ON CAYLEY IDENTITY FOR SELF-ADJOINT OPERATORS IN HILBERT SPACES

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Abstract. We prove an analogue to the Cayley identity for an arbitrary self-adjoint operator in a Hilbert space. We also provide two new ways to characterize vectors belonging to the singular spectral subspace in terms of the analytic properties of the resolvent of the operator, computed on these vectors. The latter are analogous to those used routinely in the scattering theory for the absolutely continuous subspace.

1. Introduction

If $M$ is a matrix in $\mathbb{C}^n$ and $d_M(\lambda) := \det(M - \lambda)$ is its characteristic polynomial, the celebrated Cayley identity says that

$$d_M(M) \equiv 0.$$  

In [25] we have studied the “almost Hermitian” spectral subspace of a nonself-adjoint, non-dissipative operator $L$. The following criterion has been established: a nonself-adjoint operator possesses almost Hermitian spectrum (i.e., its almost Hermitian spectral subspace coincides with the Hilbert space $H$) iff a natural generalization of Cayley identity hold both for the operator itself and its adjoint.

This generalization of Cayley identity is formulated in terms of the so-called weak outer annihilation. The following definition of it has been suggested:

**Definition 1.1.** Let $\gamma(\lambda)$ be an outer [21] in the upper half-plane $\mathbb{C}_+$ uniformly bounded scalar analytic function. We call this function a *weak annihilator* of an operator $L$, if

$$w - \lim_{\varepsilon \downarrow 0} \gamma(L + i\varepsilon) = 0. \tag{1.1}$$
As a by-product of the aforementioned analysis of nonself-adjoint operators and using essentially nonself-adjoint techniques (i.e., the dilation of a dissipative operator, see [2]) we have further been able to prove, that a self-adjoint operator $A$ has trivial absolutely continuous subspace if and only if $A$ is weakly annihilated in the sense of the above definition.

Moreover, the corresponding outer analytic function admits an explicit choice, i.e., it can be chosen to be equal to the perturbation determinant $D_{A/A-iV}(\lambda)$ of the pair $A, A-iV$ [24], where $V$ is an auxiliary non-negative trace class operator.

The natural question on the possibility to formulate a “local” version of this criterion for self-adjoint operators with mixed spectrum, i.e., how to ascertain in similar terms whether the spectrum of a given self-adjoint operator $A$ is purely singular inside some Borel set of the real line, was posed some time ago by Prof. David Pearson. The present paper is an attempt to give an (in our view, so far incomplete) answer to this question. We prove the result envisaged in two quite different flavours: the one that we prefer (but analytic difficulties then only allow us to give the proof under rather restrictive assumptions on the operators’ spectrum, see below) and the one that actually allows to give a rigorous proof in the most general case.

The paper is organized as follows.

Since the functional model of a nonself-adjoint operator is of crucial importance for our approach and the proof of our main result relies heavily upon the symmetric form of the Nagy-Foiaş functional model due to Pavlov [26, 3] (see also the paper [6] by Naboko), we continue with a brief introduction to the main concepts and results obtained in this area in Section 2.

Section 3 contains our main result, which may be viewed as a generalization of the Cayley identity to self-adjoint operators with an arbitrary spectral structure.

Finally, in Section 4 we derive two new characterizations of vectors belonging to the singular spectral subspace of a self-adjoint operator in terms of the analytic properties of the resolvent of the operator, computed on these vectors. The latter are analogous to those used routinely in the scattering theory for the absolutely continuous subspace.

At this time, we have elected to postpone the discussion of possible applications, since the meaningful examples we have in mind, i.e., the examples in which the singularity of the spectrum in a given set is either unknown or cannot be obtained by some simpler classical techniques, do require substantial an non-trivial analysis to be fully considered.
2. The functional model of a dissipative operator

In the present section we briefly recall the functional model of a nonself-adjoint operator constructed in [2, 3] in the dissipative case and then extended in [4, 5, 6, 7] to the case of a wide class of non-dissipative operators. We consider a class of nonself-adjoint operators of the form

\[ L = A + iV, \]

where \( A \) is a self-adjoint operator in \( H \) defined on the domain \( D(A) \) and the perturbation \( V \) admits the factorization \( V = \alpha J \alpha / 2 \), where \( \alpha \) is a non-negative self-adjoint operator in \( H \) and \( J \) is a unitary operator in an auxiliary Hilbert space \( E \), defined as the closed range of the operator \( \alpha \): \( E \equiv R(\alpha) \). This factorization corresponds to the polar decomposition of the operator \( V \). It can also be easily generalized to the “node” case [8], where \( J \) acts in an auxiliary Hilbert space \( \mathcal{H} \) and \( V = \alpha^* J \alpha / 2, \alpha \) being an operator acting from \( \mathcal{H} \) to \( \mathcal{H} \). In order that the expression \( A + iV \) be meaningful, we impose the condition that \( V \) be \((A)\)-bounded with relative bound less than 1, i.e., \( D(A) \subset D(V) \) and for some \( a \) and \( b \) \((a < 1)\) the condition \( \|Vu\| \leq a\|Au\| + b\|u\|, \ u \in D(A) \) is satisfied, see [9]. Then the operator \( L \) is well-defined on the domain \( D(L) = D(A) \).

Alongside with the operator \( L \) we are going to consider the maximal dissipative operator \( L_{\|=} = A + i\alpha^2/2 \) and the one adjoint to it, \( L_{\|=}^* = A - i\alpha^2/2 \). Since the functional model for the dissipative operator \( L_{\|=} \) will be used below, we require that \( L_{\|=} \) is completely nonself-adjoint, i.e., that it has no reducing self-adjoint parts. This requirement is not restrictive in our case due to Proposition 1 in [6].

