Nilpotent Gauging of $SL(2,R)WZNW$ Models, and Liouville Field

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Abstract

We consider the gauging of $SL(2,R)$ WZNW model by its nilpotent subgroup $E(1)$. The resulting space-time of the corresponding sigma model is seen to collapse to a one dimensional field theory of Liouville. Gauging the diagonal subgroup $E(1) \times U(1)$ of $SL(2,R) \times U(1)$ theory yields an extremal three dimensional black string. We show that these solutions are obtained from the two dimensional black hole of Witten and the three dimensional black string of Horne and Horowitz by boosting the gauge group.
1 Introduction

There has been a great deal of interest in the gauged Wess-Zumino-Novikov-Witten models as candidates for description of non trivial geometric back grounds for string theory, since the discovery by Witten [1] that the target manifold of the conformal field theory of the gauged $SL(2, R) WZNW$ model was that of a two dimensional black hole [2-12]. The original model which had a singularity and a horizon was obtained by gauging a non-compact $U(1)$ subgroup of $SL(2, R)$ whereas gauging a compact subgroup would yields a non singular manifold.

The extension of the construction to higher dimensions produced charged black holes[3-10]. In particular Horne and Horowitz [4], considered the simplest extension to $SL(2, R) \times U(1)$ and constructed various three dimensional black strings, carrying charge. The duality relation between to two types of gauging, the vector and axial gauging has also been extensively studied [2,3,12].

Gauging two conjugate subgroups of a $WZNW$ model yields identical theories. But guaging it with two subgroups of different conjugacy class should yield different target spaces. Thus the Lorentzian two dimensional black hole solution of Witten [1], with singularity and horizon, obtained from gauging the non-compact subgroup of $SL(2, R)$, is distinct from the Euclidean black hole target manifold obtained by gauging a compact subgroup, which is in a different conjugacy class of the subgroups of $SL(2, R)$. Therefore it is interesting to consider gauging by an element of the only other conjugacy class of $SL(2, R)$, the class of nilpotent subgroup $E(1)$.

In this paper we undertake to study the target manifold of the $SL(2, R)$ and $SL(2, R) \times U(1)$ $WZNW$ models gauged by their $E(1)$ subgroups. In the first case we find an unexpected result , that is the resultant manifold is only one dimensional. In fact the effective action turns out to be the Liouville field action. To understand the result we study the Witten black hole with the gauged subgroup boosted. In the limit of infinite boost we obtain the Liouville field as expected from gauging the $E(1)$ subgroup. We will also discuss the connection of our result with the standard Hamiltonian reduction [13-15].

Next we consider the gauging of the $SL(2, R) \times U(1)$ $WZNW$ model where the gauge group is the diagonal $U(1) \times E(1)$ subgroup . In this case no such reduction of the degrees of freedom occurs ; instead, we obtain the extremal three dimensional black string solution which is again found by an infinite boost of the noncompact gauging.
In the $SL(2, R)$ case, for large but finite boost parameter, the theory remains two dimensional and may be interpreted as a $c = 1$ matter coupled to Liouville, which acts as a slowly varying back ground [16].

In section 2 we discuss all the possible gaugings of $SL(2, R)$ and boosting of the black hole solutions. In section 3 the $E(1)$ gauging of the $SL(2, R) \times U(1)$ WZNW model is considered and its singularity structure discussed. In section 4 we will make additional comments about the symmetries of the $SL(2, R)/E(1)$ theory and discuss possible reasons for its reduction of degrees of freedom.

2 Different Gaugings of $SL(2, R)$ WZNW Model

In this section we consider gauging the $G = SL(2, R)$ Wess-Zumino-Witten models, by one of its subgroups $H$. $SL(2, R)$ has three distinct conjugacy classes of subgroups isomorphic to: rotations in two dimension $SO(2)$, Lorentz transformations in two dimension $SO(1, 1)$, and the isotropy group of light-like vectors, i.e. the one dimensional euclidean group $E(1)$. In four dimensional Lorentz group, the corresponding three subgroups are $SO(3)$, $SO(2, 1)$ and $E(2)$, the last of which is the group of motions of the two dimensional plane and is the symmetry of a massless particle.

We take $\sigma_1, i\sigma_2$, and $\sigma_3$, where $\sigma_i$ are the Pauli matrices, as the generators of $SL(2, R)$. Then the subgroup $SO(2)$ is generated by $i\sigma_2$, the $SO(1, 1)$ by $\sigma_3$, and $E(1)$ is generated by $\sigma^+ = \sigma_3 + i\sigma_2$. Observe that $\sigma^{+2} = 0$ and thus $E(1)$ is nilpotent.

