Integration of the geodesic equations via Noether Symmetries

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(Dated: September 27, 2022)

Through this article, I will overview the use of Noether symmetry approach in discussing the integration of geodesic equations for the geodesic Lagrangians of spacetimes. I will also give some examples to reveal the efficiency of Noether symmetry approach by finding the first integrals related for the geodesic Lagrangians of the Gödel-type, Schwarzschild, Reissner-Nordström and Kerr spacetimes. After obtaining the approximate Noether symmetries of the Schwarzschild, Reissner-Nordström and Kerr spacetimes, the first integrals associated with each of approximate Noether symmetries have been integrated to find a general solution of geodesic equations in terms of the arc length $s$.

PACS numbers: 04.20.-q, 11.30.-j, 45.05.+x

I. INTRODUCTION

The differential equations can be deduced from a Lagrangian function through a variational technique. Noether symmetries [1] which are the special classes of Lie symmetries are intimately connected with conservation laws, or the first integrals in the case of ordinary differential equations (ODEs) derived from the corresponding Lagrangian. The equations of geodesic motion are expressed in terms of the configuration space variables, i.e., the metric coefficients. Therefore, the configuration space of our model is a 4-dimensional Riemannian manifold with coordinates $x^i$ ($i = 0, 1, 2, 3$), in which we construct a point-like geodesic Lagrangian to produce the geodesic equations of motion. The geodesic equations are a system of second order ODEs and can be derived from a Lagrangian function $L(\tau, x^i, \dot{x}^i)$ of the system related to the geodesic motion. Here the dot represents the derivative with respect to an affine parameter $\tau$, and this is the arc length $s$ in most of the spacetimes. Note that $Q = \{x^i, i = 0, 1, 2, 3\}$ is the configuration space from which it is possible to derive the corresponding tangent space $TQ = \{x^i, \dot{x}^i\}$ where the Lagrangian $L$ is defined. Taking the variation of the geodesic Lagrangian

$$L(s, x^k, \dot{x}^k) = \frac{1}{2}g_{ij}(x^k)\dot{x}^i\dot{x}^j - V(s, x^k),$$

with respect to the coordinates $x^i$, it follows the geodesic equations of motion,

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = F^i(s, x^\ell),$$

where $\Gamma^i_{jk}(x^\ell)$ are the connection coefficients of the metric and $F^i = g^{ij}V_j$ is the conservative force field. The energy functional associated with $L$ is

$$E_L = \dot{x}^i \frac{\partial L}{\partial \dot{x}^j} - L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + V(s, x^k),$$

which is the Hamiltonian of the system.

Noether symmetries are associated with differential equations possessing a Lagrangian, and they describe physical features of differential equations in terms of conservation laws admitted by them [2]. Thus one can use the geodesic Lagrangian associated with the geodesic motion for spacetimes to integrate the geodesic equations of motion if it is possible. It is capable to construct analytical solutions of geodesic equations by reducing their complexity, using not only the Noether symmetry but also the approximate Noether symmetry approach. In order to find out analytical solutions of geodesic equations for the considered geodesic Lagrangian, one can use the obtained first integrals of motion which are Noether constants. Recently the Noether symmetries of geodesic Lagrangian for some spacetimes have been calculated, and classified according to their Noether symmetry generators [3]-[11]. In this study we consider Noether symmetries (the approximate Noether symmetries) of the geodesic Lagrangian (the perturbed geodesic Lagrangian) for the line elements of some known spacetimes, rather than those of the geodesic equations.
We design this study as follows. In the following section, we aim to give some examples of Noether symmetries for the geodesic Lagrangian \( L \) of some well-known spacetimes. In Section III, we present a detailed analysis of the approximate Noether symmetries of Schwarzschild, Reissner-Nordström and Kerr spacetimes. Finally, we conclude with a brief summary and discussions in Section IV.

II. NOETHER SYMMETRIES FOR THE GEODESIC LAGRANGIANS

The Noether symmetry (NS) generator for the geodesic Lagrangian associated with the system of ODEs in \( (2) \) is

\[
X = \xi(s, x^k) \frac{\partial}{\partial \tau} + \eta^i(s, x^k) \frac{\partial}{\partial x^i},
\]

if there exists a function \( f(s, x^k) \) which is sometimes called a gauge function, and the NS condition

\[
X[1]L + L(D_s \xi) = D_s f,
\]

is satisfied, where \( D_s = \partial/\partial s + \dot{x}^j \partial/\partial x^j \) is the total derivative operator and \( X[1] \) is the first prolongation of NS generator \( X \), i.e.

\[
X[1] = X + \dot{\eta}^i(s, x^k, \dot{x}^j) \frac{\partial}{\partial \dot{x}^j},
\]

with \( \dot{\eta}^j(s, x^k, \dot{x}^j) = D_s \eta^j - \dot{x}^j D_s \xi \). It has to be noted here that the NS condition \( (5) \) takes the alternative form

\[
\xi, i = 0, \quad g_{ij} \eta^j_s = f_s, \quad L_\eta g_{ij} = \xi_s g_{ij}, \quad L_\eta V = -\xi_s V - f_s,
\]

where \( L_\eta \) is the Lie derivative operator along \( \eta \), and the set of all NSs form a finite dimensional Lie algebra denoted by \( \mathcal{N} \).

For every NS, there is a conserved quantity (or a first integral) of the system of equations \( (2) \) given by

\[
I = -\xi E_L + g_{ij} \eta^j \dot{x}^i - f,
\]

where the energy functional \( E_L \) of the geodesic Lagrangian is the same as in Eq. \( (3) \).

Now let us recall the spacetime symmetries. A vector field \( K \), satisfying the equation \( (9) \)

\[
L_K g_{ij} = 2\psi g_{ij},
\]

is called a conformal Killing vector (CKV), where \( L_K \) is the Lie derivative operator along the vector field \( K \), and \( \psi = \psi(x^i) \) is a conformal factor. For \( \psi_{;ij} \neq 0 \), the CKV field \( K \) is said to be proper, otherwise it is a special conformal Killing vector (SCKV) field \( \psi_{;ij} = 0 \). The vector field \( K \) is a homothetic vector (HV) for \( \psi_{;i} = 0 \), and it is an isometry or a Killing vector (KV) field for \( \psi = 0 \). The set of all CKV (respectively SCKV, HKV and KV) form a finite-dimensional Lie algebra denoted by \( \mathcal{C} \) (respectively \( \mathcal{S}, \mathcal{H} \) and \( \mathcal{G} \)).

