FIRST BETTI NUMBERS OF THE ORBITS OF MORSE FUNCTIONS ON SURFACES

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Let $M$ be a connected compact orientable surface and let $P$ be either the real line $\mathbb{R}$ or a circle $S^1$. The group $D(M)$ of diffeomorphisms on $M$ acts in the space of smooth mappings $C^\infty(M, P)$ by the rule $(f, h) \mapsto f \circ h$, where $h \in D(M)$ and $f \in C^\infty(M, P)$. For $f \in C^\infty(M, P)$, let $O(f)$ denote the orbit of $f$ relative to the specified action. By $M(M, P)$ we denote the set of isomorphism classes for the fundamental groups $\pi_1O(f)$ of orbits of all Morse mappings $f : M \to P$. S. Maksymenko and B. Feshchenko studied the sets of isomorphism classes $B$ and $T$ of groups generated by direct products and certain wreath products. They proved the inclusions $M(M, P) \subset B$ valid under the condition that $M$ differs from the 2-sphere $S^2$ and 2-torus $T^2$ and $M(T^2, \mathbb{R}) \subset T$. We show that these inclusions are equalities and describe some subclasses of $M(M, P)$ under certain restrictions imposed on the behavior of functions on the boundary $\partial M$. We also prove that, for any group $G \in B$ ($G \in T$), the center $Z(G)$ and the quotient group with respect to the commutator subgroup $G/[G, G]$ are free Abelian groups of the same rank, which can be easily found by using the geometric properties of a Morse mapping $f$ such that $\pi_1O(f) \simeq G$. In particular, this rank is the first Betti number of the orbit $O(f)$ of the mapping $f$.

1. Introduction

Assume that $M$ is a compact surface, $D(M)$ is a group of $C^\infty$-diffeomorphisms on $M$, and $P$ is either a real line $\mathbb{R}$ or a circle $S^1$. We define the natural right action of the group $D(M)$ on the space of smooth mappings $C^\infty(M, P)$ by the rule $(f, h) \mapsto f \circ h$, where $h \in D(M)$ and $f \in C^\infty(M, P)$. Let

$$O(f) = \{f \circ h \mid h \in D(M)\}$$

be an orbit of $f$ with respect to this action and let $S(f) = \{h \in D(M) \mid f \circ h \equiv f\}$ be a stabilizer of $f$.

We equip the spaces $D(M)$ and $C^\infty(M, P)$ with the corresponding Whitney $C^\infty$-topologies. By $D(M, X)$ we denote the group of diffeomorphisms on $M$ fixed on a closed subset $X \subset M$. Moreover, by $O(f, X)$ and $S(f, X)$, respectively, we denote the orbit and stabilizer of $f$ under the action of $D(M, X)$. Also let $O_f(f)$ and $O_f(f, X)$ denote the connected components of $f$ in $O(f)$ and $O(f, X)$, respectively.

Since $\pi_1O(f)$ and $\pi_1O(f)$ are isomorphic at the point $f$, for the sake of simplicity, we denote $\pi_1O(f)$ and $\pi_1O(f, X)$ by $\pi_1O(f)$ and $\pi_1O(f, X)$, respectively.

**Definition 1.1.** By $F(M, P)$ we denote the space of smooth mappings $f \in C^\infty(M, P)$ satisfying the following conditions:

(i) the mapping $f$ takes constant values in each connected component of the boundary $\partial M$ and does not have critical points on $\partial M$;

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(ii) for each critical point $z$ of the mapping $f$, there exists a local mapping $f_z : \mathbb{R}^2 \to \mathbb{R}$ of the function $f$ near $z$ such that $f_z - f_z(0)$ is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

A mapping $f \in C^\infty(M, P)$ is called a Morse mapping if it satisfies condition (i) and all its critical points are nondegenerate. The set of all Morse mappings from $M$ into $P$ is denoted by $\text{Morse}(M, P)$. We say that a Morse mapping $f$ is a mapping of general position if $f$ takes different values at different critical points.

By virtue of the Morse lemma, each Morse mapping $f$ satisfies condition (ii) with homogeneous polynomials $f_z - f_z(0) = \pm x^2 \pm y^2$ at each critical point $z$ and, hence,

$$\text{Morse}(M, P) \subset \mathcal{F}(M, P).$$

**Definition 1.2.** Every mapping $f \in \mathcal{F}(M, P)$ can be associated with a (continuous) function $\varepsilon_f$ from the set of connected components of the boundary $\partial M$ in $\{ \pm 1 \}$ that takes the value $-1$ if $f$ has a local minimum on the component of the boundary or the value $+1$ if $f$ has a local maximum on the component of the boundary.

Let $\mathcal{E}_M$ be the set of all continuous functions $\varepsilon : \partial M \to \{ \pm 1 \}$.

For $\varepsilon \in \mathcal{E}_M$, by $\mathcal{F}(M, P, \varepsilon)$ ($\text{Morse}(M, P, \varepsilon)$) we denote a subset of the class $\mathcal{F}(M, P)$ ($\text{Morse}(M, P)$) of functions $f$ for which $\varepsilon_f = \varepsilon$.

The homotopic types of stabilizers and orbits of functions from the class $\mathcal{F}(M, P)$ were computed by S. Maksymenko [4, 5] and B. Feshchenko [1, 6]. For Morse functions, they were found in the works by O. Kudryavtseva [7–10]. In particular, in [5], it was shown that if $M \neq S^2$ or $\mathbb{R}P^2$, then $\mathcal{O}_f(f)$ is aspherical.* Moreover, if $M$ is a Morse mapping of the general position, then the orbit $\mathcal{O}_f(f)$ is homotopically equivalent to $(S^1)^k \simeq \mathbb{Z}^k$ for some $k$. If $M = S^2$ or $\mathbb{R}P^2$, then $\pi_i\mathcal{O}_f(f) \simeq \pi_i\mathbb{SO}(3)$ for $i \geq 2$ and if $f$ is a function of general position, then $\mathcal{O}_f(f)$ is homotopically equivalent to $(S^1)^k \times \mathbb{SO}(3)$ for some $k$. Later, Kudryavtseva generalized these results and showed that if $M$ is oriented and $f : M \to \mathbb{R}$ is a Morse function, then there exists a free action of a certain finite group $H$ on a $k$-torus $(S^1)^k$ such that $\mathcal{O}_f(f)$ is homotopically equivalent to $((S^1)^k / H) \times \mathbb{SO}(3)$ for $M = S^2$.

Let $M$ be an oriented surface distinct from a 2-sphere and a 2-torus. As already indicated, $\mathcal{O}_f(f)$ is aspherical and, in particular, its homotopic type is determined solely by the fundamental group $\pi_1\mathcal{O}_f(f)$. The exact algebraic structure of these groups was described in [4]. To formulate these results, we consider the group classes $\mathcal{B}$ and $\mathcal{T}$.

**Classes $\mathcal{B}$ and $\mathcal{T}$**

Let $G$ be a group and let $n, m \in \mathbb{N}$. We define ineffective right actions $\alpha : G^{nm} \times \mathbb{Z}^2 \to G^{nm}$ and $\beta : G^n \times \mathbb{Z} \to G^n$ of the group $\mathbb{Z}^2$ upon $G^{nm}$ and $\mathbb{Z}$ upon $G^n$ by cyclic shifts of coordinates as follows:

$$\alpha((g_{i,j})_{i,j=1}^{n,m}, (a, b)) = (g_{i+a,j+b})_{i,j=1}^{n,m}, \quad \beta((g_i)_{i=1}^n, a) = (g_{i+a})_{i=1}^n,$$

where $a, b \in \mathbb{Z}$. This enables us to introduce the structures of groups on the Cartesian products of the sets $G^{nm} \times \mathbb{Z}$ and $G^n \times \mathbb{Z}$ by the following standard formulas:

$$(g_1, a_1, b_1) \cdot (g_2, a_2, b_2) = (\alpha(g_1, a_2, b_2)g_2, a_1 + a_2, b_1 + b_2), \quad g_1, g_2 \in G^{nm}, \quad a_1, a_2, b_1, b_2 \in \mathbb{Z},$$

$$(g_1, a_1) \cdot (g_2, a_2) = (\beta(g_1, a_2)g_2, a_1 + a_2), \quad g_1, g_2 \in G^n, \quad a_1, a_2 \in \mathbb{Z}.$$  

We denote the obtained groups by $G \wr_{n,m} \mathbb{Z}^2$ and $G \wr_{n} \mathbb{Z}$, respectively. Note that these groups are wreath products of $G$ by $\mathbb{Z}^2$ and $G$ by $\mathbb{Z}$ under the actions $\alpha$ and $\beta$.

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* This means that $\pi_i\mathcal{O}_f(f) \simeq 0$ for all $i \geq 2$. 

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Note that we have the following isomorphisms:

\[ G \wr \mathbb{Z}_1 \cong G \times \mathbb{Z}, \quad 1 \wr \mathbb{Z}_n \cong \mathbb{Z}, \]
\[ G \wr \mathbb{Z}_2 \cong G \times \mathbb{Z}^2, \quad 1 \wr \mathbb{Z}_2 \cong \mathbb{Z}^2. \]

**Definition 1.3.** Assume that \( \mathcal{B} \) is the minimal set of isomorphism classes of the groups satisfying the following conditions:

(i) \( 1 \in \mathcal{B} \);

(ii) if \( G_1, G_2 \in \mathcal{B} \), then \( G_1 \times G_2 \in \mathcal{B} \);

(iii) if \( G \in \mathcal{B} \) and \( n \geq 1 \), then \( G \wr \mathbb{Z}_n \in \mathcal{B} \).

Also let \( \mathcal{T} \) be the set of isomorphism classes of the groups formed by groups of the form \( G \wr \mathbb{Z}_n \), where \( G \in \mathcal{B} \) and \( n, m \geq 1 \).

In addition, let \( \mathcal{B}^O \) be a subclass of \( \mathcal{B} \) formed by the groups \((A \times B) \wr \mathbb{Z}_n\), where \( A, B \in \mathcal{B} \setminus \{1\} \) and \( n \geq 1 \). Moreover, let \( \mathcal{B}^O \subset \mathcal{B} \subset \mathcal{T} \).

**Remark 1.1.** It is easy to see that \( G \) belongs to the class \( \mathcal{B} \) (\( \mathcal{T} \)) if and only if \( G \) is obtained from the trivial group by finitely many operations \( \times \) and \( \wr \mathbb{Z}_n \) (with the last operation \( \wr \mathbb{Z}_n \) for the class \( \mathcal{T} \)). Thus, each group \( G \in \mathcal{B} \) (\( G \in \mathcal{T} \)) can be represented in the form of a word in the alphabet

\[ \mathcal{A}_B = \{1, \mathbb{Z}, (,), \times, l_2, l_3, l_4, \ldots\}, \quad (\mathcal{A}_T = \{1, \mathbb{Z}, (,), \times, l_2, l_3, \ldots, l_1, 1, l_1, 2, \ldots\}). \]

This word is called a realization of the group \( G \) in the alphabet \( \mathcal{A}_B \) (\( \mathcal{A}_T \)). It is clear that this realization is not uniquely defined. Thus, the following realizations of the same group exist:

\[ \left( \frac{1 \wr \mathbb{Z}_3}{3} \right) \times \mathbb{Z} = \mathbb{Z} \times \left( \frac{1 \wr \mathbb{Z}_3}{3} \right) = \mathbb{Z} \times \mathbb{Z} = 1 \times \mathbb{Z} \times \mathbb{Z}. \]

**Definition 1.4.** Let \( \Delta \) be a partition of the manifold \( M \) into connected components of the level sets of the mapping \( f \). The quotient space \( \Gamma_f = M / \Delta \) is called a Kronrod–Reeb graph with respect to the mapping \( f \).

