GALOIS ORDERS

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Abstract. We introduce a new class of noncommutative rings - Galois orders, realized as certain subrings of invariants in skew semigroup rings, and develop their structure theory. The class of Galois orders generalizes classical orders in noncommutative rings and contains many classical objects, such as the Generalized Weyl algebras, the universal enveloping algebra of the general linear Lie algebra, associated Yangians and finite $W$-algebras and certain rings of invariant differential operators on algebraic varieties.

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1. Introduction

Let $\Gamma$ be an integral domain and $U \supset \Gamma$ an associative noncommutative algebra over a base field $k$. A motivation for the study of pairs "algebra-subalgebra" comes from the representation theory of Lie algebras. In particular, in the theory of Harish-Chandra modules $U$ is the universal
enveloping algebra of a reductive finite dimensional Lie algebra \( L \) and \( \Gamma \) is the universal enveloping algebra of some reductive Lie subalgebra \( L' \subset L \). For instance, the case when \( \Gamma \) is the universal enveloping algebra of a Cartan subalgebra leads to the theory of Harish-Chandra modules with respect to this Cartan algebra - weight modules. Another important example is a pair \((U, \Gamma)\), where \( U \) is the universal enveloping algebra and \( \Gamma \) is a certain maximal commutative subalgebra of \( U \), called Gelfand-Tsetlin subalgebra. In the case \( U = U(\mathfrak{sl}_n) \) the analogs of Harish-Chandra modules - Gelfand-Tsetlin modules - were studied in [DFO1]. Similarly, Okounkov and Vershik ([OV]) showed that representation theory of the symmetric group \( S_n \) is associated with a pair \((U, \Gamma)\), where \( U \) is the group algebra of \( S_n \) and \( \Gamma \) is the maximal commutative subalgebra generated by the Jucys-Murphy elements.

An attempt to understand the phenomena related to the Gelfand-Tsetlin formulae ([GTs]) was the paper [DFO2] where the notion of Harish-Chandra subalgebra of an associative algebra and the corresponding notion of a Harish-Chandra module were introduced. In particular, in [DFO2] the categories of Harish-Chandra modules were described as categories of modules over some explicitly constructed categories. This construction is a broad generalization of the presentation of finite dimensional associative algebras by quivers and relations. This techniques was applied to the study of Gelfand-Tsetlin modules for \( \mathfrak{gl}_n \).

Current paper can be viewed on one hand as a development of the ideas of [DFO2] in the "semi-commutative case", i.e. noncommutative algebra and commutative subalgebra and, on the other hand, as an attempt to understand the role of skew group algebras in the representation theory of infinite dimensional algebras (e.g. [Bl], [Ba], [BavO], [Ex]). Recall, that the algebras \( A_1, U(\mathfrak{sl}_2) \) and their quantum analogues are unified by the notion of a generalized Weyl algebra. Their irreducible modules are completely described modulo classification of irreducible elements in a skew polynomial ring in one variable over a skew field. The main property of a generalized Weyl algebra \( U \) is the existence of a commutative subalgebra \( \Gamma \subset U \) such that the localization of \( U \) by \( S = \Gamma \setminus \{0\} \) is the skew polynomial algebra. On the other hand this technique can not be applied in case of more complicated algebras such as the universal enveloping algebras of simple Lie algebras of rank \( \geq 2 \).

We make an important observation that the Gelfand-Tsetlin formulae for \( \mathfrak{gl}_n \) define an embedding of the corresponding universal enveloping algebra into a skew group algebra of a free abelian group over some field of rational functions \( L \) (see also [Kh]). A remarkable fact is that this field \( L \) is a Galois extension of the field of fractions of the corresponding Gelfand-Tsetlin subalgebra of the universal enveloping algebra. This fact leads to a concept of Galois orders defined as certain subrings of invariants in skew semigroup rings.

We propose a notion of a "noncommutative order" as a pair \((U, \Gamma)\) where \( U \) is a ring, \( \Gamma \subset U \) a commutative subring such that the set \( S = \Gamma \setminus \{0\} \) is left and right Ore subset in \( U \) and the corresponding ring of fractions \( \mathcal{U} \) is a simple algebra (in general, \( \Gamma \) is not central in \( U \)). Galois orders introduced in the paper are examples of such noncommutative orders.

Let \( \Gamma \) be a commutative finitely generated domain, \( K \) the field of fractions of \( \Gamma \), \( K \subset L \) a finite Galois extension, \( G = G(L/K) \) the corresponding Galois group, \( \mathcal{M} \subset \text{Aut} L \) a submonoid. Assume that \( G \) belongs to the normalizer of \( \mathcal{M} \) in \( \text{Aut} L \) and for \( m_1, m_2 \in \mathcal{M} \) their double \( G \)-cosets coincide if and only if \( m_1 = gm_2g^{-1} \) for some \( g \in G \). If \( \mathcal{M} \) is a group the last condition can be rewritten as \( \mathcal{M} \cap G = \{e\} \). If \( G \) acts on \( \mathcal{M} \) by conjugation then \( G \) acts on the skew group algebra \( L \ast \mathcal{M} \) by automorphisms: \( g \cdot (am) = (g \cdot a)(g \cdot m) \). Let \( \mathcal{K} = (L \ast \mathcal{M})^G \) be the subalgebra of \( G \)-invariants in \( L \ast \mathcal{M} \).

We will say that an associative ring \( U \) is a \( \Gamma \)-ring, provided there is a fixed embedding \( \iota : \Gamma \to U \).

**Definition 1.** A finitely generated (over \( \Gamma \)) \( \Gamma \)-subring \( U \subset \mathcal{K} \) is called a Galois \( \Gamma \)-ring (or Galois ring over \( \Gamma \)) if \( KU = UK = \mathcal{K} \).

If \( \Gamma \) is fixed then we simply say that \( U \) is a Galois ring.

In this case \( U \cap K \) is a maximal commutative subring in \( U \) and the center of \( U \) coincides with \( \mathcal{M} \)-invariants in \( U \cap K \) (Theorem 4.1). Moreover, the set \( S = \Gamma \setminus \{0\} \) is an Ore multiplicative set (both from the left and from the right) and the corresponding localizations \( U[S^{-1}] \) and \( S^{-1}U \) are canonically isomorphic to \( \mathcal{K} \) (Proposition 4.2). Note that the algebra \( \mathcal{K} \) has a canonical
decomposition into a sum of pairwise non-isomorphic finite dimensional left and right \(K\)-bimodules (cf. (2.6)). We introduce a class of Galois orders with some integrality conditions - integral Galois rings or Galois orders. These rings satisfy some local finiteness condition and they are defined as follows.

**Definition 2.** A Galois ring \(U\) over \(\Gamma\) is called right (respectively left) integral Galois ring, or Galois order over \(\Gamma\), if for any finite dimensional right (respectively left) \(K\)-subspace \(W \subset U[S^{-1}]\) (respectively \(W \subset [S^{-1}]U\)), \(W \cap U\) is a finitely generated right (respectively left) \(\Gamma\)-module. A Galois ring is Galois order if it is both right and left Galois order.

A concept of a Galois order over \(\Gamma\) is a natural noncommutative generalization of a classical notion of \(\Gamma\)-order in skew group ring \(K\) since we do not require the centrality of \(\Gamma\) in \(U\) (cf. [MCR], chapter 5, 3.5). We note the difference of our definition from the notion of order given in [MCR] (chapter 3, 1.2), [HGK] (section 9).

How big is the class of Galois rings and orders? We note that any commutative algebra is Galois over itself. If \(\Gamma \subset U \subset K \subset L\) and \(U\) is finitely generated over \(\Gamma\), then \(U\) is a Galois \(\Gamma\)-ring. If \(\Gamma\) is noetherian then \(U\) is an order if and only if \(U\) lies in the integral closure of \(\Gamma\) in \(K\). In Section 3 we study so-called balanced \(\Gamma\)-bimodules. This approach, based on the bimodule theory, allows to understand the structure and to construct Galois rings. Another important tool in the study of Galois rings is their Gelfand-Kirillov dimension, cf. Section 6. Using the results of Section 6 we show in Section 7 that the following algebras are integral Galois orders in corresponding skew group rings:

- Generalized Weyl algebras over integral domains with infinite order automorphisms which include many classical algebras, such as \(n\)-th Weyl algebra \(A_n\), quantum plane, \(q\)-deformed Heisenberg algebra, quantized Weyl algebras, Witten-Woronowicz algebra among the others [Ba], [BavO];
- The universal enveloping algebra \(U(\mathfrak{gl}_n)\) over its Gelfand-Tsetlin subalgebra;
- Some rings of invariant differential operators, e.g. symmetric and orthogonal differential operators on \(n\)-dimensional torus (cf. Section 7.3);
- It is shown in [FMO],[FMO1] that shifted Yangians and finite \(W\)-algebras associated with \(\mathfrak{gl}_n\) are Galois orders over corresponding Gelfand-Tsetlin subalgebras.

We emphasize that the theory of Galois orders unifies the representation theories of universal enveloping algebras and generalized Weyl algebras. On one hand the Gelfand-Tsetlin formulae give an embedding of \(U(\mathfrak{gl}_n)\) into a certain localization of the Weyl algebra \(A_m\) for \(m = n(n+1)/2\) (cf. Remark 7.1 and [Kh]). On the other hand the intrinsic reason for such unification is a similar hidden skew group ring structure of these algebras as Galois orders. We believe that the concept of a Galois order will have a strong impact on the representation theory of infinite-dimensional associative algebras. We will discuss the representation theory of Galois rings in a subsequent paper. Preliminary version of this paper appeared in the preprint form [FO].

2. Preliminaries

All fields in the paper contain the base field \(k\), which is algebraically closed of characteristic 0. All algebras in the paper are \(k\)-algebras.

2.1. Integral extensions. Let \(A\) be an integral domain, \(K\) its field of fractions and \(\tilde{A}\) the integral closure of \(A\) in \(K\). Recall that the ring \(\tilde{A}\) is called normal if \(A = \tilde{A}\). Let \(A\) be a normal noetherian ring, \(K \subset L\) a finite Galois extension, \(\tilde{A}\) the integral closure of \(A\) in \(L\).

**Proposition 2.1.**

- If \(\tilde{A}\) is noetherian then \(\tilde{A}\) is finite over \(A\).
- If \(A\) is a finitely generated \(k\)-algebra then \(\tilde{A}\) is finite over \(A\). In particular, \(\tilde{A}\) is finite over \(A\).

The following statement is probably well known but we include the proof for the convenience of the reader.
Proposition 2.2. Let $i : A \hookrightarrow B$ be an embedding of integral domains with a regular $A$. Assume the induced morphism of varieties $i^* : \text{Spec} B \to \text{Spec} A$ is surjective (e.g. $A \subset B$ is an integral extension). If $b \in B$ and $ab = a'$ for some nonzero $a \in A$ then $b \in A$.

Proof. In this case $i$ induces an epimorphism of the $\text{Spec} B$ onto $\text{Spec} A$. Fix $m \in \text{Specm} A$. Assume $ab = a' \in A$. Since the localization $A_m$ is a unique factorization domain, we can assume that $a_m b = a'_m$, where $a_m, a'_m \in A_m$ are coprime. If $a_m$ is invertible in $A_m$ then $b \in A_m$. If $a_m$ is not invertible in $A_m$ then there exists $P \in \text{Spec} A_m$ such that $a_m \in P$ and $a'_m \not\in P$. It shows that $P$ does not lift to $\text{Spec} B_m$. Hence $b \in A_m$ for every $m \in \text{Specm} A$, which implies $b \in A$ ([Mat], Theorem 4.7).

\[\Box\]

2.2. Skew (semi)group rings. If a semigroup $M$ acts on a set $S$, $M \times S \rightarrow S$, then $\varphi(m, s)$ will be denoted either by $m \cdot s$, or $ms$, or $s^m$. In particular $s^{mm'} = (s^m)^m$, $m, m' \in M$, $s \in S$. By $S^M$ we denote the subset of all $M$-invariant elements in $S$.

Let $R$ be a ring with a unit, $M$ a semigroup and $f : M \rightarrow \text{Aut}(R)$ a homomorphism. Then $M$ acts naturally on $R$ (from the left): $g \cdot r = f(g)(r)$ for $g \in M$, $r \in R$. The skew semigroup ring\(^1\), $R \rtimes M$, associated with the left action of $M$ on $R$, is a free left $R$-module, $\bigoplus_{m \in M} Rm$, with a basis $M$ and with the multiplication defined as follows
\[
(r_1 m_1) \cdot (r_2 m_2) = (r_1 r_2^{m_1})(m_1 m_2), \quad m_1, m_2 \in M, \quad r_1, r_2 \in R.
\]

Assume that a finite group $G$ acts on $R$ by automorphisms and on $M$ by conjugation. For every pair $g \in G$, $m \in M$ fix an element $\alpha_{g,m} \in R$ and define a map
\[
G \times (R \rtimes M) \rightarrow R \rtimes M, \quad (g, rm) \mapsto \alpha_{g,m} r^g m^g, \quad r \in R, m \in M, g \in G.
\]

This map defines an action of $G$ on $R \rtimes M$ by automorphisms if and only if $\alpha_{g,m}$ satisfy the following conditions:
\begin{align}
\alpha_{g,m_1 m_2} &= \alpha_{g,m_1} \alpha_{g,m_2}^m, \quad g \in G, \quad m_1, m_2 \in M; \\
\alpha_{g_1 g_2,m} &= \alpha_{g_1,g_2(m)} \alpha_{g_2,m}^{g_1}, \quad g_1, g_2 \in G, \quad \alpha_{e,m} = 1, m \in M.
\end{align}

In this case we say that the action of $G$ on $R \rtimes M$ is monomial.

If $\alpha_{g,m} = 1$ for all $g \in G$ and $m \in M$ then $g(rm) = r^g m^g$ for all $g \in G$, $r \in R$, $m \in M$. For simplicity in this paper we will work with a trivial $\alpha$ but all the results remain valid for a general monomial action of $G$ satisfying the conditions above.

If $x \in R \rtimes M$ then we write it in the form
\[
x = \sum_{m \in M} x_m m,
\]
where only finitely many $x_m \in M$ are nonzero. We call the finite set
\[
\text{supp} x = \{m \in M | x_m \neq 0\}
\]
the support of $x$. Hence $x \in (R \rtimes M)^G$ if and only if $x_{m^g} = x_m^g$ for all $m \in M, g \in G$. If $x \in (R \rtimes M)^G$ then supp $x$ is a finite $G$-invariant subset in $M$. For $\varphi \in \text{Aut} R$ set
\[
H_\varphi = \{h \in G | \varphi^h = \varphi\}, \quad O_\varphi = \{\varphi^g | g \in G\}.
\]

If $G$ is a finite group and $H$ is its subgroup then the notation $F = \sum_{g \in G/H} F(g)$ means that $g$ runs over a set of representatives of the quotient $G/H$ and $F(g)$ does not depend on the choice of these representatives. In particular, $F$ is well defined.

