THE GAUSS IMAGE PROBLEM WITH WEAK ALEKSANDROV CONDITION

VADIM SEMENOV

Abstract. We introduce a relaxation of the Aleksandrov condition for the Gauss Image Problem. This weaker condition turns out to be a necessary condition for two measures to be related by a convex body. We provide several properties of the new condition. A solution to the Gauss Image Problem is obtained for the case when one of the measures is assumed to be discrete and the another measure is assumed to be absolutely continuous, under the new relaxed assumption.

Contents

1. Introduction 2
2. Preliminaries 5
3. Weak Aleksandrov Condition 8
4. Essential Estimates and Partial Rescaling of a Polytope 10
5. The Partial Rescaling of a Polytope and the Functional 14
6. Proof of the Main Result 22
References 27

Date: October 1, 2024.
2010 Mathematics Subject Classification. 52A20, 52A38, 52A40, 52B11, 35J20, 35J96.

Key words and phrases. Convex Geometry, The Gauss Image Problem, Aleksandrov Condition, Monge-Ampère equation, Aleksandrov Problem.
1. Introduction

The Gauss Image Problem, introduced in [13], is a natural extension to the classical Aleksandrov question of finding a body with the prescribed Aleksandrov’s integral curvature [1–3]. This problem is a part of the study of Minkowski problems, a vital area of research in convex geometry. The study of these problems has led to the formulation of the log-Brunn-Minkowski conjecture [5, 12, 17, 21, 36] and to the sharp affine $L^p$ Sobolev inequality [28]. The latter has also inspired many other sharp affine isoperimetric inequalities [16, 25, 28].

Readers are referred to Chapters 8 and 9 of Schneider’s textbook [37] for an introduction to Minkowski problems and to the articles [7, 10, 11, 18–20, 22, 24–27, 29, 32–34, 38, 40–44] for an overview of the recent developments. Additionally, we acknowledge the works [4, 6, 31, 35, 39] related to the regularity of Minkowski problems.

Given two measures $\mu$ and $\lambda$ on $S^{n-1}$, the Gauss Image Problem asks about the existence of a convex body $K$, containing the origin in its interior, such that $\mu = \lambda(K, \cdot)$, where by $\lambda(K, \cdot)$ we denote the pullback of $\lambda$ under the radial Gauss Image map of $K$: a composition of the multivalued Gauss map of $K$ and the radial map of $K$. More formally, given a Borel set $\omega \subset S^{n-1}$ we define:

\[
\lambda(K, \omega) := \lambda \left( \bigcup_{u \in \omega} N(K, \rho_K(u)u) \right),
\]

where $N(K, v)$ is the normal cone of a boundary point $v \in \text{bd}(K)$ and $\rho_K(\cdot)$ is the radial function of $K$.

Many significant measures can be described as pullbacks of a certain $\lambda$ under the Gauss Image map. For instance, when $\lambda$ is the spherical Lebesgue measure, $\lambda(K, \cdot)$ is known as Aleksandrov’s integral curvature of the body $K$ [3]. When $\lambda$ is Federer’s $(n-1)$th curvature measure, $\lambda(K, \cdot)$ is the surface area measure of Aleksandrov-Fenchel-Jessen [2]. Finally, the more recently defined dual curvature measure is also a pullback of a certain $\lambda$ under the Gauss Image map [19]. All of these examples motivate the necessity for a systematic study of how measures transfer to each other through the radial Gauss Image Map, that is, the Gauss Image Problem:

**The Gauss Image Problem** (Raised in [13]) Suppose $\lambda$ is a measure defined on the Lebesgue measurable subsets of $S^{n-1}$, and $\mu$ is a Borel measure on $S^{n-1}$. What are the necessary and sufficient conditions on $\lambda$ and $\mu$, so that there exists a convex body $K$ with the origin in its interior such that

\[
\mu = \lambda(K, \cdot)?
\]

If such a convex body exists, to what extent is it unique?

When $\lambda$ is a spherical Lebesgue measure, we recover the original Aleksandrov problem, which Aleksandrov first studied in [1–3]. Different proofs of the Aleksandrov problem were given by Oliker [34] and Bertrand [8]. The $L^p$ analogs of the Aleksandrov problem were considered by Huang, Lutwak, Yang, and Zhang in [18], by Mui in [30], and by Zhao in [42].

When one of the measures is assumed to be absolutely continuous, the Gauss Image Problem was studied in [13] by Böröczky, Lutwak, Yang, Zhang, and Zhao. There, the Aleksandrov relation was introduced to attack the problem:

**Definition 1.1.** Two Borel measures $\mu$ and $\lambda$ on $S^{n-1}$ are called Aleksandrov related if

\[
\lambda(S^{n-1}) = \mu(S^{n-1}) > \mu(\omega) + \lambda(\omega^*)
\]
for each compact, spherically convex set $\omega \subset S^{n-1}$, where the set $\omega^* \subset S^{n-1}$ is defined as the spherical polar set to $\omega$:

$$\omega^* := \bigcap_{u \in \omega} \{ v \in S^{n-1} : u \cdot v \leq 0 \}. \tag{1.4}$$

Equivalently, one can define two Borel measures $\mu$ and $\lambda$ on $S^{n-1}$ to be Aleksandrov related if $\mu(S^{n-1}) = \lambda(S^{n-1})$ and for each compact, spherically convex set $\omega \subset S^{n-1}$,

$$\mu(\omega) < \lambda(\omega^*_\pi), \tag{1.5}$$

where

$$\omega^*_\pi := \bigcup_{u \in \omega} \{ v \in S^{n-1} : u \cdot v > 0 \}. \tag{1.6}$$

With this new condition, the following solution to the Gauss Image Problem was obtained:

**Theorem 1.2** (K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang and Y. Zhao [13]). Suppose $\mu$ and $\lambda$ are Borel measures on $S^{n-1}$, and $\lambda$ is absolutely continuous. If $\mu$ and $\lambda$ are Aleksandrov related, then there exists a convex body $K$ containing the origin in its interior, such that $\mu = \lambda(K, \cdot)$.

Moreover, if the absolutely continuous measure $\lambda$ is strictly positive on open sets, it was shown that the Aleksandrov relation is a necessary assumption for the existence of a solution to the Gauss Image Problem. In this case, a solution to the Gauss Image Problem was shown to be unique up to a dilation. We refer the reader to [13] for this result and an introduction to the Gauss Image Problem. Additionally, let us also mention Theorem 1.7 and Remark 4.9 in Bertrand [8], which also imply Theorem 1.2 using a very different method.

While the Aleksandrov relation is a natural assumption when one of the measures is assumed to be positive on open sets, it turns out that there are numerous examples of measures $\mu$ and $\lambda$ satisfying $\mu = \lambda(K, \cdot)$ that are not Aleksandrov related. For instance, let $K = B^n$, a unit ball centered at the origin, and $\mu$ and $\lambda$ to be any even, absolutely continuous, identical measures supported on small symmetric spherical caps $\omega$ and $-\omega$, where $\omega \subset S^{n-1}$ is a cap around the north pole and $-\omega \subset S^{n-1}$ is a cap around the south pole. Then, $\mu = \lambda = \lambda(K, \cdot)$, and $\mu(\omega) + \lambda(\omega^*) = \lambda(S^{n-1})$, which violates the Aleksandrov relation. Moreover, starting with the body $K = B^n$, we can perturb it along the equator while preserving the convexity. We thereby obtain a family of convex bodies such that every member still solves the Gauss Image Problem for fixed measures $\mu$ and $\lambda$. This observation indicates that, in general, the solution to the Gauss Image Problem may be highly non-unique.

Based on these considerations, we introduce a relaxation of the Aleksandrov relation for the Gauss Image Problem. This relaxation turns out to be a necessary assumption for the two measures to be related by a convex body. That is, for the existence of a convex body $K$ with origin in its interior such that $\mu = \lambda(K, \cdot)$. See Proposition 3.1.

**Definition 1.3.** Given Borel measures $\mu$ and $\lambda$ on $S^{n-1}$, we say that $\mu$ is weakly Aleksandrov related to $\lambda$ if $\mu(S^{n-1}) = \lambda(S^{n-1})$ and for each closed set $\omega \subset S^{n-1}$, contained in a closed hemisphere, there exists $\alpha \in (0, \pi/2)$ such that

$$\mu(\omega) \leq \lambda(\omega^*_\pi - \omega), \tag{1.7}$$

where
where
\[
\omega_{\frac{\pi}{2}, -\alpha} := \bigcup_{u \in \omega} \{ v \in S^{n-1} : u \cdot v > \cos \left( \frac{\pi}{2} - \alpha \right) \}.
\]

Besides showing that the weak Aleksandrov relation is a necessary assumption for the existence of a solution to the Gauss Image Problem, we also show that the classical Aleksandrov relation implies the weak Aleksandrov relation. The nature of the constant \( \alpha \) is addressed in Section 3. In particular, if \( \mu = \lambda(K, \cdot) \), then the constant \( \alpha \) in the weak Aleksandrov condition is closely related to the inner to outer radius ratio of the body \( K \). See Proposition 3.1 and the discussion after it.

Now, with an appropriate necessary condition, we are ready to state the main result of the paper. In the following, a measure \( \mu \) is a discrete measure if it can be expressed as
\[
\mu = \sum_{i=1}^{m} \mu_i \delta_{v_i}
\]
where \( \mu_i \) are some positive coefficients, and \( \delta_{v_i} \) is a Dirac measure of the set \( \{ v_i \} \). For the measure \( \mu \), we also define the set \( \mathcal{P}_\mu \) of polytopes as
\[
\mathcal{P}_\mu = \{ \text{conv} \{ \beta_i v_i \mid 1 \leq i \leq m \} \mid (\beta_1, \ldots, \beta_m) \in \mathbb{R}_{>0}^m \}.
\]

**Theorem 1.4.** Suppose \( \mu \) and \( \lambda \) are Borel measures on \( S^{n-1} \), so that \( \mu \) is discrete and not concentrated on a closed hemisphere, and \( \lambda \) is absolutely continuous. If \( \mu \) is weakly Aleksandrov related to \( \lambda \), then there exists a polytope \( P \in \mathcal{P}_\mu \) such that \( \mu = \lambda(P, \cdot) \).

In particular, we establish that given a discrete measure \( \mu \) and an absolutely continuous measure \( \lambda \), the weak Aleksandrov relation is a necessary and sufficient condition for the existence of a solution to the Gauss Image Problem.

In conclusion, we would like to comment on the differences between the methods introduced in this paper and those presented in [13]. The proof of Theorem 1.2 in [13] has the following structure: First, it is shown that any convex body that maximizes the specific functional on convex bodies (defined below, see (2.15)) is a solution to the Gauss Image Problem. Then, by analyzing the classical Aleksandrov relation for specific measures, it is proven that any sequence of convex bodies maximizing this functional exhibits a bound on the inner to outer radius ratio of its elements. This bound is arguably the most challenging aspect of the paper [13]. From the Blaschke selection theorem, the authors then deduce that this sequence contains a convergent subsequence that converges to a non-degenerate convex body \( K \) maximizing the functional (2.15). The limiting body \( K \), in turn, solves the Gauss Image Problem.

The main challenge and difference in the proof of Theorem 1.4, as compared to the main result of [13], is that the weak Aleksandrov relation does not impose a bound on the inner to outer radius ratio for the possible solution, unlike its stronger counterpart. Going back to the previously mentioned example of spherical caps, for any scalars \( \lambda_1, \lambda_2 > 0 \), define \( K_{\lambda_1, \lambda_2} \) to be the convex hull in \( \mathbb{R}^n \) of \( \lambda_1 \omega \subset \lambda_1 S^{n-1} \) and \( -\lambda_2 \omega \subset \lambda_2 S^{n-1} \). Note that any \( K_{\lambda_1, \lambda_2} \) is a solution to \( \mu = \lambda(K, \cdot) \), where \( \mu \) and \( \lambda \) are defined as before. (This is true because the normal cones of \( K_{\lambda_1, \lambda_2} \) do not change for radial directions contained in the support of \( \mu \), when we vary \( \lambda_1 \) and \( \lambda_2 \). See Section 2 for the definitions.) Hence, in contrast to the classical Aleksandrov relation assumption, the solution body may contain parts that one can dilate independently. Consequently, unlike the case when the classical Aleksandrov relation
is assumed, a sequence of convex bodies, say \((K_n)_{n=1}^\infty\) such that \(K_n \subseteq rB^n\) for all \(n\) and some \(r\), may maximize the functional while converging to a degenerate convex body. This makes the proof of the main theorem in this paper vastly differ from that in [13], as not every sequence of normalized convex bodies maximizing the functional is suitable for the proof. To construct this sequence and to overcome these challenges, we invoke a new process that we call the partial rescaling of convex bodies. See (4.17).

