A direct blowing-up and rescaling argument on the fractional Laplacian equation

Wenxiong Chen∗ Congming Li† Yan Li

June 5, 2015

Abstract

In this paper, we develop a direct blowing-up and rescaling argument for a nonlinear equation involving the fractional Laplacian operator. Instead of using the conventional extension method introduced by Caffarelli and Silvestre, we work directly on the nonlocal operator. Using the integral defining the nonlocal elliptic operator, by an elementary approach, we carry on a blowing-up and rescaling argument directly on nonlocal equations and thus obtain a priori estimates on the positive solutions for a semi-linear equation involving the fractional Laplacian.

We believe that the ideas introduced here can be applied to problems involving more general nonlocal operators.

Key words The fractional Laplacian, blowing-up, rescaling, a priori estimates.

∗Partially supported by the Simons Foundation Collaboration Grant for Mathematicians 245486.
 †Corresponding author, partially supported by NSF DMS-1405175 and NSFC-11271166.
1 Introduction

The fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudo-differential operator, assuming the form

$$(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz,$$

(1)

$$= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz,$$

(2)

where $\alpha$ is any real number between 0 and 2. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^n$. In this space, it can also be equivalently defined in terms of the Fourier transform

$$(-\hat{\Delta})^{\alpha/2} \hat{u}(\xi) = |\xi|^{\alpha} \hat{u}(\xi),$$

where $\hat{u}$ is the Fourier transform of $u$. One can extend this operator to a wider space of functions.

Let

$$L_\alpha = \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \}.$$

Then it is easy to verify that for $u \in L_\alpha \cap C_{loc}^{1,1}$, the integral on the right-hand side of (2) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

The non-locality of the fractional Laplacian makes it difficult to investigate. To circumvent this difficulty, Caffarelli and Silvestre [CS] introduced the extension method that reduced a nonlocal problem into a local one in higher dimensions. For a function $u : \mathbb{R}^n \to \mathbb{R}$, consider the extension $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases}
\text{div}(y^{1-\alpha} \nabla U) = 0, & (x, y) \in \mathbb{R}^n \times [0, \infty), \\
U(x, 0) = u(x).
\end{cases}$$

Then

$$(-\Delta)^{\alpha/2} u(x) = -C_{n,\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \ x \in \mathbb{R}^n.$$

This extension method has been applied successfully to problems involving the fractional Laplacian, and a series of fruitful results have been obtained (see [BCPS] [BCPS1] [CZ] and the references therein).
The main purpose of this paper is to introduce a direct approach to study these nonlocal operators. We will carry on the blowing up and rescaling arguments on the nonlocal problem directly to obtain the a priori estimates on the solutions and to obtain the following.

**Theorem 1** For $0 < \alpha < 2$ and a bounded domain $\Omega \subset \mathbb{R}^n$, consider

$$\begin{cases}
(\nabla)^{\alpha/2}u(x) = u^p(x), & x \in \Omega, \\
u(x) \equiv 0, & x \notin \Omega.
\end{cases} \tag{3}$$

Assume that $u \in L_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\Omega)$ and $1 < p < \frac{n+\alpha}{n-\alpha}$, then there exists some constant $C$, independent of $u$, such that

$$\| u \|_{L^\infty(\Omega)} \leq C. \tag{4}$$

It is well-known that a priori estimates play important roles in establishing the existence of solutions. Once there is such an a priori estimate, then one can use various methods, such as continuation or topological degrees, to derive the existence of solutions.

**2 Proof of Theorem 1**

**Proof.** Suppose (4) does not hold, then there exists a sequence of solutions $\{u_k\}$ to (3) and a sequence of points $\{x^k\} \subset \Omega$ such that

$$u_k(x^k) = \max_{\Omega_k} u_k := m_k \to \infty. \tag{5}$$

Let

$$\lambda_k = m_k^{-\frac{\alpha}{n-\alpha}} \text{ and } v_k(x) = \frac{1}{m_k} u_k(\lambda_k x + x^k), \tag{6}$$

then we have

$$(-\nabla)^{\alpha/2}v_k(x) = v_k^p(x), \quad x \in \Omega_k := \{x \in \mathbb{R}^n \mid x = \frac{y-x^k}{\lambda_k}, \ y \in \Omega\}. \tag{7}$$

Let $d_k = \text{dist}(x^k, \partial \Omega)$. We will carry out the proof using the contradiction argument while exhausting all three possibilities.
Case i. \( \lim_{k \to \infty} \frac{d_k}{\lambda_k} = \infty \).

