On the classical $W_{N}^{(l)}$ algebras

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We analyze the $W_{N}^{(l)}$ algebras according to their conjectured realization as the second Hamiltonian structure of the integrable hierarchy resulting from the interchange of $x$ and $t$ in the $l^{th}$ flow of the $sl(N)$ KdV hierarchy. The $W_{4}^{(3)}$ algebra is derived explicitly along these lines, thus providing further support for the conjecture. This algebra is found to be equivalent to that obtained by the method of Hamiltonian reduction. Furthermore, its twisted version reproduces the algebra associated to a certain non-principal embedding of $sl(2)$ into $sl(4)$, or equivalently, the $u(2)$ quasi-superconformal algebra. General aspects of the $W_{N}^{(l)}$ algebras are also presented. We point out in particular that the $x \leftrightarrow t$ interchange approach of the $W_{N}^{(l)}$ algebra appears straightforward only when $N$ and $l$ are coprime.
1. Introduction

Given a hierarchy of two-dimensional evolution equations, one can interchange the role of the independent variables $x$ and $t$ for any member of the hierarchy, thus producing a new integrable hierarchy of evolution equations. Furthermore, such an interchange in two different equations of a given hierarchy produces two new independent hierarchies. The idea of interchanging the roles of $x$ and $t$ for integrable equations goes back to [1]. There it was shown with simple examples (KdV, mKdV, sine-Gordon) that the resulting hierarchy is also integrable and bi-Hamiltonian. For the KdV and mKdV cases, Kupershmidt [2] has obtained the same conclusion independently by considering a more general transformation (GL(2)) of the independent variables. The Boussinesq equation in $x$-evolution has been constructed in [3] and was shown to be bi-Hamiltonian and in fact equivalent to the fractional KdV hierarchy of [4].

These statements have now been fully generalized in [5][6] where various extensions of the Drinfeld-Sokolov [7] approach to KdV-type equations have been worked out, including hierarchies obtained by $x \leftrightarrow t$ interchange.

The interest for these new hierarchies obtained by $x \leftrightarrow t$ interchange is motivated by the potential conformal character of their second Hamiltonian structure. Since the Hamiltonian character of an evolution equation depends crucially upon which of the variables is chosen for the evolution, one expects a priori that the interchange of $x$ and $t$ will substantially modify the Hamiltonian properties (i.e. the form of the Poisson brackets) of a given integrable system. New conformal Poisson algebras would yield new extended conformal algebras upon quantization, in the same way as the second Hamiltonian structure of the usual generalized KdV hierarchies are related to $W$-algebras [8]. Recall that a Hamiltonian structure is said to be conformal if it contains the Poisson bracket characterizing the second Hamiltonian structure of the KdV equation, namely

$$\{u(x), u(y)\} = (\partial^3 + 4u\partial + 2u_x) \delta(x - y),$$  \hspace{1cm} (1.1)$$

where the fields on the RHS are evaluated at $x$ and $\partial \equiv \partial_x$. The Fourier transform of this bracket yields the Virasoro algebra realized in terms of Poisson brackets [9], hence the name conformal. Actually, the classical $W_3^{(2)}$ algebra of Polyakov [10], has been shown [8] to be equivalent to the second Hamiltonian structure of the Boussinesq (or sl(3) KdV) hierarchy with $x$ and $t$ interchanged at the level of the Boussinesq equation itself.
It was conjectured in [3] that the second Hamiltonian structure of the hierarchy obtained by interchanging $x$ and $t$ in the $l^{th}$ flow of the $\text{sl}(N)$ KdV hierarchy (with $l < N$) would produce a new conformal algebra, called $W_N^{(l)}$ (with $W_N^{(1)} \equiv W_N$). See also [5][6] for similar results and conjectures. We will call this new hierarchy the $\text{sl}(N)_l$ hierarchy.

It is simple to show that the restriction $l < N$ is necessary to produce a conformal $W_N^{(l)}$ algebra. The $\text{sl}(N)$ KdV hierarchy is characterized by the scalar Lax operator

$$L = \partial^N + u_2 \partial^{N-2} + \ldots + u_N.$$  \hspace{1cm} (1.2)

The evolution equations are

$$\partial_{t_l} L = [(L^{1/N})_+, L],$$ \hspace{1cm} (1.3)

where + denotes the differential part of a pseudo-differential operator. Interchanging $x$ and $t$ in the $l^{th}$ flow amounts to interchanging $t_l$ and $t_1 = x$. In the newly produced hierarchy $t_l$ plays the role of the space variable. In the normalization where dim$(x) = -1$, $t_l$ has dimensions $-l$. To renormalize the dimensions of the new space variable to $-1$, one has to divide all the dimensions by $l$. Thus the new algebra will contain bosonic fields of fractional dimension (multiples of $1/l$). In order to be conformal, it must contain a field of spin 2 (after dimensional renormalization). Now in (1.3), the evolution of the highest spin field takes the form

$$\partial_{t_l} u_N = c u_2^{(N+l-2)} + \ldots, \quad c = \text{constant}, \quad u^{(i)} = (\partial^i u).$$  \hspace{1cm} (1.4)

As it will become clear below, the new set of independent fields required to describe the system obtained by interchange of $t_1$ and $t_l$ can be chosen generically to include $u_2, u_{2x}, u_{2xx}, \ldots, u_2^{(N+l-3)}$. Since $u_2^{(i)}$ originally had dimension $2 + i$, its new dimension will be $(2 + i)/l$. Thus the field $u_2^{(2l-2)}$ will have new dimension 2, and it belongs to the above sequence only if $l < N$.

Before pursuing the discussion of the second Hamiltonian structure of the $\text{sl}(N)$ hierarchy, one should settle the question of its integrability. This is most naturally discussed in terms of a zero curvature condition. The usual $\text{sl}(N)$ hierarchy can be described by the scattering problem

$$\Phi_{t_l} = V^{(l)} \Phi$$  \hspace{1cm} (1.5)

where $\Phi = (\phi, \phi_x, \ldots)^T$, $V^{(l)} = (L)^{l/N}$ and $L \phi = \lambda \phi$, $\lambda$ being the spectral parameter, i.e.

$$V^{(l)}_x - V^{(1)}_{t_l} + [V^{(1)}, V^{(l)}] = 0$$  \hspace{1cm} (1.6)
with $x = t_1$. The interchange $t_1 \leftrightarrow t_l$ amounts to interchange the roles of $V^{(1)}$ and $V^{(l)}$. But the point is that we stick to a zero curvature formulation, hence this operation manifestly preserves the integrability property (this is proved rigorously in [5]). In $V^{(l)}$ there is a constant piece, equal to

$$\Lambda^{(l)}_N = \lambda^i \begin{pmatrix} 0 & I_{N+j-l} \\ I_j & 0 \end{pmatrix} \quad N i + j = l \quad (1.7)$$

where $I_k$ is the $k \times k$ unit matrix. Hence for $l < N$, we find $i = 0$ and $j = l$ and the two matrices entering in (1.6) are linear in $\lambda$.

One strategy to obtain the second Hamiltonian structure of the sl$(N)_l$ KdV hierarchies is the following [3]. One first writes down the $l^{th}$ flow of the sl$(N)$ KdV and mKdV hierarchies. These equations are related by a Miura transformation which characterizes the usual sl$(N)$ hierarchy. The Miura map gives the free field representation of the classical $W_N$ algebra, which is the second Hamiltonian structure of the sl$(N)$ KdV hierarchy. Let us denote the corresponding Hamiltonian operator by $P_2$. Similarly, denote by $\Theta$ the natural Hamiltonian operator of the sl$(N)$ mKdV hierarchy$^1$. The canonical character of the Miura map translates into the statement that [11][12]

$$P_2 = D \Theta D^\dagger \quad (1.8)$$

where $D$ is the Fréchet derivative of the KdV fields with respect to the modified KdV fields. It is computed from the Miura map. $D^\dagger$ is its formal adjoint. Thus the $l^{th}$ flows of the sl$(N)$ KdV and mKdV hierarchies read as

$$u_{t_l} = P_2 \nabla_u H , \quad u = (u_2, u_3, \ldots, u_N)^T ,$$
$$p_{t_l} = \Theta \nabla_p H_m , \quad p = (p_1, p_2, \ldots, p_{N-1})^T , \quad (1.9)$$

where $\nabla_u = (\delta/\delta u_2, \ldots, \delta/\delta u_N)^T$ and similarly for $\nabla_p$. $H$ is the appropriate Hamiltonian for the $l^{th}$ flow and $H_m$ is the expression of the same Hamiltonian in terms of the modified fields. By interchanging $t_1$ and $t_l$, one gets

$$\tilde{u}_{t_l} = \tilde{P}_2 \nabla_{\tilde{u}} \tilde{H} ,$$
$$\tilde{p}_{t_l} = \tilde{\Theta} \nabla_{\tilde{p}} \tilde{H}_m , \quad (1.10)$$

