ZETA FUNCTIONS FOR BIVARIATE LAURENT POLYNOMIALS
OVER $p$-ADIC FIELDS

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Abstract. We continue the study of local zeta functions for Laurent polynomials started in [12]. We give an explicit formula for the local zeta function attached to a bivariate Laurent polynomial $f$ over a $p$-adic field, when $f$ is weakly non-degenerate with respect to a Newton polytope associated to $f$.

1. Introduction

Local zeta functions were introduced in the 60’s by Israel Gel’fand and André Weil. Gel’fand study this functions over $\mathbb{R}$ (Archimedean case) with the aim of showing the existence of fundamental solutions to certain partial differential equations with constant coefficients, see [5]. Meanwhile Weil studied $p$–adic local zeta functions in order to generalize some results of Siegel about quadratic forms, e.g. the Siegel-Poisson formula, see [16]. Significant contributions have since been made, specially in the last three decades, see e.g. [4, 6, 7, 13] and the references therein.

One of the most powerful tools in the theory is the resolution of singularities, in particular a lot of results has been obtained for non-degenerate polynomials (more generally analytic functions); see e.g. [14, 5, 11] for the Archimedean case, and [2, 3, 15, 17], among others, in the non-Archimedean case. In [12] we begin the study of local zeta functions attached to Laurent polynomials over $p$–adic fields, we introduced there a non-degeneracy condition with respect to a Newton polytope $\Gamma_\infty$ associated to the Laurent polynomial, then we used that condition to show the existence of a meromorphic continuation of the attached local zeta functions as rational functions of $q^{-s}$. In contrast with classical Igusa’s zeta functions, the meromorphic continuation of zeta functions for Laurent polynomials have poles with positive and negative real parts. We also obtain asymptotic expansions for $p$–adic oscillatory integrals attached to Laurent polynomials and give bounds for the size of ‘tubular neighborhoods’ attached to the polynomials.

The main tool used in [12] for the meromorphic continuation of the local zeta functions is a variation of toric resolution of singularities. In the classical case of polynomials one constructs a toric manifold from a decomposition in simple cones of $\mathbb{R}^n$, called fan, which is subordinated to $\Gamma_\infty$, see [9]. We use this fan to construct a new simple decomposition $\mathcal{F}$ of $\mathbb{R}^n_+$ subordinated to $\Gamma_\infty$ and then proceed to construct a toric manifold and a toric resolution of singularities, see [12, Sections 2–3]. But in general the set of generators of $\mathcal{F}$ may contain extra rays coming from the intersection of $\mathbb{R}^n_+$ with the cones in the original fan, see Remark [11]. This could lead to superfluous candidate poles for the local zeta function.

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In this paper we show that in dimension 2, i.e. the case of bivariate Laurent polynomials, the generators of $F$ are just the normal vectors to $\Gamma_\infty$ contained in $\mathbb{R}^2_+$ and the vectors in the canonical basis of $\mathbb{R}^2$, see Lemma 2. We also construct a conical partition of $\mathbb{R}^2_+$ into open cones and subordinated to $\Gamma_\infty$. Our main result, Theorem 2 employs this partition to give an explicit formula, as in [3], for the local zeta function attached to bivariate Laurent polynomials.

We provide some examples to clarify our approach to the study of this new type of local zeta functions for Laurent polynomials.

2. Newton Polytopes, and Non-degeneracy Conditions

In this section, we review some basic results about Newton polytopes as in [14] and [9].

2.1. Newton Polytopes. We take $\mathbb{R}_+ := \{ x \in \mathbb{R}; x \geq 0 \}$. If $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^2$, we identify the dual space of $\mathbb{R}^2$ with $\mathbb{R}^2$ itself by means of it.

Let $K$ be a local field of characteristic zero. Let $f(x, y) = \sum_{(i,j) \in S} f_{i,j} x^i y^j \in K[x^{\pm 1}, y^{\pm 1}]$, with $S$ a finite subset of $\mathbb{Z}^2$ and $f_{i,j} \in K \setminus \{0\}$ for all $(i, j) \in S$. The set $S$ is called the support of $f$. We define the Newton polytope $\Gamma_\infty(f) := \Gamma_\infty$ of $f$ at infinity as the convex hull of $S$ in $\mathbb{R}^2$. In combinatorics, a set like $\Gamma_\infty$ is typically called a rational (or lattice) polytope (i.e. a compact polyhedron). From now on, we will use just polytope to mean rational polytope and assume that $\dim \Gamma_\infty = 2$.

