Closure operators on dcpos

France Dacar, Jožef Stefan Institute
France.Dacar@ijs.si
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Abstract
We examine collective properties of closure operators on posets that are at least dcpos. The first theorem sets the tone of the paper: it tells how a set of preclosure maps on a dcpo determines the least closure operator above them, and pronounces the related induction principle, and its companion, the obverse induction principle. Using this theorem we prove that the poset of closure operators on a dcpo is a complete lattice, and then provide a constructive proof of the Tarski’s theorem for dcpos. We go on to construct the joins in the complete lattice of Scott-continuous closure operators on a dcpo, and to prove that the complete lattice of nuclei on a preframe is a frame, giving some constructions in the special case of the frame of all nuclei on a frame. In the rather drawn-out proof if the Hofmann–Mislove–Johnstone theorem we show off the utility of the obverse induction, applying it in the proof of the crucial lemma. After that we shift a viewpoint and prove some results, analogous to results about dcpos, for posets in which certain special subsets have enough maximal elements; these results actually specialize to dcpos, but at the price of using the axiom of choice. We conclude by pointing out two convex geometries associated with closure operators on a dcpo.

1 Preliminaries
In this section we present notation and terminology, and state some basic facts.

For every set $X$ we denote the set of all subsets of $X$ by $\mathcal{P}X$.

If $X$ and $Y$ are sets, then $\text{Fun}(X,Y)$ denotes the set of all functions from $X$ to $Y$. We write $\text{Fun}(X,X)$ as $\text{Fun}(X)$.

If $X$ and $Y$ are sets, $A \subseteq X$, and $F \subseteq \text{Fun}(X,Y)$, then we write $F(A)$ for the set $\{f(a) \mid f \in F \text{ and } a \in A\} \subseteq B$. We write $\{f\}(A)$ as $f(A)$ and $F(\{a\})$ as $F(a)$.

Let $X$ be a set and $F$ a set of functions $X \to X$.

A fixed point, or fixpoint, of $F$ is an $x \in X$ such that $f(x) = x$ for every $f \in F$. We denote the set of all fixed points of $F$ by $\text{Fix}(F)$, and call it the fixpoint set of $F$. We write $\text{Fix}(\{f\})$ as $\text{Fix}(f)$; always $\text{Fix}(f) \subseteq f(X)$, and $\text{Fix}(f) = f(X)$ iff $f$ is an idempotent function. Note that $\text{Fix}(F) = \bigcap_{f \in F} \text{Fix}(f)$; in particular $\text{Fix}(\emptyset) = X$.

Let $A$ be a subset of $X$. The subset $A$ is said to be closed under $F$ if $F(A) \subseteq A$; as a special case, $a \in X$ is a fixed point of $F$ iff $\{a\}$ is closed under $F$. The subset $A$ is
said to be **inversely closed under** $F$ if $f^{-1}(A) \subseteq A$ for every $f \in F$. The subset $A$ is closed under $F$ iff its complement $X \setminus A$ is inversely closed under $F$; the statement $(\forall x \in X)(\forall f \in F)(x \in A \Rightarrow f(x) \in A)$ says that $A$ is closed under $F$; it is equivalent to the statement $(\forall x \in X)(\forall f \in F)(f(x) \notin A \Rightarrow x \notin A)$, which says that $X \setminus A$ is inversely closed under $F$.

Let $P$ be a poset.

If a subset of the poset $P$ has a least (greatest) element, we shall write it as $\bot(A)$ (resp. $\top(A)$). If the whole poset $P$ has a least (greatest) element, we shall rather write it as $\bot = \bot_P$ (resp. $\top = \top_P$).

Let $x \in P$ and $A \subseteq P$. We shall denote by $A \uparrow x$ the set $\{y \in A \mid y \geq x\}$ of all elements of $A$ above $x$, and by $A \downarrow x$ the set $\{y \in A \mid y \leq x\}$ of all elements of $A$ below $x$. In particular, $\uparrow x = P \uparrow x$ is the principal filter of $P$ generated by $x$, and $\downarrow x = P \downarrow x$ is the principal ideal of $P$ generated by $x$. We shall write $x \geq A$ ($x \leq A$) to mean that $x \geq a$ ($x \leq a$) for every $a \in A$, that is, that $x$ is an upper (lower) bound of $A$.

A **lower set** of the poset $P$ is a subset $A$ of $P$ such that $x \in A$, $y \in P$, and $x \geq y$ imply $y \in A$. Dually, an **upper set** of the poset $P$ is a subset $A$ of $P$ such that $x \in A$, $y \in P$, and $x \leq y$ imply $y \in A$. A subset $A$ of the poset $P$ is a lower set of $P$ if and only if its complement $P \setminus A$ is an upper set of $P$.

Let $X$ be a subset of the poset $P$. We shall denote by $\downarrow X$ the lower set of $P$ generated by $X$ (which is the least lower set of $P$ that contains $X$): $\downarrow X = \bigcup_{x \in X} \downarrow x$. Dually, we shall denote by $\uparrow X$ the upper set generated by $X$.

A poset $D$ is said to be **(upward) directed** if it is nonempty and every two elements of $D$ have an upper bound in $D$, or equivalently, if every finite (possibly empty) subset of $D$ has an upper bound in $D$. A subset of the poset $P$ is **(upward) directed** if it is directed as a subposet of $P$. Dually there is the notion of a **downward directed** (subset of a) poset. An **ideal** of the poset $P$ is a directed lower set of $P$, and dually, a **filter** of the poset $P$ is a downward directed upper set of $P$.

A **directed complete poset**, or a **dcpo** for short, is a poset in which every directed subset has a join. A dcpo is said to be **pointed** if it has a least element.

Let $A$ be a subset of a dcpo $P$. The subset $A$ is said to be **closed under directed joins**, or shorter, **directed-closed**, if for every directed subset $D$ of $A$ its join $\bigvee D$ in $P$ lies in $A$. The subset $A$ is said to be **inaccessible by directed joins** if every directed subset $D$ of $P$ whose join lies in $A$ has at least an element in $A$ (that is, intersects $A$). The subset $A$ is inaccessible by directed joins if and only if its complement $P \setminus A$ is closed under directed joins.

A function $f$ from a poset $P$ to a poset $Q$ is **increasing** (decreasing) if $x \leq y$ in $P$ always implies $f(x) \leq f(y)$ ($f(x) \geq f(y)$) in $Q$. If $f : P \to Q$ and $g : Q \to R$ are increasing functions between posets, then the composite $gf = g \circ f : P \to R$ is increasing. The set of all increasing functions $P \to Q$ shall be denoted by $\text{Inc}(P,Q)$, and $\text{Inc}(P,P)$ will be shortened to $\text{Inc}(P)$. If $P$ is a poset ordered by $\leq$, then the poset on the same set of elements, but ordered by $\geq$, is said to be the **opposite** of the poset $P$ and is
denoted by $P^{\text{op}}$. If $P$ and $Q$ are posets, then an increasing function $P \to Q$ is also an increasing function $P^{\text{op}} \to Q^{\text{op}}$, and a decreasing function $P \to Q$ is the same thing as an increasing function $P \to Q^{\text{op}}$ or an increasing function $P^{\text{op}} \to Q$.

For any set $X$ and any poset $P$ the set $\text{Fun}(X,P)$ of all functions $X \to P$ is made into a poset with the pointwise partial ordering: if $f, g : X \to P$, then we let $f \leq g$ iff $f(x) \leq g(x)$ for every $x \in X$. In particular, for any posets $P$ and $Q$ the set $\text{Inc}(P,Q)$ is made into a poset in this way. Let $P$, $Q$, $R$ be posets. The composition

$$\text{Inc}(Q,R) \times \text{Inc}(P,Q) \rightarrow \text{Inc}(P,R) : (g,f) \mapsto gf$$

is increasing in both operands: if $f_1 \leq f_2$ in $\text{Inc}(P,Q)$ and $g \in \text{Inc}(Q,R)$, then $gf_1 \leq gf_2$ because $\text{Inc}(P,Q)$ and $\text{Inc}(P,R)$ are ordered pointwise and $g$ is increasing, and if we have $f \in \text{Inc}(P,Q)$ and $g_1 \leq g_2$ in $\text{Inc}(Q,R)$, then $g_1f \leq g_2f$ because $\text{Inc}(Q,R)$ and $\text{Inc}(P,R)$ are ordered pointwise. In particular, $\text{Inc}(P)$ is an ordered composition monoid.

Let $f$ be an endofunction on a poset $P$. We say that $f$ ascends (descends) on $x \in P$ if $x \leq f(x)$ ($x \geq f(x)$). The function $f$ is ascending (descending) if $f$ ascends (descends) on every element of $P$. An ascending and increasing function $f$ is called a preclosure map (a.k.a. an inflationary map, as in Escardó [1]). The set $\text{Asc}(P)$ of all ascending maps on a poset $P$ is a composition monoid; it is also a poset (ordered pointwise), but it is not, in general, an ordered monoid. However, the set $\text{Precl}(P)$ of all preclosure maps on a poset $P$ is a submonoid, and hence a sub-(ordered monoid)\footnote{Every submonoid of an ordered monoid $M$ becomes a sub-(ordered monoid) when we equip it with the partial order induced from $M$.} of the ordered monoid $\text{Inc}(P)$.

A closure operator on a poset $P$ is an idempotent preclosure map on $P$, that is, it is an endofunction on $P$ that is ascending, increasing, and idempotent; the poset of all closure operators on $P$, ordered pointwise, is denoted by $\text{Cl}(P)$. Dually, an interior operator on a poset $P$ is a closure operator on $P^{\text{op}}$, that is, it is an endofunction on $P^{\text{op}}$ that is descending, increasing, and idempotent.

If $g$ is a preclosure map on $P$ and $h$ is a closure operator on $P$, then the inequality $g \leq h$ is equivalent to either of the equalities $gh = h$, $hg = h$.

A closure system in a poset $P$ is a fixpoint set of some closure operator on $P$. A subset $C$ of $P$ is a closure system in $P$ if and only if for every $x \in P$ the set $C \uparrow x$ has a least element. If $C$ is a closure system in $P$, then the endofunction $\gamma$ on $P$, which sends every $x \in P$ to the least element $\gamma(x)$ of $C \uparrow x$, is a closure operator denoted by $\text{cl}_C$. We denote by $\text{ClSys}(P)$ the poset of all closure systems in $P$ ordered by inclusion. The map $\text{Cl}(P) \to \text{ClSys}(P) : \gamma \mapsto \text{Fix}(\gamma)$ is an antiisomorphism of posets, with the inverse $\text{ClSys}(P) \to \text{Cl}(P) : C \mapsto \text{cl}_C$. The notion of an interior system is dual to that of a closure system; that is, an interior system in a poset $P$ is a fixpoint set of some interior operator on $P$.

Let $\gamma$ be a closure operator on $P$. If a subset $S$ of the closure system $\gamma(P)$ has a join in the subposet $\gamma(P)$ of $P$, we write the join as $\bigvee^\gamma S$. For any subset $S$ of $P$ which has a join $\bigvee S$ in the poset $P$, the subset $\gamma(S)$ of $\gamma(P)$ has in the subposet $\gamma(P)$ the join

\[\text{Inc}(Q,R) \times \text{Inc}(P,Q) \rightarrow \text{Inc}(P,R) : (g,f) \mapsto gf\]
\(\vee \gamma(S) = \gamma(\vee S).\) In particular, if \(S \subseteq \gamma(P)\) has a join \(\vee S\) in \(P\), then \(S = \gamma(S)\) has the join \(\vee \gamma(S) = \gamma(\vee S)\) in \(\gamma(P)\). \(^2\) Let \(\gamma': P \rightarrow \gamma(P)\) be the closure operator \(\gamma\) with its codomain restricted to the subposet \(\gamma(P)\) of \(P\); then \(\gamma'\) preserves all joins which exist in \(P\).

Let \(\beta\) and \(\gamma\) be closure operators on a poset \(P\) such that \(\beta \gamma \leq \gamma \beta\); then \(\gamma \beta\) is a closure operator on \(P\), and it is the join of the closure operators \(\gamma\) and \(\beta\) in the poset \(\text{cl}(P)\). Indeed, \(\gamma \beta\) is a preclosure map, which is idempotent since \(\gamma \beta \gamma \beta \leq \gamma \gamma \beta = \gamma \beta\); thus \(\gamma \beta\) is a closure operator. If \(\delta\) is a closure operator and \(\{\beta, \gamma\} \leq \delta\), then \(\gamma \beta \leq \delta \delta = \delta\), and so \(\gamma \beta\) is the least upper bound of \(\{\beta, \gamma\}\) in \(\text{cl}(P)\). In symbols, \(\gamma \beta = \beta \vee \gamma\) in \(\text{cl}(P)\).

Let \(L\) be a complete lattice.

A subset of \(L\) is a closure system if and only if it is closed under arbitrary meets. The set \(\text{ClSys}(L)\) of all closure systems in \(L\) is closed under arbitrary intersections, and is therefore a closure system in the complete lattice \(\mathcal{P}L\); consequently, the poset \(\text{ClSys}(L)\) is a complete lattice. The poset \(\text{Cl}(L)\), which is antiisomorphic to the poset \(\text{ClSys}(L)\), is likewise a complete lattice. Let \(\Gamma \subseteq \text{Cl}(L)\). The meet of \(\Gamma\) in \(\text{Cl}(L)\) is computed pointwise: \((\wedge \Gamma)(x) = \bigwedge \Gamma(x)\) for \(x \in L\). The join \(\vee \Gamma\) in \(\text{Cl}(L)\) is the closure operator whose fixpoint set is \(\text{Fix}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Fix}(\gamma)\).

Let \(X\) be a subset of \(L\). The subset \(X\) generates the closure system \(\text{clsys}(X)\), the least closure system in \(L\) that contains \(X\); the closure system \(\text{clsys}(X)\) is the set of the meets of all subsets of \(X\). The closure operator \(\text{cl}(X)\) on \(L\) whose fixpoint set is \(\text{clsys}(X)\) is given by \(\text{cl}(X)(y) = \bigwedge \text{clsys}(X) (X \uparrow y) = \bigwedge (X \uparrow y)\) for \(y \in L\).

We now turn to closure operators on, and closure systems in, the complete lattice \(\mathcal{P}E\) of all subsets of some set \(E\).

One way to determine a closure system in \(\mathcal{P}E\) is by a set of closure rules.

A closure rule on \(E\) is a pair \((B, c) \in \text{ClRul}(E) := (\mathcal{P}E) \times E\), which is usually written as \(B \rightarrow c\), with the set \(B\) called the body and the element \(c\) called the head of the closure rule. If \(R\) is a set of closure rules on \(E\), then we write \((B, c) \in R\) as \(R: B \rightarrow c\). When the ‘ambient set’ \(E\) is known and we write \(B \rightarrow c\), it is understood that \(B\) is a subset of \(E\) and \(c\) is an element of \(E\).

A closure theory on \(E\) is a set \(T\) of closure rules on \(E\) which is

- reflexive: for all \(B \subseteq E\) and all \(b \in E\),
  - if \(b \in B\) then \(T: B \rightarrow b\), and
- transitive: for all \(B, C \subseteq E\) and all \(d \in E\),
  - if \(T: B \rightarrow c\) for every \(c \in C\) and \(T: C \rightarrow d\), then \(T: B \rightarrow d\).

We denote by \(\text{ClTheor}(E)\) the poset of all closure theories on \(E\) ordered by inclusion. \(^3\)

We say that a subset \(X\) of \(E\) obeys a closure rule \(B \rightarrow c\) on \(E\), or that the closure rule \(B \rightarrow c\) is obeyed by the subset \(X\), if \(B \subseteq X\) implies \(c \in X\). If \(\mathcal{X}\) is a set of subsets

\(^2\) A subset of \(\gamma(P)\) may have a join in \(\gamma(P)\) without having a join in \(P\).
\(^3\) A set of closure rules is also known as an implicational system, while a closure theory is referred to as a complete implicational system.
of $E$ and $R$ is a set of closure rules on $E$, then we say that $\mathcal{X}$ obeys $R$, or that $R$ is obeyed by $\mathcal{X}$, if every set in $\mathcal{X}$ obeys every rule in $R$.

The relation “$\mathcal{X}$ obeys $B \rightarrow c$” between a subset $X$ of $E$ and a closure rule $B \rightarrow c$ on $E$ gives rise to a contravariant Galois connection

$$\sigma, \varrho : \mathcal{P} \mathrm{ClRul}(E) \leftrightarrow (\mathcal{P} E)^{\text{op}},$$

where for every $R \subseteq \mathcal{P} \mathrm{ClRul}(E)$, $\sigma(R)$ is the set of all subsets of $E$ that obey $R$, and for every $\mathcal{X} \subseteq \mathcal{P} E$, $\varrho(\mathcal{X})$ is the set of all closure rules on $E$ obeyed by $\mathcal{X}$.

It is easy to verify that $\sigma(R)$ is a closure system in the complete lattice $\mathcal{P} E$ for every $R \subseteq \mathcal{P} \mathrm{ClRul}(E)$, and that $\varrho(\mathcal{X})$ is a closure theory on $E$ for every $\mathcal{X} \subseteq \mathcal{P} E$. The converse is also true: every closure system in $\mathcal{P} E$ is of the form $\sigma(R)$ for some $R \subseteq \mathcal{P} \mathrm{ClRul}(E)$, and every closure theory on $E$ is of the form $\varrho(\mathcal{X})$ for some $\mathcal{X} \subseteq \mathcal{P} E$. In short: $\sigma(\mathcal{P} \mathrm{ClRul}(E)) = \mathcal{P} \mathrm{Sys}(E)$ and $\varrho(\mathcal{P} E) = \mathcal{P} \mathrm{Theor}(E)$.

