Certified Rational Parametric Approximation of Real Algebraic Space Curves with Local Generic Position Method

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Abstract
In this paper, an algorithm to compute a certified $G^1$ rational parametric approximation for algebraic space curves is given by extending the local generic position method for solving zero dimensional polynomial equation systems to the case of dimension one. By certified, we mean the approximation curve and the original curve have the same topology and their Hausdorff distance is smaller than a given precision. Thus, the method also gives a new algorithm to compute the topology for space algebraic curves. The main advantage of the algorithm, inhering from the local generic method, is that topology computation and approximation for a space curve is directly reduced to the same tasks for two plane curves. In particular, the error bound of the approximation space curve is obtained from the error bounds of the approximation plane curves explicitly. Nontrivial examples are used to show the effectivity of the method.

Key words: Real algebraic space curve, topology preserving, rational approximation parametrization, local generic position

1. Introduction

Algebraic space curves have many applications in computer aided geometric design, computer aided design, and geometric modeling. For example, the algebraic space curves defined by two quadrics are widely used in geometric modeling. One can have an exact parametrization for these algebraic space curves. However, exact parametrization representation for general algebraic space curves do not exist. And usually, we are interested in a rational parametric representation. So the use of approximate techniques is unavoidable for parametrization of algebraic space curves. Some approximate techniques are able to reproduce exact rational parameterizations, if those are available. Otherwise, one usually approximates an algebraic space curves with piecewise rational curves under a given precision. Moreover, sometimes, one requires that the approximation curves preserve the topology of the original algebraic space curve. We call the approximation certified (at precision $\epsilon$) if it has the same topology as the original one and the Hausdorff distance between the curve and its approximation is upper-bounded (by $\epsilon$) simultaneously.

There are several difficulties for approximate parametrization of algebraic space curves. The first one is to preserve the topology of the algebraic space curve. In fact, there already exist some related work of computing the topology of algebraic space curves, reduced or non-reduced, for example [3,16,30,24,33]. Most of them require the curve to be in a generic position. For the space curves which are not in a generic position, one need to take a coordinate transformation such that the new space curve is in a generic position. Thus some geometric information of the original space curve is lost. Some non-singular critical points of the new space curve may not correspond to the non-singular critical points of the original space curve. One needs additional computation to get these points in the original coordinate system. Subdivision method can preserve the topology of the curve in a theoretical sense. But it is rather difficult to reach the required bound in practice currently [11,32]. Even if one gets the topology of the given curve, the approximation curve may have different topology as the original curve when two or more curve segments are very close (see Figure 2). The second difficulty is the error control of the approximation curve. Some error functions is...
reliable but it is not easy to compute in practice, for example [15]. We need to find a reliable and efficient method to control the error during the approximation. The third one is the continuity of the approximation curve. We usually require the approximation to be $C^1$ (or $G^1$) continuous or higher in practice. Doing so, we need compute the tangent direction of algebraic space curve at some points. It is not a trivial task especially when the component considered is non-reduced. Its tangent direction can not be decided by the normal directions $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, $(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z})$ of the two surfaces $f = 0$ and $g = 0$ at the given point. In the non-reduced case, two normal directions are parallel or at least one of them does not exist at the point.

There exist nice work about approximating of algebraic space curves. Exact parametrization of the intersection of algebraic surfaces is obtained in [1,10,17–19,22,27,39,41,43,45]. Of course, it is topology preserving. The approximation of the intersection of generic algebraic surfaces with numeric method is also considered [5,6,31,34,36]. Usually, numeric method cannot guarantee the topology of the original algebraic space curve.

In [7], the authors considered approximating of the regular algebraic space curves with circular arcs by numeric method combining with the subdivision method. It works well for low degree algebraic space curves. In [23], the authors present an algorithm to approximate an irreducible space curves under a given precision. It based on the fact that there exists a bilateral map between the projection curve $C$ for some direction and the irreducible algebraic space curve $S$. But an irreducible decomposition of a given two polynomials system is not an easy task. And we need to consider the intersection of two or more irreducible space curves after we decompose a reducible space curve. Other type of intersection of surfaces can be found in [34,36] and related references.

In [10], the authors present an algorithm to approximate an algebraic space curve, defined by $f = g = 0$, in the generic position with Ferguson’s cubic $p(t) = (x(t), y(t), z(t)), t \in [0,1]$ and by minimizing an integral to control the error. They compute the topology of the space curve at first, so it is topology preserving. But they do not check whether the approximation curve exactly preserves the topology of the original space curve. And it works well for regular space curve. From the formula above, we can find that if some segments of the algebraic space curve is not regular, the method may fail.

In [37], the authors consider the irreducible algebraic space curve in generic position such that its projection is birational. They use a genus 0 plane algebraic curve to approximate the projection plane curve under a given precision if it exists. Thus they have a rational approximation space curve for the original space curve. The method is not topology preserving.

In this paper, we present a new algorithm to compute a certified $G^1$ rational parametric approximation for algebraic space curves, which solves the three difficulties mentioned above nicely. The algorithm is certified in the sense that the approximation curve and the original curve have the same topology and their Hausdorff distance is smaller than a given precision. The algorithm works for algebraic space curves which need not to be regular or in generic positions. The key idea is to extend the local generic position method [13,12] for zero-dimensional polynomial systems to one-dimensional algebraic space curves. The algorithm consists of four major steps.

Firstly, the space curve $S$, which is the intersection of $f(x, y, z) = 0$ and $g(x, y, z) = 0$, is projected to the $xy$-plane as a plane curve $C_1$ and $C_1$ is approximated piecewisely with functions of the form $h_1(x), x \in [a, b]$.

Secondly, we find a number $s > 0$ such that under the shear transformation $\varphi : (x, y, z) \rightarrow (x, y + s z, z), \varphi(f)$ and $\varphi(g)$ are in a generic position in the sense that there is one to one correspondence between the curve segments of $S$ and that of their projection curve $C_2$ to the $xy$-plane. The plane curve $C_2$ is also approximated piecewisely with functions of the form $h_2(x), x \in [a, b]$.

Thirdly, we choose $s$ such that $C_2$ is in a local generic position to $C_1$ in the following sense.

- The plane curves $C_1$ and $C_2$ can be divided into segments such that each segment of $C_2$ corresponds to a segment of $C_1$.
- Let $h_1(x), h_2(x), x \in [a, b]$ be the approximations for a segment $C_1$ of $C_1$ and the corresponding curve segment $C_2$ of $C_2$ with precisions $\epsilon_1$ and $\epsilon_2$ respectively. Then the space curve segment $S$ corresponding to $C_2$ can be approximated by $(x, h_1(x), h_2(x))/s$ with precision $\sqrt{\epsilon_1^2 + \epsilon_2^2}/s$.

In other words, if $C_2$ is in a local generic position to $C_1$, then each segment of the space curve can be represented as a linear combination of corresponding segments of $C_1$ and $C_2$. As a consequence, a certified parametrization for the space curve can be computed from that of $C_1$ and $C_2$ directly. This step is the main contribution of the paper.

Finally, we show that a plane curve can be approximated such that the piecewise approximation curve for the space curve has $G^1$ continuity and usually has the following forms: $(x, a_1 x + b_1 + \frac{c_1}{d_1 x + 1}, a_1 x + b_1 + \frac{c_1}{d_1 x + 1} + \frac{c_2}{d_2 x + 1}, a_2 y + b_2 + \frac{c_2}{d_2 y + 1})$.

The topology of the space curve is obtained directly from the two projection steps. Thus this new method can not only compute the topology of the space curve but approximate the algebraic space curve under any given precision.

The paper is organized as below. In the next section, we will consider the certified approximation of plane algebraic curve under a given precision. In Section 3, we will show the theory and algorithm for certified approximation of algebraic space curves. In Section 4, we will show some examples to illustrate the effectivity of our method. We draw a conclusion in the last section.
2. Approximate parametrization of plane algebraic curves

Given a plane algebraic curve defined by a square free polynomial \( f \in \mathbb{Q}[x, y] \), our aim is to give a piecewise \( C^1 \)-continuous approximation of \( C = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0 \} \) in a given box \( B = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d \} \) such that each piece of the approximation curve has the form \((x, h(x))\) and the approximation error is bounded by a given precision \( \varepsilon > 0 \), where \( \mathbb{Q}, \mathbb{R} \) are the fields of rational numbers and real numbers, respectively. And the whole approximation curve has the same topology as \( C \).