In a WZNW model with the group $G$, two different ways of gauging an abelian subgroup $H$ has been considered : axial and vector gauging. For the axial gauging the group action on the $SL(2, R)$ is

$$g \rightarrow hgh, \hspace{1em} geG, \hspace{1em} heH.$$  \hspace{1em} (1)

Then the action,

$$I(g, A) = I_{WZNW} + \frac{k}{2\pi} \int d^2z \text{tr}(A^{-1}\partial g + A\overline{\partial}gg^{-1} + A\overline{A} + g^{-1}Ag\overline{A}),$$

$$I_{WZNW} = \frac{k}{4\pi} \int d^2z \text{tr}(g^{-1}\partial gg^{-1}\overline{\partial}g) - \frac{k}{12\pi} \int_B d^3x \text{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$$ \hspace{1em} (2)

is invariant under (2.1) together with,

$$A \rightarrow h(A + \partial)h^{-1},$$

$$\overline{A} \rightarrow h^{-1}(\overline{A} + \overline{\partial})h,$$ \hspace{1em} (3)
where $A$ and $\overline{A}$ are the complex components of the gauge field and take their values in the Lie algebra of $H$. We will denote the coefficient of the Lie algebra generator by $A$.

Similarly, the action

$$I(g, A) = I_{WZNW} + \frac{k}{2\pi} \int d^2z \text{tr}(A \overline{A} g g^{-1} - \overline{A} g^{-1} \partial g + A \overline{A} - g^{-1} A g \overline{A})$$

is invariant under the vector gauge transformation ,

$$g \rightarrow hgh^{-1}$$

$$A \rightarrow h^{-1}(A + \partial)h$$

$$\overline{A} \rightarrow h^{-1}(\overline{A} + \partial)h$$

In the following we will first consider the axial gauging (1-3) for $G = SL(2, R)$ and its three different subgroups. Then we will take up the corresponding vector gaugings, and at last we will discuss the duality between the two gaugings.

Let us take the parametrization

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \quad ab + uv = 1$$

for the group element $g \in SL(2, R)$ and gauge the action (2) with the noncompact subgroup $SO(1, 1)$ generated by $\sigma_3$. Using the gauge freedom we can take

$$a + b = 0$$

when $ab < 0$ , and

$$a - b = 0$$

when $ab > 0$. Note that $u$ and $v$ are gauge invariant parameters. Substituting in (2.2), then yields

$$I(g, A) = \frac{2k}{\pi} \int d^2z (1-uv)[A - \frac{u\partial v - v\partial u}{4(1-uv)}] [\overline{A} + \frac{u\overline{\partial}v - v\overline{\partial}u}{4(1-uv)}] - \frac{k}{4\pi} \int d^2z \frac{\partial u\overline{\partial}v - \partial v\overline{\partial}u}{1-uv}$$

The second term, when compared to a target space model, gives the two dimensional metric

$$ds^2 = -\frac{k}{2} \frac{dudv}{1-uv}$$

The first term, after integration over the gauge field, gives the dilaton field

$$\Phi = \ln(1-uv) + a$$
where \( a \) is a constant. The above metric and dilaton fields can also be verified to satisfy the equations of motion of the target space action.

We have used the following target space action which we set down for future reference,

\[
I = -\frac{1}{4\pi \alpha'} \int d^2 \sigma \sqrt{h(h^{\alpha\beta} g_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) \partial_{\alpha} X^\mu \partial_{\beta} X^\nu} + \frac{1}{4\pi} \int d^2 \sigma \sqrt{h} \left( \frac{1}{2} h^{\alpha\beta} \partial_{\alpha} \Phi \partial_{\beta} \Phi + \Phi R(2) \right)
\]

(12)

The equation to solve for the dilaton is

\[
R_{\mu\nu} = D_\mu D_\nu \Phi
\]

(13)

The solution (2.10) has been discussed in great detail and corresponds to a two dimensional black hole with the singularity at \( uv = 1 \), and horizon at \( u = 0 \) and \( v = 0 \). (Fig.1). As seen from Eq.(2.11), the string coupling vanishes at spatial infinity and blows up near the singularity. The black hole’s mass is obtained by ADM procedure and is

\[
M = \sqrt{\frac{2}{k}} e^a
\]

(14)

Next we consider the compact subgroup \( SO(2) \) generated by \( i\sigma_2 \). The gauge freedom (2.1) allows us to fix

\[
u + v = 0
\]

(15)

in the notation of (2.6). Then in terms of the gauge invariant parameters \( x = u - v \) and \( y = a - b \), the resultant target space metric and dilaton fields become

\[
ds^2 = \frac{k}{2} \left( \frac{dx^2 + dy^2}{4 + x^2 + y^2} \right)
\]

(16)

\[
\Phi = \ln(4 + x^2 + y^2) + \text{const.}
\]

(17)

This solution is of course nonsingular and is termed Euclidean blackhole.

