Cauchy flights in confining potentials

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We analyze confining mechanisms for Lévy flights evolving under the influence of external potentials. Given a stationary probability density function (pdf), we address the reverse engineering problem: design a jump-type stochastic process whose target pdf (eventually asymptotic) equals the preselected one. To this end, dynamically distinct jump-type processes can be employed. We demonstrate that one "targeted stochasticity" scenario involves Langevin systems with a symmetric stable noise. Another derives from the Lévy-Schrödinger semigroup dynamics (closely linked with topologically induced super-diffusions), which has no standard Langevin representation. For computational and visualization purposes, the Cauchy driver is employed to exemplify our considerations.

I. MOTIVATION

We consider a subclass of so-called Lévy flights that is mathematically identifiable as symmetric stable stochastic processes, [1]-[4]. These non-Gaussian processes are of the jump-type and, in contrast to familiar Gaussian diffusion-type processes, involve a number of obstacles.

One of them is a shortage of explicit analytic solutions, e.g. explicit probability density functions (pdfs) and transition probability densities. Another comes from theoretically admitted existence of arbitrarily small jumps and fat tails of the pdf which typically preclude the existence of (in the least the second) moments. In the presence of confining potentials the resultant pdfs may admit higher moments, but merely a finite number of them may exist.

Third, some care is needed in any computer-assisted analysis of Lévy flights, since imposing a lower and upper bound on the size of jumps, sets the problem within the ramifications of the central limit theorem which implies the standard Gaussian limit, [5], albeit the simulated process is non-Gaussian by definition.

In the present paper we set general confinement criterions for symmetric stable processes, with special emphasis on the analytically tractable case of the Cauchy noise [6]-[10]. With regard to a specific response to external potentials, we pay special attention to two classes of jump-type processes: those related to the Langevin equation and those induced by the Cauchy-Schrödinger semigroup dynamics (that involves a fractional analog of the generalized diffusion equation). We leave for a separate study another interesting possibility, that is based on Bochner’s concept of subordination, c.f. [17].

Lévy flights in confining potentials with a standard (mostly for an additive noise) Langevin representation have received ample attention, [6]-[14] and [13, 16]. For Cauchy-Langevin processes, a manipulation with the forward drift and/or its (external conservative force) potential sets rules of the game, e.g. directly leads to stationary probability densities. They never have a Gibbs-Boltzmann form, characteristic for Gaussian diffusion processes, [17].

Another class of Lévy processes, that are driven by dynamical semigroups, was analyzed in detail in [18,19]. Cauchy driver has received there special attention. The semigroup-driven processes independently reappeared (in the context of systems with topological complexity like folded polymers or complex networks) in Refs. [20]-[23]. In Refs. [18,19], external potentials appear as additive perturbations of the noise generator and under the name of effective potentials they appear in [20] as well. In the “topological” literature, the resultant semigroup dynamics has been implemented via local modifications of jump rates of the associated jump-type process.

The semigroup-driven (and topologically-induced) processes appear to have no standard Langevin representation, irrespective of whether we adopt an additive or multiplicative noise. Only under special circumstances a connection with the multiplicative noise has been established in Ref. [20], but it is not a generic property of semigroup-driven processes, see e.g. also [25].

Although we formulate a framework incorporating symmetric stable processes in general, we strongly rely on a mathematical theory of the Cauchy semigroup-driven processes. This theory, without any "topological" context, has been formulated in Ref. [19]. No explicit analytic examples of confining potentials, nor pdfs were given there.

We shall demonstrate that the Langevin-driven and semigroup-driven Lévy processes stay in affinity and may share common for both stationary pdf. A super-diffusive dynamical pattern of behavior is generically expected to arise. An asymptotic approach towards a stationary pdf is then in principle possible.

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This motivates our "targeted stochasticity" discussion whose original formulation for Langevin-driven Lévy systems can be found in Ref. [17], the reverse engineering problem being included. The necessity of considering the semigroup dynamics in this reconstruction problem points first, to an independent dynamical mechanism, and second, to the fact that only the properly tailored semigroup (and thus the semigroup potential) may guarantee that the prescribed invariant density actually is an asymptotic target of the process. This point was left aside in "topological" references [20, 23], where the pertinent invariant densities were postulated to have a Gibbsian form, hence a tacit assumption was made about extrremely strong confining properties of the topologically-induced process.

