SINGULARITIES OF SECANT VARIETIES

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Abstract. We study the singularities of the secant variety $\Sigma(X, L)$ associated to a smooth variety $X$ embedded by a sufficiently positive adjoint bundle $L$. We show that $\Sigma(X, L)$ is always Du Bois singular. We also give a necessary and sufficient condition for $\Sigma(X, L)$ to have rational singularities.

1. Introduction

Given a smooth projective variety $X$ of dimension $n$ over an algebraically closed field of characteristic zero, and a very ample line bundle $L$ that embeds $X$ into a projective space $\mathbb{P}^N$. The (first) secant variety $\Sigma(X, L)$ is defined as the Zariski closure of the union of 2-secant lines to $X$ in $\mathbb{P}^N$. Recently, Ullery [18] gave a sufficient condition on $L$ for the normality of the secant variety $\Sigma(X, L)$, completing the results of Vermeire. She showed that, among other things, when $X$ is a curve, $\Sigma(X, L)$ is normal if $\deg L \geq 2g + 3$; when $n \geq 2$, $\Sigma(X, L)$ is normal if $L = \omega_X \otimes A^{2(n+1)} \otimes B$, where $A$, $B$ are very ample and nef line bundles, respectively.

Inspired by [18], we study the singularities of $\Sigma(X, L)$ from the cohomological point of view. To state results in a uniform way, throughout the paper we always make the following assumption on $L$ unless otherwise stated.

Assumption 1.1. For $n \geq 2$, we assume that $L = \omega_X \otimes A^{2(n+1)} \otimes B$, where $A$ and $B$ are very ample and nef line bundles respectively. For $X$ being a curve, we assume that $\deg L \geq 2g + 3$, where $g$ is the genus of $X$.

According to a result of Ein and Lazarsfeld [6] (in case $n \geq 2$), such $L$ satisfies Property $N_{n+1}$, i.e. $X$ embeds in $\mathbb{P}^N$ under $|L|$ as a projectively normal variety, the homogeneous ideal of $X$ is generated quadrics, and the minimal graded free resolution of $O_X$ is linear up to $(n+1)$-th step. One may expect that the singularities of $\Sigma(X, L)$ will be somewhat well behaved if $L$ satisfies the assumption. Our first result confirms this expectation in the sense that

Theorem 1.2. (= Theorem 3.3) $\Sigma(X, L)$ has Du Bois singularities.

Key words and phrases. secant varieties, Du Bois singularities, Cohen-Macaulayness, rational singularities.
The notion of Du Bois singularities is originated from complex geometry and plays an important role in classification of algebraic varieties as shown in [11]. Roughly speaking, a variety has Du Bois singularities if its cohomological behavior is the same as a simple normal crossing variety. From this point of view, the notion of Du Bois singularities is a generalization of rational singularities.

As a result, it is natural to ask whether a secant variety has rational singularities in general. Before answering this question, we recall the Kempf’s criterion for rational singularities: given a normal variety $Z$ and a resolution of singularities $f: Y \rightarrow Z$, then $Z$ has rational singularities if and only if $Z$ is Cohen-Macaulay and $f^\ast \omega_Y = \omega_Z$. So we prove the following two theorems.

**Theorem 1.3.** (=*Theorem 4.2*) For any closed point $x \in X \subset \Sigma(X, L)$, we have

\[
\text{depth}_x \Sigma(X, L) = n + 2 + \max \{i \mid i \leq n - 1, \text{ and } H^j(X, O_X) = 0, 1 \leq j \leq i\}
\]

Here we adopt the convention that if the set $\{i \mid i \leq n - 1, \text{ and } H^j(X, O_X) = 0, 1 \leq j \leq i\}$ is empty, then the max is 0.

Note that with the assumption on $L$ we know that $\Sigma(X, L) \setminus X$ is smooth and $\dim \Sigma(X, L) = 2n + 1$, so it follows from Theorem [1,3] that $\Sigma(X, L)$ is Cohen-Macaulay if and only if $H^i(X, O_X) = 0$, for all $1 \leq i \leq n - 1$. On the other hand, the second condition of rational singularities is controlled by the top cohomology group as the following theorem shows.

**Theorem 1.4.** (=*Theorem 4.8*) Let $t: \mathbb{P}(E_L) \rightarrow \Sigma(X, L)$ be a natural resolution of singularities (see §2). Then $t^\ast \omega_{\mathbb{P}(E_L)} \simeq \omega_{\Sigma(X, L)}$ if and only if $H^n(X, O_X) = 0$.

Note that in Theorem [1,4] we do not need $\Sigma(X, L)$ to be Cohen-Macaulay. Here $\omega_{\Sigma(X, L)}$ is the $-(2n + 1)$ cohomology of the dualizing complex $\omega_{\Sigma(X, L)}^\bullet$. Combing the two theorems above, we conclude that $\Sigma(X, L)$ has rational singularities if and only if $H^i(X, O_X) = 0$, for all $1 \leq i \leq n$, which was observed by Vermeire [21] in case $n = 1$.

