A note on the GFF with one free boundary condition

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Abstract

In this note, we shall prove an explicit formula on the probability of the level line of the Gaussian Free Field (GFF) with mixed boundary condition terminating at the free boundary, which generalizes the results of GFF with Dirichlet boundary condition.

1 Background

Let us first recall some basic knowledge on Loewner chains ([Law05, LSW01, RS05]). Given a continuous function $W$ from $\mathbb{R}_+$ to $\mathbb{R}$: $t \rightarrow W_t$, the Loewner’s ordinary differential equation (ODE) is defined by

$$\frac{\partial}{\partial t}g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z \in \mathbb{H}. \quad (1.1)$$

For $z \in \mathbb{H} \setminus \{0\}$, let $\tau(z)$ be the blowup time of the Loewner’s ODE above. For $t \geq 0$, let $K_t$ be the set of $z \in \mathbb{H}$ with $\tau(z) \leq t$. Then the collection \{ $K_t : t \in \mathbb{R}_+$ \} forms an increasing family of compact subsets of $\mathbb{H}$ such that for each $t$, $g_t$ is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ satisfying $g_t(z) = z + 2t/z + O(1/|z|^2)$ as $z \rightarrow \infty$. The conformal maps $(g_t)_{t \geq 0}$ (or the compact hulls $(K_t)_{t \geq 0}$) are called the Loewner chain driven by $W$. For $\kappa > 0$ and $B_t$ being the one dimensional standard Brownian motion, the random Loewner chain driven by $W_t = \sqrt{\kappa}B_t$ is called the Schramm Loewner Evolution (SLE$_\kappa$) process. The SLE$_\kappa$ process is generated by a curve $\gamma$ from 0 to $\infty$, i.e., for each $t \geq 0$, $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. There is an variant of SLE$_\kappa$ which is known as the SLE$_\kappa(\rho)$ process. For $\rho \geq 0$, and $y \leq x \in \mathbb{R}$, consider the following Stochastic differential equation system:

$$dW_t = \sqrt{\kappa}dB_t + \frac{\rho}{W_t - V_t}dt, \quad dV_t = \frac{2}{V_t - W_t}, \quad W_0 = x, \quad V_0 = y. \quad (1.2)$$

The random Loewner chain driven by $W$ defined in (1.2) is called the SLE$_\kappa(\rho)$ process from $x$ to $\infty$ with force point $y$. More properties of SLE$_\kappa(\rho)$ process can be found in [MS16].

Next, let us give a brief introduction to the GFF ([DS11, MS16]). For $a \in \mathbb{R}$, the Green function with Dirichlet boundary condition on $(a, \infty)$ and Neumann boundary condition (also called free boundary condition) on $(-\infty, a)$ is given by

$$G_{\text{mix}}(z, w) := \frac{1}{2\pi} \Re \left( \frac{(\sqrt{z - a} + \sqrt{w - a})}{(\sqrt{z - a} - \sqrt{w - a})} \frac{(\sqrt{z - a} - \sqrt{w - a})}{(\sqrt{z - a} + \sqrt{w - a})} \right), \quad z, w \in \mathbb{H}, \quad (1.3)$$

where we take the branch of the square root such that it takes value in the upper half plane.

The GFF with Dirichlet boundary condition on $(a, \infty)$ and Neumann boundary condition (also called free boundary condition) on $(-\infty, a)$ is a centered Gaussian process $\Gamma$ indexed by the set of continuous functions with compact support in $\mathbb{H}$ such that

$$E[\Gamma(f)\Gamma(g)] = \int_{\partial \mathbb{H} \times \partial \mathbb{H}} f(z)G_{\text{mix}}(z, w)g(w)dzdw.$$
2 Results

Given \( n + 1 \) points \( a < b_1 < b_2 < \ldots < b_n \). Let \( \phi \) be the harmonic function on \( \mathbb{H} \) satisfying the Neumann boundary condition on \( (-\infty, a) \) and taking values \( \pm \lambda \) alternatively in the intervals \((b_i, b_{i+1})\) for \( i = 0, 1, 2, \ldots, n \), where we take \( b_0 = a, b_{n+1} = +\infty \) and \( \lambda = \sqrt{\pi/8} \) (see the following figure).

The GFF (denoted by \( h \)) with free boundary condition on \( (\infty, a) \) and alternating boundary conditions \( \pm \lambda \) on \((b_i, b_{i+1})\) for \( i = 1, 2, \ldots, n \) is defined as the Gaussian process \( \Gamma + \phi \), where \( \Gamma \) is defined as the previous section.

