The exact proof that Maxwell equations with the 3D $E$ and $B$ are not Lorentz covariant equations. The new Lorentz invariant field equations

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In this paper it will be exactly proved both in the geometric algebra and tensor formalisms that the usual Maxwell equations with the three-dimensional (3D) vectors of the electric and magnetic fields, $E$ and $B$ respectively, are not, contrary to the general opinion, Lorentz covariant equations. Consequently they are not equivalent to the field equations with the observer independent quantities, the electromagnetic field tensor $F^{ab}$ (tensor formalism) or with the bivector field $F$ (the geometric algebra formalism). Different 4D algebraic objects are used to represent the standard observer dependent and the new observer independent electric and magnetic fields. The proof of a fundamental disagreement between the standard electromagnetism and the special relativity does not depend on the character of the 4D algebraic objects used to represent the electric and magnetic fields. The Lorentz invariant field equations are presented with 1-vectors $E$ and $B$, bivectors $E_{HL}$ and $B_{HL}$ and the abstract tensors, the 4-vectors $E^a$ and $B^a$. All these quantities are defined without reference frames. Such field equations are in a complete agreement with experiments.

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I. INTRODUCTION

Recently an exact proof is presented that the standard transformations (ST) [1,2] (see also the standard textbooks, e.g. [3,4]) of the three-dimensional (3D) vectors of the electric and magnetic fields, $E$ and $B$ respectively, are not relativistically correct. This proof is given both in the tensor formalism [5] and the geometric (Clifford) algebra formalism [6]. It is shown in both formalisms that these ST of $E$ and $B$ drastically differ from the correct Lorentz transformations (LT) of the corresponding 4D algebraic objects representing the electric and magnetic fields. The fundamental difference is that in the ST, e.g., the components of the transformed 3D $E'$ are expressed by the mixture of components of the 3D $E$ and $B$, and similarly for $B'$. However, the correct LT always transform, e.g., the electric field (i.e., the 4D algebraic object representing the electric field) only to the electric field, and similarly for the magnetic field. The mentioned proof from [5,6] implies that the usual Maxwell equations (ME) with the 3D $E$ and $B$ are not Lorentz covariant equations. Consequently they are not equivalent to the field equations with the electromagnetic field tensor $F^{ab}$ (tensor formalism) or to those with the bivector field $F$ (the geometric algebra formalism). In this paper the above statement will be exactly proved both in the geometric algebra and tensor formalisms. Different 4D algebraic objects are used to represent the standard observer dependent and the new observer independent electric and magnetic fields. First the electric and magnetic fields are represented by the observer dependent 1-vectors $E_f$ and $B_f$ defined in the $\gamma_0$ - frame. The usual ME in the component form are derived in section II.A. and their LT are considered in section II.B. It is explicitly shown in II.B., using the correct LT of $E_f$ and $B_f$, that the Lorentz transformed ME are not of the same form as the original ones. This proves that, contrary to the general opinion, the usual ME are not Lorentz covariant equations. In section II.C. the ST of the usual ME are considered taking into account the ST of the components of the 3D $E$ and $B$. It is proved that both the ST of the usual ME and the ST of the 3D $E$ and $B$ have nothing in common with the correct LT. The new Lorentz invariant field equations are constructed in section II.D. in which the electric and magnetic fields are represented by the observer independent, i.e., defined without reference frames, 1-vectors $E$ and $B$. In sections III. to III.D. the whole consideration is repeated but dealing with the observer dependent bivectors $E_H$ and $B_H$ defined in the $\gamma_0$ - frame and the observer independent bivectors $E_{HL}$ and $B_{HL}$. In the geometric algebra formalism the active LT are used. Comparing the derivations in sections II. to II.D. and sections III. to III.D. one concludes that the formulation with 1-vectors is simpler than the approach with bivectors and also it is much closer to the classical formulation of the electromagnetism with the 3D vectors $E$ and $B$. In sections IV. to IV.D. the proof is presented in the tensor formalism using the observer dependent 4-vectors $E^0$ and $B^0$ defined in the $\gamma_0$ - frame and the observer independent 4-vectors $E^a$ and $B^a$. In the tensor formalism the passive LT are used. All quantities in the Lorentz invariant field equations derived with the use of 1-vectors $E$ and $B$, bivectors $E_{HL}$ and $B_{HL}$ and the abstract 4-vectors $E^a$ and $B^a$ are geometric, coordinate-free quantities, i.e., quantities that are defined without reference frames. All such equations are completely equivalent to
the field equations with \( F \) (given, e.g. in [7-9] and discussed in detail in [10]) or with \( F^{\mu \nu} \) (already presented, e.g., in [11]). It can be concluded from the consideration presented in all mentioned sections that the proof of a fundamental disagreement between the standard electromagnetism and the special relativity (SR) does not depend on the character of the 4D algebraic objects used to represent the electric and magnetic fields. The discussion and a short comparison with some experiments are given in section V. (We note that the comparison of the geometric approach to SR and the standard formulation of SR with experiments that test SR is also given in detail in [12].) The summary and conclusions are presented in section VI.

II. THE PROOF IN THE GEOMETRIC ALGEBRA FORMALISM USING 1-VECTORS \( E \) AND \( B \)

For the standard formulation of electrodynamics with the Clifford multivectors, see, e.g., [7-9]. (A modern and very stimulating mathematical treatment of the Clifford algebra and the geometric calculus is presented in [13].) In [7-9] the electromagnetic field is represented by a bivector-valued function \( F = F(x) \) on the spacetime. The source of the field is the electromagnetic current \( j \) which is a 1-vector field and the gradient operator \( \partial \) is also 1-vector. A single field equation for \( F \) is first given by M. Riesz [14] as

\[
\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c.
\]

(1)

The trivector part is identically zero in the absence of magnetic charge. The geometric (Clifford) product is written by simply juxtaposing multivectors \( AB \). The dot \( \cdot \) and wedge \( \wedge \) in (1) denote the inner and outer products respectively. All quantities in (1) are defined without reference frames; they are observer independent quantities, i.e., they are independent of the reference frame and the chosen system of coordinates in that frame. Consequently the equation (1) is a Lorentz invariant equation. In fact, it is independent of even an indirect reference to an inertial system. In the geometric algebra formalism (as in the tensor formalism as well) one mainly deals either with 4D quantities that are defined without reference frames, e.g., Clifford multivector \( F \) (the abstract tensor \( F^{\mu \nu} \)) or, when some basis has been introduced, with coordinate-based geometric quantity (CBGQ) that comprises both components and a basis. The SR that exclusively deals with quantities defined without reference frames or, equivalently, with CBGQs, can be called the invariant SR. The reason for this name is that upon the passive LT any CBGQ remains unchanged. The invariance of some 4D CBGQ upon the passive LT reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. It is taken in the invariant SR that such 4D geometric quantities are well-defined not only mathematically but also experimentally, as measurable quantities with real physical meaning. Thus they do have an independent physical reality. The invariant SR is discussed in [11,12] in the tensor formalism and in [10,15] in the Clifford algebra formalism. It is explicitly shown in [12] that the true agreement with experiments that test SR exists when the theory deals with well-defined 4D quantities, i.e., the quantities that are invariant upon the passive LT. The generally accepted agreement between these experiments and the standard formulation of SR is only an “apparent” agreement caused by the fact that in the standard treatments only parts of the relevant 4D quantities are considered and thus not the whole 4D quantities, see [12].

In the usual geometric algebra formalism, e.g., [7-9], instead of working only with such observer independent quantities one introduces (in order to get a more familiar form for (1)) a space-time split and the relative vectors in the \( \gamma_0 \) frame, i.e., a particular time-like direction \( \gamma_0 \) is singled out. \( \gamma_0 \) is tangent to the world line of an observer at rest in the \( \gamma_0 \) frame.

(The generators of the spacetime algebra are four basis vectors \( \{ \gamma_\mu \}, \mu = 0..3 \), satisfying \( \gamma_\mu \cdot \gamma_\nu = \eta_{\mu \nu} = \text{diag}(+--). \) This basis is a right-handed orthonormal frame of vectors in the Minkowski spacetime \( M^4 \) with \( \gamma_0 \) in the forward light cone. The \( \gamma_k \) \( (k = 1, 2, 3) \) are spacelike vectors. The \( \gamma_\mu \) generate by multiplication a complete basis, the standard basis, for spacetime algebra: \( 1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_5, \gamma_5 \) (16 independent elements). \( \gamma_5 \) is the pseudoscalar for the frame \( \{ \gamma_\mu \} \). It is worth noting that the standard basis corresponds, in fact, to the specific system of coordinates, i.e., to Einstein’s system of coordinates. In the Einstein system of coordinates the Einstein synchronization [2] of distant clocks and Cartesian space coordinates \( x^i \) are used in the chosen inertial frame of reference. However different systems of coordinates of an inertial frame of reference are allowed and they are all equivalent in the description of physical phenomena. For example, in [11] two very different, but completely equivalent systems of coordinates, the Einstein system of coordinates and ”radio” (”r”) system of coordinates, are exposed and exploited throughout the paper. The connection between the basis vectors in the ”r” and in the Einstein system of coordinates is given as \( r_0 = \gamma_0, \ r_i = \gamma_0 + \gamma_i \). Thence the metric tensor \( g_{\mu \nu,r} \) in the ”r” system of coordinates is given as \( g_{00,r} = g_{00,r} = g_{i}\gamma_r (i \neq j) = -1, \ g_{ii,r} = 0 \); the metric tensor \( g_{\mu \nu,r} \) is not the same as the Minkowski metric tensor \( \eta_{\mu \nu} = \text{diag}(+--). \) We note that in SR, i.e., in the theory of flat spacetime, any specific \( g_{\mu \nu} \) (for the specific system of coordinates) can be transformed to the Minkowski metric tensor \( \eta_{\mu \nu} \); for example, \( g_{\mu \nu,r} \) is transformed by the matrix \( (T^u_{\nu,r})^{-1} \) given in [11] to \( \eta_{\mu \nu} \). The coordinate system in which \( g_{00} = 0 \) at every point in 4D spacetime is called
time-orthogonal since in it the time axis is everywhere orthogonal to the spatial coordinate curves. This happens in the cases when in some inertial frame of reference the Einstein synchronization is chosen together with, e.g., Cartesian, or polar, or spherical, etc., spatial coordinates. However it is not the case when the "r" synchronization is chosen. It is almost always tacitly assumed in both geometric algebra and tensor formalisms that, e.g., for the spacetime algebra, [7] Space-Time Calculus: "a given inertial system is completely characterized by a single future-pointing, timelike unit vector." In this case it refers to the unit vector in the time direction, $\gamma_0$ basis vector, and the inertial system characterized by $\gamma_0$ is refered to as the $\gamma_0$ - frame, or the $\gamma_0$ - system. The preceding discussion shows that the above claim from [7] is not true in general. Namely $\gamma_0$ and $r_0$ are the same vectors ($\gamma_0$, i.e., $r_0$, is the unit vector directed along the world line of the clock at the origin), but the spatial basis vectors $\gamma_i$ and $r_i$ are very different and moreover $r_0$ is not orthogonal to $r_i$. (The spatial basis vectors by definition connect simultaneous events, the event "clock at rest at origin reads 0 time" with the event "clock at rest at unit distance from the origin reads 0 time," and thus they are synchronization-dependent. The spatial basis vector $e_i$ connects two above mentioned simultaneous events when Einstein's synchronization of distant clocks is used. The spatial basis vector $r_i$ connects two above mentioned simultaneous events when "radio" clock synchronization of distant clocks is used. All this is explained in more detail in [11].) This means that the usual space-time split and the relative vectors, e.g., [7-9], are obtained not only by singling out a particular time-like direction $\gamma_0$ but also implicitly assuming that the whole standard basis $\{\gamma_\mu\}$ (i.e., the Einstein system of coordinates) is chosen. In this paper, for the sake of brevity and of clearness of the whole exposition, we shall also work only with standard basis $\{\gamma_\mu\}$, but remembering that the approach with 4D quantities that are defined without reference frames holds for any choice of the basis.)

The bivector field $F$ is decomposed in the $\gamma_0$ - frame into electric and magnetic parts using different algebraic objects to represent these fields. The explicit appearance of $\gamma_0$ in these expressions implies that the space-time split is observer dependent and thus all quantities obtained by the space-time split in the $\gamma_0$ - frame are observer dependent quantities. In [7,8] the observer independent $F$ field from [11] is expressed in terms of observer dependent quantities, i.e., as the sum of a relative vector $E_H$ and a relative bivector $\gamma_5 B_H$

$$F = E_H + c\gamma_5 B_H, \ E_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F\gamma_0), \ 
\gamma_5 B_H = (1/c)(F \wedge \gamma_0)\gamma_0 = (1/2c)(F + \gamma_0 F\gamma_0). \ (2)$$

(The subscript 'H' is for - Hestenes.) Both $E_H$ and $B_H$ are, in fact, bivectors. Similarly in [9] $F$ is decomposed in terms of observer dependent quantities, 1-vector $E_J$ and a bivector $B_J$ (the subscript 'J' is for - Jancewicz) as $F = \gamma_0 \wedge E_J - cB_J$, where $E_J = F \cdot \gamma_0$ and $B_J = -(1/c)(F \wedge \gamma_0)\gamma_0$. The $F$ field can be also decomposed in terms of another algebraic objects; the observer dependent electric and magnetic parts of $F$ are represented with 1-vectors that are denoted as $E_f$ and $B_f$ (see also [6] and [15]). The physical description with 1-vectors $E_f$ and $B_f$ is simpler but completely equivalent to the description with the bivectors $E_H, B_H$ [7,8] or with 1-vector $E_J$ and a bivector $B_J$ [9]. Such decomposition of $F$ is not only simpler but also much closer to the classical representation of the electric and magnetic fields by the 3D vectors $E$ and $B$ than those used in [7-9]. Thus

$$F = E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0, \ E_f = F \cdot \gamma_0, \ B_f = -(1/c)\gamma_5(F \wedge \gamma_0). \ (3)$$

Having at our disposal different decompositions of $F$ into observer dependent quantities we proceed to present the proof that the classical electromagnetism and the SR are not in agreement first using the decomposition [3] and then [2]. (We shall not deal with the decomposition of $F$ into $E_J$ and $B_J$ from [9] since both the procedure and the results are completely the same as with [3] and [2].)

