In this article, we consider an ad-hoc deformation of the EPRL model for quantum gravity by a cosmological constant term. This sort of deformation has been first introduced by Han for the case of the 4-simplex. In this article, we generalise the deformation to the case of arbitrary vertices, and compute its large-$j$-asymptotics. We show that, if the boundary data corresponds to a 4d polyhedron $P$, then the asymptotic formula gives the usual Regge action plus a cosmological constant term. We pay particular attention to the determinant of the Hessian matrix, and show that it can be related to the one of the undeformed vertex.

I. MOTIVATION

Spin Foam models are tentative proposals for a path integral formulation of quantum gravity. They are a very active research subject, and have many connection points with state sum models, tensor field theories, and loop quantum gravity ([1] and references therein).

One of the most widely studied model ist the one by Engle, Pereira, Rovelli and Livine [2]. It provides the definition of a so-called vertex amplitude $A_v$, which assigns a transition amplitude to a spin network state, which is interpreted as 3d boundary geometry. The boundary is that of a small piece of 4d space-time (a “vertex”), while the whole path integral is defined by glueing many of these vertices to one large 2-complex (e.g. described in [7–9]).

Each of the local boundary spin network states is defined on a graph $\Gamma$. The spin network states on $\Gamma$ form a Hilbert space $\mathcal{H}_\Gamma$, and the amplitude can be regarded as linear form on this space. The original EPRL amplitude was defined on a complete graph $K_5$, corresponding to the boundary of a 4-simplex. The model has been generalized to arbitrary graphs [10], although it can be argued that the model would have to be amended to include the correct implementation of the volume simplicity constraints [11, 12].

There is an asymptotic regime of the amplitude, in which one can show it to be connected to the exponential of the Regge action, i.e. a discrete analogue of the Einstein Hilbert action for general relativity [13]. This, among others, is an indication of the connection between the EPRL model and the path integral for quantum gravity.

There are several versions of deformation of this model to include a non-zero cosmological constant. The most technically clean one is a deformation of the underlying group $SU(2)$ to a quantum group $SU(2)_q$, with $q = e^{2\pi i k}$ a root of unity, where $\Lambda = 6\pi/(\ell_P^2 k)$. [14–17]. One of the earliest deformations of the model, however, was still on the level of classical groups, by deforming $A_v$, keeping $\mathcal{H}_\Gamma$ unchanged. The definition was given by Han [15], for the case of a 4-simplex, and a partial analysis of the asymptotic regime was given, which demonstrated the emergence of the Regge action plus a cosmological constant term.

While this ad-hoc deformation of the EPRL model shows no obvious connection to the later definitions with quantum groups, it is a useful tool for calculations. In particular, in recent calculations concerning the RG flow of the EPRL model (see [18–20]), it turned out to be desirable to have a running cosmological constant. The boundary graphs are more complicated in that case, so a generalisation of Han’s deformation to more complicated graphs is needed. This is what is going to be undertaken in this article.

The plan of this article is as follows: First, we will remind the reader of the definition of the undeformed EPRL amplitude in section II. We will then give a general definition of the deformed amplitude in section III. This will be a straightforward generalisation of Han’s formula. The main part of the article will be in the following section IV where we will consider the asymptotic analysis of this amplitude in the general case. In particular, we will prove the (highly non-trivial) statement, that the asymptotic of the deformed amplitude coincides with the undeformed one, apart from a cosmological constant term. This requires a very careful handling of the determinant of the Hessian matrix in the stationary phase approximation, and we will divert some of the (rather technical) details to the appendix B.
II. THE UNDEFORMED MODEL

We consider a general spin foam vertex, for the Riemannian signature EPRL-FK model, for Barbero-Immirizí parameter $\gamma \in (0, 1)$. The amplitude is a linear map on the boundary Hilbert space. A state in that Hilbert space is given by boundary data, which is completely described by a directed graph $\Gamma \subset S^3$ embedded into a three-sphere. For example, for a 4-simplex the graph is given by the complete graph with five nodes, with the knotting as in figure 1. The boundary graph of a 4-simplex.

A boundary geometry on $\Gamma$ is given by a collection of spins $j_L \in \frac{1}{2}N$ associated to the links $L \in \text{Links}(\Gamma)$ of $\Gamma$, and a collection of 3d unit vectors $n_{NL}$ associated to pairs of nodes $N \in \text{Nodes}(\Gamma)$ of the graph, and links $L$ which are connected to $N$. For all $L \supset N$, the corresponding unit vectors are chosen such that they satisfy

$$G_N := \sum_{L \supset N} j_L n_{NL} - \sum_{L \to N} j_L n_{NL} = 0. \quad (1)$$

Here, the notation $L \to N$ denotes all links which are ingoing to the node $N$, while $L \leftarrow N$ denotes the outgoing links.

For Riemannian signature, the local gauge group is $\text{Spin}(4)$. We use the Hodge duality in four dimensions, under which its Lie algebra decomposes into $\text{spin}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, two commuting $SU(2)$-subalgebras, which are the eigenspaces under the Hodge $*$ for eigenvalues $\pm 1$. Consequently, one has the group isomorphism $\text{Spin}(4) \simeq SU(2) \times SU(2)$, and an irreducible representation of $\text{Spin}(4)$ can therefore be depicted as pair $(j^+, j^-)$ of half-integers.