We also note that the functional model in the general case of operators with not necessarily additive imaginary part and with non-empty resolvent set has been developed in [7].

Now we are going to briefly describe a construction of the self-adjoint dilation of the completely nonself-adjoint dissipative operator \( L_{\|=} \), following [2, 3], see also [6].

The characteristic function \( S(\lambda) \) of the operator \( L_{\|=} \) is a contractive, analytic operator-valued function acting in the Hilbert space \( E \), defined for \( Im \lambda > 0 \) by

\[ S(\lambda) = I + i\alpha(L_{\|=} - \lambda)^{-1}\alpha. \quad (2.1) \]

In the case of an unbounded \( \alpha \) the characteristic function is first defined by the latter expression on the manifold \( E \cap D(\alpha) \) and then extended by continuity to the whole space \( E \). The definition given above makes it possible to consider \( S(\lambda) \) for \( Im \lambda < 0 \) with \( S(\lambda) = (S^*(\lambda))^{-1} \) provided that the inverse exists at the point \( \lambda \). Finally, \( S(\lambda) \) possesses boundary
values on the real axis in the strong topology sense: $S(k) \equiv S(k + i0)$, $k \in \mathbb{R}$ (see [2]).

Consider the model space $\mathcal{H} = L_2(\mathbb{R}, S^*)$, which is defined in [3] (see also [10] for description of general coordinate-free models) as Hilbert space of two-component vector-functions $(\tilde{g}, g)$ on the axis $(\tilde{g}(k), g(k) \in E, k \in \mathbb{R})$ with metric

$$\langle (\tilde{g}, g), (\tilde{g}, g) \rangle = \int_{-\infty}^{\infty} \langle (I S(k) S^*(k)) (\tilde{g}(k)), (\tilde{g}(k)) \rangle_{E \oplus E} \, dk.$$ 

It is assumed here that the set of two-component functions has been factored by the set of elements with norm equal to zero. Although we consider $(\tilde{g}, g)$ as a symbol only, the formal expressions

$$g_- := (\tilde{g} + S^*g)$$

and

$$g_+ := (S\tilde{g} + g)$$

(the motivation for the choice of notation is self-evident from what follows) can be shown to represent some true $L_2(E)$-functions on the real line. In what follows we plan to deal mostly with these functions.

Define the following orthogonal subspaces in $\mathcal{H}$:

$$D_- \equiv \begin{pmatrix} 0 \\ H^2(E) \end{pmatrix}, \quad D_+ \equiv \begin{pmatrix} H^2_+(E) \\ 0 \end{pmatrix}, \quad K \equiv \mathcal{H} \ominus (D_- \oplus D_+),$$

where $H^2_{+(-)}(E)$ denotes the Hardy class [2] of analytic functions $f$ in the upper (lower) half-plane taking values in the Hilbert space $E$. These subspaces are “incoming” and “outgoing” subspaces, respectively, in the sense of [11].

The subspace $K$ can be described as $K = \{(\tilde{g}, g) \in \mathcal{H} : g_- \equiv \tilde{g} + S^*g \in H^2_+(E), g_+ \equiv S\tilde{g} + g \in H^2_-(E)\}$. Let $P_K$ be the orthogonal projection of the space $\mathcal{H}$ onto $K$, then

$$P_K(\tilde{g}, g) = \begin{pmatrix} \tilde{g} - P_+ (\tilde{g} + S^*g) \\ g - P_- (S\tilde{g} + g) \end{pmatrix},$$

where $P_{\pm}$ are the orthogonal Riesz projections of the space $L_2(E)$ onto $H^2_{\pm}(E)$.

The following Theorem holds [2, 3]:

**Theorem 2.1.** The operator $(L \| - \lambda_0)^{-1}$ is unitarily equivalent to the operator $P_K(k - \lambda_0)^{-1}|_K$ in the space $K$ for all $\lambda_0, \text{Im} \lambda_0 < 0$.

This means, that the operator of multiplication by $k$ in $\mathcal{H}$ serves as a minimal $(\text{clos}_{\text{Im} \lambda \neq 0} (k - \lambda)^{-1} K = \mathcal{H})$ self-adjoint dilation [2] of the operator $L \|$.
3. Characterization of singular spectrum in terms of weak annihilation

In the present section, we attempt to provide a localized criterion of pure singularity of the spectrum of a general self-adjoint operator inside a given set of the real line, building upon the technique and approach developed in [25]. It is worth mentioning that not only the proofs of our results in this direction exploit essentially nonself-adjoint (in particular, functional model related) techniques, but even certain crucial objects of the nonself-adjoint spectral theory appear already in their statements.

Our next Theorem in our view constitutes the most natural localization of the corresponding “global” result of [25]. Unfortunately, we are only able to prove the result in this natural form in the case when both ends of the interval $\Delta$ where one wants to ascertain pure singularity of the spectrum are located inside a spectral gap. Any attempt to get rid of this rather horrible restriction requiring crucial a-priori information on the spectral structure fails due to the lack of control over the annihilating function at the endpoints of the interval $\Delta$. It seems that in the general setting one has to resort to a quite different (and less natural) definition of annihilation (see Theorem 3.5 below).

**Theorem 3.1.** Let $A$ be a (possibly, unbounded) self-adjoint operator in the Hilbert space $H$. Let a point $\lambda_0 \in \mathbb{R}$ belong together with some neighborhood $\Delta$ to the resolvent set of $A$. Then the following two statements are equivalent.

