Spectra of 2D H-donors in magnetic field and their Zeeman splitting.

Maen Odeh
UGRU, UAEU,
Al-Ain, Abu Dhabi,
UAE.
email: maenodeh@emirates.net.ae

Omar Mustafa
Department of Physics, Eastern Mediterranean University
G. Magusa, North Cyprus, Mersin 10 - Turkey
email: omar.mustafa@emu.edu.tr

March 28, 2022

Abstract

The pseudoperturbative shifted - $l$ expansion technique PSLET [20-25] is employed to study the spectra of 2D H-donors in magnetic field. The Zeeman effect is investigated and results are found to be excellent compared to those in ref.[14].

1 Introduction

1
The most prominent feature of technology in the last decades was the realization of low dimensional structures. This achievement urged a huge body of theoretical and experimental research to study different kinds of interactions constrained by the new structures [1-14]. Perhaps, one of the most important and simplest is that of hydrogenic impurity in bulk semiconductors in a constant magnetic field. In such systems, the energy scales of both coulombic and magnetic potentials are altered because of the dielectric function of the medium, $\epsilon \neq 1$, and the effective mass of the electron $m^*/m \neq 1$. In semiconductors, typical values of the effective mass $m^*$ and dielectric constant $\epsilon$ make the effective Rydberg $R^* (= 13.6 \frac{m^*/m}{\epsilon^2} \text{eV})$ about $10^3 - 10^4$ times smaller and the dimensionless magnetic field $\gamma = \hbar \omega_c / 2R^*$ (in a fixed magnetic field $B$) about $10^4 - 10^5$ times larger than those of the free hydrogenic atom [2,13]. As a result, the situation where $\gamma \sim 1$ (i.e., the intermediate magnetic field regime, where neither the Coulombic nor the confining potentials can be considered small relative to each other) is of special physical interest. Therefore, many methods had been employed to resolve its significance [14].

However, all calculations have shown that the donor states strongly confined by quantum wells can be correctly described by two dimensional (2D) hydrogenic-like atoms with properly variational parameters [9-12]. We can thus study 2D hydrogenic donors in a magnetic field to understand the behavior of donors strongly confined by a quantum wells in different magnetic fields. Although the Hamiltonian models in these settings can be separable, one is still prevented from obtaining an analytically exact solution. Only for the two extreme limits of the magnetic field ( $\gamma = 0$ and $\gamma \to \infty$) the problem can be treated exactly [1,3]. Recently, Taut [15,16] has obtained (conditionally) exact solutions for specific values of $\gamma$, however no ground-state solutions were obtained.

The adiabatic method [1,3] had been employed in $\gamma \gg 1$ regime. How-
ever, calculations based on this method have usually been restricted to the
ground state and the first few excited states. Other work has been established
using variational method. Makado and McGill [4] have based their approach
on that of Aldich and Greene [2] with different set of basis and they got quite
good results. Martin et al [11] have used a two-point quasifractional approx-
imation and found excellent interpolation between the weak and strong mag-
netic field perturbation expansions. MacDonald and Ritchie [13] have used
a two-point Pade’ approximation. The curves they obtained from different
Pade’ approximation are very different so that no regular pattern appears
and the results become unreliable [4]. Zhu et al [14] have used a power se-
ries expansion method. Villalbe and Pino [17] have used a finite-difference
scheme with a linear mesh of up to 2000 points and failed to provide a good
estimation of the ground state for the 2D hydrogen atom. Mustafa [18] and
Quigora et al [19] have used the shifted N-expansion technique and their
results were in good agreement with those of Martin et al [11].

Recently, we have introduced a pseudoperturbative shifted-$l$ ($l$ is the an-
gular momentum quantum number ) expansion technique ( PSLET ) to solve
Schrödinger equation for states with arbitrary number of nodal zeros. It sim-
ply consists of using $1/\bar{l}$ as a pseudoperturbation parameter, where $\bar{l} = l - \beta$,
and $\beta$ is a suitable shift. The shift $\beta$ is vital for it removes the poles that
would emerge, at the lowest orbital states with $l = 0$, in our proposed expan-
sion below. Our new analytical, often semi-analytical, methodical proposal
PSLET has been successfully applied to quasi - relativistic harmonic oscil-
lator [20], spiked harmonic oscillator [21], anharmonic oscillators [22], two
dimensional hydrogenic atom in an arbitrary magnetic field [23], two elec-
trons in quantum parabolic dots [24], truncated coulombic potentials [25],
...etc [26].

