Percolation induced effects in two-dimensional coined quantum walks: analytic asymptotic solutions

To cite this article: B Kollár et al 2014 New J. Phys. 16 023002

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16
(2014) 023002
doi:10.1088/1367-2630/16/2/023002

Abstract

Quantum walks on graphs can model physical processes and serve as efficient tools in quantum information theory. Once we admit random variations in the connectivity of the underlying graph, we arrive at the problem of percolation, where the long-time behaviour appears untreatable with direct numerical methods. We develop novel analytic methods based on the theory of random unitary operations which help us to determine explicitly the asymptotic dynamics of quantum walks on two-dimensional finite integer lattices with percolation. Based on this theory, we find new unexpected features of percolated walks like asymptotic position inhomogeneity or special directional symmetry breaking.

1. Introduction

The dynamics of particles is one of the central problems of physics. The rules formulated for the motion of particles reflect our knowledge at an elementary level. Surprisingly, on a larger scale we can recover certain properties of complex physical systems with models based on simple rules.

Author to whom any correspondence should be addressed.

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New Journal of Physics 16 (2014) 023002
1367-2630/14/023002+19$33.00 © 2014 IOP Publishing Ltd and Deutsche Physikalische Gesellschaft
Recently, a novel elementary model of motion has been introduced for quantum particles: quantum walks (QWs) [1–3] have been defined in analogy to classical random walks. Classical walks reflect the fact that the motion of particles is given by sequences of tiny collisions of inherently random character. QWs describe the motion of a quantum particle on a graph and capture its wave character [4]. The particle wave function is repeatedly split, and the partial waves mutually interfere forming a characteristic interference pattern. This pattern is radically different from its classical counterpart, leading to rich behaviour from exponential speedup to localization. In turn, these may lead to interesting physical effects in models of coherent transport. Much like their classical counterpart, QWs are suitable tools for computation, thus are perfectly fitting for designing quantum algorithms [5–10]. Moreover, QWs have been proven to be universal computational primitives [11, 12]. Due to their potential, QWs have attracted considerable experimental interest. A number of experiments demonstrating QWs have been reported with trapped atoms [13, 14], trapped ions [15, 16], photons in optical waveguides [17–20], in an optical realization [21] and in optical feedback loops [22–25].

Any realistic model has to respect the simple fact that the world around us differs to a certain degree from the models we form to describe it. These differences can be viewed as perturbations or imperfections. While imperfections are usually expected to destroy the ideal situation, they can also lead to completely new effects which are not present in the original model. QWs are, in this respect, no exception [26–29]. When focusing on the so-called discrete QWs, we can study the ideal situation and then analyse the effect of different perturbations. They can lead to diffusion-like behaviour representing the final state in a transition from quantum ballistic motion to classical diffusion [26]. When static phase randomness is present, the QW exhibits Anderson type localization. This effect has been demonstrated experimentally [19, 23].

In the present paper, we will focus on perturbations of the underlying graph. Defects of the graph can be viewed as a classical percolation problem combined with quantum dynamics [27, 30, 31]. Since continuous time QWs serve as a model for electron transport [32], these randomizations of the underlying graph occur naturally as a model for disordered media [33]. Even fairly small changes in the graph may lead to a radical alteration of the dynamics [34]. The static percolation problem (where the disorder does not change throughout the time evolution) for the continuous time QWs is sometimes referred to as statistical networks [35, 36]. Dynamical percolation, i.e. changing the graph during the time evolution repeatedly, has recently been considered for the continuous time case [37]. Discrete QWs are usually defined on a regular lattice and the motion of the particle is defined by the evolution operator which is the product of the coin and the step operator. The defects of the graph determine the form of the evolution operator [30].

In a previous paper [38], we introduced a general formalism describing the evolution of QWs on graphs with dynamically occurring defects: percolation graphs [39–41]. We applied the general method of random unitary operations in order to determine the possible asymptotic states. This approach allowed us to identify the so-called attractor space and to specify the long-time behaviour of the system. Besides the completely mixed state, we found other stationary states as well as asymptotically oscillating solutions. The application of these results to higher-dimensional graphs is not a trivial one: while the general formalism is in principle independent of the dimension of the underlying graph, in practice carrying out the prescribed calculations directly is very involved, seriously limiting its use for more complicated cases.

In this paper we improve significantly our method which, in certain cases, leads to a considerably more efficient way to construct the attractor space and the asymptotic dynamics.
and is particularly suitable for dynamically percolated QWs on integer lattices. We point out that static percolation is not the subject of our treatment. We demonstrate that under rather general conditions one can construct the attractor space using pure quantum states. In this way, we have a much better physical insight into the asymptotic behaviour through the analysis of the properties of the pure states’ constituents.

Coined QWs in two dimensions exhibit a number of interesting effects \[42\] that are not present in one-dimensional (1D) systems. A prominent example is localization or trapping at the origin \[43–46\] for an initially spatially localized walker. Here we report on a number of novel features for two-dimensional (2D) QWs on percolation lattices. We illustrate the rich spectrum of properties of 2D percolated walks for three special types of coins. The first one is the Hadamard coin extended to 2D walks. In striking contrast to the 1D case, here a non-uniform position distribution can be obtained asymptotically. Another new effect is the breaking of the directional symmetry: should we rotate the initial state and the underlying graph by 90° in a certain unpercolated walk, the obtained position distribution will also be rotated. However, if we add percolation, then the above directional symmetry will be broken. Next, we show that for percolated walks driven by the Grover coin, trapping (localization) can be found. Our last example is the Fourier coin, where we prove that the asymptotic position distribution is always uniform, independent of the initial state.

All the obtained results shed light on open system dynamics in connection with QWs and are of theoretical interest presenting novel mathematical methods allowing for unexpected results. In addition, the present results are not of purely theoretical value but form the basis for experimental implementations. In view of recent developments in the optical realization of 2D walks \[25\], the predicted effects could be experimentally observed.

The paper is organized as follows. First, we introduce our notation and define the model of QWs on percolation graphs. Then, we review the general method for obtaining the asymptotic states of the model. In section 4, we present a procedure using the pure eigenstate ansatz. In section 5, we work out the case of QWs on 2D percolation graphs in detail and present three explicit examples: the Hadamard walk, the Grover walk and the Fourier walk. Finally, we draw some conclusions in section 6.

2. Definitions

Given a \(d\)-regular finite simple graph \(G(V, E)\), the Hilbert space \(\mathcal{H}\) of a discrete time QW is defined as the tensor product of the position space \(\mathcal{H}_P\) and the coin space \(\mathcal{H}_C\). Here, the position space is spanned by state vectors corresponding to the vertices of the graph \(G\), whereas the coin space is spanned by state vectors corresponding to the directions of possible nearest neighbour hops between adjacent vertices, forming a \(d\)-dimensional space.