Now we proceed to the new case of the nilpotent subgroup \( E(1) \) of \( SL(2, R) \), generated by \( \sigma = \sigma_3 + i\sigma_2 \). The parameters \( x = u - v \) and \( w = a - b - u - v \) remain invariant under the axial gauge transformation \( g \rightarrow hgh \); and provide a gauge invariant parametrization of the quotient manifold \( SL(2, R)/E(1) \). We fix the gauge condition

\[
a + b = 0
\]

(18)

which is possible only when \( w \neq 0 \). However for the region near \( w = 0 \), another gauge fixing is possible:

\[
u + v = 0
\]

(19)
which describes a one dimensional section of the manifold. Substituting for various terms in the action (2.2), taking account of the condition $a + b = 0$, we get:

$$I(g, A) = -\frac{k}{2\pi} \int d^2z \frac{1}{w^2} \left| A + \frac{1}{2w^2}(x\partial w - w\partial x) \right|^2 + \frac{k}{2\pi} \int d^2z \frac{|\partial w|^2}{w^2}$$

(20)

Then integrating over the gauge field $A$, results in the target space metric

$$ds^2 = \frac{k}{2} \frac{(dw)^2}{w^2}$$

(21)

and the dilation field

$$\Phi = 2\ln w + a$$

(22)

where $a$ is a constant. The dilaton form (2.22) can also be obtained from the differential equation (13).

The striking feature of our result (21) and (22) is that, both the metric and the dilaton field are one-dimensional, that is they depend only on $w$ and not on the other dimension $x$ of the manifold. One of the dimensions of the geometric manifold $SL(2, R)/E(1)$ has evaporated.

Even more interesting, the effective target model action, where $w = \pm e^{\varphi}$ depending on the sign of $w$,

$$I_{eff} = \frac{k}{2\pi} \int d^2z (\partial \varphi \overline{\partial} \varphi + 2\varphi),$$

(23)

is nothing but the action of a one dimensional bosonic field with background charge and without cosmological constant which can be interpreted as Liouville field in view of the limiting form of its coupling to the coordinate $X$ when the boost parameter considered in the following goes to infinity. Although Liouville field has been obtained before, as a quotient of $SL(2, R)$ by its subgroups, or as a hamiltonian reduction of the $SL(2, R)$ WZNW model, we must emphasize that the above result is different and unexpected. This is because in these approaches two degrees of freedom are removed by either a constraint together with a gauge fixing [2,13-15], or by taking independent left and right group action $g \rightarrow h_L gh_R$ for the gauge transformations [13] where the left and right actions are effected by independent upper and lower triangular matrices. But here we do not do anymore than the usual gauged WZNW models. In the present case the geometric manifold $SL(2, R)/E(1)$ has two dimensions whereas the string moving on it sees a one dimensional manifold.

The reduction in the degrees of freedom in the metric (21) is plausible on account of the behaviour of corresponding metrics (10) and (16) for the Lorentzian and Euclidean
cases, since in appropriate variables [1], there is a change in the signature of the metric when crossing from the Euclidean to the Lorentzian metric. However, the complete disappearance of the coordinate $x$ from the action (2.23) needs a more exact explanation.

To understand this phenomenon of the disappearance of one of the dimensions, and to understand what has happened to the missing degree of freedom we will obtain our solution (2.23) from the standard black hole solution (10) and (11). We thus boost the subgroup $SO(1,1)$, generated by $\sigma_3$, in the direction of $\sigma_1$ by a parameter $t$. We let $t$ go to infinity. As boosting the subgroup only shifts it in the same conjugacy class, the effect on the blackhole will simply be an isometric translation of the manifold. However at the limit of the infinite boost, the subgroup $SO(1,1)$ will degenerate to the nilpotent subgroup $E(1)$ and the black hole should correspondingly degenerate into our solution (23), i.e., to the pure Liouville field. Note that infinite boosting is a singular transformation and can transform elements of different conjugacy classes to each other.

Thus consider,

$$\sigma_3^t = e^{-\frac{t}{2}\sigma_1}\sigma_3 e^{\frac{t}{2}\sigma_1}$$

$$= cht \sigma_3 + sht i\sigma_2$$

and use the group generated by $\sigma_3^t$ to gauge $SL(2,R)$ WZNW model, we get the standard black hole solution (10) and (11) boosted, which for large $t$ gives the following action,

$$I_{eff} = \frac{k}{2\pi} \int d^2 z [\partial \varphi \partial \varphi + 2\varphi - 4\epsilon e^{-2\varphi} \partial x \partial x - 4\epsilon e^{-2\varphi} (x^2 + 4)(\partial \varphi \partial \varphi + 1)],$$

where $\epsilon = e^{-2t}$.

We have chosen the gauge fixing condition

$$a + b = 0$$

which is valid for

$$|(a - b)coht - (u + v)sht| > |a + b|$$

Otherwise the other gauge fixing condition,

$$(a - b)coht = (u + v)sht$$

should be used.

As it can be seen from (2.25), at the limit of $t \rightarrow \infty$, $(\epsilon \rightarrow 0)$, the black hole effective action approaches the Liouville action (2.23), which we found as the target space action of the $SL(2,R)$ WZNW model gauged by its nilpotent subgroup $E(1)$.
If we take the other region of the manifold described by the gauge condition (2.28), we find an effective action,

\[ I_{\text{eff}} = \frac{k}{2\pi} \int d^2z \frac{\overline{\partial} x \partial x}{4 + x^2}. \tag{29} \]

Now we may trace the cause of the elimination of the extra degree of freedom \( x \), when \( t \to \infty \): As it was pointed out before, for finite \( t \) there are two distinct regions of \((u, v)\) plane determined by the gauge conditions (2.26) and (2.28). When \( t \) increases, the region described by the gauge condition (2.26), \( a + b = 0 \), narrows and tends to the one dimensional line, parametrized by \( \varphi \) above. What then happens to the other region determined by the second gauge condition (2.27)? The answer is that, exactly at \( t = \infty \), \((\sigma, \text{nilpotent gauging})\), the condition \( a + b = 0 \) suffices to describe all the regions; i.e., at \( t = \infty \) the gauge orbits of the two region connect to yield a single gauge orbit of the nilpotent subgroup \( E(1) \), thus rendering the theory one dimensional.

