THE FORMULA FOR THE REGULARIZED TRACE OF
THE STURM-LIOUVILLE OPERATOR WITH
A LOGARITHMIC POTENTIAL

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Abstract. We have obtained a regularized trace formula for the Sturm-Liouville operator on a semi-axis with a logarithmic potential.

1. Introduction

Let the function \( q \) be defined on \((0, +\infty)\), real-valued, summable on any finite interval \((0, b)\), \( b > 0 \), and

\[
\lim_{x \to +\infty} q(x) = +\infty.
\]

Further, let \( L \) be the operator generated in \( L^2((0, +\infty)) \) by the differential expression \( ly := -y'' + qy \) and the boundary condition \( y(0) = 0 \) [1, Ch. V, § 18]. According to the well-known A. M. Molchanov theorem [2], operator \( L \) has a discrete spectrum. In the paper [3] an asymptotic equation for the spectrum of the operator \( L \) whose potential can grow arbitrarily slowly was obtained. This equation allows us to calculate the first few (up to a summable remainder) terms of the asymptotic series for the eigenvalues in the case

\[
q = \log \ldots \log x, \ m \in \mathbb{N}, \ a = \text{const} > \begin{cases} e^{m-2}, & m \geq 2, \\ 0, & m = 1. \end{cases}
\]

In particular, when \( q = \log(x + a) \)

\[
\lambda_k = s_k + O\left(k^{-1}(\log k)^{-3/2}\right), \quad \lambda_k = s_k + O\left(\log(2\sqrt{\pi}k) - k^{-1}\left(\frac{a}{\pi} \sqrt{\log k + k_0 - 1/4} + \frac{c_0}{\sqrt{\log k}}\right)\right),
\]

where \( c_0 = \frac{a}{2\pi} (1 + \log(2\sqrt{\pi}/a)) \). A parameter \( k_0 \) is some positive integer (regularization defect) that ensures the convergence of the series

\[
\sum_{k=2}^{\infty} (\lambda_k - s_k).
\]

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The purpose of this paper is to find the value of $k_0$ and to calculate the sum of the series (3), which is called the regularized trace of the operator $L_a$ with the potential $q = \log(x + a)$.

The first $L$ operator with the potential $q(x) = \log^+ x := \max\{\log x, 0\}$ was considered in [4] as an example of Feynman–Kac theory application [5, §X.11] (in combination with Karamata's Tauberian theorem [6, Ch. XII, § 7]) to the problem of finding the asymptotics of the function

$$N(\lambda) := \sum_{k_1 < \lambda} 1.$$  

(4)

This result was summarized by K. Kh. Boymatov [7] on Sturm–Liouville operators with a matrix potential that allows growth of order $\log \ldots \log x$ ($m \in \mathbb{N}$).

Regularized traces of the form (3) in the case of power growth potentials are well studied (see [8] and references therein). From the formulas (1) – (2) it is clear that for any $n$ operator $L_a^{-n}$ is not a trace-class operator, therefore the classical method of zeta-functions [9, 10] is not applicable in this situation. On the other hand, due to the exponential growth of the function (4) $\theta$-function of the operator $L_a$

$$\Theta_a(t) := \sum_{k=1}^{\infty} e^{-t\lambda_k}.$$  

(5)

is determined only on the half-plane $\Re t > c$ with some positive $c$. Therefore, the parabolic equation method [11] using asymptotics $\Theta_a(t)$ as $t \to +0$ seems inapplicable in this case. However, formulas (1) – (2) allow us to construct an analytic continuation of the function $\Theta_a$ on the half-plane $\Re t > 0$. Using this fact, it is possible to find the sum of a series (3).

2. Asymptotics of the function $\Theta_a$

By virtue of formulas (1) – (2) and (5) the function $\Theta_a$ holomorphic in the half-plane $\Re t > 1$.