We add dynamical edge (bond) percolation to the model. Roughly speaking, the graph of the walk is randomly changing at each step of the walk. Every edge of the graph \(G\) has an independent, but identical, \(p\) probability of being perfect (present), and the complementary probability \(1 – p\) of being broken (missing). Such a percolation graph is shown in figure 1. Here we note, that although the presented methods are valid more generally, in this paper we will treat only 2D integer lattices with a lattice size greater than 2 in each direction. Preceding every discrete step of a QW we randomly choose an edge configuration \(\mathcal{K} \subseteq E\), thus making the percolation change in time (dynamical percolation). This dynamical percolation serves as a source of decoherence, induced by classical randomness—a lack of control over the system.
One can ask immediately the following question: how do we define the actual time evolution of a QW on an imperfect (percolation) graph? Our approach is to keep the time evolution unitary. We introduce a reflection operator $R$ to alter the definition of a step (shift) operator $S$ of the walk. The role of the reflection is simple: whenever the walker faces a broken edge, instead of stepping through, it undergoes a reflection in its internal coin degree of freedom (see figure 1). This reflection is carried out by the action of the reflection operator $R$. Employing all of these criteria, we can finally define a $K$ (edge configuration) dependent unitary shift operator

$$S_K = \sum_{(m,m\oplus c) \notin K} |m\rangle_P \langle m| \otimes |c\rangle_C \langle c| R + \sum_{(m,m\oplus c) \in K} |m \oplus c\rangle_P \langle m| \otimes |c\rangle_C \langle c|, \quad \text{(1)}$$

where $m \oplus c$ denotes the nearest neighbour of the graph vertex $m$ in the direction labelled by the coin state $c$. Thus, the unitary time evolution of a single step on a given edge configuration $K$ (imperfect graph) has the form of

$$U_K = S_K (I_P \otimes C), \quad \text{(2)}$$

where $I_P$ is the identity operator acting on the position space and $C$ is the (special) unitary coin operator acting only on the internal degree of freedom. The usage of $I_P$ reflects the fact that in this paper we look at the case where the system is homogeneous, i.e. at every vertex the coin operator is the same.

From the building blocks we defined above, the full-time evolution can be constructed. The change of the underlying percolation graph after each iteration is caused by classical randomness, thus it introduces mixing during the time evolution. Hence, we use the density operator formalism to describe the complete state $\hat{\rho}(t)$ of the walk at discrete time $t$. The complete time evolution takes the form of

$$\hat{\rho}(t+1) = \sum_K \pi_K(p) U_K \hat{\rho}(t) U_K^\dagger \equiv \Phi(\hat{\rho}(t)), \quad \text{(3)}$$
where we introduced the linear superoperator $\Phi()$. With $\pi_K(p)$, we denote the probability that the actual configuration $K$ occurs. By its construction, the superoperator $\Phi()$ belongs to the family of random unitary operations—RUO maps.

3. Asymptotics—general method

The long time behaviour rendered by such RUO maps are studied in detail in [47–49]. Asymptotic states and their dynamics are governed by the so-called attractor space spanned by operators $X_{\lambda,i} \in B(\mathcal{H})$, where all $X_{\lambda,i}$ satisfy the following set of equations:

$$
X_{\lambda,i} U_K = \lambda U_K X_{\lambda,i},
$$

where $|\lambda| = 1$ for all $K \subseteq E$. With $B(\mathcal{H})$ we denote the space of linear operators acting on the Hilbert-space $(\mathcal{H})$ of the system in question. Note that $X_{\lambda,i}$ are not necessarily valid density matrices. The search for the attractors is usually quite tedious. However, for certain cases a shortcut exists and the attractor space can be constructed with the help of pure eigenstates, which we discuss in section 4.

The asymptotics can be readily determined with the following formula:

$$
\hat{\rho}_{as}(t \gg 0) = \sum_{|\lambda|=1; i} \lambda^i X_{\lambda,i} \text{Tr}(\hat{\rho}(t = 0)X_{\lambda,i}^\dagger),
$$

where the phases of $\lambda$ are responsible for the appearance of non-monotonous asymptotic dynamics, e.g. limit cycles. Moreover, we require that $X_{\lambda,i}$ form an orthonormal basis. In general, the asymptotic behaviour of dynamics generated by RUO maps is independent of the probabilities by which unitary operations are applied, except for the extremal cases when $p = 0$ or 1 [48].

For percolated QWs, the very demanding work of finding the attractor space matrices (4) can be simplified to a great extent with the use of the method proposed in our previous paper [38]. Let us summarize the essential main steps.

We rewrite conditions set by (4) into the separated form

$$
(IP \otimes C)X_{\lambda,i} (IP \otimes C^\dagger)\lambda^* = \lambda X_{\lambda,i} S_{\lambda;i} S_{\lambda;i}.
$$

This form allows us to gain the solution in two steps. We have to find a subspace defined by shift operators corresponding to different configurations

$$
S_{\lambda;i} X_{\lambda,i} S_{\lambda;i} = S_{\lambda;i} X_{\lambda,i} S_{\lambda;i}
$$

for all $\mathcal{K} / \mathcal{K} \subseteq E$. We call the conditions imposed by shift operators the shift conditions, i.e. conditions connecting coin blocks (submatrices corresponding to the same position values) of attractor space matrices $X_{\lambda,i}$. For QWs on graphs with a high degree of symmetry, these conditions are rather simple.

Moving forward, on the subspace determined by the shift conditions only a single equation of type (6) should be fulfilled. The most convenient choice is the graph where all of the edges are missing. This results in the following local condition:

$$
(IP \otimes RC)X_{\lambda,i} (IP \otimes C^\dagger R^\dagger) = \lambda X_{\lambda,i}.
$$
Note that the latter condition acts only on local coin subspaces, and is homogeneous spatially. Thus, it can be decomposed into a set of equations acting only on coin blocks of $X_{\lambda,i,j}$ attractor matrices. The importance of this coin condition is twofold. Firstly, it determines the actual shape of the attractor space matrices. Secondly, the $|\lambda| = 1$ eigenvalues are determined by this equation. Therefore, one arrives at the conclusion that the form of the attractor space is highly dependent on the coin operator $C$. The topology of the graph is reflected by the shift conditions (7), while the coin condition of (8) is responsible for the appearance of specific asymptotic behaviour, e.g. the limit cycle.

The $|\lambda| = 1$ eigenvalues of (4) are responsible for the type of asymptotic dynamics that build up, i.e. monotonous or limit cycle. While a rich variety of asymptotic behaviours are available in the full quantum state picture, the asymptotic position density operator (and position distribution) of the QWs under consideration are strictly stationary in time. This can be seen readily.

