Extremal decomposition problems in the Euclidean space

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Abstract

Composition principles for reduced moduli are extended to the case of domains in the $n$-dimensional Euclidean space, $n > 2$. As a consequence analogues of extremal decomposition theorems of Kufarev, Dubinin and Kirillova in the planer case are obtained.

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1 Introduction and notations

Extremal decomposition problems have a rich history and go back to M.A. Lavrentiev’s inequality for the product of conformal radii of nonoverlapping domains. There exist two methods of their study: the extremal-metric method and the capacitive method. The first one has been systematically developed in papers by G.V. Kuz’mina, E.G. Emel’yanov, A.Yu. Solynin, A. Vasil’ev, and Ch. Pommerenke \cite{9 14 6 11}. The second approach is developed mainly in works of V.N. Dubinin and his students \cite{4 5 2 3}. In particular, a series of well-known results about extremal decomposition follows one way from composition principles for generalized reduced moduli (see \cite{1} p. 56 and \cite{12}). In the present paper we extend the mentioned composition principles to the case of spatial domains. As a consequence we get theorem about extremal decomposition for the harmonic radius \cite{7} obtained earlier in \cite{5}.

Throughout the paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space consisting of points $x = (x_1, \ldots, x_n)$, $n \geq 3$, and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the length of a vector $x \in \mathbb{R}^n$. We introduce the following notations:

\begin{align*}
B(a, r) & = \{x \in \mathbb{R}^n : |a - x| < r\}, \\
S(a, r) & = \{x \in \mathbb{R}^n : |a - x| = r\}, \quad a \in \mathbb{R}^n;
\end{align*}

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\[ \omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2) \] is the area of the unit sphere \( S(0, 1) \);
\[ \lambda_n = ((n-2)\omega_{n-1})^{-1}. \]

\( D \) is a bounded domain in \( \mathbb{R}^n \), \( \Gamma \) is a closed subset of \( \partial D \). The pair \((D, \Gamma)\) is admissible if there exists the Robin function, \( g_\Gamma(z, z_0, D) \) harmonic in \( D \setminus \{z_0\} \), continuous in \( \overline{D} \setminus \{z_0\} \) and

\[
\frac{\partial g_\Gamma}{\partial n} = 0 \text{ on } (\partial D) \setminus \Gamma, \tag{1}
\]
\[
g_\Gamma = 0 \text{ on } \Gamma, \tag{2}
\]

and in a neighborhood of \( z_0 \) there is an expansion

\[
g_\Gamma(z, z_0, D) = \lambda_n \left( |z - z_0|^{2-n} - r(D, z_0, \Gamma)^{2-n} + o(1) \right), \quad z \to z_0, \tag{3}
\]

where \( \partial/\partial n \) means the inward normal derivative on the boundary. In what follows all such pairs are assumed to be admissible.

In the case \( \Gamma = \emptyset \) we change the condition (1) by the condition

\[
\frac{\partial g_\Gamma}{\partial n} = \frac{1}{\mu_{n-1}(\partial D)} \text{ on } \partial D,
\]

where \( \mu_{n-1}(\partial D) \) is the area of boundary.

By analogy with the definition of the Robin radius for plain domains from the paper [3] we will call the constant \( r(D, z_0, \Gamma) \) the Robin radius of the domain \( D \) and the set \( \Gamma \). Note that in the case of \( \Gamma = \partial D \) we get the harmonic radius [7, 10, 5].

