Decomposing graphs into interval colorable subgraphs and no-wait multi-stage schedules

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A B S T R A C T
A graph $G$ is called interval colorable if it has a proper edge coloring with colors $1, 2, 3, \ldots$ such that the colors of the edges incident to every vertex of $G$ form an interval of integers. Not all graphs are interval colorable; in fact, quite few families have been proved to admit interval colorings. In this paper we introduce and investigate a new notion, the interval coloring thickness of a graph $G$, denoted $\theta_{\text{int}}(G)$, which is the minimum number of interval colorable edge-disjoint subgraphs of $G$ whose union is $G$.

Our investigation is motivated by scheduling problems with compactness requirements, in particular, problems whose solution may consist of several schedules, but where each schedule must not contain any waiting periods or idle times for all involved parties. We first prove that every connected properly 3-edge colorable graph with maximum degree 3 is interval colorable, and using this result, we deduce an upper bound on $\theta_{\text{int}}(G)$ for general graphs $G$. We demonstrate that this upper bound can be improved in the case when $G$ is bipartite, planar or complete multipartite and consider some applications in timetabling.

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1. Introduction

1.1. Background, motivation and our contribution

The classical graph coloring problem is the problem of assigning positive integers, identified as colors, to the vertices or edges of a graph so that no two adjacent vertices/edges receive the same color. A variety of topics in computer science and operations research such as scheduling, frequency assignment, and register allocation may be formulated as graph coloring problems, see e.g. [8,15,26] and references therein. Many concrete applications involve extra constraints. Consider, for example, the classic school timetabling problem with an additional compactness requirement.

Example 1. In a school we want to schedule lectures so that they are consecutive for both teachers and classes. Constructing a schedule with $t$ time periods satisfying these requirements is equivalent to the problem of finding an edge coloring of the graph $G$, with vertices for teachers and classes, and where every edge represents a lecture given by a certain teacher to a certain class, with colors $1, 2, \ldots, t$ such that the colors of the edges incident to every vertex of $G$ are distinct and form an interval of integers. Such a coloring is called an interval $t$-coloring of $G$. 

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The notion of interval colorings was introduced by Asratian and Kamalian in 1987 [5] (available in English as [6]), motivated by the problem of constructing such timetables. A graph is interval colorable if it has an interval $t$-coloring for some integer $t$. Not all graphs are interval colorable, a simple example is the complete graph $K_3$ with 3 vertices.

Now consider another scenario.

**Example 2.** Suppose that a provincial children’s soccer competition is to take place in a town during a couple of days. In such a competition, teams from all over the province participate and each team takes part in a limited number of matches against competing teams. Moreover, since the competition involves teams from all over the province, for practical reasons, we require that every team’s matches should be consecutive without any waiting periods during each of the competition days. Is it possible to schedule all matches during $k \geq 1$ days so that the matches are consecutive for every team during each day?

In general, a solution to such a scheduling problem as in Example 2, where we ask for a schedule in a total of $t$ time units partitioned into $k \geq 1$ stages, i.e. disjoint time periods, so that a “no-wait” condition holds at each stage, we call a no-wait multi-stage schedule.

We can model Example 2 in graph theoretical terms by forming a graph where vertices represent teams and edges represent matches. (Note that this graph is not complete, since not every team plays against every other team). Then a no-wait multi-stage schedule with $k$ stages exists if and only if there is a decomposition of $G$ into $k$ interval colorable subgraphs, that is, a list of $k$ subgraphs of $G$ such that every edge of $G$ appears in exactly one subgraph in the list. Thus, in general terms, the question of the existence of a no-wait multi-stage schedule can be formulated in graph theoretical terms as follows.

**Problem 1.1.** Let $G$ be a graph and $k$ a positive integer. Is there a decomposition of $G$ into $k$ interval colorable subgraphs?

The minimum integer $k$ for which $G$ admits such a decomposition we call the interval coloring thickness of $G$ and denote by $\theta_{int}(G)$.

In this paper, we introduce and investigate the parameter $\theta_{int}(G)$; our investigation is motivated by scheduling problems whose solutions permit a schedule partitioned into several stages, but each such stage must not contain any waiting periods or idle times for each involved party.

For instance, in the timetabling problem described above, we might ask for a weekly school timetable, where the schedule for each day satisfies the no-wait condition; this application is investigated in some detail in Section 3.2.2. Naturally, there are many more scheduling problems where a no-wait multi-stage schedule is desirable; let us here just consider two further concrete examples. Firstly, we have the following variation of the well-known open shop problem [19] with an additional compactness requirement.

**Example 3.** We are given $m$ processors $P_1, \ldots, P_m$ and $n$ jobs $J_1, \ldots, J_n$ to be processed within a period of $t$ consecutive time units. Each job $J_i$ consists of tasks $T_{i1}, \ldots, T_{im}$ which has to be processed on $P_1, \ldots, P_m$, respectively. For simplicity, we assume that the processing time of each task is 0 or 1. Different tasks of the same job cannot be processed simultaneously and no processor can work on two tasks at the same time. Furthermore, we assume that all tasks $T_{i1}, \ldots, T_{im}$ of $J_i$ are to be executed contiguously, and, similarly, there should be no waiting periods for the processors, i.e. all tasks of $P_j$ should be executed contiguously.

Is it possible to schedule all jobs within a total of $t$ consecutive time units, partitioned into $k$ stages, so that the no-wait condition holds at each stage? That is, rather than requiring that the whole schedule satisfies a no-wait condition, which is a quite strong requirement, we ask for a no-wait multi-stage schedule. Naturally, such a schedule is desirable when a production process may be partitioned into several disjoint time periods, e.g. days. A solution with $k$ stages exists if and only if the bipartite graph $B$ with vertices $J_1, \ldots, J_n$, $P_1, \ldots, P_m$, where the vertices $J_i$ and $P_j$ are joined by an edge if the processing time of the task $T_{ij}$ is 1, satisfies $\theta_{int}(B) \leq k$.

Finally, let us consider the following variation of a problem described by Bodur and Luedtke [7].

**Example 4.** Suppose that some firms organize job interviews for possible candidates during a couple of days. We need to provide the schedule of job interviews where neither firm representatives nor candidates wait between their meetings during these days. If we construct a bipartite graph $H$ with parts $F$ and $C$, where vertices in $F$ represent firms and vertices in $C$ represent candidates, and edges represent the required interviews, then the minimum number of days needed for a schedule of job interviews without waiting periods is precisely equal to $\theta_{int}(H)$.