The original reverse engineering problem reads: given a stationary pdf, can we tailor a drift function so that the system Langevin dynamics would admit the predefined as an asymptotic target? In the course of our discussion, we in fact extend the range of applicability of the original "targeted stochasticity" scenario and demonstrate that, for a symmetric stable driver, a priori chosen stationary pdf may serve as a target density for both Langevin and semigroup-driven jump-type processes. Even though their detailed dynamical patterns of behavior are different. In the near-equilibrium regime this dynamical distinction becomes immaterial. For analytically tractable and visualization insights, we shall basically refer to Cauchy processes in confining potentials.

For the record we point out that the term "equilibrium" needs to be addressed with some care for non-Gaussian processes. No physical thermalization mechanisms have ever been proposed for Lévy flights. Moreover, their physical "reason" (origin of noise) appears to be exterior to the physical system, with no reliable kinetic theory background, and therefore no fluctuation-dissipation response theory could have been set for any stable noise.

To the contrary, the noise "reason" is definitely an intrinsic feature of the environment-particle coupling in case of the standard Brownian motion, based on the kinetic theory derivations. All traditional fluctuation-dissipation relationships find their place in the Brownian framework. None of them has been reproduced in the context of Lévy flights.

II. RESPONSE OF GAUSSIAN NOISE TO CONFINING POTENTIALS: SMOLUCHOWSKI PROCESSES

Albeit we are primarily interested in jump-type stochastic processes, certain useful intuitions can be borrowed from the standard theory of Brownian motion. Namely, let us consider a one-dimensional Smoluchowski diffusion process [20], with the Langevin representation

\[ \dot{x} = b(x,t) + A(t) \]  

(1)

where \( \langle A(s) \rangle = 0, \langle A(s)A(s') \rangle = 2D\delta(s-s') \) and \( b(x) \) is a forward drift of the process having the gradient form \( b = 2D\nabla \Phi \), where \( D \) stands for a diffusion constant.

If an initial probability density \( \rho_0(x) \) is given, then the diffusion process obeys the Fokker-Planck equation

\[ \partial_t \rho = D\Delta \rho - \nabla (b \cdot \rho) . \]  

(2)

We introduce an osmotic velocity field \( u = D \ln \rho \), together with the current velocity field \( v = b - u \). The latter obeys the continuity equation \( \partial_t \rho = -\nabla j \), where \( j = v \cdot \rho \) has a standard interpretation of a probability current.

Presently we pass to time-independent drifts of the diffusion process, that are induced by external (conservative, Newtonian) force fields \( f = -\nabla V \). One arrives at Smoluchowski diffusion processes by setting

\[ b = \frac{f}{m\beta} = -\frac{1}{m\beta} \nabla V . \]  

(3)

This expression accounts for the fully-fledged phase-space derivation of the spatial process, in the regime of large \( \beta \). It is taken for granted that the fluctuation-dissipation balance gives rise to the standard form \( D = k_B T / m\beta \) of the diffusion coefficient.

Let us consider a stationary asymptotic regime, where \( j \to j_\ast = 0 \). We denote an (a priori assumed to exist) invariant density \( \rho_\ast = \rho_\ast(x) \). Since \( v_\ast = 0 \) and \( b = f / m\beta \), by its very definition, does not depend functionally on the probability density, there holds

\[ b = b_\ast = u_\ast = D\nabla \ln \rho_\ast . \]  

(4)

Consequently, we have \( \rho_\ast(x) = (1/Z) \exp[-V(x)/k_B T] \), where \( 1/Z \) is a normalization constant. Our outcome has the familiar Gibbs-Boltzmann form.

Following a standard procedure [20, 29] we transform the Fokker-Planck equation into an associated Hermitian (Schrödinger-type) problem by means of a redefinition

\[ \rho(x,t) = \Psi(x,t)\rho_\ast^{1/2}(x) \]  

(5)
that takes the Fokker-Plank equation into a parabolic one, often called a generalized diffusion equation:

$$\partial_t \Psi = D \Delta \Psi - \nabla \Psi.$$  \hspace{1cm} (6)

Its potential \( V \) derives, as a function of the drift \( b(x) \), from a compatibility condition

$$V(x) = (1/2)[b^2/(2D) + \nabla b].$$  \hspace{1cm} (7)