The undeformed vertex amplitude $A_\Gamma$ is constructed in the following way: Define

$$j_L^\pm := \frac{1 \pm \gamma}{2} j_L. \quad (2)$$

For a unit vector $n \in S^2$, define the coherent states

$$|j, n\rangle := D_j(g_n)|j, j\rangle, \quad (3)$$

i.e. the action on the highest weight vector with a group element $g_n$, which is such that $g_n e_z = n$, with $e_z$ being the unit vector in $z$-direction.

Define the boosting map

$$\beta_L : V_{j_L} \rightarrow V_{j_L^+} \otimes V_{j_L^-} \quad (4)$$

is the isometric embedding of $j_L$ into the corresponding subspace of the Clebsch-Gordon decomposition of

$$V_{j_L^+} \otimes V_{j_L^-} \simeq V_{j_L} \simeq V_{j_L} \uparrow \oplus \cdots V_{j_L+j_L}.$$ \quad (5)

Also, denote by $\mathcal{P} : \mathcal{H} \rightarrow \text{Inv}_{SU(2) \times SU(2)}(\mathcal{H})$ the operator

$$\mathcal{H} := \left( \bigotimes_{L \leftarrow N} V_{j_L^+} \otimes V_{j_L^-} \right) \otimes \left( \bigotimes_{L \to N} V_{j_L^+} \otimes V_{j_L^-} \right), \quad (6)$$

which is the projector onto the invariant subspace of the Hilbert space $\mathcal{H}$.

With this, one defines the boosted Livine-Speziale-intertwiners

$$\iota^+_N := \mathcal{P} \left[ \bigotimes_{L \leftarrow N} \beta_L |j_L, n_{NL}\rangle \otimes \bigotimes_{L \to N} |j_L, n_{NL}\rangle \beta_L^\dagger \right]. \quad (7)$$

As a result of this definition, the tensor product of all boosted Livine-Speziale intertwiners is an endomorphism on the tensor product of all representation spaces over the links, i.e.

$$\bigotimes_N \iota^+_N : \bigotimes_L \left( V_{j_L^+} \otimes V_{j_L^-} \right) \longrightarrow \bigotimes_L \left( V_{j_L^+} \otimes V_{j_L^-} \right), \quad (8)$$

where the $i^\pm = (i^+, i^-)$ factorise due to $\gamma < 1$. The vertex amplitude $A_v$ is defined as the trace of this map, i.e.

$$A_v := \text{tr} \left( \bigotimes_N \iota^+_N \right) = A^+_v A^-_v. \quad (9)$$

---

2 In general, it is suspected that the knotting of the graph is important for the formula of the vertex amplitude, as soon as quantum groups are involved. As it turns out, however, Han’s heuristic deformation does not seem to depend on the precise knotting class of the graph $\Gamma$.

3 With the definition, one has to demand that all three $j_L, j_L^\pm$ are half-integers, which puts severe restrictions on the Barbero-Immirizí parameter $\gamma$. This is a pathology of the Riemannian model, which does not occur in the Lorentzian context.

4 Given $n \in S^2$, the corresponding $g_n$ is only defined up to a $U(1) \subset SU(2)$-subgroup. Different choices amount to states $|j, n\rangle$ which differ by a complex phase. For one vertex amplitude, this phase is not important, while for larger triangulations, the relative phases of these states in neighbouring vertices have to be taken care of, since they encode the 4d curvature.
III. DEFORMATION OF THE MODEL

We now deform the model with a cosmological constant term. The state-of-the-art method to do this is to resort to replacing the group $SU(2)$, which features prominently in the construction of the EPRL-FK model, by its quantum group counterpart $SU_q(2)$, with $q = e^{i \pi / 2}$ a root of unity, where $\Lambda = 6\pi / (\ell_p^2 k)$.

There is a heuristic alternative to this, which relies on a deformation of the EPRL-FK model, which stays purely on the classical level. This was proposed by Han in the case of a 4-simplex [15]. Here we generalise Han’s result to arbitrary vertices, and perform the large-$j$-asymptotics, including the treatment of the Hessian matrix.

Given the definition of the vertex amplitude $A_\Gamma$, the deformation is given in terms of a parameter $\omega \in \mathbb{R}$. It is constructed as follows: The graph $\Gamma$ needs to be projected down to the $2d$ plane, where it can be depicted with crossings (see figure 3 for the example of a hypercubic graph). For each crossing $C$ in the graph between two links $L, L’$ with spins $k_L, k_{L’}$, define the crossing operator

$$R_C := e^{i \omega \sigma(C)} V_C$$

were $\omega \in \mathbb{R}$ is the deformation parameter, $\sigma(C) = \pm 1$ is the type of crossing (ove- or under-crossing, see figure 2), and with

$$V_C := \sum_{\epsilon = \pm} \frac{\epsilon^4}{(1 + \gamma)^2} \sum_{I=1}^{3} D(\epsilon_L)(X_I^+) \otimes D(\epsilon_{L’})(X_I^-),$$

where the $X_I^+$ ($X_I^-$) are an orthonormal basis of the self-dual (anti-self-dual) $\mathfrak{su}(2)$. The operator $R_C$ acts as endomorphism on $V_{(j_L^+, j_L^-)} \otimes V_{(j_{L’}^+, j_{L’}^-)}$ [9]. By tensoring $\otimes_C R_C$ with the identity operator for all links in $\Gamma$ which do not appear in a crossing, we obtain an endomorphism on $H^\Gamma$. The deformed vertex amplitude $A^\omega_\Gamma$ is then defined as

$$A^\omega_\Gamma := \text{tr} \left( \otimes_N \left( 1 \otimes \otimes_C R_C \right) \right).$$

Note that while $R_C$ depends on the choice of orthonormal basis, the amplitude $A^\omega_\Gamma$ does not, due to the gauge-invariance of each boosted Livine-Speziale intertwiner.