$^1$ It is also called the mKdV first Hamiltonian structure for historical reasons (except in [7]). Also, it has lower dimensions than the second one. Its naturalness is due to the fact that the second Hamiltonian structure is both complicated and non-local.
where \( \tilde{u} = (\tilde{u}_2, \tilde{u}_3, \ldots, \tilde{u}_{lN-l+1})^T \) and \( \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{lN-l})^T \) are the new independent fields, whose number depends on \( l \) (a canonical set of independent fields will be displayed later). We want to find \( \tilde{P}_2 \). Notice that we know \( \tilde{H} \) since any conserved density \( h \) for the \( \text{sl}(N) \) hierarchy satisfies \( \partial_t h = \partial_x \tilde{h} \). Thus \( \tilde{h} \) is a conserved density of the new system. In (1.10), \( \tilde{H} \) is the conservation law of appropriate dimension. We also know the Miura map relating \( \tilde{p} \) to \( \tilde{u} \): it is simply a rewriting of the usual Miura map where the \( x \)-derivatives of the modified fields are eliminated by means of the \( l \text{th} \) \( \text{sl}(N) \) mKdV equation. Hence we also know \( \tilde{H}_m \). Now as it will be illustrated below, there is a natural way to write the first equation of the \( \text{sl}(N)_l \) mKdV hierarchy (i.e. a choice of new modified fields \( \tilde{p} \)) which makes \( \tilde{\Theta} \) obtainable by inspection in a totally straightforward way. (It only contains \( \partial_t \) and constants, which makes its Hamiltonian character manifest). Having \( \tilde{\Theta} \) and \( \tilde{D} \), the Fréchet derivative of \( \tilde{u} \) with respect to \( \tilde{p} \), one can reconstruct \( \tilde{P}_2 \) by
\[
\tilde{P}_2 = \tilde{D} \tilde{\Theta} \tilde{D}^\dagger.
\]
(1.11)
The Hamiltonian property of \( \tilde{P}_2 \) is thus inherited from that of \( \tilde{\Theta} \). \( \tilde{P}_2 \) corresponds to the classical \( W_N^{(l)} \) algebra.

The advantage of this construction, apart from its conceptual simplicity, is that it gives directly the free field representation of the \( W_N^{(l)} \) algebra. A minor drawback is that \( \tilde{\Theta} \) must be obtained by inspection. Now the above procedure is totally straightforward in the cases where \( N \) and \( l \) are coprime (\( ((N,l) = 1) \). However, when such is not the case, the first \( \text{sl}(N)_l \) equation appears under the form of a constrained system (see e.g. (5.7)). We defer to another publication the detailed analysis of such cases, for which the simplest example is \( W_4^{(2)} \).

Here we work out in details a new example of a \( W_N^{(l)} \) algebra, namely the \( W_4^{(3)} \) case.

This gives the first explicit form of a \( W_N^{(l)} \) algebra for \( N > 3 \), derived from the point of view of integrable hierarchies. We will also check that the same algebra can be obtained directly by the method of Hamiltonian reduction and the corresponding flows can be extracted by reduction from \( \text{sl}(4) \) self-dual Yang-Mills equations, as was the case for \( W_3^{(2)} \). After having worked out a new non-trivial example of \( W_N^{(l)} \) algebra, we will be in position to present a set of general remarks concerning the structure of these algebras for \( (N,l) = 1 \), including their spin content. But before, we illustrate the method by a simple example.

\( W_4^{(2)} \) is presented from the point of view of Hamiltonian reduction in [13]. After field redefinition and twisting, it is equivalent to the particular \( W \) algebra derived in [14] by considering the embedding of \( \text{sl}(2) \) into \( \text{sl}(4) \) fixed by the decomposition \( 4 \to 2 + 2 \) of the fundamental representation.
2. A simple example: Interchange of $x$ and $t$ for the usual KdV equation

Let us write the KdV equation under the form

\[ u_t = u_{xxx} + 6uu_x . \quad (2.1) \]

Its two Hamiltonian structures are

\[ u_t = P_2 \nabla \frac{1}{2} \int u^2 dx \quad (2.2a) \]
\[ = P_1 \nabla \frac{1}{2} \int (2u^3 - u_x^2) dx , \quad (2.2b) \]

with

\[ P_2 = \partial^3 + 4u\partial + 2u_x \quad \text{and} \quad P_1 = \partial , \quad (2.3) \]

The KdV equation in $x$-evolution is obtained as follows: one first introduces two new independent fields

\[ v = u_x \quad \text{and} \quad w = u_{xx} . \quad (2.4) \]

so that (2.1) can be rewritten as

\[ \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} v \\ w \\ u_t - 6uv \end{pmatrix} . \quad (2.5) \]

This is the KdV equation in $x$-evolution or equivalently the first flow in the $\text{sl}(2)_3$ hierarchy.

One proceeds similarly with the mKdV equation

\[ p_t = p_{xxx} - 6p^2p_x \quad (2.6a) \]
\[ = \Theta_1 \nabla \frac{1}{2} \int (p_x^2 + p^4) \quad (2.6b) \]
\[ = \Theta_2 \nabla \frac{1}{2} \int p^2 , \quad (2.6c) \]

with

\[ \Theta_1 = -\partial \quad \text{and} \quad \Theta_2 = \partial^3 - 4\partial p \partial^{-1} p \partial . \quad (2.7) \]

The Miura transformation

\[ u = p_x - p^2 \quad (2.8) \]
is a canonical map from $\Theta_1$ to $P_2$. Indeed the Fréchet derivative of $u$ with respect to $p$ is $\partial - 2p$ so that $D^\dagger = -\partial - 2p$ and 

$$P_2 = (\partial - 2p)(-\partial)(-\partial - 2p) .$$

(2.9)

Introducing

$$q = p_x, \ r = p_{xx} ,$$

(2.10)

one can rewrite the mKdV equations in the form

$$
\begin{pmatrix}
  p \\
  q \\
  r \\
\end{pmatrix}_x =
\begin{pmatrix}
  q \\
  r \\
  p_t + 6p^2q \\
\end{pmatrix} . 
$$

(2.11)

Now in order to find $\tilde{P}_2$, the second Hamiltonian structure for (2.5), we will need $\tilde{\Theta}_1$, the first Hamiltonian structure for (2.11). As already pointed out this must be found by inspection. We now show that with a simple field redefinition, this step is straightforward. The trick is to look for the field transformation which simplifies maximally the equation of the highest degree in (2.11). Here this amounts to introducing a new variable $s$ linearly related to $r$ such that $s_x = p_t$. Thus we choose $s = r - 2p^3$ and (2.11) becomes

$$
\begin{pmatrix}
  p \\
  q \\
  s \\
\end{pmatrix}_x =
\begin{pmatrix}
  q \\
  s + 2p^3 \\
  p_t \\
\end{pmatrix} . 
$$

(2.12)

We want to write the RHS under the form $\tilde{\Theta}_1 \nabla \tilde{H}_m$. $\tilde{H}_m$ is the $x \leftrightarrow t$ interchanged version of $\int (p^4 + p_x^2)dx$, that is $\int (p^4 + \ldots)dt$. Hence we look for the density $(p^4 + \ldots)$ such that $\partial_x (p^4 + \ldots) = \partial_t (\ldots)$, where in the RHS one has a usual mKdV conserved density. Dimensionally one sees that it is $p^2$. Since $\partial_t p^2 = \partial_x (2pp_{xx} - p^2 - 3p^4)$, and $\tilde{H}_m$ is only defined up to a multiplicative constant, we choose $\tilde{H}_m$ as

$$\tilde{H}_m = \int \left( \frac{3}{2}p^4 + \frac{1}{2}p_x^2 - pp_{xx} \right)dt = \int \left( -\frac{1}{2}p^4 + \frac{1}{2}q^2 - ps \right)dt. $$

(2.13)

Of course, using (2.12), it is simple to check explicitly that $$(\tilde{H}_m)_x = 0.$$ An even more direct approach is the following: we know that the KdV conservation laws can be obtained as $\int \text{Res} L^{k/2}dx$ with $L = \partial^2 + u$. Writing $\text{Res} L^{k/2}$ in terms of the new fields, one gets directly the new conserved densities. For example, $\text{Res} L^{3/2} = \frac{1}{8}(u^2 + \frac{1}{3}u_{xx})$ so that one can take $\tilde{H}$ to be $\frac{3}{2} \int (u^2 + \frac{1}{3}w)dt$ which gives directly the above $\tilde{H}_m$, using the Miura
transformation presented below in (2.16). We now search for a matrix differential operator \( \tilde{\Theta}_1 \) such that
\[
\begin{pmatrix}
q \\
s + 2p^3 \\
p_t
\end{pmatrix} = \tilde{\Theta}_1 \begin{pmatrix}
-s - 2p^3 \\
q \\
-p
\end{pmatrix}.
\]
(2.14)

The elements of \( \tilde{\Theta}_1 \) are easily found by inspection to be
\[
\tilde{\Theta}_1 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -\partial_t
\end{pmatrix}.
\]
(2.15)