A. Noether Symmetries of the Gödel-type Spacetimes

In cylindrical coordinates \( x^i = (t, r, \phi, z) \), \( i = 0, 1, 2, 3 \), the line element for the Gödel-type spacetimes can be written as

\[
 ds^2 = [dt + H(r) d\phi]^2 - dr^2 - D(r)^2 f \phi^2 - dz^2.
\]

The necessary and sufficient conditions for a Gödel-type manifold to be spacetime homogeneous (STH) are given by

\[
\frac{D''}{D} = \text{const.} \equiv m^2, \quad \frac{H'}{D} = \text{const.} \equiv -2 \omega,
\]

where prime (‘) denotes derivative with respect to the radial coordinate \( r \). All STH Riemannian manifolds endowed with the Gödel-type spacetime \( (10) \) are obtained by solving equations in \( (11) \) as follows:

**Class I:** \( m^2 > 0, \omega \neq 0 \),

\[
H(r) = \frac{2 \omega}{m^2} [1 - \cosh(mr)], \quad D(r) = \frac{1}{m} \sinh(mr).
\]

**Class II:** \( m^2 < 0, \omega \neq 0 \),

\[
H(r) = \frac{-2 \omega}{m^2} [1 - \cosh(mr)], \quad D(r) = \frac{1}{m} \sinh(mr).
\]

**Class III:** \( m = 0, \omega \neq 0 \),

\[
H(r) = -\omega^2/r^2, \quad D(r) = 0.
\]

**Class IV:** \( m^2 = 0, \omega = 0 \),

\[
H(r) = 0, \quad D(r) = 0.
\]
Class II: \( m^2 = 0, \omega \neq 0 \).

\[
H(r) = -\omega r^2, \quad D(r) = r.
\]  

(13)

Class III: \( m^2 \equiv -\mu^2 < 0, \omega \neq 0 \).

\[
H(r) = \frac{2\omega}{\mu^2} [\cos(\mu r) - 1], \quad D(r) = \frac{1}{\mu} \sin(\mu r).
\]  

(14)

Class IV: \( m^2 \neq 0, \omega = 0 \). This class yields a degenerate Gödel-type manifolds, since the cross term related to the rotation \( \omega \) in the line element vanishes. One can make \( H = 0 \) by a trivial coordinate transformation with \( D(r) \) given as in Classes I and III depending on whether \( m^2 > 0 \) or \( m^2 \equiv -\mu^2 < 0 \).

Using the Gödel-type spacetime (10), the geodesic Lagrangian takes such a form

\[
\mathcal{L} = \frac{1}{2} \left[ t^2 - r^2 - z^2 + (H(r)^2 - D(r)^2)\dot{\phi}^2 \right] + H(r)\dot{\phi} - V(t, r, \phi, z).
\]  

Then it follows for this Lagrangian that the energy functional is

\[
E_\mathcal{L} = \frac{1}{2} \left[ t^2 - r^2 - z^2 + (H(r)^2 - D(r)^2)\dot{\phi}^2 \right] + H(r)\dot{\phi} + V(t, r, \phi, z),
\]  

and the conserved quantity associated with NS generator \( X \) is

\[
I = -\xi E_\mathcal{L} + \left( \eta^0 + H\eta^2 \right) \dot{t} - \eta^1 \dot{r} + \left[ H\eta^0 + (H^2 - D^2)\eta^2 \right] \dot{\phi} - \eta^3 \dot{z} - f(s, t, r, \phi, z).
\]  

(17)

The complete NS analysis of Gödel-type spacetimes for classes I, II, III and IV has been given by Camci[8, 9]. Let us briefly summarize the results. The geodesic Lagrangian \( \mathcal{L} \) of Gödel-type spacetimes for classes I, II, III and IV yields 7 NS generators. Thus, the Gödel-type spacetimes corresponding to those classes admit the algebra \( N_7 \supset G_7 \). In special class I (where \( m^2 = 4w^2 \)) and class IV, it is found 9 NS generators. The NS algebra admitted by the special class I is \( N_9 \supset G_7 \) while the Gödel-type spacetime in class IV admits the algebra \( N_9 \supset G_9 \). The first integrals have been obtained by using the geodesic Lagrangians for the Gödel-type spacetimes of each classes I, II, III and IV, due to the existence of NS vector fields including the KVs. Using the first integrals obtained in all classes of Gödel-type spacetimes, the analytical solutions of geodesic equations have been derived. This result represents the usefulness of the NSs.

As an example, we give only the obtained NSs and associated first integrals for class I as follows (See Ref. [8] for the details of calculation). For class I there are seven NSs such that \( X_1, ..., X_5 \) are KVs,

\[
X_1 = \partial_t, \quad X_2 = \partial_z, \quad X_3 = \frac{2\omega}{m}\partial_t - m\partial_\phi,
\]

\[
X_4 = -\frac{H}{D}\sin\phi\partial_t + \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi,
\]

\[
X_5 = -\frac{H}{D}\cos\phi\partial_t - \sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi,
\]  

(18)

and \( Y_1, Y_2 \) are two non-Killing NSs,

\[
Y_1 = \partial_s, \quad Y_2 = s\partial_z \quad \text{with } f = -z.
\]  

(19)

Then the first integrals associated with \( X_1, ..., X_5, Y_1 \) and \( Y_2 \) are found by the relation (17) as

\[
I_1 = \dot{t} + H\dot{\phi}, \quad I_2 = -\dot{z}, \quad I_3 = \frac{2w}{m}I_1 - m \left[ H\dot{t} + (H^2 - D^2)\dot{\phi} \right],
\]  

(20)

\[
I_4 = -\frac{\sin\phi}{D} \left\{ H(1 + D')\dot{t} + \left[ H^2 + (H^2 - D^2)D' \right] \dot{\phi} \right\} - \cos\phi\dot{r},
\]  

(21)

\[
I_5 = -\frac{\cos\phi}{D} \left\{ H(1 + D')\dot{t} + \left[ H^2 + (H^2 - D^2)D' \right] \dot{\phi} \right\} + \sin\phi\dot{r},
\]  

(22)

\[
I_6 = -E_\mathcal{L}, \quad I_7 = -s\dot{z} + z,
\]  

(23)
respectively, and we have introduced the parameters $p = 1 - \beta^2 - 2w\gamma$ such as $\eta > 0$, $1 - \beta^2 - 2w\gamma^2 \geq 0$, $(1 + p^2) > q^2$, $t_0 = t(0)$ and $\phi_0 = \phi(0)$. Here the constants of motion $p_t = I_1$, $p_\phi = 2\omega I_1/m^2$ and $p_z = I_2$ represent the conservation of energy, angular momentum and $z$ component of momentum, respectively, and we have introduced the parameters $p$ and $q$ such as

$$p := \frac{1 - \beta^2 + 2w\gamma}{2\eta}, \quad q := \sqrt{p^2 - \frac{m^2\gamma^2}{4\eta}},$$

where $p^2 \geq m^2\gamma^2/4\eta$.

**B. Noether Symmetries of the Spherically Symmetric Spacetimes**

The field of a spherically symmetric gravitational source at rest at the origin is given by the Schwarzschild line element

$$ds^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2r}\right)} - r^2d\Omega^2,$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, $G$ is Newton’s gravitational constant, $M$ is the mass of the point gravitational source and $c$ is the speed of light in vacuum.

The geodesic Lagrangian of the Schwarzschild metric

$$\mathcal{L} = \frac{1}{2} \left[\left(1 - \frac{2GM}{c^2r}\right)c^2t^2 - \frac{r^2}{\left(1 - \frac{2GM}{c^2r}\right)} - r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2)\right] - V(t, r, \theta, \phi),$$

has five NSs $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ for constant potential, which are four KVs corresponding the conservation of energy and angular momentum only, i.e.,

$$\mathbf{K}_0 = \partial_t, \quad \mathbf{K}_1 = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \quad \mathbf{K}_2 = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \quad \mathbf{K}_3 = \partial_\phi,$$

and the translation of the arc length $s$, i.e., $\mathbf{Y}_0 = \partial_s$. Note that conservations of linear momentum and the spin angular momentum are lost.

