Abstract

We study the complexity of Destructive Shift Bribery. In this problem, we are given an election with a set of candidates and a set of voters (each ranking the candidates from the best to the worst), a despised candidate \( d \), a budget \( B \), and prices for shifting \( d \) back in the voters’ rankings. The goal is to ensure that \( d \) is not a winner of the election.

We show that this problem is polynomial-time solvable for scoring protocols (encoded in unary), the Bucklin and Simplified Bucklin rules, and the Maximin rule, but is NP-hard for the Copeland rule. This stands in contrast to the results for the constructive setting (known from the literature), for which the problem is polynomial-time solvable for \( k \)-Approval family of rules, but is NP-hard for the Borda, Copeland, and Maximin rules. We complement the analysis of the Copeland rule showing W-hardness for the parameterization by the budget value, and by the number of affected voters. We prove that the problem is W-hard when parameterized by the number of voters even for unit prices. From the positive perspective we provide an efficient algorithm for solving the problem parameterized by the combined parameter the number of candidates and the maximum bribery price (alternatively the number of different bribery prices).

1 Introduction

We study the complexity of the destructive variant of the Shift Bribery problem. We consider the family of all (unary encoded) scoring protocols (including the Borda rule and all \( k \)-Approval rules) and the Bucklin, Simplified Bucklin, Copeland, and Maximin rules. It turns out that for all of them—except for the Copeland rule—the problem can be solved in polynomial time. This stands in sharp contrast to the constructive case, where the problem is NP-hard [16] (and hard in the parameterized sense [4]) for the Borda, Copeland, and Maximin rules (however, Shift Bribery is in P for the \( k \)-Approval family of rules and the Simplified Bucklin and Bucklin rules [35]).

The Shift Bribery problem was introduced by Elkind et al. [16] to model (a kind of) campaign management problem in elections. The problem is as follows: We are given

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an election, that is, a set of candidates and a set of voters that rank the candidates from the most to the least desirable one, a preferred candidate \( p \), a budget \( B \), and the costs for shifting \( p \) forward in voters’ rankings. Our goal is to decide if it is possible to ensure \( p \)’s victory by shifting \( p \) forward, without exceeding the budget. In this paper we study the destructive variant of the problem, where the goal is to ensure that a given despised candidate \( d \) does not win the election, by shifting him or her back in voters’ rankings (but, again, each shift comes at a cost and we cannot exceed the given budget).

Studying destructive variants of election problems, where the goal is to change the current winner (such as manipulation [9], control [23], and bribery [18, 25, 7, 37]), is a common practice in computational social choice, but our setting is somewhat special. So far, all the destructive variants of the problems were defined by changing the goal from “ensure the victory of the preferred candidate” to “preclude the despised candidate from winning,” but the set of available actions remained unaffected (e.g., in both the constructive and the destructive problem of control by adding voters, we can add some voters from a given pool of voters and in both the constructive and the destructive bribery problem, we can pay voters to change their votes in some way). In our case, we feel that it is natural to divert from this practice and change the ability of “shifting the distinguished candidate forward” to the ability of “shifting the distinguished candidate back”. Below we explain why.

If, when defining our destructive problem, we stuck with the “ability to shift forward,” as in the Constructive Shift Bribery problem, we would get the following problem: Ensure that a given despised candidate does not win the election by shifting him or her forward in some of the votes (without exceeding the budget). However, for a monotone voting rule, if a candidate is already winning an election, then shifting him or her forward certainly cannot preclude him or her from winning. This would make our problem interesting only for a relatively small set of nonmonotone rules. Further, the problem certainly would not be modeling what one would intuitively think of as negative, destructive campaigning. Thus, while it certainly would be interesting to study how to exploit nonmonotonicity of rules such as STV and Dodgson (or various multiwinner rules) to preclude someone from winning, it would not be the most practical way of defining Destructive Shift Bribery.

1.1 Related Work

In recent years, Constructive Shift Bribery received quite some attention. The problem was defined by Elkind et al. [16], as a simplified variant of Swap Bribery (which itself received some attention, for example, in the works of Bredereck et al. [5], Dorn and Schlotter [13], Faliszewski et al. [19], Knop et al. [24], and papers regarding combinatorial domains, such as those of Mattei et al. [27] and Dorn and Krüger [12]; importantly, Shiryaev

\footnote{Nonetheless, there are interesting nonmonotone rules, such as the single transferable rule (STV) and the Dodgson rule. It is also quite common for multiwinner rules to not be monotone (see the works of Elkind et al. [15] and Faliszewski et al. [24]), so studying this variant of the Shift Bribery problem for multiwinner voting rules may be interesting.}
et al. [36] studied a variant of Destructive Swap Bribery and we comment on their work later). Elkind et al. [16] have shown that Constructive Shift Bribery is NP-hard for the Borda, Copeland, and Maximin voting rules, but polynomial-time solvable for the \( k \)-Approval family of rules. They also gave a 2-approximation algorithm for the case of Borda, which was later generalized to the case of all scoring rules by Elkind and Faliszewski [14]. Chen et al. [4] considered parametrized complexity of Constructive Shift Bribery, and have shown a varied set of results (in general, parametrization by the number of positions by which the preferred candidate is shifted tends to lead to FPT algorithms, parametrization by the number of affected voters tends to lead to hardness results, and parametrization by the available budget gives results between these two extremes). Recently, Maushagen et al. [29] studied both constructive and destructive shift bribery (using our model with backward shifts) for the case of round-based rules such as STV (there referred as Hare), Coombs, Baldwin, and Nanson.

Shift Bribery belongs to the family of bribery-related problems, that were first studied by Faliszewski et al. [17] (see also the work of Hazon et al. [22] for a similar problem), and that received significant attention within computational social choice literature (see the survey of Faliszewski and Rothe [20]). Briefly put, in the regular Bribery problem the goal is to ensure that a given candidate is a winner of an election by modifying—in an arbitrary way—up to \( k \) votes, where \( k \) is part of the input. Many types of bribery problems were already studied, including—in addition to Swap and Shift Bribery—Support Bribery [35], Extension Bribery [2, 19], and others (e.g., in judgment aggregation [1], and in the setting of voting in combinatorial domains [27, 12, 26]). However, from our point of view the most interesting variant of the problem is Destructive Bribery, first studied by Faliszewski et al. [18] and also, independently, by Magrino et al. [25] and Cary [7] under the name Margin of Victory (this problem was also studied by Xia [37] and Dey and Narahari [11]). The idea behind Margin of Victory is that it can be very helpful in validating election results: If it is possible to change the election result by changing (bribing) relatively few votes, then one may suspect that—possibly—the election was tampered with. Bribery problems are also related to lobbying problems [8, 3, 30].

A variant of Destructive Swap Bribery was studied by Shiryaev et al. [36] in their work on testing how robust are winners of given elections. They presented an analysis of the case where every swap has a unit price and they showed that the problem is easy for scoring protocols and for the Condorcet rule.\(^2\) This work is very closely related to ours, but the definition of the problem is somewhat different (more general types of swaps but less general price functions). Very recently, Bredereck et al. [5] studied an analogous setting

\(^2\)The authors’ definition of the Condorcet rule slightly differs from a standard one usually seen in the literature. They assume that if there is no unique Condorcet winner the rule returns the whole set of candidates instead of an empty set (well established as a return value for this case in the literature.)
(Destructive Swap Bribery with unit prices) for the case of multiwinner voting rules.

1.2 Our Contribution.

We believe that Destructive Shift Bribery is worth studying for three main reasons. First, it simply is a natural variant of the Constructive Shift Bribery problem and as the constructive variant received significant attention, we feel that it is interesting to know how the destructive variant behaves (indeed, the work of Maushagen et al. [29] is a sign that the destructive setting also attracts attention). Second, it models natural negative campaigning actions, aimed at decreasing the popularity of a given candidate. Third, it serves a similar purpose as the Margin of Victory problem [25, 37]: If it is possible to preclude a given candidate from winning through a low-cost destructive shift bribery, then it can be taken as a signal that the election might have been tampered with, or that some agent performed a possibly illegal form of campaigning. Below we summarize the main contributions of this paper:

1. We define the Destructive Shift Bribery problem and justify why a definition that diverts from the usual way of defining destructive election problems is appropriate in this case.

