Inhomogeneous spacetimes in Einstein-äether cosmology

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Abstract
We investigate the existence of inhomogeneous Szekeres spacetimes in Einstein-äether theory. We show that inhomogeneous solutions which can be seen as extension of the Szekeres solutions existing in Einstein-äether gravity only for a specific relation between the dimensionless coefficients which defines the coupling between the æther field with gravity. The two Szekeres classes of solutions are derived. Also a class of inhomogeneous FLRW-like spacetimes is allowed by the theory for arbitrary values of the dimensionless coefficients of the æther field. The stability of the solutions obtained is performed from where we find that the field equations evolve more variously in Einstein-äther than in general relativity, where isotropic spacetimes and Kantowski–Sachs spacetimes are found to be attractors.

Keywords: inhomogeneous spacetime, Szekeres, Einstein-Aether, exact solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

Inhomogeneous spacetimes are of special interest in the gravitational theory, because in general they are exact solutions of Einstein’s general relativity (GR) without any symmetries. Inhomogeneous cosmological models are those which do not satisfy the cosmological principle, but they provide the limit of Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime [1]. There are many applications of the inhomogeneous spacetimes in cosmological studies which cover all the different epochs of the Universe. Indeed, inhomogeneous Universes can be seen as the limit of FLRW spacetimes with inhomogeneous perturbations, which can describe the CMB anisotropies, as also the rest of the structure formation [2]. As far as the very
early Universe is concerned, inhomogeneous spacetimes can have singularities of many kinds, i.e. isotropic, cigar, pancake, oscillatory and nonscalar. For more details we refer the reader to the discussion given in [3].

One of the most well-known inhomogeneous spacetimes is the Lemaître–Tolman–Bondi (LTB) metric which has the spherical symmetry. There are various cosmological applications of the LTB spacetime, and more specifically as toy models in cosmological studies [4–8]. LTB spacetimes belong to the more general family of Szekeres spacetimes [9]. The latter spacetimes are exact solutions of GR with an inhomogeneous fluid source where in general the metric depends on two scale factors. Szekeres spacetimes are categorized in two classes, the FRLW-like spacetimes where LTB metric belongs.

The recent observation of gravitational waves [10, 11] and the direct observation of the black hole at the center of the Galaxy M87 by the event horizon telescope [12, 13] indicates the validity of GR. However, in very-large scales, GR is challenged by the cosmological observations [14–18]. During the last decades cosmologists have worked on two main directions on the explanation of the cosmological observations. The first direction is based on the introduction of an energy–momentum tensor in Einstein’s field equations, where the matter source is described by an exotic matter source, such that Chaplygin gas, quintessence, $k$–essence and others [19–25]. The alternative direction introduced by cosmologists is based on the introduction of new terms in the Einstein–Hilbert action, the role of these new terms is to drive the dynamics of the modified field equations such that to explain the observable phenomena. These kinds of theories are called alternative/modified theories of gravity [26–34]. Therefore, it becomes necessary to study the existence of cosmological evolution of inhomogeneous spacetimes in these extensions of general relativity.

A family of theories which have drawn the attention of cosmologists are the Lorentz violated theories. In this work, we are interested in the existence of inhomogeneous cosmological exact solutions in the Einstein–æther theory [35, 36]. In this specific theory, the kinematic quantities of a unitary time-like vector field coupled to gravity are introduced in the Einstein–Hilbert action. That vector field is called æther and defines a preferred frame at each point in the spacetime.

Although the gravitational field equations in Einstein-æther theory are of second-order because of the introduction of the nonlinear terms which follow by the æther field, there are few known exact solutions in the literature, some exact cosmological solutions presented recently in [37], while exact solutions which correspond to critical points on the phase space of the dynamical system are determined in [38–45]. Moreover, exact solutions in the presence of a scalar field coupled to the æther were found in [46, 47].

As far as the inhomogeneous spacetimes in Einstein-æther theory are concerned, some exact solutions determined in [45], where the spacetime admits the spherical symmetry, while cosmological perturbations in Einstein-æther theory have been studied before [49, 50]. For spacetimes where they do not admit any isometry, there are not known exact solutions in the literature. That is specific the problem that we investigate in this work.

In this study, we focus on the field equations of the Einstein-æther field for the four-dimensional spacetime which provides the Szekeres spacetimes in GR. We prove the existence of generalized Szekeres solutions in the context of Einstein-æther theory. Furthermore, we investigate the general evolution of the field equations for the case of Szekeres–Szafron spacetimes in Einstein-æther theory. In particular we write the dynamic equation by using the $1 + 3$ decomposition and we study the existence of critical points as also their stability of different values of the Einstein-æther free parameters. The paper is structured as follows.

In section 2, we present the field equations in the Einstein-æther theory. Section 3 includes the main analysis of our work where we solve the field equations of Einstein-æther theory in
the context of an inhomogeneous spacetime which provides the Szekeres family of solutions. The stability of the exact spacetimes is studied in section 4. We found that as opposed to GR, in Einstein-æther theory the field equations admit more critical points while the stability of the limits of GR changes such that spacetimes of special interests to be found as attractors. Finally, in section 5 we discuss the results and we draw our conclusions.

2. Einstein-æther gravity

Einstein-æther theory is a Lorentz violated gravitational theory which consists of GR coupled at second derivative order to a dynamical timelike unitary vector field, the æther field, \( u^\mu \). This vector can be thought as the four-velocity of the preferred frame.

The gravitational action integral is defined as

\[
S_{AE} = \int \sqrt{\frac{\det g}{g}} R - \int \sqrt{\frac{\det g}{g}} \left( K^{\alpha\beta\mu\nu} u^\mu u^\nu \eta_{\alpha\beta} - \lambda (u^\mu u^\nu + 1) \right),
\]

(1)

where \( R \) is Ricciscalar of the underlying spacetime with line metric tensor \( g_{\alpha\beta} \), and \( K^{\alpha\beta\mu\nu} \) describes the coupling between the æther field and the gravity, defined as

\[
K^{\alpha\beta\mu\nu} \equiv c_1 g^{\alpha\beta} g^{\mu\nu} + c_2 g^{\alpha\mu} g^{\beta\nu} + c_3 g^{\alpha\nu} g^{\beta\mu} + c_4 g^{\mu\nu} u^\alpha u^\beta.
\]

(2)

Function \( \lambda \) is a Lagrangemultiplier which constraints \( u^\mu \) to be a unitary vector field. Parameters \( c_1, c_2, c_3 \) and \( c_4 \) are dimensionless constants and define the coupling between the æther field with gravity. Indeed when constants \( c_Z, Z = 1, 2, 3, 4 \) vanish then action integral (1) becomes that of the Einstein–Hilbert action.

