Free-fall Rainbow BTZ Black Hole

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Abstract

Doubly special relativity (DSR) is an effective model for encoding quantum gravity in flat spacetime. To incorporate DSR into general relativity, one could use “Gravity’s rainbow”, where the spacetime background felt by a test particle would depend on its energy. For a black hole, there are two natural orthonormal frames, the stationary one hovering above it and freely falling one along geodesics. Since the rainbow metric is the metric that the radiated particles “see”, a more natural orthonormal frame is the one anchored to the particles. And the cases with the stationary orthonormal frame have been extensively studied in the literature. In this paper, we investigate properties of rainbow BTZ black holes in the scenario with the free-fall orthonormal frame. We first review the thermodynamic properties of a BTZ black hole. Furthermore, we obtain the Free-fall (FF) rainbow BTZ black hole and then calculate its Hawking temperature via the Hamilton-Jacobi method. Finally, we discuss the thermodynamic properties of a FF stationary rainbow BTZ black hole.

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I. INTRODUCTION

Shortly after Einstein proposed general relativity in 1915, Schwarzschild derived the first black hole solution (Schwarzschild metric) from Einstein’s equations [1]. From then on, the study of black hole has become an important field of modern physics. Due to the strong gravitational pull from a black hole, the classical theory of a black hole predicts that anything, including light, cannot escape from the black hole. However, considering the quantum effect, Stephen Hawking demonstrated that black holes radiate a thermal flux of quantum particles [2].

Shortly after this discovery, it was realized that there is the trans-Planckian problem in Hawking’s calculations [3]. Hawking radiation appears to originate from a model with a large initial frequency, far beyond the Planck mass $m_p$, which experiences an exponentially high gravitational redshift near the horizon. Therefore, Hawking’s prediction relies on the validity of quantum field theory for arbitrary high energy in curved spacetime. On the other hand, quantum field theory is considered to be more of an effective field theory whose nature remains unknown. This observation raises the question of whether any unknown physics at the Planck scale could strongly influence Hawking radiation. It is widely believed
that trans-Planckian physics can be expressed in certain modifications of existing models.

Although a full theory of quantum gravity has yet to available, various theories of quantum gravity, such as loop quantum gravity, string theory, and quantum geometry, predicts that the transformation laws of special relativity are modified at very high energies. In particular, Deformed Special Relativity (DSR) theory makes the Planck length a new invariant scale and guarantees the nonlinear Lorentz transformation in the momentum spacetime $[4–7]$. Specifically, the modified energy-momentum dispersion relation of a particle of energy $E$ and momentum $p$ in DSR can take the form of

$$E^2 f^2 (E/m_p) - p^2 g^2 (E/m_p) = m^2, \quad (1)$$

where $m_p$ is the Planck mass, and $f(x)$ and $g(x)$ are two general functions with the following properties:

$$\lim_{x \to 0} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0} g(x) = 1. \quad (2)$$

The modified dispersion relation (MDR) might play an important role in astronomical and cosmological observations, such as the threshold anomalies of ultra high energy cosmic rays and TeV photons $[8–13]$. In phenomenological physics, ground observations and astrophysical experiments have tested the predictions of MDR theory $[14–17]$. One of the most popular choice for the functions $f(x)$ and $g(x)$ has been proposed by Amelino-Camelia et al. $[18, 19]$, which gives

$$f(x) = 1 \quad \text{and} \quad g(x) = \sqrt{1 - \eta x^n}. \quad (3)$$

As shown in $[19]$, this formula is compatible with some of the results obtained in the Loop-Quantum-Gravity approach and reflects the results obtained in $\kappa$-Minkowski and other noncommutative spacetimes. Phenomenological implications of this “Amelino-Camelia (AC) dispersion relation” are also reviewed in $[19]$.

