JUMPING CONICS
ON A SMOOTH QUADRIC IN \( \mathbb{P}_3 \)

SUKMOON HUH

Abstract. We investigate the jumping conics of stable vector bundles \( E \) of rank 2 on a smooth quadric surface \( Q \) with the first Chern class \( c_1 = \mathcal{O}_Q(-1, -1) \) with respect to the ample line bundle \( \mathcal{O}_Q(1, 1) \). We show that the set of jumping conics of \( E \) is a hypersurface of degree \( c_2(E) - 1 \) in \( \mathbb{P}_3 \). Using these hypersurfaces, we describe moduli spaces of stable vector bundles of rank 2 on \( Q \) in the cases of lower \( c_2(E) \).

1. Introduction

The moduli space of stable sheaves on surfaces has been studied by many people. Especially, over the projective plane, the moduli space of stable sheaves of rank 2 was studied by W.Barth \[1\] and K.Hulek \[9\], using the jumping lines and jumping lines of the second kind. In \[15\], this idea was generalized to the jumping conics on the projective plane. In this article, we use the concept of jumping conics on the smooth quadric surface.

Let \( Q \) be a smooth quadric in \( \mathbb{P}_3 = \mathbb{P}(V) \), where \( V \) is a 4-dimensional vector space over complex numbers \( \mathbb{C} \), and \( \mathcal{M}(k) \) be the moduli space of stable vector bundles of rank 2 on \( Q \) with the Chern classes \( c_1 = \mathcal{O}_Q(-1, -1) \) and \( c_2 = k \) with respect to the ample line bundle \( H = \mathcal{O}_Q(1, 1) \). \( \mathcal{M}(k) \) form an open Zariski subset of the projective variety \( \overline{\mathcal{M}}(k) \) whose points correspond to the semi-stable sheaves on \( Q \) with the same numerical invariants. The Zariski tangent space of \( \mathcal{M}(k) \) at \( E \), is naturally isomorphic to \( H^1(Q, \text{End}(E)) \) and so the dimension of \( \mathcal{M}(k) \) is equal to \( h^1(Q, \text{End}(E)) = 4k - 5 \), since \( E \) is simple.

Using the Beilinson-type theorem on \( Q \) \[3\], we obtain the following monad for \( E \in \mathcal{M}(k) \),

\[
0 \rightarrow \mathbb{C}^{k-1} \otimes \mathcal{O}_Q(-1, -1) \rightarrow \mathbb{C}^k \otimes (\mathcal{O}_Q(0, -1) \oplus \mathcal{O}_Q(-1, 0)) \rightarrow \mathbb{C}^{k-1} \otimes \mathcal{O}_Q \rightarrow 0,
\]

with the cohomology sheaf \( E \), where the first injective map derives a map

\[
\delta : H^1(E(-1, -1)) \otimes V^* \rightarrow H^1(E).
\]

As in \[2\], we similarly define \( S(E) \subset \mathbb{P}^*_3 \), the set of jumping conics of \( E \), and prove that \( S(E) \) is a hypersurface in \( \mathbb{P}^*_3 \) of degree \( k - 1 \) whose equation is

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given by \( \det \delta(z) = 0, \) where \( \delta(z) \) is a symmetric \((k - 1) \times (k - 1)\)-matrix. We give a criterion for \( H \in \mathbb{P}^3 \) to be a singular point of \( S(E) \) and calculate the exact number of singular points of \( S(E) \) when \( E \) is a Hulsbergen bundle, i.e. \( E \) admits the following exact sequence,

\[
0 \to O_Q \to E(1, 1) \to I_Z(1, 1) \to 0,
\]

where \( Z \) is a 0-cycle on \( Q \) with length \( k \) whose support is in general position.

In Section 4, we describe the above results in the cases \( c_2 \leq 3 \) by investigating the map \( S : \mathcal{M}(k) \to |O_{\mathbb{P}^3}(k - 1)|, \) sending \( E \) to \( S(E) \). When \( c_2 = 2 \), \( S(E) \) is a hypersurface in \( \mathbb{P}^3 \) and \( \mathcal{M}(2) \) is isomorphic to \( \mathbb{P}^3 \setminus Q \) via \( S \), which was already shown in [8]. In the case of \( c_2 = 3 \), we investigate the surjective map from \( \mathcal{M}(3) \) to \( \mathbb{P}^3 \), sending \( E \) to the vertex point of the quadric cone \( S(E) \subset \mathbb{P}^3 \) to give an explicit description of \( \mathcal{M}(3) \). In fact, the generic fibre of this map over \( H \in \mathbb{P}^3 \) is isomorphic to the set of smooth conics which are Poncelet related to the smooth conic \( H \cap Q \). As a result, we can observe that \( S \) is an isomorphism from \( \mathcal{M}(3) \) to its image and in particular, when \( c_2 = 2, 3 \), the set of jumping conics, \( S(E) \), determines \( E \) uniquely.

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2. The Beilinson Theorem and Jumping Conics

2.1. The Beilinson Theorem. Let \( V_1 \) and \( V_2 \) be two 2-dimensional vector spaces with the coordinate \([x_1]\) and \([x_2]\), respectively. Let \( Q \) be a smooth quadric isomorphic to \( \mathbb{P}(V_1) \times \mathbb{P}(V_2) \) and then it is embedded into \( \mathbb{P}_3 \simeq \mathbb{P}(V) \) by the Segre map, where \( V = V_1 \otimes V_2 \). Let us denote \( f^*O_{\mathbb{P}_3}(a) \otimes g^*O_{\mathbb{P}_3}(b) \) by \( O_Q(a,b) \) and \( E \otimes O_Q(a,b) \) by \( E(a,b) \) for coherent sheaves \( E \) on \( Q \), where \( f \) and \( g \) are the projections from \( Q \) to each factors. Then the canonical line bundle \( K_Q \) of \( Q \) is \( O_Q(-2,-2) \).

Definition 2.1. For a fixed ample line bundle \( H \) on \( Q \), a torsion free sheaf \( E \) of rank \( r \) on \( Q \) is called stable (resp. semi-stable) with respect to \( H \) if

\[
\chi(F \otimes O_Q(mH)) \bigg/ r' < (resp. \leq) \chi(E \otimes O_Q(mH)) \bigg/ r,
\]

for all non-zero subsheaves \( F \subset E \) of rank \( r' \).

Let \( \overline{\mathcal{M}}(k) \) be the moduli space of semi-stable sheaves of rank 2 on \( Q \) with the Chern classes \( c_1 = O_Q(-1,-1) \) and \( c_2 = k \) with respect to the ample line bundle \( H = O_Q(1,1) \). The existence and the projectivity of \( \overline{\mathcal{M}}(k) \) is known in [6] and it has an open Zariski subset \( \mathcal{M}(k) \) which consists of the stable vector bundles with the given numeric invariants. By the Bogomolov theorem, \( \mathcal{M}(k) \) is empty if \( 4k < c_1^2 = 2 \) and in particular, we can consider
only the case $k \geq 1$. Note that $E \simeq E^*(-1, -1)$ and by the Riemann-Roch theorem, we have
\[
\chi_E(m) := \chi(E(m, m)) = 2m^2 + 2m + 1 - k,
\]
for $E \in \mathcal{M}(k)$.

Using the same trick as in the proof of the Beilinson theorem on the vector bundles over the projective space [13], we can obtain similar statement over $Q$.