In view of Eq. (4), its equivalent form is

$$V(x) = D \frac{\Delta \rho^{1/2}_s}{\rho^{1/2}_s}.$$  \hspace{1cm} (8)

If the \((1/2mD \text{ rescaled})\) Schrödinger-type Hamiltonian \( \hat{H} = -D \Delta + V \) is a bounded from below, self-adjoint operator in a suitable Hilbert space, then one arrives at a dynamical semigroup \( \exp(-t\hat{H}) \), with the dynamical rule 

$$\Psi(x,t) = \{\exp(-t\hat{H})\Psi\}(x,0),$$

pushing forward in time the initial data \( \Psi(x,0) \). The semigroup is contractive, hence asymptotically \( \Psi(x,t) \to \rho^{1/2}_s \). Accordingly, \( \rho(x,t) \to \rho_s(x) \).

The above Schrödinger semigroup (parabolic) reformulation of the Fokker-Planck equation refers to the very same diffusion process and the dynamics of \( \rho(x,t) \) does not depend on the theoretical framework of choice. In below we shall demonstrate that for non-Gaussian processes, the semigroup-driven and Langevin-induced dynamics refer to inequivalent dynamical patterns of behavior. Even if both are associated with a common stationary (target) pdf.

### III. RESPONSE OF LÉVY FLIGHTS TO CONFINING POTENTIALS

#### A. Lévy driver

Let us set general rules of the game with respect to the response of any symmetric stable noise to external potentials. We recall that a characteristic function of a random variable \( X \) completely determines a probability distribution of that variable. If this distribution admits a density \( \rho(x) \), we can write \( \langle \exp(ipX) \rangle = \int_R \rho(x) \exp(ipx)dx \) which, for infinitely divisible probability laws, gives rise to the famous Lévy-Khintchine formula (see, e.g. [3]).

From now on, we concentrate on the integral part of the Lévy-Khintchine formula, which is responsible for arbitrary stochastic jump features:

$$F(p) = -\int_{-\infty}^{+\infty} \left[ \exp(ipy) - 1 - \frac{ipy}{1 + y^2} \right] \nu(dy),$$  \hspace{1cm} (9)

where \( \nu(dy) \) stands for the appropriate Lévy measure. The corresponding non-Gaussian Markov process is characterized by \( \langle \exp(ipX_t) \rangle = \exp[-tF(p)] \) and, upon setting \( \tilde{p} = -i\nabla \) instead of \( p \), yields an operator \( F(\tilde{p}) = \hat{H} \) which is a direct analog of the free Schrödinger Hamiltonian.

We restrict further considerations to non-Gaussian random variables whose probability densities are centered and symmetric, e.g. a subclass of stable distributions characterized by

$$F(p) = \lambda |p|^\mu \Rightarrow \hat{H} \doteq \lambda |\Delta|^{\mu/2}.$$  \hspace{1cm} (10)

(In passing, we note that the adopted definition of a pseudo-differential operator may be replaced by the negative of a suitable Riesz fractional derivative.) In the above, \( \mu < 2 \) and \( \lambda > 0 \) stands for the intensity parameter of the Lévy process. The fractional Hamiltonian \( \hat{H} \), which is a non-local pseudo-differential operator, by construction is positive and self-adjoint on a properly tailored domain. A sufficient and necessary condition for both these properties to hold true is that the pdf of the Lévy process is symmetric. \([2]\).

The associated jump-type dynamics is interpreted in terms of Lévy flights. In particular

$$F(p) = \lambda |p| \to \hat{H} = F(\tilde{p}) = \lambda |\nabla| \doteq \lambda(-\Delta)^{1/2}$$

refers to the Cauchy process, see e.g. \([8, 18, 19]\).

The pseudo-differential Fokker-Planck equation, which corresponds to the fractional Hamiltonian \( \hat{H}_\mu \) and the fractional semigroup \( \exp(-t\hat{H}_\mu) = \exp(-\lambda|\Delta|^{\mu/2}) \), reads

$$\partial_t \rho = -\lambda|\Delta|^{\mu/2} \rho,$$  \hspace{1cm} (12)
to be compared with the Fokker-Planck equation for freely diffusing particle \( \partial_t \rho = D \Delta \rho \).

