Gromov–Witten invariants of Fano threefolds
of genera 6 and 8

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Abstract. The aim of this paper is to prove Golyshev’s conjecture in the cases of Fano threefolds \(V_{10}\) and \(V_{14}\). This conjecture states modularity of \(D3\) equations for smooth Fano threefolds with Picard group \(\mathbb{Z}\). More precisely, we find counting matrices of prime two-pointed Gromov–Witten invariants for them. For this we use the method that lets us find Gromov–Witten invariants of complete intersections in varieties whose invariants are (partially) known.

In this work we use the following method to find two-pointed Gromov–Witten invariants for \(V_{10}\) and \(V_{14}\). According to S. Mukai, these varieties are complete intersections in grassmannians (Theorem 3.1.1). Find their generating series for one-pointed invariants with descendants (the so called \(I\)-series). By Quantum Lefschetz Theorem (see 4.2.2) find \(I\)-series for \(V_{10}\) and \(V_{14}\). Finally, by divisor axiom and topological recursion relations find polynomial expressions for the coefficients of \(I\)-series in terms of two-pointed prime invariants (Proposition 6.2.2). Finally, inverse these expressions and find two-pointed invariants.

The paper is organized as follows. In §1 we give definitions of Gromov–Witten invariants and stacks of stable maps of genus 0 with marked points. In §2 we state Golyshev’s conjecture. In §3 we consider \(I\)-series for grassmannians. §4 contains Quantum Lefschetz Theorem. The relations between one- and multi-pointed invariants are in §5. Finally, in §6 we prove the main theorem of this paper (Theorem 6.1.1), in which we explicitly find the specific counting matrices for threefolds \(V_{10}\) and \(V_{14}\).

Bibliography. Gromov–Witten invariants were introduced for counting the numbers of curves of different genera on different varieties, which intersect given homological classes. Such numbers are called prime invariants. Axiomatic treatment of them was given in [KMI]. After that these invariants were explicitly constructed. The generalization (the so called invariants with descendants) was obtained in [BM1] and [Beh].

The idea to express the Gromov–Witten invariants of hypersurfaces in terms of the invariants of the ambient variety (the so called Mirror Formula), the main (up to now) tool of finding of quantum cohomology, seems to belong to A. Givental (see, for instance, [Gi]). We use it in the form given by A. Gathmann (see [Ga2]). The formula for \(I\)-series for grassmannians (the Hori–Vafa conjecture) was proved by A. Bertram, I. Ciocan-Fontanine, B. Kim in [BCK]. A. Bertram with H. Kley in [BK] and Y.-P. Lee with R. Pandharipande in [LP] express multipointed invariants in terms of one-pointed ones.

Fano threefolds of genera 6 and 8 were studied in [Gu1], [Gu2], [Gu3], [Is1], [Is2], [I], [IP], [Lo1], [Lo2], [M], [Mar]. Gromov–Witten invariants of such varieties that count the lines are computed in [Mar], see also [IP]. The number of conics on the threefold of genus 6 that pass through a general point is found in [Lo1].

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Gromov–Witten invariants of Fano threefolds first was studied in [BM2]. Quantum cohomology of some threefolds was found in [Ci] (some blow-ups of projective spaces and quadrics), [AM], [Ba] (toric varieties), [QR] (vector bundles over $\mathbb{P}^n$), and [Bea] (some complete intersections).

Conjecture on modularity of smooth Fano threefolds with Picard Group $\mathbb{Z}$ is stated by V. Golyshin in [Go1].

Conventions and notations. Everything is over $\mathbb{C}$. Throughout the paper:

A grassmannian $G(r,n)$ is the variety of $r$-dimensional linear subspaces in the $n$-dimensional linear space.

In the sequel the Gromov–Witten invariants (GW-invariants) mean the genus 0 invariants (i.e. the invariants that correspond to rational curves).

The cohomology algebra $H^*(X,\mathbb{Q})$ of variety $X$ we denote as $H^*(X)$.

Poincaré dual class to the class $\gamma \in H^*(X)$ on $X$ we denote as $\gamma^\vee$.

The cohomology algebra of $H^*(X)$ we denote as $H^*_{alg}(X)$.

The subset of classes of effective curves in $H_2(X,\mathbb{Z})$ we denote by $H^2_2(X)$.

1. Main definitions

Let $X$ be a smooth projective variety with Picard group $\mathbb{Z}$ such that $-K_X$ is nef.

We mention only those definitions and axioms that will be used in the sequel.

1.1. Moduli spaces.

**Definition 1.1.1.** The genus of curve $C$ is the number $h^1(\mathcal{O}_C)$.

It is easy to see that the curve is of genus 0 if and only if it is a tree of rational curves.

