INJECTIVE AND PROJECTIVE HILBERT C*-MODULES, AND C*-ALGEBRAS OF COMPACT OPERATORS

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ABSTRACT. We consider projectivity and injectivity of Hilbert C*-modules in the categories of Hilbert C*-(bi-)modules over a fixed C*-algebra of coefficients (and another fixed C*-algebra represented as bounded module operators) and bounded (bi-)module morphisms, either necessarily adjointable or arbitrary ones. As a consequence of these investigations, we obtain a set of equivalent conditions characterizing C*-subalgebras of C*-algebras of compact operators on Hilbert spaces in terms of general properties of Hilbert C*-modules over them. Our results complement results recently obtained by B. Magajna, J. Schweizer and M. Kusuda. In particular, all Hilbert C*-(bi-)modules over C*-algebras of compact operators on Hilbert spaces are both injective and projective in the categories we consider. For more general C*-algebras we obtain classes of injective and projective Hilbert C*-(bi-)modules.

The goal of this paper is to determine the injective and projective Hilbert C*-modules over a fixed C*-algebra, when one allows the maps between C*-modules to be bounded. Most prior work on injectivity has focused on the case of contractive maps. To better understand our motivations and the distinction between this work and the work of others, we first review the concept of injectivity and some history of the subject.

To give a definition of the term, injective, that is useful for our purposes, we need a category, consisting of objects that are sets and morphisms between them that are functions, and for each object, \( \mathcal{N} \), certain subsets, \( \mathcal{M} \subseteq \mathcal{N} \) that are also objects in the category, which we call the subobjects of \( \mathcal{N} \). Then an object \( \mathcal{I} \) in this category is called injective, provided that for every object \( \mathcal{N} \), every subobject, \( \mathcal{M} \subseteq \mathcal{N} \) and every morphism, \( \phi : \mathcal{M} \to \mathcal{I} \), there is a morphism \( \psi : \mathcal{N} \to \mathcal{I} \), that extends \( \phi \). Note that if we keep the objects and morphisms the same,

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but change the subobjects, then it is possible that the injectives might change. Among the differences between this definition of injectivity and the categorical definition that is given in say, [33], is that our definition allows for the possibility that the inclusion map of the subobject into the object is not a morphism in the category. Many definitions of injectivity, implicitly only consider subobjects with the property that the inclusion map is a morphism.

If one fixes a ring \( R \), and considers the category whose objects are left \( R \)-modules, subobjects are left \( R \)-submodules and morphisms are left \( R \)-module maps, then the above concept of injective reduces to the classical definition of an injective left \( R \)-module. In this case the inclusions of subobjects into objects are always morphisms.

Consider the category whose objects are Banach spaces, subobjects are closed subspaces and whose morphisms are the contractive, linear maps. Then it is easy to see, by a simple scaling, that a Banach space, \( I \), is injective in this setting if and only if for every Banach space, \( N \), every closed subspace, \( M \subseteq N \) and every bounded linear map, \( \phi : M \to I \), there is an extension, \( \psi : N \to I \), of \( \phi \) with \( \|\phi\| = \|\psi\| \).

A classic result, often called the Nachbin-Goodner-Kelley theorem, is that a Banach space is injective in this category if and only if it is isometrically isomorphic to the space of continuous functions on an extremally disconnected, compact Hausdorff space [26].

However, if one changes the category slightly, keeping the objects to be Banach spaces and subobjects to be closed subspaces, but allowing the morphisms to be all bounded, linear maps, then a Banach space is injective in this category if and only if every bounded, linear map on a subspace has a bounded, linear extension to the whole space, but not necessarily of the same norm. A complete understanding of the injective Banach spaces in this setting is still unknown [?], but it is fairly easy to see that Hilbert spaces are not injective.

Thus, generally, when one allows bounded maps to be the morphisms instead of contractive maps, then the problems become more difficult.

On the other hand, if we now keep our morphisms to be bounded, linear maps, but restrict our objects, by considering only Hilbert spaces, then it is fairly easy to show that every Hilbert space \( \mathcal{H} \) is injective. This follows, since the extension can be achieved by projecting onto the subspace. Thus, in this restricted category, every object is now injective.

One way to generalize Hilbert spaces, is to consider the category whose objects are Hilbert C*-modules over a fixed C*-algebra, \( A \), subobjects are Hilbert C*-submodules and morphisms are the bounded,
A-module maps. When $A = \mathbb{C}$, then this category reduces back down to the category of Hilbert spaces and bounded linear maps.

Thus, in parallel with the Banach space case, we wish to determine the injective objects in this setting. The first question that we shall address is characterizing the C*-algebras $A$, such that, like $\mathbb{C}$, every Hilbert $A$-module is injective.

As is the case with Banach spaces, if one restricts the morphisms to be the contractive, module maps, then the theory of injective Hilbert $A$-modules is somewhat simpler and is largely worked out in the work of Huaxin Lin [31, 32] and Zhou Tian Xu [55]. In some of Lin’s work, he studies injectivity, where in our language, the objects are Hilbert C*-modules, subobjects are Hilbert C*-submodules, and the morphisms are adjointable, contractive module maps. In this case the inclusion map of a submodule into the larger module is, generally, not a morphism, since it need not be adjointable. Thus, to encompass the type of “injectivity” studied by Lin, one needs the more general definition of injectivity given above and some care must be taken when citing general facts about injectives from category theory in his context. As we will show later, if the inclusion map is required to be adjointable, i.e., if one restricts the subobjects, then the submodules are necessarily orthogonally complemented and every object is injective (Theorem 3.1). Thus, the differences between our results and those due to H. Lin [31, 32] are caused by differences in the categories that we consider.

So far we have only discussed injectivity, but similar comments apply to projectivity, which in many ways is a dual theory to injectivity. For the concept of projectivity, in addition to specifying the morphisms, one needs to specify the quotients. We will make precise definitions of projectivity in Section 4.

We shall also answer many parallel questions about characterizing projective modules.

In the settings that we shall consider, the set of objects of all categories under consideration consists of Hilbert C*-modules $\{\mathcal{M}, \langle.,.\rangle\}$ over some fixed C*-algebra $A$, i.e. (left) $A$-modules $\mathcal{M}$ equipped with an $A$-valued inner product $\langle.,.\rangle : \mathcal{M} \times \mathcal{M} \to A$, cf. [29]. We specify a second C*-algebra $B$ that is supposed to act on $\mathcal{M}$ as a set of bounded adjointable operators via module-specific *-representations. Thus, $\mathcal{M}$ is an $A$-$B$-bimodule, with the right action of $B$ given by bounded adjointable maps on $\mathcal{M}$, so that, in particular, $\langle am_1b,m_2\rangle = a\langle m_1,m_2b^*\rangle$, for every, $a \in A,b \in B$, and $m_1,m_2 \in \mathcal{M}$. We call $\mathcal{M}$ a Hilbert $A$-$B$-bimodule. The requirement of the existence of a second action by $B$ changes the unitary equivalence classes of Hilbert $A$-$B$-bimodules, i.e., the notion of equivalence in the categories under consideration. Note
that every Hilbert $A$-module is automatically a Hilbert $A$-$C$-module, where $C$ denotes the complex numbers.

The sets of morphisms that we study will consist of either all bounded bimodule morphisms between the objects, or all adjointable bounded bimodule morphisms between them. We shall denote these two categories by $\mathcal{B}(A, B)$ and $\mathcal{B}^*(A, B)$, respectively.

The subobjects that we will consider will be, generally, all Hilbert $A$-$B$-submodules and occasionally the orthogonally complemented $A$-$B$-submodules.

The primary goal of the present paper is the investigation of two problems: (i) characterize the C*-algebras $A$ and $B$ for which any Hilbert $A$-$B$ bimodule is injective or projective for one of the sets of morphisms under consideration and one of the two concepts of subobjects; (ii) find suitable sets of projective, or injective, Hilbert $A$-$B$ bimodules for given C*-algebras $A$ and $B$ and fixed morphism sets.

In most cases the action of the C*-algebra $B$ of bounded $A$-linear operators on the Hilbert $A$-modules $\mathcal{M}$ turns out to play a minor role. So we can concentrate on the C*-algebra $A$ of module coefficients, on the Hilbert $A$-$B$ modules. We obtain a full characterization of the C*-algebras $A$ for which any Hilbert $A$-$B$ bimodule is injective for both the principal categories. For the category with only bounded adjointable $A$-$B$ bimodule maps as morphisms any C*-algebra of coefficients $A$ (and any C*-algebra of bounded adjointable operators $B$) will suffice, whereas for the category with all bounded $A$-$B$ bimodule maps as morphisms only C*-algebras $A$ of compact operators (and arbitrary C*-algebras $B$) have this property. If the C*-algebra of coefficients $A$ is monotone complete then a Hilbert $A$-$B$ bimodule is injective in the category with the set of bounded $A$-$B$ module maps if and only if it is self-dual.

In the case of projectivity, we show that every Hilbert $A$-$B$ bimodule is projective in the category $\mathcal{B}^*(A, B)$, for every C*-algebra $A$. We prove that when $A$ is a C*-algebra of compact operators, then every Hilbert $A$-$B$ bimodule is projective in $\mathcal{B}(A, B)$, but we are unable to resolve if these are the only C*-algebras with this property. A characterization of such algebras is not available at present. Even more, the question whether all Hilbert C*-modules are projective in the larger category, or not, remains open.

However, we do prove that all Hilbert $A$-$B$ bimodules over a certain C*-algebra are projective in the categories investigated if and only if the kernel of every surjective bounded module map between Hilbert $A$-modules is a topological direct summand of the domain. Moreover, we identify a family of projective C*-modules over unital C*-algebras. We
show that the finitely generated Hilbert $C^*$-modules over unital $C^*$-algebras are projective objects of the categories under consideration.

There are some parallels between our results on projectivity and research in progress on extensions of Hilbert $C^*$-modules and on projectivity of Hilbert $C^*$-modules in this different category by Damir Bakic and Boris Guljas, [2, 3].

Another way to modify the categories under consideration would be to restrict the set of objects to self-dual (or orthogonally comparable) Hilbert $C^*$-modules. Recall that a Hilbert $C^*$-module $M$ is orthogonally comparable provided that any time $\phi : M \to N$ is an isometric module map, then $\phi(M)$ is orthogonally complemented in $N$. However, this choice implies the adjointability of all bounded module morphisms between them and, consequently, that any Hilbert $C^*$-submodule is an orthogonal summand, cf. [13]. So our questions would have an immediate answer: in these latter categories all objects are projective and injective for arbitrary $C^*$-algebras $A$ and $B$.