Let \( \Delta = \{\delta_k\}_k^m \) be a collection of real numbers and \( Z = \{z_k\}_k^m \) be points of the domain \( D \). For \( \Gamma = \emptyset \) we additionally require

\[
\sum_{k=1}^m \delta_k = 0.
\]

Define the potential function for the domain \( D \), the set \( \Gamma \), the collection of points \( Z \), and numbers \( \Delta \):

\[
u(z) = u(z; Z, D, \Gamma, \Delta) = \sum_{k=1}^m \delta_k g_\Gamma(z, z_k, D).
\]

Note that for \( \Gamma = \emptyset \) the function \( g_\Gamma(z, z_k, D) \) is defined up to an additive constant. Nevertheless, the function \( u(z) \) is defined uniquely and characterized by the condition

\[
\frac{\partial u}{\partial n} = 0 \text{ on } \partial D.
\]
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It is clear from the definition of the potential function that in a neighborhood of $z_k$ we have

$$u(z) = \delta_k \lambda_n |z - z_k|^{2-n} + a_k + o(1), \ k = 1, \ldots, m,$$

where

$$a_k = -\delta_k \lambda_n r(D, z_k, \Gamma)^{2-n} + \sum_{l \neq k} \delta_l g_\Gamma(z_l, z_k, D).$$

Now if we introduce the following notation

$$g_\Gamma(z_k, z, D) = -\lambda_n r(D, z_k, \Gamma)^{2-n},$$

then the constant in the expansion of the potential function in a neighborhood of $z_k$ is

$$a_k = \sum_{l = 1}^{m} \delta_l g_\Gamma(z_l, z_k, D). \ (4)$$

A function $v(z)$ is admissible for $D, Z, \Delta, \text{and} \Gamma$ if $v(z) \in \text{Lip}$ in a neighborhood of each point of $D$ except maybe finitely many such points, continuous in $D \setminus \bigcup_{k=1}^{m} \{z_k\}$, $v(z) = 0$ on $\Gamma$, and in neighborhood of $z_k$ there is an expansion

$$v(z) = \delta_k \lambda_n |z - z_k|^{2-n} + b_k + o(1), \ z \to z_k. \ (5)$$

The Dirichlet integral is the following

$$I(f, D) = \int_D |\nabla f|^2 \, d\mu,$$

where $d\mu = dx_1 \ldots dx_n$.

2 Main results

Lemma 2.1 The asymptotic formula

$$I(u, D_r) = \left( \sum_{k=1}^{m} \delta_k^2 \right) \lambda_n r^{2-n} + \sum_{k=1}^{m} \delta_k a_k + o(1), \ r \to 0,$$

is true, where $u$ is the potential function and $a_k, \ k = 1, \ldots, m$ are defined in $H$ and $D_r = D \setminus \bigcup_{k=1}^{m} B(z_k, r)$.

Proof. The Green’s identity

$$\int_V |\nabla u|^2 \, d\mu = -\int_{\partial V} u \frac{\partial u}{\partial n} \, ds.$$
gives
\[ I(u, D_r) = -\int_{\partial D_r} u \frac{\partial u}{\partial n} ds = -\sum_{k=1}^{m} \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds. \] \tag{6}

The second equality in (6) holds because \( u \frac{\partial u}{\partial n} = 0 \) on \( \partial D \). Note that
\[ u = \delta_k \lambda_n r^{2-n} + a_k + o(1), \quad z \to z_k \]
in a neighbourhood of \( z_k \).

We calculate the integral \( \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds \). Let \( u(z) = h(z) + g(z) \), where \( h(z) = \lambda_n \delta_k |z - z_k|^{2-n} \) and \( g(z) \) is a harmonic function. Note that \( g(z_k) = a_k \).

For \( |z - z_k| = r \) we have the following correlations
\[ r^{n-1} u \frac{\partial u}{\partial n} = r^{n-1} \left( h \frac{\partial h}{\partial n} + h \frac{\partial g}{\partial n} + g \frac{\partial h}{\partial n} + g \frac{\partial g}{\partial n} \right) \]
\[ = (2 - n) \lambda_n \delta_k r^{2-n} + r \lambda_n \delta_k \frac{\partial g}{\partial n} + (2 - n) g \lambda_n \delta_k + g \frac{\partial g}{\partial n} r^{n-1} \]
\[ = -\frac{\lambda_n \delta_k^2}{\omega_{n-1}} r^{2-n} - g(z_k) \delta_k r^{n-1} + o(1), r \to 0. \]