In the effective action (25), the nonleading terms have the form of the interactions of a \( c = 1 \) matter field \( x \) with the Liouville field \( \varphi \). In fact if the field \( x \) varies rapidly compared with \( \varphi \), (which can be expected as \( \varphi \) is a background field) then the last term can be ignored and the action becomes that of a two dimensional gravity

\[ I'_{\text{eff}} = \frac{k}{2\pi} \int d^2z [\overline{\partial} \varphi \partial \varphi + 2\varphi - 4\epsilon e^{-2\varphi} \overline{\partial} x \partial x], \tag{30} \]

provided \( x \) is rescaled to absorb \( \epsilon \).

We may also boost the compact subgroup \( SO(2) \) generated by \( i\sigma_2 \); and study its behaviour as the boost parameter \( t \) goes to infinity. The calculations are similar to the Lorentzian black hole yielding,

\[ I_{\text{eff}} = \frac{k}{2\pi} \int d^2z [\overline{\partial} \varphi \partial \varphi + 2\varphi + 4\epsilon e^{-2\varphi} \overline{\partial} x \partial x + 4\epsilon e^{-2\varphi} (x^2 + 4)(\overline{\partial} \varphi \partial \varphi + 1)] \tag{31} \]

with the important difference of the sign for the \( x \) field terms indicating that again for large but finite \( t \), there is a possible \( c = 1 \) interpretation, albeit a Euclidean one. Note that the difference of sign of the \( x \) field contribution when approaching the \( E(1) \) limit naturally leads to the expectation that exactly at the \( E(1) \) limit, the \( x \) field should disappear and the one dimensional Liouville field \( \varphi \) remain.

The vector gauging of \( SL(2, R) \) WZNW model Eq.(2.4) goes through the same way with the result that for the noncompact subgroup generated by \( \sigma_3 \), the following gauge fixing conditions are allowed

\[ u + v = 0 \quad \text{for } uv < 0 \tag{32} \]
\[ u - v = 0 \quad \text{for} \quad uv > 0 \quad (33) \]

and the target space metric and dilaton field become exactly of the same form as the axial gauge case

\[ ds^2 = -\frac{k}{2} \frac{dadb}{1 - ab} \quad (34) \]
\[ \Phi = \ln(1 - ab) + \text{const.} \quad (35) \]

with \( u \) and \( v \) replaced by \( a \) and \( b \).

For the compact subgroup generated by \( i\sigma_2 \), we get a similar result when gauging the vector symmetry as the axial one,

\[ ds^2 = \frac{k}{2} \frac{dx'^2 + dy'^2}{x'^2 + y'^2 - 4} \quad (36) \]
\[ \Phi = \ln(x'^2 + y'^2 - 4) + \text{const.} \quad (37) \]

where \( x' = u + v, \ y' = a - b \).

The case of nilpotent subgroup \( E(1) \) generated by \( \sigma^+ \) is even simpler. The gauge fixing,

\[ u - v = 0 \quad \text{for} \quad w \neq 0 \quad (38) \]

leaves the two gauge invariant parameters \( w \) and \( a + b \) to describe the quotient manifold (for \( w \neq 0 \)). However as in the case of the axial gauging, the effective action becomes one dimensional, that of the Liouville theory,

\[ I_{\text{eff}} = \frac{k}{2\pi} \int d^2z (\partial \bar{\varphi} \partial \varphi + 2\varphi), \quad (39) \]

exactly as in (2.32), when (2.37) gauge is used and

\[ I_{\text{eff}} = \frac{k}{2\pi} \int \frac{\partial x \partial \bar{x}}{4 + x^2}, \quad (40) \]

when we use the gauge condition (2.33). The infinite boost and the limiting procedure described above also applies in this case. It is remarkable that the result of axial and vector gauging become identical, thus satisfying duality trivially. The duality relation between the two gaugings collapses to a selfduality for a manifold with less dimensions. We will comment on this later in section IV.
3 \( SL(2, R) \times U(1) \) Gauged by \( E(1) \) subgroup

We saw in the previous section that the target manifold of \( SL(2, R) \) WZNW model gauged by its \( E(1) \) subgroup, generated by the nilpotent element \( \sigma^+ = \sigma_3 + i\sigma_2 \), degenerated into a one dimensional field which we identified with the Liouville field, thus removing all of the geometric structure associated with the black hole solution obtained from gauging a non-compact subgroup \( SO(1, 1) \) of \( SL(2, R) \). To gain insight into the geometric consequences of gauging a nilpotent subgroup, we shall therefore consider a group larger than \( SL(2, R) \). The simplest example to take is \( SL(2, R) \times U(1) \) as done in Ref [4], where a black string was discovered. We will gauge the diagonal \( E(1) \times U(1) \) subgroup rather than the usual noncompact \( U(1) \).

