A simple approach to rigorous approximation of invariant measures

Stefano Galatolo, Isaia Nisoli

February 1, 2012

Abstract

We describe a general result on the approximation of fixed points of operators between normed vector spaces allowing an explicit estimation of the error. This result is particularly suited for the approximation of invariant measures in dynamical systems and in particular by the Ulam method.

We apply this result to implement an algorithm for the rigorous computation of invariant densities of piecewise expanding maps up to some error in the $L^1$ distance.

We show how several related computational and numerical issues can be solved and show some computer experiment calculating rigorous approximation of invariant measure and entropy of some one dimensional maps.

1 Introduction

Overview As it is well known, dynamical systems, are often very good models of natural phenomena. In many cases the presence of chaos is a problem for the simulation and the predictive understanding of the behavior of the system. On the other hand it is also known that the statistical behavior of the evolution of a system is quite stable and often from this point of view the evolution of the system obeys to some precise law which can be predicted and obtained from the description of the system.

Several important features of the statistical behavior of a given system are “encoded” in the so called Physical Invariant Measure. Having quantitative information on the invariant measure can give information on the statistical behavior for the long time evolution of the system. Physical invariant measures are the ones which (in some sense that will be precised below) represent the statistical behavior of a large set of initial conditions.

The problem of the existence and properties of such invariant measures has become a central area of research in the modern theory of Dynamical Systems.
A big part of the result is abstract and gives no quantitative precise information on the measure. This is of course a significant limitation in the applications.

What is said above strongly motivates the search for algorithms which are able to compute quantitative information on the physical measure. The problem of approximating invariant measure of dynamical systems was quite widely studied in the literature, some algorithm are proved to converge to the real invariant measure (up to errors in some given metrics) in some classes of systems, but results giving a rigorous bound on the error are relatively few.

In this paper we will describe an algorithm which is able to approximate interesting invariant measures with a precise bound on the error of the approximation and its practical implementation. The main theoretical ideas behind are quite general and in some sense simplify some previous approach ([18]). The practical implementation of the method and the necessary precise estimates are described here for the class of piecewise expanding maps. We also present some real computer experiment performing the rigorous computation on interval maps, and our solution to the nontrivial computational/numeric issues arising.

**Invariant measure and statistical properties** Let $X$ be a metric space, $T : X \rightarrow X$ a Borel measurable map and $\mu$ a $T$-invariant Borel probability measure. An invariant measure is a Borel probability measure $\mu$ on $X$ such that for each measurable set $A$ it holds $\mu(A) = \mu(T^{-1}(A))$. They represent *equilibrium* states, in the sense that probabilities of events do not change in time.

A set $A$ is called $T$-invariant if $T^{-1}(A) = A \ (mod \ 0)$. The system $(X, T, \mu)$ is said to be ergodic if each $T$-invariant set has total or null measure. In such systems the famous Birkhoff ergodic theorem says that time averages computed along $\mu$-typical orbits coincides with space average with respect to $\mu$. More precisely, for any $f \in L^1(X, \mu)$ it holds

$$\lim_{n \to \infty} \frac{S_n^f(x)}{n} = \int f \ d\mu,$$

for $\mu$ almost each $x$, where $S_n^f = f + f \circ T + \ldots + f \circ T^{n-1}$.

This shows that in an ergodic system, the statistical behavior of observables, under typical realizations of the system is given by the spatial average of the observable with respect to $\mu$.

We say that a point $x$ belongs to the basin of an invariant measure $\mu$ if (1) holds at $x$ for each bounded continuous $f$. In case $X$ is a manifold (possibly with boundary), a physical measure is an invariant measure whose basin has positive Lebesgue measure (for more details and a general survey see [25]).

**The transfer operator** Let us consider the space $SPM(X)$ of signed Borel probability measures on $X$. A function $T$ between metric spaces naturally induces a function $L : SPM(X) \rightarrow SPM(X)$ which is linear and is called transfer operator (associated to $T$). Let us define $L$: if $\mu \in SPM(X)$ then $L[\mu]$ is such that
\[ L[\mu](A) = \mu(T^{-1}(A)). \]

Measures which are invariant for \( T \) are fixed points of \( L \), hence the computation of invariant measures (and many other dynamical quantities) very often is done by computing the fixed points (or other spectral information) of this operator (restricted to a suitable Banach subspace). The most applied and studied strategy is to find a finite dimensional approximation for \( L \) (restricted to a suitable function space) reducing the problem to the computation of the corresponding relevant eigenvectors of a finite matrix.

An example of this is done by discretizing the space \( X \) by a partition and replacing the system by a (finite state) Markov Chain representing the dynamics between sets of the partition (see precise definition in section 3). Taking finer and finer partitions it is possible to obtain in some cases that the finite dimensional model will converge to the real one (and its natural invariant measure to the physical measure of the original system). In some case there is an estimation for this speed of convergence (see e.g. [9] for a discussion), but a rigorous bound on the error (and then a real rigorous computation) is known only in a few cases (piecewise expanding or expanding maps, see [3, 18]).

Another approach is to consider a perturbation of the system by a small noise. The resulting transfer operator has a kernel and then can be approximated by a finite dimensional one, by a kind of Faedo-Galerkin method and relevant eigenvectors are calculated (see e.g. [4, 5]).

Variations on the method of partitions are given in [6, 7], while in [23] a different method, fastly converging, based on periodic points is exploited for piecewise analytic Markov maps. In some interesting examples we can obtain the physical measure as limit of iterates of the Lebesgue measure \( \mu = \lim_{n \to \infty} L^n_T(m) \). To obtain rigorous bounds on the error the main point is to explicitly estimate the speed of convergence to the limit. This sometimes can be done using techniques related to decay of correlations ([12, 14]). General, abstract results on the computability of invariant measures are given in [13] (see also [11]). It is worth to remark that in these papers are shown also some negative result. Indeed, there are examples of computable systems without computable invariant measures.

**Plan of the paper**  In section 2 we show a general result regarding the approximation of fixed points for linear operators between Banach spaces. In this result fixed points are approximated extracting and exploiting as much information as possible from the approximated operator. This general result can be applied to the Ulam approximation method. In Section 3 we show how this can be done and we show an algorithm for the approximation of invariant measures. In particular we perform all the required estimations in the case of piecewise expanding maps (with bounded derivative).

In Section 4 we show how to implement the algorithm in practice. In particular we have to show a way to rapidly compute the steady state of a large

---

1 Computable, here means that the dynamics can be approximated at any accuracy by an algorithm, see e.g. [13] for precise definition.
Markov chain up to a prescribed error. We also discuss several other computational and programming issues, explaining how we have implemented the algorithm to perform real rigorous computations on some example of piecewise expanding maps.

In Section 6 we show the result of some experiments. Here we consider piecewise expanding maps, and the invariant measure is computed up to an error of less than 1% with respect to the $L_1$ distance. As an application we show a rigorous estimation of the entropy (by the Lyapunov exponent) of such maps. These estimations can be used as a benchmark for the validation of statistical methods to compute entropy from time series. We remark that for the experimental validation of these methods to understand how they converge fast to the real value of the entropy an exact estimate for the value is needed. Usually some simple system is considered, where the entropy value can be computed analytically. We give a method which can produce such estimation on nontrivial systems, where, due to the lack of a Markov structure, the convergence of statistical, symbolic methods is slower (see [15] e.g.).

Acknowledgements. The authors wish to thank C. Liverani for his encouragement and fundamental suggestions on the method we are going to describe. We also would like to thank S. Luzzatto and ICTP (Trieste) for support and encouragement.

The first author wishes also to thank “Gruppo Nazionale per l’Analisi Matematica, la Probabilitá e le loro Applicazioni” (GNAMPA, INDAM), for financial support.

The second author wishes to thank the CNPQ (Conselho Nacional de Pesquisas Científicas e Tecnológicas CNPq-Brazil), and IM-UFRJ (Instituto de Matemáticas of Universidade Federal do Rio do Janeiro).

The authors acknowledge the CINECA Award N. HP10C42W9Q, 2011 for the availability of high performance computing resources and support.

2 A general result on the approximation of fixed points

Let us consider a restriction of the transfer operator to an invariant normed subspace (often a Banach space of regular measures) $B \subseteq SPM(X)$ and let us still denote it by $L: B \rightarrow B$. Suppose that is possible to approximate $L$ in a suitable way by another operator $L_\delta$ of which we can calculate fixed points and other properties.

