Hirota’s Solitons in the Affine and the Conformal Affine Toda Models

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ABSTRACT

We use Hirota’s method formulated as a recursive scheme to construct complete set of soliton solutions for the affine Toda field theory based on an arbitrary Lie algebra. Our solutions include a new class of solitons connected with two different type of degeneracies encountered in the Hirota’s perturbation approach.

We also derive an universal mass formula for all Hirota’s solutions to the Affine Toda model valid for all underlying Lie groups. Embedding of the Affine Toda model in the Conformal Affine Toda model plays a crucial role in this analysis.

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1Work supported in part by U.S. Department of Energy, contract DE-FG02-84ER40173 and by NSF, grant no. INT-9015799
2Supported by Fapesp
3Work supported in part by CNPq
1 Introduction

Toda field theories are among the most important examples to study integrability and conformal invariance. They can be classified according to an underlying Lie algebra. The conformal Toda (CT) theories are conformally invariant and completely integrable models associated to finite dimensional Lie algebra \( \mathcal{G} \). The affine Toda (AT) models can be regarded as the conformal Toda perturbed in such a way that although conformal invariance is broken, complete integrability is preserved. The underlying Lie algebra is the Loop algebra \( \hat{\mathcal{G}} \) and these models are known to possess soliton solutions. The third class, the conformal affine Toda (CAT) theories, are obtained by recovering the conformal invariance in the AT models by introducing two extra fields. These fields can be shown to be related to adding central extensions to the Loop algebra leading to a full Kac-Moody algebra underlying the CAT model.

The interest in the soliton solutions of the Affine Toda (AT) field theoretical model is partly motivated by the well-known results for the simpler case of the Sine Gordon field theory and partly by the special status of the AT theory in view of its interpretation as deformed unitary minimal theory.

This paper is devoted to obtaining, in a systematic manner, soliton solutions of the AT model and linking them to solutions and properties of the CAT. This is accomplished by using the Hirota’s tau-function method which was already successfully been applied to the AT field theory, although not all solutions were found, to the cases of the Lie algebras \( A_r \), \( C_r \) and other algebras \[1\]. The method involves introducing one extra tau-function apart from those assigned to each AT field. It was within the context of the CAT model in \[2\] that this fact was made clear in view of the new fields introduced in order to restore conformal invariance.

There are three main (but related) goals which we have accomplished in this paper: (i) A description of soliton solutions obtained by Hirota’s method for the AT model defined on an arbitrary Lie algebra. (ii) Analysis of the Hirota’s method as a perturbation method based on recurrence relations. (iii) Application of the link between Conformal Affine Toda (CAT) model and its integrable deformation represented by AT theory to reveal new features of the latter. One of the results we obtain here from general arguments of conformal field theory, is a general mass-formula valid for soliton solutions. A parallel and complementary approach to find soliton solutions for the AT model was developed in \[4\] using the Leznov-Saveliev method, see also \[5\] for the Bäklund’s method approach to the problem in the case of \( A_r \).

The Hirota’s method employed in this paper constitute essentially a perturbative approach based on recurrence relations which can be conveniently characterized in terms of the eigenvalue problem of a matrix \( L_{ij} = l^{\psi}_i K_{ij} \) where \( K_{ij} \) is the extended Cartan matrix and \( l^{\psi}_i \) are integers in the expansion of the highest coroot \( \frac{\psi^*}{\psi^{**}} \) in terms of simple coroots \( \alpha^*_i \) of an algebra \( \mathcal{G} \). A soliton solution is assigned to each eigenvalue of \( L_{ij} \) and there can be at most rank \( \mathcal{G} + 1 \) when no degeneracy is present. The fact that the \( L_{ij} \) is singular is crucial in truncating the perturbative series leading to an exact solution. For the purpose of this study we have established a simple way of solving this eigenvalue problem for \( L_{ij} \) matrix in terms of Chebyshev polynomials and their special properties. The situation is changed however when there is a degeneracy. There are two separate sources of degeneracy. One comes from the
degeneracy of the matrix $L_{ij}$. Any linear combination of the degenerate eigenvectors gives rise to a soliton solution within the Hirota’s method. This feature clearly demonstrates the nonlinear superposition principle underlying the method. Another degeneracy is an intrinsic feature of Hirota’s perturbation scheme itself. The relevant recurrence relation is based on adding to the $n$-th order perturbed solution non trivial elements of the kernel of $L_{ij} - n^2 \lambda \delta_{ij}$ (for $\lambda$ an eigenvalue of $L$) which exists in some cases ($SU(6p)$ and $SP(3p)$ with $p$ integer). Remarkably the Hirota series truncates in these cases too, producing a new class of solutions. It is generally true that the soliton solutions obtained in the degenerate cases truncate at higher orders of the tau-functions.

There are two related ways of understanding AT model and its relation to CAT model. One way of thinking introduces AT model as a deformation of the CAT model preserving integrability of the theory. Such a deformation modifies the underlying $W_\infty$ symmetry structure of CAT but still maintain enough symmetry to keep the AT massive structure both attractive and tractable for physicists. On the other hand there is also a parallel approach, introduced first into the literature in [2], which views AT model as a version of the CAT model with conformal symmetry being gauge-fixed. This second approach proves to be very useful in this paper defining the precise limit in which one model goes into another. One particular consequence of the connection between the CAT and AT models is a special simple relation between their respective Energy-Momentum (EM) tensors. The modified EM tensor of CAT model [6] in some special limit consists of a sum of EM tensor of AT model and the extra surface term. Since there is no mass scale in the CAT model the only contribution to the soliton masses of the AT model comes from the pure surface term which ensures enormous simplicity of the final universal expression for the soliton masses.

The outline of the paper is as follows. In Section 2 we discuss a connection between AT and CAT models and relation between their EM tensors. In Section 3 the Hirota’s method is discussed as a perturbation approach and the general features of this perturbation and recurrence relations are described. We also derive the general mass formula and discuss topological charges. In Sections 4, 5, 6 and 7 this theory is used to find and describe soliton solutions of the AT model with the underlying Lie algebras $A_r = SU(r+1)$, $C_r = Sp(r)$, $D_r = SO(2r)$ and $B_r = SO(2r+1)$, respectively. In Sections 8, 9 and 10 this analysis is repeated for AT models with exceptional Lie algebras $G_2$, $F_4$ and finally we make a few comments for the $E_n$ with $n = 6, 7, 8$. In Appendix A we list for reader’s convenience a number of technical properties of Chebyshev’s polynomials.

## 2 The CAT and AT Models

The equations of motion of the CAT model associated with a simple Lie algebra $G$ are given by [2, 3, 4]:

\[
\begin{align*}
\partial_\pm \varphi^a &= \frac{1}{q} \left( q^a e^{q K_a \varphi^b} - q^0 \psi e^{-q K_a \psi} \right) e^{\eta} \\
\partial_\pm \eta &= 0 \\
\partial_\pm \nu &= \frac{2}{\psi} \frac{q^0}{q} e^{-q K_a \psi} e^{\eta}
\end{align*}
\]  

(2.1)  

(2.2)  

(2.3)
where $K_{ab} = 2\alpha_a\alpha_b/\alpha_0^2$ is the Cartan Matrix of $G$, $a, b = 1, ..., \text{rank } G = r$, $\psi$ is the highest root of $G$, $K_{\psi b} = 2\psi_\alpha \alpha_b/\alpha_0^2$, $\rho_0^G$ are positive integers appearing in the expansion $t_\psi = \rho_0^G \alpha_a/\alpha_0^2$, where $\alpha_a$ are the simple roots of $G$ and $q^a$, $q^0$ and $\bar{q}$ are coupling constants. The derivatives $\partial_\pm$ are w.r.t. the light cone coordinates $x^\pm = x \pm t$.

These equations are invariant under the conformal transformations

$$x_+ \to \tilde{x}_+ = f(x_+), \quad x_- \to \tilde{x}_- = g(x_-)$$ (2.4)

and

$$e^{-\varphi^a(x_+, x_-)} \to e^{-\tilde{\varphi^a}(\tilde{x}_+, \tilde{x}_-)} = e^{-\varphi^a(x_+, x_-)}$$ (2.5)
$$e^{-\nu(x_+, x_-)} \to e^{-\tilde{\nu}(\tilde{x}_+, \tilde{x}_-)} = (\frac{df}{dx_+})^B (\frac{dg}{dx_-})^B e^{-\nu(x_+, x_-)}$$ (2.6)
$$e^{-\eta(x_+, x_-)} \to e^{-\tilde{\eta}(\tilde{x}_+, \tilde{x}_-)} = (\frac{df}{dx_+})^\frac{1}{2} (\frac{dg}{dx_-})^\frac{1}{2} e^{-\eta(x_+, x_-)}$$ (2.7)

where $f$ and $g$ are analytic functions and $B$ is an arbitrary number. Therefore $e^{\varphi^a}$ are scalars under conformal transformations and $e^\nu$ can also be arranged to be a scalar by setting $B = 0$.

The equations (2.1)-(2.3) can be written in the form of a zero curvature condition (for the associated linear system) \cite{2, 7, 6}

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0$$ (2.8)

where

$$A_+ = \partial_+ \Phi + e^{ad} \mathcal{E}_+, \quad A_- = -\partial_- \Phi + e^{-ad} \mathcal{E}_-$$ (2.9)

and

$$\Phi = \frac{\bar{q}}{2} \left( \sum_{a=1}^{\text{rank } G} \varphi^a H_a^0 + \eta T_3 + \nu C \right)$$ (2.10)
$$\mathcal{E}_+ = \sum_{a=1}^{\text{rank } G} E_{\alpha a}^0 + E_{-\psi}^1, \quad \mathcal{E}_- = \sum_{a=1}^{\text{rank } G} q^a E_{-\alpha a}^0 + q^0 E_{-\psi}^1$$ (2.11)

where $T_3 = 2\hat{\delta}.H_0 + hD$, with $\hat{\delta} = \frac{1}{2} \sum_{\alpha > 0} \frac{2}{\alpha^2}$ (with $\alpha$ being the positive roots of $G$), is the generator used to perform the so called homogeneous grading of a Kac-Moody algebra. $H_a^0$, $D$ and $C$ are the generators of the Cartan subalgebra of the affine Kac-Moody algebra $\hat{G}$ associated to $G$, $E_{\alpha a}^0$ ($E_{-\alpha a}^0$), $a = 1, 2, ..., \text{rank } G = r$, $E_{-\psi}^1$ ($E_{-\psi}^{-1}$) are the positive (negative) simple root step operators of $\hat{G}$.

The Lagrangian for the CAT model is given by

$$\mathcal{L} = \frac{\bar{q}^2}{4} \sum_{a, b=1}^{r} 2 \alpha_a^2 K_{ab} \partial_\rho \varphi^a \partial^\rho \varphi^b + \frac{2}{2} \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \partial_\rho \varphi^a \partial^\rho \eta + \frac{q^2}{2} \partial_\rho \eta \partial^\rho \nu - U(\varphi, \eta)$$ (2.12)
where $h$ is the Coxeter number of $G$ and the potential being

$$U(\varphi, \eta) = \sum_{a=1}^{r} \frac{4q^a}{\alpha_a^2} \epsilon^{q(K_{ab} \varphi^b + \eta)} + \frac{4q^0}{\psi^2} \epsilon^{q(-K_{\psi \varphi} + \eta)}$$  \hfill (2.13)

Let us introduce, for convenience the vector

$$\varphi \equiv \sum_{a=1}^{r} \frac{2\alpha_a}{\alpha_a^2} \varphi^a$$  \hfill (2.14)

The potential (2.13) can then be written as

$$U(\varphi, \eta) = \sum_{j=0}^{4} \frac{4q^j}{\alpha_j^2} \epsilon^{q(\alpha_j \cdot \varphi + \eta)}$$  \hfill (2.15)

where we have denoted $\alpha_0 = -\psi$ as the extra simple root of the affine Kac-Moody algebra $\hat{G}$.

The potential (2.15) is invariant under the transformation

$$\varphi \rightarrow \varphi + \frac{2\pi i}{\hat{q}} \mu^v ; \eta \rightarrow \eta + \frac{2\pi i}{\hat{q}} n$$  \hfill (2.16)

where $\mu^v$ is a coweight of $G$, i.e. $\mu^v = \sum_{a=1}^{r} m_a \frac{2\lambda_a}{\alpha_a^2}$ where $m_a$ and $n$ are integers. Such transformation is complex and therefore in the regime where the coupling constant $\hat{q}$ is pure imaginary the real vacuum solutions are degenerate. This generalizes to any simple Lie algebra the degenerate vacua of the sine Gordon model where the minima of the potential are identified with (co-) weight lattice of $SU(2)$. Such degenerate vacua are responsible for the appearance of topological soliton solutions in the AT and CAT models as we describe below.

Notice that the potential (2.15) and the CAT model equations of motion (2.1)-(2.3) are invariant under the transformation

$$\varphi \rightarrow \varphi + \zeta \hat{t} ; \eta \rightarrow \eta - \zeta ; \nu \rightarrow \nu + b \zeta$$  \hfill (2.17)

where $b$ is an arbitrary constant and $\zeta$ is an harmonic function, i.e., $\partial_+ \partial_- \zeta = 0$, and

$$\hat{t} \equiv \sum_{j=0}^{r} \frac{2\hat{\lambda}_j}{\alpha_j^2}$$  \hfill (2.18)

where $\hat{\lambda}_j$ are the fundamental weights of the Kac-Moody algebra $\hat{G}$, i.e. $\hat{\lambda}_0 = (0, \frac{\psi^2}{t_0}, 0)$, $\hat{\lambda}_a = (\lambda_a, \frac{\psi^2}{t_0}, 0)$, $a = 1, 2, ..., r$, with $\lambda_a$ being the fundamental weights of the simple Lie algebra $G$ and the scalar product defined as $\hat{a} \cdot \hat{b} \equiv a \cdot b + a_c b_d + a_D b_C$ for $\hat{a} = (a, a_C, a_D)$ and $\hat{b} = (b, b_C, b_D)$. When evaluating the scalar product we consider the simple roots $\alpha_j$ as $r + 2$-component vectors, i.e. $\alpha_a \equiv (\alpha_a, 0, 0)$, $a = 1, 2, ..., r$ and $\alpha_0 = (-\psi, 0, 1)$. 

