ADHM and D-instantons in orbifold AdS/CFT duality

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Abstract: We consider ADHM instantons in product group gauge theories that arise from D3-branes located at points in the orbifold $\mathbb{R}^6/\mathbb{Z}_p$. At finite $N$ we argue that the ADHM construction and collective coordinate integration measure can be deduced from the dynamics of D-instantons in the D3-brane background. For the large-$N$ conformal field theories of this type, we compute a saddle-point approximation of the ADHM integration measure and show that it is proportional to the partition function of D-instantons in the dual $AdS_5 \times S^5/\mathbb{Z}_p$ background, in agreement with the orbifold AdS/CFT correspondence. Matching the expected behaviour of D-instantons, we find that when $S^5/\mathbb{Z}_p$ is smooth a saddle-point solution only exists in the sector where the instanton charges in each gauge group factor are the same. However, when $S^5/\mathbb{Z}_p$ is singular, the instanton charges at large $N$ need not be the same and the space of saddle-point solutions has a number of distinct branches which represent the possible fractionations of D-instantons at the singularity. For the theories with a type 0B dual the saddle-point solutions manifest two types of D-instantons.

Keywords: Solitons Monopoles and Instantons, $1/N$ Expansion, Duality in Gauge Field Theories, Supersymmetry and Duality.
1. Introduction

There is a natural relation between D-branes and the ADHM construction of multi-instanton solutions in gauge theory which arises from considering a configuration of \( k \) Dp-branes embedded in \( N \) coincident D\((p + 4)\)-branes. The embedding is described by a charge \( k \) instanton solution in the four-dimensional SU\((N)\) gauge theory associated to the four transverse directions of the Dp-branes in the D\((p + 4)\)-brane world volume [1–4]. This correspondence is valid for arbitrary values of \( N \), but in the large-\( N \) limit it simplifies dramatically [4] leading to powerful instanton tests [4–6] of the AdS/CFT duality in \( \mathcal{N} = 4 \) gauge theory [7, 8]. Thus both at finite \( N \) and in the large-\( N \) limit there is a very appealing relation between ADHM and D-instantons. It is this relation in more general theories that we want to further investigate in this paper.

To see how the collective coordinate integration measure for ADHM multi-instantons emerges from D-brane dynamics, consider \( k \) D\((-1)\)-branes (D-instantons) embedded in \( N \) coincident D3-branes. The theory on the world volume of \( N \) D3-branes is four-dimensional \( \mathcal{N} = 4 \) supersymmetric SU\((N)\) gauge theory. The \( k \) D-instantons correspond to ‘point-like’ topological defects identified with Yang-Mills instantons of charge \( k \). The theory on the world volume of the D-instantons describes the Yang-Mills instanton dynamics. This is a zero-dimensional matrix theory whose partition function in the strong coupling \( \alpha' \to 0 \) limit gives the ADHM collective coordinate integration measure (weighted with the instanton action). This relation holds for arbitrary \( N \) and the \( \alpha' \to 0 \) limit is taken in order to decouple the world volume theory from gravity in the bulk. This finite-\( N \) approach was used in Sec. IV.2 of [4] to derive directly from D-brane dynamics the ADHM measure in \( \mathcal{N} = 4 \) supersymmetric SU\((N)\) gauge theory. The resulting expression is in precise agreement with the expression previously deduced in Ref. [9, 10] from the field theory considerations alone.

The matrix model describing the D-instantons inside the D3-branes can be viewed as a dimensional reduction to zero dimensions of the effective theory describing D5-branes inside D9-branes. The theory on the world volume of the \( k \) D5-branes is pure \( \mathcal{N} = (1,1) \) supersymmetric U\((k)\) gauge theory in six dimensions with \( N \), the number of D9-branes, additional hypermultiplets (and so the resulting theory actually only has \( \mathcal{N} = (1,0) \) supersymmetry). What is particularly striking is that the auxiliary degrees-of-freedom \( \chi \) introduced in [4, 5] to bi-linearize the four-fermion interaction in the ADHM instanton action arise now in a very natural way as the scalars corresponding to the six-dimensional gauge field [4] and describe the freedom for the D-instantons to be ejected from the D3-branes. This geometrical interpretation of \( \chi \) variables deserves a comment. One might think that since we have started with D5-branes inside D9-branes and then dimensionally reduced six dimensions common to the both types of branes, we must end up with D-instantons lying inside the D3-branes, however, this static reasoning however is naïve. The six-dimensional gauge field \( \chi \) living on the world volume of the D5-branes turns into six scalar ‘fields’ after the dimensional reduction. The six scalars \( \chi \) specify excitations of the D-instantons transverse to the D3 world volume. Thus, when \( \chi \) are non-zero
the D-instantons are in fact ejected from the world volume of the D3-branes. In the large-\(N\) limit, it turns out that \(\chi\) gains a ‘VEV’ which constrains its length and an \(S^5\) is generated!

Ref. [4] went on the consider the large-\(N\) limit of this measure. Using a steepest descent approximation the result is very simple to state. The large-\(N\) gauge invariant ADHM measure is proportional to the partition function of the six-dimensional pure \(\mathcal{N} = (1, 1)\) supersymmetric U(\(k\)) gauge theory, with no additional matter, dimensionally reduced to zero dimensions. Alternatively, this can be described as the ten-dimensional \(\mathcal{N} = 1\) supersymmetric pure U(\(k\)) gauge theory dimensionally reduced to zero dimensions. This is precisely what one expects for D-instantons in a flat background with no D3 branes present, where the U(1) \(\subset\) U(\(k\)) components of the ten-dimensional gauge field are interpreted as the position of the charge-\(k\) D-instanton in \(R^{10}\). The only difference with the large-\(N\) measure is that the U(1) components of the gauge field are now interpreted as the position of the charge-\(k\) D-instanton in \(AdS_5 \times S^5\), the near horizon geometry of the \(N \to \infty\) D3-branes.

To make this more precise it is convenient to view the gauge-invariant ADHM integration measure as a product of two factors: the centre-of-mass measure and the reduced measure. The centre-of-mass measure includes only the global collective coordinates—the position, the scale-size, the supersymmetric and superconformal fermion zero modes, and the \(\chi\)-coordinates of the multi-instanton configuration as the whole. The second factor—the reduced measure—includes integrations over the remaining relative collective coordinates (relative positions, relative scale sizes, etc.). It follows from the analysis of Ref. [4] that the reduced gauge-invariant ADHM measure in the large-\(N\) limit equals the non-abelian part of the D-instanton measure in the flat background, while the centre-of-mass ADHM measure gives the volume element of \(AdS_5 \times S^5\).

This concludes the review of the finite-\(N\) and large-\(N\) relations between D-instantons and Yang-Mills instantons in \(\mathcal{N} = 4\) gauge theory. We are now ready to generalize this picture other theories with \(\mathcal{N} < 4\). An interesting class of generalizations of the AdS/CFT correspondence [7,8] to four dimensional theories with less than \(\mathcal{N} = 4\) supersymmetry, follows from considering string theory on a background where the \(S^5\) is replaced by the quotient \(S^5/\Gamma\), where in general the finite group \(\Gamma\) does not necessarily act freely and so the resulting space has singularities [11–13]. For simplicity here we shall only consider the abelian cases \(\Gamma = \mathbb{Z}_p\), although our results have an obvious generalization to the non-abelian cases. The relevant string theory is either type IIB, or the non-supersymmetric type 0B, depending upon the action of \(\mathbb{Z}_p\) on the fields of the theory. There is substantial evidence that the dual gauge theory is simply a certain \(\mathbb{Z}_p\)-projection of the \(\mathcal{N} = 4\) theory, where \(\mathbb{Z}_p\) acts both on gauge and on the SU(4)\(_R\) indices of the various fields [12]. The resulting theories can have either \(\mathcal{N} = 0, 1\) or 2 supersymmetries.

Before summarizing our findings, we briefly review the description of the \(\mathbb{Z}_p\)-projected gauge theories and their relation with D-branes on orbifolds.
1.1 The projected $\mathcal{N} = 4$ gauge theories and D-branes on orbifolds

The theories that we will consider can be defined as projections of an $\mathcal{N} = 4$ supersymmetric gauge theory with a unitary gauge group $U(n)$ [11–13]. The projection is obtained by an action of a finite group $\Gamma$ which is embedded in both the gauge group and the $SU(4)_R$ group of $R$-symmetries. The embeddings $\Gamma \subset U(n)$ and $\Gamma \subset SU(4)_R$ are specified by the decomposition of the adjoint representation of the gauge group and the 4 of SU(4)$_R$, respectively, into the irreducible representations of $\Gamma$. The projected theory is then defined by the same action of the $\mathcal{N} = 4$ theory with all the $\Gamma$ non-invariant fields set to zero.

The construction outlined above works for any such $\Gamma$, however for simplicity, we shall concentrate on the abelian cases where $\Gamma = \mathbb{Z}_p$. Without loss of generality we can take the embedding $\mathbb{Z}_p \subset U(n)$ to be generated by

$$\sigma_{\{N_q\}} = \begin{pmatrix} 1_{[N_1] \times [N_1]} & \ldots & \ldots & e^{2\pi i/p}1_{[N_p] \times [N_p]} \\ e^{2\pi i/p}1_{[N_2] \times [N_2]} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & e^{2(p-1)\pi i/p}1_{[N_p] \times [N_p]} \end{pmatrix}.$$ (1.1)

The embedding $\mathbb{Z}_p \subset SU(4)_R$ can be defined by a set of four phases, or equivalently four integers $\{q_1, q_2, q_3, q_4\}$ defined modulo $p$, subject to the condition

$$q_1 + q_2 + q_3 + q_4 = 0 \mod p.$$ (1.2)

The gauge field $v_n$ is an SU(4)$_R$ singlet and hence invariant under $\mathbb{Z}_p \subset SU(4)_R$. On the fermions $\lambda^A$, which transform as a 4 of SU(4)$_R$, the action of $\mathbb{Z}_p \subset SU(4)_R$ is generated by

$$\lambda^A \rightarrow e^{2\pi i q_A/p} \lambda^A.$$ (1.3)

The action on the scalar fields which form a 6 of SU(4)$_R$ (or a vector of SO(6) $\subset SU(4)_R$) is given most easily by writing them as an anti-symmetric tensor $A^{AB}$, subject to the reality condition

$$(A^{AB})^\dagger = \frac{1}{2} \epsilon_{ABCD} A^{CD}.$$ (1.4)

In this basis the action of $\mathbb{Z}_p \subset SU(4)_R$ is generated by

$$A^{AB} \rightarrow e^{2\pi i (q_A + q_B)/p} A^{AB}.$$ (1.5)

The $\mathbb{Z}_p$ action on SO(6) vectors as specified above may not give a faithful representation of $\mathbb{Z}_p$. This happens when all the combinations $q_A + q_B$ are even and when $p$ itself is even, in which case $\mathbb{Z}_p$ is not a faithful representation of SO(6). In particular the inner product $A_a B_a \equiv \epsilon_{ABCD} A^{AB} B^{CD}$.

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1Here $\dagger$ acts only on gauge indices and not on SU(4)$_R$ indices. We frequently pass between the antisymmetric tensor representation $A^{AB}$ and explicit SO(6) vector representation $A_a$, via $A^{AB} = (1/\sqrt{8}) \Sigma_a A_a$, where the coefficients are defined in the Appendix of [4]. In particular the inner product $A_a B_a \equiv \epsilon_{ABCD} A^{AB} B^{CD}$.
case only the subgroup $\mathbb{Z}_{p/2} \subset \mathbb{Z}_p$ is faithfully represented on vectors, a subtlety which will prove to be important.

