THE $p$-CYCLE OF HOLOGOMIC $\mathcal{D}$-MODULES AND AUTOMORPHISMS OF THE WEYL ALGEBRA

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ABSTRACT. Let $X = \mathbb{A}^n$ be complex affine space, and let $T^*X$ be its cotangent bundle. For any smooth Lagrangian $\mathcal{L} \subset T^*X$ satisfying some cohomology vanishing assumptions, we construct a holonomic $\mathcal{D}_X$-module $M$ associated to $\mathcal{L}$ in a canonical way. This association goes through positive characteristic—namely, after reducing $M$ mod $p$ for sufficiently large $p$, we have that the “$p$-support” of $M$ is equal to $\mathcal{L}$. As a consequence, we deduce that the group of Morita autoequivalences of the $n$-th Weyl algebra is isomorphic to the group of symplectomorphisms of $T^*\mathbb{A}^n$. This generalizes an old theorem of Dixmier (in the case $n = 1$) and settles a conjecture of Belov-Kanel and Kontsevich in general.

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1. Introduction

Let \( X_\mathbb{C} \) be a smooth complex algebraic variety. As is well known, the sheaf of differential operators on \( X_\mathbb{C} \), \( \mathcal{D}_{X_\mathbb{C}} \), is a filtered quantization of the cotangent bundle \( T^*X_\mathbb{C} \). As a consequence, to any coherent \( \mathcal{D}_{X_\mathbb{C}} \) module \( M_\mathbb{C} \), one can associate an algebraic cycle, called the singular support \( SS(M_\mathbb{C}) \), in \( T^*X_\mathbb{C} \). The underlying variety of \( SS(M_\mathbb{C}) \) has two features that we should like to highlight. The first is that it is invariant with respect to the natural scaling action of \( \mathbb{G}_m \) on \( T^*X_\mathbb{C} \); this is a direct consequence of its construction. The second is that it is a coisotropic subvariety; this is a deep theorem that was first proved in [SKK], and then again in [Ga]. Every component of a coisotropic subvariety of \( \mathbb{G}_m \)-equivariant, this is a deep theorem that was first proved in [SKK], and then again in [Ga]. Every component of a coisotropic subvariety of \( T^*X_\mathbb{C} \) has dimension at least \( \text{dim}(X) \). Those \( \mathcal{D}_{X_\mathbb{C}} \)-modules whose supports have dimension equal to \( \text{dim}(X) \) (and hence are Lagrangian cycles) form an abelian subcategory- the category of holonomic \( \mathcal{D}_{X_\mathbb{C}} \)-modules- which has many remarkable properties (c.f. [Be] or [HTT] for detailed introductions).

Though the singular support is used to define holonomicity, when one turns to the problem of classifying holonomic \( \mathcal{D}_{X_\mathbb{C}} \)-modules, one finds that it is a rather unrefined invariant. For instance, every vector bundle with flat connection on \( X_\mathbb{C} \) may be regarded as a holonomic \( \mathcal{D}_{X_\mathbb{C}} \)-module, and in this case \( SS(M_\mathbb{C}) \) can be computed easily- its underlying variety is simply the zero section \( X_\mathbb{C} \subset T^*X_\mathbb{C} \). The theory of flat connections is, on the other hand, quite rich. For instance, there is a fully faithful functor from the category of (finite dimensional, complex) representations of the topological fundamental group \( \pi_1(X_\mathbb{C}) \) to the category of vector bundles with flat connection on \( X_\mathbb{C} \). If \( X_\mathbb{C} \) is projective over \( \mathbb{C} \), this is an equivalence. If not, the category of vector bundles with flat connection on \( X_\mathbb{C} \) is even larger than that of representations of \( \pi_1 \) (which could, of course, already be quite complicated).

Thus, it seems natural to look for some sort of more refined cycle which one could associate to a holonomic \( \mathcal{D}_{X_\mathbb{C}} \)-module. A little thought shows that, in order to get anywhere, one must drop one of the two main properties of the singular support (either the \( \mathbb{G}_m \)-invariance or the co-isotropicity). The coisotropic nature of \( SS(M_\mathbb{C}) \) reflects the deep connection between \( \mathcal{D}_{X_\mathbb{C}} \) and \( T^*X_\mathbb{C} \); a piece of structure which it seems unwise to discard. Thus one is led to search for a way to associate to a holonomic \( \mathcal{D}_{X_\mathbb{C}} \)-module a Lagrangian cycle which is not necessarily conical. This can be (and has been) done in various ways; one prominent example is the non-abelian Hodge theory of Corlette, Simpson, Sabbah, and Mochizuki.

In this paper, we will focus on a different sort of cycle, following the ideas of M. Kontsevich in [Ko] and A. Belov-Kanel in [BKKo], and whose origin goes back, at least, to [ML] (similar ideas appear also in [Ts]) The main idea is as follows: one may select a commutative ring \( R \), which is a finitely generated \( \mathbb{Z} \)-algebra, and a scheme \( X \) which is smooth over \( S = \text{Spec}(R) \), such that \( X \times_S \text{Spec}(\mathbb{C}) \supseteq X_\mathbb{C} \). Localizing \( R \) if necessary, one may also find a sheaf \( M \) of \( \mathcal{D}_{X/S} \)-modules such that \( M \otimes_R \mathbb{C} = M_\mathbb{C} \). Then, for any perfect field \( k \) of positive characteristic, and any point \( \text{Spec}(k) \to S \), one may base change to obtain a \( \mathcal{D}_{X_k} \) module \( M_k \). As will be explained below, to such a module one can associate a cycle in \( T^*X_k^{(1)} \) (the Frobenius twist of \( T^*X_k \)), called the \( p \)-cycle. It is a theorem of T. Bitoun that the variety underlying this cycle, called the \( p \)-support, is Lagrangian. In [Ko] Kontsevich outlines a program

\[ ^1 \text{and perhaps impossible} \]
in which these Lagrangian subvarieties can be “fit together” to form some kind of cycle for $M_C$, which he calls the “arithmetic support.” Furthermore, he conjectures that this cycle provides, in various senses, a very refined invariant of $M_C$.

Most of the conjectures in [Ko] remain far out of reach. However, there are some cases in which the ideas of the previous paragraph can be approached. In particular, consider a Lagrangian $\mathcal{L}_C \subset T^*X_C$. After choosing an appropriate ring $R$ as in the previous paragraph, we may suppose $\mathcal{L}_C=\mathcal{L} \times_S \text{Spec}(\mathbb{C})$ for some $\mathcal{L} \subset T^*(X/S)$ which is smooth over $S$. Then, after base changing to $\text{Spec}(k)$, we can look at $\mathcal{L}_k^{(1)} \subset T^*X_k^{(1)}$. We say that $M_C$ has constant arithmetic support equal to $\mathcal{L}_C$ if, for any $R$ and $k$ as above (after possibly localizing at finitely many elements) the $p$-support of $M_k$ is equal to $\mathcal{L}_k^{(1)}$. It is not hard to see that if this condition for one such $R$, it is true for all of them. Clearly, if a $D$ module $M_C$ has constant arithmetic support, the variety $\mathcal{L}_C$ should be thought of as the variety underlying the arithmetic support of $M_C$ (we are ignoring multiplicity for now, though it cannot, and will not, be ignored below). Then the question arises: which Lagrangian subvarieties occur as the constant arithmetic support of a holonomic $D$-module? In [Ko], Kontsevich has conjectured some sufficient conditions. This paper gives the proof of one of these conditions, in a special case. The main result proved here is:

**Theorem 1.** Let $X_C$ be a smooth affine variety. Let $\mathcal{L}_C \subset T^*X_C$ be a smooth Lagrangian subvariety. Suppose

a) The projection $\pi: \mathcal{L}_C \to X_C$ is dominant.

b) The singular homology $H_1^{\text{sing}}(\mathcal{L}_C, \mathbb{Z}) = 0$.

c) There exists a smooth projective compactification $\overline{\mathcal{L}}_C$ of $\mathcal{L}_C$ such that the Hodge cohomology group $H^{0,2}(\overline{\mathcal{L}}_C) = 0$.

Then there exists a unique irreducible holonomic $D_{X_C}$-module $M_C$ which has constant arithmetic support equal to $\mathcal{L}_C$, with multiplicity 1.

In the case where $X = \mathbb{A}^m_\mathbb{C}$, this result implies more; indeed, the condition that $\pi: \mathcal{L}_C \to X_C$ is dominant can be removed by making use of the action of the symplectic group $\text{Sp}_{2m}$ on $T^*\mathbb{A}^m_\mathbb{C}$. Thus we have

**Corollary 2.** Let $\mathcal{L}_C \subset T^*\mathbb{A}^m_\mathbb{C}$ be a smooth Lagrangian subvariety satisfying the cohomological conditions b and c) of theorem 1. Then there exists a unique irreducible holonomic $D_{\mathbb{A}^m_\mathbb{C}}$-module $M_C$ which has constant arithmetic support equal to $\mathcal{L}_C$, with multiplicity 1.

As indicated in [BKKo], this higher dimensional version has a number of interesting consequences, including a description of the Picard group of the Weyl algebra. Let us briefly explain (some details are provided in section §6 below). Recall that $\text{Pic}(D)$, where $D$ is any algebra, is the group of isomorphism classes of invertible $D-D$-bimodules (the group operation is tensor product). On the other hand, consider the group of algebraic symplectomorphisms of the symplectic variety $T^*\mathbb{A}^m_\mathbb{C}$. Taking the graph such a morphism, $\phi$, yields a Lagrangian subvariety $\mathcal{L}_C^{\phi}$ of $T^*\mathbb{A}^m_\mathbb{C} \times T^*\mathbb{A}^m_\mathbb{C} \simeq T^*\mathbb{A}^{2m}_\mathbb{C}$. By construction $\mathcal{L}_C^{\phi} = \mathcal{L}_C^{\phi^*}$ and so satisfies the required cohomological assumptions. Thus we obtain from 2 a unique $D_{2m} = D_m \otimes D_m$-module $M_C^{\phi}$ corresponding to $\mathcal{L}_C^{\phi}$. One verifies that the bimodule corresponding to the Lagrangian $\mathcal{L}_C^{\phi^{-1}}$ is the inverse bimodule to $M_C^{\phi}$. Combining this fact with the ideas of [Ts] and [BKKo], we conclude...
Theorem 3. There is an isomorphism of groups

$$\text{Pic}(\mathcal{D}_m) \cong \text{Symp}(T^* \mathbb{A}^m_1)$$

In the case $m = 1$, it is known that $\text{Pic}(\mathcal{D}_1) = \text{Aut}(\mathcal{D}_1)$. In this case the theorem is due to Dixmier in [Di]. Its reproof in [ML] using positive characteristic techniques is, in a sense, the starting point for everything done here.

Before proceeding, we indicate the main steps of the argument of the paper, and outline the contents. Let $\mathcal{L}_C \subset T^* X_C$ be as in theorem 1. Section 2 contains general preliminaries on differential operators in positive characteristic, the $p$-curvature, and in particular the Azumaya property of $\mathcal{D}_{X_k}$. We explain there how to obtain a $\mathcal{D}_{X_k}$-module $M_k$ whose $p$-support is precisely $\mathcal{L}_k^{(1)}$. Thus we are led to the question of lifting this object to a $\mathcal{D}$-module in characteristic zero.

In section 3 we discuss this lifting problem in detail for connections on varieties; we give cohomological criteria for the obstruction to (infinitesimal) lifting, as well as a description of the set of lifts. We also include several variants (for Higgs bundles and for $\lambda$-connections).

In section 4 we prove theorem 1 in the case where $X_C$ is a curve. In this case, the (co)homology vanishing assumptions imply that the Lagrangian $\mathcal{L}_C \to \mathbb{A}^1_1$ (this is the only affine smooth curve whose first singular homology vanishes). Furthermore, since the projection $\pi : \mathcal{L}_C \to X_C$ is dominant, we deduce that $X_C \to \mathbb{A}^1_1$ as well, and therefore that $\pi$ is actually surjective; this, in turn, implies that the positive-characteristic $\mathcal{D}$-module $M_k$ constructed earlier is a connection on $\mathbb{A}^1_1$. We may complete this to a bundle $\overline{M}_k$ on $\mathbb{P}^1_k$ with a connection which degenerates at the point at infinity, and we explain that there is a preferred choice of such. Using the lifting of connections developed in the previous section, we deduce that $M_k$ admits, for each Witt ring $W_n(k)$, a unique lift to a connection over $\mathbb{A}^1_{W_n(k)}$ with suitable properties; namely, we control the behavior of the connection at the point at $\{\infty\}$ in $\mathbb{A}^1_{W_n(k)}$. Taking the limit yields, for each prime $p >> 0$, a connection over a particular field in characteristic zero (e.g., if our initial ring was $R = \mathbb{Z}$, then we would get a connection over each $p$-adic field $\mathbb{Q}_p$). To show that these connections all agree, we must invoke the theory of rigid irreducible connections, due (in the irregular case) to Bloch-Esnault and Arinkin, expanding earlier work of Katz (in the regular case). This theory implies that, under suitable circumstances, a connection on $\mathbb{A}^1_k$ (where $K$ is a field of characteristic zero) is uniquely determined by its formal type at $\{\infty\} = \mathbb{P}^1_k \setminus \mathbb{A}^1_k$; we compute the relevant formal type in the case of interest to us, and deduce the theorem from this. At the end of the section, we point out the relationship between this case of the theorem and the classical Abhyankar-Moh theorem.

In section 5, we deal with the higher dimensional case. In order to work with connections (instead of arbitrary holonomic $\mathcal{D}$-modules), we pull back over a projective birational morphism $\varphi : \tilde{X}_C \to X_C$ which resolves the subvariety of $X_C$ over which the map $\pi$ is not finite etale; further, we can compactify to obtain a smooth variety $\overline{X}_C$. Since the morphism is birational, we have an open subset $U_C$ over which $\varphi$ is an isomorphism. We show that the finite etale morphism $\mathcal{L}_{U_C} \to U_C$ extends to a finite flat map $\tilde{\mathcal{L}}_C \to \tilde{X}_C$. Now we look mod $p$. The previous theory enables us to construct a bundle $\overline{M}_k$ on $\tilde{X}_k$ for which the connection degenerates in a preferred way at the “divisor at infinity.” We can, as before, study the lifting of such objects to characteristic zero. However, the theory of rigidity is not as
well-behaved in higher dimensions as it is in dimension $1$; namely, there is no theorem (of which I am aware) that would guarantee that a connection is determined, globally, by its formal behavior at a given divisor. So, we have to go in a different direction; namely, we consider not just a connection $\overline{\nabla}_k$ but a $\lambda$-connection $\overline{\nabla}_{\lambda,k}$, which degenerates, at $\lambda = 0$, to a Higgs bundle supported on the Lagrangian $\mathcal{L}_C$ (see below for details and definitions). We can then show a uniqueness property for these objects by making use of a rigidity property for the closure of $\mathcal{E}|_{U_C}$ inside $T^*X_C$, and, making use of the appropriate Hilbert scheme, deduce the required uniqueness for lifts of the bundle $\overline{\nabla}_k$; yielding a well defined vector bundle $\overline{\nabla}_C$ with a connection which degenerates at some divisor on $\check{X}_C$. Then we finish the argument by pushing forward (under $\varphi$) to $\check{X}_C$.

Finally in section 6 we look at the case of higher dimensional affine space and prove 2 and theorem 3 as indicated above.

### 1.1. Notations and Conventions.
Throughout, $R$ will be an integral domain of finite type over $\mathbb{Z}$, taken with an embedding $R \to \mathbb{C}$. Undecorated letters such as $X, Y, Z$ will denote schemes over $S = \text{Spec}(R)$. $k$ will denote a perfect field of positive characteristic. Decorated letters such as $X_C$ and $X_k$ will denote the base change of $X$ to $C$, and, relative to a morphism $k \to \text{Spec}(R)$, the base change of $X$ to $k$.

If $M$ is an object on $X$ (such as a coherent sheaf or $\mathcal{D}$-module), the letters $M_C$ and $M_k$ will denote the base change (to $C$ and $k$, respectively).

If $\varphi : X \to Y$ is a morphism, we will denote by the same letter the induced morphism $\varphi : X_C \to Y_C$ and $\varphi : X_k \to Y_k$; since we always decorate the varieties, this should (hopefully) cause no confusion. If $Z \to Y$ is another morphism, we shall often denote the base change by $Z_Y$.

If $X \to S$ is a smooth map, then we have the relative tangent sheaf $\mathcal{T}_{X/S}$ and the relative differential forms $\Omega^1_{X/S}$; these will be vector bundles on $X$, and, taking the relative spectrum yields the relative cotangent and tangent bundles, respectively. The enveloping algebra of the sheaf $\mathcal{T}_{X/S}$ is denoted $\mathcal{D}_{X/S}$, the sheaf of relative differential operators.

Over $\mathbb{C}$, we shall follow the notations and conventions of [HTT] concerning algebraic $\mathcal{D}$ modules and the functors between them; the one exception is that, if $j : U_C \subset X_C$ is an open inclusion of smooth complex varieties, and $M_{U_C}$ is an irreducible holonomic $\mathcal{D}_{U_C}$-module, we denote by $j_*(M_{U_C})$ the unique irreducible holonomic $\mathcal{D}_{X_C}$-module extending $M_{U_C}$.

Over a field, the cotangent bundle is equipped with a standard one-form, we shall denote $\alpha$; the two-form $d\alpha = \omega$ is given in local coordinates by $\sum_{i=1}^m dx_i \wedge dy_i$ (where $m = \text{dim}(X)$).

Finally, we shall often encounter the following situation: we have a map $\varphi : X_F \to Y_F$ (where $F$ is a field of any characteristic) between smooth $F$-varieties. This yields a correspondence:

$$T^*Y_F \leftarrow Y_F \times_{X_F} T^*X_F \to T^*X_F$$

We shall denote these maps $d\varphi : Y_F \times_{X_F} T^*X_F \to T^*X_F$ and $\text{pr}_\varphi : Y_F \times_{X_F} T^*X_F \to T^*X_F$, respectively. We remark that if $\varphi$ is finite etale, the map $d\varphi$ is an isomorphism, and thus we can (and shall) speak of the induced map $T^*Y_F \to T^*X_F$. 


2. Basic Results and Background

In this section, we describe the basic results on reduction mod \( p \) and the \( p \)-support which we shall use throughout the paper. The basic references for the material in this section are [BKKö, Ko, Bi, Ts], and [VdB]; and also [HTT] and [Be] for basic facts about \( \mathcal{D} \)-modules (over \( \mathbb{C} \)).

2.1. Basic definitions and notions. In this section, we let \( X_\mathbb{C} \) be a smooth algebraic variety over \( \mathbb{C} \). We may choose a commutative ring \( R \), finitely generated over \( \mathbb{Z} \), so that, with \( S = \text{Spec}(R) \), there exists a scheme \( X \) with a smooth morphism \( X \to S \), satisfying
\[
X \times_S \text{Spec}(\mathbb{C}) = X_\mathbb{C}
\]
Similarly, for any morphism \( X_\mathbb{C} \to Y_\mathbb{C} \) of complex varieties, we may choose such a ring \( R \) and smooth \( S \)-schemes \( X \) and \( Y \) so that \( X_\mathbb{C} \to Y_\mathbb{C} \) is induced by base change to \( \mathbb{C} \) from a morphism of schemes over \( S = \text{Spec}(R) \), \( X \to Y \).

The sheaf of relative differential operators \( \mathcal{D}_{X/S} \) is a coherent sheaf of algebras on \( X \) satisfying \( \mathcal{D}_{X/S} \otimes_R \mathbb{C} = \mathcal{D}_{X_\mathbb{C}} \). We shall also make use of the algebra \( S(\mathcal{T}_{X/S}) \), the symmetric algebra of the sheaf of tangent vectors; this is isomorphic as an algebra to \( \pi_1((\mathcal{O}_{T^*\times(X/S)}) \) where \( \pi : T^*(X/S) \to X \) denotes the natural projection from the cotangent bundle.

A \( \mathcal{D}_{X/S} \) module will be a quasicoherent sheaf \( M \) which is a quasicoherent module over \( \mathcal{D}_{X/S} \). The data of an action of \( \mathcal{D}_{X/S} \) on \( M \) is equivalent to the data of a flat connection:
\[
\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_{X/S}
\]
where \( \nabla \) is an \( \mathcal{O}_X \)-linear map satisfying the Leibniz rule: \( \nabla(df \cdot m) = df \cdot m + f \cdot \nabla(m) \); and flatness: \( \nabla \circ \nabla = 0 \).

Similarly, a Higgs sheaf \( M \) is a quasicoherent sheaf \( M \) which is a quasicoherent module over \( S(\mathcal{T}_{X/S}) \). The data of an action of \( S(\mathcal{T}_{X/S}) \) on \( M \) is equivalent to the data of a Higgs field
\[
\Theta : M \to M \otimes_{\mathcal{O}_X} \Omega^1_{X/S}
\]
where \( \Theta \) is an \( \mathcal{O}_X \)-linear map satisfying flatness: \( \Theta \circ \Theta = 0 \).

Finally, we shall also use the intermediary notion of a \( \lambda \)-connection: namely, a quasicoherent sheaf \( M_\lambda \) on the variety \( X \times_S \mathbb{A}^1_S \), which is flat over \( O(\mathbb{A}^1_S) = O(S)|[\lambda] \), equipped with an \( O(S)|[\lambda]\)-linear operator
\[
\nabla : M_\lambda \to M_\lambda \otimes_{O_X \times_S \mathbb{A}^1_S} \Omega^1_{X \times_S \mathbb{A}^1_S/S}
\]
satisfying the rule \( \nabla(f \cdot m) = \lambda df \cdot m + f \cdot \nabla(m) \) and flatness: \( \nabla \circ \nabla = 0 \). As above, the category of these objects is equivalent to the category of quasi-coherent sheaves over an algebra on \( X/S \); namely, we define the sheaf of algebras \( \mathcal{D}_{\lambda,X/S} \) as follows: over an affine open subset \( U \), let \( \{\xi_i\} \) generate the module of derivations \( \text{Der}(U/S) \). We consider the algebra
\[
\mathcal{D}_{\lambda,U/S} := O(U)|[\lambda]<\xi_i>/<(\xi_i,f) - \lambda \xi_i(f),[\xi_i,\xi_j])
\]
where \( f \) is any element of \( O(U) \). Then one verifies directly that this construction sheafifies to yield a coherent sheaf of algebras \( \mathcal{D}_{\lambda,X/S} \); the category of quasi-coherent modules over this algebra is then equivalent to the category of quasi-coherent sheaves with \( \lambda \)-connection. Let us note that there is an isomorphism of sheaves of algebras
\[
\mathcal{D}_{X/S}[\lambda,\lambda^{-1}] \cong \mathcal{D}_{\lambda,X/S}[\lambda^{-1}]
\]
defined, in local coordinates, as the identity on \(O(U)[\lambda]\) and sending \(\xi_i \rightarrow \lambda^{-1}\xi_i\). This after inverting \(\lambda\), a \(\lambda\)-connection is simply a relative connection over \(S \times (\mathbb{A}^1_k \setminus \{0\})\).

Now, let \(M_C\) be a finitely generated \(D_{X_C}\)-module. As \(M_C\) is locally finitely generated on \(X_C\), we may, after possibly localizing the ring \(R\), assume that there exists an \(S\)-flat \(D_{X/S}\)-module \(M_S\) so that \(M_S \otimes_R C = M_C\). Therefore, if \(k\) is a perfect field of positive characteristic, to any \(k\)-point of \(S\) we may associate the scheme \(X_k = X_S \times_S \text{Spec}(k)\), and the \(D_{X_k}\)-module \(M_k := M_S \otimes_R k\). Since \(X_S \rightarrow S\) is smooth, \(X_k\) is a smooth \(k\)-variety for all such \(k\). We remark that, despite the notation, the variety \(X_k\) and the module \(M_k\) will, in general, depend on all of the choices made in the construction (and not just on the field \(k\)).

2.2. The \(p\)-support. In this section, we give a brief reminder on \(D\)-modules in positive characteristic, following [BMR] and [OV]. We shall suppose until further notice that \(X_k \rightarrow \text{Spec}(k)\) is a smooth variety over a perfect field \(k\), of dimension \(n\). We write \(\text{Fr}_X\) for the absolute Frobenius morphism of \(X_k\). Then we have the diagram

\[
X_k \xrightarrow{F} X_k^{(1)} \xrightarrow{\sigma} X_k \xrightarrow{\text{Spec}(k) \rightarrow \text{Spec}(k)}
\]

in which \(X_k^{(1)} := X_k \times_{\text{Spec}(k)} \text{Spec}(k)\) (where \(\text{Spec}(k) \rightarrow \text{Spec}(k)\) is the Frobenius) and, since \(k\) is perfect, \(\sigma : X_k^{(1)} \rightarrow X_k\) is an isomorphism. Since the variety \(X_k\) is smooth, we have \(F^{-1}(O(X_k^{(1)})) \cong O(X_k)^p\) as sheaves on \(X_k\); and we will typically simply regard this as being an inclusion \(O(X_k^{(1)}) = O(X_k)^p \subset O(X_k)\), since the underlying topological spaces are identified under \(F\).

In this situation, the sheaf of differential operators \(D_{X_k}\) has a large center: as sheaves of algebras over \(X_k\) we have \(Z(D_{X_k}) = \pi^{(1)}_* (O(T^* X_k^{(1)}))\), where \(\pi^{(1)}\) is the natural map \(T^* X_k^{(1)} \rightarrow X_k^{(1)}\). Let us explain how this isomorphism is constructed: for any open \(U_k \subset X_k\) and any derivation \(\partial \in T_{U_k}\), there is another derivation \(\partial^{[p]}\) called the \(p\)th iterate - obtained by applying \(\partial\) to \(O(U_k)\) \(p\) times. We also have the \(p\)th power element \(\partial^p \in D_{U_k}\). One verifies that the map

\[
\partial \rightarrow \partial^p - \partial^{[p]}
\]

is a morphism of sheaves of \(O_{X_k^{(1)}}\)-modules \(T_{U_k^{(1)}} \rightarrow D_{U_k}\) which extends to a morphism of sheaves of algebras

\[
O_{T^* X_k^{(1)}} = \text{Sym}_{O_{X_k^{(1)}}}(T_{X_k^{(1)}}) \rightarrow D_{X_k^{(1)}}
\]

which realizes \(O_{T^* X_k^{(1)}}\) as \(Z(D_{X_k})\).

It follows from this description that \(D_{X_k}\) is finite and flat over \(Z(D_{X_k})\) ([BMR], chapter 2). Since \(\pi^{(1)}\) is an affine map, the category of sheaves of coherent modules over \(\pi^{(1)}_* (O(T^* X_k^{(1)}))\) is equivalent to the category of coherent sheaves on \(T^* X_k^{(1)}\), and so we may regard \(F_* (D_{X_k})\) as a coherent sheaf of algebras on \(T^* X_k^{(1)}\). Then the principle theorem about this algebra is
**Theorem 4.** ([BMR], Theorem 2.2.3) The algebra $F_*(\mathcal{D}_{X_k})$ is an Azumaya algebra over $T^*X_k^{(1)}$. If $m > 0$, this algebra is not split.

This theorem is the key to a large body of work on $\mathcal{D}$ modules in positive characteristic, including (but not limited to) [BMR], [OV], [BB], [N], [Bi], [VdB]. A much-exploited feature of this algebra is that, although $\mathcal{D}_{X_k}$ does not split on $T^*X_k^{(1)}$, it does split on many subvarieties of $T^*X_k^{(1)}$. For example, the module $O_{X_k}$, with the standard flat connection, is a splitting module for $(\mathcal{D}_{X_k})|_{X_k^{(1)}}$ - indeed, it is not difficult to compute in local coordinates the isomorphism $(\mathcal{D}_{X_k})|_{X_k^{(1)}} \rightarrow \mathcal{E}nd_{O_{X_k}}(O_{X_k})$

(Here $\mathcal{E}nd$ denote the sheaf endomorphism ring). This result yields the fact that the functor

$$M \rightarrow O_{X_k} \otimes_{O_{X_k^{(1)}}} M = F^*M$$

is an equivalence of categories between $\text{Coh}(X_k^{(1)})$ and $\text{Mod}((\mathcal{D}_{X_k})|_{X_k^{(1)}}).$ This, in turn, is a restatement of the classical Cartier descent (c.f. [Ka], theorem 5.1).

With this result in mind, we make the

**Definition 5.** Let $M_k$ be a coherent $\mathcal{D}_{X_k}$-module. As $M$ is a coherent module over $Z(\mathcal{D}_{X_k})$, we may consider the support $\text{supp}(Z(\mathcal{D}_{X_k}))(M_k)$ as a subvariety of $T^*X_k^{(1)}$; this variety is called the $p$-support of $M$. For each component $C$ of this variety we can consider the multiplicity $\text{mult}(M|_C).$ We define $m_C(M) := \frac{1}{p^m} \text{mult}(M|_C)$.

The algebraic cycle $\sum m_C(M)[C]$ is called the $p$-cycle of $M$.

The invariant $m_C$ is a renormalized multiplicity; in particular, things are set up so that we have $m_{X_k^{(1)}}(O_{X_k}) = 1$.

Let us also remark at this point that one can make a similar construction for $\mathcal{D}_{\lambda,X_k}.$. We have an isomorphism $Z(\mathcal{D}_{\lambda,X_k}) \cong \pi^{(1)}_*(\mathcal{O}(T^*X_k^{(1)}) \otimes \mathcal{O}(\mathbb{A}_k^1))$, where the $\mathbb{A}_k^1$-component corresponds to the central variable $\lambda$. The algebra $\mathcal{D}_{\lambda,X_k}$ is not Azumaya, however, we note the

**Corollary 6.** The algebra $\mathcal{D}_{\lambda,X_k}[\lambda^{-1}] \cong \mathcal{D}_{X_k}[\lambda,\lambda^{-1}]$ is Azumaya over $T^*X_k^{(1)} \times (\mathbb{A}_k^1 \setminus \{0\})$.

This follows from the proof of the Azumaya property in [OV], section 2.1.

In addition to the notion of $p$-support, we shall also have occasion to make use of the, essentially equivalent, notion of the $p$-curvature of a connection (or $\lambda$-connection) in positive characteristic. In the case of a connection $M_k$, we shall write

$$\Psi : M_k \rightarrow M_k \otimes F^* (\Omega^1_{X_k^{(1)}})$$

for the operator obtained by dualizing from the morphism

$$\mathcal{T}_{X_k^{(1)}} \rightarrow F_* \mathcal{E}nd_{O_X}(M_k)$$

given by the action of the centre.

Now suppose $M_{\lambda,k}$ is a $\lambda$-connection. Then the isomorphism $Z(\mathcal{D}_{\lambda,X_k}) \cong O(T^*X_k^{(1)}) \times \mathbb{A}_k^1$ is realized via the morphism

$$\partial \rightarrow \partial^p - \lambda^{p-1} \partial^p$$
(c.f., [LP] section 3.5). Dualizing yields a morphism
\[ \Psi : M_{\lambda,k} \to M_{\lambda,k} \otimes_{\mathcal{O}_{X_1}} [\lambda] F^*(\Omega^1_{(X^{(1)}_1 \times X^{(1)})/k^{(1)}}) \]
which is again called the $p$-curvature.

Now, we return to the situation in which $X \to S$ is a smooth morphism over
the spectrum of a finite-type $\mathbb{Z}$-algebra. For a coherent $\mathcal{D}_S$ module $M_S$, the base
change $M_C := M_S \otimes_R \mathbb{C}$ is a coherent $\mathcal{D}_C$-module. The main theorem of [Bi] and
[VdB] compares the behavior of the singular support of $M_C$ with the behavior of
the $p$-support of $M_k$, for a general $k$-point of $S$. More precisely, we have

**Theorem 7.** ([Bi], [VdB]) Suppose that $M_C$ is a holonomic $\mathcal{D}_C$ module. Then, af-
ter possibly replacing $S$ by an open subset, we have that for any perfect field $k$ of
positive characteristic, and any $k$ point $\text{Spec}(k) \to S$, the $p$-support $\text{supp}_{Z(\mathcal{D}_X)}(M_k)$ is a
Lagrangian subvariety of $T^* X^{(1)}_k$. In particular, each component of $\text{supp}_{Z(\mathcal{D}_X)}(M_k)$
has dimension $m$.

2.3. The $p$-support and the standard functors. In this subsection, we shall
collect some results about push-forward and pull-back of $\mathcal{D}$-modules in various char-
acteristics, and, in particular, we shall recall an important result of Bezrukavnikov
and Braverman concerning the behavior of the $p$-support under these functors.

Let us begin by recalling the

**Definition 8.** Let $\varphi : X \to Y$ be a morphism of smooth schemes over $S$. If $M$
is any coherent $\mathcal{D}_{Y/S}$-module, then we define a natural $\mathcal{D}_{X/S}$-module structure on
$\varphi^* M$ as follows: since the map $\varphi$ gives rise to an exact sequence
\[ \varphi^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \]
the connection $M \to M \otimes \Omega^1_{Y/S}$ pulls back to
\[ \varphi^* M \to \varphi^* M \otimes \varphi^* \Omega^1_{Y/S} \to \varphi^* M \otimes \Omega^1_{X/S} \]
yielding a natural flat connection (over $S$) on $\varphi^* M$; this is the pull-back for $\mathcal{D}$-
modules. Taking the left derived functor of this functor yields the derived pullback
$L\varphi^*$. Replacing the base scheme $S$ by $\text{Spec}(\mathbb{C})$ or $\text{Spec}(k)$ gives the definition of the
pullback of $\mathcal{D}_C$ and $\mathcal{D}_{X_1}$-modules, respectively.

By the fact that tensor products of sheaves commute with base change, one
deduces directly from the definitions the

**Lemma 9.** With $\varphi : X \to Y$ as above, we have $\varphi^*(M) \otimes_R \mathbb{C} \to \varphi^*(M \otimes_R \mathbb{C})$ and
$\varphi^*(M) \otimes_R k \to \varphi^*(M \otimes_R k)$.

Now we can recall the definition of push-forward: for any $\varphi : X \to Y$ as above,
8 implies that $\varphi^*(\mathcal{D}_{Y/S})$ is naturally a $\mathcal{D}_{X/S}$-module. By the formalism of side-
changing (c.f., e.g. [HTT] chapter 1) we can form the right $\mathcal{D}_{X/S}$-module $\mathcal{D}_{Y,X} :=
\varphi^*(\mathcal{D}_{Y/S}) \otimes_{\mathcal{O}_X} \omega_{X/S}$. This module also carries the structure of a left $\varphi^*(\mathcal{D}_{Y/S})$
-module induced from the natural action of $\mathcal{D}_{Y/S}$ on itself. With this in hand, we make the

**Definition 10.** Let $\varphi : X \to Y$ be a proper morphism of smooth schemes over $S$.
Then for any coherent $\mathcal{D}_{X/S}$ module $M$, we define
\[ \int \varphi M := R\varphi_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_{X/S}} M) \]
as an object of $D^b_{coh}(\mathcal{D}_{Y/S})$. The $i$-th cohomology of this object will be denoted by $\int^i \varphi M$. Replacing the base scheme $S$ by $\text{Spec}(\mathbb{C})$ or $\text{Spec}(k)$ gives the definition of the push-forward of $\mathcal{D}_{X_c}$ and $\mathcal{D}_{X_e}$-modules, respectively.

**Remark 11.** In the case that the map $\varphi$ is a finite map, then over the base field $\mathbb{C}$ the functor $\int^i \varphi$ is an exact functor; i.e., it sends objects in homological degree zero to objects in homological degree zero. In this case we will denote $\int^0 \varphi$ simply by $\int$.

Let us note that this definition certainly makes sense for an arbitrary (not necessarily proper) morphism of schemes. However, in the case where the morphism is not proper, this functor will not preserve coherence; and therefore will not necessarily behave well with respect to base change. Thus we will reserve the notation for the proper case.

In the proper case, let us note the:

**Lemma 12.** With $\varphi : X \to Y$ a proper morphism as above, we have that for any $M \in \text{Mod}(\mathcal{D}_{X/S})$, $\int^i \varphi M \in \text{Coh}(\mathcal{D}_{Y/S})$.

With this in hand, let us recall the base-change for proper direct images:

**Proposition 13.** With $\varphi : X \to Y$ a proper morphism as above, we have, after possibly restricting to a dense open subset of the base scheme $S$, that the natural maps

$$\int^i M \otimes_R \mathbb{C} \to \int^i (M \otimes_R \mathbb{C})$$

and

$$\int^i M \otimes_R k \to \int^i (M \otimes_R k)$$

are isomorphisms.