Given a set $R$ of closure rules, we shall say that the closure system $\sigma(R)$ in $\mathcal{P} E$ is determined by $R$, and that the closure operator on $\mathcal{P} E$ which has $\operatorname{Fix}(\gamma) = \sigma(R)$ is determined by $R$.

The restriction $\mathcal{P} \mathrm{Theor}(E) \rightarrow \mathcal{P} \mathrm{Sys}(\mathcal{P} E) : T \mapsto \mathrm{sys}(T)$ of the mapping $\sigma$ is an anti-isomorphism of complete lattices, with the inverse $\mathcal{P} \mathrm{Sys}(\mathcal{P} E) \rightarrow \mathcal{P} \mathrm{Theor}(E) : C \mapsto \mathrm{rul}(C)$, which is the restriction of the mapping $\varrho$.

The isomorphism $\mathcal{P} \mathrm{Cl}(\mathcal{P} E) \sim \mathcal{P} \mathrm{Sys}(\mathcal{P} E)^{\text{op}} : \gamma \mapsto \operatorname{Fix}(\gamma)$ composes with the isomorphism $\mathcal{P} \mathrm{Sys}(\mathcal{P} E)^{\text{op}} \sim \mathcal{P} \mathrm{Theor}(E)$ to yield the isomorphism of complete lattices $\mathcal{P} \mathrm{Cl}(\mathcal{P} E) \sim \mathcal{P} \mathrm{Theor}(E)$ which sends each closure operator $\gamma$ on $\mathcal{P} E$ to the closure theory $\mathrm{rul}(\gamma) := \{ B \rightarrow c \mid c \in \gamma(B) \}$ on $E$. The isomorphism $\mathcal{P} \mathrm{Theor}(E) \sim \mathcal{P} \mathrm{Sys}(\mathcal{P} E)^{\text{op}} \sim \mathcal{P} \mathrm{Cl}(\mathcal{P} E) : C \mapsto \mathrm{cl}_C$, yieding the isomorphism of complete lattices $\mathcal{P} \mathrm{Theor}(E) \sim \mathcal{P} \mathrm{Cl}(\mathcal{P} E)$, the inverse of the preceding composite isomorphism, which sends each closure theory $T$ on $E$ to the closure operator $\mathrm{cl}_T$ on $\mathcal{P} E$ given by $\mathrm{cl}_T(X) = \{ y \in E \mid T : X \rightarrow y \}$ for $X \subseteq E$.

## 2 Overview of the paper

The central theme of the paper are the properties of the poset $\mathcal{C}l(P)$ of all closure operators on a poset $P$ that is at least a dcpo, and of subposets of $\mathcal{C}l(P)$ consisting of some special kind of closure operators, for instance of the subposet $\mathcal{S}c\mathcal{C}l(P)$ of all Scott-continuous closure operators. The results of the paper are, among other things, subsuming and substantially extending almost every known fixed point theorem for posets.\(^4\)

The basic fact is that, for a dcpo $P$, the poset $\mathcal{C}l(P)$ is a complete lattice. More is true, actually: the complete lattice $\mathcal{C}l\mathcal{S}ys(P)$ of all closure systems in $P$, which is antistable to the complete lattice $\mathcal{C}l(P)$, is a closure system in the powerset lattice $\mathcal{P} P$.

\(^4\)An exception are the fixpoint theorems for $\omega$-complete posets, which may not be dcpos and are for this reason not considered in the paper.
(which means that the intersection of a set of closure systems in \( P \) is always a closure system in \( P \)); what is more, the fixpoint set of any preclosure map on \( P \) is a closure system. These results are established in section 3 as consequences of the main theorem, Theorem 1 which describes how a set \( G \) of preclosure maps on a dcpo \( P \) determines the least closure operator \( \overline{G} \) above them.

The main theorem of section 3 also pronounces the induction principle: if a subset of \( P \) is closed under directed joins in \( P \) and is closed under \( G \), then it is closed under \( \overline{G} \). The induction principle has a dual, the obverse induction principle, but the passage from the former to the latter involves the law of excluded middle (EM); since the one application of the obverse induction principle in the paper is in a proof which intentionally avoids using EM, the obverse induction principle is stated and proved on its own.

Section 3 concludes with the version of Tarski’s fixed point theorem for dcpos, which is proved using Theorem 1 and its consequences.

Section 4 considers the poset \( \text{ScCl}(P) \) of all Scott-continuous closure operators on a dcpo \( P \). The main result of the section is Theorem 13 if \( G \) is a set of Scott-continuous preclosure maps on \( P \), then the least closure operator \( \overline{G} \) above \( G \) is the directed pointwise join of the composition monoid \( G^* \) generated by \( G \), and it is Scott-continuous. Consequently, the poset \( \text{ScCl}(P) \) is an interior system in the complete lattice \( \text{Cl}(P) \) and so is itself a complete lattice with the joins inherited from \( \text{Cl}(P) \), while the poset \( \text{DcClSys}(P) \) of all closure systems in \( P \) that are closed under directed joins (which is antiisomorphic to the complete lattice \( \text{ScCl}(P) \) and is therefore a complete lattice) is a closure system in \( \text{ClSys}(P) \) as well as in \( \mathcal{RP} \).

For every closure operator \( \gamma \) on \( P \) there exists the greatest Scott-continuous closure operator \( \text{sc} \gamma \) below \( \gamma \), called the Scott-continuous core of the closure operator \( \gamma \). Dually, for every closure system \( C \) in \( P \) there exists the least directed-closed closure system \( \text{dcclsys}(C) \) that contains \( C \). Not much can be said about \( \text{sc} \gamma \) and \( \text{dcclsys}(C) \) for a general dcpo \( P \), but if \( P \) is a domain (a continuous dcpo), then both can be constructed: Proposition 18 gives the construction of the Scott-continuous core of a closure operator on a domain \( P \), while Proposition 19 has the construction of \( \text{dcclsys}(C) \) for a closure system \( C \) in a domain \( P \) (and also of \( \text{dcclsys}(X) \) for any subset \( X \) of \( P \)).

In section 5 we carry out the project that is only sketched in Escardó [1]: we prove (when the time comes) that the poset of all nuclei on a preframe is a frame.

A frame is a complete lattice in which binary meets distribute over arbitrary joins. A preframe is a meet-semilattice that is also a dcpo and in which binary meets distribute over directed joins. Given a meet-semilattice \( P \), a nucleus (prenucleus) on \( P \) is a closure operator (a preclosure map) on \( P \) that preserves binary meets; the fixpoint set of a nucleus on \( P \) is called a nuclear system in \( P \); \( \text{Nuc}(P) \) denotes the poset of all nuclei on \( P \) and \( \text{NucSys}(P) \) denotes the poset of all nuclear systems in \( P \).

The main result of the section is Theorem 21 which states that for any set \( \Gamma \) of prenuclei in a preframe \( P \) the least closure operator \( \overline{\Gamma} \) above the set of preclosure maps \( \Gamma \) is a nucleus. It follows that for a preframe \( P \) the poset \( \text{Nuc}(P) \) is closed under all joins in the complete lattice \( \text{Cl}(P) \) and is thus a complete lattice; besides this, \( \text{Nuc}(P) \) is closed
under the (pointwise calculated) binary meets in $\text{Cl}(P)$. For every closure operator $\gamma$ on a preframe $P$ there exists the largest nucleus $\text{nuc}\gamma$ below $\gamma$, the nuclear core of $\gamma$.

Proposition 23 tells us that on a preframe $P$ the complete lattice $\text{Nuc}(P)$ is actually a frame. Both Theorem 21 and Proposition 23 are proved using the induction principle.

We conclude the section by taking a quick look at Scott-continuous nuclei on $P$.

In section 6 we look at the nuclei on a frame $L$. Since a frame is a special preframe, all results for preframes specialize to the frame $L$. But, since a frame is very special preframe, we can say much more about the frame of nuclei $\text{Nuc}(L)$ on the frame $L$ than about the frame of nuclei on a mere preframe. For instance, by Proposition 28, the subset $\text{Nuc}(L)$ of $\text{Cl}(L)$ is closed not only under arbitrary joins in $\text{Cl}(L)$ but also under arbitrary (not just binary) meets in $\text{Cl}(L)$; that is, $\text{Nuc}(L)$ is a sub-(complete lattice) of the complete lattice $\text{Cl}(L)$.

A frame is relatively pseudocomplemented, that is, it is a complete Heyting algebra. Corollary 34 at the end of the section gives a formula, which uses the operation $\Rightarrow$ of relative pseudocomplementation, for the nuclear core $\text{nuc}\gamma$ of a closure operator $\gamma$ on $L$.

In [1] Escardó shows off the utility of join induction by using it in a proof of the Hofmann–Mislove–Johnstone theorem. In section 7 we prove the HMJ theorem in a way that demonstrates the power of the obverse induction principle. Our proof of the HMJ theorem is spread through proofs of three lemmas, with parts of it reasoned out in the connecting text; the short concluding reasoning then ties everything together. The obverse induction principle is used in the proof of Lemma 40, which is for that very reason short and transparent.

For a dcpo $P$, the complete lattice $\text{ClSys}(P)$ of all closure systems in $\mathcal{P}P$ is a closure system in the powerset lattice $\mathcal{P}P$ and so it is determined by a set of closure rules on $P$. In section 8 we prove that $\text{ClSys}(P)$ is determined by the set of all default closure rules associated with $P$, which are the closure rules on $P$ of the form $B \to c$, where $c \in P$ is a maximal lower bound of $B \subseteq P$. The proof of this result, however, requires the axiom of choice (AC), since in a set theory without AC it implies AC.

We actually develop a little theory which operates with maximal elements. We prove several assertions of the following form: if in every subset of $P$ of some special kind every element has a maximal element of the subset above it, then $P$ has a certain property. For example, if every lower bound of any subset of $P$ is below some maximal lower bound of the subset, then the closure systems in $P$ are determined by the set of all default closure rules; in short, if $P$ has ‘enough’ default closure rules associated with it — if it is default enabled — then the closure system $\text{ClSys}(P)$ of $\mathcal{P}P$ is determined by the default closure rules. Similarly, by requiring that a meet-semilattice possesses ‘enough’ closure rules of a certain form, we can prove that $\text{Nuc}(P)$ is an interior system in $\text{Cl}(P)$, and requiring existence of even more closure rules we are able to prove that $\text{Nuc}(P)$ is a frame. Interestingly, we can prove all this without ever invoking AC. These results mimick results in section 5 and they in fact imply them by specialization, at the price of being forced to use AC to accomplish it.
The class of the default-enabled posets is strictly larger than the class of the dcpos. A default-enabled poset $P$ shares with posets the properties that the set of all closure systems in $P$ is a closure system in $\mathcal{P}P$, and that the fixpoint set of every preclosure map on $P$ is a closure system in $P$. Here is a project that may turn out to be more of an adventure than it appears: characterize, in structural terms, the posets that have the one, or the other, or both of these properties.

In section 9 we prove that for every dcpo $P$, the closure operator $\text{clsys}_P$ on $\mathcal{P}P$ (which for each $X \subseteq P$ yields the least closure system in $P$ that contains $X$) and the closure operator $\text{dclsys}_P$ on $\mathcal{P}P$ (which for each $X \subseteq P$ yields the least directed-closed closure system in $P$ that contains $X$) are convex, meaning that they satisfy the anti-exchange axiom. Since all that we need to obtain these two results is the property of every dcpo $P$ that $\text{ClSys}(P)$ is a closure system in $P$, the results are valid also for every default-enabled poset $P$.

3 The complete lattice of closure operators on a dcpo

Let $P$ be a dcpo, and let $M := \text{Precl}(P)$ be the pointwise-ordered composition monoid of all preclosure maps on $P$. In $M$ all directed joins exist, and they are calculated pointwise: if $F$ is a directed subset of $M$, then at each $x \in P$ the set $F(x)$ is directed, thus the map $\varphi: P \to P : x \mapsto \bigvee F(x)$ is well defined, and one easily verifies that it is a preclosure map; it follows that $\varphi = \bigvee F$ in the poset $M$. The ordered monoid $M$ is therefore a dcpo; moreover, $M$ is a pointed dcpo since the identity map $\text{id}_P$ is its least element. Mark that every submonoid of $M$ is a directed subset of $M$ because $f, g \leq fg$ for any $f, g \in M$: $f \leq fg$ because $g$ is ascending and $f$ is increasing, and $g \leq fg$ because $f$ is ascending.

The following theorem describes how a set of preclosure maps on a dcpo determines the least closure operator above them.

**Theorem 1.** Let $P$ be a dcpo, and let $G$ be a set of preclosure maps on $P$. Then $\text{Fix}(G)$ is a closure system in $P$, and the corresponding closure operator $\overline{G}$ on $P$, which has $\text{Fix}(\overline{G}) = \text{Fix}(G)$, is the least of all closure operators on $P$ that are above $G$.

The **induction principle** holds: if a subset of $P$ is closed under directed joins in $P$ and is closed under $G$, then it is closed under $\overline{G}$.

Moreover, the **obverse induction principle** holds: if a subset of $P$ is inaccessible by directed joins and is inversely closed under $G$, then it is inversely closed under $\overline{G}$.

**Proof.** In the pointwise-ordered composition monoid $M$ of all preclosure maps on $P$, let $H$ be the intersection of all submonoids that contain $G$ and are closed under directed joins in $M$; $H$ is the least such submonoid. Since $H$ is a directed subset of $M$, the join $h = \bigvee H$ in $M$ exists. Then $h \in H$, because $H$ is closed under directed joins, thus $h$ is the greatest element of $H$. Since $H$ is a submonoid of $M$, we have $hh \in H$, hence $hh \leq h$, which shows that $h$ is a closure operator. We have $h \geq G$ because $H$
contains \( G \). Let \( k \geq G \) be a closure operator; then \( k \in M \). The set \( K = M \downarrow k \) is a submonoid of \( M \) since \( \text{id}_P \in K \) and since \( f, f' \in M \) and \( f, f' \leq k \) imply \( ff' \leq kk = k \); \( K \) is evidently closed under directed joins in \( M \) (it is closed under all existing joins in \( M \)) and contains \( G \), thus it contains \( H \), whence \( h \leq k \).

The induction principle. Let \( A \) be a subset of \( P \) that is closed under directed joins and is closed under \( G \). Let \( F \) be the set of all \( f \in M \) such that \( f(A) \subseteq A \). Then \( F \) is a submonoid of \( M \) and contains \( G \); \( F \) is closed under directed joins in \( M \), because \( A \) is closed under directed joins in \( P \) and because the directed joins in \( M \) are calculated pointwise. It follows that \( H \subseteq F \), hence \( h \in F \), that is, \( h(A) \subseteq A \).

For every \( g \in G \) we have \( gh = h \), which implies that every element of \( h(P) = \text{Fix}(h) \) is a fixed point of \( G \). Conversely, if \( a \) is a fixed point of \( G \), then the set \( \{a\} \) is closed under \( G \), and since it is evidently closed under directed joins in \( P \), it is closed under \( h \), thus \( h(a) = a \in \text{Fix}(h) \).

The closure operator \( \overline{G} := h \) has the properties stated in the first assertion of the proposition.

The obverse induction principle. Suppose that \( A \subseteq P \) is inversely closed under \( G \) and that it is inaccessible by directed joins. Let \( F \) be the set of all \( f \in M \) such that \( f^{-1}(A) \subseteq A \); \( F \) contains \( G \) and it is a submonoid of \( M \). Let \( E \) be a directed subset of \( F \); we shall show that \( \bigvee E \in F \). Let \( x \in P \), and suppose that \( (\bigvee E)(x) = \bigvee E(x) \in A \); since \( A \) is inaccessible by directed joins, there exists \( e \in E \) with \( e(x) \in A \), and we have \( x \in A \) because \( A \) is inversely closed under \( e \). It follows that \( A \) is inversely closed under \( \bigvee E \). We see that \( F \) is closed under directed joins, so \( F \) contains \( H \) and with it the closure operator \( \overline{G} \), whence \( A \) is inversely closed under \( \overline{G} \).

In the classical logic, which uses the law of excluded middle (EM for short) with abandon, the obverse induction principle for a subset \( A \) of \( P \) is just a rephrasing of the induction principle for the complement \( P \setminus A \). Since we want to apply the obverse induction principle in situations where EM is not admissible, we proved it on its own.

Let \( P, G \), and \( \overline{G} \) be as in Theorem \([\square]\). We shall say that the closure operator \( \overline{G} \) is generated by the set \( G \) of preclosure maps.

The special case of Theorem \([\square]\) where \( G = \{g\} \) is of interest on its own.

**Corollary 2.** If \( g \) is a preclosure map on a dcpo \( P \), then \( \text{Fix}(g) \) is a closure system in \( P \), and the closure operator \( \overline{g} \) on \( P \) that has \( \text{Fix}(\overline{g}) = \text{Fix}(g) \) is the least of all closure operators on \( P \) that are above \( g \).

The induction principle and the obverse induction principle of course hold in the special case featuring a single preclosure map; there is no need to restate them.

As a consequence of Theorem \([\square]\) if \( P \) is a dcpo, then in the poset \( \text{Cl}(P) \) every subset has a join, therefore \( \text{Cl}(P) \) is a complete lattice.
Corollary 3. The poset \( \text{Cl}(P) \) of all closure operators on a dcpo \( P \) is a complete lattice, and so is the poset \( \text{ClSys}(P) \) of all closure systems in \( P \). In \( \text{Cl}(P) \), the join of a set \( G \) of closure operators is the closure operator \( \overline{G} \) generated by \( G \), while in \( \text{ClSys}(P) \), the meet of a set \( C \) of closure systems is the intersection \( \bigcap C \).

