Research Article

Existence Solutions for a Class of Schrödinger-Maxwell Systems with Steep Well Potential

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In this paper, we are concerned with the system of the Schrödinger-Maxwell equations

\[
\begin{aligned}
\Delta u + \lambda V(x) u + b K(x) \phi u &= |u|^{p-2} u, \text{ in } \mathbb{R}^3, \\
\Delta \phi &= K(x) u^2, \text{ in } \mathbb{R}^3,
\end{aligned}
\]

where \(\lambda, b > 0\) are constants, and \(3 < p < 6\). Under appropriate assumptions on \(V\) and \(K\), we prove the existence of positive solutions in the case \(3 < p < 4\) via the truncation technique. Moreover, suppose that \(V\) may change sign, we also obtain the multiplicity of solutions for the case \(4 < p < 6\).

1. Introduction and Main Results

Consider the following system of Schrödinger-Maxwell equations:

\[
\begin{aligned}
\Delta u + \lambda V(x) u + b K(x) \phi u &= |u|^{p-2} u, \text{ in } \mathbb{R}^3, \\
\Delta \phi &= K(x) u^2, \text{ in } \mathbb{R}^3,
\end{aligned}
\]

(1)

where \(\lambda, b > 0\) are constants, and \(3 < p < 6\). Such a system is also called the Schrödinger-Poisson equation which is obtained while looking for the existence of standing waves for nonlinear Schrödinger equations interacting with an unknown electrostatic field. For more details on the physical aspects, we refer the reader to [1] and the references therein.

On the potential \(V\), we make the following assumptions:

(V1) \(V \in C(\mathbb{R}^3, \mathbb{R})\) and \(V\) is bounded below.

(V2) \(V\) there exists a constant \(c > 0\) such that the set \(\{x \in \mathbb{R}^3 : V(x) \leq c\}\) is nonempty and meas\(\{x \in \mathbb{R}^3 : V(x) \leq c\} < +\infty\), where meas denote the Lebesgue measure in \(\mathbb{R}^3\).

(V3) \(\Omega = int V^{-1}(0)\) is nonempty and has smooth boundary and \(\Omega = V^{-1}(0)\).

In recent years, system (1) has been widely studied under various conditions on \(V\) and \(K\). The greatest part of the literature focuses on the study of the system for \(V\) and \(K\) being constants or radially symmetric functions. We refer the reader to [2–9].

When the potential \(V(x)\) is neither a constant nor radially symmetric, in [10, 11], the existence of ground state solutions is proved for \(3 < p < 6\). In [12–14], the existence of nontrivial solution is obtained via the variational techniques in a standard way under the following condition:

\[
(V_{\lambda}) \quad V \in C(\mathbb{R}^3, \mathbb{R}), \quad \inf_{x \in \mathbb{R}^3} V(x) \geq a_0 > 0, \text{ where } a_0 > 0 \text{ is a constant. Moreover, for any } M > 0, \text{ meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < +\infty, \text{ where } \text{meas} \text{ denote the Lebesgue measure in } \mathbb{R}^3.
\]

It is worth mentioning that conditions \((V_{\lambda})\) were first introduced by Bartsch and Wang [15] to guarantee the compact embedding of the functional space. If replacing \((V_{\lambda})\) by more general assumptions \((V_1)\) and \((V_2)\), the compactness of the embedding fails and this situation becomes more complicated. Recently, [16, 17] considered this case. The authors studied the following problem

\[
\begin{aligned}
\Delta u + \lambda V(x) u + K(x) \phi u &= f(x, u), \text{ in } \mathbb{R}^3, \\
\Delta \phi &= K(x) u^2, \text{ in } \mathbb{R}^3,
\end{aligned}
\]

(2)

where \(\lambda > 0\) is a parameter, the potential \(V\) may change sign and \(f\) is either the superlinear or sublinear in \(u\) as \(|u| \rightarrow \infty\).
Very recently, Liu and Mosconi [18] considered the following system with a coercive sign-changing potential and a 3-sublinear nonlinearity:

\[
\begin{cases}
-\Delta u + V(x)u + \lambda \phi u = f(u), \text{ in } \mathbb{R}^3, \\
-\Delta \phi = u^2, \text{ in } \mathbb{R}^3.
\end{cases}
\]

By using a linking theorem, the authors obtained the existence of nontrivial solutions. Nextly, Gu, Jin, and Zhang [19] investigated the existence of sign-changing solutions for system (3). By using the method of invariant sets of descending flow, the multiple radial sign-changing solutions are obtained in the subquadratic case as \(\lambda\) small. For more results about the Schrödinger-Poisson systems, we refer the reader to [20–23] and the reference therein.