The Reissner-Nordström (RN) metric is a static, spherically symmetric and asymptotically flat spacetime

$$ds^2 = \left(1 - \frac{2GM}{c^2r} + \frac{GQ^2}{c^4r^2}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{2GM}{c^2r} + \frac{GQ^2}{c^4r^2}} - r^2d\Omega^2,$$

where $Q$ is the electric charge of the point gravitational source.
The geodesic Lagrangian of the Reissner-Nordström metric

\[ \mathcal{L} = \frac{1}{2} \left[ \left( 1 - \frac{2GM}{c^2r} + \frac{GQ^2}{c^4r^2} \right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{\left( 1 - \frac{2GM}{c^2r} + \frac{GQ^2}{c^4r^2} \right)} - r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] - V(t, r, \theta, \phi), \]

has also five NSs [14] for constant potential, which are four KVs given in [20] for the Schwarzschild metric and the translation symmetry \( Y_0 = \partial_s \).

### C. Noether Symmetries of the Kerr Spacetime

Here I will use the signature \((-, +, +, +)\) for the Kerr spacetime, in which the line element in Boyer-Lindquist coordinates is given by

\[ ds^2 = - \left( 1 - \frac{2GMr}{\Sigma c^2} \right) c^2 dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{4GMa}{\Sigma c^2} \sin^2 \theta \, dtd\phi + \left( \frac{r^2 + a^2}{c^2} \right) - \frac{a^2}{c^2} \Delta \sin^2 \theta \frac{\sin^2 \theta}{\Sigma} \, d\phi^2, \]

where \( \Sigma = r^2 + \frac{a^2}{c^2} \cos^2 \theta \) and \( \Delta = r^2 + \frac{a^2}{c^2} - \frac{2GMr}{c^2} \) with the mass \( M \) and the spin \( a = J/(Mc) \) (units of length) of the gravitating source.

The geodesic Lagrangian \((1)\) for the Kerr metric \((28)\) is

\[ \mathcal{L} = \frac{1}{2} \left[ - \left( 1 - \frac{2GMr}{\Sigma c^2} \right) c^2 \dot{t}^2 + \frac{\Sigma}{\Delta} \dot{r}^2 + \Sigma \dot{\theta}^2 \right. \]

\[ + \left. \left( \frac{r^2 + a^2}{c^2} \right) - \frac{a^2}{c^2} \Delta \sin^2 \theta \right] \frac{\sin^2 \theta}{\Sigma} \dot{\phi}^2 - \frac{4GMa \sin^2 \theta}{\Sigma c^2} \dot{\phi} \right] - V(t, r, \theta, \phi), \]

Solving the NS equations for the geodesic Lagrangian of the Kerr metric we get two isometries and the translation of the geodesic parameter as NS generators \([17]\)

\[ K_0 = \partial_t, \quad K_3 = \partial_{\phi}, \quad Y_0 = \partial_s, \]

(29)

corresponding to the conservation of total energy, conservation of the angular momentum per unit mass at azimuthal direction, and the translation of the arc length, respectively.

### III. Approximate Noether Symmetries for the Geodesic Lagrangians

In this section, we introduce the approximate Noether symmetry (ANS) approach of the first-order perturbed Lagrangian extending the procedure of obtaining ANSs until the \( n^{th} \)-order. A perturbed Lagrangian to \( n^{th} \)-order can be written as

\[ \mathcal{L}(s, x^i, \dot{x}^i; \epsilon) = \mathcal{L}_0(s, x^i, \dot{x}^i) + \epsilon \mathcal{L}_1(s, x^i, \dot{x}^i) + \ldots + \epsilon^n \mathcal{L}_n(s, x^i, \dot{x}^i) + O(\epsilon^{n+1}). \]

(30)

Then an ANS generator related with the above Lagrangian is given by

\[ X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \ldots + \epsilon^n X_n, \]

(31)

up to the gauge function

\[ f(s, x^i; \epsilon) = f_0(s, x^i) + \epsilon f_1(s, x^i) + \epsilon^2 f_2(s, x^i) + \ldots + \epsilon^n f_n(s, x^i), \]

if the ANS generator \((31)\) satisfies the approximate Noether symmetry conditions

\[ X_0^{[1]} \mathcal{L}_0 + \mathcal{L}_0(\partial_s x_0) = D_s f_0, \]

\[ X_0^{[1]} \mathcal{L}_0 + X_0^{[1]} \mathcal{L}_1 + \mathcal{L}_0(\partial_s x_0) + \mathcal{L}_1(\partial_s x_0) = D_s f_1, \]

\[ X_0^{[1]} \mathcal{L}_0 + X_0^{[1]} \mathcal{L}_1 + X_0^{[1]} \mathcal{L}_2 + \mathcal{L}_0(\partial_s x_0) + \mathcal{L}_1(\partial_s x_0) + \mathcal{L}_2(\partial_s x_0) = D_s f_2, \]

\[ \ldots \]

\[ X_0^{[1]} \mathcal{L}_0 + X_0^{[1]} \mathcal{L}_1 + \ldots + X_0^{[1]} \mathcal{L}_n + \mathcal{L}_0(\partial_s x_0) + \mathcal{L}_1(\partial_s x_0) + \ldots + \mathcal{L}_n(\partial_s x_0) = D_s f_n, \]

\[ \ldots \]
where \( n \geq 1 \), and \( X_0 \) is the exact NS generator, \( X_1, X_2, \ldots, X_n \) are the first-order, second-order, \( \ldots, \) \( n^{th} \)-order ANS generators, respectively, which are defined as

\[
X_\alpha = \xi_\alpha \frac{\partial}{\partial s} + \eta^i_\alpha \frac{\partial}{\partial x^i}, \quad (\alpha = 0, 1, 2, \ldots, n),
\]

\[
X_\alpha^{[1]} = X_\alpha + \eta_i^{(1)} \frac{\partial}{\partial x^i}, \quad \eta_i^{(1)} = D_s \eta^i_\alpha - \dot{x}^i D_s \xi_\alpha.
\]

The spacetime metric \( g_{ij} \) can be decomposed up to \( n^{th} \)-order as follows:

\[
g_{ij} = \gamma_{ij} + \epsilon h_{ij} + \epsilon^2 \sigma_{ij} + \ldots + \epsilon^n \lambda_{ij},
\]

which means by (30) and (32) that the exact and perturbed geodesic Lagrangians of motion have the form

\[
\mathcal{L}_0(s, x^k, \dot{x}^k) = \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j - V_0(s, x^k),
\]

\[
\mathcal{L}_1(s, x^k, \dot{x}^k) = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j - V_1(s, x^k),
\]

\[
\mathcal{L}_2(s, x^k, \dot{x}^k) = \frac{1}{2} \sigma_{ij} \dot{x}^i \dot{x}^j - V_2(s, x^k),
\]

\[
\mathcal{L}_n(s, x^k, \dot{x}^k) = \frac{1}{2} \lambda_{ij} \dot{x}^i \dot{x}^j - V_n(s, x^k),
\]

where \( \gamma_{ij}, h_{ij}, \sigma_{ij} \) and \( \lambda_{ij} \) are the exact, the first-order, the second-order and the \( n^{th} \)-order perturbed metrics, respectively. The metric \( \gamma_{ij} \) should be non-degenerate (i.e., \( \det(\gamma_{ij}) \neq 0 \)). But the other metrics \( h_{ij}, \sigma_{ij}, \ldots, \lambda_{ij} \) can be degenerate (i.e., \( \det(h_{ij}) = 0, \det(\sigma_{ij}) = 0, \ldots, \det(\lambda_{ij}) = 0 \)) or non-degenerate, and they represent slight deviations from flat spacetime geometry if the metric \( \gamma_{ij} \) represents flat geometry.