2.1. Notations

In this subsection, we will introduce some notations.

A point \( P = (x_0, y_0) \) is said to be a singular point on \( C \) if \( f(x_0, y_0) = f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \), where \( f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y} \). A non-singular point is called a regular point. An \( x \)-critical (\( y \)-critical) point \( P = (x_0, y_0) \) is a point satisfying \( f(x_0, y_0) = f_y(x_0, y_0) = 0 \) \((f(x_0, y_0) = f_x(x_0, y_0) = 0) \). So a singular point is both \( x \)-critical and \( y \)-critical points. The inflexion points or flexes of \( C \) are its non-singular points satisfying its Hessian equation \( H(f) = 0 \) (see [44]).

A regular curve segment \( C \) of \( C \) is a connected part of \( C \) with two endpoints \( P_0(x_0, y_0) \) and \( P_1(x_1, y_1) \) \((x_0 \neq x_1) \), both \( P_0, P_1 \) are bounded) and there are no \( x \)-critical points, \( y \)-critical points, flex on \( C \) except for \( P_0, P_1 \). So a regular curve segment is convex, monotonous w.r.t. \( x \) or \( y \), and inside a triangle defined by its endpoints and their tangent directions. Let \( \Delta \) be the triangle defined by \( P_0, P_1 \) and their tangent lines. An endpoint of a regular curve segment is called a vertical tangent point, VT point for short, if the regular curve segment has a vertical tangent line at this endpoint.

A parametric curve is said to be \( C^1 \)-continuous (\( G^1 \)-continuous) if the curves are joined and the first derivatives are continuous (the curves also share a common tangent direction at the join point).

2.2. Curve segmentation of a real plane algebraic curve

In this subsection, we will show how to divide a plane curve inside a box \( B \), denoted as \( C_B \), into regular curve segments with the form \( [P_0(x_0, y_0), P_1(x_1, y_1), T_0(1, k_0), T_1(1, k_1)] \), where \( P_0, P_1 \) are endpoints and \( T_0, T_1 \) are tangent directions at the endpoints.

We will follow the steps below.

At first, Compute the topology of \( C_B \). There are many related work to solve this problem, such as [2,4,8,14,20,28,38,40]. Some methods work well, but they need a coordinate system transformation. We prefer the methods which do not take a coordinate system transformation.

Second, Compute all the flexes, \( x \)-critical and \( y \)-critical points of \( C_B \). \( y \)-critical points are computed before. \( x \)-critical points and flexes can be obtained by solving the corresponding equations.

Third, we split the plane curve into regular curve segments at \( x(y) \)-critical points or flexes of \( C_B \). An easy way to solve the problem is as follows. For all the \( x \)-coordinates of \( x \)-critical points and flexes, lifting them to split \( C_B \) at these intersection points. We can find all these endpoints.

Finally, we represent the tangent direction of any non-VT point as \( (1, k) \), \( k \in \mathbb{R} \setminus \{+\infty, -\infty\} \). It is convenient for our approximation. The tangent direction of a VT point is defined to be \((1, \infty)\).

Tangent direction computation of singularities. We compute the tangent direction of a point close to a singularity on the regular curve segment to replace the tangent direction of the singularity. It is easy to compute. Let \( P(\alpha, \beta) \) be a singularity of a planar algebraic curve \( h(x, y) = 0 \) and \( C : (x, y(x)) \), \( x \in [\alpha, \gamma] \) a regular curve segment originating from right side (left side is similar) of \( P \). Then the tangent direction of \( C \) at \( P \) is \((1, t) = (1, \lim_{x \rightarrow \alpha^+} y(x)) \). In practice, we can take some point very close to \( P \) on \( C \), which is a regular point. Let \([a, b]\) be the isolating interval of \( \alpha(\neq a, \neq b) \). We can use the tangent direction of some regular point to replace the tangent direction of \( C \) at \( P \). For instance, \((1, \frac{\partial y(b)}{\partial x}) = (1, \frac{1}{t}) \) can be regarded as the tangent direction of \( C \) at \( P \). For the regular curve segments shall the same tangent direction, we can take the average value of them as their tangent directions, and the center of the isolating box of \( P \) as the singularity, as shown in Figure 1. When \( C \) has a vertical tangent direction, \( t = \infty \). If \(|\frac{\partial y(b)}{\partial x}| > N \), for example, \( N = 100 \), we can regard the regular curve segment have a vertical tangent direction.

If we cannot distinguish the tangent directions of two groups of regular curve segments, we can refine \([a, b]\) to a narrower one and recompute the tangent directions again until we can distinguish them or they are less than some given bounded value \( \tau \) such that \(|k - k'| < \tau \), where \( k, k' \) are tangent directions.

![Fig. 1. Approximate the tangent direction of regular curve segments at a singularity](image1)

![Fig. 2. The approximation curves change the topology of original ones](image2)

2.3. Approximation of a regular curve segment

We will give an approximate parametrization of real plane algebraic curves. Though Gao and Li [23] have obtained a rational quadratic approximation of real plane algebraic curves with B-splines, we need to derive a piecewise
approximation curve as \((x, h(x))\) of real plane algebraic curves in order to approximately parameterize real space algebraic curves in a different way. Let \(C\) be the regular curve segment defined by two points \(P_0(x_0, y_0), P_1(x_1, y_1)\). We divide the approximation problem into two cases.

**C not containing a VT point.** Given a regular curve segment \(C\), its two endpoints \(P_0(x_0, y_0), P_1(x_1, y_1)\) does not contain a VT point. And the tangent directions of the regular curve segment at \(P_0, P_1\) are \((1, k_0), (1, k_1)\), respectively. We will construct an explicit rational quadratic function \(Y_1(x)\) to approximate \(C\) such that \(Y_1(x_i) = y_i, Y_1'(x_i) = k_i, i = 0, 1\), where \(Y_1'(x) = \frac{\partial Y_1}{\partial x}(x)\).

Assuming that
\[
Y_1(x) = \frac{ax^2 + bx + c}{dx + 1},
\]
and \(x_0 = 0, x_1 = 1\), we have
\[
Y_1(0) = y_0, Y_1(1) = y_1, Y_1'(0) = k_0, Y_1'(1) = k_1.
\]

Solving \(a, b, c, d\) from the equations above, we have
\[
a = -\frac{2y_0y_1 + y_1^2}{y_1 + k_1 + y_0},
b = -\frac{2y_0y_1 + y_1^2 + 2y_0k_1 + y_0 - y_0k_1}{y_1 + k_1 + y_0},
c = y_0,
d = -\frac{2y_1 + k_0 + k_1 + 2y_0}{y_1 + k_1 + y_0}.
\]

From the representation, we need to require
\[
-y_1 + k_1 + y_0 \neq 0,
\]
and \(d \times 1 + 1\) has no roots in \([0, 1]\), that is, \(d > -1\), from which we can derive that
\[
(−y_1 + k_0 + y_0)(−y_1 + k_1 + y_0) < 0.
\]

From the mean value theorem, we know that \(y_1 - y_0 = k_x\), where \(k_x\) is the tangent direction of some \(x \in (0, 1)\). Since \(C\) is monotonous, so \(k_x\) is some value between \(k_0\) and \(k_1\). Thus conditions (2) and (3) are satisfied directly. We can easily transform the interval \([0, 1]\) to \([x_0, x_1]\) by setting \(x = \frac{X - x_0}{x_1 - x_0}\), where \(x \in [0, 1]\) when \(X \in [x_0, x_1]\).

Furthermore, when \(d \neq 0\), that is to say \(-2y_1 + k_0 + k_1 + 2y_0 \neq 0\). Then expression (1) can be transformed into
\[
Y_1(x) = \frac{ax + b + \frac{c}{d}}{dx + 1}.
\]

Though equation (4) is equivalent to equation (1) when \(d \neq 0\) essentially, it has a simpler form and can reduce computation when evaluation. When \(d = 0\), equation (1) is a polynomial of degree two. And we have simple expressions for parameters \(a, b, c\), that is a = \(\frac{k_1 - k_0}{y_0}\), \(b = k_0, c = y_0\).

