Isospin particle systems on quaternionic projective spaces

Vahagn Yeghikyan,  
S. Bellucci, S. Krivonos, A. Nersessian  

Yerevan State University  

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$S^2 \rightarrow \mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$

Landau problem on $\mathbb{H}P^n$ from a free particle on $\mathbb{C}P^{2n+1}$

Oscillator

Conclusion
By definition the complex (quaternionic) projective space is the set of all the lines in $\mathbb{C}^{n+1}(\mathbb{H}^{n+1})$ through the origin. Namely, we identify two points $(v_1, \ldots, v_{n+1})$ and $(v'_1, \ldots, v'_{n+1})$ if

$$\exists g \in \mathbb{C} \text{ or } \mathbb{H}, \text{ so that}$$

$$(v_1, \ldots, v_{n+1}) = g(v'_1, \ldots, v'_{n+1}).$$

Or, equivalently $v \overline{v} = 1 \Rightarrow g \overline{g} = 1$.

$$(v_1, \ldots, v_{n+1}) \in S^{2n+1} \text{ for } \mathbb{C} \text{ or } S^{4n+3} \text{ for } \mathbb{H}.$$ 

$$g \in S^1 \text{ for } \mathbb{C} \text{ or } S^3 \text{ for } \mathbb{H}.$$ 

Thus, we come to two families of (Hopf) fibrations

$$S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n, \quad S^3 \rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n.$$ 

$$S^2 \rightarrow \mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n.$$
Let us consider $\mathbb{C}^{2n+2} \simeq \mathbb{H}^{n+1}$ with complex $\lambda$ and quaternionic $v$ coordinates:

$$v_i = \lambda_{2i-1} + j\lambda_{2i}, \quad i = 1, \ldots, n+1.$$ 

The charts on the projective manifolds:

$$z_{(\nu)}^{(\mu)} = \lambda_{\mu} \lambda_{\nu}^{-1}, \quad z_\mu, \lambda_\mu \in \mathbb{C}, \quad q_{(\beta)}^{(\alpha)} = v_\alpha v_{\bar{\beta}}^{-1}, \quad q_\alpha, v_\alpha \in \mathbb{H}.$$ 

$\mu, \nu = 1, \ldots, 2n+2, \quad \alpha, \beta = 1, \ldots n+1.$

It is obvious that the functions $z_i$ and $q_i$ are invariant under the global transformation

$$\lambda_\mu \rightarrow g_\lambda \lambda_\mu, \quad g_\lambda \bar{g}_\lambda = 1, \quad g_\lambda \in \mathbb{C},$$

$$v_i \rightarrow g_v v_i, \quad g_v \bar{g}_v = 1, \quad g_v \in \mathbb{H}.$$
By definition

\[ q_\alpha = v_\alpha v_{n+1}^{-1} \Rightarrow v_\alpha = q_\alpha v_{n+1} = q_\alpha (\lambda_{2n+1} + j\lambda_{2n+2}) = \lambda_{2\alpha-1} + j\lambda_{2\alpha} \]

Dividing both sides of the last equality by \( \lambda_{2n+2} \) we find

\[ q_\alpha (u + j) = z_{2\alpha-1} + jz_{2\alpha}, \quad \alpha = 1, \ldots, n \]

where

\[ z_r = \lambda_r / \lambda_{2n+2}, \quad z_{2n+1} = \lambda_{2n+1} / \lambda_{2n+2} \equiv u \]

Thus:

- \( z_1, \ldots, z_{2n+1} \) parameterize \( \mathbb{C}P^{2n+1} \)
- \( q_1, \ldots, q_n \) parameterize \( \mathbb{H}P^n \)
- \( u \) parameterizes \( \mathbb{C}P^1 \sim S^2 \).
Definitions:

\[ q_\alpha = w_{2\alpha-1} + jw_{2\alpha} \quad w \in \mathbb{C} \]

\[
(\Omega_{\mu\nu}) = \begin{pmatrix}
\varepsilon & 0 & 0 & \ldots \\
0 & \varepsilon & 0 & \ldots \\
0 & 0 & \varepsilon & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}, \quad \varepsilon = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

\[ z^\mu = uw^\mu + \Omega^{\mu\nu} \bar{w}_\nu, \quad z_{2n+1} = z^{2n+1} = u, \quad \mu, \nu = 1, \ldots, 2n. \]

- \( w^\mu \) has upper index, while its conjugate \( \bar{w}_\mu \) has a lower one.
- \( \Omega_{\mu\nu} \Omega^{\nu\lambda} = \delta^\lambda_\mu \),
- \( w_\mu = \Omega_{\mu\nu} w^\nu, \quad \bar{w}^\mu = \Omega^{\mu\nu} \bar{w}_\nu \)