2. We show that Destructive Shift Bribery is a significantly easier problem than constructive shift bribery. To this end, we show polynomial time algorithms for Destructive Shift Bribery for all scoring rules with unary encoded scores (including the Borda rule and $k$-Approval family of rules), Simplified Bucklin, Bucklin, and Maximin.

3. We show that in spite of our easiness results, there still are voting rules for which the problem is computationally hard. We exemplify this by proving NP-hardness and $W[1]$-hardness (for several parameters) for the case of Copeland family of rules.

The paper is organized as follows. In Section 2 we formally define elections, present the voting rules used in the paper, define our problem, and briefly review necessary notions regarding parametrized complexity. In Section 3 we present our results, with one subsection for each of the studied rules. We conclude in Section 4 with an overview of our work, tables of results, and suggestions for future research.

2 Preliminaries

For each positive integer $t$, by $[t]$ we mean the set $\{1, 2, \ldots, t\}$. We assume that the reader is familiar with standard notions regarding algorithms and complexity theory, as presented in the textbook of Papadimitriou [34].
2.1 Elections and Election Rules

An election is a pair $E = (C, V)$, where $C = \{c_1, c_2, \ldots, c_m\}$ is a set of candidates and $V = \{v_1, v_2, \ldots, v_n\}$ is a multiset of voters. Each voter is associated with his or her preference order $\succ_i$, that is, a strict ranking of the candidates from the best to the worst (according to this voter). For example, we may have election $E = (C, V)$ with $C = \{c_1, c_2, c_3\}$ and $V = \{v_1, v_2\}$, where $v_1$ has preference order $v_1: c_1 \succ c_2 \succ c_3$. If $c$ is a candidate and $v$ is a voter, we write $\text{pos}_v(c)$ to denote the position of $c$ in $v$’s ranking (e.g., in the preceding example we would have $\text{pos}_{v_1}(c_1) = 1$). Given an election $E = (C, V)$ and two distinct candidates $c$ and $c'$, by $N_E(c, c')$ we mean the number of voters who prefer $c$ to $c'$.

An election rule $R$ is a function that given an election $E = (C, V)$ outputs a set $W \subseteq C$ of tied election winners (typically, we expect to have a single winner, but due to symmetries in the profile it is necessary to allow for the possibility of ties). We use the unique-winner model, that is, we require a candidate to be the only member of $R(E)$ to be considered $E$’s winner (see the works of Obraztsova et al. for various other tie-breaking mechanisms and algorithmic consequences of implementing them; indeed, there are situations where the choice of the tie-breaking rule affects the complexity of election-related problems).

We consider the following voting rules (for the description below, we consider election $E = (C, V)$ with $m$ candidates; for each rule we describe the way it computes candidates’ scores, so that the candidates with the highest score are the winners, unless explicitly stated otherwise):

**Scoring protocols.** A scoring protocol is defined through vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonincreasing, nonnegative integers. The $\alpha$-score of a candidate $c \in C$ is defined as $\sum_{v \in V} \alpha_{\text{pos}_v(c)}$. The most popular scoring protocols include the family of $k$-Approval rules (for each $k$, $k$-Approval scoring protocol is defined through a vector of $k$ ones followed by zeros) and the Borda rule, defined by vector $(m - 1, m - 2, \ldots, 0)$. The 1-Approval rule is known as Plurality.

**Bucklin and Simplified Bucklin.** A Bucklin winning round is the smallest value $\ell$ such that there is a candidate whose $\ell$-Approval score is greater or equal to $|V|/2 + 1$ (in other words, if there is a candidate ranked among top $\ell$ positions by a strict majority of the voters). All candidates whose $\ell$-Approval score is at least this value are the winners under the Simplified Buckling rule. The Bucklin score of a candidate is his or her $\ell$-Approval score, where $\ell$ is the Bucklin winning round of the election. The candidates with the highest Bucklin score are the Bucklin winners (note that the set of Bucklin winners is a subset of the set of Simplified Bucklin winners).

**Copeland.** Let $\alpha$, $0 \leq \alpha \leq 1$, be a rational number. The Copeland$^\alpha$ score of a candidate $c$ is defined as:

$$\left|\{d \in C \setminus \{c\} : N_E(c, d) > N_E(d, c)\}\right| + \alpha \left|\{d \in C \setminus \{c\} : N_E(c, d) = N_E(d, c)\}\right|.$$

In other words, candidate $c$ receives one point for each candidate whom he or she defeats in their head-to-head contest (i.e., for each candidate over whom $c$ is preferred.
by a majority of the voters) and $\alpha$ points for each candidate with whom $c$ ties their head-to-head contest.

**Maximin.** The Maximin score of candidate $c$ is defined as $\min_{d \in C \setminus \{c\}} N_E(c, d)$.

We write $\text{score}_E(c)$ to denote the score of candidate $c$ in election $E$ (the election rule will always be clear from the context).

**Definition 1.** For election $E = (C, V)$, a Condorcet winner is a candidate $c \in C$ who defeats all the other candidates in head-to-head contests. We call an election rule $R$ Condorcet-consistent if, for every possible election $E$, it selects a Condorcet winner when one exists.

Among all the rules we consider only Maximin and Copeland are Condorcet-consistent. However, even though they both share this property, the results we obtained show that they differ significantly when considered from our problem’s perspective.

### 2.2 Destructive Shift Bribery

The **Destructive Shift Bribery** problem for a given election rule $R$ is defined as follows. We are given an election $E = (C, V)$, a despised candidate $d \in C$ (typically, the current election winner), a budget $B$ (a nonnegative integer), and the prices for shifting $d$ backward for each of the voters (see below). The goal is to ensure that $d$ is not the unique $R$-winner of the election by shifting him or her backward in the voters’ preference orders without exceeding the budget.

We model the “prices for shifting $d$ backward” as destructive shift-bribery price functions. Let us fix an election $E = (C, V)$, with $C = \{c_1, c_2, \ldots, c_m\}$, $V = \{v_1, v_2, \ldots, v_n\}$, and let the despised candidate be $d$. Let $v$ be some voter and let $j = \text{pos}_v(d)$. Function $\rho : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ is a destructive shift-bribery price function for voter $v$ if it satisfies the following conditions:

1. $\rho(0) = 0$,
2. for each two $i, i', i < i' \leq m - j$, it holds that $\rho(i) \leq \rho(i')$ for $i < i' \leq m - j$, and
3. $\rho(i) = +\infty$ for $i > m - j$

For each $i$, we interpret value $\rho(i)$ as the price of shifting $d$ back by $i$ positions in $v$’s preference order. Value $+\infty$ is used for the cases where shifting $d$ back by $i$ positions is impossible (due to $d$’s position in the vote). We assume that in each instance, the price functions are encoded by simply listing their values for all arguments for which they are not $+\infty$.

For our hardness results, we focus on the case of unit price functions, where, for each voter, shifting the despised candidate backward by $i$ positions costs $i$ units of the budget. Sometimes we also consider all-or-nothing price functions, where for each voter $v$ there is a value $c_v$ (which can also be set to $\infty$) such that the cost of shifting the despised candidate $i$ positions backward always costs $c_v$, irrespective of $i$ (except for $i = 0$, where the cost is zero by definition).
Example 1. Let us consider an instance of Destructive Shift Bribery for the Borda rule with the following election:

\[ v_1: b \succ a \succ c \succ d, \quad v_2: d \succ b \succ a \succ c, \quad v_3: d \succ c \succ a \succ b, \quad v_4: d \succ a \succ b \succ c. \]

We take \( d \) to be the despised candidate, and assume that we have unit price functions. In this election, candidate \( d \) wins with 9 points, and the second-best candidate, \( a \), has 6 points. However, if we shifted \( d \) two positions back in \( v_4 \)'s preference order, then \( d \)'s score would decrease to 7 and \( a \)'s score would increase to 7. Thus, \( d \) would no longer be the unique winner. In consequence, for \( B = 2 \) we have a “yes”-instance of Destructive Shift Bribery. On the other hand, shifting \( d \) back by one position only (in whichever vote) cannot preclude \( d \) from winning. So, for \( B = 1 \) we have a “no”-instance.