It would be very interesting to see whether one could prove the result of the Theorem 1.2, the main results of [13], under the weak Aleksandrov relation assumption instead of the classical Aleksandrov relation. Our paper can be viewed as a step towards this direction. We state this in the Conjecture 6.3.

**Acknowledgements** The author is extremely grateful to the editor and the referee for their valuable and extensive comments, which significantly improved readability of the paper.

2. Preliminaries

By \(\mathcal{K}^n\) we denote the set of convex bodies (compact, convex subsets with nonempty interior in \(\mathbb{R}^n\)). By \(\mathcal{K}^n_o \subset \mathcal{K}^n\) we denote those convex bodies that contain the origin in their interiors. Given \(K \in \mathcal{K}^n_o\), let \(x \in \partial K\) be a boundary point. The normal cone at \(x\) is defined by

\[
N(K, x) = \{v \in S^{n-1}: (y - x) \cdot v \leq 0 \text{ for all } y \in K\},
\]

which parametrizes all unit normals at a given boundary point. For \(K \in \mathcal{K}^n_o\), the radial map \(r_K: S^{n-1} \to \partial K\) of \(K\) is defined for \(u \in S^{n-1}\) by \(r_K(u) = ru \in \partial K\), where \(r > 0\). Given a subset \(\omega\) of \(S^{n-1}\), the radial Gauss Image of \(\omega\) is defined as follows:

\[
\alpha_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K, x) \subset S^{n-1}.
\]

The radial Gauss Image map, \(\alpha_K\), maps sets of \(S^{n-1}\) to sets of \(S^{n-1}\). Outside of a spherical set of Lebesgue measure zero, the multivalued map \(\alpha_K\) is singular valued. It is known that \(\alpha_K\) maps Borel measurable sets to Lebesgue measurable sets. See p.88–89 in [37] for both of these results. We denote the restriction of \(\alpha_K\) to the corresponding singular valued map by \(\alpha_K\). For additional details, we refer the reader to [13].

The radial function \(\rho_K: S^{n-1} \to \mathbb{R}\) is defined by:

\[
\rho_K(u) = \max\{a : au \in K\}.
\]

In this case, \(r_K(u) = \rho_K(u)u\). The support function of the body \(K\) is defined by:

\[
h_K(x) = \max\{x \cdot y : y \in K\}.
\]

For \(K \in \mathcal{K}^n_o\), we define its polar body \(K^* \in \mathcal{K}^n_o\) as the convex body with support function given by \(h_{K^*} := \frac{1}{\rho_K}\).

We denote by \(r_K\) the radius of the largest ball contained in \(K\) and centered at \(o\). Similarly, we denote \(\mathcal{R}_K\) to be the radius of the smallest ball containing \(K\) and centered at \(o\). We will refer to \(r_K\) as the inner radius of the body \(K\), to \(\mathcal{R}_K\) as the outer radius of \(K\), and to the ratio \(\frac{r_K}{\mathcal{R}_K}\) as the inner to outer radius ratio of the body \(K\).

It is important to note that for any \(K \in \mathcal{K}^n_o\), the following identity holds:

\[
\min \rho_K = \min h_K = r_K \leq \mathcal{R}_K = \max \rho_K = \max h_K.
\]
The support hyperplane to $K$ with an outer unit normal $v \in S^{n-1}$ is defined as
\begin{equation}
H_K(v) = \{x : x \cdot v = h_K(v)\}.
\end{equation}
By $H^-(\alpha, v)$ we denote the halfspace $\{x : x \cdot v \leq \alpha\}$ and by $H(\alpha, v)$ we denote the hyperplane $\{x : x \cdot v = \alpha\}$. Given a set $S \subset \mathbb{R}^n$ we write its convex hull as
\begin{equation}
\text{conv}(S).
\end{equation}
For a set $\omega \subset S^{n-1}$, we define cone $\omega$ as the cone that $\omega$ generates in $\mathbb{R}^n$, that is
\begin{equation}
\text{cone } \omega = \{tu : t \geq 0 \text{ and } u \in \omega\}.
\end{equation}
We say that $\omega \subset S^{n-1}$ is spherically convex if the cone that $\omega$ generates is a nonempty, proper, convex subset of $\mathbb{R}^n$. Therefore, a spherically convex set in $S^{n-1}$ is always nonempty and contained in a closed hemisphere of $S^{n-1}$. Given $\omega \subset S^{n-1}$ contained in a closed hemisphere, the polar set $\omega^*$ is defined by:
\begin{equation}
\omega^* = \bigcap_{u \in \omega}\{v \in S^{n-1} : u \cdot v \leq 0\}.
\end{equation}
We note that the polar set is always spherically convex. If $\omega \subset S^{n-1}$ is a closed set, we define its outer parallel set $\omega_\alpha$ for some $\alpha \in (0, \frac{\pi}{2})$ to be
\begin{equation}
\omega_\alpha = \bigcup_{u \in \omega}\{v \in S^{n-1} : u \cdot v > \cos \alpha\}.
\end{equation}
For notational convenience, if $\omega$ contains a single vector, say, $\omega = \{v\}$ where $v \in S^{n-1}$, we are going to simply write $v_\alpha$ instead of $\{v\}_\alpha$.

As mentioned previously, $\alpha_K$ maps Borel measurable sets to Lebesgue measurable sets. Given a Borel measure $\lambda$ we, as in [13], define the Gauss Image measure of $\lambda$ via $K$ as
\begin{equation}
\lambda(K, \omega) := \lambda(\alpha_K(\omega))
\end{equation}
for each Borel $\omega \in S^{n-1}$. Note, however, that the naming is a bit misleading as, in general, $\lambda(K, \cdot)$ does not necessarily have to be a measure. For example, in the dimension 2, let $K$ be a square centered at the origin with sides perpendicular to unit vectors $u_1, u_2, u_3, u_4$. Let $\lambda = \sum_{i=1}^4 \delta_{u_i}$ where $\delta_{u_i}$ are Dirac measures of sets $\{u_i\}$. Let unit vectors $v_1$ and $v_2$ be such that $r_K(v_1), r_K(v_2)$ are in the interior of the side of $K$ perpendicular to $u_1$. Then,
\begin{equation}
\alpha_K(\{v_1\}) = \alpha_K(\{v_2\}) = \alpha_K(\{v_1, v_2\}) = \{u_1\}.
\end{equation}
Implying that:
\begin{equation}
1 = \lambda(K, \{v_1\}) = \lambda(K, \{v_2\}) = \lambda(K, \{v_1, v_2\})
\end{equation}
which establishes that $\lambda(K, \cdot)$ is not countably additive.

On the other hand, if $\lambda$ is an absolutely continuous Borel measure, which is the case of this work, $\lambda(K, \cdot)$ is always a measure. For this and related results, see [13]. We also point out Lemma 3.3 in [13], which states that:

**Lemma 2.1.** If $\lambda$ is an absolutely continuous Borel measure, and $K \in \mathcal{K}_b^n$, then
\begin{equation}
\int_{S^{n-1}} f(u)d\lambda(K, \cdot) = \int_{S^{n-1}} f(\alpha_K(u))d\lambda(v)
\end{equation}
for each bounded Borel measurable function $f : S^{n-1} \to \mathbb{R}$. 

We note that if for a pair of a given $\mu$ and $\lambda$, there exists $K \in \mathcal{K}_o^n$ such that $\mu = \lambda(K, \cdot)$, then we say that the measures $\mu$ and $\lambda$ are related by the convex body $K$. For $K \in \mathcal{K}_o^n$ and $\lambda$ absolutely continuous, we define the functional $\Phi(K, \mu, \lambda)$ by

\begin{equation}
\Phi(K, \mu, \lambda) := \int_{S^{n-1}} \log \rho_K d\mu + \int_{S^{n-1}} \log \rho_K d\lambda.
\end{equation}

Sometimes, we will write $\Phi(K)$, suppressing $\mu$ and $\lambda$. Note that $\Phi(K, \mu, \lambda)$ corresponds to $\Phi_{\mu, \lambda}(K^*)$ in the notation of [13]. This functional is intimately associated with the Gauss Image Problem. For example, Theorem 8.2 in [13] shows that if $\mu$ is a Borel measure and $\lambda$ is an absolutely continuous Borel measure such that

\begin{equation}
\Phi(K, \mu, \lambda) = \sup_{K' \in \mathcal{K}_o^n} \Phi(K', \mu, \lambda)
\end{equation}

for $K \in \mathcal{K}_o^n$, then $\mu = \lambda(K, \cdot)$. It is important to stress that:

\begin{equation}
\text{If } \mu = \lambda(K, \cdot), \text{ then } \mu = \lambda(cK, \cdot) \text{ for any } c > 0.
\end{equation}

That is, the nature of the problem is not sensitive to the rescaling of the convex bodies.

The Aleksandrov relation, as well as the weak Aleksandrov relation, were defined in the Introduction. We simply note the interchangeable use of the terms “Aleksandrov condition” and “Aleksandrov relation”.

A measure $\mu$ is called discrete if it takes the form:

\begin{equation}
\mu = \sum_{i=1}^m \mu_i \delta_{v_i}
\end{equation}

where $\delta_{v_i}$ are Dirac measures of sets $\{v_i\}$ containing a single vector $v_i \in S^{n-1}$ and $\mu_i$ are strictly positive coefficients. Aside from Proposition 3.3, the measure $\mu$ will always be assumed to be discrete and written as in (2.18) with letters $v$ and $m$ reserved specifically for $\mu$.

Given a discrete measure $\mu$ not concentrated on a closed hemisphere, we define $\mathcal{P}_\mu$ to be the set of the convex hull of points $\{\beta_i v_i\}$ with $\beta_i > 0$. Given any $P \in \mathcal{P}_\mu$, since $\mu$ is not concentrated on a closed hemisphere, $P$ contains the origin in its interior. Therefore, $\mathcal{P}_\mu \subset \mathcal{K}_o^n$. Moreover, any $P \in \mathcal{P}_\mu$ is a polytope, such that each vertex of $P$ is located in a radial direction $v_i$ for some $i \in \{1, \ldots, m\}$. Note, however, that sometimes a polytope $P \in \mathcal{P}_\mu$ might have fewer than $m$ vertices corresponding to some $\beta_j v_j$ contained inside the convex hull of the remaining points.

The next part of notations can be viewed as a discrete analog to the standard concepts of the support function and the Wulff shape. Given $P \in \mathcal{P}_\mu$, we define its representation to be an $m$-tuple of positive numbers

\begin{equation}
\alpha = (\alpha_1, \ldots, \alpha_m) := (h_{P^*}(v_1), \ldots, h_{P^*}(v_m)).
\end{equation}

Note that if $\alpha$ is the representation of $P$ then

\begin{equation}
P = \text{conv}\{v_i/\alpha_i \mid 1 \leq i \leq m\},
\end{equation}

\begin{equation}
P^* = \bigcap_{i=1}^m H^-(\alpha_i, v_i).
\end{equation}
Conversely, suppose we start with some \( m \)-tuple of positive numbers, \( \gamma \). We define \( P_\gamma \in P_\mu \) to be the following:

\[
P_\gamma = \left( \bigcap_{i=1}^{m} H^-(\gamma_i, v_i) \right)^*.
\]

We call such \( P \) a dual Wulff Shape of the \( m \)-tuple \( \gamma \). We refer to the representation of \( P_\gamma \) as the Wulff tuple of \( \gamma \). In particular, one has that if \( \alpha \) is a Wulff tuple of \( \gamma \), then

\[
\alpha_i \leq \gamma_i.
\]

Moreover, if \( \alpha_i < \gamma_i \), then the facet of \( P_\gamma^* \) in the direction \( v_i \) is degenerate.

