Maximal Entanglement of Nonorthogonal States: Classification

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Abstract

A necessary and sufficient condition for the maximal entanglement of bipartite nonorthogonal pure states is found. The condition is applied to the maximal entanglement of coherent states. Some new classes of maximally entangled coherent states are explicitly constructed; their limits give rise to maximally entangled Bell-like states.

1 Introduction

Quantum entanglement plays an important role in such areas of quantum information processing as quantum teleportation [1], superdense coding [2], quantum key distribution [3] and telecloning [4]. Recently, entangled nonorthogonal states have attracted much attention in quantum cryptography [5]. Bosonic entangled coherent states [6–10] and $su(2)$ and $su(1,1)$ entangled coherent states [11] are typical examples of entangled nonorthogonal states. For general bipartite nonorthogonal states some conditions have been found for maximal entanglement [12].

In this letter we address the problem of finding necessary and sufficient conditions for the maximal entanglement of bipartite nonorthogonal pure states [13,14]

$$|\psi\rangle = \mu|\alpha\rangle \otimes |\beta\rangle + \nu|\gamma\rangle \otimes |\delta\rangle,$$

(1.1)

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where $\mu, \nu$ are two complex numbers, $|\alpha\rangle$ and $|\gamma\rangle$ are linearly independent normalized states of the systems 1, and $|\beta\rangle$ and $|\delta\rangle$ are linearly independent normalized states of the system 2. Here we are interested in the non-orthogonal case in which the overlaps $\langle \alpha | \gamma \rangle$ and $\langle \beta | \delta \rangle$ are non-vanishing. Note that the normalization constant is not $|\mu|^2 + |\nu|^2$, but

$$
\left[ |\mu|^2 + |\nu|^2 + \mu \nu^* \langle \gamma | \alpha \rangle \langle \delta | \beta \rangle + \mu^* \nu \langle \alpha | \gamma \rangle \langle \beta | \delta \rangle \right]^{-1/2}.
$$

(1.2)

We first give a classification of maximal entanglement for the state (1.1). A necessary and sufficient condition for maximal entanglement is found. We then apply the condition to coherent states and find a simple condition for maximally entangled coherent states in terms of the coherence parameters. Some new classes of maximally entangled coherent states, with relative phase different from the previously-noted $\pi$, are explicitely constructed. In the limit, these states give rise to maximally entangled Bell-like states.

2 Maximal entanglement: classification

The entanglement of a quantum system can be measured by the entanglement of formation, or simply entanglement, which is defined as the entropy of either one of the two subsystems 1 or 2 [15]. It has been pointed out that the entanglement of a two-qubit state $|\Psi\rangle$ can be expressed as a function of the concurrence [16]

$$
C \equiv |\langle \Psi | \sigma_y \otimes \sigma_y | \Psi^* \rangle|,
$$

(2.1)

where $\sigma_y$ is the spin-flip operator and $|\Psi^*\rangle$ is the complex conjugate of $|\Psi\rangle$. Concurrence itself can also be regarded as a measure of entanglement which ranges from 0 to 1 [16]. One may readily obtain the concurrence for the state (1.1) by introducing an orthogonal normalized basis in the subspace spanned by $|\alpha\rangle$ and $|\gamma\rangle$ and by $|\beta\rangle$ and $|\delta\rangle$; it has the value [12]

$$
C = \frac{2 |\mu \nu^* \langle \gamma | \alpha \rangle \langle \delta | \beta \rangle + \mu^* \nu \langle \alpha | \gamma \rangle \langle \beta | \delta \rangle|}{|\mu|^2 + |\nu|^2 + \mu \nu^* \langle \gamma | \alpha \rangle \langle \delta | \beta \rangle + \mu^* \nu \langle \alpha | \gamma \rangle \langle \beta | \delta \rangle}.
$$

(2.2)

The state (1.1) is referred to as a Maximally Entangled State (MES) when $C = 1$.