**Proposition 2.2.** [3] For any holomorphic bundle $E$ on $Q$, there is a spectral sequence
\[
E_{1}^{p,q} \Rightarrow E_{\infty}^{p+q} = \begin{cases} E, & \text{if } p + q = 0; \\ 0, & \text{otherwise}, \end{cases}
\]
with
\[
\begin{align*}
E_{1}^{p,q} &= 0, \quad |p + 1| > 1 \\
E_{0}^{0,q} &= H^q(E) \otimes O_Q \\
E_{1}^{-2,q} &= H^q(E(-1, -1)) \otimes O_Q(-1, -1),
\end{align*}
\]
and an exact sequence
\[
\cdots \to H^q(E(0, -1)) \otimes O_Q(0, -1) \to E_{1}^{-1,q} \to H^q(E(-1, 0)) \otimes O_Q(-1, 0) \to \cdots.
\]

**Proof.** Let $p_1$ and $p_2$ be the projections from $Q \times Q$ to each factors and denote $p_1^*O_Q(a, b) \otimes p_2^*O_Q(c, d)$ by $O(a, b)(c, d)'$. If we let $\Delta$ be the diagonal of $Q \times Q$, we have the following Koszul complex,
\[
\begin{align*}
1 \quad 0 \to O(-1, -1)(-1, -1)' \to \bigoplus_{i=0}^{1} O(-i, 1 - i)(-i, 1 - i)' \to O \to O_\Delta.
\end{align*}
\]
If we tensor it with $p_2^*E$, then we have a locally free resolution of $p_2^*E|_\Delta$. If we take higher direct images under $p_1$, we get the assertion by the standard argument on the spectral sequence. \qed

From the stability condition of $E \in \mathcal{M}(k)$, we have $H^0(E(a, b)) = 0$ whenever $a + b \leq 0$. Hence $E_{1}^{p,q} = 0$ for $p = -2, -1, 0$ and $q = 0, 2$ and thus the proposition gives us a monad
\[
M : 0 \to K_{1,1} \otimes O_Q(-1, -1) \to E_{1}^{-1,1} \to K_{0,0} \otimes O_Q \to 0,
\]
with the cohomology sheaf $E(M) = E$, where $K_{a,b} = H^1(E(-a, -b))$ and $E_{1}^{-1,1}$ fits into the following exact sequence,
\[
0 \to K_{0,1} \otimes O_Q(0, -1) \to E_{1}^{-1,1} \to K_{1,0} \otimes O_Q(-1, 0) \to 0.
\]
Since $H^1(O_Q(1, -1)) = 0$, this exact sequence splits. Thus we have the following corollary.
Corollary 2.3. Let $E \in \mathcal{M}(k)$. Then $E$ becomes the cohomology sheaf of the following monad:

$$
M(E) : 0 \rightarrow K_{1,1} \otimes O_Q(-1, -1) \rightarrow \bigoplus_{i=0}^{1}(K_{i,1-i} \otimes O_Q(-i, -1+i)) \rightarrow K_{0,0} \otimes O_Q \rightarrow 0.
$$

Note that $k_{1,1} = k_{0,0} = k - 1$ and $k_{1,0} = k_{0,1} = k$, where $k_{i,j} = \dim K_{a,b}$. Let us denote by $a$, the first injective map in the monad in the corollary (2.3). Since $E \simeq E^*(-1, -1)$, the last surjective map is the dual of $a$, twisted by $O_Q(-1, -1)$ and thus the monad $M$ is completely determined by $a$. The monomorphism $a$ corresponds to an element $\alpha$ in

$$
K_{1,1}^* \otimes ((K_{0,1} \otimes V_1) \oplus (K_{1,0} \otimes V_2)),
$$
i.e. $\alpha = (\alpha_1, \alpha_2)$, where $\alpha_2 \in \text{Hom}(V_1^*, \text{Hom}(K_{1,1}, K_{1,-2-i}))$. Since $K_{1,1}^* \simeq K_{0,0}$ and $K_{1,0} \simeq K_{0,1}$, we can obtain a map

$$
\delta : V_1^* \otimes V_2^* \rightarrow \text{Hom}(K_{1,1}, K_{0,0}),
$$
defined by $\delta := \alpha_2^t \circ \alpha_1 + \alpha_1^t \circ \alpha_2$. So $\delta(z) \in K_{0,0} \otimes K_{0,0}$ since $K_{1,1}^* \simeq K_{0,0}$. Again from the self-duality of $E$, i.e. $E \simeq E^*(-1, -1)$, we have $\delta(z) = \delta(z)^t$.

In other words, $\delta(z)$ is an element in $\text{Sym}^2(K_{0,0})$ for all $z$.

2.2. Jumping Conics. Let $H$ be a general hyperplane section of $\mathbb{P}_3$ and then $C_H := Q \cap H$ is a conic on $H$. Let $E$ be a vector bundle of rank $r$ on $Q$. If we choose an isomorphism $f : \mathbb{P}_1 \rightarrow C_H$, then due to Grothendieck, we have

$$
f^*E|_{C_H} \simeq O_{\mathbb{P}_1}(a_1) \oplus \cdots \oplus O_{\mathbb{P}_1}(a_r),
$$
where $a_{E,H} := (a_1, \cdots, a_r) \in \mathbb{Z}^r$ such that $a_1 \geq \cdots \geq a_r$. Here, $a_{E,H}$ is called the splitting type of $E|_{C_H}$.

Definition 2.4. A conic $C_H = Q \cap H$ on $Q$, is called a jumping conic of $E$ if the splitting type $a_{E,H}$ of $E|_{C_H}$ is different from the generic splitting type $a_E$. We will denote the set of jumping conics of $E$ by $S(E) \subset \mathbb{P}_3^*.$

Remark 2.5. The above definition is valid only for the general hyperplane sections $H$. Later, we give an equivalent definition for the jumping conics for arbitrary case, using the cohomological criterion.

From the theorem (0.2) in [12], we have $a_i - a_{i+1} \leq 2$ for all $i$ since the degree of $Q \subset \mathbb{P}_3$ is 2. From the following proposition, we know that this upper bound can be sharpened to be 1.

Proposition 2.6. If $E$ is a stable vector bundle on $Q$ of rank $r$, we have

$$
a_i - a_{i+1} \leq 1, \text{ for all } i,
$$
where $a_E = (a_1, \cdots, a_r)$.
We consider the incidence variety \( \mathbf{I} = \{(x, H) \in \mathbb{P}^3 \times \mathbb{P}^3 \mid x \in C_H \} \) with the projections \( \pi_1 \) and \( \pi_2 \) to each factors. Suppose that \( i \) is the first index such that \( a_i - a_{i+1} \geq 2 \). Moreover, we can assume that \( a_i = 0 \). Now, let us consider the natural map
\[
\pi_2^* \pi_2^* \pi_1^* E \to \pi_1^* E
\]
and \( E_1 \) be the image of this map. Then \( E_1 \) is a subsheaf of \( \pi_1^*(E) \) on \( \mathbf{I} \) of rank \( i \) such that
\[
f^* E_1 \mid_{\pi_2^{-1}(H)} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_i),
\]
for a general \( H \in \mathbb{P}^3 \) and an isomorphism \( f : \mathbb{P}^1 \to C_H \). Then the quotient sheaf \( E_2 := \pi_1^*(E)/E_1 \) is of rank \( r - i \) with
\[
f^* E_2 \mid_{\pi_2^{-1}(H)} \simeq \mathcal{O}_{\mathbb{P}^1}(a_{i+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r),
\]
for a general \( H \in \mathbb{P}^3 \). From the following lemma, the pull-back of the relative tangent bundle \( T_{\mathbf{I}|Q} \) to \( \mathbb{P}^1 \), is \( \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \). Hence, the restriction of the sheaf \( \text{Hom}(T_{\mathbf{I}|Q}, \text{Hom}(E_1, E_2)) \) to \( C_H \) is isomorphic to the direct sum of \( \mathcal{O}_{\mathbb{P}^1}(a_{j_1} - a_{j_2} + 1)^{\oplus 2} \) where \( j_1 \geq i + 1 \) and \( j_2 \leq i \). In particular, we have
\[
\text{Hom}(T_{\mathbf{I}|Q}, \text{Hom}(E_1, E_2)) = 0.
\]
By the Descente-Lemma \cite{13}, there exists a subsheaf of \( E' \) of \( E \) on \( Q \) such that \( \pi_1^* E' = E_1 \) and it would make the contradiction to the stability of \( E \).