2.1.1. Faces. Let $H$ be the line $\{ x \in \mathbb{R}^2; \langle a, x \rangle = b \}$. Then $H$ determines two closed half-spaces:

$$H^+ := \{ x \in \mathbb{R}^2; \langle a, x \rangle \geq b \} \quad \text{and} \quad H^- := \{ x \in \mathbb{R}^2; \langle a, x \rangle \leq b \}.$$ 

We say that $H$ is a supporting line of $\Gamma_\infty$, if $\Gamma_\infty \cap H \neq \emptyset$ and $\Gamma_\infty \subset H^+$ or $\Gamma_\infty \subset H^-$. A face of $\Gamma_\infty$ is the intersection of $\Gamma_\infty$ with a supporting line. Faces of $\Gamma_\infty$ can be subdivided according to their dimension as vertices, edges, and $\Gamma_\infty$ itself. We denote by $\text{Vert}(\Gamma_\infty)$ the set of vertices of $\Gamma_\infty$.

Given $a \in \mathbb{R}^2$, we define

$$d(a) = \inf \{ \langle a, x \rangle; x \in \Gamma_\infty \}. \quad (2.1)$$

In fact $d(a) = \min \{ \langle a, x \rangle; x \in \text{Vert}(\Gamma_\infty) \}$, and $d(a) = \langle a, x_0 \rangle$ for some $x_0 \in \text{Vert}(\Gamma_\infty)$.

Now, given a supporting line $H$ of $\Gamma_\infty$ containing an edge of $\Gamma_\infty$, there exists a unique vector $a \in \mathbb{Z}^2 \setminus \{0\}$ perpendicular to $H$ and directed into the polytope. We will call this vector, the inward normal to $H$. A vector $a = (a_1, a_2) \in \mathbb{Z}^2$ is called primitive if $\text{g.c.d.}(a_1, a_2) = 1$. It follows that every edge of $\Gamma_\infty$ has a unique primitive inward vector, the set of such vectors is denoted by $\mathcal{D}(\Gamma_\infty)$. 

2.2. Cones and Fans. An explicit formula for the local zeta function attached to a Laurent polynomial \( f \) requires polyhedral subdivisions of \( \mathbb{R}^2 \setminus \{0\} \) subordinated to \( \Gamma_\infty \). In what follows we give some details about this construction by using ideas of [12] and [3]; there the reader may find the proofs of the results given here.

For \( a_1, \ldots, a_k \in \mathbb{R}^n \) \((k \leq n)\) we call

\[
\Delta = \{ \lambda_1 a_1 + \cdots + \lambda_k a_k; \; \lambda_i \in \mathbb{R}, \lambda_i > 0 \}
\]

the cone strictly spanned by the vectors \( a_1, \ldots, a_k \); when the generators \( a_1, \ldots, a_k \) are linearly independent over \( \mathbb{R} \), the cone is called simplicial, and when \( a_1, \ldots, a_k \in \mathbb{Z}^n \) the cone is called rational. If \( \{a_1, \ldots, a_k\} \) is a subset of a basis of the \( \mathbb{Z} \)-module \( \mathbb{Z}^n \), we call \( \Delta \) a simple cone.

Now, take \( a \in \mathbb{R}^2 \), we define the first meet locus of \( a \) as

\[
F(a) = \{ x \in \Gamma_\infty; \langle a, x \rangle = d(a) \}.
\]

Note that \( F(a) \) is a face of \( \Gamma_\infty \), and that \( F(0) = \Gamma_\infty \). We use this notion to define an equivalence relation on \( \mathbb{R}^2 \) by taking

\[
a \sim a' \iff F(a) = F(a').
\]

In order to describe the equivalence classes of \( \sim \) we define the cone associated to \( \tau \) (a given face of \( \Gamma_\infty \)) as

\[
\Delta_\tau = \{ a \in \mathbb{R}^2; F(a) = \tau \}.
\]

Note that \( \Delta_{\Gamma_\infty} = \{0\} \). The other equivalence classes are described in the next Lemma, which also provides the relation between (2.2) and (2.3).

**Lemma 1.** Let \( \tau \) be a face of \( \Gamma_\infty \), \( \tau \neq \Gamma_\infty \), then

(i) The topological closure \( \overline{\Delta_\tau} \) of \( \Delta_\tau \) is a rational polyhedral cone and

\[
\overline{\Delta_\tau} = \{ a \in \mathbb{R}^2; F(a) \supset \tau \}.
\]

(ii) Let \( \gamma_1, \gamma_l, l = 1 \text{ or } 2 \), be the edges of \( \Gamma_\infty \) containing \( \tau \). Let \( a_1, a_l \in \mathbb{Z}^2 \setminus \{0\} \) be the unique primitive perpendicular inward vectors to \( \gamma_1, \gamma_l \) respectively. Then

\[
\Delta_\tau = \{ \lambda_1 a_1 + \lambda_l a_l; \; \lambda_1, \lambda_l > 0 \}
\]

and

\[
\overline{\Delta_\tau} = \{ \lambda_1 a_1 + \lambda_l a_l; \; \lambda_1, \lambda_l \in \mathbb{R} \text{ and } \lambda_1, \lambda_l \geq 0 \}.
\]

(iii) \( \dim \Delta_\tau = \dim \overline{\Delta_\tau} = 2 - \dim \tau \).