Our extent is to exploit as much as possible the information contained in $L_\delta$ to approximate fixed points of $L$. Let us hence suppose that $f, f_\delta \in B$ are fixed points, respectively of $L$ and $L_\delta$.

\textbf{Theorem 1.} Suppose that:

a) $\|L_\delta f - Lf\|_B \leq \epsilon$
\( \exists N \) such that \( \| L_\delta^N (f_\delta - f) \|_B \leq \frac{1}{2} \| f_\delta - f \|_B \)

c) \( L_\delta \) is continuous on \( B \); \( \exists C \) s.t. \( \forall g \in B, \| L_\delta g \|_B \leq C \| g \|_B \).

Then
\[
\| f_\delta - f \|_B \leq 2 \epsilon \sum_{i \in [0,N-1]} C^i
\]

**Remark 2.** In the following we show how the above items a), b), c) are natural, in the context of approximating a fixed point of the transfer operator: a) means that in some sense \( L_\delta \) is an approximation of \( L \). About b), the required \( N \) will be calculated from a description of \( L_\delta \) exploiting the fact that, under natural assumptions \( L_\delta \) asymptotically contracts the space of zero average signed measures in \( B \). Remark that b) also means that there is no “projection” of \( f \) on other fixed points of \( L_\delta \) than \( f_\delta \). c) will be obtained from the way \( L_\delta \) is defined.

**Proof.** (of Theorem 1)
\[
\| f_\delta - f \|_B \leq \| L_\delta^N f_\delta - L_\delta^N f \|_B
\]
\[
\leq \| L_\delta^N f_\delta - L_\delta^N f \|_B + \| L_\delta^N f - L_\delta^N f \|_B
\]
\[
\leq \| L_\delta^N (f_\delta - f) \|_B + \| L_\delta^N f - L_\delta^N f \|_B
\]
\[
\leq \frac{1}{2} \| f_\delta - f \|_B + \| L_\delta^N f - L_\delta^N f \|_B
\]

(applying item b)). Hence
\[
\| f_\delta - f \|_B \leq 2 \| L_\delta^N f_\delta - L_\delta^N f \|_B
\]

but
\[
L_\delta^N - L_\delta^N = \sum_{k=1}^{N} L_\delta^{N-k}(L_\delta - L)L^{k-1}
\]

hence
\[
(L_\delta^N - L_\delta^N)f = \sum_{k=1}^{N} L_\delta^{N-k}(L_\delta - L)L^{k-1} f
\]
\[
= \sum_{k=1}^{N} L_\delta^{N-k}(L_\delta - L)f
\]

by item c), hence
\[
\| (L_\delta^N - L_\delta^N)f \|_B \leq \sum_{k=1}^{N} C^{N-k} \| (L_\delta - L)f \|_B
\]
\[
\leq \epsilon \sum_{i \in [0,N-1]} C^i
\]

by item a), and then
\[
\| f_\delta - f \|_B \leq 2 \epsilon \sum_{i \in [0,N-1]} C^i.
\]

\( \square \)
3 Estimation with $L_1$ norm and Ulam method

Let us suppose now that $X$ is a manifold with boundary. Let us describe better the Ulam’s Discretization method. In this method the space $X$ is discretized by a partition $I_δ$ (with $k$ elements) and the system is approximated by a (finite state) Markov Chain with transition probabilities

$$P_{ij} = m(T^{-1}(I_j) \cap I_i)/m(I_i)$$ (2)

(where $m$ is the normalized Lebesgue measure on $X$) and defining a corresponding finite-dimensional operator $L_δ$ ($L_δ$ depend on the whole chosen partition but simplifying we will indicate it with a parameter $δ$ related to the size of the elements of the partition) we remark that in this way, to $L_δ$ it corresponds a matrix $P_k = (P_{ij})$.

We remark that $L_δ$ can be seen in the following way: let $F_δ$ be the $\sigma$–algebra associated to the partition $I_δ$, then:

$$L_δ(f) = E(L(E(f|F_δ))|F_δ),$$ (3)

(see also [18], notes 9 and 10 for some more explanations). Taking finer and finer partitions, in certain systems including for example piecewise expanding one-dimensional maps, the finite dimensional model converges to the real one and its natural invariant measure to the physical measure of the original system, see e.g. [8, 9, 18].

For the sake of simplicity we will suppose that all sets $I_j$ have the same measure: $m(I_j) = \frac{1}{k}$. This will simplify some notation. The general case can be treated similarly.

We now apply Theorem 1 to a more concrete case. Suppose that:

- $L_δ$ is the Ulam approximation of $L$ as defined above.
- $B = L_1(X)$,
- $L$ satisfies a Lasota Yorke inequality of the type

$$||L^n g||_{B'} \leq \lambda^n ||g||_{B'} + B ||g||_{L_1},$$ (4)

where $B'$ is another Banach subspace of $L_1$ and $\lambda < 1$. Inequalities of this type hold for several systems having some form of expansiveness or hyperbolicity (see [3, 17, 20, 24] and Theorem 8  e.g.).

**Remark 3.** A useful remark is that the L–Y inequality gives us an upper bound for the $B'$ norm of a fixed point. Indeed, if $f$ is a fixed point, we have that

$$||f||_{B'} = ||L^n f||_{B'} \leq \lambda^n ||f||_{B'} + B ||f||_{L_1}.$$

Now, supposing $||f||_{L_1} = 1$ and letting $n$ go to infinity, we have that

$$||f||_{B'} \leq B.$$
Let us outline the main points which allow the application of Theorem 1 in this case to construct an algorithm to approximate invariant measures.

Remark 4. To estimate $\|L_\delta f - Lf\|_{L_1}$ as required in item a) of Theorem 1 we remark that

$$\|L_\delta f - Lf\|_{L_1} \leq \|L_\delta - L\|_{B'\to L_1} \|f\|_{B'};$$

(where $\|\cdot\|_{B'\to L_1}$ is the operator norm, as an operator $B' \to L_1$) hence we can write the conclusion of the theorem in this way:

$$\|f_\delta - f\|_{L_1} \leq 2N \sum_{i=0}^{N-1} C_i \|L_\delta - L\|_{B'\to L_1} \|f\|_{B'}.$$

Another way to estimate $\|L_\delta f - Lf\|_{L_1}$ which is slightly more complicated but can give better estimations is explained in Subsection 3.3.4.

In this setting, under suitable assumptions:

I1 the norm $\|f\|_{B'}$ is estimated by the coefficients of the L-Y inequality, indeed as seen in Remark 3:

$$\|f\|_{B'} \leq B$$

(see also Section 3.3.1 below);

I2 under suitable assumptions $\|L_\delta - L\|_{B'\to L_1}$ is estimated a priori by the method of approximation (see Section 3.3.2 below);

I3 the integer $N$ relative to item b) in Theorem 1 can be estimated by the matrix $P_k$ relative to $L_\delta$. In practice what we need to do is to calculate the first $N$ such that $\|P_k^N\|_1 < \frac{1}{2}$. Here $\|\cdot\|_1$ is the 1-operator norm of $P_k$, and $V$ is the space of vectors with zero mean (see Section 3.1 below).

I4 Since $\mathcal{B} = L_1(X)$ and we consider the Ulam approximation then $C = 1$ (see Section 3.3.3 below).

Now let us discuss more precisely Item I3, which is central in this approach and whose discussion is general. We discuss the other Items in the Subsection 3.3 with precise estimations related to a particular family of cases: the piecewise expanding maps.

3.1 About item I3

To compute $N$ we consider $V = \{\mu \in \mathcal{B}|\mu(X) = 0\}$ and $\|L_\delta^N\|_{V} \|_{L_1\to L_1}$. Since $f - f_\delta \in V$, if we prove

$$\|L_\delta^N\|_{V} \|_{L_1\to L_1} < \frac{1}{2}$$

we imply Item b) of theorem 1. In the Ulam approximation $L_\delta$ is a finite rank operator, hence, once we fix a basis this is given by a matrix. The natural basis $\{f_1, \ldots, f_k\}$ to consider is the set of characteristic functions of the sets in the partition $I_\delta$. If $I_\delta = \{I_1, \ldots, I_k\}$ then $f_i = 1_{I_i}$; after the choice of this basis, the
set of linear combinations of such characteristic functions can be identified with \( \mathbb{R}^k \). By a small abuse of notation we will also indicate by \( V \) the set of zero average vectors in \( \mathbb{R}^k \).