We are now in a position to implement the $\mathbb{Z}_p$ projection on the fields. Taking a combination of the gauge transformation (1.1) and the action on $SU(4)_R$ indices (1.3):

$$\sigma_{\{N_q\}} v_n \sigma_{\{N_q\}}^{-1} = v_n, \quad \sigma_{\{N_q\}}^A \sigma_{\{N_q\}}^{-1} = e^{-2\pi i A/p} \lambda^A, \quad \sigma_{\{N_q\}} A^{AB} \sigma_{\{N_q\}}^{-1} = e^{-2\pi i (q_A + q_B)/p} A^{AB}.$$  

(1.6)

Hence the gauge group of the projected theory is $U(N_1) \times \cdots \times U(N_p)$ with $n = \sum_{q=1}^p N_q$. An element of the gauge group of the projected theory has the block diagonal form

$$
\begin{pmatrix}
U^{(1)}_{[N_1] \times [N_1]} \\
U^{(2)}_{[N_2] \times [N_2]} \\
\vdots \\
U^{(p)}_{[N_p] \times [N_p]} 
\end{pmatrix},
$$

(1.7)

where $U^{(q)} \in U(N_q)$. The abelian components of the $U(N_q)$ factors actually decouple in the infra-red and so effectively we can take the gauge group to be $SU(N_1) \times \cdots \times SU(N_p)$.\(^2\) In order to write down the projected fields we introduce a block-form notation for the $U(n)$ adjoint-valued fields of the parent theory. Let $E_{q r}$ be the $p \times p$ matrix with a one in position $(q, r)$ and zeros elsewhere.\(^3\) The gauge field of the projected theory has the block-diagonal form

$$v_n = \sum_{q=1}^p v_n^{(q)} \otimes E_{q q},$$

(1.8)

where the block $v_n^{(q)}$ is the gauge field of the $U(N_q)$ subgroup of the gauge group. The fermions and scalars have off-diagonal components, since the scalar and fermion fields in the original theory have non-trivial transformations under the $R$-symmetry:

$$\lambda^A = \sum_{q=1}^p \lambda^{(q)A} \otimes E_{qq+q_A}, \quad A^{AB} = \sum_{q=1}^p A^{(q)AB} \otimes E_{qq+q_A+q_B}.$$  

(1.9)

One can see that the spectrum of fields in the theory consists of the gauge bosons of the product gauge group along with various matter fields that transform in bi-fundamental representations $(N_q, \bar{N}_r)$ of a pair of the group factors, or adjoint representations of a single $U(N_q)$ factor. The amount of supersymmetry depends on the set of integers $\{q_1, q_2, q_3, q_4\}$. We can take (subject to (1.2)):

\(^2\)This decoupling is valid in the context of a four dimensional theory, however, we shall also use the same kind of projection in lower dimensional theories where the decoupling does not occur.

\(^3\)We will always think of the labels $q, r, q_A,$ etc., as being defined modulo $p$. 

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(i) \( \{0, 0, q_3, q_4\} \) (without loss of generality we can take \( q_3 = -q_4 = 1 \)). These models have \( \mathcal{N} = 2 \) supersymmetry and \( \mathbb{Z}_p \) leaves two components of an SO(6) vector fixed (in SU(4) language the two components \( A^{12} \) and \( A^{34} \)).

(ii) \( \{0, q_2, q_3, q_4\} \) with \( q_2, q_3, q_4 \neq 0 \mod p \). These models have \( \mathcal{N} = 1 \) supersymmetry.

(iii) All the other case have \( \mathcal{N} = 0 \) supersymmetry.

These theories describe the low energy dynamics of D3-branes on the orbifold \( \mathbb{R}^4 \times \mathbb{R}^6 / \mathbb{Z}_p \), where the D3-branes lie along the \( \mathbb{R}^4 \) factor and so at a point on the \( \mathbb{R}^6 / \mathbb{Z}_p \) orbifold [14]. One way to analyze this set-up is to consider the D3-branes on the covering space \( \mathbb{R}^{10} \) using a method of images. Each D3-brane will then come with its images under \( \mathbb{Z}_p \). So each collection of \( N \) D3-branes at the same point in \( \mathbb{R}^6 \) will have \( p \) images. This naturally gives rise to the gauge theory described above but with \( N \equiv N_1 = \cdots = N_p \). This is situation where the adjoint of the original \( U(n) \equiv U(pN) \) theory decomposes into \( N \) copies of the regular representation of \( \mathbb{Z}_p \). However, there is more freedom if the D3-branes are at the singularity of the orbifold [14]. In that case, there are no images and consequently one can permit the more general situation with gauge group \( U(N_1) \times \cdots \times U(N_p) \). Each \( U(N_q) \) factor describes \( N_q \) ‘fractional’ D3-branes which are confined to move on the orbifold singularity [14–16]. In this case \( p \) fractional D3-branes of each of the \( p \) types can form a genuine D3-brane which can then move away from the singularity.\(^4\)

It turns out that it is precisely the theories where the D3-branes can roam on the orbifold, i.e. \( N \equiv N_1 = \cdots = N_p \), which are relevant for large-\( N \) limit and an AdS/CFT duality, because these theories are, for \( \mathcal{N} = 1 \) and 2, conformal, and, for \( \mathcal{N} = 0 \), conformal at leading order in \( 1/N \) [13]. For the supersymmetric theories the large-\( N \) dual is conjectured to be type IIB superstring theory on \( AdS_5 \times S^5 / \mathbb{Z}_p \) [11]. For \( \mathcal{N} = 2 \) supersymmetry the action of \( \mathbb{Z}_p \) fixes an \( S^1 \subset S^5 \) and so the resulting spacetime has an orbifold singularity and consequently there is no supergravity description (unless the singularity is blown up in some way). On the contrary for the \( \mathcal{N} = 1 \) theories there are no singularities and we expect to have a supergravity description in the appropriate limit. Notice that in this case there is an important difference between the flat orbifold \( \mathbb{R}^4 \times \mathbb{R}^6 / \mathbb{Z}_p \) and the near horizon geometry \( AdS_5 \times S^5 / \mathbb{Z}_p \). The former has an orbifold singularity at the origin of \( \mathbb{R}^6 \), whereas there is no such singularity in the latter since the radius of the \( S^5 \) is constant over \( AdS_5 \) [11]. This fact will turn out to have an important implication for the large-\( N \) limit of the instanton measure.

For the \( \mathcal{N} = 0 \) theories the situation is slightly more complicated because there are two cases depending on whether \( \mathbb{Z}_p \subset SO(6) \subset SU(4) \) or not. In general only the subgroup \( \mathbb{Z}_p/2 \subset \mathbb{Z}_p \) acts faithfully on SO(6) vectors. In this case the element whose action on SO(6) vectors is not faithfully represented acts on the fields as \( (-1)^F \), where \( F \) is the fermion number.

\(^4\)It is interesting to view these fractional D3-branes in a T-dual set-up [17] where they correspond to a segment of D4-brane suspended between two NS5-branes as in [18].
operator. When $Z_p$ does act faithfully then the large-$N$ dual is expected to be the type IIB superstring on $AdS_5 \times S^5/Z_p$, as for the supersymmetric cases. However, when only $Z_{p/2}$ acts faithfully, the appearance of the factor $(-1)^F$ is a clue that in this case the dual theory is not the type IIB superstring but rather is the non-supersymmetric type 0B string. In particular, when $p = 2$, this is precisely the projection of the $\mathcal{N} = 4$ which is thought to be dual to the non-supersymmetric type 0B string on $AdS_5 \times S^5$ [19,20]. In general we expect the dual theory to be the type 0B string on $AdS_5' \times S^5/Z_{p/2}$.

1.2 Summary of results

As in Ref. [4] we can investigate the general $Z_p$-projected theories at finite $N$ by considering the dynamics of D3-branes and D-instantons in the flat orbifold background $\mathbb{R}^4 \times \mathbb{R}^6/Z_p$, where the D3-branes lie along $\mathbb{R}^4$ and hence at a point on the orbifold $\mathbb{R}^6/Z_p$. In particular if the D3-branes lie at the orbifold singularity then the resulting gauge theory can have the general product group structure $U(N_1) \times \cdots \times U(N_p)$ with matter fields transforming in bi-fundamental representations generalizing to a six-dimensional orbifold the set-up in Ref. [14]. This theory is the $Z_p$-projection of the $\mathcal{N} = 4$ supersymmetric gauge theory with gauge group $U(n)$, $n = \sum_{q=1}^{p} N_q$, described in the last section. We will argue that the matrix theory of D-instantons which lie at the orbifold singularity in the presence of the D3-branes is a certain $Z_p$-projection of the $U(K)$ matrix theory describing the D-instantons in the $\mathcal{N} = 4$ case, as described in more detail in Sec. 3.2.\footnote{In our notation $k_q$ is the instanton charge for the $U(N_q)$ factor of the gauge group. We also define $K = k_1 + \cdots + k_p$ to be the total instanton charge.} The discrete group $Z_p$ is embedded in the $U(K)$ group of the matrix theory as $\sigma_{\{k_q\}}$ and in the $U(n)$ ‘flavour’ group as $\sigma_{\{N_q\}}$. The group also acts, as in (1.3), on $SU(4)_R$, which in the matrix model is the covering group of the Lorentz group of the parent six-dimensional theory. The resulting theory consequently has symmetry groups $U(k_1) \times \cdots \times U(k_p)$ and $U(N_1) \times \cdots \times U(N_p)$. We argue that the partition function of the resulting matrix model in the decoupling limit $\alpha' \to 0$ is precisely the ADHM measure in the gauge theory in the instanton charge sector $\mathcal{C} = \{k_1, \ldots, k_p\}$, generalizing the $\mathcal{N} = 4$ case in [4] in an obvious way. In particular, in the $\mathcal{N} = 2$ case the resulting theories arise from the dimensional reduction of the D3 and D7-brane configurations in the $\mathbb{R}^4/Z_p$ orbifold background described in [14].

We then consider the large-$N$ gauge invariant measure for ADHM instanton in the conformal theories with gauge group $U(N) \times \cdots \times U(N)$ ($p$-times) and find the following results. Firstly, in (i) and (ii) below, for the theories whose dual is the type IIB theory (i.e. all the supersymmetric theories and the $\mathcal{N} = 0$ theories where $Z_p$ acts faithfully on SO(6) vectors):

(i) When $Z_p$ acts freely on SO(6) vectors, a solution of the large-$N$ saddle-point equations for the instanton measure only exists in the sector where all the instanton charges are the same. In other words when $S^5/Z_p$ is smooth in the dual theory, only the $\mathcal{C} = \{k, \ldots, k\}$ charge sectors contribute at leading order in $1/N$. The saddle-point solution describes $k$ point-like objects, the
D-instantons of the string theory, moving in $AdS_5 \times S^5 / \mathbb{Z}_p$. The large-$N$ instanton measure has the form of the partition function of the $\mathbb{Z}_p$-projected $\mathcal{N} = 1$ supersymmetric $U(K) \equiv U(pk)$ gauge theory in ten dimensions, dimensionally reduced to zero dimensions. This theory has gauge group $U(k) \times \cdots \times U(k)$. Equivalently we may describe it as the $\mathbb{Z}_p$-projection of the $\mathcal{N} = (1,1)$ pure six-dimensional $U(K)$ gauge theory dimensionally reduced to zero dimensions. The saddle-point solution corresponds to the coulomb branch of the matrix theory where the $U(k) \times \cdots \times U(k)$ gauge symmetry is generically broken to $U(1)^k_{\text{diag}}$.\(^6\)

(ii) When $\mathbb{Z}_p$ does not act freely on $SO(6)$ vectors a solution of the saddle-point equations exists in the general charge sector where the instanton numbers can be different $C = \{ k_1, \ldots, k_p \}$. In this case, the saddle-point solutions have multiple branches labelled by $\tilde{k} = 0, \ldots, \min(k_q)$. The branches describe $\tilde{k}$ D-instantons moving in $AdS_5 \times S^5 / \mathbb{Z}_p$ and $k_q - \tilde{k}$ fractional D-instantons of type $q$ moving in $AdS_5 \times (S^5 / \mathbb{Z}_p)_{\text{sing}}$. (Here $(S^5 / \mathbb{Z}_p)_{\text{sing}}$ is the subspace of $S^5$ fixed by the action of $\mathbb{Z}_p$.) The large-$N$ instanton measure is then described by the partition function of the $\mathbb{Z}_p$-projected $\mathcal{N} = 1$ ten-dimensional $U(K)$ gauge theory dimensionally reduced to zero dimensions, as in (i), but with the more general $\mathbb{Z}_p$-projection which permits the gauge group $U(k_1) \times \cdots \times U(k_p)$ (or as in (i) equivalently the $\mathbb{Z}_p$-projection of the $\mathcal{N} = (1,1)$ pure six-dimensional $U(K)$ gauge theory dimensionally reduced to zero dimensions). The saddle-point solutions correspond to a series of coulomb branches of the matrix theory where the symmetry $U(k_1) \times \cdots \times U(k_p)$ is generically broken to $U(1)^{\tilde{k}}_{\text{diag}} \times U(1)^{K-p\tilde{k}}$. When $\tilde{k} = k_1 = \cdots = k_p$ this is identical to the coulomb branch in (i) above.