We recall that a fan \( \mathcal{L} \) is a finite collection of rational polyhedral cones \( \{ \Lambda_i; i \in I \} \) in \( \mathbb{R}^n \) such that: (i) if \( \Lambda_i \in \mathcal{L} \) and \( \Lambda \) is a face of \( \Lambda_i \), then \( \Lambda \in \mathcal{L} \); (ii) if \( \Lambda_1, \Lambda_2 \in \mathcal{L} \), then \( \Lambda_1 \cap \Lambda_2 \) is a face of \( \Lambda_1 \) and \( \Lambda_2 \). The support of \( \mathcal{L} \) is \( \cup_{i \in I} \Lambda_i \). A fan \( \mathcal{L} \) is called simplicial (resp. simple) if all its cones are simplicial (resp. simple). A fan \( \mathcal{L} \) is called subordinated to \( \Gamma_\infty \), if every cone in \( \mathcal{L} \) is contained in an equivalence class of \( \sim \). We denote by \( \text{gen}(\mathcal{L}) \), the set of all generators of the cones in \( \mathcal{L} \).

We are ready for the conical subdivision of \( \mathbb{R}^2 \setminus \{0\} \). Set \( \{e_1, e_2\} \) for the canonical basis of \( \mathbb{R}^2 \) and denote by \( \Delta_i \) the cone strictly spanned by \( e_i \); \( e_i \in \{e_1, e_2\} \). Note that if \( \Delta_1 \cap \mathbb{R}^2_+ \neq \emptyset \), then \( \Delta_1 \cap \mathbb{R}^2_+ \) is a cone of form (2.2), therefore we have the following Lemma.

**Lemma 2.** (1) \( \{\Delta_\tau \mid \tau \text{ is a face of } \Gamma_\infty\} := \{\Delta_\tau\} \) is a conical partition of \( \mathbb{R}^2 \) into open cones.
(2) If $\text{Int}(\Delta)$ denotes the topological interior of a cone $\Delta$, then
\[ C := \{ \text{Int}(\Delta_r \cap \mathbb{R}^2_+) \cup \Delta_r \cap \mathbb{R}^2_+ \neq 0 \} \cup \Delta_1 \cup \Delta_2 \]
forms a conical partition of $\mathbb{R}^2_+ \setminus \{0\}$ into open cones.

(3) $\{ \Delta_{\tau} \}$ (resp. $\overline{C} := (\overline{\Delta} \cap \mathbb{R}^2_+) \cup \overline{\Delta_1} \cup \overline{\Delta_2}$) is a fan subordinated to $\Gamma_\infty$ with support $\mathbb{R}^2$ (resp. $\mathbb{R}^2_+$).

Each cone in $C$ can be partitioned into a finite number of simplicial cones. By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones, see e.g. [8]. In this way we may find simple fans containing $\{ \Delta_{\tau} \}$ (resp. $\overline{C}$) and subordinated to $\Gamma_\infty$.

From now on, we fix a simple fan $\mathcal{F}$ subordinated to $\Gamma_\infty$ with support $\mathbb{R}^2_+$. We set $\mathcal{F}_0$ to be the cone $\mathbb{R}^2_+$ and its faces. We will say that $\mathcal{F}$ is trivial if $\mathcal{F} = \mathcal{F}_0$.

**Example 1.** Given arbitrary $n, m \in \mathbb{N}$, set $f(x, y) = (y^{-1} + x)^n + y^m \in K[x^{\pm 1}, y^{\pm 1}]$. Then $\Gamma_\infty$ and the conical partition of $\mathbb{R}^2$ induced by it are depicted in the following pictures.

![Diagram](image)

Note that $C = \mathbb{R}^2_+ \setminus \{0\}$ and $\overline{C}$ is trivial, by introducing the ray $(1, 1) \in \mathbb{R}^2_+$, we get a nontrivial and simple fan $\mathcal{F}$ subordinated to $\Gamma_\infty$ and supported on $\mathbb{R}^2_+$.

A relevant fact when considering local zeta functions of bivariate Laurent polynomials is that we have an explicit description for the generators of the simple fan $\mathcal{F}$ subordinated to $\Gamma_\infty$. Indeed
\[ \text{gen}(\mathcal{F}) = \mathcal{D}(\mathcal{F}) \cup \mathcal{E}(\mathcal{F}) \cup \mathcal{E}'(\mathcal{F}), \]
where $\mathcal{D}(\mathcal{F}) \subseteq \mathcal{D}(\Gamma_\infty)$, $\mathcal{E}(\mathcal{F}) \subseteq \{ e_1, e_2 \}$ and $\mathcal{E}'(\mathcal{F})$ is a finite set of primitive vectors corresponding to the extra rays induced by the subdivision into simple cones.

