On the quantum superalgebra $U_q(gl(m, n))$
and its representations at roots of 1

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1 Introduction

Let $p$ be a prime, and let $\eta$ be a $p$th root of unity in the complex field $\mathbb{C}$. Let $\mathfrak{g}$ be the Lie algebra of a semisimple and simply connected algebraic group $G$ defined over $\mathbb{F}_p$. Lusztig conjectured ([14]) that the representation theory of $G$ and that of $U_\eta(\mathfrak{g})$ are the same under certain range of weights. The conjecture is now known to hold for $p$ larger than certain number (see [1, 6]).

The main purpose of the article is to extend the above conjecture to the super case where $\mathfrak{g}$ is the general linear Lie superalgebra $gl(m, n)$ and $G$ is the linear algebraic supergroup $GL(m, n)$. The definition in [3, 13] states that $G$ is the functor from the category of commutative superalgebras to the category of groups defined on a commutative superalgebra $A$ by letting $G(A)$ be the group of all invertible $(m+n) \times (m+n)$ matrices of the form

$$g = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where $W$ is an $m \times m$ matrix with entries in $A_0$, $X$ is an $m \times n$ matrix with entries in $A_1$, $Y$ is an $n \times m$ with entries in $A_1$, and $Z$ is an $n \times n$ matrix with entries in $A_0$. The relation between the category of $G$-modules and category of $\mathfrak{g}$-modules given in [3, 13] is similar to the one between the category of modules for algebraic groups and the category of modules for their Lie algebras [9].

We adopt here the definition of the quantum supergroup $U_q(gl(m, n))$ from [19]. First we prove the PBW theorem, then we give a description of the $A$-form $U_A$ in terms of generators and relations following [14]. With these results, we propose a conjecture for the super case, and prove that the conjecture follows from the
Lusztig’s conjecture provided that the highest weight is \( p \)-typical (see Sec.2.2 for definition).

The paper is arranged as follows. Sec.2 is the preliminaries. In Sec. 3, we study the algebra \( \tilde{U}_q \). Sec 4 is about the relations in \( U_q(gl(m, n)) \). In Sec. 5, we study the highest weight simple modules for \( U_q(gl(m, n)) \). In Sec. 6, we define the \( \mathcal{A} \)-form \( U_{\mathcal{A}} \) for the quantum supergroup \( U_q(gl(m, n)) \), using which we prove in Sec. 7 the PBW theorem. In Sec.8, we give a description of \( U_{\mathcal{A}} \) in terms of generators and relations. In Sec.9, we extend the Lusztig’s conjecture to the super case and prove that the conjecture follows from the Lusztig’s conjecture in case of a \( p \)-typical weight. In Sec.10, we prove the Lusztig’s tensor product theorem for \( U_q(gl(m, n)) \).

## 2 Preliminaries

### 2.1 Notation

Throughout the paper we use the following notation.

\[
\begin{align*}
[1, m + n] &= \{1, 2, \cdots, m + n - 1\}, \\
[1, m + n] &= \{1, 2, \cdots, m + n\}, \\
[0, l] &= \{0, 1, \cdots, l - 1\}, \\
[0, l]^{m+n} &= \text{the set of all } m + n\text{-tuples } z = (z_1 \cdots z_{m+n}) \text{ with } 0 \leq z_i < l \text{ for all } i = 1, \cdots, m+n \\
\mathcal{I}_0 &= \{(i,j) | 1 \leq i < j \leq m \text{ or } m + 1 \leq i < j \leq m + n\} \\
\mathcal{I}_1 &= \{(i,j) | 1 \leq i \leq j \leq m + n\} \\
\mathcal{I} &= \mathcal{I}_0 \cup \mathcal{I}_1 \\
\mathcal{A}^B &= \text{the set of all tuples } \psi = (\psi_{ij})_{(i,j)\in B} \text{ with } \psi_{ij} \in \mathcal{A}, \text{ where } B = \mathcal{I}_0 \text{ or } B = \mathcal{I}_1 \\
\mathcal{A} &= \mathbb{Z}[q, q^{-1}] \text{ where } q \text{ is an indeterminate} \\
\mathcal{A}' &= Q(q) \text{ the quotient field of } \mathcal{A} \\
h(V) &= \text{the set of all homogeneous elements in a } \mathbb{Z}_2\text{-graded vector space} \\
V = V_0 \oplus V_1 &= \text{the universal enveloping superalgebra for the Lie superalgebra } L. \\
\bar{x} &= \text{the parity of the homogeneous element } x \in V = V_0 \oplus V_1. \\
U(L) &= \text{the universal enveloping superalgebra for the Lie superalgebra } L.
\end{align*}
\]

Note: Subalgebras of a superalgebra and modules over a superalgebra will be assumed to be \( \mathbb{Z}_2\)-graded.
2.2 The quantum deformation of $gl(m, n)$

The Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has a basis $\{e_{ij} \mid 1 \leq i, j \leq m + n\}$. We denote $e_{ij}$ with $i < j$ also by $f_{ij}$. Then we get $\mathfrak{g}_1 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_1 = \langle e_{ij} \mid (i, j) \in \mathcal{I}_1 \rangle, \quad \mathfrak{g}_{-1} = \langle f_{ij} \mid (i, j) \in \mathcal{I}_1 \rangle.$$ 

Denote by $\mathfrak{g}^+$ the subalgebra $\mathfrak{g}_0 + \mathfrak{g}_1$ of $\mathfrak{g}$. Let $H = \langle e_{ii} \mid 1 \leq i \leq m + n \rangle$, and let $T$ be the linear algebraic group consisting of $(m + n) \times (m + n)$ invertible diagonal matrices. Then we have $\text{Lie}(T) = H$. The set of positive roots of $\mathfrak{g}$ relative to $T$ is $\Phi^+ = \Phi^+_0 \cup \Phi^+_1$, where

$$\Phi^+_0 = \{ \epsilon_i - \epsilon_j \mid (i, j) \in \mathcal{I}_0 \}, \quad \Phi^+_1 = \{ \epsilon_i - \epsilon_j \mid (i, j) \in \mathcal{I}_1 \}.$$ 

Let $\Lambda =: X(T) = \mathbb{Z} e_1 + \mathbb{Z} e_2 + \cdots + \mathbb{Z} e_{m+n}$. Let $\check{e}_i$ be the 1-psg: $G_m \to T$ such that each $t \in G_m$ is mapped into a diagonal matrix with all entries equal to 1 but the $i$th equal to $t$ if $i \leq m$, and $t^{-1}$ if $i > m$. Then the 1-psg's $\check{e}_i$ form a $\mathbb{Z}$-basis of $Y(T)$. The nondegenerate paring $X(T) \times Y(T) \to \mathbb{Z}$: $(\lambda, \mu) \mapsto \langle \lambda, \mu \rangle$ induces a symmetric bilinear form on $\Lambda$ defined by (see [7])

$$(\epsilon_i, \epsilon_j) = \langle \epsilon_i, \check{e}_j \rangle = \begin{cases} \delta_{ij}, & i \leq m \\ -\delta_{ij}, & i > m. \end{cases}$$

Let

$$\rho_0 = 1/2 \sum_{\alpha \in \Phi^+_0} \alpha, \quad \rho_1 = 1/2 \sum_{\alpha \in \Phi^+_1} \alpha,$$

and set $\rho =: \rho_0 - \rho_1 \in \Lambda$. Defining $P(\lambda) = \prod_{\alpha \in \Phi^+_1} (\lambda + \rho, \alpha)$ for $\lambda \in \Lambda$, we have $P(\lambda) \in \mathbb{Z}$ for each $\lambda \in \Lambda$. An element $\lambda \in \Lambda$ is called typical (resp. $p$-typical) if $P(\lambda) \neq 0$ ($P(\lambda) \notin p\mathbb{Z}$).

Let $\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{m+n} e_{m+n} \in \Lambda$. For each $(i, j) \in \mathcal{I}_1$, set $c(i, j) = i + j - 2m - 1$. A direct computation shows that $\lambda$ is typical if and only if

$$\lambda_i + \lambda_j \neq c(i, j)$$

for all $(i, j) \in \mathcal{I}_1$.

Note: (1) For each $\mu \in \mathbb{Z}^+$, there is a typical weight $\lambda = \sum \lambda_i e_i \in \Lambda$ such that $\lambda_i - \lambda_{i+1} \geq \mu$ for all $i \in [1, m+n) \setminus m$. First, let $\lambda_{m+i} = (n-i)\mu$, for $i = 1, \ldots, n$. To choose $\lambda_i$ for $i \in [1, m]$, we proceed by induction on $i$. Let $\lambda_m$ be such that $\lambda_m + \lambda_{m+i} \neq c(m, m+i)$ for all $i \in [1, n]$. Assume we have chosen $\lambda_i$ for $1 < i \leq m$. Let $\lambda_{i-1}$ be such that $\lambda_{i-1} + \lambda_{m+j} \neq c(i-1, m+j)$ for all $j \in [1, n]$ and $\lambda_{i-1} \geq \lambda_i + \mu$. Then we obtain $\lambda \in \Lambda$ as desired.

(2) Assume the Lie superalgebra $\mathfrak{g}$ is defined over a field $k$. By identifying $H^* \otimes_{\mathbb{Z}} k$, the bilinear form on $\Lambda$ is extended naturally to $H^*$. For each $\lambda \in \Lambda$, we denote $\lambda \otimes 1 \in \Lambda \otimes_{\mathbb{Z}} k = H^*$ by $\bar{\lambda}$. 

3
Put 
\[ h_{\alpha_i} = e_{ii} - (-1)^{\delta_{im}} e_{i+1,i+1}, \quad e_{\alpha_i} = e_{i,i+1}, \quad f_{\alpha_i} = e_{i+1,i} \]
for \( i \in [1, m+n] \), as well as \( h_{\alpha_m} = e_{m+n,m+n} \). The Distinguished Cartan matrix (see [5, p.344]) of the Lie superalgebra \( \mathfrak{g} \) is \( A = (a_{ij})_{1 \leq i,j \leq m+n-1} \) with

\[
a_{ij} = \begin{cases} 
2, & \text{if } i = j \neq m \\
1, & \text{if } (i, j) = (m, m+1) \\
-1, & \text{if } |i-j| = 1 \text{ and } (i,j) \neq (m, m+1) \\
0, & \text{otherwise.} 
\end{cases}
\]

The augmented Cartan matrix, denoted by \( \tilde{A} \), is the \((m+n) \times (m+n-1)\) matrix whose first \( m+n-1 \) rows are the rows of \( A \) and whose last row is \((0, \ldots, 0, -1)\).

Let \( U(\mathfrak{g}) \) be the universal enveloping superalgebra of \( \mathfrak{g} = gl(m,n) \). The Serre-type relations for the universal enveloping superalgebra of the special linear superalgebra \( sl(m,n) \) are given in [17], from which one can easily show that \( U(\mathfrak{g}) \) is generated by the elements \( e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_j}, i \in [1, m+n], j \in [1, m+n] \) and relations

\[
\begin{align*}
(a1) \quad & h_{\alpha_i} h_{\alpha_j} = h_{\alpha_j} h_{\alpha_i} \\
(a2) \quad & h_{\alpha_i} e_{\alpha_j} - e_{\alpha_j} h_{\alpha_i} = a_{ij} e_{\alpha_j}, \quad h_{\alpha_i} f_{\alpha_j} - f_{\alpha_j} h_{\alpha_i} = -a_{ij} f_{\alpha_j}, \\
(a3) \quad & e_{\alpha_i} f_{\alpha_j} - (-1)^{\delta_{im}} f_{\alpha_j} e_{\alpha_i} = \delta_{ij} h_{\alpha_i}, \\
(a4) \quad & e_{\alpha_i} e_{\alpha_j} = e_{\alpha_j} e_{\alpha_i}, \quad f_{\alpha_i} f_{\alpha_j} = f_{\alpha_j} f_{\alpha_i}, \quad \text{if } |i-j| > 1, \\
(a5) \quad & e_{\alpha_i}^2 e_{\alpha_j} - 2 e_{\alpha_i} e_{\alpha_j} e_{\alpha_i} + e_{\alpha_j} e_{\alpha_i}^2 = 0, \text{if } |i-j| = 1, i \neq m, \\
(a6) \quad & f_{\alpha_i}^2 f_{\alpha_j} - 2 f_{\alpha_i} f_{\alpha_j} f_{\alpha_i} + f_{\alpha_j} f_{\alpha_i}^2 = 0, \text{if } |i-j| = 1, i \neq m, \\
(a7) \quad & e_{\alpha_i}^2 = f_{\alpha_i}^2 = 0, \\
(a8) \quad & [e_{\alpha_m}, [e_{\alpha_{m-1}}, e_{\alpha_m}, e_{\alpha_{m+1}}]] = 0, [f_{\alpha_m}, [f_{\alpha_{m-1}}, f_{\alpha_m}, f_{\alpha_{m+1}}]] = 0.
\end{align*}
\]

Let \( M \) be a \( U(\mathfrak{g}_0) \)-module. For \( \mu \in H^* \), define the \( \mu \)-weight space of \( M \) by

\[ M_\mu = \{ m \in M \mid h m = \mu(h) m \quad \text{for all}\quad h \in H \}. \]

A nonzero vector \( v^+ \in M_\mu \) is said to be maximal if \( e_{ij} v^+ = 0 \) for all \( (i, j) \in I_0 \).