4
The canonical energy-momentum tensor corresponding to the CAT model Lagrangian (2.12) is given by

\[
\Theta_{\rho\sigma} = \frac{q^2}{2} \sum_{a,b=1}^{r} \frac{2}{\alpha_a^2} K_{ab} \left[ \partial_\rho \varphi^a \partial_\sigma \varphi^b - \frac{1}{2} g_{\rho\sigma} \partial_\mu \varphi^a \partial^\mu \varphi^b \right] + \frac{q^2}{2} \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \left[ \partial_\rho \varphi^a \partial_\sigma \eta + \partial_\sigma \varphi^a \partial_\rho \eta - g_{\rho\sigma} \partial_\mu \varphi^a \partial^\mu \eta \right] + \frac{q^2 \hbar}{2} \left[ \partial_\rho \eta \partial_\sigma \nu + \partial_\sigma \eta \partial_\rho \nu - g_{\rho\sigma} \partial_\mu \eta \partial^\mu \nu \right] + q g_{\rho\sigma} \left[ \sum_{a=1}^{r} \frac{4q^a}{\alpha_a^2} e^{q(K_{ab} \varphi^b + \eta)} + \frac{4q^0}{\psi \gamma^2} e^{q(-K_{ab} \varphi^b + \eta)} \right]
\]

(2.19)

where \(\rho, \sigma = 0,1\) are space-time indices (\(\partial_0 \equiv \partial_t, \partial_1 \equiv \partial_x\)) and \(g_{00} = 1, g_{11} = -1, g_{01} = g_{10} = 0\).

One finds that this E-M tensor has a non-zero trace given by

\[
\Theta^\rho_\rho = 2U(\varphi, \eta)
\]

(2.20)

Since the CAT model is conformally invariant its E-M tensor can be made traceless. The usual procedure is to add a term \(W_{\rho\sigma} = (\partial_\rho \partial_\sigma - g_{\rho\sigma} \partial^2) f\) with an arbitrary function \(f\). Clearly the addition of such a term does not violate the conservation of \(\Theta_{\rho\sigma}\). The trace of the extra term is given by \(W^\rho_\rho = -\partial^2 f\) and this fixes the choice of \(f\) which renders the modified E-M tensor traceless. Accordingly one verifies that the modified traceless E-M tensor of the CAT model is given by \[3\]

\[
\Theta^{\text{CAT}}_{\rho\sigma} = \Theta_{\rho\sigma} - q \left( \partial_\rho \partial_\sigma - g_{\rho\sigma} \partial^2 \right) \left( \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \varphi^a + h \nu \right)
\]

(2.21)

The CAT model can be obtained via a Hamiltonian reduction procedure from the two-loop WZNW model \[3\]. The modification of the E-M tensor described above is equivalent to the one performed on the Sugawara tensor of the two-loop WZNW model in order for reduction procedure to respect the conformal invariance \[3\].

The Affine Toda model (AT) associated to a simple Lie algebra \(G\) possesses rank-\(G\) fields \(\varphi^a, a = 1, 2, \ldots\) rank-\(G\); and it is not conformally invariant. Its equations of motions correspond to eq. (2.11) for \(\eta = 0\). As shown in \[2\] the AT models constitute a “gauge fixed” version of the CAT models. The \(\eta\) field can be “gauged away” by a conformal transformation (2.4) with \(f' = e^{\eta_+(x_+)}\) and \(g' = e^{\eta_-(x_-)}\) where \(\eta_+\) and \(\eta_-\) are the parameters of the solution of the free field \(\eta\), i.e. \(\eta(x_+, x_-) = \eta_+(x_+) + \eta_-(x_-)\). Therefore for every regular solution of the \(\eta\) field the CAT model defined on a space-time \((x_+, x_-)\) corresponds to the AT model with the extra field \(\nu\) defined on a space-time \((\tilde{x}_+, \tilde{x}_-)\) where \(\tilde{x}_+ = f^{x_+} dy_+ e^{\eta_+(y_+)}\) and \(\tilde{x}_- = f^{x_-} dy_- e^{\eta_-(y_-)}\). Such connection allows one to calculate exactly the perturbation of the \(\eta\) field on the dynamics of the AT model \[2\].

For the particular solution \(\eta = 0\) the CAT and AT are defined on the same space-time and the dynamics of the fields \(\varphi^a\) are the same on both models. In such case the E-M tensor
of these theories are related by

$$\Theta_{\rho\sigma}^{AT} = \Theta_{\rho\sigma}^{CAT} \mid_{\eta=0} + \bar{q} \left( \partial_{\rho} \partial_{\sigma} - g_{\rho\sigma} \partial^2 \right) \left( \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \varphi^a + h \nu \right)$$

(2.22)

This relation provides a very interesting insight on the role of the broken conformal symmetry of the AT models. If one considers classical solutions of soliton type which can be put at rest at some Lorentz frame, then the energy of the solution can be interpreted as the rest mass of the soliton. Such mass should be proportional to some mass scale of the theory. Due to the conformal (scale) invariance such mass scale does not exist in the CAT model and therefore the soliton mass should vanish. Alternatively, this can be seen by taking the matrix element of $\Theta_{\rho\sigma}^{CAT}$ between one soliton states of momentum $p$ and $p'$, obtaining

$$\langle p \mid \int dx \Theta_{\rho\sigma}^{CAT} \mid p' \rangle = A P_\rho P_\sigma + B \left( K_\rho K_\sigma - K^2 g_{\rho\sigma} \right)$$

(2.23)

where $P \equiv \frac{p+p'}{2}$ and $K \equiv p - p'$. Relation (2.23) follows from the symmetry of $\Theta_{\rho\sigma}^{CAT}$ and its conservation $\partial^i \Theta_{\rho\sigma}^{CAT} = 0$. If in the classical limit, which is obtained by considering the tree diagrams of the vertex (2.23), $B$ has no poles at $K^2 = 0$, we shall have in the forward direction

$$\langle p \mid \int dx \Theta_{\rho\sigma}^{CAT} \mid p \rangle = A P_\rho P_\sigma$$

(2.24)

and because $\Theta_{\rho\sigma}^{CAT}$ is traceless it follows we must have either $p^2 = 0$ or $A = 0$ for $p^2 \neq 0$. Therefore $\Theta_{\rho\sigma}^{CAT}$ does not give any contribution for the case of a soliton of mass different from zero ($p^2 \neq 0$).

However in the AT model the same is not true, and from (2.22) one observes that the contribution to the soliton mass in the AT models comes from a total divergence contained in the second term of (2.22). Indeed, denoting by $M$ the soliton mass and $v$ its velocity (in units of light velocity) one gets

$$\frac{M v}{\sqrt{1 - v^2}} = \int_{-\infty}^{\infty} dx \Theta_{01}^{AT} = \bar{q} \int_{-\infty}^{\infty} dx \partial_x \partial_t \left( \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \varphi^a + h \nu \right)$$

(2.25)

This universal formula will be used to determine the masses of the solitons we will find below, using Hirota’s method. This result was reported independently by Olive and we learned while typing this paper that his results was obtained using very similar arguments to ours.
# 3 Hirota’s Method

We now construct solitons solutions for CAT and AT models associated to any simple Lie algebra $G$ using the Hirota’s method \([10]\). We introduce the $\tau$-functions as

$$\varphi^a = \frac{1}{q} \left( -\ln \tau_a \frac{\partial}{\partial \tau_0} + \vartheta_a \right) \quad \nu = \frac{1}{q} \sqrt{\psi} \left( \sigma - \ln \tau_0 \right) \quad (3.1)$$

with $a = 1, 2, \ldots, \text{rank-}G$, and where \([2]\)

$$\vartheta_a = \sum_{b=1}^{\text{rank}G} (R^*)^{-1}_{ab} \ln \left( \frac{q^0 \psi_a^b}{q^b} \right) \quad (3.2)$$

where $R$ is the matrix with entries $R_{ab} = \delta_{ab} + n^\psi_a$, with $n^\psi_a$ being the integers in the expansion $\psi = n^\psi_a \alpha_a$, and so $n^\psi_a = \frac{a^2}{\sqrt{\psi}} n_a$, and

$$(R^{-1})_{ab} = \delta_{ab} - \frac{n^\psi_b}{h} \quad (3.3)$$

Substituting these definitions in (2.1) and (2.3) for $\eta = 0$ one obtains that the resulting equations can be decoupled into

$$\triangle (\tau_j) = \beta l^\psi_j \left( 1 - \prod_{k=0}^{\text{rank}G} \tau_k^{-K_{jk}} \right) \quad (3.4)$$

$$\partial_+ \partial_- \sigma = \beta \quad (3.5)$$

where $K_{ij}$, $i, j = 0, 1, 2, \ldots, \text{rank-}G$, is the extended Cartan Matrix of the Affine Kac-Moody algebra $\hat{G}$ associated to $G$ (obtained from the ordinary Cartan matrix of $G$ by adding one extra row and column corresponding to the simple root $\alpha_0 = -\psi$). Furthermore

$$\triangle (F) \equiv \partial_+ \partial_- \ln F = \frac{\partial_+ \partial_- F}{F} - \frac{\partial_+ F \partial_- F}{F^2} \quad (3.6)$$

and

$$\beta = \frac{q^j}{l^j} e^{K_{jk} \vartheta_k} \quad \text{for any } j = 0, 1, \ldots, r \quad (3.7)$$

\(^4\text{From (2.3) and (2.4) one sees that, for } B = 0, \text{ the } \tau\text{-functions are primary fields of conformal weight } (0, 0). \text{ Notice this definition of the } \tau\text{-functions is related to that of ref. [2] by } \tau_j \rightarrow (\tau_j)^{l^\psi_j}. \text{ Also, these } \tau\text{-functions are related to the Leznov-Saveliev [11] solution of the CAT model [2]. Comparing (3.1) with eqs. (14) and (15) of ref. [2] (where } \bar{q} = 1) \text{ one can write (on shell and for } \eta = 0)\)\)

$$\tau_a = e^{l^\psi \vartheta_a} \langle \lambda(a) | e^{\sum_{k=0}^{\text{rank}G} K_{jk} \tau_k} M_+(x_+) M_-^{-1}(x-) e^{-\sum_{k=0}^{\text{rank}G} K_{jk} \tau_k} | \lambda(a) \rangle$$

$$\tau_0 = e^{\sqrt{\psi} \vartheta} \langle \lambda(0) | e^{\sum_{k=0}^{\text{rank}G} K_{jk} \tau_k} M_+(x_+) M_-^{-1}(x-) e^{-\sum_{k=0}^{\text{rank}G} K_{jk} \tau_k} | \lambda(0) \rangle$$
is a constant independent of the index \( j \). We have also set \( \vartheta_0 = 0 \). In the calculation we have used the fact that \( l^\psi_j, j = 0, 1, 2, \ldots, \text{rank-\( G \)}, \) with \( l^\psi_0 = 1 \), constitute a null vector of the extended Cartan matrix. Indeed, \( \sum_{j=0}^r K_{ij} l^\psi_j = \sum_{a=1}^{r+1} \frac{2a_1 a_0}{a_0^2} l^\psi_a = 0 \), since \( a_0 = -\psi \) and \( \frac{\psi}{\vartheta} = \sum_{a=1}^r l^\psi_a \frac{a_0}{a_a^2} \).

The solution to (3.5) gives \( \sigma \) as

\[
\sigma(x_+, x_-) = \beta x_+ x_- + F(x_+) + G(x_-)
\]

with \( F \) and \( G \) being arbitrary functions.

Reference [1] provided for the first time the Hirota’s solution for the Affine Toda models associated to \( A_r \equiv SU(r+1) \). The technical manipulations, required in order to write down the consistent Hirota’s equation for the AT model, relied on the \( \tau \)-functions which exceeded by one the number of fields physically associated to the model. The origin of the extra \( \tau \)-function, namely \( \tau_0 \), can be traced back, in view of the analysis in [2], to the \( \nu \) field revealing intrinsic connection of the Hirota’s equations (3.4) to the structure of the CAT model.

In the spirit of the Hirota’s method we expand the \( \tau \)-functions in a formal power series in a parameter \( \epsilon \) as

\[
\tau_i = 1 + \epsilon \tau_i^{(1)} + \ldots + \epsilon^{N_i} \tau_i^{(N_i)}
\]

(3.9)

We will be interested in solutions were the space-time dependence of the \( \tau \)-functions is given by

\[
\tau_i^{(n)} = \delta_i^{(n)} e^{n \Gamma} \quad (3.10)
\]

with

\[
\Gamma = \gamma_+ x_+ + \gamma_- x_- + \xi = \gamma (x - vt) + \xi
\]

(3.11)

where \( \delta_i^{(n)} \) are constant vectors to be found from Hirota’s equations, and \( \gamma_+ = \frac{\gamma}{2}(1 + v) \), \( \gamma_- = \frac{\gamma}{2}(1 - v) \) and \( \xi \) are parameters of the solution.

The basic idea is to expand Hirota’s eqs. (3.4) in powers of \( \epsilon \) and solve them order by order. The method gives an exact solution if the series truncates at some finite order in \( \epsilon \). The actual value of the parameter \( \epsilon \) is irrelevant in the procedure and it can be set to unity at the end of the calculation. Below we give a detailed procedure to solve Hirota’s equations. But before that we would like to discuss some general properties of the method.

**Theorem 3.1** Let us suppose one has a solution of the Hirota’s eqs. and let us denote by \( N_i \) the highest power of \( \epsilon \) in the Hirota’s expansion of \( \tau_i, i = 0, 1, 2, \ldots, \text{rank-\( G \)} \). Then \( N_i \) are the components of a null vector of the extended Cartan matrix:

\[
K_{ij} N_j = 0
\]

(3.12)

Since the zero eigenvalue of \( K_{ij} \) is non degenerate and since \( l^\psi_i \) is a null vector of \( K \), it follows that \( N_i = \kappa l^\psi_i \), where \( \kappa \) is some positive integer.

**Proof.** Since \( K_{ii} = 2 \) and \( K_{ij} \leq 0 \) for \( i \neq j \) one observes that by multiplying both sides of (3.4) by \( \tau_j^2 \) the powers of \( \tau \)'s will all be positive. After multiplication the term of highest power in \( \epsilon \) on the l.h.s. of (3.4) will be \( \epsilon^{N_i K_{ii}} \left( \tau_i^{(N_i)} \right)^2 A \left( \tau_i^{(N_i)} \right) \) which is zero because of the
ansatz (3.10). Notice now that on the r.h.s there is only one unique term contributing to the highest power of \( \epsilon \) leading to

\[
\left( \tau_{i}^{(n_i)} \right)^{K_{ii}} \epsilon^{N_i K_{ii}} = \prod_{k=0,k \neq i}^{\text{rank } G} \left( \tau_{k}^{(n_k)} \right)^{-K_{ik}} \epsilon^{-N_k K_{ik}}
\]

(3.13)

Since the sum of the powers of \( \epsilon \) on the r.h.s. of the above equation should equal \( N_i K_{ii} \) one obtains (3.12) .

Therefore a necessary condition for the expansion (3.9) to truncate at some finite order (when ansatz (3.10) is used) is that the matrix \( K_{ij} \) appearing in the Hirota’s eqs. (3.4) should be singular.

