Modelling Quantum Mechanics by a Quantumlike Description of Electric Signal Propagation in Transmission Lines

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Abstract

It is shown that transmission line technology can be suitably used for simulating quantum mechanics. Using manageable technology, several quantum mechanical problems can be simulated. The electric signal envelope propagation through the line is shown to be governed by a Schrödinger-like equation for a complex function, representing the low-frequency component of the signal.

1. Introduction

There exist several purely classical systems whose behaviour can be described with models which are fully similar to the formal apparatus of quantum mechanics, although their nature have nothing to do with the quantum one. These models are usually referred to as “quantumlike models” [1,2]. Typically they are governed by a Schrödinger-like equation where \( h \) is replaced by another physical parameter of the particular classical system considered. In particular, quantumlike models have been successfully proposed in the transport of electromagnetic radiation beams in linear and nonlinear regimes [3,4] (see, for instance, the theory of optical fibres [5,6] and related subjects), in sound wave theory [7], in plasma physics [8], in the theories of transport and dynamics of charged-particle beams [9] and in the dynamics of ocean waves [10]. To describe the analytic signal theory [11,12], a quantumlike formalism has been used; recently it was employed to translate new results of quantum mechanics into the theory of analytic signals [13,14]. Quantumlike models also received a great deal of attention in the literature to describe nonlinear electric signal propagation in transmission lines in a number of papers. In particular, this subject received an important development in literature in connection with some experimental investigations of modulational instability and soliton formation in nonlinear transmission lines [15].

Transmission line theory is also important for a number of scientific and technological applications. In particular, it is widely used in accelerator physics to model several parts of a given accelerating machine [16]. Also one should point out that the transmission line can be considered as a continuous set of short circuits with capacitance \( C \), inductance \( L \), and resistance \( R \). A separate short circuit and two interacting short circuits were considered in the quantum limit in Ref. [18]. A chain of circuits was studied in the quantum domain in Ref. [19].

The quantumlike treatment of the transmission line is the goal of our paper. For instance, in this modelling operation, electric signals, propagating through a transmission line, modelling a piece of accelerating machines, may be analyzed to study their characteristic impedances. It seems to be clear that in the accelerator theory two “classical” quantumlike problems are combined, namely, the analytic signal theory and the propagation through the transmission lines.

The aim of this work consists in the following steps: (i) to point out that the propagation of an electric signal through a transmission line can be described with a quantumlike model in terms of a complex wavefunction whose evolution is governed by a Schrödinger-like equation; (ii) to point out that our results are relevant for simulating quantum effects with manageable transmission line technology.

The article is organized as follows. In the next section, we briefly introduce the concept of a transmission line and give the wave equation governing the propagation of a signal as a current perturbation in space and time. The effective refractive index of the line is also introduced. In Section 3, taking the slowly-varying amplitude approximation, a Schrödinger-like equation for a complex function is derived from the wave equation. The quantumlike formalism obtained in this way allows us in Section 4 to discuss the signal propagation in the cases of some simple space profiles of the effective refractive index that can simulate some processes of quantum mechanics. For instance, we consider the case a quadratic space-profile of the refractive index to show that the quantum harmonic oscillator problem and the related coherent states can be simulated by employing a transmission line. In Section 5, the perspective of how to produce a simulation of some quantum topics, such as the Frank–Condon principle and the Ramsauer effect, is drawn stressing the advantages that their simulation can be obtained by using a transmission line. Finally, conclusions and remarks are presented in this section.