To determine \( N \) we have to consider the matrix \( P_k|_V \) associated to the action of \( L_\delta \) on the space of zero mean vectors with respect to this basis and compute its operator norm \( \|P_k|_V\|_1 \) where

\[
\|P_k|_V\|_1 = \sup_{|v|_1 = 1} |P(v)|_1.
\]

By Equation 3 the behavior of \( L_\delta \) and its relation with \( P_k \) is described by

\[
f \xrightarrow{E|F_\delta I^{-1}} v \xrightarrow{P_k} v' \xrightarrow{I} f' = L_\delta(f)
\]

where \( I: \mathbb{R}^k \to L_1 \) is the trivial identification of a vector in \( \mathbb{R}^k \) with a piecewise constant function given by the choice of the basis. This implies that \( \forall f \in L_1 \)

\[
\|L_\delta\|_{L_1 \to L_1} \leq \|P_k\|_1.
\]

Indeed if \( f \in L_1 \), \( \|E(f|F_\delta)\|_{L_1} \leq \|f\|_{L_1} \) and \( I \) is trivially an isometry.

Remark that if \( \int f \, dm = 0 \), then \( \int E(f|F_\delta) \, dm = 0 \) and converse, and hence

\[
\|L_\delta|_V\|_{L_1} \leq \|P_k|_{I^{-1}(V)}\|_1.
\]

Trivially the matrix corresponding to \( L_\delta^N \) is \( P_k^N \). And then

\[
\|L_\delta^N|_V\|_{L_1} \leq \|P_k^N|_{I^{-1}(V)}\|_1.
\]

Summarizing, we can have an estimation of \( \|L_\delta^N|_V\|_{L_1 \to L_1} \) by calculating a matrix \( M \) approximating \( P_k|_{I^{-1}(V)} \) and \( \|MN\|_1 \).

The algorithm will hence calculate \( \|M^j\|_1 \) for each integer \( j > 0 \), computing \( M^j \) iteratively from \( M^{j-1} \), until it finds some \( j \) for which \( \|M^j\|_1 < \frac{1}{2} \) and output this \( j \) as the \( N \) required in item b) of Theorem 1.

### 3.2 The algorithm

We now present informally the general algorithm which arises from the previous considerations for the approximation of invariant measures. More details on the implementation of each step are given in the following subsections.

**Algorithm 5.** The algorithm hence works as follows:

1. Input the map and the partition.

2. Compute the matrix \( P_k \) and the corresponding approximated fixed point \( \tilde{f}_\delta \) of \( L_\delta \) up to some required approximation \( \epsilon_1 \)

3. Compute \( \Delta L \), an estimation for \( \|L_\delta f - Lf\|_{L_1} \) up to some error \( \epsilon_2 \)

\( \|\cdot\|_1 \) will denote the \( L_1 \) norm on \( \mathbb{R}^n \).
4. Compute $N$ such that item b) of Theorem 1 is verified as described in item I3 above.

5. If all computations end successfully, output $\tilde{f}_\delta$.

All was said before allows us to state the following

**Proposition 6.** $I^{-1}(\tilde{f}_\delta)$ is an approximation of one invariant measure of $L$ up to an error $\epsilon$ given by:

$$\epsilon \leq \epsilon_1 + 2N(\Delta L + \epsilon_2)$$

in the $L_1$ norm.

### 3.3 Details in the piecewise expanding case

We now enter in more details, showing how the previously explained algorithm works in a concrete but nontrivial family of cases, where all the required computations and estimations can be done.

Let

$$||\mu|| = \sup_{\phi \in C^1_{[0,1]}, |\phi| = 1} |\mu(\phi')|$$

this is related to bounded variation\(^3\) if $\mu$ has density $f$ then it is easy to show that $||\mu|| \leq 2\text{var}(f) + 2|f|_\infty$.

In this case $X = [0,1], B' = \{\mu, ||\mu|| < \infty\}$. The dynamics we will consider is defined by a map satisfying the following requirements:

**Definition 7.** A nonsingular function $T : ([0,1], m) \to ([0,1], m)$ is said piecewise expanding if

- There is a finite set of points $d_1 = 0, d_2, ..., d_n = 1$ such that $T|_{(d_i, d_{i+1})}$ is $C^2$ and for each $i$, $\int \frac{|T''|}{|T'|} dx < \infty$.

- $\inf_{x \in [0,1]} |D_x T| > 2$ on the set where it is defined.

We remark that usually it is supposed $\inf_{x \in [0,1]} |D_x T| > 1$. We can always suppose that the derivative is bigger than 2 by considering some iterate of $T$ (of course the physical measure of the iterate is the same).

We suppose that the map is computable, in the sense that we can compute the probabilities $P_{ij}$ defined in \(^2\) up to any given accuracy. This is the case for example, if the map is given by branches which are given by analytic functions with computable coefficients.

It is now well known that piecewise expanding maps have a finite set of ergodic absolutely continuous invariant measure with bounded variation density.

Such densities are also fixed points of the (Perron Frobenius) operator\(^4\) $L : L^1([0,1]) \to L^1([0,1])$ defined by

\(^3\) Recall that the variation of a function $g$ is defined as $\text{var}(g) = \sup_{(x_i) \text{ Finite subdivisions of } [0,1]} \sum_{i=1}^n |g(x_i) - g(x_{i+1})| < \infty$

\(^4\) Note that this operator corresponds to the above defined transfer operator, but it acts on densities instead of measures.
\[ (Lf)(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{T'(y)} \]

We now explain how to face all the points raised in the concrete implementation of Algorithm 5.

### 3.3.1 About Item I1

In this section we obtain an explicit estimation of the coefficients of the Lasota Yorke inequality for piecewise expanding maps. We follow the approach of [19], trying to optimize the size of the constants.

**Theorem 8.** If \( T \) is piecewise expanding and \( \mu \) is a measure on \([0,1]\)

\[
||L\mu|| \leq \frac{2}{\inf T'} ||\mu|| + \frac{2}{\min(d_i - d_{i+1})} \mu(1) + 2\mu(\frac{T''}{(T')^2}).
\]

**Proof.** Remark that

\[
L\mu(\phi') = \sum_{Z \in \{(d_i,d_{i+1})|i\in(1,...,n-1)} L\mu(\phi'\chi_Z)
\]

since \( L\mu \) gives zero weight to the points \( d_i \) (\( L\mu \) is absolutely continuous).

For each such \( Z \) define \( \phi_{cont} \) to be linear and such that \( \phi_Z = \phi - \phi_{cont} \), on \( Z \), and extend it to \([0,1]\) by setting it to zero outside \( Z \). This is a continuous function. Moreover for each \( x \in Z \)

\[
|\phi_{cont}'| \leq \frac{2|\phi|_{\infty}}{\min(d_i - d_{i+1})}
\]

Thus

\[
|L\mu(\phi')| \leq |\sum Z \mu((\psi_Z \circ T)\chi_{T^{-1}(Z)})| + |\sum Z \mu((\psi_Z \circ T) T'' (T')^{-2})\chi_{T^{-1}(Z)})| + \frac{2|\phi|_{\infty}}{\min(d_i - d_{i+1})} \mu(1)
\]

\[
\leq |\mu((\psi_Z \circ T)\chi_{T^{-1}(Z)})| + 2|\phi|_{\infty} \mu(\frac{T''}{(T')^2}) + \frac{2|\phi|_{\infty}}{\min(d_i - d_{i+1})} \mu(1).
\]

\( \sum_{Z} \frac{\psi_Z T'}{T'} \) is not \( C^1 \), but it can be approximated as well as wanted by a \( C^1 \) function \( \psi_e \) such that \( |\psi_e - \sum Z (\psi_Z \circ T')|_{\infty} \) and \( \mu(|\psi_e - \sum Z (\psi_Z \circ T)|) \) are as small as wanted. Hence

\[
|\mu((\psi_E \circ T)\chi_{T^{-1}(Z)})| \leq ||\mu|| |\psi_E \circ T|_{\infty} \leq ||\mu|| \frac{2}{\inf T'} |\phi|_{\infty}
\]

10
\begin{align*}
|L\mu(\phi^i)| &\leq |\mu| - \frac{2}{\inf T} ||\phi||_\infty + 2|\phi||_\infty \mu\left(\frac{T''}{(T')^2}\right) + \frac{2|\phi||_\infty}{\min(d_i - d_{i+1})} \mu(1) \\
||L\mu|| &\leq \frac{2}{\inf T} ||\mu|| + \frac{2}{\min(d_i - d_{i+1})} \mu(1) + 2\mu\left(\frac{T''}{(T')^2}\right) 
\end{align*}

\begin{remark}
We remark that from
\begin{equation}
||L\mu|| \leq \frac{2}{\inf T} ||\mu|| + \frac{2}{\min(d_i - d_{i+1})} \mu(1) + 2\mu\left(\frac{T''}{(T')^2}\right)
\end{equation}

it is easy to extract
\begin{equation}
||L\mu|| \leq \frac{2}{\inf T} ||\mu|| + \left(\frac{2}{\min(d_i - d_{i+1})} + 2\frac{T''}{(T')^2}\right) ||\mu||_1
\end{equation}

Where \(||\mu||_1 = \sup \{|\mu(\phi)|\} \) coincides with the \(L_1\) norm for a density of \(\mu\).
\end{remark}

\begin{remark}
From now on, the following notation is going to be used throughout the paper
\begin{equation}
B' := \frac{2}{\min(d_i - d_{i+1})} + 2\frac{T''}{(T')^2}_\infty.
\end{equation}

This constant plays a central role in our treatment and is the biggest obstruction in getting good estimates for the rigorous error.