Let \( M(\lambda) \) be a simple \( U(\mathfrak{g}_0) \)-module generated by a maximal vector of weight \( \lambda \in H^* \). We can view \( M_0(\lambda) \) as a \( U(\mathfrak{g}^+) \)-module by letting \( \mathfrak{g}_1 \) act trivially on it. Then the induced \( U(\mathfrak{g}) \)-module

\[ \mathcal{K}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} M_0(\lambda) \]

is called a Kac module. In case \( \mathfrak{g} \) is defined over \( \mathbb{C} \), [10, Prop. 2.9] says that \( \mathcal{K}(\lambda) \) is simple if and only if \( \lambda \) is typical.
Let $\mathbb{F}$ be a field with char. $\mathbb{F} = 0$, and let $q$ be an indeterminate over $\mathbb{F}$. Then the quantum supergroup $U_q(\mathfrak{g})$ (see [19, p.1237]) is defined to be the $\mathbb{F}(q)$-superalgebra with the generators $K_j, K_j^{-1}, j \in [1, m+n], E_{i,i+1}, F_{i,i+1}, i \in [1, m+n]$, and relations

\begin{align*}
(R1) \quad & K_i K_j = K_j K_i, K_i K_i^{-1} = 1, \\
(R2) \quad & K_i E_{j,j+1} K_i^{-1} = q_i^{(\delta_{ij} - \delta_{i,j+1})} E_{j,j+1}, \quad K_i F_{j,j+1} K_i^{-1} = q_i^{-(\delta_{ij} - \delta_{i,j+1})} F_{j,j+1}, \\
(R3) \quad & [E_{i,i+1}, F_{j,j+1}] = \delta_{ij} \frac{K_i K_i^{-1} - K_i^{-1} K_i}{q_i - q_i^{-1}}, \\
(R4) \quad & E_{m,m+1}^2 = F_{m,m+1}^2 = 0, \\
(R5) \quad & E_{i,i+1} E_{j,j+1} + E_{j,j+1} E_{i,i+1} = F_{j,j+1} F_{i,i+1} + F_{i,i+1} F_{j,j+1}, \quad |i - j| > 1, \\
(R6) \quad & E_{i,i+1}^2 E_{j,j+1} - (q + q^{-1}) E_{i,i+1} E_{j,j+1} E_{i,i+1} + E_{j,j+1} E_{i,i+1}^2 = 0 \quad (|i - j| = 1, i \neq m), \\
(R7) \quad & F_{i,i+1}^2 F_{j,j+1} - (q + q^{-1}) F_{i,i+1} F_{j,j+1} F_{i,i+1} + F_{j,j+1} F_{i,i+1}^2 = 0 \quad (|i - j| = 1, i \neq m), \\
(R8) \quad & [E_{m-1,m+2}, E_{m,m+1}] = [F_{m-1,m+2}, F_{m,m+1}] = 0,
\end{align*}

where

$$q_i = \begin{cases} q, & \text{if } i \leq m \\ q^{-1}, & \text{if } i > m. \end{cases}$$

Most often, we shall use $E_{\alpha_i}$ (resp. $F_{\alpha_i}; K_{\alpha_i}$) to denote $E_{i,i+1}$ (resp. $F_{i,i+1}; K_i K_i^{-1}$) for $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Remark: (1) For each pair of indices $(i, j)$ with $1 \leq i < j \leq m+n$, the notation $E_{ij}, F_{ij}$ are defined by (see [19])

\begin{align*}
E_{ij} &= E_{ic} E_{cj} - q_c^{-1} E_{cj} E_{ic}, \quad i < c < j, \\
F_{ij} &= -q_c E_{ic} F_{cj} + F_{cj} E_{ic}.
\end{align*}

(2) The parity of the elements $E_{ij}, F_{ij}, K_{s}^{\pm 1}$ is defined by $E_{ij} = F_{ij} = \tilde{e}_{ij} \in \mathbb{Z}_2$, $K_{s}^{\pm 1} = 0$.

(3) The bracket product (denote $[,]$ in [19]) in $U_q(\mathfrak{g})$ is defined by

\begin{equation}
[x, y] = xy - (-1)^{\tilde{e}_{ij}y}yx, \quad x, y \in h(U_q(\mathfrak{g})).
\end{equation}

Therefore the relations (R8) can be written as (see [12])

\begin{align*}
E_{\alpha_{m-1}} E_{\alpha_m} E_{\alpha_{m+1}} E_{\alpha_m} + E_{\alpha_{m+1}} E_{\alpha_m} E_{\alpha_{m+1}} + E_{\alpha_{m-1}} E_{\alpha_m} E_{\alpha_{m+1}} E_{\alpha_m} + E_{\alpha_{m-1}} E_{\alpha_{m+1}} E_{\alpha_m} E_{\alpha_{m+1}} + E_{\alpha_{m+1}} E_{\alpha_m} E_{\alpha_{m+1}} E_{\alpha_m} &= 0, \\
F_{\alpha_{m-1}} F_{\alpha_m} F_{\alpha_{m+1}} F_{\alpha_m} + F_{\alpha_{m+1}} F_{\alpha_m} F_{\alpha_{m+1}} + F_{\alpha_{m-1}} F_{\alpha_m} F_{\alpha_{m+1}} + F_{\alpha_{m+1}} F_{\alpha_m} F_{\alpha_{m+1}} F_{\alpha_m} + F_{\alpha_{m+1}} F_{\alpha_m} F_{\alpha_{m+1}} F_{\alpha_m} &= 0.
\end{align*}

Let $U_q(\mathfrak{g}_{0})$ be the quantum deformation of the Lie algebra $\mathfrak{g}_{0}$ defined by the even generators $E_{\alpha_i}, F_{\alpha_i}, i \in [1, m+n] \setminus m, K_{s}^{\pm 1}, j \in [1, m+n]$ and relations (R1)-(R3), (R5)-(R7) involving only even generators. Then it is easy to see that

$$U_q(\mathfrak{g}_{0}) = U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n).$$

In the remainder of the paper, we shall abbreviate $U_q(\mathfrak{g})$ to $U_q$. 

5
3 The Hopf superalgebra $\hat{U}_q$

Let $\hat{U}_q$ be the $\mathbb{F}(q)$-superalgebra defined by the generators $E_{i,i+1}, F_{i,i+1}, K_j, K_j^{-1}$ (for all $i \in [1, m+n], j \in [1, m+n]$) and relations (R1)-(R3). Then $U_q$ is a quotient of $\hat{U}_q$. Denote by $\hat{U}_q^+$ (resp. $\hat{U}_q^-$) the subalgebra of $\hat{U}_q$ generated by the elements $E_{i,i+1}$ (resp. $F_{i,i+1}$; $K_j^\pm 1$), $i \in [1, m+n], j \in [1, m+n]$. We use notation like $E_{ij}, F_{ij}$, etc, for the corresponding elements in $\hat{U}_q$ and $U_q$; it will be clear from the context what is meant.

A bijective (even)$\mathbb{F}$-linear map $f$ from a $\mathbb{F}$-superalgebra $A$ into itself is called an anti-automorphism (resp. $\mathbb{Z}_2$-graded anti-automorphism) if $f(xy) = f(y)f(x)$ (resp. $f(xy) = (-1)^{xy} f(y)f(x)$) for any $x, y \in h(A)$.

Remark: (1) Let $A = A_0 \oplus A_1$ be a superalgebra. Then the product in $A \otimes A$ is given by

\[(a \otimes b)(c \otimes d) = (-1)^{bc} ac \otimes bd, a, b, c, d \in h(A).\]

(2) In a Hopf superalgebra $A = A_0 \oplus A_1$, the antipode $S$ is a $\mathbb{Z}_2$-graded anti-automorphism.

**Lemma 3.1.** There is on $\hat{U}_q$ a unique structure $(\Delta, \epsilon, S)$ of a Hopf superalgebra such that for all $i \in [1, m+n], j \in [1, m+n]$

\[
\begin{align*}
\Delta(E_{\alpha_i}) &= E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i}, \\
\Delta(F_{\alpha_i}) &= F_{\alpha_i} \otimes 1 + K_{\alpha_i}^{-1} \otimes F_{\alpha_i}, \\
\Delta(K_j) &= K_j \otimes K_j; \\
S(E_{\alpha_i}) &= -E_{\alpha_i}K_{\alpha_i}^{-1}, \\
S(F_{\alpha_i}) &= -K_{\alpha_i}F_{\alpha_i}, \\
S(K_j) &= K_j^{-1}; \\
\epsilon(E_{\alpha_i}) &= \epsilon(F_{\alpha_i}) = 0, \epsilon(K_j) = \epsilon(K_j^{-1}) = 1.
\end{align*}
\]

**Proof.** To prove the lemma, one needs show that relations (R1)-(R3) are satisfied by the images of the generators under the homomorphisms $\Delta, \epsilon, S$. We first prove the case $i = j = m$ in (R3).

We have

\[
\begin{align*}
\Delta([E_{\alpha_m}, F_{\alpha_m}]) &= [E_{\alpha_m} \otimes K_{\alpha_m} + 1 \otimes E_{\alpha_m}, F_{\alpha_m} \otimes 1 + K_{\alpha_m}^{-1} \otimes F_{\alpha_m}] \\
&= [E_{\alpha_m} \otimes K_{\alpha_m}, F_{\alpha_m} \otimes 1] + [1 \otimes E_{\alpha_m}, F_{\alpha_m} \otimes 1] \\
&\quad + [E_{\alpha_m} \otimes K_{\alpha_m}^{-1}, K_{\alpha_m}^{-1} \otimes F_{\alpha_m}] + [1 \otimes E_{\alpha_m}, K_{\alpha_m}^{-1} \otimes F_{\alpha_m}] \\
&= \frac{K_{\alpha_m} - K_{\alpha_m}^{-1}}{q_m - q_m^{-1}} \otimes K_{\alpha_m} + (1 \otimes E_{\alpha_m})(F_{\alpha_m} \otimes 1) + (F_{\alpha_m} \otimes 1)(1 \otimes E_{\alpha_m})
\end{align*}
\]
+E_{\alpha_m}K^{-1}_{\alpha_m} \otimes K_{\alpha_m}F_{\alpha_m} - K^{-1}_{\alpha_m}E_{\alpha_m} \otimes F_{\alpha_m}K_{\alpha_m} + K^{-1}_{\alpha_m} \otimes K_{\alpha_m} - K^{-1}_{\alpha_m}\\
= \frac{K_{\alpha_m} \otimes K_{\alpha_m} - K^{-1}_{\alpha_m} \otimes K^{-1}_{\alpha_m}}{q_m - q_m^{-1}}\\
= \Delta\left(\frac{K_{\alpha_m} - K_{\alpha_m}^{-1}}{q_m - q_m^{-1}}\right)\\
S([E_{\alpha_m}, F_{\alpha_m}]) = -(S(E_{\alpha_m})S(F_{\alpha_m}) + S(F_{\alpha_m})S(E_{\alpha_m}))\\
= -(E_{\alpha_m}F_{\alpha_m} + F_{\alpha_m}E_{\alpha_m})\\
= \frac{K_{\alpha_m} - K_{\alpha_m}^{-1}}{q_m - q_m^{-1}}\\
= S\left(\frac{K_{\alpha_m} - K_{\alpha_m}^{-1}}{q_m - q_m^{-1}}\right)\\
\epsilon([E_{\alpha_m}, F_{\alpha_m}]) = 0 = \epsilon\left(\frac{K_{\alpha_m} - K_{\alpha_m}^{-1}}{q_m - q_m^{-1}}\right).

The remainder of the proof is similar (cf. [8, 4.8]) and is therefore omitted. \qed

It is easy to see that

**Lemma 3.2.** There are \(\mathbb{Z}_2\)-graded anti-automorphism \(\Psi\) and anti-automorphism \(\Omega\) of \(\tilde{U}_q\) such that

\[
\Psi(E_{\alpha_i}) = E_{\alpha_i}, \quad \Psi(F_{\alpha_i}) = F_{\alpha_i}, \quad \Psi(K_j) = K_j, \quad \Psi(q) = q^{-1},
\]

\[
\Omega(E_{\alpha_i}) = F_{\alpha_i}, \quad \Omega(F_{\alpha_i}) = E_{\alpha_i}, \quad \Omega(K_j) = K_j^{-1}, \quad \Omega(q) = q^{-1},
\]

for all \(i \in [1, m+n], j \in [1, m+n]\).

**Lemma 3.3.** Let \(\bar{\Omega}\) be the \(\mathbb{F}\)-linear map from the \(\mathbb{F}\)-superalgebra \(\tilde{U}_q \otimes \tilde{U}_q\) into itself defined by

\[
\bar{\Omega}(a \otimes b) = \Omega(b) \otimes \Omega(a).\]

Then we have

(a) \(\bar{\Omega}(u_1u_2) = \bar{\Omega}(u_2)\bar{\Omega}(u_1), u_1, u_2 \in h(\tilde{U}_q \otimes \tilde{U}_q)\),

(b) \(\bar{\Omega}\Delta = \Delta\bar{\Omega}\).

