Superconductivity of neutral modes in quantum Hall edges

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(Dated: October 22, 2021)

Edges of quantum Hall phases give rise to a multitude of exotic modes supporting quasiparticles of different values of charge and quantum statistics. Among these are neutralons (chargeless anyons with semion statistics), which were found to be ubiquitous in fractional quantum Hall matter. Studying and manipulating the neutral sector is an intriguing and interesting challenge, all the more so since these particles are accessible experimentally. Here we address the limit of strongly-interacting neutralons giving rise to neutralon superconductivity, where pairing is replaced by a quarteting mechanism. We discuss several manifestations of this effect, realizable in existing experimental platforms. Furthermore, this superconducting gapping mechanism may be exploited to facilitate the observation of interference of the accompanying charged anyons.

Introduction. A two dimensional electron gas in a fractional quantum Hall (FQH) state can host exotic anyonic quasiparticles and boundary modes [11–5]. The boundary modes may be used as building blocks for realizing anyonic transport and designing interference experiments to observe fractional quantum statistics beyond bosons and fermions [6]. Major recent experimental developments involve the observation of Hong-Ou-Mandel anyonic correlations as well as a demonstration of anyonic interferometry [13–20].

Some of the exotic boundary modes arise as renormalized bare edge modes. Examples include neutral modes which have been experimentally detected through thermometry [21] and the generation of upstream charge noise [22–26]. The paradigmatic model of gapless neutral modes was introduced by Kane, Fisher, and Polchinski (KFP) [27] for the ν = 2/3 FQH edge which hosts counterpropagating ν = 1 and ν = 1/3 chiral bare modes [28, 29]. Random backscattering and Coulomb interaction between the modes drive the edge to a new low-energy fixed point that hosts a charge-2e/3 mode and a counterpropagating “upstream” neutral mode. In the original KFP model the neutral modes satisfy SU(2) symmetry, but more complex edges (for example, due to edge reconstruction) may give rise to more elaborate structures such as SU(3) symmetric modes [30].

So far these studies have focused on non-interacting neutral modes. Exploring the physics of interacting neutral mode system opens the door to new exotic phases. Here, we show that interaction within the neutral mode sector may give rise to “neutralon superconductivity”, relying on amalgamating (hereafter “pairing”) together a quartet of neutral quasiparticles.

The neutral modes are chiral, so in order to open a superconducting gap, a counterpropagating partner needs to be introduced [31]. Here, we take advantage of a recent material engineering breakthrough [32] and theoretically investigate a suitably designed FQH bilayer, where two counterpropagating copies of the ν = 2/3 FQH neutral mode appear, see also [33]. We consider the limit of weak neutralon-neutralon interactions. In that limit, in the presence of disorder-induced tunneling between the counterpropagating neutralons, Anderson localization is suppressed [34], and hence will not compete against opening a superconducting gap. Uniform backscattering competes with uniform pairing, both being marginal operators (for weak interactions). However, neutralons are charge dipoles, hence subject to a weak attractive density-density interaction which favors pairing. Interestingly, due to the semionic nature of neutralons, the pairing must involve four of them. This pairing conserves momentum and is marginally relevant even in the presence of edge disorder.

The novel type of superconductivity in the neutral sector has experimental manifestations that involve measurement of the charge modes. Tunneling of electrons across a quantum point contact bridging fractional quantum Hall states will generally excite neutral modes. When the neutral modes are gapped by pairing, electron tunneling is highly suppressed at low energies, which may be observed in the low-bias I – V characteristics and the shot noise Fano factor. Another signature involves a confined quantum dot or anti-dot geometry, where unpaired neutralons come at a cost of pairing energy, which is manifest in Coulomb blockade peak spacings [35].

Furthermore, our analysis concerns the design of anyonic interferometers. It is known that gapless neutralons may act as “which-path” detectors. Their ubiquity [26, 50] leads to dephasing, hence suppression of interference [37]. We therefore propose that when neutralons condense to a gapped state, the sensitivity of anyonic interferometry will improve. Below, we outline interferometer designs that could be used to gap out the harmful neutralons while leaving the charge excitations gapless, thus improving interferometer performance. The observation of superconducting neutralon phase may have far-reaching impact on the quest of anyonic interference.

Our theoretical analysis follows these steps: We will
dau levels, our edge theory therefore consists of four chi-

Both the top and bottom components have an interface described by the KFP theory for the $\nu = 2/3$ FQH state. The filling fractions are chosen in a way that produces opposite spin polarizations for the top and bottom layers. b) Edge spectrum of the clean limit. Correlated intralayer backscattering process (green curved arrows) that contributes to pairing of neutralons at low energies. c) The characteristic temperature scales and the two-step RG flow towards low temperature (large white arrows). The bare edge modes at high temperatures on the left, with Coulomb interaction (yellow wiggly line), random backscattering (solid line with a cross), and correlated intralayer backscattering (curved arrows connected by a dashed line). As temperature is lowered below $D_{KFP}$, the edge is described by the KFP low-energy theory (middle panel) of $2e/3$ charge modes and disordered neutral modes with a pairing interaction (green wiggly double line). At temperatures below $\Delta_n$, quartets of neutralons become gapped (blue ellipse) and only the charge modes remain (right panel). Their opposite spin polarization prevents backscattering.

Consider a bilayer of counterpropagating neutral modes. The latter are represented by bosonic fields as described in the original work by KFP [27]. The corresponding action, Eq. (3), is derived without inter-layer interactions. We then include weak inter-layer perturbations, pairing and backscattering, see Eq. (5). Employing the perturbative renormalization group, we identify the parameter regime where pairing becomes the most relevant perturbation. The energy scale at which such pairing becomes non-perturbative (strongly coupled) is the superconducting gap $\Delta_n$, at energies below which the pairing interaction leads to a quartet superconductivity. Finally, we discuss the experimental manifestations of the superconducting phase.

