LOGARITHMIC SOBOLEV INEQUALITY REVISITED

HOAI-MINH NGUYEN AND MARCO SQUASSINA

Abstract. We provide a new characterization of the logarithmic Sobolev inequality.

1. Introduction

The classical Sobolev inequality translates information about the derivatives of a function into information about the size of the function itself. Precisely, for a function $u$ with square summable gradient in dimension $N$ one obtains that $u$ is $2N/(N-2)$-summable, that is a gain in summability which depends on $N$ and which tends to deteriorate as $N \to \infty$. On the other hand, since the middle fifties, people have started looking at possible replacements of the Sobolev inequality in order to provide an improvement in the summability independent of the dimension $N$, which can be done in terms of integrability properties of $u^2 \log u^2$. This was firstly done by Stam [23] who proved the logarithmic Sobolev inequality with Gauss measure

$$
\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_2^2} \, d\mathcal{G} \leq \frac{1}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 \, d\mathcal{G}, \quad d\mathcal{G} = e^{-\pi|x|^2} \, dx.
$$

The formula was originally discovered in quantum field theory in order to handle estimates which are uniform in the space dimension, for systems with a large number of variables. A different proof and further insight was obtained by Gross in [17]. See also the work of Adams and Clarke [1] for an elementary proof of the previous inequality. These properties are widely used in statistical mechanics, quantum field theory and differential geometry. A variant of the logarithmic Sobolev inequality with Gauss measure is given by the following one parameter family of euclidean inequalities [18, Theorem 8.14]

$$
\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_2^2} \, dx + N(1 + \log a)\|u\|_2^2 \leq \frac{a^2}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.
$$

for any $u \in H^1(\mathbb{R}^N)$ and $a > 0$. A version of this inequality for fractional Sobolev spaces $H^s(\mathbb{R}^N)$ can be found in [13]. Recently some new characterization of the Sobolev spaces were provided in [4,19,21] (see also [2,3,5-9,20]) in terms of the following family of nonlocal functionals

$$
I_\delta(u) := \int \int_{\{|u(y) - u(x)| > \delta\}} \frac{\delta^2}{|x - y|^{N+2}} \, dxdy, \quad \delta > 0,
$$

where $u$ is a measurable function on $\mathbb{R}^N$. In particular, if $N \geq 3$ and $I_\delta(u) < \infty$ for some $\delta > 0$, then in [21] it was proved that

$$
\int_{\{|u| > \lambda_N \delta\}} |u|^{2N/(N-2)} \, dx \leq C_N I_\delta(u)^{N/(N-2)}, \tag{1.1}
$$

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The second author is member of Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
for some positive constants $C_N$ and $\lambda_N$. This is a sort of nonlocal improvement of the classical Sobolev inequality and it is also possible to show that in the singular limit $\delta \searrow 0$ one recovers the classical Sobolev result, since $I_{\delta}$ converges to the Dirichlet energy up to a normalization constant. The aim of this note is to remark that in this context also a logarithmic type estimate holds. Thus we have the summability gain independent of $N$ can be controlled in terms of $I_{\delta}(u)$.

More precisely, we have the following

**Theorem 1.1.** Let $u \in L^2(\mathbb{R}^N)$ $(N \geq 3)$. There is a positive constant $C_N$ such that

$$
\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \, dx + \frac{N}{2} \log \|u\|_2^2 \leq \frac{N}{2} \log \left( C_N \delta^N \|u\|_2^{2N-4} + C_N I_{\delta}(u) \right),
$$

for all $\delta > 0$. In particular, if $u \in L^2(\mathbb{R}^N)$ is such that $I_{\delta}(u) < \infty$ for some $\delta > 0$, then

$$
(1.2) \quad \int_{\mathbb{R}^N} u^2 \log u^2 \, dx < +\infty.
$$

**Proof.** By a simple normalization argument, we may reduce the assertion to proving that

$$
(1.3) \quad \int_{\mathbb{R}^N} u^2 \log u^2 \, dx \leq \frac{N}{2} \log \left( C_N \delta^N + C_N I_{\delta}(u) \right), \quad \text{for all } \delta > 0,
$$

for any $u \in L^2(\mathbb{R}^N)$ such that $\|u\|_2 = 1$. Considering the normalized outer measure

$$
\mu(E) := \int_E u^2(x) \, dx, \quad \mu(\mathbb{R}^N) = 1,
$$

and using Jensen’s inequality for concave nonlinearities and with measure $\mu$, we have

$$
(1.4) \quad \log \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \, dx \right) = \log \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \, d\mu \right) \geq \int_{\mathbb{R}^N} \log |u|^{\frac{2N}{N-2}} \, d\mu = \frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx.
$$