Here, we should point out that for the power-type nonlinearity \(f(u) = |u|^{p-2}u\), in order to get the boundedness of a (PS) sequence, the methods heavily rely on the restriction \(p \in (4, 6)\). Meanwhile, the condition \((V_1) - (V_3)\) cannot guarantee the compactness of the embedding of \(H^1(\mathbb{R}^3)\) into the Lebesgue spaces \(L^s(\mathbb{R}^3)\), \(s \in [2, 6)\). This prevents from using the variational techniques in a standard way. Motivated by the works mentioned above and [24–28], in the present paper, we are mostly interested in sign-changing potentials and consider system (1) with more general potential \(V\), \(K\), and the range of \(p\). Our main results are as follows:

**Theorem 1.** Suppose that \(V \geq 0\), \((V_1) - (V_3)\), \(K \in (L^1(\mathbb{R}^3) \cap L^{10}(\mathbb{R}^3)) \cup L^\infty(\mathbb{R}^3)\), \(K(x) \geq 0\), and \(3 < p < 4\) hold. Then, system (1) possesses at least one nontrivial solution for \(b\) small and \(\lambda\) large.

**Remark 2.** It is known that it is difficult to get the boundedness of a (PS) sequence when dealing with the case \(p \in (3, 4)\). To overcome the difficulty, motivated by [24, 25], we use the truncation technique to obtain a bounded Cerami sequence for \(b\) small. In this case, the conditions \((V_4)\) and \((K_1)\) of Theorem 1.3 in [23] cannot be used. Moreover, in the process of proving the convergence of a bounded Cerami sequence, we use the observation that the condition \(K \in L^3(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)\) makes the less strong influence of the nonlocal term \(K(x) \phi u\) (The conclusions remain valid if \(K \in L^2(\mathbb{R}^3)\)). In this sense, Theorem 1 can be viewed as an improvement of Theorem 1.3 in Zhao et al. [23].

**Theorem 3.** Suppose that \((V_1), (V_2), K \in L^3(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)\), \(K(x) \geq 0\), and \(4 < p < 6\) hold. Then, system (1) possesses infinitely many distinct pairs of nontrivial solutions whenever \(\lambda > 0\) is sufficiently large.

**Remark 4.** In Theorem 3, \(V\) is allowed to be sign-changing for \(p \in (4, 6)\). We obtain the multiplicity of solutions for (1).

### 2. Preliminaries

Let

\[
H^1(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \}.
\]

is the usual Sobolev space with the standard inner product and norm

\[
(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx, ||u|| = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{1/2}.
\]

In our problem, we work in the space defined by

\[
E_\lambda = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V^+(x) u^2) dx < +\infty \right\}
\]

with the inner product and the norm

\[
\langle u, v \rangle_{E_\lambda} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \lambda V^+(x) uv) dx, ||u||_{E_\lambda} = \langle u, u \rangle_{E_\lambda}^{1/2},
\]

where \(V^+(x) = \max \{ \pm V(x), 0 \}\), \(V(x) = V^+(x) - V^-(x)\). Obviously, it follows from \((V_1), (V_2)\) that the embedding \(E_\lambda \hookrightarrow H^1(\mathbb{R}^3)\) is continuous.

As in [26], let

\[
F_\lambda = \{ u \in E_\lambda : \text{supp} u \subset V^{-1}(\{0, \infty\}) \}
\]

and denote the orthogonal complement of \(F_\lambda\) in \(E_\lambda\) by \(F_\lambda^\perp\). Consider the eigenvalue problem

\[
-\Delta u + \lambda V^+(x) u = \mu V^-(x) u, u \in F_\lambda^\perp.
\]

In view of \((V_1), (V_2)\), the quadratic form \(u \mapsto \int_{\mathbb{R}^3} V^-(x) u^2 dx\) is weakly continuous. We have the following proposition.

**Proposition 5** ([26], Lemma 8). Suppose \((V_1), (V_2), \) and \(V \neq 0\). Then, for each fixed \(j\),

(i) \(\mu_j(\lambda) \to 0\) as \(\lambda \to +\infty\)

(ii) \(\mu_j(\lambda)\) is a nonincreasing continuous function of \(\lambda\), where \(\mu_j(\lambda) = \inf_{\dim M_\lambda = j, M_\lambda \subset F_\lambda^\perp} \sup\{ \| u \|_{E_\lambda}^2 : u \in M, \int_{\mathbb{R}^3} V^-(x) u^2 dx = 1 \} \) is sequence of positive eigenvalues of problem (P) satisfying \(\mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots \mu_j(\lambda) \to \infty\) as \(j \to \infty\) and the corresponding eigenfunctions \(\{ e_j(\lambda) \}_{j=1}^{\infty}\).