Model of a single edge. We start from the description of a single composite interface depicted in Fig. 1. Both the top and bottom components have an interface similar to a $\nu = 2/3$ edge. Assuming spin-polarized Landau levels, our edge theory therefore consists of four chiral bosonic fields $\phi_{+1/3,\uparrow}, \phi_{-1,\uparrow}, \phi_{-1/3,\downarrow}, \phi_{+1,\downarrow}$ where the subscripts indicate the chirality ("+$" denotes a right mover), charge, and spin. The imaginary time action is [27] ($a$ is a short-distance cutoff)

$$S^{(0)} = \int d\tau dx \frac{1}{4\pi} \left[ \partial_\tau \Phi K \partial_x \Phi + \partial_x \Phi V \partial_\tau \Phi \right]$$

$$+ \frac{1}{a} \int d\tau dx \sum_{i=1,2} \left[ \xi_i(x) e^{i\phi_i} \Phi_i^+ + \xi_i^\dagger(x) e^{-i\phi_i} \Phi_i \right],$$

where we introduced the 4-component vector $\Phi = (\phi_+, \phi_-)$ in this basis the matrix $V$ is almost block-diagonal [35], describing the velocities and short-range screened Coulomb interactions between the modes; the matrix $K = \text{diag}(3,-1,-3,1)$ describes the commutation relations [39]. We may use the same action to describe other interfaces such as those depicted in Fig. 2 [38]. On the second line of Eq. (1) we have included random intralayer backscattering of electrons between the counterpropagating 1/3 and 1 modes; here $c_{l(t)} = (-1)(3,1)$ and $\xi_l$ is a $\delta$-correlated random coefficient, $(\xi_l(x) \xi_{l'}^*(x')) = a^{-1} W_l \delta(x-x')$, with zero average. This term is a relevant perturbation under the renormalization group [40] (RG) and leads to a non-trivial renormalization of the edge theory [27]. In Eq. (1) we neglect inter-layer perturbations which will be included later, see Eq. (5).

Below a temperature scale $T \sim D_{KFP}$, the disorder strength $W_l$ becomes large [41] and the edge action can be diagonalized in terms of spinless neutralons and spinful charge-$2e/3$ modes given by the respective linear combinations (Fig. 1b),

$$\phi_{+,0} = \frac{3\phi_{+1/3,\uparrow} + \phi_{-1,\uparrow}}{\sqrt{2}}, \phi_{-,2/3,\uparrow} = \frac{\sqrt{3}}{2} [\phi_{+1/3,\uparrow} + \phi_{-1,\uparrow}],$$

and similarly for the bottom edge. Introducing $\phi = (\phi_{+2/3,\downarrow}, \phi_{-2/3,\downarrow}, \phi_{+0,\uparrow}, \phi_{-0,\downarrow})$, the low-energy action is

$$S_{KFP}^{(0)} = \int d\tau dx \frac{1}{4\pi} \left[ \partial_\tau \Phi_{KFP} i \partial_x \Phi + \partial_x \Phi_{KFP} \partial_\tau \Phi \right]$$

$$+ \frac{1}{a} \int d\tau dx \left[ \xi_l(x) e^{i\phi_{+,0}} + \xi_l^\dagger(x) e^{-i\phi_{+,0}} + \text{h.c.} \right],$$

with $\Phi_{KFP} = \text{diag}(1,-1,1,-1)$ and $V_{KFP}$ is a block diagonal matrix. The block diagonality of $V_{KFP}$ is a result of the random intralayer backscattering, which makes neutralon-chargon interactions $\partial_x \phi_{+,0} \partial_x \phi_{+,2/3}$ irrelevant [27]. However, $V_{KFP}$ includes a neutralon-neutralon interaction $v_{0,0}$ which is not irrelevant for layer-correlated disorder (considered below) [38]. The second line in Eq. (3) introduces random phases into the neutral sector but does not give rise to a gap [27].

Inter-layer tunneling. Let us next include weak inter-layer tunneling to the action $S_{KFP}^{(0)}$, Eq. (5). This introduces the leading (in the RG sense) perturbations in
the neutral sector: pairing [depicted in Fig. 1b], \( O_p = e^{i\sqrt{2}\phi_{\tau,\sigma}^a - \phi_{\tau,\sigma}^b} \) and backscattering, \( O_b = e^{i\sqrt{2}\phi_{\tau,\sigma}^a + \phi_{\tau,\sigma}^b} \).

In the absence of neutral-neutral interactions \( (v_{0,0} = 0) \), both operators have a scaling dimension \( \delta = 2 \) and they are thus marginal (to leading order) as homogeneous perturbations \( \epsilon \). However, for \( v_{0,0} \neq 0 \), one of the two operators is favored: for negative (positive) \( v_{0,0} \), pairing (backscattering) becomes relevant while backscattering (pairing) becomes irrelevant. The relevant pairing term gives rise to a gap in the neutralon spectrum, \( \Delta_n \approx D_{KFP}/|\lambda_p|^{v_0/(2|v_{0,0}|)} \), in the limit \( \lambda_p \ll |v_{0,0}|/v_0 \ll 1 \) where \( \lambda_p \) is the dimensionless pairing amplitude \( 38 \).

We show below that in the case when \( v_{0,0}/v_0 \) is comparable to the pairing and backscattering amplitudes, all three interactions get significantly renormalized but the general conclusion of a gap remains. The backscattering operator does not conserve momentum (unlike pairing), and is expected to be less relevant when the neutralons have a finite density (as depicted in Fig. 1b).

In the charge sector, backscattering is forbidden by spin conservation. The pairing of the charge modes is highly irrelevant \( 13 \) and also forbidden by charge conservation in the absence of an external superconductor \( 14 \).

Next we will analyze the interlayer pairing in the neutral sector. It is convenient to introduce the \( SU(2)_1 \) current operators \( 22 \) \( \tilde{J}_\tau \):

\[
\tilde{J}_\tau = \frac{1}{2\sqrt{2}} \partial_x \phi_{\tau,0}, \quad J_\tau^\pm = \frac{\pm i e^{\pm i\sqrt{2}\phi_{\tau,0}}}{2\alpha}, \quad \tau = t, b = +, - ,
\]

and \( J_\tau^z = J_\tau^x + i J_\tau^y \). In terms of the currents, we have \( O_p = 2\alpha J_t^x J_b^x \) and \( O_b = 2\alpha J_t^y J_b^y \). We can write the combined neutralon inter-layer Hamiltonian in the form

\[
H_{\text{p+bs}} = 2\pi v_0 \int dx \sum_{x,y,z} \lambda^z J_t^z J_b^z , \tag{5}
\]