Encouraged by its satisfactory performance, we extend, in this paper,
PSLET recipe (in section 2) and introduce its 2D version to treat the problem of 2D hydrogenic donors in magnetic fields. In section 3, we discuss the results of PSLET. We conclude our work in section 4.

2 Hamiltonian model and PSLET recipe

2.1 Hamiltonian model

The problem of a 2D hydrogenic donor in the presence of a magnetic field that is perpendicular to the 2D plane, in the effective mass approximation, can be described by the Hamiltonian [14]

\[ \hat{H} = -\frac{\hbar^2}{2\mu} \nabla_2^2 + \frac{e^2}{2\mu c^2} \vec{A} \cdot \vec{A} + i \frac{\hbar e}{\mu c} \vec{A} \cdot \nabla_2 - \frac{we^2}{\epsilon \rho} \]  

(1)

where \( \vec{A} \) is the vector potential, \( \mu \) is the electron reduced mass, \( \epsilon \) is the static dielectric constant, \( w \) is equal to 0 or 1 and \( \nabla_2 \) represents \((i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y})\). If we choose the cylindrical gauge such that

\[ \vec{A} = \frac{1}{2} \vec{B} \times \vec{r} \]  

(2)

and the wavefunction \( \Psi(\rho, \varphi) \) to be in the form

\[ \Psi(\rho, \varphi) = e^{im\varphi} \frac{u(\rho)}{\sqrt{\rho}} \]  

(3)

then the Schrödinger like equation that describes the system reads

\[ \left[ -\frac{d^2}{d\rho^2} + \frac{m^2}{\rho^2} - \frac{1}{4} + \left( \frac{\gamma^2}{4} - \frac{2w}{\rho} \right) \right] u(\rho) = [E(m) - m\gamma] u(\rho) \]  

(4)
where \( m \) is a well-defined magnetic quantum number of the electron in the cylindrically symmetric magnetic and 2D Coulomb potentials, \( E(m) \) is the eigenenergy in units of \( R^* = \mu e^4/2\hbar^2 \epsilon^2 \), the effective Rydberg, \( \gamma = \hbar \omega_c/2R^* \), \( \omega_c (= eB/\mu c) \) is the cyclotron frequency and \( \rho \) is in units of the effective Bohr radius \( (a^* = \epsilon \hbar^2/e^2 \mu) \).

### 2.2 PSLET recipe

The 2D version of PSLET starts with shifting the magnetic quantum number through \( \tilde{l} = l - \beta \), where \( l = |m| \), and \( \beta \) is a suitable shift introduced technically to avoid the trivial case \( l = 0 \), and to be determined below. Eq.(4) thus becomes

\[
\left[ -\frac{d^2}{d\rho^2} + \frac{(\tilde{l} + \beta + 1/2)(\tilde{l} + \beta - 1/2)}{\rho^2} + \frac{\tilde{l}^2}{Q} \right] u(\rho) = \varepsilon \ u(\rho) \quad (5)
\]

where \( \varepsilon = E(m) - m\gamma \), \( V(\rho) = \frac{1}{4} \gamma^2 \rho^2 - \frac{2w}{\rho} \), and \( Q \) is a scaling factor that is set equal to \( \tilde{l}^2 \) at the end of the calculations.

We employ Taylor’s theorem in expanding Eq.(5) about \( \rho_o \), an arbitrary point on the \( \rho \) axis. It is convenient then to transform the coordinates in Eq.(5) via \( x = \tilde{l}^{1/2}(\rho - \rho_o)/\rho_o \). Expansions about \( x = 0 \) (i.e., \( \rho = \rho_o \)) yield

\[
\left[ -\frac{d^2}{dx^2} + \tilde{V}(x(\rho)) \right] u(x) = \frac{\rho_o^2}{l} \varepsilon u(x), \quad (6)
\]

where

\[
\tilde{V}(x(\rho)) = \rho_o^2 \left[ \frac{1}{\rho_o^2} + \frac{V(\rho_o)}{Q} \right] + \tilde{l}^{1/2} \left[ -2 + \frac{V'(\rho_o)\rho_o^3}{Q} \right] x + \left[ 3 + \frac{V''(\rho_o)\rho_o^4}{2Q} \right] x^2 + 2\beta \sum_{n=1}^{\infty} (-1)^n (n + 1)x^n \tilde{l}^{-n/2}
\]
$$+ \sum_{n=3}^{\infty} \left[ (-1)^n(n+1)x^n + \left( \frac{d^nV(\rho_\circ)}{d\rho_\circ^n} \right) \frac{\rho_\circ^2(\rho_\circ x)^n}{n!Q} \right] \tilde{l}^{-(n-2)/2}$$

$$+ \left( \beta^2 - \frac{1}{4} \right) \sum_{n=0}^{\infty} (-1)^n(n+1)x^n \tilde{l}^{-(n+2)/2} + 2\beta,$$