(iii) In the non-supersymmetric theories with a type 0B dual where only $\mathbb{Z}_{p/2} \subset \mathbb{Z}_p$ acts faithfully on $SO(6)$ vectors, the solution of the saddle-point equations is slightly more complicated. In this case, we split the instantons into two sets with charges

$$C_+ = \{ k_2, k_4, \ldots, k_p \} \quad \text{and} \quad C_- = \{ k_1, k_3, \ldots, k_{p-1} \}. \quad (1.10)$$

From the point-of-view of the saddle-point analysis the two sets of instantons are completely decoupled.\(^7\) For each set $C_{\pm}$ separately the solutions of the saddle-point equations is analogous to (i) and (ii) above. So if $\mathbb{Z}_{p/2}$ acts freely on $S^5$ then a saddle-point solution only exists if each of the instanton charges in each sets $C_{\pm}$ is the same: $C_{\pm} = \{ k_{\pm}, \ldots, k_{\pm} \}$. In this case the solution can be interpreted as describing two kinds of D-instantons in $AdS_5 \times S^5 / \mathbb{Z}_{p/2}$. On the contrary, if $\mathbb{Z}_{p/2}$ does not act freely on $S^5$, then saddle-point solutions exist in the general $\{ k_1, \ldots, k_p \}$ charge sectors. As in the type IIB cases the solutions now exhibit multiple branches describing the fractionation of the two kinds of D-instantons.

Beginning with the type IIB cases, our results are exactly what one would expect from generalizing the result of the same analysis in the $\mathcal{N} = 4$ theory \cite{4}. In other words we find that the large-$N$ ADHM instanton measure is identical to the D-instanton partition function

\(^6\)This is the maximal abelian subgroup of the diagonal subgroup $U(k)_{\text{diag}} \subset U(k) \times \cdots \times U(k)$.

\(^7\)The point is that the two sets only communicate through fermionic variables which play no rôle in the saddle-point analysis.
on the flat orbifold $\mathbb{R}^4 \times \mathbb{R}^6/\mathbb{Z}_p$. This is to be expected: as in the $\mathcal{N} = 4$ theory the partition function for D-instantons on $AdS_5 \times S^5$ is identical to that in flat space. However, there is an important restriction in the cases when $S^5/\mathbb{Z}_p$ is smooth. In the corresponding flat orbifold, describing the finite-$N$ solution, the D-instantons could lie at the singularity and one would have contributions from the general charge sector $\{k_1, \ldots, k_p\}$ corresponding to fractional D-instantons. However the near horizon geometry $AdS_5 \times S^5/\mathbb{Z}_p$ is smooth and so we expect only the charge sector $\{k, \ldots, k\}$ can contribute at large $N$ and this is precisely what we find from our saddle-point analysis for case (i). When $S^5/\mathbb{Z}_p$ has a singularity there is no such constraint on the charge sector as we find from the saddle-point analysis in case (ii). In this case the space of solutions has a number of different branches which represents the possible fractionations of D-instantons at the singularity $(S^5/\mathbb{Z}_p)_{\text{sing}}$.

For the theories whose duals are the type 0B string theory the situation is essentially doubled up: in this case the instantons in the even/odd groups in the product $U(N) \times \cdots \times U(N)$ are independently interpreted as in the last paragraph with D-instantons in the string theory. It is interesting that this is exactly what is expected from the type 0B string theory since the number of Ramond-Ramond fields is doubled up compared with the type IIB theory. This means that each D-brane comes in two varieties: ‘magnetic’ and ‘electric’. It is pleasing that this feature also emerges as the outcome of a saddle-point analysis of large-$N$ gauge theory instantons.

2. The ADHM Construction for Product Groups

In this section we consider the construction of multi-instanton solutions for the projected $\mathcal{N} = 4$ theory. The construction is a very obvious generalization of the ADHM formalism set out in [4]. The new feature is that some of the matter fields transform in bi-fundamental representations of a pair of the gauge group factors. Fortunately this problem was considered in the early instanton literature [21] as part of a program to construct solutions for fields transforming in arbitrary representations of a gauge group in the back-ground of an ADHM instanton. The resulting formalism is such an obvious generalization of that for an adjoint-valued field that we will keep our discussion brief. Note that as far as the instanton solutions are concerned there is no difference between $U(N_q)$ and $SU(N_q)$ since the instanton is embedded in the non-abelian part of the gauge group.

2.1 The gauge fields

First of all consider the gauge fields. To leading order in the coupling one simply ignores the matter fields and considers the gauge fields in isolation. At this order, there are no couplings between the different $SU(N_q)$ group factors; in order words, one constructs a general instanton
solution by embedding instanton solutions with charges $k_q$, $q = 1, \ldots, p$, in each of the SU($N_q$) factors. The ADHM construction then involves the familiar objects $U^{(q)}$, $\tilde{U}^{(q)}$, $\Delta^{(q)}$, $a^{(q)}$, $b^{(q)}$, $f^{(q)}$, $a_n^{(q)}$, $w_n^{(q)}$, etc., for each of the groups. As with the gauge fields of the theory, it is convenient to assemble the ADHM parameters for the product group into a $p \times p$ block diagonal matrix where each block pertains to a single SU($N_q$) factor. So, for example

$$a = \sum_{q=1}^{p} a^{(q)} \otimes E_{qq}. \quad (2.1)$$

The matrix $a^{(q)}$ is then the ADHM matrix of the $q^{th}$ gauge group factor, and hence is $(N_q + 2k_q) \times 2k_q$ dimensional.

The ADHM variables, modulo the ADHM constraints, parameterize the moduli space of the instanton solution, up to an auxiliary symmetry

$$U(k_1) \times \cdots \times U(k_p), \quad (2.2)$$

where each factor $U(k_q)$ acts on the variables of the $q^{th}$ block in the way described in [4].

### 2.2 The fermion fields

Up till now we have simply used $p$ independent copies of the ADHM construction one for each of the SU($N_q$) group factors. This is obvious because the gauge fields of each SU($N_q$) factor of the instanton solution are completely decoupled. For any matter field which is in an adjoint representation of the gauge group, an eventuality that will occur if some $q_A = 0$, for the fermions, or some $q_A + q_B = 0$, for the scalars, the construction of the solution is exactly as [4]. The matter fields, however, can communicate between the different factors since some of them are in bi-fundamental representations.

Let us consider the general problem of a fermion $\lambda_\alpha$ transforming in the bi-fundamental $(N_q, \bar{N}_r)$ of SU($N_q$) × SU($N_r$). The equation we have to solve is the Dirac equation $\bar{\Phi}^{\dot{\alpha} \alpha} \lambda_\alpha = 0$ for $\lambda_\alpha$ in the background of an instanton solution with charge $k_q$ and $k_r$ in each of the two group factors, respectively. The solution generalizes that for an adjoint fermion [21] and has the form

$$\lambda_\alpha = U^{(q)} M^{(qr)} f^{(r)} \bar{b}^{(r)} U^{(r)} - \tilde{U}^{(q)} b^{(q)} f^{(q)} \tilde{M}^{(qr)} U^{(r)}, \quad (2.3)$$

where $U^{(q)}$ and $U^{(r)}$, etc., refer to objects from the ADHM construction for the groups SU($N_q$) and SU($N_r$), respectively. The ‘off-diagonal’ objects $M^{(qr)}$ and $\tilde{M}^{(qr)}$ are constant Grassmann matrices of dimension $(N_q + 2k_q) \times k_r$ and $k_q \times (N_r + 2k_r)$, respectively. The Dirac equation is then satisfied by virtue of the fermionic ADHM constraints

$$\bar{\Delta}^{(q)} \lambda_\alpha = \bar{\Delta}^{(q)} M^{(qr)} \Delta^{(r)} + \tilde{M}^{(qr)} \Delta^{(r)} = 0. \quad (2.4)$$

\[8\] We will assume that the reader is familiar with Sec. II of [4].
Equations (2.3), (2.4) correctly reproduce the adjoint fermion zero mode construction \cite{4, 21, 22} when \( N_q = N_r \) and \( k_q = k_r \), while when \( k_r = 0 \) the SU\((N_q)\) fundamental fermion zero mode construction is recovered \cite{21, 22}. In the latter case \( \Delta_{\dot{\alpha}}^{(r)} \) and \( \mathcal{M}^{(qr)} \) are trivial, and the collective coordinates \( \tilde{\mathcal{M}}^{(qr)} \) are unconstrained, just as required.

It is now rather easy to see how to generalize the construction when we have a collection of fermion fields that transform in a set of adjoint and bi-fundamental representations of the product group as in (1.9). The trick is to write the fermionic collective coordinates in terms of the block-form matrices already introduced in (1.9). To this end we introduce \( \mathcal{M}^A \) and \( \tilde{\mathcal{M}}^A \) that are four \((n+2K) \times K\) and \( K \times (n+2K)\) matrices of constant Grassmann numbers with non-zero elements in particular blocks:

\[
\mathcal{M}^A = \sum_{q=1}^{p} \mathcal{M}^{(q)A} \otimes E_{q,q_A}, \tag{2.5}
\]

where \( \mathcal{M}^{(q)A} \equiv \mathcal{M}^{(q+q_A)A} \) and \( \tilde{\mathcal{M}}^{(q)A} \equiv \tilde{\mathcal{M}}^{(q+q_A)A} \). The ADHM constraints for all the fermions can therefore be written in a single equation as

\[
\bar{\Delta}_{\dot{\alpha}} \mathcal{M}^A + \bar{\mathcal{M}}^A \Delta_{\dot{\alpha}} = 0. \tag{2.6}
\]

### 2.3 The scalar fields

In this section, we consider the ADHM formalism for the scalar fields. The construction for the scalar fields in the background of an instanton solution follows the same pattern as in the \( \mathcal{N} = 2 \) theories \cite{22} and in the \( \mathcal{N} = 4 \) theories \cite{4}.

To leading order, the scalar fields satisfy the covariant Klein-Gordon equation, with a source term bi-linear in fermions coming from the Yukawa interactions. Fortunately, the most obvious generalization of the solution in the \( \mathcal{N} = 4 \) theory provides the solution in the projected theories. As in the \( \mathcal{N} = 4 \) theory it is convenient to imagine a set of collective coordinates for the scalar fields, even though no actual moduli exist. In our theories, these pseudo collective coordinates take the block form

\[
\mathcal{A}^{AB} = \sum_{q=1}^{p} \mathcal{A}^{(q)AB} \otimes E_{q,q_A+q_B}, \tag{2.7}
\]

where \( \mathcal{A}^{(q)AB} \) is a \( k_q \times k_{q_A+q_B} \) matrix. The pseudo collective coordinates are then eliminated—at some later stage—by the algebraic equations

\[
\mathbf{L} \cdot \mathcal{A}^{AB} = \Lambda^{AB}. \tag{2.8}
\]

Here the right-hand side involves a bilinear in the fermionic collective coordinates

\[
\Lambda^{AB} = \frac{1}{2\sqrt{2}} \left( \tilde{\mathcal{M}}^A \mathcal{M}^B - \bar{\mathcal{M}}^B \mathcal{M}^A \right) \tag{2.9}
\]
and reflects the Yukawa source term. In (2.9), the linear operator \( \mathbf{L} \) on \( K \times K \) matrices is defined by

\[
\mathbf{L} \cdot \Omega = \frac{1}{2} \{ \Omega, W^0 \} + [a_n', [a_n', \Omega]] ,
\]

where \( K \times K \) hermitian matrix \( W^0 \) has the block-diagonal components

\[
W^{(q)0} = \bar{w}^{(q)\bar{\alpha}} w^{(q)}_{\bar{\alpha}} ,
\]

For later use, we define the components of \( \mathbf{L} \) with respect to the block-form basis:

\[
\mathbf{L} \cdot \sum_{q=1}^{p} \Omega^{(q)} \otimes E_{qq+r} = \sum_{q=1}^{p} \mathbf{L}^{(qq+r)} . \Omega^{(q)} \otimes E_{qq+r}
\]

from which we deduce that on \( k_q \times k_r \) matrices \( \Phi \)

\[
\mathbf{L}^{(qr)} \cdot \Phi = (\frac{1}{2} W^{(q)0} + a_n^{(q)} a_n^{(q)}) \Phi + \Phi (\frac{1}{2} W^{(r)0} + a_n^{(r)} a_n^{(r)}) - 2a_n^{(q)} \Phi a_n^{(r)} .
\]

### 2.4 The multi-instanton action

When the expressions for the gauge, fermions and scalar fields are substituted into the action the result is not simply a constant, but contains an interaction term which depends on the fermionic collective coordinates:

\[
S_{\text{inst}} = \frac{8\pi^2 K}{g^2} - iK \theta + S_{\text{quad}} ,
\]

where \( S_{\text{quad}} \) is a fermion quadrilinear interaction

\[
S_{\text{quad}} = \frac{\pi^2}{g^2} \epsilon_{ABCD} \text{tr}_K \Lambda^{AB} \Lambda^{CD} = \frac{\pi^2}{g^2} \epsilon_{ABCD} \text{tr}_K \Lambda^{AB} L^{-1} \Lambda^{CD} .
\]

The expression for \( S_{\text{quad}} \) is derived in a completely analogous way to the \( \mathcal{N} = 4 \) theory [4].

