ESSENTIAL DIMENSION OF GROUP SCHEMES
OVER A LOCAL SCHEME

DAJANO TOSSICI

Abstract. In this paper we develop the theory of essential dimension of group schemes over an integral base. Shortly we concentrate over a local base. As a consequence of our theory we give a result of invariance of the essential dimension over a field. The case of group schemes over a discrete valuation ring is discussed.

1. Introduction

The notion of essential dimension of a finite group over a field \( k \) was introduced by Buhler and Reichstein [BR97]. It was later extended to various contexts. First Reichstein generalized it to linear algebraic groups [Rei00] in characteristic zero; afterwards Merkurjev gave a general definition for functors from the category of extension fields of the base field \( k \) to the category of sets [BF03]. In particular one can consider the essential dimension of group schemes over a field (see Definition 1.1).

In this paper we would like to extend the notion of essential dimension of a group scheme over a base scheme more general than a field.

If \( G \) is an affine flat group scheme of locally finite presentation over \( S \), a \( G \)-torsor over \( X \) is an \( S \)-scheme \( Y \) with a left \( G \)-action by \( X \)-automorphisms and a faithfully flat and locally of finite presentation morphism \( Y \rightarrow X \) over \( S \) such that the map \( G \times_S Y \rightarrow Y \times_X Y \) given by \((g,y) \mapsto (gy,y)\) is an isomorphism. We recall that isomorphism classes of \( G \)-torsors over \( X \) are classified by the pointed set \( H^1(X, G) \) [Mil80, III, Theorem 4.3]. If \( G \) is commutative, then \( H^1(X, G) \) is a group, and coincides with the cohomology group of \( G \) in the fppf topology.

Definition 1.1. Let \( G \) be an affine group scheme of finite type over a field \( k \). Let \( k \subseteq K \) be an extension field and \([\xi] \in H^1(\text{Spec}(K), G)\) be the class of a \( G \)-torsor \( \xi \). Then the essential dimension of \( \xi \) over \( k \), which we denote by \( \text{ed}_k \xi \), is the smallest non-negative integer \( n \) such that

(i) there exists a subfield \( L \) of \( K \) containing \( k \), with \( \text{trdeg}(L/k) = n \),

(ii) such that \([\xi]\) is in the image of the morphism

\[
H^1(\text{Spec}(L), G) \rightarrow H^1(\text{Spec}(K), G).
\]

The essential dimension of \( G \) over \( k \), which we denote by \( \text{ed}_k G \), is the supremum of \( \text{ed}_k \xi \), where \( K/k \) ranges through all the extension of \( K \), and \( \xi \) ranges through all the \( G \)-torsors over \( \text{Spec}(K) \).

Moreover there is another possible definition, more geometric, for the essential dimension of group schemes. See [BR97, Definition 2.5] and [BF03, Definition 6.8].
Definition 1.2. Let $G$ be an affine group scheme of finite type over a field $k$. The essential dimension of a $G$-torsor $f : Y \to X$ is the smallest dimension of a scheme $X'$ over $k$ such that

(i) there exists a $G$-torsor $f' : Y' \to X'$,

(ii) and a commutative diagram of $k$-rational maps

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & Y' \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{h} & X'
\end{array}
$$

with $g$ dominant and $G$-equivariant. It will be denoted by $\text{ed}_S f$. The essential dimension of $G$ over $k$ is $\sup_{f} \text{ed}_S f$ where $f$ varies between all $G$-torsors. It will be denoted by $\text{ed}_k G$.

We are going to generalize this second definition over a base more general than a field. At a very first sight, over a field, the new definition could look slightly different from Definition 1.2. However we will prove that they are equivalent.

Very soon we specialize to the case that the base is local. And we generalize some standard basic results true over a field. For instance we prove the existence, under some conditions on $G$, of a classifying torsor whose essential dimension gives the essential dimension of the group scheme $G$.

As a natural application of this theory we prove the following result.

Theorem 4.1. Let $G$ be an affine faithfully flat group scheme of finite presentation over an integral locally noetherian scheme $S$. Then there exists a non-empty open subscheme $U$ of $S$ such that for any $s \in U$, with residue field $k(s)$,

$$
\text{ed}_{k(s)}(G_{k(s)}) \leq \text{ed}_{k(S)}(G_{k(S)}) = \text{ed}_{\mathcal{O}_{S,s}} G_{\mathcal{O}_{S,s}}.
$$

In particular we obtain the following corollary.

Corollary 4.4. Let $k$ be a field, let $X$ be an integral scheme of finite type over $k$ with fraction field $k(X)$ and let $G$ be an affine group scheme of finite type over $k$. There exists a non-empty open subscheme $U$ of $X$ such that for any $x \in U$, with residue field $k(x)$,

$$
\text{ed}_{k(x)}(G_{k(x)}) \leq \text{ed}_{k(X)}(G_{k(X)}) = \text{ed}_{\mathcal{O}_{X,x}} G_{\mathcal{O}_{X,x}}.
$$

In particular if the set of $k$-rational points of $X$ is Zariski dense then

$$
\text{ed}_{k}(G) = \text{ed}_{k(X)}(G_{k(X)}).
$$

If $X$ is the affine line the last part of the Corollary is the homotopy invariance theorem of Berhui and Favi [BF03, Theorem 8.4]. Of course by induction it works over $\mathbb{A}^n_k$. It seems to us that the strategy of that proof can not be immediately generalized to the general case because of technical Lemma [BF03, Lemma 8.3].

As A. Vistoli pointed to us, the last part of the Corollary was also proven in the unpublished result [BRV07, Proposition 2.14] in the case $X = \mathbb{A}^n_k$ or $k$ algebraically closed. In this case it seems that their argument can be generalized to (geometrically integral) varieties with the set of $k$-rational points which is Zariski dense.

It would be interesting to know if the converse of the last part of the Corollary 4.4 is true, more precisely
ESSENTIAL DIMENSION OF GROUP SCHEMES OVER A LOCAL SCHEME

(Q) if $X$ is an integral scheme of finite type over $k$ and $ed_k(G) = ed_k(G_{k(X)})$ for any affine group scheme $G$ of finite type over $k$, is the set of $k$-rational point of $X$ Zariski dense?

Some considerations are given at the end of section §4. At the moment the question is open, except for finite fields (proven in [Tân13, Proposition 3.6 and Lemma 4.5]).

ACKNOWLEDGMENTS

I would like to thank Q. Liu, C. Pepin, M. Romagny, J. Tong and A. Vistoli for very useful comments and conversations. Finally I also thank the referee for useful comments. I have been partially supported by the project ANR-10-JCJC 0107 from the Agence Nationale de la Recherche.

2. $S$-rational maps

In this section we recall some notions about rational morphisms. Here $S$ will denote a general scheme. Starting from the next section we will add some hypotheses on it. We will put some details since definitions we give here are slightly different from classical references such as [BLR90] §2.5 or [Gro67] §20. We will point out the differences later on. We begin with the notion of schematically dominant.

Definition 2.1. Let $f : X \to Y$ be a morphism of schemes. We say that $f$ is schematically dominant if $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is injective. We say that $f : X \to Y$ is schematically dense if it is schematically dominant and an open immersion.

Remark 2.2. If $Y$ is reduced then we recover the usual definitions of dominant and dense [Gro66, Proposition 11.10.4]. Without other assumptions the above definition does not mean that a morphism is schematically dominant if and only if the schematic image is $Y$. This is true if the morphism is quasi-compact (see [GW12, Proposition 10.30]).

We also have the relative version.

Definition 2.3. Let $f : X \to Y$ be a morphism of $S$-schemes. We say that $f$ is $S$-dominant if, for any $s \in S$, the morphism $X_s \to Y_s$ is schematically dominant, where $Y_s := Y \times_S \text{Spec}(k(s))$ and $X_s := X \times_S \text{Spec}(k(s))$ are the fibers over $S$. If moreover $f$ is an open immersion we say that $X$ is $S$-dense in $Y$.

We say that $f$ is $S$-universally dominant if it is $S'$-dominant under any base change $S' \to S$. In the case of an open immersion we say that $f$ is $S$-universally dense.

We remark that $S$-universally schematically dominant implies $S$-schematically dominant. Under some mild conditions the converse is true.

Proposition 2.4. Let $f : X \to Y$ be a morphism of $S$-schemes with $X$ flat over $S$. In any of these two situations

(i) $Y$ locally noetherian,

(ii) $f$ is open immersion and $Y$ flat locally of finite presentation over $S$,

then $f$ is $S$-dominant if and only if it is $S$-universally dominant.

Remark 2.5. We remark that from the Proposition it follows that, under the same hypotheses, $f$ is $S$-schematically dominant if and only if $f_T$ is $T$-schematically dominant for any base change $T \to S$. 
Proof. [Gro66, Théorème 11.10.9, Proposition 11.10.10] □

Let $X$ and $Y$ be two $S$-schemes. Let $U$ and $U'$ be two $S$-dense open subschemes of $X$. If we have two morphisms $f : U \to Y$ and $f' : U' \to Y$ we say that they are equivalent if there exists an open subscheme $V \subseteq U' \cap U$, which is $S$-dense in $X$, such that $f$ and $f'$ coincide over $V$. One easily verifies that it is an equivalence relation.

Definition 2.6. An $S$-rational map between two $S$-schemes $X$ and $Y$ is the equivalence class of an $S$-morphism $f : U \to Y$ where $U$ is $S$-dense in $X$. An $S$-rational morphism is denoted by $f : X \dashrightarrow Y$ and for any $U$ as above we say that $f$ is defined over $U$.

