COMPLETELY DECOMPOSABLE DIRECT SUMMANDS OF TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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ABSTRACT. Let $A$ be a finite rank torsion–free abelian group. Then there exist direct decompositions $A = B \oplus C$ where $B$ is completely decomposable and $C$ has no rank 1 direct summand. In such a decomposition $B$ is unique up to isomorphism and $C$ unique up to near–isomorphism.

1. Introduction

Torsion-free abelian groups of finite rank (tffr groups) are best thought of as additive subgroups of finite dimensional $\mathbb{Q}$–vector spaces. All “groups” in this article are torsion-free abelian groups of finite rank. The rank of a group $A$ is the dimension of the vector space $\mathbb{Q}A$ that $A$ generates. By reason of rank, such groups always have “indecomposable decompositions”, meaning direct decompositions with indecomposable summands. Although as shown in [16], a group has only finitely many non–isomorphic summands, its indecomposable decompositions can be highly non–unique, (see for example [15, Section 90]), and a group may have such decompositions in which the number of summands or the ranks of the summands differ. A particularly striking result in this direction is due to A.L.S. Corner [12], [13].

Let $P = (r_1, \ldots, r_t)$ be a partition of $n$, i.e., $r_1 \geq 1$ and $r_1 + \cdots + r_t = n$. Then $G$ realizes $P$ if there is an indecomposable decomposition $G = G_1 \oplus \cdots \oplus G_t$ such that for all $i$, $r_i = \text{rank}(G_i)$.

Corner’s Theorem. Given integers $n \geq k \geq 1$, there exists a group $G$ of rank $n$ such that $G$ realizes every partition of $n$ into $k$ parts $n = r_1 + \cdots + r_k$.

Corner’s Theorem is related to two problems posed by Fuchs [15, Problems 67 and 68], namely

1. Given an integer $m$, find all sequences $n_1 < \cdots < n_s$ for which there is a tffr group of rank $m$ having indecomposable decompositions into $n_1, \ldots, n_s$ summands,

2. Given partitions $r_1 + \cdots + r_k = n = r'_1 + \cdots + r'_\ell$ of a positive integer $n$, under what conditions does there exist a tffr group with indecomposable decompositions with summands of ranks $r_1, \ldots, r_k$ and $r'_1, \ldots, r'_\ell$ respectively?

The second problem of Fuchs was solved by Blagoveshchenskaya, [20, Theorem 13.1.19] for a restricted class $C$ of groups: let $P$ and $Q$ be partitions of $n$. There is a group $G \in C$ realising $P$ and $Q$ if and only if the sum of the largest part of each and the number of parts of the other does not exceed $n + 1$.  

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More generally, one can pose the

**Question:** Characterize the families \( \mathcal{P} \) of partitions of \( n \) that can be realized by a tffr group.

Corner’s Theorem shows that families of partitions of \( n \) of fixed length \( k \) can be realized. On the other hand, he comments that

\[
\ldots \text{it can be shown quite readily that an equation such as } 1+1+2 = 1+3 \ldots \text{cannot be realized.}
\]

A more general question was settled by Lee Lady for almost completely decomposable groups (defined below) [17, Corollary 7], [20, Theorem 9.2.7]. A group \( G \) is clipped if it has no direct summands of rank 1. Lady’s “Main Decomposition Theorem” says that every almost completely decomposable group \( G \) has a decomposition \( G = G_{cd} \oplus G_{cl} \) where \( G_{cd} \) is completely decomposable, \( G_{cd} \) is unique up to isomorphism, and \( G_{cl} \) is unique up to near–isomorphism. Near isomorphism is a weakening of isomorphism due to Lady [18]. There are several equivalent definitions, see for example [20, Chapter 9], the most useful one for us being that a group \( A \) is nearly isomorphic to \( B \), denoted \( A \cong_{nr} B \), if there exists a group \( K \) such that \( A \oplus K \cong_{nr} B \oplus K \).

It follows from this definition that near isomorphism is an equivalence relation on the class of groups. Moreover rank and the property of being clipped are invariants of near isomorphism classes.

An important result due to Arnold [1, 12.9], [20, Theorem 12.2.5], is that if \( A \cong_{nr} A' \) and \( A = X \oplus Y \), then \( A' = X' \oplus Y' \) with \( X \cong_{nr} X' \) and \( Y \cong_{nr} Y' \). Conversely, if \( X \cong_{nr} X' \) and \( Y \cong_{nr} Y' \), then \( X \oplus Y \cong_{nr} X' \oplus Y' \).