To incorporate DSR into the framework of general relativity, Magueijo and Smolin $[20]$ proposed the “Gravity’s rainbow”, where the spacetime background felt by a test particle would depend on its energy. Consequently, the energy of the test particle deforms the background geometry and hence the dispersion relation. As regards the metric, it would be replaced by a one parameter family of metrics which depends on the energy of the test particle, forming a “rainbow metric”. Specifically, for the BTZ black hole, the corresponding “rainbow metric” solution to the rainbow Einstein’s Field equations in an stationary orthonormal frame is given in $[20]$:
Later, Alsaleh specifically studied the thermodynamic properties of the rainbow BTZ black hole in a stationary orthonormal frame [21]. Since the rainbow metric is the metric that the radiated particles “see”, a more natural orthonormal frame is the one anchored to the particles. Actually, in section III we will show that the rainbow BTZ black hole in the free-fall orthonormal frame is given by

\[
\begin{align*}
    ds^2 &= \frac{-h(r)}{g^2(E/m_p)} dt^2 + \frac{dx^2}{h(r)} + r^2 \left[ N_\phi(r) dt + d\phi \right]^2, \\
    &= \left[ \frac{1}{g^2(E/m_p)} - \frac{1}{f^2(E/m_p)} \right] h(r) dt^2 + \frac{-h(r) dt^2 + \frac{dx^2}{h(r)} + r^2 \left[ N_\phi(r) dt + d\phi \right]^2}{g^2(E/m_p)}. 
\end{align*}
\]  

(4)

In the remainder of this article, we dub the rainbow BTZ black holes (4) and (5) as stationary frames (SF) and free-fall frames (FF) rainbow BTZ black holes, respectively. In this paper, we aim to explore the thermodynamic properties of the FF Rainbow BTZ black hole.

There are some methods proposed to understand Hawking radiation [22–27]. Recently, a semi-classical method of modeling Hawking radiation as a tunneling process has been developed and attracted much attention. This method was first proposed by Kraus and Wilczek [28, 29], which is known as the null geodesic method. Later, the tunneling behaviors of particles were investigated using the Hamilton-Jacobi method [30–32]. Furthermore, taking the effects of quantum gravity into account, the Hamilton-Jacobi equation was modified, and the modified Hawking temperature was derived [33–38]. These motivate us to use the Hamilton-Jacobi method to study the gravity rainbow effect of Hawking radiation [39, 40].

The cases with the stationary orthonormal frame have been extensively studied by many authors [41–49].

The BTZ black hole is a solution of Einstein field equations in three-dimensional curved space, which describes a rotating AdS geometry [50]. It plays an important role in Field theory and string theory. In this paper, we will study the quantum gravity effect on BTZ black hole in the framework of gravity rainbow theory with the free-fall orthonormal frame. The remainder of our paper is organized as follows. In section III the thermodynamic properties of BTZ black holes are briefly reviewed. In section III the metric of a FF rainbow BTZ black hole is derived, and its Hawking temperature is obtained using the Hamilton-Jacobi method. The temperature and entropy of a FF rainbow BTZ black hole are computed. Finally, the thermodynamic properties of the FF static rainbow BTZ black
hole will be studied. Section IV is devoted to our discussion and conclusion. Throughout
the paper we take geometrized units \( c = 8 G = k_B = 1 \), where the Planck constant \( \hbar \)
is square of the Planck mass \( m_p \).

II. BTZ BLACK HOLE

The BTZ black hole is an important solution to Einstein field equation in a \((2 + 1)\)
dimensional space with a negative cosmological constant. The action is

\[
S = \frac{1}{2\pi} \int \sqrt{-g} \left[ R + 2\Lambda \right],
\]

(6)

where \( \Lambda = -\frac{l^2}{2} \) is cosmological constant, and \( l \) is AdS radius. The BTZ black hole solution
to the action (6) is \[50\]

\[
ds^2 = -h(r) \, dt^2 + \frac{1}{h(r)} \, dr^2 + r^2 \left[ N_\phi(r) \, dt + d\phi \right]^2,
\]

(7)

where

\[
h(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2},
\]

\[
N_\phi(r) = -\frac{J}{2r^2}.
\]