A. The field equations in the $\gamma_0$ - frame. The Maxwell equations

When [3] is introduced into the field equation for $F$ [11] we find

$$\partial[(F \cdot \gamma_0) \wedge \gamma_0 + (F \wedge \gamma_0) \cdot \gamma_0] = j/\varepsilon_0 c, \ 
\partial(E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0) = j/\varepsilon_0 c. \ (4)$$

The equations [3] can be now written as coordinate-based geometric equations in the standard basis $\{\gamma_\mu\}$ and the second equation becomes

$$\{\partial_\alpha [\delta^{\alpha\beta}_{\mu\nu} E^\mu_f(\gamma_0)^\nu + \varepsilon^{\alpha\beta\mu\nu}(\gamma_0)_{\mu} B_{f,\nu}] - (j^\beta/\varepsilon_0)\} \gamma_\beta + \partial_\alpha [\delta^{\alpha\beta}_{\mu\nu}(\gamma_0)^\mu c B_f^\nu + \varepsilon^{\alpha\beta\mu\nu}(\gamma_0)_{\mu} E_{f,\nu}] \gamma_5 \gamma_\beta = 0, \ (5)$$
The relation (7) is nothing else than the standard identification of the components $\gamma$ with $(\gamma_0)^\mu = (1, 0, 0, 0)$ and

$$E_f = E_f^\mu \gamma_\mu = 0 \gamma_0 + F^{0i} \gamma_i,$$
$$B_f = B_f^\mu \gamma_\mu = 0 \gamma_0 + (-1/2c) \varepsilon^{0kli} F_{kl} \gamma_i.$$  \hspace{1cm} \text{(6)}

Thence the components of $E_f$ and $B_f$ in the $\{\gamma_\mu\}$ basis (i.e., in the Einstein system of coordinates) are

$$E_f^i = F^{0i}, \quad B_f^i = (-1/2c) \varepsilon^{0kli} F_{kl}.$$  \hspace{1cm} \text{(7)}

The relation (7) is nothing else than the standard identification of the components $F^{\mu\nu}$ with the components of the 3D vectors $E$ and $B$, see, e.g., [3, 4]. (It is worth noting that Einstein’s fundamental work [16] is the earliest reference on covariant electrodynamics and on the identification of some components of $F^{\alpha\beta}$ with the components of the 3D $E$ and $B$.) We see that if the standard basis $\{\gamma_\mu\}$ is chosen in an inertial frame of reference, the $\gamma_0$ - frame, in which the observers who measure the basis components $E_f^\mu$ and $B_f^\mu$ are at rest, i.e., their velocity $v$ is $v = c \gamma_0$, or in the components $v^\alpha = (c, 0, 0, 0)$, then in the $\gamma_0$ - frame $E_f$ and $B_f$ do not have the temporal components $E_f^0 = B_f^0 = 0$. Thus $E_f$ and $B_f$ actually refer to the 3D subspace orthogonal to the specific timelike direction $\gamma_0$. Notice that we can select a particular - but otherwise arbitrary - inertial frame of reference as the $\gamma_0$ - frame, to which we shall refer as the frame of our ‘fiducial’ observers (for this name see [17]). The subscript ‘$f$’ in the above relations stands for - fiducial - and denotes the explicit dependence of these quantities on the $\gamma_0$, i.e., ‘fiducial’ - observer. Using that $E_f^i = B_f^0 = 0$ and $(\gamma_0)^\mu = (1, 0, 0)$ the equation (6) becomes

$$\begin{align*}
(\partial_h E_f^k - j^0/c\varepsilon_0) \gamma_0 + (\partial_h E_f^k + c\varepsilon^{ijkl} \partial_j B_{fk} - j^i/c\varepsilon_0) \gamma_i + \\
(-c\partial_h B_f^k) \gamma_5 \gamma_i + (c\partial_h B_f^k + c\varepsilon^{ijkl} \partial_j E_{fk}) \gamma_5 \gamma_i = 0.
\end{align*}$$  \hspace{1cm} \text{(8)}

The first part (with $\gamma_\alpha$) in (8) is from the 1-vector part of (1), i.e., (5), while the second one (with $\gamma_5\gamma_\alpha$) is from the trivector (pseudovector) part of (4), i.e., (5). Both parts in (8) are written as coordinate-based geometric equations in the standard basis $\{\gamma_\mu\}$ and cannot be further simplified as geometric equations. In the first part (with $\gamma_\alpha$) in (8) one recognizes two Maxwell equations in the component form, the Gauss law for the electric field (the first bracket, with $\gamma_0$) and the Ampère-Maxwell law (the second bracket, with $\gamma_i$). Similarly from the second part (with $\gamma_5\gamma_\alpha$) in (8) we recognize the component form of another two Maxwell equations, the Gauss law for the magnetic field (with $\gamma_5\gamma_0$) and Faraday’s law (with $\gamma_5\gamma_i$).

B. Lorentz transformations of the Maxwell equations

Let us now apply the active Lorentz transformations upon (5), or (4). We write (5), or (4), in the form

$$a^\alpha \gamma_\alpha + b^\alpha (\gamma_5 \gamma_\alpha) = 0.$$  \hspace{1cm} \text{(9)}

The coefficients $a^\alpha$ and $b^\alpha$ are clear from (5), or (4); they are the usual Maxwell equations in the component form. In the Clifford algebra formalism, e.g., [7-9], the LT are considered as active transformations; the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis $\{\gamma_\mu\}$) are transformed into the components of a new 1-vector relative to the same frame (the basis $\{\gamma_\mu\}$ is not changed). Furthermore the LT are described with rotors $R$, $RR = 1$, in the usual way as $p \rightarrow p' = RpRR = p'\gamma_\mu$. To an observer in the $\{\gamma_\mu\}$ frame the vector $p'$ appears the same as the vector $p$ appears to an observer in the $\{\gamma'_\mu\}$ frame. For boosts in the direction $\gamma_1$ the rotor $R$ is given by the relation

$$R = (1 + \gamma + \gamma_5 \gamma_0 \gamma_1)/(2(1 + \gamma))^{1/2},$$  \hspace{1cm} \text{(10)}

$\beta$ is the scalar velocity in units of $c$, $\gamma = (1 - \beta^2)^{-1/2}$. Then the LT of (4) are given as

$$R\{\partial[(F \cdot \gamma_0) \wedge \gamma_0 + (F \wedge \gamma_0) \cdot \gamma_0] - j/\varepsilon_0 c\} \bar{R} = 0,$$
$$R\{\partial[E_f \wedge \gamma_0 + c(\gamma_5 B_f) \cdot \gamma_0] - j/\varepsilon_0 c\} \bar{R} = 0,$$  \hspace{1cm} \text{(11)}

where $R$ is given by (10). (A coordinate-free form of the LT is also given in the Clifford algebra formalism in [15] and in the tensor formalism in [11]. The form presented in [15] does not need to use rotors but, of course, it can be expressed by rotors as well.) Then the LT of (5) are

$$R\{a^\alpha \gamma_\alpha + b^\alpha (\gamma_5 \gamma_\alpha)\} \bar{R} = 0.$$  \hspace{1cm} \text{(12)}
Performing the LT we find the explicit expression for (12) as
\[
\gamma_0(\gamma a^0 - \beta \gamma a^1) + \gamma_1(\gamma a^1 - \beta \gamma a^0) + \gamma_2 a^2 + \gamma_3 a^3 + \\
\gamma_5\gamma_0(\gamma b^0 - \beta \gamma b^1) + \gamma_5\gamma_1(\gamma b^1 - \beta \gamma b^0) + \gamma_5\gamma_2 b^2 + \gamma_5\gamma_3 b^3 = 0.
\]
(13)

It can be simply written as
\[
a^\alpha \gamma_\alpha + b^\alpha (\gamma_5 \gamma_\alpha) = 0,
\]
(14)

where, e.g., \(a^0 = \gamma_0 a^0 - \beta \gamma a^1\) and, as it is said, \(a^\alpha\) and \(b^\alpha\) are the usual Maxwell equations in the component form given in (5), or (14). This result (13), i.e., (14), is exactly the usual result for the active LT of a 1-vector and of a pseudovector. It is important to note that, e.g., the Gauss law for the electric field \(a^0\) does not transform by the LT again to the Gauss law but to \(a^0\), which is a combination of the Gauss law and a part of the Ampère-Maxwell law \((a^1)\).

The second equation in (11) can be expressed in terms of Lorentz transformed derivatives and Lorentz transformed 1-vectors \(E'_f\) and \(B'_f\) as
\[
\partial'[E'_f \wedge \gamma_0 + c(\gamma_5 B'_f) \cdot \gamma_0] - j'/\varepsilon_0 c = 0,
\]
(15)

where \(\partial' = R\partial R\), \(\gamma_0 = R\gamma_0 R = \gamma_\gamma - \beta \gamma_\gamma\) and (see also [6]) the Lorentz transformed \(E'_f\) is
\[
E'_f = R(F \cdot \gamma_0) \tilde{R} = RE_f \tilde{R} = R(F^{\alpha 0} \gamma_0) \tilde{R} = E'_f \gamma_\mu = \\
= -\beta \gamma E'_f \gamma_1 + \gamma E'_f \gamma_2 + E'_f \gamma_3,
\]
(16)

what is the usual form for the active LT of the 1-vector \(E_f\). Similarly is obtained for \(B'_f\)
\[
B'_f = R[-(1/c)\gamma_5 (F \wedge \gamma_0)] \tilde{R} = RB_f \tilde{R} = R[-(1/(2c))\varepsilon_{0kl} F_{kl} \gamma_i] \tilde{R} = \\
= B'_f \gamma_\mu = -\beta \gamma B'_f \gamma_1 + \gamma B'_f \gamma_2 + B'_f \gamma_3 + B'_f \gamma_5.
\]
(17)

It is worth noting that \(E'_f\) and \(B'_f\) are not more orthogonal to \(\gamma_0\), i.e., they do have the temporal components \(\neq 0\). Furthermore the components \(E''_f\) \((\overline{B''_f})\) transform upon the active LT again to the components \(E'_f\) \((\overline{B'_f})\) as seen from (10) and (17): there is no mixing of components. When (16) is written in an expanded form as a coordinate-based geometric equation in the standard basis \(\{\gamma_\mu\}\) it takes the form of (14) but now the coefficients \(a^\alpha\) are written by means of the Lorentz transformed components \(\partial'_k, E'_f^k\) and \(B'_f^k\) (for simplicity only the term \(a^0\gamma_0\) is presented)
\[
a^0 \gamma_0 = \{(\gamma (\partial'_k E'_f^k) - j^0/c\varepsilon_0) + \beta \gamma (\partial'_1 E'_f^0 \gamma_1 + c(\partial'_2 B'_f \gamma_3 - \partial'_3 B'_f \gamma_2))\}\gamma_0,
\]
(18)

and it substantially differs in form from the term \(a^0\gamma_0 = (\partial_k E'_f^k - j^0/c\varepsilon_0)\gamma_0\) in (8). As explained above the coefficient \(a^0\) is the Gauss law for the electric field written in the component form. It is clear from (18) that the LT do not transform the Gauss law into the ‘primed’ Gauss law but into quite different law (15); \(a^0\) contains the time component \(E'_f\) (while \(E'_f = 0\)), and also the new “Gauss law” includes the derivatives of the magnetic field. The same situation happens with other Lorentz transformed terms, which explicitly shows that the Lorentz transformed ME (15) with (18) are not of the same form as the original ones (8). This is a fundamental result which reveals that, contrary to the previous derivations, e.g., [2,16], [3,4], [7-9], and contrary to the general opinion, the usual ME are not Lorentz covariant equations. The physical consequences of this achievement will be very important and they will be carefully examined.