where \( \lambda^z = \lambda_p + \lambda_b, \quad \lambda^y = \lambda_b - \lambda_p, \quad \lambda_p, \lambda_b \) are the dimensionless pairing and backscattering amplitudes and \( v_0 \) is the neutralon velocity. The neutralon density-density interaction from Eq. (4) is included in the ZZ term, \( \lambda_0^z = 2|v_{0,0}|/v_0 \). Upon reducing the bandwidth, these coupling constants get renormalized. In the absence of disorder [the second line in Eq. (3)], the perturbative RG equations for \( \lambda^z, \lambda^y \) are\( 10 \):\n
\[
\frac{d}{dl} \lambda^z = \lambda^y \lambda^z, \quad \frac{d}{dl} \lambda^y = \lambda^x \lambda^z, \tag{6}
\]

\[
\frac{d}{dl} \lambda^z = \lambda^x \lambda^y, \quad (l = \ln D_{KFP}/D) , \tag{7}
\]

where \( D \ll D_{KFP} \) is the reduced bandwidth. We solve the above equations for \( \lambda^i(D) \) with the initial condition \( \lambda(D_{KFP}) = (\lambda_p + \lambda_b, \lambda_b - \lambda_p, \lambda_b)^T \). We assume that pairing and backscattering are weak, so that \( |\lambda_0^z| > |\lambda^x|, |\lambda^y| \).

Then, the sign of \( \lambda_0^z \) determines the low-energy RG fixed point: when \( \lambda_0^z > 0 \), the fixed point corresponds to strong backscattering \( (\lambda^x = \lambda^y = \pm \lambda^z) \), while if \( \lambda_0^z < 0 \), the fixed point is of strong pairing type \( (\lambda^x = -\lambda^y = \pm \lambda^z) \).

Within each type, the fixed point is further determined by the sign of \( \lambda_b \) or \( \lambda_p \); for example, in the strong pairing case \( \lambda_p > 0 \) flows to \( (\lambda^x = -\lambda^y = -\lambda^z > 0) \) while \( \lambda_p < 0 \) flows to \( (\lambda^x = -\lambda^y = +\lambda^z < 0) \).

To estimate the strong coupling energy scale \( \Delta_n \), we set \( |\lambda^i(D_n)| \gg 1 \). We find \( \Delta_n \approx D_{KFP} \left( 2 |\lambda^x_0|/\lambda^y_0 \right)^{-1} \) in the limit \( \lambda_p \ll |\lambda_0^z| \ll 1 \) and \( \Delta_n \approx D_{KFP} e^{-\pi/2|\lambda_p|} \) in the limit \( |\lambda_0^z| \ll |\lambda_p| \ll 1 \).

At temperatures \( T \ll \Delta_n \), the neutral excitations are gapped and only the charge modes remain from Eq. (3). Next, we will show that the random terms \( \propto \xi_t(x), \xi_b(x) \) in Eq. (3) do not modify our conclusions.

Interpreting the current operators \( J_{t,b} \) as spin densities, the second line of Eq. (3) can be regarded as a random “in-plane magnetic field”: the Hamiltonian corresponding to Eq. (3) reads

\[
H_{\text{neutral}} = 2\pi v_0 \int dx \left( \frac{1}{3} v_0 J_t^x + \xi_t(x) J_t^+ + \xi_b(x) J_b^- \right). \tag{8}
\]

The random magnetic field can be cancelled by the following gauge transformation, that preserves the \( SU(2)_1 \) algebra \( 15 \) for \( \tau = t, b \),

\[
J_\tau^+ = S_\tau^z J_\tau^+ + \frac{1}{8\pi} \epsilon^{ijk} [S_\tau^j \partial_x S_\tau^i]^j, \tag{9}
\]

where \( S_\tau(x) \) is a suitably chosen \( 38 \) real orthogonal matrix. For generic disorder, Eq. (9) does not keep the pairing term invariant and finding the ground state configuration is difficult. However, in the simple and realistic case of layer-correlated disorder, \( \xi_t = \xi_b = \xi \) we have \( 38 \)

\[
J_t^+ \eta J_b = \tilde{J}_t^+ \eta \tilde{J}_b, \quad \text{where} \quad \eta = \text{diag}(1, -1, -1). \tag{10}
\]

Thus, the rotation (10) makes the Hamiltonian independent of disorder,

\[
H_{\text{neutral}} + H_{\text{pairing}} = 2\pi v_0 \int dx \left[ \frac{1}{3} \sum_{\tau=t,b} \tilde{J}_\tau^z + \lambda \tilde{J}_\tau^+ \eta \tilde{J}_\tau^- \right], \tag{11}
\]

as long as we have \( \lambda = (\lambda_t, -\lambda_b, -\lambda) \) in Eq. (9). We can therefore use Eqs. (10)–(7), derived in the absence of disorder, to study Eq. (11). With \( \lambda > 0 \), we find a strong-pairing RG fixed point which preserves the direction of the vector \( \lambda \). We expect that the fixed point with \( \lambda < 0 \) is similarly stable to disorder \( 38 \).

We have shown that, under certain assumptions, the disorder term in Eq. (8) can be gauged away and the same low-energy fixed points as in the clean system can be reached. When \( \lambda_0^z < 0 \), we identified two stable strong-pairing fixed points corresponding to \( \lambda^x > 0 \) and \( \lambda^x < 0 \).
Next, we study the low-energy properties of the charge excitations near a fixed point where the neutralons are paired.

**Experimental manifestations.** The gapping of neutral modes at low energies has a number of implications to transport experiments. Signatures of neutral mode gap can be found in tunneling across a QPC, see Fig. 2. Tunneling of fractional charge between the charge-2/e/3 eigenmodes at low bias voltage may be impeded in several ways depending on the filling factors of the left and right sides of the QPC as well as the filling \((\nu_L, \nu_R)_{M}\) of the middle section. Most conducive to fractional charge tunneling is having fractional \((\nu_L, \nu_R)_{M}\) (see Fig. 2b); in this case tunneling of 2e/3 and e/3 quasiparticles is allowed. The latter involves the gapped neutralons and is thus suppressed (similarly to the case of charge-\(e\) tunneling discussed below) but the former is not. Indeed, the tunneling operator \(O_{2/3,↑} = e^{-\sqrt{\frac{2}{3}} \phi_{-2/3,↑,L}} e^{i \sqrt{\frac{2}{3}} \phi_{-2/3,↑,R}}\) creates (annihilates) a charge-2e/3 eigenmode on the right (left) side of the QPC. The scaling dimension of \(O_{2/3,↑}\) is \(\delta = 2/3\) and the tunneling current shows the corresponding zero-bias anomaly, \(I \propto V^{2\delta - 1}\) (keeping \(eV \gg T\)). The fractional tunneling charge also has a noise signature [8, 47, 48]: tunneling charges \(2e/3\) leads to a shot noise Fano factor 2/3.