Substituting the expansion (3.9) into Hirota’s eqs. (3.4) and using (3.10) and (3.11) one obtains that the resulting equation in order zero in \( \epsilon \) is trivially satisfied whilst the first order equation leads to

\[
L_{ij} \delta_{j}^{(1)} = \lambda \delta_{i}^{(1)}
\]

(3.14)

where \( L_{ij} = l^\psi K_{ij} \), and \( \lambda = \frac{\gamma_i \gamma_j}{\beta} = \frac{\gamma_i^2 (1-v^2)}{4 \beta} \). Therefore the parameters of the solution are restricted by the possible eigenvalues of the matrix \( L_{ij} \). As we shall show below, except for the algebras \( SU(6p) \) and \( SP(3p) \) with \( p \) a positive integer, the higher terms \( \delta^{(n)} \) \( (n \geq 2) \), are uniquely determined by \( \delta^{(1)} \). Therefore, if the eigenvalues are non degenerate there can be at most rank-\( G + 1 \) one-soliton solutions. However if there are degeneracies we can have many more solutions. Since a right null vector of \( K_{ij} \) is also a right null vector of \( L_{ij} \) and vice versa, it follows that there will always be a zero eigenvalue. This eigenvalue is not degenerate and the corresponding soliton solution is trivial in the sense that the \( \varphi \)'s fields (but not \( \nu \)) are constants.

### 3.1 Soliton Masses

From (3.1) one gets

\[
\bar{q} \left( \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \varphi^a + h \nu \right) = -\sum_{j=0}^{r} \frac{2}{\alpha_j^2} \log \tau_j + \frac{2h}{\psi^2} \hbar \sigma + \sum_{a=1}^{r} \frac{2}{\alpha_a^2} \varphi_a
\]

(3.15)

where we have used the fact that \( \sum_{a=1}^{r} \frac{2}{\alpha_a^2} l^\psi = \frac{2}{\psi^2} (h - 1) \). Therefore from the relation (2.25) one obtains

\[
\frac{M v}{\sqrt{1 - v^2}} = -\sum_{j=0}^{r} \frac{2}{\alpha_j^2} \frac{\tau_j}{\tau_j} \bigg|_{-\infty}^{\infty} + \frac{2h \sigma}{\psi^2} \bigg|_{-\infty}^{\infty} = -\sum_{j=0}^{r} \frac{2}{\alpha_j^2} \frac{\tau_j}{\tau_j} \bigg|_{-\infty}^{\infty}
\]

(3.16)

where we have chosen the functions \( F \) and \( G \) in the definition (3.8) in such way that the contribution to the mass from the \( \sigma \) field vanishes. Substituting (3.9) and (3.10) into (3.16) one gets

\[
\frac{M v}{\sqrt{1 - v^2}} = \gamma v \sum_{i=0}^{r} \frac{2}{\alpha_i^2} \frac{\epsilon \delta_i^{(1)} e^\Gamma + \epsilon^2 \delta_i^{(2)} e^\Gamma + \ldots + \epsilon^{N_i} \delta_i^{(N)} e^{N_i \Gamma}}{1 + \epsilon \delta_i^{(1)} e^\Gamma + \ldots + \epsilon^{N_i} \delta_i^{(N)} e^{N_i \Gamma}} \bigg|_{-\infty}^{\infty}
\]

(3.17)

Recalling that \( \Gamma = \gamma (x - vt) + \xi \) we see that by taking \( \gamma > 0 \) the lower limit \( x \to -\infty \) does not contribute and the limit \( x \to \infty \) is finite. For \( \gamma < 0 \) the converse happens. Therefore the
mass is proportional to $|\gamma|$. From Theorem (3.1) we have $N_j = \kappa l_j^\beta$ where $\kappa$ is a positive integer. Therefore

$$\frac{Mv}{\sqrt{1 - v^2}} = |\gamma| \sum_{j=0}^r \frac{2}{\alpha_j^2} N_j = |\gamma| \frac{2 \hbar \kappa}{\psi^2}$$

with $h$ being the Coxeter number of $G$. Recalling that $\lambda = \gamma^2(1 - v^2)/4\beta$ we therefore arrive at the following expression for the soliton masses in the AT model

$$M = \frac{4\hbar \kappa}{\psi^2} m \sqrt{\lambda}$$

where we have denoted $m \equiv \sqrt{\beta}$. This mass formula is true for any algebra and for any solution constructed using the ansatz (3.10). Notice that the masses of solitons associated with a given eigenvalue $\lambda$ are quantized in units of $\frac{4\hbar}{\psi} m \sqrt{\lambda}$. In addition the soliton masses are proportional to the masses of the fundamental particles of the AT model.

### 3.2 Soliton charges

An important consequence of Theorem (3.1) is that the asymptotic values of the $\varphi$’s fields are always finite for the solutions constructed through the ansatz (3.10). From (3.1), (3.9) and (3.10) we see that, for $\gamma > 0$, $\varphi_a(-\infty) = \frac{\theta_a}{\bar{q}}$ and $N_i = \kappa l_i^\beta$ with $\varphi_i = 1$ we see that the limit of $\tau_a/\tau_0^\beta$ as $x \to \infty$ is finite and so from (3.1) $\varphi_a(\infty)$ is also finite. For $\gamma < 0$ the limits are interchanged. Since the masses of the solitons, as shown above are finite, it follows that such asymptotic values of $\varphi$’s are vacua of the potential (2.13). As discussed before, in the regime where $\bar{q}$ is purely imaginary the real minima of the potential are degenerate. We then introduce the topological charge of the solitons as $(i\bar{q} \equiv \bar{q})$

$$Q \equiv \frac{\bar{q}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \varphi = \frac{\bar{q}}{2\pi} (\varphi(\infty) - \varphi(-\infty))$$

The actual calculation of the charges presents some ambiguities which we now discuss. From (3.1), (2.14) and (3.9) we have

$$\varphi = \sum_{a=1}^r \frac{2\alpha_a}{\alpha_a^2} \varphi^a = -\frac{1}{i\bar{q}} \sum_{a=1}^r \frac{2\alpha_a}{\alpha_a^2} \log \frac{\tau_a}{\tau_0^\beta} + \vartheta = -\frac{1}{i\bar{q}} \sum_{j=0}^r \frac{2\alpha_j}{\alpha_j^2} \log \tau_j + \vartheta$$

where $\vartheta = \frac{1}{i\bar{q}} \sum_{a=1}^r \frac{2\alpha_a}{\alpha_a^2} \theta_a$. The $\tau$-functions are in general complex and therefore the logarithm function may not be single valued. One could therefore use the prescription, as in [1], that $\log z = \log |z| + i\bar{\theta}$ for $z = |z| e^{i\theta}$, where $0 \leq \bar{\theta} < 2\pi$ and $\theta - \bar{\theta}$ is multiple of $2\pi$. However the implementation of such prescription requires also a criterion on the use of the identity $\log AB = \log A + \log B$. The reason is that the result depends on the order in which the bar prescription and the identity are used. For instance, one could get $\log AB = \log A + \log B = \log |A| + \log |B| + i\arg A + i\arg B$ or $\log AB = \log |AB| + i\arg A + \arg B$. Due to such ambiguities the values of the charges one obtains will differ by a sum of co-roots of $G$.

\[\text{We would like to point out that for } x \to \pm\infty, \tau_a/\tau_0^\beta \text{ becomes a phase, i.e. } \lim_{x \to \pm\infty} |\tau_a/\tau_0^\beta| = 1\]
Therefore it is quite clear which coset in $\Lambda^v_W/\Lambda^v_R$ the charge lies (where $\Lambda^v_W$ and $\Lambda^v_R$ denote the co-weight and co-root lattices of $\mathcal{G}$ respectively), but the determination of the possible values of the charge, inside the coset, that a given soliton solution can have is quite delicate. For this reason we will give below the charges of the solutions we constructed, up to a sum of co-roots of $\mathcal{G}$. Following (3.20) and (3.21) we will use the formula (for $\gamma > 0$)

$$Q = -\frac{1}{2\pi i} \lim_{x \to \infty} \sum_{a=1}^{r} \frac{2\alpha_a}{\alpha_a^2} \log \frac{\tau_a}{\tau_0^a} = -\frac{1}{2\pi i} \sum_{a=1}^{r} \frac{2\alpha_a}{\alpha_a^2} \log \frac{\delta_a^{(N_a)}}{(\delta_0^{(N_0)})^{\tau_a}}$$

(3.22)

where $N_j$ is the highest power of $\epsilon$ in the Hirota’s expansion of $\tau_j$, $j = 0, 1, 2, \ldots, r$. For $\gamma < 0$ the sign of the charge reverses.

### 3.3 Recurrence Method

We now describe a method to solve Hirota’s equations recursively. We show how to determine the higher $\delta^{(n)}$ ($n \geq 2$) from $\delta^{(1)}$. We write Hirota’s equation (3.4) as:

$$G_i = \beta F_i$$

(3.23)

by introducing the auxiliary quantities:

$$G_i \equiv \tau_i^2 \Delta (\tau_i)$$

(3.24)

$$F_i \equiv l_i^\psi \left( \tau_i^2 - \prod_{k=0, k \neq i}^{\text{rank} \mathcal{G}} \tau_k^{-K_{ik}} \right)$$

(3.25)

and possessing the following $\epsilon$ expansion:

$$G_i = \epsilon G_i^{(1)} + \epsilon^2 G_i^{(2)} + \ldots$$

$$F_i = \epsilon F_i^{(1)} + \epsilon^2 F_i^{(2)} + \ldots$$

Clearly (3.23) must hold for each order i.e.

$$G_i^{(n)} = \beta F_i^{(n)}$$

(3.26)

where both sides of (3.26) can be rewritten as

$$G_i^{(n)} = \partial_+ \partial_- \tau_i^{(n)} + \mathcal{A}_i^{(n)} (\tau^{(n-1)}, \ldots, \tau^{(1)})$$

(3.27)

$$F_i^{(n)} = L_{ij} \tau_j^{(n)} + \mathcal{B}_i^{(n)} (\tau^{(n-1)}, \ldots, \tau^{(1)})$$

(3.28)

where $L_{ij} = l_i^\psi K_{ij}$ and $\mathcal{A}^{(n)}$ and $\mathcal{B}^{(n)}$ are sums of products of $\tau$‘s such that the sum of orders of $\tau$‘s for each term is $n$. From now on we choose the ansatz (3.10) for the $\tau$-functions. Substituting (3.27) and (3.28) into (3.26) we get

$$\left( L_{ij} - n^2 \lambda \delta_{ij} \right) \delta_j^{(n)} = V_i^{(n-1)} \quad n = 1, 2, \ldots$$

(3.29)
where $\lambda \equiv \gamma_+\gamma_-/\beta$ and where we grouped together the lower order terms $\delta^{(k)}$'s with $k < n$ into one term $V_i^{(n-1)}$ nonlinear in $\delta$'s and defined by

$$V_i^{(n-1)} \left( \delta^{(n-1)}, \ldots, \delta^{(1)} \right) = \frac{A_i^{(n)}/\beta - B_i^{(n)}}{\exp n\Gamma} = \lambda a_i^{(n-1)} - b_i^{(n-1)} \tag{3.30}$$

For convenience we have introduced above new quantities $b_i^{(n-1)} = B_i^{(n)}/\exp n\Gamma$ and $a_i^{(n-1)} = A_i^{(n)}/(\gamma_+\gamma_-\exp n\Gamma)$. Notice that $V_i^{(n-1=0)} = 0$ due to:

$$a_i^{(0)} = b_i^{(0)} = 0 \quad ; \quad i = 0, \ldots, r \tag{3.31}$$

Hence the expansion of the $\tau$ function must always start at the first order with $\delta_i^{(1)}$ being an eigenvector of $L_{ij}$ matrix as we explained in (3.14). This forces the parameter $\lambda$ of Hirota’s equation to be one of the eigenvalues $\lambda^{[k]}$ of the $L_{ij}$ matrix. Since $V_j^{(n-1)}$ depends on $\delta^{(k)}$ with $k < n$ as seen from its definition (3.30), equation (3.29) is a recursive relation and can be used to determine higher order $\delta$'s. If $n^2\lambda$ is not equal to any eigenvalue of $L_{ij}$ then the recurrence relation becomes

$$\delta_i^{(n)} = S_{ij}^{(n-1)} V_j^{(n-1)} \quad ; \quad S_{ij}^{(n)} \equiv L_{ij} - n^2\lambda\delta_{ij} \tag{3.32}$$

If $n^2\lambda$ equals one of the eigenvalues of $L_{ij}$ we have to be more careful. In fact, such kind of “degeneracy” opens the way for constructing new solutions of Hirota’s equation. Expanding both $V_j^{(n-1)}$ and $\delta_i^{(n)}$

$$\delta_i^{(n)} = \sum_k d_{ij}^{(n)} [k] v_i^{[k]} \quad ; \quad V_i^{(n-1)} = \sum_k c_{ij}^{(n-1)} [k] v_i^{[k]} \tag{3.33}$$

in terms of the eigenvectors $v_j^{[k]}$ of the $L_{ij}$ matrix:

$$L_{ij}v_j^{[k]} = \lambda^{[k]}v_i^{[k]} \tag{3.34}$$

and plugging this expansion back into eq. (3.29) results in

$$\left( \lambda^{[k]} - n^2\lambda \right) d_{ij}^{(n)} [k] = c_{ij}^{(n-1)} [k] \tag{3.35}$$

So, if $\lambda^{[k]} \neq n^2\lambda$ then $d_{ij}^{(n)} [k]$ is determined from (3.33). However if $\lambda^{[k]} = n^2\lambda$, $d_{ij}^{(n)} [k]$ is undetermined and we can choose it arbitrarily. Notice however that for consistency $c_{ij}^{(n-1)} [k]$ has to vanish in such case (in fact, in all cases where the kernel is non trivial we found that $c_{ij}^{(n-1)} [k]$ consistently vanishes). Such non trivial kernel of $L_{ij} - n^2\lambda\delta_{ij}$ introduces new parameters in the procedure and we will show below that it leads to extra soliton solutions of the AT and CAT models.

As we are going to show in the following sections, the eigenvalues of the matrix $L_{ij}$ for the classical algebras can be written as

$$\lambda^{[j]} = 4c_{ij} \sin^2 \left( \frac{j\pi}{h} \right) \quad ; \quad j = 0, 1, 2, \ldots, r_{ij} \tag{3.36}$$
where \( c_G = 1 \) and \( r_G = \text{rank}\, G \) for \( G = SU(r+1) \), and \( Sp(r) \), and \( c_G = 2 \) for \( G = SO(2r) \) and \( SO(2r+1) \), whilst \( r_G = \text{rank}\, G - 1 \) for \( G = SO(2r+1) \) and \( r_G = \text{rank}\, G - 2 \) for \( G = SO(2r) \). \( SO(2r+1) \) has an eigenvalue \( \lambda = 2 \) and \( SO(2r) \) has two eigenvalues \( \lambda = 2, 2 \).