**Remark 1.** In dimensions greater than two the set $\text{gen}(\mathcal{F})$ may contain additional rays coming from the intersection of $\mathbb{R}^n_+$ with some of the cones in the corresponding simplicial fan subordinated to $\Gamma_\infty$ and supported on $\mathbb{R}^n$, see [12], Remark 3.5]. For example, take $f(x, y, z) = x^{-1} + y^{-1} + z^{-1}(1 + y) + (xz)^{-1}y \in K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. Then the set of inward normal vectors of $\Gamma_\infty$ is $\mathcal{D}(\Gamma_\infty) = \{(1, 1, 1), (0, 0, 1), (0, -1, -1), (-1, -1, -2), (-1, 0, 0)\}$. The cone generated by vectors $(0, 0, 1)$, $(1, 1, 1)$ and $(-1, 0, 0)$ belongs to the original simple fan supported on $\mathbb{R}^3$, but its intersection with $\mathbb{R}^3_+$ is a $3$-dimensional cone generated by vectors $(0, 0, 1), (1, 1, 1)$ and $(0, 1/2, 1/2)$. The last vector is in neither set of (2.4).
2.3. **Non-degeneracy Conditions.** Given \(a \in \mathbb{R}^2_+\), we define the face function of \(f(x, y) = \sum_{(i,j) \in S} f_{i,j} x^i y^j\) with respect to \(a\) as

\[
f_a(x, y) = \sum_{(i,j) \in S \cap F(a, \Gamma_\infty)} f_{i,j} x^i y^j.
\]

We set \(T^2 = T^2(K) := \{(x_1, x_2) \in K^2; x_1 \cdot x_2 \neq 0\}\), for the 2-dimensional torus considered as a \(K\)-analytic manifold.

**Definition 1.** Let \(f(x, y) = \sum_{(i,j) \in S} f_{i,j} x^i y^j \in K[x^{\pm 1}, y^{\pm 1}]\), be a non-constant Laurent polynomial with Newton polytope \(\Gamma_\infty\). We say that \(f\) is non-degenerate with respect to \(a \in \mathbb{R}^2_+, \) if the system of equations

\[
\{f_a(x, y) = 0, \nabla f_a(x, y) = 0\}
\]

has no solutions in \(T^2(K)\). We say that \(f\) is weakly non-degenerate with respect to \(\Gamma_\infty\), if \(f_a\) is non-degenerate with respect to any \(a \in \mathbb{R}^2_+\).

**Remark 2.** By allowing \(a\) to take values on \(\mathbb{R}^2\), we have that the property of being non-degenerate with respect to any \(a \in \mathbb{R}^2\) is precisely the standard non-degeneracy condition of Khovanskii, see \([10]\).

**Example 2.** Let \(f(x, y)\) as in Example \([3]\). Then \(f\) is non-degenerate with respect to \(\Gamma_\infty\) but \(f\) is weakly non-degenerate with respect to \(\Gamma_\infty\).

3. **Local Zeta Functions for Laurent Polynomials**

In this section we review the definition of the local zeta function attached to a Laurent polynomial in two variables, in the sense of \([12]\). Let \(K\) be a \(p\)-adic field, i.e. \([K : \mathbb{Q}_p] < \infty\), where \(\mathbb{Q}_p\) denotes the field of \(p\)-adic numbers. Let \(R_K\) be the valuation ring of \(K\), \(P_K\) the maximal ideal of \(R_K\), and \(\overline{K} = R_K/P_K\) the residue field of \(K\). The cardinality of the residue field of \(K\) is denoted by \(q\), thus \(\overline{K} = \mathbb{F}_q\).

For \(z \in K\), \(\text{ord}(z) \in \mathbb{Z} \cup \{+\infty\}\) denotes the valuation of \(z\), and \(|z|_K = q^{-\text{ord}(z)}\).

Take \(p\) a fixed uniformizing parameter of \(R_K\).

We equip \(K^2\) with the norm \(\| (x, y) \|_K := \max \{|x|_K, |y|_K\}\). Then \((K^2, \| \cdot \|_K)\) is a complete metric space and the metric topology is equal to the product topology.

**Definition 2.** Given \(f\) a Laurent polynomial in two variables as before, the local zeta function attached to \(f\) is given by:

\[
Z(s, f) = \int_{T^2(K)} |f(x, y)|_K^s \, |dx dy|,
\]

where \(|dx dy|\) is the normalized Haar measure of \(K^2\), which is the measure induced by a 2-degree differential form \(dx \wedge dy\).

