STRONGLY HOMOTOPY LIE ALGEBRAS AND DEFORMATIONS OF CALIBRATED SUBMANIFOLDS

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Abstract. For an element $\Psi$ in the graded vector space $\Omega^*(M, TM)$ of tangent bundle valued forms on a smooth manifold $M$, a $\Psi$-submanifold is defined as a submanifold $N$ of $M$ such that $\Psi|_N \in \Omega^*(N, TN)$. The class of $\Psi$-submanifolds encompasses calibrated submanifolds, complex submanifolds and all Lie subgroups in compact Lie groups. The graded vector space $\Omega^*(M, TM)$ carries a natural graded Lie algebra structure, given by the Frölicher-Nijenhuis bracket $[\cdot, \cdot]^{FN}$. When $\Psi$ is an odd degree element with $[\Psi, \Psi]^{FN} = 0$, we associate to a $\Psi$-submanifold $N$ a strongly homotopy Lie algebra, which governs the formal deformations of $N$ as a $\Psi$-submanifold. As application we revisit formal and smooth deformation theory of complex closed submanifolds and of $\phi$-calibrated closed submanifolds, where $\phi$ is a parallel form in a Riemannian manifold.

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1. INTRODUCTION

Let \((M,g)\) be a Riemannian manifold and \(\nabla\) the Levi-Civita connection. A differential form \(\varphi\) is called parallel if \(\nabla \varphi = 0\). In this case we shall write \((M,g,\varphi)\). When we want to stress that \(\varphi\) has degree \(p\) we shall write \(\varphi^p\) for \(\varphi\). If \(\varphi\) is parallel, \(\varphi\) is closed and its comass is constant. Normalizing the comass of \(\varphi\), we regard \(\varphi\) as a calibration. All important calibrated submanifolds are \(\varphi\)-calibrated submanifolds for some parallel differential form \(\varphi\) \cite{Dao1977, HL1982, Le1990, McLean1998, Joyce2007}. On another hand, \(\varphi\)-calibrated submanifolds play an important role in the geometry of manifolds with special holonomy, in higher dimensional gauge theory and in string theory as “super-symmetric cycles” or “branes” \cite{SYZ1996, DT1998, Tian2000, GYZ2003, AW2003, Joyce2007, DS2011, Walpuski2012, Walpuski2014}. Note that manifolds with special holonomy always admit parallel forms, see Subsection 2.1 below.

Deformation theory of closed calibrated submanifolds has been initiated by McLean \cite{McLean1998} inspired by similarities between calibrated submanifolds and complex submanifolds. McLean considered deformations of special Lagrangian, associative, coassociative and Cayley submanifolds. In \cite{LV2017} Lê-Vanżura observed that any \(\varphi\)-calibrated submanifold \(L^k\) in a Riemannian manifold \((M,g)\) considered by McLean (as well as any Kähler submanifold) satisfies the following Harvey-Lawson identity \cite{LV2017, Definition 1.1} \[ (1.1) \quad |\varphi(\xi)|^2 + |\Psi_E(\xi)|^2 = |\xi|^2 \quad \text{for all} \quad x \in M, \]

and \(\xi\) an element in the Grassmannian of unit decomposable \(k\)-vectors in \(T_x M\), for some \(E\)-valued form \(\Psi_E \in \Omega^k(M, E)\), where \(E\) is a Riemannian vector bundle over \(M\). In this case the defining equation of \(\varphi\)-calibrated submanifolds \(L^k\) is equivalent to \(\langle \Psi_E \rangle_{L^k} = 0\). McLean showed that, in the reformulation of \cite{LV2017} using the Harvey-Lawson identity, the equation \(\langle \Psi_E \rangle_{L^k} = 0\) is essentially elliptic for special Lagrangian and coassociative submanifolds and using the standard elliptic theory he proved that deformations of those submanifolds are unobstructed. Additionally, he proved that the equation \(\langle \Psi_E \rangle_{L^k} = 0\) is elliptic for associative and Cayley submanifolds \(L^k\) but deformation of those calibrated submanifolds may be obstructed.

Further works on deformations of calibrated submanifolds are devoted to the smoothness and the Zariski tangent space to the moduli space of closed submanifolds.

\footnote{In \cite{Dao1977} Dao, based on the previous works by Federer and Lawson, proposed to use parallel differential forms to study area-minimizing real currents, but he did not invent the word “calibration”.
}
calibrated submanifolds that are special Lagrangian, associative, coassociative and Cayley in (tamed) almost/nearly Calabi-Yau, $G_2$ and $\text{Spin}(7)$-manifolds \cite{AS2008, AS2008a, GIP2003, Gayet2014, Kawai2017, Ohst2014}, or to similar questions concerning calibrated submanifolds with elliptic boundary condition \cite{Butscher2003, KL2009, GW2011, Ohst2014} and non-compact calibrated submanifolds of certain type \cite{JS2005, KL2012, Lotay2009}.

In the present paper we propose a new approach to deformation of calibrated submanifolds. Firstly, we do not look for a Harvey-Lawson type identity. Instead, using the first cousin principle we characterize $\varphi$-calibrated submanifolds up to first order via the vector-valued form $\hat{\varphi} \in \Omega^*(M, TM)$ that is obtained from $\varphi$ by contraction with the metric (Lemma 3.1). Motivated by Lemma 3.1 we introduce the notion of a $\Psi$-submanifold (Definition 3.3) and develop a general deformation theory for closed $\Psi$-submanifolds for any square-zero element $\Psi$ of odd degree in the graded Lie algebra $\Omega^*(M, TM)$, using strongly homotopy Lie algebras (Proposition 5.7). This generalizes the assignment of a strongly homotopy Lie algebra to a complex submanifold (Remark 6.15). Further we observe that the deformation problem for $\varphi$-calibrated submanifolds in $(M, g, \varphi)$ is equivalent to the deformation problem of $\varphi \wedge dt$-calibrated submanifolds in $(M \times S^1, g + dt^2, \varphi \wedge dt)$, which, in its turn, is equivalent to the deformation problem of $\varphi \wedge dt$-submanifolds in $M \times S^1$. As a result, using further analytic and an (over determined) elliptic property of the defining equation for $\hat{\varphi}$-submanifolds, we prove that both the formal and the smooth deformation problems for a closed $\varphi$-calibrated submanifold in $(M, g, \varphi)$ are encoded in its associated $L_\infty$-algebra (Theorem 6.4).

The paper is organized as follows. In Section 2 we collect known results concerning parallel differential forms and Frölicher-Nijenhuis bracket that are important for the main part of the paper. In Section 3 we introduce the notion of a $\Psi$-submanifold (Definition 3.3) which seems a good notion to understand deformations of calibrated submanifolds (Corollary 3.5). In Section 4 we assign to each $\Psi$-submanifold a canonical strongly homotopy Lie algebra, if $\Psi$ is a square-zero element of odd degree in the graded Lie algebra $(\Omega^*(M, TM), [-, -]^{FN})$ (Theorem 4.1). In Section 5 we define the deformation problem for $\Psi$-submanifolds and study formal deformations using the strongly homotopy Lie algebra (Proposition 5.7). In Section 6 we study infinitesimal, smooth and formal deformations of calibrated submanifolds in details (Proposition 6.1, Theorem 6.4) and revisit deformation theory of complex submanifolds (Theorem 6.14, Remark 6.15).

Notations and conventions.

- In this paper manifolds and their submanifolds are denoted by capital Latin letters $M, L$, etc. When we want to emphasize the dimension of a manifold $M$ (resp. a submanifold $L$) we write $M^m$ (resp. $L^l$). The tangent map to a smooth map $f : M \rightarrow N$ is denoted by $Tf : TM \rightarrow TN$, and its value at the point $x \in M$ by $T_xf : T_xM \rightarrow T_{f(x)}N$. 
• Small Greek letters usually denote scalar valued forms and capital Greek letters denote vector valued forms.

• For a scalar valued form \( \varphi \) on \( M \) we denote by \( \hat{\varphi} \) the associated \( TM \)-valued form on \( M \) obtained from \( \varphi \) by contraction with the metric (see (2.6) and the sentence that follows for explanation).

• For a (finite dimensional or infinite dimensional) vector space \( V \) we denote by \( 0 \in V \) the origin of \( V \). If \( V \) is the space of \( (C^k \text{ or } L^2_k) \) sections of a vector bundle \( E \) over a manifold \( L \) then we also denote by \( 0 \) the zero section of \( E \).

• We adopt Getzler’s conventions about \( L_\infty \)-algebras [Getzler2009].

2. Preliminaries

2.1 Parallel differential forms on a Riemannian manifold. In this section we recall the classification of parallel differential forms on a Riemannian manifold \((M, g)\), described in Tables 1, 2, 3, 4 from [Besse1987, Chapter 10].

Let \( \varphi \) be a parallel form on \((M, g)\) such that \( \varphi \) is not a multiple of the volume form. Then the restricted holonomy group \( \text{Hol}^0(M, g) \) is contained in the stabilizer \( \text{Stab}(\varphi) \) and therefore is strictly smaller than the group \( O(m) \). Since locally a Riemannian manifold \((M, g)\) is a product of Riemannian manifolds whose holonomy group action on the tangent bundle is irreducible, the classification of parallel forms on \((M, g)\) is reduced to the case of irreducible Riemannian manifolds \((M, g)\). Symmetric Riemannian spaces are examples of manifolds admitting parallel forms.

• The algebra of parallel forms on an irreducible symmetric space \( M = G/H \) is isomorphic to the algebra of \( Ad_H \)-invariant forms on \( T_e G/H \). In particular, if \( M = G/H \) is compact then the algebra of parallel forms is isomorphic to the de Rham cohomology algebra \( H^*(M, \mathbb{R}) \). A list of the Poincaré polynomials of all the simply connected compact irreducible symmetric spaces has been compiled by Takeuchi in [Takeuchi1962].

In 1955, Marcel Berger proved that if \((M, g)\) is a simply-connected Riemannian manifold with irreducible holonomy group and nonsymmetric, then \( \text{Hol}^0(M, g) \) must be one of \( SO(n) \), \( U(m) \) (Kähler manifolds) , \( SU(m) \) (special Kähler manifolds, in particular Calabi-Yau manifolds), \( Sp(m) \) (hyper-Kähler manifolds), \( Sp(m) \times Sp(1) \) (quaternionic Kähler manifolds), \( G_2 \) (\( G_2 \)-manifolds) or \( \text{Spin}(7) \) (\( \text{Spin}(7) \)-manifolds).

• The algebra of parallel forms on a Kähler manifold is generated by the Kähler 2-form \( \omega \).

• The algebra of parallel forms on a special Kähler manifold is generated by the Kähler 2-form \( \omega^2 \), the real and imaginary part of the complex volume form \( \text{Re} \text{vol}_\mathbb{C}, \text{Im} \text{vol}_\mathbb{C} \). The latter are called special Lagrangian forms, abbreviated as SL-forms.

• The algebra of parallel forms on a quaternionic Kähler manifold is generated by the quaternionic 4-form \( \psi \).
• The algebra of parallel forms on a hyper-Kähler manifold is generated by the three Kähler 2-forms.
• The algebra of parallel forms on a $G_2$-manifold is generated by the associative 3-form $\varphi$ (and its dual coassociative 4-form $*\varphi$).
• The algebra of parallel forms on a Spin(7)-manifold is generated by the self-dual Cayley 4-form $\kappa$.

We also refer the reader to [Bryant1987, Salamon1989] for geometry of parallel forms on manifolds with special holonomy.

2.2. Frölicher-Nijenhuis bracket. Let us recall the definition of the Frölicher-Nijenhuis bracket on $\Omega^*(M, TM)$ following [KMS1993 §8], see also [KLS2017a §2.1] for a short account.