(8)

The parameters \( M \) and \( J \) can be interpreted as the mass and the angular momentum of the
BTZ black hole. The BTZ black hole has two horizons located \( r = r_\pm \), which are determined
by \( h(r) = 0 \):

\[
r_\pm^2 = \frac{Ml^2 \pm \sqrt{M^2l^4 - J^2l^2}}{2}.
\]

(9)

In terms of \( r_\pm \), we can rewrite \( h(r) \), \( M \) and \( J \) as

\[
h(r) = \frac{1}{l^2r^2} \left( r^2 - r_\pm^2 \right) \left( r^2 - r_-^2 \right), \quad M = \frac{r_+^2 + r_-^2}{l^2} \text{ and } J = \frac{2r_+r_-}{l}.
\]

(10)

The surface gravity \( \kappa \) of the BTZ black hole at the outer horizon \( r = r_+ \) is

\[
\kappa = \frac{r_+^2 - r_-^2}{l^2r_+}.
\]

(11)

So the Hawking temperature \( T_h \) of the BTZ black hole is \[51 54\]

\[
T_h = \frac{\hbar\kappa}{2\pi} = \frac{\hbar \left( r_+^2 - r_-^2 \right)}{2\pi l^2 r_+}.
\]

(12)
The first law of thermodynamics for the BTZ black hole was obtained in \[55\], which reads

\[
dM = T_h dS + \Omega_H dJ, \tag{13}
\]

where \( S \) and \( \Omega_H = \frac{J}{2r_+} = \frac{r}{r_+} \) are the entropy and the angular velocity of the BTZ black hole, respectively. The eqn. (13) can be rewritten in the form:

\[
dS = \frac{dM}{T_h} - \frac{\Omega_H dJ}{T_h} = \frac{4\pi dr_+}{\hbar}, \tag{14}
\]

which, by integration, leads to

\[
S = \frac{4\pi r_+}{\hbar}. \tag{15}
\]

Note that the BTZ black hole entropy \( S \) was also computed using the Euclidean action method \[50, 52, 54, 56, 57\]. The heat capacity of the BTZ black hole is

\[
C_J = T_h \left( \frac{\partial S}{\partial T_h} \right)_J = \frac{4\pi r_+ \Delta}{2 - \Delta}, \tag{16}
\]

where \( \Delta = \sqrt{1 - \frac{l^2 J^2}{M^4}}. \) Since \( 0 \leq \Delta \leq 1 \), the BTZ black hole always has a positive heat capacity, which implies the thermodynamic system for the BTZ black hole is stable.

### III. FREE-FALL RAINBOW BTZ BLACK HOLE

In this section, we obtain the Free-fall (FF) rainbow BTZ black hole and then calculate its Hawking temperature via the Hamilton-Jacobi method. Finally, we discuss the rainbow corrections to the entropy of the black hole.

#### A. Free-fall Rainbow BTZ Metric

First, we generalize the energy-independent BTZ metric (7) to the energy-dependent BTZ rainbow metric. Generally, the energy-independent metric is given by

\[
ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu, \tag{17}
\]

can be rewritten in terms of a set of energy-independent orthonormal frame fields \( e_a \):

\[
ds^2 = \eta^{ab} e_a \otimes e_b, \tag{18}
\]
where \( a = (0, i) \) and \( i \) is the spatial index. For the MDR (1), the energy-dependent rainbow counterpart for the energy-independent metric (17) can be obtained using equivalence principle \([20]\), which gives

\[
d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu \otimes dx^\nu = \eta^{ab} \tilde{e}_a \otimes \tilde{e}_b = \left( \frac{1}{g^2(E/m_p)} - \frac{1}{f^2(E/m_p)} \right) e_0 \otimes e_0 + \frac{ds^2}{g^2(E/m_p)}, \tag{19}\]

where the energy-dependent orthonormal frame fields are

\[
\tilde{e}_0 = \frac{e_0}{f(E/m_p)} \text{ and } \tilde{e}_i = \frac{e_i}{g(E/m_p)}. \tag{20}\]

Note that a tilde is used for an energy-dependent quantity.

