SOME OBSERVATIONS CONCERNING REDUCIBILITY OF QUADRINOMIALS

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Abstract. In a recent paper [4], Jankauskas proved some interesting results concerning the reducibility of quadrinomials of the form \( f(x, x) \), where \( f(a, x) = x^n + x^m + x^k + a \). He also obtained some examples of reducible quadrinomials \( f(a, x) \) with \( a \in \mathbb{Z} \), such that all the irreducible factors of \( f(a, x) \) are of degree \( \geq 3 \).

In this paper we perform a more systematic approach to the problem and ask about reducibility of \( f(a, x) \) with \( a \in \mathbb{Q} \). In particular by computing the set of rational points on some genus two curves we characterize in several cases all quadrinomials \( f(a, x) \) with degree \( \leq 6 \) and divisible by a quadratic polynomial. We also give further examples of reducible \( f(a, x) \), \( a \in \mathbb{Q} \), such that all irreducible factors are of degree \( \geq 3 \).

1. Introduction

Let \( f(x) \) be a polynomial with rational coefficients. We say that the polynomial \( f \) is primitive if it is not of the form \( g(x^l) \) for some \( l \geq 2 \). Throughout the paper, reducibility of a polynomial will mean reducibility over \( \mathbb{Q} \).

At the West Coast Number Theory conference in 2007, P.G. Walsh posed the following question. Let \( n > m > k \) be positive integers. Does there exist an irreducible polynomial \( f(x) = x^n + x^m + x^k + 4 \), \( \deg f > 17 \), such that for some integer \( l > 1 \), the polynomial \( f(x^l) \) is reducible in \( \mathbb{Z}[x] \)? The answer to this question was given by J. Jankauskas in [4]. He proved that the only primitive quadrinomial of the form \( f(x) = x^n + x^m + x^k + 4 \), such that \( f(x^l) \) is reducible for some \( l > 1 \), is the polynomial \( f(x) = x^4 + x^3 + x^2 + 4 \). In this case, \( l = 2g \), where \( g \in \mathbb{N}_+ \). In particular, when \( l = 2 \),

\[
f(x^2) = (x^4 - x^3 + x^2 - 2x + 2)(x^4 + x^3 + x^2 + 2x + 2).\]

Let \( a \in \mathbb{Q} \) and define the quadrinomial

\[
f_{n,m,k}(a, x) = x^n + x^m + x^k + a.
\]

Walsh also asked for examples of reducible primitive \( f_{n,m,k}(a, x) \) with integer constant coefficient \( a > 4 \), but which have no linear or quadratic factor. He gave one such example: \( x^7 + x^5 + x^3 + 8 = (x^3 - x^2 - x + 2)(x^4 + x^3 + 3x^2 + 2x + 4) \).

Jankauskas in the cited paper also considered this problem using a computational
Theorem 2.1. Let $a, x \in \mathbb{Q}$, such that $a > 0$ and $m, k \in \mathbb{N}$ with $n > m > k$ and “small” $n$, such that $f_{n,m,k}(a, x)$ is divisible by a quadratic polynomial. Although this case is far from general, it is highly nontrivial. Indeed, as we shall see for fixed $n, m, k \in \mathbb{N}$ with $n > 5$, the problem reduces to finding all rational points on a curve of genus $\geq 2$.

Remark 1.1. We are interested in this paper in the reducibility of $f_{n,m,k}(a, x)$, but it is worth pointing out that it is easy to show the existence of infinitely many irreducible such for a fixed degree $n \geq 3$, as follows.

Lemma 1.2. Let $p \geq 5$ be prime. Then the quadrinomial $x^n + x^m + x^k + p$, $n > m > k \geq 1$, is irreducible over $\mathbb{Q}$.

Proof. Suppose $x^n + x^m + x^k + p = f_1(x)f_2(x)$ in $\mathbb{Z}[x]$, with $n > \deg(f_1), \deg(f_2) \geq 1$. The constant coefficient of $f_1$ is without loss of generality $\pm p$, so that the constant coefficient of $f_2$ is $\pm 1$. Not all roots of $f_2$ can therefore have absolute value greater than 1, so take $z \in \mathbb{C}$ as root of $f_2$ satisfying $|z| \leq 1$. Then $p = |z^n + z^m + z^k| \leq |z|^n + |z|^m + |z|^k \leq 3$.

More on (ir)reducibility of general quadrinomials can be found in [2].

2. REDUCIBLE QUADRINOMIALS $f_{n,m,k}(a, x)$ WITH DEG $f = 4$

In this section we characterize all $a \in \mathbb{Q} \setminus \{0\}$ and pairs of integers $m, k \in \mathbb{N}$ satisfying $4 > m > k$ such that $f_{4,m,k}(a, x)$ (of degree 4) is divisible by a quadratic polynomial.

Theorem 2.1. Let $f_{4,m,k}(a, x) = x^4 + x^m + x^k + a$, where $4 > m > k \geq 1$ and $a \in \mathbb{Q}$. Then $f_{4,m,k}(a, x)$ has a quadratic factor precisely in the following cases:

1. If $(m, k) = (2, 1)$ then $f_{4,2,1}(a, x)$ is divisible by $x^2 + px + q$ if and only if $a = \frac{(p^3 + p - 1)(p^3 + p + 1)}{4p^2}$, $q = \frac{p^3 + p - 1}{2p}$.

In this case we have

$$f_{4,2,1}(a, x) = \left(x^2 - px + \frac{p^3 + p + 1}{2p}\right) \left(x^2 + px + \frac{p^3 + p - 1}{2p}\right).$$

2. If $(m, k) = (3, 1)$ then $f_{4,3,1}(a, x)$ is divisible by $x^2 + px + q$ if and only if $a = \frac{(p^3 - 2p^2 + p + 1)(p^3 - p^2 - 1)}{(2p - 1)^2}$, $q = \frac{p^3 - p^2 - 1}{2p - 1}$.

In this case we have

$$f_{4,3,1}(a, x) = \left(x^2 - (p - 1)x + \frac{p^3 - 2p^2 + p + 1}{2p - 1}\right) \left(x^2 + px + \frac{p^3 - p^2 - 1}{2p - 1}\right).$$
(3) If \((m,k) = (3,2)\), then \(f_{4,3,2}(a,x)\) is divisible by \(x^2 + px + q\) if and only if

\[
a = \frac{(p-1)p(p^2-p+1)^2}{(2p-1)^2}, \quad q = \frac{p(p^2-p+1)}{2p-1}.
\]

In this case we have

\[
f_{4,3,2}(a,x) = \left(x^2 - (p-1)x + \frac{(p-1)(p^2-p+1)}{2p-1}\right) \left(x^2 + px + \frac{p(p^2-p+1)}{2p-1}\right).
\]

Proof. The proof of the theorem is very easy and we only give a sketch of the reasoning. We compute for fixed \((m,k) \in \{(2,1), (3,1), (3,2)\}\) the polynomial \(f_{4,m,k}(a,x) \mod x^2 + px + q\). This polynomial is of the form \(r(x) = A_{m,k}x + B_{m,k}\), where \(A_{m,k}\), \(B_{m,k} \in \mathbb{Z}[a,p,q]\). So if \(r(x) \equiv 0\), then \(f_{4,m,k}(a,x)\) is divisible by \(x^2 + px + q\). The system of equations \(A_{m,k} = 0\), \(B_{m,k} = 0\) is triangular with respect to \(a\), \(q\), so it can be easily solved. For example if \((m,k) = (2,1)\), then

\[
A_{2,1} = -p^3 - p + 1 + 2pq, \quad B_{2,1} = a - (p^2 + 1)q + q^2.
\]

and we get the expressions for \(a\), \(q\) displayed in the Theorem. The remaining cases follow analogously.

Remark 2.2. Recall that a polynomial is said to have reducibility type \((n_1,n_2,...,n_k)\) if there exists a factorization of the polynomial into irreducible polynomials of degrees \(n_1, n_2,...,n_k\). Types are ordered so that \(n_1 \leq n_2 \leq ... \leq n_k\). In a recent paper [4] we study the type of reducibility of trinomials and get several new results which complement the results obtained by Schinzel in the series of papers [6], [7], [8] (many corrections and some additional material can be found [9]). The results contained in Theorem 2.1 can be used to characterize those \(a \in \mathbb{Q}\) such that the quadrinomial \(f_{4,m,k}(a,x)\) has reducibility type \((1,1,2)\). For example, if \((m,k) = (2,1)\) then the set of those \(a \in \mathbb{Q}\) such that \(f_{4,2,1}(a,x)\) has reducibility type \((1,1,2)\) is parameterized by the rational points of the genus 1 curve

\[
C_{2,1} : \quad r^2 = -p(p^3 + 2p + 2),
\]

which is birationally equivalent to the elliptic curve \(E_{2,1} : \quad y^2 = x^3 + x^2 - x - 5\) of rank 0 and trivial torsion. Similarly, if \((m,k) = (3,1)\) we are led to the curve

\[
C_{3,1} : \quad r^2 = -(p+1)(2p-1)(2p^2 - 5p + 5)
\]

birationally equivalent to the elliptic curve \(E_{3,1} : \quad y^2 + xy = x^3 - x^2 - 5\) of rank 0 and \(\text{Tors}(E_{3,1}(\mathbb{Q})) = \{O,(2,-1)\}\). The point \((2,-1)\) leads to the quadrinomial

\[
x^4 + x^3 + x + 1 = (x + 1)^2(x^2 - x + 1).
\]

Finally, if \((m,k) = (3,2)\) then we get the curve

\[
C_{3,2} : \quad r^2 = -(p-1)(2p-1)(2p^2 - p + 3),
\]

birationally equivalent to the elliptic curve \(E_{3,2} : \quad y^2 + xy + y = x^3 - x - 1\) of rank 0 and \(\text{Tors}(E_{3,2}(\mathbb{Q})) = \{O,(1,-1)\}\). However, the point \((1,1)\) leads to \(a = 0\).