**C containing a VT point.** When a given regular curve segment contains a VT point, it means the tangent line at \(P_0\) or \(P_1\) is a vertical line \(x = x_0 = 0\) or \(x = x_1 = 0\). In this case, the method above does not work. But we can use part of an ellipse or a hyperbola \(\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1 = 0(a > 0, b > 0)\) to derive an approximate parametrization of a real plane algebraic curve. Note that a regular curve segment containing a VT point has four cases which exactly correspond to the four parts of an ellipse or a hyperbola: the vertical line is \(x = x_0 = 0\) or \(x = x_1 = 0\) and \(y \geq y_0\) or \(y \leq y_0\). We consider the case that \(C\) has a vertical tangent line at \(P_0(x_0, y_0)\) and \(C\) monotonously increases from \(P_0\) to \(P_1\). And we assume that the tangent line at \(P_1\) is \(k_1(x - x_1) - (y - y_1) = 0(k_1 \geq 0)\). The approximate curve is \(Y_2(x) = y_0 + \frac{b\sqrt{a^2 - (x - x_0 - a)^2}}{a}\). Note that we have \(x_o = x_0 \pm a, y_o = y_0\) from the property of the ellipse or the hyperbola. So we have
\[
Y_2(x) = y_0 + \frac{b\sqrt{a^2 - (x - x_0 - a)^2}}{a}.
\]

And \(Y_2(x_1) = y_1, Y_2'(x_1) = k_1\). Solving it, we have
\[
a = \frac{(x_1 - x_0)(x_0k_1 + y_1 - y_0 - x_1k_1)}{y_1 - y_0 - 2x_1k_1 + 2x_0k_1},
b = \frac{(x_0k_1 + y_1 - y_0 - x_1k_1)\sqrt{y_1 - y_0 - 2x_1k_1 + 2x_0k_1}}{y_1 - y_0 - 2x_1k_1 + 2x_0k_1}.
\]

From the representation, we can find that \(a, b\) are well defined if \(k_1 < \frac{y_1 - y_0}{2(x_1 - x_0)}\) for an ellipse or \(k_1 > \frac{y_1 - y_0}{2(x_1 - x_0)}\) for a hyperbola. So we can choose \((x_1, y_1)\) on the regular curve segment such that \(k_1 \neq \frac{y_1 - y_0}{2(x_1 - x_0)}\). Then we can use part of an ellipse or a hyperbola to approximate the regular curve segments with VT points. The other three cases can be solved in a similar way.

**Lemma 1** The two kinds of approximation curves above, say \(\bar{C} : (x, h(x)), x \in [x_0, x_1]\) for \(C\) are inside the triangle formed by the endpoints and the tangent directions at the endpoints of \(C\), denoted as \(\Delta\).

**Proof.** For both cases, \(\bar{C}\) is part of a quadric curve. And the curve intersects all three edges of \(\Delta\) at least twice (including the multiplicities). If \(\bar{C}\) goes out of \(\Delta\), it will intersect the edge(s) at least three times (including multiplicities). But it is not possible since \(\bar{C}\) is part of a quadric curve. So the lemma is true. \(\square\)

**Topology preserving approximation.** After we get the approximation regular curve segments, we need to check whether the approximation curve change the topology of the original curve. Even if we get the correct topology of the given algebraic planar curve, the approximation curve may have a different topology as the original curve, especially when two regular curve segments are very close, for example, see Figure 2. So we need to ensure that our numeric approximation curve has the same topology as the original one. We need only ensure that any two approximation curves, say \(C_1, C_2 : (x, p(x)), (x, q(x)), x \in [a, b]\), are disjoint. If \(p(x) - q(x) = 0\) has no real roots in \((a, b)\), then the two approximation regular curve segments are disjoint. There are two kinds of approximation curves, say \(Y_1(x), Y_2(x)\). So we need to consider:

Case one: two approximation curves are both rational ones
as $Y_1(x)$. Then $T(x) = p(x) - q(x)$ can be simplified into a cubic univariate polynomial. It is easy to check whether it contains a real roots in $(a, b)$ by its coefficients.

Case two: one is as $Y_1(x)$ and the other is as $Y_2(x)$. Then $T(x) = p(x) - q(x)$ can be simplified into a quartic univariate polynomial. It is also easy to check whether it contains a real roots in $(a, b)$ by its coefficients.

Case three: both approximation curves are as $Y_2(x)$. They both are parts of quadric algebraic curves. Considering the intersection of two quadric algebraic curves, we can judge whether $C_1, C_2$ are disjoint or not.

Doing so as above, our approximation is exactly topology preserving.

### 2.4. Error control of the approximation

We will show the error control of the plane approximation curve in this subsection. In geometry, the approximation error should be defined as the following Hausdorff distance between the segment $S$ and its approximation $S_a$:

$$e(S, S_a) = \text{dis}(S, S_a) = \max_{P \in S} \min_{P' \in S_a} d(P, P').$$

(6)

However such a distance is difficult to compute. As an implement, the distance from an approximation parametric curve $P(t) = (x(t), y(t)), 0 \leq t \leq 1$ to the implicit defined curve $C : f(x, y) = 0$ is taken in the following form, which is called the error function [15],

$$e(t) = \frac{f(x(t), y(t))}{\sqrt{f_x(x(t), y(t))^2 + f_y(x(t), y(t))^2}}.$$  

(7)

The approximation error between $P(t)$ and $C$ is set as an optimization problem

$$e(P(t), C) = \max_{0 \leq t \leq 1} (e(t)).$$

Let $C : (x, \tilde{y}(x)), x \in [x_0, x_1]$ be the regular curve segment and $\overline{C} := (x, Y(x)), x \in [x_0, x_1]$ its approximation curve. It is not difficult to find that the following bound is an upper bound of the Hausdorff distance between the segment $C$ and its approximation curve $\overline{C}$ from (6):

$$\max_{x \in [x_0, x_1]} |Y(x) - \tilde{y}(x)|.$$  

(8)

We use Newton-Raphson method to obtain $\tilde{y}(x_i^0)$ at some point $x_i^0 \in [x_0, x_1]$ in practice and $Y(x_i^0)$ is the start point. If we fail to get a point with Newton-Raphson method or the point satisfying $|Y(x_i^0) - \tilde{y}(x_i^0)| \leq \delta$, we can divide the regular curve segment into two ones. The approximation error is bounded by $\max_i |Y(x_i^0) - \tilde{y}(x_i^0)|$.

In practice, we sample $x_i^0$ as $x_i^0 = x_0 + i/n(x_1 - x_0), 0 \leq i \leq n$, for a proper value of $n$.

In order to control the error under a given precision, we need to divide the regular curve segment into two or more regular curve segments recursively until the error requirement satisfied. We subdivide the regular curve segments into two or more regular curve segments uniformly in the $x$ coordinate. For any regular curve segment $C : (x, \tilde{y}(x)), x \in [a, b]$, we denote the endpoints as $P_0(x_0, y_0), P_1(x_1, y_1)$ and the tangent directions as $(1, k_i), i = 0, 1$. We can find that $P_0, P_1$ and two tangent directions form a triangle. One can subdivide the regular curve segment into two (or more) ones, for example, $C_1 : x \in [x_0, (x_0 + x_1)/2], C_2 : x \in [(x_0 + x_1)/2, x_1]$, if the precision is not satisfied. For the approximation curve of our method, we will prove that it can achieve any given precision.

**Theorem 2** Let $P_0, P_1$ be the endpoints of a regular curve segment $C$ and $\Delta$ the triangle related to the regular curve segment as defined before. The Hausdorff distance between $C$ and its approximation curve(s) tends to zero if we subdivide $C$ into two or more regular curve segments recursively.

