Generic properties of purely $\alpha$-Hausdorff continuous operators

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Abstract

It is shown that the time-average quantum return probability of (Baire) generic states of systems with purely $\alpha$-Hausdorff continuous spectrum has an oscillating behaviour between a (maximum) fast power-law decay and a (minimum) slow power-law decay. As an application, a new result on the quantum dynamics of Schrödinger operators with limit-periodic potentials is presented.

1 Introduction

Let $T$ be a self-adjoint operator in a separable complex Hilbert space $\mathcal{H}$. The (time-average) quantum return probability, which gives the (time-average) probability of finding the system at time $t > 0$ in its initial state $\psi \in \mathcal{H}$, is given by

$$W_T^\psi(t) := \frac{1}{t} \int_0^t |\langle \psi, e^{-isT} \psi \rangle|^2 ds.$$ 

The study of the large time behaviour of this dynamic quantity plays an important role in spectral theory and quantum dynamics; for instance, “good” continuity properties of the spectral measure $\mu_\psi^T$ of the pair $(T, \psi)$ are associated with “good” transport properties, in the sense of the following theorem. Recall that $\mu_\psi^T$ is uniformly $\alpha$-Hölder continuous (denoted $U\alpha$H) if there exists a constant $C > 0$ such that, for every interval $I \subset \mathbb{R}$ with $|I| < 1$, one has $\mu_\psi^T(I) < C|I|^\alpha$, where $| \cdot |$ denotes the Lebesgue measure.

Theorem 1.1 (Theorem 3.1 in [20]). Let $\alpha \in [0, 1]$.

(i) If $\mu_\psi^T$ is $U\alpha$H, then there exists a positive constant $C_\psi$ such that, for every $t > 0$,

$$W_T^\psi(t) < C_\psi t^{-\alpha}.$$

(ii) If there exists a positive constant $C_\psi$ such that, for every $t > 0$,

$$W_T^\psi(t) < C_\psi t^{-\alpha},$$

then $\mu_\psi^T$ is $U\frac{\alpha}{2}$H.

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Part (i) of Theorem 1.1 is, indeed, a particular case of a result by Strichartz [21].

This notion of $\alpha$-Hölder continuity is, in some sense, related to the notion of $\alpha$-Hausdorff continuity. Given $\alpha \in [0, 1]$, recall that a positive Borel measure $\mu$ on $\mathbb{R}$ is $\alpha$-Hausdorff continuous (denoted $\alpha c$) if $\mu(\Lambda) = 0$ for any Borel set $\Lambda \subset \mathbb{R}$ with $h^\alpha(\Lambda) = 0$, and that $\mu$ is $\alpha$-Hausdorff singular (denoted $\alpha s$) if there exists a Borel set $\Lambda$ with $h^\alpha(\Lambda) = 0$ such that $\mu(\mathbb{R}\setminus\Lambda) = 0$, where $h^\alpha$ denoted the $\alpha$-dimensional Hausdorff (exterior) measure [18].

Definition 1.1. Let $T$ be a self-adjoint operator. We say that $T$ has purely $\alpha$-Hausdorff continuous spectrum if, for each $\psi \in \mathcal{H}$, the spectral measure $\mu^T_\psi$ of the pair $(T, \psi)$ is $\alpha c$-Hausdorff continuous.

For $\alpha \in [0, 1]$ and a self-adjoint operator $T$, the $\alpha$-Hausdorff continuous and $\alpha$-Hausdorff singular subspaces induced by $T$ are defined, respectively, by

$$H^T_{\text{ac}} := \{ \psi \in \mathcal{H} | \mu^T_\psi \text{ is } \alpha c \} \quad \text{and} \quad H^T_{\text{as}} := \{ \psi \in \mathcal{H} | \mu^T_\psi \text{ is } \alpha s \}.$$  

Such subspaces are closed (in norm), mutually orthogonal, invariant under $T$ and $\mathcal{H} = H^T_{\text{ac}} \oplus H^T_{\text{as}}$ (see [20] for details). Set

$$H^T_{\text{uH}}(\alpha) := \{ \psi \in \mathcal{H} | \mu^T_\psi \text{ is } \text{UH} \alpha \}.$$  

The next result, due to Last, establishes the aforementioned connection between $\text{UH}^\alpha$ and $\alpha c$ measures.

Theorem 1.2. Let $\alpha \in [0, 1]$ and let $T$ be a bounded self-adjoint operator. Then, $H^T_{\text{uH}}(\alpha)$ is a subspace of $\mathcal{H}$ and

$$\overline{H^T_{\text{uH}}(\alpha)} = H^T_{\text{ac}},$$

where $\overline{\cdot}$ denotes the norm-closure in $\mathcal{H}$.