This is somewhat of a surprise: one might have expected the action to be a constant if we have a solution of the Euler-Lagrange equations. As explained in [4], the ADHM expressions for the gauge, fermion and scalar fields are not, in fact an exact solution to the Euler-Lagrange equations when all the fermion modes are ‘turned-on’. The fact is that although the fermion zero modes are Dirac zero-modes, the majority of them are lifted at tree level by the Yukawa interactions with the scalars. The philosophy that we adopt, and explained at length in [4], is that retaining collective coordinates for the lifted modes provide a convenient way of including the perturbative (tree level) effects of the modes. In this point-of-view, the only exact fermion zero modes, not lifted by (2.15), are the supersymmetric and superconformal zero modes. These modes are defined in terms of \( \mathcal{M}^A \) and \( \bar{\mathcal{M}}^A \), as in the \( \mathcal{N} = 4 \) case [4], but only for \( q_A = 0 \), giving 8, 4 and 0 modes, for \( \mathcal{N} = 2, 1 \) and 0, respectively.
3. The Collective Coordinate Measure

In this section we write down the integration measure on the space of ADHM collective coordinates and then show how for $N_q \geq 2k_q$ it can be reduced to a measure on the set of gauge invariant variables. Then in this new set of variables, the ADHM constraints will be explicitly resolved. Note that the restriction $N_q \geq 2k_q$ is certainly consistent with the large-$N$ limit.

3.1 The flat measure

In order to calculate physical quantities we need to know how to integrate on the space of ADHM variables. This is the measure induced from the full functional integral of the field theory. Thankfully, repeating the argument of Refs. [4, 9, 10] it turns out that the measure is remarkably simple when written in terms of the complete set of bosonic and fermionic ADHM variables: it is just the flat measure for all the variables with all algebraic constraints imposed via explicit delta functions. In order to define the physical measure, we must divide by the volume of the auxiliary group (2.2):

$$\int d\mu^{\{k\}}_{\text{phys}} \sim \frac{1}{\prod_{q=1}^{p} \text{Vol} U(k_q)} \int da' d\bar{w} dw \prod_{A=1,2,3,4} d\mathcal{M}^A d\bar{\mu}^A \prod_{B=2,3,4} dA^{1B}$$

$$\times \prod_{c=1,2,3} \delta^{[0]}(tr_2 \tau^c \bar{a}a) \prod_{A=1,2,3,4} \prod_{a=1,2} \delta^{[qA]}(\bar{a}_a M^A a_a + \bar{a}_a M^A) \prod_{B=2,3,4} \delta^{[q_1+qB]}(L \cdot A^{1B} - \Lambda^{1B}) .$$

(3.1)

The $\sim$ above indicates that in contrast with our previous work [4, 9, 10] we are not going to keep track of the overall normalization of the measure. In the above, both the integrals and the delta functions are defined with respect to a particular basis of matrices. A given $K \times K$ matrix quantity with block-form $M = \sum_{q=1}^{p} M^{(q)} \otimes E_{q,q+r}$, where $M^{(q)}$ is a $k_q \times k_{q+r}$ matrix, can be expanded in the basis of $K \times K$ matrices $T^{(r)}_a$:

$$M = \sum_{a=1}^{n_r} M_a T^{(r)}_a ,$$

(3.2)

where $n_r = \sum_{q=1}^{p} k_q k_{q+r}$ and the basis $T^{(r)}_a$, with $T^{(r)}_a \dagger = T^{(-r)}_a$, is normalized by $\text{tr}_K T^{(r)}_a T^{(-s)}_b = \delta^{rs} \delta_{ab}$. The delta functions in (3.1) are defined as

$$\delta^{[r]}(M) = \prod_{a=1}^{n_r} \delta(\text{tr}_K T^{(-r)}_a M) .$$

(3.3)

The pseudo collective coordinates for the scalar fields $A^{AB}$ can be explicitly integrated out:

$$\int \prod_{B=2,3,4} dA^{1B} \prod_{B=2,3,4} \delta^{[q_1+qB]}(L \cdot A^{1B} - \Lambda^{1B})$$

$$= \prod_{p} \prod_{q=1}^{p} \left(\det L^{(q,q+q_1+qB)}\right)^{-1} = \prod_{p} \prod_{q=1}^{p} \left(\det L^{(q,q+qA+qB)}\right)^{-1/2} .$$

(3.4)
The last equality follows from (1.2) and the fact that \( \det L^{(q,r)} = \det L^{(r,q)} \).

The measure (3.1) must be augmented with the instanton action, \( \exp -S_{\text{inst}} \) and, as we discuss in Sec. 3.3, the ratio of the fluctuation determinants in the \( \mathcal{N} = 0 \) cases. Following the \( \mathcal{N} = 4 \) example, it is convenient to bi-linearize the fermion quadrilinear interaction by introducing a set of auxiliary bosonic ‘collective coordinates’ \( \chi_{AB} \):

\[
\prod_{q=1}^{p} \prod_{A<B} (\det L^{(q,qA+qB)})^{-1/2} \exp -S_{\text{quad}} 
\sim \int d\chi \exp [-\text{tr}_K \chi_a L \chi_a + 4\pi i g^{-1} \text{tr}_K \chi_{AB} A^{AB}] .
\] (3.5)

Notice that this transformation absorbs the determinant factors appearing in (3.4). The auxiliary variables \( \chi_{AB} \) form an antisymmetric pseudo real tensor of SO(6) whose elements are \( K \times K \) matrices subject to

\[
\frac{1}{2} \epsilon^{ABCD} \chi_{CD} = \chi_{AB}^\dagger ,
\] (3.6)

where \( \dagger \) acts only on instanton indices. The variables \( \chi_{AB} \) can be written as an explicit SO(6)_R vector \( \chi_a, a = 1, \ldots, 6 \), by using the coefficients \( \Sigma_{AB}^a \) defined in the Appendix of [4]:

\[
\chi_{AB} = \frac{1}{\sqrt{8}} \Sigma_{AB}^a \chi_a .
\] (3.7)

The matrices \( \chi_{AB} \) have the following block structure

\[
\chi_{AB} = \sum_{q=1}^{p} \chi_{AB}^{(q)} \otimes E_{qq-qA-qB} .
\] (3.8)

3.2 The ADHM Measure and D-branes

In this section, we argue that the measure that we have constructed in terms of the bosonic variables \( \{a'_n, \chi_{AB}, w_\dot{\alpha}, \bar{w}^{\dot{\alpha}}\} \) and the fermionic variables \( \{\mathcal{M}^A_{\alpha}, \mu^A, \bar{\mu}^A\} \), can be derived by considering the dynamics of D-instantons in the background of D3-branes.

Let us briefly re-cap the same story in the \( \mathcal{N} = 4 \) theory described in Sec. IV.2 of Ref. [4]. In order to describe the D(−1)-D3 system one starts with the \( D5 - D9 \) system and then dimensionally reduces. We are interested in the world-volume theory of the D5-branes which on dimensional reduction will describe the matrix model of the D-instantons. The six-dimensional theory describing the \( K \) D5-branes consists of a U(\( K \)) vector multiplet of \( \mathcal{N} = (1,1) \) supersymmetry with fields \( \{\chi_a, \mathcal{M}^A_{\alpha}, \lambda_{\dot{\alpha}}^A, a'_n, D^c\} \) and \( n \) fundamental hypermultiplets of \( \mathcal{N} = (1,0) \) supersymmetry with fields \( \{w_\dot{\alpha}, \bar{w}^{\dot{\alpha}}, \mu^A, \bar{\mu}^A\} \).\(^9\)

\(^9\) The indices are defined as follows: \( a = 1, \ldots, 6 \) and \( A = 1, \ldots, 4 \) are spacetime SU(4) vector and spinor indices; \( \alpha = 1, 2 \) and \( \dot{\alpha} = 1, 2 \) are spinor indices of the SU(2)_L × SU(2)_R R-symmetry group of the theory; \( n = 1, \ldots, 4 \) is a vector index of SO(4) ⊂ SU(2)_L × SU(2)_R and finally \( c = 1, 2, 3 \) labels the adjoint representation of SU(2)_R.
In Ref. [4] we showed that the partition function of the matrix theory resulting from the dimensional reduction to zero dimensions of the six-dimensional theory whose field content is described above in the strong coupling limit (corresponding to \( \alpha' \to 0 \)) is precisely the ADHM \( k \)-instanton measure (weighted with the action). The relation between the fields of the six-dimensional theory and the ADHM construction is manifest, except for the bosonic variables \( D^c \) and the fermionic variables \( \lambda^\dot{\alpha}_A \). These fields may be eliminated by their equations-of-motion and they simply enforce the bosonic and fermionic ADHM constraints by producing the explicit delta functions as in our ADHM measure (3.1).

We now describe how this relation extends to the projected theories. The idea is a simple extension of the \( \mathcal{N} = 4 \) case to the case where the D-instantons and D3-branes move on the orbifold spacetime \( \mathbb{R}^4 \times \mathbb{R}^6 / \mathbb{Z}_p \). The D3-branes lie along \( \mathbb{R}^4 \) and at the singularity of the orbifold. The world-volume theory of the D3-branes is the \( \mathbb{Z}_p \)-projected U(\( n \)) gauge theory as described in Sec. 1.2. Now we want to consider the matrix theory of \( K \) D-instantons moving in the D3-brane world volume. Not surprisingly the resulting theory will be identical to the \( \mathcal{N} = 4 \) case, but with a \( \mathbb{Z}_p \) projection. It is easy to see how the projection must act on the matrices. Firstly it is embedded in the U(\( K \)) symmetry (the remnant of the six-dimensional gauge symmetry) as \( \sigma_{\{k_q\}} \) and in the U(\( n \)) symmetry (the remnant of the flavour symmetry of the six-dimensional theory) as \( \sigma_{\{N_q\}} \). Finally it acts on \( SU(4) \) spacetime indices of the six-dimensional parent theory as in (1.3). Explicitly on the bosonic ADHM variables

\[
\sigma_{\{k_q\}} a_n^{\prime} \sigma_{\{k_q\}}^{-1} = a_n^{\prime}, \quad \sigma_{\{N_q\}} w_{\dot{\alpha}}^{\prime} \sigma_{\{N_q\}}^{-1} = w_{\dot{\alpha}},
\]

\[
\sigma_{\{k_q\}} \bar{w}^\dot{\alpha} \sigma_{\{N_q\}}^{-1} = \bar{w}^\dot{\alpha}, \quad \sigma_{\{k_q\}} \chi_{AB} \sigma_{\{k_q\}}^{-1} = e^{2\pi i (q_A + q_B)/p} \chi_{AB},
\]

(3.9)

and on the fermionic ADHM variables

\[
\sigma_{\{k_q\}} \mathcal{M}_{\dot{\alpha}}^A \sigma_{\{k_q\}}^{-1} = e^{-2\pi i q_A/p} \mathcal{M}_{\dot{\alpha}}^A, \quad \sigma_{\{N_q\}} \mu^A \sigma_{\{N_q\}}^{-1} = e^{-2\pi i q_A/p} \mu^A, \quad \sigma_{\{k_q\}} \bar{\mu}^A \sigma_{\{k_q\}}^{-1} = e^{-2\pi i q_A/p} \bar{\mu}^A.
\]

(3.10)

Finally on the Lagrange multipliers for the ADHM constraints

\[
\sigma_{\{k_q\}} D^c \sigma_{\{k_q\}}^{-1} = D^c, \quad \sigma_{\{k_q\}} \lambda^\dot{\alpha}_A \sigma_{\{k_q\}}^{-1} = e^{2\pi i q_A/p} \lambda^\dot{\alpha}_A.
\]

(3.11)

The resulting theory has the expected symmetries: U(\( k_1 \)) \( \times \cdots \times U(k_p) \), the auxiliary symmetry of the ADHM construction (2.2), and U(\( N_1 \)) \( \times \cdots \times U(N_p) \), the gauge symmetry of the original \( \mathbb{Z}_p \)-projected gauge theory. Notice that, as in the \( \mathcal{N} = 4 \) case, the variables \( \chi_{AB} \) describe the freedom for the D-instantons to be ejected from the D3-branes in the orbifold directions, the six directions orthogonal to the D3-brane world-volume.