**Proof.** (a) One can assume \(u_1 = x_1 \otimes x_2, u_2 = y_1 \otimes y_2\), where \(x_1, x_2, y_1, y_2\) are homogeneous elements in \(\tilde{U}_q\). Note that \(\Omega\) is an even map; that is, \(\Omega(x) = \bar{x}\) for any \(x \in h(\tilde{U}_q)\). Then (a) follows from a straightforward computation.

(b) It suffices to show that both functions \(\bar{\Omega}\Delta\) and \(\Delta\bar{\Omega}\) have the same images at the generators of \(\tilde{U}_q\). For \(i \in [1, m+n]\), we have

\[
\bar{\Omega}\Delta(E_{\alpha_i}) = \bar{\Omega}(E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i})
\]

\[
= K_{\alpha_i}^{-1} \otimes F_{\alpha_i} + F_{\alpha_i} \otimes 1 = \Delta(F_{\alpha_i}) = \Delta\Omega(E_{\alpha_i}).
\]

Similarly one proves that \(\bar{\Omega}\Delta(F_{\alpha_i}) = \Delta\Omega(F_{\alpha_i})\) and \(\bar{\Omega}\Delta(K_j^{\pm 1}) = \Delta\Omega(K_j^{\pm 1}), j \in [1, m+n]\). \qed
Recall the notation $\Lambda$ and $\Phi^+$. For each $\mu = l_1 \epsilon_1 + \cdots + l_m \epsilon_m \in \Lambda$, set $K_\mu = \prod_{i=1}^{m+n} K_i^{l_i} \in \bar{U}_q$, so that $K_\nu = \Pi K_i^{k_i}$ for $\nu = \sum k_i \alpha_i \in \mathbb{Z} \Phi^+$. For each finite sequence $I = (\alpha_1, \ldots, \alpha_r)$ of simple roots, we denote

$$E_I =: E_{\alpha_1} \cdots E_{\alpha_r}, F_I = F_{\alpha_1} \cdots F_{\alpha_r}, wtI = \alpha_1 + \cdots + \alpha_r.$$ 

In particular, we let $E_\emptyset = F_\emptyset = 1$. Clearly the parity of the element $E_I$ (resp. $F_I$) is $E_I = \sum_{i=1}^r E_{\alpha_i}$ (resp. $F_I = \sum_{i=1}^r F_{\alpha_i}$).

**Lemma 3.4.** Let $I$ be a sequence as above. We can find elements $C^I_{A,B} \in \mathcal{A}$ indexed by finite sequences of simple roots $A$ and $B$ with $wtI = wtA + wtB$ such that in $\bar{U}_q$ and in $U_q$

$$\Delta(E_I) = \sum_{A,B} C^I_{A,B}(q) E_A \otimes K_{wtA} E_B,$$

$$\Delta(F_I) = \sum_{A,B} C^I_{A,B}(q^{-1}) F_A K_{wtB}^{-1} \otimes F_B.$$ 

We have $c_{A,\emptyset} = \delta_{A,I}$ and $c_{\emptyset,B} = \delta_{B,I}$.

**Proof.** By Lemma 3.3(b), it suffices to prove the first identity. Note that the following identities hold in $\bar{U}_q$:

$$K_{\alpha_i} E_{\alpha_j} = q^{(\alpha_i, \alpha_j)} E_{\alpha_j} K_{\alpha_i}, K_{\alpha_i} F_{\alpha_j} = q^{-(\alpha_i, \alpha_j)} F_{\alpha_j} K_{\alpha_i},$$

using which one proves the first identity exactly as that in [8, Lemma 4.12]. \qed

Consider for each extension field $k \supset \mathbb{F}(q)$ a unitary free associative $k$-superalgebra $M_k = (M_k)_0 \oplus (M_k)_1$ with the homogeneous generators $\xi_i$, $i \in [1, m+n]$, for which the parity is defined by $\tilde{\xi}_i = \tilde{\delta}_{im} \in \mathbb{Z}_2$.

Let $k^*$ be the set of nonzero numbers in the field $k$.

**Lemma 3.5.** For each $c = (c_1, c_2, \ldots, c_{m+n}) \in (k^*)^{m+n}$, there is on $M_k$ a structure as a $\bar{U}_q$-module such that for all $i \in [1, m+n], j \in [1, m+n]$ and all finite product $\xi_{i_1} \cdots \xi_{i_r} \in M_k$

$$F_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = \xi_i \xi_{i_1} \cdots \xi_{i_r},$$

$$K_j \xi_{i_1} \cdots \xi_{i_r} = c_j q^{-(\epsilon_j, \alpha_{i_1} + \cdots + \alpha_{i_r})} \xi_{i_1} \cdots \xi_{i_r},$$

$$E_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = \sum_{1 \leq s \leq r, i_s = i} (-1)^{s-1} \xi_i \sum_{i=1}^s \xi_{i_s},$$

$$c_i c_{i+1}^{-1} q^{-(\alpha_i, \alpha_{i+1} + \cdots + \alpha_r)} - c_i^{-1} c_{i+1} q^{(\alpha_i, \alpha_{i+1} + \cdots + \alpha_r)} \xi_{i_1} \cdots \xi_{i_r}. $$

**Proof.** These formulas define endomorphisms $f_{\alpha_i}, k_j$ and $e_{\alpha_i}$ of $M_k$. It is clear that $k_j^{-1}$ is defined by $k_j^{-1} \xi_{i_1} \cdots \xi_{i_r} = c_j^{-1} q^{(\epsilon_j, \sum_{s=1}^r \alpha_{i_s})} \xi_{i_1} \cdots \xi_{i_r}$. To prove the lemma, we need show that these endomorphisms satisfy the relations (R1)-(R3). The relations (R1) and (R2) follow from a straightforward computation. We are left with (R3).
In case \( i \neq j \), we have
\[
e_{\alpha_i} f_{\alpha_j} \xi_{i_1} \cdots \xi_{i_r} = e_{\alpha_i} \xi_j \xi_{i_1} \cdots \xi_{i_r} = \sum_{1 \leq s \leq r, i_s = i} (-1)^{s} (\xi_{i_1} + \xi_j) E_{\alpha_i} q_{i_s}^{-1} \xi_j \xi_{i_1} \cdots \hat{\xi}_{i_s} \cdots \xi_{i_r}
\]
and
\[
f_{\alpha_j} e_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = \sum_{1 \leq s \leq r, i_s = i} (-1)^{s} (\xi_{i_1} + \xi_j) \tilde{E}_{\alpha_i} q_{i_s}^{-1} \xi_j \xi_{i_1} \cdots \hat{\xi}_{i_s} \cdots \xi_{i_r}.
\]
Since \( i \neq j \), either \( \xi_i = \tilde{E}_{\alpha_i} = 0 \) or \( \xi_j = 0 \), so that \( e_{\alpha_i} f_{\alpha_j} = f_{\alpha_j} e_{\alpha_i} \).

In case \( i = j \), we have
\[
e_{\alpha_i} f_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = e_{\alpha_i} \xi_i \xi_{i_1} \cdots \xi_{i_r} = \frac{c_i c_{i+1} q^{-(\alpha_i, s_{i+1})} - c_i^{-1} c_{i+1} q^{-(\alpha_i, s_{i+1})}}{q_i - q_i^{-1}} \xi_{i_1} \cdots \hat{\xi}_{i_i} \cdots \xi_{i_r} + (-1)^{r} f_{\alpha_i} e_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = \sum_{1 \leq s \leq r, i_s = i} (-1)^{s} (\xi_{i_1} + \xi_j) \tilde{E}_{\alpha_i} q_{i_s}^{-1} \xi_j \xi_{i_1} \cdots \hat{\xi}_{i_s} \cdots \xi_{i_r}.
\]
Thus, the relation (R3) is satisfied.

We denote this module by \( M_k(c) \).

For each \( c \in (k^*)^{m+n} \), one can show similarly that there is on \( M_k \) a unique structure as a \( \tilde{U}_q \)-module such that for all \( i \in [1, m+n] \), \( j \in [1, m+n] \) and all finite product \( \xi_{i_1} \cdots \xi_{i_r} \in M_k \)
\[
E_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = \xi_i \xi_{i_1} \cdots \xi_{i_r},
\]
\[
K_j \xi_{i_1} \cdots \xi_{i_r} = c_j q^{(\varepsilon_{j, \alpha_i})} \xi_{i_1} \cdots \xi_{i_r},
\]
\[
F_{\alpha_i} \xi_{i_1} \cdots \xi_{i_r} = \sum_{1 \leq s \leq r, i_s = i} (-1)^{s} c_{i+1} q^{-(\alpha_i, s_{i+1})} \xi_{i_1} \cdots \hat{\xi}_{i_i} \cdots \xi_{i_r}.
\]
We denote this \( \tilde{U}_q \)-module by \( M_k'(c) \).

With Lemma 3.4 and the \( \tilde{U}_q \)-modules \( M_k(c) \), \( M_k'(c) \), Jantzen’s argument ([8, Prop. 4.16]) can be applied almost verbatim to obtain
Proposition 3.6. The elements $F_1K_\mu E_J$ with $\mu \in \Lambda$ and $I,J$ finite sequences of simple roots are a basis of $\tilde{U}_q$.

It follows that the map

$$\theta : \tilde{U}_q \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+ \longrightarrow \tilde{U}_q$$

$$u_1 \otimes u_2 \otimes u_3 \mapsto u_1u_2u_3$$

is an isomorphism of $\mathbb{F}(q)$-vector spaces.

In $\tilde{U}_q$, set $u_{ex}^+ = [E_{m-1,m+2}, E_{m,m+1}]$; given $i, j \in [1, m + n)$, $i \neq j$, set $u_{ij}^+ =$:

$$\begin{cases}
E_{i,i+1}^2E_{j,j+1} - (q + q^{-1})E_{i,i+1}E_{j,j+1}E_{i,i+1} + E_{j,j+1}E_{i,i+1}^2, & \text{if } |i - j| = 1, i \neq m \\
E_{i,i+1}E_{j,j+1} - E_{j,j+1}E_{i,i+1}, & \text{if } |i - j| > 1.
\end{cases}$$

Let $u_{ex}^- = \Omega(u_{ex}^+)$ and let $u_{ij}^- = \Omega(u_{ij}^+)$ for all $i, j \in [1, m + n)$, $i \neq j$.

Lemma 3.7. The following identities hold in $\tilde{U}_q$.

$$[F_{s,s+1}, u_{ij}^+] = 0, \quad [F_{s,s+1}, E_{m,m+1}^2] = 0,$$

for all $s, i, j \in [1, m + n)$, $i \neq j$.

$$[F_{s,s+1}, u_{ex}^+] = 0 \quad \text{for all} \quad s \in [1, m + n) \setminus m + 1, \quad [F_{m,m+1}, E_{m-1,m+2}] = 0.$$  

The proof of the lemma follows from a straightforward computation, we leave it to the interested reader.

Let $\langle E_{m,m+1}^2 \rangle$ be the two-sided ideal of $\tilde{U}_q^+$ generated by the element $E_{m,m+1}^2$. Using the fact that $q_m = q_{m+1}^{-1}$, one can show that

$$E_{m,m+1}E_{m,m+2} + q_mE_{m,m+2}E_{m,m+1} \in \langle E_{m,m+1}^2 \rangle$$

and

$$E_{m,m+1}E_{m-1,m+1} + q_{m+1}^{-1}E_{m-1,m+1}E_{m,m+1} \in \langle E_{m,m+1}^2 \rangle.$$

Applying these identities, one verifies easily that

Lemma 3.8.  

$$[F_{m+1,m+2}, u_{ex}^+] \in \langle E_{m,m+1}^2 \rangle.$$  

Let $\mathcal{J}$ (resp. $\mathcal{J}^+$, $\mathcal{J}^-$) be the two-sided ideal of $\tilde{U}_q$ (resp. $\tilde{U}_q^+$, $\tilde{U}_q^-$) generated by the homogeneous elements

$$u_{ij}^+, E_{m,m+1}^2, u_{ex}^+ \quad \text{resp.} \quad u_{ij}^+, E_{m,m+1}^2, u_{ex}^+, u_{ij}^+, E_{m,m+1}^2, u_{ex}^-,$$

where $i, j \in [1, m + n), i \neq j$.

Lemma 3.9. The two-sided ideal in $\tilde{U}_q$ generated by the elements $u_{ij}^+(i, j \in [1, m + n), i \neq j)$, $E_{m,m+1}^2$, $u_{ex}^+$ is equal to the image of $\tilde{U}_q^0 \otimes \tilde{U}_q^0 \otimes \mathcal{J}^+$ under the map $\theta$ defined above.
Proof. Denote the image of $\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+$ by $V$. It is contained in the two-sided ideal in $\tilde{U}_q$ generated by all $u_{ij}^+ E_{m,m+1} E_I$, $u_{ex}^+ E_I$. It suffices to show that $V$ is a two-sided ideal of $\tilde{U}_q$. From the relations (R1)-(R3) we see that $V$ is stable under the left multiplication by the homogeneous element $u \in \tilde{U}_q^- \cup \tilde{U}_q^0 \cup \tilde{U}_q^+$, so that $V$ is a left ideal in $\tilde{U}_q$.

As a vector space $V$ is spanned by the homogeneous elements

$$uu_{ij}^+ E_I, u E_{m,m+1} E_I, uu_{ex}^+ E_I$$

with $u \in \tilde{U}_q$, with all $i, j \in [1, m + n), i \neq j$, and with all sequences $E_I$ as preceding Lemma 3.4. Then it is clear that the right multiplication of $V$ by the homogeneous elements $u \in \tilde{U}_q^+ \cup \tilde{U}_q^0$ stabilizes $V$. To complete the proof, we must show that $V$ is stable under the right multiplication by $F_{s,s+1}$ for all $s \in [1, m + n)$.