Thus if \(f_{4,m,k}(a,x)\) has reducibility type \((1,1,2)\), then \((m,k) = (3,1)\) and \(a = 1\).
3. Reducible quadrinomials \( f_{n,m,k}(a,x) \) with \( \deg f = 5 \)

In this section we characterize all \( a \in \mathbb{Q} \setminus \{0\} \) and pairs of integers \( m, k \in \mathbb{N} \) satisfying \( 5 > m > k \) such that the quadrinomial \( f_{n,m,k}(a,x) \) is divisible by a quadratic polynomial. However, before stating the results we use a lemma that allows computation of the polynomials \( f_{n,m,k}(a,x) \) mod \( x^2 + px + q \). More precisely we prove the following:

**Lemma 3.1.** Let \( n \in \mathbb{N} \). Then we have
\[
x^n \mod (x^2 + px + q) = A_n(p,q)x + B_n(p,q),
\]
where \( A_0(p,q) = 0 \), \( A_1(p,q) = 1 \), \( B_0(p,q) = 1 \), \( B_1(p,q) = 0 \), and where for \( n \geq 2 \):
\[
A_n(p,q) = -pA_{n-1}(p,q) - qA_{n-2}(p,q), \quad B_n(p,q) = -pB_{n-1}(p,q) - qB_{n-2}(p,q).
\]

**Proof.** Define \( x^n \mod x^2 + px + q = A_n(p,q)x + B_n(p,q) \). The expressions for \( A_0, A_1, B_0, B_1 \), are clear. In order to shorten the notation put \( A_n = A_n(p,q) \) and \( B_n = B_n(p,q) \). Then
\[
x^{n+1} \mod (x^2 + px + q) = A_n x^2 + B_n x = A_n(-px - q) + B_n x = A_{n+1}x + B_{n+1}.
\]

From the last equality, \( A_{n+1} = -pA_n + B_n \) and \( B_{n+1} = -qA_n \). Eliminating \( B_n \) from the first equation we get the recurrence relation for \( A_n(p,q) \) displayed in the statement of the Lemma. Similar reasoning gives the recurrence relation for \( B_n(p,q) \).

From the above Lemma we get the following:

**Corollary 3.2.** If \( f_{n,m,k}(a,x) \) is divisible by \( x^2 + px + q \) for some \( a, p, q \in \mathbb{Q} \) with \( a \neq 0 \) then
\[
A_n(p,q) + A_m(p,q) + A_k(p,q) = 0 \quad \text{and} \quad a = -B_n(p,q) - B_m(p,q) - B_k(p,q).
\]

From this Corollary it follows that divisibility of \( f_{n,m,k}(a,x) \) by \( x^2 + px + q \) for a fixed triple of exponents \( n, m, k \) is equivalent to the characterization of rational points on the curve
\[
C_{n,m,k} : A_n(p,q) + A_m(p,q) + A_k(p,q) = 0.
\]

Note that from the recurrence relation for \( A_n(p,q) \), it follows easily that \( \deg_p A_n = n - 1 \) and \( \deg_q A_n = \left\lfloor \frac{n-3}{2} \right\rfloor + 1 \). Define the polynomial
\[
F_{n,m,k}(p,q) = A_n(p,q) + A_m(p,q) + A_k(p,q)
\]
and let \( \overline{F}_{n,m,k}(p,q,r) = r^n F_{n,m,k}(p/r, q/r) \) be the homogenization of the polynomial \( F_{n,m,k} \). We also define:
\[
C_{n,m,k}(\mathbb{Q}) = \{(p : q : r) \in \mathbb{P}^2(\mathbb{Q}) : \overline{F}_{n,m,k}(p,q,r) = 0\},
\]
the set of rational points on the curve \( C_{n,m,k} \) together with the points at infinity.

The study of the curve \( C_{n,m,k} \) or more precisely its birational models which are hyperelliptic, and the corresponding set \( C_{n,m,k}(\mathbb{Q}) \) will be the main object of study in this paper.
Remark 3.3. The recurrence relations satisfied by $A_n(p, q)$ and $B_n(p, q)$ allow us to get Binet type formula for these polynomials. Indeed, using the standard method and initial conditions for the sequence $A_n(p, q)$ we get that

$$A_n(p, q) = \frac{1}{\sqrt{p^2 - 4q}} \left( \left( \frac{\sqrt{p^2 - 4q} - p}{2} \right)^n - \left( \frac{-\sqrt{p^2 - 4q} - p}{2} \right)^n \right).$$

A similar result can be given for $B_n(p, q)$ because $B_n(p, q) = -qA_{n-1}(p, q)$.

Now we are ready to prove the following:

**Theorem 3.4.** Let $f_{5,m,k}(a, x) = x^5 + x^m + x^k + a$ with $m, k \in \mathbb{N}$ and $5 > m > k \geq 1$. Then the following holds:

1. If $(m, k) \in \{(2, 1), (4, 1), (4, 2), (4, 3)\}$ then the set $C_{5,m,k}(\mathbb{Q})$ is finite. The only reducible $f_{5,m,k}(a, x)$ with quadratic factor is given by $(m, k) = (4, 3)$ and $a = -1$ with factor $x^2 + 1$.

2. If $(m, k) \in \{(3, 1), (3, 2)\}$ then the set $C_{5,m,k}(\mathbb{Q})$ is infinite and is parameterized by the rational points on a certain elliptic curve of positive rank.

**Proof.** We perform case by case analysis.

Case $(m, k) = (2, 1)$. Solving the equation $F_{5,2,1}(p, q) = 0$ in rational numbers $p, q$ is equivalent to the study of rational points on the hyperelliptic quartic curve $H_{5,2,1} : s^2 = 5p^4 + 4p - 4$.

where $s = \pm(2q - 3p^2)$. Setting

$$(x, y) = ((-10p^3 + 20p^2 + 1)/s^2 - (25p^6 - 100p^5 - 50p^3 + 100p^2 - 80p - 2 - s^3)/(2s^3)),$$

there results $E_{5,2,1} : y^2 + y = x^3 + 5x + 1$, an elliptic curve of rank 0 and trivial torsion. It follows that the set $H_{5,2,1}(\mathbb{Q})$ is empty.

Case $(m, k) = (3, 1)$. Solving the equation $F_{5,3,1}(p, q) = 0$ in rational numbers $p, q$ is equivalent to the study of rational points on the hyperelliptic quartic curve $H_{5,3,1} : s^2 = (p^2 + 1)(5p^2 - 3)$,

where $s = \pm(2q - 3p^2 - 1)$. Using the point $(p, s) = (1, 2)$ as a zero point, then $H_{5,3,1}$ is birationally equivalent to the elliptic curve $E_{5,3,1} : Y^2 = X^3 - X^2 + 4X$

under the mapping

$$(p, s) = \varphi(X, Y) = \left( \frac{X + 4 - Y}{3X - Y - 4}, \frac{2(X^3 - 12X^2 + 4X + 16Y - 16)}{(3X - Y - 4)^2} \right).$$

We have that $\text{Tors}(E_{5,3,1}(\mathbb{Q})) \cong \{0, (0,0)\}$ and the rank of $E_{5,3,1}$ is one, with generator $P = (1, 2)$. Our reasoning shows that if $f_{5,3,1}(a, x)$ is divisible by $x^2 + px + q$ then $p = \varphi_1(X, Y)$, where $\varphi_1$ is the first coordinate of the function $\varphi$, and $q$'s can be computed (if needed) from the relation $\varphi_2(X, Y) = \pm(2q - 1 - 3\varphi_1(X, Y)^2)$. The corresponding $a \in \mathbb{Q}$ can be computed from the expression given in Corollary 3.2. In particular there are infinitely many such $a$ corresponding to the integer multiples of $P$. For example, the point $-P$ corresponds to $p = -1, q = 3$ and $a = -12$, which leads to the factorization

$$x^5 + x^3 + x - 12 = (x^2 - x + 3)(x^3 + x^2 - x - 4).$$
where $s$ is equivalent to the study of rational points on the hyperelliptic quartic curve $H_{5,4,1} : s^2 = 5p^4 - 8p^3 + 4p^2 - 4$, where $s = \pm(2q - 3p^2 + 2p)$. However, the curve $H_{5,4,1}$ is not locally solvable at the prime 2, and thus the set $H_{5,4,1}(\mathbb{Q})$ is empty.

Case $(m,k) = (3,2)$. Solving the equation $F_{5,3,2}(p,q) = 0$ in rational numbers $p,q$ is equivalent to the study of rational points on the hyperelliptic quartic curve $H_{5,3,2} : s^2 = 5p^4 + 2p^2 + 4p + 1$, where $s = \pm(2q - 3p^2 - 1)$. Using the point $(p,s) = (0,1)$ as a point at infinity we find that $H_{5,3,2}$ is birationally equivalent to the elliptic curve $E_{5,3,2} : Y^2 + Y = X^3 - X^2 - X + 1$ under the mapping $$(p,s) = \varphi(X,Y) = \left( \frac{X - 1}{Y - X + 1}, \frac{Y + X^3 - 3X^2 + 3X - 1}{(Y - X + 1)^2} \right).$$ The torsion subgroup of $E_{5,3,2}(\mathbb{Q})$ is trivial and the rank of the curve is one, with generator $P = (1,0)$. The point $P$ corresponds to $a = 1$, and the point $2P$, for example, leads to $a = 6$ and $a = -363$. The latter gives the factorization $$x^5 + x^3 + x^2 - 363 = (x^2 - 2x + 11)(x^3 + 2x^2 - 6x - 33).$$

Case $(m,k) = (4,2)$. Solving the equation $F_{5,4,2}(p,q) = 0$ in rational numbers $p,q$ is equivalent to the study of rational points on the hyperelliptic quartic curve $H_{5,4,2} : s^2 = p(5p^3 - 8p^2 + 4p + 4)$, where $s = \pm(2q - 3p^2 + 2p)$. Using the point $(p,s) = (0,0)$ as a point at infinity we find that $H_{5,4,2}$ is birationally equivalent to the elliptic curve $E_{5,4,2} : Y^2 + Y = X^3 + X^2 - 2X + 1$, which has trivial torsion and rank 0. It follows that $H_{5,4,2}(\mathbb{Q}) = \{(0,0)\}$. The point $(0,0)$ leads to $a = 0$, with no corresponding quadrinomial.