Tunneling is much more restricted when the middle region consists of an integer filling fraction state, c.f. \(\nu_m\) in the middle section of Fig. 2b. In this case, only electrons (charge-\(e\)) are allowed to tunnel through the middle section. However, tunneling single electrons would excite the neutral modes and therefore come at the high energy cost of order \[35\] \((voltage bias \ eV \ll \Delta_n). Tunneling of a pair of electrons (3 charge-2e/3 quasiparticles) does not excite the neutrals and is allowed. (Also, tunneling of a “Cooper pair” of counterpropagating neutralons would be allowed but will not transfer charge.) Tunneling of a pair of electrons has a scaling dimension \(\delta = 4\), suppressing the tunneling current at low bias, \(I \propto V^7\). Thus, when the tunnel barrier (in either bottom or top layer) has an integer filling fraction state, the low-bias tunneling current becomes highly suppressed. The tunneling current shot noise Fano factor in this case is expected to be 2, yet its observation may be challenging due to the smallness of the current. Gapped neutralons cannot propagate along the edge and thus are not expected to produce noise.

A complementary signature of neutralon pairing can be found in Coulomb blockaded quantum dots or antidots [35]. In [38] we show that neutralon pairing gap leads to a unique signature in the Coulomb blockade peak spacings.

The bilayer geometry where the neutral modes become gapped allows one to consider anyonic Mach-Zehnder or Fabry-Perot interferometers free of neutral mode dephasing, c.f. Fig. 2. As discussed above, the configurations with fractional filling factors \((\nu_L, \nu_R)_{M}\) depicted in Fig. 2a are most suitable for constructing such an interferometer since they allow tunneling of fractional charge quasiparticles. The size of the neutral mode gap \(\Delta_n\) imposes some limitations to the interferometer design. For example, the distance \(L\) between the QPCs should be large enough, \(L \gg v_0/\Delta_n\), and the bias voltage low enough, \(eV \ll \Delta_n\), so that neutral modes cannot propagate through the interferometer causing dephasing. To observe interference, the length of the edge should not exceed the full incoherence length scale \[35\].

**Discussion.** We showed that counterpropagating neutral modes in a suitably designed FQH interface can be renormalized to a new type of superconducting phase with a neutralon quartet pairing. We focused on engineered bilayer interfaces whose edge structure is similar to the \(\nu = 2/3\) KFP edge theory [27]. In this case, the neutralons are semions and the superconductivity arises from neutralon quarteting. We expect our mechanism to also apply to reconstructed edges with...
emergent chiral modes \[30\] and other filling fractions, as long as these edges come with counterpropagating neutral modes. With different types of edge structures other unconventional neutralon statistics may arise, and we anticipate even more exotic (superconducting) phases of strongly-interacting neutralon matter.

Acknowledgements. We thank Jinhong Park for useful discussions. M.G. was supported by the Israel Science Foundation (Grant No. 227/15) and the US-Israel Binational Science Foundation (Grant No. 2016224). Y.G. was supported by DFG RO 2247/11-1, MI 658/10-2 and CRC 183 (project C01), the Minerva Foundation, the German Israeli Foundation (GIF I-1505-303.10/2019), the Helmholtz International Fellow Award, and by the Italia-Israel QUANTRA grant.
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In this Supplementary Material, we present details of the actions (1) and (3), the RG equations (6)–(7), the orthogonal transformation (9), discuss the neutralon pairing term and its signatures in a quantum antidot, and show alternative QPC designs to Fig. 2.

A. V-matrices

In this Section, we give explicit expressions for the $V$-matrices introduced in the main text. We also discuss the lowest-order RG equations for the pairing term, valid in the limit of relatively strong neutralon-neutralon interaction.

The $V$-matrix in Eq. (1) is given by

$$
V = \begin{pmatrix}
\mathbf{u} & \mathbf{v}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}^T & \mathbf{v}
\end{pmatrix},
$$

where $u_1$ and $v_1$ are the velocities of the $\nu = 1/3$ and $\nu = 1$ modes and $v_{1/3,1}$ is their interaction strength. We assume that these quantities are the same for both top and bottom layers. The $2 \times 2$ matrix $\mathbf{u}$ characterizes the inter-layer repulsive interactions; we assume its matrix elements are small in comparison to the intralayer terms $\mathbf{v}$, and the RG flow is analogous to the one in KFP theory [27].

At the KFP fixed point, in Eq. (5) we have

$$
V_{KFP} = \begin{pmatrix}
\begin{pmatrix}
v_{2/3} & v_{2/3,3/3}
v_{2,3/3} & v_{2/3}
\end{pmatrix} & 0 \\
0 & \begin{pmatrix}
v_0 & v_{0,0}
v_{0,0} & v_0
\end{pmatrix}
\end{pmatrix},
$$

where $v_0$ is the neutral mode velocity. Due to the randomness in the neutral sector, the interactions that couple to the neutralons are generally irrelevant perturbations and can be left out. We will keep the neutral-neutralon interaction $v_{0,0}$ which, in the case of layer-correlated disorder, is not irrelevant. The bare value of this interaction is $v_{0,0} = \frac{1}{2}(u_{1/3} + u_1 - 2u_{1/3,1})$ and can be positive (repulsion) or negative (attraction). In most designs we have $1/3$ and $1$ modes closest together so that $u_{1/3,1} > u_1$, $u_{1/3}$ and therefore $v_{0,0} < 0$ would be expected. The neutralons are charge dipoles so their attraction is not entirely surprising. The sign determines the relevant gap opening perturbation in the neutral sector and $v_{0,0} < 0$ makes pairing the relevant perturbation. In the charge sector, $v_{2/3}$ is the velocity of both $\nu = 2/3$ charge modes and $v_{2/3,2/3}$ denotes their interaction strength. [Like $v_{0,0}$, also $v_{2/3,2/3}$ can be obtained from the matrix $\mathbf{u}$ with the help of Eq. (2) of the main text.]