(7)

and the prime on $V(\rho_\circ)$ denotes the derivative with respect to $\rho_\circ$. It is convenient to expand $\varepsilon$ as

$$\varepsilon = \sum_{n=-2}^{\infty} E^{(n)}_{n,\beta,\tilde{l}} \tilde{l}^{-n}.$$  

(8)

Equation (6) is exactly of the type of Schrödinger equation for one-dimensional anharmonic oscillator. This leads to the identification of the leading correction in the energy series, Eq.(8), namely

$$E^{(-2)}_{n,\beta,\tilde{l}} = \frac{1}{\rho_\circ^2} + \frac{V(\rho_\circ)}{Q}.$$  

(9)

Here $\rho_\circ$ is chosen to minimize $E^{(-2)}_{n,\beta,\tilde{l}}$, i.e.

$$\frac{dE^{(-2)}_{n,\beta,\tilde{l}}}{d\rho_\circ} = 0 \quad \text{and} \quad \frac{d^2E^{(-2)}_{n,\beta,\tilde{l}}}{d\rho_\circ^2} > 0,$$

(10)

which in turn gives, with $\tilde{l} = \sqrt{Q}$,

$$l - \beta = \sqrt{\rho_\circ^3V'(\rho_\circ)/2}.$$  

(11)

Eq.(11) is an explicit equation in $\rho_\circ$. However, closed form solutions are usually hard to be found, if not impossible. Thus we use numerical solutions to solve for $\rho_\circ$ (hence, the technique is often called semi-analytical
The shifting parameter $\beta$ is determined by choosing $\tilde{E}_{n_{\rho},l}^{(-1)} = 0$. This choice is physically motivated. It requires not only the agreements between PSLET eigenvalues and the exact known ones for the harmonic oscillator and Coulomb potentials but also between the eigenfunctions as well. Hence

$$\beta = -\frac{1}{2} (n_{\rho} + \frac{1}{2}) \Omega, \quad (12)$$

where

$$\Omega = 2 \sqrt{3 + \frac{\rho_{o} V''(\rho_{o})}{V'(\rho_{o})}}. \quad (13)$$

Equation (7) thus becomes

$$\tilde{V}(x(\rho)) = \rho_{o}^{2} \tilde{l} \left[ \frac{1}{\rho_{o}^{2}} + \frac{V(\rho_{o})}{Q} \right] + \sum_{n=0}^{\infty} v^{(n)}(x) \tilde{l}^{-n/2}, \quad (14)$$

where

$$v^{(0)}(x) = \frac{1}{4} \Omega^{2} x^{2} + 2 \beta, \quad (15)$$

$$v^{(1)}(x) = -4 \beta x - 4x^{3} + \frac{\rho_{o}^{5} V'''(\rho_{o})}{6 Q} x^{3}, \quad (16)$$

and for $n \geq 2$

$$v^{(n)}(x) = (-1)^{n} 2 \beta (n + 1) x^{n} + (-1)^{n} (\beta^{2} - 1/4)(n - 1) x^{(n-2)} + B_{n} x^{n+2}, \quad (17)$$
\[ B_n = (-1)^n(n+3) + \frac{\rho_0^{(n+4)}}{Q(n+2)!} \frac{d^{n+2} V(\rho_0)}{d\rho_0^{n+2}} \quad (18) \]

Equation (6) thus becomes

\[
\left[ -\frac{d^2}{dx^2} + \sum_{n=0}^{\infty} u^{(n)}(x) \bar{l}^{-n/2} \right] u(x) = \rho_o^2 \left[ \sum_{n=1}^{\infty} E_{n,\bar{l}}^{(n-1)} \bar{l}^{-n} \right] u(x) \quad (19)
\]

Up to this point, one would conclude that the above procedure is nothing but an animation of the eminent shifted large-N expansion (SLNT). However, because of the limited capabilities of SLNT in handling large-order corrections via the standard Rayleigh-Schrödinger perturbation theory, only low-order corrections have been reported, sacrificing in effect its preciseness. Therefore, one should seek for an alternative and proceed by setting the wave functions with any number of nodes as

\[ u(x) = F_{n,\bar{l}}(x) \exp(U_{n,\bar{l}}(x)) \quad (20) \]