Suppose that $T$ has purely $\alpha$-Hausdorff continuous spectrum and let $\psi \in \mathcal{H}$. Taking into account Theorems 1.1 and 1.2, one may ask the following questions about the (Baire) typical behaviour of the spectral measure $\mu^T_\psi$:

(i) What can be said about the modulus of continuity of the distribution of $\mu^T_\psi$?

(ii) What can be said about the asymptotic behaviour of $W^T_\psi(t)$ as $t \to \infty$?

In this work, we show that for a (Baire) typical $\psi \in \mathcal{H}$, $\mu^T_\psi$ does not have any kind of Hölder regularity and that the (time-average) quantum return probability $W^T_\psi(t)$ has an oscillating behaviour between a (maximum) fast power-law decay and a (minimum) slow power-law decay. More precisely, we prove the following result.

Theorem 1.3. Let $\alpha \in [0, 1]$ and $T$ be a bounded self-adjoint operator with purely $\alpha$-Hausdorff continuous spectrum. Then, the set of $\psi \in \mathcal{H}$ such that, for all $k \in \mathbb{N}$,

(i) $\mu^T_\psi$ is not $\text{U}(1/k)\mathcal{H}$,

(ii) $$\liminf_{t \to \infty} t^{\alpha-1/k}W^T_\psi(t) = 0 \quad \text{and} \quad \limsup_{t \to \infty} t^{1/k}W^T_\psi(t) = \infty,$$

is generic in $\mathcal{H}$ (that is, such set contains a $G_\delta$ dense subset of $\mathcal{H}$).
**Remark 1.1.** We note that due to Theorems 1.1 and 1.2, the states in the $\alpha$-Hausdorff continuous subspace $\mathcal{H}_{\alpha c}$ may be well approximated by states for which $W^T_\psi(t)$ has a fast power-law decay, that is,

$$\mathcal{H}_{\alpha c} \subset \{ \psi \mid \sup_{t>0} t^\alpha W^T_\psi(t) < \infty \},$$

where $\sup$ denotes the norm-closure in $\mathcal{H}$. Nevertheless, Theorem 1.3 shows that if $\eta \in \{ \psi \mid \sup_{t>0} t^\alpha W^T_\psi(t) < \infty \}$ and $\xi \in \mathcal{H}_{\alpha c}$, then the asymptotic behaviours of $W^T_\eta(t)$ and $W^T_\xi(t)$ can be quite different; in particular, it is not possible to remove the closure in the inclusion above.

Before we proceed, some words about notation are required: $\mathcal{H}$ will always denote an infinite dimensional and separable complex Hilbert space, and if $T$ is a self-adjoint operator in $\mathcal{H}$, then $\mathcal{D}(T)$, $\text{rng} T$ and $\sigma(T)$ denote, respectively, its domain, range and spectrum. For each Borel set $\Lambda \subset \mathbb{R}$, $P^T(\Lambda)$ represents the spectral resolution of $T$ over $\Lambda$. $\mu$ always indicates a finite positive Borel measure on $\mathbb{R}$. For each $x \in \mathbb{R}$ and each $\epsilon > 0$, $B(x; \epsilon)$ denotes the open interval $(x - \epsilon, x + \epsilon)$.

In order to study the asymptotic behaviour of $W^T_\psi(t)$, we recall the notions of lower and upper $q$-generalized fractal dimensions of $\mu$, $q > 0$ and $q \neq 1$, defined respectively as

$$D^-_\mu(q) := \liminf_{\epsilon \downarrow 0} \frac{\ln \left( \int \mu(B(x; \epsilon))^{q-1} d\mu(x) \right)}{(q - 1) \ln \epsilon} \quad \text{and} \quad D^+_\mu(q) := \limsup_{\epsilon \downarrow 0} \frac{\ln \left( \int \mu(B(x; \epsilon))^{q-1} d\mu(x) \right)}{(q - 1) \ln \epsilon},$$

with the integrals taken over the support of $\mu$.

The next identities, extracted from [6, 19], show that the correlation dimensions (i.e., generalized fractal dimensions for $q = 2$) rule the time-average power-law decay rates of $W^T_\psi(t)$, that is,

$$\liminf_{t \to \infty} \frac{\ln W^T_\psi(t)}{\ln t} = -D^+_\mu(2),$$

$$\limsup_{t \to \infty} \frac{\ln W^T_\psi(t)}{\ln t} = -D^-_\mu(2). \quad (1)$$

As a direct consequence of Theorem 1.3 and the above identities, one has the following result.