From a theoretical point of view, the problem of determining whether a (bipartite) graph $G$ has an interval coloring (or, equivalently, whether $\theta_{int}(G) = 1$) is NP-complete [35]. However some classes of graphs have been proved to admit interval colorings; it is known, for example, that trees, regular and complete bipartite graphs [5,20], bipartite graphs with maximum degree at most three [20], doubly convex bipartite graphs [4,23], grids [16], outerplanar bipartite graphs [17] and some classes of biregular bipartite graphs [3,9,10,20,21,24,33,39] have interval colorings.

Due to the NP-hardness of computing the parameter $\theta_{int}(G)$, in this paper we focus on locating tractable instances and proving constructive upper bounds on $\theta_{int}$ for different families of graphs. First we prove that every connected 3-edge...
A colorable graph with maximum degree 3 is interval colorable, and using this result we deduce a general upper bound on the interval coloring thickness of an arbitrary graph. Then we investigate the parameter $\theta_{int}$ for various families of graphs; in particular, we demonstrate how this general upper bound on $\theta_{int}(G)$ can be improved in the case when $G$ is bipartite, planar or complete $r$-partite. We conclude the paper by pointing to some open problems.

1.2. Graph theoretical preliminaries

We use [38] for terminology and notation not defined here. All graphs considered are finite, undirected, allow multiple edges and contain no loops, unless otherwise stated.

A simple graph is a graph without loops and multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees of the vertices of a graph $G$, respectively, and by $d_G(v)$ (or just $d(v)$) the degree of a vertex $v$ in $G$. A graph $G$ is called subcubic if $\Delta(G) \leq 3$. The distance $d_G(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest path between $u$ and $v$ in $G$.

A 2-factor of a graph $G$ (where loops are allowed) is a 2-regular spanning subgraph of $G$. Petersen’s well-known theorem [29] asserts that every 2-regular graph (where loops are allowed) can be decomposed into edge-disjoint 2-factors.

A proper $t$-edge-coloring of a graph $G$ is a mapping $\alpha : E(G) \rightarrow \{1, \ldots, t\}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e$ and $e'$ in $G$. If $e \in E(G)$ and $\alpha(e) = k$ then we say that the edge $e$ is colored $k$. The chromatic index $\chi'(G)$ of a graph $G$ is the minimum number $t$ for which there exists a proper $t$-edge coloring of $G$. The two famous theorems on edge coloring are König’s theorem [25], which states that $\chi'(G) = \Delta(G)$ if $G$ is bipartite, and Vizing’s theorem [37] which asserts that $\chi'(G) \leq \Delta(G) + 1$ for any simple graph $G$. In addition, Shannon’s theorem [36] states that $\chi'(G) \leq \frac{3}{2}\Delta(G)$ for any graph $G$. Vizing’s theorem partitions the set of simple graphs into two sets, namely the graphs $G$ that satisfy $\chi'(G) = \Delta(G)$, and $\chi'(G) = \Delta(G) + 1$, respectively; the latter family of graphs is called Class 2 and the former Class 1.

If $\alpha$ is an edge coloring of $G$ and $v \in V(G)$, then $S_\alpha(v, \alpha)$ (or $S(v, \alpha)$) denotes the set of colors appearing on edges incident to $v$. For two positive integers $a$ and $b$ with $a \leq b$, we denote by $[a, b]$ the interval of integers $\{a, a+1, \ldots, b\}$.

2. A new class of interval colorable graphs

It was shown in [6] that the condition $\chi'(G) = \Delta(G)$ is necessary (but not sufficient) for a graph $G$ to be interval colorable. Here we show that this condition is sufficient for $G$ to be interval colorable if $G$ is a connected subcubic graph; this generalizes the result of Hansen [20] that all bipartite subcubic graphs are interval colorable.

**Theorem 2.1.** Let $G$ be a graph with $\chi'(G) = \Delta(G) \leq 3$. If no component of $G$ is an odd cycle, then $G$ is interval colorable.

Clearly, the condition on odd cycles is necessary. Before proving the theorem, let us also note that it is sharp with respect to the maximum degree; the graph obtained from a triangle $v_1v_2v_3v_1$ by adding a path of length two with a new internal vertex $v_0$ between any pair of distinct vertices $v_i, v_j$ satisfies the condition $\chi'(G) = \Delta(G) = 4$, but is not interval colorable.

**Proof of Theorem 2.1.** If $G$ has maximum degree at most 2, then $G$ is trivially interval colorable, so assume that $\Delta(G) = 3$. Since $\chi'(G) = \Delta(G)$, there is a proper 3-edge coloring of $G$ using colors 1, 2, 3. Let $M$ be the set of edges colored 1.

From this edge coloring of $G$, we define an edge-colored graph $T$ as follows: a vertex $v$ is in $V(T)$ if and only if $v \in V(G)$ and $v$ is a vertex of degree 1 in $G - M$, or $v$ is incident with an edge of $M$ (or both). The edge set of $T$ consists of blue, red and green edges:

- every edge of $M$ is in $E(T)$ and is colored red;
- two vertices of degree 1 that are endpoints of a (maximal) path in $G - M$ of even length are joined by an blue edge in $T$;
- two vertices of degree 1 that are endpoints of a (maximal) path in $G - M$ of odd length are joined by an green edge in $T$.

Note that $T$ is a bipartite graph with maximum degree 2. Moreover, if $C$ is a cycle of $T$, then every second edge of $T$ is red.

**Claim 2.2.** There is a vertex coloring $c : V(T) \rightarrow \{A, B\}$ such that every edge that is in a path or in a cycle of $T$ with an even number of blue edges, satisfies that

(i) both endpoints of a red or green edge are colored by the same color ($A$ or $B$);
(ii) the endpoints of a blue edge are colored by different colors.

Moreover, every edge of a cycle $C$ with an odd number of blue edges satisfies (i) and (ii) except for one red edge $e$ of $C$ whose endpoints are colored by different colors. Additionally, we can select this edge $e$ to be any of the red edges of $C$. 27
**Proof.** The claim trivially holds for any path or cycle with an even number of blue edges; just color the vertices of every component of $T$ sequentially so that they satisfy the conclusion of the claim. That the statement holds for cycles of $T$ with an odd number of blue edges is a straightforward exercise left to the reader. □

We continue the proof of the theorem. Let $c$ be a coloring of $T$ satisfying the conclusion of the claim.