The total set field equations follow by variation of the action integral (1) with the metric tensor, the æther field and the Lagrange multiplier \( \lambda \). The latter condition, \( \delta S_{AE} / \delta \lambda = 0 \) provides the constraint condition

\[
u^\mu u_\mu + 1 = 0,
\]

(3)

for the unitarity of the æther field. Variation with respect to the æther field \( \delta S_{AE} / \delta u^\mu \), gives the equation of motion for the vector field \( u^\mu \), that is,

\[
c_4 g^{\mu\nu} u^\alpha u^\beta u_{\rho\sigma} g^{\rho\sigma} - c_4 g^{\mu\nu} g^{\lambda\kappa} u_{\lambda\beta} u^\beta u_{\rho\sigma} - c_4 g^{\mu\nu} u_\beta u^\mu u_{\rho\sigma} - K^{\alpha\beta\mu\nu} u_{\mu\alpha\beta} - \lambda g^{\mu\nu} u_\alpha = 0.
\]

(4)

Therefore, the modified gravitational field equations follows by variation with respect to the metric tensor \( \delta S_{AE} / \delta g_{\mu\nu} = 0 \), that is,

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}^a.
\]

(5)

The lhs of the latter expression is the Einstein tensor, while the rhs is the contribution of the æther field in the gravitational field equations which are presented by the æther energy–momentum tensor \( T_{\mu\nu}^a \) defined as [45]

\[
T_{ab}^a = 2c_1 (u^\mu u_{\nu\rho} - u_{\nu\rho} u^\lambda) + 2\lambda u_{ab} g^{\mu\nu} + g_{ab} \Phi_u - 2 (u^\mu J_{\nu\rho\lambda} \Phi_a - (u^\mu J_{\nu\rho\lambda})_{\rho\lambda}) - c_4 (u^\mu u^\nu) (u_{\nu\rho} u_{\rho\lambda}),
\]

(6)

in which \( J^a_m = -K^{ab}_{mn} u^a_{\nu\rho} \), \( \Phi_u = -K^{ab}_{mn} u^a_{\nu\rho} u^d_{\nu\rho} \).
An equivalent way to write the action integral (1) is with the use of the kinematic quantities for the æther field $u^{\mu}$. Indeed at the $1 + 3$ decomposition quantity $u^{\mu}_{\nu}$ can be written as

$$u^{\mu}_{\nu} = \sigma^{\mu}_{\nu} + \omega^{\mu}_{\nu} + \frac{1}{3} \theta h^{\mu}_{\nu} - \alpha^{\mu} u^{\nu}$$

(7)

where

$$\alpha^{\mu} = u^{\mu}_{\nu} u^{\nu}, \quad \theta = u^{\mu}_{\nu} h^{\mu}_{\nu}, \quad \sigma^{\mu}_{\nu} = u^{\mu}_{(\alpha,\beta)} h^{\alpha}_{\mu} h^{\beta}_{\nu} - \frac{1}{3} \theta h^{\mu}_{\nu}, \quad \omega^{\mu}_{\nu} = u^{\mu}_{(\alpha,\beta)} h^{\alpha}_{\mu} h^{\beta}_{\nu}$$

(8)

in which $h^{\mu}_{\nu} = g^{\mu}_{\nu} - \frac{1}{2} u^{\mu} u_{\nu}$.

Thus, by using the latter expression the action integral (1) is simplified as

$$S_{EA} = \int \sqrt{-g} d^4x \left( R + c_{\theta} \theta^2 + c_{\sigma} \sigma^2 + c_{\omega} \omega^2 + c_{\alpha} \alpha^2 \right)$$

(9)

where the æther field $u^{\mu}$ has been assumed to be unitary and the new coefficient constants are defined as [48]

$$c_{\theta} = \frac{1}{3} (3 c_2 + c_1 + c_3), \quad c_{\sigma} = c_1 + c_3, \quad c_{\omega} = c_1 - c_3, \quad c_{\alpha} = c_4 - c_1,$$

(10)

and $\sigma^2 = \sigma^{\mu}_{\nu} \sigma_{\mu\nu}$, $\omega^2 = \omega^{\mu}_{\nu} \omega_{\mu\nu}$.

We proceed our analysis by studying the field equations in the case of inhomogeneous spacetimes.

### 3. Szekeres spacetimes

Consider now the inhomogeneous spacetime with line element

$$ds^2 = -dt^2 + e^{2a(t,x,y,z)} dx^2 + e^{2b(t,x,y,z)} (dy^2 + dz^2).$$

(11)

In the context of GR with an inhomogeneous pressureless fluid source with energy momentum tensor $T_{\mu\nu} = \rho_m (t, x, y, z) v^{\mu} v_{\nu}$, where $v_{\mu} = \delta^{\mu}_t$, the line element (11) provides the inhomogeneous Szekeres spacetimes. Szekeres spacetimes are exact solutions of Einstein’s GR which lack any symmetry in general, while Szekeres spacetimes have been characterized as ‘partially’ locally rotational spacetimes [51].

The magnetic part of the Weyl tensor is zero and since there is not any pressure component there is no information dissemination with gravitational or sound waves between the world-lines of neighboring fluid elements, that is why Szekeres spacetimes belong to the family of silent Universes [52, 53]. Furthermore, the rotation and acceleration of the fluid source are identical zero, while in general the spacetimes are anisotropic which means that the shear is nonzero as also the expansion rate is nonzero.

In inhomogeneous cosmology, the large-scale structure of the Universe is described by exact solutions, unlike cosmological perturbation theory to explain the structure formation [2]. In addition in [68] proved that small inhomogeneities in the spacetime does not affect necessary the existence of expansion phases of the Universe. Therefore Szekeres spacetimes can have applications in the description of the Universe in the pre- and after-inflationary epochs. In
Szekeres spacetimes have been applied as exact perturbations models of an FLRW background in order to describe the structure formation of the Universe. While in [73, 74] the authors proved that a wide class of inhomogeneous geometries, including the Szekeres spacetimes, can evolve into homogeneous FLRW geometries for specific initial conditions. A similar result was found in [75] without imposing an inflationary era in the evolution of the Universe.

Szekeres spacetimes are categorized in two subfamilies. Subfamily (A) with $b_s = 0$, describes inhomogeneous spacetimes with two independent scale factors whose time derivatives satisfy the field equations of Kantowski–Sachs spacetimes. The second subfamily (B) is characterized by the condition $b_s \neq 0$ and corresponds to inhomogeneous FLRW spacetime with only one free time-dependent scale factor. Szekeres spacetimes have been generalized by assuming homogeneous fluid source [54], cosmological constant [55], heat flow [56], electromagnetic field [56, 58], viscosity [59–61] and others [2]. Recently exact solutions for the line element (11) with an inflaton have been determined in [62] while some cyclic Szekeres spacetimes have been applied as exact perturbation models of an FLRW background in order to describe the structure formation of the Universe. While in [73, 74] Szekeres spacetimes have been generalized to inhomogeneous geometries with two independent scale factors whose time derivates satisfy the field equation of Kantowski–Sachs spacetimes. The second subfamily (B) is characterized by the condition $b_s \neq 0$ and corresponds to inhomogeneous FLRW spacetime with only one free time-dependent scale factor. Szekeres spacetimes have been generalized by assuming homogeneous fluid source [54], cosmological constant [55], heat flow [56], electromagnetic field [56, 58], viscosity [59–61] and others [2]. Recently exact solutions for the line element (11) with an inflaton have been determined in [62] while some cyclic Szekeres spacetimes were found in [63] by considering the existence of a second phantom ideal gas. We continue by investigating the existence of exact solutions for the line element (11) in the case of Einstein–æther theory.

