SPACE-TIME EVENTS AND RELATIVISTIC
PARTICLE LOCALIZATION

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ABSTRACT

A relation expressing the covariant transformation properties of a relativistic
position operator is derived. This relation differs from the one existing in the
literature expressing manifest covariance by some factor ordering. The relation is
derived in order for the localization of a particle to represent a space-time event.
It is shown that there exists a conflict between this relation and the hermiticity of
a positive energy position operator.

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1. Introduction

In classical physics one has a realization of the concept of a space-time event [1]. Namely, the fact that a particle is localized in \( x \) at time \( t \) represents the space-time event \( x \). The aim of this article is to investigate whether one can have in quantum mechanics, at least in principle, the same situation. Can the fact that a particle is localized in space at a given time represent a space-time event? A space-time event is characterized, in a flat space, by its transformation properties under the Poincaré group,

\[
x^\mu \rightarrow \Lambda_\nu^\mu x^\nu + \alpha^\mu.
\]

On the other hand, a particle, in quantum theory, is defined as a realization of an irreducible unitary representation of the Poincaré group [2]. It seems difficult to include interactions in a relativistic invariant manner for these irreducible representations. So the particle is described, at the first quantized level, by a reducible representation having both signs of the energy. At the second quantized level, positive energy irreducible representations characterize the one particle sector of the Hilbert space of the theory.

The requirement of relativistic invariance says nothing about the localization properties of the particle. We have a Hilbert space carrying a representation of the Poincaré group and all the operators that act on this space. How can we construct observables in this space?

In non-relativistic quantum theory we have a more or less well defined procedure: the principle of equivalence. There one begins with the fundamental classical observables, the position and the momentum and turns them into operators satisfying the Heisenberg commutation relations but having the same physical meaning of the classical observables. A way to solve the problem in a relativistic theory is to use the geometrical origin of the Poincaré transformations and to impose on the observables certain constraints arising from their preimposed transformation properties and hope that these do determine the observables completely. This is the way
that relativistic position operators have been determined. For instance in reference [3] it was shown that the requirement that the position operator transforms in a given way under translations, rotations, space and time inversions determines it completely if one searches for a self-adjoint operator with commuting components.

Pryce [4] and later on Currie, Jordan and Sudarshan [5], in the same spirit, gave relations that characterize covariant position operators. However they used a semi-classical reasoning. They found the classical relations and then converted the Poisson Brackets into commutators with a given ordering prescription, namely they used the symmetric ordering of operators.

The aim of this article is to reexamine this relation in order to see whether this ordering prescription is justified. In order to do that we rely on our above definition of covariance. Namely a position operator is defined to be manifestly covariant if its eigenstates transform under a pure Lorentz transformation as does a space-time event.

In section 2 classical manifest covariance is briefly reviewed. The transformation of the Newton-Wigner eigenstates under a boost is the subject of section 3. In section 4 the relation expressing the covariance of the quantum localization is derived. The solution to this relation, for the scalar particle, is discussed in section 5. Finally we conclude with some general and speculative remarks.

2. Classical manifest covariance

In order to have relativistic invariance it is necessary and sufficient to have a collection of generators $J_i, K_i$ representing the Lorentz Lie algebra on phase space by means of the Poisson Brackets. Relativistic invariance does not dictate the functional form of the transformation of the position variables. We can add the requirement of covariant transformation by giving the form of the transformed coordinates. We will do that in the following.

Suppose the particle is described by the position $q_i(t)$ in a given inertial frame $\mathcal{R}$. 
In a frame related to the first by an infinitesimal Lorentz transformation the position is described by

\[ q_j'(t) = q_j(t) - \epsilon^i \{ K_i, q_j(t) \} \]  

(2.1)

Note that time is not transformed in this equation.