### 3.3 Fluctuation determinants for \( \mathcal{N} = 0 \)

The ADHM collective coordinate measure that we have constructed in the non-supersymmetric cases needs to be supplemented with the determinants of the fluctuations of the various fields,
gauge, fermion, scalar and ghosts, around the instanton solution. Fortunately, in a supersymmetric theory, it was shown by D’Adda and Di Vecchia [23], that the determinants all cancel in a self-dual, i.e. instanton background. The purpose of this section is to consider the determinants in the $\mathcal{N} = 0$ theories and show that they will not affect the large-$N$ saddle-point analysis undertaken in Sec. 4. This is not surprising as the theories we consider have the vanishing beta-function in the large-$N$ limit.

Let us define $\Delta^{(v,f,s)}_R$ to be the appropriate generalized Laplacian operators that govern the fluctuations in the vector, Weyl fermion and complex scalar fields, respectively, around the instanton solution, in the representation $R$ of the gauge group.\footnote{For the vector fields, of course, only the adjoint representation is relevant.} For each of these operators we will define a suitably regularized determinant:

$$\det \Delta^{(v,f,s)}_R = \exp -\Gamma^{(v,f,s)}_R . \quad (3.12)$$

The details of the regularization procedure will not be relevant for our purposes. The fluctuations for all the fields will contribute

$$\mathcal{F} = \exp \left( \Gamma^{(v)}_{\text{adj}} - \Gamma^{(f)}_R + \Gamma^{(s)}_R \right) \quad (3.13)$$

to the classical measure. In the above $R_f$ and $R_s$ are the representations of the gauge group of the Weyl fermions and complex scalar fields, respectively.

D’Adda and Di Vecchia [23] proved in an instanton background

$$\Gamma^{(v)}_{\text{adj}} = \Gamma^{(s)}_{\text{adj}}, \quad \Gamma^{(f)}_R = \Gamma^{(s)}_R \quad (3.14)$$

(in the vector case the ghost contribution is included) and consequently the contribution (3.13) is

$$\mathcal{F} = \exp \left( \Gamma^{(s)}_{\text{adj}} - \Gamma^{(s)}_R + \Gamma^{(s)}_R \right) . \quad (3.15)$$

In a supersymmetric theory there is a Weyl fermion superpartner to the gauge field—the gluino—and a Weyl fermionic superpartner to each complex scalar field; consequently $R_f = \text{adj} + R_s$ and the determinants all cancel, i.e. $\mathcal{F} = 1$.

The fluctuations will only contribute to measure in the $\mathcal{N} = 0$ cases where in order to evaluate the contribution we need the expression for $\Gamma^{(s)}_R$, where $R$ is either an adjoint representation of SU($N_q$) or a bi-fundamental representation ($\mathbf{N}_q, \mathbf{N}_r$) of SU($N_q$) $\times$ SU($N_r$). Fortunately, expressions for these determinants were calculated some time ago [24]. For a bi-fundamental representation

$$\Gamma^{(s)}_{(\mathbf{N}_q, \mathbf{N}_r)} = N_q \Gamma^{(s)}_{\mathbf{N}_q} + N_r \Gamma^{(s)}_{\mathbf{N}_r} - \log \det L^{(q \cdot r)} + \frac{1}{16\pi^2} \int d^4x \log \det f^{(q)} \Box^2 \log \det f^{(r)} . \quad (3.16)$$
In the above, \( f^{(q)} \), is the \( k_q \times k_q \) ADHM matrix of the \( SU(N_q) \) factor. The operator \( L^{(q \tau)} \) is defined in (2.13). The result for the adjoint representation of \( SU(N_q) \) follows by a restriction [24]:

\[
\Gamma^{(s)}_{\text{adj}} = 2N_q \Gamma^{(s)}_{N_q} - \log \det L^{(qq)} + \frac{1}{16\pi^2} \int d^4x \log \det f^{(q)} \Box^2 \log \det f^{(q)} . \tag{3.17}
\]

Notice that in both cases (3.16) and (3.17), the only \( N_q \) dependence comes from the first term only, involving the determinant of the \( N_q \) representation. It is fortunate that the cumbersome regularization dependent expression for \( \Gamma^{(s)}_{N_q} \) will not required (see [25–27] for details).

To get the fluctuation contribution to the measure, we need to consider the product over the various fields. Although, the final result is non-vanishing for the non-supersymmetric theories, significantly the \( N_q \) dependent and regularization dependent parts of (3.16) and (3.17) cancel between the bosons and fermions to leave the following—\( N_q \) independent—contribution to the measure:

\[
\mathcal{F} = \prod_{q=1}^{p} \left\{ (F^{(qq)})^{-1} \prod_{A<B} (F^{(qq+q\Lambda+q\Delta)})^{-1/2} \prod_{A} F^{(qq+q\Lambda)} \right\} , \tag{3.18}
\]

where

\[
F^{(qr)} = \det L^{(qr)} \exp \left[ -\frac{1}{16\pi^2} \int d^4x \log \det f^{(q)} \Box^2 \log \det f^{(r)} \right] . \tag{3.19}
\]

For the supersymmetric theories, since \( q_1 = 0 \), we have

\[
\prod_{A} F^{(qq+q\Lambda)} = F^{(qq)} \prod_{A<B} (F^{(qq+q\Lambda+q\Delta)})^{1/2} , \tag{3.20}
\]

and so \( \mathcal{F} = 1 \), as expected from the general analysis of D’Adda and Di Vecchia [23].

The main result that we take away from this section is that in the \( \mathcal{N} = 0 \) theories where the fluctuation determinants contribute to the measure, there is no \( N_q \) dependence in the final result and so these terms play no rôle in the large-\( N \) saddle-point approximation of the measure undertaken in Sec. 4.

### 3.4 The gauge invariant measure

Since the measure is used to calculate the correlation functions of gauge invariant operator insertions, it is convenient, following [4], to change variables to a set of gauge invariant parameters and to explicitly integrate over the gauge degrees-of-freedom which parameterize a coset space. As explained in [4], this brings along a very significant advantage; namely the non-linear bosonic ADHM constraints become trivial and can be explicitly integrated out.
A natural set of gauge-invariant collective coordinates, for \( N_q \geq 2k_q \), is obtained by constructing bosonic bilinear variables \( W \) with diagonal blocks:

\[
W^{(q)\hat{\alpha}\hat{\beta}} = \bar{w}^{(q)\hat{\alpha}} w^{(q)\hat{\beta}}. \tag{3.21}
\]

In terms of the \( 2K \times 2K \) matrix \( W \), we then define

\[
W^0 = \text{tr}_2 W, \quad W^c = \text{tr}_2 \tau^c W, \quad c = 1, 2, 3, \tag{3.22}
\]

where \( W^0 \) appeared previously in (2.11).

The gauge invariant measure then follows from the identity [4]

\[
\int_{\text{gauge coset}} dw^{(q)} d\bar{w}^{(q)} \sim (\det_{2k_q} W^{(q)})^{N_q-2k_q} dW^{(q)0} \prod_{c=1,2,3} dW^{(q)c}, \tag{3.23}
\]

valid for \( N_q \geq 2k_q \), applied to each diagonal block. The constant of proportionality was derived in Ref. [4] but will not be needed for our present purposes. As already mentioned earlier, going to a gauge invariant measure brings a very significant advantage: we can integrate out the ADHM \( \delta \)-functions explicitly. In the bosonic sector, the ADHM constraints can be written succinctly in terms of the gauge invariant variables as [4]

\[
0 = W^c - i[a'_m, a'_m] \bar{\eta}^c_{nm}. \tag{3.24}
\]

The remarkable feature of this re-writing of the constraints is that it is linear in \( W^c \) and consequently the \( W^c \) integrals simply remove the bosonic ADHM \( \delta \)-functions in (3.1).

There is a similar story in the fermionic sector. First of all we decompose

\[
\mu^A = w_\hat{\alpha} \zeta^{\hat{\alpha}A} + \nu^A, \quad \bar{\mu}^A = \bar{\zeta}_{\hat{\alpha}} \bar{w}^{\hat{\alpha}} + \bar{\nu}^A. \tag{3.25}
\]

Here \( \zeta^{\hat{\alpha}A} \) and \( \nu^A \) have the same block diagonal form as \( \mathcal{M}^A \) as in (2.5). So \( \zeta^{(q)\hat{\alpha}A} \) and \( \bar{\zeta}^{(q)\hat{\alpha}} \) have dimension \( k_q \times k_{q+q_A} \), while \( \nu^{(q)A} \) and \( \bar{\nu}^{(q)A} \) have dimension \( (N_q-2k_q) \times k_{q+q_A} \) and \( k_q \times (N_{q+q_A} - 2k_{q+q_A}) \), respectively. The variables \( \nu^A \) lie in the orthogonal subspace to \( w_{\hat{\alpha}} \), in the sense that

\[
\bar{w}^{\hat{\alpha}} \nu^A = 0, \quad \bar{\nu}^A w_{\hat{\alpha}} = 0. \tag{3.26}
\]

In terms of the new variables the fermionic ADHM constraints (2.6) are

\[
\bar{\zeta}^A_{\hat{\alpha}} W^A_{\hat{\beta}} + W_{\hat{\alpha}\hat{\beta}} \zeta^{\hat{\beta}A} + [\mathcal{M}^{\alpha A}, a'_m] = 0. \tag{3.27}
\]

The \( \bar{\zeta}^A_{\hat{\alpha}} \) integrals then remove the fermionic ADHM \( \delta \)-functions in (3.1).

Furthermore, in the fermionic sector we can integrate out the \( \nu^A \) and \( \bar{\nu}^A \) variables; this being the fermionic version of integrating over the gauge coset [4]. In order to do this, it is useful to split the fermion bilinear (2.9) that couples to \( \chi_{AB} \) in (3.5) as

\[
\Lambda^{AB} = \hat{\Lambda}^{AB} + \tilde{\Lambda}^{AB}, \tag{3.28}
\]

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where the first term has components just depending on \( \{ \nu^A, \bar{\nu}^A \} \):

\[
\hat{\Lambda}^{AB} = \frac{1}{2\sqrt{2}} (\bar{\nu}^A \nu^B - \bar{\nu}^B \nu^A),
\]

and the second term depends on the remaining variables

\[
\hat{\Lambda}^{AB} = \frac{1}{2\sqrt{2}} \left( \bar{\zeta}^A W^\alpha_\beta \zeta^B - \bar{\zeta}^B W^\alpha_\beta \zeta^A + \{ M^\alpha A, M^B_\beta \} \right),
\]

where \( \bar{\zeta}^A \) is eliminated by the fermionic ADHM constraints (3.27). We can now explicitly integrate out the \( \nu^A \)'s and \( \bar{\nu}^A \)'s. In general, unless \( k_1 = \cdots = k_p \), the result is rather cumbersome to write down and for the large-\( N \) calculation with \( N \equiv N_1 = \cdots = N_p \), to be undertaken in Sec. 4, we need only note

\[
\int \prod_{A=1,2,3,4} d\nu^A d\bar{\nu}^A \exp \left[ \sqrt{8\pi} i g^{-1} \text{tr}_K \chi_{AB} \bar{\nu}^A \nu^B \right] \sim (\text{det}_4 K)^N.
\]

4. The Large-\( N \) Instanton Measure

In this section we investigate the ADHM gauge-invariant instanton measure for the conformal field theories with \( N \equiv N_1 = \cdots = N_p \) in the large-\( N \) limit by a steepest descent method following Ref. [4]. Our primary purpose is to determine the dependence of the large-\( N \) measure on the variables of the ADHM construction and so we shall not keep track of the numerical pre-factor as we did in the \( N = 4 \) theory.