Case $(m,k) = (4,3)$. Solving the equation $F_{5,4,3}(p,q) = 0$ in rational numbers $p,q$ is equivalent to the study of rational points on the hyperelliptic quartic curve $H_{5,4,3} : s^2 = (p - 1)(5p^3 - 3p^2 + 3p - 1)$, where $s = \pm(2q - 3p^2 + 2p - 1)$. Using the point $(p,s) = (1,0)$ as a point at infinity we find that $H_{5,4,3}$ is birationally equivalent to the elliptic curve $E_{5,4,3} : Y^2 + Y = X^3$ with rank 0 and $\text{Tors}(E_{5,4,3}(\mathbb{Q})) = \{O,(0,0),(0,-1)\}$. It follows that $H_{5,4,3}(\mathbb{Q}) = \{(0,-1),(1,0)\}$. The point $(0,-1)$ leads to $a = -1$ and the point $(1,0)$ leads to $a = 0$.

Tying now all the results together we get the statement of the Theorem. □
4. Reducible quadrinomials \( f_{n,m,k}(a, x) \) with \( \deg f = 6 \)

In the previous section the problem of finding reducible quadrinomials \( f_{n,m,k}(a, x) \) with \( \deg f = 5 \) and \( a \in \mathbb{Q} \) resulted in the examination of the set of rational points on several curves of genus one. In this section we examine the divisibility of a sextic \( f_{6,m,k}(a, x) \) by the quadratic polynomial \( x^2 + px + q \), which results in the investigation of several curves of genus two. Computing all the rational points on any specific curve of genus 2 is unfortunately still very much an open problem. In some cases, in particular if the Jacobian of the curve has rank at most 1, then Chabauty arguments can work. Otherwise elliptic Chabauty techniques can sometimes apply, occasionally with a great deal of associated work, allowing all rational points to be determined. But in other cases, the methods do not fall to computation. If a curve takes the form \( y^2 = g(x) \) for an irreducible quintic polynomial \( g \), and has Jacobian of rank at least 2, then the only approach known to us is to factor over the quintic field defined by a root of \( g \), and try to apply elliptic Chabauty techniques. But the associated arithmetic often leaves a machine churning. In what follows, some results are absolute, when we have been able to determine all the rational points on a given curve; and some results are conjectural, when we have been unable to prove a given set of points (usually found by search up to a height bound of \( 10^6 \)) is complete. We introduce an Asterisked Theorem, where cases without asterisk are absolute, and cases with asterisk represent instances where the reader will know that the result depends upon a set of listed points on some curve of genus 2 being complete.

**Theorem** 4.1. Let \( f_{6,m,k}(a, x) = x^6 + x^m + x^k + a, m, k \in \mathbb{N} \) with \( 6 > m > k \geq 1 \) and suppose that \( f_{6,m,k}(a, x) \) is divisible by \( x^2 + px + q \). Then the following hold, with quadratic factors listed respectively:

1. If \((m, k) = (2, 1)\) then \( a \in \{-\frac{5795}{1728}, -\frac{3655}{1728}, -\frac{1115}{1728}, \frac{10}{27}, \frac{51}{64}\} \), with factors \( x^2 - \frac{3x}{2} + \frac{19}{12} \), \( x^2 - \frac{3x}{2} + \frac{17}{12} \), \( x^2 + \frac{x}{2} - \frac{5}{12} \), \( x^2 + x + \frac{1}{3} \), \( x^2 + \frac{x}{2} + \frac{3}{4} \).

2. If \((m, k) = (3, 1)\) then \( a \in \{-\frac{4881488}{209000}, -\frac{2899}{1728}, \frac{353}{64}, \frac{639}{64}\} \), with factors \( x^2 - \frac{5x}{2} + \frac{373}{60} \), \( x^2 + \frac{x}{2} - \frac{13}{12} \), \( x^2 + \frac{x}{2} + \frac{3}{4} \), \( x^2 - \frac{5x}{2} + \frac{9}{4} \).

3. If \((m, k) = (4, 1)\) then \( a \in \{-\frac{4881}{8}, \frac{1441}{8}\} \), with factors \( x^2 - 4x + \frac{33}{2} \), \( x^2 - 4x + \frac{11}{2} \).

4. If \((m, k) = (5, 1)\) then \( a \in \{-\frac{2869705313}{912}, -21, -\frac{3}{8}, 1, 206682\} \), with factors \( x^2 - 13x + \frac{1403}{8} \), \( x^2 + 2x + 3 \), \( x^2 + x - \frac{1}{2} \), \( (x + 1)^2 \), \( x^2 - 13x + 57 \).

5. If \((m, k) = (3, 2)\) then \( a \in \{-\frac{474281}{10099}, \frac{1}{64}\} \), with factors \( x^2 + \frac{3x}{2} + \frac{73}{36} \), \( x^2 + \frac{3x}{2} + \frac{3}{4} \).

6. If \((m, k) = (4, 2)\) then \( a = q^3 - q^2 + q, q \in \mathbb{Q}\) with factor \( x^2 + q \).

7. If \((m, k) = (5, 2)\) then \( a \in \{-1, -\frac{9}{64}, -\frac{29147243090237}{21956433224192}, -\frac{1983235848957}{14467923638863}\} \), with factors \( x^2 - x + 1 \), \( x^2 - \frac{x}{4} + \frac{3}{2} \), \( x^2 + x \frac{3709}{27848} \), \( x^2 + \frac{x}{59} + \frac{3267}{24367} \).
Chabauty arguments are therefore unavailable. One can factor over the quintic

\[ H \]

are not applicable, and we are unable to determine explicitly all the rational points.

The rank of the Jacobian variety is 2, so traditional Chabauty techniques is equivalent to the study of rational points on the hyperelliptic sextic curve

\[ P, q \]

method to work, and have been unable to find explicitly all points on \( H \). Elliptic Chabauty methods will work. However, we were unsuccessful in getting this

is equivalent to the study of rational points on the hyperelliptic sextic curve

\[ s \]

Case \((m, k) = (2, 1)\). Solving the equation \( F_{6,2,1}(p, q) = 0 \) in rational numbers \( p, q \), is equivalent to the study of rational points on the hyperelliptic sextic curve of genus 2

\[ H_{6,2,1} : s^2 = p(p^5 - 3p + 3), \]

where \( s = \pm(3pq - 2p^3) \). On the curve \( H_{6,2,1} \) we have rational points with \( p \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety of \( H_{6,2,1} \) is equal to 2, and traditional Chabauty arguments are therefore unavailable. One can factor over the quintic field \( \mathbb{Q}(\theta) \), \( \theta^5 - 3\theta + 3 = 0 \), to obtain

\[ 3(3P + (\theta^4 - 3))(3P^4 - \theta^4 P^3 - \theta^3 P^2 - \theta^2 P - \theta) = \Box \]

with \( P = 1/p \). This allows deduction of elliptic quartics of the type

\[ 3P^4 - \theta^4 P^3 - \theta^3 P^2 - \theta^2 P - \theta = \delta \Box \]

for a finite number of \( \delta \in \mathbb{Q}(\theta) \), and there is a possibility that an approach using elliptic Chabauty methods will work. However, we were unsuccessful in getting this method to work, and have been unable to find explicitly all points on \( H_{6,2,1} \).

Case \((m, k) = (3, 1)\). Solving the equation \( F_{6,3,1}(p, q) = 0 \) in rational numbers \( p, q \), is equivalent to the study of rational points on the hyperelliptic sextic curve

\[ H_{6,3,1} : s^2 = 4p^6 + 4p^3 + 12p + 1, \]

where \( s = \pm(6pq - 4p^3 + 1) \). On the curve \( H_{6,3,1} \) we have rational points with \( p \in \{-\frac{1}{2}, 0, \frac{1}{2}\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 2, so traditional Chabauty techniques are not applicable, and we are unable to determine explicitly all the rational points on \( H_{6,3,1} \).

Case \((m, k) = (4, 1)\). Solving the equation \( F_{6,4,1}(p, q) = 0 \) in rational numbers \( p, q \), is equivalent to the study of rational points on the hyperelliptic sextic curve

\[ H_{6,4,1} : s^2 = p(p + 1)(p^4 - p^3 + 2p^2 - 2p + 3), \]

Proof. As before, we perform a case by case analysis.

Case \((m, k) = (2, 1)\). Solving the equation \( F_{6,2,1}(p, q) = 0 \) in rational numbers \( p, q \), is equivalent to the study of rational points on the hyperelliptic sextic curve of genus 2

\[ H_{6,2,1} : s^2 = p(p^5 - 3p + 3), \]

where \( s = \pm(3pq - 2p^3) \). On the curve \( H_{6,2,1} \) we have rational points with \( p \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety of \( H_{6,2,1} \) is equal to 2, and traditional Chabauty arguments are therefore unavailable. One can factor over the quintic field \( \mathbb{Q}(\theta) \), \( \theta^5 - 3\theta + 3 = 0 \), to obtain

\[ 3(3P + (\theta^4 - 3))(3P^4 - \theta^4 P^3 - \theta^3 P^2 - \theta^2 P - \theta) = \Box \]

with \( P = 1/p \). This allows deduction of elliptic quartics of the type

\[ 3P^4 - \theta^4 P^3 - \theta^3 P^2 - \theta^2 P - \theta = \delta \Box \]

for a finite number of \( \delta \in \mathbb{Q}(\theta) \), and there is a possibility that an approach using elliptic Chabauty methods will work. However, we were unsuccessful in getting this method to work, and have been unable to find explicitly all points on \( H_{6,2,1} \).

Case \((m, k) = (3, 1)\). Solving the equation \( F_{6,3,1}(p, q) = 0 \) in rational numbers \( p, q \), is equivalent to the study of rational points on the hyperelliptic sextic curve

\[ H_{6,3,1} : s^2 = 4p^6 + 4p^3 + 12p + 1, \]

where \( s = \pm(6pq - 4p^3 + 1) \). On the curve \( H_{6,3,1} \) we have rational points with \( p \in \{-\frac{1}{2}, 0, \frac{1}{2}\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 2, so traditional Chabauty techniques are not applicable, and we are unable to determine explicitly all the rational points on \( H_{6,3,1} \).