Eq.(19) readily transforms into the following Riccati type equation

\[
- \left[ F''_{n,\bar{l}}(x) + 2F'_{n,\bar{l}}(x)U'_{n,\bar{l}}(x) \right] + F_{n,\bar{l}}(x) \{ - [U''_{n,\bar{l}}(x) + (U'_{n,\bar{l}}(x))^2] \\
+ 2\beta + \frac{1}{4} \Omega^2 x^2 + \sum_{n=1}^{\infty} u^{(n)}(x) \bar{l}^{-n/2} \} = \rho_o^2 F_{n,\bar{l}}(x) \sum_{n=1}^{\infty} E_{n,\bar{l}}^{(n-1)} \bar{l}^{-n} \]

where primes denotes derivatives with respect to \( x \). It is evident that (21) admits solutions of the form

\[ F_{n,\bar{l}}(x) = x^{\nu} + \sum_{n=0}^{\infty} \sum_{p=0}^{\nu} a_{p,n}^{(n)} x^{p} \bar{l}^{-n/2} \quad (22) \]
\[ U'_{n,\rho}(x) = \sum_{n=0}^{\infty} U(n)_{n,\rho}(x)\tilde{l}^{-n/2} + \sum_{n=0}^{\infty} G(n)_{n,\rho}(x)\tilde{l}^{-(n+1)/2}, \tag{23} \]

where

\[ U(n)_{n,\rho}(x) = \sum_{m=0}^{n+1} D_{m,n,n}(x) x^{2m-1} \quad ; \quad D_{0,n,\rho} = 0, \tag{24} \]

\[ G(n)_{n,\rho}(x) = \sum_{m=0}^{n+1} C_{m,n,n}(x) x^{2m}. \tag{25} \]

Substituting equations (22)-(25) into equation (21) implies

\[
\begin{align*}
F'_{n,\rho,l}(x) [ \sum_{n=0}^{\infty} & U(n)_{n,\rho}(x)\tilde{l}^{-n/2} + \sum_{n=0}^{\infty} G(n)_{n,\rho}(x)\tilde{l}^{-(n+1)/2} \sum_{n=0}^{\infty} U(m)_{n,\rho}(x)U(n-m)_{n,\rho}(x)\tilde{l}^{-n/2} ] \\
+ & G(m)_{n,\rho}(x)G(n-m)_{n,\rho}(x)\tilde{l}^{-(n+2)/2} + 2U^{(m)}_{n,\rho}(x)G(n-m)_{n,\rho}(x)\tilde{l}^{-(n+1)/2} + \sum_{n=0}^{\infty} U(n)_{n,\rho}(x)\tilde{l}^{-n/2} \\
- & \rho_0^2 \sum_{n=1}^{\infty} E(n-1)_{n,\rho,l}(x) - 2F_{n,\rho,l}(x) [ \sum_{n=0}^{\infty} U(n)_{n,\rho}(x)\tilde{l}^{-n/2} ] \\
+ & G(n)_{n,\rho}(x)\tilde{l}^{-(n+1)/2}] - 2F_{n,\rho,l}(x) = 0 \tag{26} \end{align*}
\]

Obviously, the solution of equation (26) follows from the uniqueness of power series representation. Therefore, for a given \( n, \) we equate the coefficients of the same powers of \( \tilde{l} \) and \( n, \) respectively. One can then calculate the energy eigenvalue and eigenfunctions from the knowledge of \( C_{m,n,n}, D_{m,n,n} \) and \( a(n)_{p,n,\rho} \) in a hierarchical manner. Nevertheless, the procedure just described is suitable for a software package such as MAPLE to determine the energy
eigenvalue and eigenfunction up to any order of the pseudoperturbation series.