**Lemma 2.7.** Let \( f : \mathbb{P}^1 \to C_H \) be an isomorphism. Then we have an isomorphism
\[
f^* T_{\mathbf{I}|Q}|_{C_H} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}.
\]

**Proof.** Let \( \mathbf{I}' \) be the incidence variety in \( \mathbb{P}^3 \times \mathbb{P}^3 \), i.e. \( \mathbf{I}' \simeq \mathbb{P}(T_{\mathbb{P}^3}^*) \). Then we have the universal exact sequence of \( \mathbb{P}(T_{\mathbb{P}^3}^*) \),
\[
0 \to M \to \pi_1^* T_{\mathbb{P}^3}^* \to N \to 0,
\]
where \( M \) and \( N \) are vector bundles on \( \mathbf{I}' \) of rank 1 and 2, respectively. If we restrict the universal sequence to \( \pi_2^{-1}(\{H\}) \), then we obtain
\[
0 \to N^*_{H|\mathbb{P}^3} \to T_{\mathbb{P}^3}^*|_{H} \to T_H^* \to 0,
\]
where \( N_{H|\mathbb{P}^3} \simeq \mathcal{O}_H(1) \) is the normal bundle of \( H \) in \( \mathbb{P}^3 \). Note that
\[
T_{\mathbf{I}'|\mathbb{P}^3}|_H \simeq \text{Hom}(M, N)|_H \simeq N_{H|\mathbb{P}^3} \otimes T_H^*.
\]
Since \( f^* N_{H|\mathbb{P}^3} \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1}(2) \), it is enough to prove that
\[
f^* T_H^* \simeq f^* \Omega_H \simeq \mathcal{O}_{\mathbb{P}^1}(-3)^{\oplus 2}
\]
since \( T_{\mathbf{I}'|\mathbb{P}^3} \) is the restriction of \( T_{\mathbf{I}'|\mathbb{P}^3} \) to \( \mathbf{I} \). If we tensor the following exact sequence,
\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{C_H} \to 0,
\]
with \( \Omega_{\mathbb{P}^2}(1) \) (recall that \( H \simeq \mathbb{P}^2 \) and \( C_H \) is the image conic of \( \mathbb{P}^1 \) by \( f \)) and take the long exact sequence of cohomology, then we obtain \( H^0(\Omega_{\mathbb{P}^2}(1)|_{C_H}) = 0 \) from the Bott theorem \([13]\). Thus we have

\[
h^0(f^*(\Omega_{\mathbb{P}^2}(1))) = h^0(\Omega_{\mathbb{P}^2}(1)|_{C_H}) = 0.
\]

Since \( c_1(f^*(\Omega_{\mathbb{P}^2}(1))) = -2 \), the only possibility is that \( f^*(\Omega_{\mathbb{P}^2}(1)) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus 2 \) and so \( f^*T_H \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \oplus 2 \).

**Remark 2.8.** From the theorem 1 in \([15]\), we know that for a semistable vector bundle of rank 2 on \( \mathbb{P}^2 \) and a general smooth conic \( f: \mathbb{P}^1 \to \mathbb{P}^2 \), we have

\[
f^*E \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus 2, & \text{if } c_1(E) = 0; \\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus 2, & \text{if } c_1(E) = -1. \end{cases}
\]

Since \( T_{\mathbb{P}^2} \) is semistable, we can also obtain \( f^*T_{\mathbb{P}^2} \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \oplus 2 \) for a general conic. In fact, this is true for all conics and \( T_{\mathbb{P}^2} \) is the only vector bundle of rank 2 on \( \mathbb{P}^2 \) up to twists with the same splitting type over all smooth conics except a direct sum of two line bundles \([11]\).

Let us assume that \( E \in \mathcal{M}(k) \) be a stable vector bundle on \( Q \). As an immediate consequence of the above proposition, we get the following corollary.

**Corollary 2.9.** For a general conic \( C_H \) on \( Q \), we have

\[
E|_{C_H} \simeq \mathcal{O}_{C_H}(-p) \oplus \mathcal{O}_{C_H}(-p),
\]

where \( p \) is a point on \( C_H \).

In particular, the jumping conics of \( E \) can be characterized by

\[
h^0(\mathcal{O}_{C_H}) \neq 0,
\]

and we will use this cohomological criterion as the definition of the jumping conics of \( E \).

We consider the exact sequence,

\[
0 \to E(-1, -1) \to E \to E|_{C_H} \to 0,
\]

to derive the following long exact sequence,

\[
0 \to H^0(E|_{C_H}) \to H^1(E(-1, -1)) \to H^1(E),
\]

where the last map is given by \( \delta(z) = \alpha_2^* \otimes \alpha_1 + \alpha_1^* \otimes \alpha_2 \), where \( z \) is the coordinates determining the hyperplane section \( H \). Hence \( C_H \) is a jumping conic if and only if \( \det(\delta) = 0 \). Note that \( \det(\delta) \) is a homogeneous polynomial of degree \( c_2(E) - 1 \) with the coordinates of \( V_1^* \otimes V_2^* \). This determinant does not vanish identically due to the proposition \([2.6]\), and so we obtain that \( S(E) \) is a hypersurface of degree \( c_2(E) - 1 \) in \( \mathbb{P}_3^* \). From the fact that \( \delta(z) = \delta(z)^4 \), we obtain the following statement.

**Theorem 2.10.** \( S(E) \) is a symmetric determinantal hypersurface of degree \( c_2(E) - 1 \) in \( \mathbb{P}_3^* \).
The natural question on $S(E)$ is the smoothness and the next proposition will give an answer to this question.

**Proposition 2.11.** If $h^0(E|_{C_H}) \geq 2$, then $H \in \mathbb{P}_3^*$ is a singular point of $S(E)$.

**Proof.** The statement is clear from the theory on the singular locus of symmetric determinantal varieties [7]. Indeed, let $M = M_0$ denote the projective space $\mathbb{P}_N$ of $(k - 1) \times (k - 1)$ matrices up to scalars and $M_i$ be the locus of matrices of corank $i$ or more. Let us consider a map $\varphi : \mathbb{P}_3^* \to M_i$ determined naturally by $\delta$. If we let $S_i$ be the preimage of $M_i$ via $\varphi$, then we have

$$T_pS_2 = d\varphi^{-1}(T_q\varphi(S_2)) = d\varphi^{-1}(T_qM_2 \cap T_q\varphi(\mathbb{P}_3^*)) = d\varphi^{-1}(M \cap T_q\varphi(\mathbb{P}_3^*)),$$

where $q = \varphi(p)$ and $p \in S_2$. In particular, $S_2$ is the singular locus of $S_1 = S(E)$. \qed

**Remark 2.12.** Let $f : \mathbb{P}_1 \to C_H \subset Q$ be a smooth conic on $Q$ and assume that we have

$$f^*E|_{C_H} \simeq \mathcal{O}_{\mathbb{P}_1}(-1 - i) \oplus \mathcal{O}_{\mathbb{P}_1}(-1 + i),$$

where $i$ is a nonnegative integer. Note that $k = h^0(E|_{C_H}) = \text{corank}(\delta(z))$, where $z$ is the coordinates of $H$. If $i \geq 2$, then $H \in \mathbb{P}_3^*$ is a singular point of $S(E)$.