In [11] (Lemma 3.1), it is shown that, for any function \( f \in \mathcal{F}(T^2, \mathbb{R}) \) on a 2-torus, the Kronrod–Reeb graph \( \Gamma_f \) is either a tree or has a unique cycle.

We introduce the notation

\[ \mathcal{G}_X(M, P) := \{ \pi_1 \mathcal{O}(f, X) \mid f \in \mathcal{F}(M, P) \}, \]
\[ \mathcal{M}_X(M, P) := \{ \pi_1 \mathcal{O}(f, X) \mid f \in \text{Morse}(M, P) \}, \]
\[ \mathcal{G}_X(M, P, \varepsilon) := \{ \pi_1 \mathcal{O}(f, X) \mid f \in \mathcal{F}(M, P, \varepsilon) \}, \]
\[ \mathcal{M}_X(M, P, \varepsilon) := \{ \pi_1 \mathcal{O}(f, X) \mid f \in \text{Morse}(M, P, \varepsilon) \}, \]
\begin{align*}
\mathcal{G}^\Psi & := \{ \pi_1 \mathcal{O}(f) \mid f \in \mathcal{F}(T^2, \mathbb{R}), \ \Gamma_f \text{ is a tree} \}, \\
\mathcal{M}^\Psi & := \{ \pi_1 \mathcal{O}(f) \mid f \in \text{Morse}(T^2, \mathbb{R}), \ \Gamma_f \text{ is a tree} \}, \\
\mathcal{G}^O & := \{ \pi_1 \mathcal{O}(f) \mid f \in \mathcal{F}(T^2, \mathbb{R}), \ \Gamma_f \text{ contains a unique cycle} \}, \\
\mathcal{M}^O & := \{ \pi_1 \mathcal{O}(f) \mid f \in \text{Morse}(T^2, \mathbb{R}), \ \Gamma_f \text{ contains a unique cycle} \}.
\end{align*}

It is clear that
\begin{align*}
\mathcal{M}_X(M, P) & \subset \mathcal{G}_X(M, P), \quad \mathcal{M}^\Psi \subset \mathcal{G}^\Psi, \\
\mathcal{M}_X(M, P, \varepsilon) & \subset \mathcal{G}_X(M, P, \varepsilon), \quad \mathcal{M}^O \subset \mathcal{G}^O.
\end{align*}

The following results were obtained in [1–4]:

**Proposition 1.1** [2]. Let $M$ be a connected compact surface and let $f \in \mathcal{F}(M, P)$. Then
\[ \mathcal{O}_f(f) = \mathcal{O}_f(f, \partial M) \]
and, in particular,
\[ \pi_1 \mathcal{O}(f) = \pi_1 \mathcal{O}(f, \partial M). \]
Hence,
\[ \mathcal{G}_{\partial M}(M, P) = \mathcal{G}(M, P). \]

**Proposition 1.2.** Let $M$ be a connected compact oriented surface distinct from a 2-sphere.

1. If $M$ is also distinct from a 2-torus, then $\mathcal{G}_{\partial M}(M, P) \subset \mathcal{B}$ [4].

2. If $M$ is a 2-torus, then
   \begin{enumerate}
   \item $\mathcal{G}^\Psi \subset \mathcal{T}$ [1],
   \item $\mathcal{G}^O \subset \mathcal{B}$ [3, 12].
   \end{enumerate}

More exact formulations of Proposition 1.2 are presented in Propositions 2.4, 4.1, and 4.2.

**Main Results**

By $Z(G)$ and $[G, G]$ we denote the center and the commutant of $G$, respectively. The theorem presented below shows that the number of symbols $Z$ in the realization of the group $G \in \mathcal{B}$ in the alphabet $\mathcal{A}_G$ (of the group $G \in \mathcal{T}$ in the alphabet $\mathcal{A}_T$) is uniquely determined by the group $G$.

**Theorem 1.1.** Suppose that $G \in \mathcal{B}$ ($G \in \mathcal{T}$), $\omega$ is an arbitrary realization of $G$ in the alphabet $\mathcal{A}_G$ ($\mathcal{A}_T$), and $\beta_1(\omega)$ is the number of symbols $\Xi$ in the realization $\omega$. Then the following isomorphisms take place:
\[ Z(G) \cong G/[G, G] \cong \Xi^{\beta_1(\omega)}. \]

In particular, the number $\beta_1(\omega)$ depends only on the group $G$. 
By using Proposition 1.2 and Theorem 1.1, we arrive at the following corollary:

**Corollary 1.1.** Let $M$ be a connected compact oriented surface other than a 2-sphere and let $f \in F(M, P)$, where $P = \mathbb{R}$ for $M = T^2$. Also let $G = \pi_1 \mathcal{O}(f)$, let $\omega$ be an arbitrary realization of $G$ in the alphabet $A_B$ for $M \neq T^2$ and in the alphabet $A_T$ for $M = T^2$, and let $\beta_1(\omega)$ be the number of symbols $\mathbb{Z}$ in the realization $\omega$. Then the first integer-valued homology group $H_1(\mathcal{O}_f(f), \mathbb{Z})$ of the orbit $\mathcal{O}_f(f)$ is a free Abelian group of rank $\beta_1(\omega)$:

$$H_1(\mathcal{O}_f(f), \mathbb{Z}) \simeq \mathbb{Z}^{\beta_1(\omega)}.$$

In particular, $\beta_1(\omega)$ is the first Betti number of the orbit $\mathcal{O}_f(f)$.

**Proof.** We apply the well-known Gurevich theorem [13] according to which, for any linearly connected topological space $X$, the isomorphism

$$H_1(X, \mathbb{Z}) \simeq \pi_1 X / [\pi_1 X, \pi_1 X]$$

is true. By virtue of the Gurevich theorem, Propositions 1.1 and 1.2, and Theorem 1.1, we get

$$H_1(\mathcal{O}_f(f), \mathbb{Z}) \simeq \pi_1 \mathcal{O}(f) / [\pi_1 \mathcal{O}(f), \pi_1 \mathcal{O}(f)] = G / [G, G] \simeq \mathbb{Z}^{\beta_1(\omega)}.$$

**Theorem 1.2.**

1. Suppose that $M$ is a connected compact oriented surface other than a 2-torus and a 2-sphere and that $\varepsilon : \partial M \to \{\pm 1\}$ is an arbitrary mapping from $\mathcal{E}_M$. The following assertions are true:

   (a) if $M = S^1 \times [0, 1]$ and $\varepsilon$ is a constant, i.e., takes equal values in the components of the boundary $\partial M$, then

   $$\mathcal{M}_{\partial A}(A, P, \varepsilon) = \mathcal{G}_{\partial A}(A, P, \varepsilon) = B \setminus \{1\};$$

   (b) if $M = S^1 \times [0, 1]$ and $\varepsilon$ takes different values in the components of the boundary $\partial M$ or $M \neq S^1 \times [0, 1]$, then

   $$\mathcal{M}_{\partial M}(M, P, \varepsilon) = \mathcal{G}_{\partial M}(M, P, \varepsilon) = B.$$

2. The following identities are valid:

   $$\mathcal{M}^\Psi = \mathcal{G}^\Psi = T \quad \text{and} \quad \mathcal{M}^O = \mathcal{G}^O = B^O.$$

The present paper is organized as follows:

In Sec. 2, we find $\pi_1 \mathcal{O}(f)$ for the mappings $f$ on connected compact oriented surfaces $M$ other than a 2-torus and a 2-sphere. In particular, we present the main structures used to prove Proposition 1.2(1).

Theorem 1.2(1) is proved in Sec. 3.

In Sec. 4, we find $\pi_1 \mathcal{O}(f)$ for the functions $f$ on a torus. In particular, we present the main structures used in the proof of Proposition 1.2(2). Moreover, we prove Theorem 1.2(2).

In Sec. 5, we prove Theorem 5.1 on the centers of wreath products of arbitrary groups $A$ and $B$ in the case of ineffective action of $B$ upon the set $X$. By Theorems 5.2, the center of an arbitrary group $G$ from the classes $B$ and $T$ is isomorphic to $\mathbb{Z}^{\beta_1(\omega)}$, where $\omega$ is an arbitrary realization of $G$. This theorem is the first part of one of our main results, namely, of Theorem 1.1.
In Sec. 6, we determine the commutant of the group $G \triangleright Z$ and the quotient group $G \triangleright Z/[G \triangleright Z, G \triangleright Z]$ (Theorem 6.1 and Theorem 6.2, respectively). By Theorem 6.3, the quotient group $G/[G, G]$, where $G \in \mathcal{B}$ ($G \in \mathcal{T}$), is also isomorphic to $\mathbb{Z}^{\beta_1(\omega)}$, which is the second part of Theorem 1.1.

2. Construction of a Group According to a Given Mapping

In this section, we describe the main structures used to prove Proposition 1.2(1). Let $M$ be a surface and let $f \in \mathcal{F}(M, P)$.

**Definition 2.1.** The level sets of $f$ and their connected components are called critical if they contain at least one critical point and regular if they do not contain critical points.

**Definition 2.2.** Let $X$ be a connected component of a level set of the mapping $f$. A submanifold $R \subset M$ is called an $f$-regular neighborhood of $X$ if:

(i) $R$ is connected,

(ii) the connected components of the boundary $\partial R$ are connected components of some regular level sets of the mapping $f$,

(iii) $R$ contains $X$ and $R \setminus X$ does not contain critical points of $f$.

More generally, let $X = \bigcup_{i=1}^{k} X_i$ be a disconnected union of connected components $X_i$ of the level sets of the mapping $f$. For any $X_i$, $i = 1, \ldots, k$, we choose its $f$-regular neighborhood such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Then the union $U = \bigcup_{i=1}^{k} U_i$ is called an $f$-regular neighborhood of $X$.

**Definition 2.3.** Let $X$ be a connected component of a level set of the mapping $f$, let $R_X$ be an $f$-regular neighborhood of $X$, and let $D_1, \ldots, D_q$ be connected components of $M \setminus R_X$ diffeomorphic to 2-disks. Then the union $N_X = R_X \cup D_1 \cup \ldots \cup D_q$ is called a canonical neighborhood of $X$.

In [4], it was shown that the procedure of determination of the fundamental groups of orbits of mappings from the class $\mathcal{F}(M, P)$ is reduced to the determination of these groups for mappings given on disks and cylinders. Namely, the following statement was proved:

**Proposition 2.1 ([4], Theorem 5.4).** Let $M$ be a connected compact oriented surface with negative Euler characteristic and let $f \in \mathcal{F}(M, P)$. Suppose that $K$ is the union of all nonextreme critical components of the level sets of $f$ in which the Euler characteristic of their canonical neighborhoods is negative, $R$ is an $f$-regular neighborhood of $K$, and $X_1, \ldots, X_k$ are the connected components of $M \setminus R$. Then:

(i) $X_i$ is either a 2-disk or cylinder for any $i$, $i = 1, \ldots, k$,

(ii) $\pi_1(\mathcal{O}(f, \partial M) \simeq \pi_1(\mathcal{O}(f, \partial M \cup R)) \simeq \prod_{i=1}^{k} \pi_1(\mathcal{O}(f|_{X_i}, \partial X_i))$.

