SINGULARITIES OF GAUSS MAPS OF WAVE FRONTS WITH NON-DEGENERATE SINGULAR POINTS

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Abstract. We study singularities of Gauss maps of fronts and give characterizations of types of singularities of Gauss maps by geometric properties of fronts which are related to behavior of bounded principal curvatures. Moreover, we investigate relation between a kind of boundedness of Gaussian curvatures near cuspidal edges and types of singularities of Gauss maps of cuspidal edges. Further, we consider extended height functions on fronts with non-degenerate singular points.

1. Introduction

In study of (extrinsic) differential geometry of surfaces in the Euclidean 3-space $\mathbb{R}^3$, Gauss maps play important roles. For instance, singular points of the Gauss map coincide with the parabolic points, namely, points at which the Gaussian curvature vanishes. On the other hand, it is known that height functions in the normal direction of surfaces have singularities. Height functions on surfaces measure types of contact of surfaces and planes. Generically, height functions on surfaces have $A_k$ singularities ([17]). In particular, height functions on surfaces have $D_4$ singularities at flat umbilic points (cf. [6]).

In this paper, we study singularities of Gauss maps of (wave) fronts and height functions. In general, the Gaussian curvature of a front is unbounded near singular points. However, the Gaussian curvature of a front is rationally bounded, which is a kind of boundedness introduced by Martins, Saji, Umehara and Yamada [26], at a singular point which is also a singular point of the Gauss map. Therefore studying singularities of Gauss maps of fronts might be related to investigate the behavior of the Gaussian curvature of a front near a singular point. In fact, we will show relationships between rational boundedness and contact order of a singular curve and a parabolic curve for a cuspidal edge. To do this, we need to consider the behavior of a bounded principal curvature near a singular point of a front. It is known that a bounded principal curvature of a front coincides with the limiting normal curvature $\kappa_\nu$ at a non-degenerate singular point, which contains a cuspidal edge or a swallowtail. Moreover, a non-degenerate singular point is also a singular point of the corresponding Gauss map of a front if and only if $\kappa_\nu$ vanishes at this point. Thus, to study types of singularities of a Gauss map, we consider properties of a bounded principal curvature of a front at non-degenerate singular point (Theorem 3.3). Further, we consider contact between the singular curve and the singular set of the Gauss map for a cuspidal edge. We give some
geometric interpretations of a rational boundedness and a rational continuity of
the Gaussian curvature of a cuspidal edge in terms of contact properties of two
curves (Corollaries 3.7 and 3.10).

In addition, we study height functions on fronts with non-degenerate singular
points of the second kind, which do not contain cuspidal edges. In particular, we
consider contact between fronts with non-degenerate singular points of the second
kind and their limiting tangent planes and show a condition that corresponding
height functions have $D_4$ singularities in terms of differential geometric properties
of initial fronts (Theorem 4.3).

2. Preliminaries

2.1. Fronts. We review some notions of (wave) fronts. For details, see [1,17,22,
26,37,38].

Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a $C^\infty$ map, where $\Sigma \subset (\mathbb{R}^2;u,v)$ is a domain. We then
say that $f$ is a frontal if there exists a unit vector field $\nu$ along $f$ such that
$\langle df_q(X_q), \nu(q) \rangle = 0$ for any $q \in \Sigma$ and $X_q \in T_q\Sigma$, where $\langle \cdot, \cdot \rangle$ is a usual inner
product of $\mathbb{R}^3$. This vector field $\nu$ is called a unit normal vector or the Gauss
map of $f$. A frontal $f$ is called a (wave) front if the pair of mappings $L_f = (f, \nu) : \Sigma \rightarrow \mathbb{R}^3 \times S^2$ gives an immersion, where $S^2$ denotes the unit sphere in $\mathbb{R}^3$.

We fix a frontal $f$. A point $p \in \Sigma$ is a singular point if rank $df_p < 2$ holds. We
denote by $S(f)$ the set of singular points of $f$ and call the image $f(S(f))$ of $S(f)$
by $f$ the singular locus. We set a function $\lambda : \Sigma \rightarrow \mathbb{R}$ by

$$\lambda(u,v) = \det(f_u,f_v,\nu)(u,v),$$

where det is the usual determinant of $3 \times 3$ matrices, $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$. We call this function $\lambda$ the signed area density function. It is obviously
$S(f) = \lambda^{-1}(0)$. Take a singular point $p \in S(f)$. Then $p$ is non-degenerate if
d$\lambda(p) \neq 0$, that is, $(\lambda_u(p),\lambda_v(p)) \neq (0,0)$. If $p$ is a non-degenerate singular
point of $f$, then there exist a neighborhood $V \subset \Sigma$ of $p$ and a regular curve
$\gamma : (\varepsilon,\varepsilon) \ni t \mapsto \gamma(t) \in V$ ($\varepsilon > 0$) with $\gamma(0) = p$ such that $\Im(\gamma) = S(f) \cap V
by the implicit function theorem, where $\Im(\gamma)$ is the image of $\gamma$. Moreover, there
exists a never vanishing vector field $\eta$ on $S(f) \cap V$ such that for any $q \in S(f) \cap V,
df_q(\eta_q) = 0$ holds. We call $\gamma$, $\tilde{\gamma} = f \circ \gamma$ and $\eta$ a singular curve, a singular locus and a null vector field, respectively. If a vector field $\tilde{\eta}$ on $V$ satisfies $\tilde{\eta}|_{S(f) \cap V} = \eta$, then $\tilde{\eta}$ is called an extended null vector field [36,37].

A non-degenerate singular point $p$ is called the first kind if $\det(\gamma',\eta)(0) \neq 0,$
where $' = d/dt$. Otherwise, $p$ is called the second kind (26).

Definition 2.1. Let $f,g : (\mathbb{R}^2,0) \rightarrow (\mathbb{R}^3,0)$ be $C^\infty$ map germs. Then $f$ and $g$
are $\mathcal{A}$-equivalent if there exist diffeomorphism germs $\theta : (\mathbb{R}^2,0) \rightarrow (\mathbb{R}^2,0)$ and
$\Theta : (\mathbb{R}^3,0) \rightarrow (\mathbb{R}^3,0)$ such that $\Theta \circ f = g \circ \theta$ holds.

Definition 2.2. Let $f : (\Sigma,p) \rightarrow (\mathbb{R}^3,f(p))$ be a map germ around $p$. Then
$f$ at $p$ is a cuspidal edge if the map germ $f$ is $\mathcal{A}$-equivalent to the map germ
$(u,v) \mapsto (u,v^2,v^3)$ at $0$, $f$ at $p$ is a swallowtail if the map germ $f$ is $\mathcal{A}$-equivalent
to the map germ $(u,v) \mapsto (u,3v^4+uv^2,4v^3+2uv)$ at $0$. 
These singularities are *generic singularities* of fronts in $\mathbf{R}^3$ \cite{1}. Moreover, a cuspidal edge is non-degenerate singular point of the first kind, and a swallowtail is of the second kind \cite{26}.

Criteria for these singularities are known.

**Fact 2.3** \cite{22} \cite{36}. Let $f : (\Sigma, p) \to (\mathbf{R}^3, f(p))$ be a front germ and $p \in \Sigma$ a singular point of $f$. Then the following assertions hold:

1. $f$ at $p$ is $A$-equivalent to the cuspidal edge if and only if $\eta \lambda(p) \neq 0$.
2. $f$ at $p$ is $A$-equivalent to the swallowtail if and only if $d\lambda(p) \neq 0$, $\eta \lambda(p) = 0$ and $\eta \eta \lambda(p) \neq 0$.

We note that criteria for other singularities of fronts and frontals are known \cite{10,18,19,34,25,26}.

### 2.2. Geometric invariants of fronts

**Definition 2.4.** Let $f : \Sigma \to \mathbf{R}^3$ be a front, $\nu$ its unit normal vector and $p \in \Sigma$ a non-degenerate singular point of $f$. Then a local coordinate system $(U; u, v)$ centered at $p$.

- the $u$-axis gives a singular curve,
- $\eta = \partial_v$ (resp. $\eta = \partial_u + \varepsilon(u)\partial_v$ with $\varepsilon(0) = 0$) gives a null vector field on the $u$-axis, and
- there are no singular point other than the $u$-axis.