**Corollary 1.1.** Let $\alpha \in [0, 1]$ and $T$ be a bounded self-adjoint operator with purely $\alpha$-Hausdorff continuous spectrum. Then,

$$\{ \psi \in \mathcal{H} \mid D^+_\mu(2) \geq \alpha \quad \text{and} \quad D^-_\mu(2) = 0 \}$$

is generic in $\mathcal{H}$.

**Remark 1.2.** For $\alpha = 1$ (that is, when $T$ has purely absolutely continuous spectrum), Theorem 1.3 states that the typical dynamical situation is characterized by extreme values of the power-law decay rates of the (time-average) quantum return probabilities. Here, for a typical vector $\psi$ one has $D^-_\mu(2) = 0$, and so the return probability decays (power-law) as slow as possible for a sequence of time going to infinity, though the spectrum is purely absolutely continuous. Note that Corollary 1.1 for purely absolutely continuous spectrum can be seen as the counterpart of the following situation detailed in [2]: for operators with pure point spectrum dense in an interval, there exists a generic set of states $\psi$ whose spectral measures have maximal upper generalized dimension $D^+_\mu(q) = 1$ ($0 < q < 1$); such states are, therefore, (weakly) delocalized (see [2, 4, 15, 17]).
There are in the literature (see, for instance, [9, 10, 12]), numerous examples for which our general results apply. As an illustration, we present examples of Schrödinger operators for which the hypotheses of Theorem 1.3 and Corollary 1.1 are satisfied.

Example 1.1 (Free Hamiltonian). Let $H_0 : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be given by the law $(H_0 \psi)(x) = -\psi''(x)$, and set $H := H_0P_{H_0}([0, 1])$ (a bounded self-adjoint operator). $H_0$ has purely absolutely continuous spectrum; see [14] for details.

Example 1.2 (Sturmian operators). Consider the class of Schrödinger operators with sturmian potentials, that is, discrete one-dimensional Schrödinger operators, denoted by $H_{\lambda, \theta, \beta}$, on $\ell^2(\mathbb{Z})$ whose actions are given by

$$(H_{\lambda, \theta, \beta} \psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda v_n^{\theta, \beta} \psi_n,$$

where

$$\lambda v_n^{\theta, \beta} = \chi_{[1-\theta, 1)}(n\theta + \beta \mod 1),$$

with coupling constant $\lambda \in \mathbb{R}$, rotation number $\theta \in \mathbb{T}$ and phase $\beta \in \mathbb{T}$.

Let $\theta \in \mathbb{T}$ be a (irrational) number of bounded density. Then for every $\lambda \neq 0$, there exists $0 < \alpha(\lambda, \theta) < 1$ such that for every $\beta \in \mathbb{T}$, $H_{\lambda, \theta, \beta}$ has purely $\alpha$-Hausdorff continuous spectrum; see [12] for details.

Application to limit-periodic operators. Let the class of Schrödinger operators with limit-periodic potentials, that is, discrete one-dimensional ergodic Schrödinger operator defined on $\ell^2(\mathbb{Z})$ by the law

$$(H^{w}_{g, \tau} \psi)_n = \psi_{n+1} + \psi_{n-1} + v_n(w)\psi_n,$$  \hfill (3)

where

$$v_n(w) = g(\tau^n(w));$$  \hfill (4)

here, $w$ belongs to a Cantor group $\Omega$ (an abelian topological group which is compact, totally disconnected and perfect, i.e., it has no isolated points), $\tau : \Omega \to \Omega$ is a minimal translation on $\Omega$ and $g : \Omega \to \mathbb{R}$ is a continuous sampling function (that is, $g \in C(\Omega, \mathbb{R})$, where $C(\Omega, \mathbb{R})$ is endowed of the norm of uniform convergence). For more details, see [5, 7, 10]. For each $w \in \Omega$, let $X_w$ be the set of limit-periodic operators $H^{w}_{g, \tau}$ given by (3) and (4), endowed with the metric

$$d(H^{w}_{g, \tau}, H^{w'}_{g', \tau}) = \|g - g'\|_{\infty}.$$

Theorem 1.4. There exists a generic set $\mathcal{M} \subset \ell^2(\mathbb{Z})$ so that, for each $\psi \in \mathcal{M}$, the set of operators $H^{w}_{g, \tau} \in X_w$ such that, for each $k \geq 1$,

$$\lim_{t \to \infty} t^{1-1/k}W^{\psi}_{\tau}(t) = 0 \quad \text{and} \quad \limsup_{t \to \infty} t^{1/k}W^{\psi}_{\tau}(t) = \infty,$$

is generic in $X_w$.