In the minimal model program, typical types of singularities are Kawamata log terminal (klt) and log canonical (lc). And it is well known that klt and lc implies rational and Du Bois singularities, respectively. (cf. [12] and [11]). So we ask the following question:

**Question 1.5.** Can one find some boundary divisor $\Delta$ on $\Sigma(X, L)$ such that $(\Sigma(X, L), \Delta)$ is a log canonical pair, or even more, Kawamata log terminal in the case that $\Sigma(X, L)$ has rational singularities?

As an example (cf. Theorem 3.5) we show that when $X \simeq \mathbb{P}^2$, $\Sigma(X, L)$ is log canonical with some boundary $\Delta$. 
We remark that the singularities of classical varieties are of great interest to algebraic geometers. For example, it has been shown that generic determinantal varieties, Schubert varieties and Richardson varieties are klt ([19], [17], [2] and [15]). In particular, they all have rational singularities. The result of this paper provides examples of classical varieties having Du Bois singularities but not rational singularities.

Besides the singularities mentioned above, people are also interested in the notions of singularities from the view point of arc space. Mather-Jacobian (MJ) discrepancy and related singularities were introduced and have been explored by several authors. Wenbo Niu pointed out to us that the answer to Question 1.5 is negative in the context of MJ discrepancy: if $L$ is sufficiently ample, then for any point $x \in X$, $\dim T_x \Sigma(X, L) = \dim T_x \mathbb{P}^N \gg 2n$, so by [5, Proposition 3.3], $\Sigma(X, L)$ cannot be MJ-log canonical.

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2. Notation and preliminaries

We will use the constructions and notations in [18] and we recall them here for the reader’s convenience. We refer the interested reader to [4] and [20] for constructions in curves and higher dimension varieties respectively.

Let $X$ be a smooth variety of dimension $n$ embedded to $\mathbb{P}^N$ by a very ample line bundle $L$. From now on, we will simply write $\Sigma(X)$ or $\Sigma$ for $\Sigma(X, L)$ when the context is clear. $L$ is said to be $k$-very ample if the natural map $H^0(L) \to H^0(L \otimes O_\xi)$ is surjective for every length $k + 1$ closed subscheme $\xi$ of $X$. Consider a line bundle of the form $L = \omega_X \otimes A^k \otimes B$, where $A, B$ are very ample and nef line bundles respectively and $k \geq 1$. As shown in [18], $L$ is 3-very ample for $k \geq n + 4$, and $\Sigma$ is normal for $k \geq 2(n + 1)$ in case $n \geq 2$. We remark that $L$ can fail to be 3-very ample if $k \leq n + 3$. An example is the Veronese embedding $j : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ under $[O_{\mathbb{P}^2}(2)]$. In this case $\dim \Sigma(\mathbb{P}^2, O_{\mathbb{P}^2}(2)) < 5$ (cf. [3, p. 43]), and hence $O_{\mathbb{P}^2}(2)$ cannot be 3-very ample.

Denote by $X^{[2]}$ the Hilbert scheme of two points on $X$. The universal family $\Phi \subset X^{[2]} \times X$ is isomorphic to the blowing up $\pi : Bl_\Delta X^2 \to X^2$ over $X^2$, where $\Delta$ is the diagonal on $X^2$. Denote the two projections from $\Phi$ by $\sigma$ and $q$, as shown below

\begin{center}
\begin{tikzpicture}
\node at (0,0) {$X^{[2]}$};
\node at (3,0) {$X$};
\node at (1.5,1.5) {$\Phi$};
\draw[->] (0,0) -- (1.5,1.5) node[below] {$\sigma$};
\draw[->] (3,0) -- (1.5,1.5) node[above] {$q$};
\end{tikzpicture}
\end{center}
$q$ is the composition of $\pi$ and the first projection $p_1 : X^2 \to X$.

One can define the tautological vector bundle $E_L$ on $X^{[2]}$ by $E_L = \sigma_*(q^*L)$. The rank two bundle $E_L$ is tautological in the sense that for any length two closed subscheme $\xi$ of $X$, $E_L \otimes k([\xi]) = H^0(X, L \otimes O_\xi)$, and $H^0(X, L) \cong H^0(X^{[2]}, E_L)$.

