Superfluidity as a Tool for Constraining the Energy–Momentum Relation

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The present work explores the possibilities that superfluidity could offer in the context of quantum gravity phenomenology, at least in the realm of deformed dispersion relations. The experimental proposal involves a Bose–condensed sodium gas trapped by an isotropic harmonic oscillator potential. A deformed dispersion relation for the particles of the system is considered and the consequences of this assumption upon the possible region of superfluidity of this system is analyzed. It will be shown that in this sense the effects of quantum gravity could be detected resorting to experiments of superfluidity in Bose–Einstein condensates. Finally, using the current experimental results in this direction an upper bound for the corresponding phenomenological parameters will be also obtained.

Keywords: Landau criterion, sodium gas

I. INTRODUCTION

Quantum gravity phenomenology [1–3] emerges as an answer of the community to the mathematical and physical difficulties plaguing all theoretical models behind a quantum theory of gravity [4, 5]. These efforts embody new physical effects, for instance, deformed versions of the dispersion relation, deviations from the $1/r$–potential and violations of the equivalence principle. At this point it is noteworthy to comment that these cases do not exhaust the extant possibilities.

In the quest for a solution to this long–standing puzzle in modern physics we have some efforts which entail, unavoidably, the breakdown of Lorentz symmetry [6–8]. Lorentz symmetry is a the bedrock of modern physics and, therefore, it has been subjected to some of the highest precision tests in Physics [9–11]. The current experimental results show no evidence of a violation of this symmetry, nevertheless this last fact does not discard it and, in consequence, further work is required. At this point it has to be clearly stated that the phrase violation of Lorentz symmetry has several meanings, i.e., it embodies several characteristics. For instance, Local Lorentz Invariance, or Local Position Invariance [12]. For us it will mean a modification of the dispersion relation. Let us state this phrase in a mathematical language. As mentioned above several quantum–gravity models predict a modified dispersion relation [6–8], the one can be characterized, phenomenologically, through corrections hinging upon Planck’s energy, $E_p$

$$E^2 = p^2 c^2 \left[ 1 - \alpha \left( E/E_p \right)^n \right] + (mc^2)^2.$$  \hspace{1cm} (1)

Here $\alpha$ is a coefficient, whose precise value depends upon the considered quantum–gravity model, while $n$, the lowest power in Planck’s length leading to a non–vanishing contribution, is also model dependent.

The quest in this direction has already considered interferometry as a tool [13–17]. Clearly, all these previous works in quantum gravity phenomenology involve different fields of physics [18], i.e., the search includes now many areas of modern physics. One of these topics is condensed matter physics. Indeed, the use of cold atoms, either bosonic or fermionic, is a point already considered [19, 20]. In particular the possibility of constraining the energy–momentum relation resorting to cold atoms has already shown us that this kind of systems could open up new landscapes in the context of gravitational physics [21]. Of course, this last topic in the context of phenomenology of quantum gravity leads us to ask if there are additional low–temperature effects that could be used as trackers for new effects. This question implies the quest for the detection of these kind of effects in the realm of condensed matter physics, i.e., a broadening of the current attempts. Clearly, another low–temperature effect is the phenomenon of superfluidity [22] and, therefore, we wonder if this case could offer a new window for our search. In the present work we explore this situation, namely, the possibilities that superfluidity has to offer in the context of quantum gravity phenomenology.

Having stated our goal we must discuss, though briefly, the physics behind the emergence of viscosity in a flow. Concerning the phenomenon of superfluidity the first experimental results can be found in the work of Kamerling Onnes of 1911 in which he detected that if cooled below 2.2 °K He I did not contract but rather expand [23]. The current work has been able to provide a coherent picture to the subjacent Physics [24, 25]. Bose–Einstein Condensation (BEC) is also connected to the presence of very low temperatures. Fritz London [26] put forward the idea of a connection between these two effects asserting that the transition from He I (the high temperature phase of liquid helium) and He II (the low temperature phase) should be considered an example of a BEC. Taking into account London’s idea and, joining it to the previous work BEC–quantum gravity phenomenology, once

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again, we are confronted with the question about a possible use of superfluidity as a tool in our quest. The concept of elementary excitations in the realm of superfluidity was first introduced by Landau [27] (within the two-fluid model proposed by Tisza [28]) as a core feature in the description of the behavior of He II. Landau asserted [29] that the normal fluid (the non-superfluid component) could be regarded as a dilute gas whose components are weakly-interacting elementary excitations which move in a background defined by the superfluid component. Along these ideas the phenomenon of superfluidity appears when the velocity of the corresponding flow lies below a certain threshold value given by

\[ v_{(\text{crit})} = \min \left( \frac{\epsilon(p)}{p} \right). \tag{2} \]

In this last expression \( \epsilon(p) \) denotes the energy of an elementary excitation and \( p \) the corresponding momentum.