2. The wave equation of a linear dispersive transmission line

Let us consider the usual representation of an electromagnetic (e.m.) transmission line as a series of R-L-C short circuits (series of moduli). Each modulus exhibits inductance, capacitance and resistance per unity length, \( L', C' \)
and $R'$, respectively. Let us denote with $x$ and $t$ the longitudinal space coordinate and the time, respectively. Thus, it is easy to see that the perturbation voltage and current signals, $\delta v(x, t)$ and $\delta i(x, t)$, appearing at the end of an arbitrary modulus of the line, located at the longitudinal position range $x, x + dx$, obey the following coupled short-circuit equations:

$$\frac{\partial \delta v}{\partial x} = L \frac{\partial \delta i}{\partial t} + R' \delta i, \tag{1}$$

$$\frac{\partial \delta i}{\partial x} = C \frac{\partial \delta v}{\partial t}. \tag{2}$$

Combining (1) and (2), and solving for $\delta i$ we get the following usual wave equation for the transmission line:

$$\frac{\partial^2 \delta i}{\partial x^2} = - \frac{1}{L'C} \frac{\partial^2 \delta i}{\partial t^2} + \frac{R'}{L} \frac{\partial \delta i}{\partial t} = 0. \tag{3}$$

This wave equation accounts for the dissipations along the line of ohmic nature.

For the sake of simplicity, let us assume that the ohmic dissipations are negligible. Consequently, Eq. (3) becomes:

$$\frac{\partial^2 \delta i}{\partial x^2} - V^2 \frac{\partial^2 \delta i}{\partial x^2} = 0, \tag{4}$$

where we have introduced the phase velocity

$$V = \sqrt{\frac{1}{LC}}. \tag{5}$$

When the parameters $L'$ and $C'$ of the line are homogeneous, the phase velocity does not depend on the coordinates, i.e.,

$$V_0 = \sqrt{\frac{1}{L_0'C_0}}, \tag{6}$$

where $L_0$ and $C_0$ are some unperturbed values.

In order to take into account very slow space and time modulations of $L'$ and $C'$, let us assume:

$$L' = L_0 f_1(x, t), \tag{7}$$

$$C' = C_0 f_2(x, t), \tag{8}$$

where $f_1(x, t)$ and $f_2(x, t)$ are specific functions that account for the inhomogeneity profile in space and time. Thus,

$$V^2 = \frac{1}{L_0 C_0} \frac{f_1(x, t)}{f_2(x, t)} = \frac{V_0^2}{N^2(x, t)}, \tag{9}$$

where it is clear that $N(x, t)$ accounts for the refractive index, say $n(x, t)$ of the medium.

Note that: $V^2/c^2 = V_0^2/c^2 N^2$ ($c$ being the light speed), thus

$$n(x, t) = n_0 N(x, t), \tag{10}$$

where $n_0$ is the unperturbed refractive index (namely, the one of the homogeneous case). Consequently,

$$N(x, t) = \frac{n(x, t)}{n_0}, \tag{11}$$

is the relative refractive index.

By taking into account Eq. (11), Eq. (4) becomes:

$$n^2(\eta, t) \frac{\partial^2 \delta i}{\partial \eta^2} - \frac{\partial^2 \delta i}{\partial \eta^2} = 0, \tag{12}$$

where $\eta = x/c$.

3. A Schrödinger-like equation for weakly-dispersive transmission lines

3.1. The telegraphist’s equation

Consider the case of a time-independent refractive index, i.e., $n = n(\eta)$. By taking a solution of (12) of the form

$$\delta i = (\xi)t_0 \exp(-\omega00), \tag{13}$$

we obtain the following Helmholtz-like equation (usually referred to as “the telegraphist’s equation”):

$$\frac{\partial^2 \delta i}{\partial \eta^2} + c^2 K^2(\eta) \delta i = 0, \tag{13}$$

where $K^2(\eta) = \omega0^2 n^2(\eta)/c^2$. A number of problems in linear-transmission lines have been described by means of Eq. (13) [16].