C. Standard transformations of the Maxwell equations

In contrast to the correct active Lorentz transformations of \(E_f\) (10) and \(B_f\) (17) it is wrongly assumed in the usual derivations of the ST for \(E'_{st}\) and \(B'_{st}\) (the subscript - st - is for - standard) that the quantities obtained by the active LT of \(E_f\) and \(B_f\) are again in the 3D subspace of the \(\gamma_0\) - observer (see also [6]). This means that it is wrongly assumed in all standard derivations, e.g., in the Clifford algebra formalism [7,9] (and in the tensor formalism [3,4] as well), that one can again perform the same identification of the transformed components \(F^{\mu\nu}\) with the components of the 3D \(E'\) and \(B'\) as in (7). Thus it is taken in standard derivations that for the transformed \(E'_{st}\) and \(B'_{st}\) hold \(E'_{st} = B'_{st} = 0\) as for \(E_f\) and \(B_f\),
\[
E'_{st} = (RF \tilde{R}) \cdot \gamma_0 = F' \cdot \gamma_0 = F^{\alpha 0} \gamma_i = E'_{st} \gamma_i = \\
E'_f \gamma_1 + (\gamma E'_f \beta \gamma c B'_f) \gamma_2 + (\gamma E'_f + \beta \gamma c B'_f) \gamma_3,
\]
(19)
where \( F' = R F \tilde{R} \), and similarly for \( B_{st}' \)

\[
B_{st}' = -(1/c) \gamma_5 (F' \wedge \gamma_0) = -(1/2c) \varepsilon^{0kl} F'_{kl} \gamma_i = B_{st}' \gamma_i = B_1' \gamma_1 + (\gamma B_j^2 + \beta \gamma E_j^3/c) \gamma_2 + (\gamma B_j^1 - \beta \gamma E_j^2/c) \gamma_3. \tag{20}
\]

From the relativistically incorrect transformations \([19]\) and \([20]\), one simply finds the transformations of the spatial components \( E_{st}'i \) and \( B_{st}'i \)

\[
E_{st}'i = F'^{i0}, \quad B_{st}'i = -(1/2c) \varepsilon^{0kl} F'_{kl}. \tag{21}
\]

As can be seen from \([19]\) and \([20]\), i.e., from \([21]\), the transformations for \( E_{st}'i \) and \( B_{st}'i \) are exactly the ST of components of the 3D vectors \( E \) and \( B \) that are quoted in almost every textbook and paper on relativistic electrodynamics including \([2]\) and \([3,4]\). These relations are explicitly derived and given in the Clifford algebra formalism, e.g., in \([7]\), Space-Time Algebra (eq. (18.22)), New Foundations for Classical Mechanics (Ch. 9 eqs. (3.51a,b)), in \([8]\) Geometric algebra for physicists (Ch. 7.1.2 eq. (7.33)) and in \([9]\) (Ch. 7 eqs. (20a,b)). Notice that, in contrast to the active Lorentz transformations \([16] \) and \([17] \), according to the ST \([19] \), i.e., \([21] \), the transformed components \( E_{st}'i \) are expressed by the mixture of components \( E'f \) and \( B'f \), and \([20] \) shows that the same holds for \( B_{st}'i \). In all previous treatments of SR, e.g., \([7-9]\) and \([2-4]\) the transformations for \( E_{st}'i \) and \( B_{st}'i \) are considered to be the Lorentz transformations of the 3D electric and magnetic fields. However the above analysis, and \([5,6]\) as well, show that the transformations for \( E_{st}'i \) and \( B_{st}'i \) \([21]\) are derived from the relativistically incorrect transformations \([19] \) and \([20] \), which are not the Lorentz transformations; the Lorentz transformations are given by the relations \([16] \) and \([17] \).

It is also argued in all previous works, starting in the year 1905 with Einstein’s fundamental paper on SR \([2]\), that the usual ME with the 3D \( E \) and \( B \) are Lorentz covariant equations. The relation \([15] \) together with \([18] \) shows that it is not true; the Lorentz transformed ME are not of the same form as the original ones. Here we explicitly show that in the standard derivations the ME remain unchanged in form not upon the LT but upon some transformations which, strictly speaking, have nothing to do with the LT of the equation \([11] \), i.e., of the usual ME \([8] \). The difference between the Lorentz transformed ME, given by \([11] \) or finally by \([15] \) with \([18] \) (or by \([15] \)) and the equations (given below) obtained by applying the LT is the same as it is the difference between the LT of \( E_f \) (\( B_f \)) given by \([15] \) \([17] \) and their ST given by \([19] \) \([20] \). Thus the ST of the equation \([11] \) are

\[
(R \tilde{R})([(RF \tilde{R}) \cdot \gamma_0] \wedge \gamma_0 + [(RF \tilde{R}) \cdot \gamma_0] \cdot \gamma_0) - (R \tilde{R})/\varepsilon_0 c = 0,
\]

\[
\partial' \{E_{st} \wedge \gamma_0 + c(\gamma B_{st}') \cdot \gamma_0 \} - j'/\varepsilon_0 c = 0, \tag{22}
\]

where \( E_{st}'i \) and \( B_{st}'i \) are defined by \([19] \) and \([20] \). Notice that, in contrast to the correct LT \([11] \) or \([15] \), \( \gamma_0 \) is not transformed in \([22] \). The second equation in \([22] \) is of the same form as the second equation in \([11] \) but with primed derivative \( \partial' \), \( E_{st}'i \) and \( B_{st}'i \) fields and the primed current \( j' \) replacing the corresponding unprimed quantities. When this second equation in \([22] \) is written as a coordinate-based geometric equation in the standard basis \{\( \gamma_\mu \)\} it becomes

\[
(\delta'_{st} E^{st}_{k0} - j^{k0}/c \varepsilon_0 \gamma_0 + (\partial'_{st} E_{st}^{i0} + c \varepsilon^{ijk0} \partial'_{st} B_{st,k} - j^{i0}/c \varepsilon_0) \gamma_i + (c \partial'_{st} B_{st}^{i0} + \varepsilon^{ijk0} \partial'_{st} E_{st,k}) \gamma_0 \gamma_i = 0. \tag{23}
\]

The equation \([23] \) is of the same form as the original ME \([8] \) but the electric and magnetic fields are not transformed by the LT than by the ST. Therefore, as can be seen from \([22] \) (together with \([19] \) and \([20] \)), the equation \([23] \) is not the LT of the original ME \([8] \); the LT of the ME \([8] \) are the equations \([15] \) with \([18] \) (i.e., \([14] \)) where the Lorentz transformed electric and magnetic fields are given by the relations \([16] \) and \([17] \).

### D. Lorentz invariant field equations with 1-vectors \( E \) and \( B \)

Instead of decomposing \( F \) into the observer dependent \( E_f \) and \( B_f \) in the \( \gamma_0 \) - frame, as in \([14] \), we present here an observer independent decomposition of \( F \) into 1-vectors of the electric \( E \) and magnetic \( B \) fields that are defined without reference frames, i.e., they are independent of the chosen reference frame and of the chosen system of coordinates in it, see also \([15] \). We define

\[
F = (1/c) E \wedge v + (e_5 B) \cdot v, \\
E = (1/c) F \cdot v, \quad e_5 B = (1/c^2) F \wedge v, \quad B = -(1/c^2) e_5 (F \wedge v), \tag{24}
\]

where the pseudoscalar \( e_5 \) of some basis \{\( e_\mu \)\}, that does not need to be the standard basis \{\( \gamma_\mu \)\}, is defined as \( e_5 = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \). It holds that \( E \cdot v = B \cdot v = 0 \) (since \( F \) is skew-symmetric). \( v \) in \([24] \) can be interpreted as
the velocity (1-vector) of a family of observers who measure \( E \) and \( B \) fields. The velocity \( v \) and all other quantities entering into (24) are defined without reference frames. \( v \) characterizes some general observer. Thus both relations in (24) hold for any observer. However it has to be emphasized that (24) is not a physical definition of \( E \) and \( B \); the physical definition has to be given in terms of the Lorentz force and Newton’s second law as, e.g., in [15]. The relations (24) actually establish the equivalence of the formulation of electrodynamics with the field bivector \( F \) and the formulation with 1-vectors of the electric and magnetic fields. (Recently [10] I have presented a complete formulation of the electrodynamics using exclusively the bivector field \( F \).) Both formulations, with \( F \) and \( E, B \) fields, are equivalent formulations, but every of them is a complete, consistent and self-contained formulation. When (24) is used the field equation for \( F \) becomes

\[
\partial((1/c)E \wedge v + (e_5B) \cdot v) = j/\varepsilon_0c. \quad (25)
\]

In contrast to the field equation (11), that holds only for the \( \gamma_0 \)-observer, the field equation (25) holds for any observer; the quantities entering into (25) are all defined without reference frames. The equation (25) is physically completely equivalent to the field equation for \( F \) (11). In some basis \( \{e_\mu\} \), that does not need to be the standard basis \( \{\gamma_\mu\} \), the field equation (25) can be written as a coordinate-based geometric equation

\[
[\partial_\alpha (\delta^\alpha_\beta \mu\nu E^\mu c^\nu - \varepsilon^{\alpha\beta\mu\nu} v_\mu c B_\nu) - (j^\beta / \varepsilon_0)] e_\beta + \\
\partial_\alpha (\delta^\alpha_\beta \mu\nu v^\mu c B^\nu + \varepsilon^{\alpha\beta\mu\nu} v_\mu E_\nu) e_\beta = 0,
\]

(26)

where \( E^\alpha \) and \( B^\alpha \) are the basis components of the electric and magnetic 1-vectors \( E \) and \( B \); and \( \delta^\alpha_\beta \mu\nu = \delta^\alpha_\beta \delta^\mu_\nu - \delta^\mu_\beta \delta^\alpha_\nu \). The first part in (26) (it contains sources) emerges from \( \partial \cdot F = j/\varepsilon_0c \) and the second one (the source-free part) is obtained from \( \partial \wedge F = 0 \), see also [15]. Instead of that in (25) one can equivalently use the \( E, B \) formulation with the field equation (25), or in the \( \{e_\mu\} \) basis (26). (The complete \( E, B \)-formulation of the relativistic electrodynamics will be reported elsewhere.) We remark that (26) follows from (25) for those systems of coordinates for which the basis 1-vectors \( e_\mu \), constant, e.g., the standard basis \( \{\gamma_\mu\} \) (the Einstein system of coordinates). For a nonconstant basis, for example, when one uses polar or spherical basis 1-vectors (and, e.g., the Einstein synchronization) then one must also differentiate these nonconstant basis 1-vectors. Furthermore one can completely forget the manner in which the equation with \( E \) and \( B \) is obtained, i.e., the field equation with \( F \) (11) and consider the equation with \( E \) and \( B \) (26), which is defined without reference frames, or the corresponding coordinate-based geometric equation (26), as the primary and fundamental equations for the whole classical electromagnetism. In such correct relativistic formulation of the electromagnetism the field equation with 1-vectors \( E \) and \( B \) (25) takes over the role of the usual ME with the 3D \( E \) and \( B \), i.e., of the ME (8). We note that the equivalence of formulations of electromagnetism with tensors \( E^\alpha \) and \( B^\alpha \) is reported in [11,18] while the component form in the Einstein system of coordinates is given in [17,19] and [20].

Let us now take that in (26) the standard basis \( \{\gamma_\mu\} \) is used instead of some general basis \( \{e_\mu\} \). Then (26) can be written as \( C^\alpha \gamma_\beta + D^\beta \gamma_5 \gamma_\beta = 0 \), where \( C^\beta = \partial_\alpha (\delta^\alpha_\beta \mu\nu E^\nu - \varepsilon^{\alpha\beta\mu\nu} v_\mu c B_\nu) - j^\beta / \varepsilon_0 \) and \( D^\beta = \partial_\alpha (\delta^\alpha_\beta \mu\nu v^\mu c B^\nu + \varepsilon^{\alpha\beta\mu\nu} v_\mu E_\nu) \). When the active LT are applied upon such (26) with the \( \{\gamma_\mu\} \) basis the equation remains of the same form but with primed quantities replacing the unprimed ones (of course the basis is unchanged). This can be immediately seen since the equation (26) is written in a manifestly covariant form. Thus the Lorentz transformed (26) is

\[
R(C^\alpha \gamma_\beta + D^\beta \gamma_5 \gamma_\beta) \tilde{R} = 0,
\]

\[
C^\alpha \gamma_\beta + D^\beta \gamma_5 \gamma_\beta = 0,
\]

(27)

where, e.g., \( C^\alpha = \partial_\alpha (\delta^\alpha_\mu \mu\nu v^\nu c B^\nu - \varepsilon^{\alpha\beta\nu\mu} v^\mu c B^\nu) - j^\beta / \varepsilon_0 \). Obviously such formulation of the electromagnetism with fundamental equation (26) or (25) is a relativistically correct formulation.

What is the relation between the relativistically correct field equation (25) or (26) and the usual ME (8)? From the above discussion and from section II.A. one concludes that if in (25) we specify the velocity \( v \) of the observers who measure \( E \) and \( B \) fields to be \( v = c \gamma_0 \), then the equation (25) becomes the equation (11). Further choosing the standard basis \( \{\gamma_\mu\} \) in the \( \gamma_0 \)-frame, in which \( v = c \gamma_0 \), or in the components \( v^\alpha = (c, 0, 0, 0) \), then in that \( \gamma_0 \)-frame \( E \) and \( B \) become \( E_\perp \) and \( B_\perp \) and they do not have the temporal components \( E_0^\perp = B_0^\perp = 0 \). The coordinate-based geometric equation (26) becomes the usual Maxwell equations (8). Thus the usual Clifford algebra treatments of the electromagnetism [7-9] with the space-time split in the \( \gamma_0 \)-frame and the usual ME (8) are simply obtained from our observer independent formulation with field equation (26) or (25) choosing that \( v = c \gamma_0 \) and choosing the standard basis \( \{\gamma_\mu\} \). We see that the correspondence principle is simply satisfied in this formulation with \( E \) and \( B \) fields; all results obtained in the previous treatments from the usual Maxwell equations with the 3D \( E \) and \( B \) remain valid in the formulation with the 1-vectors \( E \) and \( B \) if physical phenomena are considered only in one inertial frame of reference. Namely the selected inertial frame of reference can be chosen to be the \( \gamma_0 \)-frame with the \( \{\gamma_\mu\} \) basis. Then there, as
explained above, the coordinate-based geometric equation (20) can be reduced to the equations containing only the components, the four Maxwell equations in the component form, the ME \( S \). Thus for observers who are at rest in the \( \gamma_0 \)-frame (\( v = c_0 \)) the components of the 3D \( E \) and \( B \) can be simply replaced by the space components of the 1-vectors \( E \) and \( B \) in the \( \{\gamma_\mu\} \) basis. We remark that just such observers are usually considered in the conventional formulation with the 3D \( E \) and \( B \). The dependence of the field equations (26) on \( v \) reflects the arbitrariness in the selection of the \( \gamma_0 \)-frame but at the same time it makes the equations (26) independent of that choice. The \( \gamma_0 \)-frame can be selected at our disposal, which proves that we don’t have a kind of the “preferred” frame theory. All experimental results that are obtained in one inertial frame of reference can be equally well explained by our geometric formulation of electrodynamics with the 1-vectors \( E \) and \( B \) as they are explained by the usual ME with the 3D \( E \) and \( B \).