Now we specialize to the case where the base is $\text{Spec}(k)$. Then, there is a result of Bezrukavnikov and Braverman which explains how the $p$-support behaves under the functors $\varphi^*$ and $\int^i \varphi$. Namely, we have the following:

**Theorem 14.** (Bezrukavnikov-Braverman) Let $\varphi : X_k \to Y_k$ be a morphism of smooth varieties over $k$. Then the variety $X_k^{(1)} \times_{Y_k^{(1)}} T^*Y_k^{(1)}$ carries the Azumaya algebras $(d\varphi^{(1)})^*(\mathcal{D}_{X_k})$ and $pr_{Y_k}^*(\mathcal{D}_{Y_k})$.

These Azumaya algebras are equivalent; i.e., there is a $(pr_{Y_k}^*(\mathcal{D}_{Y_k}), (d\varphi^{(1)})^*(\mathcal{D}_{X_k}))$-bimodule (which is a vector bundle on $X_k^{(1)} \times_{Y_k^{(1)}} T^*Y_k^{(1)}$) yielding an equivalence of categories $F : \text{Mod}^{f,g}((d\varphi^{(1)})^*(\mathcal{D}_{X_k})) \to \text{Mod}^{f,g}(pr_{Y_k}^*(\mathcal{D}_{Y_k}))$. The pushforward functor is then expressed as

$$M \to (pr_{Y_k})_* \circ F \circ (d\varphi^{(1)})^* M$$

For later use, we note that this theorem enables us to explain the behavior of the $p$-support, as a cycle, under a finite push-forward. Since the functor $F$ is, in this case, given by tensoring by a line bundle on $X_k^{(1)} \times_{Y_k^{(1)}} T^*Y_k^{(1)}$, we see that, for $M$ whose $p$-cycle is $\sum m_C(M)[C]$, the $p$-cycle of $\int^i \varphi M$ is given by

$$(pr_{Y_k})_* \left( \sum m_C(M)[(d\varphi^{(1)})^{-1} C] \right),$$

where $(pr_{Y_k})_*$ is the finite pushforward of cycles, and the class of $[(d\varphi^{(1)})^{-1} C]$ is given by the inverse-image subscheme of $O_{C^{(1)}}$ in
$X^{(1)}_k \times_{Y^{(1)}_k} T^{*} Y^{(1)}_k$, counted with multiplicity. This is the case of the theorem we shall need in this version of the paper.

2.4. $\mathcal{D}$-modules of constant arithmetic support. In this section, we recall some key observations of M. Kontsevich. In his paper [Ko], Kontsevich has defined a notion of arithmetic support of a holonomic $\mathcal{D}$-module in characteristic zero. The main theorems of this paper concern a particular case of this notion, which we now define. To set things up, let $\mathcal{L}_C \subset T^{*}(X_C)$ be a Lagrangian subvariety of the cotangent bundle of $X_C$, equipped with the usual symplectic structure. Then, we form an appropriate choice of the ring $R$, we may suppose that there exists $\mathcal{L} \subset T^{*}(X/S)$, so that $\mathcal{L} \to S$ is smooth, and $\mathcal{L} \times_S \text{Spec}(\mathbb{C}) = \mathcal{L}_C$. Then, for each perfect field $k$ of positive characteristic, and each $k$-point $\text{Spec}(k) \to S$ we obtain $\mathcal{L}_k \subset T^{*}(X_k)$, and, under the isomorphism $T^{*}(X_k) \to T^{*}(X^{(1)}_k)$, a smooth variety $\mathcal{L}^{(1)}_k \subset T^{*}(X^{(1)}_k)$. We have $m_C(M) = 1$ for each component $C$ of $\mathcal{L}^{(1)}_k$, we say that $M_C$ has arithmetic support equal to $\mathcal{L}_C$ with multiplicity 1.

A priori, this definition seems quite unwieldy, since it requires something to be checked for all rings $R$ and models $M$. But in fact a standard argument shows the

**Definition 15.** Let $M_C$ be a holonomic $\mathcal{D}$-module on $X_C$. Given $\mathcal{L}_C$ and $S$ as above, after a finite extension and localization, we may suppose there exists $M$ flat over $S$ with $M \otimes_R \mathbb{C} = M_C$. Then we say that $M_C$ has arithmetic support equal to $\mathcal{L}_C$, if, for any such $S$, $M$, and $\mathcal{L}$, there is a generic open subset $U \subset S$ such that for all $k$-points $\text{Spec}(k) \to U$, we have $\text{supp}_{\mathcal{D}(X_k)}(M_k) = \mathcal{L}^{(1)}_k$. If, furthermore, we have $m_C(M) = 1$ for each component $C$ of $\mathcal{L}^{(1)}_k$, we say that $M_C$ has arithmetic support equal to $\mathcal{L}_C$ with multiplicity 1.

$\mathcal{D}$-modules of constant arithmetic support are quite unusual; in fact, the only known constructions of them up to now are elaborations of the following:

**Example 17.** (Exponential $\mathcal{D}$-modules) Let $f : X_C \to \mathbb{A}^1$ be a regular function. Then we define the $\mathcal{D}_{X_C}$-module $e_f$ as follows: as a coherent sheaf this is simply $O(X)$. For a local function $g$ on $X$, and a local vector field $\xi$, we set $\xi \cdot g = \xi(f)g + \xi(g)$. This gives $e_f$ the structure of a holonomic $\mathcal{D}_{X_C}$-module (indeed, it is coherent as an $O(X)$-module). Set $\mathcal{L}(f) = \Gamma(df)$, the graph of $df$ in the cotangent bundle $T^{*}(X_C)$. This is a smooth Lagrangian subvariety. Then we have the

**Claim 18.** For a prime $p >> 0$ such that $X_k$ is smooth and $f : X_k \to \mathbb{A}^1_k$ is well-defined, the $\mathcal{D}_{X_k}$-module $e_f$ has $p$-support equal to $\Gamma(df)^{(1)}$, and $e_f$ is a splitting bundle for the Azumaya algebra $\mathcal{D}_{X_k}$ along this subvariety. Thus the $\mathcal{D}_{X_C}$-module $e_f$ has arithmetic support equal to $\mathcal{L}(f)_C$, with multiplicity 1.

**Proof.** We shall proceed by writing $e_f$ in terms of generators and relations. Since $X_k$ is smooth, we may cover it with open affine subvarieties $\{U_i\}$ on which the sheaf of vector fields is trivialized, i.e., such that $\mathcal{D}(U_i) = O(U_i)[\partial_1, \ldots, \partial_n]$. On such an open subset, the module $e_f(U_i)$ is given as

$$e_f(U_i) = \mathcal{D}(U_i)(\partial_i - \partial_i(f))$$

---

2In fact, we give details in the case of interest to us below in lemma 60.
For any ring \( R \) as above so that \( f \) and the \( U_i \) are defined over \( R \), we see that for any perfect field of positive characteristic \( k \), and any \( \text{Spec}(k) \rightarrow S = \text{Spec}(R) \), the center \( Z(D(U_i)_k) \) is generated over \( k \) by \( O(U_i)_k \) and \( \{ \partial_i \} \). It follows from this that the annihilator of \( e^f(U_i)_k \) in \( Z(D(U_i)_k) \) is the ideal \( I \) generated by \( \{ \partial_i - (\partial_i(f))^p \} \); which implies the claim about the support. In addition, since \( e^f \) is a vector bundle \( X_k \), \( F_\star(e^f) \) is a vector bundle on \( X_k^{(1)} \) and the description of the annihilator implies that this bundle has rank \( p^n \), which implies that it is a splitting bundle for the Azumaya algebra \( D_X \). The result follows.

With this example in hand, one can construct more examples of \( D \)-modules of constant arithmetic support: namely, using the theorem of theorem 14, one shows that applying the push-forward or pull-back of a morphism to a \( D \)-module of constant arithmetic support yields another \( D \)-module of constant arithmetic support. Therefore, the push-forward or pull-back of any exponential \( D \)-module by a morphism is a \( D \)-module of constant arithmetic support. In fact, Kontsevich has conjectured in [Ko] that, up to taking summands and extensions of \( D \)-modules, all \( D \)-modules of constant arithmetic support arise this way, a claim which, in general, seems far out of reach at this time.

On the other hand, we will make some use of exponential \( D \)-modules here. In order to make use of the above observations, let us consider a Lagrangian \( L_C \subset T^* X_C \) as in theorem 1- recall that in particular this means that the natural projection \( L_C \rightarrow X_C \) is dominant. Then there is an open subset \( U_C \subset X_C \), whose compliment has codimension \( \geq 1 \), so that the restriction \( L_{U_C} \rightarrow U_C \) is finite etale; here we have denoted the inverse image of \( U_C \) in \( L_C \) by \( L_{U_C} \). Suppose this finite etale cover has degree \( r \). Then we have:

**Lemma 19.** Suppose that \( X_C \) and \( L_C \) are as above. Let \( \alpha \) be the 1-form on \( L_C \) given by pulling back the canonical one form on \( T^*(X_C) \). Then there exists \( f \in \Gamma(O(L_C)) \) such that \( df = \alpha \). The \( D_{L_C} \)-module \( e^f \) has constant arithmetic support equal to the graph of \( \alpha \) in \( T^*(L_C) \).

**Proof.** Since \( L_C \) is smooth Lagrangian, \( d\alpha = \omega \mid_{L_C} = 0 \). By Grothendieck’s algebraic DeRham theorem, we have that \( H^1_{\text{dR}}(L_C) = H^1(L_C, \mathbb{Q}) \otimes \mathbb{Q} = 0 \); therefore there exists \( f \in \Gamma(O(L_C)) \) so that \( df = \alpha \). Now the last sentence follows from example 17. \( \square \)

Now, because \( L_{U_C} \rightarrow U_C \) is finite etale, we obtain a flat connection on \( U_C \) of rank \( r \) by taking the pushforward \( \int_\pi(e^f) \). We then have the

**Lemma 20.** The connection \( \int_\pi(e^f) \) is an irreducible flat connection with constant arithmetic support equal to \( L_{U_C} \). Over a field \( k \) of positive characteristic, \( \int_\pi(e^f)_k \) is a splitting bundle for \( L_C^{(1)} \subset T^* U_C^{(1)} \).

**Proof.** For a given point \( l \in L_{U_C} \) we may write \( i(l) = (x_0, y_0) \in T^* U_C \), so that \( \pi_C(l) = x_0 \in U_C \). We have an identification \( T_{x_0} U_C \cong \text{Span}_C \{ dx_1, \ldots, dx_r \} \). Thus the point \( i(l) \in T_{x_0} U_C \) has the form \( \sum_{i=1}^n y_i^0 dx_i \), where the tuple \( (y_1^0, \ldots, y_n^0) \) are the
coordinates of the point \( y_0 \in \mathbb{A}^n \). Now, we have the projection map
\[ d\pi : T^*_p \mathcal{L} \to T^*_p \mathcal{U} \] (which is an isomorphism) satisfying
\[ d\pi(i^* (\sum y_i dx_i))(l) = d\pi((\sum y_i^0 dx_i)|_{T^*_p \mathcal{C}}) = \sum y_i^0 dx_i \]
But this says precisely that \((d\pi)(\Gamma(df)) = \mathcal{L}_U \). Now, applying the remark following theorem 14 to the map \( \pi \) yields that \( \pi_*(e^f) \) has constant arithmetic support equal to \( \mathcal{L}_U \), with multiplicity 1. The fact that \( \pi_*(e^f) \) is a splitting bundle follows from a rank computation: the rank of \( F_k(\pi_*(e^f)) \) as a bundle on \( U^{(1)}_k \) is \( p^m \); and since \( \mathcal{L}_U^{(1)} \to U^{(1)}_k \) is finite etale of rank, we see that \( F_k(\pi_*(e^f)) \) is a bundle of rank \( p^m \) on \( U^{(1)}_k \).

Finally, the irreducibility of \( \pi_*(e^f) \) follows from the lemma below. \( \square \)

To complete the proof, and for later use as well, we record the

**Lemma 21.** Let \( M_C \) be a holonomic \( \mathcal{D} \)-module on a variety \( X_C \), which has constant arithmetic support, with multiplicity 1, equal to \( \mathcal{L}_C \) - an irreducible Lagrangian subvariety of \( T^*X_C \). Then \( M_C \) is an irreducible \( \mathcal{D} \)-module.

**Proof.** To deduce the irreducibility of \( M_C \), we apply the theorem of Bitoun and Van den Bergh, theorem 7, as follows: if \( M_C \) had a strict submodule \( N_C \), we could choose a ring \( R \), finitely generated over \( \mathbb{Z} \), so that there exists a smooth map \( U \to S = \text{Spec}(R) \), and such that \( N_C \) was induced by base change from \( N \), defined over \( R \), and also so that \( M_C \) was induced from \( M_U \), defined over \( R \). Then, for any perfect field of positive characteristic \( k \) and any \( \text{Spec}(k) \to S \), we would have that \( N_k \subset M_k \), and so theorem 7 would imply that the \( p \)-support of \( N_k \) would be a Lagrangian subvariety of \( \mathcal{L}_k^{(1)} \); therefore, the \( p \)-support of \( N_k \) would be equal to \( \mathcal{L}_k^{(1)} \). This would then imply that the \( p \)-support of \( M_k \) would have multiplicity at least 2 - contradiction. \( \square \)

2.5. The \( \mathcal{D} \)-module \( M_k \). In this subsection, we shall give the construction of the holonomic \( \mathcal{D}_{X_k} \)-module \( M_k \), which we shall eventually show is the reduction mod \( p \) of a \( \mathcal{D} \)-module \( M_C \) for \( p >> 0 \).

In the previous section, we have constructed such a module over an open subset of the variety \( X_C \). Unfortunately, the natural extension of this connection to a holonomic \( \mathcal{D} \)-module on \( X_C \) is not the \( \mathcal{D} \)-module of the theorem; indeed, consider the following:

**Example 22.** \(^3\)(Airy Equation) We consider \( X_C = \mathbb{A}^3_3 \) and \( \mathcal{L}_C = \{ y^2 - x = 0 \} \). Then the module \( M_C \) is the \( \mathcal{D} \)-module corresponding to the Airy equation:
\[ M_C = \mathcal{D}/(t \frac{d}{dx})^2 - x \]
is follows either from a direct computation, or, more efficiently, from noting that \( M_C \) is the Fourier transform of \( N_C := e^{x^3/3} \). Since, by the previous section, \( N_C \) has constant arithmetic support (equal to \( \{ y - x^2 \} \)), one can see that the same is true of \( M_C \), whose arithmetic support is simply the “90 degree rotation” of the arithmetic support of \( N_C \).

\(^3\)Thanks to MK
This example highlights several key features of the problem: since, in this example $L \to X_C$ is surjective, the $D$-module we are looking for this case is actually a connection on $X_C$. Indeed, we have the

**Proposition 23.** Let $\text{char}(k) > 0$, such that the reduction of $\int_{\pi} (e^f)$ (from lemma 20) exists over $k$. Then $\int_{\pi} (e^f)$ admits an extension to a coherent $D_{X_k}$-module $M_k$ such that $F_*(M_k)$, as a coherent sheaf on $T^*X^{(1)}$, is a vector bundle on $L^{(1)}_k$. The set of such extensions is indexed by $\text{Pic}(L^{(1)}_k)$.

Therefore, if $L^{(1)}_k \to X^{(1)}_k$ is finite flat, this implies that $F_*(M_k)$ is a vector bundle on $X^{(1)}_k$, and therefore that $M_k$ is a vector bundle on $X_k$.

**Proof.** It follows from 17 and the fact that the map $\pi$ is etale over $U_k$ that the module $\int_{\pi} (e^f)$ is a splitting bundle for the Azumaya algebra $D_{U^{(1)}_k}$ over $L^{(1)}_k$. Since $L^{(1)}_k$ is an open subset of the smooth variety $L^{(1)}_k$, it follows from the theory of the Brauer group (in particular [Mi], corollary 2.6) that $D_{X_k}$ is split on $L^{(1)}_k$. This shows the existence of the claimed extension; since splittings for any Azumaya algebra on a smooth variety are indexed by Pic, the next sentence follows. Further, any extension of $\int_{\pi} (e^f)$ which, as a sheaf on $T^*X^{(1)}_k$ is a bundle on $L^{(1)}_k$ is a bundle of rank $p^m$ (because this is the rank of $\int_{\pi} (e^f)$ on $L^{(1)}_{U^{(1)}_k}$) and is therefore a splitting bundle, hence the uniqueness. The last sentence follows directly. $\square$

Let us apply this proposition in the case of a curve: we observe that the cohomology vanishing condition $H^1_{\text{sing}}(L_C) = 0$ implies directly that $L_C \to \mathbb{A}^1_C$. Since $\pi : L_C \to X_C$ is dominant, this implies $X_C$ is an affine open subset of $\mathbb{A}^1_C$, since any non constant morphism from $\mathbb{A}^1_C$ to $\mathbb{A}^1_C$ is surjective, we deduce that in fact $X_C = \mathbb{A}^1_C$. Therefore

**Lemma 24.** Let $\text{char}(k) > 0$, and suppose $X_k \to \mathbb{A}^1_k$; let $M_k$ be as in 23. Then $M_k$ is a vector bundle with connection on $X_k$.

This is an immediate consequence of 23. It implies that we cannot expect $M_C$ of the theorem to be singular along any divisor in $\mathbb{A}^1_C$; if it were, the reduction mod $p$ for $p >> 0$ would also have singularities.

Now the problem for us is that it is far from clear that the $D_{X_k}$-module $M_k$ lifts to characteristic zero in a fashion that is independent of the prime $p$. Showing that this is indeed the case will, essentially, occupy the rest of the paper.

Before proceeding, we record some useful information about the $D_{X_k}$ module $M_k$. Let $E_k$ denote the coherent sheaf $\text{End}(M_k)$ with its induced $D_{X_k}$-module structure; this connection will play a major role in the next sections. Then, because the differential of the deRham complex for $E_k$ is linear over $O_{X^{(1)}_k}$, the deRham cohomology groups $\mathcal{H}^i_{dR}(E_k)$ are naturally coherent sheaves on $X^{(1)}_k$. For later use we record some information about these sheaves.

We start with the rather general:

**Lemma 25.** Let $\phi : U'_k \to U_k$ be a finite etale cover. Let $E_k$ be any vector bundle with connection on $U_k$. Then we have an isomorphism $(\phi^{(1)})^* \mathcal{H}^i_{dR}(E_k) \to \mathcal{H}^i_{dR}(\phi^*(E_k))$
Proof. We have a natural map of connections $\mathcal{E}_k \to \int_\phi \phi^* \mathcal{E}_k$, which thus yields, over each affine open subset of $U_k$, a natural map $H^0(\mathcal{H}^i_{dR}(\mathcal{E}_k)) \to H^0(\mathcal{H}^i_{dR}(\phi^*(\mathcal{E}_k)))$. As this is easily seen to be a morphism of sheaves over $O(U_k^{(1)})$, we get a natural map $(\phi^*_1)^* \mathcal{H}^i_{dR}(\mathcal{E}_k) \to \mathcal{H}^i_{dR}(\phi^*(\mathcal{E}_k))$.

It suffices to show that this map is an isomorphism after taking the formal completion of these sheaves at any point. However, since $\phi$ is etale, it becomes an isomorphism between the formal neighborhood of any point $x$ in $U_k$ and the formal neighborhood of $\phi(x)$; this immediately implies the result. □

Now, returning to $M_k$ and $\mathcal{E}_k$ as above, we show the:

**Corollary 26.** The sheaves $\mathcal{H}^i_{dR}(\mathcal{E}_k)$ ($i \geq 0$) are vector bundles on $U_k^{(1)}$.

**Proof.** We may choose a finite etale cover $\phi : W_k \to U_k$ such that the variety $L_{U_k} \times_{U_k} W_k$ is isomorphic to a disjoint union of copies of $W_k$ and the projection $pr : L_{U_k} \times_{U_k} W_k \to W_k$ is a Zariski local isomorphism. Indeed, taking $W_k = L_{U_k}$ and $\phi = \pi$ suffices for this.\footnote{For a proof see lemma 36 below}

By the previous lemma it suffices to show that $\mathcal{H}^i_{dR}(pr^* \mathcal{E}_k)$ are vector bundles on $W_k$. However, by theorem 14 the $p$-support of the sheaf $pr^* M_k$ is isomorphic to $L_{U_k}^{(1)} \times_{U_k^{(1)}} W_k^{(1)}$; indeed one sees that as a subvariety of $T^* W_k^{(1)}$, this $p$-support is equal to the disjoint union of the graphs $\{\Gamma(df_{i})\}_{i=1}^r$, such that we have $f_i = f \circ p \circ s_i$, where $s_i : W_k \to L_{U_k} \times_{U_k} W_k$ are the $r$ distinct sections of the projection, and $p : L_{U_k} \times_{U_k} W_k \to L_{U_k}$ is the projection.

By decomposing $pr^* M_k$ via the action of the center $Z(D_{L_{U_k}^{(1)} \times_{U_k^{(1)}} W_k^{(1)}})$, we see that $pr^* M_k$ is the direct sum of $r$ line bundles with connection on $L_{U_k} \times_{U_k} W_k$, with mutually disjoint $p$-support.

So, we may write

$$pr^* \mathcal{E}_k = \bigoplus_{j,l=1}^r L_j \otimes L_l^*$$

where, for $j \neq l$, the $p$-support of $L_j \otimes L_l^*$ is a smooth variety inside $T^*(L_{U_k}^{(1)} \times_{U_k^{(1)}} W_k^{(1)})$ which does not intersect the zero section and whose projection to $L_{U_k}^{(1)} \times_{U_k^{(1)}} W_k^{(1)}$ is an isomorphism; and for $j = l$ we have $L_j \otimes L_l = O_{L_{U_k} \times_{U_k} W_k}$. Cartier’s theorem (c.f. [Ka], chapter 7) assures us that $\mathcal{H}^i_{dR}(O_{L_{U_k} \times_{U_k} W_k})$ is a vector bundle over $L_{U_k}^{(1)} \times_{U_k^{(1)}} W_k^{(1)}$. To finish the proof we shall show that

$$\mathcal{H}^i_{dR}(L_j \otimes L_l^*) = 0$$

for all $i$ whenever $j \neq l$. The $p$-support of $L_j \otimes L_l^*$ is by construction the variety $\Gamma(d(f_j - f_l))^{(1)}$, which never intersects the zero-section of $W_k^{(1)}$. By definition the $i$th deRham cohomology sheaf is equal to

$$\mathcal{R}Hom^i_{D_{W_k}}(O_{W_k}, L_j \otimes L_l^*)$$

Since the module $O_{W_k}$ has $p$-support $W_k^{(1)} \subset T^* W_k^{(1)}$ by considering the action of the center on $\mathcal{R}Hom^i_{D_{W_k}}(O_{W_k}, L_j \otimes L_l^*)$ we see that this group must vanish. □
3. Lifting of Connections

In this section, we develop the general formalism for lifting coherent $\mathcal{D}$-modules from positive characteristic to characteristic zero; this has some overlap with the standard results of deformation theory of vector bundles as in [Hart], chapter 7; and in the hopes of keeping the paper accessible we shall give down to earth proofs in the style of that text. At the end of the section we shall give some variants on the same concept; one flavor being lifting Higgs bundles and $\lambda$-connections, and another being the deformation of Higgs bundles to infinitesimal $\lambda$-connections.

Now, let $X_k$ be a smooth scheme over the perfect field $k$ of positive characteristic, and $M_k$ a vector bundle with flat connection. Let $E_k := \mathcal{E}nd(M_k)$ with its induced connection. We may form the sheaves $H^i_{dR}(E_k)$, which we regard as being coherent sheaves on $X_k(1)$. In particular, the cohomology of these sheaves may be computed as Cech cohomology.

Now suppose given an Artinian local ring $A$ with residue field $k$, a scheme $X_A$ which lifts $X_k$, and a coherent $\mathcal{D}_{X_A}$-module lifting $M_k$.

Now consider an Artinian local ring $A'$ such that we have $0 \to J \to A' \to A \to 0$ where the ideal $J$ satisfies $J^2 = 0$. Suppose that we are given a flat lift $X_{A'}$ of $X_A$ to $A'$. We consider the question of lifting $M_A$ to $X_{A'}$. Let $U_A$ be an open subset of $X_A$.

By a lift of $M_A(U_A)$ we shall mean a coherent $\mathcal{D}_{X_A'}$-module $M_{A'}$ on $U_{A'}$, which is flat over $A'$, together with an isomorphism of connections $M_{A'}/J(U_{A'}) \cong M_A(U_A)$.

An isomorphism of lifts is an isomorphism of connections which preserves this extra morphism. In this section we work out the structure of the set of (equivalence classes of) such lifts.

**Theorem 27.** With notations as above

a) The set of lifts of $M_A$, if nonempty, is in bijection with

$$\text{Ext}^1_{\mathcal{D}_{X_k}}(M_k, M_k) \otimes_k J \cong H^1_{dR}(E_k) \otimes_k J$$

b) The obstruction to lifting $M_A$ is a class in

$$\text{Ext}^2_{\mathcal{D}_{X_k}}(M_k, M_k) \otimes_k J \cong H^2_{dR}(E_k) \otimes_k J$$

The proof will take a number of steps, some local, and others global in nature. As above we let $E_k$ denote the vector bundle $\mathcal{E}nd(M_k)$ on $X_k$ (where $M_k := M_A \otimes_k k$) equipped with its natural connection. We begin by recalling some useful standard formulas about the action of the differential on $E_k$:

**Lemma 28.** Let $U_k$ be an open subset of $X_k$ such that $M_k|U_k$ is a trivial bundle. Choosing a basis, we may write the connection as $d + \Theta$ where $d$ is the differential which annihilates the chosen basis. We write $D$ for the induced connection on $E_k$.

Then we have

a) Let $\eta \in \mathcal{E}_k(U_k)$. Then $D(\eta) = d\eta + \Theta \eta - \eta \Theta$

b) Let $\alpha \in \mathcal{E}_k \otimes \Omega^1_{U_k}$. Then $D(\alpha) = d\alpha - (\alpha \Theta + \Theta \alpha)$

In both of these expressions, the multiplication is the one induced from the composition product on $\mathcal{E}_k$ and the wedge product on forms.

The proofs of these formulas are the standard (long) computations in differential geometry and will be omitted.
We shall say that $M_A$ is \textit{locally liftable} if there exists an open affine cover $\{U_i\}$ such that $M_A(U_i)$ is liftable. Then we have:

**Lemma 29.** There is a class in $H^0(\mathcal{H}^2_{dR}(\mathcal{E}_k)) \otimes_k J$ which vanishes iff $M_A$ is locally liftable.

**Proof.** Choose an open cover $\{U_i\}$ so that $M_A(U_i)$ are trivial bundles. We work over a given $U_i$. Writing the connection as $d + \Theta$ in some basis $\{e_i\}$, we consider the free module $M_A(U_i)$ as a lift of $M_A(U_i)$, along with the operator $d + \tilde{\Theta}$ where $\tilde{\Theta}$ is any lift of $\Theta$. This operator is a flat connection iff the curvature

$$\tilde{C} = d\tilde{\Theta} - \tilde{\Theta}d\tilde{\Theta} = 0$$

by the Cartan structure equation (c.f. [GH], page 75). Since this expression vanishes mod $J$ by assumption, we may consider it as an element of $H^0(\mathcal{E}_A \otimes \Omega^2_{X_A'}(U_i))$ (here $\mathcal{E}_A' := \text{End}(M_A)$). Since by construction we have an isomorphism $M_A(U_i)/J \to M_A(U_i)$, we may further identify $H^0(\mathcal{E}_A' \otimes \Omega^1_{X_A'})(U_i)$ with $J \otimes_k \mathcal{E}_k \otimes \Omega^1_{X_k}(U_i)$. By the Bianchi identity, (c.f. [GH], page 76) we have that $D(\tilde{C}) = 0$, so that in fact we may consider $\tilde{C} \in H^2_{dR}(\mathcal{E}_k)(U_i) \otimes_k J$. First, we claim that this is independent of the choice of $\tilde{\Theta}$: if we choose $\tilde{\Theta}'$ which also lifts $\Theta$, then the difference $\tilde{\Theta}' - \tilde{\Theta} = \alpha$ may be regarded as an element of $J \otimes_k \mathcal{E}_k \otimes \Omega^1_{X_k}(U_i)$. Then

$$d\tilde{\Theta}' - \tilde{\Theta}d\tilde{\Theta}' = d\tilde{\Theta} - \tilde{\Theta}d\tilde{\Theta} + d\alpha - (\alpha \tilde{\Theta} + \tilde{\Theta}\alpha)$$

since $\alpha \alpha = 0$ because $\alpha \in J(\mathcal{E}_A') \otimes \Omega^1_{X_A'}(U_i)$. But the expression $d\alpha - (\alpha \tilde{\Theta} + \tilde{\Theta}\alpha)$ is, by lemma 28, equal to $D(\alpha)$; making the class of $\tilde{C} \in H^2_{dR}(\mathcal{E}_k)(U_i) \otimes_k J$ independent.

Furthermore, consider $g \in GL(M_A)$. Then we may choose a lift to $\tilde{g} \in GL(M_A')$, and the curvature $\tilde{C}$ in the basis $\{\tilde{g}e_i\}$ is given by $\tilde{g}C\tilde{g}^{-1}$ (c.f. [GH], page 75). Thus the element $\tilde{C} \in H^2_{dR}(\mathcal{E}_k)(U_i) \otimes_k J$ is invariant under change of basis and hence globalizes to give an element of $H^0(\mathcal{H}^2_{dR}(\mathcal{E}_k)) \otimes_k J$.

Finally, note that if the class $\tilde{C}$ vanishes, then we may, in local coordinates, find $\alpha \in J \otimes_k \mathcal{E}_k \otimes \Omega^1_{X_k}(U_i)$ so that $D(\alpha) = -\tilde{C}$. Changing $\Theta$ to $\Theta + \alpha$ then shows the existence of local lifts. Conversely if local lifts exist then by construction $\tilde{C}$ vanishes over every open set in an open cover, and hence everywhere. $\square$

Now we shall describe the local situation in detail:

**Proposition 30.** a) Let $U \subset X_A$ be any open subset. As above let $D : \mathcal{E}_k \otimes \Omega^1_{U_k} \otimes_k J \to \mathcal{E}_k \otimes \Omega^2_{U_k} \otimes_k J$ denote the differential. There is an action of the group $\text{ker}(D)(U_k^{(1)})$ on the set of isomorphism classes of lifts of $M_A(U)$, this action is compatible with restriction of open subsets.

b) Let $U$ be affine. Then the action of part a) descends to an action of the group $\mathcal{H}^1_{dR}(\mathcal{E}_k)(U_k^{(1)})$ on the set of isomorphism classes of lifts of $M_A(U)$. If, furthermore, $M_A(U)$ is the trivial vector bundle, this action makes the set of lifts a pseudotorsor over $\mathcal{H}^1_{dR}(\mathcal{E}_k)(U_k^{(1)})$.

Note that this proves theorem 27 in the case of a trivial vector bundle on an affine scheme: by the previous lemma, the only obstruction to lifting in this case is the obstruction class in $H^0(\mathcal{H}^2_{dR}(\mathcal{E}_k))$, and this proposition asserts that the set of lifts, if nonempty, is a torsor over $H^0(\mathcal{H}^1_{dR}(\mathcal{E}_k))$; this is exactly the content of theorem 27 in this case. Now let us proceed to the
Proof. a) We first construct an action of $\text{Ker}(D)(U_k^{(1)}) \otimes_k J$ on the set of lifts of $M_A(U)$. We let $M_{A'}(U)$ be such a lift, with connection $\nabla$, and $\alpha$ an element of $E_k \otimes \Omega^1_{U_k} \otimes_k J$. We have the inclusion $J \subset A'$, and since our lift comes with an isomorphism $M_{A'}/J \rightarrow M_A$, we may identify it with an element of $E_k \otimes \Omega^1_{U_k} \otimes_k J$. Thus we may regard $\alpha$ as an element of $E_{A'} \otimes \Omega^1_{U_{A'}}$ (here $E_{A'} := \text{End}(M_{A'})$); or equivalently, an element of $\text{Hom}(M_{A'}, M_{A'} \otimes \Omega^1_{U_{A'}})$.

Now suppose that $D(\alpha) = 0$ (where, by abuse of notation, we denote by $D$ the map $D \otimes 1$ on $E_k \otimes \Omega^1_{U_k} \otimes_k J$). Then we also have $D(\alpha) = 0$ when $\alpha$ is considered as an element of $E_{A'} \otimes \Omega^1_{U_{A'}}$. Therefore we may define the map

$$\nabla + \alpha : M_{A'} \rightarrow M_{A'} \otimes \Omega^1_{U_{A'}}$$

and the condition $D(\alpha) = 0$, along with the fact that $\alpha$ takes values in $J M_{A'}$ so that $\alpha \alpha$ implies that this is a new flat connection on $M_{A'}$. Since $\alpha \equiv 0 \mod J$, this is also a lift of $M_A$ on $U$. We claim that this gives an action of ker($D$) on the set of isomorphism classes of lifts of $M_A$; to see this we must check that any isomorphism of lifts remains an isomorphism after modifying the connections via $\nabla \rightarrow \nabla + \alpha$.

So, let $\phi : M_{A'} \rightarrow M'_{A'}$ be an isomorphism of lifts and let $\nabla$ and $\nabla'$ be the connections on $M_{A'}$ and $M'_{A'}$, respectively. We denote by $\alpha_1$ and $\alpha_2$ the actions of $\alpha$ on $M_{A'}$ and $M'_{A'}$, respectively; recalling here that the action of $\alpha$ is determined by the data of the isomorphisms $M_{A'}/J \rightarrow M_A$ and $M'_{A'}/J \rightarrow M_A$. Then we compute that

$$(\phi \otimes 1)(\nabla + \alpha_1) = \nabla' + (\phi \otimes 1)(\alpha_1(\phi^{-1}))$$

But since $\phi$ is an isomorphism of lifts, it preserves the data of the isomorphisms $M_{A'}/J \rightarrow M_A$ and $M'_{A'}/J \rightarrow M_A$; implying that in fact

$$\alpha_2 = (\phi \otimes 1)(\alpha_1(\phi^{-1}))$$

so that $\nabla + \alpha_1 \rightarrow \nabla' + \alpha_2$ as lifts; as required. That this action is compatible with localization follows directly from the definition.

b) Now suppose that $U$ is affine. Then we have that that $H^1_{dR}(E_k)(U_k^{(1)}) = \text{Ker}(D)/\text{Im}(D)$. So, suppose that there is an element $\eta \in E_k \otimes_k J$ such that $D(\eta) = \alpha$. Then again regarding $\eta \in E_{A'}$, we have the automorphism $I + \eta$ of $M_{A'}$, with inverse $I - \eta$ (and $\eta \eta = 0$ as $\eta$ takes values in $J M_{A'}$); this becomes the identity map mod $J$. Then one computes that

$$((I + \eta) \otimes 1) \circ \nabla \circ (I - \eta) = \nabla - D(\eta) = \nabla - \alpha$$

(this follows by computing in local coordinates from lemma 28 part a) so that any $\alpha$ in the image of the connection $D$ acts trivially on the set of isomorphism classes of lifts; thus we get an action of $H^1_{dR}(E_k)(U_k^{(1)}) \otimes_k J$ on the set of isomorphism classes of lifts of $M_A$.

Now, suppose $U$ is affine and $M_A(U)$ is the trivial bundle; write the connection as $d + \Theta$. Suppose this connection is liftable. We may identify the set of isomorphism classes of lifts with the set of connections on the free module $M_{A'}(U)$ of the form $d + \Theta$ where $\Theta$ is a lift of $\Theta$. The difference of any two such connections is an element of $\text{Hom}(M_{A'}(U), M_{A'} \otimes \Omega^1_{A'}(U))$ whose image lies in $J M_{A'} \otimes \Omega^1_{A'}(U)$; i.e., we may identify it with an element $\alpha$ of $E_k \otimes \Omega^1_{U_k} \otimes_k J$ which satisfies $D(\alpha) = 0$ (since the connections are flat). This shows that the action constructed above is transitive on the set of lifts of $M_A(U)$. 
Now we note that two connections of the form $d + \Theta$ are identified as lifts when they are conjugate under the action of an element of $GL(M_A)$ of the form $I + \eta$; where $\eta$ is an element of $\text{End}(M_A)$ whose image lies in $JM_A$ so that $\eta$ may be identified with an element of $\mathcal{E}_k \otimes J$. As above the conjugation takes a connection $\nabla$ to $\nabla - D\eta$; thus two lifts are equivalent iff their difference lies in the exact forms inside $\mathcal{E}_k \otimes \Omega^1_{U_k} \otimes_k J$; this shows the lemma. 