Proof. For every \( G \subseteq \text{Cl}(P) \), \( \overline{G} \) is the join of \( G \) in \( \text{Cl}(G) \) by Theorem 1: \( \forall G = \overline{G} \). Since the mapping \( \text{Cl}(P) \rightarrow \text{ClSys}(P) : g \mapsto \text{Fix}(g) \) is an antiisomorphism of complete lattices, for every \( G \subseteq \text{Cl}(P) \) we have \( \bigwedge_{g \in G} \text{Fix}(g) = \text{Fix}(\bigvee G) = \text{Fix}(\overline{G}) = \text{Fix}(G) = \bigcap_{g \in G} \text{Fix}(g) \), and it follows that all meets in \( \text{ClSys}(P) \) exist and that they are calculated as intersections.

Let \( P \) be a dcpo. The set \( \text{ClSys}(P) \) of all closure systems in \( P \) is a closure system in the complete lattice \( \mathcal{P} P \). The corresponding closure operator on \( \mathcal{P} P \) maps each subset \( X \) of \( P \) to the closure system \( \text{clsys}(X) = \text{clsys}_P(X) \), which is the least of all closure systems in \( P \) that contain \( X \).

Proposition 4. If \( h \) is a closure operator on a dcpo \( P \), then the subposet \( h(P) = \text{Fix}(h) \) of \( P \) is a dcpo. The restriction \( h' : P \rightarrow h(P) \) of the closure operator \( h \) preserves directed joins.

Proof. Let \( S \) be a directed subset of \( h(P) \). The set \( S \) is also directed in \( P \), hence it has a join \( \bigvee S \) in \( P \), and then \( h(\bigvee S) \) is the join of \( S \) in \( h(P) \). The second assertion of the proposition holds because for any closure operator \( h \) on an arbitrary poset \( P \) the restriction \( h' : P \rightarrow h(P) \) preserves all existing joins. \( \square \)

Tarski’s fixed point theorem, a version for dcpos, easily follows from Theorem 1.

Theorem 5. Let \( f \) be an increasing map on a dcpo \( P \). The fixpoint set of \( f \), as a sub-poset of \( P \), is a dcpo, and for every \( x \in P \) on which \( f \) ascends there exists a least fixed point of \( f \) above \( x \). If \( P \) has a least element, then \( f \) has a least fixed point.

Proof. Let \( A := \{ x \in P \mid x \leq f(x) \} \); that is, \( A \) is the set of all elements of \( P \) on which \( f \) ascends. It is clear that \( A \) contains all fixed points of \( f \). Since \( x \leq f(x) \) implies \( f(x) \leq f(f(x)) \), the set \( A \) is closed under \( f \). If \( S \subseteq A \) is directed, then the join \( \bigvee S \) in \( P \) exists, and \( s \leq f(s) \leq f(\bigvee S) \) for every \( s \in S \), hence \( \bigvee S \leq f(\bigvee S) \), so \( A \) is closed under directed joins. Thus the sub-poset \( A \) is a dcpo and the restriction \( g : A \rightarrow A \) of \( f \) is a preclosure map on \( A \). The closure operator \( g \) on \( A \) maps each \( x \in A \) to the least element above \( x \) in the set \( \text{Fix}(g) = \text{Fix}(g) = \text{Fix}(f) \), and this set, as a sub-poset of \( A \), and hence of \( P \), is a dcpo.

If \( P \) has a least element \( \bot \), then \( \bot \in A \), and \( g(\bot) \) is the least fixed point of \( f \). \( \square \)
The last statement of Theorem 5 is the bare-bones Tarski’s fixed point theorem for dcpos; let us restate it on its own.

**Corollary 6.** Every increasing endomap on a pointed dcpo has a least fixed point.

It can be proved, with a generous help from the axiom of choice, that the bare-bones Tarski’s fixed point property, stated in the corollary, in fact characterizes pointed dcpos. See, for example, Theorem 11 (after consulting Corollary 2 of Theorem 1) in Markowsky [6].

### 4 Scott-continuous closure operators on dcpos

To begin with we establish a general result about preservation of some special joins by the pointwise join of functions that preserve those special joins.

For any poset $P$ we denote by $\mathcal{J}P$ the set of all subsets of $P$ that have a join in $P$.

Let $P$ and $Q$ be posets, and let $A \subseteq \mathcal{J}P$.

We shall say that a function $f: P \to Q$ preserves $A$-joins if for every $A \in A$ we have $f(A) \in \mathcal{J}Q$ and $f(\bigsqcup A) = \bigsqcup f(A)$. Let $F$ be a set of functions $P \to Q$. We shall say that $F$ preserves $A$-joins if every function in $F$ preserves $A$-joins. We shall say that $F$ has a pointwise join if $F(x) \in \mathcal{J}Q$ for every $x \in P$. Whenever $F$ has a pointwise join we define the pointwise join $\bigsqcup \cdot F: P \to Q$ of $F$ by $(\bigsqcup \cdot F)(x) := \bigsqcup F(x)$ for every $x \in P$.

Note that if $A$ contains all subsets $\{x, y\}$ of $P$ with $x < y$, then every function $P \to Q$ which preserves $A$-joins is increasing.

**Lemma 7.** Let $P$ and $Q$ be posets, let $F$ be a set of functions $P \to Q$, and $A \subseteq \mathcal{J}P$. If $F$ preserves $A$-joins and has a pointwise join, then the pointwise join $\bigsqcup F$ preserves $A$-joins. Moreover, for every $A \in A$ we have $F(A) \in \mathcal{J}Q$ and

$$\bigsqcup (\bigsqcup F)(A) = \bigsqcup (\bigsqcup F)(A) = \bigsqcup F(A). \quad (1)$$

**Proof.** Consider any $A \in A$. By assumption the join $\bigsqcup A$ exists and the join $\bigsqcup F(x)$ exists for every $x \in P$, therefore the element $(\bigsqcup F)(\bigsqcup A) = \bigsqcup F(\bigsqcup A)$ of $Q$ and the subset $(\bigsqcup F)(A) = \{\bigsqcup F(a) \mid a \in A\}$ of $Q$ are well-defined. Let $y$ be an arbitrary element of $Q$. The chain of equivalences

$$y \geq (\bigsqcup F)(\bigsqcup A) \iff y \geq \bigsqcup F(\bigsqcup A) \text{ for every } f \in F,$$

$$\iff y \geq \bigsqcup f(A) \text{ for every } f \in F \quad \text{(since } f(\bigsqcup A) = \bigsqcup f(A)),$$

$$\iff y \geq f(a) \text{ for every } a \in A \text{ and for every } f \in F,$$

$$\iff y \geq F(A),$$

proves that $F(A) \in \mathcal{J}Q$ and that $\bigsqcup F(A) = (\bigsqcup F)(\bigsqcup A)$. Then

$$y \geq (\bigsqcup F)(A) \iff y \geq \bigsqcup F(a) \text{ for every } a \in A,$$

$$\iff y \geq f(a) \text{ for every } f \in F \text{ and for every } a \in A,$$

$$\iff y \geq F(A),$$

clinches the proof of the equalities $(1)$. □
Lemma 6 is so general with a reason: it makes perfectly clear that \( F \) preserving \( \mathcal{A} \)-joins and \( F \) having a pointwise join are two independent properties of the set of functions \( F \); in a certain sense these two properties are orthogonal to each other. For example, suppose that \( P \) and \( Q \) are dcpo and that \( \mathcal{A} \) is the set of all directed subsets of \( P \); in this case \( F \) preserving \( \mathcal{A} \)-joins means that \( F \) preserves directed joins. The set of functions \( F \) need not be directed (in the pointwise ordering). It surely helps if \( F \) is directed, since then the sets \( F(x), x \in P \), are directed subsets of \( Q \) and have joins in \( Q \), thus we know that \( F \) has a pointwise join because it is directed and \( Q \) is a dcpo, and we can conclude that the pointwise join \( \bigvee F \) preserves directed joins. But suppose that \( Q \) is a complete lattice (with \( P \) still just any dcpo): then every set \( F \) of functions \( P \rightarrow Q \) that preserve directed joins has a pointwise join \( \bigvee F \) which preserves directed joins.

A function between posets \( f: P \rightarrow Q \) is said to be \textbf{Scott-continuous} if it preserves all existing directed joins.\(^9\) In detail, \( f \) is Scott-continuous if and only if for every directed subset \( Y \) of \( P \) which has a join \( \bigvee Y \) in \( P \), the \( f \)-image of this join is the join in \( Q \) of the \( f \)-image of the set \( Y \).\(^9\) That is, \( f(\bigvee Y) = \bigvee f(Y) \). In particular, a Scott-continuous function preserves joins of all pairs of comparable elements of \( P \), which implies that \( f \) is increasing, therefore for every directed subset \( Y \) of \( P \) its image \( f(Y) \) is a directed subset of \( Q \). From this it follows that if \( f: P \rightarrow Q \) and \( g: Q \rightarrow R \) are Scott-continuous functions between posets, then the composite function \( gf: P \rightarrow R \) is Scott-continuous.

For any posets \( P \) and \( Q \) we let \( \text{Sc}(P,Q) \) denote the poset of all Scott-continuous functions \( P \rightarrow Q \) with the pointwise ordering.

For any poset \( P \) we denote by \( \text{ScPrecl}(P) \) the pointwise ordered poset of all Scott-continuous preclosure maps on \( P \), and by \( \text{ScCl}(P) \) the pointwise ordered poset of all Scott-continuous closure operators on \( P \).

The following proposition follows from Lemma\(^7\) by specialization.

**Proposition 8.** Let \( P \) and \( Q \) be dcpos. If \( F \) is a directed subset of \( \text{Sc}(P,Q) \), then the pointwise join \( \bigvee F \) exists and is Scott-continuous.

**Corollary 9.** If \( P \) and \( Q \) are dcpos, then \( \text{Sc}(P,Q) \) is a dcpo in which directed joins are calculated pointwise.

**Corollary 10.** If \( P \) is a dcpo, then \( \text{ScPrecl}(P) \) is a pointed dcpo in which directed joins are calculated pointwise.

**Proof.** If \( F \) is a directed set of Scott-continuous preclosure maps on \( P \), then the pointwise join \( \bigvee F \) is is a preclosure map which is Scott-continuous by Proposition\(^8\) Therefore \( \text{ScPrecl}(P) \) is a dcpo with directed joins that are calculated pointwise, and it is a pointed dcpo since the identity map \( \text{id}_P \) is its bottom element. \( \Box \)

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\(^9\)In other words, \( f \) is Scott-continuous iff it preserves \( \mathcal{D} \)-joins, where \( \mathcal{D} \) is the set of all directed subsets of \( P \) that have a join in \( P \).

\(^{10}\)There are those who prefer the more long-winded “the \( f \)-image of the set \( Y \) has a join in \( Q \) which is equal to the \( f \)-image of the join of \( Y \) in \( P \)”.
Let $P$, $Q$, $R$ be posets. For any sets of functions $F \subseteq \text{Inc}(P,Q)$ and $G \subseteq \text{Inc}(Q,R)$ we write $GF := \{gf \mid f \in F, g \in G\}$; when both $F$ and $G$ are directed, $GF$ is easily seen to be directed (recall that composition of increasing maps is increasing in both operands).

**Proposition 11.** Let $P$, $Q$, and $R$ be dcpos. If $F \subseteq \text{Sc}(P,Q)$ and $G \subseteq \text{Sc}(Q,R)$ are directed, then $GF \subseteq \text{Sc}(P,R)$ is directed and

$$(\forall G)(\forall F) = \forall(GF) .$$

**Proof.** Let $x \in P$. The set $F(x)$ is directed, every $g \in G$ preserves directed joins, and $G$, being directed, has a pointwise join, and so we calculate

$$
\begin{align*}
((\forall G)(\forall F))(x) &= (\forall G)((\forall F)(x)) = (\forall G)(\forall F(x)) \quad \text{(apply Lemma 7)} \\
&= \forall G(F(x)) = \forall(GF)(x) = (\forall(GF))(x) \\
&= (\forall(GF))(x).
\end{align*}
$$

This proves the equality (2). \hfill \square

**Corollary 12.** Let $P$ be a dcpo. The dcpo $\text{Sc}(P)$ is a composition monoid in which the composition distributes over directed joins: if $F$ and $G$ are directed subsets of $\text{Sc}(P)$, then $GF$ is a directed subset of $\text{Sc}(P)$ and $(\forall G)(\forall F) = \forall(GF)$. Likewise the pointed dcpo $\text{ScPrecl}(P)$ is an ordered composition monoid in which the composition distributes over directed joins.

Let $P$ be a dcpo. Every submonoid of the ordered composition monoid $\text{ScPrecl}(P)$ is a submonoid of the ordered composition monoid $\text{Precl}(P)$ and is therefore directed.

**Theorem 13.** Let $P$ be a dcpo, and let $G$ be a set of Scott-continuous preclosure maps on $P$; we denote by $G^*$ the submonoid of $\text{ScPrecl}(P)$ generated by $G$. The directed join $h := \bigvee G^*$ of $G^*$ in $\text{ScPrecl}(P)$ (which is calculated pointwise) is a Scott-continuous closure operator on $P$. Moreover, $h$ is also the closure operator on $P$ generated by the set $G$ of preclosure maps on $P$, therefore $\text{Fix}(h) = \text{Fix}(G)$.

**Proof.** Since the submonoid $G^*$ of $\text{ScPrecl}(P)$ is directed, the join $h = \bigvee G^* = \bigvee G^*$ in $\text{ScPrecl}(P)$ exists. According to Corollary 12 we have

$$hh = (\bigvee G^*)(\bigvee G^*) = \bigvee(G^*G^*) = \bigvee G^* = h ,$$

thus $h$ is a Scott-continuous closure operator on $P$, and clearly $h \geq G$.

Now let $k$ be a closure operator on $P$ and $k \geq G$. (Note that we are not assuming that $k$ is Scott-continuous.) First, $k \geq \text{id}_P \in G^*$. Next, if $u \in G^*$ is a composite of $n \geq 1$ functions in $G$, then $u \leq k^n = k$. Thus $k \geq G^*$, whence $k \geq \bigvee G^* = \bigvee G^* = h$.

It follows that $h$ is the closure operator on $P$ generated by the set $G$ of preclosure maps on $P$, therefore $h$ is, according to Theorem 1 the (unique) closure operator on $P$ that has $\text{Fix}(h) = \text{Fix}(G)$. \hfill \square
**Corollary 14.** If \( g \) is a Scott-continuous preclosure map on a dcpo \( P \), then \( \text{Fix}(g) \) is the fixpoint set of the Scott-continuous closure operator \( \bigvee_{k \in \mathbb{N}} g^k \) (a join of a nonempty chain), which is the least of all closure operators on \( P \) that are above \( g \).

The join in the corollary is of course of the composition monoid \( \{g\}^* = \{g^k \mid k \in \mathbb{N}\} \) generated by a single Scott-continuous preclosure map \( g \).

**Corollary 15.** Let \( P \) be a dcpo, and let \( G \) be a set of Scott-continuous closure operators on \( P \). The pointwise directed join \( \bigvee G^\ast \) is the join \( \bigvee G \) of \( G \) in \( \text{ScCl}(P) \), and it is also the join of \( G \) in \( \text{Cl}(P) \) so that \( \text{Fix}(\bigvee G) = \text{Fix}(G) = \bigcap_{g \in G} \text{Fix}(g) \).

Let \( P \) be a dcpo. Corollary 15 tells us that the subset \( \text{ScCl}(P) \) (of all Scott-continuous closure operators on \( P \)) of the set \( \text{Cl}(P) \) (of all closure operators on \( P \)) is closed under all joins of the complete lattice \( \text{Cl}(P) \) and is therefore itself a complete lattice whose joins are inherited from \( \text{Cl}(P) \). Correspondingly, the set of the fixpoint sets of all Scott-continuous closure operators on \( P \) is a closure system in the complete lattice \( \text{ClSys}(P) \) of all closure systems in \( P \), and is therefore a closure system in the powerset lattice \( \mathcal{P}P \), that is, it is closed under arbitrary intersections. There is a less roundabout way to see this, using an explicit characterization, given below in Lemma 17 of fixpoint sets of the Scott-continuous closure operators on a dcpo.

But first an auxiliary lemma, almost trivial, though still worth telling on its own.

**Lemma 16.** Let \( P \) be a poset, \( \gamma \) a closure operator on \( P \), and \( X \) a subset of \( P \). If both \( \bigvee X \) and \( \bigvee \gamma(X) \) exist, then \( \gamma(\bigvee X) = \bigvee \gamma(\gamma(X)) \).

**Proof.** We get the asserted identity by applying \( \gamma \) to \( \bigvee X \leq \bigvee \gamma(X) \leq \gamma(\bigvee X) \). \( \Box \)

And here is the promised characterization.

**Lemma 17.** A closure operator \( \gamma \) on a dcpo \( P \) is Scott-continuous if and only if \( \text{Fix}(\gamma) \) is closed under directed joins.

**Proof.** Write \( C := \text{Fix}(\gamma) \).

Suppose \( \gamma \) is Scott-continuous. If \( Y \subseteq C \) is directed, then \( \gamma(\bigvee Y) = \bigvee \gamma(Y) = \bigvee Y \), therefore \( \bigvee Y \in C \).