Consider a cycle $C$ in $T$ with an odd number of blue edges. If $C$ corresponds to an even cycle in $G$, then $C$ is good. If $C$ corresponds to an odd cycle $C'$ in $G$, then by assumption, some edge of $M$ must be incident with a vertex of $C'$; that is, some vertex of $C'$ of degree two in $G - M$ is in $T$. Consider a maximal path $P$ in $G - M$ that is in $C'$ and such that some internal vertex $v$ of $P$ is incident with an edge of $M$. We choose the coloring $c$ on $C$ such that a red edge incident with an endpoint $v$ of $P$ is colored by different colors. Moreover, without loss of generality, we assume that $u$ is an internal vertex of $P$ incident with an edge of $M$, such that the distance between $v$ and $u$ is minimal. Finally, assume that if $d_{c}(u, v)$ is odd, then $c(u) \neq c(v)$, and if $d_{c}(u, v)$ is even, then $c(u) = c(v)$. (This can be achieved by possibly swapping all colors under $c$ on the component $C$ if necessary; note that $u$ is not in $C$)

Thus, we may assume that for every cycle $C$ in $T$ with an odd number of blue edges, if $C$ is not good, then the following holds:

There is exactly one red edge $e$ of $C$ whose endpoints are colored differently. Moreover, one endpoint of $e$ is an endpoint of a component $P$ in $G - M$ that is a path, which satisfies that some internal vertex of $P$ is incident with an edge from $M$. Additionally, if $u$ is the internal vertex of $P$ with shortest distance to $v$ that is incident with an edge from $M$, then $u$ and $v$ are colored differently (under $c$) if $d_{p}(u, v)$ is odd, and they are colored by the same color (under $c$) if $d_{p}(u, v)$ is even.

We are now ready to color the edges of $G$; since $G - M$ is a bipartite graph with maximum degree two, its components are paths and even cycles.

For every component $Q$ of $G - M$, we color the edges alternately by colors 2 and 3. If $Q$ is a cycle then this yields a proper coloring of the edges of $Q$: if $Q$ is a path then we start with color 2 at a vertex colored $A$ if $Q$, and color 3 at a vertex of $Q$ colored $B$. This yields a proper coloring of $G - M$ since every odd maximal path of $G - M$ corresponds to a green edge in $T$, and every even maximal path of $G - M$ corresponds to a blue edge.

An edge of $M$ is colored by color 1 if both its endpoints are colored $A$, and by color 4 if both its endpoints are colored $B$. Since an endpoint $x$ of a path $P$ in $G - M$ is incident to an edge colored 2 from $P$ if $c(x) = A$, and an edge colored 3 if $c(x) = B$, this yields an interval coloring of the hitherto edge-colored subgraph of $G$.

It remains to color the edges of $M$ that correspond to red edges whose endpoints are colored differently under $c$. Every such red edge is in a unique cycle in $T$ with an odd number of blue edges. So consider such a cycle $C$ in $T$. If $C$ is good, then we recolor all its edges by colors 2 and 3 alternately. If $C$ is not good, then we consider some cases.

Let $e = uv$ be an edge of $M$ in $C$, whose endpoints are colored differently. Then $v$ is an endpoint of a path $P$ in $G - M$ containing an internal vertex $u$, that is incident with an edge from $M$, and satisfying that $d_{p}(u, v)$ is minimal w.r.t to this property.

- If $d_{p}(u, v)$ is even and $c(u) = c(v) = A$, then we recolor the edges on the portion of $P$ from $u$ to $v$ by colors 0 and 1 alternately, and starting with color 0 at $u$, and color $uv$ by the color 2; if $c(u) = c(v) = B$, then we proceed similarly, but use colors 5 and 4 on the portion of $P$ from $u$ to $v$, and starting with color 5 at $u$, and color $uv$ by the color 3;
- If $d_{p}(u, v)$ is odd and $c(u) = A$ and $c(v) = B$, then we recolor the edges on the portion of $P$ from $u$ to $v$ by colors 0 and 1 alternately, and starting with color 0 at $u$, and color $uv$ by the color 1; if $c(u) = B$ and $c(v) = A$, then we proceed similarly, but use colors 5 and 4 on the portion of $P$ from $u$ to $v$, and starting with color 5 at $u$, and color $uv$ by the color 4.

Recall that red edges of $T$ whose endpoints are colored differently under $c$ are all in different components of $T$. Hence, by repeating this process for every red edge of $T$ whose endpoints are colored differently (under $c$) we obtain an interval edge coloring of $G$. □

**Corollary 2.3.** A connected subcubic graph $G$ is interval colorable if and only if $\chi'(G) = \Delta(G)$.

By König’s edge coloring theorem [25], $\chi'(G) = \Delta(G)$ for any bipartite graph $G$. This and Theorem 2.1 imply the following:

**Corollary 2.4 ([20]).** If $G$ is a bipartite graph with maximum degree at most 3, then $G$ has an interval coloring.

Other families of graphs that satisfy the hypothesis of Theorem 2.1 include the so-called series–parallel graphs, that is, graphs that can be obtained recursively from a single edge by the operations of subdividing and doubling edges. Every simple series–parallel graph with maximum degree at least 3 is Class 1 [28]; thus every such graph is interval colorable. Since every outerplanar graph is series–parallel, this generalizes a result of [31].

### 3. Upper bounds on the interval coloring thickness of graphs

A natural strategy for proving upper bounds on the interval coloring thickness of graphs is to consider decompositions into graphs that are known to admit interval colorings. In this section we present some results in this vein.
3.1. A general upper bound on $\theta_{\text{int}}(G)$

In this section we prove a general upper bound on $\theta_{\text{int}}(G)$ for an arbitrary graph $G$ in terms of its chromatic index. For the proof of this result we need the following lemma, the proof of which is left to the reader.

**Lemma 3.1.** Let $H$ be an interval colorable graph. If $G$ is obtained by

(i) adding a new pendant edge to $H$, or
(ii) adding a cycle $C$ with exactly one common vertex $v$ with $H$ that has degree 1 in $H$,

then $G$ is interval colorable.

Note that Lemma 3.1 implies the following:

**Proposition 3.2.** If $G$ is a connected graph with $\Delta(G) \geq 3$ where any two cycles are vertex-disjoint, then $G$ admits an interval coloring.

Let us now prove the main result of this section.

**Theorem 3.3.** For any graph $G$, $\theta_{\text{int}}(G) \leq 2 \left\lceil \frac{\chi'(G)}{5} \right\rceil$.

**Proof.** Without loss of generality we assume that $G$ is connected.

**Case 1.** $\chi'(G) \leq 5$.

We will show that $\theta_{\text{int}}(G) \leq 2$. Consider a proper 5-edge coloring of $G$ and let $M_i$ denote the set of edges colored $i$, for $i = 1, \ldots, 5$. Furthermore, let $H$ and $F$ be the subgraphs of $G$ induced by the sets $M_1 \cup M_2$ and $M_3 \cup M_4 \cup M_5$, respectively. Clearly, $\Delta(F) \leq 3$ and $H$ is a bipartite graph with $\Delta(H) \leq 2$.

