Scaling Cosmology

Winfried Zimdahl
Fachbereich Physik, Universität Konstanz
PF M678, D-78457 Konstanz, Germany
and
Diego Pavón
Departamento de Física, Universidad Autónoma de Barcelona
08193 Bellaterra (Barcelona), Spain

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Abstract

We show that with the help of a suitable coupling between dark energy and cold dark matter it is possible to reproduce any scaling solution \( \rho_X \propto \rho_M a^\xi \), where \( \rho_X \) and \( \rho_M \) are the densities of dark energy and dark matter, respectively. We demonstrate how the case \( \xi = 1 \) alleviates the coincidence problem. Future observations of supernovae at high redshift as well as quasar pairs which are planned to discriminate between different cosmological models will also provide direct constraints on the coupling between dark matter and dark energy.

1 Introduction

As is widely known, current observational evidence heavily favors an accelerating and spatially flat Friedmann-Lemaître-Robertson-Walker universe (for a pedagogical short update see [1]). Since normal matter fulfills the strong energy condition and cannot drive cosmic acceleration, recourse is often made either to a small cosmological constant (ΛCDM model) or to an almost evenly distributed source of energy called “dark energy” or “quintessence” with equation of state \( p_X = w_X \rho_X \) where \( -1 \leq w_X < 0 \), such that it makes the pressure negative enough to render the deceleration parameter negative (see e.g. [2]). (Obviously, the quantity \( w_X \) depends on the particular form assumed by the potential of the self-interacting quintessence scalar field). Since cold dark matter (i.e., dust) and quintessence decay with the expansion at different rates the question arises: “why the ratio between CDM and quintessence energies should be of the same order today”? In other words, “where the relationship
\[ \dfrac{\rho_M}{\rho_X}_0 = \mathcal{O}(1) \] comes from?” This is in essence the “coincidence problem” \[ (1) \]. As usual, the zero subindex means present time.

Certain kind of models invoke that both components (dark matter and dark energy) may not be separately conserved due to some (unknown) coupling between each other. This proposal has been explored \[ (1) \] and looks promising as a suitable mutual interaction can make both components redshift coherently. However, because of problems of their own (as the inability to recover the dust era when going back in time) neither of these proposals, as they stand, can be regarded as the final answer.

As suggested by Dalal et al. it seems rather more advisable to use the scanty observational information we possess to constrain the quintessence field from a minimum of theoretical input than trying to get a detailed fit to these data from any specific potential \[ (1) \]. These authors introduced a generalized class of dark energy models characterized by a non-canonical scaling of the ratio of the densities of dark matter and dark energy with the scale factor of the Robertson-Walker metric. They suggest a phenomenological form

\[ \dfrac{\rho_M}{\rho_X} \propto a^{-\xi} \]

for the ratio of the dark matter density \( \rho_M \) to the density \( \rho_X \) of the dark energy, where the scaling parameter \( \xi \) is regarded as a new variable. For an equation of state \( p_X = -\rho_X \) of the dark energy component a value \( \xi = 3 \) amounts to the \( \Lambda \)CDM model. A value \( \xi = 0 \) represents a stationary ratio \( \rho_M/\rho_X = \text{const} \).

If the cosmological dynamics admits a stable, stationary solution \( \rho_M/\rho_X = \text{const} \), corresponding to \( \xi = 0 \) and the present universe is already close to this state, there will be no coincidence problem. Consequently, according to \[ (1) \], the deviation of the parameter \( \xi \) from \( \xi = 0 \) quantifies the severity of the problem. But it is not only the stationary solution which deserves interest. Any solution which deviates from \( \xi = -3w_X \) represents a testable, non-standard cosmological model and any solution with a scaling parameter \( \xi < 3 \) will make the coincidence problem less severe. It is therefore desirable to have a physical mechanism that could give rise to such kind of deviations from the standard dynamics.

The purpose of this paper is to show that a departure from the standard \( \xi = -3w_X \) case can be obtained if cold dark matter and quintessential dark energy are no longer assumed to be separately conserved. More precisely, we shall demonstrate that a suitable interaction between dark matter and dark energy is able to produce any desired scaling. The specific parameter choice \( w_X = -3 \) and \( \xi = 1 \) is used to establish an exactly solvable toy model for a non-standard cosmological dynamics. Upcoming observations which will constrain cosmological models in a \( \xi - w_X \) plane, as discussed in \[ (1) \], will also put limits on such type of interactions.
2 Scaling solutions

We investigate a two–component system of cold dark matter (subindex $M$) and dark energy (subindex $X$) where

$$\rho = \rho_M + \rho_X \quad \text{and} \quad p = p_M + p_X$$  \hspace{1cm} (2)

are the total energy density and the total pressure, respectively. The components are assumed to possess the equations of state

$$p_M \ll \rho_M \quad \text{and} \quad p_X = w_X \rho_X .$$  \hspace{1cm} (3)

