Matching Condition on the Event Horizon and Holography Principle

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Abstract

It is shown that the event horizon of 4D black hole or \( ds^2 = 0 \) surfaces of multidimensional wormhole-like solutions reduce the amount of information necessary for determining the whole spacetime and hence satisfy the Holography principle. This leads to the fact that by matching two metrics on a \( ds^2 = 0 \) surface (an event horizon for 4D black holes) we can match only the metric components but not their derivatives. For example, this allows us to obtain a composite wormhole inserting a 5D wormhole-like flux tube between two Reissner-Nordström black holes and matching them on the event horizon. Using the Holography principle, the entropy of a black hole from the algorithm theory is obtained.

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I. INTRODUCTION

Matching two metrics which are solutions of the Einstein equation leads to the fact that surface stress-energy appears on the matching surface. This is a consequence of the Einstein equations. A detail explanation of this can be found, for example, in Refs [1], [2], [3]. A cause of this is evident: the Riemann tensor contains second derivatives of the metric which lead to a $\delta$-function in the left-hand side of the Einstein equations, hence in the right-hand side there should be $\delta$-like surface stress-energy.

But the Holography principle proposed in Refs [4], [5], [6] tells us that there is a surface which essentially cuts down the number of degrees of freedom. It can suggest that matching two metrics on this surface can substantially change the matching procedure on such a surface. For this purpose we propose the Lorentz invariant surface on which $ds^2 = 0$.

Further we consider several solutions of 4D and vacuum multidimensional (MD) gravity: two solutions are the Reissner-Nordström and Yang-Mills black holes (BH) and two solutions are wormhole-like (WH) solutions which in some sense are dual to the above-mentioned BHs.

To begin with, we bring the definition of the Holography principle according to [6]: "...a full description of nature requires only a two-dimensional lattice at the spatial boundaries of the world ...". Our aim is to show that this principle works on the $ds^2 = 0$ surface.

II. 4D CASE

A. Event horizon as a Holography surface for the Reissner-Nordström BH

The metric in this case is

$$ds^2 = \Delta(r)dt^2 - \frac{dr^2}{\Delta(r)} - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),$$

the electromagnetic potential is

$$A_\mu = u = u = \{\omega(r), 0, 0, 0\}.$$

The Einstein - Maxwell equations can be written as

$$-\frac{\Delta'}{r} + \frac{1 - \Delta}{r^2} = \frac{\kappa}{2} \omega'^2,$$

$$-\frac{\Delta''}{2} - \frac{\Delta'}{r} = -\frac{\kappa}{2} \omega'^2,$$

$$\omega' = \frac{\omega}{r^2}.$$

\footnote{often this is an event horizon (EH) but for some wormhole-like non-asymptotically flat multidimensional solutions it is not so.}

\footnote{This duality means that static regions of the 4D BHs are given by $r \geq r_g$ but the static regions of the MD wormhole-like solutions by $|r| \leq r_g$.}
It is easy to prove that Eq. (4) is a consequence of (3) and (5). For the Reissner-Nordström BH the event horizon (EH) is defined by the condition $\Delta(r_g) = 0$, where $r_g$ is the radius of the EH. Hence in this case we see that on the EH
\[ \Delta_g' = \frac{1}{r_g} - \frac{\kappa}{2} r_g \omega_g'^2, \] (6)
here (g) means that the corresponding value is taken on the EH. Thus, Eq. (3), which is the Einstein equation, is a first-order differential equations in the whole spacetime ($r \geq r_g$). The condition (3) tells us that the derivative of the metric on the EH is expressed through the metric value on the EH. And this means that Holography principle works here and is connected with the presence of the EH. In passing we remark that this allows us to calculate the BH entropy from an algorithmical point of view [7] without any quantum-mechanical calculations, which we will do in [11].

**B. Event horizon as a Holography surface for an SU(2) Yang-Mills BH**

Here we use the following metric
\[ ds^2 = e^{2\nu(r)} \Delta(r) dt^2 - \frac{dr^2}{\Delta(r)} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \] (7)