First we deal with cuspidal edges. Let $f : \Sigma \to \mathbf{R}^3$ be a front, $\nu$ its unit normal vector and $p \in \Sigma$ a cuspidal edge. Let $\kappa_s$, $\kappa_\nu$, $\kappa_c$, $\kappa_t$ and $\kappa_i$ denote the *singular curvature* \cite{37}, the *limiting normal curvature* \cite{37}, the *cuspidal curvature* \cite{26}, the *cusp-directional torsion* \cite{25} and the *edge inflectional curvature* \cite{26}, respectively. If we take an adapted coordinate system $(U; u, v)$ around $p$, then

\begin{equation}
\kappa_s = \text{sgn}(\lambda_\nu) \frac{\det(f_u, f_{uu}, \nu)}{|f_u|^3}, \quad \kappa_\nu = \frac{f_{uu}, \nu}{|f_u|^2}, \quad \kappa_c = \frac{|f_u|^3/2 \det(f_u, f_{vv}, f_{vuv})}{|f_u| f_{uu} f_{vuv}}, \quad \kappa_i = \frac{\det(f_u, f_{vv}, f_{vuv}) - \det(f_u, f_{vv}, f_{uu})\langle f_u, f_{vv} \rangle}{|f_u|^2 |f_u| f_{uu} f_{vuv}}, \quad \kappa_i = -3 \frac{\langle f_u, f_{vv} \rangle \det(f_u, f_{vv}, f_{uu})}{|f_u|^5 |f_u| f_{uu} f_{vuv}} \quad \text{hold on the } u\text{-axis, where } | \cdot | \text{ denotes the standard norm of } \mathbf{R}^3. \end{equation}

We note that $\kappa_c$ does not vanish along the singular curve if it consists of cuspidal edges \cite[Proposition 3.11]{26}. Moreover, $\kappa_s$ is an intrinsic invariant of a cuspidal edge which relates to the convexity and concavity \cite{15,37}. For other geometric meanings of these invariants, see \cite{15,16,25,26,37}.

Take an adapted coordinate system $(U; u, v)$ centered at $p$. Since $df(\eta) = f_v = 0$ on the $u$-axis, there exists a $C^\infty$ map $h : U \to \mathbf{R}^3 \setminus \{ 0 \}$ such that $f_v = vh$ on $U$. We note that $\{ f_u, h, \nu \}$ gives a frame on $U$, and we may take $\nu$ as $\nu = (f_u \times h)/|f_u \times h|$ (cf. \cite{25,26,28}). Under the adapted coordinate system $(U; u, v)$ centered at $p$ with
\[ \lambda_0(u, 0) = \det(f_u, h, \nu)(u, 0) > 0, \] 
invariants \( \kappa_s, \kappa_{\nu}, \kappa_c \) and \( \kappa_t \) as in (2.1) can be 
written as follows (15 (2.17) and 40 Lemma 2.7):

\[ \kappa_s = \frac{2E_{\nu}E - E_{\nu}E_{\nu} - E_{\nu}F}{2E^{3/2}\sqrt{EG - F^2}}, \quad \kappa_{\nu} = \frac{L}{E}, \]

(2.2)

\[ \kappa_c = 2\tilde{N} \left( \frac{E}{\sqrt{EG - F^2}} \right)^{3/4}, \quad \kappa_t = \frac{\tilde{E}M - \tilde{F}L}{\sqrt{EG - F^2}} \]

along the \( u \)-axis, where

\[ \tilde{E} = |f_u|^2, \quad \tilde{F} = \langle f_u, h \rangle, \quad \tilde{G} = |h|^2; \]

\[ \tilde{L} = -\langle f_u, \nu_u \rangle, \quad \tilde{M} = -\langle h, \nu_u \rangle, \quad \tilde{N} = -\langle h, \nu_v \rangle. \]

We consider the representation of \( \kappa_i \) like as (2.2). We now introduce the following functions:

\[ \tilde{\Gamma}^1_{11} = \frac{G\tilde{E}_u - 2\tilde{F}\tilde{F}_u + 2\tilde{F} \langle f_u, h_u \rangle}{2(\tilde{E}G - \tilde{F}^2)}, \quad \tilde{\Gamma}^2_{11} = \frac{2\tilde{E}\tilde{F}_u - 2\tilde{E} \langle f_u, h_u \rangle - \tilde{F}\tilde{E}_u}{2(\tilde{E}G - \tilde{F}^2)} \]

(2.4)

\[ \tilde{\Gamma}^1_{12} = \frac{2G \langle f_u, h_u \rangle - \tilde{F}\tilde{G}_u}{2(\tilde{E}G - \tilde{F}^2)}, \quad \tilde{\Gamma}^2_{12} = \frac{\tilde{E}\tilde{G}_u - 2\tilde{F} \langle f_u, h_u \rangle}{2(\tilde{E}G - \tilde{F}^2)} \]

\[ \tilde{\Gamma}^1_{22} = \frac{2G\tilde{F}_v - \nu\tilde{G}\tilde{G}_u - \tilde{F}\tilde{G}_v}{2(\tilde{E}G - \tilde{F}^2)}, \quad \tilde{\Gamma}^2_{22} = \frac{\tilde{E}\tilde{G}_v - 2\tilde{F}\tilde{F}_v + \nu\tilde{G}\tilde{G}_u}{2(\tilde{E}G - \tilde{F}^2)} \]

where the functions \( \tilde{E}, \tilde{F} \) and \( \tilde{G} \) are defined in (2.3) and \( \tilde{E}_v = 2\nu \langle f_u, h_u \rangle \) holds. We call functions \( \tilde{\Gamma}^i_{jk} \) as in (2.4) modified Christoffel symbols.

**Lemma 2.5.** Under the above notations, we have

\[ f_{uu} = \tilde{\Gamma}^1_{11} f_u + \tilde{\Gamma}^2_{11} h + \tilde{L} \nu, \]

\[ f_{uv} = \nu \tilde{\Gamma}^1_{12} f_u + \nu \tilde{\Gamma}^2_{12} h + \nu \tilde{M} \nu, \]

\[ f_{vv} = \nu \tilde{\Gamma}^1_{22} f_u + \left( 1 + \nu \tilde{\Gamma}^2_{22} \right) h + \nu \tilde{N} \nu. \]

**Proof.** We now set the following:

\[ f_{uu} = X_1 f_u + X_2 h + X_3 \nu, \]

\[ f_{uv} = \nu h_u = Y_1 f_u + Y_2 h + Y_3 \nu, \]

\[ f_{vv} = \nu h_v = Z_1 f_u + Z_2 h + Z_3 \nu, \]

where \( X_i, Y_i, Z_i : U \to \mathbb{R} \) \((i = 1, 2, 3)\) are \( C^\infty \) functions.

First we consider \( f_{uu} \). By the definition of \( \tilde{L} \) and \( \langle f_u, \nu \rangle = 0 \), we have \( X_3 = \tilde{L} \). Let us determine the functions \( X_1 \) and \( X_2 \). By direct calculations, we have

\[ \langle f_{uu}, f_u \rangle = \tilde{E} X_1 + \tilde{F} X_2, \quad \langle f_{uu}, h \rangle = \tilde{F} X_1 + \tilde{G} X_2. \]

Differentiating \( \tilde{E} = \langle f_u, f_u \rangle \) by \( u \), we obtain \( \langle f_{uu}, f_u \rangle = \tilde{E}_u/2 \). Moreover, since \( \tilde{F} = \langle f_u, h \rangle \), we have \( \langle f_{uu}, h \rangle = \tilde{F}_u - \langle f_u, h_u \rangle \). Thus the above equations can be
rewritten as
\[
\begin{pmatrix}
\frac{\tilde{E}_u}{2} - \langle h, u \rangle \\
\tilde{F}_u - \langle h, u \rangle
\end{pmatrix}
= \begin{pmatrix}
\tilde{E} & \tilde{F} \\
\tilde{F} & \tilde{G}
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}.
\]
Since \(\tilde{E}\tilde{G} - \tilde{F}^2 > 0\), one can solve this equation and get \(X_i = \tilde{\Gamma}_{i1} \) (\(i = 1, 2\)).