We now give a necessary and sufficient condition for maximal entanglement ($C = 1$). Let $\mu = k \nu e^{i\theta}$, where $k$ and $\theta$ be real parameters with $k > 0$. Noting
that $|\langle \alpha|\gamma \rangle| \leq 1$ and $|\langle \beta|\delta \rangle| \leq 1$, we write
\[ \langle \alpha|\gamma \rangle = \sin a e^{i\theta_1}, \quad \langle \beta|\delta \rangle = \sin b e^{i\theta_2}, \] (2.3)

where $a, b$ and $\theta_1, \theta_2$ are all real parameters and $0 \leq a, b \leq \pi/2$. Then we may rewrite the condition $C = 1$ as
\[ k' = 2 \cos a \cos b - 2 \sin a \sin b \cos(\theta - \theta_1 - \theta_2), \] (2.4)

where $k' \equiv (k^2 + 1)/k$. We consider two different cases.

**Case 1:** $-1 \leq \cos(\theta - \theta_1 - \theta_2) \leq 0$. In this case we have
\[ k' \leq 2 \cos a \cos b - 2 \sin a \sin b = 2 \cos(a - b) \leq 2, \] (2.5)

denoting $(k - 1)^2 \leq 0$, or $k = 1$, $k' = 2$. Inserting $k' = 2$ into relation (2.5) we also have $\cos(a - b) = 1$, namely $a = b + 2m\pi$ ($m$ an integer). Then from Eq.(2.4) we have
\[ \cos(\theta - \theta_1 - \theta_2) = -1, \] (2.6)

denoting $\theta - \theta_1 - \theta_2 = \pi$. So in this case the MES condition is obtained as
\[ \mu = \nu e^{i\theta}, \quad \langle \alpha|\gamma \rangle = -(\beta|\delta)^* e^{i\theta}. \] (2.7)

**Case 2:** $0 \leq \cos(\theta - \theta_1 - \theta_2) \leq 1$. In this case the second term in (2.4) is always non-negative and thus
\[ k' \leq 2 \cos a \cos b \leq 2, \] (2.8)

which leads to $k = 1$ and $k' = 2$ as in case 1. Then the relation (2.8) is valid only when
\[ \cos a = \cos b = 1 \text{ or } \sin a = \sin b = 0. \] (2.9)

So the MES condition in this case is
\[ |\mu| = |\nu|, \quad \langle \alpha|\gamma \rangle = \langle \beta|\delta \rangle = 0, \] (2.10)

which is clearly the orthogonal case.

In summary we obtain the following theorem
Theorem 1 The states (1.1) are MES if and only if one of the following conditions is satisfied

(1) \( \mu = \nu e^{i\theta} \) and \( \langle \alpha | \gamma \rangle = -\langle \beta | \delta \rangle^* e^{i\theta} \) (\( \theta \) is a real parameter) for the nonorthogonal case;
(2) \( |\mu| = |\nu| \) for the orthogonal case \( \langle \alpha | \gamma \rangle = \langle \beta | \delta \rangle = 0 \).

The necessity can be verified directly.

Before closing this section we remark that the state (1.1) is disentangled \((C = 0)\) if and only if one of the following is true: (1) \( \mu = 0 \); (2) \( \nu = 0 \); (3) \( |\alpha\rangle = \pm|\gamma\rangle \); (4) \( |\beta\rangle = \pm|\delta\rangle \).

3 Maximally entangled states

Now we turn to the explicit construction of MES. It is easy to see that condition (2.7) is satisfied when

\[
|\gamma\rangle = e^{i\vartheta}|\beta\rangle, \quad |\delta\rangle = e^{i(\theta+\pi-\vartheta)}|\alpha\rangle,
\]

where \( \vartheta \) is an arbitrary phase. In this case, the normalized MES is obtained as

\[
|\psi\rangle = \frac{\mu}{|\mu| \sqrt{2(1-|\langle \alpha | \beta \rangle|^2)}} (|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle),
\]

which is just the antisymmetric MES given in paper [12]. Note that the phase \( e^{i\theta} \) does not enter the anti-symmetric state. It is natural to ask if there exist MES other than the antisymmetric MES. The answer is positive.