The space $\text{Der}(\Omega^*(M))$ of graded derivations of the graded commutative algebra $\Omega^*(M)$ is a graded Lie algebra. First we recall the definition of algebraic graded derivations in $\text{Der}(\Omega^*(M))$. They are defined by insertions $\iota_K$ for $K \in \Omega^*(M, TM)$. For $K = \alpha^k \otimes X$ we define $\iota_K \in \text{Der}(\Omega^*(M))$ as follows

$$\iota_{\alpha^k \otimes X}^l : = \alpha^k \wedge (\iota_X \beta^l) \in \Omega^{k+l-1}(M).$$

Next we define the linear map

$$L : \Omega^*(M, TM) \to \text{Der}(\Omega^*(M)), \ K \mapsto L_K,$$

(2.1)

$$L_K := L(K) := [\iota_K, d] \in \text{Der}(\Omega^*(M)).$$

Proposition 2.1. ([KMS1993 Theorem 8.3, p. 69]) For any graded derivation $D \in \text{Der}(\Omega^*(M))$ there are unique $K \in \Omega^*(M, TM)$ and $K' \in \Omega^*(M, TM)$ such that

$$D = L_K + \iota_{K'}.$$

We have $K' = 0$ if and only if $[D, d] = 0$ and $D$ is algebraic if and only if $K = 0$.

It follows from Proposition 2.1 that the map $L$ is injective and its image $L(\Omega^*(M, TM))$ is the centralizer of $d$ in $\text{Der}(\Omega^*(M))$.

(2.2)

$$L(\Omega^*(M, TM)) = \{D \in \text{Der}(\Omega^*(M)) \mid [D, d] = 0\}.$$

Hence $L(\Omega^*(M, TM))$ is closed under the graded Lie bracket $[-, -]$ on $\text{Der}(\Omega^*(M))$. Then we define the Frölicher-Nijenhuis bracket $[-, -]^F_N$ on $\Omega^*(M, TM)$ as the pull-back of the graded Lie bracket on $\text{Der}(\Omega^*(M))$ via the linear embedding $L$, i.e.,

(2.3)

$$L_{[K,L]}^F_N := [L_K, L_L].$$

Thus the Frölicher-Nijenhuis bracket provides $\Omega^*(M, TM)$ with a structure of $\mathbb{Z}$-graded (hence $\mathbb{Z}_2$-graded) Lie algebra.

Furthermore the Frölicher-Nijenhuis bracket enjoys the following functoriality with respect to local diffeomorphisms. First of all, for a local diffeomorphism $f : M \to N$ and any $K \in \Omega^*(N, TN)$, the pull-back of $K$ by $f$ is...
defined as follows
\[(2.4) \quad (f^* K)_x (X_1, \cdots, X_k) := (T_x f)^{-1} K_{f(x)} (T_x f \cdot X_1, \cdots, T_x f \cdot X_k).\]
Then we have [KMS1993, 8.16, p. 74]
\[(2.5) \quad f^* [K, L]^{FN} = [f^* K, f^* L]^{FN}.
\]
Let \((M, g)\) be a Riemannian manifold. Recall that the contraction \(\partial_g : \Lambda^k T^* M \to \Lambda^{k-1} T^* M \otimes TM\) is defined pointwise as follows [KLS2017a, (2.5)]
\[(2.6) \quad \partial_g (\alpha^k) := (\iota_{e_i} \alpha^k) \otimes (e^i)^\#,
\]
where the sum is taken pointwise over some basis \((e_i)\) of \(T_x M\) with dual basis \((e^i)^\#)\).
We also abbreviate \(\partial_g (\varphi)\) by \(\hat{\varphi}\). A straightforward computation via geodesic normal coordinates yields the following

**Proposition 2.2.** (cf. [KLS2017a, Proposition 2.2]) For any parallel differential form \(\varphi\) on a Riemannian manifold \((M, g)\) we have \([\hat{\varphi}, \hat{\varphi}]^{FN} = 0\).

**Definition 2.3.** We say that an element \(\Psi \in \Omega^{2k+1}(M, TM)\) is of square-zero, if \([\Psi, \Psi]^{FN} = 0\).

### 3. \(\varphi\)-Calibrated Submanifolds and \(\Psi\)-Submanifolds

In this section, motivated by geometry of calibrated submanifolds (Lemma 3.1), we introduce the notion of a \(\Psi\)-submanifold for any \(\Psi \in \Omega^*(M, TM)\) (Definition 3.3). We provide examples of \(\Psi\)-submanifolds that are not calibrated submanifolds (Example 3.6), including all complex submanifolds as well as all Lie subgroups in compact Lie groups. We show that if \(\varphi^k\) is a parallel \(k\)-form on a Riemannian manifold \((M, g)\) then \(\varphi^k\)-submanifolds \(L^k\) are minimal submanifolds if the restriction of \(\varphi^k\) to \(L^k\) does not vanish (Theorem 3.4).

For a submanifold \(L\) in a manifold \(M\) we denote by \(NL\) the normal bundle of \(L\) and by \(pr : TM|_L \to NL\) the canonical projection. If \(M\) is endowed with a Riemannian metric \(g\) then we also identify \(NL\) with the (Riemannian) normal bundle of \(L\) that is the orthogonal complement to the tangent bundle \(TL\).

**Lemma 3.1.** Let \(\varphi^k\) be a calibration on a Riemannian manifold \((M, g)\) and \(L\) a \(\varphi^k\)-calibrated submanifold. Then \(pr \circ \varphi^k|_L = 0 \in \Omega^*(L, NL)\).

**Proof.** Let \(L\) be a \(\varphi^k\)-calibrated submanifold. Let \((e_i)\) be an orthonormal basis in \(T_x L\) and \((f_j)\) - an orthonormal basis in \(N_x L\). Then for any \(i \in [1, k]\) we have
\[(3.1) \quad pr \circ \varphi (e_1 \wedge \cdots \wedge e_i \wedge \cdots \wedge e_k) = \sum_{j=1}^{n-k} \varphi (f_j \wedge e_1 \wedge \cdots \wedge e_i \wedge \cdots \wedge e_k) \otimes f_j.
\]
By the first cousin principle for calibrated submanifolds [HL1982, HM1986, Le1990] the right hand side of (3.1) vanishes. This completes the proof. □

Remark 3.2. Let us denote by $G_k(T_xM)$ the Grassmannian of all unit decomposable $k$-vectors in $T_xM$ and by $\tilde{T}_xL$ the unit $k$-vector associated to the oriented tangent space $T_xL$ whose orientation is defined by the volume form $\omega |_{L}$. The Grassmanian $G_k(T_xM)$ has the natural Riemannian metric induced from the Riemannian metric on $T_xM$. Note that the tangent space $T_{\tilde{T}_xL} G_k(T_xM)$ has an orthogonal basis consisting of $k$-vectors of the form $f_j \wedge e_1 \wedge \cdots e_i \wedge \cdots \wedge e_k$. Let $\tilde{\varphi}^k(x)$ denote the restriction of $\varphi^k(x)$ to $G_k(T_xM)$. Then we have

$$\langle \text{pr} \circ \varphi^k(e_1 \wedge \cdots e_i \wedge \cdots \wedge e_k), f_j \rangle = \langle d_{e_1 \wedge \cdots \wedge e_k} \tilde{\varphi}^k(x), f_j \wedge e_1 \wedge \cdots e_i \wedge \cdots \wedge e_k \rangle$$

where the pairing in the LHS of (3.2) is defined via the Riemannian metric.

Lemma 3.1 motivates the following

Definition 3.3. Let $M$ be a smooth manifold and $\Psi \in \Omega^k(M, TM)$. A submanifold $L \subseteq M$, where $l \geq k$, will be called a $\Psi$-submanifold, if $\text{pr} \circ \Psi|_L = 0 \in \Omega^k(L^l, NL^l)$, or equivalently, $\Psi|_L \in \Omega^k(L^l, TL^l)$.

Theorem 3.4. (cf. [Robles2012, Theorem 1.2]) Assume that $\varphi^k$ is a parallel form on a Riemannian manifold $(M, g)$. Then a $\tilde{\varphi}^k$-submanifold $L^k$ is a minimal submanifold if $\varphi^k|_{L^k} \neq 0$.

Proof. Theorem 3.4 is equivalent to Theorem 1.2 in [Robles2012], which has been proved by using moving frame method. A version of Theorem 3.4 is stated in [Le2013] as Lemma 6.5, referring to [Le1990, Lemma 1.1]. We provide below a short proof, using the argument in the proof of Lemma 1.1 in [Le1990].

Let $L^k$ be a $\tilde{\varphi}^k$-submanifold in $(M, g)$. We shall compute the mean curvature $H$ of $L^k$. Let $V_1, \ldots, V_k$ be local vector fields on $N^k$ around a point $p \in L^k$ such that $|V_1 \wedge \cdots \wedge V_k| = 1$. By (3.2), for each $x \in L$ the unit $k$-vector $T_x L$ is a critical point of the function $\tilde{\varphi}^k(x)$. Hence we have

$$\varphi^k(T_x L) = c$$

for some constant $c$ (this follows from the classification of parallel forms on $(M, g)$, see e.g. [Besse1987, Theorem 10.108, Corollary 10.110] and Subsection 2.1.1). Recall that $c \neq 0$ by the assumption of Theorem 3.4.
Let $X$ be a normal vector on $L$. Using the argument in the proof of Lemma 1.1 in [Le1990] we compute
\[
0 = (\iota_X d\varphi^k(V_1, \cdots, V_n)) = \sum_{i=1}^k (-1)^i V_i (\varphi^k(X, V_1, \cdots, V_i, \cdots, V_n))
\]
\[
- X(\varphi^k(V_1, \cdots, V_k)) + \sum_{1 \leq i < j \leq n} (-1)^{i+j} \varphi^k([V_i, V_j], X, \cdots, V_i, \cdots, V_j, \cdots, V_k)
\]
\[
+ \sum_{i=1}^k (-1)^i \varphi^k([X, V_i], \cdots, V_i, \cdots, V_k).
\]
(3.4)

The first and third terms in (3.4) are zero, since $L$ is a $\hat{\varphi}$-submanifold. The second term is zero by (3.3). Hence we get from (3.4)
\[
0 = \sum_{i=1}^k (-1)^i \varphi^k([X, V_i], \cdots, V_i, \cdots, V_k) = c \sum_{i=1}^k \langle [X, V_i], V_i \rangle = c\langle -H, X \rangle.
\]
Since $c \neq 0$ we obtain $H = 0$. This proves Theorem 3.4. □

From Theorem 3.4 we obtain immediately the following

**Corollary 3.5.** A deformation of a $\varphi$-calibrated submanifold inside the class of $\hat{\varphi}$-submanifolds remains in the subclass of $\varphi$-calibrated submanifolds.

**Example 3.6.** 1. By Lemma 3.1 each $\varphi$-calibrated submanifold is a $\hat{\varphi}$-submanifold. In particular, every associative submanifold $L^3$ in a $G_2$-manifold $M^7$ is a $\hat{\varphi}$-submanifold, where $\varphi$ is the associative 3-form on $M^7$. We claim that every 3-dimensional $\hat{\varphi}$-submanifold is an associative submanifold. To prove this assertion we regard $\hat{\varphi} \in \Omega^2(M^7, TM^7)$ as the 2-fold cross product $TM^7 \times TM^7 \to TM^7$: $\varphi(X, Y, Z) = \langle X \times Y, Z \rangle$ where $\times$ denotes the cross product [HL1982, KLS2017a]. Then our assertion follows from the first cousin principle for $\hat{\varphi}$-submanifolds and the observation that a 3-plane is associative if and only if it is invariant under the 2-fold cross product [HL1982].

2. Let us consider a complex manifold $(M, g, J)$. We regard $J$ as an element in $\Omega^1(M, TM)$. Clearly a submanifold $L$ in $M$ is a $J$-submanifold if and only if it is a complex submanifold.

3. Let $\star \varphi$ be the coassociative 4-form on a $G_2$-manifold $M^7$. The associated form $\tilde{\varphi} \in \Omega^3(M^7, TM^7)$ is often denoted by $\chi$ and called the 3-fold cross product $[HL1982, KLS2017a]$. (a) It is shown in the proof of Lemma 5.6 in [KLS2018] that a 3-submanifold $L^3 \subset M^7$ is a $\chi$-submanifold, if and only if it is a coassociative submanifold.