(i) The spectrum of $A$ to the left of the point $\lambda_0$ is purely singular;

(ii) There exists an outer bounded in the upper half-plane non-trivial (i.e., non-constant) scalar function $\gamma(\lambda)$ with real boundary values almost everywhere on $(\lambda_0, +\infty)$ and non-real boundary values almost everywhere on $(-\infty, \lambda_0)$, weakly annihilating the operator $A$, i.e.,

$$w - \lim_{\varepsilon \downarrow 0} (\gamma(A + i\varepsilon) - \gamma_*(A - i\varepsilon)) = 0,$$

where $\gamma_*(\lambda) := \overline{\gamma(\overline{\lambda})}$ is an outer bounded in the lower half-plane analytic function.

**Proof.** Choose $V$ to be a trace class non-negative self-adjoint operator in the Hilbert space $H$ such that

$$\bigvee_{\text{Im } \lambda \neq 0} (A - \lambda)^{-1}VH = H.$$  \hfill (3.1)

Clearly, such choice is always possible.
We follow the approach developed in [6] for the operators $L$ admitting the representation $L_\kappa = A + \alpha \kappa / 2$, where $\alpha \geq 0$ is a non-negative operator in the Hilbert space $H$ and $\kappa$ is a bounded operator in the subspace $E$, being the closure of the range of $\alpha$. Choose $\alpha$ to be a Hilbert-Schmidt class operator defined by the formula $\alpha = \sqrt{2} V \in S_2$.

Then the operator $L_\kappa$ is well-defined on the domain $D(L_\kappa) = D(A)$. Moreover, $L_\kappa \equiv A$ when $\kappa = 0$, i.e., $L_0 \equiv A$. Consider the dissipative operator $L_\| \equiv A + iV$ (this operator coincides with $L_{iI}$). Clearly, it is a maximal dissipative operator in $H$; moreover, it is easy to see that the condition (3.1) guarantees that it is also completely nonself-adjoint.

Construct the functional model based on the operator $L_\|$ (see Section 1 above). In the corresponding dilation space $H$ the following formulae describe the action of the resolvent $(A - \lambda)^{-1}$ on all vectors $(\tilde{g}, g) \in K$, as above $K$ being the model image of $H$ (see [6]):

\[
(A - \lambda)^{-1} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \frac{1}{(k - \lambda)} \begin{pmatrix} \tilde{g} \\ g - \frac{1}{2} \left( (S(\lambda) - I) \frac{1}{2} \right)^{-1} g_-(\lambda) \end{pmatrix}, \quad Im \lambda < 0 \tag{3.2}
\]

\[
(A - \lambda)^{-1} \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = P_K \frac{1}{(k - \lambda)} \begin{pmatrix} \tilde{g} - \frac{1}{2} \left( (S(\lambda) - I) \frac{1}{2} \right)^{-1} g_+(\lambda) \\ g \end{pmatrix}, \quad Im \lambda > 0. \tag{3.3}
\]

Here $S(\lambda)$ is the characteristic function of completely nonself-adjoint maximal dissipative operator $L_\|$, all the other notation has already been introduced above.

We introduce the following notation for the operator functions, appearing in this representation: $\Theta_A(\lambda) := I + (S(\lambda) - I) \frac{1}{2}$ and $\Theta'_A(\lambda) := I + (S^*(\lambda) - I) \frac{1}{2}$. The functions $\Theta_A$ and $\Theta'_A$ are bounded analytic operator functions in half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

Recall that the characteristic function $S(\lambda)$ is a contraction in the upper half plane. It follows that, since $\Theta_A(\lambda) = (I + S(\lambda))/2$ and $\Theta'_A(\lambda) = (I + S^*(\lambda))/2$, they are outer contractions (see [2]) in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

By definition of $S(\lambda)$, both operator functions also have well-defined outer [19, 2] determinants, $\gamma_A, \gamma'_A$, bounded (in fact, contractive) in their respective half-planes. It is also clear that $\gamma'_A(\lambda) = \gamma_A(\lambda)$. We remark, that $\gamma_A(\lambda)$ is a clearly non-zero function since \( \lim_{\tau \to +\infty} \gamma_A(i\tau) = 1 \).

\[^1\] It is easy to see that $\gamma_A(\lambda)$ in fact coincides with the perturbation determinant $D_{A/A-iV}(\lambda)$ of the pair $A, A - iV$ [24].
W.l.o.g. assume, that the point $\lambda = 0$ together with its neighborhood $\Delta$ belongs to the resolvent set of the operator $A$. It follows that since the operator $L^\parallel$ is completely nonself-adjoint and dissipative, the same neighborhood also belongs to its resolvent set. Thus the function $S(\lambda)$ admits analytic continuation to $\mathbb{C}_-$ through the named neighborhood of zero and the determinant $\gamma_A(\lambda)$ is $C^\infty$ there.

Since $\gamma(\lambda)$ is an outer function, it admits the following representation in terms of the logarithm of its boundary values on the real line:

$$\gamma_A(\lambda) = e^{ic} \exp \left\{ \frac{i}{\pi} \int_{\mathbb{R}} \left( \frac{1}{\lambda - t} + \frac{t}{1 + t^2} \right) \log |\gamma(t)| dt \right\},$$

where $\gamma(t) := \gamma(t + i0)$ are the boundary values of the function $\gamma$ from above and $c$ is some real constant. From (3.2) it further follows, that $\gamma_A(\lambda)$ is separated from zero on $\Delta$.

Fix a point $\delta_0$ such that $-\delta_0 \in \Delta$ and let

$$\gamma_1(\lambda) = e^{ic} \exp \left\{ \frac{i}{\pi} \int_{-\infty}^{-\delta_0} \left( \frac{1}{\lambda - t} + \frac{t}{1 + t^2} \right) \log |\gamma_A(t)| dt \right\}$$

be the new outer (by construction), bounded in the upper half-plane function. Let further $\varphi(t)$ be the harmonic conjugate (or, in other words, the Hilbert transform) of the function $f(t)$, equal to $\log |\gamma_A(t)|$ on $(-\infty, -\delta_0)$ and to 0 elsewhere. Clearly, the function $\varphi(t)$ is itself infinitely smooth on any interval $[-\delta_1, +\infty)$ provided that $\delta_1 < \delta_0$.