Fix \( k \in \{1, 2, 3, \ldots, n\} \), let \( I_k \) be the subset of \( \{1, 2, 3, \ldots, n\} \) that consists of numbers having the same parity as \( k \) and \( J_k = \{1, 2, 3, \ldots, n\} \setminus I_k \). In [IK13, Proposition 6], the authors gave a coupling between the GFF in the strip with arbitrary mixed boundary condition which can be stated as following:

**Theorem 2.1.** There exists a coupling between \( h \) and the random Loewner chain started from \( b_k \) driven by the following SDE systems:

\[
\begin{align*}
\begin{cases}
db_k(t) &= 2dB_t + \frac{1}{b_k(t)-a(t)} dt + \sum_{i \in I_k} F(b_k(t), b_i(t), a(t)) dt - \sum_{i \in J_k} F(b_k(t), b_j(t), a(t)) dt \\
da(t) &= \frac{2}{a(t)-b_k(t)} dt, \quad db_k(t) = \frac{2}{b_k(t)-b_k(t)} dt, \quad \forall i \neq k,
\end{cases}
\end{align*}
\]

where \( F \) is defined as

\[
F(x, y, z) := \frac{2}{x-y} \sqrt{\frac{y-z}{x-z}}, \quad z < x, z < y.
\]

Now we state our main result.

**Theorem 2.2.** The Loewner chain driven by \( b_k(t) \) in (2.1) is almost surely generated by a simple curve \( \gamma[0, T] \to \mathbb{H} \) such that

\[
\gamma(0) = b_k, \quad \gamma(0, T) \subset \mathbb{H}, \quad \gamma(T) = b_i \text{ for some } i \in I_k \text{ or } \gamma(T) \in (-\infty, a).
\]

Moreover, we have

\[
\mathbb{P}(\gamma(T) \in (-\infty, a)) = \prod_{i \in I_k} \frac{1 + \frac{b_i-a}{b_i-a}}{1 - \frac{b_i-a}{b_i-a}} \prod_{i \in J_k} \frac{1 + \frac{b_i-a}{b_i-a}}{1 - \frac{b_i-a}{b_i-a}} \prod_{i \in J_k} \frac{1 - \frac{b_i-a}{b_i-a}}{1 + \frac{b_i-a}{b_i-a}}.
\]

We begin the proof with the following lemma.

**Lemma 2.3.** Suppose that \((g_t)_{t \geq 0}\) is SLE_4(1) process from \( b_k \) to \( \infty \) with force point \( a \). Define \( b_j(t) := g_t(b_j) \) for \( j \neq k \) and \( a(t) = g_t(a) \). Let \( F \) be the function defined as (2.2),

\[
J_t := \sum_{i \in I_k} F(b_k(t), b_i(t), a(t)) dt - \sum_{i \in J_k} F(b_k(t), b_j(t), a(t)) dt,
\]

and

\[
Z_t := \left( \prod_{i \in I_k} \frac{\sqrt{b_k(t) - a(t)} - \sqrt{b_k(t) - a(t)}}{\sqrt{b_k(t) - a(t)} + \sqrt{b_k(t) - a(t)}} \prod_{j \in J_k} \frac{\sqrt{b_j(t) - a(t)} + \sqrt{b_k(t) - a(t)}}{\sqrt{b_j(t) - a(t)} - \sqrt{b_k(t) - a(t)}} \right)^2.
\]

Then \( \log(Z_t) \) is a local martingale and \( d \log(Z_t) = 2J_t dB_t \).
Proof. By [MS16, Theorem 1.3] SLE$_4(-1)$ is generated by a simple curve. So $b_j(t): j \neq k$ and $a(t)$ are well-defined. By definition we have
\[
\frac{1}{2} \log(Z_t) = \sum_{i \in I_k} \log \left( \sqrt{b_i(t) - a(t)} - \sqrt{b_k(t) - a(t)} \right) - \log \left( \sqrt{b_i(t) - a(t)} + \sqrt{b_k(t) - a(t)} \right) + \sum_{j \in J_k} \log \left( \sqrt{b_j(t) - a(t)} + \sqrt{b_k(t) - a(t)} \right) - \log \left( \sqrt{b_j(t) - a(t)} - \sqrt{b_k(t) - a(t)} \right).
\]
So from Itô’s formula, we have
\[
d\log(Z_t) = \sum_{i \in I_k} \frac{-2}{(b_i(t) - b_k(t))(b_k(t) - a(t))} \frac{b_i(t) - a(t)}{b_k(t) - a(t)} dt + \frac{2}{b_k(t) - b_i(t)} \frac{b_i(t) - a(t)}{b_k(t) - a(t)} \left( 2dB_t + \frac{-1}{b_k(t) - a(t)} \right) + \sum_{j \in J_k} \frac{2}{(b_j(t) - b_k(t))(b_k(t) - a(t))} \frac{b_j(t) - a(t)}{b_k(t) - a(t)} dt - \frac{2}{b_k(t) - b_j(t)} \frac{b_j(t) - a(t)}{b_k(t) - a(t)} \left( 2dB_t + \frac{-1}{b_k(t) - a(t)} \right) = 2J_t dB_t.
\]