Consequently the kernel of \( L_{ij} - n^2 \lambda \delta_{ij} \) for the classical algebras is non trivial when there exists \( j, j' \) and \( n \) such that \( \sin^2 \left( \frac{j}{h} \right) = n^2 \sin^2 \left( \frac{j'}{h} \right) \). The solutions for such equation are very scarce and can be obtained from the following more general theorem \([12]\) which we here present without proof.

**Theorem 3.2** The only solutions of the equation

\[
\sin (a\pi) = c \sin (b\pi)
\]

with \( a, b \) and \( c \) non-zero rational numbers are when either \( c = \pm 1 \) or \( \{ \sin (a\pi), \sin (b\pi) \} \subset \{ 1, -1, \frac{1}{2}, -\frac{1}{2} \} \).

Therefore the kernel is non trivial for \( SU(6p) \) and \( Sp(3p) \) only, with \( p \) a positive integer, (where in both cases \( h = 6p \)) with \( j = 3p \), \( j' = p \) and \( n = 2 \). We will show later that in such cases we obtain new soliton solutions.

For the exceptional algebras one can check that the kernel of \( L_{ij} - n^2 \lambda \delta_{ij} \) is always trivial.

Summarizing, using the above method we can find the \( \delta^{(n)}_i \) recursively using (3.32) and/or (3.33). We expect to find the soliton solutions for the cases where this series truncates. That is, we have to have:

\[
V_i^{(n-1)} = 0 \quad \text{for} \quad n > N_i
\]

since from now on all \( \delta_i^{(n)} \) will be zero.

### 3.4 Calculation of \( V_i^{(n-1)} \)

The calculation of \( V_i^{(n-1)} \) defined in (3.30) can be easily performed using the Leibniz rule

\[
\frac{d^n(AB)}{dx^n} = \sum_{l=0}^{n} \frac{d^l A}{dx^l} \frac{d^{n-l} B}{dx^{n-l}}
\]

and the fact that for an arbitrary function \( H \) of the tau-functions we have \( H^{(n)} = d^n H / d\epsilon^n |_{\epsilon = 0} / n! \). Let us first give a derivation of \( a^{(n-1)}_i \) used to determine \( V_i^{(n-1)} \) in (3.31). From (3.6) and (3.24) we have \( G_i = \tau_i \partial_+ \partial_- \tau_i - \partial_+ \tau_i \partial_- \tau_i \). Therefore

\[
G_i^{(n)} = \gamma_+ \gamma_- \sum_{l=0}^{n} \left( (n-l)^2 - l(n-l) \right) \delta_i^{(l)} \delta_i^{(n-l)} e^{n\Gamma}
\]

Dividing by \( \gamma_+ \gamma_- \exp(n\Gamma) \) and subtracting the term \( n^2 \delta_i^{(n)} \) corresponding to \( \partial_+ \partial_- \tau_i \) we obtain

\[
a_i^{(n-1)} = \sum_{l=1}^{n-1} \left( n^2 - 3nl + 2l^2 \right) \delta_i^{(l)} \delta_i^{(n-l)}
\]

The final form of \( V_i^{(n-1)} \) depends through \( b_i^{(n-1)} \) on the Cartan matrix and varies therefore from algebra to algebra. However there are few generic terms which always appear in \( F_i^{(n)} \).

As seen from (3.25) these terms are: \( (\tau_i^2)^{(n)} \), \( (\tau_{k-1} \tau_{k+1})^{(n)} \), \( (\tau_i \tau_j^2)^{(n)} \) and \( (\tau_i \tau_j \tau_k)^{(n)} \). Therefore
we give here the results for generic quadratic and cubic terms in $\tau$-functions. Applying the same procedure as for the $G_i$ function we arrive at

$$\left(\tau_i \tau_j\right)^{(n)}/e^{n\Gamma} = \delta_i^{(n)} + \delta_j^{(n)} + \sum_{l=1}^{n-1} \delta_i^{(l)} \delta_j^{(n-l)}$$

(3.41)

and similarly

$$\left(\tau_i \tau_j \tau_k\right)^{(n)}/e^{n\Gamma} = \delta_i^{(n)} + \delta_j^{(n)} + \delta_k^{(n)} + \sum_{l=1}^{n-1} \delta_i^{(l)} \delta_j^{(n-l)} + \sum_{l=1}^{n-1} \delta_j^{(l)} \delta_k^{(n-l)} + \sum_{l=1}^{n-1} \delta_i^{(l)} \delta_k^{(n-l)}$$

(3.42)

Based on these results we will complete the calculation of $V_i^{(n-1)}$ below for the relevant algebras.

4 $A_r \sim SU(r+1)$

The extended Cartan Matrix for $SU(r+1)$ is given as

$$K = \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 & -1 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & -1 & 2 & -1 & 0 \\
-1 & 0 & 0 & \ldots & 0 & -1 & 2
\end{pmatrix}$$

(4.1)

and Hirota’s equation (3.4) reads

$$\tau_j^2 \triangle (\tau_j) = \beta \left(\tau_j^2 - \tau_{j+1} \tau_{j-1}\right) \quad \text{for } j = 0, 1, 2, \ldots, r$$

(4.2)

where, due to the periodicity of the extended Dynkin diagram, it is understood that $\tau_{j+r+1} = \tau_j$. The first order Hirota solution is obtained from the eigenvectors of $L_{ij} = l_i^\varphi K_{ij}$ as in (3.14), namely

$$L_{ij} v_j = \lambda v_i$$

(4.3)

Since $l_i^\varphi = 1$ for all $i$’s, this yields the system of equations

$$v_{j-1} - (2 - \lambda) v_j + v_{j+1} = 0 \quad \text{for } j = 0, 1, \ldots, r$$

(4.4)

with

$$v_{j+r+1} = v_j$$

(4.5)

Eq. (1.3) can be recognized as the recurrence relation for the Chebyshev polynomials with $2 - \lambda = 2x$ (see appendix A). Therefore any linear combination of the Chebyshev polynomials of type I and II satisfy (4.4). The relation (4.5) is equivalent, in fact, to the secular equation for
and determines the eigenvalues. Using the trigonometric representation of the Chebyshev polynomials given in appendix A we get

$$\lambda_j = 4 \sin^2 \left( \frac{j \pi}{r+1} \right) \quad \text{for } j = 0, 1, 2, \ldots, r \quad (4.6)$$

By inspection $\lambda_j = \lambda_{r+1-j}$ which clearly indicates that all eigenvalues, apart from $\lambda_0 = 0$ and $\lambda_{(r+1)/2} = 4$ (for even $r+1$), are degenerate. The corresponding eigenvectors are

$$v_k^{[0]} = 1$$
$$v_k^{[\lambda_i]} = \exp \left( \frac{2\pi i l k}{r+1} \right) \quad ; \quad v_k^{[\lambda_i]} = \exp \left( -\frac{2\pi i l k}{r+1} \right) \quad l = 1, 2, \ldots, \left[ \frac{r}{2} \right]$$

$$v_k^{[\lambda_{(r+1)/2}]} = (-1)^k \quad \text{for } r+1 \text{ even} \quad (4.7)$$

where $k = 0, 1, 2, \ldots, r$ and

$$\left[ \frac{r}{2} \right] = \begin{cases} \frac{(r-1)}{2}, & \text{if } r \text{ is odd} ; \\ r/2, & \text{otherwise.} \end{cases}$$

Let us now apply our Hirota perturbation method to $SU(r+1)$. From (3.30), (3.40) and (3.41) we find a closed expression for $V_i^{(n-1)}$ to be

$$V_i^{(n-1)} = -\sum_{l=1}^{n} \left( 1 - \lambda \left( n^2 - 3nl + 2l^2 \right) \right) \delta_j^{(l)} \delta_j^{(n-l)} - \delta_{j+1}^{(n-1)} \delta_{j-1}^{(l)} \quad (4.8)$$

According to (3.14), $\delta^{(1)}$ has to be an eigenvector of $L$. We start with the eigenvalues $\lambda_0 = 0$ and $\lambda_{(r+1)/2} = 4$ (for even $r+1$) which are not degenerate, and so we have $\delta_j^{(1)} = 1$ and $\delta_j^{(1)} = (-1)^j$ respectively. Substituting these vectors into expression (4.8) for $n = 2$ we see that $V_j^{(1)} = 0$. Consequently from (3.32) we have $\delta_j^{(2)} = 0$, which leads to $V_j^{(2)} = 0$ and $\delta_j^{(3)} = 0$ and so on. Hence the Hirota perturbation series truncates at first order. For $\lambda_0 = 0$ we then get $\tau_j = 1 + e^\Gamma$, which is a trivial solution in the sense that the $\varphi$’s fields are constants (see (3.1)). For $\lambda_{(r+1)/2} = 4 = \gamma + \gamma - \beta$ (for $r+1$ even) we get

$$\tau_j = 1 + (-1)^j e^\Gamma \quad (4.9)$$

According to (3.14) the mass of such solution is $M = \frac{8(r+1)}{\psi^2}m$ (for $r+1$ even).

For the remaining set of (degenerate) eigenvalues we consider the following general linear combination:

$$\delta_j^{(1)} = y_1 \exp \left( \frac{2\pi i l j}{r+1} \right) + y_2 \exp \left( -\frac{2\pi i l j}{r+1} \right) \quad l = 1, 2, \ldots, \left[ \frac{r}{2} \right] \quad (4.10)$$

where $j = 0, 1, 2, \ldots, r$, $y_1$ and $y_2$ are arbitrary constants. The calculation based on (1.8) gives:

$$V_j^{(1)} = -2y_1y_2 \left( 1 - \cos \left( \frac{4\pi l}{r+1} \right) \right) = -y_1y_2\lambda_l (4 - \lambda_l) \quad l = 1, 2, \ldots, \left[ \frac{r}{2} \right] \quad (4.11)$$
where we used expression (4.6) for the eigenvalues \( \lambda_l \). Rewriting (4.11) as \( V(1) = y_1 y_2 \lambda_l (4 - \lambda_l) v^{[0]} \) and expanding \( \delta^{(2)} = y_1 y_2 v^{[0]}(1 - \frac{1}{4} \lambda_l) \) in terms of the eigenvectors from (4.7) we get after substitution into (3.35)

\[
\delta^{(2)} = y_1 y_2 \left( 1 - \frac{1}{4} \lambda_l \right) v^{[0]} \tag{4.12}
\]

Substituting this into (4.8) we find \( V(2) = 0 \) and consequently \( \delta^{(3)} = 0 \), which in turn leads to \( V^{(3)} = 0 \) and \( \delta^{(4)} = 0 \). One can verify that \( V^{(n)} = 0 \) with \( n \geq 4 \). Therefore \( \delta^{(n)} = 0 \) for \( n \geq 3 \).

To summarize, the general solution of Hirota’s equations for \( A_r \sim SU(r + 1) \) is given by the tau-function

\[
\tau_j = 1 + \left( y_1 \exp \left( \frac{2 \pi i l j}{r + 1} \right) + y_2 \exp \left( \frac{-2 \pi i l j}{r + 1} \right) \right) e^{r} + y_1 y_2 \left( 1 - \frac{1}{4} \lambda_l \right) e^{2r} \tag{4.13}
\]

with \( \Gamma \) given by (3.11), \( \frac{\alpha \gamma + \beta}{\beta} = \lambda_l = 4 \sin^2 \left( \frac{l \pi}{r + 1} \right) \) and \( l = 1, \ldots, \left[ \frac{r}{2} \right] \). The solutions given in (4.13) contain those obtained by Hollowood when either \( y_1 = 0 \) or \( y_2 = 0 \) where the corresponding masses are given by eqns. (3.19) with \( \kappa = 1 \), i.e.

\[
M_l = \frac{8(r + 1)}{\psi^2} m \sin \left( \frac{l \pi}{r + 1} \right) \tag{4.14}
\]

However when considering both \( y_1 \) and \( y_2 \neq 0 \), the solution truncates only in the second order in \( \epsilon \) yielding a solution with mass twice as large. We should point out that in such case we have a “two-soliton-like” solution surviving in the static limit.

If we consider the topological charge given in (3.22) we find that when \( y_1 = 0 \) the corresponding charge is

\[
Q_{-l} = - \frac{1}{2 \pi} \sum_{a=1}^{r} \frac{2 \alpha_a}{\alpha_a^2} \left( \frac{2 \pi i l a}{r + 1} \right) = \omega_{r+1-l} + \beta_+ \tag{4.15}
\]

for some \( \beta_+ \in \Lambda_R \). Taking now \( y_2 = 0 \) the charge is

\[
Q_l = - \frac{1}{2 \pi} \sum_{a=1}^{r} \frac{2 \alpha_a}{\alpha_a^2} \left( \frac{2 \pi i l a}{r + 1} \right) = \omega_l + \beta_+ \tag{4.16}
\]

for some \( \beta_+ \in \Lambda_R \) and \( l = 1, \ldots, \left[ \frac{r}{2} \right] \). \( \omega_i \) \( i = 1, \ldots, r \) are the fundamental weights of \( SU(r + 1) \).

The case of solution (4.13) when \( y_1, y_2 \neq 0 \), under such prescription correspond to charges lying in the root lattice \( \Lambda_R \).

### 4.1 Second type of degeneracy for \( SU(6p) \)

As noted in section 3.3, for the eigenvalue \( \lambda_p = 1 \) of \( SU(6p) \) the higher order \( \delta^{(n)} \)'s \( n \geq 2 \) are not uniquely determined by \( \delta^{(1)} \) since the kernel of \( L_{ij} - n^2 \lambda \delta_{ij} \) is not trivial for \( n = 2 \). We therefore can add to \( \delta^{(2)} \) a term proportional to the eigenvector corresponding to \( \lambda_{3p} = 4 \),
introducing a new parameter in the solution. The first order contribution to the tau-function corresponding to \( \lambda_p = 1 \) is

\[
\delta^{(1)}_j = y_1 \exp \left( \frac{2\pi ipj}{6p} \right) + y_2 \exp \left( -\frac{2\pi ipj}{6p} \right) = y_1 \exp \left( \frac{i\pi j}{3} \right) + y_2 \exp \left( -\frac{i\pi j}{3} \right) \tag{4.17}
\]

where \( y_1 \) and \( y_2 \) are arbitrary constants reflecting the ordinary degeneracy between eigenvalues \( \lambda_p \) and \( \lambda_{6p-p} \). From (4.11) we get

\[
V_j^{(1)} = -3y_1y_2 \tag{4.18}
\]

and it is easy to see that the linear combination

\[
\delta^{(2)} = \frac{3y_1y_2}{4} v_{[\lambda=0]} + z v_{[\lambda_{3p}]} = \frac{3y_1y_2}{4} + z(-1)^j \tag{4.19}
\]

for arbitrary \( z \) satisfies the second order recurrence relation: \( (L - 2^2\lambda_p I)\delta^{(2)} = V^{(1)} \) recalling that \( 2^2\lambda_p = \lambda_{3p} \). Plugging now (4.19) into (4.8) we obtain

\[
V_j^{(2)} = (-1)^{j+1} z \left( \delta^{(1)}_j + \delta^{(1)}_{j+1} + \delta^{(1)}_{j-1} \right) = -2z \left( y_1 u_{[2]}^{[\lambda_{3p}]} + y_2 u_{[1]}^{[\lambda_{3p}]} \right) \tag{4.20}
\]

hence \( V_j^{(2)} \) is a linear combination of two eigenvectors corresponding to the degenerate eigenvalue \( \lambda_{2p} = 3 \). From recurrence relation \( (L - 3^2\lambda_p I)\delta^{(3)} = V^{(2)} \) we now find \( \delta^{(3)}_j = z(-1)^j \delta^{(1)}_j / 3 \). One more step of recurrence procedure yields \( V^{(3)} = -zy_1y_2(-1)^j \) and \( \delta^{(4)}_j = (-1)^j zy_1y_2/12 \). At this point the recurrence procedure terminates and the final solution is given by

\[
\tau_j = 1 + \left( y_1 \exp \left( \frac{i\pi j}{3} \right) + y_2 \exp \left( -\frac{i\pi j}{3} \right) \right) e^\Gamma + \left( \frac{3}{4} y_1y_2 + (-1)^j z \right) e^{2\Gamma} + \frac{z}{3}(-1)^j \left( y_1 \exp \left( \frac{i\pi j}{3} \right) + y_2 \exp \left( -\frac{i\pi j}{3} \right) \right) e^{3\Gamma} + (-1)^j \frac{zy_1y_2}{12} e^{4\Gamma} \tag{4.21}
\]

which of course reproduces (4.13) for \( z \to 0 \). The mass in such case is given by the mass formula (3.19) with \( \kappa = 4 \), namely \( M = \frac{96m_p}{\Lambda} \). The topological charge in this case can be verified to lie in the coset \( \omega_{3p} + \Lambda_R \) of \( \Lambda_W/\Lambda_R \).