Proof. Let $C_{ac} := \{T \in X_w \mid \sigma(T) \text{ is with purely absolutely continuous}\}$. Since $(X_w, d)$ is a separable space, $C_{ac} \subset X_w$ (endowed with the induced topology) is also separable. Let $(T_j)$ be a
dense sequence in $C_{ac}$ (being, therefore, dense in $X_w$, since $C_{ac}$ is dense in $X_w$ by Theorem 1.1 in [11]). If $\mu_{\psi}'$ denotes the spectral measure of the pair $(T_j, \psi)$, it follows from Corollary 1.1 that

$$M = \bigcap_j \{ \psi \in \ell^2(\mathbb{Z}) \mid D^+_{\mu_{\psi}'} (2) = 1 \text{ and } D^-_{\mu_{\psi}'} (2) = 0 \}$$

is generic in $\ell^2(\mathbb{Z})$. Since, by Proposition 2.4 in [4], for every $\psi \in M$,

$$X_{01}(\psi) := \{ T \mid D^+_{\mu_{\psi}'} (2) = 1 \text{ and } D^-_{\mu_{\psi}'} (2) = 0 \} \supset \cup_j \{ T_j \}$$

is a set $G_\delta$ in $X_w$, it follows that $X_{01}(\psi)$ is also a generic in $X_w$.

The proof is now a direct consequence of identities (1) and (2).

\[ \square \]

**Remark 1.3.** It is worth underlying that if it is true that there exists a dense subset of $C(\Omega, \mathbb{R})$ such that $H^w$ has pure point spectrum for every $w \in \Omega$, then the result in Theorem 1.4 will be a direct consequence of Theorem 1.2 in [4]. But, to the best of our knowledge, this is still an open problem, even for (Haar-)a.e. $w \in \Omega$.

The organization of this paper is as follows. In Section 2, we present a study of the typical local scale spectral properties of self-adjoint operators with purely continuous spectrum. More precisely, we show that the local scale properties of a purely continuous spectral measure, typically in Baire sense, display a erratic behaviour (see Theorem 2.1 ahead). In Section 3, we use such result to prove Theorem 1.3.

## 2 Local properties of spectral measures and dynamics

Recall that the pointwise lower and upper scaling exponents of the measure $\mu$ at $x \in \mathbb{R}$ are defined, respectively, by

$$d^-_\mu (x) := \liminf_{\epsilon \downarrow 0} \frac{\ln \mu(B(x; \epsilon))}{\ln \epsilon} \quad \text{and} \quad d^+_\mu (x) := \limsup_{\epsilon \downarrow 0} \frac{\ln \mu(B(x; \epsilon))}{\ln \epsilon}$$

if, for all small enough $\epsilon > 0$, $\mu(B(w, \epsilon)) > 0$; one sets $d^\pm_\mu (x) := \infty$, otherwise.

Note that, for each $x \in \mathbb{R}$ and each $\epsilon > 0$,

$$\int_{\mathbb{R}} e^{-2t|x-y|} d\mu(y) \geq \int_{B(x; 1/t)} e^{-2t|x-y|} d\mu(y) \geq e^{-2t} \mu(B(x; 1/t)).$$

On the other hand, for each $0 < \delta < 1$ and each $t > 0$,

$$\int_{\mathbb{R}} e^{-2t|x-y|} d\mu(y) = \int_{\mu(B(x; \frac{1}{1-t}))} e^{-2t|x-y|} d\mu(y) + \int_{\mu(B(x; \frac{1}{1-t}))^c} e^{-2t|x-y|} d\mu(y)$$

$$\leq \mu(B(x, 1/(1-t)\delta)) + e^{-t\delta} \mu(\mathbb{R}).$$

Thus, at least when $\mu$ has a certain local regularity (with respect to the Lebesgue measure), we expect that $\int_{\mathbb{R}} e^{-2t|x-y|} d\mu(y)$ and $\mu(B(x; 1/t))$ are asymptotically comparable as $t \to \infty$. In this sense, the following identities are expected:

$$\liminf_{t \to \infty} \frac{\ln \int_{\mathbb{R}} e^{-2t|x-y|} d\mu(y)}{\ln t} = -d^+_\mu (w),$$

(6)
\[
\limsup_{t \to \infty} \frac{\ln \left[ \int_{\mathbb{R}} e^{-2t|w-y|}d\mu(y) \right]}{\ln t} = -d^-(\nu). \quad (7)
\]

Indeed, these identities were proven in [3] (note that since it is not possible to compare directly the two terms on the right-hand side of (5), some caution should be exercised when checking (6) and (7)). We use such identities in the proof of Theorem 2.1.