It’s not difficult to see that \( \Omega_k \to \mathbb{R}^n \) as \( k \to \infty \).

We’ll prove that there exists a function \( v \) such that as \( k \to \infty \),

\[
v_k(x) \to v(x) \text{ and } (-\Delta)^{\alpha/2} v_k(x) \to (-\Delta)^{\alpha/2} v(x),
\]

thus

\[
(-\Delta)^{\alpha/2} v(x) = v^p(x), \quad x \in \mathbb{R}^n.
\]

It has been proved that (9) has no positive solution (see ). Meanwhile, (10) indicates that

\[
v(0) = \lim_{k \to \infty} v_k(0) = 1.
\]

This is a contradiction. Hence \( u \) must be uniformly bounded in \( \Omega \).

To obtain (8), we need the following propositions to boost the regularity of \( v_k \).

**Proposition 2.1** [Si] Let \( u \in C^{k,\sigma} \) and suppose that \( k + \sigma - \alpha \) is not an integer. Then \( (-\Delta)^{\alpha/2} u \in C^{l,\gamma} \) where \( l \) is the integer part of \( k + \sigma - \alpha \) and \( \gamma = k + \sigma - \alpha - l \).

**Proposition 2.2** [Si] Let \( w = (-\Delta)^{\alpha/2} u \). Assume that \( w \in L^\infty(\mathbb{R}^n) \) and \( u \in L^\infty(\mathbb{R}^n) \) for \( \alpha > 0 \).

- If \( \alpha \leq 1 \), then \( u \in C^{0,\sigma}(\mathbb{R}^n) \) for any \( \sigma < \alpha \). Moreover,

\[
\| u \|_{C^{0,\sigma}(\mathbb{R}^n)} \leq C(\| u \|_{L^\infty} + \| w \|_{L^\infty})
\]

for a constant \( C \) depending only on \( n, \alpha \) and \( \sigma \).

- If \( \alpha > 1 \), then \( u \in C^{1,\sigma}(\mathbb{R}^n) \) for any \( \sigma < \alpha - 1 \). Moreover,

\[
\| u \|_{C^{1,\sigma}(\mathbb{R}^n)} \leq C(\| u \|_{L^\infty} + \| w \|_{L^\infty})
\]

for a constant \( C \) depending only on \( n, \alpha \) and \( \sigma \).

**Proposition 2.3** [Si] Let \( w = (-\Delta)^{\alpha/2} u \). Assume that \( w \in C^{0,\sigma}(\mathbb{R}^n) \) and \( u \in L^\infty \) for \( \sigma \in (0,1] \) and \( \alpha > 0 \).
• If $\sigma + \alpha \leq 1$, then $u \in C^{0,\sigma + \alpha}(\mathbb{R}^n)$. Moreover,
  $$\| u \|_{C^{0,\sigma + \alpha}(\mathbb{R}^n)} \leq C(\| u \|_{L^\infty} + \| w \|_{C^{0,\sigma}})$$
  for a constant $C$ depending only on $n, \alpha$ and $\sigma$.

• If $\sigma + \alpha > 1$, then $u \in C^{1,\sigma + \alpha - 1}(\mathbb{R}^n)$. Moreover,
  $$\| u \|_{C^{1,\sigma + \alpha - 1}(\mathbb{R}^n)} \leq C(\| u \|_{L^\infty} + \| w \|_{C^{0,\sigma}})$$
  for a constant $C$ depending only on $n, \alpha$ and $\sigma$.