### 5 Sp(r) Case

The Cartan matrix is given by

\[
K = \begin{pmatrix}
2 & -2 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & \ldots & 0 & -2 & 2 \\
\end{pmatrix} \tag{5.1}
\]
\( l_i^\psi = 1 \) for \( i = 0, 1, 2, \ldots, r \), and yields the following Hirota’s equations

\[
\begin{align*}
\tau_0^2 \triangle (\tau_0) &= \beta (\tau_0^2 - \tau_1^2) \\
\tau_a^2 \triangle (\tau_a) &= \beta (\tau_a^2 - \tau_{a+1} \tau_{a-1}) \quad \text{for} \ a = 1, 2, \ldots, r - 1 \\
\tau_r^2 \triangle (\tau_r) &= \beta (\tau_r^2 - \tau_{r-1}^2)
\end{align*}
\]

(5.2)

The corresponding eigenvalue equation \( L_{ij} v_j = K_{ij} v_j = \lambda v_i \) reads in components as

\[
\begin{align*}
(2 - \lambda) v_0 - 2 v_1 &= 0 \quad \text{(5.3)} \\
-v_{a-1} + (2 - \lambda) v_a - v_{a+1} &= 0 \quad \text{for} \ a = 1, 2, \ldots, r - 1 \quad \text{(5.4)} \\
-2v_{r-1} + (2 - \lambda)v_r &= 0 \quad \text{(5.5)}
\end{align*}
\]

From (3.14), \( \delta^{(1)} \) has to be one of the eigenvalues of \( L \equiv K \). Using the results of section 3.4 the function \( V_i^{(2)} \) from (3.30) reads

\[
\begin{align*}
V_0^{(2)}(\delta^{(1)}) &= (\delta_0^{(1)})^2 - (\delta_1^{(1)})^2 \\
V_a^{(2)}(\delta^{(1)}) &= (\delta_a^{(1)})^2 - \delta_{a+1}^{(1)} \delta_{a-1}^{(1)} \quad \text{for} \ a = 1, 2, \ldots, r - 1 \\
V_r^{(2)}(\delta^{(1)}) &= (\delta_r^{(1)})^2 - (\delta_{r-1}^{(1)})^2
\end{align*}
\]

(5.6) (5.7) (5.8)

Our soliton construction is based on two results:

1. The eigenvalues of the extended Cartan matrix are \( \lambda_j = 4 \sin^2 (j \pi / 2r) \), \( j = 0, \ldots, r \)

2. \( V_i^{(2)}(\delta^{(1)}) = (1 - x^2)(\delta_0^{(1)})^2 \), where \( x = (2 - \lambda)/2 \) and \( V_i^{(n)} = 0 \) for \( n \geq 3 \).

The proof of these results is based on the basic properties of Chebyshev polynomials [13, 14] listed in Appendix A. Let us start on the proof by first introducing the variable \( x = (2 - \lambda)/2 \) into (5.3), (5.4) and (5.5) and rewriting

\[
v_i = T_i(x)v_0 \quad ; \ i = 0, \ldots, r
\]

(5.9)

Now equations (5.3) and (5.4) are trivially satisfied due to the basic recurrence relation (A.1). The remaining equation (5.5) becomes

\[
xT_r(x) - T_{r-1}(x) = 0
\]

(5.10)

This equation is not satisfied automatically by the Chebyshev’s polynomials. It is in fact equivalent to imposing the secular equation for the \( SP(r) \) Cartan matrix. Using (A.2) we see that (5.10) is equivalent to

\[
(1 - x^2)T'_r(x) = 0
\]

(5.11)

So the \( (r + 1) \) solutions to (5.10) are given by the extrema of \( T_r \) from (A.3). This corresponds to

\[
\lambda_j = 4 \sin^2 \left( \frac{j \pi}{2r} \right)
\]

(5.12)
which proves the first observation. Next, substituting (5.9) into (5.6) we get

\[ V_0(2) = (1 - x^2)(\delta_0^{(1)})^2 \]  

(5.13)

Using (A.3) we get for (5.7)

\[ V_a(2) = (T_a^2 - T_{a+1}T_{a-1}) (\delta_0^{(1)})^2 = \frac{1}{2} (T_0 - T_2) (\delta_0^{(1)})^2 = (1 - x^2)(\delta_0^{(1)})^2 \]  

(5.14)

Using (5.10) we get

\[ V_r(2) = (T_r^2 - T_{r-1}^2) (\delta_0^{(1)})^2 = (1 - x^2)T_r^2 (\delta_0^{(1)})^2 = (1 - x^2)(\delta_0^{(1)})^2 \]  

(5.15)

where we used the fact that eigenvalues correspond to extremas for \( T_r \) (see (A.6)). Therefore (5.13) and (5.14) hold for any value of \( x \), while (5.15) is valid only when \( x \) corresponds to an eigenvalue of the \( Sp(r) \) Cartan Matrix. So the first part of point 2 is proved. We now prove the rest. The eigenvector corresponding to \( \lambda = 0 \) (\( x = 1 \)) is \( v_{\lambda=0} \), as can be seen from (A.7) or direct inspection. Therefore \( V(2) \) is proportional to \( v_{\lambda=0} \). From (3.33) and (3.35) we see that \( \delta^{(2)} \) is also proportional to \( v_{\lambda=0} \)

\[ \delta^{(2)} = \left( \frac{1 + x}{8} \right) (\delta_0^{(1)})^2 \]  

(5.16)

(eespecially all components are equal). Note, that \( \delta^{(2)} \) vanishes for \( \lambda = 0 \) (\( x = 1 \)), due to indeterminacy in (3.35) we don’t get this result from (5.16).

From direct calculation we get

\[ V_0^{(3)} = (2 - \lambda)\delta_0^{(1)}\delta_0^{(2)} - 2\delta_1^{(1)}\delta_1^{(2)} \]  

(5.17)

\[ V_a^{(3)} = (2 - \lambda)\delta_a^{(1)}\delta_a^{(2)} - \delta_{a-1}^{(1)}\delta_{a+1}^{(2)} - \delta_{a+1}^{(1)}\delta_{a-1}^{(2)} \]  

for \( a = 1, 2, \ldots, r - 1 \)  

(5.18)

\[ V_r^{(3)} = (2 - \lambda)\delta_r^{(1)}\delta_r^{(2)} - 2\delta_{r-1}^{(1)}\delta_{r-1}^{(2)} \]  

(5.19)

After factorizing the equal components of \( \delta^{(2)} \) we are left with the eigenvalue equation for \( \delta^{(1)} \) and so \( V^{(3)} \) vanishes. From (3.33) also \( \delta^{(3)} \) must vanish. Hence \( V^{(4)} \) will be the same as \( V^{(2)} \) with \( \delta^{(1)} \) replaced by \( \delta^{(2)} \). Again since all components of \( \delta^{(2)} \) are equal \( V^{(4)} \) vanishes. Since all \( V^{(n)} \) are quadratic in \( \delta \)'s and since the highest nonvanishing \( \delta \) is \( \delta^{(2)} \) it follows that \( V^{(n>4)} = 0 \).

From the above considerations we can write the general soliton solution for \( SP(r) \) as

\[ \tau_i = 1 + T_i(x)\delta_0^{(1)} e^\Gamma + \left( \frac{1 + x}{8} \right) (\delta_0^{(1)})^2 T_i(1)e^{2\Gamma} \]  

(5.20)

where \( x = (2 - \lambda)/2 \) and \( \lambda = \gamma_+\gamma_-/\beta \). Notices that for \( x = -1 \) the series breaks at the first order and this corresponds to known solutions of \( Sp(r) \) [2]. For \( x = 1 \) we have

\[ \tau_i = 1 + T_i(1)\delta_0^{(1)} e^\Gamma \]  

(5.21)

and so all the \( \varphi \) fields are constant since all \( \tau \)'s are equal.
Masses can be easily analyzed from the universal soliton mass formula \((3.19)\). Clearly, \(\kappa = 2\) for the new solutions in \((5.20)\) and taking into account the value of the Coxeter number for \(Sp(r) \quad (h = 2r)\) we find

\[
M_j = \frac{8r}{\psi^2} m \sqrt{\lambda_j} = \frac{16r}{\psi^2} m \sin \left(\frac{\pi j}{2r}\right) \quad ; \quad j = 0, \ldots, r \quad (5.22)
\]

For solutions with \(x = -1\) the mass would be half as large.

The topological charge defined in \((3.22)\) lies in the coroot lattice \(\Lambda^v_R\) for \(x \neq -1\) since all \(\tau\)'s possess in common the same coefficient of the highest power in \(e^\Gamma\). For \(x = -1\) the corresponding topological charges given from \((5.20)\) is

\[
Q_m = -\frac{1}{2\pi} \sum_{a=1}^r \frac{2\alpha_a}{\alpha_a^2} (\pi m) = \omega_1 + \hat{\beta}_m \quad (5.23)
\]

where \(\omega_1\) is a fundamental weight of \(Sp(r)\) leading to the defining representation. Again the solutions \((5.20)\) provide solitons with topological charges lying in the coset \(\Lambda^v_W/\Lambda^v_R\) where \(\Lambda^v_W\) and \(\Lambda^v_R\) denote the coweight and coroot lattices of \(Sp(r)\) respectively.

\section{6 \quad D_r = SO(2r)}

The case of \(D_4 = SO(8)\) has a special interest and so we start by discussing it separately.

\subsection{6.1 \quad D_4 = SO(8)}

The matrix \(L_{ij} = l^\psi_i K_{ij}\) is given by

\[
L = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 \\
-2 & -2 & 4 & -2 & -2 \\
0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & 0 & 2
\end{pmatrix} \quad (6.1)
\]

and the integers \(l^\psi_i\) are \(1, 1, 2, 1, 1\). The eigenvalues of the matrix \(L\) are \((0, 2, 2, 2, 6)\) and the corresponding right eigenvectors are

\[
v^{\lambda=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad v^{\lambda=2}_{[1]} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}; \quad v^{\lambda=2}_{[2]} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}; \quad v^{\lambda=2}_{[3]} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}; \quad v^{\lambda=6} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{pmatrix} \quad (6.2)
\]

The Hirota’s equations read

\[
\tau_0^2 \triangle (\tau_0) = \beta (\tau_0^2 - \tau_2)
\]
\[ \tau_1^2 \triangle (\tau_1) = \beta \left( \tau_1^2 - \tau_2 \right) \]
\[ \tau_2^2 \triangle (\tau_2) = 2\beta \left( \tau_2^2 - \tau_0 \tau_1 \tau_3 \tau_4 \right) \]
\[ \tau_3^2 \triangle (\tau_3) = \beta \left( \tau_3^2 - \tau_2 \right) \]
\[ \tau_4^2 \triangle (\tau_4) = \beta \left( \tau_4^2 - \tau_2 \right) \]  \hspace{1cm} (6.3)

The solution corresponding to \( \frac{\tau_1 \tau_2 \tau_3 \tau_4}{\beta} = \lambda = 6 \) is given

\[ \delta^{(1)} = (1, 1, -4, 1, 1) \]
\[ \delta^{(2)} = (0, 0, 1, 0, 0) \]  \hspace{1cm} (6.4)

and all the remaining \( \delta \)'s vanish. Therefore according to Theorem 3.1 we have \( \kappa = 1 \). So, from (3.19) we get the mass of such soliton is \( M = \frac{2}{\psi^2} 12\sqrt{6} m \).

The eigenvalue \( \lambda = 2 \) has multiplicity 3. We then apply Hirota's method by taking \( \delta^{(1)} \) to be a general linear combination of the three eigevectors. The solution is then given by

\[ \delta^{(1)} = \left( \begin{array}{c}
 x_1 + x_2 + x_3 \\
 -x_1 + x_2 - x_3 \\
 0 \\
 x_1 - x_2 - x_3 \\
 -x_1 - x_2 + x_3
\end{array} \right) ; \quad \delta^{(2)} = \frac{1}{3} \left( \begin{array}{c}
 x_1 x_2 + x_1 x_3 + x_2 x_3 \\
 -x_1 x_2 + x_1 x_3 - x_2 x_3 \\
 3(x_1^2 + x_2^2 + x_3^2) \\
 -x_1 x_2 - x_1 x_3 + x_2 x_3 \\
 x_1 x_2 - x_1 x_3 - x_2 x_3
\end{array} \right) \]

\[ \delta^{(3)} = P_3 \left( \begin{array}{c}
 1 \\
 1 \\
 -16 \\
 1 \\
 1
\end{array} \right) ; \quad \delta^{(4)} = P_4 \left( \begin{array}{c}
 0 \\
 0 \\
 1 \\
 0 \\
 0
\end{array} \right) ; \quad \delta^{(5)} = \left( \begin{array}{c}
 0 \\
 0 \\
 0 \\
 0 \\
 0
\end{array} \right) ; \quad \delta^{(6)} = P_3^2 \left( \begin{array}{c}
 0 \\
 0 \\
 1 \\
 0 \\
 0
\end{array} \right) \]  \hspace{1cm} (6.5)

and the higher \( \delta \)'s vanish and where \( P_3 = x_1 x_2 x_3 / 27 \) and \( P_4 = (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) / 9 \). The solution holds true for any value (even complex) of the parameters \( x_1, x_2 \) and \( x_3 \) used in the linear combination.