For a pseudo-differential operator \(|\Delta|^{\mu/2}\), the action on a function from its domain is greatly simplified, in view of the properties of the Lévy measure \( \nu_\mu(dx) \). Namely, remembering that we overcome a singularity at 0 by means of the principal value of the integral, we have \([18, 19]\):

\[
(|\Delta|^{\mu/2}f)(x) = -\int [f(x+y) - f(x)]\nu_\mu(dy) .
\] (13)

By changing an integration variable \( y = x+y \) and employing a direct connection with the Riesz fractional derivative of the \( \mu \)-th order, \([13]\), we arrive at

\[
(|\Delta|^{\mu/2}f)(x) = -\frac{\Gamma(\mu + 1)}{\pi} \frac{\sin(\pi \mu/2)}{||x-x||^{1+\mu}} \int f(z) - f(x) \nu_\mu(dz)
\] (14)

with \((|\Delta|^{\mu/2}f)(x) = -\partial^\mu f(x)/\partial|x|^\mu \). The case of \( \mu = 1 \) refers to the Cauchy driver (e.g. noise).

We note a systematic sign difference between our notation for a pseudo-differential operator \(|\Delta|^{\mu/2}\) and this based on the fractional derivative notion, like e.g. \( \Delta^{\mu/2} = \partial^{\mu}/\partial|x|^\mu \) of Refs. \([20, 22]\).

### B. Langevin scenario

In case of jump-type (Lévy) processes a response to external perturbations by conservative force fields appears to be particularly interesting. On the one hand, one encounters a widely accepted reasoning (Refs. \([6]-[13]\)) where the Langevin equation, with additive deterministic and Lévy ”white noise” terms, is found to imply a fractional Fokker-Planck equation, whose form faithfully parallels the Brownian version, e.g. (c.f. \([6]\), see also \([8]\))

\[
\dot{x} = b(x) + A^\mu(t)
\]

\[
\downarrow
\]

\[
\partial_t \rho = -\nabla (b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho .
\] (15)

We emphasize a difference in sign in the second term, if compared with Eq. (4) of Ref. \([6]\). There, the minus sign is absorbed in the adopted definition of the (Riesz) fractional derivative. Apart from the formal resemblance of operator symbols, we do not directly employ fractional derivatives in our formalism.

Let us assume that the fractional Fokker-Planck equation \((15)\) admits a stationary pdf \( \rho_*(x) \). Then, a functional form of the drift \( b(x) \) can be reconstructed by means of an indefinite integral

\[
b(x) = -\lambda \int |\Delta|^{\mu/2} \rho_*(x) dx .
\] (16)

This is the reverse engineering problem of \([17]\).

### C. Lévy-Schrödinger semigroup

On the other hand, by mimicking the previous Gaussian strategy, we can directly refer to the Hamiltonian framework and dynamical semigroups with Lévy generators being additively perturbed by a suitable potential, see e.g. \([18, 19]\). For example, assuming that the functional form of \( V(x) \) guarantees that \( \hat{H}_\mu = \lambda |\Delta|^{\mu/2} + V \) is self-adjoint and bounded from below in a suitable Hilbert space, we may readily pass to the fractional (non-Gaussian, jump process) analog of the generalized diffusion equation:

\[
\partial_t \Psi = -\lambda |\Delta|^{\mu/2} \Psi - V \Psi .
\] (17)

The dynamical semigroup reads \( \exp(-t\hat{H}_\mu) \) and the compatibility condition affine to that of Eq. \([8]\), typically takes the form of the time-adjoint equation for an auxiliary function \( \theta(x,t) \):

\[
\partial_t \theta = \lambda |\Delta|^{\mu/2} \theta + V \theta .
\] (18)
General theory \cite{18,19} tells us that a (properly normalized) product \(\theta^*(x, t)\theta(x, t)\) determines a probability density \(\rho(x, t)\) of a Markov process that interpolates between the boundary data \(\rho(x, 0)\) and \(\rho(x, T)\), in the time span \(t \in [0, T]\).