**Theorem 2.1.** Let \( T \) be a bounded self-adjoint operator with purely continuous spectrum. Then, for each \( x \in \sigma(T) \), the set

\[
\mathcal{G}(x) := \{ \psi \in \mathcal{H} \mid d^{-}_{\nu}(x) = 0 \text{ and } d^{+}_{\nu}(x) = \infty \}
\]

is generic in \( \mathcal{H} \).

**Remark 2.1.** Let \( 0 \neq \psi \in \mathcal{H} \). Since, by dominated convergence, for every \( t > 0 \) the mapping

\[
\sigma(T) \ni x \mapsto \int_{\mathbb{R}} e^{-2t|x-y|}d\mu_{\psi}(y)
\]

is continuous, it follows that

\[
\mathcal{J}_{\psi} := \{ x \in \sigma(T) \mid d^{+}_{\mu_{\psi}}(x) = \infty \text{ and } d^{-}_{\mu_{\psi}}(x) = 0 \} = A_- \cap A_+,
\]

is a \( G_\delta \) set in \( \sigma(T) \), since

\[
A_- = \bigcap_{l \geq 1} \bigcap_{n \geq 1} \bigcap_{k \geq 1} \bigcup \{ x \in \sigma(T) \mid t^l \int_{\mathbb{R}} e^{-2t|x-y|}d\mu_{\psi}(y) < \frac{1}{n} \}
\]

and

\[
A_+ = \bigcap_{l \geq 1} \bigcap_{n \geq 1} \bigcap_{k \geq 1} \bigcup \{ x \in \sigma(T) \mid t^{1/l} \int_{\mathbb{R}} e^{-2t|x-y|}d\mu_{\psi}(y) > n \}.
\]

Now let \( (x_n)_{n \in \mathbb{N}} \subset \sigma(T) \) be a dense sequence in \( \sigma(T) \). So, if

\[
\psi \in \bigcap_{n} \mathcal{G}(x_n) = \{ \phi \mid d^{-}_{\mu_{\psi}}(x_n) = 0 \text{ and } d^{+}_{\mu_{\psi}}(x_n) = \infty, \text{ for each } n \in \mathbb{N} \},
\]

it follows that \( \mathcal{J}_{\psi} \) is generic in \( \sigma(T) \). We note that this remark is particularly interesting when \( T \) has purely absolutely continuous spectrum, since it shows the striking difference between the typical behaviour of \( d^{\pm}_{\mu_{\psi}} \) from the topological and measure points of view; namely, if \( \mu_{\psi} \) is purely absolutely continuous, then \( \mu_{\psi}^{T\text{-ess}} \inf d^{-}_{\mu_{\psi}} = 1 \) (see [18] for details).

**Example 2.1.** Let \( \Delta : \mathcal{H}^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \ n \geq 5, \) be the self-adjoint realization of the Laplacian. \( \Delta \) has purely absolutely continuous spectrum [14], and so, the hypotheses of Theorems 1.3 and 2.1 are satisfied. We note that, in this example, reasonably localized functions, in general, do not belong to the sets given in the previous results. For instance, \( \mathcal{G}(0) \cap \mathcal{S}(\mathbb{R}^n) = \emptyset \), where \( \mathcal{S}(\mathbb{R}^n) \) denote the Schwartz space. Namely, consider the following statement.

**Claim:** If \( n \geq 5 \), then \( \mathcal{S}(\mathbb{R}^n) \) is a subspace of \( \text{rng} \Delta \).
Proof. Let $M_0 : D(M_0) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $D(M_0) = \{u \in L^2(\mathbb{R}^n) \mid |x|^2 u \in L^2(\mathbb{R}^n)\}$, be defined, for each $u \in D(M_0)$, by the action $(M_0 u)(x) = |x|^2 u$, with $x \in \mathbb{R}^n$.

For each $f \in \mathcal{S}(\mathbb{R}^n)$, let $u(x) = \frac{f(x)}{|x|^2}$, $u(0) = 0$. We assert that $u \in L^2(\mathbb{R}^n)$; indeed,

$$
\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^4} \, dx \leq \|f\|_\infty^2 \int_{B(0;1)} \frac{1}{|x|^4} \, dx + \int_{B(0;1)^c} \frac{|f(x)|^2}{|x|^4} \, dx \leq \|f\|_\infty^2 w_n \int_0^r \frac{r^{n-1}}{x^4} \, dr + \|f\|_{L^2(\mathbb{R}^n)}^2 < \infty,
$$

since $n \geq 5$. Therefore, $u \in D(M_0)$ and $M_0 u = f$, from which follows that $\mathcal{S}(\mathbb{R}^n) \subset \text{rng} \, M_0$.