The properties of the solution seem to be insensitive to the actual values of the parameters \( x \)'s as long as they do not vanish. However when one or two of them vanish the solution changes substantially. According to Theorem 3.1 we have \( \kappa = 1, 2, 3 \) when two, one and none of the \( x \)'s vanish respectively. Then from (3.19) the masses are given by

\[ M_1 = \frac{2}{\psi^2} 12\sqrt{2} m \quad \text{; when two } x \text{'s vanish} \]
\[ M_2 = \frac{2}{\psi^2} 24\sqrt{2} m \quad \text{; when one of the } x \text{'s vanishes} \]
\[ M_3 = \frac{2}{\psi^2} 36\sqrt{2} m \quad \text{; when none of the } x \text{'s vanishes} \]

The topological charge defined in (3.22) yields for the solution (6.4), \( Q \in \Lambda_R \), the root lattice of \( SO(8) \). For those solutions given in (3.3), the topological charge depends upon the
parameters \( x_1, x_2 \) and \( x_3 \). By explicit calculation it can be shown that

\[
Q(x_1, 0, 0) \in \lambda_v + \Lambda_R
\]
\[
Q(0, x_2, 0) \in \lambda_s + \Lambda_R
\]
\[
Q(0, 0, x_3) \in \tilde{\lambda}_s + \Lambda_R
\]
\[
Q(x_1, x_2, x_3) \in \Lambda_R
\]

where \( \lambda_v, \lambda_s \) and \( \tilde{\lambda}_s \) are the vector and the two spinor weights of \( SO(8) \) respectively.

6.2 \( D_r = SO(2r) \) for \( r \geq 5 \)

The \( L \) matrix is here given by

\[
L = \begin{pmatrix}
2 & 0 & -1 & 0 & \ldots & 0 & 0 \\
0 & 2 & -1 & 0 & \ldots & 0 & 0 \\
-2 & -2 & 4 & -2 & \ldots & 0 & 0 \\
0 & 0 & -2 & 4 & -2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
0 & 0 & \ldots & -2 & 4 & -2 & -2 \\
0 & 0 & \ldots & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & \ldots & -1 & 0 & 2
\end{pmatrix}
\]

(6.7)

with the integers \( l_i^\psi \) given by \((1, 1, 2, 2, \ldots, 2, 1, 1)\). To analyze the eigenvalue equation \( L_{ij} v_j = \lambda v_i \) we follow the usual procedure of first introducing variable \( x = (4 - \lambda)/4 \) and expressing the components \( v_i \) in terms of Chebyshev polynomials. As usual the generic part of the eigenvalue problem takes the form of the recurrence relations for Chebyshev polynomials (A.1). Trying to fit an ansatz in form of linear combination of Chebychev polynomials into all eigenvalue equations produces the following result (in case \( \lambda \neq 2 \) or \( x \neq \frac{1}{2} \)):

\[
v_1 = v_0, \quad v_a = 2(U_{a-1} - U_{a-2}) v_0 \quad ; \quad a = 2, 3, \ldots, r - 2 \\
v_r = v_{r-1} = (U_{r-2} - U_{r-3}) v_0
\]

(6.8)

with the consistency condition, playing the role of the secular equation,

\[
v_{r-2} = (4x - 2) v_r \quad \text{or} \quad (x - 1)T'_{r-1}(x) = 0
\]

(6.9)

to be imposed on the solution (6.8). The solutions to (6.9) take the following form

\[
\lambda_k = 8 \sin^2 \left( \frac{k\pi}{2(r - 1)} \right) \quad k = 0, 1, \ldots, r - 2 \quad (\lambda \neq 2)
\]

(6.10)

The case \( \lambda = 2 \) has to be treated separately. One finds that the eigenvalue \( \lambda = 2 \) is twofold degenerated with eigenvectors:

\[
v^\lambda=2_{[1]} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v^\lambda=2_{[2]} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{pmatrix}
\]

(6.11)
In case where \( r - 1 \) is a multiple of \( 3 \) \((r - 1 = 3N)\) degeneracy becomes threefold and the extra eigenvector has components given in terms of three components \( v_0, v_1, v_{r-1} \) playing role of free parameters

\[
v_{3+3k} = v_{4+3k} = -v_{6+3k} = -v_{7+3k} = -(v_0 + v_1) \quad k = 0, 2, 4, 6, \ldots
\]

\[
v_{2+3k} = 0 \quad k = 0, 1, \ldots, N
\]

\[
v_r = -v_{r-1} + (-1)^{N+1}(v_0 + v_1)
\]

(6.12)

The Hirota's equations read

\[
\tau_0^2 \triangle (\tau_0) = \beta (\tau_0^2 - \tau_2)
\]

\[
\tau_1^2 \triangle (\tau_1) = \beta (\tau_1^2 - \tau_2)
\]

\[
\tau_2^2 \triangle (\tau_2) = 2\beta (\tau_2^2 - \tau_0 \tau_1 \tau_3)
\]

\[
\tau_a^2 \triangle (\tau_a) = 2\beta (\tau_a^2 - \tau_{a-1} \tau_{a+1}) \quad a = 3, 4, \ldots, r - 3
\]

\[
\tau_{r-2}^2 \triangle (\tau_{r-2}) = 2\beta (\tau_{r-2}^2 - \tau_{r-3} \tau_{r-1} \tau_r)
\]

\[
\tau_{r-1}^2 \triangle (\tau_{r-1}) = \beta (\tau_{r-1}^2 - \tau_{r-2})
\]

\[
\tau_r^2 \triangle (\tau_r) = \beta (\tau_r^2 - \tau_{r-2})
\]

(6.13)

Using the results of section 3.4 one obtains that the quantities \( V^{(n-1)}_i \) defined in (3.30) are given by

\[
V^{(n-1)}_0 = -\sum_{l=1}^{n-1} (1 - \lambda(n^2 - 3nl + 2l^2)) \delta_0^{(l)} \delta_0^{(n-l)}
\]

\[
V^{(n-1)}_1 = -\sum_{l=1}^{n-1} (1 - \lambda(n^2 - 3nl + 2l^2)) \delta_1^{(l)} \delta_1^{(n-l)}
\]

\[
V^{(n-1)}_2 = -\sum_{l=1}^{n-1} (2 - \lambda(n^2 - 3nl + 2l^2)) \delta_2^{(l)} \delta_2^{(n-l)} + 2 \sum_{l=1}^{n-1} \left( \delta_0^{(l)} \delta_1^{(n-l)} + \delta_1^{(l)} \delta_3^{(n-l)} + \delta_0^{(l)} \delta_3^{(n-l)} + \sum_{m=1}^{l-1} \delta_0^{(m)} \delta_1^{(l-m)} \delta_3^{(n-l)} \right)
\]

\[
V^{(n-1)}_a = -\sum_{l=1}^{n-1} \left( (2 - \lambda(n^2 - 3nl + 2l^2)) \delta_a^{(l)} \delta_a^{(n-l)} - 2\delta_{a-1}^{(l)} \delta_{a+1}^{(n-l)} \right) ; \quad a = 3, 4, \ldots, r - 3
\]

\[
V^{(n-1)}_{r-2} = -\sum_{l=1}^{n-1} (2 - \lambda(n^2 - 3nl + 2l^2)) \delta_{r-2}^{(l)} \delta_{r-2}^{(n-l)}
\]

\[
+ 2 \sum_{l=1}^{n-1} \left( \delta_{r-3}^{(l)} \delta_{r-1}^{(n-l)} + \delta_{r-1}^{(l)} \delta_{r}^{(n-l)} + \delta_{r-3}^{(l)} \delta_{r}^{(n-l)} + \sum_{m=1}^{l-1} \delta_{r-3}^{(m)} \delta_{r-1}^{(l-m)} \delta_{r}^{(n-l)} \right)
\]

\[
V^{(n-1)}_{r-1} = -\sum_{l=1}^{n-1} (1 - \lambda(n^2 - 3nl + 2l^2)) \delta_{r-1}^{(l)} \delta_{r-1}^{(n-l)}
\]

\[
V^{(n-1)}_r = -\sum_{l=1}^{n-1} (1 - \lambda(n^2 - 3nl + 2l^2)) \delta_r^{(l)} \delta_r^{(n-l)}
\]

(6.14)
Applying the recurrence method described in section 3.3 one can now construct the solutions.

For the eigenvalues \( \lambda_k = 8 \sin^2 \frac{k \pi}{2(r-1)} = \frac{2 \gamma^2}{\beta} \neq 2, \ k = 1, 2, \ldots, r-2 \), we obtain the solution

\[
\begin{align*}
\tau_0 &= 1 + e^\Gamma \\
\tau_1 &= 1 + e^\Gamma \\
\tau_a &= 1 + 2 \cos \theta_k e^{\lambda} + e^{2\lambda} ; \quad a = 2, 3, \ldots, r-2 \\
\tau_{r-1} &= 1 + (-1)^k e^\Gamma \\
\tau_r &= 1 + (-1)^k e^\Gamma 
\end{align*}
\]  

(6.15)

where \( \theta_k = \frac{k \pi}{2(r-1)} \) and where we have used the fact that the Chebyshev polynomials satisfy

\[
U_{a-1}(x_k) - U_{a-2}(x_k) = \frac{\cos (2a-1)\theta_k}{\cos \theta_k} \text{ with } x_k = \cos 2\theta_k. \quad \text{According to Theorem 3.3 we see that} \\
\kappa = 1 \text{ in this case and therefore form } (6.13) \text{ the masses are given by}
\]

\[
M_k = \frac{2}{\psi^2} 8(r-1) \sqrt{2} m \sin \frac{k \pi}{2(r-1)} ; \quad k = 1, 2, \ldots, r-2
\]

(6.16)

These solutions correspond to topological charges lying in the coset \( Q_k \in \Lambda_u + \Lambda_R \) for \( k \) odd and \( Q_k \in \Lambda_R \) for \( k \) even, where \( \lambda_i \) denote the vector weight of \( SO(2r) \).

For \( \lambda = 2 \) we have to consider the case when \( r-1 \) is a multiple of 3 separately since as shown above the eigenvalue has multiplicity three. Let us consider first the case \( r-1 \neq 3 \) multiple of 3. The eigenvalue \( \lambda = 2 \) has multiplicity 2 and the eigenvectors are given by

\[
(6.11). \quad \text{So, taking } \delta^{(i)} \text{ to be a linear combination, with parameters } y_1 \text{ and } y_2, \text{ of these eigenvectors and applying the procedure of section 3.3 we obtain the following solution}
\]

\[
\begin{align*}
\tau_0 &= 1 + y_1 e^\Gamma + c(y_1, y_2) e^{2\Gamma} \\
\tau_1 &= 1 - y_1 e^\Gamma + c(y_1, y_2) e^{2\Gamma} \\
\tau_a &= 1 + d_a(y_1, y_2) e^{2\Gamma} + c(y_1, y_2)^2 e^{4\Gamma} ; \quad a = 2, 3, \ldots, r-2 \\
\tau_{r-1} &= 1 - y_2 e^\Gamma + (-1)^{r-1} c(y_1, y_2) e^{2\Gamma} \\
\tau_r &= 1 + y_2 e^\Gamma + (-1)^{r-1} c(y_1, y_2) e^{2\Gamma}
\end{align*}
\]  

(6.17)

where \( \Gamma \) is defined in (6.11), \( \lambda = 2 = \frac{2 \gamma^2}{\beta} \), \( c(y_1, y_2) = (y_1^2 + (-1)^{r-1} y_2^2)/4(r-1) \) and \( d_a = (-1)^a ((4r-a) - 2) y_1^2 + (-1)^r (4a-2)y_2^2)/4(r-1), \ a = 2, 3, \ldots, r-2 \).

There are two special solutions which correspond to the choices \( y_2 = \pm i r y_1 \) since in such case \( c(y_1, y_2) = 0 \). Then according to Theorem 3.1 we have \( \kappa = 1 \) and from (6.13) the masses of these two solutions are the same and equal to

\[
M_1 = \frac{2}{\psi^2} 4(r-1) \sqrt{2} m ; \quad \text{for } y_2 = \pm i r y_1
\]

(6.18)

For \( y_2 \neq \pm i r y_1 \) we have \( \kappa = 2 \) and therefore the mass is

\[
M_2 = \frac{2}{\psi^2} 8(r-1) \sqrt{2} m ; \quad \text{for } y_2 \neq \pm i r y_1
\]

(6.19)
For \( y_2 = i^* y_1 \) we can show that the topological charge for either \( r = 2k \) or \( r = 2k + 1 \) lie in the cosets, \( Q \in \lambda_s + \Lambda_R \) for even \( k \) or \( Q \in \lambda_s + \Lambda_R \) for odd \( k \). When \( y_2 = -i^* y_1 \) the situation is reversed, i.e. \( Q \in \lambda_s + \Lambda_R \) for odd \( k \) or \( Q \in \bar{\lambda}_s + \Lambda_R \) for even \( k \).

For the case where \( r - 1 \) is multiple of 3, the eigenvalue \( \lambda = 2 \) has multiplicity 3. Analogously, we then take \( \delta^{(1)} \) to be a linear combination of the three eigenvectors \((6.11)\) and \((6.12)\) and apply the method described in section 3.3. The calculations are a bit cumbersome and we give here the results for the simplest example corresponding to \( SO(3) \). However for \( SO(8) \) the degeneracy of \( \lambda = 2 \) is related to the symmetries of the Dynkin diagram whilst for \( SO(6p + 2) \) \( (p > 1) \) the degeneracy is accidental. Indeed, the solutions are different.