Suppose \( C \) is closed under directed joins, and let \( Y \subseteq P \) be directed. Then \( \gamma(Y) \) is a directed subset of \( C \), thus \( \bigvee \gamma(Y) \in C \), and \( \gamma(\bigvee Y) = \bigvee \gamma(\gamma(Y)) = \bigvee \gamma(Y) \). \( \Box \)

Let \( P \) be a dcpo.

We know that the poset \( \text{Cl}(P) \) of all closure operators on \( P \), ordered pointwise, and the poset \( \text{ClSys}(P) \) of all closure systems on \( P \), ordered by inclusion, are complete lattices, where \( \text{ClSys}(P) \) is a closure system in the powerset lattice \( \mathcal{P}P \), meaning that the intersection of any set of closure systems in \( P \) is a closure system in \( P \).

Let us denote by \( \text{DcClSys}(P) \) the subposet of \( \text{ClSys}(P) \) consisting of all directed-closed closure systems in \( P \). If \( C \) is any subset of \( \text{DcClSys}(P) \), then the intersection \( \bigcap C \)
is a closure system closed under directed joins, so it belongs to $\text{DeClSys}(P)$. We see that the set $\text{DeClSys}(P)$ is a closure system in the complete lattice $\mathcal{P}P$, and is also a closure system in the complete lattice $\text{ClSys}(P)$. The poset $\text{DeClSys}(P)$ is thus a complete lattice in which all meets are intersections. The isomorphism of complete lattices $\text{Cl}(P) \rightarrow \text{ClSys}(P)^{\text{op}}$ restricts to the isomorphism of complete lattices $\text{ScCl}(P) \rightarrow \text{DeClSys}(P)^{\text{op}}$.

Since $\text{DeClSys}(P)$ is a closure system in $\mathcal{P}P$, for every subset $X$ of $P$ there exists the least of all directed-closed closure systems that contain $X$, which we denote by $\text{dcclsys}(X)$. The endomapping $\text{dcclsys}$ on $\mathcal{P}P$ is a closure operator on $\mathcal{P}P$, and it restricts to a closure operator on $\text{ClSys}(P)$. Since $\text{ScCl}(P)$ is an interior system in $\text{Cl}(P)$, for every closure operator $\gamma$ on $P$ there exists the greatest of all Scott-continuous closure operators on $P$ that are below $\gamma$, which we denote by $\text{sc}\gamma$ and call it the Scott-continuous core of the closure operator $\gamma$. The endomapping $\text{sc}$ on $\text{Cl}(P)$ is an interior operator on $\text{Cl}(P)$. Via the isomorphism $\text{Cl}(P) \rightarrow \text{ClSys}(P)^{\text{op}}$ of any closure operator $\gamma$ for every closure system $\gamma$ on $P$ there exists the greatest of all Scott-continuous closure operators on $P$ that are below $\gamma$, which we denote by $\text{sc}\gamma$ and call it the Scott-continuous core of the closure operator $\gamma$. The endomapping $\text{sc}$ on $\text{Cl}(P)$ is an interior operator on $\text{Cl}(P)$. Via the isomorphism $\text{Cl}(P) \rightarrow \text{ClSys}(P)^{\text{op}}$ of any closure operator $\gamma$ on $P$ there exists the greatest of all Scott-continuous closure operators on $P$ that are below $\gamma$, which we denote by $\text{sc}\gamma$ and call it the Scott-continuous core of the closure operator $\gamma$.

For a general dcpo $P$ we cannot say much either about the closure operator $\text{dcclsys}$ on $\mathcal{P}P$ (and its restriction to $\text{ClSys}(P)$) or about the interior operator $\text{sc}$ on $\text{Cl}(P)$. However, if $P$ is a domain, then there exist explicit constructions of the Scott-continuous core $\text{sc}\gamma$ of any closure operator $\gamma$ on $P$ and of the directed-closed closure system $\text{dcclsys}(C)$ generated by any closure system $C$ in $P$; these two constructions are described in Proposition 18 and Proposition 19.

The following definitions are from Continuous Lattices and Domains [2] (CLaD).

Let $P$ be a poset.

For any elements $x$ and $y$ of $P$ we say that $x$ is way below $y$, and write $x \ll y$, if for every directed subset $D$ of $P$ that has a join in $P$, the relation $y \leq \bigvee D$ implies that $x \leq d$ for some $d \in D$.

For every $x \in P$ we define the set $\downarrow x := \{ u \in P \mid u \ll x \}$, which is a lower set contained in the principal ideal $\downarrow x$.

The poset $P$ is said to be continuous if it satisfies the axiom of approximation: for every $x \in P$ the set $\downarrow x$ is directed and has in $P$ the join $\bigvee \downarrow x = x$.

A domain is a continuous dcpo.

We shall silently use the basic properties of the way-below relation. Besides those we will also need the following two results from CLaD [2].

The first result is the interpolation property of the way-below relation on a continuous poset $P$ (Theorem I-1.9(ii)): for any $x, z \in P$ such that $x \ll z$ there exists $y \in P$ so that $x \ll y \ll z$.

The second result is a characterization of Scott-continuous functions between domains (Proposition II-2.1(5)): a function $f : P \rightarrow Q$, where $P$ and $Q$ are domains, is Scott-continuous iff $f(\downarrow x) = \bigvee f(\downarrow x)$ for every $x \in P$ (i.e., $f(x)$ is the join of $f(\downarrow x)$ in $Q$, or, more long-windedly, the join of $f(\downarrow x)$ in $Q$ exists and is equal to $f(x)$).
And here is the first of the promised constructions, namely the construction of the Scott-continuous core of a closure operator on a domain.

**Proposition 18.** Let $\gamma$ be a closure operator on a domain $P$. Then for all $x \in P$ we have $(\text{sc } \gamma)(x) = \bigvee \gamma(\downarrow x)$.

**Proof.** We define the endomap $\gamma^\circ$ on $P$ by $\gamma^\circ(x) := \bigvee \gamma(\downarrow x)$ for $x \in P$.

$\gamma^\circ$ is ascending: $\gamma^\circ(x) = \bigvee \gamma(\downarrow x) \geq \bigvee \downarrow x = x$.

$\gamma^\circ$ is increasing. If $x \leq y$, then $\downarrow x \subseteq \downarrow y$, which clearly implies $\gamma^\circ(x) \leq \gamma^\circ(y)$.

$\gamma^\circ$ is idempotent. It suffices to prove that $\gamma^\circ(\gamma^\circ(x)) \leq \gamma^\circ(x)$, and to prove this inequality it suffices to prove that every element of $P$ which is way below the left hand side is below the right hand side. So let $u \ll \gamma^\circ(\gamma^\circ(x)) = \bigvee \gamma(\downarrow \gamma^\circ(x))$. The join is directed, thus there exists $v \in \downarrow \gamma^\circ(x)$ such that $u \leq \gamma(v)$. Now $v \ll \gamma^\circ(x) = \bigvee \gamma(\downarrow x)$, where the join is directed, thus $v \leq \gamma(w)$ for some $w \ll x$. It follows that $u \leq \gamma(v) \leq \gamma(w) \leq \bigvee \gamma(\downarrow x) = \gamma^\circ(x)$.

We have proved that $\gamma^\circ$ is a closure operator on $P$.

Evidently $\gamma^\circ \leq \gamma$.

The closure operator $\gamma^\circ$ is Scott-continuous. Since $P$ is a domain, we will prove that $\gamma^\circ$ is Scott-continuous when we prove that $\bigvee \gamma^\circ(\downarrow x) = \gamma^\circ(x)$ for every $x \in P$. It suffices to prove the inequality $\geq$. By the definition of $\gamma^\circ$ we have

\[
\gamma^\circ(x) = \bigvee_{u \ll x} \gamma(u), \quad (3)
\]

\[
\bigvee \gamma^\circ(\downarrow x) = \bigvee_{v \ll x} \gamma^\circ(v) = \bigvee_{v \ll x} \bigvee_{u \ll v} \gamma(u). \quad (4)
\]

Given any $u$ way below $x$, there exists, because of the interpolation property, an element $v \in P$ such that $u \ll v \ll x$, which shows that the term $\gamma(u)$ in the join in (3) appears also in the double join in (4). This proves the inequality $\bigvee \gamma^\circ(\downarrow x) \geq \gamma^\circ(x)$.

Let $\beta$ be a Scott-continuous closure operator on $P$ such that $\beta \leq \gamma$. Then for every $x \in P$ we have $\beta(x) = \bigvee \beta(\downarrow x) \leq \bigvee \gamma(\downarrow x) = \gamma^\circ(x)$.

We conclude that $\gamma^\circ = \text{sc } \gamma$.

For the second construction, that of a directed-closed closure system generated by a closure system in a domain, we have to introduce an operation.

For each subset $X$ of a dcpo $P$ we let $\text{dj}(X)$ denote the set of the joins of all directed subsets of $X$. We have $X \subseteq \text{dj}(X)$ because one-element sets are directed, thus the mapping $\text{dj}: \mathcal{P}P \to \mathcal{P}P$ is ascending. It is clear that the mapping $\text{dj}$ is increasing. But $\text{dj}$ is in general not idempotent; more often than not it is very far from being a closure operator. The following proposition thus comes as a slight surprise.

**Proposition 19.** If $C$ is a closure system in a domain $P$, then $\text{dclsys}(C) = \text{dj}(C)$; therefore, if $X$ is any subset of $P$, then $\text{dclsys}(X) = \text{dj}(\text{clsys}(X))$.

**Proof.** Let $P$ be a domain, $C$ a closure system in $P$, and $\gamma$ a closure operator on $P$ with $\text{Fix}(\gamma) = C$. Since $\text{dclsys}(C)$ contains $C$ and is closed under directed joins, it contains $\text{dj}(C)$. On the other hand, $\text{dclsys}(C) = \text{dclsys}(\text{Fix}(\gamma)) = \text{Fix}(\text{sc } \gamma) = (\text{sc } \gamma)(P)$.
is the set of the closures \((sc\gamma)(x) = \bigvee\gamma(\downarrow x)\) for all \(x \in P\). Since for each \(x \in P\) the set \(\gamma(\downarrow x)\) is a directed subset of \(C\), its join belongs to \(dj(C)\); this proves the inclusion \(dclsys(C) \subseteq dj(C)\).

If \(X\) is any subset of \(P\), then applying \(dclsys\) to \(X \subseteq clsys(X) \subseteq dclsys(X)\) we get \(dclsys(X) = dclsys(clsys(X)) = dj(clsys(X))\). \(\Box\)

Proposition 19 generalizes Theorem 4-1.22 in Lattice Theory: Special Topics and Applications [3] (LT-STA-2).

5 The frame of nuclei on a preframe

In this section we carry out the project that is only sketched in broad outline at the end of Section 3 in Escardó [1].

A preframe is a dcpo \(P\) that is also a meet-semilattice, in which binary meets distribute over directed joins; that is, the directed distributive law holds:

\[ x \land \bigvee Y = \bigvee_{y \in Y} (x \land y), \quad x \in P, \ directed \ Y \subseteq P. \]

Note that if \(x \in P\) and \(Y \subseteq P\) is directed, then also \(\{x \land y \mid y \in Y\}\) is directed.

For a while, let \(P\) be any meet-semilattice.

In the poset \(Fun(P)\) of all endofunctions on \(P\) ordered pointwise, any two endofunctions \(\gamma\) and \(\delta\) have a meet \(\gamma \land \delta\), which is calculated pointwise, and the following is true:

1. If \(\gamma\) and \(\delta\) are ascending, so is \(\gamma \land \delta\).
2. If \(\gamma\) and \(\delta\) are increasing, so is \(\gamma \land \delta\).
3. If \(\gamma\) and \(\delta\) are closure operators, so is \(\gamma \land \delta\).
4. If \(\gamma\) and \(\delta\) preserve binary meets, so does \(\gamma \land \delta\).

Properties (i) and (ii) are easily verified. For (iii), assume \(\beta\) and \(\gamma\) are closure operators on \(P\) and put \(\beta = \gamma \land \delta\). Then \(\beta\) is increasing and ascending by (i) and (ii). Since

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11Preframes are also known as meet continuous semilattices. See Definition O-4.1 in CLaD [2].
12A meet-semilattice is a poset in which any two elements have a meet, or equivalently, in which every nonempty finite subset has a meet. Some authors require that a meet-semilattice is to have a top element; for instance, see Stone Spaces [4].
13In Escardó [1], a preframe is understood as a dcpo that is also a meet-semilattice with a top element in which binary meets distribute over directed joins. As a preframe is defined here (and as a meet continuous semilattice is defined in CLaD [2]), it is not required to possess a top element. This additional generality seems trivial, but it isn’t.
14Equivalently, the meet shift \(P \to P : y \mapsto x \land y\) is Scott-continuous for every \(x \in P\). This is the justification for calling the semilattice \(P\) meet continuous.
Finally, to prove (iv), assume that \( \gamma \) and \( \delta \) preserve binary meets and let \( x, y \in P \); then
\[
(\gamma \land \delta)(x \land y) = \gamma(x \land y) \land \delta(x \land y) = \gamma(x) \land \gamma(y) \land \delta(x) \land \delta(y) \\
= (\gamma \land \delta)(x) \land (\gamma \land \delta)(y).
\]
Because of (i), (ii), and (iii), in the posets \( \text{Precl}(P) \) and \( \text{Cl}(P) \) all binary meets exists and they are calculated pointwise.

**Proposition 20.** If \( \gamma \) and \( \delta \) are closure operators on a meet-semilattice \( P \), and \( \gamma \land \delta \) is their meet in \( \text{Cl}(P) \), then \( \text{Fix}(\gamma \land \delta) = \{ x \land y \mid x \in \text{Fix}(\gamma), y \in \text{Fix}(\delta) \} \).

**Proof.** If \( z \in \text{Fix}(\gamma \land \delta) \), then \( z = (\gamma \land \delta)(z) = \gamma(z) \land \delta(z) \), where \( \gamma(z) \in \gamma(P) = \text{Fix}(\gamma) \) and \( \delta(z) \in \delta(P) = \text{Fix}(\delta) \).

If \( x \in \text{Fix}(\gamma) \) and \( y \in \text{Fix}(\delta) \), then \( (\gamma \land \delta)(x \land y) = \gamma(x \land y) \land \delta(x \land y) \leq \gamma(x) \land \delta(y) = x \land y \), and it follows that \( x \land y \in \text{Fix}(\gamma \land \delta) \).

A closure operator on the meet-semilattice \( P \) that preserves binary meets (hence preserves nonempty finite meets\(^{15}\)) is called a **nucleus** on \( P \). A preclosure map on \( P \) that preserves binary meets is called a **prenucleus** on \( P \). A map \( \gamma: P \to P \) is a prenucleus iff it is ascending and preserves binary meets, and it is a nucleus iff it is ascending and idempotent and preserves binary meets (in both cases \( \gamma \) is increasing since it preserves binary meets). If \( \gamma \) is a prenucleus, and \( P \) possesses a top element \( \top \), then \( \gamma(\top) = \top \) because \( \gamma \) is ascending; that is, if the empty meet\(^{16}\) in \( P \) exists, then \( \gamma \) preserves it.

We let \( \text{Prenuc}(P) \) and \( \text{Nuc}(P) \) denote the pointwise-ordered sets of all prenuclei resp. all nuclei on a meet-semilattice \( P \). The fixpoint set of a nucleus on \( P \) shall be called a **nuclear system** in \( P \), and the poset of all nuclear systems in \( P \), ordered by inclusion, shall be denoted by \( \text{NucSys}(P) \).

From (i)–(iv) above it follows that the pointwise meet of two prenuclei on the meet-semilattice \( P \) is a prenucleus on \( P \), and that the pointwise meet of two nuclei on \( P \) is a nucleus on \( P \). Therefore, in the poset \( \text{Prenuc}(P) \) all binary meets exist and they are calculated pointwise, and the same is true for the poset \( \text{Nuc}(P) \).

From now on let \( P \) be a preframe.

How about the joins of sets of nuclei on the preframe \( P \)? They always exist, and they are calculated in the complete lattice \( \text{Cl}(P) \) of all closure operators on \( P \). We give a slightly more general result.

**Theorem 21.** Let \( P \) be a preframe, and let \( \Gamma \) be a set of prenuclei on \( P \); then the closure operator \( \bar{\Gamma} \), generated by the set \( \Gamma \) of preclosure maps, is a nucleus. In particular, if \( \Gamma \) is a set of nuclei on \( P \), then the join \( \bigvee \Gamma \), taken in the complete lattice \( \text{Cl}(P) \), is a nucleus.

\(^{15}\) A “nonempty finite meet” is short for a “meet of a nonempty finite set”.

\(^{16}\) When we say “the empty meet” we mean “the meet of the empty set”.

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Proof. Put \( \delta = \Gamma \). Let \( x, y \in P \). Since \( \delta \) is increasing, we have \( \delta(x \land y) \leq \delta(x) \land \delta(y) \); we must prove that the converse inequality also holds.

We will first prove the weaker assertion \( x \land \delta(y) \leq \delta(x \land y) \). Let \( A \) be the set of all \( z \in P \) such that \( x \land z \leq \delta(x \land y) \). The set \( A \) contains \( y \), and it is closed under directed joins by directed distributivity. For any \( z \in A \) and any \( \gamma \in \Gamma \) we have

\[
x \land \gamma(z) \leq \gamma(x) \land \gamma(z) = \gamma(x \land z) \leq \gamma(\delta(x \land y)) = \delta(x \land y),
\]

hence \( \gamma(z) \in A \); thus \( A \) is closed under \( \Gamma \). The induction principle gives \( \delta(y) \in A \).