The convergence of the integral in \((3.1)\) is not a straightforward matter, this is an important difference with the classical case, see \([12]\). We introduce some additional notation in order to clarify this subject.

For \(a = (a_1, a_2) \in \mathbb{Z}^2 \setminus \{0\}\), set \(\|a\| = a_1 + a_2\) and let \(d(a)\) as before. Then

\[
\mathcal{P}(a) = \begin{cases} \left\{-\frac{\|a\|}{d(a)} + \frac{2\pi \sqrt{-1}}{d(a) \ln q}\right\} & \text{if } d(a) \neq 0, \\ \emptyset & \text{if } d(a) = 0. \end{cases}
\]
Let \( \mathcal{F} \) be the fixed simple fan subordinated to \( \Gamma_\infty \) and supported in \( \mathbb{R}_2^+ \) as before. Denote by \( \text{gen}(\mathcal{F}) \), the set of all generators of the cones in \( \mathcal{F} \) as before. Set
\[
A(\mathcal{F}) := \bigcup_{a \in \text{gen}(\mathcal{F}), \, d(a) \neq 0} \left\{ \frac{\|a\|}{-d(a)} ; \, d(a) < 0 \right\}, \quad B(\mathcal{F}) := \bigcup_{a \in \text{gen}(\mathcal{F}), \, d(a) \neq 0} \left\{ \frac{\|a\|}{-d(a)} ; \, d(a) > 0 \right\},
\]
\[
\alpha := \alpha(\mathcal{F}) = \begin{cases} \min_{\gamma \in A(\mathcal{F})} \gamma & \text{if } A(\mathcal{F}) \neq \emptyset \\ +\infty & \text{if } A(\mathcal{F}) = \emptyset \end{cases}
\]
and \( \beta := \beta(\mathcal{F}) = \max_{\gamma \in B(\mathcal{F}) \cup \{-1\}} \gamma \).

The next Theorem contains the main properties of \( Z(s, f) \).

**Theorem 1** ([12 Theorem 3.3]). Let \( f \) be a weakly non-degenerate Laurent polynomial with respect to \( \Gamma_\infty \), and let \( \mathcal{F} \) be a fixed simple and non-trivial fan subordinated to \( \Gamma_\infty \). Then the following assertions hold:

(i) \( Z(s, f) \) converges for \( \Re(s) \in (\beta, \alpha) \);

(ii) \( Z(s, f) \) has a meromorphic continuation to the whole complex plane as a rational function of \( q^{-s} \), and the poles belong to the set
\[
\bigcup_{a \in \text{gen}(\mathcal{F}), \, d(a) \neq 0} \mathcal{P}(a) \cup \left\{ -1 + \frac{2\pi \sqrt{-1}}{\ln q} Z \right\}.
\]

In addition, the multiplicity of any pole is \( \leq 2 \).

Proof of Theorem 1 given in [12] is based on a variation of toric resolution of singularities, we refer the reader there for further details.

**Remark 3.** Due to formula \( 2.3 \) we have that the set of candidate poles for \( Z(s, f) \) can be read off immediately from the geometry of \( \Gamma_\infty \).

**Example 3.** Take \( f(x, y) = (y^{-1} + x)^n + y^n \in K[x^{\pm 1}, y^{\pm 1}] \), for arbitrary \( n, m \in \mathbb{N} \) as in Example 1. Assume that the coefficients of \( f \) are in \( (pR_K)^2 \), then
\[
Z(s, f) = \int_{(pR_K \setminus \{0\})^2} |f(x, y)|_K^s \, |dx\,dy| = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \int_{p^aR_K^* \times p^bR_K^*} |f(x, y)|_K^s \, |dx\,dy|
\]
\[
= \left( 1 - q^{-1} \right) q^{-2+s} \frac{1 - q^{-1+ns}}{1 - q^{-1+ns}}.
\]

Note that the integral converges for \( \Re(s) < \frac{1}{n} \). Thus local zeta functions may have poles with positive real parts.

**Remark 4.** Furthermore, if \( f(x, y) \) is a weakly non-degenerate Laurent polynomial with coefficients in \( R_K^* \) and \( \mathcal{F} \) is the trivial fan, then the first meet locus of any integer vector in \( \mathbb{R}_2^+ \) is a point, say \( m = (m_1, m_2) \). Thus
\[
Z(s, f) = \sum_{(a_1, a_2) \in \mathbb{N}^2} \int_{p^{a_1}R_K^* \times p^{a_2}R_K^*} |f(x, y)|_K^s \, |dx\,dy|
\]
\[
= \left( 1 - q^{-1} \right)^2 \sum_{(a_1, a_2) \in \mathbb{N}^2} q^{-\|a\| - <a, m> s} = \frac{(1 - q^{-1})^2}{(1 - q^{-1-m_1}s)(1 - q^{-1-m_2}s)}.
\]
4. Explicit Formulas

Let $L \subseteq \mathbb{C}$ be a number field, and let $A_L$ be its ring of integers. Take a maximal ideal $\mathfrak{p}$ of $A_L$, and denote by $K := K_{\mathfrak{p}}$ the completion of $L$ with respect to the $\mathfrak{p}$-adic valuation. Denote by $\mathbb{F}_q$ the residue field of $\mathfrak{p}$. We will use all the notation for $p$-adic fields introduced before.