When $L$ is 3-very ample, $E_L$ is generated by global sections and $\Sigma(X, L)$ is singular only along $X$. In this case, due to Betram [4] in curve case and Vermeire [20] in general, the projective bundle $\mathbb{P}(E_L) \subset X^{[2]} \times \mathbb{P}(H^0(X, L))$ together with second projection provides a natural resolution of singularities $t : \mathbb{P}(E_L) \to \Sigma(X, L)$. And the exceptional divisor of $t$ is $\Phi$. In particular, given a closed point $x \in X$ the fiber $F_x$ of $t$ is isomorphic to $\text{Bl}_x(X)$, the blowing up of $X$ at $x$. As a result, we have the Cartesian diagram below

$$
\begin{array}{ccc}
F_x & \rightarrow & \Phi \downarrow \\
\Phi & \rightarrow & \mathbb{P}(E_L) \downarrow \\
\{x\} & \rightarrow & X \rightarrow \Sigma(X, L)
\end{array}
$$

The next easy lemmas will be used several times in the sequel.

**Lemma 2.1.** The map $q : \Phi \to X$ is smooth.

**Proof.** Since both $X$ and $\Phi$ are smooth and every fiber of $q$ has dimension $n$, $q$ is flat [10, III, Ex.10.9]. For any closed point $y \in \Phi$, let $x = q(y)$. Then $\dim_{k(y)}(\Omega_{\Phi/X} \otimes k(y)) = \dim_{k(y)}(\Omega_{F_x/k(x)} \otimes k(y)) = n$. Thus $\Omega_{\Phi/X}$ is locally free of rank $n$. So $q$ is smooth of relative dimension $n$. \qed

**Lemma 2.2.** For all $i \geq 0$,

$$R^iq_*O_\Phi \cong H^i(X, O_X) \otimes O_X.$$

**Proof.** Since $q = p_1 \circ \pi$ and $R^j\pi_*O_\Phi = 0$ for all $j > 0$, one has $R^iq_*O_\Phi \cong R^ip_1_*O_{X^2} \cong H^i(X, O_X) \otimes O_X$. \qed

We recall local duality (§ V. Theorem 6.2 in [9]), which will be the main tool when we study the dualizing sheaf of the secant variety. Let $(R, p)$ be a local ring. An injective hull $I$ of the residue field $k = R/p$ is an injective $R$ module such that for any non-zero submodule $N \subset I$ we have $N \cap k \neq 0$. Or equivalently, the injective hull $I$ is the minimal injective module containing $k$.

**Theorem 2.3.** (Local duality) Let $(R, p)$ be a local ring and $\mathcal{F}^* \in D^+_{\text{coh}}(R)$. Then

$$R\Gamma_p(\mathcal{F}^*) \to R\text{Hom}(R\text{Hom}(\mathcal{F}^*, \omega_R^*), I)$$

is an isomorphism.
In particular, if we take $i$-th cohomology on both sides, we have
\[(2.1) \quad H^i_p(F^* \simeq \text{Hom}(H^{-i}(R\text{Hom}(F^*, \omega_R^*)), I) \simeq \text{Hom}(E_{\text{Ext}}^{-i}(F^*, \omega^*_R)), I).\]

The $-i$ comes from switching the cohomology functor $H^i(\cdot)$ and $\text{Hom}(\cdot, I)$. In this paper we will only use the case when $F^*$ is a module, as in chapter V. Corollary 6.3 in [9].

**Lemma 2.4.** Given a close point $x \in \Sigma$, let $O_{\Sigma_x}$ be the local ring of $O_{\Sigma}$ at $x$ and $\Sigma_x = \text{spec} O_{\Sigma_x}$ be the local scheme. Then we have the following equation
\[(\omega^*_\Sigma) \otimes O_{\Sigma_x} \simeq \omega^*_x,
\]
where $\omega^*$ denote dualizing complexes.

**Proof.** Well known to experts. See for example Lemma 3.1 in [13]. □

### 3. Du Bois Singularities

In this section, we prove $\Sigma(X, L)$ is always Du Bois and give a partial answer to Question 1.5. It turns out that both results are related to the positivity property of $N_{F_x/P(E_L)}^*$, the conormal sheaf of $F_x$ in $P(E_L)$. The following important lemma is due to Ullery [18, Lemma 2.2].

**Lemma 3.1.** Let $n = \dim X$. Assume $L$ is a 3-very ample line bundle on $X$. Then for all $x \in X$,
\[N_{F_x/P(E_L)}^* \simeq O_{F_x}^n \oplus b_x^*L(-2E_x).\]
Here $F_x \simeq \text{Bl}_x X$ and $E_x$ is the exceptional divisor.

The technical result below will be used in many places later on.

**Proposition 3.2.** $R^it_*O_{P(E_L)}(-\Phi) = 0$ for all $i > 0$.

**Proof.** Since $t : P(E_L) \to \Sigma$ is an isomorphism over $\Sigma \setminus X$, it suffices to prove the statement at any closed point $x \in X$.

By the theorem on formal functions (cf. [10, III. 11]), one has the isomorphism
\[R^it_*\widehat{O}_{P(E_L)}(-\Phi)_x \simeq \lim_{k \to \infty} H^i(F_x, O_{P(E_L)}(-\Phi) \otimes O_{P(E_L)}/\mathcal{J}_{F_x}^k),\]
so it suffices to show that $H^i(F_x, O_{P(E_L)}(-\Phi) \otimes O_{P(E_L)}/\mathcal{J}_{F_x}^k) = 0$ for all $k > 0$.