We remark that once an inequality of the form
\begin{align*}
||Lg||_{B'} \leq 2\lambda ||g||_{B'} + B'||g||_1.
\end{align*}
is established (with \(2\lambda < 1\)) then, iterating, we have
\begin{align*}
||L^2g||_{B'} \leq 2\lambda ||Lg||_{B'} + B'||Lg||_1 \leq 2\lambda(2\lambda ||Lg||_{B'} + B'||g||_1) + B'||g||_1
\end{align*}
and thus
\begin{align*}
||L^n g||_{B'} \leq 2^n \lambda^n ||Lg||_{B'} + \frac{1}{1 - 2\lambda} B'||g||_1
\end{align*}
obtaining the inequality in the form required at \(4\) and the coefficient

\begin{equation*}
B = \frac{1}{1 - 2\lambda} B'
\end{equation*}

which is important to estimate \(||f||\) in our algorithm.
3.3.2 About item I2

As outlined before, on the interval $[0, 1]$ we consider a partition made of intervals having length $\delta$. As remarked in Item I2 we need an estimation on the quality of approximation by Ulam discretization (a similar estimation was given in [18, Lemma 4.1]).

**Lemma 11.** For piecewise expanding maps, if $L_\delta$ is given by the Ulam discretization as explained before, for every $f$ in $BV[0, 1]$ we have that

$$||L f - L_\delta f||_{L_1} \leq \delta(2\lambda + 1 + B')||f||,$$

where $B'$ is defined in Remark 10.

**Proof.** First of all, we have that trivially

$$\text{var}(E(f|F)) \leq \text{var}(f),$$

moreover, for $f \in BV[0,1]$, holds

$$||E(f|F_\delta) - f||_{L_1} \leq \delta \cdot \text{var}(f).$$

Thus for all $f$ in $BV[0,1]$ we have

$$||(L - L_\delta)f||_{L_1} \leq \delta ||E(L(E(f|F_\delta)|F_\delta)) - L(E(f|F_\delta))||_{L_1} + ||E(f|F_\delta) - f||_{L_1},$$

which implies, from the remark above

$$||(L - L_\delta)f||_{L_1} \leq \delta \cdot \text{var}(L E(f|F_\delta)) + ||E(f|F_\delta) - f||_{L_1}.$$

Using now Theorem 8 we have that

$$||(L - L_\delta)f||_{L_1} \leq \delta 2\lambda||E(f|F_\delta)|| + \delta B' ||E(f|F_\delta)||_{L_1} + ||E(f|F_\delta) - f||_{L_1}, \tag{6}$$

but, from the properties of $E(f|F)$ stated above we have that, since $||f|| \geq \text{var}(f)$ and $||f|| \geq ||f||_{L_1}$

$$||(L - L_\delta)f||_{L_1} \leq \delta 2\lambda||f|| + \delta B' ||f||_{L_1} + \delta \cdot Var(f) \leq \delta(2\lambda + 1 + B')||f||.$$

This gives us the last estimate which is needed by Item 3 of algorithm 5.

**Remark 12.** Working with partitions subdividing the interval $[0,1]$ in $k$ equal subintervals, and recalling Remark 3 we that, when $f$ is an invariant measure, the inequality takes the form

$$||L f - L_\delta f||_{L_1} \leq \frac{(2\lambda + 1 + B')}{k} B.$$
3.3.3 About item I4

It is easy to see that if \( L_\delta \) is given by the Ulam method

\[
\|L_\delta f\|_{L_1} \leq \|f\|_{L_1};
\]

indeed \( \|L f\|_{L_1} \leq \|f\|_{L_1} \) and \( \|E(f|F_\delta)\|_{L_1} \leq \|f\|_{L_1} \) as seen in Section 3.1 and \( L_\delta \) comes from the composition of such functions.

3.3.4 A little complication for a better estimation in our case

We remark that from Equation 6, we can also obtain in the same way, recalling that \( \|f\|_{L_1} = 1 \)

\[
\|(L - L_\delta)f\|_{L_1} \leq \delta(2\lambda + 1)\|f\| + \delta B'.
\]

By this (see remark 12)

\[
\|(L - L_\delta)f\|_{L_1} \leq \frac{2\lambda B + B + B'}{k}.
\]

This inequality can also be used in our algorithm to improve the precision of the estimation in the case of \( L_1 \) approximation by the Ulam approximation scheme, obtaining:

\[
\|f - f_\delta\|_{L_1} \leq 2N \frac{2\lambda B + B + B'}{k}.
\]

4 Implementing the algorithm

In this section we explain the details in the implementation of our algorithm in the piecewise expanding case and some related numerical issue. The main points are the computation of a rigorous approximation of the related Markov chain and a fast method to approximate rigorously its steady state. We include some implementation and numerical supplementary remarks, which can be skipped at a first reading.

4.1 Computing the Ulam approximation

To compute the matrix of the Ulam approximation, we have developed an algorithm that computes the entries of the matrix \( P_k \) with a rigorous error bound \( \varepsilon \). The computed matrix will be called \( \Pi \). Our algorithm computes each entry and the error associated to each entry; the maximum of all these errors is \( \varepsilon \).

If we partition the interval \([0, 1]\) in \( k \) intervals \( I_1, \ldots, I_k \) the \( i, j \)-th entry of \( P_k \) is given by equation 2.

To compute the \((i, j)\)-th entry, we partition the interval \( I_i \) in \( m \) smaller intervals \( J_1, \ldots, J_m \). This \( m \) has to be chosen to be bigger than the derivative of \( T|_{I_i} \) to ensure that the images \( T(J_l) \) are shorter than the target interval \( I_j \).

For each \( l = 1, \ldots, m \) we look at the images \( T(J_l) \) and we test whether this images are contained in \( I_j \). Clearly there are three possible situations:
1. $T(J_l)$ is contained in $I_j$;

2. $T(J_l)$ is contained in the complement of $I_j$;

3. $T(J_l)$ overlaps $I_j$ but it is not contained.

Clearly, 1 and 2 define unambiguously whether the interval belongs to $T^{-1}I_j \cap I_i$ or not. In case 1 we have that $J_l$ belongs to $T^{-1}I_j$, while in case 2 it belongs to the complement. Since we are computing $m(T^{-1}I_j \cap I_i)$ in case 1 we take into account the measure of $J_l$, while in case 2 we simply forget about $J_l$.

The tricky part is how to deal with case 3. We would like to understand how much of the mass of the interval is sent into $I_j$ by $T$. To perform the computation with a rigorous bound on the error we developed a simple iterative algorithm, with a stopping criterion, that computes the coefficients of the Ulam approximation keeping track of the error done. First we compute a matrix $\Pi'$ which approximates $P_k$, then we “normalize” $\Pi'$ in a way that the resulting matrix $\Pi$ represents a Markov chain. If $k$ is big enough we can assume we are in one of the two following cases: the function is monotone on $J_l$ or has one discontinuity point in $J_l$.