**Definition 1.1.2.** The connected curve $C$ with $n \geq 0$ marked points $p_1,\ldots,p_n \in C$ is called prestable if it has at most ordinary double points as singularities and $p_1,\ldots,p_n$ are distinct smooth points (see [Ma], III–2.1). The map $f : C \to X$ of connected curve of genus 0 with $n$ marked points are called stable if $C$ is prestable and there are at least three marked or singular points on every contracted component of $C$ ([Ma], V–1.3.2).

In the other words, a stable map of connected curve is the map that has only finite number of infinitesimal automorphisms.

**Definition 1.1.3.** The family of stable maps (over the scheme $S$) of curves of genus 0 with $n$ marked points is the collection $(\pi : C \to S, p_1,\ldots,p_n, f : C \to X)$, where $\pi$ is the following map. It is a smooth projective map with $n$ sections $p_1,\ldots,p_n$. Its geometric fibers $(C_s, p_1(s),\ldots,p_n(s))$ are prestable curves of genus 0 with $n$ marked points. Finally, the restriction $f|_{C_s}$ on each fiber is a stable map.

Two families over $S$

$$(\pi : C \to S, p_1,\ldots,p_n, f), \quad (\pi' : C' \to S, p'_1,\ldots,p'_n, f')$$

are called isomorphic if there is an isomorphism $\tau : C \to C'$ such that $\pi = \pi' \circ \tau, p'_i = \tau \circ p_i, f = f' \circ \tau$.

Let $\beta \in H^2_2(X)$. Consider the following (contravariant) functor $\overline{M}_n(X,\beta)$ from the category of (complex algebraic) schemes to the category of sets. Let $\overline{M}_n(X,\beta)(S)$ be the set of isomorphism classes of families of stable maps of genus 0 curves with $n$ marked points $(\pi : C \to S, p_1,\ldots,p_n, f)$ such that $f_*([C_s]) = \beta$, where $[C_s]$ is the fundamental class of $C_s$. 

**Definition 1.1.4.** The moduli space of stable maps of genus 0 curves of class $\beta \in H_2^+(X)$ with $n$ marked points to $X$ is the Deligne–Mumford stack (see [Ma], V–5.5) which is the coarse moduli space that represents $\overline{M}_n(X, \beta)$. This space is denoted by $\overline{M}_n(X, \beta)$.

The stack $\overline{M}_n(X, \beta)$ is compact and smooth, that locally it is a quotient of a smooth variety by a finite group. Hence, we can consider an intersection theory on it (see [Vi]). In the general case $\overline{M}_n(X, \beta)$ has a “wrong” dimension, so it is equipped with the virtual fundamental class $[\overline{M}_n(X, \beta)]^{\operatorname{virt}}$ of virtual dimension $\operatorname{vdim} \overline{M}_n(X, \beta) = \dim X - \deg_{K_X} \beta + n - 3$. Let $X$ be a convex variety (i. e. for any map $\mu : \mathbb{P}^1 \to X$ the equality $H^1(\mathbb{P}^1, \mu^*(TX)) = 0$ holds). Then $\overline{M}_n(X, \beta)$ is projective normal variety of pure dimension $\dim X - \deg_{K_X} \beta + n - 3$. Thus, the virtual fundamental class is the usual one. More in [Ma], VI–1.1.

1.2. Gromov–Witten invariants. Consider the evaluation maps $\psi_i : \overline{M}_n(X, \beta) \to X$, given by $\psi_i(C; p_1, \ldots, p_n, f) = f(p_i)$. Let $\pi_{n+1} : \overline{M}_{n+1}(X, \beta) \to \overline{M}_n(X, \beta)$ be the forgetful map at the point $p_{n+1}$ (which forget this point and contract unstable component after it); consider the sections $\sigma_i : \overline{M}_n(X, \beta) \to \overline{M}_{n+1}(X, \beta)$, which coincide with the points $p_i$. The image of a curve $(C; p_1, \ldots, p_n, f)$ under $\sigma_i$ is a curve $(C'; p_1, \ldots, p_{n+1}, f')$. Here $C' = C \bigcup C_0$, $C_0 \simeq \mathbb{P}^1$, $C_0$ and $C$ intersect at the non-marked point $p_i$ on $C'$, and $p_{n+1}$ and $p_i$ lie on $C_0$. The map $f'$ contracts $C_0$ to the point and $f'|_{C'} = f$.

Consider the sheaf $L_i$ given by $L_i = \sigma_i^* \omega_{\overline{M}_{n+1}}$, where $\omega_{\overline{M}_{n+1}}$ is a relative dualizing sheaf of $\pi_{n+1}$. Its fiber over the point $(C; p_1, \ldots, p_n, f)$ is $T_{p_i} C$. Put cotangent line class $\psi_i = c_1(L_i) \in H^2(\overline{M}_n(X, \beta))$.