We claim that once the fields are chosen such that the highest degree modified equation has the form \( \phi_x = \psi_t, \tilde{\Theta}_1 \) is always obtained as simply as above. Now let us work out the Miura transformation:
\[
u = p_x - p^2 = q - p^2,
\]
\[
v = u_x = p_{xx} - 2pp_x = s + 2p^3 - 2pq,
\]
\[
w = u_{xx} = p_{xxx} - 2p_x^2 - 2pp_{xx} = p_t - 2q^2 - 2ps - 4p^4 + 6p^2q.
\]
(2.16)

The Fréchet derivative of \((u,v,w)^T\) with respect to \((p,q,s)^T\) is found to be
\[
\tilde{D} = \begin{pmatrix}
-2p & 1 & 0 \\
6p^2 - 2q & -2p & 1 \\
\partial_t - 2s - 16p^3 + 12pq & -4q + 6p^2 & -2p
\end{pmatrix}
\]
(2.17)

and
\[
\tilde{D}^\dagger = \begin{pmatrix}
-2p & 6p^2 - 2q & -\partial_t - 2s - 16p^3 + 12pq \\
1 & -2p & -4q + 6p^2 \\
0 & 1 & -2p
\end{pmatrix}.
\]
(2.18)

Now it is simple to obtain \( \tilde{P}_2 \), from the matrix product (1.11). The result is [1][2][3]
\[
\tilde{P}_2 = \begin{pmatrix}
0 & 2u & \partial_t + 2v \\
-2u & -\partial_t & 2w + 12u^2 \\
\partial_t - 2v & -2w - 12u^2 & -8u\partial_t - 4w
\end{pmatrix}.
\]
(2.19)

The first Hamiltonian structure can be obtained by shifting \( u \) by a constant factor, i.e. with \( u \to u + \lambda, \tilde{P}_2 \to \tilde{P}_2 + 2\lambda\tilde{P}_1 \) where
\[
\tilde{P}_1 = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 12u \\
0 & -12u & -4\partial_t
\end{pmatrix}.
\]
(2.20)
The field redefinition \( w \rightarrow w + 6u^2 \) transforms \( \tilde{P}_1 \) into a manifestly Hamiltonian form similar to (2.15). Notice that to recover the first Hamiltonian structure, one should not shift the field of highest spin in the new set of independent fields, but merely the highest spin field of the original set. On the other hand, given \( \tilde{\Theta}_1 \), one can calculate \( \tilde{\Theta}_2 \) as follows: the master-symmetries for (2.12) can be obtained directly from those of the mKdV equation with \( x \) and \( t \) interchanged. Let \( T \) be the analogue, for (2.12), of the first time dependent symmetry for the mKdV equation. Then \( \tilde{\Theta}_2 \) is, up to a multiplicative factor, the Lie derivative of \( \tilde{\Theta}_1 \) with respect to \( T \).

Thus the operator (2.19) characterizes the second Hamiltonian structure of the sl(2)\(_3\) KdV hierarchy. This operator clearly does not define a \( W \)-algebra, since it is not conformal. Indeed, by rescaling the dimensions such that \( \dim(\partial_t) = 1 \), so that one must divide the dimension of all the fields by 3, one gets \( \dim(u, v, w) = 2/3, 1, 4/3 \). Hence there is no spin-2 field.

As a final remark, we stress that all the equations of the sl(2)\(_3\) hierarchy can be obtained systematically from those of the sl(2) hierarchy by interchanging \( t_l = t_3 \) and \( t_1 = x \) at the level of the KdV equation itself. For instance, the \( j^{th} \) flow in the ordinary KdV hierarchy takes the form

\[
    u_{t_j} = f_j(u, u_x, u_{xx}, u_{xxx}, \ldots) \tag{2.21}
\]

The \( x \)-derivatives are eliminated by means of (2.4) and (2.1) with the result that (2.2) is transformed into

\[
    u_{t_j} = g_j(u, v, w, u_{t_3}, v_{t_3}, w_{t_3}, \ldots) \tag{2.22}
\]

and similar expressions for \( v_{t_j}, w_{t_j} \).

3. The classical \( W_4^{(3)} \) algebra by \( x \leftrightarrow t \) interchange.

3.1. Generalities on the second Hamiltonian structure of scalar Lax equations.

Introduce the pseudo-differential operator

\[
    F^{(l)} = \sum_{k=1}^{N} \partial^{-k} f^{(l)}_{N-k+1} \tag{3.1}
\]

where \( f^{(l)}_{1} \) is fixed by the condition

\[
    \text{Res} \left[ F^{(l)} , L \right] = 0 \tag{3.2}
\]
with $L$ the scalar Lax operator (1.1) and $\text{Res} \sum_i a_i \partial^i = a_{-1}$. The second Hamiltonian structure of the sl($N$) KdV hierarchy takes the form

$$
\partial_t L = (LF^{(l)})_+ L - L(F^{(l)}L)_+ ,
$$

which translates into

$$
(u_i)_{t_l} = (P_2)_{ij} f_j^{(l)} .
$$

$P_2$ gives the Poisson brackets of the different fields, i.e.

$$
\{u_i(x), u_j(y)\} = (P_2(x))_{ij} \delta(x - y) .
$$

The Miura transformation, which furnished the free field realization of this Poisson algebra, can be obtained as follows: one first factorizes $L$ as

$$
L = (\partial + \phi_{N-1})(\partial + \phi_{N-2}) \ldots (\partial + \phi_1)(\partial + \phi_0) ,
$$

where $\sum_{0}^{N-1} \phi_i = 0$. The $\phi_i$’s are then expressed in terms of a set of linearly independent fields $p_i$’s by

$$
\phi_k = \sum_{i=1}^{N-1} \omega^{ki} p_i , \quad \omega = e^{2i\pi/N} .
$$

The Poisson structure for the $p_i$’s is

$$
\{p_i(x), p_j(y)\} = -\frac{1}{N} \delta_{N-i,j} \delta_x(x - y) .
$$

These brackets can be diagonalized, by introducing the fields

$$
r_j = \begin{cases} 
\sqrt{\frac{N}{2}} (p_j + p_{N-j}) , & j < \frac{N}{2} \\
\sqrt{N} p_j , & j = \frac{N}{2} \\
- i \sqrt{\frac{N}{2}} (p_j - p_{N-j}) , & j > \frac{N}{2}
\end{cases}
$$

so that

$$
\{r_i(x), r_j(y)\} = -\delta_{ij} \delta_x(x - y) .
$$
3.2. Specialization to the sl(4) case.

The sl(4) scalar Lax operator is

\[ L = \partial^4 + u_2 \partial^2 + u_3 \partial + u_4 \, . \] (3.11)

The components \((i, j)\) of the corresponding operator \(P_2\) are then found to be

\begin{align*}
(2, 2) & : \quad 5\partial^3 + u_2 \partial + \partial u_2 \\
(2, 3) & : \quad -5\partial^4 - 2\partial^2 u_2 + 2\partial u_3 + u_3 \partial \\
(2, 4) & : \quad \frac{3}{2} \partial^5 + \frac{3}{2} \partial^3 u_2 - \frac{3}{2} \partial^2 u_3 + 3\partial u_4 + u_4 \partial \\
(3, 3) & : \quad -6\partial^5 - 2(\partial^3 u_2 + u_2 \partial^3) + (\partial^2 u_3 - u_3 \partial^2) + 2(u_4 \partial + \partial u_4) - \frac{1}{2}(u_2 \partial + \partial u_2) \\
(3, 4) & : \quad 2\partial^6 + 2\partial^4 u_2 + \frac{3}{2} u_2 \partial^4 - 2\partial^3 u_3 + 3\partial^2 u_4 - u_4 \partial^2 + \frac{1}{2} u_2 \partial^2 u_2 - \frac{1}{2} u_2 \partial u_3 \\
(4, 4) & : \quad \frac{3}{4} \partial^7 + \frac{3}{4} (u_2 \partial^6 + \partial^5 u_2) + \frac{3}{4} (u_3 \partial^4 - \partial^4 u_3) + (u_4 \partial^3 + \partial^3 u_4) + \frac{3}{4} u_2 \partial^3 u_2 + u_2 u_4 \partial + \partial u_2 u_4 - \frac{3}{4} u_3 \partial u_3 + \frac{3}{4} (u_3 \partial^2 u_2 - u_2 \partial^2 u_3)
\end{align*}

(3.12)