In Ref [4] the subgroup \( U(1) \) generated by \( \sigma_3 \) was axially gauged for the \( SL(2, R) \times U(1) \) WZNW model. The action

\[
I(g, A) = I(g) + \frac{k}{2\pi} \int d^2 z \ tr[A\overline{\partial}gg^{-1} + \overline{A}g^{-1}\partial g + A\overline{A} + Ag\overline{A}g^{-1}]
\]

\[
+ \frac{k}{2\pi} \int d^2 z \left[ \frac{A_{c}}{k}(A\overline{\partial}f + \overline{A}\partial f) + \frac{8c^2}{k}A\overline{A} \right] + \frac{1}{\pi} \int d^2 \partial f \bar{\partial} f
\]

is invariant under the transformation

\[
\begin{align*}
g &\rightarrow hgh, \quad g \in SL(2, R); \quad f &\rightarrow h^c fh, \quad f \in U(1) \\
A &\rightarrow h(A + \partial)h^{-1}, \quad \overline{A} &\rightarrow h^{-1}(\overline{A} - \overline{\partial})h
\end{align*}
\]

here \( h \) is an element of the subgroup \( SO(1, 1) \) generated by \( \sigma_3 \), and \( c \) is a constant specifying the charge of the field \( f \) under the gauge group. By \( h^c \) we mean \( e^{ic} \) when \( h = e^{i\sigma_3} \). It is worth noting that this group is nilpotent only when it act on \( SL(2, R) \). We have used \( A = A\sigma_3 \), to distinguish the Lie algebra valued gauge field \( A \) from the c-number gauge fields \( A \). Using the gauge fixing condition \( a + b = 0 \), (or \( a - b = 0 \)) and integrating the gauge fields, it was found that the resultant target manifold action

\[
I_{eff} = \frac{1}{\pi} \int d^2 z \left[ \frac{k}{8r^2(1 - \frac{1}{r})(1 + \frac{1}{r})} \partial r \bar{\partial} r - (1 - \frac{1 + \lambda}{r}) \partial \tau \overline{\partial} \tau + (1 - \frac{1}{r}) \partial f \overline{\partial} f 
\]

\[
+ \sqrt{\frac{1}{1 + \lambda}}(1 - \frac{1 + \lambda}{r})(\partial f \overline{\partial} \tau - \overline{\partial} f \partial \tau)
\]

describes the metric, where \( \lambda = \frac{2c^2}{k} \)

\[
ds^2 = -(1 - \frac{1 + \lambda}{r})d\tau^2 + (1 - \frac{\lambda}{r})df^2 + (1 - \frac{1 + \lambda}{r})^{-1}(1 - \frac{\lambda}{r})^{-1}\frac{kdr^2}{8r^2},
\]

\( k < 0 \).
where $\lambda = \frac{2c^2}{k}$ and the axion field

$$B_{\tau f} = \sqrt{1 + \lambda} \left( 1 - \frac{1 + \lambda}{r} \right)$$

(5)

Here

$$u = e^{\sqrt{\frac{r}{2k(r+\lambda)}}} \sqrt{r - (1 + \lambda)}$$

$$v = -e^{-\sqrt{\frac{r}{2k(r+\lambda)}}} \sqrt{r - (1 + \lambda)}$$

(6)

The dilaton field is

$$\Phi = \ln r + a$$

(7)

The metric (3.4) corresponds to the black string,

$$ds^2 = -(1 - \frac{M}{r'})d\tau^2 + (1 - \frac{Q^2}{Mr'})^{-1}df^2 + (1 - \frac{M}{r'})^{-1}(1 - \frac{Q^2}{Mr'})^{-1}\frac{kdr^2}{8r^2},$$

(8)

where the axionic charge $Q$ is,

$$Q = e^a \sqrt{\frac{2c}{k}(1 + \lambda)}$$

(9)

and mass per unit length of the black string (3.4) is

$$M = e^a \sqrt{\frac{2c}{k}(1 + \lambda)}$$

(10)

In (3.8), $r' = e^a \sqrt{\frac{r}{k}}$. The black string manifold with metric (3.4) has a singularity at $r = 0$ and an event horizon at $r = 1 + \lambda$. Notice that $\lambda$ is a free parameter corresponding to the representation of $f$ under $SO(1,1)$.