The contraction is done, as usual, between upper and lower indices. So,

\[ z \bar{z} = z^\mu \bar{z}_\mu = -z_\mu \bar{z}^\mu \]
Fubini-Study metric

\[ ds^2 = \frac{dzd\bar{z}}{1 + z\bar{z}} - \frac{(\bar{z}dz)(d\bar{z}z)}{(1 + z\bar{z})^2} \]

\[ z \to (q, u) \]

\[ dqd\bar{q} - \frac{(\bar{q}dq)(d\bar{q}q)}{(1 + q\bar{q})^2} + \frac{(du + A)(d\bar{u} + \bar{A})}{(1 + u\bar{u})^2}, \]

where

\[ A = \frac{u(\bar{w}dw - wd\bar{w}) - (\bar{w}\Omega d\bar{w}) - u^2(w\Omega dw)}{1 + w\bar{w}} \]

\( \mathbb{C}P^N \) is a symmetric Riemannian manifold \( SU(N + 1)/U(1) \times SU(N) \) and thus, the isometries of the metric form a representation of algebra \( su(2n + 2) \).

\[ R_i = \partial_i + \bar{z}_i(\bar{z}\partial), \quad J^i_j = \iota (z^j\partial_i - \bar{z}_i\bar{\partial}^j) + \iota \delta^i_j ((z\partial) - (\bar{z}\bar{\partial})) \]
The system of a free particle on $\mathbb{C}P^{n+1}$ is defined by a triple

$$(H_0 = (g^{-1})_i^j p_j \bar{p}^i, \quad \omega = dp_i \wedge dz^i + d\bar{p}_i \wedge d\bar{z}^i, \quad T^*\mathbb{C}P^{2n+1})$$

The Noether constants defined by the $su(n+2)$ generators look as follows

$$R_i = p_i + \bar{z}_i(\bar{z}\bar{p}), \quad J_{i}^{j} = \imath \left( z^{j} p_i - \bar{z}_i \bar{p}^j \right) + \imath \delta_{i}^{j} ((zp) - (\bar{z}\bar{p}))$$
Complete canonical transformations:

\[ z^\mu = uw^\mu + \bar{w}^\mu, \quad z_{2n+1} = z^{2n+1} = u, \quad \mu, \nu = 1, \ldots, 2n. \]

\[ p_\mu = \frac{\bar{u}}{1 + uu} \pi_\mu + \frac{1}{1 + uu} \bar{\pi}_\mu, \quad p_{2n+1} = p_u - \frac{1}{1 + uu} (\bar{u}w^\mu \pi_\mu - w^\mu \bar{\pi}_\mu) \]

Rewriting the Hamiltonian via coordinates \((w, u, \pi, p_u)\) gives

\[ H_0 = (g^{-1})_\mu^\nu \bar{P}_\mu P_\nu + (1 + uu)^2 p_u \bar{p}_u, \]

where we introduced the inverse metric to

\[ (g^{-1})_\mu^\nu = (1 + w\bar{w})(\delta_\mu^\nu + \bar{w}_\mu w^\nu + w_\mu \bar{w}^\nu), \]

and the covariant momenta

\[ P_\mu = \pi_\mu - w_\mu \frac{l_3}{1 + w\bar{w}} - \bar{w}_\mu \frac{l_+}{1 + w\bar{w}}, \]

with the \(su(2)\) generators \(l_\pm, l_3\) defining the isometries of \(S^2\):

\[ l_3 = -i(\bar{u}p_u - \bar{\bar{p}}_u), \quad l_+ = \bar{p}_u + u^2 p_u, \quad l_- = p_u + \bar{u}^2 \bar{p}_u. \]
The Poisson brackets

\[ \{ l_3, l_\pm \} = \pm \imath l_\pm, \quad \{ l_+, l_- \} = 2 \imath l_3 \]

\[ \{ w^\mu, P_\nu \} = \delta^\mu_\nu, \quad \{ P_\mu, P_\nu \} = -2 \frac{\Omega_{\mu\nu}}{1 + w \bar{w}} l_+, \quad \{ P_\mu, \bar{P}^\nu \} = \imath \frac{\delta^\nu_\mu l_3}{(1 + w \bar{w})^2}. \]

\[ \{ P_\mu, l_+ \} = \frac{\bar{w}_\mu l_+}{1 + w \bar{w}}, \quad \{ P_\mu, l_- \} = -\frac{\bar{w}_\mu l_-}{1 + w \bar{w}} - 2 \imath \frac{w_\mu l_3}{1 + w \bar{w}}, \]

\[ \{ P_\mu, l_3 \} = \frac{\imath w_\mu l_+}{1 + w \bar{w}}. \]

\[ \{ l_3, p_u \} = -\imath p_u, \quad \{ l_+, p_u \} = 2 up_u, \quad \{ l_-, p_u \} = 0, \]

\[ \{ P_\mu, p_u \} = -\frac{p_u}{1 + w \bar{w}} (\bar{w}_\mu + 2uw_\mu). \]
It is easy to see that the Casimir operator of $su(2)$ coincides with the second summand of the Hamiltonian:

$$I^2 = I_+ I_- + I_3^2 = (1 + u\bar{u})^2 p_u \bar{p}_u$$

Fix $I^2 = s^2 = \text{const}$

$$I_+ = s \frac{2x}{1 + x\bar{x}}, \quad I_3 = s \frac{1 - x\bar{x}}{1 + x\bar{x}}, \quad \{x, \bar{x}\} = \frac{i}{2s} (1 + x\bar{x})^2, \quad s \in \mathbb{R}. $$