3.2. Slowly-varying amplitude approximation in weakly-dispersive transmission lines

Now, in order to describe the propagation of electric signal envelopes, let us assume that $n(\eta, t)$ is weakly perturbed, i.e.,

$$n(\eta, t) \approx n_0 + \delta n(\eta, t), \tag{14}$$

where $|\delta n| \ll n_0$. We look for a solution of (12) which can be taken in the form:

$$\delta i(\eta, t) = \Phi(\eta, t) \exp(-\omega00), \tag{15}$$

where $\Phi(\eta, t)$ is a very slow function of $t$ compared to the phase term variation, i.e.,

$$\left| \frac{\partial \Phi}{\partial t} \right| \ll \omega0|\Phi|. \tag{16}$$

The slow function $\Phi$ accounts for an amplitude modulation of the signal, in such a way that $\delta i$ plays the role of a wave envelope (electric signal envelope). Substituting Eq. (15) and Eq. (14) in Eq. (12) and taking into account the first-order quantities only, we get the following Schrödinger-like equation for the current perturbation envelope:

$$i\omega0 \frac{\partial \Phi}{\partial t} = -\frac{1}{2n_0^2} \frac{\partial^2 \Phi}{\partial \eta^2} - \omega0^2 \frac{\delta n}{n_0} \Phi - \frac{\omega0^2}{2} \Phi, \tag{17}$$

Putting

$$\Phi(\eta, t) = \Psi(\eta, t) \exp \left( \frac{i\omega00}{2} \right), \tag{18}$$

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we finally get the following equation

\[
\frac{1}{i} \frac{\partial \Psi}{\partial s} = -\frac{1}{2n_0} \frac{\partial^2 \Psi}{\partial s^2} - \frac{\delta n}{n_0} \Psi, \tag{19}
\]

where the dimensionless variables \( s = \omega t \) and \( \tau = \omega n \) have been introduced.

Note that Eq. (19) corresponds to the usual one-dimensional Schrödinger equation with \( \hbar = 1 \), and where \( s \) and \( \tau \) replace the time \( t \) and the space-coordinate \( x \), respectively. Consequently, we realize that the relative refractive index perturbation \(-\delta n/n_0\) plays the role of an effective potential and \( n_0^2 \) plays the role of an effective mass. Thus, Eq. (19) can be cast as

\[
\frac{1}{i} \frac{\partial \Psi}{\partial s} = -\frac{1}{2n_0^2} \frac{\partial^2 \Psi}{\partial s^2} + U(\tau, s) \Psi, \tag{20}
\]

where

\[
U(x, s) = -\frac{\delta n}{n_0}. \tag{21}
\]

It is easy to see that (21) can be also cast as:

\[
U(\tau, s) = \frac{L_s C_0}{L(\tau, s) C(\tau, s)} - 1. \tag{22}
\]

For the sake of simplicity, in the following we fix the constant \( n_0 = 1 \). Note that, as in Quantum Mechanics the wavefunction of an elementary particle is a solution of the Schrödinger equation, the complex function \( \Psi(\tau, t) \) involved in Eq. (20) represents the analog of the quantum wavefunction which is associated with the electric signal envelope propagating through the transmission line. In this paper we conventionally call this complex function the signal envelope wavefunction (SEW). Once the signal envelope wavefunction is normalized, i.e.,

\[
\int_{-\infty}^{\infty} |\Psi(\tau, s)|^2 \, d\tau = 1, \tag{23}
\]

the quantity \( |\Psi(\tau, s)|^2 \) plays the role of probability density to find the electric signal envelope at the location \( x = c\tau \) and at the time \( t = s/c \). We may syntetically call \( |\Psi|^2 \) the probability density associated with the electric signal envelope.

4. Propagation through a line with a quadratic space-profile refractive index

In this section we consider a special case of effective refractive index that may be useful to simulate some of quantum problems. In fact, we examine the propagation of an electric pulse through a transmission line with an effective quadratic space-profile refractive index.