Because of the close relation of the Magajna-Schweizer theorem ([34, 51]) to the circle of questions studied in the present paper, $C^*$-algebras $A$ of the form $A = c_0 \sum_\alpha \oplus K(H_\alpha)$ are of special interest. Here the symbol $K(H_\alpha)$ denotes the $C^*$-algebra of all compact operators on some Hilbert space $H_\alpha$, and the $c_0$-sum is either a finite block-diagonal sum or a block-diagonal sum with a $c_0$-convergence condition on the $C^*$-algebra components $K(H_\alpha)$. The $c_0$-sum may possess arbitrary cardinality. These $C^*$-algebras have been precisely characterized by W. Arveson [1, §I.4, Th. I.4.5] as $C^*$-subalgebras of (full) $C^*$-algebras of compact operators on Hilbert spaces. We give a number of further equivalent characterizations of this class of $C^*$-algebras in terms of general properties of Hilbert $C^*$-modules over them which are of separate interest. Throughout the present paper we refer to these $C^*$-algebras as $C^*$-algebras of compact operators on certain Hilbert spaces.

1. Preliminaries

In this section we give some definitions and basic facts from Hilbert $C^*$-module theory needed for our investigations. The papers [39, 25, 10, 30, 31, 13], some chapters in [23, 53], and the books by E. C. Lance [29] and by I. Raeburn and D. P. Williams [47] are used as standard reference sources. We make the convention that all $C^*$-modules of the present paper are left modules by definition. A pre-Hilbert $A$-module over a $C^*$-algebra $A$ is an $A$-module $M$ equipped with an $A$-valued mapping $\langle \cdot, \cdot \rangle : M \times M \to A$ which is $A$-linear in the first argument
and has the properties:
\[ \langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \quad \text{with equality iff} \quad x = 0. \]
The mapping \( \langle \cdot, \cdot \rangle \) is said to be the \( A \)-valued inner product on \( \mathcal{M} \). A pre-Hilbert \( A \)-module \( \{ \mathcal{M}, \langle \cdot, \cdot \rangle \} \) is Hilbert if and only if it is complete with respect to the norm \( \| \cdot \| = \| \langle \cdot, \cdot \rangle \|_A^{1/2} \). We always assume that the linear structures of \( A \) and \( \mathcal{M} \) are compatible. Two Hilbert \( A \)-modules are isomorphic if they are isometrically isomorphic as Banach \( A \)-modules, if and only if they are unitarily isomorphic, [29]. We would like to point out that an \( A \)-module can carry unitarily non-isomorphic \( A \)-valued inner products which induce equivalent complete norms, [13]. Two Hilbert \( A-B \)-modules are isomorphic if and only if they are unitarily isomorphic as Hilbert \( A \)-modules in such a way that the isomorphism intertwines the \( * \)-representations of \( B \) on them.

Hilbert C*-submodules of Hilbert C*-modules might not be direct summands, and if they are direct summands then they might be merely topological, but not orthogonal summands. We say that a Hilbert C*-module \( \mathcal{N} \) is a topological summand of a Hilbert C*-module \( \mathcal{M} \) which contains \( \mathcal{N} \) as a Banach C*-submodule in case \( \mathcal{M} \) can be decomposed into the direct sum of the Banach C*-submodule \( \mathcal{N} \) and of another Banach C*-submodule \( \mathcal{K} \). The denotation is \( \mathcal{M} = \mathcal{N} + \mathcal{K} \). If, moreover, the decomposition can be arranged as an orthogonal one (i.e. \( \mathcal{N} \perp \mathcal{K} \)) then the Hilbert C*-submodule \( \mathcal{N} \subseteq \mathcal{M} \) is an orthogonal summand of the Hilbert C*-module \( \mathcal{M} \) i.e. \( \mathcal{M} = \mathcal{K} \oplus \mathcal{L} \). Examples where these situations appear can be found e.g. in [13].

A Hilbert \( A \)-module \( \{ \mathcal{M}, \langle \cdot, \cdot \rangle \} \) over a C*-algebra \( A \) is said to be self-dual if and only if every bounded module map \( r : \mathcal{M} \to A \) is of the form \( \langle \cdot, x_r \rangle \) for some element \( x_r \in \mathcal{M} \). The set of all bounded module maps \( r : \mathcal{M} \to A \) forms a Banach \( A \)-module \( \mathcal{M}' \). The module action of \( A \) on \( \mathcal{M}' \) is defined by the formula \( (a \cdot r)(x) = r(x)a^* \) for any \( x \in \mathcal{M} \), each \( a \in A \) and \( r \in \mathcal{M}' \). A Hilbert \( A \)-module is called C*-reflexive (or more precisely, \( A \)-reflexive) if and only if the map \( \Omega \) defined by the formula \( \Omega(x)[r] = r(x) \) for each \( x \in \mathcal{M} \), every \( r \in \mathcal{M}' \), is a surjective module map of \( \mathcal{M} \) onto the Banach \( A \)-module \( \mathcal{M}'' \), where \( \mathcal{M}'' \) consists of all bounded module maps from \( \mathcal{M}' \) to \( A \). Note that the property of being self-dual does not depend on the choice of the C*-algebra of coefficients \( A \) within \( \langle \mathcal{M}, \mathcal{M} \rangle \subseteq A \subseteq M(\langle \mathcal{M}, \mathcal{M} \rangle) \), whereas the property of being \( A \)-reflexive sometimes does.

As an example consider the C*-algebra \( A = c_0 \) of all sequences converging to zero and set \( \mathcal{M} = c_0 \) with the standard \( A \)-valued inner product. Consider \( \mathcal{M} \) both as a Hilbert \( A \)-module and as a Hilbert \( M(A) \)-module. The multiplier C*-algebra of \( A = c_0 \) is \( M(A) = l_\infty \),
the set of all bounded sequences. Then $\mathcal{M}'$ equals $l_\infty$ as a one-sided $A$-module, independently of the choice of sets of coefficients. In contrast, the set of all bounded $A$-linear maps of $\mathcal{M}'$ to $A$ can be identified with $A = c_0$, whereas the set of all bounded $M(A)$-linear maps of $\mathcal{M}'$ to $M(A)$ can be identified with $l_\infty$. Generally speaking, the $A$-dual Banach $A$-module $\mathcal{M}'$ of a Hilbert $A$-module $\mathcal{M}$ can be described as the completion of the linear hull of the unit ball of $\mathcal{M}$ with respect to the topology induced by the seminorms, $\{\|\langle x, \cdot \rangle\|_A : x \in \mathcal{M}, \|x\| \leq 1\}$ [13, Th. 6.4]. The process of forming higher order $C^*$-dual Banach $C^*$-modules of a given Hilbert $C^*$-module $\mathcal{M}$ stabilizes after the second step since $\mathcal{M}' \equiv \mathcal{M}''$. We have the standard chain of isometric Banach $C^*$-module embeddings $\mathcal{M} \subseteq \mathcal{M}'' \subseteq \mathcal{M}'$ by [39, 40].

Furthermore, we are going to consider various bounded $C^*$-linear operators $T$ between Hilbert $C^*$-modules $\mathcal{M}, \mathcal{N}$ with one and the same $C^*$-algebra of coefficients. Quite regularly those operators $T$ may not admit an adjoint bounded $C^*$-linear operator $T^* : \mathcal{N} \to \mathcal{M}$ fulfilling the equality $\langle T(x), y \rangle_\mathcal{N} = \langle x, T^*(y) \rangle_\mathcal{M}$ for any $x \in \mathcal{M}$, any $y \in \mathcal{N}$. We denote the $C^*$-algebra of all bounded $C^*$-linear adjointable operators on a given Hilbert $A$-module $\mathcal{M}$ by $\text{End}^*_A(\mathcal{M})$. The Banach algebra of all bounded $A$-linear operators on $\mathcal{M}$ is denoted by $\text{End}_A(\mathcal{M})$. For more detailed information on these situations we refer to [13].

A result that we shall use often is a bounded closed graph theorem for Hilbert $C^*$-modules that is a variant of N. E. Wegge-Olsen’s result. We show how the bounded closed graph theorem can be derived from his result. In contrast, an example of E.C. Lance shows that there is no analogue of the unbounded closed graph theorem for general Hilbert $C^*$-modules.

**Proposition 1.1.** (N. E. Wegge-Olsen [53, Th. 15.3.8])

Let $A$ be a $C^*$-algebra, $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ be a Hilbert $A$-module and $T$ be an adjointable bounded module operator on $\mathcal{M}$. If $T$ has closed range then $T^*, (T^*T)^{1/2}$ and $(TT^*)^{1/2}$ have also closed ranges and

$$
\mathcal{M} = \text{Ker}(T) \oplus T^*(\mathcal{M}) = \text{Ker}(T^*) \oplus T(\mathcal{M})
$$

$$
= \text{Ker}(|T|) \oplus |T|(\text{Ker}(T^*)) \oplus |T^*|(\mathcal{M}).
$$

In particular, each orthogonal summand appearing on the right is automatically norm-closed and coincides with its bi-orthogonal complement inside $\mathcal{M}$. Moreover, $T$ and $T^*$ have polar decomposition.

**Corollary 1.2.** (bounded closed graph theorem)

Let $A$ be a $C^*$-algebra and $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$, $\{\mathcal{N}, \langle \cdot, \cdot \rangle\}$ be two Hilbert $A$-modules. The graph of every bounded $A$-linear operator $T$ coincides
with its bi-orthogonal complement in \( M \oplus \mathcal{N} \), and it is always a topological summand with topological complement \{(0, z) : z \in \mathcal{N}\}. A bounded \( A \)-linear operator \( T : M \to \mathcal{N} \) possesses an adjoint operator \( T^* : \mathcal{N} \to M \) if and only if the graph of \( T \) is an orthogonal summand of the Hilbert \( A \)-module \( M \oplus \mathcal{N} \).

**Remark 1.3.** By a counterexample due to E. C. Lance ([29, pp. 102-104]) the graph of a closed, self-adjoint, densely defined, unbounded module operator need not coincide with its bi-orthogonal complement.

**Proof.** Since the inequality \( \|T(x)\| \leq \|T\|\|x\| \) is valid for every \( x \in M \) the graph of \( T \) is a norm-closed Hilbert \( A \)-submodule of the Hilbert \( A \)-module \( M \oplus \mathcal{N} \). Moreover, since the graph of \( T \) is the kernel of the bounded module operator \( S : (x, y) \to (0, T(x) - y) \) on \( M \oplus \mathcal{N} \) it coincides with its bi-orthogonal complement there, [12, Cor. 2.7.2]. If \( T \) has an adjoint then the operator \( T^* : (x, y) \to (x, T(x)) \) is adjointable on \( M \oplus \mathcal{N} \). By Proposition [11] the graph of \( T \) is an orthogonal summand.

