A lower bound on the solutions of Kapustin–Witten equations

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Abstract In this article, we consider the Kapustin–Witten equations on a closed four-manifold. We study certain analytic properties of solutions to the equations on a closed manifold. The main result is that there exists an $L^2$-lower bound on the extra fields over a closed four-manifold satisfying certain conditions if the connections are not ASD connections. Furthermore, we also obtain a similar result about the Vafa–Witten equations.

Keywords Kapustin–Witten equations · ASD connections · Flat $G_C$-connections

Mathematics Subject Classification 53C07 · 58E15

1 Introduction

Let $X$ be an oriented four-manifold with a given Riemannian metric $g$. We use the metric on $X$ to define the Hodge star operator on $\Lambda^* T^* X$ and then write the bundle of 2-forms as the direct sum $\Omega^2 T^* X = \Omega^+ \oplus \Omega^-$ with $\Omega^+$ denoting the bundle of self-dual 2-forms and with $\Omega^-$ denoting the bundle of anti-self-dual 2-forms, with respect to this Hodge star. If $\omega$ denotes a given 2-form, then its respective self-dual and anti-self-dual parts are denoted by $\omega^+$ and $\omega^-$. Let $P$ be a principal bundle over $X$ with structure group $G$. Supposing that $A$ is a connection on $P$, then $F_A$ is used below to denote its curvature 2-form. This is a 2-form on $X$ with values in the bundle associated to $P$ with fiber the Lie algebra
of \( G \) denoted by \( \mathfrak{g} \). We define by \( d_A \) the exterior covariant derivative on section of \( \Lambda^*T^*X \otimes (P \times_G \mathfrak{g}) \) with respect to the connection \( A \).

The Kapustin–Witten equations are defined on a Riemannian four-manifold given a principal bundle \( P \). For most our considerations, \( G \) is taken to be \( SO(3) \). The equations require a pair \((A, \phi)\) of connection on \( P \) and section of \( T^*X \otimes (P \times_G \mathfrak{g}) \) to satisfy

\[
(F_A - \phi \wedge \phi)^+ = 0 \quad \text{and} \quad (d_A \phi)^- = 0 \quad \text{and} \quad d_A \ast \phi = 0.
\] (1.1)

Witten [9, 11, 23–26] and also Haydys [10] proposed that certain linear combinations of the equations in (1.1) and the version with the self and anti-self dual forms interchanged should also be considered. The latter are parametrized by \( \tau \in [0, 1] \) which can be written as:

\[
\tau(F_A - \phi \wedge \phi)^+ = (1 - \tau)(d_A \phi)^+,
\]

\[
(1 - \tau)(F_A - \phi \wedge \phi)^- = -\tau(d_A \phi)^-,
\]

\[
d_A \ast \phi = 0.
\] (1.2)

The \( \tau = 0 \) version of (1.2) and the \( \tau = 1 \) version of (1.2) are the version of (1.1) that is defined on \( X \) with its same metric but with its orientation reversed. In the case when \( X \) is compact, Kapustin and Witten [11] proved that the solution of (1.1) with \( \tau \in (0, 1) \) exists only in the case when \( P \times_G \mathfrak{g} \) has zero first Pontrjagin number, and if so, the solutions are such that \( A + \sqrt{-1} \phi \) defining a flat \( PSL(2, \mathbb{C}) \) connection. A nice discussions of there equations can be found in [8].

If \( X \) is compact, and \((A, \phi)\) obeys (1.1), then

\[
YM_{\mathbb{C}}(A + \sqrt{-1} \phi) = \int_X \left( |(F_A - \phi \wedge \phi)|^2 + |d_A \phi|^2 \right) \text{dvol}_g
\]

\[
= \int_X \left( |F_A|^2 - 2 \langle F_A, \phi \wedge \phi \rangle + |\phi \wedge \phi|^2 - \langle \ast [F_A, \phi], \phi \rangle \right) \text{dvol}_g
\]

\[
= \int_X \left( |F_A|^2 - 4 |F_A^+|^2 \right) \text{dvol}_g
\]

\[
= -4\pi^2 p_1(P \times_{SO(3)} \mathfrak{so}(3)).
\]

where \( \mathfrak{so}(3) \) is the Lie algebra of \( SO(3) \) and \( p_1(P) \) is the first Pontrjagin number, the second line we used the fact (2.5). This identity implies, among other things, that there are no solutions to (1.1) in the case when \( X \) is compact and the first Pontrjagin number is positive. It also implies that \((A, \phi)\) solves (1.1) when \( X \) is compact and \( p_1(P \times_{SO(3)} \mathfrak{so}(3)) = 0 \) if and only if \( A + \sqrt{-1} \phi \) defines a flat \( SL(2, \mathbb{C}) \) connection on \( X \).

In [16], Taubes studied the Uhlenbeck style compactness problem for \( SL(2, \mathbb{C}) \) connections, including solutions to the above equations, on four-manifolds (see also [17,18]). In [13], Tanaka observed that Eq. (1.1) on a compact Kähler surface are the same as Simpson’s equations, and proved that the singular set introduced by Taubes for the case of Simpson’s equations has a structure of a holomorphic subvariety.

We define the configuration spaces

\[
\mathcal{C} := A_P \times \Omega^1(X, g_P),
\]

\[
\mathcal{C}' := \Omega^{2,\pm}(X, g_P) \times \Omega^{2,\mp}(X, g_P).
\]
We also define the gauge-equivariant map

\[ \text{KW} : \mathcal{C} \to \mathcal{C}', \]
\[ \text{KW}(A, \phi) = \left( F_A^+ - (\phi \wedge \phi)^+, (dA\phi)^- \right). \]

Mimicking the setup of Donaldson theory, the KW-moduli space is

\[ M_{\text{KW}}(P, g) := \{(A, \phi) | \text{KW}(A, \phi) = 0\}/G_P. \]

The moduli space \( M_{\text{ASD}} \) of all ASD connections can be embedded into \( M_{\text{KW}} \) via \( A \mapsto (A, 0) \), where \( A \) is an ASD connection. In particular, Taubes [14,15] proved the existence of ASD connections on certain four manifolds satisfying extra conditions.

For any positive real constant \( C \in \mathbb{R}^+ \), we define the \( C \)-truncated moduli space

\[ M^C_{\text{KW}}(P, g) := \{(A, \phi) \in M_{\text{KW}}(P, g) | \|\phi\|_{L^2(X)} \leq C\}. \]

In this article, we obtain that there exists an \( L^2 \)-lower bound on the extra fields \( \phi \) over a closed, oriented, four-manifold satisfying certain conditions if the connections are not an ASD connections.

**Theorem 1.1** (Main Theorem) Let \( X \) be a closed, oriented, four-dimensional Riemannian manifold with Riemannian metric \( g \); let \( P \to X \) be a principal \( G \)-bundle with \( G \) being a compact Lie group with \( p_1(P) \) negative and be such that there exist \( \mu, \delta > 0 \) with the property that \( \mu(A) \geq \mu \) for all \( A \in \mathcal{B}_\delta(P, g) \), where \( \mathcal{B}_\delta(P, g) := \{[A] : \|F_A^+\|_{L^2(X)} < \delta\} \) and \( \mu(A) \) is as in (4.1). There exists a positive constant, \( C \), with the following significance. If \( (A, \phi) \) is an \( L^2_1 \) solution of \( M^C_{\text{KW}}(P, g) \), then \( A \) is anti-self-dual with respect to the metric \( g \).

**Corollary 1.2** Let \( X \) be a closed, oriented, four-dimension Riemannian manifold with Riemannian metric \( g \) and let \( P \) be a principal \( SO(3) \) bundle with \( P \times_{SO(3)} so(3) \) has negative first Pontrjagin class over \( X \). Then, there is an open dense subset, \( \mathcal{C}(X, p_1(P)) \), of the Banach space, \( \mathcal{C}(X) \), of conformal equivalence classes, \( [g] \), of \( C^r \) Riemannian metrics on \( X \) (for some integer \( r \geq 3 \)) with the following significance. If \( [g] \in \mathcal{C}(X, p_1(P)) \), then there exists a positive constant, \( C \), with the following significance. Suppose that \( P \) and \( X \) obeys one of the following sets of conditions:

1. \( b^+(X) = 0 \); or
2. \( b^+(X) > 0 \) and the second Stiefel–Whitney class, \( w_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \), is non-trivial.