Now by rewriting \(L\) under the form

\[ L = (\partial + \phi_3)(\partial + \phi_2)(\partial + \phi_1)(\partial + \phi_0) \, , \] (3.13)

one can express the fields \(u_i\) in terms of the \(\phi_i\)'s and ultimately using (3.7) and (3.9), in terms of the fields \(r_i\)'s. The result turns out to be

\begin{align*}
u_2 &= -\frac{1}{2} (r_1^2 + r_2^2 + r_3^2) + \sqrt{2}r_1 x + r_2 x - \sqrt{2}r_3 x \\
u_3 &= \frac{3}{\sqrt{2}} r_{1xx} + r_{2xx} - \frac{1}{\sqrt{2}} r_{3xx} - \frac{3}{2} r_1 r_{1x} - \frac{1}{\sqrt{2}} r_1 r_{2x} + \frac{1}{2} r_1 r_{3x} - \frac{1}{2} \sqrt{2} r_{1r_2x} - \frac{r_2 r_{2x}}{\sqrt{6}} - \frac{1}{2} r_2 r_{3x} - \frac{1}{2} r_1 r_{3x} - \frac{1}{2} \sqrt{2} r_{2r_3x} - \frac{1}{2} r_2 r_{3x} + \frac{r_1^2 r_2}{2} - \frac{r_2^2 r_3}{2} \\
u_4 &= \frac{1}{\sqrt{2}} r_{1xxx} + \frac{1}{2} r_{2xxx} - \frac{1}{2} r_1 r_{1xx} - \frac{1}{2} \sqrt{2} r_2 r_{1xx} - \frac{1}{2} r_1 r_{2xx} - \frac{r_2 r_{2xx}}{2} - r_1 r_{3xx} - \frac{1}{2} \sqrt{2} r_2 r_{3xx} + \frac{3}{4} r_1 r_{1r_2x} + \\
&\quad \frac{1}{2} r_2 r_{2xx} - \frac{1}{2} r_1 r_{3xx} - \frac{1}{2} \sqrt{2} r_2 r_{3xx} + \frac{3}{4} r_1 r_{1r_2x} + \\
&\quad \frac{1}{2} \sqrt{2} r_1 r_{1r_3} + \frac{1}{4} r_1 r_2 r_3 - \frac{1}{2} \sqrt{2} r_1 r_{r_3x}^2 + \frac{1}{2} r_1^2 r_{2xx} + \frac{1}{2} \sqrt{2} r_1 r_{r_2x} -
\end{align*}
The first Hamiltonian structure for these modified fields is given by the Hamiltonian operator $\Theta$ of (3.10) (we omit the subscript 1)

$$\Theta = \begin{pmatrix} -\partial & 0 & 0 \\ 0 & -\partial & 0 \\ 0 & 0 & -\partial \end{pmatrix}$$  \quad (3.15)$$

Again one can check explicitly the canonical character of the above Miura map by checking directly the identity $P_2 = D\Theta D^\dagger$.

In the following we will be interested more particularly in the third flow of the sl(4) KdV hierarchy. It can be computed from

$$L_t = [(L^{3/4})_+, L] \quad \text{where } t = t_3$$

and

$$L^{3/4} = \partial^3 + \frac{3}{4} u_2 \partial + (\frac{3}{4} u_3 - \frac{3}{8} u_{2x}) + \frac{3}{4} (u_4 - \frac{1}{2} u_{3x} + \frac{1}{12} u_{2xx} - \frac{1}{8} u_2^2) \partial^{-1} + \ldots$$  \quad (3.16)$$

One obtains

$$u_{2t} = \frac{1}{4} u_{2xxx} - \frac{3}{2} u_{3xx} + 3 u_{4x} - \frac{3}{4} u_2 u_{2x},$$

$$u_{3t} = \frac{3}{4} u_{2xxxx} - 2 u_{3xxx} + 3 u_{4xx} - \frac{3}{4} u_2 u_{3x} - \frac{3}{4} u_2 u_3,$$

$$u_{4t} = \frac{3}{8} u_{2xxxxx} - \frac{3}{4} u_{3xxxx} + u_{4xxx} + \frac{3}{8} u_2 u_{2xxx} - \frac{3}{4} u_2 u_{3xx} + \frac{3}{8} u_2 u_3 + \frac{3}{4} u_2 u_{4x} - \frac{3}{4} u_3 u_{3x}. \quad (3.17)$$

These equations have the following Hamiltonian formulation:

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = P_2 \nabla_u \int (u_4 - \frac{1}{8} u_2^2) dx,$$
the Hamiltonian density being \( \text{Res } L^{3/4} \) up to total derivatives. We will also be interested in the modified version of this equation. Rewriting the above Hamiltonian in terms of the modified fields, one has

\[
H_m = \int \left[ -\frac{1}{4} r_{1x} r_{1x} + \frac{1}{8} r_{2x} r_{2x} - \frac{1}{4} r_{3x} r_{3x} - \frac{3}{4} r_{1x} r_{2x} r_{3x} - \frac{1}{32} (r_1^4 + 6r_1^2 r_2^2 - 6r_1 r_2^2 - r_2^4 + 6r_2^2 r_3^2 + r_3^4) \right],
\]

and the corresponding mKdV equations are

\[
\begin{pmatrix}
  r_1 \\
  r_2 \\
  r_3
\end{pmatrix}_t = - \begin{pmatrix}
  \frac{1}{2} r_{1xx} - \frac{3}{4} r_{2x} r_{3x} - \frac{1}{8} r_1^3 - \frac{3}{8} r_1 r_2^2 + \frac{3}{8} r_1 r_3^2 \\
  -\frac{1}{4} r_{2xx} + \frac{3}{4} (r_1 r_3)_x - \frac{3}{8} r_1^2 r_2 + \frac{1}{8} r_2^3 - \frac{3}{8} r_2 r_3^2 \\
  \frac{1}{2} r_{3xx} - \frac{3}{4} r_1 r_{2x} + \frac{3}{8} r_1^2 r_3 - \frac{3}{8} r_2 r_3 - \frac{1}{8} r_3^3
\end{pmatrix},
\]

(3.19)

3.3. The Hamiltonian structure of the \( \text{sl}(4)_3 \) hierarchy.

We now want to rewrite (3.17) and (3.19) as evolution equations with respect to \( x \). Let us start with (3.17), and introduce the fields

\[
\begin{align*}
  u_{ix} &= v_i, \\
  u_{ixx} &= w_i, \\
  &i = 2, 3, 4.
\end{align*}
\]

(3.20)

One finds

\[
\begin{align*}
  u_{ix} &= v_i, \\
  v_{ix} &= w_i, \\
  w_{2x} &= 4u_{2t} + 6w_3 - 12v_4 + 3u_2 v_2, \\
  w_{3x} &= -\frac{6}{5} v_{2t} + \frac{2}{5} u_{3t} + \frac{12}{5} w_4 - \frac{9}{10} v_2^2 - \frac{9}{10} u_2 w_2 + \frac{3}{10} u_2 v_3 + \frac{3}{10} v_2 u_3, \\
  w_{4x} &= 3w_{2t} - 6v_{3t} + 10u_{4t} - 6u_2 u_{2t} - 6u_2 w_3 + \frac{21}{2} u_2 v_4 - \frac{9}{2} u_2^2 v_2 + \frac{27}{4} v_2 w_2 - 9v_2 v_3 + \frac{15}{2} u_3 v_3 - \frac{33}{4} w_2 u_3.
\end{align*}
\]

(3.21)

The \( x \leftrightarrow t \) interchange version of \( H_m \) read off from \( \text{Res } L^{3/4} \) in (3.16) and reexpressed in terms of the above fields is

\[
\tilde{H} = \int (u_4 - \frac{1}{8} u_2^2 - \frac{1}{2} v_3 + \frac{1}{12} w_2) dt,
\]

(3.22)

and we are looking for the corresponding Hamiltonian operator \( \tilde{P}_2 \) which allows the rewriting of the above system in the form (1.10). For this we need first the modified version of (3.21) and its natural Hamiltonian structure. To write (3.19) in \( x \)-evolution we introduce
the variables \( r_{ix} = s_i, r_{ixx} = \tilde{q}_i \). However, using the hindsight gained from the study of the KdV and the Boussinesq cases, we choose new variables \( q_i \) so as to keep the highest field equations in the form \( q_{ix} = r_{it} \). The explicit form of the \( q_i \)'s can then be read off directly from (3.19), i.e. \( q_1 = -\frac{1}{2}r_{1xx} + \frac{3}{4}r_{2x}r_3 + \ldots \), and the sl(4)_3 version of (3.19) is

\[
\begin{align}
\begin{cases}
    r_{1x} &= s_1 \\
    s_{1x} &= -2q_1 - 2\left[ -\frac{3}{4}s_2r_3 - \frac{1}{8}r_1^3 - \frac{3}{8}r_1r_2^2 + \frac{3}{8}r_1r_3^2 \right] \\
    q_{1x} &= r_{1t} \\
    r_{2x} &= s_2 \\
    s_{2x} &= 4q_2 + 4\left[ 3\frac{3}{4}s_1r_3 + \frac{3}{8}r_1s_3 - \frac{3}{8}r_1r_2^2 + \frac{1}{8}r_2^2 - \frac{3}{8}r_2r_3^2 \right] \\
    q_{2x} &= r_{2t} \\
    r_{3x} &= s_3 \\
    s_{3x} &= -2q_3 - 2\left[ -\frac{3}{4}s_1r_2 + \frac{3}{4}r_1^2r_3 - \frac{3}{4}r_2r_3^2 - \frac{1}{8}r_3^3 \right] \\
    q_{3x} &= r_{3t}
\end{cases}
\end{align}
\]

(3.23)