The above perturbed Lagrangian (30) yields a \( n^{th} \)-order (in \( \epsilon \)) perturbed system of ODEs. If \( X_\alpha \) are the ANSs corresponding to the perturbed geodesic Lagrangians \( \mathcal{L}_\alpha(s, x^i, \dot{x}^i) \), then

\[
I_0 = -\xi_0 E_{\mathcal{L}_0} + \eta^0_{\alpha} \gamma_{ij} \dot{x}^j - f_0,
\]

\[
I_1 = -\xi_0 E_{\mathcal{L}_1} - \xi_1 E_{\mathcal{L}_0} + (\eta^0_{\alpha} h_{ij} + \eta^1_{\alpha} \gamma_{ij} \dot{x}^j - f_1,
\]

\[
I_2 = -\xi_0 E_{\mathcal{L}_2} - \xi_1 E_{\mathcal{L}_1} - \xi_2 E_{\mathcal{L}_0} + (\eta^0_{\alpha} \sigma_{ij} + \eta^1_{\alpha} h_{ij} + \eta^2_{\alpha} \gamma_{ij} \dot{x}^j - f_2,
\]

\[
\ldots
\]

\[
I_n = -\xi_0 E_{\mathcal{L}_n} - \xi_1 E_{\mathcal{L}_{n-1}} - \ldots - \xi_n E_{\mathcal{L}_0} + \left( \eta^0_{\alpha} \lambda_{ij} + \ldots + \eta^n_{\alpha} \sigma_{ij} + \eta^{n-1}_{\alpha} h_{ij} + \eta^n_{\alpha} \gamma_{ij} \right) \dot{x}^j - f_n,
\]

are the first integrals associated with ANSs \( X_\alpha \), \( (\alpha = 0, 1, 2, \ldots, n) \). Here the exact and the perturbed energy functionals for the perturbed Lagrangian (30) are

\[
E_{\mathcal{L}_0} = \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j + V_0, \quad E_{\mathcal{L}_1} = \frac{1}{2} h_{ij} \dot{x}^i \dot{x}^j + V_1, \quad E_{\mathcal{L}_2} = \frac{1}{2} \sigma_{ij} \dot{x}^i \dot{x}^j + V_2, \ldots, \quad E_{\mathcal{L}_n} = \frac{1}{2} \lambda_{ij} \dot{x}^i \dot{x}^j + V_n.
\]

It has been investigated the ANSs and conservation laws of the geodesic equations without the potential function for the Schwarzschild [13] and the RN [14] the spacetimes. Constructing the geometrical set of equations corresponding to the ANS equations with an arbitrary potential function, the ANSs of the geodesic Lagrangian for the Schwarzschild, the RN and the Bardeen spacetimes have been determined by Camci [15, 16]. Hussain et al. [17] have recovered all the lost conservation laws as trivial second-order approximate conservation laws of a Lagrangian for the geodesic equations by using the ANS approach in the Kerr and the charged-Kerr spacetimes. They have also discussed the problem of energy in cylindrical and plane gravitational wave spacetimes using approximate Noether symmetry method [18]. Ali and Feroze [19] have generalized the work in Ref. [17] such that the ANS of the most general plane symmetric static spacetime are obtained.

### A. Approximate Symmetries of the Schwarzschild Spacetime

First, we will look at the ANSs of geodesic Lagrangian for the Schwarzschild metric. In that context, we consider the Schwarzschild line element given in (26). Setting \( 2GMc^{-2} = \tau_0 \) and using

\[
\left(1 - \frac{2GM}{c^2 r}\right)^{-1} = 1 + \frac{\tau_0}{r} + O(\epsilon^2),
\]
the first-order perturbed geodesic Lagrangian of Schwarzschild metric is given by
\[
\mathcal{L} = \frac{1}{2} \left[ t^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] - \frac{\epsilon r_0}{2r} (\dot{t}^2 + \dot{r}^2) - V(t, r, \theta, \phi) + O(\epsilon^2),
\] (34)
which yields
\[
\mathcal{L}_0 = \frac{1}{2} \left[ t^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] - V_0, \quad \mathcal{L}_1 = -\frac{r_0}{2r} (\dot{t}^2 + \dot{r}^2) - V_1,
\] (35)
where \( r_0 \) is a dimensional parameter (units of length), \( \gamma_{ij} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta) \) is called as the Minkowski metric, and \( h_{ij} = \text{diag}(-r_0/r, -r_0/r, 0, 0) \). Moreover, the above Lagrangian reduces to the geodesic Lagrangian of the Minkowski metric in the limit \( \epsilon = 0 \).

Applying the ANS approach to these exact and perturbed metrics \( \gamma_{ij} \) and \( h_{ij} \) for the Schwarzschild spacetime, we find from the exact and first-order ANS equations that for the constant potential, e.g. \( V(t, r, \theta, \phi) = V_0 + \epsilon V_1 \) where \( V_0, V_1 \) are constants, we find 5 exact ANSs and 17 first-order ANSs which includes also exact ones. Here the exact ANSs are the four KVs given in (??) and (??) and one non-Killing vector field \( \mathbf{Y}_0 = \partial_s \) which gives translation in geodetic parameter \( s \) and it always exists for the canonical geodesic Lagrangian \( \mathbf{Y}_0 = \partial_s \). The remaining first-order nontrivial ANSs are
\[
\begin{align*}
\mathbf{Y}_1 &= \sin \theta \cos \phi \partial_r + \frac{\cos \theta \cos \phi}{r} \partial_t - \frac{\csc \theta \sin \phi}{r} \partial_\theta, \\
\mathbf{Y}_2 &= \sin \theta \sin \phi \partial_r + \frac{\cos \theta \sin \phi}{r} \partial_t + \frac{\csc \theta \cos \phi}{r} \partial_\theta, \\
\mathbf{Y}_3 &= \cos \theta \partial_\theta - \frac{\sin \theta}{r} \partial_\phi, \\
\mathbf{Y}_4 &= r \sin \theta \cos \phi \partial_t + t \mathbf{Y}_1, \\
\mathbf{Y}_5 &= r \sin \theta \sin \phi \partial_t + t \mathbf{Y}_2, \\
\mathbf{Y}_6 &= r \cos \theta \partial_\theta + t \mathbf{Y}_3, \\
\mathbf{Y}_7 &= s \partial_s + \frac{1}{2} (t \partial_t + r \partial_r), \quad \text{with } f_1 = -V_0 s, \\
\mathbf{Y}_8 &= s \mathbf{K}_0, \quad \text{with } f_1 = t, \\
\mathbf{Y}_9 &= s \mathbf{Y}_1, \quad \text{with } f_1 = -r \sin \theta \cos \phi, \\
\mathbf{Y}_{10} &= s \mathbf{Y}_2, \quad \text{with } f_1 = -r \sin \theta \sin \phi, \\
\mathbf{Y}_{11} &= s \mathbf{Y}_3, \quad \text{with } f_1 = -r \cos \theta, \\
\mathbf{Y}_{12} &= s (s \partial_s + t \partial_t + r \partial_r), \quad \text{with } f_1 = \frac{1}{2} (t^2 - r^2 - 2V_0 s^2). 
\end{align*}
\] (36)

For the Schwarzschild spacetime considered as a first perturbation of the Minkowski metric, three nontrivial ANS generators \( \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3 \) provide the conservation of linear momentum, and three nontrivial ANS generators \( \mathbf{Y}_4, \mathbf{Y}_5, \mathbf{Y}_6 \) give the conservation of spin angular momentum due to Lorentz invariance.