If \( (A, \phi) \) is an \( L^2_1 \) solution of \( M^C_{\text{KW}}(P, g) \), then \( A \) is anti-self-dual with respect to the metric \( g \).

The organization of this paper is as follows. In Sect. 2, we first review some estimates of the Kapustin–Witten equations. Thanks to Uhlenbeck’s work, we observe that \( A \) must be a flat connection if \( \|\phi\|_{L^2} \) is sufficiently small, when \( A + \sqrt{-1}\phi \) is a flat \( G_C \)-connection and \( \phi \in \ker d^*_A \). In Sect. 3, we study certain analytic properties of solutions.
to the Kapustin–Witten equations on the closed manifolds. In Sect. 4, we generalize the previous observation to the case of Kapustin–Witten equations. More precisely, we show that if $X$ satisfies certain conditions and $\|\phi\|_{L^2}$ is sufficiently small, there exists an ASD connection near given connection $A$ measured by $\|F_A^+\|_{L^2}$, then we complete the proof of main theorem by the similar way for the case of flat $G_C$-connections. In the last section, we also obtain similar result about the Vafa–Witten equations.

2 Fundamental preliminaries

We shall generally adhere to the now standard gauge-theory conventions and notation of Donaldson and Kronheimer [2]. Throughout our article, $G$ denotes a compact Lie group and $P$ a smooth principal $G$-bundle over a compact Riemannian manifold $X$ of dimension $n \geq 2$ and endowed with Riemannian metric $g$. For $u \in L^p(X, g_P)$, where $1 \leq p < \infty$ and $k$ is an integer, we denote

$$
\|u\|_{L^p_k(X)} := \left( \sum_{j=0}^k \int_X |\nabla^j_A u|^p \text{dvol}_g \right)^{1/p},
$$

(2.1)

where $\nabla_A : C^\infty(X, \Omega^l(g_P)) \to C^\infty(X, T^*X \otimes \Omega^l(g_P))$ is the covariant derivative induced by the connection, $A$, on $P$ and Levi–Civita connection defined by the Riemannian metric, $g$, on $T^*X$, and all associated vector bundle over $X$, and $\nabla^j_A := \nabla_A \circ \cdots \circ \nabla_A$ (repeated $j$ times for $j \geq 0$). The Banach spaces, $L^p_{k, A}(X, \Omega^l(g_P))$, are the completions of $\Omega^l(X, g_P)$ with respect to the norms (2.1). For $p = \infty$, we denote

$$
\|u\|_{L^\infty_{k, A}(X)} := \sum_{j=0}^k \text{ess sup}_X |\nabla^j_A u|.
$$

(2.2)

For $p \in [1, \infty)$ and nonnegative integer $k$, Banach space duality to define

$$
L^p'_{-k, A}(X, \Omega^l(g_P)) := \left( L^p_{k, A}(X, \Omega^l(g_P)) \right)^*,
$$

where $p' \in [1, \infty)$ is the dual exponent defined by $1/p + 1/p' = 1$.

2.1 Identities for the solutions

This section derives some basic identities that are obeyed by solutions to Kapustin–Witten equations.

**Theorem 2.1** (Weitezenböck formula)

$$
d^*_A d_A + d_A d_A^* = \nabla_A^* \nabla_A + \text{Ric}(\cdot) + \ast[\ast F_A, \cdot] \text{ on } \Omega^l(X, g_P)
$$

(2.3)

where Ric is the Ricci tensor.
As a simple application of Weitzenböck formula, we have the following proposition:

**Proposition 2.2** If \((A, \phi)\) is a solution of Kapustin–Witten equations, then

\[
\nabla_A^* \nabla_A \phi + \text{Ric} \circ \phi + 2 \left[ (\phi \wedge \phi)^+, \phi \right] = 0.
\]

(2.4)

**Proof** From \((d_A \phi)^- = 0\), we have \(d_A \phi = *d_A \phi\). Then, we have

\[
d_A d_A \phi = d_A *d_A \phi.
\]

Since \(d_A^* = -*d_A\), we obtain

\[
d_A^* d_A \phi = -* [F_A, \phi].
\]

(2.5)

From (2.3) and (2.5), we complete the proof of Proposition 2.2. \(\square\)

Using a technique of Taubes ([14], p. 166), also described in ([12], p. 23–24), we combine the Weitzenböck formula with Morrey’s mean-value inequality to deduce a bound on \(\|\phi\|_{L^\infty}\) in terms of \(\|\phi\|_{L^2}\). First, we recall a mean value inequality as follows:

**Theorem 2.3** ([12] Theorem 3.1.2) Let \(X\) be a smooth closed Riemannian manifold. For all \(\lambda > 0\) there are constants \(\{C_\lambda\}\) with the following property. Let \(V \to X\) be any real vector bundle equipped with a metric, \(A \in \mathcal{A}_V\) be any smooth metric-compatible connection, and \(\sigma \in \Omega^0(X, V)\) be any smooth section which satisfies the pointwise inequality

\[
\langle \sigma, \nabla_A^* \nabla_A \sigma \rangle \leq \lambda |\sigma|^2.
\]

Then, \(\sigma\) satisfies the estimate

\[
\|\sigma\|_{L^\infty} \leq C_\lambda \|\sigma\|_{L^2}.
\]

**Theorem 2.4** Let \(X\) be a compact four-dimensional Riemannian manifold. There exists a constant, \(C = C(X)\), with the following property. For any principal bundle \(P \to X\) and any \(L^1_1\) solution \((A, \phi)\) to the Kapustin–Witten equations,

\[
\|\phi\|_{L^\infty(X)} \leq C \|\phi\|_{L^2(X)}.
\]

**Proof** By Theorem 3.7, we may assume that \((A, \phi)\) is smooth. From (2.4), in pointwise,

\[
\langle \nabla_A^* \nabla_A \phi, \phi \rangle = -\left( \langle \text{Ric} \circ \phi, \phi \rangle + 4 |(\phi \wedge \phi)^+|^2 \right).
\]

(2.6)

Since \(X\) is compact, we get a pointwise bound of the form

\[
\langle \nabla_A^* \nabla_A \phi, \phi \rangle \leq \lambda |\phi|^2
\]

(2.7)
for a constant $\lambda$ depending on Riemannian curvature of $X$. From (2.7) and Theorem 2.3, we obtain
$$\|\phi\|_{L^\infty(X)} \leq C\|\phi\|_{L^2(X)}.$$ 
\qed

### 2.2 Flat $G_C$-connections

Let $P \to X$ be a principal $G$-bundle with $G$ being a compact Lie group with $p_1(P)$ is zero, then the solutions $(A, \phi)$ to the Kapustin–Witten equations are flat $G_C$-connections:
$$F_A - \phi \wedge \phi = 0 \text{ and } dA\phi = 0 \text{ and } dA^*\phi = 0.$$ 

First, we review a key result due to Uhlenbeck for the connections with $L^p$-small curvature ($2p > n$) [20].

**Theorem 2.5** ([20] Corollary 4.3) Let $X$ be a closed, smooth manifold of dimension $n \geq 2$ and endowed with a Riemannian metric, $g$, and $G$ be a compact Lie group, and $2p > n$. Then, there are constants, $\varepsilon = \varepsilon(n, g, G, p) \in (0, 1]$ and $C = C(n, g, G, p) \in [1, \infty)$, with the following significance. Let $A$ be a $L^p_1$ connection on a principal $G$-bundle $P$ over $X$. If
$$\|F_A\|_{L^p(X)} \leq \varepsilon,$$
then there exists a flat connection, $\Gamma$, on $P$ and a gauge transformation $g \in L^p_2(X)$ such that

1. $d^p_1(g^*(A) - \Gamma) = 0$ on $X$,
2. $\|g^*(A) - \Gamma\|_{L^p_{1,1}} \leq C\|F_A\|_{L^p(X)}$ and
3. $\|g^*(A) - \Gamma\|_{L^{n/2}_{1,1}} \leq C\|F_A\|_{L^{n/2}(X)}$.