Using this agreement we denote
\[
[a_\varphi] := \sum_{g \in G/H_\varphi} a_\varphi^g \varphi^g \in (R \rtimes M)^G, \quad \varphi \in M, \quad a \in R_{H_\varphi},
\]
\[\text{In a subsequent publication we will consider a more general case of the crossed product of } R \text{ and } M.\]
and set \((R \ast \mathcal{M})^G_G = \{ [a \varphi] \mid a \in R^H_{H \varphi} \}\). Then we have the following decomposition of \((R \ast \mathcal{M})^G\) into a direct sum of left (right) \(R^G\)-subbimodules

\[(2.6) \quad (R \ast \mathcal{M})^G = \bigoplus_{\varphi \in \mathcal{G} \setminus \mathcal{M}} (R \ast \mathcal{M})^G_{\varphi},\]

where \(R^G\) acts on \((R \ast \mathcal{M})^G_G\) as follows

\[(2.7) \quad \gamma \cdot [a \varphi] = [(a \gamma) \varphi], \quad [a \varphi] \cdot \gamma = [(a \gamma^\varphi) \varphi], \quad \gamma \in R^G.\]

Thus every \(x \in (R \ast \mathcal{M})^G\) can be uniquely written in the form \(\sum_{\varphi \in \mathcal{M} \setminus G} [x \varphi], x \in R^H_{H \varphi}\). We have for \(\gamma \in \Gamma\)

\[ [a_1 \varphi_1] [a_2 \varphi_2] = \sum_{\tau \in \mathcal{G}_1 \setminus \mathcal{G}_2} \left( \sum_{\varphi_1 = \tau} a^{\varphi_1} \gamma \varphi_1 a^{\varphi_2} \right). \]

For \(a, b \in R^H_{H \varphi}\) denote

\[(2.8) \quad [a \varphi b] = \sum g \in R^{H \varphi} a^{\varphi g} b^{\varphi g}, \quad \text{so for } \gamma \in R^G \text{ holds}
\]

\[ \gamma [a \varphi b] = [a \varphi (b \gamma^{-1}) \varphi], \quad [a \varphi b] \gamma = [(\gamma^{\varphi} a) \varphi b], \]

since \(\varphi(R^G), \varphi^{-1}(R^G) \subset R^H_{H \varphi}\). Note that in obvious way \([a \varphi] = [\varphi^{\varphi^{-1}}], a \in R, \varphi \in \text{Aut } R\).

### 2.3. Separation actions

If \(R = L\) is a field, \(K \subset L\) be a finite Galois extension of fields, \(G = G(L/K)\) the Galois group and \(\iota\) the canonical embedding \(K \hookrightarrow L\). Then \(K = L^G\) and

\[(2.9) \quad \text{dim}_K^L \mathcal{K}_\varphi = \text{dim}_K^L \mathcal{K}_\varphi = |L^H_{H \varphi} : K| = |G : H_\varphi| = |0_\varphi|,\]

where \(\text{dim}_K^L \mathcal{K}_\varphi, \text{dim}_K^L \mathcal{K}_\varphi\) are right and left \(K\)-dimensions.

**Definition 3.**

1. A monoid \(\mathcal{M} \subset \text{Aut } L\) is called separating (with respect to \(K\)) if for any \(m_1, m_2 \in \mathcal{M}\) the equality \(m_1|_K = m_2|_K\) implies \(m_1 = m_2\).
2. An automorphism \(\varphi : L \to L\) is called separating (with respect to \(K\)) if the monoid generated by \(\{\varphi^g \mid g \in G\}\) in \(\text{Aut } L\) is separating.

**Lemma 2.1.** Let monoid \(\mathcal{M}\) be separating with respect to \(K\). Then

1. \(\mathcal{M} \cap G = \{e\}\).
2. For any \(m \in \mathcal{M}, m \neq e\) there exists \(\gamma \in K\) such that \(\gamma^m \neq \gamma\).
3. If \(Gm_1 \varphi = Gm_2 \varphi\) for some \(m_1, m_2 \in \mathcal{M}\), then there exists \(g \in G\) such that \(m_1 = m_2^g\).
4. If \(\mathcal{M}\) is a group, then the statements (1), (2), (3) are equivalent and each of them imply that \(\mathcal{M}\) is separating.

**Proof.** We prove the statement (3), other statements are trivial. \(Gm_1 \varphi = Gm_2 \varphi\) if and only if for some \(g, g' \in G\) holds \(m_1^g = m_2 g'\). Then \(m_1^g\) and \(m_2\) acts in the same way on \(K\), hence \(m_1^g = m_2\).

Let \(\iota : K \hookrightarrow L\) be an embedding. Denote \(\text{St}(\iota) = \{ g \in G \mid \iota g = \iota \}\).

**Lemma 2.2.** Let \(\varphi \in \mathcal{M}, \varphi = \varphi_1\). Then

1. If \(\varphi\) is separating, then \(H_\varphi = \text{St}(\iota)\).
2. \(K(\varphi) = L^{\text{St}(\iota)}\), in particular, if \(\varphi\) is separating \(K(\varphi) = L^H_{H \varphi}\).

**Proof.** Obviously \(H_\varphi \subset \text{St}(\iota)\). Conversely, if \(g \varphi = \varphi_1\), then \(\varphi^{-1} g \varphi = g_1 \in G\) and \(\varphi^{-1} (g \varphi g^{-1}) = g_1 g^{-1}\). Thus \(\varphi\) and \(g \varphi g^{-1}\) coincide on \(K\), implying \(g \varphi g^{-1} = \varphi\) and (1). Note that \(g \in G(L/K \varphi(K)) \cap G\) if and only if \(g|_{\varphi(K)} = \text{id}\) (i.e. \(g \in \text{St}(\iota)\)), implying (2).
3. Bimodules

3.1. Balanced bimodules. For commutative \( k \)-algebras \( A \) and \( B \) we will denote by \((A - B) - \text{bimod}\) the category of finitely generated \( A - B \)-bimodules. If \( A = B \) we will simply write \( A - \text{bimod}\).

**Proposition 3.1.** Let \( K \subset L \) be a finite field extension. The full subcategories of \( K - \text{bimod}, (K - L) - \text{bimod} \) or \((L - K) - \text{bimod}\) consisting of objects, which are finite dimensional as left or right modules are Jordan-Hölder and Krull-Schmidt categories.

**Proof.** It follows from the finiteness of the length of the objects of these categories.

In this section all bimodules over fields are assumed to be finite dimensional from both sides and \( k \)-central (unless the contrary is stated).

**Definition 4.** A homomorphism of algebras \( \varphi : A \to B \) endows \( B \) with the structure of \( B - A \)-bimodule \( B_\varphi \) such that for \( a \in A, b \in B, b' \in B_\varphi \) holds \( b \cdot b' \cdot a = bb'\varphi(a) \).

**Remark 3.1.** (1) In opposite, an \( B - A \)-bimodule \( V \), which is free of rank 1 from the left, defines a homomorphism \( \varphi = \varphi_V : A \to B \) by \( va = \varphi(a)v \), where \( v \in V \) is a right free generator of \( V \).

(2) If \( \varphi : A \to B \) and \( \psi : B \to C \) are homomorphisms of algebras then there exists an isomorphism of \( C - A \)-bimodules

\[
C_\psi \otimes_B B_\varphi \simeq C_{\varphi \psi}, \quad c \otimes b \mapsto c\psi(b), c \in C, b \in B.
\]

Let \( K \subset L \) be an extension and \( 1_K \) the canonical embedding \( K \subset L \). We will write \( 1 \) instead of \( 1_K \) when the field \( K \) is fixed. If \( V = K V_K \) is a \( K \)-bimodule then denote \( K V_L = V \otimes K L, L V_K = L \otimes_K V \) and \( L V_L = L \otimes_K K V_L \).

Let \( K \subset L \) be a Galois extension with the Galois group \( G = G(L/K) \), then the group \( G \times G \) acts on \( L V_L \) as

\[
(g_1, g_2) \cdot (l_1 \otimes v \otimes l_2) \mapsto l_1^{g_1} \otimes v \otimes l_2^{g_2^{-1}}, \quad (g_1, g_2) \in G \times G, v \in V, l_1, l_2 \in L
\]

by automorphism of \( K \)-bimodules. The \( K \)-bimodule of invariants is canonically isomorphic to \( V \). If we restrict the action of \( G \times G \) to the action of \( G \) from the left (from the right), by automorphisms of \( K - L (L - K) \) bimodules, then the invariants will be \( K V_L (L V_K) \).

Analogously, \( G \) acts naturally from the left on the \( L - K \)-bimodule \( L V_K \) by automorphisms of \( K \)-bimodule,

\[
g \cdot (l \otimes v) \mapsto l^g \otimes v, g \in G, v \in V, l \in L \quad \text{and} \quad (L V_K)^G \simeq K V_K.
\]

Assume that the right action of \( K \) on \( V \) is \( L \)-diagonalizable from the left. It means \( L V_K \) splits into a sum of \( L - K \)-bimodules, which are one dimensional as right \( L \)-modules. By Remark 3.1, (1) such one dimensional \( L - K \)-bimodule is of the form \( L_\varphi \) for some field embedding \( \varphi : K \to L \).

**Definition 5.** A \( K \)-bimodule \( K V_K \) is called \( L \)-balanced over a finite Galois extension \( K \subset L \) if \( L V_L \) is a direct sum of one-dimensional from the left and from the right \( L \)-bimodules, i.e. bimodules of the form \( L_\varphi \) for \( \varphi \in \text{Aut} L \). A \( K \)-bimodule \( K V_K \) is called \( L \)-balanced if it is \( L \)-balanced over some finite Galois extension \( K \subset L \).

3.2. Monoidal category of balanced bimodules. Denote by \( K - \text{bimod}_L \) the full subcategory in \( K - \text{bimod} \) consisting of all \( L \)-balanced \( K \)-bimodules.

**Remark 3.2.** The category \( L - \text{bimod}_L \) is by definition semisimple and its isoclasses of simples are represented by the bimodules \( L_\varphi \), were \( \varphi : L \to L \) is an automorphism.

**Theorem 3.1.** The category \( K - \text{bimod}_L \) is an abelian semisimple monoidal category.

**Proof.** Note that by Remarks 3.1, (2) and by Remark 3.2 above the category \( L - \text{bimod}_L \) satisfies the theorem.

Let \( V, W \) be \( L \)-balanced \( K \)-bimodules, \( \pi : V \to W \) an \( K \)-bimodule epimorphism, \( \pi_L : L V_L \to L W_L \) the induced epimorphism of \( L \)-bimodules. Since \( G \) acts trivially on \( K \) the map \( \pi_L \) is a homomorphism of \((K \otimes_k K)[G \times G]\)-bimodules.
On the other hand \( p_L \) admits the right inverse \( L - L \)-bimodule monomorphism

\[
s_L : lW_L \rightarrow lV_L, \quad p_L s_L = \text{id}_{lW_L}.
\]

Since \( G \) acts trivially on \( K \) for every \( g = (g_1, g_2) \in G \times G \) the morphism

\[
gs_L g^{-1} : lW_L \rightarrow lV_L, \quad l_1 \otimes w \otimes l_2 \mapsto g_1 \cdot s_L(l_1^{-1} \otimes w \otimes l_2^2) \cdot g_2^{-1}
\]

are \( K \)-bimodule homomorphisms. Then the \( K \)-bimodule homomorphism

\[
\sigma_L = \frac{1}{|G|^2} \sum_{g \in G \times G} gs_L g^{-1}
\]

commutes with the action \( G \times G \), hence both \( \sigma_L \) and \( \pi_L \) are \( (K \otimes K)\left[G \times G\right]-\text{bimodule homomorphisms.} \)

We have

\[
p_L \sigma_L = \frac{1}{|G|^2} \sum_{g \in G \times G} p_L gs_L g^{-1} = \frac{1}{|G|^2} \sum_{g \in G \times G} g p_L s_L g^{-1} = \text{id}_{lW_L}.
\]

Since \( \sigma_L \) maps \( lW_L^{G \times G} \) to \( lV_L^{G \times G} \), it induces a \( K \)-bimodule homomorphism \( \sigma : W \rightarrow V \), which splits \( p \). Hence \( K - \text{bimod}_L \) is semisimple.

Consider the standard \( K \)-bimodule monomorphism

\[
i : V \otimes K W \rightarrow V \otimes_K L \otimes_K W, \quad v \otimes w \mapsto v \otimes 1 \otimes w.
\]

Then the induced \( L \)-bimodule homomorphism

\[
L(V \otimes_K W)_L \rightarrow L \otimes_K V \otimes_K L \otimes_K W \otimes_K L \simeq lV_L \otimes lW_L,
\]

is a monomorphism. Since \( lW_L \) and \( lV_L \) are isomorphic to the sums of simple one-dimensional \( L \)-bimodules, the same is true for their tensor product over \( L \) and for its subbimodule \( L(V \otimes_K W)_L \).

Note also that \( K \) is a weak unit with respect to \( \otimes_K \) in \( K - \text{bimod}_L \). \( \square \)

3.3. Simple balanced bimodules. In this section we describe simple objects in \( K - \text{bimod}_L \).

**Lemma 3.1.** Let \( K \subset L \) be a Galois extension, \( G = G(L/K) \).

1. ([DK], Ch. 5.1) If for a field \( F \) holds \( K \subset F \subset L \), \( H = G(L/F) \) and \( i_F : F \rightarrow L \) is the canonical embedding, then as \( L - F \)-bimodule

\[
L \otimes_K F \simeq \biguplus_{g \in G/H} L_{g i_F}, \text{ in particular } L \otimes_K L \simeq \biguplus_{g \in G} L_g.
\]

2. A \( K \)-bimodule \( V \) is \( L \)-balanced if and only if the \( L - K \)-bimodule \( lV_K \) is a direct sum of modules of the form \( L_{\varphi_1}, \varphi \in \text{Aut} L \).

3. The right and the left \( K \)-dimensions of a balanced bimodule coincide.

**Proof.** To prove the statement (1) we present \( F \) as a simple extension \( F = K[\alpha], \alpha \in F \). Let \( f(X) \) be a minimal polynomial of \( \alpha \) over \( K \), \( \alpha = \alpha_1, \ldots, \alpha_k \in L \) all roots of \( f(X) \). Then \( F \simeq K[X]/(f(X)) \) and

\[
L \otimes_K F \simeq L \otimes_K K[X]/(f(X)) \simeq L[X]/(f(X)) \simeq \prod_{i=1}^k L[X]/(X - \alpha_i).
\]

The right \( F \)-module structure on \( L[X]/(X - \alpha_i) \) is defined by multiplication on \( X \), that proves (1).