For the SU(2) Yang-Mills gauge field we choose the following monopole-like ansatz as in [8]
\[ A^a_i = \frac{\epsilon^a_{ij} r^j}{r^2} (1 - f(r)); \] (8)
\[ A^a_t = \frac{x^a}{r^2} \nu(r), \] (9)
here $a = 1, 2, 3$ is the internal index; $i = 1, 2, 3$ is the spatial index. For simplicity we consider the case $v = 0$. Thus, we have the following set of Einstein - Yang - Mills equations
\[ - \frac{\Delta'}{r} + \frac{1 - \Delta}{r^2} = \frac{\kappa}{r^2} \left[ \Delta f'^2 + \frac{1}{2r^2} \left( f^2 - 1 \right)^2 \right], \] (10)
\[ \nu' = \frac{\kappa}{r} f'^2, \] (11)
\[ R^g_{\theta \theta} - \frac{1}{2} R = \kappa T^g_{\theta \theta}, \] (12)
\[ \Delta f'' + \Delta f' \nu' + f' \Delta' = \frac{f}{r^2} \left( f^2 - 1 \right). \] (13)
Due to the Bianchi identities Eq. (12) is a consequence of the other equations. From (11) we have
\[ \nu(r) = \kappa \int_r^\infty \frac{f'^2}{r} dr, \] (14)
here we choose the time so that $\nu_{r \rightarrow \infty} = 0$. So the value on the EH is
\[ \nu_g = \kappa \int_{r_g}^{\infty} \frac{f'^2}{r} \, dr. \]  

(15)

The presence of an EH means that \( \Delta(r_g) = 0 \), hence close to an EH

\[ \Delta = \Delta_1 x + \Delta_2 \frac{x^2}{2} + \cdots, \]  

(16)

\[ f = f_0 + f_1 x + f_2 \frac{x^2}{2} + \cdots, \]  

(17)

here \( x = r - r_g \). Then from the Einstein - Yang - Mills equations we have:

\[ f_2 = -\frac{\kappa}{r_g^3} f_1^2, \]  

(18)

\[ f_1 = \frac{r_g^2}{r_g^2 \Delta_1} \left( f_0^2 - 1 \right), \]  

(19)

\[ \Delta_1 = \frac{1}{r_g} - \frac{\kappa}{2r_g^3} \left( f_0^2 - 1 \right)^2. \]  

(20)

Thus we have only one physically significant parameter \( f_0 = f(r_g) \). Again we have the Holography principle on the EH. In this case even more: the Yang - Mills equation satisfy the Holography principle since the first derivative of \( f(r) \) on the EH is expressed through the value of \( f_0 \) on the EH, which is seen from Eq. (19).

III. MULTIDIMENSIONAL WORMHOLE-LIKE CASES

A. \( ds^2 = 0 \) surface as Hologram surface for 5D WH-like solution

Let us consider the 5D WH-like metric

\[ ds^2 = \frac{1}{\Delta(r)} dt^2 - R_0^2 \Delta(r) \left[ d\chi + \omega(r)dt \right]^2 - dr^2 - a(r) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]  

(21)

here \( \chi \) is the 5\textsuperscript{th} coordinate, \( r, \theta \) and \( \varphi \) are the ordinary spherical coordinates, \( R_0 \) is some constant. The 5D Einstein equations are

\[ \Delta \Delta'' - \Delta'^2 + \Delta \Delta' \frac{\omega'}{a} + R_0^2 \Delta^4 \omega'^2 = 0, \]  

(22)

\[ \left( a \Delta^2 \omega' \right)' = 0, \]  

(23)

\[ a'' = 2. \]  

(24)

The solution is \[ \]  

\[ \]  

\[ \]  

\[ 3 \text{we can exclude } r_g \text{ making all the magnitudes dimensionless } (r^* = r/r_g; \kappa' = \kappa/r_g^2). \]
\[ a = r^2 + r_0^2 \]  
(25)

\[ \Delta = \frac{q}{2r_0} \frac{r^2 - r^2}{r_0^2 + r^2}, \]  
(26)

\[ \omega = \frac{4r_0^2}{q} \frac{r}{r_0^2 - r^2}. \]  
(27)

Here \( q \) and \( r_0 \) are some constants. It is easy to prove that \( G_{tt}(\pm r_0) = \Delta^{-1}(\pm r_0) - R_0^2 \Delta(\pm r_0) \omega^2(\pm r_0) = 0 \) and \( ds^2 = 0 \) on surfaces \( r = \pm r_0 \). In this sense the surface \( r = r_0 \) is analogous to an EH. On the surface \( ds^2 = 0 \) \( \Delta(\pm r_0) = 0 \), and therefore from Eq. (22) we have

\[ \Delta'_{0} = \pm \frac{q}{a_0} = \pm \frac{q}{2r_0^2}. \]  
(28)

The signs (±) correspond, respectively, to \( (r = \mp r_0) \) where the surfaces \( ds^2 = 0 \) are located. This also indicates that the surface \( ds^2 = 0 \) works here according to the Holography principle.