In fact, we have

$$uu_{ij}^+ E_I F_{s,s+1} = (-1)(E_{I, s+1} u_{ij}^+) F_{s,s+1} u F_{s,s+1} u_{ij}^+ E_I$$

$$+uu_{ij}^+ [E_I, F_{s,s+1}] + (-1) E_{I, s+1} u [u_{ij}^+, F_{s,s+1}] E_I,$$

here the first summand is in $V$; the commutator $[E_I, F_{s,s+1}]$ is in $\tilde{U}_q^0 \tilde{U}_q^+$ by (R3), so that the second summand is in $V$; the third term is equal to 0 by Lemma 3.7. Therefore, $uu_{ij}^+ E_I F_{s,s+1} \in V$.

Applying Lemma 3.7 and 3.8, one proves similarly that

$$u E_{m,m+1} E_I F_{s,s+1}, uu_{ex}^+ E_I F_{s,s+1} \in V$$

for all $s \in [1, m + n)$. Therefore $V$ is a two-sided ideal. \qed

By applying $\Omega$, one gets

Lemma 3.10. The two-sided ideal in $\tilde{U}_q$ generated by the elements $u_{ij}^+ (i, j \in [1, m + n), i \neq j)$, $E_{m,m+1}^2, u_{ex}^+$ is equal to the image of $J^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+$ under the map $\theta$.

By Lemma 3.9, 3.10, we get

$$J = \theta(\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+) - \theta(\tilde{U}_q^- \otimes \tilde{U}_q^0 \otimes \tilde{U}_q^+).$$

This gives an induced vector space isomorphism

$$U_q = \tilde{U}_q / J \cong \tilde{U}_q^- / J^- \otimes \tilde{U}_q^- / J^- \otimes \tilde{U}_q^+ / J^+$$

with $U_q^0 \cong \tilde{U}_q^0$. To summarize, one gets

Corollary 3.11. (1) The multiplication map

$$\bar{\theta} : U_q^- \otimes U_q^0 \otimes U_q^+ \rightarrow U_q, u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$$
is an isomorphism of vector spaces.

(2) $U_q^+$ is isomorphic to the superalgebra generated by the elements $E_{i,i+1}$, $i \in [1, m+n)$ and relations

$$u_{ij}^+ = 0 (i, j \in [1, m+n], i \neq j), E_{m,m+1}^2 = 0, u_{ex}^+ = 0.$$  

(3) $U_q^-$ is isomorphic to the superalgebra generated by the elements $F_{i,i+1}$, $i \in [1, m+n)$ and relations

$$u_{ij}^- = 0 (i, j \in [1, m+n], i \neq j), F_{m,m+1}^2 = 0, u_{ex}^- = 0.$$  

(4) The $K_\mu$ with $\mu \in \Lambda$ are a basis of $U_q^0$.

Let $U'_q(\mathfrak{g}_0)$ (resp. $U'_q(\mathfrak{g}_0)^+$; $U'_q(\mathfrak{g}_0)^-$; $U'_q(\mathfrak{g}_0)^0$) be the subalgebra of $U_q$ generated by the even generators

$$E_{i,i+1}, F_{i,i+1}, K_j^{\pm 1} (\text{resp.} E_{i,i+1}, F_{i,i+1}, K_j^{\pm 1}), i \in [1, m+n) \setminus m, j \in [1, m+n].$$  

Let $\tilde{U}_q(\mathfrak{g}_0)$ be the algebra generated by the above elements with relations (R1)-(R3) in 2.2. Denote by $\tilde{U}_q(\mathfrak{g}_0)^+$ (resp. $\tilde{U}_q(\mathfrak{g}_0)^-$; $\tilde{U}_q(\mathfrak{g}_0)^0$) the subalgebra of $\tilde{U}_q(\mathfrak{g}_0)$ generated by the elements $E_{i,i+1}$ (resp. $F_{i,i+1}; K_j^{\pm 1}$) for all $i \in [1, m+n) \setminus m, j \in [1, m+n]$.

In the remainder of this subsection, we use [8, 4.16, 4.21] which hold in $U_q(sl_m \oplus sl_n)$ and can be easily generalized to $U_q(\mathfrak{g}_0)$ (see 2.2). By [8, 4.16], there is an isomorphism of $\mathbb{F}(q)$-vector spaces

$$\tilde{U}_q(\mathfrak{g}_0)^- \otimes \tilde{U}_q(\mathfrak{g}_0)^0 \otimes \tilde{U}_q(\mathfrak{g}_0)^+ \longrightarrow \tilde{U}_q(\mathfrak{g}_0).$$  

Clearly the quantum group $U_q(\mathfrak{g}_0)$ is a quotient of $\tilde{U}_q(\mathfrak{g}_0)$.

**Proposition 3.12.** There is an isomorphism of $\mathbb{F}(q)$-algebras: $U_q(\mathfrak{g}_0) \cong U'_q(\mathfrak{g}_0)$.

**Proof.** Let $\tilde{U}'_q(\mathfrak{g}_0)$ be the subalgebra of $\tilde{U}_q$ generated by the even generators. Then there is an epimorphism of $\mathbb{F}(q)$-algebras $\pi : \tilde{U}_q(\mathfrak{g}_0) \longrightarrow \tilde{U}'_q(\mathfrak{g}_0)$. Using (R1)-(R3), we obtain that $\tilde{U}'_q(\mathfrak{g}_0)$ is spanned by the elements $F_i K_\mu E_J$, with $I, J$ finite sequences of simple even roots, which by Prop. 3.6 becomes a basis of $\tilde{U}'_q(\mathfrak{g}_0)$; while $\tilde{U}_q(\mathfrak{g}_0)$, by [8, 4.16], has an analogous basis, so that $\pi$ is an isomorphism. Thus, we can identify $\tilde{U}_q(\mathfrak{g}_0)$ with the subalgebra $\tilde{U}'_q(\mathfrak{g}_0)$ of $\tilde{U}_q$.

Denote by $f$ the canonical epimorphism from $\tilde{U}_q$ into $U_q$. Then we obtain by Coro. 3.11(4) that $f$ maps $\tilde{U}_q(\mathfrak{g}_0)^0$ isomorphically onto $U_q^0$. Moreover, we have

$$U'_q(\mathfrak{g}_0)^+ = f(\tilde{U}_q(\mathfrak{g}_0)^+) = (\tilde{U}_q(\mathfrak{g}_0)^+ + \mathcal{J}^+)/\mathcal{J}^+ \cong \tilde{U}_q(\mathfrak{g}_0)^+/(\tilde{U}_q(\mathfrak{g}_0)^+ \cap \mathcal{J}^+).$$
Using Prop. 3.6, we obtain
\[ \tilde{U}_q(g_0)^+ \cap J^+ = \sum_{m \notin \{i, j\}} \tilde{U}_q(g_0)^+ u_{ij} \tilde{U}_q(g_0)^+ . \]
Let
\[ U_q(g_0) \cong U_q(g_0)^- \otimes U_q^0 \otimes U_q(g_0)^+ \]
be the triangular decomposition of \( U_q(g_0) \). By [8, 4.21(b)],
\[ U_q(g_0)^+ \cong \tilde{U}_q(g_0)^+ / \sum_{m \notin \{i, j\}} \tilde{U}_q(g_0)^+ u_{ij} \tilde{U}_q(g_0)^+ \]
\[ \cong \tilde{U}_q'(g_0)^+ . \]
Similarly one proves that \( U_q(g_0)^- \cong U_q'(g_0)^- \). This establishes the proposition. \( \square \)

We shall identify \( U_q(g_0) \) with \( U_q'(g_0) \) in the following.

4 The structure of \( U_q \)

4.1 The braid group action on \( U_q \)

For \( i \in [1, m + n) \setminus m \), the automorphism \( T_{\alpha_i} \) of \( U_q \) is defined by (see [19, Appendix A] and also [14, 1.3])
\[
T_{\alpha_i}(E_{\alpha_j}) = \begin{cases} 
-F_{\alpha_i}K_{\alpha_i} & \text{if } i = j, \\
E_{\alpha_j} & \text{if } a_{ij} = 0, \\
-E_{\alpha_i}E_{\alpha_j} + q_i^{-1}E_{\alpha_j}E_{\alpha_i} & \text{if } a_{ij} = -1.
\end{cases}
\]
\[
T_{\alpha_i}F_{\alpha_j} = \begin{cases} 
-K_{\alpha_i}^{-1}E_{\alpha_i} & \text{if } i = j, \\
F_{\alpha_j} & \text{if } a_{ij} = 0, \\
-F_{\alpha_j}F_{\alpha_i} + q_iF_{\alpha_i}F_{\alpha_j} & \text{if } a_{ij} = -1.
\end{cases}
\]
\[
T_{\alpha_i}K_j = \begin{cases} 
K_{i+1} & \text{if } j = i, \\
K_i & \text{if } j = i + 1, \\
K_j & \text{if } j \neq i, i + 1.
\end{cases}
\]
It is pointed out in [19] that each \( T_{\alpha_i} \) is a \( \mathbb{Z}_2 \)-graded automorphism of \( U_q \), which means (see [19, Appendix. A])
\[
T_{\alpha_i}(uv) = (-1)^{\bar{a}_i}T_{\alpha_i}(u)T_{\alpha_i}(v), \quad u, v \in h(U_q).
\]
But a straightforward computation shows that \( T_{\alpha_i} \) is an even automorphism for \( U_q \), that is,
\[
T_{\alpha_i}(uv) = T_{\alpha_i}(u)T_{\alpha_i}(v), \quad \text{for all } \ u, v \in h(U_q).
\]
In fact, one can see this by checking that \( T_\alpha(s \in [1, m + n) \setminus m) \) preserves the relation (R3) in the case \( i = j = m \).

By a straightforward computation ([19, A3]), one obtains for each \( i \in [1, m + n) \setminus m \) the inverse map \( T_\alpha^{-1} \):

\[
T_\alpha^{-1}E_{ij} = \begin{cases} 
-K_\alpha^{-1}F_{\alpha_i}, & \text{if } i = j, \\
E_{\alpha_j}, & \text{if } a_{ij} = 0, \\
-E_{\alpha_j}E_{\alpha_i} + q_i^{-1}E_{\alpha_i}E_{\alpha_j}, & \text{if } a_{ij} = -1.
\end{cases}
\]

\[
T_\alpha^{-1}F_{ij} = \begin{cases} 
-E_{\alpha_i}K_{\alpha_i}, & \text{if } i = j, \\
F_{\alpha_j}, & \text{if } a_{ij} = 0, \\
-F_{\alpha_i}F_{\alpha_j} + q_iF_{\alpha_j}F_{\alpha_i}, & \text{if } a_{ij} = -1.
\end{cases}
\]

\[
T_\alpha^{-1}K_j = \begin{cases} 
K_{i+1}, & \text{if } j = i, \\
K_i, & \text{if } j = i + 1, \\
K_j, & \text{if } j \neq i, i + 1.
\end{cases}
\]

There are \( \mathbb{Z}_{q2} \)-graded algebra automorphism \( \Psi \) and antiautomorphism \( \Omega \) of \( U_q \) inherited from \( \tilde{U}_q \) (see Lemma 3.2). Then according to [19], we have

\[
(*) \quad \Omega T_\alpha = T_\alpha \Omega.
\]

Suppose \( i < k < k + 1 < j \). The following identities, given in [19], can be verified easily by induction:

\[
(b1) \quad E_{i,j} = (-1)^{j-i-1}T_\alpha T_{\alpha_i+1} \cdots T_{\alpha_k-1} T_{\alpha_{j-2}}^{-1} \cdots T_{\alpha_{k+1}}^{-1} E_{k,k+1},
\]

\[
(b2) \quad F_{i,j} = (-1)^{j-i-1}T_\alpha T_{\alpha_i+1} \cdots T_{\alpha_k-1} T_{\alpha_{j-2}}^{-1} \cdots T_{\alpha_{k+1}}^{-1} F_{k,k+1}.
\]

Applying the formula \((*)\) above we get \( \Omega(E_{i,j}) = F_{i,j}((i, j) \in \mathcal{I}) \). It then follows from the formulas \((b1), (b2)\) that \( E_{i,j}^2 = F_{i,j}^2 = 0 \) for \((i, j) \in \mathcal{I}_1\).

### 4.2 Some formulas in \( U_q \)

In this subsection we give more relations in \( U_q \).

**Lemma 4.1.** Assume \( i < s < t < j \). Then

\[
(a) \quad [E_{i,j}, E_{s,t}] = 0, \quad (b) \quad [F_{i,j}, F_{s,t}] = 0.
\]

**Proof.** Note that \((b)\) follows from \((a)\) using the involution \( \Omega \), so it suffices to prove \((a)\). We proceed by induction on \( t - s \). The case \( t - s = 1 \) follows immediately from [19, Lemma 1]. Now assume \( t - s \geq 2 \).

