Ricci Flow of 3-D Manifolds with One Killing Vector

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Abstract

We implement a suggestion by Bakas and consider the Ricci flow of 3-d manifolds with one Killing vector by dimensional reduction to the corresponding flow of a 2-d manifold plus scalar (dilaton) field. By suitably modifying the flow equations in order to make them manifestly parabolic, we are able to show that the equations for the 2-d geometry can be put in the form explicitly solved by Bakas using a continual analogue of the Toda field equations. The only remaining equation, namely that of the scale factor of the extra dimension, is a linear equation that can be readily solved using standard techniques once the 2-geometry is specified. We illustrate the method with a couple of specific examples.

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1 Introduction

Ricci flow of 2-D manifolds is interesting because of its relationship to the renormalization group equations of generalized 2-D sigma models and because it provides a proof of the uniformization theorem in 2-D. The uniformization theorem in two dimensions states that every closed orientable two-dimensional manifold with handle number 0, 1, or > 1, admits uniquely the constant curvature geometry with positive, zero, or negative curvatures, respectively.

Recently, Bakas has shown that the 2-D Ricci flow equations in conformal gauge provide a continual analogue of the Toda field equations. Using this algebraic approach he was able to write down the general solution.

The potential importance of a 3D uniformization theorem is evident particularly in the context of (super)membrane physics and three-dimensional quantum gravity where one should be able to perform path-integral quantization via a similar procedure to that in two dimensions. Unfortunately, there is no uniformization theorem in three dimensions, only a conjecture due to W.P. Thurston.

Recently there has been speculation that Perelman has overcome some roadblocks in Hamilton’s program to prove the conjecture using the ‘Ricci flow’. It is therefore important to understand in detail the properties of this flow.

In the following, we follow up on a suggestion by Bakas to use his 2-D results in order to analyze the flow equations for 3-D manifolds with a single Killing vector. This provides a tractable midisuperspace approach. We will show that the suitably modified flow equations reduce to the infinite dimensional generalization of the Toda equation for the conformal factor of the invariant 2-d submanifold plus a linear equation for the scale factor of the extra dimension. Note that since the latter scale factor depends on the coordinates of the invariant subspace, our manifolds are not simple direct products. We will analyze two exact analytic solution in detail and show that it has the expected behaviour.

The paper is organized as follows. Section II reviews 2-D flow equations and Bakas’ results, Section III presents the flow equations in the chosen coordinate system. It shows as well how to modify the equations to make them manifestly parabolic. The resulting equations have a particularly simple form. Section IV presents specific solutions and Section V ends with conclusions and prospects for future work.
2 The 2D Case

We now summarize the methodology and results of Bakas[2] since they play a crucial role in the following. The Ricci flow equations, for arbitrary 2-metric \( g_{AB} \) are:

\[
\frac{\partial g_{ij}}{\partial t} = -R_{ij} + \nabla_i \xi_j + \nabla_j \xi_i. \tag{1}
\]

The last two terms (the so-called “De Turck” terms) incorporate the effects of all possible diffeomorphisms and can be chosen arbitrarily in order to simplify the equations and/or optimize convergence. Originally, De Turck [8] chose the vector field

\[
\xi^i := g^{jk} (\Gamma^i_{jk} - \Delta^i_{jk}), \tag{2}
\]

where \( \Gamma^i_{jk} \) is the Christoffel connection with respect to the Riemannian metric \( g_{ij} \) and \( \Delta^i_{jk} \) is a fixed ‘background connection’. The purpose was to replace the Ricci flow, which is only weakly parabolic, by an equivalent flow which is strongly parabolic.