Now we substitute \( \delta(x) \) for \( x \) in \( x \land \delta(y) \leq \delta(x \land y) \) and get

\[
\delta(x) \land \delta(y) \leq \delta(\delta(x) \land y) \leq \delta(\delta(x \land y)) = \delta(x \land y),
\]

where the second inequality holds because \( \delta(x) \land y \leq \delta(x \land y) \) and \( \delta \) is increasing. \( \square \)

Corollary 22. For any preframe \( P \), the subset \( \text{Nuc}(P) \) of \( \text{Cl}(P) \) is closed under arbitrary joins in the complete lattice \( \text{Cl}(P) \), hence it is a complete lattice.\(^{17}\) Also, \( \text{Nuc}(P) \) is closed under binary meets (and hence under nonempty finite meets) in \( \text{Cl}(P) \), since binary meets in \( \text{Nuc}(P) \) are calculated pointwise, same as they are calculated in \( \text{Cl}(P) \).

If \( P \) possesses a top element \( \top \), then the top nucleus is the same as the top closure operator, which is the constant map \( P \to P : x \mapsto \top \). However, if \( P \) does not have a top element, then the top nucleus might be strictly smaller than the top closure operator. That is, though a subset \( \text{Nuc}(P) \) of \( \text{Cl}(P) \) is closed under all joins and also under all nonempty finite meets, both taken in the complete lattice \( \text{Cl}(P) \), it might not be closed under the empty meet taken in \( \text{Cl}(P) \). This can already happen in a finite preframe. Since every finite directed set has a greatest element, which is its join, every finite poset is a dcpo and every finite meet-semilattice is a preframe. The meet-semilattice \( P_1 \) in the left panel of Figure 1 is the simplest possible example: there are four closure operators corresponding to the four closure systems exhibited in the right panel, while there is only one nucleus, namely the identity map.

Since \( \text{Nuc}(P) \) is an interior system in the complete lattice \( \text{Cl}(P) \), for every closure operator \( \gamma \) on the preframe \( P \) there exists the largest nucleus below \( \gamma \), the nuclear core \( \text{nuc} \gamma \) of the closure operator \( \gamma \).

A frame is a complete lattice \( L \) in which finite meets distribute over arbitrary joins, which means that the following **infinite distributivity law** holds in \( L \):

\[
x \land \bigvee Y = \bigvee_{y \in Y} (x \land y), \quad x \in L, \ Y \subseteq L.
\]

\(^{17}\)Here we see that not requiring that preframes have top elements is not so innocent as it seems. For a preframe \( P \) with a top element it is trivial that there is a greatest nucleus on \( P \), namely the constant map sending every element of \( P \) to its top element. In contrast, the existence of a greatest nucleus on a preframe \( P \) which lacks a top element is a quite nontrivial matter; and the greatest nucleus on \( P \) does exist, since it is the top element of the complete lattice \( \text{Nuc}(P) \).

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Figure 1. The only nuclear system in the preframe $P_1$ is the whole $P_1$
(represented by the black dot in the right panel), so the only nucleus on $P_1$
is the identity map, which is different from the top closure operator on $P_1$.

As an ordered structure, a frame is the same thing as a complete Heyting algebra.
The difference is in the structural features that are perceived as basic and have to be
preserved by morphisms. For frames the basic operations are finite meets and arbitrary
joins, and morphisms of frames are maps that preserve these operations; correspond-
ingly, a subframe is a subset of a frame closed under finite meets and arbitrary joins,
equipped with the frame structure induced from the ‘ambient’ frame. A complete Heyt-
ing algebra has another basic operation, the implication $\Rightarrow$ (also known as the relative
pseudocomplementation); a morphism of complete Heyting algebras is a morphism of
frames that in addition preserves implications.

**Proposition 23.** For any preframe $P$ the complete lattice $\text{Nuc}(P)$ is a frame.

**Proof.** Let $\beta \in \text{Nuc}(P)$ and $\Gamma \subseteq \text{Nuc}(P)$, and write $\delta := \bigvee \Gamma$, $\delta' := \bigvee_{\gamma \in \Gamma} (\beta \land \gamma)$ (joins taken in $\text{Nuc}(P)$, hence in $\text{Cl}(P)$). We must show that $\beta \land \delta = \delta'$. The inequality $\beta \land \delta \geq \delta'$ holds because $\beta \land \delta \geq \beta \land \gamma$ for every $\gamma \in \Gamma$.

To prove the other inequality, let $x \in P$ and put $A := \{ z \in P \mid \beta(x) \land z \leq \delta'(x) \}$. Evidently $x \in A$, and $A$ is closed under directed joins by directed distributivity in $P$.

In order to see that $A$ is closed under $\Gamma$, we consider any $\gamma \in \Gamma$ and any $z \in A$, exhibit the following chain of equalities and inequalities,

\[
\beta(x) \land \gamma(z) = \beta(x) \land \gamma \beta(x) \land \gamma(z) = \beta(x) \land \gamma(\beta(x) \land z) \leq \beta(x) \land \gamma(\delta'(x)) \\
\leq \beta(\delta'(x)) \land \gamma(\delta'(x)) = (\beta \land \gamma)(\delta'(x)) \\
= \delta'(x),
\]

and then from the inequality between the first and the last expression in the chain
conclude that $\gamma(z) \in A$. By the induction principle it then follows that $\delta(x) \in A$,
that is, that $(\beta \land \delta)(x) = \beta(x) \land \delta(x) \leq \delta'(x)$.

In conclusion of this nuclear-themed section we take a quick look at Scott-continuous
nuclei on a preframe. Given a preframe $P$, we denote by $\text{ScNuc}(P)$ the poset of all
Scott-continuous nuclei on $P$, ordered pointwise.
Theorem 24. Let $\Gamma$ be a set of Scott-continuous prenuclei on a preframe $P$. The pointwise join $\delta := \bigvee \Gamma^*$ is a Scott-continuous nucleus on $P$. Moreover, $\delta$ is the closure operator on the dcpo $P$ that is generated by the set $\Gamma$ of preclosure maps on $P$.

If $\Gamma$ is a set of Scott-continuous nuclei on $P$, then the pointwise join of $\Gamma^*$ is the join of $\Gamma$ in $\text{ScNuc}(P)$ as well as in $\text{Nuc}(P)$ and in $\text{Cl}(P)$. The set $\text{ScNuc}(P)$ is therefore an interior system in the complete lattice $\text{Nuc}(P)$ and also in the complete lattice $\text{Cl}(P)$, and so is itself a complete lattice since it is a subposet of $\text{Nuc}(P)$ and hence of $\text{Cl}(P)$.

Proof. According to Theorem 13, $\delta$ is a Scott-continuous closure operator on $P$, and it is also the closure operator on $P$ generated by the set $\Gamma$ of preclosure maps. By Theorem 21 it then follows that $\delta$ is a nucleus. □

Proposition 25. The pointwise meet of two Scott-continuous nuclei on a preframe $P$ is a Scott-continuous nucleus on $P$.

Proof. Let $\gamma$ and $\delta$ be Scott-continuous nuclei on $P$. We know that the pointwise meet $\gamma \land \delta$ is a nucleus, so it remains to prove that the meet is Scott-continuous.

Let $Y$ be a directed subset of $P$; we shall prove that $(\gamma \land \delta)(\bigvee Y) = \bigvee(\gamma \land \delta)(Y)$. It suffices to prove the inequality $\leq$. We calculate:

\[
(\gamma \land \delta)(\bigvee Y) = \gamma(\bigvee Y) \land \delta(\bigvee Y) = \bigvee(\gamma(y) \land \delta(y))
\]

the last equality holds because of directed distributivity. Now if $y_1, y_2 \in Y$, then there exists $y \in Y$ such that $\{y_1, y_2\} \subseteq y$, and it follows that $\gamma(y_1) \land \delta(y_2) \leq \gamma(y) \land \delta(y) = (\gamma \land \delta)(y) \leq \bigvee(\gamma \land \delta)(Y)$. The last join in (5) is therefore $\leq \bigvee(\gamma \land \delta)(Y)$, and we have the desired inequality. □

The following is a straightforward consequence of Theorem 24 and Proposition 25.

Corollary 26. Let $P$ be a preframe. The subposet $\text{ScNuc}(P)$ of the frame $\text{Nuc}(P)$ is in $\text{Nuc}(P)$ closed under all joins and under binary meets, therefore it is itself a frame, whose arbitrary joins and nonempty finite meets are inherited from $\text{Nuc}(P)$.

It is an open question whether there exists a preframe $P$ such that the top element of $\text{ScNuc}(P)$ is different (hence strictly below) the top element of $\text{Nuc}(P)$. If such a preframe exists, it lacks a top element, and it is by necessity infinite since every increasing function between finite poset is Scott-continuous and so every nucleus on a finite preframe (that is, on a finite meet-semilattice) is Scott-continuous.

6 Nuclei on frames

In this section we consider frames, as preframes with special properties. For starters we specialize Proposition 23 to frames.

Corollary 27. For any frame $L$ the complete lattice $\text{Nuc}(L)$ is a frame.
Proof. Suppose nucleus with Fix($y$) point of $\gamma C$ only if Let Proposition 29.

Let $a$ the infinite distributivity implies that ($\gamma C$) comparisons well with the long-winded proof of Proposition II.2.5 in Stone Spaces [4].

From now on, to th end of the section, we let $L$ be a frame.

Arbitrary joins and finite meets in the complete lattice Nuc($L$) are calculated in the complete lattice Cl($L$), as in any preframe; but, $L$ being a frame, more is true.

**Proposition 28.** Let $L$ be a frame. Arbitrary meets in the complete lattice Nuc($L$) are calculated pointwise, same as they are calculated in the complete lattice Cl($L$). As arbitrary joins in Nuc($L$) also are calculated in Cl($L$), the complete lattice Nuc($L$) is a sub-(complete lattice) of the complete lattice Cl($L$).

**Proof.** We have to prove that for any set $\Gamma$ of nuclei its pointwise meet $\alpha := \bigwedge \Gamma$, which is a closure operator, preserves binary meets; but this is a straightforward consequence of the associativity-cum-commutativity of arbitrary meets in the complete lattice $L$. □

We have mentioned that a frame is a complete Heyting algebra: for any two elements $a, b \in L$ there exists the relative pseudo-complement of $a$ with respect to $b$, which is the unique element $(a \Rightarrow b) \in L$ such that

$$x \land a \leq b \iff x \leq (a \Rightarrow b) \quad \text{ for every } x \in L;$$

the infinite distributivity implies that $(a \Rightarrow b) = \bigvee \{x \mid x \land a \leq b\}$.

The following proposition characterizes nuclear systems in a frame

**Proposition 29.** Let $L$ be a frame. A subset $C$ of $L$ is a nuclear system in $L$ if and only if $C$ is a closure system in $L$ (i.e., it is closed under arbitrary meets), and $x \in L$, $y \in C$ together imply $(x \Rightarrow y) \in C$.

**Proof.** Suppose $C$ is a nuclear system. Then $C$ is a closure system. Let $\gamma$ be the nucleus with Fix($\gamma$) = $C$. Let $x \in L$ and $y \in C$; we have to prove that $x \Rightarrow y$ is a fixed point of $\gamma$. It suffices to prove that $\gamma(x \Rightarrow y) \leq (x \Rightarrow y)$. Since $x \land (x \Rightarrow y) \leq y$ and $\gamma(y) = y$ we have

$$x \land \gamma(x \Rightarrow y) \leq \gamma(x) \land \gamma(x \Rightarrow y) = \gamma(x \land (x \Rightarrow y)) \leq \gamma(y) = y,$$

and by the defining property of $x \Rightarrow y$ it follows that $\gamma(x \Rightarrow y) \leq (x \Rightarrow y)$, as required.

Conversely, suppose that $C$ is a closure system, and that $x \in L$, $y \in C$ always imply $(x \Rightarrow y) \in C$. Let $\gamma$ be the closure operator with Fix($\gamma$) = $\gamma(L)$ = $C$; we shall prove that $\gamma$ preserves binary meets. We need only prove that $x \land \gamma(y) \leq \gamma(x \land y)$ for all $x, y \in L$.

Let $z := (x \Rightarrow \gamma(x \land y)) \in C$. From $x \land y \leq \gamma(x \land y)$ we get $y \leq (x \Rightarrow \gamma(x \land y)) = z$; but then $\gamma(y) \leq \gamma(z) = z$, whence $x \land \gamma(y) \leq x \land z \leq \gamma(x \land y)$.

\[18\] A nuclear system in a frame (alias locale) is also known as a sublocale.

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For any two subsets $A$ and $B$ of a frame $L$ we shall write

$$(A \Rightarrow B) := \{a \Rightarrow b \mid a \in A, b \in B\}.$$ 

Using this notation, Proposition 29 says that a closure system $C$ in the frame $L$ is a nuclear system iff $(L \Rightarrow C) \subseteq C$ (which implies $(L \Rightarrow C) = C$).

Every subset $X$ of a complete lattice generates the smallest closure system $\text{clsys}(X)$ in the lattice that contains $X$; the closure system $\text{clsys}(X)$ consists of the meets of all subsets of $X$. Likewise every subset $X$ of the frame $L$ generates the smallest nuclear system $\text{nucsys}(X)$ in $L$ that contains $X$; can we somehow construct $\text{nucsys}(X)$?

**Proposition 30.** If $X$ is any subset of a frame $L$, then

$$\text{nucsys}(X) = \text{clsys}(L \Rightarrow X).$$

**Proof.** The endofunction $(L \Rightarrow -) : X \mapsto (L \Rightarrow X)$ on $\mathcal{P}L$ is a closure operator on the powerset lattice $\mathcal{P}L$: it is ascending because $(\top \Rightarrow x) = x$ for every $x \in X$; it is evidently increasing; and it is idempotent because $(a \Rightarrow (b \Rightarrow x)) = ((a \wedge b) \Rightarrow x)$ for all $a, b \in L$ and every $x \in X$. In view of Proposition 29 the closure operator $\text{nucsys}$ on $\mathcal{P}L$ is the join, in the complete lattice $\text{Cl}(\mathcal{P}L)$, of the closure operators $\text{clsys}$ and $(L \Rightarrow -)$.

Let $X$ be any subset of $L$. We shall prove the inclusion

$$(L \Rightarrow \text{clsys}(X)) \subseteq \text{clsys}(L \Rightarrow X),$$

from which it will follow that $\text{clsys}(L \Rightarrow -) = \text{clsys} \lor (L \Rightarrow -) = \text{nucsys}$. Consider a general element of $\text{clsys}(X)$, which is of the form $\bigwedge_{i \in I} x_i$ for some elements $x_i (i \in I)$ of the set $X$; also let $y \in L$. Then

$$(y \Rightarrow \bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (y \Rightarrow x_i) \in \text{clsys}(L \Rightarrow X),$$

which proves the asserted inclusion. $\square$

Mark that we do not obtain a shortcut when constructing $\text{nucsys}(C)$ for a closure system $C$ since we still have to construct $\text{nucsys}(C)$ as $\text{clsys}(L \Rightarrow C)$, which resists simplification.

However, there is an important special case where $\text{nucsys}(X)$ does simplify.

**Proposition 31.** If $x$ is any element of a frame $L$, then

$$\text{nucsys}\{x\} = (L \Rightarrow x).$$

**Proof.** It suffices to prove that $L \Rightarrow x$ is already closed under all meets, so that $\text{nucsys}\{x\} = \text{clsys}(L \Rightarrow x) = (L \Rightarrow x)$. And indeed, if $y_i, i \in I$, are any elements of $L$, then $\bigwedge_{i \in I} (y_i \Rightarrow x) = ((\bigvee_{i \in I} y_i) \Rightarrow x) \in (L \Rightarrow x)$. $\square
We are not yet satisfied. Now that we have constructed $\text{nucsys}(X)$, the nuclear system generated by a subset $X$ of a frame, we are curious what the corresponding nucleus $\text{nuc}_X$ looks like, the one whose fixpoint set is $\text{nucsys}(X)$.

**Proposition 32.** If $X$ is any subset of a frame $L$, then

$$\text{nuc}_X(y) = \bigwedge_{x \in X} ((y \Rightarrow x) \Rightarrow x) \quad \text{for } y \in L.$$  

**Proof.** For every $y \in L$ we have

$$\text{nuc}_X(y) = \bigwedge \text{nucsys}(X) \uparrow y = \bigwedge (\text{clsys}(L \Rightarrow X)) \uparrow y = \bigwedge (L \Rightarrow X) \uparrow y.$$  

Fix $x \in X$, and let $u \in L$. Then $y \leq (u \Rightarrow x)$ iff $u \wedge y \leq x$ iff $u \leq (y \Rightarrow x)$, and for every $u \leq (y \Rightarrow x)$ we have $(u \Rightarrow x) \geq (y \Rightarrow x) \Rightarrow x$. The meet of all terms of the form $(u \Rightarrow x)$ in $\bigwedge ((L \Rightarrow X) \uparrow y)$ is $((y \Rightarrow x) \Rightarrow x)$. Now we release $x$ to run through the whole set $X$ and obtain the formula for $\text{nuc}_X(y)$ given in the proposition. \hfill $\square$

Let us exhibit two special cases of Proposition $32$, one with $X = \{x\}$, and another one where $X$ is a closure system.