(b) By Lemma 3.1 every coassociative submanifold $L^4$ is a $\chi$-submanifold. We claim that a 4-dimensional $\chi$-submanifold is a coassociative submanifold. To prove this it suffices to show that the coassociative plane (up to orientation) is the only critical point(s) of the function $\tilde{\varphi}$ defined in Remark 3.2. This assertion is equivalent to the statement that the associative plane (up
to orientation) is the only critical point(s) of the function \( \varphi \), which has been proved in Example 3.6.

It is not hard to conclude from (a) and (b) that a \( \chi \)-submanifold in a \( G_2 \)-manifold is either an associative submanifold or a coassociative submanifold. Thus we regard \( \chi \) as an analogue of the complex form \( J \in \Omega^3(M, TM) \) in complex geometry. In [KLS2017a] Kawai-Lê-Schwachhöfer gave another interpretation of this fact, proving that a \( G_2 \)-structure is torsion-free if and only if \([\chi, \chi]^{FN}\) vanishes.

4. Let \( \alpha := \text{Re}(\text{vol}_C) \) be the SL-calibration on a Calabi-Yau manifold \((M, g, \omega, \text{vol}_C)\). Lemma 3.1 implies that every special Lagrangian submanifold \( L \subset M \) is a \( \hat{\alpha} \)-submanifold.

5. Let \( M^7 \) be a \( G_2 \)-manifold and \( \varphi \) the defining associative 3-form. In [LV2017] Lê-Vanžura define a form \( \tau \in \Omega^4(M^7, TM^7) \) as follows. For \( x, y, z, w \in TM^7 \) we set (HL1982, (1.17), Theorem 1.18, p. 117), see also [LV2017, Remark 4.2]

\[
(3.5) \quad \tau(x, y, z, w) := - (\varphi(y, z, w)x + \varphi(z, x, w)y + \varphi(x, y, w)z + \varphi(y, x, z)w).
\]

Then any 4-submanifold in \( M^7 \) is a \( \tau \)-submanifold.

6. Let \( M^8 \) be a Spin(7)-manifold and \( \psi^4 \) its defining Cayley form. Recall that \( \psi^4(X, Y, Z, W) = \langle P(X, Y, Z), W \rangle \) where \( P \) is the 3-fold vector cross product, see e.g. [Fernandez1986]. By Lemma 3.1 every Cayley submanifold is a \( \hat{\psi} \)-submanifold. Since any \( \hat{\psi} \)-submanifold \( L \) is invariant under the triple product \( P \), \( L \) must be a Cayley submanifold.

7. Let \( G \) be a compact Lie group provided with the Killing metric. Denote by \( \omega^3 \) the Cartan 3-form on \( G \). The calibration \( \omega^3 \) has been first considered by Dao in [Dao1977] and later by Tasaki [Tasaki1985]. By Theorem 3.1 in [Le1990] any 3-dimensional Lie subgroup in \( G \) is a \( \hat{\omega} \)-submanifold. Since the tangent space \( T_e G \) is invariant under the Lie bracket, any Lie subgroup in \( G \) is a \( \hat{\omega} \)-submanifold. In [Le1990, Section 3] Lê classified stably minimal 3-dimensional subgroups in compact semi-simple Lie groups of classical type, see also [Le1990b] for the classification of all stably minimal simple Lie subgroups in classical Lie groups. Clearly non-stably minimal Lie subgroups cannot be calibrated submanifolds.

8. Let \( \theta^3, \theta^5, \ldots, \theta^{2n-1} \) be bi-invariant forms on \( SU(m) \). By Theorem 3.4 in [Le1990] for any \( n < m \) the standard subgroup \( SU(n) \subset SU(m) \) is a \( \hat{\phi} \)-submanifold for \( \phi = \theta^1 \wedge \cdots \wedge \theta^{2n-1} \).

### 4. The \( L_\infty \)-algebra associated to a \( \Psi \)-submanifold

In this section, using Voronov’s derived bracket construction [Voronov2005], we prove the following

**Theorem 4.1.** Let \( \Psi \in \Omega^*(M, TM) \) be an odd degree element which is square-zero, i.e., such that \([\Psi, \Psi]^{FN} = 0\), and let \( L \) be a \( \Psi \)-submanifold.
Corollary 4.2. 1. Assume that $\varphi^k$ is a parallel $k$-form on a Riemannian manifold $(M, g)$ and $L^k$ is a closed $\varphi^k$-calibrated submanifold. If $k$ is even, then there is a canonical $\mathbb{Z}_2$-graded $L_\infty$-algebra structure on $\Omega^*(L^k, NL^k)[-1]$. If $k$ is odd, then there is a canonical $\mathbb{Z}_2$-graded $L_\infty$-algebra structure on $\Omega^*(L^k \times S^1, N(L^k \times S^1))[-1]$.

2. (cf. Manetti2007) For any closed complex submanifold $L$ in a complex manifold $M$ there is a canonical $\mathbb{Z}$-graded $L_\infty$-algebra structure on $\Omega^*(L, NL)[-1]$.

3. For every closed associative submanifold $L^3$ in a $G_2$-manifold $(M^7, \varphi^3)$ there are canonical $\mathbb{Z}_2$-graded $L_\infty$-algebra structures both on $\Omega^*(L^3, NL^3)[-1]$ and on $\Omega^*(L^3 \times S^1, N(L^3 \times S^1))[-1]$.

Proof. Let $L^k$ be a closed $\varphi^k$-calibrated submanifold of the Riemannian manifold $(M, g)$. If $k$ is even, then $L^k$ is a $\varphi^k$-submanifold of $M$. If $k$ is odd, then $L^k \times S^1$ as a $\varphi^k \wedge dt$-submanifold of $M \times S^1$. This proves statement 1. Statement 2 is immediate, as any complex submanifold is a $J$-submanifold. Finally any associative submanifold $L^3$ of a $G_2$-manifold $(M^7, \varphi^3)$ is a $\varphi^3$-calibrated submanifold, so we have an $L_\infty$-structure on $\Omega^*(L^3 \times S^1, N(L^3 \times S^1))[-1]$ by statement 1. On the other hand, it has been showed in KLS2018 that $L^3$ is a $\varphi^3$-submanifold (see Example 3.63). This proves statement 3. \qed

The remainder of this section is devoted to the proof of Theorem 4.1.

First let us recall Voronov’s construction of a $\mathbb{Z}_2$-graded $L_\infty$-algebra from a set of $V$-data. A set of $V$-data is a quintuple $(g, \mathfrak{a}, \iota, P, \triangle)$, where

- $g = g_0 \oplus g_1$ is a $\mathbb{Z}_2$-graded Lie algebra (with Lie bracket by $[-,-]$),
- $\mathfrak{a}$ is an abelian Lie algebra;
- $\iota : \mathfrak{a} \to g$ is a Lie algebra inclusion;
- $P : g \to \mathfrak{a}$ is a (not necessarily bracket preserving) projection, inverting $\iota$ from the left and such that $\ker P \subseteq g$ is a Lie subalgebra,
- $\triangle \in (\ker P) \cap g_1$ is an element such that $[\triangle, \triangle] = 0$.

Proposition 4.3. ([Voronov2005 Theorem 1, Corollary 1]). Let $(L, \mathfrak{a}, \iota, P, \triangle)$ be a set of V-data. Then $\mathfrak{a}[-1]$ is a $\mathbb{Z}_2$-graded $L_\infty$-algebra with multibrackets

$$l_n(a_1, \cdots, a_n) = (-)^n P[\cdots ([\triangle, \iota(a_1)], \iota(a_2)], \cdots, \iota(a_n)].$$

where

$$* = (n-1)|a_1| + (n-2)|a_2| + \cdots + |a_{n-1}| + \frac{n(n+1)}{2},$$

and the vertical bars $| - |$ denote the degree.
Replacing \(\mathbb{Z}_2\) by \(\mathbb{Z}\) in the definition of \(V\)-data, Formula (4.1) gives a \(\mathbb{Z}\)-graded \(L_\infty\)-algebra. A homotopy Lie theoretic interpretation of the Voronov’s \(L_\infty\)-algebra structure on \(\frak{a}[-1]\) can be found in [Bandiera2015].

The proof of Theorem [4.1] will now go through several steps. The first step consists in associating \(V\)-data to a \(\Psi\)-manifold \(L\) equipped with a tubular neighborhood \(\tau\).

**Definition 4.4.** Let \(\Psi \in \Omega^*(M,TM)\) be an odd degree element with \([\Psi,\Psi]^{FN} = 0\), let \(j: L \hookrightarrow M\) be a \(\Psi\)-submanifold and \(\tau: NL \to U \subset M\) be a tubular neighborhood of \(L\) in \(M\), i.e. a diffeomorphism \(\tau: NL \to U \subset M\) onto an open neighborhood of \(L\) such that \(\tau \circ 0 = j\). Denote by \(\pi: NL \to N\) the projection. The 5-ple \((gL, aL, \iotaL, PL, \DeltaL, \tau)\) is defined as follows:

- The graded Lie algebra \(gL\) is \(\Omega^*(NL,TNL)\) with the \(FN\) bracket;
- The abelian graded Lie algebra \(aL\) is the graded vector space \(\Omega^*(L,NL)\) endowed with the zero bracket;
- The graded vector space morphism \(\iotaL: aL \to gL\) is defined on decomposable elements as \(\iotaL(\omega \otimes X) = \pi^*(\omega) \otimes X\), where \(X\) is the canonical vertical lift of \(X\) given by the natural identification \(N_{\pi(x)}L \cong \ker(\pi_*: T_xNL \to T_{\pi(x)}L)\);
- The graded vector space morphism \(PL: gL \to aL\) is the composition

  \[\Omega^*(NL,TNL) \xrightarrow{|L} \Omega^*(L,TNL|L) \xrightarrow{pr} \Omega^*(L,NL),\]

  where the rightmost arrow is the natural projection induced by the canonical splitting \(TNL|L = TL \oplus NL\);
- The element \(\DeltaL, \tau\) in \(gL, \tau\) is \(\DeltaL, \tau = \tau^*\Psi\), where

  \[\tau^*: \Omega^*(M,TM) \to \Omega^*(NL,TNL)\]

  is the pullback of tensors along the local diffeomorphism \(\tau\).

**Remark 4.5.** Notice that, as the notation suggests, \(\DeltaL, \tau\) is the unique component of the 5-ple \((gL, aL, \iotaL, PL, \DeltaL, \tau)\) which is actually dependent on the tubular neighborhood \(\tau\).

**Proposition 4.6.** The 5-ple \((gL, aL, \iotaL, PL, \DeltaL, \tau)\) associated with a \(\Psi\)-manifold is a 5-ple of \(V\)-data. As a consequence the graded vector space \(aL[-1] = \Omega^*(L,NL)[-1]\) carries a \(\mathbb{Z}_2\)-graded \(L_\infty\)-algebra structure induced by these data. When \(\Psi\) has degree 1, this is actually a \(\mathbb{Z}\)-graded \(L_\infty\)-algebra structure.