Let

\[
E_\lambda = \text{span} \{ e_j(\lambda) : \mu_j(\lambda) \leq 1 \}, E_\lambda^\perp = \text{span} \{ e_j(\lambda) : \mu_j(\lambda) > 1 \}.
\]


Then,
\[ E_\lambda = E_{\lambda}^g \oplus F_{\lambda} \oplus E_{\lambda}^g. \]  

Moreover \( \dim E_{\lambda} < +\infty \) for every fixed \( \lambda > 0 \).

In the sequel, we denote the usual \( L^p \)-norm by \( \| \cdot \|_p \), and \( C \) stands for the continuity of the following embedding
\[ H^s(R^3) \hookrightarrow L^p(R^3), \]
there are constants \( a_i > 0 \) and \( a > 0 \) such that
\[ \| u \|_s \leq a_1 \| u \|_{E_{\lambda}^g}, \| \phi \|_6 \leq a \| \phi \|_{D^{1,2}(R^3)}, \forall u \in E_{\lambda}, \forall \phi \in D^{1,2}(R^3). \]

It is well known that system (1) is the Euler-Lagrange equation of the functional \( J : E_{\lambda} \times D^{1,2}(R^3) \rightarrow R \) defined by
\[ J(u, \phi) = \frac{1}{2} \left( \int_{R^3} \lambda V(x) \phi^2 dx - \frac{1}{4} \int_{R^3} \left| \nabla \phi \right|^2 dx \right) + \frac{b}{2} \int_{R^3} K(x) \phi u^2 dx - \frac{1}{p} \int_{R^3} |u|^p dx. \]

Evidently, the action functional \( J \) belongs to \( C^1(E_{\lambda} \times D^{1,2}(R^3), R) \) and its critical points are the solutions of system (1). It is easy to know that \( J \) exhibits a strong indefiniteness, namely, it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [29], by which we are led to study a one variable functional that does not present such a strongly indefiniteness.

Actually, considering for all \( u \in E_{\lambda} \), the linear functional \( L_u \) defined in \( D^{1,2}(R^3) \) by
\[ L_u(v) = \int_{R^3} K(x) u^2 v dx. \]

If \( K \in L^\infty(R^3) \), the H"older inequality and the Sobolev inequality imply
\[ |L_u(v)| \leq \int_{R^3} \left| K(x) u^2 v \right| dx \leq \| K \|_{L^\infty} \| u \|^2_{12/5} \| v \|_6 \leq C \| K \|_{L^\infty} \| u \|^2_{12/5} \| v \|_{D^{1,2}}, \]
while for \( K \in L^1(R^3) \), we have
\[ |L_u(v)| \leq \int_{R^3} \left| K(x) u^2 v \right| dx \leq \| K \|_{L^1} \| u \|^2_{12/5} \| v \|_6 \leq C \| K \|_{L^1} \| u \|^2_{12/5} \| v \|_{D^{1,2}}. \]

Hence, by the Lax-Milgram theorem, there exists a unique \( \phi_u \in D^{1,2}(R^3) \) such that
\[ \int_{R^3} \nabla \phi_u \nabla v dx = \int_{R^3} K(x) u^2 v dx, \forall v \in D^{1,2}(R^3). \]

Moreover, we can write an integral expression for \( \phi_u \) in the form:
\[ \phi_u(x) = \frac{1}{4\pi} \int_{R^3} \frac{K(y)}{|x-y|} u^2(y) dy, \forall u \in E_{\lambda}. \]

So, we can consider the functional \( I_\lambda : E_{\lambda} \rightarrow R \) defined by \( I_\lambda(u) = J(u, \phi_u) \). Then,
\[ I_\lambda(u) = \frac{1}{2} \| u \|^2_{E_{\lambda}} - \frac{1}{2} \int_{R^3} \lambda V(x) u^2 dx + \frac{b}{4} \int_{R^3} K(x) \phi_u u^2 dx - \frac{1}{p} \int_{R^3} |u|^p dx. \]

It follows from (16), (17), (18), and the Sobolev inequality that
\[ \| \phi_u \|_{D^{1,2}} = \| L_u \| \leq C \| K \|_{L^\infty} \| u \|_{12/5} \| u \|^2_{12/5} \int_{R^3} K(x) u^2 \phi_u(x) dx \leq C \| K \|_{L^\infty} \| u \|, \]
\[ \| \phi_u \|_{D^{1,2}} = \| L_u \| \leq C \| K \|_{L^1} \| u \|^2_{12/5} \int_{R^3} K(x) u^2 \phi_u(x) dx \leq C \| K \|_{L^1} \| u \|_{12/5}. \]