In (2) we prove firstly the statement “if”. Applying Remark 3.1, (2) we obtain the following isomorphisms of \( L - K \)-bimodules, which proves the statement.

\[
L \otimes_K L_{\varphi_1} \simeq L \otimes_K (L \otimes_L L_{\varphi_1}) \simeq (L \otimes_K L) \otimes_L L_{\varphi_1} \simeq \biguplus_{g \in G} L_g \varphi_1.
\]
Hence there exist \( \sum \) consider 0 \( \neq \) the maps \( K \) by Lemma 2.2, (2), implying (2). The following steps. If with the structure of a simple left 

To prove the simplicity of \( K \), we will use the corresponding structure of left \( A \) \( K \) (2) are obvious. Theorem 3.2. (1) \( V \) has a structure of \( K \)-bimodule since \( K \subset L^H \). It turned out, that \( V(\varphi) \), \( \varphi \in Aut L \) cover all simples in \( K \) – \text{bimod}_L.

**Theorem 3.2.**

1. \( L \otimes_K V(\varphi) \simeq \bigoplus_{g \in G/H} L_{g\varphi} \) as a \( L – K \)-bimodule, i.e. \( V(\varphi) \) is \( L \)-balanced.

2. \( V(\varphi) \) is a simple \( K \)-bimodule.

3. Any simple object in \( K – \text{bimod}_L \) is isomorphic to \( V(\varphi) \) for some \( \varphi \in Aut L \).

4. Let \( \varphi, \varphi' \in Aut L \). Then \( V(\varphi) \simeq V(\varphi') \) if and only if \( G\varphi|_K = G\varphi'|_K \), equivalently \( G\varphi G = G\varphi' G \).

5. Assume \( \varphi \in M \) for a separating monoid \( M \subset Aut L, a \in L^H \), \( v = [a \varphi] \in X \). (2.5) Then \( K v K \simeq V(\varphi) \) as \( K \)-bimodule.

**Proof.** Consider \( V(\varphi) \) as \( L^H – K \) bimodule. Using Lemma 3.1, (1) and Remark 3.1, (2) we obtain the following isomorphisms of \( L – K \)-bimodules, which proves (1)

\[
\begin{align*}
L \otimes_K V(\varphi) &= L \otimes_K L^H_j \simeq L \otimes_K (L^H \otimes_{L^H} L^H_j) \simeq (L \otimes_K L^H) \otimes_{L^H} L^H \simeq \\
&= \bigoplus_{g \in G/H} L_g \otimes_{L^H} L^H_j \simeq \bigoplus_{g \in G/H} (L_g \otimes_{L^H} L^H) \simeq \bigoplus_{g \in G/H} L_g.
\end{align*}
\]

To prove the simplicity of \( V(\varphi) \) consider any nonzero \( x \in L^H \). Then \( K \cdot x : K \cdot x \varphi(K) = K \), by Lemma 2.2, (2), implying (2).

Now we prove (3). Let \( V \) be a simple \( L \)-balanced \( K \)-bimodule. We divide the proof in the the following steps. If \( A \) is a \( k \)-algebra, then in the proofs below instead of structure of \( A – K \)-bimodule we will use the corresponding structure of left \( A \otimes_k \)-module.

**Step 1.** The equality \( (l'g \otimes k) \cdot (l \otimes v) = l' l g \otimes k v, k \in K, g \in G, l, l' \in L, v \in V \) endows \( L K \) with the structure of a simple left \( (L * G) \otimes_k K \)-module.

The correctness of \( (L * G) \otimes_k K \)-module structure is checked immediately. To prove the simplicity consider \( 0 \neq x \in L K, x = \sum_{g \in G} l_g \otimes v_g \), where \( v_g \in V, g \in G \) and \( \{ l_g \mid l \in L, g \in G \} \) is a normal \( K \)-basis of \( L \). Consider \( g' \in G \) such that \( v_{g'} \neq 0 \). By the theorem of independence of characters the maps \( w_{g'} : G \rightarrow L, w_{g'}(g_1) = l_{g_1 g}, g \in G \) form a basis in the \( L \)-vector space of maps \( G \rightarrow L \). Hence there exist \( \sum_{g \in G} \lambda_g g \in L * G, \lambda_g \in L, \) such that

\[
(\sum_{g \in G} \lambda_g g) \cdot x = \sum_{g \in G} \left( \sum_{g_1 \in G} \lambda_{g_1} l_{g_1 g} \right) \otimes v_g = 1 \otimes v_{g'},
\]
which obviously generates $V$ as $K$-bimodule and $L V_K$ as $L - K$-bimodule.

**Step 2.** $L V_K \simeq \bigoplus_{j \in G/H} L^d_{g_j}$ for some $d \geq 1$, where $j = \varphi i$ for some $\varphi \in \text{Aut} L$. Besides every $L_{g_j}$ is a simple $(L \ast H) \otimes_k K$-submodule in $L V_K$, where $H = \text{St}(g_j)$.

By definition $L V_K \simeq \bigoplus_{j \in S} L^d_j$ as a $L - K$-module for some pairwise non-isomorphic $L_j$. Since $g(L_j) \simeq L_{g_j}$ and $L V_K$ is simple as $(L \ast G) \otimes_k K$-module, we have $L V_K \simeq \bigoplus_{j \in G/H} L^d_{g_j}$ as a $L - K$-bimodule. The $L - K$-submodule $L_{g_j}$ of $L V_K$ is $H$-invariant, hence it is a $(L \ast H) \otimes_k K$-module, $H = H_{g_j}$. Besides, $L_{g_j}$ is irreducible as $L - K$-bimodule.

**Step 3.** $d = 1$.

Note that $(L \ast G) \otimes_k K$ is a free right $(L \ast H) \otimes_k K$-module of rank $[G : H]$. The canonical embedding of $(L \ast H) \otimes_k K$-modules $L_j \rightarrow L V_K$ induces a homomorphism of $(L \ast G) \otimes_k K$-modules $\Phi : (L \ast G) \otimes L \ast H L_j \rightarrow L V_K$, which is an epimorphism, since $\Phi \neq 0$ and $L V_K$ is simple. On the other hand for the left $K$-dimensions $\dim^L K$ holds

$$\dim^L K (L \ast G \otimes L \ast H L_j) = [L : K][G : H], \quad \dim^L K L V_K = d[L : K][G : H].$$

Hence, $d = 1$ and $\Phi$ is an isomorphism.

**Step 4.** The mapping

$$\psi : K[G] \times L_j \rightarrow (L \ast G) \otimes L \ast H L_j, \quad (kg, l) \mapsto kg \otimes l, \quad k \in K, g \in G, l \in L_j$$

induces an isomorphism of left $K[G] \otimes K$-modules

$$\Psi : K[G] \otimes K[H] L_j \rightarrow (L \ast G) \otimes L \ast H L_j.$$

Indeed, $\psi$ is $K[H]$-bilinear and commutes with the action of $K[G]$ from the left and with the action of $K$ from the right. Again a comparison of $K$-dimensions implies the statement.

**Step 5.** $V \simeq V(\varphi)$.

Steps (3) and (4) shows, that the composition

$$\Psi \circ \Phi : K[G] \otimes K[H] L_j \rightarrow L V_K$$

is an isomorphism of $K[G] \otimes K$-modules. By the Frobenius reciprocity for left $K[H]$-module $L_j$ we obtain the chain of $K$-bimodule isomorphisms

$$V \simeq (L V_K)^G \simeq (K[G] \otimes K[H] L_j)^G \simeq \text{Hom}_{K[G]}(K, K[G] \otimes K[H] L_j) \simeq \text{Hom}_{K[H]}(K, K[H] L_j) \simeq L^H.$$

It leaves to prove (5). Assume $V(\varphi) \simeq V(\varphi')$. Then $L \otimes K V(\varphi) \simeq L \otimes K V(\varphi')$. Hence from (1), $\varphi' = g \varphi$ for some $g \in G$ and thus $G \varphi = G \varphi'$ and $G \varphi | K = G \varphi' | K$ The converse statement easily follows.

Using (2.8) and Lemma 2.2, (2) we obtain $K[a \varphi] K = [K \varphi(K) a \varphi] = [L^H a \varphi]$ which immediately implies the isomorphism $[L^H a \varphi] \simeq V(\varphi)$ and hence the last statement. 

### 3.4. Grotendieck ring of category balanced bimodules and Hecke algebra

Let $K_0(K, L)$ be the Grothendieck ring of the category $K \ast \text{bimod}_L$ and for $V \in K \ast \text{bimod}_L [V]$ the class of $V$ in $K_0(K, L)$. Theorem 3.2 shows that simple $L$-balanced $K$-bimodules can be enumerated by the double cosets $G \varphi G$ or by the $G$-orbits $G \varphi$. We show that the ring structure on $K_0(K, L)$ is closely related with some Hecke algebra (Corollary 3.3).

To calculate in $K_0(K, L)$ we need some preliminaries. A family of elements $S$ of a set $T$ is the mapping $S : \mathcal{J} \rightarrow T$, where $\mathcal{J}$ in the set of indices. If the group $G$ acts on $\mathcal{J}$ and $T$, then we say $S$ is $G$-invariant provided that $S$ is a map of $G$-sets. To simplify the notation we will write $i$ instead of $S(i), i \in \mathcal{J}$. By $S / G$ we denote the induced map of factor sets $S / G : \mathcal{J} / G \rightarrow T / G$. In particular, $S / G$ is a family of elements of $T / G$, indexed by $\mathcal{J} / G$. 

Denote \( \text{Hom}_{k}(-, \text{fields})(K, L) \) the set of all field \( k \)-embeddings \( K \rightarrow L \), and
\[
\mathcal{B}(K, L) = \{ S \mid S \subset \text{Hom}_{k}(-, \text{fields})(K, L), |S| < \infty, GS = S \}.
\]
Then by Lemma 3.1, (2) we can correspond to a finitely generated balanced \( K \)-bimodule \( V \) a \( G \)-invariant family \( S_{V} : \mathcal{J}_{V} \rightarrow \text{Hom}_{k}(K, L) \), such that \( L_{V}K \simeq \bigoplus_{\tau \in \mathcal{J}_{V}} L_{S_{V}(\tau)} \). In obvious way factorization by \( G \) induces the family \( s_{V} = S_{V}/G : \mathcal{J}_{V}/G \rightarrow \mathcal{B}(K, L) \). Obviously, the image of \( s_{V} \) defines the \( K \)-bimodule \( V \) uniquely up to isomorphism and we can write \( L_{V}K \simeq \bigoplus_{\tau \in \mathcal{J}_{V}/G} L_{s_{V}(\tau)} \).

In particular, by Theorem 3.2 (1), we can choose \( \mathcal{J}_{V}(\varphi) \) the set \( G/H, S_{V}(gH) = g\varphi \). Then \( \mathcal{J}_{V}(\varphi)/G \) is one-element and and the image of \( s_{V} \) is the subset \( \{ g\varphi \mid g \in G/H, H = \text{St}(\varphi) \} \). A double coset \( C = G\varphi G \subset G \backslash \text{Aut} L/G \) defines an
\[
b_C = b_{\varphi} = \sum_{\psi \in C} \psi = \sum_{g \in G/H_{\varphi}} \sum_{\tau \in \varphi \mathbb{Q}[\text{Aut} L]} \tau \in \mathbb{Q}[\text{Aut} L],
\]
If \( x = \sum_{\varphi \in G \backslash \text{Aut} L/G} n_{\varphi} b_{\varphi} \in \mathbb{Q}[\text{Aut} L], n_{\varphi} \in \mathbb{N} \), then denote \( V(x) = \bigoplus_{\varphi \in G \backslash \text{Aut} L/G} V(\varphi)^{n_{\varphi}} \). In particular, \( V(b_{\varphi}) \simeq V(\varphi) \).

**Corollary 3.1.** Let \( V \) be an object of \( K \)-bimod_{L} and in the notation above \( V \simeq \bigoplus_{\tau \in \mathcal{J}_{V}/G} V(s_{\tau}(\tau)) \).

1. For \( \varphi \in \text{Aut} L \) the multiplicity \( n_{\varphi} \) of \( V(\varphi) \) in \( V \) is given by
\[
n_{\varphi} = \sum_{\tau \in \mathcal{J}_{V}, S_{\tau}(\tau) = \varphi} \frac{|\text{St}(\varphi)|}{|G|},
\]
2. \( [V] = \sum_{\tau \in \mathcal{J}_{V}} \frac{|\text{St}(S_{\tau}(\tau))|}{|G|} [V(S_{\tau}(\tau))]. \)

**Proof.** The statement (2) follows from (1) The proof follows from Theorem 3.2, (1).

Recall, if \( G_{1} \) is a group, \( G \subset G_{1} \) is a finite subgroup and \( A \) is a commutative ring, then the Hecke algebra \( \mathcal{H}_{A}(G_{1}; G) \subset A[G_{1}] \) is a free module over \( A \) with a basis \( h_{G_{1}} \) labeled by double cosets in \( G_{1} \backslash G_{1}/G \). For details on Hecke algebras we refer to [Kr]. We will need the following result from [Kr] (Theorem 1.6.6) slightly adapted to our conditions.

**Theorem 3.3.** Let \( \Omega = \text{Aut} L \). Then

1. \( e_{\Omega} = \frac{1}{|G|} \sum_{g \in G} g \) is an idempotent in the group algebra \( \mathbb{Q}[\Omega] \).
2. One has \( e_{G}\varphi e_{G} = \frac{|H_{\varphi}|}{|G|^{2}} b_{\varphi} \) for all \( \varphi \in \Omega \) and \( e_{G}\mathbb{Q}[\Omega] e_{G} \) becomes a subalgebra of \( \mathbb{Q}[\Omega] \) with \( e_{G} \) as its identity element.
3. The mapping \( \Phi : \mathcal{H}_{Q}(\Omega; G) \rightarrow e_{G}\mathbb{Q}[\Omega] e_{G} \subset \mathbb{Q}[\Omega] \), where
\[
\sum_{\varphi \in G \backslash \Omega/G} n_{\varphi} h_{G_{1}} \varphi \in G \rightarrow \frac{1}{|G|} \sum_{\varphi \in G \backslash \Omega/G} b_{\varphi}
\]
is an isomorphism of \( Q \)-algebras.

We will identify the Hecke algebra \( \mathcal{H}_{Q}(\Omega; G) \) with \( \text{Im} \Phi \subset \mathbb{Q}[\Omega] \). Given \( \varphi, \psi \in \text{Aut} L \), introduce an equivalence relation \( \sim \) \( (\sim \sim \varphi, \psi) \) on \( G \) as follows:
\[
g \sim g' \text{ if and only if } G\varphi g \psi G = G\varphi' \psi G.
\]
For \( \varphi \in \text{Aut} L \) denote by \( H_{\varphi} = \text{St}(\varphi) \), where \( i : K \rightarrow L \) is the canonical embedding.

