BCOV INVARIANT FOR CALABI-YAU PAIRS

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ABSTRACT. We construct BCOV invariant for Calabi-Yau pairs. The construction covers the classical BCOV invariant and certain equivariant BCOV invariant. The BCOV invariant obtained is expected to be well-behaved under birational equivalence.

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0. INTRODUCTION

The BCOV torsion is an invariant for Calabi-Yau manifolds. Bershadsky, Cecotti, Ooguri and Vafa initiated the study of BCOV torsion for Calabi-Yau threefold in the outstanding papers [1, 2]. Their work extended the mirror symmetry conjecture of Candelas, de la Ossa, Green and Parkes [11]. Fang and Lu [15] studied the BCOV torsion for Calabi-Yau manifolds of arbitrary dimension.

The BCOV invariant is another invariant for Calabi-Yau manifolds, which could be viewed as a normalization of the BCOV torsion. Fang, Lu and Yoshikawa [16] constructed and studied the BCOV invariant for Calabi-Yau threefolds. Their work confirmed a conjecture of Bershadsky, Cecotti, Ooguri and Vafa [1, 2] concerning the BCOV torsion of quintic mirror threefolds. Eriksson, Freixas and Mourougane [14] extended these constructions to Calabi-Yau manifolds of arbitrary dimension.

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Since the BCOV torsion is defined as a product of certain Quillen metrics, the works of Bismut, Gillet and Soulé [5, 6, 7] on the Quillen metric is of fundamental importance in the study of BCOV invariant. The formula of Bismut and Lebeau [8] is another powerful tool in the study of BCOV invariant.

In this paper, we extend the BCOV invariant to Calabi-Yau pairs. Let $X$ be a Kähler manifold. Let $K_X$ be its canonical bundle. Let $m \in \mathbb{Z} \setminus \{0, -1\}$. Let $K_X^m$ be the $m$-th tensor power of $K_X$. We assume that $H^0(X, K_X^m) \neq 0$. Let $\gamma \in H^0(X, K_X^m) \setminus \{0\}$. Let $Y$ be the zero locus of $\gamma$, which we call a $m$-canonical divisor. We assume that $Y$ is a smooth reduced divisor. We call $(X, Y)$ a Calabi-Yau pair. We will construct a real number $\tau(X, Y)$ depending only on the complex structure of $(X, Y)$. In particular, if $X$ is Calabi-Yau, the real number $\tau(X, \emptyset)$ is exactly the BCOV invariant of $X$.

**Curvature of $\tau(X, Y)$**. Let $\pi_X : \mathcal{D} \to S$ be a holomorphic fibration. We denote $X_s = \pi_X^{-1}(s)$ for $s \in S$. Let $\mathcal{Y} \subseteq \mathcal{D}$ be a complex hypersurface such that the restricted map $\pi_{\mathcal{Y}} := \pi_{\mathcal{D}} |_{\mathcal{Y}}$ is also a fibration. For $s \in S$, we denote $Y_s = \pi_{\mathcal{Y}}^{-1}(s)$. We assume that $(X_s, Y_s)$ is a Calabi-Yau pair for each $s \in S$. Let $\tau(X, Y)$ be the function $s \mapsto \tau(X_s, Y_s)$.

Our central result is a formula relating the BCOV torsion $\tau(X, Y)$ to the Hodge form and the Weil-Petersson form.

First we introduce the Hodge form. Let $g^{H^{p,q}(X)}$ be Hermitian metrics on the holomorphic vector bundles $H^{p,q}(X)$ over $S$ such that $g^{H^{p,q}(X)}(u, u) = g^{H^{p,q}(X)}(\pi, \pi)$ for $u \in H^{p,q}(X)$. The Hodge form associated with $H^\bullet(X)$ can be defined by

$$
\omega_{H^\bullet(X)} = \frac{1}{2} \sum_{0 \leq p, q \leq \dim X} (-1)^{p+q}(p-q)c_1(H^{p,q}(X), g^{H^{p,q}(X)}) \in \Omega^{1,1}(S),
$$

which is independent of the Hermitian metrics $g^{H^{p,q}(X)}$. In [1.2] we will give an intrinsic definition of the Hodge form, which does not involve any metric. We remark that (0.1) is exactly the Hermitian form of the Hodge metric considered in [15]. The Hodge form $\omega_{H^\bullet(Y)} \in \Omega^{1,1}(S)$ associated with $H^\bullet(Y)$ can be defined the same way.

Now we introduce the Weil-Petersson form. Locally we have a holomorphic map $s \mapsto \gamma_s \in H^0(X_s, K_{X_s}^m)$ such that $Y_s$ is the zero locus of $\gamma_s$. The function

$$
P : s \mapsto \log \int_{X_s \setminus Y_s} |\gamma_s \bar{\gamma_s}|^{1/m}
$$

is well-defined up to harmonic functions, where $|\gamma_s \bar{\gamma_s}|^{1/m}$ is the unique real positive volume form on $X_s \setminus Y_s$ such that $m$-th power equals $i^{n^2} \gamma_s \wedge \bar{\gamma_s}$. The Weil-Petersson form is defined by

$$
\omega_{\pi_{\mathcal{D}}, \pi_{\mathcal{Y}}} = -\frac{\partial \bar{\partial} P}{2\pi i} \in \Omega^{1,1}(S).
$$

In fact, the Weil-Petersson form $\omega_{\pi_{\mathcal{D}}, \pi_{\mathcal{Y}}}$ is the Kähler form of the Weil-Petersson metric ([12, 22], cf. [17]).

We denote

$$
w(X, m) = \frac{\chi(X)}{12} - \frac{\chi(Y)}{12(m+1)} = \frac{1}{12} \int_X c_n(TX) - \frac{1}{12(m+1)} \int_Y c_{n-1}(TY).$$
Theorem 0.1. The function $\tau(X, Y)$ satisfies the following equation,

\begin{equation}
\frac{\partial^2}{\partial \tau} \tau(X, Y) = \omega_{H^*} - \frac{1}{m+1} \omega_{H^*} - w(X, m) \omega_{\pi_X, \pi_Y}.
\end{equation}

Relation with Yoshikawa’s equivariant BCOV invariant and Borcherds product.
We consider the case $m = -2$, $\dim X = 2$. Then $X$ admits a ramified 2-cover $X' \to X$ whose branch locus is $Y$. Moreover, $X'$ is a 2-elementary K3 surface, i.e., $X'$ is a K3 surface equipped with an involution $\iota$ commuting with $X' \to X$. Yoshikawa [25] constructed an equivariant BCOV invariant for 2-elementary K3 surfaces, which we denote by $\tau(X', \iota)$.

Theorem 0.2. For $m = -2$ and $X$ a del Pezzo surface with $K_X^2$ very ample, we have

\begin{equation}
\tau(X, Y) = -\tau(X', \iota) + \nu(X),
\end{equation}

where $\nu(X)$ depends only on $X$.

The proof of Theorem 0.2 is based on Theorem 0.1 and [8, Theorem 0.1]. Let $(\cdot, \cdot)$ be the intersection form on $\text{Pic}(X)$. By [18, Chapter III, 3.4 Proposition], if $\deg X := (K_X, K_X) \geq 2$, then $K_X^2$ is very ample.

Let $g$ be the genus of the curve $Y$. By [19, Theorem 0.1], the function

\begin{equation}
(X', \iota) \mapsto \exp \left( -2g(2g+1)\tau(X', \iota) \right)
\end{equation}

on the moduli space of $(X', \iota)$ is the product of a Borcherds product [9] and a Siegel modular form. By Theorem 0.2, the same result holds for $\tau(X, Y)$ if $X$ is rigid, i.e., $X$ admits no deformation, which holds for $\deg X \geq 5$.

Behavior of $\tau(X, Y)$ under blowing up. Let $(X, Y)$ be a Calabi-Yau pair with $m = 1$ and $Z \subseteq X$ be a sub manifold of codimension 2 such that $Z \cap Y = \emptyset$. Let $X'$ be the blowing up of $X$ along $Z$. We denote by $f : X' \to X$ the canonical projection. Set $Y' = f^{-1}(Y \cup Z) \subseteq X'$. Then $(X', Y')$ is also a Calabi-Yau pair with $m = 1$. We are interested in the value of

\begin{equation}
\tau(X', Y') - \tau(X, Y).
\end{equation}

Here the technical conditions $Z \cap Y = \emptyset$ and $\text{codim} Z = 2$ are due to our hypothesis that the canonical divisor $Y'$ is smooth and reduced.

Theorem 0.3. There exists $\nu \in \mathbb{R}$ such that for $X, Y, Z, X', Y'$ as above with $\dim X = 2$ and $Z$ a single point, we have

\begin{equation}
\tau(X', Y') - \tau(X, Y) = \nu.
\end{equation}

In other words, $\nu$ is a universal constant.

The proof of Theorem 0.3 is based on [8, Theorem 0.1] and [4, Theorem 8.10].

This paper is organized as follows. In [11] we introduce several fundamental notions and constructions. In [2] we construct the BCOV invariant $\tau(X, Y)$ and establish
Theorem 0.1. In §3, we establish Theorem 0.2. In §4, we establish Theorem 0.3 together with a weak result about (0.8) in arbitrary dimension. In the appendix §5, we explicitly calculate several Bott-Chern forms, which will be used in §3.