Let $f(x, y) \in A_L[x^{\pm 1}, y^{\pm 1}]$ be a non-constant Laurent polynomial, which is non-degenerate with respect to $\Gamma_\infty$ over $\mathbb{C}$. Then, by the Weak Nullstellensatz, for almost all $\mathfrak{p}$, i.e. for $q$ big enough, the system of equations

\begin{equation}
\{ f_a (x, y) = 0, \ \nabla f_a (x, y) = 0 \}
\end{equation}

has no solutions in $(R_K^\mathfrak{p})^2$, for any $a$ in $\mathbb{R}_+^2$. If we denote by $\overline{f}$ the reduction modulo $\mathfrak{p}$ of $f$, by Hensel’s Lemma, condition (4.1) is equivalent to the following: the system of equations

\begin{equation}
\{ \overline{f}_a (x, y) = 0, \ \nabla \overline{f}_a (x, y) = 0 \}
\end{equation}

has no solutions in $(\mathbb{F}_q^\times)^2$, for any $a$ in $\mathbb{R}_+^2$. In this case, we will say that $f$ (or that $\overline{f}$) is weakly non-degenerate with respect to $\Gamma_\infty$ over $\mathbb{F}_q$.

**Lemma 3.** Let $f$ be a weakly non-degenerate Laurent polynomial with respect to $\Gamma_\infty$ over $\mathbb{F}_q$, such that $f_a(x, y) = \frac{\hat{f}_a(x, y)}{a^{n-m}}$, with $\hat{f}_a(x, y) \in K[x, y]$ and $n, m \in \mathbb{N}$. For any $a$ in $\mathbb{R}_+^2$, the two following assertions are equivalent:

1. the system of equations $\{ f_a (x, y) = 0, \nabla f_a (x, y) = 0 \}$ has no solutions in $(R_K^\mathfrak{p})^2$;
2. the system of equations $\{ \hat{f}_a (x, y) = 0, \nabla \hat{f}_a (x, y) = 0 \}$ has no solutions in $(R_K^\mathfrak{p})^2$.

**Proof.** The equivalence follows from the following calculation:

\[
\frac{\partial \hat{f}_a}{\partial x} (x, y) = \begin{cases} y^m \frac{\partial \hat{f}_a}{\partial x} (x, y), & \text{if } n = 0, \\ x^n y^m \frac{\partial \hat{f}_a}{\partial x} (x, y) + n x^{n-1} y^m f_a(x, y), & \text{if } n \geq 1. \end{cases}
\]

Since $q$ is big enough, we have $\pi \neq 0$, for any $n$. A similar equation holds for $\frac{\partial \hat{f}_a}{\partial y} (x, y)$.

**Remark 5.** Let $g \in R_K[x^{\pm 1}, y^{\pm 1}]$ be a non-constant Laurent polynomial satisfying (i) $\overline{g} \in \mathbb{F}_q[x^{\pm 1}, y^{\pm 1}] \setminus \mathbb{F}_q$, and (ii) the system of equations

\[ \{ \overline{g} (x, y) = 0, \nabla \overline{g} (x, y) = 0 \} \]

has no solutions in $(\mathbb{F}_q^\times)^2$. Then

\[
\int_{(R_K^\mathfrak{p})^2} |g(x, y)|_K^s \ |dx dy| = q^{-2} \left\{ (q - 1)^2 + N \left( \frac{q^s - 1}{1 - q^{s-1}} \right) \right\}, \ \text{for } \text{Re}(s) > 0,
\]

where $N := \text{Card} \left\{ (x, y) \in (\mathbb{F}_q^\times)^2 : \overline{g} (x, y) = 0 \right\}$. The formula follows from Lemma 3 by using the Stationary Phase Formula, see [7] Theorem 10.2.1.
Lemma 4. Let \( f \) be a weakly non-degenerate Laurent polynomial with respect to \( \Gamma_\infty \) over \( \mathbb{F}_q \). For \( a \) in \( \mathbb{R}_+^2 \), set
\[
g(x, y; a) := f_a(x, y) + \sum_{(i, j) \in S \cap p(x, y)} \left( p_i^{(x, y)} f_{i, j} x^i y^j = f_a(x, y) + p^{(x, y)} f_{i, j} x^i y^j,\right.
\]
where \( p' > 0 \), and \( l_{i, j} \geq 0 \) for every \( (i, j) \). Then
\[
\int \frac{|g(x, y; a)|_K^s \, dx \, dy}{(R_\infty^s)^2} = q^{-2} \left\{ (q - 1)^2 + N_a \left( \frac{q^{-s} - 1}{1 - q^{-s} - 1} \right) \right\}, \text{ for } \Re(s) > 0,
\]
where \( N_a := \Card \left\{ (x, y) \in (\mathbb{F}_q^s)^2 : f_a(x, y) = 0 \right\}. \)