Therefore
\[ \int_{S(z_k, r)} u \frac{\partial u}{\partial n} ds = \int_{S(0, 1)} u \frac{\partial u}{\partial n} r^{n-1} ds \]
\[ = -\lambda_n \delta_k^2 r^{2-n} - \delta_k a_k + o(1), \quad r \to 0. \]

Substituting it in (6) we get the lemma. \( \square \)

**Lemma 2.2** For an admissible function \( v \) and the potential function \( u \) we have
\[ I(v - u, D_r) = I(v, D_r) - I(u, D_r) - 2 \sum_{k=1}^{m} \delta_k (b_k - a_k) + o(1), \quad r \to 0. \]

**Proof.** One may observe that
\[
I(v - u, D_r) = \int_{D_r} (|\nabla v|^2 + |\nabla u|^2 - 2\nabla u \cdot \nabla v) \, d\mu \\
= \int_{D_r} (|\nabla v|^2 - |\nabla u|^2) \, d\mu + 2 \int_{\partial D_r} (v - u) \frac{\partial u}{\partial n} \, ds \\
= I(v, D_r) - I(u, D_r) + 2 \sum_{k=1}^{m} \int_{S(z_k, r)} (v - u) \frac{\partial u}{\partial n} \, ds \\
= I(v, D_r) - I(u, D_r) - 2 \sum_{k=1}^{m} \delta_k (b_k - a_k) + o(1), \quad r \to 0.
\]
Here we calculated the integral \( \int_{S(\mathbf{z}, r)} (v - u) \frac{\partial u}{\partial n} \) a similar way as in the proof of lemma 2.1 and used the Green’s identity
\[
\int_{D_r} (\nabla u \cdot \nabla v) d\mu = - \int_{\partial D_r} v \frac{\partial u}{\partial n} ds,
\]
where \( n \) is the inner normal vector. □

The quantity
\[
\sum_{k=1}^{m} \delta_k a_k = \sum_{k=1}^{m} \sum_{l=1}^{m} \delta_k \delta_l g_{\Gamma}(z_l, z_k, D)
\]
we call the reduced modulus and denote it by \( M(D, \Gamma, Z, \Delta) \). According to lemma 2.1
\[
M(D, \Gamma, Z, \Delta) = \lim_{r \to 0} \left( I(u, D_r) - \left( \sum_{k=1}^{n} \delta_k^2 \right) \lambda_n r^{2-n} \right).
\]

**Theorem 2.3** Let sets \( D, \Gamma, \) collections \( Z = \{z_k\}_{k=1}^m, \Delta = \{\delta_k\}_{k=1}^m, \) be as in the definition of the reduced modulus \( M = M(D, \Gamma, Z, \Delta) \), \( u(z) \) be the potential function for \( D, \Gamma, Z, \Delta \), and let \( D_i \subset D \) be pairwise non-overlapping subdomains of \( D, \Gamma_i, Z_i = \{z_{ij}\}_{j=1}^{n_i}, \Delta_i = \{\delta_{ij}\}_{j=1}^{n_i}, \) be from the definition of the reduced moduli \( M_i = M(B_i, \Gamma_i, Z_i, \Delta_i) \), \( u_i(z) \) be the potential function for \( D_i, \Gamma_i, Z_i, \Delta_i, i = 1, ..., p \). Assume that the following conditions are fulfilled:
1) \((D \cap \partial D_i) \subset \Gamma_i, i = 1, ..., p;\)
2) \( \Gamma \subset (\bigcup_{i=1}^{p} \Gamma_i) \bigcup \left[ \mathbb{R}^n \setminus \left( \bigcup_{i=1}^{p} \overline{D_i} \right) \right]; \)
3) \( Z = \bigcup_{i=1}^{p} Z_i, \) that is each point \( z_k \in Z \) coincides with some point \( z_{ij} \in Z_i \) for \( k = k(i, j) \) and vice versa;
4) \( \delta_k = \delta_{ij} \) for \( k = k(i, j) \).
Then the inequality
\[
M \geq \sum_{i=1}^{p} M_i + \sum_{i=1}^{p} I(u - u_i, D_i) \geq \sum_{i=1}^{p} M_i
\]
holds.