**Theorem 1.** The function $\Theta_a$ admits a meromorphic (with a single pole at $t = 1$) continuation to the half-plane $\Re t > 0$ and as $t \to +0$ the following asymptotic decomposition

$$\Theta_a(t) \sim -\frac{a}{2\sqrt{\pi}} t^{-1/2} - (k_0 + 1/4) + \theta_0 t^{1/2} + \theta_1 t + O(t^{3/2}),$$  

(6)

where

$$\theta_0 = \frac{a(\log a - 1)}{2\sqrt{\pi}},$$

$$\theta_1 = \lambda_1 + \sigma(2) + \frac{a}{\pi} \int_1^{\infty} \{x \left( (\log x)^{1/2} x^{-1} \right) \}' dx$$

$$+ \left( k_0 - \frac{1}{4} \right) (\gamma - 1) - \left( k_0 + \frac{5}{4} \right) \log 2\sqrt{\pi} + \frac{1}{2} \log 2\pi$$
\[
+ \frac{a}{2\pi} \left( 1 + \log \frac{2\sqrt{\pi}}{a} \right) \int_1^\infty x \left( \log x \right)^{-1/2} x^{-1} \, dx,
\]

\(\gamma - Euler's \, constant.\)

**Proof.** We transform the decomposition (1) – (2) to a form, which is more convenient for finding the asymptotics of the function \(\Theta_a: \)

\[
\lambda_k = \log k + c_1 + c_2 (\log k)^{1/2} k^{-1} + c_3 k^{-1} + c_0 (\log k)^{-1/2} k^{-1} + r_k,
\]

where \( c_1 = \log 2\sqrt{\pi}, \quad c_2 = \frac{a}{\pi}, \quad c_3 = k_0 - \frac{1}{4}, \quad c_0 = \frac{a}{2\pi} \left( 1 + \log(2\sqrt{\pi}/a) \right), \)

\[
r_k = O(k^{-1}(\log k)^{-3/2}), \quad k \to \infty.
\]

Then

\[
\Theta_a(t) = \sum_{k=1}^\infty e^{-t\lambda_k} = e^{-t\lambda_1} + \sum_{k=2}^\infty \sum_{r} e^{-t(\log k + c_2(\log k)^{1/2} k^{-1} + c_3 k^{-1} + c_0 (\log k)^{-1/2} k^{-1} + r_k)}
\]

\[
= e^{-t\lambda_1} + \sum_{k=2}^\infty \sum_{r} e^{-c_2 t (\log k)^{1/2} k^{-1}} e^{-c_3 t k^{-1}} e^{-c_0 t (\log k)^{-1/2} k^{-1}} e^{-tr_k}.
\]

Putting each exponent according to the Taylor formula, we get

\[
e^{-c_0 t k^{-1}} = 1 - c_3 t k^{-1} + O(t^2 k^{-2});
\]

\[
e^{-c_2 t (\log k)^{1/2} k^{-1}} = 1 - c_0 t (\log k)^{-1/2} k^{-1} + O(t^2 (\log k)^{-1} k^{-2});
\]

\[
e^{-c_2 t (\log k)^{1/2} k^{-1}} = 1 - c_2 t (\log k)^{1/2} k^{-1} + O(t^2 (\log k)^{-2});
\]

\[
e^{-tr_k} = 1 - tr_k + O(t^2 r_k^2).
\]

Substituting these expansions into (9), we have

\[
\Theta_a(t) = e^{-t\lambda_1} + e^{-tc_1} \left\{ \zeta(t) - 1 - t \left[ c_2 \varphi_1(t) + c_3 (\zeta(t + 1) - 1) \right. \right.
\]

\[
\left. + \left. c_0 \varphi_2(t) + \varphi_3(t) \right] \right\} + \varphi_4(t),
\]

where

\[
\varphi_1(t) = \sum_{k=2}^\infty (\log k)^{1/2} k^{-1-t}, \quad \varphi_2(t) = \sum_{k=2}^\infty (\log k)^{-1/2} k^{-1-t},
\]

\[
\varphi_3(t) = \sum_{k=2}^\infty k^{-t} r_k, \quad \varphi_4(t) = e^{-tc_1} \sum_{k=2}^\infty k^{-t} R_k(t),
\]

\[
R_k(t) = O \left( t^2 (\log k)^k \right).
\]