Let $F_0$ be the subgraph of $F$ obtained from $F$ by removing all components from $F$ that are odd cycles. Then $F_0$ is interval colorable by Theorem 2.1.

Now, if $F$ contains no odd cycles, then we are done, because $H$ is interval colorable. Otherwise, the nontrivial components of $F - E(F_0)$ are $F_1$, $\ldots$, $F_t$ where $t \geq 1$ uncolored components $C_i$ that are odd cycles.

Since $G = H \cup F$ is connected, there is a path $P_1$ in $H$ where one endpoint $x_1$ is in $V(F_0)$, and the other endpoint $y_1$ is in one of the odd cycles $C_1, \ldots, C_t$, say $C_1$. Moreover, we assume that $C_1$ is the odd cycle from $\{C_1, \ldots, C_t\}$ with shortest distance to $F_0$ in $G$; then $P_1$ is disjoint from all the cycles $C_2, \ldots, C_t$.

By Lemma 3.1, the graph $F_0 \cup C_1 \cup P_1$ is interval colorable. Put $F_1 = F_0 \cup C_1 \cup P_1$ and $H_1 = H - E(P_1)$. Then $H_1$ is interval colorable as well. The number of components that are odd cycles in $F - V(F_1)$ is $l - 1$; thus by continuing this process we will obtain an interval colorable graph $G$ containing all odd cycles of $F$. Moreover, the subgraph $H_l$ of $G$ is interval colorable because it is bipartite and has maximum degree at most 2. Hence, $\theta_{\text{int}}(G) \leq 2$.

**Case 2.** $\chi'(G) \geq 6$.

Consider a proper edge coloring of $G$ with colors $1, 2, \ldots, \chi'(G)$. Let $t = \left\lceil \frac{\chi'(G)}{5} \right\rceil$ and let $M_i$ denote the set of edges colored $i$, for $i = 1, \ldots, \chi'(G)$. We define the subgraphs $G_1, \ldots, G_t$ of $G$, where $k = \left\lceil \frac{\chi'(G)}{5} \right\rceil$ as follows: For each $j = 1, \ldots, t$, let $G_j$ be the subgraph induced by the set of edges $M_{t-j+1} \cup M_{t-j+2} \cup \cdots \cup M_{t}$; if $t = k$, then $G_k = \bigcup_{i=1}^{k} G_i$; otherwise if $t \neq k$, then $t + 1 = k$. In this case we define $G_k$ as the subgraph induced by the set $\bigcup_{i=1}^{k} M_i$. For each $i = 1, \ldots, k$, $G_i$ is a graph with $\Delta(G_i) \leq 5$ colored with 5 colors. Therefore, by the result in Case 1, $\theta_{\text{int}}(G_i) \leq 2$. Thus $\theta_{\text{int}}(G) \leq 2k = 2 \left\lceil \frac{\chi'(G)}{5} \right\rceil$. □

By the well-known edge coloring theorems by Shannon [36] and Vizing [37] we have the following consequence of Theorem 3.3.

**Corollary 3.4.** (i) For any graph $G$, $\theta_{\text{int}}(G) \leq 2 \left\lceil \frac{\Delta(G)}{10} \right\rceil$.

(ii) For any simple Class 1 graph $G$, $\theta_{\text{int}}(G) \leq 2 \left\lceil \frac{\Delta(G)}{10} \right\rceil$.

(iii) For any simple Class 2 graph $G$, $\theta_{\text{int}}(G) \leq 2 \left\lceil \frac{\Delta(G)+1}{5} \right\rceil$.

It is well-known that the upper bounds in Vizing’s and Shannon’s theorems can be achieved by polynomial-time algorithms (see e.g. [34]), so the proof of Theorem 3.3 combined with these proofs yield efficient algorithms for finding decompositions and colorings attaining the upper bounds in Corollary 3.4.

3.2. Bipartite graphs

There is a prominent line of research on interval colorings of bipartite graphs. This family of graphs is also particularly interesting due to applications in scheduling problems and timetabling. In this section we prove upper bounds on $\theta_{\text{int}}(G)$ for bipartite graphs $G$ and also consider a specific application involving weekly school timetables.
Theorem 3.7. If $G$ is an Eulerian bipartite graph, then $\theta_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{3} \right\rceil$.

Proof of Theorem 3.7. Let $G$ be a bipartite graph with $\Delta(G) = 3$. By Corollary 2.4, $G$ is interval colorable. It is also easy to show that $\theta_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{3} \right\rceil$.

We prove here a slightly stronger result which can be useful in scheduling problems where several schedules of roughly equal "size" is desirable (see Proposition 3.10).

Proposition 3.5. If $G$ is a bipartite graph with $\Delta(G) \geq 4$, then $\theta_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Moreover $G$ can be decomposed into $\left\lceil \frac{\Delta(G)}{2} \right\rceil$ edge-disjoint interval colorable subgraphs in such a way that at each vertex the numbers of incident edges, in any pair of subgraphs, differ by at most one.

Proof. Let $k = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Define a new bipartite graph $H$ by splitting each vertex $v$ of degree at least $k + 1$ into $\left\lceil \frac{d(v)}{k} \right\rceil$ vertices of degree $k$ and possibly (if $d(v) > k \cdot \left\lceil \frac{d(v)}{k} \right\rceil$) one extra vertex of degree $d(v) - k \cdot \left\lceil \frac{d(v)}{k} \right\rceil$. The partitioning of the edges in this splitting is arbitrary, other than ensuring that each vertex receives the correct degree. Since the graph $H$ has maximum degree $k$, it has, by König’s edge coloring theorem [25], a proper $k$-edge coloring. Now collapse $H$ back into $G$ and consider the induced coloring of the edges of $G$. Since each vertex of $G$ is split into at most $3$ vertices, every $v \in V(G)$ can be incident to at most $\left\lceil \frac{d(v)}{2} \right\rceil$ edges of each color. On the other hand, corresponding to each vertex $v$ there were $\left\lceil \frac{d(v)}{k} \right\rceil$ vertices of degree $k$. Thus there must be at least $\left\lceil \frac{d(v)}{k} \right\rceil$ edges of each color incident to $v$. This means that the obtained coloring of $G$ satisfies the following condition: at each vertex the numbers of incident edges, in any pair of colors, differ by at most $1$. Let $G_i$ be the subgraph of $G$ induced by the edges of color $i$, $1 \leq i \leq k$. By construction, the degree of each vertex in $G_i$ does not exceed $3$. Then, by Corollary 2.4, $G_i$ is interval colorable, for $i = 1, 2, \ldots, k$. Therefore $\theta_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. □

We remark that since there are well-known efficient algorithms for finding an optimal proper edge coloring of a bipartite graph, the proof of the preceding proposition combined with the proof of Theorem 2.1, yields a polynomial-time algorithm for finding a decomposition, and interval colorings of the corresponding parts, attaining the upper bound $\theta_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$. This also holds for our next result, which is an improvement of the preceding proposition for Eulerian bipartite graphs. To prove it we need the following lemma.

