MEASURES ON POLYHEDRAL CONES:
CHARACTERIZATIONS AND KINEMATIC FORMULAS

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ABSTRACT. This paper is about conic intrinsic volumes and their associated integral geometry. We pay special attention to the biconic localizations of the conic intrinsic volumes, the so-called support measures. An analysis of these quantities has so far been confined to the PhD thesis of Stefan Glasauer (1995). We rederive the results from this thesis with novel streamlined proofs and expand them in several ways. Additionally, we introduce a new class of functionals on polyhedral cones lying between the intrinsic volumes and the well-studied \( f \)-vector, which counts the equidimensional faces of a cone, and derive a characterization and kinematic formulas for these functionals as well.

1. INTRODUCTION

Spherical integral geometry has so far led a niche existence, outshone by the classical theory of Euclidean integral geometry whose roots date back to the 18th century with Buffon’s famous needle problem [KR97]. Although these two settings are closely related, they are different in some decisive aspects, and a discussion of the spherical context as an exotic version of the Euclidean one almost certainly falls short of providing a full view on this independent theory. Furthermore, recent developments [VS92, DT05, DT09, ALMT14] have shown that spherical, or conic, integral geometry can be extremely useful in some applied areas of mathematics such as numerical optimization and compressed sensing. This paper is aimed at laying out the foundations of a theory of conic valuations and their (conic and biconic) localizations. The deepest results so far have been achieved by Glasauer [Gla95, Gla96]. But, although parts of his work has been included in [SW08, Sec. 6.5], the proofs of his results on support measures can solely be found in his thesis. Additionally, he adopts the spherical viewpoint, which makes the theory of localizations incomplete and raises unnecessary hurdles for the proofs. By consistently adopting the conic viewpoint we obtain a complete theory of localizations with streamlined proofs, which we present in a self-contained manner.

In the following sections we introduce the conic intrinsic volumes, describe their surprisingly numerous parallels to the \( f \)-vector of a cone, and give a summary of the main results. Section 1.4 provides a detailed view on the specific contributions of this paper.

1.1. Counting and measuring faces. A set \( C \subseteq \mathbb{R}^d \) is a (convex) polyhedral cone if it can be described as the intersection of finitely many closed half-spaces, that is, \( C = \{x \in \mathbb{R}^d : Ax \leq 0\} \) for some matrix \( A \in \mathbb{R}^{m \times d} \). Equivalently, \( C \subseteq \mathbb{R}^d \) is a polyhedral cone if it is the nonnegative linear hull of a finite set of vectors in \( \mathbb{R}^d \), \( C = \{By : y \in \mathbb{R}^k, y \geq 0\} \) for some matrix \( B \in \mathbb{R}^{d \times k} \). We denote the set of polyhedral cones in \( \mathbb{R}^d \) by \( \mathcal{P}(\mathbb{R}^d) \).

A subset \( F \subseteq C \) of a polyhedral cone \( C \) is a face of \( C \) if there exists \( z \in \mathbb{R}^d \) such that
\[
\langle x, z \rangle \leq 0 \text{ for all } x \in C \quad \text{and} \quad F = \{x \in C : \langle x, z \rangle = 0\}.
\]
Clearly, every face of a polyhedral cone is again a polyhedral cone, and choosing \( z = 0 \) shows that in particular \( C \) itself is a face of \( C \). A polyhedral cone has finitely many faces, and the \( f \)-vector counts these faces according to their dimensions: for \( C \in \mathcal{P}(\mathbb{R}^d) \),
\[
f(C) = (f_0(C), \ldots, f_d(C)), \quad f_k(C) := |\{F : F \text{ face of } C \text{ with } \dim(\text{span}(F)) = k\}|
\]
where \( \text{span}(F) := F - F \) denotes the linear span of \( F \). Although the \( f \)-vector will not be part of the investigations of this paper, we recall some of its fundamental properties [Zie95]:
(0) linear invariance: The $f$-vector is invariant under nondegenerate linear transformations, $f(TC) = f(C)$ for $C \in \mathcal{P}(\mathbb{R}^d)$ and $T \in \text{GL}_d$.

(1) subspaces: If $L \subseteq \mathbb{R}^d$ is an $m$-dimensional linear subspace, then $f_m(L) = 1$ and $f_k(L) = 0$ for all $k \neq m$.

(2) products: The $f$-vector of the product $C \times D$ of polyhedral cones $C, D$ is given by the convolution of the $f$-vectors of $C$ and $D$,

$$f_k(C \times D) = \sum_{i+j=k} f_i(C) f_j(D).$$

(3) polarity: Denoting the polar cone of $C \in \mathcal{P}(\mathbb{R}^d)$ by $C^\circ := \{z \in \mathbb{R}^d : (x, z) \leq 0 \text{ for all } x \in C\} \in \mathcal{P}(\mathbb{R}^d)$, the $f$-vector of $C^\circ$ is the reverse $f$-vector of $C$,

$$f_k(C^\circ) = f_{d-k}(C).$$

(4) Euler property: The Euler characteristic yields for every polyhedral $C \in \mathcal{P}(\mathbb{R}^d)$, which is not a linear subspace,

$$\sum_{k=0}^{d} (-1)^k f_k(C) = 0.$$  

This paper is not about the $f$-vector\footnote{See [Swa14] for a recent survey on this impressive theory.} but about certain variants of it, which may be conceived as “weighted versions” of the $f$-vector. For this we introduce the following notation: the set of linear spans of the ($k$-dimensional) faces of $C \in \mathcal{P}(\mathbb{R}^d)$ shall be denoted by

$$\mathcal{L}(C) := \{\text{span}(F) : F \text{ face of } C\}, \hspace{1cm} \mathcal{L}_k(C) := \{L \in \mathcal{L}(C) : \dim(L) = k\}.$$ 

Note that $L \cap C$ with $L \in \mathcal{L}(C)$ is a face of $C$, and all faces of $C$ are of this form. In particular, $\mathcal{L}(C)$ is a finite set and $f_k(C) = |\mathcal{L}_k(C)|$. The supporting planes of a cone $C \in \mathcal{P}(\mathbb{R}^d)$ and of its polar $C^\circ$ are related via the bijections

$$\mathcal{L}_k(C) \to \mathcal{L}_{d-k}(C^\circ), \hspace{1cm} L \mapsto L^\perp,$$

where $L^\perp := \{z \in \mathbb{R}^d : (x, z) = 0 \text{ for all } x \in L\} (= L^\circ)$ denotes the orthogonal complement of $L$.

One of the most important, yet entirely simple, results, which we will make heavy use of in this paper, is that a polyhedral cone decomposes disjointly into the relative interiors of its faces, i.e.,

$$C = \bigcup_{L \in \mathcal{L}(C)} \text{int}_L(C \cap L) = \bigcup_{k=0}^{d} S_k(C), \hspace{1cm} S_k(C) := \bigcup_{L \in \mathcal{L}_k(C)} \text{int}_L(C \cap L),$$

where $\text{int}_L$ shall denote the interior with respect to the relative topology in $L$. The set $S_k(C)$ is called the $k$-skeleton of $C$. Furthermore, we denote the standard Gaussian measure on $L$ by $\gamma_L$ and abbreviate $\gamma_d := \gamma_{\mathbb{R}^d}$. In the case $\dim L = 0$ this coincides with the Dirac measure supported at the origin, for which we use the notation $\Delta = \gamma_0$.

The $u$-vector of $C \in \mathcal{P}(\mathbb{R}^d)$ collects the sums of the Gaussian volumes of equidimensional faces of $C$:

$$u(C) = (u_0(C), \ldots, u_d(C)), \hspace{1cm} u_k(C) := \sum_{L \in \mathcal{L}_k(C)} \gamma_L(C \cap L).$$

Note that $u_0(C) = f_0(C)$ and $u_d(C) = \gamma_d(C)$. The $u$-vector is a weaker invariant than the $f$-vector; clearly, linear transformations can change the Gaussian volumes of the faces. However, we still have orthogonal invariance:

(0') orthogonal invariance: The $u$-vector is invariant under orthogonal transformations, $u(TQ(C) = u(C)$ for $C \in \mathcal{P}(\mathbb{R}^d)$ and $Q \in O(d)$. 

The \( \mathbf{u} \)-vector satisfies the inequalities \( 0 \leq u_k \leq f_k \), and for linear subspaces the \( \mathbf{u} \)- and the \( \mathbf{f} \)-vector coincide, so that the \( \mathbf{u} \)-vector also has property (1). Furthermore, the \( \mathbf{u} \)-vector also satisfies the product rule (2), cf. (1.3).

On the other hand, the \( \mathbf{u} \)-vector possesses neither a polarity property as in (1.4) nor an Euler property as in (1.5). The \( \mathbf{v} \)-vector, to be introduced next, may be thought of as the straightforward way to reestablish the polarity relation, which, almost mysteriously, not only reestablishes the Euler property as well, but also yields further fundamental properties.

The \( \mathbf{v} \)-vector of \( C \in \mathcal{P}(\mathbb{R}^d) \) collects the \textit{(conic) intrinsic volumes} of \( C \):

\[
\nu(C) = (\nu_0(C), \ldots, \nu_d(C)), \quad \nu_k(C) := \sum_{L \in \mathcal{F}_k(C)} \gamma_L(C \cap L) \gamma_L(C^0 \cap L^\perp). \tag{1.9}
\]

Alternatively, one can define the intrinsic volume of a polyhedral cone by combining the decomposition (1.7) with the projection map \( \Pi_C : \mathbb{R}^d \to C, \Pi_C(x) = \arg \min \{\|x - y\| : y \in C\} \): denoting by \( \mathbf{g} \in \mathbb{R}^d \) a standard Gaussian vector, we have

\[
\nu_k(C) = \mathbb{P} \{ \Pi_C(\mathbf{g}) \in S_k(C) \}. \tag{1.10}
\]

From the definition (1.9) it is immediate that \( 0 \leq \nu_k \leq u_k \). Note also that from the characterization (1.10) one directly obtains \( \nu_0(C) + \cdots + \nu_d(C) = 1 \). So the \( \mathbf{v} \)-vector is actually a discrete probability distribution.\(^2\) Furthermore, the \( \mathbf{v} \)-vector is orthogonal invariant, i.e., it satisfies property (0*), and it satisfies the subspace and the product rules (1) and (2), just as the \( \mathbf{u} \)-vector does. Unlike the \( \mathbf{u} \)-vector, the \( \mathbf{v} \)-vector also satisfies the polarity property (3), which follows from the bijection (1.6) between the supporting subspaces of \( C \) and \( C^\circ \). Less obvious is the fact that the \( \mathbf{v} \)-vector even satisfies the Euler property (4). But more than that, the \( \mathbf{v} \)-vector possesses two additional remarkable properties, which are not shared by the \( \mathbf{f} \)-vector:

\begin{enumerate}
  \item \textit{additivity}: If \( C, D \in \mathcal{P}(\mathbb{R}^d) \) are such that \( C + D = C \cup D \), or equivalently, \( C \cup D \in \mathcal{P}(\mathbb{R}^d) \), then
    \[
    \nu(C \cup D) + \nu(C \cap D) = \nu(C) + \nu(D). \tag{1.11}
    \]
  \item \textit{continuity}: If \( C_i \in \mathcal{P}(\mathbb{R}^d), i \in \mathbb{N} \), such that \( \lim_{i \to \infty} C_i = C \in \mathcal{P}(\mathbb{R}^d) \) with respect to the conic Hausdorff metric\(^3\), then
    \[
    \lim_{i \to \infty} \nu(C_i) = \nu(C). \tag{1.12}
    \]
\end{enumerate}

\textbf{Remark 1.1.} The weaker invariance of the \( \mathbf{u} \)- and \( \mathbf{v} \)-vector may be bemoaned, but in fact this weaker invariance is a necessary requirement for the features, which we will discuss next, the kinematic formulas. These formulas treat, for example, the \( \mathbf{u} \)- or \( \mathbf{v} \)-vector of the (random) intersection of one cone with a randomly rotated second cone. It turns out that the expectation of these random vectors can again be expressed in terms of the \( \mathbf{u} \)- and \( \mathbf{v} \)-vectors, respectively, of the components. Formulas of this kind do not hold for the \( \mathbf{f} \)-vector simply because of its linear invariance: taking linear transformations of the components does not change their \( \mathbf{f} \)-vectors, but it does of course change the probabilities for the random intersections. So in a sense we have traded the strong invariance of the \( \mathbf{f} \)-vector for new probabilistic formulas. Strangely enough, we will see that through the kinematic formulas we will also regain linear invariance \textit{in expectation}, cf. (1.15) below.

See Table 1.1 for an overview of the properties of the \( \mathbf{f} / \mathbf{u} / \mathbf{v} \)-vectors.

\(^2\) A loose but intuitive interpretation of \( \nu_k(C) \) is that it describes the “amount of \( k \)-dimensionality” of \( C \); if \( \nu_7(C) = 0.23 \) then \( C \) is 23% 7-dimensional.

\(^3\) The conic Hausdorff metric is just the spherical Hausdorff metric which is obtained by replacing a cone by its intersection with the unit sphere, cf. [Ame11, Sec. 3.2].
Theorem 1.2 (Kinematic formulas for $u$ and $v$). Let $C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d)$ and $T_0, \ldots, T_n \in \text{GL}_d$, and let $k > 0$. Then for $Q_0, \ldots, Q_n \in O(d)$ i.i.d uniformly at random,

\[
\mathbb{E} \left[ u_k \left( \bigcap_{i=0}^n T_i Q_i C_i \right) \right] = u_{nd+k}(C_0 \times \cdots \times C_n),
\]

and

\[
\mathbb{E} \left[ v_k \left( \bigcap_{i=0}^n T_i Q_i C_i \right) \right] = v_{nd+k}(C_0 \times \cdots \times C_n).
\]

Note that as special cases of (1.13) and (1.14), we obtain for $C \in \mathcal{P}(\mathbb{R}^d)$, $T \in \text{GL}_d$, and $Q \in O(d)$ uniformly at random,

\[
\mathbb{E} \left[ u(TQ) \right] = u(C), \quad \mathbb{E} \left[ v(TQ) \right] = v(C).
\]

So although $u$ and $v$ in general both fail to be linear invariants, they are still linear invariants in expectation.

The additional polarity property of the $v$-vector has important consequences for the kinematics. For example, since $(C \cap D)^\circ = C^\circ + D^\circ$ we immediately obtain the following corollary from Theorem 1.2 (see Section 2.1 for some subtleties involving the linear transformations $T_0, \ldots, T_n$).
Corollary 1.3 (Polar kinematic formula for $\nu$). Let the notation and assumptions be as in Theorem 1.2. Then

$$E \left[ v_{d-k} \left( \sum_{i=0}^{n} T_i Q_i C_i \right) \right] = v_{d-k}(C_0 \times \cdots \times C_n).$$ (1.16)

Moreover, the fact that $\nu$ is a probability distribution, i.e., $\nu_0 + \cdots + \nu_d = 1$, and linearity of expectation yield the following formulas for the boundary cases in (1.14) and (1.16).

Corollary 1.4 (Boundary cases for $\nu$). Let the notation be as in Theorem 1.2. Then

$$E \left[ \nu_0 \left( \bigcap_{i=0}^{n} T_i Q_i C_i \right) \right] = \sum_{j=0}^{nd} v_j(C_0 \times \cdots \times C_n), \quad E \left[ \nu_d \left( \bigcup_{i=0}^{n} T_i Q_i C_i \right) \right] = \sum_{j=0}^{nd} v_{d+j}(C_0 \times \cdots \times C_n).$$ (1.17)

Remark 1.5. If one combines the kinematic formulas for the conic intrinsic volumes with the Euler property $\sum_k (-1)^k v_k(C) = 0$ if $C$ is not a linear subspace, then one obtains the so-called Crofton formulas, which describe the intersection probabilities of randomly rotated cones. These Crofton formulas form the link between the theory of conic integral geometry and the applications in optimization and compressed sensing. See [ALMT14, MT14] for further details.

The different kinematic formulas for $\nu$ given in (1.14), (1.16), (1.17) can in fact be seen as special cases\(^4\) of a general kinematic formula, which we formulate next.

If $F(X_0, \ldots, X_n)$ denotes a Boolean formula in the variables $X_0, \ldots, X_n$, and if $C_0, \ldots, C_n \in \mathcal{P}^{d}$, we define the evaluation $F(C_0, \ldots, C_n) \in \mathcal{P}^{d}$ to be the result of the following replacements in the formula $F(X_0, \ldots, X_n)$:

| $\gamma(\cdots)$ | $\wedge$ | $\vee$ |
|-------------------|---------|-------|
| replace $X_i$ by $C_i$ |

So if, for example, $F(X_0, X_1, X_2) = \gamma(X_0 \wedge X_1) \vee X_2$, then $F(C_0, C_1, C_2) = (C_0 \cap C_1)^\circ + C_2$. Note that $C + D = C \cup D$ if the latter is a convex cone, which leads to the more familiar case where the logical $\vee$ corresponds to $\cup$.

A Boolean formula in which every variable appears at most once is called a Boolean read-once formula. If $F(X_0, \ldots, X_n)$ is a Boolean read-once formula and if $L_0, \ldots, L_0 \subseteq \mathbb{R}^d$ are linear subspaces in general position,\(^5\) then $F(L_0, \ldots, L_n)$ is again a linear subspace in general position whose dimension only depends on $F$, the ambient dimension $d$, and the dimensions of $L_0, \ldots, L_n$. We may thus define for every read-once formula $F(X_0, \ldots, X_n)$ and for any dimension $d$ the function

$$\dim^F_d : \{0, \ldots, d\}^{n+1} \rightarrow \{0, \ldots, d\}, \quad \dim^F_d(k_0, \ldots, k_n) = \dim \{F(L_0, \ldots, L_n)\},$$ (1.18)

where $L_0, \ldots, L_n \subseteq \mathbb{R}^d$ are linear subspaces in general position with $\dim(L_i) = k_i$. For example, in the cases $F(X_0, \ldots, X_n) = X_0 \wedge \cdots \wedge X_n$ and $F(X_0, \ldots, X_n) = X_0 \vee \cdots \vee X_n$ we obtain

$$\dim^F_d(k_0, \ldots, k_n) = \max \{0, k_0 + \cdots + k_n - nd\} \quad \min \{d, k_0 + \cdots + k_n\}.$$ (1.18)

In the following theorem we formulate a generalization of the kinematic formulas for $\nu$ given in (1.14), (1.16), and (1.17), which we prove in the special case $T_0, \ldots, T_n \in O(d)$. Of course, if $T_i \in O(d)$ fixed and $Q_i \in O(d)$ uniformly at random, then $T_i Q_i \in O(d)$ uniformly at random, so we might as well drop the $T_i$ here. But we include them nevertheless to ease the comparison with the other formulas and also because the restriction $T_i Q_i \in O(d)$ should be seen as an (unsubstantial) artefact of the proof, cf. Remark 1.7.