**Proof.** We will consider divide $C$ into two regular curve segments for the proof since dividing them into more regular curve segments are similar. Let $P((x_0 + x_1)/2, \tilde{y})$ be a point on $C$. Denote the triangles formed by $P_1, P, P_2$ and the tangent directions of $C$ at these points as $\Delta_1(\Delta_2)$. Let the lengths of the line segments $P_1P, PP_2$ be $L_1^1, L_2^1$ and the heights of the triangles $\Delta_1, \Delta_2$ corresponding to the edges $P_1P, PP_2$ are $H_1^1, H_2^1$ as shown in Figure 3. Subdividing the regular curve segments recursively in a similar way, denoting the length of the edges and heights as $L_j^1, H_j^1$ of the triangles, we have the sum of the areas of these triangles are

$$A = \sum_j (L_j^1 H_j^1/2) < \frac{1}{2} \sum_j L_j^1 \max_j H_j^1.$$  

Assume that one edge $L_j^1 = P_j \overline{P}_j, P_j(x_j^1, y_j^1), P_j(x_{j+1}^1, y_{j+1}^1)$. Since the given regular curve segment is bounded, $|y_j^1 - y_{j+1}^1|$ tends to zero when $|x_j^1 - x_{j+1}^1|$ tends to zero. And $\sum_j L_j^1$ tends to the length of arc, say $L$, of the given regular curve segment and $H_j^1$ tends to zero when all corresponding $|x_j^1 - x_{j+1}^1|$ of $L_j^1$ tends to zero. Thus $A$ tends to zero. From the result of Lemma 1, we can have the opinion that we have a proof of the theorem. □

There are two ways to find the subdivision points on a given regular curve segment $C : (x, \tilde{y}(x)), x \in [a, b]$. Since we get the topology of the plane projection curve, we know the order of the given regular curve segment among all the regular curve segments of the projection curve $h(x, y) = 0$ when $x$ changes from $a$ to $b$. That is, we can find the point on $C$ for a fixed $x$ coordinate, say $x_0 \in (a, b)$. It is the real root with the same order of $h(x_0, y) = 0$ in a fixed interval (or $(-\infty, +\infty)$).

Another way is a local method. We can trace the regular curve segment to find the point on $C$ with given $x$ coordinate since the regular curve segments are monotonous and convex. From the endpoint of the regular curve segment, compute the tangent line of the regular curve segment at some point $P$, find a point $Q$ on the tangent line by increasing the $x$ coordinate such that the line segment $\overline{PQ}$ has no intersection with the projection curve. Then fix the $x$ coordinate of $Q$, to find a point on the regular curve segment.

Note that we know the direction to find the point from the
position or negativeness of the tangent direction. Doing so recursively, we can find the point that we want, as shown in Figure 4.

![Fig. 3. Splitting a regular curve][2] ![Fig. 4. Finding subdivision point by tracing][3]

With the preparation above, we have the following algorithm to approximate a plane algebraic curve.

**Algorithm 3** The inputs are \( C : f(x, y) = 0 \), a bounding box \( B \) and an error bound \( \delta > 0 \). The outputs are parametric curves \( C_1 := \{ B_i(x) = (x, y(x)), a_i \leq x \leq b_i, (i = 1, ..., N) \} \), such that they give a \( C^1 \)-continuous and topology preserving approximation to \( C_B \) with \( e(C_1, C) < \delta \).

(i) Regular curve segmentation of \( C_B \) as in Section 2.2.
(ii) Regular curve segment approximation as in Section 2.3 with error control of the approximation as in Section 2.4.

The correctness of the algorithm is clear from the analysis above. The termination of the algorithm is guaranteed by Lemma 1 and Theorem 2.

### 3. Certified approximate parametrization of algebraic space curves

In this section, we will consider approximate parametrization of algebraic space curves defined by \( f, g \in \mathbb{Q}[x, y, z] \) such that two assumptions hold:

- For any \( x_0 \in \mathbb{R} \), \( f(x_0, y, z) = g(x_0, y, z) = 0 \) has a finite number of solutions; and
- the leading coefficients of \( f, g \) w.r.t. \( z \) have no common factors only in \( x \).

The assumptions are to ensure that we can use local generic position method to recover the points on the space curve from the points on two plane projection curves. The first assumption ensures that the algebraic space curve defined by \( f = g = 0 \) does not have a plane curve on the plane \( x = x_0 \). The second assumption ensures that we can find a generic position only by taking a shear map \( (x, y, z) \rightarrow (x, y + s z, z) \). If the two projection curves have no factors only involving \( x \), the two assumptions hold.

In fact, most of the problems we considered satisfy the condition. Note that we can exchange \( x, y, z \) freely. And another coordinate system transformation \( (x, y, z) \rightarrow (x + s z, y, z) \) can help us to find out the missing regular curve segments in the first transformation, even when the algebraic space curve containing vertical lines. Thus we can remove the assumptions with the method mentioned here. But we still assume the two assumptions holds in this section.

#### 3.1. Definition of local generic position

In order to reduce the 3D approximation of space curves into 2D approximation of plane curves, we need the concept of local generic position. We recall the related definitions for zero-dimensional bivariate polynomial system [12]. Let \( \mathbb{C} \) be the field of complex numbers. Let \( f, g \in \mathbb{Q}[x, y] \). We say two plane curves defined by two polynomials \( f, g \) such that \( \gcd(f, g) = 1 \) are in a **generic position** w.r.t. \( y \) if

1. The leading coefficients of \( f \) and \( g \) w.r.t. \( y \) have no common factors.
2. Let \( h \) be the resultant of \( f \) and \( g \) w.r.t. \( y \). For any \( \alpha \in \mathbb{C} \) such that \( h(\alpha) = 0 \), \( f(\alpha, y), g(\alpha, y) \) have only one common zero in \( \mathbb{C} \).

Then we will introduce the technique of local generic position (LGP for short) method.

Given \( f, g \in \mathbb{Q}[x, y] \), not necessarily to be in generic position, we can take a coordinate system transformation \( \phi : (x, y) \rightarrow (x + s y, y), s \in \mathbb{Q} \) such that

- \( \phi(f), \phi(g) \) are in a generic position w.r.t. \( x \).
- Let \( \tilde{h} \), \( h \) be the resultant of \( f \) and \( g \) w.r.t. \( y \), respectively. Each root \( \alpha \) of \( h(x) = 0 \) has a neighbor interval \( H_\alpha \) such that \( H_\alpha \cap H_\beta = \emptyset \) for roots \( \beta \neq \alpha \) of \( h = 0 \). And any root \( (\gamma, \eta) \) of \( f = g = 0 \) which has a same \( x \)-coordinate \( \gamma \), is mapped to \( (\gamma', \gamma + s \eta) \in H_{\gamma'} \), where \( h(\gamma') = 0, \tilde{h}(\gamma') = 0 \), as shown in Figure 5. Thus we can recover \( \eta = \frac{\gamma - \gamma'}{s} \).

We can find the method has two nice properties: 1) The 2D solving problem is transformed into a 1D solving problem. 2) The error control of the solutions is easier.

#### 3.2. Basic idea

Now we want to extend this technique to 3-D case. Let \( f \land g \) denote the algebraic space curve defined by \( f = g = 0 \). Denote \( \pi_x : (x, y, z) \rightarrow (x, y) \) and \( h = \pi_z(f \land g) \). Let \( \varphi : (x, y, z) \rightarrow (x, y + s z, z) \), \( h = \pi_z(\varphi(f) \land \varphi(g)) \). For a proper regular curve segment \( C : (x, p(x)), x \in [x_0, x_1] \) of the plane curve defined by \( h = 0 \), it corresponds to one (the corresponding space regular curve segment may be at infinity) or more space regular curve segment(s), denoted as \( S_1, \ldots, S_t \). If we can choose a proper \( s \) such that \( \varphi(f), \varphi(g) \)
is in “a good” position, and it has some local property, that is, the corresponding projection regular curve segments of \( S_1, \ldots, S_t \), say \( C_i : (x, q_i(x)), x \in [x_0, x_1] (i = 1, \ldots, t) \), are in a fixed neighborhood of \( C \), then we can recover the \( z \)-coordinate of \( S_1 : z = \frac{a(x)-p(x)}{s} (i = 1, \ldots, t) \). But to make \( C_i \) in a fixed neighborhood of \( C \) is an unreachable task sometimes. Fortunately, what we need is to find out the correspondence between the regular curve segments of the plane curves defined by \( h = 0 \) and \( \bar{h} = 0 \). We can choose some sample points on \( x \)-axis, say \( x_i^{(0)}, i = 1, \ldots, n \), a proper \( s \) such that \( \varphi(f(x_i^{(0)}, y, z)), \varphi(g(x_i^{(0)}, y, z)) \) are in a generic position. Then we can figure out the correspondence between the regular curve segments of \( h = 0 \) and \( \bar{h} = 0 \), as shown in Figure 6.