These flows can be written in a Hamiltonian form as follows. The Hamiltonian can be easily obtained from the above \( \tilde{H} \), where we express the KdV fields in terms of the modified fields, using the Miura map (3.14) and the equations (3.23) to eliminate the \( x \)-derivative of the modified KdV fields. We get

\[
\tilde{H}_m = \int \left[ r_1q_1 + r_2q_2 + r_3q_3 + \frac{1}{4}s_1^2 - \frac{1}{8}r_1^3 + \frac{1}{4}s_2^2 - \frac{1}{32}r_1^4 - \frac{3}{16}r_1^2r_2^2 + \frac{3}{16}r_1^2r_3^2 + \frac{1}{32}r_2^4 - \frac{3}{16}r_2^2r_3^2 - \frac{1}{32}r_3^4 \right] dt
\]

(3.24)

We determine \( \tilde{\Theta} \) by inspection, writing \((r, s, q)_x^T = \tilde{\Theta}(\nabla_r \tilde{H}_m, \nabla_s \tilde{H}_m, \nabla_q \tilde{H}_m)^T\), with the result:

\[
\tilde{\Theta} = s \begin{pmatrix}
    0 & -4 & 0 \\
    -2 & 6r_3 & 6r_1 \\
    -2 & -6r_1 & \partial_t
\end{pmatrix},
\]

(3.25)
This operator can be further simplified by introducing the new fields \( \tilde{s}_i = s_i - 3\delta_{2i}r_1r_3 \), 
\( 2\tilde{r}_1 = r_1, -4\tilde{r}_2 = r_2, 2\tilde{r}_3 = r_3 \), so that \( \tilde{\Theta} \) takes the form

\[
\tilde{\Theta} = \begin{pmatrix}
\tilde{r} & \tilde{s} & q \\
0 & I_3 & 0 \\
-I_3 & 0 & 0 \\
0 & 0 & I_3 \partial_t
\end{pmatrix}.
\] (3.26)

Since this is antisymmetric and field independent (so that the Jacobi identities are automatically satisfied), this operator is manifestly Hamiltonian. Now, having \( \tilde{\Theta} \) and the Miura map, which leads to \( \tilde{D} \), one can calculate \( \tilde{P}_2 \) by (1.11). The explicit form of \( \tilde{P}_2 \) obtained this way is not manifestly conformal. However, after some field redefinitions given in the next section, it can be transformed into a conformal algebra, to be presented in that same section. Exactly the same algebra can be derived by the method of Hamiltonian reduction to which we now turn.

4. The classical \( W_4^{(3)} \) algebra by the method of Hamiltonian reduction.

4.1. \( W_4^{(3)} \) by Hamiltonian reduction.

In this section, we derive the \( W_4^{(3)} \) algebra by the method of Hamiltonian reduction. This method is by now standard and will not be reviewed here [15]. We start from a 1-dim connection that depends on a coordinate \( t \), taking values in the algebras of \( \text{sl}(4) \). In the matrix representation we constrain the (1, 4) element of the connection to be -1, i.e.

\[
A(t) = \begin{pmatrix}
J_{11} & J_{12} & J_{13} & -1 \\
J_{21} & J_{22} & J_{23} & J_{24} \\
J_{31} & J_{32} & J_{33} & J_{34} \\
J_{41} & J_{42} & J_{43} & J_{44}
\end{pmatrix}, \quad J_{44} = -\sum_{i=1}^{3} J_{ii}.
\] (4.1)

The form of the constraint is preserved by the gauge transformations

\[
g^{-1}(\partial_t + A(t))g = \partial_t + A^g(t),
\] (4.2)

where \( g \) is a lower triangular matrix with 1’s on the diagonal. We cannot completely fix this gauge invariance in a local way. However, restricting ourselves to the subalgebra with
$g_{23} = 0$, we can fix the gauge invariance with respect to this subalgebra; using (4.2), we bring $A(t)$ to the canonical form

$$Q(t) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ A_1 & U_1 & Z & 0 \\ B_1 & A_2 & U_2 & 0 \\ T & B_2 & A_3 & -U_1 - U_2 \end{pmatrix} .$$

(4.3)

Using the expression of the fields appearing in $Q$ in terms of the original currents of $A(t)$ (which form an sl($N$) current algebra), we obtain an algebra which we call the $W_4^{(3)}$ algebra. The superscript 3 in this context corresponds to the fact that the third upper diagonal has been set to $-1$. The spin content of the algebra can be read off directly from (4.3) since the dimensions are constant along the diagonals and by moving from the top right corner to the bottom left corner, they increase by units of $1/3$. Since a constant has dimension 0, we find dim($Z, U_i, A_i, B_i, T$) = (2/3, 1, 4/3, 5/3, 2).

After introducing $T_0 = T - A_2 Z - U_1^2 - U_2^2 - U_1 U_2 + \frac{2}{3} U_1 t + \frac{1}{3} U_2 t$, which makes $T_0$ into a classical energy-momentum tensor, we get the algebra

$$\{Z, U_1\} = Z \delta , \quad \{Z, U_2\} = -Z \delta , \quad \{Z, A_2\} = -\delta_t + (U_2 - U_1) \delta$$

$$\{Z, B_1\} = -A_1 \delta , \quad \{Z, B_2\} = A_3 \delta , \quad \{U_1, U_1\} = -\frac{3}{4} \delta_t , \quad \{U_1, U_2\} = \frac{1}{4} \delta_t$$

$$\{U_1, A_1\} = -A_1 \delta , \quad \{U_1, A_2\} = A_2 \delta , \quad \{U_1, B_2\} = B_2 \delta , \quad \{U_2, U_2\} = -\frac{3}{4} \delta_t$$

$$\{U_2, A_2\} = -A_2 \delta , \quad \{U_2, A_3\} = A_3 \delta , \quad \{U_2, B_1\} = -B_1 \delta , \quad \{A_1, A_2\} = B_1 \delta$$

$$\{A_1, A_3\} = 2Z \delta_t + (Z_t + 2(U_1 + U_2) Z) \delta , \quad \{A_2, A_3\} = B_2 \delta$$

$$\{A_1, B_2\} = \delta_{tt} + (3U_1 + U_2) \delta_t +$$

$$\left(\frac{4}{3} U_{1t} + \frac{2}{3} U_{2t} + T_0 + 2ZA_2 + 3U_1^2 + 2U_1 U_2 + U_2^2\right) \delta$$

$$\{A_3, B_1\} = -\delta_{tt} + (U_1 + 3U_2) \delta_t +$$

$$\left(\frac{2}{3} U_{1t} + \frac{4}{3} U_{2t} + T_0 - 2ZA_2 - U_1^2 - 2U_1 U_2 - 3U_2^2\right) \delta$$

$$\{B_1, B_2\} = 2A_2 \delta_t + (A_2 t + 2(U_1 + U_2) A_2) \delta$$

$$\{T_0, Z\} = \frac{2}{3} Z \delta_t - \frac{1}{3} Z_t \delta$$

$$\{T_0, U_1\} = -\frac{1}{6} \delta_{tt} + U_1 \delta_t , \quad \{T_0, U_2\} = \frac{1}{6} \delta_{tt} + U_2 \delta_t$$

$$\{T_0, A_1\} = \frac{4}{3} A_1 \delta_t + \frac{1}{3} A_{1t} \delta$$

$$\{T_0, A_3\} = \frac{5}{3} B_1 \delta_t + \frac{2}{3} B_{1t} \delta$$

$$\{T_0, T_0\} = \frac{5}{9} \delta_{ttt} + 2T_0 \delta_t + T_{0t} \delta$$

(4.4)
All other brackets vanish. Here the two fields in the Poisson brackets are evaluated at \( t \) and \( t' \) respectively; all the fields on the RHS are evaluated at \( t \) and \( \delta = \delta(t - t') \). We have checked that the Jacobi identities are satisfied for this algebra.

4.2. A twisted version of the \( W_4^{(3)} \) algebra and the relation to covariantly coupled algebras and quasi-superconformal algebras.

Note the \( \{ T_0, U_i \} \) relations in (4.4), which display the non-primary character of the \( U_i \) fields. (Recall that a field \( \phi \) is primary, with dimension \( h \) if it satisfies \( \{ T_0, \phi \} = h\phi \delta_t + (h - 1)\phi \delta \). This could be cured by taking \( T_N = T_0 - \frac{1}{6}U_{1t} + \frac{1}{6}U_{2t} \) as the energy-momentum tensor. Then all the fields are primary, but their spins, as read off their Poisson brackets with \( T_N \), is no longer equivalent to the grading under which the soliton equations that will be presented in the next section are homogeneous: \( Z, U_{1,2}, A_2 \) now have spin 1, \( A_{1,3}, B_{1,2} \) spin \( 3/2 \) and \( T_N \) spin 2.