Conversely, if the graph of \( T \) is an orthogonal summand of \( M \oplus \mathcal{N} \) then its orthogonal complement consists precisely of the pairs of elements \( \{(x, y) : x = -T^*(y), y \in \mathcal{N}\} \). To see this consider the equality \( \langle z, x \rangle_M + \langle T(z), y \rangle_N = 0 \) which has to be valid for any \( z \in M \) and any pair \( (x, y) \) of elements of the orthogonal complement of the graph of \( T \). The assumption of the existence of two pairs \( (x_1, y) \) and \( (x_2, y) \) in this complement forces \( \langle z, x_1 \rangle_M = \langle z, x_2 \rangle_M \) for any \( z \in M \), and therefore, \( x_1 = x_2 \). Hence, \( T^*(y)(z) = \langle z, (-x) \rangle_M \) for any \( z \in M \) and for \( T^* : \mathcal{N} \to M' \). So \( T^* \) is everywhere defined on \( \mathcal{N} \) taking values exclusively in \( M \subseteq M' \). This shows the existence of the adjoint operator \( T^* \) of \( T \) in the sense of its definition.

The property of the graph of a bounded module operator to be a topological summand with topological complement \( \{(0, z) : z \in \mathcal{N}\} \) follows from the decomposition \( (x, y) = (x, T(x)) + (0, y - T(x)) \) for every \( x \in M \), \( y \in \mathcal{N} \). Since \( T(0) = 0 \) for any linear operator \( T \) the intersection of the graph with the \( A\)-\( B \)-submodule \( \{(0, z) : z \in \mathcal{N}\} \) is always trivial. \( \square \)

There is still one open problem about complements whose solution for (at least, surjective) bounded module mappings would give us insight into the solution of the main question of the fourth section concerning projective Hilbert \( C^* \)-modules.
Problem 1.4. Suppose, a bounded module operator between Hilbert C*-modules has a norm-closed image which is either a topological summand or merely coincides with its biorthogonal complement with respect to the image Hilbert C*-module. Is the kernel of such an operator always a topological summand, or are there counterexamples?

The difficulties surrounding this problem are illuminated by an example constructed by V. M. Manuilov in [35].

2. C*-algebras of compact operators and the Magajna-Schweizer theorem

In this section, we prove that the class of C*-algebras of compact operators on certain Hilbert spaces and their C*-subalgebras can be characterized by the appearance of certain properties common to all Hilbert C*-modules over them. The different aspects shown below enable us to establish classes of injective and projective Hilbert A-modules for these and other C*-algebras of coefficients A in forthcoming sections. Our starting point is the following result by Bojan Magajna and Jürgen Schweizer:

Theorem 2.1. (B. Magajna, J. Schweizer [34, 51])
Let A be a C*-algebra. The following three conditions are equivalent:
(i) A is of $c_0 - \sum_i \oplus K(H_i)$-type, i.e. it has a faithful $\ast$-representation as a C*-algebra of compact operators on some Hilbert space.
(ii) For every Hilbert A-module $\mathcal{M}$ every Hilbert A-submodule $\mathcal{N} \subseteq \mathcal{M}$ is automatically orthogonally complemented in $\mathcal{M}$, i.e. $\mathcal{N}$ is an orthogonal summand of $\mathcal{M}$.
(iii) For every Hilbert A-module $\mathcal{M}$ every Hilbert A-submodule $\mathcal{N} \subseteq \mathcal{M}$ that coincides with its bi-orthogonal complement $\mathcal{N}^{\perp \perp} \subseteq \mathcal{M}$ is automatically orthogonally complemented in $\mathcal{M}$.

Based on the Magajna-Schweizer theorem further investigations were made for the identification of generic general properties of Hilbert C*-modules which characterize entire classes of C*-algebras of coefficients, cf. [17]. Many of these generic properties turned out to characterize C*-algebras of compact operators in case they are common for all Hilbert C*-modules over a certain C*-algebra of coefficients. We present these properties here as a list of equivalent conditions that extend the conditions of Magajna-Schweizer.

Proposition 2.2. [17] Let A be a C*-algebra. The following seven conditions are equivalent:
(i) A is of $c_0 - \sum_i \oplus K(H_i)$-type, i.e., it has a faithful $\ast$-representation as a C*-algebra of compact operators on some Hilbert space.
(iv) For every Hilbert $A$-module $\mathcal{M}$ and every bounded $A$-linear map $T : \mathcal{M} \to \mathcal{M}$ there exists an adjoint bounded $A$-linear map $T^* : \mathcal{M} \to \mathcal{M}$.

(v) For every pair of Hilbert $A$-modules $\mathcal{M}, \mathcal{N}$ and every bounded $A$-linear map $T : \mathcal{M} \to \mathcal{N}$ there exists an adjoint bounded $A$-linear map $T^* : \mathcal{N} \to \mathcal{M}$.

(vi) The kernels of all bounded $A$-linear operators between arbitrary Hilbert $A$-modules are orthogonal summands.

(vii) The images of all bounded $A$-linear operators with norm-closed range between arbitrary Hilbert $A$-modules are orthogonal summands.

(viii) For every Hilbert $A$-module every Hilbert $A$-submodule is automatically topologically complemented, i.e., it is a topological summand.

(ix) For every (maximal) norm-closed left ideal $I$ of $A$ the corresponding open projection $p \in A^{**}$ is an element of the multiplier $C^*$-algebra $M(A)$ of $A$.

We will see in the following sections that some of these equivalent conditions force Hilbert $C^*$-modules over $C^*$-algebras of compact operators to be projective or injective. The investigation of these generic categorical properties of Hilbert $C^*$-modules revealed, however, a problem that is still unsolved. It is related to the identification of (non-)injective and (non-)projective Hilbert $C^*$-modules, and so we list it here:

**Problem 2.3.** Characterize those $C^*$-algebras $A$ for which the following condition holds: For every Hilbert $A$-module, every Hilbert $A$-submodule that coincides with its bi-orthogonal complement is automatically topologically complemented there.

Problem 2.3 revisits the difference between B. Magajna’s theorem and J. Schweizer’s theorem on the level of topological summands. The results by M. Kusuda [28] indicate that the solutions of these problems has to be similar to his results on orthogonal summands. M. Kusuda considered the problem in [28] in 2005 and has got a number of results towards a solution in the spirit of the Magajna-Schweizer theorem using spectral methods for $C^*$-algebras. However, his results indicate that the final solution of Problem 2.3 might not have a simple formulation but might consist of a rather extended list of cases to be distinguished.
3. Injectivity

Let $A$ and $B$ be two fixed C*-algebras. We consider two categories. In both categories the objects will be the Hilbert $A$-$B$-bimodules. The sets of morphisms that we study will consist of either all bounded bimodule morphisms between the objects, or all adjointable, bounded bimodule morphisms between them. In both cases, the subobjects will be the set of all Hilbert $A$-$B$-submodules, that is, norm closed subspaces which are invariant under both the module actions.

We shall denote these two categories, together with the specified sets of subobjects, by $\mathcal{B}(A, B)$ and $\mathcal{B}^*(A, B)$, respectively. Note that every left $A$-module is always equipped with a (right) action by the complex numbers, $\mathbb{C}$. Thus, $\mathcal{B}(A, \mathbb{C})$ (respectively, $\mathcal{B}^*(A, \mathbb{C})$) is just the category of left Hilbert $A$-modules and bounded (respectively, bounded, adjointable) maps.

So, in summary, a Hilbert $A$-$B$-bimodule $E$ is injective in $\mathcal{B}(A, B)$, (respectively, $\mathcal{B}^*(A, B)$) if and only if for every Hilbert $A$-$B$-bimodule, $N$, and every Hilbert $A$-$B$-subbimodule, $M$ of $N$, and every bounded, (respectively, bounded, adjointable) bimodule map, $\phi : M \to E$, there is a bounded (respectively, bounded, adjointable), bimodule map $\psi : N \to E$ that extends $\phi$. In other words, a Hilbert $A$-$B$ bimodule $E$ is injective if and only if the diagram

\[ \begin{array}{ccc} N & \uparrow T & E \\ \downarrow & \phi & \\ M & \rightarrow & \end{array} \]

can be completed to a commutative one by an $A$-$B$ bimodule morphism $\psi : N \to E$ of the selected category.

Before beginning our study of injectivity, we first point out what happens when the set of subobjects is required to be smaller.

The following theorem should be contrasted with H. Lin’s result [31, Th. 2.14], which applies to the category of left Hilbert $A$-modules with morphisms the contractive adjointable maps, but a larger family of subobjects was allowed, namely, all the submodules were considered subobjects. Thus, the inclusion maps were not in general morphisms. H. Lin obtained that, in this setting, a Hilbert $A$-module is injective if and only if it is orthogonally comparable as a Hilbert $A$-module. We can demonstrate that even expanding the morphisms to the bounded adjointable $A$-$B$-bimodule maps, but requiring the inclusion maps to be morphisms, changes the picture rather significantly.
Theorem 3.1. Let $A$ be an arbitrary $C^*$-algebra and \{${\mathcal E}, \langle \cdot, \cdot \rangle$\} be a Hilbert $A$-module. Let $B$ be another $C^*$-algebra admitting a $*$-representation in $\text{End}_A^*(\mathcal E)$. Then $\mathcal E$ is an injective object in the category whose objects are the Hilbert $A$-$B$-bimodules, whose morphisms are either the (adjointable) contractive or (adjointable) bounded bimodule maps and whose subobjects are the $A$-$B$-submodules whose inclusion maps are adjointable. Consequently, every element of those categories is injective.

Proof. Since by assumption the inclusion $T : \mathcal M \hookrightarrow \mathcal N$ is an adjointable bounded $A$-$B$-bimodule map the map $T^*$ is a surjective bounded $A$-$B$-bimodule map and Proposition 1.1 applies: the image set $T(\mathcal M) \subseteq \mathcal N$ is an orthogonal summand. Moreover, the map $T^{-1} : T(\mathcal M) \to \mathcal M$ defined as $T^{-1}(T(x)) = x$ for $x \in \mathcal M$ is everywhere defined on $T(\mathcal M) \subseteq \mathcal N$ and bijective, so it is bounded and $A$-$B$-bilinear by definition. It can be extended to a map defined on $\mathcal N$ simply setting it to be the zero map on the orthogonal complement of $T(\mathcal M)$ in $\mathcal N$. Preserving the denotation $T^{-1}$ for this extension, setting $\psi = \phi \circ T^{-1}$ yields the desired extension of $\phi$ to $\mathcal N$. Consequently, the Hilbert $A$-$B$-bimodule $\mathcal E$ is automatically injective in the category under consideration. \hfill \Box

We now focus on the two categories that are our principal interest.

To make further progress in identifying the injective objects of the category $\mathcal B(A, B)$ we consider consequences of the definition of injectivity.