From equations (4) and (5), it is apparent that for all asymptotic $\hat{\rho}(t)_{as}$ quantum states a set of equations

$$\hat{\rho}(t+1)_{as} = U_{K}\hat{\rho}(t)_{as}U_{K}^\dagger$$

for all $K \subseteq E$ holds. Thus, the equation describing the evolution on a graph with all edges broken

$$\hat{\rho}(t+1)_{as} = U_{\{\}}\hat{\rho}(t)_{as}U_{\{\}}^\dagger = (I_P \otimes RC_C)\hat{\rho}(t)_{as} (I_P \otimes RC_C)^\dagger$$

is also satisfied. Note that this last expression is local: parts of the density operator corresponding to different graph vertices cannot interfere. Thus, all asymptotic position density operators satisfy the following equation:

$$(\hat{\rho}(t+1)_{as})_{p} \equiv \text{Tr}_C \left( \hat{\rho}(t+1)_{as} \right)$$

$$= \text{Tr}_C \left\{ (I_P \otimes RC_C)\hat{\rho}(t)_{as} (I_P \otimes RC_C)^\dagger \right\}$$

$$= \text{Tr}_C \left( \hat{\rho}(t)_{as} \right) \equiv \left( \hat{\rho}(t)_{as} \right)_{p} .$$

(11)

Therefore, all asymptotic position density operators—thus, the asymptotic position distributions—do not change with additional interactions: the spatial distribution is stationary.

The general conditions (7) and (8) considerably simplify the finding of asymptotics. However, especially for higher dimensions, this is still a formidable task. The problem can be significantly simplified by the following procedure.

### 4. Pure eigenstate ansatz

Let us assume that we have the basis $\{|\phi_{\alpha,i_{\alpha}}\rangle\}$ of common eigenstates satisfying

$$U_{K}|\phi_{\alpha,i_{\alpha}}\rangle = \alpha|\phi_{\alpha,i_{\alpha}}\rangle$$

for all $K$, where we use the index $i_{\alpha}$ to respect the possible degeneracy of the eigenvalue $\alpha$. It is straightforward to see that any matrix from the span of $\{|\phi_{\alpha,i_{\alpha}}\rangle\langle\phi_{\beta,i_{\beta}}|\}$ with a fixed eigenvalue product $\alpha^* \beta = \lambda$, i.e.

$$Y_{\lambda,i} = \sum_{\alpha^* \beta = \lambda, i_{\alpha}, i_{\beta}} A_{\alpha^* \beta}^{i_{\alpha}, i_{\beta}}|\phi_{\alpha,i_{\alpha}}\rangle\langle\phi_{\beta,i_{\beta}}|$$

(13)
is an attractor corresponding to the superoperator eigenvalue $\lambda$ with matrix elements $A_{\alpha,i}^{\alpha,i}$. Let us call these attractors $p$-attractors.

One can notice that $p$-attractors in the form (13) satisfy equations

$$Y_{\lambda,i}U_K = \lambda U_{K'}Y_{\lambda,i},$$

(14)

where $|\lambda| = 1$ for all $K, K' \subseteq E$. This equation, in comparison with the condition for general attractor matrices (4), is more restrictive. Remarkably, the opposite implication is also true: whenever a non-zero matrix $X$ satisfies all equations (14), then $X$ can always be written in the form (13) with $|\lambda| = 1$ superoperator eigenvalue.

The first statement is trivial, and to prove the second statement we first rewrite the set of equations (14) in the vector form

$$U_K \otimes U_{K'}^*|y_{\lambda,i}\rangle = \lambda|y_{\lambda,i}\rangle$$

(15)

with $\langle p, q|y_{\lambda,i}\rangle \equiv \langle p|Y_{\lambda,i}|q\rangle$ expanded in the natural (position $\otimes$ coin) basis. Multiplying both sides by the probabilities of the unitaries occurring during time evolution $\pi_K(p), \pi_{K'}(p)$ and summing over all unitaries, we transform (15) into an eigenvalue problem

$$B \otimes B^*|y_{\lambda,i}\rangle = \lambda|y_{\lambda,i}\rangle$$

(16)

with $B \equiv \sum_j \pi_K(p)U_K$ and $|\lambda| = 1$. The spectral radius of the map $B$ is bounded by one, the restriction of this map to Hilbert subspace corresponding to eigenvalues $|\alpha| = 1$ is diagonalizable and any eigenvector corresponding to an eigenvalue $|\alpha| = 1$ is a common eigenstate of unitaries $U_K$ and vice versa. Hence, a general solution of (16) is

$$|y_{\lambda,i}\rangle = \sum_{\alpha^{\pm} = \lambda, i, \alpha} A_{\alpha,i}^{\alpha,i} |\phi_{\alpha,i}\rangle \otimes (|\phi_{\alpha,i}\rangle)^*$$

(17)

which in matrix notation takes the form (13).

We can conclude that an attractor $X_{\lambda,i}$ satisfying the set of equations $X_{\lambda,i}U_K = \lambda U_K X_{\lambda,i}$ can be constructed from common eigenstates, if and only if an attractor $X_{\lambda,i}$ satisfies the more-restrictive set of equations $X_{\lambda,i}U_K = \lambda U_K X_{\lambda,i}$. A direct consequence follows immediately: the trivial attractor proportional to the identity operator is clearly an attractor but not a $p$-attractor (except in the trivial case when an RUO map describes unitary evolution). Thus, the space of attractors always contains—as a minimal subspace—the span of $p$-attractors and identity. Surprisingly, in number of non-trivial cases this minimal subspace forms the whole attractor set and the asymptotic dynamics can be analysed readily.

Indeed, one can assume an orthogonal projection $P$ onto the subspace of the common eigenstates of unitaries $\{U_K\}$. Let $\tilde{P}$ be its orthogonal complement satisfying $P + \tilde{P} = I$. In all cases, both projections are fixed points of the RUO map $\Phi()$. Thus, the asymptotic dynamics after a sufficient number of iterations can be written as

$$\hat{\rho}(t \gg 1) = U_K^t \mathcal{P} \hat{\rho}(t = 0) \mathcal{P}(U_K^t)^t + \tilde{P} \frac{\text{Tr}[\hat{\rho}(t = 0) \tilde{P}]}{\text{Tr} \tilde{P}}$$

(18)

for all unitaries corresponding to all pairs of indices $K, K'$. The asymptotic evolution in this case is an incoherent (statistical) mixture of a purely unitary dynamics inside the common eigenstates and a maximally mixed state living on the orthogonal subspace.
With the just-proposed procedure, one can find elements of attractor space for a broad class of RUO time evolutions. Computation of $p$-attractors is much easier usually than the computation of general attractors, since finding common eigenvectors is an easier task than to find general elements of the attractor space. For arbitrary dimensional graphs, the number of equations determining the general attractors scales quadratically with the size of the problem (which has dimension of the Hilbert space) while the proposed method scales linearly. In addition, the process of searching for attractors is more intuitive. The common eigenstates also form a decoherence free subspace, moreover they carry a physical meaning—as elements of the Hilbert space—while the general attractors do not necessarily have this property, as they are elements of $B(\mathcal{H})$. In the following, we show that 2D percolated QWs can be successfully treated with the just-described pure state method.