We first study the monotone case. Since $J_l$ is an “undecided” interval and the function is monotone, we know that there exists a point $\alpha$ which partitions $J_l$ into two subintervals; one of those two subintervals is sent into $I_j$ by $T$ and the other is sent into the complement of $I_j$. Clearly, this $\alpha$ may not be computable exactly; to solve this issue what we do is to use a sort of bisection principle. We divide $J_l$ in $m$ smaller intervals; some of these intervals are sent into $I_j$ by $T$, some others are sent into the complement of $I_j$ and one of them is going to be “undecided”. Now, we can divide this interval into $m$ smaller intervals; some of these intervals are sent into $I_j$ by $T$ and so on. We iterate this process and, if $s$ is the number of steps, at the $s$-th step the “undecided” interval has length

$$\frac{1}{k \cdot m^s}.$$ 

The algorithm stops when this length is shorter than a prescribed length, leaving a small undecided interval; the length of this “undecided” interval adds up to a number $\varepsilon_{ij}$.
Numerical Remark 1. To compute the images of the endpoints of the intervals, we are working with a library that computes exactly rational numbers (the GiNaC library \[2\]). Since all our examples are, in their continuity intervals, rational maps and the intervals have rational endpoints, the computation of the matrix is rigorous.

Numerical Remark 2. The matrix has a special form which mirrors the graph of the function, i.e., if the interval does not contain a discontinuity the non zero entries of the matrix are distributed on each row near the image of the mid-point of the interval. As we already said, the number of nonzeros on a row is bounded from above by two times the maximum of the derivative in the interval. This a priori knowledge of the structure of the matrix permits us to fasten the computation, computing only the intervals “near” the image of the midpoint of the interval. To be more precise, we compute all the \(2\Xi\) intervals which are near the one containing the image of the midpoint, where \(\Xi\) is an integer that bounds from above the derivative of the function in \([0, 1]\).

In a similar way we study the case of the interval crossing a discontinuity point: if the discontinuity point belongs to \(J_l\) we divide \(J_l\) in smaller intervals. Only one of these intervals will contain the discontinuity point and we divide it into even smaller intervals; we continue this process until the interval that crosses the discontinuity has length smaller than the prescribed length. As before, the length of this “undecided” interval adds up to a number \(\varepsilon_{ij}\) such that

\[|\Pi'_{ij} - P_{ij}| < \varepsilon_{ij}.\]

Numerical Remark 3. If the interval \(J_l\) contains a discontinuity point, we can, again, infer the distribution of nonzeros on the row by thinking about how the maps behave: the only intervals for which the entries are non zero are those near to the images of the endpoints of the interval \(J_l\).

The maximum of all the \(\varepsilon_{ij}\) is really important for all our estimates: we are going to denote it by \(\varepsilon\).

We recall that we denote the matrix containing the computed coefficients by \(\Pi'\), to distinguish it from \(P_k\), the actual matrix of the Ulam discretization. In general, this matrix is a sparse matrix, whose number of nonzero in the \(j\)-th row depends on the derivative of the function in the interval \(I_j\). To be sure that \(\Pi'\) is a good approximation of a stochastic matrix we compute the sum of the elements for each row, subtract this number to 1 and spread the result uniformly on each of the nonzero elements of the row obtaining a new “markovized” matrix \(\Pi\). This assures us that the matrix \(\Pi\) represents a Markov chain. If we look at the graph associated to \(\Pi'\), i.e., the graph where each nonzero element \(\Pi'_{ij}\) is a vertex and there is an directed edge between \(i\) and \(j\), it is easily seen that \(\Pi\) represents a Markov chain with exactly the same graph.

If \(\varepsilon\) is the maximum of the errors \(|\Pi'_{ij} - P_{ij}|\), we have that for each row, if we denote by \(\text{nnz}\) the number of nonzero elements of the row, that the sum of the elements of the row differs from 1 by at most \(\text{nnz} \cdot \varepsilon\). So, if we spread the result uniformly on each of the nonzero elements of the row we have a new matrix \(\Pi\)
such that
\[ |\Pi_{ij} - P_{ij}| < 2 \cdot \epsilon, \]
therefore, the matrix \( \Pi \) is such that
\[ ||P_k - \Pi||_1 < 2 \cdot k \cdot \epsilon. \]

**Numerical Remark 4.** The markovization process is done numerically working with integers; this is done taking advantage of the representation of floating points in double precision floating point arithmetics. Each one of the coefficients of the matrix is between 0 and 1; we multiply each coefficient of the matrix by \( 2^{50} \) and take its integer part. We then subtract all these integers to \( 2^{50} \) obtaining a number \( z \); we divide the integer \( z \) by the number of nonzeros and we sum the result on each of the nonzeros. The remainder of the division of \( z \) by the number of nonzeros is then added to the first nonzero of the row. Then, we divide every coefficient by \( 2^{50} \); this permits us to ensure that the matrix has row sum 1. This procedure implies that, in practice, our markovization algorithm produces a matrix \( \Pi \) such that
\[ ||P_k - \Pi||_1 < 3 \cdot k \cdot \epsilon. \]

The matrix \( \Pi \) is the matrix we are going to work with and the “markovization” process ensures that the biggest eigenvalue of \( \Pi \) is 1.

### 4.2 Computing rigorously the steady state vector and the error

In this section we only consider the case of a transitive Markov chain; the results here work fine also with non recurrent Markov chains but if the Markov chain is reducible there are some issues that can be solved but are not treated here. For us, mostly, irreducibility means that there is only one invariant density.

**Numerical Remark 5.** Remark that if the Markov chain is reducible, we can still decompose the Markov chain using tools from graph theory to identify the different connected components of the graph of the Markov chain; this is not implemented in our software since the examples to which we are going to apply it are all topologically transitive. This implies transitivity in the Markov chain approximating them. Indeed, let \( \hat{I}_i \) and \( \hat{I}_j \) be the interior of two intervals of the partition, since the map is topologically transitive and the derivative is bounded away from zero, there exists an \( N_{ij} \) such that \( T^{N_{ij}}(\hat{I}_i) \cap \hat{I}_j \neq \emptyset \), and this intersection is a union of intervals, with nonzero measure. Therefore, if we call \( \hat{N} \) the maximum of all these \( N_{ij} \) the matrix \( P^\hat{N}_k \) has strictly positive entries and therefore the matrix \( P_k \) represents an irreducible Markov chain. By the Perron-Frobenius theorem this implies that the steady state of the Markov chain is unique.

We want to find the left eigenvector associated with the eigenvalue 1 of the matrix \( \Pi \); we know that \( \Pi \) has largest eigenvalue 1 since it has row sum 1 and therefore it admits as right eigenvector the vector \((1, \ldots, 1)^T\). Moreover, since \( \Pi \)
represents a transitive Markov chain, the eigenvalue 1 has algebraic multiplicity 1.

To compute this eigenvector we use the power iteration method; given any initial condition $b_0$, if we set
\[ b_{l+1} = b_l \cdot \Pi, \]
we have that $b_l$ converges to the eigenvector we are looking for; we want to find a stopping criterion that permits us to bound the numerical error of this operation from above.

To do this we use build an enclosure for the eigenvector using an idea from the proof of the Perron-Frobenius theorem. The main idea of the proof of the Perron-Frobenius theorem [3, Theorem 1.1] is that a Markov matrix $A$ (aperiodic, irreducible) contracts the simplex $\Lambda$ of the vectors $v$ having 1-norm 1.

This simplex is given by the convex combinations of the vectors $e_1, \ldots, e_k$ of the base; after $l$ iterations, the diameter of the simplex is going to be bounded by
\[ ||A^l e_i - A^l e_j||_1 = ||A^l (e_i - e_j)||_1, \]
for $i, j = 1, \ldots, k$ and $i \neq j$.

Therefore, to build an enclosure for the eigenvector, we iterate the matrix on the vectors $\{e_i - e_j\}$ for $i, j = 1, \ldots, k$ and $i \neq j$; in practice, thanks to the triangle inequality
\[ ||A^l (e_i - e_j)||_1 \leq ||A^l (e_1 - e_j)||_1 + ||A^l (e_1 - e_i)||_1, \]
we can iterate only the vectors of the form $\{e_1 - e_j\}$ for $j = 2, \ldots, k$.

**Remark 13.** Please remark that $\{e_1 - e_j\}$ for $j = 2, \ldots, k$ is a base for the space of average 0 vectors; therefore, while we build the enclosure we can also compute the integer $N$ used in Theorem 2 as explained in 3.1. Later, in this section, we explain some subtle points about the computation of $N$.