Next we consider \(f_{uv}\). It follows that \(\langle f_{uv}, \nu \rangle = v \langle h, \nu \rangle = -v \langle h, \nu_u \rangle = v\tilde{M} = Y_3\) since \(\langle h, \nu \rangle = 0\). For \(Y_1\) and \(Y_2\), by the similar computations as above, we get the following equation:
\[
v \left( \frac{\langle f_u, h_u \rangle}{2} \right) = \begin{pmatrix}
\tilde{E} & \tilde{F} \\
\tilde{F} & \tilde{G}
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix}.
\]
Therefore we have \(Y_i = v\tilde{\Gamma}_{i2}^{1} \) (\(i = 1, 2\)).

Finally we show the case of \(f_{vv}\). \(f_{vv}\) can be written as \(f_{vv} = h + vh\) since \(f_v = vh\). Thus the inner product of \(f_{vv}\) and \(\nu\) is calculated as \(\langle f_{vv}, \nu \rangle = v \langle h, \nu \rangle = -v \langle h, \nu_u \rangle = v\tilde{N} = Y_3\) since \(\langle h, \nu \rangle = 0\) and \(\langle \nu, \nu \rangle = 1\). For \(Z_i \) (\(i = 1, 2\)), we have the following equation
\[
\begin{pmatrix}
\tilde{F} + v \left( \frac{\tilde{F} - vG_u}{2} \right) \\
\tilde{G} + \frac{vG_v}{2}
\end{pmatrix}
= \begin{pmatrix}
\tilde{E} & \tilde{F} \\
\tilde{F} & \tilde{G}
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
\]
by the similar calculations as above. Solving this equation, we have \(Z_1 = v\tilde{\Gamma}_{21}^{1}\) and \(Z_2 = 1 + v\tilde{\Gamma}_{22}^{2}\).

Using Lemma 2.5, we formulate the edge inflectional curvature \(\kappa_i\) in our setting.

**Lemma 2.6.** Under the above settings with \(\eta\lambda(u, 0) > 0\), \(\kappa_i\) can be expressed as
\[
(2.5) \quad \kappa_i = \frac{(\tilde{E}\tilde{M} - \tilde{F}\tilde{L})(2\tilde{F}_u - \tilde{E}\tilde{E}_{vv} - \tilde{E}_u\tilde{F})}{2\tilde{E}^{5/2}(\tilde{E}\tilde{G} - \tilde{F}^2)} + \frac{\tilde{E}\tilde{L}_u - \tilde{E}_u\tilde{L}}{\tilde{E}^{5/2}}
\]
along the \(u\)-axis.

**Proof.** We take an adapted coordinate system \((U; u, v)\) around a cuspidal edge \(p\) with \(\eta\lambda(u, 0) = \det(f_u, h, \nu)(u, 0) > 0\). By Lemma 2.5, \(f_{uuu}\) is given by
\[
f_{uuu} = *_1f_u + *_2h + \left( \tilde{\Gamma}_{11}^1\tilde{L} + \tilde{\Gamma}_{11}^2\tilde{M} + \tilde{L}_u \right) \nu,
\]
where \(*_i \) (\(i = 1, 2\)) are some functions. Thus it follows that
\[
\det(f_u, f_{vv}, f_{uuu}) = \left( \tilde{\Gamma}_{11}^1\tilde{L} + \tilde{\Gamma}_{11}^2\tilde{M} + \tilde{L}_u \right) \det(f_u, h, \nu)
\]
along the \(u\)-axis. Moreover, we have
\[
\frac{\det(f_u, f_{vv}, f_{uuu}) \langle f_u, f_{uuu} \rangle}{|f_u \times f_{vv}|} = \frac{\tilde{L}\tilde{E}_u}{2}.
\]
Therefore we obtain
\[
\kappa_i(u) = \frac{\det(f_u, f_{uv}, f_{uuu})}{|f_u|^3 |f_u \times f_{uv}|}(u,0) - 3 \frac{\det(f_u, f_{uv}, f_{uu})}{|f_u|^5 |f_u \times f_{uv}|}(u,0)
\]
\[
= \left( \frac{\widetilde{\Gamma}_{i1} \tilde{L} + \widetilde{\Gamma}_{i2} \tilde{M} + \tilde{L}_u}{E^{3/2}} - \frac{3 \tilde{E} \tilde{E}_u}{2E^{5/2}} \right)(u,0)
\]
\[
= \left( \frac{(\tilde{E} \tilde{M} - \tilde{F} \tilde{L})(2 \tilde{F}_u \tilde{E} - \tilde{E} \tilde{E}_{uv} - \tilde{E}_u \tilde{F})}{2E^{5/2}(\tilde{E} \tilde{G} - \tilde{F}^2)} + \frac{\tilde{E} \tilde{L}_u - \tilde{E}_u \tilde{L}}{E^{5/2}} \right)(u,0),
\]
where we used the relation \( \tilde{E}_v = 2 \langle f_u, h_u \rangle \) along the \( u \)-axis and the expressions of \( \widetilde{\Gamma}_{i1} \) (\( i = 1, 2 \)) as in \( \text{[2.4]} \).

By \( \text{[2.2]} \) and \( \text{[2.5]} \), we see that

\[
\kappa_i = \kappa_t \kappa_s + \frac{\kappa_u}{\sqrt{E}}
\]
along the \( u \)-axis. In particular, if \( \kappa_u(p) = 0 \), then we have \( \kappa_i(p) = \kappa_t(p) \kappa_s(p) \).

Next, we consider the non-degenerate singular point of the second kind. Let \( f : \Sigma \to \mathbf{R}^3 \) be a front, \( \nu \) its unit normal vector and \( p \) a non-degenerate singular point of the second kind. Take an adapted coordinate system \( (U; u, v) \) around \( p \), and denote by \( \tilde{H} \) the mean curvature of \( f \) defined on \( U \setminus \{ v = 0 \} \). Then the normalized cuspidal curvature \( \mu_c(p) \) is defined by \( \mu_c(p) = 2\tilde{H}(p) \), where \( \tilde{H} = v \tilde{H} \) (cf. \([26]) \). It is known that \( f \) is a front at \( p \) if and only if \( \mu_c(p) \neq 0 \) holds (\([26] \text{ Proposition 4.2}] \).

### 2.3. Principal curvatures of fronts.

We recall behavior of principal curvature of fronts near non-degenerate singular points. Let \( f : \Sigma \to \mathbf{R}^3 \) be a front, \( \nu \) a unit normal vector of \( f \) and \( p \in \Sigma \) a non-degenerate singular point of \( f \). Then we take an adapted coordinate system \( (U; u, v) \) around \( p \). Under this coordinate system, there exists a \( C^\infty \) map \( h : U \to \mathbf{R}^3 \setminus \{ 0 \} \) such that \( df(\eta) = v h \) and \( \langle h, \nu \rangle = 0 \). If \( p \) is a cuspidal edge, then \( df(\eta) = f_v = v h \) and the pair \( \{ f_u, h, \nu \} \) gives a frame. On the other hand, if \( p \) is of the second kind, \( f_u = v h - \varepsilon(u)f_v \) and \( \{ h, f_v, \nu \} \) gives a frame since \( df(\eta) = f_u + \varepsilon(u)f_v \) holds.

Let \( K \) and \( \tilde{H} \) be the Gaussian and the mean curvature of \( f \) defined on \( U \setminus \{ v = 0 \} \), namely, the set of regular points of \( f \) on \( U \). Then these functions are bounded \( C^\infty \) functions. Using these functions, we define two functions \( \kappa_j \) (\( j = 1, 2 \)) as

\[
\kappa_1 = H + \sqrt{H^2 - K}, \quad \kappa_2 = H - \sqrt{H^2 - K}
\]
on \( U \setminus \{ v = 0 \} \). These functions are dealt with principal curvatures of \( f \). It is known that one of \( \kappa_j \) (\( j = 1, 2 \)) can be extended as a bounded \( C^\infty \) function on \( U \) (\([40] \text{ Theorem 3.1}] \), see also \([27]) \). Moreover, one can take a principal vector \( \mathbf{V} \) with respect to bounded principal curvature (\([40] \text{ (3.2)} \) and (3.3)])

We write \( \kappa \) as the bounded principal curvature of \( f \) on \( \tilde{U} \) and \( \kappa \) as the unbounded one. We remark that \( \kappa(p) = \kappa_u(p) \) holds. On the other hand, for a bounded principal curvature \( \kappa \), we have the following.