The following results were obtained for the mappings on disks and cylinders:

**Proposition 2.2 ([4], Theorem 5.6).** Assume that $f \in \mathcal{F}(D^2, P)$ has a unique critical point $z$. Then $z$ is a local extremum and, in addition:

(i) if $z$ is nondegenerate, then $\pi_1(\mathcal{O}(f, \partial D^2)$ is a trivial group,

(ii) if $z$ is degenerate, then $\pi_1(\mathcal{O}(f, \partial D^2) \simeq \mathbb{Z}$.
**Proposition 2.3** ([4], Theorem 5.5). Let \( f \in \mathcal{F}(S^1 \times [0, 1], P) \). Then the following assertions are true:

(i) if \( f \) does not have critical points, then \( \pi_1 \mathcal{O}(f, S^1 \times 0) \) is a trivial group.

(ii) \( \pi_1 \mathcal{O}(f, \partial(S^1 \times [0, 1])) \cong \pi_1 \mathcal{O}(f, S^1 \times 0) \),

(iii) let \( M \) be a 2-disk or a cylinder, let \( V \) be connected component of \( \partial M \), and let \( f_M \in \mathcal{F}(M, P) \). Also let \( W \) be a regular connected component of a certain level set of the mapping \( f_M \) that splits \( M \) into two connected components and let \( A \) and \( B \) be the closures of these components; moreover, suppose that \( h(W) = W \) for all \( h \in S(f_M, V) \) and that \( V \subset A \); then \( A \) is a cylinder, \( B \) is a 2-disk or a cylinder, and the isomorphism

\[
\pi_1 \mathcal{O}(f_M, \partial M) \cong \pi_1 \mathcal{O}(f_M|_A, \partial A) \times \pi_1 \mathcal{O}(f_M|_B, \partial B)
\]

takes place.

**Corollary 2.1.** Let \( f \in \mathcal{F}(S^1 \times [0, 1], P) \), let \( R \) be an \( f \)-regular neighborhood of a connected component of \( \partial(S^1 \times [0, 1]) \), and let \( A = S^1 \times [0, 1] \setminus R \). Then

\[
\pi_1 \mathcal{O}(f, \partial(S^1 \times [0, 1])) \cong \pi_1 \mathcal{O}(f|_A, \partial A).
\]

We introduce the notation:

\((M, V)\) is one of the pairs \((D^2, \partial D^2)\) or \((S^1 \times [0, 1], S^1 \times 0)\), \( f \in \mathcal{F}(M, P) \).

\(K\) is the critical connected component of a certain level set of \( f \) “closest” to \( V \), i.e., \( K \) is the unique critical component such that the connected component \( M \setminus K \) containing \( V \) does not contain critical points.

\(R_K\) is an \( f \)-regular neighborhood of \( K \).

\(Z\) is the set of connected components of \(M \setminus R_K\). Since \( h(R_K) = R_K\) for any \( h \in S(f, V) \), \( h\) rearranges the elements of \(Z\), i.e., we observe the appearance of a natural action of \( S(f, V) \) upon \(Z\).

\(S(Z) = \{ h \in S(f, V) \mid h(Z) = Z \ \text{for each} \ Z \in Z \}\) is the inefficiency kernel of the action of \( S(f, V) \) upon \(Z\). Thus, the action of \( S(f, V)/S(Z) \) upon \(Z\) is efficient.

\(Z^{\text{fix}} = \{ X_0, X_1, \ldots, X_a \}\) is the set of elements of \(Z\) invariant under the action of \( S(f, V)\). We enumerate \( X_i \) so that \( X_0 \) is a cylinder that contains \( V \), and \( X_1 \) is a cylinder that contains the component of the boundary \( S^1 \times 1 \) in the case where \( M = S^1 \times [0, 1] \). If \( M = D^2 \), then \( X_i \) are disks for \( i = 1, \ldots, n \). At the same time, if \( M = S^1 \times [0, 1] \), then they are disks for \( i = 2, \ldots, n \).

\(Z^{\text{reg}} = Z \setminus Z^{\text{fix}} = \{ Y_1, Y_2, \ldots, Y_b \}\).

**Proposition 2.4** ([4], Theorem 5.8).

1. Let \( Z^{\text{reg}} = \emptyset \), i.e., \( Z = Z^{\text{fix}} = \{ X_0, X_1, \ldots, X_a \} \). Then

\[
\pi_1 \mathcal{O}(f, \partial M) \cong \prod_{i=1}^{a} \pi_1 \mathcal{O}(f|_{X_i}, \partial X_i) \times \mathbb{Z}.
\]
2. Let $\mathbf{Z}^\text{reg} = \{Y_i\}_{i=1}^k \neq \emptyset$. Then $S(f,V)/S(\mathbf{Z}) \simeq \mathbb{Z}_m$ for some $m \geq 2$. Moreover, $\mathbb{Z}_m$ semifreely acts upon $\mathbf{Z}$, i.e., freely acts upon $\mathbf{Z}^\text{reg}$. Hence, $m$ divides $b$ and this action has $c = b/m$ orbits. We fix any 2-disks $Y_1, Y_2, \ldots, Y_c$ that belong to pairwise different orbits of the action $\mathbb{Z}_m$. Then

\[
corresponds \quad \text{either } \mathbf{Z}^\text{fix} = \{X_0\} \quad \text{or} \quad \mathbf{Z}^\text{fix} = \{X_0, X_1\}.
\]

Moreover, the following properties are true:

(a) If $\mathbf{Z}^\text{fix} = \{X_0\}$, then

\[
\pi_1O(f, \partial M) \simeq \prod_{i=1}^c \pi_1O(f|_{Y_i}, \partial Y_i) \times \mathbb{Z};
\]

(b) if $\mathbf{Z}^\text{fix} = \{X_0, X_1\}$, then

\[
\pi_1O(f, \partial M) \simeq \prod_{i=1}^c \pi_1O(f|_{Y_i}, \partial Y_i) \times \mathbb{Z} \times (\pi_1O(f|_{X_1}, \partial X_1)).
\]

**Corollary 2.2.** Let $M = S^1 \times [0, 1]$ and $f \in \mathcal{F}(M, P, \varepsilon)$, where $\varepsilon \equiv 1$ or $\varepsilon \equiv -1$. Then $\pi_1O(f, \partial M) \in \mathcal{B} \setminus \{1\}$.

Thus, in [4], it was described how to obtain a group from the class $\mathcal{B}$ according to a given mapping. In the present work, we describe the procedure of construction of mappings

\[
f_1 \in \mathcal{F}(D^2, P) \quad \text{and} \quad f_2 \in \mathcal{F}(S^1 \times [0, 1], P)
\]

such that

\[
\pi_1O(f_1, \partial D^2) \simeq G \quad \text{and} \quad \pi_1O(f_2, S^1 \times 0) \simeq G
\]

according to a given group $G \in \mathcal{B}$.

3. **Construction of a Mapping with Given Fundamental Group of the Orbit on a Surface**

**Lemma 3.1.** Suppose that $C = S^1 \times [0, 1]$, $V = S^1 \times 0$, $f \in \mathcal{F}(C, P)$, $h \in S(f, V)$, and $W$ is a regular connected component of some level set of the mapping $f$. If $W$ is isotopic to $V$, then $h(W) = W$.

**Proof.** Denote $h(W) = W'$. Since $W$ is isotopic to $V$, $W$ splits $C$ into cylinders $A_0 \supset V$ and $A_1$. Let $h(A_0) = B_0$ and $h(A_1) = B_1$. Then $W'$ splits $C$ into cylinders $B_0 \supset V$ and $B_1$.

Assume that $W' \neq W$. Since $W$ and $W'$ are connected components of a level set of the mapping $f$, we conclude that $W \cap W' = \emptyset$. Moreover, since $W$ and $W'$ are isotopic to $V$, we have either $A_0 \subset B_0$ or $B_0 \subset A_0$. Without loss of generality, we can assume that $A_0 \subset B_0$. Let $C' = B_0 \setminus A_0$. Then $C'$ is a cylinder whose components of the boundary are $W$ and $W'$. Since $h \in S(f, V)$, we conclude that $W$ and $W'$ belong to the common level set of the mapping $f$. Hence, $A_0$ and $B_0 = A_0 \cup C'$ contain different numbers of critical points of the mapping $f$, which is impossible. Therefore, $h(W) = W$.

Lemma 3.1 is proved.
Example 3.1. Let $G = A \times B$, where $A, B \in B$, and let $C = S^1 \times [-1,1]$ be a cylinder. Assume that, for any $\varepsilon_1, \varepsilon_2 \in \mathcal{E}_C$, there exist mappings $f_A \in \mathcal{F}(C, P, \varepsilon_1)$ and $f_B \in \mathcal{F}(C, P, \varepsilon_2)$ ($f_A \in \text{Morse}(C, P, \varepsilon_1)$ and $f_B \in \text{Morse}(C, P, \varepsilon_2)$) such that the following isomorphisms are true:

$$\pi_1 O(f_A, \partial C) \simeq A \quad \text{and} \quad \pi_1 O(f_B, \partial C) \simeq B.$$ 

We now show that, for any $\varepsilon \in \mathcal{E}_C$, there exists a mapping $f \in \mathcal{F}(C, P, \varepsilon)$ ($f \in \text{Morse}(C, P, \varepsilon)$) such that

$$\pi_1 O(f, \partial C) \simeq A \times B.$$ 

One can easily construct mappings

$$f_A \in \mathcal{F}(S^1 \times [-1,0], P, \varepsilon_A), \quad f_B \in \mathcal{F}(S^1 \times [0,1], P, \varepsilon_B), \quad f'_B \in \mathcal{F}(S^1 \times [0,1], P, \varepsilon'_B)$$

$$\left(f_A \in \text{Morse}(S^1 \times [-1,0], P, \varepsilon_A), \quad f_B \in \text{Morse}(S^1 \times [0,1], P, \varepsilon_B), \quad f'_B \in \text{Morse}(S^1 \times [0,1], P, \varepsilon'_B)\right)$$

satisfying the conditions:

(i) $\pi_1 O(f_A, \partial(S^1 \times [-1,0])) \simeq A$ and

$$\pi_1 O(f_B, \partial(S^1 \times [0,1])) \simeq \pi_1 O(f'_B, \partial(S^1 \times [0,1])) \simeq B;$$

(ii) $\varepsilon_A(S^1 \times (-1)) = -1, \varepsilon_A(S^1 \times 0) = 1, \varepsilon_B(S^1 \times 0) = -1, \varepsilon_B(S^1 \times 1) = 1, \varepsilon'_B(S^1 \times 0) = -1, \varepsilon'_B(S^1 \times 1) = -1$;

(iii) $f_A, f_B, \text{and } f'_B$ coincide with the projection of $\varphi(x, t) = t$ in the vicinity of $S^1 \times 0$.