Let \( A \) be a group. We say that an indecomposable decomposition \( A = \bigoplus_{i \in [n]} A_i \) of \( A \) is **unique up to near isomorphism** if whenever \( A = \bigoplus_{j \in [m]} B_j \) is an indecomposable decomposition of \( A \), then \( n = m \) and there is a permutation \( \sigma \) of \([n]\) such that \( A_i \cong_{nr} B_{\sigma(i)} \) for all \( i \in [n] \).

By Arnold’s Theorem, nearly isomorphic groups of rank \( n \) realize the same partitions of \( n \).

Denote the partition \((m, 1, \ldots, 1)\) where there are \( k \) 1s, by \((m, 1^k)\). Since indecomposable groups are certainly clipped, if an almost completely decomposable group of rank \( n \) realizes partitions \((m, 1^{n-m})\) and \((m', 1^{n-m'} )\), then \( m = m' \).

Our main result is the generalization of the Main Decomposition Theorem to arbitrary torsion-free groups of finite rank (Theorem 2.5) which then settles Corner’s remark.

It may be asked to describe the isomorphism classes of indecomposable groups of a given rank. Rank–1 groups are indecomposable and have been classified by means of types ([19], [15]) and there are \( 2^{\aleph_0} \) isomorphism classes. It is also possible to describe the indecomposable almost completely decomposable groups of rank 2 (see [20 Section 12.3]) but in general this task must be accepted as being hopeless.

A **completely decomposable** group is a direct sum of rank–1 groups, and completely decomposable groups were classified in terms of cardinal invariants by Baer [15, Section 86, page 113]. In particular, their decompositions into rank–1 summands are unique up to isomorphism.

**Almost completely decomposable groups** are finite extensions of completely decomposable groups of finite rank. This class of groups was introduced and first studied by Lee Lady [17], see [20] for a comprehensive exposition. An almost
completely decomposable group $X$ contains special completely decomposable subgroups, namely those of minimal index in $X$, the regulating subgroups of $X$. Rolf Burkhardt [11] showed that the intersection of all regulating subgroups is again a completely decomposable subgroup of finite index in $X$. This group, that is fully invariant in $X$, is the regulator $R(X)$ of $X$.

Most published examples of groups with non–unique decompositions are almost completely decomposable groups. It is also noteworthy that for an almost completely decomposable group $X$ with non–unique indecomposable decompositions the index $[X : R(X)]$ is a composite number. On the other hand if $[X : R(X)]$ is the power of a prime $p$, then Faticoni and Schultz proved that the indecomposable decompositions of $X$ are unique up to near–isomorphism, [13, 20 Corollary 10.4.6]. The problem then remains to determine the near–isomorphism classes of indecomposables. For an almost completely decomposable group $X$, write $R(X) = \bigoplus_{\rho \in T_{cr}(X)} R_{\rho}$ with $\rho$–homogeneous components $R_{\rho}$ and $R_{\rho} \neq 0$. Then $T_{cr}(X)$ is called the critical typeset of $X$. The problem has been largely solved when the critical typeset is an inverted forest in a number of papers by Arnold–Mader–Mutzbauer–Solak ([2], [3], [4], [5], [6], [7], [8], [9], [10]) using representations of posets as a tool.

2. Main Decomposition

A rank–1 group is a group isomorphic with an additive subgroup of $\mathbb{Q}$. A type is the isomorphism class of a rank–1 group. It is easy to see that every rank–1 group is isomorphic to a rational group by which we designate an additive subgroup of $\mathbb{Q}$ that contains $1$. If $A$ is a rank–1 group, then $\text{type}(A)$ denotes the type of $A$, i.e., the isomorphism class containing $A$. Types are commonly denoted by $\sigma, \tau, \ldots$. We will also use $\sigma, \tau, \ldots$ to mean a rational group of type $\sigma, \tau, \ldots$ It will always be clear from the context whether $\tau$ is a rational group or a type. The advantage is that any completely decomposable group $A$ of finite rank $r$ can be written as $A = \sigma v_1 \oplus \cdots \oplus \tau v_r$ with $v_i \in A$ because $1 \in \tau_i$, and $\text{type}(\tau_i) = \tau_i$. In this case $\{v_1, \ldots, v_r\}$ is called a decomposition basis of $A$.

A completely decomposable group is called $\tau$–homogeneous if it is the direct sum of rank–1 groups of type $\tau$, and homogeneous if it is $\tau$–homogeneous for some type $\tau$. It is known [13, 86.6] that pure subgroups of homogeneous completely decomposable groups are direct summands.

Definition 2.1. A group $G$ is $\tau$–clipped if $G$ does not possess a rank–1 summand of type $\tau$.