**Corollary 33.** If $x$ is any element of a frame $L$, then

$$\text{nuc}_{\{x\}}(y) = ((y \Rightarrow x) \Rightarrow x) \quad \text{for } y \in L.$$  

**Corollary 34.** If $\gamma$ is a closure operator on a frame $L$, then

$$\text{nuc}_{\gamma(L)}(y) = \bigwedge_{u \in L} ((y \Rightarrow \gamma(u)) \Rightarrow \gamma(u)) \quad \text{for } y \in L.$$  

The nucleus $\text{nuc} \gamma := \text{nuc}_{\gamma(L)}$ is the nuclear core of the closure operator $\gamma$, that is, it is the greatest of all nuclei on the frame $L$ that are below $\gamma$.

It is high time we introduce, for each $x \in L$, the **regular nucleus** $r_x := \text{nuc}_{\{x\}}$ on $L$; by Proposition $31$ the fixpoint set of the regular nucleus $r_x$ is $L \Rightarrow x$. The mapping $L \to \text{NucSys}(L) : x \mapsto (L \Rightarrow x)$ is injective because $x$ is the least element of $L \Rightarrow x$, and it follows that also the mapping $L \to \text{Nuc}(L) : x \mapsto r_x$ is injective.

**Proposition 35.** If $\nu$ is a nucleus on a frame $L$ and $x \in L$, then $\nu \leq r_x$ iff $x \in \text{Fix}(\nu)$. In particular, if $x, y \in L$, then $r_x \leq r_y$ iff $y \in (L \Rightarrow x)$.

**Proof.** Let $\nu$ be a nucleus on a frame $L$ and $x \in L$. Then $\nu \leq r_x$ iff $\text{Fix}(r_x) \subseteq \text{Fix}(\nu)$, iff $(L \Rightarrow x) \subseteq \text{Fix}(\nu)$, iff $x \in \text{Fix}(\nu)$; the last equivalence holds since $x = (\top \Rightarrow x)$ is in $(L \Rightarrow x)$, and because $x \in \text{Fix}(\nu)$ implies $(L \Rightarrow x) \subseteq \text{Fix}(\nu)$. \hfill $\square$

By Proposition $32$ the set of all regular nuclei on $L$ meet-generates the complete lattice of all nuclei on $L$. Moreover, if $\nu$ is a nucleus and $X$ is a subset of $L$, then $\nu = \bigwedge_{x \in X} r_x$ iff $X$ generates the nuclear system $\text{Fix}(\nu)$; in particular, $\nu = \bigwedge \{r_x \mid x \in \text{Fix}(\nu)\}$.  

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The sub-(complete lattice) \( \text{Nuc}(L) \) of \( \text{Cl}(L) \) is not only an interior system in \( \text{Cl}(L) \), which gives us for every closure operator on \( L \) the largest nucleus below it, it is also a closure system in \( \text{Cl}(L) \), and so for any given closure operator \( \gamma \) on \( L \) there exists the least nucleus \( \nu \) above it, which is the (pointwise) meet of all nuclei above \( \gamma \); but since \( \text{Nuc}(L) \) is meet-generated by the regular nuclei, it follows that \( \nu = \bigwedge \{ r_x \mid x \in L, \ \gamma \leq r_x \} \). For a closure operator \( \gamma \) on \( L \) and the least nucleus \( \nu \) above it, let \( C := \text{Fix}(\gamma) \) and \( N := \text{Fix}(\nu) \). Since for every \( x \in L \) we have \( \nu \leq r_x \) iff \( \gamma \leq r_x \), by Proposition \( 35 \) it follows that \( N := \{ x \in L \mid (L \Rightarrow x) \subseteq C \} \); since \( N \subseteq C \) we have also \( N = \{ x \in C \mid (L \Rightarrow x) \subseteq C \} \), and therefore \( \nu = \bigwedge \{ r_x \mid x \in C, \ \gamma \leq r_x \} \).

**Proposition 36.** If \( \gamma \) is a closure operator on a frame \( L \) and \( \nu \) is the least nucleus above \( \gamma \), then

\[
\nu = \bigwedge \{ r_x \mid x \in C, \ \gamma \leq r_x \}, \quad \text{Fix}(\nu) = \{ x \in C \mid (L \Rightarrow x) \subseteq \text{Fix}(\gamma) \},
\]

where the meet in the formula for \( \nu \) is calculated pointwise.

### 7 The Hofmann–Mislove–Johnstone theorem

In [1] Escardó shows off the utility of join induction by using it in a proof of the Hofmann–Mislove–Johnstone theorem. In this section we use the HMJ theorem as a training wheel on which we try out an application of the obverse induction principle. The proof of the HMJ theorem is spread through proofs of three lemmas, with parts of it reasoned out in the connecting text; the short concluding reasoning then ties everything together. The obverse induction principle gets its chance in the proof of Lemma \( 40 \) where it performs admirably, simplifying proofs of the corresponding results in Johnstone \( 5 \) and Escardó \( 11 \) and shortening them to five easy lines of the proof proper (after the introductory line).

**Theorem 37 (Johnstone).** The compact fitted quotient frames of any frame are in order-reversing bijective correspondence\(^{19}\) with the Scott-open filters of the frame.

If this sounds all Greek to you, do not panic; everything will be explained below—slowly and in sickening detail—before we embark on the actual proof of the theorem, which will be short and quite painless.

Frames we have already defined: a frame is a complete lattice in which finite meets distribute over arbitrary joins, and a frame morphism is a mapping from a frame to a frame that preserves finite meets and arbitrary joins.

Henceforward let \( L \) be an arbitrary frame.

\(^{19}\) An “order-reversing bijective correspondence” means an antiisomorphism of posets, that is, a bijection between posets such that both the bijection itself and its inverse are order-reversing.
Consider a nucleus $\gamma$ on $L$. The subposet $\gamma(L) = \text{Fix}(\gamma)$ of $L$ is a complete lattice in which meets are calculated in $L$ and the join of a subset $S$ of $\gamma(L)$ is $\bigvee S = \gamma(\bigvee S)$. Moreover, the infinite distributivity law holds in the complete lattice $\gamma(L)$, so it is in fact a frame: given any $x \in \gamma(L)$ and any $Y \subseteq \gamma(L)$, we have $x \wedge \bigvee Y = \gamma(x) \wedge \gamma(\bigvee Y) = \gamma(x \wedge \bigvee Y) = \gamma(\bigvee_{y \in Y} (x \wedge y)) = \bigvee_{y \in Y} \gamma(x \wedge y)$. The restriction $\gamma': L \to \gamma(L)$ of $\gamma$ preserves finite meets because $\gamma$ preserves them, and it preserves joins because $\gamma$ is a closure operator; thus $\gamma'$ is a surjective frame morphism. This is why the nuclear system $\gamma(L)$ is also called a quotient frame of $L$.

Let $f: L \to K$ be a morphism of frames. Since $f$ preserves all joins, it has a right adjoint $g: K \to L$, which preserves all meets, thus the closure operator $\gamma := gf$ on $L$ preserves finite meets, that is, it is a nucleus on $L$. Now suppose that $f$ is surjective, and hence $g$ is injective and $fg = \text{id}_K$. Denoting by $\gamma': L \to \gamma(L) = g(K)$ the restriction of $\gamma$ and by $h: g(K) \to K$ the restriction of $f$, we have an isomorphism $h$ of frames such that $h\gamma' = f$. Therefore, every surjective frame morphism from $L$ is isomorphic to an ‘inner’ surjective frame morphism from $L$ associated with a nucleus on $L$.

A frame is said to be compact if its top element is inaccessible by directed joins. Spelled out: a frame $K$, with a top element $\top$, is compact if and only if every directed subset $S$ of $K$ whose join is $\top$ already contains $\top$.

So we now know what is a compact quotient frame. “Fitted” comes next.

Let $a \in L$. The principal ideal $\downarrow a$ is a frame, the map $f_a: L \to \downarrow a : x \mapsto x \wedge a$ is a surjective frame morphism, and the defining property of $(\Rightarrow)\Rightarrow$ shows that the right adjoint of $f_a$ is the map $g_a: \downarrow a \to L : y \mapsto (a \Rightarrow y)$; the nucleus $a^\circ := g_a f_a$ on $L$ maps $x \in L$ to $a^\circ(x) = (a \Rightarrow (x \wedge a)) = (a \Rightarrow x)$. The nucleus $a^\circ$ is called the open nucleus associated with $a$; the corresponding nuclear system is $a^\circ(L) = (a \Rightarrow L)$. The restriction of the mapping $g_a$ to $\downarrow a \to (a \Rightarrow L)$ is an isomorphism of posets and hence of frames.

A nucleus $\gamma$ on $L$, and the corresponding nuclear system $\gamma(L) = \text{Fix}(\gamma)$, are said to be fitted, if $\gamma$ is a join of open nuclei (with the join taken in the complete lattice Nuc($L$)). We shall denote by $\text{Nuc}_{\text{fit}}(L)$ the set of all fitted nuclei on $L$, and by $\text{Nuc}_{\text{Sysfit}}(L)$ the set of all fitted nuclear systems (that is, fitted quotient frames) on $L$. Subposet $\text{Nuc}_{\text{fit}}(L)$ of Nuc($L$) is a complete lattice because it is evidently closed under joins in Nuc($L$). Correspondingly, the subposet $\text{Nuc}_{\text{Sysfit}}(L)$ of NucSys($L$) is a complete lattice; it is closed under meets in NucSys($L$), and since meets in NucSys($L$) are intersections, $\text{Nuc}_{\text{Sysfit}}(L)$ is a closure system in $\mathcal{P}L$.

Below any nucleus $\gamma \in \text{Nuc}(L)$ there exists the greatest fitted nucleus $\gamma^\phi \in \text{Nuc}_{\text{fit}}(L)$; $\gamma^\phi$ is simply the join of all open nuclei below $\gamma$. The mapping $\gamma \mapsto \gamma^\phi$ is an interior operator on Nuc($L$); it is fittingly called the fitting of nuclei on $L$.

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20 We can verify directly that $a^\circ = (a \Rightarrow \_)$ is a nucleus: it is ascending and increasing; it is idempotent, $(a \Rightarrow (a \Rightarrow x)) = ((a \wedge a) \Rightarrow x) = (a \Rightarrow x)$; it preserves binary meets, $(a \Rightarrow (x \wedge y)) = (a \Rightarrow x) \wedge (a \Rightarrow y)$.

21 Actually, $\text{Nuc}_{\text{fit}}(L)$ is a subframe of Nuc($L$), that is, it is also closed under finite meets, since $\perp^\circ : x \mapsto \top$ is the greatest nucleus, and $a^\circ \wedge b^\circ = (a \vee b)^\circ$ for all $a, b \in L$. 

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Given a nucleus \( \gamma \) on \( L \), which open nuclei on \( L \) are below \( \gamma \)? Lemma 38 has the answer. In the proof of this lemma we are going to use the following inequality satisfied by an endomap \( f \) on the frame \( L \) that preserves binary meets, and hence is increasing: for all \( x, y \in L \), \( f(x \Rightarrow y) \leq (f(x) \Rightarrow f(y)) \). The inequality follows from the inequality between the first and the last expressions in \( f(x) \land f(x \Rightarrow y) = f(x \land (x \Rightarrow y)) \leq f(y) \).

**Lemma 38.** Let \( L \) be a frame. If \( a \in L \) and \( \gamma \in \text{Nuc}(L) \), then \( a^o \leq \gamma \) iff \( \gamma(a) = \top \).

**Proof.** If \( a^o \leq \gamma \), then \( \top = (a \Rightarrow a) = a^o(a) \leq \gamma(a) \). Conversely, if \( \gamma(a) = \top \), then for every \( x \in L \), \( a^o(x) = (a \Rightarrow x) \leq \gamma(a \Rightarrow x) \leq (\gamma(a) \Rightarrow \gamma(x)) = (\top \Rightarrow \gamma(x)) = \gamma(x) \).

We can now write down the following formula for the fitting of a nucleus:

\[
\gamma^\delta = \bigvee \{ a^o \mid \gamma(a) = \top \} \quad \text{for every } \gamma \in \text{Nuc}(L).
\] (6)

So far we completely understand one side of the bijection mentioned in Theorem 37. There is not much left to understand on the other side.

Recall that a filter of a poset \( P \) is a downward directed upper set of \( P \). In our frame \( L \) a filter is an upper set closed under finite meets (including the empty meet, that is, a filter always contains \( \top \)). Every filter \( V \) of \( L \) obeys the *modus ponens* rule: for all \( a, b \in L \), if \( a \in V \) and \( (a \Rightarrow b) \in V \), then \( b \in V \) because \( b \geq a \land (a \Rightarrow b) \in V \).

For any nucleus \( \gamma \) on \( L \), the set \( \gamma^{-1}(\top) \) is a filter of \( L \); we shall call filters of this form **nuclear filters** of \( L \), and will denote by \( \text{NucFilt}(L) \) the poset of all nuclear filters of \( L \) ordered by inclusion. Since

\[
(\bigwedge \Gamma)^{-1}(\top) = \bigcap \{ \gamma^{-1}(\top) \mid \gamma \in \Gamma \} \quad \text{for every } \Gamma \subseteq \text{Nuc}(L)
\]

(recall that all meets of nuclei are calculated pointwise), it follows that \( \text{NucFilt}(L) \) is a closure system in \( \mathcal{P}L \). Indeed, given a set \( \mathcal{V} \) of nuclear filters, let \( \Gamma \) be the set of all nuclei \( \gamma \) such that \( \gamma^{-1}(\top) \in \mathcal{V} \). Since every filter in \( \mathcal{V} \) is of the form \( \gamma^{-1}(\top) \) for some nucleus \( \gamma \) in \( \Gamma \), the intersection \( \bigcap \mathcal{V} = \bigcap_{\gamma \in \Gamma} \gamma^{-1}(\top) = (\bigwedge \Gamma)^{-1}(\top) \) is a nuclear filter.

By definition, a Scott-open subset of a poset is an upper set inaccessible by directed joins. Since every filter is an upper set by definition, a filter is Scott-open if and only if it is inaccessible by directed joins.

We have everything ready to relate compact fitted quotient frames to Scott-open filters. The following lemma is Lemma 4.4 in Escardó [1], which in turn is Lemma 3.4(i) in Johnstone [5]; its proof is almost verbatim as in Escardó [1], which in turn is lifted from Johnstone [5]. Anyway, this lemma is not very deep, it is an immediate consequence of the relationship between joins in a frame and joins in a quotient frame of the frame.

**Lemma 39.** Let \( \gamma \) be a nucleus on a frame \( L \). Then the quotient frame \( \gamma(L) \) is compact if and only if the nuclear filter \( \gamma^{-1}(\top) \) is Scott-open.

**Proof.** \((\Rightarrow)\) Suppose \( \gamma(L) \) is compact, and let \( S \subseteq L \) be directed with \( \bigvee S \in \gamma^{-1}(\top) \). Since \( \bigvee \gamma(S) = \gamma(\bigvee S) = \top \) and \( \gamma(L) \) is compact, there is some \( s \in S \) with \( \gamma(s) = \top \), that is, with \( s \in \gamma^{-1}(\top) \).

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(\iff) Suppose \( \gamma^{-1}(\top) \) is Scott-open, and let \( S \subseteq \gamma(L) \) be directed with \( \bigvee \gamma S = \top \). Since \( \bigvee \gamma S = \gamma(\bigvee S) \), we have \( \bigvee S \in \gamma^{-1}(\top) \), and since \( \gamma^{-1}(\top) \) is Scott-open, there is some \( s \in S \) such that \( s \in \gamma^{-1}(\top) \), that is, such that \( s = \gamma(s) = \top \). \qed

The fitting of nuclei, the interior operator \( \gamma \mapsto \gamma^\phi \) on \( \text{Nuc}(L) \), is a counit of a certain (covariant) Galois connection, which we now proceed to describe.

For every \( \gamma \in \text{Nuc}(L) \) put \( \nabla \gamma := \gamma^{-1}(\top) \), and for every \( S \in \mathcal{P}L \) put \( \Delta S := \bigvee_{s \in S} s^\circ \). For any \( \gamma \in \text{Nuc}(L) \) and any \( S \in \mathcal{P}L \) the chain of equivalences

\[
\Delta S \leq \gamma \iff (\forall s \in S)(s^\circ \leq \gamma) \\
\iff (\forall s \in S)(\gamma(s) = \top) \quad \text{(by Lemma 38)} \\
\iff S \subseteq \nabla \gamma
\]

shows that \( (\Delta, \nabla) \) is a Galois connection \( \mathcal{P}L \rightleftharpoons \text{Nuc}(L) \). By our definitions, \( \mathcal{P}L \) is the set \( \text{Nuc}_{\text{fit}}(L) \) of all fitted nuclei on \( L \), while \( \nabla \text{Nuc}(L) \) is the set \( \text{Nuc}_{\text{Filt}}(L) \) of all nuclear filters of \( L \). From the general properties of Galois connections it at once follows that \( \nabla \Delta \) is an interior operator on \( \text{Nuc}(L) \) and that for any nucleus \( \gamma \), \( \nabla \Delta \gamma = \gamma^\phi \) (see [6]) is the greatest fitted nucleus below \( \gamma \) (all of which we already know), while on the other side, \( \text{nucfilt} := \nabla \Delta \) is a closure operator on \( \mathcal{P}L \), where for any subset \( S \) of \( L \), \( \text{nucfilt}(S) \) is the least nuclear filter of \( L \) that contains \( S \). We have the identities \( \nabla \Delta \nabla = \nabla \) and \( \nabla \Delta \Delta = \Delta \), meaning, respectively, that \( (\gamma^\phi)^{-1}(\top) = \gamma^{-1}(\top) \) for every \( \gamma \in \text{Nuc}(L) \) and that \( \bigvee \{ s^\circ \mid s \in \text{nucfilt}(S) \} = \bigvee \{ s^\circ \mid s \in S \} \) for every \( S \subseteq L \). And, restricting \( \Delta \) to \( \text{Nuc}_{\text{Filt}}(L) \to \text{Nuc}_{\text{fit}}(L) \) and \( \nabla \) to \( \text{Nuc}_{\text{fit}}(L) \to \text{Nuc}_{\text{Filt}}(L) \), we obtain two isomorphisms of complete lattices which are inverses to each other.