Choose yet another function $\gamma_2(\lambda)$ as follows:

$$\gamma_2(\lambda) = e^{-ic} \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{+\infty} \left( \frac{1}{\lambda - t} + \frac{t}{1 + t^2} \right) \varphi_1(t) dt \right\},$$

where $\varphi_1(t)$ is any $C^1(\mathbb{R})$ function such that $\varphi_1(t) \equiv \varphi(t)$, $t \geq 0$. As it is easily seen, on the right half-line $\arg \gamma_2(t + i0) = - \arg \gamma_1(t + i0)$ almost everywhere. What’s more, since $\varphi_1(t)$ is $C^1$ on the real line, its harmonic conjugate is continuous and thus bounded [22]. It follows, that the function $\gamma_2(\lambda)$ is itself bounded and outer in the upper half-plane, admitting the following representation:

$$\gamma_2(\lambda) = e^{-ic} \exp \left\{ \frac{i}{\pi} \int_{-\infty}^{+\infty} \left( \frac{1}{\lambda - t} + \frac{t}{1 + t^2} \right) (-\tilde{\varphi}_1(t)) dt \right\},$$

where $\tilde{\varphi}_1(t)$ is the harmonic conjugate of the function $\varphi_1$.

Consider the function $\gamma(\lambda) := \gamma_1(\lambda)\gamma_2(\lambda)$. By virtue of its construction, it is a bounded outer function in the upper half-plane with almost everywhere real boundary values on the right half-line. What’s more, together with its first factor it cancels out the zeroes of the function

...
\( \gamma_A(\lambda) \) on the interval \((-\infty, -\delta_0)\): \(|\gamma(t + i\varepsilon)/\gamma_A(t + i\varepsilon)| \leq C \) uniformly in \(\varepsilon\) for some finite constant \(C\) and every \(t \leq -\delta_0\). It remains to be seen that this function can be chosen in a way such that its boundary values to the left of the point 0 are non-real almost everywhere. In fact, this can be safely assumed w.l.o.g.: if not, denote by \(\Omega \subset (-\infty, 0)\) the set of points where the corresponding boundary values are real almost everywhere. Then consider a non-negative smooth enough function \(g(k)\) having its support equal to the closure of \(\Omega\) and define \(\hat{g}(\lambda) := \int g(k) \frac{1}{k - \lambda} dk\). Multiplication by this outer bounded factor clearly equips the function \(\gamma(\lambda)\) with the properties required by the Theorem.

We will now prove that \(\gamma(\lambda)\) weakly annihilates the self-adjoint operator \(A\) in the sense of the Theorem.

First, let \(u\) belong to the spectral subspace \(E_A(0, +\infty)H\) of the operator \(A\), where \(E_A(\cdot)\) is the operator-valued spectral measure associated with \(A\). Then by the spectral theorem and by Lebesgue dominated convergence theorem it is easy to see that

\[
\lim_{\varepsilon \downarrow 0} \langle (\gamma(A+i\varepsilon) - \gamma(A-i\varepsilon))u, v \rangle = \int_0^{+\infty} (\gamma(k+i0) - \gamma(k+i0))d\mu_{u,v}(k) = 0
\]

for all \(v\) in \(H\) (in a nutshell, we have used the fact that the function \(\gamma\) by its construction is an analytic continuation of the function \(\gamma\) to the lower half-plane).

It remains to be seen that if \(u = E_A(-\infty, 0)H\), then

\[
\lim_{\varepsilon \downarrow 0} \langle \gamma(A+i\varepsilon)u, v \rangle = 0
\]

and

\[
\lim_{\varepsilon \downarrow 0} \langle \gamma(A-i\varepsilon)u, v \rangle = 0
\]

for all \(v\) in \(H\), provided that the spectrum of the operator \(A\) is purely singular to the left of the point zero. We will check the first identity above, the second being verified analogously.

The bounded (due to v. Neumann inequality \[2\] or, alternatively, due to the spectral theorem) operator \(\gamma(A+i\varepsilon)\) is defined by the Riesz-Dunford integral,

\[
\langle \gamma(A+i\varepsilon)u, v \rangle = \frac{1}{2\pi i} \left( \int_{-\infty+3i\varepsilon/2}^{+\delta_0+3i\varepsilon/2} - \int_{-\infty+3i\varepsilon/2}^{-\delta_0+3i\varepsilon/2} \right) \gamma_A(\lambda) \langle (A + i\varepsilon - \lambda)^{-1}u, v \rangle d\lambda.
\]

Using the model representation (3.2) we then immediately obtain:

\[
\left\langle \gamma(A + i\varepsilon) \left( \hat{g} \frac{g}{f} \right), \left( \hat{f} \frac{g}{f} \right) \right\rangle = \left\langle \gamma(k + i\varepsilon) \left( \hat{g} \frac{g}{f} \right), \left( \hat{f} \frac{g}{f} \right) \right\rangle +
\]
Here for all \((\tilde{g}, g)\) a meaningful object. Due to analytic properties of the functions \(g\) following form:

\[
\frac{1}{2\pi i} \int_{-\infty}^{-\delta_0} \gamma(t + i\varepsilon) \left\langle \frac{1}{k - (t + i\varepsilon)} \left( \frac{1}{2} \Theta_A^{-1}(t - i\varepsilon)g_-(t - i\varepsilon) \right), \left( \frac{f}{g} \right) \right\rangle \, dt - \\
\frac{1}{2\pi i} \int_{-\infty}^{-\delta_0} \gamma(t + i\varepsilon) \left\langle \frac{1}{k - (t + i\varepsilon)} \left( \frac{1}{2} \Theta_A^{-1}(t + i\varepsilon)g_+(t + i\varepsilon) \right), \left( \frac{f}{g} \right) \right\rangle \, dt.
\]

