Quantum gravity contributions to the cosmological constant

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Abstract. In the standard QFT approach to the calculation of the cosmological constant contributions from the gravitational sector are usually neglected. Using a simple two-dimensional scenario, but in connection with higher dimensions, we discuss the possibility that quantum gravity can play an important role in the calculation of $\Lambda$, providing large quantum corrections and renormalizations. We reproduce the QFT result as a sub-case, while allowing a very large set of values.

1. Introduction
The cosmological constant problem [1, 2] and references therein, in its striking simplicity, is perhaps the strongest evidence that the present understanding of everything we generally refer to as “quantum gravity” is far from being satisfactory: while we know with a great accuracy that $\Lambda$ is very small in Planck units ($\Lambda \sim 10^{-123}$) [3, 4], there is no theoretical approach that seems to be able to predict such a measurement in a convincing fashion. In the standard approach the cosmological constant is calculated in the semiclassical approximation as the vacuum expectation value of the stress energy tensor of the Standard Model of particle physics, but the results of this calculation turn out to be catastrophic, with a discrepancy between the theoretical and experimental value of $\Lambda$ between 60 and 120 orders of magnitude. There is no debate that something has to be improved to overcome this situation.

A point of concern with the standard QFT analysis of the vacuum energy is represented by the semi-classical framework. Contributions to $\Lambda$ are calculated on a flat background geometry, discarding completely any contribution which could be given by the gravitational sector. While it is true that the vacuum fluctuations of a quantum gravitational field are suppressed by the Planck scale with respect to (w.r.t.) the vacuum fluctuations of the matter fields, the dynamics of quantum gravity is completely ignored, precluding the possibility of some mechanism induced by quantum gravity that might explain the value of $\Lambda$ that we measure today.

In this paper we discuss how the cosmological constant can be determined in a quantum theory of gravity, accounting for quantum dynamics of matter, gauge and the gravitational fields, including their vacuum fluctuations, possibly improving our understanding of what $\Lambda$ is really made of. In order to put to practice our idea we will focus on the case of two-dimensional dilaton gravity, taking advantage of the simplified framework. Most of the technical results are contained in [5, 6, 7, 8].
This paper is organized as follows: in Section 2 we introduce two-dimensional dilaton gravity, its dual action in terms of Liouville fields, the inclusion of scalar matter. The BRST formulation of the model is briefly discussed. In Section 3 we deal with the quantization of the theory, determining the central extensions of the Virasoro algebra, while Section 4 presents the realization of the quantum constraints and describes the contributions to the cosmological constant. We summarize and discuss our results in Section 5.

2. 2d Dilaton Maxwell gravity coupled with scalar matter

We consider the most general action principle of two-dimensional dilaton gravity in the presence of a Maxwell field:

\[ S_{DM} = \int dx^2 \sqrt{-g} \left( XR - U(X)X_\mu X^\mu - 2V(X) - \frac{1}{4} G(X) F_{\mu\nu} F^{\mu\nu} \right), \]

(1)

where \( X \) is the dilaton, \( U(X) \), \( V(X) \) and \( G(X) \) are arbitrary functions of \( X \), and \( F_{\mu\nu} \) is the usual field strength for the vector gauge field \( A_\mu \). Commas denote derivatives. With specific restrictions on the \( U, G \) potentials, there exists a dual formulation of (1) in terms of two decoupled Liouville fields on flat space-time [6]. In particular the dual action principle takes the form:

\[ S_{\text{dual}} = \int d^2x \sqrt{-g} \left[ \frac{1}{2} (Z_{,\mu} Z^{,\mu} - Y_{,\mu} Y^{,\mu}) - 2\Lambda e^{Z/\xi} - \frac{e^{-Y/\xi}}{16q\Lambda} F_{\mu\nu} F^{\mu\nu} + \xi (Z - Y) R_\flat \right], \]

(2)

where \( Z \) and \( Y \) are two real scalar fields, \( \Lambda \) is the cosmological constant and \( q \) and \( \xi \) are non-vanishing (coupling) constants. The line element reads:

\[ dx^2 = -\lambda_0 \lambda_1 dt^2 + (\lambda_0 - \lambda_1) dt \, ds + ds^2, \]

(3)

which is pure gauge. The subscript \( \flat \) indicates gravitational quantities that are calculated w.r.t. this metric. In this formulation the cosmological constant can always be reabsorbed by a constant shift of \( Z, Y \), as in \( Z \rightarrow Z - \xi \ln \Lambda \), but it is left explicit in the following.