Lemma 3.2. Let $A$, $B$ be $C^*$-algebras and \{${\mathcal E}, \langle \cdot, \cdot \rangle$\} be an injective Hilbert $A$-$B$-bimodule in one of the two categories under consideration. If $\mathcal E \subseteq \mathcal N$ is an $A$-$B$-submodule, then the Hilbert $A$-$B$-bimodule $\mathcal E$ is a topological summand of the Hilbert $A$-$B$-bimodule $\mathcal N$.

Moreover, $\mathcal E$ is $A$-reflexive as a Hilbert $A$-module, and whenever $\mathcal E$ is a Hilbert $A$-submodule of another Hilbert $A$-module $\mathcal M$ with $\mathcal E^\perp = \{0\}$ then $\mathcal E = \mathcal E^{\perp \perp}$ in $\mathcal M$.

Proof. In the definition of injectivity, let $\mathcal M = \mathcal E$, let $T$ denote the inclusion of $\mathcal E$ into $\mathcal N$ and let $\phi = \text{id}_{\mathcal E}$. By supposition there exists an $A$-$B$ bimodule morphism $\psi : \mathcal N \to \mathcal E$ such that $\psi \circ T = \text{id}_{\mathcal E}$. By [23, Lemma 3.1.8(2)] we have the set identities $\mathcal N = \psi^{-1}(\mathcal E) = \text{Im}(T) + \text{Ker}(\psi)$ and $\{0\} = T(\text{Ker}(\text{id}_{\mathcal E})) = \text{Im}(T) \cap \text{Ker}(\psi)$. Therefore, $\mathcal E$ has to be a topological summand with topological complement $\text{Ker}(\psi)$ there, i.e. $\mathcal N = T(\mathcal E) + \text{Ker}(\psi)$.

To derive the $A$-reflexivity of injective Hilbert $A$-modules consider the definition of injectivity with $\mathcal M = \mathcal E$, $\mathcal N = \mathcal E''$ and $\phi = \text{id}_{\mathcal E}$. By [10, Prop. 2.1] the $A$-valued inner product on $\mathcal E$ extends to an $A$-valued inner product on its $A$-bidual Banach $A$-module $\mathcal E''$. Moreover, the
-representation of $B$ on $\mathcal{E}$ turns into a $\ast$-representation of $B$ on $\mathcal{E}''$ via the canonical isometric embedding $\mathcal{E} \subseteq \mathcal{E}''$, since every bounded module operator on $\mathcal{E}$ extends to a bounded module operator on $\mathcal{E}''$ in a unique way by [40]. However, the embedded copy of $\mathcal{E}$ is a topological summand of $\mathcal{E}''$ if and only if both they coincide. Indeed, since we have the chain of isometric embeddings $\mathcal{E} \subseteq \mathcal{E}'' \subseteq \mathcal{E}'$ by [39, 40] the assumption of $\mathcal{E}$ being a non-trivial topological summand of $\mathcal{E}''$ would lead to the non-uniqueness of the representation of the zero map on $\mathcal{E}$ in $\mathcal{E}'$, a contradiction to the definition of this set. The last statement above is a consequence of the injectivity and $A$-reflexivity of $\mathcal{E}$ and of [15, Lemma 3.1].

In many categories the converse of Lemma 3.2 holds too, that is, if an object is complemented in every object that it is a subobject of, then it is injective. This holds any time that there are enough injective elements in the category that the object can be embedded via a morphism as a subobject of an injective object. Often these splitting properties serve as an alternative means to define injectivity, [24]. However, in Proposition 3.3 below we indicate that for a large number of unital, monotone incomplete $C^*$-algebras $A$ the $C^*$-algebra $A$ itself is not injective in the category of all Hilbert $A$-modules and bounded module maps, despite the fact that unital $C^*$-algebras $A$ are always orthogonally comparable as Hilbert $A$-modules. The same holds for certain non-unital $C^*$-algebras $A$ provided $M(A) = LM(A)$.

**Proposition 3.3.** Let $A$ be a $C^*$-algebra and $A^N$ be the standard Hilbert $A$-module of all $N$-tuples of elements of $A$ for given positive integers $N$. The following are equivalent:

1. $A^N$ is injective in $\mathcal{B}(A, C)$ for one $N \in \mathbb{N}$,
2. $A^N$ is injective in $\mathcal{B}(A, C)$ for every $N \in \mathbb{N}$,
3. $A$ is injective in $\mathcal{B}(A, C)$,
4. $M(A)$ is a monotone complete $C^*$-algebra.

**Proof.** Let $\mathcal{M} \subseteq \mathcal{N}$ be a subobject and let $\phi : \mathcal{M} \to A^N$ be a bounded $A$-module map. Note that $\phi = (\phi_1, \ldots, \phi_N)$ where $\phi_i : \mathcal{M} \to A$ are bounded $A$-module maps. The map $\psi : \mathcal{N} \to A^N$ that extends $\phi$ exists if and only if there exist bounded $A$-module maps $\psi_i : \mathcal{N} \to A(i)$ coinciding with $\phi_i$ on $\mathcal{M}$, where the index $(i)$ denotes $i$-th coordinate of $A^N$. This shows the equivalence of (1), (2) and (3).

Also, we see that such an extension exists if and only if a generalized Hahn-Banach type theorem is valid for arbitrary pairs of Hilbert $A$-modules $\mathcal{M} \subseteq \mathcal{N}$ and arbitrary bounded $A$-linear functionals $r : \mathcal{M} \to C$. This results in $\phi$ being a monomorphism.
A. By [15, Th. 2] this takes place if and only if $M(A)$ is monotone complete.

**Proposition 3.4.** Let $A$ be a unital C*-algebra. If there exist any full Hilbert $A$-modules that are injective in $\mathcal{B}(A, \mathbb{C})$, then $A$ is monotone complete. Hence, if $A$ is simple, unital and not monotone complete, then there are no non-zero injective Hilbert $A$-modules in $\mathcal{B}(A, \mathbb{C})$.

**Proof.** Let $\mathcal{E}$ be a full injective Hilbert $A$-module. By [36, Lemma 2.4.3] there exists a finite positive integer $n$ and a subset of elements $\{e_1, ..., e_n\}$ of $E$ such that $\sum_{i=1}^{n} \langle e_i, e_i \rangle_A = 1$ because the Hilbert $A$-module $E$ is full. Note, that $E^n$ is injective whenever $E$ is injective and $n$ is a finite positive integer. Therefore, we have an isometric left $A$-module map, $\phi : A \to \mathcal{E}^n$ given by $\phi(a) = \sum_{i=1}^{n} ae_i$. Since, $A$ is orthogonally comparable by [13, Prop. 6.2, Th. 6.3], there exists a bounded $A$-module map, $\psi : \mathcal{E}^n \to \phi(A)$. From this it follows easily that $\phi(A)$ is injective in $\mathcal{B}(A, \mathbb{C})$. Since $A$ and $\phi(A)$ are isomorphic, $A$ is injective in $\mathcal{B}(A, \mathbb{C})$. Hence, by Proposition 3.3, $M(A) = A$ is monotone complete.

To see the final assertion, note that since $A$ is simple and unital, every non-zero Hilbert $A$-module is full, since the range of its' $A$-valued inner product is a norm-closed two-sided ideal in $A$. □

When $A$ is unital, not simple and not monotone complete, it is possible to have injectives in $\mathcal{B}(A, \mathbb{C})$, as the following example shows. However, we will show below that when $A$ is unital but not monotone complete, then there are not enough injectives, so that every Hilbert $A$-module can be embedded in an injective.

**Example 3.5.** Let $A = \mathbb{C} \oplus B$, where $B$ is a unital C*-algebra that is assumed to be not monotone complete. Thus, $A$ is unital and not monotone complete. Note, that every Hilbert space $\mathcal{K}$ is a (non-full) Hilbert $A$-module, with $(0 \oplus B)\mathcal{K} = 0$. We claim that $\mathcal{K}$ is an injective Hilbert $A$-module in $\mathcal{B}(A, \mathbb{C})$.

Indeed, in case $\mathcal{E}$ is a Hilbert $A$-module and $\mathcal{H} = (\mathbb{C} \oplus 0)\mathcal{E}$ and $\mathcal{F} = (0 \oplus B)\mathcal{E}$ are its submodules, then $\mathcal{E} = \mathcal{H} \oplus \mathcal{F}$ is an orthogonal direct sum decomposition. Moreover, any $A$-module map from $\mathcal{E}$ into $\mathcal{K}$ is zero on $\mathcal{F}$, and it is a linear map on $\mathcal{H}$. The fact that $\mathcal{K}$ is injective in $\mathcal{B}(A, \mathbb{C})$ now follows easily from that it is injective in the category of Hilbert spaces and bounded linear maps.

By the above results, the category $\mathcal{B}(A, \mathbb{C})$ does not contain any non-zero injective object for setting $A$ to be one of the following C*-algebras, among others (cf. [3]):
• the reduced group C*-algebra $C^*_r(F_2)$ of the free group on two generators $F_2$,
• the irrational rotation algebras $A_\theta$, $\theta \in (0, 1)$ - irrational,
• the Cuntz algebras $O_n$, $n \in \mathbb{N}$ and the Cuntz-Krieger algebra $O_\infty$,
• the Bunce-Deddens algebras $B(\{n_k\})$, $n, k \in \mathbb{N}$,
• Blackadar’s projectionless unital simple C*-algebra.

The following result, shows that even in cases when injectives do exist for $A$ unital and not monotone complete, there cannot exist “enough”. We say that a Hilbert $A$-module, $E$ can be boundedly embedded in a Hilbert $A$-module $F$, provided that there exists a module map, $T : E \to F$ that is bounded above and below, i.e., there are constants, $0 < C_1 \leq C_2$ such that $C_1\|e\| \leq \|T(e)\| \leq C_2\|e\|$.

Often a category is said to have enough injectives if every object can be embedded into an injective object. The following result shows that if $A$ is unital, then the only time that $B(A, \mathbb{C})$ can have enough injectives is when $A$ is monotone complete.

**Proposition 3.6.** Let $A$ be a unital C*-algebra. If every Hilbert $A$-module can be boundedly embedded into a Hilbert $A$-module that is injective in $B(A, \mathbb{C})$, then $A$ is monotone complete.

**Proof.** By hypothesis, there exists an injective Hilbert $A$-module $\mathcal{F}$ and a bounded embedding $T : A \to \mathcal{F}$ with constants $C_1, C_2$ as above. Since $T$ is an $A$-module map, there exists an element $f \in \mathcal{F}$ such that $T(a) = af$.