Recall that

\[
E_{s,t} = E_{s,c}E_{c,t} - q_c^{-1}E_{c,t}E_{s,c}, s < c < t.
\]
Using induction hypothesis and the identity $\bar{E}_{s,t} = \bar{E}_{s,c} + \bar{E}_{c,t}$, we get

$$[E_{i,j}, E_{s,t}] = E_{i,j}E_{s,t} - (-1)^{E_{i,j}E_{s,t}}E_{s,t}E_{i,j}$$
$$= [E_{i,j}, E_{s,c}]E_{c,t} - q_{c}^{-1}[E_{i,j}, E_{c,t}]E_{s,c}$$
$$= 0.$$ 

The following list of formulas will be useful.

**Lemma 4.2.** [19, p. 1238-1239]

1. $[E_{i,j}, F_{c,c+1}] = \delta_{c+1,i}E_{i,c}K_{c}K_{c+1}^{-1}q_{c}^{-1} - \delta_{i,c}(-1)^{\delta_{c,m}}E_{c+1,j}K_{c}^{-1}K_{c+1},$

   $i < j, i \neq c,$ or $j \neq c + 1.$

2. $E_{a,i}E_{s,j} = (-1)^{E_{a,i}E_{s,j}}q_{a}E_{s,j}E_{a,i}, \quad s < i < j.$

3. $E_{js}E_{is} = (-1)^{E_{js}E_{is}}q_{s}^{-1}E_{is}E_{js}, \quad i < j < s.$

It follows from the formula (1) that $[E_{i,j}, F_{c,c+1}] = 0,$ if $i < c < c + 1 < j.$ Applying a similar proof as that for Lemma 4.1, one gets, for $i < s < t < j,$

4. $[E_{i,j}, F_{s,t}] = 0,$

and hence

5. $[F_{i,j}, E_{s,t}] = 0.$

The formula([19, (i)])

6. $[E_{a,b}, E_{c,d}] = (q_{b} - q_{b}^{-1})E_{a,d}E_{c,b}, \quad a < c < b < d$

can be easily verified by using the formula (2) and the fact $E_{ab} = E_{ac}E_{cb} - q_{c}^{-1}E_{cb}E_{ac}.$ Note that the original assumption in [19] is imprecise.

Set

$S^{+} = \{E_{ij}|(i,j) \in I\}, S_{0}^{+} = \{E_{ij}|(i,j) \in I_{0}\}, S_{1}^{+} = \{E_{ij}|(i,j) \in I_{1}\},$

$\mathcal{H} = \{K_{i}|1 \leq i \leq m + n\}, \quad S^{-} = \Omega(S^{+}), \quad S_{0}^{-} = \Omega(S_{0}^{+}), \quad S_{1}^{-} = \Omega(S_{1}^{+}).$

For $x, y \in S := S^{-} \cup \mathcal{H} \cup S^{+},$ we write $x < y$ if one of the following conditions holds:

(i) $x \in S^{-}, \quad y \in S_{0}^{-} \cup \mathcal{H} \cup S^{+},$

(ii) $x \in S^{-}, \quad y \in \mathcal{H} \cup S^{+},$

(iii) $x \in \mathcal{H}, \quad y \in S^{+},$

(IV) $x \in S_{0}^{+}$ and $y \in S_{1}^{+},$
(V) \(x = E_{i,j} \in S_i^+, y = E_{s,t} \in S_t^+, i \in \{0, 1\}\), where \(i < s\) or, \(i = s\) and \(j < t\).

(VI) \(x, y \in S_i^-\) with \(\Omega(y) < \Omega(x)\).

For \(x, y \in S\), we write \(x \lesssim y\) if \(x < y\) or \(x = y\). The order \(<\) (but not \(\lesssim\)) can be extended naturally to a larger set \(\overline{S} = \{x^n| x \in S, n \in \mathbb{Z}^+\}\) by letting \(x^n < y^m\) if and only if \(x < y\). We call a product \(x_1x_2 \cdots x_n \in U_q(x_i \in S)\) a standard monomial if \(x_i < x_j\) whenever \(i < j\).

**Lemma 4.3.** Let \(x, y \in S^+\) with \(x \lesssim y\). Then

\[
yx = \sum_{x_i \lesssim y_j} c_i x_i y_i, \quad c_i \in A.
\]

**Proof.** Let \(x = E_{i,j}, y = E_{s,t}\). Suppose both \(x\) and \(y\) are contained in the same \(S_i^+, i = 0, 1\). In view of the formulas from Lemma 4.1, 4.2, we need only verify the case \(i < s < j < t\). By the formula (6), we get

\[
yx = E_{s,t}E_{i,j} = (-1)^{E_{i,j}E_{s,t}}[E_{i,j}E_{s,t} - (q_j - q_j^{-1})E_{i,t}E_{s,j}].
\]

Since \(E_{i,j} < E_{i,t} < E_{s,j} < E_{s,t}\), the lemma follows immediately.

Suppose \(x \in S_0^+\) and \(y \in S_1^+\). It suffices to verify the cases \(i < s < j \leq m < t\) and \(s < m < i < t < j\). In case \(i < s < j \leq m < t\), we use the identity \((*)\) above. In this case we have \(E_{i,j} < E_{s,j} < E_{i,t} < E_{s,t}\). By Lemma 4.1, we have \(E_{i,t}E_{s,j} = (-1)^{E_{i,t}E_{s,j}}E_{s,j}E_{i,t}\), so that the lemma follows. The case \(s < m < i < t < j\) can be proved similarly.

Let \(E_i^d\) denote the standard monomial \(\Pi_{(i,j) \in I_1} E_{i,j}^{d_{ij}}, d_{ij} \in \{0, 1\}\). Set \(|d| = \sum d_{ij}\). For \(k \geq 0\), let \(\mathcal{N}_1^{(k)} = \{E_i^d||d| = k\}\). Since \(E_{i,j}^2 = 0\) for \((i, j) \in I_1\), we have \(\mathcal{N}_1^{(k)} = 0\) for any \(k > nm = |I_1|\). Set

\[\mathcal{N}_1 = \sum_{k \geq 0} \mathcal{N}_1^{(k)}, \quad \mathcal{N}_1^+ = \sum_{k > 0} \mathcal{N}_1^{(k)}\]

If \(E_{ij} \ll E_{st}\) with \((i, j) \in I_1\), then we get by definition that \((s, t) \in I_1\). From Lemma 4.3, it then follows that \(\mathcal{N}_1^{(i)}\mathcal{N}_1^{(j)} \subseteq \mathcal{N}_1^{(i+j)}\), and hence \((\mathcal{N}_1^+)^{nm+1} = 0\). Let

\[\mathcal{N}_{-1} =: \Omega(\mathcal{N}_1)\]

Using Coro. 3.11(1) and Lemma 4.3, one obtains easily that

\[U_q = \mathcal{N}_{-1}U_q(\mathfrak{g}_0)\mathcal{N}_1\]

**Lemma 4.4.** Let \(k \geq 0\). Then \(\mathcal{N}_1^{(k)}U_q(\mathfrak{g}_0) \subseteq U_q(\mathfrak{g}_0)^{\mathcal{N}_1^{(k)}}\)

**Proof.** We proceed with induction on \(k\). The case \(k = 0\) is trivial. Assume \(k > 1\). To apply the induction hypotheses, it is sufficient to show that \(E_{ij}U_q(\mathfrak{g}_0) \subseteq U_q(\mathfrak{g}_0)^{\mathcal{N}_1^{(1)}}\) for any fixed \(E_{ij}\) with \((i, j) \in I_1\). Since \(U_q(\mathfrak{g}_0)\) is generated by the
elements $E_{s,s+1}, F_{s,s+1}, K_t^{\pm 1}, s \in [1, m+n) \setminus m, t \in [1, m+n]$, the proof reduces to showing that $E_{ij}x \in U_q(\mathfrak{g}_0)\mathcal{N}_1^{(1)}$ with $x$ being one of the above generators. The case $x = K_t^{\pm 1}$ is obvious; the case $x = F_{s,s+1}$ follows from Lemma 4.2(1); the case $x = E_{s,s+1}$ is given by Lemma 4.1, Lemma 4.2(2),(3) and the first equation provided by Remark 2.2(1).

From the lemma it follows that $U_q(\mathfrak{g}_0)\mathcal{N}_1^+$ is a nilpotent ideal of the subalgebra $U_q(\mathfrak{g}_0)\mathcal{N}_1$.

5 Highest weight modules for $U_q$

In this section, we shall construct simple highest weight $U_q$-modules following the procedure in [16].

Recall the $\tilde{U}_q$-module $M_k(\mathfrak{c})$. Let $k = \mathbb{F}(q)$ and denote $M_k(\mathfrak{c})$ simply by $M(\mathfrak{c})$. Set

$$\phi_{ij} =:\begin{cases} 
\xi^2 \xi_j - (q + q^{-1})\xi_i \xi_j \xi_i + \xi_j \xi_i^2, & \text{if } |i - j| = 1, i \neq m \\
\xi \xi_i - \xi_j \xi_i, & \text{if } |i - j| > 1,
\end{cases}$$

$$\phi_m =: \xi_m^2, \quad \phi_{ex} =: \xi_m - 1 \xi_m \xi_{m+1} + \xi_{m+1} \xi_m - 1 \xi_{m+1} \xi_m + 1 \xi_m \xi_{m-1} \xi_m + 1 \xi_m \xi_{m+1} \xi_m - 1$$

Let $N = N_0 \oplus N_1$ be the two-sided ideal of $M(\mathfrak{c})$ generated by these homogeneous elements. Recall the endomorphisms $E_{\alpha_i}, F_{\alpha_i}, K_j^{\pm 1}, i \in [1, m+n), j \in [1, m+n]$ given in Lemma 3.5.

**Lemma 5.1.** $N$ is stable under these endomorphisms.

**Proof.** As a vector space, $N$ is spanned by homogeneous elements of the form $u_1 \phi_{ij} u_2, u_1 \phi_m u_2, u_1 \phi_{ex} u_2, u_1, u_2 \in h(M(\mathfrak{c}))$. It follows immediately from definition that $N$ is stable under all $F_{\alpha_i}(i \in [1, m+n)), K_j^{\pm 1}(j \in [1, m+n])$. We can assume $u_1 = \xi_{i_1} \cdots \xi_{i_k}$. Recall the notion $u_{ij}(1 \leq i \neq j \leq m+n), u_{ex}$. Then we get from Lemma 3.5 that, for all $t \in [1, m+n)$,

$$E_{\alpha_i} u_1 \phi_{ij} u_2 = E_{\alpha_i} \xi_{i_1} \cdots \xi_{i_k} \phi_{ij} u_2$$

$$= E_{\alpha_i} F_{\alpha_{i_1}} \cdots F_{\alpha_{i_k}} u_{ij} u_2$$

$$= (-1)^{E_{\alpha_i}} \sum_{s=1}^k F_{\alpha_{is}} + \bar{u}_{ij}) F_{\alpha_{i_1}} \cdots F_{\alpha_{i_k}} u_{ij} E_{\alpha_i} u_2 + [E_{\alpha_i}, F_{\alpha_{i_1}} \cdots F_{\alpha_{i_k}}] u_{ij} u_2$$

The first summand is obviously contained in $N$. Since the commutator $[E_{\alpha_i}, F_{\alpha_{i_1}} \cdots F_{\alpha_{i_k}}]$ is contained in $\tilde{U}_q - \tilde{U}_q^0$, the second summand is in $N$, while Lemma 3.7, applied with $\Omega$, implies that the third summand is equal to zero. So we get $E_{\alpha_i} u_1 \phi_{ij} u_2 \in N$.

Similarly one can show that the endomorphism $E_{\alpha_i}$ maps the elements $u_1 \phi_m u_2, u_1 \phi_{ex} u_2$ into $N$, and the proof is complete.
Lemma 5.1 immediately yields a $\tilde{U}_q$-module $\tilde{M}(c) =: M(c)/N$.

**Lemma 5.2.** The $k$-linear maps $E_{a_i}, F_{a_i}, K_\pm^1$: $\tilde{M}(c) \rightarrow \tilde{M}(c)$ induced by the analogous maps in Lemma 3.5 satisfy the relations (R4)-(R8) in 2.2, hence define a $U_q$-module $\tilde{M}(c)$.

**Proof.** The proof follows from a similar arguments as in non-super case(cf. [8]). To illustrate, we show that the relation (R8) is satisfied by the $k$-linear maps $E_{a_i}, F_{a_i}$. By Remark 2.2(3), the relation (R8) can be written as $u_{ex}^\pm = 0$. Note that $u_{ex}^\pm \in (\tilde{U}_q)_0$. Then we have

$$u_{ex}^- \xi_{i_1} \cdots \xi_{i_r} = \phi_{ex} \xi_{i_1} \cdots \xi_{i_r} \in N.$$

$$u_{ex}^+ \xi_{i_1} \cdots \xi_{i_r} = u_{ex}^+ f_{i_1} \cdots f_{i_r} \cdot 1$$

$$= f_{i_1} \cdots f_{i_r} u_{ex}^+ \cdot 1 + [u_{ex}^+, f_{i_1} \cdots f_{i_r}] \cdot 1$$

$$= \sum_{k=1}^r f_{i_1} \cdots [u_{ex}^+, f_{i_k}] \cdots f_{i_r} \cdot 1.$$ 

Since the commutator $[u_{ex}^+, f_{i_k}]$ is equal to $u_1 \phi_{m} u_2$ with $u_1, u_2 \in M(c)$ by Lemma 3.7, 3.8, the summation above is contained in $N$. Therefore we have $u_{ex}^\pm = 0$ on $M(c)$. \qed

Let $M = M_0 \oplus M_1$ be a $U_q$-module. For any $c = (c_1, \ldots, c_{m+n}) \in (k^*)^{m+n}$, set $M_c = \{ x \in M | Kix = c_ix, i = 1, \ldots, m+n \}$ and $(M_c)_j = M_c \cap M_j, \quad j \in \mathbb{Z}_2$. Then it is easy to prove that $M_c = (M_c)_0 \oplus (M_c)_1$ and $\sum_c M_c$ is a direct sum.