**Proposition 2.7.** Under the above setting, the unbounded principal curvature \( \kappa \) changes the sign across the singular curve.
Proof. We set \( \hat{\kappa} = \lambda \tilde{\kappa} \), where \( \lambda = \det(f_u, f_v, \nu) \). Then by [40, Remark 3.2], if \( p \) is a cuspidal edge of \( f \), \( \hat{\kappa} \) is proportional to a nonzero functional multiple of the cuspidal curvature \( \kappa_c \) along the singular curve. On the other hand, if \( p \) is of the second kind, \( \hat{\kappa}(p) \) is proportional to a nonzero real multiple of the normalized cuspidal curvature \( \mu_c(p)(\neq 0) \). Thus \( \hat{\kappa}(p) \neq 0 \) holds, and hence we may assume that \( \hat{\kappa} > 0 \) near \( p \). The function \( \lambda \) changes the sign across the singular curve. Therefore we have the assertion. \( \square \)

**Definition 2.8.** Under the above settings, a point \( p \) is a ridge point if \( V\kappa(p) = 0 \) holds, where \( V\kappa \) means directional derivative of \( \kappa \) in the direction \( V \). Moreover, \( p \) is a \( k \)-th order ridge point if \( V^{(m)}\kappa(p) = 0 \) \((1 \leq m \leq k)\) and \( V^{(k+1)}\kappa(p) \neq 0 \) holds, where \( V^{(m)}\kappa \) means \( m \)-th order directional derivative of \( \kappa \) in the direction \( V \).

Porteous introduced the concepts of ridge points on regular surfaces. He showed that a ridge point on a surface corresponds to an \( A_3 \) singularity of the distance squared functions on it. For more other characterizations of ridge points on surfaces, see [2,7,11,30,31].

### 3. Gauss maps of fronts

#### 3.1. Singularities of maps from the plane into the plane.

We recall singularities of a map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) to consider singularities of Gauss maps of fronts. Whitney showed that generic singularities of these mappings are a fold and a cusp, which are \( \mathcal{A} \)-equivalent to the germs \((u, v) \mapsto (u, v^2)\) and \((u, v) \mapsto (u, v^3 + uv)\) at the origin, respectively (see Figure 1). Moreover, Rieger [32] classified singularities of \( C^\infty \) map germs from a plane into a plane with corank 1 and \( \mathcal{A}_e \)-codimension \( \leq 6 \). Singularities called lips, beaks and swallowtail are the map germs \( \mathcal{A} \)-equivalent to \((u, v) \mapsto (u, v^3 + u^2v), (u, v^3 - u^2v)\) and \((u, v^4 + uv)\) at the origin, respectively (see Figure 2). These singularities are \( \mathcal{A}_e \)-codimension 1.

![Figure 1. Fold and cusp.](image1.png)

![Figure 2. Lips, beaks and swallowtail.](image2.png)
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^\infty$ map. Then we set a function $\Lambda : \mathbb{R}^2 \to \mathbb{R}$ by

$$\Lambda(u, v) = \det(f_u, f_v)(u, v)$$

for some local coordinates $(u, v)$. We call $\Lambda$ the identifier of singularity of $f$. By the definition of $\Lambda$, we see that the set of singular points $S(f)$ of $f$ coincides with $\Lambda^{-1}(0)$. Take a corank 1 singular point $p$ of $f$. Then there exist a neighborhood $V$ of $p$ and a never vanishing vector field $\eta \in \mathfrak{X}(V)$ such that $df_\eta(\eta_\theta) = 0$ holds for any $q \in S(f) \cap V$. We call $\eta$ the null vector field. A singular point $p$ is called a non-degenerate singular point if $d\Lambda(p) \neq 0$.

**Fact 3.1** [33, 42]. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^\infty$ map and $p \in \mathbb{R}^2$ a singular point of $f$. Then

1. $f$ at $p$ is a fold if and only if $\eta\Lambda(p) \neq 0$.
2. $f$ at $p$ is a cusp if and only if $d\Lambda(p) \neq 0$, $\eta\Lambda(p) = 0$ and $\eta\eta\Lambda(p) \neq 0$.
3. $f$ at $p$ is a swallowtail if and only if $d\Lambda(p) \neq 0$, $\eta\Lambda(p) = \eta\eta\Lambda(p) = 0$ and $\eta\eta\eta\Lambda(p) \neq 0$.
4. $f$ at $p$ is a lips if and only if $d\Lambda(p) = 0$ and $\det\text{Hess}(\Lambda(p)) > 0$.
5. $f$ at $p$ is a beaks if and only if $d\Lambda(p) = 0$, $\det\text{Hess}(\Lambda(p)) < 0$ and $\eta\eta\Lambda(p) \neq 0$.

Criteria for more degenerate corank 1 singularities of maps from the plane into the plane are given by Kabata [21].

### 3.2. Singularities of Gauss maps of fronts.

Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu : \Sigma \to S^2$ the Gauss map of $f$ and $p \in \Sigma$ a non-degenerate singular point. It is obviously that $\nu$ is a $C^\infty$ map between 2-dimensional manifolds.

We consider the singularities of Gauss map of a front.

**Proposition 3.2.** Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ the Gauss map and $p \in \Sigma$ a non-degenerate singular point of $f$. Then the Gauss map $\nu$ is also singular at $p$ if and only if $\kappa(p) = 0$, where $\kappa$ is a bounded principal curvature near $p$. Moreover, $p$ is a non-degenerate singular point of $\nu$ if and only if $p$ is not a singular point of $\kappa$.

**Proof.** Since $p$ is a non-degenerate singular point of a front $f$, there exist a neighborhood $V \subset \Sigma$ of $p$ and a regular curve $\gamma : (-\varepsilon, \varepsilon) \to V$ such that $\text{Im}(\gamma) = S(f) \cap V$. Let $(u, v)$ be a coordinate system of $V$. Then an identifier of singularity $\Lambda : V \to \mathbb{R}$ of $\nu$ is given by

$$\Lambda(u, v) = \det(\nu_u, \nu_v, \nu)(u, v).$$

By the Weingarten formula, the function $\Lambda$ as in (3.1) can be expressed as

$$\Lambda = K\kappa = \kappa \hat{\kappa},$$

where $\hat{\kappa} = \lambda\kappa$. Since $\hat{\kappa}(p) \neq 0$, $p$ is a singular point of $\nu$ if and only if $\kappa(p) = 0$ holds.

By this argument, we can regard $\tilde{\Lambda} = \kappa$ as the identifier of singularity of $\nu$. Hence $p$ is a non-degenerate singular point of the Gauss map $\nu$ if and only if $\kappa(p) = 0$ and $(\partial_u \kappa(p), \partial_v \kappa(p)) \neq (0, 0)$, where $\partial_u = \partial/\partial u$ and $\partial_v = \partial/\partial v$. □
It is known that a non-degenerate singular point of a front is also a singular point of its Gauss map if and only if the limiting normal curvature $\kappa_\nu$ vanishes at $p$ (cf. [26]). Moreover, in such a case, the Gaussian curvature of a front is \textit{rationally bounded} (see [26, Theorem B and Corollary C]). For detailed definition, see [26, Definition 3.4].

By Proposition 3.2, we may assume that the set of singular points $S(\nu)$ of the Gauss map $\nu$ is given as $S(\nu) = \{ q \in U \mid \kappa(q) = 0 \}$, and $\nu$ has only corank 1 singularities at $p$. As in the case of regular surfaces, we call a point $q \in S(\nu) = \kappa^{-1}(0)$ and a curve given by $\kappa^{-1}(0)$ a parabolic point and a parabolic curve of $f$, respectively.

**Theorem 3.3.** Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ the Gauss map, $p \in \Sigma$ a non-degenerate singular point of $f$ and $\kappa$ a bounded principal curvature at $p$. Assume that $p$ is a parabolic point of $f$. Then the following assertions hold.

1. Suppose that $p$ is a regular point of $\kappa$.
   - $p$ is a fold of $\nu$ if and only if $p$ is not a ridge point.
   - $p$ is a cusp of $\nu$ if and only if $p$ is a first order ridge point.
   - $p$ is a swallowtail of $\nu$ if and only if $p$ is a second order ridge point.