2.3 Parameterized Complexity

In the theory of parameterized complexity, the goal is to study the computational difficulty of problems with respect to both their input length, as in classic computational complexity theory, and some additional “parameters.” For example, for Destructive Shift Bribery problem, the parameters may be the numbers of voters or candidates in the input election, the budget values, the maximum numbers of shifts available for a bribery etc. (we take the parameters to be numbers). For a parameter \( k \), we say that a problem parametrized by \( k \) is fixed-parameter tractable (is in FPT) if there is an algorithm that solves it in time \( f(k) \cdot |I|^{O(1)} \), where \( |I| \) is the length of the encoding of a given instance and \( f \) is an arbitrary computable function (that depends on the parameter value only). Intuitively, if a problem is in FPT for some parameter \( k \), then we can hope that its instances where \( k \) is small can be solved efficiently.

Parameterized complexity theory also offers a theory of intractability. In particular, it is widely believed that if a problem is \( W[1] \)-hard with respect to some parameter \( k \), then there is no FPT algorithm for this problem for this parametrization. The original definition of the class \( W[1] \) is quite involved and it is currently common to define the class by providing one of its complete problems (CLIQUE parametrized by the size of the clique is the most common example) and the notion of a parametrized reduction. In our case the situation is even simpler: All our \( W[1] \)-hardness proofs give polynomial-time many-one reductions from well-known \( W[1] \)-hard problems and guarantee that the values of the parameters in the reduced-to instances depend only on the values of the parameter in the reduced-from instances (in other words, our proofs do not use full power of parametrized reductions).

We point readers interesting in more detailed treatments of parametrized complexity theory to the textbooks of Niedermeier [31] and Cygan et al. [10].

3 Results

In this section we present our main results. We show polynomial-time algorithms for the \( k \)-Approval family of rules, the Borda rule, all scoring protocols (provided that they are
encoded in unary or that the destructive shift bribery price functions are encoded in unary),
the Bucklin family of rules, and the Maximin rule. For the Copeland* family of rules, we
prove NP-hardness and several results regarding its parameterized complexity.

As a warm-up, we start with an observation regarding an upper bound of the complexity
of the Destructive Shift Bribery problem.

**Observation 1.** The Destructive Shift Bribery problem is in NP for every voting
rule for which winner determination is in P.

*Proof.* Given an instance of the problem, we guess in which votes to sh ift the despised
candidate and by how many positions. Then we check if the cost of these shifts does not
exceed the budget and if implementing them ensures that the despised candidate is not the
unique winner of the election.

Every voting rule that we consider in this paper is polynomia l-time computable and,
thus, Observation 1 guarantees that the Destructive Shift Bribery problem for each of
our rules is in NP (in particular, all our NP-hardness proofs in fact show NP-completeness).

### 3.1 The k-Approval Family of Rules

We start with the k-Approval family of rules. In this case, our algorithm is very simple:
If d is the despised candidate then in each vote we should either not shift d at all or shift
him or her from one of the top k positions (where each candidate receives a single point)
to the (k + 1)-st position (where he or she would receive no points), in consequence also
shifting the candidate previously at the (k + 1)-st position one place forward (to receive
a point). Choosing which action to do for each particular voter is easy via the following
greedy/brute-force algorithm (our algorithm is based on a similar idea as that of Elkind et
al. [16] for the constructive case).

**Theorem 1.** For each k ∈ N, the Destructive Shift Bribery problem for the k-
Approval rule is in P.

*Proof.* Let E = (C, V) be the input election with C = {c1, c2, ..., cm} and V =
{v1, v2, ..., vn}, let d = c1 be the despised candidate, let B be the budget, and let
{ρ1, ρ2, ..., ρn} be the destructive shift-bribery price functions for the voters. Our algo-

rithm works as follows.

For every candidate c ∈ C \ {d}, we test if it is possible to guarantee that the score of c
is at least as high as that of d, by spending at most B units of budget, as follows:

1. We partition the voters into three groups, V_{d,c}, V_d, and V’, such that: V_{d,c} contains
   exactly the voters that rank c on the (k + 1)-st position and that rank d above c, V_d
   contains the remaining voters that rank d among top k positions, and V’ contains the
   other remaining voters.

2. We guess two numbers, a and b, such that |V_{d,c}| ≤ a and |V_d| ≤ b.
3. We pick a voters from $V_{d,c}$ for whom shifting $d$ to the $(k+1)$-st position is least expensive and we pick $b$ voters from $V_d$ for whom shifting $d$ to the $(k+1)$-st position is least expensive. We shift $d$ to the $(k+1)$-st position in the chosen votes.

4. If $d$ is not the unique winner in the resulting election and the total cost of performed shifts is smaller than or equal to budget $B$, then we accept. Otherwise, we either try a different candidate $c$ or different values of $a$ and $b$. After trying all possible combinations, we reject.

The algorithm runs in polynomial time: It requires trying at most $O(m)$ different candidates and $O(n^2)$ different values of $a$ and $b$. All the other parts of the algorithm require polynomial time (in fact, a careful implementation can achieve running time $O(mn^2)$).

To show correctness of the algorithm we start with an observation that it is never beneficial to shift $d$ below position $k+1$. Further, an optimal solution, after which some candidate $c$ has at least as high a score as $d$, can consist solely of actions that shift $d$ backward from one of the top $k$ positions to the $(k+1)$-st one, in effect either promoting $c$ to the $k$-th position (shifts in voter group $V_{d,c}$), or promoting some other candidate (shifts in voter group $V_d$). We guess how many shifts of each type to perform and execute the least costly ones.

3.2 The Borda Rule and All Scoring Rules

Next we consider the Borda rule. We also obtain a polynomial-time algorithm, but this time we resort to dynamic programming. After proving Theorem 2 below, we will show that our algorithm generalizes to all scoring protocols (provided that either the scores are encoded in unary or the price functions are encoded in unary).

**Theorem 2.** The **Destructive Shift Bribery** problem for the Borda rule is in $P$.

**Proof.** Let $E = (C, V)$ be the input election with $C = \{c_1, c_2, \ldots, c_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, let $d = c_1$ be the despised candidate, let $B$ be the budget, and let $\{\rho_1, \rho_2, \ldots, \rho_n\}$ be the destructive shift-bribery price functions for the voters. As in the case of the proof of Theorem 1, we give an algorithm that first guesses some candidate $c \in C \setminus \{d\}$ and then checks if it is possible to ensure—by shifting $d$ backward without exceeding budget $B$—that $c$ has at least as high a score as $d$. The algorithm performing this test is based on dynamic programming.

We fix $c$ to be the candidate that we want to have a score at least as high as $d$. Further, for each $j \in [n]$ and $k \in [m]$, we set $A(j, k)$ to be 1 if $v_j$ ranks $c$ among first $k$ positions below $d$, and we set it to be 0 otherwise. Finally, we write $s$ to denote the difference between the scores of $d$ and $c$, that is, $s = \text{score}_E(d) - \text{score}_E(c)$ (it must be that $s > 0$; otherwise we could accept immediately since $d$ would not be the unique winner of the election).

For each $j \in [n]$ and each positive integer $k$, we define $f(j, k)$ to be the smallest cost for shifting $d$ backward in the preference orders of the voters from the set $\{v_1, v_2, \ldots, v_j\}$, so that, if $E'$ is the resulting election, it holds that:

$$s - (\text{score}_{E'}(d) - \text{score}_{E'}(c)) \geq k.$$
In other words, \( f(j, k) \) is the lowest cost of shifting \( d \) backward in the preference orders of voters from the set \( \{v_1, v_2, \ldots, v_j\} \) so that the relative score of \( c \) with respect to \( d \) increases by at least \( k \) points. Our goal is to compute \( f(n, s) \); if it is at most \( B \) then it means that we can ensure that \( c \) has at least as high a score as \( d \) by spending at most \( B \) units of the budget. We make the following observation.

**Observation 2.** Shifting candidate \( d \) backward by some \( k \) positions decreases the score of \( d \) by \( k \) points and, if \( d \) passes \( c \), increases the score of \( c \) by one point. In effect, relative to \( d \), \( c \) gains, respectively, \( k \) or \( k + 1 \) points.

Now, we can express function \( f \) as follows. We have that \( f(0, 0) = 0 \) and for each positive \( k \) we have \( f(k, 0) = \infty \) (for technical reasons, whenever \( k < 0 \), we take \( f(j, k) = 0 \)). For each \( j \in [n] \) and \( k \in [s] \), we have:

\[
f(j, k) = \min_{k' \leq k} f(j - 1, k - (k' + A(j, k')) + \rho_j(k').
\]

To explain the formula, we observe that:

1. The minimum is taken over the value \( k' \); \( k' \) gives the number of positions by which we shift \( d \) backward in vote \( v_j \).
2. When we shift \( d \) backward by \( k' \) positions, relative to \( d \), candidate \( c \) gains \( k' + A(j, k') \) points.
3. The cost of this shift is \( \rho_j(k') \).