Let us consider entangled coherent states; namely, all four states in (1.1) are coherent states

\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\]

where \( \alpha \) is an arbitrary complex number. The overlap between two coherent states \( |\alpha\rangle \) and \( |\gamma\rangle \) is

\[
\langle \alpha | \gamma \rangle = \exp \left[ -\frac{1}{2} \left( |\alpha|^2 + |\gamma|^2 - 2\alpha^* \gamma \right) \right].
\]
Then the MES condition (2.7) simplifies to

\[ |\alpha^2 + |\gamma|^2 - 2 \alpha^* \gamma| = |\beta|^2 + |\delta|^2 - 2 \beta \delta^* - 2i(\theta + \pi), \quad (3.5) \]

from which we conclude that

**Theorem 2**  
**Coherent states are maximally entangled if and only if**  
\[ \mu = \nu \exp(i\theta) \]  
**and both sides of Eq. (3.5) have the same real part**

\[ |\alpha^2 + |\gamma|^2 - 2 \text{Re}(\alpha^* \gamma)| = |\beta|^2 + |\delta|^2 - 2 \text{Re}(\beta \delta^*). \quad (3.6) \]

The difference of their imaginary parts gives rise to a relative phase \( \theta \)

\[ \theta = \text{Im}(\alpha^* \gamma) - \text{Im}(\beta \delta^*) - \pi. \quad (3.7) \]

As an example we can choose \( \alpha \) and \( \delta \) such that they have the same phase, as do \( \beta \) and \( \delta \). In this case \( \alpha^* \gamma = |\alpha||\gamma|, \beta \delta^* = |\beta||\delta| \) and \( \theta = -\pi \). Then the MES condition is obtained as

\[ |\alpha| - |\gamma| = \pm(|\beta| - |\delta|) = \lambda', \quad (3.8) \]

where \( \lambda' \) is a real parameter, and the corresponding MES are

\[ |\alpha\rangle \otimes |\beta\rangle - |(1 - \lambda'/|\alpha|)\alpha\rangle \otimes |(1 \mp \lambda'/|\beta|)\beta\rangle. \quad (3.9) \]

If we further choose \( \beta = -\alpha \) and \( \lambda' = 2|\alpha| \), we obtain the well-known MES

\[ |\alpha\rangle \otimes |-\alpha\rangle - |-\alpha\rangle \otimes |\alpha\rangle \quad (3.10) \]

and a new MES

\[ |\alpha\rangle \otimes |-\alpha\rangle - |-\alpha\rangle \otimes |-3\alpha\rangle. \quad (3.11) \]

Both states (3.10) and (3.11) have the same normalization constant \( [2(1 - e^{-4|\alpha|^2})]^{-\frac{1}{2}} \).

We now give an example in which the relative phase is not \( \pi \). Suppose that \( \alpha \) and \( \gamma \) have a phase difference of \( \pi/2 \), as therefore do \( \beta \) and \( \gamma \), namely;

\[ \frac{\gamma}{|\gamma|} = \frac{\alpha}{|\alpha|} e^{i\pi/2}, \quad \frac{\delta}{|\delta|} = \frac{\beta}{|\beta|} e^{\pm i\pi/2}, \quad (3.12) \]
Then $\alpha^*\gamma$ and $\beta\delta^*$ are pure imaginary

$$\alpha^*\gamma = i|\alpha\gamma|, \quad \beta\delta^* = \mp i|\beta\delta|. \quad (3.13)$$