The definition here is stronger than EGA’s definition of $S$-pseudo morphism [Gro67, 20.2.1]. In fact there it is not required that the open subscheme is schematically dense on each fiber of $X$. This difference will be very important in the definition of compressions (§3), in order to have a theory of essential dimension over a general base which is compatible with the theory we obtain when we restrict to a point. While in [BLR90, §2.5] the definition of rational morphisms is the same except the fact that all schemes considered are $S$-smooth and so $S$-dense means just Zariski dense on the fibers.

Definition 2.7. An $S$-rational map $f : X \dashrightarrow Y$ is $S$-dominant if it can be represented by an $S$-dominant morphism.

In fact, by the following lemma, the above definition can be restated saying that $f$ is $S$-dominant if any of its representatives is $S$-dominant.

Lemma 2.8. Let $f : X \dashrightarrow Y$ be an $S$-rational map. Let us consider two representatives $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$. Then $f_1$ is $S$-dominant if and only if $f_2$ is $S$-dominant.

Proof. By symmetry it is sufficient to prove just one implication. Moreover, by definition, we can reduce to the case $S$ is the spectrum of a field and in this case $S$-dense is simply schematically dense. Now let $V$ be an open subscheme $V \subseteq U_1 \cap U_2$ and schematically dense in $X$ such that $f_1$ and $f_2$ coincide over $V$, and we call $g$ the restriction. Let us suppose that $f_1$ is schematically dominant. Then the morphism $O_Y \to (f_1)_* O_{U_1}$ is injective. Since $V$ is schematically dominant in $X$ it is also schematically dominant in $U_1$. So, using the following diagram, one can easily see that $g$ is schematically dominant and therefore $f_2$ is schematically dominant.

\[
\begin{array}{ccc}
(f_1)_* O_{U_1} & \rightarrow & (f_2)_* O_{U_2} \\
\downarrow & & \downarrow \\
O_Y & \rightarrow & g_* O_V \\
\end{array}
\]

Definition 2.9. Let $f : X \dashrightarrow Y$ be an $S$-rational map. We say that it is $S$-birational if it is $S$-dominant and there exists a representative $f_1 : U \to Y$ which is an open immersion. We will say that $X$ and $Y$ are $S$-birational if there exists an $S$-birational map between $X$ and $Y$. 

Definition 2.10. Let $S$ be a scheme, $f : X \to Y$ and $g : Y \to Z$ two $S$-rational maps. We call $g \circ f : X \to Z$, if it exists, the rational map represented by $g \circ f_U$, where $U$ is an $S$-dense open subscheme where $f$ is defined and such that $g$ is defined over $f(U)$.

In general it is not possible to define the composition of two $S$-rational maps, even if they are $S$-schematically dominant and the schemes are irreducible. This is possible if we use the classical definition of dominant. Here is an example where the composition does not work.

Example 2.11. Let $k$ be a field. Let us consider $X = \text{Spec}(k[x, y]/(xy, x^2))$, $Y = \text{Spec}(k[x, y]/(x^2))$ and $Z$ equal to $Y$ minus the origin. Then let $f : X \to Y$ be the natural inclusion and let $g : Y \to Z$ the birational morphism induced by the identity over $Z \subseteq Y$. Then the composition $g \circ f$ is defined over the open subscheme $X$ minus the origin. But this open subscheme, which is the maximal where $g \circ f$ is defined, is not schematically dense since the embedded point $(x, y)$ does not belong to it.

However we can define the composition of $S$-rational maps in some cases.

Lemma 2.12. Let $S$ be a scheme, $f : X \to Y$ and $g : Y \to Z$ two $S$-rational maps. In the following cases the composition $g \circ f$ exists.

(i) $g$ is a morphism,

(ii) $f$ is a flat locally of finite presentation morphism and $Y$ is locally noetherian,

(iii) $X \to S$ has integral fibers and $f$ is $S$-dominant.

Proof. The first case is clear. For the other two situations take an $S$-dense $V$ of $Y$ where $g$ is defined. We will prove that $U := f^{-1}(V)$ is $S$-dense in $X$ and so the composition exists. Clearly we can suppose that $S$ is a point. For (ii) we remark that since $f$ is flat locally of finite presentation then $f$ is an open map. So $f(X)$ is open. Then it intersects $V$, which is schematically dense. Therefore $f^{-1}(V)$ is non-empty. Now since $f$ is flat then $f^{-1}(V)$ is schematically dense in $X$, by [Sta16, Lemma 28.24.13, Tag 081H], since any open subscheme of a locally noetherian scheme is retrocompact.

For (iii) we observe that $f^{-1}(V)$ is non-empty since $f$ is schematically dominant. Now if $f^{-1}(V)$ is a non-empty open set of $X$ then it is dense, since $X$ is irreducible. But then it is also schematically dense since $X$ is reduced. □

It is easy to see that the composition is well defined, i.e. does not depend on the representative of $f$.

Lemma 2.13. Let $S$ be a scheme and let $f : Y \to Z$ and $g : W \to Z$ be morphisms of $S$-schemes. For any $S$-rational maps $h_1 : T \to Y$ and $h_2 : T \to W$ such that $f \circ h_1$ is equal to $g \circ h_2$ as $S$-rational maps then there exists a unique $S$-rational map $h : T \to Y \times_Z W$ such that $p_Y \circ h = h_1$ and $p_W \circ h = h_2$ where $p_Y$ and $p_W$ are the projections over $Y$ and $W$.

Proof. Easy to prove using the universal property of cartesian product over an $S$-dense open subscheme where $f \circ h_1$ and $g \circ h_2$ are equal. □

Lemma 2.14. Let $f : X \to Y$ be an $S$-rational map with $X$ flat locally of finite presentation over $S$. For any morphism $T \to S$, we have a $T$-rational map $f_T : X_T \to Y_T$ obtained by base change.
Proof. If \( f \) is defined over an \( S \)-dense \( U \) then \( U_T \) is \( T \)-schematically dense in \( X_T \) by Proposition 2.4. Therefore, using Lemma 2.13 we have a \( T \)-rational map \( f_T : X_T \rightarrow Y_T \).

\[\square\]

3. Definitions and General Results

In the following \( S \) will be an integral locally noetherian scheme. And till the end of the paper, if not differently specified, for group scheme we will mean an affine faithfully flat group scheme of finite presentation over the base. Let \( f : X \rightarrow S \) be a faithfully flat morphism of locally finite type. If \( \eta \) is the generic point of \( S \) we call \( \dim(f^{-1}\eta) \) the relative dimension of \( X \) over \( S \) and we denote it by \( \dim_S X \).

If \( X \) is also irreducible then \( f : X \rightarrow S \) is equidimensional, i.e. for any \( x \in X \), \( \dim(f^{-1}\eta) = \dim_x f^{-1}(f(x)) \) (see \cite{Gro65} Corollaire 6.1.1, Proposition 13.2.3).

For any scheme \( T \) we call \( \mathcal{C}_T \) the full subcategory of \( \text{Sch}/T \) given by faithfully flat schemes \( X \) of locally finite presentation over \( T \) with geometrically integral fibers.

**Lemma 3.1.** Let \( f : X \rightarrow T \) be a flat morphism.

(i) If \( f \) is of locally finite presentation, \( T \) irreducible and any fiber is irreducible then \( X \) is irreducible.

(ii) If \( X \) and \( T \) are locally noetherian, \( T \) is reduced and any fiber of \( f \) is reduced then \( X \) is reduced.

**Proof.** (i) Since \( f \) is flat of locally finite presentation then \( f \) is open. Now let \( U \) and \( V \) be two open sets of \( Y \). Then \( f(U) \) and \( f(V) \) are open since \( f \) is open. Moreover their intersection is non-empty since \( T \) is irreducible. Let \( t \) be a point of this intersection. So \( U \) and \( V \) intersect the fiber over \( t \). Since any fiber of \( f \) is irreducible then there is point over \( t \) contained in \( U \) and \( V \). So \( X \) is irreducible.

(ii) This is \cite{Gro65} Corollaire 3.3.5. \[\square\]

By the previous Lemma any object of \( \mathcal{C}_T \) with \( T \) integral is integral.

And if \( T' \) is another scheme with a morphism \( \pi : T' \rightarrow T \) and \( X \) is an object of \( \mathcal{C}_T \) then \( X_{T'} := X \times_T T' \) is an object of \( \mathcal{C}_{T'} \). In fact if \( t' \) is a point of \( T \) then \( X_{t'} \) is isomorphic to \( X_t \times_{\text{Spec}(k(t))} \text{Spec}(k(t')) \) where \( \pi(t') = t \). So \( X_{t'} \) is geometrically integral.

**Definition 3.2.** Let \( G \) be a group scheme over \( S \). Let \( f : Y \rightarrow X \) and \( f' : Y' \rightarrow X' \) be two \( G \)-torsors with \( X \) and \( X' \) objects of \( \mathcal{C}_S \). We say that \( f' \) is an \( S \)-weak compression of \( f \) if there exists a diagram over \( S \)

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y' \\
\downarrow{f} & & \downarrow{f'} \\
X & \xrightarrow{h} & X'
\end{array}
\]

which is commutative (i.e. \( f' \circ g \) and \( h \circ f \) are equal as \( S \)-rational maps), where \( g \) and \( h \) are \( S \)-rational and \( g \) is \( G \)-equivariant (i.e. there exists an open \( S \)-dense subscheme \( U \) of \( Y \) stable by \( G \) such that \( g_U : U \rightarrow X \) represents \( g \) and it is \( G \)-equivariant).