Let $p = \langle f, f \rangle^{1/2}$. By the above inequalities, $C_1^2\|a\|^2 \leq \|ap^2a^*\| \leq C_2^2\|a\|^2$ for every $a \in A$. Taking $a = g(p)$ for $g$ some continuous function on the spectrum of $p$ and using the fact that $p$ is a positive element of $A$, we get by the spectral mapping theorem, that $C_1\|g(t)\|_{\infty} \leq \|tg(t)\|_{\infty} \leq C_2\|g(t)\|_{\infty}$, where $\| \cdot \|_{\infty}$ denotes the supremum over the spectrum of $p$. These inequalities imply that the spectrum of $p$ is contained in the interval, $[C_1, C_2]$, and hence $p$ is invertible in $A$.

Therefore, $\mathcal{F}$ is an injective full Hilbert $A$-module and by Proposition 3.4, $A$ is monotone complete. \hfill \Box

**Problem 3.7.** We do not have a complete set of analogous results for C*-bimodules. For example, if $A$ and $B$ are unital, simple C*-algebras, and neither one is monotone complete, can there exist any full Hilbert $A$-$B$-bimodules that are injective in $B(A, B)$? Under what conditions on unital C*-algebras $A$ and $B$, can there exist enough injectives in $B(A, B)$?
In contrast, for non-unital C*-algebras there are often enough injectives.

The following result characterizes the C*-algebras for which every Hilbert A-module is injective in \( \mathcal{B}(A, \mathbb{C}) \).

**Theorem 3.8.** Let \( A \) be a C*-algebra of compact operators on some Hilbert space. Let \( \{ \mathcal{E}, \langle ., . \rangle \} \) be a Hilbert \( A \)-module and \( B \) be another C*-algebra admitting a *-representation on \( \mathcal{E} \). Then \( \mathcal{E} \) is an injective object in \( \mathcal{B}(A, B) \).

Conversely, let \( A \) be a C*-algebra. If every Hilbert \( A \)-module is injective in \( \mathcal{B}(A, \mathbb{C}) \), then \( A \) is *-isomorphic to a C*-algebra of compact operators on some Hilbert space.

**Proof.** Referring to Theorem 2.1 and Proposition 2.2 we see that every bounded \( A \)-linear map between Hilbert \( A \)-modules over a C*-algebra \( A \) of type \( c_0 - \sum_i \oplus K(H_i) \) possesses an adjoint. So every inclusion map is adjointable and we are in the situation of Theorem 3.1. This shows the first assertion.

To demonstrate the converse implication consider a maximal left-sided ideal \( I \) of the C*-algebra \( A \). The \( A \)-valued inner product on \( I \) is that one inherited from \( A \). Setting \( \mathcal{E} = \mathcal{M} = I, \mathcal{N} = A, \phi = \text{id}_I \) and taking the standard \( A \)-linear embedding of \( I \) into \( A \) in the definition of injectivity, we see that the existence of an \( A \)-module map \( \psi : A \to I \) extending \( \phi \) is equivalent to the existence of an orthogonal projection \( p_I \in \mathcal{M}(A) \) such that \( I = Ap_I \). So by Proposition 2.2 (ix) the C*-algebra \( A \) has to be of type \( c_0 - \sum_i \oplus K(H_i) \) as to be shown. \( \square \)

Let us remark that maximal left-sided ideals \( I \) of C*-algebras \( A \) may admit trivial sets of bounded module operators \( \text{End}_A(I) = \mathbb{C} \). For example, consider the case of \( A \) being the matrix algebra of complex \( 2 \times 2 \) matrices. Consequently, the validity of the second assertion for the more general setting of Hilbert \( A-B \)-bimodules heavily depends on the structure of the C*-algebra \( B \) and of the resulting collection of objects of \( \mathcal{B}(A, B) \).

When \( A \) is a monotone complete C*-algebra we can characterize the injectivity of Hilbert \( A-B \) bimodules in terms of self-duality. This strengthens a result of H. Lin ([31, Th. 2.2,Prop. 3.10]) that he obtained in the contractive morphism situation, since every module that is injective in the contractive morphism situation is automatically injective in the setting of bounded morphisms. Also, our result complements a result by D. P. Blecher and V. I. Paulsen on the injective envelope of
an operator bimodule stating that it has to be a self-dual Hilbert C*-module over an injective (and hence, monotone complete) C*-algebra, cf. [6].

**Theorem 3.9.** Let \( A \) be a monotone complete C*-algebra, \( \{M, \langle \cdot, \cdot \rangle\} \) be a Hilbert \( A \)-module. Let \( B \) be a C*-algebra admitting a *-representation in \( \text{End}_A^*(M) \). Then \( M \) is injective in \( \mathcal{B}(A, B) \) if and only if \( M \) is self-dual as a Hilbert \( A \)-module.

**Proof.** Suppose \( M \) is injective in \( \mathcal{B}(A, B) \), and consider the canonical isometric embedding of \( M \) into its \( A \)-dual Banach \( A \)-module \( M' \). By [11, Th. 4.7] the \( A \)-valued inner product on \( M \) can be continued to an \( A \)-valued inner product on \( M' \) in a manner compatible with the canonical embedding \( M \hookrightarrow M' \). For the action on the right, the *-representation of \( B \) on \( M \) induces a *-representation of \( B \) on \( M' \) via the canonical embedding, since every bounded module operator on \( M \) extends to a unique bounded module operator on \( M' \) by [39]. Finally, the copy of \( M \) in \( M' \) is a topological summand there if and only if both the sets coincide, since the zero functional on \( M \) would admit several representations in \( M' \) otherwise. So \( M \) has to be self-dual.

To establish the converse implication consider the diagram

\[
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{T} & M' \\
\uparrow & & \\
\mathcal{K} & \xrightarrow{\phi} & M
\end{array}
\]

with \( T \) an isometric \( A-B \)-bilinear embedding and \( \phi \) a bounded \( A-B \)-bilinear map. In this diagram we can replace \( \phi \) by \( \phi/\|\phi\| \), a contractive map. Then there exists a bounded \( A \)-linear map \( \psi: \mathcal{L} \to M \) such that \( \langle \phi/\|\phi\| \rangle = \psi \circ T \) by [31, Th. 2.2]. Since \( \phi \) and \( T \) are \( B \)-linear the map \( \psi \) turns out to be \( B \)-linear, too. Multiplying both sides by the constant \( \|\phi\| \) we obtain the map \( \|\phi\|\psi \) that completes the diagram above to a commutative one. So \( M \) is injective in the selected category. \( \square \)

When the C*-algebra of coefficients of a Hilbert C*-module \( \mathcal{E} \) is not a unital C*-algebra and the Hilbert C*-module \( \mathcal{E} \) is full, i.e. its C*-algebra of coefficients \( A \) is the minimal admissible one, then we can consider \( \mathcal{E} \) as a Hilbert C*-module over larger C*-algebras, reasonably over C*-algebras containing the C*-algebra of coefficients \( A \) as an ideal and belonging to the multiplier algebra \( M(A) \) of \( A \). However, a construction by D. Bakić and B. Guljaš in [2] gives us the opportunity to establish a necessary condition on those Hilbert \( A \)-modules to be injective in the category of Hilbert \( M(A) \)-modules.

Let \( A \) be a (non-unital) C*-algebra and \( M \) be a full Hilbert \( A \)-module equipped with an \( A \)-valued inner product \( \langle \cdot, \cdot \rangle \). If \( A \) is equipped with
the standard $A$-valued inner product defined by the rule $\langle a, b \rangle_A = ab^*$, then the Hilbert $M(A)$-module $\text{End}_A^*(A, M)$ of all adjointable bounded $A$-linear maps from $A$ to $M$ is denoted by $\mathcal{M}_d$. The $M(A)$-valued inner product on $\mathcal{M}_d$ is defined by $\langle r, s \rangle = s^* \circ r$ for any $r, s \in \mathcal{M}_d$. One of the remarkable properties of this construction is the existence of an isometric embedding $\Gamma$ of $M$ into $\mathcal{M}_d$. It is defined by the formula $\Gamma(x)(a) = ax$ for any $a \in A$, each $x \in M$. The image $\Gamma(M) \subseteq \mathcal{M}_d$ coincides with the subset $A \cdot \mathcal{M}_d$. Note, that the construction depends on the unitary equivalence classes of both the $A$-valued inner products on $A$ and on $M$. Furthermore, $\mathcal{M}_d$ can be characterized topologically as the linear hull of the completion of the unit ball of $M$ with respect to the strict topology, where the strict topology is induced by the set of semi-norms $\{\|\langle \cdot, x \rangle\|_A : x \in M\} \cup \{\|b \cdot \|_M : b \in A\}$. So the described extension turns out to be a closure operation, i.e., $\mathcal{M}_d \equiv \mathcal{M}_{dd}$ for any Hilbert C*-module $M$. Finally, the closure operation obeys orthogonal decompositions, i.e. $(\mathcal{M} \oplus \mathcal{N})_d = \mathcal{M}_d \oplus \mathcal{N}_d$, and the sets of all adjointable bounded module maps on $M$ and on $\mathcal{M}_d$ are always $\ast$-isomorphic, simply by restricting operators on $\mathcal{M}_d$ to the $M(A)$-invariant subset $\Gamma(M) \subseteq \mathcal{M}_d$ that is isometrically isomorphic to $M$. For all these results we refer to [2].

**Proposition 3.10.** Let $A$ be a non-unital C*-algebra and $E$ be a full Hilbert $A$-module. Let $B$ be another C*-algebra that admits a $\ast$-representation on $E$. If $E$ is injective in $\mathcal{B}(M(A), B)$, then $E \equiv E_d$.

**Proof.** Note, that the isomorphism of the sets of all adjointable bounded module maps on both the Hilbert $A$-module and on its strict closure turns the strict closure into a $M(A)$-$B$ bimodule, too. So $E_d$ is contained in the same category under consideration.

Referring to the definition of injectivity, set $\mathcal{M} = E$, $\mathcal{N} = E_d$, identify $E$ with its image, $\Gamma(E) \subseteq E_d$ and $\phi = \text{id}_E$. Since $\Gamma(E)$ is injective, there is a bounded $M(A)$-$B$-bimodule map, $\psi : E_d \rightarrow \Gamma(E)$ extending the identity map. Furthermore, by [13] Th. 6.4] we have the canonical isometric inclusions $E \hookrightarrow E_d \hookrightarrow E'$, and the $M(A)$-linear bounded identity operator on $E$ has a unique extension to the identity operator on $E'$ preserving the norm. In particular, the identity operator on $E$ extends uniquely to the identity operator on $E_d$. Therefore, $E \equiv E_d$. \hfill $\square$

We have that $E \equiv E_d$ for a Hilbert $A$-module $E$ provided that either the C*-algebra $A$ of coefficients or the C*-algebra $K_A(E)$ is unital. Whether only direct orthogonal sums of Hilbert $A$-modules of these two types can possess this closure property is an open problem at present, cf. [4].
Corollary 3.11. Let $A$ be a C*-algebra. If $A$ is injective in the category $\mathcal{B}(M(A), \mathbb{C})$, then $A$ has to be unital (i.e. $A = M(A)$) and monotone complete. Moreover, if $A^N$ is injective for some $N \in \mathbb{N}$, then $A^N$ is injective for any $N \in \mathbb{N}$, in particular for $N = 1$.