In order to have covariant transformation properties of the position we must have

\[ q_j'(t') = q_j(t) - \epsilon_j t, \]

\[ t' = t - \epsilon^i q_i(t). \]  

(2.2)

These relations simply state that the fact that the particle is localized in a given position at a given time represents a space-time event. From relations (2.2) one can deduce how the positions in the two frames are related at the same time, to first order in \( \epsilon \),

\[ q_j'(t) = q_j'(t') - (t - t') \{ H, q_j'(t') \} = q_j(t) - \epsilon_j t - \epsilon^i q_i(t) \{ H, q_j(t) \}. \]  

(2.3)

\( H \) is the Hamiltonian of the system. Comparing the two equations (2.1) and (2.3) we find the relation,

\[ \{ K_j, q_i \} = t \delta_{ij} + q_j \{ H, q_i \}. \]  

(2.4)

This is the relation expressing the covariance of the classical position variable [4,5]. Define \( N_j \) as usual by

\[ N_j = K_j + t p_j, \]  

(2.5)

where \( p_j \) is the translation generator satisfying \( \{ q_j, p_i \} = \delta_{ij} \). Our final relation
becomes,
\[
\{N_j, q_i\} = q_j \{H, q_i\}.
\]  
(2.6)

A solution to this equation, for the spinless particle, satisfying as well the correct
transformation properties under the other Poincaré generators is
\[
q_i = \frac{N_i}{H}.
\]  
(2.7)

3. Transformation of the Newton-Wigner eigenstates

The Newton-Wigner position operator was introduced as the unique positive
energy operator satisfying the axioms of localizability of Newton and Wigner [3,6].
These axioms require the commutativity of the components, the hermiticity, and
the correct transformation properties under the translations, rotations, parity and
time reversal. The Newton-Wigner operator does not exist for massless spinning
particles. It plays an important role in the study of the non-relativistic limit [7]
and the semi-classical approximation [8]. The transformation under boosts of the
Newton-Wigner operator were not postulated and is the subject of this section.

Consider a free scalar particle, a basis of its Hilbert space is given by the
momentum eigenstates so that a general state may be written as
\[
|\psi> = \int d\tilde{p} \psi(p) |p>.
\]  
(3.1)

Here the measure is the invariant one,
\[
d\tilde{P} = \frac{d^3p}{(2\pi)^\frac{3}{2}\omega(p)}, \quad \omega(p) = \sqrt{p^2 + m^2}.
\]  
(3.2)

The scalar product is such that,
\[
< p | p' > = (2\pi)^\frac{3}{2}\omega(p) \delta^3(p - p').
\]  
(3.2)
This reproduces the scalar product of Bargmann and Wigner [9],

$$< \phi | \psi > = \int d\tilde{p} \, \phi^*(p) \psi(p). \quad (3.3)$$

The eigenstates of the Newton-Wigner position operator are given by,

$$\psi_{q_0}(p) = \sqrt{\omega(p)} e^{-ip.q_0}, \quad (3.3)$$

this state represents a particle localized in \( q_0 \) in a frame \( \mathcal{R} \) at time \( t \). We are interested in the state of the particle viewed from a frame \( \mathcal{R}' \), related to \( \mathcal{R} \) by a boost with velocity \( \beta \). Let \( t', q_0' \) be the Lorentz transform of \( t, q_0 \). The state at time \( t' \) is given, in configuration space, by,

$$< q | \psi >' = \psi'(q) = \frac{\sqrt{\gamma}}{(2\pi)^3} \int dp \sqrt{\left( 1 - \frac{\beta.p}{\omega} \right)} e^{ip.(q-q_0')} \quad (3.4)$$

Here, \( \gamma \) is defined by,

$$\gamma = \sqrt{1 - \beta^2}, \quad (3.5)$$

and \( |q> \) is an eigenstate of the Newton-Wigner operator.

The function (3.4) cannot vanish outside a bounded domain because it is the Fourier transform of a non analytic function. This is due to the presence of square roots in the integrand. The state in the frame \( \mathcal{R}' \) is not localised at time \( t' \). One may easily verify that it is not localised at any time. This proves that the Newton-Wigner operator is not manifestly covariant. The integral may be exactly evaluated in the massless one-dimensional case. The result is,

$$\psi'(q) = \frac{\sqrt{\gamma}}{2} \left( \sqrt{1 + \beta} + \sqrt{1 - \beta} \right) \delta(q - q_0') + \frac{i\sqrt{\gamma}}{2\pi} \left( \sqrt{1 - \beta} - \sqrt{1 + \beta} \right) \frac{PP}{q - q_0'}, \quad (3.6)$$

where \( PP \) denotes the principal part.
We conclude that the localisation, in the Newton-Wigner sense, of a particle at a given time does not represent a space-time event. This fact has been noted several times in the literature [3,10,11], but to the knowledge of the author no explicit calculation has been given.