Lemma 3.6. If $G$ is a bipartite graph with $\Delta(G) = 2r$ and the degree of every vertex in $G$ is 1,2 or $2r$, then $G$ has an interval $2r$-coloring $\alpha$ such that for each $v \in V(G)$ with $d_G(v) = 2$, $\delta_G(v, \alpha) = (2i - 1, 2i)$ for some $1 \leq i \leq r$.

The proof of this lemma is similar to the proof of Theorem 2.1 in [1], so we shall omit it.

Theorem 3.7. If $G$ is an Eulerian bipartite graph, then $\theta_{\text{int}}(G) \leq \left\lceil \frac{\Delta(G)}{4} \right\rceil$.

Proof of Theorem 3.7. For the proof, we construct a new graph $G^*$ as follows: for each vertex $u \in V(G)$ of degree $2k$, we add $\frac{\Delta(G)}{2} - k$ loops at $u$ ($1 \leq k < \frac{\Delta(G)}{2}$). Clearly, $G^*$ is a $\Delta(G)$-regular graph. By Petersen’s theorem [29], $G^*$ can be decomposed into a union of edge-disjoint 2-factors $F_1, \ldots, F_{\frac{\Delta(G)}{2}}$. By removing all loops from the 2-factors $F_1, \ldots, F_{\frac{\Delta(G)}{2}}$, we obtain that the resulting graph $G$ is a union of edge-disjoint subgraphs $F_1', \ldots, F_{\frac{\Delta(G)}{2}}'$, where each $F_i'$ is a collection of even cycles in $G$.

Since the maximum degree in $G$ is even, $\Delta(G) = 4l - 2$ or $\Delta(G) = 4l$, for some integer $l$. We now define the subgraphs $Q_1, \ldots, Q_l$ by setting $Q_i = F_{2i - 1} \cup F_{2i}$, $i = 1, \ldots, l - 1$, and setting $Q_l = F_{2l - 1} \cup F_{2l}$ if $\Delta(G) = 4l$, and $Q_l = F_{2l - 1}$ if $\Delta(G) = 4l - 2$. By Lemma 3.6, each $Q_i$ has an interval coloring; thus $\theta_{\text{int}}(G) \leq l$. □

For a bipartite graph $G$ with parts $X$ and $Y$, we denote the maximum degree of the vertices in $X$ by $\Delta(X)$; $\delta(X)$ denotes the minimum degree of these vertices.

For bipartite graphs where one of the parts $X$ and $Y$ has small maximum degree, the following upper bound is useful.

Proposition 3.8. If $G$ is a bipartite graph with parts $X$ and $Y$, then $\theta_{\text{int}}(G) \leq \min(\Delta(X), \Delta(Y))$.

Proof. Let $G$ be a bipartite graph with parts $X$ and $Y$. We prove that $\theta_{\text{int}}(G) \leq \Delta(X)$; the proof is by induction on $\Delta(X)$.

If $\Delta(X) = 1$, then $G$ is a disjoint union of stars, and thus $\theta_{\text{int}}(G) = 1$. If $\Delta(X) \geq k$, then we form a subgraph $H$ of $G$ by picking one edge incident to each vertex of degree $\Delta(X)$ in $G$. The graph $H$ is a disjoint union of stars, and hence interval colorable. Since $\Delta(G - E(H)) = \Delta(G) - 1$, the desired result now follows by induction. □

The upper bound in Proposition 3.8 is tight, since there are bipartite graphs without interval colorings where all vertices in one part have degree two (see e.g. [32]).
3.2.2. An application in timetabling

In this section we consider a detailed application of multi-stage no-wait schedules in school timetabling.

In a school, there are $m$ teachers $P_1, \ldots, P_m$ and $n$ classes $J_1, \ldots, J_n$. A class consists of a set of students who follow exactly the same program. We are given an $n \times m$ requirement matrix $B = (b_{ij})$ where $b_{ij}$ is the number of lectures involving class $J_i$ and teacher $P_j$. We shall assume that all lectures have the same duration (say one period or one hour) and for every day of the week the lessons can take place in periods 1, 2, 3, \ldots.

A weekly timetable for $k$ days, corresponding to the requirement matrix $B$, is a sequence $S = (S_1, S_2, \ldots, S_k)$ of $k$ arrays with $n$ rows satisfying the following conditions:

(i) each entry of $S_l$ is either one of the members of the set $\{P_1, \ldots, P_m\}$ or is empty, $l = 1, \ldots, k$;
(ii) the total number occurrences of the symbol $P_j$ in the $i$th row of the arrays $S_1, \ldots, S_k$ is precisely $b_{ij}$, for $j = 1, \ldots, m$;
(iii) in each column of $S_l$ all non-empty symbols are different, $l = 1, \ldots, k$.

Each of the arrays $S_1, \ldots, S_k$ is called a daily timetable. If $S_l = (s_{ih}^l)$ and $s_{ih}^l$ is empty then the class $J_i$ has a free lesson in the $h$th period. If however $s_{ih}^l = P_j$ then class $J_i$ has a lesson with teacher $P_j$ that period. In fact, if $k = 1$ then the array $S$ is a daily and weekly timetable simultaneously.

Some results on weekly timetables were found by de Werra [12, 13]. In particular he showed that for any $k \geq 2$ there exists a weekly timetable for $k$ days in which the lessons for each class and each teacher are spread throughout the week as evenly as possible.

Let us now consider timetables without interruptions. We say that the class $J_i$ (the teacher $P_j$) has an interruption in a weekly timetable if there are two periods $j_1$ and $j_2$ at some day of the week, such that $j_1 + 1 < j_2$, the class $J_i$ (the teacher $P_j$) has lessons at the periods $j_1$ and $j_2$, but it is free at the period $j_1 + 1$.

If there are no interruption for each classes and teachers in the daily timetable $S_i$, for $i = 1, \ldots, k$, then we say that $(S_1, S_2, \ldots, S_k)$ is a weekly timetable without interruptions.