5. Two-dimensional quantum walks

Let us employ our method now on the case of a 2D QW on a percolation lattice. We will work out three specific examples of coin operators. We consider 2D Cartesian (square) lattices (see figure 1) with two different boundary conditions. We investigate the cases of $M \times N$ tori (periodic boundaries) and carpets (reflecting boundary conditions). We note that the method presented in previous sections is applicable also for other types of boundary conditions.

On percolated 2D systems, the coin spaces corresponding to the possible shift directions are spanned by the vectors $|L\rangle_C, |D\rangle_C, |U\rangle_C, |R\rangle_C$. The unitarity of the time evolution on an imperfect graph is ensured by the use of the reflection operator. Throughout this paper, we use the transposition matrix $\sigma_s \otimes \sigma_c$ as the reflection operator $R$, and we choose the coin matrix $C$ from the $SU(4)$ group. Let us introduce a shorthand $|x, y, c\rangle \equiv |m = (x; y)\rangle_P \otimes |c\rangle_C$ for denoting state vectors living on the composite Hilbert-space $\mathcal{H}$ of 2D QWs.

We apply the general method reviewed in section 3. To find the asymptotics of 2D QWs on percolation graphs, we expand an attractor space matrix in the natural basis

$$X_{s,i} = \sum_{s_1, \ldots, s_2, t_1, t_2} W_{s_2, t_2, c_2}^{s_1, t_1, c_1} |s_1, t_1, c_1\rangle \langle s_2, t_2, c_2|,$$

where $W_{s_2, t_2, c_2}^{s_1, t_1, c_1}$ denotes the matrix elements. We also introduce the following shorthand corresponding to coin states $|L\rangle_C, |D\rangle_C, |U\rangle_C, |R\rangle_C$ acting on indices corresponding to position states:

$$L(s, t) = (s - 1, t), \quad D(s, t) = (s, t - 1),$$
$$U(s, t) = (s, t + 1), \quad R(s, t) = (s + 1, t)$$

for all $s, t$ which are well defined with respect to the given topology of the underlying graph, i.e. taking boundary conditions into account. We define an involution $\sim$ acting on coin indices as $|\bar{L}\rangle_C = |R\rangle_C$ and $|\bar{D}\rangle_C = |U\rangle_C$.

Using the notation we have just introduced, the shift conditions of (7) take the following form:

$$W_{c(s,t),c}^{c(s,t),c} = W_{s,t,\tilde{c}}^{s,t,\tilde{c}}$$

$$W_{c(s,t),c}^{s,t,\tilde{c}} = W_{s,t,\tilde{c}}^{c(s,t),c}$$

(21) (22)
Note that indices $s, t, c$ run on their corresponding abstract space, i.e. $s, t$ on the sites of the 2D graph and $c$ on coin state labels $L, D, U, R$. When $c(s_1, t_1) = (s_2, t_2)$ or $\tilde{c} = d$ is not satisfied, all elements satisfy

$$W^{c(s_1, t_1), c}_{d(s_2, t_2), d} = W^{s_1, t_1, \tilde{c}}_{s_2, t_2, \tilde{d}} = W^{s_1, t_1, \tilde{c}}_{d(s_2, t_2), d} = W^{c(s_1, t_1), c}_{s_2, t_2, \tilde{d}}. \quad (23)$$

If $c(s_1, t_1)$ or $d(s_2, t_2)$ is not well defined, i.e. at least one of the points belong to a reflecting boundary (in the case of the carpet), the corresponding equations must be omitted from the set of equations defined above. In summary, all matrix elements of an attractor space matrix satisfy (21) and (22), and non-diagonal matrix elements must satisfy the stricter condition of (23). A natural way to solve these equations is first to find a subspace satisfying (23) and then to enlarge this subspace allowing (21) and (22). Next, the condition imposed by (8) must be solved on the now-obtained abstract subspace, resulting in the full attractor space.

In the following, we show that with the help of the pure state ansatz proposed in section 4, the above strategy can be simplified considerably. The condition defining $p$-attractors (14) for percolated QWs can be separated similarly as in the case of general attractors (6)

$$(I_P \otimes C) Y_{\lambda, i} (I_P \otimes C^\dagger) = \lambda S^T_{K} Y_{\lambda, i} S_{K'} \quad (24)$$

for all $K, K' \subseteq E$. Note that in comparison with (6), in (24), only the right side differs. This implies that the only difference comes from the shift conditions, namely

$$S^T_{K} Y_{\lambda, i} S_{K'} = S^T_{K} Y_{\lambda, i} S_{L'} \quad (25)$$

for all $K, K', L, L' \subseteq E$. Considering this, the rule for matrix elements $V^{s_1, t_1, c}_{s_2, t_2, c_2} = \langle s_1, t_1, c_1 | Y_{\lambda, i} | s_2, t_2, c_2 \rangle$ of a possible $p$-attractor $Y_{\lambda, i}$ of 2D QWs can be obtained

$$V^{c(s_1, t_1), c}_{d(s_2, t_2), d} = V^{s_1, t_1, \tilde{c}}_{s_2, t_2, \tilde{d}} = V^{s_1, t_1, \tilde{c}}_{d(s_2, t_2), d} = V^{c(s_1, t_1), c}_{s_2, t_2, \tilde{d}} \quad (26)$$

which is the same rule as in (23), but here it applies to all elements including diagonal elements. Thus, our strategy is simplified further: firstly, we find all $p$-attractors. Secondly, we enlarge the found subspace by allowing (21) and (22). In this way at least one additional attractor—the trivial one, proportional to the identity—can be found.

Since $p$-attractors can be constructed from the common eigenstates of a QW, they can be found rather easily. These states are determined by the equations

$$I_P \otimes C |\phi_{a, i_a}\rangle = \alpha S^T_{K} |\phi_{a, i_a}\rangle \quad (27)$$

for all $K \subseteq E$. Firstly, as we have seen previously, the coin can be separated. Thus, all common eigenstates are confined to a subspace satisfying the shift conditions

$$S^T_{K} |\phi_{a, i_a}\rangle = S^T_{K'} |\phi_{a, i_a}\rangle \quad (28)$$

for all $K, K' \subseteq E$. These shift conditions can be rewritten in the natural basis $|\phi\rangle = \sum_{s, t, c} \phi_{s, t, c} |s, t, c\rangle$, taking the elegant form of

$$\phi_{s, t, \tilde{c}} = \phi_{c(s, t), c}, \quad (29)$$

where the topology of the underlying graph (boundary conditions) again must be taken into account. Secondly, in the subspace spanned by the latter shift conditions, an ordinary and local
eigenvalue problem determines the form of the coin states, and the possible spectrum

\[ I_P \otimes RC | \phi_{\alpha,i} \rangle = \alpha | \phi_{\alpha,i} \rangle. \]  

(30)

In the following, we employ the just-described procedure to explicitly solve certain percolated 2D QWs.