If a polytope has an index \( a \), such as \( \hat{P}_a \), we are going to write its coefficients in representation as:

\[
\alpha_a = (\alpha_{a,1}, \ldots, \alpha_{a,m})
\]

Given \( P \in P_\mu \) with its representation denoted by the \( m \)-tuple \( \alpha \) and a nonempty and not full index set \( I \subset \{1, \ldots, m\} \), we will denote by \( U(I), L(I), U^*(I), L^*(I) \) the following quantities:

\[
\begin{align*}
U(I) &:= \max_{i \in I} \alpha_i, \\
L(I) &:= \min_{i \in I} \alpha_i, \\
U^*(I) &:= \max_{i \notin I} \alpha_i, \\
L^*(I) &:= \min_{i \notin I} \alpha_i.
\end{align*}
\]

It will usually be the case that:

\[
0 < L^*(I) \leq U^*(I) \leq L(I) \leq U(I) = 1.
\]

Outside Proposition 6.1 and Proposition 6.2, the index set \( I \) remains the same through each proposition. In this case we simply write

\[
U, L, U^*, L^*
\]

to denote the same quantities. If a polytope has an index \( t \), such as \( P_t \), we are going to write

\[
P_t, \alpha_t, L_t, U_t, L_t^*, U_t^*
\]

to denote the same quantities for \( P_t \).

We use the books of Schneider [37] as our standard reference. The books of Gruber and Gardner are also good alternatives [14, 15].

### 3. Weak Aleksandrov Condition

Let us start by showing that the weak Aleksandrov relation is a necessary condition for Borel measures to be related by a convex body.

**Proposition 3.1.** Given \( K \in K^n_0 \), suppose \( \lambda \) and \( \lambda(K, \cdot) \) are Borel measures. Then, \( \lambda \) is weakly Aleksandrov related to \( \lambda(K, \cdot) \).

**Remark.** Note that if \( \lambda \) is an absolutely continuous Borel measure, then \( \lambda(K, \cdot) \) automatically becomes a Borel measure. For more details, see Section 2 and Lemma 3.3 in [13].
Proof. Since $K \in \mathcal{K}_o^n$, there exists $c > 0$ such that $\frac{r_K}{\Omega_K} > c$. Consider some $u \in S^{n-1}$ and $v \in \alpha_K(u)$. Then from (2.5),
\begin{equation}
(3.1) \quad r_K \leq h_K(v) = \rho_K(u) u \cdot v \leq \Omega_K u \cdot v.
\end{equation}
Hence, $c < \frac{r_K}{\Omega_K} \leq u \cdot v$. Therefore, for each $u \in S^{n-1}$, we have:
\begin{equation}
(3.2) \quad \alpha_K(u) \subset u_{\arccos(c)} \subset u_{\frac{\pi}{2} - \alpha}
\end{equation}
for some $\alpha$, where $0 < \alpha < \frac{\pi}{2}$. Therefore, for any closed set $\omega$ contained in a closed hemisphere, since $\omega_{\frac{\pi}{2} - \alpha} = \bigcup_{u \in \omega} u_{\frac{\pi}{2} - \alpha}$, we obtain:
\begin{equation}
(3.3) \quad \lambda(K, \omega) = \lambda(\alpha_K(\omega)) \leq \lambda(\omega_{\frac{\pi}{2} - \alpha}).
\end{equation}
\hfill \Box

In particular, the above proof shows that given measures $\lambda$ and $\lambda(K, \cdot)$, the constant $\alpha$ in the weak Aleksandrov relation does not depend on the choice of a closed set $\omega$ contained in a closed hemisphere. Moreover, the constant $\alpha$ in the above proof encompasses the lower bound on $\frac{r_K}{\Omega_K}$. That is, the bound on the inner to outer radius ratio of the body $K$. The following Proposition is the step in the opposite direction. Recall the notation for a discrete measure $\mu$ in (2.18).

**Proposition 3.2.** Suppose that a discrete Borel measure $\mu$ is weakly Aleksandrov related to a Borel measure $\lambda$. Then, there exists a uniform constant $\alpha \in (0, \frac{\pi}{2})$, such that for any a closed set contained in a closed hemisphere $\omega \subset S^{n-1}$:
\begin{equation}
(3.4) \quad \mu(\omega) \leq \lambda(\omega_{\frac{\pi}{2} - \alpha}).
\end{equation}

**Remark.** We refer to this $\alpha$ as the uniform weak Aleksandrov constant for measures $\mu$ and $\lambda$. \hfill \triangle

**Proof.** Consider all possible $I \subset \{1, \ldots, m\}$ such that $\{v_i\}_{i \in I}$ are contained in a closed hemisphere. Let $\omega^I = \bigcup_{i \in I} v_i$. Since $\mu$ and $\lambda$ are weak Aleksandrov related for each $\omega^I$, we have $\mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2} - \alpha_I})$ for some $\alpha_I$. Since there are only finitely many of those $I$ satisfying the assumption, we can choose $\alpha > 0$ to be the minimum of the $\alpha_I$’s. Now for any closed set $\omega$ contained in a closed hemisphere, we obtain that for some $I \subset \{1, \ldots, m\}$:
\begin{equation}
(3.5) \quad \mu(\omega) = \mu(\omega \cap \{v_i\}_{i \in \{1, \ldots, m\}}) = \mu(\omega^I) \leq \lambda(\omega^I_{\frac{\pi}{2} - \alpha}) \leq \lambda(\omega_{\frac{\pi}{2} - \alpha}),
\end{equation}
where the last step follows from set inclusion. \hfill \Box

Finally, we note that the classical Aleksandrov relation easily implies the weak Aleksandrov relation. In the following, $\mu$ is not necessarily a discrete measure.

**Proposition 3.3.** Suppose that a Borel measure $\mu$ is Aleksandrov related to a Borel measure $\lambda$. Then, $\mu$ is weakly Aleksandrov related to $\lambda$.

**Proof.** Since $\mu$ is Aleksandrov related to $\lambda$, for each compact, spherically convex set $\omega \subset S^{n-1}$, we obtain:
\begin{equation}
(3.6) \quad \mu(\omega) < \lambda(S^{n-1}) - \lambda(\omega^*) = \lambda(\omega_{\frac{\pi}{2}}).
\end{equation}
Now, consider any closed set $\gamma$ contained in a closed hemisphere. Let
\begin{equation}
(3.7) \quad \omega = \langle \gamma \rangle := S^{n-1} \cap \text{conv}(\text{cone}\gamma),
\end{equation}
where \( \text{conv} (\text{cone } \gamma) \) denotes the convex hull of cone \( \gamma \). Note that the convex hull of any set \( S \in \mathbb{R}^n \) is given by all finite convex combinations of elements in \( S \). Thus, recalling the definition of a cone, we obtain the following: for each \( v \in \omega \), there exist vectors \( \{v_i\}_{i \in I} \subset \gamma \) with \( I \) finite index set such that
\[
(3.8) \quad v = \sum_{i \in I} \sigma_i v_i \text{ for some } \sigma_i > 0.
\]

First we want to establish that \( \omega^\pi_2 \subset \gamma^\pi_2 \). Choose any \( u \in \omega^\pi_2 \). Then for some \( v \in \omega \), \( u \cdot v > 0 \). Hence, for some \( \{v_i\}_{i \in I} \subset \gamma \) with \( I \) finite index set,
\[
(3.9) \quad u \cdot v = \sum_{i \in I} \sigma_i (u \cdot v_i) > 0, \text{ where } \sigma_i > 0.
\]
Thus, at least for one \( i \in I \), we have that \( u \cdot v_i > 0 \). Hence, \( u \in v_i^\pi_2 \subset \gamma^\pi_2 \). We obtained the desired.

Combining, we obtain the following chain of inequalities:
\[
(3.10) \quad \mu(\gamma) \leq \mu(\omega) < \lambda(\omega^\pi_2) \leq \lambda(\gamma^\pi_2).
\]
In particular, obtaining strict inequality \( \mu(\gamma) < \lambda(\gamma^\pi_2) \). By the continuity of measure \( \lambda \), \( \lambda(\gamma^\pi_{-\alpha}) \to \lambda(\gamma^\pi_2) \) as \( \alpha \to 0 \). Hence, for a given closed set \( \gamma \) contained in a closed hemisphere, there exists an \( \alpha \) such that
\[
(3.11) \quad \mu(\gamma) < \lambda(\gamma^\pi_{-\alpha}).
\]
The weak Aleksandrov condition follows.

One might wonder whether we can define the weak Aleksandrov relation merely by restricting the definition to the collection of compact spherically convex sets instead of closed sets contained in a closed hemisphere. We have not investigated this question, leaving it to the reader if they are interested.

4. Essential Estimates and Partial Rescaling of a Polytope

For the rest of the paper, we will assume that \( \mu \) is a discrete Borel measure written as in (2.18), which is not concentrated on a closed hemisphere. We will also assume that \( \lambda \) is an absolutely continuous measure. We begin with the following lemma, which enables us to concentrate our attention exclusively on polytopes.

**Lemma 4.1.** Given any \( K \in \mathcal{K}_0^n \), there exists a polytope \( P \in \mathcal{P}_\mu \) such that:
\[
(4.1) \quad \Phi(P, \mu, \lambda) \geq \Phi(K, \mu, \lambda).
\]

**Proof.** Choose any \( K \in \mathcal{K}_0^n \). Define \( P \) as:
\[
(4.2) \quad P := \text{conv}\{\rho_K(v_i)v_i \mid 1 \leq i \leq m\}.
\]
Clearly, \( P \in \mathcal{P}_\mu \). Moreover, since \( P \) is a convex hull of a subset of \( K \), \( P \subset K \). Hence, \( h_P \leq h_K \), which implies that:
\[
(4.3) \quad \int_{S^{n-1}} \log \rho_K \cdot d\lambda \leq \int_{S^{n-1}} \log \rho_P \cdot d\lambda.
\]
Simultaneously, by the definition of \( P \), for any \( i \) we have that \( \rho_P(v_i) \geq \rho_K(v_i) \). Since \( P \subset K \), we also have that \( \rho_P(v_i) \leq \rho_K(v_i) \). Therefore, for all \( i \), \( \rho_P(v_i) = \rho_K(v_i) \), and, thus,
(4.4) \[
\int_{S^{n-1}} \log \rho_K d\mu = \int_{S^{n-1}} \log \rho_P d\mu.
\]
Combining both equations (4.2) and (4.4), we obtain that \(\Phi(K, \mu, \lambda) \leq \Phi(P, \mu, \lambda).\) \(\square\)

Theorem 8.2 in [13] shows that if \(K \in \mathcal{K}_n\) maximizes the functional \(\Phi(\cdot, \mu, \lambda)\) for a Borel measure \(\mu\) and an absolutely continuous measure \(\lambda\), then \(K\) solves the Gauss Image Problem. Thus, to prove Theorem 1.4, it is sufficient to establish the existence of a \(K \in \mathcal{K}_n\) that maximizes the functional. Lemma 4.1 permits us to restrict bodies to polytopes of the above form, allowing us to work exclusively within the class \(P_\mu\).

We start with some lemmas concerning the class \(P_\mu\). For the rest of the article, we are going to work with the notation defined in (1.10)-(2.24). Recall that for a given \(P \in P_\mu\), we call an \(m\)-tuple \(\alpha\) to be a representation of \(P\) if

\[\alpha = (\alpha_1, \ldots, \alpha_m) = (h_{P^*}(v_1), \ldots, h_{P^*}(v_m)).\]

**Lemma 4.2.** Given \(P \in P_\mu\), let \(\alpha\) be its representation. Then, for any \(u \in S^{n-1}:
\]

\[
\rho_{P^*}(u) = \min \left\{ \frac{\alpha_i}{u \cdot v_i} \bigg| i \in \{1, \ldots, m\}, u \cdot v_i > 0 \right\}.
\]

Moreover, given \(u \in S^{n-1}\)

\[
\rho_{P^*}(u) = \frac{\alpha_i}{u \cdot v_i}
\]
if and only if for some \(i\), \(r_{P^*}(u) \in H_{P^*}(v_i)\).