IV. LARGE-$j$ ASYMPTOTICS OF THE DEFORMED AMPLITUDE

In the case of $\gamma \in (0,1)$, the undeformed amplitude $A_\Gamma = A^+_\Gamma A^-_\Gamma$ factorises over the two sectors (self-dual and anti-self-dual). Since the respective generators $[X_I^+, X_J^-] = 0$ commute, so do the $R_C = R^+_C R^-_C$, of course. Hence, also the deformed amplitude factorises:

$$A^\omega_\Gamma = A^\omega^+_\Gamma A^\omega^-_\Gamma.$$ (13)

First we note that, due to the factorisation property, it is enough to look at only the $+$-part. To simplify notation, in what follows we abbreviate $j_L^+ \rightarrow j$, $j_{L’}^+ \rightarrow j’$, $D_{j_L^+}(X_I^+) \rightarrow X_I$, $g_a^+ \rightarrow g_a$, etc.

In particular, we have that the (undeformed) $+$-amplitude is given by

$$A^+_\Gamma = \int_{SU(2)^N} dg_a \prod_{b \rightarrow a} \langle j_{ab}, n_{ab} | (g_a)^{-1} g_b | j_{ab}, n_{ba} \rangle$$

(14)

where the product ranges over all links, where in the formula $b$ is the starting point (source) of the link, and $a$ is the end point (target).

Now assume that there is a crossing between the link $b \rightarrow a$ and $b’ \rightarrow a’$. Then, in the deformed amplitude, the two corresponding factors in the product (14) are replaced by

$$\langle \Psi | \exp \left( \frac{4 i \omega \sigma(C)}{(1 + \gamma)^2} \sum_{I=1}^{3} X_I \otimes X_I \right) | \Phi \rangle$$

(15)

with

$$\langle \Psi | = \langle j_{ab}, n_{ab} | (g_a)^{-1} \otimes (j_{a’b’}, n_{a’b’}) | (g_{a’})^{-1}$$

$$| \Phi \rangle = g_b | j_{ab}, n_{ba} \rangle \otimes g_b’ | j_{a’b’}, n_{b’a’} \rangle$$

The expression (15) can be expanded to

---

5 Technically, by this definition, the graph has to be such that
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{4i\omega}{(1+\gamma)^2} \right)^n \prod_{I_1, I_2, \ldots, I_n=1} \langle j_{ab}, n_{ab} | (g_a)^{-1} X_{I_1} X_{I_2} \cdots X_{I_n} g_b | j_{ab}, n_{ba} \rangle \]

To consider the stationary phase of an individual term, we use the resolution of identity
\[ (2j + 1) \int_{S^2} d^2 n \langle j, n | j, n \rangle = 1 \nu_j \]
\[ n - 1 \text{ times, and write} \]
\[ \langle j_{ab}, n_{ab} | (g_a)^{-1} X_{I_1} X_{I_2} \cdots X_{I_n} g_b | j_{ab}, n_{ba} \rangle \]
\[ = (2j + 1)^{n-1} \int_{(S^2)^{n-1}} d^2 n_i \langle j_{ab} | (g_a)^{-1} X_{I_i} | j, n_i \rangle \]
\[ \times \langle j, n_1 | X_{I_2} | j, n_2 \rangle \cdots \langle j, n_{n-1} | X_{I_{n-1}} g_b | j, n_{ba} \rangle \]

The \( X_I \) are the generators of the \( \mathfrak{su}(2) \) Lie algebra \([X_I, X_J] = i\epsilon_{IJK} X_K\), which is why, in the spin-1/2 representation, we have \( X_I = \sigma_I / 2 \) in terms of the Pauli matrices \( \sigma_I \). We have therefore
\[ \langle j, n | X_I | j, n' \rangle = j \langle n | \sigma_I | n' \rangle \langle n' | n \rangle^{2j-1} \]
\[ \text{where} \ | n \rangle := | 1/2, n \rangle. \]

With this, we can write
\[ \langle j_{ab}, n_{ab} | (g_a)^{-1} X_{I_1} X_{I_2} \cdots X_{I_n} g_b | j_{ab}, n_{ba} \rangle \]
\[ = \int_{(S^2)^{n-1}} d^2 n_i a(n_i, g_a, g_b) e^{S(n_i, g_a, g_b)} \]
\[ \text{with} \]
\[ a(n_i, g_a, g_b) = (2j + 1)^{n-1} |n_{ab}| (g_a)^{-1} \sigma_I \langle n_1 | n_2 | \cdots | n_{n-1} | g_b | n_{ba} \rangle \]
\[ S(n_i, g_a, g_b) = 2j \left( \ln |n_{ab}| (g_a)^{-1} n_1 \right) + \ln n_1 | n_2 | \cdots + \ln n_{n-1} | g_b | n_{ba} \right) \]

This is now in a form where one can perform the (extended) stationary phase approximation. Note that this is for one term in the sum \[16\] only, and the variables are all the \( g_a \) and, for every crossing, \( n_i \) and \( n'_i \) with \( i = 1, \ldots, n - 1 \) (the \( n'_i \) vectors come from the term similar to \[19\]), with the dashed nodes \( a', b' \). First, we note that the criticality condition \( \text{Re} S = 0 \) (where we consider the whole action for \( A_\gamma \) now), is equivalent to
\[ g_a n_{ab} = g_b n_{ba} \]
\[ \text{and} \]
\[ n_i = g_b n_{ba}, \quad n'_i = g_b n'_{ba} \quad \text{for all } i. \]

One should note that the criticality equations \[21\] for the group elements \( g_a \) are precisely the ones for the deformed amplitude. The criticality equations for the unit vectors \( n_i, n'_i \) (remember that, per crossing, there are \( 2(n-1) \) unit vectors), are such that, on each edge which participates in some crossing, all vectors have to be equal, and coincide with the two normal vectors \( g_a n_{ab} = g_b n_{ba} \).