At last, here comes the punch line — or should it be the punch lemma?

The following lemma is Lemma 3.4(ii) in Johnstone [5], reappearing as Lemma 4.3(2) in Escardó [1]. We give a short and simple proof, which uses the obverse induction principle instead of ordinals and transfinite induction in Johnstone [5], and instead of the join induction principle in Escardó [1].

**Lemma 40.** Every Scott-open filter of a frame \( L \) is nuclear.

**Proof.** Let \( V \) be a Scott-open filter of a frame \( L \), and let \( \gamma = \Delta V = \bigvee \{ v^\circ \mid v \in V \} \).

The filter \( V \), being Scott open, is inaccessible by directed joins. If \( v \in V \), and \( x \in L \) is such that \( v^\circ(x) = (v \Rightarrow x) \in V \), then \( x \in V \) by modus ponens, which means that \( V \) is inversely closed under \( \{ v^\circ \mid v \in V \} \). By the obverse induction principle, \( V \) is inversely closed under \( \gamma \), so certainly \( \text{nucfilt}(V) = \nabla \gamma = \gamma^{-1}(\top) \subseteq V \). Since also \( V \subseteq \text{nucfilt}(V) \), we see that \( V = \text{nucfilt}(V) \) is a nuclear filter. \qed

After all the preparations, Theorem 37 is easy to prove.

**Proof of Theorem 37.** Let \( \mathcal{F} \) be the poset of all Scott-open filters of \( L \) ordered by inclusion, let \( \mathcal{Q} \) be the subposet of \( \text{NucSys}_{\text{fit}}(L) \) consisting of all compact fitted quotient frames on \( L \), and let \( \mathcal{G} \) be the subposet of \( \text{Nuc}_{\text{fit}}(L) \) consisting of all nuclei \( \gamma \) on \( L \).
such that $\gamma(P) \in Q$. Now Lemma \[40\] and Lemma \[39\] tell us that $F$ is a subposet of $\text{NucFilt}(L)$ and that the isomorphisms of complete lattices

$$ \text{NucFilt}(L) \rightarrow \text{NucFilt}(L) \rightarrow \text{NucSys}_\text{fit}(L) ^\text{op} : V \mapsto \Delta V \mapsto (\Delta V)(L)$$

restrict to isomorphism of posets $F \rightarrow G \rightarrow Q ^\text{op}$. □

8 Doing it with maximal elements

Let $P$ be a dcpo. Since the poset $\text{ClSys}(P)$ of all closure systems in $P$ is a closure system in $\mathcal{P}P$, it is determined by a set of closure rules on $P$. One such set of closure rules is, of course, the full-fledged closure theory consisting of all closure rules obeyed by $\text{ClSys}(P)$. But this closure theory is too large; we want some smaller set of closure rules that determines the closure system $\text{ClSys}(P)$, and moreover, we want a set of closure rules which can be described in terms of the structure of the dcpo $P$.

We shall obtain a suitable set of closure rules using the approach in Francesco Ranzato’s paper \[7\]. We will not follow the exposition in the paper; our treatment will be more streamlined, and we will obtain some results that are not in the paper.

A default closure rule associated with a poset $P$ is a closure rule $B \mapsto c$ on the set $P$ (that is, $B \subseteq P$ and $c \in P$) where $c$ is a maximal lower bound of $B$. We shall denote by $R_{\text{df}}(P)$ the set of all default closure rules associated with a poset $P$. We shall write $B \mapsto_{\text{df}} c$ to mean that $B \mapsto c$ is a default closure rule, that is, that $R_{\text{df}}(P) : B \mapsto c$.

A default closure rule can be reflexive (which means that it has $c \in B$). A default closure rule $B \mapsto c$ is reflexive if and only if $c$ is the least element of $B$. In more detail: if $B \mapsto c$ is a default closure rule and $c \in B$, then $c$ is the least element of $B$; if $B$ has a least element $c$, then $B \mapsto c$ is the unique default closure rule with the body $B$.

**Lemma 41.** If $f$ is a preclosure map on a poset $P$, then $\text{Fix}(f)$ obeys $R_{\text{df}}(P)$.

**Proof.** Let $B \mapsto_{\text{df}} c$ with $B \subseteq \text{Fix}(f)$. For any $b \in B$ we have $f(c) \leq f(b) = b$, thus $f(c)$ is a lower bound of $B$. Since $c \leq f(c)$ and $c$ is a maximal lower bound of $B$, it follows that $f(c) = c \in \text{Fix}(f)$. □

Let $P$ be a poset.

We shall say that $P$ has a ceiling if for every element $x$ of $P$ there exists a maximal element $y$ of $P$ such that $x \leq y$. We shall say that a subset $A$ of $P$ has a ceiling if the subposet $A$ of $P$ has a ceiling. Mark that the empty poset has a ceiling.

We shall say that $P$ is default-enabled if for every subset $X$ of $P$ the set of all lower bounds of $X$ in $P$ has a ceiling (that is, every lower bound of $X$ is below some maximal lower bound of $X$).\[22\] If $P$ is default enabled, then in particular the set $P$ itself, which is the set of all lower bounds of the empty subset, has a ceiling.

\[22\] A default-enabled poset is in Ranzato \[7\] called a relatively maximal lower bound complete poset, which is rather a mothful, so Ranzato shortens it to rmlb-complete poset, which is not very mnemonic.
The following lemma tells us that a default-enabled poset has enough default closure rules associated with it to determine the closure systems in the poset.

**Lemma 42.** Let $P$ be a default-enabled poset. If a subset $C$ of $P$ obeys $R_{df}(P)$, then $C$ is a closure system in $P$.

**Proof.** Let $x$ be an arbitrary element of $P$; it suffices to prove that the set $B := C \uparrow x$ has a least element. The element $x$ is a lower bound of $B$, thus $x \leq u$ for some maximal lower bound of $B$ because $P$ is default-enabled. Then $B \rightarrow_{df} u$, therefore $u \in C$ because $C$ obeys $R_{df}(P)$, whence $u \in C \uparrow x = B$ is the least element of $B$. $\square$

The proofs of Lemma 41 and Lemma 42 correspond to the two parts of the proof of Theorem 4.4 in Ranzato [7] (where Lemma 41 is slightly more general than the first part of Theorem 4.4).

We have the following consequence of Lemma 41 and Lemma 42:

**Proposition 43.** Let $P$ be a default-enabled poset. A subset of $P$ is a closure system in $P$ if and only if it obeys $R_{df}(P)$. Consequently, the family $\text{ClSys}(P)$ of all closure systems in $P$ is a closure system in $\mathcal{P}P$, and so is itself a complete lattice in which all meets are intersections. Also the poset $\text{Cl}(P)$ of all closure operators on $P$, being antiisomorphic to the poset $\text{ClSys}(P)$, is a complete lattice.

We want to cook up for default-enabled posets a theorem that would resemble Theorem 1 for dcpos. With this aim in mind we introduce the following notion:

Let us say that a subset $A$ of a poset $P$ is **default-enabled within $P$** if it satisfies the following two conditions:

(i) the subposet $A$ is default-enabled;

(ii) for every $x \in P$ the set $A \downarrow x$ has a ceiling.

The condition (i) is a property of the structure of the subposet $A$ alone, independent of the rest of the structure of the ‘ambient’ poset $P$, while the condition (ii) prescribes how the subposet $A$ has to ‘sit’ inside the poset $P$.

And here is the theorem mimicking Theorem 1 with it we wander beyond Ranzato [7].

**Theorem 44.** Let $P$ be a default-enabled poset, and let $G$ be a set of preclosure maps on $P$. The set $\text{Fix}(G)$ is a closure system in $P$, and the closure operator $\overline{G}$ on $P$ which has $\text{Fix}(\overline{G}) = \text{Fix}(G)$ is the least closure operator on $P$ that is above $G$.

The following induction principle holds: if a subset $A$ of $P$ is default-enabled within $P$ and is closed under $G$, then it is closed under $\overline{G}$.

**Proof.** For every $g \in G$ the fixpoint set $\text{Fix}(g)$ obeys $R_{df}(P)$ by Lemma 41, thus $\text{Fix}(g) \in \text{ClSys}(P)$ by Lemma 42. Since $\text{ClSys}(P)$ is closed under arbitrary intersections, by Proposition 43 the set $\text{Fix}(G) = \bigcap_{g \in G} \text{Fix}(g)$ is a closure system, so there exists...
a (unique) closure operator $h$ on $P$ which has $\text{Fix}(h) = \text{Fix}(G)$. If $g \in G$, then $h(P) = \text{Fix}(h) \subseteq \text{Fix}(g)$, whence $g(x) \leq g(h(x)) = h(x)$ for every $x \in P$; it follows that $h \geq G$. Let $k$ be a closure operator $\geq G$. For every $g \in G$ we have $gk = k$. Therefore $\text{Fix}(k) = k(P) \subseteq \text{Fix}(g)$ for every $g \in G$; but then $\text{Fix}(k) \subseteq \text{Fix}(G) = \text{Fix}(h)$, which means that $k \geq h$. The closure operator $\bar{G} := h$ has the properties stated in the theorem.

The induction principle.

Assume that $A \subseteq P$ is default-enabled within $P$ and closed under $G$.

Let $G_A$ be the set of restrictions $g_A : A \to A$ of the maps $g \in G$. We obtained a set $G_A$ of preclosure maps on a default-enabled subposet $A$, thus there is (by the first part of the proof above, applied to the poset $A$) a closure operator $h'$ on $A$ such that $\text{Fix}(h') = \text{Fix}(G_A)$, where $\text{Fix}(G_A) = A \cap \text{Fix}(G) = A \cap \text{Fix}(h) \subseteq \text{Fix}(h)$. If $a \in A$, then $a \leq h'(a) \in \text{Fix}(h') = \text{Fix}(G_A) \subseteq \text{Fix}(h)$, thus $h(a) \leq h(h'(a)) = h'(a)$.

We shall prove that for every $a \in A$ also $h(a) \geq h'(a)$, and then we will be able to conclude that $h(a) = h'(a) \in A$.

So let $a \in A$. By assumption $A \downarrow h(a)$ has a ceiling. Since $a \in A \downarrow h(a)$, there exists in $A \downarrow h(a)$ a maximal element $a'$ such that $a \leq a'$. For every $g \in G$ we have $g(a') \leq g(h(a)) = h(a)$ and $g(a') \in A$, thus $g(a') \in A \downarrow h(a)$; now since $a' \leq g(a')$ and $a'$ is maximal in $A \downarrow h(a)$, it follows that $g(a') = a'$. Thus we have $a \leq a'$ in $A$, where $a'$ is fixed by $g_A$ for every $g \in G$, therefore $a'$ is fixed by $h'$, and it follows that $h'(a) \leq h'(a') = a' \leq h(a)$.

\[ \square \]

Every dcpo is a default-enabled poset. First, every nonempty dcpo has a maximal element, by Zorn’s lemma. Next, if $P$ is a dcpo and $x$ is any element of $P$, then the principal filter $\uparrow x$ is a sub-dcpo of $P$ and hence has a maximal element which is also a maximal element of $P$; it follows that $P$ has a ceiling. Finally, if $X$ is any subset of a dcpo $P$, then the set of all lower bounds of $X$ in $P$ is a sub-dcpo of $P$, thus it has a ceiling, and we see that $P$ is default-enabled.

Suppose that a subset $A$ of a dcpo $P$ is closed under directed joins in $P$. If $x$ is any element of $A$, then $A \downarrow x = A \cap \downarrow x$ is the intersection of two sub-dcpos of $P$, thus it is a sub-dcpo of $P$, so it has a ceiling. The sub-dcpo $A$ is default-enabled within $P$.

Theorem 44 therefore specializes to Theorem 11 but we need the axiom of choice to do this. We in fact cannot do the specialization without invoking the axiom of choice, since it is not hard to prove that the assertion that every dcpo has a ceiling, and hence that every nonempty dcpo has a maximal element, implies the axiom of choice (in the theory of sets without the axiom of choice).

Luckily we do not need the help of the axiom of choice in order to specialize Theorem 44 to Theorem 11 since we already proved the latter theorem on its own.

The obvious question to ask at this point is whether the class of default-enabled posets is strictly larger than the class of dcpos. The answer is yes, it is strictly larger: the poset $P_3$ in Figure 2 (reproduced from Ranzato 4) is default-enabled while it is not a dcpo. This poset $P_3$, though it answers the question in the affirmative, is not very exciting, since the only closure operator on it is the identity map. Here is a challenge:
Figure 2. A default-enabled poset which is not a dcpo.

describe a class of interesting default-enabled posets that are far from being dcpos and whose complete lattices of closure operators are quite nontrivial.

Now we are going to travel farther beyond Ranzato [7]. Let $P$ be a meet-semilattice.

For any two elements $a$ and $b$ of $P$ we define the set $(a \Rightarrow b) := \{ x \in P \mid x \land a \leq b \}$, and then define the set $(a \Rightarrow b)$ as the set of all maximal elements of the set $(a \Rightarrow b)$. Note that the set $(a \Rightarrow b)$ is always nonempty as it contains the element $b$; however, $(a \Rightarrow b)$ may not have any maximal elements, so the set $(a \Rightarrow b)$ may be empty.

A nuclear closure rule associated with $P$ is a unary closure rule $b \rightarrow c$, where $c \in (a \Rightarrow b)$ for some $a \in P$. The set of all nuclear closure rules on $P$ shall be denoted by $R_{\text{nuc}}(P)$. We shall write $b \rightarrow_{\text{nuc}} c$ to mean that the closure rule $b \rightarrow c$ is nuclear; that is, $b \rightarrow_{\text{nuc}} c$ is synonymous with $R_{\text{nuc}}(P): b \rightarrow_{\text{nuc}} c$. A subset $X$ of $P$ obeys $R_{\text{nuc}}(P)$ if and only if $(a \Rightarrow x) \subseteq X$ for all $a \in P$ and all $x \in X$.

**Lemma 45.** If $\gamma$ is a prenucleus on a meet-semilattice $P$, then $\text{Fix}(\gamma)$ obeys $R_{\text{nuc}}(P)$.

**Proof.** Suppose that $b \rightarrow_{\text{nuc}} c$ with $b \in \text{Fix}(\gamma)$; we shall prove that $c \in \text{Fix}(\gamma)$. There exists $a \in P$ such that $c \in (a \Rightarrow b)$. Since $c \land a \leq b$ and $\gamma(b) = b$, we have

$$\gamma(c) \land a \leq \gamma(c) \land \gamma(a) = \gamma(c \land a) \leq \gamma(b) = b,$$

thus $\gamma(c) \in (a \Rightarrow b)$. Since $c \leq \gamma(c)$ and $c$ is maximal in $(a \Rightarrow b)$, we have $\gamma(c) = c$. □

A sort of strong converse of Lemma 45 holds if for all elements $a, b$ of a meet-semilattice $P$ the set $(a \Rightarrow b)$ has a ceiling.

**Lemma 46.** Let $P$ be a meet-semilattice in which every set $(a \Rightarrow b)$ with $a, b \in P$ has a ceiling. Let $\gamma$ be a closure operator on $P$. If $(a \Rightarrow b) \subseteq \text{Fix}(\gamma)$ for all $a \in P$ and all $b \in \text{Fix}(\gamma)$, then $\gamma$ preserves binary meets, that is, it is a nucleus.

---

23Which means no cheap tricks. For example, we can stand the poset $P_2$ on top of any dcpo and obtain a default-enabled poset which is not a dcpo — but such a poset is as uninteresting as is the poset $P_2$.  

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Proof. Let \( a, b \in P \). The inequality \( \gamma(a) \land \gamma(b) \geq \gamma(a \land b) \) holds since \( \gamma \) is increasing. For the converse inequality it suffices to prove the inequality \( a \land \gamma(b) \leq \gamma(a \land b) \)\(^{24}\). Since \( a \land b \leq \gamma(a \land b) \), \( b \) lies in \( (a \Rightarrow \gamma(a \land b)) \). Since \( (a \Rightarrow \gamma(a \land b)) \) has a ceiling, there exists \( c \in (a \Rightarrow \gamma(a \land b)) \) such that \( b \leq c \). Since \( \gamma(a \land b) \in \text{Fix}(\gamma) \), it follows from our assumption about \( \text{Fix}(\gamma) \) that \( c \in \text{Fix}(\gamma) \), therefore \( \gamma(b) \leq \gamma(c) = c \), and we conclude that \( a \land \gamma(b) \leq a \land c \leq \gamma(a \land b) \).

Let us say that a meet-semilattice \( P \) is **nuclear-enabled** if it is default-enabled and every set \( (a \Rightarrow b) \) with \( a, b \in P \) has a ceiling. The following proposition is a consequence of Lemma \([45]\) and Lemma \([46]\)

**Proposition 47.** If \( P \) is a nuclear-enabled meet-semilattice, then \( \text{NucSys}(P) \) is a closure system in \( \mathcal{P}P \) determined by the set of closure rules \( R_{df}(P) \cup R_{\text{nuc}}(P) \).

Let \( P \) be a nuclear-enabled meet-semilattice.