**Proof.** Recall from (2.20) that \(P^*\) can be written as,

\[P^* = \bigcap_{i=1}^{m} H^-(\alpha_i, v_i).\]

Fix some \(u \in S^{n-1}\). Suppose for some \(i\), \(u \cdot v_i \leq 0\). Then \(H^-(\alpha_i, v_i)\) contains the entire ray in the direction of \(u\) starting at the origin. Suppose now for a given \(i\), \(u \cdot v_i > 0\). Then a ray in the direction of \(u\) starting at the origin intersects the hyperplane \(H(\alpha_i, v_i)\). In this case, it is straightforward to verify that the distance from the origin to the intersection will be:

\[
\frac{\alpha_i}{u \cdot v_i}.
\]
Therefore, looking back at the equation (4.8), we obtain that

\[
\rho_{P^*}(u) = \min \left\{ \frac{\alpha_i}{u \cdot v_i} \bigg| i \in \{1, \ldots, m\}, u \cdot v_i > 0 \right\}.
\]

The last part of the statement follows from equation (4.10) and (4.8). \(\square\)

The following lemma is a core estimate, which will later be used to properly rescale the polytopes without decreasing the value of the functional.

**Lemma 4.3.** Given \(P \in P_\mu\), let \(\alpha\) be its representation. If we are given a nonempty and not full index set \(I \subset \{1, \ldots, m\}\), with \(U^* < L\), then:

\[
\bigcup_{i \notin I} (v_i)_{\arccos \frac{U^*}{L}} \subset \alpha_P \left( \bigcup_{i \notin I} v_i \right).
\]
Proof. Suppose that for a given direction \( u \in S^{n-1} \), we have \( \rho_{P^*}(u) < L \). Then, using Lemma 4.2 we obtain

\[
\min \left\{ \frac{\alpha_i}{u \cdot v_i} \left| i \in \{1, \ldots, m\}, u \cdot v_i > 0 \right. \right\} = \rho_{P^*}(u) < L.
\]  

(4.12)

Suppose the minimum is achieved for some index \( j \). Then, from the previous equation, using the definition of \( L \), we obtain:

\[
\frac{\alpha_j}{u \cdot v_j} < L = \min_{i \in I} \alpha_i.
\]  

(4.13)

Thus, since \( 0 < u \cdot v_j \leq 1 \), we obtain that \( \alpha_j < \min_{i \in I} \alpha_i \), and, hence, \( j \notin I \). Now, applying the second part of Lemma 4.2 we obtain that

\[
\rho_{P^*}(u) = \frac{\alpha_j}{u \cdot v_j} \iff r_{P^*}(u) \in H_{P^*}(v_j) \iff r_P(v_j) \in H_P(u) \iff u \in \alpha_P(v_j).
\]  

(4.14)

Since \( j \notin I \), we obtain \( u \in \alpha_P(\bigcup_{i \notin I} v_i) \).

Thus, we have established that if for a given direction \( u \in S^{n-1} \), we have \( \rho_{P^*}(u) < L \), then \( u \in \alpha_P(\bigcup_{i \notin I} v_i) \). Now, pick any \( u \in (v_j)_{\arccos \frac{U^*}{L}} \) for some \( j \notin I \). We obtain \( u \cdot v_j > \frac{U^*}{L} \).

Thus, combining this with equation (4.12) and the fact that \( j \notin I \), we obtain:

\[
\rho_{P^*}(u) = \min \left\{ \frac{\alpha_i}{u \cdot v_i} \left| i \in \{1, \ldots, m\}, u \cdot v_i > 0 \right. \right\} \leq \frac{\alpha_j}{u \cdot v_j} \leq \frac{U^*}{u \cdot v_j} < L.
\]  

(4.15)

Thus, \( \rho_{P^*}(u) < L \) and the claim follows from the first part of the proof.

\[ \square \]

In the upcoming proof of Theorem 1.4, we will utilize what we refer to as the partial rescaling of a polytope. Let us now describe this construction. Suppose we are given a polytope \( P \in \mathcal{P}_\mu \), along with a nonempty and not full index set \( I \subset \{1, \ldots, m\} \). Let \( \alpha \) denote the representation of \( P \). Recall that \( P \) and \( P^* \) can be written as

\[
P = \text{conv}\{\frac{v_i}{\alpha_i} | 1 \leq i \leq m\},
\]  

(4.16)

\[
P^* = \bigcap_{i=1}^m H^-(\alpha_i, v_i).
\]

We would like to rescale the half spaces that correspond to the index set \( I \) in the second formula by a factor \( t > 0 \). This procedure will be called a partial rescaling of polytope \( P \) with respect to index set \( I \). In terms of the preceding formula, the partial rescaling of a polytope \( P \) can be written as

\[
P_t = \text{conv}\left(\{\frac{v_i}{\alpha_i} | i \in I\} \cup \{\frac{v_i}{t\alpha_i} | i \notin I\}\right),
\]  

(4.17)

\[
P^*_t = \bigcap_{i \in I} H^-(\alpha_i, v_i) \bigcap_{i \notin I} H^-(t\alpha_i, v_i).
\]
In fact, $P_t$ is the dual Wulff shape of the $m$-tuple $\gamma_t$ defined by:

\begin{equation}
\begin{aligned}
\gamma_{t,i} &= \alpha_i \text{ if } i \in I, \\
\gamma_{t,i} &= t\alpha_i \text{ if } i \notin I.
\end{aligned}
\end{equation}

See the Preliminaries section for the definitions. We will always assume that $t \in (0, 1]$. The most important point to make about the partial rescaling is that the representation of $P_t$ is not necessarily equal to $\gamma_t$. For example, if the set $\{v_i \mid i \notin I\}$ is not contained in a closed hemisphere, $P_t^*$ will approach the origin in the Hausdorff distance as $t \to 0$, and, thus, for all $i \in \{1, \ldots, m\}$ we will have $\alpha_{t,i} \to 0$ while $\gamma_{t,i} = \alpha_i$ will remain constant. In the end, if $\alpha_t$ is the representation of $P_t$, we can only claim that:

\begin{equation}
\alpha_{t,i} \leq \gamma_{t,i},
\end{equation}

as mentioned in (2.22).

The following lemma characterizes the behavior of the partial rescaling. It can be seen as a discrete analog to the classical results about Wulff shapes.

**Lemma 4.4.** Suppose $P \in \mathcal{P}_\mu$ and $I \subset \{1, \ldots, m\}$ is a nonempty and not full index set. Let $\alpha$ be its dual representation. Consider the $m$-tuple $\gamma_t$ defined by:

\begin{equation}
\begin{aligned}
\gamma_{t,i} &= \alpha_i \text{ if } i \in I, \\
\gamma_{t,i} &= t\alpha_i \text{ if } i \notin I.
\end{aligned}
\end{equation}

Let $P_t$ be the dual Wulff shape of $\gamma_t$, where $t \in (0, 1]$. Let $\alpha_t$ denote its representation. Then,

\begin{equation}
\begin{aligned}
t\alpha_i &\leq \alpha_{t,i} \leq \alpha_i \text{ for } i \in I, \\
\alpha_{t,i} &= t\alpha_i \text{ if } i \notin I.
\end{aligned}
\end{equation}

**Proof.** Let $i \notin I$. By the definition of the representation,

\begin{equation}
\alpha_{t,i} = h_{P_t^*}(v_i).
\end{equation}

From (4.17), we obtain that $tP^* \subset P_t^*$. Thus,

\begin{equation}
\alpha_{t,i} = h_{P_t^*}(v_i) \geq th_{P^*}(v_i) = t\alpha_i.
\end{equation}

On the other hand, from the definition of the dual Wulff shape we have that

\begin{equation}
P_t^* \subset P^* \cap H^-(t\alpha_i, v_i),
\end{equation}

which implies that

\begin{equation}
\alpha_{t,i} = h_{P_t^*}(v_i) \leq t\alpha_i.
\end{equation}

Combining, (4.23) and (4.25) we obtain that for $i \notin I$

\begin{equation}
\alpha_{t,i} = t\alpha_i.
\end{equation}

If $i \in I$, then, as $tP^* \subset P_t^* \subset P^*$, we obtain

\begin{equation}
t\alpha_i \leq \alpha_{t,i} \leq \alpha_i.
\end{equation}
5. THE PARTIAL RESCALING OF A POLYTOPE AND THE FUNCTIONAL

We now turn our attention to the key lemma related to the partial rescaling. Under the assumption of the weak Aleksandrov relation, as we have noted, we can no longer claim that for any sequence maximizing the functional, there is a lower bound on the inner to outer radius ratio. The following lemma provides us with the tool to overcome this difficulty. It establishes that, with proper assumptions, the partial rescaling does not decrease the value of the functional.

**Lemma 5.1.** Suppose \( P \in \mathcal{P}_\mu \) and \( I \subset \{1, \ldots, m\} \) is a nonempty and not full index set. Let \( \alpha \) be its dual representation. Consider the \( m \)-tuple \( \gamma_t \) defined by

\[
\gamma_{t,i} = \begin{cases} 
\alpha_i & \text{if } i \in I, \\
t\alpha_i & \text{if } i \notin I.
\end{cases}
\]

Let \( P_t \) be the dual Wulff shape of \( \gamma_t \). Suppose for some \( 0 < t_0 < 1 \) the following holds:

\[
\mu(\bigcup_{i \notin I} v_i) \geq \lambda(\alpha_P(\bigcup_{i \notin I} v_i)).
\]

Then, \( \Phi(P_{t_0}) \geq \Phi(P) \).

**Proof.** Before we start to compare \( \Phi(P_{t_0}) \) with \( \Phi(P) \), let us first analyze the behavior of the radial functions under the partial rescaling. Let \( t \) be any value between \( t_0 \) and 1. First, we claim that for any \( u \in S^{n-1} \):

\[
\rho_{P_t}(u) = \min \left\{ \frac{\alpha_i}{u \cdot v_i} \mid i \in I, u \cdot v_i > 0 \right\} \bigcup \left\{ \frac{t\alpha_i}{u \cdot v_i} \mid i \notin I, u \cdot v_i > 0 \right\}.
\]

The proof of this claim is the same as the proof of Lemma 4.2, where instead of (4.8) one should use (4.17). We also notice that the equation (4.17) implies the following relation:

\[
\alpha_P(\bigcup_{i \notin I} v_i) \subset \alpha_P(\bigcup_{i \notin I} v_i).
\]

for \( 0 < t_1 < t_2 \leq 1 \). In fact, from (4.17) it is straightforward to verify that

\[
\bigcup_{0 < t \leq 1} \alpha_P(\bigcup_{i \notin I} v_i) = \bigcup_{i \notin I} (v_i)^{\frac{1}{t}}.
\]

Now, examining the equations (5.3), (5.4), and (5.5), we can separate three possible behaviors of \( \rho_{P_t}(u) \) as a function of \( t \) for \( t \leq 1 \):

- If \( u \in \alpha_P(\bigcup_{i \notin I} v_i) \), then \( u \in \alpha_P(v_j) \) for some \( j \notin I \). Thus, \( r_P(v_j) \in H_P(u) \) which is equivalent to \( r_{P_t}(u) \in H_{P_t}(v_j) \). Thus, from Lemma 4.2 we obtain

\[
\rho_{P_t}(u) = \frac{\alpha_j}{u \cdot v_j},
\]

where \( j \notin I \), and, in particular, the minimum in (5.3) is attained at \( j \notin I \). Therefore, from (5.3) we obtain that:

\[
\rho_{P_t}(u) = t \rho_{P_1}(u).
\]

- If \( u \notin \bigcup_{i \notin I} (v_i)^{\frac{1}{t}} \), then \( u \cdot v_i \leq 0 \) for all \( i \notin I \). Thus, from equation (5.3) (or equation (5.5)) we obtain that:

\[
\rho_{P_t}(u) = \rho_{P_1}(u).\]
• If \( u \in \bigcup_{i \notin I} (v_i) \setminus A_P(\bigcup_{i \notin I} v_i) \), then by applying equations (5.3), (5.4), and (5.5) in a way similar to the previous two cases, we obtain that as \( t \) decreases, \( \rho_{P_t}^*(u) \) remains constant up until the first moment when \( u \in A_P(\bigcup_{i \notin I} v_i) \), after which it starts to scale. Let \( t(u) \) denote the maximum \( t \) for which \( u \in A_P(\bigcup_{i \notin I} v_i) \). We obtain:

\[
\rho_{P_t}^*(u) = \rho_{P^*}(u) \quad \text{when} \quad t \in [t(u), 1],
\]

\[
\rho_{P_t}^*(u) = \frac{t}{t(u)} \rho_{P^*}(u) \quad \text{when} \quad t \in (0, t(u)].
\]

By looking at the dual, using Lemma 4.4 and the convex hull equation (4.17) for \( P \), we obtain the following behavior for \( \rho_{P_t}(v_i) \):

• If \( i \notin I \), then

\[
\rho_{P_t}(v_i) = \frac{\rho_P(v_i)}{t}.
\]