Thus, \( I \) is a well-defined \( C^1 \) functional with derivative given by
\[ \langle I'(u), v \rangle = \int_{R^3} \nabla u \nabla v dx + \int_{R^3} \lambda V(x) u v dx \]
\[ + b \int_{R^3} K(x) \phi_u u v dx - \int_{R^3} |u|^{p-2} u v dx. \]

By the proposition 2.3 in [12], we know that \( u, \phi \in E_{\lambda} \times D^{1,2}(R^3) \) is a critical point of \( J \) if and only if \( u \) is a critical point of \( I \) and \( \phi = \phi_u \).

To complete the proof of our theorem, we need the following results.

**Theorem 6** (see [30]). Let \( X \) be a real Banach space with its dual space \( X^* \), and suppose that \( J \in C^1(X, R) \) satisfies
\[ \max \{ J(0), J(e) \} \leq \mu < \eta \leq \inf_{\| u \|=\rho} J(u), \]
for some \( \mu < \eta, \rho > 0 \) and \( e \in X \) with \( \| e \| > \rho \). Let \( c \geq \eta \) be characterized by
\[ c = \inf_{y \in \Gamma} \max_{y(t) \in [0, 1]} J(y(t)), \]
where \( \Gamma = \{ y \in C([0, 1], X) : y(0) = 0, y(1) = e \} \) is the set of continuous paths joining 0 and e. Then, there exists a
sequence \( \{u_n\} \subset X \) such that
\[
J(u_n) \longrightarrow c \geq \eta, (1 + \|u_n\|)\|f'(u_n)\|_X \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\]
(26)

**Theorem 7** see ([31], Theorem 9.12). Let \( E \) be an infinite dimensional Banach space and \( \Gamma \subset C^1(E, R) \) be even, satisfy (PS), and \( I(0) = 0 \). If \( E = V \oplus X \), where \( V \) is finite dimensional and \( \Gamma \) satisfies

(1) there are constants \( \rho, \alpha > 0 \) such that \( |\Gamma|_{\emptyset, \Omega} \geq \alpha \), and

(2) for each finite dimensional subspace \( \tilde{E} \subset E \), there is an \( \tilde{R} = R(\tilde{E}) \) such that \( I \leq 0 \) on \( \tilde{E} \setminus B_{\tilde{R}(\tilde{E})} \),

then, \( I \) possesses an unbounded sequence of critical values.

**Lemma 8** ([17], Lemma 2.3). The function \( \phi_u \) possess the following properties:

(1) The mapping \( u \longrightarrow \phi_u \) maps bounded sets of \( E_\lambda \) into bounded sets of \( D^{1,2}(R^3) \);

(2) If \( u_n \rightharpoonup u \) in \( E_\lambda \), then \( \phi_{u_n} \rightharpoonup \phi_u \) in \( D^{1,2}(R^3) \);

(3) The mapping \( u \longrightarrow \phi_u : E_\lambda \longrightarrow D^{1,2}(R^3) \) is continuous.

### 3. Proof Of Theorem 1

In this section, we give the proof of Theorem 2 by using Theorem 6. Let \( \eta \in C^\infty([0, \infty), R) \) be a cut-off function satisfying \( 0 \leq \eta \leq 1, \eta(t) = 1 \) if \( 0 \leq t \leq 1, \eta(t) = 0 \) if \( t \geq 2, \max_{t \geq 0} |\eta'(t)| \leq 2 \) and \( \eta'(t) \leq 0 \) for each \( t > 0 \). For every \( T > 0 \) we consider the truncated functional \( J : E_\lambda \longrightarrow R \) defined by

\[
J(u) = \frac{1}{2} \|u\|_{E_\lambda}^2 + \frac{b}{4} \left( 1 + \|u\|_{E_\lambda}^2 / T^2 \right) \int_{R^3} K(x) \phi_u u^2 dx - \frac{1}{p} \int_{R^3} |u|^p dx.
\]
(27)

It is easy to see that \( J \) is of \( C^1 \). Moreover, for each \( u, v \in E_\lambda \), we have

\[
\langle f'(u), v \rangle = \langle u, v \rangle_{E_\lambda} + b \eta \left( 1 + \|u\|_{E_\lambda}^2 / T^2 \right) \int_{R^3} K(x) \phi_u(x) u v dx + \frac{b}{2 T^2} \left( 1 + \|u\|_{E_\lambda}^2 / T^2 \right) \int_{R^3} K(x) \phi_u(x) u v dx - \int_{R^3} |u|^p u v dx.
\]
(28)

**Lemma 9.** Suppose that \( 3 < p < 4 \) and \( (V_1), (V_2) \) hold. Then, there exists \( \alpha > 0 \) such that \( c_{T, \lambda} \geq \alpha > 0 \).