We implemented the enclosure algorithm in the following way; the program get as an input a threshold $\varepsilon_{\text{num}}$ and we iterate the vectors $\{e_1 - e_j\}$, with $j = 2, \ldots, n$ and look at their 1-norm. For each $j$, we denote by $l_j$ the integer such that
\[ ||A^{l_j} (e_1 - e_j)|| \leq \varepsilon_{\text{num}}. \]

Let $l$ the maximum over $j$ of all the $l_j$: we have that
\[ \text{diam}(A^l(\Lambda)) \leq \max_{i,j=1,\ldots,n} ||A^{l_j} (e_i - e_j)||_1 \]
\[ \leq ||A^{l_j} (e_1 - e_j)||_1 + ||A^{l_j} (e_1 - e_i)||_1 < 2\varepsilon_{\text{num}}. \]

Therefore, for any initial condition $b_0$, iterating it (renormalizing it) $l$ times we get a vector contained in $A^l(\Lambda)$, whose numerical error is enclosed by $2\varepsilon_{\text{num}}$. 

17
Numerical Remark 6. To take into account the numerical error of the row-column multiplication we used a really rough estimate, knowing that

$$||\text{float}(Av) - Av||_1 \leq k \cdot \epsilon_{mach},$$

where $\epsilon_{mach}$ is the roundoff error, and by a easy induction

$$||\text{float}(A^jv) - A^jv||_1 \leq l \cdot k \cdot \epsilon_{mach}.$$ 

Indeed, to take into account the numerical error, for each $j$, we take an $l_j$ such that

$$||A^j(e_1 - e_j)||_1 + l_j \cdot k \cdot \epsilon_{mach} < \epsilon_{num}.$$ 

Therefore, we computed the eigenvector with a rigorous bound on the numerical error; what is left is to compute the rigorous error. 

From Section 3.3.2 we have an estimate for the quantity

$$||L_f - L_\delta f||_{L^1}$$

depending on constants that, once the map is known, we can compute by hand.

The main issue that remains to be solved is the computation of the number of iterations $N$ needed for the Ulam approximation $L_\delta$ to contract to $1/2$ the space of average 0 vectors as explained in Section 3.1. 

In some way, we already assessed this question while we were computing the iterations of the simplex; the vectors $e_1 - e_j$, with $j = 1, \ldots, k$ are a base for the space of average 0 vectors, so, while computing rigorously the eigenvector, we can compute also the number of iterations. But we have to be careful since we do not know the matrix $P_k$ of the Ulam approximation $L_\delta$ explicitly but we know only its approximation $\Pi$. To compute the number of iterations needed by $L_\delta$ to contract the space of average 0 vectors to $1/2$, we need to use a chain of inequalities to give an upper bound to the norm of $L_\delta^j|V$, for $j$ an integer.

Indeed (see Section 3.1)

$$||L_\delta^j|V|| \leq ||(P_k^j - \Pi^j + \Pi^j)|V||_1 \leq ||(P_k^j - \Pi^j)|V||_1 + ||\Pi^j|V||_1.$$ 

By how we constructed the matrix $\Pi$ we have that $||P_k - \Pi||_1 < 2 \cdot k \cdot \epsilon$ where $k$ is the size of the partition, $\epsilon$ is the maximum of the errors in the computation of the coefficient of the matrix and

$$||P_k^j - \Pi^j|V||_1 = \sum_{i=1}^{j} P_k^{j-i}(P_k - \Pi)\Pi^{i-1}|V||_1 \leq \sum_{i=1}^{j} ||P_k^{j-i}|V||_1 \cdot ||P_k - \Pi||_1 \cdot ||\Pi^{i-1}|V||_1 \leq 2 \cdot j \cdot k \cdot \epsilon,$$

since $||P_k - \Pi||_1 < 2 \cdot k \cdot \epsilon$, $||P_k^j|V||_1 \leq 1$ and $||\Pi^h||_1 \leq 1$ for every $j, h$. Therefore

$$||P_k^j|V|| \leq 2 \cdot j \cdot k \cdot \epsilon + ||\Pi^j|V||_1.$$
Numerical Remark 7. Please note that, if the approximation error is big and the size of the discretization is big, then, after a short number of iterates, $2 \cdot j \cdot k \cdot \varepsilon$ may be so big that the whole computation is destroyed. Therefore, we have to be careful, while choosing the error allowed on each coefficient of the matrix, to choose it small enough that no problem arises in this step.

We recall now the sources of error in our computation, to make clear the last step of our program:

1. the discretization error, coming from the Ulam Discretization of the transfer operator;
2. the approximation error: since we cannot compute exactly the matrix $P_k$, we have to approximate it by computing a matrix $\Pi$;
3. the numerical error in the computation of the eigenvector.

The rigorous error for the invariant measure is computed from all these errors, using the algorithm we will explain below.

Remark 14. Please note that in the experiments we have chosen the diameter of the enclosure to be small enough so that the numerical error is small and the biggest source of uncertainty is the discretization error.

Now, if we denote by $f$ the invariant measure of the system, by $v_k$ the eigenvector of the Ulam approximation $P_k$, by $v_\varepsilon$ the eigenvector of $\Pi$ and by $\hat{v}$ the eigenvector computed using $\Pi$, we have

$$
\|f - \hat{v}\|_1 \leq \|f - v_k\|_1 + \|v_k - v_\varepsilon\|_1 + \|v_\varepsilon - \hat{v}\|_1.
$$

In the first part of the section we showed how to bound the numerical error

$$
\|v_\varepsilon - \hat{v}\|_1,
$$

item 3 in our list.

We recall that, since we have satisfied items I1, I2, I3 and I4 we get that the discretization error (item 1) $\|f - v_k\|_1$ is, by our theoretical estimates:

$$
\|f - v_k\|_1 \leq 2 \cdot N \|L f - L \delta f\|_L.
$$

The last thing we need to compute to get our rigorous estimate is a bound for

$$
\|v_k - v_\varepsilon\|_1,
$$

the approximation error, item 2 in our list. To do this we use again Theorem 11, computing the number of iterates $N_\varepsilon^5$ needed for $\Pi$ to contract to $1/2$ the space of average 0 vectors; then

$$
\|v_k - v_\varepsilon\|_1 \leq 2 N_\varepsilon \|P_k - \Pi\|_1 \|v_k\|_1 \leq 4 N_\varepsilon \cdot k \cdot \varepsilon.
$$

\[^5\]please note that, if $\varepsilon$ is small, $N_\varepsilon = N$ is expected. In the program we compute the two values indipendently, even if in general $N_\varepsilon \leq N$. 

19
Finally, we have that
\[ \|f - \tilde{v}\|_1 \leq 2 \cdot N \|L_f - L_{\tilde{v}}\|_{L^1} + 4N\varepsilon \cdot k \cdot \varepsilon + \|v_\varepsilon - \tilde{v}\|_1, \]
or explicitly
\[ \|f - \tilde{v}\|_1 \leq 2 \cdot N \cdot \frac{2\lambda + 1 + B'}{k} \cdot B + 4N\varepsilon \cdot k \cdot \varepsilon + \|v_\varepsilon - \tilde{v}\|_1. \]

Remark 15. Please note that, if we use the estimate in section 3.3.4, the last inequality becomes
\[ \|f - \tilde{v}\|_1 \leq 2N \frac{2\lambda B + B' + B}{k} + 4N\varepsilon \cdot k \cdot \varepsilon + \|v_\varepsilon - \tilde{v}\|_1. \]

5 Rigorous computation of the Lyapunov exponent

One of the uses of a rigorously computed density is the rigorous computation of the Lyapunov exponent of a map. The Lyapunov exponent at a point \( x \), denoted by \( \lambda(x) \), of a piecewise expanding map is defined by

\[ \lambda(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n} \log(|(T^i)'(x)|); \]

by Birkhoff ergodic theorem, we have that, relative to an invariant measure \( \mu \), for \( \mu \)-a.e. \( x \) we have that

\[ \lambda(x) = \int_0^1 \log(|T'|)d\mu = \lambda. \]

Our algorithm permits us to compute the density of an invariant measure with a rigorous error bound; by Young’s inequality we have that

\[ \left| \int_0^1 \log(|T'(x)|)f(x)dx - \int_0^1 \log(|T'(x)|)\tilde{v}(x)dx \right| \leq \max_{x \in [0,1]} (\log|T'(x)|)||f - \tilde{v}||_1, \]

where by a small abuse of notation \( \tilde{v} \) is identified with by a piecewise constant function representing it, see Section 3.1. Please note that since we only know \(||f - \tilde{v}||_1\), which is a global quantity, there are no naive ideas to have a better estimation.