In the experimental realm the quest for this critical velocity has been carried out in a sodium–BEC, and the results show a possible velocity threshold located around the value of 1.6 mm/s [30]. Sodium is a system that can be condensed and the speed of sound in it has already been detected [31, 32]. At this point we may now state clearly the ideas contained in the present work and why they have been chosen. The main purpose is to obtain a prediction for the critical velocity for a BEC in which the relation energy–momentum of the particles of the gas has been deformed along some possibilities contained in several quantum gravity models. The system is a sodium gas trapped by an isotropic harmonic oscillator potential. The reason for this particular choice stems from the fact that several of its properties have already been detected and measured, for instance, evidence for a critical velocity [30] and speed of sound [31, 32]. In other words, our proposal is the following one: use a BEC-condensed sodium gas and measure the region in which superfluidity is present. Compare the size of this region against the theoretical prediction here obtained and deduce the corresponding bound for our parameter containing the breakdown of Lorentz symmetry. Up to now, superfluidity has not been considered a relevant element in quantum gravity phenomenology and the present work shall be considered as an analysis of the perspectives that this topic could offer.

II. MATHEMATICAL MODEL

Even the simplest mathematical model for a BEC (the Gross–Pitaevskii equation) trapped by a harmonic oscillator potential has no analytical solution, yet [33]. This last fact implies that in this topic we must resort to approximation methods, etc. In our case, the theoretical model will describe the features of the BEC resorting to an approximation method in which the presence of interactions among the particles produces a change in the frequency of the trap rendering a smaller value than the one provided by the trap, for the case of repulsive interactions. This assumption will allow us to calculate the energy of the ground state and of the thermal cloud. The energy and momentum of the elementary excitations, according to Bogoliubov ideas, are a function of the energy of the excited particles, and this parameter can be computed from our assumptions. Finally, the critical velocity is deduced as a function of our phenomenological variables and compared against the measurement readouts. From this comparison an upper bound for our model values will appear.

From a fundamental point of view our mathematical model can be defined by an \( N \)-particle Hamiltonian the one in the formalism of second quantization is [34]

\[
\hat{H} = \int d\vec{r} \left[ -\hat{\psi}^\dagger(\vec{r}, t) \frac{\hbar^2}{2m} \nabla^2 \hat{\psi}(\vec{r}, t) + V(\vec{r}) \hat{\psi}^\dagger(\vec{r}, t) \hat{\psi}(\vec{r}, t) + \frac{U_0}{2} \hat{\psi}^\dagger(\vec{r}, t) \hat{\psi}^\dagger(\vec{r}, t) \hat{\psi}(\vec{r}, t) \hat{\psi}(\vec{r}, t) \right]. \tag{3}
\]

In this Hamiltonian \( \hat{\psi}^\dagger(\vec{r}, t) \) and \( \hat{\psi}(\vec{r}, t) \) represent bosonic creation and annihilation operators, respectively. It is valid only at low energies and momenta and implies that the interaction among the particles is, as usual, codified by the scattering length parameter \( a \), i.e., \( U_0 = \frac{4\pi a \hbar^2}{m} \). The corresponding trapping potential \( (V(\vec{r})) \) is an isotropic harmonic oscillator whose frequency reads \( \omega \). Moreover, there are \( N \) particles in the gas, each of them with mass \( m \), the volume occupied by the system is \( V \).

The mathematical assumptions in the present model read:

(i) Only two states are populated, namely, ground and the first excited state. We may provide to this assumption a realistic physical meaning recalling that for a bosonic system, with chemical potential \( \mu \) and energy levels of single–particle \( \epsilon \), the occupation number in thermal equilibrium is given by [35] \((\beta = 1/(kT))\)

\[
< n(\epsilon) > = \frac{1}{e^{(\epsilon-\mu)\beta} - 1}. \tag{4}
\]

Clearly, it is a monotonic decreasing function of \( \epsilon \), and this feature justifies the present assumption.