**Proof.** Consider the function
\[
v(z) = \begin{cases} u_i(z), & z \in D_i, \\ 0, & z \in D \setminus (\bigcup_{i=1}^{p} D_i). \end{cases}
\]
The condition 1) guarantees that the function \( v(z) \) is continuous in \( \overline{D} \setminus \bigcup_{k=1}^{m} \{z_k\} \). From the conditions 2) and 3) it follows that \( v(z) = 0 \) for \( z \in \Gamma \) and in a neighbourhood of \( z_k, k = 1, ..., m, \) there is the expansion (5). Applying lemma 2.2 we get
\[
I(v - u, D) = I(v, D_r) - I(u, D_r) - 2 \sum_{k=1}^{p} \delta_k (b_k - a_k) + o(1), \quad r \to 0,
\]
(7)
here $a_k$ and $b_k$ from (4) and (5) respectively. By lemma 2.1

$$I(v, D_r) = \lambda_n r^{2-n} \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij}^2 + \sum_{i=1}^{p} M_i + o(1) = \lambda_n r^{2-n} \sum_{k=1}^{m} \delta_k^2 + \sum_{i=1}^{p} M_i + o(1),$$

$$I(u, D_r) = \lambda_n r^{2-n} \sum_{k=1}^{m} \delta_k^2 + M + o(1), \quad r \to 0,$$

taking into account 3), we have

$$\sum_{k=1}^{m} \delta_k (b_k - a_k) = \sum_{i=1}^{p} M_i - M.$$

Substituting the obtained correlations in (7), we see that the inequality

$$\sum_{i=1}^{p} I(u - u_i, D_i) \leq I(v - u, D) = M - \sum_{i=1}^{p} M_i + o(1), \quad r \to 0,$$

is true. Theorem is proved. □

**Theorem 2.4** Let sets $D, \Gamma$, collections $Z = \{z_k\}_{k=1}^{m}$, $\Delta = \{\delta_k\}_{k=1}^{m}$, be as in the definition of the reduced modulus $M := M(D, \Gamma, Z, \Delta)$, $u(z)$ be the potential function for $D, \Gamma, Z, \Delta$, and let $D_i \subset D, i = 1, \ldots, p$, be pairwise non-overlapping domains, $\Gamma_i, Z_i = \{z_{ij}\}_{j=1}^{n_i}, \Delta_i = \{\delta_{ij}\}_{j=1}^{n_i}$, be from the definition of the reduced moduli $M_i = M(D_i, \Gamma_i, Z_i, \Delta_i)$, $u_i(z)$ be the potential function for $D_i, \Gamma_i, Z_i, \Delta_i, i = 1, \ldots, p$. Assume that $\Gamma_i \subset \Gamma, i = 1, \ldots, p$, $Z = \bigcup_{i=1}^{m} Z_i$, (that is each point $z_k \in Z$ coincides with some point $z_{ij} \in Z_i$ for $k = k(i, j)$ and vice versa), $\delta_k = \delta_{ij}$. Then the inequality

$$\sum_{i=1}^{p} M_i \geq M + \sum_{i=1}^{p} I(u - u_i, D_i) \geq M$$

holds.