This shows that, using the same gauge symmetry as in the undeformed case, setting one \( g_a = \mathbb{1} \), all critical points are isolated, when they are also isolated in the undeformed case \[7\].

The stationary points are equally easily identified, and they are, as in the undeformed case, the closure condition for each node, and \( n_i = g_b n_{ba}, n'_i = g_b n'_{ba} \) for all \( i \).

In particular, this means that, after gauge-fixing, the critical and stationary points of the deformed and the undeformed amplitude are in one-to-one correspondence. Furthermore, it is easy to see that the value of the respective actions, evaluated at corresponding critical stationary points, coincide.

\[7\] This is, indeed, the generic case, e.g. in the case of the \( n_{ab} \) forming a Regge boundary geometry at the 4-simplex \[12\], in all cases of the hypercuboid \[21\], or the hyperfrustum, if \( \alpha \in (\pi/4, 3\pi/4) \). \[19\]
A. The Hessian matrix

To make the notation easier, we assume that there is only one crossing. The general case with many crossings can be treated similarly, though. We also assume that there is at least one critical, stationary point \( g_a^{(c)} \) for the undeformed (gauge-fixed) amplitude. Before we continue, we perform a coordinate transformation on the \( g_a, n_i, n_i' \) variables, via

\[
g_a \to g_a, \quad n_i \to g_b n_i, \quad n_i' \to g_b n_i'.
\]

(23)

Since \( SU(2) \) acts via rotations on \( S^2 \), the Jacobi matrix for this transformations is equal to unity. The action after the coordinate transformation is then given by

\[
S(g_a, n_i, n_i') = \sum_{cd \neq ab, a'b'} 2j_{cd} \ln(n_{cd}) (g_c)^{-1} g_d n_{dc}
\]

+ \( 2j_{ab} \left( \ln(n_{ab}) (g_a)^{-1} g_b n_1 + \ln(n_1 n_2) + \cdots + \ln(n_{n-1} n_{ba}) \right) \)

+ \( 2j_{a'b'} \left( \ln(n_{a'b'}) (g_{a'})^{-1} g_b n'_1 + \ln(n'_1 n'_2) + \cdots + \ln(n'_{n-1} n_{b'a'}) \right) \).

(24)

Note that the first two lines in (24) are the same as in the undeformed case, while the remaining two come from the deformation due to the crossing. We compute the Hessian matrix for the deformed amplitude, at the critical stationary point

\[
g_a = g_a^{(c)}, \quad n_i = n_{ba}, \quad n_i' = n_{b'a'}.
\]

(25)

In particular, we introduce coordinates around this point via \( g_a = e^{i\pi d} g_a^{(c)} \) and

\[
n_i = g_{na} \begin{pmatrix} \sin \phi_i & \cos \phi_i \sin \chi_i \\ \cos \phi_i & \cos \chi_i \end{pmatrix}, \quad n_i' = g_{na'} \begin{pmatrix} \sin \xi_i & \cos \xi_i \cos \chi_i \\ \cos \xi_i & \cos \xi_i \cos \chi_i \end{pmatrix},
\]

(26)

where the angles take values in \( \phi_i, \xi_i \in (-\pi, \pi) \) and \( \theta_i, \chi_i \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). The critical and stationary point is assumed at \( x_i^c = 0, \phi_i = \xi_i = \theta_i = \chi_i = 0 \). The vectors \( |n_i\rangle \) are then given by

\[
|n_i\rangle = g_{na} \exp \left( i \frac{\phi_i}{2} \sigma_1 \right) \exp \left( -i \frac{\theta_i}{2} \sigma_2 \right) |e_z\rangle
\]

\[
= g_{na} \left[ \left( \cos \frac{\phi_i}{2} \cos \frac{\theta_i}{2} + i \sin \frac{\phi_i}{2} \sin \frac{\theta_i}{2} \right) |\uparrow\rangle \\
+ \left( \cos \frac{\phi_i}{2} \sin \frac{\theta_i}{2} + i \sin \frac{\phi_i}{2} \cos \frac{\theta_i}{2} \right) |\downarrow\rangle \right],
\]

where \( |e_z\rangle = |\uparrow\rangle \) is the highest weight vector in the spin \( \frac{1}{2} \)-representation. A similar formula holds for \( |n_i'\rangle \). This leads to

\[
\frac{\partial}{\partial \phi_i} \langle n_i | n_{i+1} \rangle_{\text{crit, stat}} = 0,
\]

(27)

and similar relations for the other angles. Also,

\[
\langle n_i | n_{i+1} \rangle_{\text{crit, stat}} = 1.
\]

(28)