Obviously, different choices of the orthonormal frame can give different rainbow metrics. In fact, the form of the rainbow metric crucially depends on the time component of the orthonormal frame, \( e_0 \). For the BTZ metric (7), \( e_0 \) in the literature \([21, 58, 59]\) is usually chosen to be

\[
e_0 = \sqrt{h(r)} dt, \tag{21}\]

where this orthonormal frame basis is hovering above the black hole. On the other hand, the particles radiated from the black hole travel along the geodesics. Since the rainbow metric is the metric that the radiated particles "see", a more natural orthonormal frame is the one anchored to the particles.

For massive particles moving along the geodesics, \((e_0)^\mu\) is just the 3-velocity vector \( u^\mu \) of the geodesics. To compute the 3-velocity vector, we consider \( p_t \) and \( p_\phi \), which are conserved along geodesics (since the metric does not depend explicitly on \( t \) and \( \phi \)). It leads immediately to the first integrals of the \( t \)- and \( \phi \)- equations. These are given by

\[
\begin{align*}
    u_t &= g_{tt} u^t + g_{t\phi} u^\phi = [-h(r) + r^2 N_\phi^2(r)] \dot{t} + r^2 N_\phi \dot{\phi} = -E/m \\
    u_\phi &= g_{\phi t} u^t + g_{\phi\phi} u^\phi = r^2 N_\phi \dot{t} + \dot{r}^2 \dot{\phi} = L/m
\end{align*} \tag{22}\]

where \( m, E \) and \( L \) are the mass, the energy and the angular momentum of the particles. Solving eqns. (22) for \( \dot{t} \) and \( \dot{\phi} \) gives

\[
\begin{align*}
    u^t &= \dot{t} = \frac{N_\phi(r) L + E}{mh(r)}, \\
    u^\phi &= \dot{\phi} = \frac{L}{mr^2} - \frac{N_\phi(r) [N_\phi(r) L + E]}{mh(r)}. \tag{23}\]
\]
We then use $g^\mu \nu u_\mu u_\nu = -1$ to find $u^\tau$. So the 3-velocity vector of the radiated particle is

$$u^\mu = \left( \frac{N_\phi (r) L + E}{m h (r)} , \sqrt{\frac{[E + N_\phi (r) L]^2}{m^2} - h (r) \left( 1 + \frac{L^2}{m^2 r^2} \right)^2} , \frac{L}{m r^2} - \frac{N_\phi (r) [N_\phi (r) L + E]}{m h (r)} \right).$$

(24)

Therefore, the time component of the orthonormal frame anchored to the radiated particles is then given by

$$e_0 = u_\mu dx^\mu = - \frac{E + N_\phi (r) L}{m} dt + \sqrt{\frac{[E + N_\phi (r) L]^2}{m^2} - \frac{h (r) \left( 1 + \frac{L^2}{m^2 r^2} \right)^2}{h (r)}} dr.$$

(25)

From eqn. (19), the rainbow BTZ metric is

$$ds^2 = \left[ \frac{1}{g^2 (E/m_p)} - \frac{1}{f^2 (E/m_p)} \right] e_0 \otimes e_0 + \frac{-h (r) dt^2 + \frac{dr^2}{h (r)} + r^2 (N_\phi (r) dt + d\phi)^2}{g^2 (E/m_p)},$$

(26)

where $e_0$ is given by eqn. (25). The rainbow BTZ metric (26) is dubbed as "Free-fall (FF) rainbow BTZ black hole."