Proof. This follows from the facts that $A_d = M(A)$ and $(A^N)_d = M(A)^N$ for any $N \in \mathbb{N}$ by construction. So $A = M(A)$ by the previous proposition. Furthermore, Proposition 3.3 and Theorem 3.9 force $A$ to be monotone complete. □

4. Projectivity

Let $A$ and $B$ be two fixed C*-algebras. We consider the categories $\mathcal{B}(A, B)$ (respectively, $\mathcal{B}^*(A, B)$) consisting of Hilbert $A$-$B$-bimodules as objects and (adjointable) bounded $A$-$B$-bilinear maps as morphisms, where $A$ serves as the C*-algebra of coefficients and $B$ admits a *-representation in the C*-algebras of all adjointable bounded operators on Hilbert $A$-modules that are objects. By definition a Hilbert $A$-$B$-bimodule $\mathcal{F}$ is projective in $\mathcal{B}(A, B)$ (respectively, $\mathcal{B}^*(A, B)$) if and only if the diagram

\[
\begin{array}{cc}
\mathcal{N} & \\
\downarrow T & \\
\mathcal{M} & \leftarrow \mathcal{F}
\end{array}
\]

(2)

can be completed to a commutative one by an $A$-$B$-(respectively, adjointable) bimodule morphism $\psi : \mathcal{F} \to \mathcal{N}$, whenever $T$ is a surjective $A$-$B$-bimodule (respectively, adjointable) morphism and $\phi$ is a (respectively, adjointable) $A$-$B$-bimodule morphism between Hilbert $A$-$B$-bimodules.

It is fairly easy to prove (and we do) that every object is projective in $\mathcal{B}^*(A, B)$. We do not know if the same is true for $\mathcal{B}(A, B)$, but we identify a family of C*-algebras for which every object in $\mathcal{B}(A, B)$ is projective.

We begin by disposing of the $\mathcal{B}^*(A, B)$ case.

Theorem 4.1. Let $A$ be an arbitrary C*-algebra and $\{\mathcal{F}, \langle \cdot, \cdot \rangle\}$ be a Hilbert $A$-module. Let $B$ be another C*-algebra that admits a *-representation in $\text{End}_A^\ast(\mathcal{F})$. Then $\mathcal{F}$ is a projective object in the category $\mathcal{B}^*(A, B)$.

Proof. Consider an adjointable surjective bounded $A$-$B$-bimodule map $T : \mathcal{N} \to \mathcal{M}$ of two Hilbert $A$-$B$-bimodules $\mathcal{M}$ and $\mathcal{N}$. Since $T$ possesses closed range by definition, the range of $T^* : \mathcal{M} \to \mathcal{N}$ is closed in $\mathcal{N}$ and an orthogonal summand by Proposition 4.1. Since $T$
is surjective, $T^*$ has to be injective, and we have the decomposition $\mathcal{N} = T^*(\mathcal{M}) \oplus \ker(T)$. Both these orthogonal summands are $A$-$B$-invariant by construction. Every element $x \in \mathcal{M}$ possesses a unique pre-image $T^{-1}(x) \in T^*(\mathcal{M})$. The operator $T^{-1} : \mathcal{M} \to T^*(\mathcal{M}) \subseteq \mathcal{N}$ defined this way is everywhere defined on $\mathcal{M}$ and possesses a closed range, hence, it is bounded. Moreover, it is $A$-$B$-linear. Setting $\psi : \mathcal{F} \to \mathcal{N}$ to be defined by the rule $\psi(f) = T^{-1}(\phi(f)) \in T^*(\mathcal{M}) \subseteq \mathcal{N}$ for $f \in \mathcal{F}$ we obtain a bounded $A$-$B$-bilinear map $\psi$ completing the diagram (2) to a commutative one.

The following test for projectivity is often useful.

**Theorem 4.2.** Let $A$ and $B$ be arbitrary $C^*$-algebras and $\{\mathcal{F}, \langle \cdot, \cdot \rangle\}$ be a Hilbert $A$-$B$-bimodule. Then the following statements are equivalent:

(i) $\mathcal{F}$ is projective in $\mathcal{B}(A, B)$,

(ii) every bounded, surjective bimodule map, $T : \mathcal{N} \to \mathcal{F}$ has a right inverse, $S : \mathcal{F} \to \mathcal{M}$ that is a bounded bimodule map,

(iii) whenever, $T : \mathcal{N} \to \mathcal{F}$ is a bounded, surjective bimodule map, $\ker(T)$ is a topological summand of $\mathcal{N}$ with a complementary space that is a bimodule, i.e. is a topological bimodule summand.

**Proof.** The equivalence of (ii) and (iii) is clear.

Assume that $\mathcal{F}$ is projective. By definition there exists an $A$-$B$-bimodule morphism $\psi : \mathcal{F} \to \mathcal{N}$ such that $T \circ \psi = \text{id}_\mathcal{F}$. By [24, Lemma 3.1.8(2)] we have the set identities $\mathcal{N} = T^{-1}(\mathcal{F}) = \text{Im}(\psi) + \ker(T)$ and $\{0\} = \psi(\ker(\text{id}_\mathcal{F})) = \text{Im}(\psi) \cap \ker(T)$. Therefore, the Hilbert $A$-$B$-bimodule $\ker(T)$ is a topological summand with topological complement $\text{Im}(\psi)$ there, i.e. $\mathcal{N} = \psi(\mathcal{F}) \oplus \ker(T)$. The invariance of $\ker(T)$ under the action of $B$ is caused by the $A$-$B$-bilinearity of the operator $T$. Thus, (i) implies (iii).

Conversely, assume that (ii) holds and consider the situation of diagram (1). Let $\mathcal{L} = \{(f, n) \in \mathcal{F} \oplus \mathcal{N} : \phi(f) = T(n)\}$, which is an $A$-$B$-submodule of $\mathcal{F} \oplus \mathcal{N}$. The map $R : \mathcal{L} \to \mathcal{F}$, defined by $R((f, n)) = f$ is a bounded bimodule surjection and hence has a right inverse, $S : \mathcal{F} \to \mathcal{L}$. Let $P : \mathcal{L} \to \mathcal{N}$ be defined by $P((f, n)) = n$, so $P$ is also a bounded bimodule map and $\psi = P \circ S : \mathcal{S} \to \mathcal{N}$ is the desired lifting of $\phi$. \hfill \Box

Theorem 4.2 indicates a way to find non-projective Hilbert $A$-modules if such Hilbert $C^*$-modules exist at all.

**Theorem 4.3.** Let $A$ be a $C^*$-algebra of type $c_0 \oplus \sum_i \mathbb{K}(H_i)$, i.e. a $C^*$-algebra of compact operators on a certain Hilbert space. Let $\{\mathcal{F}, \langle \cdot, \cdot \rangle\}$
be a Hilbert $A$-module and $B$ be another $C^*$-algebra admitting a $*$-representation in $\text{End}^*_A(\mathcal{F})$. Then $\mathcal{F}$ is a projective object in $\mathcal{B}(A, B)$.

Proof. Referring to Theorem 2.1 and Proposition 2.2 we see that every kernel is an orthogonal summand. When the map is a bimodule map, the kernel and its orthogonal complement are both bimodules. □

Problem 4.4. Are the $C^*$-algebras $A$ of type $c_0-\sum_i K(H_i)$ the only $C^*$-algebras $A$ for which all Hilbert $A$-modules are projective in $\mathcal{B}(A, C)$, or not?

In fact, we do not even know whether or not every Hilbert $A$-$B$-bimodule is projective in $\mathcal{B}(A, B)$ for every pair of $C^*$-algebras, $A$ and $B$. The investigations of the authors did not reveal any counterexample, so we state the question as a problem to the readers:

Problem 4.5. Does there exist a $C^*$-algebra for which there is a non-projective Hilbert $A$-module in the category $\mathcal{B}(A, C)$? Does there exist a pair of $C^*$-algebras $A$, $B$ and a non-projective Hilbert $A$-$B$-bimodule in $\mathcal{B}(A, B)$?

By 4.2 the above problem is equivalent to determining whether or not every surjective bimodule map between Hilbert bimodules has a right inverse that is a bimodule map.

The following general result partially links the final solution of the projectivity problem to the solution of Problem 2.3 above:

Corollary 4.6. Let $A$ be a $C^*$-algebra. Every Hilbert $A$-module is projective in the category $\mathcal{B}(A, C)$ if and only if the kernel of every surjective bounded $A$-linear map between Hilbert $A$-modules is a topological summand.

Proof. Apply 4.2(iii). □

We now take a closer look at projectivity in the case of unital $C^*$-algebras and its connection with Kasparov’s stabilization theorem.

Proposition 4.7. Let $A$ be a unital $C^*$-algebra. Then for every $N \in \mathbb{N}$ the Hilbert $A$-module, $A^N$, is projective in $\mathcal{B}(A, C)$.

Proof. Given a Hilbert $A$-module $\mathcal{N}$ and a bounded surjective module map, $T : \mathcal{N} \to A^N$, choose elements, $x_j \in \mathcal{N}$, such that $T(x_j) = e_j$, where $e_j$ denotes the element that is $1_A$ in the $j$-th component and 0, elsewhere. The map $R : A^N \to \mathcal{N}$, defined by $R((a_1, ..., a_N)) = \sum_j a_jx_j$ is a right inverse for $T$. □

Problem 4.8. When is a non-unital $C^*$-algebra, $A$, a projective object in $\mathcal{B}(A, C)$?
By 4.3, some non-unital C*-algebras are projective in $\mathcal{B}(A, \mathbb{C})$. Also, by the above result it is easy to see that any time $A$ is projective, then $A^N$ is projective.

The corresponding infinite dimensional version of $A^N$ is $\ell^2(A) = \{(a_1, a_2, ...) : \sum_{n=1}^{\infty} a_n a_n^* \in A\}$, where the convergence is in the norm sense.

**Proposition 4.9.** If $\ell^2(A)$ is projective in $\mathcal{B}(A, \mathbb{C})$, then every countably generated Hilbert $A$-module is projective in $\mathcal{B}(A, \mathbb{C})$.