To this aim, we do induction on $k$. For $k = 1$,
\[O_{P(E_L)}(-\Phi) \otimes O_{P(E_L)}/\mathcal{J}_{F_x} \simeq N_{\Phi/P(E_L)}^*|_{F_x} \simeq b_x^*L(-2E_x),\]
where the last isomorphism is by [18 p. 8], and $E_x$ is the exceptional divisor of the blowing up of $X$ at $x$, $b_x : F_x \to X$.

We now argue in case $n \geq 2$, but it is evident that the statement below is also valid for $n = 1$. 

For any \( j \geq 1 \), we have
\[
b_x^*L^j(-2jE_x)) \simeq \omega_{F_x} \otimes P^{n+1} \otimes Q,
\]
where
\[
P = b_x^*A^2(-E_x), \quad \text{and} \quad Q = \left( \omega_{F_x} \otimes P^{n+1} \otimes b_x^*B \right)^{j-1} \otimes b_x^*B.
\]
It is well known that \( P \) is very ample, so \( \omega_{F_x} \otimes P^{n+1} \otimes b_x^*B \) is very ample, cf.\([6, \text{p. 57}]\). It follows that \( Q \) is nef and \( P^{n+1} \otimes Q \) is ample. Then by Kodaira vanishing, we have
\[
H^i(F_x, b_x^*L^j(-2jE_x)) = 0 \quad \text{for all } i > 0.
\]
In particular, the vanishing above for \( j = 1 \) is desired for \( k = 1 \).

For any \( k > 1 \), consider the exact sequence
\[
0 \to O_{P(E_L)}(-\Phi) \otimes \mathcal{F}_x^k \otimes \mathcal{F}_{F_x}^{k+1} \to O_{P(E_L)}(-\Phi) \otimes O_{P(E_L)}(1) \otimes \mathcal{F}_{F_x}^k \to O_{P(E_L)}(-\Phi) \otimes O_{P(E_L)}(1) \otimes \mathcal{F}_{F_x}^k \to 0.
\]
We observe that
\[
O_{P(E_L)}(-\Phi) \otimes \mathcal{F}_x^k \otimes \mathcal{F}_{F_x}^{k+1} \simeq b_x^*L(-2E_x) \otimes S^kN^*_{F_x/P(E_L)}.
\]
By Lemma 3.1
\[
S^kN^*_{F_x/P(E_L)} \simeq \bigoplus_{j=0}^{k} \bigoplus_{i=0}^{n-k-j-1} b_x^*L^i(-2jE_x),
\]
so (3.2) is a direct sum with summands
\[
b_x^*L^{i+1}(-2(j+1)E_x), 0 \leq j \leq k.
\]
Then it is immediate by (3.1) that
\[
H^i(F_x, O_{P(E_L)}(-\Phi) \otimes \mathcal{F}_x^k \otimes \mathcal{F}_{F_x}^{k+1}) = 0,
\]
which completes the proof together with the induction hypothesis. \( \square \)

With this proposition, we are ready to prove our first main theorem.

**Theorem 3.3.** If a smooth variety \( X \) of dimension \( n \) is embedded by a line bundle \( L \) defined as Assumption 1.1, then the secant variety \( \Sigma(X, L) \) has Du Bois singularities.

**Proof.** First we claim that \( t_*O_{P(E_L)}(-\Phi) \simeq \mathcal{I}_{X/\Sigma} \), where \( \mathcal{I}_{X/\Sigma} \) denotes the ideal sheaf of \( X \) in \( \Sigma \). By the assumption on \( L \), the main result in \([18]\) implies that \( t_*O_{P(E_L)} = O_{\Sigma} \). So from the exact sequence
\[
0 \to O_{P(E_L)}(-\Phi) \to O_{P(E_L)} \to O_{\Phi} \to 0,
\]
we see that the claim follows from the fact \( O_{X} \simeq q_*O_{\Phi} \), see Lemma 2.2.

With Proposition 3.2, this implies that \( Rt_*O_{P(E_L)}(-\Phi) \simeq \mathcal{I}_{X/\Sigma} \) in derived category. In particular, we have the following splitting sequence
\[
\mathcal{I}_{X/\Sigma} \to Rt_*O_{P(E_L)}(-\Phi) \to \mathcal{I}_{X/\Sigma}.
\]
Since $X$, $\Phi$ and $\mathbb{P}(E_L)$ are all smooth, $\Sigma(X, L)$ has Du Bois singularities by Theorem 3.4.