**B. The surface \( ds^2 = 0 \) as a Holography surface for a 7D WH-like solution**

Here we work with gravity on the principal bundle as in Ref. [10], i.e. the base of the bundle is an ordinary 4D Einstein spacetime and the fibre of the bundle is the SU(2) gauge group. In our case we have gravity on the SU(2) principal bundle with the SU(2) structural group (simultaneously it gives the extra coordinates). This group as the sphere \( S^3 \) is the space of the extra dimensions. Thus, the dimension of our MD gravity is 7.

The gravity equations are:

\[ R_{a\mu} = 0, \]  
(29)

\[ R_{a}^a = R_4^4 + R_5^5 + R_6^6 = 0, \]  
(30)

here \( A = 0, 1, 2, \ldots, 6 \) is a MD index on the total space of the bundle, \( \mu = 0, 1, 2, 3 \) is the index on the base of the bundle, \( a = 4, 5, 6 \) is the index on the fibre of the bundle. For MD gravity on the principal bundle we have the following theorem [11,12]:

Let \( G \) be the group fibre of the principal bundle. Then there is a one-to-one correspondence between the \( G \)-invariant metrics on the total space \( \mathcal{X} \) and the triples \((g_{\mu\nu}, A_{a}^\mu, h_{\gamma ab})\). Here \( g_{\mu\nu} \) is Einstein’s pseudo-Riemannian metric on the base; \( A_{a}^\mu \) is the gauge field of the group \( G \) (the nondiagonal components of the multidimensional metric); \( h_{\gamma ab} \) is the symmetric metric on the fibre.

According to this theorem 7D metric has the following form

\[ ds^2 = \frac{\Sigma^2(r)}{w^2(r)} dt^2 - R_0^2 u(r) \left( \sigma^a + A_{a}^\mu dx^\mu \right)^2 - dr^2 - a(r) \left( d\theta^2 + \sin \theta d\varphi^2 \right), \]  
(31)

here \( A_{a}^\mu \) is the above-mentioned SU(2) gauge field, the one-forms \( \sigma^a \) on the SU(2) group can be written as follows:
\[\sigma^1 = \frac{1}{2}(\sin \alpha \beta - \sin \beta \cos \alpha \gamma), \quad (32)\]
\[\sigma^2 = -\frac{1}{2}(\cos \alpha \beta + \sin \beta \sin \alpha \gamma), \quad (33)\]
\[\sigma^3 = \frac{1}{2}(d\alpha + \cos \beta d\gamma), \quad (34)\]

here we have introduced Euler's angles \(\alpha, \beta, \gamma\) on the fibre (\(SU(2)\) group) and \(0 \leq \beta \leq \pi, 0 \leq \gamma \leq 2\pi, 0 \leq \alpha \leq 4\pi\).

An ansatz for the gauge potential \(A^\mu_a\) is taken as for the monopole (as in section II B). For simplicity we examine the case \(f(r) = 0\) only. Then we have the following vacuum gravity equations

\[v'' - \frac{\Sigma' v'}{\Sigma} + 4 \frac{u' v'}{u} + \frac{a' v'}{a} = 0, \quad (35)\]
\[u'' + \frac{\Sigma' u'}{u} - \frac{u'^2}{u} + \frac{4}{R_0^2} - \frac{1}{12} \frac{R_0^2 u}{a^2} + \frac{1}{3} \frac{R_0^2 a^4}{\Sigma^2} v^2 = 0, \quad (36)\]
\[\frac{\Sigma''}{\Sigma} + \frac{a' \Sigma'}{a \Sigma} - \frac{6}{R_0^2} + \frac{R_0^2 u}{8 a^2} = 0. \quad (37)\]

In Ref. [13] an approximative solution has been found. Here we are interested only in what happens close to the surface \(ds^2(\pm r_0) = 0\). In order that a surface \(ds^2(\pm r_0) = 0\) exist it is necessary that the following condition be satisfied:

\[G_{tt} = \frac{\Sigma_0^2}{u_0^2} - \frac{R_0^2 u_0 v_0^2}{v_0^2} = 0, \quad (39)\]

here the index \((0)\) means that the corresponding quantities are taken on the surface \(r = \pm r_0\).