Therefore, the function \(\Theta_a\) admits a meromorphic extension to the half-plane \(\Re \, t > 0\) with a single pole at \(t = 1\) (because of the term \(\zeta(t)\)). Find the asymptotics \(\Theta_a(t)\) as \(t \to +0.\)
Let \( \rho(x) = x - \{x\} \), where \( \{x\} \) is fractional part of the number \( x \). Then \( \forall \ 0 < \varepsilon < 1 \)

\[
\varphi_1(t) = \int_{1+\varepsilon}^{\infty} (\log x)^{1/2} x^{-1-t} d\rho(x) = \int_{1+\varepsilon}^{\infty} (\log x)^{1/2} x^{-1-t} dx + \varepsilon \left( \log(1+\varepsilon) \right)^{1/2} (1 + \varepsilon)^{-1-t} + \int_{1+\varepsilon}^{\infty} \{x\} \left( (\log x)^{1/2} x^{-1-t} \right)' dx,
\]

whence as \( \varepsilon \to +0 \) we get

\[
\varphi_1(t) = \int_{1}^{\infty} (\log x)^{1/2} x^{-1-t} d t + \int_{1}^{\infty} \{x\} \left( (\log x)^{1/2} x^{-1-t} \right)' dx =: \varphi_{11}(t) + \varphi_{12}(t).
\]

Direct calculation gives

\[
\varphi_{11}(t) = \frac{\sqrt{\pi}}{2} t^{-3/2}.
\]

Further,

\[
\varphi_{12}(t) = \int_{1}^{\infty} \{x\} \left[ -(1+t)(\log x)^{1/2} + \frac{1}{2}(\log x)^{-1/2} \right] x^{-2-t} dx.
\]

Denote the integrand by \( f(x,t) \). Then \( \forall k \)

\[
\frac{\partial^k}{\partial t^k} f(x,t) = O \left( (\log x)^{-1/2+k} x^{-2} \right), \quad x \geq 1,
\]

uniformly in \( \Re t > 0 \), therefore \( \varphi_{12} \) is holomorphic at zero, so

\[
\varphi_{1,2}(t) = \varphi_{1,2}(0) + O(t), \quad t \to 0.
\]

Therefore,

\[
\varphi_1(t) = \frac{\sqrt{\pi}}{2} t^{-3/2} + \int_{1}^{\infty} \{x\} \left( (\log x)^{1/2} x^{-1-t} \right)' dx + O(t), \quad t \to +0. \tag{11}
\]

Similarly, it is proved that

\[
\varphi_2(t) = \sqrt{\pi} t^{-1/2} + \int_{1}^{\infty} \{x\} \left( (\log x)^{-1/2} x^{-1} \right)' dx. \tag{12}
\]

Let us introduce the function

\[
\sigma(x) = \sum_{k\geq x} r_k.
\]

From (1) – (2) it follows that

\[
\sigma(x) = O \left( (\log x)^{-1/2} \right), \quad x \to \infty.
\]

Then

\[
\varphi_3(t) = -\int_{2}^{\infty} x^{-t} d \sigma(x) = \frac{\sigma(2)}{2t} - t \int_{2}^{\infty} x^{-t-1} \sigma(x) dx = \sigma(2) + O(t^{1/2}), \ t \to +0. \tag{13}
\]
Further, due to the well-known properties of $\zeta$-functions (see, for example, [12, Ch. II, 10])

\[
\begin{align*}
\zeta(t) &= -\frac{1}{2} - \frac{1}{2} \log(2\pi) t + O(t^2), \\
\zeta(t + 1) &= \frac{1}{t} + \gamma + O(t).
\end{align*}
\]

Now substituting (11) – (15) into (10), after simple calculations we get (6). The theorem is proved.