Now for later use, let us define a sheaf supported on $S(E)$. As in [1], we can see that $S(E)$ is the support of the $\mathcal{O}_{\mathbb{P}_3^*}$-sheaf $\vartheta_E(1)$ defined by the following exact sequence,

$$0 \to K_{1,1} \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \to K_{0,0} \otimes \mathcal{O}_{\mathbb{P}_3^*} \to \vartheta_E(1) \to 0. \tag{8}$$

The first injective map is composed of

$$K_{1,1} \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \to K_{1,1} \otimes (V_1^* \otimes V_2^*) \otimes \mathcal{O}_{\mathbb{P}_3^*} \to K_{0,0} \otimes \mathcal{O}_{\mathbb{P}_3^*},$$

where the first map is from the Euler sequence over $\mathbb{P}_3^*$ and the second map is from the map $\delta$. So $\vartheta_E$ is an $\mathcal{O}_{S(E)}$-sheaf.

From the incidence variety $I \subset Q \times \mathbb{P}_3^*$, we obtain

$$0 \to \pi_1^*\mathcal{O}_Q(-1, -1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}_3^*}(-1) \to \mathcal{O}_{Q \times \mathbb{P}_3^*} \to \mathcal{O}_I \to 0.$$

If we tensor it with $\pi_1^*E$ and take the direct image of it, we obtain,

$$0 \to K_{1,1} \otimes \mathcal{O}_{\mathbb{P}_3^*}(-1) \to K_{0,0} \otimes \mathcal{O}_{\mathbb{P}_3^*} \to R^1\pi_2*\pi_1^*E \to 0.$$

Since this exact sequence coincide with the sequence [3], we have

**Lemma 2.13.** $\vartheta_E(1) \simeq R^1\pi_2*\pi_1^*E$. 

3. Examples

Let \( \mathcal{M}(k) \) be the moduli space of stable vector bundles of rank 2 on \( Q \) with the Chern classes \( c_1 = \mathcal{O}_Q(-1,-1) \) and \( c_2 = k \) with respect to the ample line bundle \( \mathcal{O}_Q(1,1) \). The dimension of \( \mathcal{M}(k) \) can be computed to be \( 4k - 5 \). By sending \( E \in \mathcal{M}(k) \) to the set of jumping conics of \( E \), we can define a morphism

\[
S : \mathcal{M}(k) \to |\mathcal{O}_Q^\ast(k-1)| \simeq \mathbb{P}_N,
\]

where \( N = \binom{k+2}{3} - 1 \).

Let \( Z = \{x_1, \ldots, x_k\} \) be a 0-dimensional subscheme of \( Q \) with length \( k \) in general position. If \( E \) is a stable vector bundle fitted into the exact sequence,

\[
0 \to \mathcal{O}_Q \to E(1,1) \to I_Z(1,1) \to 0,
\]

which is called a 

Hulsbergen bundle,

then \( E \) is in \( \mathcal{M}(k) \). Note that if \( k \leq 4 \), then \( E \in \mathcal{M}(k) \) admits the above exact sequence. Conversely, let us consider the above extension. It is parametrized by

\[
\mathbb{P}(Z) := \mathbb{P}\text{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \simeq \mathbb{P}H^0(\mathcal{O}_Z)^\ast.
\]

If we give \( \mathbb{P}(Z) \) the coordinate system \((c_1, \ldots, c_k)\) corresponding to \( Z \), then by the lemma (5.1.2) in Chapter 1 \[13\] or \[4\], the bundle \( E \) corresponding to \((c_1, \ldots, c_k)\) is locally free if and only if \( c_i \neq 0 \) for all \( i \).

Now by the theorem \[2.10\], \( S(E) \subset \mathbb{P}_3^\ast \) is a hypersurface of degree \( k - 1 \).

Lemma 3.1.

(1) If \( |Z \cap H| \geq 3 \), then \( h^0(E|_{C_H}) \geq 2 \).

(2) If \( |Z \cap H| \leq 2 \), then \( h^0(E|_{C_H}) \leq 1 \).

Proof. Let \( m = |Z \cap H| \geq 3 \). If \( C_H \) is a smooth conic, then by tensoring the above exact sequence with \( \mathcal{O}_H \), we have \( E|_{C_H} \simeq \mathcal{O}_{C_H} ((m-1)p) \oplus \mathcal{O}_{C_H} (-mp) \) since \( \text{Ext}^1(\mathcal{O}_{C_H} (-mp), \mathcal{O}_{C_H} ((m-1)p)) = 0 \). Thus, \( h^0(E|_{C_H}) \geq 2 \).

Let us assume that \( C_H = l_1 + l_2 \), i.e. \( H \) is a tangent plane of \( Q \). Note that

\[
h^0(C_H, \mathcal{O}(a_1, a_2)) = \begin{cases} 
0, & \text{if } a_i < 0; \\
a_i, & \text{if } a_i \geq 0, a_j < 0; \\
a_1 + a_2 + 1, & \text{if } a_i \geq 0
\end{cases}
\]

where \( \mathcal{O}(a_1, a_2) := \mathcal{O}_{l_1}(a_1) \cup \mathcal{O}_{l_2}(a_2) \). From the lemma (2.1) in \[11\], it is clear that \( h^0(E_{C_H}) \geq 2 \). For example, when \( m = 3 \) and \( Z \cap H = \{x, y, z\} \), \( x, y \in l_1, z \in l_2 \) and \( q = l_1 \cap l_2 \notin Z \), we have

\[
0 \to \mathcal{O}_{l_1}(1) \cup \mathcal{O}_{l_2} \to E \to \mathcal{O}_{l_1}(-2) \cup \mathcal{O}_{l_2}(-1) \to 0,
\]

and in particular the filtrations in the lemma (2.1) of \[11\], coincide in \( q \).

Thus, \( h^0(E|_{C_H}) = 2 \).

Assume that \( |Z \cap H| \leq 2 \). If \( C_H \) is smooth, we obtain in a similar way as above that \( E_{C_H} \) is either \( \mathcal{O}_{C_H} (-2p) \oplus \mathcal{O}_{C_H} \) or \( \mathcal{O}_{C_H} (-p) \oplus \mathcal{O}_{C_H} (-p) \) and thus \( h^0(E|_{C_H}) \leq 1 \). When \( H \) is a tangent plane section at \( q \in Q \), we can
also similarly show that $h^0(E|_{CH}) \leq 1$, except when $Z \cap H = \{x, y\}$ and $y = q$, say $x \in l_1$. In this case, we have

$$E|_{l_1} \simeq \mathcal{O}_{l_1}(1) \oplus \mathcal{O}_{l_1}(-2), \text{ and}$$

$$E|_{l_2} \simeq \mathcal{O}_{l_2} \oplus \mathcal{O}_{l_2}(-1).$$

Since $y = q$ is the intersection point of $l_1$ and $l_2$, the sub-bundles $\mathcal{O}_{l_1}(1)$ and $\mathcal{O}_{l_2}$ do not coincide at $y$. So $h^0(E|_{CH}) = 1$. □

Since we have $\binom{k}{3}$ hyperplanes that meet $Z$ at 3 points and thus $S(E)$ has at least $\binom{k}{3}$ singular points. Thus we have the following statement.

**Proposition 3.2.** For a Hulsbergen bundle $E \in \mathcal{M}(k)$, $S(E)$ is a hypersurface of degree $k - 1$ in $\mathbb{P}_3$ with $\binom{k}{3}$ singular points.