Finally, consider the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Then, since $\mathcal{F}^{-1} (\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}(\mathbb{R}^n)$ and $-\Delta = \mathcal{F}^{-1} M_0 \mathcal{F}$, the result follows. \hfill $\Box$

It has been shown in [1] (Proposition 3.1) that for every negative self-adjoint operator $T$, each $\eta \in \text{rng} \, T$ and each $\psi \in T^{-1}\{\eta\}$, one has for $t > 0$,

$$
\|e^{tT} \eta\|_{\mathcal{H}} \leq \frac{\|\psi\|_H}{t e}.
$$

Thus, by Claim, if $n \geq 5$ then, for each $f \in \mathcal{S}(\mathbb{R}^n)$ and $t > 0$,

$$
\|e^{t\Delta} f\|_{L^2(\mathbb{R}^n)} \leq \frac{\|u\|_{L^2(\mathbb{R}^n)}}{t e},
$$

and therefore, by the Spectral Theorem and (7), it follows that for each $f \in \mathcal{S}(\mathbb{R}^n)$, $d^{-\infty}_{\mu^f}(0) \geq 2$; consequently, $\mathcal{G}(0) \cap \mathcal{S}(\mathbb{R}^n) = \emptyset$.

Remark 2.2. Taking into account Example 2.1, it is an interesting open problem to obtain explicit examples of Schrödinger operators for which the sets presented in the statements of Theorems 1.3, 2.1 and Corollary 1.1 contain reasonably localized functions. We recall [16] that given a finite Borel measure $\mu$ on $[a, b] \subset \mathbb{R}$, there exists a continuous half-line Schrödinger operator for which the spectral measure coincides with $\mu$ on $[a, b]$; therefore, such operators do exist.

Proof of Theorem 2.1

The lemma below plays a fundamental role in the proof of Theorem 2.1.

Lemma 2.1 (Lemma 2.1 in [3]). Let $T$ be a negative self-adjoint operator with $0 \in \sigma(T)$, and let $\alpha : \mathbb{R}_+ \to (0, \infty)$ be such that

$$
\lim_{t \to \infty} \alpha(t) = \infty.
$$

Then, there exist $\eta \in \mathcal{H}$ and a sequence $t_j \to \infty$ such that, for sufficiently large $j$,

$$
\mu_\eta^T(B(0; 1/t_j)) \geq \frac{1}{\alpha(t_j)}.
$$

We divide the proof in 4 steps.

Step 1. Let us show that, for each $\rho > 0$ and each $x \in \sigma(T)$,

$$
\{\psi \in \mathcal{H} \mid d^+_{\mu^f}(x) \geq d^-_{\mu^f}(x) \geq \rho\}
$$
is dense in $\mathcal{H}$. Namely, let for each $n \in \mathbb{N}$ and each $y \in \mathbb{R}$,
\[
f_{n,\rho}(x, y) := \left(1 - e^{-n|x-y|^\rho}\right)^{1/2},
\]
and for each $\psi \neq 0$, let $\psi_n := f_{n,\rho}(x, T)\psi$, where $f_{n,\rho}(x, T) := P_T(f_{n,\rho}(x, \cdot))$. Note that since $\mu_\psi^T$ is purely continuous, one gets by the Spectral Theorem and dominated convergence that
\[
\|\psi_n - \psi\|^2 = \|f_{n,\rho}(x, T)\psi - \psi\|^2 \leq \int_{\mathbb{R}} \left(\left(1 - e^{-n|x-y|^\rho}\right)^{1/2} - 1\right)^2 d\mu_\psi^T(y) \leq \mu_\psi^T(\{x\}) + \int_{\mathbb{R}\setminus\{x\}} \left(\left(1 - e^{-n|x-y|^\rho}\right)^{1/2} - 1\right)^2 d\mu_\psi^T(y)
\]
as $n \to \infty$, that is, $\psi_n \to \psi$ in $\mathcal{H}$.

Now, by Fubini’s Theorem,
\[
\int_{\mathbb{R}} e^{-2t|x-y|} d\mu_\psi^T(y) = \int_{\mathbb{R}} e^{-2t|x-y|} d\mu_{f_{n,\rho}(x, T)\psi}^T(y) = \int_{\mathbb{R}} e^{-2t|x-y|} |f_{n,\rho}(x, y)|^2 d\mu_\psi^T(y) \leq \rho \int_{\mathbb{R}} \frac{(1 - e^{-n|x-y|^\rho})}{|x-y|^\rho} d\mu_\psi^T(y) \leq \frac{n\rho}{2\rho t^\rho} \|\psi\|^2.
\]
Thus, it follows from identity (7) that for each $n \geq 1$, $d_{\mu_\psi}^T(x) \geq \rho$, and therefore, that
\[
\{ \psi \in \mathcal{H} \mid d_{\mu_\psi}^T(x) \geq \rho \}
\]
is dense in $\mathcal{H}$.