**Proof.** The map \(\iotaL\) is injective, the map \(PL\) is surjective, and one manifestly has \(PL \circ \iotaL = id_{aL}\) so we are left with showing \([\iotaL aL, \iotaL aL] = 0\), that \(\ker PL\) is a Lie subalgebra of \(gL\), that \(\DeltaL, \tau \in \ker PL\) and \([\DeltaL, \tau, \DeltaL, \tau] = 0\). To this aim, consider the composition

\[\hat{PL} = \iotaL \circ PL: \Omega^*(NL,TNL) \to \Omega^*(NL,TNL).\]

It is shown in [KLS2018] that the image of \(\hat{PL}\) is an abelian subalgebra of the graded Lie algebra \((\Omega^*(NL,TNL), [-,-]^{FN})\) and that \(\ker \hat{PL}\) is closed.
under the Fr"olicher-Nijenhuis bracket. As $P_L$ is surjective, the image of $\tilde{P}_L$ coincides with the image of $\iota_L$, so that $\iota_L(a_L)$ is an abelian subalgebra of $\mathfrak{g}_L$. As $\iota_L$ is injective, we have $\ker \tilde{P}_L = \ker P_L$, and so $\ker P_L$ is a Lie subalgebra of $\mathfrak{g}_L$. By the naturality of the Fr"olicher-Nijenhuis bracket, we have

$$[\triangle_{L,\tau}, \triangle_{L,\tau}] = [\tau^*\Psi, \tau^*\Psi]^{FN} = \tau^*[\Psi, \Psi]^{FN} = 0.$$  

Finally, as $L$ is a $\Psi$-manifold in $M$ and $\tau$ is a diffeomorphism relative to $L$ in a neighborhood of $L$ (identified with the zero section in $NL$), we have that $L$ is a $\triangle_{L,\tau}$-manifold in $NL$. Therefore, $P_L \triangle_{L,\tau} = 0$, by definition of $\triangle_{L,\tau}$-manifold.

The underlying graded vector space of the $L_\infty$-algebra structure induced on $\Omega^*(L, NL)[-1]$ by Proposition 4.6 is independent of $\tau$. Our next step will consists in showing that also the $L_\infty$-algebra structure is actually independent of $\tau$, up to isomorphism. To begin with, let us show that a reparameterization of the tubular neighborhood leaves the $L_\infty$-algebra structure unchanged up to isomorphism.

**Lemma 4.7.** Let $\tau_0$ and $\tau_1$ be two tubular neighborhoods of $L$ in $M$ such that $\tau_1 = \tau_0 \circ \psi$ for some diffeomorphism $\psi$ of $NL$ relative to $L$. Then $\psi$ induces an isomorphism of $V$-data between $(\mathfrak{g}_L, a_L, \iota_L, P_L, \Delta_{L, \tau_0})$ and $(\mathfrak{g}_L, a_L, \iota_L, P_L, \Delta_{L, \tau_1})$. In particular $(\mathfrak{g}_L, a_L, \iota_L, P_L, \Delta_{L, \tau_0})$ and $(\mathfrak{g}_L, a_L, \iota_L, P_L, \Delta_{L, \tau_1})$ induce isomorphic $L_\infty$-algebra structures on $\Omega^*(L, NL)[-1]$.

**Proof.** As $\psi$ is a diffeomorphism of $NL$ relative to $L$ the pullback along $\psi$ induces commutative diagrams

$$\begin{array}{ccc}
\Omega^*(L, NL) & \xrightarrow{\psi^*} & \Omega^*(NL, TNL) \\
\downarrow & & \downarrow \\
\Omega^*(L, NL) & \xrightarrow{\psi^*} & \Omega^*(NL, TNL)
\end{array} \quad \text{;} \quad \begin{array}{ccc}
\Omega^*(NL, TNL) & \xrightarrow{P_L} & \Omega^*(L, NL) \\
\downarrow & & \downarrow \\
\Omega^*(NL, TNL) & \xrightarrow{P_L} & \Omega^*(L, NL)
\end{array}$$

Finally, we have

$$\psi^* \Delta_{L, \tau_0} = (\tau_0^{-1} \circ \tau_1)^*(\tau_0^* \Psi) = \tau_1^* \Psi = \Delta_{L, \tau_1}.$$  

In order to prove that the $L_\infty$-algebra structure $\Omega^*(L, NL)[-1]$ is generally independent of $\tau$, up to isomorphism, as we can not directly compare two distinct tubular neighborhoods of $L$ in $M$ it is convenient to pass to formal neighborhoods.

**Definition 4.8.** Let $\Psi \in \Omega^*(M, TM)$ be an odd degree element with $[\Psi, \Psi]^{FN} = 0$, let $L \subset \overline{M}$ be a $\Psi$-submanifold and $\tau : NL \to U \subset M$ be a tubular neighborhood of $L$ in $M$. Finally, let $NL_{for} \hookrightarrow NL$ be the formal neighborhood of $L$ in $NL$ via the zero section embedding $s_0 : L \hookrightarrow NL$. In the same notation as Proposition 4.6 the 5-ple $(\mathfrak{g}_L^{for}, a_L^{for}, \iota_L^{for}, P_L^{for}, \Delta_L^{for})$ is the restriction to $NL_{for}$ of the 5-ple $(\mathfrak{g}_L, a_L, P_L, \iota_L, \Delta_{L, \tau})$. 


Remark 4.9. Notice that the graded abelian Lie algebras $\mathfrak{a}_L$ and $\mathfrak{a}_L^{for}$ actually coincide: they both are the graded vector space $\Omega^*(L,NL)$ endowed with the zero bracket. In particular the restriction to $NL_{for}$ is the identity morphism on $\Omega^*(L,NL)$.

Corollary 4.10. The 5-ple $(\mathfrak{g}_L^{for}, \mathfrak{a}_L^{for}, \iota_L^{for}, \phi_L^{for}, \Delta_L^{for})$ is a set of V-data and so induces a $\mathbb{Z}_2$-graded $L_\infty$-algebra structure on $\mathfrak{a}_L^{for}[-1] = \Omega^*(L,NL)[-1]$. Moreover this $\mathbb{Z}_2$-graded $L_\infty$-algebra structure coincides with that induced on $\Omega^*(L,NL)$ by the V-data $(\mathfrak{g}_L, \mathfrak{a}_L, P_L, \iota_L, \Delta_L, \tau)$.

Lemma 4.11. Let $\tau_0$ and $\tau_1$ be two isotopic tubular neighborhoods of $L$ in $M$. Then $\Delta_{L,\tau_0}^{for}$ and $\Delta_{L,\tau_1}^{for}$ are gauge equivalent square-zero elements in $\mathfrak{g}_L^{for}$. In particular the V-data $(\mathfrak{g}_L^{for}, \mathfrak{a}_L^{for}, P_L^{for}, \Delta_L^{for}) = (\mathfrak{g}_L, \mathfrak{a}_L, P_L, \iota_L, \Delta_L, \tau)$ induce isomorphic $L_\infty$-algebra structures on $\mathfrak{a}_L^{for}[-1] = \Omega^*(L,NL)[-1]$.

Proof. By definition of isotopic tubular neighborhoods, there exist a smooth family $\Phi_t$ of maps $\Phi_t: NL \to M$, with $t \in [0,1]$, which are diffeomorphisms on their images and such that $\Phi_t \circ s_0 = j$ for every $t \in [0,1]$, such that $\Phi_0 = \tau_0$ and $\Phi_1 = \tau_1$. Let $\tilde{\Phi}_t$ be the composition of $\Phi_t$ with the embedding $NL_{for} \hookrightarrow NL$ of the formal neighborhood $NL_{for}$ of $L$ into $NL$. Then $\tilde{\Phi}_t$ is a formal diffeomorphism between $NL_{for}$ and the formal neighborhood $\hat{L}_M$ of $L$ inside $M$. Let $\Delta_t^{for} = \tilde{\Phi}_t^*(\Psi|_{\hat{L}_M})$. Then $\Delta_{\tau_0}^{for} = \Delta_t^{for}$ and $\Delta_{\tau_1}^{for} = \Delta_{\tau_0}^{for}$. Moreover, writing $\hat{\Delta}_t^{for}$ for the formal diffeomorphism of $NL_{for}$ relative to $L$ given by $\hat{\Delta}_t^{for} = \tilde{\Phi}_t^{-1} \circ \Phi_t$, we have

$$\Delta_t^{for} = \hat{\Delta}_t^{for} = \hat{\Delta}_t^{for} (\hat{\Delta}_t^{-1} \circ \Phi_t)^*(\Psi|_{\hat{L}_M}) = \hat{\Delta}_t^{for} \Delta_0^{for}$$

As $\hat{\Delta}_0^{for} = id_{NL_{for}}$, differentiating the above equation with respect to $t$ we find

$$\frac{d}{dt} \Delta_t^{for} = L_{\xi_t} \Delta_t^{for},$$

where $L_{\xi_t}$ is the Lie derivative of the tensor field $\Delta_t^{for}$ with respect to the vector field $\xi_t = \frac{d}{dt} \hat{\Delta}_t^{for}$. For every $t$, the vector field $\xi_t$ is an element in $\Omega^0(NL_{for}, TNL_{for}) = (\mathfrak{g}_L^{for})_0$. Moreover, $L_{\xi_t} \Delta_t^{for} = [\xi_t, \Delta_t^{for}]^{FN}$. Thus, the family of elements $\Delta_t^{for}$ satisfies

$$\frac{d}{dt} \Delta_t^{for} = [\xi_t, \Delta_t^{for}]^{FN}$$

$$\Delta_0^{for} = \Delta_{\tau_0}^{for}$$

$$\Delta_1^{for} = \Delta_{\tau_1}^{for}$$

and it is therefore a gauge equivalence between $\Delta_{\tau_0}^{for}$ and $\Delta_{\tau_1}^{for}$ in $\mathfrak{g}_L^{for}$. The final part of the statement follows from the following
Proposition 4.12 (Cattaneo & Schätz, cf. [CS2008, Theorem 3.2]). Let \((g, a, \iota, P, \Delta)\) and \((g, a, \iota, P, \Delta_1)\) be V-data, and let \(a[-1]_0\) and \(a[-1]_1\) be the associated \(L_\infty\)-algebras. If \(\Delta_0\) and \(\Delta_1\) are gauge equivalent and they are intertwined by a gauge transformation preserving \(\ker P\), then \(a[-1]_0\) and \(a[-1]_1\) are \(L_\infty\)-isomorphic.

\[
\text{Definition 5.1.} \text{A smooth one-parameter family } L_{NL}\text{ is an infinitesimal } \Psi\text{-deformation.}
\]

Corollary 4.13. Let \(\tau_0\) and \(\tau_1\) be two isotopic tubular neighborhoods of \(L\) in \(M\). Then the V-data \((g_L, a_L, \iota_L, P_L, \Delta_L, \tau_0)\) and \((g_L, a_L, \iota_L, P_L, \Delta_L, \tau_1)\) induce isomorphic \(L_\infty\)-algebra structures on \(\Omega^*(L, NL)[-1]\).

**Proof.** Immediate from Corollary 4.10 and Lemma 4.11.

Putting Corollary and Corollary 4.13 together, we obtain the following statement, which is a rephrasing of Theorem 4.11.

Proposition 4.14. Let \(\Psi \in \Omega^*(M, TM)\) be an odd square-zero element, and \(L\) a \(\Psi\)-submanifold of \(M\). Then the \(\mathbb{Z}_2\)-graded \(L_\infty\)-algebra structure on \(\Omega^*(L, NL)[-1]\) induced by the V-data \((g_L, a_L, \iota_L, P_L, \Delta_L, \tau)\) is independent of the tubular neighborhood \(\tau\), up to isomorphism.

**Proof.** Given two tubular neighborhoods \(\tau_0\) and \(\tau_1\) of \(L\) in \(M\), there always exists a third tubular neighborhood \(\hat{\tau}_1\) such that \(\tau_0\) and \(\hat{\tau}_1\) are isotopic relative to \(L\) and \(\hat{\tau}_1 = \tau_1 \circ \psi\) for a suitable diffeomorphism of \(NL\) relative to \(L\), see, e.g., [H1997, Theorem 5.3].

5. Deformations of \(\Psi\)-submanifolds

5.1. **Smooth and infinitesimal deformations of \(\Psi\)-submanifolds.** Let \(\Psi \in \Omega^{2k-1}(M, TM)\) be an odd degree, square zero element and \(L\) a closed \(\Psi\)-submanifold. From now on, in this section, we use a tubular neighborhood \(\tau : NL \to U \subset M\) to identify the normal bundle \(NL\) with an open neighborhood \(U\) of \(L\) in \(M\) as in the proof of Theorem 4.11 and we work on \(NL\). For instance, abusing the notation, we regard \(\Psi\) (instead of \(\tau^*(\Psi)\)) as a square zero element in \(\Omega^*(NL, TNL)\). A \(C^1\)-small deformation of \(L\) in \(NL\) can be identified with a section \(L \to NL\). We say that a section \(s : L \to NL\) is a \(\Psi\)-section, if its image \(s(L)\) is a \(\Psi\)-submanifold in \(NL\).