Based on these observations, we conclude that the formula is correct. It is clear that using this formula and standard dynamic programming techniques, we can compute \( f(n, s) \) in polynomial time with respect to \( n \) and \( m \). This means that we can test in polynomial time, for a given candidate \( c \), if it is possible to ensure that \( c \)'s score is at least as high as that of \( d \).

Our algorithm for **Destructive Shift Bribery** for Borda simply considers each candidate \( c \in C \setminus \{d\} \) and tests if, within budget \( B \), it is possible to ensure that \( c \) has at least as high a score as \( d \). If this test succeeds for some \( c \), we accept. Otherwise we reject.

A careful inspection of the above proof shows that there is not much in it that is specific to the Borda rule (as opposed to other scoring protocols). Indeed, there are only the following two dependencies:

1. The fact that the difference between the scores of candidate \( d \) and candidate \( c \) (value \( s \)) can be bounded by a polynomial of the number of voters and the number of candidates (dynamic programming requires us to store a number of values of the function \( f \) that is proportional to \( s \), so it is important that this value is polynomially bounded).
2. The way we compute the increase of the score of \( c \), relative to \( d \), in the recursive formula for \( f(j, k) \).
Since it is easy to modify the formula for \( f(j, k) \) to work for an arbitrary scoring protocol \( \alpha \), we have the following corollary (the assumption about unary encoding of the input scoring protocol ensures that the first point in the above list of dependencies is not violated).

**Corollary 1.** There exists an algorithm that, given as input a scoring protocol \( \alpha \) (for \( m \) candidates) encoded in unary and an instance \( I \) of the Destructive Shift Bribery problem (with the same number \( m \) of candidates), tests in polynomial time if \( I \) is a “yes”-instance for the voting rule defined by the scoring protocol \( \alpha \).

What can we do if, in fact, our scoring protocol is impractical to encode in unary (e.g., if our scoring protocol is of the form \((2^m - 1, 2^{m-2}, \ldots, 1)\))? In this case, it is easy to show the following result.

**Corollary 2.** There exists an algorithm that, given as input a scoring protocol \( \alpha \) (for \( m \) candidates) and an instance \( I \) of the Destructive Shift Bribery problem (with the same number \( m \) of candidates), with price functions encoded in unary, tests in polynomial time if \( I \) is a “yes”-instance for the voting rule defined by the scoring protocol \( \alpha \).

To prove this result, we use the same argument as before, but now function \( f \) has slightly different arguments: \( f(j, t) \) is the maximum increase of the score of \( c \), relative to \( d \), that one can achieve by spending at most \( t \) units of budget (using such “dual” formulation is standard for bribery problems and was applied, for example, by Faliszewski et al. [17, Theorem 3.8] in the first paper regarding the complexity of bribery problems).

On the other hand, if both the scoring protocol and the price functions are encoded in binary, and the scoring protocol is part of the input, then the problem is NP-complete by a reduction from the Partition problem.

**Definition 2.** In the Partition problem, we are given a sequence of positive integers and we ask if it can be partitioned into two subsequences whose elements sum up to the same value.

**Theorem 3.** The Destructive Shift Bribery problem is NP-complete if both the scoring protocol and the price functions are encoded in binary, and the scoring protocol is part of the input.

**Proof.** Consider an instance of the Partition problem with sequence \( S = (s_1, s_2, \ldots, s_n) \) and let \( s = \sum_{i=1}^{n} s_i \) be the sum of the elements from \( S \). Without loss of generality, we assume that for each \( i \in [n-1] \) we have \( s_i \geq s_{i+1} \). We also assume that \( s \) is even and that \( s_1 < \frac{s}{2} \) (for \( s_1 = \frac{s}{2} \) an instance is polynomial-time solvable, for \( s_1 > \frac{s}{2} \) we get a “no”-instance of Partition). We form an instance of Destructive Shift Bribery as follows:

1. We introduce candidates \( d, p_1, \ldots, p_n \), and dummy candidates \( \{c^i_j \mid i, j \in [n] \} \).

2. We form a scoring vector \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n^2+n+1}) \) such that \( \alpha_1 = \frac{s}{2} \), for each \( i \in [n] \) we have \( \alpha_{i+1} = s_i \), and for all \( i \geq n+2 \) we have \( \alpha_i = 0 \). By our assumptions regarding the sequence \( S \), we have that \( \alpha \) is a legal scoring vector.
3. Our election consists of \( n \) votes, each representing an element of \( S \). For each element \( s_i \), we construct vote \( v_i \) by placing candidate \( p_i \) on the first position and candidate \( d \) on the \( (i+1) \)-st position. We fill the remaining \( n \) places among the top \( n+2 \) positions in vote \( v_i \) with candidates from the set \( \{c'_j : j \in [n]\} \) in an arbitrary way. Then we complete vote \( v_i \) by placing the remaining candidates on positions \( n+3, n+4, \ldots \) in an arbitrary order.

4. We set the budget \( B \) to be \( \frac{s}{2} \). For every voter \( v_i \), we define bribery function \( \rho_i \) such that \( \rho_i(0) = 0 \), for each \( t \in [n - i + 1] \) we have \( \rho_i(t) = s_i \), and \( \rho_i(t) = B + 1 \) for all other possible values of \( t \).

We illustrate this construction in the example below.

**Example 2.** Consider a Partition instance with sequence \( S = (5, 4, 2, 2, 1) \). Our reduction forms an election with candidate set \( C = \{d\} \cup \{p_1, \ldots, p_5\} \cup \{c'_i | i, j \in [5]\} \) and votes \( v_1, \ldots, v_5 \). The scoring protocol \( \alpha \) is \( (7, 5, 4, 2, 2, 1, 0, \ldots, 0) \) and budget is \( B = 7 \). The preference orders of the voters are:

\[
\begin{align*}
v_1: & \quad p_1 > d > c_1^1 > c_1^2 > c_1^3 > c_1^4 > c_1^5 > \ldots \\
v_2: & \quad p_2 > c_2^1 > d > c_2^2 > c_2^3 > c_2^4 > c_2^5 > \ldots \\
v_3: & \quad p_3 > c_3^1 > d > c_3^2 > c_3^3 > c_3^4 > c_3^5 > \ldots \\
v_4: & \quad p_4 > c_4^1 > c_4^2 > c_4^3 > d > c_4^4 > c_4^5 > \ldots \\
v_5: & \quad p_5 > c_5^1 > c_5^2 > c_5^3 > c_5^4 > d > c_5^5 > \ldots 
\end{align*}
\]

For every voter \( v_i \), its bribery function \( \rho_i \) has the following values: \( \rho_i(0) = 0 \), \( \rho_i(t) = s_i \) for \( t \in [6 - i] \), and \( \rho_i(t) = 8 \) for all other possible values of \( t \).

The unique winner of the election constructed by our reduction is \( d \), with score \( s \). For each candidate \( p_i \), we have \( \text{score}(p_i) = \frac{s}{2} \), and for all \( i, j \in [n] \) we have \( \text{score}(c_i^j) < \frac{s}{2} \). Scores of the \( p_i \) candidates cannot change due to shifting \( d \) backward within budget, because either they are ranked ahead of \( d \) or they are ranked too far behind \( d \) (so it is impossible to shift \( d \) behind them within budget). Similarly, no dummy candidate can achieve a score larger than \( \frac{s}{2} \). Thus to prevent \( d \) from winning the election, one has to decrease his or her score at least by \( \frac{s}{2} \) (so that the \( p_i \) candidates have at least as high scores as \( d \) has).

We observe that by bribing some voter \( v_i \), we are not able to decrease candidate \( d \)’s score by a value greater than the number of units of budget spent. Moreover, the budget is \( \frac{s}{2} \), which is also the number of points by which \( d \)’s initial score has to be reduced. Hence, if we choose to bribe some voter \( v_i \), then we have to shift \( d \) backward to the \((n+2)\)-nd position, decreasing his or her score by \( s_i \) points at the same time. Together, these observations mean that in a successful bribery we have to reduce \( d \)’s score by a sum of some elements \( s_1, \ldots, s_n \). Further, this sum has to equal exactly \( \frac{s}{2} \). Therefore, there exists a solution to the created Destructive Shift Bribery instance if and only if there exists a solution to the initial Partition instance.
The reduction is computable in polynomial time and (by Observation 1) the Destructive Shift Bribery problem belongs to NP, which means that the problem is NP-complete.