**Step 2.** Let us show that for every $x \in \sigma(T)$, there exists $\eta \in \mathcal{H}$ such that $d_{\mu_\psi}^T(x) = 0$. Set $L_x = (-\infty, x] \cap \sigma(T)$, $T_x = T P_T(L_x)$ and $T_x^0 = T_x - xI$. So, by Lemma 2.1, there exist $\eta \in \mathcal{H}$ and $\varepsilon_j \to 0$ such that, for sufficiently large $j$,
\[ \mu^T_\eta (B(0; \varepsilon_j)) \geq \frac{1}{-\ln(\varepsilon_j)} \Rightarrow \mu^T_\eta (B(x; \varepsilon_j)) \geq \frac{1}{-\ln(\varepsilon_j)} \]

\[ \Rightarrow \ln \left( \mu^T_\eta (B(x; \varepsilon_j)) \right) \geq \ln \left( \mu^T_\eta (B(x; \varepsilon_j) \cap L_x) \right) = \ln \left( \mu^T_\eta (B(x; \varepsilon_j)) \right) \geq \ln \left( \frac{1}{-\ln(\varepsilon_j)} \right) \]

\[ \Rightarrow \frac{\ln \left( \mu^T_\eta (B(x; \varepsilon_j)) \right)}{\ln \varepsilon_j} \leq \frac{\ln \left( \frac{1}{-\ln(\varepsilon_j)} \right)}{\ln \varepsilon_j} \Rightarrow d^-_{\mu^T_\eta} (x) = 0. \]

**Step 3.** Let us show that for every \( x \in \sigma(T) \),

\[ \{ \psi \in \mathcal{H} \mid d^-_{\mu^T_\psi} (x) = 0 \} \]

is dense in \( \mathcal{H} \). Namely, let \( x \in \sigma(T) \) and set, for every \( n \geq 1 \),

\[ S_n := (-\infty, x - \frac{1}{n}) \cup \{x\} \cup (x + \frac{1}{n}, \infty). \]

Set also, for each \( \psi \in \mathcal{H} \) and each \( n \geq 1 \),

\[ \psi_n := P^T(S_n)\psi + \frac{1}{n} \eta, \]

where \( \eta \) is given by **Step 2**. One has that \( \psi_n \rightarrow \psi \) in \( \mathcal{H} \), since \( P^T(S_n) \rightarrow 1 \) in the strong sense. Moreover, for each \( n \geq 1 \) and each \( 0 < \varepsilon < \frac{1}{n} \), one has

\[ \mu^T_{\psi_n} (B(x; \varepsilon)) = \langle P^T(B(x; \varepsilon))\psi_n, \psi_n \rangle \]

\[ = \langle P^T(B(x; \varepsilon))P^T(S_n)\psi, \psi_n \rangle + \frac{1}{n} \langle P^T(B(x; \varepsilon))\eta, \psi_n \rangle \]

\[ = \langle P^T(B(x; \varepsilon) \cap S_n)\psi, \psi_n \rangle + \frac{1}{n} \langle P^T(B(x; \varepsilon))\eta, \psi_n \rangle \]

\[ = \langle P^T(\{x\})\psi, \psi_n \rangle + \frac{1}{n} \langle P^T(B(x; \varepsilon))\eta, P^T(S_n)\psi \rangle + \frac{1}{n} \langle P^T(B(x; \varepsilon))\eta, \psi \rangle \]

\[ = \langle P^T(\{x\})\psi, \psi_n \rangle + \frac{1}{n} \langle P^T(\{x\})\eta, \psi \rangle + \frac{1}{n} \langle P^T(B(x; \varepsilon))\eta, \eta \rangle \]

\[ = \frac{1}{n^2} \langle P^T(B(x; \varepsilon))\eta, \eta \rangle \]

\[ = \frac{1}{n^2} \mu^T_\eta (B(x; \varepsilon)), \]

from which follows that

\[ d^-_{\mu^T_{\psi_n}} (x) = \liminf_{\varepsilon \downarrow 0} \frac{\ln \mu^T_{\psi_n} (B(x; \varepsilon))}{\ln \varepsilon} = \liminf_{\varepsilon \downarrow 0} \frac{\ln \mu^T_\eta (B(x; \varepsilon))}{\ln \varepsilon} = d^-_{\mu^T_\psi} (x) = 0. \]

Thus,

\[ \{ \psi \in \mathcal{H} \mid d^-_{\mu^T_\psi} (x) = 0 \} \]

is dense in \( \mathcal{H} \).