**Definition 1.2.1 ([Ma], VI–2.1).** Consider $\gamma_1, \ldots, \gamma_n \in H^*(X)$ and let $d_1, \ldots, d_n$ be non-negative integers. Then the Gromov–Witten invariant (correlator) is the number given by

$$\langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n \rangle_\beta = \psi_1^{d_1} \cdot \psi_2^{d_2} \cdot \ldots \cdot \psi_n^{d_n} \cdot [\overline{M}_n(X, \beta)]^{\operatorname{virt}}$$

if $\sum \operatorname{codim} \gamma_i + \sum d_i = \operatorname{vdim} \overline{M}_n(X, \beta)$ and 0 otherwise. The number $\sum d_i$ is called the degree of invariant with respect to the descendants. The invariants of degree 0 are called prime (and symbols $\tau_0$ are omitted).

After this definition axioms, which define Gromov–Witten invariants, become their properties. We will refer to two of them.

- **Divisor axiom ([Ma], V–5.4).**
  Let $\gamma_0 \in H^2(X)$. Then

$$\langle \gamma_0, \tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n \rangle_\beta = \langle \gamma_0 \cdot \beta \rangle \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n \rangle_\beta + \sum_{k, d_k \geq 1} \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k-1} (\gamma_0 \cdot \gamma_k), \ldots, \tau_{d_n} \gamma_n \rangle_\beta. \quad (1)$$
The only exception is the case $\beta = 0$ and $n = 2$. In this case $\langle \gamma_0, \gamma_1, \gamma_2 \rangle_0 = \gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdot [X]$. 

- **Fundamental class axiom** ([Ma], V–5.1).

Let $1$ be the fundamental class of $X$. Then

\[
\langle 1, \tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n \rangle_\beta = \sum_{k,d \geq 1} \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k-1} \gamma_k, \ldots, \tau_{d_n} \gamma_n \rangle_\beta.
\]

(2)

The only exception is the case $\beta = 0$ and $n = 2$. In this case $\langle 1, \gamma_1, \gamma_2 \rangle_0 = \gamma_1 \cdot \gamma_2 \cdot [X]$.

**Remark 1.2.2.** In this paper Gromov–Witten invariants are considered for varieties of Picard rank 1. In this case the class of the curve $\beta \in H_+^2(X)$ is determined by its (anticanonical) degree $d$, and we will often write $\langle \ldots \rangle_d$ instead of $\langle \ldots \rangle_\beta$. For the case of greater Picard rank one should use multidegree.

1.3. **I-series** ([Ga2]). Let $R \in H^*(X)$ be Poincaré self-dual subalgebra (that is, for any class $\gamma \in R$ we have $\gamma^V \in R$). Let $1 = \gamma_0, \ldots, \gamma_N$ be the basis in $R$. Let $\beta \in H_+^2(X)$ be an effective curve of degree $d$ (with respect to the positive Picard group generator). Put

\[
I^X_{d,R} = I^X_{\beta,R} = \sum_{i,j} \langle \tau_i \gamma_j \rangle_\beta \gamma^V_j.
\]

**Definition 1.3.1** ([Ga2]). *I-series* $I^X_R$ is given by the following.

\[
I^X_R = \sum_{d \geq 0} I^X_{d,R} \cdot q^d.
\]

We denote $I$-series $I^X_R$ for $R = H^*(X)$ by $I^X$; for $R = H^*_{alg}(X)$ by $I^X_{alg}$. Let $Y \subset X$ be a complete intersection and $\pi: H^*(X) \to H^*(Y)$ be a natural restriction homomorphism. Then for $R = \pi(H^*(X))$ the series $I^X_R$ we denote by $I^X_{\text{rest}}$.

**Definition 1.3.2.** Let $I \in H^*(X) \otimes \mathbb{C}[[q]]$. Put

\[
I = \sum_{0 \leq i \leq N} \gamma_i \otimes I^{(i)}.
\]

The term $I^{(0)}$ we denote by $I^H_0$.