Notice that $|v_k| \leq 1$, applying the propositions above to (7) gives:

1. if $\alpha \leq 1$, then for all $0 < \sigma < \alpha$,
   
   I. $\| v_k \|_{C^{0,\alpha + \sigma}(\mathbb{R}^n)} < C_{n,\alpha,\sigma}$, when $\sigma + \alpha \leq 1$.
   
   II. $\| v_k \|_{C^{1,\alpha + \sigma - 1}(\mathbb{R}^n)} < C_{n,\alpha,\sigma}$, when $\sigma + \alpha > 1$.

2. if $\alpha > 1$, then for all $0 < \sigma < \alpha - 1$,
   
   III. $\| v_k \|_{C^{1,\gamma}(\mathbb{R}^n)} < C_{n,\alpha,\sigma}$, where $l$ is the integer part of $1 + \alpha + \sigma$ and $
   \gamma = 1 + \alpha + \sigma - l$.

Part I and part II can be derived directly from Proposition 2.2 and Proposition 2.3. To obtain part III, we first apply Proposition 2.2 and it gives

$$\| v_k \|_{C^{1,\alpha}(\mathbb{R}^n)} < C_{n,\alpha,\sigma}.$$ 

Then we show that $v_k$ has the regularity claimed above in a neighborhood of the origin. Because such reasoning can be done in a neighborhood of any point in $\mathbb{R}^n$, we conclude that $v_k$ has the desired uniform regularity.

Let $\varphi$ be a smooth cutoff function such that $\varphi(x) \in [0, 1]$ in $\mathbb{R}^n$, $\text{supp} \varphi \subset B_2$ and $\varphi(x) \equiv 1$ in $B_1$. Let $(-\Delta)^{\alpha/2}v_k = g_k$. Define

$$v_k^0(x) := c_{n, -\alpha} \int_{\mathbb{R}^n} \frac{\varphi(y)g_k(y)}{|x - y|^{n-\alpha}} dy = (-\Delta)^{-\alpha/2}(\varphi g_k)(x).$$

Then

$$(-\Delta)^{\alpha/2}v_k^0(x) = g_k(x) = (-\Delta)^{\alpha/2}v_k(x), \ x \in B_1.$$
Hence $v_0^k - v_k$ is a smooth function in $B_1$. In addition, its $C^{1,\sigma}$-norm can be controlled by the $L^\infty$-norms of $v_0^k$ and $v_k$.

Given that

$$v_0^k = (-\Delta)^{-\alpha/2}(\varphi g_k) = (-\Delta)^{1-\alpha/2} \circ (-\Delta)^{-1}(\varphi g_k),$$

it then follows from the $C^{2,\sigma}$-estimates for the poisson equation (see [GT]) that $(-\Delta)^{-1}(\varphi g_k) \in C^{3,\sigma}$. Also, its norm is solely dependent of $\| v_k \|_{C^{1,\sigma}}$. Thus applying Proposition 2.1 gives $v_0^k \in C^{\gamma,\sigma}$ and $l > \alpha$. This proves III.

Due to the equi-continuity of $\{v_k\}$ in $\mathbb{R}^n$, we are able to employ the Arzelà-Ascoli theorem and claim the existence of a converging sequence $\{v_{1m}\}$ in $B_1(0)$. For sure, one can find a subsequence of $\{v_{1m}\}$, denoted by $\{v_{2m}\}$, that converges in $B_2(0)$. Then another subsequence of $\{v_{2m}\}$, denoted as $\{v_{3m}\}$, converges in $B_3(0)$. By induction, we get a chain of sequences

$$\{v_{jm}\} \supset \{v_{2m}\} \supset \{v_{3m}\} \cdots$$

such that $\{v_{jm}\}$ converges in $B_j(0)$. Now form the diagonal sequence $\{v_{jj}\}$, whose $j$th term is the $j$th term in the $j$th subsequence $\{v_{jm}\}$. Such $\{v_{jj}\}$ converges at all points in any $B_R(0)$. Thus we have constructed a sequence of solutions that converges point-wise in $\mathbb{R}^n$.