Bakas chose to work in the conformal gauge:

\[
ds^2 = g_{ij} dx^i dx^j = \frac{1}{2} \exp(\Phi)(dX^2 + dY^2). \tag{3}
\]

In this gauge, without the need to add De Turck terms takes the form of a non-linear “heat equation”:

\[
\frac{\partial}{\partial t} e^\Phi = \nabla^2 \Phi. \tag{4}
\]

The Toda equations describe the integrable interactions of a collection of two dimensional fields \( \Phi_i(X,Y) \) coupled via the Cartan matrix \( K_{ij} \):

\[
\sum_j K_{ij} e^{\Phi_j(X,Y)} = \nabla^2 \Phi_i(X,Y). \tag{5}
\]

Bakas argues that Eq.(4) is a continual analogue of the above, with the Cartan matrix replaced by the kernel:

\[
K_{ij} \rightarrow K(t, t') = \frac{\partial}{\partial t} \delta(t, t'). \tag{6}
\]

This leads to a general solution to (4) in terms of a power series around the free field expanded in path ordered exponentials. Although the resulting expression is difficult to work with explicitly, it does provide a formal complete solution to the 2d flow equations.
In the next section we will show that a similar formal solution also applies to the flow equations for three dimensional metrics with at least one Killing field.

3 The 3D Case with One Killing Vector Field

We consider the case where there the metric admits at least one Killing vector field\(^1\). We choose coordinates \((w, x^B)\) such that \(\partial/\partial w\) is the Killing vector field. The upper case latin indices take the values 1, 2. The metric can be written in the form

\[
ds^2 = e^\sigma (dw^2 + A_B dx^B)^2 + e^\phi \delta_{BC} dx^B dx^C.
\]

(7)

The functions \(\sigma, A_B, \phi\) depend only on the coordinates \(x^B\).

In this coordinate system, the naive Ricci flow equations for the metric:

\[
\dot{g}_{ij} = -2R_{ij},
\]

(8)

do not preserve the diagonal form of the metric, nor are they manifestly parabolic. We therefore consider a metric flow equation suggested by the DeTurck modifications:

\[
\dot{g}_{ij} = -2R_{ij} - e^{-\sigma/2} L_\xi g_{ij}.
\]

(9)

The vector field \(\xi^m := g^{mn} \partial_n e^{\sigma/2}\). This can be written entirely in terms of tensors since

\[
\left| \frac{\partial}{\partial w} \right|^2 = e^\sigma.
\]

(10)

Note, however, that unless \(\sigma\) is constant, the extra term does not correspond precisely to a diffeomorphism of the metric.

Using MAPLE/GRtensor, it follows that the explicit form of the flow for the metric above is

\[
\partial_t e^\phi = \Delta \phi + e^{\sigma - \phi} F,
\]

(11)

\[
\partial_t F = e^{-\sigma} \left[ \Delta \left( e^{\sigma - \phi} F \right) - \nabla \cdot \nabla \left( e^{\sigma - \phi} F \right) \right],
\]

(12)

\[
\partial_t e^\sigma = e^{\sigma - \phi} \Delta \sigma - \left( e^{\sigma - \phi} F \right)^2.
\]

(13)

\(^1\)This is the ‘midisuperspace flow’ examined by Isenberg and Jackson.\[3\]
In the above, all vector operators are with respect to the 2D Euclidean metric. The quantity \( F := \partial_1 A_2 - \partial_2 A_1 \) is the curl of the ‘twist potential’ \( A_B \).

We now examine the special case where the twist is trivial, that is, \( F = 0 \). We also write \( x = x^1, y = x^2 \). The flow reduces to the Toda flow for \( \phi \):

\[
\partial_t e^\phi = \Delta \phi
\]

plus the flow

\[
e^\phi \partial_t \sigma = \Delta \sigma
\]

The latter is not quite the linearized Toda equation. But given a solution of the Toda equation for \( \phi \), we can in principle solve the second for \( \sigma \).

We look at two classes of solutions of the flow equations. The first is obtained by choosing \( \sigma = \phi \), so that the metric is conformally flat, and the flow for the \( \sigma \) field is just another copy of the Toda equation.

Hence for the sausage solution [2] of the Toda equation:

\[
e^\phi = \frac{2 \sinh [2 \gamma (t_0 - t)]}{\gamma (\cosh [2 \gamma (t_0 - t)] + \cosh (2y))}.
\]

In the limit as \( t \to \infty \), \( e^\phi \to 2/\gamma \), and the Ricci tensor goes to zero. Hence, in this limit, the geometry is flat. On the other hand, in the limit \( t \to t_0^+ \), \( e^\phi \to 2(t - t_0)/\cosh^2 y \). In the last limit, we find that the Ricci-scalar \( R \sim \frac{1}{2(t-t_0)}(5 - \cosh(2y)) \). So, if we flow the highly curved non-homogeneous metric

\[
g^{(0)}(x, y, z; t = t_0 + \epsilon) := \frac{2 \epsilon}{\cosh^2 (y)} \delta_{ij},
\]

we end up at \( t \to \infty \) with the flat metric. This is consistent with Thurston’s conjecture.