We say that \( f' \) is an \( S \)-compression of \( f \) if moreover \( g \) is an \( S \)-dominant map. And we say that a weak \( S \)-compression (resp. \( S \)-compression) \( f' \) is defined everywhere if \( g \) and \( h \) are morphisms.

We have the following easy result.
Lemma 3.3. Let $X, X'$ be objects of $\mathfrak{C}_S$. If $f': Y' \to X'$ is a weak $S$-compression of $f: Y \to X$ then there exists an $S$-dense open subscheme $U$ of $X$ such that $f': Y' \to X'$ is a defined everywhere weak $S$-compression of $f_U: Y_U \to U$.

If $f'$ is an $S$-compression then for any $S$-dense open subscheme $U'$ of $X'$ there exists an $S$-dense open subscheme $U$ of $X$ such that $f_U': Y'_U \to U'$ is a defined everywhere $S$-compression of $f_U: Y_U \to U$.

Proof. We use notation of the definition. Take an $S$-dense open subscheme $W$ in $Y$ which is $G$-stable and such that $g$ is defined over $W$. Then $W \to f(W)$ is a $G$-torsor and $f(W)$ is an $S$-dense open subscheme. The induced morphism $f(W) \to X'$ clearly represents $h$. If we set $f(W) = U$ we have that $f'$ is a weak compression of $f_U$.

The last part follows remarking that if $U'$ is an $S$-dense open subscheme of $Y$ then $f_U': Y'_U \to U'$ is a weak $S$-compression of $f$ since $g$ is dominant. □

Definition 3.4. The essential dimension of a $G$-torsor $f: Y \to X$ is the smallest relative dimension of $X'$ over $S$ in a weak compression $f': Y' \to X'$ of $f$, where $X'$ is an object of $\mathfrak{C}_S$ and it will be denoted by $ed_S f$. The essential dimension of $G$ over $S$ is $\sup_f ed_S f$ where $f$ varies between all $G$-torsors over objects of $\mathfrak{C}_S$. It will be denoted by $ed_S G$.

We recall that, apparently, the above definition is different in the case $S$ is a field, from the usual definition which uses compressions (Definition 1.2). There are three differences. First usually one considers compressions instead of weak compressions. Over a field $k$ this is not a problem: if we have a $k$-weak compression

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & Y' \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{h} & X'
\end{array}
$$

then we have that $f_Z': Y'_Z \to Z$ is a $k$-compression of $f$, where $Z$ is the schematic image of $h$. In fact it is clear that $h: X \to Z$ is $k$-dominant. Since $f_Z'$ is faithfully flat then also $g: Y \to Y'_Z$ is $k$-dominant.

The second difference is the fact that we are supposing that the scheme $X'$ is geometrically integral. Thirdly we take locally finite presentation schemes, but this does not cause problems since the essential dimension of an algebraic affine group scheme is finite.

If $S$ is the spectrum of a field then by [BF03 Lemma 6.11 and Remark 6.12] the classical definition using compressions [BF03 Definition 6.8] of essential dimension is equivalent to the functorial definition. We will prove later in Proposition 3.18 that over a field they are both equivalent to our definition.

Lemma 3.5. Let $f: Y \to X$, $f': Y' \to X'$ and $f'' : Y'' \to X''$ be three $G$-torsors over $S$ with $X, X', X''$ objects of $\mathfrak{C}_S$.

(i) If $f''$ is a defined everywhere weak $S$-compression of $f'$ and $f'$ is a weak $S$-compression of $f$ then $f''$ is a weak $S$-compression of $f$.

(ii) If $f''$ is a weak $S$-compression of $f'$ and $f'$ is an $S$-compression of $f$ then $f''$ is a weak $S$-compression of $f$.

(iii) If $f''$ is an $S$-compression of $f'$ and $f'$ is an $S$-compression of $f$ then $f''$ is an $S$-compression of $f$. 
Proof. The proof of (i) is immediate, since in this case the composition of the involved rational maps is well defined.

Now since $f''$ is a weak $S$-compression of $f'$ then, by Lemma 3.3 there exists an $S$-dense open subscheme $U'$ of $X'$ such that $f''$ is a defined everywhere weak $S$-compression of $f''_U$. Therefore, by the last part of Lemma 3.3 there exists an $S$-dense open subscheme $U$ of $X$ such that $f''_U$ is an $S$-defined everywhere compression of $f_U$. Applying (i) we obtain (ii). Moreover we also obtain (iii) since composition of dominant maps is dominant.

**Lemma 3.6.** Let $f : Y \to X$ be a $G$-torsor over $S$ with $X$ an object of $\mathfrak{C}_S$ which is integral. For any morphism $q : T \to S$, with $T$ integral locally noetherian, then $\text{ed}_S f \geq \text{ed}_T f_T$.

Proof. We can clearly suppose that $\text{ed}_S f$ is finite. Let us consider a weak $S$-compression

$$
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y'' \\
\downarrow f' & & \downarrow f'' \\
X' & \xrightarrow{h'} & X''
\end{array}
$$

So, using Lemma 2.14 we obtain, by base change over $T$, a weak $T$-compression

$$
\begin{array}{ccc}
Y'_T & \xrightarrow{g'_T} & Y''_T \\
\downarrow f'_T & & \downarrow f''_T \\
X'_T & \xrightarrow{h'_T} & X''_T
\end{array}
$$

So the above diagram gives a weak $T$-compression. Let us take $X'$ such that $\text{ed}_S f = \dim_S X'$. By Lemma 3.4 $X'$ is irreducible so, as remarked at the beginning of the section, the morphism $f : X' \to S$ is equidimensional. We have now to compute $\dim_T X'_T = \dim \pi^{-1}_T \xi$ where $\pi : X'_T \to T$ is the structural morphism and $\xi$ is the generic point of $T$. If $s = q(\xi)$ then, as schemes, $\pi^{-1}_T \xi$ is isomorphic to $X'_s \times_s \xi$. So $\dim_T X'_T = \dim X'_s$ since the dimension of a finite type scheme over a field does not change when extending the field. Since $X'$ is equidimensional we also have $\dim X'_s = \dim_S X'$. So

$$\text{ed}_S f = \dim_S X' = \dim_T X'_T \geq \text{ed}_T f_T.$$

**Definition 3.7.** Let $f : Y \to X$ be a $G$-torsor over $S$ with $X$ an object of $\mathfrak{C}_S$. We will say that it is **$S$-classifying** if for any $S$-dense open subscheme $U$ of $X$ and for any $G$-torsor $f' : Y' \to X'$ over $S$, where $X'$ is an object of $\mathfrak{C}_S$ with positive $S$-dimension, then $f_U : Y_U \to U$ is a weak compression of $f'$.

**Lemma 3.8.** The compression of an $S$-classifying $G$-torsor is $S$-classifying.

Proof. Let $f : Y \to X$ be an $S$-classifying $G$-torsor, let $f'' : Y'' \to X''$ be a compression of $f$ and let $f' : Y' \to X'$ be any $G$-torsor over $S$ with $X'$ of positive $S$-dimension. We have to prove that for any $S$-dense open subscheme $V$ of $X'$ then $f''_V$ is a weak compression of $f'$. But, by Lemma 3.3, $f''_V$ is an $S$-defined everywhere compression of $f_V$ for some $S$-dense open subscheme $U$ of $X$. Since $f$ is classifying then $f_U$ is a weak compression of $f'$ and so, by Lemma 3.5 (i), we have that $f''_V$ is a weak $S$-compression of $f'$. □
Proposition 3.9. The essential dimension of a group scheme $G$ over $S$ is equal to the essential dimension of an $S$-classifying $G$-torsor, if it exists.

Proof. Clearly the essential dimension of $G$ is greater than or equal to the essential dimension of any $G$-torsor, in particular of an $S$-classifying $G$-torsor. We will now prove that the essential dimension of any $G$-torsor is at most the essential dimension of an $S$-classifying $G$-torsor.

Let $f' : Y' \to X'$ be a $G$-torsor and let $f : Y \to X$ be an $S$-classifying $G$-torsor. Let $n$ be the essential dimension of $f$. Moreover we can suppose that the $S$-dimension of $X'$ is strictly positive otherwise there is nothing to prove. So let us consider a weak $S$-compression $f'' : Y'' \to X''$ of $f$ such that $\dim_S X'' = n$. By Lemma 3.9 there exists an $S$-dense open subscheme $V$ of $X$ such that $f''$ is a defined everywhere weak $S$-compression of $f_V$. Now $f$ is an $S$-classifying $G$-torsor so $f_V$ is a weak $S$-compression of $f'$, then $f''$ is a weak compression of $f'$ by Lemma 3.9 (i). So the essential dimension of $f'$ is less than or equal to $n$. \hfill $\square$

We now give a condition for the existence of an $S$-classifying torsor, in the case $S$ is local.

Proposition 3.10. Let us suppose $S$ is local. Let $G$ be a group scheme and let us suppose that $G$ acts linearly on $\mathbb{A}^n_S$ and that there exists an $S$-dense $G$-stable open subscheme $Y$ of $\mathbb{A}^n_S$ such that we have an induced $G$-torsor $f : Y \to X$, with $X$ an object of $\mathfrak{C}_S$. Then $f$ is an $S$-classifying $G$-torsor. In particular the essential dimension of $G$ is finite and less than or equal to $n - \dim_S G$.

Proof. The last statement is clear.