To realize the aim above, there are two key steps. One is how to find an approximation \( s \), the other is how to find the correspondence between the regular curve segments of \( h = 0 \) and \( \bar{h} = 0 \). We need some preparations at first.

We say that two algebraic surfaces defined by \( f, g \in \mathbb{Q}[x, y, z] \) such that \( \gcd(f, g) = 1 \) are in a \( z \)-generic position if

1) The leading coefficients of \( f \) and \( g \) w.r.t. \( z \) have no common factors.

2) Let \( h \) be the resultant of \( f \) and \( g \) w.r.t. \( z \). There are only a finite number of zeros \((\alpha, \beta) \in \mathbb{C}^2 \) such that \( (\alpha, \beta) \) is not a \( q \)-critical point, \( h(\alpha, \beta) = 0 \) and \( f(\alpha, \beta, z), g(\alpha, \beta, z) \) have more than one distinct common zeros in \( \mathbb{C} \).

The definition is similar to the definition of pseudo-generic position in [16]. In Theorem 4 of [16], the authors also provide a method to check whether two given surfaces are in a pseudo-generic position or not.

**Computing \( s \).** We will show how to find \( s \) mentioned before. Let \( \pi_y : (x, y) \to (x) \). Denote the real roots of \( \pi_y(h) = 0 \) and the \( x \)-coordinates of the flexes and \( x \)-critical points of \( h = 0 \) as \( \alpha_1, \alpha_3, \ldots, \alpha_{2t-1} \). Find two rational numbers less than \( \alpha_1 \) and larger than \( \alpha_{2t-1} \), denoted as \( \alpha_0, \alpha_2t \) respectively. For any two adjacent real roots \( \alpha_{2i-1}, \alpha_{2i+1} \) of \( \pi_y(h) = 0 \), we can find a rational number, say \( \alpha_{2i} \). Then we obtain a sequence \( \alpha_i(i = 0, \ldots, 2t) \). Assume the real roots of \( h(\alpha_i, y) = 0 \) are \( \beta_{i,j}(j = 0, \ldots, t_i) \) which are listed in increasing order. We can find out that \((\alpha_i, \beta_{i,j})\) divide the plane curve \( h = 0 \) in the region \([\alpha_0, \alpha_2t] \times \mathbb{R} \) into regular curve segments. Let

\[
R = \max_{0 \leq s \leq 2t} \min_{0 \leq \alpha_i \leq 1} RB_s(f(\alpha_i, \beta_{i,j}, z)),
\]

\[
r = \min_{0 \leq s \leq 2t} \max_{0 \leq \beta_{i,j} \leq 1} (\beta_{i,j+1} - \beta_{i,j}),
\]

\[
0 < s < \frac{R}{2}, s \in \mathbb{Q}.
\]

where \( \beta_{i,-1} = -\infty \), \( RB_s(f(\alpha_i, \beta_{i,j}, z)) \) is the root bound of \( f(\alpha_i, \beta_{i,j}, z) \) in \( z \), \( f \) can be replaced by \( g \). Since it is probability 1 under condition (9) to obtain such an \( s \) that \( \varphi(f(\alpha_i, y, z)), \varphi(g(\alpha_i, y, z)) \) are in a generic position with the assumptions, it is probability 1 that \( \varphi(f(\alpha_i, y, z)), \varphi(g(\alpha_i, y, z)) \) are in a local generic position for all \( \alpha_i (i = 0, \ldots, 2t) \). And we can ensure this by checking whether \( \varphi(f) \wedge \varphi(g) \) is in a \( z \)-generic position.

**Finding the correspondence.** With local generic position method and the assumptions, we can recover the points of \( f \wedge g \) corresponding to \((\alpha_i, \beta_{i,j})\), say \((\alpha_i, \beta_{i,j}, z_{i,j,k}) \) \( (1 \leq k \leq t_{i,j}) \). As shown in Figure 6, from \( B(\alpha, \beta), B_1(\alpha, \beta) \), we can find out the point corresponding to \( B, B_1 \) in 3D space: \((\alpha, \beta, z_{i,j,k})\). Note that \( B_1 \) is in a neighborhood \( \alpha \times (\beta - r/2, \beta + r/2) \) of \( B \).

Let \( \alpha, \alpha' \) be any \( \alpha_i, \alpha_{i+1} \). We will classify the piece of curves inside \((\alpha, \alpha') \times \mathbb{R} \) into two cases by considering whether they contain singularities.

If the two endpoints of a regular curve segment \( C \) of \( h = 0 \) are in the fixed neighborhood of the endpoints of a regular curve segment \( \bar{C} \) of \( h = 0 \) respectively, we know that \( C \) corresponds to \( \bar{C} \), see \( \bar{R}_iU_i(i = 1, 2) \) and \( \bar{R}U \) in Figure 6 for example.

There are two cases for the singular points of \( \bar{h} = \pi_z(\varphi(f) \wedge \varphi(g)) = 0 \) in \((\alpha, \alpha') \times \mathbb{R} \): One case is that some correspond to singularities of \( f \wedge g \). Two or more space regular curve segments of \( f \wedge g \) intersect on a cylinder surface defined by some factor(s) of \( h = 0 \). So this kind of singularities of \( f \wedge g \) may not correspond to singularities of \( h = 0 \). If two or more left (right) branches of a singularity of \( \bar{h} = 0 \) correspond to a same regular curve segment of \( h = 0 \), we can judge that it is a true singularity of \( f \wedge g \); see the point \( D \) in Figure 6 for example. The regular curve segments \( A_1DL_2, A_2DL_1, A_3EF_2L_3 \) belong to \( \bar{h} = 0 \) all correspond to \( A\bar{R} \) since \( A_1, A_2, A_3(R_1, R_2, R_3) \) are in a neighborhood of \( A(\bar{R}) \).

The other case is that they are not true singularities of \( f \wedge g \). Thus the curve branches of \( \bar{h} = 0 \) which pass through these singular points of \( \bar{h} = 0 \) with different tangent lines correspond to disjoint space curves of \( \varphi(f) \wedge \varphi(g) \), then we can find out the correspondence of \( C \) and \( \bar{C} \), see points \( E, F \) in Figure 6 for example. Note that the continuous space curve maps to a continuous plane curve by \( \pi_z \). If there exist two or more curve branches having same tangent lines at a singular point of \( \bar{h} = 0 \) in \((\alpha, \alpha') \times \mathbb{R} \), see point \( G \) in Figure 6 for example, we call it tangent false singularity. If \( C_i \) contains only one tangent false singularity in \((\alpha, \alpha') \), we can still find out the correspondence of \( C_i \) and its corresponding regular curve segment \( C \) following the correspondence of the endpoints of \( C \) and \( C_i \). Note that if the endpoints of two regular curve segments of \( f \wedge g \) have same \( x, y \) coordinates, their corresponding projection curves overlap in \( h = 0 \), and the projection curve of their corresponding regular curve segment in \( \varphi(f) \wedge \varphi(g) \) are disjoint in \( h = 0 \) (except the endpoints). For \( G \) in Figure 6, we know the endpoints of \( \bar{R}_iG\bar{U}_i \) are in the fixed neighborhood of \( R \) and \( U \), so \( \bar{R}_3G, \bar{G}_3 \) correspond to \( RU \). But if \( \bar{C}_i \) contains two or more tangent false singularities in \((\alpha, \alpha') \), we can not determine the correspondence of the part(s) of \( C_i \) between these tangent false singularities and \( C \) (or other regular curve segment of \( h = 0 \)). As shown in Figure 6, \( H, K \) are two tangent false singularities and we do not know the correspondence of the two regular curve segments between them. We can find back the correspondence in the following
way. Let \((p_0, p_1)\) and \((q_0, q_1)\) be two adjacent tangent false singularities on \(C_1\) in \((\alpha, \alpha')\). Choose a rational number \(\gamma\) such that \(p_0 < \gamma < q_0\). Solving \(f(\gamma, y, z) = g(\gamma, y, z) = 0\), we can get some real points on \(f \cap g\). Solving \(h(\gamma, y) = 0\), we can get some real points on \(h = 0\). Since \(\varphi(f) \land \varphi(g)\) is in a \(z\)-generic position, \(\varphi(f(\gamma, y, z)) \land \varphi(g(\gamma, y, z))\) is in a generic position. So two group of points have a one-to-one map. Thus we can find out the correspondence between them. So we can decide the correspondence between \(C\) and (parts of) \(C_1\).