The method of the lemma also shows the:

**Claim 31.** Let $M_A$ be a lift of $M_A$. The sheaf of groups of automorphisms of this lift is isomorphic to $\mathcal{H}_{dR}^0(\mathcal{E}_k) \otimes_k J$.

**Proof.** As above any automorphism of a lift may be written as $I + \eta$ where we may regard $\eta \in \mathcal{E}_k \otimes_k J$. The condition for conjugation by such an element to take the connection $\nabla$ to itself is simply $D\eta = 0$, the result follows. 

Finally, before proving theorem 27 we recall some facts about the hypercohomology of the deRham complex in positive characteristic. Namely, since the cohomology sheaves $\mathcal{H}_{dR}^0(\mathcal{E}_k)$ are coherent over $X^{(1)}_k$, we have a convergent spectral sequence of hypercohomology

$$H^i(\mathcal{H}_{dR}^j(\mathcal{E}_k)) \rightarrow \mathbb{H}_{dR}^{i+j}(\mathcal{E}_k)$$

which is known in this context as the conjugate spectral sequence (c.f. [K3], chapter 2). In the low degrees, the general machinery of the cohomology spectral sequence tells us rather explicitly what is happening:

**Lemma 32.** a) The conjugate spectral sequence yields a two term filtration of $\mathbb{H}_{dR}^1(\mathcal{E}_k)$ whose bottom term is isomorphic to $H^1(\mathcal{H}_{dR}^0(\mathcal{E}_k))$ and whose top term is the kernel of a natural map $H^0(\mathcal{H}_{dR}^1(\mathcal{E}_k)) \rightarrow H^2(\mathcal{H}_{dR}^0(\mathcal{E}_k))$.

b) The conjugate spectral sequence yields a three term filtration of $\mathbb{H}_{dR}^2(\mathcal{E}_k)$ whose bottom term is isomorphic to $H^2(\mathcal{H}_{dR}^0(\mathcal{E}_k))$, middle term is a submodule of $H^1(\mathcal{H}_{dR}^1(\mathcal{E}_k))$, and whose top term is a submodule of $H^0(\mathcal{H}_{dR}^2(\mathcal{E}_k))$.

With this in hand, we may combine the preliminaries to give the

**Proof.** (of theorem 27) We first consider part a). We suppose that lifts exist; let $\{U_i\}$ be an affine open cover of $X_A$. Then any such lift can be considered as a set of modules $M_A(U_i)$ along with isomorphisms of lifts $\gamma_i : M_A(U_i) \rightarrow M_A(U_j)|_{U_{ij}}$. We may compose any such map with an element $\alpha_{ij}$ of the group $\text{Aut}(M_A(U_{ij})) = \mathcal{H}_{dR}^0(\mathcal{E}_k)(U_{ij}) \otimes_k J$; and if we choose such elements for each $U_{ij}$, we see that they produce a new globally defined lift of $M_A$ iff they satisfy the cocycle condition $\alpha_{ij}\alpha_{jk}\alpha_{ki}^{-1} = 1$. This therefore yields an action of the cohomology group $H^1(\mathcal{H}_{dR}^0(\mathcal{E}_k)) \otimes_k J$ on the set of lifts, and one checks directly that an element of $H^1(\mathcal{H}_{dR}^0(\mathcal{E}_k)) \otimes_k J$ acts trivially iff it is trivial.

From the construction, we have that the orbit of a given lift $M_{A'}$ of the group $H^1(\mathcal{H}_{dR}^0(\mathcal{E}_k)) \otimes_k J$ is precisely the set of lifts which are locally isomorphic to $M_A$.

Now consider an arbitrary second lift $M_{A''}$. By 30 we can find a set of elements $\alpha_i \in \text{Ker}(D)(U_i) \otimes_k J$ so that $\alpha_i \circ M_{A'}(U_i) = M_{A''}(U_i)$; on the overlaps $U_{ij}$ the difference $\alpha_i - \alpha_j \in \text{Im}(D)$. Taking the images of the elements $\alpha_i \in \text{Ker}(D)(U_i) \rightarrow \mathcal{H}_{dR}^1(\mathcal{E}_k)(U_i^{(1)})$ yields a global section in $H^0(\mathcal{H}_{dR}^0(\mathcal{E}_k)) \otimes_k J$, whose image under the natural map to $H^1(\text{Ker}(D))$ is zero. This global section is trivial iff the two lifts $M_{A'}$ and $M_{A''}$ are locally isomorphic; therefore the quotient of the set of lifts
by the action of $H^1(H^0_{dR}(E_k)) \otimes_k J$ is identified with the subset of $H^0(H^1_{dR}(E_k))$ whose image under the natural map to $H^1(\text{Ker}(D))$ is zero. Comparing this with lemma 32 part a) yields the result.

Now we look at obstructions; we have already seen that the obstruction to lifting locally is a class in $H^0(H^2_{dR}(E_k)) \otimes_k J$. By construction, this class comes from a class in $\text{Ker}(D) \subset E_k \otimes_k \Omega^2_{X_k} \otimes_k J$. From the filtration in lemma 32 part b), we can find an element in $\mathbb{H}_2^{dR}(E_k) \otimes_k J$ which maps to this element, if nonzero.

On the other hand, supposing that this element vanishes, choose local lifts on a set $\{U_i\}$ of open affines such that $M_A(U_i)$ is trivial. On any intersection $U_{ij} = U_i \cap U_j$ these lifts give, by 30, an element $\alpha_{ij}$ of the group $H^1_{dR}(E_k)(U_{ij}) \otimes_k J$, and on triple overlaps $U_{ijk}$ we have $\alpha_{ijk} = \alpha_{ij} + \alpha_{jk}$ by the torsor property. Thus $\{\alpha_{ij}\}$ is a one-cocycle and defines an element in $H^1(H^1_{dR}(E_k)) \otimes_k J$. By lifting the elements $\alpha_{ijk}$ to $\text{Ker}(D)$ as in the last paragraph, one sees that this element is in the image of $F_1(\mathbb{H}_2^{dR}(E_k) \otimes_k J) \to H^1(H^1_{dR}(E_k)) \otimes_k J$ (here $F_2$ denotes the conjugate filtration as above).

If this element is trivial then $\alpha_{ij} = \beta_i - \beta_j$ for some $\beta_i \in H^1_{dR}(E_k)(U_i) \otimes_k J$. Fix one index $i_0$. On $U_j$ (for $j \neq i_0$) we modify $M_A(U_j)$ by acting on it by $\beta_j$ (under the action of $H^1_{dR}(E_k)(U_j)$), and on $U_{i_0}$ we modify by acting by $-\beta_{i_0}$; this yields lifts on $\{U_i\}$ which are isomorphic on each overlap. Thus the element in $H^1(H^1_{dR}(E_k)) \otimes_k J$ vanishes iff these local lifts are isomorphic on overlaps.

So, suppose that this happens. Then these isomorphisms give elements $\gamma_{ij} \in \text{Aut}(M_A(U_{ij})) = H^0_{dR}(E_k)(U_{ij}) \otimes_k J = F_0(\mathbb{H}_2^{dR}(E_k) \otimes_k J)$. From these we obtain, on the triple overlaps $U_{ijk}$, the elements $\gamma_{ij} \gamma_{jk} \gamma_{ki}^{-1}$, and thus a Cech two-cocycle. Then it follows directly that the associated element in $H^2(H^0_{dR}(E_k)) \otimes_k J$ vanishes iff the isomorphisms between lifts on the overlaps $U_{ij}$ can be chosen to glue on $U_{ijk}$ to give a sheaf on all of $X_A$; this completes the proof of part b) and hence the theorem.

We close this subsection with some brief general remarks about the cohomology groups appearing in the theorem. In many situations, these groups are quite amenable to computation. For instance, it is often the case that one can show that the conjugate spectral sequence degenerates at $E_2$; indeed, we shall see that in the cases of interest to us, this is how we compute the cohomology groups we are interested in.

3.1. Lifting of $\lambda$-connections and Higgs bundles. In this subsection, we shall explain several variations of the above picture which will be used in the sequel. In all cases, the proofs are completely analogous to what has been done above; we shall indicate the modifications when necessary.

The first variation we shall consider is the theory lifting of $\lambda$-connections. So, we consider a vector bundle with flat $\lambda$-connection $M_{\lambda,k}$ where again $k$ is a perfect field of positive characteristic.

Flatness implies that the operator $\nabla$ extends to operators

$$\nabla^i : M_{\lambda,k} \otimes \Omega^i_X \otimes \Lambda_k^1 \to M_{\lambda,k} \otimes \Omega^{i+1}_X \otimes \Lambda_k^1$$

for $0 \leq i \leq n - 1$, and the sheaf cohomology of the resulting complex will be denoted $H^i_{dR}(M_{\lambda,k})$. We regard the sheaves $H^i_{dR}(M_{\lambda,k})$ as (coherent) sheaves on $X_k^{(1)} \times_k \Lambda_k^1$. 


Now, we proceed as before: we consider Artinian local rings \( A' \) and \( A \), with residue field \( k \), such that we have
\[
0 \to J \to A' \to A \to 0
\]
where the ideal \( J \) satisfies \( J^2 = 0 \). Suppose that we are given a flat lift \( X_{A'} \) of \( X_A \) to \( A' \). We consider the question of lifting \( M_{A, \lambda} \) to \( X_{A'} \). We shall lift \( M_{A, \lambda} \) to \( X_{A'} \times_{A'} A'[\lambda] \).

Let \( U_A \) be an open subset of \( X_A \). By a lift of \( M_{A, \lambda}(U_A) := M_{A, \lambda}(U_A \times A[\lambda]) \), we shall mean a vector bundle with flat \( \lambda \)-connection \( M_{A', \lambda} \) on \( U_{A'} \times A'[\lambda] \), together with an isomorphism of \( \lambda \)-connections \( M_{A', \lambda}/J(U_A) \overset{\sim}{\to} M_{A, \lambda}(U_A) \). As before, an isomorphism of lifts is an isomorphism of connections which preserves this extra morphism. We set \( \mathcal{E}_{\lambda,k} := \mathcal{E}_{\text{End}}(M_{\lambda,k}) \); this has an induced \( \lambda \)-connection by the usual formula.

Then the theorem we want reads:

**Theorem 33.** With notations as above

- a) The set of lifts of \( M_{A} \) forms a pseudotorsor \( \mathbb{H}^1_{dR}(\mathcal{E}_{k,\lambda}) \otimes_k J \).
- b) The obstruction to lifting \( M_{A} \) is a class in \( \mathbb{H}^2_{dR}(\mathcal{E}_{k,\lambda}) \otimes_k J \).

The proof proceeds by an identical sequence of steps to the proof of theorem 27; replacing everywhere the action of the connection on \( E_k \) with the action of the \( \lambda \)-connection on \( \mathcal{E}_{\lambda,k} \). We shall discuss the structure of the groups \( \mathbb{H}^i_{dR}(\mathcal{E}_{\lambda,k}) \), in certain situations, later in the paper.

Next, we can consider the lifting question for Higgs sheaves themselves. For any Higgs bundle, \((H_k, \Theta)\), we have the Higgs cohomology sheaves \( H^i_{\text{Higgs}}(H_k) \) which are given by the cohomology of the complex
\[
\Theta^i : H_k \otimes \Omega^i_{X_k} \to H_k \otimes \Omega^{i+1}_{X_k}
\]
these are coherent sheaves on \( X_k \) itself, since the operator \( \Theta \) is a morphism of coherent sheaves.

We now consider a coherent Higgs sheaf \((H_k, \Theta)\), which can be be lifted to \((H_{A'}, \Theta)\). We wish to describe the lifts to a Higgs sheaf \((H_{A'}, \Theta)\). If \( H_k \) is a vector bundle over \( X_k \), we denote by \( F_k \) the endomorphism bundle of \( H_k \), equipped with its natural Higgs field. The theorem now reads

**Theorem 34.** With notations as above

- a) The set of lifts of \( M_{A} \) forms a pseudotorsor over \( \mathbb{H}^1_{\text{Higgs}}(F_k) \otimes_k J \).
- b) The obstruction to lifting \( M_{A} \) is a class in \( \mathbb{H}^2_{\text{Higgs}}(F_k) \otimes_k J \).

The proof is again identical to the proof of theorem 27.

Finally, we should like to discuss a somewhat different lifting problem which also fits into the same framework. In particular, we are interested in the infinitesimal deformations of a Higgs sheaves to \( \lambda \)-connections. Consider now the scheme \( X \), smooth over an affine base scheme \( S \). In particular, we keep open the possibility that \( S \) is not a field (and if it is, we allow it to have any characteristic). Now, we may define the notion of a vector bundle with \( \lambda \)-connection over \( S \times \mathbb{Z}[\lambda]/\lambda^n \);
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namely, it is a coherent module over $\mathcal{D}_{X/S,\lambda}/\lambda^n$ which is a vector bundle over $X_S[\lambda]$. Given such a sheaf $M_{\lambda,n}$, we wish to consider the set of isomorphism classes of deformations of it; i.e. the set of isomorphism classes of $\lambda$-connections $M_{\lambda,n+1}$ (over $\mathbb{Z}[\lambda]/\lambda^{n+1}$), which are equipped with an isomorphism

$$M_{\lambda,n+1}/\lambda^n \cong M_{\lambda,n}$$

We consider now a Higgs sheaf $H_S$ on $X$, and suppose given a deformation $M_{\lambda,n}$. Then the theorem reads

**Theorem 35.** With notations as above

a) The set of deformations of $M_{\lambda,n}$ to $M_{\lambda,n+1}$ forms a pseudotorsor over $H^1_{\mathrm{Higgs}}(\mathcal{F}_S)$

b) The obstruction to lifting $M_{\lambda,n}$ is a class in $H^2_{\mathrm{Higgs}}(\mathcal{F}_S)$

This is proved in exactly the same way as the analogous statement for the lifting problem.

4. THE ONE-DIMENSIONAL CASE

In this section, we return to our main problem; and we consider, in particular, the one-dimensional case. As noted above, this implies $X \rightarrow \mathcal{L} \rightarrow \mathbb{A}_S^1$. Recall from 23 and lemma 24 that we have constructed, for each perfect field $k$ of positive characteristic and $k \rightarrow S$, a connection $M_k$ on $X_k$ whose $p$-support is equal to $\mathcal{L}_k^{(1)}$ (with multiplicity 1). Let us further recall that by definition, $M_k|_{U_k} = \pi_*(e^f)|_{U_k}$, where $U_k \subset X_k$ is the open subset on which $\mathcal{L}_k \rightarrow X_k$ is finite etale. The degree of this map is denoted $r$. Denote by $\mathbb{L}$ and $\mathbb{X}$ copies of $\mathbb{P}^1$ which contain $\mathcal{L}$ and $X$, respectively, as standard open affines.

Let $p$ be any prime number such that $p$ is unramified in $R$. Then if $k$ is the perfect closure of the field of fractions of $R/pR$, we have embeddings

$$R/p^nR \rightarrow W_n(k)$$

for every $n \geq 0$; here $W_n(k)$ is the ring of length $n$ Witt vectors over $k$. We shall therefore work with lifts of $M_k$ to $W_n(k)$ for each $n \geq 0$. For any object over $k$ (scheme, vector bundle, etc) we denote lifts of this object to $W_n(k)$ with a subscript $n$. $W(k)$ shall denote the inverse limit of the $W_n(k)$. As outlined in the introduction, the first step in proving the one-dimensional case (and, for that matter, the higher dimensional case as well) is a detailed study of the lifting of $M_k$ to connections over $W_n(k)$ for all $n \geq 0$. We proceed to that now.

4.1. The preferred lift. Our goal is to construct lifts which have certain preferred behavior at the divisor at infinity in $X_k$. To specify what this behavior is, let us denote by $(\mathcal{H}_C, \Theta)$ the Higgs bundle on $X_C$ obtained by pushforward of the Lagrangian $\mathcal{L}_X$ (as usual, the structure of $\mathcal{L}_X$ as a subvariety of $T^*X_C$ yields the Higgs field $\Theta$).

We now note that the morphism $\pi : \mathcal{L}_C \rightarrow X_C$ extends unique to a map $\overline{\pi} : \mathbb{L}_C \rightarrow \mathbb{X}_C$. This map takes the divisor at infinity in $\mathbb{L}_C$ to the divisor at infinity in $\mathbb{X}_C$, and is a finite branched cover of rank $r$ in a neighborhood of this divisor. Consider the Higgs bundle $(\pi^*\mathcal{H}, \pi^*\Theta)$ on $\mathcal{L}_C$. Then we have the
Lemma 36. On $\mathcal{L}_{U_c}$ the Higgs bundle $(\pi^*\mathcal{H}, \pi^*\Theta)$ decomposes as a direct sum of one dimensional, pairwise non-isomorphic Higgs bundles, $(\mathcal{H}_i, \Theta_i)$.

Proof. The support (inside $T^*\mathcal{L}_{X_C}$) of $(\pi^*\mathcal{H}, \pi^*\Theta)$ is, over the open subset $\mathcal{L}_{U_c}$, isomorphic to the variety $\mathcal{L}_{U_c} \times_{U_c} \mathcal{L}_{U_c}$, which is finite etale over $\mathcal{L}_{U_c}$. We claim that $\mathcal{L}_{U_c} \times_{U_c} \mathcal{L}_{U_c}$ is isomorphic to a disjoint union of $r$ copies of $\mathcal{L}_{U_c}$. To see this, note that if we take the base change to the generic point of $U_C$, then this is just the fact that, if $F \subset L$ is a separable field extension of degree $r$, then $L \otimes_K L \cong \bigoplus_{i=1}^r L$.

But $\mathcal{L}_{U_c} \times_{U_c} \mathcal{L}_{U_c}$ is a smooth variety, so it must be a union of $r$ disjoint varieties if this is so after some localization; this implies the claim. But now the lemma follows by decomposing $(\pi^*\mathcal{H}, \pi^*\Theta)$ over its support (as a $T^*X_C$-module) just as in 26. □

As usual, we may take the above decomposition to be defined over $R$; we denote the operators $\Theta_i$ by the same letter over any ring to which $R$ maps. As indicated above, our goal in this section is to construct some natural lifts of $M_k$. In the previous section, we have considered the lifting problem for connections on arbitrary smooth varieties; now we must consider an extension of this problem to include the case where the connection degenerates at some divisor. At the time of this writing, the most general picture for such a lifting problem is unclear (to me, anyway). However, in the situation of interest here, we have, due to the previous lemma, very good control over the behavior of the connection at the divisor where it degenerates, and we may use this to give a very natural extension of theorem 27 to this situation. We start with the rather general

Definition 37. a) A meromorphic connection on $X_R$ is a bundle $M_R$ on $X_R$ equipped with an $R$-linear map

$$\nabla : M_R \to M_R \otimes_{\mathcal{O}_{X_R}} \Omega^1_{X_R}(D)$$

where $D$ is an $R$-divisor on $X_R$. We demand that $\nabla$ satisfy the Leibniz rule and the integrability condition.

b) A meromorphic Higgs bundle on $X_R$ is a bundle $H_R$ on $X_R$ equipped with an $R$-linear map

$$\Theta : H_R \to H_R \otimes_{\mathcal{O}_{X_R}} \Omega^1_{X_R}(D)$$

where $D$ is an $R$-divisor on $X_R$. We demand that $\Theta$ be $\mathcal{O}_{X_R}$-linear and satisfy the integrability condition.

c) A meromorphic $\lambda$-connection on $X_R$ is a bundle $M_R$ on $X_R \times_R \mathbb{A}^1_R$ equipped with an $R$-linear map

$$\nabla : M_R \to M_R \otimes_{\mathcal{O}_{X_R}} \Omega^1_{X_R}(D)[\lambda]$$

where $D$ is an $R$-divisor on $X_R$. We demand that $\nabla$ satisfy the $\lambda$-Leibniz rule and the integrability condition.

Of course, these definitions are the exactly the same when we replace $R$ by any of the rings or fields to which it maps. We would like to show that $M_k$ admits a preferred extension to a meromorphic connection on $X_k = \mathbb{P}^1_k$. To do, we first examine the situation for the Higgs bundle $\mathcal{H}$. We start by working over the field $C$.

Let $\{\epsilon_i\}_{i=1}^r$ denote a basis of $\pi^*\mathcal{H}|_{U_C}$ on which the Higgs fields acts diagonally (which exists by lemma 36). Denote by $\widetilde{\pi^*\mathcal{H}}$ the extension of $\pi^*\mathcal{H}|_{U_C}$ to $\mathcal{L}_{U_c} \cup$
\{\infty\} defined by extending these basis elements; i.e., we take the free module over \(O_{\mathcal{L}_U}\{\infty\}\) generated by \(\{e_i\}_{i=1}^r\). Gluing this with the Higgs bundle \(\pi^*\mathcal{H}_C\) on the open set \(\mathcal{L}_U\) yields a meromorphic Higgs bundle \(\pi^*\mathcal{H}_R\) on \(\mathcal{L}_R\). We note that this is an abuse of notation - at this point \(\pi^*\mathcal{H}_R\) depends on the choice of diagonalizing basis.

Note that the projection \(\pi : \mathcal{L}_C \to X_C\) admits the compactification \(\bar{\pi} : \bar{\mathcal{L}}_C \to \bar{X}_C\). This map takes \(\{\infty\}\) to \(\{\infty\}\), and in the formal neighborhood of \(\{\infty\}\) is a branched cyclic cover of rank \(r\). This cover has cyclic Galois group \(G_r\). Again invoking lemma 36, we see that we may (and shall) demand that the group \(G_r\) acts simply transitively on the set \(\{e_i\}_{i=1}^r\); this determines the bundle \(\pi^*\mathcal{H}_C\) up to a rescaling by a power of a local coordinate at \(\{\infty\}\).

Thus we may define an extension of \(\mathcal{H}_C\) to \(\bar{X}_C\), \(\bar{\mathcal{H}}_C\) as follows: we have a bundle \((\bar{\pi}^*\mathcal{H}_C)_{\{\infty\}}\) defined by restricting \(\pi^*\mathcal{H}_C\) to the formal neighborhood of \(\{\infty\}\) in \(\bar{\mathcal{L}}_C\). Then we define a bundle

\[
(\bar{\mathcal{H}})_{\{\infty\}} := (\bar{\pi}^*\mathcal{H}_C)_{\{\infty\}} \cap (\mathcal{H}_C)[\{\infty\}]
\]

where the meaning of this is as follows: \((\bar{\mathcal{H}})[\{\infty\}]\) denotes the formal completion at infinity of an arbitrary extension of \(\mathcal{H}_C\) to a bundle on \(\bar{X}_C\); followed by inversion of a local coordinate at \(\{\infty\}\). This object is independent of the extension chosen; and we have the inclusion

\[
(\mathcal{H}_C)[\{\infty\}] \subset (\pi^*\mathcal{H}_C)[\{\infty\}]
\]

where the object on the right is defined analogously (on \(\bar{\mathcal{L}}_C\)). Thus there is also an inclusion

\[
(\bar{\pi}^*\mathcal{H}_C)_{\{\infty\}} \subset (\pi^*\mathcal{H}_C)[\{\infty\}]
\]

and we are taking the intersection inside \((\bar{\pi}^*\mathcal{H}_C)[\{\infty\}]\); it is easily verified that the result is a bundle on \(\bar{X}_\{\infty\}\), and thus provides an extension of \(\mathcal{H}_C\) to a bundle on all of \(\bar{X}_C\). We note that this extension can also be described as

\[
(\bar{\mathcal{H}})_{\{\infty\}} = (\pi^*\mathcal{H}_C)[\{\infty\})^{G_r}\]

This extension is, however, still not well defined - i.e., it depends on the choice of \(\pi^*\mathcal{H}_C\).

Further, let us introduce the following bundles: we can consider \(\bigwedge^k \pi^*\mathcal{H}_C\) for any \(k \geq 0\); this is a bundle with meromorphic Higgs field on \(\bar{\mathcal{L}}_C\). Let us define

\[
\bigwedge^k \mathcal{H}_C
\]

to be the extension of \(\bigwedge^k \mathcal{H}_C\) to \(\bar{X}_C\) obtained, at \(\{\infty\}\), by taking the intersection with \((\bigwedge^k \pi^*\mathcal{H}_C)_{\{\infty\}}\).

We may now define \(\bar{\mathcal{H}}_R\) and \(\bigwedge^k \mathcal{H}_R\) to be bundles on \(\bar{X}_R\) obtained by spreading out; similarly for \(\pi^*\mathcal{H}_R\) on \(\bar{\mathcal{L}}_R\). We can then, for \(p >> 0\), define the analogous bundles over \(k\). It is also easy to see that one can obtain the same collection of
bundles over $\overline{X}_k$ and $\overline{L}_k$ by repeating the arguments that led to their construction over $\mathbb{C}$.

Now, we examine in more detail the local behavior at $\{\infty\}$:

**Lemma 38.** Let $\overline{H}_R$ be an extension of the type described above. Then we have a short exact sequence

$$0 \rightarrow \pi^*\overline{H}_R \rightarrow \pi^*\overline{H}_R \rightarrow \bigoplus_{i=1}^{r-1} O_i(\infty) \rightarrow 0$$

of sheaves on $\overline{L}_R$.

**Proof.** It suffices to work over $\mathbb{C}$ and then spread out. By definition the two bundles $\pi^*\overline{H}_C$ and $\pi^*\overline{H}_C$ are equal over $\mathcal{L}_R$. Consider a basis $\{e_i\}_{i=1}^r$ for $((\pi^*\overline{H}_C))_{\{\infty\}}$. Then in fact a basis for $((\pi^*\overline{H}_C))_{\{\infty\}}$ is given by

$$\{z^{k/r}(\sum_{i=1}^r (\xi^k)^{i-1}e_i))_{k=0}^{r-1}$$

where $z$ is a local coordinate for $\overline{X}_C$ at $\{\infty\}$, and we have chosen an $r$th root $z^{1/r}$ as a basis for $\overline{L}_R$ at $\{\infty\}$, and a primitive $r$th root of unity $\xi$. One checks directly that this is a set of $G$-invariants which is spanning and maximal. The result follows. □

Given this, we have the

**Lemma 39.** There is a unique choice of Higgs bundle $\overline{H}_R$ satisfying

$$\bigwedge^r H_R \equiv (O_{\overline{X}_R}, \tilde{f})$$

where $\tilde{f}$ is a regular function on $X_R$.

**Proof.** The bundle $\bigwedge^r H_R$ is a line bundle on $\overline{X}_R$ whose Higgs field is non-degenerate on $X_R$. To compute the Higgs field, we note that at the generic point $\xi$ of $X_R$, the Higgs field $\Theta$ may be regarded as an endomorphism of the finite dimensional vector space $H_\xi$. After passing to the finite separable field extension $\mathcal{L}_\xi$, this endomorphism has distinct eigenvalues which are the functions $df_i$ of 26. The trace of this endomorphism is therefore a function $\tilde{f}$ which is defined over $X_\xi$. The non-degeneracy of the Higgs field on $X_R$ now implies that $\tilde{f}$ is a regular function on $X_R$.

So, for any choice of $\overline{H}_R$, we have an isomorphism

$$\bigwedge^r H_R \equiv (\mathcal{M}_{\overline{X}_R}, \tilde{f})$$

where $\mathcal{M}_{\overline{X}_R}$ is a line bundle. On the other hand, as observed above, the choice of $\overline{H}_R$ is determined up to a rescale (by a power of $z^{1/r}$) on $\overline{L}_R$. Furthermore, by the previous lemma, we have the exact sequence

$$0 \rightarrow \pi^*\overline{H}_R \rightarrow \pi^*\overline{H}_R \rightarrow \bigoplus_{i=1}^{r-1} O_i(\infty) \rightarrow 0$$
where $O_{i\{\infty\}}$ is the structure sheaf of the divisor $i\{\infty\}$. Taking $\bigwedge^r$ of this sequence shows that we have an exact sequence

$$0 \to \bigwedge^r(\pi^*H_R) \to \bigwedge^r(\pi^*H_R) \to O_{N\{\infty\}} \to 0$$

where $N = \sum_{i=1}^{r-1} i = \frac{r(r - 1)}{2}$. Further, from the fact that $\bigwedge^r H_R$ is a line bundle, we obtain

$$\pi^*(\bigwedge^r H_R) \cong \bigwedge^r(\pi^*H_R)$$

Therefore we have the equation

$$\deg(\pi^*(\bigwedge^r H_R)) = \deg(\pi^*H_R) + N$$

and so if we choose the unique rescale so that $\deg(\pi^*H_R) = -N$ the result follows.

Choose an eigenbasis for the Higgs field of $\pi^*H_R$ in the local ring at $\{\infty\}$. We let $[\Theta]$ denote the matrix of the Higgs field in this basis; and we use the same letter to denote the base change of this matrix to any field to which $R$ maps.

We wish to consider the analogue of the above extension for the connection $M_k$. In fact, by 26, as in the case of the Higgs bundle $H_R$ we have an isomorphism

$$\pi^* M_k|_{L_U} \cong \bigoplus_{i=1}^{r} O_{L_U} e_i$$

where $e_i$ are such that $\nabla(e_i) = \Theta_i e_i$. The $\{e_i\}$ are determined up to an action of $\text{Gl}_n(O_{L_U}(1))$.

Write $\pi^*\! M_k$ for any extension of this connection to a bundle with meromorphic connection which is generated, in the local ring at $\{\infty\}$, by a set of eigenvectors $\{e_i\}$ with $\nabla(e_i) = \Theta_i e_i$. After completing at $\{\infty\}$, we can further demand that the cyclic group $G_r$ acts transitively on the $e_i$; if we do this then such a basis is determined up to an action of $(z^{1/r})^p$, where, as above $z^{1/r}$ is a local coordinate of $(\mathbb{C}^*)_{\{\infty\}}$.

Then we write $\overline{M}_k$ for the extension of $M_k$ which satisfies

$$\overline{(M_k)}_{\{\infty\}} = (\pi^*\! M_k)_{\{\infty\}} \cap (\overline{M_k})^{\star \{\infty\}}$$

As with the Higgs case, we obtain also the bundles

$$\bigwedge^l M_k$$

for any $l \geq 0$. Then the result reads:

**Lemma 40.** The connection $M_k$ admits a unique extension to a meromorphic connection $\overline{M}_k$ (of the type described above) such that $\bigwedge^l M_k \cong (e^f)$ as meromorphic connections; where $(e^f)$ denotes the meromorphic connection $(O_{P^1}, d + df)$.

**Proof.** This is completely analogous to the proof of lemma 39. \qed

Now, the main result which is to be proved in this section is the
Theorem 41. There exists a unique inverse system of meromorphic connections $\{M_n\}$ on $X_{W_n(k)}$ such that

1) $\mathfrak{M}_n = \mathfrak{M}_k$

2) For each $n \geq 1$ there exists an extension of $\pi^*M_n$ to $L_{W_n(k)}$, denoted $\pi^*\mathfrak{M}_n$, whose localization at infinity satisfies admits an eigenbasis for $\nabla$, denoted $\{e_i\}_{i=1}^r$ such that the $\{e_i\}$ satisfy

$$\nabla e_i - \Theta e_i \in \Omega^1_{L_{W_n(k)}} \otimes \pi^*\mathfrak{M}_n$$

Further, the bundle $\mathfrak{M}_n$ satisfies $\mathfrak{M}_n$[∞] = $\mathfrak{M}_n$[∞] ∩ $\mathfrak{M}_n$[⋆∞].

We shall construct this inverse system one step at a time, by lifting. We would like to isolate condition 2; we call any meromorphic connection satisfying it $\Theta$-regular at infinity. Before proceeding, let us note a fact which shows that this condition is reasonable:

Lemma 42. Let $Y_k$ be a smooth variety over $k$, and $M_k$ a vector bundle with flat connection on $Y_k$. Suppose that

$$M_k = \bigoplus_{i=1}^l M_{i,k}$$

where $M_{i,k}$ are bundles with connection whose p-supports are mutually non-intersecting.

Now suppose given a lift $Y_n$ over $W_n(k)$ and a vector bundle $M_n$ with connection which lifts $M_k$. Then there is a decomposition

$$M_n = \bigoplus_{i=1}^l M_{i,n}$$

where the $M_{i,n}$ are vector bundles with connection which lift $M_{i,k}$.

Proof. We give two proofs; the first of which is by induction on $n$. Suppose we know this decomposition for $M_{n-1}$.

First proof. We have that lifts of $M_{n-1}$ are, by theorem 27, classified by $H^1_{dR}(\mathfrak{M}_{n-1})$. The assumption on the p-support implies

$$\text{Ext}^1_{\mathfrak{M}_{X_k}}(M_{i,k}, M_{j,k}) = 0$$

if $i \neq j$ (just as in 26). Therefore

$$H^1_{dR}(\mathfrak{M}_k) = \bigoplus_{i=1}^l \text{Ext}^1(M_{i,k}, M_{i,k})$$

so that each extension is a direct sum of extensions of the $M_{i,k}$, as claimed.

Second proof. (sketch) We start by working locally on an affine subset $U_k$. In local coordinates let $\{\partial_1, \ldots, \partial_n\}$ be a basis for the tangent sheaf, so that $\partial_i(y_j) = \delta_{ij}$ for some regular sequence $\{y_i\}$. Then we can consider the collection of differential operators

$$\{\partial_i^{p^n}\}$$

and one computes directly that these operators annihilate the ring of functions on $Y_n$. Their reductions mod $p$ are simply powers of the operators defining the p-curvature, and therefore act on the $M_{i,k}$ with mutually non-intersecting support inside $T^*X_k^{(n)} \times X_k^{(n)} X_k$ where the $?^{(n)}$ denotes the nth iterate of the Frobenius map.
Thus the operators \( \{ \partial_i^n \} \) also act with mutually non-intersecting support inside \( \text{Spec}(O(U_n)[[\partial_1^n, \ldots, \partial_m^n]]) \) and hence split the bundle \( M_n \). By computing the change of basis which results from changing local coordinates, one can show that this decomposition is independent of the choices.

Therefore, any for any lift \( M_n \) as in the theorem, the module \( \pi^* M_n(U) \) admits an eigenbasis for the connection. Thus we see that we are demanding that the extension \( \pi^* M_n \) be compatible with the basis in a suitable sense. In order to construct such extension, we shall need a slight extension of our previous lifting theorems.

To state this theorem; we let \( \mathcal{E}_k := \mathcal{E}nd(\mathcal{M}_k) \). This is a bundle which is an extension of \( \mathcal{E}_k \) to \( \mathcal{X}_k \). This bundle has a meromorphic connection (induced from the connection on \( \mathcal{M}_k \)), and so we can consider the naive deRham cohomology sheaves, i.e., the kernel and cokernel of

\[
\mathcal{E}_k \to \mathcal{E}_k \otimes \Omega^1_{\mathcal{X}_k}(D)
\]

Certainly, the cokernel of this map is an extension to \( \mathcal{X}_k^{(1)} \) of the sheaf \( \mathcal{H}_{dR}(\mathcal{E}_k) \); however, this sheaf does not classify lifts of \( \mathcal{M}_k \) which are \( \Theta \)-regular at \( \{ \infty \} \).

To see the issue, we first consider the analogous problem for \( \pi^*(\mathcal{M}_k) \), the set of lifts of \( \pi^*(\mathcal{M}_k) \) which are \( \Theta \)-regular at \( \{ \infty \} \) is the trivial connection without singularity at \( \{ \infty \} \), namely, if \( \{ e_i \} \) are a basis of \( \pi^*(\mathcal{M}_k) \) so that \( \nabla(x) = \Theta(x)e_i \), then this summand is the span of \( \{ e_i \otimes e^*_i \} \). Write

\[
\pi^*(\mathcal{E}_k) = \nabla_k \bigoplus \mathcal{N}_k
\]

where \( \nabla_k \) is the trivial connection over the local ring at \( \{ \infty \} \) in \( \mathcal{E}_k \). Then we have the

**Lemma 43.** Suppose \( \mathcal{M}_k \) is as in theorem 41. Then, in a neighborhood \( \mathcal{N}_k \) of \( \{ \infty \} \), the set of lifts of \( \pi^*(\mathcal{M}_k) \) which are \( \Theta \)-regular at infinity form a torsor over \( \mathcal{H}_{dR}(\mathcal{N}_k)(\mathcal{N}_k^{(1)}) \).