Now define $N_t := \exp\left\{-\frac{1}{8} \int_0^t J_s^2 ds\right\}$, where $J_t$ is defined as (2.5). And it can be checked that $M_t := Z_t^4 N_t$ is a local martingale up to the first time $T$ that one of $\{a, b_1, ..., b_{k-1}, b_{k+1}, ..., b_n\}$ is swallowed and $dM_t = \frac{1}{2} M_t J_t dB_t$. In fact,
\[
T := \inf \{t: b_k(t) - a(t) = 0 \text{ or } b_k(t) - g_i(b_i) = 0 \text{ for some } i \neq k\}.
\]
Now define
\[
T_n^1 := \inf \left\{t : \frac{b_k(t) - a(t)}{b_i(t) - a(t)} \leq \frac{1}{n} \text{ or } \frac{b_k(t) - a(t)}{b_i(t) - a(t)} \geq 1 - \frac{1}{n} \text{ for some } i > k\right\}. \tag{2.6}
\]
\[
T_n^2 := \inf \left\{t : \frac{b_k(t) - a(t)}{b_i(t) - a(t)} \leq \frac{1}{n} \text{ or } \frac{b_k(t) - a(t)}{b_i(t) - a(t)} \geq 1 - \frac{1}{n} \text{ for some } i < k\right\}. \tag{2.7}
\]
And $T_n := T_n^1 \land T_n^2$. Then we have $(M_t : 0 \leq t \leq T_n)$ is a bounded martingale for any $n$. Since the Loewner chain driven by $\{2.1\}$ can be obtained by weighting SLE$_4(-1)$ with the local martingale $M_t$. We can see that the Loewner chain driven by $\{2.1\}$ is absolutely continuous with respect to SLE$_4(-1)$ up to $T_n$ and therefore the Loewner chain driven by $\{2.1\}$ is generated by a continuous curve up to $T_n$. Since $n$ is arbitrary, we have the Loewner chain driven by $\{2.1\}$ is generated by a continuous curve up to $T$, which we will denote by $\gamma$. We need to show that as $t \to T$, $\gamma(t)$ converges almost surely. Notice that when $\gamma(t)$ accumulates at $\{b_i : i \in I_k\}$, the local martingale is uniformly bounded and therefore it is absolutely continuous with respect to SLE$_4(-1)$ up to and include $T$. But SLE$_4(-1)$ is a continuous curve from $b_k$ to $(-\infty, a)$. So $\gamma$ can not accumulate at $\{b_i : i \in I_k\}$. The same reason excludes the event that $\gamma$ accumulates at $b_k$. So almost surely as $t \to T$, $\gamma(t)$ accumulates at $(-\infty, a)$. Let $E$ be the event that $\gamma(t)$ has accumulation point in $(-\infty, a)$. Then on the event $E$, $M_t$ is bounded and $\gamma$ is absolutely continuous with respect to SLE$_4(-1)$ and therefore is a continuous curve from $b_k$ to $(-\infty, a)$. We have finished the first part of Theorem 2.2.

Now we prove (2.4). Define
\[
g(a, b_1, b_2, ..., b_n) = \prod_{i \in I_k, i < k} \left( 1 - \frac{b_k-a}{b_i-a} \right) \prod_{i \in I_k, i > k} \left( 1 + \frac{b_k-a}{b_i-a} \right) \prod_{i \in J_k, i < k} \left( 1 + \frac{b_k-a}{b_i-a} \right) \prod_{i \in J_k, i > k} \left( 1 - \frac{b_k-a}{b_i-a} \right). \tag{2.9}
\]
One can check that $\tilde{M}_t := g(a(t), b_1(t), b_2(t), ..., b_n(t))$ is a local martingale up to $T$. 