For each $\mu = \mu_1 \epsilon_1 + \mu_2 \epsilon_2 + \cdots + \mu_{m+n} \epsilon_{m+n} \in \Lambda$, set $q^\mu = (q^{\mu_1}, \ldots, q^{\mu_{m+n}}) \in (k^*)^{m+n}$.

By the relation (R2) from 2.2, one gets, for all $i \in [1, m+n)$,

$$E_{a_i} M_c \subseteq M_{cq^{a_i}}, \quad F_{a_i} M_c \subseteq M_{cq^{-a_i}}.$$ 

Let $c_1, c_2 \in (k^*)^{m+n}$. We define $c_2 \leq c_1$ to mean that

$$c_1 c_2^{-1} = q^{\sum l_i a_i}, l_i \in \mathbb{N}.$$ 

It is easy to see that this is a well defined partial order. A homogeneous nonzero element $x \in M_c$ is called maximal if $E_{a_i} x = 0$ for all $i \in [1, m+n)$. We call $M = M_0 \oplus M_1$ a highest weight module if there is a maximal vector $v^+ \in M_c$ such that $M = U_q v^+$.

Note that any proper $\mathbb{Z}_2$-graded $U_q$-submodule of $M$ is contained in the $\mathbb{Z}_2$-graded subspace $\sum_{c^' \leq c} M_{c'}$, hence $M$ has a unique maximal $\mathbb{Z}_2$-graded submodule and a unique simple $(\mathbb{Z}_2$-graded) quotient.

For the $U_q$-module $\tilde{M}(c)$ as above, since the image of $1 \in M(c)$ is nonzero, it is a maximal vector of $\tilde{M}(c)$ which generates $\tilde{M}(c)$ as a $U_q$-module. Then the unique simple quotient of $\tilde{M}(c)$ is again a highest weight $U_q$-module of highest weight $c$. 

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**Theorem 5.3.** For each \( c \in (k^*)^{m+n} \), there exists a simple \( U_q \)-module of highest weight \( c \). It is unique up to isomorphism and contains a unique maximal vector, up to scalar multiple.

**Proof.** The existence of the simple module as claimed is given above, and the uniqueness of the maximal vector is proved exactly as in [16, 2.6].

To prove the uniqueness of the simple module, let \( I \) the left ideal of \( U_q \) generated by the elements

\[
E_{i,i+1}, i \in [1, m+n), K_j - c_j, j \in [1, m+n].
\]

By Coro. 3.11(1), we have \( U_q/I \cong U_q^- \). It is clear that \( U_q/I \) is a highest weight \( U_q \)-module of highest weight \( c \). Furthermore, each simple module of highest weight \( c \) is a homomorphic image of \( U_q/I \). Then the uniqueness of the simple quotient of \( U_q/I \) implies that any two simple modules of highest weight \( c \) are isomorphic. \( \blacksquare \)

6 \hspace{1em} The superalgebra \( U_A \)

Recall the notion \( K_{\alpha_i} =: K_iK_i^{-1}, i \in [1, m+n) \). To unify notation, put \( K_{\alpha_{m+n}} =: K_{m+n} \). Set

\[
\left[ K_{\alpha_i}; c \right]_t = \prod_{s=1}^{t} \frac{K_{\alpha_i}q_i^{e_s+1} - K_{\alpha_i}^{-1}q_i^{-c+1}}{q_i^s - q_i^{-s}}
\]

for each \( i \in [1, m+n], c \in \mathbb{Z}, t \in \mathbb{N} \). Let \( U_A \) be the \( A \)-subalgebra(with 1) of \( U_q \) generated by the homogeneous elements

\[
E_{\alpha_i}^{(l)} = [l]!^{-1}E_{\alpha_i}^l, F_{\alpha_i}^{(l)} = [l]!^{-1}F_{\alpha_i}^l, K_{\alpha_i}^{\pm 1}, \left[ K_{\alpha_i}; c \right]_t,
\]

\( l \in \mathbb{N}, i \in [1, m+n), j \in [1, m+n], s = m, m+n, c \in \mathbb{Z}, t \in \mathbb{N} \). By [16, 4.3.1] we get \( \left[ K_{\alpha_i}; c \right]_t \in U_A( i \in [1, m+n) \setminus m, c \in \mathbb{Z}, t \in \mathbb{N} ) \), so that \( \left[ K_{\alpha_i}; c \right]_t \in U_A \) for all \( i \in [1, m+n], c \in \mathbb{Z}, t \in \mathbb{N} \).

Denote by \( \text{U}_A^+ \) (resp. \( \text{U}_A^-; \text{U}_A^0 \)) the \( A \)-subalgebra of \( U_A \) generated by the elements

\[
E_{\alpha_i}^{(l)} (\text{resp. } F_{\alpha_i}^{(l)}, K_{\alpha_i}^{\pm 1}, \left[ K_{\alpha_i}; c \right]_t), i \in [1, m+n), j \in [1, m+n], c \in \mathbb{Z}, t \in \mathbb{N}.
\]

Let \( U_q(\frak{g}_0)_A \) (resp. \( (\frak{N}_-)_A; (\frak{N}_1)_A \)) be the \( A \)-subalgebra of \( U_A \) generated by the homogeneous elements

\[
E_{\alpha_i}^{(l)} F_{\alpha_i}^{(l)}, l \in \mathbb{N}, i \in [1, m+n) \setminus m, K_{\alpha_i}^{\pm 1}, \left[ K_{\alpha_i}; c \right]_t, j \in [1, m+n], c \in \mathbb{Z}, t \in \mathbb{N}
\]

(resp. \( E_{ij}^{(l)}, l = 0, 1, (i, j) \in \mathbb{Z}_4; F_{ij}^{(l)}, l = 0, 1, (i, j) \in \mathbb{Z}_4 \)).

It is easy to see that

\[
\Omega(U_A) = U_A \hspace{1em} \Omega(U_q(\frak{g}_0)_A) = U_q(\frak{g}_0)_A
\]
\[ \Omega(U_\mathcal{A}^\pm) = U_\mathcal{A}^\pm \quad \Omega((\mathcal{N}_1)_\mathcal{A}) = (\mathcal{N}_1)_\mathcal{A} \quad \Omega((\mathcal{N}_{-1})_\mathcal{A}) = (\mathcal{N}_{-1})_\mathcal{A}. \]

Recall the augmented Cartan matrix \( \bar{A} = (a_{ij}) \). By a short computation, we get, for \( l \in \mathbb{N}, i \in [1, m+n] \) and \( j \in [1, m+n] \),

\[
(h1) \quad \left[ K_{\alpha_i}; c \right] E_{\alpha_j}^{(l)} = E_{\alpha_j}^{(l)} \left[ K_{\alpha_i}; c + l a_{ij} \right].
\]

\[
(h2) \quad \left[ K_{\alpha_i}; c \right] E_{\alpha_j}^{(l)} = E_{\alpha_j}^{(l)} \left[ K_{\alpha_i}; c - l a_{ij} \right].
\]

Recall that \( E_{ij}^2 = 0 \) in case \( \bar{E}_{i,j} = \bar{1} \). So we denote \( E_{i,j}^{(N)} = [N]^{-1} E_{ij}^N \) only for \( N = 0, 1 \) if \( \bar{E}_{i,j} = \bar{1} \) but for all \( N \in \mathbb{N} \) if \( \bar{E}_{ij} = 0 \).

**Lemma 6.1.**

\[
\begin{align*}
(e1) & \quad E_{i,j}^{(N)} E_{i,j}^{(M)} = \binom{M + N}{N} E_{i,j}^{(N+M)}; \\
(e2) & \quad E_{i,j}^{(N)} E_{s,t}^{(M)} = (-1)^{N M} E_{i,s}^{(N)} E_{t,j}^{(M)} E_{i,j}^{(N)},
\end{align*}
\]

for \( i < s < t < j \) or \( s < t < i < j \);

\[
\begin{align*}
(e3) & \quad E_{t,a}^{(N)} E_{t,b}^{(M)} = \left((-1)^{\bar{E}_{t,a} \bar{q}_t} E_{t,a}^{(N)} E_{t,b}^{(M)} \right) E_{t,a}^{(N)}, \quad t < a < b; \\
(e4) & \quad E_{b,t}^{(N)} E_{a,t}^{(M)} = \left((-1)^{\bar{E}_{b,t} \bar{q}_t} E_{b,t}^{(N)} E_{a,t}^{(M)} \right) E_{a,t}^{(N)}, \quad a < b < t;
\end{align*}
\]

\[
(e5) \quad E_{i,c}^{(N)} E_{c,j}^{(M)} = \sum_{0 \leq k \leq \min\{N,M\}} q_{c}^{(M-k)(N-k)} E_{c,j}^{(M-k)} E_{i,c}^{(N-k)}, \quad i < c < j;
\]

\[
(e6) \quad E_{c,i+1}^{(M)} E_{j,j+1}^{(N)} = E_{j,j+1}^{(N)} E_{i,i+1}^{(M)} \quad \text{if} \quad i \neq j.
\]

**Proof.** (e1) is obvious. (e2) follows from induction with Lemma 4.1(a) and the relation (R5). (e3) and (e4) are given by induction with Lemma 4.2(2), (3). (e5) is given by induction on \( N, M \) with the formula from Remark 2.2(1). (e6) is immediate from the relation (R3).

With only minor adjustments of the Kac’s formula in [16, 4.3], one obtains

\[
(e7) \quad E_{\alpha_i}^{(N)} E_{\alpha_i}^{(M)} = \sum_{0 \leq t \leq \min\{M,N\}} (-1)^{\delta_{i,m} N M (t-1)} E_{\alpha_i}^{(M-t)} F_{\alpha_i}^{(N-t)} \left[ K_{\alpha_i}; 2t - N - M \right] E_{\alpha_i}^{(N-t)}.
\]

Using the formulas (e6)-(e7) above, together with (h1), (h2), we get \( U_\mathcal{A} = U_\mathcal{A}^- U_\mathcal{A}^0 U_\mathcal{A}^+ \), and which, together with Lemma 4.3, implies that

\[ U_\mathcal{A} = (\mathcal{N}_{-1})_\mathcal{A} U_\mathcal{A} (\mathcal{N}_0)_\mathcal{A} (\mathcal{N}_1)_\mathcal{A}. \]
7 The PBW theorem

Assume \((r_1, \ldots, r_{m+n}) \in \mathbb{Z}^{m+n}\) with \(r_i - r_{i+1} \in \mathbb{N}\) for all \(i \in [1, m+n] \setminus m\). Let 
\(M = M_0 \oplus M_1\) be a simple \(U_q\)-module with highest weight \(c = (q_1^{r_1}, \ldots, q_{m+n}^{r_{m+n}})\), and 
let \(x \in M_c\) be a maximal vector. Set \(M_A = U_A x \subseteq M\). Then by a proof similar to 
that of [16, Prop. 4.2], one gets

**Lemma 7.1.**
(a) \(M_A\) is a \(U_A\)-submodule of \(M\).
(b) \(\mathbb{F}(q) \otimes_A M_A \rightarrow M\) is an (even) isomorphism of \(\mathbb{F}(q)\)-vector spaces.
(c) \(M_A\) is the direct sum of \(M_A \cap M_{c'}\) with each \(M_A \cap M_{c'}\) a finite generated free \(A\)-module of finite rank.

Recall from Sec. 4 that

\[
U_q(\mathfrak{g}_0) \mathcal{N}_1 = U_q(\mathfrak{g}_0) + U_q(\mathfrak{g}_0) \mathcal{N}_1^+
\]
and \(U_q(\mathfrak{g}_0) \mathcal{N}_1^+\) is a nilpotent (\(\mathbb{Z}_2\)-graded) ideal of \(U_q(\mathfrak{g}_0) \mathcal{N}_1\). Since \(U_q(\mathfrak{g}_0)\) contains no zero divisors, the sum is direct. Let \(N = N_0 \oplus N_1\) be a simple \(U_q(\mathfrak{g}_0) \mathcal{N}_1\)-module. Then \(U_q(\mathfrak{g}_0) \mathcal{N}_1^+ N\) is a graded submodule of \(N\) which must be 0 since \(U_q(\mathfrak{g}_0) \mathcal{N}_1^+\) is nilpotent. It follows that \(N\) is also simple as a \(U_q(\mathfrak{g}_0)\)-module, so that \(N = N_0\) or \(N = N_1\). Therefore, each simple \(U_q(\mathfrak{g}_0) \mathcal{N}_1\)-module is a simple \(U_q(\mathfrak{g}_0)\)-module (concentrated in 0 or 1) annihilated by \(U_q(\mathfrak{g}_0) \mathcal{N}_1^+\).