5.1. The 2D Hadamard walk: breaking the directional symmetry

The 2D Hadamard walk is a direct generalization of the 1D Hadamard walk, using the tensor product form coin

\[ H^{(2D)} = H \otimes H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \]

where

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]  

(32)

is the well-known coin operator of the 1D Hadamard walk. In the undisturbed case, this coin exhibits a spreading behaviour, which is characterized by peaks propagating from the origin at a constant velocity. In the percolation case, first we solve (30) to gain the spectrum of \( p \)-states resulting in the set of eigenvalues \( \{ i, -i, 1, 1 \} \). The corresponding eigenvectors of the \( RC \) operator are \( |v_1\rangle_C = \frac{1}{2}(1, -i, -i, -1)^T \), \( |v_2\rangle_C = \frac{1}{2}(1, i, i, -1)^T \), \( |v_3\rangle_C = \frac{1}{2}(1, 0, 0, 1)^T \) and \( |v_4\rangle_C = \frac{1}{2\sqrt{3}}(0, 1, -1, 0)^T \), respectively. We find the following orthonormal basis on the percolated \( M \times N \) carpet:

\[ |\phi_1\rangle = \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} \frac{(-1)^s}{\sqrt{MN}} |s, t\rangle_P \otimes |v_1\rangle_C, \]  

(33)

\[ |\phi_2\rangle = \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} \frac{(-1)^s}{\sqrt{MN}} |s, t\rangle_P \otimes |v_2\rangle_C, \]  

(34)

\[ |\phi_3(t)\rangle = \sum_{s=0}^{M-1} \frac{1}{\sqrt{M}} |s, t\rangle_P \otimes |v_3\rangle_C, \]  

(35)

\[ |\phi_4(s)\rangle = \sum_{t=0}^{N-1} \frac{(-1)^t}{\sqrt{N}} |s, t\rangle_P \otimes |v_4\rangle_C. \]  

(36)

The next step is to prove the completeness. For the \( \alpha = i \) eigenvalue, a general common eigenstate must have the form \( |\phi\rangle = \sum_{s,t} a_{s,t} |s, t\rangle_P \otimes |v_1\rangle_C \). Employing shift conditions (29), we get \( a_{s+1,t} = -a_{s,t} \) and \( a_{s,t+1} = a_{s,t} \), thus a single eigenvector is found and it takes the form (33). Similarly, for \( \alpha = -i \), a single vector (34) is found. For \( \alpha = 1 \), the general form of a common eigenstate is \( |\phi\rangle = \sum_{s,t} |s, t\rangle_P \otimes (a_{s,t} |v_3\rangle_C + b_{s,t} |v_4\rangle_C) \). Applying the shift conditions, we find \( a_{s,t} = a_{s-1,t} \) and \( b_{s,t} = -b_{s,t-1} \). This means \( M + N \) free parameters, thus an \( M + N \) dimensional subspace of common eigenstates with basis vectors (35) and (36). Now, we have to determine
the remaining attractors which cannot be constructed from common eigenstates. We repeat, that from (21) and (22) we know that non-\(p\)-attractors can be searched in a diagonal form. Thus, solving the local equation (8) for a diagonal coin block for \(\lambda = 1\) imposes

\[
B = \begin{pmatrix}
a & c & C & A \\
d & b & B & D \\
-D & B & b & -d \\
A & -C & -c & a
\end{pmatrix},
\]

where \(a - A = b + B\) and \(D - d = c + C\). As we require only diagonal coin blocks to be non-zero, and also for them \(A = B = C = D = c = d = 0\), thus \(a = b\), This means that all diagonal coin blocks are proportional to identity. Due to shift conditions of (21), all \(a_{s,t}\) are equal, thus a single attractor is revealed to be proportional to identity, i.e. we found the trivial attractor as the only additional non-\(p\)-attractor. Similarly, one can show that for the other possible attractor eigenvalues there are no additional non-\(p\)-attractors. In summary, all attractors can be constructed employing (13) and adding the trivial attractor proportional to identity. Thus, the solution presented here is complete.

Let us have a look at the influence of boundary conditions on the available eigenvectors. It should be noted that \(|\phi_1\rangle\) and \(|\phi_2\rangle\) are not available for periodic boundary condition in the \(s\)-direction with odd \(M\). In a similar way, \(|\phi_4\rangle\) is not a common eigenstate for a periodic boundary condition in the \(t\)-direction with odd \(N\). From the \(\alpha = \{i, -i, 1, 1\}\) pure state eigenvalues, the possible attractor space eigenvalues are \(\lambda = \{1, -1, i, -i\}\). For the \(\lambda = 1\) eigenvalue, for all boundary conditions

\[
X_0 = I, \quad (38)
\]

\[
X_1(t_1, t_2) = |\phi_3(t_1)\rangle\langle\phi_3(t_2)| \quad (39)
\]

are valid attractors, spanning a \(1 + N^2\) dimensional space. For even \(M\) on periodic boundary conditions in the \(s\) direction, or open boundaries in the \(s\) direction, additional attractors

\[
X_2 = |\phi_1\rangle\langle\phi_1|, \quad (40)
\]

\[
X_3 = |\phi_2\rangle\langle\phi_2| \quad (41)
\]

form a 2D space. When in the \(t\)-direction the system is open or periodic with even \(N\) the additional

\[
X_4(s_1, s_2) = |\phi_4(s_1)\rangle\langle\phi_4(s_2)|, \quad (42)
\]

\[
X_5(s_1, t_2) = |\phi_4(s_1)\rangle\langle\phi_3(t_2)|, \quad (43)
\]

\[
X_6(t_1, s_2) = |\phi_3(t_1)\rangle\langle\phi_4(s_2)| \quad (44)
\]

attractors become available, forming an \(M^2 + 2MN\) dimensional space.

For the superoperator eigenvalue \(\lambda = i\), for even \(M\) in the \(s\)-direction or open boundaries in the \(s\)-direction

\[
Y_1(t_2) = |\phi_1\rangle\langle\phi_3(t_2)|, \quad (45)
\]

\[
Y_2(t_1) = |\phi_3(t_1)\rangle\langle\phi_2| \quad (46)
\]
attractors are available spanning a $2N$ dimensional space. The following attractors appear in addition if either we have an open boundary condition in the direction $t$ or we have a periodic boundary condition for $t$ with even $N$

\begin{align}
Y_3(s_2) &= |\phi_1\rangle\langle\phi_4(s_2)|, \quad (47) \\
Y_4(s_1) &= |\phi_4(s_1)\rangle\langle\phi_2|.
\end{align}

In that case, the dimension of the attractor space is increased by $2M$. The form of definition (4) implies that the attractors corresponding to the conjugate $\lambda = -i$ eigenvalue are simply the Hermitian conjugate of the attractor space matrices corresponding to $\lambda = i$.

The last possible superoperator eigenvalue is $\lambda = -1$, with the attractors

\begin{align}
Z_1 &= |\phi_1\rangle\langle\phi_2|, \quad (49) \\
Z_2 &= |\phi_2\rangle\langle\phi_1|.
\end{align}

available when direction $s$ is open or periodic with even $M$, adding a 2D space to the attractor space. Altogether, the maximal number of attractors for carpet (open boundaries) or for even $\times$ even torus (periodic boundaries) are $(M+N+2)^2+1$.