However there is a fundamental difference between the standard approach with the 3D \( E \) and \( B \) and the approach with 4D quantities \( E \) and \( B \) that are defined without reference frames. It is considered in all standard treatments that the equation (23) is the LT of the original ME \( S \). But, as shown here, the equation (23) is not the LT of the original ME \( S \); the LT of the ME \( S \) are the equations (13) (i.e., (14) with (18), or (15)). The ME \( S \) are obtained from our field equation (20) by putting that \( v = c_0 \) and choosing the standard basis \( \{\gamma_\mu\} \). In the same way the equations (24), which are the LT of the equations (26), become the LT of the ME \( S \), that is, the equations (13) (or (14) with (18), or (15)), when in (24) it is taken that \( v' = R(c_0)\bar{R}, \bar{R}' = R\bar{R}, E' = RE_f \bar{R} = E_f, B' = RB_f \bar{R} = B_f \). We recall from section II.B. that to an observer in the \( \{\gamma_\mu\} \) frame the vector \( p' = (p' = R p \bar{R}) = p'\gamma_\mu \) appears the same as the vector \( p = (p = p\gamma_\mu) \) appears to an observer in the \( \{\gamma'_\mu\} \) frame. This, together with the preceding discussion, show that the usual ME with the 3D \( E \) and \( B \), i.e., the equations \( S \) and the equation (24) obtained by the ST from \( S \), cannot be used for the explanation of any experiment that test SR, i.e., in which relatively moving observers have to compare their data obtained by measurements on the same physical object. In contrast to the description of the electromagnetism with the 3D \( E \) and \( B \), the description with 4D fields \( E \) and \( B \), i.e., with the equations (26) and (27), is correct not only in the \( \gamma_0 \)-frame with the standard basis \( \{\gamma_\mu\} \) but in all other relatively moving frames and it holds for any permissible choice of coordinates, i.e., basis \( \{e_\mu\} \). We see that the relativistically correct fields \( E \) and \( B \) and the new field equations (25) and (26) do not have the same physical interpretation as the usual 3D fields \( E \) and \( B \) and the usual 3D ME \( S \) except in the \( \gamma_0 \)-frame with the \( \{\gamma_\mu\} \) basis in which \( E^0 = B^0 = 0 \). This consideration completely defines the relation between our approach with 4D \( E \) and \( B \) and all previous approaches.

III. THE PROOF IN THE GEOMETRIC ALGEBRA FORMALISM

USING BIVECTORS \( E_H \) AND \( B_H \)

A. The field equations in the \( \gamma_0 \)-frame. The Maxwell equations

The same proof and the whole consideration as in section II. can be repeated using in the \( \gamma_0 \)-frame with the \( \{\gamma_\mu\} \) basis the decomposition of \( F \) into the bivectors \( E_H \) and \( B_H \) (2) instead of the decomposition of \( F \) into 1-vectors \( E_f \) and \( B_f \). It will be seen that the type of the algebraic object chosen to represent the electric and magnetic fields is irrelevant for the whole consideration and for the obtained results. We shall briefly repeat the main results from section II. but starting with \( E_H \) and \( B_H \) instead of \( E_f \) and \( B_f \). When the decomposition (2) is substituted into the field equations (1) we find

\[
\partial[(F \cdot \gamma_0) \wedge \gamma_0 + (F \wedge \gamma_0) \cdot \gamma_0] = j/\varepsilon_0c, \\
\partial(E_H + c\gamma_0 B_H) = j/\varepsilon_0c.
\]

All quantities in (28) can be written as CBGQs in the standard basis \( \{\gamma_\mu\} \), see \( \{6\} \),

\[
E_H = F^{i0}\gamma_i \wedge \gamma_0, \quad B_H = (1/2c)e^{k0l}F_{kl}\gamma_i \wedge \gamma_0.
\]

It is seen from (29) that both bivectors \( E_H \) and \( B_H \) are parallel to \( \gamma_0 \), that is, it holds that \( E_H \wedge \gamma_0 = B_H \wedge \gamma_0 = 0 \). Further it follows from (29) that the components of \( E_H \) and \( B_H \) in the \( \{\gamma_\mu\} \) basis give rise to the tensor (components) \( (E_H)^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot E_H) = (\gamma^\nu \wedge \gamma^\mu) \cdot E_H \), (and the same for \( (B_H)^{\mu\nu} \)) which, written out as a matrix, have entries

\[
(E_H)^{i0} = F^{i0} = - (E_H)^{0i} = E^i, \quad (E_H)^{ij} = 0, \\
(B_H)^{i0} = (1/2c)e^{k0l}F_{kl} = -(B_H)^{0i} = B^i, \quad (B_H)^{ij} = 0.
\]
Then (29) becomes

\[ E_H = (E_H)^0 \gamma_i \land \gamma_0 = E^i \gamma_i \land \gamma_0, \]
\[ B_H = (B_H)^0 \gamma_i \land \gamma_0 = B^i \gamma_i \land \gamma_0. \]  

(31)

Multiplying (25) by \( \gamma_0 \) and using (26) and (30) we write the resulting equations as a coordinate-based geometric equation

\[ (\partial_k E^k - j^0 / c \varepsilon_0) + (\partial_0 E^i - c e^{i j k \theta} \partial_j B_k + j^i / c \varepsilon_0)(\gamma_i \land \gamma_0) + \\
(\alpha \partial_k B^k) \gamma_5 + (\alpha \partial_0 B^i + c e^{i j k \theta} \partial_j E_k)(\gamma_5 \gamma_i \land \gamma_0) = 0. \]  

(32)

The equation (32) is exactly the same as the equations obtained in the standard geometric algebra formalism, e.g., (8.5) and (8.6a-8.6d) in [7] Space-Time Algebra, but now written as a coordinate-based geometric equation. (25) encodes all four ME in the component form in the same way as it happens with the equation (3). It is worth noting that this step, the multiplication of (25) by \( \gamma_0 \), in order to get the usual ME, is unnecessary in the formulation from section II. with 1-vectors \( E_f \) and \( B_f \). This shows that the approach with 1-vectors \( E_f \) and \( B_f \) is simpler than the approach with bivectors \( E_H \) and \( B_H \) and also it is much closer to the classical formulation of the electromagnetism with the 3D vectors \( E \) and \( B \).

B. Lorentz transformations of the Maxwell equations

Let us now apply the active LT (using (10) to (32). First we rewrite (32) in the form

\[ a^0 + a^i (\gamma_i \land \gamma_0) + b^0 \gamma_5 + b^i \gamma_5 (\gamma_i \land \gamma_0) = 0. \]  

(33)

The coefficients \( a^0, a^i \) and \( b^0, b^i \) are clear from (32); they are the usual ME in the component form. As it is said the usual ME (32), i.e., (33), are obtained multiplying the equations (25) by \( \gamma_0 \) The LT of the resulting equations (after multiplication by \( \gamma_0 \)) are \( R \{ \gamma_0 [\partial (\gamma \land \gamma_0) + (\gamma \land \gamma_0) \cdot \gamma] / \varepsilon_0 c \} R \), that is,

\[ R \{ \gamma_0 [\partial (E_H + c \gamma_5 B_H) - j / \varepsilon_0 c \} R = 0. \]  

(34)

Then after applying the LT upon (32), i.e., (33), we find

\[ a^0 + R [a^i (\gamma_i \land \gamma_0)] R + b^0 \gamma_5 + R [b^i \gamma_5 (\gamma_i \land \gamma_0)] R = 0, \]  

(35)

where \( R [a^i (\gamma_i \land \gamma_0)] R = a^1 (\gamma_1 \land \gamma_0) + \gamma [a^2 (\gamma_2 \land \gamma_0) + a^3 (\gamma_3 \land \gamma_0)] - \beta \gamma [a^1 (\gamma_1 \land \gamma_0) + a^2 (\gamma_2 \land \gamma_0)] \) and \( R [b^i \gamma_5 (\gamma_i \land \gamma_0)] R = b^1 (\gamma_1 \land \gamma_2) + \gamma [-b^2 (\gamma_2 \land \gamma_1) + b^3 (\gamma_2 \land \gamma_1)] + \beta \gamma [b^1 (\gamma_1 \land \gamma_0) - b^3 (\gamma_2 \land \gamma_0)]. \) This result (35) is the usual result for the active LT of a multivector from (33).

The above equation (35) can be expressed in terms of Lorentz transformed derivatives and Lorentz transformed \( E_H \) and \( B_H \) as

\[ \gamma'_0 [\partial' (E'_H + c \gamma_5 B'_H) - j' / \varepsilon_0 c] = 0, \]  

(36)

where \( \gamma'_0 = R \gamma_0 R, \partial' = R \partial R, \) and (see also [6]) the Lorentz transformed bivectors are \( E'_H \) and \( B'_H \). This \( E'_H \) is

\[ E'_H = R [(F \cdot \gamma_0 \gamma_0)] R = R E_H \tilde{R} E^1 \gamma_1 \land \gamma_0 + \gamma (E^2 \gamma_2 \land \gamma_0 + \\
E^3 \gamma_3 \land \gamma_0) - \beta \gamma (E^2 \gamma_2 \land \gamma_1 + E^3 \gamma_3 \land \gamma_1). \]  

(37)

The components \( (E'_H)^{\mu \nu} \) that are different from zero are \( (E'_H)^{10} = E^1, (E'_H)^{20} = \gamma E^2, (E'_H)^{30} = \gamma E^3, (E'_H)^{12} = \beta \gamma E^2, (E'_H)^{13} = \beta \gamma E^3. \) \( (E'_H)^{\mu \nu} \) is antisymmetric, i.e., \( (E'_H)^{\mu \nu} = - (E'_H)^{\nu \mu} \) and we denoted, as in (30), \( E^i = F^{i0} \).

Similarly we find for \( B'_H \)

\[ B'_H = R [- (1 / c) \gamma_5 (F \land \gamma_0) / \varepsilon_0 c] R = R B_H \tilde{R} = B^1 \gamma_1 \land \gamma_0 + \\
\gamma (B^2 \gamma_2 \land \gamma_0 + B^3 \gamma_3 \land \gamma_0) - \beta \gamma (B^1 \gamma_1 \land \gamma_1 + B^3 \gamma_3 \land \gamma_1). \]  

(38)

The components \( (B'_H)^{\mu \nu} \) that are different from zero are \( (B'_H)^{10} = B^1, (B'_H)^{20} = \gamma B^2, (B'_H)^{30} = \gamma B^3, (B'_H)^{12} = \beta \gamma B^2, (B'_H)^{13} = \beta \gamma B^3. \) \( (B'_H)^{\mu \nu} \) is antisymmetric, i.e., \( (B'_H)^{\mu \nu} = - (B'_H)^{\nu \mu} \) and we denoted, as in (30), \( B^i =
\((1/2c)\epsilon^{k\ell\mu\nu}F_{k\ell}\). Both (37) and (38) are the familiar forms for the active LT of bivectors, here \(E_H\) and \(B_H\). It is worth noting that \(E_H'\) and \(B_H'\), in contrast to \(E_H\) and \(B_H\), are not parallel to \(\gamma_0\), i.e., it does not hold that \(E_H' \land \gamma_0 = B_H' \land \gamma_0 = 0\) and thus there are \((E_{H'})^{ij} \neq 0\) and \((B_{H'})^{ij} \neq 0\). Further, as it happens for \(E_f\) and \(B_f\), see (19) and (21), the components \((E_{H'})^{\mu\nu}\) transform upon the active LT again to the components \((B_{H'})^{\mu\nu}\); there is no mixing of components. Thus by the active LT \(E_H\) transforms to \(E_H'\) and \(B_H\) to \(B_H'\). Actually, as we said, this is the way in which every bivector transforms upon the active LT. The last form of the Lorentz transformed field equation, (37), can be written as a coordinate-based geometric equation in the standard basis \(\{\gamma_\mu\}\), but for simplicity we only quote the scalar term \(a^0\)

\[
a^0 = -\beta \gamma [\partial_0 (E_H')^{10} + \gamma [\partial_k (E_H')^{k0}] + \beta \gamma [\partial_k (E_H')^{k1}] + \partial_0 (E_H')^{31}] - (\gamma j^0 - \beta \gamma j^1)/\varepsilon_0 c
\]

(39)

Comparing \(a^0\) with \(a^0\) from the usual ME (22), i.e., \(a^0 = \partial_k (E_H)^{k0} - j^0/\varepsilon_0\), we again see, as in section II.B with \(E_f\) and \(B_f\), that \(a^0\) substantially differs in form from the term \(a^0\) in (22). Again the same situation happens with other transformed terms, which shows, as in section II.B, that the Lorentz transformed ME, (38), with (39), are not of the same form as the original ones (22), i.e., (21). This is a fundamental result which once again reveals that, contrary to the previous derivations, e.g., \([2,16], [3,4], [7-9]\), and contrary to the generally accepted belief, the usual ME are not Lorentz covariant equations.