Notations. For $p, q \in \mathbb{N}$ and a complex vector bundle $F$ over a complex manifold $S$, we denote by $\Omega^{p,q}(S, F)$ (resp. $A^{p,q}(S, F)$) the vector space of $(p, q)$-forms (resp. $(p, q)$-current) on $S$ with values in $F$. For ease of notation, we denote $\Omega^{p,q}(S) = \Omega^{p,q}(S, \mathbb{C})$ (resp. $A^{p,q}(S) = A^{p,q}(S, \mathbb{C})$). For a differential form (resp. current) $\omega$ on $S$, its component of degree $(p, q)$ is denoted by $\{\omega\}^{(p,q)}$.

For $k \in \mathbb{N}$ and a complex manifold $S$, we denote by $H^k(S)$ the $k$-th de Rham cohomology of $S$ with coefficients in $\mathbb{C}$. For $p, q \in \mathbb{N}$ and a complex manifold $S$, we denote $H^{p,q}(S) = H^q(S, \Omega^p_S)$. If $S$ is a compact Kähler manifold, we identify $H^{p,q}(S)$ with a sub vector space of $H^{p+q}(S)$ via the Hodge theory.

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1. Preliminary

1.1. Chern form and Bott-Chern form. Let $S$ be a complex manifold. Let $E$ be a holomorphic vector bundle over $S$. Let $g^E$ be a Hermitian metric on $E$. Let

$$R^E \in \Omega^{1,1}(S, \text{End}(E))$$

be the Chern curvature of $(E, g^E)$. For $k \in \mathbb{N}$, we denote by $c_k$ the $k$-th elementary symmetric polynomial. The $k$-th Chern form of $(E, g^E)$ is defined by

$$c_k(E, g^E) := c_k \left( -\frac{R^E}{2\pi i} \right) \in \Omega^{k,k}(S).$$

The $k$-th Chern class of $E$ is defined by

$$c_k(E) := [c_k(E, g^E)] \in H^{2k}(S),$$

which is independent of $g^E$.

We denote

$$c(E, g^E) = 1 + c_1(E, g^E) + c_2(E, g^E) + ... \in \bigoplus_{k \in \mathbb{N}} \Omega^{k,k}(S).$$

The total Chern class is defined by

$$c(E) := [c(E, g^E)] \in H^\bullet(S).$$

For a short exact sequence of holomorphic vector bundles over $S$,

$$0 \to E' \to E \to E'' \to 0,$$
we have

\( c(E) = c(E')c(E'') \in H^*(S) \).

Let \( g^E \) be a Hermitian metric on \( E \). Let \( g^{E'} \) be the Hermitian metric on \( E' \) induced by \( g^E \) via the embedding \( E' \to E \). Let \( g^{E''} \) be the quotient Hermitian metric on \( E'' \) induced by \( g^E \) via the surjection \( E \to E'' \). The Bott-Chern form \([5, \text{Section 1f}]\)

\[
\tilde{c}(E, E', g^E) \in \bigoplus_{k \in \mathbb{N}} \Omega^{k,k}(S) \\
\partial \Omega^{k-1,k}(S) + \bar{\partial} \Omega^{k,k-1}(S)
\]
is such that

\[
\frac{\partial \bar{\partial}}{2\pi i} \tilde{c}(E, E', g^E) = c(E, g^E) - c(E', g^{E'})c(E'', g^{E''}).
\]

1.2. Hodge form. Let \( S \) be a complex manifold. Let \( H^\bullet_Z \) be a local system of finitely generated graded \( \mathbb{Z} \)-module over \( S \). We assume that \( H^k_Z = 0 \) for \( k < 0 \) and \( k > n \). We denote \( H^\bullet_C = H^\bullet_Z \otimes \mathbb{C} \), which a graded flat complex vector bundle over \( S \).

For \( k = 0, \ldots, n \), let

\[
H^k_C = F^0 H^k_C \supseteq F^1 H^k_C \supseteq \cdots \supseteq F^k H^k_C \supseteq F^{k+1} H^k_C = 0
\]
be a filtration of \( H^k_C \) by holomorphic sub vector bundles. We assume that there exists a decomposition of \( H^k_C \) by smooth complex sub vector bundles

\[
H^k_C = \bigoplus_{0 \leq p+q \leq n} H^{p,q}_C
\]
such that

\[
F^p H^k_C = \bigoplus_{p' = p}^k H^{p',k-p'}_C, \quad H^{p,q}_C = \overline{H^{p',q'}_C}.
\]

We remark that the decomposition (1.11) is uniquely determined by the filtration (1.10) via the following identity,

\[
H^{p,q}_C = F^p H^{p+q}_C \cap F^{q+1} H^{p+q}_C.
\]

Moreover, the identification

\[
H^{p,q}_C = F^p H^{p+q}_C / F^{p+1} H^{p+q}_C
\]
induces a holomorphic structure on \( H^{p,q}_C \). We call \( H^\bullet := (H^\bullet_Z, F^\bullet H^\bullet_C) \) a variation of Hodge structure over \( S \).

Set

\[
\lambda = \bigotimes_{0 \leq p+q \leq n} \left( \det H^{p,q}_C \right)^{(-1)^{p+q}p}, \quad \lambda_{\text{dR}} = \bigotimes_{k=1}^n \left( \det H^k_C \right)^{(-1)^k}.
\]

Then \( \lambda \) (resp. \( \lambda_{\text{dR}} \)) is a holomorphic (resp. flat) line bundle over \( S \). We have

\[
\lambda_{\text{dR}} = \lambda \otimes \lambda.
\]
Let $U \subseteq S$ be a small open subset. Let $\tau \in H^0(U, \lambda)$ be a nowhere vanishing holomorphic section. Let $\sigma \in \mathcal{C}^\infty(U, \lambda_{\operatorname{dR}})$ be a non zero constant section. There exists $f \in \mathcal{C}^\infty(U, \mathbb{C})$ such that

\begin{equation}
\sigma = e^\tau \otimes \tau .
\end{equation}

Then $\overline{\partial \partial} \operatorname{Ref} \in \Omega^{1,1}(U)$ is independent of $\tau$ and $\sigma$. The Hodge form $\omega_{H^\bullet} \in \Omega^{1,1}(S)$ associated with the variation of Hodge structure $H^\bullet$ is defined by

\begin{equation}
\omega_{H^\bullet}|_U = \overline{\partial \partial} \operatorname{Ref} / 2\pi i .
\end{equation}

Let $g^{H^\xi}$ be a Hermitian metric on $H^\xi_C$ such that

\begin{equation}
g^{H^\xi}(u, v) = 0 , \quad \text{for } u \in H^{p,q}_C, v \in H^{p',q'}_C \text{ with } (p, q) \neq (p', q') ,
\end{equation}

\begin{equation}
g^{H^\bullet}(u, u) = g^{H^\bullet}(\pi, \pi) , \quad \text{for } u \in H^\bullet_C .
\end{equation}

Let $g^{H^{\xi q}}$ be the restriction of $g^{H^\xi}$ to $H^{p,q}_C$. Let $c_1(H^{p,q}_C, g^{H^{p,q}}) \in \Omega^{1,1}(S)$ be the first Chern form of $(H^{p,q}_C, g^{H^{p,q}})$.

**Proposition 1.1.** The following identity holds,

\begin{equation}
\omega_{H^\bullet} = \frac{1}{2} \sum_{0 \leq p, q \leq n} (-1)^{p+q} (p - q) c_1(H^{p,q}_C, g^{H^{p,q}}) .
\end{equation}

**Proof.** Let $\| \cdot \|_\lambda$ (resp. $\| \cdot \|_{\lambda_{\operatorname{dR}}}$) be the norm on $\lambda$ (resp. $\lambda_{\operatorname{dR}}$) induced by $g^{H^\xi}$.