The first integrals associated with the 5 exact ANSs are
\[
I^0_0 = -E_{\mathcal{L}_0}, \quad I^2_0 = i, \quad I^3_0 = -r^2 \sin^2 \theta \dot{\phi}, \quad I^0_0 = -r^2 \left[ \cos \phi \dot{\theta} - \sin \theta \cos \theta \sin \phi \dot{\phi} \right], \quad I^0_0 = -r^2 \left[ \sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi} \right], \quad (37)
\]
where the exact energy functional \( E_{\mathcal{L}_0} \) is
\[
E_{\mathcal{L}_0} = \frac{1}{2} \left[ \dot{t}^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] + V_0. \quad (38)
\]
Then the first integrals associated with the 17 first-order ANSs are given by

\[ I_1^1 = -(E_{L_0} + E_{L_1}), \quad I_1^2 = \left(1 - \frac{r_0}{r}\right) t, \quad I_1^3 = -r^2 \sin^2 \theta \phi, \]

\[ I_1^4 = -r^2 \left[ \cos \phi \dot{\theta} - \sin \theta \cos \theta \sin \phi \dot{\phi} \right], \quad I_1^5 = -r^2 \left[ \sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi} \right], \]

\[ I_1^6 = -\sin \theta \cos \phi \dot{r} - r \cos \theta \cos \phi \dot{\theta} + r \sin \theta \sin \phi \dot{\phi}, \]

\[ I_1^7 = -\sin \theta \sin \phi \dot{r} - r \cos \theta \sin \phi \dot{\theta} - r \sin \theta \cos \phi \dot{\phi}, \]

\[ I_1^10 = r \sin \theta \sin \phi \dot{t} + I_1^11, \quad I_1^{11} = r \cos \theta \dot{t} + I_1^3 t, \]

\[ I_1^{12} = s \dot{t} - t, \quad I_1^{13} = -(E_{L_0} - V_0) s + \frac{1}{2} (t \dot{t} - r \dot{r}), \]

\[ I_1^{14} = -(E_{L_0} - V_0) s^2 + s (t \dot{t} - r \dot{r}) - \frac{1}{2} (q^2 - r^2), \]

\[ I_1^{15} = I_1^3 s + r \sin \theta \cos \phi, \quad I_1^{16} = I_1^7 s + r \sin \theta \sin \phi, \quad I_1^{17} = I_1^3 s + r \cos \theta, \]

where the first-order energy functional \( E_{L_1} \) is

\[ E_{L_1} = -\frac{r_m}{2r} \left( I^2 + \dot{r}^2 \right) + V_1. \]

Defining the Noether constants as \( I^2 = -E, \ I^3 = L_1, \ I^4 = a_1, \ I^5 = a_2, \ I^6 = a_3, \ I^7 = a_4, \ I^8 = a_5, \ I^9 = a_6, \ I^{10} = a_7, \ I^{11} = a_8, \ I_1^1 = b_1, \ I_1^{12} = b_2, \ I_1^{13} = b_3, \ I_1^{14} = b_4, \ I_1^{15} = b_5, \ I_1^{16} = b_6, \ I_1^{17} = b_7 \), and using the first integrals (39), it follows that

\[ E_{L_0} + E_{L_1} = -b_1, \quad E = -t_0 \left(1 - \frac{r_0}{r}\right), \quad L_z = -r^2 \sin^2 \theta \phi, \]

\[ t(s) = t_0 s - b_2, \quad t_0 = \frac{1}{b_2} (a_3 b_5 + a_4 b_6 + a_5 b_7 - 2 b_3), \]

\[ r \sin \theta \cos \phi = -a_3 s + b_5, \quad r \sin \theta \sin \phi = -a_4 s + b_6, \quad r \cos \theta = -a_5 s + b_7, \]

\[ a_1 = a_3 b_7 - a_5 b_5, \quad a_2 = a_4 b_7 - a_5 b_6, \quad b_3 = \frac{1}{2} \left( b_5^2 + b_6^2 + b_7^2 - b_2^2 \right), \]

\[ a_6 = t_0 b_5 - a_3 b_2, \quad a_7 = t_0 b_6 - a_4 b_2, \quad a_8 = t_0 b_7 - a_5 b_2, \]

where \( t_0 \) is a constant of integration and \( b_2 \neq 0 \). Thus, Eq. (38) yields

\[ r(s) = \sqrt{(a_3 s - b_5)^2 + (a_4 s - b_6)^2 + (a_5 s - b_7)^2}, \]

\[ \theta(s) = \tan^{-1} \left( \frac{\sqrt{(a_3 s - b_5)^2 + (a_4 s - b_6)^2}}{-a_5 s + b_7} \right), \quad \phi(s) = \tan^{-1} \left( \frac{a_4 s - b_6}{a_3 s - b_5} \right). \]

After considering Eqs. (38) and (40), we have found the exact and the first-order energy functionals as

\[ E_{L_0} = \frac{1}{2} \left( t_0^2 - a_3^2 - a_4^2 - a_5^2 \right) + V_0, \]

\[ E_{L_1} = -\frac{r_0}{2r} \left[ t_0^2 + \frac{1}{r^2} \left( (a_3^2 + a_4^2 + a_5^2) s - (a_4 b_5 + a_5 b_6 + a_5 b_7) \right) \right] + V_1. \]

Further, using Eqs. (16) and (17), it follows from Eq. (41) that the component of angular momentum \( L_z \) becomes a constant such as \( L_z = a_4 b_5 - a_5 b_6 \). We point out the fact that the exact energy functional \( E_{L_0} \) given in (38) is already a constant. It is seen from Eq. (49) if \( a_3^2 + a_4^2 + a_5^2 = 0 \) and \( a_3 b_5 + a_4 b_6 + a_5 b_7 = 0 \), i.e., this means \( r = \sqrt{b_5^2 + b_6^2 + b_7^2} = \text{const.} \), then the first-order energy functional \( E_{L_1} \) becomes constant, i.e. \( E_{L_1} = -\frac{r_0 t_0^2}{2r} + V_1 \).