2. Counting matrices for Fano threefolds

2.1. Counting matrices.

**Definition 2.1.1** ([Go1]). Let $X$ be a smooth threefold Fano variety such that $\text{Pic}(X) \cong \mathbb{Z}$ and $K = -K_X$. Counting matrix is the matrix of its $GW$-invariants, namely the following matrix $A \in \text{Mat}(4 \times 4)$.

\[
\begin{pmatrix}
  a_{00} & a_{01} & a_{02} & a_{03} \\
  1 & a_{11} & a_{12} & a_{13} \\
  0 & 1 & a_{22} & a_{23} \\
  0 & 0 & 1 & a_{33}
\end{pmatrix}.
\]

Numeration of rows and columns starts from 0 and the elements are given by

\[
a_{ij} = \frac{\langle K^{3-i}, K^j, K \rangle_{j-i+1}}{\deg X} = \frac{j - i + 1}{\deg X} \cdot \langle K^{3-i}, K^j \rangle_{j-i+1}
\]

(the degree is taken with respect to the anticanonical class).
It is easy to see that the matrix $A$ is symmetric with respect to the secondary diagonal: $a_{ij} = a_{3-j,3-i}$. By definition, $a_{ij} = 0$ if $j - i + 1 < 0$. If $j - i + 1 = 0$, then $a_{ij} = 1$, because it is just a number of intersection points of $K^{3-i}$, $K^j$, and $K$, which is $\deg X$; $a_{00} = a_{33} = 0$. For the other coefficients $a_{ij}$’s are “expected” numbers of rational curves of degree $j - i + 1$ passing through $K^{3-i}$ and $K^j$, multiplied by $\frac{j-i+1}{\deg X}$. The only exception is the following: by divisor axiom

$$a_{01} = 2 \cdot (2 \cdot \text{ind}(X) \cdot \text{[the number of conics passing through the general point]})$$

2.2. Golyshev’s conjecture. It is more convenient to use a family $A^\lambda = A + \lambda E$, where $E$ is the identity matrix. Thus, the element of the family $A^\lambda$ is given by six parameters: five different GW–invariants $a_{ij}$ and $\lambda$.

Consider the one-dimensional torus $\mathbb{G}_m = \text{Spec } \mathbb{C}[t, t^{-1}]$ and the differential operator $D = t \frac{\partial}{\partial t}$. Construct the family of matrices $M^\lambda$ in the following way. Put its elements $m_{kl}^\lambda$ as follows:

$$m_{kl}^\lambda = \begin{cases} 0, & \text{if } k > l + 1, \\ 1, & \text{if } k = l + 1, \\ a_{kl}^\lambda \cdot (Dt)^{l-k+1}, & \text{if } k < l + 1. \end{cases}$$

Now consider the family of differential operators

$$\tilde{L}^\lambda = \text{det}_{\text{right}}(DE - M^\lambda),$$

where $\text{det}_{\text{right}}$ means “right determinant”, i.e. the determinant, which is calculated with respect to the rightmost column; all minors are calculated in the same way. Dividing $\tilde{L}^\lambda$ on the left by $D$, we get the family of operators $L^\lambda$, so $\tilde{L}^\lambda = DL^\lambda$.

**Definition 2.2.1** ([Go1], 1.8). The equation of the family $L^\lambda[\Phi(t)] = 0$ is called counting equation $D3$.

**Conjecture 2.2.2** (V. Golyshev, [Go1]). The solution of the $D3$ equation for smooth Fano threefold is modular. More precisely, let $X$ be such variety, $i_X$ be its index, and $N = \frac{\deg X}{2i_X}$. Then in the family of counting equations for $X$ there is one, $L^\lambda_X[\Phi(t)] = 0$, whose solution is an Eisenstein series of weight $2$ on $X_0(N)$.

Based on this conjecture, V. Golyshev in [Go1] gives a list of predictions for counting matrices of Fano threefolds. So we should find all of the counting matrices for Fano varieties to check it.

### 3. $I$-series and Grassmannians

3.1. Mukai Theorem.

**Theorem 3.1.1** ([M1]). A smooth Fano threefold $V_{10}$ of genus 6 (and anticanonical degree 10) is a section of grassmannian $G(2, 5)$ by a linear subspace of codimension 2 and a quadric in the Plücker embedding.

A smooth Fano threefold $V_{14}$ of genus 8 (and anticanonical degree 14) is a section of grassmannian $G(2, 6)$ by a linear subspace of codimension 5 in the Plücker embedding.

This theorem is a particular case of the general Mukai Theorem, which describes all smooth Fano threefolds with Picard number 1 as sections of certain sheaves on grassmannians.
3.2. $I$-series for grassmannians.

**Theorem 3.2.1** (The Hori–Vafa conjecture ([HV], Appendix A), proof in [BCK]). Let $x_1, \ldots, x_r$ be the Chern roots of the dual to the tautological subbundle $S^*$ on $G = G(r, n)$ and $r > 1$. Then

$$I_G = \sum_{d \geq 0} (-1)^{r-1} \sum_{d_1 + \ldots + d_r = d} \frac{\prod_{1 \leq i < j \leq r} (x_i + d_i - x_j - d_j)}{\prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \prod_{l=1}^{d_i} (x_i + l)^n} q^d.$$ 

In the case $r = 1$ (i.e. for projective space)

$$I^{P_n-1} = \sum_{d \geq 0} \prod_{i=1}^d \frac{q^d}{(H + i)^n},$$

where $H$ is Poincaré dual to the class of hyperplane section.