**Proof.** If $\mathcal{M}$ is countably generated then by Kasparov’s stabilization theorem [25], $\mathcal{M} \oplus \ell^2(A)$ is $A$-module isomorphic to $\ell^2(A)$. Thus, $\mathcal{M}$ is isomorphic to an orthogonally complemented submodule of $\ell^2(A)$. Now an elementary diagram chase shows that an orthogonally complemented submodule of a projective module is projective. □

**Problem 4.10.** Let $A$ be a C*-algebra, when is $\ell^2(A)$ projective in $\mathcal{B}(A, \mathbb{C})$?

We will make some progress on this question below. For these results we will need some concepts from operator spaces. Given any Hilbert C*-module, $\mathcal{M}$ we can represent it as operators on a Hilbert space. This allows us to make sense of the norms of matrices over the Hilbert C*-module and these norms turn out to be canonical, i.e. to only depend on the inner product. For our purposes, we will only need to refer to $M_\infty(A)$ which denotes the set of $\infty \times \infty$ matrices over $A$ which are bounded, i.e. such that $\|(a_{i,j})\| \equiv \sup_n \|(a_{i,j})^n_{i,j=1}\| < +\infty$ and to $C_\infty(\mathcal{M}) = \{(m_1, m_2, ...) : (\langle m_i, m_j \rangle) \in M_\infty(A)\}$.

**Proposition 4.11.** Let $\phi : \ell^2(A) \to \mathcal{M}$ be defined by $\phi((a_1, a_2, ...)) = \sum_n a_n m_n$. Then $\phi$ defines a bounded $A$-module map if and only if $\|(\langle m_i, m_j \rangle)\| \equiv \sup_n \|(a_{i,j})^n_{i,j=1}\| \approx +\infty$ is finite. Moreover, in this case, $\|\phi\| = \|(\langle m_i, m_j \rangle)\|$.

**Proof.** For any finitely supported tuple, we have $\|\phi((a_1, ..., a_n, 0, 0...))\| = \| \sum_{i,j=1}^{n} a_i (m_i, m_j) a_j^* \|$. But for any $(p_{i,j}) \in M_n(A)$, we have that

$$\|\phi(p_{i,j})\| = \sup \{ \| \sum_{i,j=1}^{n} a_i p_{i,j} a_j^* \| : \sum_{j=1}^{n} a_j a_j^* \leq 1_A \},$$

from which the result follows. □

**Theorem 4.12.** Let $A$ be a unital C*-algebra. Then $\ell^2(A)$ is projective in $\mathcal{B}(A, \mathbb{C})$ if and only if for every pair of Hilbert $A$-modules, $\mathcal{N}$, $\mathcal{M}$ and every bounded, surjective module map, $T : \mathcal{N} \to \mathcal{M}$, the induced map $T_\infty : C_\infty(\mathcal{N}) \to C_\infty(\mathcal{M})$, is surjective.
Proof. Assume that we are in the setting of diagram (1). Since the map \( \phi : \ell^2(A) \to \mathcal{M} \) is bounded, we have \((m_1, m_2, \ldots)^t \in C_\infty(\mathcal{M})\), with \(\phi((a_1, \ldots)) = a_1 m_1 + \ldots\), and in order to lift \(\phi\) to a map \(\psi\) we must find \((n_1, \ldots)^t \in C_\infty(\mathcal{N})\), with \(T(n_i) = m_i\), for all \(i\). □

Note that we do not require the map \(T_\infty\) to be bounded in the above result, only onto.

We now take a closer look at what projectivity implies for non-unital C*-algebras.

**Corollary 4.13.** Let \(A\) be a non-unital C*-algebra. If \(A\) equipped with the canonical \(A\)-valued inner product is a projective Hilbert \(A\)-module in the category \(\mathcal{B}(A, \mathcal{C})\), then every element \(t \in \text{LM}(A)\) that induces a surjective map \(T : A \to A\) by the formula \(T(a) = at^*\), admits a right inverse that is an element of \(\text{LM}(A)\), and the kernel of \(T\) is a topological summand of \(A\). Moreover, every surjective bounded module map \(T : A \to A\) is realized by multiplication by a left multiplier in the way indicated.

If \(\text{M}(A) = \text{LM}(A)\) for the C*-algebra under consideration, then these conditions are automatically fulfilled.

**Proof.** Consider the diagram (2) setting \(\mathcal{N} = \mathcal{M} = A\) and \(\phi = \text{id}_A\). Since \(A\) is supposed to be a projective Hilbert \(A\)-module there exists a map \(\psi : A \to A\) which is implemented by the rule \(\psi(a) = as^*\) for some \(s \in \text{LM}(A)\) by the existing canonical identification of \(\text{End}_A(A)\) with \(\text{LM}(A)\), cf. [30]. Note, that \(T \circ \psi = \phi\) by the choice of \(\psi\). Consequently, \(1_A = 1_{\text{LM}(A)} = s^*t^* = ts\) since \(a = 1_A\) is a possible choice for the free variable. So \(stst = s(tst) = st\) and the element \(p = st\) is an idempotent element of \(\text{LM}(A)\). So \(s \in \text{LM}(A)\) is the right inverse of \(t \in \text{LM}(A)\). Note, that the idempotent \((1_A - p) \in \text{LM}(A)\) maps \(A\) onto the kernel of the map \(T\) which becomes a topological summand of the Hilbert \(A\)-module \(A\). The last two statements follow from the canonical identification of \(\text{End}_A(A)\) with \(\text{LM}(A)\) and from spectral decomposition in \(\text{M}(A)\), cf. [30] and Proposition 1.1. □

We close by looking at what projectivity means in the purely algebraic category consisting of all \(A\)-modules and of all \(A\)-linear maps for respective Hilbert \(A\)-modules. We show that finitely generated Hilbert \(A\)-modules are in fact also projective in all the categories of Hilbert \(A\)-\(B\) bimodules under consideration, as one might expect.

**Theorem 4.14.** Let \(A\) be a unital C*-algebra and \(\mathcal{F}\) be a finitely generated Hilbert \(A\)-module (which is automatically a projective object in the category consisting of all \(A\)-modules over a fixed C*-algebra \(A\) and
of all $A$-linear maps by \cite[15.4.8]{53}). Let $B$ be a $C^*$-algebra represented as a $C^*$-algebra of bounded adjointable operators on $F$. Then $F$ is an orthogonal summand of some Hilbert $A$-module $A^n$, $n < \infty$, and $F$ is projective in the categories $\mathcal{B}(A, B)$ and in $\mathcal{B}^*(A, B)$.

Proof. Fix an $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $F$. By \cite[Cor. 15.4.8]{53} and by the definition of projective $A$-modules in algebra $F$ has to be finitely generated, and every finitely generated Hilbert $A$-module is projective in the purely algebraic sense. Consider the diagram (2) again. By supposition there exists an $A$-linear map $\psi : F \rightarrow N$ such that $\phi = T \circ \psi$. Let us show that $\psi$ is bounded. By \cite[19]{10} there exists a finite algebraic set of generators $\{x_1, ..., x_n\}$ of $F$ such that the reconstruction formula $x = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$ is valid for every $x \in F$. This set $\{x_1, ..., x_n\}$ of generators is called a normalized tight frame of $F$ with respect to the fixed $A$-valued inner product $\langle \cdot, \cdot \rangle$. Therefore, $\psi(x) = \sum_{i=1}^{n} \langle x, x_i \rangle \psi(x_i)$ for any $x \in F$. Using the Cauchy-Schwarz inequality for Hilbert $C^*$-modules \cite[Prop. 1.1]{29} we obtain the inequality

$$
\|\psi(x)\| = \left\| \sum_{i=1}^{n} \langle x, x_i \rangle \psi(x_i), \psi(x) \right\|_N \\
= \left\| \sum_{i=1}^{n} \langle x, x_i \rangle \langle \psi(x_i), \psi(x) \rangle \right\|_N \\
\leq \sum_{i=1}^{n} \|x\|^{1/2} \|x_i\|^{1/2} \|\psi(x_i)\|^{1/2}_N \|\psi(x)\|^{1/2}_N \\
= \left( \sum_{i=1}^{n} \|x_i\|^{1/2} \|\psi(x_i)\|^{1/2}_N \right) \|x\|^{1/2} \|\psi(x)\|^{1/2}_N.
$$

Cutting by $\|\psi(x)\|^{1/2}_N$ the boundedness of $\psi$ and, hence, the assertion of the theorem becomes obvious. \hfill \Box

**Problem 4.15.** Prove or disprove that selfdual Hilbert $C^*$-modules are projective objects in the categories under consideration.

The problem of finding non-projective Hilbert $C^*$-modules in the category $\mathcal{B}(A, C)$ is closely related to the problem of characterizing surjective bounded $A$-linear maps between Hilbert $C^*$-modules that do not admit right inverses in the set of all bounded $A$-linear maps. Note, that in case the domain and the range of the surjective maps are identified such maps can be considered as special left multipliers of the
C*-algebra of all 'compact' operators on the underlying Hilbert C*-module, which turns the problem into an open C*-algebraic problem of left multiplier algebras of C*-algebras.

In fact, if a bounded module map $T : \mathcal{M} \to \mathcal{N}$ is surjective then the right ideal $T \cdot K_A(\mathcal{M} \oplus \mathcal{N})$ of the C*-algebra of 'compact' operators $K_A(\mathcal{M} \oplus \mathcal{N})$ is closed and hence,

$$0 \to T \cdot K_A(\mathcal{M} \oplus \mathcal{N}) \to K_A(\mathcal{M} \oplus \mathcal{N}) \to K_A(\mathcal{M} \oplus \mathcal{N})/T \cdot K_A(\mathcal{M} \oplus \mathcal{N}) \to 0$$

is a short exact sequence of Hilbert $K_A(\mathcal{M} \oplus \mathcal{N})$-modules. (Here $T$ is identified with the operator $T \oplus 0$ of $\text{End}_A(\mathcal{M} \oplus \mathcal{N})$, and the multiplier C*-algebra of $K_A(\mathcal{M} \oplus \mathcal{N})$ is identified with the set of all adjointable bounded module maps on a copy of itself, cf. [25].) The sequence above would be split, i.e. the Hilbert $K_A(\mathcal{M} \oplus \mathcal{N})$-submodule $T \cdot K_A(\mathcal{M} \oplus \mathcal{N})$ would be a topological summand of the Hilbert $K_A(\mathcal{M} \oplus \mathcal{N})$-module $K_A(\mathcal{M} \oplus \mathcal{N})$, if and only if the operator $T$ would admit a right inverse in the Banach algebra of all bounded module maps on $K_A(\mathcal{M} \oplus \mathcal{N})$, if and only if the norm-closed left ideal $T \cdot K_A(\mathcal{M} \oplus \mathcal{N})$ can alternatively be characterized as an ideal of the form $P \cdot K_A(\mathcal{M} \oplus \mathcal{N})$ for some idempotent bounded module map $P$ on the Hilbert C*-module $\mathcal{M} \oplus \mathcal{N}$, cf. [30]. For the case of adjointable surjective bounded module maps $T$ the situation is well-known: generally speaking, adjointable bounded operators $S$ on Hilbert C*-modules $\mathcal{L}$ have a norm-closed range if and only if they possess a generalized inverse $S^+$ fulfilling $SS^+S = S$, $S^+SS^+ = S^+$, if and only if the right ideal $S \cdot K_A(\mathcal{L})$ is norm-closed, [56, 57]. So a surjective operator $T$ admits a generalized inverse in the C*-algebra of all bounded modular operators on the Hilbert $K_A(\mathcal{M} \oplus \mathcal{N})$-module since the image of $T$, the set $\{0\} \oplus \mathcal{N}$, is obviously an orthogonal summand of the Hilbert C*-module $\mathcal{M} \oplus \mathcal{N}$. For the proofs of these facts and for the ideas on the one-sided multiplier situation see Lun Chuan Zhang’s publications [56, 57].