3. Operator $e^{-tL_a}$ and its trace

3.1. Estimation of the resolvent kernel for the operator $L_a$

We introduce the notation. Let be

\[
\Omega_a = \mathbb{C} \setminus \{\log a, +\infty\}, \quad \Omega_{r_\sigma} = \{\sigma \leq \arg(\lambda - \log a - r) \leq 2\pi - \sigma\}, \quad r > 0, 0 < \sigma < \pi,
\]

$\phi(x, \lambda)$ and $\psi(x, \lambda)$ – solutions of the equation

\[
-y'' + qa y = \lambda y
\]

that satisfy the following conditions:

\[
\begin{align*}
\phi(0, \lambda) &= 0, \quad \frac{\partial \phi}{\partial x}(0, \lambda) = 1, \\
\psi(x, \lambda) &\sim (qa(x) - \lambda)^{-1/4} \exp\left(- \int_0^x (qa(t) - \lambda)^{1/2} dt\right), \quad x \to +\infty.
\end{align*}
\]

Hereinafter, the expression $z^{1/n}$ will mean that branch of the root $\sqrt[n]{z}$, which is positive for positive $z$. Since for each fixed $\lambda \notin [qa(b), +\infty)$ the function

\[
\alpha(x, \lambda) = \frac{1}{8} \frac{qa''(x)}{(qa(x) - \lambda)^{3/2}} - \frac{5}{32} \frac{qa''(x)}{(qa(x) - \lambda)^{5/2}}
\]

is summable on $(b, +\infty)$, there exists a unique solution $\psi$, satisfying (17),[13, Ch. II, § 6].

The resolvent kernel for the operator $L_a$ has the form

\[
G(x, y, \lambda) = \frac{1}{\psi(0, \lambda)} \left\{ \begin{array}{ll}
\psi(x, \lambda)\phi(t, \lambda), & x < t < \infty, \\
\phi(x, \lambda)\psi(t, \lambda), & 0 < t < x.
\end{array} \right.
\]

Therefore,

\[
G(x, x, \lambda) = \frac{1}{\psi(0, \lambda)} \phi(x, \lambda)\psi(x, \lambda).
\]

**Lemma 1.** For each $\epsilon > 0$ there is a constant $r_\epsilon > 0$ such that if $\epsilon \leq \sigma < \pi$, $r \geq r_\epsilon$, then for all $x \geq 0$ and $\lambda \in \Omega_{r_\sigma}$ function $\psi(x, \lambda)$ permits the representation

\[
\psi(x, \lambda) \sim (qa(x) - \lambda)^{-1/4} \exp\left(- \int_0^x (qa(t) - \lambda)^{1/2} dt\right) \times
\]
\[ x \left[ 1 + \int_{x}^{+\infty} \left( \alpha' + \frac{\alpha'}{2\sqrt{q-\lambda}} + R \right) dt \right], \quad (21) \]

where \( \alpha \) is defined by the formula (18) and

\[ \sup_{x \geq 0, \lambda \in \Omega_{r\sigma}} |R(x, \lambda)(x + a)^3(q - \lambda)^{5/2}| < \infty. \quad (22) \]

**Proof.** Substituting

\[ \psi(x, \lambda) = (q_a(x) - \lambda)^{-1/4} \exp \left( - \int_0^x (q_a(t) - \lambda)^{1/2} dt + \int_{x_0}^x (-\alpha + \beta) dt \right), \quad (23) \]

we get the equation for \( \beta \)

\[ \beta' - 2(\beta_0 + \beta_1 + \alpha)\beta + 2\beta_1\alpha + \alpha^2 - \alpha' + \beta^2 = 0, \]

where

\[ \beta_0 = \sqrt{q_a - \lambda}, \quad \beta_1 = \frac{1}{4} \frac{q'_a}{q_a - \lambda}. \]