**Corollary 3.2.2** ([BCK], Proposition 3.5). The constant term of $I^G$ for $G = G(2, n)$ is

$$\sum_{d \geq 0} \sum_{(d!)} \sum_{d = 0}^d \left( \frac{d}{m} \right)^n (n(d - 2m)(\gamma(m) - \gamma(d - m)) + 2),$$

where $\gamma(m) = \sum_{j=1}^m \frac{1}{j}$ and $\gamma(0) = 0$.

4. Quantum Lefschetz Theorem

The variety $X$ is smooth projective and with Picard group $\mathbb{Z}$ as before.

4.1. $I$-series of hypersurfaces.

**Lemma 4.1.1** ([Ga1], Lemma 5.5 or proof of Lemma 1 in [LP]). Let $Y \subset X$ be a complete intersection and $\varphi: H^*(X) \to H^*(Y)$ be the restriction homomorphism. Let $\tilde{\gamma}_1 \in \varphi(H^*(X))^\perp$ and $\gamma_2, \ldots, \gamma_l \in \varphi(H^*(X))$. Then for each $\beta \in \varphi(H_2^+(X)) \subset H_2^+(Y)$ the Gromov–Witten invariant on $Y$ of the form

$$\langle \tau_{d_1} \tilde{\gamma}_1, \tau_{d_2} \gamma_2, \ldots, \tau_{d_l} \gamma_l \rangle^\beta$$

vanishes.

Thus, by divisor axiom (1) and this lemma $I^Y = I^{Y \text{ rest}}$.

**Remark 4.1.2.** In contrast to the one-pointed invariants, the two-pointed ones, which correspond the classes in $\varphi(H^*(X))^\perp$, can be non-zero (see [Bea], Proposition 1).

4.2. Mirror Formula.

**Theorem 4.2.1** (Mirror Formula, [Ga2], Corollary 1.13). Let $Y \subset X$ be a hypersurface and $-K_Y \geq 0$. Then there exist series $R(q) \in H^*(X)[[q]]$ and $S(q) \in H^*(X)[[q]]$ such that

$$\sum_{\beta} \prod_{i=0}^{Y, \beta} (Y + i) \cdot I^X_{\beta} \cdot q^{Y, \beta} = R(q) \cdot \sum_{\beta} I^Y_{\beta} \cdot \tilde{q}^{Y, \beta},$$

where $\tilde{q} = q \cdot e^{S(q)}$.

A. Gathmann describes these series in Definition 1.11 and Lemma 1.12 from [Ga2]. In the case of Fano varieties the Mirror Formula may be simplified.
Corollary 4.2.2. Let $I = \text{ind} Y \in \mathbb{N}$ be the index of Fano variety $Y$. If $Y$ is complete intersection of hypersurfaces $Y_1, \ldots, Y_k$ of degrees $d_1, \ldots, d_k$ in $X$, then

$$\sum_{\beta} \prod_{j=1}^{k} \prod_{i=0}^{Y_j} (Y_j + i) \cdot I_{i,j}^{X} \cdot q^{Y \cdot \beta} = e^{\alpha_Y \cdot q^{\deg Y}} \cdot \sum_{\beta} I_{i,j}^{Y} \cdot q^{Y \cdot \beta},$$

(3)

where $\alpha_Y = \prod d_i! \cdot I_{i,j}^{X}$ if $I = 1$ and $\alpha_Y = 0$ if $I \geq 2$.

Proof. The form of series $R(q)$ and $S(q)$ involves that for Fano varieties $R(q) = e^{\alpha_Y \cdot q^{\deg Y}}$ and $S(q) = 0$. For $I > 1$ we have $R(q) = 1$. To find the number $\alpha_Y$ for the other case notice that by the dimensional argument $\langle H^{\dim Y} \rangle = 0$, where $H$ is a hyperplane section of $X$. Comparing the coefficients of the formula (3) at $q^{\deg Y}$ and $H^k$, we get

$$\alpha_Y = \prod_{i=1}^{k} d_i! \cdot I_{i,j}^{X}.$$ 

\[ \square \]

Remark 4.2.3. Conjecture 4.2.2 may be generalized in the following way. Put $\alpha_Y = \lambda_Y$. Then the equation $L^{X}[\Phi(t)] = 0$ has an Eisenstein series of weight 2 on $X_0(N)$ as a solution.