Moreover we remark that since $Y$ is an open subscheme of an affine space it is an object of $\mathfrak{C}_S$. The same is then true for $X$ since $Y \to X$ is $S$-dominant (since faithfully flat). We have now to prove that, for any $V$ open subscheme of $X$ faithfully flat over $S$ and for any $G$-torsor $f' : Y' \to X'$, with $X'$ an object of $\mathfrak{C}_S$ of positive dimension, that $f_V$ is a weak compression of $f'$. We can clearly suppose $V = X$.

We proceed exactly as in [Mer09] Theorem 4.1 where the result is proved in the case of a field. We repeat the proof just to point out where the hypothesis that $S$ is local is used.

Let us consider the $G \times_S G$-torsor $Y' \times_S Y \to X' \times_S X$. If we quotient by the diagonal we have a $G$-torsor $Y' \times Y \to Z$. Now we have that $Y' \times_S Y \to Y'$ is an open subscheme of the trivial vector bundle $Y' \times_S \mathbb{A}^n_S \to Y'$. Since $G$ acts linearly on $\mathbb{A}^n_S$ then we have that $Y' \times_S \mathbb{A}^n_S \to Y'$ descends to a vector bundle $W$ over $X'$ which contains $Z$ as an open subscheme. For any point $x$ of $X'$ there exists an open subscheme $U'$ containing $x$ such that the vector bundle is trivial. Let us take $x$ in the preimage of the closed point of $S$ under the morphism $\pi : X' \to S$. Since $\pi(U')$ is open, $S$ is local and $\pi(U')$ contains the closed point of $S$ then it is equal to $S$. So $U'$ intersects any fiber of $X'$ over $S$. Since $X'$ has integral fibers we have that $U'$ is $S$-dense. Then by the following Lemma we have a rational section $s$, defined over $U'$, of the vector bundle which factorizes through $Z$. Then we have finally, by
Lemma 2.13 a weak $S$-compression given by
\[
\begin{array}{ccc}
Y' & \xrightarrow{\phi} & Y \\
\downarrow f' & & \downarrow f \\
X' & \xrightarrow{\psi} & X
\end{array}
\]
So we are done using Lemma 3.5(i).

Lemma 3.11. Let us suppose $S$ local and $X \to S$ is a faithfully flat morphism locally of finite presentation with integral fibers of positive dimension. For any $S$-dense open subscheme $V$ of a linear affine space $\mathbb{A}^n_S$ there exists an affine $S$-dense open subscheme $U$ of $X$ with a morphism $U \to V$.

Proof. We can suppose that $X$ is affine taking an affine open subscheme with non-empty fiber over the closed point of $S$. Since $S$ is local, as in the proof of the Proposition, one proves that this open subscheme maps onto $S$ and so, since fibers of $X \to S$ are integral, it is an $S$-dense open subscheme of $X$. In particular it is faithfully flat over $S$. Moreover we can assume that $V$ is a principal open. So let $X = \text{Spec}(A)$ and $V = \text{Spec}(R[T_1, \ldots, T_n]_f)$ with $f \in R[T_1, \ldots, T_n]$ but with at least one invertible coefficient. Let $m$ be the maximal ideal of $R$. Since $A/mA$ is infinite then there exist $a_1, \ldots, a_n \in A$ such that $h := f(a_1, \ldots, a_n)$ is nonzero modulo $m$. So, $T_i \mapsto a_i$ for $i = 1, \ldots, n$, gives a morphism $\text{Spec}(A_h) \to V$ and $\text{Spec}(A_h)$ is faithfully flat over $S$.

The following result generalizes [BF03, Remark 4.12], which works over a field.

Corollary 3.12. Let us suppose $S$ local. If

(i) $G$ is a closed subgroup scheme of $GL_{n,S}$ and

(ii) there exists an $S$-dense open subscheme $U$ of $GL_{n,S}$ such that the schematic quotient $U/G$ exists and $U \to U/G$ is a $G$-torsor,

then $U \to U/G$ is an $S$-classifying $G$-torsor. In particular the essential dimension of $G$ is finite and less than or equal to $n^2 - \dim_S G$.

Proof. Let us consider $GL_{n,S}$ contained in $\mathbb{A}^n_S$. If we view $\mathbb{A}^n_S$ as the scheme which represents the functor of square matrices of order $n$ then $GL_{n,S}$ acts on it by multiplication on the right. In fact it acts freely, i.e. it acts freely on $T$-points, with $T$ any $S$-scheme. Now by condition (ii), $U \to U/G$ is a $G$-torsor. It is easy to verify that $U/G$ is an object of $\mathcal{E}_S$. So by the above proposition it is a classifying $G$-torsor.

Remark 3.13. We recall that any affine flat group scheme of finite type over an affine regular noetherian scheme $T$ of dimension $\leq 2$ is isomorphic to a closed subgroup scheme of $GL_{n,T}$ for some $n$ (see [GD11, Exposé VI, Proposition 13.2]). So any group scheme (recall our conventions at the beginning of the section) over a regular noetherian local scheme of dimension $\leq 2$ satisfies the first condition. Moreover condition (i) is always satisfied for finite group schemes over $S$. In fact the usual proof which works for fields (see for instance [Wat79, §3.4]) works also for finite group schemes of finite presentation over $S$. The main point is to find a finite free $O_S$-module $M$ where $G$ acts faithfully, i.e. the morphism of sheaves $G \to GL_n(M)$ is injective. But in fact firstly one takes $O_G$. Since $G$ is flat and of finite presentation over $S$ then $O_G$ is projective as $O_S$-module [Bou07 §1.5, Corollaire]. So it is
a direct factor of a finite (since it is finitely generated) free $\mathcal{O}_S$-module $M$. Then $G$ acts faithfully on $M$.

We remark that, supposing the first condition is verified, the second one is satisfied if $S$ is of dimension $\leq 1$. This follows from \cite[Theorem 4.C and Theorem 3.1.1]{Ana73}. In fact one can take $U = \text{GL}_{m,S}$. In general, if \text{(i)} is satisfied, there exists always an open subscheme $U$ with that property, by \cite[Exposé V, Théorème 10.4.2]{GD11}, since the action of $G$ over $\text{GL}_{m,S}$ is free. But we do not know if we can take it $S$-dense in general.

**Definition 3.14.** We will call a standard torsor any torsor which satisfies conditions of Corollary (even if the base is not local).

Since the hypotheses of Corollary 3.12 are stable by base change then if $G$ has a standard torsor over $S$ then it has a standard torsor, by base change, over $T$ for any morphism $T \to S$.

**Corollary 3.15.** Let us suppose $S$ local and let us suppose there exists a standard $G$-torsor over $S$. Let $T \to S$ be a morphism of schemes with $T$ local, integral and noetherian. Then $\text{ed}_T G_T \leq \text{ed}_S G_S$.

**Remark 3.16.** If $S$ and $T$ are spectra of fields this corollary is exactly \cite[Proposition 1.5]{BF03}, even if the proof is different. The above corollary can be applied, for instance, with $T$ a point of $S$.

**Proof.** Since the pull-back of a standard torsor is a standard torsor then the result follows by Lemma 3.6 and Proposition 3.9. 

We obtain the following result.

**Corollary 3.17.** Let $k$ be a field, let $G$ be a $k$-group scheme and let us suppose that $S$ is a local $k$-scheme. For any point $x$ of $S$ with residue field $k(x)$

$$\text{ed}_k G \geq \text{ed}_S G_S \geq \text{ed}_{k(x)} G_{k(x)}.$$ 

In particular if the closed point is $k$-rational then

$$\text{ed}_k G = \text{ed}_S G_S.$$

**Proof.** This follows by the above corollary, since $G_S \times_S \text{Spec}(k(x)) \simeq G_{k(x)}$. For the last part, since $x$ is $k$-rational, then $k(x) = k$ and $G_{k(x)} = G$.

We finally prove that if $S$ is a field we recover the usual definition of essential dimension.

**Proposition 3.18.** If $k$ is a field then Definition 3.4 is equivalent to the usual definition of essential dimension of a group scheme (Definition 1.1).

**Proof.** In \cite[Corollary 6.16 and Lemma 6.11]{BF03} it is proved that the essential dimension of $G$ (as in Definition 1.1) is the dimension of $X'$ where $Y' \to X'$ is a $k$-compression of $Y \to X$ and $Y \to X$ is a classifying $G$-torsor. By \cite[Remark 4.12]{BF03} we can suppose it is standard. We remark that in \cite[see also Definition 1.2]{BF03} it is not assumed that $X'$ is an object of $\mathcal{C}_k$. So, a priori, it could be not geometrically integral. But since $Y$ is an open subscheme of an affine space it is geometrically integral. The same is then true for $X$ since $Y \to X$ is schematically dominant (since faithfully flat). Finally since also $X \to X'$ is schematically dominant, being dominant between reduced schemes, then also $X'$ is geometrically
integral. We also remark that in Definition 3.4 we admit $k$-weak compressions. But this is not at all a problem, since over a field, as explained after Definition 3.4 we can suppose to work with $k$-compressions.

We have this unsurprising result.

**Lemma 3.19.** Let $S$ be a local. Let $H$ be a closed (faithfully flat) $S$-subgroup scheme of a group scheme $G$ over $S$ and suppose that $G$ has a standard torsor. Then

$$\text{ed}_S H + \dim_S H \leq \text{ed}_S G + \dim_S G.$$  

*Proof.* The proof follows [BP03 Theorem 6.19], which gives the result over a field.