The second type of solution is of the Liouville type. We set

\[
e^{\phi(x,y;t)} = T(t) e^{\psi(x,y)}
\]

Now for \( t \geq t_0 \), we find that

\[
T(t) = \beta (t - t_0)
\]

\[
\Delta \psi = \beta e^\psi = 0,
\]

where \( \beta \) is a separation constant. The second of the above equations is the Liouville equation, so the two dimensional part of the metric, \( e^\phi (dx^2 + dy^2) \) has constant negative curvature (for \( t \geq t_0 \)).
One case has $\sigma = a\phi$ for a constant $a$. In this case
\[ ds^2 = e^{a\phi} dw^2 + e^\phi (dx^2 + dy^2). \] (20)
The second case is that $\sigma = H(t)\rho(x, y)$. There are two solutions:
\[ H(t) = \pm H_0(t - t_0)^{\gamma/\beta}, \] (21)
with the $+$ (respectively $-$) sign corresponding to positive (respectively negative) sign for the separation constant $\beta$. The function $\rho$ satisfies the linear partial differential equation
\[ \Delta \rho - \gamma e^\psi \rho = 0. \] (22)
Hence the metric is
\[ ds^2 = e^{\pm H_0(t - t_0)^{\gamma/\beta}} dw^2 \pm \beta(t - t_0)e^\psi (dx^2 + dy^2). \] (23)
The quantity $\psi$ is a solution of the Liouville equation.

Consider first the case $t \leq t_0$. If $\gamma \geq \beta$, then the flow starts from some highly curved non-homogeneous metric as $t \to -\infty$. As $t \to t_0^-$, we have
\[ R_{AB} \sim \frac{1}{2(t_0 - t)} g_{AB}, \]
\[ R_{ww} \sim 0, \] (24)
with $A, B, \ldots = x, y$. Hence, the geometry is asymptotically that of the homogeneous, but anisotropic geometry $S^2 \times E^1$.

Similarly, for the case of $t \geq t_0$, the flow is ‘backwards’ from a non-homogeneous geometry for $t > t_0$ to the homogeneous geometry $H^2 \times E^1$.

Thus the flow is consistent with the Thurston conjecture.

4 Conclusions

We have shown that the Ricci flow equations for 3d metrics with at least one Killing vector can be integrated in precisely the same manner as the 2d equations, providing a particular choice of De Turck term is made. It is interesting to speculate whether these techniques could work for more general 3 metrics.
Consider, without loss of generality, a diagonal metric

\[ ds^2 = e^{\phi_1(x,t)}(dx^1)^2 + e^{\phi_2(x,t)}(dx^2)^2 + e^{\phi_3(x,t)}(dx^3)^2 \]  

where the functions \( \phi_i(x,t) \) depend on all 3 coordinates \( x^i \). The resulting bare Ricci flow is again not manifestly elliptic, and the equations have non-trivial off-diagonal terms on the RHS that make direct integration difficult. Since in three dimensions any metric can be made diagonal with a suitable coordinate transformation, it is reasonable to assume that there exists a modified flow:

\[ \dot{g}_{ij} = -2 \left( [R_{ij}(g) + \Phi (\nabla_i \xi_j + \nabla_j \xi_i)] \right), \]  

where \( \Phi \) is some scalar field. The extra term would ensure that diagonal metrics evolve into diagonal metrics. That is, one should look for fields \( (\Phi, \xi) \) that diagonalize the right hand side of the above for arbitrary diagonal metric \( g_{ij} \). We have as yet not succeeded in this, but if such a flow did exist, it is possible that the resulting three flow equations for each of the three scale factors would take a form similar to what we have found above, albeit with non-trivial coupling. It may therefore provide a basis for solving the 3-d flow equations in a more general setting.

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