On the other hand, applying (1.1), we derive that, for all $\delta > 0$,

$$
\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx \leq \log \left( D_N \delta^N + C_N I_{\delta}(u)^{\frac{N}{N-2}} \right),
$$

for some positive constant $D_N$, which implies (1.3). Here we used the fact that

$$
\int_{\{|u| \leq \lambda_N \delta\}} |u|^{\frac{2N}{N-2}} \, dx \leq \lambda_N^{\frac{4}{N-2}} \delta^{\frac{4}{N-2}},
$$

since $\int_{\mathbb{R}^N} u^2 \, dx = 1$. \hfill \Box

Defining a notion of *entropy* as typical in statistical mechanics:

$$
\text{Ent}_\mu(f) := \int_{\mathbb{R}^N} \frac{f}{\|f\|_{1,\mu}} \log \frac{f}{\|f\|_{1,\mu}} \, d\mu + \frac{N}{2} \log \|f\|_{1,\mu}, \quad f \geq 0, \quad \|f\|_{1,\mu} := \int f \, d\mu,
$$

the conclusion of the previous results reads as

$$
u \in L^2(\mathbb{R}^N), \quad \exists \delta > 0 : I_{\delta}(u) < +\infty \implies \text{Ent}_{\mathbb{R}^N}(u^2) < +\infty.
$$

**Remark 1.2** (Logarithmic NLS). If $u \in H^1(\mathbb{R}^N)$, then the results of [19] show that

$$
(1.5) \quad \lim_{\delta \searrow 0} I_{\delta}(u) = Q_N \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,
$$
for some constant $Q_N > 0$. Hence, passing to the limit as $\delta \searrow 0$ in the inequality of Theorem 1.1 one recovers classical forms of the logarithmic inequality. The logarithmic Schrödinger equation
\begin{equation}
(1.6)
\begin{align*}
\partial_t \phi + \Delta \phi + \phi \log |\phi|^2 = 0, \quad \phi : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}, \quad N \geq 3,
\end{align*}
\end{equation}
admits applications to quantum mechanics, quantum optics, transport and diffusion phenomena, theory of superfluidity and Bose-Einstein condensation (see \cite{25} and \cite{10–12}). The standing waves solutions of (1.6) solve the following semi-linear elliptic problem
\begin{equation}
(1.7)
- \Delta u + \omega u = u \log u^2, \quad u \in H^1(\mathbb{R}^N).
\end{equation}
These equations were recently investigated in \cite{14,24}. From a variational point of view, the search of solutions to (1.7) can be associated with the study of critical points (in a nonsmooth sense) of the lower semi-continuous functional $J : H^1(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by
\begin{equation}
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx,
\end{equation}
which is well defined by the logarithmic Sobolev inequality. Due to Theorem 1.1 and (1.5), one could handle a kind of nonlocal approximations of (1.7), formally defined for $\delta > 0$ by
\begin{equation}
I^A_\delta(u) + \omega u = u \log u^2,
\end{equation}
which are associated with the energy functional $J^A_\delta : H^1(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by
\begin{equation}
J^A_\delta(u) = I^A_\delta(u) + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx.
\end{equation}
Since there holds $I^A_\delta(u) \leq C_N \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$ for all $\delta > 0$ and $u \in H^1(\mathbb{R}^N)$ (cf. \cite[Theorem 2]{19}) the energy functional $J^A_\delta$ is well defined, for every $\delta > 0$.

**Remark 1.3** (Magnetic case). If $A : \mathbb{R}^N \to \mathbb{R}^N$ is locally bounded and $u : \mathbb{R}^N \to \mathbb{C}$, we set
\begin{equation}
\Psi_u(x, y) := e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y), \quad x, y \in \mathbb{R}^N.
\end{equation}
It was observed in \cite{15} that the following **Diamagnetic inequality** holds
\begin{equation}
||u(x)| - |u(y)|| \leq |\Psi_u(x, x) - \Psi_u(x, y)|, \quad \text{for a.e. } x, y \in \mathbb{R}^N.
\end{equation}
In turn, by defining
\begin{equation}
I^A_\delta(u) := \int_{\{||\Psi_u(x, y) - \Psi_u(x, x)|| > \delta\}} \frac{\delta^2}{|x-y|^{N+2}} \, dx \, dy,
\end{equation}
we have
\begin{equation}
(1.8)
I^A_\delta(|u|) \leq I^A_\delta(u), \quad \text{for all } \delta > 0 \text{ and all measurable } u : \mathbb{R}^N \to \mathbb{C}.
\end{equation}
Then, Theorem 1.1 yields the following **Magnetic logarithmic Sobolev inequality**. For $u \in L^2(\mathbb{R}^N)$, there is a positive constant $C_N$ such that
\begin{equation}
\int_{\mathbb{R}^N} \frac{|u|^2}{\|u\|^2} \frac{|\nabla |u||^2}{\|u\|^2} \, dx + \frac{N}{2} \log \|u\|^2 \leq \frac{N}{2} \log \left( C_N \delta^\frac{2}{N+1} \|u\|_2^{\frac{2N}{N+1}} + C_N I^A_\delta(u) \right).
\end{equation}
Notice that, since $I^A_\delta(|u|) \approx \|\nabla |u||^2$ as $\delta \searrow 0$ \cite{19} and $I^A_\delta(u) \approx \|\nabla u - iAu\|_2^2$ as $\delta \searrow 0$ \cite{22}, from inequality (1.8) one recovers $\|\nabla |u||_2 \leq \|\nabla u - iAu\|_2$ which follows from the well-know diamagnetic inequality for the gradients $|\nabla u| \leq |\nabla u - iAu|$, see \cite{18}.