Let us assume that there exists an invariant (stationary) pdf \(\rho_*\) of this Cauchy semigroup-induced process. If we demand that \(\theta(x, t)\) actually does not depend on time, and adopt a decomposition \(\rho = \Psi \rho_*^{1/2}\), c.f. Eq. (5), we are allowed to set \(\theta \equiv \rho_*^{1/2}\) and remove limitations upon the time interval. Ultimately, we arrive at a compatibility condition that is a direct fractional version of Eq. (5):

\[
\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}.
\]

(19)

In the present case, we can readily evaluate the dynamics of \(\rho(x, t) = \Psi(x, t)\rho_*^{1/2}(x)\):

\[
\partial_t \rho = \rho_*^{1/2} \partial_t \Psi = -\lambda \rho_*^{1/2} |\Delta|^{\mu/2} [\rho_*^{-1/2} \rho] + \mathcal{V} \cdot \rho.
\]

(20)

This is a departure point for the reverse engineering procedure: given the stationary pdf, find the semigroup potential \(\mathcal{V}(x)\).

It is interesting to observe that by making cosmetic changes: set \(\lambda = 1\), formally identify \(\rho_*^{1/2} = \exp[-\beta V(x)]\), with whatever \(V\) and \(\beta = 1/k_BT\), we and up with a familiar form of the transport equation previously introduced in a number of papers:

\[
\partial_t \rho = -\exp(-\beta V/2) |\Delta|^{\mu/2} \exp(\beta V/2) \rho + \rho \exp(\beta V/2) |\Delta|^{\mu/2} \exp(-\beta V/2),
\]

c.f. formula (6) in \cite{22}, formula (5) in \cite{23} and formula (36) in \cite{20}. There, the investigated process was named a topologically induced super-diffusion. We point out a systematic sign difference between our \(|\Delta|^{\mu/2}\) and the corresponding fractional derivative \(\Delta^{\mu/2}\) of \cite{21,22,23}. Graphically these symbols look similar, but have different origin.

Remark: It is of some interest to invoke an independent approach of Refs. \cite{21,22} where one modifies jumping rates by suitable local factors, to arrive at a response mechanism that is characteristic of the previously outlined semigroup dynamics. In view of (14), the free transport equation \(\partial_t \rho = -\lambda |\Delta|^{\mu/2} \rho\) can be re-written as a master equation \(\partial_t \rho(x) = \int [w(x|z)\rho(z) - w(z|x)\rho(x)] \nu_\mu(dz)\). The jump rate \(w(x|y) \sim 1/|x-y|^{1+\mu}\) is an even function, \(w(x|z) = w(x|z)\).

If we replace the jump rate \(w(x|y)\) of the free fractional dynamics by the expression \(w_\phi(x|y) \sim \exp[\Phi(x) - \Phi(y)]\) and account for the fact that \(w_\phi(x|z) \neq w_\phi(z|x)\), then the master equation takes the form: \((1/\lambda) \partial_t \rho = |\Delta|^{\mu/2} f = -(\exp \Phi)|\Delta|^{\mu/2} \exp(\Phi) \rho + \rho \exp(\Phi)|\Delta|^{\mu/2} \exp(\Phi)\). Whatever \(\Phi(x)\) has been chosen (up to a normalization factor), then formally \(\rho_* = \exp 2\Phi\) is a stationary solution of that transport equation. We note that a physically attractive point in the topologically-induced dynamics pattern was an assumption that \(\exp 2\Phi\) sets a Gibbsian form of the pdf. Accounting for the normalization factor \(1/Z\) one presumes that \(\rho_* = (1/Z) \exp(-V_*/k_BT)\) with an external potential \(V_* = -k_BT \ln(Z \rho_*\), whose physical origin is based on a crude phenomenology, c.f. \cite{21,22}. With these re-definitions, the above transport equation takes the form (21).

IV. REVERSE ENGINEERING FOR CAUCHY FLIGHTS

By choosing \(\mu = 1\) in the above, we narrow down the whole discussion to Cauchy processes, when e.g. (all integrals are evaluated by means of their Cauchy principal value)

\[
(\Delta|^{\mu/2} f)(x) = - \int_R \left[ f(x + y) - f(x) - \frac{y \nabla f(x)}{1 + y^2} \right] \nu_\mu(dy)
\]

\[
\Downarrow
\]

\[
(\Delta|^{\mu/2} f)(x) = - \int [f(x + y) - f(x)] \nu_\mu(dy).
\]

(22)

The Cauchy-Lévy measure, associated with the Cauchy semigroup generator \(|\Delta|^{1/2} = \nabla\), reads

\[
\nu_{1/2}(dy) = \frac{1}{\pi \cdot y^2}.
\]

(23)
By changing an integration variable \( y \rightarrow z = x + y \), we give Eq. (20) the familiar form
\[
(\|\nabla f\|)(x) = -\frac{1}{\pi} \int \frac{f(z) - f(x)}{|z - x|^2} \, dz
\]  
(24)
where \( 1/\pi|z - x|^2 \) has an interpretation of an intensity with which jumps of the size \( |z - x| \) occur.