We suppose that in this region there is the following behaviour \((r = +r_0)\)

\[u(r) = u_0 \left(1 - \frac{r}{r_0}\right)^{1/2} + \cdots, \quad (40)\]
\[\Sigma(r) = \Sigma_0 + \Sigma_1 \left(1 - \frac{r}{r_0}\right)^{3/2} + \cdots, \quad (41)\]
\[v(r) = \frac{q \Sigma_0}{a_0 u_0^3} \left(1 - \frac{r}{r_0}\right) + \cdots. \quad (42)\]

This leads to the following result:

\[u_0^2 = \sqrt{\frac{2}{3}} \frac{|q| r_0}{a_0}, \quad (43)\]
\[R_0 = \sqrt{\frac{2}{3}} u_0 r_0, \quad (44)\]
\[\frac{\Sigma_1}{\Sigma_0} = \frac{12}{a_0}. \quad (45)\]

Here we also see the Holography principle: \(u_0\) and \(\Sigma'(r_0) = \Sigma_1\) are not independent initial data, they are determined from the dimensionless magnitudes \(q/r_0\) and \(a_0/r_0^2\).
IV. DISCUSSION

Thus we see that at least for static spherically symmetric solutions in 4D and vacuum MD gravity the Holography principle leads from the presence of the $ds^2 = 0$ surface (event horizon for the 4D gravity). For researchers working with 4D Einstein - Yang - Mills black holes this is well known: the condition (13) is necessary for numerical calculations (see, for example, [8]).

These results allow us to say that (at least for the static spherically symmetric cases) on the surface $ds^2 = 0$ the Holography principle changes and simplifies the matching conditions due to reduction of the physical degrees of freedom. Roughly speaking, close to this surface the Einstein differential equations of the second order are reduced to first-order equations\(^4\). In this case it is evident that: matching of two metrics on the surface $ds^2 = 0$ does not lead to $\delta$–functions and hence the appearance of an additional surface stress-energy.

A. Composite WH with 5D WH-like solution and two Reissner-Nordström black holes

For example, having a 5D WH-like solution, we can match to it two Reissner-Nordström black holes on the two $ds^2 = 0$ surfaces [14]. This can be done since ordinary 4D electro-gravity can be considered as 5D vacuum gravity in the initial Kaluza sense\(^5\) and on the EH we join fibre to fibre and base to base Reissner - Nordström and 5D WH-like solutions. In this case we have to match on the surface $ds^2 = 0$ (an EH for an observer at infinity) only the following quantities

- The area of the surface $ds^2 = 0$ of the 5D WH-like solution with the area of the EH of the Reissner - Nordström BH:

$$4\pi a = 4\pi r_g^2,$$

here the left-hand side is 5D and the right-hand side is 4D.

- Let we compare the 5D equation $R_{15} = 0$

$$\left(4\pi \omega' e^{-2\nu}\right)' = 0,$$

with the Maxwell equation

$$\left(4\pi r^2 E\right)' = 0.$$

In both cases $4\pi a$ or $4\pi r^2$ is the area of an 2-sphere and Eqs (17) and (18) tell us that the electrical field flux is preserved. Hence we can make a conclusion that $\omega' e^{-2\nu}$

\(^4\)for the 4D spherically symmetric case it is an exact result.

\(^5\)when $G_{55} = 1$ and the Lagrangian is not varied with respect to $G_{55}$. 7
is a "5D electrical" field, $E_5 = \omega' e^{-2\nu}$ and for the 4D case we have the conventional definition of the electrical field $E$. Hence on the $ds^2 = 0$ matching surface

$$\omega_0' e^{-2\nu_0} = E_g,$$

where (0) means that the corresponding 5D quantity is on the $ds^2 = 0$ surface and (g) means that this 4D quantity is taken on the EH.

- We do not match $G_{rr}$ and $g_{rr}$ since these components of 4D and 5D metrics are arbitrary: they depend only on the choice of the radial coordinate.

In fact we see that we have only two matching conditions and this is evident: a Reissner-Nordström BH is characterized only by two physical quantities: the electrical charge $Q$ and the mass $m$. Also the 5D WH-like solution (25) - (27) is characterized only by two physical quantities: the constants $q$ and $r_0$. It is remarkable that for these 4D and 5D physical quantities we have only the matching conditions (46) and (49) as a consequence of the Holography principle.