We define the maps $g \in \mathcal{F}(C, P)$ and $g' \in \mathcal{F}(C, P)$ ($g \in \text{Morse}(C, P)$ and $g' \in \text{Morse}(C, P)$) depicted in Fig. 1 so that

$$g|_{S^1 \times [-1,0]} = g'|_{S^1 \times [-1,0]} = f_A, \quad g|_{S^1 \times [0,1]} = f_B, \quad g'|_{S^1 \times [0,1]} = f'_B.$$
Then
\[ \varepsilon_g(S^1 \times (-1)) = -1, \quad \varepsilon_{g'}(S^1 \times (-1)) = -1, \]
\[ \varepsilon_g(S^1 \times 1) = 1, \quad \varepsilon_{g'}(S^1 \times 1) = -1. \]

Similarly, \( -g \in \mathcal{F}(C, P) \) and \( -g' \in \mathcal{F}(C, P) \) (\( -g \in \text{Morse}(C, P) \) and \( -g' \in \text{Morse}(C, P) \)) and, in addition,
\[ \varepsilon_{-g} = -\varepsilon_g \quad \text{and} \quad \varepsilon_{-g'} = -\varepsilon_{g'}. \]

Thus, by Proposition 2.3(3), we have the following isomorphisms:
\[ \pi_1 \mathcal{O}(g, \partial C) \simeq \pi_1 \mathcal{O}(g', \partial C) \simeq \pi_1 \mathcal{O}(-g, \partial C) \simeq \pi_1 \mathcal{O}(-g', \partial C) \simeq A \times B. \]

Thus, for any \( \varepsilon \in \mathcal{E}_C \), there exists a mapping \( f \in \mathcal{F}(C, P, \varepsilon) \) (\( f \in \text{Morse}(C, P, \varepsilon) \)) such that
\[ \pi_1 \mathcal{O}(f, \partial C) \simeq A \times B. \]

**Example 3.2.** Let \( G = A \delta_n \mathbb{Z} \), where \( A \subseteq B \), and let \( C = S^1 \times [0, 1] \) be a cylinder. Assume that there exists a mapping \( f_{DA} \in \mathcal{F}(D_A, P) \) (\( f_{DA} \in \text{Morse}(D_A, P) \)) from the disk \( D_A \) into \( P \) for which \( \pi_1 \mathcal{O}(f_{DA}, \partial D_A) \simeq A \).

Then, for any \( \varepsilon \in \mathcal{E}_C \), we can easily construct a mapping \( f \in \mathcal{F}(C, P, \varepsilon) \) (\( f \in \text{Morse}(C, P, \varepsilon) \)) such that:

(i) \( f \) has the critical level \( K \) depicted in Fig. 2 and containing \( n \) nondegenerate saddle points;

(ii) the remaining critical points lie in the disks \( D_0, \ldots, D_{n-1} \);

(iii) \( \pi_1 \mathcal{O}(f_{D_i}, \partial D_i) \simeq A, \quad i = 0, \ldots, n-1 \);

(iv) there exists an isomorphism \( h \in \mathcal{S}(f, S^1 \times 0) \) such that \( h(D_i) = D_{(i+1) \mod n}, \quad i = 0, \ldots, n-1 \).

Hence, by virtue of Proposition 2.4, we have the isomorphism \( \pi_1 \mathcal{O}(f, \partial C) \simeq A \delta_n \mathbb{Z} \).

Recall that, for any function \( \varepsilon \in \mathcal{E}_M \), the following classes are defined:
\[ G_X(M, P, \varepsilon) := \{ \pi_1 \mathcal{O}(f, X) \mid f \in \mathcal{F}(M, P, \varepsilon) \}, \quad (1) \]
\[ M_X(M, P, \varepsilon) := \{ \pi_1 \mathcal{O}(f, X) \mid f \in \text{Morse}(M, P, \varepsilon) \}. \quad (2) \]
Theorem 1.2.

1. Suppose that $M$ is a connected compact oriented surface other than a 2-torus and a 2-sphere and that $\varepsilon \in E_M$. Then the following assertions are true:

   (i) if $M$ is a cylinder and $\varepsilon \equiv 1$ or $\varepsilon \equiv -1$, then 
   \[ M_{\partial A}(A, P, \varepsilon) = G_{\partial A}(A, P, \varepsilon) = B \setminus \{1\}, \]

   (ii) if $M$ is a cylinder and $\varepsilon(\partial M) = \{\pm 1\}$ or $M$ is not a cylinder, then 
   \[ M_{\partial M}(M, P, \varepsilon) = G_{\partial M}(M, P, \varepsilon) = B. \]

Proof. The inclusion $G_{\partial M}(M, P, \varepsilon) \subset B$ and, hence, the inclusion $M_{\partial M}(M, P, \varepsilon) \subset B$ directly follow from Proposition 2.4. We now prove the opposite inclusion in case (ii) and the inclusions $B \setminus \{1\} \subset G_{\partial M}(M, P, \varepsilon)$ and $B \setminus \{1\} \subset M_{\partial M}(M, P, \varepsilon)$ in case (i).

1. We now show that $\{1\} \subset G_{\partial M}(M, P, \varepsilon)$, where $M = S^1 \times [0, 1]$ is a cylinder and $\varepsilon(\partial M) = \{\pm 1\}$. Suppose that $f \in \mathcal{F}(S^1 \times [0, 1], P, \varepsilon)$ does not have critical points. Then $\varepsilon(\partial(S^1 \times [0, 1])) = \{\pm 1\}$ and, by Proposition 2.3(1), (2), we get 
   \[ \pi_1 \mathcal{O}(f, \partial(S^1 \times [0, 1])) \simeq \pi_1 \mathcal{O}(f, S^1 \times 0) \simeq \{1\}. \]

2. We show that $B \subset G_{\partial D^2}(D^2, P, \varepsilon)$ and $B \setminus \{1\} \subset G_{\partial M}(M, P, \varepsilon)$, where $M$ is a cylinder.

Let $G \in B \setminus \{1\}$. Since $1 \in \mathbb{Z} = \mathbb{Z}$ and $B \times 1 = B$ for any $B \in B$, there exists a realization $\omega$ of the group $G$ in the form 
   \[ G = G_1 \times G_2 \times \ldots \times G_n, \]

where each group $G_i$ either has the form $G_i = \mathbb{Z}$ or $G_i = H_i \times l_i \mathbb{Z}$, if $H_i \in B \setminus \{1\}$.

By $s(\omega)$ we denote the total number of symbols $\times$ and $l_i$ for all $n \geq 1$ in the realization $\omega$.

Further, we proceed by induction on $s(\omega)$.

Base of Induction

Construction of a Mapping from $D^2$ into $P$ by the Trivial Group. Let $f \in \mathcal{F}(D^2, P)$ have a nondegenerate critical point. Then, by Proposition 2.2(1), we have \[ \pi_1 \mathcal{O}(f, \partial D^2) \simeq \{1\}. \]

Construction of a Mapping from $D^2$ into $P$ by the Group $\mathbb{Z}$. Let $f \in \mathcal{F}(D^2, P)$ have a unique degenerate critical point. Then, by virtue of Proposition 2.2(2), we have \[ \pi_1 \mathcal{O}(f, \partial D^2) \simeq \mathbb{Z}. \]

Construction of a Mapping from the Cylinder $C$ into $P$ by the Group $\mathbb{Z}$. Assume that $f \in \mathcal{F}(D^2, P)$ has a unique nondegenerate critical point, i.e., \[ \pi_1 \mathcal{O}(f, \partial D^2) \simeq \{1\}. \] It follows from Example 3.2 for $A = \{1\}$ that, for any $\varepsilon' \in E_C$, it is possible to construct a mapping $f_C \in \mathcal{F}(C, P, \varepsilon')$ such that \[ \pi_1 \mathcal{O}(f_C, \partial C) \simeq l_n \mathbb{Z} \simeq \mathbb{Z}. \]

Inductive Hypothesis

Assume that if $s(\omega) < n$, then $G \in G_{\partial D^2}(D^2, P, \varepsilon)$ and $G \in G_{\partial M}(M, P, \varepsilon)$, where $M$ is a cylinder.
Inductive Transition

It is necessary to show that if $s(\omega) = n$, then $G \in \mathcal{G}_{\partial D^2}(D^2, P, \varepsilon)$ and $G \in \mathcal{G}_{\partial M}(M, P, \varepsilon)$, where $M$ is a cylinder. In the case where $M$ is a cylinder, this follows from Examples 3.1 and 3.2. We now show that this statement remains true for a disk.

Given a mapping $f_{C'} \in \mathcal{F}(C', P, \varepsilon')$ from the cylinder $C'$ into $P$, where $\varepsilon' \in \mathcal{E}_{C'}$, and knowing how to construct a mapping from the disk into $P$ by using the trivial group, one can construct a mapping $f \in \mathcal{F}(D, P)$ from the disk $D$ into $P$ such that

$$\pi_1 \mathcal{O}(f, \partial D) \simeq \pi_1 \mathcal{O}(f_{C'}, \partial C').$$

Thus, we can find the mapping $f \in \mathcal{F}(D, P)$ from the disk $D$ of radius 2 into $P$ such that:

(i) $f(\tilde{D}) = [1, 2]$ and $f(D \setminus \tilde{D}) = [0, 1]$, where $\tilde{D} \subset D$ is a disk of radius 1 centered at the center of the disk $D$,

(ii) $\pi_1 \mathcal{O}(f|_{\tilde{D}}, \partial(\tilde{D})) \simeq \{1\}$ and $\pi_1 \mathcal{O}(f|_{D \setminus \tilde{D}}, \partial(D \setminus \tilde{D})) \simeq \pi_1 \mathcal{O}(f_{C'}, \partial C').$ 

Note that the preimage $f^{-1}(1) = S^1 \times 1$ is invariant under any $h \in \mathcal{S}(f, \partial D)$. Therefore, by Proposition 2.3(3),

$$\pi_1 \mathcal{O}(f, \partial D) \simeq \pi_1 \mathcal{O}(f|_{D \setminus \tilde{D}}, \partial(D \setminus \tilde{D})) \times \pi_1 \mathcal{O}(f_{\tilde{D}}, \partial\tilde{D}) \simeq \pi_1 \mathcal{O}(f_{C'}, \partial C') \times \{1\}.$$

Hence, $G \in \mathcal{G}_{\partial D^2}(D^2, P, \varepsilon)$ and $G \in \mathcal{G}_{\partial M}(M, P, \varepsilon)$, where $M$ is a cylinder.

3. Assume that the surface $M$ is neither a disk, nor a cylinder. It is necessary to show that $B \subset \mathcal{G}_{\partial M}(M, P, \varepsilon)$.