2. Suppose that $p$ is a singular point of $\kappa$.
   - $p$ is a lips of $\nu$ if and only if $\det \text{Hess}(\kappa(p)) > 0$.
   - $p$ is a beaks of $\nu$ if and only if $\det \text{Hess}(\kappa(p)) < 0$ and $p$ is a first order ridge point.

**Proof.** Assume that $p$ is a non-degenerate singular point of the second kind, and take an adapted coordinate system $(U; u, v)$ centered at $p$. Then there exists a map $h : U \to \mathbb{R}^3 \setminus \{0\}$ such that $f_u = vh - \varepsilon(u)f_v$. We set functions as follows:

$$
\tilde{E} = |h|^2, \quad \tilde{F} = \langle h, f_v \rangle, \quad \tilde{G} = |f_v|^2, \quad \tilde{L} = -\langle h, \nu_u \rangle, \quad \tilde{M} = -\langle h, \nu_v \rangle, \quad \tilde{N} = -\langle f_v, \nu_v \rangle,
$$

where $|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$ for any $\mathbf{x} \in \mathbb{R}^3$. Note that $\tilde{E}\tilde{G} - \tilde{F}^2 \neq 0$ and $\tilde{L}(p) \neq 0$ hold. We assume that $\tilde{L}(p) > 0$. In this case, principal curvatures $\kappa$ and $\tilde{\kappa}$ can be written as

$$
\kappa = \frac{2((\tilde{L} + \varepsilon(u)\tilde{M})\tilde{N} - v\tilde{M}^2)}{\tilde{A} + \tilde{B}}, \quad \tilde{\kappa} = \frac{2((\tilde{L} + \varepsilon(u)\tilde{M})\tilde{N} - v\tilde{M}^2)}{\tilde{A} - \tilde{B}},
$$

where

$$
\tilde{A} = \tilde{G}(\tilde{L} + \varepsilon(u)\tilde{M}) - 2v\tilde{F}\tilde{M} + v\tilde{E}\tilde{N},
$$

$$
\tilde{B} = \sqrt{\tilde{A}^2 - 4v(\tilde{E}\tilde{G} - \tilde{F}^2)((\tilde{L} + \varepsilon(u)\tilde{M})\tilde{N} - v\tilde{M}^2)}
$$

(cf. [40, Theorem 3.1]). Moreover, by using functions as in (3.2), $\nu_u$ and $\nu_v$ are expressed as

$$
\nu_u = \frac{\tilde{F}(\varepsilon(u)\tilde{M} - \varepsilon\tilde{N}) - \tilde{G}\tilde{L}}{\tilde{E}\tilde{G} - \tilde{F}^2} h + \frac{\tilde{F}\tilde{L} - \tilde{E}(v\tilde{M} - \varepsilon\tilde{N})}{\tilde{E}\tilde{G} - \tilde{F}^2} f_v,
$$

$$
\nu_v = \frac{\tilde{F}\tilde{N} - \tilde{G}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2} h + \frac{\tilde{F}\tilde{M} - \tilde{E}\tilde{N}}{\tilde{E}\tilde{G} - \tilde{F}^2} f_v,
$$

respectively.
can be extended on $U$ of regular surfaces is studied in [3]. We also remark that relationship between Gauss maps for regular surfaces are given (cf. [2,17]), and stability of Gauss maps Definition 3.4.

Let $\alpha$ be a regular plane curve. Let $\beta$ be another plane curve defined by the zero set of a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$. We say that $\alpha$ has $(k+1)$-point contact at $t_0 \in I$ with $\beta$ if the composite function $g(t) = F(\alpha(t))$ satisfies

$$g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0, \quad g^{(k+1)}(t_0) \neq 0,$$

where $g^{(i)} = d^i g/dt^i$ $(1 \leq i \leq k+1)$ (see Figure 3). Moreover, $\alpha$ has at least $(k+1)$-point contact at $t_0$ with $\beta$ if the function $g(t) = F(\alpha(t))$ satisfies

$$g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0.$$

In this case, we call the integer $k$ the order of contact.

It is known that the curve $\alpha$ has $(k+1)$-point contact at $t_0$ with $\beta$ if and only if the composite function $g$ has an $A_k$ singularity at $t_0$, where a function $f : \mathbb{R} \to \mathbb{R}$ has an $A_k$ singularity at $t_0 \in \mathbb{R}$ if $f^{(i)}(t_0) = 0$ $(1 \leq i \leq k)$ and $f^{(k+1)}(t_0) \neq 0$ hold (cf. [4,17]).

First, we show the following lemma.
Lemma 3.5. Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ its Gauss map and $p \in \Sigma$ a cuspidal edge of $f$. Let $\kappa$ be a bounded principal curvature near $p$. Then the parabolic curve defined by $\kappa = 0$ is regular at $p$ if and only if either $\kappa'_\nu \neq 0$ or $4\kappa_t^2 + \kappa_s\kappa_c^2 \neq 0$ holds at $p$.

Proof. Let $(U; u, v)$ be an adapted coordinate system around $p$ satisfying $\eta \lambda > 0$ on the $u$-axis. Then the parabolic curve defined by $\kappa = 0$ is regular at $p$ if and only if $(\partial_u \kappa, \partial_v \kappa) \neq (0, 0)$ at $p$. We now remark that $\kappa(u, 0) = \kappa_\nu(u)$. Thus $\partial_u \kappa(p) \neq 0$ if and only if $\kappa'_\nu(p) \neq 0$ holds. On the other hand, we obtain

$$\left. \partial_v \kappa \right|_{(p)} = -\frac{1}{2\kappa_c} \left( 4\kappa_t^2 + \kappa_s\kappa_c^2 \right) \left( \frac{\tilde{E}\tilde{G} - \tilde{F}^2}{\tilde{E}} \right)^{1/4}$$

along the $u$-axis by [41, Proposition 2.8]. Thus we have the assertion. \qed

We note that the condition $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$ implies that $p$ is not a sub-parabolic point of $f$ (see [41]).

Proposition 3.6. Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ the Gauss map of $f$ and $p \in \Sigma$ a cuspidal edge of $f$. Let $\kappa$ be the bounded principal curvature near $p$. Suppose that $\kappa(p) = 0$ and $(\partial_u \kappa(p), \partial_v \kappa(p)) \neq (0, 0)$. Then the singular curve $\gamma$ passing through $p$ has $(k + 1)$-point contact ($k \geq 1$) with the parabolic curve defined by $\kappa^{-1}(0)$ at $p$ if and only if $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$ and

$$\kappa_\nu(p) = \kappa'_\nu(p) = \cdots = \kappa_v^{(k)}(p) = 0 \text{ and } \kappa_v^{(k+1)}(p) \neq 0$$

hold.

Proof. Take an adapted coordinate system $(U; u, v)$ centered at $p$. Then the singular curve $\gamma$ is given as $\gamma(u) = (u, 0)$. Moreover, it follows that $\kappa(u, 0) = \kappa_\nu(u)$ holds ([40, Theorem 3.1]). When $\partial_u \kappa(p) = \kappa'_\nu(p) = 0$, the parabolic curve $\kappa^{-1}(0)$ is regular at $p$ if and only if $\partial_v \kappa(p) \neq 0$. This is equivalent to $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$ by Lemma [3.5]. Thus we have the assertion by the definition of contact of two curves (see Definition 3.4). \qed

By this proposition, we have the following.

Corollary 3.7. Let $f : \Sigma \to \mathbb{R}^3$ be a front and $p \in \Sigma$ a cuspidal edge of $f$. Then the Gaussian curvature $K$ of $f$ is rationally bounded at $p$ if and only if a parabolic
curve passes through \( p \). Moreover, \( K \) is rationally continuous at \( p \) if the singular curve passing through \( p \) has at least 2-point contact with a parabolic curve at \( p \).

Proof. It is known that the Gaussian curvature \( K \) of \( f \) is rationally bounded (resp. rationally continuous) if and only if \( \kappa_\nu(p) = 0 \) (resp. \( \kappa_\nu(p) = \kappa'_\nu(p) = 0 \)) \cite[Corollary 3.12]{26}. Thus the assertion follows from Proposition \ref{3.6} \( \square \)

This implies that behavior of a bounded principal curvature of a cuspidal edge is closely related to rational boundedness of the Gaussian curvature.

Conversely, the following assertion holds.