We will now consider gauging the $E(1)\times U(1)$ subgroup of $SL(2,R)\times U(1)$, generated by $\sigma^+ = \sigma_3 + i\sigma_2$. We notice that the same action is still invariant under the axial gauge transformation (3.2) even if $E(1)$ is nilpotent. Note however, that the term $\mathbf{A}\mathbf{A}$ in (3.1) is absent now. Again, gauge fixing by setting $a + b = 0$ (when $w \neq 0$), and integrating over the gauge field, we obtain,

$$I_{eff} = \frac{1}{\pi} \int d^2z \left\{ \frac{w^2}{w^2 - 4\lambda} \partial f \overline{\partial f} + \frac{c}{w^2 - 4\lambda} \left[ x(\partial f \overline{\partial w} - \partial w \overline{\partial f}) - w(\partial f \overline{\partial \overline{w}} - \partial \overline{w} \overline{\partial f}) \right] + \frac{k}{2w^2(w^2 - 4\lambda)} \left[ (w^2 - \lambda x^2 - 4\lambda) \partial w \overline{\partial w} - \lambda w^2 \partial x \overline{\partial x} + \lambda w x(\partial x \overline{\partial w} + \partial w \overline{\partial x}) \right] \right\}$$

(11)

where $w = a - b - u - v$ and $x = u - v$. As this is not diagonal for the symmetric part, we will make the change of variable,

$$y = \frac{x}{w}$$

(12)
and end up with the final form of the effective action,

$$I_{\text{eff}} = \frac{1}{\pi} \int d^2z \left[ \frac{k}{2} \frac{\partial w \overline{\partial w}}{w^2} + \frac{w^2}{w^2 - 4\lambda} (\partial f \overline{\partial f} - c^2 \partial y \overline{\partial y}) + \frac{cw^2}{w^2 - 4\lambda} (\partial f \overline{\partial y} - \partial y \overline{\partial f}) \right]$$  \hspace{1cm} (13)

Note that we are working in the gauge $w \neq 0$, $a + b = 0$; and therefore the transformation (3.12) is well defined.

From (3.13), we find the metric and the axionic antisymmetric field,

$$ds^2 = k \frac{d w^2}{w^2} + \frac{w^2}{w^2 - 4\lambda} (df^2 - c^2 dy^2)$$  \hspace{1cm} (14)

$$B_{fy} = -\frac{w^2}{w^2 - 4\lambda}.\hspace{1cm} (15)$$

The dilalon field comes out as,

$$\Phi = \ln(w^2 - 4\lambda) + a$$  \hspace{1cm} (16)

depending only on $w$. Note that for $\lambda = 0$, the metric (3.14) reduces to the $SL(2, R)/E(1)$ metric (2.21) plus a free bosonic degree of freedom $f$,

$$ds^2 \xrightarrow{\lambda \to 0} k \frac{d w^2}{w^2} + df^2 \hspace{1cm} \text{as} \hspace{0.5cm} \lambda \to 0$$ \hspace{1cm} (17)

However, for $\lambda \neq 0$, the additional field $y$ remains in contrast to the reduction of the degrees of freedom encountered for $SL(2, R)/E(1)$, Eq. (2.21).

There is also another difference between this case and the $SL(2, R)/E(1)$. As the dilaton field $\Phi$ (in 3.16) is not linear in the field $\varphi$ (which is defined in $w = \pm e^\varphi$), in contradistinction with $SL(2, R)/E(1)$ case, we can not interpret our solutions as Liouville field. Observe also that at large distances $w \to \infty$, the metric (3.14) becomes flat,

$$ds^2 \to \frac{k}{2} d\varphi^2 + df^2 - dy^2, \hspace{1cm} \text{as} \hspace{0.5cm} w \to \infty$$  \hspace{1cm} (18)

The physical interpretation of our solution is that of an extremal charged blackhole:

It can be seen from the metric (3.14) that the curvature is

$$R = -\frac{64\lambda}{k} \frac{w^2 + 3\lambda}{(w^2 - 4\lambda)^2}$$  \hspace{1cm} (19)

We can calculate its mass $M$ and charge $Q$ following Ref.[4]. Notice that the asymptotic metric is a three dimensional Lorentzian metric $\eta_{\mu\nu}$ which if we expand our metric (3.14) around it, $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$, and use the relation

$$M_{\text{tot}} = \frac{1}{2} \int e^\Phi [\partial^i \gamma_{ij} - \partial_i \gamma_{jj} + \gamma_{ij} \partial^j \Phi] ds^i \hspace{1cm} (20)$$
we obtain, for mass per unit length of the black hole
\[ M = 4\lambda e^a \sqrt{\frac{2}{k}} \tag{21} \]

The axionic charge can be obtained from (3.15) using \( Q = \frac{1}{2} e^{\Phi} H^* \), where \( H \) is the antisymmetric third rank tensor of the axion field \( B \). The result is
\[ Q = -4\lambda e^a \sqrt{\frac{2}{k}} \tag{22} \]
Thus leading to an extremal black hole, \( |Q| = M \).

The singularity of the black hole is at
\[ w = \pm 2\sqrt{\lambda}, \tag{23} \]
from (3.19), and the horizon can easily be seen to be at \( w = 0 \), which is in the extended region of our manifold (Fig.3). Note that at \( \lambda = 0 \) we obtain a flat, \( R = 0 \), one dimensional space, as expected from the decoupling of \( SL(2, R) \) from the \( U(1) \) factor. It should be noted that although we started from arbitrary \( \lambda \), we landed with the special extremal solution \( M = |Q| \) of Horne and Horowitz.