The inclusion of massless scalar matter directly in the dual formulation is trivial and it is simply done by adding an arbitrary number \( N \) of terms in the form:

\[ S_\phi = -\frac{1}{2} \int_M d^2x \sqrt{-g} \phi_{,\mu} \phi^{,\mu}. \]

(4)

The existence of this duality is of fundamental importance: a highly coupled non-linear system is cast into a much simpler decoupled one, opening the way for canonical non-perturbative quantization. This comes at the fair price of a restriction on the \( U, G \) potentials. In particular one requires:

\[ U(X) = \frac{1}{2} \frac{\partial^2}{\partial X^2} - \partial_X \ln(V(X)), \]

(5)

\[ G(X) = \frac{\xi^2 e^X/\xi^2}{4qV(X)}. \]

(6)

Furthermore we also ask \( V(X) \neq 0 \). Such a restriction allows still for enough freedom to cover a large classes of dilaton gravity models, for instance a subset of the so-called \( ab \)-family (including the Witten black hole and the CGHS models [9 10]) and Liouville gravity [11].

We will focus on a model described by (2) with the addition of \( N \) massless real scalar fields (4). In order to take advantage of the formal equivalence of field modes and quantum harmonic oscillators, which will allow to express quantum field operators in terms of creation
and annihilation operators on a suitably defined Fock space of quantum states, we will restrict our attention to the case of a space-time with a cylindrical topology. In particular we will consider $\mathcal{M} = \mathbb{R} \otimes S^1$, where the time coordinate $t$ takes values on the real line and the space coordinate $s$ is replaced by an angular coordinate limited to the interval $[0, 2\pi)$. In this process we are implicitly introducing a length scale $\ell_c$ characteristic of the size of the compactification. In the following we will work in units which give $\ell_c = 1$, therefore we can simply replace the spatial coordinate $s \in \mathbb{R}$ with $s \in [0, 2\pi)$ with no ambiguities. All fields will be then required to be periodic in the space coordinate, so that for any field $x$ one has $x(t, s = 0) = x(t, s = 2\pi)$. To deal with the gauge invariances it is useful to look at the BRST formulation of the model \cite{12}; after the introduction of ghost degree of freedom and the BRST charge, the BRST extended constraints are readily calculated and, as expected from the properties of the duality, it is easy to check that their algebra is first class. In particular the (smearred) Poisson brackets read:

$$\{L^\pm(f), L^\pm(g)\} = \pm L^\pm(g_sf - f_sg) ,
\{L^\pm(f), L^0(g)\} = 0 ,
\{L^\pm(f), L^\mp(g)\} = -2q(\epsilon^Y/\xi P_A1, L^0)(fg) ,$$

(7)

where $\pm$ label the two Virasoro generators and $\emptyset$ indicates the constraint related to the $U(1)$ gauge invariance, while $P_{A1}$ is the momentum conjugate to the component $A_1$ of the Maxwell field. As expected these brackets are vanishing on the constraints surface. In the following we will choose to work in the conformal gauge for the gravitational sector, $\lambda_0 = \lambda_1 = 1$, and the Coulomb gauge in the $U(1)$ sector, $A_0 = A_{1,s} = 0$. Gauge fixing is implemented by a suitable choice of the gauge fixing fermion.

Per effect of the decoupling, the BRST extended Virasoro generators are a sum of terms from the different sectors of the model:

$$L^\pm = L^\pm Z + L^\pm Y + L^\pm g + \sum_{n=1}^{N} L_{n}^\pm \phi ,$$

(8)

where the $g$ superscript indicates the contributions of the BRST ghosts. This feature is of paramount importance in the quantization procedure, since it allows to quantize each sector separately. The different contributions take the form:

$$L_{Z}^\pm = -\frac{1}{4} (P_Z \mp Z_s) ^2 + \xi (P_Z \mp Z_s)s + \Lambda e^Z/\xi ,$$

(9)

$$L_{Y}^\pm = \frac{1}{4} (P_Y \mp Y_s) ^2 + \xi (P_Y \pm Y_s)s + q\Lambda e^Y/\xi P_{A1}^2 ,$$

(10)

$$L_{\phi}^\pm = \frac{1}{4} (P_\phi \mp \phi_s) ^2 ,$$

(11)

$$L_{g}^\pm = \frac{1}{2\pi} (c_s^\pm b_{\mp,s} + 2c_{s^\pm}b_{\pm} )$$

(12)

where $P$ are the conjugate momenta of the fields and $c_s^\pm$, $b_s^\pm$ are the ghost fields associated with the Virasoro generators, satisfying:

$$\{c^\pm(s), b_s^\pm(s')\}^+ = -2i\pi \delta_{2\pi}(s - s') ,$$

(13)

with $\delta_{2\pi}(s - s')$ being the $2\pi$-periodic Dirac $\delta$ distribution on the unit circle. Gauss’ constraint on the other hand has no BRST extension and is simply given by $L^0 = P_{A1,s}$.  