Therefore, for all second derivatives which have at least one derivative w.r.t. one of the angles, the ln can be left out, e.g.:

\[
\frac{\partial^2}{\partial \phi_i \partial \phi_{i+1}} \ln \langle n_i | n_{i+1} \rangle_{\text{crit, stat}} = \frac{\partial^2}{\partial \phi_i \partial \phi_{i+1}} (n_i | n_{i+1} \rangle_{\text{crit, stat}},
\]

and similar relations for all other varying types of angles. Thus we get, at the stationary and critical points:

\[
\frac{\partial^2 S}{\partial \phi_i^2} = \frac{\partial^2 S}{\partial \phi_i^2} = -j_{ab},
\]

(29)

\[
\frac{\partial^2 S}{\partial \phi_i \partial \phi_{i+1}} = \frac{\partial^2 S}{\partial \phi_i \partial \phi_{i+1}} = -j_{a'b'},
\]

Also, we get

\[
\frac{\partial^2 S}{\partial \theta_i \partial \theta_{i+1}} = \frac{\partial^2 S}{\partial \theta_i \partial \theta_{i+1}} = \frac{j_{ab}}{2}, \quad \frac{\partial^2 S}{\partial \phi_i \partial \theta_i} = 0,
\]

(30)

All other mixed \( \phi, \theta \) angle derivatives are zero. For \( \xi, \chi \) angles similar relations hold. Furthermore, we have

\[
\langle n_{ab} (g_a)^{-1} g_b | n_1 \rangle_{\text{crit, stat}} = e^{i\psi}.
\]

(31)

Using this and \( g_a g_{na} = g_b g_{na} e^{-i\psi} \), we get on the critical and stationary point that
\[
\frac{\partial^2 S}{\partial x_i^d \partial \phi_1} = 2j_{ab} \frac{\partial^2}{\partial x_i^d \partial \phi_1} \ln(n_{ab}((g_a)_{-1} e^{i \sigma_j} g_b g_{n_{ab}} e^{i \phi_1} \sigma_1 | e^{-i \theta_i} \sigma_2 | ))
\]

\[
= 2j_{ab} e^{-i \psi} \frac{\partial^2}{\partial x_i^d \partial \phi_1} (n_{ab}((g_a)_{-1} e^{i \sigma_j} g_b g_{n_{ab}} e^{i \phi_1} \sigma_1 | e^{-i \theta_i} \sigma_2 | ))
\]

\[
= j_{ab} \left( IV_2 - V_1 I \right),
\]

where in the end we have taken all angles \( \phi_i = \theta_i = 0 \).

Also, \( V_I \) is the I-th component of the image of the J-th unit vector under the rotation \( G := (g_b g_{n_{ab}})^{-1} \), i.e.

\[
G \sigma J G^{-1} = V_I \sigma_I.
\]

Furthermore, we have

\[
\frac{\partial^2 S}{\partial x_i^d \partial \phi_1} = - \frac{\partial^2 S}{\partial x_i^j \partial \phi_1}
\]

and

\[
\frac{\partial^2 S}{\partial x_i^d \partial \theta_1} = j_{ab} \left( IV_1 + V_2 I \right)
\]

\[
= - \frac{\partial^2 S}{\partial x_i^j \partial \theta_1} = \frac{1}{i} \frac{\partial^2 S}{\partial x_i^j \partial \phi_1}
\]

Also, there are, again, equivalent relations for the \( \xi_1 \) and \( \chi_1 \) angles, where \( a \rightarrow a', b \rightarrow b' \). Finally, it is not hard to see that the matrix of second derivatives of \( x_i^d \)

\[
\tilde{H}_{ij} := \frac{\partial^2 S}{\partial x_i^d \partial x_j^d}
\]

at the critical and stationary point coincides precisely with the matrix in the undeformed case - even if (cd) = (ab) or (a'b'). The determinants of the Hessian matrix \( H \) of the whole integral evaluates to

\[
\det(H) = (j_{ab} j_{a'b'})^{2(n-1)} \det(\tilde{H}).
\]

This is shown in appendix B.

From the analysis, it is clear that that the case of more than one crossing can simply be computed by an induction over the number of crossings \( C \), and reduced to

\[
\det(H) = \det(\tilde{H}) \prod_C (j_{ab} j_{a'b'})^{2(n-1)}.
\]

**B. Putting everything together**

We now replace \( j_{cd} \rightarrow \lambda j_{cd} \), and consider the asymptotic expression for \( \lambda \rightarrow \infty \). Using the normalized measure on \( S^2 \) in \( \phi, \theta \)-coordinates \[26\], we get

\[
dn = \frac{1}{4\pi} d\phi_i d\theta_i \cos \theta_i, \quad dn' = \frac{1}{4\pi} d\xi_i d\chi_i \cos \chi_i
\]