We admit interactions between both components according to

$$\dot{\rho}_M + 3H \rho_M = Q$$  \hspace{1cm} (4)

and

$$\dot{\rho}_X + 3H (1 + w_X) \rho_X = -Q ,$$  \hspace{1cm} (5)

where the coupling term $Q$ is to be determined below. It is convenient to introduce the quantities $\Pi_M$ and $\Pi_X$ by

$$Q \equiv -3H \Pi_M \equiv 3H \Pi_X ,$$  \hspace{1cm} (6)

with the help of which we can write ($A = M, X$)

$$\dot{\rho}_A + 3H (\rho_A + P_A) = 0 , \quad P_A = p_A + \Pi_A .$$  \hspace{1cm} (7)

The coupling is then included via $\Pi_M = -\Pi_X$.

To derive a specific expression for the interaction term let us consider the time evolution of the ratio $\rho_M/\rho_X$,

$$\left( \frac{\rho_M}{\rho_X} \right) ' = \frac{\rho_M}{\rho_X} \left[ \frac{\dot{\rho}_M - \dot{\rho}_X}{\rho_M - \rho_X} \right] .$$  \hspace{1cm} (8)

From Eqs. (4)-(6) we obtain

$$\left( \frac{\rho_M}{\rho_X} \right) ' = 3H \frac{\rho_M}{\rho_X} \left[ w_X - \frac{\rho}{\rho_M \rho_X} \Pi_M \right] .$$  \hspace{1cm} (9)

We look for solutions with the scaling behavior

$$\frac{\rho_M}{\rho_X} = r \left( \frac{a_0}{a} \right)^\xi .$$  \hspace{1cm} (10)
Here, $r$ denotes the ratio of both components at the present time, i.e., at $a = a_0$, and the parameter $\xi$ is a constant. Inserting (10) into (9) and solving for $\Pi_M$ we find

$$\Pi_M = -\Pi_X = \left[ \frac{\xi}{3} + w_X \right] \frac{a^\xi}{a^\xi + ra_0^\xi} \rho_M = \frac{\frac{\xi}{3} + w_X}{1 + r(1 + z)^\xi} \rho_M, \quad (11)$$

where $1 + z \equiv a_0/a$. This generalizes previous investigations for the case $\xi = 0$. There is a transfer of energy from the scalar field to the matter, i.e., $Q > 0$, for $w_X + (\xi/3) < 0$.

$\Pi_M$ and $\Pi_X$ are the effective pressures, equivalent to those interaction between both components given by the quantity $Q$ in (3), which guarantee a scaling solution (10). We arrive at the conclusion that by a suitable choice of the interaction between both components we may produce any desired scaling behavior of the energy densities.

The uncoupled case corresponding to $\Pi_M = 0$ is given by

$$\frac{\xi}{3} + w_X = 0. \quad (12)$$

The $\Lambda$CDM model is recovered as the special case with $w_X = -1$ and $\xi = 3$. The interacting models are parametrized by deviations from the standard case of separately conserved quantities. For $1 \ll z$, i.e., when $\rho_X \ll \rho_M$ (according to (10)), we have

$$\frac{|\Pi_M|}{\rho_M} \ll 1 \quad (1 \ll z), \quad (15)$$

and

$$-\frac{\Pi_X}{\rho_X} = r \left( \frac{\xi}{3} + w_X \right) \frac{(1 + z)^\xi}{1 + r(1 + z)^\xi}. \quad (14)$$

In the following we shall assume $\xi \neq -3w_X$ and $\xi > 0$, i.e., we consider departures from the standard case of separately conserved quantities. For $1 \ll z$, i.e., when $\rho_X \ll \rho_M$ (according to (10)), we have