For the æther field we do the simplest selection and we assume that it is the comoving observer $u^\mu = \delta^\mu_0$, which is normalized, i.e. $u^\mu u_\mu = -1$ [64, 65]. For such a selection and for the line element (11) we calculate $\omega = 0$ and $\alpha = 0$, consequently the coefficient constants $c_\omega$ and $c_\alpha$ do not play any role in the evolution of the dynamical system. For the matter source we consider the energy momentum tensor $T_{\mu\nu} = \rho_m (t, x, y, z) u_\mu u_\nu$. The physical reason that we have assumed the æther field to be the comoving observer is in order the FLRW limit to exists and our solutions to describe inhomogeneous cosmological solutions.

The energy momentum tensor for the æther field is calculated to be diagonal with components

$$T^{\text{ae}}_t = (c_1 + c_2 + c_3) \left( a_j \right)^2 + 2 (c_1 + 2 c_2 + c_3) \left( b_j \right)^2 + 4 c_2 \left( a_j b_j \right),$$

(12)

$$T^{\text{ae}}_x = (c_1 + c_2 + c_3) \left( 2 a_s + \left( a_j \right)^2 + 4 a_s b_j \right) + 4 c_2 b_x - 2 (c_1 - 2 c_2 + c_3) \left( b_j \right)^2,$$

(13)

$$T^{\text{ae}}_y = T^{\text{ae}}_z = 2 c_2 a_s - (c_1 - c_2 + c_3) \left( a_j \right)^2 + 2 (c_1 + 2 c_2 + c_3) \left( b_j + b_j \right)^2 + a_j b_j,$$

(14)

or equivalently

$$T^{\text{ae}}_{\mu\nu} = \rho^a u_\mu u_\nu + p^a \delta^{\mu\nu}_0 + 2 q^a_{\mu\nu} u_\mu + \pi^a_{\mu\nu}$$

(15)

in which the physical quantities are defined as [45]

$$\rho^a = -c_d \theta^2 - 6 c_\sigma \sigma^2,$$

(16)

$$p^a = c_d \left( 2 \theta_j + \theta^2 \right) - 6 c_\sigma \sigma^2,$$

$$q^a_{\mu\nu} = 0,$$  \quad $$\pi^a_{\mu\nu} = -2 \pi^a_{\mu} = -2 \pi^a_{\nu} = -4 c_\sigma \left( \sigma_j + \theta \sigma \right),$$

(17)

where the shear $\sigma$ and the expansion rate $\theta$ are given by the following expressions

$$\theta (t, x, y, z) = a_j + 2 b_j, \quad \sigma (t, x, y, z) = \frac{a_j - b_j}{3}.$$
The equation of motion for the æther field (4) provides the following components

\[
\left( c_0 + \frac{c_\sigma}{6} \right) \theta_{,\sigma} - 3c_\sigma \sigma_{,\sigma} - 9c_\sigma \sigma_{,\sigma} = 0, \quad (19)
\]

\[
\left( c_0 + \frac{c_\sigma}{6} \right) \theta_{,A} + \frac{3}{2} c_\sigma \sigma_{,A} + \frac{9}{2} c_\sigma \sigma a_{A} = 0, \quad (20)
\]

where \( A, B = y \text{ or } z \) with \( A \neq B \).

As we can see for this specific selection of the æther field there are not any nondiagonal terms at the energy momentum tensor \( T_{\mu \nu}^\text{eff} \) while the equation of motion for the æther field (4) provides constraints on the space independent variables for the unknown functions \( a(t,x,y,z) \) and \( b(t,x,y,z) \).

The diagonal field equations (5) are

\[
0 = 2(1 + 2c_2)b_{,\sigma} + 2(c_1 + c_2)a_{,\sigma} - (2c_1 - 4c_2 - 3)(b_{,\sigma})^2 + (c_1 + c_2) \left( (a_{,\sigma})^2 + 4a_{,\sigma}b_{,\xi} \right) - 4 \left( b_{,\xi \xi} \right) e^{-2b} - (b_{,\sigma})^2 e^{-2a} \quad (21)
\]

\[
0 = (1 + 2c_1 + 4c_2)b_{,\sigma} + (1 + 2(c_1 + c_2))a_{,\sigma} - (-2c_1 - 4c_2 - 1)(b_{,\sigma})^2 + (1 - c_1 + c_2) (a_{,\sigma})^2 + (1 + 2c_1 + 4c_2)a_{,\sigma}b_{,\xi} - e^{-2a} (b_{,\lambda \lambda} + (b_{,\sigma})^2 - a_{,\lambda}b_{,\lambda}) - 2e^{-2b} (a_{,\lambda \xi} + a_{,\xi} \xi) \quad (22)
\]

\[
0 = (1 + c_1 + 2c_2) \left( 4b_{,\sigma} + 2a_{,\sigma} \right) (6 + 4c_1 + 16c_2)(b_{,\sigma})^2 + (2 + 4c_2) (a_{,\sigma})^2 + 4(1 + 2c_1 + 4c_2)b_{,\sigma}a_{,\sigma} - \rho_m(t,x,y,z) + 2e^{-2a} (2a_{,\sigma}b_{,\xi} - 3b_{,\xi}^2 - 2b_{,\sigma}x) - 8e^{-2b} (a_{,\lambda \xi} + a_{,\xi} \xi + b_{,\xi} \xi) . \quad (23)
\]

where without loss of generality we have assumed \( c_3 = 0 \) and \( \xi = y + iz \). Specifically, equation (21) is the \( xx \) component of the field equations, \( G^\text{eff}_{\xi \xi} = e \xi T^\text{eff}_{\xi \xi} \), equation (22) correspond to the \( \xi \xi \) component \( G^\text{eff}_{\xi \xi} = e \xi T^\text{eff}_{\xi \xi} \), while equation (23) is \( G^\text{eff}_{\xi} = e \xi T^\text{eff}_{\xi} \), in which \( e \xi T^\text{eff}_{\mu \nu} \) is the effective energy momentum tensor \( e \xi T^\text{eff}_{\mu \nu} = T^{(\text{m})}_{\mu \nu} + T^{\text{ae}}_{\mu \nu} \).

The nondiagonal field equations are the constraint equations presented in [9], they are

\[
b_{,\lambda \lambda} - a_{,\lambda} b_{,\lambda} + b_{,\lambda} b_{,\lambda} = 0, \quad (24)
\]

\[
a_{,\lambda} + b_{,\lambda} + a_{,\xi} a_{,\lambda} - b_{,\xi} a_{,A} = 0, \quad (25)
\]

\[
b_{,\xi A} - b_{,\xi} a_{,A} = 0, \quad (26)
\]

\[
a_{,\lambda \xi} + (a_{,\xi})^2 - 2a_{,\xi} b_{,\xi} = 0. \quad (27)
\]

where now, \( \xi = y + iz, \quad \bar{\xi} = y - iz \) and \( A = \xi \text{ or } \bar{\xi} \).

We continue our analysis by assuming the two possible cases (A) \( b_{,\xi} = 0 \) and (B) \( b_{,\xi} \neq 0 \).