Although the energy series Eq.(8), could appear divergent, or at best, asymptotic for small \( \tilde{l} \) one can still calculate the eigen series to a very good accuracy by performing the sophisticated \([N,M]\) Pade’ approximation [25],

\[
P^M_N(1/\tilde{l}) = \frac{P_0 + P_1/\tilde{l} + \cdots + P_M/\tilde{l}^N}{1 + q_1/\tilde{l} + \cdots + q_N/\tilde{l}^M}
\] (27)

to the energy series, Eq(8). The energy series, Eq(8), is calculated up to \( E^{(8)}_{n,\rho,l}/\tilde{l}^8 \) by

\[
\varepsilon_{n,\rho,l} = \tilde{l}^2 E^{(-2)}_{n,\rho,l} + E^{(0)}_{n,\rho,l} + \cdots + E^{(8)}_{n,\rho,l}/\tilde{l}^8 + O(1/\tilde{l}^9),
\] (28)

and with the \( P^4_{4}(1/\tilde{l}) \) Pade’ approximant it becomes

\[
\varepsilon_{n,\rho,l}[4, 4] = \tilde{l}^2 E^{(-2)}_{n,\rho,l} + P^4_{4}(1/\tilde{l}).
\] (29)

Our method is therefore well prescribed.

3 Results and Discussion

For the potential in hand, Eq.(13) reads

\[
\Omega = 2\sqrt{2} \sqrt{\frac{2\gamma^2 \rho_o^3 + 2w}{\gamma^2 \rho_o^4 + 4w}}
\] (30)

Eq(11) in turn yields

\[
l + \sqrt{2}(\rho_o + \frac{1}{2}) \sqrt{\frac{2\gamma^2 \rho_o^3 + 2w}{\gamma^2 \rho_o^4 + 4w}} = \sqrt{\frac{1}{4} \gamma^2 \rho_o^4 + w \rho_o}
\] (31)
Once Eq.(31) is solved for given values of \( \gamma, w, n_\rho \) and \( l \), the coefficients \( C_{m,n,n_\rho}, D_{m,n,n_\rho} \) and \( a_{p,n_\rho}^{(n)} \) are obtained in a sequential manner. Then the eigenvalues and eigenfunctions are calculated in one batch for each value of \( n_\rho, l, m, \gamma \) and \( w \).

It worths mentioning that the method is able to reproduce the exact know results for both pure Coulombic (\( \gamma = 0 \)) and pure magnetic (\( w = 0 \)) interactions. This is evident from Eq.(9). For the pure Coulomb interaction (\( \gamma = 0 \)), Eq.(9) reads

\[
\bar{l}^2 E_{n_\rho,l}^{(-2)} = -(n_\rho + |m| + 1/2)^{-2}, \tag{32}
\]

while for the pure magnetic interaction (\( w = 0 \)), Eq.(9) reads

\[
\bar{l}^2 E_{n_\rho,l}^{(-2)} = (2n_\rho + m + |m| + 1)\gamma \tag{33}
\]

with identically vanishing higher order corrections.

Table 1 shows the results of PSLET for the energy spectra of 1s, 2p, 2s, 3d, 3p, 3s and 4d states excluding the Zeeman term \( m\gamma \), for wide range of the dimensionless magnetic field \( \gamma \). We have plotted the quantum levels of 2D donor states without the Zeeman term in Figure 1. It is evident that as \( \gamma \) approaches zero, the quantum levels approach those of Eq.(32). Our results compare exactly with those obtained by Zhu et al [14]. It is evidently seen in Figure 1 that the energy of the quantum levels increases with \( \gamma \). Increasing \( \gamma \) results in a narrower quantum well and hence stronger binding energy. It is interesting to note that the energies of the excited states increases more rapidly with \( \gamma \) than those of the ground states and the energy differences between these states increase with the applied field.

To study the effect of the Zeeman term, \( m\gamma \), we have obtained the energy spectra for small values of the \( \gamma \). These are displayed in Table 2. Figure 2 shows the results in table 2 including the Zeeman term. Clearly as the
magnetic field is turned on, the states with $m \neq 0$, i.e., $2p, 3d, 3p, 4d...$, split because of their negative and positive $m$. As $\gamma$ increases continuously, the energy values of positive and negative $m$ states close to those of lower and higher $m = 0$ states and the levels with negative $m$ cross those of positive $m$. For example, as $\gamma$ is turned on the $3d$ states split into $3d_+$ and $3d_-$ levels. As $\gamma$ increases, the $3d_-$ state gets closer to the $2s$ state and crosses the $2p_+$ state. While the $3d_+$ state gets closer to the $3s$ state and crosses the $4d_-$ state. Clearly $2p_-$ and $3d_-$ states admit minima due to the competition between the Zeeman term and the parabolic quantum well in this weak range of $\gamma$.