To understand the reason for the appearance of the special extremal solution we go back to the Horne and Horowitz gauged \( WZNW \) model and boost the subgroup \( SO(1,1) \) used there. Thus we follow the procedure of the last section and boost the generator \( \sigma_3 \) as in (2.24) and look at the limiting case of the boost parameter \( t \to \infty \). However, there is a subtlety related to the transformation of the \( U(1) \) factor under the boost. The \( U(1) \) transforms as a one dimensional representation with weight \( c \) under the \( SO(1,1) \) generated by \( \sigma_3 \), (Eq.3.2). A priori there is no connection between \( c \) and the weight \( c_t \) of the \( U(1) \) transformation under the \( SO(1,1) \) generated by \( \sigma_3^t \) of Eq.(2.24). However, if the boost is to be an isometry of the manifold, leaving the effective action (3.3) invariant, we must choose
\[ c_t = ce^t \tag{24} \]
With this caveat we can find the limit \( t \to \infty \) of the \( SL(2, R) \times U(1) \), gauged by the subgroup generated by \( \sigma_3^t \), and observe that the region specified by the gauge condition \( a + b = 0 \), goes to our effective action (3.13) of the \( SL(2, R) \times U(1)/E(1) \) theory. We also find that the singularity of the Horne and Horowitz solution, (3.4), at \( r = 0 \), goes to our singularity of (3.14), at \( w_a = 4\lambda \). The horizon at \( r = 1 + \lambda \) going to our horizon at
\( w = 0 \). Finally the charge and mass going to the same value \( |Q| = M = 4\lambda \sqrt{k} e^a \) giving the expected extremal black hole solution. We can now see the reason for this: the mass, charge ratio depends on \( \lambda \) in the Horne and Horwitz solution (Eqs.(3.9) and (3.10)). But as \( \lambda \) depends on the boost parameter \( t \) and diverges as \( t \rightarrow \infty \), the ratio \( |Q|/M \) goes to unity, giving the extremal solution.

The region near \( w = 0 \), \( a - b = 0 \), can also be similarly treated. In this gauge, the action (3.1) becomes

\[ I = \frac{k}{2\pi} \int d^2 \! z \left[ \frac{\partial x^2}{4 + x^2} + \frac{2}{k} |\partial f|^2 \right] + \frac{2c}{\pi} \int d^2 \! z (A \partial f + A \partial f + 2cA) \]  

(25)

which clearly shows the decoupling of the \( U(1) \) factor from \( SL(2, R) \). Integration over the gauge field \( A \) removes \( f \), as it appears in the action as \( |\partial f + 2cA|^2 \), and the effective action becomes one dimensional and identical to \( SL(2, R) \) case in section II,

\[ I_{\text{eff}} = \frac{k}{2\pi} \int d^2 \! z \frac{\partial x \partial x}{4 + x^2} \]  

(26)

We will next consider vector gauging the \( E(1) \) subgroup of \( SL(2, R) \times U(1) \). In the region determined by the gauge condition \( u - v = 0, w \neq 0 \), we obtain the following effective action

\[ I_{\text{eff}} = \frac{1}{\pi} \int d^2 \! z \left[ \frac{k}{2} \frac{\partial w \partial w}{w^2} + \frac{w^2}{w^2 + 4\lambda} (\partial f \partial f - c^2 \partial y \partial y) + \frac{cw^2}{w^2 + 4\lambda} (\partial f \partial y - \partial y \partial f) \right] \]  

(27)

which is very similar to the axial gauge case Eq. (3.13), except for the change of signs. We can read off the metric and the axionic field \( B \) from (3.26),

\[ ds^2 = \frac{k}{2} \frac{dw^2}{w^2} + \frac{w^2}{w^2 + 4\lambda} (df^2 - c^2 dy^2) \]  

(28)

\[ B_{xy} = \frac{w^2}{w^2 + 4\lambda} \]  

(29)

The dilaton field can be similarly obtained,

\[ \Phi = \ln(w^2 + 4\lambda) + a \]  

(30)

In contrast to the \( SL(2, R) \) case, duality between the axial and vector gauging is not trivial. In fact, to obtain the metric in the case of vector gauging from that of the axial gauging, we must let

\[ \lambda \rightarrow -\lambda, \]  

(31)
which however has to accompany a change of the sign of the $B$ field and consequently
the sign of charge, in order to obtain Eq.(3.28). But a change in sign of $\lambda = \frac{2e^2}{k}$ requires
complexification of the representation weight $c$! The reason for the difficulty is that in
the vector gauging $g \rightarrow h^{-1} gh$, the transform $f \rightarrow h^{-c}fh^c$ leaves $f$ invariant, while
our original action

$$I(g, A) = I(g) + \frac{k}{2\pi} \int d^2 z \text{tr}[A A g^{-1} - \overline{A}g^{-1}\partial g + \overline{A}A - Ag\overline{A}g^{-1}]$$

$$+ \frac{k}{2\pi} \int d^2 z \left[ \frac{4c}{k}(A\overline{f} + \overline{A}\partial f) + \frac{8e^2}{k}A\overline{A} \right]$$

obtained as a modification of (3.1), is invariant only with $f \rightarrow f + 2\epsilon c$.