**Theorem 2.6** Let $X$ be a closed, oriented, four-dimensional Riemannian manifold with Riemannian metric $g$; let $P \to X$ be a principal $G$-bundle with $G$ being a compact Lie group with $p_1(P) = 0$. There exists a positive constant, $C$, with the following significance. If $(A, \phi)$ is an $L^2_1$ solution of $M^C_{KW}(P, g)$, then $A$ is a flat connection.

**Proof** By Theorem 3.7, we can assume $(A, \phi)$ is smooth. From Theorem 2.4, we have
$$\|\phi\|_{L^\infty(X)} \leq C_7\|\phi\|_{L^2(X)},$$
where $C_7$ is only dependent on manifold. Since $(A, \phi)$ is a solution of the Kapustin–Witten equations, we have
$$\|F_A\|_{L^p(X)} \leq \|\phi \wedge \phi\|_{L^p(X)} \leq C_8\|\phi\|_{L^2(X)}^2,$$
where $p > 2$. We can choose $\|\phi\|_{L^2(X)} \leq C$ sufficiently small such that $C_8 C^2 \leq \varepsilon$, where $\varepsilon$ is the constant in the hypothesis of Theorem 2.5. Then, from Theorem 2.5, there exists a flat connection $\Gamma$ such that

$$\|A - \Gamma\|_{L^2(X)} \leq C_9 \|F_A\|_{L^2(X)}.$$ 

Using the Weitzenböck formula, we have identities

$$(d^*_\Gamma d + d^*_\Gamma d)\phi = \nabla^*_\Gamma \nabla \phi + \text{Ric} \circ \phi,$$  \quad (2.8)$$
and

$$(d^*_A d_A + d_A d^*_A)\phi = \nabla^*_A \nabla A \phi + \text{Ric} \circ \phi + \ast[\ast F_A, \phi],$$  \quad (2.9)$$
which lead to the following two integral inequalities:

$$\|\nabla_\Gamma \phi\|_{L^2(X)}^2 + \int_X \langle \text{Ric} \circ \phi, \phi \rangle \geq 0.$$  \quad (2.10)$$
and

$$\|\nabla A \phi\|_{L^2(X)}^2 + \int_X \langle \text{Ric} \circ \phi, \phi \rangle + 2\|F_A\|^2 = 0.$$  \quad (2.11)$$
On the other hand, we also have another integral inequality

$$\|\nabla A \phi - \nabla_\Gamma \phi\|_{L^2(X)}^2 \leq \|[A - \Gamma, \phi]\|_{L^2(X)}^2 \leq C_{10} \|A - \Gamma\|_{L^4(X)}^2 \|\phi\|_{L^4(X)}^2 \leq C_{11} \|F_A\|_{L^2(X)}^2 \|\phi\|_{L^2(X)}^2.$$  \quad (2.12)$$
Combing these inequalities, we arrive at

$$0 \leq \|\nabla_\Gamma \phi\|_{L^2(X)}^2 + \int_X \langle \text{Ric} \circ \phi, \phi \rangle \leq \|\nabla A \phi\|_{L^2(X)}^2 + \int_X \langle \text{Ric} \circ \phi, \phi \rangle + \|\nabla A \phi - \nabla_\Gamma \phi\|_{L^2(X)}^2 \leq (C_{12} \|\phi\|_{L^2(X)}^2 - 2) \|F_A\|_{L^2(X)}^2.$$  \quad (2.13)$$
We can choose $\|\phi\|_{L^2(X)} \leq C$ sufficiently small such that $C_{12} C^2 \leq 1$, then $F_A$ vanishes. \hfill \Box

2.3 A vanishing theorem on extra fields

As usual, we define the stabilizer group $\Gamma_A$ of $A$ in the gauge group $G_P$ by

$$\Gamma_A := \{g \in G_P \mid g^*(A) = A\}.$$ 

**Definition 2.7** A connection $A$ is said to be *irreducible* if the stabilizer group $\Gamma_A$ is isomorphic to the centre of $G$, and $A$ is called *reducible* otherwise.
Lemma 2.8 ([2] Lemma 4.3.21) If \( A \) is an irreducible \( SU(2) \) or \( SO(3) \) anti-self-dual connection on a bundle \( E \) over a simply connected four-manifold \( X \), then the restriction of \( A \) to any non-empty open set in \( X \) is also irreducible.

Theorem 2.9 Let \( X \) be a simply connected Riemannian four-manifold; let \( P \rightarrow X \) be an \( SU(2) \) or \( SO(3) \) principal bundle. If \( A \in A_P \) is irreducible, anti-self-dual connection and \( \phi \in \Omega^1(X, \mathfrak{g}_P) \) satisfy

\[
\phi \wedge \phi = 0 \quad \text{and} \quad d_A \phi = 0 \quad \text{and} \quad d^*_A \phi = 0
\]

then \( \phi = 0 \).

Proof Since \( F^+_A = 0 \), \( \phi \wedge \phi = 0 \), then \( \phi \) has at most rank one. Let \( Z_c \) denote the complement of the zero of \( \phi \). By unique continuation of the elliptic equation \((d_A + d^*_A)\phi = 0\), \( Z_c \) is either empty or dense.

The Lie algebra of \( SU(2) \) or \( SO(3) \) is three-dimensional, with basis \( \{\sigma^i\}_{i=1,2,3} \) and Lie brackets

\[
[\sigma^i, \sigma^j] = 2\epsilon_{ijk} \sigma^k.
\]

In a local coordinate, we can set \( \phi = \sum_{i=1}^3 \phi_i \sigma^i \), where \( \phi_i \in \Omega^1(X) \). Then

\[
0 = \phi \wedge \phi = 2(\phi_1 \wedge \phi_2)\sigma^3 + 2(\phi_3 \wedge \phi_1)\sigma^2 + 2(\phi_2 \wedge \phi_3)\sigma^1.
\]

We have

\[
0 = \phi_1 \wedge \phi_2 = \phi_3 \wedge \phi_1 = \phi_2 \wedge \phi_3. \tag{2.13}
\]

On \( Z_c \), \( \phi \) is non-zero, then without loss of generality we can assume that \( \phi_1 \) is non-zero. From (2.13), there exist functions \( \mu \) and \( \nu \) such that

\[
\phi_2 = \mu \phi_1 \quad \text{and} \quad \phi_3 = \nu \phi_1.
\]

Hence,

\[
\phi = \phi_1 (\sigma^1 + \mu \sigma^2 + \nu \sigma^3) = \phi_1 (1 + \mu^2 + \nu^2)^{1/2} \left( \frac{\sigma^1 + \mu \sigma^2 + \nu \sigma^3}{\sqrt{1+\mu^2+\nu^2}} \right).
\]

Then, on \( Z_c \) write \( \phi = \xi \otimes \omega \) for \( \xi \in \Omega^0(Z_c, \mathfrak{g}_P) \) with \( \langle \xi, \xi \rangle = 1 \), and \( \omega \in \Omega^1(Z_c) \). We compute

\[
0 = d_A(\xi \otimes \omega) = d_A \xi \wedge \omega - \xi \otimes d \omega,
\]

\[
0 = d_A *(\xi \otimes \omega) = d_A \xi \wedge * \omega - \xi \otimes d * \omega.
\]

Taking the inner product with \( \xi \) and using the consequence of \( \langle \xi, \xi \rangle = 1 \) that \( \langle \xi, d_A \xi \rangle = 0 \), we get \( d \omega = d^* \omega = 0 \). It follows that \( d_A \xi \wedge \omega = 0 \). Since \( \omega \) is nowhere zero along \( Z_c \), we must have \( d_A \xi = 0 \) along \( Z_c \). Therefore, \( A \) is reducible along \( Z_c \). However according to [2] Lemma 4.3.21, \( A \) is irreducible along \( Z_c \). This is a contradiction unless \( Z_c \) is empty. Therefore \( Z = X \), so \( \phi \) is identically zero. \( \square \)
3 Analytic results

3.1 The Kuranishi complex

The most fundamental tool for understanding moduli space of anti-self-dual connection is the complex associated with an anti-self-dual connection $A_{\text{asd}}$ given by

$$0 \rightarrow \Omega^0(g_P) \xrightarrow{d A_{\text{asd}}} \Omega^1(g_P) \xrightarrow{d^+ A_{\text{asd}}} \Omega^2_{++}(g_P) \rightarrow 0.$$ 

The complex associated with Kapustin–Witten equations is the form:

$$0 \rightarrow \Omega^1(g_P) \xrightarrow{d^0_{(A, \phi)}} \Omega^1(g_P) \times \Omega^1(g_P) \xrightarrow{d^1_{(A, \phi)}} \Omega^2_{-}(g_P) \times \Omega^2_{+}(g_P) \rightarrow 0,$$

where $d^1_{(A, \phi)}$ is the linearization of KW at the configuration $(A, \phi)$, and $d^0_{(A, \phi)}$ gives the action of infinitesimal gauge transformations. These maps $d^0_{(A, \phi)}$ and $d^1_{(A, \phi)}$ form a complex whenever $\text{KW}(A, \phi) = 0$.