**B. Effective Hawking Temperature**

We now use the Hamilton-Jacobi method to calculate the Hawking temperature of the FF rainbow black hole (26). In the Hamilton-Jacobi method, one ignores the self-gravitation of emitted particles and assumes that their action satisfies the relativistic Hamilton-Jacobi equation. The tunneling probability for the classically forbidden trajectory from inside to outside the horizon is obtained by using the Hamilton-Jacobi equation to calculate the imaginary part of the action for the tunneling process. In [39], it showed that, in the rainbow metric $ds^2 = \tilde{g}_{\mu \nu} dx^\mu dx^\nu$, the Hamilton-Jacobi equations for massive particles can simply be written as

$$\tilde{g}_{\mu \nu} \partial^\mu I \partial^\nu I = m^2,$$

(27)

where $I$ is the radiated particle’s classical action. Since the FF rainbow black hole (26) is independent of $t$ and $\phi$, we can employ the following ansatz for the action $I$

$$I = -Et + W (r) + L\phi,$$

(28)

where, as above defined, $E$ and $L$ are the radiated particle’s energy and angular momentum, respectively. Defining $p_r (r) \equiv W' (r)$, we can use the FF rainbow black hole (26) to rewrite
the Hamilton-Jacobi equation \([27]\) as the equation in terms of \(p_r\), which is too long to put here. Solving the Hamilton-Jacobi equation for \(p_r\), we obtain

\[ p_r^\pm (r) = \frac{A_r^\pm (r)}{h(r) \tilde{H}(r)} \]

(29)

where +/- denotes the outgoing/ingoing solutions. Here, \(A_r^\pm (r)\) are regular functions of \(r\) without poles, the detailed forms of which are rather complicated and not relevant. In the denominator of \(p_r^\pm (r)\), we define

\[ \tilde{H}(r) \equiv h(r) - \epsilon (E/m_p) \left[ E + LN_\phi(r) \right]^2/m^2, \]

\[ \epsilon (E/m_p) \equiv 1 - \frac{g^2 (E/m_p)}{f^2 (E/m_p)}. \]

(30)

The corresponding action is

\[ I_\pm = -Et + \int p_r^\pm (r) \, dr + L\phi, \]

(31)

where the imaginary part of \(I_\pm\) comes from the integral of \(p_r^\pm (r)\).

According to the Hamilton-Jacobi method, the residues of \(p_r^\pm (r)\) lead to the Hawking temperature of the radiation. So the poles of \(p_r^\pm (r)\) correspond to the locations of the horizons of the FF rainbow black hole \([26]\). Eqn. (29) shows that the poles of \(p_r^\pm (r)\) are at the location where \(h(r) = 0\) or \(\tilde{H}(r) = 0\). For \(h(r) = 0\), one simply has \(r = r_\pm\). Moreover, it can show that there are also two solutions to \(\tilde{H}(r) = 0\), namely \(r = \tilde{r}_\pm\) with \(\tilde{r}_+ \geq \tilde{r}_-\). The ordering of \(r_\pm\) and \(\tilde{r}_\pm\) depends on the sign of \(\epsilon (E/m_p)\). Since \(h'(r) > 0\) for \(r \geq r_+\) and \(h'(r) < 0\) for \(r \leq r_-\), we then have \(\tilde{r}_+ \geq r_+ \geq r_- \geq \tilde{r}_-\) when \(\epsilon (E/m_p) > 0\). In this case, \(\tilde{r}_+\) is the radius of the outermost horizon. Similarly, in the case with \(\epsilon (E/m_p) < 0\), the radius of the outermost horizon is \(r_+\).

To find the Hawking temperature on the outermost horizon, we need to calculate the imaginary part of \(I_+\) by integrating \(p_r^\pm (r)\) along the semicircle around the outermost horizon. As shown in \([60, 61]\), the probability of a particle tunneling from inside to outside the horizon is

\[ P_{\text{emit}} \propto \exp \left[ -\frac{2}{\hbar} (\text{Im} \, I_+ - \text{Im} \, I_-) \right]. \]

(32)

For a particle of energy \(E\) and angular momentum \(L\) residing in a system with temperature \(T\) and angular velocity \(\omega\), the Maxwell–Boltzmann distribution is \([62]\)

\[ P \propto \exp \left[ -\frac{E - \omega L}{T} \right]. \]