**Proof.** For any \( u \in E_\lambda \), we have

\[
\phi_{u}(ty) = \frac{1}{4 \pi} \int_{R^3} \frac{K(y)}{|x - y|} t^4 u^2(y) dy \leq \frac{1}{4 \pi} \|K\|_{L^\infty} \int_{R^3} \frac{1}{|x - y|} t^4 u^2(y) dy = \frac{1}{4 \pi} \|K\|_{L^\infty} \int_{R^3} \frac{1}{|x - z/t|} t^4 u^2(z) \frac{1}{t^4} dz = \frac{t^2}{4 \pi} \|K\|_{L^\infty} \int_{R^3} \frac{K(z)}{|tx - z|} u^2(z) dz = \frac{t^2}{4 \pi} \|K\|_{L^\infty}^2 \phi_t(t^2 x). \]
(29)

Let \( x_0 \in \Omega \) and \( B_{r_0}(x_0) \subset \Omega \) for some \( r_0 > 0 \). Let \( u \in C^{\infty}_0(\Omega) \) be such that \( \text{supp} u \subset B_{r_0}(0) \). Then, \( \text{supp} v \subset B_{r_0}(x_0) \) for \( t > 1 \). Hence, it is easy to see that

\[
J(v_t) = \frac{t^4}{2} \int_{B_{r_0}(0)} |\nabla u|^2 dx + \frac{t^3}{4} \|K\|_{L^\infty}^2 \int_{B_{r_0}(0)} \phi_t u^2 dx - \frac{t^{2p-3}}{p} \int_{B_{r_0}(0)} |u|^p dx.
\]
(30)

Since \( p > 3 \), we have that \( J(v_t) \longrightarrow -\infty \) as \( t \longrightarrow +\infty \). Taking \( v = v_t \) for \( t \) large enough, we have \( J(v) < 0 \).

(iii) since \( c_{T, \lambda} \geq \max_{t \geq 0} \phi_t(v) \), there exists a constant \( M > 0 \) (independent of \( T, \lambda \) and \( b \)) such that \( c_{T, \lambda} \geq M \).

**Lemma 10.** Suppose that \( 3 < p < 4 \) and \( (V_1), (V_2) \) hold. Then, there exists \( \alpha > 0 \) such that \( c_{T, \lambda} \geq \alpha > 0 \).

**Proof.** For any \( u \in E_\lambda \), we have

\[
J(u) = \frac{1}{2} \|u\|_{E_\lambda}^2 + \frac{b}{4} \eta \left( 1 + \|u\|_{E_\lambda}^2 / T^2 \right) \int_{R^3} K(x) \phi_u u^2 dx - \frac{1}{p} \int_{R^3} |u|^p dx \geq \frac{1}{2} \|u\|_{E_\lambda}^2 - \frac{1}{p} \int_{R^3} |u|^p dx \geq \frac{1}{2} \|u\|_{E_\lambda}^2 - \frac{1}{p} \|u\|_{E_\lambda}^2 \left( 1 - C \|u\|_{E_\lambda}^2 \right).
\]
(31)

Since \( p > 3 \), we conclude that there exists \( \rho > 0 \) such that \( u(\rho) \geq 0 \) for all \( u \in E_\lambda \) with \( \|u\|_{E_\lambda} = \rho \).

From Lemmas 9 and 10 and Theorem 6, we thus deduce that there exist a Cerami sequence \( \{u_n\} \subset E_\lambda \) such that

\[
J(u_n) \longrightarrow c_{T, \lambda}, \quad \left( 1 + \|u_n\|_{E_\lambda} \right) f'(u_n) \longrightarrow 0.
\]
(32)

**Lemma 11.** Let \( 3 < p < 4 \), \( V(x) \geq 0 \), \( (V_1), (V_2), (K_x) \geq 0 \), and \( K \in L^3(R^3) \cup L^\infty(R^3) \) be satisfied. Then, there exists \( \Lambda > 1 \) such that for each \( \lambda \in (\Lambda, \infty), \) if \( \{u_n\} \subset E_\lambda \) is sequence
satisfying (32), then \( \{u_n\} \) has a convergent subsequence in \( E_k \) for \( b \) small enough.