1 We are entitled to this choice in the two-dimensional case: the natural units are usually chosen by fixing $c = h = G = 1$. In two dimensions, however, Newton’s constant is dimensionless, so that instead of fixing $G$ we can fix the compactification scale $\ell_c$. 


3. The central charge of the Virasoro algebra

Quantization follows the usual canonical procedure, by choosing a polarization for phase space and a Hilbert space, by promoting observables to operators (using normal ordering when required, with the annihilation operators to the right of the creation ones) and by substituting Poisson brackets with commutators or anti-commutators, as the case may be, inclusive of the extra factor multiplying the values of the corresponding classical brackets.

We will require the classical symmetries to be unbroken at the quantum level. The $L^0$ constraint has identically vanishing brackets with the two Virasoro generators and itself, hence the corresponding operator is quantum mechanically trivial. The Virasoro algebra, on the other hand will exhibit the typical central extension in each different sector $x$. Looking at the Fourier modes we will have:

$$[L_r^{\pm x}, L_q^{\pm x}] = (r - q)\hbar_s L_{r+q}^{\pm x} + \hbar_s^2 c_x \delta_{r+q},$$  \hspace{1cm} (14)

where $\hbar_s$ is a place holder for $\hbar = 1$. The ghosts and scalar matter degrees of freedom reproduce the usual results known from string theory [13, 14]:

$$c_g = -\frac{13}{6} r^3 + \frac{1}{6} r,$$  \hspace{1cm} (15)

$$c_\phi = \frac{1}{12} r^3 + \frac{1}{6} r.$$  \hspace{1cm} (16)

In the Liouville sector, in order to ensure the closure of the quantum algebra, the possibility of quantum corrections to the coupling constant $\xi$ needs to be considered [13], in a manner dependent on the fields. As a matter of fact, only terms of $[9, 10]$ involving the fields linearly need to be corrected, with the replacements $\xi \to \xi_Z = \xi + \delta Z$ and $\xi \to \xi_Y = \xi + \delta Y$ for the corresponding couplings, respectively. The factor $\xi$ appearing in the exponential Liouville term contributions to $[9, 10]$ remains unchanged. The commutators can be calculated explicitly by using expansions of the fields in terms of creation and annihilation operators and by fixing the value of the quantum corrections we can get a Virasoro algebra with a central extension. By choosing $\xi_Z = \xi_Y = \xi - \frac{\hbar_s}{8\pi}$ we obtain the central charges:

$$c_Z = -\left(\frac{1}{12} + \frac{4\pi}{\hbar_s} \left(\xi - \frac{\hbar_s}{8\pi}\right)^2\right) r^3 - \frac{1}{6} r,$$  \hspace{1cm} (16)

$$c_Y = \left(\frac{1}{12} + \frac{4\pi}{\hbar_s} \left(\xi - \frac{\hbar_s}{8\pi}\right)^2\right) r^3 + \frac{1}{6} r.$$  \hspace{1cm} (17)

Note how the central charges in the $Z$ and $Y$ sector are identical in absolute value but appear with opposite signs. When summing up all contributions to the central charge the $c_Z$ and $c_Y$ cancel out and the only contributions are then given by the ghosts and the scalars:

$$c = \frac{N - 26}{12} r^3 + \frac{N + 1}{6} r.$$  \hspace{1cm} (18)

The central charge $c$ breaks the Virasoro algebra at the quantum level. However, it is possible to eliminate the $r^3$ term in $c$ by tuning the number of scalars to $N = 26$, in the same way as the number of space-time dimensions is tuned in bosonic string theory. In this way the cubic term of the central charge is cancelled and the remaining linear term can be reabsorbed with a shift of the zero modes $L_0^{\pm} \to L_0^{\pm} - \hbar_s 9/4$ so that the quantum Virasoro algebra is closed.

Such a shift of the zero modes is nothing else than the contributions of the quantum vacuum fluctuations to the total energy of the system, which are finite in two dimensions.

4. Quantum constraints

Once the quantum Virasoro algebra is obtained, it is possible to find the quantum realization of the constraints on Hilbert space, following the usual Dirac prescription that physical states have
to be annihilated by the constraints. As a matter of fact the cosmological constant $\Lambda$ and the coupling constants $\xi$ and $q$ are still free parameters: by requiring certain quantum states to be physical $\Lambda$ can be constrained to take a specific value.