Let us now analyse the conclusions one can draw from the explicit form of the eigenvectors (33)–(36) for the asymptotic behaviour of the walks. The common eigenvectors $|\phi_1\rangle$ and $|\phi_2\rangle$ in (33) and (34) are uniform in position. When the asymptotic state can be expanded by these, then the asymptotics will be uniform in position. In contrast, the other two families of eigenvectors $|\phi_3(t)\rangle$ and $|\phi_4(s)\rangle$ in (35) and (36) are spatially non-uniform. The asymptotic states built by them will have ridge-like stripes. Therefore, the boundary conditions for which $|\phi_3(t)\rangle$ or $|\phi_4(s)\rangle$ are allowed can lead to a non-uniform asymptotic position distribution. While dynamical percolation means spatially a homogeneous source of decoherence, it may result in a spatially inhomogeneous asymptotic distribution.

Further analysing the asymptotically inhomogeneous solutions, we find that percolation can cause the breaking of the directional symmetry in the following sense. Taking a certain initial state, the unpercolated 2D Hadamard walk may show a directional symmetry for the position distribution: if both the graph and the initial state are rotated by $90^\circ$ the resulting position distribution will also be a rotated version of the original position distribution at all times. In a numerical example, we demonstrate that introducing percolation in this system can break the above directional symmetry.

Let us consider the example of a torus with size even $\times$ odd. A QW with percolation on such a torus will have an attractor space with dimension $(N+2)^2+1$. In contrast, if we rotate the graph (odd $\times$ even torus) while keeping the coin operator the same, we find an attractor space with dimension $(N+M)^2+1$. This change in the dimension of the attractor space clearly demonstrates the symmetry breaking. Furthermore, by examining the corresponding eigenvalues we find that in the second case (odd $\times$ even torus) only the $\lambda = 1$ eigenvalue occurs, leading to stationary asymptotic states. Whereas in the first case (even $\times$ odd torus), also the $\lambda = \{-1, \pm i\}$ eigenvalues will be included in the solution, possibly allowing for oscillations in the asymptotic state of the system. In figure 2, we plot the asymptotic marginal position distributions for the two cases. Numerical simulations of a Hadamard walk on tori without percolation show no difference between even $\times$ odd and rotated odd $\times$ even systems within numerical precision. Thus, we conclude that the directional symmetry breaking is induced by percolation. This new phenomenon is one of the main results of this paper.
Figure 2. Asymptotic position probability distributions \( P \) of the 2D Hadamard walk on the torus graphs, starting from the initial state: \( |7, 7⟩_P \otimes \frac{1}{\sqrt{2}}(|L⟩_C + |D⟩_C) \). The left plot corresponds to the \( 15 \times 16 \) percolation torus and carpet and the right plot corresponds to the \( 16 \times 15 \) percolation torus. Due to the rotation of the underlying graph (and the initial state), the position distribution changes considerably. However, in the case of carpets, the position distribution rotates and remains otherwise unchanged. The position distribution of the \( 15 \times 16 \) percolation torus and carpet (rotated carpet) are identical (rotated). For the unpercolated (unitary) case, the symmetry breaking is not observable within numerical precision.

5.2. The Grover walk: trapping at the origin

In this part, we analyse the properties of the percolated walk governed by the Grover diffusion operator \( G_{i,j} = 2/d - \delta_{i,j} \), that is

\[
G = \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix}
\]  

(51)

for 2D QWs. The undisturbed Grover walk is characterized by the effect of trapping (localization), i.e. the probability of finding a particle in its initial position does not decay to zero during the time evolution. In the 2D Grover walk, there exists a single well defined localized initial coin state \( |\psi^{\text{spread}}⟩ = \frac{1}{2}(1, -1, -1, 1)^T \) for which the trapping effect is avoided, i.e. the wave function spreads freely.

In the following, we determine the attractor space of the Grover walk. We omit the details of the analytical proof as it is fairly analogous to the case of the Hadamard walk and hence can be easily reconstructed. Similarly to the previous Hadamard case, we solve (30) to gain the spectrum of the common eigenstates. After a lengthy, but straightforward, calculation we find the explicit form of the common eigenstates

\[
|\phi⟩ = \frac{1}{\sqrt{4MN}} \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} |s, t⟩_P \otimes |v_1⟩_C,
\]  

(52)
\[
|\phi_2(s, t)\rangle = \frac{1}{\sqrt{8}}\{ |s, t\rangle_P \otimes |v_2\rangle_C + |s, t \oplus 1\rangle_P \otimes (|v_2\rangle_C + |v_3\rangle_C) + |s \oplus 1, t\rangle_P \otimes (|v_2\rangle_C + |v_4\rangle_C) \\
+ |s \oplus 1, t \oplus 1\rangle_P \otimes (|v_2\rangle_C + |v_3\rangle_C + |v_4\rangle_C) \},
\]

(53)

\[
|\phi_3(s)\rangle = \sum_{t=0}^{N-1} \frac{(-1)^t}{\sqrt{2N}} |s, t\rangle_P \otimes |v_3\rangle_C,
\]

(54)

\[
|\phi_4(t)\rangle = \sum_{s=0}^{M-1} \frac{(-1)^s}{\sqrt{2N}} |s, t\rangle_P \otimes |v_4\rangle_C,
\]

(55)

where \( |v_1\rangle_C = (1, -1, -1, 1)^T, \quad |v_2\rangle_C = (1, 1, 0, 0)^T, \quad |v_3\rangle_C = (0, -1, 1, 0)^T \) and \( |v_4\rangle_C = (-1, 0, 0, 1)^T \). By \( \oplus \) we denote an addition that takes the boundary conditions into account. For open boundary conditions (carpet) the part leaning over the boundary of the graph should be omitted (its amplitude is zero and the corresponding superposition state is normalized accordingly), and for periodic boundary conditions, the addition \( \oplus \) corresponds to modulo operations with respect to the graph size. These common eigenstates correspond to the eigenvalues \( \alpha = \{-1, 1, 1, 1\} \), respectively.

From the general theory, we know that all \( p \)-attractors can be constructed from the above common eigenstates, employing (13). More importantly, the attractors that do not belong to the subspace of \( p \)-attractors should be found. However, it turns out that similarly to the Hadamard case, the only non-\( p \)-attractor is the trivial one, proportional to identity. Thus, the total number of attractors is \( (MN + M + N + 1)^2 + 1 \) for all carpets, and \( (MN + 1)^2 + 1 \) for tori where \( M \) or \( N \) are odd. In the latter case (54) and (55), are restricted by the periodic boundary conditions—they are not common eigenstates. However, when \( M \) and \( N \) are both even in the case of tori, a single additional state from (54) or (55) can be chosen as an additional common eigenstate, resulting in an attractor space with a total number of attractors \( (MN + 2)^2 + 1 \). Note that in all cases the attractor space is fully symmetric under rotations, thus in contrast to the 2D Hadamard walk percolation-induced symmetry breaking cannot be observed.