**Proof.** Let \( \{u_n\} \) be a Cerami sequence satisfying (32). Let \( T = \sqrt{2p(M+1)/p-2} \), we show that \( \|u_n\|_{E_k} \leq T \). We first prove that \( \|u_n\|_{E_k} \leq \sqrt{2T} \). Suppose by contradiction that there exist a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such that \( \|u_n\|_{E_k} > \sqrt{2T} \), we obtain

\[
c^2 \lambda_k = \lim_{n \to \infty} \left( I(u_n) - \frac{1}{p} \left( \|u_n\|_{E_k} - \frac{b}{2p} \|u_n\|_{E_k}^2 \right) \phi(u_n) \right)
= \lim_{n \to \infty} \left[ \frac{1}{p} \|u_n\|_{E_k}^2 - \frac{b}{2p} \|u_n\|_{E_k}^4 \right] + \frac{b}{2pT_p} \|u_n\|_{E_k}^4 \phi(u_n)
\geq \liminf_{n \to \infty} \left[ \frac{1}{2} \|u_n\|_{E_k}^2 \right] \geq 2(M+1),
\]

which is a contradiction by Lemma 9. Suppose that there exists no subsequence of \( \{u_n\} \) which is uniformly bounded by \( B \). Then, we deduce that \( \|u_n\|_{E_k} \leq \sqrt{2T} \). We handle the case of \( K \in L^1(R^3) \) (The conclusions remain valid if \( K \in L^{\infty}(R^3) \).) By (22), we obtain

\[
c^2 \lambda_k = \left( I(u_n) - \frac{1}{p} \left( \|u_n\|_{E_k} - \frac{b}{2p} \|u_n\|_{E_k}^2 \right) \phi(u_n) \right)
= \lim_{n \to \infty} \left[ \frac{1}{p} \|u_n\|_{E_k}^2 - \frac{b}{2p} \|u_n\|_{E_k}^4 \right] + \frac{b}{2pT_p} \|u_n\|_{E_k}^4 \phi(u_n)
\geq \liminf_{n \to \infty} \left[ \frac{1}{2} \|u_n\|_{E_k}^2 \right] \geq 2(M+1),
\]

Furthermore, for any \( \varphi \in E_k \), we have

\[
\left| \int_{R^3} K(x) \phi_{un}(x) u_n(x) dx - \int_{R^3} K(x) \phi_{un}(x) u dx \right| \to 0, \quad \text{as} \quad n \to \infty.
\]

In fact, if \( K \in L^{\infty}(R^3) \), we just need to show that

\[
\left| \int_{R^3} \phi_{un}(x) u_n(x) dx - \int_{R^3} \phi_{un}(x) u dx \right| \to 0, \quad \text{as} \quad n \to \infty.
\]

Note that

\[
\int_{R^3} \phi_{un}(x) u_n(x) dx - \int_{R^3} \phi_{un}(x) u dx = \int \left( \phi_{un} - \phi_{u} \right) u dx.
\]

The first limit on the right is 0 by the fact \( \phi_{un}(u_n - u) \to 0 \) in \( L^2(R^3) \) and so is the second limit because \( \phi_{un} - \phi_u \to 0 \) in \( L^6(R^3) \) and \( u \varphi \in L^{6/5}(R^3) \). While for \( K \in L^1(R^3) \), on the one hand, since \( Ku \in L^{6/5}(R^3) \), using (35) (c), we have

\[
\left| \int_{R^3} K(x) \phi_{un}(x) - \phi_{u}(x) dx \right| \to 0, \quad \text{as} \quad n \to \infty.
\]

as \( n \to \infty \). On the other hand, since \( K \in L^1(R^3) \), for any \( \epsilon_1 > 0 \) there exists \( \rho_1 = \rho_1(\epsilon_1) > 0 \) such that

\[
\|K\|_{L^1(R^3)} < \epsilon_1.
\]

Moreover, in view of the Sobolev embedding theorem, \( u_n \to u \) implies that

\[
u_n \to u \quad \text{in} \quad L^\infty(R^3), \quad 2 \leq \nu < 6.
\]

Hence, for large \( n \), we obtain

\[
\|u_n - u\|_{4,\nu} < \epsilon_1.
\]

Consequently,

\[
\left| \int_{R^3} K(x) \phi_{un}(x) (u_n(x) - u(x)) u(x) dx \right| \leq \|K\|_1 \|\phi_{un}\|_6 \|u\|_4 \epsilon_1
+ \epsilon_1 \|\phi_{un}\|_6 \|u\|_4 \|u_n - u\|_4 \leq C \epsilon_1.
\]

From above inequality and (39), one has

\[
\left| \int_{R^3} K(x) \phi_{un}(x) u_n(x) dx - \int_{R^3} K(x) \phi_{un}(x) u dx \right| \to 0, \quad \text{as} \quad n \to \infty.
\]
Then, $J'(u_n) \to 0$ implies that
\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} \lambda V(x) u_n^2 + b \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u_n^2 dx - \mu \int_{\mathbb{R}^3} u_n^p dx = 0.
\]  
(45)