The method of variation of constants leads to the equation

\[ \beta = \gamma + \int_{x}^{+\infty} e^{-2 \int_{x}^{t} (\beta_0 + a)} dt (q_a(x) - \lambda)^{1/2} (q_a(t) - \lambda)^{-1/2} \beta^2 dt, \quad (24) \]

where

\[ \gamma = \int_{x}^{+\infty} e^{-2 \int_{x}^{t} (\beta_0 + a)} dt (q_a(x) - \lambda)^{1/2} (q_a(t) - \lambda)^{-1/2} (2\beta_1 \alpha - \alpha' + \alpha^2) dt. \]

By direct calculations, it is easy to verify that for every \( r > 0, \) \( 0 < \sigma < \pi \)

\[ |q_a(t) - \lambda| > \sin \sigma |q_a(x) - \lambda| \quad \forall \ t \geq x, \lambda \in \Omega_{r\sigma}. \]

Integrating in parts and taking into account the last inequality, we get

\[ \gamma = -\frac{\alpha'}{2\sqrt{q_a - \lambda}} + O \left( (x + a)^{-3}(q_a(x) - \lambda)^{-5/2} \right) \]

uniformly in \( x \geq 0 \) and \( \lambda \in \Omega_{r\sigma} \) for all \( r > 0, \) \( 0 < \sigma < \pi. \)

Replacing

\[ R = (x + a)^3(q_a(x) - \lambda)^{5/2} \left( \beta + \frac{\alpha'}{2\sqrt{q_a - \lambda}} \right) \quad (25) \]

converts the equation (24) to the form

\[ R(x, \lambda) = \tilde{\gamma}(x, \lambda) + \int_{x}^{+\infty} K(x, t, \lambda) R^2(t, \lambda) dt, \quad (26) \]
where functions $\tilde{\gamma}(x, \lambda)$ and $K(x, t, \lambda)$ are continuous on $[0, +\infty) \times \Omega_a$ and $[0, +\infty)^2 \times \Omega_a$ respectively. Moreover,

$$\tilde{\gamma}(x, \lambda) = O(1), \quad K(x, t, \lambda) = O\left((x+a)^{-3}(q_a(x) - \lambda)^{-5/2}\right)$$

uniformly in $x \geq 0$ and $\lambda \in \Omega_{r\sigma}$ (for all $r > 0, 0 < \sigma < \pi$). According to the second estimate for any $\varepsilon > 0$ there is a sufficiently large $r_\varepsilon > 0$ such that for all $r \geq r_\varepsilon$ and $\varepsilon \leq \sigma < \pi$ the integral operator on the right side of (26) is a contraction in a Banach space $C([0, +\infty) \times \Omega_{r\sigma})$. The method of successive approximations shows that for all $r \geq r_\varepsilon$ and $\varepsilon \leq \sigma < \pi$, the equation (26) has a unique solution $R \in C([0, +\infty) \times \Omega_{r\sigma})$. Hence, according to the equalities (25) and (23), for $x_0 = +\infty$, the representation (21) with the estimate (22) follows. The lemma is proved.

According to the proof of the lemma it follows that if $\varepsilon \leq \sigma < \pi, r \geq r_\varepsilon$ then the function $\psi(x, \lambda)$ for every $\lambda \in \Omega_{r\sigma}$ does not have zeros on $[0, +\infty)$. Therefore, the solution $\varphi(x, \lambda)$ can be accepted in the form

$$\varphi(x, \lambda) = \psi(0, \lambda)\psi(x, \lambda) \int_0^x \psi^{-2}(t, \lambda)dt. \quad (27)$$

Substituting this expression into (20), we get

$$G(x, x, \lambda) = \psi^2(x, \lambda) \int_0^x \psi^{-2}(t, \lambda)dt. \quad (28)$$

We set

$$\Psi(x, \lambda) = \int_0^x \psi^{-2}(t, \lambda)dt, \quad x > 0, \lambda \in \Omega_{r\sigma}, \varepsilon \leq \sigma < \pi, r \geq r_\varepsilon. \quad (29)$$