(ii) The Hartree approximation will be employed for the mathematical description of the two occupied states. In other words, the ground state of the interacting system is deduced by a Ginzburg–Pitaevskii–Gross energy functional [36], and it entails that the ground state wavefunction corresponds to the case of a harmonic oscillator situation but the frequency is modified due to the fact...
that the system has a non–vanishing scattering length [37], such that the fundamental length parameter reads.

\[ R = \left( \frac{2}{\pi} \right)^{1/10} \left( \frac{Na}{l} \right)^{1/5}. \]  

(5)

In this last expression \( l \) is the radius related to the trap given by the isotropic harmonic oscillator of the trap

\[ l = \sqrt{\frac{\hbar}{m\omega}}. \]  

(6)

Of course, we end up with an effective frequency

\[ \tilde{\omega} = \frac{\hbar}{mR^2}. \]  

(7)

The experimental conditions entail \( R > l \) [30] and, therefore, \( \tilde{\omega} < \omega \).

The order parameter of those particles in the ground state is

\[ \psi(\vec{r}) = \sqrt{\frac{N(0)}{(R\sqrt{\pi})^3}} \exp \left[ -\frac{x^2}{2R^2} \right]. \]  

(8)

Here \( N(0) \) denotes the number of particles in the lowest energy state. The presence of a non–vanishing scattering length implies that in the ground state not all the particles can have zero–momentum, the reason for this lies in the fact that the two–body interaction mixes in components with atoms in other states [35] and

\[ N(0) = N \left[ 1 - \frac{8}{3} \sqrt{\frac{N\alpha^3}{\pi V}} \right]. \]  

(9)

Clearly,

\[ N(0) = \int (\psi_0(\vec{r}))^2 d^3r, \]  

(10)

\[ V = \frac{4\pi}{3} R^3. \]  

(11)

Concerning the thermal cloud, the core of this part is also comprised by the Hartree approximation. We assume that all excited particles are in the same state and, since the temperature is very low, it corresponds to the first excited state of a particle trapped by a harmonic oscillator with a frequency given by (7). If \( \Psi(1) \) denotes the wave function of the thermal cloud, \( \phi(1) \) the wavefunction of the first excited state of a single particle in our effective trap, and \( N(c) \) the number of particles in the cloud then

\[ \Psi(1)(\vec{r}) = \sqrt{N(c)} \phi(1)(\vec{r}). \]  

(12)

From the symmetry of the system we may conclude that an excited particle can have vanishing momentum along the \( x \) and \( y \) axes but a non–zero one in the \( z \)–direction, or vanishing momentum along the \( z \) and \( y \) axes but non–zero one in the \( x \)–direction, or, finally, vanishing momentum along the \( x \) and \( z \) axes and larger than zero along \( y \)–direction. The symmetry of our trap and of the scattering length entail that one third of the excited particles will have non–vanishing momentum along the \( x \)–axis, one third along the \( y \)–axis, and the remaining third along the \( z \)–direction. Mathematically this corresponds to the following expressions

\[ \psi(1^i)(\vec{r}) = \frac{8}{\sqrt{27\pi}} V \sqrt{\frac{N\alpha^3}{\pi V}} \frac{x(1^i)}{R} \exp \left[ -\frac{x^2}{2R^2} \right]. \]  

(13)

Here \( x(1^1) = x \), \( x(1^2) = y \), and \( x(1^3) = z \).

Of course, (13) must be related to the total number of particles in excited states \( N(1^i) = \frac{8}{3} N \sqrt{\frac{2\pi}{2\pi}} \), a condition that becomes [35]

\[ N(c) = \int \left[ \sum_{i=1}^{3} (\psi(1^i)(\vec{r}))^2 \right] d^3r. \]  

(14)

The three expressions in (13) will be used for the definition of the wavefunction \( \Psi(1)(\vec{r}) \) of the thermal cloud, i.e.,

\[ \Psi(1)(\vec{r}) = \psi(1^1)(\vec{r}) + \psi(1^2)(\vec{r}) + \psi(1^3)(\vec{r}). \]  

(15)

III. ELEMENTARY EXCITATIONS AND DEFORMED DISPERSION RELATIONS

Having stated our assumptions we proceed to compute the critical velocity [22]. The deduction of the energy of an elementary excitation and of its corresponding momentum requires the knowledge of the energy of a single–particle in the first excited state [34]. The thermal cloud contains particles in the first excited state of an isotropic harmonic oscillator whose frequency is (7) therefore the energy of an excited particle is given by this assumption and easily calculated as a function of the effective frequency of our variational procedure

\[ \tilde{\epsilon}^{(0)} = \frac{5}{2} \hbar \tilde{\omega}. \]  

(16)

This is the case in which no deformed dispersion relation has been considered. Introducing this quantum gravity parameter we have
\[ \tilde{\epsilon} = \frac{5}{2} \hbar \omega + \alpha p^n. \] (17)

In this last expression \( \alpha \) and \( n \) are parameters stemming from the considered quantum gravity model [18].