Let $U \subseteq S$ be a small open subset. Let $\tau \in H^0(U, \lambda)$ be a nowhere vanishing holomorphic section. Let $\sigma \in \mathcal{C}^\infty(U, \lambda_{\operatorname{dR}})$ be a non zero constant section. Let $f \in \mathcal{C}^\infty(U, \mathbb{C})$ be as in (1.17). By (1.17) and (1.19), we have

\begin{equation}
\operatorname{Ref} = - \log \| \tau \|^2_\lambda + \frac{1}{2} \log \| \sigma \|^2_{\lambda_{\operatorname{dR}}} .
\end{equation}

By the Poincaré-Lelong formula, (1.18) and (1.21), we have

\begin{equation}
\omega_{H^\bullet} = c_1(\lambda, \| \cdot \|_\lambda) - \frac{1}{2} c_1(\lambda_{\operatorname{dR}}, \| \cdot \|_{\lambda_{\operatorname{dR}}}) .
\end{equation}

On the other hand, by (1.19), we have

\begin{equation}
c_1(\lambda, \| \cdot \|_\lambda) = \sum_{0 \leq p, q \leq n} (-1)^{p+q} p c_1(H^{p,q}_C, g^{H^{p,q}}) ,
\end{equation}

\begin{equation}
c_1(\lambda_{\operatorname{dR}}, \| \cdot \|_{\lambda_{\operatorname{dR}}}) = \sum_{0 \leq p, q \leq n} (-1)^{p+q} (p + q) c_1(H^{p,q}_C, g^{H^{p,q}}) .
\end{equation}

By (1.22) and (1.23), we obtain (1.20). This completes the proof. \hfill \square

For $r \in \mathbb{N}$, we denote by $H^\bullet[r]$ the $r$-th right shift of $H^\bullet$, i.e.,

\begin{equation}
H^k_Z[r] = H^{k-2r}_Z , \quad H^{p,q}_C[r] = H^{p-r,q-r}_C .
\end{equation}

**Proposition 1.2.** The following identity holds,

\begin{equation}
\omega_{H^\bullet} = \omega_{H^\bullet[r]} .
\end{equation}

**Proof.** The right hand side of (1.20) is invariant under right shift. \hfill \square
1.3. BCOV torsion. Let \( X \) be a compact Kähler manifold. Let \( n = \dim X \). Let \( \omega \) be a Kähler form on \( X \). For \( p = 1, \cdots, n \), set

\[
\lambda_p(X) = \bigotimes_{q=0}^{n} \left( \det H^{p,q}(X) \right)^{(-1)^q}.
\]

Let \( \| \cdot \|_{\lambda_p(X),\omega} \) be the Quillen metric [21, 5] on \( \lambda_p(X) \) associated with \( \omega \). Set

\[
\lambda(X) = \bigotimes_{p=1}^{n} \left( \lambda_p(X) \right)^{(-1)^p} = \bigotimes_{0 \leq p,q \leq n} \left( \det H^{p,q}(X) \right)^{(-1)^{p+q}}.
\]

Let \( \| \cdot \|_{\lambda(X),\omega} \) be the metric on \( \lambda(X) \) induced by \( \| \cdot \|_{\lambda_p(X),\omega} \). Set

\[
\lambda_{\text{dR}}(X) = \lambda(X) \otimes \bar{\nabla}(X) = \bigotimes_{k=1}^{n} \left( \det H^k(X) \right)^{(-1)^k}.
\]

Let \( \| \cdot \|_{\lambda_{\text{dR}}(X),\omega} \) be the metric on \( \lambda_{\text{dR}}(X) \) induced by \( \| \cdot \|_{\lambda(X),\omega} \).

Let \( \sigma_{k,1}, \cdots, \sigma_{k,m_k} \in \text{Im} \left( H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{R}) \right) \) be a basis of the lattice. Set

\[
\sigma_k = \sigma_{k,1} \wedge \cdots \wedge \sigma_{k,m_k} \in \det H^k(X),
\]

which is well-defined up to \( \pm 1 \). Set

\[
\sigma = \bigotimes_{k=1}^{n} \sigma_k^{(-1)^k} \in \lambda_{\text{dR}}(X).
\]

The BCOV torsion of \((X,\omega)\) is defined by

\[
\tau_{\text{BCOV}}(X,\omega) = \log \| \sigma \|_{\lambda_{\text{dR}}(X),\omega}.
\]

Now let \( \pi_{\mathcal{X}} : \mathcal{X} \to S \) be a holomorphic fibration. We denote \( X_s = \pi_{\mathcal{X}}^{-1}(s) \) for \( s \in S \). Let \( \omega \in \Omega^{1,1}(\mathcal{X}) \) be a fiberwise Kähler form on \( \mathcal{X} \), i.e., the restriction of \( \omega \) to each fiber is a Kähler form. Let \( g^{TX} \) be the fiberwise Kähler metric induced by \( \omega \). Let \( \omega_{H^\ast}(X) \in \Omega^{1,1}(S) \) be the Hodge form associated with the variation of Hodge structure \( H^\ast(X) \) over \( S \). We denote by \( \tau_{\text{BCOV}}(X,\omega) \) the function \( s \mapsto \tau_{\text{BCOV}}(X_s,\omega|_{X_s}) \) on \( S \).

**Theorem 1.3.** We have

\[
\frac{\overline{\partial \partial}}{2\pi i} \tau_{\text{BCOV}}(X,\omega) = \omega_{H^\ast}(X) + \frac{1}{12} \int_X c_1(TX,g^{TX}) c_n(TX,g^{TX}).
\]

**Proof.** We may assume that \( S \) a unit disc in \( \mathbb{C} \). Let \( \tau \in H^0(S,\lambda(X)) \) be a nowhere vanishing holomorphic section. Let \( f \in \mathcal{C}^\infty(S,\mathbb{R}) \) such that \( \sigma = \pm e^f \tau \otimes \bar{\tau} \). By the definition of Hodge form in [1,2], the definition of BCOV torsion and the Poincaré-Lelong formula, we have

\[
\frac{\overline{\partial \partial}}{2\pi i} \tau_{\text{BCOV}}(X,\omega) = \frac{\overline{\partial \partial} f}{2\pi i} + \frac{\overline{\partial \partial} \log \| \tau \|_{\lambda(X),\omega}^2}{2\pi i} = \omega_{H^\ast}(X) - c_1(\lambda(X),\| \cdot \|_{\lambda(X),\omega}).
\]
By [5] Theorem 0.1] and [2, page 374], we have

\[
c_1(\lambda(X), \| \cdot \|_{\lambda(X), \omega})
= \sum_{k=1}^{n} (-1)^k k \left( \int_X Td(TX, g^{TX}) \text{ch}(A^k(T^*X), g^{T^*X}) \right)_{(1,1)}
= -\frac{1}{12} \int_X c_1(TX, g^{TX}) c_n(TX, g^{TX}) \cdot
\]

(1.35)

By (1.34) and (1.35), we obtain (1.33). This completes the proof. 

2. BCOV INVARINANT FOR CALABI-YAU PAIRS

2.1. Construction of \( \tau(X, Y) \). Let \( m, X, \gamma \) and \( Y \) be as in the introduction. We denote \( n = \dim X \).
Let \( \omega \) be a Kähler form on \( X \). Let \( \| \cdot \|_{\omega} \) be the norm on \( K_X^m \) induced by \( \omega \). Let \( g^{TX} \) be the metric on \( TX \) induced by \( \omega \). Recall that we defined Chern forms in \( \S 1.1 \). Set

\[
a_X(\gamma, \omega) = \frac{1}{12} \int_X c_n(TX, g^{TX}) \log \| \gamma \|_{\omega}^2.
\]

(2.1)

Let \( N_Y \) be the normal bundle of \( Y \subseteq X \). Let \( \nabla \) be a connection on \( K_X^m \). Set

\[
\gamma' = \nabla \gamma \bigg|_Y \in H^0(Y, N_Y^{-1} \otimes K_X^m) = H^0(Y, N_Y^{-m-1} \otimes K_Y^m),
\]

(2.2)

which is independent of \( \nabla \). Let \( \| \cdot \|_{\omega} \) be the norm on \( N_Y^{-m-1} \otimes K_Y^m \) induced by \( \omega \). Let \( g^{TY} \) be the metric on \( TY \) induced by \( \omega \). Set

\[
a_Y(\gamma, \omega) = \frac{1}{12} \int_Y c_{n-1}(TY, g^{TY}) \log \| \gamma' \|_{\omega}^2.
\]

(2.3)

Recall that we defined Bott-Chern forms in \( \S 1.1 \). Set

\[
b_Y(\omega) = \frac{1}{12} \int_Y \bar{c}(TX \big|_Y, TY, g^{TX} \big|_Y). \]

(2.4)

Recall that we defined BCOV torsion in \( \S 1.3 \). Recall that we defined \( w(X, m) \in \mathbb{N} \) in \( \S 0.4 \). Set

\[
\tau(X, \gamma, \omega) = \tau_{BCOV}(X, \omega) - \frac{1}{m+1} \tau_{BCOV}(Y, \omega|_Y)
- \frac{1}{m} a_X(\gamma, \omega) + \frac{1}{m(m+1)} a_Y(\gamma, \omega) + \frac{1}{m} b_Y(\omega)
\]

+ \( w(X, m) \log \int_{X \setminus Y} |\gamma|^{1/m} \).

(2.5)

Theorem 2.1. The real number \( \tau(X, \gamma, \omega) \) is independent of \( \omega \).