3.3 The Bucklin Family of Rules

We continue our analysis of the Destructive Shift Bribery problem by considering the Bucklin and Simplified Bucklin rules. We find that for both rules our problem is polynomial-time solvable (via a dynamic programming approach). Before presenting our solution, it is helpful to define some additional notation. For election \( E = (C, V) \), we denote the \( k \)-Approval score of some candidate \( c \in C \) by \( \text{score}^k(c) \). For some fixed \( k \), we frequently use the term \( k \)-Approval point referring to any single point a candidate gets during the computation of his or her \( k \)-Approval score. Consequently, for some fixed \( k \), a candidate may lose or gain a \( k \)-Approval point as an effect of a shift action of some voter. We use \( \text{maj}(V) = \left\lfloor \frac{|V|}{2} \right\rfloor + 1 \) to denote the strict majority threshold for voters \( V \).

Theorem 4. The Destructive Shift Bribery problem for the Bucklin rule is in \( P \).

Proof. Let us fix budget \( B \) and election \( E = (C, V) \), where \( C = \{c_1, c_2, \ldots, c_m\} \) and \( V = \{v_1, v_2, \ldots, v_n\} \). We also have shift-bribery prices functions \( \{\rho_1, \rho_2, \ldots, \rho_n\} \). Without loss of generality, we assume that \( c_1 = d \) is the despised candidate.

To prove our theorem, we give an algorithm which first guesses candidate \( c_i \neq d \) and the Bucklin winning round \( k \) and, then, checks whether \( c_i \) is able to preclude \( d \) from being the unique winner of the election while ensuring that \( k \) is the Bucklin winning round (or the Bucklin winning round is even earlier). We start by defining an appropriate helper function, then we show that we can solve our problem using this function, and eventually we show that this function is polynomial-time computable.

For each candidate \( c_i \in C \setminus \{d\} \) and each \( k \in [m] \), we define the following function \( f^k_i \). For each \( w \in \{0\} \cup [n] \) and each \( p, q, q' \in [m] \), we let \( f^k_i(w, p, q, q') \) be the lowest cost of bribing at most first \( w \) voters so that \( \text{score}^k(c_i) = p \), \( \text{score}^k(d) = q \), and \( \text{score}^{k-1}(d) = q' \). In other words, the function gives the cost of setting up the scores of candidate \( c_i \) and \( d \) in round \( k \) (and just before round \( k \), for \( d \)).

To check whether there is a successful bribery (i.e., one that ensures that \( d \) is not a unique winner of the election) of cost at most \( B \), it suffices to check if there is candidate \( c_i \), \( c_i \neq d \), round number \( k \in [m] \), and values \( p, q, q' \) such that:

1. \( q' < \text{maj}(V) \),
2. \( p \geq \text{maj}(V) \),
3. \( p \geq q \), and

\( f^k_i(n, p, q, q') \leq B \). Condition (1) guarantees that \( d \) does not win in any round prior to \( k \), Condition (2) ensures that \( k \) is the largest possible Bucklin winning round, and Condition (3) imposes that \( d \) is not a unique winner if \( k \) is the Bucklin winning round (note that the actual
winning round number might be smaller than \(k\) but then \(d\) certainly does not win). If such a set of values exists, then we accept. Otherwise we reject.

The above algorithm requires computing \(O(n^3)\) values of at most \(O(m^2)\) functions \(f_k^i\). Thus, to show that the algorithm runs in polynomial time, it suffices to show that the values of the \(f_k^i\) functions are polynomial-time computable, which we now do.

We compute values \(f_k^i(n, p, q, q')\) using dynamic programming. For each \(p, q, q' \in [n]\), \(f_k^i(0, p, q, q') = 0\) if prior to any bribery we have \(\text{score}^k(c_i) = p\), \(\text{score}^k(d) = q\), and \(\text{score}^{k-1}(d) = q'\). Otherwise, we have \(f_k^i(0, p, q, q') = \infty\). To compute \(f_k^i(w, p, q, q')\) for \(w > 0\), we express it using a recursive formula. To present this formula compactly, for each voter \(v_w\) and each \(k, 1 \leq k < m\), we define:

1. Function \(L_k^w\) such that for each \(j \in \{0\} \cup [k]\) we have \(L_k^w(j) = 1\) if candidate \(d\) is ranked below the \(k\)-th position after he or she is shifted backward by \(j\) positions, and we have \(L_k^w(j) = 0\) otherwise.

2. Function \(G_k^w\) such that for each \(i, j \in [m]\), we have \(G_k^w(i, j) = 1\) if after shifting candidate \(d\) backward by \(j\) positions, candidate \(c_i\) is ranked among top \(k\) positions, and we have \(G_k^w(i, j) = 0\) otherwise.

Using this notation, we express function \(f_k^i\) as follows:

\[
    f_k^i(w, p, q, q') = \min_{j \in \{0\} \cup [k]} \{f_k^i(w-1, p - G_k^w(i, j), q - L_k^w(i, j), r - L_k^{k-1}(j)) + \rho_w(j)\}
\]

Intuitively, for given \(k\) and \(i\), the above formula checks all possible values of \(j\) for which we shift \(d\) backward by \(j\) positions in vote \(v_w\), and uses shifts in the preceding votes to fulfill the function’s definition. This formula, together with standard dynamic programming techniques, allows us to compute the values of function \(f_k^i\) in polynomial time. This completes the proof.

By slightly changing the function presented in the above proof and using a similar algorithm, we obtain an algorithm for the Simplified Bucklin rule.

**Theorem 5.** The **Destructive Shift Bribery** problem for the Simplified Bucklin rule is in P.

**Proof.** We adjust the proof of Theorem 4 to match the case of the Simplified Bucklin rule. For this rule, the exact score of a candidate in the Bucklin winning round is not taken into account while determining the winners (provided that it is greater than the majority of the voters). Therefore, we can simplify function \(f_k^i(w, p, q, q')\) from the proof of Theorem 4 by removing parameter \(q\). Consequently, we remove the constraint associated with parameter \(q\) while checking for a successful bribery. The rest of the proof remains the same, so **Destructive Shift Bribery** for the Simplified Bucklin rule is in P as well. \(\square\)

\(\text{For function } f_k^i \text{ it is never necessary to shift } d \text{ to position lower than } k + 1, \text{ which is why we consider } j \in \{0\} \cup [k].\)
3.4 The Copeland Rule

Let us now move on to the case of Copeland^\alpha family of rules. We show that irrespective of the choice of \alpha, 0 \leq \alpha \leq 1, the Destructive Shift Bribery problem is NP-complete for Copeland^\alpha. We also show W[1]-hardness of the problem for parametrizations by the budget, the number of voters, and the number of bribed voters. On the other hand, an FPT algorithm for the parametrization by the number of candidates was given by Knop et al. [24] (formally, they did not study Destructive Shift Bribery, but it is immediate to see that their proof extends to this setting). To show our hardness results, we use reductions from one of the classic NP-complete problems, CLIQUE (which is also W[1]-hard for the parameterization by the clique size).

**Definition 3.** In the CLIQUE problem we are given a graph and an integer k. We ask whether it is possible to find a k-clique, that is, a size-k set of pairwise adjacent vertices, in the given graph.

We use the following notation in the proofs in this section. By putting some set S in a preference order, we mean listing the contents of the set in an arbitrary but fixed order. To denote the reversed order we use \overrightarrow{S} (e.g., for set S = \{a, b, c\}, writing S in a preference order could mean b \succ c \succ a; then, putting \overrightarrow{S} in a preference order would mean a \succ c \succ b).

**Theorem 6.** For each (rational) value of \alpha, 0 \leq \alpha \leq 1, the Destructive Shift Bribery problem for the Copeland^\alpha rule is NP-complete and W[1]-hard for the parameterization by the budget value and for the parameterization by the number of affected of voters.

**Proof.** Since winner determination for Copeland^\alpha is in P, by Observation 1 we see that our problem is in NP and it remains to show its NP-hardness. We give a reduction from CLIQUE.