To show the other part of (8), we turn to the definition of the fractional Laplacian.

$$(-\Delta)^{\alpha/2} v_k(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\alpha}} dy$$

$$= \frac{1}{2} \left[ \int_{\mathbb{R}^n \setminus B_1(0)} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\alpha}} dy ight]$$

$$+ \int_{B_1(0)} \frac{2v_k(x) - v_k(x+y) - v_k(x-y)}{|y|^{n+\alpha}} dy$$

$$= \frac{1}{2} (I_1 + I_2).$$

Henceforth we use $C$ to denote positive constants whose values may vary from line to line. Doing Taylor expansion near $x$ to $v_k$ in $I_2$, the equi-continuity
of $v_k$ gives

$$|I_2| \leq \int_{B_1(0)} \frac{|2v_k(x) - v_k(x + y) - v_k(x - y)|}{|y|^{n+\alpha}} dy$$

$$\leq \int_{B_1(0)} \frac{C|y|^\alpha}{|y|^{n+\alpha}} dy$$

$$= \int_{B_1(0)} \frac{C}{|y|^{\alpha - \epsilon}} dy$$

$$< \infty.$$

Meanwhile, since $|v_k| \leq 1$,

$$|I_1| \leq \int_{\mathbb{R}^n \setminus B_1(0)} \frac{|2v_k(x) - v_k(x + y) - v_k(x - y)|}{|y|^{n+\alpha}} dy$$

$$\leq \int_{\mathbb{R}^n \setminus B_1(0)} \frac{4C}{1 + |y|^{n+\alpha}} dy$$

$$< \infty.$$

Therefore,

$$\int_{\mathbb{R}^n} \frac{|2v_k(x) - v_k(x + y) - v_k(x - y)|}{|y|^{n+\alpha}} dy$$

$$\leq \int_{\mathbb{R}^n} \frac{C}{1 + |y|^{\alpha - \epsilon}} \left(1 + \frac{1}{|y|^{\alpha - \epsilon}}\right) dy$$

$$< \infty.$$

It then follows from the Lebesgue’s dominated convergence theorem that

$$\lim_{k \to \infty} (-\Delta)^{\alpha/2} v_k(x)$$

$$= \lim_{k \to \infty} \frac{1}{2} \int_{\mathbb{R}^n} \frac{2v_k(x) - v_k(x + y) - v_k(x - y)}{|y|^{n+\alpha}} dy$$

$$= \frac{1}{2} \int_{\mathbb{R}^n} \lim_{k \to \infty} \frac{2v_k(x) - v_k(x + y) - v_k(x - y)}{|y|^{n+\alpha}} dy$$

$$= (-\Delta)^{\alpha/2} v(x).$$

This proves (8) and (9).

**Case ii.** $\lim_{k \to \infty} \frac{d_k}{\lambda_k} = C > 0$. 

7
In this case,

$$\Omega_k \rightarrow R^n_{+C} := \{x_n \geq -C \mid x \in R^n\} \text{ as } k \rightarrow \infty.$$ 

Similar to Case i, here we’re able to establish the existence of a function $v$ such that as $k \rightarrow \infty$,

$$v_k(x) \rightarrow v(x) \text{ and } (-\Delta)^{\alpha/2}v_k(x) \rightarrow (-\Delta)^{\alpha/2}v(x), \quad (10)$$

and thus

$$(-\Delta)^{\alpha/2}v(x) = v^p(x), \quad x \in R^n_{+C}. \quad (11)$$

It’s known that (11) has no positive solution (see ). Meanwhile, it follows from (6) that

$$v(0) = \lim_{k \rightarrow \infty} v_k(0) = 1.$$ 

This is a contradiction. Hence (4) is true. Next we prove (10) and (11).