C. Standard transformations of the Maxwell equations

As can be easily shown, see also \([6]\), the ST for \(E_{H,ST}\) and \(B_{H,ST}\) are derived wrongly assuming that the quantities obtained by the active LT of \(E_H\) and \(B_H\) are again parallel to \(\gamma_0\), i.e., that again holds \(E_H' \land \gamma_0 = B_H' \land \gamma_0 = 0\) and consequently that \((E_{H,ST}')^{ij} = (B_{H,ST}')^{ij} = 0\). Thence, in contrast to the correct LT of \(E_H\) and \(B_H\), (37) and (38) respectively, it is taken in standard derivations \([7,\text{Space-Time Algebra (eq. (18.22))}, \text{New Foundations for Classical Mechanics (Ch. 9 eqs. (3.51a,b))}, [8] \text{Geometric algebra for physicists (Ch. 7.1.2 eq. (7.33)))}\) that

\[
E_{H,ST} = (F' \land \gamma_0)\gamma_0 = (E_{H,ST})^{0\gamma_0} \gamma_0 = E_{ST,\gamma_0} \land \gamma_0 = E_1 \gamma_0 \land (\gamma E^2 - \beta \gamma c B^3) \gamma_2 \land \gamma_0 + (\gamma E^3 + \beta \gamma c B^2) \gamma_3 \land \gamma_0,
\]

(40)

where \(F' = RF\tilde{R}\). Similarly we find for \(B_{H,ST}^\prime\)

\[
B_{H,ST}^\prime = (-1/c)\gamma_0 [(F' \land \gamma_0) \cdot \gamma_0] = (B_{H,ST})^{0\gamma_0} \gamma_0 = B_{ST,\gamma_0} \land \gamma_0 = B^1 \gamma_0 \land (\gamma E^2 + \beta \gamma E^3/c) \gamma_2 \land \gamma_0 + (\gamma B^3 - \beta \gamma E^2/c) \gamma_3 \land \gamma_0.
\]

(41)

The relations (19) and (21) immediately give the familiar expressions for the ST of the 3D vectors \(E\) and \(B\). Now, in contrast to the correct LT of \(E_H\) and \(B_H\), (37) and (38) respectively, the components of the transformed \(E_{H,ST}^\prime\) are expressed by the mixture of components \(E^i\) and \(B^i\), and the same holds for \(B_{H,ST}^\prime\).

Here we again explicitly show that in the standard derivations \([7-9]\) the ME (32) remain unchanged in form not upon the LT but upon some transformations which, strictly speaking, have nothing to do with the LT of the equation (32). Namely the ST of the second equation in (32) (after multiplication by \(\gamma_0\)) are given as

\[
\gamma_0 [\partial_0 (E_{H,ST}) + c\gamma_0 (B_{H,ST}) - j'/\varepsilon_0 c] = 0,
\]

(42)

where \(E_{H,ST}\) and \(B_{H,ST}\) are determined by (19) and (21). Notice again that, in contrast to the correct LT \(E_1\) or \(B_1\), \(\gamma_0\) is not transformed in (42), as it is not transformed in the ST of the electric and magnetic fields \(E_1\) and \(B_1\). When (42) is written as a coordinate-based geometric equation in the standard basis \(\{\gamma_\mu\}\) it becomes

\[
(\partial_k E_{ST}^{i0} - j^0/\varepsilon_0) + (\partial_0 E_{ST}^{i0} - c\epsilon^{ijk\ell} \partial_j B_{ST,k}^\ell + j^i/\varepsilon_0)(\gamma_i \land \gamma_0) + (c\partial_0 B_{ST}^{i0} \gamma_0 + (c\partial_i B_{ST}^{i0} + \epsilon^{ijk\ell} \partial_j B_{ST,k}^\ell) \gamma_5(\gamma_i \land \gamma_0) = 0.
\]

(43)

The equation (13) is of the same form as the original ME (32) but the electric and magnetic fields are not transformed by the LT than by the ST. As seen from (42) (together with (19) and (21)) the equation (43) is not the LT of the original ME (32); the LT of the ME (32) is the equation (38) with (39) (i.e., (21) or (22)), where the Lorentz transformed electric and magnetic fields are given by the relations (37) and (38).

D. Lorentz invariant field equations with bivectors \(E_{HL}\) and \(B_{HL}\)
As explained in the preceding sections the observer independent \( F \) field is decomposed in \( \gamma_0 = H \) (see [7,8]) in terms of observer dependent quantities, i.e., as the sum of a relative vector \( E_H \) and a relative bivector \( \gamma_3 B_H \), by making the space-time split in the \( \gamma_0 \)-frame. But, similarly as in section II.D., we present here an observer independent decomposition of \( F \) into bivectors \( E_{HL} \) and \( B_{HL} \) that are defined without reference frames, i.e., which are independent of the chosen reference frame and of the chosen system of coordinates in it. We define

\[
F = E_{HL} + c\gamma_5 B_{HL}, \quad E_{HL} = (1/c^2)(F \cdot v) \wedge v \\
B_{HL} = -(1/c^3)\gamma_5[(F \wedge v) \cdot v], \quad \gamma_3 B_{HL} = (1/c^3)(F \wedge v) \cdot v
\]

(44)

(The subscript ‘HL’ is for - Hestenes, Lasenby, see [21]). Of course, as in II.D., formulation with 1-vectors \( E \) the space-time split in the \( (44) \) entering into \( (44) \) and \( B \) coordinate-based geometric equation, and it looks much more complicated than the equation (26) with 1-vectors \( E \), the field equation for \( (26) \) with 1-vectors \( E \) the quantities entering into formulation with bivectors \( F \) equivalent to the field equation for \( (26) \) with 1-vectors \( E \) and all other quantities entering into \( (44) \) are defined without reference frames. Consequently \( (44) \) holds for any observer. When \( (44) \) is used the field equation for \( F \) \( (11) \), after multiplication by \( v/c \) (instead of by \( \gamma_0 \)), becomes

\[
(v/c)\{\partial(E_{HL} + c\gamma_5 B_{HL}) - j/\varepsilon_0 c\} = 0.
\]

(45)

In contrast to the field equation \( (23) \) that holds only for the \( \gamma_0 \)-observer, the field equation \( (15) \) holds for any observer; the quantities entering into \( (15) \) are all defined without reference frames. The equation \( (15) \) is physically completely equivalent to the field equation for \( F \) \( (11) \), i.e., to the field equation with 1-vectors \( E \) and \( B \). The equation \( (23) \) corresponds to the equation \( (11) \) while \( (15) \) corresponds to \( (23) \). The field equation \( (15) \) can be written as a coordinate-based geometric equation, and it looks much more complicated than the equation \( (23) \) with 1-vectors \( E \) and \( B \). We write it (for better comparison) as two equations; the first one will yield the scalar and bivector parts of \( (32) \) when \( v/c = \gamma_0 \). It is

\[
(1/c)v_\beta \partial_\alpha(E_{HL})^{\alpha \beta} + [(1/2c)v^\sigma \partial_\alpha(E_{HL})^{\beta \sigma} - (1/2)c^2v^\sigma \partial_\alpha(B_{HL})_{\mu \nu}]\gamma_3 \wedge \gamma_\sigma
\]

\[
= (1/\varepsilon_0 c^2)(v_\alpha j^\alpha + v^\beta j^\beta \gamma_3 \wedge \gamma_\sigma).
\]

(46)

The second equation will yield the pseudoscalar and pseudobivector parts of \( (32) \) when \( v/c = \gamma_0 \) and it is

\[
v_\beta \partial_\alpha(B_{HL})^{\alpha \beta} \gamma_5 + (1/2)v^\sigma \partial_\alpha(B_{HL})^{\mu \nu} \gamma_5(\gamma_\mu \wedge \gamma_\nu) + (v_\beta \partial^\alpha - v^\sigma \partial_\beta)(E_{HL})_{\alpha \sigma} \gamma_3 \wedge \gamma_\nu = 0.
\]

(47)

The equation \( (15) \) is with sources and it emerges from \( \partial \wedge F = j/\varepsilon_0 c \), while \( (17) \) is the source-free equation and it emerges from \( \partial \wedge F = 0 \). Comparing \( (15) \) and \( (17) \) in the \( E_{HL}, B_{HL} \) - formulation with the corresponding parts in \( (23) \) with 1-vectors \( E \) and \( B \) we see that the formulation with \( E \) and \( B \) is much simpler and more elegant than the formulation with bivectors \( E_{HL} \) and \( B_{HL} \); the physical content is completely equivalent.

The equations \( (15) \) and \( (17) \) are written in a manifestly covariant form. This means that when the active LT are applied upon such \( (15) \) and \( (17) \) the equations remain of the same form but with primed quantities replacing the unprimed ones (of course the basis is unchanged).

The whole discussion from section II.D. (with 1-vectors \( E \) and \( B \)) about the correspondence principle applies in the same measure to the formulation with bivectors \( E_{HL} \) and \( B_{HL} \). The only difference is the simplicity of the formulation with 1-vectors \( E \) and \( B \).

The same conclusions hold for the formulation with 1-vector \( E_I \) and a bivector \( B_J \) from [9], but for the sake of brevity that formulation will not be considered here.

IV. THE PROOF IN THE TENSOR FORMALISM USING 4-VECTORS \( E^a \) AND \( B^a \)

The same proof that the classical electromagnetism and SR are not in agreement can be given in the tensor formalism as well. The important parts of this issue are already treated in two papers, [11] and [5].

Let us start with some general definitions. The electromagnetic field tensor \( F^{ab} \) is defined without reference frames, i.e., it is an abstract tensor, a geometric quantity; Latin indices \( a,b,c \) are to be read according to the abstract index notation, as in [22] and [11,12], [18]. When some reference frame (a physical object) is introduced and the system of coordinates (a mathematical object) is adopted in it, then \( F^{ab} \) can be written as a CBGQ containing components and a basis. As already said in the invariant formulation of SR that uses 4D quantities defined without reference frames [11,12], [18] and [5] in the tensor formalism, and [10,15] and [6] in the Clifford algebra formalism, any permissible system of coordinates, not necessary the Einstein system of coordinates, i.e., the standard basis \( \{\gamma_\mu\} \), can be used on an equal footing. However, for simplicity, in this part we shall deal only with the standard basis \( \{\gamma_\mu\} \). When \( F^{ab} \) is written as a CBGQ it becomes \( F^{ab} = F^{\mu \nu} \gamma_\mu \otimes \gamma_\nu \), where Greek indices \( \mu, \nu \) in \( F^{\mu \nu} \) run from 0 to 3 and they denote the components of the geometric object \( F^{ab} \) in some system of coordinates, here the standard basis \( \{\gamma_\mu\} \). In the
tensor formalism $\gamma_\mu$ denote the basis 4-vectors (not components) forming the standard basis $\{\gamma_\mu\}$ and $\otimes$ denotes the tensor product of the basis 4-vectors. In the tensor formalism I shall often denote the unit 4-vector in the time direction $\gamma_0$ as $t^b$ as well. Then in some reference frame with the standard basis $\{\gamma_\mu\}$ $t^b$ can be also written as a CBGQ, $t^b = t^\mu \gamma_\mu$, where $t^\mu$ is a set of components of the unit 4-vector in the time direction ($t^\mu = (1, 0, 0, 0)$). Almost always in the standard covariant approaches to SR one considers only the components of the geometric quantities taken in the $\{\gamma_\mu\}$ basis and thus not the whole tensor. However the components are coordinate quantities and they do not contain the whole information about the physical quantity.

**A. The field equations in the $\gamma_0$-frame. The Maxwell equations**

In the abstract index notation the field equations are given as

\[
(-g)^{-1/2} \partial_\mu ((-g)^{1/2} F^{ab}) = j^b / \varepsilon_0 c, \quad \varepsilon^{abcd} \partial_a F_{cd} = 0 \tag{48}
\]

where $g$ is the determinant of the metric tensor $g_{ab}$ and $\partial_\mu$ is an ordinary derivative operator. Now there are two field equations while in the geometric algebra formalism they are united in only one field equation. When written in the $\{\gamma_\mu\}$ basis as coordinate-based geometric equations the relations (45) become

\[
\partial_\mu F^{\alpha\beta} \gamma_\beta = (1/\varepsilon_0 c) j^\alpha \gamma_\beta, \quad \partial_\mu * F^{\alpha\beta} \gamma_\beta = 0. \tag{49}
\]

Notice that from (49) one simply finds the usual covariant form (the component form in the $\{\gamma_\mu\}$ basis) of the field equations with $F^{\alpha\beta}$ and its dual $* F^{\alpha\beta}$

\[
\partial_\mu F^{\alpha\beta} = j^\beta / \varepsilon_0 c, \quad \partial_\mu * F^{\alpha\beta} = 0, \tag{50}
\]

where $* F^{\alpha\beta} = (1/2) \varepsilon^\alpha\beta\gamma^\delta F_{\gamma\delta}$. In analogy with the geometric algebra formalism, $F^{ab}$ can be decomposed in terms of the observer dependent 4-vectors $E^a_j$ and $B^a_j$ by singling out a particular time-like direction $t^b$. (This corresponds to the decomposition of $F$ into 1-vectors $E_f$ and $B_f$ (3).) Thus

\[
F^{ab} = \delta^{ab} c^d E_{f}^d + \varepsilon^{abcd} t_a B_{f,d},
\]

\[
E^a_f = F^{ab} t_b, \quad B^a_f = (1/2c) \varepsilon^{abcd} t_b F_{cd}. \tag{51}
\]