The poset \( \text{NucSys}(P) \), being a closure system in the complete lattice \( \mathcal{P}P \), is a complete lattice; it is also a closure system in the complete lattice \( \text{ClSys}(P) \). The meets in \( \text{NucSys}(P) \), as well as in \( \text{ClSys}(P) \), are intersections. Correspondingly, \( \text{Nuc}(P) \) is an interior system in \( \text{Cl}(P) \), and hence is a complete lattice with joins inherited from the complete lattice \( \text{Cl}(P) \): for every subset \( \Gamma \) of \( \text{Nuc}(P) \) the join \( \bigvee \Gamma \), taken in \( \text{Cl}(P) \), is a nucleus, therefore is a join of \( \Gamma \) in \( \text{Nuc}(P) \); moreover, \( \text{Fix}(\bigvee \Gamma) = \bigcap_{\gamma \in \Gamma} \text{Fix}(\gamma) \) by the antiisomorphism between \( \text{Cl}(P) \) and \( \text{ClSys}(P) \).

Let \( \Gamma \subseteq \text{Nuc}(P) \), and set \( C := \{ \text{Fix}(\gamma) \mid \gamma \in \Gamma \} \). The join of \( C \) in \( \text{NucSys}(P) \) is \( B := nucsys(\bigcup C) \), where \( nucsys = \bigvee \text{nucsys}_P \) is the closure operator on \( \mathcal{P}P \) determined by the closure rules \( R_{df}(P) \cup R_{\text{nuc}}(P) \). If \( \beta \) is the meet of \( \Gamma \) in \( \text{Nuc}(P) \), then \( \text{Fix}(\beta) = B \).

The nonempty finite meets in \( \text{Cl}(P) \) as well as in \( \text{Nuc}(P) \) are calculated pointwise. If \( P \) has a top element \( \top \), then the constant map \( P \rightarrow P : x \mapsto \top \) is the top element of both \( \text{Cl}(P) \) and \( \text{Nuc}(P) \). If \( P \) does not have a top element, then the top element of \( \text{Nuc}(P) \) may be different (thus strictly smaller) than the top element of \( \text{Cl}(P) \).

**Proposition 48.** Let \( P \) be a nuclear-enabled meet-semilattice, and let \( \Gamma \) be a set of prenuclei on \( P \). The closure operator \( \chi \) on \( P \) which has \( \text{Fix}(\chi) = \text{Fix}(\Gamma) \), the least closure operator on \( P \) that is above \( \Gamma \), is a nucleus.

**Proof.** For every \( \gamma \in \Gamma \) the set \( \text{Fix}(\gamma) \) obeys \( R_{\text{nuc}}(P) \) by Lemma \([45]\); therefore \( \text{Fix}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Fix}(\gamma) \) obeys \( R_{\text{nuc}}(P) \). By Lemma \([46]\) the closure operator \( \chi \) preserves binary meets, that is, its is a nucleus. By Theorem \([44]\) the closure operator \( \chi \) is the least closure operator on \( P \) that is above \( \Gamma \).

The following proposition is a do-it-by-maximal-elements analogue of Proposition \([23]\).

**Proposition 49.** If \( P \) is a default-enabled meet-semilattice, and for all \( a, b \in P \) the set \( (a \Rightarrow b) \) is default-enabled within \( P \) (so it certainly has a ceiling), then the complete lattice \( \text{Nuc}(P) \) is a frame.

\(^{24}\)Then \( \gamma(a) \land \gamma(b) \leq \gamma(\gamma(a) \land b) \leq \gamma(\gamma(a \land b)) = \gamma(a \land b) \).
Proof. We shall prove that for all $\beta \in \text{Nuc}(P)$ and all $\Gamma \subseteq \text{Nuc}(P)$ the following identity holds:

$$\beta \land \bigvee \Gamma = \bigvee_{\gamma \in \Gamma} (\beta \land \gamma).$$

The inequality $\geq$ is clear, so it remains to prove the converse inequality $\leq$. We write $\delta := \bigvee \Gamma$ and $\delta' := \bigvee_{\gamma \in \Gamma} (\beta \land \gamma)$. We have to prove that

$$\beta(x) \land \delta(x) = (\beta \land \delta)(x) \leq \delta'(x) \quad \text{for every } x \in P$$

(recall that finite meets in Nuc($P$) are calculated pointwise). Let $A := (\beta(x) \Rightarrow \delta'(x)) = \{ z \in P \mid \beta(x) \land z \leq \delta'(x) \}$. Clearly $x \in A$, and $A$ is by assumption default-enabled within $P$. We prove that the set $A$ is closed under $\Gamma$ precisely as we did in the proof of Proposition 23. By the induction principle, formulated in Theorem 44, it follows that $A$ is closed under $\delta$, and hence that $\delta(x) \in A$, which means that $\beta(x) \land \delta(x) \leq \delta'(x)$.

The last two propositions above specialize to propositions about preframes since in a preframe every set of the form $(a \Rightarrow b)$ is a subdcpo (in fact it is a Scott-closed subset) and as such it is default-enabled within the preframe. The act of specialization requires the use of the axiom of choice, so we are lucky, again, that we have already proved the specialized propositions.

9 Two convex geometries associated with a dcpo

A convex geometry is a structure $(E, \gamma)$ where $E$ is a set and $\gamma$ is a convex closure operator on $\mathcal{P}E$, which means that $\gamma$ satisfies the following anti-exchange axiom:

\begin{itemize}
  \item [(AE)] For every subset $A$ of $E$ and all elements $x, y$ of $E$,
  \begin{itemize}
    \item if $x, y \notin \gamma(A)$ and $x \neq y$ and $x \in \gamma(A \cup \{y\})$, then $y \notin \gamma(A \cup \{x\})$.
  \end{itemize}
\end{itemize}

The anti-exchange axiom is equivalent to the following condition:

\begin{itemize}
  \item [(CAS)] For every $\gamma$-closed subset $C$ of $E$ and all elements $x, y$ of $E$,
  \begin{itemize}
    \item if $x, y \notin C$ and $\gamma(C \cup \{y\}) = \gamma(C \cup \{x\})$, then $x = y$.
  \end{itemize}
\end{itemize}

Let $\gamma$ be an arbitrary closure operator on $\mathcal{P}E$. For every subset $A$ of $E$, the closure operator $\gamma$ induces the preorder $\leq_A$ on the set $E \setminus A$, where $x \leq_A y$ iff $x \in \gamma(A \cup \{y\})$. The condition (CAS) requires that for every $\gamma$-closed subset $C$ of $E$ the preorder $\leq_C$ is antisymmetric, that is, that it is a partial order.

The following proposition is the main result of this section. It generalizes Proposition 5-5.1 in LT-STA-2 [3]. It will be proved in due time.

Proposition 50. If $P$ is a dcpo, then $(P, \text{clsys}_P)$ and $(P, \text{dcclsys}_P)$ are convex geometries.

We start with some very general observations.
Lemma 51. Let $P$ be a poset. If $C$ is a closure system in $P$ and $A$ is a lower set of $P$, then $C \cup A$ is a closure system in $P$.

Proof. Let $\gamma$ be the closure operator on $P$ with $\text{Fix}(\gamma) = C$. We define the endomapping $\gamma^A$ on $P$ by
\[
\gamma^A(x) := \begin{cases} 
  x & \text{if } x \in A, \\
  \gamma(x) & \text{otherwise}. 
\end{cases}
\]

It is clear that $\gamma^A$ is ascending and idempotent. Let $x \leq y$ in $P$; we shall prove that $\gamma^A(x) \leq \gamma^A(y)$. This is clear if $x \notin A$ or $y \in A$. Suppose that $x \in A$ and $y \notin A$; then $\gamma^A(x) = x \leq y \leq \gamma(y) = \gamma^A(y)$. Thus $\gamma^A$ is a closure operator on $P$. Since $\text{Fix}(\gamma^A) = C \cup A$, we are done. \hfill $\Box$

Lemma 52. Let $P$ be a poset. If $C$ is a closure system in $P$ that is closed under existing directed joins in $P$, and $A$ is a finitely generated lower set of $P$, then $C \cup A$ is a closure system in $P$ that is closed under existing directed joins in $P$.

The set $C \cup A$ is a closure system by Lemma 51. It remains to prove that $C \cup A$ is closed under existing directed joins. Since $A$ is a union of finitely many principal ideals, and every principal ideal is closed under all existing joins hence under all existing directed joins, the desired result is a consequence of the following lemma.

Lemma 53. If subsets $A$ and $B$ of a poset $P$ are closed under existing directed joins in $P$, then the subset $A \cup B$ is closed under existing directed joins in $P$.

Proof. Let a directed subset $Y$ of $A \cup B$ have a join $u$ in $P$. We consider two cases.

Case 1: $Y \cap A$ is a cofinal subset of $Y$. The set $Y \cap A$ has the same upper bounds in $P$ as the set $Y$, thus the join $u$ of $Y$ in $P$ is also the join of $Y \cap A$ in $P$. Since $Y \cap A$ is a directed subset of $A$ and $A$ is closed under existing directed joins, we have $u \in A$.

Case 2: $Y \cap A$ is not a cofinal subset of $Y$. There exists $b \in Y$ such that the set $Y \uparrow b$ is disjoint with $A$ and is, therefore, contained in $B$. Since $Y \uparrow b$ is a cofinal subset of $Y$ it follows that $u \in B$. \hfill $\Box$

Proposition 54. Let $P$ be a poset. If $\text{ClSys}(P)$ is a closure system in $\mathcal{P}P$, then $(P, \text{clsys}_P)$ is a convex geometry.

Proof. Let $C$ be a closure system in $P$, and suppose that $x, y \in P$ are not in $C$ and that $\text{clsys}_P(C \cup \{y\}) = \text{clsys}_P(C \cup \{x\})$. The set $C \cup \downarrow y$ is, according to Lemma 51, a closure system, and it contains $C \cup \{y\}$, thus it contains $\text{clsys}_P(C \cup \{y\})$. Now from $x \in \text{clsys}_P(C \cup \{y\}) \subseteq C \cup \downarrow y$ and $x \notin C$ it follows that $x \in \downarrow y$, that is, that $x \leq y$. Likewise we see that $y \leq x$, and we conclude that $x = y$. The closure operator $\text{clsys}_P$ on $\mathcal{P}P$ satisfies the condition (CAS). \hfill $\Box$

Let $P$ be a poset.

We denote by $\text{DcClSys}(P)$ the subposet of $\mathcal{P}P$ consisting of all closure systems in $P$ that are closed under existing directed joins in $P$ (shorter: are directed-closed in $P$). This extends the notation $\text{DcClSys}(P)$ for a dcpo $P$ introduced in section 4.
Suppose that ClSys($P$) is a closure system in $\mathcal{P}P$. Then DcClSys($P$) is a closure system in $\mathcal{P}P$, too. Indeed, if $C$ is any set of directed-closed closure systems in $P$, then the intersection $\bigcap C$ is a closure system by assumption, and it is also directed-closed. Therefore, for every subset $X$ of $P$ there exists not only the least closure system clsys($X$) that contains $X$ (this one by assumption), but there exists also the least directed-closed closure system dcclsys($X$) that contains $X$.

**Proposition 55.** Let $P$ be a poset. If ClSys($P$) is a closure system in $\mathcal{P}P$, then $(P, \text{dcclsys}_P)$ is a convex geometry.

The proof is the same as that of Proposition 54, except that it uses Lemma 52 instead of Lemma 51.

**Proof of Proposition 50.** As $P$ is a dcpo, ClSys($P$) is a closure system in $\mathcal{P}P$. Now apply Proposition 54 and Proposition 55. □

The following proposition has essentially the same proof as Proposition 50.

**Proposition 56.** If $P$ is a default-enabled poset, then $(P, \text{clsys}_P)$ and $(P, \text{dcclsys}_P)$ are convex geometries.

We are not done yet. For a dcpo $P$, the closure operator clsys$_P$ is convex for a reason, the reason being that this closure operator is acyclic. Below we give the definition of acyclic closure operators, but only after the definition of a funnel for a closure operator.

Let $E$ be a set and $\gamma$ a closure operator on $\mathcal{P}E$.

We shall say that a preorder $\leq$ on $E$ is a funnel for the closure operator $\gamma$, or that $\gamma$ has a funnel $\leq$, if for every $X \subseteq E$ and every $y \in \gamma(X)$ there exists a subset $Z$ of $X$ such that $y \leq Z$ and $y \in \gamma(Z)$. Mark that a preorder $\leq$ on $E$ is a funnel for $\gamma$ iff for every $X \subseteq E$ and every $y \in \gamma(X)$ it follows that $y \in \gamma(X \uparrow y)$.

We shall say that the closure operator $\gamma$ is acyclic if it has an antisymmetric funnel, that is, a funnel which is a partial order on $E$.

**Proposition 57.** Let $E$ be a set, $\gamma$ a closure operator on $\mathcal{P}E$, and $\leq$ a preorder on $E$. The following are equivalent:

1. $\leq$ is a funnel for $\gamma$;
2. for all $X, U \subseteq E$, if $U$ is an upper set of $(E, \leq)$, then $\gamma(X) \cap U \subseteq \gamma(X \cap U)$;
3. for all $X \subseteq E$ and all $y \in E$ we have $\gamma(X) \uparrow y \subseteq \gamma(X \uparrow y)$.

**Proof.** (1) $\implies$ (2). Assume (1), and let $X, U \subseteq E$ with $U$ an upper set of $(E, \leq)$. Let $u \in \gamma(X) \cap U$ and write $Z := X \uparrow u$. Then $Z \subseteq U$ because $U$ is an upper set, and $u \in \gamma(Z)$ since $\leq$ is a funnel for $\gamma$, thus $Z \subseteq X \cap U$ and $u \in \gamma(Z) \subseteq \gamma(X \cap U)$.

(2) $\implies$ (3) holds by specialization ($U = \uparrow y$).

(3) $\implies$ (1). Assuming (3), suppose that $y \in \gamma(X)$; then $y \in \gamma(X) \uparrow y \subseteq \gamma(X \uparrow y)$. □
And why are acyclic closure operators so interesting? This is why:

**Proposition 58.** Let \( E \) be a set, \( \leq \) a partial order on \( E \), and \( \gamma \) a closure operator on \( \mathcal{P}E \). If \( \leq \) is a funnel for \( \gamma \), then the following statements are true:

(i) For all \( A \subseteq E \) and all \( x, y \in E \), if \( x \notin \gamma(A) \) and \( x \in \gamma(A \cup \{y\}) \), then \( x \leq y \).

(ii) The closure operator \( \gamma \) is convex.

**Proof.** (i) Assume that \( A, x, y \) satisfy the premises. Since \( x \in \gamma(A \cup \{y\}) \) and \( \leq \) is a funnel for \( \gamma \), it follows that \( x \in \gamma((A \cup \{y\}) \uparrow x) \). Now the set \((A \cup \{y\}) \uparrow x\) must contain \( y \) since otherwise we would have \((A \cup \{y\}) \uparrow x = A \uparrow x \) and \( x \in \gamma(A \uparrow x) \subseteq \gamma(A) \), contrary to assumptions. That is, we have \( x \leq y \).

(ii) For every \( \gamma \)-closed \( C \subseteq P \) and for all \( x, y \in P \setminus C \), if \( \gamma(C \cup \{x\}) = \gamma(C \cup \{y\}) \), then by part (i) it follows that \( x \leq y \) and \( y \leq x \), whence \( x = y \). The closure operator \( \gamma \) satisfies the condition (CAS).

Proposition 58 generalizes Lemma 8-3.23 in LT-STA-2 [3], from algebraic closure operators of poset type to arbitrary acyclic closure operators. The proof of the proposition is not completely modeled after the proof of Lemma 8-3.23, since the latter proof uses Lemma 8-3.2 which provides a useful consequence of algebraicity of the closure operator, and the proof above has no use (and no need) for such a lemma.

**Proposition 59.** Let \( P \) be a poset. If \( \text{ClSys}(P) \) is a closure system in \( \mathcal{P}P \), then the partial order of \( P \) is a funnel for the closure operator \( \text{clsys}_P \), which is therefore acyclic.

**Proof.** Let \( X \subseteq P \) and \( y \in \text{clsys}_P(X) \); we have to prove that \( y \in \text{clsys}_P(X \uparrow y) \).

The set \( \text{clsys}_P(X \uparrow y) \cup (P \setminus \uparrow y) \) is by Lemma 51 a closure system in \( P \); it contains the set \( X \), so it contains the closure system \( \text{clsys}_P(X) \) and hence contains the element \( y \); since \( y \notin P \setminus \uparrow y \), we conclude that \( y \in \text{clsys}_P(X \uparrow y) \).

**Corollary 60.** If \( P \) is a dcpo or a default-enabled poset, then \( \text{clsys}_P \) is acyclic.

Therefore, if \( P \) is a dcpo or a default-enabled poset, then the closure operator \( \text{clsys}_P \) on \( \mathcal{P}P \) is convex because it is acyclic, in view of Proposition 58. This proves again the first halves of Proposition 50 and Proposition 56, but the original direct proofs of those halves were markedly simpler, so one can be excused for not seeing the point of the new proofs that go the roundabout way through acyclicity. However, Corollary 60 is of independent interest. For example, the part of the corollary about default-enabled posets is the special case of Lemma 4.3 in Ranzato [7] for a default-enabled poset, which (i.e., the special case) is then used in the proof of Theorem 5.2 in [7].

\[^{25}\text{Mark that the partial order in Proposition 58 is opposite to the partial order in Lemma 8-3.23.}\]
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