Let $D_1 = B_1(0) \cap \{x_n \geq 0\}$. Through an argument similar to that in Case i, we have the equicontinuity of $v_k$ and

$$|v_k(x) - v_k(y)| \leq C_{D_1}|x - y|^\alpha + \epsilon, \quad x, y \in D_1, \quad 0 < \epsilon < 1.$$ 

Together with $|v_k(x)| \leq 1$, we are able to show the existence of a converging subsequence $\{v_{1k}\}$ of $\{v_k\}$ such that for each given $x \in D_1$,

$$v_{1k}(x) \rightarrow v(x) \text{ and } (-\Delta)^{\alpha/2}v_{1k}(x) \rightarrow (-\Delta)^{\alpha/2}v(x).$$

Let $D_2 = B_2(0) \cap \{x_n \geq -\frac{C}{2}\}$. We can still have

$$|v_k(x) - v_k(y)| \leq C_{D_2}|x - y|^\alpha + \epsilon, \quad x, y \in D_2.$$ 

This again gives a converging subsequence $\{v_{2k}\}$ of $\{v_{1k}\}$ such that for each given $x \in D_2$,

$$v_{2k}(x) \rightarrow v(x) \text{ and } (-\Delta)^{\alpha/2}v_{2k}(x) \rightarrow (-\Delta)^{\alpha/2}v(x).$$

Let $D_3 = B_3(0) \cap \{x_n \geq -\frac{2C}{3}\}$. There exists a subsequence $\{v_{3k}\}$ of $\{v_{2k}\}$ such that for each given $x \in D_3$,

$$v_{3k}(x) \rightarrow v(x) \text{ and } (-\Delta)^{\alpha/2}v_{3k}(x) \rightarrow (-\Delta)^{\alpha/2}v(x).$$

Repeating the process above for $m$ times and it gives a pointwise converging subsequence $\{v_{mk}\}$ in $D_m = B_m(0) \cap \{x_n \geq -\frac{(m-1)C}{m}\}$. Now we take the
diagonal sequence \( \{v_{mm}\} \) with \( v_{mm}(x) \) being the \( m \)th term in \( \{v_{mk}\} \). It’s easy to see that for a fixed \( x_0 \in \mathbb{R}^n_+ \), there exists some \( m_{x_0} \) such that for \( m > m_{x_0} + 1 \),

\[
v_{mm}(x) \rightarrow v(x) \text{ and } (-\Delta)^{\alpha/2}v_{mm}(x) \rightarrow (-\Delta)^{\alpha/2}v(x).
\]

Thus verifies (10) and (11).

**Case iii.** \[ \lim_{k \to \infty} \frac{d_k}{\lambda_k} = 0. \]

In this case, there exists a point \( x^o \in \partial \Omega \) such that

\[ x^k \rightarrow x^o, \quad k \rightarrow \infty. \]

Let \( \tilde{x}^o = \frac{x^o - x_k}{\lambda_k} \). Obviously,

\[ |\tilde{x}^o| \rightarrow 0, \quad k \rightarrow \infty. \]

Then we will show that \( v_k \) is uniformly Hölder continuous near \( \tilde{x}^o \), i.e.

\[ |v_k(x) - v_k(\tilde{x}^o)| \leq C|x - \tilde{x}^o|^\alpha/2. \] (12)

We postpone the proof of (12) for a moment. Notice that

\[ v_k(0) - v_k(\tilde{x}^o) = 1. \] (13)

On the other hand, it follows from (12) that

\[ |v_k(x) - v_k(\tilde{x}^o)| \rightarrow 0, \text{ as } x \rightarrow 0. \] (14)

This contradicts (13). Therefore **Case iii** will not happen.

To prove (12), we introduce an auxiliary function \( \varphi \) such that for all \( x \) near \( \tilde{x}^o \), it holds that

\[ |v_k(x) - v_k(\tilde{x}^o)| \leq |\varphi(x) - \varphi(\tilde{x}^o)| \leq C|x - \tilde{x}^o|^\alpha/2. \] (15)

Let

\[ \psi_1(x) = \begin{cases} C_{n,\alpha}(1 - |x - z|^2)^{\alpha/2}, & x \in B_1(z), \\ 0, & x \in \overline{B_1^c(z)} \end{cases}, \] (16)
where $z = \frac{\tilde{x}_o}{|\tilde{x}_o|}(1 + \tilde{x})$ and $B_1^c(z) = \mathbb{R}^n \setminus B_1(z)$. To simplify the calculation, without any loss of generality, we set $z$ to be the origin, thus

$$
\psi_1(x) = \begin{cases} 
C_{n,\alpha}(1 - |x|^2)^{\alpha/2}, & x \in B_1(0), \\
0, & x \in B_1^c(0).
\end{cases} \tag{17}
$$