**Definition 5.1.** A smooth one-parameter family \(\{s_t\}\) of smooth sections of the vector bundle \(NL \to L\) starting with \(t = 0\) from the zero section will be called a smooth \(\Psi\)-deformation of \(L\) if each section in the family is a \(\Psi\)-section. A section \(s : L \to NL\) will be called an infinitesimal \(\Psi\)-deformation of \(L\), if \(\varepsilon s\) is a \(\Psi\)-section up to infinitesimals \(O(\varepsilon^2)\), where \(\varepsilon\) is a formal parameter.

Clearly if \(\{s_t\}\) is a smooth \(\Psi\)-deformation, then the section \(\frac{ds_t}{dt}|_{t=0} : L \to NL\) is an infinitesimal \(\Psi\)-deformation.

- Denote by \(\pi : NL \to L\) the projection.
- Identify \(L\) with the image of the zero section \(0 : L \to NL\).
Recall $\iota_L : \Omega^*(L,NL) \to \Omega^*(NL,TNL)$ and $P_L : \Omega^*(NL,TNL) \to \Omega^*(L,NL)$ from the previous section.

- Given a section $s : L \to NL$ let $\{\psi_t\}$ be the flow on $NL$ generated by the vector field $\iota_L(s)$ and denote $\exp\iota_L(s) := \psi_1$.
- We define a map $F_{\Psi}: \Gamma(NL) \to \Omega^*(L,NL)$ as follows

\begin{equation}
F_{\Psi}(s) := P_L(\exp\iota_L(-s)^*\Psi).
\end{equation}

**Proposition 5.2.** Let $s : L \to NL$ be a section. Then $s$ is a $\Psi$-section, if and only if $F_{\Psi}(s) = 0 \in \Omega^*(L,NL)$.

**Proof.** Let $x \in L$. We begin with two simple remarks. First of all, for $v \in T_x L$ we have

\begin{equation}
\exp\iota_L(s)_*v = T_x s \cdot v.
\end{equation}

Second, let $w \in T_{s(x)} NL$. Then $w$ can be uniquely written as $w = w_s + w_N$ where $w_s$ is tangent to $s(L)$ and $w_N$ is a tangent vector vertical with respect to projection $NL \to L$. In particular $w_N$ is the vertical lift of a, necessarily unique, vector in $N_x L$ that we denote $w^L_N$. Finally, we have

\begin{equation}
\text{pr} \exp\iota_L(-s)_*w = w^L_N.
\end{equation}

Both (5.2) and (5.3) can be easily checked, e.g. in local coordinates. Now, we compute $F_{\Psi}(s)$ explicitly. So, let $v_1, \ldots, v_{2k-1} \in T_x L$. Then

\begin{align*}
F_{\Psi}(s)_x(v_1, \ldots, v_{2k-1}) &= P_L(\exp\iota_L(-s)^*\Psi)_x(v_1, \ldots, v_{2k-1}) \\
&= \text{pr} \exp\iota_L(-s)_*(\Psi_{s(x)}(\exp\iota_L(s)_*v_1, \ldots, \exp\iota_L(s)_*v_{2k-1})) \\
&= \text{pr} \exp\iota_L(-s)_*(\Psi_{s(x)}(T_x s \cdot v_1, \ldots, T_x s \cdot v_{2k-1})) \\
&= \Psi_{s(x)}(T_x s \cdot v_1, \ldots, T_x s \cdot v_{2k-1})^L_N.
\end{align*}

This shows that $F_{\Psi}(s) = 0$ if and only if $\Psi(w_1, \ldots, w_{2k-1})$ is tangent to $s(L)$ for all $w_1, \ldots, w_{2k-1}$ tangent to $s(L)$, i.e. $s(L)$ is a $\Psi$-submanifold. \hfill $\square$

**Corollary 5.3.** A smooth section $s : L \to NL$ is an infinitesimal $\Psi$-deformation of $L$ if and only if $\iota_1(s) = 0$.

For $\Psi \in \Omega(M,TM)$ denote by $\text{Diff}_\Psi(M)$ the subgroup of the diffeomorphism group $\text{Diff}(M)$ whose elements preserve $\Psi$.

**Definition 5.4.** Given $\Psi \in \Omega^*(M,TM)$ and a closed $\Psi$-submanifold $L$ we denote by $\mathcal{M}_\Psi(L)$ the set of all closed submanifolds in $M$ that are obtained from $L$ by smooth $\Psi$-deformations. We shall call $\mathcal{M}_\Psi(L)$ the pre-moduli space of $\Psi$-submanifolds in the connected component of $L$. The quotient $\mathcal{M}_\Psi(L)/\text{Diff}_\Psi(M)$ is the moduli space of $\Psi$-submanifolds in the connected component of $L$.

We will work with $\mathcal{M}_\Psi(L)$ only and will not discuss the moduli problem. But note that in most applications, $\text{Diff}_\Psi(M)$ is a (finite dimensional) Lie group.
Being a differential operator, the map $F_{\Psi} : \Gamma(NL) \to \Omega^1(L,NL)$ extends to a smooth map between the completion of $\Gamma(NL)$ and $\Omega^1(L,NL)$ with respect to $C^k$-norms, see e.g. Proposition [6.5] below. Hence the pre-moduli space $M_{\Psi}(L)$ inherits the $C^k$-topology from $C^k\Gamma(NL)$ for any $k \in [0,\infty]$.

5.2. Formal deformations of $\Psi$-submanifolds. Let $\varepsilon$ be a formal parameter.

Let us recall that a formal series $s(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i s_i \in \Gamma(NL)[[\varepsilon]]$, $s_i \in \Gamma(NS)$ such that $s_0 = 0$ is called a formal deformation of $L$.

Denote by $\mathfrak{X}(NL)$ the space of smooth vector fields on $NL$ and interpret it as the Lie algebra of derivations of the commutative algebra of smooth functions $C^\infty(NL)$. Then formal series $\iota_L(s(\varepsilon)) := \sum_{i=1}^{\infty} \varepsilon^i \iota_L(s_i) \in \mathfrak{X}(NL)[[\varepsilon]]$ extends to a derivation of formal functions $f(\varepsilon) := \sum_{i=0}^{\infty} \varepsilon^i f_i$, $f_i \in C^\infty(NL)$ as follows. For $\xi(\varepsilon) := \sum_{i=0}^{\infty} \varepsilon^i \xi_i$, where $\xi_i \in \mathfrak{X}(NL)$ we set

$$L_{\xi(\varepsilon)} f(\varepsilon) := \sum_{i=1}^{\infty} \varepsilon^k \sum_{i+j=k} L_{\xi_i} f_k.$$

We define the exponential of the Lie derivative $L_{\xi(\varepsilon)}$ as the following formal power series

$$\exp L_{\xi(\varepsilon)} := \sum_{n=0}^{\infty} \frac{1}{n!} L_{\xi(\varepsilon)}^n.$$

Proposition [5.2] motivates the following

**Definition 5.5.** A formal deformation of $L$ is said a $\Psi$-formal deformation, if $F_{\Psi}(s(\varepsilon)) := P(\exp L_{\iota_L(s(\varepsilon))}\Psi) = 0 \in \Omega^*(L,NL)[[\varepsilon]]$.

**Remark 5.6.** (cf. [LO2016], §10, [LOT2014], Remark 4.8, [LS2014], Definition 4.8) Let $s(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i s_i$ be a formal $\Psi$-deformation of $L$. Then $s_1$ is an infinitesimal $\Psi$-deformation. Given an infinitesimal $\Psi$-deformation $s$, we say that $s$ is unobstructed, if there exists a formal $\Psi$-deformation whose first coefficient $s_1$ is $s$. If all infinitesimal deformations are unobstructed, then we say that the formal deformation problem is unobstructed. Otherwise it is obstructed. Smooth obstructedness/unobstructedness are defined in a similar, obvious way.

**Proposition 5.7.** A formal deformation $s(\varepsilon)$ of $L$ is a formal $\Psi$-deformation if and only if $-s(\varepsilon)$ is a solution of the (formal) Maurer-Cartan equation

$$MC(s(\varepsilon)) := \sum_{k=1}^{\infty} I_k(s(\varepsilon), \ldots , s(\varepsilon)) = 0.$$

**Proof.** The expression $MC(s(\varepsilon)) = 0$ should be interpreted as an element of $\Omega^*(L,NL)[[\varepsilon]]$. Proposition [5.7] then follows from (5.4), Definition [5.5] and the following identity

$$P(L_{\iota_L(\xi)}^k \Psi) = I_k(\xi, \ldots , \xi), \quad k \geq 1,$$
for \( \xi \in \Gamma(NL) \), which immediately follows from the definition of \( I_k \) in Proposition 4.3.

\[ \square \]

6. Deformations of \( \varphi \)-calibrated submanifolds

In this section we consider smooth \( \hat{\varphi} \)-deformations of a closed \( \varphi \)-calibrated submanifold \( L \) where \( \varphi \) is a parallel calibration on a Riemannian manifold \( (M, g) \). First we prove that the space of all infinitesimal \( \hat{\varphi} \)-deformations of \( L \) coincides with the space of Jacobi vector fields on \( L \) regarding \( L \) as a minimal submanifold (Proposition 6.1). Then we prove our main theorem stating that the formal and smooth deformations of a closed \( \varphi \)-calibrated submanifold are encoded in its canonically associated \( \mathbb{Z}_2 \)-graded strongly homotopy Lie algebra (Theorem 6.4). In Remark 6.13 we discuss some related results. In the last subsection we revisit the deformation theory of complex submanifolds using the developed method in the present paper.

6.1. Infinitesimal deformations of \( \hat{\varphi} \)-submanifolds. Given a \( \varphi \)-calibrated submanifold \( L \) recall that \( M_{\hat{\varphi}}(L) \) is the pre-moduli space of all \( \varphi \)-calibrated submanifolds that are obtained from \( L \) by smooth \( \hat{\varphi} \)-deformations. By Corollary 3.5, \( M_{\hat{\varphi}}(L) \) consists of all closed minimal submanifolds that are obtained from \( L \) by smooth deformations. The following Lemma is an infinitesimal version of Corollary 3.5.

**Proposition 6.1.** The space \( J_{\hat{\varphi}}^2(L) \) of infinitesimal \( \hat{\varphi} \)-deformations of \( L \) coincides with the space \( J(L) \) of Jacobi fields on \( L \), regarding \( L \) as a minimal submanifold.

**Proof.** (1) Assume that \( L \) is a closed \( \varphi \)-calibrated submanifold and \( s \in \Gamma(NL) \) is an infinitesimal \( \hat{\varphi} \)-deformation. Let us recall that \( \psi_t = \exp(\iota_L(ts)) \).

Since \( L \) is compact there exist a positive number \( A \) and a positive number \( \varepsilon_0 \) such that, for any \( x \in L \)

\[ |pr((\psi_t)^*\hat{\varphi})|_L(x)| \leq A \cdot t^2 \]  

for any \( t \leq \varepsilon_0 \). Denote by \( G_\varphi(x) \) the space of unit decomposable \( l \)-vectors \( w \) in \( G_l(T_xM) \) such that \( \varphi(w) = 1 \). Denote by \( \rho \) the distance on the Grassmannian \( G_l(T_xM) \) induced by the Riemannian metric on \( T_xM \).

We shall abbreviate \( L_t := \psi_t(L), x_t := \psi_t(x) \) and \( \hat{\varphi}_t := (\psi_t)^*\hat{\varphi} \).