(3.4)

Rewriting \(\Theta_A'(\lambda) = \Omega'(\lambda)/\gamma_A(\lambda)\) and \(\Theta_A(\lambda) = \Omega(\lambda)/\gamma_A(\lambda)\) with bounded in the lower (resp., upper) half-plane operator function \(\Omega'(\lambda)\) (resp., \(\Omega\)), it is now easy to see that the last expression assumes the following form:

\[
\left\langle \gamma(A + i\varepsilon) \left( \frac{\tilde{g}}{g} \right), \left( \frac{f}{f} \right) \right\rangle = \left\langle \gamma(k + i\varepsilon) \left( \frac{\tilde{g}}{g} \right), \left( \frac{f}{f} \right) \right\rangle + \\
\int_{-\infty}^{-\delta_0} \frac{\gamma(t + i\varepsilon)}{\gamma_A(t + i\varepsilon)} \left\langle \frac{1}{2} \Omega(t - i\varepsilon)g_-(t - i\varepsilon), f_+(t + i\varepsilon) \right\rangle \, dt - \\
\int_{-\infty}^{+\infty} \frac{\gamma(t + i\varepsilon)}{\gamma_A(t + i\varepsilon)} \left\langle \frac{1}{2} \Omega(t + i\varepsilon)g_+(t + i\varepsilon), f_-(t - i\varepsilon) \right\rangle \, dt. \quad (3.5)
\]

Due to analytic properties of the functions \(g_{\pm} \in H^+_2(E)\), \(f_{\pm} \in H^+_2(E)\) the latter expression has a limit as \(\varepsilon\) tends to 0 and by Lebesgue dominated convergence theorem and Schwartz inequality

\[
\lim_{\varepsilon \to 0} \left\langle \gamma(A + i\varepsilon) \left( \frac{\tilde{g}}{g} \right), \left( \frac{f}{f} \right) \right\rangle = \\
\int_{-\infty}^{-\delta_0} [\langle \gamma \tilde{f}, g_- \rangle + \langle \gamma f, g_+ \rangle + \frac{\gamma(t)}{\gamma_A(t)} \langle \frac{1}{2} \Omega g_+, f_- \rangle + \frac{\gamma(t)}{\gamma_A(t)} \langle \frac{1}{2} \Omega g_-, f_+ \rangle] \, dt.
\]

(3.6)

Here \(\int [\langle \gamma \tilde{f}, g_- \rangle + \langle \gamma f, g_+ \rangle] \, dt = \langle \gamma(\tilde{g}, g), (\tilde{f}, f) \rangle\) and therefore represents a meaningful object.

In order to prove that this limit is actually equal to zero, we recall [19, 20] that for all \((\tilde{g}, g) \in H_s(A)\) and for all \((\tilde{f}, f) \in K\)

\[
\left\langle \left[ (L - k - i\varepsilon)^{-1} - (L - k + i\varepsilon)^{-1} \right] \left( \frac{\tilde{g}}{g} \right), \left( \frac{f}{f} \right) \right\rangle \xrightarrow{\varepsilon \to 0} 0 \quad (3.7)
\]

for a.a. real \(k\). Again taking into account formulae describing the action of the resolvent of the operator \(L\) in the model representation in upper and lower half-planes, consider the following expression for arbitrary vectors \((\tilde{g}, g) \in H_s(A)\), \((\tilde{f}, f) \in K \equiv H\):

\[
\frac{1}{2\pi i} \gamma(t + i\varepsilon) \left\langle \left[ (A - t - i\varepsilon)^{-1} - (A - t + i\varepsilon)^{-1} \right] \left( \frac{\tilde{g}}{g} \right), \left( \frac{f}{f} \right) \right\rangle =
\]


\[ \frac{\gamma(t + i\varepsilon)}{2\pi i} \int_{-\infty}^{-\delta_0} \frac{2i\varepsilon}{(k-t)^2 + \varepsilon^2} \left\langle \begin{pmatrix} \tilde{g} \\
 f \end{pmatrix}, \begin{pmatrix} f \\
 \tilde{f} \end{pmatrix} \right\rangle dk + \frac{\gamma(t + i\varepsilon)}{\gamma_A(t + i\varepsilon)} \left\langle \frac{1}{2} \Omega(t + i\varepsilon)g_+(t + i\varepsilon), f(t - i\varepsilon) \right\rangle + \frac{\gamma(t + i\varepsilon)}{\gamma_A(t + i\varepsilon)} \left\langle \frac{1}{2} \Omega'(t - i\varepsilon)g_-(t - i\varepsilon), f_+(t + i\varepsilon) \right\rangle \]

(cf. (3.5)). The latter expression has a limit for a. a. \( t \in \mathbb{R} \), equal to the integrand in (3.6). On the other hand, from (3.7) it follows, that this limit is identically equal to zero for a. a. \( t \). This observation completes the proof.