Let $v_n = u_n - u$. It follows from $\langle V_1 \rangle$, $\langle V_2 \rangle$ that
\[
\||v_n||^2_2 = \int_{\mathbb{R}^3} v_n^2 + \int_{\mathbb{R}^3} v_n^2 dx \leq \frac{1}{\lambda} \|v_n\|^2_{E_\lambda} + o(1). 
\]  
(46)

Moreover, let $0 < \alpha < \min \{ (6 - p)/2, 1 \}, 2 < p < 6$. Then, $2 < (2 - \alpha)/2 - 6 < 0$. By the Sobolev inequalities and Hölder inequality, one has
\[
\||v_n||_p^p \leq \left( \int_{\mathbb{R}^3} \|v_n\|^2_{E_\lambda} \right)^{\frac{p}{2}} \left( \int_{\mathbb{R}^3} v_n^2 dx \right)^{\frac{p-2}{2}} \leq \|v_n\|^2_{E_\lambda} + o(1).
\]  
(47)

We know
\[
o(1) = \left( J'(u_n), u_n \right) = \|u_n\|^2_{E_\lambda} + b \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u_n^2 dx - \|u_n\|^2_p - \|u_{n/2}\|^2_p - \|u_n\|^2_p - o(1) \geq \|v_n\|^2_{E_\lambda} - \|v_n\|^2_{E_\lambda} + o(1) \geq \left[ 2(\alpha - 2) \alpha \frac{p-2}{p} \right] \|v_n\|^2_{E_\lambda} + o(1).
\]  
(48)

Letting $\Lambda > 0$ be so large that the term in the brackets above is positive when $\lambda \geq \Lambda$, we get $v_n \to 0$ in $E_{\lambda}$. Since $v_n = u_n - u$ and $v_n \to 0$, it follows that $u_n \to u$ in $E_{\lambda}$. This completes the proof.

Proof of Theorem 2. Note that $\{u_n\}$ is also a Cerami sequence of $I_{\lambda}$ satisfying $\|u_n\|_{E_{\lambda}} \leq T$, the conclusion follows from Lemmas 9, 10, 11 and Theorem 6.

4. Proof Of Theorem 3

In this section, while $V$ is sign-changing, we study the existence of solutions of (1) for the case $4 < p < 6$ and give the proof of Theorem 3. Without loss of generality, we assume that $b = 1$.

Lemma 13. Let $\langle V_1 \rangle$, $\langle V_2 \rangle$, $K(x) \geq 0$, and $K \in L^1(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ be satisfied. Then, there exist $\alpha, \rho > 0$ such that $I_{\lambda}(u) \geq \alpha$ for all $u \in E_{\lambda} \oplus F_{\lambda}$ with $\|u\|_{E_{\lambda}} = \rho$.

Proof. By proposition 5, for each fixed $\lambda > \Lambda$, there exists a positive integer $n_{\lambda}$ such that $\mu_j(\lambda) \leq 1$ for $j \geq n_{\lambda}$ and $\mu_j(\lambda) > 1$ for $j \geq n_{\lambda}$. Thus, for any $u = u_1 + u_2 \in E_{\lambda} \oplus F_{\lambda}$, we have
\[
I_{\lambda}(u) = \frac{1}{2} \|u\|_{E_{\lambda}}^2 - \frac{1}{2} \int_{\mathbb{R}^3} \lambda V^-(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_{u_1} u^2 dx
\]
\[
- \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \leq \frac{1}{2} \|u\|_{E_{\lambda}}^2 + C_{K} \|u\|_{E_{\lambda}}^4 - \frac{1}{p} \|u\|_{E_{\lambda}}^p
\]
\[
\leq \frac{1}{2} \|u\|_{E_{\lambda}}^2 + C_{K} \|u\|_{E_{\lambda}}^4 - C \|u\|_{E_{\lambda}}^p.
\]  
(52)

Since $p > 4$, there is a large $r > 0$ such that $I(u) < 0$ on $E \backslash B_r(0)$.

Lemma 14. Let $\langle V_1 \rangle$, $\langle V_2 \rangle$, $K(x) \geq 0$, and $K \in L^1(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ be satisfied. Then, for any finite dimensional subspace $E \subset E_{\lambda}$, there is a large $r > 0$ such that $I_{\lambda}(u) < 0$ on $E \backslash B_r(0)$.