**Theorem 3.4.** ([11, Theorem 1.6]) Let $f : Y \to X$ be a proper morphism between reduced schemes of finite type over $\mathbb{C}$. Let $W \subseteq X$ and $F := f^{-1}(W) \subseteq Y$ be closed reduced subschemes with ideal sheaves $\mathcal{I}_{W/X}$ and $\mathcal{I}_{F/Y}$. Assume that the natural map $\rho : \mathcal{I}_{W/X} \longrightarrow \mathcal{R}t_*\mathcal{I}_{F/Y}$ admits a left inverse $\rho'$, that is, $\rho' \circ \rho = id_{\mathcal{I}_{W/X}}$. Then if $Y$, $F$ and $W$ all have Du Bois singularities, then so does $X$.

Next we show that in some special cases, $\Sigma(X, L)$ has singularities more directly related to the minimal model program. For example $X = \mathbb{P}^2$, satisfies the hypothesis in the following theorem.

**Theorem 3.5.** If for any $x \in X$, $F_x = Bl_x X$ is a Fano variety, then on $\Sigma(X, L)$ there exists a boundary $\Delta$ such that $(\Sigma(X, L), \Delta)$ is a log canonical pair.

**Proof.** For any $x \in X$, $F_x$ is a fiber of $q : \Phi \to X$, so by adjunction formula we have

$$K_{\mathbb{P}(E_L)} + \Phi |_{F_x} = K_{F_x}.$$ 

In other words, $-(K_{\mathbb{P}(E_L)} + \Phi)$ is relative ample over $\Sigma(X, L)$. (Recall that $t : \mathbb{P}(E_L) \to \Sigma(X, L)$ is isomorphic outside of $\Phi$.)

Pick an ample line $H$ on $\Sigma(X, L)$ so that $N = -(K_{\mathbb{P}(E_L)} + \Phi) + t^*H$ is ample. For $m \gg 0$, we can assume $mN \sim D$, where $D$ is a reduced divisor intersecting $\Phi$ transversely. Let $\Delta' = \Phi + \frac{1}{m}D$, then we see that

$$K_{\mathbb{P}(E_L)} + \Delta' \sim_{l, \mathbb{Q}} 0.$$ 

Note that $\Phi$ is a reduced and smooth divisor on $\mathbb{P}(E_L)$, so $\Delta'$ is a linear combination of simple normal crossing divisors with coefficients at most one. In particular $(\mathbb{P}(E_L), \Delta')$ is a log canonical pair. Then by Lemma 1.1 in [7], we see that on $\Sigma(X, L)$ there exists a boundary $\Delta$ such that $(\Sigma(X, L), \Delta)$ is a log canonical pair.

**4. On Cohen-Macaulayness**

In this section, we calculate the depth of local rings of $\Sigma$ which measures how far $\Sigma$ is from being Cohen-Macaulay. We also study the sheaf $t_*\omega_{\mathbb{P}(E_L)}$ using the language of derived category and Grothendieck Duality theorem.

Since we are dealing with the projective case, a basic lemma for our purpose is

**Lemma 4.1.** Let $X$ be a projective scheme over an algebraically closed field $K$ of pure dimension $n$ with an ample divisor $D$. Let $F$ be a coherent sheaf
on $X$ such that support of $\mathcal{F}$ has pure dimension $n$. Then $\text{depth}_x \mathcal{F} \geq k$ for all closed point $x \in X$ if and only if $H^i(X, \mathcal{F}(-rD)) = 0$ for all $i < k$ and $r \gg 0$.

Proof. See [1, Lemma 2.3].

With the lemma above, we will prove

**Theorem 4.2.** If $L$ is defined as Assumption [1.1] then

$$\text{depth}_x \Sigma(X, L) = n + 2 + \max \{i \mid i \leq n - 1, \text{and } H^j(X, \mathcal{O}_X) = 0, 1 \leq j \leq i\}.$$  

Here we adopt the convention that if the set $\{i \mid i \leq n - 1, \text{and } H^j(X, \mathcal{O}_X) = 0, 1 \leq j \leq i\}$ is empty, then the max is 0.

**Corollary 4.3.** If $X$ is a curve, then $\Sigma(X, L)$ is Cohen-Macaulay.

Remark 4.4. Sidman and Vermeire [16] proved a stronger result that for a curve of genus $g$ embedded as linear normal curve by a line bundle of degree $d \geq 2g + 3$, $\Sigma$ is arithmetically Cohen-Macaulay.

**Corollary 4.5.** If $X$ can be embedded in a projective space as a scheme-theoretic complete intersection, then for sufficiently ample $L$, $\Sigma(X, L)$ is CM.

Proof. Since $X$ is a complete intersection, $H^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq n - 1$. Theorem 4.2 implies $\text{depth}_x \Sigma(X, L) = 2n + 1$.