We need also check whether our approximation space curve changes the topology of original space curve or not. Since the plane approximation curve does not change the topology of the plane projection curve, we need only to check whether two approximation space regular curve segments having the same \(y\) coordinate are disjoint or not. We assume that the two approximation space regular curve segments are

\[
C_1 : (x, y(x), \frac{y_2(x) - y_1(x)}{s}, C_2 : (x, y(x), \frac{y_1(x) - y(x)}{s}), x \in [a, b].
\]

They are disjoint if \(\frac{y_2(x) - y_1(x)}{s} - \frac{y_1(x) - y(x)}{s} = 0\) has no real roots in \((a, b)\). We use the similar method as we check two plane approximation regular curve segments in Section 2.3.

3.3. Error control of the approximation space curves

In this subsection, we will consider how to control the error of the approximation space curve.

**Theorem 5** Use the notations as before. If we approximate the plane curves \(h = 0\) and \(\bar{h} = 0\) with errors \(\epsilon_1, \epsilon_2\), respectively, the error of each coordinate of the approximating curve of the algebraic space curve \(f \land g\) is bounded by \(\max(\epsilon_1, \frac{\epsilon_2}{s})\), and the Hausdorff distance error of the approximating curve is bounded by \(\sqrt{\epsilon_1^2 + (\epsilon_1 + \epsilon_2)^2}\).

**Proof.** Let \(C : (x, y(x))\) \((C_1 : (x, y_1(x)))\) be the regular curve segment of \(h = 0\) (\(\bar{h} = 0\)) and \(\bar{C} : (x, p(x))\) \((\bar{C}_1 : (x, q(x)))\) its approximation curve, \(x \in [x_0, x_1]\). \(C_1\) corresponds to \(C\). Let \(S_i : (x, y_i(x), \tilde{y}(x))\) (exact representation) and \(\bar{S}_i : (x, p(x), \frac{q(x) - p(x)}{s})\), \(x \in [x_0, x_1]\) be a space regular curve segment and its approximation. From the condition, we have \(e(C, \overline{C}) < \epsilon_1, e(C_1, \overline{C}_1) < \epsilon_2\). The error here is defined by (8). Let us consider the three coordinates of one part of the approximation curve \(S : (x, p(x), \frac{q(x) - p(x)}{s}), x \in [x_0, x_1]\).

The errors of the first and second coordinates are 0 and \(\epsilon_1, \epsilon_2\), respectively. For the third coordinate, we have \(\tilde{z}(x) = \frac{y(x) - y_i(x)}{s}\). Thus

\[
\frac{|\tilde{z}(x) - \frac{q(x) - p(x)}{s}|}{s} \leq \frac{|\tilde{z}(x) - \frac{q(x) - p(x)}{s}|}{s} \leq \frac{|\tilde{z}(x) - \frac{q(x) - p(x)}{s}|}{s} \leq \frac{\epsilon_1 + \epsilon_2}{s}.
\]

So the third coordinate is bounded by \(\frac{\epsilon_1 + \epsilon_2}{s}\) from (8). From the definition of Hausdorff distance (6), we have the Hausdorff distance of \(S_i\) and \(\bar{S}_i\):

\[
e(S_i, \bar{S}_i) = \max_{P \in S_i} \min_{P' \in \bar{S}_i} d(P, P') \leq \max_{P \in S_i, P' \subseteq P} d(P, P') \leq \sqrt{\epsilon_1^2 + (\epsilon_1 + \epsilon_2)^2}.
\]

This ends the proof. □

If the required precision for the approximation curve is \(\epsilon\), we can approximate the plane algebraic curves \(h = 0\) and \(\bar{h} = 0\) with precision \(\frac{s}{\sqrt{\epsilon_1^2 + (\epsilon_1 + \epsilon_2)^2}}\) from the theorem.

Fig. 6. Two projection curves and their correspondences
3.4. $G^1$-continuous rational approximation space curve

We will derive approximation space curve from plane approximation curve. And we will re-parameterize the non-rational parametric curve into rational ones. Thus the obtained approximation space parametric curves are $G^1$-continuous and rational.

**Lemma 6** Use the notations as before. If we approximate the plane curves $h = 0$ and $\tilde{h} = 0$ with $C^1$-continuous parametric curve, the approximation curve of the algebraic space curve $f \land y$ is $C^1$-continuous.

**Proof.** Let $(x, p(x)), (x, q(x)), x \in [x_0, x_1]$ be two corresponding approximation curves of the regular curve segments of $h = 0$ and $\tilde{h} = 0$ and $p(x), q(x)$ are $C^1$-continuous in $[x_0, x_1]$. We can obtain the approximating curve of the space regular curve segment: $S : (x, p(x)), (x, q(x)), x \in [x_0, x_1]$. The tangent direction of $S$ at any $x$ is $(1, \frac{\partial p}{\partial x}, \frac{\partial q}{\partial x})$. From the definition of $C^1$-continuous, we can find that $S$ is $C^1$-continuous since $(x, p(x)), (x, q(x)), x \in [x_0, x_1]$ is $C^1$-continuous. For the 3D point $P$ of $f \land y$ corresponding to a VT point, if we require the approximating space curve is $C^1$-continuous at $P$, then the whole approximating space curve is also $C^1$-continuous. □

When re-parameterizing the approximation space regular curve segments into rational ones, we need to know the tangent directions of the endpoints of space regular curve segments. For the endpoints corresponding to non-VT points, we can directly get it from the tangent directions of the plane curves. For the endpoints corresponding to VT points, we can get the tangent directions as follows. At first, we assume that $(x, p(x)), (x, q(x)), x \in [x_0, x_1]$ are parametric plane regular curve segments of exact algebraic regular curve segments $(x, \tilde{y}_1(x)), (x, \tilde{y}_2(x)), x \in [x_0, x_1]$ and $x_0$ corresponds to a VT point. The exact tangent direction of the algebraic regular curve segment at $x_0$ is $(1, \frac{\tilde{y}_1'(x_0)}{\tilde{y}_2'(x_0)})^T$ from the parametric representation. Note that $(1, \infty)$ corresponds to $(0, 1)$ for plane regular curve segments. So for the approximation tangent direction at $x_0$: $(1, (\frac{\partial p(x_0)}{\partial x} - \frac{\partial q(x_0)}{\partial x})/(s \frac{\partial q(x_0)}{\partial x}))$, if $\frac{\partial q(x_0)}{\partial x}$ is larger than (or less than) some given value, for example, 100 or -100, we can reset the tangent direction as $(0, 1, (\frac{\partial p(x_0)}{\partial x} - \frac{\partial q(x_0)}{\partial x})/(s \frac{\partial q(x_0)}{\partial x}))$. Moreover, if $100(\frac{\partial p(x_0)}{\partial x} - \frac{\partial q(x_0)}{\partial x})/(s \frac{\partial q(x_0)}{\partial x})$ is larger than (or less than) some given value, we can set the tangent direction as $(0, 0, 1)$. So the tangent directions at $x_0$ are $(0, 1, p), p \neq 0$ or $(0, 0, \pm 1)$.