Case \((m, k) = (4, 1)\). Solving the equation \( F_{6,4,1}(p, q) = 0 \) in rational numbers \( p, q \), is equivalent to the study of rational points on the hyperelliptic sextic curve

\[ H_{6,4,1} : s^2 = p(p + 1)(p^4 - p^3 + 2p^2 - 2p + 3), \]
where \( s = \pm (3pq - 2p^3 - p) \). On the curve \( H_{6,4,1} \) we have rational points with \( p = -4, -1, 0 \) and these lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety of \( H_{6,4,1} \) is 1, and Chabauty’s method as implemented in Magma \([5]\) is able to obtain the explicit list of rational points on \( H_{6,4,1} \), precisely the three above.

Case \((m, k) = (5, 1)\). Solving the equation \( F_{6,5,1}(p, q) = 0 \) in rational numbers \( p, q \) is equivalent to the study of rational points on the hyperelliptic sextic curve

\[
H_{6,5,1} : s^2 = 4p^6 - 8p^5 + 5p^4 + 12p - 4,
\]

where \( s = \pm (2(-3p + 1)q + 4p^3 - 3p^2) \). On the curve \( H_{6,5,1} \) we have rational points with \( p \in \{-1, \frac{1}{3}, 1, 2, -13\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 3, and we have not attempted to find explicitly all the rational points.

Case \((m, k) = (3, 2)\). Solving the equation \( F_{6,3,2}(p, q) = 0 \) in rational numbers \( p, q \) is equivalent to the study of rational points on the hyperelliptic sextic curve

\[
H_{6,3,2} : s^2 = 4p^6 + 4p^3 - 12p^2 + 1,
\]

where \( s = \pm (6pq - 4p^3 + 1) \). On the curve \( H_{6,3,2} \) we have rational points with \( p \in \{0, \frac{3}{2}\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 1, and Magma’s Chabauty routines determine the full set of rational points as precisely those given above.

Case \((m, k) = (4, 2)\). Solving the equation \( F_{6,4,2}(p, q) = 0 \) in rational numbers \( p, q \) leads to \( p^2(p^2 - 1)(p^2 + 2) = 0 \), so leads either to \( p = 0 \), or to study of rational points on the hyperelliptic quartic curve

\[
H_{6,4,2} : s^2 = (p + 1)(p - 1)(p^2 + 2),
\]

where \( s = \pm (3q - 2p^2 - 1) \). In this latter case, the curve is birationally equivalent to the elliptic curve

\[
E_{6,4,2} : Y^2 = X^3 + X^2 + 8X + 8,
\]

of rank 0 and \( \text{Tors}(E_{6,4,2}(\mathbb{Q})) = \{O, (2, -6), (2, 6), (-1, 0)\} \). The torsion points all correspond to \( a = 0 \).

In the first case, then \( p = 0, a = q^3 - q^2 + q \), and the cubic \( x^3 + x^2 + x + a \) has the rational root \( x = -q \).

Case \((m, k) = (5, 2)\). Solving the equation \( F_{6,5,2}(p, q) = 0 \) in rational numbers \( p, q \) is equivalent to the study of rational points on the hyperelliptic sextic curve

\[
H_{6,5,2} : s^2 = p(4p^5 - 8p^4 + 5p^3 - 12p + 4),
\]

where \( s = \pm (2(-3p + 1)q + 4p^3 - 3p^2) \). On the curve \( H_{6,5,2} \) we have rational points with \( p \in \{-1, 0, \frac{1}{3}, \frac{1}{13}\} \) and these numbers lead to the \( a \)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 3, and traditional Chabauty arguments do not apply. As in the case \((m, k) = (2, 1)\) we attempted an attack using elliptic Chabauty methods, but were unsuccessful; so have been unable to find explicitly all points on \( H_{6,5,2} \).
Case \((m, k) = (4, 3)\). Solving the equation \(F_{6,4,3}(p, q) = 0\) in rational numbers \(p, q\) is equivalent to the study of rational points on the hyperelliptic sextic curve

\[
H_{6,4,3} : s^2 = 4p^6 + 4p^4 + 4p^3 + 4p^2 - 4p + 1,
\]

where \(s = \pm(6pq - 4p^3 - 2p + 1)\). On the curve \(H_{6,4,3}\) we have rational points with \(p = 0, 2\) and this number leads to the \(a\)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 1, and Magma’s Chabauty routines is successful in finding all the rational points, which are precisely as above.

Case \((m, k) = (5, 3)\). Solving the equation \(F_{6,5,3}(p, q) = 0\) in rational numbers \(p, q\) is equivalent to the study of rational points on the hyperelliptic sextic curve

\[
H_{6,5,3} : s^2 = 4p^6 - 8p^5 + 5p^4 + 4p^3 + 2p^2 + 1,
\]

where \(s = \pm(2(-3p + 1)q + 4p^3 - 3p^2 - 1)\). On the curve \(H_{6,5,3}\) we have rational points with \(p \in \{-1, -\frac{3}{2}, 0, \frac{1}{2}, 2\}\) and these numbers lead to the \(a\)'s displayed in the statement of the theorem. The rank of the Jacobian variety is 2, and we are unable to determine explicitly all the rational points.

Case \((m, k) = (5, 4)\). Solving the equation \(F_{6,5,4}(p, q) = 0\) in rational numbers \(p, q\) leads to \(p^2(4p^4 - 8p^3 + 9p^2 - 8p + 4) = \Box\), so leads to \(p = 0\), which in turn leads to \(q = 0\), or to the study of rational points on the hyperelliptic quartic curve

\[
H_{6,5,4} : s^2 = 4p^4 - 8p^3 + 9p^2 - 8p + 4,
\]

where \(s = \pm(2(-3p + 1)q/p + 4p^2 - 3p + 2)\). Taking the point \((0, 2)\) as a point at infinity, we get that \(H_{6,5,4}\) is birationally equivalent to the elliptic curve

\[
E_{6,5,4} : Y^2 + XY + Y = X^3 - X^2 - 2X
\]

We have that \(\text{Tors}(E_{6,5,4}(\mathbb{Q})) = \{O, (-1, 0)\}\), and that the rank of \(E_{6,5,4}(\mathbb{Q})\) is 1, with generator \(P = (0, 0)\). For example, the point \(2P = (3, 2)\) leads to \(a = \frac{13}{512}\). We have the following factorization in the case \(a = 18\):

\[
x^6 + x^5 + x^4 + 18 = (x^2 + 3x + 3)(x^4 - 2x^3 + 4x^2 - 6x + 6).
\]

Tying now all these results together we get the statement of the theorem. \(\Box\)

**Remark 4.2.** There are some instances where \(f_{6,m,k}(a, x)\) can split as the product of two cubics, for example when \((m, k) = (5, 3)\):

\[
\begin{align*}
(x^3 - x^2 + x - \frac{1}{3})(x^3 + 2x^2 + x + \frac{1}{3}) &= x^6 + x^5 + x^3 - \frac{1}{9}, \\
(x^3 + \frac{1}{2}x + \frac{1}{4})(x^3 + x^2 - \frac{1}{2}x + \frac{1}{4}) &= x^6 + x^5 + x^3 + \frac{1}{16}, \\
(x^3 - \frac{1}{4}x^2 - \frac{3}{32}x + \frac{117}{512})(x^3 + \frac{5}{4}x^2 + \frac{17}{32}x + \frac{507}{512}) &= x^6 + x^5 + x^3 + \frac{59319}{262144}.
\end{align*}
\]

For all \((m, k)\) except \((4, 2)\) and \((5, 4)\), the variety that parameterizes such examples is a non-hyperelliptic curve of genus 4, and so other than searching for small points (resulting in the above examples) we have not carried investigation further here. In the remaining two cases, we can describe precisely when the sextic splits as the product of two cubics.
Theorem 4.3. The quadrinomial $x^6 + x^4 + x^2 + a$ is the product of two cubics precisely when $a = -\frac{(u-1)^2(u+1)(3+u^2)^2}{64u^2}$, for $u \in \mathbb{Q} \setminus \{0\}$. The quadrinomial $x^6 + x^5 + x^4 + a$, $a \neq 0$, cannot split as the product of two cubics.

Proof. Case $(m, k) = (4, 2)$. We have $f_{6,4,2}(a,x) = g(x^2)$ for a cubic polynomial $g$, and from Lemma 29 of [6]
$f_{6,4,2}(a,x) = -h(x)h(-x)$,
where $h(x) = x^3 + ux^2 + vx + w$, say. Comparing coefficients gives the (triangular) system of equations
$$1 + u^2 - 2v = 0, \quad 1 - v^2 + 2uw = 0, \quad a + w^2 = 0;$$
solving this system with respect to $v, w$ and $a$,
$$v = \frac{u^2 + 1}{2}, \quad w = \frac{(u - 1)(u + 1)(u^2 + 3)}{8u}, \quad a = -\frac{(u - 1)^2(u + 1)^2(3 + u^2)^2}{64u^2}.$$
Then $f_{6,4,2}(a,x) = -h(x)h(-x)$, where
$$h(x) = x^3 + ux^2 + \frac{u^2 + 1}{2}x + \frac{(u - 1)(u + 1)(u^2 + 3)}{8u}.$$

Case $(m, k) = (5, 4)$. Suppose that
$$x^6 + x^5 + x^4 + a = (x^3 + px^2 + qx + r)(x^3 + sx^2 + tx + u).$$
Comparing coefficients,
$$-1 + p + s = 0, \quad -1 + q + ps + t = 0, \quad r + qs + pt + u = 0, \quad rs + qt + pu = 0, \quad rt + qu = 0.$$
Eliminating $r, s, t, u,$
$$-p^2(p^2 - p + 1)^2 + (p^2 - p + 1)(3p^2 - p + 1)q - (3p^2 - p + 2)q^2 + 2q^3 = 0,$$
a curve of genus 2. Under the mapping
$$(X,Y) = (p^2 - p + 1 - q)/q,$$
we obtain a hyperelliptic model
$$C : Y^2 = X(4X^4 + X^3 + 6X^2 + X + 4).$$
Now the Jacobian of $C$ has rank 1, and Magma’s Chabauty routines determine that the only finite rational points on $C$ are $(X,\pm Y) = (0,0), (1,4)$. This in turn gives $(p, q) = (0,0), (1,1), (\frac{1}{2}, \frac{1}{2})$ as the complete set of finite rational points on $C$, leading only to $a = 0$. □