Note that since \( \nabla_k \) is a connection without singularity at \( \{ \infty \} \), this group is not the same as

\[
\text{coker}(\pi^*(\mathcal{E}_k) \rightarrow \pi^*(\mathcal{E}_k) \otimes \Omega^1_{\mathcal{X}_k}(D))(\mathcal{N}_k^{(1)})
\]

The group \( \mathcal{H}_{dR}(\mathcal{N}_k)(\mathcal{N}_k^{(1)}) \) is, instead, the quotient of \( \mathcal{H}_{dR}(\mathcal{E}_k) \otimes \Omega^1_{\mathcal{X}_k} \) by the image of \( \nabla_k \). In particular, these two groups differ by \( \nabla_k(D)/\nabla_k \). With this in mind, we give the

**Proof.** (of lemma 43) The argument is almost the same as that of 30. We may trivialize the connection on \( \pi^*(\mathcal{M}_k) \) as \( d + \Theta' \), where \( \Theta' \) is a Higgs field with values in \( \pi^*(\mathcal{M}_k) \otimes \Omega^1_{\mathcal{X}_k}(D) \). The set of all lifts is identified with the set of \( d + \Theta \) where \( \Theta \) is a meromorphic Higgs field lifting \( \Theta' \). As before, the difference between two lifts is an element in \( J \cdot (\mathcal{E}_A \otimes \Omega^1_{\mathcal{N}_A}(D)) \). If both of these lifts are supposed to be \( \Theta \)-regular at \( \{ \infty \} \), we must demand that the difference element lives in \( J \cdot (\pi^*(\mathcal{E}_A) \otimes \Omega^1_{\mathcal{N}_A}) \approx \pi^*(\mathcal{E}_k) \otimes \Omega^1_{\mathcal{N}_k} \).

As before, two lifts are identified iff the difference between their Higgs fields is the differential of an element of \( J \cdot \pi^*(\mathcal{E}_A) \). Thus the group we want is
the quotient of $\pi^*(E_k) \otimes \Omega^1_{W_k}$ by the image of $\nabla$; by the remarks above thus yield the lemma.

Now we wish to explain the analogous group over $\overline{X}_k$. Recall that the map $\pi$ becomes a branched cover of degree $r$ at the point $\{\infty\}$. Since $p \nmid r$ this is a tamely ramified map. Thus if $G_r$ is the cyclic group of size $r$, we have

\[ \widehat{O}(\overline{L}_k) \cong \widehat{O}(X_k) \]

Recall that on $L^{(1)}_k$ we have the complex

\[ \nabla : \overline{V}_k \oplus \overline{N}_k \to (\overline{V}_k \otimes \Omega^1_{\overline{L}_k}) \oplus (\overline{N}_k \otimes \Omega^1_{\overline{L}_k}(D)) \]

whose cokernel was the group that appeared above in lemma 43 (namely, we choose $D$ so that $\nabla$ is a surjection from $\overline{N}_k$ to $(\overline{N}_k \otimes \Omega^1_{\overline{L}_k}(D))$). Completing at the point at $\{\infty\}$ and taking $G_r$ invariants\(^5\) (an exact operation) yields a complex on $X_k$:

\[ (4.1) \]

\[ \nabla : \overline{E}_k \to \overline{E}_k \otimes \Omega^1_X \]

where $\Omega^1_X$ is an extension of $\Omega^1_{\overline{X}_k}$ which is not equal to $\Omega^1_{\overline{X}_k}(D)$ for any $D$. We denote the cokernel of this map by $H^1_{dR}(\overline{E}_k)$, and the kernel by $\overline{H}^0_{dR}(\overline{E}_k)$. Then we have the

**Lemma 44.** Suppose $M_n$ is as in theorem 41. Then, in a neighborhood $N_k$ of $\{\infty\}$, the set of lifts of $M_n$ which are $\Theta$-regular at infinity form a torsor over $\overline{H}^1_{dR}(\overline{E}_k)(N_k^{(1)})$.

The proof is immediate from lemma 43 and the definition of $\overline{H}^1_{dR}(\overline{E}_k)$.

Now, we suppose that we have found $M_n$ as in the statement of the theorem 41. We consider, over any open subset $N$ of $\overline{X}_{n+1}$, the set $L(W)$ of isomorphism classes of lifts $M_{n+1}(W)$ of $M_n(W)$ to a bundle with connection which is $\Theta$-regular at infinity.

Then the result is:

**Theorem 45.** The set $L(\overline{X}_{n+1})$ forms a pseudotorsor over $\overline{H}^1_{dR}(\overline{E}_k)$ (the hypercohomology of the complex equation (4.1)). The obstruction to the global existence of $M_{n+1}$ lives in $H^1(\overline{H}^1_{dR}(\overline{E}_k))$ (since $X_k$ is a curve).

**Proof.** The argument has exactly the same structure as the proof of theorem 27; indeed, all of the statements there follow formally from the fact that the set of lifts over an affine open subset on which $M$ is trivial form a torsor over $\overline{H}^1_{dR}(\overline{E}_k)$. The analogous statement in this case is that the set of lifts which are $\Theta$-regular at infinity forms a torsor over $\overline{H}^1_{dR}(\overline{E}_k)$. Of course, this is the same statement over an open subset not containing $\{\infty\}$, and it follows from lemma 44 in the case of an open subset containing $\{\infty\}$. \[\square\]

So, to solve the lifting problem, we need to describe the cohomology. The key result there is the

\(^5\)Or, equivalently, taking the intersection with $\overline{X}_k$ as before
**Theorem 46.** The trace map \( t : \mathcal{E}_k \to O_k \) induces isomorphisms
\[
H^i(\mathcal{H}_{dR}^1(\mathcal{E}_k)) \cong H^i(\mathcal{H}_{dR}^1(O_k))
\]
(for \( i = 0, 1 \)). This map takes the obstruction class for \( \mathcal{M}_{n+1} \) to the class for \( (e') \).

In order to prove this, we shall relate the cohomology of \( \mathcal{E}_k \) to the cohomology of the Higgs bundle \( \mathcal{F}_k^{(1)} := \text{End}(\mathcal{H}_k^{(1)}) \). We define the sheaves \( \mathcal{H}_{Higgs}^0(\mathcal{F}_k^{(1)}) \) and \( \mathcal{H}_{Higgs}^1(\mathcal{F}_k^{(1)}) \) in a completely analogous way as for \( \mathcal{E}_k \); we first define the complex over \( L_k \) and then descend to \( X_k \).

**Lemma 47.** There are isomorphisms
\[
\mathcal{H}_{dR}^0(\mathcal{E}_k) \cong \mathcal{H}_{Higgs}^0(\mathcal{F}_k^{(1)})
\]
and
\[
\mathcal{H}_{dR}^1(\mathcal{E}_k) \cong \mathcal{H}_{Higgs}^1(\mathcal{F}_k^{(1)})
\]
The analogous isomorphisms holds for \( \pi^* \mathcal{E}_k \) and \( \pi^* \mathcal{F}_k \).

**Proof.** First, we show that this is true on \( X_k \). Let \( \Psi \) be the \( p \)-curvature operator on \( \mathcal{E}_k \). The kernel and cokernel of \( \Psi \) are both \( \mathcal{D}_X \)-modules with \( p \)-curvature zero. From this fact and Cartier descent we deduce
\[
\ker(\Psi) \cong F^*(\mathcal{H}_{dR}^0(\mathcal{E}_k))
\]
and
\[
coker(\Psi) \cong F^*(\mathcal{H}_{dR}^1(\mathcal{E}_k))
\]
On the other hand, in a suitable basis \( \Psi \) is the matrix of \( p \)th powers of \( \Theta \); which is the Higgs field of \( \mathcal{F}_k^{(1)} \). Since \( F^* \) is exact we get
\[
\ker(\Psi) \cong F^*\mathcal{H}_{Higgs}^0(\mathcal{F}_k^{(1)})
\]
and
\[
coker(\Psi) \cong F^*\mathcal{H}_{Higgs}^1(\mathcal{F}_k^{(1)})
\]
and thus the isomorphisms hold on \( X_k^{(1)} \). The same argument yields the analogous isomorphisms for \( \pi^* \mathcal{E}_k \) and \( \pi^* \mathcal{F}_k \) on \( L_k^{(1)} \).

Now, if we pull back to \( L_k^{(1)} \) and look on \( L_{U_k}^{(1)} \); there we have
\[
\pi^*(\mathcal{E}_k)|_{L_{U_k}} \cong \mathcal{V}_{L_{U_k}} \bigoplus N_{L_{U_k}}
\]
where \( \mathcal{V}_{L_{U_k}} \) is a trivial bundle with connection and \( N_{L_{U_k}} \) is a bundle with no deRham cohomology (this follows from the proof of 26). This extends to the decomposition
\[
\pi^*\mathcal{E}_k \cong \mathcal{V}_k \bigoplus N_k
\]
on \( \mathcal{L}_k \). We have the analogous decomposition \( \pi^*\mathcal{F}_k = \mathcal{T}_k \bigoplus \mathcal{G}_k \) of Higgs bundles, where \( \mathcal{T}_k \) is a bundle with zero Higgs field, and \( \mathcal{G}_k \) is a Higgs bundle with trivial Higgs cohomology over \( L_{U_k} \). The isomorphisms constructed in the previous paragraph (on \( L_k^{(1)} \)) restrict on \( L_{U_k}^{(1)} \) to the Cartier isomorphisms
\[
\mathcal{H}_{dR}(\mathcal{V}_k)(L_{U_k}^{(1)}) \cong \mathcal{H}_{Higgs}^i(\mathcal{T}_k)(L_{U_k}^{(1)})
\]
for \( i = 0, 1 \). Since the bundle \( \pi^* \mathcal{E}_k \) is generated by eigenvectors for the connection, we deduce that these isomorphisms extend to
\[
H^i_{dR}(V_k)(\mathcal{L}^{(1)}_{U_k}) \cong H^i_{\text{Higgs}}(T_k)(\mathcal{L}^{(1)}_{U_k})
\]
thus proving the result for \( \pi^*(\mathcal{E}_k) \); the case of \( \mathcal{E}_k \) follows by descent. \( \square \)

Now we can give the

**Proof.** (of theorem 46) We claim that
\[
H^i_{\text{Higgs}}(\mathcal{F}^{(1)}_k) = \pi_*^{(1)}(O_{\mathcal{L}^{(1)}_k} \otimes \Omega^i_{\mathcal{L}_k})
\]
Over \( X_k \), this is a standard identity for Higgs bundles, namely:
\[
H^i_{\text{Higgs}}(\mathcal{F}^{(1)}_k) \cong \text{Hom}^i(\mathcal{H}_k, \mathcal{H}_k) = \pi_*^{(1)}(O_{\mathcal{L}_k} \otimes \Omega^i_{\mathcal{L}_k})
\]
It extends over \( \{\infty\} \) by the definition of the cohomology \( H^i_{\text{Higgs}}(\mathcal{F}^{(1)}_k) \); namely, it is the descent to \( \overline{X}_k \) of a sheaf on \( \mathcal{L}^{(1)}_k \) which by construction is
\[
\pi_*^{(1)}(O_{\mathcal{L}^{(1)}_k} \otimes \Omega^i_{\mathcal{L}^{(1)}_k} \times \mathcal{L}^{(1)}_k)
\]
in a neighborhood of \( \{\infty\} \) (here \( \pi \) is the induced projection \( \mathcal{L}^{(1)}_k \times \mathcal{L}^{(1)}_k \to \mathcal{L}^{(1)}_k \)).

Therefore we have that \( \dim H^j(\mathcal{H}^i_{\text{Higgs}}(\mathcal{F}^{(1)}_k)) = 1 \) if \( i + j = 0 \) or \( i + j = 2 \) and 0 for \( i + j = 1 \). This implies the same for \( H^1_{dR}(\mathcal{E}_k) \); which implies the degeneration of the hypercohomology spectral sequence, and therefore the uniqueness of lifts. Since the trace map becomes multiplication by \( r = \dim(M_k) \) (which is nonzero since \( p \nmid r \)), the isomorphism of the theorem follows. The fact about the obstruction class follows from \( \bigwedge^r M_k \cong (e^\mathcal{F}) \). \( \square \)

Now let us note that theorem 41 is an immediate consequence; since the connection \( (e^\mathcal{F}) \) possesses a unique lift for each \( n \) (by its very construction); we see that the same is true for \( \overline{\mathcal{M}}_n \), which is the content of the theorem.

### 4.2. Algebrization

In this section, we shall examine the inverse limit of the connections constructed above and show that the resulting formal object can be algebrized in characteristic zero. We start by briefly recalling the general results on algebrization of formal objects in the context that we shall need them; to this end; we refer to Illusie’s article [Il] for all definitions and details. So, let \( \mathcal{X} \) be a smooth projective formal scheme over the Witt ring \( W(k) \), where \( k \) is a perfect field of positive characteristic. We suppose that \( \mathcal{D} \) is a divisor in \( \mathcal{X} \). Then we have Grothendieck’s existence theorem in this context (c.f. [Il]):

**Theorem 48.** There exists a unique smooth projective scheme \( X \) over \( W(k) \), and a divisor \( \mathcal{D} \) in \( X \), such that the formal completions \( \hat{X} = \mathcal{X} \) and \( \hat{\mathcal{D}} = \mathcal{D} \). Furthermore the functor of completion
\[
\text{Coh}(X) \to \text{Coh}(\mathcal{X})
\]
from coherent sheaves on \( X \) to those on \( \mathcal{X} \) is an equivalence of categories.
In our case the formal scheme $\mathcal{X}$ will simply be the projective line over $W(k)$ (as a formal scheme), and so $X$ will be the projective line over $W(k)$ (as a scheme), and the divisor $D$ will be the complement of a standard affine space.

We wish to bring connections into the picture. Suppose $F$ is a bundle with meromorphic connection on the formal scheme $\mathcal{X}$. In this situation we have the

**Corollary 49.** Let $M$ be the unique algebraic coherent sheaf with $\tilde{M}=F$. This coherent sheaf is equipped with a unique meromorphic connection $\nabla$ whose completion agrees with the meromorphic connection on $F$.

This is proved exactly as in the complex situation (c.f., e.g. [Ma], theorem 5.1).

Now we apply this in the situation of the previous sections: taking the inverse limit of meromorphic connections $\mathcal{M}_n$ from theorem 41 yield a formal meromorphic connection $\tilde{M}$ on $\mathbb{P}^l$, which, by the previous corollary, is the completion of a unique algebraic meromorphic connection $\tilde{M}$ on $\mathbb{P}^l$.

Now let us note in addition the

**Lemma 50.** Let $\pi: \tilde{X}_{W(k)} \to X_{W(k)}$ be the usual projection. Consider the completion of $\tilde{X}_{W(k)}$ at the divisor $\{\infty\}$. The meromorphic connection $(\pi^*\tilde{M}_{W(k)})(\infty)$ admits an eigenbasis for the connection $\{f_i\}$ in which $g_i = \Theta_i(f_i) \mod(p)$. Therefore there exists a rational function of the form

$$r(t) = c_k t^{-r_k} + c_{k-1} t^{-r(k-1)} + \ldots$$

(where $t$ is a local coordinate at $\{\infty\}$) so that the connection $\nabla - r(t) \text{Id} \otimes dt$ has its most negative term (in with respect to the valuation of $t$) has $r$ distinct eigenvectors.

Now consider the connection $\nabla' = \nabla - r(t) \text{Id}$. In the chosen basis $\{f_i\}$ write

$$\nabla' = t^{-k} A_l + t^{l+1} A_{l+1} + \ldots$$

By construction we have that $A_l \mod(p)$ is invertible with distinct eigenvalues. Thus the same is true for $A_l$ over $W(k)$. But then we may apply the "spectral splitting lemma" ([BV], lemma 6.2.1)\footnote{The proof given there is for a field, but it applies mutatis mutandis to $W(k)$; c.f. also [BV], section 4.3} to see that the formal completion $(\pi^*\tilde{M})(\infty)$ is a direct sum of one-dimensional meromorphic connections. We choose a compatible basis with the splitting, which by abuse we call also $\{f_i\}$.

Taking the reduction mod $p^n$ for any $n \geq 1$, we get a basis, also called $\{f_i\}$, for $(\pi^*\tilde{M})(\infty) \otimes_{W(k)} W(k)/p^n$, which is an eigenbasis for $\nabla$. We know further by the $\Theta$-regularity condition that there exists an eigenbasis $\{e_i\}$ such that $(\nabla - \Theta_i) e_i$ has no poles at $\{\infty\}$. Thus, we have that the $\{e_i\}$ and $\{f_i\}$ are two basis for the given meromorphic extension of $\pi^*\mathcal{M}_n$ on which $\nabla$ acts diagonally (with distinct eigenvalues); therefore they must be equal up to rescaling by invertible elements. Thus we see that, in each $\pi^*\mathcal{M}_n$, we have the inclusion $(\nabla - \Theta_i)(f_i) \subset (\pi^*\mathcal{M}_n)(\infty)$; the result follows.
Finally, putting it all together and inverting the prime $p$, we have the:

**Theorem 51.** The vector bundle with connection $\hat{M}$ is algebrizable. After inverting the prime $p$, the resulting sheaf is extendable to a vector bundle with meromorphic connection on $\mathbb{P}^1_K (K = \operatorname{Frac}(W(k)))$ which is singular only at the divisor $\{\infty\}$; furthermore this connection is the $\Theta$-regular at infinity.

4.3. **Rigidity.** In this section, we show that the algebraic vector bundle with meromorphic connection constructed in the previous section is a rigid $\mathcal{D}$-module in a suitable sense; furthermore, we determine the “formal structure at infinity” of this object. Then we make use of the theory due originally to Katz [Ka2], in the regular case, and then developed in the irregular case by Bloch-Esnault and Arinkin (c.f. [A], [BE]) of rigid irreducible connections on the affine line to show that the connection of theorem 51 is the same, for every field $K = \operatorname{Frac}(W(k))$ with $\operatorname{char}(k) > 0$.

We begin by recalling a few definitions from the theory of rigid irreducible connections. The case of interest to us is that of a geometrically irreducible algebraic connection $M_K$ over $\mathbb{A}^1_K$ where $K$ is a field of characteristic zero. If $l : \mathbb{A}^1_K \to \mathbb{P}^1_K$ denotes the inclusion, then we have the geometrically irreducible $\mathcal{D}_{\mathbb{P}^1_K}$-module $l_*(M_K)$. We also have the $\mathcal{D}_{\mathbb{P}^1_K}$-module $l_*(\mathcal{E}_K)$ where as above $\mathcal{E}_K = \mathcal{E}nd(M_K)$. We recall the

**Definition 52.** The geometrically irreducible $\mathcal{D}_{\mathbb{A}^1_K}$-module $M_K$ is said to be rigid if the Euler characteristic

$$\chi(l_*(\mathcal{E}_K)) = 2$$

where for any holonomic $\mathcal{D}_{\mathbb{P}^1_K}$-module $N_K$ we have

$$\chi(N_K) := h^0_{dR}(N_K) + h^2_{dR}(N_K) - h^1_{dR}(N_K)$$

By Poincare duality, and the irreducibility of $M_K$, this condition is equivalent to $h^1_{dR}(l_*(\mathcal{E}_K)) = 0$.

The main result about rigid $\mathcal{D}_{\mathbb{A}^1_K}$-modules is the equivalence of the above definition with another characterization; recall that the formal type at infinity of a holonomic $\mathcal{D}_{\mathbb{A}^1_K}$-module $N_K$ denotes the isomorphism class of the formal completion of $N_K$ at the point at infinity; this is naturally a coherent $\mathcal{D}_{K((z))}$-module. Then we have the

**Theorem 53.** (Katz, Bloch-Esnault) Let $M_K$ be a geometrically irreducible connection on $\mathbb{A}^1_K$. Then $M_K$ is rigid iff, for any other geometrically irreducible connection $N_K$ on $\mathbb{A}^1_K$ with the same formal type at infinity, we have $M_K \cong N_K$ on $\mathbb{A}^1_K$.

This was first proved for regular connections in chapter 1 of [Ka2], and then for irregular connections in section 4 of [BE]. This is the result that we shall use to glue our different lifts of $M_K$ in characteristic zero. From now on we let $M_K$ be as in theorem 51. Our first concern is calculating the Euler characteristic of $l_*(\mathcal{E}_K)$. In fact we have the

**Lemma 54.** $\chi(l_*(\mathcal{E}_K)) = \chi(l_*(\mathcal{E}_K)) + 1$

**Proof.** This follows from formal local analysis of the connections at $\{\infty\}$; namely, we have the inclusion $l_*(\mathcal{E}_K) \subset l_*(\mathcal{E}_K)$. The cokernel of this map a $\mathcal{D}$-module supported at $\{\infty\}$, which is therefore a finite direct sum of simple modules of
the form $\delta_\infty$. The number of terms in this sum is the rank of the vector space $H^1_{dR}(E_\infty)$. Thus we must show that this rank is 1.

First we consider $(\pi^* M_K)_\{\infty\}$. We know that this is a direct sum of $r$ rank one pairwise distinct connections. The space of (continuous) endomorphisms of $(\pi^* M_K)_\{\infty\}$ which preserve the connection is therefore a rank $r$ vector space.

Now, by construction Galois group $G_r$ acts on $(\pi^* M_K)_\{\infty\}$ by shifting the rank one summands; i.e., if $\sigma$ is a generator of $G_r$ we have $\sigma f_i = f_{i+1}$ where the indices are taken mod $r$. Thus the space of endomorphisms which preserve this action is a one-dimensional vector space.

Thus the fact that $H^1_{dR}(E_\infty) = H^1_{dR}(\pi^* E_\infty)^{G_r}$ yields the result. \qed

In order to make use of this lemma, we shall use the results of the previous section to compute $\chi(l_*(E_K))$. We have constructed a complex $\nabla : E_K \to E_k \otimes \Omega^1_{X_k}$; by our theory this is liftable to $K$; namely, $E_K$ has been constructed as the main result of the previous section. Further, by lemma 50, we recall that the formal meromorphic connection $\pi^* (E_K)_\{\infty\}$ admits a direct sum into an $r$-dimensional trivial piece and an $r^2 - r$ dimensional complement. Thus we may construct $E_K \otimes \Omega^1_{X_K}$ exactly as it was done in the discussion above lemma 44.

Now recall from [BE], section 3, that Deligne has given a theory of good lattices for meromorphic connections, which allows one to compute the deRham cohomology of the various extensions of the connection over the complement of a point. We use this theory to show:

**Lemma 55.** The Euler characteristic of the complex $\nabla : E_K \to E_K \otimes \Omega^1_{X_K}$ is equal to the Euler characteristic of $l_*(E_K)$.

**Proof.** We first consider $\pi^*(E_K)$. We work as above over the formal completion at $\{\infty\}$. There we have the splitting $\pi^*(E_K) = \tilde{V}_K \oplus \tilde{N}_K$ (with $\tilde{V}_K$ the trivial part). To this we associate the complex

\[(4.2) \quad \nabla : \tilde{V}_K \oplus \tilde{N}_K \to \tilde{V}_K(D_\infty) \oplus \tilde{N}_K((r + 1)D_\infty)\]

where $D_\infty$ is the reduced divisor at $\{\infty\}$; note that $\nabla : \tilde{N}_K \to \tilde{N}_K((r + 1)D_\infty)$ is surjective.

Recall from (the proof of) [BE], lemma 3.3, that the hypercohomology of this complex is equal to the deRham cohomology of $l_*(\pi^* E_K)$ (where $l : L_K \to E_K$). The complex we are interested in is

\[(4.3) \quad \nabla : \tilde{V}_K \oplus \tilde{N}_K \to \tilde{V}_K \oplus \tilde{N}_K(kD_\infty)\]

So that there is an inclusion of complex 4.3 into complex 4.2; the cone of this inclusion is the skyscraper sheaf of rank $r$ at $\{\infty\}$.

On the other hand, by construction both of these complexes (and the inclusion map) admit descent along the map $\pi$, the cokernel of the resulting inclusion is one dimensional (by the argument of lemma 54); and the hypercohomology of the descent of equation (4.2) is the deRham cohomology of $l_*(E_K)$ (again by the argument of [BE], lemma 3.3). Now the result follows from the previous lemma. \qed

Therefore the results of the previous section imply the
**Proposition 56.** For each prime $p$, the connection $M_K$ (over $K = \text{Frac}(W(k))$) is rigid.

*Proof.* By theorem 46 we have the the Euler characteristic of $\nabla : \mathcal{E}_K \to \mathcal{E}_K \otimes \Omega^1_{X_K}$ is 2. Since, by construction, $\nabla : \mathcal{E}_K \to \mathcal{E}_K \otimes \Omega^1_{X_K}$ is the base change to the generic fibre of a complex defined over $W(k)$ whose reduction mod $p$ is equal to $\nabla : \mathcal{E}_k \to \mathcal{E}_k \otimes \Omega^1_{X_k}$, the previous lemma and invariance of Euler characteristic imply $\chi(h^*(\mathcal{E}_K)) = 2$.

Now we must show that $M_K$ is geometrically irreducible. After replacing the base field $K$ by a finite extension if necessary, suppose there is a subconnection, $M'_K$. Since $M_K$ has an $A$-model (where $A$ is a dvr inside $K$ whose special fibre is a perfect field $k$ of positive characteristic) $M_A$, we see that $M'_A := M_A \cap M'_A$ is a sub $\mathcal{D}_{X_A}$-module of $M_A$; the quotient is a connection over $X_A$ as well. Thus the base change to the special fibre $M_k$ has a sub and quotient connection as well. But this is clearly impossible since $M_k$ has $p$-support $\mathcal{L}^{(1)}_k$ with multiplicity 1. \(\square\)

Finally, we wish to consider which rigid connections on $\mathbb{A}^1_K$ have a formal type at $\{\infty\}$ which is compatible with the condition of $\Theta$-regularity. In fact we have the

**Lemma 57.** There is exactly one geometrically irreducible connection on $\mathbb{A}^1_K$ which has an extension to a meromorphic connection which is $\Theta$-regular at $\{\infty\}$.

*Proof.* The condition of being $\Theta$-regular at $\{\infty\}$ characterizes the formal type $\pi^*M_K$, but not necessarily that of $M_K$. The ambiguity is as follows: if the summands of $(\pi^*M_K)_{\{\infty\}}$ are of the form

$$\nabla(e_i) = \tilde{\theta}_i e_i$$

(for some function $\tilde{\theta}_i$) then if we modify the action by

$$\nabla(e_i) = (\tilde{\theta}_i + a_i z^{-1}) e_i$$

in such a way that $a_i = \xi^{i-1}a_1$ (for $i \in \{1, \ldots, r\}$) then this will descend to another formal connection whose pullback by $\pi$ has the same formal type. This means that we may tensor the connection $M_K$ by the line bundle with connection $z^{-1}$ on $G_a$. This modification is never a connection at $\{0\}$ unless $a = 0$. Thus any other geometrically irreducible connection on $\mathbb{A}^1_K$ which has an extension which is $\Theta$-regular at $\{\infty\}$ has the same formal type at $\{\infty\}$ as the connection $M_K$; by rigidity they are isomorphic. \(\square\)

4.4. **Uniqueness.** In this section, we use Galois descent to construct a connection over $F = \text{Frac}(R)$ which agrees, after base change, with the rigid connections constructed in the previous sections. We deduce that this connection has constant $p$-support equal to $\mathcal{L}_k$ (with multiplicity 1) and is the unique such connection to have this property.

The main claim is the following descent argument for rigid local systems:

**Theorem 58.** Suppose $K \subset L$ is a field extension of fields of characteristic zero. Suppose $M_L$ is a rigid connection on $\mathbb{A}^1_L$ whose formal type at infinity is induced by base change from a formal connection over $K((z))$. Then there exists a unique geometrically irreducible connection $M_K$ over $\mathbb{A}^1_K$ such that $M_K \otimes_K L = M_L$. 

Proof. Since $M_L$ is a finite type $\mathcal{D}$-module, we may choose a subfield $K \subset L' \subset L$ over which $M_L$ is defined such that $K \subset L'$ is a finite type field extension.

Thus we may in fact assume $K \subset L$ is generated by a single element. For any $g \in \text{Gal}(L/K)$, we can consider the induced automorphism of $\mathbb{A}_1^1$ which we shall also denote by $g$. Since $g^*M_L$ is a rigid connection whose formal type at infinity is, by assumption, the same as that of $M_L$, we see that there is an isomorphism $\lambda_g : M_L \to g^*M_L$. Thus for any two elements $g, h$, we get an element of $\text{End}(M_L)$ given by $g^*\lambda_h \circ \lambda_g \circ \lambda_{gh}^{-1}$. This is a two-cocycle for the group $G$ with values in the representation $L$, and so by Hilbert’s theorem 90 it is the differential of a one-cocycle $\beta : G \to L$. But then, we may modify the $\lambda_g$ and consider the isomorphisms $\lambda_g : \beta(g)^{-1}$; these then define an action of $G$ on the bundle $M_L$; and we deduce that $M'_L := M_K$ is a bundle over $\mathbb{A}_1^1$ such that $M_K \otimes_K L = M_L$; the uniqueness follows since $M_K$ is rigid.

We deduce directly the:

Corollary 59. There exists a unique geometrically irreducible connection $M_F$ over $F = \text{Frac}(R)$ such that, for each $K = \text{Frac}(W(k))$, we have $M_K = K \otimes_F M_F$.

Finally, we must complete the circle by looking at the $p$-curvature of $M_F$. Extending $R$ if necessary, we may assume that $M$ is defined over $R$ by a model $\tilde{M}_R$. For each $p$ unramified in $R$, we have the inclusion $R \subset W(k)$ (where $k$ is the perfect closure of $R/pR$). Thus the connection

$$\tilde{M}_{W(k)} := W(k) \otimes_R \tilde{M}_R$$

is a $W(k)$-model for the connection $M_K$ constructed above. So we only need to recall the:

Lemma 60. The connection $\tilde{M}_{W(k)}/p$ has $p$-support $L^{(1)}_k$ with multiplicity 1.

Proof. Since $W(k)$ is a dvr with uniformizing parameter $p$ we may suppose that $\tilde{M}_{W(k)} \subset M_{W(k)} \subset p^{-1}M_{W(k)}$. We have the exact sequence

$$0 \to \tilde{M}_{W(k)}/pM_{W(k)} \to M_{W(k)}/pM_{W(k)} \to M_{W(k)}/\tilde{M}_{W(k)} \to 0$$

and since $F_*(M_{W(k)}/pM_{W(k)})$ is a vector bundle on $L^{(1)}_k$ which is a splitting bundle for $D|_{L^{(1)}_k}$, we see that the inclusion is generically an isomorphism and the quotient is supported on a finite subset of $L^{(1)}_k$. We also have

$$0 \to \tilde{M}_{W(k)}/pM_{W(k)} \to \tilde{M}_{W(k)}/p\tilde{M}_{W(k)} \to M_{W(k)}/\tilde{M}_{W(k)} \to 0$$

Which shows that $\tilde{M}_{W(k)}/p\tilde{M}_{W(k)}$ generically a bundle on $L^{(1)}_k$; since it is also a bundle on $X^{(1)}_k$ it follows that it is a bundle on $L^{(1)}_k$ and therefore that its $p$-support is exactly $L^{(1)}_k$ with multiplicity 1.

Let us point out now that theorem 1 has been proved. Certainly we have constructed a connection of the required type. To see the uniqueness, let $N_K$ be any other such. Then it is geometrically irreducible by lemma 21. Furthermore, one shows (as in section 3.2):

Lemma 61. $N_K$ admits an extension to a meromorphic connection which is $\Theta$-regular at $\{\infty\}$.

Therefore the uniqueness part of the theorem follows from lemma 57.
4.5. **Abhyankar-Moh Theorem.** In this section we make some brief remarks about the relation of the \(m=1\) case of the theorem 1 and the Abhyankar-Moh theorem about embeddings of \(\mathbb{A}^1_C\) into \(\mathbb{A}^2_C\). As it turns out, the two theorems are essentially equivalent. To see why, we recall (one version of) the statement of the theorem: let \(\{x, y\}\) be standard coordinates on \(\mathbb{A}^2_C\). Then:

**Theorem 62. (Abhyankar-Moh)** Consider any embedding \(i: \mathbb{A}^1_C \to \mathbb{A}^2_C\). Then there exists an automorphism \(a\) of \(\mathbb{A}^2_C\) such that \(a \circ i\) is the standard embedding of the \(x\)-axis. We can choose \(a\) so that the Jacobian \(J(a) = 1\).

This is essentially ([AM] theorem 1.6); we remark that the claim about choosing \(J(a) = 1\) is not mentioned there; however, if one has found an \(a\) as in the statement of the first sentence of the theorem, then \(J(a) = c\) is necessarily a constant function on \(\mathbb{A}^2_C\). Multiplying \(a\) on the right by the transformation given by \(y \to y\) and \(x \to c^{-1}x\) yields an automorphism with \(J = 1\) whose action on the \(x\)-axis is the same as that of \(a\).

Further, we assume throughout the rest of this section that the image of \(i\), called \(\mathcal{L}_C\), has a dominant projection to the \(x\)-axis \(\mathbb{A}^1_C\). If not, the image of \(i\) is simply a line in \(\mathbb{A}^2_C\) for which the theorem is easy.

To see why this applies to our situation, we recall that there is a natural map

\[
\text{Aut}(D_{\mathbb{A}^m_C}) \to \text{Aut}_{\text{Symp}}(\mathbb{A}^{2m}_C)
\]

constructed in [ML] (for \(m = 1\)) and then [Ts] and [BKKo] in general (see section §6 for more on this). Here, the group on the right is the group of algebraic symplectomorphisms of \(\mathbb{A}^{2m}_C\) when equipped with the standard symplectic form.

In the case \(m = 1\), this map is known to be an automorphism from [ML]; in addition, in this case, the group on the right is simply the group of automorphisms of \(\mathbb{A}^2_C\) whose Jacobian is equal to 1. Furthermore, in this case the construction of the inverse map

\[
\text{Aut}_{\text{Symp}}(\mathbb{A}^2_C) \to \text{Aut}(D_{\mathbb{A}^1_C})
\]

can be done quite explicitly; namely, it is known\(^7\) that the group \(\text{Aut}_{\text{Symp}}(\mathbb{A}^2_C)\) is generated by \(\text{SL}_2(C)\) and transformations of the form

\[
\begin{cases}
  x \to x \\
  y \to y + f(x)
\end{cases}
\]

where \(f(x)\) is an arbitrary polynomial. One sees quite directly that both \(\text{SL}_2(C)\) and all of these elements act on \(D_{\mathbb{A}^1_C}\) by replacing \(y\) with \(\frac{d}{dx}\) everywhere in the formulas; after checking some relations this yields the required map. Furthermore, one may see quite directly that, given an automorphism \(a\) of \(\mathbb{A}^2_C\) which goes to \(a' \in \text{Aut}(D_{\mathbb{A}^1_C})\), after reduction mod \(p\), \(a'\) acts on \(Z(D_{\mathbb{A}^1_C}) \cong k[x^p, (\frac{d}{dx})^p]\) by \(a^{(1)}\).

Indeed, this is the key to the proof of the automorphism given in [ML].

So, given a curve \(\mathcal{L}_C\) isomorphic to \(\mathbb{A}^1_C\) inside \(\mathbb{A}^2_C\), the element \(a\) from the previous theorem may be used to construct a \(\mathcal{D}\)-module: namely, we take \(a^*O_X\), which, by construction, will have constant arithmetic support equal to \(\mathcal{L}_C\).

On the other hand, the one dimensional case of theorem 1, combined with the main theorem of Arinkin’s paper [A], may also be used to reprove the Abhyankar-Moh theorem. Namely, Arinkin extends Katz’s constructive algorithm for rigid

\(^7\)Due, I believe, to Dixmier and reproved, e.g., in [ML]
irreducible connections to the irregular case, thus yielding a finite procedure which constructs \( M_C \) from \( O_X \). Let us recall corollary 2.5 of that paper:

**Theorem 63.** Let \( M_C \) be a rigid irreducible connection on some open subset \( U_C \subset \mathbb{P}^1_C \). Then \( M_C \) can be constructed out of \( O_{\mathbb{A}^1_C} \) by a sequence of operations of the following types:

- a) Tensor by a rank one connection.
- b) Pullback by an automorphism of \( \mathbb{P}^1_C \).
- c) Fourier transform.

Because \( M_C \) has no singularities except at \( \{ \infty \} \), the only allowable rank one local systems have no singularities except at \( \{ \infty \} \), and hence are of the form \( e^f \) for some \( f \in O(\mathbb{A}^1_C) \). Similarly, the only allowable automorphisms of \( \mathbb{P}^1_C \) are those which fix \( \{ \infty \} \); i.e., the multiplication by a constant.