Let us fix some arbitrary value of \alpha and an instance of CLIQUE with a given graph G = (V(G), E(G)) and an integer k. We construct an instance of Destructive Shift Bribery with an election E = (C, V), unit bribery prices and budget B = 3\binom{k}{2}. Let the set of candidates C be \{d, p\} \cup L \cup L' \cup V(G) \cup E(G) \cup S (members of V(G) and E(G) are both vertices and edges in G and corresponding candidates in our election). Sets L, L', and S contain dummy candidates, where L and L' consist of |V| \cdot |E| \cdot B candidates each, and S consists of \binom{k}{2} + k + 1 candidates. We introduce the following voters:

1. For each edge \{u, v\} = e \in \overline{E}, we introduce two voters with the following preference orders:

   1 vote: \overrightarrow{d \succ u \succ v \succ e \succ L \succ L' \succ p \succ E(G) \setminus \{e\} \succ V(G) \setminus \{u, v\} \succ S}

   1 vote: \overrightarrow{\overrightarrow{S} \succ \overrightarrow{V(G) \setminus \{u, v\}} \succ \overrightarrow{E(G) \setminus \{e\}} \succ \overrightarrow{L'} \succ \overrightarrow{L} \succ e \succ v \succ u \succ d.}
| candidate(s) | score |
|-------------|-------|
| \( d \)     | \( |V| + |E| + |L| + |L'| + |S| \) |
| \( p \)     | \( |V| + |E| + |L| + |L'| + 1 \) |
| \( S \)     | \( \leq |V| + |E| + |L'| + |S| \) |
| \( L \)     | \( \leq |V| + |E| + |L| + |S| - 1 \) |
| \( L' \)    | \( \leq |V| + |E| + |L| + |L'| - 1 \) |
| \( V \)     | \( \leq |E| + |V| - 1 \) |
| \( E \)     | \( \leq |E| - 1 \) |

Table 1: Scores of all the candidates in the constructed election. We indicate upper bounds by \( \leq \) where scores relate to groups of candidates.

2. We introduce \( 2k - 3 \) voters with the following preference orders:

- \( k - 2 \) votes: \( E(G) \succ d \succ L \succ S \succ p \succ L' \succ V(G) \),
- \( k - 2 \) votes: \( p \succ L' \succ d \succ L \succ V(G) \succ S \succ E(G) \),
- 1 vote: \( S \succ p \succ d \succ L' \succ V(G) \succ L \succ E(G) \).

3. We introduce \( 6k^2 \) voters with the following preference orders:

- \( 3k^2 \) votes: \( d \succ L \succ L' \succ C \setminus (\{d\} \cup L \cup L') \),
- \( 3k^2 \) votes: \( C \setminus (\{d\} \cup L \cup L') \succ d \succ L' \succ L \).

Note that the number of voters is odd, so the value of \( \alpha \) is irrelevant.

We present the scores of the candidates, prior to bribery, in Table 1 (for some candidates we only provide upper bounds on their scores; to verify the values in the table, it is helpful to note that the preference orders of the pairs of voters in the first group are reverses of each other, and thus one can disregard them when calculating scores). We see that candidate \( d \) is the winner.

By shifting \( d \) backward without exceeding the budget, we can only change the outcome of head-to-head contests between \( d \) and the candidates from \( V(G) \) and \( E(G) \). All the other candidates are either ranked too far away from \( d \) or, as in the case for \( L \) and \( L' \), \( d \) has too large advantage over them. For each candidate \( v \in V(G) \), we have \( N_E(d,v) - N_E(v,d) = 2k - 3 \), and for each \( e \in E(G) \), we have \( N_E(d,e) - N_E(e,d) = 1 \). Thus, for \( d \) to lose a head-to-head contest against a candidate \( v \in V(G) \), \( d \) has to be shifted behind \( v \) in at least \( k - 1 \) votes, and to lose a head-to-head contest against a candidate \( e \in E(G) \), \( d \) has to be shifted behind \( e \) in at least one vote (note that the scores of candidates in \( V(G) \) and \( E(G) \) are so low that after a shift bribery that does not exceed the budget, neither of them can be a winner).

Candidate \( p \) has the second highest score in our election and we have \( \text{score}_E(d) - \text{score}_E(p) = \binom{k}{2} + k \). Hence, to prevent \( d \) from being a winner, we need to lower \( d \)'s
score by at least \( \binom{k}{2} + k \). If our graph contains a clique of size \( k \) that consists of edges \( Q = \{e_1, \ldots, e_{\binom{k}{2}}\} \), then for each voter from the first group that corresponds to one of these edges we shift \( d \) backward by three positions. This costs \( 3\binom{k}{2} \) units of budget. Candidate \( d \) loses one point for each edge from the clique and one point for each vertex from the clique (because \( d \) passes each of them exactly \( k - 1 \) times). Altogether, \( d \) loses \( \binom{k}{2} + k \) points and ceases to be the unique winner.

For the other direction, assume that there is a shift bribery of cost at most \( B = 3\binom{k}{2} \) that ensures that \( d \) is not the unique winner. By the observations from the previous paragraphs, we can assume that the bribery affects voters in the first group only (and for each pair of voters there, it affects the first one). We claim that the bribery has to shift \( d \) behind \( \binom{k}{2} \) edge candidates in the first group of voters. To see why this is the case, assume that it shifts \( d \) behind \( y = \binom{k}{2} - x \) (distinct) edge candidates, where \( x \) is a positive integer smaller or equal to \( \binom{k}{2} \). To make \( d \) lose \( \binom{k}{2} + k \) points, we need to shift \( d \) backwards behind at least \( \binom{k}{2} + k - y = k + x \) vertex candidates at least \( k - 1 \) times. That means that altogether the number of unit shifts that we need to make is at least:

\[
3 \left( \binom{k}{2} - x \right) + \frac{(k + x)(k - 1)}{2} - 2 \left( \binom{k}{2} - x \right).
\]

This value is equal to:

\[
\binom{k}{2} - x + (k + x)(k - 1) = \binom{k}{2} - x + k(k - 1) + x(k - 1) = 3\binom{k}{2} - x + x(k - 1) = 3\binom{k}{2} + x(k - 2),
\]

which is greater than \( B = 3\binom{k}{2} \) for \( k > 2 \) and \( x > 1 \). Since we can assume that \( k > 2 \) without loss of generality, we see that a successful shift bribery that does not exceed the budget has to guarantee that \( d \) passes exactly \( \binom{k}{2} \) edge candidates. One can verify that this leads to \( d \) passing \( k \) vertex candidates \( k - 1 \) times each only if these edges form a size-\( k \) clique.

\textbf{Clique} is \( W[1] \)-hard for the parameter \( k \). As our budget is a function of \( k \) and the number of affected voters is exactly \( \binom{k}{2} \), our reduction shows \( W[1] \)-hardness with respect to the budget and with respect to the number of affected voters.

The problem remains hard also for the parametrization by the number of voters.

\textbf{Theorem 7.} Parameterized by the number of voters, the Destructive Shift Bribery problem is \( W[1] \)-hard, even for the case of unit prices.

\textbf{Proof.} We give a parameterized reduction from the Multicolored Independent Set problem, where we are given a graph \( G = (V(G), E(G)) \) with each vertex colored with one out of \( h \) colors, and we ask if there is a size-\( h \) subset of vertices \( I \subseteq V(G) \) such that every vertex has a different color and for each two \( u, v \in I \), there is no edge \( \{u, v\} \) in the graph.
We assume without loss of generality that the number of vertices of every color is the same, there are no edges between vertices of the same color, and there exists at least one vertex with non-zero degree among vertices of each color. Let $q$ be the number of vertices of each color. We denote the maximum degree among vertices of graph $G$ by $\Delta$. Let $\delta(v)$ be the degree of vertex $v \in V(G)$, and let $E(v)$ be the set of edges adjacent to a given vertex $v \in V(G)$. By $V(i)$ we mean the set of all vertices of color $i$.