(33)
From eqn. (33), the effective Hawking temperature can be read off from the Boltzmann factor in $P_{\text{emit}}$:

$$\tilde{T}_h = \frac{\hbar [E + N_\phi (r_h) L]}{2 (\text{Im } I_+ - \text{Im } I_-)}, \quad (34)$$

where $-N_\phi (r_h) = -\tilde{g}_{t\phi}/\tilde{g}_{\phi\phi} = -g_{t\phi}/g_{\phi\phi}$ is the angular velocity of the FF rainbow BTZ black hole, and $r_h = r_+$ or $\tilde{r}_+$ is the outermost horizon. For the $\epsilon (E/m_p) < 0$ and $\epsilon (E/m_p) > 0$ cases, we find

- $\epsilon (E/m_p) < 0$: The outermost horizon is at $r = r_+$. Using the residue theory for the semi circle around $r = r_+$, we get

$$\text{Im } I_+ = \text{Im } I_- = \frac{\pi [E + N_\phi (r_+ L)]}{\tilde{H}' (r_+)} \quad (35)$$

which gives the effective Hawking temperature

$$\tilde{T}_h = \infty. \quad (36)$$

- $\epsilon (E/m_p) > 0$: The outermost horizon is at $r = \tilde{r}_+$. Using the residue theory for the semi circle around $r = \tilde{r}_+$, we get

$$\text{Im } I_+ = 0 \quad (37)$$

$$\text{Im } I_- = -\frac{2\pi [E + N_\phi (r_+ L)]}{\tilde{H}' (\tilde{r}_+)} \sqrt{\frac{g^2 (E/m_p) (1 + \frac{L^2}{m^2 r_+^2})}{f^2 (E/m_p)} - \frac{L^2}{m^2 r_+^2}}, \quad (38)$$

which gives the effective Hawking temperature

$$\tilde{T}_h = \frac{\hbar \tilde{H}' (\tilde{r}_+)}{4\pi \sqrt{1 - \epsilon (E/m_p) (1 + \frac{L^2}{m^2 r_+^2})}} \quad (39)$$

When $g (E/m_p) = f (E/m_p) = 1$, one has $\epsilon (E/m_p) = 0$, $\tilde{r}_+ = r_+$ and $\tilde{H} (r) = h (r)$, which means that, as expected, $\tilde{T}_h$ of the FF rainbow BTZ black hole would reduce to $T_h$ of the BTZ black hole, given in eqn. (12). To express $\tilde{T}_h$ in terms of $E$ and $L$, we need to solve $\tilde{H} (r) = 0$ for $\tilde{r}_+$, whose expression is quite complicated. However for $\epsilon (E/m_p) \ll 1$, one has that, to $O (\epsilon (E/m_p))$,

$$\tilde{T}_h \approx T_h \left\{ 1 + \frac{\epsilon (E/m_p)}{2} \left[ 1 + \frac{L^2}{m^2 r_+^2} + \frac{\hbar^2 \left( E - \frac{3L^2}{2r_+^2} \right)^2}{8\pi^2 \tilde{T}_h^2 m^2} - \frac{\hbar \left( E - \frac{3L^2}{2r_+^2} \right)}{\pi T_h m^2 r_+^2} \right] \right\} \quad (40)$$
Since the Hawking radiation spectrum is dominated by low angular momentum modes \[63\], we can set \( L = 0 \) for simplicity. In this case, the effective Hawking temperature becomes

\[
\tilde{T}_h \approx T_h \left\{ 1 + \frac{\epsilon (E/m_p)}{2} \left[ 1 + \frac{\hbar^2 E^2 \left( \frac{2}{7\pi} + \frac{3J^2}{2\pi^2} \right)}{8\pi^2 T_h^2 m^2} \right] \right\}, \tag{41}
\]

which shows that the rainbow gravity correction tends to increase the Hawking temperature of the black hole.