**Theorem 3.4.** Let \( \varphi, \psi \in \text{Aut} L \). Then
\[
V(\varphi) \otimes_{K} V(\psi) \simeq \bigoplus_{c_{\varphi} \in G/\sim} V((\varphi \psi)^{c_{\varphi} \psi})^{c_{\varphi}}.
\]
where \( c_g \) is the equivalence class of \( g, |c_g| \) its size and \( s^g_{\varphi\psi} = \frac{|H_{\varphi g\psi}|}{|H_{\varphi}||H_{\psi}|} \).

**Proof.** Let \( \varphi, \psi \in \text{Aut} L \). Then by Theorem 3.2, (1) and Remark 3.1, (2)

\[
L \otimes_K V(\varphi) \otimes_K V(\psi) \simeq \bigoplus_{g \in G/H_{\varphi}} L_{g \varphi_1} \otimes_K V(\psi) \simeq \bigoplus_{g \in G/H_{\varphi}} (L_{g \varphi} \otimes_L L) \otimes_K V(\psi) \simeq \bigoplus_{g \in G/H_{\varphi}} L_{g \varphi} \otimes_L (L \otimes_K V(\psi)) \simeq \bigoplus_{g \in G/H_{\varphi}} \bigoplus_{g' \in G/H_{\psi}} L_{g \varphi} \otimes L_{g' \psi}.
\]

Then by Corollary 3.1

\[
(V(\varphi) \otimes_K V(\psi)) = \sum_{g \in G/H_{\varphi}} |H_{g \varphi}g\psi| |V(g \varphi g\psi)| = \sum_{g \in G/\sim} s^g_{\varphi\psi} |c_g| |V(\varphi g\psi)|.
\]

which completes the proof.

\[
(3.10) \quad [V(\varphi) \otimes_K V(\psi)] = \sum_{g \in G/H_{\varphi}} |H_{g \varphi}g\psi| |V(g \varphi g\psi)| = \sum_{c_g \in G/\sim} s^g_{\varphi\psi} |c_g| |V(\varphi g\psi)|.
\]

**Corollary 3.2.** Let \( \varphi, \psi \in \text{Aut} L \). Then \( \frac{1}{|G|} b_\varphi b_\psi \in \mathbb{Z}[\text{Aut} L] \) and

\[
V(b_\varphi) \otimes_K V(b_\psi) \simeq V(\frac{1}{|G|} b_\varphi \cdot b_\psi).
\]

**Proof.** Clearly,

\[
\frac{1}{|G|} b_\varphi b_\psi = \sum_{g_1 g_2 \in G} g_1 \varphi g_2 \psi g_2,
\]

which proves the first statement. On the other hand we have the following equalities in \( \mathbb{Q}[\text{Aut} L] \):

\[
b_\varphi \cdot b_\psi = (\sum_{g \in G/H_{\varphi}} g \varphi g' \sum_{g \in G/H_{\psi}} g \psi g') = \frac{|G|}{|H_{\varphi}||H_{\psi}|} \sum_{g \in G} |H_{\varphi g\psi}| b_{\varphi g\psi}.
\]

Comparison with (3.10) we complete the proof.

\[
\square
\]

**Corollary 3.3.** The map

\[
\Psi : \mathbb{Q} \otimes_{\mathbb{Z}} K_0(K, L) \to \mathcal{H}_Q(\text{Aut} L; G), \quad \Psi([V(\varphi)]) = \frac{1}{|G|} b_\varphi
\]

is an isomorphism of \( \mathbb{Q} \)-algebras.

**Proof.** Since the classes \([V(\varphi)]\) and the elements \( \frac{1}{|G|} b_\varphi, \varphi \in G \setminus \text{Aut} L/G \), form the \( \mathbb{Q} \)-bases in

\[
\mathbb{Q} \otimes_{\mathbb{Z}} K_0(K, L) \text{ and in } \mathcal{H}_Q(\text{Aut} L; G) \text{ respectively, then } \Psi \text{ is an isomorphism of } \mathbb{Q} \text{-vector spaces. The fact that } \Psi \text{ is an algebra homomorphism follows immediately from Corollary 3.2.}
\]

\[
\square
\]

4. Galois rings

4.1. Notation and some examples. For the rest of the paper we will assume that \( \Gamma \) is an integral domain, \( K \) the field of fractions of \( \Gamma \), \( K \subset L \) is a finite Galois extension with the Galois group \( G, \ i : K \to L \) is a natural embedding, \( \mathcal{M} \subset \text{Aut} L \) is a separating monoid on which \( G \) acts by conjugations, \( \Gamma \) is the integral closure of \( \Gamma \) in \( L, \mathcal{K} = (L * \mathcal{M})^G \).

Recall from the introduction that an associative ring \( U \subset \mathcal{K} \) containing \( \Gamma \) is called a Galois \( \Gamma \)-ring if it is finitely generated over \( \Gamma \) and \( KU = \mathcal{K}, UK = \mathcal{K} \). Note that following Lemma 4.1 below both equalities in this definition are equivalent.
Corollary 4.1. Let \( T \) be a simple \( K \)-module. Then for \( L = K, G = \{e\} \) the algebra \( U = \Gamma[x; \sigma] \) is a Galois \( \Gamma \)-ring in \( K \cdot M \), when \( x \) is identified with \( 1 * \sigma \in K \cdot M \).

Characterization of a Galois ring.

4.2. Let \( \Gamma = k[x_1, \ldots, x_n] \) and \( \sigma_1, \ldots, \sigma_n \in \text{Aut} \Gamma \), such that \( \sigma_i \sigma_j = \sigma_j \sigma_i \), \( i, j = 1, \ldots, n \). Then the skew group ring \( \Gamma \cdot M \) is a Galois ring over \( \Gamma \) with trivial \( G \).

More examples will be given in Section 7.

4.2. Characterization of a Galois ring. A \( \Gamma \)-submodule of \( \mathcal{K} \) which for every \( m \in M \) contains \([b_1 m], \ldots, [b_k m]\) where \( b_1, \ldots, b_k \) is a \( K \)-basis in \( L^H \) will be called a \( \Gamma \)-form of \( \mathcal{K} \). We will show that any Galois subring in \( \mathcal{K} \) is its \( \Gamma \)-form.

Lemma 4.1. Let \( u \in U \) be a nonzero element, \( T = \text{supp} u \), \( u = \sum_{m \in T/G} [a_m m] \). Then

\[
K(\Gamma u \Gamma) = (\Gamma u \Gamma) K = KuK \cong \bigoplus_{m \in T/G} V(m).
\]

In particular \( U \) is a \( \Gamma \)-form of \( \mathcal{K} \). Besides, \( L(\Gamma u \Gamma) = (\Gamma u \Gamma) L = LuL = \sum_{m \in T} Lm \subset L \cdot M \).

Proof. Note that by Theorem 3.2, (5) and Lemma 2.1, (3) the modules \( V(m), m \in T/G \), are pairwise non-isomorphic simple \( K \)-bimodules. Since by Lemma 2.2, (2) \( K[m]K = KK^m[m] \cong V(m), m \in T/G \),

we have

\[
KuK \subset \bigoplus_{m \in T/G} K[a_m m]K = \bigoplus_{m \in T/G} K[a_m m]K \cong \bigoplus_{m \in T/G} K[m]K \cong \bigoplus_{m \in T/G} V(m).
\]

Since all \( V(m) \) are simple, then the image of \( KuK \) in \( W = \bigoplus_{m \in T/G} V(m) \) generates \( W \) as a \( K \)-bimodule. Hence \( KuK \cong W \) and therefore \( K[a_m m]K \subset KuK \) for all \( m \in T/G \).

For the rest of the proof it is enough to consider \( u = [am] \). Then

\[
\Gamma [am] \Gamma = [\Gamma \cdot m(\Gamma) am] \text{ and } KTm(\Gamma) = Km(K).
\]

The statement \( K(\Gamma u \Gamma) = (\Gamma u \Gamma) K = KuK \) now follows from Lemma 2.2, (2).

Obviously \( L[am] \) is a \( L \)-submodule in \( \sum_{m' \in T'} Lm \). Since this is a direct sum of non-isomorphic simple \( L \)-bimodules, any submodule has the form \( \sum_{m \in T'} Lm, T' \subset T \). On the other hand \( \text{supp}[am] = T \), and thus \( L[am] = \sum_{m \in T} Lm \).

Corollary 4.1. Let \([a \varphi], [b \psi] \in \mathcal{K} \). Then

\[
\text{supp} [a \varphi][b \psi] = \text{supp} [a \varphi] \text{supp} [b \psi] = \emptyset \varphi \emptyset \psi.
\]
Proof. Multiplication on $L$ does not change the support. Then applying Lemma 4.1
\[
\text{supp } [a\varphi] \Gamma [b\psi] = \text{supp } L([a\varphi] \Gamma [b\psi]) = \text{supp } L(K[a\varphi] \Gamma [b\psi]) = \text{supp } (\sum_{m \in O_{\varphi}} Lm)[b\psi] = \mathcal{O}_{\varphi} \mathcal{O}_{\psi}.
\]
\[\square\]

Proposition 4.1. Assume a ring $U \subset \mathcal{K}$ is generated over $\Gamma$ by $u_1, \ldots, u_k \in U$.

(1) If $\bigcup_{i=1}^{k} \text{supp } u_i$ generate $M$ as a semigroup, then $U$ is a Galois ring.

(2) If $LU = L \ast M$, then $U$ is a Galois ring.

Proof. The statement (2) follows from (1). Consider a $K$-submodule $Ku_1K + \cdots + Ku_kK$ in $\mathcal{K}$. By Lemma 4.1, this bimodule contains the elements $[a_1\varphi_1], \ldots, [a_N\varphi_N]$, where $\varphi_1, \ldots, \varphi_N$ is a set of generators of $M$. By Corollary 4.1 $\text{supp } ([a_1m_1][a_2m_2]) = \text{supp } [a_1m_1] \cap \text{supp } [a_2m_2]$ for $[a_1m_1], [a_2m_2] \in U$, then for every $m \in M$ there exists a nonzero $a_m \in L^Hm$ such that $[a_m] \in U$.

Theorem 4.1. Let $U$ be a Galois ring, $e \in M$ the unit element and $U_e = U \cap Le$. Then

(1) For every $x \in U$ holds $x_e \in K$ and $U_e \subset K e$.

(2) The $k$-subalgebra in $L \ast M$ generated by $U$ and $L$ coincides with $L \ast M$.

(3) $U \cap K$ is a maximal commutative $k$-subalgebra in $U$.

(4) The center $Z(U)$ of algebra $U$ equals $U \cap K^M$.

Proof. Let $x \in U$ and $x_e = \lambda$, $\lambda \in L$. Then for any $g \in G$ holds $\lambda = x_e = (x^g)_e = \lambda^g$. Hence $\lambda \in L^G = K$. The statement (2) follows from Lemma 4.1.

Consider any $x \in L \ast M$ such that $x\gamma = \gamma x$ for all $\gamma \in \Gamma$. Assume $x_{\varphi} \neq 0$ for some $\varphi \in M$, $\varphi \neq e$. Since the action of $M$ is separating, there exists $\gamma \in \Gamma$ such that $\gamma^\varphi \neq \gamma$. Then $(\gamma x)_{\varphi} = \gamma x_{\varphi} \neq \gamma^\varphi x_{\varphi} = (x\gamma)_\varphi$ which is a contradiction. Hence $x \in U \cap Le = U_e \subset K \text{ which completes the proof of (3)}$.

To prove (4) consider a nonzero $z \in Z(U)$. It follows from the proof of (3) that $z \in U \cap K$. Besides, $z \in \Gamma \cap Z(U)$ if and only if for every $[a\varphi] \in U$ holds $z[a\varphi] = [a\varphi]z$, i.e. $z = z^\varphi$. \[\square\]

Theorem 4.1, (3) in particular shows that an noncommutative associative algebra is never a Galois ring over its center. For the same reason the universal enveloping algebra of a simple finite-dimensional Lie algebra is not a Galois ring over the enveloping algebra of its Cartan subalgebra.

A submonoid $H$ of $M$ is called an ideal of $M$ if $MH \subset H$ and $HM \subset H$.

Corollary 4.2. There is one-to-one correspondence between the two-sided ideals in $\mathcal{K}$ and the $G$-invariant ideals in the monoid $M$. This correspondence is given by the following bijection
\begin{equation}
I \mapsto J = J(I) = \bigcup_{u \in I} \text{supp } u, \quad J \mapsto I = I(J) = \sum_{\varphi \in J} K[\varphi]K,
\end{equation}
where $I \subset \mathcal{K}$, $J \subset M$ are ideals, $I \neq 0$, $J$ is $G$-invariant. In particular, if $M$ is a group then $\mathcal{K}$ is a simple ring.

Proof. Let $I$ be a nonzero ideal in $\mathcal{K}$. If $0 \neq u \in I$ then $KuK \simeq \sum_{\varphi \in \text{supp } u/G} K[\varphi]K$
by Lemma 4.1 and for every $m \in M$ holds $(K[m]K)(KuK) \subset I$, $(KuK)(K[m]K) \subset I$. By Corollary 4.1 for every $m \in M$ and $\varphi \in \text{supp } u$ there exist $u', u'' \in I$ such that $m\varphi \in \text{supp } u'$ and $\varphi m \in \text{supp } u''$. This gives the map $I \mapsto J(I)$. Analogously, $I(J)$ is a two-sided ideal in $\mathcal{K}$ and both maps are mutually inverse. \[\square\]

Proposition 4.2. Let $U$ be a Galois ring over $\Gamma$, $S = \Gamma \setminus \{0\}$.

(1) The multiplicative set $S$ satisfies both left and right Ore condition.
(2) The canonical embedding \( U \hookrightarrow \mathcal{K} \) induced the isomorphisms of rings of fractions \( [S^{-1}U] \simeq \mathcal{K}, U[S^{-1}] \simeq \mathcal{K} \).