If \( \delta n \) is time-independent, an interesting case to be considered is the one where \(-\delta n/n_0\) is a parabolic function of \( \tau \), namely \( n_0(\tau) = 1 \)

\[
\frac{1}{i} \frac{\partial \Psi(\tau, s)}{\partial s} = -\frac{1}{2} \frac{\partial^2}{\partial \tau^2} \Psi(\tau, s) + \frac{1}{2} \frac{\tau^2}{\tau^2} \Psi(\tau, s), \tag{24}
\]

where we have assumed \(-\delta n/n_0 = k \tau^2/2\), with \( k > 0 \). It is easy to show that Eq. (24) admits the following orthonormal discrete modes for the SEW

\[
\Psi_{n}(\tau, s) = \frac{1}{\sqrt{2\pi n!}} H_n \left(\frac{\tau}{\sqrt{2}\sigma(s)}\right) \times \exp\left[-\frac{\tau^2}{4\sigma^2(s)} + i \frac{\tau^2}{2\rho(s)} + i(1 + 2n)\phi(s)\right]
\]

\[n = 0, 1, 2, \ldots\] \tag{25}

Here \( H_n \) are the Hermite polynomials, \( \sigma(s) \) obeys to the following envelope equation

\[
\sigma'' + k \sigma - \frac{1}{4\sigma^2} = 0, \tag{26}
\]

and

\[
\frac{1}{\rho} = \frac{\sigma'}{\sigma}, \tag{27}
\]

\[
\phi' = -\frac{1}{4\sigma^2}. \tag{28}
\]

Here each prime denotes the derivative with respect to \( s \). It is easy to see that \( \sigma(s) \), appearing in Eqs. (25)–(28), coincides with the r.m.s. of the fundamental mode \( \Psi_{0}(\tau, s) \); in general, an arbitrary SEW \( \Psi \) has a r.m.s. \( \sigma(s) \) defined by:

\[
\sigma(s) = \left(\int_{-\infty}^{\infty} \tau^2 |\Phi_{0}(\tau, s)|^2 \, d\tau\right)^{1/2}. \tag{29}
\]

In addition, we can also define the expectation value for the transverse linear momentum associated to an arbitrary SEW \( \Psi(\tau, s) \)

\[
\sigma_p(s) = \left(\int_{-\infty}^{\infty} |\frac{\partial \Psi_{0}(\tau, s)}{\partial \tau}|^2 \, d\tau\right)^{1/2}. \tag{30}
\]

Here \( \hat{p} = -i\partial/\partial \tau \); in fact, \( \tau \) and \( p \equiv d\tau/ds \) play the role of conjugate variables.

It is suitable to introduce the following matrix

\[
\hat{T}(s) = \begin{pmatrix} \sigma_0^2(s) & -\sigma_0(s)\sigma'(s) \\ -\sigma_0(s)\sigma'(s) & \sigma_0^2(s) \end{pmatrix}, \tag{31}
\]

whose determinant is an invariant, namely

\[
\sigma_p^2 \sigma^2 - (\sigma \sigma')^2 = \frac{1}{2} = \text{const}. \tag{32}
\]

It is easy to prove that

\[
\sigma(s)\sigma'(s) = \int_{-\infty}^{\infty} \Psi^*_{0}(\tau, s) \left(\hat{p} + \frac{\hat{p}^2}{2}\right) \Psi_{0}(\tau, s) \, d\tau
\]