4.1 The saddle-point equations and their solution

In order to take the large \( N \) limit of the multi-instanton measure, we have to gather together all the terms which involve the exponential of a quantity times \( N \). As explained in [4], after rescaling \( \chi_{AB} \rightarrow \sqrt{N} \chi_{AB} \), there are three terms that contribute to what can be viewed as an ‘effective action’:

\[
S_{\text{eff}} = -2K(1 + 3 \log 2) - \log \det_2 K W - \log \det_4 K \chi + \text{tr}_K \chi_a L \chi_a.
\]

The second and third terms come from (3.23) and (3.31) while the final term comes from (3.5). Fortunately, as we have explained in Sec. 3.3, the fluctuation determinants give an \( N \) independent contribution and so do not play any rôle in the saddle-point analysis.

We can now perform a steepest descent approximation of the measure which involves finding the minima of the effective action with respect to the variables \( \chi_{AB}, W^0 \) and \( a'_n \). The resulting
saddle-point equations are\textsuperscript{11}
\begin{align}
\epsilon^{ABCD} (L \cdot \chi_{AB}) \chi_{CE} &= \frac{1}{2} \delta^D_5 \mathbb{1}_{[K] \times [K]} , \\
\chi_\alpha \chi_\alpha &= \frac{1}{2} (W^{-1})^0 , \\
[\chi_\alpha, [\chi_\alpha, a'_n]] &= i \tilde{\eta}^{c'}_{nm} [a'_m, (W^{-1})^c] .
\end{align}
(4.2)
where we have introduced the matrices
\begin{align}
(W^{-1})^0 &= \text{tr}_2 W^{-1} , \\
(W^{-1})^c &= \text{tr}_2 \tau^c W^{-1} .
\end{align}
(4.3)

We note that the expression for the effective action (4.1) and the saddle point equations (4.2) look identical to those derived in [4] for the unprojected $\mathcal{N} = 4$ theory. The solutions of these equations, however, will not be the same as they have to be invariant under the $\mathbb{Z}_p$ projection. This means that $W^0$ and $a'_n$ have the block-diagonal form (2.1), while $\chi_{AB}$ is generally off-diagonal (3.8). With this in mind we shall use the analysis of the $\mathcal{N} = 4$ theory as a guide and draw heavily on the results derived in [4]. As in the $\mathcal{N} = 4$ case we look for a solution with $W^c = 0, c = 1, 2, 3$, which means that the instantons are embedded in mutually commuting $\text{SU}(2)$ subgroups of the gauge group. In this case equations (4.2) are equivalent to
\begin{align}
[a'_n, a'_m] &= [a'_n, \chi_{AB}] = [\chi_{AB}, \chi_{CD}] = 0 , \\
W^0 &= \frac{1}{2} (\chi_\alpha \chi_\alpha)^{-1} .
\end{align}
(4.4)
The final equation can be viewed as giving the value of $W^0$, the instanton sizes, at the saddle-point and clearly $\chi_\alpha \chi_\alpha$ and $W^0$ must be non-degenerate.

In the appendix we prove that in the type IIB case when $S^5/\mathbb{Z}_p$ is smooth there is no solution to (4.4) unless $k \equiv k_1 = \cdots = k_p$. For the analogous type 0B case, when $S^5/\mathbb{Z}_{p/2}$ is smooth, a solution only exists in the charge sector with $k_+ \equiv k_2 = k_4 = \cdots = k_p$ and $k_- \equiv k_1 = k_3 = \cdots = k_{p-1}$. For the type IIB case, the general solution, up to the auxiliary symmetry, has the block-form
\begin{align}
a'^{(q)}_n &= \text{diag} \left( -X^1_n , \ldots , -X^k_n \right) , \\
\chi^{(q)}_{AB} &= \text{diag} \left( \rho_1^{-1} \hat{\Omega}^1_{AB} , \ldots , \rho_k^{-1} \hat{\Omega}^k_{AB} \right) .
\end{align}
(4.5)
Here $\hat{\Omega}^i_{AB}$ are unit six-vectors:
\begin{align}
\epsilon^{ABCD} \hat{\Omega}^i_{AB} \hat{\Omega}^j_{CD} &= 1 \quad \text{or} \quad \hat{\Omega}^i_a \hat{\Omega}^i_a = 1 ,
\end{align}
(4.6)
for each $i$. Notice that the discrete transformation $\sigma_{\{k,\ldots,k\}} \in U(k) \times \cdots \times U(k)$ fixes the form of (4.5) and implies some discrete identifications of the $\hat{\Omega}^i_{AB}$, explicitly $\hat{\Omega}^i_{AB} \sim e^{-2\pi i (q_{AB} + q_B)/p} \hat{\Omega}^i_{AB}$. So the coordinates $\hat{\Omega}^i_{AB}$ are valued on the quotient $S^5/\mathbb{Z}_p$. The solution parameterizes the positions $\{\rho_i, X^i_n, \hat{\Omega}^i_{AB}\}$ of $k$ D-instantons in $AdS_5 \times S^5/\mathbb{Z}_p$. Notice that the solution breaks the auxiliary symmetry to the maximal abelian subgroup $U(1)^k_{\text{diag}}$ of the diagonal subgroup $U(k)_{\text{diag}} \subset U(k) \times \cdots \times U(k)$.

\textsuperscript{11}As usual we shall frequently swap between the two representations $\chi_\alpha$ and $\chi_{AB}$ for $\text{SO}(6)$ vectors.
Now we turn to the solution when $S^5/Z_p$ has a singularity. In this case solutions exist in all charge sectors and there are multiple branches in the solution space labelled by $\tilde{k} = 0, \ldots, \min(k)$:

$$a^{(q)}_n = \text{diag}( -X^1_n, \ldots, -X^k_n, -X^{1,q}_n, \ldots, -X^{k_q-k,q}_n),$$

$$\chi^{(q)}_{AB} = \begin{cases} 
\text{diag}(\hat{\rho}_1^{-1}\hat{\Omega}_1^{\hat{A}B}, \ldots, \hat{\rho}_k^{-1}\hat{\Omega}_k^{\hat{A}B}, \hat{\rho}_{1,q}^{-1}\hat{\Omega}_{1,q}^{\hat{A}B}, \ldots, \hat{\rho}_{k_q-k,q}^{-1}\hat{\Omega}_{k_q-k,q}^{\hat{A}B}) & q_A + q_B = 0, \\
\text{diag}(\hat{\rho}_1^{-1}\hat{\Omega}_1^{\hat{A}B}, \ldots, \hat{\rho}_k^{-1}\hat{\Omega}_k^{\hat{A}B}, 0, \ldots, 0) & q_A + q_B \neq 0.
\end{cases}$$  \tag{4.7}

In the above the $\hat{\Theta}^{i,q}_{AB}$ are unit vectors lying in $(S^5/Z_p)^{\text{sing}}$, in other words subject to (4.6) but with $\hat{\Theta}^{i,q}_{AB} = 0$, for $q_A + q_B \neq 0$. In this case the solution represents $\tilde{k}$ D-instantons with positions $\{\rho_i, X^i_n, \hat{\Omega}_i^{\hat{A}B}\}$ in $AdS_5 \times S^5/Z_p$ along with $k_q-k$ fractional D-instantons of type $q$ with positions $\{\rho_{i,q}, X^{i,q}_n, \hat{\Theta}^{i,q}_{AB}\}$ in $AdS_5 \times (S^5/Z_p)^{\text{sing}}$. The solution breaks the auxiliary symmetry to $U(1)^d \times U(1)^{K-pk}$. In the case $k \equiv k_1 = \cdots = k_p$ on the branch with $\tilde{k} = k$ we recover the solution (4.5).

In the type 0B theories, the solutions are generalizations of those above with the following difference. There are now two kinds of D-instanton associated to the even and odd blocks. In particular, when $S^5/Z_{p/2}$ is smooth a solution only exist in the charge sector $\{k_-, k_+, k_-, \ldots, k_+\}$. The fact that D-instantons now come in two types matches precisely our expectation of the type 0B string theory.

\subsection*{4.2 The large-$N$ instanton measure}

In this section we construct the large-$N$ instanton measure. In principle, we have to expand the effective action around the general solutions written down in the last section to sufficient order to ensure that the fluctuation integrals converge. In general because the Gaussian form has zeros whenever two D-instantons coincide one has to go to quartic order in the fluctuations. Fortunately, as explained in [4], we do not need to expand about the most general solution to the saddle-point equations to quartic order since this is equivalent to expanding to the same order around the most degenerate solution where all the D-instantons (in our present case both fractional and non-fractional) are at the same point in $AdS_5 \times S^5/Z_p$. The resulting quartic action has flat directions corresponding to the relative positions of the D-instantons. We can then recover the original expansion that we wanted by re-expanding this action around some general point along a flat direction. It can then be established \textit{ex post facto} that this is a consistent procedure.

When $S^5/Z_p$ has a singularity, the maximally degenerate solution is (4.7) with all the instantons at the same point. In particular, unless $k_1 = \cdots = k_p$ there are fractional D-instantons and this means that only the components $\chi_{AB}$ with $q_A + q_B = 0$ are non-zero:

$$W^0 = 2\rho^2 1_{[K] \times [K]}, \quad a_n^i = -X_n 1_{[K] \times [K]}, \quad \chi_{AB} = \rho^{-1}\hat{\Theta}_{AB} 1_{[K] \times [K]},$$  \tag{4.8}
where as before $\hat{\Theta}_{AB}$ is a unit vector in the directions fixed by $\mathbb{Z}_p$. The desired result for the large-$N$ measure is immediate because on the saddle-point solution (4.8) all the variables are proportional to the identity matrix as in the $N = 4$ case and one can essentially copy the analysis of Ref. [4] verbatim. The result is that the large-$N$ measure is simply, up to a normalization constant, the partition function of the $\mathbb{Z}_p$-projection of $\mathcal{N} = 1$ supersymmetric ten-dimensional $\mathrm{U}(K)$ gauge theory dimensionally reduced to zero dimensions. The ten-dimensional gauge field has

$$A_\mu = \left( \rho^{-1} a'_n, \rho \chi_{AB} \right),$$

and the ten-dimensional Majorana-Weyl fermion has components

$$\Psi = \sqrt{\frac{\pi}{2g}} \left( \rho^{-1/2} \mathcal{M}_\alpha^A, \rho^{1/2} \zeta^{\hat{\alpha}A} \right).$$

The $\mathbb{Z}_p$-projection acts as on the $\mathrm{U}(K)$-valued variables as

$$\sigma_{\{k_q\}} a'_n \sigma_{\{k_q\}}^{-1} = a'_n, \quad \sigma_{\{k_q\}} \chi_{AB} \sigma_{\{k_q\}}^{-1} = e^{2\pi i (q_A + q_B)/p} \chi_{AB},$$

$$\sigma_{\{k_q\}} \mathcal{M}_\alpha^A \sigma_{\{k_q\}}^{-1} = e^{-2\pi i q_A/p} \mathcal{M}_\alpha^A, \quad \sigma_{\{k_q\}} \zeta^{\hat{\alpha}A} \sigma_{\{k_q\}}^{-1} = e^{-2\pi i q_A/p} \zeta^{\hat{\alpha}A}.$$  

(4.11)

The partition function is then defined as the integral over the projected variables of the ten-dimensional gauge theory with an action where all the derivative terms are set to zero:

$$Z = \int_{\mathrm{U}(K):\mathbb{Z}_p} d^{10} A d^{16} \Psi e^{-NS(A_\mu, \Psi)},$$

(4.12)

with

$$S(A_\mu, \Psi) = -\frac{1}{2} \mathrm{tr}_K [A_\mu, A_\nu]^2 + N^{-1/2} \mathrm{tr}_K \Psi \Gamma_\mu [A_\mu, \Psi].$$

(4.13)

The result for the large-$N$ measure when $S^5/\mathbb{Z}_p$ is smooth is also the partition function of the corresponding $\mathbb{Z}_p$-projected gauge theory, but only for the appropriate charge sectors, i.e. $\{k, \ldots, k\}$, for type IIB, and $\{k_-, k_+, k_-, \ldots, k_+\}$, for type 0B. We consider the former case first.\(^\text{12}\) The arguments leading to this conclusion are more complicated because the maximally degenerate saddle-point solution for $\chi_{AB}$ is not simply proportional to the identity. Below we sketch the proof.