We now construct a mapping $f \in \mathcal{F}(M, P, \varepsilon)$ with at least one point of local maximum such that all its saddle points lie on the common connected component $K$ of some level set of the mapping $f$ and all its local extrema are nondegenerate. Since $B \subset \mathcal{G}_{\partial D^2}(D^2, P)$, for $G \in B$, there exists a mapping $f_{D^2} \in \mathcal{F}(D^2, P)$ from the disk $D^2$ into $P$ such that $\pi_1 \mathcal{O}(f_{D^2}, \partial D^2) \simeq G$. We change the mapping $f$ in an $f$-regular neighborhood $D_m$ of a certain point of local maximum in order to guarantee that $\pi_1 \mathcal{O}(f|_{D_m}, \partial D_m) \simeq G$. Also let $N$ be a canonical neighborhood of the component $K$ containing all connected components of the boundary $\partial M$. Then $N = M$ and, hence, $N$ has a negative Euler characteristic. Let $R$ be an $f$-regular neighborhood of the component $K$ that contains all connected components of the boundary $\partial M$ and is such that $D_m$ is a connected component of the closure $\overline{M \setminus R}$. All other connected components of $\overline{M \setminus R}$ are also disks. We denote them by $D_1, D_2, \ldots, D_n$. Thus, by Proposition 2.1, we have the following isomorphism:

$$\pi_1 \mathcal{O}(f, \partial M) \simeq \pi_1 \mathcal{O}(f|_{D_m}, \partial D_m) \times \prod_{i=1}^{n} \pi_1 \mathcal{O}(f|_{D_i}, \partial D_i).$$

By Proposition 2.2, for any $i = 1, \ldots, n$, we obtain $\pi_1 \mathcal{O}(f|_{D_i}, \partial D_i) \simeq \{1\}$. Hence,

$$\pi_1 \mathcal{O}(f, \partial M) \simeq \pi_1 \mathcal{O}(f|_{D_m}, \partial D_m) \simeq G.$$

4. It is easy to see that not only the established inclusions but also similar inclusions for the fundamental groups of orbits of the Morse functions are true, namely,

(i) $\{1\} \subset \mathcal{M}_{\partial M}(M, P, \varepsilon)$, where $M = S^1 \times [0, 1]$ is a cylinder and $\varepsilon(\partial M) = \{\pm 1\};$
Construction of a Group According to a Given Function on the Torus

To this end, it suffices to repeat the reasoning presented above by replacing the mapping by the group \( \mathbb{Z} \) from \( D^2 \) into \( P \) as follows:

**Construction of a Mapping by the Group \( \mathbb{Z} \) from \( D^2 \) into \( P \)**. Let \( f \in \text{Morse}(D_A, P) \) be a mapping from the 2-disk \( D_A \) into \( P \) with a unique nondegenerate critical point, i.e., \( \pi_1 O(f, \partial D_A) \cong \{1\} \). As in Example 3.2, for \( A = \{1\} \), we can construct a mapping \( f_{D^2} \in \text{Morse}(D^2, P) \) from the disk \( D^2 \) into \( P \) such that

\[
\pi_1 O(f_{D^2}, \partial D^2) \cong 1 \cdot n \cdot \mathbb{Z} \cong \mathbb{Z}.
\]

Theorem 1.2(1) is proved.

**Remark 3.1.** By virtue of Proposition 2.3(1), (3), in the case where \( M \) is a cylinder and \( \varepsilon \equiv 1 \) or \( \varepsilon \equiv -1 \), it is possible to construct \( f \in \mathcal{F}(M, P, \varepsilon) \) (\( f \in \text{Morse}(M, P, \varepsilon) \)) such that \( f(\partial M) = \text{const} \).

4. Construction of a Group According to a Given Function on the Torus

Let \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) be a 2-torus, let \( f \in \mathcal{F}(T^2, \mathbb{R}) \), and let \( \Gamma_f \) be the Kronrod–Reeb graph of the function \( f \). By \( p_f : T^2 \to \Gamma_f \) we denote the projection of \( T^2 \) onto \( \Gamma_f \). Let \( D_{\text{id}}(T^2) \) be a connected component of the identity mapping \( \text{id}_{T^2} \) in \( D(T^2) \) and let \( S'(f) = S(f) \cap D_{\text{id}}(T^2) \).

Recall that \( \Gamma_f \) is either a tree or contains a unique cycle. In both cases, the structure of \( \pi_1 O(f) \) is described in Propositions 4.1 and 4.2.

1. Let \( \Gamma_f \) be a tree. In this case, by Lemma 2.4 in [1], there exists a unique critical level \( K \) of the function \( f \) such that, for an \( f \)-regular neighborhood \( N_K \) of the level \( K \) invariant under \( S'(f) \), all connected components of the closure \( T^2 \setminus N_K \) are 2-disks. We denote these 2-disks by \( D_1, \ldots, D_b \) and say that \( K \) is a special level. Since \( N_K \) is invariant under \( S'(f) \), the disks \( D_1, \ldots, D_b \) are also invariant under \( S'(f) \). Hence, every \( h \in S'(f) \) induces a permutation \( \rho(h) \) of the disks \( D_1, \ldots, D_b \). Thus, we get the homomorphism \( \rho : S'(f) \to \Sigma \{D_i\}_{i=1}^b \) from \( S'(f) \) into the group of permutations of the disks \( D_1, \ldots, D_b \). Let \( G_K := \rho(S'(f)) \) be a subgroup of \( \Sigma \{D_i\}_{i=1}^b \).

**Proposition 4.1** [1, 6, 14]. The following properties are true:

(i) the isomorphism \( G_K \cong \mathbb{Z}_m \times \mathbb{Z}_n \) takes place for some \( n, m \geq 1 \);

(ii) there exists a section \( s : G_K \to S'(f) \) of the homomorphism \( \rho \), i.e., a homomorphism such that

\[
\rho \circ s = \text{id}_{G_K};
\]

(iii) the subgroup \( s(G_K) \cong \mathbb{Z}_m \times \mathbb{Z}_n \) freely acts on \( T^2 \) and, hence, also on \( D_1, \ldots, D_b \); if \( r \) is the number of orbits of this action, then each orbit consists of the same number of disks \( mn \) and, therefore, \( b = mn r \);

(iv) if one disk is chosen in each orbit of the free action of \( \mathbb{Z}_m \times \mathbb{Z}_n \) and the disks obtained as a result are denoted by \( D_1, \ldots, D_r \), then the following isomorphism takes place:

\[
\pi_1 O(f) \cong \left( \prod_{i=1}^r \pi_1 O(f|_{D_i}, \partial D_i) \right) \mathbb{Z}_m \times \mathbb{Z}_n.
\]
2. Assume that \( \Gamma_f \) contains a unique cycle. We choose any point \( x \) on an arbitrary edge of this cycle. Then \( C = p_f^{-1}(x) \) is a regular connected component of a level set of the function \( f \) that does not split \( T^2 \). Let

\[
C = \{ h(C) \mid h \in S'(f) \} = \{ C_0 = C, C_1, \ldots, C_{n-1} \}.
\]

Since each \( C_i, i = 0, \ldots, n - 1 \), does not split the torus and all \( C_i \) are mutually disjoint, they are pairwise isotopic. Thus, for \( n \geq 2 \) and each \( i = 0, \ldots, n - 1 \), these curves can be redefined so that the curves \( C_i \) and \( C_{(i+1) \mod n} \) bound a cylinder that does not contain any other curves from \( C \).

Let \( R_{C_i} \) be \( f \)-regular neighborhoods of \( C_i, i = 0, \ldots, n - 1 \), such that

\[
\{ h(R_C) \mid h \in S'(f) \} = \{ R_{C_0}, R_{C_1}, \ldots, R_{C_{n-1}} \}.
\]

Then the connected components of the closure \( T^2 \setminus \bigcup_{i=0}^{n-1} R_{C_i} \) are cylinders. By \( Q \) we denote an arbitrary cylinder of this type.

**Proposition 4.2** [3, 12].

1. There exists \( h \in S'(f) \) such that:

   (a) \( h \) does not have fixed points,
   (b) \( h^n = \text{id}_{T^2} \),
   (c) \( h(C_i) = C_{(i+1) \mod n}, i = 0, \ldots, n - 1 \).

Thus, \( h \) induces a free action of \( \mathbb{Z}_n \) on \( T^2 \) that preserves \( f \), is invariant, and cyclically rearranges the components \( C_i, i = 0, \ldots, n - 1 \). This gives the following commutative diagram:

\[
\begin{array}{ccc}
T^2 & \xrightarrow{f} & \mathbb{R} \\
p & & g \\
\downarrow & & \downarrow \\
T^2/\mathbb{Z}_n & \xrightarrow{p} & T^2/\mathbb{Z}_n,
\end{array}
\]

where the factor mapping \( p : T^2 \to T^2/\mathbb{Z}_n \) in an \( n \)-sheeted covering of the torus \( T^2/\mathbb{Z}_n \) and the function \( g \in \mathcal{F}(T^2/\mathbb{Z}_n, \mathbb{R}) \) has a Kronrod–Reeb graph with unique cycle.

2. The following isomorphism takes place:

\[
\pi_1 \mathcal{O}(f) \simeq \pi_1 \mathcal{O}(f|_Q, \partial Q) \wr_n \mathbb{Z}.
\]

Recall that \( \mathcal{T} \) is the set of classes of groups isomorphisms formed by groups of the form \( G \wr_{n,m} \mathbb{Z}^2 \), where \( G \in \mathcal{B} \) and \( n, m \geq 1 \). Moreover, \( \mathcal{B}^O \) is a subclass of \( \mathcal{B} \) formed by the groups \( (A \times B) \wr_n \mathbb{Z} \), where \( A, B \in \mathcal{B} \setminus \{1\} \) and \( n \geq 1 \).

**Theorem 1.2.** 2. The identities

\[
\mathcal{M}^\Psi = \mathcal{G}^\Psi = \mathcal{T} \quad \text{and} \quad \mathcal{M}^O = \mathcal{G}^O = \mathcal{B}^O
\]

are true.
Proof. By using Proposition 4.1, we arrive at the inclusions
\[ G^\Psi \subset \mathcal{T}, \quad \mathcal{M}^\Psi \subset \mathcal{T}. \]

1. We now show that the inclusions \( G^O \subset B^O \) and \( \mathcal{M}^O \subset B^O \) are true.

Let \( f \in \mathcal{F}(T^2, \mathbb{R}) \) be a function on the torus such that \( \Gamma_f \) contains a single cycle. Then, by Proposition 4.2, we have the isomorphism
\[ \pi_1 O(f) \simeq \pi_1 (f|_Q, \partial Q) \wr \mathbb{Z}, \]
where \( Q \subset T^2 \) is the cylinder defined above. By \( \partial Q^{-1} \) and \( \partial Q^1 \) we denote the connected components of the boundary \( \partial Q \). Then \( f(\partial Q^{-1}) < f(C) \) and \( f(\partial Q^1) < f(C) \), where \( C \in \mathcal{C} \). Note that \( \epsilon_{f|_Q}(\partial Q) = \pm 1 \). Thus, by the theorem on mean value, there exists a connected component \( W \) of the level set of mapping \( f \) for which \( f(W) = f(C) \). By \( C_{-1} \) and \( C_1 \) we denote the cylinders into which \( W \) splits \( Q \). Thus, by virtue of Proposition 2.3(3), we arrive at the isomorphism
\[ \pi_1 O(f|_Q, \partial Q) \simeq \pi_1 O(f|_{C_{-1}}, \partial C_{-1}) \times \pi_1 O(f|_{C_1}, \partial C_1). \]

Moreover, \( \epsilon_{f|_{C_{-1}}} (\partial C_{-1}) \equiv -1 \) and \( \epsilon_{f|_{C_1}} (\partial C_1) \equiv 1 \). Hence, it follows from Corollary 2.2 that
\[ \pi_1 O(f|_{C_{-1}}, \partial C_{-1}) \quad \text{and} \quad \pi_1 O(f|_{C_1}, \partial C_1) \in B^O. \]

Therefore, \( G^O \subset B^O \). This yields \( \mathcal{M}^O \subset B^O \).

2. We now show that the following inclusions are true:
\[ \mathcal{T} \subset G^\Psi, \quad \mathcal{T} \subset \mathcal{M}^\Psi, \quad B^O \subset G^O, \quad B^O \subset \mathcal{M}^O. \]

For the group \( G \), we construct a function \( f \in \mathcal{F}(T^2, \mathbb{R}) \) \((f \in \text{Morse}(T^2, \mathbb{R}))\) such that \( \pi_1 O(f) \simeq G \).

\( \Gamma_f \) is a tree. Let \( G \in \mathcal{T} \). Then \( G = A_{m,n} \mathbb{Z} \), where \( A \in \mathcal{B}, \ n, m \geq 1 \).