Now let us consider the effect on the arithmetic support of each of the three operations in the corollary. So, suppose we have a \( \mathcal{D} \)-module \( N_C \) with constant arithmetic support \( L_C \). Tensoring by \( e^f \) corresponds to the operation \( L_C \rightarrow L_C + df \)

Now consider the automorphism \( a_c \) defined by multiplication by \( c \) on \( \mathbb{A}^2_C \). Let \( \tilde{a}_C \) be the automorphism of \( \mathbb{A}^2_C \) defined by \( x \rightarrow cx \) and \( y \rightarrow c^{-1}y \). Then the pullback by \( a_c \) corresponds to the operation \( L_C \rightarrow \tilde{a}_C^*(L_C) \)

Finally, the Fourier transform corresponds to \( L_C \rightarrow r^*(L_C) \) where \( r \) is the automorphism \( x \rightarrow y, y \rightarrow -x \) of \( \mathbb{A}^2_C \). Each of these three operations, therefore, moves the arithmetic support by the action of an automorphism of \( \mathbb{A}^2_C \) (and it follows from the previous discussion that these elements generate the automorphism group). Therefore, our proof of the one-dimensional case of theorem 1 combined with Arinkin’s theorem, reproves the Abhyankar-Moh theorem.

Of course, this proof of the theorem is vastly more complicated than the original. However, the main ingredients will generalize to higher dimensions, allowing us to do much more in that case. This is what we shall turn to in the next section.

5. The Higher Dimensional Case

In this section, we turn to the higher dimensional case of theorem 1. In particular, we carry out the outline for this case given in the introduction.

So we suppose that \( X_C \) is a smooth affine variety and \( L_C \subset T^*X_C \) is such that \( \pi : L_C \rightarrow X_C \) is dominant. As in 26, we have \( U_C \subset X_C \) so that \( L_{U_C} \rightarrow U_C \) is finite etale, and we have the \( \mathcal{D}_{U_C} \)-module \( \pi_*(e^f) \). After reduction mod \( p \), this yields \( \pi_*(e^f) \) on \( U_k \) which is a splitting bundle for \( D_{(U_{t})_{U}}L_{U_k} \). Exactly as in 23 we have that \( \pi_*(e^f) \) admits a unique extension to a coherent \( \mathcal{D}_{X_k} \)-module \( M_k \) such that \( F_*(M_k) \), as a coherent sheaf on \( T^*X^{(1)}_k \), is a vector bundle on \( L^{(1)}_k \).

Unlike in the dimension 1 case, however, it might not be true that this \( M_k \) is a bundle on \( X_k \). This makes the lifting problem more difficult to control. If \( D_k = X_k \setminus U_k \), we should like to keep track of the “asymptotic behavior” of the connection as it degenerates at \( D \). To accomplish this, we use an old method in \( \mathcal{D} \)-module theory, going back to the origins of the subject:
Let \( \varphi : \tilde{X}_C \rightarrow X_C \) be a projective morphism so that the inverse image \( \varphi^{-1}(D_C) := \tilde{D}_C \) is a global normal crossings divisor, and \( \tilde{X}_C \) is smooth. Such morphisms exist according to Hironaka’s theorem. In addition, we can further compactify \( \tilde{X}_C \) to \( \bar{X}_C \), such that \( \bar{X}_C \backslash \tilde{X}_C \) is also a normal crossings divisor, whose union with \( \tilde{D}_C \), denoted \( \bar{E}_C \), is normal crossings as well.

Now, instead of considering \( M_k \) on \( X_k \), we look instead at extending \( \varphi^*(M_k|_{U_k}) \) (which is “equal” to \( M|_{U_k} \) since \( \varphi \) is an isomorphism on \( U_k \)) on \( \tilde{X}_k \). As before, we shall consider extensions of \( M|_{U_k} \) to meromorphic connections on \( \bar{X}_C \). To do this, we must first look at the \( p \)-supports. In the first subsection, we shall show

**Theorem 64.** Let \( \varphi : \tilde{X}_C \rightarrow X_C \) and \( \tilde{X}_C \subset \bar{X}_C \) be as above. Then there exists a normal compactification \( \bar{\mathcal{L}}_C \) of \( \tilde{\mathcal{L}}_C \) so that

1) The finite etale map \( \pi : \mathcal{L}_{U_C} \rightarrow U_C \) extends to a finite flat map \( \bar{\pi} : \bar{\mathcal{L}}_C \rightarrow \bar{X}_C \).

2) We have \( H^1_{\text{sing}}(\bar{\mathcal{L}}_C; \mathbb{Z}) = 0 \). In addition, we have\(^8\) \( H^0(\Omega^2_{\bar{\mathcal{L}}_C}) = 0 = H^2(\Omega^2_{\bar{\mathcal{L}}_C}) \).

As in the previous chapter, we shall then proceed to construct a family of meromorphic Higgs bundles on \( \bar{X}_C \) which extends the Higgs bundle associated to \( \mathcal{L}_{U_C} \subset T^*U_C \). The set of such bundles is indexed by \( \text{Pic}(\bar{\mathcal{L}}_C) \); we denote a member of this set by \( \bar{\mathcal{M}}_C \). Each \( \bar{\mathcal{M}}_C \) gives rise to a meromorphic Higgs bundle, \( \mathcal{F}_C \), which extends the bundle \( \mathcal{End}(\mathcal{H}_{U_C}) \). We then define an appropriate notion of Higgs complex for bundles of the type \( \bar{\mathcal{M}}_C \) or \( \mathcal{F}_C \). We have the

**Proposition 65.** We have isomorphisms \( \mathbb{H}^1_H(\mathcal{F}_C) \cong \mathbb{H}^1_H(\mathcal{O}_{\bar{\mathcal{L}}_C}) \cong \mathbb{H}^1_dH(\mathcal{O}_{\bar{\mathcal{L}}_C}) \).

Taking reduction mod \( p \), we obtain the analogous results over \( k \). After this, we can form a bundle with meromorphic \( \lambda \)-connection, denoted \( \bar{\mathcal{M}}_{\lambda,k} \), so that \( \bar{\mathcal{M}}_{\lambda,k}/\lambda \cong \mathcal{F}_C \), and whose \( p \)-support satisfies certain conditions. We have analogues of all of the theorems for \( \mathcal{M}_k \) in the one-dimensional case; in particular, we show that \( \bar{\mathcal{M}}_{\lambda,k} \) satisfies strong cohomology vanishing assumptions; indeed, using the previous proposition, we show that the appropriate \( \mathbb{H}^1 \) vanishes. We then use our general lifting theory to show that there exists a unique lift of this object to a \( \lambda \)-connection \( \bar{\mathcal{M}}_{\lambda,k} \), where now \( \lambda = \text{Frac}(W(k)) \). Then we show that there is a rigidity theory for such objects, i.e., we give a uniqueness theorem analogous to lemma 57, and then we can argue as in the previous section to obtain a unique \( \bar{\mathcal{M}}_F \) (where \( F = \text{Frac}(R) \)).

Finally we finish the proof of theorem 1 by studying the pushforward \( \int_{\varphi} (j_\ast M_F) \) of the irreducible holonomic \( D \)-module \( j_\ast (M_F) \) which extends to \( \bar{X}_F \) the connection \( M_F \).

### 5.1. The variety \( \mathcal{L}_C \) and the Higgs bundle \( \mathcal{H}_C \)

As indicated above, the first step in the argument is to invoke resolution of singularities: we choose a birational projective map \( \varphi : \tilde{X}_C \rightarrow X_C \) so that \( \varphi^{-1}(D_C) \) is a global normal crossings divisor in \( \tilde{X}_C \). The map \( \varphi \) is an isomorphism over \( U_C \), and we shall abuse notation by writing \( U_C \subset \tilde{X}_C \) for \( \varphi^{-1}(U_C) \). This implies that \( T^*U_C \) is an open dense subset inside \( T^*\tilde{X}_C \).

\(^8\)For the definition of differential forms on a normal variety that we use, c.f. the discussion above 71 below
Recall that we are given \( \mathcal{L}_C \subset T^*X_C \). Then we first consider the variety \( \mathcal{L}^d_C \subset X_C \times_{X_C} T^*X_C \) which we define as follows: the variety \( \tilde{X}_C \) is constructed out of \( X_C \) by a finite sequence of blowups along smooth subvarieties \([W]\). Since the cotangent bundle of a variety is smooth over the variety, we have that \( \tilde{X}_C \times_{X_C} T^*X_C \) is constructed out of \( T^*X_C \) by a finite sequence of blowups along smooth subvarieties. Then we define \( \mathcal{L}^d_C \) to be the strict transform of \( \mathcal{L}_C \) inside \( \tilde{X}_C \times_{X_C} T^*X_C \). Then we have:

**Lemma 66.** The variety \( \mathcal{L}^d_C \) is the blowup of \( \mathcal{L}_C \) along an ideal sheaf; hence it is a reduced irreducible variety. The morphism \( \mathcal{L}^d_C \to \mathcal{L}_C \) is projective. Since \( \mathcal{L}_C \) is smooth, there is an open subset \( V_C \subset L_C \) whose complement has codimension \( \geq 2 \) on which the map \( \mathcal{L}^d_C \to \mathcal{L}_C \) is an isomorphism.

This follows immediately from the general properties of blowups proved, e.g., in \([H]\), section 2.7; the only possible exception being the last sentence. To see this, observe that the blowup of a smooth variety along an ideal whose support is a subvariety of codimension 1 is generically an isomorphism along that support; this is because the blowup of an ideal generated by a single element is the identity map.

In fact, it is worth noting that this holds in greater generality: if \( Z_C \to Y_C \) is any projective birational morphism of varieties where \( Y_C \) is smooth, there is an open subset of \( Y_C \), whose compliment has codimension \( \geq 2 \), on which the map is an isomorphism. This follows by the same logic, noting that any such morphism is the blowup of some ideal sheaf (by \([H]\), theorem 7.7).

Now, since \( \mathcal{L}_C \) is a smooth variety with \( H_1^{\text{sing}}(L_C; \mathbb{Z}) = 0 \), the fact that \( V_C \) has a compliment of codimension \( \geq 2 \) implies that \( H_1^{\text{sing}}(V_C; \mathbb{Z}) = 0 \). In fact, this property turns out to almost be a birational invariant; the catch is that it is so only for normal projective varieties.

Thus we turn to proving part of theorem 64. Recall that \( \mathcal{L}_{U_C} \subset T^*U_C \) is the graph of the differential of the “multi-valued function” \( f \in O(L_{U_C}) \). Such a function may be analyzed using Abhyankar’s Theorem: to state this result, consider a point \( x \in \tilde{D}_C = \bigcup_{i \in J} Y_{i,C} \), where \( \{Y_{i,C}\} \) are the components of the normal crossings divisor \( \tilde{E}_C \). Suppose \( x \in \bigcap_{i \in J} Y_{i,C} \) for some index set \( J \). We choose local coordinates \( \{x_1, \ldots, x_m\} \) in an open affine neighborhood \( \mathcal{N}_x,C \) of \( x \) so that \( Y_{i,C} = \{x_i = 0\} \) in \( \mathcal{N}_x,C \).

For any integer \( l > 0 \), define the affine scheme \( \mathcal{N}_x,C^{(l)} \) as the Spec of \( O(\mathcal{N}_x,C)[y_i]_{i \in J}/(y_i - x_i^l) \). Then we have finite flat morphism \( p_l : \mathcal{N}_x,C^{(l)} \to \mathcal{N}_x,C \). We shall abuse notation and denote the inverse images of \( \tilde{E}_C \) and \( U_C \) under this morphism again by \( \tilde{E}_C \) and \( U_C \). Then the theorem reads:

**Theorem 67.** (Abhyankar’s Theorem) There exists \( l > 0 \) so that the base change of \( \mathcal{L}_{U_C} \to \mathcal{N}_x,C \cap U_C \) over the map \( p_l : \mathcal{N}_x,C^{(l)} \to \mathcal{N}_x,C \) extends to a finite etale cover \( \tilde{p}_l : \mathcal{L}_C^{(l)} \to \mathcal{N}_x,C^{(l)} \).

The proof may be found in \([SGA1]\), chapter 13, appendix 1. The function \( f \in O(L_{U_C}) \) may thus be regarded as a meromorphic function on \( \mathcal{L}_C^{(l)} \), which is singular only at some components of \( \tilde{p}_l^{-1}(\tilde{D}_C) \). For later use we record some facts and
notation: the map $N_{x,l}^{(l)} \to N_{x,c}$ is a finite Galois cover with Galois group $G$ a product of cyclic groups. Let us make the

Claim 68. The $G$-action on $N_{x,l}^{(l)}$ extends to a $G$-action on $\hat{L}_{c}^{(l)}$; i.e., there is a $G$-action making the morphism $\hat{\pi}_{l}$ a $G$-equivariant map.

Proof. The open subset $\hat{L}_{U_{c}}^{(l)}$ has a $G$-action by construction. Since the variety $\hat{L}_{c}^{(l)}$ is smooth, for each component of it the ring of functions is the integral closure of the ring of functions on $N_{x,l}^{(l)}$ inside the fraction field of the given component of $\hat{L}_{c}^{(l)}$.

The existence of a $G$-action on the open dense subset $\hat{L}_{U_{c}}^{(l)}$ implies the existence of a $G$-action on the total quotient ring of $\hat{L}_{c}^{(l)}$, which restricts to the $G$-action on the ring of functions on $N_{x,l}^{(l)}$ by definition. Thus the integral closure of this ring inherits a $G$-action as well. □

So, we may consider the ring $O(\hat{L}_{c}^{(l)})^{G}$; denote the Spec of this ring by $\overline{L}_{x,c}$. Then we have the

Lemma 69. We have that $\overline{L}_{x}$ is the normalization of $N_{x,c}$ inside $K(\mathcal{L}_{U_{c}})$.\hfill □

Proof. By the preceding claim, we may identify the $G$-invariants inside the total quotient ring of $\hat{L}_{c}^{(l)}$ with $K(\mathcal{L}_{U_{c}})$. Since the normalization of $O(N_{x,l}^{(l)})$ inside the total quotient ring of $\hat{L}_{c}^{(l)}$ is $O(\hat{L}_{c}^{(l)})$, we deduce the result. □

Now we define the variety $\overline{L}_{c}$ as the normalization of $\overline{X}_{c}$ inside $K(\mathcal{L}_{U_{c}})$. By construction this is a normal projective variety equipped with a finite map $\overline{\pi}: \overline{L}_{c} \to \overline{X}_{c}$. In this case we have the

Lemma 70. The map $\overline{\pi}$ is flat. □

Proof. By construction the map $\hat{\pi}_{l}: \hat{L}_{c}^{(l)} \to N_{x,l}^{(l)}$ is finite etale, and hence flat. By lemma 69, the map $\overline{\pi}: \overline{L}_{x,c} \to N_{x,c}$ is obtained from $\hat{\pi}_{l}$ by taking $G$-invariants. Since $G$ is a finite group, the result follows. □

Thus we have proved every statement of theorem 64, except for the cohomology vanishing. To do so, we shall make use of a regularity property of $\overline{L}_{c}$ which holds by the construction:

Recall that a $V$-manifold is a topological space which may be written, locally, as the quotient of an open subset of Euclidean space by a finite group. In the context of algebraic varieties, a $V$-manifold is a variety which may be written, locally in the Zariski topology, as the quotient of a smooth variety by a finite group. A $V$-manifold is therefore automatically a normal variety. Since we obtained $\overline{L}_{x,c}$ as such a quotient, it follows that $\overline{L}_{c}$ is a $V$-manifold.

$V$-manifolds satisfy many properties of smooth varieties; to state the one relevant to us, we recall that for any normal variety $Z_{c}$, the sheaf $\Omega_{Z_{c}}^{0}$ is defined as the push-forward from the smooth locus $Z_{c}^{sm}$ of $\Omega_{Z_{c}^{sm}}^{0}$; by the normality of $Z_{c}$ this is reflexive coherent sheaf on $\Omega_{Z_{c}}^{0}$. Using the $V$-manifold property, it is not difficult to see that if $Z_{c}$ is a $V$-manifold the sheaf $\Omega_{Z_{c}}^{0}$ is actually a bundle on $Z_{c}$. Then we have:
Proposition 71. a) The Hodge decomposition holds for a projective $V$-manifold, $Z_C$:

$$H^i_{dR}(Z_C) = \bigoplus_{i+j=l} H^i(\Omega^j_{Z_C})$$

b) We have

$$H^i(\Omega^j_{Z_C}) = H^j(\Omega^i_{Z_C})$$

where $\overline{\cdot}$ is the complex conjugation on deRham cohomology.

To finish the proof of theorem 64 we must compare the cohomology of the variety $\overline{L}_C$ with that of $\mathcal{L}_C$. We have a birational morphism $\mathcal{L}_C \to \overline{L}_C$, and, since these varieties are quasi-projective, we may find a variety $\overline{L}_C$ so that $\mathcal{L}_C \to \overline{L}_C$ factors as an open immersion $\mathcal{L}_C \to \overline{L}_C$ followed by a projective morphism $\overline{L}_C \to \overline{L}_C$. Since an open immersion is birational, the morphism $\overline{L}_C \to \overline{L}_C$ is as well. Note also that we may now embed the open smooth subvariety $V_C$ into $\overline{L}_C$.

Now, applying resolution of singularities to $\overline{L}_C$, we may find a projective birational morphism $(\overline{L}_C)' \to \overline{L}_C$ which is an isomorphism on the smooth locus of $\overline{L}_C$. Thus we now have a projective birational morphism $(\overline{L}_C)' \to \overline{L}_C$, where $(\overline{L}_C)'$ is smooth and contains $V_C$ as an open subvariety. We now have the

Lemma 72. Let $Z_C$ be a normal complex quasi-projective variety.

a) Suppose there is an open subvariety $W_C$ satisfying $H^1_{sing}(W_C; \mathbb{Z}) = 0$. Then $H^1_{sing}(Z_C; \mathbb{Z}) = 0$ as well.

b) Suppose that $W_C$ admits a normal compactification $\overline{W}_C$ such that $H^0(\Omega^2_{\overline{W}_C}) = 0$. Then any normal compactification of $Z_C$, $\overline{Z}_C$, satisfies $H^0(\Omega^2_{\overline{Z}_C})$.

Proof. a) By the Riemann existence theorem and the fact that $H^1_{sing} = \pi_1/[\pi_1, \pi_1]$, the condition $H^1_{sing} = 0$ is equivalent to the statement that any finite etale cover with an abelian group of deck transformations is trivial. But on a normal variety, any etale cover whose restriction to the generic point is trivial is itself trivial.

b) The rational map $\overline{Z}_C \dashrightarrow \overline{W}_C$ is, by normality, defined outside of codimension 2 in $\overline{Z}_C$. Since $\overline{W}_C$ is projective we can, as in [H], section 2.7, resolve the indeterminacies of this map by blowing up $\overline{Z}_C$; i.e., there exists a blow-up of $\overline{Z}_C$, $\overline{Z}_C$, such that there is a morphism $\overline{Z}_C \to \overline{W}_C$ extending the given rational map. Now, arguing as in lemma 66, we have that there is an open subset $W'_C \subset \overline{W}_C$ whose compliment has codimension $\geq 2$ over which the map $\overline{Z}_C \to \overline{W}_C$ is an isomorphism. Since $\Omega^2_{\overline{W}_C}$ is a reflexive coherent sheaf on $\overline{W}_C$, we have

$$0 = H^0(\Omega^2_{\overline{Z}_C}) = H^0(\Omega^2_{\overline{W}_C})$$

On the other hand, any nonzero global section of $\Omega^2_{\overline{Z}_C}$ would restrict to a nonzero global section of $\Omega^2_{\overline{W}_C}$ since $W'_C$ is (isomorphic to) an open subset of $\overline{Z}_C$. Thus

$$H^0(\Omega^2_{\overline{Z}_C}) = 0$$

Finally, the birational projective morphism $\overline{Z}_C \to \overline{Z}_C$ induces an injective map

$$H^0(\Omega^2_{\overline{Z}_C}) \to H^0(\Omega^2_{\overline{Z}_C}) = 0$$

yielding the conclusion of part b).  

$\square$
From this we conclude that in fact \( H^1_sing(\mathcal{L}_C^1) = 0 = H^0(\Omega^2_{\mathcal{L}_C^1}) \). Now we also have the

**Lemma 73.** Let \( \varphi : Y_C \to Z_C \) be a birational projective morphism of normal complex varieties. Then if \( H^1_sing(Y_C) = 0 \) then \( H^1_sing(Z_C) = 0 \). If \( H^0(\Omega^2_{Y_C}) = 0 \), then \( H^0(\Omega^2_{Z_C}) = 0 \); where \( Y_C \) and \( Z_C \) are normal compactifications of \( Y_C \) and \( Z_C \), respectively.

**Proof.** Consider the first sentence: as before, we consider any abelian cover of \( Z_C \). The pullback to \( Y_C \) is also an abelian cover, which by the assumption is trivial. But \( \varphi \) is an isomorphism over an open subset of \( Z_C \), and so the cover is trivial on this open subset, and hence trivial over all of \( Z_C \). The second sentence is proved by the same argument as part b) of the previous lemma. \( \square \)

Applying this, along with 71 to the morphism \( \mathcal{L}_C^1 \to \mathcal{L}_C \) thus completes the proof of theorem 64.

Before proceeding, we would like to record for later use the

**Lemma 74.** Consider the open subset \( W_C \subset \mathcal{X}_C \) on which the divisor \( E_C \) is smooth. Then \( \mathcal{L}_C \) is a smooth variety over \( W_C \). The inverse image \( \pi^{-1}(\mathcal{E}_C) \) is a smooth divisor over \( W_C \) as well. Further, the closure of \( \mathcal{L}_{U_C} \) inside \( T^*W_C \), denoted \( \mathcal{L}_{W_C} \) is also smooth, and can be regarded as an open subvariety of \( \mathcal{L}_C \).

**Proof.** This is a local statement; consider any point \( x \in D_i \) on which the irreducible component \( D_i \) is smooth. Then, over the neighborhood \( N_{x,C} \) the variety \( \mathcal{L}_C \) is by construction the quotient of a smooth variety by the action of a cyclic group which looks, locally, like \((z_1, z_2, \ldots, z_m) \to (z_1, z_2, \ldots, z_m)\). Such a quotient is smooth, as is the divisor \( \{z_1 = 0\} \).

Similarly, we may identify \( \mathcal{L}_{W_C} \) as follows: work locally on \( N_{x,C} \) where \( x \in W_C \). We have the finite etale morphism

\[ T^*\mathcal{L}_C^{(i)} \to T^*\mathcal{N}_x^{(i)} \]

and the function \( f \in O(\mathcal{L}_{U_C}) \) may thus be regarded as a meromorphic function on \( \mathcal{L}_C^{(i)} \); which possibly has poles on some components of the inverse image of \( E_C \) (which is a smooth divisor by the assumption \( x \in W_C \)). Then \( \mathcal{L}_{W_C} \) becomes identified with the quotient by the action of \( G \) on the variety \( \Gamma(df) \subset T^*\mathcal{L}_C^{(i)} \); since this can be identified with an open subvariety of \( \mathcal{L}_C^{(i)} \) the result follows as in the first paragraph. \( \square \)

### 5.1.1. Higgs Sheaves.

Now we shall consider the Higgs sheaves associated to these Lagrangian subvarieties. As in the one-dimensional case, we shall have occasion to consider Higgs sheaves on both \( \mathcal{X}_C \) and \( \mathcal{L}_C \). We recall that \( \Omega^{1}_{\mathcal{L}_C} \) is defined to be the bundle obtained by pushing forward the sheaf of differential forms on \( (\mathcal{L}_C)^{sm} \).

A Higgs field on a sheaf \( \mathcal{H}_C \) is an \( O \)-linear morphism

\[ \Theta : \mathcal{H}_C \to \mathcal{H}_C \otimes \Omega^{1}_{\mathcal{L}_C} \]

which satisfies the integrability condition \( \Theta \circ \Theta = 0 \).
Now let us consider the situation on $\overline{X}_C$: the inclusion $\tilde{\mathcal{L}}_{W^c} \subset T^*(W^c)$ yields a Higgs sheaf $\mathcal{H}_C = \pi_* (O_{\tilde{\mathcal{L}}_{W^c}})$ with its induced Higgs field. Since the projection is only quasifinite, this is only a quasi-coherent sheaf. In order to construct a coherent sheaf, we have to show how to “fill in the gaps.” In addition, we need to extend our sheaf over the codimension 2 subvariety $\overline{X}_C \setminus W^c$.

We consider first a model of our problem in a rather general situation: let $Z_C$ be a smooth variety, with a normal crossings divisor $Y_C$; with components $\{Y_i, C\}$. Let $\mathcal{H}_C$ be a Higgs bundle on $U_C = Z_C \setminus Y_C$.

Now suppose we are given a finite flat morphism $\pi : Z'_C \to Z_C$, which is finite etale over $U_C$. We have that the pullback of $Y_C$ defines a divisor, which we denote $Y'_C$ and we consider $U'_C = Z'_C \setminus Y'_C$. The pullback $(\pi)^*_{U_C} (\mathcal{H}_C)$ defines a Higgs bundle $\mathcal{H}'_C$ on $U'_C$.

Let $Y_{i,C} = \bigcap_{i \in J} Y_{i,C}$ be an irreducible subvariety defined as the intersection of some collection of components of $Y_C$. Let $\{P\}$ be any point of the scheme $\overline{X}_C$-not necessarily closed- such that $\{P\}$ is contained in $\{Y_j\}_{j \in J}$ but none of the other components of $Y_C$. Let $\{P'\}$ be any point of $L_C$ which lies over $\{P\}$. Denote by $\tilde{\mathcal{O}}_{Y_{i,C}}$ and $\tilde{\mathcal{O}}_{Y_{i,C}}$ the completions of the local rings $O_{Z_C}$ and $O_{Z'_C}$, respectively, at $\{P\}$ and $\{P'\}$, respectively. In an affine neighborhood of the generic point of $Y_{i,C}$, we write the ideal of $Y_{i,C}$ as $(y_i)_{i \in J}$, where the $\{y_i\}$ are a regular sequence which locally generate the components of $Y_C$.

We shall work with the rings $\tilde{\mathcal{O}}_{Y_{i,C}}[\star Y_{i,C}]$ and $\tilde{\mathcal{O}}_{Y_{i,C}}[\star Y'_{i,C}]$, which are defined to be the localization at $\{y_i\}_{i \in J}$ of the rings $\tilde{\mathcal{O}}_{Y_{i,C}}$ and $\tilde{\mathcal{O}}_{Y'_{i,C}}$.

Now, since $\mathcal{H}'_C$ is a bundle, we may, for each $i$ as above, prolong it to a locally free sheaf $\hat{\mathcal{H}}'_i$ over $\tilde{\mathcal{O}}_{Y_{i,C}}[\star Y'_{i,C}]$, which is equipped with meromorphic Higgs field (c.f. [Ma], section 2). Now, we suppose that, for each such $J$, there is a decomposition:

\begin{equation}
\hat{\mathcal{H}}'_C = \bigoplus_{i=1}^n \hat{\mathcal{H}}_{i,C} \otimes e_i[\star Y']
\end{equation}

where $\hat{\mathcal{H}}_{i}$ is a vector bundle with Higgs field (not meromorphic; i.e., the Higgs field has no poles) and $\Theta e_i = \theta_i e_i$ for some one-forms $\{\theta_i\}$; possibly with poles along $Y_{i,C}$.

Of course, such a decomposition will be far from unique; indeed, it is clear that, at the very least, we can “rescale” each subspace $\hat{\mathcal{H}}_{i,C}$ by multiplying by a power of a local coordinate function $\{y_i\}$. To help clear up the ambiguity we make the

**Definition 75.** Suppose that $Z_C, Z'_C$ and $\mathcal{H}_C$ are as above. Let $\{Q\}$ be any point contained in the closure of $\{P\}$; and $\{Q'\}$ lying over it contained in the closure of $\{P'\}$. We have the collection of decompositions equation (5.1). Let $K \subset J$ be the set of indices so that $\{Y_k\}_{k \in K}$ is the set of divisors contained $\{Q\}$. Then we have the flat inclusion of rings

$$
\tilde{\mathcal{O}}_{\{Q\}}[\star Y_{K,C}] [y_j^{-1}] \to \tilde{\mathcal{O}}_{\{P\}}[\star Y_{J,C}]
$$

obtained via completion. Thus we obtain two direct sum decompositions of the locally free sheaf $\mathcal{H}'_C$ over $\tilde{\mathcal{O}}_{Y_{J,C}}[\star Y_{J,C}]$; the first being the given decomposition, and the second obtained by localizing and completing the decomposition for $Y_{K,C}$. Then
this collection of decompositions is said to be compatible if, for all \( \{ P, Q \} \) as above, the given decomposition for \( \hat{O}_{\{P\}}[\ast Y_{I,C}] \) is a refinement of decomposition obtained by localizing and completing, i.e., the summands in the given decomposition are all summands of the summands in the decomposition obtained by localizing and completing.

When we have a compatible collection of decompositions as above; then we see, by looking over the generic point of each \( Y_{I,C} \), that there is a well-defined vector bundle \( \mathcal{H}_C \) in codimension 2 obtained by extending \( \mathcal{H}_C' \) so that the completion is isomorphic to \( \bigoplus_{i=1}^n \mathcal{H}_{i,C} \). Thus there is a unique reflexive coherent sheaf \( \mathcal{H}_C \) on \( Z_C' \) which extends this bundle. So we make the

**Definition 76.** A reflexive coherent meromorphic Higgs sheaf \( \mathcal{H}_C \) will be called an eigenvector extension of \( \mathcal{H}_C \) if it is obtained from a decomposition as above by taking the intersection

\[
\mathcal{H}_C = \mathcal{H}_C' \cap \mathcal{H}_C
\]

In the case where the bundles \( \mathcal{H}_{i,C} \) are of rank 1- so that \( \{ e_i \} \) is a basis, such an eigenvector extension will be called complete.

Now, consider the completion of \( \mathcal{H}_C \) along any closed point \( x \) in the divisor \( Y_C' \). The map \( (\hat{Z}_C)_x \to \hat{Z}_{x(x)} \) is a Galois cover, with Galois group \( G \), a product of cyclic groups. Then, in the above situation, a basis \( \{ e_i \} \) will be called \( G \)-adjusted if, for each such completion, the cyclic group \( G \) preserves the set \( \{ e_i \} \). It suffices to check this at only finitely many points in \( Z_C' \) (in particular, at one point in each intersection of components of \( Y_C' \)).

Now, let us record the

**Lemma 77.** Suppose that \( \mathcal{H}_C \) and \( \mathcal{G}_C \) are eigenvector extensions of \( \mathcal{H}_C \) and \( \mathcal{G}_C \), respectively.

1. There is a well-defined eigenvector extension \( \mathcal{H}_C \otimes \mathcal{G}_C \) of \( \mathcal{H}_C \otimes \mathcal{G}_C \); if \( \mathcal{H}_C \) and \( \mathcal{G}_C \) are \( G \)-adjusted then this extension is \( G \)-adjusted for appropriate \( G \); if the extensions of both \( \mathcal{H}_C \) and \( \mathcal{G}_C \) are complete then the extension for \( \mathcal{H}_C \otimes \mathcal{G}_C \) is as well.

2. There are well defined eigenvector extensions of \( \mathcal{H}_C \otimes \mathcal{G}_C \); which are \( G \)-adjusted (and complete) if \( \mathcal{H}_C \) and \( \mathcal{G}_C \) are.

In particular, \( \mathcal{E} \text{End}(\mathcal{H}_C) \) has a well-defined \( G \)-adjusted eigenvector extension \( \mathcal{E} \text{End}(\mathcal{H}_C) \).

**Proof.** If the completion of the pullback under \( Z_C' \to Z_C \) has a decomposition as in equation (5.1), the same is true after any further pullback; and the pullback of a \( G \)-adjusted bundle is clearly \( G \)-adjusted for the new Galois group. So, we choose a single pullback which works for both \( \mathcal{H}_C \) and \( \mathcal{G}_C \); if \( \{ e_i \} \) and \( \{ f_i \} \) are the chosen sets of eigenvectors then \( \{ e_i \otimes f_j \} \) provides a set of eigenvectors for \( (\pi^{(i)})^*(\mathcal{H}_C \otimes \mathcal{G}_C) \), which we can then descent to a lattice for \( \mathcal{H}_C \otimes \mathcal{G}_C \). This proves a). Part b) follows similarly from the fact that the dual basis to an eigenbasis is an eigenbasis. Part c) follows from the fact that \( \mathcal{E} \text{End}(\mathcal{H}_C) = \mathcal{H}_C \otimes \mathcal{H}_C^\ast \).

Furthermore, we would like to consider the notion of the Higgs complex for \( \mathcal{H}_C \). On \( U_C \), we have the usual complex

\[
\mathcal{H}_C \to \mathcal{H}_C \otimes \Omega_{U_C}^1 \to \mathcal{H}_C \otimes \Omega_{U_C}^2 \to \cdots \to \mathcal{H}_C \otimes \Omega_{U_C}^n
\]
which computes Higgs cohomology. Since \( \pi \) is finite etale on \( U_\mathbb{C} \), we may pull-back by it to obtain
\[
(\pi)^* \mathcal{H}_\mathbb{C} \to (\pi)^* \mathcal{H}_\mathbb{C} \otimes \Omega^1_{Y_\mathbb{C}} \to (\pi)^* \mathcal{H}_\mathbb{C} \otimes \Omega^2_{Y_\mathbb{C}} \to \cdots \to (\pi)^* \mathcal{H}_\mathbb{C} \otimes \Omega^n_{Y_\mathbb{C}}.
\]
We consider extensions of this to a complex on all of \( Z'_\mathbb{C} \). Pass to \( \mathcal{O}_{Y_j, \mathbb{C}}[\ast Y'_{j, \mathbb{C}}] \).
Consider the bundle \( \widetilde{\mathcal{H}}_{i, \mathbb{C}} \otimes e_i \) with its meromorphic Higgs field; suppose that the eigenvalue \( \theta_i \) has a pole of order \( m_{ij} \) on the component \( Y_j, \mathbb{C} \). Then we can consider the complex
\[
(\widetilde{\mathcal{H}}_{i, \mathbb{C}} \otimes e_i) \to (\widetilde{\mathcal{H}}_{i, \mathbb{C}} \otimes e_i) \otimes \Omega^1_{Z'_\mathbb{C}}(\sum_j m_{ij} Y_{j, \mathbb{C}}) \to \cdots \to (\widetilde{\mathcal{H}}_{i, \mathbb{C}} \otimes e_i) \otimes \Omega^m_{Z'_\mathbb{C}}(\sum_j n m_{ij} Y_{j, \mathbb{C}})
\]
which is obtained from the action of \( \Theta \); we call this the Higgs complex of \( \widetilde{\mathcal{H}}_{i, \mathbb{C}} \otimes e_i \) (here \( m = \dim(Z'_\mathbb{C}) \)). Since we have the decomposition equation (5.1), taking the direct sum over the Higgs complexes of the \( \widetilde{\mathcal{H}}_{i, \mathbb{C}} \otimes e_i \) yields a complex, which we call the Higgs complex for the extension \( \widetilde{\mathcal{H}}_{i, \mathbb{C}} \). The compatibility condition ensures that this complex descends to a complex of vector bundles (in codimension 2) on \( Z'_\mathbb{C} \); and therefore a complex of reflexive coherent sheaves on all of \( Z'_\mathbb{C} \).

Taking the intersection with the Higgs complex on \( U_\mathbb{C} \) yields a complex on \( X_\mathbb{C} \); which we call the Higgs complex for \( \mathcal{H}_\mathbb{C} \); it depends on the choice of eigenvalue extension.

Now we return to the situation of \( \mathcal{H}_\mathbb{C} \) on \( X_\mathbb{C} \): as in the previous section we consider a neighborhood \( N_{x, \mathbb{C}} \) of \( x \in \mathcal{E}_\mathbb{C} \). We have the finite flat map \( \overline{\pi} : \overline{\mathcal{L}}_{x, \mathbb{C}} \to N_{x, \mathbb{C}} \). Let \( \mathcal{H}_{U_\mathbb{C}} \) be the Higgs bundle \( \pi_\ast (\mathcal{O}_{U_\mathbb{C}}) \) on \( U_\mathbb{C} \). Then we have the

**Lemma 78.** The bundle \( \mathcal{H}_{U_\mathbb{C}} \) admits eigenvector extensions for the morphism \( \pi : \overline{\mathcal{L}}_\mathbb{C} \to \overline{X}_\mathbb{C} \). These may be chosen to be \( G \)-adjusted eigenvector extensions.