According to the ideas of Bogoliubov [34, 38] the energy of an elementary excitation, here denoted by \( \epsilon \), is a function of the energy of the excited particles of the BEC, namely,

\[ \epsilon = \sum \sqrt{\langle \hat{\epsilon} \rangle^2 + \frac{2 N U(0)}{V} \hat{\epsilon}}. \] (18)

The contribution to the energy of all the elementary excitations turns out to be [34, 38]

\[ \tilde{E} = \sum \sqrt{\langle \hat{\epsilon} \rangle^2 + \frac{2 N U(0)}{V} \hat{\epsilon}} < \tilde{n}_\epsilon >. \] (19)

We have defined \(< \tilde{n}_\epsilon > = \langle n_\epsilon \rangle > 1 + \langle n_\epsilon \rangle \). (20)

Since our model has to be consistent with the present experimental technology at this point we resort to the laboratory values related to the detection of a critical velocity in a sodium condensed gas [30] in which the occupation number of the particles in the first excited state fulfills the condition \( N_e \sim 10^2 > 1 \). Therefore, \(< \tilde{n}_{e(1)} > = 1 \). In addition, \(< \tilde{n}_{e(i)} > = 0, \forall i > 1 \). Indeed, we have considered that the thermal cloud is comprised by particles which occupy only the first excited state, in other words, \(< n_{e(i)} > = 0, \forall i > 1 \). Introducing this condition into (20) leads us to the aforementioned result for the occupation number of the elementary excitations.

The order of magnitude of this deformed dispersion relation has to be very small, otherwise, it would have already been detected. This means that we must expect the fulfillment of

\[ \epsilon^{(0)} >> \alpha p^n. \] (21)

We now cast (18) in a different form, and for this we resort to the effective volume \( V = 4 \pi R^3/3 \), use (5), (6), and (7) and keep only terms linear in \( \alpha \). The final result reads

\[ \epsilon = \left( \frac{4 \pi}{3} \right)^{1/3} \frac{\alpha}{m V^{2/3}} \sqrt{\frac{25}{4} \left( \frac{4 \pi}{3} \right)^{2/3} + \frac{20 \pi N a}{V^{1/3}}} \] (22)

\[ + \frac{\tilde{\alpha}}{2} \sqrt{\frac{4 \pi N a}{5 V^{1/3}}} \frac{1}{V^{n/3}}. \]

\[ \tilde{\alpha} = \left( \frac{4 \pi}{3} \right)^{2/3} \left[ \sqrt{24 \pi} \left( \frac{4 \pi}{3} \right)^{1/3} \frac{1}{\hbar} \right]^n \alpha. \] (23)

According to Landau [29], in order to find the critical velocity we must now deduce the momentum of this elementary excitation. These physical variables, which define the normal component of the fluid, can be regarded as a bosonic gas whose components are weakly-interacting and moving in a region in which a constant potential exists, and this potential is defined by a mean field approach [34]. According to this interpretation we may rewrite (22) in the same form as in the case in which our BEC is a homogeneous one [34]. In other words, take (22) impose the condition \( \alpha = 0 \) and compare the result against

\[ \epsilon = \frac{\hbar^2 k}{2 m} \sqrt{k^2 + \frac{16 \pi N a}{V}}. \] (24)

This last argument allows us to deduce the wavenumber related to our elementary excitation and, therefore, its momentum.

\[ k = \left( \frac{4 \pi}{3} \right)^{1/3} \sqrt{5 \frac{1}{V^{1/3}}}. \] (25)

\[ p = \left( \frac{4 \pi}{3} \right)^{1/3} \sqrt{\frac{5 \hbar}{V^{1/3}}}. \] (26)

Resorting to Landau criterion (2) we obtain that the critical velocity is given by

\[ v_{(cr)} = \frac{1}{\sqrt{3}} \frac{\hbar}{m V^{1/3}} \sqrt{\frac{25}{4} \left( \frac{4 \pi}{3} \right)^{2/3} + \frac{20 \pi N a}{V^{1/3}}} + 1 \sqrt{20} \left( \frac{4 \pi}{3} \right)^{1/3} \] (27)

\[ \times \left[ \sqrt{24 \pi} \left( \frac{4 \pi}{3} \right)^{1/3} \right]^{n} \sqrt{\frac{4 \pi N a}{5 V^{1/3}}} \left( \frac{\hbar}{V^{1/3}} \right)^{n-1} \alpha. \] (27)