**Reparameterization of space curve.** If the tangent direction at $x_0$ is $(0, 1, p)$, we can re-parameterize the space curve segment with the form

$$P(t) = (\frac{a_{1}t^2 + b_{1}t + c_{1}}{d_{1}t + 1}, t, \frac{a_{2}t^2 + b_{2}t + c_{2}}{d_{2}t + 1} + \frac{c_{3}}{d_{3}t + 1}), t \in [0, 1].$$

such that it is $G^1$-continuous with other regular curve segments at the endpoints. Assume that the two endpoints are $(x_i, y_i, z_i), i = 0, 1$ and the given tangent directions at two endpoints are $(x'_i, y'_i, z'_i), i = 0, 1$. Thus $x'_0 = 0$. Here for simplicity, we assume that $y_0 = 0, y_1 = 1$ since we can set $t = \frac{y - y_0}{y_1 - y_0}$. Bisecting the regular curve segment ensures that $y'_1 \neq 0$ since the regular curve segment is monotonous.

We require that the parametric space curve satisfying $G^0$ and $G^1$ conditions at the two endpoints. So we have eight valid equations from the following equations.

$$P(t)|_{t=0} = (x_0, y_0, z_0), P(t)|_{t=1} = (x_1, y_1, z_1),$$

$$\frac{\partial P(t)}{\partial t}|_{t=0} = (0, 1, p), \frac{\partial P(t)}{\partial t}|_{t=1} = (1, x'_1, y'_1, z'_1).$$

Solving them, we have one solution as below.

$$a_1 = \frac{x_0^2 - 2x_0x_1 + x_1^2}{-x_1 + x'_1 + x_0}$$

$$a_2 = \frac{1}{d_3} \left( x'_1d_3 + x'_1d_2d_3 - x_1d_3 - d_3x_0 + x'_0 + x'_1d_2 - x_1d_2 + d_2x_0 + x'_1 \right)$$

$$b_1 = \frac{-x_0(-2x_0 - 2x_1 + x'_1)}{-x_1 + x'_1 + x_0}$$

$$b_2 = -\frac{1}{d_3^2} \left( -2x_1d_2 + d_2x_0 + x'_1d_2 - 2x_1d_2 + d_2x_0d_3 + x'_1d_2d_3 \right)$$

$$c_1 = x_0$$

$$c_2 = -\frac{1}{d_3^2} \left( -2x_1d_2 + d_2x_0 + x'_1d_2 + 2x_1d_2d_3 + x'_1d_2 + x'_0 - 2x_1 \right)$$

$$c_3 = -\frac{1}{d_3^2} \left( -2x_1d_2 + d_2x_0 + x'_1d_2 + 2x_1d_2d_3 + x'_1d_2 + x'_0 - 2x_1 \right)$$

$$d_1 = \frac{2x_0 - 2x_1 + x'_1}{-x_1 + x'_1 + x_0},$$

where $d_2, d_3$ are free variables. At first, we require that $x'_1 - x_1 + x_0 \neq 0$, $d_3(d_2 - d_3) \neq 0$ since they are denominators. Second, we require that $d_i t + 1 = 0(i = 1, 2, 3)$ in $t$ has no root in $[0, 1]$, that is, $d_i > -1$. For $i = 1$, we have equal conditions:

$$(x_0 - x_1)(x_0 - 1 + x'_1) < 0.$$  

(12)

Since the given planar regular curve segment is monotonous (w.r.t. both $x$ and $y$), the first condition and (12) hold directly. We can choose proper $d_2, d_3$ such that conditions hold.

For the case of tangent direction is $(0, 0, 1)$, we can set the parametric regular curve segment as

$$P(t) = (\frac{a_{1}t^2 + b_{1}t + c_{1}}{d_{1}t + 1}, \frac{a_{2}t^2 + b_{2}t + c_{2}}{d_{2}t + 1} + \frac{c_{3}}{d_{3}t + 1}, t), t \in [0, 1],$$

and solve a similar equation system to get the parametric regular curve segments.

The left problem is to control the precision. Let $\epsilon$ be the required precision for the whole approximation parametric curve. If the non-rational parametric curve $S_1 : (x, p(x), q(x)), x \in [x_0, x_1]$ approximates the regular curve segment of algebraic space curve $S$ with precision $\epsilon/2$, and
the new rational parametric curve \( S_2(x(t), y(t), z(t)) \) approximate \( S_1(x, p(x), q(x)) \) with precision \( \epsilon/2 \), then \( S_2 \) approximate \( S \) with precision \( \epsilon \).

We need to control the approximation precision of \( S_2 \) to \( S_1 \). In [42], the authors consider the approximation of 3-D parametric curve with rational Bézier curves. For our problem, we need rational curve. For any fixed \( p \) we can derive a univariate polynomial equation in \( t \) of degree 2 by \( p(x(t)) = y(t) \). Solving it, we have two real solutions (the solutions do exist). Choose the one such that \( x(t) \) close to \( x_0 \), say \( t_0 \). Denote the distance between \( (x(t_0), p(x(t_0)), q(x(t_0))) \) and \( (x(t_0), y(t_0), z(t_0)) \) as \( D(x_0) \). From the definition (6), we can find that max\( \forall x_0 \mid x_0, x_1 \rangle D(x_0) \geq \epsilon(S_1, S_2) \) is an upper bound of the Hausdorff distance of \( S_1 \) and \( S_2 \). We can choose some sample points to estimate the error between \( S_1 \) and \( S_2 \).

Thus, in the end, we get a \( G^1 \)-continuous piecewise rational approximation space curve under a given precision. When we approximate a regular curve segment containing a VT point in practice, we usually select a short distance for it since the error control is much easier.

### 4. Algorithm and examples

In this section, we will give the main algorithm to approximate algebraic space curves and use some non-trivial examples to illustrate the effectiveness of our algorithm.

**Algorithm 7** The inputs are \( f, g \in \mathbb{Q}[x, y, z] \) such that gcd\( f, g \) = 1 and satisfying the two assumptions, a bounding box \( B = [X_1, X_2] \times [Y_1, Y_2] \times [Z_1, Z_2] \) and an error bound \( \epsilon > 0 \). The outputs are piecewise rational parametric curve \( C_i := \{(x, y(x), z(x)) \mid (x, y), z, a_i \leq x (or \ y) \leq b_i, (i = 1, ..., N) \} \), which give a \( G^1 \)-continuous approximation to \( f \& g \) in \( B \) with precision \( \epsilon \).

(i) Topology determination and regular curve segmentation of the plane curve defined by \( C_1 : \pi_z(f \& g) \).

(ii) Compute a rational number \( s \) as mentioned in Theorem 4.

(iii) Let \( \varphi_z : (x, y, z) \rightarrow (x, y + s, z) \). Topology determination and regular curve segmentation of the plane curve defined by \( C_2 : \pi_z(\varphi(f) \& \varphi_z(g)) \).

(iv) Find out the correspondence between the regular curve segments of \( C_1 \) and \( C_2 \).

(v) Approximate the regular curve segments without VT point of \( C_1 \) and \( C_2 \) with \( \epsilon_0 < \frac{\epsilon}{\sqrt{3} \sqrt{2} \sqrt{1 + 4}} \) and the ones with VT point with precision \( \epsilon_0 < \frac{\epsilon}{\sqrt{3} \sqrt{2} \sqrt{1 + 4}} \).

(vi) Recover the space approximation regular curve segments of \( f \& g \) with formula (10).

(vii) Re-parameterize the non-rational approximation curves to rational approximation curves under the error control if there exist.

(viii) Output the piecewise approximation regular curve segments.

We will show several examples to illustrate our algorithm.

**Example 1** Consider the algebraic space curve defined by the system \( \{f, g\} = \{x^2 + y^2 + z^2 - 4, (z - 1) (x^2 + y^2 - 3 z^2)\} \).

In fact, they are two plane circles with \( z = \pm 1 \) as shown in Figure 7 (green ones). The space curve is not irreducible, not regular, and not in a generic position. We will approximate it with rational curves under precision \( 10^{-2} \). Following Algorithm 7, we have

(i) Compute the resultant of \( f, g \) w.r.t. \( z \), we have \( C_1 : h = x^2 + y^2 - 3 = 0 \), as the red circle in Figure 7. We split \( C_1 \) into eight regular curve segments with \( x \)-coordinates \([-1.732050808, -1.0, 0, 1.0, 1.732050808]: \) And the tangent directions of the points all are \((1, -\frac{\partial h}{\partial x}/\frac{\partial h}{\partial y})\) evaluated at these points. Note that \( x = 0 \) corresponds \( x \)-critical points of \( C_1 \) and \((\pm 1.732050808, 0) \) correspond to VT points.