\(^4\)We only prove the general kinematic formula (1.19) with the restriction $T_0, \ldots, T_n \in O(d)$, but we are convinced that this formula holds in greater generality, cf. Remark 1.7.

\(^5\)See Appendix B for more details on these genericity assumptions.
Theorem 1.6 (General kinematic formula for $\nu$). Let $C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d)$ and $T_0, \ldots, T_n \in O(d)$, and let $0 \leq k \leq d$. Furthermore, let $F(X_0, \ldots, X_n)$ be a Boolean read-once formula. Then for $Q_0, \ldots, Q_n \in O(d) \text{id} \text{ uniformly at random},$

$$
\mathbb{E} \left[ \nu_k \left( F(T_0Q_0C_0, \ldots, T_nQ_nC_n) \right) \right] = \sum_{\dim_{\nu}(k_0, \ldots, k_n) = k} \nu_{k_0}(C_0) \cdots \nu_{k_n}(C_n).
$$

(1.19)

Note that (1.14) is (1.19) in the special case $F(X_0, \ldots, X_n) = X_0 \wedge \cdots \wedge X_n$, while (1.16) is the case $F(X_0, \ldots, X_n) = X_0 \vee \cdots \vee X_n$. Likewise, the boundary cases in (1.17) are covered by these choices for $F$.

Remark 1.7. Only in dimension $d = 1$ the general kinematic formula (1.19) is true for every Boolean formula; for $d \geq 2$ one can find counterexamples, like the formula $F(X_0, X_1) = (X_0 \vee X_1) \wedge \neg X_0$, for which the general kinematic formula (1.19) fails. However, the restriction to read-once formulas is probably too strict as well as the restriction $T_0, \ldots, T_n \in O(d)$. In fact, one can show the following: the $(d$-dimensional) conic Hadwiger conjecture asserts that the conic intrinsic volumes form a basis for the vector space of orthogonal invariant continuous valuation on $\mathcal{P}(\mathbb{R}^d)$; if this conjecture is true, then (1.19), for general $T_0, \ldots, T_n \in \text{GL}_d$, holds in dimension $d$ for any monotone Boolean formula, i.e., any Boolean formula, which does not contain any negations. We refer to the upcoming paper [AB] for the details.

1.3. Localizations. The key to proving the kinematic formulas provided in the previous section is to consider localizations of $u$ and $v$. In this section we give the definitions of these localizations and also provide a description of the general proof strategy for the kinematic formulas. For details about the involved characterization theorems we refer to the following sections.

We say that a Borel set $M \subseteq \mathbb{R}^d$ is a conic Borel set if it is invariant under positive scaling, i.e., $\lambda M = M$ for all $\lambda > 0$. The set of conic Borel sets is called the conic (Borel) $\sigma$-algebra on $\mathbb{R}^d$, denoted

$$
\mathcal{B}(\mathbb{R}^d) := \{M \subseteq \mathbb{R}^d \text{ Borel set} : \lambda M = M \text{ for all } \lambda > 0\}.
$$

(1.20)

We denote the set of $\sigma$-additive real functions defined on this algebra by $\mathcal{M}(\mathbb{R}^d)$, the set of conic measures. The most important conic measures are the Dirac measure $\Delta$ supported at the origin and the (standard) Gaussian measure $\gamma_d$. Every orthogonal invariant conic measure is a linear combination of $\Delta$ and $\gamma_d$, cf. Lemma 2.2.

To get the connection to the $u$- and $v$-vectors, one considers families of conic measures, which are parametrized by polyhedral cones. The polyhedral measures $\Psi_k(C, \cdot)$ and the curvature measures $\Phi_k(C, \cdot)$, where $C \in \mathcal{P}(\mathbb{R}^d)$ and $0 \leq k \leq d$, are defined by

$$
\Psi_k(C, M) := \sum_{L \in \mathcal{L}_k(C)} \gamma_L(C \cap L \cap M),
$$

(1.21)

$$
\Phi_k(C, M) := \sum_{L \in \mathcal{L}_k(C)} \gamma_L(C \cap L \cap M) \gamma_{L^\perp}(C^\perp \cap L^\perp),
$$

(1.22)

where $M \in \mathcal{B}(\mathbb{R}^d)$. Note that $\Psi_k(C, \mathbb{R}^d) = u_k$ and $\Phi_k(C, \mathbb{R}^d) = v_k$, so the polyhedral measures and the curvature measures are localizations of the $u$- and $v$-vector, respectively. An alternative characterization for the curvature measures is given by

$$
\Phi_k(C, M) = \mathbb{P}\{\Pi_C(g) \in S_k(C) \cap M\},
$$

(1.23)

where $g \in \mathbb{R}^d$ denotes a standard Gaussian vector, cf. (1.10).

The general proof strategy for the kinematic formula of the $u$-vector (1.13) consists of proving a characterization theorem for the polyhedral measures (see Theorem 3.5), and to use this to prove a kinematic formula for the polyhedral measures (see Theorem 3.7), which specializes to (1.13). As for the intrinsic volumes $\nu$ we could proceed similarly with the curvature measures. But a characterization of these measures is a bit more complicated, which makes the curvature measures a less favorable tool. So instead, we will use some
other measures which require introducing a bit more notation, but which share much of the simplicity and naturalness of the polyhedral measures.

The biconic (Borel) $\sigma$-algebra on $\mathbb{R}^d \times \mathbb{R}^d =: \mathbb{R}^{d+d}$ is defined by

$$\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) := (\mathcal{M} \subseteq \mathbb{R}^{d+d} \text{ Borel set} : (\lambda, \lambda') \mathcal{M} = \mathcal{M} \text{ for all } \lambda, \lambda' > 0),$$

(1.24)

where $\langle \lambda, \lambda' \rangle \mathcal{M} := \{ (\lambda x, \lambda' x') : (x, x') \in \mathcal{M} \}$. Again, we consider families of biconic measures parametrized by polyhedral cones. The support measures $\Theta_k(C, \cdot)$, where $C \in \mathcal{P}(\mathbb{R}^d)$ and $0 \leq k \leq d$, are defined by

$$\Theta_k(C, \cdot) := \{ (\Pi_C(g)) \in \mathcal{S}_k(C) \text{ and } \{ \Pi_C(g), \Pi_C'(g) \} \in \mathcal{M} \},$$

(1.25)

where $\Pi_C : \mathbb{R}^d \to C$ denotes again the orthogonal projection map, and where $g \in \mathbb{R}^d$ is a standard Gaussian vector. Comparing this with (1.10) we see that $\nu_k(C) = \Theta_k(C, \mathbb{R}^{d+d})$, so also the support measures are localizations (biconic this time) of the intrinsic volumes. Moreover, if the biconic set is a direct product $\mathcal{M} = M \times M'$, then we obtain

$$\Theta_k(C, M \times M') = \sum_{L \in \mathcal{L}_k(C)} \gamma_L(C \cap L \cap M) \gamma_L(C^o \cap L^\perp \cap M').$$

(1.26)

Therefore, $\Theta_k(C, M \times \mathbb{R}^d) = \Phi_k(C, M)$, so the support measures also localize the curvature measures. In fact, the support measures, due to their inherent symmetry between the primal cone $C$ and its polar $C^o$, appear to be the more natural choice for a localization of the intrinsic volumes than the curvature measures. This impression is further supported by the specific form of the characterization theorem for the support measures, which shares much of the simplicity with that of the polyhedral measures; its proof is almost as elementary. We will exploit this characterization in a similar way as for the polyhedral measures and derive a corresponding (biconically) localized version of (1.14) in Theorem 4.4.

### 1.4. Outline and contributions.

The organization of the paper is as follows. In Section 2 we will recall some well-known properties of the polyhedral cones that we will use in the following sections, and we will introduce the conic and biconic $\sigma$-algebras and the associated conic and biconic measures in greater detail than in Section 1.3. Sections 3 and 4 are devoted to the polyhedral and curvature measures, and to the support measures, respectively. In both sections we first discuss the corresponding measures and their elementary properties, then we show that they are characterized through some of these properties, and then we use these characterizations to deduce kinematic formulas. Section 5 treats the general kinematic formula and contains the proof of Theorem 1.6. These sections constitute the main body of the paper.

We supplement this with three appendices. The first of these is devoted to the Steiner formulas, which play an important role in establishing the theory around the intrinsic volumes, and the curvature and support measures for general convex cones. Since we limit the discussion in this paper to polyhedral cones, we will not make use of these formulas (with the effect that some commonly employed arguments have to be replaced). In Appendix A we shortly present the Steiner formulas and explain how they can be used to generalize some of the results to general closed convex cones. Appendix B is devoted to settling some subtleties that arise in the proofs of the kinematic formulas, and in Appendix C we provide a concise proof for the characterization of the curvature measures. This characterization goes back to a result by Schneider [Sch78, Thm. 6.2], and is contained in [SW08, Thms. 6.5.4/14.4.7]. We chose to include this proof here for three reasons: (1) to allow a direct comparison with the characterization of the support measures, (2) to ease the incorporation of this result in the conic theory (in [SW08] the authors argue in the spherical setting), and (3) to remedy a minor inaccuracy, which is contained in all the previous proofs.

As already mentioned right from the start, the bulk of this paper is based on material from Glasauer’s thesis [Gla95, Gla96]. However, we have carried out a number of changes and additions to justify a separate paper. The most apparent change is that we do not use the spherical setting but argue in a conical context using the concepts of the conic and biconic
\(\sigma\)-algebras, which were introduced in [AB14]. This seemingly superficial change has some important consequences. First of all, the conic versions of the curvature measures and the support measures also cover the boundary cases, which are excluded in the spherical theory. So the conic viewpoint allows a more complete theory. Furthermore, the proofs allow significant simplifications and streamlining, which makes the conic theory more accessible.

The \(\nu\)-vector and its localization given by the polyhedral measures are new concepts, which are apparently introduced here for the first time; the corresponding characterization and kinematic formulas are to that effect also new. These new measures also show that the support measures are more natural localizations of the intrinsic volumes than the curvature measures. This becomes clearest when comparing the simplicity of the arguments in Section 3.2 and Section 4.2 with the arguments in Appendix C.

The equality
\[E[\nu(TQC)] = \nu(C)\]
in (1.15) has first been observed by Mike McCoy and Joel Tropp [MT13]. Incorporating this newly discovered invariance (in expectation) into the kinematic formulas requires a new proof strategy in Sections 3.3/4.3 than typically used, cf. for example [McC13, Sec. 5.3.1]. This subtle point is best observed in the proof of Theorem 1.6 provided in Section 5. The search for a most general conic kinematic formula will not be settled in this paper and we refer to the upcoming paper [AB] for further work in this direction.

1.5. **Acknowledgments.** Part of this research was carried out while spending a year at the mathematics department of The University of Manchester. Special thanks go to my collaborator and host in Manchester Martin Lotz for his broad support and for many useful and interesting discussions. I would also like to thank Mike McCoy and Joel Tropp as well as Peter Bürgisser for useful discussions on conic intrinsic volumes. This research was partly supported by DFG grant AM 386/1-2 and EPSRC grant EP/I01912X/1-CF05.

2. **Preliminaries**

In this section we provide some preliminaries about polyhedral cones in general, and about the conic and biconic \(\sigma\)-algebras.

2.1. **Polyhedral cones.** Most of the relevant notation and properties of the polyhedral cones, which we will use in this paper, has already been introduced in Section 1. In this section we supplement some further aspects.

The first of these aspects concerns (nonsingular) linear transformations \(T \in \text{Gl}_d\). The \(f\)-vector is invariant under these transformations, \(f(TC) = f(C)\). More precisely, the faces of \(TC\) are all of the form \(TF\) for some face \(F\) of \(C\), or in terms of the corresponding linear subspaces,
\[
L_k( TC ) = \{ TL : L \in L_k(C) \},
\]
for \(k = 0, \ldots, d\). Recall that Corollary 1.3 is supposed to follow immediately from Theorem 1.2 via polarity. More precisely, to deduce this one needs to know that the polar of \(TC\) is given by
\[
(TC)^\circ = T^\circ C^\circ, \quad \text{where} \quad T^\circ := (T^{-1})^T = (T^T)^{-1}. \quad (2.1)
\]
Note that we have for \(S, T \in \text{Gl}_d, Q \in O(d)\),
\[
(T^\circ)^\circ = T, \quad (ST)^\circ = S^\circ T^\circ, \quad Q^\circ = Q.
\]
Replacing \(C_0, \ldots, C_n\) and \(T_0, \ldots, T_n\) in Theorem 1.2 by \(C_0^\circ, \ldots, C_n^\circ\) and \(T_0^\circ, \ldots, T_n^\circ\), respectively, and applying polarity yields Corollary 1.3.

The next important aspect we need to address is that of direct products. The faces of \(C \times D\), where \(C, D \in \mathcal{P}(\mathbb{R}^d)\), are given by the products of the faces of \(C\) and \(D\). In terms of the corresponding linear subspaces, we obtain
\[
L_k( C \times D ) = \bigcup_{i+j=k} \{ L_0 \times L_1 : L_0 \in L_i(C), L_1 \in L_j(D) \}. \quad (2.2)
\]
For the skeletons of a direct product we have the following useful formula:

\[ S_k(C \times D) = \bigcup_{i+j=k} S_i(C) \times S_j(D). \]  

(2.3)

These formulas (2.2) and (2.3) are verified easily.

The final aspect we will address here concerns the largest linear subspace contained in the cone and the dimension of the linear span of the cone. The largest linear subspace contained in \( C \) is given by \( C \cap (-C) \), and its dimension is known as the lineality of the cone,

\[ \text{lin}(C) = \dim(C \cap (-C)). \]  

(2.4)

The dimension of the linear span is connected with the lineality via polarity, as it is easy to verify that

\[ \dim(\text{span}(C)) + \text{lin}(C^\circ) = d. \]

To avoid redundancy we therefore do not introduce a separate notation for the dimension of the linear span of a cone. Note that the lineality of a product is given by \( \text{lin}(C \times D) = \text{lin}(C) + \text{lin}(D) \).

It turns out that the lineality fits perfectly into our theory if we define the \( \ell \)-vector of a cone in the following way: for \( C \in \mathcal{P}(\mathbb{R}^d) \),

\[ \ell(C) = \{ \ell_0(C), \ldots, \ell_d(C) \}, \quad \ell_k(C) = \begin{cases} 1 & \text{if } \text{lin}(C) = k \\ 0 & \text{else}. \end{cases} \]  

(2.5)

Note that if we used the dimension of the linear span instead of the lineality for this vector construction, then the resulting vector would be the reversal of \( \ell(C^\circ) \). Note also that

\[ \ell_0(C) = f_0(C) = u_0(C) \quad (\neq v_0(C) \text{ in general}). \]  

(2.6)

The \( \ell \)-vector (and its localization, to be introduced in Section 2.2 below) will play a role in the characterization of the polyhedral measures, cf. Section 3.2.

We note that the \( \ell \)-vector satisfies the properties (0)–(2) of the \( f \)-vector, i.e., linear invariance: \( \ell(TC) = \ell(C) \) for all \( T \in \text{GL}_d \), subspace-property: \( \ell(L) = f(L) = u(L) = v(L) \) for linear subspaces \( L \subseteq \mathbb{R}^d \), and product rule: \( \ell_k(C \times D) = \sum_{i+j=k} \ell_i(C) \ell_j(D) \). It also satisfies a “strong kinematic formula”, as shown in the following proposition, cp. Theorem 1.2.

**Proposition 2.1.** Let \( C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d) \) and \( T_0, \ldots, T_n \in \text{GL}_d \). Then for almost all \( (Q_0, \ldots, Q_n) \in O(d)^{n+1} \) and for \( k > 0 \),

\[ \ell_k \left( \bigcap_{i=0}^n T_i Q_i C_i \right) = \ell_{nd+k}(C_0 \times \cdots \times C_n), \quad \ell_0 \left( \bigcap_{i=0}^n T_i Q_i C_i \right) = \sum_{j=0}^{nd} \ell_j(C_0 \times \cdots \times C_n). \]  

(2.7)

Note that the second equation in (2.7) in connection with the observation (2.6) settles the boundary case in (1.13).

**Proof of Proposition 2.1.** Let \( L_i := C_i \cap (-C_i) \), denote the largest subspace contained in \( C_i \). Then the largest subspace contained in \( C := \bigcap_{i=0}^n T_i Q_i C_i \) is given by \( C \cap (-C) = \bigcap_{i=0}^n T_i Q_i L_i \).

The dimension of this intersection is almost surely given by \( \max(0, k_0 + \cdots + k_n - nd) \), where \( k_i := \dim(L_i) = \dim(C_i), \ i = 0, \ldots, n \), cf. Lemma B.2. Hence, if \( k > 0 \), then almost surely \( \ell_k \left( \bigcap_{i=0}^n T_i Q_i C_i \right) = 1 \) iff \( nd+k = k_0+\cdots+k_n \) and zero else. The same holds for \( \ell_{nd+k}(C_0 \times \cdots \times C_n) \), as the lineality is additive: \( \text{lin}(C_0 \times \cdots \times C_n) = k_0+\cdots+k_n \). This settles the case \( k > 0 \). One argues analogously in the case \( k = 0 \). \( \square \)

### 2.2. Conic Borel sets and measures

Given a conic Borel set \( M \in \mathcal{B}(\mathbb{R}^d) \), i.e., \( \lambda M = M \) for all \( \lambda > 0 \), almost all information about this set is contained in the intersection \( M \cap S_\mathbb{R}^{d-1} \) with the unit sphere, except for the information if the origin is contained in the set or not. This observation shows that \( \mathcal{B}(\mathbb{R}^d) \) decomposes disjointly into:

\[ \mathcal{B}(\mathbb{R}^d) = \{ M \in \mathcal{B}(\mathbb{R}^d) : 0 \not\in M \} \cup \{ M \in \mathcal{B}(\mathbb{R}^d) : 0 \in M \}, \]  

(2.8)
where both parts are equivalent to the Borel algebra on the unit sphere, \( \{ M \in \mathcal{B}(\mathbb{R}^d) : \mathbf{0} \not\in M \} = \{ M \in \mathcal{B}(\mathbb{R}^d) : \mathbf{0} \in M \} = \mathcal{B}(S^{d-1}) \). The following convention turns out to be particularly convenient: for \( M \in \mathcal{B}(\mathbb{R}^d) \),

\[
M_* := M \setminus \{\mathbf{0}\}.
\]

We denote the embedding of the set of spherical measures into the set of conic measures by

\[
\nu \in \mathcal{M}(S^{d-1}) \mapsto \hat{\nu} \in \hat{\mathcal{M}}(\mathbb{R}^d), \quad \hat{\nu}(M) := \nu(M \cap S^{d-1}) \text{ for } M \in \mathcal{B}(\mathbb{R}^d).
\]

Given a conic measure \( \mu \in \hat{\mathcal{M}}(\mathbb{R}^d) \), the decomposition (2.8) implies that

\[
\mu - \mu(\{\mathbf{0}\}) \Delta = \hat{\nu}
\]

for some spherical measures \( \nu \in \mathcal{M}(S^{d-1}) \). This shows that every conic measure can be written uniquely as the sum of a (lifted) spherical measure and a scaled Dirac measure.