Let

$$
\psi_2(x) = \frac{1}{|x|^{n-\alpha}} \psi_1\left(\frac{x}{|x|^2}\right)
$$

be the Kelvin transform of $\psi_1(x)$. Then

$$
(-\triangle)^{\alpha/2} \psi_2(x) = \frac{C_{n,\alpha}}{|x|^{n+\alpha}}, \quad x \in B_1^c(0).
$$

Let $\xi(x)$ be a smooth cutoff function such that $\xi(x) \in [0, 1]$ in $\mathbb{R}^n$ and $\xi(x) = 0$ in $B_1(0)$ and $\xi(x) = 1$ in $B_3^c(0)$. Then we have

$$
(-\triangle)^{\alpha/2} \xi(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{2 \xi(x) - \xi(x+y) - \xi(x-y)}{|y|^{n+\alpha}} dy \geq -C.
$$

For $k > 0$, let

$$
\varphi(x) = k\psi_2(x) + \xi(x),
$$

and $D = (B_3(0) \setminus B_1(0)) \cap \Omega_k$. For $k$ sufficiently large and a fixed $x \in D$,

$$
(-\triangle)^{\alpha/2} \varphi(x) = k(-\triangle)^{\alpha/2} \psi_2(x) + (-\triangle)^{\alpha/2} \xi(x) \geq \frac{kC_{n,\alpha}}{|x|^{n+\alpha}} - C \geq 1.
$$

Thus

$$
\begin{cases} 
(-\triangle)^{\alpha/2}(\varphi - v_k) \geq 0, & x \in D, \\
\varphi(x) - v_k(x) > 0, & x \in D^c.
\end{cases} \tag{18}
$$

Applying the maximum principle (see [Si]) to (18), it gives

$$
\varphi(x) \geq v_k(x), \quad x \in D.
$$

To show that $\varphi(x)$ is Hölder continuous near $\tilde{x}$ in $D$, it suffices to show that $\psi_2(x)$ has such Hölder continuity. Indeed,

$$
\psi_2(x) - \psi_2(\tilde{x}) = \frac{1}{|x|^{n}}(|x| - |\tilde{x}|)^{\alpha/2}(|x| + 1)^{\alpha/2} \leq C(|x| - 1)^{\alpha/2}.
$$
Hence

\[ 0 \leq v_k(x) - v_k(\tilde{x}^o) = v_k(x) \leq \varphi(x) = \varphi(x) - \varphi(\tilde{x}^o) \leq C|x - \tilde{x}^o|^\alpha/2. \] (19)

This proves (15) and completes the proof of Case iii.

References

[BCPS] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, Proc Royal Soc. of Edinburgh, 143(2013) 39-71.

[BCPS1] B. Barrios, E. Colorado, A. de Pablo, and U. Sánchez, On some critical problems for the fractional Laplacian operator, J. Diff. Equa. 252(2012) 6133-6162.

[CFY] W. Chen, Y. Fang, and R. Yang, Liouville theorems involving the fractional Laplacian on a half space, Advances in Math. in press, 2014.

[CL] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63(1991) 615-622.

[CL1] W. Chen and C. Li, Methods on Nonlinear Elliptic Equations, AIMS book series, vol. 4, 2010.

[CLO] W. Chen, C. Li, and B. Ou, Classification of solutions for an integral equation, CPAM, 59(2006) 330-343.

[CLO1] W. Chen, C. Li and B. Ou, Qualitative properties of solutions for an integral equation, Disc. Cont. Dyn. Sys. 12(2005) 347-354.

[CS] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. PDE. 32(2007) 1245–1260.

[CZ] W. Chen and J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, arXiv: 1404.1640v1, 2014.