Take a standard $G$-torsor $f : U \to X$. By Proposition 3.9 and Corollary 3.12 the essential dimension of $G$ over $S$ is equal to the essential dimension of this torsor. We remark that $g : U \to U/H$ is a standard classifying $H$-torsor. Now let $f' : U' \to X'$ be an $S$-weak compression of $f$ such that $\text{ed}_S G = \dim_S X'$. Then $g' : U' \to U'/H$ is an $S$-weak compression of $g$. Therefore

$$\text{ed}_S H \leq \dim_S(U'/H) = \dim_S U' - \dim_S H = \dim_S G + \dim_S X' - \dim_S H = \text{ed}_S G + \dim_S G - \dim_S H$$

and we are done. □

### 4. Invariance of Essential Dimension by Base Change

In this section we see when the essential dimension over a field remains invariant if we change base field. In the following $S$ is, as above, an integral locally noetherian scheme. And see the beginning of section §3 for the assumptions on group schemes. We first prove the following result.

**Theorem 4.1.** Let $G$ be a group scheme over $S$. Then there exists a non-empty open subscheme $U$ of $S$ such that for any $s \in U$, with residue field $k(s)$,

$$\text{ed}_{k(s)}(G_{k(s)}) \leq \text{ed}_{k(s)}(G_{k(s)}) = \text{ed}_{O_{S,s}} G_{O_{S,s}}.$$  

*Proof.* By Lemma 3.6 we have just to prove that there exists a non-empty open subscheme $U$ of $S$ such that, for any $s \in U$, $\text{ed}_{O_{S,s}}(G_{O_{S,s}}) \leq \text{ed}_{k(s)}(G_{k(s)})$. We know that $G_{k(s)}$ has a closed immersion $i$ in some $\text{GL}_{n,k(S)}$. We observe that $\text{Spec}(k(s)) = \lim U$ where $U$ ranges through affine open subschemes of $S$. By [GD11 Exposé VI Proposition 10.16] and [Gro66 Théorème 8.8.10] there exists an affine open subscheme $V$ of $S$ such that $i$ extends (uniquely up to restrict the open subscheme) to a closed immersion $G_V \to \text{GL}_{n,V}$. The closed immersion $i$ gives a standard torus $f : Y \to X$ over $\text{Spec}(k(S))$. By [Gro66 Théorème 8.8.1] up to restrict $V$ we can suppose that $f$ extends to a morphism $Y' \to X'$ over $V$. Again by [Gro66 Théorème 8.8.2] up to restrict $V$ we can suppose that the $X$-action of $G_{k(S)}$ over $Y$ extends to a $X'_V$-action of $G_V$ over $X'_V$. Finally, by [GD11 Exposé VI Proposition 10.16] up to restrict again $V$, we can suppose that $\tilde{f} : Y'_V \to Z'_V$ is a $G_V$-torsor. We remark that it is a standard $G_V$-torsor associated to the above extension of $i$.

Let $g : W \to Z$ be a $k(S)$-weak compression of $f_{k(S)}$ such that $\dim_{k(S)} Z = \text{ed}_{k(S)} G_{k(S)}$, which is possible since $f_{k(S)}$ is classifying. Reasoning as above this $k(S)$-weak compression extends to a weak $U$-compression of $\tilde{f}_U$, where $U$ is an
Then non-empty open subscheme $U$ such that $\dim \text{dim (tangent space)}$ is an upper semicontinuous function. Then Theorem 1.2], we have that $\text{ed}_{\mathcal{O}_{S,x}} \mathcal{O}_{S,x} \leq \text{ed}_{k(S)} G_{k(S)}$ as wanted. □

**Example 4.2.** If $G$ is a group scheme over $\mathbb{Z}$ we have that

$$\text{ed}_q G_q \geq \text{ed}_{\mathbb{F}_p} G_{\mathbb{F}_p}$$

for all except possibly finitely many primes $p$.

Now we state some corollaries. We recall that a group scheme over a field $L$ is called almost special (as defined in [TV13] Theorem 1.2] has been proved that this is the minimal value which can be obtained. For instance trigonalizable group schemes of height $\leq 1$ are almost special (see [TV13 Corollary 4.5]). We recall that a group scheme over a field of characteristic $p$ is of height $\leq n$ if it is killed by $F^n$ where $F$ is the Frobenius. We do not know examples of trigonalizable group schemes not almost special even if we think that there are a lot of them. By the previous Theorem it follows the following corollary.

**Corollary 4.3.** Let $G$ be a group scheme over $S$ such that $G_{k(S)}$ is almost special. Then $\text{ed}_{k(S)} G_{k(S)} \leq \text{ed}_{k(s)} G_s$ for any $s \in S$ and there exists a non-empty open subscheme $U$ of $S$ such that, for any $s \in U$, $G_s$ is almost special and $\text{ed}_{k(s)} G_s = \text{ed}_{k(S)} G_{k(S)}$.

**Proof.** We have $\text{ed}_{k(S)} G_{k(S)} = \text{dim}_{k(S)} \text{Lie} G_{k(S)} - \text{dim} G_{k(S)}$. By [TV13 Theorem 1.2], $\text{ed}_{k(S)} G_s \geq \text{dim}_{k(s)} \text{Lie} G_s - \text{dim}_{k(s)} G_s$ for any $s \in S$. Moreover, the dimension of the tangent space is an upper semicontinuous function. Then $\text{dim}_{k(s)} \text{Lie} G_s \geq \text{dim}_{k(S)} \text{Lie} G_{k(S)}$ for any $s \in S$ and there exists a non-empty open subscheme $V$ of $S$ such that $\text{dim}_{k(s)} \text{Lie} G_s = \text{dim}_{k(S)} \text{Lie} G_{k(S)}$. So for any $s \in V$, $\text{ed}_{k(s)} G_s \geq \text{ed}_{k(S)} G_{k(S)}$ and $\text{ed}_{k(S)} G = \text{dim}_{k(s)} \text{Lie} G_s - \text{dim} G_s$ for any $s \in V$. Finally, by the Theorem [4.4] there exists a non-empty open subscheme $U \subseteq V$ of $S$ such that $\text{ed}_{k(s)} G_s \leq \text{ed}_{k(S)} G_{k(S)}$. Then, for any $s \in U$, $G_s$ is almost special and $\text{ed}_{k(s)} G_s = \text{ed}_{k(S)} G_{k(S)}$. □

We also have this corollary.

**Corollary 4.4.** Let $k$ be a field, let $X$ be an integral scheme of finite type over $k$ with fraction field $k(X)$ and let $G$ be a group scheme over $k$. There exists a non-empty open subscheme $U$ of $X$ such that for any $x \in U$, with residue field $k(x)$,

$$\text{ed}_{k(x)} (G_{k(x)}) \leq \text{ed}_{k(X)} (G_{k(X)}) = \text{ed}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}.$$

In particular if the set of $k$-rational points of $X$ is Zariski dense then

$$\text{ed}_k G = \text{ed}_{k(X)} (G_{k(X)}).$$

**Proof.** We have just to apply Theorem [4.4] to $G_X$ over $X$ and we are done. The last statement is clear using Corollary [3.17]. □

**Remark 4.5.** (i) The requirement about rational points is necessary. In fact, otherwise, one can take a finite extension of fields $k \subseteq k'$ and a group $G$ such that $\text{ed}_k G > \text{ed}_{k'} G$. However we do not have a counterexample with $X$ geometrically integral over $k$ of positive dimension. See also discussion before Proposition [4.11].
(ii) If the set of $k$-rational points are dense then, in particular, there exists a Zariski-dense set of closed points $x$, given by $k$-rational points, such that $\text{ed}_k(x)G = \text{ed}_{k(X)}G$. One could ask if this works in general, i.e. does there exist a Zariski-dense set $C$ of closed points such that for any $x \in C$ we have $\text{ed}_k(x)G_{k(x)} = \text{ed}_{k(X)}G_{k(X)}$? The answer is no. In fact take a geometrically integral variety $X$ over $\mathbb{R}$ without $\mathbb{R}$-rational points, e.g. the affine curve over $\mathbb{R}$ defined by $x^2 + y^2 = -1$. Then, by [BF03, Theorem 7.6], we have that $\text{ed}_\mathbb{R}(X)\mathbb{Z}/4\mathbb{Z} = \text{ed}_\mathbb{R}\mathbb{Z}/4 = 2$ since $\mathbb{R}$ and $\mathbb{R}(X)$ have no square roots of $-1$. But any closed point of $X$ has $\mathbb{C}$ as residue field. And we have that $\text{ed}_\mathbb{C}\mathbb{Z}/4\mathbb{Z} = \text{ed}_\mathbb{C}\mu_4 = 1$.

We recall that a field $k$ is pseudo algebraically closed if any (non-empty) integral geometrically irreducible scheme $X$ of finite type over $k$ has at least $k$-rational point. It is clear that this is equivalent to say that for any such an $X$ the set of $k$-rational point of $X$ is dense.

**Corollary 4.6.** If $k$ is a pseudo algebraically closed field and $X$ an integral geometrically irreducible scheme of finite type over $k$ with fraction field $k(X)$ then
\[
\text{ed}_k(G) = \text{ed}_{k(X)}(G_{k(X)}),
\]
for any group scheme $G$ over $k$.

**Proof.** This follows just by definition of pseudo algebraically closed and by the last part of the Corollary 4.4. \hfill \Box

**Corollary 4.7.** Let $G$ be a group scheme over a field $k$. Then
\[
\text{ed}_k G_k \leq \text{ed}_K G_K
\]
for any extension $K$ of $k$, where $\bar{k}$ is an algebraic closure of $k$.