Thus we find:

$$\{P_\mu, x\} = \frac{\bar{w}_\mu x}{1 + w\bar{w}} - i \frac{x^2 \bar{w}_\mu}{1 + w\bar{w}},$$

$$\{P_\mu, \bar{x}\} = -\frac{\bar{w}_\mu \bar{x} + iw_\mu}{1 + w\bar{w}}, \quad \{w^\mu, x\} = \{w^\mu, \bar{x}\} = 0$$

and

$$H_0 = (g^{-1})_\mu^\nu \bar{P}_\mu P_\nu + s^2.$$
With such a substitution the generators $R$ and $J$ reduce to the $sp(n+1)$ generators of the isometries of $\mathbb{H}P^n$:

\[
L_{\mu}^{\nu} = J_{\mu}^{\nu} + J^{\nu}_{\mu} = \imath (w^{\nu} \pi_{\mu} - \bar{w}_{\mu} \bar{\pi}^{\nu}) + \imath (w_{\mu} \pi^{\nu} - \bar{w}^{\nu} \bar{\pi}_{\mu})
\]

\[
L_3 = J_{2n+1}^{2n+1} = l_3 + \frac{\imath}{2} ((\pi w) - (\bar{\pi} \bar{w})) ,
\]

\[
L_- = R_{2n+1} = (\bar{\pi}^{\mu} w_{\mu}) + l_- = \bar{L}_+
\]

\[
L_{\mu} = \imath J_{2n+1}^{\mu} - R^{\mu} = \bar{\pi}^{\mu} - ((\bar{\pi}^{\nu} w_{\nu}) - l_-) \bar{w}^{\mu} + ((\pi w) - \imath l_3) w^{\mu}
\]
\[
\{ L_{\mu\nu}, L_{\rho\sigma} \} = -\imath (\Omega_{\mu\rho} L_{\sigma\nu} + \Omega_{\nu\rho} L_{\sigma\mu} + \Omega_{\nu\sigma} L_{\mu\rho} + \Omega_{\mu\sigma} L_{\nu\rho}) \\
\{ L_+, L_- \} = 2\imath L_3, \quad \{ L_3, L_{\pm} \} = \pm \imath L_{\pm}, \quad \{ L_{\mu\nu}, L_{\pm} \} = \{ L_{\mu\nu}, L_3 \} = 0 \\
\{ L^\mu, L_- \} = 0, \quad \{ L^\mu, L_+ \} = \overline{L}^\mu, \quad \{ L^\mu, L_3 \} = \frac{\imath}{2} L^\mu, \\
\{ L^\mu, L^\rho \} = -\imath \delta^\mu_\rho L^\sigma - \imath L_\rho \Omega_{\mu\sigma}, \quad \{ L^\mu, L^\nu \} = 2\Omega^{\mu\nu} L_- , \\
\{ \overline{L}_\mu, \overline{L}_\nu \} = -2\Omega_{\mu\nu} L_+, \quad \{ L^\mu, \overline{L}_\nu \} = \imath L^\nu_\mu + 2\imath \delta^\mu_\nu L_3 \]
A more complicated but still integrable extension of the considered system is the following (S. Bellucci and A. Nersessian, Phys. Rev. D 67, 065013 (2003)):

\[ H_{\text{osc}} = H_{\mathbb{H}P^n} + \omega_0^2 w \bar{w} \]

This extension, of course does not preserve the whole Noether symmetries, however, it has some additional hidden symmetry generators:

\[ I_{\nu}^{\mu} = L^{\mu} \bar{L}_\nu - \bar{L}^{\mu} L_\nu + \omega_0^2 (w^{\mu} \bar{w}_\nu - \bar{w}^{\mu} w_\nu) \]

\[ T_{\mu\nu} = I_- (\bar{\pi}_\mu w_\nu - \pi_\nu w_\mu) + I_+ (\pi_\mu \bar{w}_\nu - \bar{w}_\nu \pi_\mu) - \nu I_3 (\pi_\mu w_\nu - \pi_\nu w_\mu + \bar{\pi}_\mu \bar{w}_\nu - \bar{w}_\nu \bar{\pi}_\mu) \]
Conclusion

- We have explicitly constructed the projection map for the fibration $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$.
- Using this map we reduced the free particle system on $\mathbb{C}P^{2n+1}$ to an isospin particle on $\mathbb{H}P^n$.
- We have presented an integrable oscillator extension of the considered system as well as studied its (cubic) symmetry algebra.