5. Reducible quadrinomials $f_{n,m,k}(a,x)$ with $\deg f \geq 7$

The problem of divisibility of $f(a,x) = x^n + x^{n-m} + x^{n-2m} + a$ by $x^2 + px + q$ for some $p, q, a \in \mathbb{Q}$ can be reduced to the study of rational points on certain hyperelliptic curves. More precisely:
Theorem 5.1. Let \( f(a, x) = x^n + x^{n-m} + x^{n-2m} + a \), \( n > 2m \geq 3 \) and suppose that \( f(a, x) \) is divisible by \( x^2 + px + q \) for some \( a, p, q \in \mathbb{Q} \) with \( a \neq 0 \). Then there exists \( t \in \mathbb{Q} \) such that \( q = tp^2 \) and

\[
A_{n-m}(1, t)^2 - 4A_{n-2m}(1, t)A_n(1, t) = \square.
\]

In particular if \( m = 1 \), we have the following results:

1. If \( n = 7 \) then \( a \in \{ -2, 1 \} \) with respective factorizations

\[
x^7 + x^6 + x^5 - 2 = (x^2 - x + 1)(x^5 + 2x^4 + 2x^3 - 2x - 2),
\]

\[
x^7 + x^6 + x^5 + 1 = (x + 1)(x^2 + 1)(x^4 - x + 1).
\]

2. * If \( n = 8 \) then

\[
a \in \left\{ 2, 419928937515451125000, -13149874643832399539673, -1262324516855259464728576 \right\},
\]

with respective factors \( x^2 + x + \frac{1}{3}, x^2 + \frac{90}{17}x + \frac{174150}{83697} \), and \( x^2 + \frac{223}{728}x + \frac{263891}{1059968} \).

Proof. It is clear that for any given pair \( p, q \) of rational numbers one can find a rational number \( t \) such that \( q = tp^2 \). Now, recall the formula for \( A_n(p, q) \):

\[
A_n(p, q) = \frac{1}{\sqrt{p^2 - 4q}} \left( \left( \frac{\sqrt{p^2 - 4q} - p}{2} \right)^n - \left( \frac{-\sqrt{p^2 - 4q} - p}{2} \right)^n \right),
\]

and note that \( A_n(p, q) = A_n(p, p^2t) = p^nA_n(1, t) \). Thus we have that

\[
A_n(p, q) + A_{n-m}(p, q) + A_{n-2m}(p, q) = p^{n-2m-1}(p^{2m}A_n(1, t) + p^mA_{n-1}(1, t) + A_{n-2}(1, t)).
\]

In order to find all rational solutions of the equation \( A_n(p, q) + A_{n-m}(p, q) + A_{n-2m}(p, q) = 0 \) it is enough now to characterize the set of rational points on the hyperelliptic curve

\[
H_{n,n-m,n-2m} : s^2 = A_{n-m}(1, t)^2 - 4A_{n-2m}(1, t)A_n(1, t),
\]

where \( s = \pm(2A_n(1, t)p + A_{n-m}(1, t)) \).

Suppose \( m = 1 \), and consider those \( n \) such that the curve \( H_{n,n-1,n-2} \) has genus two; it is straightforward to check that \( n = 7 \) and \( n = 8 \) are the only cases.

Case \( n = 7 \). Here, the problem of divisibility of \( f(a, x) \) by \( x^2 + px + q \) is equivalent to the study of rational points on the genus 2 hyperelliptic curve:

\[
H_{7,6,5} : s^2 = 4t^5 - 27t^4 + 72t^3 - 66t^2 + 24t - 3,
\]

where \( s = \pm(2(t^3 - 6t^2 + 5t - 1)p + (t - 1)(3t - 1)) \). The rank of the Jacobian variety is 1, and Magma’s Chabauty routines determine that the only finite rational points on \( H_{7,6,5} \) are \((t, \pm s) = (1, 2)\). These return \( a = -2 \); the infinite point returns \( a = 1 \).

Case \( n = 8 \). Here, the problem of divisibility of \( f(a, x) \) by \( x^2 + px + q \) is equivalent to the study of the rational points on the genus 2 hyperelliptic curve:

\[
H_{8,7,6} : s^2 = t^6 + 36t^5 - 138t^4 + 186t^3 - 111t^2 + 30t - 3,
\]

where \( s = \pm 2((2t - 1)(2t^2 - 4t + 1)p - t^3 + 6t^2 - 5t + 1) \). There are points at \( t = \frac{1}{3}, \frac{1}{2}, 1, \frac{5}{2} \). The rank of the Jacobian variety is 2, so standard Chabauty arguments do not apply. The sextic factors over \( \mathbb{Q}(\sqrt{3}) \):

\[
s^2 = (t^3 + 18t^2 - 15t + 3)^2 - 3(12t^2 - 10t + 2)^2,
\]
solving the first, the second and the fourth equation with the respect to
Theorem 5.3.

Then following system of equations

\[ L \]

Factoring over finite number of \( \delta \) over a sextic field. But again, we were unsuccessful in completing this approach. Consequently, we are unable to determine explicitly all the rational points on \( H_{8,7,6} \).

\[ \square \]

Although we have not proved the following, it seems plausible:

**Conjecture 5.2.** Let \( m, n \in \mathbb{N}_+ \) with \( n > 2m \) and define the polynomial

\[ F_{m,n}(t) = A_{n-m}(1,t)^2 - 4A_{n-2m}(1,t)A_{n}(1,t). \]

Then \( F_{m,n} \) has no multiple roots.

**Theorem 5.3.** Let \( f(a,x) = x^8 + x^m + x^k + a \), with \( k, m \) even. Suppose that \( f(a,x) \) is reducible and \( x^4 + x^m + x^k + a \) is irreducible. We have the following results:

1. If \( (m,k) = (4,2) \), then \( t \in \left\{ \frac{1}{4}, \frac{625}{4} \right\} \) with respective factorizations

\[ x^8 + x^4 + x^2 + \frac{1}{4} = \left( x^4 - 2x^3 + 2x^2 - x + \frac{1}{2} \right) \left( x^4 + 2x^3 + 2x^2 + x + \frac{1}{2} \right), \]

\[ x^8 + x^4 + x^2 + \frac{625}{4} = \left( x^4 - 2x^3 + 2x^2 - 7x + \frac{25}{2} \right) \left( x^4 + 2x^3 + 2x^2 + 7x + \frac{25}{2} \right). \]

2. If \( (m,k) = (6,2) \), then there does not exist \( a \in \mathbb{Q} \) satisfying the required properties.

3. If \( (m,k) = (6,4) \), then \( a = 1 \) with the factorization

\[ x^8 + x^6 + x^4 + x^2 = (x^4 - x^3 + x^2 - 2x + 2)(x^4 + x^3 + x^2 + 2x + 2). \]

**Proof.** Case \( (m,k) = (4,2) \). By Lemma 29 in [4] we know that \( f(a,x) = h(x)h(-x) \), where, say,

\[ h(x) = x^4 + px^3 + qx^2 + rx + s. \]

Comparing now the coefficients in the equality \( f(a,x) = h(x)h(-x) \) we get the following system of equations

\[ p^2 - 2q = 0, \quad 1 - q^2 + 2pr - 2s = 0, \quad 1 + r^2 - 2qs = 0, \quad a - s^2 = 0. \]

Solving the first, the second and the fourth equation with the respect to \( a, q, r \) we get

\[ q = \frac{p^2}{2}, \quad r = \frac{-4 + p^4 + 8s}{8p}, \quad a = s^2. \]

We are thus left with finding rational points on the genus two curve given by

\[ H_{8,4,2} : v^2 = 2(p^6 + 4p^2 - 8), \]

where \( v = \pm \left( \frac{8s - 3p^4 - 4}{2p} \right) \). The rank of the Jacobian variety is 2, so standard Chabauty techniques are unavailable to us. However, we can work over the cubic number field \( L \) defined by \( \theta^3 + 4\theta - 8 = 0 \), when the equation of the curve takes the form

\[ v^2 = 2(p^2 - \theta)(p^4 + \theta p^2 + \theta^2 + 4). \]

Factoring over \( L \) results in two cases to consider, one of which is locally unsolvable, and the other of which is amenable to an argument using elliptic Chabauty
techniques. The complete set of rational points is \((\pm p, \pm q) = (2, 12)\), which return \(a = \frac{1}{4}, \frac{9pq}{4}\).

Case \((m, k) = (6, 2)\). From Lemma 29 in [3] we know that \(f(a, x) = h(x)h(-x)\), where as before \(h(x) = x^4 + px^3 + qx^2 + rx + s\). Comparing coefficients in the equality \(f(a, x) = h(x)h(-x)\) we get the following system of equations

\[
1 + p^2 - 2q = 0, \quad -q^2 + 2pr - 2s = 0, \quad 1 + r^2 - 2qs = 0, \quad a - s^2 = 0.
\]

Solving the first, the second and the fourth equation with the respect to \(a, q, r\) we get

\[
q = \frac{p^2 + 1}{2}, \quad r = \frac{1 + 2p^2 + p^4 + 8s}{8p}, \quad a = s^2.
\]

We are thus left with finding the rational points on the genus two curve

\[
H_{8,6,2} : v^2 = 2(p^6 + p^4 - p^2 - 9),
\]

where \(v = \pm \left(\frac{8s - 3p^3 - 2p^2 + 1}{2p}\right)\). There is an obvious map with \(p^2 = X\) to the elliptic curve

\[
E_{8,6,2} : Y^2 = 2(X^3 + X^2 - X - 9),
\]

which has rank 0 and trivial torsion. Thus there are no rational points on \(H_{8,6,2}\).