**Proof.** We have the finite etale morphism \( \pi : \mathcal{L}_{U_\mathbb{C}} \to U_\mathbb{C} \). Pulling back the Higgs bundle \( \mathcal{H}_{U_\mathbb{C}} \) to \( \mathcal{L}_{U_\mathbb{C}} \) yields as in 26 a Higgs bundle which is a direct sum of line bundles with Higgs field; write these line bundles as \( (\mathcal{M}_i, \theta_i) \). Each \( \theta_i \) is a one form which might have poles along the exceptional divisor \( Y_\mathbb{C} \). Now restrict to the localization of some \( \{P\} \) which is contained in \( Y_j, \mathbb{C} \); and let \( \{P'\} \) live over \( \{P\} \).

We say \( \theta_i \) and \( \theta_j \) have the same polar type; and write \( \theta_i \sim \theta_j \), if \( \theta_i - \theta_j \in \Omega_{\mathcal{L}(P'), \mathbb{C}} \).

This is an equivalence relation, and we write \( \{P'\} \) for the set of equivalence classes. Then we have an isomorphism
\[
\pi^\ast (\mathcal{H}_{U_\mathbb{C}})|_{\mathcal{E}_{(P'), \mathbb{C}}} \ast Y_{j, \mathbb{C}} \cong \bigoplus_{P_j} \bigoplus_{(\theta_i, \in P_j)} (\mathcal{M}_i, \theta_i) \ast Y_{j, \mathbb{C}} \cong \bigoplus_{P_j} \pi^\ast (\mathcal{H}_j) \otimes e_j \ast Y_{j, \mathbb{C}}
\]
where \( \mathcal{H}_j \) is a summand of \( \mathcal{H}_{U_\mathbb{C}} \), such that the Higgs field on \( \pi^\ast (\mathcal{H}_j) \) has no pole along each divisor \( \{Y_{j, \mathbb{C}}\}_{j \in J} \) and \( e_j \) is a eigenvector for the Higgs field; and, as above, the hat denotes completion at the local ring of \( \{P\} \).
To deduce the right hand isomorphism one uses definition of the equivalence relation: indeed, this definition implies a decomposition over $\hat{\mathcal{O}}_{Y_{j,C}}$ of the type

$$\pi^*([\mathcal{H}_{C}])_{\hat{\mathcal{L}}_{U_{C}}^{n\mathcal{L}_{r,c}}[\star Y_{C}]} \cong \bigoplus_{p_j} \hat{L}_j \otimes e_j \star Y_{C}$$

where $\hat{L}_j$ is a Higgs bundle on $\hat{\mathcal{O}}_{Y_{j,C}}$ whose Higgs field has no pole; and then we define $\hat{\mathcal{H}}_j$ to be the intersection $\hat{L}_j \cap \hat{\mathcal{H}}_{U_{C} \cap N_{r,c}}$. $\hat{\mathcal{H}}_j$ is therefore a Higgs bundle whose Higgs field has no pole after pullback to $\mathcal{L}_{U_{C}}$.

Therefore we may select extensions $\hat{\mathcal{H}}_i$ of each $\hat{\mathcal{H}}_i$ to a reflexive coherent Higgs sheaf on Spec$(\mathcal{O}(P))$ whose pullback to Spec$(\mathcal{O}(P'))$ has no pole. Thus the bundle

$$\bigoplus_{p_j} \pi^*([\mathcal{H}_j]) \otimes e_j$$

(with the Higgs field induced from pullback on $\pi^*([\mathcal{H}_j])$) provides an extension of the required type at $Y_{j,C}$.

Now we have to show that this collection of decompositions can be chosen to be compatible. To see this, we first work over the open subset $W_C$ on which the divisor $Y_C$ is smooth. There, no component of the divisor intersects another, and so, given decompositions of the form equation (5.2), we can select eigenvector extensions $\hat{\mathcal{H}}_i$ compatible. To see this, we first work over the open subset $W_C$ when we are over open subset $L$ as follows: recall that the field extension $K(\pi^*\hat{\mathcal{H}}_C) \subset K(\hat{\mathcal{L}}_C)$ is finite and separable. We let $K^{cl}$ be the Galois closure of this extension. Taking the integral closure of $\hat{\mathcal{L}}_C$ in this field yields a variety $(\hat{\mathcal{L}}_C)^{cl}$ equipped with a finite map to $\hat{\mathcal{L}}_C$, and hence a finite map to $\pi^*\hat{\mathcal{H}}_C$. This map makes $\pi^*\hat{\mathcal{H}}_C$ the quotient of $(\pi^*\hat{\mathcal{H}}_C)^{cl}$ by the action of a finite group, $G$. After pulling back $\bigoplus_{i=1}^r \mathcal{M}_i$ to $(\pi^*\hat{\mathcal{H}}_C)^{cl}$, we demand that the chosen basis of $\bigoplus_{i=1}^r \mathcal{M}_i$ be invariant with respect to $G$ (this is possible since $\bigoplus_{i=1}^r \mathcal{M}_i$ is the pullback of a bundle on $\pi^*\hat{\mathcal{H}}_C$). This then ensures that the extension $\hat{\mathcal{H}}_C$ is $G$-adjusted.

\textbf{Remark 79.} It seems worth pointing out what is happening in a more explicit way when we are over open subset $W_C$ on which $Y_C$ is smooth. Suppose we look locally about one component $Y_{i,C} \subset Y_C \cap W_C$. We have the Lagrangian $\hat{\mathcal{L}}_{W_C} \subset T^*W_C$, and after removing a closed subset of codimension $\geq 2$ and and completing at $Y_{i,C}$ we
have the decomposition
\[ \widehat{L}_{\mathcal{W}_C} = (\widehat{L}_{\mathcal{W}_C})_{fin} \bigcup (\widehat{L}_{\mathcal{W}_C})_{inf} \]
where \((\widehat{L}_{\mathcal{W}_C})_{fin}\) extends to a Lagrangian over \(Y_{i,C}\) and \((\widehat{L}_{\mathcal{W}_C})_{inf}\) does not. This then yields the decomposition
\[ \widehat{H}[\pi Y_{i,C}] = \widehat{H}_{fin}[\pi Y_{i,C}] \oplus \widehat{H}_{inf}[\pi Y_{i,C}] \]
which extends over \(Y_{i,C}\) to an isomorphism of quasi-coherent Higgs sheaves
\[ \pi_*(O_{\mathcal{W}_C^{-}}) \cong \widehat{H}_{fin} \oplus \widehat{H}_{inf}[\pi Y_{i,C}] \]
on the formal scheme given by completing \(T^*W_C\) along \(Y_{i,C}\); here \(\widehat{H}_{fin} = \pi_*(O_{\mathcal{W}_C})_{fin}\).

Now define an extension of \(\pi_*(O_{\mathcal{W}_C^{-}})\) to be given as the direct sum of two pieces:

the first is simply \(\widehat{H}_{fin}\). The second is obtained by pulling back \(\widehat{H}_{inf}[\pi Y_{i,C}]\) under \(\pi\), choosing an eigenbasis, and then descending down to \(\widehat{X}_C\).

Any extension of \(H_{U_C}\) to a bundle \(\overline{H}_C\) on \(\widehat{X}_C\) obtained as in the lemma will be called a \(\Theta\)-regular extension. If \(x\) is a closed point in \(\widehat{L}_C\), the letter \(\Theta_x\) will denote the matrix of the Higgs field of \(\overline{H}_x\) in an adjusted eigenbasis as above. We would like to describe all of the \(\Theta\)-regular extensions. Indeed, we have the

**Proposition 80.** The set of \(\Theta\)-regular meromorphic Higgs bundles \(\overline{H}_C\) is in natural bijection with \(\text{Pic}(\widehat{L}_C)\).

This will be proved after some preparation. Let us introduce some terminology:

Let \(Y_{i,C}\) be any component of the divisor \(\widehat{E}_C\). Then \(\pi^{-1}(Y_{i,C})\) is a union of divisors; call the components \(Y_{ij,C}\). We have the meromorphic function \(f\) on \(\widehat{L}_C\). We call \(Y_{ij,C}\) a finite divisor if \(f\) extends over \(Y_{ij,C}\); and we call \(Y_{ij,C}\) an infinite divisor if \(f\) has a pole at \(Y_{ij,C}\). Let \(\widehat{N}_{ij,C}\) is a neighborhood of \(Y_{ij,C}|W_C\); i.e., we throw out all the intersections of divisors and then take a neighborhood of a divisor in this space. Denote by \(\widehat{N}_{ij,C}\) the completion along \(Y_{ij,C}\). In this formal neighborhood, the map \(\pi\) becomes a branched cover of degree \(l_{ij}\).

We begin with the

**Lemma 81.** a) Suppose that \(Y_{ij,C}\) is an infinite divisor. Then there is a short exact sequence
\[ 0 \to (\pi|_{\widehat{N}_{ij}})^* \widehat{H}_{ij,C} \to \bigoplus_l (O_{\widehat{N}_{ij}}; \Theta^{(l)}_{ij}) \to \bigoplus_{k=1}^{l_{ij}-1} O_{kY_{ij,C}} \to 0 \]
as meromorphic Higgs bundles, for certain one-forms \(\Theta^{(l)}_{ij}\).

b) Suppose that \(Y_{ij,C}\) is a finite divisor. Then there is a short exact sequence
\[ 0 \to \bigoplus_l (O_{\widehat{N}_{ij}}; \Theta^{(l)}_{ij}) \to (\pi|_{\widehat{N}_{ij}})^* \widehat{H}_{ij,C} \to \bigoplus_{k=1}^{l_{ij}-1} O_{kY_{ij,C}} \to 0 \]

\(^9\)It is not difficult to verify that this is necessary; i.e., if one pulls back the finite piece, extends by choosing an eigenbasis, and then descends one does not necessarily obtain a bundle preserved under the Higgs field for \(\widehat{X}_C\).
as meromorphic Higgs bundles, for certain one-forms $\Theta^{(i)}_{ij}$.

In both cases, the one forms $\Theta^{(i)}_{ij}$ are the Galois conjugates of $df|_{N_{ij}}$ under the action of the Galois group $G_{ij}$ (after completion at any point in $Y_{ij,\mathbb{C}}$).

**Proof.** Part a) follows directly from the definition of an eigenvector extension; exactly as in lemma 38. Part b) is the same local computation applied in the case where the Higgs bundle extends over the divisor. □

Now we consider the Higgs bundle $\pi^*\mathcal{H}_C$ on $\tilde{L}_U$. We know from 26 that this sheaf splits as a direct sum of line bundles with Higgs field, and that the support is isomorphic to

$$\tilde{L}_U \times_U \tilde{L}_U$$

Thus $\pi^*\mathcal{H}_C$ automatically admits an adjusted eigenvector extension to $\tilde{L}_C$, which we denote $\pi^*\tilde{\mathcal{H}}_C$.

Now we deduce the:

**Lemma 82.** a) There is a short exact sequence of Higgs sheaves

$$0 \to (\pi^*)^i\mathcal{H}_C \to \pi^*\mathcal{H}_C \to \bigoplus_{Y_{ij,\mathbb{C}}} \bigoplus_{j'}^{l_{ij}-1} \bigoplus_{k=1}^{r} O_{K_{Y_{ij',\mathbb{C}}}} \to 0$$

where the first sum is over all divisors $Y_{ij,\mathbb{C}}$, and the second sum is over the infinite divisors $Y_{ij',\mathbb{C}}$.

b) There is a short exact sequence of Higgs sheaves

$$0 \to \bigoplus_{i=1}^{r} \mathcal{M}_{i,C} \to \pi^*\mathcal{H}_C \to \bigoplus_{Y_{ij,\mathbb{C}}} \bigoplus_{j'}^{l_{ij}-1} \bigoplus_{k=1}^{r} O_{K_{Y_{ij',\mathbb{C}}}} \to 0$$

where the first sum is over all divisors $Y_{ij,\mathbb{C}}$, and the second sum is over the finite divisors $Y_{ij',\mathbb{C}}$. The $\mathcal{M}_{i,C}$ are line bundles on $\overline{L}_C$, equipped with meromorphic Higgs fields.

**Proof.** Part a) is an immediate consequence of part a) of the previous lemma. As for part b), from both parts of the previous lemma we see that there is locally (near each divisor $Y_{ij,\mathbb{C}}$) an inclusion of this form. Now, we have the splitting

$$\pi^*\mathcal{H}_{U_C} = \bigoplus_{i=1}^{r} \mathcal{M}_{i,\mathcal{L}_U}$$

where the $\mathcal{M}_{i,\mathcal{L}_U}$ are line bundles with Higgs field on $\mathcal{L}_U$. Taking the intersection of each such line bundle with the sheaf $\pi^*\tilde{\mathcal{H}}_C$ yields the extensions $\mathcal{M}_{i,C}$; the result follows from the fact that both $\bigoplus_{i=1}^{r} \mathcal{M}_{i,C}$ and $\pi^*\tilde{\mathcal{H}}_C$ are reflexive sheaves, so that the quotient is supported on a pure codimension 1 subset. □

This is the higher dimensional analogue of the short exact sequence in lemma 39.

We would like to investigate the structure of the line bundles $\mathcal{M}_{C}$ on $\overline{L}_C$ appearing in the lemma. To give the construction in the most general form, recall that the field extension $K(\overline{X}_C) \subset K(\overline{L}_C)$ is finite and separable. We let $K^{cd}$ be the Galois closure of this extension. Taking the integral closure of $\overline{L}_C$ in this field
yields a variety $\tilde{L}_C$ equipped with a finite map to $\tilde{X}_C$, and hence a finite map to $\tilde{X}_C$. This map makes $\tilde{X}_C$ the quotient of $(\tilde{L}_C)^{cl}$ by the action of a finite group.

Now, consider any morphism $\tilde{f} : \tilde{L}_C \to \tilde{X}_C$. We say that $\tilde{f}$ is a Galois conjugate of $f$ if this is so after pullback to $(\tilde{L}_C)^{cl}$. Let $(\mathcal{M}_C, df)$ be any line bundle, equipped with meromorphic Higgs field $df : \mathcal{M}_C \to \mathcal{M}_C \otimes \Omega^1_{L_C}$. We say that this Higgs bundle is Galois conjugate to a Higgs bundle $(\mathcal{M}_C, df)$ if, after pullback to $(\tilde{L}_C)^{cl}$, we have $\mathcal{M}_C = g^*\mathcal{M}_C$ and $\tilde{f} = f \circ g$ for some element $g$ of the Galois group.

We have the

**Lemma 83.** Let $\mathcal{H}_C$ be any $\Theta$-regular extension. The collection of Higgs line bundles $\bigoplus_{i=1}^{r} \mathcal{M}_{i,C}$ from part b) of lemma 82 associated to $\mathcal{H}_C$ consists of Galois conjugates of $(\mathcal{M}_C, df)$.

Thus bijection with $\text{Pic}(\mathcal{L}_C)$ on the set of $\Theta$-regular extensions (from 80) is given by identifying $\mathcal{H}_C$ with the unique line bundle $\mathcal{M}_C$ whose associated Higgs field is $df$.

**Proof.** By adjunction, we have a morphism $(\mathcal{M}_C, df) \to \pi_!\pi_*\mathcal{M}_C$, so that the line bundle $(\mathcal{M}_C, df)$ is included amongst the $\mathcal{M}_{i,C}$. We must examine the other members of this set.

First suppose that $\tilde{L}_C$ is already Galois over $\tilde{X}_C$. Then in fact $\pi_!\pi_*\mathcal{M}_C = \pi_!\pi_*(g^*\mathcal{M}_C)$ for any element $g$ of the Galois group; this follows directly from the definitions. Thus each Galois conjugate of $(\mathcal{M}_C, df)$ is included in the sum, which shows the lemma in this case.

For the general case, one argues in exactly the same way but with $\pi_! : (\tilde{L}_C)^{cl} \to \tilde{X}_C$ instead of $\tilde{L}_C \to \tilde{X}_C$. There is a collection of line bundles on $(\tilde{L}_C)^{cl}$ associated, exactly as in lemma 81 and lemma 82, to the Higgs sheaf $(\pi_!)^*(\pi)_*(\mathcal{M}_C)$, and the line bundles $(\pi_!)^*(\mathcal{M}_{i,C})$ are amongst this collection; so the result follows as in the previous paragraph.

Finally, the last sentence follows from the construction of the $(\mathcal{M}_C, df)$. $\square$

Now we wish to analyze the endomorphism sheaf of $\mathcal{H}_C$, and the associated Higgs complex. Since $\mathcal{H}_C$ is an eigenvector extension, lemma 77 yields an extension

$$\mathcal{E}_{\text{nd}}(\mathcal{H})_C := \mathcal{F}_C$$

to a meromorphic reflexive coherent Higgs sheaf on $\tilde{X}_C$.

Therefore we can consider the Higgs complex for $\mathcal{F}_C$, denoted $\text{Higgs}(\mathcal{F}_C)$, defined just below lemma 77. We denote the sheaf cohomology of this complex by $\mathcal{H}^i_{\text{Higgs}}(\mathcal{F}_C)$; we have the

**Proposition 84.** a) $\mathcal{H}^1(\text{Higgs}(\mathcal{F}_C)) = 0$

b) $\mathcal{H}^2(\text{Higgs}(\mathcal{F}_C)) = \mathcal{H}^1(\Omega^1_{L_C})$.

**Proof.** (sketch) By the definition of $\mathcal{H}_C = \pi_*(\mathcal{M}_C)$ (for $\mathcal{M}_C$ a line bundle on $\tilde{L}_C$) we have an inclusion of Higgs sheaves

$$\pi_*(\mathcal{O}_{\tilde{L}_C}) \to \mathcal{F}_C$$
where the first sheaf has trivial Higgs field. This induces an isomorphism
\[ \pi_*(\Omega^0_{\tilde{L}_C}) = \mathcal{H}^0_{\text{Higgs}}(\pi_*(\mathcal{O}_{\tilde{L}_C})) \rightarrow \mathcal{H}^0_{\text{Higgs}}(\mathcal{F}_C) \]

of quasicoherent sheaves on \( \tilde{X}_C \). Considering this map locally about each divisor where the Higgs bundle is meromorphic, we deduce as in lemma 47, by looking at the generic point of each \( Y_{f,C} \), that it extends to a map
\[ \pi_*(\Omega^0_{\tilde{L}_C}) \rightarrow \mathcal{F}_C \]
where, again, the first sheaf has trivial Higgs field. This induces isomorphisms
\[ \pi_*(\Omega^0_{\tilde{L}_C}) = \mathcal{H}^0_{\text{Higgs}}(\pi_*(\mathcal{O}_{\tilde{L}_C})) \rightarrow \mathcal{H}^0_{\text{Higgs}}(\mathcal{F}_C) \]

Therefore the result follows from the assumptions on the cohomology of \( \tilde{L}_C \). \( \square \)

Finally, we end this subsection with a few comments on spreading out and liftings of \( \tilde{L}_C \). Since \( \tilde{L}_C \) is a normal projective variety, we have that the identity component of the Picard scheme has dimension \( H^0(\Omega^1_{\tilde{L}_C}) \), which in this case is \( 0 \). Thus Pic is a discrete, finitely generated abelian group. Therefore we may choose \( R \) so that every line bundle on \( \tilde{L}_C \) is defined on \( \tilde{L}_R \); it follows further that \( \text{Pic}(\tilde{L}_k) \rightarrow \text{Pic}(\tilde{L}_C) \) for \( \text{char}(k) \gg 0 \). We further demand that \( \tilde{L}_R \rightarrow \tilde{X}_R \) is finite flat, both varieties are normal (and \( \tilde{X}_R \) is regular), and the divisor \( Y_R \subset \tilde{X}_R \) is normal crossings.

Now we make the

**Definition 85.** Let \( (\mathcal{G}_R, \Psi) \) be a meromorphic Higgs bundle on \( \tilde{X}_R \), whose Higgs field has no poles on \( U_R \). Then \( (\mathcal{G}_R, \Psi) \) is said to be \( \Theta \)-regular at infinity if

1) After pullback \( \pi : \mathcal{L}_U_R \rightarrow U_R \), the Higgs bundle \( \pi^*(\mathcal{G}_R) \) is isomorphic to a direct sum of Higgs line bundles.

2) Let \( Y_{J,R} \) be an intersection of components of the exceptional divisor, and let \( \{P\} \) be any point contained in \( Y_{J,R} \). Suppose that \( \{Y_{J,R}\}_{J \in J} \) are the only divisors containing \( \{P\} \). Consider \( \pi : \mathcal{L}_R \rightarrow \tilde{X}_R \); let \( \{P'\} \) be any point lying above \( \{P\} \).

We demand that there exists a reflexive coherent sheaf \( \mathcal{G}_{\tilde{L}} \) on \( \tilde{L}_R \), so that \( \pi^*(\mathcal{G}_{\tilde{L}}) \), after localizing and completing at \( P' \), and then localizing at \( \{y_i\}_{i \in J} \), admits a decomposition of the form
\[ \pi^*(\mathcal{G}_{\tilde{L}})[*Y_{J,R}] = \bigoplus_{i=1}^{n} \mathcal{G}_{i,R} \otimes e_i[*Y_{J,R}] \]

where \( \mathcal{G}_{i,R} \) are modules (over \( \mathcal{O}_{\tilde{L}_R}[*Y_{J,R}] \)) with a Higgs field which admits no pole, \( e_i \) are eigenvectors for the Higgs field, and \( \bigoplus_{i=1}^{n} \mathcal{G}_{i,R} \) is the completion at \( \{Y_{J,R}\} \) of \( \mathcal{G}_{\tilde{L}} \).

3) We have that \( \mathcal{G}_R = \bigoplus_{i=1}^{n} \mathcal{G}_{i,R} \cap \mathcal{G}_{U,R} \).

4) If \( \Theta \) is the matrix of \( \mathcal{H}_R \) in an adjusted eigenbasis as in lemma 78, we have
\[ (\Psi - \Theta)(e_i) \subset \Omega^1_{\tilde{L}} \]
i.e., \( \Psi - \Theta \) has no poles.
This definition may be applied verbatim after replacing $R$ by any of the rings of fields to which it maps.

In other words, these are meromorphic Higgs bundles which share the same asymptotics as $\mathcal{H}_R$. We have the

**Proposition 86.** Suppose that $\mathcal{H}_{W_n(k)}$ is a lift of $\mathcal{H}_k$ to a Higgs bundle which is $\Theta$-regular at infinity. Then there is a unique lift of $\mathcal{H}_{W_n(k)}$ to $\mathcal{H}_{W_{n+1}(k)}$ which is also $\Theta$-regular at infinity. In particular, any $\mathcal{H}_{W_n(k)}$ which is $\Theta$-regular at infinity is simply the specialization to $W_n(k)$ of $\mathcal{H}_R$.

**Proof.** (sketch) We apply theorem 34 to the current situation; indeed, everything goes through without change except we use the Higgs cohomology $\mathcal{H}_{Higgs}(\mathcal{F}_k)$ instead of the usual Higgs cohomology (this is completely analogous to lemma 44). Then we apply the cohomology calculation of the previous proposition. \(\square\)

5.2. The $\mathcal{D}$-module $M_k$ and the $\lambda$-connection $\mathcal{M}_{\lambda,k}$. In this section we shall construct a $\lambda$-connection which deforms the Higgs bundle $\mathcal{H}_k$; and then we shall explain the associated lifting theory, eventually yielding a $\lambda$-connection over a field of characteristic zero which we shall see to be unique.

5.2.1. The $\lambda$-connection $\mathcal{M}_{\lambda,k}$. In this section we construct a meromorphic $\lambda$-connection which specializes, at $\lambda = 1$ to a connection $\mathcal{N}_k$ whose $p$-support is $L_k^{(1)}$, and at $\lambda = 0$, to the Higgs bundle $\mathcal{H}_k$. We shall examine the deformation theory of this connection, and show that obstructions to lifting are determined, in a strong sense, by the choice of line bundle used in defining $\mathcal{N}_k$. We shall then see that there is a preferred choice, called $\mathcal{M}_{\lambda,k}$, which lifts all the way to characteristic zero, thus solving the lifting problem after setting $\lambda = 1$.

Our strategy is similar to what we have done before: consider the bundle $O_{L_{kU}}[\lambda]$ with $\lambda$-connection $\nabla(e) = df \cdot e$ for some generator $e$. Then $\mathfrak{m}_e(O_{L_{kU}}[\lambda])$ is naturally a $\lambda$-connection on $U_k$; denoted $N_{\lambda,U_k}$. At $\lambda = 0$, the associated Higgs field is simply $\mathfrak{m}_e(O_{L_{kU}})$; at $\lambda = 1$ we recover $M_{U_k}$.

It will also be necessary to consider a completed version of the situation: namely the bundle $O_{\hat{L}_{kU}}[[\lambda]]$ with $\lambda$-connection $\nabla(e) = df \cdot e$ for some generator $e$; we may take $\mathfrak{m}_e(O_{\hat{L}_{kU}}[[\lambda]])$ (denoted $N_{\lambda,U_k}$) which again specializes to $\mathfrak{m}_e(O_{L_{kU}})$ at $\lambda = 0$.

We can consider extensions of these objects to $\mathcal{M}_k$. This proceeds mostly analogously to the construction of the Higgs bundle $\mathcal{H}_C$ in the previous section.

We begin with the

**Proposition 87.** a) There is an extension of $\mathfrak{m}_e(O_{L_{kU}}[[\lambda]]/[\lambda]^{-1})$ to a $\mathcal{D}_{W_k}[\lambda,\lambda^{-1}]$-module, denoted $N_{\lambda,W_k}[[\lambda]^{-1}]$. The sheaf $F_*((N_{\lambda,W_k}[[\lambda]^{-1}])$ is a vector bundle on $L_{W_k}^{(1)} \times \text{Spec}(k[[\lambda]]).$ The set of such extensions is a torsor over $\text{Pic}(L_{W_k}^{(1)}).

a') Similarly, there is an extension of $\mathfrak{m}_e(O_{L_{kU}}((\lambda))$ to a $\mathcal{D}_{W_k}(\lambda))$-module, $N_{\lambda,W_k}[[\lambda]^{-1}]$. The sheaf $F_*((N_{\lambda,W_k}[[\lambda]^{-1}])$ is a vector bundle on $L_{W_k}^{(1)} \times \text{Spec}(k((\lambda))).$ The set of such extensions is a torsor over $\text{Pic}(L_{W_k}^{(1)}).

b) Each $N_{\lambda,W_k}[[\lambda]^{-1}]$ admits an extension to a quasicoherent sheaf with meromorphic $\lambda$-connection on $W_k$, $N_{\lambda,W_k}$. The sheaf $F_*((N_{\lambda,W_k})$ is a vector bundle on $L_{W_k}^{(1)} \times \mathbb{A}_k^1$. At $\lambda = 1$, this specializes to a connection, $N_{W_k}$, and at $\lambda = 0$, it is
isomorphic, as a quasicoherent sheaf with Higgs field, to \( \pi_*(\mathcal{M}_k) \), where \( \mathcal{M}_k \) is a line bundle on \( \hat{L}_{W_k} \).

b) Each \( N_{\lambda,W_k}^{-1} \) admits an extension to a quasicoherent sheaf with meromorphic \( \lambda \)-connection on \( W_k \), \( N_{\lambda,W_k} \). The sheaf \( F_*(N_{\lambda,W_k}^{-1}) \) is a vector bundle on \( \hat{L}_{W_k}^{(1)} \times \text{Spec}(k[[t]]) \). At \( \lambda = 0 \), it is isomorphic, as a quasicoherent sheaf with Higgs field, to \( \pi_*(\mathcal{M}_k) \), where \( \mathcal{M}_k \) is a line bundle on \( \hat{L}_{W_k} \).

Proof. a) We recall from 6 that \( \mathcal{D}_{W_k}[\lambda, \lambda^{-1}] \) is an Azumaya algebra over \( T^*W_k(1) \times_k (\mathbb{A}^1 \setminus \{0\}) \). Then the subvariety \( \hat{L}_{W_k} \times_k (\mathbb{A}^1 \setminus \{0\}) \) is smooth. Since \( F_*(\mathcal{D}_{W_k}[\lambda, \lambda^{-1}]) \) splits on \( \hat{L}_{W_k}^{(1)} \times_k (\mathbb{A}^1 \setminus \{0\}) \), we deduce as before from [Mi], corollary 2.6 that \( F_*(\mathcal{D}_{\hat{X}_k}[\lambda, \lambda^{-1}]) \) splits on \( \hat{L}_{W_k}^{(1)} \times_k (\mathbb{A}^1 \setminus \{0\}) \). Thus we may choose an extension of \( M|_{U_k} \) to \( W_k \); different such extensions are parametrized by \( \text{Pic}(\hat{L}_{W_k}^{(1)} \times (k\setminus\{0\})) = \text{Pic}(\hat{L}_{W_k}^{(1)}) \).

b) We have embeddings \( \pi_*(O_{\hat{L}_{U_k}}[\lambda]) \subset \pi_*(O_{\hat{L}_{U_k}})[\lambda^{-1}] \) and \( N_{\lambda,W_k}[\lambda^{-1}] \subset \pi_*(O_{\hat{L}_{U_k}}[\lambda^{-1}]) \); the first given by localization at \( \lambda \), and the second by localization at the divisor \( W_k \setminus U_k \). Define \( N_{\lambda,W_k} := \pi_*(O_{\hat{L}_{U_k}}[\lambda]) \cap N_{\lambda,W_k}[\lambda^{-1}] \). Then the claim of the proposition is an easy exercise.

Parts a’) and b’) are proved completely analogously.

Now we wish to consider extensions to \( \hat{N}_{\lambda,k} \), a vector bundle with meromorphic \( \lambda \)-connection on \( \hat{X}_k \). We start on \( W_k \); and produce extensions in a method completely analogous to 79.

Namely, by decomposing under the action of the \( p \)-curvature operator, we see that, on the punctured formal neighborhood \( \hat{N}_{i,k} \setminus Y_{i,k} \) (inside \( W_k \setminus Y_{i,k} \)), we have a direct sum decomposition

\[
N_{\lambda,W_k} \cong \bigoplus \hat{N}_{ij,k}
\]

where \( N_{ij,k} \) is supported on punctured the formal neighborhood of a divisor \( Y_{ij,k}^{(1)} \subset \hat{L}_{W_k}^{(1)} \). We denote the restriction of the \( p \)-curvature \( \Psi \) to \( \hat{N}_{ij,k} \) by \( \Psi_{ij} \).

Now we can mimic the construction of the previous section in this case: if \( Y_{ij,k} \) is a finite divisor, then it is contained in \( \hat{L}_{W_k} \subset T^*W_k \). Thus \( \pi_*(F_*(N_{\lambda,W_k})|_{N_{ij,k}^{(1)}}) \) is a vector bundle on \( \hat{O}_{N_{ij,k}^{(1)}} \), which is a sub-sheaf of \( F_*(\hat{N}_{ij,k}) \); in fact by construction it is a \( \hat{O}_{N_{ij,k}^{(1)}} \) -submodule which is preserved under the connection; and we define

\[
\hat{N}_{ij,k} := \pi_*(F_*(N_{\lambda,W_k})|_{N_{ij,k}^{(1)}})
\]

If, on the other hand \( Y_{ij,k} \) is an infinite divisor, we have

\[
(\hat{N}_{ij,k})^{(1)} \cong \bigoplus_{k=1}^l \hat{O}_{N_{ij,k}} \cdot e_k
\]

where the \( \{e_k\} \) are eigenvectors for the connection:

\[
\nabla(e_l) = \theta_l e_l
\]

By computing the \( p \)-curvature one sees that \( \theta_l - \Theta^{(1)}_{ij} \) has no pole (where \( \Theta^{(1)}_{ij} \) is as in lemma 81). We note in particular that the “polar part” of \( \nabla e_l \) does not depend on \( \lambda \).
We now define the lattice $\hat{N}_{ij,k}$ by taking the $G_{ij}$-invariants inside a lattice generated by the $\{e_k\}$; this yields a bundle $\hat{N}_{i,k}$. As in the previous section this is not unique; we are allowed to rescale by the $p^t$th power of a local coordinate at each infinite divisor.

Since the complement of $W_k$ in $\overline{X}_k$ has codimension $\geq 2$, there is a unique extension of $\hat{N}_{i,k}$ to a reflexive coherent sheaf $\overline{N}_{i,k}$ with connection on $\overline{X}_k$. We denote by $\overline{\mathcal{H}}_k$ the specialization $\overline{N}_{i,k}/\lambda$. This is a $\Theta$-regular Higgs sheaf on $\overline{X}_k$.

Before proceeding let us make the

**Definition 88.** Let $F$ be either $\mathbb{C}$ or a perfect field of positive characteristic.

a) Consider the restriction of the meromorphic Higgs bundle $\overline{\mathcal{H}}_F$ to a neighborhood $\mathcal{N}_{i,F}$; so that the Higgs field is singular only on the smooth divisor $Y_{i,F} \subset \mathcal{N}_{i,F}$.

We write, uniquely,

$$\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}} = (\mathcal{H}_F)_{\text{inf}} \oplus (\mathcal{H}_F)_{\text{inf}}$$

where $(\mathcal{H}_F)_{\text{inf}}$ is a Higgs bundle with non-singular Higgs field, and $(\mathcal{H}_F)_{\text{inf}}$ is a Higgs bundle whose Higgs field is singular along $Y_{i,F}$. We define

$$\text{supp}(\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}}) = \text{supp}((\mathcal{H}_F)_{\text{inf}}) \bigcup \text{supp}((\mathcal{H}_F)_{\text{inf}}_{\mid \mathcal{N}_{i,F} \setminus Y_{i,F}})$$

b) We define $\text{supp}(\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}})$ to be the unique variety obtained by gluing $\mathcal{L}_U$ with the subvarieties of $(T^*\mathcal{N}_{i,F})$ (the completion along $((T^*\mathcal{N}_{i,F})_{\mid Y_{i,F}}$ obtained in part a).

c) We define the support of $\overline{\mathcal{H}}_F$, $\text{supp}(\overline{\mathcal{H}}_F)$ to be the closure of $\text{supp}(\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}})$ inside $T^*(\mathcal{N}_{i,F})$.

d) We define the p-support of $\mathcal{N}_{k}$ or $\mathcal{N}_{\lambda,k}$ in a completely analogous fashion.

Now we may note the

**Claim 89.** The $p$-support of $\overline{N}_{i,k}$ is equal to $\text{supp}(\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}})_{\text{inf}} \times A^1_k$. The $p$-support of $\mathcal{N}_{\lambda,k}$ is equal to $\text{supp}(\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}})_{\text{inf}} \times A^1_k$.

After completing at $(\lambda)$ we see that the support of $\mathcal{N}_{\lambda,k}$ is equal to $\text{supp}(\overline{\mathcal{H}}_{\mid \mathcal{N}_{i,F}})_{\text{inf}} \times \text{Spec}(k[[\lambda]])$.

This follows directly from 87. In addition, we record the

**Proposition 90.** a) The set of sheaves $\overline{N}_{\lambda,k}$ is in bijection with $\text{Pic}(\overline{\mathcal{L}}_{W_k})_{(1)}$. The same is true as the set of sheaves $\overline{N}_{\lambda,k}$.