Proposition 3.8. Let \( f : \Sigma \to \mathbb{R}^3 \) be a front, \( \nu : \Sigma \to S^2 \) the Gauss map of \( f \) and \( p \) a cuspidal edge of \( f \). Suppose that the Gaussian curvature \( K \) of \( f \) is rationally continuous at \( p \). Then \( p \) is a cusp of \( \nu \) if and only if \( \kappa_i(p) = 0 \) and \( \kappa_s(p)\kappa'_s(p) \neq 0 \).

Proof. Let \((U; u, v)\) be an adapted coordinate system around \( p \) with \( \eta \lambda(u, 0) > 0 \).

We denote by \( \kappa \) and \( \mathbf{V} = V_1 \partial_u + V_2 \partial_v \) a bounded \( C^\infty \) principal curvature of \( f \) on \( U \) and a principal vector relative to \( \kappa \), respectively. We note that \( V_1(p) \neq 0 \).

Since \( K \) is rationally continuous at \( p \), \( \kappa_\nu = \kappa'_\nu = 0 \) at \( p \). This implies that \( \kappa(p) = \partial_u\kappa(p) = 0 \). In this case, the parabolic curve \( \kappa^{-1}(0) \) is regular at \( p \) if and only if \( \partial_u\kappa(p) \neq 0 \), that is, \( 4\kappa_i^2 + \kappa_s\kappa'_s(p) \neq 0 \) at \( p \) by Lemma \ref{3.5} Thus it follows that

\[
\mathbf{V}_\kappa = V_1 \partial_u \kappa + V_2 \partial_v \kappa = V_2 \partial_v \kappa
\]

at \( p \). Since \( V_2 = -\kappa_i \sqrt{\bar{E}G - \bar{F}^2} \) holds along the \( u \)-axis \cite[Proposition 3.3]{40}, \( \mathbf{V}_\kappa(p) = 0 \) if and only if \( \kappa_s(p) = 0 \).

We calculate second order directional derivative \( \mathbf{V}^{(2)}\kappa(p) \) under the assumptions that \( \kappa_\nu = \kappa'_\nu = \kappa_i = 0 \) hold at \( p \). By a direct computation, we have

\[
\mathbf{V}^{(2)}\kappa = V_1 \partial_u (\mathbf{V}_\kappa) + V_2 \partial_v (\mathbf{V}_\kappa).
\]

Since \( V_2 = 0 \) at \( p \),

\[
\mathbf{V}^{(2)}\kappa(p) = V_1(p) \partial_u (\mathbf{V}_\kappa)(p)
\]

holds. By \( \ref{2.6} \) and \( \ref{3.4} \), \( \mathbf{V}_\kappa \) can be written as

\[
\mathbf{V} = \frac{1}{2\kappa_c} \left( 4\kappa_i^3 + \kappa_i \kappa'_s \right) \left( \frac{(\bar{E}G - \bar{F}^2)^3}{\bar{E}} \right)^{1/4}
\]

along the \( u \)-axis. Thus \( \partial_u(\mathbf{V}_\kappa) \neq 0 \) at \( p \) if and only if \( 12\kappa_i^2 \kappa'_s + \kappa'_i \kappa'_s + 2\kappa_i \kappa_c \kappa'_s \neq 0 \) at \( p \) by the assumptions above. When \( \kappa'_s(p) = 0 \), \( \kappa_i(p) = \kappa_s(p)\kappa_i(p) \) holds \( \ref{2.6} \), see also \cite[Theorem 4.4]{25}. Hence \( \partial_u(\mathbf{V}_\kappa) \neq 0 \) at \( p \) if and only if \( \kappa'_i(p) \neq 0 \). Summing up, we have the assertion. \( \square \)

For a cuspidal edge, it is known that the singular locus \( \hat{\gamma} = f \circ \gamma \) is a line of curvature of \( f \) if and only if the cusp-directional torsion \( \kappa_t \) identically vanishes along the singular curve \( \gamma \) \cite[Proposition 3.3]{40}, see also \cite{20}. This is equivalent to the case that the direction of the principal vector \( \mathbf{V} \) with respect to the bounded principal curvature \( \kappa \) is parallel to the direction of \( \gamma' \) along the singular curve \( \gamma \).

We now consider relationships between contactness of a singular curve with a parabolic curve at the cuspidal edge point and types of singularities of the Gauss map under the assumption that the singular locus \( \hat{\gamma} \) is a line of curvature.
Proposition 3.9. Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ its Gauss map and $p \in \Sigma$ a cuspidal edge. Let $\gamma$ be a singular curve passing through $p$. Suppose that $\kappa$ is bounded near $p$, $\kappa^{-1}(0)$ is a regular curve near $p$ and $\hat{\gamma} = f \circ \gamma$ is a line of curvature. Then

1. $p$ is a fold of $\nu$ if and only if $\kappa'_\nu \neq 0$ at $p$.
2. $p$ is a cusp of $\nu$ if and only if $\kappa'_\nu = 0$, $\kappa''_\nu \neq 0$ and $\kappa_s \neq 0$ at $p$.
3. $p$ is a swallowtail of $\nu$ if and only if $\kappa'_\nu = \kappa''_\nu = 0$, $\kappa'''_\nu \neq 0$ and $\kappa_s \neq 0$ at $p$.

Proof. Take an adapted coordinate system $(U; u, v)$ centered at $p$. Let us denote the principal vector with respect to $\kappa$ by $V = V_1(u, v)\partial_u + V_2(u, v)\partial_v$. Since $\hat{\gamma}(u) = f(u, 0)$ is a line of curvature, $V_2(u, 0) = 0$ (Proposition 3.2). Then by the Malgrange preparation theorem (cf. [40, Chapter IV]), there exists a function $W : U \to \mathbb{R}$ such that $V_2(u,v) = vW(u,v)$. Thus the principal vector $V$ is rewritten as $V = V_1(u,v)\partial_u + vW(u,v)\partial_v$. We note that $V_1(u,0) = \tilde{N}(u,0) = -h(u,0), vW(u,0)) \neq 0$ by [40, (3.3)], where $h : U \to \mathbb{R}^3 \setminus \{0\}$ is a $C^\infty$ map satisfying $f_v = vh$.

Under this setting, first, second and third order directional derivatives of $\kappa$ in the direction of $V$ are

$$
\begin{align*}
V\kappa &= V_1(\partial_u\kappa), \\
V^{(2)}\kappa &= V_1^2(\partial_u^2\kappa) + (\partial_uV_1)(\partial_v\kappa), \\
V^{(3)}\kappa &= V_1(V_1^2(\partial_u^2\kappa) + 3V_1(\partial_uV_1)(\partial_v^2\kappa) + ((\partial_uV_1)^2 + V_1(\partial_u^2V_1))(\partial_v\kappa))
\end{align*}
$$

at $p$, where $\partial_i^u = \partial_i^u/d\partial_i^u$ ($i = 1, 2, 3$). Thus $p$ is not a ridge point if and only if $\partial_u\kappa \neq 0$ at $p$, and $p$ is a first (resp. second) order ridge point if and only if $\partial_u\kappa = 0$ and $\partial_u^2\kappa \neq 0$ (resp. $\partial_v\kappa = \partial_v^2\kappa = 0$ and $\partial_v^3\kappa \neq 0$) at $p$. If $\partial_u\kappa(p) = 0$, then a parabolic curve $\kappa^{-1}(0)$ through $p$ is regular if and only if $\partial_v\kappa(p) \neq 0$. This is equivalent to $\kappa_s(p)\kappa_c(p)^2 \neq 0$ by Lemma 3.5. Since $\kappa_c(p) \neq 0$, $\kappa^{-1}(0)$ passing through $p$ is regular if and only if $\kappa_s(p) \neq 0$ when $\partial_u\kappa(p) = 0$.

On the other hand, we have $\partial_u\kappa = \kappa'_\nu$, $\partial_u^2\kappa = \kappa''_\nu$ and $\partial_u^3\kappa = \kappa'''_\nu$ at $p$ since $\kappa = \kappa_\nu$ on the $u$-axis. By Theorem 3.3 we have the conclusions.

Propositions 3.6, 3.9 and Corollary 3.7 yield the following assertion.