**Proof.** We first restrict to finitely generated extensions $K$ of $k$. Let $X$ be an integral scheme of finite type over $k$ with fraction field $K$. Then by Corollary 4.4 there exists a closed point $x$ of $X$, since closed points are dense in $X$, such that $\text{ed}_k(x)G_{k(x)} \leq \text{ed}_K G_K$. But since $x$ is closed then $k(x)$ is a finite extension of $k$, so
\[
\text{ed}_k G_k \leq \text{ed}_k(x)G_{k(x)} \leq \text{ed}_K G_K
\]
Now the general case follows from the following lemma. \hfill \Box

**Lemma 4.8.** Let $G$ be a group scheme over $k$ and $K$ an extension of $k$. Then there exists a finitely generated extension $L$ of $k$ such that
\[
\text{ed}_K G_K = \text{ed}_L G_L.
\]

**Proof.** Take a standard classifying $G$-torsor $f : Y \to X$. Then we have that $f_K : Y_K \to X_K$ is a standard classifying $G_L$-torsor. Let $Y' \to X'$ a compression of $f_K$ such that $\dim_K X' = \text{ed}_K G_K$. The $G_K$-torsor $Y' \to X'$ and the compression are in fact defined over a finitely generated extension $L$ of $k$. So we obtain a compression of $f_L : Y_L \to X_L$. This means that $\text{ed}_L G_L \leq \dim_K X' = \text{ed}_K G_K$. Since we always have the opposite inequality we are done. \hfill \Box

**Corollary 4.9.** Let $G$ be a group scheme over a field $k$. Then the following are equivalent:

---

14 DAJANO TOSSECI
(i) \[ \text{ed}_k G = \text{ed}_{k'} G_{k'}, \]

for any finite extension \( k' \) of \( k \).

(ii) \[ \text{ed}_k G = \text{ed}_K G_K, \]

for any extension \( K \) of \( k \).

(iii) \[ \text{ed}_k G = \text{ed}_{\bar{k}} G_{\bar{k}}. \]

**Proof.** It is clear that (ii) \( \Rightarrow \) (iii). Now (iii) \( \Rightarrow \) (i) since any finite extension of \( k \) is included in \( \bar{k} \).

We finally prove (i) \( \Rightarrow \) (ii). The proof is similar to that one of Corollary 4.7.

By Lemma 4.8 we can suppose \( K \) finitely generated over \( k \). Then \( K \) is the fraction field of an integral variety \( X \). Then by Corollary 4.4 there exists a closed point \( x \) of \( X \) such that \( \text{ed}_{k(x)} G_{k(x)} \leq \text{ed}_K G_K \leq \text{ed}_K G_k \). But since \( x \) is closed then \( k(x) \) is a finite extension of \( k \), so we are done. \( \square \)

**Example 4.10.** If \( k \) is algebraically closed and \( G \) is a group scheme over \( k \) then \( \text{ed}_k G = \text{ed}_K G_K \) for any extension \( K \) of \( k \). This is also proven in [BRV07, Proposition 2.14].

As explained in the introduction it would be interesting to answer the following question.

(Q) if \( X \) is an integral scheme of finite type over \( k \) and \( \text{ed}_k (G) = \text{ed}_{k(X)} (G_{k(X)}) \) for any group scheme \( G \), is the set of \( k \)-rational points of \( X \) Zariski-dense?

If the answer was positive (at least over number fields), and we are not so optimistic, the Lang conjecture, i.e. the set of rational points of varieties of general type over a number field is not Zariski-dense, could be rephrased in terms of essential dimension, giving, possibly, a new point of view. Namely the Lang conjecture would be rewritten as:

(L) if \( X \) is a variety of general type over a number field \( k \) and with fraction field \( K \) then there exists a group scheme \( G \) over \( k \) such that \( \text{ed}_k G \neq \text{ed}_K G_K \).

Nevertheless we remark that the above statement (L) always implies, using Corollary 4.4 Lang Conjecture. Only the converse is linked to the question (Q).

The positive answer to the question (Q) for varieties over \( \mathbb{F}_p \) (generalizable to finite fields) is given essentially in [Tân13, Prop 3.6 and Lemma 4.5]. We remark that the set of rational points of any positive dimensional variety over a finite field is not Zariski dense. The following result is just a refinement of [Tân13, Prop 3.6]. The idea of the proof is the same, we just slightly improved it to include, for instance, the essential dimension of \((\mathbb{Z}/p\mathbb{Z})^2\) over \( \mathbb{F}_p \).

**Proposition 4.11.** Let \( k \) be a field of characteristic \( p \). Then

\[ \text{ed}_K(\mathbb{Z}/p\mathbb{Z})^r = \begin{cases} 
2 & \text{if } K \text{ is finite of order less than } p^r \\
1 & \text{otherwise.} 
\end{cases} \]

In particular if \( K \) is the function field of a positive dimensional variety over \( \mathbb{F}_p \) and \( q = p^r \) then \( \text{ed}_q(\mathbb{Z}/p\mathbb{Z})^s > \text{ed}_K(\mathbb{Z}/p\mathbb{Z})^s \) if \( s > r \).

**Proof.** Suppose that \( K \) is finite of order greater than or equal to \( p^r \). Then it contains an \( \mathbb{F}_p \)-vector space of dimension \( p^r \) and so we can embed \((\mathbb{Z}/p\mathbb{Z})^r\) in \( \mathbb{G}_{a,K} \), which gives \( \text{ed}_K((\mathbb{Z}/p\mathbb{Z})^r) \leq \dim_K \mathbb{G}_{a,K} + \text{ed}_K \mathbb{G}_{a,K} = 1 \) (see [Led07, Proposition 5] for a more general statement).
Now we suppose we are in the other situation. By [BF03, Lemma 7.2], we have that if \( ed_K(\mathbb{Z}/p\mathbb{Z})^r = 1 \) then \( (\mathbb{Z}/p\mathbb{Z})^r \) is isomorphic to a subgroup of \( \text{PGL}_2(K) \). Let \( q \) be the order of \( K \). It is easy to see that \( \text{PGL}_2(K) \) has order \( (q+1)(q^2-q) = q(q^2-1) \). Therefore it has no \( p \)-subgroups of order greater than \( q \). Therefore \( ed_K(\mathbb{Z}/p\mathbb{Z})^r > 1 \). On the other hand by [Tân13, Lemma 3.5] we have that the essential dimension of an elementary \( p \)-abelian group in characteristic \( p \) should be less than or equal to 2. So we are done.

The last part is clear. \( \square \)

5. Essential dimension over a discrete valuation ring

In this section let \( R \) be a discrete valuation ring with residue field \( k \) of characteristic \( p > 0 \) and fraction field \( K \). We set \( S = \text{Spec}(R) \). We recall that for group scheme over \( S \) we will mean an affine faithfully flat group scheme of finite presentation over \( S \). So in particular any group scheme will be automatically flat over the base. If \( G \) is a group scheme over \( K \), a model of \( G \) is a group scheme \( \mathcal{G} \) over \( S \) with an isomorphism \( \mathcal{G} \times_S K \to G \). If \( G \) finite we require \( \mathcal{G} \) finite over \( S \), if not differently specified. We observe that any finite (flat) group scheme over \( S \) has a standard \( \mathbb{G}_a \)-torsor (see Remark 3.13).

We know that \( \mathbb{G}_a \) and \( \mathbb{G}_m \) have essential dimension zero over any field. We have the following result.

**Proposition 5.1.** A model of \( \mathbb{G}_m,K \) or \( \mathbb{G}_a,K \) has essential dimension zero if and only if it is smooth with connected fibers.

**Proof.** We first prove the if part. It is known by [WW80, Theorem 2.2] that \( \mathbb{G}_a,S \) is the unique smooth model with connected fibers of \( \mathbb{G}_a,K \) and by [WW80, Theorem 2.5] that any smooth model with connected fibers of \( \mathbb{G}_m,K \) is isomorphic to

\[
\mathcal{G}^\lambda = \text{Spec}(R[T, \frac{1}{1+\lambda T}])
\]

for some \( \lambda \in R \setminus \{0\} \), where the law group is the unique one such that the morphism

\[
\mathcal{G}^\lambda \to \mathbb{G}_m,S
\]

given by \( T \mapsto 1+\lambda T \) is a morphism of group schemes. This is clearly an isomorphism on the generic fiber. These groups depend only on the valuation of \( \lambda \). We remark that if \( \lambda = 0 \) then \( \mathcal{G}^\lambda \simeq \mathbb{G}_a,S \).

Now let \( Y \to X \) be a \( \mathbb{G}_a,S \)-torsor with \( X \) object of \( \mathfrak{C}_S \). It is well known that this torsor is locally trivial in the Zariski topology. Here the important point in fact is that it is trivial over an \( S \)-dense, using the argument already used in the proof of Proposition 3.10. Therefore the essential dimension is zero.

Now, also a \( \mathcal{G}^\lambda \)-torsor, with \( \lambda \neq 0 \), is locally trivial in the Zariski topology. This has been proved in [Tos08, Proposition 2.3.1]. So, as above, it is trivial over an \( S \)-dense and therefore the essential dimension is zero.

Just to be precise in [Tos08, Proposition 2.3.1] it has been proved that the first group of cohomology of \( \mathcal{G}^\lambda \) in the small fppf site is the same of the first group of cohomology in the small Zariski site over a scheme \( X \). Small fppf site means that the category you are considering is that one of flat of locally finite presentation schemes over \( X \). Similarly for the small Zariski site you are considering just Zariski
open sets of $X$. But by [Mi80, III Proposition 3.1] the cohomology is the same if you take the small site or the big site (i.e. you consider the category of all schemes locally of finite presentation over $X$).