### 3.1. Class A with \( b_{,\xi} = 0 \)

The first class of spacetimes follow by the condition \( b_{,\xi} = 0 \). Indeed, by replacing

\[
b = \ln(\Phi(t)) + \nu (\xi, \bar{\xi}), \quad a = \ln(R(t,x) + \Phi(t) \nu (x, \xi, \bar{\xi})) \quad (28)
\]
in the constraint conditions (20) it follows
\[(c_1 + c_2) \mu_A (\Phi R_t - \Phi_x R) = 0, \tag{29}\]
while from (19) we have \(\xi\)
\[(c_1 + c_2) (R_{xx}(R + \Phi) - \mu \Phi R_x - \mu_x (\Phi R_t - \Phi_x R) - R_x R_t) = 0. \tag{30}\]

From the latter conditions we get the subclasses (i) \(c_1 + c_2 = 0\), (ii) \(R(t, x) = \Phi(t) \omega(x)\) and (iii) \(\mu (x, \xi, \bar{\xi}) = \chi(x)\), \(R(t, x) = \Phi^6(t) \chi(x)\).

3.1.1. **Subclass** \(A_0\). In the first class where \(c_1 + c_2 = 0\), by replacing (28) in (21) we find
\[(1 + 2c_2) (2\Phi R_{xx} + \Phi^2_x) - 4e^{-\nu} \nu_{\xi\bar{\xi}} = 0 \tag{31}\]
therefore it follows that \(e^{-\nu} \nu_{\xi\bar{\xi}} = \nu_0\), from where we find that
\[\nu (\xi, \bar{\xi}) = -2 \ln \left(1 + \frac{k}{4} (\xi - \xi_0) (\bar{\xi} - \bar{\xi}_0)\right) \tag{32}\]
where without loss of generality we select \(\xi_0 = 0\), \(\bar{\xi}_0 = 0\). Moreover, from (22) it follows
\[2(1 + 2c_2) \left( \Phi R_{xx} + \Phi_x R_t + \frac{R}{2\Phi} (\Phi_x)^2 \right) - k(1 + 4c_2) \frac{R}{\Phi} - (1 + 2c_2) (2e^{-2\nu} \nu_{\xi\bar{\xi}} + k\mu) = 0. \tag{33}\]
Hence, with the use of the constraint equations (27) it follows
\[\mu (x, \xi, \bar{\xi}) = \left( \frac{U(x)}{2} \xi \bar{\xi} + U_1(x) \xi + U_2(x) \bar{\xi} + W(x) \right) e^{(\xi, \bar{\xi})} \tag{34}\]
in which equations (31) and (32) are simplified
\[(1 + 2c_2) (2\Phi R_{xx} + \Phi^2_x) = k = 0, \tag{35}\]
\[2(1 + 2c_2) \left( \Phi R_{xx} + \Phi_x R_t + \frac{R}{2\Phi} (\Phi_x)^2 \right) - k(1 + 4c_2) \frac{R}{\Phi} - (1 + 2c_2) (2U(x) + kW(x)) = 0. \tag{36}\]

System (35), (36) can be easily seen that is integrable. From equation (35) we find that \(\Phi(t)\) is expressed in terms of elliptic integrals, while then equation (36) is a linear equation for \(R(t, x)\) in terms of derivatives of \(t\), which is a well-known integrable.

For \(k = 0\), a closed-form solution can be easily obtained with the use of power-law exponents as follows
\[\Phi(t) = \Phi_0 t^\frac{k}{2}, \quad R(t, x) = \frac{9U(x)}{10\Phi_0} t^\frac{k}{4} + R_1(x) t^\frac{k}{4} + R_2(x) t^\frac{k}{4}. \tag{37}\]
3.1.2. Subclass $A_{ii}$. For the second subclass where $R(t,x) = \Phi(t)\omega(x)$, the line element (11) becomes
\[ ds^2 = -dt^2 + \Phi^2(t) \left[ \left( \omega(x) + \mu(x,\xi,\tilde{\xi}) \right)^2 dx^2 + e^{2\nu(t,\xi)} \left( dy^2 + dz^2 \right) \right] . \] (38)
and by following the same procedure as before we find that $\nu(\xi,\tilde{\xi})$ and $\mu(x,\xi,\tilde{\xi})$ are given by the expressions (32) and (34) while function $\Phi(t)$ satisfies the second-order ordinary differential equation
\[ 2(1 + c_1 + 3c_2) \Phi\Phi_{yt} + (1 + 4c_1 - 2c_2) (\Phi_y)^2 + k = 0. \] (39)
Furthermore, from (22) and with the use of (39) the constraint equation it follows
\[ -4(2U + k(W + \omega)) + (4 + k\xi\tilde{\xi}) (\Phi_y)^2 (c_1 + c_2) (\xi\tilde{\xi} (2U + \omega) + U_1 \xi + W_2 \xi + 4(W + \omega) = 0. \] (40)
from where we can infer that all the functions on the parameter $x$ are zero. Hence, the spacetime (38) is the homogeneous FLRW spacetime, consequently from (23) it follows that the energy density is homogeneous. The generic solution of the later system was recently presented in [37].

3.1.3. Subclass $A_{ii}$. For the third subclass the line element (11) is simplified
\[ ds^2 = -dt^2 + \Phi^K(t) + \Phi(t)^2 \chi^2(x) dx^2 + \Phi^2(t) e^{2\nu(t,\xi)} \left( dy^2 + dz^2 \right) . \] (41)
where without loss of generality we can select $\chi^2(x) = 1$. Function $\nu(\xi,\tilde{\xi})$ is determined by expression (32). However, the two equations (21) and (22) are in consistency if and only if $K = 1$, from where the latter spacetime reduces to the homogeneous spacetime (38). Hence, there is not any new solution in that consideration. Before we proceed with the next class of solutions, we summarize our results in the following statement.

For the Szekeres Einstein-æther gravity there exist inhomogeneous solutions with $b_x$ only when the coefficient constants of the æther field satisfy the algebraic condition $c_1 + c_2 = 0$. Otherwise the spacetime reduces to the homogeneous FLRW geometry.

3.2. Class $B$ with $b_x \neq 0$

For the second class it holds $b_x \neq 0$, where from the constraint equations (24)–(27) it follows
\[ a = \ln \left( h(x) \left( \Phi_x + \Phi \nu_x \right) \right), \quad b = \ln (R\Phi) + \nu \] (42)
in which $R = R(t,x)$ and
\[ \nu(x,\xi,\tilde{\xi}) = -\ln \left( 1 + \frac{U(x)}{4} \xi \xi + \frac{U_1(x)}{2} \xi + \frac{U_2(x)}{2} \xi + W(x) \right). \] (43)
Hence, there is only one free time dependent function in the spacetime. Moreover, without loss of generality we can select $h(x) = 1$.

We continue by substituting (42) in the equations of motion for the æther field (19) and (20) from where we infer the two subclasses (i) $c_1 + c_2 = 0$ and (ii) $\Phi(t,x) = \Phi(t)\omega(x)$.

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3.2.1. Subclass $B_{(i)}$. For the first subclass it follows that

$$2 (1 + 2c_2) \Phi_{,tt} + (1 + 2c_2) (\Phi_{,x})^2 + K(x) = 0,$$  \hspace{1cm} (44)

where

$$K(x) = U(x) (1 + W(x)) - U_2(x) U_1(x).$$  \hspace{1cm} (45)

Equation (44) is the modified second Friedman equation in Einstein-æther theory.