**Lemma 6.2.** The inequality (6.1) is equivalent to the existence of a positive number \( B \) and a positive number \( \varepsilon_1 \) such that

\[ \rho\left(T_{x_t}L_t, G_\varphi(x_t)\right) \leq B \cdot t^2 \]  

for all \( t \in (0, \varepsilon_1) \) (Recall that \( T_{x_t}L_t \) is the unit \( l \)-vector associated to the oriented tangent space \( T_xL \)).
Proof. Since $\psi_0 = Id$ we observe that $|\text{pr} \hat{\varphi}_t| = O(t^2)$ if and only $|\text{pr} \hat{\varphi}_{t,L}| = O(t^2)$. Recall that we denoted by $\hat{\varphi}(x)$ the form $\varphi$ (at the point $x$) regarded as a function on the Grassmannian $G_l(T_x M)$ of unit decomposable $l$-vectors. Then function $|d_w \hat{\varphi}(x)|$ is smooth in the variable $w \in G_l(T_x M)$. Since $d_w \hat{\varphi}(x) = 0$ if $w \in G_{\varphi}(x)$, this implies that there exist positive constants $C_1, C_2$ such that

$$C_1 \cdot |d_w \hat{\varphi}(x)| \leq \rho(w, G_{\varphi}(x)) \leq C_2 \cdot |d_w \hat{\varphi}(x)|.$$  

Now, Lemma 6.2 follows from the following easy equality, which is a corollary of (6.2),

$$|\text{pr} \hat{\varphi}|_{T_{x_t} L_t} = |d_{T_{x_t} L_t} \hat{\varphi}(x_t)|.$$  

It follows from Lemma 6.2 that there exist constant $C_3$ and a positive number $\varepsilon_2 < \varepsilon_1$ such that for all $x \in L$ and all $t \in (0, \varepsilon_2)$ we have

$$1 - C_3 t^4 \leq \langle \varphi, T_x L_t \rangle \leq 1 + C_3 t^4.$$  

Since $\psi_0 = Id$, there exist constants $C_4$ and a positive number $\varepsilon_3 < \varepsilon_2$ such that, for all $x \in L$ and all $t \in (0, \varepsilon_3)$, we have

$$1 - C_4 t \leq |(\psi_t)_* T_x L| \leq 1 + C_4 t.$$  

It follows from (6.5) and (6.6) that there exists a constant $C_5$ such that for all $x \in L$ and all $t \in (0, \varepsilon_3)$, we have

$$(1 - C_5 t^3) \langle \varphi, (\psi_t)_* (T_x L) \rangle \leq |(\psi_t)_* T_x L| \leq (1 + C_5 t^3) \langle \varphi, (\psi_t)_* (T_x L) \rangle.$$  

Finally, it follows from (6.7), and $\langle \varphi, (\psi_t)_* (T_x L) \rangle = \langle \psi_t^*(\varphi), T_x L \rangle$, that

$$\frac{d^2}{dt^2} |_{t=0} \text{vol}(\psi_t(L)) = \int_L \frac{d^2}{dt^2} |_{t=0} |(\psi_t)_* (T_x L)| \text{dvol}_L = \int_L \frac{d^2}{dt^2} |_{t=0} (\psi_t)^* \varphi = 0.$$  

Hence $s$ is a Jacobi vector field. This proves that $J_{\hat{\varphi}}(L) \subset J(L)$ as claimed.

(2) Conversely, assume that $s$ is a Jacobi vector field on $L$. Since $L$ is $\varphi$-calibrated by Remark 2.3 in [LV2017], $s$ is an infinitesimal deformation of $L$ as a $\varphi$-calibrated submanifolds. This is the same to say that (6.2) holds for same $B$ and $\varepsilon_0$. By Lemma 6.2 this implies that $s \in J_{\hat{\varphi}}(L)$. This concludes the proof of Proposition 6.1. \hfill \Box

Motivated by Proposition 6.1, in what follows, we shall call an infinitesimal $\hat{\varphi}$-deformation of a $L$, simply a Jacobi field on $L$. Recall that $J(L)$ is finite dimensional and its dimension is called the nullity of $L$ [Simons1968].

From Proposition 6.1 we obtain Corollary 6.3 below, which has been first proved by Simons in [Simons1968] Theorem 3.5.1] by computing the Jacobi operator on a compact Kähler submanifold $L$. Simons’ computation has
been generalized by McLean [McLean1998] for calibrated submanifolds, and simplified by Lê-Vanţura [LV2017], using different methods.

**Corollary 6.3.** Let \( L \) be a compact and closed Kähler submanifold in a Kähler manifold \((M, g, \omega^2)\). Then the nullity of \( L \) is equal to the dimension of the space of globally defined holomorphic sections in \( NL \).

### 6.2. Formal and smooth deformations of \( \varphi \)-calibrated submanifolds

The purpose of this section is to show the following result.

**Theorem 6.4 (Main Theorem).** Let \( L \) be a closed \( \varphi \)-calibrated submanifold in \((M, g, \varphi)\), where \( \varphi \) is a parallel calibration in a real analytic Riemannian manifold \((M, g)\).

1. The premoduli space \( \mathcal{M}_\varphi(L) \) is locally finite dimensional analytic variety.
2. A formally unobstructed Jacobi field \( s \in J(L) \) is smoothly unobstructed.
3. There is a canonical \( \mathbb{Z}_2 \)-graded strongly homotopy Lie algebra that governs formal and smooth deformations of \( L \) in the class of \( \varphi \)-calibrated submanifolds.

The assumption of the real analyticity of \((M, g)\) is rather mild. Indeed, the existence of a parallel form \( \varphi \) implies that the holonomy of \((M, g)\) is restricted. Thus, if \((M, g)\) is locally irreducible, this already implies the real analyticity of \((M, g)\). However, if, for instance, \( M = M_1 \times M_2 \) is a Riemannian product, then one could have a calibration of the form

\[
\text{vol}_{M_1} \wedge \alpha^k + \beta^{n_1+k},
\]

where \( \alpha^k, \beta^{n_1+k} \) are parallel forms on \( M_2 \). In this case, the analyticity of \((M_1, g_1)\) needs to be imposed.

**Proof of Theorem 6.4.** (1) We shall prove the first assertion of Theorem 6.4 in three steps. In the first step we assume that \((M, g, \varphi)\) is multi-symplectic, and construct a local analytical chart for \( \mathcal{M}_\varphi(L) \). In the second step we shall show that the coordinate transitions are analytic under these hypotheses, and in the third step, we show that the assumptions of multi-symplecticity of \( \varphi \) can be dropped.

Step 1. Assuming that \((M, g, \varphi)\) is analytic, a \( \varphi \)-calibrated submanifold \( L \subset M \) is minimal and whence analytic by the Morrey regularity theorem [Morrey1954, Morrey1958, Morrey2008].

We shall construct a local analytic chart on \( \mathcal{M}_\varphi(L) \) using the Inverse Function Theorem (IFT) for analytic mappings between real analytic Banach manifolds [Douady1966], see [LS2014 Appendix] for a short account, and employing many technical tools in the proof of Theorem 4.9 in [LS2014], see also the pioneering paper by Koiso [Koiso1983] for a close idea, which is different from [LS2014] (and the current proof) in technical tools, partly because we are working with different moduli problems. One of important technical points in our proof is to show the analyticity of certain mappings.
between the Banach spaces under consideration (Proposition 6.6). This will be done in three stages. First, using the analyticity of $\varphi$, we establish pointwise estimates which imply the analyticity of the associated mapping between certain Banach spaces (Lemma 6.8, Proposition 6.5). Then using the obtained estimates we prove the analyticity of the desired map. Finally we reduce the IFT from the infinite dimensional setting to a finite dimensional setting, using the overdetermined ellipticity of the map $F_\varphi$ (Lemma 6.10), and therefore complete the first step in the proof of the first assertion of Theorem 6.4.

It is well-known that there are two equivalent approaches to the definition of the $C^k$-norm (resp. $L^2_k$-norm) on the space $\Gamma(E)$ of smooth sections of a vector bundle $E$ over a compact Riemannian manifold $L$, see e.g. [DK1990, Appendix, Sobolev spaces, p. 421]. In the present paper, as in [LS2014], we choose the following approach. Choose local coordinates on $L$ and bundle trivializations of $E$ and for $p \in L$ and $s \in C^k\Gamma(E)$ set

$$(6.8) \quad \|s\|_{C^k,p} := \sum_{|I| \leq k} \|D_I s\|_p, \quad \text{and} \quad \|s\|_{C^k} := \sup_{p \in L} \|s\|_{C^k,p}$$

where the sum is taken over all multi-indexes $I$, and $D_I$ denotes multiple partial derivatives with respect to the given coordinates around $p$.

In what follows we denote by $C^k\Gamma(E)$ (resp. $L^2_k\Gamma(E)$) the completion of the space $\Gamma(E)$ with respect to the $C^k$-norm (resp. the $L^2_k$-norm).

Next for each $k \geq 1$ we shall define an analytic map $F_k$ from an open neighborhood of $0$ in $L^2_k\Gamma(NL)$ to $L^2_{k-1}\Omega^1(L,NL)$, whose zero set in a $C^1$-neighborhood of $0 \in \Gamma(NL)$ consists precisely of $\hat{\varphi}$-sections of $NL$ that are $C^1$-close to $0$. Moreover $F_{k+1} = F_k \circ e_{k+1,k}$ where $e_{k+1,k}$ denotes the natural embedding $L^2_{k+1}\Gamma(F) \to L^2_k\Gamma(F)$. Thus, later, we shall abuse the notation and write $F$ instead of $F_k$. We shall choose the map $F_k$ as the completion of a map $F_{\hat{\varphi}}$ (see (5.1)).

We begin noticing that, for $k \geq 1$, and for each $s \in C^k\Gamma(NL)$ the $C^k$-vector field $\iota_L(s)$ defines a $C^{k+1}$ flow, which is also denoted by $\{\psi_t\}$, on $NL$.

Further we denote again by $\exp \iota_L(s)$ the time-one map $\psi_1$ and define the map, abusing notation,

$$(6.9) \quad F_{\hat{\varphi}} : C^k\Gamma(NL) \to C^{k-1}\Omega^1(L,NL), \quad s \mapsto P_L(\exp \iota_L(-s)^*\hat{\varphi}).$$

**Proposition 6.5.** For each $k$ the map $F_{\hat{\varphi}}$ is analytic in an open neighborhood of $0 \in C^k\Gamma(NL)$.

**Proposition 6.6.** For each $k$ the map $F_{\hat{\varphi}} : C^k\Gamma(NL) \to C^{k-1}\Omega^1(L,NL)$ extends uniquely to a map $F : L^2_k\Gamma(NL) \to L^2_{k-1}\Omega^1(L,NL)$ which is analytic in an open neighborhood of $0 \in L^2_k\Gamma(NL)$.

**Proof of Proposition 6.5.** To prove Proposition 6.5, taking into account Proposition 6.3, rewritten as Lemma 6.7 below, it suffices to prove Lemma 6.8 below.
Lemma 6.7. ([LS2014, Lemma 6.2]) Let \( U \) be an open subset of a Banach space \( E \). A smooth mapping \( f \) from \( U \) to a Banach space \( F \) is analytic at a point \( x \in U \) if and only if there exists a positive number \( r \) depending on \( x \) such that the following holds. For any affine line \( l \) through \( x \) the restriction of \( f \) to \( l \cap U \) is analytic at \( x \) with radius of convergence at least \( r \).