**Lemma 2.2.** Let \( \mu \in \hat{\mathcal{M}}(\mathbb{R}^d) \) be an orthogonal invariant conic measure, i.e., \( \mu(QM) = \mu(M) \) for all \( M \in \mathcal{B}(\mathbb{R}^d) \), \( Q \in O(d) \). Then

\[
\mu = \mu(\{\mathbf{0}\}) \Delta + \mu(\mathbb{R}^d_+) \gamma_d.
\]

**Proof.** If \( \mu \in \hat{\mathcal{M}}(\mathbb{R}^d) \) then it can be written as \( \mu = \mu(\{\mathbf{0}\}) \Delta + \hat{\nu} \) for some spherical measure \( \nu \in \mathcal{M}(S^{d-1}) \). The spherical measure \( \nu \) is orthogonal invariant, since

\[
\nu(QM) = \nu(M) = \nu(M) = \nu(M)
\]

for any spherical Borel set \( M \in \mathcal{B}(S^{d-1}) \) and the corresponding conic borel set \( M = \{ \lambda \mathbf{x} : \lambda > 0, \mathbf{x} \in \hat{M} \} \). The Lebesgue measure is up to scaling the only orthogonal invariant Borel measure on the sphere, see for example [SW08, Ch. 13]. This implies \( \hat{\nu} = \mu(\mathbb{R}^d_+) \gamma_d \). \( \square \)

### 2.3. Biconic Borel sets and measures.

Recall that the biconic (Borel) \( \sigma \)-algebra on \( \mathbb{R}^{d+d} = \mathbb{R}^d \times \mathbb{R}^d \) is defined by

\[
\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) = \{ \mathcal{M} \subseteq \mathbb{R}^{d+d} \text{ Borel set} : (\lambda, \lambda') \mathcal{M} = \mathcal{M} \text{ for all } \lambda, \lambda' > 0 \}
\]

where \( (\lambda, \lambda') \mathcal{M} = \{ (\lambda \mathbf{x}, \lambda' \mathbf{x}') : (\mathbf{x}, \mathbf{x}') \in \mathcal{M} \} \). We denote the corresponding set of biconic measures by \( \hat{\mathcal{M}}(\mathbb{R}^d, \mathbb{R}^d) \). The (biconic) Dirac measure on \( \mathbb{R}^{d+d} \) supported in \( \{\mathbf{0}\} \) is denoted by \( \Delta \).

Sets of the form \( M \times M' \) with \( M, M' \in \mathcal{B}(\mathbb{R}^d) \) belong to \( \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) \), but not all elements in \( \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) \) are of this form. However, direct products generate the biconic \( \sigma \)-algebra, as we will show in Proposition 2.5 below that the biconic \( \sigma \)-algebra is the product of the conic \( \sigma \)-algebras,

\[
\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d).
\]

We will show this elementary result at the end of this section in Proposition 2.5.

The set of polyhedral cones can be embedded into the biconic \( \sigma \)-algebra by the map

\[
\text{BL}: \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d), \quad \text{BL}(C) := \{(x, z) \in C \times C^\circ : \langle x, z \rangle = 0\}.
\]

We call \( \text{BL}(C) \) the biconic lift of \( C \).\(^6\) Note that for a linear subspace \( L \subseteq \mathbb{R}^d \) we obtain \( \text{BL}(L) = L \times L^\perp \), but in general \( \text{BL}(C) \) is not a direct product. Combining the facial decomposition (1.7) with the polarity relation (1.6) yields the following two disjoint decompositions of the biconic lift

\[
\text{BL}(C) = \bigcup_{L \in \mathcal{L}(C)} (\text{int}_L(C \cap L) \times (C^\circ \cap L^\perp)) = \bigcup_{L \in \mathcal{L}(C)} ((C \cap L) \times \text{int}_{L^\perp}(C^\circ \cap L^\perp)).
\]

These decompositions will be important for the proof of the Characterization Theorem 4.3 in Section 4.2.

Besides lifting the whole cone into the biconic \( \sigma \)-algebra, it is also convenient to lift the \( k \)-skeleton via

\[
\mathcal{S}_k(C) := \bigcup_{L \in \mathcal{L}_k(C)} (\text{int}_L(C \cap L) \times \text{int}_{L^\perp}(C^\circ \cap L^\perp)), \quad \text{BL}^*(C) := \bigcup_{k=0}^d \mathcal{S}_k(C).
\]

\(^6\) A related notion in algebraic geometry is the conormal variety, cf. [RS10].
Except in the case where $C$ is a linear subspace, $\mathbf{B}^+(C)$ is a proper subset of $\mathbf{B}(C)$. But the following simple proposition shows that $\mathbf{B}(C)$ makes up the “essential part” of $\mathbf{B}(C)$.

**Proposition 2.3.** Let $C \in \mathcal{D}(\mathbb{R}^d)$ and let its projection map be denoted by $\Pi_C: \mathbb{R}^d \to C$, $\Pi_C(x) = \text{arg min}\{\|x - y\| : y \in C\}$. Then for every $x \in \mathbb{R}^d$,

$$
\hat{\Pi}_C(x) := (\Pi_C(x), \Pi_C^-(x)) \in \mathbf{B}(C).
$$

(2.14)

Moreover, if $g \in \mathbb{R}^d$ denotes a Gaussian random vector, then almost surely

$$
\hat{\Pi}_C(g) \in \mathbf{B}^+(C).
$$

**Proof.** The first claim is a well-known result by J.-J. Moreau, cf. for example [HUL01, Sec. 3.2]: the projection onto the primal cone $\Pi_C(x) = y$ and the projection onto the polar cone $\Pi_C^-(x) = y'$ satisfy $\langle y, y' \rangle = 0$ (and $y + y' = x$).

As for the second claim, it is easily seen that if $(\Pi_C(x), \Pi_C^-(x))$ does not lie in the union of the $\mathcal{S}_k(C)$, $k = 0, \ldots, d$, then $x$ lies in a subspace of the form $L + L'$ with $L \in \mathcal{L}_m(C)$, $L' \in \mathcal{L}_n(C^*)$, and $m + n < d$. The union of all these (finitely many) subspaces, has Gaussian measure zero, which shows the second claim. 

Proposition 2.3 implies that, using the notation from (2.14), we can write the support measures in the following way

$$\Theta_k(C, \mathcal{M}) = \mathbb{P}\{\hat{\Pi}_C(g) \in \mathcal{S}_k(C) \cap \mathcal{M}\},$$

(2.15)

cp. the analogous characterizations of the intrinsic volumes (1.10) and the curvature measures (1.23). In fact, Moreau’s decomposition theorem states that

$$\text{add} \circ \hat{\Pi}_C = I_d, \quad \text{add} : \mathbb{R}^{d+d} \to \mathbb{R}^d, \quad \text{add}(x, x') := x + x'.$$

(2.16)

Using this notation, we can write the support measures in the following form in which the projection map is completely eliminated,

$$\Theta_k(C, \mathcal{M}) = \gamma_d\left(\text{add}\{\mathcal{S}_k(C) \cap \mathcal{M}\}\right).$$

(2.17)

The biconic structure naturally admits an involution, which we call the reversal map,

$$\text{rev} : \hat{\mathcal{S}}(\mathbb{R}^d, \mathbb{R}^d) \to \hat{\mathcal{S}}(\mathbb{R}^d, \mathbb{R}^d), \quad \text{rev}(\mathcal{M}) := \{\langle x, x' \rangle : (x, x') \in \mathcal{M}\}. $$

(2.18)

The reversal of the biconic lift is the biconic lift of the polar, $\text{rev}(\mathbf{B}(C)) = \mathbf{B}(C^*)$, and for the lifted $k$-skeletons we obtain $\text{rev}(\mathcal{S}_k(C)) = \mathcal{S}_{d-k}(C^*)$. Composing the reversal map with the biconic projection (2.14) yields $\text{rev} \circ \hat{\Pi}_C = \hat{\Pi}_{C^*}$.

Another natural definition is the following action of the general linear group. Recall that $(TC)^* = T^*C^*$. We define the action of the general linear group on the biconic $\sigma$-algebra via

$$T \cdot \mathcal{M} := \{(Tx, T^*x') : (x, x') \in \mathcal{M}\}. $$

(2.19)

This action has the following relations to the other structures we have introduced so far,

$$\text{BL}(TC) = T \text{BL}(C), \quad \mathcal{S}_k(TC) = T \mathcal{S}_k(C), \quad \text{rev}(T \cdot \mathcal{M}) = T^0 \text{rev}(\mathcal{M}), \quad T \hat{\Pi}_C = \hat{\Pi}_{TC}.$$

When forming the product of biconic sets of the form $M \times M'$ and $N \times N'$ it makes sense to take the product of the first components as the first component and the products of the second components as the second component, i.e., take $M \times N \times M' \times N'$. We call the corresponding construction for general biconic sets the biconic product: for $\mathcal{M} \in \hat{\mathcal{S}}(\mathbb{R}^d, \mathbb{R}^d)$, $\mathcal{N} \in \hat{\mathcal{S}}(\mathbb{R}^e, \mathbb{R}^e)$,

$$\mathcal{M} \hat{\times} \mathcal{N} := \{(x, y, x', y') : (x, x') \in \mathcal{M}, (y, y') \in \mathcal{N}\} \in \hat{\mathcal{S}}(\mathbb{R}^{d+e}, \mathbb{R}^{d+e}).$$

(2.20)

Note that we indeed have $(M \times M') \hat{\times} (N \times N') = M \times N \times M' \times N'$, and

$$\text{BL}(C \times D) = \text{BL}(C) \hat{\times} \text{BL}(D), \quad \mathcal{S}_k(C \times D) = \bigcup_{i+j=k} \mathcal{S}_i(C) \hat{\times} \mathcal{S}_j(D), \quad \hat{\Pi}_{C \times D} = \hat{\Pi}_C \hat{\times} \hat{\Pi}_D,$$

$$\text{rev}(\mathcal{M} \hat{\times} \mathcal{N}) = \text{rev}(\mathcal{M}) \hat{\times} \text{rev}(\mathcal{N}), \quad T(\mathcal{M} \hat{\times} \mathcal{N}) = T(\mathcal{M}) \hat{\times} T(\mathcal{N}).$$

(2.21)
The final structure, which we introduce on $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$, is the biconic version of intersection, and its associated reverse operation; we use the neutral terms of conjunction $\land$ and disjunction $\lor$: for $\mathcal{M}, \mathcal{N} \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$,
\begin{align}
\mathcal{M} \land \mathcal{N} & := \{(x, x') + y') \in \mathcal{M} \cup \mathcal{M}_1, (x, y') \in \mathcal{N}\}, \\
\mathcal{M} \lor \mathcal{N} & := \{(x + y, x') \in \mathcal{M}, (y, x') \in \mathcal{N}\}.
\end{align}
(2.22) (2.23)

For direct products $\mathcal{M} = M \times M'$, $\mathcal{N} = N \times N'$ we obtain
\begin{align}
\mathcal{M} \land \mathcal{N} & := (M \cap N) \times (M' + N'), \\
\mathcal{M} \lor \mathcal{N} & := (M + N) \times (M' \cap N').
\end{align}
(2.24)

The conjunction and disjunction naturally extend the lattice structure from $\mathcal{B}(\mathbb{R}^d)$ to the biconic $\sigma$-algebra, since
\begin{align}
\text{BL}(C) \land \text{BL}(D) & = \text{BL}(C \cap D), \\
\text{BL}(C) \lor \text{BL}(D) & = \text{BL}(C + D).
\end{align}

However, it should be noted that although the biconic $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ is now endowed with a similar number of operations, (rev, $\land$, $\lor$), as the set of polyhedral cones $\mathcal{P}(\mathbb{R}^d)$, $(\cup, \cap, +)$, the structure of $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ is significantly weaker than the structure of $\mathcal{P}(\mathbb{R}^d)$. In fact, for $d \geq 2$ the biconic $\sigma$-algebra even fails to be a lattice, as, for example, the idempotency axiom $\mathcal{M} \land \mathcal{M} = \mathcal{M}$ is in general not satisfied. The following proposition lists some further important properties of this structure, like De Morgan’s Law.

**Proposition 2.4.** Let $\mathcal{M}, \mathcal{N}, \mathcal{M}_0, \mathcal{M}_1 \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$. Then
\begin{align}
\text{rev}(\mathcal{M} \land \mathcal{N}) & = \text{rev}(\mathcal{M}) \lor \text{rev}(\mathcal{N}), \\
(\mathcal{M}_0 \cup \mathcal{M}_1) \land \mathcal{N} & = (\mathcal{M}_0 \land \mathcal{N}) \cup (\mathcal{M}_1 \land \mathcal{N}), \\
(\mathcal{M}_0 \cap \mathcal{M}_1) \land \mathcal{N} & \subseteq (\mathcal{M}_0 \land \mathcal{N}) \cap (\mathcal{M}_1 \land \mathcal{N}).
\end{align}
(2.24)

If $d \geq 2$, then in general, $(\mathcal{M}_0 \cap \mathcal{M}_1) \land \mathcal{M} \neq (\mathcal{M}_0 \land \mathcal{N}) \cap (\mathcal{M}_1 \land \mathcal{N})$, and $\mathcal{M}_0 \cap \mathcal{M}_1 = \emptyset$ does not imply $(\mathcal{M}_0 \land \mathcal{N}) \cap (\mathcal{M}_1 \land \mathcal{N}) = \emptyset$.

**Proof.** De Morgan’s Law follows directly from the definitions of $\land$, $\lor$, rev. Furthermore, we have
\begin{align}
(\mathcal{M}_0 \cup \mathcal{M}_1) \land \mathcal{N} & = \{(x, x') + y') : (x, x') \in \mathcal{M}_0 \cup \mathcal{M}_1, (x, y') \in \mathcal{N}\} \\
& = \{(x, x') : (x, x') \in \mathcal{M}_0, (x, y') \in \mathcal{N}\} \cup \{(x, x') : (x, x') \in \mathcal{M}_1, (x, y') \in \mathcal{N}\} \\
& = (\mathcal{M}_0 \land \mathcal{N}) \cup (\mathcal{M}_1 \land \mathcal{N}),
\end{align}
(2.25)

\begin{align}
(\mathcal{M}_0 \cap \mathcal{M}_1) \land \mathcal{N} & = \{(x, x') + y') : (x, x') \in \mathcal{M}_0 \cap \mathcal{M}_1, (x, y') \in \mathcal{N}\} \\
& \subseteq \{(x, x') : (x, x') \in \mathcal{M}_0, (x, y') \in \mathcal{N}\} \cap \{(x, x') : (x, x') \in \mathcal{M}_1, (x, y') \in \mathcal{N}\} \\
& = (\mathcal{M}_0 \land \mathcal{N}) \cap (\mathcal{M}_1 \land \mathcal{N}).
\end{align}

If $\mathcal{M}_0 = M_0 \times M_0', \mathcal{M}_1 = M_1 \times M_1'$, $\mathcal{N} = N \times N'$, then
\begin{align}
(\mathcal{M}_0 \cap \mathcal{M}_1) \land \mathcal{N} & = (M_0 \cap M_1) \cap N \times ((M_0' \cap M_1') \cap N') = (\mathcal{M}_0 \land \mathcal{N}) \cap (\mathcal{M}_1 \land \mathcal{N}),
\end{align}

where the inequality (+) holds for example in the case where $M_0 = N_1 = N \neq \emptyset$ and $M_0', M_1', N'$ are pairwise linear independent lines lying in a 2-dimensional linear space $L$ in which case $(M_0 \cap M_1') + N' = N' \neq L = (M_0' \cap M_1') + M_1$. Replacing in the above example $M_0', M_1', N'$ by $M_0' \setminus \{0\}, M_1' \setminus \{0\}, N' \setminus \{0\}$ yields $\mathcal{M}_0 \cap \mathcal{M}_1 = \emptyset$ but $(\mathcal{M}_0 \land \mathcal{N}) \cap (\mathcal{M}_1 \land \mathcal{N}) = M \times (L \setminus \{0\}) \neq \emptyset$. 

We finish this section with the announced result about the product structure of the biconic $\sigma$-algebra.

**Proposition 2.5.** The biconic $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ is the product algebra
\begin{align}
\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d).
\end{align}
(2.25)

From this proposition we can deduce the following useful lemma about biconic measures.
Lemma 2.6. If $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ are such that $\mu_1(M \times M') = \mu_2(M \times M')$ for all $M, M' \in \mathcal{B}(\mathbb{R}^d)$, then $\mu_1 = \mu_2$.

Proof. The class $\{M \times M' : M, M' \in \mathcal{B}(\mathbb{R}^d)\}$ is closed under finite intersections. If two measures $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^d)$ coincide on such a class, then they also coincide on the $\sigma$-algebra generated by it, cf. [Bog07, Lem. 1.9.4], which is $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ by Proposition 2.5. □

Proof of Proposition 2.5. The inclusion $\subseteq$ in (2.25) follows from the fact that $\mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ is a $\sigma$-algebra, that contains all sets of the form $M \times M'$ with $M, M' \in \mathcal{B}(\mathbb{R}^d)$. For the other inclusion let $\mathcal{M} \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ and decompose it into

$$\mathcal{M} = \mathcal{M}_s \cup \mathcal{M}_1 \cup \mathcal{M}_2,$$

$$\mathcal{M}_s := \mathcal{M} \cap \{R^d \times R^d\},$$

$$\mathcal{M}_1 := \mathcal{M} \cap \{R^d \times \{0\}\},$$

$$\mathcal{M}_2 := \mathcal{M} \cap \{\{0\} \times R^d\}.$$

Clearly, $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$, as both are of the form $M \times M'$ with $M, M' \in \mathcal{B}(\mathbb{R}^d)$. As for the remaining set $\mathcal{M}_s$, we make the following definition

$$\mathcal{B}^s(\mathbb{R}^d) := \{M \in \mathcal{B}(\mathbb{R}^d) : \mathcal{M} \subseteq \{0\}\},$$

$$\mathcal{B}^s(\mathbb{R}^d, \mathbb{R}^d) := \{\mathcal{M} \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) : \mathcal{M} \subseteq \{0\} \times \{0\}\}.$$

Using the functions $R^d \to S^{d-1}, x \mapsto \frac{x}{\|x\|}$, and $R^d \times R^d \to S^{d-1} \times S^{d-1}, (x, x') \mapsto \left(\frac{x}{\|x\|}, \frac{x'}{\|x\|}\right)$, we see that

$$\mathcal{B}^s(\mathbb{R}^d) = \mathcal{B}(S^{d-1}), \quad \mathcal{B}^s(\mathbb{R}^d, \mathbb{R}^d) = \mathcal{B}(S^{d-1} \times S^{d-1}).$$

We have $\mathcal{B}(S^{d-1} \times S^{d-1}) = \mathcal{B}(S^{d-1}) \otimes \mathcal{B}(S^{d-1})$, cf. [Bog07, Lem. 6.4.2], so that

$$\mathcal{B}^s(\mathbb{R}^d, \mathbb{R}^d) = \mathcal{B}(S^{d-1} \times S^{d-1}) = \mathcal{B}(S^{d-1}) \otimes \mathcal{B}(S^{d-1}) = \mathcal{B}^s(\mathbb{R}^d) \otimes \mathcal{B}^s(\mathbb{R}^d) \subseteq \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d).$$

This shows that $\mathcal{M}_s \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$, which finishes the proof. □

3. Conic localization

In this section we first discuss the properties of the polyhedral measures and the curvature measures, then we prove that the polyhedral measures are characterized by some of these properties, and then we use this characterization to prove a kinematic formula, which generalizes the kinematic formula for the $u$-vector, (1.13).