• If \( i \in I \), then \( \rho_{P_t}(v_i) \) is non-decreasing as \( t \) decreases and

\[
\rho_{P_t}(v_i) \leq \frac{\rho_P(v_i)}{t}.
\]

Now with the help of equations (5.7)-(5.11) regarding the behavior of \( \rho_{P_t} \) and \( \rho_{P_t}^* \), we would like to compute \( \Phi(P_{t_1}) - \Phi(P_{t_2}) \) for some \( 0 < t_1 < t_2 \leq 1 \). From equations (5.4) and (5.5), we can separate \( \Phi(P_{t_1}) - \Phi(P_{t_2}) \) into the following terms:

\[
\Phi(P_{t_1}) - \Phi(P_{t_2}) =
\int_{S^{n-1}} \log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}})d\mu + \int_{A_{P_{t_1}}(\bigcup_{i \notin I} v_i)} \log(\rho_{P_{t_1}^*}) - \log(\rho_{P_{t_2}^*})d\lambda + \int_{S^{n-1}\setminus \bigcup_{i \notin I} (v_i)} \log(\rho_{P_{t_1}}) - \log(\rho_{P_{t_2}})d\lambda + \int_{\bigcup_{i \notin I} (v_i) \setminus A_{P_{t_1}}(\bigcup_{i \notin I} v_i)} \log(\rho_{P_{t_1}^*}) - \log(\rho_{P_{t_2}^*})d\lambda + \int_{A_{P_{t_1}}(\bigcup_{i \notin I} v_i) \setminus A_{P_{t_2}}(\bigcup_{i \notin I} v_i)} \log(\rho_{P_{t_1}^*}) - \log(\rho_{P_{t_2}^*})d\lambda.
\]

Now, we are going to estimate each term in the above equation.
The first term
First, we look at the integral with respect to $\mu$. From (5.10) and since $\rho_{P_i}(v_i)$ is non-decreasing for $i \in I$, see (5.11), we obtain that:

$$\int_{S^{n-1}} \log(\rho_{P_1}) - \log(\rho_{P_2}) d\mu = \sum_{i \in I} \left( \log(\rho_{P_1}(v_i)) - \log(\rho_{P_2}(v_i)) \right) \mu(v_i) + \sum_{i \notin I} \left( \log(\rho_{P_1}(v_i)) - \log(\rho_{P_2}(v_i)) \right) \mu(v_i) \geq \log(\frac{t_2}{t_1}) \mu(\bigcup_{i \notin I} v_i).$$

(5.13)

The second term
There are two possibilities here. First, recalling (5.4), we have that

$$\alpha_{P}(\bigcup_{i \notin I} v_i) \subset \alpha_{P_2}(\bigcup_{i \in I} v_i) \subset \alpha_{P_1}(\bigcup_{i \in I} v_i).$$

(5.14)

If we suppose that $u \in \alpha_{P}(\bigcup_{i \notin I} v_i)$ we immediately obtain from (5.7) that

$$\log(\rho_{P_2}^*(u)) - \log(\rho_{P_1}^*(u)) = \log(\frac{t_1}{t_2}).$$

(5.15)

Alternatively, if $u \in \alpha_{P_2}(\bigcup_{i \notin I} v_i) \setminus \alpha_{P}(\bigcup_{i \notin I} v_i)$, then $t_1 \leq t(u)$ and $t_2 \leq t(u)$ and we use (5.9) to obtain the same result:

$$\log(\rho_{P_2}^*(u)) - \log(\rho_{P_1}^*(u)) = \log(\frac{t_1}{t(u)}\rho_{P_1}^*(u)) - \log(\frac{t_2}{t(u)}\rho_{P_2}^*(u)) = \log(\frac{t_1}{t_2}).$$

(5.16)

Combining the two, for the second term, we obtain

$$\int_{\alpha_{P_2}(\bigcup_{i \notin I} v_i)} \log(\rho_{P_2}^*) - \log(\rho_{P_1}^*) d\lambda = \log(\frac{t_1}{t_2}) \lambda(\alpha_{P_2}(\bigcup_{i \notin I} v_i)).$$

(5.17)

The third term
From (5.8), the third term is computed as:

$$\int_{S^{n-1} \setminus \bigcup_{i \notin I} v_i} \log(\rho_{P_1}^*) - \log(\rho_{P_2}^*) d\lambda = 0.$$

(5.18)
The fourth and the fifth terms

Now, we want to estimate the last two terms. As in (5.9), given
\begin{equation}
      u \in \bigcup_{i \notin I} (v_i)_{\frac{\pi}{2}} \setminus \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right),
\end{equation}
recall that \( t(u) \) denote the maximum \( t \) such that \( u \in \alpha_P \left( \bigcup_{i \notin I} v_i \right) \). Notice that if
\begin{equation}
      u \in \bigcup_{i \notin I} (v_i)_{\frac{\pi}{2}} \setminus \alpha_{P_1} \left( \bigcup_{i \notin I} v_i \right),
\end{equation}
from (5.4) and (5.5) we obtain that \( t(u) < t_1 < t_2 \). And, thus, since \( t_1, t_2 \in [t(u), 1) \), from (5.9) we obtain:
\begin{equation}
    \log(\rho_{P_2}(u)) - \log(\rho_{P_2}(u)) = \log(\rho_{P_1}(u)) - \log(\rho_{P_1}(u)) = 0,
\end{equation}
which implies that the fourth term is zero. On the other hand, if
\begin{equation}
    u \in \alpha_{P_1} \left( \bigcup_{i \notin I} v_i \right) \setminus \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right),
\end{equation}
then \( t(u) \in [t_1, t_2) \). Thus, from (5.9) we obtain:
\begin{equation}
    \left| \int_{\alpha_{P_1} \left( \bigcup_{i \notin I} v_i \right) \setminus \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right)} \log(\rho_{P_1}) - \log(\rho_{P_2}) \, d\lambda \right| =
\end{equation}
\begin{equation}
    \left| \int_{\alpha_{P_1} \left( \bigcup_{i \notin I} v_i \right) \setminus \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right)} \log(\frac{t_1}{t(u)}) \, d\lambda \right| 
    \leq \lambda \left( \alpha_{P_1} \left( \bigcup_{i \notin I} v_i \right) \setminus \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right) \right) \left| \log(\frac{t_1}{t_2}) \right|.
\end{equation}

Overall, we established (5.23) using the assumption that \( 0 < t_1 < t_2 \leq 1 \). Let us now investigate (5.23) under the additional assumption that \( t_1 \) is approximately \( t_2 \).

From the continuity of measure \( \lambda \) and since \( \lambda \) an is absolutely continuous measure, as \( t_1 \to t_2 \) (with \( t_1 < t_2 \)), we find that
\begin{equation}
    \lambda \left( \alpha_{P_1} \left( \bigcup_{i \notin I} v_i \right) \setminus \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right) \right) \to \lambda \left( \partial \left( \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right) \right) \right) = 0
\end{equation}
where \( \partial \) denotes the boundary of the set in \( S^{n-1} \). So, in particular, given any \( \varepsilon > 0 \) for all \( t_1 \) sufficiently close to \( t_2 \), the right side at the end of (5.23) is less than \( \varepsilon |\log(\frac{t_1}{t_2})| \).

Combining all of the estimates for different terms of (5.12), we obtain that given any \( t_2 \in (0, 1] \) and \( \varepsilon > 0 \), we can pick \( \delta > 0 \) with \( t_2 - \delta > 0 \) such that for all \( t_1 \in [t_2 - \delta, t_2] \):
\begin{equation}
    \Phi(P_{t_1}) - \Phi(P_{t_2}) \geq \log(\frac{t_2}{t_1}) \mu \left( \bigcup_{i \notin I} v_i \right) + \log(\frac{t_1}{t_2}) \lambda \left( \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right) \right) - \varepsilon \left| \log(\frac{t_1}{t_2}) \right|
\end{equation}
\begin{equation}
    = \log(\frac{t_2}{t_1}) \left( \mu \left( \bigcup_{i \notin I} v_i \right) - \lambda \left( \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right) \right) - \varepsilon \right).
\end{equation}

Now, we are going to analyze two special cases of previous equation guided by the assumption (5.2).

Case 1. Given \( t_2 \in (0, 1] \), suppose \( \mu \left( \bigcup_{i \notin I} v_i \right) > \lambda \left( \alpha_{P_2} \left( \bigcup_{i \notin I} v_i \right) \right) \).
Then (5.25), implies that we can pick \( \delta > 0 \) such that for all \( t_1 \in [t_2 - \delta, t_2) \), \( \Phi(P_{t_1}) > \Phi(P_{t_2}) \).

**Case 2.** Given \( t_2 \in (0, 1] \), suppose \( \mu(\bigcup_{i \in I} v_i) = \lambda(\alpha_{P_{t_2}}(\bigcup_{i \in I} v_i)) \).

Then, from (5.2), (5.4) and (5.5), for \( t_1 \in (0, t_2] \) :

\[
\lambda(\alpha_{P_{t_1}}(\bigcup_{i \notin I} v_i) \setminus \alpha_{P_{t_2}}(\bigcup_{i \notin I} v_i)) = 0.
\]

This forces the last part of (5.23) to be equal to zero. Which in turn implies that the equation (5.25), under this particular assumptions, refines to the following:

\[
\Phi(P_{t_1}) - \Phi(P_{t_2}) \geq \log(t_2) \mu(\bigcup_{i \notin I} v_i) + \log(t_2) \lambda(\alpha_{P_{t_2}}(\bigcup_{i \notin I} v_i))
\]

\[
= \log(t_2)(\mu(\bigcup_{i \notin I} v_i) - \lambda(\alpha_{P_{t_2}}(\bigcup_{i \notin I} v_i))) = 0.
\]

Therefore, in this particular case, for all \( t_1 \in (0, t_2] \) we obtain that \( \Phi(P_{t_1}) \geq \Phi(P_{t_2}) \).

We are now fully prepared to conclude the proof. Recall that we were given \( t_0 < 1 \) such that

\[
\mu(\bigcup_{i \notin I} v_i) \geq \lambda(\alpha_{P_{t_0}}(\bigcup_{i \notin I} v_i)).
\]

From (5.4), we obtain that the same statement holds for any \( t_2 \in [t_0, 1] \):

\[
\mu(\bigcup_{i \notin I} v_i) \geq \lambda(\alpha_{P_{t_2}}(\bigcup_{i \notin I} v_i)).
\]

Suppose now that \( \Phi(P_{t_0}) < \Phi(P_1) \). Since \( \Phi(P_1) \) is continuous, let \( t_2 \in (t_0, 1] \) be the smallest value such that \( \Phi(P_{t_2}) = \Phi(P_1) \). Then, for all \( t_1 \in [t_0, t_2) \), \( \Phi(P_{t_1}) < \Phi(P_{t_2}) \). If

\[
\mu(\bigcup_{i \notin I} v_i) > \lambda(\alpha_{P_{t_2}}(\bigcup_{i \notin I} v_i))
\]

then a contradiction arises from Case 1. If

\[
\mu(\bigcup_{i \notin I} v_i) = \lambda(\alpha_{P_{t_2}}(\bigcup_{i \notin I} v_i))
\]

then, from Case 2, we would obtain that \( \Phi(P_{t_0}) = \Phi(P_{t_2}) \), which is also a contradiction. Therefore, \( \Phi(P_{t_0}) \geq \Phi(P_1) \), which was the desired. \( \square \)

The next Lemma is a core component of the proof of Theorem 1.4, where we utilize two previous technical results: Lemma 4.3 and Lemma 5.1. Before we begin with the proof, let us first try to explain its statement in a more intuitive way. Suppose we are given some \( P \in P_\mu \), and a nonempty and not full index set \( I \subset \{1, \ldots, m\} \). Suppose for this index set we have the following:

\[
0 < L^* \leq U^* < L \leq U = 1
\]

\[
\frac{L}{U} \approx 1, \quad \frac{U^*}{L} \approx 0, \quad \frac{L^*}{U^*} \approx 1.
\]
Suppose \( \mu \) is weakly Aleksandrov related to \( \lambda \) and \( \alpha \) is the uniform weak Aleksandrov constant for \( \mu \) and \( \lambda \) given by Proposition 3.2. Then, Lemma 5.2 claims that we can find a new polytope \( P_t \in \mathcal{P}_\mu \) with \( \Phi(P_t) \geq \Phi(P) \) such that:

\[
0 < L_t^* \leq U_t^* < L_t \leq U_t = 1
\]

\[
\frac{L_t}{U_t} \approx 1, \quad \frac{U_t^*}{L_t} \approx \cos\left(\frac{\pi}{2} - \alpha\right), \quad \frac{L_t^*}{U_t^*} \approx 1.
\]

Vaguely speaking, Lemma 5.2 allows us to construct a new polytope with coefficients that are closer together, without decreasing the functional value. Even though the Aleksandrov condition is not stated explicitly, it is embedded in the following Lemma as part of the assumption about:

\[
\mu\left(\bigcup_{i \not\in I} v_i\right) \leq \lambda\left(\bigcup_{i \not\in I} (v_i)_{\frac{\pi}{2} - \alpha}\right).
\]

We again recall that we use the notation established in (1.10)-(2.24).