The action of $g \in G_P$ on $(A, \phi) \in \mathcal{A}_P \times \Omega^1(g_P)$, and the corresponding infinitesimal action of $\xi \in \Omega^0(g_P)$ is

$$d^0_{(A, \phi)} : \Omega^0(g_P) \rightarrow \Omega^1(g_P) \times \Omega^1(g_P),$$

$$d^0_{(A, \phi)}(\xi) = (-d_A \xi, [\xi, \phi]).$$

The linearization of KW at the point $(A, \phi)$ is given by

$$d^1_{(A, \phi)}(a, b) = ((d_A b + [a, \phi])^-, (d_A a + [b, \phi])^+).$$

We compute

$$d^1_{(A, \phi)} \circ d^0_{(A, \phi)}(\xi) = ([\xi, (d_A \phi)^-], [\xi, (F_A + \phi \wedge \phi)^+]).$$

The dual complex is

$$0 \rightarrow \Omega^2_{-}(g_P) \times \Omega^2_{+}(g_P) \xrightarrow{d^1_{(A, \phi)}} \Omega^1(g_P) \times \Omega^1(g_P) \xrightarrow{d^0_{(A, \phi)}} \Omega^0(g_P) \rightarrow 0.$$

The codifferentials are

$$d^1_{(A, \phi)}(a', b') = (d^* A a' - *[\phi, a'], d^* A b' + *[\phi, b']),$$

and

$$d^0_{(A, \phi)}(a, b) = (-d^* A a + *[\phi, b]).$$
Proposition 3.1  The map $KW(A, \phi)$ has an exact quadratic expansion given by

$$KW(A + a, \phi + b) = KW(A, \phi) + d^1_{A, \phi}(a, b) + \{(a, b), (a, b)\},$$

where $\{(a, b), (a, b)\}$ is the symmetric quadratic form given by

$$\{(a, b), (a, b)\} := ([a, b]^{-}, (a \wedge a + [b, b])^+).$$

Given fixed $(A_0, \phi_0)$, we look for solutions to the inhomogeneous equation $KW(A_0 + a, \phi_0 + b) = \psi_0$. By Proposition 3.1, this equation is equivalent to

$$d^1_{A_0, \phi_0} + \{(a, b), (a, b)\} = \psi_0 - KW(A_0, \phi_0). \tag{3.1}$$

To make this equation elliptic, it is natural to impose the gauge-fixing condition

$$d^{0,*}_{(A_0, \phi_0)}(a, b) = \zeta.$$

If we define

$$\mathcal{D}_{(A_0, \phi_0)} := d^{0,*}_{(A_0, \phi_0)} + d^1_{(A_0, \phi_0)},$$

$$\psi = \psi_0 - KW(A_0, \phi_0),$$

then the elliptic system can be rewritten as:

$$\mathcal{D}_{(A_0, \phi_0)} + \{(a, b), (a, b)\} = (\zeta, \psi). \tag{3.2}$$

This situation is consider in [6] equation 3.2 in the context of $PU(2)$ monopoles.

### 3.2 Regularity and elliptic estimates

First, we summarize the result of [6], which applies verbatim to Kapustin–Witten equations upon replacing the $PU(2)$ spinor $\Phi$ by $\phi$.

Proposition 3.2 ([6] Corollary 3.4) Let $X$ be a closed, oriented, four-dimensional Riemannian manifold, let $P \to X$ be a principal bundle with compact structure group, and let $(A_0, \phi_0)$ be a $C^\infty$ configuration in $\mathcal{C}_P$. Then, there exists a positive constant $\epsilon = \epsilon(A_0, \phi_0)$ such that if $(a, b)$ is an $L^2_k$ solution to (3.2) over $X$, where $(\zeta, \psi)$ is in $L^2_k$ and $\|(a, b)\|_{L^4} < \epsilon$, and $k \geq 0$ is an integer, then $(a, b) \in L^2_{k+1}$ and there is a polynomial $Q_k(x, y)$, with positive real coefficients, depending at most on $(A_0, \phi_0)$, $k$ such that $Q_k(0, 0) = 0$ and

$$\|(a, b)\|_{L^2_{k+1, A_0}(X)} \leq Q_k \left( \|(\zeta, \psi)\|_{L^2_{k, A_0}(X)}, \|(a, b)\|_{L^2(X)} \right). \tag{3.3}$$

In particular, if $(\zeta, \psi)$ is in $C^\infty$ and if $(\zeta, \psi) = 0$, then

$$\|(a, b)\|_{L^2_{k+1, A_0}(X)} \leq C \|(a, b)\|_{L^2(X)}.$$
Proposition 3.3 Let $X$ be a closed, oriented, four-dimensional Riemannian manifold; let $P \rightarrow X$ be a principal bundle with compact structure group. Suppose $\Omega \subset X$ is an open subset such that $P|\Omega$ is trivial, and $\Gamma$ is a smooth flat connection. Then, there exists a positive constant $\epsilon = \epsilon(\Omega)$ with the following significance. Suppose that $(a, b)$ is an $L^2_2(\Omega)$ solution to the elliptic system (3.2) over $\Omega$ with $(A_0, B_0) = (\Gamma, 0)$, where $(\xi, \psi)$ is in $L^2_2(\Omega)$, $k \geq 1$ is an integer, and $\|(a, b)\|_{L^2} < \epsilon$. Let $\Omega' \Subset \Omega$ be a precompact open subset. Then, $(a, b)$ is in $L^2_{k+1}(\Omega')$ and there is a polynomial $Q_k(x, y)$, with positive real coefficients, depending at most on $k, \Omega, \Omega'$ such that $Q_k(0, 0) = 0$ and

$$\|(a, b)\|_{L^2_{k+1, r}(\Omega')} \leq Q_k\left(\|(\xi, \psi)\|_{L^2_{k, A_0}(X)}, \|(a, b)\|_{L^2(X)}\right). \tag{3.4}$$

If $(\xi, \psi)$ is in $C^\infty(\Omega)$ then $(\xi, \psi)$ is in $C^\infty(\Omega')$ and if $(\xi, \psi) = 0$, then

$$\|(a, b)\|_{L^2_{k+1, r}(\Omega')} \leq C\|(a, b)\|_{L^2(\Omega)}.$$ 

We assume that $\int_{B_r(x)} |F_A|^2 \leq \epsilon$ any $x \in X$ and a $0 < r \leq \delta$. We then use the following version of the Uhlenbeck theorem (the original appeared in [19]) stated in Remark 6.2a of [22] and proved in Pages 105–106 of [22] by Wehrheim.

Theorem 3.4 Let $X$ be a compact four-dimensional manifold, let $P \rightarrow X$ be a principal bundle with compact structure group, and let $2 \leq p < 4$. Let $B_r(x)$ denote the geodesic ball of radius $r$ centred at $x$. Then, there exists a constant $C, \epsilon > 0$ such that the following holds:

For every point $x \in X$, there exists a positive radius $r_x$ such that for all $r \in (0, r_x]$, all smooth flat connections $\Gamma \in \Omega^1(B_r(x), \mathfrak{g}_P)$, and all $L^p_1$ connection $A \in \Omega^1(B_r(x), \mathfrak{g}_P)$ with $\|F_A\|_{L^p(B_r(x))} \leq \epsilon$, there exists a gauge transformation $g \in \mathcal{G}^{2, p}(\mathcal{B}_x)$ such that

1. $d^*_1(g^*A - \Gamma) = 0$ on $B_r(x)$ and $\frac{\partial}{\partial r} g^*A - \Gamma = 0$ on $\partial B_r(x)$, and
2. $\|g^*A - \Gamma\|_{L^2_1(B_r(x))} \leq C\|F_A\|_{L^2(B_r(x))}.$

At this point, we must deviate slightly from [6], since we have no estimate of the form $|\phi|^4 \leq C|\phi \wedge \phi|^2$ (c.f. [6] Lemma 2.26), so $F^+_A$ does not bound $\phi$. Instead, we get the following analogue of [6] Corollary 3.15 by combining Proposition 3.3 and Theorem 3.4.