**Lemma 2.** Let $\varepsilon \leq \sigma < \pi, r \geq r_\varepsilon$. Then

$$\Psi(x, \lambda) \sim \frac{1}{2} \exp\left(2 \int_0^x (q_a(t) - \lambda)^{1/2}dt\right) \times \left[1 - 2 \int_x^{+\infty} \frac{\alpha' + \alpha^'/2}{\sqrt{q - \lambda}} dt + Q(x, \lambda)\right], \quad (30)$$

where

$$\sup_{x \geq 0, \lambda \in \Omega_{r\sigma}} |Q(x, \lambda)(x + a)^2(q - \lambda)^{5/2}| < \infty. \quad (31)$$

**Proof.** Fix $\varepsilon > 0$ and everywhere until the end of the proof of the lemma we assume that $\varepsilon < \sigma < \pi, \lambda \in \Omega_{r\sigma}, x > 0$. The function $\Psi$ can be represented as

$$\Psi(x, \lambda) = \int_0^{x/2} \psi^{-2}(t, \lambda)dt + \int_{x/2}^x \psi^{-2}(t, \lambda)dt =: \Psi_1(x, \lambda) + \Psi_2(x, \lambda). \quad (32)$$

Since for all $\sigma - \pi < \arg q_a(x) - \lambda < \pi - \sigma$ then $\forall \ 0 \leq t < x/2$

$$\Re\left(\int_t^x \sqrt{q_a - \lambda}dt\right) > \Re\left(\int_{x/2}^x \sqrt{q_a - \lambda}dt\right)$$

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Further, since $\forall \ t \in [x/2,x] \ |q_a(x) - q_a(t)| < \log 2$ and $|q_a(x) - \lambda| > r \sin \sigma$ then for all $r > 2\log 2/\sin \sigma$
\begin{equation}
\frac{1}{2} < \left| \frac{q_a(x) - \lambda}{q_a(t) - \lambda} \right| < \frac{3}{2} \quad \forall \ t \in [x/2,x], \lambda \in \Omega_{r\sigma}.
\end{equation}
Hence, taking into account (33), we conclude
\begin{equation}
\Re \left( \int_{t}^{x} \sqrt{q_a - \lambda} \, d\tau \right) \geq C |q_a(x) - \lambda|^{1/2} x.
\end{equation}
Therefore, for all $x \geq 0, \lambda \in \Omega_{r\sigma}$, where $r > 2\log 2/\sin \epsilon$,
\begin{equation}
|\Psi_1(x, \lambda)| \leq C(r) \left| e^{2\int_{t}^{x}(q_a(t) - \lambda)^{1/2} \, dt} (x + a)^{-2} (q(x) - \lambda)^{-5/2} \right|,
\end{equation}
$C(\epsilon) > 0$ depends only on $\epsilon$.

Next, we substitute the expression (21) into the formula for $\Psi_2$ and integrate the latter in parts. Consequently, according to (34), we obtain (30), (31).

**Lemma 3.** Let $0 < \epsilon \leq \sigma < \pi$. Then there exists $r_\epsilon > 0$ so that the representation is true
\begin{equation}
G(x, x, \lambda) = \frac{1}{2} (q_a(x) - \lambda)^{-1/2} \left( 1 - e^{-2\int_{t}^{x} \sqrt{q_a(t) - \lambda} \, dt} + g(x, \lambda) \right),
\end{equation}
\begin{equation}
|g(x, \lambda)| \leq C(\epsilon) (x + a)^{-2} |q(x) - \lambda|^{-5/2} \quad \forall \ x \geq 0, \lambda \in \Omega_{r\sigma},
\end{equation}
where $r > r_\epsilon, C(\epsilon) > 0$ depends only on $\epsilon$.