4. Quantum manifest covariance

In this section we are interested in the relation expressing the covariance property of the quantum relativistic position operator. It is known that many relativistic position operators considered in the literature [4,12,13] have non-commuting components. So one may argue that localization may be meaningless in a relativistic theory, why then search for a covariant localization? The point is that although the position operator may have non-commuting components, one may find, as was recently been proved for massless particles [12], states localized within an arbitrary small region in space. The question is now whether this localization is covariant or not. The effect of having non-commuting components will be neglected in the following, but this may easily be cured by considering states localized in an arbitrary small region instead of exactly localised states. Anyway, we may restrict ourselves to scalar particles and verify at the end that a solution having commuting components exists.

Suppose the particle is described by a state vector localized in $x$ at time $t$ in an inertial frame $\mathcal{R}$,

$$|\psi>_t = |x> \otimes |\alpha>.$$ (4.1)

Where $\alpha$ represents internal degrees of freedom (spin, ...). The observer in a frame $\mathcal{R'}$ related to $\mathcal{R}$ by an infinitesimal boost would describe the particle, at the instant $t$, by the state vector

$$|\psi>_t' = |\psi>_t - ie^{i\hat{K}_i}|\psi>_t.$$ (4.2)

Note that, as in the classical case, time is unchanged.
We say that we have manifest covariance if the observer in $\mathcal{R}'$ sees the particle localized in $x'$ at time $t'$, the latter being related to $x$ and $t$ by a Lorentz transformation. In other words manifest covariance is the requirement that the localized particle at a given time represents a space-time event. This is realised if we have

$$|\psi >'_t = |x' > \otimes |\alpha' > .$$

(4.3)

Since we are interested only in the manifestly covariant localization we do not impose any requirement on the internal variables $\alpha'$. Our aim is now to find the relation satisfied by the position operator in order for this to be true. We will have to translate in time the state vector $|\psi >'_t$ to get it at time $t'$ and then impose the relation (4.3). The particle is described at time $t'$ in the frame $\mathcal{R}'$ by the wavefunction given by

$$|\psi >'_t = |\psi >'_t - i(t' - t)\hat{H}|\psi >'_t .$$

(4.4)

Here $\hat{H}$ is the Hamiltonian of the particle. Using the relation (4.2) and the value for $t' - t$ given in (2.2) we get, to first order in $\epsilon$,

$$|\psi >'_t = |\psi >_t - i\epsilon^j \hat{K}_j |\psi >_t + i\epsilon^j \hat{H} \hat{x}_j |\psi >_t .$$

(4.5)

We have used the fact that the particle is localised in $\mathcal{R}$ in order to get the last term on the right hand side of the above relation. We now impose relation (4.3) by requiring

$$\hat{x}_i |\psi >'_t = x'_i |\psi >'_t = (x_i - \epsilon_i t) |\psi >'_t .$$

(4.6)

Substituting for $|\psi >'_t$ in the above relation its value given by (4.5) we get the following relation,

$$\left(\hat{x}_i \hat{K}_j - \hat{x}_i \hat{H} \hat{x}_j \right) |\psi >_t = \left( - i\delta_{ij} t + x_i \hat{K}_j - x_i \hat{H} \hat{x}_j \right) |\psi >_t .$$

(4.7)

We now replace $x_i |\psi >_t$ by $\hat{x}_i |\psi >_t$ and we use the definition $\hat{N}_i = \hat{K}_i + t\hat{p}_i$ to get the required relation,
\[ [\hat{x}_i, \hat{N}_j] = [\hat{x}_i, \hat{H}\hat{x}_j]. \] (4.8)

This is the main result of this article.

It is important to note that relation (4.8) is realized for a self-adjoint position operator only if the latter commutes with \([\hat{H}, \hat{x}_i]\). This is realized for the Dirac position operator. A solution to equation (4.8), for a scalar particle, satisfying as well the correct transformation properties under the other Poincaré generators is given by,

\[ \hat{X}_i = \frac{1}{\hat{H}} \hat{N}_i. \] (4.9)

This operator is not self-adjoint since \(\hat{H}\) and \(\hat{N}_i\) do not commute. The physical relevance of having non self-adjoint position operators will be discussed in the next section. An important property of the above operator is that it does not couple positive and negative energy states because it is constructed out of the generators of the Poincaré group. Another property of this operator is that it has commuting components.