We can diagonalize the neutral sector $V$-matrix with the rotation

$$
\begin{pmatrix}
\phi_{+,0} \\
\phi_{-,0}
\end{pmatrix} = \begin{pmatrix}
\cosh \chi & \sinh \chi \\
\sinh \chi & \cosh \chi
\end{pmatrix} \begin{pmatrix}
\phi_{+,0} \\
\phi_{-,0}
\end{pmatrix} = e^{i\sqrt{2}([\phi_{+,0}-\phi_{-,0}])},
$$

which yields a diagonal $V$-matrix with equal velocities $\sqrt{v_{0,0}^2 - v_{0,0}^2}$ for the modes $\phi_{\pm}$. In the new basis, the pairing operator is

$$
O_p = e^{i\sqrt{2}\phi_{+,0}-\phi_{-,0}} = e^{i\sqrt{2}(\cosh \chi - \sinh \chi)[\phi_{+,0}-\phi_{-,0}]},
$$

and its scaling dimension is $\delta = 2(\cosh \chi - \sinh \chi)^2$. In the limit of weak interaction, $\chi \approx -v_{0,0}/(2v_0)$ and $\delta \approx 2(1 + \frac{v_{0,0}}{2v_0})$. The pairing is relevant, $\delta < 2$, when $v_{0,0} < 0$ (attractive interaction). The backscattering operator $O_b = e^{i\sqrt{2}([\phi_{+,0}+\phi_{-,0}])}$ has $\delta = 2(\cosh \chi + \sinh \chi)^2$ and is irrelevant when $v_{0,0} < 0$.

Ignoring the renormalization of $v_{0,0}$, the RG equation for the dimensionless pairing amplitude $\lambda_p$ is

$$
\frac{d}{dt} \lambda_p = (2 - \delta) \lambda_p
$$

and thus $\lambda_p(l) = e^{(2-\delta)l} \lambda_p(0)$. The strong coupling scale $l_\Delta$ is found from $\lambda_p(l_\Delta) \sim 1$. Writing $D = D_{KFP} e^{-l}$, we find

$$
\Delta_n \sim D_{KFP} |\lambda_p(0)|^{-1/(\delta-2)} \sim D_{KFP} |\lambda_p(0)|^{-v_{0,0}/(2v_{0,0})} \sim D_{KFP} |\lambda_p(0)|^{1/|\lambda_0^5|}
$$

where we introduced $\lambda_0^5 = 2v_{0,0}/v_0$ and took $v_{0,0} < 0$ and $v_{0,0}/v_0 \ll 1$. 

SM1. SUPPLEMENTARY MATERIAL TO “SUPERCONDUCTIVITY OF NEUTRAL MODES IN QUANTUM HALL EDGES”
B. Solution of the RG equations (6)–(7)

In this Section, we give details on how to solve Eqs. (6)–(7) of the main text. For completeness, we replicate the equations below:

$$\frac{d}{dl} \lambda^x = \lambda^y \lambda^x, \quad \frac{d}{dl} \lambda^y = \lambda^x \lambda^y, \quad \frac{d}{dl} \lambda^z = \lambda^x \lambda^y, \quad (l = \ln D_{\text{KFP}} / D),$$

where $D < D_{\text{KFP}}$ is the reduced bandwidth and we take the initial condition $\lambda(D_{\text{KFP}}) = (\lambda_p + \lambda_b, \lambda_b - \lambda_p, \lambda_0)^T$.

The above equation [Eqs. (6)–(7) of the main text] can be solved after identifying the two integrals of motion, $(\lambda^x)^2 - (\lambda^y)^2 = c_x$ and $(\lambda^x)^2 - (\lambda^y)^2 = c_y$, with $c_x, y$ constants. For illustration, we provide the solution in two limits $|\lambda_p| \ll |\lambda_0|$ and $|\lambda_p| \gg |\lambda_0|$. To illustrate the first limit, we take initial condition $\lambda_0 |\ll| \lambda_0|$, we find $\lambda^x(l) = -\lambda^y(l) = -\sqrt{c_x} \sinh \left( \int \sqrt{c_x} - \frac{1}{\lambda_0} \right)$ and $\lambda^z(l) = \sqrt{c_x} \coth \left( \int \sqrt{c_x} - \frac{1}{\lambda_0} \right)$. We can define the strong coupling limit as $|\lambda^x(l_\Delta)| \approx |\lambda^z(l_\Delta)|$ which yields $l_\Delta \approx \frac{\ln 2}{\lambda_0 \lambda_0}$ in the perturbative regime, $\lambda_p |\ll| \lambda_0|$. From this, we obtain the scale $\Delta_n = D_{\text{KFP}} \exp^{-l_\Delta} \approx D_{\text{KFP}} (\frac{\ln 2}{\lambda_0 \lambda_0})^{- \frac{1}{\lambda_0 \lambda_0}}$. In the limit $|\lambda_p| \gg |\lambda_0|$, we find $\lambda^x(l) = -\lambda^y(l) = \lambda_p \sec(l |\lambda_p|)$ and $\lambda^z(l) = -|\lambda_p| \tan(l |\lambda_p|)$. Solving the strong-coupling condition yields $l_\Delta \approx \frac{\pi}{2 \lambda_0 \lambda_0}$ in the perturbative regime, $|\lambda_p| \ll 1$. From this, we obtain the scale $\Delta_n \approx D_{\text{KFP}} \exp^{-\pi / |\lambda_p|}$. In the limit $|\lambda_p| \sim |\lambda_0|$, the two expressions for $\Delta_n$ approximately agree, $\Delta_n \sim D_{\text{KFP}} \exp^{-c / |\lambda_p|}$ with $c \sim 1$.

C. Orthogonal transformation $S$

In this Section, we give more details on the transformation, Eq. (9) of the main text, that is used to gauge out the disorder term of the neutralons.