Lemma 2.4. Take the notations as above, \((\tilde{M}_t : 0 \leq t \leq T)\) is a martingale and we have \(\lim_{t \to T} \tilde{M}_t = 1\) if 
\(\gamma(T) \in (-\infty, a)\); \(\lim_{t \to T} \tilde{M}_t = 0\) otherwise.

Proof. Suppose that \(\gamma(T) = b_{i_0}\) for some \(i_0 \in J_k\). Without lost of generality, we may assume \(b_{i_0} < b_k\). Then as \(t \to T\) we have \(b_j(t) - b_k(t) \to 0\) for any \(i_0 \leq j \leq k\). Hence we have
\[
\prod_{i \in J_k, i > k} \frac{1 + \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)} \prod_{i \in J_k, i < i_0} \frac{1 + \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)}, \quad \text{and} \quad \prod_{i \in J_k, i < i_0} \frac{1 - \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)}
\]
We are left to investigate
\[
I := \prod_{i \in J_k, i_0 \leq i < k} \frac{1 + \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)} \prod_{i \in J_k, i_0 \leq i < k} \frac{1 - \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)},
\]
We can put the products in pairs \((i_0, i_0 + 1), (i_0 + 2, i_0 + 3), \ldots, (k - 3, k - 2), (k - 1)\). For each pair \((j, j + 1),\) we have
\[
\frac{1 - \sqrt{b_j(t) - a(t)}}{b_j(t) - a(t)} \cdot \frac{1 + \sqrt{b_{j+1}(t) - a(t)}}{b_{j+1}(t) - a(t)} = \left(1 + \frac{b_{j+1}(t) - a(t)}{b_j(t) - a(t)}\right)^2 \frac{b_k(t) - b_j(t)}{b_j(t) - b_{j+1}(t)} \to 1 \quad \forall i_0 \leq j < k - 1.
\]
So
\[
\lim_{t \to T} \prod_{i \in J_k, i_0 \leq i < k} \frac{1 + \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)} \prod_{i \in J_k, i_0 \leq i < k} \frac{1 - \sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)} = \lim_{t \to T} \frac{1 - \sqrt{b_{j+1}(t) - a(t)}}{b_j(t) - a(t)} = 0.
\]
We get \(\tilde{M}_t \to 0\) as \(t \to T\) when \(\gamma(T) = b_{i_0}\) for some \(i_0 \in J_k\).

Now suppose that we have \(\gamma(T) \in (-\infty, a)\). Then as \(t \to T\), \(a(t) - b_k(t) \to 0\) and \(b_i(t) - b_k(t) \to 0\) for \(i < k\). So
\[
\prod_{i \in J_k, i > k} 1 - \frac{\sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)} \cdot \prod_{i \in J_k, i > k} 1 + \frac{\sqrt{b_i(t) - a(t)}}{b_i(t) - a(t)} \to 1.
\]
By Lemma B.2 of [PW19], we have as \(t \to T\),
\[
\frac{b_i(t) - a(t)}{b_k(t) - a(t)} \to 0, \quad \forall i < k.
\]
Therefore if \(\gamma(T) \in (-\infty, a)\), \(\tilde{M}_t \to 1\) as \(t \to T\). So \((\tilde{M}_t : 0 \leq t \leq T)\) is a uniformly bounded martingale and by the stopping theorem we can get (2.4).

Remark 2.5. Since \(k\) is arbitrary, we can get all the probabilities that the level line started from \(b_k\) terminates at the free boundary arc \((-\infty, a)\). Let \(a\) go to \(\infty\), we can get the crossing probabilities of the GFF with Dirichlet boundary condition which is the Theorem 1.4 of [PW19]. For example, take \(k = 1\) and \(n = 2N - 1\), we have
\[
g(a, b_1, b_2, \ldots, b_n) = \prod_{i \in J_k, i > k} 1 + \frac{b_i - a}{b_i - a} \prod_{i \in J_k, i > k} 1 - \frac{b_i - a}{b_i - a}
\]
By taking \(a \to \infty\), the limit above is equal to
\[
\frac{b_2 - b_1}{b_3 - b_1} \cdot \frac{b_4 - b_1}{b_5 - b_1} \cdot \ldots \cdot \frac{b_{2N-2} - b_1}{b_{2N-1} - b_1},
\]
which is equation (1.8) of [PW19].
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