\[= \frac{\tau^2}{2}. \tag{33}
\]
Consequently, from (32) it follows that
\[
\langle \hat{\tau}^2 \rangle \hat{\rho}^2 = \left( \frac{\hat{\tau}^2 + \hat{\rho}^2}{2} \right)^2 = \frac{1}{4},
\]
(34)
which is formally identical to the Robertson–Schrödinger uncertainty relation [20], [21] (see also Ref. [22]) for partial cases when \( \langle \tau \rangle = \langle p \rangle = 0 \). Note that (32), or equivalently (34), gives the usual form of the Heisenberg-like uncertainty principle which is analogous to the Heisenberg uncertainty relation in quantum mechanics (again for \( \langle \tau \rangle = \langle p \rangle = 0 \))
\[
\sigma_{\tau} \sigma_{p} \geq \frac{1}{2}.
\]
(35)
The equilibrium solution of (26) \( (d^2 \sigma(s)/ds^2 = 0) \), namely
\[
\sigma_{0}^2 = \frac{1}{2 \sqrt{k}},
\]
implies that the set of Hermite–Gauss modes (25) reduces to the Hamiltonian eigenstates of the harmonic oscillator
\[
\Psi_n^0(\tau, s) = \frac{1}{(2\pi \sigma_0^2(e^2n^2))^{1/4}} 
\times \exp\left(-\frac{\tau^2}{4\sigma_0^2} + i(1 + 2n)\phi_0(s)\right) H_n\left(\frac{\tau}{\sqrt{2\sigma_0}}\right),
\]
(37)
where \( n = 0, 1, 2, \ldots \),
\[
\phi_0(s) = -\sqrt{k}s^2/2.
\]
(38)
and the energy values \( \epsilon_n^0 \), given by averaging the Hamiltonian of the system with the wavefunction (37), are the analogs of the Hamiltonian eigenvalues of the quantum harmonic oscillator
\[
\epsilon_n^0 = (n + \frac{1}{2}) \sqrt{k}.
\]
(39)
In particular, for \( n = 0 \), Eqs. (37), (38) and (39) give the ground-like state
\[
\Psi_0^0(\tau, s) = \frac{1}{(2\pi \sigma_0^2)^{1/4}} \exp\left(-\frac{\tau^2}{4\sigma_0^2} + i\phi_0(s)\right),
\]
(40)
which is purely Gaussian and the lowest energy reachable by the electric signal envelope is \( \epsilon_0^0 = (1/2) \sqrt{k} \). This means that \( \langle \tau \rangle \) and \( \langle p \rangle \) are equal to zero at this state of the electric signal envelope. In these conditions the uncertainty relation is minimized as
\[
\sigma_{\tau} \sigma_{p} \approx \frac{1}{2}.
\]
(41)
Equation (41) holds also during the evolution of the electric signal, because, in addition to Eq. (36), we have \( \sigma_{\tau}(s) = \sigma_p = \text{const} \). In summary, we conclude that if we initially prepare the SEW according to the matching conditions (36), its evolution is ruled by a quantumlike behaviour in terms of a ground-like state which minimizes the uncertainty relation and corresponds to the lowest accessible energy \( (1/2) \sqrt{k} \) of the electric signal envelope.

As it is well known, SEW (40) belongs to the infinite series of coherent state functions, labelled by a complex number \( \sigma = \sigma_1 + i\sigma_2 \), and widely used in quantum mechanics and quantum optics [23–25].

5. Discussion and perspectives

Let us discuss what physical consequences can be extracted from the observation that the electric signal in the transmission line can be associated with the Schrödinger-like equation. We got the result that the transmission line can be considered as a quantumlike system; the distributed along the line conductance \( C \) and inductance \( L \) can, in principle, be considered as inhomogeneous and time-dependent functions, i.e., \( C = C(\tau, s) \) and \( L = L(\tau, s) \). They account for the effective potential-energy function \( U = U(\tau, s) \). In fact, in quantumlike systems, such as a light ray in an optical fibre or an electron beam in an accelerator in the framework of a thermal-wave model, the refractive index profile plays the role of an effective potential-energy function.

Let us consider now the interesting situation in the presence of some filtering of the modes in the transmission line. To this end, let us consider a transmission line whose refractive index gives an effective potential well \( U \) which has a rectangular structure in the domain of nonhomogeneity of the transmission line. In this case, an electric signal in the transmission line is propagating along the line in complete analogy with an electromagnetic wave in a waveguide. The solutions to the Schrödinger-like equations (modes of the signal in transmission line) can be treated as wave functions which are reflected or transmitted by a potential barrier. There are effects involved in the above propagation that we discuss qualitatively in the following. We may consider this problem as an analog of two known quantum subjects: The Ramsauer effect and the Frank–Condon principle.

The Ramsauer effect shows that the cross section of scattering of an electron beam by atoms varies with electron energy. This variation, corresponds to a variation of the electron de Broglie wavelength in comparison with the effective size of the atomic potential acting on the electrons. For some energy (wavelength), the atoms are more transparent for electron waves, for other energies they are darker.