3.1. If $c_2 = 1$, then there is no stable vector bundles. In fact, it can be shown [5] that there exists a unique strictly semi-stable vector bundle $E_0 := \mathcal{O}_Q(-1, 0) \oplus \mathcal{O}_Q(0, -1)$. Since $h^0(E_0) = h^1(E_0(-1, -1)) = 0$, we have $h^0(E_0|_{CH}) = 0$ for all $H \in \mathbb{P}_3$. Hence, if we extend the concept of the jumping conic to semi-stable bundles, we can say that there is no jumping conic of $E_0$. It is consistent with the fact that $S(E_0)$ is a hypersurface of degree 0 in $\mathbb{P}_3$.

3.2. If $c_2 = 2$, then $S(E)$ is a hyperplane in $\mathbb{P}_3$. So the map $S$ is from $\mathcal{M}(2)$ to $\mathbb{P}_3$. It was shown in [5] that $S$ extends to an isomorphism

$$\overline{S}: \overline{\mathcal{M}}(2) \to \mathbb{P}_3,$$

where $\overline{\mathcal{M}}(2)$ is the compactification of $\mathcal{M}(2)$ in the sense of Gieseker [3], whose boundary consists of non-locally free sheaves with the same numeric invariants. In fact, for $E \in \overline{\mathcal{M}}(2)$, we have $h^0(E(1, 1)) = 3$ and can define a morphism from $\mathbb{P}_2 \simeq \mathbb{P}H^0(E(1, 1))$ to the Grassmannian $Gr(1, 3)$, sending a section $s$ to the line in $\mathbb{P}_3$ containing the two zeros of $s$. The image of this map can be shown to be a 2-cycles of $Gr(1, 3)$ corresponding to the unique point in $\mathbb{P}_3$. $\overline{S}$ maps $E$ to this uniquely determined point. Moreover, $\mathcal{M}(2)$ maps to $\mathbb{P}_3 \setminus Q$ via $S$ and in particular, $S(E)$ determines $E$ completely. Let $Z$ be a 0-cycle on $Q$ with length 2 such that the support of $Z$ does not lie on a line in $Q$ and consider an extension family $\mathbb{P}(Z)$ of $E$, admitting the following exact sequence,

$$0 \to \mathcal{O}_Q \to E(1, 1) \to I_Z(1, 1) \to 0.$$

Then, $\mathbb{P}(Z) \simeq \mathbb{P}_1$ is the secant line of $Q$ passing through the support of $Z$. From this description, it can be easily checked that $H \in S(E)$ if and only if $E|_{CH} \simeq \mathcal{O}_{CH} \oplus \mathcal{O}_{CH}(-2p)$, which is consistent with the fact that $S(E)$ is smooth.
3.3. If $c_2 = 3$, we have a map $S : M(3) \to |\mathcal{O}_{P_3^*}(2)| \simeq \mathbb{P}_9$, where $S(E)$ is a quadric in $\mathbb{P}_3^*$, $E(1, 1)$ is fitted into the following exact sequence,

$$0 \to \mathcal{O}_Q \to E(1, 1) \to I_Z(1, 1) \to 0,$$

with a 0-cycle $Z$ on $Q$ with length 3. If $Z$ is contained in a line on $Q$, then $E$ contains $\mathcal{O}_Q(0, -1)$ or $\mathcal{O}_Q(-1, 0)$ as a sub-bundle, contradicting to the stability of $E$. Thus there exists a unique hyperplane $H$ in $\mathbb{P}_3$ containing $Z$.

**Remark 3.3.** Conversely, if $Z$ is not contained in any line on $Q$, then it can be easily shown from the standard computation that any sheaf $E$ admitting an exact sequence (10) is semi-stable. In fact, if a subscheme scheme of length 2 of $Z$ is contained in a line on $Q$, any line $E$ admitting (10) is strictly semi-stable.

Now let us consider a map

$$\eta_E : \mathbb{P}_1 \simeq \mathbb{P}H^0(E(1, 1)) \to \text{Gr}(2, 3) \simeq \mathbb{P}_3^*,$$

sending a section $s \in H^0(E(1, 1))$ to the projective plane in $\mathbb{P}_3$ containing a 0-cycle $Z$ in the exact sequence (10), which is obtained from $s$. Before proving that $\eta_E$ is a constant map, we suggest a different proof of the fact that $S(E)$ is a quadric cone in $\mathbb{P}_3^*$.

**Proposition 3.4.** For $E \in M(3)$, $S(E)$ is a quadric cone in $\mathbb{P}_3^*$ with a vertex point.

**Proof.** Let $s$ be a section of $E(1, 1)$ from which $E(1, 1)$ admits an exact sequence (10) for a 0-dimensional cycle $Z$ of length 3. Let $Z = \{z_1, z_2, z_3\}$. If $H_s$ be a hyperplane in $\mathbb{P}_3$ containing $Z$, then $E|_{C_{H_s}}$ admits an exact sequence,

$$0 \to \mathcal{O}_{C_{H_s}}(p) \to E|_{C_{H_s}} \to \mathcal{O}_{C_{H_s}}(-3p) \to 0,$$

where $p$ is a point on $C_{H_s}$. It splits since $H^1(\mathcal{O}_{C_{H_s}}(-6p)) = 0$. Thus $E|_{C_{H_s}}$ is isomorphic to $\mathcal{O}_{C_{H_s}}(p) \oplus \mathcal{O}_{C_{H_s}}(-3p)$ and in particular, $H_s \in S(E)$. Similarly, if $H$ contains only 2 points of $Z$, then $H \in S(E)$. It can be shown that $H \notin S(E)$ if $H$ contains only 1 point of $Z$. Let us consider a hyperplane $H(z_1)$ in $\mathbb{P}_3^*$ whose points correspond to the hyperplanes in $\mathbb{P}_3$ containing $z_1$. From the previous argument, we know that the intersection of $H(z_1)$ with $S(E)$ consists of 2 straight lines whose intersection point corresponds to the hyperplane $H_3$. If $S(E)$ is a smooth quadric, then $H(z_1)$ is the tangent plane of $S(E)$ at $H_s$. Similarly, we can define $H(z_i)$, $i = 2, 3$, and they would also become the tangent plane of $S(E)$ at $H_s$, which is absurd. We can similarly derive a contradiction in the case when $Q$ is a hyperplane in $\mathbb{P}_3^*$ with multiplicity 2. Let us assume that $S(E)$ consists of two hyperplanes meeting at a line $l$. Clearly, $H_s$ lies in $l$. There are 3 lines on $S(E)$ corresponding to the hyperplanes containing 2 points of $Z$ and they are exactly the intersection of $H(z_i)$’s. Hence there is a hyperplane of $S(E)$ that contains two intersecting lines of $H(z_i)$’s. It is impossible since the two intersecting lines of $H(z_i)$ with $S(E)$ lie on different components of $S(E)$. Thus $Q$ is a quadric cone with a vertex point. \qed
Corollary 3.5. For $E \in \mathcal{M}(3)$, the map $\eta_E$ is a constant map to the vertex point of $S(E)$.

Proof. Using the notation in the proof of the preceding proposition, the planes $H(zi)$ meet with $S(E)$ at two different lines. The only possibility is that $H_s$ is the vertex point of $S(E)$. Now we get the assertion since this argument is valid for all sections of $E(1,1)$. □

Remark 3.6. The hyperplane $H$ corresponding to the vertex point of $S(E)$ is the unique hyperplane for which $E|_{C_H}$ is isomorphic to $\mathcal{O}_{C_H}(-3p) \oplus \mathcal{O}_{C_H}(p)$, where $p$ is a point on $Q$. For the other hyperplanes in $S(E)$, $E|_{C_H}$ become $\mathcal{O}_{C_H}(-2p) \oplus \mathcal{O}_{C_H}$.