Analysing the structure of the eigenstates, we find another interesting effect related to trapping. The common eigenstates \( |\phi_2(s, t)\rangle \) have finite support. The importance of this observation is twofold: firstly, these states cannot be sensitive to boundary conditions, thus one can expect that they remain common eigenstates even on an infinite system. Secondly, these states demonstrate the survival of the trapping (localization) effect: even under the particularly strong decoherence effect of dynamical percolation, the localized \( |\phi_2(s, t)\rangle \) states are invariants of the time evolution, thus they survive. Consequently, an initially localized state overlapping with a \( |\phi_2(s, t)\rangle \) state preserves its trapping property. A set of such robust states may be perfect candidates for information storage, as they form a decoherence-free subspace and are separated spatially (except the neighbouring ones). This trapping effect for the percolation graph is illustrated in figure 3.

The above results motivate us to extend the definition of trapping. The eigenstates with finite support, if they are independent of the boundaries, represent trapping in a general sense for QWs. As we have seen, in particular cases this general trapping property can survive decoherence effects in an open system.
Figure 3. Position probability distribution $P$ of Grover (51) walks on the $15 \times 15$ torus. The left plot corresponds to 1000 steps of unitary time evolution (perfect graph) starting from $|7, 7\rangle_p \otimes \frac{1}{2}((|L\rangle_C + |D\rangle_C + |U\rangle_C + |R\rangle_C)$, whereas the right plot corresponds to the asymptotic distribution on the percolation case from the same initial state as the unperturbed one. It is apparent that the trapping (localization) property of the walk is preserved due to the common eigenstates (53) with finite support. We should point out, however, that the localization effect is not as pronounced—the height of the peak is not as high as in the unperturbed case.

5.3. The Fourier walk

In our last example, we consider the Fourier coin operator as an example that has also been examined in the context of 2D QWs. The Fourier coin does not have trapping initial states if started from a single position, but its Pólya number can be 1, i.e. the walker can be recurrent [45, 46].

The Fourier coin in the natural basis takes the form of the discrete Fourier transformation matrix

$$F = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \quad (56)$$

On the regular 2D lattice, the undisturbed walk driven by the Fourier coin produces a position probability distribution that is dominated by a slowly propagating peak and a ring-like area. For a family of localized initial states $|\psi^{\text{ring}}\rangle_C = (a, b, a, -b)^T$ with $|a|^2 + |b|^2 = \frac{1}{2}$, the central peak disappears and one can observe a propagating ring-like distribution.

Let us determine the attractor space of the Fourier walk on a 2D percolation lattice. The spectrum of common eigenstates is $\{\alpha_n = \exp(i\mu_n) \mid \mu_n = \frac{\pi}{8}(3 + 4n)\}$, and the common eigenstates themselves take the form of

$$|\phi_n\rangle = \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x_s^\mu y_t^n |s, t\rangle_p \otimes |v_n\rangle_C, \quad (57)$$

where

$$|v_n\rangle_C = (v_{n,1}, v_{n,2}, v_{n,3}, v_{n,4})^T = \frac{1}{\mathcal{N}} ((\alpha_n + 1)^2, \alpha_n^2 + i, \alpha_n^2 - 1, 2\alpha_n^3 + \alpha_n^2 - 1)^T \quad (58)$$
with $N^2 = 16 + 12 \cos(\mu_n) - 4 \sin(3 \mu_n) \), $x_n = v_{n,2}/v_{n,3}$ and $y_n = v_{n,1}/v_{n,4}$. Since the derivation consists of very similar steps as we detailed for the 2D Hadamard case, we do not repeat them here.

From the structure of the common eigenstates, it is apparent that these states are not available on periodic boundary conditions ($M \times N$ tori) except when both $M$ and $N$ are a multiple of 16. The $p$-attractors can be readily constructed by using (13). The possible asymptotic superoperator eigenvalues are $\lambda \in \{1, -1, i, -i\}$. Now we have to determine the number of non-$p$-attractors. Similarly to the previous cases, a straightforward analysis concludes that the only non-$p$-attractor available is the trivial one, which is proportional to identity. Thus, the attractor space has the dimension of $16 + 1 = 17$ for carpets, and for $M \times N$ tori only the trivial identity is a valid attractor, except when both $M$ and $N$ are a multiple of 16. In the latter case, the attractor structure is concurrent with the carpet.

A careful second look at the states of (57) reveals that they all correspond to a flat distribution in position. Thus, this coin induces a uniform asymptotic distribution and the asymptotic limit cycles, determined by the superoperator eigenvalues $\lambda \in \{-1, i, -i\}$ are only observable in the coin degree of freedom. Nevertheless, the appearance of such coins in the 2D cases by their connection with the similar behaviour observed for 1D walks [38] is interesting.

All the previous case studies display quite different families of asymptotic behaviours. The 2D Hadamard walk exhibits a percolation-induced symmetry breaking effect, which is an effect so far unobserved for Hadamard walks. The system also shows ridge-like asymptotic states in position. This phenomenon is quite a departure from the always uniform class of 1D QWs [38] including the Hadamard coin. The Grover walk keeps its trapping (localization) property despite the strong decoherence caused by the dynamical percolation. The last showcased walk (Fourier coin) shows non-trivial behaviour only in the internal degree of freedom. The position distribution is uniform, similar to the above class of 1D walks. It is important to note that all of these effects are controlled only by the selection of the coin operator.

6. Conclusions

QWs can be used to model a variety of processes. The ideal walk is described by a unitary evolution. Disturbances of this ideal description lead to decoherence with a number of unexpected new effects. In the present paper we studied a special case of decoherence caused by dynamically broken links of the underlying graph defining the position space of the walk. This kind of perturbation is referred to as dynamical percolation.

Using the method of random unitary operations, we solved the asymptotic dynamics of percolated 2D walks. We have extended the method sketched in our previous paper [38] and have shown that explicit analytic forms can be obtained. This is facilitated by the fact that a considerable part of the attractor space elements can be determined using the pure eigenstate ansatz. Here we emphasize that this ansatz is a universal tool applicable for general RUO maps to determine a decoherence-free subspace, however, it does not necessarily provide us with the complete solution for the attractors’ space.

Based on the presented analytic tools, we derived a number of results which are characteristic for percolated QWs in general, and we discussed some of the results that are dependent on the topology (boundaries) of the underlying graph. Using the general formalism, we have proven that for all percolated QWs, the asymptotic position distribution is always stationary in time. For special choices of the boundary conditions, we showed that
percolation-induced symmetry breaking can be observed for a walk driven by the 2D Hadamard coin. Our extensive study also revealed position non-uniformity in the asymptotics, an effect that is not possible for 1D Hadamard walks on percolation graphs. For the 2D Grover walk, we observed trapping caused by the existence of a family of robust eigenstates with finite support, which form a decoherence-free subspace. Based on this observation, one can extended the concept of trapping (localization) to finite or open systems. All the asymptotic properties of the studied systems can be controlled by the careful selection of coin, thus the results of the present paper motivate further studies on the classification of coin operators.