IV. CRITICAL VELOCITY

At this point we proceed to check our model, and in order to do this we consider the case in which \( \alpha = 0 \). The experimental parameters [30] to be used are: (i) a critical speed of \( v_{(cr)} = 1.6 \text{ mm/s} \); (ii) the number of particles in this experiment has a minimum of \( N = 3 \times 10^6 \) and a maximum of \( N = 12 \times 10^6 \), and for the evaluation of our expression we will take the arithmetic average, i.e., \( N = 7.5 \times 10^6 \); (iii) the effective volume is that of an ellipsoid whose axes are \( l_1 = 45 \times 10^{-6} \text{m} \) and \( l_1 = 150 \times 10^{-6} \text{m} \) such that \( V = 4 \pi l_1^2 l_2 \), and, finally, (iv) a scattering length \( a = 2.75 \times 10^{-6} \text{m} \).
Introducing these values into (27) (setting $\alpha = 0$) implies

$$v^{(m)} = 1.95 \text{ mm/s}. \quad (28)$$

The reported critical speed is [30]

$$v^{(c)} = 1.6 \text{ mm/s} \quad (29)$$

The ensuing error is less that 18 percent

$$\left| v^{(c)} - v^{(m)} \right| / v^{(m)} = 0.179. \quad (30)$$

In other words, our model provides a very good description of the experiment and, hence, the analysis of the deformed case within the present framework seems a reasonable assumption.

V. CONCLUSIONS

In the present work the analysis of the options that superfluidity could offer in the context of quantum gravity phenomenology has been done. The model has been a Bose-condensed sodium gas trapped by an isotropic harmonic oscillator in which the energy–momentum relation for the particles has been deformed along the proposals emerging from some quantum gravity models. Afterwards, along a perturbative approach, the energy and momentum of the elementary excitations generated by the particles in the thermal cloud have been calculated. Finally, we introduce these last two physical parameters in Landau criterion associated to superfluidity and find that the breakdown of Lorentz symmetry, in the form of a deformed dispersion relation, implies a modification of the region in which, for a sodium BEC, superfluidity may exist. If $\alpha > 0$, then the aforementioned region grows (compared to the case in which $\alpha = 0$), whereas, if $\alpha < 0$, then this region becomes smaller. Indeed, the allowed superfluidity velocities are those falling into the interval $(0, v_{(crit)})$. and, clearly, this interval becomes larger for $\alpha > 0$.

In relation with a bound for our phenomenological parameter $\alpha$, a fleeting glimpse at (27) tells us that we must first choose a value for $n$. As an example we take $n = 1$ (a condition that implies that $\alpha$ has units of speed) and consider the experimental values related to the evidence of a critical velocity in a sodium condensed gas [30]. Clearly a choice has to be made in connection with the number of particles since this physical variable changed from experiment to experiment; for the sake of concreteness we consider the highest value, i.e., $N = 12 \times 10^6$, taking the lowest case ($N = 3 \times 10^6$) does not modify the order of magnitude of the ensuing bound.

The measurement readouts [30] are given up to units of tenths of mm/s and, in consequence, the smallest scale can be considered as an approximation for the experimental error [39]. In other words, if $\Delta v$ denotes the experimental error of the measuring device, then the aforementioned argument implies $\Delta v \sim 0.1 \text{ mm/s}$. The experimental error has to be equal or larger than the term containing the effects of the breakdown of Lorentz symmetry, this phrase means for this situation

$$14 \times 10^{-5} \text{ mm/s} \geq \alpha. \quad (31)$$

Additional cases ($n = 2$ does not imply new physics, it entails only a redefinition of the inertial mass) can be analyzed in the same manner. The present argument tells us that we may deduce an upper bound for deformed dispersion relations associated to the structure given by (1), and this expression reads

$$\Delta v \geq \left[ \sqrt{24\pi} \left( \frac{4\pi}{3} \right)^{1/3} n \right] n^{-1} \frac{\sqrt{4\pi N a}}{5\sqrt{1/3}} \left( \frac{h}{V^{1/3}} \right)^{n-1} \alpha. \quad (32)$$

Let us comment that in the present model the deformed dispersion relation has been introduced only in the context of particles (see (17)) but not in connection with the corresponding elementary excitations also called quasi–particles, expression (22)). Clearly, an additional possibility is the introduction of the breakdown of Lorentz symmetry at the level of the kinematics of the quasi–particles. This second option implies the introduction of a second pair of phenomenological parameters, since the quantum gravity modifications could be particle–dependent. This case can be, without any further problem, be considered in the present framework.

Summing up, we have shown that superfluidity in sodium–condensed gases offers a new window in the realm of quantum gravity phenomenology and that the present experimental results are enough to deduce some rough bound for the involved parameters.

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