(ii) Since \( \pi_y(h) = x^2 - 3 \), we can obtain \( \alpha_0 = -2, \alpha_1 = -1.732050808, \alpha_2 = 0, \alpha_3 = 1.732050808, \alpha_4 = 2 \). And we can get \( r = 3.464101616 \). Following Theorem 4, we have \( R = 1.0 \) when we choose \( g \) to compute \( R \). We can select \( s = 1 < \frac{1}{R} \) = 1.732050808.

(iii) Compute the resultant of \( \varphi(f) = f(x, y + z, z), \varphi(g) = g(x, y + z, z) \) w.r.t. \( z \), we have \( C_2 : \hat{h} = (x^2 + y^2 - 2 + 2 y) (-2 + x^2 - 2 y + y^2) = 0 \), as two blue circles in Figure 7. Since \( \pi_y(\hat{h}) = (x^2 - 3) (x^2 - 2) = 0 \), we split \( C_2 \) into 16 regular curve segments at \( x = (-1.732050808, -1.414213562, 0, 1.414213562, 1.732050808) \). And the tangent directions at the endpoints of these regular curve segments are \((1, -\frac{\partial h}{\partial x}/\frac{\partial h}{\partial y})\) evaluated at the points close to these points. We can get the approximating tangent directions. And we can find that \((\pm 1.732050808, \pm 1) \) are VT points since the absolute values of \( \frac{\partial h}{\partial x}/\frac{\partial h}{\partial y} \) evaluated at \((1.73204, \pm 0.006118660), (\pm 1.73204, \pm 0.9938813400) \) are larger than 200.

(iv) As shown in Figure 7, the critical points of \( C_1 \) are \( P, Q \). Choose a vertical line which intersect \( C_1 \) at \( W \), \( K(K = P, Q, W, T) \) are points on \( C_1 \) and \( K_1, K_2 \) are corresponding points of \( K \) on \( C_2 \). Consider \( W(0, 1.732050808), W_1(0, 1.732050808), W_2(0, 0.732050808) \) for example. We can find that \( W_1, W_2 \) are on the line \( x = 0 \) in a neighborhood with radius 1.732050808 centered at \( W \). So we can conclude that \( W_1, W_2 \) correspond to \( W \) with local generic position method. The correspondence of other points are similar.

(v) Approximate \( C_1, C_2 \) respectively. In order to derive the required precision \( 10^{-2} \), we use precision \( \epsilon_3 = 0.0044 < \frac{1}{\sqrt{1 + 4}} \cdot 10^{-2} \) for the regular curve segments of \( C_1, C_2 \) without VT point(s), and we use precision \( \epsilon_2 = \)
0.0022 < \frac{1}{2y - 1} 10^{-2} for the regular curve segments with VT point. Consider a regular curve segments on $C_1$, $(-1.732050808, 0), (1.60, 0.663324958)$ are the endpoints for the one, denoted as $C_2$. And it has no VT point. $(-1.60, 0.6633249580), (1.40, 1.019803903)$ are endpoints for the other, denoted as $C_2$. It has no VT point. The approximation of $C_1$ is $(x, 0.9999999959 \sqrt{-x^2 + 0.00000464 x + 3.00000806}, x \in [-1.732050808, -1.60])$ and the error is very small. The approximation for $C_2$ is $(x, 0.6106757885 x + 3.01809554 - 0.1270414345/(0.5015084499 x + 1.0), x \in [-1.60, -1.40])$ and the error is 0.0004 < $\epsilon_1$. For the regular curve segments on $C_2$ with endpoints: $([-1.732050808, 1.0], (1.60, 1.6633249580], (1.414213562, 2.0)$ are endpoints for the other, denoted as $C_4$, without VT point. Similarly as $C_1$, the approximation for $C_3$ is $(x, 1.0 + 0.9999999054 \sqrt{-x^2 + 0.00000466 x + 3.00000809}, x \in [-1.732050808, -1.60])$. The approximation for $C_4$ is $(x, 0.6301674595 x + 3.32409999 - 1.265242345/(0.508634591 x + 1.0), x \in [-1.60, -1.40]$ and the error is 0.0002 < $\epsilon_2$. We can find that parts of $C_1, C_2$ and $C_3, C_4$ are correspond.

(vii) Recover the approximation space curves of $f \wedge g$ by the formula $z = g(x) - g(y)$. The space regular curve segment corresponding to $C_1$ and $C_3$, we have its approximation parametric space regular curve segment for $x \in [-1.732050808, -1.60]$

\[
(x, 0.9999999959 \sqrt{-x^2 + 0.00000464 x + 3.00000806}, 1.0 + 0.9999999054 \sqrt{-x^2 + 0.00000466 x + 3.00000809}, -0.9999999959 \sqrt{-x^2 + 0.00000464 x + 3.00000806}).
\]

The approximation space curve is not rational, denoted as $S_1$. The approximation corresponding to $C_2, C_4$ for $x \in [-1.60, -1.414213562]$, denoted as $S_2$, is

\[
(x, 0.6106757885 x + 2.310809554 - 0.1270414345/(0.508634591 x + 1.0), 0.191401640 x + 0.103250445 - 0.1270414345/(0.5015084499 x + 1.0), 0.1270414345/(0.5015084499 x + 1.0)).
\]

(viii) We will re-parameterize $S_1$ into rational one. At first, we can find that the $y$ coordinate of $S_1$ changes from 0 to 0.6633249580. Its two endpoints are $P_0(-1.732050808, 0, 1.0), P_1(-1.60, 0.6633249580, 1.0)$. The tangent direction of $S_2$ at $P_2$ is $(1, 2.412090757, 0.0)$. By approximating the tangent direction of $S_1$ at $P_1$, we have $(1, 283.0783218, 0.0)$. And there is another regular curve segment which shares a same tangent direction with $S_1$ at $P_1$. Taking their average value, we can set the tangent direction of $S_1$ at $P_1$ as $(0, 1, 0)$. Using Formula (11), we can easily obtain the rational approximation regular curve segment for $S_1$ is

\[
(-2.412090984 y - 22.71853197 + 20.98438311 z, y, 1.0).
\]

The error in $x$-direction is 0.0020563160 < $\epsilon_2$. (We take 19 sample points besides endpoints to compute the error.). So the approximation rational curve satisfies the error requirement.

(viii) Output the piecewise approximation curve.

Example 2 Approximate the algebraic space curve defined by $f = g = 0$, where $f = x^2 + y^2 + z^2 - 4, g = (x^2 + y^2 + 2y - z^2) (z - x - 4y)$. It is a space curve with singular point. The approximation space curve is as the left part of Figure 8 and the error is 0.013. The color differs the different approximating piecewise regular space curve segments.

Fig. 8. Approximation curve and projection curves in Example 2.

Example 3 In this example, we will approximate the algebraic space curve defined by $f = g = 0$ inside $[-2, 2] \times [-2, 2]$ with error $\epsilon = 0.014$, where $f = 95 - 94 x^3 + 64 x^2 y + 28 x^2 z - 61 x^2 + 69 x y^2 - 53 x y z - 59 x y + 28 x z^2 - 15 x z - 83 x - 3 y^3 + 59 y^2 z + 49 y^2 + 4 y z^2 + 11 y z + 9 y - 81 z^3 - 8 z^2 - 9 z, g = 49 + 7 x^3 - 46 x^2 y + 87 x^2 z + 94 x^2 + 73 x y^2 + 93 x y z - 3 x y - 27 x z^2 + 56 x z + 70 x + 72 y^3 - 37 y^2 z - 20 y^2 + 79 y z^2 - 78 y z - 3 y + 94 z^3 + 30 z^2 + 47 z$. The approximation space curve is as Figure 9.

Fig. 9. Approximation curve and projection curves in Example 3.

5. Conclusion

We introduce a local generic position method to compute the topology as well as the piecewise approximation curves of algebraic space curves. Especially, we present an algorithm to approximate algebraic space curve by piecewise rational curves with correct topology and under any given precision. The method is effective.

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References and Notes

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