### A. Ornstein-Uhlenbeck process

In case of the Ornstein-Uhlenbeck-Cauchy (OUC) process, the drift is given by \( b(x) = -\gamma x \), and an asymptotic invariant density associated with
\[
\partial_t \rho = -\lambda \nabla \rho + \nabla[(\gamma x) \rho] 
\]  
(25)
reads:
\[
\rho_\ast(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)} 
\]  
(26)
where \( \sigma = \lambda/\gamma \), c.f. Eq. (9) in Ref. [8].

A characteristic function of this density reads
\[
F(p) = \frac{-\sigma|p|}{|p|} 
\]  
and gives account of a non-thermal fluctuation-dissipation balance. The modified noise intensity parameter \( \sigma \) is a ratio of an intensity parameter \( \lambda \) of the free Cauchy noise and of the friction coefficient \( \gamma \).

From the start we know what is the drift \( b(x) = -\lambda x \) which directs the process towards a target (stationary) pdf. To deduce the Feynman-Kac potential \( V \) for the OUC process, we need to evaluate
\[
V(x) = \frac{\lambda}{\pi} (\sigma^2 + x^2)^{1/2} \int \left[ \frac{1}{\sqrt{\sigma^2 + (x + y)^2}} - \frac{1}{\sqrt{\sigma^2 + x^2}} \right] \frac{dy}{y^2}. 
\]  
(27)
In the notation \( a = \sigma^2 + x^2 \), \( b = 2x \), \( R(y) = \sigma^2 + (x + y)^2 \) indefinite integral reads, [31]:
\[
\frac{\lambda}{\pi} \left[ \frac{\sqrt{a}}{y^2 \sqrt{R(y)}} - \frac{1}{y^2} \right] \frac{dy}{y} = \frac{\lambda}{\pi} \left[ \frac{\sqrt{R(y)}}{y \sqrt{a}} + \frac{b}{2a} \text{Arsh} \left( \frac{2a + by}{2\sigma|y|} \right) + \frac{1}{y} \right]. 
\]  
(28)
Because of the singularity at \( y = 0 \), we must handle the integral in terms of its principal value, i.e. by resorting to \( \int \rightarrow \int_{-\infty}^{\epsilon} + \int_{\epsilon}^{+\infty} \), and next performing the \( \epsilon \rightarrow 0 \) limit.

Taking into account that \( \text{arsinh} \, x = \ln(x + \sqrt{1 + x^2}) \), [32], we ultimately get
\[
V(x) = \frac{\lambda}{\pi} \left[ \frac{2}{\sqrt{a}} + \frac{x}{a} \ln \frac{\sqrt{a} x^2}{\sqrt{a} - x} \right]. 
\]  
(29)
\( V(x) \) is bounded both from below and above, with the asymptotics \( (2/|x|) \ln |x| \) at infinities, well fitting to the general mathematical construction of (topological) Cauchy processes in external potentials, [19].

Accordingly, we know for sure that there exists a jump-type process driven by the Cauchy semigroup with the potential function \( V \), Eq. (29), whose invariant density coincides with that for the Langevin-supported OUC process. This form of the semigroup potential, gives a guarantee that \( \rho_\ast \) actually is an asymptotic invariant density of the process. In Fig. 1 we reproduce the functional shape of the potential (29), [33].

### B. Confined Cauchy process

The OUC process is not confined, since for the Cauchy density its second moment is nonexistent. We shall adopt the OUC discussion to Cauchy-type processes whose invariant densities admit the second moment. Let us consider the quadratic Cauchy density:
\[
\rho_\ast(x) = \frac{2}{\pi} \frac{1}{(1 + x^2)^2}. 
\]  
(30)
FIG. 1: Upper panel: the coordinate dependence of semigroup potentials $V(x)$: (29) for different $\sigma$ and (33) (inset). Lower panel: time dependent variance $X^2(t)$ for Langevin-type (solid line) and semigroup-driven (dashed line) processes associated with the pre-defined target pdf (33). Points correspond to numerical calculation.

The action of $|\nabla|$ upon this density can be evaluated by recourse to the free Cauchy evolution.