We construct a function \( f_0 \in \mathcal{F}(T^2, \mathbb{R}) \) \((f_0 \in \text{Morse}(T^2, \mathbb{R}))\) depicted in Fig. 3 with two saddle critical points on the critical level \( K \), namely, a nondegenerate maximum \( v_0 \), and a nondegenerate minimum \( v_1 \).
Then, for an $f_0$-regular vicinity $R_K$ of the level $K$ invariant under $S'(f_0)$, the closure $\overline{T^2 \setminus R_K}$ consists of two 2-disks: a disk $D_0$ containing the point of maximum and a disk $D_1$ containing the point of minimum. We change the function $f_0$ in an $f_0$-regular vicinity $D_0$ of the point of maximum so that

\[ \pi_1 \mathcal{O}(f|_{D_0}, \partial D_0) \simeq A, \]

which is possible by virtue of Theorem 1.2(1).

We take an $mn$-sheet covering $p: T^2 \to T^2$ given by the formula $p(x, y) = (mx \mod 1, ny \mod 1)$, where $x \in \mathbb{R}/\mathbb{Z}$ and $y \in \mathbb{R}/\mathbb{Z}$, and depicted in Fig. 4 for $m = 3$ and $n = 4$. Let

\[ p^{-1}(D_0) = \{ D_{i,j}^{0,m,n} \}_{i,j=1} \quad \text{and} \quad p^{-1}(D_1) = \{ D_{i,j}^{1,m,n} \}_{i,j=1} \]

be the sets of connected components of the preimages of the disks $D_0$ and $D_1$. Then the function $f: T^2 \to \mathbb{R}$ given by the formula $f := f_0 \circ p$ is just the required function. Indeed, it is easy to see that $\Gamma_f$ is a tree and $K$ is a special level. By virtue of Proposition 4.1, for some $k, l \geq 1$, we obtain $G_K \simeq \mathbb{Z}_k \times \mathbb{Z}_l$.

Further, we show that $G_K \simeq \mathbb{Z}_m \times \mathbb{Z}_n$. Note that the diffeomorphisms of the torus

\[ h_{i,j}(x, y) = \left( x + \frac{i}{m} \mod 1, y + \frac{j}{n} \mod 1 \right), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n, \]

belong to $S'(f)$. Hence, the group of these diffeomorphisms $\mathcal{H} = \{ h_{i,j} \}_{i,j=1}^{m,n}$ is a subgroup of $S'(f)$.

Note that $\rho(\mathcal{H})$ is injectively embedded in $G_K$ and the sets

\[ \mathbb{D}_0 = \{ D_{i,j}^{0,m,n} \}_{i,j=1}^{m,n} \quad \text{and} \quad \mathbb{D}_1 = \{ D_{i,j}^{1,m,n} \}_{i,j=1}^{m,n} \]

are invariant under $S'(f)$. Since, by Proposition 4.1, the group $s(G_K)$ freely acts upon $\{ D_{i,j}^{0,m,n} \}_{i,j=1}^{m,n}$, the number of elements in $s(G_K)$ and $G_K$ coincides with the number of disks on the orbit of the disk $D_0^{1,1}$, i.e., is equal to $mn$. The number of elements in $\rho(\mathcal{H})$ is also equal to $mn$. Hence,

\[ G_K = \rho(\mathcal{H}) \simeq H \simeq \mathbb{Z}_m \times \mathbb{Z}_n. \]
Thus, by Propositions 4.1 and 2.2, we find
\[
\pi_1 \mathcal{O}(f) \simeq \left( \pi_1 \mathcal{O}(f|_{D_0^{1,1}}) \times \pi_1 \mathcal{O}(f|_{D_1^{1,1}}) \right) \lambda_{m,n} \mathbb{Z}^2 \\
\simeq (A \times 1) \lambda_{m,n} \mathbb{Z}^2 \simeq A \lambda_{m,n} \mathbb{Z}^2.
\]

**Γ_{f} is not a tree.** Let \( G \in B^O \). Then \( G = (A \times B) \lambda_n \mathbb{Z} \), where \( A, B \in B \setminus \{1\} \) and \( n \geq 1 \). We construct a function \( g \in \mathcal{F}(T^2, \mathbb{R}) \) (\( g \in \text{Morse}(T^2, \mathbb{R}) \)) with two saddle critical points (one maximum and one minimum). Let \( x \) be a point on the edge of this cycle, let \( C = p_f^{-1}(x) \) be the corresponding regular connected component of some level set of the function \( g \), and let \( C' \) be the second connected component of this level set. By \( R_C \) and \( R_{C'} \) we denote regular vicinities of the components \( C \) and \( C' \), respectively. Then the connected components \( C_A \) and \( C_B \) of the closure \( \overline{T^2 \setminus (R_C \cup R_{C'})} \) are cylinders and \( g|_{C_A} \in \mathcal{F}(C_A, \mathbb{R}, +1) \), \( g|_{C_B} \in \mathcal{F}(C_B, \mathbb{R}, -1) \). Moreover, \( g(\partial C_A) = a \) and \( g(\partial C_B) = b \) for some \( a, b \in \mathbb{R} \).

By Theorem 1.2(1) and Remark 3.1, we can choose functions \( f_A \in \mathcal{F}(C_A, \mathbb{R}, -1) \), \( f_B \in \mathcal{F}(C_B, \mathbb{R}, +1) \) (\( f_A \in \text{Morse}(C_A, \mathbb{R}, -1) \), and \( f_B \in \text{Morse}(C_B, \mathbb{R}, +1) \)) such that
\[
\pi_1 \mathcal{O}(f_A) = A \quad \text{and} \quad \pi_1 \mathcal{O}(f_B) = B.
\]
Replacing \( g|_{C_A} \) and \( g|_{C_B} \) with \( f_A \) and \( f_B \), respectively, we get \( g \in \mathcal{F}(T^2, \mathbb{R}) \) (\( g \in \text{Morse}(T^2, \mathbb{R}) \)).

We now take an \( n \)-sheet covering \( p: T^2 \to T^2 \) given by the formula
\[
p(x, y) = (nx \mod 1, y),
\]
where \( x \in \mathbb{R}/\mathbb{Z} \) and \( y \in \mathbb{R}/\mathbb{Z} \). Note that all connected components of the closure \( \overline{T^2 \setminus p^{-1}(R_C)} \) are cylinders. By \( Q \) we denote an arbitrary cylinder of this kind.

Thus, by Proposition 4.2, \( f := g \circ p \) is the required function, i.e., \( f \in \mathcal{F}(T^2, \mathbb{R}) \) (\( f \in \text{Morse}(T^2, \mathbb{R}) \)), \( \Gamma_f \) contains a unique cycle, and
\[
\pi_1 \mathcal{O}(f) \simeq \pi_1 \mathcal{O}(f|_Q, \partial Q) \lambda_n \mathbb{Z} \overset{\text{Corollary 2.1}}{\simeq} \lambda_n \mathbb{Z} \simeq (A \times B) \lambda_n \mathbb{Z}.
\]

Theorem 1.2(2) is proved.

5. Centers of Wreath Products

Let \( A \) and \( B \) be two groups. Assume that \( B \) acts upon the set \( X \). In other words, we have a homomorphism \( \varphi \) from \( B \) into the permutation group \( \Sigma(X) \). For \( b \in B \), by \( \varphi_b: X \to X \) we denote the corresponding permutation. Also let \( \text{Map}(X, A) \) be the group of all mappings \( f: X \to A \) with respect to the operation of pointwise multiplication. Then the group \( B \) acts upon \( \text{Map}(X, A) \) according to the following rule: the result of action of \( b \in B \) upon \( f \in \text{Map}(X, A) \) is the composition
\[
f \circ \varphi_b: X \to X \to A.
\]

The semidirect product \( \text{Map}(X, A) \ltimes \varphi B \) with respect to this action is called the *unbounded wreath product* of \( A \) and \( B \) and denoted by \( \text{AWr}_X B \). Thus, this is, in fact, the direct product \( \text{Map}(X, A) \times B \) with the operation
of multiplication given by the formula

\[(f_1, b_1) \cdot (f_2, b_2) = ((f_1 \circ \varphi_{b_2}) \cdot f_2, b_1 \cdot b_2)\]

for \((f_1, b_1), (f_2, b_2) \in \text{Map}(X, A) \times \varphi B\).

By \(\sigma(f)\) we denote the support of the function \(f \in \text{Map}(X, A)\):

\[\sigma(f) = \{x \in X \mid f(x) \neq e, \text{ where } e \text{ is the identity of the group } A\}.\]

By \(\text{Map}_{\text{fin}}(X, A)\) we denote the subset \(\text{Map}(X, A)\) formed solely by functions with finite support \(|\sigma(f)| < \infty\). The semidirect product \(\text{Map}_{\text{fin}}(X, A) \times \varphi B\) is called bounded wreath product and denoted by \(\text{Awr}_X B\).

**Remark 5.1.** If \(X = \mathbb{Z}_n\) and \(B = \mathbb{Z}\) acts upon \(X\) by cyclic shifts, then \(\text{Awr}_{\mathbb{Z}_n} \mathbb{Z}\) is the wreath product \(A \wr \mathbb{Z}\). If \(X = \mathbb{Z}_n \times \mathbb{Z}_m\) and \(B = \mathbb{Z}^2\) acts upon \(X\) by 2-cyclic shifts, then \(\text{Awr}_{\mathbb{Z}_n \times \mathbb{Z}_m} \mathbb{Z}^2\) is the wreath product \(A \wr_{\mathbb{Z}_n \times \mathbb{Z}_m} \mathbb{Z}^2\).

By \(Z(A)\) we denote the center of the group \(A\). Let \(\tilde{D}(A)\) be a subgroup \(\text{Map}(X, Z(A))\) of functions \(h : X \rightarrow Z(A)\) constant on each orbit of the action of \(B\) upon \(X\) and let \(D(A)\) be the subgroup \(\text{Map}_{\text{fin}}(X, Z(A))\) of functions with the same property.

Theorem 4.2 in [15] yields the isomorphisms

\[Z(\text{Awr}_X B) \cong \tilde{D}(A), \quad Z(\text{Awr}_X B) \cong D(A),\]

where the action of the group \(B\) upon \(X\) is efficient.

In the case where the action of \(B\) upon \(X\) is inefficient, for arbitrary groups \(A\) and \(B\), we get a more general situation. In this section, we generalize Theorem 4.2 in [15] and consider the case of inefficient action of \(B\) upon \(X\).

By \(\mathcal{O}\) we denote the set of all orbits of \(B\) on \(X\). Further, by \(\mathcal{O}_{\text{fin}}\) we denote the set of finite orbits. Recall that the direct product of the sets \(V\) indexed by an infinite set consists of all infinite sequences of elements from \(V\), whereas the direct sum contains only sequences with finitely many nonzero elements.

**Theorem 5.1.** The following isomorphisms take place:

\[Z(\text{Awr}_X B) = \tilde{D}(A) \times (\ker \varphi \cap Z(B)) \cong \left( \prod_{\lambda \in \mathcal{O}} Z(A) \right) \times (\ker \varphi \cap Z(B)), \quad (3)\]

\[Z(\text{Awr}_X B) = D(A) \times (\ker \varphi \cap Z(B)) \cong \left( \bigoplus_{\lambda \in \mathcal{O}_{\text{fin}}} Z(A) \right) \times (\ker \varphi \cap Z(B)). \quad (4)\]

Let \(\tilde{Q}\) be a subgroup of \(\text{Awr}_X B\) whose elements \((f, l)\) satisfy the conditions:

(a) \(f\) is constant on each orbit of \(B\) on \(X\), i.e., \(f(x) = a_\lambda\) for any \(x \in O_\lambda, O_\lambda \in \mathcal{O}\), and each \(a_\lambda\) is an element of the center \(Z(A)\),

(b) \(l \in \ker \varphi \cap Z(B)\), where \(Z(B)\) is the center of \(B\).
It is clear that if an element from $AWr_X B$ satisfies conditions (a) and (b), then this element belongs to the center and, therefore, $\tilde{Q} \subset Z(AWr_X B)$.