Corollary 3.10. Under the same settings as Proposition 3.9, a point $p$ is a cusp (resp. swallowtail) of $\nu$ if and only if $\gamma$ has $2$-point contact (resp. $3$-point contact) at $p$ with $\kappa^{-1}(0)$. Moreover, the following assertions hold:

1. The Gaussian curvature $K$ of $f$ is rationally bounded but not rationally continuous at $p$ if $p$ is a fold of $\nu$.
2. $K$ is rationally continuous at $p$ if $p$ is a cusp or a swallowtail of $\nu$.

Example 3.11. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a $C^\infty$ map given by

$$
f(u,v) = \left( u, \frac{1}{2}u^2 + \frac{1}{2}v^2, \frac{1}{3}v^3 + u^4 \right).
$$

This gives a cuspidal edge with $S(f) = \{v = 0\}$ and $\eta = \partial_v$. We set

$$
\nu(u,v) = \frac{1}{\sqrt{1 + v^2 + (-4u^3 + uv)^2}}(-4u^3 + uv, -v, 1).
$$
Then $\nu$ is the Gauss map of $f$. For $f$, it follows that

$$\kappa_{\nu}(u) = \frac{12u^2}{\sqrt{1 + 16u^6(1 + u^2 + 16u^6)}},$$

$\kappa_s(0) = 1 \neq 0$ and $\kappa_t = 0$ on the $u$-axis, namely, $f(u, 0)$ is a line of curvature. Moreover, we see that $\Lambda(u, v) = 12u^2 - v$ is an identifier of singularity of $\nu$. Thus the parabolic curve $\kappa^{-1}(0)$ of $f$ is given by the equation $12u^2 - v = 0$, where $\kappa$ is a bounded $C^\infty$ principal curvature of $f$. Further, the origin is a non-degenerate singular point of $\nu$. By a direct computation, it follows that the singular curve has 2-point contact at the origin with $\kappa^{-1}(0)$, that is, $\kappa_{\nu}(0) = \kappa'_{\nu}(0) = 0$ and $\kappa''_{\nu}(0) \neq 0$ hold. From Proposition 3.9, the origin is a cusp of $\nu$ (see Figure 4).

Moreover, we see that $\tilde{\Lambda}(u, v) = 12u^2 - v$ is an identifier of singularity of $\nu$. Thus the parabolic curve $\kappa^{-1}(0)$ of $f$ is given by the equation $12u^2 - v = 0$, where $\kappa$ is a bounded $C^\infty$ principal curvature of $f$. Further, the origin is a non-degenerate singular point of $\nu$. By a direct computation, it follows that the singular curve has 2-point contact at the origin with $\kappa^{-1}(0)$, that is, $\kappa_{\nu}(0) = \kappa'_{\nu}(0) = 0$ and $\kappa''_{\nu}(0) \neq 0$ hold. From Proposition 3.9, the origin is a cusp of $\nu$ (see Figure 4).

Moreover, we see that $\tilde{\Lambda}(u, v) = 12u^2 - v$ is an identifier of singularity of $\nu$. Thus the parabolic curve $\kappa^{-1}(0)$ of $f$ is given by the equation $12u^2 - v = 0$, where $\kappa$ is a bounded $C^\infty$ principal curvature of $f$. Further, the origin is a non-degenerate singular point of $\nu$. By a direct computation, it follows that the singular curve has 2-point contact at the origin with $\kappa^{-1}(0)$, that is, $\kappa_{\nu}(0) = \kappa'_{\nu}(0) = 0$ and $\kappa''_{\nu}(0) \neq 0$ hold. From Proposition 3.9, the origin is a cusp of $\nu$ (see Figure 4). Moreover, the Gaussian curvature $K$ of $f$ is rationally continuous at the origin by Corollary 3.10.

**Figure 4.** The images of cuspidal edge (left) and its Gauss map (right) in Example 3.11. Thick curves on the cuspidal edge and the Gauss map are the image of the parabolic curve via $f$ and $\nu$, respectively.

### 3.4. Cuspidal edges with bounded Gaussian curvatures and their Gauss maps.

We consider a special case that the Gaussian curvature of a cuspidal edge is bounded. Let $f : \Sigma \to \mathbb{R}^3$ be a front, $p \in \Sigma$ a cuspidal edge of $f$ and $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ a singular curve through $p$. Let $\nu$ the Gauss map of $f$. Then it is known that the Gaussian curvature $K$ of $f$ is bounded near $p$ if and only if the limiting normal curvature $\kappa_{\nu} \equiv 0$ along $\gamma$ (see [26, Fact 2.12] and [37, Theorem 3.1]). In this case, the set of singular points of $\nu$ coincides with $S(f)$ near $p$. More precisely, the bounded principal curvature $\kappa$ of $f$ defined near $p$ vanishes along $\gamma$. By Lemma 3.5, $p$ is a non-degenerate singular point of $\nu$ if and only if $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$ holds.

**Proposition 3.12.** Let $f : \Sigma \to \mathbb{R}^3$ be a front, $p \in \Sigma$ a cuspidal edge of $f$ and $\nu : \Sigma \to S^2$ the Gauss map of $f$. Suppose that the Gaussian curvature $K$ of $f$ is bounded sufficiently small neighborhood of $p$. Then

1. $p$ is a fold of $\nu$ if and only if $\kappa_t(p) = 0$ and $4\kappa_t(p)^2 + \kappa_s(p)\kappa_c(p)^2 \neq 0$ hold.
2. $p$ is a cusp of $\nu$ if and only if $\kappa_t(p) = 0$, $\kappa'_t(p) \neq 0$ and $\kappa_s(p) \neq 0$ hold.

**Proof.** Take an adapted coordinate system $(U; u, v)$ around $p$. Let denote by $\kappa$ and $V$ the bounded principal curvature on $U$ and the principal vector with respect
to $\kappa$, respectively. Then by the assumption $\kappa(u,0) = \kappa_v(u) = 0$ hold. Since $\partial_u\kappa(p) = \partial_v^2\kappa(p) = 0$, we have

$$V\kappa(p) = V_2(p)(\partial_u\kappa(p)).$$

Since $V_2(p) \neq 0$ is equivalent to $\kappa_t(p) \neq 0$, and $\partial_u\kappa(p) \neq 0$ is equivalent to $4\kappa_t(p)^2 + \kappa_u(p)\kappa_v(p)^2 \neq 0$, we have the first assertion.

We show the second assertion. We assume that $V\kappa(p) = 0$. Since $p$ is a non-degenerate singular point of $\nu$, $\partial_u\nu(p) \neq 0$. This implies that $V_2(p) = 0$, namely, $\kappa_t(p) = 0$. Under these conditions, we have

$$V^{(2)}\kappa(p) = V_1(p)(\partial_uV_2(p))(\partial_v\kappa(p)).$$

Since $V_1(p) \neq 0$ and $\partial_u\nu(p) \neq 0$, $p$ is a cusp of $\nu$ if and only if $\partial_uV_2(p) \neq 0$. This is equivalent to $\kappa'_t(p) \neq 0$. Thus we have the second assertion. \hfill $\square$

We remark that similar characterizations for flat fronts in the hyperbolic 3-space $H^3$ and the de Sitter 3-space $S^3_1$ and for linear Weingarten fronts in $H^3$ are known ([23,35]). In [23], Kokubu and Umehara studied global properties of linear Weingarten fronts of Bryant type by meromorphic representation formulae.

4. Extended height functions on fronts with non-degenerate singular points of the second kind

Let $f : \Sigma \to \mathbb{R}^3$ be a front, $\nu$ its unit normal and $p \in \Sigma$ a non-degenerate singular point of the second kind of $f$. Then we take an adapted coordinate system $(U;u,v)$ centered at $p$ satisfying $\lambda_u = \det(h, f_u, \nu) > 0$ on the $u$-axis. It is known that there exists a strongly adapted coordinate system $(U;u,v)$ centered at $p$ which is an adapted coordinate system with $(f_{uv}(p), f_v(p)) = 0$ (cf. [26]). Thus we assume that $(U;u,v)$ is a strongly adapted coordinate system centered at $p$ for later calculations.