The orthogonal matrix $S_r$ is found by requiring that the linear-in-$\mathbf{j}_r$ terms cancel in the Hamiltonian $H_{\text{neutral}} + H_{\text{pairing}}$, see Eqs. (5), (8). One finds that $S_r$ satisfies the condition

$$\frac{1}{8\pi} \sum_{jk} \varepsilon^{ijk} [S_r \partial_x S_r]_{T r}^j k = v_0^{-1} \frac{\lambda_p \xi_i^x(x) - \frac{3}{2} \xi_i^x(x)}{(3)^2 - (\frac{1}{\pi} \lambda^y)^2}, \quad (i = x, y, z)$$

where $\xi_i^x = 0$, $\xi_i^y = \xi_r + \xi_r^*, \xi_i^y = i(\xi_r - \xi_r^*)$. The matrix $S_r \partial_x S_r = -(\partial_x S_r) S_r^T$ is antisymmetric, and we find

$$\partial_x S_r = -U_r S_r, \quad U_{l m}^r (x) \equiv -\sum_i \varepsilon^{l m} 4\pi v_0^{-1} \frac{3}{2} \xi_i^x(x) - \frac{1}{\pi} \lambda_i^x \xi_i^y(x)}{(3)^2 - (\frac{1}{\pi} \lambda^y)^2}.$$  

The solution $S_r(x)$ can be written as a path-ordered exponential,

$$S_r(x) = T_x \exp \left[ \int_{x_0}^x dx' U_r(x') \right] S_r(x_0),$$

and $x = x_0$ denotes the starting point of the disordered region; we will thus take $S_r(x_0) = 1$.

Let us next focus on the realistic case where disorder couples to the adjacent top and bottom layers in equal strengths, $\xi_l = \xi_b$. We then have $\xi_{lip} = \xi_b^y$ and $\xi_{lip} = -\xi_b^y$ and

$$U_r(x) = 4\pi v_0^{-1} \begin{pmatrix} 0 & 0 & -\frac{1}{\pi} \lambda^y \xi_b^x(x) \\ 0 & 0 & -\frac{1}{\pi} \lambda^y \xi_b^y(x) \\ -\frac{1}{\pi} \lambda^x \xi_b^x(x) & -\frac{1}{\pi} \lambda^x \xi_b^y(x) & 0 \end{pmatrix}.$$  

Due to the layer-correlated disorder, we have the non-trivial property $S_r(x)^T \text{diag}(-1, 1, 1) S_r(x) = \text{diag}(-1, 1, 1)$. In particular, the pairing term $\mathbf{j}_T^3 \text{diag}(-\lambda, \lambda, \lambda) \mathbf{j}_b$ is invariant under the transformation (9) of the main text.

We still cannot evaluate $S(x)$ in explicit form, but we can obtain its average properties. The matrix $S(x)$ describes a sequence (on the $x$-axis) of random rotations by a random angle $\sqrt{\xi_x^x + \xi_y^y}$ about a random axis $(\xi_x, \xi_y, 0)$. We will now assume that the correlation length of the angle distribution is much shorter than that of the axis direction. In
this limit we can carry out the average in two steps, first over the angle and then over the axis direction. For a fixed axis, the path-ordering in Eq. \((10)\) of the main text can be removed and \(S_\tau(x)\) can be obtained explicitly. For example, for a rotation about the \(x\)-axis, we can set \(\xi_b(x) = 0\) and find

\[
S_\tau(x) = S_b(x) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x') - \sin \tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x') & 0 \\
0 & 0 & \cos \tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x')
\end{pmatrix}.
\]

(S12)

We denote here \(\tilde{\nu}_0^{-1} = \nu_0^{-1} \frac{4\pi}{\xi_x(x')}\). Averaging over the angle with \(\langle \xi^2(x) \xi^2(x') \rangle = 2\nu^{-1}W\delta(x-x')\) yields (we denote \(S = S_t = S_b\))

\[
\langle S(x) \rangle = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-\tilde{\nu}_0^{-2}a^{-1}W(x-x_0)} & 0 \\
0 & 0 & e^{-\tilde{\nu}_0^{-2}a^{-1}W(x-x_0)}
\end{pmatrix}.
\]

(S13)

Taking a Gaussian distribution for \(\xi_x\) in Eq. \((12)\), we can similarly find the higher moments such as \(\langle S^{ij}(x)S^{kl}(x) \rangle\).

After averaging over the angle, we can average over the axis vector in the \(x-y\) plane, which yields

\[
\langle S(x) \rangle = \begin{pmatrix}
\frac{1}{2}(1 + e^{-\tilde{\nu}_0^{-2}a^{-1}W(x-x_0)}) & 0 & 0 \\
0 & \frac{1}{2}(1 + e^{-\tilde{\nu}_0^{-2}a^{-1}W(x-x_0)}) & 0 \\
0 & 0 & e^{-\tilde{\nu}_0^{-2}a^{-1}W(x-x_0)}
\end{pmatrix}.
\]

(S14)

In the limit \((x-x_0) \gg \nu_0^2/W\), we find then \(\langle S(x) \rangle = \text{diag}(\frac{1}{2}, \frac{1}{2}, 0)\). For the matrix \(M^{ij}(x) = S^{ij}_t(x)S^{ij}_b(x)\) [see Eq. \((10)\) of the main text], we find

\[
M(x) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \sin^2 \tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x') & \frac{1}{2} \sin 2\tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x') \\
0 & \frac{1}{2} \sin 2\tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x') & \cos^2 \tilde{\nu}_0^{-1} \int_{x_0}^{x} dx' \xi_x(x')
\end{pmatrix},
\]

(S15)

and \(\langle M \rangle = \frac{1}{4} \text{diag}(0, 1 - e^{-4\nu_0^{-2}a^{-1}W(x-x_0)}, 1 + e^{-4\nu_0^{-2}a^{-1}W(x-x_0)})\). For the variance of \(M(x')\), we find [in the limit \(x(t) - x_0 \gg \nu_0^2/W\)]

\[
\langle M(x)M(x') \rangle - \langle M \rangle^2 = \begin{cases}
0, & x-x' \gg \nu_0^2/W, \\
\langle M \rangle, & x-x' < \nu_0^2/W.
\end{cases}
\]

(S16)

Upon averaging over the axis and taking the limit \((x-x_0) \gg \nu_0^2/W\), we find

\[
\langle M \rangle = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

(S17)

This form was used to obtain the average of \(M = M - \text{diag}(0, 0, 1)\) in the main text, Eq. \((11)\).

Even though this result was obtained in the limit where the angle \(\sqrt{\xi_x^2 + \xi_y^2}\) varies much faster than the axis direction, we expect the result to hold more generally since for random disorder we sample all points on the Bloch sphere in an uncorrelated way.