**Proof.** Assume \( s \in S, u \in U \). Following Lemma 4.1, \( U \) contains a right \( K \)-basis \( u_1, \ldots, u_k \) of \( KuK \), hence in \( \mathcal{K} \) holds
\[
s^{-1}u = \sum_{i=1}^{k} u_i \gamma_i s_i^{-1} \text{ for some } s_i \in S, \gamma_i \in \Gamma, i = 1, \ldots, k.
\]
Then in \( U \) holds
\[
u \cdot (s_1 \ldots s_k) = s \cdot (\sum_{i=1}^{k} u_i \gamma_i s_1 \ldots s_{i-1}s_{i+1} \ldots s_k),
\]
which shows (1). Besides \( S \) acts on \( U \) torsion free both from the left and from the right. Then there exist the right and left rings of fractions \( U[S^{-1}], [S^{-1}]U \). Following Lemma 4.1, the canonical embedding \( U \hookrightarrow \mathcal{K} \) satisfies the conditions for the ring of fractions ((i),(ii), (iii), [MCR], 2.1.3). Hence (2) follows. \( \square \)

**Theorem 4.2.** The tensor product of two Galois rings is a Galois ring.

**Proof.** Let \( U_i \) be a Galois \( \Gamma_i \)-subring in the skew-group algebra \( L_i \rtimes M_i \) with fraction fields \( K_i \), \( G_i = G(L_i/K_i) \) \( i = 1, 2 \). Then \( M = M_1 \times M_2 \) acts on \( L_1 \otimes_k L_2, (m_1, m_2) \cdot (l_1 \otimes l_2) = m_1 l_1 \otimes m_2 l_2 \). Since \( k \) is algebraically closed, \( L_1 \otimes_k L_2 \) is a domain, hence \( M \) acts on its field of fractions \( L \). Let \( K \subset L \) be the field of fractions of \( K_1 \otimes_k K_2 \). The extension \( K \subset L \) is a finite Galois extension with the Galois group \( G = G_1 \times G_2 \). Consider the composition
\[
i : U_1 \otimes_k U_2 \longrightarrow (L_1 \rtimes M_1) \otimes_k (L_2 \rtimes M_2) \xrightarrow{\Phi} (L_1 \otimes_k L_2) \rtimes (M_1 \times M_2) \rightarrow L \rtimes M.
\]
We identify \( U_1 \otimes_k U_2 \) with its image. To endow \( U_1 \otimes_k U_2 \) with the structure of a Galois ring we shall prove that \( L(U_1 \otimes_k U_2) = L \rtimes M \) (Proposition 4.1). But \( L(U_1 \otimes_k U_2) \supset L_1 U_1 \otimes_k L_2 U_2 = (L_1 \rtimes M_1) \otimes_k (L_2 \rtimes M_2) \), which contains \( \Phi^{-1}(M_1 \times M_2) \). \( \square \)

5. Galois orders

5.1. Characterization of Galois orders. Let \( M \) be a right \( \Gamma \)-submodule in a torsion free right \( \Gamma \)-module \( N \). Consider the right submodule in \( N \)
\[
\mathbb{D}_{r,N}(M) = \{ x \in N \mid \text{there exists } \gamma \in \Gamma, \gamma \neq 0 \text{ such that } x \cdot \gamma \in M \},
\]
which is clearly a right \( \Gamma \)-module. For the left modules \( M \subset N \) analogously is defined \( \mathbb{D}_{l,N}(M) \).

If \( N \) is a Galois order \( U \) over \( \Gamma \), we write \( \mathbb{D}_r(M) \) and \( \mathbb{D}_r(M) \).

**Lemma 5.1.** For right \( \Gamma \)-submodules of \( U \) holds the following:

1. \( M \subset \mathbb{D}_r(M), \mathbb{D}_r(\mathbb{D}_r(M)) = \mathbb{D}_r(M) \).
2. \( \mathbb{D}_r(M) = MK \cap U \).
3. If \( N \subset M \) then \( \mathbb{D}_r(N) \subset \mathbb{D}_r(M) \).
4. \( \mathbb{D}_r(\Gamma) = U_e \).

**Proof.** Statements (1) and (3) are obvious. Statement (2) follows from the fact that \( U \) is torsion free over \( \Gamma \). Theorem 4.1 (1) claims that \( U_e \subset K \), implying (4). \( \square \)

Lemma 5.1, (2) gives the following characterization of Galois orders (cf. Definition 2).

**Corollary 5.1.** A Galois ring \( U \) over a noetherian \( \Gamma \) is right Galois order if and only if for every finitely generated right \( \Gamma \)-module \( M \subset U \), the right \( \Gamma \)-module \( \mathbb{D}_r(M) \) is finitely generated.

**Corollary 5.2.** If a Galois ring \( U \) over a noetherian domain \( \Gamma \) is projective as a right (left) \( \Gamma \)-module then \( U \) is a right (left) Galois order.
Remark 5.1. Let \( K \) skew polynomial algebra order, such that \( \varphi \) Chandra subalgebra in \( U \). Corollary 5.4. Clearly, \( U \subset M \). Consider the following finitely generated right \( \Gamma \)-module \( \Gamma \)-submodule of \( U \). is not finitely generated. On the other hand, \( \Gamma \) holds \( \{ \] holds \( \Gamma \subset \Gamma \). Moreover, it is finitely generated as left and right \( \Gamma \)-module simultaneously. The statement follows from Proposition 2.1.

We will show in Theorem 5.2, (2) that the converse statement holds when \( M \) is a group.

5.2. Harish-Chandra subalgebras. Following [DFO2] a commutative subalgebra \( \Gamma \subset U \) is called a Harish-Chandra subalgebra in \( U \) if for any \( u \in U \), the \( \Gamma \)-bimodule \( \Gamma u \Gamma \) is finitely generated both as a left and as a right \( \Gamma \)-module. Assume \( \Gamma \) and some family \( \{ u_i \in U \} \) generates \( U \) as \( k \)-algebra and every \( \Gamma u_i \Gamma, i \in I \) is left and right finitely generated. Then it is easy to see, that \( \Gamma \) is a Harish-Chandra subalgebra in \( U \).

Proposition 5.1. Assume \( \Gamma \) is finitely generated algebra over \( k \), \( U \) is a Galois ring. Then \( \Gamma \) is a Harish-Chandra subalgebra in \( U \) if and only if \( m \cdot \Gamma = \Gamma \) for every \( m \in M \).

Proof. Note that \( \bar{\Gamma} \) is finitely generated as \( \Gamma \)-module (Proposition 2.1). Suppose first \( m \cdot \bar{\Gamma} = \bar{\Gamma} \) for every \( m \in M \). It is enough to prove that \( \Gamma[am] \Gamma \) is finitely generated as a left (right) \( \Gamma \)-module for any \( m \in M, a \in L \). But following (2.7)

\[
\Gamma[am]\Gamma = [\Gamma \cdot m(\Gamma)am] = [am\Gamma \cdot m^{-1}(\Gamma)]
\]

is finitely generated over \( \Gamma \) from the left, since \( \Gamma m(\Gamma) \subset \bar{\Gamma} \), and it is finitely generated from the right, since \( \Gamma m^{-1}(\Gamma) \subset \bar{\Gamma} \). Conversely, assume \( \Gamma[am] \Gamma \) is finitely generated right \( \Gamma \)-module for any \( [am] \in U \). It means that \( \Gamma \cdot m^{-1}(\Gamma) \) is finite over \( \Gamma \), i.e. \( m^{-1}(\Gamma) \subset \bar{\Gamma} \). Analogously, \( m(\Gamma) \subset \bar{\Gamma} \).

Proposition 5.2. If \( U \) is a right (left) Galois order over a noetherian \( \Gamma \) then for any \( m \in M \) holds \( m^{-1}(\Gamma) \subset \Gamma \) \( (m(\Gamma) \subset \bar{\Gamma}) \).

Proof. Let \( U \) be right Galois order, \( [am] \in U \), \( \gamma \in \Gamma \). Assume \( x = m^{-1}(\gamma) \notin \bar{\Gamma} \). Then the right \( \Gamma \)-submodule of \( U \)

\[
M = \sum_{i=0}^{\infty} \gamma^i[am] \Gamma = \sum_{i=0}^{\infty} [amx^i \Gamma]
\]

is not finitely generated. On the other hand, \( x \) is an algebraic element over \( K \). Let

\[
\gamma_0 x^n + \gamma_1 x^{n-1} + \cdots + \gamma_n = 0, \gamma_i \in \Gamma, \gamma_0 \neq 0.
\]

Consider the following finitely generated right \( \Gamma \)-module \( N = \sum_{i=0}^{n-1} \gamma^i[am] \Gamma = \sum_{i=0}^{n-1} [amx^i \Gamma] \). But \( M \subset D_r(N) \) which is a contradiction. The case of left order treated analogously.

From Proposition 5.2 and Proposition 5.1 we immediately obtain

Corollary 5.4. Let \( \Gamma \) be a noetherian domain and \( U \) a Galois order over \( \Gamma \). Then \( \Gamma \) is a Harish-Chandra subalgebra in \( U \).

Remark 5.1. Let \( \Gamma \) be integrally closed in \( K \) and \( \varphi : K \rightarrow K \) an automorphism of infinite order, such that \( \varphi(\Gamma) \notin \Gamma \). Set \( L = K, M = \{ \varphi^n | n \geq 0 \} \). Then \( L \ast M \) is isomorphic to the skew polynomial algebra \( K[x; \varphi] ([MCR]) \). Its subalgebra \( U \) generated by \( \Gamma \) and \( x \) is a Galois ring. Clearly, \( U \) is left Galois order (but not right Galois order).
5.3. Properties of Galois orders. Let $U$ be a Galois ring over $\Gamma$, $S \subset M$ a finite $\Gamma$-invariant subset. Denote
\begin{equation}
U(S) = \{ u \in U \mid \text{supp}\ u \subset S \}.
\end{equation}

Obviously, it is a $\Gamma$-submodule in $U$ and $D_r(U(S)) = D_r(U(S)) = U(S)$. This notion will give us one more characterization of Galois orders (Theorem 5.1).

It will be convenient to consider the $\Gamma$-bimodule structure of $U$ as a $\Gamma \otimes_k \Gamma$-module structure.

For every $f \in \Gamma$ define $f^S_G \in \Gamma \otimes_k L$ (respectively $f^S_L \in L \otimes_k \Gamma$) as follows
\begin{equation}
f^S_G = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{-i}) = \sum_{i=0}^{[S]} f^{[S]-i} \otimes h_i, \ h_0 = 1,
\end{equation}
\begin{equation}
(f^S_L) = \prod_{s \in S} (f^i \otimes 1 - 1 \otimes f) = \sum_{i=0}^{[S]} h_i' \otimes f^{[S]-i}, \ h_0' = 1).
\end{equation}

Since $S$ is $\Gamma$-invariant, then all $h_i$ and $h_i'$ are $\Gamma$-invariant expressions in $f^m, m \in M$, they belong to $K$. If $U$ is right (left) integral, then $h_i^S \in \Gamma \otimes \Gamma$ ($h_i^S \in \Gamma \otimes \Gamma$). We will consider the properties of $f_S = f^S_G$, the case of $f^S_L$ can be treated analogously.

Lemma 5.2. Let $m^{-1}(\Gamma) \subset \Gamma$ for any $m \in M$, $S \subset M$ a $\Gamma$-invariant subset, $u \in U, f \in \Gamma$.

(1) $u \in U(S)$ if and only if $f_S \cdot u = 0$ for every $f \in \Gamma$.

(2) If $T = \text{supp}\ u \subset S$ then $f_T \cdot u \in U(S)$ for every $f \in \Gamma$.

(3) If $f_S = \sum_{i=1}^{n} f_i \otimes g_i, [am] \in L \ast M$ then $f_S \cdot [am] = [\sum_{i=1}^{n} f_i g_i^m a]m = [\prod_{s \in S} (f \otimes f^{-m}) am]$.

(4) Let $S$ be a $\Gamma$-orbit and $T$ an $\Gamma$-invariant subset in $M$. The $\Gamma$-bimodule homomorphism $P^S_T(U(T)) \to U(S), u \mapsto f_T \cdot u, f \in \Gamma$ is either zero or $\text{Ker}\ P^S_T = U(T \setminus S)$ (both cases are possible, cf. (1)).

(5) Let $S = S_1 \cup \cdots \cup S_n$ be the decomposition of $S$ in $\Gamma$-orbits and $P^S_{S_i} : U(S) \to U(S_i)$ for some $f_i \in \Gamma, i = 1, \ldots, n$ are defined in (4) nonzero homomorphisms. Then the homomorphism
\begin{equation}
P^S : U(S) \to \bigoplus_{i=1}^{n} U(S_i), \ P^S = (P^S_{S_1}, \ldots, P^S_{S_n}),
\end{equation}
is a monomorphism.

(6) The statements above hold, if $\Gamma$ is a Harish-Chandra subalgebra in $U$.

Proof. Consider any $[am] \in L \ast M, s \in \text{Aut} L$. Then
\begin{equation}
(f \otimes 1 - 1 \otimes f^s) \cdot [am] = [fam] - [amf^s] = [(f \otimes f^{-ms}) am],
\end{equation}
\begin{equation}
f_S \cdot [am] = \prod_{s \in S} (f \otimes 1 - 1 \otimes f^{-i}) \cdot [am] = \prod_{s \in S} (f \otimes f^{-ms}) am).
\end{equation}

If $m \in S$, then one of $f - f^{-ms}$ equals zero, hence, $f_S \cdot [am] = 0$. To prove the converse we show that for any $m \not\in S$ there exists $f \in \Gamma$ such that $f \neq f^{-ms}$ for all $s \in S$. Following Lemma 2.1, (2) for every $m \in M, m \neq e$, the space of $m$-invariants $\Gamma^m \neq \Gamma$. But the $k$-vector space $\Gamma$ can not be covered by finitely many proper subspaces $\Gamma^{ms}, s \in S$, that completes the proof of (1).

Obviously, $f_{\text{supp}\ u} \cdot u = 0$ for any $f \in \Gamma$. Then statement (2) follows from (1) and from the equality $f_{\text{supp}\ u} = f_S f_T$. Statement (3) follows from the formulas (2.8), 2.2. By (3), $f_T \setminus S \neq 0$ if and only if $\sum_{i=1}^{n} f_i g_i^m \neq 0$, and in this case $f_T \setminus S$ acts on $U(S)$ injectively, that proves (4).

Finally, (5) follows from (4), since $\bigcap_{i=1}^{n} \text{Ker}\ P^S_{S_i} = 0$ and (6) follows from the definition. \(\square\)

Theorem 5.1. Let $U$ be a Galois ring over a noetherian Harish-Chandra subalgebra $\Gamma$. Then the following statements are equivalent:

(1) $U$ is right (respectively left) Galois order.
\((2)\) \(U(S)\) is finitely generated right (respectively left) \(\Gamma\)-module for any finite \(G\)-invariant subset \(S \subset \mathcal{M}\).

\((3)\) \(U(O_m)\) is finitely generated right (respectively left) \(\Gamma\)-module for any \(m \in \mathcal{M}\).

**Proof.** Assume \(U\) is right Galois order. Let \(S\) be a finite \(G\)-invariant subset of \(\mathcal{M}\), and \(u_1, \ldots, u_k \in U(S)\) a basis of \(U(S)K\) as a right \(K\)-space. Then
\[
\mathbb{D}_r(\sum_{i=1}^{k} u_i \Gamma) = (\sum_{i=1}^{k} u_i \Gamma)K \cap U = U(S)K \cap U = \mathbb{D}_r(U(S)) = U(S).
\]
Therefore, \(U(S) = \mathbb{D}_r(\sum_{i=1}^{k} u_i \Gamma)\), which proves \((2)\). Obviously, \((2)\) implies \((3)\). Assume \((3)\) holds.