Denote by \( B \) the large-\( j \)-expression for the undeformed \(+\)-amplitude \[14\], and by \( B' \) its deformation. Then, because the critical and stationary points are in one-to-one correspondence, and the Hessian matrix \( \det(\tilde{H}) \) for the undeformed case can be pulled out of the sum, we have \( B' = BC \) with

\[
C = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{4i \omega \sigma(C)}{(1 + \gamma)^2} \right)^n \left( \frac{1}{(4\pi)^2} \left( \frac{2\pi}{\lambda} \right)^{2(n-1)} \right) \times \prod_{i=1}^{n} \left( \lambda n_{ba} \right)^{2(n-1)} \left( \lambda n_{b'a'} \right)^{2(n-1)}
\]

\[
= \prod_{n=0}^{\infty} \left( \lambda n_{ba} \right)^{2(n-1)} \left( \lambda n_{b'a'} \right)^{2(n-1)} \left( \lambda n_{b'a'} \right)^{2(n-1)}
\]

\[
e^{i \omega \lambda^2 \sigma(C) X_{ab} Y_{a'b'}}
\]

with the vectors \( n_{ab} = g_a n_{ab} \), and

\[
X_{ab} = k_{ab} n_{ab}, \quad Y_{a'b'} = k_{a'b'} n_{a'b'}
\]

\[
\text{with } 1 + \gamma k_{cd} = j_{cd} \text{ and } j_{cd} \text{ scaling by } \lambda. \text{ This stays finite if, additionally,}\]

\[
\text{to scaling } j_{cd} \text{ up by } \lambda, \text{ scaling the deformation parameter as } \omega \rightarrow \omega \lambda^{-2} \text{ at the same time.}
\]

This is the computation of the \(+\)-part, i.e. \( C^+ \). It is noteworthy that \( C^- \) is the same expression, just with a minus sign in the exponential, i.e. \( C^- = (C^+)^{-1} \).

Expression \[40\] is for one crossing. The case of many crossings is straightforward, however, since we demanded that each edge is part of at most one crossing. For many crossings, one gets

\[
C = e^{i \omega \sum C \sigma(C) X_{ab} Y_{a'b'}}
\]

**C. Relation to the cosmological constant**

We now relate our final result \[40\] to the cosmological constant. For this, we assume an amplitude in which there are two distinct solutions to the stationary phase equations \[21\] \[8\]. We denote these as \( g_a^{(i)} \), with \( i = 1, 2 \).

\[8\] This seems to be the case whenever the boundary data allows for a unique, non-degenerate 4-geometry \[19\] \[21\] \[22\].
FIG. 3. The boundary graph of a hypercuboid (or a hyperfrustum). This graph has eight nodes, 24 links, and six crossings. The lines going to infinity all meet at node 7.

We denote the asymptotic expression for the undeformed amplitude by

$$\mathcal{A}_\Gamma^\pm \rightarrow B_+^{(1)} + B_+^{(2)},$$

and from this and (40) one gets that

$$\mathcal{A}_\Gamma^\pm \rightarrow (B_+^{(1)} C_+^{(1)} + B_+^{(2)} C_+^{(2)})(B_-^{(1)} C_-^{(1)} + B_-^{(2)} C_-^{(2)})$$

$$= B_+^{(1)} B_-^{(1)} + B_+^{(2)} B_-^{(2)} + (B_+^{(1)} B_-^{(2)} C_+^{(1)} C_-^{(2)} + B_+^{(2)} B_-^{(1)} C_+^{(2)} C_-^{(1)}).$$

The terms $B_+^{(1)} B_-^{(2)}$ and $B_+^{(2)} B_-^{(1)}$ evaluated on the same solution, have been called "weird terms", and one can see that they remain unchanged under the deformation of the model. The mixed terms however do get changed, and one has

$$C_+^{(1)} C_-^{(2)} = (C_+^{(2)} C_-^{(1)})^{-1}$$

$$= \exp \left( i \omega \sum_C \sigma(C) \left( X_{ab}^{(1)} \cdot Y_{a'b'}^{(1)} - X_{ab}^{(2)} \cdot Y_{a'b'}^{(2)} \right) \right)$$

$$= \exp \left( 12 i \omega \sum_C \sigma(C) \ast (B_{ab} \wedge B_{a'b'}) \right).$$

Here $\ast$ denotes the Hodge dual, $B_{ab} = (X_{ab}^{(1)}, X_{ab}^{(2)})$ and $B_{a'b'} = (Y_{a'b'}^{(1)}, Y_{a'b'}^{(2)})$ are the bivectors in $\mathbb{R}^4 \wedge \mathbb{R}^4 \simeq \mathfrak{so}(4) \simeq \mathbb{R}^3 \oplus \mathbb{R}^3$ associated to the edges $(ab)$ and $(a'b')$, which are constructed from the two distinct solutions $g_a^{(i)}$. See appendix A for details.

In the case of a 4-simplex, the expression in (45) has been shown to be proportional to the 4-volume of such a simplex, given by the boundary data [15]. For the case of the hyperfrustum, which has a boundary graph depicted in figure 3, the critical and stationary equations have been solved in [19], and the solution can be shown, with the notation from that article, to be

$$\sum_C \sigma(C) \ast (B_{ab} \wedge B_{a'b'})$$

$$= \frac{k(j_1 + j_2)}{2} \sqrt{1 - \frac{(j_1 - j_2)^2}{8k^2}} = V_{\text{frustum}},$$

where $V_{\text{frustum}}$ is the 4-volume of the hyperfrustum. In the case of a hypercuboid, a similar calculation can be carried out. With the notation from [12] and the conventions in appendix A one finds

$$\sum_C \sigma(C) \ast (B_{ab} \wedge B_{a'b'}) = \frac{j_1 j_6 + j_2 j_5 + j_3 j_4}{3},$$

which coincides with $V_{\text{hypercuboid}}$ if the geometricity conditions $j_1 j_6 = j_2 j_5 = j_3 j_4$ are satisfied. See [12] for a closer discussion of this point, and the relation to the volume simplicity constraints within the EPRL model.