An action invariant under $f \rightarrow f$, has been considered by Ishibashi et.al. [8], with
the relevant $U(1)$ part,

$$I_f = \frac{1}{\pi} \int d^2 z [\partial f \overline{A}f + 2ic(\overline{A}\partial f - A\overline{f})]$$

Using this action we could again consider gauging with the $E(1)$ subgroup and find
various physical quantities such as the metric, the dilaton field, and the background
gauge fields. But, a clearer duality does not appear here either. The problem is that the
appropriate action for vector gauging is Eq.(3.1), while the appropriate action for axial
gauging is Eq.(3.33) with (3.34) replaced for the $U(1)$ part; and these are fundamentally
different; thus duality is muddled.

4 Conclusion.

In this work we have studied the coset models $G/H$, as conformal field theories obtained by
gauging the subgroup $H$ of the $WZNW$ model for $G = SL(2, R)$ or $SL(2, R) \times U(1)$, spe-
cializing $H$ to the non semisimple subgroup $E(1)$ of $SL(2, R)$. We found that the two di-
menstional coset $SL(2, R)/E(1)$ gives a target space theory of only one dimension which we
identified with the Liouville field theory with vanishing cosmological constant, Eq.(2.23).
The same type of dimensional reduction did not appear for $SL(2, R) \times U(1)/E(1)$, which
resulted in a three dimensional target space theory and exhibited on extremal black hole
(or rather black string), $|Q| = M$, structure (Eq.3.13).
To see how the above structure is formed, we took the $SL(2, R)/U(1)$ black hole of Witten [1], and the $SL(2, R) \times U(1)/U(1)$ of Horne and Horowitz [4], and boosted the $U(1)$ gauged subgroup and studied their behaviour as the boost parameter tended to infinity. As expected, at the limit of infinite boost, the $SL(2, R)/U(1)$ black hole reduced to our Liouville Field of Eq.(2.23), and the $SL(2, R) \times U(1)$ charged black string reduced to our extremal charged black string of Eq.(3.13). We argued that in the $SL(2, R)$ case, there is enlargening of the orbit of the gauge group when the boost tends to infinity and that region inside the black hole shrinks to a one dimensional slice in this limit; thus results an one dimensional theory, a phenomenon which is prevented by the behaviour of $\lambda$ when boosted, for the $SL(2, R) \times U(1)$ case. In the following we will pursue the study of the course of reduction of the extra degree of freedom for $SL(2, R)$ and hope to shed further light on this phenomenon.

We would like to argue that the disappearance of the parameter $x$ from the apparently two parameter space of the action in Eq. (2.2), is the consequence of a sudden enlargement of its symmetry, as the gauge subgroup $U(1)$ tends to $E(1)$ under boost.

In fact, the gauge part of the action

$$I_{\text{gauge}} = \frac{k}{2\pi} \int d^2 z tr [A \bar{\sigma} g^{-1} + A \bar{\sigma} g + A A + A \bar{\sigma} g g^{-1}]$$

written in terms of $\sigma^\pm = \sigma_3 \pm i\sigma_2$, takes the form

$$I_{\text{gauge}} = \frac{k}{2\pi} \int d^2 z tr [A \sigma^+ \bar{\sigma} g^{-1} + A \bar{\sigma} + A \bar{\sigma} + A \sigma^+ g g^{-1}]$$

$$+ \frac{k}{2\pi} e^{-2t} \int d^2 z tr [A \sigma^- \bar{\sigma} g^{-1} + A \sigma^- g^{-1} \partial g + 2A \bar{A} + A A (\sigma^- g \sigma^- g^{-1})]$$

$$+ O(e^{-4t})$$

(2)

where we denote by $A$ and $\bar{A}$ the c-number gauge fields scaled by $e^t$. Now if we let $\bar{A} \rightarrow -\bar{A}$ in (4.1), this action changes into the gauge part of the action for vector gauging (2.4), except for $A \bar{A}$ term which has the wrong sign. But Eq.(4.2) shows that to leading order in $e^{-t}$, the $A \bar{A}$ term does not appear, and in the infinite $t$ limit, the action has an effective vector gauge symmetry

$$A \rightarrow A - h^{-1} \partial h$$

$$\bar{A} \rightarrow \bar{A} + h^{-1} \bar{\partial} h$$

(3)
Although this transformation violates reality of $A$, it is formally a symmetry of our theory and is respected by the equations of motion and survives path integration. It is not difficult to check that the vector and axial gauge transformations are compatible and that the axial gauge condition is respected by the vector gauge transformation and vice versa. Then the only variable which is both vector and axial gauge invariant is $w = tr\sigma^+ g$, thus giving us a one dimensional field theory in terms of $w$.

The phenomenon of the reduction of the effective degree of freedom in gauged $WZNW$ models, is not restricted to our example and, we conjecture it occurs whenever the gauged subgroup contains nilpotent factors. There are other specific examples [17] and the general question is under investigation by the authors.

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