Proof. Let \( \omega \) be a fiberwise Kähler form on \( X \times \mathbb{C}P^1 \). Then \( a_X(\gamma, \omega), a_Y(\gamma, \omega), b(\gamma, \omega) \) and \( \tau(X, \gamma, \omega) \) become real functions on \( \mathbb{C}P^1 \). It suffices to show that \( \tau(X, \gamma, \omega) \) is a constant functions on \( \mathbb{C}P^1 \).
By (1.33), we have
\[
\frac{\partial \partial }{2 \pi i} \tau_{BCOV}(X, \omega | X) = \frac{1}{12} \int_X c_1(TX, g^{TX}) c_n(TX, g^{TX}) ,
\]
(2.6)
\[
\frac{\partial \partial }{2 \pi i} \tau_{BCOV}(Y, \omega | Y) = \frac{1}{12} \int_Y c_1(TY, g^{TY}) c_{n-1}(TY, g^{TY}) .
\]

By the Poincaré-Lelong formula, we have
\[
\frac{\partial \partial }{2 \pi i} a_X(\gamma, \omega) = \frac{m}{12} \int_X c_n(TX, g^{TX}) + \frac{1}{12} \int_Y c_n(TX, g^{TX}) ,
\]
(2.7)
\[
\frac{\partial \partial }{2 \pi i} a_Y(\gamma, \omega) = \frac{1}{12} \int_Y c_{n-1}(TY, g^{TY}) (m c_1(TY, g^{TY}) + (m + 1) c_1(N_Y, g^{N_Y}) ) .
\]

By the definition of Bott-Chern form in §1.1, we have
\[
\frac{\partial \partial }{2 \pi i} b_Y(\omega) = \frac{1}{12} \int_Y c_n(TX, g^{TX}) - \frac{1}{12} \int_Y c_{n-1}(TY, g^{TY}) c_1(N_Y, g^{N_Y}) .
\]
(2.8)

By (2.6)-(2.8), we get
\[
\frac{\partial \partial }{2 \pi i} \tau(X, \gamma, \omega) = 0 .
\]
(2.9)
Since CP1 is compact, \( \tau(X, \gamma, \omega) \) is constant on CP1. This completes the proof. \( \square \)

For \( z \in \mathbb{C}^* \), a direct calculation yields
\[
a_X(z \gamma, \omega) - a_X(\gamma, \omega) = \frac{\chi(X)}{12} \log |z|^2 ,
\]
(2.10)
\[
a_Y(z \gamma, \omega) - a_Y(\gamma, \omega) = \frac{\chi(Y)}{12} \log |z|^2 ,
\]
\[
\log \int_{X \setminus Y} |z^{m-g} \gamma|^{1/m} = \log \int_{X \setminus Y} |\gamma|^{1/m} = \frac{1}{m} \log |z|^2 .
\]

As a consequence, we have
\[
\tau(X, z \gamma, \omega) = \tau(X, \gamma, \omega) .
\]
(2.11)

Definition 2.2. The BCOV invariant of \((X, Y)\) is defined by
\[
\tau(X, Y) = \tau(X, \gamma, \omega) .
\]
(2.12)
By Theorem 2.1 and (2.11), the BCOV invariant \( \tau(X, Y) \) is well-defined.

2.2. Curvature of \( \tau(X, Y) \). Recall that \( \omega_{H^{\bullet}(X)}, \omega_{H^{\bullet}(Y)} \in \Omega^{1,1}(S) \) are Hodge forms (see §1.2) associated with the variations of Hodge structure \( H^{\bullet}(X) \) and \( H^{\bullet}(Y) \) over \( S \).

Proof of Theorem 0.1. We may suppose that \( S \) is a unit disc in \( \mathbb{C} \). Let \( \omega \) be a fiberwise Kähler form on \( X \). By (1.33), we have
\[
\frac{\partial \partial }{2 \pi i} \tau_{BCOV}(X, \omega | X) = \omega_{H^{\bullet}(X)} + \frac{1}{12} \int_X c_1(TX, g^{TX}) c_n(TX, g^{TX}) ,
\]
(2.13)
\[
\frac{\partial \partial }{2 \pi i} \tau_{BCOV}(Y, \omega | Y) = \omega_{H^{\bullet}(Y)} + \frac{1}{12} \int_Y c_1(TY, g^{TY}) c_{n-1}(TY, g^{TY}) .
\]
Noticing that (2.7) and (2.8) equally hold for $(\pi_{\mathcal{O}}, \pi_{\mathcal{O}^*})$, formula (0.5) follows from (2.7), (2.8), (2.13) and the definition of the Weil-Petersson form (see (0.3)).

3. Example

In the whole section, we assume that $X$ is a del Pezzo surface, i.e., $\dim X = 2$ and $K_X^{-1}$ is ample. We also assume that $Y \subseteq X$ is a smooth reduced $(-2)$-canonical divisor.

3.1. Boundary behavior. Let $\gamma \in H^0(X, K_X^{-2})$ with zero locus $Y$. We have

$$\gamma' := \nabla \gamma|_Y \in H^0(Y, N_Y^{-1} \otimes K_X^{-2}) = H^0(Y, K_Y^{-1} \otimes K_X^{-1}).$$

Let

$$\gamma'^{-1} \in H^0(Y, K_Y \otimes K_X)$$

be the inverse of $\gamma'$.

Let $\gamma : Y \to X$ be the canonical embedding. We have a short exact sequence of coherent sheaves on $X$,

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X^{-1}) \to j_* \mathcal{O}_Y(K_Y) \to 0,$$

where $\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X^{-1})$ is defined by $\gamma$, $\mathcal{O}_X(K_X^{-1}) \to j_* \mathcal{O}_Y(K_Y)$ is defined by $(\gamma')^{-1}$. Since $K_X^{-1}$ is ample, by the Kodaira-Nakano vanishing theorem and Serre duality, we have

$$H^{<2}(X, K_X) = H^{>0}(X, K_X^{-1} \otimes K_X)^* = 0,$$

$$H^{>0}(X, K_X^{-1}) = H^{>0}(X, K_X^{-2} \otimes K_X) = 0.$$

Taking the long exact sequence associated with (3.3) and applying (3.4), we get the following isomorphisms,

$$H^0(X, K_X^{-1}) \to H^{1,0}(Y), \quad H^{1,1}(Y) \to H^{2,2}(X).$$

The first isomorphism in (3.5) could be explicitly calculated as follows,

$$H^0(X, K_X^{-1}) \to H^{1,0}(Y) \quad \phi \mapsto (\gamma')^{-1} \phi|_Y.$$

The second isomorphism in (3.5) is just the dual of the canonical identification

$$H^{0,0}(X) = H^{0,0}(Y) = \mathbb{C}.$$

Set

$$\lambda_1 = \bigotimes_{q=0}^2 (\det H^q(X, K_X^{-1}))^{(-1)^q} = \det H^0(X, K_X^{-1}),$$

$$\lambda_2 = \bigotimes_{q=0}^2 (\det H^q(X, K_X))^{(-1)^q} = H^{2,2}(X).$$

Let $\omega$ be a Kähler form on $X$. Let $\| \cdot \|_{\lambda_1, \omega}$ (resp. $\| \cdot \|_{\lambda_2, \omega}$) be the Quillen metric (see [21, 5]) on $\lambda_1$ (resp. $\lambda_2$) associated with $\omega$.

Let $g$ be the genus of $Y$. By the first isomorphism in (3.5), we have

$$g = \dim H^0(X, K_X^{-1}).$$
We fix a basis \( \phi_1, \ldots, \phi_g \in H^0(X, K_X^{-1}) \). Set
\[
(3.10) \quad \phi_X = \phi_1 \wedge \cdots \wedge \phi_g \in \det H^0(X, K_X^{-1}) = \lambda_1.
\]
Let \( \varphi_1, \ldots, \varphi_g \in H^1(Y) \) be the images of \( \phi_1, \ldots, \phi_g \in H^0(X, K_X^{-1}) \) via the first isomorphism in (3.5). Let \( \langle \cdot, \cdot \rangle \) be the intersection form on \( H^1(Y) \). Set
\[
(3.11) \quad J(\gamma, \phi_X) = \det \left( \langle \varphi_i, \bar{\varphi}_j \rangle_{1 \leq i, j \leq g} \right),
\]
which is determined by \( \gamma \) and \( \phi_X \).

Let \( 1_X \in H^{0,0}(X) \) be the constant function 1 on \( X \). Let
\[
(3.12) \quad 1_X^* \in H^{2,2}(X) = \lambda_2
\]
be the dual of \( 1_X \), i.e., \( 1_X^* \) is represented by a volume form on \( X \) of volume 1.

We denote
\[
\alpha_X(\gamma, \omega) = \frac{1}{12} \int_X c_1^2(TX, g^{TX}) \log \| \gamma \|_\omega^2,
\]
\[
\alpha_Y(\gamma, \omega) = \frac{1}{12} \int_Y c_1(TX, g^{TX}) \log \| \gamma \|_\omega^2,
\]
\[
\beta_Y(\gamma, \omega) = \frac{1}{12} \int_Y \log \| \gamma \|_\omega^2 \frac{\bar{\partial} \partial}{\partial \tau} \log \| \gamma \|_\omega^2.
\]

**Proposition 3.1.** We have
\[
\tau(X, Y) = \tau_{\text{BCOV}}(X, \omega) + \log \| 1_X^* \|_{\omega_{\lambda_2, \omega}}^2 - \log \| \phi_X \|_{\lambda_1, \omega}^2 + \log J(\gamma, \phi_X)
\]
\[
+ 3\alpha_X(\gamma, \omega) + \frac{3}{2} \alpha_Y(\gamma, \omega) + \frac{9}{2} \beta_Y(\gamma, \omega) - \frac{3}{2} \beta_Y(\gamma, \omega)
\]
\[
+ w(X, -2) \log \int_X |\gamma|^{-1/2} + \text{constant}.
\]

Here 'constant' means a number depending only on the topology of \( X \).