All quantities from (51) can be written as CBGQs in the standard basis $\{\gamma_\mu\}$. Then in the tensor formalism we find the same equations as the equations (3) in the geometric algebra formalism with 1-vectors $E_f$ and $B_f$. They are

\[
E^a_f = E^a_f \gamma_\mu = 0 \gamma_0 + F^{kl} \gamma_k,
\]

\[
B^a_f = B^a_f \gamma_\mu = 0 \gamma_0 + (-1/2c) \varepsilon^{kli} F_{kl} \gamma_i, \tag{52}
\]

whence we get the relation $E^a_f \gamma_0 = F^{0a}$, $B^a_f \gamma_0 = (1/2c) \varepsilon^{kli} F_{lk}$, which is, as already said, not the standard identification of the components $F^{\nu\mu}$ with the components of the 3D vectors $E$ and $B$, see, e.g., [16], [3, 4]. (As mentioned previously Einstein’s fundamental work [16] is the earliest reference on generally covariant electrodynamics and on the identification of some components of $F^{ab}$ (actually $F^{ab}$) with the components of $E$ and $B$. He introduces an electromagnetic potential 4-vector (in component form) and from this constructs $F^{ab}$, the component of the $F^{ab}$ tensor. Then he writes the equations (3) and shows that these equations correspond to the usual Maxwell equations with $E$ and $B$ if he makes the identification given in the equations (4). It has to be mentioned that Einstein actually worked with the equations for basis components in the $\{\gamma_\mu\}$ basis and thus not with the abstract tensors, defined without reference frames, or with coordinate-based geometric equations (see, e.g., [23] for the comparison of Einstein’s view of spacetime and the modern view).) In fact, the whole discussion in connection with the relations (3) and (4) applies in the same measure to (52). Thus in the rest frame of ‘fiducial’ observers (we again call that frame - the $\gamma_0$-frame) $E^a_f$ and $B^a_f$ do not have the temporal components $E^a_0 = B^a_0 = 0$; in the $\gamma_0$-frame $t^\mu$ can be interpreted as the 4-velocity (the components in the $\{\gamma_\mu\}$ basis) of the observers that are at rest there. In the standard treatments the 3-vectors $E$ and $B$, as geometric quantities in the 3D space, are constructed from the spatial components $E^i$ and $B^i$ from (22), i.e., (4), and the unit 3-vectors $i$, $j$, $k$, e.g., $E = F^{10}i + F^{20}j + F^{30}k$. These results are quoted in numerous textbooks and papers treating relativistic electrodynamics in the tensor formalism, see, e.g., [16], [3, 4]. Actually in the usual covariant approaches, e.g., [16], [3, 4], one forgets about $E^0$ and $B^0$ components and simply makes the identification of six independent components of $F^{\mu\nu}$ with three components $E^i$, $E^i = F^{i0}$, and three components $B^i$, $B^i = (1/2) \varepsilon^{ijkl} F_{jk}$. Since in SR we work with the 4D spacetime the mapping the between the components of $F^{\mu\nu}$ and the components of the 3D vectors $E$ and $B$ is mathematically better founded by the relations
4-vectors coordinate-based geometric equations with $t^\beta$. Using (52) and procedure is made in an inertial frame of reference with the standard basis $\{\gamma_\mu\}$. Therefore we proceed the consideration using (52). Note again that the whole geometric quantities would need to be multiplied with the unit 4-vectors $\gamma_i$ and not with the unit 3-vectors.

Substituting (51) (but written in the $\{\gamma_\mu\}$ basis, where $F^{\alpha\beta} = \delta^{\alpha\beta} - \delta^{\alpha\beta} t^\mu B_f,\nu$ into (50) we find the coordinate-based geometric equations with $E^{\mu}_f$, $B^{\mu\nu}_f$ and $t^\nu$ as

$$
\partial_\alpha (\delta^{\alpha\beta} - \delta^{\alpha\beta} t^\mu B_f,\nu) \gamma_\beta = (1/\epsilon_0) \gamma_\beta
$$

$$
\partial_\alpha (\delta^{\alpha\beta} - \delta^{\alpha\beta} t^\mu c B_f^\nu + \epsilon^{\alpha\beta\mu\nu} E_f,\nu) \gamma_\beta = 0.
$$

Using (52) and $t^\alpha = (1, 0, 0, 0)$ in (53) these equations become the same equations as (8), that is, the usual Maxwell equations in the component form. They are

$$
(\partial_0 E_f^\mu - j^0/c_0) \gamma_0 + (-\partial_\mu E_f^\nu + \epsilon^{\nu\lambda\sigma} \partial_\lambda B_f,\sigma - j^i/c_0) \gamma_i = 0
$$

$$
(-c \partial_0 B_f^\nu) \gamma_0 + (c \partial_\mu B_f^\nu + \epsilon^{\nu\lambda\sigma} \partial_\lambda E_f,\sigma) \gamma_i = 0.
$$

The same discussion holds for (51) (53) as for (8) (5).

**B. Lorentz transformations of the Maxwell equations**

Let us now apply the passive LT to the equations (53), or (54) in the tensor formalism we shall deal with the passive LT. Upon the passive LT the sets of components $E_f^\mu$ and $B_f^{\mu\nu}$ determined in the $\gamma_0$ - frame (the $S$ frame) from (52) transform to $E_f^\mu$ and $B_f^{\mu\nu}$ in the relatively moving IFR $S'$. From (57) and (58) into (60) we find the

$$
E_f^\mu = F^{\mu\nu} v_\nu', \quad B_f^{\mu\nu} = (1/2) \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} v_\nu' = (F^*)_\nu v_\nu',
$$

$$
E_f'^\mu = (-\beta \gamma E_1^1, \gamma E_1^1, E_2, E_3), \quad B_f'^{\mu\nu} = (-\beta \gamma B_1^1, \gamma B_1^1, B_2^2, B_3^3),
$$

where $v_\nu' = (\gamma, \beta \gamma, 0, 0)$, and $v_\nu'$ is not in the time direction in $S'$, i.e., it is not $t_\nu'$. The unit 4-vector (the components) $t^\mu$ in the time direction in $S$ transforms upon the LT into the unit 4-vector $v_0'$, the 4-velocity of the moving observers, that contains not only the temporal component but also $\neq 0$ spatial components. Hence, the LT transform the set of components (52) into (55). Note that $E_f^\mu$ and $B_f^{\mu\nu}$ do have the temporal components as well. Further the components $E_f^{\mu\nu}$ ($B_f^{\mu\nu}$) in $S$ transform upon the LT again to the components $E_f'^{\mu\nu}$ ($B_f'^{\mu\nu}$) in $S'$; there is no mixing of components. Actually this is the way in which every well-defined 4-vector (the components) transforms upon the LT. A geometric quantity, an abstract tensor $E^\alpha$, can be represented by CBGQs in $S$ and $S'$ (both with the Einstein system of coordinates) as $E_f^{\mu\gamma_\mu}$ and $E_f'^{\mu\gamma_\mu}$, where $E_f^\mu$ and $E_f'^\mu$ are given by the relations (52) and (55) respectively. All the primed quantities (components and the basis) are obtained from the corresponding unprimed quantities through the LT. Of course it must hold that

$$
E^\alpha = E_f^{\mu\gamma_\mu} = E_f'^{\mu\gamma_\mu},
$$

since the components $E_f^\mu$ transform by the LT, while the basis $\gamma_\mu$ transforms by the inverse LT, thus leaving the whole CBGQ invariant upon the passive LT. The invariance of some 4D CBGQ upon the passive LT is the crucial requirement that must be satisfied by any well-defined 4D quantity. It reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. The use of CBGQs enables us to have clearly and correctly defined the concept of sameness of a physical system for different observers. The importance of this concept in SR was first pointed out in [24,25]. However they also worked with components in the Einstein system of coordinates (the covariant quantities) and not with geometric quantities (the invariant quantities). It is worth noting that in all other standard treatments, e.g., [2-4] (and [7-9] in the geometric algebra formalism), the importance of such concept is completely overlooked what caused many difficulties in understanding SR. It can be easily checked by the direct inspection that (56) holds when $E_f^\mu$ and $E_f'^\mu$ are given by (52) and (55). (The same holds for $B^\alpha$.)

The equations (53), or (54), can be written as $a^\alpha \gamma_\alpha = 0$ and $b^\alpha \gamma_\alpha = 0$, similarly to the equation (5). The coefficients $a^\alpha$ and $b^\alpha$ are clear from the first and second equation respectively in (53), or (54): they are the usual Maxwell equations in the component form. Then upon the passive LT the equations (53), or (54), transform to

$$
\begin{align*}
a^\alpha \gamma_\alpha' &= 0, \quad b^\alpha \gamma_\alpha' = 0,
\end{align*}
$$

(57)
and it holds, for any 4-vector (a geometric quantity), that \( a^\alpha \gamma^\alpha = a^\alpha \gamma_\alpha \), and \( b^\alpha \gamma^\alpha = b^\alpha \gamma_\alpha \); the coefficients transform by the LT as \( a^0 = \gamma a^0 - \beta \gamma a^1 \), \( a^1 = \gamma a^1 - \beta \gamma a^0 \), \( a^2 = a^2 \), \( a^3 = a^3 \) (and the same for \( b^\alpha \)), while the basis 4-vectors transform by the inverse LT as \( \gamma^i = \gamma_\alpha + \beta \gamma \gamma_\alpha \), \( \gamma^1 = \gamma_1 + \beta \gamma \gamma_3 \), \( \gamma^2 = \gamma_2 \), \( \gamma^3 = \gamma_3 \). Of course \( i^\nu \) transforms to \( i^\nu \) and \( E^\mu_i \), \( B^\mu_i \) are given by (55). (The equation (57) corresponds to the equation (10) in the geometric algebra formalism with 1-vectors \( E_f \) and \( B_f \).) Again we see that, e.g., the Gauss law for the electric field \( a^0 \) does not transform by the LT again to the Gauss law but to \( a^0 \), which is a combination of the Gauss law and a part of the Ampère-Maxwell law (\( a^1 \)). When the coefficients \( a^\alpha \) and \( b^\alpha \) are written in terms of the primed quantities (from the \( S' \) frame) they become (for simplicity only the coefficient \( a^0 \) is written) \( a^0 = \gamma (\partial_k E^\mu_k) + \beta \gamma (\partial_k E^\mu_k + c(\partial_2 B_{f3} - \partial_3 B_{f2}]) - j^0 / \varepsilon_0 \), (58) and it is completely different in form than the coefficient \( a^0 = (\partial_k E^\mu_k - j^0 / \varepsilon_0) \) in (54). (55) corresponds to (15). Again, it can be concluded from (58) that the LT do not transform the Gauss law into the primed Gauss law but into a quite different law; \( a^0 \) contains the time component \( E^\mu_0 \) while the starting, unprimed \( E^\mu_0 \) is \( E^\mu_0 = 0 \). Also the new "Gauss law" includes the derivatives of the magnetic field. The same situation happens with the other Lorentz transformed terms, which once again explicitly shows that neither in the tensor formalism the Lorentz transformed ME (57) with (58) are of the same form as the original ones (54). As discussed in section II.B. this fundamental result reveals, in the tensor formalism as well, that, contrary to all previous derivations, e.g., [2-4], and contrary to the generally accepted opinion, the usual ME are not Lorentz covariant equations.

### C. Standard transformations of the Maxwell equations

In this section we present the derivation of the ST of the ME in the tensor formalism which is in a complete analogy with the derivation in section II.C. In all usual treatments, e.g., [3] and [4] eqs. (3.5) and (3.24), in \( S' \) one again simply makes the identification of six independent components of \( F^{\mu\nu} \) with three components \( E^\mu_i \), \( E^\mu_i = F^{\mu_0} \), and three components \( B^\mu_i \), \( B^\mu_i = (1/2) \varepsilon_{ikl} F^{\mu_{ikl}} \). This means that standard treatments assume that under the passive LT the set of components \( t^\nu = (1, 0, 0, 0) \) from \( S \) transforms to \( t'^\nu = (1, 0, 0, 0) \) (\( t'^\nu \) are the components of the unit 4-vector in the time direction in \( S' \) and in the Einstein system of coordinates), and consequently that \( E^\mu_i \) and \( B^\mu_i \) from \( (52) \) transform to \( E^\mu_i \), \( B^\mu_i \) in \( S' \),

\[
E^\mu_{st.} = F^{\mu\nu} t^\nu, \quad B^\mu_{st.} = (F^\ast)_i t^\nu, \quad E^\mu_{st.} = (0, E^1, E^2, E^3), \quad B^\mu_{st.} = (0, B_1, B_2, B_3),
\]

where the subscript - st. is for - standard. The temporal components of \( E^\mu_{st.} \) and \( B^\mu_{st.} \) in \( S' \) are again zero as are the temporal components of \( E^\mu_i \) and \( B^\mu_i \) in \( S \). This fact clearly shows that the transformations given by the relation \( (59) \) are not the LT of some well-defined 4D quantities; the LT cannot transform a 4-vector for which the temporal component is zero in one frame \( S \) to the 4-vector with the same property in relatively moving frame \( S' \); i.e., they cannot transform the unit 4-vector in one frame \( S \) to the unit 4-vector in the time direction in another relatively moving frame \( S' \). Obviously \( E^\mu_{st.} \) and \( B^\mu_{st.} \) are completely different quantities than \( E^\mu_i \) and \( B^\mu_i \) that are obtained by the correct LT. We can easily check that

\[
E^\mu_{st.} \gamma_{\mu} \neq E^\mu_i \gamma_{\mu}, \quad B^\mu_{st.} \gamma_{\mu} \neq B^\mu_i \gamma_{\mu}.
\]