Let \(M_0\) be a simple \(U_q(\mathfrak{g}_0)\)-module of highest weight \(c = (q_1^{r_1}, \ldots, q_{m+n}^{r_{m+n}})\). We view \(M_0\) as a \(U_q(\mathfrak{g}_0) \mathcal{N}_1\)-module annihilated by \(U_q(\mathfrak{g}_0) \mathcal{N}_1^+\). Define the induced \(U_q\)-module

\[
K(c) = U_q \otimes_{U_q(\mathfrak{g}_0) \mathcal{N}_1} M_0.
\]

Now let \(v^+ \in (M_0)_c\) be a maximal vector. We regard \(\mathbb{F}\) as an \(A\)-module by letting \(q\) act as multiplication by 1. Set

\[
K_A = U_A v^+ \subseteq K(c), \quad \bar{K}(c) = \mathbb{F} \otimes_A K_A.
\]

Let \(e_{i,i+1}, f_{i,i+1}, i \in [1, m+n], h_{\alpha_j}, K_j, j \in [1, m+n]\) denote respectively the endomorphisms of \(\bar{K}(c)\) induced by the elements

\[
E_{i,i+1}, F_{i,i+1}, \left[ K_{\alpha_j} ; \begin{array}{c} 0 \\ 1 \end{array} \right], K_j.
\]

Then we have

**Lemma 7.2.**
(1) \(\bar{K}_j = 1\).

(2) The elements \(e_{\alpha_i} = e_{i,i+1}, f_{\alpha_i} = f_{i,i+1}, h_{\alpha_i}\) satisfy the relations for the universal enveloping algebra \(U(\mathfrak{g})\) of the Lie superalgebra \(\mathfrak{g} = gl(m,n)\) (see 2.2).

(3) The element \(h_{\alpha_i}\) acts on \(\bar{K}(c)_c\) as multiplication by

\[
\begin{align*}
    r_i - (-1)^{b_i} r_{i+1}, & \quad i \in [1, m+n] \\
    r_{m+n}, & \quad i = m+n.
\end{align*}
\]
(4) In case of \( \mathbb{F} = \mathbb{C} \), the \( U(\mathfrak{g}) \)-module \( \tilde{K}(c) \) is a homomorphic image of the Kac module \( \mathcal{K}(\lambda) \) (see 2.2) with \( \lambda = \sum_{i=1}^{m+n} r_i \epsilon_i \in \Lambda \).

Proof. (1)-(3). We first verify that \( [h_{\alpha_m}, e_{m+1,m+2}] = e_{m+1,m+2} \). Indeed, we have

\[
\frac{K_m K_{m+1}^{-1} - K_m^{-1} K_{m+1}}{q_m - q_m^{-1}} E_{m+1,m+2} - E_{m+1,m+2} \frac{K_m K_{m+1}^{-1} - K_m^{-1} K_{m+1}}{q_m - q_m^{-1}} = E_{m+1,m+2} ((q_m - 1) \frac{K_m K_{m+1}^{-1} - K_m^{-1} K_{m+1}}{q_m - q_m^{-1}} + K_m^{-1} K_{m+1})
\]

in \( U_A \), which gives us

\[
h_{\alpha_m} e_{m+1,m+2} - e_{m+1,m+2} h_{\alpha_m} = e_{m+1,m+2}
\]
on \( \tilde{K}(c) \). The remaining relations can be proved similarly (cf. [16, 4.11]).

(4) Let \( v^+ \in K(c) \) be a maximal vector. Since \( K_A = U_A v^+ = (N_{-1})_A U_q(\mathfrak{g}_0)_A (N_1)_A v^+ = (N_{-1})_A U_q(\mathfrak{g}_0)_A v^+ \), we have

\[
\tilde{K}(c) = U(\mathfrak{g}_-)_A U(\mathfrak{g}_0)_A v^+ = U(\mathfrak{g}_-)_A U(sl_m \oplus sl_n) v^+.
\]

By the assumption on \( c \), the \( U(\mathfrak{g}_0) \)-submodule \( U(\mathfrak{g}_0)_A v^+ \) is integrable. Then it is a semisimple \( U(sl_m \oplus sl_n) \)-module by [11, 10.7]. Since it is generated by a unique maximal vector \( v^+ \), it is a simple \( U(sl_m \oplus sl_n) \)-module, and hence a simple \( U(\mathfrak{g}_0) \)-module. Thus, \( \tilde{K}(c) \) is a homomorphic image of the Kac module \( \mathcal{K}(\lambda) \).

Recall the notion \( E_1^d, d \in \{0,1\}^{\mathbb{Z}_1} \). Denote by \( E_0^\psi \) the standard monomial \( \Pi_{(i,j) \in \mathbb{Z}_0} E_i^\psi_j \) for each \( \psi = (\psi_{ij})_{(i,j) \in \mathbb{Z}_0} \in \mathbb{N}^{\mathbb{Z}_0} \). Set

\[
F_1^d = \Omega(E_1^d), d \in \{0,1\}^{\mathbb{Z}_1}, \quad F_0^\psi = \Omega(E_0^\psi), \psi \in \mathbb{N}^{\mathbb{Z}_0}.
\]

**Theorem 7.3.** The set of elements

\[ B = \{ F_1^d F_0^\psi | d \in \{0,1\}^{\mathbb{Z}_1}, \psi \in \mathbb{N}^{\mathbb{Z}_0} \} \]

is a \( \mathbb{F}(q) \)-basis of the subalgebra \( U_q^- \).

**Proof.** Let \( B_1 \) be any finite subset of \( B \). It suffices to show that the set \( B_1 \) is linearly independent. It’s no loss of generality to assume \( \mathbb{F} = \mathbb{C} \). Choose an integer \( \mu > 0 \) such that \( \psi_{ij} \leq \mu \) for all \((i,j) \in \mathbb{Z}_0\) and all \( \psi \) with \( F_1^d F_0^\psi \in B_1 \). By the representation theory of \( U(\mathfrak{g}_0) \), there is a finite dimensional simple \( U(\mathfrak{g}_0) \)-module with such a highest weight \( \lambda = \sum_{i=1}^{m+n} r_i \epsilon_i \) that \( \mu \leq r_i - r_{i+1} \) for all \( i \in [1,m+n) \setminus m \). We may also assume \( \lambda \) is typical by the note (1) in Sec. 2.2.

Let \( c = (q_{m+1}^r, \ldots, q_{m+n}^r) \), and let \( \tilde{K}(c) \) be as defined above. According to [10, Prop. 2.9], the \( U(\mathfrak{g}) \)-module \( \mathcal{K}(\lambda) \) is simple. Then Lemma 7.2(4) shows that \( \tilde{K}(c) \cong \mathcal{K}(\lambda) \). Therefore, our assumption implies that the elements in \( B_1 \) induce linearly independent endomorphisms of the Kac module \( \tilde{K}(c) \), so that the set \( B_1 \) is linearly independent. \( \square \)
Corollary 7.4. (PBW theorem) The following elements

\[ F_i^d q^\psi F_0^e K_\mu E_0^d E_1^d, \quad d, d' \in \{0, 1\}^I, \psi, \psi' \in N^{\mathbb{T}_0}, \mu \in \Lambda \]

form a \( \mathbb{F}(q) \)-basis of \( U_q \).

Consequently, we get an isomorphism of \( \mathbb{F}(q) \)-vector spaces:

\[ \mathcal{N}_{-1} \otimes U_q(\mathfrak{g}_0) \otimes \mathcal{N}_1 \rightarrow U_q, \quad u^- \otimes u_0 \otimes u^+ \rightarrow u^-u_0u^+, \quad u^\pm \in \mathcal{N}_{\pm 1}, u_0 \in U_q(\mathfrak{g}_0). \]

8 Generators and relations of \( U_A \)

In this subsection, we shall give a description of the \( A \)-superalgebra \( U_A \) in terms of generators and relations.

We shall consider the set consisting of the following variables:

\[ (a) \quad E_{ij}^{(N)}((i, j) \in \mathcal{I}, N \in \{\mathbb{N}, 0\}, \text{if } (i, j) \in \mathcal{I}_0 \), \]
\[ (b) \quad F_{ij}^{(N)}((i, j) \in \mathcal{I}, N \in \{0, 1\}, \text{if } (i, j) \in \mathcal{I}_1 \), \]
\[ (c) \quad K_{\alpha_i}, K_{\alpha_i}^{-1}, \left[ K_{\alpha_i}; c \right] (i \in [1, m + n], c \in \mathbb{Z}, t \in \mathbb{N}). \]

The parity of the variable is defined naturally. We denote the variable \( E_{i, i+1}^{(N)} \) (resp. \( F_{i, i+1}^{(N)} \)) also by \( E_{ii}^{(N)} \) (resp. \( F_{ii}^{(N)} \), \( i \in [1, m + n] \). The variable \( E_{ij}^{(1)} \) (resp. \( F_{ij}^{(1)} \)), \( (i, j) \in \mathcal{I} \) is also denoted by \( E_{ij} \) (resp. \( F_{ij} \)).

Let \( \mathcal{V}^+ \) be the \( A \)-superalgebra defined by the homogeneous generators \( (a) \) and relations

\[ (e0) \quad E_{ij}^{(0)} = 1, (i, j) \in \mathcal{I}, \quad E_{ij}^{(2)} = 0, (i, j) \in \mathcal{I}_1, \]
\[ (e1) - (e5) : \quad (e1)-(e5) \text{ in Lemma 6.1}. \]

Let \( \mathcal{V}^- \) be the \( A \)-superalgebra defined by the homogeneous generators \( (b) \) and relations

\[ (f0) \quad F_{i,j}^{(0)} = 1, (i, j) \in \mathcal{I}, \quad F_{ij}^{(2)} = 0, (i, j) \in \mathcal{I}_1, \]
\[ (f1) \quad F_{ij}^{(N)} F_{ij}^{(M)} = \left[ M + N \right]_{F_{ij}^{(N+M)}}, \]
\[ (f2) \quad F_{i,j}^{(N)} F_{s,t}^{(M)} = (-1)^{NMF_{ij}F_{st}F_{si}^{(M)}F_{ij}^{(N)}}, \]

for \( i < s < t < j \) or \( s < t < i < j \),

\[ (f3) \quad F_{t,a}^{(N)} F_{t,b}^{(M)} = [(-1)^{F_{t,a}q_{dt}}]^{NM} F_{t,b}^{(N)} F_{t,a}^{(M)} , \quad t < a < b, \]
\[ (f4) \quad F_{b,t}^{(N)} F_{a,t}^{(M)} = [(-1)^{F_{b,t}q_{dt}^{-1}}]^{NM} F_{a,t}^{(N)} F_{b,t}^{(M)} , \quad a < b < t, \]
\[ (f5) \quad F_{i,j}^{(M)} F_{i,c}^{(N)} = \sum_{0 \leq k \leq \min\{N,M\}} d_{c}^{M-k} F_{i,c}^{(N-k)} F_{i,j}^{(k)} F_{i,j}^{(M-k)}, \quad i < c < j. \]
Let $\mathcal{V}^0$ be the $\mathcal{A}$-algebra defined by the generators $(c)$ and relations [14, 2.3(g1)-(g5)] with $K_{\alpha_i}$ and $q_i$ in place of $K_i$ and $v$ respectively.

Recall the augmented Cartan matrix $\tilde{A}$. Let $\mathcal{V}$ be the $\mathcal{A}$-superalgebra defined by the homogeneous generators $(a)$, $(b)$ and $(c)$ and relations listed above together with relations (h1)-(h6) below:

\[(h1) \quad \left[ \frac{K_{\alpha_i}; c}{t} \right] E_{\alpha_j}^{(l)} = E_{\alpha_j}^{(l)} \left[ \frac{K_{\alpha_i}; c + la_{ij}}{t} \right] \]
\[(h2) \quad \left[ \frac{K_{\alpha_i}; c}{t} \right] F_{\alpha_j}^{(l)} = F_{\alpha_j}^{(l)} \left[ \frac{K_{\alpha_i}; c - la_{ij}}{t} \right] \]

for $l \in \mathbb{N}, i \in [1, m + n]$ and $j \in [1, m + n]$.

\[(h3) \quad E_{i,i+1}^{(M)} F_{j,j+1}^{(N)} = F_{j,j+1}^{(N)} E_{i,i+1}^{(M)} \quad \text{if} \quad i \neq j.\]

\[(h4) \quad E_{\alpha_i}^{(N)} F_{\alpha_i}^{(M)} = \sum_{0 \leq t \leq \min(M,N)} (-1)^{\delta_{m,N}(t-1)} F_{\alpha_i}^{(M-t)} \left[ \frac{K_{\alpha_i}; 2t - N - M}{t} \right] E_{\alpha_i}^{(N-t)}.\]

\[(h5) \quad K_{\alpha_i}^\epsilon E_{\alpha_j}^{(N)} = q_i^{\epsilon N a_{ij}} E_{\alpha_j}^{(N)} K_{\alpha_i}^\epsilon, \quad \epsilon = \pm 1, a_{ij} \in \tilde{A}.\]

\[(h6) \quad K_{\alpha_i}^\epsilon F_{\alpha_j}^{(N)} = q_i^{\epsilon N a_{ij}} F_{\alpha_j}^{(N)} K_{\alpha_i}^\epsilon.\]

Lemma 8.1. Let $i < c < j$. Then the following identities hold in $\mathcal{V}^+$.

\[(1) \quad E_{i,c}^{(N)} = \sum_{k=0}^{N} (-1)^{k} q_c^{-k} E_{c,c}^{(k)} E_{i,c} E_{c,j}^{(N-k)}.\]

\[(2) \quad E_{i,c}^{(M)} E_{c,j}^{(M+N)} E_{i,c}^{(N)} = E_{c,j}^{(N)} E_{i,c}^{(M+N)} E_{i,c}^{(M)}.\]

\[(3) \quad E_{i,j}^{(N)} = \sum_{k=0}^{N} (-1)^{k} q_c^{-k} E_{c,c}^{(N-k)} E_{i,j}^{(N)} E_{i,c}^{(k)}.\]

\[(4) \quad E_{c,j}^{(N)} E_{i,c}^{(M)} = \sum_{0 \leq k \leq \min(N,M)} (-1)^{k} q_{c}^{k+(N-k)(M-k)} E_{i,c}^{(M-k)} E_{c,j}^{(N-k)} E_{i,c}^{(k)}.\]

\[(5) \quad [E_{ij}, E_{st}] = (q_j - q_j^{-1}) E_{it} E_{sj}, i < s < j < t.\]

Proof. (1) In the righthand side we substitute $E_{i,c}^{(N)} E_{c,j}^{(N-k)}$ by the expression provided by (e5); applying the formula [8, 0.2 (4)], we get the left-hand side.