**Step 4.** Finally, in this step, we finish the proof. Since, for each \( x \in \mathbb{R} \) and each \( t > 0 \), the mapping

\[ \mathcal{H} \ni \psi \mapsto \int_{\mathbb{R}} e^{-2t|x-y|}d\mu^T_\psi (y) = \langle g_t(T, x)\psi, \psi \rangle, \]

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with \( g_t(y, x) = e^{-2t|x-y|} \), is continuous, it follows that for every \( x \in \mathbb{R} \), each one of the sets

\[
B_-(x) := \{ \psi \in \mathcal{H} \mid d^+_{\mu^T_\psi}(x) = \infty \} = \bigcap_{l \geq 1} \bigcap_{n \geq 1} \bigcap_{k \geq 1} \bigcup_{t \geq k} \left\{ \psi \in \mathcal{H} \mid t_l \int_{\mathbb{R}} e^{-2t|x-y|} d\mu^T_\psi(y) < 1/n \right\}
\]

and

\[
B_+(x) := \{ \psi \in \mathcal{H} \mid d^-_{\mu^T_\psi}(x) = 0 \} = \bigcap_{l \geq 1} \bigcap_{n \geq 1} \bigcap_{k \geq 1} \bigcup_{t \geq k} \left\{ \psi \in \mathcal{H} \mid t_l \int_{\mathbb{R}} e^{-2t|x-y|} d\mu^T_\psi(y) > n \right\}
\]

is a \( G_\delta \) set in \( \mathcal{H} \). Thus, it follows from Steps 1. and 3. that for each \( x \in \sigma(T) \), both \( B_-(x) \) and \( B_+(x) \) are generic sets in \( \mathcal{H} \), and so

\[
G(x) = \bigcap_{n \geq 1} \left\{ \psi \in \mathcal{H} \mid d^-_{\mu^T_\psi}(x) = 0 \text{ and } d^+_{\mu^T_\psi}(x) \geq n \right\}
\]

is also generic in \( \mathcal{H} \).

### 3 Proof of Theorem 1.3

Let \( \alpha \in [0, 1] \). We need the following result.

**Claim.** If \( x \in \sigma(T) \), then \( G(x) \subset \{ \psi \in \mathcal{H} \mid \mu^T_\psi \text{ is not } U(1/k)H, \forall k \in \mathbb{N} \} \).

Namely, it is enough to note that, if \( \mu^T_\psi \) is UoH, then for each \( x \in \sigma(T) \), \( d^+_{\mu^T_\psi}(x) \geq \alpha \).

(i) This is a direct consequence of Theorem 2.1 and Claim.

(ii) For \( x \in \sigma(T) \), if \( \psi \in G(x) \), then by Claim, for each \( k \geq 1 \), \( \mu^T_\psi \) is not \( U(1/2k)H \). Thus, it follows from Theorem 1.1 that for each \( k \geq 1 \),

\[
\limsup_{t \to \infty} t^{1/k} W^T_\psi(t) = \infty
\]

from which follows, again from Theorem 2.1, that for each \( k \geq 1 \), the set

\[
\{ \psi \in \mathcal{H} \mid \limsup_{t \to \infty} t^{1/k} W^T_\psi(t) = \infty \} \supset G(x)
\]

is generic in \( \mathcal{H} \).

It remains to prove that for each \( k \geq 1 \), the set

\[
A_k := \{ \psi \in \mathcal{H} \mid \liminf_{t \to \infty} t^{\alpha-1/k} W^T_\psi(t) = 0 \}
\]

is generic in \( \mathcal{H} \). The proof that for each \( k \geq 1 \), \( A_k \) is a \( G_\delta \) subset of \( \mathcal{H} \) follows closely the arguments presented in Remark 2.1. On the other hand it follows from Theorem 1.1(i) that for each \( k \geq 1 \),

\[
\mathcal{H}^T_{\text{uhH}}(\alpha) \subset A_k.
\]

Finally, since, by Theorem 1.2,

\[
\overline{\mathcal{H}^T_{\text{uhH}}(\alpha)} = \mathcal{H}^T_{\text{ac}},
\]

it follows that for each \( k \geq 1 \), \( A_k \) is a dense \( G_\delta \) subset of \( \mathcal{H} \) (recall that \( T \) has purely \( \alpha \)-Hausdorff continuous spectrum, by hypothesis).
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