First of all, the maximally degenerate solution is

$$W^0 = 2 \rho^2 1_{[K] \times [K]}, \quad a'_n = -X_n 1_{[K] \times [K]}, \quad \chi_{AB} = \rho^{-1} \hat{\Omega}_{AB} 1_{[k] \times [k]} \otimes \sum_{q=1}^p E_{q - qA - qB}.$$  

(4.14)

\(^\text{12}\)The following analysis is also valid in the same charge sector when $S^5/\mathbb{Z}_p$ has a singularity.
Notice that $\chi_{AB}$, for $q_A + q_B \neq 0$, is off-diagonal. It is useful to introduce the notation $S_{AB}$ for the saddle-point solution for $\chi_{AB}$. For each of the quantities $v \in \{a'_n, \chi_{AB}\}$ we define the following decomposition

$$v = v_0 + \tilde{v} + \hat{v},$$

(4.15)

where $v_0$ is the saddle-point value of the variable, $\tilde{v}$ are the components which do not commute with at least one $S_{AB}$. The remaining variables $\hat{v}$ then commute with all the components $S_{AB}$.

Explicitly this means that

$$v_0 = v'_1[k] \otimes \sum_{q=1}^{p} E_{q} q + r, \quad \tilde{v} = \sum_{q=1}^{p} \tilde{v}^{(q)}_1[k] \otimes E_{q} q + r, \quad \hat{v} = \hat{v}^{(0)}_1[k] \otimes \sum_{q=1}^{p} E_{q} q + r.$$

(4.16)

with $\sum_{q=1}^{p} \tilde{v}^{(q)}_1[k] = 0$ and $\text{tr}_k \hat{v}^{(0)}_1[k] = 0$.

The complication in this case is that there are now fluctuations that are lifted at Gaussian order. The relevant terms in the expansion of the effective action are

$$S^{(2)} = -\text{tr}_K [S_a, \delta a'_n]^2 - \rho^4 \text{tr}_K [S_a, \delta \chi_b]^2 + \rho^4 \text{tr}_K [S_a, \delta \chi_a]^2.$$  

(4.17)

So any fluctuation in $a'_n$ and $\chi_a$ that does not commute with $S_b$ will be lifted at this order in the expansion. These are precisely the variables $\tilde{a}'_n$ and $\tilde{\chi}_a$ in the decomposition (4.15).

Actually, not quite all the fluctuations $\tilde{\chi}_a$ are lifted by (4.17) because there are exact flat directions of $S_{\text{eff}}$ generated by the action of the auxiliary $U(k) \times \cdots \times U(k)$ symmetry group on the saddle-point solution. Infinitesimally, these are the variations

$$\tilde{\chi}_a^\parallel = [S_a, \epsilon],$$

(4.18)

where $\epsilon = \sum_{q=1}^{p} \epsilon^{(q)}_q \otimes E_{q} q$ are infinitesimal parameters of the transformation. It is convenient to ‘gauge-fix’ this symmetry by integrating only over fluctuations $\tilde{\chi}_a^\perp$ orthogonal to the variations $\tilde{\chi}_a^\parallel$. These fluctuation can be specified by the conditions

$$\text{tr}_K (\tilde{\chi}_a^\perp [S_a, \epsilon]) = 0, \quad \forall \epsilon,$$

(4.19)

or equivalently $[\tilde{\chi}_a^\perp, S_a] = 0$, hence the ‘gauge-fixed’ quadratic action is therefore

$$S^{(2)}_{\text{gf}} = -\text{tr}_K [S_a, \tilde{a}'_n]^2 - \rho^4 \text{tr}_K [S_a, \tilde{\chi}_b^\perp]^2.$$  

(4.20)

Now we come to the crux of the argument. In general there could be cubic interactions that couple one tilded variable with one two hatted variables, i.e. schematically of the form $\text{tr}_K \tilde{v}_1 \hat{v}_2 \hat{v}_3$, that contribute at the same order in $1/N$ as the Gaussian terms in (4.20) and quartic terms in the hatted variables. However, because of the decompositions (4.20) such

\[^{13}\text{These are in addition to the fluctuations } \delta W^0 \text{ that are lifted at this order: see [4].}\]
couplings vanish. The quartic terms in the hatted variables follow in an identical way to the $N = 4$ calculation [4]. Finally, at leading order in $1/N$ the bosonic variables are controlled by the gauge-fixed action

$$S_{gf} = -\text{tr}_K [S_a, \hat{a}_n]^2 - \rho^4 \text{tr}_K [S_a, \hat{\chi}_b]^2 - \frac{1}{2} \text{tr}_K \left( \rho^{-4}[\hat{a}_n, \hat{a}_m]^2 + 2[\hat{\chi}_a, \hat{a}_n]^2 + \rho^4[\hat{\chi}_a, \hat{\chi}_b]^2 \right). \quad (4.21)$$

Up to terms which are sub-leading in $1/N$, we can write this as a gauge fixed version of the action of the ten-dimensional gauge field (4.9) as in (4.13). The point is that (4.21) differs from (4.13) by (i) terms quadratic in tilded variables and quadratic in hatted variables (ii) quartic in tilded variables. But both these kinds of terms are sub-leading in $1/N$. So in the resulting partition function (4.12) some of the terms in the action are actually higher order in $1/N$. A similar result is follows in the $\{k-, k+, k-, \ldots, k+\}$ charge sector of the type 0B theories when $S^5/\mathbb{Z}_p/2$ is smooth.

In retrospect we can see the relation between the general solution of the saddle-point equations and the flat directions of the gauge theory action (4.13). The point is that the vanishing commutators in (4.4) are precisely the equations-of-motion of the gauge theory action. In other words the space-of-solutions of the saddle-point equations is identical to the 'vacuum moduli space' of the matrix model. Therefore the fact that the saddle-point solutions can be identified with the moduli space of D-instantons is no accident. In the general case, the solutions describe the possible fractionations of D-instantons at the singularity $(S^5/\mathbb{Z}_p)_{\text{sing}}$. In the $N = 4$ case the integrals over the relative positions of the D-instantons are actually convergent and the charge $k$ D-instanton can be thought of as a bound-state. It would be interesting to investigate the convergence of the integrals over the relative positions of the (fractional) D-instantons in these more general theories. In particular, the convergence issue is crucial for finding the $N$-dependence of the large-$N$ instanton measure.

### 4.3 Contribution to 16 fermion correlators

In the $N = 4$ theory, the simplest correlation functions which received instanton contributions involved insertions which only saturate the Grassmann integrals over the collective coordinates corresponding to the 16 supersymmetric and superconformal zero-modes. In this section, we show that the effective large-$N$ measure in the charge sector $\{k, \ldots, k\}$ in the analogous fermionic sector is identical to the $N = 4$ case. By analogous fermionic sector we mean the sector where the integrals over the 16 variables $\xi^A$ and $\bar{\eta}^{\dot{A}}$ defined by

$$M^\alpha_{\alpha} = \xi^A_{[k]} \otimes \sum_{q=1}^{p} E_{q,q-A}, \quad \xi^{\dot{\alpha}}_{\dot{A}} = \bar{\eta}^{\dot{A}}_{[k] \times [k]} \otimes \sum_{q=1}^{p} E_{q,q-A}, \quad (4.22)$$

are saturated by insertions rather than from terms in the action of the partition function (4.12). These variables obviously include the collective coordinates $\xi^A$ and $\bar{\eta}^{\dot{A}}$, for $q_A = 0,$
corresponding to the supersymmetric and superconformal zero-modes modes.  

So the effective measure in this fermionic sector is defined by separating out the integrals over $\xi^A_\alpha$ and $\bar{\eta}^{\dot{\alpha}A}$ from (4.12) and leaving them unsaturated. The appropriate thing to do now, and which will be justified \textit{ex post facto}, is to expand the action around the branch in the solution space in (4.14). The gauge fixing described in the last section leading to the action (4.21) is valid in this context. We can integrate the fluctuations lifted at Gaussian order $\tilde{a}'_n$ and $\tilde{\chi}^\perp_a$:

$$
\int d\tilde{a}' d\tilde{\chi}^\perp \exp \left( -N \text{tr}_K [S_a, \tilde{a}'_n]^2 - N \rho^4 \text{tr}_K [S_a, \tilde{\chi}^\perp_a]^2 \right) = \rho^{-10k^2(p-1)N-9k^2(p-1)/2} (\det'_p M)^{-9k^2/2},
$$

(4.23)

where $M$ is the $p \times p$ matrix$^{15}$

$$
M = 2 \sum_{q=1}^p \epsilon^{ABCD} \hat{\Omega}_{AB} \hat{\Omega}_{CD}(E_{qq-qa-qb} - E_{qq}).
$$

(4.24)

The prime on the determinant in (4.23) implies that we should take a product only over the non-null eigenvalues of $M$ since the latter has a null eigenvector of the form $\sum_{q=1}^p e_q$. In addition, we must take into account the gauge-fixing Jacobian:

$$
J_{GF} = (\text{Vol } U(k))^{p-1} \rho^{-(p-1)k^2} \left( \det'_p M \right)^{k^2/2}.
$$

(4.25)

To arrive at our answer, we must also integrate-out the fermionic partners to the Gaussian variables, namely, $\tilde{M}^A_\alpha$ and $\tilde{\zeta}^{\dot{\alpha}A}$. These are coupled, to the saddle-point solution via the terms

$$
NS_t = \left( \frac{2\pi^2 N}{g^2} \right)^{1/2} \text{tr}_K \left( \rho [S_{AB}, \tilde{\zeta}^{\dot{\alpha}A}]\tilde{\zeta}^B + \rho^{-1} [S_{AB}, \tilde{M}^{\alpha A}]\tilde{M}^B \right).
$$

(4.26)

The fermionic integrals are then

$$
\int \prod_{A=1,2,3,4} d\tilde{\zeta}^A d\tilde{M}^A e^{NS_t} = \left( \frac{2\pi^2 N}{g^2} \right)^{4(p-1)k^2} (\det_{4p} G)^{2k^2},
$$

(4.27)

where $G$ is the $4p \times 4p$ matrix with block-form elements

$$
G_{AB} = \hat{\Omega}_{AB} \sum_{q=1}^p (E_{qq-qa-qb} - E_{qq})
$$

(4.28)

$^{14}$Actually the definition of the supersymmetric and superconformal collective coordinates differs from (4.22) by a linear shift which we may transform away: see [4].

$^{15}$It is useful to introduce a basis of vectors to match the block-form of the matrices. To this end we define the $p$-dimensional column vector $e_q$ with a 1 in the $q^{th}$ position and 0 elsewhere. These vectors have the property $E_{qr}e_s = e_q \delta_{rs}$. 

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and, as before, the prime, indicates the removal of the zero eigenvalue. Since as \( p \times p \) matrices 
\([G_{AB}, G_{CD}] = 0\), it is straightforward to show that

\[
\text{det}'_4 G = \left( \text{det}'_p \left( \frac{1}{8} \epsilon^{ABCD} G_{AB} G_{CD} \right) \right)^2 = \left( \text{det}'_p \frac{1}{8} M \right)^2 ,
\]

where \( M \) is the matrix defined in (4.24). Notice that the determinant factors of \( M \) cancel 
between (4.23), (4.25) and (4.27).

What remains are the integrals over the quartic fluctuations \( \delta'_n \) and \( \chi_a \) along with their 
fermionic partners which have the form of the partition function of ten-dimensional \( \mathcal{N} = 1 \) 
supersymmetric SU(\( k \)) gauge theory dimensionally reduced to zero dimensions. Being careful 
with the factors of \( N \) and \( g \), our final result for the large-\( N \) measure in this sector is

\[
\int d\mu_{\text{phys}}^{\{k,\ldots,k\}} e^{-S_{\text{inst}}} = g^8 \sqrt{N} e^{2\pi ikp\tau} \int \frac{d^4X}{\rho^5} \frac{d\hat{\Omega}}{d^2\xi^A d^2\bar{\eta}^A} \cdot Z_{\text{SU}(k)} + \cdots .
\]

where the ellipsis reminds us that the expression is only the part of the measure where the 16 
variables \( \xi^A_{\alpha} \) and \( \bar{\eta}^{\alpha A} \) are left un-integrated.

Notice that (4.30) is identical to the large-\( N \) measure in the charge \( k \) sector of the \( \mathcal{N} = 4 \) 
theory with \( \tau \rightarrow p\tau \). Hence charge \( k \) instanton contributions to the 16 fermion correlators 
evaluated in the \( \mathcal{N} = 4 \) theory will be identical to those from the charge \( \{k,\ldots,k\} \) sector in 
the orbifold theories.