C. Thermodynamics of FF Static Rainbow BTZ Black Hole

For simplicity, we estimate the rainbow corrected temperature and entropy for a FF static rainbow BTZ black hole, which has \( J = 0 \). First, we can use the Heisenberg uncertainty principle to estimate the momentum \( p \) of an emitted particle \(64, 65\):

\[
p \sim \delta p \sim \hbar / \delta x \sim \hbar / \tilde{r}_. \tag{42}
\]

Using \( \tilde{H} (\tilde{r}_+) = 0 \) and eqns. \[42\] and \[1\], we can use express the energy \( E \) in terms of the black hole mass \( M \). The effective Hawking temperature \(\tilde{T}_h\) then can be written as a function of \( M \), \( T (M) \), which can be interpreted as the rainbow corrected temperature of the black hole. Using the first law of black hole thermodynamics, we find that the entropy of the black hole is

\[
S (M) = \int \frac{dM}{T (M)}. \tag{43}
\]

When \( M \gg \frac{\hbar^2}{m^2 l^2} \), the corrected Hawking temperature and entropy are estimated as

\[
T (M) \sim \frac{\hbar \sqrt{M}}{2\pi l} \left[ 1 + \frac{\epsilon (m/m_p) (M + 1)}{2M} \right],
\]

\[
S (M) \sim \frac{\hbar M^{\frac{3}{2}}}{3l^2 \pi} \left[ 1 + \frac{\epsilon (m/m_p) (M + 3)}{2M} \right], \tag{44}
\]

respectively.

To numerically investigate \( T (M) \) and \( S (M) \), we focus on the Amelino-Camelia dispersion relation \[3\] with \( n = 2 \) and \( \eta > 0 \). We plot \( T (M) \) and \( S (M) \) for various values of \( \eta \) in FIG. \[1\], where we take \( m = 0.01 \). The left panel of FIG. \[1\] shows that the black hole temperature increases with increasing \( \eta \), which implies that the rainbow effects would speed up the evaporation of the black hole. Moreover, the terminal temperature is zero when \( \eta = 0 \).
while it is greater than zero when \( \eta > 0 \), which means the rainbow effects would lead to a more violent death of the black hole. On the other hand, the right panel of FIG. 1 shows the black hole entropy decreases with increasing \( \eta \). Therefore, the black hole tends to store less information when the rainbow effects are turned on.

IV. DISCUSSION AND CONCLUSION

In this paper, we considered a FF rainbow BTZ black hole and analyzed the effects of rainbow gravity on the temperature and entropy. We first derived the metric of a FF rainbow BTZ black hole. Then, we used the Hamilton-Jacobi method to obtain the effective Hawking temperature of the rainbow BTZ black hole, which was shown to depend on the energy of radiated particles. We found that, when \( \epsilon (E/m_p) > 0 \), the radiated particles experience a infinity Hawking temperature, which might shed light on the black hole firewall paradox. Finally, we employed the uncertainty principle to estimate the corrected temperature and entropy of the black hole. It showed that, for when \( \epsilon (E/m_p) < 0 \), the black hole evaporates faster and stores less information than in the usual case.

[1] Note that, for massless particles, one has

\[
\frac{E}{p} = \sqrt{1 - \epsilon (E/m_p)},
\]

which means that \( \epsilon (E/m_p) > 0 \) corresponds to the subluminal case that has \( E/p < 1 \), and that \( \epsilon (E/m_p) < 0 \) to the superluminal case that has \( E/p > 1 \).
Acknowledgments

We are grateful to Dr. H. Wu, and Prof. D. Chen for their useful discussions. This work is supported in part by NSFC (Grant No.11747171, 11005016, 11175039 and 11375121) and the Fundamental Research Funds for the Central Universities. Natural Science Foundation of Chengdu University of TCM (Grants nos. ZRYY1729 and ZRQN1656). Discipline Talent Promotion Program of /Xinglin Scholars(Grant no. QNXZ2018050) and the key fund project for Education Department of Sichuan (Grant no. 18ZA0173)

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