This means that, e.g., \( E^\mu_{st.} \gamma_{\mu} \) and \( E^\mu_{st.} \gamma_{\mu} \) are not the same quantity for observers in \( S \) and \( S' \). As far as relativity is concerned the quantities, e.g., \( E^\mu_{st.} \gamma_{\mu} \) and \( E^\mu_{st.} \gamma_{\mu} \), are not related to one another. The observers in \( S \) and \( S' \) are not looking at the same physical object but at two different objects; every observer makes measurement on its own object and such measurements are not related by the LT. The transformations \( (59) \) are not the LT and \( E^\mu_{st.} \) and \( B^\mu_{st.} \), in contrast to \( E^\mu_i \) and \( B^\mu_i \), are not well-defined 4D quantities. From the relativistically incorrect transformations \( (59) \) one simply derives the transformations of the spatial components \( E^\mu_{st.} \), \( B^\mu_{st.} \) which are the same as \( (21) \). It can be again seen from \( (59) \), or \( (21) \), that the transformations of \( E^\mu_{st.} \) and \( B^\mu_{st.} \) are exactly the ST of components of the 3-vectors \( E \) and \( B \) that are obtained by Lorentz [1] and independently by Einstein [2] and subsequently quoted in almost every textbook and paper on relativistic electrodynamics. Notice that, in the tensor formalism as well, according to the ST \( (59) \), i.e., \( (21) \), the transformed components \( E^\mu_{st.} \) and \( B^\mu_{st.} \) are expressed by the mixture of components \( E^\mu_i \) and \( B^\mu_i \). This completely differs from the correct LT \( (55) \). Both the transformations \( (59) \) and the transformations for \( E^\mu_{st.} \) and \( B^\mu_{st.} \) \( (21) \) are typical examples of the "apparent" transformations that are first discussed in [24] and [25]. The "apparent" transformations of the spatial distances (the Lorentz contraction) and the temporal distances (the dilatation of time) are elaborated in detail in [11, 12] (see also [20]). It is explicitly shown in [12] that the true
agreement with experiments that test SR exists only when the theory deals with well-defined 4D quantities, i.e., the quantities that are invariant upon the passive LT. In all previous treatments of SR, e.g., [2–4], the transformations for \( E^a_{\mu} \) and \( B^a_{\mu} \) are considered to be the LT of the 3D electric and magnetic fields. However as shown above (the comparison of (53) and (54), or (51)) the transformations for \( E^a_{\mu} \) and \( B^a_{\mu} \) are derived from the relativistically incorrect transformations (50) and moreover the 3-vectors \( E' \) and \( B' \) are again formed by an incorrect procedure in 4D spacetime, i.e., by multiplying these relativistically incorrect components with the unit 3-vectors in the \( S' \) frame.

Let us now perform the ST of the ME (53) supposing that \( E^a_{\mu} \) and \( B^a_{\mu} \) in \( S \) are transformed into \( E^a'_{\mu} \) and \( B^a'_{\mu} \) in \( S' \) according to (57) and that the set of components \( v' = (1, 0, 0, 0) \) from \( S \) transforms to \( v'' = (1, 0, 0, 0) \) in \( S' \). Then (53) transforms to the same equations but with \( E^a_{\mu} \) and \( B^a_{\mu} \) replacing \( E^a'_{\mu} \) and \( B^a'_{\mu} \) and \( v'' \) replacing \( v' \). From the transformed equations obtained in such a way one easily finds the ST of the ME (51). They are

\[
(\partial_k E^a_{\mu k} - j^a / c \varepsilon_0) \gamma_0' + (\partial^a_{\mu} E^a_{\nu} + c^a \delta^a_{\mu} \delta^a_{\nu} B^a_{\nu,k} - j^a / c \varepsilon_0) \gamma_0' = 0, \\
(\partial^a_{\nu} E^a_{\nu}) \gamma_0' + (\partial^a_{\nu} B^a_{\nu} + c^a \delta^a_{\mu} \delta^a_{\nu} B^a_{\nu}) \gamma_0' = 0.
\]

(61)

The equations (61) correspond to the equation (29) in the formalism with 1-vectors \( E_f \) and \( B_f \). They are of the same form as the original ME (51) with primed quantities replacing the corresponding unprimed ones, but, as remarked above, \( E^a_{\mu} \) and \( B^a_{\mu} \) replace \( E^a'_{\mu} \) and \( B^a'_{\mu} \) from \( S \). Then we get the same result as in the geometric algebra formalism, i.e., that the equations (61) are not the correct LT but relativistically incorrect transformations of the original ME (50); the LT of the ME (50) are the equations (57) with (55), where the Lorentz transformed electric and magnetic fields, the components \( E^a_{\mu} \) and \( B^a_{\mu} \) respectively, are given by the relations (55). We note that Einstein's derivation [2] of the ST of fields and of the ME, together with the similar derivation presented in [4], is already discussed in detail in [11] and will not be repeated here.

D. Lorentz invariant field equations with 4-vectors \( E^a \) and \( B^a \)

In a completely similar way as in section II.D. we perform here an observer independent decomposition of \( F^{ab} \) into 4-vectors of the electric \( E^a \) and magnetic \( B^a \) fields that are defined without reference frames, i.e., they are independent of the chosen reference frame and of the chosen system of coordinates in it. (This decomposition and many results quoted here are already presented and discussed in [11] and also in [18].) Formally all results here can be obtained from the equations given in sections IV.A. and IV.B. replacing in them the quantities from the rest frame of 'fiducial' observers, i.e., the \( \gamma_0 \) - frame, \( t^a \), \( E^a_f \) and \( B^a_f \), by the quantities defined without reference frames, \( v^a \), \( E^a \) and \( B^a \), respectively. Thus instead of (31) we have a Lorentz invariant decomposition

\[
F^{ab} = \delta^{ab} \epsilon^{cde} v^c d^e + \varepsilon^{abcd} v^c B^d, \\
E^a = F^{ab} v_b, \quad B^a = (1/2c)\varepsilon^{abcd} v_b F_{cd}.
\]

(62)

Inserting (32) into (33) we find the Lorentz invariant field equations with 4-vectors \( E^a \) and \( B^a \), or better to say the field equations (with \( E^a \) and \( B^a \)) that are defined without reference frames

\[
(-g)^{-1/2} \partial_a [(-g)^{1/2} \left( \delta^{ab} \epsilon^{cde} v^c d^e + \varepsilon^{abcd} v^c B^d \right)] = j^b / \varepsilon_0 c, \\
\varepsilon^{abcd} \partial_b [(E^c v_d - E_d v_c) + \varepsilon_{cdef} \nu^f B^l] = 0.
\]

(63)

where a, b, ..., f are all the abstract indices. When writing (63) as coordinate-based geometric equations in the \( \{\gamma_{\mu}\} \) basis they become

\[
\partial_\alpha (\delta^{a\beta}_{\mu} v^\mu v^\nu + \varepsilon^{a\beta\mu\nu} v^\mu B^\nu) \gamma_\beta = (j^b / \varepsilon_0) \gamma_0, \\
\partial_\alpha (\delta^{a\beta}_{\mu} \nu^\mu + \varepsilon^{a\beta\mu\nu} \nu^\mu E_\nu) \gamma_\beta = 0.
\]

(64)

(The equations from (64) correspond to (58) but with the above mentioned replacements.) It is clear from their form that the equations (64) are invariant upon the LT. The usual ME (54) are simply obtained from (64) specifying that \( v^a / c = t^a \), i.e., choosing the rest frame of 'fiducial' observers, the \( \gamma_0 \) - frame. In a relatively moving frame \( S' \) all quantities in (64) will be replaced with the primed quantities, but due to their invariance upon the LT the equations with primed quantities are exactly equal to the corresponding equations in \( S \) (given by (64)). Setting that \( v'^a \) in the transformed (54) is the LT of the components \( t^a \), i.e., \( v'^a = (\gamma c, \beta \gamma c, 0, 0) \), one easily finds the Lorentz transformed ME (57) with (58). Thus both the ME (51) and their LT (57) with (58) are obtained in a simple manner from (61).

V. DISCUSSION AND SHORT COMPARISON WITH EXPERIMENTS
The results obtained in this paper reveal that the usual formulation of the relativistic electrodynamics which uses the ST of the electric and magnetic fields and of the ME cannot be in agreement with experiments that test SR, i.e., in which the observers from two frames of reference compare their measurements of the same physical quantity. The careful analysis of the traditional experiments that test SR and their modern versions is reported in [12] and it undoubtedly shows that the usual formulation of SR is only in an "apparent" agreement with experiments. All usual explanations invoke the Lorentz contraction, the dilatation of time and/or the ST of the 3D E and B. However, as shown in [11] and [12] (see also [20]), the Lorentz contraction (the dilatation of time) refer to the comparison of two spatial (temporal) distances in two inertial frames of reference, which means that they have nothing in common with the LT; the LT cannot connect spatial (temporal) distances taken separately, see Figs. 3. and 4. in [11] for the Lorentz contraction and the dilatation of time respectively. The essential point which is illustrated by Figs. 3. and 4. is that, e.g., the Lorentz contracted length and the rest length do not refer to the same quantity in the 4D spacetime. They are different quantities in the 4D spacetime not only for different inertial frames of reference but also for different synchronizations. Only the spacetime length does have a well-defined physical sense in the 4D spacetime, see Figs. 1. and 2. in [11] for the spacetime length for a moving rod and a moving clock respectively, and also the discussion of the "Car and garage paradox" in the second paper in [20].

The ST of the 3D E and B are often derived, e.g., in the well-known textbooks on electrodynamics [26], assuming the existence of the Lorentz contraction of a moving charged system. This again shows in another way that the ST are not relativistically correct transformations. The accepted existence of the Lorentz contracted length of a moving object (in 1D case, $L' = L/\gamma$) leads many authors, e.g., [26] and [27], to the conclusion that the charge density of a moving system of charges $\lambda'$ is well-defined quantity in the 4D spacetime and consequently that it can be compared with the corresponding charge density of the same system of charges when it is at rest ($\lambda$). $\lambda' = \gamma \lambda$. Moreover the macroscopic electric charge is usually defined both in the classical (e.g., [3], [27]) and quantum field theories (e.g., [28]), by the integral of the charge density over the hypersurface $t = \text{const.}$, $Q = \int_{t = \text{const.}} \rho d^3x$ (in the quantum field theories $\rho = j^0/c$ is the charge density operator). Jackson [3], for example, explicitly argues, when discussing the invariance of electric charge that, [3] p.549, "the charge in a small volume element $d^3x$ is $\rho d^3x$. Since this is an experimental invariant, it is true that $\rho' d^3x' = \rho d^3x$." Thus the Lorentz contraction is always assumed in such conventional definition. The electric charge is an experimental invariant, but it is not correctly defined by the conventional definition. It is correctly defined as a manifestly invariant quantity (a Lorentz scalar): the total electric charge $Q$ in a three-dimensional hypersurface $H$ with two-dimensional boundary $\delta H$ is defined by the tensor equation $Q_{3H} = \left(1/c\right) \int_H j^0 t_w dH$, where $t_w$ is the unit normal to $H$. The charge-current density 4-vector $j^a$ as a coordinate-free quantity is a well-defined 4D quantity ($j^a = j^0 e_a$) and not the charge density itself. All this is discussed in much more detail in [10] and in the second paper in [20], see also the references therein.

An important result was obtained in [20] (the second paper) using the invariant definition of charge, particularly the fact that the charge density is well-defined quantity in the 4D spacetime only in the rest frame of charges. The mentioned result is that there is a second-order electric field ($\sim \nu^2/c^2$, $\nu$ is the drift speed of the conduction charges) not only outside a moving loop with steady current, as usually obtained (e.g., [27]), but also outside the same stationary loop. Of course both results refer to superconducting loops. Namely outside a normal conductor with steady current there is always a zero-order electric field (independent of $\nu$) together with usually considered magnetic field. The results from [5,6] and from this paper confirm in another way the mentioned results for the loop with steady current, since the electric field as 4D quantity always transforms by the LT again to the electric field. This means that if there is an electric field outside a moving loop with steady current than it must exist for the same but stationary loop. Such electric field is an experimentally verifiable result and has to be carefully examined. The already performed experiments [29] cannot, contrary to their claims, measure such external electric fields (in fact, quadrupole's electric moment), but they can measure only the potentials from monopoles. The reason is that they used probes directly connected with superconducting wires. The experiments in [30] are better suited for measurements of such external electric fields from steady currents but they dealt with normal conductors and not with superconductors. The authors of [30] forgot that always there is an external electric field for normal current-carrying conductors. Thus their experiment actually has nothing to do with the test of breakdown of local Lorentz invariance. However the same type of the experiment as in [30], but with the superconducting coil, could probably detect the external second-order electric fields. All this will be discussed in more detail elsewhere.