The presence of the Liouville potentials involving the $Z$ and the $Y$ fields prevents one from following the standard string theory approach, i.e., extracting the modes $L_n^{\pm}$ of the quantum constraints with a discrete Fourier transform and looking at $L_n^{\pm}|\psi_{\text{phys}}\rangle = 0$ with $n = 0, 1, 2$. However, it is sufficient to look the weaker condition:

$$\langle \psi_{\text{phys}}|L^+(s)|\psi_{\text{phys}}\rangle = 0.$$  \hfill (19)

Under the requirement that a given quantum state of the Universe is physical, we can determine the value that the cosmological constant has to take for the quantum constraints to be satisfied. Considering linear combinations of the shifted Virasoro generators, the quantum constraints for an arbitrary quantum physical state will be:

$$(L^+ + L^-) = 0, \quad (L^+ - L^-) = 0, \quad \langle L^0 \rangle = 0,$$  \hfill (20)

with:

$$L^\pm = L^{\pm, Z} + L^{\pm, Y} + \sum_{n=1}^{N} L^{\pm, \phi_m} - c_m/2,$$  \hfill (21)

inclusive of the quantum corrected coupling constants $\xi_Z = \xi_Y$ and the shift of the zero modes determined above. The ghost sector is omitted by taking advantage of the BRST invariance.

As the cosmological constant enters the expressions for $L^\pm$ only through the Liouville potential terms (cf. (9) (10)), which are identical for the $+$ and $-$ cases, only $\langle L^+ + L^- \rangle$ will depend on $\Lambda$. We have:

$$\langle L^+ + L^- \rangle = -\frac{1}{2} \langle P_Z^2 + Z_{ss}^2 \rangle + \frac{1}{2} \langle P_Y^2 + Y_{ss}^2 \rangle + \sum_{n=1}^{N} \frac{1}{2} \langle P_n^2 + \phi_{n,s}^2 \rangle + \frac{2\xi}{4\pi\xi} \left( \langle Z_{ss} \rangle - \langle Y_{ss} \rangle \right) - c_m + \Lambda \left[ \frac{2\langle e^{Z/\xi} \rangle}{\xi} + 2q \langle e^{Y/\xi} P_A^2 \rangle \right],$$

$$\langle L^+ - L^- \rangle = \langle P_Z Z_{s} \rangle + \langle P_Y Y_{s} \rangle + \sum_{n=1}^{N} \langle P_{\phi_n} \phi_{n,s} \rangle - 2 \left( \xi - \frac{\hbar_s}{8\pi\xi} \right) \left( \langle P_Z \rangle - \langle P_Y \rangle \right),$$

$$\langle L^0 \rangle = \langle P_{A_{1,s}} \rangle,$$  \hfill (22a)

where $c_m = \hbar_s 9/2$. A qualitative analysis of the quantum constraints reveals interesting features. By solving (22a) for the cosmological constant we can see that contributions to $\Lambda$ are of three kinds (up to the factor in square brackets): classical contributions (the first line in (22a)), quantum fluctuations of the vacuum (the $c_m$ term) and additional quantum contributions determined by the quantum corrections to the coupling constant (the second term in the second line of (22a)).

Schematically we can write the cosmological constant as:

$$\Lambda = - \left[ \langle Kin_{\text{grav}} \rangle + \langle Kin_{\phi} \rangle + 2 \left( \xi - \frac{\hbar_s}{8\pi\xi} \right) \langle \partial_{\text{grav}}^2 \rangle - \delta_m \right] \langle \text{Liouville} \rangle^{-1}.$$  \hfill (23)

The QFT approach is recovered if all fields are taken to be in the vacuum state, so that only vacuum fluctuations of ghosts and scalars (finite in two dimensions when the regularization is
removed) are left in the square brackets and the Liouville potential factor is unity. It is more interesting, however, to look at the cosmological constant for a generic state away from the vacuum, when all terms can contribute to the value of \( \Lambda \). We can expect the Liouville potentials, which are exponential in the fields and the inverse of the coupling constant \( \xi \), to be very sensitive to excitations of the gravitational sector and/or small values of \( \xi \) itself. For instance in the case of small \( \xi \) we have \( \langle \text{Liouville} \rangle \gg 0 \), suppressing all terms in square brackets in (23). Let us choose \( \xi = 0.0035 \) and let us require to be physical a quantum state in which all fields but \( Z \) are in the vacuum and \( \langle Z \rangle = 1 \). Then (22c) and (22c) vanish identically and (22a) gives:

\[
\Lambda \sim 1.85 \times 10^{-124} .
\]