**Lemma 4.6.** For all integers $i, r$,

$$H^i(\Sigma, \mathcal{I}_{X/\Sigma}(-r)) \simeq H^i(\mathcal{F}(E_L), \mathcal{O}_{\mathcal{F}(E_L)}(-r) \otimes \mathcal{O}_{\mathcal{F}(E_L)}(-\Phi)).$$

Proof. Since $R^i \mathcal{I}_{t,O_{\mathcal{F}(E_L)}}(-r) \otimes \mathcal{O}_{\mathcal{F}(E_L)}(-\Phi) = 0$ for $j > 0$, by Proposition 3.2 we have

$$H^i(\mathcal{F}(E_L), \mathcal{O}_{\mathcal{F}(E_L)}(-r) \otimes \mathcal{O}_{\mathcal{F}(E_L)}(-\Phi)) \simeq H^i(\Sigma, t,O_{\mathcal{F}(E_L)}(-\Phi) \otimes \mathcal{O}_{\mathcal{F}(E_L)}(-r)).$$

To finish the proof, we use the fact $t,O_{\mathcal{F}(E_L)}(-\Phi) \simeq \mathcal{I}_{X/\Sigma}$, as shown in the proof of Theorem 3.3.

**Lemma 4.7.** Let $r \gg 0$. Then

$$H^i(\Phi, \mathcal{O}_\Phi(-r)) = \begin{cases} 0 & \text{if } i < n, \\ H^n(X, L^{-r}) \otimes H^{i-n}(X, \mathcal{O}_X) & \text{if } n \leq i \leq 2n. \end{cases}$$

Proof. Since $L$ is ample, we have that if $b < n$ and $r \gg 0$, $H^b(X, R^q_{\mathcal{O}_\Phi} \otimes L^{-r}) = 0$ by Serre vanishing. It follows that the Leray spectral sequence

$$E_2^{a,b} = H^b(X, R^q_{\mathcal{O}_\Phi} \otimes L^{-r}) \Rightarrow H^{a+b}(\Phi, \mathcal{O}_\Phi(-r))$$

degenerates at the level $E_2$. We see that if $i < n$, $H^i(\Phi, \mathcal{O}_\Phi(-r)) = 0$ for $r \gg 0$. And if $n \leq i \leq 2n$,

$$H^i(\Phi, \mathcal{O}_\Phi(-r)) \simeq H^n(X, R^{i-n}_{\mathcal{O}_\Phi} \otimes L^{-r}) \simeq H^n(X, L^{-r}) \otimes H^{i-n}(X, \mathcal{O}_X),$$

where the last isomorphism is by Lemma 2.2. □
Proof of Theorem 4.2. Since $O_{b(E)}(1) = t^*O_Z(1)$, the tautological line bundle $O_{b(E)}(1)$ is big and nef. Therefore by Kawamata-Viehweg vanishing,

(4.1) \[ H^i(O_{b(E)}(r)) = 0 \quad \text{for all } i < 2n + 1 \text{ and } r > 0. \]

Consider the exact sequence

(4.2) \[ 0 \to O_{b(E)}(r) \otimes O_{b(E)}(r) \to O_{b(E)}(r) \to O_{\Phi}(r) \to 0, \]

where $O_{\Phi}(1) = q^*L$. We obtain that for $i < 2n + 1$,

\[
H^i(\Sigma, J_X/\Sigma(-r)) \cong H^i(O_{b(E)}(r) \otimes O_{b(E)}(r)) \quad \text{by Lemma 4.6} \]

\[
\cong H^{i-1}(O_{\Phi}(r)) \quad \text{by (4.1), (4.2) } (H^{-1} = 0) \]

\[
\cong \begin{cases} 
0 & \text{if } i \leq n, \\
H^n(X, L^{-r}) \otimes H^{i-1-n}(X, O_X) & \text{if } n < i < 2n + 1, 
\end{cases} 
\]

where the last isomorphism is by Lemma 4.7.

Then from the exact sequence

\[ 0 \to J_X/\Sigma(-r) \to O_{\Sigma}(r) \to O_X(-r) \to 0, \]

and applying Kodaira vanishing to $O_X(-r)$, we see

\[
H^i(O_{\Sigma}(r)) = \begin{cases} 
0 & \text{if } i < n, \\
H^n(X, L^{-r}) \otimes H^{i-1-n}(X, O_X) & \text{if } n + 1 < i < 2n + 1. 
\end{cases} 
\]

So to finish the proof, we shall show that $H^n(O_{\Sigma}(r)) = H^{n+1}(O_{\Sigma}(r)) = 0$.

In view of above results, both groups sit in the exact sequence

\[ 0 \to H^n(O_{\Sigma}(r)) \to H^n(O_X(-r)) \to H^{n+1}(\Sigma, J_X/\Sigma(-r)) \to H^{n+1}(O_{\Sigma}(r)) \to 0, \]

and hence we need to prove $\alpha$ is an isomorphism.