**Proof:** The proof consists of several steps. All the metrics involved in the proof are induced by \( \omega \).

**Step 1.** We show that the right hand side of (3.14) is independent of \( \omega \).

Let \( \omega \) be a fiberwise Kähler form on \( X \times \mathbb{C}P^1 \). Then all the terms involved become functions on \( \mathbb{C}P^1 \). By [5, Theorem 0.1], we have
\[
\frac{\bar{\partial} \partial}{2\pi i} \log \| 1_X^* \|_{\omega_{\lambda_2, \omega}}^2 - \frac{\bar{\partial} \partial}{2\pi i} \log \| \phi_X \|_{\lambda_1, \omega}^2
\]
\[
= \left\{ \int_X \text{Td}(TX, g^{TX}) \left( \text{ch}(K_X^{-1}, g^{K_X^{-1}}) - \text{ch}(K_X, g^{K_X}) \right) \right\}^{(1,1)}
\]
\[
= \frac{1}{2} \int_X c_1^3(TX, g^{TX}) + \frac{1}{6} \int_X c_1(TX, g^{TX}) c_2(TX, g^{TX}) \in \Omega^{1,1}(\mathbb{C}P^1).
\]

By the Poincaré-Lelong formula, we have
\[
\frac{\bar{\partial} \partial}{2\pi i} \alpha_X(\gamma, \omega) = \frac{1}{12} \int_Y c_1^2(TX, g^{TX}) - \frac{1}{6} \int_X c_3^2(TX, g^{TX}) \in \Omega^{1,1}(\mathbb{C}P^1).
\]
By the Poincaré-Lelong formula and (3.1), we have
\begin{equation}
\frac{\partial \bar{\partial}}{2\pi i} \log \|\gamma^\prime\|_\omega^2 = c_1(N, g^N) - 2c_1(TX, g^{TX}) - c_1(TY, g^{TY}) - c_1(TX, g^{TX}) \in \Omega^{1,1}(X \times \mathbb{C}P^1).
\end{equation}

By (2.6)-(2.8) and (3.15)-(3.17), the right hand side of (3.14), viewed as a function on \(\mathbb{C}P^1\), is harmonic. Thus it is independent of \(\omega\).

**Step 2.** We show the following identity under the assumption \(\|\gamma^\prime\|_\omega = 1\),
\begin{equation}
\tau_{BCOV}(Y, \omega|Y) = \log \left( \frac{1}{\beta^1} \right) + \log \frac{1}{\beta^2} + \log \frac{1}{\beta^3} + \cdots + \log \frac{1}{\beta^n} + \text{constant}.
\end{equation}

Let \(1_Y \in H^{0,0}(Y)\) be the constant function 1 on \(Y\). Let \(1^*_Y \in H^{1,1}(Y)\) be the dual of \(1_Y\). Let \(\alpha_1, \cdots, \alpha_{2g} \in H^1(Y)\) be a basis of the lattice \(H^1(Y, \mathbb{Z})\). Set
\begin{equation}
\alpha_Y = \alpha_1 \wedge \cdots \wedge \alpha_{2g} \in \det H^1(Y).
\end{equation}

By the definition of BCOV torsion in (1.3), we have
\begin{equation}
\tau_{BCOV}(Y, \omega|Y) = \log \left( \frac{1}{\beta^1} \right) + \log \left( \frac{1}{\beta^2} \right) + \log \left( \frac{1}{\beta^3} \right) + \cdots + \log \left( \frac{1}{\beta^n} \right) + \text{constant}.
\end{equation}

Under the isomorphism (3.5), we have
\begin{equation}
1^*_X = 1^*_Y, \quad \phi_X \otimes \overline{\phi_X} = J(\gamma, \phi_X) \alpha_Y.
\end{equation}

By (3.21) and (3.22), we have
\begin{equation}
\tau_{BCOV}(Y, \omega|Y) = \log \left( \frac{1}{\beta^1} \right) + \log \left( \frac{1}{\beta^2} \right) + \log \left( \frac{1}{\beta^3} \right) + \cdots + \log \left( \frac{1}{\beta^n} \right) + \text{constant}.
\end{equation}

We remark that the assumption \(\|\gamma^\prime\|_\omega = 1\) is equivalent to assumption(A) in [3, Definition 1.5]. Now, by [8, Theorem 0.1], (5.2) and (5.7), we have
\begin{equation}
\log \left( \frac{1}{\beta^1} \right) + \log \left( \frac{1}{\beta^2} \right) + \log \left( \frac{1}{\beta^3} \right) + \cdots + \log \left( \frac{1}{\beta^n} \right) + \text{constant}.
\end{equation}

Here the 'constant' comes from the integrations in [8, (0.5)] involving \(R(x)\). By (3.23) and (3.24), we get (3.18).

**Step 3.** We conclude.

By Step 1, it is sufficient to prove (3.14) with a Kähler form \(\omega\) satisfying \(\|\gamma^\prime\|_\omega = 1\).

This assumption implies \(\alpha_Y(\gamma, \omega) = \beta_Y(\gamma, \omega) = 0\). Now, by (2.5), (2.12) and Step 2, we obtain (3.14). This completes the proof. \(\square\)

We denote
\begin{equation}
D = \left\{ t \in \mathbb{C} : |t| < 1 \right\}, \quad D^* = \left\{ t \in \mathbb{C} : 0 < |t| < 1 \right\}.
\end{equation}
Let
\[(3.26) \quad \left( \gamma_t \in H^0(X, K_X^{-2}) \setminus \{0\} \right)_{t \in D} \]
be a holomorphic family. Let \(Y_t \subseteq X\) be the zero locus of \(\gamma_t\). Let \(l \in \mathbb{N}\). We assume that
- the family \((Y_t)_{t \in D^*}\) is smooth;
- the family \((Y_t)_{t \in D}\) has \(l\) ordinary double points at \(t = 0\).

**Proposition 3.2.** As \(t \to 0\), we have
\[(3.27) \quad \tau(X, Y_t) = \frac{1}{8} \log |t|^2 + \mathcal{O} \left( \log(- \log |t|) \right). \]

**Proof.** Let \(x_1, \ldots, x_l \in X\) be the singular points of \(Y_0 \subseteq X\). We fix a Kähler form \(\omega \in \Omega^{1,1}(X)\) such that the curvature of the Kähler metric induced by \(\omega\) vanishes near \(x_1, \ldots, x_l\). All the metrics involved in the proof are induced by \(\omega\).

We fix \(\phi_X \in \text{det} H^0(X, K_X^{-1})\). By Proposition 3.1, we have
\[(3.28) \quad \tau(X, Y_t) = \tau_{\text{BCOV}}(X, \omega) + \log \|1^*_X\|_{\lambda_2, \omega} - \log \|\phi_X\|_{\lambda_1, \omega} + \log J(\gamma_t, \phi_X)
+ 3\alpha_X(\gamma_t, \omega) + \frac{3}{2} a_X(\gamma_t, \omega) + \frac{9}{2} \alpha_Y(\gamma_t, \omega) - \frac{3}{2} b_Y(\omega) - \frac{3}{2} \beta_Y(\gamma_t, \omega)
+ w(X, -2) \log \int_X |\gamma_t|^{-1/2} + \text{constant}. \]

Since the Kähler form \(\omega\) is independent of \(t\), as \(t \to 0\), we obviously have
\[(3.29) \quad \tau_{\text{BCOV}}(X, \omega) = \mathcal{O}(1), \quad \log \|1^*_X\|_{\lambda_2, \omega} = \mathcal{O}(1), \quad \log \|\phi_X\|_{\lambda_1, \omega} = \mathcal{O}(1). \]

Since \(c_1(TX, g^{TX})\) and \(c_2(TX, g^{TX})\) vanish near \(x_1, \ldots, x_l\), as \(t \to 0\), we have
\[(3.30) \quad \alpha_X(\gamma_t, \omega) = \mathcal{O}(1), \quad \alpha_Y(\gamma_t, \omega) = \mathcal{O}(1), \quad \beta_Y(\gamma_t, \omega) = \mathcal{O}(1). \]

Proceeding in the same way as Step 3 in the proof of [26, Theorem 5.1], as \(t \to 0\), we have
\[(3.31) \quad b_Y(\omega) = \mathcal{O}(1). \]

By a direct calculation, as \(t \to 0\), we have
\[(3.32) \quad \log \int_X |\gamma_t|^{-1/2} = \mathcal{O}(1). \]

By the Hodge theory, as \(t \to 0\), we have
\[(3.33) \quad \log J(\gamma_t, \phi_X) = \mathcal{O} \left( \log(- \log |t|) \right). \]

Proceeding in the same way as in the proof of [24, Theorem 4.1], as \(t \to 0\), we have
\[(3.34) \quad \beta_Y(\gamma_t, \omega) = \frac{l}{12} \log |t|^2 + \mathcal{O}(1). \]

By (3.28)-(3.34), we obtain (3.27). This completes the proof. \(\square\)
3.2. Relation with Yoshikawa’s equivariant BCOV invariant. Let \( f : X' \to X \) be the ramified 2-cover whose branch locus is \( Y \). Let \( \iota \) be the involution on \( X' \) commuting with \( f \). Then \((X', \iota)\) is a 2-elementary K3 surface.