5. Discussion

Relation (4.8) is not the one given by Pryce [4] and used later to express the manifest covariance of position operators [5,10]. The implicit derivation of Pryce, which later was made more explicit by Currie, Jordan and Sudarshan [5] used the principle of equivalence. The method consisted in considering the classical relation and converting the Poisson Bracket into a commutator and finally ordering the products symmetrically. When one does this one gets the following relation

\[ [\hat{x}_i, \hat{N}_j] = \frac{1}{2} \left( \hat{x}_j [\hat{x}_i, \hat{H}] + [\hat{x}_i, \hat{H}]\hat{x}_j \right). \] (5.1)

Observe that this ordering prescription is somewhat arbitrary and does not
reproduce our result (4.8). It coincides with it when $\hat{x}_i$ commutes with $[\hat{x}_j, \hat{H}]$. This is the case for the Dirac position operator.

We now discuss further the solution to relation (4.8) for the spin 0 case.

Consider a free scalar particle satisfying the Klein-Gordon equation:

$$(p^2 - m^2) \phi(p) = 0.$$  \hfill (5.2)

We use the Hilbert space scalar product (3.3).

With this scalar product the Newton-Wigner position operator is given by [3],

$$\hat{q}_i = i \frac{\partial}{\partial p_i} - i \frac{p_i}{2\omega(p)^2}.$$  \hfill (5.3)

Note that this operator is self-adjoint with respect to the scalar product (3.3). The covariant position operator (4.9) is given by,

$$\hat{X}_i = i \frac{\partial}{\partial p_i}.$$  \hfill (5.4)

It is not self-adjoint, because it differs from the self-adjoint Newton-Wigner operator by a complex operator. The Newton-Wigner operator is the self-adjoint part of $\hat{X}_i$,

$$\hat{q}_i = \frac{1}{2} \left( \hat{X}_i + \hat{X}_i^\dagger \right).$$  \hfill (5.5)

One can construct the eigenstates of the covariant position operator in the momentum representation and find,

$$|x >= \int d\mathbf{p} e^{-i \mathbf{x}.\mathbf{p}} |\mathbf{p} >.$$  \hfill (5.6)

This state is a solution to the equation

$$\hat{X}_i |x >= x_i |x >.$$  \hfill (5.7)

Note that this state is obtained, at the second quantized level, by applying the
field operator $\hat{\phi}^\dagger(x)$ to the vacuum state,

$$|x> = \hat{\phi}^\dagger(x)|vac>.$$  

(5.8)

The scalar product of two such eigenstates is given by,

$$<x'|x> = \int d\tilde{p} \ e^{-i\tilde{p}.(x-x')}.$$  

(5.9)

The scalar product of two eigenstates with two different eigenvalues is not zero. This is a manifestation of the non self-adjointness of the covariant position operator. The usual interpretation of the scalar product as a probability amplitude seems to exclude the use of non hermitian operators as observables. Another manifestation of the non-self-adjointness of the covariant position operator is that we no longer have the usual closure relation in terms of the eigenstates of the position operators and their duals. The closure relation may be deduced from the relation

$$\hat{1} = \int d\tilde{p} \ |p><p|. $$  

(5.10)

After inverting relation (5.4) to get the momentum eigenstates in terms of the position eigenstates, we get the closure relation,

$$\hat{1} = \int dx dx' \ |x> \tilde{\delta}(x-x') <x'|,$$  

(5.11)

where the function $\tilde{\delta}(x-x')$ is defined by,

$$\tilde{\delta}(x-x') = \int \frac{d\tilde{p}}{(2\pi)^3} \ \omega(p)^2 e^{i\tilde{p}.(x'-x)}. $$  

(5.12)

Using this closure relation we can get the scalar product in the covariant position representation,

$$<\psi|\psi> = \int dx dx' \ \psi(x')^* \tilde{\delta}(x-x') \psi(x). $$  

(5.13)

We can see from the above relation that the interpretation of $|\psi(x)|^2 \equiv <\psi|x><x|\psi>$ as a probability density is not possible since the probabilities do not add to one.
One is thus forced to conclude that the eigenstates of this covariant position operator cannot represent physical localized states.

6. Conclusion

We have shown the impossibility of constructing localized states that transform under Lorentz transformations as do space-time events. We conclude that in a quantum theory space-time events do not have a physical realization. The motivation for introducing space-time was a classical one [1]. In a quantum theory, space-time seems not to have an exact physical reality. Its role is reduced to a parametrization of fields, permitting some important concepts such as gauge invariance and locality. One is tempted to abandon the manifold structure of space-time for other constructions that reduce to it at large length scales [14].

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