**Proof.** The function $u$ is admissible for $D_i, i = 1, \ldots, p$. Let $b_k$ be constants from the expansion of the function $u$ in a neighbourhood of $z_k$, $b_{ij} = b_k$ if $k = k(i, j)$. Applying lemmata 2.1 and 2.2 with the potential functions $u_k$
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for \( D_k \) we get

\[
\sum_{i=1}^{p} \sum_{j=1}^{n_i} (\delta_{ij}) r^{2-n} \lambda_n + \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} a_{ij} + o(1) = \sum_{i=1}^{p} I(u_i, (D_i)_r) =
\]

\[
= \sum_{i=1}^{p} \left( I(u, (D_i)_r) - 2 \sum_{j=1}^{n_i} \delta_{ij} (b_{ij} - a_{ij}) - I(u - u_i, (D_i)_r) \right) + o(1)
\]

\[
\leq I(u, D_r) - \sum_{i=1}^{p} I(u - u_i, (D_i)_r) - 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} (b_{ij} - a_{ij}) + o(1)
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{n_i} (\delta_{ij}) r^{2-n} \lambda_n + \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} b_{ij} - 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} (b_{ij} - a_{ij})
\]

\[
- \sum_{i=1}^{p} I(u - u_i, (D_i)_r) + o(1), \ r \to 0.
\]

It implies that

\[
\sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} b_{ij} \leq \sum_{i=1}^{p} \sum_{j=1}^{n_i} \delta_{ij} a_{ij} - \sum_{i=1}^{p} I(u - u_i, D_i)
\]

or equivalently

\[
\sum_{i=1}^{p} I(u - u_i, D_i) + M(D, \Gamma, Z, \Delta) \leq \sum_{i=1}^{p} M(D_i, \Gamma_i, Z_i, \Delta_i).
\]

Here we used the fact that the function \( u - u_i \) has no singularity in \( D_i \). □

Denote by \( r(D_l, x_l) = r(D_l, x_l, \partial D) \) the harmonic radius. Directly from theorem 2.3 we get theorem 2 of the paper [5]

**Corollary 2.5** For any non-overlapping domains \( D_l \subset \mathbb{R}^n, n \geq 3 \), points \( x_l \in D_l \) and real numbers \( \delta_l, l = 1, \ldots, m \) the inequality

\[
- \sum_{l=1}^{m} \delta_l^2 r(D_l, x_l)^{2-n} \leq \sum_{l=1}^{m} \sum_{p=1 \ (p \neq l)}^{m} \delta_l \delta_p |x_l - x_p|^{2-n}
\]

holds true.

**Proof.** The Green’s function of the ball \( B(0, \rho) \) is

\[
\lambda_n \left( |x - x_0|^{2-n} - \frac{|x_0||x|}{\rho} - \frac{\rho x_0}{|x_0|} \right)^{2-n}.
\]
Denote by $D_l(\rho)$ the intersection $D_l \cap B(0, \rho)$. By theorem 2.3

$$M(\rho) \geq \sum_{l=1}^{m} M_l(\rho),$$

where $M(\rho)$ is the modulus of the ball $B(0, \rho)$, the collections $\{x_l\}_{l=1}^{m}$, $\Delta = \{\delta_l\}_{l=1}^{m}$, and $\Gamma = \partial B$,

$$M_l(\rho) = -\delta_l^2 r(D_l(\rho), x_l)^{2-n} \lambda_n.$$

It is sufficient to take a limit as $\rho \to \infty$. □

Theorems 2.3 and 2.4 imply for $p = 1$ monotonicity of the quadratic form

$$\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D)$$

under extension of a domain. Following [2] we will say that a domain $\tilde{D}$ is obtained by extending a domain $D$ across a part of its boundary $\gamma \subset \partial D$ if $D \subset \tilde{D}$ and $(\partial D) \cap \tilde{D}$ lies in $\gamma$.