By sending $E \in \mathcal{M}(3)$ to the vertex point of $S(E)$, we can define a map

$$\Lambda^* : \mathcal{M}(3) \to \mathbb{P}^*_3.$$ 

Let $p$ be a point in $\mathbb{P}^*_3 \setminus Q^*$, where $Q^*$ is the dual of $Q$, whose points correspond to the tangent planes of $Q$. We can pick a stable vector bundle $E$ fitted into the exact sequence (10) for a 0-cycle $Z$ of length 3 whose support lies in the hyperplane section corresponding to $p$. Then $E$ maps to the point $p$ via $\Lambda^*$. In the case when $p \in Q^*$, we can also choose a 0-cycle $Z$ for which there exists a stable vector bundle $E$ mapping to $p$. Thus $\Lambda^*$ is surjective and its generic fibres are 4-dimensional.

Now let us consider the determinant map

$$\lambda_E : \wedge^2 H^0(E(1,1)) \to H^0(\mathcal{O}_Q(1,1)).$$ 

Since $h^0(E(1,1)) = 2$, the dimension of the domain is 1-dimensional.

Lemma 3.7. $\lambda_E$ is injective.

Proof. We follow the argument in the proof of the lemma (6.6) in [14]. Let $s_1, s_2$ be two linearly independent sections of $E(1,1)$. Assume that $s_1 \wedge s_2$ maps to 0 via $\lambda_E$. It would generate a line subbundle $L$ of $E(1,1)$ with $h^0(L) = 2$. The only choices for $L$ is $\mathcal{O}_Q(0,1)$ and $\mathcal{O}_Q(1,0)$, and both contradict the stability of $E$. □

Let us define $q_E$ to be the point in $\mathbb{P}^*_3 \simeq \mathbb{P} H^0(\mathcal{O}_Q(1,1))$ corresponding to the image of $\lambda_E$. Since $E(1,1)$ is fitted into the exact sequence (10), $H^0(E(1,1))$ can be considered as the direct sum of $H^0(\mathcal{O}_Q)$ and $H^0(I_Z(1,1))$, so $\wedge^2 H^0(E(1,1))$ is isomorphic to $H^0(I_Z(1,1))$. From the long exact sequence of cohomology of the exact sequence,

$$0 \to I_Z(1,1) \to \mathcal{O}_Q(1,1) \to \mathcal{O}_Z \to 0,$$

$H^0(I_Z(1,1))$ is embedded into $H^0(\mathcal{O}_Q(1,1))$. This embedding is determined by the injection of $H^0(\mathcal{O}_Z)^*$ into $H^0(\mathcal{O}_Q(1,1))^*$, i.e. the hyperplane in $\mathbb{P}_3$ containing $Z$. We know from the preceding corollary that this hyperplane is independent on the sections of $E(1,1)$. Thus the embedding of $H^0(I_Z(1,1))$ into $H^0(\mathcal{O}_Q(1,1))$ is independent on $Z$ and it would give the same map as $\lambda_E$. As a quick consequence of this argument, we obtain that the image
of \( \lambda_E \) corresponds to the unique hyperplane in \( \mathbb{P}_3 \) containing \( Z \). In other words, we obtain the following statement.

**Proposition 3.8.** \( q_E \) is the vertex point of \( S(E) \).

**Remark 3.9.** Let \( f_Q \) be the polar map from \( \mathbb{P}_3 \) to \( \mathbb{P}_3^* \) given by

\[
[x_0, \ldots, x_3] \mapsto \left[ \frac{\partial f}{\partial t_0}(x), \ldots, \frac{\partial f}{\partial t_3}(x) \right],
\]

where \( f \) is the homogeneous polynomial of degree 2 defining \( Q \). Then we have a surjective map from \( \mathcal{M}(3) \) to \( \mathbb{P}_3 \),

\[
\Lambda := f_Q^{-1} \circ \Lambda^* : \mathcal{M}(3) \to \mathbb{P}_3.
\]

For \( E \in \mathcal{M}(3) \), let \( H_E \) be the hyperplane of \( \mathbb{P}_3 \) corresponding to \( q_E \). Note that \( C_{H_E} = H_E \cap Q \) is the set of points \( p \in Q \) for which \( \Lambda(E) \) is contained in the tangent plane of \( Q \) at \( p \). Thus we can define the map \( \Lambda \) by sending \( E \) to the intersection point of the tangent planes at the support of \( Z \) in the exact sequence (10), which is independent on the choice of a section of \( E(1,1) \).

Recall that the set of singular quadrics in \( \mathbb{P}_3^* \) is the discriminant hypersurface \( D_2 \) in \( \mathbb{P}_9 \) defined by the equation \( \det(A) = 0 \), where \( A \) is a symmetric \( 4 \times 4 \)-matrix. By differentiating, we know that the singular points of \( D_2 \) are defined by the determinants of \( 3 \times 3 \)-minors of \( A \), i.e. the singular points of \( D_2 \) correspond to the singular quadrics of rank \( \leq 2 \). Let \( D_2^0 \) be the smooth part of \( D_2 \). Then we have the following picture,

\[
\mathcal{M}(3) \xrightarrow{S} D_2^0 \xrightarrow{\Lambda^*} \mathbb{P}_3^* \xrightarrow{\pi_{q_E}} \mathbb{P}_3^* \xrightarrow{\Lambda} \mathcal{M}(3),
\]

where \( D_2^0 \) is an open Zariski subset of a quartic hypersurface \( D_2 \) of \( \mathbb{P}_9 \) and the vertical map sends a singular quadric of rank 3 to its vertex point.

Let \( E \in (\Lambda^*)^{-1}(q_E) \) with \( q_E \notin Q^* \). Thus \( H_E \) is not a tangent plane of \( Q \) and so \( C_{H_E} \) is a smooth conic on \( H_E \). Let \( \mathbb{P}_3^* \) be the image of \( H_E \) via the polar map \( f_Q \), which is a hyperplane of \( \mathbb{P}_3^* \), not containing \( q_E \). Then \( \mathbb{P}_3^* \) contains the dual conic \( C_{H_E}^* \) of \( C_{H_E} \) via \( f_Q \). Let \( \pi_{q_E} \) be the projection map from \( \mathbb{P}_3^* \) to \( \mathbb{P}_2^* \) at \( q_E \). Then we can assign a smooth conic \( C(E) := \pi_{q_E}(S(E)) \subset \mathbb{P}_2^* \) to \( E \), i.e. we have a map

\[
\pi_{q_E} : (\Lambda^*)^{-1}(q_E) \to |O_{\mathbb{P}_2^*}(2)| \simeq \mathbb{P}_3.
\]

Clearly, \( C(E) \neq C_{H_E}^* \).