Let \(M \subset U\) be a finitely generated right \(\Gamma\)-submodule, \(S = \text{supp } M\). Then \(M \subset U(S)^\wedge\) and \(\mathbb{D}_r(M) \subset \mathbb{D}_r(U(S)) = U(S)\). By Corollary 5.1, it remains to prove that \(U(S)\) is finitely generated.

Let \(S = S_1 \sqcup \cdots \sqcup S_n\) be the decomposition of \(S\) into \(G\)-orbits. Following Lemma 5.2, \((5)\), \(P^S\) embeds \(U(S)\) into \(\bigoplus_{i=1}^{n} U(S_i)\) that completes the proof. \(\square\)

**Theorem 5.5.** Assume \(U\) is a Galois ring, \(\Gamma\) is noetherian and \(M\) is a group.

\((1)\) Assume \(m^{-1}(\Gamma) \subset \hat{\Gamma}\) (resp. \(m(\Gamma) \subset \hat{\Gamma}\)). Then \(U\) is right (resp. left) Galois order if and only if \(U_e\) is an integral extension of \(\Gamma\).

\((2)\) Assume \(\Gamma\) is a Harish-Chandra subalgebra in \(U\). Then \(U\) is a Galois order if and only if \(U_e\) is an integral extension of \(\Gamma\).

**Proof.** Obviously \((2)\) follows from \((1)\) and Proposition 5.1. The statement “only if” in \((1)\) follows from Corollary 5.3. Assume \(U_e\) is an integral extension of \(\Gamma\), \(m^{-1}(\Gamma) \subset \hat{\Gamma}\), but \(U\) is not right order. Following Theorem 5.1, \((3)\) there exists \(m \in \mathcal{M}\), such that \(U(O_m)\) is not finitely generated.

Since \(\mathcal{M}\) is a group then there exists \([bm^{-1}] \in U\) by Lemma 4.1. Since \(H_m = H_{m^{-1}}\) for any nonzero \(\gamma \in \Gamma\) holds
\[
([bm^{-1}]\gamma [ma])_e = \sum_{g \in G/H_m} b^{g \gamma (m^{-1})\gamma} a^g.
\]
Denote this expression by \(v_e(a), \gamma, a \in L^H_m\). Then \(v_e : L^H_m \to K\) is a right \(K\)-linear map and \(v_{\gamma_1} + v_{\gamma_2} = v_{\gamma_1 + \gamma_2}, \gamma_1, \gamma_2 \in \Gamma\).

Denote \(|G/H_m|\) by \(n\). Let \(\{a_i \in L^H_m \mid i = 1, \ldots, n\}\) be a basis of \(L^H_m\) over \(K\). In particular, \([ma_i], i = 1, \ldots, n\) form a right \(K\)-basis of \(KmK\). It will be convenient for us to enumerate entries of matrices both by the classes from \(G/H_m\) and the numbers \(1, \ldots, n\).

**Lemma 5.3.**

\((1)\) For any \(b \in L^H_m, b \neq 0\) the \(G/H_m \times n\) matrix over \(L\)
\[
X = (b^g a_i^g \mid g \in G/H_m; i = 1, \ldots, n)
\]
is non-degenerated.

\((2)\) There exists \(\gamma_1, \ldots, \gamma_n \in \Gamma\), such that \(n \times G/H_m\) matrix
\[
Y = (\gamma_i g^{-1} a_i \mid i = 1, \ldots, n; g \in G/H_m)
\]
is non-degenerated. Besides for \(n \times n\) matrices holds
\[
YX = (v_{\gamma_i}(a_j) \mid i, j = 1, \ldots, n).
\]

\((3)\) Let \(Z = (\mu_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, n)\) be a non-degenerated matrix over \(K\), \(b_i = \sum_{j=1}^{n} a_j \mu_{ij}, i = 1, \ldots, n\) the new basis of \(L^H_m\). Then
\[
(YX)Z = (v_{\gamma_i}(b_j) \mid i, j = 1, \ldots, n).
\]

\((4)\) In particular, if \(Z = (YX)^{-1}\) holds
\[
v_{\gamma_i}(b_j) = \delta_{ij}, i, j = 1, \ldots, n.
\]
Proof. To prove the first statement there is enough to prove the invertibility of the matrix \((a_g | g \in G/H_m; i = 1, \ldots, n)\). Assume, opposite, i.e. \((\sum_{g \in G/H_m} \lambda_g g)(a_i) = 0, \lambda_g \in L\), for some vector \((\lambda_g | g \in G/H_m) \neq 0\) and for any \(i = 1, \ldots, n\). Then \((\sum_{g \in G/H_m} \lambda_g g)\mid_{L^H_m} = 0\), which contradicts to the independence of different characters \(g\mid_{L^H_m} : L^H_m \rightarrow L, g \in G/H_m\).

Analogously all \(\{gm^{-1}g^{-1} | g \in G/H_m\}\) act differently in restriction on \(\Gamma\), hence the row rank of \(G/H_m \times \Gamma\) matrix over \(L\)

\[(\gamma^g | g \in G/H_m; \gamma \in \Gamma),\]

equals \(n\). Then its column rank of this matrix equals \(n\) as well, that finishes the proof of the second statement.

The third and fourth statement is proved by direct calculation

\[(YX)_{ij} = \sum_{g \in G/H_m} b_{\gamma_i} g m^{-1} g^{-1} a_j = v_{\gamma_i}(a_j).\]

\[((YX)Z)_{ij} = \sum_{l=1}^{n} v_{\gamma_i}(a_l) \mu_{ij} = v_{\gamma_i}(\sum_{l=1}^{n} a_l \mu_{ij}) = v_{\gamma_i}(b_j).\]

The last statement is obvious. \(\square\)

Assume \(U(\mathcal{O}_m)\) contains a strictly ascending chain of right \(\Gamma\)-submodules

\[(5.18) \quad M_k = \sum_{i=1}^{k} [mt_i]_{\Gamma}, i = 1, 2, \ldots, M = \bigcup_{k=1}^{\infty} M_k.\]

Fix \(\gamma_1, \ldots, \gamma_n\) from Lemma 5.3, (2) and the basis \(b_1, \ldots, b_n\) from Lemma 5.3, (4).

Consider the decomposition \(t_l = \sum_{i=1}^{n} \gamma_i b_j, \gamma_{ij} \in K\). Then there exists \(1 \leq l \leq n\), such that the \(\Gamma\)-module \(T_l = \sum_{i=1}^{n} \Gamma_{\gamma_{il}} \subset K\) is not finitely generated. In opposite case from notherianity of \(\Gamma\) and \(M \subset \bigoplus T_i\) follows, that \(M\) is finitely generated.

Then \((\{bm\gamma | \gamma[m^{-1}M]\})_e = v_{\gamma_i}(M) = T_l, which is not finitely generated. Let \(S = \mathcal{O}_{m^{-1}}\mathcal{O}_m\).

Since \(m^{\pm 1}(\Gamma) \subset \bar{\Gamma}\) there exists \(F = \sum_{i=1}^{n} f_i \otimes g_i \in \Gamma \otimes_k \Gamma\) (by Lemma 5.2, (3)), which defines a nonzero morphism \(P^S_e : U(S) \rightarrow U(\langle e \rangle) = U_e\). Then by Lemma 5.2, (3)

\[P^S_e ([bm] \gamma[\gamma[m^{-1}M]]) = \gamma T_l \subset U_e, \gamma = \sum_{i=1}^{n} f_i g_i,\]

which means that \(U_e\) is not finitely generated. \(\square\)

Corollary 5.5. Let \(\mathcal{M}\) be a group, \(\Gamma\) normal and noetherian, \(\mathcal{M} \cdot \Gamma = \bar{\Gamma}, \varphi_1, \ldots, \varphi_n \in \mathcal{M}\) a set of generators of \(\mathcal{M}\) as a semigroup, \(a_1, \ldots, a_n \in \bar{\Gamma}\). Then the subalgebra \(U\) in \(\mathcal{K}\) generated by \(\Gamma\) and \([a_1 \varphi_1], \ldots, [a_n \varphi_n]\) is a Galois order over \(\Gamma\).

Proof. Since \(\mathcal{M} \cdot \bar{\Gamma} = \bar{\Gamma}\) any \(u \in U\) has a form \(u = \sum_{m \in \mathcal{M}} [a_m m]\), where all \(a_m\) are in \(\bar{\Gamma}\). In particular, if \(u \in U_e\) then \(u = [a_e e]\) where \(a_e \in K \cap \bar{\Gamma}\). Since \(\Gamma\) is normal \(U_e = \Gamma\). Applying Theorem 5.2, (2) we obtain the statement. \(\square\)

The next corollary is a noncommutative analog of Proposition 2.2.

Corollary 5.6. Let \(U \subset L \ast \mathcal{M}\) be a Galois ring over noetherian \(\Gamma, \mathcal{M}\) a group and \(\Gamma\) a normal \(k\)-algebra. Then the following statements are equivalent

1. \(U\) is a Galois order.
(2) \( \Gamma \) is a Harish-Chandra subalgebra and, if for \( u \in U \) there exists a nonzero \( \gamma \in \Gamma \) such that \( \gamma u \in \Gamma \) or \( \gamma^{-1} u \in \Gamma \), then \( u \in \Gamma \).

Proof. Assume (1). Then \( \Gamma \) is a Harish-Chandra subalgebra by Corollary 5.4. If \( u \gamma \in \Gamma \) for \( u \in U \) and \( \gamma \in \Gamma \), then \( \text{supp } u = \{ e \} \), hence \( u \in U_e \). Applying Corollary 5.3 we obtain (2). To prove the converse implication consider \( u \in U_e \). Since \( U_e \subset K \) (Theorem 4.1, (1)), there exists \( \gamma \in \Gamma \), such that \( \gamma u \in \Gamma \). Thus, \( u \in \Gamma \). Theorem 5.2, (2) completes the proof. \( \square \)

5.4. Filtered Galois orders. Let \( U \) be a Galois ring over a noetherian normal \( k \)-algebra \( \Gamma \). Suppose in addition that \( U \) is an algebra over \( k \), endowed with an increasing exhausting filtration \( \{ U_i \}_{i \in \mathbb{Z}}, U_{-1} = \{ 0 \}, U_0 = k, U_i U_j \subset U_{i+j} \) and \( \text{gr} U = \bigoplus_{i=0}^{\infty} U_i / U_{i-1} \) the associated graded algebra.

The filtration on \( \Gamma \) induces a degree "deg" both on \( U \) and \( \text{gr} U \). For \( u \in U \) denote by \( \bar{u} \in \text{gr} U \) the corresponding homogeneous element and denote by \( \text{gr} \Gamma \) the image of \( \Gamma \) in \( \text{gr} U \).

Proposition 5.3. Assume \( \text{gr} U \) is a domain. If the canonical embedding \( i : \text{gr} \Gamma \rightarrow \text{gr} U \) induces an epimorphism

\[
\text{id} : \text{Specm} \text{gr} U \rightarrow \text{Specm} \text{gr} \Gamma
\]

then \( U \) is a Galois order over \( \Gamma \).

Proof. We apply Corollary 5.6. Suppose \( y = xu \neq 0, y, x \in \Gamma \), \( u \in U \setminus \Gamma \) with minimal possible \( \text{deg} y \). Then \( \bar{y} = \bar{x} \bar{u} \neq 0 \) in \( \text{gr} U \). By Proposition 2.2 \( \bar{u} \in \text{gr} \Gamma \). Hence \( \bar{u} = \bar{x} \) for some in \( z \in \Gamma \). Since \( z \neq u \), we have \( y_1 = xu_1 \) where \( u_1 = u - z, y_1 = y - xx \). Then \( x, y_1 \in \Gamma \), \( u \notin \Gamma \) and \( \text{deg} y_1 < \text{deg} y \). Obtained contradiction shows that \( u \in \Gamma \). \( \square \)

6. Gelfand-Kirillov dimension of Galois orders

In this section we assume that \( \mathcal{M} \) is a group of finite growth and \( \Gamma \) is an affine \( k \)-algebra of finite Gelfand-Kirillov dimension.

6.1. Growth of group algebras. Let \( S_s = \{ S_1 \subset S_2 \subset \cdots \subset S_N \subset \cdots \} \) be an increasing chain of finite sets. Then the growth of \( S_s \) is defined as

\[
\text{growth}(S_s) = \lim_{N \to \infty} \log_N |S_N|.
\]

For \( s \in S = \bigcup_{i=0}^{\infty} S_i \) set \( \text{deg } s = i \) if \( s \in S_i \setminus S_{i-1} \). Let \( \{ \gamma_1, \ldots, \gamma_k \} \) be a set of generators of \( \Gamma \). For \( N \in \mathbb{N} \) denote by \( \Gamma_N \subset \Gamma \) the subspace of \( \Gamma \) generated by the products \( \gamma_{i_1} \cdots \gamma_{i_t} \), for all \( t \leq N, i_1, \ldots, i_t \in \{ 1, \ldots, k \} \). Let \( d_r(N) = \dim_k \Gamma_N \) and let \( B_N(\Gamma) \) be a basis in \( \Gamma_N \) \((B_1(\Gamma) = \{ \gamma_1, \ldots, \gamma_k \})\). Fix a set of generators of \( \mathcal{M} \) of the form \( \mathcal{M}_1 = \mathcal{O}_{\varphi_1} \cup \cdots \cup \mathcal{O}_{\varphi_n} \). For \( N \geq 1 \), let \( \mathcal{M}_N \) be the set of words \( w \in \mathcal{M} \) such that \( l(w) \leq N \), where \( l \) is the length of \( w \), i.e.

\[
\mathcal{M}_{N+1} = \mathcal{M}_N \bigcup \left( \bigcup_{\varphi \in \mathcal{M}_1} \varphi \cdot \mathcal{M}_N \right).
\]

Note that all sets \( \mathcal{M}_N \) are \( G \)-invariant. Denote the cardinality of \( \mathcal{M}_N \) by \( d_{\mathcal{M}}(N) \). Let \( \mathcal{M}_* = \{ \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_N \subset \cdots \} \). Then growth(\( \mathcal{M} \)) is by definition growth(\( \mathcal{M}_* \)).