Proposition 3.5 Let $B$ be the open unit ball with centre at the origin, let $U \Subset B$ be an open subset, let $P \rightarrow B$ be a principal bundle with compact structure group, and let $\Gamma$ be a smooth flat connection on $P$. Then there is a positive constant $\epsilon$ such that for all integers $k \geq 1$ there is a constant $C(k, U)$ such that for all $L^2_1$ solution $(A, \phi)$ satisfying

$$\|F_A\|_{L^2(B)}^2 + \|\phi\|_{L^4(B)}^4 \leq \epsilon,$$
there is an \( L^2 \) gauge transformation \( g \) such that \( g^*(A, \phi) \) is in \( C^\infty(B) \) with
\[
d^*_\Gamma (g^*(A) - \Gamma) = 0
\]
over \( B \) and
\[
\|g^*(A, \phi)\|_{L^2_{\Gamma}(U)} \leq C(\|FA\|_{L^2(B)} + \|\phi\|_{L^2(B)}).
\]

**Proof** By choosing \( \epsilon \) as in Theorem 3.4, we can find \( g \) such that
\[
d^*_\Gamma (g^*(A) - \Gamma) = 0
\]
and
\[
\|g^*(A) - \Gamma\|_{L^2(B)} \leq C\|FA\|_{L^2(B)}.
\]
By the Sobolev embedding theorem,
\[
\|g^*A - \Gamma\|_{L^4(B)} \leq C\|FA\|_{L^2(B)}.
\]
Upon taking \( (a, b) = (g^*A - \Gamma, \phi) \), we are in the situation of Proposition 3.3. Thus we get the desired estimate. \( \square \)

We generalize this estimate for geodesic ball:

**Theorem 3.6** Let \( X \) be an oriented Riemannian four-manifold, and let \( P \rightarrow X \) be a principal bundle with compact structure group. Let \( B_r(x) \) denote the geodesic ball of radius \( r \) centred at \( x \), and fix any \( \alpha \in (0, 1] \). For all \( k \geq 1 \), there exist constants \( C(\alpha, k, r), \epsilon \) such that the following holds:

For all point \( x \in X \), there exists a positive radius \( r_x \) such that for all \( r \in (0, r_x] \), all smooth flat connection \( \Gamma \in \mathcal{A}_P(B_r(x)) \), and all \( L^2_{\Gamma} \) solution \( (A, \phi) \) with
\[
\|FA\|^2_{L^2(B_r(x))} + \|\phi\|^4_{L^4(B_r(x))} \leq \epsilon,
\]
there is an \( L^2(B_r(x)) \) gauge transformation \( g \) such that \( g^*(A, \phi) \) is in \( C^\infty(B_r(x)) \) with
\[
d^*_\Gamma (g^*(A) - \Gamma) = 0
\]
over \( B_r(x) \) and
\[
\|g^*(A, \phi)\|_{L^2_{\Gamma}(B_{ar}(x))} \leq C(\|FA\|_{L^2(B_r(x))} + \|\phi\|_{L^2(B_r(x))}).
\]

We will show that all \( L^2_{\Gamma} \) solutions to Kapustin–Witten equations are \( L^2 \)-gauge equivalent to a smooth solution. The way to deal with Kapustin–Witten equations is similar to that Mares [12] dealt with Vafa–Witten equations.
Theorem 3.7 Let $X$ be a closed smooth Riemannian four-manifold, $P \to X$ is a smooth principal $G$-bundle with $G$ compact and connected, $(A, \phi)$ is an $L_1^2$ configuration, and $KW(A, \phi) = 0$. Then, $(A, \phi)$ is $L_2^2$-gauge equivalent to a smooth configuration.

Proof By gauge fixing on small ball $B_r(x)$ in which the local regularity theorem applies, we get $L_2^2$ trivializations $h_{1, x}$ of $P$ over $B_r(x)$ such that $h_{1, x}(A, \phi)$ is smooth. Since the transition functions $h_{1, x}^{-1}h_{1, x}'$ intertwine smooth connections, they define a smooth principal $G$-bundle $P'$. The trivializations $h_{1, x}$ patch together to define an $L_2^2$ isomorphism $h_1: P \to P'$. The $h_{1, x}(A, \phi)$ determine a smooth configuration $(A', \phi')$ in $P'$ such that $h(A, \phi) = (A', \phi')$.

In order to prove that $(A, \phi)$ is $L_2^2$-gauge equivalent to a smooth connection, it suffices to show that there exists a smooth isomorphism $h_2: P \to P'$, for then $g := h_2^{-1}h_1 \in G_P^2$ is the desired gauge transformation. The existence of $h_2$ is a consequence of [12] Theorem 3.3.10. \hfill \square

4 Gap phenomenon for extra fields

4.1 Uniform positive lower bounds for the least eigenvalue of $d_A^+d_A^{+,*}$

Definition 4.1 ([14] Definition 3.1) Let $X$ be a compact four-dimensional Riemannain manifold and $P \to X$ be a principal $G$-bundle with $G$ being a compact Lie group. Let $A$ be a connection of Sobolev class $L_1^2$ on $P$. The least eigenvalue of $d_A^+d_A^{+,*}$ on $L^2(X; \Omega^+(g_P))$ is

$$\mu(A) := \inf_{v \in \Omega^+(g_P) \setminus \{0\}} \frac{\|d_A^+v\|^2}{\|v\|^2}. \quad (4.1)$$

For a Riemannian metric $g$ on a four-manifold, $X$, let $R_g(x)$ denote its scalar curvature at a point $x \in X$ and let $\mathcal{W}_g^{\pm} \in \text{End}(\Omega_x^{\pm})$ denote its self-dual and anti-self-dual Weyl curvature tensors at $x$, where $\Omega_x^2 = \Omega_x^+ \oplus \Omega_x^-$. Define

$$\omega_g^{\pm} := \text{Largest eigenvalue of } \mathcal{W}_g^{\pm}(x), \quad \forall x \in X.$$

We recall the following Weitenböck formula:

$$2d_A^+d_A^{+,*}v = \nabla_A^*\nabla_A v + \left( \frac{1}{3} R_g - 2\omega_g^+ \right) v + \{F_A^+, v\}, \quad \forall v \in \Omega^+(g_P). \quad (4.2)$$

We called a Riemannian metric, $g$, on $X$ positive if $\frac{1}{3} R_g - 2\omega_g^+ > 0$, that is, the operator $R_g/3 - 2\omega_g^+ \in \text{End}(\Omega^+)$ is pointwise positive definite. Then, the Weitenböck formula (4.2) ensures that the least eigenvalue function

$$\mu(\cdot): M_{\text{ASD}}(P, g) \to [0, \infty), \quad (4.3)$$

\(\square\) Springer
defined by $\mu(A)$ in (4.1), admits a uniform positive lower bound, $\mu_0$,

$$\mu(A) \geq \mu_0, \forall [A] \in M_{\text{ASD}}(P, g).$$

The existence of a uniform positive lower bound for the least eigenvalue function (4.3) is more subtle and relies on the generic metric theorem Freed and Uhlenbeck ([7], Pages 69–73), together with certain extensions due to Donaldson and Kronheimer ([2], Sections 4.3.3.). Under suitable hypotheses on $P$ and a generic Riemannian metric, $g$, on $X$, their results collectively ensure that $\mu(A) > 0$ for all $[A]$ in both $M_{\text{ASD}}(P, g)$ and every $M_{\text{ASD}}(P_l, g)$, appearing in its Uhlenbeck compactification (see [2] Definition 4.4.1, Condition 4.4.2, and Theorem 4.4.3),

$$\bar{M}_{\text{ASD}}(P, g) \subset \bigcup_{i=1}^L \left( M_{\text{ASD}}(P_l, g) \times \text{Sym}^l(X) \right),$$

where $L = L(k(P)) \geq 0$ is a sufficiently large integer.