**Corollary 2.6** If $\tilde{D}$ is obtained by extending $D$ across $\Gamma$, $\tilde{\Gamma} \subset (\Gamma \cup (\mathbb{R}^n \setminus \overline{D}))$, then for any real numbers $\delta_l$ and points $z_l \in D$

$$\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\tilde{\Gamma}}(z_l, z_p, \tilde{D}) \geq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D) + I(u - \tilde{u}, D)$$

$$\geq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D).$$

If $\tilde{D}$ is obtained by extending $D$ across the part of $(\partial D) \setminus \Gamma$, $\tilde{\Gamma} = \Gamma$, then

$$\sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\tilde{\Gamma}}(z_l, z_p, \tilde{D}) \leq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D) - I(u - \tilde{u}, D)$$

$$\leq \sum_{l=1}^{m} \sum_{p=1}^{m} \delta_l \delta_p g_{\Gamma}(z_l, z_p, D),$$

here $u$ and $\tilde{u}$ are the potential functions for $D$, $\Gamma$, $Z = \{z_l\}_{l=1}^{m}$, $\Delta = \{\delta_l\}_{l=1}^{m}$ and $\tilde{D}$, $\tilde{\Gamma}$, $\tilde{Z} = \{z_l\}_{l=1}^{m}$, $\tilde{\Delta} = \{\delta_l\}_{l=1}^{m}$, respectively.

In [3] the notion of the Robin radius

$$r(D, z_0, \Gamma) = \exp \lim_{z \to z_0} (g_D(z, z_0, \Gamma) + \log |z - z_0|)$$
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was introduced. This quantity generalized the notion of the conformal radius. An analogue of Kufarev’s theorem (see [8]) for non-overlapping domains \(D_1, D_2\) lying in the unit disk \(U\) under the condition \((\partial D_k \cap U) \subset \Gamma_k \subset \partial D_k, a_k \in D_k, k = 1, 2\) is the inequality

\[
 r(D_1, a_1, \Gamma_1) r(D_2, a_2, \Gamma_2) \leq |a_2 - a_1|^2 \left[ 1 - \frac{|a_2 - a_1|^2}{1 - |a_1 a_2|} \right]^{-1}.
\]

By setting in theorem [2.3] \(p = 2, \Gamma = \emptyset\), we obtain in \(\mathbb{R}^n\) the following inequality.

**Corollary 2.7** Let \(D_1\) and \(D_2\) be non-overlapping and lie in the ball \(U = B(0, 1)\), \(a_k \in D_k, (\partial D_k \cap U) \subset \Gamma_k \subset \partial D_k, k = 1, 2\). Then

\[
 -\lambda_n r(D_1, a_1, \Gamma_1)^2 - n - \lambda_n r(D_2, a_2, \Gamma_2)^2 - n \leq M(U, \emptyset, \{a_1, a_2\}, \{1, -1\}). (8)
\]

To calculate \(M(U, \emptyset, \{a_1, a_2\}, \{1, -1\})\) we need to know the Neumann function of the unit ball. Note that it is a quite complicated problem in \(\mathbb{R}^n\). In particular, for \(n = 3\) (see [13])

\[
g_{\emptyset}(x, y, U) = \frac{1}{4\pi} \left( \frac{1}{|x - y|} + \frac{|y|}{|x||y|^2 - y|} - \log \left| 1 - (x, y) + \frac{|x||y|^2 - y|}{|y|} \right| \right).
\]

In [13] there is an analytic view of \(g_{\emptyset}(D, x, y)\) for \(n = 4, 5\). So, for \(n = 3\) the inequality (8) has the following form

\[
 - r(D_1, a_1, \Gamma_1)^{-1} - r(D_2, a_2, \Gamma_2)^{-1} \leq -\frac{2}{|a_1 - a_2|} - \frac{2|a_2|}{|a_1|^2|a_2|^2 - a_2} \\
 + 2 \log \left| 1 - (a_1, a_2) + \frac{|a_1|^2 - a_2}{|a_2|} \right| + \frac{1}{1 - |a_1|^2} + \frac{1}{1 - |a_2|^2} - \log(4(1 - |a_1|^2)(1 - |a_2|^2)).
\]

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