The only if part follows by the following Lemma.

 Lemma 5.2. Any group scheme over a noetherian integral scheme $T$ has essential dimension greater than zero if one of its fibers is non-smooth or not connected.

 Proof. By Corollary 3.8 we have that the essential dimension over $T$ is greater than or equal to the essential dimension over any fiber. So we can conclude by [TV13, Theorem 1.2], for the non-smooth case. Moreover in the proof of [TV13, Proposition 4.3], has been proved that a group scheme over a field with essential dimension zero (so necessarily smooth) is connected. So we are done.

 Lemma 5.3. Let us suppose that $K$ has characteristic $p > 0$. Any model (not necessarily finite) of a finite closed subgroup scheme of $\mathbb{G}_{a,K}$ of order $p$ is isomorphic as $S$-group scheme to a closed subgroup scheme of $\mathbb{G}_{a,R}$.

 Proof. If models are finite this follows from more general statements about classification of group schemes, like [TO70] (if $S$ is complete) or [dJ93, Proposition 2.2]. Lemma 5.3 works however also for quasi-finite models. We give here a direct proof. Let $G$ be a model of a finite subgroup scheme of $\mathbb{G}_{a,K}$. Then there exists $x \in K[G]$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$, where $\Delta$ is the comultiplication. Then there exists $a \in R$ such that $ax \in R[G]$. Since $\Delta(ax) = ax \otimes 1 + 1 \otimes ax$ the element $ax$ gives a morphism $G \to \mathbb{G}_{a,R}$. Take the image of this morphism on the generic fiber and consider the schematic closure $G'$, which is flat over $R$. So the induced morphism $G \to G'$ is a model map, i.e. it is an isomorphism on the generic fiber. Therefore, by [WW80, Theorem 1.4], it is a composition of Néron blow-ups (also called dilatations). Since $G$ has order $p$, any blow-up is done over the trivial subgroup of the special fiber. Now, since $G' \subseteq \mathbb{G}_{a,R}$, the first blow-up is contained, by [BLR90, Proposition 2, §3.2], in the blow up of $\mathbb{G}_{a,R}$ in the trivial subgroup scheme of the special fiber. But, by the proof of [WW80, Theorem 2.2], this is isomorphic to $\mathbb{G}_{a,R}$. Now continuing this process we have that $G$ is isomorphic to a subgroup scheme of $\mathbb{G}_{a,R}$.

 In the proof of Proposition 5.1 we recalled smooth models of $\mathbb{G}_{m,K}$ with connected fibers, the so called $G^\lambda$, with $\lambda \in R \setminus \{0\}$. We consider the isogeny $G^\lambda \to G^\mu$, with $v(p) \geq (p-1)v(\lambda)$ if $R$ has characteristic zero, given by $T \mapsto ((1+\lambda T)^p-1)/p$. We note the kernel $G_{\lambda,1}$. It is a model of $\mu_{p,K}$, with natural isomorphism on the generic fiber. In fact all models of $\mu_{p,K}$ are of this type (see considerations just after [WW80, Theorem 2.5]).

 Definition 5.4. For $i \in \{1, \ldots, n\}$ let $\lambda_i \in R \setminus \{0\}$.

 1. A filtered $\mathbb{S}$-group scheme of type $(\lambda_1, \ldots, \lambda_n)$ is a tuple $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$ of (affine) smooth commutative $\mathbb{S}$-group schemes such that there exist exact sequences, with $1 \leq i \leq n$ and $\mathcal{E}_0 = 0$:

$$0 \to G^{\lambda_i} \to \mathcal{E}_i \to \mathcal{E}_{i-1} \to 0.$$
A Kummer group scheme of type $(\lambda_1, \ldots, \lambda_n)$ is a tuple $G = (G_1, \ldots, G_n)$ of finite (flat) commutative $S$-group schemes such that there exist a filtered $S$-group scheme $E = (E_1, \ldots, E_n)$ and commutative diagrams with exact rows, with $1 \leq i \leq n$ and $G_0 = 0$:

$$
\begin{array}{c}
0 \rightarrow G_{\lambda,1} \rightarrow G_i \rightarrow G_{i-1} \rightarrow 0 \\
0 \rightarrow G^\lambda \rightarrow E_i \rightarrow E_{i-1} \rightarrow 0
\end{array}
$$

We remark that if $G$ is a Kummer group scheme contained in a filtered groups scheme $E$ then $E/G$ is filtered. Sometimes, by abuse of notation, we will say that $E_n$ (resp. $G_n$), is a filtered $S$-group scheme (resp. Kummer group scheme). We also stress the fact that, even if the generic fiber is diagonalizable, almost always the special fiber is unipotent.

These group schemes have been introduced and classified, at least in the case the generic fiber is cyclic, in [MRT13]. Conjecturally they represent all models of finite diagonalizable $p$-group schemes. We have the following result.

**Lemma 5.5.** A filtered group scheme has essential dimension zero and a Kummer group scheme of order $p^n$ has essential dimension less than or equal to $n$.

**Proof.** The first statement follows by the fact that $G^\lambda$-torsors are locally Zariski trivial, as recalled in the proof of Proposition 5.1. Using the filtration and the long exact sequence of cohomological groups it is immediate to prove that any torsor under a filtered group scheme is locally Zariski trivial. So its essential dimension is zero.

The second statement is obtained using Lemma 3.19. □

**Lemma 5.6.** Let us suppose that $K$ has characteristic $p$. Let $G$ be a model (not necessarily finite) over $S$ of a finite infinitesimal unipotent group scheme over $K$. Then there exists a central decomposition series

$$
1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G
$$

of $G$ made by group schemes over $S$, such that each successive quotient $G_i/G_{i-1}$ is a subgroup scheme of $G_{a,S}$ of order $p$.

**Proof.** We remark that the generic fiber is necessarily of order $p^n$. By [DG70] Proposition IV §2, 2.5 the result is true on the generic fiber. Taking the schematic closure we obtain a central decomposition series where the quotients are models of subgroups of $G_{a,K}$. By Lemma 5.3 we are done. □

**Lemma 5.7.** Let us suppose that $K$ has characteristic $p$. Let $G$ be a model (not necessarily finite) over $S$ of a finite commutative unipotent infinitesimal group scheme. Then for any affine scheme $X$ over $S$ we have that $H^j(X, G) = 0$ if $j \geq 2$.

**Proof.** By the previous lemma we are reduced to proving it in the case of a closed subgroup scheme of $G_{a,S}$ of order $p$. First of all we remark that $G_{a,S}/G$ is isomorphic to $G_{a,S}$. In fact for the fibers this follows by [DG70] IV, §2 Proposition 1.1. Therefore $G_{a,S}/G$ is a smooth model of $G_{a,K}$ with special fiber isomorphic to $G_{a,k}$, in particular connected. Then it isomorphic to $G_{a,S}$ by [WW80] Theorem 2.2). Now it is well known that $H^i(X, G_{a,S}) = 0$, if $i \geq 1$, so the wanted result easily follows. □
Lemma 5.8. Let $X = \text{Spec}(A)$ be an affine faithfully flat scheme over $S$. If $f : G_1 \to G_2$ is an epimorphism of $S$-group schemes such that kernel has unipotent infinitesimal generic fiber and it is central in $G_1$ then $H^1(X, G_1) \to H^1(X, G_2)$ is surjective.

Proof. Let $P \to X$ be a $G_2$-torsor. By [Gir71] IV Proposition 2.5.8], the gerbe of all liftings of $P \to X$ is banded by the group scheme $p \ker f$, i.e. the group scheme obtained by $\ker f$ twisting by $P \to X$ with $G_2$ acting by conjugation. Since $\ker f$ is central in $G$ then $p \ker f \cong \ker f$. Again by [Gir71] IV Proposition 2.5.8], the torsor $P \to X$ is in the image of the map $H^1(X, G_1) \to H^1(X, G_2)$ if and only if the above gerbe is trivial. But gerbes over $\text{Sch}/X$ banded by a group $G$ are classified by $H^2(X, G)$. Now the generic fiber of $\ker f$ is unipotent infinitesimal so by Lemma 5.7 we have that $H^2(X, \ker f) = 0$. We have so proved that $H^1(X, G_1) \to H^1(X, G_2)$ is surjective.

Here we give a sort of generalization of [TV13] Theorem 1.4] over a discrete valuation ring for finite group schemes. We first prove the following Lemma, which is in the counterpart of [TV13] Lemma 3.4] in a less general form.

Lemma 5.9. Let

$$1 \to G_1 \to G \to G_2 \to 1$$

be an exact sequence of group schemes over $S$ such that $G_1$ is central in $G$ and the generic fiber of $G_1$ is unipotent infinitesimal. Then

$$\text{ed}_R(G) \leq \text{ed}_R(G_1) + \text{ed}_R(G_2).$$

Remark 5.10. In fact this Lemma, over a field, is weaker than [TV13] Lemma 3.4] also in the finite case. The point here is that to have a similar statement one should involve twisted forms of $G_1$, defined over a scheme $Y$ of dimension maybe greater than $1$. We do not have a decomposition for these group schemes and so we can not apply dévissage arguments to reduce to the case of subgroup schemes of $\mathbb{G}_a$. Moreover we need that the generic fiber of $G_1$ is infinitesimal otherwise we do not know if we can obtain a filtration with quotients which are subgroups of $\mathbb{G}_{a,S}$. More precisely one should prove that any model of a simple étale unipotent group scheme is contained in $\mathbb{G}_{a,S}$.