5. Expressions for one-pointed invariants in terms of prime two-pointed ones

Definition 5.1. Subalgebra $R \in H^*(Y)$ is called quantum self-dual if for all $\gamma \in R$, $\mu \in R^+$ and $\beta \in H_2^+(Y)$ the following holds: $\gamma^\vee \in R$ and $\langle \gamma, \mu \rangle = 0$.

Proposition 5.2. For each $n, I \in \mathbb{N}, k, d \in \mathbb{Z}_{\geq 0}$ there exist polynomial $f_k^d \in \mathbb{Q}[a_{ij}]$, $0 \leq i, j \leq n$, $j - i + 1 \leq d$ such that the following holds. Consider Fano variety $Y$ of dimension $n$ and index $I$ such that subalgebra $R \in H^*(Y)$ generated by $-K_Y = IH$ is quantum self-dual. Then

$$\langle \tau_k H^{d+n-2-k} \rangle_d = f_k^d(a_{ij}),$$

where $a_{i,j} = \langle H^{n-i}, H^\beta, H \rangle_{j-i+1} / \deg Y = \frac{i+j+1}{\deg Y} \langle H^{n-i}, H^j \rangle_{j-i+1}$. 

Proof. The strategy of finding $f_k^d$ is the following: express given one-pointed invariant in terms of three-pointed ones (with descendants) by divisor axiom. Then, by topological recursion relations, express these three-pointed invariants in terms of two-pointed ones.

Applying the divisor axiom for $H$

$$\langle \tau_k H^{d+n-2-k} \rangle_d = 1/d \cdot (\langle H, \tau_k H^{d+n-2-k} \rangle_d - \langle \tau_k H^{d+n-1-k} \rangle_d)$$

we get

$$\langle \tau_k H^{d+n-2-k} \rangle_d = \frac{1}{d} \sum_{i=0}^{k} \frac{(-1)^i}{d^i} \langle H, \tau_{k-i} H^{d+i+n-2-k} \rangle_d.$$ 

Now, by topological recursion (see, for instance, [KM2], Corollary 1.3)

$$\langle H^a, \tau_k H^{d+n-1-a-k} \rangle_d = 1/d (\sum_{d_1 \leq d} \langle H_{d-d_1+a-1}, \tau_{k-1} H^{d+n-1-a-k} \rangle_{d_1} \cdot a_{d_1} - a_{d_1} a_{d_1-a})$$

and

$$\langle H^a, \tau_{k-1} H^{d+n-a-k} \rangle_d.$$
This formula may be simplified further. If we express the last summand recursively on the right, we get

\[ \langle H^a, \tau_k H^{d+n-1-a-k} \rangle_d = \sum_{i=1,k, d_1 \leq d} \frac{(-1)^{i+1}}{d^i} a_{d_1-d+1+a, a} \langle H^{d_1-d+1+a}, \tau_k H^{d+n-2-a-k+i} \rangle_{d_1} + \frac{(-1)^k}{d^k} \langle H^a, H^{d+n-a} \rangle_d. \]

\[ \square \]

**Remark 5.3.** We make the assumption that Pic \( Y = \mathbb{Z} \) and \( R \) is generated by \( H \) for simplicity, as we use this in the following. One can find the similar expressions in the general case under the only assumption \( R \cap H^2(Y) \neq \emptyset \), using the same arguments as in the proof of Proposition 5.2.

**Example 5.4.** Consider a smooth Fano variety \( Y \) of index 1 and dimension 3, whose algebraic cohomology is generated by a generator of Picard group \( H \). Then we have:

- The constant term of \( I \)-series.
  1: \( \langle H^2 \rangle_2 = \deg Y \cdot a_{01}/4 \);
  2: \( \langle \tau H^3 \rangle_3 = \deg Y \cdot (a_{11}a_{01}/18 + a_{02}/27) \);
  3: \( \langle \tau_2 H^3 \rangle_4 = \deg Y \cdot (a_{01}^2/64 + a_{11}^2a_{01}/96 + 7a_{11}a_{02}/576 + a_{01}a_{12}/128 + a_{03}/256) \).

- The linear term of \( I \)-series with respect to \( H \).
  1: \( \langle H^2 \rangle_1 = \deg Y \cdot a_{11} \);
  2: \( \langle \tau H^2 \rangle_2 = \deg Y \cdot (a_{01}^2/4 + a_{12}/8 - a_{01}/4) \);
  3: \( \langle \tau_2 H^2 \rangle_3 = \deg Y \cdot (5a_{11}a_{01}/108 + a_{11}^2/18 + a_{11}a_{12}/12 - 2a_{02}/81) \);
  4: \( \langle \tau_3 H^2 \rangle_4 = \deg Y \cdot (13a_{11}^2a_{01}/576 + 17a_{11}a_{02}/1728 - a_{03}/256 - 3a_{01}^2/128 + a_{11}^4/96 + a_{12}^2/256 + a_{11}^2a_{12}/32) \).