**Theorem 4.2** ([3] Theorem 34.23, [4] Theorem 3.5) Let $G$ be a compact, simple Lie group and $P$ a principal $G$-bundle over a closed, connected, four-dimensional, oriented, smooth manifold, $X$. Then, there is an open dense subset, $\mathcal{C}(X, p_1(P))$, of Banach space, $\mathcal{C}(X)$, of conformal equivalence classes, $[g]$, of $C^r$ Riemannian metrics on $X$ (for some $r \geq 3$) with the following significance. Assume that $[g] \in \mathcal{C}(X)$ and at least one of the following holds:

1. $b^+(X) = 0$, the group $\pi_1(X)$ has no non-trivial representations in $G$, and $G = SU(2)$ or $SO(3)$; or
2. $b^+(X) > 0$, the group $\pi_1(X)$ has no non-trivial representations in $G$, and $G = SO(3)$ and the second Stiefel–Whitney class, $w_2(P) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, is non-trivial.

Then, every point $[A] \in M(P, g)$ has the property that $\mu(A) > 0$.

For a small enough $\varepsilon(g, k(P)) \in (0, 1]$, if $\|F_A^+\|_{L^2(X)} \leq \varepsilon$, the eigenvalue $\mu(A)$ also has a non-zero bound.

**Corollary 4.3** ([3] Corollary 34.28, [4] Corollary 3.9) Assume the hypotheses of Theorem 4.2 and that $g$ is generic. Then, these are constants, $\delta = \delta(P, g) \in (0, 1]$ and $\mu_0 = \mu_0(P, g) > 0$, such that

$$\mu(A) \geq \mu_0, [A] \in M_{\text{ASD}}(P, g),$$

$$\mu(A) \geq \frac{\mu_0}{2}, [A] \in \mathfrak{B}_\delta(P, g),$$

where $\mathfrak{B}_\delta(P, g) := \{[A] : \|F_A^+\|_{L^2(X)} < \delta\}$.

### 4.2 Uniform positive lower bounds for extra fields

Let $A$ be fixed, any connection $B$ can be written uniquely as:

$$B = A + a$$

with $a \in \Omega^1(g_P)$. 

\[ Springer \]
Therefore, if $B$ has anti-self-dual curvature, then

$$0 = F_A^+ + d^+_A a + (a \wedge a)^+. \quad (4.5)$$

Conversely, if $a \in \Gamma(\Omega^1(g_P))$ satisfies (4.5), then $B = A + a$ has anti-self-dual curvature. Because the operator $d^+_A$ is not properly elliptic, it is convenient to write $a = d^+_A u$ for $u \in \Omega^+(g_P)$ and replace (4.5) by

$$d^+_A d^+_A u + (d^+_A u \wedge d^+_A u)^+ = -F_A^+. \quad (4.6)$$

(4.6) is a properly elliptic system. Notice that if $A$ is anti-self-dual, then (4.6) automatically has a solution, namely $u = 0$. If $F_A^+$ is small in an appropriate norm, but non-zero, it is still reasonable to assume that (4.6) has a solution $u$ which is also small.

If $G(\cdot, \cdot)$ denotes the Green kernel of the Laplace operator, $d^* d$, on $\Omega^2(X)$, we define

$$\|v\|_{L^p(x)} := \sup_{x \in X} \int_X G(x, y)|v|(y) d\nu(y),$$

$$\|v\|_{L^p(X)} := \|v\|_{L^p(x)} + \|v\|_{L^p(x)}, \quad \forall v \in \Omega^2(g_P).$$

We recall that $G(x, y)$ has a singularity comparable with $\text{dist}_p(x, y)^{-2}$, when $x, y \in X$ are close [1]. The norm $\|v\|_{L^2(x)}$ is conformally invariant and $\|v\|_{L^p(x)}$ is scale invariant. One can show that $\|v\|_{L^p(x)} \leq c_p \|v\|_{L^p(x)}$ for every $p > 2$, where $c_p$ depends at most on $p$ and the Riemannian metric, $g$, on $X$.

**Theorem 4.4** ([5] Proposition 7.6) Let $G$ be a compact Lie group, $P$ a principal $G$-bundle over a compact, connected, four-dimensional manifold, $X$, with Riemannian metric, $g$, and $E_0, \mu_0 \in (0, \infty)$ constants. Then, there are constants, $C_0 = C_0(E_0, g, \mu_0) \in (0, \infty)$ and $\eta = \eta(E_0, g, \mu_0) \in (0, 1)$, with the following significance. If $A$ is a $C^\infty$ connection on $P$ such that

$$\mu(A) \geq \mu_0, \quad \|F_A^+\|_{L^2(x)} \leq \eta, \quad \|F_A\|_{L^2(x)} \leq E_0,$$

then there is an anti-self-dual connection, $A_{asd}$ on $P$, of class $C^\infty$ such that

$$\|A_{asd} - A\|_{L^2(x)} \leq C_0 \|F_A^+\|_{L^2(x)}.$$

Theorem 4.4 requires that $F_A^+$ is small in the sense that $\|F_A\|_{L^2(x)} \leq \varepsilon$, for a suitable $\varepsilon \in (0, 1]$. By the definition of $\|F_A^+\|_{L^2(x)}$, we have

$$\|F_A^+\|_{L^2(x)} \leq C(p, X) \|F_A^+\|_{L^\infty}.$$

Since $\|F_A\|_{L^2}^2 = 2 \|F_A^+\|_{L^2}^2 + 8\pi^2 k(P)$, where

$$k(P) := -\frac{1}{8\pi^2} \int_X \text{tr}(F_A \wedge F_A) \in \mathbb{Z},$$
then we can choose \( \| F_A^+ \|_{L^2(X)} \) sufficiently small such that \( \| F_A^+ \|_{L^{p,2}(X)} \) and \( \| F_A \|_{L^2} \) satisfy the conditions in Theorem 4.4.

**Corollary 4.5** Assume the hypotheses of Theorem 4.4. Then, there are constants, \( C = C(g, \mu_0) \in (0, \infty) \) and \( \varepsilon = \varepsilon(g, \mu_0) \in (0, 1] \), with the following significance. If \( A \) is a \( C^\infty \) connection on \( P \) such that

\[
\mu(A) \geq \mu_0, \\
\| F_A^+ \|_{L^{\infty}(X)} \leq \varepsilon,
\]

then there is an anti-self-dual connection, \( A_{asd} \) on \( P \), of class \( C^\infty \) such that

\[
\| A_{asd} - A \|_{L^1_2(X)} \leq C \| F_A^+ \|_{L^{\infty}(X)}.
\]

**Lemma 4.6** ([3] Lemma 34.6, [4] Lemma A.2) Let \( X \) be a closed, four-dimensional, oriented, smooth manifold with Riemannian metric, \( g \), and \( q \in [4, \infty) \). Then, there are positive constants, \( c = c(g, q) \geq 1 \) and \( \varepsilon = \varepsilon(g, q) \in (0, 1] \), with the following significance. Let \( r \in [4/3, 2) \) be defined by \( 1/r = 2/d + 1/q \). Let \( G \) be a compact Lie group and \( A \) be a connection of class \( C^\infty \) on a principal bundle \( P \) over \( X \) that obeys the curvature

\[
\| F_A \|_{L^2(X)} \leq \varepsilon.
\]

If \( v \in \Omega^{2, +}(X, g_P) \), then

\[
\| v \|_{\Omega^q(X)} \leq c(\| d_A^+ d_A^{+,*} v \|_{L^r(X)} + \| v \|_{L^r(X)}).
\]

