta f \right\rangle ds
\]
is a continuous square-integrable martingale with quadratic variation,
\[
\langle N_t(f) \rangle = \epsilon \int_0^t \left( \langle \mu^\epsilon_s, f^2 \rangle - \langle \mu^\epsilon_s, f \rangle \right)^2 ds.
\]
More detailed material on Fleming-Viot process can be found in [2] and [8]. This section derives the LDP for Fleming-Viot process as its mutation rate, $\epsilon$ is set to converge to zero.

Proof of Theorem 4 Using the same argument as in Section 5, Lemma 6 can be proven for FVP by defining a map $\psi : \mathbb{B}_{\alpha,\beta} \cap \mathcal{A} \to \mathcal{P}_\beta(\mathbb{R})$ defined as $\psi(u)((-\infty, y]) = \int_y^\infty du(y)$. The continuity of this map can be easily verified following the same steps as in Lemma 6 therefore, we proceed by identifying the rate function.
If \( I(\mu) < \infty \), there exists \( h \in L^2([0, 1] \times \mathbb{R}_+, dsda) \) such that (11) holds. For \( f \in C^3_c(\mathbb{R}) \),

\[
\langle \mu_t, f \rangle = - \langle F, f' \rangle - \int_0^t \left\langle u_s', \frac{1}{2} \Delta f \right\rangle ds \\
- \int_0^t \int_\mathbb{R} \int_0^1 h_s(a) df'(y) dy ds - \int_0^t \langle \mu_s, f \rangle \int_0^1 h_s(a) da ds
\]

\[
= \langle \mu_0, f \rangle + \int_0^t \left\langle \mu_s, \frac{1}{2} \Delta f \right\rangle ds + \int_0^t \int_\mathbb{R} f(u_s^{-1}(a)) h_s(a) ds da
\]

\[
- \int_0^t \langle \mu_s, f \rangle \int_0^1 h_s(a) da ds
\]

\[
= \langle \mu_0, f \rangle + \int_0^t \left\langle \frac{1}{2} \Delta^* \mu_s, f \right\rangle ds + \int_0^t \int_\mathbb{R} f(y) h_s(u_s(y)) du_s(y) dy ds
\]

\[
- \int_0^t \langle \mu_s, f \rangle \int_0^1 h_s(a) da ds
\]

hence, \( \mu \in H_{\mu_0} \) and

\[
\frac{\left( \mu_t - \frac{1}{2} \Delta^* \mu_t \right) (dy)}{\mu_t(dy)} = h_t(u_t(y)) - \int_0^1 h_t(a) da.
\]

If \( h \) satisfies (11) then \( h_s(a) \equiv h_s(a) - \int_0^1 h_s(a) da \) also satisfies the same equation. To minimize \( \int_0^1 |h_s(a)|^2 da \), we choose \( h \) such that \( \int_0^1 h_s(a) da = 0 \). Therefore, \( \mu \in \tilde{H}_{\mu_0} \) and

\[
\frac{\left( \mu_t - \frac{1}{2} \Delta^* \mu_t \right) (dy)}{\mu_t(dy)} = h_t(u_t(y)).
\]

Applying the same argument as in Section 5 establishes Theorem 4.

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