To this end, consider the natural commutative diagram

\[
\begin{array}{ccc}
H^n(O_{\Phi}(r)) & \to & H^{n+1}(O_{b(E)}(r) \otimes O_{b(E)}(-r)) \\
\downarrow & & \downarrow \\
H^n(O_X(-r)) & \to & H^{n+1}(\Sigma, J_X/\Sigma(-r)) \\
\end{array}
\]

Since the top and right column maps are isomorphisms, it is reduced to show that the natural map $H^n(O_X(-r)) \to H^n(O_{\Phi}(r))$ is an isomorphism. Again, this is the case by the Leray spectral sequence and Serre vanishing. \hfill \square

Theorem 4.8. \( t_*\omega_{b(E)} \cong \omega_{\Sigma} \text{ if and only if } H^n(X, O_X) = 0. \)
Proof. Assuming Claim 4.9 and pushing forward the exact sequence
\[ 0 \to \omega_{\mathcal{P}(E_l)} \to \omega_{\mathcal{P}(E_l)}(\Phi) \to \omega_\Phi \to 0 \]
by \( t \), we see that to prove the statement of this theorem is equivalent to showing
\[ t_*\omega_\Phi = 0 \text{ if and only if } H^0(X, O_X) = 0. \]

To this end, note that \( t \) is an isomorphism outside of \( X \) and \( \Phi \) is the exceptional divisor. So it suffices to show that \( t_*\omega_\Phi = 0 \) on \( X \). For a close point \( x \in X \), it is equivalent to showing \( t_*\omega_\Phi \otimes k(x) = 0 \) by Nakayama Lemma. But since \( t \) is flat when restricted to \( \Phi \), applying Grauert’s theorem we have
\[ t_*\omega_\Phi \otimes k(x) \cong H^0(F_x, \omega_\Phi|_{F_x}) = H^0(F_x, \omega_{F_x}). \]
Recall \( F_x \) is the blowing up of \( x \) at \( x \), so \( H^0(F_x, \omega_{F_x}) = H^0(X, \omega_X) \) and the last cohomology group is zero if and only if \( H^0(X, O_X) = 0 \) by Serre duality.

\[\Box\]

Claim 4.9.
\[ t_*\omega_{\mathcal{P}(E_l)}(\Phi) \cong \omega_\Sigma. \]

Proof. First note that there is a natural map \( t_*\omega_{\mathcal{P}(E_l)}(\Phi) \to \omega_\Sigma \) by the following natural maps
\[ t_*\omega_{\mathcal{P}(E_l)}(\Phi) \to j_* (t_*\omega_{\mathcal{P}(E_l)}(\Phi)|_U) \to j_* (\omega_\Sigma|_U) \cong \omega_\Sigma. \]
Here \( j : U = \Sigma \setminus X \to \Sigma \) is the open immersion and the last isomorphsim comes from the fact that \( \omega_\Sigma \) is a reflexive sheaf. So to prove the claim it suffices to prove that
\[ t_*\omega_{\mathcal{P}(E_l)}(\Phi)_x \cong \omega_{\Sigma_x} \]
for every closed point \( x \in \Sigma \). From now on we fix a closed point \( x \in X \subset \Sigma \) and denote the local ring of \( \Sigma \) at \( x \) by \( O_{\Sigma_x} \).

By Proposition 3.2 we have
\[ R^t_\ast O_{\mathcal{P}(E_l)}(-\Phi) \cong t_\ast O_{\mathcal{P}(E_l)}(-\Phi), \]
so applying the functor \( \mathcal{R}Hom_{\Sigma}(\cdot, \omega_\Sigma^* \otimes O_{\Sigma_x}) \) and denoting the local ring of \( \Sigma \) at \( x \) by \( O_{\Sigma_x} \), we get the isomorphisms of complexes
\[ R^t_* \omega_{\mathcal{P}(E_l)}(\Phi) \otimes O_{\Sigma_x} \cong \mathcal{R}Hom_{\Sigma}(R^t_* O_{\mathcal{P}(E_l)}(-\Phi), \omega_\Sigma^* \otimes O_{\Sigma_x}) \]
\[ \cong \mathcal{R}Hom_{\Sigma}(t_* O_{\mathcal{P}(E_l)}(-\Phi), \omega_\Sigma^* \otimes O_{\Sigma_x}) \]
\[ \cong \mathcal{R}Hom_{\Sigma_x}(t_* O_{\mathcal{P}(E_l)}(-\Phi)_x, \omega_{\Sigma_x}^*), \]
where the first isomorphism follows from Grothendieck Duality theorem and the last one is by Lemma 2.4. By taking the \( -i \)th cohomology we see that
\[ R^{2n+1-i} t_* \omega_{\mathcal{P}(E_l)}(\Phi) \otimes O_{\Sigma_x} \cong \mathcal{E}xt^{-i}(t_* O_{\mathcal{P}(E_l)}(-\Phi)_x, \omega_{\Sigma_x}^*). \]
In particular, when $i = 2n + 1$, we get the isomorphism
\begin{equation}
(4.3) \quad t_* \omega_{\mathcal{P}(E_L)}(\Phi) \otimes O_{\Sigma} \simeq \mathcal{E}xt^{-(2n+1)}(t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi)_x, \omega^*_{x} \Sigma).
\end{equation}