**Lemma 5.2.** Suppose \( P \in \mathcal{P}_\mu \) is such that \( \max_i \alpha_i = 1 \), and \( I \subset \{1, \ldots, m\} \) is a nonempty and not full index set. Suppose

\[
\mu\left(\bigcup_{i \not\in I} v_i\right) \leq \lambda\left(\bigcup_{i \not\in I} (v_i)_{\frac{\pi}{2} - \alpha}\right)
\]

for some \( 0 < \alpha < \frac{\pi}{2} \). Suppose \( U^* < L \cos\left(\frac{\pi}{2} - \alpha\right) \). In particular, \( 0 < L^* \leq U^* < L \leq U = 1 \). Then, there exists \( P_t \in \mathcal{P}_\mu \), such that:

\[
U_t = 1,
\]

\[
L_t = L,
\]

\[
U_t^* \leq L \cos\left(\frac{\pi}{2} - \alpha\right),
\]

\[
L_t^* = \frac{L^*}{U^*} L \cos\left(\frac{\pi}{2} - \alpha\right).
\]

Moreover, \( \alpha_i = \alpha_{i,j} \) for \( i \in I \) and \( \Phi(P_t) \geq \Phi(P) \).

**Proof.** We want to rescale \( P \) for index set \( I^c = \{1, \ldots, m\} \setminus I \) as:

\[
P_t^* = \bigcap_{i \in I^c} H^-(\alpha_i, v_i) \bigcap_{i \not\in I^c} H^-(t \alpha_i, v_i).
\]

Note that we partially rescale the polytope \( P \) with respect to index set \( I^c \), which is the opposite of the rescaling used in Lemma 5.1 and Lemma 4.4. (See (4.17) as well.) Let

\[
t_0 = \frac{U^*}{L \cos\left(\frac{\pi}{2} - \alpha\right)}.
\]

From our assumptions, \( 0 < t_0 < 1 \). Our goal is to analyze \( P_{t_0}^* \) and to confirm that \( \Phi(P_{t_0}) \geq \Phi(P) \) with the help of Lemma 5.1. Recall, the equation (5.3) from the proof of Lemma 5.1. Using it, we can write for \( u \in S^{n-1} \):

\[
\rho_{P_t^*}(u) = \min\left\{ \left\{ \frac{\alpha_i}{u \cdot v_i} \left| i \in I^c, u \cdot v_i > 0 \right\} \bigcup \left\{ \frac{t \alpha_i}{u \cdot v_i} \left| i \not\in I^c, u \cdot v_i > 0 \right\} \right. \right\}.
\]
Moreover, from Lemma 4.4 we obtain that
\[ t\alpha_i \leq \alpha_{t,i} \leq \alpha_i \text{ for } i \in I^c, \]
\[ \alpha_{t,i} = t\alpha_i \text{ if } i \notin I^c. \]
Thus, recalling the definitions of \( L^*(I), U^*(I), L_t(I), U_t(I) \), it follows from previous equation that
\[ U_t = tU, \]
\[ L_t = tL, \]
\[ U^* \geq U^*_t \geq tU^*. \]
Notice that from the previous equation and the equation (5.40):
\[ U^*_t \leq U^* = t_0 L \cos\left(\frac{\pi}{2} - \alpha\right) = L_{t_0} \cos\left(\frac{\pi}{2} - \alpha\right). \]
In particular,
\[ U^*_t < L_{t_0}. \]
Therefore, we can apply Lemma 4.3 to \( P_{t_0} \) to obtain:
\[ \lambda(\bigcup_{i \notin I} v_i) \leq \lambda(P_{t_0}\bigcup_{i \in I} v_i). \]
This, combined with the assumption (5.37) and the estimate (5.44), gives us the following:
\[ \mu(\bigcup_{i \notin I} v_i) \leq \lambda(\bigcup_{i \notin I} v_i) \leq \lambda(P_{t_0}\bigcup_{i \in I} v_i). \]
Therefore, since \( \alpha_{P_{t_0}}(\bigcup_{i \notin I} v_i) \bigcap \alpha_{P_{t_0}}(\bigcup_{i \in I} v_i) \) is a \( \lambda \) measure zero set, as it has Lebesgue measure zero, and since \( \mu \) and \( \lambda \) have equal weights, we obtain from the equation above that the reverse holds for \( \alpha_{P_{t_0}}(\bigcup_{i \in I} v_i) \), that is
\[ \mu(\bigcup_{i \in I} v_i) \geq \lambda(\bigcup_{i \in I} v_i). \]
By rewriting this to
\[ \mu(\bigcup_{i \notin I} v_i) \geq \lambda(\bigcup_{i \notin I} v_i), \]
we can apply Lemma 5.1 (again, notice that we partially rescale index set \( I^c \)) to conclude:
\[ \Phi(P_{t_0}) \geq \Phi(P). \]
We claim that \( P_t := t_0 P_{t_0} \) is the desired polytope. To prevent possible confusion, let us remark that \( t \) in \( P_t \) is just a convenient notation and does not correspond to partial rescaling of \( P \) as in (5.39) by a factor \( t \). On the other hand, \( P_{t_0} \) is a partial rescaling of \( P \) given by formula (5.39).
From (5.50) we obtain,
\[ \Phi(P_t) = \Phi(P_{t_0}) \geq \Phi(P). \]
What remains is to establish the values of \( U^*_t, L^*_t, L_t \) as well as \( \alpha_{t,i} \) for \( i \in I \).
Firstly, we notice from (2.5):
\[ L^*_t \geq \min_{u \in \mathbb{S}^{n-1}} h_{P_t}(u) = \min_{u \in \mathbb{S}^{n-1}} \rho_{P_t}(u). \]
We are interested in whether $\rho_{P_t}(u)$ decreases. Notice that for $t \geq t_0$, we have

$$t \geq t_0 = \frac{U^*}{L \cos\left(\frac{\pi}{2} - \alpha\right)} \geq \frac{L^*}{L \cos\left(\frac{\pi}{2} - \alpha\right)} > \frac{L^*}{L}. \hspace{1cm} (5.53)$$

Now, from the previous equation, for any $i \in I$ if $u \cdot v_i > 0$, since $u \cdot v_i \leq 1$, we have

$$\frac{t \alpha_i}{u \cdot v_i} > \frac{L^* \alpha_i}{L} > L^*. \hspace{1cm} (5.54)$$

We also have that for $i \notin I$ and $u \cdot v_i > 0$:

$$\frac{\alpha_i}{u \cdot v_i} \geq L^*. \hspace{1cm} (5.55)$$

Combining both previous equations and using $(5.41)$, we obtain that $\rho_{P_t}(u) \geq L^*$ for any $u \in S^{n-1}$ and for any $t \geq t_0$. Therefore, we can apply $(5.52)$ to deduce that $L^*_t \geq L^*$. This, combined with the fact that $L^*_t$ can only decrease as $t$ decreases and that $L^*_t = L^*$, imply that $L^*_t = L^*$ for $t \geq t_0$.

Summarizing this with $(5.43)$, we obtain that for $t \geq t_0$:

$$U_t = tU,$$
$$L_t = tL,$$
$$U^* \geq U^*_t \geq tU^*, \hspace{1cm} (5.56)$$
$$L^*_t = L^*.$$

Recall that $\alpha_t$ is a representation for $P_t$. Notice that

$$\max_i \alpha_{t,i} = \max(U_t, U^*_t). \hspace{1cm} (5.57)$$

Now, since $U = 1$, from the definition of $t_0$ and $(5.56)$ we obtain

$$U^*_t \leq U^* = t_0 L \cos\left(\frac{\pi}{2} - \alpha\right) < t_0 = t_0 U = U_{t_0}. \hspace{1cm} (5.58)$$

Therefore, combining two previous equations, we obtain

$$\max_i \alpha_{t,i} = \max(U_t, U^*_t) = U_t = t. \hspace{1cm} (5.59)$$

Recall that we defined $P_t$ to be equal to $t_0 P_{t_0}$. Thus, $P^*_t = \frac{P^*_{t_0}}{t_0}$, and, therefore, $\alpha_{t,i} = \frac{\alpha_{t_0,i}}{t_0}$. In particular, from $(5.59)$, we obtain that

$$\max_i \alpha_{t,i} = \frac{\max_i \alpha_{t_0,i}}{t_0} = 1. \hspace{1cm} (5.60)$$

Thus, from $(5.56)$, we obtain for $P_t = t_0 P_{t_0}$:

$$U_t = \frac{U_{t_0}}{t_0} = 1,$$
$$L_t = \frac{L_{t_0}}{t_0} = L,$$
$$U^*_t = \frac{U^*_{t_0}}{t_0} \leq \frac{U^*}{t_0} = L \cos\left(\frac{\pi}{2} - \alpha\right),$$
$$L^*_t = \frac{L^*_{t_0}}{t_0} = \frac{L^*}{U^* L \cos\left(\frac{\pi}{2} - \alpha\right)}. \hspace{1cm} (5.61)$$
Now, it only remains to show that for \( i \in I \), we have that \( \alpha_i = \alpha_{r,i} \). This immediately follows from (5.42):

\[
(5.62) \quad \alpha_{r,i} = \frac{\alpha_{t_0,i}}{t_0} = \alpha_i.
\]

\[\square\]

6. Proof of the Main Result

We are ready to start the proof Theorem 1.4. Our strategy is first to pick a sequence of polytopes that maximize the functional \( \Phi \). Then, we will use Lemma 5.2 to modify this sequence, ensuring that it converges to a non-degenerate convex polytope. The proof heavily relies on the notations from (1.10)-(2.24) for varying index sets. We recall that, given a polytope with index \( n \), as in \( P_n \), and an index set \( I \in \{1 \ldots m\} \) we write

\[
U_n(I) := \max_{i \in I} \alpha_i,
\]
\[
L_n(I) := \min_{i \in I} \alpha_i,
\]
\[
U_n^*(I) := \max_{i \notin I} \alpha_i,
\]
\[
L_n^*(I) := \min_{i \notin I} \alpha_i.
\]

Proposition 6.1. Suppose \( \mu \) is a discrete measure not concentrated on a closed hemisphere and \( \lambda \) is an absolutely continuous Borel measure. Suppose \( \mu \) is weakly Aleksandrov related to \( \lambda \). Then there exists a sequence of polytopes \( P_n \in \mathcal{P}_\mu \) maximizing \( \Phi(\cdot) \), such that it converges to some \( P \in \mathcal{P}_\mu \).

Proof. Let \( (P_n)_{n=1}^{\infty} \) be any sequence that maximizes the functional. For each \( n \), let \( \alpha_n \) be the representation of \( P_n \). Rescale each \( P_n \) so that \( \max_i \alpha_{n,i} = 1 \). Since the set \( \{v_i \mid 1 \leq i \leq m\} \) is not contained in a closed hemisphere, it follows from \( \max_i \alpha_{n,i} = 1 \) that there exists \( R > 0 \) such that for all \( n \), \( \mathfrak{R}_{P_n} < R \). Thus, we obtained a sequence that maximizes the functional and has a uniform bound on the outer radii of the duals.