3.2.2. Subclass $B_{(ii)}$. For the second subclass where $\Phi(t,x) = \Phi(t) \omega(x)$, from the field equations we find

$$-2 \Phi_{,tt} - (\Phi_{,x})^2 + \left( \frac{1 - K(x)}{\omega^2(x)} \right) = 0,$$  \hspace{1cm} (46)

from where it follows

$$K(x) = 1 - k \omega^2(x), \quad k = \text{cont.}$$  \hspace{1cm} (47)

In this case it is important to mention that the coefficients $c_1, c_2$ for the æther field do not play any role in the evolution of the dynamical system. The resulting spacetime is inhomogeneous but the scale factor $\Phi(t)$ does not depend on the space variable $x$. These kinds of spacetimes have been determined before in the case of GR with a homogeneous scalar field [62], or with an isotropic ideal gas [63].

4. Dynamical evolution

In the previous section for simplicity on the presentation of our calculations we assumed that the matter source is described by a pressureless fluid. However, if we replace the dust fluid with another ideal gas with constant equation of parameter $p_m = (\gamma - 1) \rho_m$, $\gamma = \text{const}$, where the energy momentum tensor for the matter source is $T_{m \nu}^\mu = (\rho_m (t, x, y, z) + p_m (t, x, y, z)) v_\mu v_\nu + p_m (t, x, y, z) g_{\mu \nu}$ we get similar results, that is we found extensions of the inhomogeneous Szekeres–Szafron spacetime in Einstein-æther gravity. Recall that as a Szekeres–Szafron system we refer to the extension of the Szekeres system where the dust fluid source is replaced by an ideal gas with constant equation of state parameter [54]. Moreover, by assuming a cosmological constant term in the gravitational action integral. Our analysis is still valid and similar results with that of [55] are obtained.

In order to study the stability of the solutions we determined we perform a detailed analysis of the critical points for the evolution equations. Such analysis is necessary in order to understand the general evolution of the spacetime for arbitrary initial conditions.

By using the dynamical quantities$^1$ $\rho, p$ and $\pi^{\mu \nu} = \pi^\mu e^\nu$, kinematic quantities $\theta$ and $\sigma$ and the electric component of the Weyl tensor, $E_{\mu \nu} = E e_{\mu \nu}$, the gravitational field equations are

$^1$The set of $\{w, e\}$ defines an orthogonal tetrad such that $u_\mu e^\mu = 0$; $e_\mu e^\mu = \delta^\mu_\mu + u^\mu u_\mu$, in order the components of tensors are scalar functions.

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expressed as a system of first-order algebraic differential equations

\[ \dot{\rho}_m + (\rho_m + p_m) \theta = 0, \quad (48) \]
\[ \dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2} (\rho_m + p_m) + \frac{1}{2} (\rho^e + p^e) = 0, \quad (49) \]
\[ \dot{\sigma} - \sigma^2 + \frac{2}{3} \theta \sigma + E + \frac{1}{2} \pi^e = 0, \quad (50) \]
\[ \dot{E} + \frac{1}{2} \dot{\pi}^e + (3\sigma + \theta) \left( E + \frac{1}{6} \pi^e \right) + \frac{1}{2} (\rho_m + p_m) \sigma + \frac{1}{2} (\rho^e + p^e) \sigma = 0, \quad (51) \]

where the algebraic constraint is

\[ \frac{\theta^2}{3} - 3\sigma^2 + \frac{R^e(3)}{2} - \rho_m - \rho^e = 0. \quad (52) \]

where \( \cdot \) denotes the directional derivative along the vector field \( u^\mu \), i.e. \( \cdot = u^\mu \nabla_\mu \), and \( R^e(3) \) describes the curvature of the three-dimensional hypersurface. We recall that for the line element (11) the magnetic part of the Weyl tensor and the vorticity term are identical zero. When \( p = \pi^e = 0 \), system (48)–(51) reduce to the known as Szekeres–Szafron system is recovered [52].

By using expressions (16) and (17) and the equation of state for the ideal gas, we can write the field equations as a system of four first-order ordinary differential equations of the form

\[ \dot{\theta} = \Theta (\rho_m, \theta, \sigma, E; \delta), \quad (53) \]
\[ \dot{\rho}_m = f_1(\rho_m, \theta, \sigma, E; \delta), \quad (54) \]
\[ \dot{\sigma} = f_2(\rho_m, \theta, \sigma, E; \delta), \quad (55) \]
\[ \dot{E} = f_3(\rho_m, \theta, \sigma, E; \delta), \quad (56) \]

in which \( \alpha \) contents the free parameters of our model, i.e. \( \delta = \delta(\gamma, c_1, c_2) \).

We continue by defining the dependent and independent variables

\[ \rho_m = \left( 1 + \frac{c_1}{3} + c_2 \right) \Omega_m(\tau) \theta^2, \quad R^e(3) = \left( 1 + \frac{c_1}{3} + c_2 \right) \Omega_\theta(\tau) \theta^2, \quad (57) \]
\[ \sigma(\tau) = (1 + c_1 + 3c_2) \Sigma(\tau) \theta^2 \quad \text{and} \quad E = (1 + c_1 + 3c_2) \epsilon(\tau) \theta^2, \quad dt = \theta d\tau \quad (58) \]

the modified Szekeres system (53)–(56) is written as a system of three first-order ordinary differential equations of the form

\[ \Omega_m' = F_1 (\Omega_m, \Sigma, \epsilon; \delta), \quad (59) \]
\[ \Sigma' = F_2 (\Omega_m, \Sigma, \epsilon; \delta), \quad (60) \]
\[ \epsilon' = F_3 (\Omega_m, \Sigma, \epsilon; \delta). \quad (61) \]

where prime denotes differentiation with respect to the variable \( \tau \) and functions \( F_1, F_2 \) and \( F_3 \) are defined as follows

\[ F_1 = \frac{\Omega_m}{3} \left( (2 - 3\gamma)(1 - \Omega_m) - 36(2\alpha - 1)\beta \Sigma^2 \right), \quad (62) \]
\[ \alpha F_2 = \epsilon - \frac{(2 - \alpha(4 + (2 - 3\gamma)\Omega_m))}{6}\Sigma - \beta\Sigma^2 + 6(1 - 2\alpha)\alpha\beta\Sigma^3, \quad (63) \]

\[ \alpha F_3 = -\frac{\epsilon}{3}(1 - \alpha(2(1 - \alpha) - (2 - 3\gamma)\Omega_m)) \]
\[ + \left( \frac{(2 - \alpha(8 + \alpha(\beta - 7)))}{18} + \frac{(2 - 3\gamma - 2\alpha\beta)}{18}\Omega_m \right) - \beta(2 - \alpha + 2\alpha^2)\epsilon \right)\Sigma \]
\[ + \frac{\beta}{3}(3 + \alpha(4 + 72\epsilon) - 7 - 36\epsilon))\Sigma^2 + \beta(2\beta + \alpha(2 - \beta + (\alpha\beta - 4))\Sigma^3. \quad (64) \]

The new parameters \( \alpha \) and \( \beta \) are defined as \( \alpha = 1 - c_1 \) and \( \beta = 1 + c_1 + 3c_2 \).