Let \( \Gamma[\mathcal{M}] \) be the group algebra of \( \mathcal{M} \). Assume, \( G \) acts on \( \Gamma[\mathcal{M}] \), acting by \( \mathcal{M} \) by conjugations and trivially on \( \Gamma \). Then the space \( \Gamma[\mathcal{M}] \) has a \( G \)-invariant basis

\[
B_N(\Gamma[\mathcal{M}]) = \bigcup_{i=0}^{N} \bigcup_{l(w)=N-i} B_i(\Gamma)w.
\]

and \( \text{GKdim } \Gamma[\mathcal{M}] = \text{growth } B_N(\Gamma[\mathcal{M}]) \). In particular (e.g. [MCR], Lemma 8.2.4)

\[
\text{GKdim } \Gamma[\mathcal{M}] = \text{GKdim } \Gamma + \text{growth } (\mathcal{M}).
\]
The growth of the chain $B_*(\Gamma[M])/G$ is equal to growth $B_*(\Gamma[M])$, since
\[ |B_N(\Gamma[M])/G| > |B_N(\Gamma[M])/G| \geq \frac{1}{|G|} |B_N(\Gamma[M])|. \]

Without loss of generality we can assume that the Galois ring $U$ is generated over $\Gamma$ by a set of generators $\mathcal{G} = \{a_1\varphi_1, \ldots, a_n\varphi_n\}$. Set $B_1(U) = B_1(\Gamma) \cup \mathcal{G}$. As above, define the subspaces $U_N$ and dimensions $d_U(N)$. For every $N \geq 1$ fix a basis $B_N(U)$ of $U_N$.

6.2. Gelfand-Kirillov dimension. The goal of this section is to prove (under a certain condition) an analogue of the formula (6.22) for Galois orders.

**Condition 1.** For every finite dimensional $\mathbb{k}$-vector space $V \subset \bar{\Gamma}$ the set $M \cdot V$ is contained in a finite dimensional subspace of $\bar{\Gamma}$.

**Theorem 6.1.** If $U$ is a Galois $\Gamma$-ring which satisfies Condition 1 and $M$ is a group of finite growth($M$), then
\[ (6.23) \quad \text{GKdim } U \geq \text{GKdim } \Gamma + \text{growth}(M). \]

The proof of this result is based on the following lemmas.

**Lemma 6.1.** If for some $p, q \in \mathbb{Z}$ and $C > 0$ for any $N \in \mathbb{N}$ holds
\[ (6.24) \quad d_U(pN + q) \geq C \log N \| x \|_{\Gamma[M]}(N), \]
then $\text{GKdim } U \geq \text{GKdim } \Gamma[M]$.

**Proof.**
\[
\text{GKdim } \Gamma[M] = \lim_{N \to \infty} \log_N d_{\Gamma[M]}(N) \leq \lim_{N \to \infty} \log_N d_U(pN + q) = \lim_{N \to \infty} \log_{pN + q} d_U(pN + q) \leq \lim_{N \to \infty} \log_N d_U(N) = \text{GKdim } U.
\]

Denote by $N(i), i = 1, 2, \ldots$ the minimal number such that for any $m \in M_i$, $U_{N(i)}$ contains an element of the form $[bm], b \neq 0$.

**Lemma 6.2.**
1. For every $i = 1, \ldots, n$ there exists a finite dimensional over $\mathbb{k}$ space $V_i \subset \Gamma$, such that for any $x \in U$ and $m \in \text{supp } x$ there exists $y \in \text{supp } \varphi_m \text{supp } y$. Besides $| \text{supp } y | \leq |G| | \text{supp } x |$ and $\deg y - \deg x \leq d$ for some fixed $d > 0$.
2. For every $k \geq 1$ there exists $t(k) \geq 0$ with the following property: for every $j \geq 1$ and $u \in U_j$, such that $| \text{supp } u | \leq k$ and for any $m \in \text{supp } u$ there exists a nonzero element $[bm] \in U_{j+t(k)}$.
3. The sequence $N(i + 1) - N(i), i = 1, 2, \ldots$ is bounded.

**Proof.** Let $L(G/H_{\varphi_i})$ be the vector space over $L$ with the basis, enumerated by cosets $G/H_{\varphi_i}, \varphi \in M$. We endow this space with the standard scalar product. Fix $i, 1 \leq i \leq n$ and consider the nonzero vector
\[ v(x) = (a^g_{\varphi_i} x, v_{\varphi_i}^{-1})_{g \in G/H_{\varphi_i}, v \in L(G/H_{\varphi_i})}. \]
Then for any $\gamma \in \Gamma$ immediate calculation shows, that
\[ (6.25) \quad (a^g_{\varphi_i} v_{\varphi_i})_{g \in G/H_{\varphi_i}} \in L_{H_{\varphi_i},m}. \]
Since $\varphi_i^g, g \in G/H_{\varphi}$ are different, there exist $\gamma_1, \ldots, \gamma_k \in \Gamma, k = |G/H_{\varphi_i}|$, such that the $k \times k$ matrix $(\gamma_j^g)_{g \in G/H_{\varphi_i}}$ is non-degenerated. Then we set $V_i = (\gamma_1, \ldots, \gamma_k)$. Since the multiplication on $\gamma \in \Gamma, \gamma \neq 0$ does not change the support, we obtain
\[ | \text{supp } y | \leq k | \text{supp } x | \leq |G| | \text{supp } x |. \]
As $d$ we can choose the maximum of $d_i = 1 + \max \{ \deg v | v \in V_i \}, i = 1, \ldots, n$. It proves (1).

Now we prove (2). If $\text{supp } u = G \cdot m$ then $u = [bm]$ for some $b \in L_{H_{\varphi_i}}$ and there is nothing to prove. Fix some $k \geq 2$. Assume $u = [cm] + v, m \notin \text{supp } v, | \text{supp } v | \leq k - 1$. For $f \in \Gamma_1$ consider
the polynomial \( f_S \), (subsection 5.3, (5.14)) with \( S = \text{supp} \ u \setminus \mathbf{G} \cdot \mathbf{m} \). Applying Lemma 5.2 we obtain the element
\[
 f_S \cdot u = f_S \cdot [cm] = [a \prod_{s \in S} (f - f^{ms^{-1}})m].
\]
Since nonunit elements \( ms^{-1}, s \in S \) act nontrivially on \( \Gamma \), there exists \( f \in \Gamma^1 \) such that \( f_S \cdot u \) is nonzero. Then
\[
 [bm] := f_S \cdot u = \sum_{i=0}^{|S|} T_i u f^{(S)^{-i}}, \quad \text{where} \quad T_i = \max_{T \subseteq S, \tau = (t_1, \ldots, t_i)} f^{t_1} \cdots f^{t_i} \in \Gamma.
\]

Due to Condition 1 all \( f^t, t \in S \) belong to a finite dimensional space \( V \) generated by \( \{ \psi \Gamma \mid \psi \in \mathcal{M} \} \subset \Gamma \). Hence all \( T_i \)-th belong to the finite dimensional space \( V(k) = \Gamma \cap \sum_{i=0}^k V_i \). Denote \( C_k \) the maximal degree of elements from \( V(k) \). Then
\[
 \deg [bm] \leq \max \{ \deg T_i u f^{(S)^{-i}} \mid i = 0, \ldots, |S| \} \leq C_k + j + |S|.
\]
Hence we can set \( t(k) = k + C_k \).

To prove (3) consider \( x = [cm] \in U_{N(i)} \), \( m \in \mathcal{M}_i \). By (1) for given \( \varphi_i \in \mathcal{M}_1 \) there exists \( y \in U_{N(i) + d} \) such that \( \varphi m \in \text{supp} y \) and \( \text{supp} y \leq |G| \). Then by (2) \( U_{N(i) + d + t(|G|)} \) contains an element of the form \([b \varphi_i m] \).

Now we are in the position to prove Theorem 6.1. Let \( D = d + t(|G|) \). The space \( U_1 \) contains elements \([a_i \varphi_i] \), where \( \varphi_i \) runs over \( \mathcal{M}_1 / \mathbf{G} \). Then, by Lemma 6.2, (3), \( U_{D(N(i) - 1) + 1} \) contains a set of the form \( \mathcal{M}_N \{ [cm] \mid m \in \mathcal{M}_N, c_m \neq 0 \} \), hence \( U_{D(N-1)+N+1} \) contains \( \Gamma_N \mathcal{M}_N \). All elements from \( \Gamma_N \mathcal{M}_N \) are linearly independent. But the set \( B_N(\Gamma \mathcal{M}) / G \) is embedded into \( \Gamma_N \mathcal{M}_N \) by setting \( \gamma[w] \mapsto \gamma[c_w w] \), \( \gamma \in \Gamma_N \), \( w \in \mathcal{M}_N \). Therefore,
\[
 d_U(N(D + 1) + 1 - D) \geq |B_N(\Gamma \mathcal{M}) / G| \geq \frac{1}{|G|} |B_N(\Gamma \mathcal{M})|.
\]
It remains to set \( p = D + 1, q = 1 - D, C = \frac{1}{|G|} \) and apply Lemma 6.1.

7. Examples of Galois rings and orders

7.1. Generalized Weyl algebras. Let \( \sigma \) be an automorphism of \( \Gamma \) of infinite order, \( X \) and \( Y \) generators of the bimodules \( \Gamma_{\sigma^{-1}} \) and \( \Gamma_\sigma \) respectively, \( V = \Gamma_{\sigma^{-1}} \oplus \Gamma_\sigma \), \( G = \{ e \} \) and \( \mathcal{M} \) is the cyclic group generated by \( \sigma \). Consider a Galois order \( U \) in \( K \ast \mathcal{M} \) which is the image of some homomorphism \( \tau : \Gamma[V] \to K \ast \mathcal{M} \) of the form \( \tau(X) = a_X b_X^{-1} \sigma^{-1}, \tau(Y) = a_Y b_Y^{-1} \sigma \) for some \( a_X, b_X, a_Y, b_Y \in \Gamma \setminus \{ 0 \} \). We can assume \( a_X = b_X = 1 \). The element \( a = a_Y b_Y^{-1} \) defines a 2-cocycle \( \xi : \mathbb{Z} \times \mathbb{Z} \to K^* \), such that \( \xi(-1,1) = a \). The following statement is obvious.

**Proposition 7.1.** \( U \) is a Galois order over \( \Gamma \) if and only if \( a \in \Gamma \). In this case \( U \) is isomorphic to a generalized Weyl algebra of rank 1 ([Ba], i.e. the algebra generated over \( \Gamma \) by \( X, Y \) subject to the relations
\[
 X \lambda = \lambda \sigma X, \quad Y \lambda = Y \lambda \sigma, \quad \lambda \in \Lambda; \quad YX = a, \quad XY = a^\sigma.
\]

7.2. Filtered algebras. Let \( U \) be an associative filtered algebra over \( k \).

**Theorem 7.1.** Suppose \( U \) is generated by \( u_1, \ldots, u_k \) over \( \Gamma \), if \( U \) a polynomial ring in \( N \) variables, \( \mathcal{M} \subset \text{Aut} \ L \) a group and \( f : U \to \mathcal{K} \) a homomorphism such that \( \cup \text{supp} f(u_i) \) generates \( \mathcal{M} \). If
\[
 \text{GKdim} \Gamma + \text{growth} \mathcal{M} = N
\]
then \( f \) is an embedding and \( U \) is a Galois ring over \( \Gamma \).
Proof. Note that $f(U)$ is a Galois $\Gamma$-ring by Proposition 4.1. Also

$$\text{GKdim } f(U) \geq \text{GKdim } \Gamma + \text{growth } \mathcal{M} = N$$

by Theorem 6.1. Hence it is enough to prove that $I = \text{Ker } f$ equals zero. Assume $I \neq 0$. Then

$$N = \text{GKdim } U = \text{GKdim } \text{gr } U > \text{GKdim } \text{gr } U / \text{gr } I = \text{GKdim } f(U) \geq N$$

which is a contradiction. $\square$

Below in 7.2.1 Theorem 7.1 will be applied to construct examples of Galois rings.

7.2.1. General linear Lie algebras. Let $\mathfrak{gl}_n$ be the general linear Lie algebra over $\mathfrak{k}$, $e_{ij}, i, j = 1, \ldots, n$ its standard basis, $U_n$ its universal enveloping algebra and $Z_n$ the center of $U_n$. Then we have natural embeddings on the left upper corner

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n$$

and induced embeddings $U_1 \subset U_2 \subset \ldots \subset U_n$.

The Gelfand-Tsetlin subalgebra $\Gamma$ in $U_n$ is generated by $\{Z_m \mid m = 1, \ldots, n\}$, which is a polynomial algebra in $\frac{n(n+1)}{2}$ variables. Denote by $K$ be the field of fractions of $\Gamma$. In the paper [Zh] was constructed a system of generators $\{\lambda_{ij} \mid 1 \leq j \leq i \leq n\}$ of $\Gamma$ and the Galois extension $\Lambda \supset \Gamma$ with the following properties.

1. $\Lambda$ is the algebra of polynomial functions on $\mathcal{L}$ algebra in variables $\{\lambda_{ij} \mid \ell_{ij} \in \mathfrak{k}, 1 \leq j \leq i \leq n\}$, $\mathcal{L} = \text{Specm } \Lambda$. An element $\ell = (\lambda_{ij} - \ell_{ij} \mid \ell_{ij} \in \mathfrak{k}, 1 \leq j \leq i \leq n)$ of $\mathcal{L}$ is usually written in the form of tableaux consisting of $n$ rows

$$\ell_{n1} \quad \ell_{n2} \quad \ldots \quad \ell_{nn}
\ell_{n-1,1} \quad \ldots \quad \ell_{n-1,n-1}
\ldots \quad \ldots \quad \ldots
\ell_{21} \quad \ell_{22}
\ell_{11}$$

(7.26)

2. The product of the symmetric groups $G = \prod_{i=1}^{n} S_i$ acts naturally on $\mathcal{L}$, where every $S_i$ permutes elements of $i$-th row. This action induces the action of $G$ on $\Lambda$.

3. $\Gamma$ is identified with the invariants $\Lambda^G$, such that $\gamma_{ij} = \sigma_{ij}(\gamma_{11}, \ldots, \gamma_{ii})$ where $\sigma_{ij}$ is the $j$th symmetrical polynomial in $i$ variables. Denote by $L$ the fraction field of $\Lambda$. Then $L^G = K$ and $G = G(L/K)$ is the Galois group of the field extension $K \subset L$.

4. Denote by $\delta_{ij} \in \mathcal{L}$ a tableau whose $ij$-th element equals 1 and all other elements are 0. Let $\mathcal{M} \simeq \mathbb{Z}^{\frac{n(n-1)}{2}}$ be additive free abelian group with free generators $\delta_{ij}$, $1 \leq j < i \leq n-1$. Analogously to (7.26) the elements of $\mathcal{M}$ are written as tableaux. Then $\mathcal{M}$ acts on $\mathcal{L}$ by shifts: $\delta_{ij} \cdot \ell = \ell + \delta_{ij}$, $\delta_{ij} \in \mathcal{M}$. This action of $\mathcal{M}$ on $\mathcal{L}$ induces the action on $\Lambda$ and $L$, hence we can consider $\mathcal{M}$ as a subgroup in $\text{Aut } L$. Note that $G$ acts on $\mathcal{M}$ by conjugations.

As in Section 4 denote $X = (L \ast \mathcal{M})^G$.