For every permutation \( \sigma \) in \( S_m \), where \( S_m \) represents the set of all possible permutations of \( m \) elements, we define the set \( A_\sigma \subset \mathbb{N} \) to contain all indices \( n \) such that:

\[
(6.2) \quad 1 = \alpha_{n,\sigma(1)} \geq \alpha_{n,\sigma(2)} \geq \ldots \geq \alpha_{n,\sigma(m)} > 0.
\]

Since \( \mathbb{N} = \bigcup_{\sigma \in S_m} A_\sigma \), there exists \( \sigma \in S_m \) such that one of these sets is infinite. Without loss of generality, we can assume that the set \( A_\sigma \) is infinite, corresponding to the identity permutation \( \sigma \). We then take the subsequence of \( (P_n)_{n=1}^{\infty} \) containing only elements in \( A_\sigma \). Since we will never use the original sequence, we redefine the constructed subsequence to be \( (P_n)_{n=1}^{\infty} \).

Thus, we obtain a sequence of polytopes \( (P_n)_{n=1}^{\infty} \) maximizing the functional such that for each \( n \):

\[
(6.3) \quad 1 = \alpha_{n,1} \geq \alpha_{n,2} \geq \ldots \geq \alpha_{n,m} > 0.
\]
Using Bolzano–Weierstrass theorem, we can pass to the subsequence, which we again redefine to be \( (P_n)_{n=1}^\infty \), such that \( P_n^* \) converges to some convex set \( K \), which can be written as:

\[
K = \bigcap_{i=1}^{m} H^- (\alpha_i, v_i),
\]

where \( \alpha_i \) are given by

\[
\lim_{n \to \infty} \alpha_{n,i} = \alpha_i.
\]

Moreover, through repeated application of Bolzano–Weierstrass theorem, we can assume as well that for each \( i \in \{1, \ldots, m-1\} \), there exists \( \beta_i \in [0,1] \):

\[
\lim_{n \to \infty} \frac{\alpha_{n,i+1}}{\alpha_{n,i}} = \beta_i,
\]

since \( 0 < \alpha_{n,i+1} \leq \alpha_{n,i} \) from (6.3). Notice that from (6.6) we also trivially obtain the following equation:

\[
\lim_{n \to \infty} \frac{\alpha_{n,i+k}}{\alpha_{n,i}} = \prod_{0 \leq j \leq k-1} \beta_{i+j},
\]

Finally, the following holds for \( \alpha_i \):

\[
1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m \geq 0.
\]

Note that if all coefficients \( \alpha_i > 0 \), then \( K \) contains the origin in its interior, and, thus, \( P_n \) will converge to \( K^* = P \in \mathcal{P}_\mu \), which is the desired. Suppose it is not the case. Then, in particular, from (6.8), we obtain that \( \alpha_m = 0 \). Since \( \alpha_1 = 1 \), we obtain that at least one of the variables \( \beta_i \) is equal to zero.

Now, we will construct index sets \( I_j \) that correspond to what we call as different rates of convergence of \( \alpha_{n,i} \) to zero. Suppose there are exactly \( k \) indices, \( i_0 \leq i_1 \leq \ldots \leq i_{k-1} \), for which \( \beta_{i_j} \) is equal to zero, where \( j \in \{0, \ldots, k-1\} \). For convenience, let \( i_k = m \). We define:

\[
I_0 = \{1, \ldots, i_0\},
I_1 = \{i_0 + 1, \ldots, i_1\},
\]

\[
I_j = \{i_{j-1} + 1, \ldots, m\}.
\]

Notice that by construction, these sets are nonempty and not full, and their union is \( \{1, \ldots, m\} \). Moreover, from (6.3), (6.6), and from the definitions of sets \( I_j \) and indices \( i_j \), we obtain the following inequalities:

\[
1 = U_n(I_0) \geq L_n(I_0) \geq U_n(I_1) \geq L_n(I_1) \geq \ldots \geq U_n(I_k) \geq L_n(I_k) > 0,
\]

\[
\lim_{n \to \infty} \frac{U_n(I_{j+1})}{L_n(I_j)} = \lim_{n \to \infty} \frac{\alpha_{n,i_{j+1}}}{\alpha_{n,i_j}} = \beta_{i_j} = 0,
\]

\[
\lim_{n \to \infty} \frac{L_n(I_j)}{U_n(I_j)} = \lim_{n \to \infty} \frac{\alpha_{n,i_j}}{\alpha_{n,i_{j-1}+1}} > c_j \text{ for some constant } c_j > 0.
\]

Intuitively, each \( I_j \) contains indices of elements that converge to 0 at the same rate. For example, from (6.4), (6.10)-(6.12), we see that \( i \in I_0 \) if and only if \( \alpha_i > 0 \). On the other hand, elements with an index in the set \( I_k \) converge to zero the fastest. We call such a sequence of
polytopes to be degenerate of order $k$. If we have a convergent sequence with limit $K$ that is degenerate of order 0, then for all $i \in \{1, \ldots, n\}$, $\alpha_i > 0$ and, hence, $K^* \in \mathcal{P}_\mu$. Therefore, to prove the proposition, it is sufficient to show that for a given sequence that is degenerate of order $k > 0$, we can always find a new sequence of polytopes that is degenerate of a strictly lesser order than $k$ and such that the new sequence still maximizes the functional. This will be our goal for the rest of the proof.

Let $I = I_0 \cup I_1 \ldots \cup I_{k-1}$. Note that the set $\{v_i \mid i \notin I\}$ is contained in a closed hemisphere as otherwise $P^*_n$ would converge to zero everywhere, which would contradict our assumption that $\max_i \alpha_i, n, i = 1$. Since $\mu$ and $\lambda$ are weak Aleksandrov related, using Proposition 3.2, we obtain that there exists a uniform weak Aleksandrov constant $\alpha > 0$. Therefore, since the set $\{v_i \mid i \notin I\}$ is a closed set and contained in a closed hemisphere, we can write:

$$
\mu(\bigcup_{i \notin I} v_i) \leq \lambda(\bigcup_{i \notin I} (v_i)_{2-\alpha}).
$$

(6.13)

From (6.11) we can find $N$ such that $\forall n > N,$

$$
\frac{U^*_n(I)}{L_n(I)} = \frac{U_n(I_k)}{L_n(I_{k-1})} < \cos\left(\frac{\pi}{2} - \alpha\right).
$$

(6.14)

It should also be noted that for all $n$ we have that $U_n(I) = U_n(I_0) = 1$. Thus, after combining this with (6.13) and (6.14), for each $n > N$, we can apply Lemma 5.2 to $P_n$ and the index set $I$ to construct the partially rescaled polytopes $P_{r,n}$. Let $(P_{r,n})_{n=N}^\infty$ be the newly obtained sequence of partially rescaled polytopes. Let $\alpha_{r,n}$ be the representations of $P_{r,n}$.

First of all, note that, according to Lemma 5.2, we have $\Phi(P_{r,n}) \geq \Phi(P_n)$. Therefore, the newly obtained sequence still maximizes the functional $\Phi(\cdot)$. Secondly, from Lemma 5.2, we also have that for $i \in I$,

$$
\alpha_{r,n,i} = \alpha_{n,i}.
$$

(6.15)

We also note the following identities based on Lemma 5.2:

\[U_{r,n}(I) = 1,\]
\[L_{r,n}(I) = L_n(I),\]
\[U^*_{r,n}(I) \leq L_n(I) \cos\left(\frac{\pi}{2} - \alpha\right),\]
\[L^*_{r,n}(I) = \frac{L^*_n(I)}{U^*_n(I)} L_n(I) \cos\left(\frac{\pi}{2} - \alpha\right).\]

(6.16)

From (6.3), (6.15), and (6.16), we obtain:

$$
1 = \alpha_{n,1} = \alpha_{r,n,1} \geq \alpha_{n,2} = \alpha_{r,n,2} \geq \alpha_{r,n,i_{k-1}} \geq \frac{U_{r,n}(I)}{\cos\left(\frac{\pi}{2} - \alpha\right)} > U^*_{r,n}(I) > 0.
$$

(6.17)

Therefore, sets $I_0, I_1, \ldots, I_{k-2}$, which contained coefficients with a rate of convergence up to $k - 2$, remain the same for the newly constructed sequence of polytopes $(P_{r,n})_{n=N}^\infty$. Moreover, $I_{k-1}$ still contains the coefficients that converge to zero with a rate $k - 1$. We will show that for the subsequence of the newly constructed sequence of rescaled polytopes, the coefficients from the index set $I_k$ converge with a rate $k - 1$ as well.

From (6.16) and (6.12), for coefficients in $I_k$ we have the following bound:

$$
1 \geq \frac{L^*_n(I)}{U^*_n(I)} \geq \frac{L^*_n(I)}{U^*_n(I)} = \frac{L_n(I_k)}{U_n(I_k)} > c_k > 0.
$$

(6.18)
Moreover, from (6.16) and (6.12), we have:

\[
\frac{L_{r,n}^*(I)}{L_{r,n}^*(I)} = \frac{L_{r,n}^*(I)}{L_{n}(I)} = \frac{L_{n}(I)}{U_{n}^*(I)} \cos\left(\frac{\pi}{2} - \alpha\right) > c_k \cos\left(\frac{\pi}{2} - \alpha\right) > 0.
\]

While it is true that for \( i \in I \), coefficients \( \alpha_{r,n,i} \) converge as they are equal to \( \alpha_{n,i} \), they might not do so for \( i \notin I \). By applying Bolzano-Weierstrass theorem and choosing new subsequence, we can ensure that they converge. Moreover, (6.18) guarantees that we can apply the same construction as in the beginning to ensure that all the ratios between the elements converge for \( i \notin I \) to some values in the range \([c_k, \frac{1}{c_k}]\). Since the proof does not change, for the sake of notational simplicity, we assume that all elements, as well as all their ratios, converge for \( i \notin I \) in \((P_{r,n})_{n=N}^\infty\). As at the beginning of the proof, we pick some subsequence, so that for some permutation \( \sigma \) of elements in \( I_k \), we have the following based on (6.17):

\[
1 = \alpha_{r,n,1} \geq \alpha_{r,n,2} \geq \ldots \geq \alpha_{r,n,i_{k-1}} = \alpha_{r,n,\sigma(i_{k-1}+1)} \geq \ldots \geq \alpha_{r,n,\sigma(m)} > 0.
\]

This, combined with (6.19), establishes that all coefficients with index \( i \in I_k \) converge the same as coefficient \( \alpha_{r,n,i_{k-1}} \):

\[
\lim_{n \to \infty} \frac{\alpha_{r,n,i}}{\alpha_{r,n,i_{k-1}}} \geq \lim_{n \to \infty} \frac{L_{r,n}^*(I)}{L_{r,n}^*(I)} > c_k \cos\left(\frac{\pi}{2} - \alpha\right) > 0.
\]

Since \( \alpha_{r,n,i_{k-1}} \) converges with a rate \( k-1 \), we have constructed a sequence of polytopes that is degenerate of order \( k-1 \). This finishes the proof. \( \square \)

The proof of Theorem 1.4 immediately follows by an application of Theorem 8.2 in [13].

**Theorem 1.4** Proof. By Proposition 6.1, there exists \( P \in \mathcal{P}_\mu \subset \mathcal{K}_o^n \) that maximizes the functional. Since \( \mu \) is a Borel measure and \( \lambda \) is an absolutely continuous Borel measure, we infer from Theorem 8.2 in [13] that \( \mu = \lambda(P, \cdot) \). \( \square \)

To conclude, let us prove another Proposition which characterizes the bound on the inner to outer radius ratio for the solution to the Gauss Image Problem. The existence of the uniform constant comes from Proposition 3.2.

**Proposition 6.2.** Suppose \( \mu \) is a discrete measure that is not concentrated on a closed hemisphere and \( \lambda \) is an absolutely continuous Borel measure. Suppose \( \mu \) is weak Aleksandrov related to \( \lambda \). Let \( \alpha \) be their uniform weak Aleksandrov constant. Then, there exists a polytope solution \( P \in \mathcal{P}_\mu \) to the Gauss Image Problem such that the ratio \( \frac{r_p}{\mathcal{L}_p} \) is bounded from below by a constant depending only on vectors \( v_i \) and the uniform weak Aleksandrov constant \( \alpha \). Apart from \( \alpha \) and vectors \( v_i \), this constant is independent of \( \lambda \).