Furthermore, equation \( (52) \) provides the constraint equation

\[ \Omega_R = -2\left(1 + 9(1 + 2\alpha)2\beta\Sigma^2 - \Omega_m \right). \quad (65) \]

We continue by determining the critical points of the dynamical system \( (59) \)–\( (61) \) and study the physical properties on the solution at the critical points as also the stability. In order to compare the results of Einstein-æther gravity with that of GR, let us proceed with the stability analysis of the Szekeres–Szafron [52].

### 4.1. Stability analysis for the Szekeres–Szafron system in GR

For \( c_1 = c_2 = 0 \), the dynamical system \( (59) \)–\( (61) \) reduces to that of the Szekeres–Szafron system. Every critical point \( P = (\Omega_m(P), \Sigma(P), \epsilon(P)) \) is a solution of the following algebraic system

\[ F_1(\Omega_m, \Sigma, \epsilon; \gamma, 0, 0) = 0, \quad F_2(\Omega_m, \Sigma, \epsilon; \gamma, 0, 0) = 0, \quad F_3(\Omega_m, \Sigma, \epsilon; \gamma, 0, 0) = 0. \quad (66) \]

Point \( A_1 \) with coordinates \( (0, 0, 0) \) describes a FLRW spacetime with nonzero negative curvature, i.e. \( \Omega_R = -2 \), that means the solution at point \( O \) is that of the milne Universe. The eigenvalues of the linearized system around the critical point are found to be \( e_1(A_1) = 1, e_2(A_1) = \frac{1}{3}, e_3(A_1) = \frac{1}{2} - \gamma \), from where we can infer that the solution at the point is always unstable.

Point \( A_2 \) with coordinates \( (1, 0, 0) \) describes a spatially flat FLRW Universe where, the eigenvalues are calculated to be \( e_1(A_2) = 1 - \frac{1}{\gamma}, e_2(A_2) = \frac{1}{2} - \gamma, e_3(A_2) = \frac{1}{2} - \gamma \). Hence point \( A_2 \) is a saddle point.

Point \( A_3 \) with coordinates \( (0, -\frac{1}{\gamma}, 0) \) describes a Kasner Universe, while it is an attractor since all the eigenvalues of the linearized system is always negative, that is, \( e_1(A_3) = -1, e_2(A_3) = -2, e_3(A_3) = \gamma - 2 \).

Point \( A_4 \) with coordinates \( (0, \frac{1}{6}, 0) \) describes a Kantowski–Sachs Universe with eigenvalues \( e_1(A_4) = -\frac{1}{3}, e_2(A_4) = \frac{1}{3} \), and \( e_3(A_4) = \gamma - 1 \). Point \( A_4 \) is a saddle point.

Point \( A_5 \) with coordinates \( (3(1 - \gamma), \frac{3\gamma - 2}{6}, \frac{\gamma - 1}{6}) \) is physical only when \( \gamma = 1 \) and reduces to \( A_4 \).

Point \( A_6 \) with coordinates \( (0, \frac{1}{6}, \frac{1}{6}) \) describes a Kasner Universe. The eigenvalues are \( e_1(A_6) = -\frac{1}{3}, e_2(A_6) = -\frac{1}{3}, e_3(A_6) = \gamma - 2 \), hence it is an attractor.
Table 1. Critical points for the Szekeres–Szafron system in general relativity

| Point | $(\Omega, \Sigma, \epsilon)$ | Physical | $[^{1}]R$ | Spacetime | Stability |
|-------|-----------------------------|---------|---------|----------|-----------|
| $A_1$ | $(0, 0, 0)$                 | Yes     | $< 0$  | FLRW (Milne Universe) | Unstable |
| $A_2$ | $(1, 0, 0)$                 | Yes     | $= 0$  | FLRW (spatially flat)  | Unstable |
| $A_3$ | $(0, -\frac{1}{3}, 0)$      | Yes     | $= 0$  | Bianchi I (Kasner Universe) | Stable |
| $A_4$ | $(0, \frac{\gamma}{2}, 0)$ | Yes     | $< 0$  | Kantowski–Sachs         | Unstable |
| $A_5$ | $(3 (1 - \gamma), \frac{\gamma}{2}, \frac{\gamma (1 - 3 \gamma)}{6})$ | No      |       | Kantowski–Sachs Unstable |           |
| $A_6$ | $(0, 0, \frac{\gamma}{2})$  | Yes     | $= 0$  | Bianchi I (Kasner Universe) | Stable |
| $A_7$ | $(0, -\frac{1}{3}, \frac{1}{3})$ | Yes     | $< 0$  | Kantowski–Sachs         | Unstable |
| $A_8$ | $(3 (3 - 4 \gamma), \frac{\gamma}{2} - \gamma, \frac{\gamma (3 \gamma - 2)}{6})$ | No      |       |           |           |

Figure 1. Phase portrait for the Szekeres–Szafron system in general relativity, for $\gamma = 1$. With red color are marked the two Kasner attractors while with green color are marked the unstable critical points.

Point $A_7 = (0, -\frac{1}{12}, \frac{1}{12})$ describes an unstable Kantowski–Sachs Universe; the eigenvalues are derived to be $e_1 (A_7) = \frac{5}{6}$, $e_2 (A_7) = -\frac{1}{2}$, $e_3 (A_7) = \gamma - \frac{3}{2}$, which means that the solution at point $A_7$ is always unstable.

Finally point $A_8 = (3 (3 - 4 \gamma), \frac{\gamma}{2} - \gamma, \frac{\gamma (3 \gamma - 2)}{6})$ is unphysical because $\Omega_m (A_8) < 0$. Hence we do not study its properties.

The above results are collected and presented in table 1. The phase portrait of the Szekeres–Szafron system in presented in figure 1 where the critical points are marked.
4.2. Stability analysis in the Einstein-æther gravity

We continue by performing the stability analysis for the Szekeres–Szafron system (57)–(62) in the Einstein-æther theory, i.e. \( c_1 c_2 \neq 0 \). For our analysis we use the constraint condition \( c_1 + c_2 = 0 \) which has been obtained before by the space-constraint equations. Therefore, by using the latter condition in the dynamical system (57)–(62) we find the following critical points:

Point \( B_1 = (0, 0, 0) \) which describes the Milne Universe it is an unstable, the eigenvalues are

\[
e_1(B_1) = \gamma - \frac{2}{3}, \quad e_{2,3}(B_1) = \frac{1}{3} (1 - c_1) \pm \sqrt{c_1 (c_1 + 1)}.
\]