Here we are specifically interested in whether there does exist a weekly timetable $(S_1, S_2, \ldots, S_k)$ without interruptions, corresponding to the requirement matrix $B$. The next result shows that this problem is in fact equivalent to Problem 1.1, stated in the introduction, for a bipartite graph $G = G(B)$ with parts $V_1$ and $V_2$, where $V_1 = \{J_1, \ldots, J_n\}$ and $V_2 = \{P_1, \ldots, P_m\}$, and where the vertices $J_i$ and $P_j$ are joined by $b_{ij}$ edges.

**Proposition 3.9.** A weekly school timetable for $k$ days without interruptions, corresponding to the requirement matrix $B$, exists if and only if $\theta_{mi}(G(B)) \leq k$.

**Proof.** Suppose that there exists a weekly timetable $S = (S_1, S_2, \ldots, S_k)$ without interruptions, corresponding to the requirement matrix $B$ where $S_l = (s_{ih}^l)$, $l = 1, \ldots, k$. Then we define an edge colored subgraph $G_i$ of $G$ as the subgraph formed by the edges with ends $J_i$ and $P_j$ colored $h$ under condition that $s_{ih}^l = P_j$, $h = 1, 2, \ldots$. Clearly, such a coloring of $G_i$ is an interval coloring with colors $1, 2, 3, \ldots$.

Conversely, suppose that the graph $G = G(B)$ can be decomposed into a union of $k$ interval colorable subgraphs $G_1, G_2, \ldots, G_k$. Then we can define a weekly timetable $S = (S_1, S_2, \ldots, S_k)$ with $S_l = (s_{ih}^l)$, for $l = 1, \ldots, k$, as follows: $s_{ih}^l = P_j$ if and only if an edge with ends $J_i$ and $P_j$ in $G_i$ is colored $h$, $l = 1, \ldots, k$. \[\Box\]

**Proposition 3.10.** Let $B = (b_{ij})$ be an $n \times m$ requirement matrix, let

$$
\Delta = \max \left\{ \frac{\sum_{i=1}^{n} b_{ij}}{m}, \max_{1 \leq i \leq n} \sum_{j=1}^{m} b_{ij} \right\},
$$

and set $k = \lceil \frac{\Delta}{2} \rceil$. Then there is a weekly school timetable for $k$ days without interruptions in which the lessons for each class and each teacher are spread throughout the week as evenly as possible.

3.2.3. Biregular graphs

Next, we consider biregular graphs; a bipartite graph is $(a, b)$-biregular if all vertices in one part have degree $a$ and all vertices in the other part have degree $b$. Our investigation of biregular graphs is partially motivated by a well-known conjecture which suggests that all biregular graphs have interval colorings [22]. Moreover, biregular graphs arise naturally in some scheduling problems; for instance, in a school timetable (see Section 3.2.2) where all teachers have the same number of lectures, and this also holds for all classes, the problem of constructing a (weekly) timetable can be formulated in terms of an edge coloring problem of a biregular graph.

It is known that all $(2, b)$-biregular [20,21,24] and $(3, 6)$-biregular graphs [10] admit interval colorings. In this section we obtain some upper bounds on the interval coloring thickness of some biregular graphs. We shall need the following corollary of Lemma 3.6.
**Proposition 3.12.** Let $G$ be a bipartite graph.

(i) If $G$ is $(3, 3r)$-biregular $(r \geq 2)$, then $\theta_{\text{int}}(G) \leq 2$.

(ii) If $G$ is $(k, kr)$-biregular $(k \geq 4, r \geq 2)$, then $\theta_{\text{int}}(G) \leq k - 2$.

**Proof.** We first prove the upper bound in (i).

Let $G$ be a $(3, 3r)$-biregular bipartite graph with parts $X$ and $Y$. Define a new graph $H$ from $G$ by splitting each vertex $y \in Y$ into $r$ vertices $y^{(1)}, y^{(2)}, \ldots, y^{(r)}$ of degree 3. The graph $H$ is cubic and bipartite, so by Hall’s matching theorem, it has a perfect matching $M$.

In the graph $G$, $M$ induces a subgraph $F$ in which each vertex $y \in Y$ has degree $r$ and each vertex $x \in X$ has degree 1, so $F$ is interval colorable. Moreover, since $G' = G - E(F)$ is $(2, 2r)$-biregular, it follows from Corollary 3.11 that it has an interval $2r$-coloring. Thus, $\theta_{\text{int}}(G) \leq 2$.

Next, we prove the upper bound in (ii). The proof is by induction on $k$. Let us first consider the base case $k = 4$.

Without loss of generality, we may assume that $G$ is a connected (otherwise, we consider every connected component of $G$) $(4, 4r)$-biregular graph. Since $G$ is bipartite and all vertex degrees in $G$ are even, $G$ has a closed Eulerian trail $C$ with an even number of edges. We color the edges of $G$ with colors red and blue by traversing the edges of $G$ along the trail $C$; we color an odd-indexed edge in $C$ with red color, and an even-indexed edge in $C$ with color blue. Let $G_R$ be the subgraph induced by the red edges, and $G_B$ be the graph induced by the blue edges.

Since $G$ is $(4, 4r)$-biregular, each of the subgraphs $G_R$ and $G_B$ is a $(2, 2r)$-biregular graph. Thus, by Corollary 3.11, $\theta_{\text{int}}(G) \leq 2$.

Let us now assume that $k \geq 5$ and that the statement is true for any $(k', kr)$-biregular bipartite graph $G'$, where $k' < k$. Let $G$ be a $(k, kr)$-biregular bipartite graph with parts $X$ and $Y$. We define a new graph $H$ from $G$ by splitting each vertex $y \in Y$ into $r$ vertices $y^{(1)}, y^{(2)}, \ldots, y^{(r)}$ of degree $k$. The graph $H$ is a $k$-regular and bipartite, so by Hall’s matching theorem, $H$ contains a perfect matching $M$.

In the graph $G$, $M$ induces a subgraph $F$ in which each vertex of $Y$ has degree $r$ and each vertex of $X$ has degree 1, so $F$ is interval colorable. Moreover, the graph $G' = G - E(F)$ is $(1, (k - 1)r)$-biregular, so by the induction hypothesis, $\theta_{\text{int}}(G') \leq k - 3$. Thus, $\theta_{\text{int}}(G) \leq k - 2$. □

### 3.3. Decompositions into forests and complete multipartite graphs

The **arboricity** of a graph $G$, denoted $\gamma(G)$, is the least number of edge-disjoint forests whose union is $G$. Since all trees and forests are interval colorable, $\theta_{\text{int}}(G) \leq \gamma(G)$ for every graph $G$. Thus the arboricity $\gamma(G)$ immediately yields an upper bound on $\theta_{\text{int}}(G)$. It was proved by Nash-Williams [27] that

$$\gamma(G) = \max \left\lceil \frac{|E(G[X])|}{|X| - 1} \right\rceil$$

where the maximum is taken over all nonempty subsets $X \subseteq V(G)$. This result implies several different upper bounds on $\gamma(G)$. We note below only two of them.