Let \( \tau(X', \iota) \) be Yoshikawa’s equivariant BCOV invariant for \((X', \iota)\) \cite{25, Definition 5.1}.

Proof of Theorem 0.2 Let \( S \) be a compact Riemann surface. Let \( \Delta \subseteq S \) be a finite subset. Let

\[
(3.35) \quad \left( [\gamma_s] \in \mathbb{P}(H^0(X, K_X^{-2})) \right)_{s \in S}
\]

be a holomorphic family. Let \( Y_s \subseteq X \) be the zero locus of \( \gamma_s \). We assume that

- the family \((Y_s)_{s \in S \setminus \Delta}\) is smooth;
- the family \((Y_s)_{s \in S} \) has exactly one ordinary double points at each \( s \in \Delta \).

We may view \( S \) as a curve in \( \mathbb{P}(H^0(X, K_X^{-2})) \). Since \( K_X^{-2} \) is very ample, a generic curve in \( \mathbb{P}(H^0(X, K_X^{-2})) \) satisfies the same properties as \( S \).

For \( s \in S \setminus \Delta \), we denote by \((X'_s, \iota_s)\) the 2-elementary K3 surface corresponding to \((X, Y_s)\), i.e., we have a ramified cover \( f_s : X'_s \to X \) with branch locus \( Y_s \subseteq X \).

Let \( \tau(X, Y) \) (resp. \( \tau(X', \iota) \)) be the function \( s \mapsto \tau(X, Y_s) \) (resp. \( s \mapsto \tau(X'_s, \iota_s) \)) on \( S \setminus \Delta \). It is sufficient to show that the function \( \tau(X, Y) + \tau(X', \iota) \) is constant on \( S \setminus \Delta \).

Step 1. We show that the function \( \tau(X, Y) + \tau(X', \iota) \) is harmonic.

Recall that \( g \) is the genus of \( Y \). By the Hirzebruch-Riemann-Roch formula, (3.4) and (3.5), we have

\[
(3.36) \quad g = \int_X \text{Td}(TX) \text{ch}(K_X^{-1}) = \frac{13}{12} \int_X c_1^2(TX) + \frac{1}{12} \int_X c_2(TX),
\]

\[
1 = \int_X \text{Td}(TX) \text{ch}(K_X) = \frac{1}{12} \int_X c_1^2(TX) + \frac{1}{12} \int_X c_2(TX).
\]

By (0.4) and (3.36), we have

\[
(3.37) \quad w(X, -2) = \frac{1}{12} \int_X c_1(TX) + \frac{1}{12} \int_Y c_1(TY) = \frac{13 - g}{12} + \frac{2 - 2g}{12} = \frac{5 - g}{4}.
\]

By Theorem 0.1 and (3.37), we have

\[
(3.38) \quad \frac{\partial \partial}{2\pi i} \tau(X, Y) = \omega_{H^*} + \frac{g - 5}{4} \omega_{\iota_{X'}, \iota_{Y'}}.
\]

Let \( \eta_s \in H^0(X'_s, K_{X'_s}) \) such that

\[
(3.39) \quad f^*_s \gamma_s^{-1} = \eta_s^2 \in H^0(X'_s, K_{X'_s}^2).
\]

Let \( \int_{X'_s} \left| \eta \right|^2 \) be the function \( s \mapsto \int_{X'_s} \left| \eta_s \right|^2 \) on \( S \setminus \Delta \). Recall that \( J(\gamma, \phi_X) \) was defined by (3.11). By Theorem 1.5, 5.9, equation (5.4) in \cite{25} and the paragraph between equations (5.12), (5.13) in \cite{25}, we have

\[
(3.40) \quad \frac{\partial \partial}{2\pi i} \tau(X', \iota) = -\frac{5 - g}{4} \frac{\partial \partial}{2\pi i} \log \int_{X'} \left| \eta \right|^2 - \frac{\partial \partial}{2\pi i} \log J(\gamma, \phi_X).
\]
By the definition of $\omega_{H^\bullet(Y)}$ in \[ L.2 \ (0.3) \text{ and } (3.40), \text{ we have}

\begin{equation}
\frac{\partial \bar{\partial}}{2 \pi i} \tau(X', \iota) = \frac{5 - g}{4} \omega_{\pi_{x'}, \pi_{y'}} - \omega_{H^\bullet(Y)}.
\end{equation}

By \[ (3.38) \text{ and } (3.41), \text{ the function } \tau(X, Y) + \tau(X', \iota) \text{ is harmonic.}

**Step 2.** We show that the function $\tau(X, Y) + \tau(X', \iota)$ admits at most log log-singularity near $\Delta \subseteq S$.

Let $s_0 \in \Delta \subseteq S$. We identify a neighborhood of $s_0 \in S$ with the unit disc $D$ such that $s_0$ is identified with $0 \in D$. Let $t \in D$ be the coordinate.

By Proposition \[ 3.2 \text{, as } t \to 0,

\begin{equation}
\tau(X, Y) = \frac{1}{8} \log |t|^2 + \mathcal{O}(\log(-\log |t|)).
\end{equation}

By \[ [25] \text{ Theorem 6.6, as } t \to 0,

\begin{equation}
\tau(X', \iota) = -\frac{1}{8} \log |t|^2 + \mathcal{O}(\log(-\log |t|)).
\end{equation}

By \[ (3.42) \text{ and } (3.43), \text{ the function } \tau(X, Y) + \tau(X', \iota) \text{ admits at most log log-singularity near } \Delta \subseteq S.

By Step 1 and Step 2, the function $\tau(X, Y) + \tau(X', \iota)$ is constant on $S$. This completes the proof.

\[ \square \]

4. BEHAVIOR OF $\tau(X, Y)$ UNDER BLOWING UP

In this section, we take $m = 1$. Let $X, \gamma \in H^0(X, K_X)$ and $Y \subseteq X$ be as before. We assume that $n = \dim X \geq 2$.

4.1. Vanishing of curvature. Let $S$ be a complex manifold. Let $(X_s, Y_s)_{s \in S}$ be a holomorphic family of Calabi Yau pairs with $m = 1$ and $\dim X_s \geq 2$. Let $(Z_s \subseteq X_s)_{s \in S}$ be a holomorphic family of sub complex manifolds of codimension 2. We assume that $Z_s \cap Y_s = \emptyset$ for each $s \in S$. Let $X_s'$ be the blowing up of $X_s$ along $Z_s$. Let $f_s : X_s' \to X_s$ be the canonical projection. Set $Y'_s = f_s^{-1}(Y_s \cup Z_s)$. Then $(X'_s, Y'_s)_{s \in S}$ is a holomorphic family of Calabi Yau pairs with $m = 1$.

Let $\tau(X, Y')$ (resp. $\tau(X', Y')$) be the function $s \mapsto \tau(X_s, Y_s)$ (resp. $s \mapsto \tau(X'_s, Y'_s)$) on $S$.

**Proposition 4.1.** We have

\begin{equation}
\overline{\partial} \partial \left( \tau(X', Y') - \tau(X, Y) \right) = 0.
\end{equation}

In particular, if $S$ admits a compactification $\overline{S}$ such that $\overline{S} \setminus S$ is of codimension $\geq 2$, then $\tau(X'_s, Y'_s) - \tau(X_s, Y_s)$ is independent of $s \in S$.

**Proof:** We consider the variations of Hodge structure $H^\bullet(X)$, $H^\bullet(Y)$, $H^\bullet(X')$, $H^\bullet(Y')$ and $H^\bullet(Z)$ over $S$. By \[ [23] \text{ Théorème 7.31}, \text{ we have}

\begin{equation}
H^\bullet(X') = H^\bullet(X) \oplus H^\bullet(Z)[1], \quad H^\bullet(Y') = H^\bullet(Y) \oplus H^\bullet(Z) \oplus H^\bullet(Z)[1].
\end{equation}

By Proposition \[ 1.2 \text{ and } (4.2), \text{ we have}

\begin{equation}
\omega_{H^\bullet(X')} = \omega_{H^\bullet(X)} + \omega_{H^\bullet(Z)}, \quad \omega_{H^\bullet(Y')} = \omega_{H^\bullet(Y)} + 2 \omega_{H^\bullet(Z)}.
\end{equation}
Recall that \( w(\cdot, \cdot) \) was defined in \((0.4)\). Since
\[
\chi(X'_s) = \chi(X_s) + 1, \quad \chi(Y'_s) = \chi(Y_s) + 2,
\]
we have
\[
w(X'_s, 1) = w(X_s, 1).
\]
By \((0.3)\), the Weil-Petersson forms of \((X_s, Y_s) \in S\) and \((X'_s, Y'_s) \in S\) coincide. Now, applying Theorem \(0.1\), \((4.3)\) and \((4.5)\), we obtain \((4.1)\).

\[\square\]

4.2. Case \(\dim X = 2\). Let \(X\) be a Kähler manifold of dimension 2. Let \(x \in X\). Let \(X'\) the blowing up of \(X\) along \(\{x\}\). Let \(f : X' \to X\) be the canonical projection. Set \(E = f^{-1}(x) \subseteq X'\), which is isomorphic to \(\mathbb{C}P^1\). Let \(j : E \to X'\) be the canonical embedding. Let \(N_E\) be the normal bundle of \(E \subseteq X'\).