Point \( B_2 = (1, 0, 0) \) describes a spatially flat FLRW spacetime, where the eigenvalues are found to be

\[
e_1(B_2) = \frac{2}{3} - \gamma, \quad e_{2,3}(B_2) = \frac{1}{12} \left( 10 - 9 \gamma - 4 c_1 \pm \sqrt{16 c_1^2 + 8 c_1 (3 \gamma - 4) + (2 + 3 \gamma^2)} \right)
\]

from where we can infer that the point describes a stable solution when

\[
\begin{aligned}
\left\{ \gamma \in \left(1, \frac{3 + \sqrt{3}}{3}\right), c_1 > -\frac{3 \gamma^2 - 8 \gamma + 4}{2 (\gamma - 1)} \right\} \cup \left\{ \gamma \in \left(\frac{3 + \sqrt{3}}{3}, 2\right), -\frac{3 \gamma^2 - 8 \gamma + 4}{2 (\gamma - 1)} < c_1, c_1 \neq 1 \right\}.
\end{aligned}
\]

(67)

Point \( B_3 = \left(0, \frac{1}{(6c_1 - 3)}, 0\right) \) describes a Kasner Universe and exists when \( c_1 \neq \frac{4}{7} \). The eigenvalues of the linearized system are calculated to be

\[
e_1(B_3) = \gamma - 2, \quad e_{2,3}(B_3) = -\frac{3}{2} \pm \frac{\sqrt{9 + 8 c_1}}{6},
\]

from where we can infer that the point is stable for \(-\frac{9}{2} < c_1 \) with \( c_1 \neq \frac{4}{7}, 1 \).

Point \( B_4 = \left(0, \frac{2}{27}, 1 + \frac{\sqrt{9 + 4 c_1 (7 - c_1)} - 9 c_1}{2 c_1 - 1}, \frac{9 + c_1 (45 - 4 c_1 (15 + 2 c_1 - 7 c_1)) + (3 c_1 - 4 c_1^2 + 4 c_1^3)}{5 (6 c_1 - 5)}\right)\) exists when \( c_1 \in \left(-\infty, \frac{7 - 2 \sqrt{71}}{2}\right) \cup \left(\frac{7 + 2 \sqrt{71}}{2}, \infty\right) \). The eigenvalues of point \( B_4 \) are determined numerically and the region of the parameters \( \gamma \) and \( c_1 \) where the point \( B_4 \) describes a stable solution for \( c_1 \geq -0.65 \) and \( c_1 \geq 6.5 \) independent from the value of parameter \( \gamma \).

Point \( B_5 = (\Omega_{\text{m}}(B_5), \Sigma(B_5), e(B_5)) \) with coordinates as given in appendix A. In Einstein-æther theory, point \( B_5 \) exists for specific values of the free parameters \( c_1 \) and \( \gamma \). The region of the existence of the critical point is presented in figure 2. Moreover, in figure 2 the region of the free parameters is given where the point \( B_5 \) describes a stable solution.

Point \( B_6 = \left(0, \frac{1}{6(1 + 0.5 c_1)}, \frac{1}{6 c_1}\right) \) exists when \( c_1 \neq \frac{4}{7} \). Point \( B_6 \) describes a Kasner Universe. The eigenvalues of the linearized system are derived to be

\[
e_1(B_6) = \gamma - 2, \quad e_{2,3}(B_6) = -\frac{1}{6} \left( 7 + 4 c_1 \pm \sqrt{9 + 16 c_1^2} \right).
\]

(68)

Hence point \( B_6 \) is a source when \(-\frac{2}{7} < c_1, c_1 \neq \frac{1}{7}, 1 \).
Figure 2. Region plot for the free parameter $\gamma$ and $c_1$ point $B_5$ exists (left figure) and $B_5$ is an attractor (right figure).

Figure 3. Region plot for the free parameter $\gamma$ and $c_1$ point $B_8$ it is physical accepted.

Point $B_7 = \left(0, \frac{2c_1-1+\sqrt{7c_1^2+2c_1+1}}{24c_1-1}, \frac{9-4c_1^2+6c_1^2+8c_1^2+(3+c_1-14c_1^2+4c_1^2)\sqrt{9+4c_1^2-7c_1}}{576(2c_1-1)} \right)$, which exists for $c_1 \in \left( -\infty, \frac{7-2\sqrt{10}}{2} \right) \cup \left( \frac{2+2\sqrt{10}}{2}, \infty \right)$. Point $B_7$ describes a Kantowski–Sachs Universe which is stable when $c_1 > \frac{7+2\sqrt{10}}{2}$. 

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Table 2. Critical points for the Szekeres–Szafron system in Einstein-aether gravity

| Point | Physical | \(^{(3)}R\) | Spacetime | Stability |
|-------|----------|-------------|-----------|-----------|
| \(B_1\) | Yes—always | \(< 0\) | FLRW (milne Universe) | Unstable |
| \(B_2\) | Yes—always | \(= 0\) | FLRW (spatially flat) | Stable—see equation (67) |
| \(B_3\) | Yes—\(c_1 \neq \frac{1}{2}\) | \(= 0\) | Bianchi I (Kasner Universe) | Stable for \(c_1 > -\frac{9}{8}\) |
| \(B_4\) | Yes—\(|c_1| > \frac{2}{\sqrt{10}}\) | \(< 0\) | Kantowski–Sachs | Stable for \(c_1 \leq -0.65\) and \(c_1 \geq 6.5\) |
| \(B_5\) | Yes—see figure 2 | | Kantowski–Sachs | Stable—see figure 2 |
| \(B_6\) | Yes—\(c_1 \neq \frac{1}{2}\) | \(= 0\) | Bianchi I (Kasner Universe) | Stable for \(c_1 > -\frac{7}{2}\sqrt{2}\) |
| \(B_7\) | Yes—\(|c_1| > \frac{2}{\sqrt{10}}\) | \(< 0\) | Kantowski–Sachs | Stable for \(c_1 > \frac{7}{2} + 2\sqrt{2}\) |
| \(B_8\) | Yes—see figure 3 | | Kantowski–Sachs | Unstable |

Figure 4. Phase portrait for the Szekeres–Szafron system in Einstein-aether gravity, for \((\gamma, c_1) = (1, -2)\). With red color are marked the two Kasner attractors, points \(B_3\) and \(B_6\), while with green color are marked the unstable critical points.

Point \(B_8\) with coordinates \(B_8 = (\Omega_m (B_8), \Sigma (B_8), \epsilon (B_8))\) as they are given in appendix A exists for the range of variables as they are given in figure 3 and describes a Kantowski–Sachs Universe. As far as the stability is concerned from numerical simulations we found that the point describes an unstable solution.

In table 2 we collect the results of the critical point analysis.

We can see that points \(B_1\) reduce to points \(A_1\) when \(c_1 = 0\), hence, the limit of general relativity is recovered. However, there exit two additional critical points which describe Kantowski–Sachs Universe. Point \(B_5\) describes an empty Kantowski–Sachs Universe while point \(B_8\) describes a Kantowski–Sachs Universe with matter source.
In addition, the stability of the solutions change. While in GR only the Kasner Universes are attractors that it is not true for the Szekeres–Szafron system in Einstein-æther theory. For example, the solution at point $A_2$ in GR which describes a spatially flat FLRW spacetime dominated by the ideal gas is always an unstable point, while in Einstein-æther the point can be an attractor.

Let us demonstrate the results by considering the free parameters to be $(\gamma, c_1) = (1, -2)$. In that case, the dynamical system (59)–(61) admits seven critical point, two are stable and five are unstable points. The stable points are the $B_3$ and the $B_6$ points. In figure 4 we present the phase portrait for the Einstein-æther Szekeres–Szafron system for those specific values of the free parameters.