**Lemma 4.7** Let \( G \) be a compact Lie group, let \( P \) be a principal \( G \)-bundle over a closed, four-dimensional, oriented, smooth manifold, \( X \), with Riemannian metric, \( g \), and \( \mu_0 \in (0, \infty) \) a constant; let \( p \in [2, 4) \) and \( q \in [4, \infty) \) be defined by \( 1/p = 1/4 + 1/q \). Then, there exist constants, \( \delta = \delta(g, p) \in (0, 1) \) and \( C = C(g, p, \mu) \in [1, \infty) \), with the following significance. If \( A = A_{asd} - d_A^{+,*} u \) is a connection on \( P \), such that

\[
\mu(A) \geq \mu_0, \\
\| d_A^{+,*} u \|_{L^1_2(X)} \leq \delta,
\]

where \( A_{asd} \) is an anti-self-dual connection. Then

\[
\| d_A^{+,*} u \|_{L^q(X)} \leq C \| F_A^+ \|_{L^p(X)}. \tag{4.7}
\]

**Proof** By the anti-self-dual equation, \( F^+(A + d_A^{+,*} u) = 0 \), we obtain

\[
d_A^+ d_A^{+,*} u + (d_A^{+,*} u \wedge d_A^{+,*} u)^+ = -F_A^+. \tag{4.8}
\]

Then we have

\[
\| F_A^+ \|_{L^2(X)} \leq 2\| d_A^{+,*} u \|_{L^2_1(X)}^2 + \| d_A^+ d_A^{+,*} u \|_{L^2_2(X)} \leq 2\| d_A^{+,*} u \|_{L^2_1(X)}^2 + \| d_A^{+,*} u \|_{L^2_2(X)}^2.
\]
We can choose constant $\delta$ small enough, such that

$$\| F_A^+ \|_{L^2(X)} \leq \varepsilon(g),$$

where the constant $\varepsilon(g)$ is the constant in Lemma 4.6. Then, we have a priori estimate for all $v \in \Omega^{2,+}(X, g_P)$,

$$\| v \|_{L^2_1(X)} \leq c \| d_A^+ d_A^{+,*} v \|_{L^{4/3}(X)} + \| v \|_{L^2(X)}.$$

By Sobolev imbedding $L^2_1 \hookrightarrow L^p$ ($p \leq 4$), we have

$$\| v \|_{L^p(X)} \leq c_p \| d_A^+ d_A^{+,*} v \|_{L^{4/3}(X)} + \| v \|_{L^2(X)}, \quad (4.9)$$

For $p \in [2, 4)$ and $q \in [4, \infty)$, Eq. (4.8) gives

$$\| d_A^+ d_A^{+,*} u \|_{L^p(X)} \leq \| F_A^+ \|_{L^p(X)} + 2 \| d_A^+ u \|_{L^4(X)} \| d_A^{+,*} u \|_{L^q(X)} \leq \| F_A^+ \|_{L^p(X)} + 2\delta \| d_A^{+,*} u \|_{L^q(X)} \leq \| F_A^+ \|_{L^p(X)} + 2\delta \| \nabla u \|_{L^q(X)} \leq \| F_A^+ \|_{L^p(X)} + 2\delta C_s \| \nabla u \|_{L^p_1(X)} \leq \| F_A^+ \|_{L^p(X)} + 2\delta C_s \| u \|_{L^p_{2,1}(X)}.$$

where we have applied the Sobolev embedding $L^p_1 \hookrightarrow L^q$ with embedding constant $C_s$ and Kato Inequality.

We have a priori $L^p$ estimate for the elliptic operator, $d_A^+ d_A^{+,*}$, namely

$$\| u \|_{L^p_2(X)} \leq C_1 \left( \| d_A^+ d_A^{+,*} u \|_{L^p(X)} + \| u \|_{L^p(X)} \right).$$

Since $p \in [2, 4)$, $\| u \|_{L^{4/3}(X)} \leq c \| u \|_{L^2(X)} \leq c\mu(A)^{-1} \| d_A^+ d_A^{+,*} u \|_{L^p(X)}$, then by (4.9), we obtain

$$\| u \|_{L^p(X)} \leq c_p \left( \| d_A^+ d_A^{+,*} u \|_{L^{4/3}(X)} + \| u \|_{L^2(X)} \right) \leq c_p \left( \| d_A^+ d_A^{+,*} u \|_{L^p(X)} + \| v \|_{L^2(X)} \right) \leq c_p \left( \| d_A^+ d_A^{+,*} u \|_{L^p(X)} + \mu(A)^{-1} \| d_A^+ d_A^{+,*} u \|_{L^2(X)} \right) \leq c_p (1 + \mu(A)^{-1}) \| d_A^+ d_A^{+,*} u \|_{L^p(X)}.$$

Combing the preceding inequalities gives

$$\| u \|_{L^p_2(X)} \leq C_1 \left( \| d_A^+ d_A^{+,*} u \|_{L^p(X)} + c_p (1 + \mu(A)^{-1}) \| d_A^+ d_A^{+,*} u \|_{L^2(X)} \right) \leq C_2 \| d_A^+ d_A^{+,*} u \|_{L^p(X)} \leq C_2 \left( \| F_A^+ \|_{L^p(X)} + \delta C_s \| u \|_{L^p_2(X)} \right).$$
Thus, for small enough $\delta$ such that $C_2 \delta C_s < \frac{1}{2}$, rearrangement yields

$$\|u\|_{L^p(X)} \leq 2C_2 \|F^+_A\|_{L^p(X)}. \quad (4.10)$$

The inequality $(4.7)$ follows from

$$\|e^{+,*}_A u\|_{L^q(X)} \leq \kappa_p \|d^{+,*}_A u\|_{L^1(X)} \leq \kappa_p \|u\|_{L^2(X)}.$$  

We now have all the ingredients required to conclude the

**Proof of Theorem 1.1.**  By Theorem 3.7, we can assume that $(A, \phi)$ is smooth. From Theorem 2.4, we have

$$\|\phi\|_{L^\infty(X)} \leq C_1 \|\phi\|_{L^2(X)},$$

where $C_1$ is only dependent on manifold. Since $(A, \phi)$ is a solution of Kapustin–Witten equations, then we have

$$\|F^+_A\|_{L^\infty(X)} = \|\phi \wedge \phi\|_{L^\infty(X)} \leq C_2 \|\phi\|^2_{L^2(X)},$$

where $C_2$ is only dependent on manifold. We can choose $\|\phi\|_{L^2(X)} \leq C$ sufficiently small such that $C_2 C^2 \leq \varepsilon$, where $\varepsilon$ is the constant in the hypothesis of Corollary 4.5, then

$$\|F^+_A\|_{L^\infty(X)} \leq \varepsilon.$$

From Corollary 4.5, there exists an anti-self-dual connection $A_0$ such that

$$\|A - A_0\|_{L^2(X)} \leq C_3 \|F^+_A\|_{L^\infty(X)}.$$

We can choose $\|\phi\|_{L^2(X)} \leq C$ sufficiently small such that $C_2 C^2 \leq \delta$, where $\delta$ is the constant in the hypothesis of Lemma 4.7. By Lemma 4.7, then we obtain

$$\|A - A_0\|_{L^q(X)} \leq C_3 \|F^+_A\|_{L^p(X)}.$$

where $1/p = 1/4 + 1/q$, $p \in [2, 4)$ and $q \in [4, \infty)$. Choosing $q = 4$, $p = 2$, hence, we have

$$\|A - A_0\|_{L^4(X)} \leq C_3 \|F^+_A\|_{L^2(X)}.$$

Using the Weitezenböck formula [7] (6.25), we have

$$(2d_{A_0}^{-*}d_{A_0}^- + d_{A_0}d_{A_0}^*)\phi = \nabla_{A_0}^* \nabla_{A_0} \phi + \text{Ric} \circ \phi. \quad (4.12)$$
which provides an integral inequality
\[ \|\nabla A_0 \phi\|^2_{L^2(X)} + \int_X (\text{Ric} \circ \phi, \phi) \geq 0. \] (4.13)

By the Weitezenböck formula [7] (6.25) again, we have
\[ (2d_A^{-*}d_A^* + d_Ad_A^*) \phi = \nabla A^* \nabla A \phi + \text{Ric} \circ \phi + \ast [F_A^+, \phi]. \] (4.14)

Since \((d_A \phi)^- = d_A^* \phi = 0\), we also obtain an integral equality
\[ \|\nabla A \phi\|^2_{L^2(X)} + \int_X (\text{Ric} \circ \phi, \phi) + 4\|F_A^+\|^2 = 0. \] (4.15)