3.1. Polyhedral and curvature measures. Before considering the localizations of the $u$- and $v$-vectors, we make the Dirac measure $\Delta$ supported at the origin cone dependent by defining for a function $h : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$

$$\Delta_h : \mathcal{P}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}, \quad \Delta_h(C, M) := h(C) \Delta(M).$$

An important special case is obtained by choosing for $h$ the indicator function for the linearity of $C$, which gives rise to the following definition.

Definition 3.1. For $0 \leq k \leq d$ we define the $k$th lineality measure $L_k(C, M) \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$,

$$L_k(C, M) := \Delta_{h_k}(C, M) \quad \text{with} \quad h_k(C) := \begin{cases} 1 & \text{if } \text{lin}(C) = k \\ 0 & \text{if } \text{lin}(C) \neq k. \end{cases}$$

The lineality measures localize the $\ell$-vector, as $L_k(C, \mathbb{R}^d) = \ell_k(C)$. In fact, we even have $L_k(C, M) = \ell_k(C)$ for all $C \in \mathcal{B}(\mathbb{R}^d)$ such that $0 \in M$ (and $L_k(C, M) = 0$ if $0 \notin M$). Due to this close connection, the lineality measures also inherit a number of easily seen properties, which we spare ourselves from enumerating here. Finally, note that the lineality measures decompose the Dirac measure,

$$\Delta = L_0(C, \cdot) + \cdots + L_d(C, \cdot).$$

The $u$- and $v$-vector are localized by the polyhedral measures and the curvature measures, respectively, and these are given by

$$\Psi_k(C, M) = \sum_{L \in L_k(C)} \gamma_L(C \cap L \cap M), \quad \Phi_k(C, M) = \sum_{L \in L_k(C)} \gamma_L(C \cap L \cap M) \gamma_{L^\perp}(C^\perp \cap L^\perp),$$

where $C \in \mathcal{P}(\mathbb{R}^d)$ and $0 \leq k \leq d$. 
Remark 3.2. For \( k = d \) the polyhedral measure coincides with the curvature measure, \( \Psi_d(C, M) = \Phi_d(C, M) = \gamma_d(C \cap M) \). Furthermore, if \( C \in \mathcal{P}(\mathbb{R}^d) \) is not a linear subspace, then \( \Phi_{d-1}(C, \cdot) = \frac{1}{2} \Psi_{d-1}(C, \cdot) \). At the other extreme, \( 2 \Psi_1(C, M) \) counts the number of extreme rays of \( C \) lying in \( M \); the \( 0 \)th polyhedral measure coincides with the \( 0 \)th lineality measure, \( \Psi_0 = \text{Lin}_M \), and the \( 0 \)th curvature measure is given by the Dirac measure scaled by the \( 0 \)th intrinsic volume, \( \Phi_0 = \Delta_{\nu_0} \), cf. (3.1).

These measures satisfy some fundamental properties, which, as we will show in Section 3.2, actually characterize these measures. We list these properties in the following proposition.

**Proposition 3.3.** Let \( C \in \mathcal{P}(\mathbb{R}^d) \) and \( 0 \leq k \leq d \).

1. (concentration): for all \( M \in \hat{\mathcal{B}}(\mathbb{R}^d) \),
   
   \[ \Psi_k(C, M) = \Psi_k(C, M \cap S_k(C)), \quad \Phi_k(C, M) = \Phi_k(C, M \cap S_k(C)). \]

2. (orthogonal invariance): for all \( Q \in O(d) \) and all \( M \in \hat{\mathcal{B}}(\mathbb{R}^d) \),
   
   \[ \Psi_k(QC, QM) = \Psi_k(C, M), \quad \Phi_k(QC, QM) = \Phi_k(C, M). \]

3. (locality): for all \( D \in \mathcal{P}(\mathbb{R}^d) \), \( M \in \hat{\mathcal{B}}(\mathbb{R}^d) \), and all open sets \( B \in \hat{\mathcal{B}}(\mathbb{R}^d) \),
   
   \[ \Psi_k(C, M) = \Psi_k(D, M), \quad \Phi_k(C, B) = \Phi_k(D, B), \quad \text{if } S_k(C) \cap M = S_k(D) \cap M, \quad \text{if } C \cap B = D \cap B. \]

4. (product rule): for all \( D \in \mathcal{P}(\mathbb{R}^d) \), \( M \in \hat{\mathcal{B}}(\mathbb{R}^d) \), \( N \in \hat{\mathcal{B}}(\mathbb{R}^e) \),
   
   \[ \Psi_k(C \times D, M \times N) = \sum_{i+j=k} \Psi_i(C, M) \Psi_j(D, N), \quad \Phi_k(C \times D, M \times N) = \sum_{i+j=k} \Phi_i(C, M) \Phi_j(D, N). \]

5. (additivity): for all \( D \in \mathcal{P}(\mathbb{R}^d) \) such that \( C + D = C \cup D \) and all \( M \in \hat{\mathcal{B}}(\mathbb{R}^d) \),
   
   \[ \Phi_k(C + D, M) = \Phi_k(C, M) + \Phi_k(D, M). \]

6. (weak continuity): for all open sets \( B \in \hat{\mathcal{B}}(\mathbb{R}^d) \),
   
   \[ \liminf_i \Phi_k(C_i, B) \geq \Phi_k(C, B), \quad \text{if } \{C_i : i \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{R}^d) \text{ such that } \lim_{i \to \infty} C_i = C. \]

Remark 3.4. If \( C \cap B = D \cap B \) for some open conic set \( B \in \hat{\mathcal{B}}(\mathbb{R}^d) \), then \( S_k(C) \cap B = S_k(D) \cap B \) for all \( k = 0, \ldots, d \). So the locality property of the polyhedral measures is stronger than the locality property of the curvature measures. On the other hand, except in the case \( k \in \{0, d\} \), the polyhedral measures do not satisfy the additivity property, which is satisfied by the curvature measures, and the polyhedral measures are also not weakly continuous except in the cases \( k \in \{0, d-1\} \).

**Proof.** Properties (1) and (2) follow directly from the definitions of \( \Psi_k \) and \( \Phi_k \). As for the locality, assume first that \( S_k(C) \cap M = S_k(D) \cap M \) for \( k = 0, \ldots, d \). In this case we obtain for every \( L \in \mathcal{L}_k(C) \): if \( \text{int}_L(L \cap S_k(C) \cap M) \neq \emptyset \), then \( L \in \mathcal{L}_k(D) \); and the same holds with course of \( C \) and \( D \) exchanged. Therefore,

\[ \Psi_k(C, M) = \sum_{L \in \mathcal{L}_k(C)} \gamma_L(L \cap S_k(C) \cap M) = \sum_{L \in \mathcal{L}_k(D)} \gamma_L(L \cap S_k(D) \cap M) = \Psi_k(D, M). \]

If \( C \cap B = D \cap B \) for some open conic set \( B \in \hat{\mathcal{B}}(\mathbb{R}^d) \), then \( S_k(C) \cap B = S_k(D) \cap B \) for all \( k = 0, \ldots, d \). Furthermore, if \( \Pi_C(x) = S_k(C) \cap B = S_k(D) \cap B \), then \( \Pi_D(x) = S_k(D) \cap B \); and the same holds with \( C \) and \( D \) exchanged. Therefore, using (1.23),

\[ \Phi_k(C, B) = \mathbb{P}\{\Pi_C(g) \in S_k(C) \cap B\} = \mathbb{P}\{\Pi_D(g) \in S_k(D) \cap B\} = \Phi_k(D, B). \]

The product rules follow directly from (2.2) and (2.3). The additivity and weak continuity of the curvature measures are special cases of the corresponding properties of the support measures, cf. Proposition 4.1. \( \square \)
3.2. **Characterizations.** The following two theorems show that the properties of the polyhedral measures and the curvature measures listed in Proposition 3.3 (basically) characterize these measures. In this section we will only prove the characterization theorem for the polyhedral measures, which seems to be new, and which may be regarded as the conic version of the characterization theorem for the support measures, which we will provide in Section 4.2. The characterization of the curvature measures is due to Schneider [Sch78] and its proof is considerably more involved. For completeness we will provide it in Appendix C.

**Theorem 3.5** (Characterization of polyhedral measures). Let \( \psi : P(\mathbb{R}^d) \times \hat{B}(\mathbb{R}^d) \to \mathbb{R} \) be such that for all \( C, D \in P(\mathbb{R}^d) \), \( M \in \hat{B}(\mathbb{R}^d), Q \in O(d) \):

1. \( \psi(C, \cdot) \in L^1(\mathbb{R}^d) \),
2. \( \psi(C, M) = \psi(C, M \cap C) \),
3. \( \psi(QC, QM) = \psi(C, M) \),
4. \( \psi(C, M) = \psi(D, M) \) if \( S_k(C) \cap M = S_k(D) \cap M \) for \( k = 0, \ldots, d \).

Then \( \psi \) is a linear combination of \( \gamma_{0,1}, \ldots, \gamma_{0,1}, \gamma_{0,1} \). In this case

\[
\psi = \sum_{k=0}^{d} \psi(L^{(k)}(\cdot, 0)) \gamma_{0,1} + \sum_{k=1}^{d} \psi(L^{(k)}(\cdot, \mathbb{R}^d)) \gamma_{1,1}, \tag{3.3}
\]

where \( L^{(k)}(\cdot, \mathbb{R}^d) \) denotes a \( k \)-dimensional subspace.

**Theorem 3.6** (Characterization of curvature measures). Let \( \psi : P(\mathbb{R}^d) \times \hat{B}(\mathbb{R}^d) \to \mathbb{R}_+ \) be such that for all \( C, D \in P(\mathbb{R}^d) \), \( M \in \hat{B}(\mathbb{R}^d), Q \in O(d) \), \( B \in \hat{B}(\mathbb{R}^d) \) open:

1. \( \psi(C, \cdot) \in L^1(\mathbb{R}^d) \),
2. \( \psi(C, M) = \psi(C, M \cap C) \),
3. \( \psi(QC, QM) = \psi(C, M) \),
4. \( \psi(C, B) = \psi(D, B) \) if \( C \cap B = D \cap B \),
5. \( \psi(C + D, M) + \psi(C \cap D, M) = \psi(C, M) + \psi(D, M) \), if \( C + D = C \cup D \).

Then \( \psi \) is a nonnegative linear combination of \( \Phi_{1,1}, \ldots, \Phi_{1,1} \) and \( \Delta_h \), where \( h : P(\mathbb{R}^d) \to \mathbb{R}_+ \) is given by \( h(C) = \psi(C, \{0\}) \). In this case

\[
\psi = \Delta_h + \sum_{k=1}^{d} \psi(L^{(k)}(\cdot, \mathbb{R}^d)) \Phi_{1,1}, \tag{3.4}
\]

where \( L^{(k)}(\cdot, \mathbb{R}^d) \) denotes a \( k \)-dimensional subspace.

**Proof of Theorem 3.5.** Let \( \psi : P(\mathbb{R}^d) \times \hat{B}(\mathbb{R}^d) \to \mathbb{R} \) satisfy the assumptions (0)–(3), i.e., \( \psi(C, \cdot) \) is a conic measure, which is concentrated on \( C \), \( \psi \) is invariant under (simultaneous) orthogonal transformations, and \( \psi(C, M) = \psi(D, M) \), if the skeletons of the cones \( C, D \) coincide in \( M \).

If \( L \subseteq \mathbb{R}^d \) is a linear subspace, then \( \psi(L, \cdot) \) is an orthogonal invariant conic measure on \( L \), so that by Lemma 2.2 it is a linear combination of the Dirac measure and the Gaussian measure \( \gamma_L \). Moreover, orthogonal invariance implies that, for \( k = \dim L > 0 \),

\[
\psi(L, \cdot) = \psi(L^{(k)}(\cdot, 0)) \Delta + \psi(L^{(k)}(\cdot, \mathbb{R}^d)) \gamma_L, \tag{3.5}
\]
where $L^{(k)} \subseteq \mathbb{R}^d$ some $k$-dimensional linear subspace. For $C \in \mathcal{P}(\mathbb{R}^d)$ we will use the facial decomposition (1.7). For $M \in \mathcal{B}(\mathbb{R}^d)$, we obtain

$$\psi(C, M) \overset{(1)}{=} \psi(C, M \cap C) \overset{(1.7)}{=} \sum_{k=0}^{d} \sum_{L \in \mathcal{Z}_k(C)} \psi(L, M \cap \text{int}_L(C \cap L))$$

$$\overset{(3)}{=} \sum_{k=0}^{d} \sum_{L \in \mathcal{Z}_k(C)} \psi(L, M \cap \text{int}_L(C \cap L))$$

$$\overset{(3.5)}{=} \sum_{k=0}^{d} \psi(L^{(k)}, \mathbb{0}) \sum_{L \in \mathcal{Z}_k(C)} \Delta(M \cap \text{int}_L(C \cap L)) + \sum_{k=1}^{d} \psi(L^{(k)}, \mathbb{R}^d) \sum_{L \in \mathcal{Z}_k(C)} \gamma_L(M \cap C \cap L)$$

$$\overset{(3.2)}{=} \sum_{k=0}^{d} \psi(L^{(k)}, \mathbb{0}) \text{Lin}_k(C, M) + \sum_{k=1}^{d} \psi(L^{(k)}, \mathbb{R}^d) \psi_k(C, M).$$

\[\square\]

### 3.3. Kinematic formulas for polyhedral measures

The following theorem provides a kinematic formula for the polyhedral measures, which specializes to the kinematic formula (1.13) for the $u$-vector in the case $M_0 = \cdots = M_n = \mathbb{R}^d$. So as a corollary we obtain the first half of Theorem 1.2.

**Theorem 3.7** (Kinematic formula for polyhedral measures). Let $C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d)$, $M_0, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d)$, $T_0, \ldots, T_n \in \text{Gl}_d$. Then for $Q_0, \ldots, Q_n \in O(d)$ iid uniformly at random and $k > 0$,

$$\mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} T_i Q_i C_i, \bigcap_{i=0}^{n} T_i Q_i M_i \right) \right] = \psi_{n d + k}(C, M),$$

where $C := C_0 \times \cdots \times C_n$, $M := M_0 \times \cdots \times M_n$, and where the polyhedral measures of the product are given by

$$\psi_m(C, M) = \sum_{i_0 + \cdots + i_m = m} \psi_{i_0}(C_0, M_0) \cdots \psi_{i_m}(C_n, M_n).$$

The product formula (3.7) is of course just an iteration of (4) in Proposition 3.3. For the proof of Theorem 3.7 we need a few of lemmas about properties of generic intersections of cones. In fact, at first glance it might not even be clear that the expectation in (3.6) exists (see (1) in Proposition B.5). As some of these lemmas are of technical nature while their statements are geometrically obvious, we defer their proofs to Appendix B.

**Lemma 3.8.** Let $C_1, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d)$, $M_1, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d)$, $T_0, T_n \in \text{Gl}_d$, and $k > 0$. Then for $Q_0, \ldots, Q_n \in O(d)$ iid uniformly at random, the map $\psi: \mathcal{P}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\psi(C_0, M_0) = \mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} T_i Q_i C_i, \bigcap_{i=0}^{n} T_i Q_i M_i \right) \right]$$

satisfies the assumptions (0)–(3) in Theorem 3.5.

**Proof.** To simplify the notation we abbreviate $T_i Q_i =: U_i$.

(0) **Claim:** $\psi(C_0, \cdot) \in \mathbb{K}(\mathbb{R}^d)$. For pairwise disjoint $N_j \in \mathcal{B}(\mathbb{R}^d)$, $j = 1, 2, \ldots$, we have that $U_0 N_j$, $j = 1, 2, \ldots$, and $U_0 N_j \cap \bigcap_{i=1}^{n} U_i M_j$, $j = 1, 2, \ldots$, are pairwise disjoint as well. Therefore,

$$\psi \left( C_0, \bigcup_{j=1}^{\infty} N_j \right) = \mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} U_i C_i, \bigcup_{j=1}^{\infty} U_0 N_j \right) \cap \bigcap_{i=1}^{n} U_i M_j \right] = \mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} U_i C_i, \bigcup_{j=1}^{\infty} \left( U_0 N_j \cap \bigcap_{i=1}^{n} U_i M_j \right) \right) \right]$$

$$= \mathbb{E} \left[ \sum_{j=1}^{\infty} \psi_k \left( \bigcap_{i=0}^{n} U_i C_i, U_0 N_j \cap \bigcap_{i=1}^{n} U_i M_j \right) \right] = \sum_{j=1}^{\infty} \mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} U_i C_i, U_0 N_j \cap \bigcap_{i=1}^{n} U_i M_j \right) \right] = \sum_{j=1}^{\infty} \psi(C_0, N_j),$$

where (\ast) follows from an application of the monotone convergence theorem.

(1) **Claim:** $\psi(C_0, M_0) = \psi(C_0, M_0 \cap C_0)$. This follows directly from the locality of $\psi_k$,

$$\psi(C_0, M_0 \cap C_0) = \mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} U_i C_i, U_0 M_0 \cap \bigcap_{i=1}^{n} U_i M_j \right) \right] = \mathbb{E} \left[ \psi_k \left( \bigcap_{i=0}^{n} U_i C_i, \bigcap_{i=1}^{n} U_i M_j \right) \right] = \psi(C_0, M_0).$$

(2) **Claim:** $\psi(Q C_0, Q M_0) = \psi(C_0, M_0)$ for $Q \in O(d)$. This follows from the observation that $Q_0 Q$ is uniformly at random in $O(d)$ and independent of $Q_1, \ldots, Q_n$. 


where \( L \), and therefore, using the product rule one more time, we obtain

\[
\max\{\dim \text{ subspace of dimension } m \}
\]

so that we need to show \( L \) is a linear subspace of dimension \( \dim \).