Actually, the twisted form of the algebra can be written in a rather compact way using the following notation:

\[
2J_{11} = -2J_{22} = U_1 - U_2, \quad J_{12} = A_2, \quad J_{21} = Z, \quad U = U_1 + U_2,
\]
\[
G_1^+ = A_1, \quad G_1^- = B_2, \quad G_2^- = B_1, \quad G_2^+ = A_3.
\]

One finds

\[
\{U, U\} = -\delta_t, \quad \{U, G_a^\pm\} = \pm G_a^\pm,
\]
\[
\{J_{ab}, J_{cd}\} = (\delta_{cb} J_{ad} - \delta_{ad} J_{cb}) \delta - (\delta_{ad} \delta_{cb} - \frac{1}{2} \delta_{ab} \delta_{cd}) \delta_t,
\]
\[
\{J_{ab}, G_c^+\} = (\delta_{ab} G_a^+ - \frac{1}{2} \delta_{ab} G_c^+) \delta,
\]
\[
\{J_{ab}, G_c^-\} = (-\delta_{ac} G_b^- + \frac{1}{2} \delta_{ab} G_c^-) \delta,
\]
\[
\{G_a^-, G_b^+\} = 2J_{ab} \delta_t + (J_{ab} + U J_{ab}) \delta + \delta_{ab} [\delta_{tt} + 2U \delta_t + (U_t + T_N + 2J_{ac} J_{cb} + \frac{3}{2} U^2) \delta].
\]

The brackets with \( T_N \) are those of primary fields with spins given above, and for \( T_N \) the central term is now \( \delta'''/2 \). All other brackets vanish. One sees that the algebra in the above form contains an \( sl(2) \) and a \( u(1) \) Kac-Moody algebras. The spin \( 3/2 \) fields have a definite \( u(1) \) charge and they transform in the defining representation of \( sl(2) \).

One thus recovers a particular example of the algebras constructed in [14][16][17]. In [17], it was obtained from the standard \( u(N - 2) \) superconformal algebra [18] by
changing the statistics of the fermionic fields (the resulting algebras were called quasi-superconformal). The present case corresponds to \( N = 4 \). On the other hand, in \([14][16]\), the general structure was inferred by considering the embeddings of \( \text{sl}(2) \) in \( \text{sl}(N) \) associated with the decomposition \( \underline{N} \to 2 + (N - 2) \underline{1} \) of the defining representation.

It is natural to ask whether there is a KdV-type hierarchy related directly to the quasi-superconformal algebras. By a direct relation we mean a hierarchy homogeneous with respect to the grading fixed by the quasi-superconformal algebra (4.6). In fact there exists such a hierarchy: this is exactly the bosonic version of the \( \text{u}(N) \) super KdV hierarchy introduced in the third reference of \([18]\) and which we will discuss in more details elsewhere. We just mention that its first Hamiltonian structure is deduced from the second one by the shift \( T_N \to T_N + \lambda \), so that it reads

\[
\{T_N, T_N\} = 2\delta' , \quad \{G_i^-, G_i^+\} = \delta .
\]  

(4.7)

In particular, the \( \text{u}(1) \) quasi-super KdV hierarchy corresponds to the hierarchy constructed in \([3][19]\) starting from a gradation intermediate between the principal and the homogeneous ones.

4.3. The \( \text{sl}(4)_3 \) flows by reduction of the self-dual Yang–Mills equations.

Following the method of \([20][21]\) (see also \([22]\)), we can use a reduction of self-dual Yang-Mills equations in 4-dim to obtain the fractional KdV equations corresponding to \( W_4^{(3)} \). To this end we start from a four dimensional space with signature (2,2) and metric \( ds^2 = 2dx\,dy + 2dz\,dt \). With \( \epsilon_{xyzt} = -1 \), the self-duality equations

\[
\begin{align*}
[D_x, D_t] &= 0 \\
[D_x, D_y] = [D_z, D_t] \quad \text{become} \\
[D_y, D_z] &= 0
\end{align*}
\]

\[
\begin{align*}
[\partial_t + Q, \partial_x + H] &= 0 \\
[\partial_t + Q, P] &= [B, \partial_x + H] \\
[P, B] &= 0
\end{align*}
\]  

(4.8)

where we have performed a reduction with respect to the two null Killing symmetries \( \partial_y \) and \( \partial_z \).

For the matrix \( Q \) we take (4.3), whereas for \( B \), we take

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
Z & 0 & 1 & 0
\end{pmatrix}
\]  

(4.9)
The reason for this choice will be given section 4.4.

Let us denote the matrix elements of $H$ by $h_{ij}$. We get, by consistency of (4.8), $h_{14t} = h_{24t} = h_{34t} = 0$ and

$$
\begin{align*}
    h_{13} &= h_{24}, & h_{12} &= h_{34}, & h_{23} &= h_{34} - h_{14}Z, \\
    h_{11} &= h_{24}Z, & h_{22} &= -h_{14}U_1 - h_{24}Z, & h_{33} &= -h_{14}U_2 - h_{24}Z, \\
    h_{21} &= -h_{14}A_1 - 2h_{24}(U_1 + U_2) - h_{34}Z, & h_{32} &= -h_{14}A_2 - 2h_{24}(U_1 + U_2) - 2h_{34}Z, \\
    h_{43} &= -h_{14}A_3 - h_{24}U_2 - h_{34}Z, & h_{31} &= -h_{14}B_1 - h_{24}A_2 - h_{34}(U_1 + 2U_2), \\
    h_{42} &= -h_{14}B_2 - h_{24}A_2 - h_{34}U_1, & h_{41} &= -h_{14}T + h_{24}(Z_t - B_1) - h_{34}A_1. 
\end{align*}
$$

(4.10)

The form of $P$ is given implicitly at the end of section 4.4. For the fields we find the evolutions

$$
\begin{align*}
    Z_x &= -h_{14}Z_t + h_{24}(A_1 - A_3) + h_{34}(U_1 - U_2) \\
    U_{1x} &= -h_{14}U_{1t} + h_{24}(-Z_t - 2(U_1 + U_2)Z - B_2) + h_{34}(A_1 - A_2 - 2Z^2) \\
    U_{2x} &= -h_{14}U_{2t} + h_{24}(-Z_t + 2(U_1 + U_2)Z + B_1) + h_{34}(A_2 - A_3 + 2Z^2) \\
    A_{1x} &= -h_{14}A_{1t} + \\
    &h_{24}(-\frac{4}{3}U_{1t} - \frac{2}{3}U_{2t} + 2(A_1 - A_2)Z - 3U_1^2 - 2U_1U_2 - U_2^2 - T_0) + \\
    &h_{34}(-Z_t - 2(U_1 + U_2)Z - B_1) \\
    A_{2x} &= -h_{14}A_{2t} + h_{24}(-2U_{1t} - 2U_{2t} - 2U_2^2 + 2U_1^2) + \\
    &h_{34}(-2Z_t + B_1 - B_2 - 2(U_2 - U_1)Z) \\
    A_{3x} &= -h_{14}A_{3t} + \\
    &h_{24}(-\frac{2}{3}U_{1t} - \frac{4}{3}U_{2t} + 2(A_2 - A_3)Z + U_1^2 + 2U_1U_2 + 3U_2^2 + T_0) + \\
    &h_{34}(-Z_t + B_2 + 2(U_1 + U_2)Z) \\
    B_{1x} &= -h_{14}B_{1t} + h_{24}(-A_{2t} + 2ZB_1 + 2(U_1 + U_2)(A_1 - A_2) \\
    &+ h_{34}(-\frac{1}{2}U_{1t} - \frac{3}{2}U_{2t} + 2(A_1 - A_2)Z - U_1^2 - 2U_1U_2 - 3U_2^2 - T_0) \\
    B_{2x} &= -h_{14}B_{2t} + h_{24}(-A_{2t} - 2ZB_2 + 2(U_1 + U_2)(A_2 - A_3) \\
    &+ h_{34}(-\frac{3}{2}U_{1t} - \frac{1}{2}U_{2t} + 2(A_2 - A_3)Z + 3U_1^2 + 2U_1U_2 + U_2^2 + T_0)
\end{align*}
$$

18
\[ T_{0x} = - h_{14} T_{0t} + h_{24} \left( -\frac{2}{3} (B_1 + B_2) + \frac{4}{3} (U_1 + U_2) Z \right)_t + \frac{h_{34}}{3} \left( -\frac{1}{3} (A_1 + A_2 + A_3) + \frac{1}{3} Z^2 \right)_t \]  

(4.11)

Inspection of the equations shows that the spins of \( h_{14}, h_{24} \) and \( h_{34} \) differ in ascending order by 1/3. If all three coefficients are zero, all the flows are trivial. If we set \( h_{14} = 1 \), then since \( h_{24} \) and \( h_{34} \) have spins 1/3 and 2/3 resp. and they are constant, we must set them equal to zero. For the same reason, the only other two solutions are \( (h_{14} = 0, h_{24} = 1, h_{34} = 0) \) and \( (h_{14} = 0, h_{24} = 0, h_{34} = 1) \). Each different solution gives rise to a set of equations which are Hamiltonian. Explicitly, we have

\[ h_{14} = 1, \quad H_1 = \int -T_0 dt \]
\[ h_{24} = 1, \quad H_{2/3} = \int [-(B_1 + B_2) + 2Z(U_1 + U_2)] dt \]
\[ h_{34} = 1, \quad H_{1/3} = \int [-(A_1 + A_2 + A_3) + Z^2] dt \]  

(4.12)

Note that the conformal dimension of these Hamiltonians is well defined. If we had modified the energy-momentum tensor, as in the previous section, in order to get a conformal algebra with all fields primary, we would have found \( H_{1/3} = \int -(A_1 + A_2 + A_3) + \ldots \), but since the fields \( A_i \)'s have different spins, the Hamiltonian is not dimensionally homogeneous.