**B. Approximate Symmetries of the Reissner-Nordström Spacetime**

Setting \( 2GMC^{-2} = r_0 \) and \( GQ^2e^{-4} = qe^2 \) at the RN spacetime (27), the RN metric coefficients become

\[ g_{tt} = 1 - \frac{\epsilon r_0}{r} + \frac{ke^2}{r^2} + O(\epsilon^3) \quad \text{and} \quad g_{rr} = -\left[ 1 + \frac{\epsilon r_0}{r} + (1 - q) \frac{q^2}{r^2} \right] + O(\epsilon^3). \]
Therefore, the second-order perturbed geodesic Lagrangian of the RN metric has the form:
\[ \mathcal{L} = \frac{1}{2} \left[ \dot{t}^2 - \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] - \frac{\epsilon r_0}{2r} \left( \dot{t}^2 + \dot{r}^2 \right) + \frac{\epsilon^2}{2r^2} \left[ q \dot{t}^2 + (q - 1) \dot{r}^2 \right] - V(t, r, \theta, \phi) + O(\epsilon^3), \] (50)
which yields the same \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) given for the Schwarzschild spacetime, and
\[ \mathcal{L}_2 = \frac{1}{2r^2} \left[ q \dot{t}^2 + (q - 1) \dot{r}^2 \right] - V_2, \] (51)
where \( \gamma_{ij} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta) \), \( h_{ij} = \text{diag}(-r_0/r, -r_0/r, 0, 0) \) and \( \sigma_{ij} = \text{diag}(q/r^2, (q - 1)/r^2, 0, 0) \). Retaining only the first order terms in the above Lagrangian and neglecting \( O(\epsilon^2) \), it reduces to the first-order perturbed geodesic Lagrangian for the Schwarzschild metric.

It is seen from the solutions of ANS equations of the RN metric that the exact ANSs of the Schwarzschild metric are retained, i.e., there exist 5 exact ANS generators, which are \( Y_0, K_0, K_1, K_2 \) and \( K_3 \). There exist also 5 first-order ANSs for the RN metric as for the exact ones. The lost symmetries for the RN metric appear in the second-order ANS generators which are solutions of ANS conditions with the constant potential, and the number of the second-order nontrivial ANS generators is seventeen, which are the same with \( K_0, \ldots, K_3, Y_0, \ldots, Y_{12} \) given the symmetry generators for the first-order perturbed Schwarzschild metric. The first integrals of the second-order ANSs for the RN metric have the same form given by the Schwarzschild metric. In summary, the solutions for these first integrals are as follows:
\[
\begin{align*}
t(s) &= t_0 s - b_2, \\
r(s) &= \sqrt{(a_3 s - b_5)^2 + (a_4 s - b_6)^2 + (a_5 s - b_7)^2}, \\
\theta(s) &= \tan^{-1}\left(\frac{\sqrt{(a_3 s - b_5)^2 + (a_4 s - b_6)^2}}{-a_5 s + b_7}\right), \\
\phi(s) &= \tan^{-1}\left(\frac{a_4 s - b_6}{a_3 s - b_5}\right),
\end{align*}
\] (52)
which together with the constraints depending on the Noether constants,
\[
\begin{align*}
E_{\mathcal{L}_0} + E_{\mathcal{L}_1} + E_{\mathcal{L}_2} &= -b_1, \\
E &= -t_0 \left(1 - \frac{r_0}{r} + \frac{q}{r^2}\right), \\
L_z &= a_4 b_5 - a_3 b_6, \\
t_0 &= \frac{1}{b_2} (a_3 b_5 + a_4 b_6 + a_5 b_7 - 2b_3), \\
a_1 &= a_3 b_7 - a_5 b_5, \\
a_2 &= a_4 b_7 - a_5 b_6, \\
b_4 &= \frac{1}{2} (b_3^2 + b_6^2 + b_7^2 - b_2^2), \\
a_6 &= t_0 b_5 - a_3 b_2, \\
a_7 &= t_0 b_6 - a_4 b_2, \\
a_8 &= t_0 b_7 - a_5 b_2.
\end{align*}
\] (54)
Here the exact and first-order energy functionals are the same with \( E_{\mathcal{L}_0} \) and \( E_{\mathcal{L}_1} \), and the second-order energy functional \( E_{\mathcal{L}_2} \) reads
\[ E_{\mathcal{L}_2} = \frac{1}{2r^2} \left[ q \dot{t}^2 + (q - 1) \dot{r}^2 \left( (a_3^2 + a_4^2 + a_5^2) s - (a_3 b_5 + a_4 b_6 + a_5 b_7) \right)^2 \right] + V_2, \] (58)
We point out again that for the RN metric the component of angular momentum \( L_z \) and the exact energy functional \( E_{\mathcal{L}_0} \) are constants. But, the energy \( E \) and the energy functionals \( E_{\mathcal{L}_1}, E_{\mathcal{L}_2} \) are explicitly depend on the arc length \( s \). Further, using the first and second relations given in Eq. (54), we have found the energy \( E \) as
\[
E = \frac{2}{t_0} \left[ b_1 - \frac{1}{2} (a_3^2 + a_4^2 + a_5^2) + V_0 + V_1 + V_2 \right] \\
+ \frac{1}{t_0 r(s)^3} \left( -r_0 + \frac{q}{r(s)} \right) \left[ (a_3^2 + a_4^2 + a_5^2) s - (a_3 b_5 + a_4 b_6 + a_5 b_7)^2 \right],
\] (59)
where \( r(s) \) is of the form \( \frac{2}{\epsilon^2} \).

C. Approximate Symmetries of the Kerr Spacetime

In Boyer-Lindquist coordinates the Kerr spacetime is given in \( (28) \), where
\[
\begin{align*}
\Sigma &= r^2 + \frac{a^2}{c^2} \cos^2 \theta, \\
\Delta &= r^2 + \frac{a^2}{c^2} - \frac{2GMr}{c^2},
\end{align*}
\] (60)
with the mass $M$ and the spin parameter $a$. Introducing the small parameter $\epsilon$ as
\[
\frac{2GM}{c^2} = r_0 \epsilon, \quad \frac{a}{c} = \kappa \epsilon,
\] (61)
and retaining third power of $\epsilon$ and neglecting higher powers, the metric coefficients in the Kerr spacetime become
\[
g_{tt} = -1 + \frac{\epsilon r_0}{r} - \frac{r_0 k^2}{r^3} \epsilon^3 \cos^2 \theta + O(\epsilon^4),
\] (62)
\[
g_{rr} = 1 + \frac{\epsilon r_0}{r} + \left( r_0^2 - k^2 \sin^2 \theta \right) \frac{\epsilon^2}{r^2} + \left( r_0^2 - 2k^2 + k^2 \cos^2 \theta \right) \frac{\epsilon^3}{r^3} + O(\epsilon^4),
\] (63)
\[
g_{\theta\theta} = r^2 + \epsilon^2 k^2 \cos^2 \theta, \quad g_{\phi\phi} = -\frac{k r_0}{r} \epsilon^2 \sin^2 \theta + O(\epsilon^4),
\] (64)
\[
g_{\phi\phi} = \sin^2 \theta \left( r^2 + k^2 \epsilon^2 + \frac{r_0 k^2}{r} \epsilon^3 \sin^2 \theta \right) + O(\epsilon^4).
\] (65)