Proof. Let $f : P \to X$ be a $G$-torsor. Let us consider the $G_2$-torsor $f_1 : P_1 = P/G_1 \to X$. Then there exists a $G_2$-torsor $f_2 : P_2 \to Y$ which is a weak $S$-compression of $f_1$ and such that $\text{ed}_S f_2 = \dim S Y$. Up to take the schematic closure in $X$, since we are over a discrete valuation ring, we can suppose that is an $S$-compression, not only weak. So $X \to Y$ is $S$-dominant. Now by Lemma 5.8 there exists a $G$-torsor $g : Q \to Y$ such that $Q/G_1 \to Y$ is isomorphic to $f_2$ as $G_2$-torsors. If $U$ is an $S$-dense open subscheme where $X \to Y$ is defined then $f_U : P_U \to U$ and $g_U : Q_U \to U$ have the same image in $H^1(U, G_2)$. Now, by [Gir71] III, Proposition 3.4.5], we have that $H^1(U, G_1)$ acts transitively on the fibers of $H^1(U, G) \to H^1(U, G_2)$. So let $h_1 : Z \to U$ be a $G_1$-torsor such that its class sends the class of $g_U$ in the class of $f_U$. Let $h_2 : Z' \to U'$ be an $S$-compression of $h_1$ with $\dim_S U' = \text{ed}_S h_1$. Now take $[(h_2)_{U' \times S Y}] : [g_{U' \times S Y}] \in H^1(U' \times_S Y, G)$, where $\cdot$ is the action of $H^1(U \times_S Y, G_1)$ over $H^1(U \times_S Y, G)$. The associated torsor is a weak $S$-compression of $f$. So

$$\text{ed}_S f \leq \dim_S U' + \dim_S Y = \text{ed}_S h_1 + \text{ed}_S f_2 \leq \text{ed}_S G_1 + \text{ed}_S G_2.$$
Theorem 5.11. Suppose that $R$ has characteristic $p$. Let $G$ be a finite group scheme of order $p^n$ over $S$, such that

(i) $G_K$ is the product of an unipotent infinitesimal group scheme and diagonalizable group scheme;

(ii) the closure of the diagonalizable part is Kummer.

Then the essential dimension is less than or equal to $n$.

Remark 5.12. The condition (ii) is conjecturally empty, i.e. any model of a diagonalizable group is Kummer. For instance this is the case for group scheme of order $p^2$ [Tos10] and some evidences can be found in [MRT13]. After the proof we will comment on the condition (i).

Proof. If $G$ is a Kummer group scheme then we have the result by Lemma 5.5. Now we suppose that the generic fiber is infinitesimal unipotent. We prove the result by induction on $n$. If $n = 0$ it is clear. Now we suppose $n \geq 1$. By [DG70] IV Proposition 2.5] there exists a central subgroup of the generic fiber of order $p$ and isomorphic to a subgroup of $G_{a,K}$. Take the schematic closure $H$. It is isomorphic to a closed subgroup scheme of $G_{a,S}$ finite over $S$ by Lemma 5.3. Moreover it is central in $G$. Therefore its essential dimension is less than or equal to 1 by Lemma 5.19 and Proposition 5.1. Now by Lemma 5.9 we have that

$$ed_S G \leq ed_S H + ed_S G/H \leq 1 + ed_S G/H.$$ 

So we can conclude by induction.

Now we consider the general case. We argue by induction, again. Let $n \geq 1$. We suppose the statement is true for group schemes of order $n - 1$ and we prove it for $n$. We call $G_{u,K}$ and $G_{d,K}$, respectively, the unipotent and the diagonalizable part of the generic fiber. We can suppose that $G_{u,K}$ is nontrivial by what we proved previously. By [DG70] IV Proposition 2.5] there exists a central subgroup of $G_{u,K}$ of order $p$ and isomorphic to a subgroup of $G_{u,K}$. Take the schematic closure $H$. It is isomorphic to a closed subgroup scheme of $G_{a,S}$ finite over $S$ by Lemma 5.3. Moreover it is central in $G$. Therefore its essential dimension is less than or equal to 1 by Lemma 5.19 and Proposition 5.1. Now by Lemma 5.9 we have that

$$ed_S G \leq ed_S H + ed_S G/H \leq 1 + ed_S G/H.$$ 

So we can conclude by induction. $\square$

Remark 5.13. We observe that in [TV13 Theorem 2.2] the above result is proved over a field without the hypothesis (i). The main point is that Lemma 5.9 is weaker than [TV13 Lemma 3.4] as observed in the Remark 5.10.

We remark that in Corollary 5.3 we proved that if $G$ is a model of an almost special group scheme over $K$ then

$$ed_K G_k \geq ed_K G_K.$$ 

If $K$ has characteristic $p > 0$, this applies, for instance, for models of diagonalizable group schemes with finite part of order a power of $p$, since smooth diagonalizable group schemes and diagonalizable group schemes are almost special [TV13 Example 4.4]. We remark that in general the special fiber is not diagonalizable but unipotent. If $G$ is a finite group scheme (flat) over $S$, with $S$ complete, and the order of $G$ is not divisible by $p$ then $ed_K G_K \geq ed_K G_k$ (see [BRV11 Theorem 5.11]).
There is equality if $K$ has characteristic $p$ (see [BRV11 Corollary 5.12]). In the examples after the next Corollary we will see that for finite $p$-group schemes over $S$ a general result can not exist. Now we give a more precise result for models of group schemes of height $\leq 1$.

**Corollary 5.14.** If $K$ is of positive characteristic and $G_K$ is a trigonalizable group scheme of height $\leq 1$, i.e. killed by Frobenius, and order $n$ then

$$\text{ed}_K G_K = \text{ed}_K G_K = n.$$

If moreover $G_K$ satisfies conditions of Theorem [5.11] then both quantities are equal to $\text{ed}_S G$.

**Proof.** If $G_K$ is killed by Frobenius then also $G$ is clearly killed by Frobenius. So the special fiber is of height $\leq 1$. Therefore the essential dimension of fibers is equal to $n$, by the proof of [TV13 Corollary 4.5]. Finally if we can apply Theorem 5.11 then $\text{ed}_S G \leq n$, where $n$ is order of $G$ and we are done, using Corollary 5.15. $\square$

**Example 5.15.** We give here examples which show that in general everything can happen if $G$ is a finite $p$-group scheme over $S$.

(i) Let us suppose that the characteristic of $K$ is zero and $K$ contains a primitive $p^2$-th root of unity. Then $(\mathbb{Z}/p^2\mathbb{Z})_K \simeq \mu_{p^2,K}$ so its essential dimension is 1. On the other hand $\text{ed}_K(\mathbb{Z}/p^2\mathbb{Z})_k = 2$ by [BF03 Proposition 7.10]. So

$$\text{ed}_K(\mathbb{Z}/p^2\mathbb{Z})_k > \text{ed}_K(\mathbb{Z}/p^2\mathbb{Z})_K$$

Moreover $(\mathbb{Z}/p\mathbb{Z})_k^2 \simeq \mu_{p,K}^2$ (just a primitive $p$-th root is needed). Therefore its essential dimension is 2 by [BF03 Corollary 3.9]. On the other hand, if $k$ infinite, $\text{ed}_k(\mathbb{Z}/p\mathbb{Z})_k = 1$ (see Proposition 4.11). So, if $k$ is infinite,

$$\text{ed}_K(\mathbb{Z}/p\mathbb{Z})_k^2 > \text{ed}_k(\mathbb{Z}/p\mathbb{Z})_k^2$$

(ii) Let us suppose that the characteristic of $K$ is $p$. As noted above if $G_K$ is diagonalizable then $\text{ed}_K G_k \geq \text{ed}_K G_K$. Strict inequality can happen.

Now take $G = \text{Spec}(A)$ with $A = R[T_1, T_2]/(T_1^p - T_1, T_2^p - p^{(p-1)}T_2)$, with $p$ an uniformizer of $R$. We define the multiplication by

$$T_1 \mapsto T_1 \otimes 1 + 1 \otimes T_1$$

$$T_2 \mapsto T_1 \otimes 1 + 1 \otimes T_1 + \pi \sum_{i=1}^{p-1} \frac{p}{p^i} T_2 \otimes T_2^{p-1}$$

This group schemes is such that its Néron blow-up in the subgroup $(\mathbb{Z}/p\mathbb{Z})_k$ of the special fiber is isomorphic to $(\mathbb{Z}/p^2\mathbb{Z})_R$. One can easily see that $G_k \simeq \alpha_{p,k} \times (\mathbb{Z}/p\mathbb{Z})_k$. So it is isomorphic to a subgroup scheme of $G_{\alpha,k}$, hence its essential dimension is 1. On the other hand $\text{ed}_K G_K = 2$ since $G_K$ is isomorphic to $(\mathbb{Z}/p^2\mathbb{Z})_K$. This shows that

$$\text{ed}_K G_K > \text{ed}_k G_k.$$
[Wat79] William C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979.

[WW80] William C. Waterhouse and Boris Weisfeiler, *One-dimensional affine group schemes*, J. Algebra 66 (1980), no. 2, 550–568.

Institut de Mathématiques de Bordeaux, 351 Cours de la Libération, 33 405 Talence, France

E-mail address, Tossici: Dajano.Tossici@math.u-bordeaux.fr