**Proof.** It follows from the formula (28) and Lemmas 2, 3.

### 3.2. Formula for the kernel of the operator $e^{-tL_a}$

Let $0 < \epsilon \leq \sigma < \pi, r > r_\epsilon$, where $r_\epsilon$ is the constant appearing in the formulation of the Lemma 3, $\gamma_{r\sigma}$ is the boundary of the region $\Omega_{r\sigma}$ with the counterclockwise direction$^1$.

**Lemma 4.** For any $t > 0$ the representation
\begin{equation}
e^{-tL_a} = \frac{1}{2\pi i} \int_{\gamma_{r\sigma}} e^{-t\lambda} (\lambda - L_a)^{-1} \, d\lambda
\end{equation}
holds. Here the limit is understood in the sense of convergence in the uniform operator topology.

$^1$Since $\lambda_1 > \log a$, then bypassing $\gamma_{r\sigma}$ the entire spectrum of $L_a$ remains on the left.
Proof. Since $e^{-tL_a}$ is a bounded operator, then

$$e^{-tL_a} = \lim_{n \to \infty} e^{-tL_a} P_n,$$

where $P_n$ is orthogonal projector on the linear span of the first $n$ eigenvectors of the operator $L_a$ and the limit is understood in the sense of a convergence in the norm. We have (see, for example, [14, § XII.2])

$$P_n = \frac{1}{2\pi i} \int_{\gamma_n} (\lambda - L_a)^{-1} d\lambda,$$

where $\gamma_n$ is any contour covering the first $n$ eigenvalues of the operator $L_a$, the integral is taken in the counterclockwise direction. Then

$$e^{-tL_a} P_n = \frac{1}{2\pi i} \int_{\gamma_n} e^{-tL_a}(\lambda - L_a)^{-1} d\lambda.$$

As $\gamma_n$ we take a contour composed of $l_n$ parts of $\gamma_{ra}$ lying in the half-plane $\Re \lambda < (\lambda_n + \lambda_{n+1})/2$, and the segment $l_n$ connecting the ends of $\gamma_n$.

Let

$$R_n = \int_{l_n} e^{-tL_a}(\lambda - L_a)^{-1} d\lambda.$$

Let us prove that for any $\Re t > 1$

$$\|R_n\| \to 0, \quad n \to \infty.$$  \hfill (39)

We have

$$\| (\lambda - L_a)^{-1} \| \leq \frac{1}{\text{dist}(\lambda, \sigma(L_a))}. \hfill (40)$$

Hence, for all $\lambda$ from $[N_-, N_+]$, where $N_{\pm} = (\lambda_n + \lambda_{n+1})/2 \pm i$, we will have $\| (\lambda - L_a)^{-1} \| \leq 2/(\lambda_{n+1} - \lambda_n)$. According to (1) - (2) $\lambda_{n+1} - \lambda_n \sim n^{-1}, \quad n \to \infty$. Hence, for the operator

$$r_n = \int_{[N_-, N_+]} e^{-tL_a}(\lambda - L_a)^{-1} d\lambda$$

we get

$$\| r_n \| = O\left(n^{-1+1}\right), \quad n \to \infty.$$  \hfill (41)

Further, since according to (40) $\| (\lambda - L_a)^{-1} \| < 1$ on $l_n \backslash [N_-, N_+]$, then

$$\| R_n - r_n \| = O\left(e^{-tn}\right), \quad n \to \infty.$$  \hfill (42)

Now (39) directly follows from (41) and (42).
3.3. Asymptotics of $\text{tr} \left( e^{-t L_a} \right)$

Lemma 5. For $\Re t > 1$

$$
\text{tr} \left( e^{-t L_a} \right) = -\frac{1}{2\pi i} \int_{\gamma} d\lambda \int_{\gamma^\sigma} e^{-t\lambda} G(x, x, \lambda) d\lambda.
$$

Proof. By Lemma 4

$$(e^{-t L_a} f)(x) = -\frac{1}{2\pi i} \int_{\gamma} d\lambda e^{-t\lambda} \int_{\gamma} G(x, y, \lambda) f(y) dy.$$  \hspace{1cm} (43)