4.4. Relation with the results obtained by \( x \leftrightarrow t \) interchange.

As already stated, the \( W_4^{(3)} \) algebra obtained by \( x \leftrightarrow t \) interchange and the one obtained by Hamiltonian reduction are fully equivalent. The fields in the two approaches are related by

\[ 4Z = u_2 \quad , \quad 8U_1 = -v_2 + 2u_3 \quad , \quad 8U_2 = -3v_2 + 2u_3 \quad , \]
\[ 16A_1 = 16u_4 - 2w_2 - 4v_3 - u_2^2 \quad , \quad 16A_2 = 16u_4 - 8v_3 + 3u_2^2 \]
\[ 16A_3 = 16u_4 + 6w_2 - 12v_3 - u_2^2 \]
\[ 4B_1 = -3u_2t + 4w_3 - 10v_4 + 3u_2v_2 - u_2u_3 \]
\[ 4B_2 = -5u_2t + 6w_3 - 14v_4 + 5u_2v_2 - u_2u_3 \]
\[ T_0 = \frac{1}{10} u_4 + \frac{7}{40} v_2t - \frac{1}{10} u_3^2 - \frac{61}{160} v_2^2 + \frac{7}{80} u_2w_2 + \frac{3}{40} u_2v_3 + \]
\[ \frac{33}{40} v_2u_3 + \frac{1}{16} u_3^2 - \frac{3}{8} u_3^2 \].  

(4.13)

With these field redefinitions and \( t \rightarrow -t \), the flow associated to \( H_{1/3} \) is easily checked to be equivalent to the first flow of the \( \text{sl}(4)_3 \) KdV hierarchy (3.21). At this point, we recall
that when regarding the $W_4$ algebra as a second Hamiltonian structure, we obtain the first Hamiltonian structure by shifting $u_4$ by a constant. Inspection of the field redefinitions shows that such a shift corresponds to shifting each of the $A_i$ fields by the same constant. So the first Hamiltonian structure for the $\text{sl}(4)$ flows can be obtained by such a shift, and then the usual procedure can be employed to generate an infinite hierarchy of flows and conserved quantities, recovering the integrability properties from another point of view.

Also, in [4] it was noticed that the same shift relates $Q$ to $B$, and $H$ to $P$, in the following sense: if the first and the second Hamiltonian structures are related by a shift of, say, the fields $q, \{\ldots\}2_{q+\lambda} = \{\ldots\}2_{q} + \lambda\{\ldots\}1_{q}$, then $Q(q + \lambda) = Q(q) + \lambda B$ and $H(q + \lambda) = H(q) - \lambda P$. Here we should remember that the field $T$ which enters $Q$ is related to the energy-momentum tensor $T_0$ by $T = T_0 + A_2Z + \ldots$ Therefore, we can see that a similar relation exists here between $Q$ and $B$. We observed that the same relation holds true for $H$ and $P$. This remark gives a heuristic justification for the form of the $B$ matrix we took in (4.9).

5. On the general structure of $W_N^{(l)}$ algebras.

In this section we want to present some general characteristics of the $W_N^{(l)}$ algebras which can be extracted from the $x \leftrightarrow t$ interchange and the Hamiltonian reduction methods.

5.1. Canonical basis of independent fields for the $\text{sl}(N)_l$ hierarchy and spin content of the $W_N^{(l)}$ algebra

In this section, $N$ and $l$ will be taken to be coprime. To perform the $x \leftrightarrow t$ interchange in the $l^{th}$ flow of the $\text{sl}(N)$ KdV hierarchy, one has to introduce a certain number of new independent fields which are the first few derivatives of the $\text{sl}(N)$ KdV fields $u_2, u_3, \ldots u_N$. This set must be chosen such that the $x$ derivative of every field is either another field of the set or can be expressed in terms of the time derivative of other fields of the set using the $l^{th}$ $\text{sl}(N)$ KdV equation. A convenient basis for these new independent fields is given by

$$\{u_i, u_{ix}, u_{ixx}, \ldots, u_i^{(l-1)}\} , i = 2, \ldots N.$$  \hspace{1cm} (5.1)

This will be called the canonical basis. It consists of $l(N - 1)$ fields. From such a basis one can read off directly the spin content of the fields in the $W_N^{(l)}$ algebra. These are

$$\text{spins } W_N^{(l)} = \left\{ \frac{i + k}{l}, i = 2, \ldots, N; k = 0, \ldots, l - 1 \right\}$$  \hspace{1cm} (5.2)
(Recall that spin $u_i^{(k)} = (i+k)/l$.) This spin content satisfies the sum rule (as conjectured in [13])
\[
\sum_{\text{spins } s} (2s-1) = \sum_{i=2}^{N} \sum_{k=0}^{l-1} \left( \frac{2(i+k)}{l} - 1 \right) = N^2 - 1 = \dim \mathfrak{sl}(N) . \tag{5.3}
\]
The basis (5.1) is of course not unique but any other basis has exactly the same spin content. For instance, in the $\mathfrak{sl}(3)$ case, one can write the second flow under the form
\[
\begin{align*}
  u_{2t} &= -u_{2xx} + u_{3x} \\
  u_{3t} &= u_{3xx} + u_{2u2x}.
\end{align*} \tag{5.4}
\]
The canonical basis corresponds to the choice of the new fields
\[
\begin{align*}
  v_2 &= u_{2x} \quad , \quad v_3 = u_{3x} \tag{5.5}
\end{align*}
\]
but one could also have considered
\[
\begin{align*}
  v_2 &= u_{2x} \quad , \quad w_2 = u_{2xx} \tag{5.6}
\end{align*}
\]
However these two choices yield equivalent $\mathfrak{sl}(3)_2$ hierarchies and in particular $\tilde{P}_2$ is the same for both cases, up to simple field redefinitions. This generalized to more complicated cases.

5.2. What happens when $(l, N) \neq 1$?

When $l$ and $N$ are coprime, it always appear to be possible to construct the $\mathfrak{sl}(N)_l$ hierarchy by direct $x \leftrightarrow t$ interchange. However, this is not the case when $(l, N) \neq 1$. For instance, consider the second flow of the $\mathfrak{sl}(4)$ hierarchy:
\[
\begin{align*}
  u_{2t} &= -2u_{2xx} + 2u_{3x} , \\
  u_{3t} &= -2u_{2xxx} + u_{3xx} + 2u_{4x} - u_2u_{2x} , \\
  u_{4t} &= -\frac{1}{2}u_{2xxxx} + u_{4xx} - \frac{1}{2}u_{2u2xx} - \frac{1}{2}u_{3u2x} . \tag{5.7}
\end{align*}
\]
It turns out here that for any choice of independent fields, the $x$-derivative of one of the fields is not determined by the above equation. This problem persists at the level of the modified equations. The Hamiltonian for the flow (5.7) is simply $\int u_3 dx$. When reexpressed in terms of modified fields, one easily gets the modified equations
\[
\begin{align*}
  r_{1t} &= r_{3xx} - (r_1r_2)_x , \\
  r_{2t} &= -\frac{1}{2}(r_1^2 - r_3^2)_x , \\
  r_{3t} &= -r_{1xx} + (r_2r_3)_x . \tag{5.8}
\end{align*}
\]
Therefore if we choose the new independent fields to be \( s_i = r_{ix} \), one sees that \( s_2x \) is not determined by the above equations due to the absence of a term \( r_{2xx} \).

It is simple to show that this situation is generic for \( l = 2 \) and \( N \) even. Indeed, up to non-linear terms, the evolution equation for \( u_i \) reads

\[
u_{ii} = u_{ixx} + 2u_{i+1 \, x} - \frac{2}{N} \binom{N}{i} u_{2 \, (i)} + \ldots
\]  

(5.9)

where \( \binom{N}{i} \) denotes a binomial coefficient. Let us fix the basis to be \( \{ u_2, u_{2x}, \ldots, u_{2 \, (N-1)}, u_3, u_4, \ldots, u_N \} \), in terms of which the argument is simpler. We want to show that the \( x \)-derivative of \( u_{2 \, (N-1)} \) is not determined by (5.7). From the \( N-3 \) first equations in (5.7), one expresses \( u_{kx} \) (\( k \neq 2 \)) in terms of the other fields of the above set:

\[
u_{kx} = \frac{1}{N} u_{2 \, (k-1)} \sum_{i=1}^{k-1} (-1)^{i+1} \binom{N}{N-i} \frac{1}{2^{i-1}}
\]

(5.10)

In the final equation, the coefficient of \( u_{2 \, (N)} \) is

\[
\frac{1}{n} \sum_{i=1}^{N-1} \binom{N}{i} (-1)^{N-i+1} \frac{2}{2^{N-1+1}} - \frac{2}{N} = \frac{(-1)^N - 1}{N \, 2^{N-1}}
\]

(5.11)

This vanishes when \( N \) is even, in which case \( u_{2 \, (N)} \) is not determined. The argument works for other choices of the basis. We expect this to be generic to the cases where \( N \) and \( l \) are not coprime, but we haven’t found a direct check of this statement within the above approach.