Conversely, let the self-adjoint operator \( A \) possess a weak outer bounded annihilator \( \gamma(\lambda) \) in the sense of the Theorem. Let the vector \( u \neq 0, u \in E(-\infty, 0)H \) belong to the absolutely continuous spectral subspace \( H_{ac} \). Then, again by the spectral theorem and by Lebesgue dominated convergence theorem it is easy to see that

\[ \int_\delta (\gamma(k + i0) - \tilde{\gamma}(k + i0))d\mu_{u,v}(k) = 0 \]

(by taking \( E_A(\delta)v \) instead of \( v \)) for an arbitrary Borel set \( \delta \subset (-\infty, 0) \) and the finite absolutely continuous complex measure \[ d\mu_{u,v}(k) := \langle dE_A(k)u, v \rangle, \]

where as above \( E_A \) is the operator valued spectral measure of the operator \( A \) and \( v \) is an arbitrary element of \( H \). Since boundary values of \( \gamma \) are non-zero almost everywhere on the real line and by assumption these boundary values are non-real almost everywhere, this implies that the measure \( d\mu_{u,v} \equiv 0 \) for all \( v \in H \).

This completes the proof. \( \square \)

Remark 3.2. The last Theorem can of course be easily generalized together with the proof given to the situation when the set, where one tests the singularity of the spectrum of the operator \( A \), is an arbitrary finite or infinite interval of the real line or even a finite unit of such disjoint intervals.

Remark 3.3. Note that the existence of a non-zero analytic bounded annihilator of the operator \( A \) is clearly sufficient for the pure singularity of its spectrum to the left of the point \( \lambda_0 \). Nevertheless, our Theorem asserts that this function can be chosen to be outer in \( \mathbb{C}_+ \) as well.

Remark 3.4. Suppose that the operator \( A \) is a self-adjoint operator with simple spectrum. Then the trace class operator \( V \) of the last Theorem due to (3.1) can clearly be chosen \[ 1 \] as a rank one operator in Hilbert space \( H \). In this situation, the proof of Theorem 3.1 can
be modified in the part concerning the choice of the annihilator in the following way: the function \( \gamma_A \) can be chosen as
\[
\gamma_A(\lambda) := \frac{1}{1 - i(D(\lambda) - 1)},
\]
where \( D(\lambda) := 1 + \langle (A - \lambda)^{-1} \varphi, \varphi \rangle \) is the perturbation determinant of the pair \( A, A + \langle \cdot, \varphi \rangle \varphi \) and \( \varphi \) is the generating vector for the operator \( A \).

The proof is a straightforward application of the explicit formula for the resolvent of a rank one perturbation of a self-adjoint operator, based on the Hilbert identity.

We now pass over to the general case, i.e., the case when one has no a-priori information on the spectral structure of the operator \( A \) near the endpoints of the interval under consideration. In this case one faces the necessity to modify somewhat the definition of annihilation. The following Theorem addresses this.

**Theorem 3.5.** Let \( A \) be a (possibly, unbounded) self-adjoint operator in the Hilbert space \( H \). Let \( \Delta \) be an arbitrary Borel set on the real line. Then the following two statements are equivalent.

(i) The spectrum of \( A \) in \( \Delta \) is purely singular, i.e., the intersection of absolutely continuous spectrum and the set \( \Delta \) is empty;

(ii) There exist an outer bounded in the upper half-plane non-trivial (i.e., non-zero) scalar function \( \gamma(\lambda) \) and an outer bounded in the upper half-plane non-constant scalar function \( \beta(\lambda) \) such that \( \Im \beta(\lambda) \) has non-tangential limits on the real line at every point of the latter and these limits are zero everywhere on \( \mathbb{R} \setminus \Delta \) and non-zero everywhere on \( \Delta \), weakly annihilating the operator \( A \) in the following sense:
\[
\text{w-lim}_{\varepsilon \downarrow 0} \gamma(\lambda + i\varepsilon)(\beta(\lambda + i\varepsilon) - \beta^*(\lambda - i\varepsilon)) = 0,
\]
where \( \beta^*(\lambda) := \overline{\beta(\overline{\lambda})} \) is an outer bounded in the lower half-plane analytic function.

**Proof.** We start with the proof of the implication \((i) \Rightarrow (ii)\).

To begin with, let \( \beta(\lambda) \) be a Riesz transform of a square summable non-negative function \( b(k) \) such that \( \text{supp} \ b = \Delta \):
\[
\beta(\lambda) = \int \frac{b(k)}{k - \lambda} dk.
\]
In order to satisfy the restrictions of the Theorem on the imaginary part of \( \beta \), further assume that \( \beta \) is in addition a \( C^1 \) function on the real line. Then clearly it is outer bounded in the upper half-plane (in
fact, even an $R$-function), the imaginary part of it has boundary limits everywhere on $\mathbb{R}$ \cite{22} and moreover, these boundary limits are equal to zero on $\mathbb{R} \setminus \Delta$ and are non-zero everywhere on $\Delta$.

Then $\beta_*(\lambda)$ is an outer bounded analytic continuation of $\beta$ to the lower half-plane $\mathbb{C}^-$ through the complement $\mathbb{R} \setminus \Delta$, whereas the jump of the continued function through $\Delta$, which is proportional to $\Im \beta(k + i0)$, is non-trivial everywhere on $\Delta$.

By the spectral theorem of a self-adjoint operator and then by the Lebesgue dominated convergence theorem it is now easy to see that $\beta(A + i\varepsilon) - \beta_*(A - i\varepsilon) \to \beta_0(A)$ strongly as $\varepsilon \to 0$, where $\beta_0(k) := 2i\pi \Im \beta(k + i0)$.

On the other hand, repeating the argument from the proof of the last Theorem (namely, from (3.4) to (3.6), where the integral is extended from $(-\infty, -\delta_0)$ to the whole real line) one arrives at the conclusion that $\gamma_0 := \gamma_0(A)$, where $\gamma_0$ is the same function as above, is such that the operator family $\gamma(A + i\varepsilon)$ has a weak limit as $\varepsilon \to 0$, given by (3.6) with the above-mentioned change of the limits of integration.