**Theorem 3.13 ([14]).** If $G$ is a simple graph with $q$ edges then $\gamma(G) \leq \left\lceil \sqrt{q/2} \right\rceil$.

The following is well-known.

**Proposition 3.14.** If $G$ is a simple planar graph then $\gamma(G) \leq 3$. If, additionally, $G$ is triangle-free or outerplanar, then $\gamma(G) \leq 2$.

Theorem 3.13 and Proposition 3.14 together with the fact that $\theta_{\text{int}}(G) \leq \gamma(G)$ imply the following for simple graphs:

**Proposition 3.15.**

(i) If $G$ is a simple graph with $q$ edges, then $\theta_{\text{int}}(G) \leq \left\lceil \sqrt{q/2} \right\rceil$.

(ii) If $G$ is a simple planar graph then $\theta_{\text{int}}(G) \leq 3$. If, additionally, $G$ is triangle-free or outerplanar, then $\theta_{\text{int}}(G) \leq 2$.

Note that this upper bound for outerplanar graphs is sharp, since odd cycles are not interval colorable, while it is an open question whether there are planar graphs with interval coloring thickness 3.

Note that for some graphs $G$ the number $\gamma(G) - \theta_{\text{int}}(G)$ can be very large. Consider, for example, the complete graph $K_{2n+1}$, $n \geq 1$. It is known [5] that $\theta_{\text{int}}(K_{2n+1}) > 1$. Moreover, for any $u \in V(G)$, the graph $H = K_{2n+1} - u$ is a complete graph with $2n$ vertices which is interval colorable. Since $K_{2n+1} - E(H)$ is a tree, which is an interval colorable graph, $\theta_{\text{int}}(K_{2n+1}) = 2$. 
On the other hand, by the formula of Nash-Williams for $\gamma(G)$ above

$$\gamma(K_{2n+1}) \geq \left\lceil \frac{|E(K_{2n+1})|}{|V(K_{2n+1})| - 1} \right\rceil \geq \left\lceil \frac{(2n + 1)n}{2n} \right\rceil = \left\lceil n + \frac{1}{2} \right\rceil = n + 1.$$

Therefore, for any $n \geq 1$, $\gamma(K_{2n+1}) - \theta(K_{2n+1}) \geq n - 1$.

Instead of decomposing a graph $G$ into trees, we could consider decompositions $G$ into complete bipartite graphs, which are also known to always admit interval colorings. We shall demonstrate this method for the case of complete multipartite graphs.

A simple graph is called complete $r$-partite ($r \geq 2$) if its vertices can be partitioned into $r$ nonempty independent sets $V_1, \ldots, V_r$ such that each vertex in $V_i$ is adjacent to all the other vertices in $V_j$ for $1 \leq i < j \leq r$. Let $K_{n_1,n_2,\ldots,n_r}$ denote a complete $r$-partite graph with independent sets $V_1, V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$.

Using the following proposition, which follows from the so-called Master theorem (see e.g. [11]), we shall deduce an upper bound on $\theta_{int}(K_{n_1,n_2,\ldots,n_r})$ for arbitrary $n_1, \ldots, n_r$.

**Proposition 3.16.** If $T(n)$ is a function defined by the recurrence

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil \right) + 1,$$

then $T(n) = \Theta(\log n)$.

First, we note the following proposition, which follows from the fact that a complete 4-partite graph can be decomposed into two subgraphs, each consisting of two disjoint complete bipartite graphs.

**Proposition 3.17.** If $G$ is a complete multipartite graph with at most four parts, then $\theta_{int}(G) \leq 2$.

Note that Proposition 3.17 is sharp since there are complete 4-partite graphs of Class 2.

**Theorem 3.18.** If $G$ is a complete $r$-partite graph, then $\theta_{int}(G) = O(\log r)$.

**Proof.** Let $V_1, \ldots, V_r$ be nonempty independent sets in $G = K_{n_1,n_2,\ldots,n_r}$ such that each vertex in $V_i$ is adjacent to all the other vertices in $V_j$ for $1 \leq i < j \leq r$. For any set $A \subset V(G)$ that is a union of some $t \geq 2$ subsets $V_{i_1}, \ldots, V_{i_t}$ we denote by $F(A)$ the complete bipartite graph with parts $V_{i_1} \cup \cdots \cup V_{i_t}$ and $V_{i_{t+1}} \cup \cdots \cup V_{i_r}$ where $k = \left\lceil \frac{t}{2} \right\rceil$.

By Proposition 3.16, we may assume that $r \geq 5$. We construct a decomposition of $G$ into edge-disjoint graphs $Q_1, Q_2, \ldots$ where the components of every $Q_i$ are disjoint complete bipartite graphs as follows:

1. Set $Q_1 = F(V_1 \cup V_2 \cup \cdots \cup V_r)$.

2. Set $Q_2 = F(V_1 \cup \cdots \cup V_k) \cup F(V_{1+k} \cup \cdots \cup V_r)$ where $k = \left\lfloor \frac{r}{2} \right\rfloor$.

Step i ($i \geq 3$). Suppose that the graphs $Q_1, \ldots, Q_{i-1}$ have been constructed and $Q_{i-1} = H_1 \cup \cdots \cup H_l$ where $H_j$ is a complete bipartite graph with parts $A_j$ and $B_j$, $j = 1, \ldots, l$, such that all sets $A_1, \ldots, A_l, B_1, \ldots, B_l$ are mutually disjoint.

If every $A_j$ is one of the sets $V_1, \ldots, V_r$ and every $B_j$ is also one of the sets $V_1, \ldots, V_r$, then STOP the algorithm. Otherwise we construct the bipartite graph $F(A_j)$ for each set $A_j$ containing at least 2 of subsets $V_1, \ldots, V_r$, $1 \leq j \leq l$, and the bipartite graph $F(B_j)$ for each set $B_j$ containing at least 2 of subsets $V_1, \ldots, V_r$, $1 \leq j \leq l$. (Note that each such set induces a complete multipartite graph.) Now define $Q_i$ as a union of all these bipartite graphs. Formally, we set

$$Q_i = \left( \bigcup_{j \in [V_1, \ldots, V_r]} F(A_j) \right) \cup \left( \bigcup_{j \in [V_1, \ldots, V_r]} F(B_j) \right).$$

This algorithm stops when all edges of $G$ will be in one of the complete bipartite graphs constructed by the algorithm. We denote by $T(r)$ the number of bipartite graphs $Q_1, Q_2, \ldots$ constructed by this algorithm. Clearly,

$$T(r) = T\left(\left\lceil \frac{r}{2} \right\rceil \right) + 1,$$

so by Proposition 3.16, $T(r) = \Theta(\log r)$.