5.3. A comment on the Hamiltonian structure of the modified fields for the \( \text{sl}(N) \) hierarchy.

Let us introduce the following basis for the modified fields:

\[
\{ \phi_i^{(k/l)}, i = 1, \ldots, N-1; k = 1, \ldots, l-1 \}
\]

(5.12)

where \( \phi_i^{(1/l)} \) are the usual modified fields \( r_i \) introduced previously, up to a possible scaling factor, and \( \phi_i^{(k/l)} \) is linearly related to \( r_i^{(k+1)} \). Notice that the new dimension of \( \phi_i^{(k/l)} \) is just \( k/l \). From the first few examples which have been worked out, it is tempting to guess that for suitably chosen \( \phi_i^{(k/l)} \), the Hamiltonian structure of the modified fields will read

\[
\{ \phi_i^{(k/l)}, \phi_j^{((l-k)/l)} \} = -\delta_{ij} \delta \quad k \neq 1/2
\]

\[
\{ \phi_i^{(1/2)}, \phi_j^{(1/2)} \} = -\delta_{i,N-i-1} \delta
\]

\[
\{ \phi_i^{(1)}, \phi_j^{(1)} \} = -\delta_{ij} \delta_t
\]

(5.13)

This structure is certainly ill defined for \( N \) and \( l \) even. For instance, for \( W_4^{(2)} \), this would give \( \{ r_2, r_2 \} = -\delta \), which is impossible.
5.4. Results from the Hamiltonian reduction

For the $W_N$-algebras (i.e. $W_N^{(1)}$ in our notation), the matrix $A(t)$ corresponding to (4.1) is constrained by setting the first upper diagonal to $-1$. It can always be brought by Hamiltonian reduction to the form of a matrix with zero entries everywhere except for the first upper diagonal which is set to $-1$ and the lowest row which takes the form

$$
\begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & \cdots & u_2 & 0 \\
\end{bmatrix}
$$

(5.14)

There are field redefinitions so that the corresponding $W$-algebra is conformal. It is well-known that the energy-momentum tensor takes the form

$$
T_0 = u_2 = -\frac{1}{2}trJ^2 + \frac{1}{l}\partial((n-1)J_{11} + (n-2)J_{22} + \cdots + J_{n-1,n-1})
$$

(5.15)

with $l = 1$.

For the $W_N^{(2)}$ algebras, we performed the Hamiltonian reductions up to $N = 8$ and found the following results. The constraint we impose is to set the second upper diagonal to $-1$. This constraint is preserved by gauge transformations generated by strictly lower triangular matrices. However, for $N$ even, it is not possible to fix this gauge freedom completely in a local way. So when considering gauge transformations of the form (4.2), we set $g_{21} = 0$ if $N$ is even. The Hamiltonian reduction gives the following: the “$Q$” matrix can always be reduced so that it is zero everywhere, except for the second upper diagonal which is $-1$, and the lower two rows. These rows appear as follows:

$$
\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & -1 \\
* & \cdots & T_1 & G^{(+)} & U & Z \\
* & \cdots & * & T_2 & G^{(-)} & -U \\
\end{bmatrix}
$$

(5.16)

The field $Z$ is identically zero if $N$ is odd. We find that $T_0 = T_1 + T_2 - U^2 - ZG^{(-)} + \frac{1}{2}U'$ is an energy-momentum tensor, whose expression in terms of the original fields is (5.15) with $l = 2$ modulo terms involving derivatives of the fields above the diagonal. With respect to this energy-momentum tensor, $Z$ has spin $1/2$, $U$ is quasi-primary of spin 1, etc. $T_0$ is the energy-momentum that respects the original grading of the KdV fields, as given by the $x \leftrightarrow t$ interchange.

The presence of the $Z$ field for $N$ even is a reflection, in the context of Hamiltonian reduction, of the presence of constraints in the approach where $x$ and $t$ are interchanged. From this latter point of view, since the $Z$ field has spin $1/2$, that means that it originally
had spin 1 before the interchange. Since there is no spin 1 field in the sl(4) KdV hierarchy, this indicates that such a field has to be introduced to take care of the constraints.

For the more general $W_N^{(l)}$ algebras, let us now restrict ourselves for convenience to the cases where $N$ and $l$ are coprime. For the $W_N^{(3)}$ cases, we find by Hamiltonian reduction a “$Q$” matrix that has zeros everywhere but for the third upper diagonal set to -1 and the lowest three rows, in which we find one spin-2/3 field, two spin-2 fields, ... They are arranged as follows:

$$
\begin{pmatrix}
0 & \ldots & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & \ldots & T_1 & B_1 & A_1 & U_1 & Z & 0 \\
* & \ldots & * & T_2 & B_2 & A_2 & U_2 & 0 \\
* & \ldots & * & * & T_3 & B_3 & A_3 & -U_1 - U_2
\end{pmatrix}
$$

(5.17)

for the $W_{3n+1}^{(3)}$, whereas for the $W_{3n+2}^{(3)}$, the spin 2/3 field $Z$ is in position $(N - 1, N)$. The combination $T_0 = T_1 + T_2 + T_3 - U_1^2 - U_2^2 - U_1 U_2 - A_i Z + \frac{2}{3} U_1' + \frac{1}{3} U_2'$ (with $i = 2$ for $W_{3n+1}^{(3)}$ and $i = 3$ for $W_{3n+2}^{(3)}$) forms the energy-momentum tensor. The explicit expression of this tensor in terms of the original currents is again (5.15) with $l = 3$ and terms involving derivatives of the fields above the diagonal.

It is clear from the above examples and their natural extension that for $N$ and $l$ coprime, Hamiltonian reduction leads to a $W_N^{(l)}$ spectrum that corresponds identically to the one dictated by the $x \leftrightarrow t$ interchange. In addition, the energy-momentum tensor that respects the natural grading of the KdV equations takes the form

$$
T_0(W_N^{(l)}) = -\frac{1}{2} tr J^2 - \frac{1}{l} \left( \sum_{i=1}^{N-1} (N - i) J'_{i,i} \right) + \text{(derivatives of non-diagonal elements)}
$$

(5.18)

We conclude that $\{T_0, T_0\} = \frac{(N^2 - N)}{12l^2} \delta_{ttt} + 2T_0 \delta_t + T_0 \delta$.

6. Conclusions.

In this work we have constructed a new explicit example of a $W_N^{(l)}$ algebra, $W_4^{(3)}$, from the point of view of generalized KdV hierarchies ([3],[4],[5],[6]). We have thus provided further support to the conjecture that the $W_N^{(l)}$ algebras correspond to the second Hamiltonian structure of the sl$(N)$ KdV hierarchy. The latter refers to the new hierarchy obtained from the standard sl$(N)$ KdV hierarchy by interchanging the roles of the variables $x$ and $t$ in the $l^{th}$ flow. Granted this conjecture, another original motivation was to
advocate this approach as being a simple and systematic way of constructing the $W_N^{(l)}$ algebras. From this point of view, the present analysis shows that it is not as simple as initially expected. At first we found from the outset that this approach works straightforwardly only when $N$ and $l$ are coprime. We plan to return to the cases $(N,l) \neq 1$ elsewhere, but these are certainly more complicated since one has to deal with constrained systems. Second, for the new example we have worked out, the expression we obtain for this second Hamiltonian structure is quite complicated. This Poisson structure can be somewhat simplified by introducing a new set of independent fields, namely those fields which appear naturally in a zero curvature formulation, or equivalently, the method of Hamiltonian reduction. If we modify the energy-momentum tensor to make them primary, then the spin content of the algebra is modified. At this step, it appears that the algebra acquires a much nicer form once we recognize the existence of an underlying Kac-Moody algebra organizing the whole conformal algebra. In fact we recover an example of a $W$ algebra one obtains by considering a non-principal embedding of sl(2) (the sum-embedding) into sl(N) [14], or equivalently of a $u(N - 2)$ quasi-conformal superalgebra [17]. We point out that the latter is also related to the bosonic form of the $u(N - 2)$ super KdV hierarchy. It is certainly interesting and satisfying to display explicitly the perfect equivalence of the method of $x \leftrightarrow t$ interchange and that of non-principal sl(2) embeddings, which from the outset looks conceptually rather remote.

Finally let us emphasize some favorable aspects of the method of $x \leftrightarrow t$ interchange. That we derive the second Hamiltonian structure via the modified fields gives us in one shot both the algebra and its free field representation. Also the method emphasizes the fact that everything we want to know about $W_N^{(l)}$ and the $sl(N)_l$ hierarchy can be extracted systematically from $W_N$ and the usual $sl(N)$ hierarchy. Hence, although there is no direct relation between the various $W_N^{(l)}$ algebras for $N$ fixed, the method unravels one such hidden relationship: all these algebras have the same Miura transformation, which is however written differently for different values of $l$.

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