It follows that $w - \lim_{\varepsilon \to 0} \gamma(A + i\varepsilon)(\beta(A + i\varepsilon) - \beta_*(A - i\varepsilon))$ exists and it’s only left to prove that it is equal to zero. Let first $u \in E_A(\Delta)$. Then $\langle \gamma(A + i\varepsilon)(\beta(A + i\varepsilon) - \beta_*(A - i\varepsilon))u, v \rangle \to 0$ for all $v \in H$ by the same argument as in the proof of the preceding Theorem (see (3.7) and below).

If on the other hand $u \in E_A(\mathbb{R} \setminus \Delta)H$, then the named limit is zero since $\beta_0(A)|_{E_A(\mathbb{R} \setminus \Delta)H} = 0$ due to the fact that $\Im \beta(k + i0) = 0$ for all $k \in \mathbb{R} \setminus \Delta$.

The proof of the inverse implication \((ii) \Rightarrow (i)\) is nothing but a slight modification of the corresponding implication of Theorem 3.1.

Indeed, let the vector $u \neq 0$, $u \in E_A(\Delta)H$ belong to the absolutely continuous spectral subspace $H_{ac}$. Then, again by the spectral theorem and by Lebesgue dominated convergence theorem it is easy to see that

$$\int_{\delta} \gamma(k + i0)(\beta(k + i0) - \bar{\beta}(k + i0))d\mu_{u,v}(k) = 0$$

(by taking $E_A(\delta)v$ instead of $v$) for an arbitrary Borel set $\delta \subset \delta$. Since boundary values of $\gamma$ are non-zero almost everywhere on the real line and by assumption the boundary values of the imaginary part of $\beta$ are non-zero in $\Delta$, this implies that the absolutely continuous measure $d\mu_{u,v} \equiv 0$ for all $v \in H$.

This completes the proof. \[\square\]
4. On the analytic properties of the resolvent

We take this opportunity to prove yet another result. We begin with the following observation, well-known from the mathematical scattering theory. Consider a self-adjoint operator $A$. Then there exists a linear set $\tilde{H}_{a.c.}$ dense in the absolutely continuous spectral subspace of $A$ such that

$$\int \| \beta \exp(iAt)u \|^2 dt < \infty$$

for all $u \in \tilde{H}_{a.c.}$ and any non-negative operator $\beta \in \mathcal{S}_2$ (see, e.g., [1]). Using the Fourier transform and Parseval’s identity, it’s easy to see [6] that the last condition is equivalent to:

$$\beta(A - \lambda)^{-1}u \in H^2_\pm(\text{Ran } \beta)$$

for all $u \in \tilde{H}_{a.c.}$ Taking an operator $V \in \mathcal{S}_1$ as in the proof of the previous Theorem, i.e., a non-negative trace class operator such that the condition (3.1) is satisfied, we can further obtain [6] the following description of the absolutely continuous spectral subspace of the operator $A$:

$$H_{a.c.} = \text{clos}\{u|\sqrt{V}(A - \lambda)^{-1}u \in H^2_\pm(E)\},$$

where as in Section 2 $E$ is the auxiliary Hilbert space, being the closed image of the operator $V$.

In this Section, we derive an analogous characterization for the singular spectral subspace $H_s$ of a self-adjoint operator $A$. Namely, the following Theorem holds.

**Theorem 4.1.** Let $A$ be a self-adjoint operator in the Hilbert space $H$. Let $V \in \mathcal{S}_1$ be a positive trace class operator in $H$ such that (3.1) holds. Then if the vector $u$ belongs to the singular spectral subspace $H_s$ of $A$, then the vector $\sqrt{V}(A - \lambda)^{-1}u$ belongs to vector Smirnov classes $N^2_\pm(E)$ [10], i.e., it can be represented as $h_\pm(\lambda)/\delta_\pm(\lambda)$, where $h_\pm \in H^2_\pm(E)$ and $\delta_\pm(\lambda)$ are scalar bounded outer analytic functions in half-planes $\mathbb{C}_\pm$, respectively. Here the functions $\delta_\pm$ can be chosen independently of vector $u$.

**Proof.** We again use the functional model constructed based on the dissipative operator $A + iV$.

Let now $u \in H_s$. The following identities hold (see [6]):

$$\sqrt{2\pi}g_+(\lambda) = -\Theta_A(\lambda)\alpha(A - \lambda)^{-1}u, \quad \text{Im } \lambda > 0,$$

$$\sqrt{2\pi}g_-(\lambda) = -\Theta'_A(\lambda)\alpha(A - \lambda)^{-1}u, \quad \text{Im } \lambda < 0$$

(4.1)
Here the operator-functions $\Theta_A(\lambda)$ and $\Theta'_A(\lambda)$ are defined by the identities $\Theta_A(\lambda) = (I + S(\lambda))/2$ and $\Theta'_A(\lambda) = (I + S^*(\lambda))/2$ ($S$ being the characteristic function of the dissipative operator $A + iV$), and are outer $\mathfrak{S}_1$-valued contractions in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

Within the conditions of Theorem 4.1 both operator-functions $\Theta_A(\lambda)$ and $\Theta'_A(\lambda)$ also possess outer determinants in their respective half-planes \cite{2}. Therefore, by the uniqueness theorem for scalar bounded analytic functions \cite{21}, they are invertible for almost all real $k$.