The third-order perturbed geodesic Lagrangian for the Kerr spacetime is given by
\[
\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1 + \epsilon^2 \mathcal{L}_2 + \epsilon^3 \mathcal{L}_3 + O(\epsilon^4),
\] (66)
where the exact, first-order, second-order and third-order geodesic Lagrangians are as follows:
\[
\mathcal{L}_0 = \frac{1}{2} \left[ -t^2 + r^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] - V_0, \quad \mathcal{L}_1 = \frac{r_0}{2r} \left( \dot{r}^2 + r^2 \right) - V_1,
\]
\[
\mathcal{L}_2 = \frac{1}{2} \left[ (r_0^2 - k^2 \sin^2 \theta) \dot{r}^2 + k^2 \cos^2 \theta \dot{\theta}^2 + \sin^2 \theta \left( k^2 \dot{\phi}^2 - \frac{2kr_0}{r} \dot{\phi} \right) \right] - V_2,
\]
\[
\mathcal{L}_3 = \frac{1}{2} \left[ -\frac{r_0 k^2}{r^3} \cos^2 \theta \dot{t}^2 + \left( \frac{r_0^2}{r^2} - 2k^2 + k^2 \cos^2 \theta \right) \dot{r}^2 + \frac{r_0 k^2}{r} \sin^4 \theta \dot{\phi}^2 \right] - V_3.
\]

There are only three ANS generators for the exact geodesic Lagrangian of the geodesic equations of Kerr spacetime such as
\[
K_0 = \partial_t, \quad K_3 = \partial_\phi, \quad Y_0 = \partial_s.
\] (67)
This is also pointed out by the Ref. [17], where they have proceeded the ANS to the second-order ANS of the geodesic Lagrangian for the Kerr spacetime. After proceeding the ANS to the third-order ANS, the solution of first-order ANS equations for constant potential yields three ANSs as in [67], and the two additional ANS generators arise for the second-order approximation as the following
\[
K_1 = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \quad K_2 = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi.
\] (68)
In addition to the symmetries $K_0, K_1, K_2, K_3$ and $Y_0$, the lost symmetries of the Kerr spacetime are appeared as the solution of the third-order ANS equations such that the symmetries $Y_1, \ldots, Y_7$ are the same as given in [56], and the remaining ones are
\[
Y_8 = s K_0, \quad with \quad f_3 = -t, \\
Y_9 = s Y_1, \quad with \quad f_3 = r \sin \theta \cos \phi, \\
Y_{10} = s Y_2, \quad with \quad f_3 = r \sin \theta \sin \phi, \\
Y_{11} = s Y_3, \quad with \quad f_3 = r \cos \theta, \\
Y_{12} = s (s \partial_s + t \partial_t + r \partial_r), \quad with \quad f_3 = \frac{1}{2} \left( \dot{r}^2 - t^2 - 2V_0 s^2 \right).
\] (69)
Hence, the number of third-order ANSs for the geodesic Lagrangian of the Kerr spacetime is seventeen. The first integrals associated with the three exact and first-order ANSs are
\[
I^1_0 = -E_{\mathcal{L}_0}, \quad I^2_0 = -\dot{t}, \quad I^3_0 = r^2 \sin^2 \theta \dot{\phi},
\] (70)
and
\[
I^1_1 = -(E_{\mathcal{L}_0} + E_{\mathcal{L}_1}), \quad I^2_1 = -\left( 1 - \frac{r_0}{r} \right) \dot{t}, \quad I^3_1 = r^2 \sin^2 \theta \dot{\phi},
\] (71)
where the exact and first-order energy functionals are, respectively,

\[ E_{L_0} = \frac{1}{2} \left[ -i^2 + r^2 + r^2(\theta^2 + \sin^2 \theta \phi^2) \right] + V_0 \quad \text{and} \quad E_{L_1} = \frac{r_0}{2r} \left( i^2 + r^2 \right) + V_1. \]  

(72)

The conservation laws for the second-order ANSs of Kerr spacetime are found as:

\[ I_2^1 = - (E_{L_0} + E_{L_1} + E_{L_2}) , \quad I_2^2 = \left( \frac{r_0}{r} - 1 \right) i - \frac{kr_0}{r} \sin^2 \theta \phi , \quad I_2^3 = \frac{kr_0}{r} i + (r^2 + k^2) \sin^2 \theta \phi , \]

\[ I_4^2 = r^2 [\cos \phi \dot{\theta} - \sin \theta \cos \theta \sin \phi \dot{\phi} ] , \quad I_5^2 = r^2 [\sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi} ] , \]  

(73)

where the second-order energy functional has the form

\[ E_{L_2} = \frac{1}{2} \left[ \frac{(r_0^2 - k^2 \sin^2 \theta)}{r^2} r^2 + k^2 \cos^2 \theta \dot{\theta}^2 + \sin^2 \theta \left( k^2 \phi^2 - \frac{2kr_0}{r} \phi \right) \right] + V_2. \]  

(74)

Further, the first integrals associated with the 17 third-order ANSs are

\[ I_3^1 = - (E_{L_0} + E_{L_1} + E_{L_2} + E_{L_3}) , \quad I_3^2 = \left( -1 + \frac{r_0}{r} - \frac{r_0 k^2}{r^3} \cos^2 \theta \right) i - \frac{r_0 k}{r} \sin^2 \theta \phi , \]

\[ I_3^3 = - \frac{r_0 k}{r} \sin^2 \theta i + \left( r^2 + k^2 + \frac{r_0 k^2}{r} \sin^2 \theta \right) \sin^2 \theta \phi , \]

\[ I_3^4 = r^2 [\cos \phi \dot{\theta} - \sin \theta \cos \theta \sin \phi \dot{\phi} ] , \quad I_3^5 = r^2 [\sin \phi \dot{\theta} + \sin \theta \cos \theta \cos \phi \dot{\phi} ] , \]

\[ I_3^6 = \sin \theta \cos \phi \dot{r} + r \cos \theta \cos \phi \dot{\phi} - r \sin \theta \sin \phi \phi , \quad I_3^7 = \sin \theta \sin \phi \dot{r} + r \cos \theta \sin \phi \dot{\phi} + r \sin \theta \phi , \]

\[ I_3^8 = \cos \theta \dot{\theta} - r \sin \theta \phi , \quad I_3^9 = - r \sin \theta \phi \dot{t} + i t_3^9 , \quad I_3^{10} = - r \sin \phi \dot{t} + i t_3^{10} \]

\[ I_3^{11} = - r \cos \theta \dot{t} + t_3^{11} , \quad I_3^{12} = - s \dot{t} + t , \quad I_3^{13} = - (E_{L_0} - V_0) s + \frac{1}{2} (- t \dot{t} + r \dot{r}) , \]

\[ I_3^{14} = - (E_{L_0} - V_0) s^2 + s (- t \dot{t} + r \dot{r}) - \frac{1}{2} (r^2 - t^2) , \]

\[ I_3^{15} = t_3^9 s - r \sin \theta \cos \phi , \quad I_3^{16} = t_3^7 s - r \sin \theta \sin \phi , \quad I_3^{17} = t_3^8 s - r \cos \theta , \]  

where the third-order energy functional \( E_{L_3} \) is

\[ E_{L_3} = \frac{1}{2} \left[ - \frac{r_0 k^2}{r^3} \cos^2 \theta \dot{\phi}^2 + \frac{(r_0^2 - 2k^2 + k^2 \cos^2 \theta)}{r^3} \phi^2 + \frac{r_0 k^2}{r} \sin^4 \theta \dot{\phi}^2 \right] + V_3. \]  

(76)