One can show, indeed, that for convex 4-dimensional polyhedra $P$ one has in general that

$$V_P = \sum_C \sigma(C) \ast (B_{ab} \wedge B_{a'b'}).$$

A proof will be presented in another publication.

V. SUMMARY AND CONCLUSION

In this article, we have discussed a generalisation of Han’s deformation of the Riemannian signature EPRL model for Barbero-Immirzi-parameter $\gamma \in (0, 1)$, to arbitrary vertices. It amounted to introducing an operator depending on a deformation parameter $\omega$, and we have considered the definition for arbitrary graphs as well as the corresponding asymptotic expression of the deformed amplitude $\mathcal{A}_\Gamma^\omega$. This deformation works by introducing an operator for each crossing $C$ of the graph $\Gamma$ in the formula for the amplitude.

The main statement is that the deformed amplitude $\mathcal{A}_\Gamma^\omega$ has a close connection to $\mathcal{A}_\Gamma$, the undeformed one. Firstly, the equations for the stationary critical points in the asymptotic analysis are in one-to-one correspondence. Also, we could show that the Hessian determinant can be treated, and is just a multiple of the undeformed one. This led to an expression of the asymptotic expression in terms of the original Regge action. In particular, the original expression consists of the so-called weird terms, as well as the cosine of the Regge action. Our analysis shows that the weird terms remain unchanged, while the Regge action is replaced by a term $\Lambda$, where $\Lambda = -12 \omega$, and $V$ is an expression which, if the boundary data is that of a convex, non-degenerate polyhedron, is equal to its volume.

$$\mathcal{A}_\Gamma^\omega \rightarrow W + W^\ast + \frac{2}{|D|} \cos (S_{\text{Regge}} - \Lambda V).$$

(48)
This way, the deformation provides, in a straightforward way, a generalization of the EPRL-KKL model to include a non-zero cosmological constant \( \Lambda \).

There are two points of note in this analysis:

1. There are cases in which the boundary data does not describe a vector geometry (in that there are two critical and stationary points), while not describing a 4d polyhedron. These “non-geometric” configurations have been discussed in [21, 22], and their presence can be attributed to the insufficient implementation of the volume simplicity constraint. The expression \( V \), however, still exists and is non-zero. It is unclear what its geometric interpretation is in that case.

2. The original EPRL-KKL amplitude \( \mathcal{A}_F \) is defined on a graph \( \Gamma \), but does not depend on its knotting class. As a consequence, the physical inner product therefore also does not [23]. Interestingly, the deformation \( \mathcal{A}'_F \), however, does depend on the knotting of \( \Gamma \). This is a property it shares with the quantum group deformations of the model. One can conjecture that this would lead to a physical Hilbert space in which graphs with different knotting classes are not equivalent. This could have interesting physical ramifications. [24]

It should also be noted that, while the expression \( V \) [12] is a knotting invariant in the asymptotic limit, i.e. does not depend on the way in which the graph \( \Gamma \) is presented on the plane [12], it is unknown whether the same is true for the quantum amplitude. We will return to this point in a future article.

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**Appendix A: Bivector conventions**

A bivector \( B_{ab} = -B_{ba} \in \Lambda^2 \mathbb{R}^4 \), \( a, b = 0, 1, 2, 3 \), can be dualized via the Hodge operator

\[
(*B)_{ab} := \frac{1}{2} \epsilon_{abcd} B_{cd},
\]

where indices are raised and lowered with the Kronecker delta \( \delta_{ab} \). The Killing form on \( \Lambda^2 \mathbb{R}^4 \) is taken to be positive definite as

\[
(B_1, B_2) := -\frac{1}{4} \text{tr}(B_1 B_2).
\]

The isomorphism \( \Lambda^2 \mathbb{R}^4 \simeq \mathbb{R}^3 \oplus \mathbb{R}^3 \)

\[
B \leftrightarrow (\vec{b}^+, \vec{b}^-)
\]

is given by

\[
b_{\pm, i} = \frac{1}{2} \left( B_{0i} \pm \frac{1}{2} \epsilon_{ijk} B_{jk} \right)
\]

with \( I = 1, 2, 3 \). The wedge product of two bivectors \( B \) and \( C \) is defined to be

\[
(B \wedge C)_{abcd} = \frac{1}{24} \epsilon_{abcd} \epsilon^{efgh} B_{ef} C_{gh}.
\]

Acting with the Hodge dual on this yields a number which is

\[
* (B \wedge C) = \frac{1}{4} \epsilon^{efgh} B_{ef} C_{gh}
\]

\[
= \frac{1}{12} \left( \vec{b}^+ \cdot \vec{c}^+ - \vec{b}^- \cdot \vec{c}^- \right),
\]

and which can be regarded as the expression for the 4d volume in the volume simplicity constraint [2, 12].

**Appendix B: Determinant of the Hessian**

The Hessian matrix of a term at level \( n \) of the expansion [10] is rather involved, and needs to be treated with care. Its matrix elements are given by \( \epsilon_{abcd} \epsilon^{efgh} \). Interestingly, the indices \( a, b, a', b' \) are not free, but \( (ab) \) and \( (a'b') \) label the links in the graph which are crossing. If we need free indices from the beginning of the alphabet to indicate nodes, we’ll begin with \( c, d, \ldots \).