\section{5. Relation to \( \mathcal{N} = 2 \) SU(\( N \)) Gauge Theory with \( N_F = 2N \)}

In the above we have considered the instanton contributions in the large-\( N \) limit to a class of 
\( \mathcal{N} = 2 \) theories with product group structure SU(\( N \)) \( \times \cdots \times \) SU(\( N \)) (after decoupling the abelian factors). In Ref. [28] we considered the contributions of instantons in the large-\( N \) limit to the 
\( \mathcal{N} = 2 \) conformal theory with gauge group SU(\( N \)) and with \( N_F = 2N \) fundamental hypermultiplets. Although apparently unconnected, the instanton contributions in these theories are related in the following way. Consider the charge sector \( \{k,0,\ldots,0\} \) in the \( \mathcal{N} = 2 \) supersymmetric theories with gauge group SU(\( N \)) \( \times \cdots \times \) SU(\( N \)). In this instanton background only the 
SU(\( N \)) gauge field \( v_{\alpha}^{(1)} \) is non-zero, and consequently only the adjoint fermion fields \( \lambda^{(1)A} \), 
with \( A = 1,2 \), and adjoint scalar fields \( A^{(1)AB} \), with \( AB = 12,34 \), along with the fermion fields \( \lambda^{(q)A} \), 
with \( q = 1,p \) and \( A = 3,4 \), and the scalar fields \( A^{(q)AB} \), with \( q = 1,p \) and \( AB = 13,14,23,24 \), 
which transform in either the \( \mathbf{N} \) or \( \bar{\mathbf{N}} \) representations of the SU(\( N \)) factor, are non-zero. These 
fields amass into the an SU(\( N \)) vector representation of \( \mathcal{N} = 2 \) supersymmetry and 2\( N \) fundamental hypermultiplets; precisely the theory considered in Ref. [28]. So the instantons in the 
\( \{k,0,\ldots,0\} \) charge sector of the first theory should describe the charge \( k \) instantons of the 
second theory.
Now we are in a position to interpret the results of Ref. [28] for the large-$N$ instanton measure in the context of the results in this paper. In the charge sector $\{k,0,\ldots,0\}$ the large-$N$ instanton measure involves the partition function for the $\mathbb{Z}_p$-projected $\mathcal{N}=1$ supersymmetric $U(k)$ gauge theory in ten-dimensions dimensionally reduced to zero dimensions, with $\mathbb{Z}_p$ embedded in the gauge group as $\sigma_{\{k,0,\ldots,0\}} = 1_{[N]\times[N]}$ and in $SU(4)_R$ as in (1.3). Only the $U(k)$-adjoint matrices $\chi_{AB}^{(1)}$, $AB = 12,34$, $\mathcal{M}_a^{(1)A}$ and $\zeta^{(1)\dot{\alpha}A}$, with $A = 1,2$, survive this projection. The resulting partition function can equally well be described as that of the $\mathcal{N} = (1,0)$ supersymmetric $U(1)$ theory in six dimensions dimensionally reduced to zero dimensions. This is precisely the result of Ref. [28]. In particular since from the orbifold perspective all the $k$ instantons are fractional we have a simple explanation of the observation that the saddle-point solution of Ref. [28] described a point-like object in $AdS_5 \times S^1$: the $S^1$ is precisely the orbifold singularity of $S^5/\mathbb{Z}_p$ on which the fractional D-instantons are confined to move.

6. Discussion

In the $\mathcal{N}=4$ theories much more could be achieved [4] because the measure could be used to calculate instanton contributions to particular correlation functions which could then be compared with D-instanton induced terms in the type IIB supergravity effective action [6]. In this comparison the overall numerical pre-factor in the large-$N$ ADHM measure was crucial to get the precise agreement found in [4]. For the orbifold theories we have not kept track of the numerical pre-factor since it is not clear on the dual side what to compare with. In fact a dual supergravity approximation of the string theory only exists if $S^5/\mathbb{Z}_p$ is smooth, unless the singularity is blown up in some way. It would clearly be interesting to attempt the more detailed check of the duality carried through in [4].

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Appendix A:

In this appendix we prove that solutions to the saddle-point equations (4.4) for the type IIB dual cases when $S^5/\mathbb{Z}_p$ is smooth only exist if all the $k_q$ all equal.

Our proof proceeds along the following lines. Since all the matrices $\chi_{AB}$ commute they have a basis of simultaneous eigenvectors. In order to show that there are no solutions to (4.4) unless
all the instanton numbers in each group factor are the same we will show that in the contrary situation the matrices $\chi_{AB}$ have at least one common null eigenvector and consequently (4.4) cannot be satisfied because $W^0$ is non-degenerate.

Consider, for the moment, one of the $\chi_{AB}$ matrices, which we denote generically as $A$, with the block form

$$A = \sum_{q=1}^{p} A^{(q)} \otimes E_{qq-r}.$$  \hfill (A.1)

Since $\mathbb{Z}_p$ acts freely on $S^5$ we have $r \neq 0 \mod p$. In general, if $r$ and $p$ have some common integer divisor(s) (the largest of which we denote $c$) $\mathbb{Z}_p$ does not act faithfully on $A$, rather only the subgroup $\mathbb{Z}_{p/c}$ is realized.

The first result we need, is that any non-null eigenvector of $A$ has the form

$$v = \sum_{q=1}^{p/c} v^{(rq+s)} \otimes e_{rq+s},$$  \hfill (A.2)

where all the components $v^{(rq+s)}$, $q = 1, \ldots, p/c$, are all non-vanishing. To see this consider the eigenvalue equation in the block-form basis:

$$A^{(r+s)} v^{(s)} = \lambda v^{(r+s)}.$$  \hfill (A.3)

Iterating this equation $p/c$ times we cycle back to yield an eigenvector equation for $v^{(s)}$:

$$A^{(rp/c+s)} \cdots A^{(r+s)} v^{(s)} = \lambda^{p/c} v^{(s)}.$$  \hfill (A.4)

Denoting the eigenvalue as $\mu$, we have $\lambda = \mu^{c/p}$. This means that each non-null eigenvector of $A^{(rp/c+s)} \cdots A^{(r+s)}$ gives rise to $p/c$ non-null eigenvectors of $A$ itself, the multiplicity arising from the $p/c$ choices of the roots of $\lambda = \mu^{c/p}$. The non-null eigenvectors of $A$ are consequently of the form

$$v = \sum_{q=1}^{p/c} \mu^{-cq/p} A^{(rq+s)} \cdots A^{(r+s)} v^{(s)} \otimes e_{rq+s}.$$  \hfill (A.5)

As stated, the non-null eigenvectors have the form (A.2) where all the components $v^{(rq+s)}$, $q = 1, \ldots, p/c$, are all non-vanishing.

The fact that the non-null eigenvectors have the form (A.2) implies that all the remaining eigenvectors must be null. The null eigenvectors can therefore be written as

$$v^{(q)} \otimes e_q,$$  \hfill (A.6)
with no sum on \( q \). A lower bound on the number of null eigenvectors of the form (A.6) is consequently

\[
d(r, q) = k_q - k_{\text{min}}(r, q),
\]

where \( k_{\text{min}}(r, q) \) is the smallest number in the set \( \{k_{q+nr}, n = 1, \ldots, p/c\} \).

We now denote the first matrix as \( A_1 \), with associated quantities \( r_1 \) and \( c_1 \), and introduce a second matrix \( A_2 \) of the same form:

\[
A_2 = \sum_{q=1}^{p} A_2^{(q)} \otimes E_{q-r_2}
\]

that commutes with \( A_1 \). We now proceed to establish a lower bound on the number of simultaneous null eigenvectors of \( A_1 \) and \( A_2 \). The point is that certain linear combinations of null eigenvectors of \( A_1 \) of the form can be non-null eigenvectors of \( A_2 \), and vice-versa. We must enumerate the maximum possible such combinations that can arise to establish a lower bound on the simultaneous null eigenvectors of \( A_1 \) and \( A_2 \). Consider the null eigenvectors of \( A_1 \) of the form (A.6). A combination of such null eigenvectors of the form

\[
v^{(q)} \otimes e_q + v^{(q+r_2)} \otimes e_{q+r_2} + \ldots + v^{(q+(p/c_2-1)r_2)} \otimes e_{q+(p/c_2-1)r_2},
\]

(A.9)

can be a non-null eigenvector of \( A_2 \). The number of such combinations that have this property is clearly bounded above by

\[
\min\left(d(r_1, q), d(r_1, q + r_2), \ldots, d(r_1, q + (p/c_2 - 1)r_2)\right).
\]

(A.10)

Hence a lower bound on the number of simultaneous null eigenvectors of the form \( v^{(q)} \otimes e_q \) is given by

\[
d(r_1, q)-\min\left(d(r_1, q), \ldots, d(r_1, q + (p/c_2 - 1)r_2)\right) = \max\left(0, d(r_1, q) - d(r_1, q + r_2), \ldots, d(r_1, q) - d(r_1, q + (p/c_2 - 1)r_2)\right)
\]

(A.11)

However, we can also run the argument the other way and consider the null eigenvectors of \( A_2 \) that can be non-null eigenvectors of \( A_1 \). Taken together this gives a lower bound \( X_q \) on the number of simultaneous null eigenvectors of \( A_1 \) and \( A_2 \) of the form \( v^{(q)} \otimes e_q \)

\[
X_q = \max\left(d(r_1, q) - d(r_1, q + n_2 r_2), d(r_2, q) - d(r_2, q + n_1 r_1)\right).
\]

(A.12)

The expression above is written in short-hand where the integers run over the values \( n_{1,2} = 0, \ldots, p/c_{1,2} - 1 \).

It is now easy to introduce a third matrix \( A_3 \) and establish a lower bound on the simultaneous null eigenvectors of all three matrices. The idea is to consider the maximum number of simultaneous null eigenvectors of \( A_1 \) and \( A_2 \) that can be non-null eigenvectors of \( A_3 \). For null eigenvectors of the form \( v^{(q)} \otimes e_q \), this number is bounded above by

\[
\min\left(X_q, X_{q+r_3}, \ldots, X_{q+(p/c_3-1)r_3}\right).
\]

(A.13)
Hence a lower bound on the number of simultaneous null eigenvectors of $A_1$, $A_2$ and $A_3$ of the form $v^{(q)} \otimes e_q$ is given by

$$\max\left(0, X_q - X_{q+r_3}, \ldots, X_q - X_{q+(p/c-1)r_3}\right).$$  \hfill (A.14)

This establishes a lower bound on the number of simultaneous null eigenvectors of all three matrices. The most stringent bound is then obtained by considering all three permutations of this line of reasoning. Putting all this together, one finds a lower bound

$$\max\left(d(r_a, q) - d(r_a, q + n_b r_b) - d(r_a, q + n_c r_c) + d(r_a, q + n_b r_b + n_c r_c)\right)$$  \hfill (A.15)

on the number of simultaneous null eigenvectors, where $a, b, c$ run over the 6 permutations of 1, 2, 3, and the integers $n_a = 0, \ldots, p/c_a - 1$.

Using the equations (4.4) for $\chi_{AB}$ and the reality condition (3.6), the last equation in (4.4) can be written as

$$\chi_{12}^\dagger \chi_{12} + \chi_{13}^\dagger \chi_{13} + \chi_{14}^\dagger \chi_{14} = \frac{1}{4}(W^0)^{-1}.$$  \hfill (A.16)

If $\chi_{12}$, $\chi_{13}$ and $\chi_{14}$ have a simultaneous null eigenvector then there can be no solution to this equation since $W^0$ is non-degenerate. But (A.15), with $r_1 = q_1 + q_2$, $r_2 = q_1 + q_3$ and $r_3 = q_1 + q_4$, gives a lower bound on the number of such null eigenvectors; hence for a solution (A.16) to exist, it is necessary that for each $q$ and permutation of $a, b, c$

$$d(r_a, q) - d(r_a, q + n_b r_b) - d(r_a, q + n_c r_c) + d(r_a, q + n_b r_b + n_c r_c) \leq 0.$$  \hfill (A.17)

The only way this set of inequalities can be satisfied is if $d(r_a, q + n_c r_c) - d(r_a, q + n_b r_b + n_c r_c)$ is independent of $n_c$ which further requires that all the $k_q$ have to be equal. QED.

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