Thus,

\[ I^Y = 1 + a_{11}q + (a_{01}/4 + (a_{11}/4 + a_{12}/8 - a_{01}/4)H)q^2 + ((a_{11}a_{01}/18 + a_{02}/27) + (5a_{11}a_{01}/108 + a_{11}^2/18 + a_{11}a_{12}/12 - 2a_{02}/81)H)q^3 + ((a_{01}^2/64 + a_{11}^2a_{01}/96 + 7a_{11}a_{02}/576 + a_{01}a_{12}/128 + a_{03}/256) + (13a_{11}^2a_{01}/576 + 17a_{11}a_{02}/1728 - a_{03}/256 - 3a_{01}^2/128 + a_{11}^4/96 + a_{12}^2/256 + a_{11}^2a_{12}/32)H)q^4 + \ldots \quad \text{(mod } H^2). \]

6. Main theorem

6.1. Complete intersections in grassmannians.

**Theorem 6.1.1.** The counting matrices of Fano threefolds \( V_{10} \) and \( V_{14} \) coincide with the predictions in [Go1].

1: For \( V_{10} \)

\[ M(V_{10}) = \begin{bmatrix}
0 & 156 & 3600 & 33120 \\
1 & 10 & 380 & 3600 \\
0 & 1 & 10 & 156 \\
0 & 0 & 1 & 0
\end{bmatrix}. \]

The shift \( \alpha_{V_{10}} \) is 6.

2: For \( V_{14} \)

\[ M(V_{14}) = \begin{bmatrix}
0 & 64 & 924 & 5936 \\
1 & 5 & 140 & 924 \\
0 & 1 & 5 & 64 \\
0 & 0 & 1 & 0
\end{bmatrix}. \]
The shift $\alpha_{V_{14}}$ is 4.

Proof. By Theorem 3.1 these varieties are complete intersections in $G(2,5)$ (for $V_{10}$) and $G(2,6)$ (for $V_{14}$).

Let $H$ be an effective generator of the Picard group of grassmannian $G = G(r,n)$. Put $I^G = I^G_{H^0}(q) + I^G_{H^1}(q) \cdot H \pmod{H^4(G)}$, that is

$$I^G = I^G_{H^0}(q) + I^G_{H^1}(q) \cdot H + \tilde{I},$$

where $\tilde{I} \in H^{>2}(G)$.

By Corollary 3.2.2

$$I^G_{H^0}(2,5) = 1 + 3q + \frac{19}{32}q^2 + \frac{49}{2592}q^3 + \frac{139}{884736}q^4 + \ldots,$$

$$I^G_{H^0}(2,6) = 1 + 4q + \frac{3}{4}q^2 + \frac{95}{5832}q^3 + \frac{865}{11943936}q^4 + \ldots.$$

Theorem 3.2.1 enables one to consider the $I$-series of grassmannians $G(2,5)$ and $G(2,6)$ as series in elementary symmetric functions $x_1 + x_2$ and $x_1 x_2$, where $x_1$ and $x_2$ are the Chern roots of the dual to the tautological subbundle. Then $I^G_{H^1}$ is the coefficient at the linear symmetric function $x_1 + x_2$. Thus,

$$I^G_{H^1}(2,5) = 10q + \frac{105}{32}q^2 + \frac{3115}{23328}q^3 + \frac{6875}{5308416}q^4 + \ldots,$$

$$I^G_{H^1}(2,6) = 15q + \frac{609}{128}q^2 + \frac{6197}{46656}q^3 + \frac{528737}{76441904}q^4 + \ldots.$$

By Corollary 4.2.2 $\alpha_{V_{10}} = 6$ and $\alpha_{V_{14}} = 4$. By formula (3), taken modulo $H^2$,

$$I^{V_{10}} = 1 + 10Hq + (39 + \frac{67}{2}H)q^2 + (220 + \frac{3200}{9}H)q^3 + (\frac{6291}{4} + \frac{89387}{48}H)q^4 + \ldots \pmod{H^2},$$

$$I^{V_{14}} = 1 + 5Hq + (16 + \frac{31}{4}H)q^2 + (2 + \frac{1031}{18}H)q^3 + (230 + \frac{14863}{96}H)q^4 + \ldots \pmod{H^2}.$$

It is easy to see that the expressions from Example 5.4 enable one to recover coefficients of the counting matrix of $Y$ in terms of $I^Y$ (mod $H^2$).