Let us fix a general hyperplane \( H \) of \( \mathbb{P}_3 \). For a 0-cycle \( Z \) with length 3 contained in \( C_H \simeq \mathbb{P}_1 \), we can consider an extension space \( \mathbb{P}(Z) := \mathbb{P}\text{Ext}^1(I_Z(1,1),O_Q) \simeq \mathbb{P}_2 \). Note that the Hilbert scheme parametrizing 0-cycles on \( C_H \) with length 3, \( \mathbb{P}_1^{[3]} \), is isomorphic to \( \mathbb{P}_3 \). Let us define

\[
\mathcal{U} := R^1p_{1*}(\mathcal{I} \otimes p_2^*O_Q(-1,-1)),
\]
where \( p_1, p_2 \) are the projections from \( \mathbb{P}_3 \times Q \) to each factors and \( \mathcal{I} \) is the universal ideal sheaf of \( \mathbb{P}_3 \times Q \). We can easily find that \( \mathcal{U} \) is a vector bundle on \( \mathbb{P}_3 \) of rank 3 and the fibre of \( \mathbb{P}(\mathcal{U}^*) \) at \( Z \in \mathbb{P}_3 \) is \( \mathbb{P}(Z) \). Then we have a rational map from \( \mathbb{P}(\mathcal{U}^*) \) to \( \mathcal{M}(3)_q := (\Lambda^*)^{-1}(q) \), and eventually to \( \mathbb{P}_5 \) after the composition with \( \pi_q \), where \( q \) corresponds to \( H \). In particular, the dimension of the image of \( \mathbb{P}(\mathcal{U}^*) \) is less than 5 since the dimension of \( \mathcal{M}(3)_q \) is 4.

\[
\begin{align*}
\mathbb{P}(\mathcal{U}^*) & \rightarrow \mathbb{P}_5 \\
\mathcal{M}(3)_q & \rightarrow 
\end{align*}
\]

For a general 0-cycle \( Z = \{z_1, z_2, z_3\} \) on \( C_H \), let \( p_{ij} \in \mathbb{P}_3^* \) be the point corresponding to the line containing \( z_i \) and \( z_j \). The conic \( C(E) \) contains \( p_{ij} \) and so the image of \( \mathbb{P}(Z) \) is contained in the projective plane in \( \mathbb{P}_5 \) parametrizing all the conics passing through three points \( p_{ij} \). Let \( Z^* = \{z_i^*, z_j^*, z_k^*\} \) be the dual lines on \( \mathbb{P}_3^* \) of \( Z \), then \( p_{ij} \) is the intersection point of \( z_i^* \) and \( z_j^* \). If we choose linear forms \( 0 \neq Z_i \in H^0(\mathcal{O}_{\mathbb{P}_3^*}(1)) \) which vanish on \( z_i^* \), then from the previous statement, \( \pi_q \circ S \) is defined by

\[
\pi_q \circ S : \mathbb{P}(Z) \rightarrow |\mathcal{O}_{\mathbb{P}_3^*}(2)|
\]

\[
(c_1, c_2, c_3) \mapsto f_1Z_2Z_3 + f_2Z_1Z_3 + f_3Z_1Z_2,
\]

where \( (c_1, c_2, c_3) \) is the coordinates from the identification of \( \mathbb{P}(Z) \) with \( \mathbb{P}H^0(\mathcal{O}_Z)^* \) and \( f_i \)'s are homogeneous polynomials of \( c_j \)'s.

**Proposition 3.10.** For a general 0-cycle \( Z \), the map \( \pi_q \circ S \) from \( \mathbb{P}(Z) \) to \( \mathbb{P}_5 \) sending \( E \) to \( \pi_q(S(E)) \), is a linear embedding.

**Proof.** From the previous argument, it is enough to check that \( f_i \)'s are linearly independent linear polynomials. In fact we can prove that \( f_i \equiv c_i \) for all \( i \).

Recall that \( I \) is the incidence variety in \( Q \times \mathbb{P}_3^* \) with the projections \( \pi_1 \) and \( \pi_2 \). Then we have an isomorphism,

\[
h : \mathcal{O}_{\mathbb{P}_3^*} \rightarrow \pi_{2*}\pi_1^*I_Z((0,0),3),
\]

given by the multiplication with \( Z_1Z_2Z_3 \). Here, \( \mathcal{O}_I((a,b),c) \) is the sheaf \( \pi_1^*\mathcal{O}_Q(a,b) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}_3^*}(c) \) on \( I \). Note that \( \pi_2, \pi_1^*I_Z \) is the ideal sheaf of functions on \( \mathbb{P}_3^* \), vanishing on the lines \( z_i^* \). From the canonical homomorphisms,

\[
\text{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \rightarrow \text{Ext}^1(\pi_1^*I_Z(1,1), \mathcal{O}_I)
\]

\[
\rightarrow \text{Ext}^1(\pi_1^*I_Z((0,0),3), \mathcal{O}_1((-1,-1),3)),
\]

we can assign to an element \( e \in \text{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \), an extension

\[
0 \rightarrow \mathcal{O}_I((-1,-1),3) \rightarrow \pi_1^*E((0,0),3) \rightarrow \pi_1^*I_Z((0,0),3) \rightarrow 0.
\]

(14)

From the long exact sequence of cohomology of (14), we obtain

\[
H^0(\mathcal{O}_{\mathbb{P}_3^*}) \rightarrow H^0(\pi_1^*I_Z((0,0),3)) \rightarrow H^1(\mathcal{O}_I((-1,-1),3)) \simeq H^0(\mathcal{O}_{\mathbb{P}_5^*}(2)),
\]
and let \( \pi(\varepsilon) \) be the image of \( 1 \in H^0(\mathcal{O}_{\mathbb{P}_3}) \). Then we can define a homomorphism

\[
\pi : \text{Ext}^1(I_Z(1,1), \mathcal{O}_Q) \rightarrow H^0(\mathcal{O}_{\mathbb{P}_3}(2)),
\]

by sending \( \varepsilon \) to \( \pi(\varepsilon) \).

From the inclusion \( I_{\pi} \hookrightarrow I_Z \), we have a natural injection from

\[
\text{Ext}^1(I_{\pi}(1,1), \mathcal{O}_Q) \cong \mathbb{C} \hookrightarrow \text{Ext}^1(I_Z(1,1), \mathcal{O}_Q)
\]

whose image is \( H^0(\mathcal{O}_{\mathbb{P}_3})^* \). It can be easily checked that any element in the image is mapped to \( H^0(\mathcal{O}_{\mathbb{P}_3}(2)) \) by the multiplication with \( (Z_1Z_2Z_3)/Z_i \). Thus \( \pi \) is defined by sending \((c_1, c_2, c_3)\) to \( c_1Z_2Z_3 + c_2Z_1Z_3 + c_3Z_1Z_2 \).

When we take the direct image of \((14)\), then we obtain

\[
\pi_{2n} \pi_{2n}^* E((0,0),3) \rightarrow \pi_{2n} \pi_{2n}^* I_Z((0,0),3) \rightarrow R^1\pi_{2n} \mathcal{O}_E((-1,-1),3)
\]

\[
\rightarrow R^1\pi_{2n} \pi_{2n}^* E((0,0),3) \rightarrow R^1\pi_{2n} \pi_{2n}^* I_Z((0,0),3) \rightarrow 0.
\]

Note that \( \pi_{2n} \pi_{2n}^* I_Z((0,0),3) \cong \mathcal{O}_{\mathbb{P}_3}; R^1\pi_{2n} \mathcal{O}_E((-1,-1),3) \cong \mathcal{O}_{\mathbb{P}_3}(2) \) and the second map in the sequence above, is given by the multiplication with \( \pi(\varepsilon) \). As an analogue of the result in \([9]\), we can easily check that \( R^1\pi_{2n} \pi_{2n}^* E((0,0),3) \) is isomorphic to \( \deg E(4) \) and its support is \( S(E) \). On the other hand, the support of \( R^1\pi_{2n} \pi_{2n}^* I_Z((0,0),3) \) is contained in \( \{p_{ij}\} \) and thus the support of \( S(E) \) is same as the support of \( \{\pi(\varepsilon) = 0\} \). Because of the same degree, they are the same.

\[\square\]

**Remark 3.11.** Using the argument as in the similar statement on the projective plane in \([10]\), we can prove that a sheaf \( E \in \mathbb{P}(Z) \) with the coordinates \((c_1, c_2, c_3)\) is locally free if and only if \( c_i \neq 0 \) for all \( i \). Thus from the proof of the preceding proposition, we can observe that the conic corresponding to the image of \( E \) is smooth if and only if \( E \) is locally free. Note that the secant variety \( V_3 \) of the Veronese surface in \( \mathbb{P}_5 \) is a cubic hypersurface. The intersection of the image of \( \mathbb{P}(Z) \) with \( V_3 \) are the 3 lines, which are the image of non-locally free sheaves in \( \mathbb{P}(Z) \).