Therefore, to compute the Lyapunov exponent, the only thing we have to do is to compute with a (relatively) small numerical error the integral

\[ \int_0^1 \log(|T'(x)|)\tilde{v}dx. \]

Since \( f'(x) > 1 \) for all \( x \), we have that, in each of the continuity intervals of \( T \), \( \log(|T'(x)|) \) is continuous and at least \( C^1 \).
To compute the integral we used the rectangle rule, interpolating the function by a piecewise constant function on the homogeneous partition on which we computed the Ulam approximation; in the intervals in which the function has a discontinuity point we have a bigger error, of the order of the size of the interval times the variation of the function. Since the partitions we used in our examples are really small, the numerical error coming from the method and the error arising from the discontinuities are really small compared to the rigorous error.

Numerical Remark 8. To ensure that the numerical error is much smaller than the rigorous error (so that, rounding up the rigorous error, we can simply ignore it), we used the library iRRAM (22) to evaluate the functions which we integrate. This library implements “Turing-rigorous” computation, i.e., computes rigorously the output of a function up to a prescribed error. In particular, we used this library to compute the function in such a way that all the digits of the double representation are rigorously computed. If we evaluate the function on the midpoints of the partition, we have that integrating with respect to the invariant measure is, essentially, the inner product between the vector containing the values of the density and the vector containing the values of the function. Since all the values are computed up to the machine precision, we know that the relative error of such an operation is bounded from above by $k \cdot \epsilon_{mach}$ which, with respect to our data, gives an absolute error of the order of $10^{-10}$. This is negligible w.r.t. the rigorous error, which is of the order of $10^{-3}$.

6 Numerical experiments

In this section we show the output of some complete experiments we made, where we used the programs described above.

The code is now in an hybrid state: the routines that generate the matrix are written using the BOOST Ublas library and can run on almost any computer, at the same time the main computational issue is the enclosure method for the certified computation of the eigenvector, which needs more computational power. In this step the number of matrix-vector products is proportional to the size of the partition: in our examples the size of the partition is of the order of $10^6$. This forced us to implement and run our programs in a parallel HPC environment, using the library PETSc and running them on the CINECA Cluster SP6.

The code for the programs, the matrices and the outputs of the cluster are found in the directory

http://poisson.phc.unipi.it/~nisoli/invmeasure/
6.1 The Lanford map

For our first numerical experiment we chose one of the maps which were investigated in [16]. The map \( T : [0, 1] \to [0, 1] \) given by

\[
T : x \mapsto 2x + \frac{1}{2}x(1 - x) \pmod{1}.
\]

What seems to be a good approximation of the invariant measure of the map is plotted in figure 1 of the cited article. Since this map does not comply with the hypothesis of our article, i.e. there are some points where \( 1 < |D_xT| \leq 2 \) we study the map \( T^2 := T \circ T \). Clearly, the invariant measures for \( T \) and \( T^2 \) coincide. We write the explicit form of \( T^2 \); if \( f(x) = 2x + \frac{1}{2}x(1 - x) \) and \( g(x) = 2x + \frac{1}{2}x(1 - x) - 1 \) we have

\[
T^2(x) = \begin{cases}
  f(f(x)) & 0 \leq x \leq \frac{5}{2} - \sqrt{17} + \frac{5}{4} \\
  g(f(x)) & \frac{5}{2} - \sqrt{17} + \frac{5}{4} < x \leq \frac{5}{2} - \frac{1}{2}\sqrt{17} \\
  f(g(x)) & \frac{5}{2} - \frac{1}{2}\sqrt{17} < x \leq \frac{5}{2} - \sqrt{17} - \frac{3}{4} \\
  g(g(x)) & \frac{5}{2} - \sqrt{17} - \frac{3}{4} < x \leq 1.
\end{cases}
\]

In figure 1 you can see a plot of this map.

First of all, please note that in every component where the map is continuous, the map is a polynomial. So, we can use exact arithmetics to compute the matrix \( \Pi \). Please note that the discontinuity points are irrational; this is taken care as we explained in Section 4.1.

The experiment is made on a homogeneous partition of \([0, 1]\) in 1048576 intervals.

In figure 2 you can see the plot of the invariant measure we obtain through our method.

We can compute directly from the explicit form of the mapping \( T^2 \) that \( \lambda = 4/9 \) and that \( B' \) is bounded from above by 12.48. Below, some of the data (input and outputs) of our algorithm.

\[
\begin{align*}
\varepsilon & \leq 3 \cdot 10^{-11} \\
\varepsilon_{\text{num}} & \leq 0.0001 \\
N & = 17 \\
N_e & = 18 \\
B & = 112.24 \\
l & = 25
\end{align*}
\]

With all this data and the estimate in section 3.3.4 we can compute a rigorous error bound for the computed invariant measure:

\[
||f - \tilde{v}||_1 \leq 0.0081.
\]
Using all these estimates, we can compute the Lyapunov exponent for the original Lanford map, which is $0.658 \pm 0.008$. For its second iterate, i.e., the map we studied with our experiment, the computed Lyapunov exponent is $1.315 \pm 0.015 \approx 2 \cdot (0.658 \pm 0.008)$ as we could expect from the properties of the Lyapunov exponent.

6.2 A piecewise expanding map

In this paragraph we study a sort of Lorenz map with bounded derivatives, $T : [0, 1] \to [0, 1]$ given by

$$T(x) = \begin{cases} 
\frac{1}{8} + 3x + 2x^2 & 0 \leq x \leq \frac{1}{3} \\
4(x - \frac{1}{7}) & \frac{1}{3} < x \leq \frac{4}{7} \\
4(x - \frac{1}{7}) & \frac{4}{7} < x \leq \frac{2}{7} \\
\frac{7}{8} + 3(x - 1) - 2(x - 1)^2 & \frac{2}{7} < x \leq 1.
\end{cases} \quad (7)$$

whose graph is plotted in figure 3.

Please note that also in this case, in every component where the map is continuous, the map is a polynomial. So, we can use exact arithmetics to compute the matrix $\Pi$.

The experiment is made on a homogeneous partition of $[0, 1]$ in 1048576 intervals.
The invariant measure we obtain through our method is plotted in figure 2. For this map $B' < 8.9$ and $\lambda = 1/3$. Below, some of the data (input and outputs) of our algorithm.

\[
\begin{align*}
\varepsilon & \leq 10^{-12} \\
\varepsilon_{\text{num}} & \leq 0.0001 \\
N & = 13 \\
N_{\varepsilon} & = 14 \\
B & = 26.7 \\
l & = 22
\end{align*}
\]

With all this data and the estimate in section 3.3.4 we can compute a rigorous error bound for the computed invariant measure:

\[||f - \tilde{v}||_1 \leq 0.0013.\]

Using all these estimates, we can compute the Lyapunov exponent for this piecewise expanding map, which is $1.327 \pm 0.002$. 

Figure 2: The invariant measure for the Lanford map
6.3 A map without the Markov property

The map $T : [0, 1] \to [0, 1]$ given by

$$T(x) = \frac{17}{5} x \mod 1$$

whose graph is plotted in figure 5. This map does not enjoy the Markov property: since $(17/5)^k$ is never an integer the orbit of 1 is dense. A finite partition $\{I_i\}$ of $[0, 1]$ is said to have the Markov property if $f(I_i) \cap I_j \neq \emptyset$ implies $I_j \subset f(I_i)$; clearly, the partition must contain at least an interval of the form $[\alpha_0, 1]$. This implies that the partition must contain an interval of the form $[\alpha_1, f(1)]$, which in turn implies that it contains an interval of the form $[\alpha_2, f^2(1)]$ and so on, for every $k$, it must contain an interval of the form $[\alpha_k, f^k(1)]$ and since the orbit of 1 is dense, this partition is made by an infinite number of intervals.

Please note that also in this case, in every component where the map is continuous, the map is a polynomial. So, we can use exact arithmetics to compute the matrix $\Pi$.

The experiment is made on a homogeneous partition of $[0, 1]$ in 1048576 intervals.