### 6.3 \( SO(14) \) solutions for \( \lambda = 2 \)

The three eigenvectors for the eigenvalue \( \lambda = 2 \) are

\[
\begin{align*}
v_{\lambda=2}^{[1]} &= (1, -1, 0, 0, 0, 0, 0) \\
v_{\lambda=2}^{[2]} &= (0, 0, 0, 0, 0, -1, 1) \\
v_{\lambda=2}^{[3]} &= (1, 1, 0, -2, -2, 0, 1, 1)
\end{align*}
\]

By taking \( \delta^{(1)} \) as a linear combination of these eigenvectors and applying the Hirota’s procedure described in section 3.3 we get the following solution

\[
\delta^{(1)} = \left( \begin{array}{c} y_1 + y_3 \\
- y_1 + y_3 \\
0 \\
-2 y_3 \\
-2 y_3 \\
0 \\
y_2 + y_3 \\
y_2 + y_3 \end{array} \right) ; \delta^{(2)} = \left( \begin{array}{c} \frac{y^2}{24} + \frac{y_1^2}{24} + \frac{y_3^2}{3} \\
\frac{y^2}{24} + \frac{y_1^2}{24} - \frac{y_3^2}{3} \\
\frac{y^2}{24} - \frac{y_1^2}{24} + y^2 \\
\frac{y^2}{24} - \frac{y_1^2}{24} + y^2 \\
\frac{y^2}{24} - \frac{y_1^2}{24} + y^2 \\
\frac{y^2}{24} - \frac{y_1^2}{24} + y^2 \\
\frac{y^2}{24} + \frac{y_1^2}{24} + \frac{y_3^2}{3} \\
\frac{y^2}{24} + \frac{y_1^2}{24} - \frac{y_3^2}{3} \end{array} \right) ; \delta^{(3)} = \frac{y_3}{27}
\]

\[
\delta^{(4)} = \left( \begin{array}{c} 0 \\
0 \\
0 \\
0 \\
P_4 - \frac{7 y_1^2 y_2^2}{108} + \frac{5 y_2^2 y_3^2}{36} \\
P_4 - \frac{10 y_1^2 y_2^2}{36} - \frac{7 y_2^2 y_3^2}{12} \\
P_4 + \frac{10 y_1^2 y_2^2}{36} + \frac{7 y_2^2 y_3^2}{12} \\
0 \\
0 \end{array} \right) ; \delta^{(5)} = P_5 \\
\delta^{(6)} = P_6
\]

(6.21)

where \( P_4 = (y_1^2 + y_2^2)^2 / 576, P_5 = -y_3 (y_1^2 + y_2^2)^2 / 2592 \) and \( P_6 = (y_3 (y_1^2 + y_2^2) / 216)^2 \).

We should distinguish four type of solutions:
1. We get three solutions by taking $y_3 = 0$ and $y_2 = \pm iy_1$ or $y_3 \neq 0$ and $y_1 = y_2 = 0$. According to Theorem 3.1 we have $\kappa = 1$ and then from (3.19) the masses are

$$M_1 = \frac{2}{\psi^2} 24\sqrt{2} m$$

(6.22)

The topological charges for $y_2 = iy_1$ can be shown to lie in the coset $Q \in \bar{\lambda}_s + \Lambda_R$ whilst for $y_2 = -iy_1$, $Q \in \lambda_s + \Lambda_R$. When $y_3 \neq 0$, and $y_1 = y_2 = 0$, $Q \in \Lambda_R$.

2. Taking $y_3 = 0$ and $y_2 \neq \pm iy_1$ we have $\kappa = 2$ and then

$$M_2 = \frac{2}{\psi^2} 48\sqrt{2} m$$

(6.23)

3. Taking $y_3 \neq 0$ and $y_2 = \pm iy_1$ again we have $\kappa = 2$ and then

$$M_3 = \frac{2}{\psi^2} 48\sqrt{2} m$$

(6.24)

4. Finally when $y_3 \neq 0$ and $y_2 \neq \pm iy_1$ we have $\kappa = 3$ and then

$$M_4 = \frac{2}{\psi^2} 72\sqrt{2} m$$

(6.25)

In the cases 2, 3 and 4 above the topological charges lie in $\Lambda_R$.

7 $B_r = SO(2r + 1)$

7.1 $SO(7)$

The solitons solutions for the case $B_3 = SO(7)$ are very similar to those of $SO(8)$. This is a consequence of the fact that the Dynkin diagram of $SO(7)$ can be obtained from that of $SO(8)$ by a folding procedure. The Hirota’s equations can be obtained from (6.3) by making $\tau_4 \equiv \tau_3$ and ignoring the equation for $\tau_4$. The eigenvalues of the matrix $L$ for $SO(7)$ are $(0, 2, 2, 6)$. The solution for $\lambda = 6$ for $SO(7)$ is obtained from (6.4) by deleting the last component of the vectors $\delta^{(1)}$ and $\delta^{(2)}$. Since the integers $l^\psi_i$ are $(1, 1, 2, 1)$ we get, according to theorem 3.1, that $\kappa = 1$ for this solution. Then from (3.19) the mass of the soliton is $M = \frac{2}{\psi^2} 12\sqrt{6} m$

The eigenvalue $\lambda = 2$ of $SO(7)$ has multiplicity 2 and the solution is constructed by taking $\delta^{(1)}$ as a linear combination of the corresponding eigenvectors and applying the procedures of section 3.3. The solution one gets can be obtained from (6.5) by making $x_3 \equiv x_1$ and by ignoring the last component of all $\delta$’s (the solution will then depend upon two parameters $x_1$ and $x_2$). We have $\kappa = 1, 2, 3$ for $x_1 = 0$ and $x_2 \neq 0$, $x_1 \neq 0$ and $x_2 = 0$ and $x_1, x_2 \neq 0$ respectively. The masses are then

$$M_1 = \frac{2}{\psi^2} 12\sqrt{2} m \quad \text{; when } x_1 = 0, x_2 \neq 0$$

$$M_2 = \frac{2}{\psi^2} 24\sqrt{2} m \quad \text{; when } x_1 \neq 0, x_2 = 0$$

$$M_3 = \frac{2}{\psi^2} 36\sqrt{2} m \quad \text{; when } x_1, x_2 \neq 0$$
7.2 \( B_r = SO(2r + 1) \) for \( r \geq 4 \)

The \( L_{ij} = l_i^\psi K_{ij} \) matrix is here given by

\[
L = \begin{pmatrix}
2 & 0 & -1 & 0 & \ldots & 0 & 0 \\
0 & 2 & -1 & 0 & \ldots & 0 & 0 \\
-2 & -2 & 4 & -2 & \ldots & 0 & 0 \\
0 & 0 & -2 & 4 & -2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -2 & 4 & -2 & 0 \\
0 & 0 & \ldots & 0 & -2 & 4 & -4 \\
0 & 0 & 0 & \ldots & 0 & -1 & 2
\end{pmatrix}
\] (7.1)

where the integers \( l_i^\psi \) are \((1, 1, 2, 2, \ldots, 2, 1)\). To analyze the eigenvalue equation \( L_{ij}v_j = \lambda v_i \) we follow the usual procedure of first introducing variable \( x = (4 - \lambda)/4 \) and expressing the components \( v_i \) in terms of Chebyshev polynomials. As usual the generic part of the eigenvalue problem takes the form of the recurrence relations for Chebyshev polynomials (A.1). Trying to fit an ansatz in form of linear combination of Chebysh ev polynomials into all eigenvalue equations produces the following result (in case \( \lambda \neq 2 \) or \( x \neq \frac{1}{2} \)):

\[
v_1 = v_0 \\
v_a = 2 (U_{a-1} - U_{a-2}) v_0 \quad ; \quad a = 2, 3, \ldots, r - 1 \\
v_r = (U_{r-1} - U_{r-2}) v_0
\] (7.2)

with the consistency condition, playing the role of the secular equation,

\[
v_{r-1} = (4x - 2) v_r \quad \text{or} \quad (x - 1) T'_r(x) = 0
\] (7.3)

to be imposed on the solution (7.2). The solutions to (7.3) takes the following form

\[
\lambda_k = 8 \sin^2 \left( \frac{k\pi}{2r} \right) \quad k = 0, 1, \ldots, r - 1 \quad (\lambda \neq 2)
\] (7.4)

The case \( \lambda = 2 \) has to be treated separately. One finds that the eigenvalue \( \lambda = 2 \) has corresponding eigenvector:

\[
v_{[1]}^{\lambda=2} = \begin{pmatrix}
1 \\
-1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\] (7.5)

In case where \( r = 3(N + 1) \) with some \( N = 1, 2, \ldots \) we get degeneracy with the extra eigenvector having components given in terms of two components \( v_0, v_1 \) playing a role of free parameters

\[
v_{3+3k} = v_{4+3k} = -v_{6+3k} = -v_{7+3k} = -(v_0 + v_1) \quad k = 0, 2, 4, 6, \ldots \\
v_{2+3k} = 0 \quad k = 0, 1, \ldots, N \\
v_r = (-1)^{r/3} \frac{(v_0 + v_1)}{2}
\] (7.6)
The Hirota’s equations read
\begin{align*}
\tau_0^2 \triangle (\tau_0) &= \beta (\tau_0^2 - \tau_2) \\
\tau_1^2 \triangle (\tau_1) &= \beta (\tau_1^2 - \tau_2) \\
\tau_2^2 \triangle (\tau_2) &= 2\beta (\tau_2^2 - \tau_0 \tau_1 \tau_3) \\
\tau_a^2 \triangle (\tau_a) &= 2\beta (\tau_a^2 - \tau_a-1 \tau_{a+1}) \quad a = 3, 4, ..., r-2 \\
\tau_{r-1}^2 \triangle (\tau_{r-1}) &= 2\beta (\tau_{r-1}^2 - \tau_{r-2} \tau_r^2) \\
\tau_r^2 \triangle (\tau_r) &= \beta (\tau_r^2 - \tau_{r-1})
\end{align*}
\tag{7.7}

Using the results of section 3.4 one obtains that the quantities $V_i^{(n-1)}$ defined in (3.30) are given by
\begin{align*}
V_0^{(n-1)} &= -\sum_{l=1}^{n-1} \left( 1 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_0^{(l)} \delta_0^{(n-l)} \\
V_1^{(n-1)} &= -\sum_{l=1}^{n-1} \left( 1 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_1^{(l)} \delta_1^{(n-l)} \\
V_2^{(n-1)} &= -\sum_{l=1}^{n-1} \left( 2 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_2^{(l)} \delta_2^{(n-l)} \\
&\quad + 2 \sum_{l=1}^{n-1} \left( \delta_0^{(l)} \delta_1^{(n-l)} + \delta_1^{(l)} \delta_3^{(n-l)} + \delta_0^{(l)} \delta_3^{(n-l)} + \sum_{m=1}^{l-1} \delta_0^{(m)} \delta_1^{(l-m)} \delta_3^{(n-l)} \right) \\
V_a^{(n-1)} &= -\sum_{l=1}^{n-1} \left( \left( 2 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_a^{(l)} \delta_a^{(n-l)} - 2\delta_{a-1}^{(l)} \delta_{a+1}^{(n-l)} \right) \quad a = 3, 4, ..., r-2 \\
V_{r-1}^{(n-1)} &= -\sum_{l=1}^{n-1} \left( 2 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_{r-1}^{(l)} \delta_{r-1}^{(n-l)} \\
&\quad + 2 \sum_{l=1}^{n-1} \left( \delta_r^{(l)} \delta_r^{(n-l)} + 2\delta_r^{(l)} \delta_{r-2}^{(n-l)} + \sum_{m=1}^{l-1} \delta_r^{(m)} \delta_r^{(l-m)} \delta_{r-2}^{(n-l)} \right) \\
V_r^{(n-1)} &= -\sum_{l=1}^{n-1} \left( 1 - \lambda (n^2 - 3nl + 2l^2) \right) \delta_r^{(l)} \delta_r^{(n-l)}
\end{align*}
\tag{7.8}

Applying the recurrence method described in section 3.3 one can now construct the solutions.

For the eigenvalues $\lambda_k = 8 \sin^2 \frac{k\pi}{2r} = \frac{2\sqrt{\gamma}}{\beta} \neq 2, k = 1, 2, ..., r-1$, we obtain the solution
\begin{align*}
\tau_0 &= 1 + e^\Gamma \\
\tau_1 &= 1 + e^\Gamma \\
\tau_a &= 1 + 2 \frac{\cos (2a-1) \theta_k}{\cos \theta_k} e^\Gamma + e^{2\Gamma} \quad a = 2, 3, ..., r-1 \\
\tau_r &= 1 + (-1)^k e^\Gamma
\end{align*}
\tag{7.9}

where $\theta_k = \frac{k\pi}{2r}$ and where we have used the fact that the Chebyshev polynomials satisfy $U_{a-1}(x_k) - U_{a-2}(x_k) = \frac{\cos (2a-1) \theta_k}{\cos \theta_k}$ with $x_k = \cos 2\theta_k$. According to Theorem 3.1 we see that
\(\kappa = 1\) in this case and therefore form (3.19) the masses are given by
\[
M_k = \frac{2}{\psi^2} 8r \sqrt{2} m \sin \frac{k\pi}{2r}; \quad k = 1, 2, ..., r - 1
\] (7.10)
and topological charges can be shown to lie in \(Q \in \Lambda_R\). Notice this solution can be obtained from (6.15) for \(SO(2r + 2)\) by a folding procedure. Identifying \(\tau_{r+1} \equiv \tau_r\) one observes that (6.15) (for \(r \rightarrow r + 1\)) reduces to (7.9) and the Hirota’s equations (6.13) reduces to (7.7).

Consider now the case where \(r \neq \) multiple of 3. The eigenvalue \(\lambda = 2\) is not degenerate in this case and taking \(\delta^{(1)}(1)\) to be the eigenvector (7.5) and applying the procedure of section 3.3 one obtains the following solution
\[
\begin{align*}
\tau_0 &= 1 + e^{r} + \frac{1}{4r} e^{2r} \\
\tau_1 &= 1 - e^{r} + \frac{1}{4r} e^{2r} \\
\tau_a &= 1 + (-1)^a \left(1 - \frac{4a - 2}{4r}\right) e^{2r} + \frac{1}{(4r)^2} e^{4r}; \quad a = 2, 3, ..., r - 1 \\
\tau_r &= 1 + \frac{(-1)^r}{4r} e^{2r}
\end{align*}
\] (7.11)
where \(\Gamma\) is given by (3.11) with \(\gamma_+ + \gamma_- = 2\). According to theorem 3.1 we have \(\kappa = 2\) and therefore from (3.19) the mass of such soliton is
\[
M = \frac{2}{\psi^2} 8r \sqrt{2} m
\] (7.12)
and its topological charge lie in \(Q \in \Lambda_R\). Again, the above solution can be obtained from (6.17) for \(SO(2r + 2)\) (i.e. \(r \rightarrow r + 1\)) by making \(y_1 = 1\) and \(y_2 = 0\) and identifying \(\tau_{r+1} \equiv \tau_r\).

For the case where \(r = \) multiple of 3, the eigenvalue \(\lambda = 2\) has multiplicity 2. The solutions are obtained by taking \(\delta^{(1)}\) to be a linear combination of the eigenvectors (7.3) and (7.4) and applying the procedures of section 3.3. Like in the \(SO(6p + 2)\) case the calculations are cumbersome. One can in fact obtain such solutions for \(SO(6p + 1)\) from those of \(SO(6p + 2)\) by a folding procedure. For instance, the solution for (7.13) can be obtained from (6.21) for the case of \(SO(14)\) by making \(y_2 = 0\) and neglecting the last component of all \(\delta^{(n)}\)s, \(n = 1, 2, ..., 6\). We then obtain three types of solutions: a) for \(y_1 = 0\) and \(y_3 \neq 0\) we have \(\kappa = 1\) and the mass is \(M_1 = \frac{2}{\psi^2} 24\sqrt{2} m;\) b) for \(y_1 \neq 0\) and \(y_3 = 0\) we have \(\kappa = 2\) and the mass is \(M_2 = \frac{2}{\psi^2} 48\sqrt{2} m\) and finally c) for \(y_1, y_3 \neq 0\) we have \(\kappa = 3\) and the mass is \(M_3 = \frac{2}{\psi^2} 72\sqrt{2} m\).