We can see that the same statement holds for arbitrary hyperplane section \( H \in \mathbb{P}_3 \). If \( H \in Q^* \), \( Q^* \) the dual conic of \( Q \), then \( C_H = l_1 \cup l_2 \). Because of the stability condition, our 0-cycles of length 3 associated to \( E \) with \( \Lambda^*(E) \in Q^* \) cannot have its support only on \( l_1 \) nor \( l_2 \). So the family of 0-cycles we consider, is isomorphic to the two copies of \( \mathbb{P}^{[2]}_1 \times \mathbb{P}_1 \). Let us denote

\[
\mathcal{M}(3) = \mathcal{M}^0(3) \coprod \mathcal{M}^1(3) \coprod \mathcal{M}^2(3),
\]

where \( \mathcal{M}^0(3) = (\Lambda^*)^{-1}(\mathbb{P}^*_{\lambda}(Q^*)) \) and \( \mathcal{M}^i(3) \)'s are the two irreducible components of \( (\Lambda^*)^{-1}(Q^*) \) whose 0-cycles have two points of its support on the ruling equivalent to \( l_i \).

First let us assume that \( H \not\in Q^* \). Let \( V \subset \mathbb{P}_5 \) be the image of \( \mathbb{P}(U^*) \) and \( v \in V \) be a general point in \( V \). Then there exists three points \( z_i \)'s on \( C_H \) and \( c_i \)'s for which we have \( v = c_1Z_1 + c_2Z_2 + c_3Z_3 \). Since \( z_i \in C_H \), the lines
Z_i’s are tangent to the dual conic \( C_H^* \), i.e. \( Z_i \)'s is a circumscribed triangle around \( C_H^* \). Note that \( Z_i \)'s is a inscribed triangle in \( v \). Thus \( V \) is the closure of the family of conics Poncelet related to \( C_H^* \) (see section 2 in [5]). From the classical result, \( V \) is a hypersurface in \( \mathbb{P}_5 \) and the generic fibre of the map \( \mathbb{P}(\mathcal{U}^*) \to V \) is isomorphic to \( \mathbb{P}_1 \). In fact, from the remark (2.2.3) in [5], \( V \) is isomorphic to a hypersurface of degree 4, \( H_4 \) in the space of conics, given by the condition \( c_2^2 - c_1 c_3 = 0 \), where

\[
\det(A - tI_3) = (-t)^3 + c_1(-t)^2 + c_2(-t) + c_3,
\]

is the characteristic polynomial of a symmetric matrix \( A \) defining a conic.

Let \( E \in \mathcal{M}^1(3) \) and so \( H \in Q^* \). If we define \( V \) as before and let \( v \in V \) be a general conic, then \( v \) pass through the dual point \( p_1 \in \mathbb{P}_2 \) of \( l_1 \). Let us fix a conic \( v \) passing through \( p_1 \). If we choose \( q_1 \in v \) not equal to \( p_1 \), then consider a line \( l \) passing through \( q_1 \) and the dual point \( p_2 \) of \( l_2 \). Let \( q_2 \) be the other intersection point of \( l \) with \( v \). Then the dual points corresponding to the lines \( \overline{pq_1}, \overline{pq_2}, \overline{q_2p_1} \) is a 0-cycle \( Z \) mapping to \( v \). It depends on the choice of \( q_1 \). Thus, \( V \) is isomorphic to a hyperplane in \( \mathbb{P}_5 \) and the generic fibre of the map from \( \mathbb{P}(\mathcal{U}^*) \) is again isomorphic to \( \mathbb{P}_1 \). We have the same argument for \( \mathcal{M}^2(3) \).

As a direct consequence, \( \mathcal{M}(3) \) is isomorphic to an open Zariski subset of a hyperplane in \( \mathbb{P}_5 \) and thus we have the following proposition.

**Proposition 3.12.** \( \mathcal{M}(3) \) admits a fibration over \( \mathbb{P}_3^* \) whose fibre over \( H \in \mathbb{P}_3^* \) is isomorphic to

1. an open Zariski subset \( H_4 \cap (\mathbb{P}_5 \setminus V_3) \) of a \( H_4 \), where \( V_3 \) is the secant variety of the Veronese surface \( S \subset \mathbb{P}_5 \) and \( H_4 \) is a hypersurface of degree 4 consisting of conics Poncelet related to \( Q \cap H \), if \( H \in \mathbb{P}_3^* \setminus Q^* \);
2. the union of two varieties \( H_i \cap (\mathbb{P}_5 \setminus V_3) \), \( i = 1,2 \), where \( H_i \) is the hyperplane in the space of conics which pass through a point \( p_i \) dual to the line \( l_i \subset H \), where \( Q \cap H = l_1 + l_2 \), if \( H \in Q^* \).

**Remark 3.13.** In fact, we can obtain differently the old result of Darboux on the Poncelet related conics in the case of triangles. We know that we have \( \dim V \leq \dim \mathcal{M}(3) \) = 4. Assume that \( C_H \) is a smooth conic. Let \( \triangle_2 \) be the subscheme of \( \mathcal{C}_H^i \) whose points are 0-cycles with at most 2 points as their supports. Similarly, we can define \( \triangle_3 \subset \triangle_2 \). Let \( Z \in \triangle_2 \), say \( Z = \{x, x, y\} \). The map \( \mathbb{P}(Z) \to \mathbb{P}_5 \) is naturally defined by sending \( (c_1, c_2, c_3) \) to \( (c_1 + c_2)XY + c_3X^2 \). From this observation, the image of \( \mathbb{P}_2 \)-bundle over \( \delta_3 \) is \( C_H \subset \mathbb{P}_5 \) mapped by \( |\mathcal{O}_{C_H}(2)| \). For \( Z = \{x, x, y\} \), \( \mathbb{P}(Z) \) is mapped to the line passing through \( X^2 \) and \( XY \). When \( Y \) is moving along \( C_H \), this line covers a projective plane \( \mathbb{P}_2(x) \) passing through the point \( X^2 \in C_H \subset \mathbb{P}_5 \). Let \( D \) be the union of such projective planes over \( x \) moving along \( C_H \). In particular, \( D \) is a subvariety of \( V \) with dimension 3 and all the non-locally free sheaves in \( \mathbb{P}(\mathcal{U}^*) \) map to \( D \). Also we have

\[
V_3 \cap V = D,
\]
where $V_3$ is the secant variety of the Veronese surface in $\mathbb{P}_5$. It also implies that $V$ is a subvariety of $\mathbb{P}_5$ with dimension 4.

Let us consider a fibre of the map $\mathbb{P}(\mathcal{U}^*) \to V$ over $XY$ with $x, y \in C_H$. The image of the closure of this fibre via the projection to $C_H^{[3]}$ is isomorphic to $V_3$, parametrizing 0-cycles whose supports contain $x$ and $y$. In fact, there exists a unique component of the closure of the fibre, mapping to $\mathbb{P}_1 \subset C_H^{[3]}$. It implies that the closure of the fibre over a generic conic $v$ in $V$ is isomorphic to $\mathbb{P}_1$ since there exists at most 1 point in $\mathbb{P}(Z)$ that maps to $v$.

In particular, the map $S : \mathcal{M}(k) \to \mathbb{P}_N$ is an isomorphism to its image, when $k \leq 3$.

**Theorem 3.14.** The set of jumping conics of $E \in \mathcal{M}(k)$, determines $E$ uniquely when $k \leq 3$.

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