We have another integral inequality
\[ \|\nabla A \phi - \nabla A_0 \phi\|^2_{L^2(X)} \leq \|[A - A_0, \phi]\|^2_{L^2(X)} \]
\[ \leq C_4\|A - A_0\|_{L^4(X)}^2\|\phi\|^2_{L^4(X)} \]
\[ \leq C_5\|F_A^+\|^2_{L^2(X)}\|\phi\|^2_{L^2(X)}. \] (4.16)

Combing the preceding inequalities gives
\[ 0 \leq \|\nabla A_0 \phi\|^2_{L^2(X)} + \int_X (\text{Ric} \circ \phi, \phi) \]
\[ \leq \|\nabla A \phi\|^2_{L^2(X)} + \int_X (\text{Ric} \circ \phi, \phi) + \|\nabla A \phi - \nabla A_0 \phi\|^2_{L^2(X)} \]
\[ \leq (C_6\|\phi\|^2_{L^2(X)} - 4)\|F_A^+\|^2_{L^2(X)}. \]

We choose \(\|\phi\|_{L^2(X)} \leq C\) sufficiently small such that \(C_6\|\phi\|^2 \leq 2\), then
\[ F_A^+ \equiv 0. \]

**Proof of Corollary 1.2.** The conclusion follows from Theorem 1.1 and positive uniform lower bound on \(\mu(A)\) provided by Corollary 4.3. □

### 5 Vafa–Witten equations

In search of evidence for S-duality, Vafa and Witten explored their twist of \(\mathcal{N} = 4\) supersymmetric Yang–Mills theory [21]. Vafa–Witten introduced a set of gauge-theoretic equations on a four-manifold; the moduli space of solutions to the equations is expected to produce a possibly new invariant of some kind. The equations we consider involve a triple consisting of a connection and other extra fields coming from a principle bundle over four-manifold. We consider the following equations for a triple \((A, B, C) \in \mathcal{A} \times \Omega^2_{+, +}(X, \mathfrak{g}_P) \times \Omega^0(X, \mathfrak{g}_P),\)

\[ d_A C + d_A^n B = 0, \]
\[ F_A^+ + \frac{1}{8}[B, B] + \frac{1}{2}[B, C] = 0. \]
where \([B.B] \in \Omega^{2,+}(X, g_P)\) is defined in [12] Appendix A. We also define the gauge-equivariant map

\[
VW(A, B, C) = \left( d_A C + d_A^* B, F_A^+ + \frac{1}{8}[B.B] + \frac{1}{2}[B, C] \right).
\]

Mimicking the setup of Donaldson theory, the \(VW\)-moduli space is

\[
M_{VW}(P, g) := \{(A, B, C) \mid VW(A, B, C) = 0\}/G_P.
\]

We are interesting in the case \(C = 0\), in which the equations reduce to

\[
d_A^* B = 0,
F_A^+ + \frac{1}{8}[B.B] = 0.
\]

**Theorem 5.1** ([12] Theorem 2.1.1) *Let X be a closed, oriented, four-manifold; \((A, B, C)\) be a solution of the Vafa–Witten equation. If \(A\) be an irreducible \(SU(2)\) or \(SO(3)\) connection, then \(C = 0\).*

In [12], Mares obtained a bound on \(\|B\|_{L^\infty(X)}\) in terms of \(\|B\|_{L^2(X)}\).

**Theorem 5.2** ([12] Theorem 3.1.1) *Let X be a closed, oriented, smooth, four-dimensional manifold. There exists a constant \(\lambda = \lambda(X)\) with the following property. For any principal bundle \(P \to X\) and any \(L^1_2\)-solution \((A, B, 0)\) to the Vafa–Witten equations,

\[
\|B\|_{L^\infty(X)} \leq \lambda \|B\|_{L^2(X)}.
\]

**Lemma 5.3** ([5] Lemma 6.6) *Let X be a closed, oriented, four-dimensional manifold with Riemannian metric, \(g\) Then, there are positive constants \(c = c(g)\) and \(\varepsilon = \varepsilon(g) \in [0, 1)\), with the following significance. If \(G\) is a compact Lie group, \(A\) be a connection Sobolev class \(L^2(X)\) on a principle \(P\) over \(X\) with

\[
\|F_A\|_{L^2(X)} \leq \varepsilon,
\]

and \(\nu \in \Omega^{2,+}(X, g_P)\), then

\[
\|\nu\|_{L^2_{1,2}(X)} \leq c \left( \|d^*_A \nu\|_{L^2(X)} + \|\nu\|_{L^2(X)} \right).
\]

(5.1)

For any real constant \(C \in \mathbb{R}^+\), we defined the \(C\)-truncated moduli space.

\[
M_{VW}^C := \{(A, B, 0) \in M_{VW} \mid \|B\|_{L^2(X)} \leq C\},
\]

then we have a similar result as follows:
Theorem 5.4 Let $X$ be a closed, oriented, four-dimensional manifold; and $P \to X$ be a principal $G$-bundle with $G$ being a compact Lie group with $p_1(P)$ negative and be such that there exist $\mu, \delta > 0$ with the property that $\mu(A) \geq \mu$ for all $A \in \mathfrak{B}_\delta(P, g)$, where $\mu(A)$ is as in (4.1). There exists a positive constant, $C$, with the following significance. If $(A, B, 0)$ is an $L^2_1$ solution of the Vafa–Witten equations and $\|B\|_{L^2(X)} \leq C$, then

$$M^C_{MVW} = \{(A, B, 0) \in M_{MVW} \mid F_A^+ = 0, \ B = 0\}.$$ 

Moreover, if $M_{ASD}$ is non-empty and $M_{MVW} \setminus M_{ASD}$ is also non-empty, then the moduli space $M_{MVW}$ is not connected.

Proof From Lemma 5.3 and the Definition 4.1 of $\mu(A)$, $\forall \nu \in \Omega^2(X, g_P)$, we have

$$\|v\|_{L^2_1(X)} \leq c \left(\|d_{A}^{+,*}v\|_{L^2(X)} + \|v\|_{L^2(X)} \right) \leq c \left(1 + 1/\sqrt{\mu(A)} \right) \|d_{A}^{+,*}v\|_{L^2(X)} \leq c \left(1 + 1/\sqrt{\mu/2} \right) \|d_{A}^{+,*}v\|_{L^2(X)}. \tag{5.2}$$

Since $(A, B, 0)$ is a solution of the Vafa–Witten equations, then we have

$$\|F_A^+\|_{L^2(X)} = \frac{1}{8} \|[B, B]\|_{L^2(X)} \leq \frac{1}{4} \|B\|_{L^4(X)} \leq \frac{1}{4} \|B\|_{L^\infty(X)} \|B\|_{L^2(X)} \leq \lambda(X) \|B\|_{L^2_1(X)}.$$ 

We choose $C$ sufficiently small such that $\lambda C^2 \leq \epsilon$, where $\epsilon$ is a constant as in hypotheses of Lemma 5.3, we apply a priori estimate (5.2) to $v = B$ to obtain

$$\|B\|_{L^2_1(X)} \leq c \left(1 + 1/\sqrt{\mu/2} \right) \|d_{A}^{+,*}B\|_{L^2(X)}. \tag{5.3}$$

As $0 = d_{A}^{*}B = 2d_{A}^{+,*}B$ on $X$, therefore, $B = 0$ on $X$ by (5.3).

If $M_{ASD}$ is non-empty and $M_{MVW} \setminus M_{ASD}$ is also non-empty, since the map $(A, B) \mapsto \|B\|_{L^2}$ is continuous, then the moduli space $M_{MVW}$ is not connected. \hfill $\Box$

Corollary 5.5 Assume the hypotheses of Corollary 1.2 and that $g$ is generic. There exists a positive constant, $C$, with the following significance. If $(A, B, 0)$ be an $L^2_1$ solution of the Vafa–Witten equations and $\|B\|_{L^2(X)} \leq C$, then

$$M^C_{MVW} = \{(A, B, 0) \in M_{MVW} \mid F_A^+ = 0, \ B = 0\}.$$ 

Moreover, if $M_{ASD}$ is non-empty and $M_{MVW} \setminus M_{ASD}$ is also non-empty, then the moduli space $M_{MVW}$ is not connected.
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