The case \( m \geq 0 \) is easily established: the intersection \( \bigcap_{i=0}^{n} U_i L^{(i)} \) is almost surely a linear subspace of dimension \( \max(\dim L^{(0)} + \cdots + \dim L^{(n)} - nd, 0) \), and the direct product \( L^{(0)} \times \cdots \times L^{(n)} \) is a linear subspace of dimension \( \dim L^{(0)} + \cdots + \dim L^{(n)} \), so that

\[
\text{LEFT}(L^{(0)}, \ldots, L^{(n)}; R^d) = \text{RIGHT}(L^{(0)}, \ldots, L^{(n)}; R^d)
\]

where \( L^{(0)}, \ldots, L^{(n-m)} \subseteq R^d \) linear subspaces.

The following simple lemma will be convenient in the proof of Theorem 3.7.

**Lemma 3.9.** Let \( k > 0, C, D \in \mathcal{P}(\mathbb{R}^d), M, N \in \mathcal{H}(\mathbb{R}^d) \). Then

\[
\sum_{j=1}^{d} \Psi_j(C, M) \Psi_{d+k}(L_j \times D, \mathbb{R}^d \times N) = \Psi_{d+k}(C \times D, M \times N),
\]

where \( L_j \subseteq \mathbb{R}^d \) a \( j \)-dimensional linear subspace.

**Proof.** Using the product rule in Proposition 3.3, we obtain

\[
\Psi_{d+k}(L_j \times D, \mathbb{R}^d \times N) = \sum_{\ell, m = d+k} \Psi_\ell(L_j, \mathbb{R}^d) \Psi_m(D, N) = \Psi_{d+k-j}(D, N),
\]

and therefore, using the product rule one more time,

\[
\sum_{j=1}^{d} \Psi_j(C, M) \Psi_{d+k}(L_j \times D, \mathbb{R}^d \times N) = \sum_{j=1}^{d} \Psi_j(C, M) \Psi_{d+k-j}(D, N) = \Psi_{d+k}(C \times D, M \times N).
\]

**Proof of Theorem 3.7.** To simplify the notation we abbreviate \( T_i Q_i =: U_i \). Define

\[
\text{LEFT}(C_0, \ldots, C_n; M_0, \ldots, M_n) := \mathbb{E} \left[ \Psi_k \left( \bigcap_{i=0}^{n} U_i C_i, \bigcap_{i=0}^{n} U_i M_i \right) \right],
\]

\[
\text{RIGHT}(C_0, \ldots, C_n; M_0, \ldots, M_n) := \Psi_{nd+k}(C_0 \times \cdots \times C_n, M_0 \times \cdots \times M_n),
\]

so that we need to show \( \text{LEFT}(C_0, \ldots, C_n; M_0, \ldots, M_n) = \text{RIGHT}(C_0, \ldots, C_n; M_0, \ldots, M_n) \). By induction on \( m \) we will show that

\[
\text{LEFT}(C_0, \ldots, C_{m-1}, L^{(0)}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-1}, R^d, \ldots, R^d) = \text{RIGHT}(C_0, \ldots, C_{m-1}, L^{(0)}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-1}, R^d, \ldots, R^d)
\]

where \( L^{(0)}, \ldots, L^{(n-m)} \subseteq R^d \) linear subspaces.

For the induction step, \( m \geq 1 \), we define \( \psi: \mathcal{P}(\mathbb{R}^d) \times \mathcal{H}(\mathbb{R}^d) \to \mathbb{R}, \)

\[
\psi(C, M) := \text{LEFT}(C_0, \ldots, C_{m-2}, C, L^{(0)}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-2}, M, R^d, \ldots, R^d)
\]

\[
= \mathbb{E} \left[ \Psi_k \left( \bigcup_{i=0}^{m-2} U_i C_i \cap U C \cap \bigcup_{i=0}^{m-2} U_{m+i} L^{(i)} \cap \bigcup_{i=0}^{m-2} U_i M_i \cap U M \right) \right],
\]
where we set \( \mathbf{U} := \mathbf{U}_{m-1} \) to simplify the notation. Note that \( \Psi_k(C,\emptyset) = 0 \), since \( k > 0 \), and thus \( \psi(C,\emptyset) = 0 \). By Lemma 3.8, \( \psi \) satisfies the assumptions in Theorem 3.5, so that

\[
\text{LEFT} \{ C_0, \ldots, C_{m-1}, L^{[0]}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-1}, \mathbb{R}^d, \ldots, \mathbb{R}^d \}
\]

\[
= \psi(C_{m-1}, M_{m-1}) (\text{H}) = \sum_{j=1}^{d} \psi(L_j, C_{m-1}) \text{Lin}_j(C_{m-1}, M_{m-1}) + \sum_{j=1}^{d} \psi(L_j, \mathbb{R}^d) \Psi_j(C_{m-1}, M_{m-1})
\]

\[
= \sum_{j=1}^{d} \text{LEFT} \{ C_0, \ldots, C_{m-2}, L_j, L^{[0]}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-2}, \mathbb{R}^d, \ldots, \mathbb{R}^d \} \Psi_j(C_{m-1}, M_{m-1})
\]

\[
\text{RIGHT} \{ C_0, \ldots, C_{m-1}, L^{[0]}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-1}, \mathbb{R}^d, \ldots, \mathbb{R}^d \},
\]

which shows the induction step, and thus finishes the proof. \( \square \)

4. Biconic Localizations

In this section we consider the biconic localizations of the intrinsic volumes, the support measures. As in Section 3 we start with a discussion of the general properties of the support measures in Section 4.1. In Section 4.2 we show how the support measures can be characterized through concentration, invariance, locality, and in Section 4.3 we will use this characterization to prove a kinematic formula for the support measures, which generalizes (1.14).

4.1 Support measures. Recall that the support measures localize both the \( \nu \)-vector and the curvature measures, and are given by \( \Theta_k(C,\mathcal{M}) = \mathbb{P} \{ \hat{\mathbf{U}}_C(g) \in \mathcal{R}_k(C) \cap \mathcal{M} \} \) where \( g \in \mathbb{R}^d \) is a standard Gaussian vector, cf. (2.15). By (1.26) and Lemma 2.6 we can write the support measures in the form

\[
\Theta_k(C,\cdot) = \sum_{L \in \mathcal{Z}_k(C)} \gamma_L(C \cap L \cap \cdot) \otimes \gamma_{L^\perp}(C^\circ \cap L^\perp \cap \cdot) = \sum_{L \in \mathcal{Z}_k(C)} \Psi_k(C \cap L,\cdot) \otimes \Psi_{d-k}(C^\circ \cap L^\perp,\cdot).
\]

In particular, we have (note that \( \Psi_d = \Phi_d \) and \( \text{lin}(C^\circ) = 0 \) iff \( \gamma_d(C) \neq 0 \))

\[
\Theta_d(C,\cdot) = \Phi_d(C,\cdot) \otimes \Delta, \quad \Theta_0(C,\cdot) = \Delta \otimes \Phi_d(C^\circ,\cdot).
\]

The following proposition lists further properties of the support measures, similar to Proposition 3.3.

**Proposition 4.1.** Let \( C, D \in \mathcal{P}(\mathbb{R}^d) \) and \( 0 \leq k \leq d \).

1. concentration: for all \( \mathcal{M} \in \hat{\mathcal{B}}(\mathbb{R}^d, \mathbb{R}^d) \),

\[
\Theta_k(C,\mathcal{M}) = \Theta_k(C,\mathcal{M} \cap \mathcal{R}_k(C)).
\]

In particular, \( \Theta_k(C,\mathcal{M}) = \Theta_k(C,\mathcal{M} \cap BL(C)) \).

2. orthogonal invariance: for all \( Q \in O(d) \) and all \( \mathcal{M} \in \hat{\mathcal{B}}(\mathbb{R}^d, \mathbb{R}^d) \),

\[
\Theta_k(QC, Q, \mathcal{M}) = \Theta_k(C,\mathcal{M}).
\]

3. locality: for all \( \mathcal{M} \in \hat{\mathcal{B}}(\mathbb{R}^d, \mathbb{R}^d) \),

\[
\Theta_k(C,\mathcal{M}) \leq \Theta_k(D,\mathcal{M}), \quad \text{if } \mathcal{R}_k(C) \cap \mathcal{M} \subseteq BL(D) \cap \mathcal{M}.
\]

In particular,

\[
\Theta_k(C,\mathcal{M}) = \Theta_k(D,\mathcal{M}), \quad \text{if } \mathcal{R}_k(C) \cap \mathcal{M} = \mathcal{R}_k(D) \cap \mathcal{M}
\]

or

\[
\text{BL}^*(C) \cap \mathcal{M} = \text{BL}^*(D) \cap \mathcal{M}
\]

or

\[
\text{BL}(C) \cap \mathcal{M} = \text{BL}(D) \cap \mathcal{M}.
\]
(4) product rule: for all \( \mathcal{M}, \mathcal{N} \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d) \),
\[
\Theta_k(C \times D, \mathcal{M} \times \mathcal{N}) = \sum_{i+j=k} \Theta_i(C, \mathcal{M}) \Theta_j(D, \mathcal{N}).
\]
(5) polarity: for all \( \mathcal{M} \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d) \),
\[
\Theta_k(C^\circ, \mathcal{M}) = \Theta_{d-k}(C, \text{rev}(\mathcal{M})).
\]
(6) additivity: for all \( \mathcal{M} \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d) \),
\[
\Theta_k(C + D, \mathcal{M}) + \Theta_k(C \cap D, \mathcal{M}) = \Theta_k(C, \mathcal{M}) + \Theta_k(D, \mathcal{M}), \quad \text{if } C + D = C \cup D.
\]
(7) weak continuity: for all open sets \( \Theta \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d) \),
\[
\liminf_{i \to \infty} \Theta_k(C_i, \Theta) \geq \Theta_k(C, \Theta), \quad \text{if } \{C_i : i \in \mathbb{N}\} \subseteq \mathscr{P}(\mathbb{R}^d) \text{ such that } \lim_{i \to \infty} C_i = C.
\]

Proof. Properties (1) and (2) follow, for example, from the formula (2.15) for the support measures. For the locality property (4.1) we will rely on a lemma that follows trivially from the Steiner formulas, which we discuss in Appendix A: according to Lemma A.1, \( \Theta_k(C, \mathcal{M}) = \Theta_k(D, \mathcal{M}) \) if \( \mathcal{M} \subseteq \text{BL}(C) \cap \text{BL}(D) \). Using this, we obtain from \( \mathcal{S}_k(C) \cap \mathcal{M} \subseteq \text{BL}(D) \cap \mathcal{M} \),
\[
\Theta_k(C, \mathcal{M}) = \Theta_k(C, \mathcal{S}_k(C) \cap \mathcal{M}) \leq \Theta_k(C, \text{BL}(D) \cap \mathcal{M}) = \Theta_k(C, \text{BL}(D) \cap \mathcal{M} \cap \mathcal{S}_k(C))
\]
which shows (4.1). The product rule follows for example from the formula (2.17) for the support measures and from the formula in (2.21) for the lifted \( \mathcal{S}_k(C \times D) \). The polarity formula follows from (2.17) and rev(\( \mathcal{S}_k(C) \)) = \( \mathcal{S}_{d-k}(C^\circ) \).

The easiest way to show the additivity and weak continuity of the support measures is to use the Steiner formulas, which we discuss in Appendix A. As we will use neither the additivity nor the weak continuity property, we will content ourselves with a description of the general idea in Appendix A; for more details we refer to [SW08, Sec. 6.5].

We will need the following lemma in the proof of the kinematic formula.

Lemma 4.2. Let \( C \in \mathscr{P}(\mathbb{R}^d) \) and \( \mathcal{M} \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d) \) such that \( \mathcal{M} \subseteq \text{BL}(C) \). Then for every \( 0 \leq k \leq d \),
\[
\Theta_k(C, \mathcal{M}) = \sum_{L \in \mathcal{L}(C)} \Theta_k(L \cap \text{BL}(L)).
\]

Proof. Since \( \mathcal{M} \subseteq \text{BL}(C) \) and \( \text{BL}(C) \subseteq \bigcup_{L \in \mathcal{L}(C)} \text{BL}(L) \), we have \( \mathcal{M} = \bigcup_{L \in \mathcal{L}(C)} (\mathcal{M} \cap \text{BL}(L)) \). Furthermore, for any two distinct linear subspaces \( L \neq L' \), we have \( \dim(\text{BL}(L) \cap \text{BL}(L')) < d \) so that \( \Theta_k(C, \text{BL}(L) \cap \text{BL}(L')) = 0 \). Using the inclusion-exclusion principle, we thus obtain
\[
\Theta_k(C, \mathcal{M}) = \Theta_k(C, \bigcup_{L \in \mathcal{L}(C)} (\text{BL}(L) \cap \mathcal{M})) = \sum_{L \in \mathcal{L}(C)} \Theta_k(C, \mathcal{M} \cap \text{BL}(L)).
\]

The locality property of the support measures implies \( \Theta_k(C, \mathcal{M} \cap \text{BL}(L)) = \Theta_k(L, \mathcal{M} \cap \text{BL}(L)) \), and of course \( \Theta_k(L, \cdot) = 0 \) if \( \dim L \neq k \), which finishes the proof.

4.2. Characterizations. The characterization theorem for the support measures enjoys a similar simplicity as the characterization for the polyhedral measures given in Theorem 3.5. The locality assumption is this time expressed in terms of the biconic lift.

Theorem 4.3 (Characterization of support measures). Let \( \psi : \mathscr{P}(\mathbb{R}^d) \times \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R} \) be such that for all \( C, D \in \mathscr{P}(\mathbb{R}^d), \mathcal{M} \in \mathscr{B}(\mathbb{R}^d, \mathbb{R}^d), Q \in O(d) \):

(0) \( \psi(C, \cdot) \in \mathbb{R}(\mathbb{R}^d, \mathbb{R}^d) \),
(1) \( \psi(C, \mathcal{M}) = \psi(C, \mathcal{M} \cap \text{BL}(C)) \),
(2) \( \psi(QC, Q \mathcal{M}) = \psi(C, \mathcal{M}) \),
(3) \( \psi(C, \mathcal{M}) = \psi(D, \mathcal{M}) \) if \( \text{BL}(C) \cap \mathcal{M} = \text{BL}(D) \cap \mathcal{M} \).
Then $\psi$ is a linear combination of $\hat{\lambda}$ and $\Theta_0, \ldots, \Theta_d$. In this case

$$\psi = \psi(\mathbb{R}^d, \{0\}) \hat{\lambda} + \sum_{k=0}^{d} \psi(L^{(k)}, \mathbb{R}_+^d) \Theta_k,$$

(4.3)

where $L^{(k)} \subseteq \mathbb{R}^d$ denotes a $k$-dimensional subspace.

**Proof.** The proof follows broadly the same line of arguments as the proof of the characterization of the polyhedral measures given in Theorem 3.5. An important tool is again the facial decomposition of $C$, which yields two partitions of the biconic lift, cf. (2.12).

Let $\psi: \mathcal{P}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}$ satisfy the assumptions (0)–(3), i.e., $\psi(C, \cdot)$ is a biconic measure, which is concentrated on BL($C$), $\psi$ is invariant under (simultaneous) orthogonal transformations, and $\psi(C, \mathcal{M}) = \psi(D, \mathcal{M})$, if $BL(C) \cap \mathcal{M} = BL(D) \cap \mathcal{M}$.

Again, we first consider the case of a linear subspace $L \subseteq \mathbb{R}^d$. For $M \in \hat{\mathcal{B}}(L)$ define

$$\psi_{L,M} : \hat{\mathcal{B}}(L^\perp) \to \mathbb{R}, \quad \psi_{L,M}(M') := \psi(L, M \times M').$$

Orthogonal invariance of $\psi$ implies that $\psi_{L,M}$ is an orthogonal invariant conic measure on $L^\perp$, so that by Lemma 2.2,

$$\psi_{L,M} = \psi_{0,0} + \psi_{1,1}(M)$$

for every $M' \in \hat{\mathcal{B}}(L^\perp)$. Consider now the measures

$$\psi_{L}^{(0)} : \hat{\mathcal{B}}(L) \to \mathbb{R}, \quad \psi_{L}^{(0)}(M) := \psi(L, M \times \{0\}), \quad \psi_{L}^{(1)}(M) := \psi(L, M \times L^\perp).$$

By the same reasoning as above we arrive at

$$\psi(L, M \times \{0\}) = \psi(L, M \times \{0\}) \Delta(M^\perp) + \psi(L, M \times L^\perp) \gamma_{L^\perp}(M')$$

(4.4)

for every $M' \in \hat{\mathcal{B}}(L^\perp)$.

Orthogonal invariance of $\psi$ implies that the constants in the above expressions only depend on $\dim(L)$ = $k$, which we denote for simplicity by (using the concentration property (1) of $\psi$)

$$r_0 := \psi(L, \{0\} \times \{0\}), \quad s_k := \psi(L, \mathbb{R}_+^d \times \{0\}), \quad t_k := \psi(L, \{0\} \times \mathbb{R}_+^d), \quad u_k := \psi(L, \mathbb{R}_+^d \times \mathbb{R}_+^d).$$

In fact, the locality property (3) of $\psi$ implies

$$r_0 = r_1 = \cdots = r_d = \psi(\mathbb{R}_+^d, \{0\} \times \{0\}) =: r.$$

We thus obtain from (4.4), for $M \in \hat{\mathcal{B}}(L)$, $M' \in \hat{\mathcal{B}}(L^\perp)$,

$$\psi(L, M \times M') = r \Delta(M^\perp) \Delta(M') + s_k \gamma_{L^\perp}(M^\perp) + t_k \Delta(M) \gamma_{L^\perp}(M') + u_k \gamma_{L^\perp}(M') \gamma_{L^\perp}(M').$$

(4.5)

Note that $s_0 = t_d = u_0 = u_d = 0$.