Case \((m, k) = (6, 4)\). As above, \(f(a, x) = h(x)h(-x)\), with \(h(x) = x^4 + px^3 + qx^2 + rx + s\). Comparing coefficients in the equality \(f(a, x) = h(x)h(-x)\) we get the following system of equations

\[
1 + p^2 - 2q = 0, \quad 1 - q^2 + 2pr - 2s = 0, \quad r^2 - 2qs = 0, \quad a - s^2 = 0.
\]

Solving the first, the second and the fourth equation with the respect to \(a, q, r\) we get

\[
q = \frac{p^2 + 1}{2}, \quad r = \frac{-3 + 2p^2 + p^4 + 8s}{8p}, \quad a = s^2.
\]

We are thus left with finding rational points on the genus two curve given by

\[
H_{8,6,4} : v^2 = 2(p^2 + 1)(p^4 + 3),
\]

where \(v = \pm \left(\frac{8s - 3p^3 - 2p^2 - 3}{2p}\right)\). There is an obvious map with \(p^2 = X\) to the elliptic curve

\[
E_{8,6,4} : Y^2 = 2(X + 1)(X^2 + 3),
\]

which has rank 0 and torsion group \(\{O, (1, 1), (1, -1), (0, 0)\}\). The points \((1, \pm 1)\) return \(a = 4\) and \(a = 0\).

**Theorem 5.4.** Let \(f(a, x) = x^{10} + x^6 + x^2 + a\). Then \(f(a, x)\) is reducible if and only if \(a = -u^5 - u^3 - u\) for some \(u \in \mathbb{Q}\).

**Proof.** It is clear that if \(a = -u^5 - u^3 - u\) for some \(u \in \mathbb{Q}\) then the polynomial \(f(a, x)\) is divisible by \(x^2 - u\). Now suppose that \(a\) is not of the form \(-u^5 - u^3 - u\). Then any possible factor is of degree \(\geq 3\) and from Lemma 29 in [3] we know that \(f(a, x) = -h(x)h(-x)\), where

\[
h(x) = x^5 + px^4 + qx^3 + rx^2 + sx + t.
\]

Comparing now the coefficients in the equality \(f(a, x) = -h(x)h(-x)\) we get the following system of equations

\[
p^2 - 2q = 0, \quad 1 - q^2 + 2pr - 2s = 0, \quad r^2 - 2qs + 2pt = 0, \quad a + r^2 = 0, \quad 1 - s^2 + 2rt = 0.
\]
Solving the first four equations with respect to \( q, r, t, a \) we get
\[
q = \frac{p^2}{2}, \quad t = \frac{p^2 s - r^2}{2p}, \quad r = \frac{p^4 - 4 + 8s}{8p}, \quad a = -t^2.
\]
On substitution, we are left with finding the rational points on:
\[
C_{10,6,2} : P^3 - 4(10s + 3)P^2 + 16(12s^2 + 4s - 29)P + 64(2s - 1)^3 = 0,
\]
where \( P = p^4 \). The curve \( C_{10,6,2} \) is of genus 1, and taking \((P, s) = (-4, 1)\) as zero point, a cubic model is
\[
E_{10,6,2} : Y^2 = X^3 + X^2 - 24X + 36.
\]
This curve has rank 0 and torsion group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \), with the eight points given by \( \{O, (6, \pm 12), (2, 0), (-6, 0), (0, \pm 6), (3, 0)\} \). Thus the complete set of rational points on \( C_{10,6,2} \) is
\[
(P, s) \in \{(12/5, 7/5), (-4, 3), (-4, 1), (-36, -1), (-4, 1), (0, 1/2), (12, -1)\}.
\]
None of the \( P \)-coordinates is a fourth power, which finishes the proof. \( \square \)

6. Some quadratic polynomials which divide infinitely many quadrinomials \( f_{n,m,k}(a, x) \)

It is an interesting and highly non-trivial problem as to whether a given polynomial \( h \in \mathbb{Q}[x] \) divides infinitely many quadrinomials. This problem was addressed by L. Hajdu and R. Tijdeman in [3]. They prove that the polynomial \( h(x) \) divides infinitely many quadrinomials if either \( h \) divides two different quadrinomials with the same sequence of exponents, or \( h \) lies in the set
\[
\{h \in \mathbb{Q}[x] : \exists q \in \mathbb{Q}[x], r \in \mathbb{N}_+ : \deg q \leq 3 \text{ and } h(x)|q(x^r) \text{ over } \mathbb{Q}\}.
\]
In this section we are interested in finding polynomials \( h \in \mathbb{Z}[x] \) such that \( h \) divides infinitely many quadrinomials \( f_{n,m,k}(a, x) \) with \( a \in \mathbb{Z} \) and \( n, m, k \in \mathbb{N} \) satisfying \( n > m > k > 0 \). First, note that the theorem of Hajdu and Tijdeman is of little use here because our quadrinomials are of very special form. Moreover, we shall concentrate on polynomials \( h \) of degree 2. Of course each \( h \) of degree 2 divides infinitely many (general) quadrinomials but it is unclear whether there exists even one such \( h \) that divides infinitely many quadrinomials \( f_{n,m,k}(a, x) \). We are interested only in polynomials that do not divide \( x^p \pm 1 \) for any \( p \in \mathbb{N}_+ \). It is immediate that if \( k \equiv m \equiv n \equiv 0 \mod p \) then \( f_{n,m,k}(\pm 3, x) \) is divisible by \( x^p - 1 \), and if \( k \equiv m \equiv n \equiv p \mod 2p \) then \( f_{n,m,k}(3, x) \) is divisible by \( x^p + 1 \). During computer experiments, we observed that if \( h(x) \in S \), where
\[
S = \{x^2 - 2x + 2, x^2 + 2x + 2, x^2 + 3x + 3\},
\]
then \( h(x) \) divides infinitely many quadrinomials \( f_{n,m,k}(a, x) \) for certain values of \( a \) and specific sequences of exponents \( n, m, k \). Essentially, the aim of this section is to give a precise description of the sequences of exponents \( (n, m, k) \) such that \( f_{n,m,k}(a, x) \equiv 0 \mod h(x) \) for \( h \in S \). First, note that if \( h(x) = x^2 + px + q \in \mathbb{Z}[x] \) divides infinitely many quadrinomials \( f_{n,m,k}(a, x) \), this immediately implies via Corollary [5,2] that the (exponential) Diophantine equation
\[
A_n(p, q) + A_m(p, q) + A_k(p, q) = 0,
\]
(3) together with the condition \( B_n(p, q) + B_m(p, q) + B_k(p, q) \neq 0 \), has infinitely many solutions in positive integers \( n, m, k \) with \( n > m > k \). We see that if \( h(x) \in S \)
and \( \theta_1, \theta_2 \) satisfy \( h(\theta_1) = 0 \) then \( \theta_1/\theta_2 \) is a root of unity. This is not a coincidence because if the quotient of the roots of the polynomial \( x^2 + px + q = 0 \) is not a root of unity then the equation has only finitely many solutions in positive integers \( n, m, k \) with \( n > m > k \). One can check that if \( h \in \mathbb{Z}[x] \) is a polynomial of degree 2 for which the quotient of roots is a root of unity and there is a non-zero integer \( a \) such that \( h(x)|f_{n,m,k}(a,x) \), then \( h \in S \). This property allows us to compute \( A_n(p,q) \) explicitly. We gather these computations in the following:

**Lemma 6.1.** We have the following equalities

1. If \((p, q) = (-2, 2)\) then
   
   \[
   A_{4n}(-2, 2) = 0, \quad A_{4n+1}(-2, 2) = (-1)^n 2^{2n}, \quad A_{4n+2}(-2, 2) = A_{4n+3}(-2, 2) = (-1)^n 2^{2n+1}.
   \]

2. If \((p, q) = (2, 2)\) then
   
   \[
   A_{4n}(2, 2) = 0, \quad A_{4n+1}(2, 2) = (-1)^n 2^{2n}, \quad A_{4n+2}(2, 2) = -A_{4n+3}(2, 2) = (-1)^{n+1} 2^{2n+1}.
   \]

3. If \((p, q) = (3, 3)\) then
   
   \[
   A_{6n}(3, 3) = 0, \quad A_{6n+3}(3, 3) = 2(-1)^n 3^{3n+1}, \quad A_{6n+4}(3, 3) = (-1)^n 3^{3n+2}, \quad A_{6n+5}(3, 3) = (-1)^n 3^{3n+2}.
   \]

**Proof.** The computations are immediate and follow from expressing \( A_n(p,q) \) in closed form, using the formulae at (1). For example, we find

\[
A_n(-2, 2) = 2^\frac{n}{2} \sin \left(\frac{n\pi}{4}\right).
\]

Similarly,

\[
A_n(2, 2) = (-1)^{n+1} 2^\frac{n}{2} \sin \left(\frac{n\pi}{4}\right), \quad A_n(3, 3) = 2 \cdot 3^{\frac{n-1}{2}} \sin \left(\frac{5\pi n}{6}\right).
\]

It is now straightforward to characterize sequences of exponents \((n, m, k)\) and values of \( a \) such that \( f(a, x) \equiv 0 \mod h(x) \), where \( h \in S \).

**Theorem 6.2.** Let \( n, m, k \in \mathbb{N} \) with \( n > m > k \), \( \gcd(n,m,k) = 1 \), \( a \in \mathbb{Z} \setminus \{0\} \), and suppose that \( x^n + x^m + x^k + a \equiv 0 \mod h(x) \) for \( h \in S \).

1. If \( h(x) = x^2 - 2x + 2 \) then there exists a positive integer \( s \) such that \( (n, m, k) = (4s+5, 4s+3, 4s+2) \) and \( a = (-1)^s \cdot 3 \cdot 2^{2s+1} \).