**Proof.** By Theorem 1.4, there exists \( P \in \mathcal{P}_\mu \) solving the Gauss Image Problem for measures \( \mu \) and \( \lambda \). Consider any sequence of solutions \( P_n \in \mathcal{P}_\mu \), with \( \max_i \alpha_{n,i} = 1 \), that maximizes the ratio

\[
\min_i \alpha_{n,i} \max_i \alpha_{n,i}.
\]

By compactness, there exists a subsequence converging to the body \( P \in \mathcal{P}_\mu \). Since \( P \) still maximizes the functional, it is a solution by Theorem 8.2 in [13]. Let \( \alpha \) be the representation
for $P$. We also have that $\max_i \alpha_i = 1$ and if $\beta$ is the representation for any other solution $P'$ to the Gauss Image Problem, then

$$\frac{\min_i \alpha_i}{\max_i \alpha_i} \geq \frac{\min_i \beta_i}{\max_i \beta_i}. \tag{6.23}$$

For this $P$, reorder the index set so that $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m > 0$. Define $I_l = \{1 \ldots l\}$. Let $k > 1$ be the integer such that the vectors $\left\{ v_i \mid i \notin I_k \right\}$ are contained in a closed hemisphere, but the vectors $\left\{ v_i \mid i \notin I_{k-1} \right\}$ are not. Clearly, $k < m - 1$. Then, for any index set $I_l$ with $l \geq k$, we have that $\left\{ v_i \mid i \notin I_l \right\}$ are contained in a closed hemisphere. Thus, from Proposition 3.2, we have

$$\mu(\bigcup_{i \notin I_l} v_i) \leq \lambda(\bigcup_{i \notin I_l} (v_i)_{\frac{\pi}{2} - \alpha}), \tag{6.24}$$

where $\alpha$ is the uniform weak Aleksandrov constant.

Suppose

$$U^*(I_l) < L(I_l)\cos\left(\frac{\pi}{2} - \alpha\right). \tag{6.25}$$

Then, we are able to apply Lemma 5.2 to find another body $P_r$ such that

$$L_r(I_l) = L(I_l), \tag{6.26}$$

$$L^*_r(I_l) = \frac{L^*(I_l)}{U^*(I_l)}L(I_l)\cos\left(\frac{\pi}{2} - \alpha\right) > L^*(I_l).$$

Notice that $L^*(I_l) < L(I_l)$. This, combined with (6.26), gives

$$\min(L_r(I_l), L^*_r(I_l)) > L^*(I_l). \tag{6.27}$$

So, in particular, if $\alpha_r$ is the representation for $P_r$, then

$$\min_i \alpha_{r,i} = \min(L^*_r(I_l), L_r(I_l)) > L^*(I_l) = \min_i \alpha_i. \tag{6.28}$$

Therefore, we obtain that

$$\frac{\min_i \alpha_{r,i}}{\max_i \alpha_{r,i}} > \frac{\min_i \alpha_i}{\max_i \alpha_i}. \tag{6.29}$$

Since $\Phi(P_r) \geq \Phi(P)$, we have that $P_r$ is still a solution to the Gauss Image Problem by Theorem 8.2 from [13]. Therefore, (6.29) contradicts (6.23).

Thus, we obtain that for all $l \geq k$ the opposite to (6.25) holds, that is

$$U^*(I_l) \geq L(I_l)\cos\left(\frac{\pi}{2} - \alpha\right). \tag{6.30}$$

In particular, since $1 = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m > 0$ we obtain for $l \geq k$

$$\alpha_{l+1} \geq \alpha_l \cos\left(\frac{\pi}{2} - \alpha\right). \tag{6.31}$$

Thus, we obtain

$$\alpha_m \geq \alpha_k \cos\left(\frac{\pi}{2} - \alpha\right)^{m-k}. \tag{6.32}$$

And, therefore,

$$\min_{u \in S^{n-1}} \rho_{P^*}(u) = \alpha_m \geq \alpha_k \cos\left(\frac{\pi}{2} - \alpha\right)^{m-k}. \tag{6.33}$$
Now, consider \( \{ v_i \mid i \notin I_{k-1} \} \). We define \( \gamma \) as following:

\[
\gamma = \inf \left\{ u \cdot v_i \mid u \in S^{n-1}, i \notin I_{k-1}, u \cdot v_i > 0 \right\}.
\]

Since \( \{ v_i \mid i \notin I_{k-1} \} \) are not contained in a closed hemisphere, we obtain that \( \gamma > 0 \). Thus, from Lemma 4.2 we have that for all \( u \in S^{n-1} \)

\[
\rho_p(u) = 
\min \left\{ \frac{\alpha_i}{u \cdot v_i} \mid i \in \{1, \ldots, m\}, u \cdot v_i > 0 \right\} \leq \min \left\{ \frac{\alpha_i}{u \cdot v_i} \mid i \notin I_{k-1}, u \cdot v_i > 0 \right\} \leq \frac{\alpha_k}{\gamma}.
\]

Combining this with equation (6.33) we obtain:

\[
\frac{r_P}{R_P} = \frac{r_{p^*}}{R_{p^*}} = \frac{\min \rho_{p^*}(u)}{\max \rho_{p^*}(u)} \geq \frac{\alpha_k \cos \left( \frac{\pi}{2} - \alpha \right)^{m-k}}{\alpha_k \gamma} \geq \gamma \cos \left( \frac{\pi}{2} - \alpha \right)^{m-k}.
\]

It would be interesting to consider whether the above approach can be used to solve the Gauss Image Problem with the weak Aleksandrov condition when \( \lambda \) is an absolutely continuous measure, and no additional discrete conditions are imposed on the measure \( \mu \), as was done in [13] for the classical Aleksandrov condition. The natural approach would be to discretize \( \mu \) and to try to invoke Proposition 6.2. Yet, we notice that the bound on the inner to outer radius ratio for the solution to the discrete problem, obtained in the Proposition 6.2, significantly depends on the structure of this discretization.

**Conjecture 6.3.** Suppose \( \mu \) and \( \lambda \) are Borel measures on \( S^{n-1} \), where \( \lambda \) is absolutely continuous. If \( \mu \) is not concentrated on a closed hemisphere and is weakly Aleksandrov related to \( \lambda \), then there exists a convex body \( K \) containing the origin in its interior, such that \( \mu = \lambda(K, \cdot) \).

**References**

[1] A. Aleksandrov, An application of the theorem on the invariance of the domain to existense proofs, Izv. Akad. Nauk SSSR Ser. Math. 3 no.3 (1939) 243–256.

[2] A. Aleksandrov, On the theory of Mixed Volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies. Mat. Sbornik N.S. 3 (1938), 27-46.

[3] A. Aleksandrov, Existence and uniqueness of convex surface with a given integral curvature. C. R. (Doklady) Acad. Sci. URSS (N.S.) 35 (1942), 131-134.

[4] L. Caffarelli, Interior \( W^{2,p} \)-estimates for solutions of the Monge-Ampère equation, Ann. Math 131 (1990) 135-150.

[5] S. Chen, Y. Huang, Q.-R. Li, and J. Liu, The \( L^p \)-Brunn-Minkowski inequalities for \( p < 1 \). Adv. Math 368:107166 (2020).

[6] S.-Y. Cheng and S.-T. Yau, On the regularity of the solution of the n-dimensional Minkowski problem, Commun. Pure Appl. Math. 29 (1976) 495–516.

[7] K.-S. Chou and X.-J. Wang, The \( L_p \)-Minkowski problem and the Minkowski problem in centroaffine geometry, Adv. Math. 205 (2006), 33–83.

[8] J. Bertrand, Prescription of Gauss curvature using optimal mass transport. Geom. Dedicata 183 (2016), 81–99.

[9] F. Besau and E. M. Werner, The floating body in real space forms. J. Differential Geom. 110 (2008), no.2, 187–220.
[10] K. J. Böröczky, M. Henk, and H. Pollehn, Subspace concentration of dual curvature measures of symmetric convex bodies, J. Differential Geom. 109 (2018), 411–429.
[11] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. (JAMS) 26 (2013), 831–852.
[12] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. (JAMS) 26 (2013), 831–852.
[13] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, The log-Brunn-Minkowski inequality, Adv. Math. 315 (2012) 1974–1997.
[14] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, The Gauss Image Problem, Commun. Pure Appl. Math. Vol.LXXIII (2020), 1046–1452.
[15] R.J. Gardner, Geometric Tomography, Second edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2006.
[16] P.M. Gruber, Convex and discrete geometry, Grundlehren der Mathematischen Wissenschaften, 336, Springer, Berlin, 2007.
[17] R. Van Handel, The local logarithmic Brunn-Minkowski inequality for zonoids, Preprint: arXiv:2202.09429 (2022).
[18] Y. Huang, E. Lutwak, D. Yang and G. Zhang, The $L^p$-Aleksandrov problem for $L^p$-Integral curvature, J. Differential Geometry 110 (2018), no. 1, 1–29.
[19] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, Geometric measures in the Brunn-Minkowski theory and their associated Minkowski problems, Acta Math. 216 (2016), 325–388.
[20] Y. Huang and Y. Zhao, On the $L^p$ dual Minkowski problem, Adv. Math. 332 (2018), 57–84.
[21] A. V. Kolesnikov and E. Milman, Local $L^p$-Brunn-Minkowski inequalities for $p < 1$, Mem. Amer. Math. Soc. 277 (2022).
[22] E. Lutwak, The Brunn-Minkowski-Firey Theory. I. Mixed volumes and the Minkowski problem, J. Differential Geometry 38 (1993) 131–150.
[23] E. Lutwak, Dual mixed volumes, Pacific J. Math., 58 (1975), 531-538.
[24] E. Lutwak and V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem, J. Differential Geom. 41 (1995), 227–246.
[25] E. Lutwak, D. Yang, and G. Zhang, $L^p$ affine isoperimetric inequalities, J. Differential Geom. 56 (2000), 111–132.
[26] E. Lutwak, D. Yang, and G. Zhang, On the $L^p$-Minkowski problem, Trans. Amer. Math. Soc. 356 (2004), no. 11, 4359–4370.
[27] E. Lutwak, D. Yang, and G. Zhang, Optimal Sobolev norms and the $L^p$ Minkowski problem, Int. Math. Res. Not. 2006, Art. ID 62987, 21.
[28] E. Lutwak, D. Yang, and G. Zhang, Sharp affine $L^p$ Sobolev inequalities J. Differential Geom. 62(2002), 17–38.
[29] E. Lutwak, D. Yang, and G. Zhang, $L^p$ dual curvature measures, Adv. Math. 329 (2018), 85–132.
[30] S. Mui, On the $L^p$ Aleksandrov Problem for negative p, Adv. math. 408 (2022).
[31] L. Nirenberg, The Weyl and Minkowski Problems in differential geometry in the Large, Commun. Pure Appl. Math., 6 (1953), 337–394.
[32] V. Oliker, Existence and uniqueness of convex hypersurfaces with prescribed Gaussian curvature in spaces of constant curvature, Sem. Inst. Matem. Appl. Giovanni Sansone (1983), 1–64.
[33] V. Oliker, Hypersurfaces in $R^{n+1}$ with prescribed Gaussian curvature and related equations of Monge-Ampère type, Comm. Partial Differential Equations 9 (1984), 807–838.
[34] V. Oliker, Embedding $S^{n-1}$ into $R^{n+1}$ with given integral Gauss curvature and optimal mass transport on $S^{n-1}$, Adv. Math 213 no. 2 (2007) 600-620.
[35] A.V. Pogorelov, The Minkowski Multidimensional Problem, V.H. Winston and Sons (1978) Washington, D.C.
[36] C. Saroglou, Remarks on the conjectured log-Brunn-Minkowski inequality. Geom. Dedicata, 177 (2015), 353-365.
[37] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Second Edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge (2014).
[38] A. Stancu, The discrete planar $L_0$-Minkowski problem, Adv. Math. 167 (2002), 160–174.
[39] N. S. Trudinger and X.-J. Wang, The Monge-Ampère equation and its geometric applications. *Handb. Geom. Anal.* (2008), 467–524.
[40] Y. Zhao, *The dual Minkowski problem for negative indices*, Calc. Var. Partial Differential Equations 56(2):Art. 18, 16, 2017.
[41] Y. Zhao, *Existence of solutions to the even dual Minkowski problem*, *J. Differential Geom.* 110 (2018), 543–572.
[42] Y. M. Zhao, The $L^p$ Aleksandrov problem for origin-symmetric polytopes, *Proc. Amer. Math. Soc.* 147 (2019), 4477-4492.
[43] G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math. 262 (2014), 909–931.
[44] G. Zhu, *The centro-affine Minkowski problem for polytopes*, *J. Differential Geom.* 101 (2015), 159–174.

Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA

*Email address: vs1292@nyu.edu*