Then we obtain immediately, that for all $\lambda \in \mathbb{C}_+$

$$\sqrt{2V}(A - \lambda)^{-1}u = -\sqrt{2\pi}\Theta_A^{-1}(\lambda)g_+(\lambda) = -\sqrt{2\pi}\delta_+^{-1}(\lambda)\Omega(\lambda)g_+(\lambda),$$

where $\Omega(\lambda)\Theta_A(\lambda) = \Theta_A(\lambda)\Omega(\lambda) = \delta_+(\lambda)I$ with a bounded operator-function $\Omega(\lambda)$, i.e., $\delta_+$ is the determinant of the operator function $\Theta_A$.

It remains to point out (see Section 2), that the function $g_+(\lambda)$ belongs to $H^2_{\pm}(E)$ as $u \in H$. Application of a similar argument to the vector $g_-$ completes the proof. \hfill \Box

Remark 4.2. As it is easily seen from the definition of $\Theta_A(\lambda)$ and $\Theta'_A(\lambda)$, the functions $\delta_\pm(\lambda)$ in the statement of Theorem 4.1 can be chosen so that $\delta_-(\lambda) = \bar{\delta}_+(\bar{\lambda})$.

In a similar way we are able to give a “weak” version of the previous Theorem. Indeed, one can easily ascertain (see, e.g., \cite{1, 6}) on the basis of F. and M. Riesz theorem \cite{21} that the absolutely continuous subspace of a self-adjoint operator $A$ can be alternatively characterized as follows:

$$H_{a.c.} = \text{clos}\{u|\langle (A - \lambda)^{-1}u, v \rangle \in H^2_{\pm} \} \quad \text{for all } v \in H.$$

The following Theorem gives an analogous representation for the singular spectral subspace.

**Theorem 4.3.** Let $A$ be a self-adjoint operator in the Hilbert space $H$. Then if the vector $u \in H$ belongs to the singular spectral subspace $H_s$, then the function $\langle (A - \lambda)^{-1}u, v \rangle$ belongs to Smirnov classes $N^1_{\pm}$ for all $v \in H$, i.e., it can be represented as $h_{\pm}(\lambda)/\delta_{\pm}(\lambda)$, where $h_{\pm} \in H^1_{\pm}$ and $\delta_{\pm}(\lambda)$ are bounded scalar outer analytic functions in half-planes $\mathbb{C}_{\pm}$, respectively. Here the functions $\delta_{\pm}$ are independent of $v \in H$ and can be chosen independently of vector $u$.

**Proof.** We will again use the model description of the resolvent of the operator $A$ \cite{3, 20}, from where it follows that

$$\langle (A - \lambda)^{-1}u, v \rangle = \left\langle \frac{1}{k - \lambda} \begin{pmatrix} \tilde{g} \\ \tilde{f} \end{pmatrix}, \left( \begin{pmatrix} \tilde{g} \\ \tilde{f} \end{pmatrix} \right) \right\rangle - \left\langle \frac{1}{k - \lambda} \begin{pmatrix} \Theta_A^{-1}(\lambda)g_+(\lambda) \\ 0 \end{pmatrix}, \left( \begin{pmatrix} \tilde{g} \\ \tilde{f} \end{pmatrix} \right) \right\rangle,$$
where $(\tilde{f}, f)$ is the model image of the vector $v$. Let $u \in H_s$. The first term on the right hand side is clearly the Cauchy transformation of an $L_1$-function, whereas the second one can be rewritten by residue calculation in the following way:

$$\left\langle \frac{1}{k-\lambda} \left( \frac{1}{2} \Theta_A^{-1}(\lambda) g_+(\lambda) \right), \left( \tilde{f} \right) \right\rangle = -2\pi i \left\langle \frac{1}{2} \Theta_A^{-1}(\lambda) g_+(\lambda), f_-(\lambda) \right\rangle_E.$$

By Theorem 4.1 the vector $u$ is such that $\sqrt{V}(A-\lambda)^{-1}u \in N_2^+(E)$, and therefore by (4.1) again, $\Theta_1^{-1}(\lambda)g_+(\lambda) = h_+(\lambda)/\delta_+(\lambda)$ for some $h_+ \in H_2^+(E)$ and some outer bounded in the upper half-plane function $\delta_+$. It follows that if one puts $\nu(\lambda) := 1/(\lambda + i)$,

$$\langle (A-\lambda)^{-1}u, v \rangle = k_1(\lambda) - 2\pi i \frac{1}{\delta_+(\lambda)} \left\langle \frac{1}{2} h_+(\lambda), f_-(\lambda) \right\rangle_E \equiv$$

$$\frac{1}{\delta_+(\lambda)\nu(\lambda)} \left[ k_1(\lambda) \delta_+(\lambda) \nu(\lambda) - 2\pi i \frac{1}{2} h_+(\lambda), f_-(\lambda) \right]\nu(\lambda) \in N_1^+,$$

since $k_1(\lambda) := \langle \frac{1}{\sqrt{V}} \tilde{g}, \left( \tilde{f} \right) \rangle$ and $f_-(\lambda) \in H_2^+(E)$.

An analogous argument applied in the case of $\mathbb{C}_-$ completes the proof. \qed

Remark 4.4. Note that the functions $\delta_+(\lambda)$ appearing in the proof of the last Theorem are the same as in the proof of Theorem 4.1, i.e., these can be chosen to be equal to the determinants of the operator-functions $\Theta_A(\lambda)$ and $\Theta_A'(\lambda)$, respectively. Thus, the corresponding outer factors in Theorems 4.1 and 4.3 admit the simplest (and explicitly computable) form in the situation when the spectrum of the operator $A$ is simple, see Remark 3.4.

Acknowledgements. Both authors express their gratitude to Prof. David Pearson for the interest expressed by him to their research and for the question that motivated this paper.

The first author is grateful to Prof. A. Sobolev for fruitful discussions during the author’s stay in UCL.

The first author is grateful to the Dept. of Mathematics, University College London where parts of this work were done for hospitality.

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