For each color $i$, we introduce vertex candidates $V(i) = \{v_1^i, v_2^i, \ldots, v_q^i\}$. We represent every edge in the input graph $G$ with one edge candidate. For each vertex candidate $v_j^i$, by $E_j^i$ we denote the set of edge candidates representing the edges incident to $v_j^i$ in graph $G$. We write $G_c$ to denote the set of all vertex and edge candidates. If for some $i \in [h]$ and $j \in [q]$, $|E_j^i| < \Delta$, we add the set of filler candidates $F_j^i$ of size $\Delta - \delta(v_j^i)$. We write $F$ to denote the union of all the sets of filler candidates for all the vertices. We introduce $h + 3$ sets of dummy candidates with $t = hq(\Delta + 1)$ elements each (note that $t > |V(G)| + |E(G)|$); namely, these sets are $D_1, \ldots, D_h$, $D', D''$, and $D'''$. We write $\mathcal{D}$ to denote $\bigcup_{i=1}^{h} D_i$ and $\mathcal{D}_{-i}$ to denote $\mathcal{D} \setminus D_i$. Lastly, we add candidates $d$, $p$ and $q$. To specify voters in a compact form, for every color $i \in [h]$ we introduce partial preference order:

$$P_i = v_1^i > E_1^i > F_1^i > v_2^i > \cdots > v_q^i > E_q^i > F_q^i.$$

For a given $i \in [h]$, we write $P_{-i}$ to denote an arbitrary but fixed order over all the vertex, edge and filler candidates not ranked in $P_i$. The constructed election consists of the following votes:

1. For each color $i$ we construct votes:

   $$o_i : d \succ P_i \succ D_i \succ \mathcal{D}_{-i} \succ P_{-i} \succ p \succ q \succ D' \succ D'' \succ D''',$$

   $$o_i' : d \succ \overline{P_i} \succ \overline{D_i} \succ \mathcal{D}_{-i} \succ P_{-i} \succ p \succ q \succ D' \succ D'' \succ D''' .$$

   We add votes $\overline{o_i}$ and $\overline{o_i}'$ with the reversed preference order of, respectively, $o_i$ and $o_i'$.

2. We introduce seven votes as follows:

   $$u_1 : p \succ F \succ D \succ q \succ D' \succ \overline{D''} \succ G_c \succ D''' \succ d,$$

   $$u_2 : d \succ \overline{D''} \succ \overline{G_c} \succ D'' \succ \overline{D'} \succ D \succ q \succ p \succ F,$$

   $$u_3 : p \succ q \succ d \succ \overline{D'} \succ D'' \succ \overline{D} \succ F \succ D'' \succ G_c,$$

   $$u_4 : G_c \succ d \succ D'' \succ F \succ D \succ p \succ q \succ D''' \succ D',$$

   $$u_5 : d \succ D''' \succ F \succ G_c \succ D'' \succ D \succ p \succ q \succ D',$$

   $$u_6 : q \succ p \succ D \succ D'' \succ d \succ \overline{D''} \succ D' \succ F \succ G_c,$$

   $$u_7 : q \succ D' \succ G_c \succ d \succ p \succ F \succ D \succ D'' \succ D''' .$$

Our election admits unit prices, and the budget is $B = h(q + (q - 1)\Delta)$.  

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Table 2: The scores of all of the candidates in the constructed election. We indicate upper bounds by ≤ where scores relate to groups of candidates.

| candidate | score                  |
|-----------|------------------------|
| d         | \((h + 3)t + |E| + |V| + |F| + 1\) |
| p         | \((h + 3)t + |F| + 1\)   |
| q         | \(3t + |F| + |E| + |V| + 1\) |
| \(D_i, i \in [h]\) | \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)t\) ≤ \((h + 3)
the price of a backward shift of $d$ behind $c$ is within the budget (e.g., for $c \in F^i_j$, $i \in [h]$, $j \in [q]$, the votes are $o_i$ and $o'_i$).

The final conclusion from our observations is that to prevent $d$ from being a unique winner of the election, with a shift bribery of cost at most $B$, one has to shift candidate $d$ behind every edge and vertex candidate at least once. Then candidate $d$ would no longer be the unique winner because $d$ and $p$ would tie (note that as $t > |E| + |V|$, no other candidate could threaten $d$). We claim that if one is able to shift $d$ back behind every edge and vertex candidate at least once without exceeding the budget, then the input graph $G$ has a multicolored independent set of size $h$. To show this, let us focus on some color $i$. Due to the budget constraint and votes’ construction, we have to shift $d$ back behind every vertex $v \in V(i)$ bribing only voters $o_i$ and $o'_i$. To achieve this, we have to spend at least $q + (q - 1)\Delta$ units of budget. We observe that it is also the upper bound because our argument holds for every color and we are constrained by the budget $B = h(q + (q - 1)\Delta)$. However, spending $q + (q - 1)\Delta$ units of budget for some color $i$ does not allow us to shift $d$ back behind all the candidates in $\bigcup_{j \in [q]} E^i_j$. We have to shift $d$ back behind all vertex candidates of color $i$ and the only possibility to achieve this is to leave all candidate edges of one vertex candidate not passed by $d$. Let us call this vertex a selected vertex. Now, we can see that the only case where all edge candidates are, nonetheless, passed by candidate $d$ is if the selected vertex candidates form a multicolored independent set. If this is not the case, than there exists at least one edge $e$ connecting two selected vertices. Since edge candidates of selected vertices are not passed by $d$, the edge candidate corresponding to $e$ is never passed by $d$ and, so, $d$ is still the unique winner. On the other hand, one can also verify that if there exists a multicolored independent set, then one can always find a successful bribery.

Since our reduction is a valid parameterized (indeed, a polynomial-time computable one) reduction where the number of voters is a function of the parameter $h$, and Multicolored Independent Set is W[1]-hard for this parameter, we conclude that our problem is W[1]-hard for the parameterization by the number of the voters.

For the sake of completeness, we finish our analysis by mentioning one more parameterized computational complexity result regarding Destructive Shift Bribery for Copelandα.

**Observation 3.** Parameterized by the combined parameter number of voters and budget value, the Destructive Shift Bribery problem is in FPT. Destructive Shift Bribery is in FPT with respect to the parameter number of voters for all-or-nothing bribery functions (where, for each voter, there is a single price for every possible shift).

**Proof.** For unconstrained bribery functions one can guess which voters to bribe and how many budget units to use for each bribed voter. The result holds for all-or-nothing prices functions because it suffices to guess the voters where we shift $d$ to the back. □
3.5 The Maximin Rule

In spite of the hardness results from the previous section, it certainly is not the case that Destructive Shift Bribery is hard for all Condorcet-consistent rules. We now show a polynomial time algorithm for the case of Maximin.

**Theorem 8**. The Destructive Shift Bribery problem for the Maximin rule is in P.

**Proof.** Let \( E = (C, V) \) be the input election with \( C = \{c_1, c_2, \ldots, c_m\} \) and \( V = \{v_1, v_2, \ldots, v_n\} \), let \( d \in C \) be the despised candidate, let \( B \) be the budget, and let \( \{\rho_1, \rho_2, \ldots, \rho_n\} \) be the destructive shift-bribery price functions for the voters. A solution to Destructive Shift Bribery is a vector \( S = (s_1, s_2, \ldots, s_n) \) of \( n \) integers such that candidate \( d \) does not win in the election resulting from bribing each voter \( v_i \in V \) to shift \( d \) back by \( s_i \) positions. Naturally, \( S \) is a solution if the price of all shifts it describes does not exceed the budget. Fix some solution \( S \) and the resulting election \( E' = (C, V') \), where \( V' = \{v'_1, v'_2, \ldots, v'_n\} \). In election \( E' \) there are always two important candidates: candidate \( w \), whose score is at least as high as that of \( d \) in \( E' \), and candidate \( t \) such that score\(_{E'}(d) = N_{E'}(d, t) \). Intuitively, \( w \) is the candidate that ensures that \( d \) is not a unique winner, and \( t \) is the candidate that “implements” the score of \( d \) in election \( E' \). Note that it is possible that \( w = t \). We call a solution \( S = (s_1, s_2, \ldots, s_n) \) tight if for every voter \( v'_i \) such that \( s_i \neq 0 \), it holds that either \( v'_i \) ranks \( d \) just below \( w \) or \( v'_i \) ranks \( d \) just below \( t \).

Intuitively, the solution is tight if we do not waste the budget to make unnecessary shifts that do not affect the relative order of \( d, w, \) and \( t \). Note that we are able to modify any solution \( S = (s_1, s_2, \ldots, s_n) \) so that it becomes tight, by undoing, for every voter, as many shifts as possible without changing the relative positions of \( d, w, \) and \( t \). \( S' \) is a solution because \( d, w, \) and \( t \) have the same scores after applying \( S' \) as after applying \( S \). More so, the cost of \( S' \) is at most as high as that of \( S \). Thus it suffices to focus on tight solutions.