From this point of view, a better understanding of the properties of non-adjointable surjective bounded modular mappings would give us much more information on the (non-)existence of non-projective Hilbert C*-modules and of non-split short exact sequences of Hilbert C*-modules over certain C*-algebras of coefficients. In this direction research is continuing.

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REFERENCES

[1] W. Arveson, An Invitation to C*-algebras, Springer, New York, 1976.
[2] D. Bakić, B. Guljaš, Extensions of Hilbert C*-modules I, Houston Math. J. 30 (2004), 537-558.
[3] D. Bakić, B. Guljaš, Extensions of Hilbert C*-modules II, Glas. Mat. Ser. III 38(58) (2003), 341-357.
[4] D. Bakić, A class of strictly complete Hilbert C*-modules, manuscript, University of Zagreb, Zagreb, Croatia, 2005.
[5] D. P. Blecher, The standard dual of an operator space, Pacific J. Math. 153 (1992), 15-30.
[6] D. P. Blecher, V. I. Paulsen, Multipliers of operator spaces, and the injective envelope, Pacific J. Math. 200 (2001), 1-17.
[7] H. B. Cohen, Injective envelopes of Banach spaces, Bull. Amer. Math. Soc. 70 (1964), 723-726.
[8] K. R. Davidson, C*-algebras by Example, Fields Institute Monographs v. 6, Amer. Math. Soc., Providence, R.I., 1996.
[9] E. G. Effros, Zhong-Jin Ruan, N. Ozawa, Injectivity and nuclearity of operator spaces, Duke Math. J. 110 (2001), 489-521.
[10] M. Frank, Self-duality and C*-reflexivity of Hilbert C*-modules, Zeitschr. Anal. Anwendungen 9 (1990), 165-176.
[11] M. Frank, Hilbert C*-modules over monotone complete C*-algebras, Math. Nachrichten 175 (1995), 61-83.
[12] M. Frank, Beiträge zur Theorie der Hilbert-C*-Moduln, Habilitation Thesis, (ISBN 3-8265-3217-1, Shaker Verlag, Aachen, 1997), Universität Leipzig, Leipzig, F.R.G., October 1997.
[13] M. Frank, Geometrical aspects of Hilbert C*-modules, Positivity 3 (1999), 215-243.
[14] M. Frank, D. R. Larson, A module frame concept for Hilbert C*-modules, in: Functional and Harmonic Analysis of Wavelets (San Antonio, TX, Jan. 1999), D. R. Larson, L. W. Baggett, eds., AMS, Providence, R.I., Contemp. Math. 247, 207-233, 2000.
[15] M. Frank, Hahn-Banach type theorems for Hilbert C*-modules, Internat. J. Math. 13 (2002), 675-693.
[16] M. Frank, D. R. Larson, Frames in Hilbert C*-modules and C*-algebras, J. Operator Theory 48 (2002), 273-314.
[17] M. Frank, Characterizing C*-algebras of compact operators by generic categorical properties of Hilbert C*-modules, to appear in K-Theory, 2006.
[18] A. Grothendieck, One caractérisation vectorielle-métrique des espaces L1, Canad. J. Math. 7 (1955), 552-561.
[19] D. Hadwin, V. I. Paulsen, Injectivity and projectivity in analysis and topology, preprint math.OA 0706.2995 at www.arxiv.org, June 2007.
[20] M. Hasumi, The extension property of complex Banach spaces, Tōhoku Math. J. (2) 10 (1958), 135-142.
[21] A. Ya. HELEMSKII, On the homological dimensions of normed modules over Banach algebras, Mat. Sb. (N.S.) 81 (123)(1970), 430-444; transl.: Math. USSR Sb. 10(1970), 399-411.

[22] A. Ya. HELEMSKII, Wedderburn-type theorems for operator algebras and modules: traditional and ‘quantized’ homological approaches, Topological Homology, Nova Sci. Publ., Huntington, NY, 2000, 57-92.

[23] K. K. JENSEN, K. THOMSEN, Elements of KK-Theory, (Series: Mathematics: Theory & Applications), Birkhäuser, Boston-Basel-Berlin, 1991.

[24] F. KASCH, Modula and Ringe, B. G. Teubner, Stuttgart, 1977.

[25] G. G. KASPAROV, Hilbert C*-modules: The theorems of Stinespring and Voiculescu, J. Operator Theory 4(1980), 133-150.

[26] J.L. KELLEY, Banach spaces with the extension property, Trans. Amer. Math. Soc. 72(1952), 323-326.

[27] M. KUSUDA, Discrete spectra of C*-algebras and complemented submodules in Hilbert C*-modules, Proc. Amer. Math. Soc. 131(2003), 3075-3081.

[28] M. KUSUDA, Discrete spectra of C*-algebras and orthogonally closed submodules in Hilbert C*-modules, Proc. Amer. Math. Soc. 133(2005), 3341-3344.

[29] E. C. LANCE, Hilbert C*-modules – a Toolkit for Operator Algebraists, London Math. Soc. Lecture Notes Series 210, Cambridge University Press, Cambridge, England, 1995.

[30] Huaxin LIN, Bounded module maps and pure completely positive maps, J. Operator Theory 26(1991), 121-138.

[31] Huaxin LIN, Injective Hilbert C*-modules, Pacific J. Math. 154(1992), 131-164.

[32] Huaxin LIN, Extensions of multipliers and injective Hilbert modules, Chinese Ann. Math. Ser. B 14(1993), 387-396.

[33] S. MACLANE, Categories for the working mathematician, Graduate Texts in Mathematics, Volume 5, Springer-Verlag, New York, 1971.

[34] B. MAGAJNA, Hilbert C*-modules in which all closed submodules are complemented, Proc. Amer. Math. Soc. 125(1997), 849-852.

[35] V. M. MANUILOV, An example of a noncomplemented Hilbert W*-module, Vestn. Moskov. Univ., Ser. I: Mat. Mekh., no. 5, 2000, 58-59, translated: Moscow Univ. Math. Bull. 55, 2000, 38-39.

[36] V. M. MANUILOV, E. V. TROITSKY, Hilbert C*-modules, series: Translations of Mathematical Monographs, v. 226, Amer. Math. Soc., Providence, R.I., 2005.

[37] G. J. MURPHY, C*-algebras and Operator Theory, Academic Press, Boston, 1990.

[38] T. OIKHBERG, Direct sums of operator spaces, J. London Math. Soc. (2) 64(2001), 144-160.

[39] W. L. PASCHKE, Inner product modules over B*-algebras, Trans. Amer. Math. Soc. 182(1973), 443-468.

[40] W. L. PASCHKE, The double B-dual of an inner product module over a C*-algebra B, Canad. J. Math. 26(1974), 1272-1280.

[41] V. I. PAULSEN, Relative Yoneda cohomology for operator spaces, in: Operator Algebras and Applications, Proceedings of the Aegean Conference on Operator Algebras and Applications, Phytagorio, Samos, Greece, Aug. 19-28, 1996, ed.:
A. Katavolos (NATO Advanced Study Institutes Series C: Mathematical and Physical Sciences), Kluwer Academic Publishers, Dordrecht, 1997, 358-393.

[42] V. I. Paulsen, Resolutions of Hilbert modules, Rocky Mountain J. Math. 27 (1997), 271-297.

[43] V. I. Paulsen, Relative Yoneda cohomology for operator spaces, J. Funct. Analysis 157 (1998), 358-393.

[44] D. Hadwin, V. I. Paulsen, Injectivity and projectivity in analysis and topology, preprint math.OA/0706.2995 at www.arxiv.org, June 2007.

[45] G. K. Pedersen, $C^*$-algebras and Their Automorphism Groups, Academic Press, London, 1979.

[46] G. Pisier, Introduction to Operator Space Theory, London Math. Soc. Lecture Note Series v. 294, Cambridge University Press, Cambridge, 2003.

[47] I. Raeburn, D. P. Williams, Morita Equivalence and Continuous Trace $C^*$-algebras, Math. Surveys and Monogr. v. 60, Amer. Math. Soc., Providence, R.I., 1998.

[48] H. Rosenthal, On injective Banach spaces and the spaces $C(S)$, Bull. Amer. Math. Soc. 75 (1969), 824-828.

[49] H. Rosenthal, On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures $\mu$, Acta. Math. 124 (1970), 205-248.

[50] Zhong-Jin Ruan, Injectivity of operator spaces, Trans. Amer. Math. Soc. 315 (1989), 89-104.

[51] J. Schweizer, A description of Hilbert $C^*$-modules in which all closed submodules are orthogonally closed, Proc. Amer. Math. Soc. 127 (1999), 2123-2125.

[52] E. V. Troitsky, Orthogonal complements and endomorphisms of Hilbert modules and $C^*$-elliptical complexes, in: Novikov Conjectures, Index Theorems and Rigiity, v. 2, eds.: S. C. Ferry, A. Ranicki and J. Rosenberg, London Math. Soc. Lecture Note Series 226 (1996), 309-331.

[53] N. E. Wegge-Olsen, K-theory and $C^*$-algebras: a Friendly Approach, Oxford University Press, Oxford, England, 1993.

[54] P. J. Wood, The operator biprojectivity of the Fourier algebra, Canad. J. Math. 54 (2002), 1100-1120.

[55] Zhou Tian Xu, Hilbert $C^*$-modules and $C^*$-algebras, I (Engl./Chin.), Nanjing University Journal, Mathematical Biquarterly 13 (1996), no. 1, 101-108.

[56] Lun Chuan Zhang, The theorem of factor decomposition of certain Hilbert $C^*$-module maps, preprint, Institute of Mathematics, Academia Sinica, Beijing, P. R. China, 2000.

[57] Lun Chuan Zhang, Complemented closed submodules and bounded generalized inverse module maps (Chinese), Math. Practice Theory 34 (2004), no. 2, 143-146.

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