Consider now a general polyhedral cone $C \in \mathcal{P}(\mathbb{R}^d)$. The biconic lift of $C$ has two disjoint decompositions, cf. (2.12), which we recall here for convenience:

$$BL(C) = \bigcup_{L \in \mathcal{Z}(C)} \left( \text{int}_L(C \cap L) \times (C^0 \cap L^\perp) \right) = \bigcup_{L \in \mathcal{Z}(C)} \left( (C \cap L) \times \text{int}_L(C^0 \cap L^\perp) \right).$$

(4.6)

Using the assumptions (0)–(3) and the first decomposition of BL($C$) in (4.6), we obtain for $M, M' \in \hat{\mathcal{B}}(\mathbb{R}^d)$

$$\psi(C, M \times M') \overset{(3)}{=} \psi(C, BL(C) \cap (M \times M')) \overset{(4.6)}{=} \sum_{L \in \mathcal{Z}(C)} \psi(C, \text{int}_L(C \cap L) \times (C^0 \cap L^\perp \cap M'))$$

$$\overset{(3)}{=} \sum_{L \in \mathcal{Z}(C)} \psi(L, \text{int}_L(C \cap L) \cap (C^0 \cap L^\perp \cap M'))$$

$$\overset{(4.5)}{=} r \Delta(M') \sum_{L \in \mathcal{Z}(C)} \Delta(\text{int}_L(C \cap L) \cap M) + \sum_{k=1}^{d} s_k \sum_{L \in \mathcal{Z}(C)} \gamma_{L}(C \cap L \cap M)$$

$$+ \sum_{k=0}^{d-1} t_k \sum_{L \in \mathcal{Z}(C)} \Delta(\text{int}_L(C \cap L) \cap M) \gamma_{L^\perp}(C^0 \cap L^\perp \cap M') + \sum_{k=1}^{d-1} u_k \sum_{L \in \mathcal{Z}(C)} \gamma_{L}(C \cap L \cap M) \gamma_{L^\perp}(C^0 \cap L^\perp \cap M').$$
Note that the origin lies in the relative interior of a unique face of \( C \) of dimension \( \text{lin}(C) \). Using this, we obtain
\[
\sum_{L \in \mathcal{L}(C)} \Delta(\text{int}_L(C \cap L) \cap M) \Delta(M) = \Delta(M) \Delta(M') = \hat{\Delta}(M \times M') \quad \text{and}
\]
\[
\sum_{k=0}^{d} t_k \sum_{L \in \mathcal{L}_k(C)} \Delta(\text{int}_L(C \cap L) \cap M) \gamma_{L^\perp}(C^\circ \cap L^\perp \cap M') = t_{\text{lin}(C)} \Delta(M) \sum_{L \in \mathcal{L}_{\text{lin}(C)}(C)} \gamma_{L^\perp}(C^\circ \cap L^\perp \cap M').
\]
Furthermore, note that
\[
\sum_{L \in \mathcal{L}_k(C)} \gamma_{L}(C \cap L \cap M) = \Psi_k(C, M), \quad \sum_{L \in \mathcal{L}_k(C)} \gamma_{L}(C^\circ \cap L^\perp \cap M') = \Psi_{d-k}(C^\circ, M'),
\]
\[
\sum_{L \in \mathcal{L}_k(C)} \gamma_{L}(C \cap L \cap M) \gamma_{L^\perp}(C^\circ \cap L^\perp \cap M') = \Theta_k(C, M \times M').
\]
Using Lemma 2.6, we arrive at
\[
\psi(C, \cdot) = r \hat{\Delta} + \sum_{k=1}^{d} s_k \Psi_k(C, \cdot) \otimes \Delta + \sum_{k=0}^{d-1} t_k \text{Lin}_k(C, \cdot) \otimes \Psi_{d-k}(C^\circ, \cdot) + \sum_{k=1}^{d-1} u_k \Theta_k(C, \cdot).
\] (4.7)

It remains to show that \( s_1 = \cdots = s_{d-1} = t_1 = \cdots = t_{d-1} = 0 \). This is achieved by using the second decomposition of \( \text{BL}(C) \) in (4.6). Analogously to the above,
\[
\psi(C, M \times M') = \sum_{k=0}^{d} \psi(L, C \cap L \cap M) \times (\text{int}_{L^\perp}(C^\circ \cap L^\perp \cap M'))
\]
and from this it follows
\[
\psi(C, \cdot) = r \hat{\Delta} + \sum_{k=1}^{d} s_k \Psi_k(C, \cdot) \otimes \text{Lin}_k(C^\circ, \cdot) + \sum_{k=0}^{d-1} t_k \Delta \otimes \Psi_{d-k}(C^\circ, \cdot) + \sum_{k=1}^{d-1} u_k \Theta_k(C, \cdot).
\] (4.8)

Now, let \( C \in \mathcal{P}(\mathbb{R}^d) \) such that \( \text{int}(C) \neq \emptyset \) (equivalently, \( \text{lin}(C^\circ) = 0 \)). Then for every \( L \in \mathcal{L}_m(C) \) with \( 1 \leq m \leq d-1 \), we have \( \Psi_k(C, L) = 0 \) if \( k > m \), and overall
\[
\psi(C, L_s \times \{0\}) = \begin{cases} \sum_{k=1}^{m} s_k \Psi_k(C, L) & \text{by (4.7)} \\ 0 & \text{by (4.8)}. \end{cases}
\]
Assuming \( C \) is such that \( \text{int}(C) \neq \emptyset \) and \( \mathcal{L}_m(C) \neq \emptyset \) for \( m = 1, \ldots, d-1 \), we obtain \( s_1 = \cdots = s_{d-1} = 0 \). Analogously, by considering \( \psi(C, \{0\} \times L_s) \) we obtain \( t_1 = \cdots = t_{d-1} = 0 \). Finally, note that
\[
\Psi_{d}(C, \cdot) \otimes \Delta = \Psi_{d}(C, \cdot) \otimes \text{Lin}_0(C^\circ, \cdot) = \Theta_d(C, \cdot), \quad \text{Lin}_0(C, \cdot) \otimes \Psi_{d}(C^\circ, \cdot) = \Delta \otimes \Psi_{d}(C^\circ, \cdot) = \Theta_0(C, \cdot),
\]
so that we obtain
\[
\psi = r \hat{\Delta} + s_d \Theta_d + t_0 \Theta_0 + \sum_{k=1}^{d-1} u_k \Theta_k = \psi(\mathbb{R}^d, \{0\} \times \{0\}) \hat{\Delta} + \sum_{k=0}^{d} \psi(L^{(k)}, \mathbb{R}^d_s \times \mathbb{R}^d) \Theta_k. \quad \square
\]

4.3. Kinematic formulas for support measures. In this section we will use the characterization of the support measures provided by Theorem 4.3 to prove the following theorem.

**Theorem 4.4** (Kinematic formula for support measures). Let \( C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d) \), \( M_0, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d) \) such that \( \mathcal{M}_i \subseteq \text{BL}(C_i) \) for all \( 0 \leq i \leq n \), and let \( T_0, \ldots, T_n \in \text{Gl}_d \) and \( k > 0 \). Then for \( Q_0, \ldots, Q_n \in O(d) \) iid uniformly at random,
\[
\mathbb{E} \left[ \Theta_k \left( \bigcap_{i=0}^{n} T_i Q_i C_i, \bigwedge_{i=0}^{n} T_i Q_i M_i \right) \right] = \Theta_{nd+k}(C, \mathcal{M}),
\] (4.9)
where \( C := C_0 \times \cdots \times C_n, \mathcal{M} := M_0 \times \cdots \times M_n \). The support measures of the product are given by
\[
\Theta_m(C, \mathcal{M}) = \sum_{l_0 + \cdots + l_n = m} \Theta_{l_0}(C_0, M_0) \cdots \Theta_{l_n}(C_n, M_n).
\] (4.10)
Again, the product formula (4.10) is just an iteration of (4) in Proposition 4.1. The polarity property of the support measures (4.2) immediately implies the following corollary from Theorem 4.4.

**Corollary 4.5** (Polar kinematic formula for support measures). Let the notation and assumptions be as in Theorem 4.4. Then

\[ E \left[ \Theta_{d-k} \left( \sum_{i=0}^{n} T_i Q_i C_i, \bigvee_{i=0}^{n} T_i Q_i \mathcal{M}_i \right) \right] = \Theta_{d-k}(C, \mathcal{M}). \]  

**Proof.**

Let \( \psi \) for all \( Q \in O(d) \), so that Proposition B.6 in Appendix B implies (1)

\[ (\psi \circ \Theta_{d-k}(\mathcal{M})) = \Theta_{d-k}(C, \mathcal{M}). \]

The general argumentation in the proof of Theorem 4.4 will be as in Section 3.3. However, the biconic \&-operation requires extra care, cf. Proposition 2.4, and some important steps rely on genericity arguments, which we again defer to Appendix B to ease the presentation. For example, see Proposition B.5(1) for the well-definedness of the left-hand side in (4.9).

The formulation of Theorem 4.4 is such that one assumes \( \mathcal{M}_i \subseteq BL(C_i), 0 \leq i \leq n \). One can drop this assumption if instead of (4.9) one considers the following claim:

\[ E \left[ \Theta_{k} \left( \bigcap_{i=0}^{n} T_i Q_i C_i, \bigwedge_{i=0}^{n} T_i Q_i \left( BL(C_i) \cap \mathcal{M}_i \right) \right) \right] = \Theta_{nd+k}(C, \mathcal{M}). \]  

We will use this version of Theorem 4.4 in the rest of this section.

**Lemma 4.6.** Let \( C_1, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d), \mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d), T_0, \ldots, T_n \in GL_d, \) and \( k > 0 \). Then for \( Q_0, \ldots, Q_n \in O(d) \) iid uniformly at random, the map \( \psi : \mathcal{P}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R} \) given by

\[ \psi(C_0, \mathcal{M}_0) = E \left[ \Theta_{k} \left( \bigcap_{i=0}^{n} T_i Q_i C_i, \bigwedge_{i=0}^{n} T_i Q_i \left( BL(C_i) \cap \mathcal{M}_i \right) \right) \right] \]  

satisfies the assumptions in Theorem 4.3.

**Proof.** To simplify the notation we abbreviate \( T_i Q_i =: U_i \).

(0) **Claim:** \( \psi(C_0, \mathcal{M}_0) \subseteq \mathcal{M} \). Let \( \mathcal{N}_j \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d), j = 1, 2, \ldots, \) be pairwise disjoint. Clearly, \( U_0 \left( BL(C_0) \cup \mathcal{N}_j \right), j = 1, 2, \ldots, \) are pairwise disjoint as well, so we may assume that \( \mathcal{N}_j \subseteq BL(C_0) \) for all \( j \). It is not true that \( U_0 \mathcal{N}_j \cap \mathcal{M} \), \( j = 1, 2, \ldots \), are pairwise disjoint, cf. Proposition 2.4. However, denoting \( D := \bigcap_{i=1}^{n} U_i C_i \), we have

\[ \mathcal{M} \subseteq \bigcap_{i=1}^{n} U_i \left( BL(C_i) \cap \mathcal{M}_i \right) \subseteq \bigcap_{i=1}^{n} U_i BL(C_i) = BL \left( \bigcap_{i=1}^{n} U_i C_i \right) = BL(D), \]

so that Proposition B.6 in Appendix B implies

\[ \psi(C_0, \bigcup_{j=1}^{\infty} \mathcal{N}_j) \subseteq \psi \left[ \bigcap_{i=1}^{\infty} U_0 \mathcal{N}_j \cap \mathcal{M} \right] \subseteq \psi \left[ \bigcap_{i=1}^{\infty} U_0 \mathcal{N}_j \cap \mathcal{M} \right] \]

where (\*) follows from an application of the monotone convergence theorem.

(1) **Claim:** \( \psi(C_0, \mathcal{M}_0) \subseteq \psi \left[ C_0, \mathcal{M}_0 \cap BL(C_0) \right] \). This follows directly from the definition of \( \psi \).

(2) **Claim:** \( \psi(Q C_0, Q_0 \mathcal{M}) = \psi(C_0, \mathcal{M}_0) \) for \( Q \in O(d) \). This also follows directly from the definition of \( \psi \), since \( Q_0 Q \) is uniformly at random in \( O(d) \) and independent of \( Q_1, \ldots, Q_n \).
where $L(\tilde{\beta})$.

(3) Claim: $\psi(C_0,\mathcal{M}_0) = \psi(C_0,\mathcal{M}_0)$ if $BL(C_0) \cap \mathcal{M}_0 = BL(C_0) \cap \mathcal{M}_0$. Define $\tilde{C}_i := C_i$ for $i = 1, \ldots, n$, and $C := \bigcap_{i=0}^{n} U_i C_i$, $\tilde{C} := \bigcap_{i=0}^{n} U_i \tilde{C}_i$, $\mathcal{M} := \bigcup_{i=0}^{n} U_i (BL(C_i) \cap \mathcal{M}_i)$. Then we have

$$BL(C) \cap \mathcal{M} = \left( \bigcap_{i=0}^{n} U_i BL(C_i) \right) \cap \left( \bigcap_{i=0}^{n} \left( U_i BL(C_i) \cap U_i \mathcal{M}_i \right) \right) \cup \left( \bigcup_{i=0}^{n} \left( U_i BL(C_i) \cap U_i \mathcal{M}_i \right) \right)$$

where $(\dagger)$ follows from Proposition 2.4 and $(\ddagger)$ follows from the assumption $BL(C_0) \cap \mathcal{M}_0 = BL(C_0) \cap \mathcal{M}_0$. By the locality of the support measures, cf. Proposition 4.1(3), it follows that $\Theta_k(C,\mathcal{M}) = \Theta_k(\tilde{C},\mathcal{M})$; in particular the expectations coincide.

\[ \square \]

Lemma 4.7. Let $k > 0$, $C, D \in \mathcal{P}(\mathbb{R}^d)$, $\mathcal{M}, \mathcal{N} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Then

$$\sum_{j=1}^{d} \Theta_j(C,\mathcal{M}) \Theta_{d+k}(L_j \times D, \mathbb{R}^{d+d} \times \mathcal{N}) = \Theta_{d+k}(C \times D, \mathcal{M} \times \mathcal{N})$$

where $L_j \subseteq \mathbb{R}^d$ a $j$-dimensional linear subspace.

\[ \text{Proof.} \text{ Argue exactly as in Lemma 3.9.} \]
where we set $U := U_{m-1}$ to simplify the notation. Note that $\Psi_k(C, \emptyset) = 0$, since $k > 0$, and thus $\psi(C, \emptyset) = 0$. By Lemma 4.6, $\psi$ satisfies the assumptions in Theorem 4.3, so that

\[
\begin{align*}
\text{LEFT} \bigl( & C_0, \ldots, C_{m-1}, L^{(0)}, \ldots, L^{(n-m)}, M_0, \ldots, M_{m-1}, \mathbb{R}^{d+d}, \ldots, \mathbb{R}^{d+d} \bigr) \\
= & \psi(C_{m-1}, M_{m-1}) \eta \psi^{(\emptyset)}_{\mathbb{R}^{d+d}}(\Delta, M_{m-1}) + \sum_{j=1}^{d} \psi(L_j, \mathbb{R}^{d+d}) \Theta_j(C_{m-1}, M_{m-1}) \\
= & \sum_{j=1}^{d} \text{LEFT} \bigl( C_0, \ldots, C_{m-2}, L_j, L^{(0)}, \ldots, L^{(n-m)}, M_0, \ldots, M_{m-2}, \mathbb{R}^{d+d}, \ldots, \mathbb{R}^{d+d} \bigr) \Theta_j(C_{m-1}, M_{m-1}) \\
(\text{IH}) = & \sum_{j=1}^{d} \Theta_{nd+k} \bigl( C_0 \times \cdots \times C_{m-2} \times L_j \times L^{(0)} \times \cdots \times L^{(n-m)}, M_0 \times \cdots \times M_{m-2} \times \mathbb{R}^{d+d} \times \cdots \times \mathbb{R}^{d+d} \bigr) \\
\leq & \Theta_j(C_{m-1}, M_{m-1}) \\
\text{[Lem. 4.7]} = & \text{RIGHT} \bigl( C_0, \ldots, C_{m-1}, L^{(0)}, \ldots, L^{(n-m)}; M_0, \ldots, M_{m-1}, \mathbb{R}^{d}, \ldots, \mathbb{R}^{d} \bigr).
\end{align*}
\]

which shows the induction step, and thus finishes the proof. \hfill \square

5. General kinematic formulas

In this section we provide the proof for the general kinematic formula stated in Theorem 1.6 (with the restriction $T_0, \ldots, T_n \in O(d)$). Since the proof is a simple induction on the number of indeterminates in the Boolean formula, essentially the same proof yields a general kinematic formula for the support measures, which we state next. We will only prove Theorem 1.6, as the proof translates straightforwardly to the support measures case.

If $F(X_0, \ldots, X_n)$ denotes a Boolean formula in the variables $X_0, \ldots, X_n$, and if $\mathcal{M}_0, \ldots, \mathcal{M}_n \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$, we define the evaluation $F(\mathcal{M}_0, \ldots, \mathcal{M}_n) \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)$ to be the result of replacing negation $\neg (\cdots)$ by the reversal map $\text{rev}(\cdots)$.

**Theorem 5.1** (General kinematic formula for support measures). Let $C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d)$, $\mathcal{M}_0, \ldots, \mathcal{M}_n \in \mathcal{B}(\mathbb{R}^d)$ such that $\mathcal{M}_i \subseteq \text{BL}(C_i)$ for all $0 \leq i \leq n$, and let $T_0, \ldots, T_n \in O(d)$ and $0 < k < d$. Furthermore, let $F(X_0, \ldots, X_n)$ be a Boolean read-once formula. Then for $Q_0, \ldots, Q_n \in O(d)$ iid uniformly at random,

\[
E \left[ \Theta_k \left( F(T_0 Q_0 C_0, \ldots, T_n Q_n C_n), F(T_0 Q_0 \mathcal{M}_0, \ldots, T_n Q_n \mathcal{M}_n) \right) \right] = \sum_{\dim_k(k_0, \ldots, k_n) = k} \Theta_{k_0}(C_0, \mathcal{M}_0) \cdots \Theta_{k_n}(C_n, \mathcal{M}_n). \tag{5.1}
\]

As remarked for Theorem 1.6, the transformations $T_0, \ldots, T_n$ in (5.1) can of course be dropped entirely; we included them to simplify the comparison to the other formulas. It seems reasonable to assume that (5.1) also holds for general $T_0, \ldots, T_n \in \text{GL}_d$. A possible approach to this would be to merge the reasoning for the polar kinematic formula given in Corollary 4.5 with the induction step in the proof of Theorem 4.4. However, it is by no means clear that this proof idea can be implemented successfully, due to the subtleties involving the biconic $\land$- and $\lor$-operations.

The boundary cases for the intrinsic volumes (1.17) do not localize to the support measures. Simple counter-examples can be found already in dimension one or two, which show that in general (notation as in Theorem 4.4)

\[
E \left[ \Theta_0 \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigcup_{i=0}^n T_i Q_i \mathcal{M}_i \right) \right] \neq \sum_{j=0}^{nd} \Theta_j(C, \mathcal{M}) \quad E \left[ \Theta_d \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigvee_{i=0}^n T_i Q_i \mathcal{M}_i \right) \right] \neq \sum_{j=0}^{nd} \Theta_{d+j}(C, \mathcal{M}).
\]

We finish this section with the proof of the general kinematic formula (1.19) for the intrinsic volumes, where we assume that $T_0, \ldots, T_n \in O(d)$, so that we may as well drop these transformations entirely in (1.19).
Proof of Theorem 1.6. Note first that since the lattice of polyhedral cones satisfies the De Morgan’s Laws \((C \cap D)^o = C^o + D^o\) and \((C + D)^o = C^o \cap D^o\), we may assume without loss of generality that all negations in \(F(X_0, \ldots, X_n)\) are directly at the variables. We now proceed by induction on \(n\).