This quantum state is nothing else than the vacuum used in the QFT approach supplemented by an excitation of the one gravitational degree of freedom in (1) (see [6] for details). A physical interpretation of \( \xi \) can be given from the condition (5) for the potential \( U(X) \). Going a step back, the dilaton field \( X \) in two-dimensional dilaton gravity can be seen, generally speaking, as the field that encodes the collective behaviour of the degrees of freedom that have been integrated out in the reduction process from higher dimensions: e.g. in spherically reduced gravity \( X \) is what is left of the angular degrees of freedom. On the other hand the single physical degree of freedom contained in the two-dimensional metric can always be taken to be a conformal mode, rescaling a Minkowski metric. From (5) we can see that \( \xi \) regulates the relative weight of the kinetic term for the dilaton in the dilaton gravity action (1). A larger value for \( \xi \) corresponds to a more “frozen” dynamics for \( X \) w.r.t. the dynamics of the two-dimensional metric, while for small \( \xi \) the kinetic term for \( X \) dominates the action. In the same way, from (5), we can see that a smaller \( \xi \) corresponds to a stronger coupling of the dilaton with the Maxwell sector. Explicitly, for \( \xi \ll 1 \), we can rescale \( X \to \xi X \) and \( q \to q\xi^2 \) and the classical action (1) can be approximated as:

\[
S_{\xi \ll 1} = \int_{\mathcal{M}} dx^2 \sqrt{-g} \left( -\frac{1}{2} X_{\mu} X^{\mu} - 2\Lambda - \frac{e^{X/\xi}}{16qA} F_{\mu\nu} F^{\mu\nu} + O(\xi) \right) ,
\]

which at the zeroth order is a massless scalar non-linearly coupled to a Maxwell field on a conformally flat background. In our cosmological observations this seems a good approximation: at large scales the Universe is well described by a Minkowski metric and considering the short time scale during which observation are made we can approximate the conformal mode/scale factor as a constant. The remaining degrees of freedom then dominate the dynamics, justifying a choice of \( \xi \ll 1 \).

On the other hand for small \( \xi \) an important contribution to \( \Lambda \) can be given by the second of the \( \langle \partial^2_{\text{grav}} \rangle \) terms, generated by the quantum corrections to the coupling constant. This suggests that quantum gravity contributions to the cosmological constant can play a fundamental role. If we take a closer look at the specific form of the contributions to \( \Lambda \) in (22a) we notice that interestingly the \( Z \) and \( Y \) field appear with opposite signs in all non-exponential terms, so that \( \langle Kin_{\text{grav}} \rangle \), \( \langle \partial^2_{\text{grav}} \rangle \) can have both positive or negative, allowing for cancellations of the \( c_m \) contributions to \( \Lambda \). Let us also point out that the Maxwell field \( A_1 \) exhibits a continuous spectrum in this model.

If we take the leap of thinking the models treated here as effective descriptions of highly symmetrical four-dimensional theories we can consider the possibility similar features to appear, including quantum gravity contributions to \( \Lambda \) and excitations of the gravitational degrees of freedom that would renormalize the value of the cosmological constant in the fashion of \( \langle \text{Liouville} \rangle \gg 0 \).
5. Discussion
In this work we discussed the cosmological constant problem in quantum gravity, with a particular focus on the case of two-dimensional models, intended as an effective description of highly symmetric higher dimensional models.
By requiring the expectation values of the quantum constraints to vanish for generic quantum states we were able to see that the value of the cosmological constant is determined not only by the vacuum fluctuations of the quantum fields (both in the matter and gravitational sectors, even if cancellations occur), but also by dynamical contributions, including quantum corrections to effective coupling constants, and a renormalization factor of gravitational origin. In particular for small values of the effective coupling constant $\xi$, which we argue to be an approximate description of the regime in which we can measure the cosmological constant, the renormalization factor can be very small, resulting in a very small value for $\Lambda$ in natural units. The usual result of QFT of a cosmological constant proportional to the vacuum energy is recovered if the state required to be physical is the vacuum of the theory.
While the validity of this result is restricted to the specific framework of the two-dimensional dilaton gravity models considered (even though they can be motivated from higher dimensional considerations), it is a non-trivial example that quantum gravity can in fact contribute in a fundamental way to the value of the cosmological constant. In particular we cannot exclude the possibility of non-perturbative contributions in higher dimensions, such as the quantum corrections to the coupling constants, that can deeply change the picture.
At this point we are able to determine $\Lambda$ for any given quantum states. If we believe that this approach might in fact predict a meaningful value for the cosmological constant, we are now faced with the issue of determining which one is the quantum state the Universe is in.

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