The invariant measure we obtain through our method is plotted in figure 6. For this map $B^' < 17$ and $\lambda = 5/17$. Below, some of the data (input and outputs) of our algorithm.
Figure 4: The invariant measure for map (7)

\[
\begin{align*}
\varepsilon & \leq 1.75 \cdot 10^{-10} \\
\varepsilon_{num} & \leq 0.0001 \\
N & = 13 \\
N_\varepsilon & = 14 \\
B & = 41.29 \\
l & = 20
\end{align*}
\]

With all this data and the estimate in section 3.3.4 we can compute a rigorous error bound for the computed invariant measure:

\[||f - \tilde{v}||_1 \leq 0.013.\]

Since the map is piecewise linear, we know exactly the Lyapunov exponent of the map, which is \(\ln(17) - \ln(5)\).
6.4 Another example

We study the map $T : [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} 
\frac{17}{12}x & 0 \leq x \leq \frac{5}{17} \\ \frac{33}{32}(x - \frac{5}{10})^2 + 3(x - \frac{5}{10}) & \frac{5}{10} < x \leq \frac{10}{17} \\ \frac{47}{32}(x - \frac{10}{17})^2 + 3(x - \frac{10}{17}) & \frac{10}{17} < x \leq \frac{15}{17} \\ \frac{5}{17} & \frac{15}{17} < x \leq 1 \end{cases}$$  

(9)

whose graph is plotted in figure 7. This map is really similar to map (8), but it is nonlinear in the two intervals $[5/17, 10/17]$ and $[10/17, 15/17]$, where it is defined by two branches of a polynomial of degree two.

Please note that also in this case, in every component where the map is continuous, the map is a polynomial. So, we can use exact arithmetics to compute the matrix $\Pi$.

The experiment is made on a homogeneous partition of $[0,1]$ in 1048576 intervals.

The invariant measure we obtain through our method is plotted in figure 8. Please note that, near 0.337 and 0.403 there are two small “staircase steps” which are visible only zooming the graph.

For this map $B' < 17.61$ and $\lambda = 1/3$. Below, some of the data (input and outputs) of our algorithm.
With all this data and the estimate in section 3.3.4 we can compute a rigorous error bound for the computed invariant measure:

$$||f - \tilde{v}||_1 \leq 0.004.$$  

Using all these estimates, we can compute the Lyapunov exponent for this piecewise expanding map, which is $1.219 \pm 0.006$.

### 6.5 Some data on the computations

In this section we would like to give some informations about the computations itselfs; while the code that computes the matrix is subject to revision, we are going to reuse the code that computes the eigenvector rigorously.
At the moment, the code that computes the matrix a purely serial code, which uses the rational numbers type (c1_RA) of the CLN library, exposed through the numeric type of the GiNaC library [2]; the computations ran without any problem on our desktop and laptop computers, in a relatively short amount of time (of the order of 10000 seconds on a single processor). This routine is highly parallelizable but, at the moment, the main computational issue arise from the rigorous computation of the eigenvalue, as the following data shows.

To compute rigorously the eigenvector the total number of matrix-vector multiplications needed is 19.353.649 for the Lanford example, 18.336.646 for map (7), 20.264.432 for map (8) and 20.567.470 for map (9). The Lanford experiment was run on 96 SMT cpu (2 cpus per core) and the other experiments were run on 64 SMT cpus. The total running time for the Lanford code was $6.263e+04$ seconds, the total running time for the experiment for map (7) was $6.229e+04$ seconds, for experiment (8) was $6.1086e+04$ and for map (9) was $6.1229e+04$ seconds.

As you can see, the rigorous computation of the eigenvector (we underline the fact that it is rigorous, since the computation of the eigenvector itself is a matter of few seconds), is really time consuming; this is due to the fact that we are taking the powers of a sparse matrix. The main issue with such an operation is that the powers of a sparse matrix could be, in general, non sparse.
This means that, after some relatively fast iterations, the vector that we are multiplying is dense.

There are a lot of techniques that are being developed to compute matrix powers; it seems like it is one of the most important fields of applications of GPGPU computing; in the near future we are going to experiment if our code runs faster on them.

On the other side, one of the big issues may be the speed of convergence of the Ulam method; in the future we are going to investigate other discretization schemes: if the dimension of the matrices is much smaller this would imply faster computations.

References

[1] V. Baladi Positive transfer operators and decay of correlations, Advanced Series in Nonlinear Dynamics, 16 World Sci. Publ., NJ, (2000).

[2] C. Bauer, A. Frink and R. Kreckel Introduction to the GiNaC Framework for Symbolic Computation within the C++ Programming Language J. Symb. Comp. (2002), 33: 112.
[3] W. Bahsoun, *Rigorous Numerical Approximation of Escape Rates Nonlinearity*, vol. 19, 2529-2542 (2006)

[4] M. Dellnitz, O. Junge *Set Oriented Numerical Methods for Dynamical Systems* Handbook of dynamical systems vol 2 - Elsevier, (2002).

[5] M. Dellnitz and O. Junge, *On the approximation of complicated dynamical behavior*, SIAM J. on Num. Anal. (1999), 36: 491-515.

[6] J. Ding, Q. Du and T. Y. Li, *High order approximation of the Frobenius-Perron operator*, Appl. Math. and Comp. (1993), 53: 151-171.

[7] J. Ding and A. Zhou, *The projection method for computing multidimensional absolutely continuous invariant measures*, J. Stat. Phys. (1994), 77: 899-908.

[8] G. Froyland *On Ulam approximation of the isolated spectrum and eigenfunctions of hyperbolic maps* Disc. Cont. Dyn. Sys. (2007), 17(3): 203-221.

[9] G. Froyland *Extracting dynamical behaviour via Markov models* in Alistair Mees, editor, Nonlinear Dynamics and Statistics: Proceedings, Newton Institute, Cambridge (1998): 283-324, Birkhauser, 2001.

[10] G. Froyland *Computer-assisted bounds for the rate of decay of correlations*. Comm. Math. Phys., 189(1):237-257, 1997.

[11] S. Galatolo, M. Hoyrup, C. Rojas *Dynamical systems, simulation, abstract computation* Cha. Sol. Fra. Volume 45, Issue 1, , Pages 114 (2012)

[12] S. Galatolo, M. Hoyrup and C. Rojas, *A constructive Borel-Cantelli lemma. Constructing orbits with required statistical properties*, Theor. Comput. Sci. (2009), 410: 2207-2222.

[13] S. Galatolo, M. Hoyrup, and C. Rojas. *Dynamics and abstract computability: computing invariant measures*, Disc. Cont. Dyn. Sys. (2011), 29(1): 193-212

[14] O. Ippei *Computer-Assisted Verification Method for Invariant Densities and Rates of Decay of Correlations*. SIAM J. Applied Dynamical Systems 10(2): 788-816 (2011)

[15] L. Calcagnile, S. Galatolo, G. Menconi *Non-sequential Recursive Pair Substitutions and Numerical Entropy Estimates in Symbolic Dynamical Systems*. Journal of Nonlinear Science, 20, pp 723-745 (2010)

[16] O. E. Lanford III *Informal Remarks on the Orbit Structure of Discrete Approximations to Chaotic Maps* Exp. Math. (1998), 7: 317:324

[17] A. Lasota, J. Yorke *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. (1973), 186: 481-488.
[18] C. Liverani, *Rigorous numerical investigations of the statistical properties of piecewise expanding maps—A feasibility study*, Nonlinearity (2001), 14: 463-490.

[19] C. Liverani *Invariant measures and their properties. A functional analytic point of view*, Dynamical Systems. Part II: Topological Geometrical and Ergodic Properties of Dynamics. Centro di Ricerca Matematica “Ennio De Giorgi”: Proceedings. Published by the Scuola Normale Superiore in Pisa (2004).

[20] S. Gouezel, C. Liverani *Banach spaces adapted to Anosov systems* Erg. Th. Dyn. Sys. (2006), 26: 189-217.

[21] M. Keane, R. Murray and L. S. Young, *Computing invariant measures for expanding circle maps*, Nonlinearity (1998), 11: 27-46.

[22] N. Müller, *The iRRAM: Exact Arithmetic in C++* in J. Blanck, V. Brattka and P. Hertling, Computability and Complexity in Analysis, pp. 222–252, Lecture Notes in Computer Science 2064, (2001) Springer Berlin / Heidelberg

[23] M. Pollicott and O. Jenkinson, *Computing invariant densities and metric entropy*, Comm. Math. Phys. (2000), 211: 687-703.

[24] M. Tsujii *Decay of correlations in suspension semi-flows of angle-multiplying maps* Erg. Th. Dyn. Sys. (2008), 28: 291-317.

[25] Lai-Sang Young *What are SRB measures, and which dynamical systems have them?* J. Stat. Phys. (2002), 108: 733-754.