D. The fixed point \(\lambda = (\lambda, -\lambda, \lambda)^T\) with \(\lambda < 0\) in the disordered case

The other strong-pairing fixed point discussed below Eqs. \((6)-(7)\) of the main text has \(\lambda = (\lambda, -\lambda, \lambda)^T\) with \(\lambda < 0\). This vector is not invariant under the gauge transformation Eq. \((9)\) of the main text. We have instead

\[
J^T \tau J_b = J^T \tau J_b + 2J^T M J_b,
\]

(S18)

where \(\tau = \text{diag}(1, -1, 1)\) and \(M^{ij}(x) = S^{ij}_t(x)S^{ij}_b(x) - \delta^{ij}\delta^z\) depends on position. However, for a suitable model of disorder, the matrix \(M(x)\) can be separated into a random and a uniform (position-independent) part. The random part is RG irrelevant and we can neglect it at low energies. The uniform part is \(O(2)\) symmetric, Eq. \((11)\), \(M = \text{diag}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\). Thus, upon averaging over disorder, we obtain a non-disordered Hamiltonian of the form Eq. \((11)\) of the main text, with \(\lambda = \lambda(\frac{3}{2}, 0, 0)^T\). We can then use the Eqs. \((6)-(7)\) of the main text to study the renormalization group flow. Now \(\lambda^z = 0\) initially, but flows to negative values since \(\lambda^x \lambda^y < 0\). We therefore expect to reach the strong-pairing fixed point \(\lambda = (\lambda, -\lambda, \lambda)^T\) with \(\lambda < 0\).
is the number of electrons on the edge added from outside, their numbers of neutralons, quasiparticles, and electrons. Let us label the different sectors by that facilitates the measurement of 4-neutralon pairing.

We have the operators. The operators that create an electron on the, say, top edge are given by the physical operators that can appear in the Hamiltonian are combinations of electron or quasiparticle creation operators. The operators that create an electron on the say top edge are given by the physical operators that can appear in the Hamiltonian are combinations of electron or quasiparticle creation operators.

The operators that create an electron on the, say, top edge are given by where is the total charging energy, coupling to electrons and quasiparticles, and does not change the neutralon number. Finally, we have the operators and that add an electron or quasiparticle and add/remove a neutralon. Thus, starting from a reference sector, say , we can access the sectors, where and are integers.

By using quantum dots or antidots the different neutralon sectors can be in principle accessed, see Fig. The charge states of an antidot in one layer can be labeled by , where is the number of electrons, is the number of quasiparticles, and is the number of neutralons. For the antidot, we take the charging energy Hamiltonian of Ref.:

\[ H_c = E_c(N + 3q - \frac{1}{e}V_g)^2 + E_{cq}q^2 + E_{cn}n^2. \]  

Here is the total charging energy, coupling to electrons and quasiparticles, and and are separate “charging energies” for quasiparticles and neutralons. The spectrum from Eq. is plotted in Fig. The solid/dashed arrows indicate those states whose energies would be lifted up by non-zero neutralon/electron pairing. Neutralon pairing is therefore distinct from electron pairing. In the limit of a large antidot (large circumference), the charging energies scale as \( E_c \propto 1/L \). In that limit the neutralon pairing gap is large compared to \( E_c \). The parameters used are \( E_{cq} = 0.133E_c \), \( E_{cn} = 0.075E_c \).

E. Four-neutralon pairing and neutralon sectors

In this section we show that the neutralon pairing is between 4 neutralons. We also propose a conceptual setup that facilitates the measurement of 4-neutralon pairing.

On a single edge of the bilayer system, say the top edge in Fig., the edge Fock space consists of sectors differing by their numbers of neutralons, quasiparticles, and electrons. Let us label the different sectors by where is the number of electrons on the edge added from outside, is the number of number of fractional \( e/3 \) quasiparticles added to the edge from the strongly-correlated bulk, and is the number of neutralons on the edge. The creation operator of a single neutralon is \( e^{-i\phi_{\pm,0}/\sqrt{2}} \) (where is the direction of propagation, or layer index). However, the physical operators that can appear in the Hamiltonian are combinations of electron or quasiparticle creation operators.

The operators that create a neutralon on the, say, top edge are given by \( e^{i\phi_{\pm,0}}e^{i\sqrt{2}\phi_{-2/3}} \) and \( e^{2i\phi_{-1}\phi_{1,1}} = e^{4\sqrt{2}\phi_{-2/3}} \) and both change the neutralon number by one. The neutral combination of these operators is \( e^{-i\phi_{-1}\phi_{1,1}} = e^{-i\sqrt{2}\phi_{-2/3}} \) which creates a pair of neutralons. This term appears in the KFP action, Eq. (3). The pairing term in Eq. is \( e^{-i\sqrt{2}\phi_{\pm,0}-\phi_{-1,0}} \) and creates 2 neutralons to each edge. Thus, we call it 4-neutralon pairing.

We also note that a charge-2\( e \) operator \( e^{3\phi_{\pm,0}} = e^{2i\sqrt{2}\phi_{-2/3}} \), does not change the neutralon number.

Figure S1. Left: A Coulomb blocked antidot in the top layer can be used to access a signature of the neutralon pairing. In charge transport from left to right, electrons or quasiparticles can enter the antidot. The current is highest near charge degeneracy points (crossings of Coulomb parabolas) shown on the right. Right: Spectrum of the “charging energy” Hamiltonian from Eq. as a function of gate charge. Charge degeneracy points of the ground state manifold are shown in circles. The solid/dashed arrows indicate those states whose energies would be lifted up by non-zero neutralon/electron pairing. Neutralon pairing is therefore distinct from electron pairing. In the limit of a large antidot (large circumference \( L \)), the charging energies scale as \( E_c \propto 1/L \). In that limit the neutralon pairing gap is large compared to \( E_c \). The parameters used are \( E_{cq} = 0.133E_c \), \( E_{cn} = 0.075E_c \).