$$\frac{|\Pi_M|}{\rho_M} \ll 1 \quad (1 \ll z), \quad (15)$$

and

$$-\frac{\Pi_X}{\rho_X} = \frac{\xi}{3} + w_X \quad (1 \ll z). \quad (16)$$

While the ratio $\Pi_X/\rho_X$ is constant in this limit and may be of the order of unity, the amount of $\Pi_M$ is much smaller than $\rho_M$. For $\frac{\xi}{3} + w_X < 0$ the $X$ component (that is dynamically unimportant for large $z$) looses energy which
is transferred to the matter. While $\Pi_X$ may be of the order of $\rho_X$, the quantity $|\Pi_M|$ is negligible compared with $\rho_M$. Since the fractional quantities on
the left-hand sides of Eqs. (15) and (16) quantify the amount of the coupling,
this means, the dark matter does not feel the interaction, it is (almost) uncoupled.
As the evolution proceeds, $\Pi_X/\rho_X$ changes only slightly to a present value
\[ -\frac{\Pi_X}{\rho_X} = \frac{r}{1+r} \left( \frac{\xi}{3} + w_X \right) \quad (z = 0) , \]
whereas the corresponding ratio for the dark matter component becomes
\[ \frac{\Pi_M}{\rho_M} = \frac{\xi + w_X}{1+r} \quad (z = 0) . \]
The point is that now the ratio $|\Pi_M|/\rho_M$ may also be of the order of unity,
i.e., the dark matter fluid feels the coupling as well. As far as the dark matter
is concerned, the interaction has been switched on during the cosmic evolution.
For $\xi = 0$ we recover the relations of the previously discussed stationary solution
\[ \frac{\Pi_M}{\rho_M} = 1 + r (1 + z) \xi \] .
In the latter case the interaction does not depend on $z$.

Using the source terms corresponding to (6) in the balances (7), the latter
can be integrated. For the matter energy density we find
\[ \rho_M = \rho_M (a_0) [1+z]^{3(1+w_X)+\xi} \left[ 1 + r (1 + z) \xi \right] \frac{1 - \frac{3w_X}{\xi}}{1+r} . \]
The total energy density becomes
\[ \rho = \rho_0 [1+z]^{3(1+w_X)} \left[ 1 + r (1 + z) \xi \right] \frac{3w_X}{\xi} , \]
where
\[ \rho_0 = \frac{r + 1}{r} \rho_M (a_0) . \]
Restricting ourselves to a universe with spatially flat sections, we obtain for the
Hubble rate
\[ H = \sqrt{\frac{8\pi G}{3}} \rho_0 (1 + z)^{\frac{3}{2}(1+w_X)} \left[ 1 + r (1 + z) \xi \right] \frac{3w_X}{\xi} . \]
Likewise, the deceleration parameter $q = -\ddot{a}/(aH^2)$ can be expressed as
\[ q = \frac{1 + 3w_X + r (1 + z) \xi}{2 (1 + r (1 + z) \xi)} . \]
We find accelerated expansion for
\[
1 + 3w_X + r (z + 1)^\xi < 0 .
\] (24)

The redshift \(z_{\text{acc}}\) at which the acceleration starts, is determined by
\[
1 + 3w_X + r (z_{\text{acc}} + 1)^\xi = 0 .
\] (25)

For \(w_X = -1\) (cosmological constant) we have accelerated expansion for
\[
(z + 1)^\xi < \frac{2}{r} \Rightarrow z_{\text{acc}} = \left(\frac{2}{r}\right)^\frac{1}{\xi} - 1 .
\] (26)

Both \(H\) and \(q\) depend on the set of parameters \(r\), \(w_x\), and \(\xi\). The luminosity distance
\[
d_L = (1 + z) \int_0^z \frac{dz}{H(z)} ,
\] (27)
as well as the angular distance \(d_A(z) = (1 + z)^{-2}d_L(z)\), can be expressed in terms of them. The corresponding effective magnitude is
\[
m_{B}^{\text{eff}} = M_{Bf} + 5 \log(H_0d_L),
\]
where we have chosen \(M_{Bf} = -3.4\). Figure 1 depicts the effective magnitude \(m_{B}^{\text{eff}}\) vs \(z\) for \(\xi = 1\) and different values of \(w_X\) with \(r = 3/7\). For \(\xi = 3\) almost the same figure would appear as for \(\xi = 1\). Figure 2 shows the magnitude differences for various combinations of \(\xi\) and \(w_X\). It becomes apparent that a much richer set of data (hopefully to be provided by the SNAP satellite) will be needed to discriminate between these models.

### 3 Special case of a scaling cosmology

The relations of the previous section considerably simplify for the special case \(\xi = 1\), \(w_X = -1\). The total energy density (28) reduces to
\[
\rho = \frac{\rho_0}{(1 + r)^3} \left[1 + r \frac{a_0}{a}\right]^3 .
\] (28)

For early times, \(a \ll a_0\), the energy density \(\rho\) redshifts as dust, while for late times, \(a_0 \ll a\), it tends to a constant value.
Figure 1: Effective magnitudes for $r = \frac{3}{4}$ and $\xi = 1$

Figure 2: Magnitude differences for various values of $\xi$ and $w_X$
Likewise, the expressions for the components are
\[
\rho_M = \frac{\rho_{M,0}}{(1 + r)^2} \frac{a_0}{a} \left[ 1 + r \frac{a_0}{a} \right]^2, \quad \rho_X = \frac{\rho_{M,0}}{r (1 + r)^2} \left[ 1 + r \frac{a_0}{a} \right]^2.
\] (29)