Let $I$ be the injective hull of $O_{\Sigma}$. Apply $\text{Hom}(\cdot, I)$ to the right hand side of equation \((4.3)\), then by \((2.1)\) we have
\begin{equation}
(4.4) \quad \text{Hom}(\mathcal{E}xt^{-(2n+1)}(t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi)_x, \omega^*_{x} \Sigma), I)
\end{equation}
where the second isomorphism comes from the following sequence
\begin{equation}
0 \rightarrow t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi) \simeq \mathcal{I}_{X/\Sigma} \rightarrow O_{\Sigma} \rightarrow O_X \rightarrow 0
\end{equation}
and the fact that $H^i_x(O_{X}, \mathcal{I}) = 0$ for all $i > n$.

Apply $\text{Hom}(\cdot, I)$ to equation \((4.4)\), by §V Corollary 6.5 in [9] we have an isomorphism of the completions at $x$
\begin{align*}
\mathcal{E}xt^{-(2n+1)}(t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi)_x, \omega^*_{x} \Sigma)_x^\wedge & \simeq \text{Hom}(H^{2n+1}_x(t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi)_x), I) \\
 & \simeq \text{Hom}(H^{2n+1}_x(O_{\Sigma}), I) \\
 & \simeq \mathcal{E}xt^{-(2n+1)}(O_{\Sigma}, \omega^*_{x} \Sigma)_x^\wedge \\
 & \simeq (\omega^*_{x} \Sigma)_x^\wedge.
\end{align*}
But by equation \((4.3)\) there is a natural map
\begin{equation}
\mathcal{E}xt^{-(2n+1)}(t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi)_x, \omega^*_{x} \Sigma)_x^\wedge \simeq t_* \omega_{\mathcal{P}(E_L)}(\Phi) \otimes O_{\Sigma} \rightarrow \omega_{x} \Sigma.
\end{equation}
So isomorphism of the completions implies the isomorphism of the local rings, that is,
\begin{equation}
\mathcal{E}xt^{-(2n+1)}(t_* \mathcal{O}_{\mathcal{P}(E_L)}(-\Phi)_x, \omega^*_{x} \Sigma) \simeq \omega_{x} \Sigma.
\end{equation}
So we have
\begin{equation}
t_* \omega_{\mathcal{P}(E_L)}(\Phi) \otimes O_{\Sigma} \simeq \omega_{x} \Sigma.
\end{equation}
This completes the proof of the claim. \hfill \box

**Remark 4.10.** In Theorem 4.8 we do not assume $H^i(X, O_X) = 0$ for all $1 \leq i \leq n - 1$, or equivalently, $\Sigma$ is Cohen-Macaulay. If we did, then claim 4.9 (hence Theorem 4.8) follows immediately from the main result in [14] (Theorem 3.1), since we already knew that $\Sigma$ is Du Bois.

Combine Theorem 4.2 and Theorem 4.8, we have

**Corollary 4.11.** $\Sigma(X, L)$ has rational singularities if and only if $H^i(X, O_X) = 0$ for all $i > 0$. \hfill \box
Remark 4.12. We note that to prove $\Sigma(X, L)$ has rational singularities, it suffices to push forward
\[ 0 \to O_{\mathbb{P}(E_L)}(-\Phi) \to O_{\mathbb{P}(E_L)} \to O_\Phi \to 0, \]
and apply Lemma (2.2) and Proposition (3.2). The two theorems we present in this section give a more detailed analysis of how far a secant variety is from being rational singular. Corollary 4.11 is a byproduct of them.

5. A multiplicity formula

The following proposition shows that more positivity of $L$ may lead to worse singularities in certain senses.

**Proposition 5.1.** The Samuel multiplicity of $\Sigma(X, L)$ at a closed point $x \in X$ is given by $L^n - 2^n$.

**Proof.** Since the multiplicity $\mu$ at $x$ coincides with the top Segre class of $([x], \Sigma(X, L))$, which is invariant under a birational proper morphism (cf. [8, Chap. 4]), we have
\[
\mu_x \Sigma(X, L) = s_0([x], \Sigma(X, L)) = s_n(Bl_x X, \mathbb{P}(E_L)) = s_n(N_{Bl_x X/\mathbb{P}(E_L)}) = (-1)^n(-b^*_x L + 2E)^n \quad \text{by Lemma 3.1}
\]
\[
= (-1)^n \sum_{i=0}^{n} \binom{n}{i} (-1)^i (b^*_x L)^i (2E)^{n-i}
\]
\[
= L^n + 2^n (-E)^n \quad \text{for } n > i > 0, (b^*_x L)^i E^{n-i} = 0
\]
\[
= L^n - 2^n,
\]
which completes the proof. $\square$

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