According to formulas (19), (27) and Lemmas 1 and 2

$$|G(x, y, \lambda)| \leq C |(q_a(x) - \lambda)^{-1/4}| q_a(y) - \lambda|^{-1/4} \exp \left( -\Re \int_{x}^{y} \sqrt{q_a(t) - \lambda} dt \right),$$  \hspace{1cm} (44)

where $C = \text{const} > 0$. Further, for any fix $x > 0$ and all $f \in L^2(0, +\infty)$

$$\left| (e^{-t L_a} f)(x) \right| \leq \int_{\gamma} |d\lambda| |e^{-t\lambda}| \int_{0}^{\infty} |G(x, y, \lambda) f(y) dy |
\leq \frac{1}{2\pi} \int_{\gamma} |d\lambda| |e^{-t\lambda}| \sqrt{\int_{0}^{\infty} |G(x, y, \lambda)|^2 dy} \| f \|
$$

Hence, taking into account the estimate (44), we will have

$$\left| e^{-t L_a} f(x) \right| \leq C \int_{\gamma^\sigma} |e^{-t\lambda}| |q_a(x) - \lambda|^{-1/4} |d\lambda| < \infty.$$

Therefore, we can apply the Fubini theorem to the right side of (43). As a result, we get

$$e^{-t L_a} f = -\frac{1}{2\pi i} \int_{\gamma^\sigma} d\lambda e^{-t\lambda} G(x, y, \lambda) f(y) dy.$$  \hspace{1cm} (45)

Consequently, the kernel of the operator $e^{-t L_a}$ has the form

$$H(x, y, t) = -\frac{1}{2\pi i} \int_{\gamma^\sigma} d\lambda e^{-t\lambda} G(x, y, \lambda).$$

According to formulas (1) – (2) for $\Re t > 1$ the operator $e^{-t L_a}$ is a trace-class one, therefore

$$\text{tr} \left( e^{-t L_a} \right) = \int_{0}^{\infty} H(x, x, t) dx = -\frac{1}{2\pi i} \int_{0}^{\infty} dx \int_{\gamma^\sigma} d\lambda e^{-t\lambda} G(x, x, \lambda).$$

Theorem 2. For the $\theta$-functions of the operator $L_a$ as $t \to +0$ the following decomposition is true

$$\Theta_a(t) \sim -\frac{a}{2\sqrt{\pi}} t^{-1/2} - 1/4 + \theta_0 t^{1/2} + \frac{\log a}{4} t + O(t^{3/2}).$$  \hspace{1cm} (45)

Here $\theta_0$ is defined in the same way as in (6).
**Proof.** The statement of the theorem follows directly from the Lemmas 3 and 5.

**Corollary 3.** The constant $k_0$ in formulas (1)–(2) and (6) is equal to 0.

4. Regularized trace of the operator $L_a$

The sum

$$\sigma = \lambda_1 + \sum_{k=2}^{\infty} \left[ \lambda_k - \log(2\sqrt{\pi}k) - k^{-1}\left(\frac{a}{\pi}(\log k)^{1/2} - \frac{1}{2}\log(2\sqrt{\pi}k) \right) \right],$$

where $c_0 = \frac{a}{2\pi}(1 + \log(2\sqrt{\pi}/a))$, we call the regularized trace of the operator $L_a$.

**Theorem 4.** The following formula is true

$$\sigma = \frac{1}{4}\left(\log(2\sqrt{\pi}) + \gamma - 1 + \log a \right) - \frac{a}{\pi} \int_{1}^{\infty} \left( (\log x)^{1/2} x^{-1} \right)' dx$$

$$- \frac{a}{2\pi} \left(1 + \log\left(\frac{2\sqrt{\pi}}{a}\right)\right) \int_{1}^{\infty} \left( \left(\log x\right)^{-1/2} x^{-1} \right)' dx. \quad (47)$$

**Proof.** The formula (47) follows from the formulas (6) and (45).

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