8 \quad G_2

The matrix \(L_{ij} = l_i^v K_{ij}\) in this case is given by
\[
L = \begin{pmatrix}
2 & -1 & 0 \\
-2 & 4 & -6 \\
0 & -1 & 2
\end{pmatrix}
\] (8.1)
where the integers \( l_i^\psi \) are \((1, 2, 1)\). The eigenvalues of \( L \) are \((0, 2, 6)\) and the corresponding eigenvectors are:

\[
\psi^{\lambda=0} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}; \quad \psi^{\lambda=2} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}; \quad \psi^{\lambda=6} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \tag{8.2}
\]

The Hirota’s equations for \( G_2 \) are

\[
\begin{align*}
\tau_0^2 \triangle (\tau_0) &= \beta (\tau_0^2 - \tau_1) \\
\tau_1^2 \triangle (\tau_1) &= 2\beta (\tau_1^2 - \tau_0 \tau_3) \\
\tau_2^2 \triangle (\tau_2) &= \beta (\tau_2^2 - \tau_1) \tag{8.3}
\end{align*}
\]

For \( \lambda = 6 \) the solution obtained, using ansatz (3.10) and the procedure of section 3.3, is given by

\[
\delta^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}; \quad \delta^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{8.4}
\]

We therefore have \( \kappa = 1 \) and according to (3.19) the mass is

\[
M_1 = \frac{2}{\psi^2} 12\sqrt{6} m \tag{8.5}
\]

For \( \lambda = 2 \) the solution is given by

\[
\begin{align*}
\delta^{(1)} &= \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}; \quad \delta^{(2)} = \begin{pmatrix} 1 \\ 3 \\ -1/3 \end{pmatrix}; \quad \delta^{(3)} = \frac{1}{27} \begin{pmatrix} 1 \\ -16 \\ 1 \end{pmatrix} \\
\delta^{(4)} &= \begin{pmatrix} 0 \\ 1/3 \\ 0 \end{pmatrix}; \quad \delta^{(5)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \delta^{(6)} = \left( \frac{1}{27} \right)^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{8.6}
\end{align*}
\]

For such solution we have \( \kappa = 3 \) and therefore from (3.19) the mass is

\[
M_2 = \frac{2}{\psi^2} 36\sqrt{2} m \tag{8.7}
\]

Notice that these two solutions can be obtained from the \( SO(8) \) solutions by a folding procedure. Indeed, by identifying \( \tau_1 \equiv \tau_3 \equiv \tau_4 \) one observes that the Hirota’s equations (8.3) become (8.3) (after the relabeling \( \tau_1 \leftrightarrow \tau_2 \)). In addition, under such identifications the solution (6.4) becomes (8.4) and the solution (6.5) becomes (8.6) by setting \( x_1 = x_2 = x_3 = 1 \). Both solutions yields topological charges lying in the co-root lattice of \( G_2 \).
9 \(F_4\)

The \(L\) matrix for \(F_4\) is given by

\[
L = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-2 & 4 & -2 & 0 & 0 \\
0 & -3 & 6 & -6 & 0 \\
0 & 0 & -2 & 4 & -2 \\
0 & 0 & 0 & -1 & 2 \\
\end{pmatrix}
\] (9.8)

The integers \(l^\psi\) are \((1, 2, 3, 2, 1)\). The eigenvalues of \(L\) are \((0, 6+2\sqrt{3}, 6-2\sqrt{3}, 3+\sqrt{3}, 3-\sqrt{3})\) and none of them are degenerate.

The Hirota's equations read

\[
\tau_0^2 \triangle (\tau_0) = \beta (\tau_0^2 - \tau_1) \\
\tau_1^2 \triangle (\tau_1) = 2\beta (\tau_1^2 - \tau_0\tau_2) \\
\tau_2^2 \triangle (\tau_2) = 3\beta (\tau_2^2 - \tau_1\tau_3^2) \\
\tau_3^2 \triangle (\tau_3) = 2\beta (\tau_3^2 - \tau_2\tau_4) \\
\tau_4^2 \triangle (\tau_4) = \beta (\tau_4^2 - \tau_3) \\
\] (9.9)

We construct the solutions using the ansatz (3.10) and the procedure of section 3.3. The results are listed below.

The solution for \(\lambda = 6 + 2\sqrt{3} = \frac{\gamma + \gamma -}{\beta}\) is given by

\[
\delta^{(1)} = \left(1, -2(2 + \sqrt{3}), 3(3 + 2\sqrt{3}), -2(2 + \sqrt{3}), 1\right) \\
\delta^{(2)} = \left(0, 1, 3 \left(3 + 2\sqrt{3}\right) , 1, 0\right) \\
\delta^{(3)} = \left(0, 0, 1, 0, 0\right) \\
\] (9.10)

We have \(\kappa = 1\) and therefore from (3.19) the mass of the soliton is

\[
M_1 = \frac{2}{\psi^2} 24\sqrt{6 + 2\sqrt{3}} m \\
\] (9.11)

The solution for \(\lambda = 6 - 2\sqrt{3} = \frac{\gamma + \gamma -}{\beta}\) is

\[
\delta^{(1)} = \left(1, -2(2 - \sqrt{3}), 3(3 - 2\sqrt{3}), -2(2 - \sqrt{3}), 1\right) \\
\delta^{(2)} = \left(0, 1, 3 \left(3 - 2\sqrt{3}\right) , 1, 0\right) \\
\delta^{(3)} = \left(0, 0, 1, 0, 0\right) \\
\] (9.12)

Again we have \(\kappa = 1\) and therefore the mass is

\[
M_2 = \frac{2}{\psi^2} 24\sqrt{6 - 2\sqrt{3}} m \\
\] (9.13)
The solution for $\lambda = 3 + \sqrt{3} = \frac{\gamma + \gamma - \beta}{\beta}$ is

\[
\delta^{(1)} = \left( \frac{1 - \sqrt{3}}{2}, 1, 0, -\frac{1}{2}, \frac{-1 + \sqrt{3}}{4} \right)
\]

\[
\delta^{(2)} = \left( a_1, 2 \left( 27 + 14 \sqrt{3} \right) a_1, \frac{9}{64}, 2 \left( 3 + 2 \sqrt{3} \right) a_1, a_1 \right)
\]

\[
\delta^{(3)} = b_1 \left( 0, -2, 8 \left( 1 + \sqrt{3} \right), 1, 0 \right)
\]

\[
\delta^{(4)} = a_1^2 \left( 0, 1, \left( 63 + 4 \cdot 3^\frac{3}{2} \right), 1, 0 \right)
\]

\[
\delta^{(5)} = (0, 0, 0, 0, 0)
\]

\[
\delta^{(6)} = a_1^3 (0, 0, 1, 0, 0)
\]

where

\[
a_1 = \frac{5 + 3^\frac{3}{2}}{64 \left( 71 + 41 \sqrt{3} \right)} \tag{9.15}
\]

\[
b_1 = \frac{-\left( 12417 + 7169 \sqrt{3} \right)}{128 \left( 172947 + 99851 \sqrt{3} \right)} \tag{9.16}
\]

In this case we have $\kappa = 2$ and therefore from (9.19) the mass of the soliton is

\[
M_3 = \frac{2}{\psi^2} 48 \sqrt{3 + \sqrt{3}} m \tag{9.17}
\]

The solution for $\lambda = 3 - \sqrt{3} = \frac{\gamma + \gamma - \beta}{\beta}$ is

\[
\delta^{(1)} = \left( \frac{1 + \sqrt{3}}{2}, 1, 0, -\frac{1}{2}, \frac{-1 - \sqrt{3}}{4} \right)
\]

\[
\delta^{(2)} = \left( a_2, 2 \left( 27 - 14 \sqrt{3} \right) a_2, \frac{9}{64}, 2 \left( 3 - 2 \sqrt{3} \right) a_2, a_2 \right)
\]

\[
\delta^{(3)} = b_2 \left( 0, -2, 8 \left( 1 - \sqrt{3} \right), 1, 0 \right)
\]

\[
\delta^{(4)} = a_2^2 \left( 0, 1, \left( 63 - 4 \cdot 3^\frac{3}{2} \right), 1, 0 \right)
\]

\[
\delta^{(5)} = (0, 0, 0, 0, 0)
\]

\[
\delta^{(6)} = a_2^3 (0, 0, 1, 0, 0)
\]

where

\[
a_2 = \frac{-5 + 3^\frac{3}{2}}{64 \left( -71 + 41 \sqrt{3} \right)} \tag{9.19}
\]

\[
b_2 = \frac{12417 - 7169 \sqrt{3}}{128 \left( -172947 + 99851 \sqrt{3} \right)} \tag{9.20}
\]
Again \( \kappa = 2 \) and the mass is

\[
M_4 = \frac{2}{\psi^2} 48\sqrt{3} - \sqrt{3} m
\]  

(9.21)

All solutions correspond to topological charges lying in the co-root lattice of \( F_4 \).

10 The \( E_n \) series

The one-soliton solutions for the exceptional algebras \( E_6, E_7 \) and \( E_8 \) were presented in [3]. However not all static soliton solutions were obtained there. In the case of \( E_6 \) the eigenvalues \( \lambda_1 = 6 - 2\sqrt{3} \) and \( \lambda_2 = 6 + 2\sqrt{3} \) have multiplicity 2. Therefore using our procedure of taking \( \delta^{(1)} \) as a linear combination of the eigenvectors associated to the degenerate eigenvalue, one obtains new solutions for \( E_6 \). We do not present them here, but under a folding procedure they lead to the solitons of \( F_4 \), constructed above, corresponding to the eigenvalues \( 6 - 2\sqrt{3} \) and \( 6 + 2\sqrt{3} \). The masses of these new soliton solutions of \( E_6 \) are the same as those of the corresponding solitons of \( F_4 \).

11 Conclusions

The Hirota tau-function method employed in this paper has proved to be extremely powerful in providing explicit soliton solutions for the Toda models. In particular, when dealing with degenerate eigenvalues the method prescribes how to consider a superposition of solutions for the non-linear problem. This yields, for instance, new solutions not previously discussed in the literature surviving in the static limit.

We have also given the masses of such solutions by using a mass formula obtained by general arguments of conformal field theory. For many of our solutions we have discussed the topological charges and verified that they can be classified according to the cosets \( \Lambda^u / \Lambda^R \), the quotient of the coweight and coroot lattices.

It is still unclear, however, how to use the representation theory of Lie algebras in order to classify solutions and their topological charges as conjectured by Hollowood in [1]. A very interesting and natural continuation of this work would be the study of the quantum theory of these solitons. This could be approached from several directions like bosonization, equivalence theorems and S-matrix [15].

A Chebyshev Polynomials. Definitions and Identities

Here we list for completeness the basic properties of Chebyshev polynomials [13, 14]. They play an instrumental role in solving the eigenvalue problem of the \( L_{ij} \) matrix for various Lie groups.

The defining relation for Chebyshev polynomials of both types, \( T_n(x) \) (type I) and \( U_n(x) = T_{n+1}'/(n + 1) \) (type II) is given in terms of recurrence relations:

\[
\begin{align*}
T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) &= 0 ; \quad n \geq 1 \\
U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) &= 0
\end{align*}
\]  

(A.1)
with \( T_0(x) = U_0(x) = 1 \) and \( T_1(x) = x \). Another useful recurrence relations are:

\[
(1 - x^2)T'_n(x) = -nxT_n(x) + nT_{n-1}(x)
\]

\[ T_m(x)T_n(x) = \frac{1}{2} \left( T_{m+n}(x) + T_{|m-n|}(x) \right) \quad m, n \geq 0 \]

The zeros \( \xi_j \) of \( T_n: T_n(\xi_j) = 0 \); \( j = 1, 2, \ldots, n \) are given by

\[
\xi_j = \cos \left( \frac{(2j - 1) \pi}{n} \right) \quad j = 1, 2, \ldots, n
\]

The extrema points \( \eta_k \) at which \( |T_n(x)| = 1 \) are given by

\[
\eta_k = \cos \left( \frac{k\pi}{n} \right) \quad k = 0, 1, 2, \ldots, n
\]

At these points we also have

\[
T_n(\eta_k) = (-1)^k \quad k = 0, 1, 2, \ldots, n
\]

It is clear that the points \( \eta_1, \ldots, \eta_{n-1} \) are zeros of the derivatives \( T'_n(\eta_k) = 0 \); \( k = 1, \ldots, n-1 \) and therefore also zeros of \( U_{n-1}(\eta_k) = 0 \); \( k = 1, \ldots, n-1 \). One also has

\[
T_n(1) = 1 \quad T_n(-1) = (-1)^n
\]

The Chebyshev polynomials take very simple form in the trigonometric representation:

\[
T_n(x) = \cos n \rho \quad U_n(x) = \frac{\sin(n + 1)\rho}{\sin \tau} \quad \text{with} \quad x = \cos \rho
\]

One notices that for the special choice of arguments:

\[
\rho_j = \frac{2\pi j}{N + 1} \quad j = 0, 1, 2, \ldots, N
\]

with some integer \( N \) the Chebyshev polynomials satisfy the periodicity relation

\[
T_{n+N+1}(\rho_j) = T_n(\rho_j) \quad U_{n+N+1}(\rho_j) = U_n(\rho_j)
\]

Chebyshev polynomials become very useful in the study of the eigenvalue equation \( L_{ij}v_j = \lambda v_i \) of the matrix \( L_{ij} \) related to the Cartan matrix. In components the generic equation reads as

\[
-v_{a-1} + (2 - \lambda)v_a - v_{a+1} = 0
\]

for the parameter \( a \) taking values between 2 and \( r - 1 = \text{rank} \ G - 1 \). Introducing variable \( x = (2 - \lambda)/2 \) into (A.11) and rewriting \( v_a \) as a linear combination of the Chebyshev polynomials

\[
v_a = (AT_a(x) + BU_a(x)) v_0
\]

we find that (A.11) is trivially satisfied due to (A.1). The remaining equations of \( K_{ij}v_j = \lambda v_i \) (whose not appearing in (A.11)) impose the secular equation on the \( L \) matrix. They have
to be solved separately for each Lie group and solutions can be found in terms of the zeros
and extrema of the Chebyshev polynomials as it was done in the text.

Acknowledgements:
One of us (LAF) acknowledges CNPq for financial support within CNPq/NSF Cooperative
Science Program and thanks Physics Department at the University of Illinois at Chicago for
hospitality. We thank J.C. Campuzano, C. Halliwell and W. Poetz for providing computer
facilities. We are very grateful to Prof. Henri Gillet for showing us his proof of theorem
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