Proof. Since \( g = \bar{f}_a \), we have \( \{g(x, y) = 0, \nabla g(x, y) = 0\} = \{f_a(x, y) = 0, \nabla f_a(x, y) = 0\}. \) From the non-degeneracy condition, the later system of equations has no solutions in \( (\mathbb{F}_q^s)^2 \), therefore, the hypotheses of Remark 5 hold and
\[
\int \frac{|g(x, y; a)|_K^s \, dx \, dy}{(R_\infty^s)^2} = q^{-2} \left\{ (q - 1)^2 + N \left( \frac{q^{-s} - 1}{1 - q^{-s} - 1} \right) \right\}, \text{ for } \Re(s) > 0,
\]
with \( N = N_a. \)

We are ready for establish our main result. Set
\[
Z_0(s, f) := \int \frac{|f(x, y)|_K^s \, dx \, dy}{(R_\infty^s(0))^2} \text{ for } \beta < \Re(s) < \alpha.
\]

Theorem 2. Let \( f \) be a Laurent polynomial in two variables which is weakly non-degenerate with respect to \( \Gamma_\infty \) over \( \mathbb{F}_q \). Let \( C := \{\text{Int}(\Delta_\infty \cap \mathbb{R}_+^2) ; \Delta_\infty \cap \mathbb{R}_+^2 \neq \emptyset\} \cup \Delta_1 \cup \Delta_2 \), be the simplicial partition of \( \mathbb{R}_+^2 \) \( \setminus \{0\} \) subordinated to \( \Gamma_\infty \). Denote for each \( \Delta \in C \)
\[
N_\Delta = \left\{ (x, y) \in (\mathbb{F}_q^s)^2 : f_a(x, y) = 0, \forall a \in \Delta \right\}.
\]
In addition for \( \Delta \in C \), denote by \( a_1, a_2 \), \( (l = 1 \text{ or } 2) \) the generators of the cone \( \Delta \). Then
\[
Z_0(s, f) = L_0(q^{-s}) + \sum_{\Delta \in C} L_\Delta (q^{-s}) S_\Delta (q^{-s}).
\]
Here
\[
L_\Delta (q^{-s}) = q^{-2} \left( (q - 1)^2 - \frac{N_\Delta (1 - q^{-s})}{1 - q^{-s} - 1} \right),
\]
for \( \Delta \in C \), and
\[
S_\Delta (q^{-s}) = \frac{\left( \sum_{h \in \mathbb{Z}} h\|h\| + d(h)^s \right) q^{-\sum_{j=1}^l (\|a_j\| + d(a_j)^s)}}{\prod_{j=1}^l (1 - q^{-\|a_j\| - d(a_j)^s})},
\]
where \( h \) runs through the elements of the set
\[
\mathbb{Z}^2 \cap \left\{ \sum_{j=1}^l \lambda_j a_j : 0 \leq \lambda_j < 1 \text{ for } j = 1, l \right\}.
\]
Proof. First note that
\[ \mathbb{R}_+^2 = \{0\} \cup \bigcup \Delta. \]

Then
\[ Z_0(s, f) = \sum_{(m,n) \in \mathbb{N}^2} \int_{(x,y) \in (R_K \setminus \{0\})^2 \atop \text{ord}(x) = m, \text{ord}(y) = n} |f(x,y)|_K^s |dx dy| \]
\[ = \int_{(R_K^\times)^2} |f(x,y)|_K^s |dx dy| + \sum_{\Delta \in \mathcal{C}} \sum_{(m,n) \in \Delta \cap \mathbb{N}^2} \int_{(x,y) \in (R_K^\times)^2 \atop \text{ord}(x) = m, \text{ord}(y) = n} |f(x,y)|_K^s |dx dy|. \]

Now we change variables as
\[ (x = p^m u, y = p^n v) \text{ with } (u,v) \in (R_K^\times)^2. \]