We have a short exact sequence of coherent sheaves on \(X'\),
\[
0 \to f^* \mathcal{O}_X(K_X) \to \mathcal{O}_{K'}(K_{X'}) \to j_* \mathcal{O}_E(N_E^{-1} \otimes K_E) \to 0.
\]
Since \(\mathcal{O}_E(N_E^{-1} \otimes K_E) \simeq \mathcal{O}_{\mathbb{C}P^1}(-1)\), its cohomology vanishes. Taking the long exact sequence associated with \((4.6)\), we get
\[
H^0(X', f^* K_X) = H^2\bullet(X').
\]
On the other hand, using the spectral sequence, we can show that
\[
H^2\bullet(X) = H^0\bullet(X', f^* K_X).
\]
Recall that \(\lambda_p(\cdot)\) was defined by \((1.26)\). By \((4.7)\) and \((4.8)\), we have
\[
\lambda_2(X) = \lambda_2(X').
\]
Applying the same argument to the identity \(f^* \mathcal{O}_X = \mathcal{O}_{X'}\), we get
\[
H^0\bullet(X) = H^0\bullet(X').
\]
As a consequence, we have
\[
\lambda_0(X) = \lambda_0(X').
\]
Let \(\omega\) (resp. \(\omega'\)) be a Kähler form on \(X\) (resp. \(X'\)). For \(p = 0, 2\), let \(\| \cdot \|_{\lambda_p(X), \omega}\) (resp. \(\| \cdot \|_{\lambda_p(X'), \omega'}\)) be the Quillen metric on \(\lambda_p(X)\) (resp. \(\lambda_p(X')\)) associated with \(\omega\) (resp. \(\omega'\)).

**Lemma 4.2.** The following identity holds,
\[
\tau_{BCOV}(X', \omega') - \tau_{BCOV}(X, \omega) = -\frac{1}{2} \log 2 - 2 \log \pi - \sum_{p=0, 2} (-1)^{p/2} \log \frac{\| \cdot \|_{\lambda_p(X'), \omega'}^2}{\| \cdot \|_{\lambda_p(X), \omega}^2}.
\]
Proof. Set

\[ \lambda_{\text{Eul}}(X) = \bigotimes_{k=0}^{4} \left( \det H^k(X) \right)^{(-1)^k} \]

(4.13)

\[ = \bigotimes_{0 \leq p, q \leq 2} \left( \det H^{p,q}(X) \right)^{(-1)^{p+q}} = \bigotimes_{p=0}^{2} \left( \lambda_p(X) \right)^{(-1)^p}. \]

(4.14)

Let \( \cdot \| \cdot \|_{\lambda_{\text{Eul}}(X)} \) be the metric on \( \lambda_{\text{Eul}}(X) \) induced by \( \cdot \| \cdot \|_{\lambda_p(X), \omega} \). By [7], the metric \( \cdot \| \cdot \|_{\lambda_{\text{Eul}}(X)} \) is independent of \( \omega \). Similarly to (1.31), we denote by

\[ \sigma^X_{\text{Eul}} \in \lambda_{\text{Eul}}(X) \]

the product of a basis of the lattice \( \operatorname{Im} (H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{R})) \). Then \( \sigma^X_{\text{Eul}} \) is well-defined up to \( \pm 1 \). Let \( \sigma^X \in \lambda_{\text{dR}}(X) \) be as in (1.31). For \( p = 0, 2 \), let

\[ \sigma_p \in \lambda_p(X) \]

be non zero elements such that

\[ \sigma^X = \sigma^X_{\text{Eul}} \otimes \sigma_0^{-1} \otimes \sigma_2 \otimes \sigma_1. \]

(4.15)

(4.16)

By the definition of Ray-Singer torsion (see [1, (0.3)] and (4.16), we have

\[ \tau_{\text{BCOV}}(X, \omega) = \log \| \sigma^X_{\text{Eul}} \|_{\lambda_{\text{Eul}}(X)}^2 - \sum_{p=0,2} (-1)^{p/2} \log \| \sigma_p \|_{\lambda_p(X), \omega}^2. \]

(4.17)

We equip \( \mathbb{C} \) with the obvious Hodge structure of weight 0. By [23, Théorème 7.31], we have

\[ H^*(X') = H^*(X) \oplus \mathbb{C}[1]. \]

(4.18)

Let \( 1 \in \mathbb{C}[1] \) be the image of 1 \( \in \mathbb{C} \). By (4.17) and (4.18), we have

\[ \tau_{\text{BCOV}}(X', \omega') = \log \| \sigma^X_{\text{Eul}} \otimes 1 \|_{\lambda_{\text{Eul}}(X')}^2 - \sum_{p=0,2} (-1)^{p/2} \log \| \sigma_p \|_{\lambda_p(X'), \omega'}^2. \]

(4.19)

By (4.17) and (4.19), we have

\[ \tau_{\text{BCOV}}(X', \omega') - \tau_{\text{BCOV}}(X, \omega) \]

\[ = \log \| \sigma^X_{\text{Eul}} \otimes 1 \|_{\lambda_{\text{Eul}}(X')}^2 - \log \| \sigma^X_{\text{Eul}} \|_{\lambda_{\text{Eul}}(X)}^2 - \sum_{p=0,2} (-1)^{p/2} \log \frac{\| \cdot \|_{\lambda_p(X'), \omega'}}{\| \cdot \|_{\lambda_p(X), \omega}}. \]

(4.20)

We denote

\[ \chi'(X) = \sum_{k=0}^{2} (-1)^k k \dim H^k(X). \]

(4.21)

Let \( \| \cdot \|_{\lambda_{\text{Eul}}(X), \text{RS}} \) be the Ray-Singer metric on \( \lambda_{\text{Eul}}(X) \). By the definition of Ray-Singer metric (cf. [10, (0.3)]) and the definition of Quillen metric (cf. [3, Definition 1.10]), we have

\[ \| \cdot \|_{\lambda_{\text{Eul}}(X)}^2 = 2^{\chi'(X)/\chi(X)/2} (2\pi)^{2\chi(X)} \| \cdot \|_{\lambda_{\text{Eul}}(X), \text{RS}}^2. \]

(4.22)
Let $\| \cdot \|_{\lambda_{\text{Eul}}(X), M}^2$ be the Milnor metric (cf. [10] Definition 1.6) on $\lambda_{\text{Eul}}(X)$. By the Cheeger-Müller theorem [13, 20], we have

\begin{equation}
\| \cdot \|_{\lambda_{\text{Eul}}(X), M}^2 = \| \cdot \|_{\lambda_{\text{Eul}}(X), M}^2.
\end{equation}

Let $U \subseteq X$ be a small subset containing $x$. Let $V = f^{-1}(U)$. We have

\begin{equation}
H^*(U) = H^*(\{ x \} ) = \mathbb{C}, \quad H^*(V) = H^*(E) = \mathbb{C} \oplus \mathbb{C}[1].
\end{equation}

We consider the following commutative diagram,

\begin{equation}
\cdots \to H^k(U) \to H^k(X) \to H^k(X, U) \to \cdots
\end{equation}

\begin{equation}
\cdots \to H^k(V) \to H^k(X') \to H^k(X', V) \to \cdots.
\end{equation}

Since $X \setminus U = X' \setminus V$, the map $H^k(X, U) \to H^k(X', V)$ is isomorphic. By [10] (3.71), we have

\begin{equation}
\log \| \sigma_{\text{Eul}} \otimes 1 \|_{\lambda_{\text{Eul}}(X), M}^2 - \log \| \sigma_{\text{Eul}} \|_{\lambda_{\text{Eul}}(X), M}^2 = \log \| 1 \otimes 1 \|_{\lambda_{\text{Eul}}(V), M}^2 - \log \| 1 \|_{\lambda_{\text{Eul}}(U), M}^2 = 0.
\end{equation}

By (4.20), (4.22), (4.23) and (4.26), we obtain (4.12). \hfill \Box

**Proof of Theorem 0.3** Let

\begin{equation}
\varphi : \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\} \to X
\end{equation}

be a holomorphic local chart such that $\varphi(0, 0) = x$. Let $U \subseteq X$ be the image of $\varphi$. We denote $V = f^{-1}(U)$. We assume that

\begin{equation}
\varphi^* \gamma = dz_1 \wedge dz_2, \quad \varphi^* \omega = \frac{i}{2} \sum_{k=1}^{2} dz_k \wedge d\bar{z}_k, \quad f^* \omega \big|_{X' \setminus V} = \omega' \big|_{X' \setminus V}.
\end{equation}