\[\Box\]

6.2. Expressions for two-pointed invariants in terms of one-pointed ones. The method of finding of two- (and more) pointed invariants in terms of one-pointed ones that we use in Theorem 6.1.1 may be used in the general case. More precisely, consider a variety $Y$ of dimension $n$ and quantum self-dual subalgebra $R \subset H^*(Y)$ with basis $\{\gamma_0, \ldots, \gamma_N\}$. Let $d \in \mathbb{N}$, $\langle \tau_i \gamma_j \rangle_k = f_{ij}^k(\langle \gamma_i, \gamma_j \rangle_d)$ (see Remark 5.3). Assume that one-pointed invariants of $Y$ and two-pointed prime ones that correspond to curves of degree $d$ are known. Then functions $f_{ij}^{d+1}$ express one-pointed invariants that correspond to curves of degree $d+1$ in terms of on prime two-pointed ones that correspond to curves of degree $d+1$ linearly. Moreover, one can choose a collection $\{(i,j)\}$ (where $i \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq N$, $i + j = d + n - 1$) such that a system of linear equations $\{f_{ij}^{d+1}(\langle \gamma_k, \gamma_l \rangle_{d+1}) = \langle \tau_i \gamma_j \rangle_{d+1}\}$, is given by nondegenerate upper-triangular matrix. Thus, by induction on $d$ one can find polynomial expressions for two-pointed invariants in terms of one-pointed ones.

**Theorem 6.2.1** ([BR], Theorem 5.2). Let $R \subset H^*(X)$ be the subalgebra generated by Picard group generator and $I^X = I^X_R$. Then Gromov–Witten invariants of type $\langle \tau_i \gamma_1, \ldots, \tau_n \gamma_n \rangle_d$, $\gamma_i \in R$ are completely determined by coefficients of $I$-series $I^X$. 

In fact, there is a way to recover two-pointed invariants from the constant term of the variety alone. Consider it for the threefold case example.

Consider a smooth Fano threefold \( Y \) of rank 1 and index 1. Put \( I_Y^H = 1 + d_2q^2 + d_3q^3 + d_4q^4 + d_5q^5 + d_6q^6 + \ldots \). Consider a map \( f : \mathbb{A}^5 \to \mathbb{A}^5 \), given by polynomials \( f_i \) that correspond to invariants \( \langle \tau_{i-2\text{pt}} \rangle_i = d_i \) for \( i = 2, \ldots , 6 \).

**Proposition 6.2.2.** The map \( f \) is birational.

**Proof.** Direct computations. Remark that this map is biregular if

\[-495d_3d_5 + 261d_2d_3^2 - 312d_4d_2^2 + 432d_4^2 + 56d_2^4 \neq 0,\]

which (a posteriori) holds for smooth Fano threefolds of rank 1.

\[\square\]

6.3. **Golyshev’s conjecture.** Using the same arguments as in this paper, one can reproduce the counting matrices for complete intersections in projective spaces (recovered by Golyshev in [Go2] from the corresponding D3 equations from Givental’s Theorem in [Gi]) and the Fano threefold \( V_5 \). The counting matrix for it was found by Beauville in [Bea]. Golyshev’s conjecture holds for these varieties. It may also be checked for \( V_{12} \) (which is a section of the orthogonal grassmannian \( OG(5, 10) \) by a linear subspace of codimension 7), \( V_{16} \) (which is a section of the Lagrangian grassmannian \( LG(3, 6) \) by a linear subspace of codimension 3), and \( V_{18} \) (which is a section of the grassmannian of the group \( G_2 \) by the linear subspace of codimension 2). Quantum multiplication by the divisor class of such grassmannians is computed in [FW]. Evidence in support of the conjecture for these varieties has been obtained by Golyshev. Counting matrix for \( V_{22} \) was computed by A. Kuznetsov; it also agrees with the prediction. Finally, counting matrices for double cover of \( \mathbb{P}^3 \) branched over a quartic, for double cover of \( \mathbb{P}^3 \) branched over a sextic, and for double cover of the cone over the Veronese surface branched over a cubic may be also found by this method. For this use their description as smooth hypersurfaces in weighted projective spaces and extend Givental’s formula for complete intersections in smooth toric varieties to the case of complete intersections in singular toric varieties.

6.4. **The case of Picard number strictly greater than 1.**

**Remark 6.4.1.** All the methods for varieties with Picard number 1 described above may be easily generalized to the case of Picard number greater than 1. One should use a multidegree \( d = (d_1, \ldots , d_r) \) instead of the degree and multidimensional variable \( q = (q_1, \ldots , q_r) \) instead of \( q \), and so on.

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