If \(n = 0\) then \(F(X_0) = X_0\) or \(F(X_0) = \neg X_0\). In this case we have \(\dim_{\delta}^F(k_0) = k_0\) or \(\dim_{\delta}^F(k_0) = d - k_0\), respectively. Therefore, we have

\[
\mathbb{E} \left[ v_k \left( F(Q_0 C_0) \right) \right] = v_k(C_0) = \sum_{\dim_{\delta}^F(k_0) = k} v_{k_0}(C_0), \quad \text{or} \quad \mathbb{E} \left[ v_k \left( F(Q_0 C_0) \right) \right] = v_k(C_0) = \sum_{\dim_{\delta}^F(k_0) = k} v_{k_0}(C_0),
\]

respectively. This settles the case \(n = 0\).

For \(n > 0\) we can permutate the variables \(X_0, \ldots, X_n\) in such a way that (without loss of generality) we have for some \(0 \leq m < n\)

\[
F(X_0, \ldots, X_n) = F_1(X_0, \ldots, X_m) \land F_2(X_{m+1}, \ldots, X_n), \quad \text{or} \quad F(X_0, \ldots, X_n) = F_1(X_0, \ldots, X_m) \lor F_2(X_{m+1}, \ldots, X_n).
\]

In the first case we have

\[
\dim_{\delta}^F(k_0, \ldots, k_n) = \max \left\{ 0, \dim_{\delta}^F(k_0, \ldots, k_m) + \dim_{\delta}^F(k_{m+1}, \ldots, k_n) - d \right\},
\]

and we may argue in the following way:

\[
\mathbb{E}_{Q_0, \ldots, Q_n} \left[ v_k \left( F(Q_0 C_0, \ldots, Q_n C_n) \right) \right] = \mathbb{E}_{Q_0, \ldots, Q_n} \left[ v_k \left( F_1(Q_0 C_0, \ldots, Q_m C_m) \land F_2(Q_m C_{m+1}, \ldots, Q_n C_n) \right) \right]
\]

\[
= \mathbb{E}_{Q_0, \ldots, Q_n} \left[ \mathbb{E}_{Q_Q'} \left[ v_k \left( F_1(Q_0 C_0, \ldots, Q_m C_m) \land F_2(Q_m C_{m+1}, \ldots, Q_n C_n) \right) \right] \right], \quad (5.2)
\]

where in the second step we have replaced \(Q_0, \ldots, Q_m\) and \(Q_{m+1}, \ldots, Q_n\) by \(QQ_0, \ldots, QQ_m\) and \(QQ_{m+1}, \ldots, QQ_n\), respectively, with \(Q, Q' \in O(d)\) iid uniformly at random.\(^7\) Applying the (normal) kinematic formula (1.14) to the inner expectation yields

\[
(5.2) = \mathbb{E}_{Q_0, \ldots, Q_n} \left[ \sum_{\max(i+j-d) = k} v_k \left( F_1(Q_0 C_0, \ldots, Q_m C_m) \right) v_j \left( F_2(Q_m C_{m+1}, \ldots, Q_n C_n) \right) \right]
\]

\[
= \sum_{\max(i+j-d) = k} \sum_{Q_0, \ldots, Q_m} v_k \left( F_1(Q_0 C_0, \ldots, Q_m C_m) \right) \mathbb{E}_{Q_0, \ldots, Q_m} \left[ v_j \left( F_2(Q_m C_{m+1}, \ldots, Q_n C_n) \right) \right]
\]

\[
= \sum_{\max(i+j-d) = k} \sum_{\dim_{\delta}^F(k_0, \ldots, k_m) = i} v_k \left( C_0 \right) \cdots v_k \left( C_m \right) \sum_{\dim_{\delta}^F(k_{m+1}, \ldots, k_n) = j} v_{k_{m+1}} \left( C_{m+1} \right) \cdots v_{k_n} \left( C_n \right),
\]

where \((*)\) follows from the induction hypothesis.

The second case can be reduced to the first case by using De Morgan’s Laws:

\[
F(X_0, \ldots, X_n) = \neg (\neg F_1(X_0, \ldots, X_m) \land \neg F_2(X_{m+1}, \ldots, X_n))
\]

\[
= \neg (F_1' (X_0, \ldots, X_m) \land F_2' (X_{m+1}, \ldots, X_n)),
\]

where the Boolean formulas \(F_i'\) and \(F_j'\) are obtained from \(\neg F_i\) and \(\neg F_j\) by “pulling the negation all the way down to the variables” using again De Morgan’s Laws. Applying the first case, we

\[7\text{This is the place where we make explicit use of the fact that the linear transformations } T_0, \ldots , T_n \text{ in (1.19) are in } O(d).\]
obtain
\[
E \left[ v_k \left( F(\mathbf{Q}_0 C_0, \ldots, \mathbf{Q}_n C_n) \right) \right] = E \left[ v_{d-k} \left( -F(\mathbf{Q}_0 C_0, \ldots, \mathbf{Q}_n C_n) \right) \right]
\]
\[
= \sum_{\text{dim}^E_t(k_0, \ldots, k_n) = d-k} v_{k_0}(C_0) \cdots v_{k_n}(C_n) = \sum_{\text{dim}^E_t(k_0, \ldots, k_n) = k} v_{k_0}(C_0) \cdots v_{k_n}(C_n).
\]
\[
\square
\]

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APPENDIX A. STEINER FORMULAS

In the Euclidean case the Steiner formula describes the volume of the tubular neighborhood of a convex body as a polynomial in the radius with coefficients given by (rescaled) Euclidean intrinsic volumes. In a straightforward way one obtains a spherical version of this, which has no longer the exact form of a polynomial, but a form in which the monomials are replaced by the volume functions of tubes around subspheres.

Using the conic instead of the spherical viewpoint, one obtains very elegant Steiner formulas, cf. [McC13, MT14]: Let \( C \subseteq \mathbb{R}^d \) be a closed convex cone, \( M \in \mathcal{B}(\mathbb{R}^d) \), and \( \mathcal{M} \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d) \).
Then
\[
\mathbb{P}\{\|\Pi_C(g)\|^2 \geq r\} = \sum_{k=0}^{d} \mathbb{P}\{\chi_k^2 \geq r\} \nu_k(C), 
\]
(A.1)
\[
\mathbb{P}\{\Pi_C(g) \in M \text{ and } \|\Pi_C(g)\|^2 \geq r\} = \sum_{k=0}^{d} \mathbb{P}\{\chi_k^2 \geq r\} \Phi_k(C, M), 
\]
(A.2)
\[
\mathbb{P}\{\hat{\Pi}_C(g) \in \mathcal{M} \text{ and } \|\Pi_C(g)\|^2 \geq r\} = \sum_{k=0}^{d} \mathbb{P}\{\chi_k^2 \geq r\} \Theta_k(C, \mathcal{M}), 
\]
(A.3)

where \( g \in \mathbb{R}^d \) Gaussian and \( \chi_k^2 \in \mathbb{R} \) denotes a chi-squared distributed random variable with \( k \) degrees of freedom. Since \( C \) is here allowed to be any closed convex cone, these formulas should be understood that in the nonpolyhedral case the left-hand sides can be expressed in the form given by the right-hand sides, and the intrinsic volumes \( \nu_k(C) \), the curvature measures \( \Phi_k(C, M) \) and the support measures \( \Theta_k(C, \mathcal{M}) \) can be defined in this way. A proof for (A.3) can be found in [SW08, Thm. 6.5.1], and (A.2) and (A.1) follow of course from (A.3).

Using these formulas, one can prove that the support measures are additive and weakly continuous, as listed in Proposition 4.1 under (6) and (7). Details for this proof can be found in [SW08, Thm. 6.5.2]. The following lemma follows directly from the fact that (A.3) characterizes \( \Theta_k(C, \mathcal{M}) \), \( k = 0, \ldots, d \), and from the fact that \( \hat{\Pi}_C(x) \in \text{BL}(C) \) for every \( x \in \mathbb{R}^d \).

**Lemma A.1.** Let \( C, D \in \mathcal{P}(\mathbb{R}^d) \) and \( \mathcal{M} \in \hat{\mathcal{F}}(\mathbb{R}^d, \mathbb{R}^d) \). If \( \mathcal{M} \subseteq \text{BL}(C) \cap \text{BL}(D) \), then \( \Theta_k(C, \mathcal{M}) = \Theta_k(D, \mathcal{M}) \).

**Appendix B. Genericity**

If \( L_0, \ldots, L_n \subseteq \mathbb{R}^d \) are linear subspaces, then
\[
\dim(L_0 \cap \cdots \cap L_n) \geq \max\{0, \dim(L_0) + \cdots + \dim(L_n) - nd\}. 
\]
(B.1)

We say that the tuple \((L_0, \ldots, L_n)\) is in general position if this inequality is an equality for every selection of some of these spaces, i.e., if
\[
\dim(L_{i_0} \cap \cdots \cap L_{i_k}) = \max\{0, \dim(L_{i_0}) + \cdots + \dim(L_{i_k}) - kd\} \quad \text{for all} \quad 0 \leq i_0 < i_1 < \cdots < i_k \leq n.
\]

Note that \((L_0, \ldots, L_n)\) is always in general position if \( \dim(L_i) = 0 \) for some \( i \).

For polyhedral cones \( C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d) \) we say that \((C_0, \ldots, C_n)\) is in general position if \((L_0, \ldots, L_n)\) is in general position for all \( L_i \in \mathcal{L}(C_i), \ i = 0, \ldots, n \).

**Remark B.1.** Recall that \( \mathcal{L}(F) \subseteq \mathcal{L}(C) \) if \( F \) is a face of \( C \). Therefore, if \((C_0, \ldots, C_n)\) is in general position and \( F_i \) is a face of \( C_i, \ i = 0, \ldots, n \), then \((F_0, \ldots, F_n)\) is in general position as well.

**Lemma B.2.** Let \( C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d) \) and \( T_0, \ldots, T_n \in \text{GL}_d \), and define
\[
\mathcal{D}_T(C_0, \ldots, C_n) := \{(Q_0, \ldots, Q_n) \in O(d)^{n+1} : (T_0Q_0, \ldots, T_nQ_nC_n) \text{ is in general position}\}. 
\]
(B.2)

Then \( \mathcal{D}_T(C_0, \ldots, C_n) \) is an open and dense subsets of \( O(d)^{n+1} \).

**Proof.** We can write \( \mathcal{D}_T(C_0, \ldots, C_n) \) as the finite intersection
\[
\mathcal{D}_T(C_0, \ldots, C_n) = \bigcap_{L_0 \in \mathcal{L}(C_0)} \cdots \bigcap_{L_n \in \mathcal{L}(C_n)} \mathcal{D}_T(L_0, \ldots, L_n),
\]
so it suffices to show the claim for linear subspaces \( C_i = L_i, \ 0 \leq i \leq n \). Furthermore, using polarity (and replacing \( L_i \) by \( L_i^\perp \)), we can reformulate the claim so that we need to show
\[
\dim(T_0^*Q_0L_0 + \cdots + T_n^*Q_nL_n) = \min(d, d_1 + \cdots + d_n), \quad d_i := \dim(L_i)
\]
for all \((Q_0, \ldots, Q_n)\) in some open dense subset of \( O(d)^{n+1} \). By orthogonal invariance we may assume without loss of generality that \( L_i = \mathbb{R}^{d_i} \times \{0\} \). Interpreting \( T_i^*Q_i \in \mathbb{R}^{d_i \times d} \) as a matrix and defining \( A_i \in \mathbb{R}^{d_i \times d_i} \) to consist of the first \( d_i \) columns of \( T_i^*Q_i \), the claim then becomes that the matrix \((A_0 \cdots A_n)\) has full rank. This rank condition can be expressed by the nonvanishing of the Gram determinant, and it is readily checked that this determinant is nonzero for an open dense subset of \( O(d)^{n+1} \).
Lemma B.3. Let \((C_0, \ldots, C_n), C_i \in \mathcal{P}(\mathbb{R}^d)\), be in general position and let \(k > 0\). Then the \(k\)-skeleton of the intersection \(C_0 \cap \cdots \cap C_n\) is given by the disjoint union

\[ S_k(C_0 \cap \cdots \cap C_n) = \bigcup_{k_0 + \cdots + k_n = k + nd} (S_{k_0}(C_0) \cap \cdots \cap S_{k_n}(C_n)) \tag{B.3} \]

Proof. We abbreviate \(C := C_0 \cap \cdots \cap C_n\). The genericity condition (and the assumption \(k > 0\)) implies that the linear span of every \(k\)-dimensional face of \(C\), \(L \in \mathcal{L}_k(C)\), can be written in the form \(L = L_0 \cap \cdots \cap L_n\) with \(L_i \in \mathcal{L}_k(C_i)\), \(0 \leq i \leq n\), and \(k_0 + \cdots + k_n = k + nd\). Since in this case we have

\[ \text{int}_L(C \cap L) = \text{int}_{L_0}(C_0 \cap L_0) \cap \cdots \cap \text{int}_{L_n}(C_n \cap L_n) \subseteq S_{k_0}(C_0) \cap \cdots \cap S_{k_n}(C_n), \]

it follows the inequality ‘\(\leq\)’ in (B.3). On the other hand, if \(L_i \in \mathcal{L}_k(C_i)\), \(0 \leq i \leq n\), such that \(L_0 \cap \cdots \cap L_n \cap C \neq \emptyset\), then \(\dim(L_0 \cap \cdots \cap L_n) = k_0 + \cdots + k_n - nd\), which shows the reverse inclusion. \(\Box\)

Lemma B.4. Let \((C_0, \ldots, C_n) \in \mathcal{P}(\mathbb{R}^d)\) and \(T_0, \ldots, T_n \in \text{Gl}_d\) such that \((T_0C_0, \ldots, T_nC_n)\) is in general position, and let \(k > 0\). Furthermore, let \(M_0, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d)\), \(M_0, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)\), and let \(\varphi_1, \varphi_2 : O(d)^{n+1} \to \mathbb{R}\) be defined by

\[ \varphi_1(Q_0, \ldots, Q_n) = \Psi_k \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigcap_{i=0}^n T_i Q_i M_i \right), \]

\[ \varphi_2(Q_0, \ldots, Q_n) = \Theta_k \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigcap_{i=0}^n T_i Q_i \{ \text{BL}(C_i) \cap \mathcal{M}_i \} \right). \]

Then \(\varphi_1\) and \(\varphi_2\) are locally continuous around \((I_d, \ldots, I_d)\).

Proof. Since \((T_0C_0, \ldots, T_nC_n)\) is in general position, \((I_d, \ldots, I_d) \in \mathcal{D}_T(C_0, \ldots, C_n)\), cf. (B.2), and by Lemma B.2 it follows that there exists an open ball \(\mathcal{U} \subseteq \mathcal{D}_T(C_0, \ldots, C_n)\) around \((I_d, \ldots, I_d)\). As seen in Lemma B.3, all supporting subspaces \(L \in \mathcal{L}_k(C)\), \(C := \bigcap_{i=0}^n T_i Q_i C_i\), are of the form \(L = \bigcap_{i=0}^n T_i Q_i L_i\) with \(L_i \in \mathcal{L}_k(C_i)\) and \(\sum_{i=0}^n \dim(L_i) = \dim(L) + nd\). Moreover, in the open ball \(\mathcal{U} \subseteq \mathcal{D}_T(C_0, \ldots, C_n)\) the set

\[ \mathcal{L}_Q := \left\{ (L_0, \ldots, L_n) \in \mathcal{L}_k(C_0) \times \cdots \times \mathcal{L}_k(C_n) : \bigcap_{i=0}^n T_i Q_i L_i \in \mathcal{L}_k(C) \right\} \]

is constant; otherwise there would exist \((Q_0, \ldots, Q_n) \in \mathcal{U}\) such that \((T_0 Q_0, \ldots, T_n Q_n, C)\) is not in general position. In the neighborhood \(\mathcal{U}\) of \((I_d, \ldots, I_d)\) we thus have (denoting \(\mathcal{L}_U := \mathcal{L}_Q\))

\[ \varphi_1(Q_0, \ldots, Q_n) = \sum_{(L_0, \ldots, L_n) \in \mathcal{L}_U} \gamma_L \left( \bigcap_{i=0}^n T_i Q_i(C_i \cap L_i \cap M_i) \right). \]

This is easily seen to depend continuously on the \(Q_i\), so that \(\varphi_1\) is a locally continuous function. Analogously, the locally constant face structure of \(C_Q\) implies the local continuity of \(\varphi_2\). \(\Box\)

Proposition B.5. Let \(C_0, \ldots, C_n \in \mathcal{P}(\mathbb{R}^d)\), \(M_0, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d)\), \(M_0, \ldots, M_n \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}^d)\), and let \(T_0, \ldots, T_n \in \text{Gl}_d\) and \(k > 0\). Then for \(Q_0, \ldots, Q_n \in O(d)\) iid uniformly at random the following holds:

1. The expectations

\[ \mathbb{E} \left[ \Psi_k \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigcap_{i=0}^n T_i Q_i M_i \right) \right], \quad \mathbb{E} \left[ \Theta_k \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigcap_{i=0}^n T_i Q_i \{ \text{BL}(C_i) \cap \mathcal{M}_i \} \right) \right] \]

exist and are finite.

2. If \(S_j(C_0) \cap M_0 = S_j(\tilde{C}_0) \cap M_0\) for all \(j = 0, \ldots, d\), \(\tilde{C}_0 \in \mathcal{P}(\mathbb{R}^d)\), then almost surely

\[ \Psi_k \left( \bigcap_{i=0}^n T_i Q_i C_i, \bigcap_{i=0}^n T_i Q_i M_i \right) = \Psi_k \left( \bigcap_{i=0}^n T_i Q_i \tilde{C}_i, \bigcap_{i=0}^n T_i Q_i M_i \right), \]

where \(\tilde{C}_i := C_i\) for \(i = 1, \ldots, n\).
Proof. (1) Combining Lemma B.4 and Lemma B.2 with the fact that \( O(d)^{n+1} \) is compact yields the existence of the expectations. The expectations are finite, since \( \Theta_k(C, \mathcal{M}) \leq 1 \), and \( \Psi_k(C, M) \) can be upper bounded, for example, by \( \Psi_k(C, M) \leq f_k(C) \), and \( f_k(C_0 \cap \cdots \cap C_n) \leq \Pi_{i=0}^n f_j(C_i) \).