So the MES conditions in this case are

$$|\alpha|^2 + |\gamma|^2 = |\beta|^2 + |\delta|^2, \quad (3.14)$$
$$\theta = |\alpha\gamma| \pm |\beta\delta| - \pi, \quad (3.15)$$

and the MES is obtained as

$$|\alpha\rangle \otimes |\beta\rangle - e^{-i|\alpha\gamma||\beta\delta|} |i\alpha\gamma|/|\alpha\rangle \otimes |\pm i|\beta\delta|/|\beta\rangle\rangle. \quad (3.16)$$

In the case $|\alpha| = |\beta| = |\gamma| = |\delta|$, the MES states further simplify to

$$|\alpha\rangle \otimes |\beta\rangle - e^{-i(|\alpha|^2+|\alpha|^2)} |i\alpha\rangle \otimes |\pm i\beta\rangle. \quad (3.17)$$

In particular, when $\alpha = \pm \beta$, we obtain some new types of MES

$$|\alpha\rangle \otimes |\gamma\rangle - |i\alpha\rangle \otimes |i\alpha\rangle, \quad (3.18)$$
$$|\alpha\rangle \otimes |\gamma\rangle - e^{-i|\alpha|^2} |i\alpha\rangle \otimes |\pm i\alpha\rangle, \quad (3.19)$$
$$|\alpha\rangle \otimes |\gamma\rangle - |i\alpha\rangle \otimes |\pm i\alpha\rangle, \quad (3.20)$$
$$|\alpha\rangle \otimes |\gamma\rangle - e^{-i2|\alpha|^2} |i\alpha\rangle \otimes |\pm i\alpha\rangle. \quad (3.21)$$

These all have the same normalization constant

$$\frac{1}{\sqrt{2(1 - e^{-2|\alpha|^2})}}. \quad (3.22)$$

It is interesting that the last two states (3.20) and (3.21) can be obtained from the first two states (3.18) and (3.19), respectively, by the local transformation $1 \otimes (-1)^{a_1^+a_2}$, where $a_1^+$ and $a_2$ are creation and annihilation operators of the second harmonic oscillator system respectively.

Note that the property of maximal entanglement of the above states is independent of the value of $\alpha$. Let us consider the state (3.18). We expand it in Fock space as

$$\frac{e^{-|\alpha|^2}}{\sqrt{2(1 - e^{-2|\alpha|^2})}} \sum_{m,n=0}^{\infty} \frac{\alpha^{m+n}}{\sqrt{m!n!}} \left((-1)^n - i^{m+n}\right) |m\rangle \otimes |n\rangle. \quad (3.23)$$
in which only terms with $m + n = 1$ survive in the limit $|\alpha| \to 0$ (there is no term $|0\rangle \otimes |0\rangle$). The limiting state is readily obtained as

\[
\frac{1}{\sqrt{2}} \left( e^{i\pi/4} |0\rangle \otimes |1\rangle - e^{-i\pi/4} |1\rangle \otimes |0\rangle \right),
\]

which is clearly a Bell-like MES. We may similarly show that the states (3.19 - 3.21) degenerate to the following orthogonal Bell-like MES in the limit $|\alpha| \to 0$

\[
\frac{1}{\sqrt{2}} \left( e^{i\pi/4} |0\rangle \otimes |1\rangle + e^{-i\pi/4} |1\rangle \otimes |0\rangle \right),
\]

\[
\frac{1}{\sqrt{2}} \left( |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \right),
\]

\[
\frac{1}{\sqrt{2}} \left( |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \right),
\]

respectively, and the states Eq.(3.10) and Eq.(3.11) degenerate to (3.27).

4 Conclusion

In this letter we gave necessary and sufficient conditions for the maximal entanglement of bipartite nonorthogonal pure states (1.1). We then applied these conditions to entangled coherent states and explicitly constructed some new types of maximally entangled coherent states. Apart from the antisymmetric example (3.10), these maximally entangled coherent states are novel in that they have relative phases other than $\pi$. In the limit when the coherence parameter tends to zero, these maximally entangled coherent states give rise to maximally entangled Bell-like states. We intend to generalize this formalism to the case of mixed states and consider possible applications in the area of quantum information.

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