Defining \( I_3^2 = - E, I_3^{14} = L_z, I_3^{13} = a_1, I_3^5 = a_2, I_3^6 = a_3, I_3^7 = a_4, I_3^8 = a_5, I_3^9 = a_6, I_3^{10} = a_7, I_3^{11} = a_8, I_3^{12} = b_1, \)

\( I_3^{13} = b_2, I_3^{14} = b_3, I_3^{15} = b_4, I_3^{16} = b_5, I_3^{17} = b_6 \) and \( I_3^8 = b_7 \), one can find from the first integrals given by (75) that

\[ t(s) = t_0 s + b_2 , \quad r(s) = \sqrt{(a_3 s - b_5)^2 + (a_4 s - b_6)^2 + (a_5 s - b_7)^2} , \]

\[ \theta(s) = \tan^{-1} \left( \frac{\sqrt{(a_3 s - b_5)^2 + (a_4 s - b_6)^2}}{a_5 s - b_7} \right) , \quad \phi(s) = \tan^{-1} \left( \frac{a_4 s - b_6}{a_3 s - b_5} \right) , \]  

(77)

(78)

and

\[ E = t_0 \left[ 1 - \frac{m}{r} + \frac{m k^2}{r^5} (a_5 s - b_7)^2 \right] + \frac{kr_0 L_0}{r^3} , \]

\[ L_z = \frac{k r_0}{r^3} \left( \frac{k L_0}{r^2} - t_0 \right) \left[ r^2 - (a_5 s - b_7)^2 \right] + L_0 \left( 1 + \frac{k^2}{r^2} \right) , \]

\[ E_{L_0} + E_{L_1} + E_{L_2} + E_{L_3} = - b_1 , \]

\[ t_0 = - \frac{1}{b_2} (a_3 b_5 + a_4 b_6 + a_5 b_7 + 2 b_3) , \quad b_2 \neq 0 , \quad b_4 = \frac{1}{2} (b_2^2 - b_5^2 - b_6^2 - b_7^2) , \]

\[ a_4 = a_5 b_5 - a_6 b_7 , \quad a_2 = a_5 b_5 - a_4 b_7 , \quad a_6 = t_0 b_5 + a_3 b_2 , \quad a_7 = t_0 b_6 + a_4 b_2 , \quad a_8 = t_0 b_7 + a_5 b_2 , \]  

where \( t_0 \) is a constant of integration, \( b_2 \neq 0 \) and \( L_0 \equiv a_3 b_6 - a_4 b_5 \). Using (77)-(78) in Eqs. (72), (74) and (76), the
exact, first-order, second-order and third order energy functionals for the Kerr spacetime take the following forms:

\[
E_{\mathcal{L}_0} = \frac{1}{2} \left( -t_0^2 + a_3^2 + a_4^2 + a_5^2 \right) + V_0, \quad E_{\mathcal{L}_1} = \frac{r_0}{2r} \left( t_0^2 + \dot{r}^2 \right) + V_1, \quad E_{\mathcal{L}_2} = \frac{1}{2r^2} \left[ t_0^2 - k^2 + \frac{k^2}{r^2} (a_5 s - b_7)^2 \right] - \frac{k^2 L_0^2}{2r^2} \frac{a_5 (a_5 s - b_7)^2}{r^2 - (a_5 s - b_7)^2} + \frac{kr_0 L_0}{r^3} + V_2, \quad E_{\mathcal{L}_3} = \frac{1}{2r^3} \left[ r_0 k^2 \left( L_0^2 - t_0^2 (a_5 s - b_7)^2 \right) + \left( (r_0^2 - 2k^2)r^2 + k^2 (a_5 s - b_7)^2 \right) \dot{r}^2 \right] + V_3,
\]

where \( \dot{r} = \frac{1}{r} \left[ (a_3^2 + a_4^2 + a_5^2) s - (a_5 b_5 + a_4 b_6 + a_5 b_7) \right] \), and

\[
\dot{\theta}^2 = \frac{1}{r^4} \frac{r^2 - (a_5 s - b_7)^2}{(a_5 L_1 - b_7 (a_3^2 + a_4^2)) s - a_5 (b_6^2 + b_7^2) + b_7 L_1)^2},
\]

with \( L_1 = a_5 b_5 + a_4 b_6 \). The photon orbits staying at the extrema, i.e., the circular equatorial orbits or the spherical photon orbits, imply \( \dot{r} = 0 \) and \( \ddot{r} = 0 \), from which one can find the following constraint equations:

\[
a_3^2 + a_4^2 + a_5^2 = 0, \quad a_5 b_5 + a_4 b_6 + a_5 b_7 = 0.
\]

The latter equations yield \( r = \sqrt{b_5^2 + b_6^2 + b_7^2} = \text{const.} \) Then the approximate energy functionals for the photon orbits read

\[
E_{\mathcal{L}_0} = -\frac{t_0^2}{2} + V_0, \quad E_{\mathcal{L}_1} = \frac{r_0 t_0^2}{2r} + V_1, \quad E_{\mathcal{L}_2} = \frac{k^2}{2r^2} (a_5 s - b_7)^2 \dot{\theta}^2 + \frac{k^2 L_0^2}{2r^2} \frac{a_5 (a_5 s - b_7)^2}{r^2 - (a_5 s - b_7)^2} - \frac{kr_0 L_0}{r^3} + V_2, \quad E_{\mathcal{L}_3} = \frac{r_0 k^2}{2r^3} \left[ L_0^2 - t_0^2 (a_5 s - b_7)^2 \right] + V_3,
\]

where \( \dot{\theta} = a_5 (b_6^2 + b_7^2 + b_7^2)/(r^2 - (a_5 s - b_7)^2) \). It is seen from the above relations that if \( a_5 = 0 \), then \( \theta \) is a constant and so all energy functionals become constants.

**IV. SUMMARY AND CONCLUSION**

In this study, for the Gödel-type, Schwarzschild, Reissner-Nordström and Kerr spacetimes, we reviewed the Noether symmetries of the corresponding canonical geodesic Lagrangians. To get the approximate Lagrangian in the background of some of those spacetimes, we set up a perturbed geodesic Lagrangian in terms of metric coefficients to use it in the ANS approach. Thus we considered the latter perturbed Lagrangians and used it to calculate and classify ANS generators by considering the ANS conditions for the Schwarzschild, Reissner-Nordström and Kerr spacetimes. In the previous section, the ANSs of the Schwarzschild, Reissner-Nordström and Kerr spacetimes have been calculated, and used to integrate the geodesic equations of motion by means of the first integrals that are due to the existence of ANS generators including the KVs. The analytical solutions for the perturbed geodesic equations of these black hole spacetimes have been derived by the aid of the first integrals associated with ANSs in complete detail.

Furthermore we note that the geodesic Lagrangian involves the potential function \( V(x^i) \) which is an unknown quantity. Using the Noether symmetry approach or the approximate Noether symmetry approach for the geodesic Lagrangian, the form of the unknown potential function may be determined.

**Acknowledgments**

The author would like to thank the organizers for the successful meeting “International Conference on Gravitation and Cosmology (PUICGC)”, The University of Punjab, Department of Mathematics, Lahore-Pakistan held in November 22-25, 2021. I dedicate this study to Prof. Dr. Ghulam Shabbir.

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