Lemma 2.2. Suppose that $G = D \oplus B = A \oplus C$ where $B$ and $C$ are $\tau$–clipped and $D, A$ are completely decomposable and $\tau$–homogeneous. Then $D \cong A$.

Proof. Let $\delta, \beta, \alpha, \gamma \in \text{End}(G)$ be the projections belonging to the given decompositions. Let $0 \neq x \in D$. Then $x = x\alpha + x\gamma$. Assume that $x\alpha = 0$. Then $\langle x \rangle_{\alpha}$ is a pure rank–1 subgroup of $D$ and hence a summand of $D$ and of $G$. Also $\langle x \rangle_{\alpha} \alpha = 0$ which says the $\langle x \rangle_{\alpha} \subseteq \text{Ker} \alpha = C$. It follows that $\langle x \rangle_{\alpha}$ is a rank–1 summand of $C$ of type $\tau$, contradicting the fact that $C$ is $\tau$–clipped. Hence $\alpha : D \rightarrow A$ is a monomorphism and therefore rank $D \leq$ rank $A$. By symmetry rank $A \leq$ rank $D$ and $D \cong A$ as desired. \hfill $\Box$

The direct sum of $\tau$–clipped groups need not be $\tau$–clipped as Example [23] shows.
Example 2.3. Let $p, q$ be different primes and let $\sigma, \tau$ be rational groups that are incomparable as types and such that neither $\frac{1}{p}$ nor $\frac{1}{q}$ is contained in either $\sigma$ or $\tau$. Let

\[ X_1 = (\sigma v_1 + \tau v_2) + Z_{\frac{1}{p}}(v_1 + v_2) \] and \[ X_2 = (\sigma w_1 + \tau w_2) + Z_{\frac{1}{q}}(w_1 + w_2). \]

It is easy to see that $R(X_1) = \sigma v_1 + \tau v_2$ and $R(X_2) = \sigma w_1 + \tau w_2$, and that $X_1$ and $X_2$ are indecomposable and, in particular, clipped. There exist integers $u_1, u_2$ such that $u_1 p + u_2 q = 1$. Now $\frac{1}{p}(v_1 + v_2) + \frac{1}{q}(w_1 + w_2) = \frac{1}{pq}((qv_1 + pw_1) + (qv_2 + pw_2))$. Set $v'_1 = qv_1 + pw_1$, $v'_2 = qv_2 + pw_2$, $w'_1 = -u_1 v_1 + u_2 w_1$, and $w'_2 = -u_1 v_2 + u_2 w_2$. Then (change of decomposition basis) $\sigma v_1 + \sigma w_1 = \sigma v'_1 + \sigma w'_1$ and $\tau v_2 + \tau w_2 = \tau v'_2 + \tau w'_2$. Hence $X = (\sigma v'_1 + \tau v'_2) + (\sigma v_1 + \tau v_2) + Z_{\frac{1}{pq}}(v'_1 + v'_2)$ so $X$ has rank–1 summands of type $\sigma$ and $\tau$.

However, Lemma 2.4 settles positively a special case.

Lemma 2.4. Let $G = A \oplus B$ where $A = \bigoplus_{\rho \neq \tau} A_{\rho}$ is completely decomposable and $B$ is $\tau$–clipped. Then $G$ is $\tau$–clipped.

Proof. We may assume that $\text{rank } A = 1$. In fact, if $A = A_1 \oplus \cdots \oplus A_k$ where $\text{rank } A_1 = 1$, then $A_1 \oplus B$ is $\tau$–clipped by the rank 1 case, $A_2 \oplus \cdots \oplus A_k \oplus B$ is $\tau$–clipped by induction, and $A \oplus B$ is $\tau$–clipped by the rank 1 case.

By way of contradiction assume that $G = \tau v + C = \sigma a + B$ with $\tau \neq \sigma$ (as rational groups or $\tau \neq \sigma$ as types). Let $\alpha : G \rightarrow \sigma a \subseteq G$, $\beta : G \rightarrow B \subseteq G$, $\delta : G \rightarrow \tau v \subseteq G$, and $\gamma : G \rightarrow C \subseteq G$ be the projections (considered endomorphisms of $G$) that come with the stated decompositions.

1. We have $v = va + v\beta$ uniquely. Suppose $va = 0$. Then $(\tau v)\alpha = 0$ and the summand $\tau v$ is contained in $\text{Ker } \alpha = B$. Then $\tau v$ is a summand of $B$ contradicting the fact that $B$ is $\tau$–clipped. So $\alpha : \tau v \rightarrow \sigma a$ is a monomorphism and $\tau \leq \sigma$.