2. If \( h(x) = x^2 + 2x + 2 \) then \( k \equiv 0 \mod 2 \) and the following holds:
   
   (a) If \( k = 4s - 2 \) for some \( s \in \mathbb{N} \) then \( m = 4s - 1, n = 4s + 4t \) with \( t \geq 0 \) and
   
   \[
a = 2^{2s-1}(-1)^{s+t+1}(2^{2t+1} + (-1)^{t+1}).
   \]

   (b) If \( k = 4s \) for some \( s \in \mathbb{N} \) then \( m = 4t + 2, n = 4t + 3 \) with \( t \geq s \) and
   
   \[
a = 2^{2s}(-1)^{t+1}(2^{2t-2s+1} + (-1)^{t-s}).
   \]

3. If \( h(x) = x^2 + 3x + 3 \) then \( k \equiv 0, 4 \mod 6 \) and the following holds:
   
   (a) If \( k = 6s - 2 \) for some \( s \in \mathbb{N} \) then \( m = 6s - 1, n = 6s + 6t \), with \( t \geq 0 \) and
   
   \[
a = 3^{3s-1}(-1)^{s+t+1}(3^{3t+1} + (-1)^{t+1}).
   \]

   (b) If \( k = 6s \) for some \( s \in \mathbb{N} \) then \( m = 6t + 4, n = 6t + 5 \), with \( t \geq s \) and
   
   \[
a = 3^{3s}(-1)^{t+1}(3^{3t-3s+2} + (-1)^{t-s}).
   \]
follows from the following fact: for any \( p, q \) to 2. A similar property holds for \((p, q) = (3, 3)\). We illustrate the proof in the case \((p, q) = (−2, 2)\).

First, observe that for \( r \neq 0 \mod 4 \), then the power of 2 dividing \( A_r(−2, 2) \) is exactly \( \lfloor \frac{r}{2} \rfloor \). Now if \( k \equiv 0 \mod 4 \), then equation \( 3 \) implies that \( A_n(−2, 2) + A_m(−2, 2) = 0 \), and clearly neither \( m \) nor \( n \) can be divisible by 4 (otherwise \( n, m, k \) are each divisible by 4, contradicting coprimality). Thus \( \lfloor \frac{r}{2} \rfloor = \lfloor \frac{m}{2} \rfloor \), forcing \( (n, m) = (4s + 3, 4s + 2) \) for some integer \( s \), which cannot satisfy \( 3 \). If \( k \equiv 1 \mod 2 \), then \( \lfloor \frac{r}{2} \rfloor > \lfloor \frac{m}{2} \rfloor \), which leads to an impossible congruence mod 2 \( \frac{r}{2} \) in equation \( 3 \). If \( k \equiv 2 \mod 4 \) then \( k = 4s + 2 \), say, and \( \lfloor \frac{r}{2} \rfloor \geq \lfloor \frac{m}{2} \rfloor \geq 2s + 1 \). Equation \( 3 \) forces \( \lfloor \frac{r}{2} \rfloor = 2s + 1 \), that is, \( m = 4s + 3 \). This gives \( A_n(−2, 2) + (−1)^{2s+2} = 0 \), so that \( n = 4s + 5 \). The value of \( a \) follows from \( a = −B_n(p, q) − B_m(p, q) − B_k(p, q) \), coupled with the identity \( B_n(p, q) = −qA_{n−1}(p, q) \).

A similar, but slightly more tedious, analysis can be performed for \((p, q) \in \{(2, 2), (3, 3)\} \), and we omit the details. \( \square \)

**Remark 6.3.** Note that if \( h(x) = x^2 + 2x + 2 \), then for each \( s \in \mathbb{N} \) the polynomial \( H(x) = 2^s h(x/2^s) = x^2 + 2^{s+1}x + 2^{2s+1} \) divides \( f(a, x) \) for infinitely many \( a \) and sequences of exponents \((n, m, k)\). A similar property holds for the polynomial \( H(x) = 3^s h(x/3^s) = x^2 + 2^{s+1}x + 3^{2s+1} \), where \( h(x) = x^2 + 3x + 3 \). This observation follows from the following fact: for any \( t \in \mathbb{Z} \) we have \( A_n(tp^2, q) = t^{n−1} A_n(p, q) \). Moreover, for \((p, q) \in \{(−2, 2), (2, 2)\} \) the only prime which divides \( A_n(p, q) \) is equal to 2. A similar property holds for \((p, q) = (3, 3) \). Indeed, in this case the only prime dividing \( A_n(p, q) \) is equal to 3 provided that \( n \neq 3 \mod 6 \), in which case the prime 2 can also occur, although only to exponent 1. All these remarks follows from Lemma 6.1.

We finish this section with the following conjecture.

**Conjecture 6.4.** Let \( h \in \mathbb{Z}[x] \) of degree \( \geq 3 \) and suppose that \( h \) does not divide any polynomial of the form \( x^n − 1 \). Then there are only finitely many triples of exponents \( n, m, k \in \mathbb{N} \) with \( n > m > k \) and integers \( a \) such that \( f_{n,m,k}(a, x) \equiv 0 \mod h(x) \).

### 7. Numerical results, open questions and conjectures

In this section we collect some numerical computations, questions and conjectures concerning various aspects of reducibility of the quadrinomial \( f(a, x) \) with \( a \in \mathbb{Q} \). We start with a very natural question concerning the existence of multiple roots of \( f_{n,m,k}(a, x) \). Note the example

\[
x^4 + x^3 + x + 1 = (x + 1)^2(x^2 − x + 1)
\]

which shows that there exists \( a \) such that \( f(a, x) \) has a double root. In fact, \( x = −1 \) is a double root of \( x^{m+k} + x^m + x^k + 1 \), where \( m, k \) are odd integers, \( m > k \). A question arises as to whether one can find other \( a \in \mathbb{Q} \) with \( f_{n,m,k}(a, x) \) having a multiple root. We have not found any examples. This motivates the following.

**Conjecture 7.1.** If \( a \in \mathbb{Q}^* \) and \( f_{n,m,k}(a, x) = x^n + x^m + x^k + a \) with \( n > m > k \) has multiple factors, then \( a = 1 \) and \( (n, m, k) = (d(t + u), dt, du) \) for \( d \geq 1 \) and odd integers \( t > u \geq 1 \).
Although we were unable to prove this conjecture we can offer a slightly weaker result.

**Theorem 7.2.** There does not exist \( a \in \mathbb{Q}^* \) such that the polynomial \( f_{n,m,k}(a,x) \) has a root of multiplicity \( \geq 3 \).

In view of the following Lemma, it suffices to assume that \( (n,m,k) = 1 \). We state the Lemma in the very concrete form needed for our purposes, although it can clearly be stated and proved in a more general setting.

**Lemma 7.3.** Suppose that \( f_{dn, dm, dk}(a,x) = x^{dn} + x^{dm} + x^{dk} + a \) has a root of multiplicity \( N \). Then \( f_{n,m,k}(a,x) \) has a root of multiplicity \( N \).

**Proof.** Let the \( n \) roots (all non-zero) of \( f_{n,m,k}(a,x) = 0 \) in \( \mathbb{C} \) be \( \{r_1, \ldots, r_n\} \). Let \( z \) be a fixed \( d \)-th root of unity, for example \( z = e^{2i\pi/d} \). Then the \( dn \) roots of \( f_{dn, dm, dk}(x) = f_{n,m,k}(x^{d}) = 0 \) are given by

\[
    z^{i_1 r_1^{1/d}}, z^{i_2 r_2^{1/d}}, \ldots, z^{i_n r_n^{1/d}}, \quad i_k = 0, \ldots, d-1,
\]

for a fixed \( d \)-th root \( r_i^{1/d} \) of \( r_i \), \( i = 1, \ldots, n \). Clearly for fixed \( r_i^{1/d} \), the \( d \) roots \( z^{i_k r_i^{1/d}}, i_k = 0, \ldots, d-1 \), are distinct. If we assume \( f(x^d) \) has a root of multiplicity \( N \), then without loss of generality the multiple root \( \rho \) satisfies

\[
    \rho = z^{i_1 r_1^{1/d}} = z^{i_2 r_2^{1/d}} = \ldots = z^{i_N r_N^{1/d}}.
\]

On raising to the \( d \)-th power,

\[
    r_1 = r_2 = \ldots = r_N = (\rho^d),
\]

so that \( f(x) \) has a root of multiplicity \( N \).

Without loss of generality, therefore, we may assume henceforth that \( (n,m,k) = 1 \).

**Proof of Theorem 7.2.** Suppose \( f_{n,m,k}(x) \) has a triple root \( \theta \). Certainly \( \theta \) is a double root of the first derivative \( f'_{n,m,k}(x) = nx^{n-1} + mx^{m-1} + kx^{k-1} \), so a double root of \( g(x) = nx^{n-k} + mx^{m-k} + k \). Accordingly, \( \theta \) is also a root of \( g'(x) = n(n-k)x^{n-k-1} + m(m-k)x^{m-k-1} \). This latter gives

\[
    \theta^{n-k} = -\frac{m(n-k)}{n(n-k)}.
\]

Now

\[
    g(\theta) = \theta^{m-k}(n\theta^{n-m} + m) + k = 0,
\]

so that

\[
    \theta^{m-k}(-\frac{m(n-k)}{n-k} + m) + k = 0,
\]

that is,

\[
    \theta^{m-k} = -\frac{k(n-k)}{m(n-m)}.
\]

Let \( d = \gcd(n-m, m-k) \), and set \( \Theta = \theta^d \). Then

\[
    \Theta^{n-m} = -\frac{m(n-k)}{n(n-k)} < 0, \quad \Theta^{m-k} = -\frac{k(n-k)}{m(n-m)} < 0,
\]

and

\[
    \Theta^{rac{n-m}{d}} = \Theta^{rac{m-k}{d}} = -\frac{k(n-k)}{m(n-m)} < 0.
\]
with \( \gcd\left(\frac{n-m}{d}, \frac{m-k}{d}\right) = 1 \). Thus \( \Theta \) itself is rational, \( \Theta < 0 \), and

\[
(6) \quad \frac{n-m}{d} \equiv 1 \mod 2, \quad \frac{m-k}{d} \equiv 1 \mod 2. 
\]