If one considers in this context the transmission lines, the suggested quantumlike description may provide a better understanding of the various degrees of "transparency" of the line for the electric signal.

In the case where there is non-homogeneity in the transmission line, the potential-energy term has a deformation, e.g., increasing (or decreasing) the depth of the potential well for some length \( L \). If the wavelength of the signal function satisfies the condition that \( 2L/\lambda = n \) where \( n \) is even, the potential well is transparent to the signal with wavelength \( \lambda \). It is just an analog of the Ramsauer effect in which the electron beam which scatters by atoms for some values of energy (with corresponding de Broglie wavelength) has smaller cross section for this process than for the other wavelength when the ratio is odd. In the model experiment, one can see that for some adapter a change in frequency corresponds to different reactions of the transmission line.
which can just correspond to the analog of the Ramsauer effect. By this method, one can measure the characteristics of the line (distributed inductance and capacitance) that is equivalent to the impedance measurement.

Another qualitative effect can be checked if one connects two different pieces of the transmission line which corresponds to connecting two potential barriers of different depths. The problem of penetrating the modes of the first piece and their transforming into the modes of the second piece of the transmission line is equivalent to calculating the Frank–Condon factor for electronic transitions in polyatomic molecules. This factor is the overlap integral of the wave-like functions describing the two different modes in the two pieces. \( |C_{nm}|^2 \) describes the portion of the signal energy of the \( n \)th mode in the first part of the transmission line which goes into the \( m \)th mode of the second piece of the line. Again one can pose the problem of maximality, e.g., of the transformation of the fundamental mode energy into the fundamental mode of the second piece of the line, having in mind that distributed inductance and capacitance in the second piece of the line are different from the first one. The transformation coefficient \( |C_{nm}|^2 \) which gives the probability of transforming the \( n \)th mode into the \( m \)th mode can be expressed in terms of the overlap integral of electric current modes \( \psi_n(\tau), \psi_m(\tau) \)

\[
C_{nm} = \int \psi_n^*(\tau)\psi_m(\tau) d\tau,
\]

where the mode \( \psi_n(\tau) \) is the mode in the first piece of the transmission line and the mode \( \psi_m(\tau) \) corresponds to the second piece of the transmission line. If one can vary in time the inductance and capacitance of the transmission line, the Frank–Condon factor describes the parametric excitation of the modes in the transmission line. For example, if the profile of the “refractive index” is modelled by a parabolic profile with varying frequency, the electric current propagation in the line is described by the oscillator with time-dependent frequency used for analysis, e.g., of the thermal wave model [26]. We can adopt the results of the analysis and apply them to the transmission line. Thus, we have for the fundamental mode the Gaussian solution [26]

\[
\psi_0(\tau, s) = \psi_0(\tau, s) \left( \frac{\epsilon(s)}{\epsilon(s)^{1/2}} \right)^{n/2} \frac{1}{\sqrt{2\pi n!}} H_n \left( \frac{\tau}{\|\epsilon(s)\|} \right),
\]

where \( H_n \) is Hermite polynomial. The Frank–Condon factor can be calculated in the explicit form for the model under consideration. Thus, the coefficients \( C_{nm} \) are expressed in terms of two-dimensional Hermite polynomials. The Frank–Condon factors can qualitatively be evaluated by a geometric method. As it is known, the maximally excited is the mode for which the refractive index curve after the change of the line parameters is intersecting the perpendicular taken from the rest point of the initial refractive index curve. Thus the modes are connected which have common rest points on the plot of two curves. One curve is the initial refractive index and the other one is the final refractive index.

Summarizing, we have shown that the electrical current signal in the transmission line obeys a Schrödinger-like equation. We have extracted some physical consequences from the fact that the transmission line is a quantumlike system. In our preliminary analysis, we have considered two physical situations of a transmission line that can be thought of as the quantum analog of known quantum phenomena.

Remarkably, this approach seems to be helpful and promising for providing a simple method of simulating a number of quantum-mechanical effects, using manageable technology.

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