Let us now briefly discuss, as an example, the Faraday disk, using both the conventional formulation of electrodynamics with the 3D E and B and their ST and the formulation with geometric 4D quantities, the invariant relativistic electrodynamics (here we shall deal only with the tensor formalism since it is better known). A conducting disk is turning about thin axle passing through the center at a right angle to the disk and parallel to a uniform magnetic field B. The circuit is made by connecting one end of the resistor to the axle (the spatial point A) and the other end to a sliding contact touching the external circumference (the spatial point C). The disk of radius R is rotating with angular velocity $\omega$. (For the description and the picture of the Faraday disk see, e.g., [27] Chap. 18 or the recent paper [31].) Let us determine the electromotive force (emf) in two inertial frames of reference, the laboratory frame S
in which the disk is rotating and the frame \( S' \) instantaneously co-moving with a point on the external circumstance (say \( C \), taken at some moment \( t \)). The \( x' \)-axis is along the velocity \( \mathbf{V} \) of the point \( C \) at \( t \) and it is parallel to the \( x \) axis. Actually all axes in \( S' \) are parallel to the corresponding axes in \( S \). The \( y' \)-axis is along the radius, i.e., along the segment \( AC \). First we calculate the emf using the standard formulation. In the \( S \) frame

\[
emf = \oint (\mathbf{F}_L/q) \cdot d\mathbf{l} = \int_{AC} (F_{L,y}/q) dy = \omega R^2 B/2,
\]

where \( \mathbf{F}_L \) is the Lorentz force \( \mathbf{F}_L = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} \). \( \mathbf{E} = 0 \) in \( S \), \( \mathbf{B} \) is along the \( +z \) axis, \( q\mathbf{u} \times \mathbf{B} \) is the magnetic part of the Lorentz force seen by the charges co-moving with the disk along the segment \( AC \). Actually all axes in \( S' \) is taken at the same moment \( t \). In the \( S' \) frame the standard treatments suppose that the Lorentz force becomes \( \mathbf{F}'_L = q\mathbf{E}' + q\mathbf{u}' \times \mathbf{B}' \), where the components of the 3D \( \mathbf{E}' \) and \( \mathbf{B}' \) are determined by the ST (60). Thus it is argued in the standard formulation that in \( S' \) the charges experiences the fields \( \mathbf{E}' = \gamma_V \beta_V \times e\mathbf{B} \) and \( \mathbf{B}' = \gamma_V \mathbf{B} \), where \( \beta_V = (V/c)\mathbf{i} \) and \( \gamma_V = (1 - \beta_V^2)^{-1/2} \). Then only the \( y' \) component of the force \( \mathbf{F}'_L \) remains and it is

\[
F'_{L,y} = -qcB\beta_u/\gamma_V(1 - \beta_u\beta_V).
\]

This result explicitly shows that the standard formulation is not relativistically correct formulation.

Let us now consider the same example in the invariant relativistic electrodynamics. In the tensor formalism the invariant Lorentz force \( K^a \) is investigated in [11] Sec. 6.1. In terms of \( F^{ab} \) it is \( K^a = (q/c)F^{ab}u_b \), where \( u^a \) is the 4-velocity of a charge \( q \). In the general case of an arbitrary spacetime and when \( u^a \) is different from \( v^a \) (the 4-velocity of an observer who measures \( E^a \) and \( B^a \)), i.e. when the charge and the observer have distinct world lines, \( K^a \) can be written in terms of \( E^a \) and \( B^a \) as a sum of the \( v^a \) - orthogonal component, \( K^a_{\perp} \), and \( v^a \) - parallel component, \( K^a_{\parallel} \), \( K^a = K^a_{\perp} + K^a_{\parallel} \). \( K^a_{\perp} \) is

\[
K^a_{\perp} = (q/c^2) \left[ (v^b u_b) E^a + c\varepsilon_{abc} v^b B^c \right]
\]

and \( \varepsilon_{abc} = \varepsilon_{dabc} v^d \) is the totally skew-symmetric Levi-Civita pseudotensor induced on the hypersurface orthogonal to \( v^a \), while

\[
K^a_{\parallel} = (q/c^2) \left[ (E^b u_b) v^a \right].
\]

Speaking in terms of the prerelativistic notions one can say that \( K^a_{\perp} \) plays the role of the usual Lorentz force lying on the 3D hypersurface orthogonal to \( v^a \), while \( K^a_{\parallel} \) is related to the work done by the field on the charge. However in the invariant SR only both components together, that is, \( K^a \), does have definite physical meaning and \( K^a \) defines the Lorentz force both in the theory and in experiments. Of course \( K^a \), \( K^a_{\perp} \) and \( K^a_{\parallel} \) are all 4D quantities defined without reference frames and the decomposition of \( K^a \) is an observer independent decomposition. Then we define the emf also as an invariant 4D quantity

\[
emf = \int_{\Gamma} (K^a/q) dl_a,
\]

where \( dl_a \) is the infinitesimal spacetime length and \( \Gamma \) is the spacetime curve. Let the observers are at rest in the \( S \) frame, \( u^a = (c,0,0,0) \) whence \( E^0 = B^0 = 0 \); the \( S \) frame is the rest frame of 'fiducial' observers, the \( \gamma_0 \) - frame with the \( \{\gamma_\mu\} \) basis. Thus the components of the 4-vectors (in the Einstein system of coordinates, i.e., in the \( \{\gamma_\mu\} \) basis) are \( E^\mu = (0,0,0,0), \ B^\mu = (0,0,0,B), \ u^\mu = (c,u = \omega \rho,0,0), \ dl^\mu = (0,0,dl^2 = dy,0) \). Hence \( K^a_{\parallel} = 0 \).
\[ K^0 = K^1 = K^3 = 0, K^2 = quB. \] When all quantities in (20) are written as CBGQs in the frame with the \( \{ \gamma_\mu \} \) basis we find \( emf = \omega R^2 B / 2 \). Since the expression (70) is independent of the chosen reference frame and of the chosen system of coordinates in it we shall get the same result in the relatively moving frame as well;

\[ emf = \int_{\Gamma (in \ S)} (K^\mu / q) dl_\mu = \int_{\Gamma (in \ S')} (K'^\mu / q) dl'_\mu = \omega R^2 B / 2. \] (71)

This can be checked directly performing the LT of all 4-vectors as CBGQs from \( S \) to \( S' \) including the transformation of \( v^\mu \gamma_\mu \). Obviously the approach with Lorentz invariant 4D quantities gives the relativistically correct answer in an enough simple and transparent way.

In a like manner we could come to the same conclusion for all experiments particularly to those that test SR. For example for the Trronton-Noble experiment [32] (see also [33]). In the experiment they looked for the turning motion of a charged parallel plate capacitor suspended at rest in the frame of the earth in order to measure the earth’s motion through the ether. All explanations, which are given until now (see, e.g., [34]), for the null result of the experiments [32] ([33]) are not relativistically correct, since they use ill-defined quantities in the 4D spacetime; e.g., the Lorentz contraction, the transformation equations for the usual 3D vectors \( E \) and \( B \) and for the torque as the 3D vector, the nonelectromagnetic forces of undefined nature, etc.. Thus, for instance, in the first paper in [34] it is claimed: "In particular it was seen that the potential energy of a charge distribution changes, due to Lorentz contraction of the system, when it is set in motion." Similarly in [34] two types of the "explanations" of the Trouton-Noble experiment are offered; one of them is with nonelectromagnetic forces of undefined nature, as in [34]. In both types of the "explanations" the Lorentz contraction is used \( (d^3\tau = \gamma d^3x) \) and, of course, the standard transformations of the 3D \( E \) and \( B \). Here, it has to be noted that often, both in the classical (e.g., [27], [34]) and quantum field theories (e.g., [28]), the electromagnetic energy and momentum are also defined, as in the standard definition of charge, by the integrals of the energy and momentum densities over the hypersurface \( t = const. \). It is then supposed that such hypersurface transforms by the LT to the hypersurface \( t' = const. \) in a relatively moving reference frame \( S' \), and consequently the Lorentz contraction is assumed, \( d^3x' = \gamma d^3x \). This is relativistically incorrect since the LT cannot transform the hypersurface \( t = const. \) in \( S \) to the hypersurface \( t' = const. \) in a relatively moving \( S' \). This is already examined for the classical electrodynamics (the covariant formulation in the Einstein system of coordinates) by Rohrlich [35] and using the component form of the electric and magnetic 4-vectors \( E^\alpha \) and \( B^\alpha \) (the tensor formalism) in the first paper in [20]. Recently [10] I have presented a Lorentz invariant formulation of the relativistic electrodynamics in the geometric algebra formalism. That formulation is exposed exclusively in terms of the bivector field \( F \), thus without using either the electric and magnetic fields or the electromagnetic potential. There [10] the most general, observer independent, expressions for the stress-energy vector \( T(n) \) (1-vector), the energy density \( U \) (scalar), the Poynting vector \( S \) and the momentum density \( g \) (1-vectors), the angular momentum density \( M \) (bivector) and the Lorentz force \( K \) (1-vector) are presented and directly derived from the field equations with \( F \). Thus, e.g., the stress-energy vector \( T(n) \) (which describes the flow of energy-momentum through a hypersurface with unit normal \( n = n(x) \)) is \( T(n) = Un + (1/c)S \), where the energy density \( U \) is \( U = -(\varepsilon_0 / 2) [F \cdot F] + 2(F \cdot n)^2n \) and the Poynting vector \( S \) is \( S = -\varepsilon_0 [F \cdot n] \cdot F - (F \cdot n)^2n \). When such invariant 4D quantities, i.e., the quantities defined without reference frames, or the CBGQs, are used in the comparison with experiments then, e.g., the explanation of the Trouton-Noble experiment is very simple and natural. The values of such quantities are the same in the rest frame of the capacitor and in the moving frame. Thus if there is no torque (but now as a geometric, invariant, 4D quantity) in the rest frame then the capacitor cannot appear to be rotating in a uniformly moving frame. However we will not discuss this problem in more detail here. It will be reported elsewhere.

We see that the general procedure in the invariant SR is the following. All considered quantities have to be written as geometric 4D quantities, e.g., as abstract 4D tensors, or as the Clifford multivectors, thus \textit{as quantities which are defined without reference frames}, like in (20), (60), or (62). The physical laws expressed in terms of such quantities automatically include the principle of relativity and there is no need to postulate it outside the mathematical formulation of the theory. This is a fundamental difference relative to the standard formulation [2] of the theory of relativity. Then an appropriate reference frame and a system of coordinates in it are chosen (in which the calculation is the simplest one) and the quantities are written as CBGQs in that chosen system of coordinates. The same result can be obtained in any other relatively moving inertial frame of reference and with any permissible system of coordinates in it (including different synchronizations) by performing the LT of all quantities (the form of the LT that is independent of the chosen system of coordinates is given in [11] in the tensor formalism and in [15] in the geometric algebra formalism). It is essential for this Lorentz invariant approach that \textit{all observers are looking at the same 4D physical quantity}. This is not the case for the traditional approaches which caused many misconceptions and misunderstandings of the SR.

VI. SUMMARY AND CONCLUSIONS
The covariance of the ME is considered to be one of the cornerstone of the modern relativistic field theories, both classical and quantum. Einstein [2] derived the ST of the 3D \( E \) and \( B \) assuming that the ME with \( E \) and \( B \) must have the same form in all relatively moving inertial frames of reference. In Einstein’s formulation of SR [2] the principle of relativity is a fundamental postulate that is supposed to hold for all physical laws including those expressed by 3D quantities, e.g., the ME with the 3D \( E \) and \( B \). The results presented in this paper substantially change generally accepted opinion about the covariance of the ME exactly proving in geometric algebra and tensor formalisms that the usual ME (\[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\]) change their form upon the LT (see \[\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\], or \[\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\], or \[\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\]). It is also proved that the ST of the ME (see \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], or \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], or \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\]), which leave unchanged the form of the ME, actually have nothing in common with the LT of the usual ME. The difference between the LT of the ME, e.g., \[\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\], and their ST, e.g., \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], is essentially the same as it is the difference between the LT of the electric and magnetic fields (see \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], or \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], or \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\]) and their ST (see \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], or \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\], or \[\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\`). This last difference is proved in detail in [5] and [6] and that proof is only briefly repeated in this paper. All this together reveals that, contrary to the generally accepted opinion, the principle of relativity does not hold for physical laws expressed by 3D quantities (a fundamental achievement). Any 3D quantity does not correctly transform upon the LT and thus it does not have an independent physical reality in the 4D spacetime; it is not the same quantity for relatively moving observers in the 4D spacetime (see also, e.g., Figs. 3. and 4. in [11], and [12]). Since the usual ME change their form upon the LT they cannot describe in a relativistically correct manner the experiments that test SR, i.e., the experiments in which relatively moving observers measure the same 4D physical quantity. Therefore the new field equations with geometric 4D quantities are constructed in geometric algebra formalism with 1-vectors \( E \) and \( B \) (\[\gamma_{\mu}\]), and with bivectors \( E_{HL} \) and \( B_{HL} \) (\[\gamma_{\mu}\]), and also in the tensor formalism with 4-vectors \( E^a \) and \( B^a \) (\[\gamma_{\mu}\]); the Lorentz invariant field equations in the tensor formalism are already presented in [11]. All quantities in these geometric equations are independent of the chosen reference frame and of the chosen coordinate system in it. When the \(\gamma_0\) - frame with the \(\{\gamma_\mu\}\) basis is chosen, in which the observers who measure the electric and magnetic fields are at rest, then all mentioned geometric equations become the usual ME. This result explicitly shows that the correspondence principle is naturally satisfied in the invariant SR. However, as seen here, the description with 4D geometric quantities is correct not only in the \(\gamma_0\) - frame with the \(\{\gamma_\mu\}\) basis but in all other relatively moving frames and it holds for any permissible choice of coordinates. We conclude from the results of this paper that geometric 4D quantities, defined without reference frames or as CBQGs, do have an independent physical reality and the relativistically correct physical laws must be expressed in terms of such quantities. The principle of relativity is automatically satisfied with such quantities while in the standard formulation of SR it is postulated outside the mathematical formulation of the theory. We see that the role of the principle of relativity is substantially different in the Einstein formulation of SR and in the invariant SR. The results of this paper clearly support the latter one. Furthermore we note that all observer independent quantities introduced here and the field equations written in terms of them hold in the same form both in the flat and curved spacetimes. The results obtained in this paper will have important and numerous consequences in all relativistic field theories, classical and quantum. Some of them will be soon examined.

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