We identify $N_E$ with a holomorphic sub vector bundle of $TX|_E$ in the obvious way. We assume that the decomposition $TX|_E = TE \oplus N_E$ is orthogonal with respect to $\omega'$. Let $\nabla$ be a connection on $K_X$. We denote

\begin{equation}
(f^* \gamma)' = \nabla f^* \gamma \big|_E \in H^0(E, N_E^{-1} \otimes K_X') = H^0(E, N_E^{-2} \otimes K_E).
\end{equation}

Let $\| \cdot \|_{\omega'}$ be the norm on $N_E^{-2} \otimes K_E$ induced by $\omega'$. We further assume that

\begin{equation}
\| (f^* \gamma)' \|_{\omega'} = 1.
\end{equation}

By (2.3), (2.12) and our assumptions, we have

\begin{equation}
\tau(X', Y') - \tau(X, Y) = \tau_{\text{BCOV}}(X', \omega') - \tau_{\text{BCOV}}(X, \omega) - \frac{1}{2} \tau_{\text{BCOV}}(E, \omega')|_E
\end{equation}

\begin{equation}
- \frac{1}{12} \int_V c_2(TX', g^{TX'}) \log \| f^* \gamma \|_{\omega'}^2.
\end{equation}

We denote

\begin{equation}
\lambda(f^* K_X) = \det H^*(X', f^* K_X).
\end{equation}
By (4.7) and (4.8), we have

\[(4.33) \quad \lambda_2(X) = \lambda(f^*K_X) = \lambda_2(X').\]

We equip \(TX'\) with the Kähler metric induced by \(\omega'\). We equip \(f^*K_X\) with the Hermitian metric induced by \(\omega\). Let \(\cdot \|_{\lambda(f^*K_X), \omega, \omega'}\) be the associated Quillen metric. By [4, Theorem 8.10] and our assumptions, we have

\[(4.34) \quad \log \frac{\| \cdot \|_{\lambda_0(X'), \omega'}^2}{\| \cdot \|_{\lambda_0(X), \omega}^2} - \log \frac{\| \cdot \|_{\lambda(f^*K_X), \omega, \omega'}^2}{\| \cdot \|_{\lambda_2(X), \omega}^2} = 0 .\]

We denote

\[(4.35) \quad \lambda(N_E) = \det H^•(E, N_E).\]

Since \(H^•(E, N_E) = 0\), we have

\[(4.36) \quad \lambda(N_E) = \mathbb{C}.\]

Let \(1 \in \lambda(N_E)\) be the image of \(1 \in \mathbb{C}\). We equip \(N_E\) with the Hermitian metric induced by \(\omega'\). We equip \(TE\) with the Kähler metric associated with \(\omega'|_E\). Let \(\cdot \|_{\lambda(N_E), \omega'}\) be the associated Quillen metric. Let \(\zeta\) be the Riemann zeta function. Using [8, Theorem 0.1] in the same way as in proof of Proposition 3.1, we get

\[(4.37) \quad \log \frac{\| \cdot \|_{\lambda(N_E), \omega'}^2}{\| \cdot \|_{\lambda_0(X'), \omega'}^2} + \log \frac{\| \cdot \|_{\lambda_2(X), \omega'}^2}{\| \cdot \|_{\lambda(f^*K_X), \omega, \omega'}^2} = -\frac{1}{12} \int_V c_2(TX', g^{TX'}) \log \| f^*\gamma \|_{\omega'}^2 + 2\zeta'(-1) - \frac{1}{12}.\]

Here \(2\zeta'(-1) - \frac{1}{12}\) comes from the integrations in \([8, (0.5)]\) involving \(R(x)\).

By (4.31), (4.34) and (4.37), we obtain

\[(4.38) \quad \tau(X', Y') - \tau(X, Y) = \log \frac{\| \cdot \|_{\lambda(N_E), \omega'}^2}{\| \cdot \|_{\lambda_0(X'), \omega'}^2} - \frac{1}{2} \tau_{BCOV}(E, \omega'|_E)\]

\[-\frac{1}{2} \log 2 - 2 \log \pi + \frac{1}{12} - 2\zeta'(-1).\]

The right hand side of (4.38) is a priori determined by \(\omega'\). On the other hand, since the left hand side of (4.38) is independent of \(\omega'\), so is the right hand side of (4.38). Hence (4.38) is a universal constant. This completes the proof. \(\Box\)

Remark 4.3. We can equally show that (4.38) is a universal constant by considering a fiberwise Kähler form on \(E \times \mathbb{C}P^1\) and applying Theorem 1.3.

5. Appendix

Let \((\pi_X, \pi_Y)\) be as in the introduction. We denote by \(X\) (resp. \(Y\)) the fiber of \(\pi_X\) (resp. \(\pi_Y\)). \(\pi_X\) assumes that \(\dim X = 2\) and \(Y \subseteq X\) is a \((-2)\)-canonical divisor.

Let \(\omega\) be a fiberwise Kähler form on \(\mathcal{X}\). All the metrics involved will be induced by \(\omega\).
We have
\[
\left\{ \text{Td}^{-1}(N_Y, g^{N_Y}) \text{Td}(TX|_Y, g^{TX}|_Y) - \text{Td}(TY, g^{TY}) \right\}^{(\leq 2, \leq 2)}
\]
\[
= \frac{1}{12} c_2(TX|_Y, g^{TX}|_Y) - \frac{1}{12} c_1(N_Y, g^{N_Y}) c_1(TY, g^{TY}) \in \Omega^{\leq 2, \leq 2}(\mathcal{Y}).
\]

By (5.1), we have
\[
\frac{\partial}{\partial \gamma} \frac{\partial}{\partial \gamma} \omega_Y(\omega)
\]
\[
= \left\{ \int_{\mathcal{Y}} \left( \text{Td}^{-1}(N_Y, g^{N_Y}) \text{Td}(TX, g^{TX}) - \text{Td}(TY, g^{TY}) \right) \right\}^{(1,1)} \in \Omega^{1,1}(S).
\]

We assume that \(\|\gamma\|_\omega = 1\). The assumption implies
\[
c_1(TY, g^{TY}) = -c_1(TX|_Y, g^{TX}|_Y) \in \Omega^{1,1}(\mathcal{Y}),
\]
\[
c_1(N_Y, g^{N_Y}) = 2c_1(TX|_Y, g^{TX}|_Y) \in \Omega^{1,1}(\mathcal{Y}).
\]

We have
\[
\left\{ \text{Td}(TY, g^{TY}) \text{ch}(K_Y, g^{K_Y}) \right\}^{(2,2)} = \frac{1}{12} c_1^2(TY, g^{TY}) \in \Omega^{2,2}(\mathcal{Y}).
\]

We have
\[
\left\{ \text{Td}(TX, g^{TX}) \left( \text{ch}(K_{X}^{-1}, g^{K_{X}^{-1}}) - \text{ch}(K_{X}, g^{K_{X}}) \right) \right\}^{(3,3)}
\]
\[
= \frac{1}{2} c_1^2(TX, g^{TX}) + \frac{1}{6} c_1(TX, g^{TX}) c_2(TX, g^{TX}) \in \Omega^{3,3}(\mathcal{X}).
\]

By (5.1) and (5.3)-(5.5), we have
\[
\left\{ \text{Td}^{-1}(N_Y, g^{N_Y}) \text{Td}(TX, g^{TX}) \text{ch}(K_Y, g^{K_Y}) \delta_Y \\
- \text{Td}(TX, g^{TX}) \left( \text{ch}(K_{X}^{-1}, g^{K_{X}^{-1}}) - \text{ch}(K_{X}, g^{K_{X}}) \right) \right\}^{(3,3)}
\]
\[
= \left\{ \left( \text{Td}^{-1}(N_Y, g^{N_Y}) \text{Td}(TX, g^{TX}) - \text{Td}(TY, g^{TY}) \right) \text{ch}(K_Y, g^{K_Y}) \delta_Y + \text{Td}(TY, g^{TY}) \text{ch}(K_Y, g^{K_Y}) \delta_Y \\
- \text{Td}(TX, g^{TX}) \left( \text{ch}(K_{X}^{-1}, g^{K_{X}^{-1}}) - \text{ch}(K_{X}, g^{K_{X}}) \right) \right\}^{(3,3)}
\]
\[
= \left( \delta_Y - 2c_1(TX, g^{TX}) \right) \left( \frac{1}{4} c_1^2(TX, g^{TX}) + \frac{1}{12} c_2(TX, g^{TX}) \right) \in A^{3,3}(\mathcal{X}).
\]
By (5.6), we have
\[
\frac{\partial \bar{\partial}}{2\pi i} \left\{ \frac{1}{4} \int_X c_1^2(T_X, g_T X) \log \|\gamma\|_\omega + \frac{1}{12} \int_X c_2(T_X, g_T X) \log \|\gamma\|_\omega \right\} = \left\{ \int_Y \text{Td}^{-1}(N_Y, g_Y^N) \text{Td}(T_X, g_T X) \text{ch}(K_Y, g_Y^{K_Y}) \\
- \int_X \text{Td}(T_X, g_T X) \left( \text{ch}(K_X^{-1}, g_X^{K_X^{-1}}) - \text{ch}(K_X, g_X^{K_X}) \right) \right\}^{(1,1)} \in \Omega^{1,1}(S).
\]

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