2. We have $a = a\delta + a\gamma$. Suppose that $a\delta = 0$. Then $(\sigma a)\delta = 0$ and the summand $\sigma a$ is contained in $\text{Ker } \delta = C$. Hence $C = \sigma a + C'$ for some $C'$ and $G = \tau v + \sigma a + C' = \sigma a + B$. Hence $\frac{C}{\sigma a} \cong \tau v + C' \cong B$. This contradicts the fact that $B$ is $\tau$–clipped. So $\delta : \sigma a \rightarrow \tau v$ is a monomorphism and hence $\sigma \leq \tau$.

3. By (1) and (2) we get the contradiction $\sigma = \tau$, saying that $G = \sigma a + B$ does not have a rank–1 summand of type $\tau$, and the special case is proved.

\[ \square \]

Theorem 2.5. (Main Decomposition.) Let $G$ be a torsion-free group of finite rank. Then there are decompositions $G = A_0 \oplus A_1$ in which $A_0$ is completely decomposable and $A_1$ is clipped.

Suppose that $G = A_0 \oplus A_1 = B_0 \oplus B_1$ where $A_0$ and $B_0$ are completely decomposable and $A_1$ and $B_1$ are clipped. Then $A_0 \cong B_0$ and consequently $A_1 \cong B_1$.

Proof. Let $A_0$ be a completely decomposable summand of $G$ of maximal rank. Then $G = A_0 \oplus A_1$ and $A_1$ is clipped.

Let $A_0 = \bigoplus_{\rho} A_{\rho}$ and $B_0 = \bigoplus_{\rho} B_{\rho}$ be the homogeneous decompositions of the completely decomposable groups $A_0$ and $B_0$. By allowing $A_{\rho}$ and $B_{\rho}$ to be the zero group, we may assume that the summation index ranges over all types $\rho$. 

\[ \square \]
MAIN DECOMPOSITION

We consider \( G = A_\tau \oplus \left( \bigoplus_{\rho \neq \tau} A_\rho \oplus A_1 \right) = B_\tau \oplus \left( \bigoplus_{\rho \neq \tau} B_\rho + B_1 \right) \). By Lemma 2.2, \( \bigoplus_{\rho \neq \tau} A_\rho \oplus A_1 \) and \( \bigoplus_{\rho \neq \tau} B_\rho + B_1 \) are both \( \tau \)-clipped. Hence by Lemma 2.2 we conclude that \( A_\tau \cong B_\tau \). Here \( \tau \) was an arbitrary type and the claim is clear. The fact that \( A_1 \cong_m B_1 \) follows from the isomorphism \( A_0 \cong A_1 \cong A_0 \oplus B_1 \). □

Corollary 2.6. Suppose that \( G \) has rank \( n \) and \( G \) realizes the partitions \((m, 1^{n-m})\) and \((m', 1^{n-m'})\) Then \( m = m' \).

Proof. The indecomposable summands of ranks \( m \) and \( m' \) are necessarily clipped. so by Theorem 2.5, the completely decomposable parts of the decompositions are isomorphic. □

In particular there is no group that realizes both \((1,1,2)\) and \((1,3)\).

We call a decomposition \( G = G_{cd} \oplus G_{cl} \) with \( G_{cd} \) completely decomposable and \( G_{cl} \) clipped a **Main Decomposition of \( G \).**

Corollary 2.7. Let \( C \) be a completely decomposable direct summand of a group \( G \). Then \( G \) has a Main Decomposition \( G_{cd} \oplus G_{cl} \) in which \( C \) is a direct summand of \( G_{cd} \).

Proof. Let \( G = C \oplus B \) and let \( B \) have Main Decomposition \( B = B_{cd} \oplus B_{cl} \). Then \( G = (C \oplus B_{cd}) \oplus B_{cl} \) is a Main Decomposition of \( G \). □

Main Decompositions are unique only up to near isomorphism. For example, let \( X = \tau v \oplus ((\tau v_1 + \sigma v_2) + Z_{1,5}(v_1 \oplus v_2)) \). The group \( (\tau v_1 + \sigma v_2) + Z_{1,5}(v_1 \oplus v_2) \) is indecomposable, hence clipped. We also have \( X = \tau (v+v_1) \oplus ((\tau v_1 + \sigma v_2) + Z_{1,5}(v_1 \oplus v_2)) \) and \( \tau v \neq \tau (v+v_1) \). On the other hand if \( G = G_{cd} \oplus G_{cl} \) and \( \operatorname{Hom}(G_{cd}, G_{cl}) = 0 \), then \( G_{cd} \) is unique and direct complements of \( G_{cd} \) are isomorphic ([20, Lemma 1.1.3]).

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