Lemma 6.8. For each \( s \in C^k \Gamma(NL) \) and each \( p \in L \) the map \( t \mapsto F(ts)_p \in T_pL \otimes NL \) is real analytic at \( t = 0 \). Moreover there are constants \( A, K > 0 \) such that for all \( m \geq 1 \) we have the following pointwise estimate

\[
\left\| \frac{d^m}{dt^m}|_{t=0} F(ts) \right\|_{C^{k-1,p}} \leq m! AK^m \| s \|^m_{C^k,p}.
\]

Proof of Lemma 6.8. Lemma 6.8 is an analogue of Lemma 4.10 in [LS2014] and will be proved in a similar way. For a given \( p \in L \) let us choose analytic coordinates \( (x^i, y^j) \) in a neighborhood \( O_L(p) \) of \( p \) in \( NL \) so that \( (x^i) \) are analytic coordinates on \( L \) and \( (y^j) \) are linear fiber coordinates. Locally

\[
\phi = \sum_{p+q=l-1} f_{i_1 \cdots i_p, r_1 \cdots r_q}^i(x,y)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dy^{r_1} \wedge \cdots \wedge dy^{r_q} \circ \frac{\partial}{\partial y^r} + \sum_{p+q=l-1} f_{i_1 \cdots i_p, r_1 \cdots r_q}^i(x,y)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dy^{r_1} \wedge \cdots \wedge dy^{r_q} \circ \frac{\partial}{\partial x^{i_1}},
\]
or, shortly,

\[
\phi = \sum_{|I|+|R|=l-1} dx^I \wedge dy^R \otimes \left( f^I_{1;R}(x,y) \frac{\partial}{\partial y^r} + f^I_{1;R}(x,y) \frac{\partial}{\partial x^{i_1}} \right)
\]

where \( I, R \) are skew-symmetric multi-indexes. As \( \phi \) is analytic, then functions \( f^I_{1;R}(x,y), f^I_{1;R}(x,y) \) are analytic.

Since estimates in Lemma 6.8 are derived at \( t = 0 \), without loss of generality we assume that the graph of the section \( ts \), for \( t \in [-1, 1] \), over \( O_L(p) \cap L \), belongs to \( O_L(p) \), and hence we can write \( s \) in coordinates as follows

\[
s = s^r(x) \frac{\partial}{\partial y^r}.
\]

A straightforward computation then shows that

\[
P_L(\exp(\imath L(s))^* \phi)_x
\]

\[
= \sum_{|I|+|R|=l-1-1} \left( f^I_{1;R}(x,s(x)) - f^I_{1;R}(x,s(x)) \frac{\partial s^r}{\partial x^{i_1}}(x) \right) dx^I \wedge dy^R \otimes \frac{\partial}{\partial y^r}
\]

where, for \( R = r^1 \cdots r^p \) we set

\[
ds^R = ds^{r_1} \wedge \cdots \wedge ds^{r_p} = \frac{\partial s^{r_1}}{\partial x^{i_1}} \cdots \frac{\partial s^{r_p}}{\partial x^{i_p}} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.
\]
From (6.14) and (6.15), for a fixed \( x \in L \) and a given \( s \), the map \( t \mapsto F(ts)_x \) yields an analytic curve (in fact a polynomial curve) in \( \wedge^{l-1}T^*_xL \otimes N_xL \). This proves the first assertion of Lemma 6.8.

Let us now prove the second assertion. We can assume that \( O_L(p) \) is contained in a compact subset of a larger coordinate domain so that all the local functions \( f_{I;R}(x,y) \), in (6.12) are bounded. Let us preliminarily recall the following characterization of analytic functions via estimates on its derivatives.

**Proposition 6.9.** ([KP2002] Proposition 2.2.10) Let \( f \in C^\infty(U) \) for some open \( U \subset \mathbb{R}^n \). The function \( f \) is in \( C^\omega(U) \) if and only if for each \( \alpha \in U \), there are an open ball \( V \), with \( \alpha \in V \subset U \), and constants \( C > 0 \) and \( R > 0 \) such that the derivatives of \( f \) satisfy

\[
\left\| \frac{\partial^{\mu} f}{\partial x^{\mu}}(x) \right\| \leq C \cdot \frac{\mu!}{R^{\|\mu\|}} \text{ for all } x \in V,
\]

and all multi-indexes \( \mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{Z}^+)^n \), where, as usual, \( \mu! = \mu_1! \cdots \mu_n! \), \( |\mu| = \mu_1 + \cdots + \mu_n \), and \( x^\mu = (x^1)^{\mu_1} \cdots (x^n)^{\mu_n} \).

Applying Proposition 6.9 to the local functions \( f_{I;R}(x,y) \) we conclude that there exist positive numbers \( A_1, K_1 \) independent of \( m \) and satisfying the following pointwise estimates

\[
|X^m(f_{I;R}(x,y))| \leq m! A_1 K_1^m |X|^m
\]

for all sufficiently differentiable vector field \( X = a^r(x) \frac{\partial}{\partial y^r} \). In what follows we shall apply (6.17) with \( X = \iota_L(s) \) to estimate the left hand side of (6.10). Note that for \( t \) small enough we have

\[
F(-ts)_x = \sum_{|I| + |R| = l - 1} \left( t^{[R]} f_{I;R}(x, ts(x)) - t^{[R]} f_{I;R}(x, ts(x)) \frac{\partial s^r}{\partial x^i}(x) \right) dx^I \wedge ds^R \otimes \frac{\partial}{\partial y^r}.
\]

Applying the identity

\[
\frac{d^m}{dt^m} f(x, ts(x)) = (\iota_L(s)^m f)(x, 0),
\]

we obtain

\[
\frac{d^m}{dt^m} F(-ts)_x|_{t=0} = \sum_{|I| + |R| = l - 1} \left( \frac{m!}{(m - |R|)!} (\iota_L(s)^m - |R| f_{I;R})(x, 0) - \frac{m!}{(m - |R|)!} (\iota_L(s)^m - |R| f_{I;R})(x, 0) \frac{\partial s^r}{\partial x^i}(x) \right) dx^I \wedge ds^R \otimes \frac{\partial}{\partial y^r},
\]

where we set \( m!/(m - q)! = 0 \) whenever \( q \geq m + 1 \).
By definition of the $C^k$-norm we have
\[ (6.21) \quad \left\| \frac{\partial s^r}{\partial x^i} \right\| \leq \|s^r\|_{C^k,p}. \]

Finally, the existence of the required constants $A$ and $K$ in (6.10) can be easily derived from (6.17), (6.19), (6.20), (6.21), taking into account
\[ \|s^1 \cdot s^2\|_{C^k} \leq \|s^1\|_{C^k} \cdot \|s^2\|_{C^k}. \]

This completes the proof of Lemma 6.8.

As we have noted above, the proof of Lemma 6.8 completes the proof of Proposition 6.5.

Proof of Proposition 6.6. Lemma 6.8 implies that the map $\hat{F}_\varphi$ can be expressed as a power series at a neighborhood of $0 \in C^k\Gamma(NL)$ as follows
\[ (6.22) \quad \hat{F}_\varphi(s) = \sum_{n=1}^{\infty} P_L(\mathcal{L}_n(s)^n) \]
that satisfies the following estimate at every point $p \in L$
\[ (6.23) \quad \|P_L(\mathcal{L}_n(s)^n)\|^2_{C^{k-1},p} \leq n!AK^n\|s\|_{C^k,p}. \]

It follows that $\hat{F}_\varphi$ extends to a map $F: L^2_k\Gamma(NL) \to L^2_{k-1}\Omega^1(L,NL)$ such that
\[ F(s) = \sum_{n=1}^{\infty} P_L(\mathcal{L}_n(s)^n) \]
where
\[ \|P_L(\mathcal{L}_n(s)^n)\|_{L^2_{k-1}(\Gamma(NL))} \leq n!AK^n\|s\|_{L^2_{k}\Omega^1(L,NL)}. \]

This completes the proof of Proposition 6.6.

Completion of Step 1 in the proof of Theorem 6.4 (1). Our next aim is to construct a finite dimensional model for a $C^1$-neighborhood of $L$ in $M_{\hat{\varphi}}(L)$ by applying the IFT to the analytic map $F$ in Proposition 6.6. The following Lemma plays a key role in this construction by reducing the IFT in the infinite dimensional setting to an IFT in a finite dimensional setting.

Lemma 6.10. The differential $d_0 F : \Gamma(NL) \to \Omega^{l-1}(L,NL)$ of $F$ at the zero section $0$ is an overdetermined elliptic linear operator.
Proof of Lemma 6.10 Using (6.20) we get, for $s \in \Gamma(NL)$,

\[
d_0 F(s) := -\frac{d}{dt} P_L(\exp \tau_L(ts)^* \varphi) = -\sum_{|I|=l-1} \left( \left( \tau_L(s)f_{I,0}(x,0) - f_{I,0}(x,0)\frac{\partial s^r}{\partial x^i}(x) \right) dx^I \otimes \frac{\partial}{\partial y^r} \right) - \sum_{|J|=l-2} f_{J,u}(x,0)dx^j \otimes ds^u \otimes \frac{\partial}{\partial y^r}.
\]

Now, let $\xi \in T^*_x L$, and compute the symbol

\[
(\sigma_\xi d_0 F)(s) = \sum_{|I|=l-1} \xi_i f_{1,0}(x,0)dx^I \otimes s(x) = \sum_{|J|=l-2} f_{J,u}(x,0)s^u(x)dx^j \wedge \xi \otimes \frac{\partial}{\partial y^r}.
\]

We shall show that $\ker \sigma_\xi d_0 F = 0$ for any $\xi \neq 0$. Without loss of generality we assume $(x'; y')$ are geodesic normal coordinates centered at $x$, and that $\xi = dx^1$. So, since $\varphi(y) = \text{vol}_L$, and the coordinate vectors are orthonormal at $x$ we have that $f_{1,0}(x,0) \neq 0$ whenever $1$ does not appear in $I$. It follows that, if $(\sigma_\xi d_0 F)(s) = 0$, then

\[
f_{1,0}(x,0)dx^2 \wedge \cdots \wedge dx^l \otimes \varphi = 0,
\]

hence $s(x) = 0$. This completes the proof of Lemma 6.10

Since $d_0 F$ is overdetermined elliptic, we have the following orthogonal decomposition with respect to the $L_2$-norm (see e.g. [Besse1987 Corollary 32, p. 464])

\[
L^2_{k-1} \Omega^{l-1}(L, NL) = d_0 F(L^2_k \Gamma(NL)) \oplus (\ker(d_0 F)^* \cap L^2_{k-1} \Omega^{l-1}(L, NL)).
\]

- Let $\Pi_1 : L^2_{k-1} \Omega^{l-1}(L, NL) \rightarrow d_0 F(L^2_k \Gamma(NL))$ be the orthogonal projection with respect to the decomposition in (6.26).
- Let $U_k(0)$ denote an open neighborhood of $0$ in $L^2_k(\Gamma(NL))$ such that the restriction of the map $F$ to $U_k(0)$ is analytic. The existence of $U_k(0)$ is ensured by Lemma 6.7.
- Denote by $\pi_k : L^2_k \Gamma(NL) \rightarrow J(L)$ the orthogonal projection. It is continuous, since $J$ is finite dimensional. Then we set

\[
\hat{F}_k := \pi_k \oplus (-\Pi_1 \circ F) : L^2_k(\Gamma(NL)) \supset U_k(0) \rightarrow J(L) \oplus d_0 F(L^2_k \Gamma(NL)).
\]

The map $\hat{F}_k$ is analytic in $U_k(0)$ and its differential at $0$ is an isomorphism. Therefore the IFT for analytic mappings of Banach spaces implies that there is an analytic inverse of $\hat{F}_k$

\[
G_k : V_k(0,0) \rightarrow U_k(0)
\]

where $V_k(0,0)$ is an open neighborhood of $(0,0) \in J(L) \oplus d_0 F(L^2_k \Gamma(NL))$. 


Let $V^J_k(0,0) := V_k(0,0) \cap (J(L),0)$. It is not hard to shown that $V^J_k(0,0)$ is independent of $k$. Next we define the map
\begin{equation}
\tau : V^J_k(0,0) \rightarrow J(L), \quad s \mapsto \pi_k \circ G_k(s) - i(s)
\end{equation}
where $i : (J(L),0) \rightarrow J(L)$ is the natural identification map.

**Lemma 6.11.** (1) The map $\tau$ is an analytic map.

(2) An element $y \in V^J_k(0,0)$ belongs to $\tau^{-1}(0)$ if and only if $y = i^{-1} \circ \pi(z)$ for some $z \in (\Pi_1 \circ F)^{-1}(0)$.