If we fix \( \Delta \in \mathcal{C} \), then for any \( a \in \mathcal{C} \) we have
\[ f(x,y) = f_a(x,y) + \sum_{(i,j) \in S \setminus F(a,\Gamma_\infty)} f_{i,j} x^i y^j, \]

and then \( ((m,n),(i,j)) = d(m,n) \) for any \( (i,j) \in F(a,\Gamma_\infty) \) whereas \( ((m,n),(i,j)) > d(m,n) \) for \( (i,j) \in S \setminus F(a,\Gamma_\infty) \). It follows from (4.3) and (4.4) that
\[ f(x,y) = p^{d(m,n)} f_a(u,v) + p^{d(m,n)+l'} f_{a,(m,n)}(u,v), \]

where \( l' > 0 \) and \( f_{a,(m,n)}(u,v) \in R_K^\times [u,v] \). Therefore
\[ Z_0(s, f) = L_0(q^{-s}) + \sum_{\Delta \in \mathcal{C}} \sum_{(m,n) \in \Delta \cap \mathbb{N}^2} q^{-||(m,n)||-d(m,n)s} \]
\[ \times \sum_a \int_{(R_K^\times)^2} |f_a(u,v) + p^{l'} f_{a,(m,n)}(u,v)|_K^s |du dv| \]
\[ = L_0(q^{-s}) + \sum_{\Delta \in \mathcal{C}} L_\Delta(q^{-s}) \sum_{(m,n) \in \Delta \cap \mathbb{N}^2} q^{-||(m,n)||-d(m,n)s}, \]

by Lemma (4).

It remains to show the formula for \( S_\Delta \). Note that in general \( \Delta \) may not be simple, so
\[ \Delta \cap \mathbb{N}^2 = \cup_h (h + \mathbb{N} a_1 + \mathbb{N} a_l), \text{ for } h \in \mathbb{Z}^2 \cap \left\{ \sum_{j=1}^l t_j a_j; 0 \leq \lambda_j < 1 \text{ for } j = 1, l \right\}. \]
Then
\[ S_\Delta(q^{-s}) = \sum_{(m,n) \in \Delta \cap \mathbb{N}^2} q^{-||(m,n)||-d(m,n)s} = \sum_{(m,n) \in (h+N_{a_1} + N_{a_1})} q^{-||(m,n)||-d(m,n)s} \]
\[ = \left( \sum_{h} q^{-||h||-d(h)s} \right) \left( \sum_{\lambda_1 \in \mathbb{N}} q^{-||a_1||-d(a_1)s}\lambda_1 \right) \left( \sum_{\lambda_i \in \mathbb{N}} q^{-||a_i||-d(a_i)s}\lambda_i \right) \]
\[ = \frac{\left( \sum_{h} q^{||h||+d(h)s} \right) q^{-\sum_{j=1}^l (||a_j||+d(a_j)s)}}{\prod_{j=1}^l (1 - q^{-||a_j||-d(a_j)s})}. \]

By using a simple polyhedral subdivision one obtains a slightly less complicated explicit formula in which all the terms \( \sum_h q^{||h||+d(h)s} \) are identically 1. But then in general we have to introduce new rays which give rise to superfluous candidate poles.

5. Examples

Example 4. Set \( g(x, y) = x^{-3} + y^{-2} + y^4 \in \mathbb{Z}[x, y] \). This polynomial is non-degenerate with respect to \( \Gamma_\infty(g) \) over \( \mathbb{F}_q \), for \( q \) big enough. The Newton polytope \( \Gamma_\infty(g) \) and the conical partition of \( \mathbb{R}^2 \) induced by it are as follows.

Vectors \( \{(1,0), (2,3), (1,0)\} \) are the edges of a non-trivial simplicial polyhedral subdivision of \( \mathbb{R}_+^2 \) subordinated to \( \Gamma_\infty(g) \). The data required in the explicit formula for \( Z_0(s, g) \) are the following:

\[ L_0(q^{-s}) = q^{-2} \left\{(q - 1)^2 + N_0 \left( \frac{q^{-s} - 1}{1 - q^{-s-1}} \right) \right\}, \]

with \( N_0 = \left\{(x, y) \in (\mathbb{F}_q^\times)^2 ; x^{-3} + y^{-2} + y^4 = 0 \right\} \), and
where \( N_\gamma = \text{Card} \left\{ (x, y) \in (\mathbb{F}_q^*)^2; x^3 + y^2 = 0 \right\} \).

Therefore the real parts of the poles of \( Z_0(s, g) \) belong to \( \left\{ \frac{1}{3}, \frac{1}{3}, \frac{5}{6}, -1 \right\} \). Note that only the poles \( \frac{1}{3} + \frac{2\sqrt{2}}{12 + g^2} \) come from the equation of a supporting plane, more precisely from the face \( \gamma \). Finally, note that \( \beta = -1 \) and \( \alpha = \frac{1}{3} \), and that this last datum does not come from the equations of the supporting lines of \( \Gamma_\infty(g) \).

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