We note that $(1/\sqrt{2\pi})\rho_{1/2} = (1/\pi)/(1 + x^2)$ actually is the Cauchy probability density. Let us consider $f(x) = \rho_{1/2}$ as the initial data for the free Cauchy evolution $\partial_t f = \lambda |\nabla| f$. This takes $f(x)$ into

$$f(x, t) = \frac{2}{\pi} \frac{1 + \lambda t}{[(1 + \lambda t)^2 + x^2]}.$$  \hspace{1cm} (31)

Since

$$\lambda |\nabla| f = - \lim_{t \downarrow 0} \partial_t f$$  \hspace{1cm} (32)
we end up with
\[ \mathcal{V}(x) = \lim_{t \to 0} \frac{\partial_t f(x)}{f(x)} = \lambda \frac{x^2 - 1}{x^2 + 1}. \] (33)

A minimum $-\lambda$ is achieved at $x = 0$, $\mathcal{V} = 0$ occurs for $x = \pm 1$, a maximum $+\lambda$ is reached at $x \to \pm \infty$. The functional shape of the potential is depicted as an inset in Fig. 1.

The potential is bounded both from below and above, hence can trivially be made non-negative (add $\lambda$). Therefore, the invariant density (30) is fully compatible with the general construction of the Cauchy-Schrödinger semigroup and the induced jump-type process, c.f. Corollary 2, pp. 1071 in [19]. This topological Cauchy process is induced by the Cauchy generator plus a potential function $V$ given by Eq. (49), c.f. Corollary 2, pp. 1071 in [19]. The process is of the jump-type and can be obtained as an $\epsilon \downarrow 0$ limit of a step process, e.g. jump process whose jump size is bounded from below by $\epsilon > 0$ but unbounded from above.

In connection with the reverse engineering problem of Ref. [17] let us note that if the quadratic Cauchy density we investigated in Ref. [11] with a focus on bimodality of the resultant stationary pdfs. For example a quartic potential $V$ was investigated in Ref. [19]: can be made positive (add a suitable constant), is locally bounded (e.g. is bounded on each compact set) and is measurable (e.g. can be arbitrarily well approximated by means of sequences of step functions). The induced jump-type process, c.f. Corollary 2, pp. 1071 in [19]. This topological Cauchy process is induced by the invariant density (30) is fully compatible with the general construction of the Cauchy-Schrödinger semigroup and the induced jump-type process, c.f. Corollary 2, pp. 1071 in [19]. This topological Cauchy process is induced by the Cauchy generator plus a potential function $V$ given by Eq. (49), c.f. Corollary 2, pp. 1071 in [19]. The process is of the jump-type and can be obtained as an $\epsilon \downarrow 0$ limit of a step process, e.g. jump process whose jump size is bounded from below by $\epsilon > 0$ but unbounded from above.

For clarity of discussion, in the lower panel of Fig. 1, we report a comparison of dynamical patterns of behavior for the semigroup-driven and Langevin-induced scenarios beginning from common (delta-type) initial data and approaching a common (pre-defined) target pdf (33). Our unimodal density (33) belongs to the subclass associated with the just mentioned $U(x)$.

C. Confined Cauchy family

We may consider various probability densities as trial ones. Let us pay attention to a broader class of densities that bear close affinity with the Cauchy noise. With a given continuous probability distribution $\rho$ we associate its Shannon entropy $S(\rho) = -\int \rho \ln \rho \, dx$. If an expectation value $E[\ln(1 + x^2)]$ is prescribed (e.g. fixed), the maximum entropy probability function belongs to a one-parameter family

\[ \rho_\alpha(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi} \Gamma(\alpha - 1/2)} \frac{1}{(1 + x^2)^\alpha} \] (36)

where $\alpha > 1/2$, [32].

The gamma function $\Gamma(\alpha) = \int_0^\infty \exp(-t) t^{\alpha-1} \, dt$ we specialize to integer $\alpha = n + 1$-values, with $n \geq 0$. Then $\Gamma(n + 1) = n!$ and $\Gamma(\alpha - 1/2) \to \Gamma(n + 1/2) = [(2n)!!]\sqrt{\pi}/n!2^n$.