For any $y \in X$ and $c \in A$, we define a map $g_{y,c} \in \text{Map}(X, A)$ by the formula

$$
g_{y,c}(x) = \begin{cases} c & \text{for } x = y, \\ e & \text{for } x \neq y. \end{cases}
$$

Let $S$ be a set of elements $(g_{y,c}, p)$ from $AWr_X B$, where $p \in B$. The set $S$ is also a subset of $Awr_X B$. By $\tilde{C}(S)$ and $C(S)$ we denote the centralizer of the set $S$ in $AWr_X B$ and the centralizer of the set $S$ in $Awr_X B$, respectively. It is clear that $Z(AWr_X B) \subset \tilde{C}(S)$ and $Z(Awr_X B) \subset C(S)$. Thus, we arrive at the following embedding for the group $\tilde{Q}$:

$$
\tilde{Q} \subset Z(AWr_X B) \subset \tilde{C}(S).
$$

**Lemma 5.1.** The equalities

$$Z(AWr_X B) = \tilde{C}(S) = \tilde{Q}
$$

are true.

**Proof.** It suffices to verify the embedding $\tilde{C}(S) \subset \tilde{Q}$.

Assume that $(g_{y,c}, p) \in S$ and $(f, l) \in \tilde{C}(S)$, where $f, g_{y,c} \in \text{Map}(X, A)$, $g_{y,c}$ is defined in (5), and $l, p \in B$. Thus, by definition, we get the equality

$$(f, l)(g_{y,c}, p) = (g_{y,c}, p)(f, l).$$

Therefore,

$$
((f \circ \varphi_p) \cdot g_{y,c}, lp) = ((g_{y,c} \circ \varphi_l) \cdot f, pl).
$$

Hence, $lp = pl$ for any $p$, whence it follows that $l \in Z(B)$, and we find

$$(f \circ \varphi_p(x)) \cdot g_{y,c}(x) = (g_{y,c} \circ \varphi_l(x)) \cdot f(x).$$

For $x \neq y, x \neq \varphi_l^{-1}(y)$ in (6), we get

$$f \circ \varphi_p(x) = f(x).$$

Equality (7) is true for any $p \in B$. Hence, $f$ takes the same value on the entire orbit.

Note that it is possible to choose $g_{y,c}(x)$ with another fixed element $y$. Thus, $f$ is constant on each orbit $B$ on $X$, i.e., $f(x) = a_\lambda$ for any $x \in O_\lambda$, $O_\lambda \in O$.

It remains to show that each $a_\lambda$ is an element of the center $Z(A)$ and $l \in \ker \varphi$.

To do this, we analyze equality (6) for $x = y$. There are two possible cases:

(i) if $\varphi_l(y) = y$, then

$$f(y)g_{y,c}(y) = g_{y,c}(y)f(y),$$
(ii) if \( \varphi_l(y) \neq y \), then
\[
f(y)g_{y,c}(y) = f(y).
\]
The second case is impossible because \( g_{y,c}(y) \neq e \). Hence, \( l \in \ker \varphi \). In the first case, we conclude that
\[
f(y) \in Z(A) \quad \text{for any} \quad y \in X
\]
because \( g_{y,c}(y) = c \), where \( c \) is an arbitrary element of \( A \). Thus, conditions 1 and 2 are satisfied.

**Lemma 5.2.** The center \( Z(Awr_X B) \) is the intersection of \( Z(AWr_X B) \) with \( Awr_X B \), i.e.,
\[
Z(Awr_X B) = Z(AWr_X B) \cap (Awr_X B).
\]

**Proof.** Indeed, the embedding
\[
(Z(AWr_X B) \cap (Awr_X B)) \subset Z(Awr_X B)
\]
is obvious.

We now check the opposite embedding. Assume that \((f,l) \in Z(Awr_X B)\). Since \( S \subset Awr_X B \), we get
\[
Z(Awr_X B) \subset C(S) \subset \tilde{C}(S) \cap (Awr_X B) \overset{\text{Lemma 5.1}}{=} Z(AWr_X B) \cap (Awr_X B).
\]

**Proof of Theorem 5.1**

For the unbounded wreath product \( AWr_X B \), the number of orbits in \( Z(AWr_X B) \) can be infinite. At the same time, for the bounded wreath product \( Awr_X B \), the number of orbits in \( Z(AWr_X B) \) can be only finite.

According to Lemmas 5.1 and 5.2, the bijections
\[
\psi_1 : Z(AWr_X B) \rightarrow \prod_{\lambda \in A} Z(A) \times (\ker \varphi \cap Z(B)),
\]
\[
\psi_2 : Z(Awr_X B) \rightarrow \bigoplus_{\lambda \in A_{\text{fin}}} Z(A) \times (\ker \varphi \cap Z(B))
\]
given by the formulas \( \psi_1(f,l) = (a_1, a_2, \ldots, a_{s_1}, l) \) and \( \psi_2(f,l) = (a_1, a_2, \ldots, a_{s_2}, l) \) are true. It is easy to see that \( \psi_1 \) and \( \psi_2 \) are homomorphisms.

Theorem 5.1 is proved.

**Corollary 5.1.**
\[
Z\left( A \uparrow \mathbb{Z} \right) = \{(a, a, \ldots, a, nk) | a \in Z(A), k \in \mathbb{Z}\} \cong D(A) \times n\mathbb{Z} \cong Z(A) \times \mathbb{Z},
\]
where

It is clear that if and the class in the first case, we obtain (8) and (9). Relation (10) is obvious. Thus,

\[ Z(A \times B) \cong Z(A) \times Z(B). \]  

Indeed, for the groups \( A \times n \) and \( A \times n,m \), we have only one orbit of the action \( B \) on \( X \). By virtue of Theorem 5.1, we obtain (8) and (9). Relation (10) is obvious. Thus,

\[
Z \left( \left( \mathbb{Z}_3 \times (\mathbb{Z}_5 \times \mathbb{Z}) \right) \times \mathbb{Z} \right) \cong Z \left( \left( \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z} \right) \right) \times 7\mathbb{Z} \\
\cong Z \left( \left( \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z} \right) \right) \times \mathbb{Z} \\
\cong Z \left( \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z} \right) \times \mathbb{Z} \\
\cong \mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z} \times 5\mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^4 \times \mathbb{Z}.
\]

**Theorem 5.2.** Suppose that \( G \in B \) \((G \in T)\), \( \omega \) is an arbitrary realization of \( G \) in the alphabet \( A_B \) \((A_T)\), and \( \beta_1(\omega) \) is the number of symbols \( \mathbb{Z} \) in the realization \( \omega \). Then \( Z(G) \cong \mathbb{Z}^{\beta_1(\omega)} \).

**Proof.** The assertion of the theorem follows from Corollary 5.1 by induction on the number of symbols 1 and \( \mathbb{Z} \) in the realization \( \omega \), which is denoted by \( l(\omega) \). For the sake of convenience, by \( \bar{\omega} \) we denote a group from the class \( B \) \((T)\) specified by the word \( \omega \). In particular, \( \omega \) is a realization of \( \bar{\omega} \) in the alphabet \( A \) \((or A_T)\). Since \( \bar{\omega} \) is a group isomorphic to \( G \), we conclude that

\[ Z(G) = Z(\bar{\omega}). \]

It is clear that if \( l(\omega) = 1 \), then \( \omega \) is either 1 or \( \mathbb{Z} \). Hence, \( Z(G) \cong 1 \) or \( Z(G) \cong \mathbb{Z} \), respectively. Assume that \( Z(G) \cong \mathbb{Z}^{\beta_1(\omega)} \) for all words with \( l(\omega) \leq k \). We prove this fact for \( l(\omega) = k + 1 \). In this case, the realization \( \omega \) is either (i) the direct product \( \omega_1 \times \omega_2 \) such that \( l(\omega_1) + l(\omega_2) = k + 1 \), \( l(\omega_1) \leq k \), and \( l(\omega_2) \leq k \), or (ii) the wreath product \( \omega_1 \wr_n \mathbb{Z} \), where \( l(\omega_1) = k \), or (iii) the wreath product \( \omega_1 \wr_{n,m} \mathbb{Z}^2 \), where \( l(\omega_1) = k - 1 \). By using Corollary 5.1, the inductive hypothesis, and the following obvious result:

\[ \beta_1(\omega_1) + \beta_1(\omega_2) = \beta_1(\omega_1 \times \omega_2), \]

we obtain

\[
Z(\bar{\omega_1} \times \bar{\omega_2}) \cong Z(\bar{\omega_1}) \times Z(\bar{\omega_2}) \cong \mathbb{Z}^{\beta_1(\omega_1)} \times \mathbb{Z}^{\beta_1(\omega_2)} \\
\cong \mathbb{Z}^{\beta_1(\omega_1) + \beta_1(\omega_2)} \cong \mathbb{Z}^{\beta_1(\omega_1 \times \omega_2)} \cong \mathbb{Z}^{\beta_1(\omega)}
\]

in the first case,

\[
Z(\bar{\omega}) \cong Z(\bar{\omega_1} \wr_n \mathbb{Z}) \cong Z(\bar{\omega_1}) \times Z \cong \mathbb{Z}^{\beta_1(\omega_1)} \times \mathbb{Z} \cong \mathbb{Z}^{\beta_1(\omega_1 \wr_n \mathbb{Z})} \cong \mathbb{Z}^{\beta_1(\omega)}
\]
in the second case, and
\[ Z(\tilde{w}) \simeq Z(\tilde{w}_1) \times \mathbb{Z}^2 \simeq \mathbb{Z}^{\beta_1}(\omega_1) \times \mathbb{Z}^2 \simeq \mathbb{Z}^{\beta_1}(\omega_1, n, m, \mathbb{Z}^2) \simeq \mathbb{Z}^{\beta_1}(\omega) \]
in the third case.

Theorem 5.2 is proved.

6. Commutant

Theorem 6.1. For an arbitrary group \( G \), the commutant \( G \lhd \mathbb{Z}^2 \) coincides with the group

\[ \left[ G \lhd \mathbb{Z}^2, G \lhd \mathbb{Z}^2 \right] = \left\{ \left( (g_{i,j})_{i,j=1}^{n,m}, 0, 0 \right) \mid \prod_{i,j=1}^{n,m} g_{i,j} \in [G, G] \right\}. \]

Proof. We first prove that every \( g = \left( (g_{i,j})_{i,j=1}^{n,m}, 0, 0 \right) \) such that

\[ \prod_{i,j=1}^{n,m} g_{i,j} \in [G, G] \]

belongs to \( \left[ G \lhd \mathbb{Z}^2, G \lhd \mathbb{Z}^2 \right] \).