Since every component of the graph $Q_i$ in the constructed decomposition of $G = K_{n_1,n_2,\ldots,n_r}$ is a complete bipartite graph, all $Q_i$ are interval colorable, and therefore $\theta_{int}(G) \leq T(r) = \Theta(\log r)$. \qed

Finally, let us note some further upper bounds on the interval coloring thickness of complete multipartite graphs. A complete $r$-partite graph $K_{n_1,n_2,\ldots,n_r}$ is called balanced and denoted by $K_{n,\ldots,n}$ if $n_1 = \cdots = n_r = n$.

Also, let $K_{n,r,a}$ denote the complete $(r + 1)$-partite graph with $n$ vertices in each of the first $r$ parts and $a$ vertices in the last part. In [30], the following consequence on the chromatic index of a balanced complete multipartite graph was observed.

**Proposition 3.19.** For any $n, r \in \mathbb{N}$ ($r \geq 2$), $K_{n,r,a}$ has an interval coloring if and only if $nr$ is even. Moreover, if $nr$ is even, then $K_{n,r,a}$ has an interval $(r - 1)n$-coloring.
Using the preceding proposition, we deduce the following two results.

**Proposition 3.20.** For any \( n, r \in \mathbb{N} \) \((r \geq 2)\),
\[
\theta_{\text{int}}(K_{n \times t}) = \begin{cases} 
1, & \text{if } nr \text{ is even}, \\
2, & \text{if } nr \text{ is odd}.
\end{cases}
\]

**Proposition 3.20** follows from **Proposition 3.19** together with the fact that if \( nr \) is odd then, \( K_{n \times r} \) can be decomposed into a copy of \( K_{n \times (r - 1)} \) and a complete bipartite graph.

**Proposition 3.21.** For any \( n, r \in \mathbb{N} \) \((r \geq 2)\), \( \theta_{\text{int}}(K_{n \times t, n}) \leq 3 \). Moreover, if \( nr \) is even, then \( \theta_{\text{int}}(K_{n \times t, n}) = 1 \).

The upper bound follows from the previous proposition, since \( K_{n \times t, n} \) can be decomposed into a copy of \( K_{n \times t} \) and a balanced complete bipartite graph.

The second part follows by considering the same decomposition and using **Proposition 3.20** for obtaining an interval \((r - 1)n\)-coloring of the copy of \( K_{n \times t} \), and then defining a suitable interval coloring of the balanced complete bipartite graph.

### 4. Concluding remarks and open problems

In this paper we have introduced and investigated the interval coloring thickness of a graph and considered some concrete applications from scheduling theory. We have presented a new class of interval colorable graphs and proved an upper bound on the interval coloring thickness of a general graph. Furthermore, we have proved improvements of this general upper bound for several families of graphs. Let us now point to some open problems, and also describe some further tractable instances.

There are several well-known examples of families of graphs that do not admit interval colorings, such as odd cycles and complete graphs of odd order. However, the answer to the following question remains open.

**Problem 4.1.** For any positive integer \( k \), is there a graph \( G \) with \( \theta_{\text{int}}(G) = k \)?

The version of this problem when the graph is assumed to be bipartite is perhaps of particular interest. In [2,18,32], the authors consider a variety of families of bipartite graphs without interval colorings; these families include, the so-called Malafiejski rosettes, the Sevast’janov rosettes and the generalized Hertz graphs. All graphs from these families have interval coloring thickness two, since they all decompose into two forests. Note further that any example of a bipartite graph with interval coloring thickness \( k \) must necessarily contain at least \( 3k + 1 \) vertices in each part, since every bipartite graph where one part contains at most three vertices is interval colorable [18].

Based on an example first described by Erdős, in [32] the authors present a family of bipartite graphs, constructed from projective planes, that do not admit interval colorings. The interval coloring thickness of graphs from this family is unknown. A solution to the following problem could perhaps shed some light on **Problem 4.1**.

**Problem 4.2.** Determine the interval coloring thickness of the so-called Erdős family of graphs described in Section 2.2 in [32].

Some graphs which are not interval colorable, permit a proper edge coloring that satisfy a slightly weaker condition. A proper \( t \)-edge coloring of a graph \( G \) is called a cyclic interval \( t \)-coloring if for each vertex \( v \) of \( G \) the edges incident to \( v \) are colored by consecutive colors, under the condition that color 1 is considered as consecutive to color \( t \), see e.g. [1]. We note the following proposition for graphs with a cyclic interval coloring.

**Proposition 4.3.** If \( G \) has a cyclic interval \( t \)-coloring with \( t \geq 2\Delta(G) - 2 \), then \( \theta_{\text{int}}(G) \leq 2 \).

**Proposition 4.3** can be proved as follows: Let \( \alpha \) be a cyclic interval \( t \)-coloring of \( G \); if \( \alpha \) is an interval coloring, then we are done; otherwise, there is some vertex \( v \in V(G) \) such that \( S(v, \alpha) \) is not an interval, that is \( S(v, \alpha) = \{1, \ldots , k_v\} \cup \{t - l_v + 1, \ldots , t\} \), for some integers \( k_v, l_v \). Let \( k^* \) be the maximum integer \( k_\* \) such that \( S(x, \alpha) = \{1, \ldots , k_\*\} \cup \{t - l_\* + 1, \ldots , t\} \), for some vertex \( x \), and let \( l^* \) be the maximum integer \( l^* \) such that \( S(y, \alpha) = \{1, \ldots , k_\*\} \cup \{t - l_\* + 1, \ldots , t\} \), for some vertex \( y \) (where \( S(x, \alpha) \) and \( S(y, \alpha) \) are not intervals). Since \( t \geq 2\Delta(G) - 2 \), \( k^* < t - l^* + 1 \), and thus \( G \) decomposes into two interval colorable subgraphs.

An example of a family of graphs satisfying **Proposition 4.3** is the so-called circular complete graphs \( K_{p/q} \) \((p \geq 2q)\) defined by setting \( V(K_{p/q}) = \{v_0, v_1, \ldots , v_{p-1}\} \) and
\[
E(K_{p/q}) = \{v_i v_j : v_i, v_j \in V(K_{p/q}) \text{ and } q \leq |i - j| \leq p - q\}.
\]

In general, we are interested in whether the following might be true.

**Problem 4.4.** Is it true that every graph \( G \) that admits a cyclic interval coloring satisfies \( \theta_{\text{int}}(G) \leq 2 \)?
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