(3) The restriction of the projection $\tau$ to $F^{-1}(0) \cap U_k(0)$ is injective.

In Lemma 6.11 we write $\pi$ instead of $\pi_k$ since $\pi_{k+1} = \pi_k \circ e_{k+1,k}$, but we also use the notation $\pi_k$ when it makes things more clear.

**Proof of Lemma 6.11.** (1) The first assertion holds since $\pi_k$ and $G_k$ are analytic maps.

(2) Let us prove the “if” part of the second assertion. Assume that $y = i^{-1} \circ \pi_k(z)$ and $\Pi_1 \circ F(z) = 0$. Then $\tilde{F}_k(z) = i^{-1} \circ \pi_k(z) = y$. It follows $z = G_k(y)$ and $\tau(y) = \pi_k \circ G_k(y) - i(y) = \pi_k(z) - \pi_k(z) = 0$, which proves the “if” assertion.

Now assume that $\tau(y) = 0$. Then $\pi_k \circ G_k(y) = i(y)$. Set $z = G_k(y)$. Then $\tilde{F}_k(z) = y = \pi_k(z)$ and therefore $\Pi_1 \circ F(z) = 0$.

(3) The last assertion holds since $\tilde{F}_k$ is invertible. \hfill $\square$

We now consider the restriction $F : \tau^{-1}(0) \rightarrow \ker \Pi_1$. Since $\tau^{-1}(0)$ is a real analytic variety, the ring of germs of analytic functions at $0$ is Noetherian [Frisch1967, Theorem I.9], whence $F^{-1}(0)$ is given as the $0$-set of finitely many (say $N$) analytic functions. In other words there is an analytic function
\[ \tilde{\tau}^k : V^J_k(0,0) \rightarrow J(L) \oplus \mathbb{R}^N \]
for some finite number $N$ such that $F^{-1}(y) = 0$ iff $y = i^{-1} \circ \pi(z)$ for some $z \in F^{-1}(0)$. This allows to identify the $C^1$-neighborhood $F^{-1}(0) \cap U_k(0)$ of $L$ in the pre-moduli space $\mathcal{M}_\varphi(L)$ with the pre-image $\tilde{\tau}^{-1}(0)$ in the neighborhood of $0 \in J(L)$ via the map $i^{-1} \circ \pi : F^{-1}(0) \cap U_k(0) \rightarrow \tilde{\tau}^{-1}(0)$, where $\tilde{\varphi}$ is an analytic map between open neighborhoods of finite dimensional vector spaces.

Since $F^{-1}(0) \cap U_k(0)$ is independent of $k$ and models a $C^1$-neighborhood $U(L)$ of $L$ in $\mathcal{M}_\varphi(L)$ this completes Step 1 in the proof of the first assertion of Theorem 6.4.

**Step 2.** In this step we shall prove the following

**Lemma 6.12.** Assume that $L_1$ belongs to a $C^1$-neighborhood $U(L) \subset \mathcal{M}_\varphi(L)$ provided with an analytic chart constructed as in Step 1. Then there is a $C^1$-neighborhood $U_1(L_1) \subset U(L) \subset \mathcal{M}_\varphi(L)$ of $L_1$ such that the analytic structure on $U_1(L_1)$ constructed via maps $\pi^{L_1}$, $\tau^{L_1}$ (as in Lemma 6.11 but with respect to $L_1$) is equivalent to the analytic structure induced from the one on $U(L)$. In other words any two analytic charts are compatible.
Proof. Since \( L_1 \) lies in a \( C^1 \)-neighborhood of \( L \), and \( L_1 \) is a minimal submanifold, we can write \( L_1 = s(L) \) for some analytic section \( s \) of \( NL \). Then \( s \) induces, via \( \exp \iota_L(s) \), an invertible analytic map between Sobolev spaces \( L^2_k(\Gamma(NL)) \) and \( L^2_k(\Gamma(NL_1)) \) as well as an invertible analytic map between \( L^2_k(\Omega^{l-1}(L,NL)) \) and \( L^2_k(\Omega^{l-1}(L,NL)) \), providing the required equivalence. This completes the proof of Lemma 6.12 and the proof of step 2 in the first assertion of Theorem 6.4. \( \square \)

Step 3. Let \( N_\varphi := \{ \xi \in TM \mid \iota_\xi \varphi = 0 \} \). Since \( \varphi \) is parallel, \( N_\varphi \) is a parallel distribution, and for any \( \varphi \)-calibrated submanifold \( L \subset M \), the normal bundle \( NL \) contains \( N_\varphi \) as a parallel subbundle. That is, again identifying \( NL \) with a tubular neighborhood of \( L \), we may decompose \( NL = N_\varphi \| N'L \) as a Riemannian manifold, and the restriction of \( \varphi \) to \( N'L \) is a parallel multisymplectic calibration. In particular, the vector bundle \( \pi : NL \to NL' \) whose fiber equals \( N_\varphi \) is equipped with a flat connection whose (discrete) holonomy group \( \Gamma \) acts on \( (N_\varphi)_p \) for \( p \in N'L \). We let 

\[ V \subset N_\varphi \]

be the subbundle on which \( \Gamma \) acts trivially. Clearly, the restriction \( \pi|_V : V \to N'L \) is a trivial vector bundle with a parallel flat connection and hence a canonical trivialization

\begin{equation}
V \leftrightarrow V_0 \times N'L,
\end{equation}

where \( V_0 \) is a vector space isomorphic to \( V_p \) for any \( p \in N'L \). From the definition of calibrated submanifolds, \( \hat{L} \subset NL \) is \( \varphi \)-calibrated iff \( L' := \pi(\hat{L}) \subset N'L \) is \( \varphi \)-calibrated and \( \hat{L} \) is a parallel submanifold lifting \( L' \). In particular, \( \pi|_{\hat{L}} : \hat{L} \to L' \) is a covering map. But if \( \hat{L} \in M_L(NL) \) is a perturbation of \( L \), then \( \pi|_{\hat{L}} : \hat{L} \to L' \) is a diffeomorphism, so that \( \hat{L} \to L' \) is a parallel section and hence of the form

\[ \hat{L} = \{ v \} \times L' \subset V_0 \times N'L \cong V, \]

for \( v \in V_0 \), using the identification \( \ref{eq:6.29} \) where \( L' \in M_L(N'L) \) is a \( \varphi \)-calibrated perturbation of \( L \subset NL' \). As a result, we conclude that the pre-moduli space locally has the form

\[ M_L(NL) = M_L(N'L) \times V_0, \]

and since \( \varphi|_{NL'} \) is multi-symplectic, our previous discussion yields that \( M_L(N'L) \) is an analytic space, whence so is \( M_L(NL) \). This completes the proof of (1).

(2) The second assertion of Theorem 6.4 is a corollary of the first assertion and the Artin’s approximation theorem \[\text{[Artin1968, Theorem 1.2]}, \]
which says that, in a finite dimensional analytic space, smooth and formal obstructedness are equivalent.

(3) Assume that $L$ is a $\varphi^{2k}$-calibrated submanifold. Then the last assertion of Theorem 6.4 for $L$ follows from the second assertion, Proposition 5.7 and Corollary 3.5.

Now assume that $L$ is a $\varphi^{2k-1}$-calibrated submanifold. Then $L \times S^1$ is a $\varphi^{2k-1} \land dt$-calibrated submanifold in $(M \times S^1, g + dt^2, \varphi \land dt)$. It is not hard to see that, if $\tilde{L}_t$ is a smooth deformation of $L \times S^1$ in the class of minimal submanifolds in $M \times S^1$, then $\tilde{L}_t = L_t \times S^1$ for some (family of) $\varphi$-calibrated submanifold(s) $L_t$. Hence the formal and smooth deformations of $\varphi^{2k-1}$-calibrated submanifold are governed by the $\mathbb{Z}_2$-graded strongly homotopy Lie algebra associated to $L \times S^1$. This completes the proof of Theorem 6.4.

\[\square\]

**Remark 6.13.**

1. Theorem 6.4 is also valid for open $\varphi$-calibrated submanifolds with compact support variation fields.

2. Theorem 6.4 is also valid for $\hat{\varphi}^k$-manifolds $L^k$ such that $\varphi^k_{|L^k} \neq 0$.

3. Assume that $L$ is simultaneously a $\varphi$-calibrated submanifold and a $\varphi'$-calibrated submanifold where $\varphi$ and $\varphi'$ are calibrations on $(M, g)$. Then any $\varphi$-calibrated closed submanifold $L'$ that is homologous to $L$ is also a $\varphi'$-calibrated submanifold. This implies that deformations of such calibrated submanifolds are easier to control. For example, let $(M^6, g, \omega^2, \alpha = \text{Re} \text{vol}_C)$ be a Calabi-Yau 6-manifold and $C \subset (M^6, g, \omega^2, \alpha)$ a complex curve. Clearly the product $L := S^1 \times C$ is simultaneously calibrated with respect to both the associative calibration $\varphi := dt \land \omega^2 + \alpha$ and the calibration $dt \land \omega^2$. Hence any deformation $L'$ of $L$ in the class of associative submanifolds is also calibrated by $dt \land \omega$. In particular $L'$ is invariant under the flow generated by the vector field $\partial_t$. This flow preserves the Calabi-Yau structure on each slice $\{t\} \times M^6$. We conclude that all the slices $L' \cap \{t = \text{constant}\}$ are isomorphic as complex curves in $M^6$. It follows that $L' = S^1 \times C'$, where $C'$ is a complex deformation of $C$. In particular, if $C$ is isolated then $L$ is isolated. The last assertion has been obtained in [CHNP2012, Lemma 5.11] by computing the kernel of the corresponding linearized operators that control the corresponding deformations. In [Leung2002] Leung studies deformation of simultaneous calibrated submanifolds using integral estimates. We refer the interested reader to [Le1993] for the relation between calibration method and integral estimate method in the theory of minimal submanifolds.

4. As we have noted in Corollary 4.2 there are two natural $\mathbb{Z}_2$-graded strongly homotopy Lie algebras associated to an associative submanifold $L$ in a $G_2$-manifold $(M^7, g, \varphi)$. It is known that smooth and infinitesimal $\chi$-deformation of $L$ are exactly smooth and infinitesimal deformations of $L$ as a minimal submanifold [McLean1998, LV2017]. Thus the strong homotopy Lie algebra attached to $L$ via $\chi$ also governs smooth and formal deformations of $L$ as a $\varphi$-calibrated submanifold.
5. The action of $\text{Diff}_\Psi(M)$ preserves the analytic structure of on $\mathcal{M}_\Psi(L)$, whence the moduli space $\mathcal{M}_\Psi(L)/\text{Diff}_\Psi(M)$ is a locally analytic variety as well. In particular, generic points of $\mathcal{M}_\Psi(L)$ or $\mathcal{M}_\Psi(L)/\text{Diff}_\Psi(M)$, respectively, are smooth, and hence (formally) unobstructed.

6.3. Deformations of complex submanifolds revisited.

Theorem 6.14. Assume that $L$ is a closed complex submanifold in a complex manifold $(M,J)$.

(1) There exists a $C^1$-neighborhood $U_{\mathcal{M}_\Psi}(L)$ of $L$ in $\mathcal{M}_\Psi(L)$ which has a structure of a finite dimensional analytic variety.

(2) A formally unobstructed holomorphic normal field is smoothly unobstructed.

(3) There is a canonical $\mathbb{Z}$-graded strongly homotopy Lie algebra that governs formal and smooth complex deformations of $L$.

Proof of Theorem 6.14. Theorem 6.14 is proved in the same way as Theorem 6.4 and we omit the proof. □

Remark 6.15. Deformations of complex submanifolds has been examined by Ji in [Ji2014] using his general theory of deformations of Lie subalgebroid. Since the Frölicher-Nijenhuis bracket of $J \in \Omega^*(M, TM)$ is $-i/2$-times the Dolbeault operator $\bar{\partial}$, Ji’s strongly homotopy Lie algebra is the same as ours up to an uninfluential global factor (see also [Manetti2007] for an equivalent formulation).

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