**B. The Holography principle and algorithmical complexity**

It is interesting that reduction of the order of Einstein’s differential equations near an EH and the Holography principle allows us to calculate the entropy of a BH without any quantum calculations. We shortly repeat this result obtained in [7]. In the 1960’s Kolmogorov ascertained that the algorithm theory allows us to define the probability notion for a single object. His idea is very simple: the probability is connected with the complexity of this object, ”chance“ = ”complexity“. The more complex (longer) is an algorithm describing this object the smaller probability it has. Of course there is the question: what is it the length of an algorithm? It was found that such an invariant, well pos ed definition can be given [15]:

The algorithmic complexity $K(x \mid y)$ of the object $x$ for a given object $y$ is the minimal length of the ”program” $P$ which is written as a sequence of the zeros and unities and allows one to construct $x$ from given $y$

$$K(x \mid y) = \min_{A(P,y) = x} l(P)$$

where $l(P)$ is the length of the program $P$; $A(P, y)$ is the algorithm calculating the object $x$, using the program $P$ when the object $y$ is given. Then we can determine the algorithmical complexity of a BH and the logarithm of it gives us the entropy of BH.

We write the initial equations for describing the Schwarzschild BH. The metric is

$$ds^2 = dt^2 - e^{\lambda(t,R)} dR^2 - r^2(t, R) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

such an algorithm can, for example, consist in the field equations describing a field distribution in the spacetime.
here \( t \) is time, \( R \) is radius, \( \theta \) and \( \phi \) are polar angles. In this case Einstein’s equations are

\[
- e^{-\lambda} r^2 + 2r \dot{r} + r^2 + 1 = 0, \quad (52)
\]

\[
- \frac{e^{-\lambda}}{r} (2r'' - r' \lambda') + \frac{\dot{r}}{t} \lambda + \frac{\dot{\lambda}^2}{2} + 2 \frac{\dot{r}}{r} = 0, \quad (53)
\]

\[
- \frac{e^{-\lambda}}{r^2} (2rr'' + r'^2 - rr' \lambda') + \frac{1}{r^2} \left( r \ddot{\lambda} + \dot{\lambda}^2 + 1 \right) = 0, \quad (54)
\]

\[
2 \dot{r}' - \dot{\lambda} r' = 0, \quad (55)
\]

where \((')\) and \((\cdot)\) mean, respectively, derivatives in \( t \) and \( r \). The \( \frac{\partial}{\partial t} \) Einstein’s equation for the initial data is

\[
- \frac{e^{-\lambda}}{r^2} (2rr'' + r'^2 - rr' \lambda') + \frac{1}{r^2} \left( r \ddot{\lambda} + \dot{\lambda}^2 + 1 \right) = 0. \quad (56)
\]

The Caushy hypersurface determining the whole Schwarzschild - Kruskal spacetime is \( t = 0 \). The ”quantity“ of the initial data according to the Holography principle can be essentially reduced: first, at \( t = 0 \) the first time derivative of all metric components is equal to zero. Therefore we have the following equation for the initial data

\[
2rr'' + r'^2 - rr' \lambda' - e^\lambda = 0. \quad (57)
\]

We know that the hypersurface \( t = 0 \) is a WH connecting two asymptotically flat, causally disconnected regions. This WH is symmetrical relative to centre \( r = r_g \), therefore the initial data for this equation are

\[
r'(R = 0, t = 0) = 0, \quad (58)
\]

\[
r(R = 0, t = 0) = r_g, \quad (59)
\]

where \( r_g \) is the radius of the event horizon. The conditions \((58)\) and \((59)\) are necessary for this WH to exist and this is also a consequence of Holography principle (reducing the information describing the BH). Thus, for describing the whole Schwarzschild - Kruskal spacetime we need the algorithm \((52)\) - \((55)\) and the initial data \((59)\). Therefore the algorithmical complexity of the Schwarzschild BH \( \mathcal{K} \) is defined by the given expression

\[
\mathcal{K} \approx L_{\text{initial}} \left( \frac{r_g}{r_{\text{pl}}} \right)^2 + L_{\text{Einstein equations}}, \quad (60)
\]

where \( L_{\text{initial}} \) is the length of the algorithm (program) determining the dimensionless number \( r_g^2/r_{\text{pl}}^2 \), performed on some universal machine, \( L_{\text{Einstein equations}} \) is the length of the algorithm (program) for solving the Einstein equations \((52)\) - \((55)\) using some universal machine, for example, the Turing machine.

V. CONCLUSION

Finally, we can suppose that the event horizon plays an exceptional role in nature: it is the surface on which the Holography principle is realized. We have shown that in this case an EH can divide the regions in our Universe with splitting off and nonsplitting off the extra dimensions in the above - mentioned sense as a consequence of realizing the Holography principle.
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