With this, we get that the final Hessian matrix is of the form

\[
H = \left( \begin{array}{cc}
A & B \\
B^T & C
\end{array} \right),
\]

where \( C \) is the same Hessian matrix as in the undeformed case, \( B \) is the matrix of mixed \( X_i^j \) and angle variables, and \( A \) is the quadratic matrix of two angle derivatives. We have

\[
\det(H) = \det(A) \det(C - B^T A^{-1} B).
\]

First, we consider the matrix \( 4(n-1) \times 4(n-1) \)-dimensional matrix \( A \). It is of the form

\[
A = \left( \begin{array}{cc}
D & 0 \\
0 & D'
\end{array} \right),
\]

where the order of the indices is as:

\[
\phi_1, \ldots, \phi_{n-1}, \theta_1, \ldots, \theta_{n-1}, \xi_1, \ldots, \xi_{n-1}, \chi_1, \ldots, \chi_{n-1}.
\]

\[
D = \frac{j_{ab}}{2} \left( \begin{array}{cc}
E & F \\
-F & E
\end{array} \right),
\]

\[
D' = \frac{j_{ab'}}{2} \left( \begin{array}{cc}
E & F \\
-F & E
\end{array} \right)
\]
with $E$ and $F$ being $(n-1)\times(n-1)$-dimensional matrices, with

\[
E_{rr} = -2, \quad r = 1, \ldots, n-1,
\]
\[
E_{r,r+1} = E_{r+1,r} = 1, \quad r = 1, \ldots, n-2,
\]
\[
F_{r,r+1} = i \quad r = 1, \ldots, n-2,
\]
\[
F_{r+1,r} = -i, \quad r = 1, \ldots, n-2,
\]
and all other entries being equal to zero. One readily computes

\[
\det(D) = j_{ab}^{2(n-1)}, \quad \det(D') = j_{a'b'}^{2(n-1)}, \quad (B5)
\]
as well as

\[
D^{-1} = \frac{1}{2j_{ab}} \begin{pmatrix} K & L \\ -L & K \end{pmatrix}, \quad (B6)
\]
\[
(D')^{-1} = \frac{1}{2j_{a'b'}} \begin{pmatrix} K & L \\ -L & K \end{pmatrix},
\]
with

\[
K_{rs} = -\delta_{rs} - 1, \quad L_{rs} = \begin{cases} \quad i \quad r < s \\ 0 \quad r = s \\ -i \quad r > s \end{cases} \quad (B7)
\]

with $r, s = 1, \ldots, n-1$. With this, we get

\[
\det(A) = (j_{ab}j_{a'b'})^{2(n-1)}. \quad (B8)
\]

Next we are turning our attention to the part $B^T A^{-1} B$. We note that the matrix $B$ is of dimension $4(n-1) \times 3N$, where $3N$ is the number of different values of the multi-index $(cI)$, i.e. $N$ is the number of nodes in the graph. Out of these $3N$ columns, only twelve contain (potentially) nonzero entries, namely $aI, bI, a'I, \text{ and } b'I$, with $I = 1, 2, 3$. These columns are (u runs from 1 to $4(n-1)$, in the order (B4) given above):

\[
B_{u,(aI)} = x^I \delta_{u,1} - i x^J \delta_{u,n}
\]
\[
B_{u,(bI)} = -x^I \delta_{u,1} + i x^J \delta_{u,n}
\]
\[
B_{u,(a'I)} = y^I \delta_{u,2n-1} - i y^J \delta_{u,3n-2}
\]
\[
B_{u,(b'I)} = -y^I \delta_{u,2n-1} + i y^J \delta_{u,3n-2},
\]

with $x^I = j_{ab}(iv^J_I - v^J_I)$ and $y^I = j_{a'b'}(i(V')^J_I - (V')^J_I)$. Denote

\[
M := B^T A^{-1} B, \quad (B9)
\]
then it is clear that $M$ is a $3N \times 3N$ matrix, which has zero entries until both row and column index are equal to one of the twelve combinations $(aI), \ldots (b'I)$ above. Now it is straightforward to show that also

\[
M_{(aI)(a'I)} = M_{(aI)(b'I)} = M_{(bI)(a'I)} = M_{(bI)(b'I)} = 0, \quad (B3)
\]

and similarly for other mixed combinations. This is clearly the case, since $A^{-1}$ is block-diagonal, as can be seen from (B3). The potentially nonzero entries are

\[
M_{(aI)(aJ)} = \sum_{u,v=1}^{2(n-1)} B_{u,(aI)} B_{v,(aJ)} (D^{-1})_{uv} \quad (B10)
\]
\[
= \frac{1}{2j_{ab}} x^I x^J K_{11} - 2ix^I x^J L_{11} + (ix^I) (ix^J) K_{11} \quad (B10)
\]
\[
= 0,
\]

as can be seen from (B6) and (B7). We also get

\[
M_{(aI)(bJ)} = \sum_{u,v=1}^{2(n-1)} B_{u,(aI)} B_{v,(bJ)} (D^{-1})_{uv} \quad (B10)
\]
\[
= \frac{1}{2j_{ab}} (-x^I x^J K_{11} + 2ix^I x^J F_{11} + (-ix^I) (ix^J) K_{11}) \quad (B10)
\]
\[
= 0.
\]

Similar relations hold for $M_{(a'I)(a'J)}$, etc., which lets us conclude that

\[
M = 0. \quad (B10)
\]

With (B2) and (B8), this immediately leads to

\[
\det(H) = (j_{ab}j_{a'b'})^{2(n-1)} \det(C), \quad (B11)
\]

where $C$ is the Hessien matrix of the undeformed case.

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