**Proof.** a) We work first over $W_k$. We have seen already in 87 that the set of $\mathcal{N}_{\lambda,k}$ is a torsor over $\text{Pic}(\overline{\mathcal{L}}_{W_k})_{(1)}$; choosing a splitting of the Azumaya algebra $\mathcal{D}_{X_k}$ $[\lambda, \lambda^{-1}]$ fixes the identification. Assume we have done this; let $\mathcal{M}_k$ be the line bundle associated to an $\mathcal{N}_{\lambda,k}$. We obtain extensions of $\mathcal{N}_{\lambda,k}|_{W_k}$ to $\mathcal{N}_{\lambda,k}|_{W_k}$ by looking at lattices inside the completion of $\mathcal{N}_{\lambda,k}|_{W_k}$ along each divisor $Y_{i,k}$; by such lattices are, in turn, indexed by extensions of $\mathcal{M}_k$ from $\overline{\mathcal{L}}_{W_k}$ to $\overline{\mathcal{L}}_{W_k}$, which gives the result on $W_k$ just as in 80. The result on $\overline{X}_k$ is obtained by pushing forward. $\Box$

Now, we remark that the existence of the bundle $\overline{N}_{\lambda,k}$ implies that $\overline{\mathcal{H}}_k$ is not arbitrary; in particular, it admits an infinitesimal deformation to a $\lambda$-connection (namely $\overline{N}_{\lambda,k}/\lambda^2$). We wish to exploit this condition systematically; a task to which we turn in the next subsections.
5.2.2. Obstructions for $\overline{H}_R$. In this section we work again over $R$. We wish to compute of the obstruction class in $H^1(\Omega^1_{\overline{X}_R})$ to deforming $\overline{H}_R$ to an infinitesimal $\lambda$-connection. In particular, we wish to consider deformations of $(\overline{H}_R, \Theta)$ to infinitesimal $\lambda$-connections which satisfy the appropriate regularity at infinity. We begin with the

**Definition 91.** a) Let $(\overline{N}_{\lambda,n,R}, \nabla)$ be a reflexive coherent sheaf over $\overline{X}_R \times \text{Spec}(R[\lambda]/\lambda^n)$ with meromorphic $\lambda$-connection, which is singular only at the divisors $Y_{i,R}$. Suppose that $\overline{N}_{\lambda,n,R}/\lambda = \overline{H}_R$. Then $\overline{N}_{\lambda,n,R}$ is said to be $\Theta$-regular at infinity if the following hold:

1) Let $Y_{i,R}$ be an intersection of components of the exceptional divisor, and let $\{P\}$ be any point contained in $Y_{i,R}$. Suppose that $\{Y_{j,R}\}_{j \in J}$ are the only divisors containing $\{P\}$. Consider $\pi : \overline{L}_R \to \overline{X}_R$; let $\{P'\}$ be any point lying above $\{P\}$. We demand that $\pi^*(\overline{N}_{\lambda,n,R})$, after localizing and completing at $\{P'\}$, admits a decomposition of the form

$$\pi^*(\overline{N}_{\lambda,n,R})[*Y_{j,R}] = \bigoplus_{i=1}^n \overline{N}_{i,\lambda,n,R} \otimes e_i[\ast Y_{j,R}]$$

where $\overline{N}_{i,\lambda,n,R}$ are sheaves (over $\text{Spec}(\hat{O}_{\overline{X}_R}[\lambda]/[\lambda^n])$) with a Higgs field which admits no pole; and $e_i$ are eigenvectors for the Higgs field.

2) There exists $\overline{N}_{\lambda,n,R}$, a reflexive coherent sheaf on $\overline{L}_R$ whose localization and completion at any $Y_{i,R}$ is $\bigoplus_{i=1}^n \overline{N}_{i,\lambda,n,R}$, and we have that $\overline{N}_{\lambda,n,R} = \overline{N}_{\lambda,n,R} \cap N_{U_R}$.

3) If $\Theta$ is the matrix of $\overline{H}_R$ in an adjusted eigenbasis as in lemma 78, we have

$$(\Psi - \Theta)(e_i) \subset \Omega^1_{\overline{X}_R}$$

i.e., $\Psi - \Theta$ has no poles. In particular, the “polar part” of $\overline{N}_{\lambda,n,R}$ is independent of $\lambda$.

The same definition works for $(\overline{N}_{\lambda,n,R}, \nabla)$ over $\overline{X}_R \times \text{Spec}(R[[\lambda]])$, and for $(\overline{N}_{\lambda,n,R}, \nabla)$ over $\overline{X}_R \times \text{Spec}(R[\lambda])$, and for any of the rings or fields to which $R$ maps.

b) Let $(\overline{N}_R, \nabla)$ be a reflexive coherent sheaf with connection on $\overline{X}_R$, which is singular only at the divisors $\{Y_{i,R}\}$. Let $\mathcal{N}_R$ be a neighborhood of $W'_R$ as above. The connection $\overline{N}_R$ will be said to be $\Theta$-regular at infinity if it satisfies the following

1) Suppose the localization and completion along the point $\{Y_{i,R}\}$ admits a decomposition

$$\pi^*(\overline{N}_R)[\ast Y_{i,R}] = \bigoplus_{i=1}^n \overline{N}_{i,R} \otimes e_i[\ast Y_{i,R}]$$

of connections, where $\overline{N}_{i,R}$ are sheaves (over $\text{Spec}(\hat{O}_{\overline{X}_R})$) with a connection which admits no pole; and $e_i$ are eigenvectors for the connection.

2) There exists $\overline{N}_R$, a reflexive coherent sheaf on $\overline{L}_R$ whose localization and completion at any $Y_{i,R}$ is $\bigoplus_{i=1}^n \overline{N}_{i,R}$, and we have that $\overline{N}_R = \overline{N}_R \cap N_{U_R}$. 


3) If $\Theta$ is the matrix of $H_R$ in an adjusted eigenbasis as in lemma 78, we have

$$\nabla - \Theta \in \Lambda^1_{L_R}$$

i.e., $\nabla - \Theta$ has no poles.

This definition works as well with $R$ replaced by any of the rings or fields to which it maps.

We note that the second definition seems rather less restrictive than the first—we are only demanding the regularity condition at generic point of intersections of divisors. For our purposes this is enough; indeed, it will turn out that for connections in characteristic zero these is essentially no difference between the notions.

With this definition in hand, we can consider the deformation problem for $H_R$. We begin with the

**Proposition 92.** Let $H_{\lambda,n,R}$ be a deformation of $H_R$ which is $\Theta$-regular at infinity. The obstruction to deforming $H_{\lambda,n,R}$ to a bundle $H_{\lambda,n+1,R}$ which is $\Theta$-regular at infinity is a class in

$$H^1(\text{Higgs}(F_R))$$

which, by 84 and our assumptions on $\mathcal{L}_R$, is equal to

$$H^1_{dR}(\mathcal{L}_R)$$

If a lift exists, it is unique.

**Proof.** Applying the general deformation theory of theorem 34 to those connections which are regular at infinity, we see that the deformation theory of $H_R$ is controlled by the extended Higgs complex of $F_R$ (just as in theorem 45). Thus the result follows from 84. \qed

Now we consider the deformation theory of $H_R$ to a $\lambda$-connection over $X_R \times_R R[\lambda]/\lambda^2$. By the proposition, the only thing to consider is the obstruction class in $H^1_{dR}(\mathcal{L}_R) = H^1(\text{Higgs}(F_R))$. We wish to consider it in more detail.

Let $\{M_{i,R}\}$ be the set of line bundles associated to $H_R$, as in lemma 83. We select a representative of the obstruction class in $H^1(\text{Higgs}(F_R))$; this is a collection of elements in

$$\bigoplus_{i=1}^r \text{Higgs}(\text{End}(M_{i,R}))(\pi^{-1}(U_{ij})) \cong \bigoplus_{i=1}^r \Omega^1_{\mathcal{L}_{U_{ij}}}$$

on the other hand we have

$$\bigoplus_{i=1}^r \text{Higgs}(\text{End}(M_{i,R}))(\pi^{-1}(U_{ij})) \cong \bigoplus_{i=1}^r \Omega^1_{\mathcal{L}_{U_{ij}}}$$

Taking the morphism $\pi^* : \Omega^1_{\mathcal{L}_{U_{ij}}} \rightarrow \Omega^1_{\mathcal{L}_{U_{ij}} \times_{U_{ij}} \mathcal{L}_{U_{ij}}} \cong \bigoplus_{i=1}^r \Omega^1_{\mathcal{L}_{U_{ij}}}$ on each $U_{ij}$ thus yields

a map $\pi^* : H^1(\text{Higgs}(F_R)) \rightarrow \bigoplus_{i=1}^r H^1(\Omega^1_{\mathcal{L}_R})$. Now we may prove the

**Proposition 93.** There is a unique choice of $H_R$ for which the obstruction class vanishes.
Proof. Let the image of the obstruction class for infinitesimally deforming \( \mathcal{H}_R \) under the map \( H^1(\mathcal{H}^1_{\text{Higgs}}(\mathcal{F}_R)) \to \bigoplus_{i=1}^r H^1(\Omega^1_{\mathcal{L}_R}) \) be denoted \( o' \); and consider the obstruction class for deforming the collection \( \{\mathcal{M}_{i,R}\}_{i=1}^r \) as an element of \( \bigoplus_{i=1}^r H^1(\Omega^1_{\mathcal{L}_R}) \); denote this class by \( o \); by lemma 83 this class is in the image of \( \pi^* \). Then we have the identity

\[
o - o' = \pi^* \left( \sum_{Y_{ij,R}} \sum_{j'} \sum_{k=1}^{l_{ij}} k[Y_{ij',R}] - \sum_{Y_{ij,R}} \sum_{j'} \sum_{k=1}^{l_{ij}} k[Y_{ij',R}] \right)
\]

where, in the first sum, the second term is over the infinite divisors \( Y_{ij,R} \), and in the second sum the second term is over the finite divisors \( Y_{ij,R} \). This follows from lemma 82, and second description of the obstruction by computing, over each open subset \( U_{ij,R} \), the change of basis between an eigenbasis of \( \bigoplus_{i=1}^r M_{\lambda,n,R} \) and a basis of \( (\pi)^*(\mathcal{N}_{\lambda,n,R}) \). So \( o' \) vanishes iff

\[
o = \pi^* \left( \sum_{Y_{ij,R}} \sum_{j'} \sum_{k=1}^{l_{ij}} k[Y_{ij',R}] - \sum_{Y_{ij,R}} \sum_{j'} \sum_{k=1}^{l_{ij}} k[Y_{ij',R}] \right)
\]

and, by the fact that the class \( o \) is indeed the image of a unique element under \( \pi^* \), we deduce the result. \( \square \)

Now we wish to show that this deformation extends all the way to \( R[[\lambda]] \). To do that, we return, for a moment, to the positive characteristic situation:

**Lemma 94.** Let \( \mathcal{N}_{\lambda,k} \) be an infinitesimal deformation of an \( \mathcal{H}_k \) over \( k[\lambda]/\lambda^2 \) which is \( \Theta \)-regular at infinity. Then \( \mathcal{N}_{\lambda,k} \) can be extended, in a unique way, to a deformation \( \mathcal{N}_{\lambda,k} \) over \( k[[\lambda]] \) which is \( \Theta \)-regular at infinity.

**Proof.** Applying the same logic as in the previous proof, but over \( k \) instead of \( R \), we see that the set of those \( \mathcal{H}_k \) which admit infinitesimal deformations which are \( \Theta \)-regular at infinity is a torsor over \( \text{Pic}(\mathcal{L}_k)^{(1)} \): namely, we are considering the set \( \{\mathcal{M}_{i,k}\} \) of line bundles on \( \mathcal{L}_k \) (associated to \( \mathcal{H}_k \)) which have deformation class in \( \bigoplus_{i=1}^r H^1(\Omega^1_{\mathcal{L}_k}) \) equal to

\[
\pi^* \left( \sum_{Y_{ij,R}} \sum_{j'} \sum_{k=1}^{l_{ij}} k[Y_{ij',R}] - \sum_{Y_{ij,R}} \sum_{j'} \sum_{k=1}^{l_{ij}} k[Y_{ij',R}] \right)
\]

which is manifestly a torsor over \( \text{Pic}(\mathcal{L}_k)^{(1)} \).

Now recall that the set of deformations of bundles of the form \( \mathcal{H}_k \) over \( k[[\lambda]] \) which are \( \Theta \)-regular at infinity is also a torsor over \( \text{Pic}(\mathcal{L}_k)^{(1)} \) by 90. One verifies directly that the actions of \( \text{Pic}(\mathcal{L}_k)^{(1)} \) are compatible under reduction mod \( \lambda^2 \). Since these torsors share an element (take any nontrivial \( \mathcal{N}_{\lambda,k} \) and then reduce mod \( \lambda^2 \)) the existence part of the lemma follows; for the uniqueness we apply 92. \( \square \)

Now we can prove the
Corollary 95. There is a unique choice of $\mathcal{H}_R$ which admits a deformation $\mathcal{M}_{\lambda,R}$ to a $\lambda$-connection over $R[[\lambda]]$ which is $\Theta$-regular at infinity.

Proof. It suffices to show that the Higgs bundle of 93 extends to a $\lambda$-connection over $R[[\lambda]]$ which is $\Theta$-regular at infinity. Suppose, inductively, that it extends to $\mathcal{M}_{\lambda,n,R}$ over $R[\lambda]/\lambda^n$, for $n \geq 2$. Consider the obstruction class to extending over $R[\lambda]/\lambda^{n+1}$. After reduction mod $p$ for $p > 0$, this class reduces to the obstruction class for extending $\mathcal{M}_{\lambda,n,k}$ to $k[\lambda]/\lambda^{n+1}$. By the previous lemma it vanishes for $p > 0$. The result follows. \hfill \Box

5.3. The preferred lift of the $\lambda$-connection $\mathcal{M}_{\lambda,k}$. In this section, we construct a preferred $\lambda$-connection $\mathcal{M}_{\lambda,k}$ which admits a lift to characteristic zero. We begin with the

Proposition 96. Every reflexive meromorphic $\lambda$-connection on $\mathcal{X}_k \times \text{Spec}(k[\lambda])$ whose $p$-support is equal to $\text{supp}(\mathcal{H}_k)^{(1)} \times \text{Spec}(k[\lambda])$ (with multiplicity one), and which is $\Theta$-regular at infinity, is isomorphic to one of the $\mathcal{N}_{\lambda,k}$ of 87.

Similarly, every reflexive meromorphic $\lambda$-connection on $\mathcal{X}_k \times \text{Spec}(k[[\lambda]])$ whose $p$-support is equal to $\text{supp}(\mathcal{H}_k)^{(1)} \times \text{Spec}(k[[\lambda]])$ (with multiplicity one), and which is $\Theta$-regular at infinity, is isomorphic to one of the $\mathcal{N}_{\lambda,k}$.

Proof. We consider the polynomial case $\mathcal{X}_k \times \text{Spec}(k[\lambda])$, the case of formal power series being similar.

We first consider the situation over $U_k$; let $\mathcal{P}_{\lambda,k}$ be our meromorphic $\lambda$-connection. We have $\text{supp}(\mathcal{H}_k)^{(1)} \cap T^*U_k^{(1)} = \mathcal{L}_{U_k}^{(1)}$, since $\mathcal{L}_{U_k}^{(1)} \to U_k^{(1)}$ is finite etale, there exists a line bundle $\mathcal{M}_{\lambda,k}$ with $\lambda$ connection on $\mathcal{L}_{U_k}$ such that $P_{\lambda,U_k} = \pi_*(\mathcal{M}_{\lambda,k})$ and, after possibly further localizing $U_k$, we have $\mathcal{M}_{\lambda,k}/\lambda = (O_{\mathcal{L}_{U_k}}, df)$. We wish to show that $\mathcal{M}_{\lambda,k} = e^j$, where $e^j$ is the line bundle with $\lambda$-connection whose generator $e$ satisfies $\nabla(e) = df \cdot e$.

Write $e$ for a generator of $\mathcal{M}_{\lambda,k}$; the $\lambda$-connection can be written $\nabla e = ge$, where $g|_{\lambda=0} = df$. Write $g - df = \lambda \omega$ for some one form $\omega = \sum_{i=0}^l \lambda^i \omega_i \in \Omega^2_{\mathcal{L}_{U_k}}[\lambda]$. The condition on our $\lambda$-connection $\mathcal{M}_{\lambda,k}$ is that the $p$-curvature is equal to $(df)^p$. Therefore the $p$-curvature of the line bundle with $\lambda$-connection on $\mathcal{L}_{U_k}$ defined by $\nabla'(e) = \omega e$ is equal to zero. Let us examine this condition: suppose $\{\partial_i, \ldots, \partial_n\}$ is a generating set of derivations in some neighborhood inside $U_k$. Then in these coordinates the $\lambda$-connection $\nabla'$ having vanishing $p$-curvature amounts to the condition

$$\sum_{i=0}^l \lambda^i \omega_i (\partial_j)^p + \lambda^{p-1} \sum_{j=0}^l \lambda^i \partial_j^{p-1} (\omega(\partial_j)) = 0$$

This implies that for each $i$ such that $\omega_i(\partial_j) \neq 0$, there is an $n$ with $\omega_n(\partial_j) \neq 0$ such that $n = (i - 1)p + 1$. Thus we see that $\omega_1$ is the only possible nonzero term; and further the connection (not $\lambda$-connection) defined by $\nabla'(e) = \omega_1 e$ has $p$-curvature zero.

Therefore, by Cartier descent, this line bundle is trivial; i.e. there exist an invertible function $h \in O_{\mathcal{L}_{U_k}}$ so that $\nabla''(he) = (dh + \omega_1)e = 0$, so that $dh = -\omega_1 = \frac{\omega}{\lambda}$. 
But then we have
\[ \nabla (he) = (\lambda dh + g)e = (\lambda dh + df + \lambda \omega)e = (df)e \]
which implies \( M_{\lambda,k} \cong e^f \) on some open subset \( U'_k \) of \( U_k \).

Now, consider \( \mathcal{P}_{\lambda,k} \) near the smooth locus of a component \( Y_{i,k} \). The condition on the support implies a decomposition
\[ \mathcal{P}_{\lambda,k} = (\mathcal{P}_{\lambda,k})_{\text{fin}} \oplus (\mathcal{P}_{\lambda,k})_{\text{inf}} \]
where \( (\mathcal{P}_{\lambda,k})_{\text{fin}} \) is a \( \lambda \)-connection without singularity, and \( (\mathcal{P}_{\lambda,k})_{\text{inf}} \) is a meromorphic \( \lambda \)-connection with singularity.

Consider the quasicoherent sheaf with \( \lambda \)-connection \( P_{\lambda,k} \) on \( W_k \) obtained by extending \( P_{\lambda,U'_k} \) over each \( Y_{i,k} \) via the extension \( (\mathcal{P}_{\lambda,k})_{\text{fin}} \oplus (\mathcal{P}_{\lambda,k})_{\text{inf}} \otimes [Y_{i,k}] \). After localizing at \( \mathcal{P} \) this is a \( D_{W_k}[\lambda, \lambda^{-1}] \)-module whose \( p \)-support is equal to \( \mathcal{L}_{W_k} \) with multiplicity one. Since, in addition, we have shown that \( P_{\lambda,U'_k} \cong \pi_* (e^f) \), it follows that that \( P_{\lambda,k} \) can be constructed by the procedure of 87, so that \( P_{\lambda,k} \cong N_{\lambda,k} \) for some \( N_{\lambda,k} \) as above. Since, in addition, \( \mathcal{P}_{\lambda,k} \) is \( \Theta \)-regular at infinity, we must have \( \mathcal{P}_{\lambda,k} \cong \overline{N}_{\lambda,k} \) for some \( \overline{N}_{\lambda,k} \) as required. \( \square \)

Now, with the help of a forward reference to the next section, we can prove the

**Lemma 97.** Let \( \mathcal{P}_k \) be a Higgs bundle which admits a deformation (necessarily unique) to a \( \lambda \)-connection \( \overline{N}_{\lambda,k} \) over \( k[[\lambda]] \), which is \( \Theta \)-regular at infinity. Then there is a unique \( \lambda \)-connection \( \overline{N}_{\lambda,k} \) over \( k[\lambda] \), with \( p \)-support equal to \( \text{supp}(\mathcal{P}_k)^{(1)} \times A^1_k \) whose completion at \( (\lambda) \) is isomorphic to \( \overline{N}_{\lambda,k} \).

**Proof.** Consider the \( p \)-support of \( \overline{N}_{\lambda,k} \) inside \( T^*(\mathcal{X}_k)^{(1)} \times \text{Spec}(k[[\lambda]]) \). In the next section we shall show, using the structure of the Hilbert scheme (c.f. lemma 109 and lemma 110 below) that this \( p \)-support is equal to \( \text{supp}(\mathcal{P}_k)^{(1)} \times \text{Spec}(k[[\lambda]]) \). Therefore the result follows from the previous one. \( \square \)

Given this, we define \( M_{\lambda,k} \) so that \( M_{\lambda,k} / \lambda = \mathcal{P}_k \), the specialization of the \( \mathcal{P}_R \) of 95. As above, we denote by \( \overline{M}_{\lambda,k} \) the completion of \( M_{\lambda,k} \) along the ideal generated by \( \lambda \).

We consider lifts of \( M_{\lambda,k} \) to \( \lambda \)-connections \( M_{\lambda,W_n(k)} \) over the ring of Witt vectors \( W_n(k) \).

As one might expect, the deformation theory of such connections is rather restricted. We shall elucidate its structure in this section. We begin with the

**Proposition 98.**  

a) Consider the trivial \( \lambda \)-connection \( (\mathcal{O}_{\mathcal{Z}_k}[[\lambda]], d_{\lambda}) \). We have isomorphisms
\[ \mathcal{H}^i_{dR}(\mathcal{O}_{\mathcal{Z}_k}[[\lambda]]) \cong \text{Ker}(d^i) \oplus \lambda \cdot \Omega^i_{\mathcal{Z}_k}[[\lambda]] \]
where \( d^i : \Omega^i_{\mathcal{Z}_k} \to \Omega^{i+1}_{\mathcal{Z}_k} \) is the usual differential.

b) Consider the trivial formal \( \lambda \)-connection \( (\mathcal{O}_{\mathcal{Z}_k}[[\lambda]], d_{\lambda}) \). We have isomorphisms
\[ \mathcal{H}^i_{dR}(\mathcal{O}_{\mathcal{Z}_k}[[\lambda]]) \cong \text{Ker}(d^i) \oplus \lambda \cdot \Omega^i_{\mathcal{Z}_k}[[\lambda]] \]
Proof. We prove a); part b) is completely similar. Consider the operator $d^i_\lambda : \Omega^i_{\mathbb{L}_k}[\lambda] \to \Omega^{i+1}_{\mathbb{L}_k}[\lambda]$, which is defined as $d^i(\sum_{i=0}^{n} f_i \lambda^i) = \sum_{i=0}^{n} (d^if) \lambda^{i+1}$. This formula shows that $\text{Ker}(d^i_\lambda) = \text{Ker}(d^i)[\lambda]$

While $\text{Im}(d^{i-1}) = \lambda \cdot \text{Im}(d^i-1)[\lambda]$

whence the result. \(\square\)

Now we wish to apply this to compute the appropriate cohomology groups of our $\lambda$-connection. As we did in lemma 77 and the remarks following it, we may construct an eigenvector extension of $E_{\lambda,k} = \text{End}(M_{\lambda,k})$ to a meromorphic connection $\overline{E}_{\lambda,k}$ on $\overline{X}_k$. This yields a version of the deRham complex with cohomology $H^i_{dR}(E_{\lambda,k})$; similar constructions work for the completed versions of these sheaves ($\hat{E}_{\lambda,k}$, etc).

Then, just as in 84 we have

**Proposition 99.** The $\lambda$-connection $\overline{E}_{\lambda,k}$ admits a sub-$\lambda$-connection $\overline{V}_{\lambda,k}$, such that the action of the $\lambda$-connection on $\overline{V}_{\lambda,k}$ is the trivial action. The inclusion induces isomorphisms

$$H^i_{dR}(\overline{V}_{\lambda,k}) \cong H^i_{dR}(\overline{E}_{\lambda,k})$$

In addition, we have that

$$H^i_{dR}(\overline{V}_{\lambda,k}) \cong \pi_* (H^i_{dR}(O_{\mathbb{L}_k} [\lambda]))$$

Similarly, we have isomorphisms

$$H^i_{dR}(\overline{E}_{\lambda,k}) \cong \pi_* (H^i_{dR}(O_{\mathbb{L}_k} [[\lambda]])$$

Given this, we can state and prove the

**Theorem 100.** a) Let $\overline{M}_{\lambda,W_n(k)}$ be a lift of $\overline{M}_{\lambda,k}$ over $W_n(k)$ which is $\Theta$-regular at infinity. Then the obstruction class to lifting $\overline{M}_{\lambda,W_n(k)}$ to $\overline{M}_{\lambda,W_{n+1}(k)}$ which is $\Theta$-regular at infinity lives in $H^1(H^i_{dR}(O_{\mathbb{L}_k} [\lambda]))$

A lift, if it exists, is unique.

b) The obstruction class to lifting the completion $\overline{M}_{\lambda,W_n(k)}$ (to a formal $\lambda$-connection which is $\Theta$-regular at infinity) is a class in $H^1(H^i_{dR}(O_{\mathbb{L}_k} [[\lambda]]))$, which is the image of the above class under the natural map $H^1(H^i_{dR}(O_{\mathbb{L}_k} [\lambda])) \to H^1(H^i_{dR}(O_{\mathbb{L}_k} [[\lambda]]))$

This map is an inclusion. A lift, if it exists, is unique.

**Proof.** By the previous proposition, we have isomorphisms

$$H^1_{dR}(M_{\lambda,k}) \cong \pi_* (H^1_{dR}(O_{\mathbb{L}_k} [\lambda]))$$

So, arguing as before (in, e.g., 92) the obstruction to lifting lives in $H^1(\pi_* (H^1_{dR}(O_{\mathbb{L}_k} [\lambda])))$
and the first and last groups vanish by the lemma below and 98. Similarly, the set of lifts is a torsor over
\[H^0(\pi_* (\mathcal{H}^1_{dR}(O_{\overline{\mathbb{L}_k}[[\lambda]]}))) \oplus H^1(\pi_* (\mathcal{H}^0_{dR}(O_{\overline{\mathbb{L}_k}}[[\lambda]])))\]
which vanishes by the lemma below and 98. This shows a); the first and last statements of b) are proved in an identical fashion.

Now we consider the natural map
\[H^1(\mathcal{H}^0_{dR}(O_{\overline{\mathbb{L}_k}}[[\lambda]])) \to H^1(\mathcal{H}^0_{dR}(O_{\overline{\mathbb{L}_k}}[[\lambda]]))\]
By the theorem on formal functions ([H], theorem 3.11.1) this is the map of completing along \((\lambda)\). So this is an inclusion iff the group \(H^1(\mathcal{H}^0_{dR}(O_{\overline{\mathbb{L}_k}}[[\lambda]]))\) has no \(\lambda\)-torsion. By 98 this group is isomorphic to
\[H^1(\ker(d^1)) \bigoplus \bigoplus_{i=1}^{\infty} \lambda^i H^1(\Omega^1_{\overline{\mathbb{L}_k}^{(i)}})\]
where the action of \(\lambda\) on \(H^1(\ker(d^1))\) is given by the map \(H^1(\ker(d^1)) \to H^1(\Omega^1_{\overline{\mathbb{L}_k}^{(1)}})\) induced by the Cartier isomorphism; which is injective by part c) of the lemma below.

So we see that we must compute the cohomology groups appearing in 98. This computation is

**Lemma 101.** Let \(d^i\) be the differential in the deRham complex for \(\overline{\mathbb{L}_k}\); so that \(d^i : \Omega^i_{\overline{\mathbb{L}_k}} \to \Omega^{i+1}_{\overline{\mathbb{L}_k}}\). We have

a) \(H^0(\ker(d^1)) = 0 = H^1(\ker(d^0))\)
b) \(H^0(\ker(d^2)) = 0 = H^2(\ker(d^0))\)
c) The natural map \(H^1(\ker(d^1)) \to H^1(\Omega^1_{\overline{\mathbb{L}_k}^{(1)}})\) is injective; as is the map induced by the Cartier isomorphism \(H^1(\ker(d^i)) \to H^1(\Omega^1_{\overline{\mathbb{L}_k}^{(i)}})\).

**Proof.** We have \(\ker(d^0) = O_{\overline{\mathbb{L}_k}^{(1)}}\), so both statements about this sheaf follow from theorem 64; since we have \(H^1(\Omega^1_{\overline{\mathbb{L}_k}^{(1)}}) = H^2(\Omega_{\overline{\mathbb{L}_k}^{(1)}}) = 0\). Next, note that \(\ker(d^2) \subset \Omega^2_{\overline{\mathbb{L}_k}}\) by definition, so that \(H^0(\ker(d^2)) = 0\). Finally, we have the exact sequence
\[0 \to \ker(d^1) \to \Omega^1_{\overline{\mathbb{L}_k}} \to d^1(\Omega^1_{\overline{\mathbb{L}_k}}) \to 0\]
yielding, first, \(H^0(\ker(d^1)) = 0\) and then the exact sequence
\[0 \to H^0(d^1(\Omega^1_{\overline{\mathbb{L}_k}})) \to H^1(\ker(d^1)) \to H^1(\Omega^1_{\overline{\mathbb{L}_k}})\]
However, we have the inclusion \(d^1(\Omega^1_{\overline{\mathbb{L}_k}}) \subset \Omega^2_{\overline{\mathbb{L}_k}}\) which gives \(H^0(d^1(\Omega^1_{\overline{\mathbb{L}_k}})) = 0\); yielding the first conclusion of part c). As for the second, we have the exact sequence
\[0 \to \Omega^1_{\overline{\mathbb{L}_k}} \to \Omega^2_{\overline{\mathbb{L}_k}} \to \text{Im}(d^0) \to 0\]
So, from \(H^1(\Omega^1_{\overline{\mathbb{L}_k}}) = H^2(\Omega_{\overline{\mathbb{L}_k}^{(1)}}) = 0\) we obtain \(H^1(\text{Im}(d^0)) = 0\); so that the required injectivity follows from
\[0 \to \text{Im}(d^0) \to \ker(d^1) \to \Omega^1_{\overline{\mathbb{L}_k}^{(1)}} \to 0\]
which is the exact sequence induced by the Cartier isomorphism. \(\square\)
Theorem 102. Let $n \geq 1$. The obstruction class to lifting $\mathcal{M}_{\lambda,W_n(k)}$ to a connection which is $\Theta$-regular at infinity vanishes. Thus, there is a unique reflexive meromorphic connection lifting $\mathcal{M}_{\lambda,k}$ which is $\Theta$-regular at infinity for each $n \geq 0$.

Proof. Only the first sentence remains to be shown. To see this, consider $\mathcal{H}_{W_n(k)} = \mathcal{M}_{\lambda,W_n(k)}/\lambda$. We know from 86 that since $\mathcal{H}_{W_n(k)}$ is $\Theta$-regular at infinity, it admits a unique lift to $\mathcal{H}_{W_n+1(k)}$. By our choice of $\mathcal{H}_{W_n+1(k)}$, this Higgs bundle admits a deformation to a formal $\lambda$-connection which is $\Theta$-regular at infinity (because of 95). The reduction mod $p^n$ of this bundle is a formal $\lambda$-connection which deforms $\mathcal{H}_{W_n(k)}$ and hence is necessarily isomorphic to $\mathcal{M}_{\lambda,n}$. Thus the obstruction class to lifting $\mathcal{M}_{\lambda,n}$ vanishes; and so the same is true for $\mathcal{M}_{\lambda,n}$ by theorem 100. \hfill \Box

Finally, we remark on the situation at $\lambda = 1$:

Corollary 103. The connection $\mathcal{M}_k$ admits a unique lift to a reflexive meromorphic connection $\mathcal{M}_{\lambda,k}$ which is $\Theta$-regular at infinity. Amongst the meromorphic connections $\mathcal{N}_k$ which have $p$-support equal to $\text{supp}(\mathcal{H}_k)^{(1)}$ and which are regular at infinity, $\mathcal{M}_k$ is the only one with this property.

Proof. The first sentence follows from the previous theorem: we obtain the existence of such a lift by setting $\lambda = 1$, and the uniqueness follows since the relevant cohomology group is trivial.

For the second sentence, by the construction of section 5.2.1 we have that $\mathcal{N}_k$ admits a deformation to $\mathcal{N}_{\lambda,k}$. After localizing at $\lambda$, we obtain the connection $\mathcal{N}_{\lambda,k}[\lambda^{-1}]$. This may be regarded as a flat family of connections, indexed by $\mathbb{A}^1 \setminus \{0\}$; and we know from 90 that the set of these is a torsor over $\text{Pic}(\mathcal{L}_k)$; the same is true of the set of $\mathcal{N}_k$ which have $p$-support equal to $\text{supp}(\mathcal{H}_k)^{(1)}$ (by essentially the same argument). Thus we conclude that each such family is trivial; i.e.,

$$\mathcal{N}_{\lambda,k}[\lambda^{-1}] \cong \mathcal{N}_k[\lambda,\lambda^{-1}] \tag{5.3}$$

Furthermore, we have from lemma 101 that the map

$$H^1(\ker(d^1)) \to H^1(\Omega^1_{\mathcal{L}_k})[\lambda,\lambda^{-1}]$$

is injective; this implies that lifts of $\mathcal{N}_{\lambda,k}$ are unobstructed iff lifts of $\mathcal{N}_{\lambda,k}[\lambda^{-1}]$ are unobstructed iff lifts of $\mathcal{N}_k$ are unobstructed (because of equation (5.3)). Since $\mathcal{M}_{\lambda,k}$ is the unique $\lambda$-connection which can be lifted, the result follows. \hfill \Box

5.4. Algebrization and uniqueness. Now our remaining task is, as in the one-dimensional case, to show that the inverse limit $(\mathcal{M}_{\lambda})_{W(k)}$ is algebrizable to an algebraic flat connection over $\mathcal{X}_{W(k)}$, and then to show that the resulting connection over $\mathcal{X}_K$ is the one we want.

The first statement is, as before, a direct consequence of formal GAGA; the argument in the one-dimensional case goes over without essential change to this case. So we shall simply state the

Theorem 104. There is a coherent sheaf with $\lambda$-connection $\mathcal{M}_{\lambda,W(k)}$ on $\mathcal{X}_{W(k)} \times \mathbb{A}^1_{W(k)}$, which is the push-forward of a vector bundle on $W_{W(k)} \times \mathbb{A}^1_{W(k)}$, as before.
W is the open subset obtained by discarding the intersections of components of the divisor D. The completion of this λ-connection along the maximal ideal of \( W(k) \) is precisely \( (\hat{M}_\lambda)_W(k) \). This connection satisfies the \( \Theta \)-regularity condition at the generic point of each exceptional divisor \( Y_{j,W(k)} \), and at the generic point of each intersection of such divisors.

After inverting the prime \( p \), we obtain a reflexive coherent sheaf with λ-connection \( \hat{M}_\lambda,W \) on \( \hat{X}_K \times \mathbb{A}^1_K \); this connection is \( \Theta \)-regular at infinity.

Now we devote ourselves to showing the uniqueness of this object. In fact we shall show the

**Theorem 105.** a) Let \( K \) be any field of characteristic zero; suppose that \( \hat{M}_\lambda,K \) and \( \hat{N}_\lambda,K \) are two reflexive coherent sheaves with λ-connection, which are \( \Theta \)-regular at infinity, such that

\[
\hat{M}_\lambda,K/\lambda = \hat{N}_\lambda,K/\lambda = \hat{N}_\lambda,K.
\]

Then we have, after specializing to \( \lambda = 1 \), \( \hat{N}_K = \hat{M}_K \).

b) The reflexive meromorphic connection \( \hat{M}_K \) from part a) has constant arithmetic support, equal to \( \text{supp}(\hat{H}_C) \) with multiplicity one. It is the unique such connection which is \( \Theta \)-regular at infinity.

This is the analogue of the notion of rigidity in the one-dimensional case; once it has been shown, we obtain a meromorphic connection \( \hat{M}_F \) (for \( F = \text{Frac}(R) \)) whose restriction to \( \hat{X}_F \) has constant \( p \)-support equal to \( \hat{L}_F \), just as in 59 and lemma 60.

In principle, I believe that this theorem can be proved by purely characteristic zero methods. In particular, given the cohomology vanishing for \( \hat{H}_K \), the objects that appear have a strong rigidity property which, combined with the fact that they are both λ-connections deforming the same object, should imply uniqueness. The problem is that the sorts of arguments one would need- involving moduli spaces of meromorphic connections- to make this idea rigorous do not seem to be available as of this time (although, much progress has been made; c.f., e.g., [Mo]).

Therefore, we shall take a different route, and prove the uniqueness by simultaneously proving that these connections have constant \( p \)-support; and then showing that this implies the existence of a nonzero morphism of \( \lambda \)-connections between the two objects by reduction mod \( p \).

The reason that we can do this is that, instead of using properties of moduli spaces of connections, we can use the \( p \)-support to reduce our question to properties of moduli spaces of subvarieties of a variety; i.e., properties of the Hilbert scheme; where there are no problems with existence. We start with the

**Lemma 106.** Let \( \hat{N}_{\lambda,k} \) be a meromorphic λ-connection on \( \hat{X}_k \), where \( k \) is a perfect field of positive characteristic. Suppose that \( \hat{N}_{\lambda,k}/\lambda = \hat{H}_k \). Then the \( p \)-support of \( \hat{N}_{\lambda,k} \) (defined as in 88) is an \( \mathbb{A}^1_k \)- flat family of subvarieties of \( T^*(\hat{X}_k)^{(1)} \), which specializes at \( \lambda = 0 \) to \( \mathbb{L}_k^{(1)} \).