Proof. Since all norms are equivalent in a finite dimensional space, there is a constant $b_1 > 0$ such that
\[
\|u\|_p \geq b_1 \|u\|_{E_{\lambda}}, \forall u \in E \subset E_{\lambda}.
\]  
(50)

By (22), there is a constant $C_K > 0$ such that
\[
\int_{\mathbb{R}^3} K(x) \phi_{u} u^2 dx \leq C_K \|u\|_{E_{\lambda}}^4, \forall u \in E_{\lambda}.
\]  
(51)

Hence, for all $u \in E$,
\[
I_{\lambda}(u) = \frac{1}{2} \|u\|_{E_{\lambda}}^2 - \frac{1}{2} \int_{\mathbb{R}^3} \lambda V^-(x) u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_{u} u^2 dx
\]
\[
- \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \leq \frac{1}{2} \|u\|_{E_{\lambda}}^2 + C_K \|u\|_{E_{\lambda}}^4 - \frac{1}{p} \|u\|_{E_{\lambda}}^p
\]
\[
\leq \frac{1}{2} \|u\|_{E_{\lambda}}^2 + C_K \|u\|_{E_{\lambda}}^4 - C \|u\|_{E_{\lambda}}^p.
\]  
(52)

Since $p > 4$, there is a large $r > 0$ such that $I(u) < 0$ on $E \backslash B_r(0)$.

Lemma 15. Let $\langle V_1 \rangle$, $\langle V_2 \rangle$, $K(x) \geq 0$, and $K \in L^1(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ be satisfied. Then, there exists $\Lambda > 1$ such that, for each $\lambda = (\Lambda, \infty)$, $I_{\lambda}$ satisfies the $(PS)_c$ condition.
Proof. Let \{u_n\} be a \((PS)_{c}\) sequence, that is, \(I_\lambda(u_n) \longrightarrow c\) and \(I_\lambda'(u_n) \longrightarrow 0\). If \{u_n\} is unbounded in \(E_\lambda\), up to a subsequence, we can assume that
\[
\|u_n\|_{E_\lambda} \longrightarrow +\infty, \quad I_\lambda(u_n) \longrightarrow c, \quad \|I_\lambda'(u_n)\| \longrightarrow 0; \quad (53)
\]
as \(n \longrightarrow \infty\), after passing to a subsequence. Set \(w_n = u_n/\|u_n\|_{E_\lambda}\), we can assume that \(w_n \rightharpoonup w\) in \(E_\lambda\) and \(w_n(x) \longrightarrow w(x)\) a.e. \(x \in \mathbb{R}^3\).

If \(w = 0\), since \(u \mapsto \int_{\mathbb{R}^3} V(x)u^2\,dx\) is weakly continuous, we have
\[
o(1) = \frac{1}{\|u_n\|_{E_\lambda}^p} \left( I_\lambda(u_n) - \frac{1}{p} \langle I_\lambda'(u_n), u_n \rangle \right)
  = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^3} \lambda V(x)u_n^2\,dx
  + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx \geq \frac{1}{2} - \frac{1}{p} + o(1),
\]
a contradiction. If \(w \neq 0\), then the set \(\Omega = \{x \in \mathbb{R}^3 : \omega(x) \neq 0\}\) has the positive Lebesgue measure. For \(x \in \Omega\), one has\(|u_n(x)| \longrightarrow \infty\) as \(n \longrightarrow \infty\), Fatou’s lemma shows that \(\int_{\Omega} |u_n|^p w_n^2\,dx \longrightarrow \infty\) as \(n \longrightarrow \infty\). Thus, by (22), we obtain
\[
\frac{1}{p} \int_{\mathbb{R}^3} |u_n|^{p-4} w_n^2\,dx = \frac{1}{2\|u_n\|_{E_\lambda}^4} \int_{\mathbb{R}^3} \lambda V(x)u_n^3\,dx
  + \frac{1}{p\|u_n\|_{E_\lambda}^4} \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx + o(1).
\]
\[
\leq \frac{1}{2\|u_n\|_{E_\lambda}^4} + \frac{1}{p\|u_n\|_{E_\lambda}^4} \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2\,dx + o(1).
\]
\[
\leq \frac{C}{p} + o(1). \quad (55)
\]
This is a contradiction. This implies \(\{u_n\}\) is bounded in \(E_\lambda\). Going if necessary to a subsequence, we can assume that \(u_n \rightharpoonup u\) in \(E_\lambda\). The following proof is similar to the proof of Lemma 11. We omit details of this.

Proof of Theorem 3. Obviously, \(I_\lambda(0) = 0\). Furthermore, \(I_\lambda\) is even. The conclusion follows from Lemmas 13, 14, and 15, and Theorem 7.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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