Raising (4) to the \( \frac{m-k}{d} \)-th power and (5) to the \( \frac{n-m}{d} \)-th power gives

\[
(7) \quad \left(\frac{m(m-k)}{n(n-k)}\right)^{\frac{m-k}{d}} = \left(\frac{k(n-k)}{m(n-m)}\right)^{\frac{n-m}{d}}. 
\]

This can be written

\[ k^{\frac{n-m}{d}}(n-k)^{\frac{n-m}{d}} = m^{\frac{m-k}{d}}(m-k)^{\frac{m-k}{d}}(n-m)^{\frac{n-m}{d}}, \]

from which it follows that if \( p \) is a prime with \( p \mid (n,k) \), then \( p \mid m \). Accordingly we may assume that \( (n,k) = 1 \). Write

\[
\frac{m(m-k)}{n(n-k)} = \frac{r}{s}, \quad (r,s) = 1, \quad \frac{k(n-k)}{m(n-m)} = \frac{u}{v}, \quad (u,v) = 1. 
\]

From (7),

\[
\frac{r}{s} = \left(\frac{a}{b}\right)^{\frac{n-m}{d}}, \quad \frac{u}{v} = \left(\frac{a}{b}\right)^{\frac{m-k}{d}}, \quad (a,b) = 1,
\]

so that

\[ r = a^{\frac{n-m}{d}}, \quad s = b^{\frac{n-m}{d}}, \quad u = a^{\frac{m-k}{d}}, \quad v = b^{\frac{m-k}{d}}. \]

Thus we obtain for integers \( t, w \):

\[
(8) \quad m(m-k) = a^{\frac{n-m}{d}} t, \quad n(n-k) = b^{\frac{n-m}{d}} t, \quad k(n-k) = a^{\frac{m-k}{d}} w, \quad m(n-m) = b^{\frac{m-k}{d}} w. 
\]

Now

\[ (b^{\frac{n-m}{d}} - a^{\frac{n-m}{d}}) t = n(n-k) - m(m-k) = (n-m)(n+m-k) > 0, \]

so that \( b > a \geq 1 \). Further, \( b^{\frac{n-m}{d}} \) divides \( n(n-k) \), and \( (n,n-k) = 1 \), so either \( b^{\frac{n-m}{d}} \) divides \( n \) or \( b^{\frac{n-m}{d}} \) divides \( n-k \).

**Case I:** \( b^{\frac{n-m}{d}} \mid n, \ b \nmid n-k. \)

Write \( n = b^{\frac{n-m}{d}} f N \), with \( f \geq 0 \) and \( (b,N) = 1 \). Then the second equation at (8) gives \( t = b^{f} N(n-k) \), and substituting into the three other equations at (8),

\[
m(m-k) = a^{\frac{n-m}{d}} b^{f} N(n-k), \quad k(n-k) = a^{\frac{m-k}{d}} w, \quad m(n-m) = b^{\frac{m-k}{d}} w. 
\]

The second equation tells us \( b \nmid w \); and the third equation that \( b \mid m \). Write \( m = b^{g} M, \ g \geq 1, (b,M) = 1 \). Then the first and third equations give

\[
b^{g} M(b^{g} M - k) = a^{\frac{n-m}{d}} b^{f} N(n-k), \quad b^{g} M(b^{\frac{n-m}{d}} f N - b^{2} M) = b^{m-k} w. 
\]

Comparing powers of \( b \), the first equation gives \( f = g \). From the second, since \( \frac{n-m}{d} + f > g \), we deduce \( 2g = \frac{m-k}{d} \), which contradicts (6).

**Case II:** \( b^{\frac{n-m}{d}} \mid n-k, \ b \nmid n. \)

Write \( n-k = b^{\frac{n-m}{d}} f N \), with \( f \geq 0 \) and \( (b,N) = 1 \). Then the second equation at (8) gives \( t = nb^{f} N \), and substituting into the remaining three equations at (8),

\[
m(m-k) = a^{\frac{n-m}{d}} nb^{f} N, \quad k\theta^{\frac{n-m}{d}} + f N = a^{\frac{m-k}{d}} w, \quad m(n-m) = b^{\frac{m-k}{d}} w. 
\]
Lemma 7.4. Let \( n > m \geq 1 \). The quadrinomial \( x^{3n} + x^{3m} + x^{n+m} - \frac{1}{27} \) splits into two irreducible factors in \( \mathbb{Q}[x] \), namely

\[
x^{3n} + x^{3m} + x^{n+m} - \frac{1}{27} = (x^n + x^m - \frac{1}{3})(x^{2n} - x^{n+m} + x^{2m} + x^n + n + \frac{1}{9}).
\]

Proof. The factorization follows from the identity \( a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) \) \( (a, b, c) = (x^n, x^n, -1/3) \). The first factor \( f_1(x) = x^n + x^m - \frac{1}{3} \) is irreducible, because \( 3x^n f_1(1/x) \) is an Eisenstein polynomial for the prime 3. Consider the second factor

\[
f_2(x) = x^{2n} - x^{n+m} + x^{2m} + x^n + \frac{m}{3} + \frac{1}{9} = (x^n + \omega x^m - \omega \frac{2}{3})(x^n + \omega^2 x^m - \omega^2),
\]

where \( \omega^2 + \omega + 1 = 0 \). It suffices to show that \( g(x) = x^n + \omega x^m - \frac{1}{3}, n > m \geq 1 \), is irreducible in \( \mathbb{Q}(\omega)[x] \). Equivalently, we prove \( h(x) = -3\omega x^m g(1/x) = x^n - 3\omega^2 x^{n-m} - 3\omega \) is irreducible in the ring \( \mathbb{Z}[\omega] \). Here, \( h(x) = x^n + \omega^2 \pi^2 x^{n-m} + \omega^2 \pi^2, \) where \( \pi^2 = -3, \) and \( \pi \) is a prime of \( \mathbb{Z}[\omega] \).

Suppose that \( h(x) = h_1(x) h_2(x) \), \( h_i(x) \in \mathbb{Z}[\omega][x] \). Since \( h(x) \equiv x^n \mod(\pi) \), unique factorization in the quotient ring implies

\[
\begin{align*}
h_1(x) &= x^j + \pi a_{j-1} x^{j-1} + a_{j-2} x^{j-2} + \ldots + a_1 x + a_0, \\
h_2(x) &= x^{n-j} + \pi (b_{n-j-1} x^{n-j-1} + b_{n-j-2} x^{n-j-2} + \ldots + b_1 x + b_0),
\end{align*}
\]

where without loss of generality we may suppose \( j \geq n - j \). If this inequality is strict, then the coefficient of \( x^{n-j} \) in the product \( h_1(x) h_2(x) \) is equal to

\[
\pi a_0 + \pi^2 a_1 b_{n-j-1} + \pi^2 a_2 b_{n-j-2} + \ldots + \pi^2 a_{n-j} b_0.
\]

However, on comparing with coefficients of \( h(x) \), this coefficient is a multiple of \( \pi^2 \), so that \( a_0 \equiv 0 \mod \pi \), which is impossible since \( a_0 b_0 = \omega \). We deduce that \( j = n - j \), i.e. \( n = 2j \). The coefficient of \( x^j \) in the product \( h_1 h_2 \) is now given by

\[

\pi a_0 + \pi^2 a_1 b_{j-1} + \pi^2 a_2 b_{j-2} + \ldots + \pi^2 a_{j-1} b_1 + \pi b_0.
\]

As before, this coefficient is a multiple of \( \pi^2 \), forcing \( a_0 + b_0 \equiv 0 \mod \pi \). Using \( a_0 b_0 = \omega \), we deduce \( a_0^2 \equiv -\omega \equiv -1 \mod \pi \), an impossible congruence.

Thus \( h(x) \) is irreducible in \( \mathbb{Z}[\omega][x] \), and \( f_2(x) \) is irreducible in \( \mathbb{Q}[x] \). \( \square \)
This result allows construction as follows of an irreducible quadrinomial \( f_{n,m,k}(a, x) \) such that \( f_{n,m,k}(a, x^3) \) is reducible, and each irreducible factor has degree \( \geq 3 \).

**Corollary 7.5.** Let \( f(x) = x^N + x^M + x^{\frac{N+M}{3}} - \frac{1}{27} \) with \( NM \not\equiv 0 \pmod{3} \) and \( N + M \equiv 0 \pmod{3} \). Then the polynomial \( f(x) \) is irreducible over \( \mathbb{Q} \), and \( f(x^3) \) is reducible over \( \mathbb{Q} \) with each irreducible factor of degree \( \geq N \).

**Proof.** Note that \( f(x^3) = f_{3N,3M,N+M}(-1/27, x) \), so that the second assertion follows from Lemma [7.4]. Suppose that \( f(x) \) is reducible over \( \mathbb{Q} \), i.e. \( f(x) = h_1(x)h_2(x) \) with \( h_1, h_2 \in \mathbb{Q}[x] \) with \( \deg h_i > 1 \). Then \( f(x^3) = h_1(x^3)h_2(x^3) = f_1(x)f_2(x) \), where \( f_1, f_2 \) are the irreducible factors of \( f(x^3) \) given in the statement of Lemma [7.4]. Thus \( h_1(x^3) = bf_i(x) \) for some \( i \in \{1, 2\} \) and \( b \in \mathbb{Q} \setminus \{0\} \). However, from the assumptions on \( N, M \), the left hand side is invariant under the mapping \( x \mapsto \omega x \), but the right hand side is not. Thus \( f(x) \) is irreducible. \( \square \)

We list in the table below other examples discovered, and include for completeness the previously known six examples.
We finish with a conjecture related to Walsh’s second question for quadrinomials defined over finite fields. We expect in this case that the following strong result is true.
Conjecture 7.6. Let $\mathbb{F}_q$ be a finite field with $q = p^e$ elements and fix a positive integer $M$. Then there is a constant $C$ such that for each integer $N \geq C$ there exists $a \in \mathbb{F}_q$ and triple $n, m, k \in \mathbb{N}_+$ ($n > m > k$) with $n > N$ such that each irreducible factor of the quadrinomial $f(a, x)$ is of degree $\geq M$.

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