These newly discovered effects may be demonstrated using present state experiments. The considerable degree of control on the walk is within the reach of optical feedback loop experiments implementing QWs [25]. They allow dynamic control over the QW topology, which is the basic requirement for the observation of the just-described phenomena.

Acknowledgments

We acknowledge support by MSM 6840770039, GACR 13-33906S, RVO 68407700, the Hungarian Scientific Research Fund (OTKA) under contract numbers K83858, NN109651, the Hungarian Academy of Sciences (Lendület Program, LP2011-016). BK acknowledges support by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 ‘National Excellence Program’.

References

[1] Aharonov Y, Davidovich L and Zagury N 1993 Quantum random walks Phys. Rev. A 48 1687–90
[2] Kempe J 2003 Quantum random walks: an introductory overview Contemp. Phys. 44 307–27
[3] Venegas-Andraca S E 2012 Quantum walks: a comprehensive review Quantum Inform. Process. 11 1015–106
[4] Konno N 2008 Quantum walks Quantum Potential Theory ed U Franz and M Schürmann (Berlin: Springer) pp 309–452
[5] Shenvi N, Kempe J and Whaley K B 2003 Quantum random-walk search algorithm Phys. Rev. A 67 052307
[6] Childs A M, Cleve R, Deotto E, Farhi E, Gutmann S and Spielman D A 2003 Exponential algorithmic speedup by a quantum walk STOC ’03: Proc. 35th Annual Symp. on Theory of ACM Computing (New York: ACM) pp 59–68
[7] Ambainis A 2007 Quantum walk algorithm for element distinctness SIAM J. Comput. 37 210–39
[8] Qiang X, Yang X, Wu J and Zhu X 2012 An enhanced classical approach to graph isomorphism using continuous-time quantum walk J. Phys. A: Math. Theor. 45 045305
[9] Paparo G D and Martin-Delgado M A 2012 Google in a quantum network Nature Sci. Rep. 2 444
[10] Sánchez-Burillo E, Duch J, Gómez-Gardenes J and Zueco D 2012 Quantum navigation and ranking in complex networks Nature Sci. Rep. 2 605
[11] Childs A M 2009 Universal computation by quantum walk Phys. Rev. Lett. 102 180501
[12] Lovett N B, Cooper S, Everitt M, Trevers M and Kendon V 2010 Universal quantum computation using the discrete-time quantum walk Phys. Rev. A 81 042330
[13] Karski M, Förster L, Choi J-M, Steffen A, Alt W, Meschede D and Widera A 2009 Quantum walk in position space with single optically trapped atoms Science 325 174–7
[14] Genske M, Alt W, Steffen A, Werner A H, Werner R F, Meschede D and Alberti A 2013 Electric quantum walks with individual atoms Phys. Rev. Lett. 110 190601
[15] Schmitz H, Matjeschik R, Schneider Ch, Glueckert J, Enderlein M, Huber T and Schaeetz T 2009 Quantum walk of a trapped ion in phase space Phys. Rev. Lett. 103 090504
[16] Zähringer F, Kirchmair G, Gerritsma R, Solano E, Blatt R and Roos C F 2010 Realization of a quantum walk with one and two trapped ions Phys. Rev. Lett. 104 100503
[17] Peruzzo A et al 2010 Quantum walks of correlated photons Science 329 1500–3
[18] Lahini Y, Verbin M, Huber S D, Bromberg Y, Pugatch R and Silberberg Y 2012 Quantum walk of two interacting bosons Phys. Rev. A 86 011603
[19] Crespi A, Osellame R, Ramponi R, Giovannetti V, Fazio R, Sansoni L, De Nicola F, Sciarrino F and Mataloni P 2013 Anderson localization of entangled photons in an integrated quantum walk Nature Photon 7 322–8
[20] Meinecke J D A, Poulios K, Politi A, Matthews J C F, Peruzzo A, Ismail N, Wörhoff K, O’Brien J L and Thompson M G 2013 Coherent time evolution and boundary conditions of two-photon quantum walks in waveguide arrays Phys. Rev. A 88 012308
[21] Broome M A, Fedrizzi A, Lanyon B P, Kassal I, Aspuru-Guzik A and White A G 2010 Discrete single-photon quantum walks with tunable decoherence Phys. Rev. Lett. 104 153602
[22] Schreiber A, Cassemiro K N, Potoček V, Gábris A, Mosley P J, Andersson E, Jex I and Silberhorn Ch 2010 Photons walking the line: a quantum walk with adjustable coin operations Phys. Rev. Lett. 104 050502
[23] Schreiber A, Cassemiro K N, Potoček V, Gábris A, Jex I and Silberhorn Ch 2011 Decoherence and disorder in quantum walks: from ballistic spread to localization Phys. Rev. Lett. 106 180403
[24] Svozilík J, León-Montiel R de J and Torres J P 2012 Implementation of a spatial two-dimensional quantum random walk with tunable decoherence Phys. Rev. A 86 052327
[25] Schreiber A, Gábris A, Rohde P P, Laiho K, Štefaňák M, Potoček V, Hamilton C, Jex I and Silberhorn Ch 2012 A 2D quantum walk simulation of two-particle dynamics Science 336 55–8
[26] Kesten H 1982 Percolation Theory for Mathematicians (Boston, MA: Birkhäuser)
[41] Steif J E 2009 A survey of dynamical percolation *Fractal Geometry and Stochastics IV* ed C Bandt, M Zähle and P Mörters (Berlin: Springer) pp 145–74

[42] Mackay T D, Bartlett S D, Stephenson L T and Sanders B C 2002 Quantum walks in higher dimensions *J. Phys. A: Math. Gen.* **35** 2745

[43] Tregenna B, Flanagan W, Maile R and Kendon V 2003 Controlling discrete quantum walks: coins and initial states *New J. Phys.* **5** 83

[44] Inui N, Konishi Y and Konno N 2004 Localization of two-dimensional quantum walks *Phys. Rev. A* **69** 052323

[45] Štefaňák M, Jex I and Kiss T 2008 Recurrence and Pólya number of quantum walks *Phys. Rev. Lett.* **100** 020501

[46] Štefaňák M, Kiss T and Jex I 2008 Recurrence properties of unbiased coined quantum walks on infinite $d$-dimensional lattices *Phys. Rev. A* **78** 032306

[47] Novotný J, Alber G and Jex I 2009 Random unitary dynamics of quantum networks *J. Phys. A: Math. Theor.* **42** 282003

[48] Novotný J, Alber G and Jex I 2010 Asymptotic evolution of random unitary operations *Cent. Eur. J. Phys.* **8** 1001–14

[49] Novotný J, Alber G and Jex I 2012 Asymptotic properties of quantum Markov chains *J. Phys. A: Math. Theor.* **45** 485301