As a companion to Theorem 1.1, we also have the following
Theorem 1.4. Let \( u \in L^2(\mathbb{R}^N) \) \((N \geq 3)\). Assume that there exists a non-decreasing function 
\( F : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that 
\( F(ts) \leq t^\beta F(s) \) for any \( s, t \geq 0 \) and some \( \beta > 0 \) and 
\[
1.4 \quad \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dxdy < +\infty.
\]

Then there exists a positive constant \( C_{N,F} \) such that 
\[
\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^2 \leq \frac{N}{2} \log \left( C_{N,F} \|u\|_2^2 + C_{N,F} \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dxdy \right),
\]
\( \square \)

In particular, condition (1.2) holds.

Proof. Consider the statement when \( \|u\|_2 = 1 \). In light of inequality (1.4), since by [21, Proposition 6] there exists \( C_N > 0 \) and \( \lambda_N > 0 \) such that 
\[
1.10 \quad \int_{\{|u| > \lambda_N F(1/2)\}} |u|^{2N/(N-2)} dx \leq C_N \left( \frac{1}{F(1/2)} \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dxdy \right)^{N/(N-2)},
\]
by arguing as in the previous proof, we get 
\[
\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 \leq \log \left( D_{N,F} + D_{N,F} \left( \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dxdy \right)^{N/(N-2)} \right),
\]
where we used the fact that 
\[
\int_{\{|u| \leq \lambda_N F(1/2)\}} |u|^{2N/(N-2)} dx \leq \lambda_N^{N-2} F(1/2)^{N-2},
\]
since \( \int_{\mathbb{R}^N} u^2 dx = 1 \). Then, we get 
\[
\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{N}{2} \log \left( C_{N,F} \|u\|_2^2 + C_{N,F} \frac{1}{\|u\|_2^2} \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dxdy \right).
\]
In the general case, using the sub-homogeneity condition on \( F \) yields 
\[
\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \leq \frac{N}{2} \log \left( C_{N,F} \|u\|_2^2 + C_{N,F} \frac{1}{\|u\|_2^2} \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dxdy \right),
\]
which yields the desired conclusion. \( \square \)

Remark 1.5 \((L^p(\mathbb{R}^N)\text{-version})\). If \( p > 1 \) and \( N > p \), one has a variant of (1.4), namely 
\[
1.11 \quad \log \left( \int_{\mathbb{R}^N} |u|^{N+p} dx \right) \geq \frac{p}{N-p} \int_{\mathbb{R}^N} |u|^p \log |u|^p dx.
\]

Then, by arguing as in the proofs of Theorems 1.1 and 1.4 with 
\[
1.12 \quad u \mapsto \int_{\{|u(y) - u(x)| > \delta\}} \frac{|u|^p}{|x - y|^{N+p}} dxdy, \quad u \mapsto \int_{\mathbb{R}^2 \mathbb{N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+p}} dxdy,
\]
in place of \( I_\delta(u) \) and (1.9) respectively, it is possible to get corresponding log-Sobolev inequalities as for the case \( p = 2 \), via the results of [21]. In particular, if \( u \in L^p(\mathbb{R}^N) \) and the functionals in (1.12) are finite at \( u \) for some \( \delta > 0 \), then 
\[
\int_{\mathbb{R}^N} |u|^p \log |u|^p dx < +\infty.
\]

The Euclidean logarithmic Sobolev inequalities for the \( p \)-case have been intensively studied, see e.g. the work of Del Pino and Dolbeault [16] and the references therein.
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(H.-M. Nguyen) DEPARTMENT OF MATHEMATICS
EPFL SB CAMA
STATION 8 CH-1015 LAUSANNE, SWITZERLAND
E-mail address: hoai-minh.nguyen@epfl.ch

(M. Squassina) DIPARTIMENTO DI MATHEMATICA E FISICA
UNIVERSITÀ CATTOLICA DEL SACRO CUORE
VIA DEI MUSEI 41, I-25121 BRESCIA, ITALY
E-mail address: marco.squassina@unicatt.it