5. Conclusions

In this work, we performed a detailed study on exact solutions for inhomogeneous space-times in the Einstein-æther theory. Specifically we studied the existence of exact solutions in Einstein-æther gravity which generalize the Szekeres solutions of GR.

For the æther field we did the simplest selection by assuming that it is the comoving observer. Indeed that it is not the general selection but it is required if we assume the existence of a FLRW limit in the resulting spacetimes [37]. For that specific selection of the æther field, the corresponding energy momentum tensor is calculated to be diagonal, hence the constraint conditions provided by the field equations are those of GR. Additionally, there is a new set of constraint conditions which follow by the equation of motion for the æther field.

The latter constraints provide conditions for the coefficient constants for the æther field, or constraints for the functional form of the scale factors. From the line element of our consideration and for that specific selection for the æther field we found that the coefficient constants $c_3, c_4$ for the æther field, do not contribute in the dynamical system. While when $c_1 + c_2 =0$, is the unique case where inhomogeneous solutions exist.

The exact solutions we found describe spacetimes which belong to the two classes of Szekeres, the inhomogeneous Kantowski–Sachs generalized spaces and the inhomogeneous FLRW generalized space. However, the scale factors in this case satisfy the modified field equations as given by the Einstein-æther theory for the Kantowski–Sachs and the FLRW spacetimes. On the other hand, for arbitrary value of the coefficient constants $c_1$ and $c_2$, the unique solution is that of a FRLW-like spacetime.

The stability of the solutions of the Szekeres spacetime in Einstein-æther theory studied from where we find that the field equations evolve more variously in Einstein-æther than in GR; there are new critical points which describe Kantowski–Sachs Universes, while the stability of the critical points with similar physical behavior with that of GR change in a way to have as attractors, Kasner universes, Kantowski–Sachs universes or spatially flat FLRW universes with nonzero matter contribution in the Universe. Contrary to GR the attractors describe Kasner spacetimes.

This analysis contributes to the subject of existence of exact solutions in Einstein-æther theory. The novelty of this work is that we proved for the first time in the literature the existence of inhomogeneous exact solutions in the Einstein-æther theory, by assuming extensions of the Szekeres spacetimes. Recall that the latter spacetimes in general do not admit any symmetry.

---

2 In general, the algebraic constraint is $c_1 + c_2 + c_3 = 0$, however, without loss of generality we selected $c_3 = 0$.  
In GR, Szekeres spacetimes can be seen as perturbative FLRW spaces [66], in a similar way it is consequence to consider a similar analysis. In a future work, we plan to extend our analysis for a generic æther field as also a more generic form for the spacetime which extends the Szekeres family. Last but not least, the Einstein-aether theory describes the classical limit of Horava gravity [67], which means that the solutions we found correspond and also hold for the Horava theory.

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Appendix A. Formulas and expressions

In this appendix we present expressions to which we have referred before. Coordinates of point $B_4$:

\[
\frac{2(\gamma + 1 + 2c_1 - 2)(\gamma - 2)}{(\gamma - 2)} \Omega_{\alpha} (B_4) = \frac{3(\gamma - 2)(5\gamma - 4) + 2(\gamma(6\gamma - 13) + 14)c_1^2 + (52\gamma - 48)c_1}{3\gamma - 2(2c_1 - 1)} \sqrt{9(\gamma - 2)^2(3\gamma - 2) + 4\gamma(\gamma(3\gamma - 10) + 24) - 16c_1^2}.
\]

Coordinates of point $B_5$:

\[
\frac{12(2\gamma - 1)(\gamma + 1 + 2c_1 - 2)(\gamma - 2)}{(\gamma - 2)} \Omega_{\alpha} (B_5) = \frac{\gamma(3\gamma - 8) - 2(\gamma - 2)(3\gamma - 2)c_1 - 4 + 12(2\gamma - 1)(\gamma + 1 + 2c_1 - 2)(\gamma - 2)}{\sqrt{3\gamma - 2} \sqrt{9(\gamma - 2)^2(3\gamma - 2) + 4\gamma(\gamma(3\gamma - 10) + 24) - 16c_1^2} + 4\gamma - 2(\gamma(3\gamma + 20) - 16)c_1}.
\]

\[
\frac{144(2\gamma - 1)(\gamma + 1 + 2c_1 - 2)^2 \gamma(\gamma - 2)}{\gamma - 2} \Omega_{\alpha} (B_5) = -\frac{(8 - 3\gamma^2) + 2(\gamma - 2)(3\gamma - 2)c_1 - 4 + 9(\gamma - 2)^2(3\gamma - 2) + 4\gamma(\gamma(3\gamma - 10) + 24) - 16c_1^2}{\sqrt{3\gamma - 2}} \sqrt{3\gamma - 4\gamma(3\gamma - 10) + 24 - 16c_1^2} + 4\gamma - 2(\gamma(3\gamma + 20) - 16)c_1 + 12 + \frac{3(\gamma - 4\gamma(3\gamma - 10) + 24 - 16c_1^2)}{\sqrt{3\gamma - 2}} \sqrt{3\gamma - 4\gamma(3\gamma - 10) + 24 - 16c_1^2} + 4\gamma - 2(\gamma(3\gamma + 20) - 16)c_1 + 12}.
\]

\[
\frac{2(\gamma + 1 + 2c_1 - 2)(\gamma - 2)}{(\gamma - 2)} \Omega_{\alpha} (B_6) = \frac{3(\gamma - 2)(5\gamma - 4) + 2(\gamma(6\gamma - 13) + 14)c_1^2 + (52\gamma - 48)c_1}{3\gamma - 2(1 - 2c_1)} \sqrt{9(\gamma - 2)^2(3\gamma - 2) + 4\gamma(\gamma(3\gamma - 10) + 24) - 16c_1^2}.
\]
12(2c_1 - 1)(\gamma + (\gamma + 2)c_1 - 2)\Sigma (B_3) = \gamma(3\gamma - 8) - 2(\gamma - 2)(3\gamma - 2)c_1 + 4
+ \sqrt{3\gamma - 2} 2 9(\gamma - 2)^2(3\gamma - 2) + 4(\gamma(3\gamma - 10) + 24) - 16c_1^2
+4(\gamma - 2)(\gamma(3\gamma + 10) - 16)c_1^2

-144(2c_1 - 1)(\gamma + (\gamma + 2)c_1 - 2)\Sigma (B_3) = \gamma(3\gamma - 8) - 2(\gamma - 2)(3\gamma - 2)c_1 + 4
+ \sqrt{3\gamma - 2} 2 9(\gamma - 2)^2(3\gamma - 2) + 4(\gamma(3\gamma - 10) + 24) - 16c_1^2
+4(\gamma - 2)(\gamma(3\gamma + 10) - 16)c_1^2

\times \left( -3(\gamma - 4)\gamma - 2(3\gamma + 2) - 16c_1 - 12 + 6 (\gamma^2 - 4) c_1^2 \right)
+ \sqrt{3\gamma - 2} 2 9(\gamma - 2)^2(3\gamma - 2) + 4(\gamma - 2)(\gamma(3\gamma + 20) - 16)c_1^2

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