**Proof.** The connection \( \hat{N}_{\lambda,k} \) is flat over \( \mathbb{A}^1_k \) by assumption. Hence the same is true of its annihilator in \( T^*(\hat{X}_k)^{(1)} \times \mathbb{A}^1_k \). \( \square \)

We wish to combine this with the fact that \( \hat{N}_{\lambda,k} \) is supposed to be \( \Theta \)-regular at infinity- at least at the generic points of divisors and their intersections- to obtain
strong limitations on the possible p-support. In order to take advantage of this condition, we shall work with some compactifications of $T^*(\mathcal{X}_k)$ and $T^*(\mathcal{L}_k)$. In order to lighten the notation, we shall denote these compactifications by $\overline{T^*(\mathcal{X}_k)}$ and $\overline{T^*(\mathcal{L}_k)}$. They are defined as "projectivizations" of the usual cotangent bundles; more precisely we have the

**Definition 107.** Let $Z_E$ be a smooth variety over the perfect field $E$ (which could be of any characteristic). The variety $T^*Z_E \times \mathbb{A}^1_k$ has a $\mathbb{G}_m$ action given by

$$\lambda(z, \xi, a) = (z, \lambda \xi, \lambda a)$$

which has $\{Z_E \times \{0\}\}$ as its fixed point set. Let $(T^*Z_E \times \mathbb{A}^1_k)^o := (T^*Z_E \times \mathbb{A}^1_k) \setminus Z_E \times \{0\}$. Then $\mathbb{G}_m$ acts freely on this variety and so we can define

$$\overline{T^*Z}_E := (T^*Z_E \times \mathbb{A}^1_k \setminus Z_E \times \{0\})/\mathbb{G}_m$$

As noted above, when $Z_E = \overline{\mathcal{X}_k}$ or $\overline{\mathcal{L}_k}$, we shall use the notation $\overline{T^*(\mathcal{X}_k)}$ and $\overline{T^*(\mathcal{L}_k)}$, respectively.

By construction these compactifications come with natural projection maps $p : \overline{T^*Z}_E \rightarrow Z_E$ which are smooth and have fibres isomorphic to projective space. From this we deduce the existence of the morphism of fibre-wise addition

$$+: \overline{T^*Z}_E \times_{Z_E} \overline{T^*Z}_E \rightarrow \overline{T^*Z}_E$$

which extends the usual addition on $T^*Z_E \times_{Z_E} T^*Z_E$; this map satisfies $z + y = z$ for any $z \in \overline{T^*Z}_E \setminus T^*Z_E$.

Now let us consider some subvarieties: we start with $\overline{T^*(\mathcal{L}_k)}$ and we consider the closure of $\Gamma(df) \subset T^*(\mathcal{L}_k)$. We note that $\overline{T^*(\mathcal{L}_k)}$ is identified with the subvariety of $(T^*\mathcal{L}_k \times \mathbb{A}^1_k \setminus \{0\})/\mathbb{G}_m$ on which the $\mathbb{A}^1_k$ coordinate is nonzero. So we can normalize by writing $\Gamma(df) = (x, df(x), 1)$.

Suppose that $f$ has a pole at a divisor $Y_{ij,k}$; which is defined via a local coordinate $\{y = 0\}$. Write

$$f = y^{-k}a_{-k} + y^{-k+1}a_{-k+1} + \ldots$$

Then $(x, df(x), 1) = (x, \frac{y(x)df(x)}{f(x)}, \frac{y(x)}{f(x)})$ inside $T^*(\mathcal{L}_k) \setminus Z(f)$. But $\frac{y(x)df(x)}{f(x)}$ extends to a well defined expression over $\{y = 0\}$, and it is generically nonzero there. Thus the closure of $\Gamma(df)$ is precisely the set of

$$(x, \frac{y(x)df(x)}{f(x)}, 0)$$

where this expression is well defined for $x \in Y_{ij,k}$. We see that this closure, denoted, $\overline{\Gamma(df)}$, is isomorphic to $\overline{\mathcal{L}_k}$ in an open subset whose compliment has codimension $\geq 2$; let us call this set $\mathcal{V}_k$. The same considerations hold for the Galois conjugates $\Gamma(df_i)$, and for the negatives $\Gamma(-df_i)$.

Now let us consider the set of infinitesimal deformations of such subschemes: recall from [Hart], chapter 1.2, that by definition, an infinitesimal deformation of a subscheme $Y_k \subset Z_k$ is an $\epsilon$-flat subscheme $Y'_k \subset Z'_k = Z_k \times_k k[\epsilon]$ (where $\epsilon^2 = 0$), which specializes at $\epsilon = 0$ to $Y$. If $\mathcal{I}$ denotes the ideal sheaf of $Y_k$, then the set of such deformations is in bijection with $H^0(N_\mathcal{Y}) := H^0(Hom_\mathcal{Y}((\mathcal{I}/\mathcal{I}^2, O_\mathcal{Y}))$ ([Hart], Theorem 2.4). Then we have:
Lemma 108. For each \( i \), there is a unique infinitesimal deformation of \( \Gamma(df_i) \) (namely, the trivial deformation). The same is true over \( V_k \).

Proof. We consider the morphism \( \psi : T^*V_k \to T^*V_k \) defined by \( (v, \xi) \mapsto (v, \xi + df_i) \). This map satisfies \( \psi(V_k) = \Gamma(df_i) \), and inside \( T^*U_k \) the map is an isomorphism in a neighborhood of \( L_{U_k} \). We claim that \( \psi^* \) takes infinitesimal flat deformations of \( \Gamma(df_i) \) to infinitesimal flat deformations of \( V_k \): let \( \mathcal{A}_k \) be an infinitesimal flat deformation of \( \Gamma(df_i) \); we may regard it as a vector bundle on \( \Gamma(df_i) \), equipped with a short exact sequence

\[
0 \to O_{\Gamma(df_i)} \to \mathcal{A}_k \to O_{\Gamma(df_i)} \to 0
\]

Applying \( \psi^* \) yields a vector bundle on \( V_k \), equipped with a short exact sequence

\[
O_{V_k} \to \psi^* \mathcal{A}_k \to O_{V_k} \to 0
\]

whose left hand map is injective upon restriction to \( U_k \). Since this is a morphism of bundles, it is injective everywhere, and \( \psi^* \mathcal{A}_k \) is a flat deformation of \( V_k \).

This construction yields a morphism of sheaves \( N_{\Gamma(df_i)} \to \psi_* (N_{V_k}) \); since this is in fact a morphism of torsion free sheaves which is generically injective, it is injective everywhere. Thus it is enough to show that \( V_k \) has no infinitesimal deformations. However, since \( V_k \) is a Lagrangian subvariety of \( T^*V_k \), we have an isomorphism

\[
N_{V_k} = \Omega^1_{V_k}
\]

and we have \( H^0(\Omega^1_{V_k}) = 0 \) since \( V_k \) has codimension \( \geq 2 \) in \( L_k \). The statement for \( \Gamma(df_i) \) follows, again by the fact that the normal sheaf \( N_{\Gamma(df_i)} \) is torsion free. \( \square \)

This result immediately implies, by induction, that any deformation of \( \Gamma(df_i) \) over any ring of the form \( k[[\epsilon]]/(\epsilon^n) \) is isomorphic to the trivial deformation.

Now consider any deformation of \( \bigcup_{i=1}^{r} \Gamma(df_i) \) over the ring \( k[[\epsilon]]/(\epsilon^n) \). If such a deformation is the scheme-theoretic union of deformations of the \( \Gamma(df_i) \)- i.e., its defining ideal sheaf is given by the intersection of the ideal sheaves of such deformations- then this deformation, being the union of trivial deformations, is trivial as well. The same is true for deformations over \( k[[\epsilon]] \).

With this result in hand, we should like to work out the consequences for \( \lambda \)-connections. Suppose that \( \mathcal{P}_{\lambda,k} \) be a meromorphic \( \lambda \)-connection on \( \tilde{X}_k \), which is \( \Theta \)-regular at infinity at the generic point of each \( Y_{ij,k} \), and such that \( \mathcal{P}_{\lambda,k}/\lambda = \mathcal{P}_{\lambda,k} \).

By definition, this means that there is a meromorphic \( \lambda \)-connection on \( \mathcal{L}_k \), \( \pi^* \mathcal{P}_{\lambda,k} \), whose intersection with \( P_{\lambda,k}(U_k) \) defines the extension \( \mathcal{P}_{\lambda,k} \); and which satisfies the appropriate conditions at the generic point of each divisor \( Y_{ij,k} \). Let \( \pi^* \mathcal{P}_{\lambda,k} \) be the completion at \( \lambda \). Then we have the

Lemma 109. Let \( \pi^* \mathcal{P}_{\lambda,k} \) be as above. Then, over \( V_k \), the closure of its \( p \)-support \( \Gamma^*(\mathcal{L}_k) \times \mathbb{A}^1_{x_k} \) is the scheme-theoretic union of deformations of \( \Gamma(df_i) \). This support is flat over \( \lambda \).

Proof. Denote the \( p \)-support in question by \( \mathcal{S}_\lambda \), and the closure in \( \Gamma^*(\mathcal{L}_k) \times \mathbb{A}^1_{x_k} \) by \( \mathcal{S}_\lambda^\circ \). First we want to show that \( \mathcal{S}_\lambda \) is a scheme theoretic union of deformations of \( \Gamma(df_i) \). To see this, note that over the open subset \( \mathcal{L}_{U_k} \), we have that \( \pi^* P_{\lambda,\mathcal{L}_{U_k}} \)
is a deformation of the Higgs bundle \( \bigoplus_{i=1}^{r}(O_{\mathcal{U}_k}, df_i) \). Since the support of this bundle is the disjoint union of the \( \{ \Gamma(df_i) \} \), the bundle \( \pi^* P_{\lambda}|_{\mathcal{U}_k} \) is a direct sum of line bundles with \( \lambda \)-connection (this can be proved exactly as the “first proof” of lemma 42). Thus the annihilator of this bundle (inside \( O(T^* \mathcal{L}_k^{(1)} \times \mathbb{A}_k^{1}) \)) is the intersection of the annihilators of each of the summands. Since \( \pi^* P_{\lambda} \) is a torsion free sheaf, its annihilator (inside \( O(T^* V_k \times \mathbb{A}_k^{1}) \)) is simply the set of local sections of \( O(T^* V_k \times \mathbb{A}_k^{1}) \) whose restriction to \( \mathcal{U}_k \) annihilate \( \pi^* P_{\lambda}|_{\mathcal{U}_k} \) - which is thus an intersection of ideals which deform the \( \Gamma(df_i) \).

Now we want to look at \( \mathfrak{X}_\lambda \). Over any divisor \( Y_{ij,k} \), we have an isomorphism

\[
\pi^*(\mathfrak{P}_{\lambda,k})\{Y_{ij,k}\} \cong \bigoplus_{i=1}^{r} O\{Y_{ij,k}\}[\lambda][Y_{ij,k}] \cdot e_i
\]

where the \( \{e_i\} \) satisfy \( \nabla(e_i) = \theta_i e_i + A_i \) where \( A_i \) takes values in \( \pi^*(\mathfrak{P}_{\lambda,k})\{Y_{ij,k}\} \otimes (\Omega^{1}_{\mathcal{U}_k})[\lambda][Y_{ij,k}] \) (because of the regularity at infinity). Since we already know the result “away from infinity”, we can work after completing along \( Y_{ij,k} \). If the divisor \( Y_{ij,k} \) is given by \( y = 0 \), we may apply the spectral splitting lemma of [BV], section 6 for the operator \( \frac{d}{dy} \) (as we did before in lemma 50), we obtain an isomorphism

\[
\pi^*(\mathfrak{P}_{\lambda,k})[\lambda]\{Y_{ij,k}\} \cong \bigoplus_{i=1}^{r} O\{Y_{ij,k}\}[\lambda][Y_{ij,k}] \cdot e_i
\]

where now \( \nabla e_i = (\theta_i + g_i)e_i \) where \( g_i \) takes values in \( \bigoplus_{i=1}^{r} (\Omega^{1}_{\mathcal{U}_k})[\lambda][Y_{ij,k}] \). Thus the \( p \)-support over the formal completion of \( Y_{ij,k} \) is the closure (in \( \pi^{-1}(\mathfrak{L}_{k})^{(1)}(\mathcal{U}_k \times \mathbb{A}_k^{1}) \)) of \( r \) Lagrangian subvarieties of the form \( \Gamma((df_i)^p + \hat{g}) \) where \( \hat{g} \) is a one-form which takes values in \( \bigoplus_{i=1}^{r} (\Omega^{1}_{\mathcal{U}_k})[\lambda][Y_{ij,k}] \). At \( \lambda = 0 \) this one form vanishes; the result follows directly.

Now we want to consider the situation for \( \mathfrak{P}_{\lambda,k} \). We have the diagram

\[
T^*(\mathfrak{L}_k) \hookrightarrow \mathfrak{L}_k \times_{\mathfrak{X}_k} T^*(\mathfrak{X}_k) \rightarrow T^*(\mathfrak{X}_k)
\]

We have defined already compactifications of the left and right hand side, and the compactification of the middle is given by

\[
\mathfrak{L}_k \times_{\mathfrak{X}_k} (T^*(\mathfrak{X}_k))
\]

Let us explain how to obtain

\[
T^*(\mathfrak{L}_k) \hookrightarrow \mathfrak{L}_k \times_{\mathfrak{X}_k} T^*(\mathfrak{X}_k) \rightarrow T^*(\mathfrak{X}_k)
\]

As the natural map \( \mathfrak{L}_k \times_{\mathfrak{X}_k} T^*(\mathfrak{X}_k) \rightarrow T^*(\mathfrak{L}_k) \), given by \( (x, \xi) \rightarrow (x, d^* \pi(\xi)) \) extends to a \( \mathbb{G}_m \)-equivariant map

\[
\mathfrak{L}_k \times_{\mathfrak{X}_k} T^*(\mathfrak{X}_k) \times \mathbb{A}_k^{1} \rightarrow T^*(\mathfrak{L}_k) \times \mathbb{A}_k^{1}
\]

by simply taking the identity on the \( \mathbb{A}_k^{1} \) factor. Then, removing the zero sections and taking the quotient yields the desired extension to a map \( \mathfrak{L}_k \times_{\mathfrak{X}_k} T^*(\mathfrak{X}_k) \rightarrow T^*(\mathfrak{L}_k) \).
With this in hand, we show the

**Lemma 110.** Over the image of the open subset $V_k$ in $\overline{X}_k$ (which we also denote $V_k$ by abuse), the $p$-support of $\overline{P}_{\lambda,k}$ is the trivial deformation of $\text{supp}(\overline{H}_k)$. Hence the $p$-support of $\overline{P}_{\lambda,k}$ (as in 88) is the trivial deformation of $\text{supp}(\overline{H}_k)$.

**Proof.** We prove that every infinitesimal approximation (i.e., the $p$-support of $\overline{P}_{\lambda,k}/\lambda^n$) is trivial for $n \geq 0$; this implies the result. Consider the $p$-support of any $\overline{P}_{\lambda,k}/\lambda^n$.

We may take the pullback of this support to $(\overline{L}_k \times_{\overline{X}_k} T^*(\overline{X}_k))_{V_k} \times k[\lambda]/\lambda^n$ and then the image in $T^*(\overline{L}_k)$. This procedure defines a morphism from deformations over $k[\lambda]/\lambda^n$ of $\text{supp}(\overline{H}_k)$ to deformations of $\bigcup_{i=1}^r \Gamma_{df_i}$. This morphism is injective—any deformation which goes to the trivial deformation is easily seen to be trivial over $U_k$ (where the map is finite etale) and hence trivial everywhere; the result over $V_k$ follows. The last sentence follows since the $p$-support of $\overline{P}_{\lambda,k}$, being the scheme-theoretic closure of the $p$-support of $\overline{P}_{\lambda,k}|_{V_k}$ is also a $\lambda$-flat deformation; and so is trivial because it is generically trivial. \hfill $\square$

Now we apply this to global considerations. We consider the Hilbert scheme

$$\text{Hilb}_{\text{supp}(\overline{H}_k)}$$

which is the scheme representing the functor which assigns to any $k$-scheme $T_k$ the set of subschemes of $\overline{T}^*(V_k) \times T_k$, flat over $T_k$, whose Hilbert polynomial is equal to that of $\text{cl}(\overline{L}_k)$. (c.f. [Ni]). It is easy to check, combining lemma 106 with the argument of lemma 109, that the bundle $\overline{P}_{\lambda,k}$ defines a morphism $\mathbb{A}^1_k \to \text{Hilb}_{\text{supp}(\overline{H}_k)}(1)$ (i.e., the $p$-support is flat). The completion of this map at $\{0\}$ is necessarily the trivial map. Since $\text{Hilb}_{\text{supp}(\overline{H}_k)}(1)$ is quasi-projective, we see that the morphism $\mathbb{A}^1_k \to \text{Hilb}_{\text{supp}(\overline{H}_k)}(1)$ is the trivial morphism; i.e., it maps every element in $\mathbb{A}^1_k$ to $\text{supp}(\overline{H}_k)^{(1)}$.

Now we may conclude the

**Proof.** (of theorem 105) For each $k$ with char$(k) >> 0$, we have that $\overline{N}_{\lambda,k}$ and $\overline{M}_{\lambda,k}$ are meromorphic $\lambda$-connections which are regular at infinity and which, by the above considerations, have the same $p$-support, namely $\text{supp}(\overline{H}_k)^{(1)} \times \mathbb{A}^1_k$. Therefore, 96 implies that both of these connections are obtainable by the construction of section 5.2.1. Therefore, 90 implies that they differ by the action of a line bundle in $\text{Pic}(\overline{L}_k)^{(1)}$; therefore upon setting $\lambda = 0$ we have that $\overline{N}_{\lambda,k}/\lambda$ differs from $\overline{M}_{\lambda,k}/\lambda$ by the action of the same bundle; since, however, $\overline{N}_{\lambda,k}/\lambda = \overline{M}_{\lambda,k}/\lambda$ we see that this line bundle is trivial and $\overline{N}_{\lambda,k} = \overline{M}_{\lambda,k}$. Thus they are also isomorphic after specializing $\lambda = 1$.

Now, since $\overline{N}_K$ and $\overline{M}_K$ are $\Theta$-regular at infinity, we can form the eigenvector extension for $N^*_K \otimes M_K$; and similarly over $k$. The existence of the isomorphism of the previous paragraph implies that the cohomology group $H^0_{\text{dr}}(N^*_K \otimes M_K) = H^0_{\text{dR}}(\overline{L}_K) \neq 0$. By base change, we deduce the existence of a nontrivial morphism between $\overline{N}_K$ and $\overline{M}_K$. Since both $\overline{N}_K$ and $\overline{M}_K$ are irreducible (since both have constant $p$-support; we can use lemma 21), we deduce that they are isomorphic; yielding part a).
Suppose now $\mathcal{T}_k$ is any such reflexive meromorphic connection. Then the condition on the $p$-support of the reductions $\mathcal{T}_k$, combined with the fact that $\mathcal{T}_k$ is $\Theta$-regular at infinity for $p >> 0$, (since $\mathcal{T}_k$ is) tells us that the bundle $\mathcal{T}_k$ is obtained from $\mathcal{M}_k$ by the action of $\text{Pic}(\mathcal{L}_k^{(1)})$. Since both $\mathcal{M}_k$ and $\mathcal{T}_k$ are liftable to characteristic zero, we see from 103 that $\mathcal{M}_k \approx \mathcal{T}_k$ for all $p >> 0$, and the uniqueness result follows as in the previous paragraph. \qed

5.5. Quantization of $L_C$. In this section, we return to the case of $L_C \subset T^*X_C$ such that $\pi : L_C \rightarrow X_C$ is dominant. We wish to show

**Theorem 111.** There is a unique holonomic $\mathcal{D}_{X_C}$-module $M_C$ which has constant arithmetic support (with multiplicity 1) equal to $L_C$.

In fact, we can see already that there is only one option for $M_C$: in the previous section we have constructed an irreducible connection $M_{U_C}$ with constant $p$-support equal to $L_{U_C}$; and which extends “correctly” to the compactification $X_C$. So we define

$$M_C := j_*(M_{U_C})$$

and we must check that this has constant $p$-support.

To do so, we use the projective morphism $\varphi : X_C \rightarrow X_C$. Denote by $\tilde{j} : U_C \rightarrow \tilde{X}_C$ the inclusion. Then there is a short exact sequence

$$0 \rightarrow \tilde{j}_*(M_{U_C}) \rightarrow \int_{\tilde{j}} (M_{U_C}) \rightarrow E_C \rightarrow 0$$

where $E_C$ is $\mathcal{D}$-module supported on the divisor $\tilde{D}$. Applying $\int_{\varphi}$ yields the exact sequence

$$0 \rightarrow \int_{\varphi}^{-1} E_C \rightarrow \int_{\varphi} \tilde{j}_*(M_{U_C}) \rightarrow \int_{\varphi} \tilde{j}_*(M_{U_C}) = \tilde{j}_*(M_{U_C})$$

and the image of the map $\int_{\varphi} \tilde{j}_*(M_{U_C}) \rightarrow \int_{\tilde{j}} (M_{U_C})$ is precisely $\tilde{j}_*(M_{U_C})$. So we wish to analyze $E_C$ and its $p$-support for $p >> 0$.

We start by looking at the the $p$-support of $\tilde{j}_*(M_{U_C})$. We have the

**Lemma 112.** a) The $\mathcal{D}$-module $\tilde{j}_*(M_{U_C})$ has constant arithmetic support $\text{supp}(\overline{\mathcal{H}}_{C})|_{\tilde{X}_C}$ with multiplicity 1.

b) Let $\{Y_{ij,C}\}$ be the collection of the finite divisors inside $\tilde{L}_C$. The $\mathcal{D}$-module

$$\int_{\tilde{j}} (M_{U_C})$$

has constant arithmetic support $\tilde{L}_C \cup \bigcup_{ij} N_{ij,C}$ where the $N_{ij,C}$ are defined as follows: let $N_{Y_{ij,C}}$ be the conormal bundle to the divisor $Y_{ij,C} \subset \tilde{X}_C$. Then $N_{ij,C} := N_{Y_{ij,C}} + Y_{ij,C}$ where the addition is the fibre-wise addition in $T^*\tilde{X}_C$.

c) The $\mathcal{D}$-module $E_C$ has constant arithmetic support $\bigcup_{ij} N_{ij,C}$.

**Proof.** (sketch) a) Consider the morphism $\text{supp}(\overline{\mathcal{H}}_{C})|_{\tilde{X}_C} \rightarrow \tilde{X}_C$. This is generically finite; thus the collection of all irreducible divisors inside $\text{supp}(\overline{\mathcal{H}}_{C})|_{\tilde{X}_C}$ on which the map is not finite is a finite collection. Denote the generic points of the images of these divisors by $\{P_i\}$. Suppose that a given $P_i$ is contained in the divisors $\{Y_{ij,C}\}_{j \in J}$. Let $\{P'_i\}$ be a point in $\overline{\mathcal{L}}_C$ living above $\{P_i\}$. We may take the formal
completion $\hat{O}_{(P_1')}[\star Y'_{j,k}]$; and applying the $\Theta$-regularity condition (and the spectral splitting lemma) we can deduce a direct sum decomposition

$$\hat{M}_{(P_1')}[\star Y'_{j,k}] = \bigoplus \hat{M}_j \otimes e_j$$

where the $\hat{M}_j$ admit connections with no pole, and the $e_j$ are eigenvectors for the connection with $\nabla(e_j) = \partial_j e_j$. Let $\hat{j}' : \hat{L}_{U_C} \to \hat{L}_{\hat{C}} := \hat{\pi}^{-1}(\hat{X}_C)$. Then we have

$$\hat{j}'_*(M_{U_C})_{(P_1')} = \hat{\pi}^*(\hat{j}_*(M_{U_C})_{(P_1)})$$

is the $\mathcal{D}$-module generated by the eigenvectors $\{e_i\}$.

The same decomposition will hold over $\hat{O}_{(P_1')}[\star Y'_{j,k}]$ when $\text{char}(k) >> 0$. One sees directly that the $\mathcal{D}$-module generated by the $\{e_i\}$ over $\hat{O}_{(P_1')}[\star Y'_{j,k}]$ with relations obtained by reducing mod $p$ the relations in characteristic zero- is a $\mathcal{D}$-submodule of $\hat{\pi}^*\hat{M}_{(P_1')}$. Therefore we see that $\hat{j}_*(M_{U_C})_{(P_1)}$ is a $\mathcal{D}$-submodule of $\hat{M}_{(P_1')}[\star Y'_{j,k}]$; and therefore that, generically over each $\{P_i\}$, the $p$-support of $\hat{j}_*(M_{U_C})_{(P_1)}$ is contained in the closure of $L^{(1)}_{U_C}$ inside $T^*\hat{X}_k$.

So we see that there is an open subset $W'_C$ of $\hat{X}_C$, whose inverse image in $\text{supp}((\hat{\mathcal{F}}_{C}))_{\hat{X}_C}$ has complement of codimension $\geq 2$, over which the $\hat{j}_*(M_{U_C})_{(P_1)}$ has constant $p$-support equal to $\text{supp}(\hat{\mathcal{F}}_{C})_{W'_C}$ (with multiplicity 1). Thus, if $i : W'_C \to \hat{X}_C$ is the inclusion, we deduce that $i_*(\hat{j}_*(M_{U_C})_{W'_C})$ has constant $p$-support $\text{supp}(\hat{\mathcal{F}}_{C})_{\hat{X}_C}$. Therefore lemma 21 implies that $i_*(\hat{j}_*(M_{U_C})_{W'_C}) = (\hat{j}_*(M_{U_C}))$ and we are done.

b) This is a direct computation in local coordinates over each divisor $Y_{i,j,C}$. Indeed, it is extremely to the proof of lemma 69, using Abyankhar’s lemma and the expression of $M_U$ in local coordinates- the details are omitted.

c) Follows directly from a) and b), and the definition of $E_C$ as the cokernel of the natural inclusion. □

Now we consider the pullback of the varieties $N_{i,C}$ of lemma 112. We denote by $L(d\varphi)^*$ the derived functor of the pull-back functor. Then:

**Lemma 113.** Consider the natural map

$$D_{\hat{X}_k \leftarrow X_k} \otimes \hat{j}_*(M_{U_C}) \to D_{\hat{X}_k \leftarrow X_k} \otimes \int_j M_{U_k}$$

Its image is supported on the closure of $L^{(1)}_{U_k}$ in $\hat{X}_k \times X_k T^*X_k$.

**Proof.** By the previous lemma we know that the cokernel of this map is $D_{\hat{X}_k \leftarrow X_k} \otimes E_k$, which by theorem 14 is supported on $(d\varphi^{(1)})^*(\text{supp}(E_k))$. In fact by the description of $E_k$ given there the module $D_{\hat{X}_k \leftarrow X_k} \otimes E_k$ contains every component of $D_{\hat{X}_k \leftarrow X_k} \otimes \int_j M_{U_k}$ (and with the same multiplicity) except for the closure of $L^{(1)}_{U_k}$; the result follows. □

Now we can give the

**Proof.** (of the existence part of theorem 111) The $\mathcal{D}_{X_C}$-module $R^0 \varphi_*(H^0(D_{\hat{X}_C \leftarrow X_C} \otimes \hat{j}_*(M_{U_C}))$ maps into $j_*(M_{U_C})$, and it is an isomorphism over $U_C$. Therefore the image of this map contains $j_*(M_{U_C})$, which by definition is the minimal extension of $M_{U_C}$. If we can show that this image has constant arithmetic support equal to $\hat{L}_C$ (with multiplicity 1) then we are done by lemma 21.
The kernel of the map
\[ R^0\varphi_* (H^0(D_{\mathcal{X}_C} \to X_C \otimes \tilde{j}_* (M_{U_C}))) \to j_* (M_{U_C}) \]
is the image of
\[ R^0\varphi_* (H^0(D_{\mathcal{X}_C} \to X_C \otimes E_C)) \to R\varphi_* (H^0(D_{\mathcal{X}_C} \to X_C \otimes \tilde{j}_* (M_{U_C}))) \]
so that the result follows from lemma 113.

Now we wish to deal with uniqueness. Suppose that \( N_C \) with constant arithmetic support equal to \( L_{U_C} \) (with multiplicity 1) has been found. Then we have by lemma 20 that \( N_{U_C} \) is a connection. Therefore \( N_C = j_* (N_{U_C}) \). We may also consider \( \tilde{j}_* (N_{U_C}) \) on \( \tilde{X}_C \). We have

**Proposition 114.** The \( \mathcal{D} \)-module \( \tilde{j}_* (N_{U_C}) \) has constant arithmetic support equal to \( \tilde{L}_{U_C} \).

**Proof.** Since we know that \( N_{U_C} \) has constant arithmetic support equal to \( L_{U_C} \), the only way for this to be false is if, for infinitely many \( k \), the \( p \)-support is equal to \( \tilde{L}_{k}^{(1)} \cup (\tilde{L}_{k}^{(1)})^{(1)} \) where \( (\tilde{L}_{k}^{(1)})^{(1)} \) is a Lagrangian subvariety in the complement of \( T^*U_k^{(1)} \). Now, there is an embedding
\[ \tilde{j}_* (N_{U_C}) \to j_* (N_{U_C}) \]
and since \( N_{U_C} \) has constant arithmetic support equal to \( L_{U_C} \), we see that the \( p \)-support \( j_* (N_{U_C}) \) is equal to the \( p \)-support of \( \tilde{j}_* (M_{U_C}) \). Arguing as in the previous proof, we deduce from this that the \( p \)-support of the image of \( R\varphi_* ((D_{\mathcal{X}_C} \to X_C \otimes \tilde{j}_* (N_{U_C}))) \rightarrow j_* (N_{U_C}) \) is not irreducible; indeed it contains the image of \( \varphi (d\varphi (1))^{-1} (N_{ij,k}^{(1)} \) for some \( ij \), which is not contained in \( \tilde{L}_{k}^{(1)} \). Contradiction.

Now, let \( \overline{N}_C \) be the canonical lattice of \( j_* (N_{U_C}) \) (i.e., the lattice described in [Ma]). This proposition implies that for \( p >> 0 \), the lattice \( \overline{N}_k \) has constant arithmetic support equal to \( \tilde{L}_{k}^{(1)} \). Thus, by Azumaya splitting, this meromorphic connection differs from \( \overline{M}_k \) by the action of a line bundle on \( \tilde{L}_{k}^{(1)} \). Since \( \overline{N}_k \) is liftable to characteristic zero this implies \( \overline{N}_k = \overline{M}_k \). Since this is true for all such \( k \), we get \( \overline{M}_C = \overline{N}_C \) (by the same argument as in the proof of theorem 105) and the uniqueness result follows.

6. The Case of Affine Spaces

In this section we apply the previous result to the case of \( X = \mathbb{A}^m_C \) and prove the theorem about the Picard group of \( \mathcal{D}_{m,C} \). In preparation, we recall the

**Lemma 115.** There is an action of the linear symplectic group, \( \text{Sp}_{2m} \), on \( \mathcal{D}_{m,F} \), where \( F \) is any field. If \( F = k \) is a perfect field of positive characteristic, then the restriction of this action to the center agrees with the defining action of \( \text{Sp}_{2m} \) on \( (T^*\mathbb{A}^m_k)^{(1)} \).

**Proof.** This is the content of proposition 2 of [BKKo].

We have
Theorem 116. Let $\mathcal{L}_C \subset T^* \mathbb{A}_C^m$ be a smooth Lagrangian, satisfying the cohomological assumptions of theorem 1. Then there is a unique holonomic $D_m$-module $M_C$ with constant arithmetic support equal to $\mathcal{L}_C$.

Proof. We shall reduce this case to the case of generic position, which was covered already in theorem 111.

Choose a point $x \in \mathcal{L}_C$. Then $T_x \mathcal{L}_C$ may be regarded as a linear Lagrangian inside $T^* \mathbb{A}_C^m$. Call it $V_C$. Then there is an element $\psi$ of $\text{Sp}_{2m}$ whose action on $T^* \mathbb{A}_C^m$ takes $V_C$ to $\mathbb{A}_C^m$. The projection $\psi(\mathcal{L}_C) \rightarrow \mathbb{A}_C^m$ is dominant by construction. We thus have, by theorem 111, a unique $M'_C$ associated to $\psi \mathcal{L}_C$, and since $\text{Sp}_{2m}$ acts on $D_m$, we can take $M_C = \psi M'_C$. By the previous lemma, this $D$-module has constant arithmetic support equal to $\mathcal{L}_C$.

Given any other such $D$-module $N_C$, we note that $(\psi^{-1})^* N_C$ has constant arithmetic support equal to $\psi(\mathcal{L}_C)$, again according to the previous lemma. But then $(\psi^{-1})^* N_C = M'_C$ and the result follows. $\square$

Now, following [BKKo] and [Ts], we sketch the proof of the following

Theorem 117. There is a natural map from the group $\text{Pic}(D_{m,C})$ of Morita autoequivalences of the Weyl algebra $D_{m,C}$ to the group $\text{Aut}_{\text{Symp}}(T^* \mathbb{A}_C^m)$ of symplectomorphisms of $T^* \mathbb{A}_C^m$. This map is an isomorphism.

In fact, the proof is a very quick consequence of the existence and uniqueness theorems and the results of the cited papers. Let us briefly recall the relevant set-up, following the notation of [Ts] (the way of [BKKo] is extremely similar): first, he introduces the field $\mathbb{Q}_U^{(\infty)}$ which is a subfield of the ring

$$\prod \mathbb{F}_p$$

where the product ranges over all prime numbers, and the $U$ denotes a principle ultrafilter on the set of primes. The field $\mathbb{Q}_U^{(\infty)}$ is isomorphic, non canonically, to $\mathbb{C}$. Then the main construction of [Ts] (proposition 7.1) goes as follows: given any endomorphism $\phi$ of the Weyl algebra $D_{m,C}$, one may regard it as an endomorphism of $D_{m,\mathbb{Q}_U^{(\infty)}}$. From this it follows that there exists a family of endomorphisms $\phi_p$ of $D_{m,\mathbb{F}_p}$ (for some infinite set of primes) whose limit is equal to $\phi$. But then each $\phi_p$ gives an endomorphism of $Z(D_{m,\mathbb{F}_p}) = T^*(\mathbb{A}^n(\mathbb{F}_p))^{(1)}$ which respects the symplectic form. In this way we have a natural map

$$\text{End}(D_{m,\mathbb{Q}_U^{(\infty)}}) \rightarrow \text{End}(T^* \mathbb{A}_C^m)$$

which is therefore equivalent to a map

$$\text{End}(D_{m,C}) \rightarrow \text{End}(T^* \mathbb{A}_C^m(\mathbb{C}))$$

Now, instead of starting with an endomorphism of $D_{m,C}$, we could have started with an invertible bimodule over $D_{m,C}$; i.e., a $D_{m,C}$ bimodule $A_C$ such that there exists a $D_{m,C}$ bimodule $B$ with $A \otimes_{D_{m,C}} B = D_{m,C}$. Applying the same technique, one obtains a map

$$\text{Pic}(D_{m,\mathbb{Q}_U^{(\infty)}}) \rightarrow \text{Cor}_{\text{Symp}}(T^* \mathbb{A}_C^m)$$

where the on the right hand side we have the monoid of symplectic correspondences of $T^* \mathbb{A}_C^m$, i.e., the monoid (under composition of correspondences) of coherent sheaves on $T^* \mathbb{A}_C^m(\mathbb{Q}_U^{(\infty)}) \times T^* \mathbb{A}_C^m(\mathbb{Q}_U^{(\infty)})$ whose support is a Lagrangian subvariety. The
image under this map will consist of invertible correspondences, and in fact it is not hard to show the

**Lemma 118.** The group of invertible symplectic correspondences of $T^*\mathbb{A}_k^m$ (where $k$ is any algebraically closed field) is isomorphic to the group of symplectomorphisms of $T^*\mathbb{A}_k^m$.

Thus we obtain the promised map $\text{Pic}(D_{m,\mathbb{C}}) \to \text{Aut}_{\text{symp}}(T^*\mathbb{A}_m^\mathbb{C})$. Since $T^*\mathbb{A}_m^\mathbb{C}$ is an affine space, the underlying variety of every invertible symplectic correspondence will satisfy the cohomology vanishing assumptions of intro; thus to each one we may associate a unique $D_{m,\mathbb{C}}$ bimodule by the existence and uniqueness theorems; and so we deduce that the map $\text{Pic}(D_{m,\mathbb{C}}) \to \text{Aut}_{\text{symp}}(T^*\mathbb{A}_m^\mathbb{C})$ is an isomorphism as claimed.

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