Furthermore, the ratios which quantify the interactions among dark matter and dark energy become
\[
\Pi_M \rho_M = -\frac{2}{3} \frac{1}{1 + r \frac{a_0}{a}} \text{ and } \Pi_X \rho_X = \frac{2}{3} \frac{r \frac{a_0}{a}}{1 + r \frac{a_0}{a}}.
\] (30)

The effective equation of state for the \(X\) component is
\[
P_X = -\frac{1 + \frac{1 + r \frac{a_0}{a}}{1 + r \frac{a_0}{a}}}{\rho_X} \rho_X.
\] (31)

Assuming again \(r = 3/7\), this corresponds to a change from
\[
P_X \approx -\frac{1}{3} \rho_X \quad (z \gg 1)
\] (32)

at early times to
\[
P_X \approx -\frac{4}{5} \rho_X \quad (z = 0)
\] (33)

at the present epoch. The effective pressure \(P_M\) of the matter component changes from a negligible value at \(z \gg 1\) to the present value
\[
P_M = \Pi_M \approx -\frac{7}{15} \rho_M \quad (z = 0).
\] (34)

The interaction has the effect that both components have a negative effective pressure.

The expression (28) for \(\rho\) has to be contrasted with the energy density
\[
\rho_{(\Lambda CDM)} = \frac{\rho_0}{(1 + r)} \left[ 1 + r \left( \frac{a_0}{a} \right)^3 \right]
\] (35)

for the \(\Lambda CDM\) model. The sum of different powers in the latter is replaced by the power of a sum in our present model. The interaction in our model makes \(\rho_M\) decay at a lower rate than in the uncoupled case. The dark energy density \(\rho_X\), on the other hand, which would remain constant without interaction, decays as well as a consequence of the transfer of energy to the matter component. This feature is familiar from decaying cosmological constant models (see [7], [8], [9], [10]).
The solution of the Friedmann equation with $\rho$ from (28) is

$$\frac{1}{2} \left[ \frac{1}{3} \frac{\kappa \rho_0}{(1 + r)^3} \right]^{1/2} (t - t_0) = \frac{1}{\sqrt{1 + r}} \left[ 1 - x \sqrt{\frac{1 + r}{1 + x}} \right] + \ln \left\{ \frac{\sqrt{1 + r}}{1 + \sqrt{1 + r}} \left[ x + \sqrt{x^2 + 1} \right] \right\}, \quad (36)$$

where $x = \sqrt{\frac{a}{a_0}}$. In the limit $a \ll a_0$ we consistently recover the dust behavior $a \propto t^{2/3}$, while the scale factor approaches an exponential growth for $a \gg a_0$. For the present special case $w_X = -1, \xi = 1$ the integral in (27) may be performed explicitly and yields

$$d_L = \frac{2}{H_0} (1 + z) \frac{1 + r}{r} \left[ 1 - \frac{1}{\sqrt{1 + \frac{r}{1 + r}}} \right]. \quad (37)$$

For small redshifts we obtain, up to third order in $z$,

$$H_0^{-1} d_L \approx z \left[ 1 + \frac{1 + \frac{r}{1 + r} z}{1 + r z} - \frac{1}{8} \frac{r}{(1 + r)^2} (6 + r) z^2 \right]. \quad (38)$$

Up to second order this expression coincides with the corresponding result for the $\Lambda$CDM model. Differences occur only in the $z^3$ term. Here, the factor $(6 + r)$ in the last expression replaces the factor $(10 + r)$ of the $\Lambda$CDM universe. The presently available SNIa data cannot discriminate between both models. Our model shares the merits of the $\Lambda$CDM model but at the same time alleviates the coincidence problem.

4 Conclusions

Scaling solutions of the type $\rho_M/\rho_X = r(a_0/a)^\xi$ seem to be promising tools to deeper analyze the relationship between the two forms of energy dominating the current evolution of the universe, namely, dark matter and dark energy. We showed that a suitably chosen interaction between them can lead to any scaling behavior of the mentioned form. In the specific case $w_X = -1, \xi = 1$ the dynamics can be analytically integrated. Since the luminosity distance in this model differs from that of the $\Lambda$CDM model only in third order in the redshift parameter $z$, it fits the present observations as well as the $\Lambda$CDM model does. Further, the Einstein-de Sitter expansion law for a dust universe is recovered for large redshifts. On the other hand, the coincidence problem, although not solved, is less severe than for the $\Lambda$CDM universe, which can be traced back to a continuous transfer of energy from the $X$ component to the CDM fluid.
While the available observational data are insufficient to discriminate between the models, it is to be expected that the SNAP satellite will provide us with a wealth of high redshift supernovae data able to do the job. Likewise, complementary observations regarding the angular distance between quasar pairs \[1\] and the evolution of cluster abundances \[12\] will further constraint the set of parameters entering the scaling models.

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