As an exemplary case let us consider

\[ \rho_\alpha(x) = \frac{16}{5\pi} \frac{1}{(1 + x^2)^4} \] (37)

By adopting the previous procedure, c.f. [31], and evaluating the principal value integrals, we end up with the following expression for the Cauchy semigroup potential:

\[ \mathcal{V}(x) = -\frac{\gamma}{2(1 + x^2)} (x^4 + 6x^2 - 3). \] (38)

The potential is bounded from below, its minimum at $x = 0$ equals $-3\gamma/2$. For large values of $|x|$, the potential behaves as $\sim (\gamma/2)x^2$ i.e. shows up a harmonic behavior.

Apart from the unbounded-ness of $\mathcal{V}(x)$ from above, this potential obeys the minimal requirements of Corollary 2 in Ref. [19]: can be made positive (add a suitable constant), is locally bounded (e.g. is bounded on each compact set) and is measurable (e.g. can be arbitrarily well approximated by means of sequences of step functions).
The Cauchy generator plus the potential (38) determine uniquely an associated Markov process of the jump-type and its step process approximations.

We can readily address the reverse engineering problem of Ref. [17]. For the density (37), we ultimately get:

\[ b(x) = -\frac{\gamma x}{16} (5x^6 + 21x^4 + 35x^2 + 35). \]  

This a bit discouraging expression which shows a linear friction \( b \sim -x \) for small \( x \) and a strong taming behavior \( b \sim -x^7 \) for large \( x \), still fits to the above mentioned Corollary 2 of Ref. [19].

V. SUMMARY

We have generalized the reverse engineering (targeted stochasticity) problem of Ref. [17] beyond the original Lévy-Langevin processes setting. We have demonstrated that the notion of Lévy flights in confining potentials is not limited to the Langevin scenario. The Lévy-Schrödinger semigroup involves the notion of external potentials as well. But then with no link to any standard Langevin representation.

Our version of the reverse engineering problem amounts to reconstructing from a given (target) stationary density the potential functions that either: (i) define the forward drift of the Langevin process, or (ii) enter the Schrödinger-type Hamiltonian expression in the semigroup dynamics. Both dynamical scenarios are expected to yield the same asymptotic outcome i.e. the pre-selected target pdf. This goal can be achieved in the semigroup picture (and models of an impact of inhomogeneous environments upon Lévy flights) only if suitable restrictions on the semigroup potentials are observed. The relevant mathematical hints come from Ref. [19] and were illustrated for the case of Cauchy driver.

Insightful, explicitly solvable models are scarce in theoretical studies of Lévy flights, in the presence of external potentials and/or external conservative forces. Therefore, our major task was to find novel analytically tractable examples, that would shed some light on apparent discrepancies between dynamical patterns of behavior associated with two different fractional transport equations (15) and (20) that are met in the literature on Lévy flights. Albeit the predominant part of this research is devoted to the standard Langevin modeling.

We note that a departure point for our investigation was a familiar transformation of the Fokker-Planck operator into its Hermitian (Schrödinger-type) partner, undoubtedly valid in the Gaussian setting. The Fokker-Planck and the corresponding parabolic equation (plus a compatibility condition (5)) essentially describe the same random dynamics.

An analogous transformation is non-existent for non-Gaussian processes. The two fractional Fokker-Planck equations (13) and (15), are inequivalent in the non-Gaussian setting, hence the semigroup dynamics and the Langevin dynamics with the Lévy driver (e.g. noise) refer to different random processes. This behavior we have depicted in the lower panel of Fig. 1. The main technical reason of the incongruence of the two processes seems to be rooted in that the stable noise generator is a (non-locally defined) pseudo-differential operator, while the standard Laplacian (Wiener noise generator) is locally defined. The reverse engineering problem allowed us to demonstrate that those two processes may nevertheless share the same target pdf and may interpolate between common pairs of boundary (initial and terminal) pdfs. Albeit in a different dynamical fashion.

One may wonder whether there is some symmetry principle (like e.g. a remnant of the time-symmetric formulation of the Schrödinger boundary data problem, [18] [19] [27] [28]) that allows to relate two fixed boundary densities by means of different dynamical scenarios. Actually, our observation that the semigroup-driven and Langevin-driven jump-type processes may share a common invariant pdf, that in turn is dynamically accessible from a common for both processes initial pdf, stands for an indirect proof that the involved dynamical scenarios are different. The assumption about driving mechanisms is the only freedom left in the above mentioned boundary data problem, once the initial and terminal pdf data are chosen.

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