Our algorithm tries each pair of \( t \) and \( w \) and checks if there is a tight solution for them with cost at most \( B \). It accepts if so, and it rejects if there is no solution with cost at most \( B \) for any choice of \( t \) and \( w \). Since there are at most \( O(m^2) \) pairs of candidates to try, it suffices to show a polynomial-time algorithm for computing the cheapest tight solution for given \( t \) and \( w \). Below we focus on the case where \( t \neq w \); the case where \( t = w \) is analogous.

Let us fix candidates \( t \) and \( w \) with \( t \neq w \). For each two candidates \( c_1, c_2 \), we write \( \text{pref}(c_1, c_2) \) to denote the set of voters (from \( V \)) that prefer \( c_1 \) over \( c_2 \). Let price\((v, c)\) be the cost of shifting candidate \( d \) just below \( c \) in \( v \)'s preference order. Since we are interested in changing the relative positions of candidates \( t, w, \) and \( d \), we focus on voters in the set \( \text{pref}(d, w) \cup \text{pref}(d, t) = \{v''_1, v''_2, \ldots, v''_\ell\} \) (i.e., voters who prefer \( d \) to at least one of \( w \) and \( t \)). We define function \( f_{w, t}(j, x, y) \) such that \( f_{w, t}(j, x, y) \) is the lowest cost for shifting \( d \) backward in the preference orders of the voters from \( \{v'_1, v'_2, \ldots, v'_\ell\} \) in a way that ensures that \( d \) moves \( x \) times below \( w \) and \( y \) times below \( t \) (observe that \( j, x, y \in \{0, 1, \ldots, \ell\} \)).

With function \( f_{w, t} \), we can compute the cost of the cheapest tight solution for candidates \( w, t \). It suffices to try all values of \( f_{w, t}(j, x, y) \) for pairs \( x, y \in \{0, 1, \ldots, \ell\} \). For each pair it is easy to compute the scores that candidates \( d \) and \( w \) get after shifting \( d \) back \( x \) times
slightly adapt all possible cases when computing function \( f \). Compute the values of functions \( f \).

On the other hand, if \( v \) happens already for the voters \( d \) behind \( w \), then we have:

\[
\text{Case 1: If } v'' \in \text{pref}(d, w) \setminus \text{pref}(d, t), \text{ that is, if } v'' \text{ prefers } d \text{ to } w, \text{ but not to } t, \text{ then we have:}
\]

\[
f_{w,t}(j, x, y) = \min \left\{ \begin{array}{l}
f_{w,t}(j - 1, x, y), \\
f_{w,t}(j - 1, x - 1, y) + \text{price}(v''_j, w)
\end{array} \right\}.
\]

To see the correctness of the formula, note that to achieve the fact that \( d \) moves \( x \) times behind \( w \) and \( y \) times below \( t \) for the voters \( v''_1, v''_2, \ldots, v''_{j-1} \), we either ensure that this happens already for the voters \( v''_1, v''_2, \ldots, v''_{j-1} \) and leave \( v''_j \) intact, or we ensure that \( d \) moves \( x - 1 \) times below \( w \) and \( y \) times below \( t \) for voters \( v''_1, v''_2, \ldots, v''_{j-1} \) and shift \( d \) back behind \( w \) in \( v''_j \)'s preference order (we omit such detailed descriptions below, but the general idea for each of the cases is the same).

\[
\text{Case 2: If } v''_j \text{ is in } \text{pref}(d, t) \setminus \text{pref}(d, w), \text{ then we have:}
\]

\[
f_{w,t}(j, x, y) = \min \left\{ \begin{array}{l}
f_{w,t}(j - 1, x, y), \\
f_{w,t}(j - 1, x, y - 1) + \text{price}(v''_j, t)
\end{array} \right\}.
\]

\[
\text{Case 3: If } v''_j \text{ is in } \text{pref}(d, w) \cap \text{pref}(d, t) \text{ and } v'' \text{ prefers } d \text{ to } w \text{ and } w \text{ to } t, \text{ then we have:}
\]

\[
f_{w,t}(j, x, y) = \min \left\{ \begin{array}{l}
f_{w,t}(j - 1, x, y), \\
f_{w,t}(j - 1, x - 1, y) + \text{price}(v''_j, w), \\
f_{w,t}(j - 1, x - 1, y - 1) + \text{price}(v''_j, t)
\end{array} \right\}.
\]

On the other hand, if \( v''_j \) prefers \( d \) to \( t \) and \( t \) to \( w \) then we have:

\[
f_{w,t}(j, x, y) = \min \left\{ \begin{array}{l}
f_{w,t}(j - 1, x, y), \\
f_{w,t}(j - 1, x, y - 1) + \text{price}(v''_j, t), \\
f_{w,t}(j - 1, x - 1, y - 1) + \text{price}(v''_j, w)
\end{array} \right\}.
\]

Given this discussion, using standard dynamic-programming approach it is possible to compute the values of functions \( f_{w,t} \) in polynomial time. For the case of \( w = t \), one has to slightly adapt all possible cases when computing function \( f_{w,t} \).

\[\text{If the bribery price functions allow for shifting } d \text{ back at zero cost in some votes, we shift } d \text{ as much as possible at zero cost as a preprocessing step.}\]
It is interesting to ask what feature of the Maximin rule—as opposed to the Copeland rule—leads to the fact that Destructive Shift Bribery is polynomial-time solvable. We believe that the reason is that it is safe to focus on a small number of candidates (the candidates $w$ and $t$). For Copeland elections, on the other hand, one has to keep track of all the candidates that the despised candidate passes, because each such pass could, in effect, decrease the despised candidate’s score. (Interestingly, the same is true for the Borda rule—each time the despised candidate passes a candidate, the despised candidate’s score decreases. However, in the case of Borda this process is unconditional, whereas in the case of Copeland’s rule, the decrease may or may not happen, depending on shifts in other preference orders).

## 4 Conclusions

We have introduced and studied a destructive variant of the Shift Bribery problem. In our problem, we ask if it is possible to preclude a given candidate from being the winner of an election by shifting this candidate backward in some of the votes (at a given cost, within a given budget), whereas in the constructive variant of the problem one asks if it is possible to ensure a given candidate’s victory by shifting him or her forward.

We have shown that Destructive Shift Bribery is polynomial-time solvable for the $k$-Approval family of rules (in effect, including the Plurality rule), the Borda rule, all scoring protocols (as long as either the protocol or the price functions can be assumed to be encoded in unary) the Simplified Bucklin rule, the Bucklin rule, and the Maximin rule. On the other hand, we have shown that for each rational value of $\alpha$, the problem is NP-complete for Copeland$^\alpha$. We have investigated the problem’s parameterized complexity in this case showing that it remains hard for the case of small budgets and for the case of few voters, even under unit price functions. However, the problem is in FPT for the case of few candidates $[24]$. We summarize our results on general complexity in Table 4, whereas Table 5 contains the parameterized complexity results for Copeland$^\alpha$. 

| Election rule  | Constructive Shift Bribery | Destructive Shift Bribery |
|----------------|----------------------------|---------------------------|
| Plurality      | P                          | P                         |
| $k$-Approval   | P                          | P                         |
| Borda          | NP-com.                    | P                         |
| Maximin        | NP-com.                    | P                         |
| Copeland$^\alpha$ | NP-com.                | NP-com.                  |

Table 4: The complexity of Shift Bribery for various election rules. The results for the constructive case are due to Elkind et al. [16] and the results regarding the destructive case are due to this paper.
Our work leads to several open questions. First, one could always study more election rules. Second, one can analyze the robustness of various election rules based on the number of backward shifts of the winner needed to change their outcome \[ \text{[36, 5]} \]. This direction can also be seen as studying a more fine-grained extension of the MARGIN OF VICTORY problem. Third, it would be interesting to perform an empirical test to measure how much we need to shift back the election winner to change the result under various assumptions regarding the voters’ preference orders (and in real-life elections, such as those collected in PrefLib \[ \text{[28]} \]) to complement the theoretical analysis mentioned in the second idea.

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Table 5: The complexity of Shift Bribery for Copeland\[\alpha\]. The results marked with ♦ are due to Knop et al. \[\text{[24]}\] and the results marked with ♠ follow from the work of Bredereck et al. \[\text{[4]}\]. The results regarding the destructive cases are due to this paper. The empty cell indicates that we are not aware of any work on this particular variant.
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