F. Mode expansion and the four neutralon parity sectors

In this section, we show that there are two degenerate ground states for the neutralon pairing operator. These ground states correspond to two of the four “parity” sectors, defined by the neutralon number modulo 4. In a finite-size edge, the degeneracy between the sectors is split by the Hamiltonian Eq. S19.
We introduce the mode expansion [45] for neutralon $\phi_{\pm,0}$ in a periodic edge of length $L$:

$$\phi_{\tau,0}(x) = \frac{2\pi x}{L\sqrt{2}} N_{\tau,0} - \tau \sqrt{2} \chi_{\tau,0} - i \sum_{q=2\pi n/L>0} \sqrt{\frac{2\pi}{Nq}} e^{\tau i q x} b_{q,\tau} - e^{-\tau i q x} b_{q,\tau}^\dagger,$$

where $m$ is a positive integer, $[b_q, b_q^\dagger] = \delta_{q,q'}$ and $[\chi_{\tau,0}, N_{\tau',0}] = i\delta_{\tau,\tau'}$. We can check that

$$[\phi_{\tau,0}(x), \phi_{\tau,0}(x')] = \tau i\pi \text{sgn}(x-x'),$$

by using the identity (as $\alpha \to 0$)

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{-\alpha n} \sin \left(\frac{2\pi n}{L}(x-x')\right) = \frac{\pi}{2} \text{sgn}(x-x') - \frac{\pi}{L}(x-x').$$

We will consider a homogeneous neutralon pairing operator $\cos \sqrt{2}[\phi_{+,0} - \phi_{-,0}]$. We will look for a homogeneous configuration: the pairing energy is minimized when the operator $\chi_{+,0} + \chi_{-,0}$ is pinned to a value $(n + \frac{1}{2})\pi$. Let us find which of the $\chi_{+,0} + \chi_{-,0}$-eigenstates $(n + \frac{1}{2})\pi$ can be called equivalent. For this, we note that the operator conjugate to $\chi_{+,0} + \chi_{-,0}$ is $(N_{+,0} + N_{-,0})/2$, i.e., $[\chi_{+,0} + \chi_{-,0}, N_{+,0} + N_{-,0}] = i$. In the number basis, the latter operator takes half-integer values. The operator $e^{i(\chi_{+,0}\chi_{-,0})}$ is the raising operator in the number basis: $e^{i(\chi_{+,0}\chi_{-,0})} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2}\right)^n e^{\frac{i}{2}n^2}$ as follows from the commutation relation $[e^{i(\chi_{+,0}\chi_{-,0})}, \frac{1}{2}(N_{+,0} + N_{-,0})] = -e^{i(\chi_{+,0}\chi_{-,0})}$.

In the number basis, the eigenstates of $e^{i(\chi_{+,0}\chi_{-,0})}$ are thus given by

$$|k\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}i n k} \left(\frac{1}{2}\right)^n |\frac{1}{2}n\rangle,$$

with eigenvalue $e^{ik}$. We note that $|k\rangle$ and $|k+4\rangle$ are the same state. Therefore, we can define $k$ in a “Brillouin zone” $k \in [0, 2\pi)$. Thus, we have four inequivalent eigenstates of the homogeneous pairing Hamiltonian: $|\frac{1}{2}\pi\rangle$, $|\frac{3}{2}\pi\rangle$, $|\frac{5}{2}\pi\rangle$, and $|\frac{7}{2}\pi\rangle$. Out of these states, we can construct eigenstates of neutralon number parity mod 4, $e^{i\pi\frac{1}{2}(N_{+,0}+N_{-,0})}$. In order to do this, we note that $e^{i\pi\frac{1}{2}(N_{+,0}+N_{-,0})}|k\rangle = |k-\pi\rangle$. We find,

$$|0\rangle = |\frac{1}{2}\pi\rangle + |\frac{5}{2}\pi\rangle + |\frac{3}{2}\pi\rangle + |\frac{7}{2}\pi\rangle,$$

$$|1\rangle = |\frac{1}{2}\pi\rangle - |\frac{5}{2}\pi\rangle - i|\frac{3}{2}\pi\rangle + i|\frac{7}{2}\pi\rangle,$$

$$|2\rangle = |\frac{1}{2}\pi\rangle + |\frac{3}{2}\pi\rangle - |\frac{5}{2}\pi\rangle - |\frac{7}{2}\pi\rangle,$$

$$|3\rangle = |\frac{1}{2}\pi\rangle - |\frac{3}{2}\pi\rangle + i|\frac{5}{2}\pi\rangle - i|\frac{7}{2}\pi\rangle.$$

The state $|n\rangle$ has an eigenvalue $e^{i\pi n/2}$ of “parity” mod 4, $e^{i\pi 1/2(N_{+,0}+N_{-,0})}$. However, there is an additional restriction on the states [S24]–[S27]. In order to have a homogeneous solution we require that $(\phi_{+,0} - \phi_{-,0})|x\rangle = 0$, or $N_{+,0} = N_{-,0}$. Considering states where $N_{+,0}$ are integers, we see that the neutralon number parity is $\pm 1$. Thus, only the states $|0\rangle$ and $|2\rangle$ are allowed ground states. For these two states, we have the boundary conditions for the neutralon creation operator $e^{\pm i\phi_{+,0}(x)/\sqrt{2}}$ as $e^{\pm i\phi_{+,0}(x\pm L)/\sqrt{2}} = e^{\mp i\pi n/2}/\sqrt{2}$ mod $(\phi_{+,0}(x)/\sqrt{2})$ where $n = 0, 2$. Thus, each ground state corresponds to a unique boundary condition for the neutralon creation operator. Upon changing $N_{\tau,0} \to N_{\tau,0} + 1$, the operator $e^{\pm i\phi_{+,0}(L)/\sqrt{2}}$ acquires a minus sign, which can be interpreted as a braiding phase $\pi$, resulting from the statistical angle $\pi/2$ for neutralons (semins) [21].

Adding/removing a single neutralon on/from one of the edges will violate the condition $N_{+,0} = N_{-,0}$ (and changes the state as $|n\rangle \to |n \pm 1\rangle$) and will therefore come at a cost $\Delta_n$ (the pairing energy). This energy cost can be used as a signature of neutralon pairing, as is illustrated in Fig. [S1].

G. Alternative QPC designs

We show here additional QPC designs to Fig. [2] of the main text. In Fig. [S2] we show two alternative designs. Both designs are similar to that of Fig. [2] in that only electrons are allowed to tunnel through the middle section. Hence,
in particular, similar considerations to those made for Fig. 2b show that the tunneling current across these alternative QPC designs would be strongly suppressed as well.

Figure S2. Two alternative QPC designs.