Figure 3 shows the effect of the Zeeman term at high magnetic field. Clearly the energy levels at high magnetic field are accumulated between the corresponding $m = 0$ states and the Landau levels. The order of the levels is $1s, 2p_-, 3d_- ...$ for the first Landau level and $2s, 2p_+, 3p_-, 4d_-...$ for the second Landau level, and so on. As $\gamma$ decreases, the values decrease until the hybrid, mentioned above occurs.

4 Concluding remarks

Using PSLET, we have obtained the energy spectra of the 2D hydrogenic donor in magnetic field. Calculated results have shown the quadratic effect of the magnetic field in partially lifting the degeneracy of 2D hydrogenic donor states. The linear effect of the magnetic field, i.e., the Zeeman term, has completely lifted the degeneracy causing hybrids between different states. The order of levels is obtained at high magnetic field. The results obtained here show exact agreement with those of Zhu et al [12,14]. The nonperturbative nature of PSLET makes it possible to treat the problem irrespective of the range of magnetic field. Moreover, we the wavefunctions obtained by PSLET
could be manipulated to be used as trail ones in variational approaches. The present problem suggests the investigation of excitons in parabolic quantum dots as they are known to have same behavior as 2D donors in magnetic field. We believe that all aspects of this problem could be revealed through PSLET.
References

[1] R. J. Elliott and R. Loudon, J. Phys. Chem. Solids 15, 196 (1960).

[2] C. Aldich and R. L. Greene, Phys. Status Solidi B 93, 343 (1979).

[3] G. Wunner and H. Ruder, Phys. Lett. 85A, 430 (1981)

[4] P. C. Makado and N. C. McGill, J. Phys. C 19, 873 (1986)

[5] J. M. Chamberlain et al, Solid State Commun. 11, 463 (1972)

[6] A. M. Davison et al, Solid State Science (Springer, Berlin, 1981), Vol.24, p.84

[7] G. J. Armistead et al, J. Phys. C 19, 6023 (1986)

[8] G. Bastard, Phys. Rev. B 24, 4714 (1981).

[9] S. Chaudhuri. Phys. Rev. B 28, 4480(1983).

[10] Wen-Ming Liu and J. J. Quinn, Phys. Rev. B 31, 2348 (1985).

[11] C. Mailhoit et al, Phys. Rev. B. 26, 4449 (1982).

[12] Jia-Lin Zhu, J. Phys. Condens. Matter 1, 1539 (1989).

[13] A. H. MacDonald and D. S. Ritchie, Phys. Rev. B 33, 8336 (1986), and references therein.

[14] Jia-Lin Zhu et al, Phys. Rev. B 41, 10792(1990).

[15] M. Taut, J. Phys. A 28, 2081 (1995)

[16] M. Taut, J. Phys. A 32, 5509 (1995)

[17] V. M. Villalba and R. Pino, J. Phys. Condens. Matter 8, 8067 (1996)
[18] O. Mustafa, J. Phys. Condens. Matter 5, 1327 (1993)

[19] L. Quiroga et al, J. Phys. Condens. Matter 7, 7517 (1995).

[20] O Mustafa and M Odeh, J. Phys. A 32, 6653 (1999)

[21] O Mustafa and M Odeh, J. Phys B 32, 3055 (1999)

[22] O Mustafa and M Odeh, Eur. Phys. J. B 15, 143(2000)

[23] O Mustafa and M Odeh, Commun. Theor. Phys. 33, 469 (2000).

[24] O Mustafa and M Odeh, Czech. J. Phys. 52, 795 (2002)

[25] O Mustafa and M Odeh, J. Phys. A 33, 7013 (2000)

[26] M Znojil, F Gemperle and O Mustafa, J. Phys. A 35, 5781 (2002).
   O Mustafa ans M Znojil, J. Phys. A 35, 8929 (2002).
   O Mustafa, J. Phys. A 35, 10671 (2002)

[27] C. M. Bender and S. A. Orszag "Advavnced Mathematical Methods for Scientists and Engineers" (McGraw- Hill, New York).(1978)
Figures Captions

Figure 1: The 2D donor energy in effective Rydberg units, excluding the Zeeman term $m\gamma$, is shown as a function of the normalized magnetic field $\gamma$ ranging from 0 to 1 for $1s$, $2p$, $2s$, $3d$, $3p$, $3s$ and $4d$ states.

Figure 2: The 2D donor energy in effective Rydberg units, including the Zeeman term $m\gamma$, is shown as a function of the normalized magnetic field $\gamma$ for $2s$, $2p$, $3d$, $3p$ and $4d$ states.

Figure 3: Same as figure 1 but including the Zeeman term $m\gamma$. 