[FC] Y. Fang and W. Chen, A Liouville type theorem for poly-harmonic Dirichlet problem in a half space, Advances in Math. 229(2012) 2835-2867.
[FL] RL. Frank and E. Lieb, Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality, Cal. Var. & PDEs, 39(2010) 85-99.

[FLe] RL. Frank and E. Lenzmann, Uniqueness and nondegeneracy of ground states for \((-\Delta)^{s}Q + Q - Q^{\frac{n+1}{n}} = 0\) in \(\mathbb{R}\), Accepted in Acta Math, preprint at arXiv:1009.4042, 2010.

[FLe1] RL. Frank and E. Lenzmann, On ground state for the \(L^{2}\)-critical boson star equation, arXiv: 0910.2721v2, 2010.

[FLS] RL. Frank, E. Lenzmann, and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, arXiv: 1302.2652v1, 2013.

[GNN] B. Gidas, W. Ni, L. Nirenberg, Symmetry and the related properties via the maximum principle, Comm. Math. Phys. 68(1979) 209-243.

[GT] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin, 2001.

[Ha] F. Hang, On the integral systems related to Hardy-Littlewood-Sobolev inequality, Math. Res. Lett. 14(2007) 373-383.

[HLZ] X. Han, G. Lu, and J. Zhu, Characterization of balls in terms of Bessel-potential integral equation, J. Diff. Equa. 252(2012) 1589-1602.

[HWY] F. Hang, X. Wang, and X. Yan, An integral equation in conformal geometry, Ann. H. Poincare Nonl. Anal. 26(2009) 1-21.

[JLX] T. Jin, Y. Y. Li, and J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, J. Eur. Math. Soc. 16(2014) 1111-1171.

[JW] S. Jarohs and Tobias Weth, Symmetry via antisymmetric maximum principles in nonlocal problems of variable order, Annali di Matematica Pura ed Applicata (1923-), doi:10.1007/s10231-014-0462-y.

[JX] T. Jin and J. Xiong, A fractional Yemabe flow and some applications, J. reine angew. Math. 696(2014) 187-223.
[L] N. S. Landkof, Foundations of modern potential theory, Springer-Verlag Berlin Heidelberg, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[Lei] Y. Lei, Asymptotic properties of positive solutions of the Hardy-Sobolev type equations, J. Diff. Equa. 254(2013) 1774-1799.

[LLM] Y. Lei, C. Li, and C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system of integral equations, Cal. Var. & PDEs, 45(2012) 43-61.

[LZ] G. Lu and J. Zhu, An overdetermined problem in Riesz-potential and fractional Laplacian, Nonlinear Analysis, 75(2012) 3036-3048.

[LZ1] G. Lu and J. Zhu, The axial symmetry and regularity of solutions to an integral equation in a half space, Pacific J. Math. 253(2011) 455-473.

[LZ2] G. Lu and J. Zhu, Symmetry and regularity of extremals of an integral equation related to the Hardy-Sobolev inequality, Cal. Var. & PDEs, 42(2011) 563-577.

[MC] L. Ma and D. Chen, A Liouville type theorem for an integral system, Comm. Pure Appl. Anal. 5(2006) 855-859.

[MZ] L. Ma and L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195(2010) 455-467.

[RS] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, Journal de Mathématiques Pures et Appliques, 3(2014) 275-302.

[Si] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60(2007) 67-112.

[ZCCY] R. Zhuo, W. Chen, X. Cui and Z. Yuan, A Liouville theorem for the fractional Laplacian, Accepted by Disc. Cont. Dyn. Sys.
Authors’ Addresses and E-mails:

Wenxiong Chen  
Department of Mathematical Sciences  
Yeshiva University  
New York, NY, 10033 USA  
wchen@yu.edu

Congming Li  
Department of Mathematics, INS and MOE-LSC  
Shanghai Jiao Tong University  
Shanghai, 200240, China, and  
Department of Applied Mathematics  
University of Colorado,  
Boulder CO USA  
cli@clorado.edu

Yan Li  
Department of Mathematical Sciences  
Yeshiva University  
New York, NY, 10033 USA  
yali3@mail.yu.edu