We show that the elements \( h_1 \) and \( h_2 \) of the group \( G \lhd \mathbb{Z}^2 \),

\[ h_1 = ( (g_{i,j})_{i,j=1}^{n,m}, k, s ), \]
\[ h_2 = ( (g_{1,1})_{i=1}^{n}, \ldots, (g_{i,m-2})_{i=1}^{n}, (g_{i,m-1} g_{i,m})_{i=1}^{n}, e, k, s ), \]

belong to the same conjugacy class, i.e.,

\[ h_2 = h_1 f, \quad \text{where} \quad f \in \left[ G \lhd \mathbb{Z}^2, G \lhd \mathbb{Z}^2 \right]. \] \hspace{1cm} (11)

Indeed,

\[ f = \left( (e, \ldots, e, (g_{i,m})_{i=1}^{n}, (g_{i,m}^{-1})_{i=1}^{n}), 0, 0 \right) \]
satisfies equality (11). It is easy to see that \( f \in \left[ G \lhd \mathbb{Z}^2, G \lhd \mathbb{Z}^2 \right] \).

Similarly, by induction, we conclude that the elements

\[ h_1 = \left( \left( \prod_{i,j=1}^{n,m} g_{i,j} \right), k, s \right), \quad h_3 = \left( \left( \prod_{j=1}^{m} g_{i,j} \right)_{i=1}^{n}, e, e \right), k, s \right), \]

belong to the same conjugacy class.
The elements $h_3$ and $h_4$ of the group $G_{n,m} \leq \mathbb{Z}^2$, 

$$h_4 = \begin{pmatrix}
\prod_{i,j=1}^{n,m} g_{i,j} & \ldots & e & e & e \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
e & \ldots & e & e & e
\end{pmatrix} \begin{pmatrix} k, s \end{pmatrix},$$

also belong to the same conjugacy class because, in this case, we can set

$$f = \begin{pmatrix}
e & \ldots & e & e \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\prod_{j=1}^{m} g_{n,j} & \ldots & e & e \\
(\prod_{j=1}^{m} g_{n,j})^{-1} & \ldots & e & e
\end{pmatrix} \begin{pmatrix} 0, 0 \end{pmatrix}.$$ 

Note that the elements

$$\alpha = ((a_{i,j})_{i,j=1}^{n,m}, 0, 0), \quad a_{1,1} = a, \quad a_{i,j} = e, \quad i = j \neq 1,$$

$$\beta = ((b_{i,j})_{i,j=1}^{n,m}, 0, 0), \quad b_{1,1} = b, \quad b_{i,j} = e, \quad i = j \neq 1,$$

from $G_{n,m} \leq \mathbb{Z}^2$ satisfy the equality

$$[\alpha, \beta] = \begin{pmatrix}
[a, b] & \ldots & e & e \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
e & \ldots & e & e \\
e & \ldots & e & e
\end{pmatrix} \begin{pmatrix} 0, 0 \end{pmatrix}. \quad (12)$$

Thus, each $g = ((g_{i,j})_{i,j=1}^{n,m}, 0, 0)$ such that $\prod_{i,j=1}^{n,m} g_{i,j} \in [G, G]$ belongs to the conjugacy class of the element $h_4$ for $k = 0$ and $s = 0$, and, according to (12), $g \in \left[ G_{n,m} \leq \mathbb{Z}^2, G_{n,m} \leq \mathbb{Z}^2 \right].$

Now let

$$g = ((g_{i,j})_{i,j=1}^{n,m}, k, s) \in \left[ G_{n,m} \leq \mathbb{Z}^2, G_{n,m} \leq \mathbb{Z}^2 \right]$$

and let $a = ((a_{i,j})_{i,j=1}^{n,m}, c, d)$ and $b = ((b_{i,j})_{i,j=1}^{n,m}, l, p)$ be elements from $G_{n,m} \leq \mathbb{Z}^2$. We get

$$aba^{-1}b^{-1} = ((a_{i-c,j-d}b_{i-c-l,j-d-p}a_{i-c-l,j-d-p}^{-1}b_{i-l,j-p}^{-1})_{i,j=1}^{n,m}, 0, 0). \quad (13)$$
Hence, $k = 0$. It is also clear that each element $a_{i,j}$, $b_{i,j}$, $a_{i,j}^{-1}$, $b_{i,j}^{-1}$ appears in representation (13) exactly once for each commutator $aba^{-1}b^{-1}$ and its inverse commutator.

Each $g \in \left[ G \wr \mathbb{Z}^2, G \wr \mathbb{Z}^2 \right]_{n,m}$ is generated by the commutators with the same property. Therefore, the product of its first $n$ coordinates takes the form

$$\prod_{i=1}^{n} g_i = c_1^i c_2^i \ldots c_r^i, \quad i_r \in \{\pm 1\}, \quad c_i \in G,$$

where $c_i$ are not necessarily different but the sum of exponents of the identical elements is always equal to zero.

Since $c_i c_j = c_j c_i [c_i^{-1}, c_j^{-1}]$, we can cancel all $c_i$ after necessary permutations. As a final result, we get only commutators. Hence,

$$\prod_{i=1}^{n} g_i \in [G, G].$$

Theorem 6.1 is proved.

By analogy with the proof of Theorem 6.1, we can establish the following corollary:

**Corollary 6.1.** For an arbitrary group $G$, the commutant $G \wr \mathbb{Z}$ coincides with the group

$$\left[ G \wr \mathbb{Z}, G \wr \mathbb{Z} \right]_n = \left\{ (g_1, g_2, \ldots, g_n, 0) \left| \prod_{i=1}^{n} g_i \in [G, G] \right. \right\}.$$

**Theorem 6.2.** For an arbitrary group $G$, the following isomorphisms of quotient groups take place:

$$G \wr \mathbb{Z}/[G \wr \mathbb{Z}, G \wr \mathbb{Z}] \cong G/[G, G] \times \mathbb{Z},$$

$$G \wr \mathbb{Z}^2/[G \wr \mathbb{Z}^2, G \wr \mathbb{Z}^2] \cong G/[G, G] \times \mathbb{Z}^2.$$

**Proof.** We now construct homomorphisms

$$\eta: G \wr \mathbb{Z} \to G/[G, G] \times \mathbb{Z},$$

$$\mu: G \wr \mathbb{Z}^2 \to G/[G, G] \times \mathbb{Z}^2$$

given by the formulas

$$\eta((g_i)_{i=1}^{n}, k) = \left( \left( \prod_{i=1}^{n} g_i \right) [G, G], k \right),$$

$$\mu((g_{i,j})_{i,j=1}^{n,m}, k, p) = \left( \left( \prod_{i,j=1}^{n,m} g_{i,j} \right) [G, G], k, p \right).$$
In order to check that \( \eta \) is a homomorphism, we perform necessary calculations:

\[
\eta(a_1, a_2, \ldots, a_n, k)\eta(b_1, b_2, \ldots, b_n, p) = (a_1 a_2 \ldots a_n [G, G], k) (b_1 b_2 \ldots b_n [G, G], p) = (a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n [G, G], k + p),
\]

\[
\eta((a_1, a_2, \ldots, a_n, k) (b_1, b_2, \ldots, b_n, p))
\]

\[
= \eta(a_{(1+p) mod n} b_1, a_{(2+p) mod n} b_2, \ldots, a_{(n+p) mod n} b_n, k + p)
\]

\[
= (a_{(1+p) mod n} b_1 a_{(2+p) mod n} b_2 \ldots a_{(n+p) mod n} b_n [G, G], k + p).
\]

Since \( G/[G, G] \) is Abelian, we get

\[
(a_{(1+p) mod n} b_1 a_{(2+p) mod n} b_2 \ldots a_{(n+p) mod n} b_n [G, G], k + p)
\]

\[
= (a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n [G, G], k + p).
\]

Hence, \( \eta \) is a homomorphism.

The homomorphism \( \eta \) is surjective because, for any element \((h, n) \in G/[G, G] \times \mathbb{Z}\), there exists an element \((h, e, e, \ldots, n)\) satisfying the equality

\[
\varphi(h, e, e, \ldots, n) = (h, n).
\]

Thus, the kernel \( \eta \) takes the form

\[
\ker \eta = \left\{(g_1, g_2, \ldots, g_n, 0) \mid \prod_{i=1}^{n} g_i \in [G, G]\right\}
\]

and coincides with \([G \wr_n \mathbb{Z}, G \wr_n \mathbb{Z}]\).

Similarly, we prove that \( \mu \) is also a surjective homomorphism. The kernel \( \mu \) has the form

\[
\ker \mu = \left\{(g_{i,j})_{i,j=1}^{n,m}, 0, 0 \mid \prod_{i,j=1}^{n,m} g_{i,j} \in [G, G]\right\}
\]

and coincides with \([G \wr_{n,m} \mathbb{Z}^2, G \wr_{n,m} \mathbb{Z}^2]\).

Theorem 6.2 is proved.

**Theorem 6.3.** Suppose that \( G \in \mathcal{B} \ (G \in T) \), \( \omega \) is an arbitrary realization of \( G \) in the alphabet \( A_B \ (A_T) \), and \( \beta_1(\omega) \) is the number of symbols \( \mathbb{Z} \) in the realization \( \omega \). Then \( G/[G, G] \simeq \mathbb{Z}^{\beta_1(\omega)} \).

**Proof.** The proof of the theorem is similar to the proof of Theorem 5.2. Instead of Remark 5.1, it is sufficient to use Theorem 6.2 and the fact that any two groups \( A \) and \( B \) satisfy the relation

\[
A \times B/[A \times B, A \times B] \simeq A/[A, A] \times B/[B, B].
\]
To prove (14), it suffices to show that the mapping \( \varphi : A \times B \to A/[A, A] \times B/[B, B] \) given by the relation
\[
(a, b) \mapsto (a[A, A], b[B, B])
\]
is a surjective homomorphism with the kernel \( \ker \varphi = [A \times B, A \times B] \).
Since \( \varphi \) is the product of surjective homomorphisms \( \varphi_1 \) and \( \varphi_2 \) given by the relations
\[
\varphi_1 : A \times B \to A/[A, A], \quad \varphi_1(a, b) = a[A, A],
\]
\[
\varphi_2 : A \times B \to B/[B, B], \quad \varphi_2(a, b) = b[B, B],
\]
it is also a surjective homomorphism.
Further, we note that
\[
\ker \varphi = \{(a, b) \mid a \in [A, A], \ b \in [B, B]\}.
\]
We now verify that \( \ker \varphi \subset [A \times B, A \times B] \). Indeed, let \( \prod_i [a_i, b_i], \prod_j [c_j, d_j] \in \ker \varphi \), where \( a_i, b_i \in A \) and \( c_j, d_j \in B \). Then
\[
\left( \prod_i [a_i, b_i], \prod_j [c_j, d_j] \right) = \left( \prod_i [a_i, b_i], e \right) \left( e, \prod_j [c_j, d_j] \right)
\]
\[
= \prod_i ([a_i, b_i], e) \prod_j (e, [c_j, d_j])
\]
\[
= \prod_i ([a_i, e], (b_i, e)) \prod_j [(e, c_j), (e, d_j)] \in [A \times B, A \times B].
\]
Conversely, for any commutator \( [(a, b), (c, d)] \) in \([A \times B, A \times B] \), we obtain
\[
[(a, b), (c, d)] = (a, b)(c, d)(a^{-1}, b^{-1})(c^{-1}, d^{-1}) = ([a, c], [b, d]) \in \ker \varphi.
\]
Theorem 6.3 is proved.

We can now present an obvious proof of the main result of the present paper, namely, Theorem 1.1. By using Theorem 5.2 under the same assumptions, we establish that
\[
Z(G) \simeq \mathbb{Z}^{β_1(ω)}.
\]
By virtue of Theorem 6.3, we get
\[
G/[G, G] \simeq \mathbb{Z}^{β_1(ω)}.
\]
Thus, it is clear that
\[
Z(G) \cong G/[G, G] \cong \mathbb{Z}^{β_1(ω)}.
\]
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