(2) To simplify notation let \( C \coloneqq \cap_{i=0}^n T_i Q_i C_i, \bar{C} \coloneqq \cap_{i=0}^n T_i Q_i C_i, M \coloneqq \cap_{i=0}^n T_i Q_i M_i \). Combining Lemma B.2 and Lemma B.3 implies that almost surely

\[
M \cap S_k(C) = M \cap \bigcup_{k_0 + \cdots + k_n = k + nd} \left( \bigcap_{i=0}^n S_k(T_i Q_i C_i) \right) = \bigcup_{k_0 + \cdots + k_n = k + nd} \left( \bigcap_{i=0}^n T_i Q_i (M \cap S_k(C_i)) \right) = M \cap S_k(\bar{C}).
\]

Therefore, the locality of the polyhedral measures, cf. Proposition 3.3(3), implies that almost surely \( \Psi_k(C, M) = \Psi_k(C, M) \ldots \)

\[
\text{Proposition B.6. Let } C, D \in \mathcal{P}(\mathbb{R}^d), \mathcal{M}, \mathcal{N}_j \in \mathcal{P}(\mathbb{R}^d, \mathbb{R}^d), j = 1, 2, \ldots, \text{ such that } \mathcal{N}_j \subseteq \text{BL}(C) \text{ for all } j, \mathcal{N}_j \cap \mathcal{N}_{j'} = \emptyset \text{ for all } j \neq j', \text{ and } \mathcal{M} \subseteq \text{BL}(D). \text{ Furthermore, let } T \in \text{GL}_d \text{ and } k > 0. \text{ Then for almost all } Q \in O(d),
\]

\[
\Theta_k(T QC \cap D, \left( \bigcup_{j=1}^n T Q N_j \right) \cdot \mathcal{M}) = \sum_{j=1}^n \Theta_k(T QC \cap D, T Q N_j \cdot \mathcal{M}). \tag{B.4}
\]

Proof. By Lemma B.2 we have that \( (T QC, D) \) is almost surely in general position. So assuming that \((C, D)\) is in general position, what we will do in the rest of the proof, it suffices to show that (B.4) holds for \( T = Q = I_d \). Using Proposition 2.4 and Lemma 4.2, we have

\[
\Theta_k(C \cap D, \left( \bigcup_{j=1}^n N_j \right) \cdot \mathcal{M}) = \sum_{L \in \mathcal{L}_k(C \cap D)} \Theta_k \left( L, \bigcup_{j=1}^n (N_j \cdot \mathcal{M}) \cap \text{BL}(L) \right)
\]

\[
= \sum_{L_0 \in \mathcal{L}_k(C), L_1 \in \mathcal{L}_k(D)} \Theta_k \left( L_0 \cap L_1, \bigcup_{j=1}^n (N_j \cdot \mathcal{M}) \cap \left( \text{BL}(L_0) \cap \text{BL}(L_1) \right) \right).
\]

where the second equality follows from the assumption that \((C, D)\) are in general position. Decomposing \( N_j = \bigcup_{L_0 \in \mathcal{L}_k(C)} (N_j \cap \text{BL}(L_0)) \) and \( \mathcal{M} = \bigcup_{L_1 \in \mathcal{L}_k(C)} (\mathcal{M} \cap \text{BL}(L_1)) \), and arguing as in Lemma 4.2,

\[
\Theta_k \left( L_0 \cap L_1, \bigcup_{j=1}^n (N_j \cdot \mathcal{M}) \cap \left( \text{BL}(L_0) \cap \text{BL}(L_1) \right) \right)
\]

\[
= \sum_{L_0 \in \mathcal{L}_k(C), L_1 \in \mathcal{L}_k(D)} \Theta_k \left( L_0 \cap L_1, \bigcup_{j=1}^n (N_j \cap \text{BL}(L_0)) \cap \left( \mathcal{M} \cap \text{BL}(L_1) \right) \right)
\]

\[
= \Theta_k \left( L_0 \cap L_1, \bigcup_{j=1}^n (N_j \cap \text{BL}(L_0)) \cap \left( \mathcal{N} \cap \text{BL}(L_1) \right) \right) = \gamma_d \left( \text{add} \left( \bigcup_{j=1}^n N_j \right) \right),
\]

where \( \cdot \mathcal{N}_j := (N_j \cap \text{BL}(L_0)) \cap (\mathcal{M} \cap \text{BL}(L_1)) \) and add: \( \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) is given by \( \text{add}(x, x') = x + x' \), cf. (2.16). Since \( \mathbb{R}^d = (L_0 \cap L_1) + L_0^1 + L_1^1 \) with \( \text{dim}(L_0 \cap L_1) + \text{dim} L_0^1 + \text{dim} L_1^1 = d \), every element \( z \in \mathbb{R}^d \) can be uniquely written as \( z = x + y_0 + y_1 \) with \( x \in (L_0 \cap L_1), y_0 \in L_0^1, y_1 \in L_1^1 \). This shows that \( \cdot \mathcal{N}_j \) are mutually disjoint as \( \cdot \mathcal{N}_j \) are, which implies

\[
\gamma_d \left( \text{add} \left( \bigcup_{j=1}^n N_j \right) \right) = \sum_{j=1}^n \gamma_d \left( \text{add}(\cdot \mathcal{N}_j) \right).
\]
Putting things together, we obtain
\[
\Theta_k \left( C \cap D, \left( \bigcup_{j=1}^{\infty} N_j \right) \wedge \mathcal{M} \right) = \sum_{L_0 \in L_0(C), i_1 \in L_1(D)} \sum_{j=1}^{\infty} \Theta_k \left( L_0 \cap L_1, \left( N_j \cap BL(L_0) \right) \wedge \left( \mathcal{M} \cap BL(L_1) \right) \right) \\
= \sum_{j=1}^{\infty} \Theta_k \left( C \cap D, N_j \wedge \mathcal{M} \right).
\]

APPENDIX C. CHARACTERIZATION OF CURVATURE MEASURES

The characterization of the curvature measures, Theorem 3.6, relies on a characterization of the spherical Lebesgue measure by Schneider [Sch78, Thm. 6.2]. We restate this result in the conic setting and for completeness also include the proof, thereby correcting a minor inaccuracy. The following fact forms an integral part of Schneider’s proof.

**Fact C.1.** Let \( f : O(d) \rightarrow \mathbb{R} \) (Haar) measurable, and let \( \varepsilon > 0 \). Then there exist \( U_1, \ldots, U_{n_c} \in O(d) \) such that for all \( U \in O(d) \)
\[
\left| \mathbb{E} \left[ f(Q) \right] - \frac{1}{n_c} \sum_{i=1}^{n_c} f(U_i U) \right| < \varepsilon,
\]
where \( Q \in O(d) \) uniformly at random.

This fact follows from von Neumann’s construction of the Haar integral, which is described in detail in [Pon39, §25] for continuous functions. The generalization to measurable functions follows from a simple approximation procedure.

The following proposition, which says that (up to scaling) the Gaussian measure is the only nonnegative additive functional on \( \mathcal{P}(\mathbb{R}^d) \), which is orthogonal invariant and vanishes on lower-dimensional cones, is from Schneider [Sch78, Thm. 6.2].

**Proposition C.2.** Let \( \mu : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R}_+ \) such that for all \( U \in O(d), C, D \in \mathcal{P}(\mathbb{R}^d) \)

1. \( \mu(U C) = \mu(C) \),
2. \( \mu(C) = 0 \), if \( \text{int}(C) = \emptyset \),
3. \( \mu(C \cup D) + \mu(C \cap D) = \mu(C) + \mu(D) \), if \( C \cup D \in \mathcal{P}(\mathbb{R}^d) \).

Then \( \mu = \mu(\mathbb{R}^d) \gamma_d \).

**Proof.** Note that properties (1)–(3) are preserved when \( \mu \) is extended via additivity to finite unions of polyhedral cones. Furthermore, if \( C \subseteq D \), then the nonnegativity of \( \mu \) implies that \( \mu(C) \leq \mu(D) \), as \( D \) can be filled up with \( C \) and further polyhedral cones, such that pairwise their interiors do not intersect.

Now, let \( \varepsilon > 0 \), let \( x_0 \in \mathbb{R}^d \setminus \{0\} \), and fix a cone \( C \in \mathcal{P}(\mathbb{R}^d) \) with nonempty interior. Applying Fact C.1 to the function
\[
f \colon O(d) \rightarrow \mathbb{R}, \quad f(U) := \begin{cases} 
1 & \text{if } Ux_0 \in C \\
0 & \text{else}
\end{cases}
\]
yields the existence of \( U_1, \ldots, U_{n_c} \in O(d) \) such that for all \( U \in O(d) \)
\[
\varepsilon > \left| \mathbb{E} \left[ f(Q) \right] - \frac{1}{n_c} \sum_{i=1}^{n_c} f(U_i U) \right| = \gamma_d(C) - \frac{N(Ux_0)}{n_c}, \quad N(x) := |\{i : x \in U_i C\}|.
\]
For \( I \subseteq \{1, \ldots, n_c\} \) let \( C_I := \bigcap_{i \in I} U_i C \). Furthermore, for \( 1 \leq k \leq n_c \) let
\[
M_k := \bigcup_{I \subseteq \{1, \ldots, n_c\} \mid |I| = k} C_I = \{x \in \mathbb{R}^d : N(x) \geq k\}.
\]
Note that $M_k \equiv M_\ell$ for $k \leq \ell$. As $\mu$ is orthogonal invariant (1) and simple (2),

$$\mu(C) = \frac{1}{n_\epsilon} \sum_{i=1}^{n_\epsilon} \mu(U_{i,C}) = \frac{n_\epsilon \mu(M_{n_\epsilon}) + (n_\epsilon - 1)\mu(M_{n_\epsilon-1}) - \mu(M_{n_\epsilon}) + \cdots + \mu(M_1) - \mu(M_2))}{n_\epsilon}$$

$$= \frac{1}{n_\epsilon} \sum_{k=1}^{n_\epsilon} \mu(M_k). \tag{C.2}$$

Let $x_{\text{min}} \in \arg \min \{N(x) : x \in \mathbb{R}^d \setminus \{0\}\}$ and $x_{\text{max}} \in \arg \max \{N(x) : x \in \mathbb{R}^d \setminus \{0\}\}$. Then $M_k = \mathbb{R}^d$ for all $k < N(x_{\text{min}})$ and $M_k = \emptyset$ for all $k > N(x_{\text{max}})$. The nonnegativity and monotonicity of $\mu$ implies

$$\left(\frac{N(x_{\text{min}}) - 1}{n_\epsilon} \mu(\mathbb{R}^d)\right) \leq \frac{1}{n_\epsilon} \sum_{k=1}^{N(x_{\text{max}})} \mu(M_k) \leq \frac{N(x_{\text{max}})}{n_\epsilon} \mu(\mathbb{R}^d).$$

It follows that

$$\frac{\mu(C)}{\mu(\mathbb{R}^d)} \in \left[\frac{N(x_{\text{min}}) - 1}{n_\epsilon}, \frac{N(x_{\text{max}})}{n_\epsilon}\right] \subseteq [\gamma_d(C) - \varepsilon, \gamma_d(C) + \varepsilon].$$

To finish the proof it remains to show that for $C \in \mathcal{P}(\mathbb{R}^d)$ with $\text{int}(C) \neq \emptyset$ and $C \neq \mathbb{R}^d$ the application of Fact C.1 in (C.1) yields a dependence of $n_\epsilon$ on $\varepsilon$ such that $n_\epsilon \to \infty$ as $\varepsilon \to 0$. Assume on the contrary that $n_\epsilon \leq N$ for all $\varepsilon > 0$. Using the compactness of the orthogonal group and applying the Bolzano-Weierstrass Theorem we find elements $U_1, \ldots, U_N$ such that $\{i : U_0 \in U_i \cap C\} = N \gamma_d(C)$ for all $U \in O(d)$. In other words, every nonzero point in $\mathbb{R}^d$ lies in exactly the same number of sets $U_i C$. Assuming that $d \geq 2$ (the other cases are trivial), the closedness of these sets yields a contradiction: let $c : [0, 1] \to \mathbb{R}^d$ be a curve which misses the origin and connects $c(0) \in C \setminus \bigcup_{i=1}^{N} U_i \delta C$ with $c(1) \in \mathbb{R}^d \setminus C$. The index set $I(t) := \{i : c(t) \in U_i \cap C\}$ satisfies $I(t) = I(0)$ for small enough $t > 0$, and $I(1) \neq I(0)$. Setting $t_{\text{inf}} := \inf\{t : I(t) \neq I(0)\}$, the closedness of $U_i C$ implies $|I(t_{\text{inf}})| \neq |I(0)|$, which is a contradiction. \hfill \Box

**Proof of Theorem 3.6.** Let $\psi : \mathcal{P}(\mathbb{R}^d) \times \mathcal{H}(\mathbb{R}^d) \to \mathbb{R}_+$ satisfy the assumptions (0)–(4), i.e., $\psi(C, \cdot)$ is a conic measure, which is concentrated on $C$, orthogonal invariant, weakly local, and additive in $C$. Since every conic measure can be written as the sum of a lifted spherical measure and a scaled Dirac measure, cf. Section 2.2, we have for $C \in \mathcal{P}(\mathbb{R}^d)$,

$$\psi(C, \cdot) = h(C) \Lambda + \psi(C, \cdot), \tag{C.3}$$

with $h(C) = \psi(C, \{0\})$, and where $\psi(C, \cdot)$ satisfies $\psi(C, M) = \psi(C, M_\ast) = \psi(M, \cdot) \geq 0$ and, by the concentration property, $\psi(\{0\}, M) = 0$. The function $\psi$ will be decomposed into $\Phi_1, \ldots, \Phi_d$; this is the nontrivial part of the theorem. We proceed by induction on $d$.

For $d = 1$ every conic set is a union of some of the sets $\{0\}, \mathbb{R}_+, \mathbb{R}_{-}$, so that we need to show $\psi(C, M) = \psi(\mathbb{R}_+, \mathbb{R}_-) \Phi_1(C, M)$ only for $M \in \{\{0\}, \mathbb{R}_+, \mathbb{R}_-\}$. For $\Phi_1$ we have the following values on these sets

| $\Phi_1(C, M)$ | $M = \{0\}$ | $M = \mathbb{R}_+$ | $M = \mathbb{R}_-$ |
|----------------|-------------|-----------------|-----------------|
| $C = \{0\}$    | 0           | 0               | 0               |
| $C = \mathbb{R}_+$ | 0           | 1/2             | 0               |
| $C = \mathbb{R}_-$ | 0           | 0               | 1/2             |
| $C = \mathbb{R}$  | 0           | 1/2             | 1/2             |

As stated above, we have $\psi(C, \{0\}) = \psi(\{0\}, M) = 0$. The concentration property further implies $\psi(\mathbb{R}_+, \mathbb{R}_-) = \psi(\mathbb{R}_-, \mathbb{R}_{++}) = 0$, and, by orthogonal invariance, $\psi(\mathbb{R}_+, \mathbb{R}_{++}) = \psi(\mathbb{R}_+, \mathbb{R}_{--}) = \frac{1}{2} \psi(\mathbb{R}, \mathbb{R}_+) = \frac{1}{2} \psi(\mathbb{R}, \mathbb{R}_-)$. The additivity property finally yields $\psi(\mathbb{R}_+, \mathbb{R}_{++}) = \psi(\mathbb{R}, \mathbb{R}_{++}) - \psi(\mathbb{R}, \mathbb{R}_{--}) = \psi(\{0\}, \mathbb{R}_+) = \psi(\mathbb{R}, \mathbb{R}_{++})$, and similarly $\psi(\mathbb{R}_-, \mathbb{R}_{--}) = \psi(\mathbb{R}, \mathbb{R}_{--})$. This settles the case $d = 1$.

Let $d > 1$ and define for a hyperplane $H \subset \mathbb{R}^d$ the restriction

$$\psi_H : \mathcal{P}(\mathbb{R}^d) \times \mathcal{H}(\mathbb{R}^d) \to \mathbb{R}_+, \quad \psi_H(M) := \psi(C \cap H, M \cap H).$$

\[\text{At this point there is a small inaccuracy in the proof in [Sch78], as } M_k = \mathbb{R}^d \text{ does not necessarily hold for } k = N(x_{\text{min}}).\]
Interpreting $\hat{\psi}_H$ as conic functional on $\mathcal{P}(H)$, we may apply the induction hypothesis and conclude that $\hat{\psi}_H$ is a linear combination of $\Phi_1, \ldots, \Phi_{d-1}$. Orthogonal invariance implies that the constants do not depend on the specific hyperplane $H$, so that we obtain

$$\hat{\psi}_H = a_1\Phi_1 + \cdots + a_{d-1}\Phi_{d-1}$$

for some constants $a_1, \ldots, a_{d-1} \geq 0$. More precisely, $a_k = \psi(L^{(k)}_*, L^{(k)}_0)$ where $L_k \subset \mathbb{R}^d$ a $k$-dimensional linear subspace. Let

$$\bar{\psi} := \hat{\psi} - a_1\Phi_1 - \cdots - a_{d-1}\Phi_{d-1}.$$ 

It remains to show that $\bar{\psi}(C, M) = a_d\Phi_d(C, M) = a_d\gamma_d(C \cap M)$.

Since the boundary $\partial C$ of $C$ is contained in a finite union of hyperplanes, the functional $\bar{\psi}(C, \cdot)$ vanishes there. Furthermore, $\bar{\psi} \geq 0$ as

$$\bar{\psi}(C, M) = \bar{\psi}(C, M \cap \partial C) + \bar{\psi}(C, M \setminus \partial C) = \bar{\psi}(C, M \setminus \partial C) \geq 0.$$ 

The functional $\bar{\psi}(\cdot, M)$ is additive and simple, and for $M = \mathbb{R}^d$ it is also orthogonal invariant. Proposition C.2 implies that $\bar{\psi}(\cdot, \mathbb{R}^d)$ is a nonnegatively weighted Gaussian measure, i.e., for all $C \in \mathcal{P}({\mathbb{R}^d})$

$$\bar{\psi}(C, C) = \bar{\psi}(C, \mathbb{R}^d) = \bar{\psi}(\mathbb{R}^d, \mathbb{R}^d) \gamma_d(C) = \psi(\mathbb{R}^d, \mathbb{R}^d) \gamma_d(C) = a_d\Phi_d(C, C)$$

with $a_d \geq 0$. By locality, it follows that $\bar{\psi}(C, M) = a_d\Phi_d(C, M)$ if $M$ is a finite union of polyhedral cones. The set of finite unions of polyhedral cones is closed under finite intersections. It follows, cf. [Bog07, Lem. 1.9.4], that $\bar{\psi}(C, M) = a_d\Phi_d(C, M)$ for all $M$ in the $\sigma$-algebra generated by the set of finite unions of polyhedral cones, and this is $\mathcal{B}(\mathbb{R}^d)$. \qed