We now define the following function:

$$\varphi : U \to \mathbb{R}, \quad \varphi(u,v) = \langle f(u,v), v \rangle - r,$$

where $v \in S^2$ is a constant vector and $r \in \mathbb{R}$. We call this function $\varphi$ the extended height function on $f$ in the direction $v$. For other properties of height functions on regular/singular surfaces, see [3,6,12,13,17,24,29,39]. In particular, Oset Sinha and Tari [29] studied contact between cuspidal edges and planes using height functions, and they characterized singularities of height functions by geometric invariants of cuspidal edges. Moreover, Francisco [9] investigated functions on a swallowtail and gave a classification of functions on a swallowtail.

**Lemma 4.1.** The extended height function $\varphi$ as in (4.1) is singular at $p$ if $v = \pm \nu(p)$.

**Proof.** By direct computation, we have $\varphi_u = \langle f_u, v \rangle = \langle vh - \varepsilon(u)f_v, v \rangle$ and $\varphi_v = \langle f_v, v \rangle$. Thus we have the assertion. \hfill $\square$

Taking $v = \nu(p)$ and $r = \langle f(p), \nu(p) \rangle$, it follows that $\varphi(p) = \varphi_u(p) = \varphi_v(p) = 0$ by (4.1) and Lemma 4.1. Thus we assume that $v = \nu(p)$ and $r = \langle f(p), \nu(p) \rangle$ in what follows. In this case, $\varphi$ measures types of contact of $f$ with the limiting
tangent plane at $p$, where the *limiting tangent plane* is a plane perpendicular to the unit normal vector $\nu$.

For a function germ $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ such that $0$ is a singular point of $\varphi$, corank of the function $\varphi$ at $0$ is given by $\text{corank}(\varphi) = 2 - \text{rank Hess}(\varphi)$ at $0$ (cf. [4][17]).

**Proposition 4.2.** Let $f : \Sigma \to \mathbb{R}^3$ be a front, $p$ a non-degenerate singular point of the second kind, $\nu$ the unit normal vector to $f$ and $\varphi$ the extended height function on $f$ as in (4.1) with $v = \nu(p)$ and $r = \langle f(p), \nu(p) \rangle$. Suppose that $\kappa$ is a bounded principal curvature of $f$ near $p$. Then

1. $p$ is a corank 1 singular point of $\varphi$ if and only if $p$ is not a parabolic point.
2. $p$ is a corank 2 singular point of $\varphi$ if and only if $p$ is a parabolic point.

**Proof.** We take an adapted coordinate system $(U; u, v)$ centered at $p$. By Lemma 4.1 we have $\varphi_u(p) = \varphi_v(p) = 0$. We consider the Hessian matrix $\text{Hess}(\varphi)$ of $\varphi$ at $p$. Since $f_{uu} = \nu_h - \varepsilon f_v - \varepsilon f_{uv}$ and $f_{uv} = \nu_h - \varepsilon f_v - \varepsilon f_{vv}$, we have

$$\varphi_{uu}(p) = \varphi_{uv}(p) = 0, \quad \varphi_{vv}(p) = \langle f_{vv}(p), \nu(p) \rangle = \tilde{N}(p).$$

Hence the Hessian matrix of $\varphi$ can be written as

$$\text{Hess}(\varphi(p)) = \tilde{N}(p) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

On the other hand, $\kappa(p) = \kappa_v(p) = \tilde{N}(p)/\tilde{G}(p)$ holds by (3.3). Thus the results follow. \hfill $\Box$

This proposition is a special case of [24, Theorem 2.11]. In fact, Martins and Nuño-Ballesteros considered distance squared functions and height functions on a class of surfaces with corank 1 singular points which contain cuspidal edges and swallowtails in [24] (see also [9][29][39][40]). For a regular surface $S$, we note that the rank of the Hessian matrix of a height function on $S$ is zero at $p \in S$ if and only if $p$ is a flat umbilic point of $S$ ([17, Proposition 2.5]).

We assume that $p$ is a parabolic point in the following. This is the case that the Gauss map $\nu$ is singular at $p$, namely, the Gaussian curvature is rationally bounded at $p$. We now set the number $\Delta_{\varphi}$ defined as

$$\Delta_{\varphi} = \left( (\varphi_{uuu})^2(\varphi_{vv})^2 - 6\varphi_{uuu}\varphi_{uvv}\varphi_{uv}\varphi_{vv} \
- 3(\varphi_{uv})^2(\varphi_{uv})^2 + 4(\varphi_{uu})^3\varphi_{vv} + 4\varphi_{uu}(\varphi_{uvv})^3 \right)(p).$$

It is known that a $C^\infty$ function germ $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is $\mathcal{R}$-equivalent to $u^3 + uv^2$ (resp. $u^3 - uv^2$) at $0$ (see Figure 5), that is, $0$ is a $D_4^+$ singularity (resp. $D_4^-$ singularity) of $g$ if and only if $j^2g = 0$ and $\Delta_{\varphi} > 0$ (resp. $\Delta_{\varphi} < 0$), where $j^2g$ is the 2-jet of $g$ at $0$ (see [34, Lemma 3.1], see also [11]). By Lemma 4.1 and Proposition 4.2 we see that $j^2\varphi = 0$ holds.

**Theorem 4.3.** Under the above conditions, the function $\varphi$ as in (4.1) with $v = \nu(p)$ and $r = \langle f(p), \nu(p) \rangle$ has a $D_4$ singularity at $p$ if and only if $p$ is a parabolic point and not a ridge point of a front $f$.

**Proof.** Take a strongly adapted coordinate system $(U; u, v)$ centered at $p$. Since $f$ is a front at $p$, $\hat{L}(p) \neq 0$ holds. Thus we may assume that $\hat{L}(p) > 0$. By
Proposition 4.2, \( j^2 \varphi = 0 \) if and only if \( p \) is a parabolic point. Hence we suppose that \( p \) is a parabolic point, namely \( \hat{N}(p) = 0 \).

First, we consider the number \( \Delta \varphi \) as in (4.3). Since \( f_{uuu} = vh_{uuu} - \varepsilon'' f_v - 2\varepsilon' f_{uv} - \varepsilon f_{uv} \) and \( f_{uv} = h + vh_v - \varepsilon \), we have \( \varphi_{uuu}(p) = 0 \). Thus it holds that

\[
\Delta \varphi = (\varphi_{uuu})^2(p)(4\varphi_{uuu}\varphi_{vvv} - 3(\varphi_{uvv})^2(p)).
\]

By direct computations, we see that 
\( \varphi_{uuu} = \hat{L} \), \( \varphi_{uvv} = \hat{2M} \), \( \varphi_{vvv} = \hat{N}_v - \frac{\hat{M}}{\hat{L}}(\hat{N}_u - 2\hat{M}) \) at \( p \). Therefore we have

\[
(4.4) \quad \Delta \varphi = 4\hat{L}(p)^2(\hat{L}(p)\hat{N}_v(p) - \hat{M}(p)(\hat{N}_u(p) + \hat{M}(p))).
\]

Next, we consider the condition of ridge points. Under the above settings, \( \kappa \) is given by (3.3). The differentials \( \partial_u \kappa = \partial \kappa / \partial u \) and \( \partial_v \kappa = \partial \kappa / \partial v \) are

\[
\partial_u \kappa(p) = \frac{\hat{N}_u(p)}{G(p)}, \quad \partial_v \kappa(p) = \frac{\hat{L}(p)\hat{N}_v(p) - \hat{M}(p)^2}{G(p)\hat{L}(p)}.
\]

The principal vector \( \mathbf{V} \) relative to \( \kappa \) is \( \mathbf{V} = (-\hat{M}, \hat{L}) \) at \( p \). Thus the directional derivative \( \mathbf{V} \kappa \) of \( \kappa \) in the direction \( \mathbf{V} \) at \( p \) is given by

\[
(4.5) \quad \mathbf{V} \kappa(p) = \frac{1}{G(p)}(\hat{L}(p)\hat{N}_v(p) - \hat{M}(p)(\hat{N}_u(p) + \hat{M}(p))).
\]

Comparing (4.4) and (4.5), we get the conclusion.

By Theorems 3.3 and 4.3, we have the following assertion.

**Corollary 4.4.** Under the same conditions as Theorem 4.3, the function \( \varphi \) as in (4.1) with \( v = \nu(p) \) and \( r = \langle f(p), \nu(p) \rangle \) has a \( D_4 \) singularity at \( p \) if and only if the Gauss map \( \nu \) has a fold singularity at \( p \).

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