In [Zh], Ch. X.70, Theorem 7, the Gelfand-Tsetlin formulae (in Zhelobenko form) are given for the action of generators of $\mathfrak{gl}_n$ on a Gelfand-Tsetlin basis of a finite dimensional irreducible representation. We show that these formulae in fact endow $U(\mathfrak{gl}_n)$ with a structure of a Galois order (Proposition 7.2). We need the following corollary from the Gelfand-Tsetlin formulae (see [BL] or [DF02]).

**Theorem 7.2.** Let $\Omega \subset \mathcal{L}$ be a set of tableaux $\ell = (\ell_{ij})$ such that $\ell_{ij} - \ell_{ij'} \notin \mathbb{Z}$ for all possible pairs $i, i', j, j', (i, j) \neq (i', j')$. Consider a $\mathfrak{k}$-vector space $T_\ell$ with the basis $\mathcal{M}$ and with the action of $E_k^+ = e_{k,k+1}, E_k^- = e_{k+1,k}, k = 1, \ldots, n-1$, given by the formulae

$$E_k^\pm \cdot m = \sum_{i=1}^{k} a_{ki}^\pm(\ell)(m \pm \delta^{ki}),$$
where $m \in \mathcal{M}$ and
\begin{equation}
(7.27)
a_{ki}^\pm(\ell) = \prod_{j \neq i}(\ell_{k+1,j} - \ell_{ki}) \prod_{j \neq i}(\ell_{kj} - \ell_{ki}).
\end{equation}

The action of an element $\gamma \in \Gamma$ on the basis vector $[\ell]$ is just the multiplication on $\gamma(\ell) \in k$.

Analogously to [O] we show, that the formulae (7.28) defines a homomorphism of $U_n$ to $\mathcal{K}$.

**Proposition 7.2.** $U_n$ is a Galois ring over $\Gamma$. This structure is defined by the embedding $\iota : U \rightarrow \mathcal{K}$ where
\begin{equation}
(7.28)
i(e_{kk+1}) = \sum_{i=1}^k \delta^{ki}a_{ki}^+ = [\delta^{ki}a_{ki}^+],
i(e_{k+1,k}) = \sum_{i=1}^k (-\delta^{ki})a_{ki}^- = [(-\delta^{ki})a_{ki}^-],
a_{ki}^+ = \prod_{j \neq i}(\lambda_{k+1,j} - \lambda_{ki}),
a_{ki}^- = \prod_{j \neq i}(\lambda_{kj} - \lambda_{ki}),
\end{equation}

for $k = 1, \ldots, n$.

**Proof.** Let $S$ be the multiplicative $\mathcal{M}$-invariant subset in $\Gamma$, generated by $\lambda_{ij} - \lambda_{ij'} - k$ for all possible $i, i', j, j'$ with $(i, j) \neq (i', j')$, where $k$ running $\mathbb{Z}$, and $\Lambda_S$ the corresponding localization. Then $\Lambda_S \circ \mathcal{M}$ has a structure of a $\Lambda_S \circ \mathcal{M}$-bimodule and for every $\ell \in \Omega = \text{Specm} \Lambda_S$ is defined a left $\Lambda_S \circ \mathcal{M}$-module
\[ V_\ell = (\Lambda_S \circ \mathcal{M}) \otimes_{\Lambda_S} (\Lambda_S/\ell). \]

Analogously the action from the left by elements $\sum_{i=1}^k (\pm \delta^{ki})a_{ki}^+(\lambda), k = 1, \ldots, n - 1$ defines on $V(\ell)$ the structures of the left $U$-module, isomorphic to the module $T_\ell$ from Theorem 7.2. These module structures define homomorphisms of $k$-algebras
\[ \tau_\ell : U \rightarrow \text{End}_k(V_\ell) \quad \text{and} \quad \rho_\ell : \Lambda_S \circ \mathcal{M} \rightarrow \text{End}_k(V_\ell), \]

besides $\text{Im} \tau_\ell \subset \text{Im} \rho_\ell$. Hence we have the diagonal homomorphisms of $k$-algebras
\[ \Delta_+: U \rightarrow \prod_{\ell \in \Omega} \text{End}_k(V_\ell) \quad \text{and} \quad \Delta_- : \Lambda_S \circ \mathcal{M} \rightarrow \prod_{\ell \in \Omega} \text{End}_k(V_\ell), \]

again $\text{Im} \Delta_+ \subset \text{Im} \Delta_-$. But $\Delta_+$ is an embedding, since for every nonzero $x \in \Lambda_S \circ \mathcal{M}$ there exists $V_\ell$, such that $x \cdot V_\ell \neq 0$. Hence the mappings (7.28) from Proposition 7.2 defines the homomorphism $i : U \rightarrow \Lambda_S \circ \mathcal{M}$. Note, that the elements in (7.28) belongs to $\mathcal{K}$, hence $i$ defines $i : U \rightarrow \mathcal{K}$. To prove, that $U$ is a Galois ring note, that $U = U(\mathfrak{gl}_n)$ is a filtered algebra, $\text{GKdim} U = n^2$ and
\[ \text{GKdim} \Gamma + \text{growth} \mathcal{M} = \frac{n(n + 1)}{2} + \frac{n(n - 1)}{2} = n^2. \]

Applying Theorem 7.1 we conclude that $i$ is an embedding and thus $U$ is a Galois ring.

Now we give here two different proofs of the fact that $U$ is a Galois order.

First method to prove that $U = U(\mathfrak{gl}_n)$ is a Galois order is based on Proposition 5.3. Let $X = (x_{ij})$ be $n \times n$-matrix with indeterminates $x_{ij}, X_k$ its submatrix of size $k \times k$, formed by the intersection of the first $k$ rows and the first $k$ columns of $X$, $\chi_{ki}$ ($i \leq k$) $i$-th coefficient of the characteristic polynomial of $X_k$. In the case of $U(\mathfrak{gl}_n)$ corresponding graded algebra $\overline{U}$ can be identified with the polynomial algebra in the variables $x_{ij}, 1 \leq i, j \leq n$ and the image of the canonical embedding $\iota : \text{gr} \Gamma \rightarrow \text{gr} U$ (see Proposition 5.3) is generated by $\chi_{ki}, 1 \leq k \leq n, 1 \leq i \leq k$. The Specm $\text{gr} U$ in a natural way can be interpreted as the space $n \times n$ matrices. Besides the induces map $\iota^* : \text{Specm} \text{gr} U \rightarrow \text{Specm} \text{gr} \Gamma$ is the map
\[ \mathbb{C}^{n^2} \rightarrow \mathbb{C}^{n(n+1)/2}, A \mapsto (\chi_{ki}(A_k) \mid k = 1, \ldots, n; i = 1, \ldots, k), \]

defined in [KW]. It is known, that this map is an epimorphism ([KW], Theorem 1). Then Proposition 5.3 implies that $U$ is a Galois order.

Another method is based on the paper [O1], where is was shown that the variety $(\iota^*)^{-1}(0)$ is an equidimensional variety of dimension $\frac{n(n - 1)}{2}$. Further, from this fact in [FO1] it is deduced that
Remark 7.2. The developed techniques can be used effectively in the case of finite $g$ for $j$ into a product of localized Weyl algebras constructed in [Kh].

Remark 7.2. The developed techniques can be used effectively in the case of finite $W$-algebras. Let $g = gl_n$, $f \in g$, $\mathfrak{g} = \bigoplus_{j \leq 2} \mathfrak{g}_j$ a good grading for $f$, i.e., $f \in \mathfrak{g}_2$ and $ad f$ is injective on $\mathfrak{g}_j$ for $j \leq -1$ and surjective for $j \geq -1$. A non-degenerate invariant symmetric bilinear form $(\ldots)$ on $\mathfrak{g}$ induces a non-degenerate skew-symmetric form on $\mathfrak{g}_{-1}$ defined by $\langle x, y \rangle = ([x, y], f)$. Let $I \subset \mathfrak{g}_{-1}$ be a maximal isotropic subspace and set $t = \bigoplus_{j \leq -1} \mathfrak{g}_j \oplus I$. Let $\chi : U(t) \to C$ be the one-dimensional representation such that $x \mapsto (x, f)$ for any $x \in t$, $I_x = \text{Ker} \chi$ and $Q_x = U(\mathfrak{g})/U(\mathfrak{g})I_x$. Then

\[ \text{End}_{U(\mathfrak{g})}(Q_x)^{op}. \]

is the finite $W$-algebra associated to the nilpotent element $f \in \mathfrak{g}$. It was shown in [BK] that any finite $W$-algebra (of type $A$) is isomorphic to a certain quotient of the shifted Yangian. It is parametrized by a sequence $\pi = (p_1, \ldots, p_n)$ with $p_1 \leq \cdots \leq p_n$. We denote the corresponding $W$-algebra by $W(\pi)$. Let $\pi_k = (p_1, \ldots, p_k)$, $k \in \{1, \ldots, n\}$. Then we have the chain of subalgebras

\[ W(\pi_1) \subset W(\pi_2) \subset \cdots \subset W(\pi_n) = W(\pi). \]

Denote by $\Gamma$ the subalgebra of $W(\pi)$ generated by the centers of $W(\pi_k)$ for $k = 1, \ldots, n$.

Theorem 7.3 ([FMO], Theorem 6.6). $W(\pi)$ is a Galois order over $\Gamma$.

7.3. Rings of invariant differential operators. In this section we construct some Galois rings of invariant differential operators on $n$-dimensional torus $k^n \setminus \{0\}$. Let $A_1$ be the first Weyl algebra over $k$ generated by $x$ and $\partial$ and $\tilde{A}_1$ its localization by $x$. Denote $t = \partial x$. Then

\[ \tilde{A}_1 \cong k[t, \sigma^{\pm 1}] \cong k[t] \ast Z, \]

where $\sigma \in \text{Aut} k[t]$, $\sigma(t) = t - 1$ and the first isomorphism is given by: $x \mapsto \sigma$, $\partial \mapsto t\sigma^{-1}$. Let $\tilde{A}_n$ be the $n$-th tensor power of $\tilde{A}_1$,

\[ \tilde{A}_n \cong k[t_1, \ldots, t_n, \sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1}] \cong k[t_1, \ldots, t_n] \ast Z^n, \]

where $x_i, \partial_i$ are natural generators of the $n$-th Weyl algebra $A_n$, $t_i = \partial_i x_i$, $\sigma_i(t_j) = t_j - \delta_{ij}$, $i = 1, \ldots, n$. Let $S = k[t_1, \ldots, t_n] \setminus \{0\}$. Then in particular we have

\[ A_n[S^{-1}] \cong k(t_1, \ldots, t_n) \ast Z^n. \]

7.3.1. Symmetric differential operators on a torus. The symmetric group $S_n$ acts naturally on $\tilde{A}_n$ by permutations. Denote $\Gamma = k[t_1, \ldots, t_n]^{S_n}$. Then we immediately have

Proposition 7.3. $\tilde{A}_n^{S_n}$ is a Galois ring over $\Gamma$ in $(k(t_1, \ldots, t_n) \ast Z^n)^{S_n}$, where $Z^n$ acts on the field of rational functions by corresponding shifts.

7.3.2. Orthogonal differential operators on a torus. The algebra $\tilde{A}_1$ has an involution $\varepsilon$ such that $\varepsilon(x) = x^{-1}$ and $\varepsilon(\partial) = -x^2 \partial$. On the other hand $k[t] \ast Z$ has an involution: $\sigma \mapsto \sigma$, $t \mapsto 2 - t$. Then $\tilde{A}_1$ and $k[t] \ast Z$ are isomorphic as involutive algebras and the isomorphism is given by: $x \mapsto \sigma$, $\partial \mapsto \sigma^{-1} + 1 - \sigma^{-2}$. Similarly we have an isomorphism of involutive algebras $\tilde{A}_n \cong k[t_1, \ldots, t_n, \sigma_1^{\pm 1}, \ldots, \sigma_n^{\pm 1}]$ and $k[t_1, \ldots, t_n] \ast Z^n$. Let $W_n$ be the Weyl group of the orthogonal Lie algebra $O_n$. If $n = 2p + 1$ then the group $W_{2p+1} = S_p \ltimes \mathbb{Z}_2^p$ acts on $\tilde{A}_p$ where $S_p$ acts by the permutations of the components and the normal subgroup $\mathbb{Z}_2^p$ is generated by the involutions described above. Consider a homomorphism $\tau : \mathbb{Z}_2^p \to \mathbb{Z}_2$ such that $(g_1, \ldots, g_p) \mapsto g_1 + \cdots + g_p$ and and let $N = \text{Ker} \tau \cong \mathbb{Z}_2^{-1}$. If $n = 2p$
then $W_{2p} \simeq S_p \ltimes N$ with a natural action on $\tilde{A}_p$. These actions induce an action of $W_n$ on $k(t_1, \ldots, t_n) \ast \mathbb{Z}^n$ for any $n$. Let $\Gamma = k[t_1, \ldots, t_n]^{W_n}$. Then we immediately have

**Proposition 7.4.** Algebra $\tilde{A}_n^{W_n}$ of orthogonal differential operators on a torus is a Galois ring over $\Gamma$ in $(k(t_1, \ldots, t_n) \ast \mathbb{Z}^n)^{W_n}$, where $\mathbb{Z}^n$ acts on the field of rational functions by corresponding shifts.

7.4. **Galois orders of finite rank.** The following example provides a link between the theory of Galois orders and the theory of orders in the classical sense.

Let $\Lambda$ be a commutative domain integrally closed in its fraction field $L$, $\mathcal{G} \subset \text{Aut} \, L$ a finite subgroup, which splits into a semi-direct product of its subgroups $\mathcal{G} \cong G \rtimes M$. Denote $\Gamma = \Lambda^G$ and $K = L^G$. Then $\Lambda$ is just the integral closure of $\Gamma$ in $L$ and the action of $G$ on $L \ast M$ is defined. A Galois order $U \subset \mathcal{K}$ over $\Gamma$ will be called a *Galois order of finite rank*.

**Proposition 7.5.** Let $U \subset \mathcal{K}$ be a Galois algebra of finite rank over $\Gamma$ and $E = L^G$. Then $\mathcal{K}$ is a simple central algebra over $E$ and $\dim_E \mathcal{K} = |M|^2$.

**Proof.** Theorem 4.1, (4) gives the statement about the center, while Corollary 4.2 gives the statement about the simplicity. From (2.6), (2.9) and subsection 2.3 we obtain

\begin{equation}
\dim \mathcal{K} = \sum_{\varphi \in \mathcal{M}/G} \dim \mathcal{K} (K \ast M)^G_{\varphi} = \sum_{\varphi \in \mathcal{M}/G} |O_{\varphi}| = |M|
\end{equation}

both as a left and as a right $K$-space structure. On other hand, $\dim_E K = |M|$, that completes the proof. $\square$

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