(2) Substitute $E_{i,c}^{(M)} E_{c,j}^{(M+N)}$ from the left-hand side and $E_{i,c}^{(M+N)} E_{c,j}^{(M)}$ from the righthand side by the expression provided by (e5), we get equal expressions.

(3) follows immediately from (1) and (2).

(4) If $E_{ic} = E_{ic}^{(N)}$ (resp. $E_{c,j} = E_{c,j}^{(N)}$), then we get $M = 1$ (resp. $N = 1$) by our convention. In the righthand side of (4), we substitute $E_{i,c}^{(M)} E_{c,j}^{(N)}$ by the expressions provided by (e5), then applying (e4)(resp. (e3)), we get the left-hand side of (4). Suppose $E_{ic} = E_{i,c}^{(N)} = 0$. In the righthand side of (4), we apply (e3), then substitute
$E_{ic}^{(M-k)}E_{cj}^{(N-k)}$ by the expression provided by (e5); performing cancelations with the formula $[8, 0.2 (4)]$, we get the left-hand side of (4).

(5) is the formula (6) following Lemma 4.2. From Sec.4 it follows from the identities

$$E_{ij} = E_{ic}E_{cj} - q_c^{-1}E_{cj}E_{ic}, \quad E_{si}E_{sj} = (-1)^{E_{si}q_s}E_{sj}E_{si}, s < i < j$$

of which the first one is given by (e5) with $N = M = 1$, and the second one is given by (e3).

We introduce in $\mathcal{V}$ the products

$$E_0^{(\psi)}E_1^d =: \Pi_{(i,j)\in I_0}E_{i,j}^{(\psi)}\Pi_{(i,j)\in I_1}E_{i,j}^{(d)}, \quad F_0^{(\psi)}F_1^d =: \Pi_{(i,j)\in I_0}F_{i,j}^{(d)}\Pi_{(i,j)\in I_1}F_{i,j}^{(\psi)},$$

$$\psi = (\psi_{ij})(i,j)\in \mathbb{Z}_0, d = (d_{ij})(i,j)\in \mathcal{I}_1 \in \{0, 1\}^{\mathcal{I}_1}$$

in the order given earlier.

Take a product $\xi_1\xi_2\cdots\xi_L$ in $\mathcal{V}^+$, with each $\xi_i$ in the form $E_{ij}^{(N_{ij})}, N_{ij} \geq 1$. Using (e1), two adjacent elements $\xi_i = E_{ab}^{(M)}, \xi_{i+1} = E_{cd}^{(N)}$ are always assumed to satisfy $(a, b) \neq (c, d)$, so that either $\xi_i \prec \xi_{i+1}$ or $\xi_{i+1} \prec \xi_i$. A product $\xi_1\xi_2\cdots\xi_L$ is said to be in good order if there is $s, 1 \leq s \leq L + 1$, such that $\bar{\xi}_i = 0$ for all $s \leq i \leq L + 1$, where by $s = L + 1$ (resp. $s = 1$) we mean that $\bar{\xi}_1 = \cdots = \bar{\xi}_L = 0$ (resp. $\bar{\xi}_1 = \cdots = \bar{\xi}_L = 1$).

Lemma 8.2. Each $\xi_1\xi_2\cdots\xi_L$ in $\mathcal{V}^+$ is equal to an $A$-linear combination of products in good order.

Proof. We first prove the case that $L = 2$, $\bar{\xi}_1 = 1$ and $\bar{\xi}_2 = 0$.

Let $\xi_1 = E_{st}$ with $(s, t) \in \mathcal{I}_1$, and let $\xi_2 = E_{ij}^{(N)}$ with $(i, j) \in \mathcal{I}_0, N \geq 1$. By the relations (e2)-(e4), we need only check the following cases:

(1) $t = i$. We have

$$\xi_1\xi_2 = E_{si}E_{ij}^{(N)}$$

(by (e5)) = $q_i^{-N}E_{ij}^{(N)}E_{si} + E_{ij}^{(N-1)}E_{sj}$,

where $E_{si} = 1$ and hence $E_{sj} = 1$.

(2) $s = j$. Using Lemma 8.1(4), one verifies this case similarly as in (1).
(3) \( s < i < t < j \). In this case we must have \( m < i \), since \( \bar{E}_{ij} = 0 \). Then

\[
\xi_1 \xi_2 = E_{st} E_{ij}^{(N)}
\]

(by Lemma 8.1(1))

\[
(\text{by Case 1}) = \sum_{k=0}^{N} (-1)^k q_t^k E_{st} E_{ij}^{(N)} E_{it}^{(N-k)} E_{tj}^{(N-k)}
\]

\[
(\text{using (e2),(e4)}) = \sum_{k=0}^{N} f_k(q) E_{tj}^{(k)} E_{it}^{(N)} E_{st} E_{tj}^{(N-k)} + \sum_{k=0}^{N} g_k(q) E_{tj}^{(k-1)} E_{it}^{(N)} E_{tj}^{(N-k)} E_{sj},
\]

where \( f_k(q), g_k(q) \in \mathcal{A} \).

From Case 1 we see that the first summation is equal to an \( \mathcal{A} \)-linear combination of elements in the form \( E_{tj}^{(k)} E_{it}^{(N)} E_{st} E_{tj}^{(N-k)} \) and \( E_{tj}^{(k)} E_{it}^{(N)} E_{tj}^{(N-k)} E_{sj} \). Clearly \( \bar{E}_{st} = \bar{E}_{sj} = \bar{1} \), and \( \bar{E}_{ij} = \bar{E}_{it} = 0 \).

(4) \( i < s < j < t \). In this case we must have \( j \leq m \). Using Lemma 8.1(4), the remainder of the proof is similar to that in Case 3 and is therefore omitted.

In summary, \( \xi_1 \xi_2 \) is an \( \mathcal{A} \)-linear combination of the products \( \xi_1 \cdots \xi_k \) such that \( \xi_1 = \cdots = \bar{\xi}_{k-1} = 0, \bar{\xi}_k = \bar{1} \). Then the case \( L > 2 \) follows from induction on the number of \( \xi_k \), \( 1 \leq i \leq L \), such that \( \xi_i = \bar{1} \). \( \square \)

**Proposition 8.3.** (a) \( \mathcal{V}^+ \) is generated as an \( \mathcal{A} \)-superalgebra by the elements \( E_{\alpha_i}^{(N)} = E_{i,i+1}^{(N)} \) \( (i \in [1, m+n), N \geq 0) \).

(b) \( \mathcal{V}^+ \) is generated as an \( \mathcal{A} \)-module by the monomials \( E_0^{(\psi)} E_1^d, \psi \in \mathbb{N}^{T_0}, d \in \{0, 1\}^{T_1} \).

*Proof. (a)* is an immediate consequence of Lemma 8.1(1).

(b) Clearly \( \mathcal{V}^+ \) is spanned as an \( \mathcal{A} \)-module by the products \( \xi_1 \xi_2 \cdots \xi_L \) as above. To prove (b), we must show that each \( \xi_1 \xi_2 \cdots \xi_L \) is an \( \mathcal{A} \)-linear combination of the monomials \( E_0^{(\psi)} E_1^d, \psi \in \mathbb{N}^{T_0}, d \in \{0, 1\}^{T_1} \). By the preceding lemma, we need only consider the following two cases.

Case 1. \( \bar{\xi}_1 = \cdots = \bar{\xi}_L = \bar{0} \). In this case we show that \( \xi_1 \cdots \xi_L \) is equal to an \( \mathcal{A} \)-linear combination of elements \( E_0^{(\psi)} \). With respect to the order given earlier, let \( \xi_l = E_{ij}^{(N)} \) be the minimal in \( \{\xi_1, \ldots, \xi_L\} \) (In case the minimal element is not unique, let \( \xi_l \) be the one with the largest \( l \)). We proceed by induction on the order of \( \xi_l \).

If \( \xi_l \) has the maximal order, say \( \xi_l = E_{m+n-1,m+n}^{(N)} \) for some \( N \geq 1 \) (in case \( n \geq 2 \)), then we must have \( L = 1 \), so that the product is \( \xi_1 \xi_2 \cdots \xi_L \) and already in the form as desired. Assume each product with the minimal elements \( \triangleright E_{ij} \) is equal to an \( \mathcal{A} \)-linear combination of \( E_0^{(\psi)} \)'s. Let \( \xi_1 \cdots \xi_L \) be any fixed product with the minimal element \( \xi_l = E_{ij}^{(N)} \) for some \( N \geq 1 \). We first claim that \( \xi_1 \xi_2 \cdots \xi_L \) is an \( \mathcal{A} \)-linear combination of monomials \( \xi_1^1 \xi_2^2 \cdots \xi_K^K \), each of which satisfies either \( \xi_1^1 = E_{i,j}^{(N')} \)
with \(1 \leq N'\) and \(\xi_j' > \xi_l\) for all \(2 \leq j \leq K\), or \(\xi_j' > \xi_l\) for all \(1 \leq j \leq K\). Once the claim is established, the induction hypotheses leads to (b).

To prove the claim, we proceed by induction on \(l\). The case \(l = 1\) is trivial. Assume \(l > 1\) and assume \(\xi_{l-1} = E_{st}^{(M)}\) with \(M \geq 1\), so that \(\xi_{l-1} > \xi_l\). We have by (e2)-(e4) that

\[
\xi_{l-1}\xi_l = E_{st}^{(M)} E_{ij}^{(N)} = f(q)\xi_l\xi_{l-1}, f(q) \in A
\]

in following cases:

\[
s = i; \quad t = j; \quad i < s < t < j; \quad j < s.
\]

In case \(j = s\), we get by Lemma 8.1(4) that

\[
\xi_{l-1}\xi_l = E_{jt}^{(M)} E_{ij}^{(N)} = \sum_{k \leq N,M} (-1)^k q_j k^{(M-k)(N-k)} E_{ij}^{(N-k)} E_{it}^{(M-k)} E_{jt}^{(M-k)}.
\]

Clearly we have \(\xi_l < E_{it}^{(k)}\), \(\xi_l < E_{jt}^{(M-k)}\) whenever \(k \geq 1\) and \(M - k \geq 1\).

We are left only with the case \(i < s < j < t\), in which we have

\[
\xi_{l-1}\xi_l = E_{st}^{(M)} E_{ij}^{(N)}
\]

(using Lemma 8.1(3) and (e4))

\[
= \sum_{k=0}^N f_k(q) E_{sj}^{(M-k)} E_{jt}^{(M)} E_{ij}^{(N)} E_{st}^{(k)}
\]

(using Lemma 8.1(4))

\[
= \sum_{k,N,M} f_{k,k'}(q) E_{sj}^{(M-k)} E_{ij}^{(N-k)} E_{it}^{(k')} E_{jt}^{(M-k')} E_{sj}^{(k')}
\]

(using (e4))

\[
= \sum f_{\bar{k},\bar{k}'}(q) E_{sj}^{(N-k')} E_{ij}^{(M-k')} E_{it}^{(k')} E_{jt}^{(M-k')} E_{sj}^{(k')},
\]

where \(f_k(q), f_{k,k'}(q), \bar{f}_{k,k'}(q) \in A\). Clearly we have (see Sec. 2)

\[
E_{ij} < E_{sj}, E_{ij} < E_{it}, E_{ij} < E_{jt}.
\]

Substituting \(\xi_{l-1}\xi_l\) in the product \(\xi_1 \cdots \xi_L\) by the expression provided by above formulas, combining adjacent terms using (e1) if necessary, we obtain that \(\xi_1 \cdots \xi_L\) is equal to an \(A\)-linear combination of products \(\xi_1' \cdots \xi_K'\), in each of which the minimal element \(\xi_s'\) either is \(E_{ij}^{(N')}\) for some \(N' \geq 1\) with \(s < l\) or satisfies \(\xi_s' = \xi_l\). Then the claim follows from induction hypotheses on \(l\). This establishes Case 1.

Case 2. \(\bar{\xi}_1 = \cdots = \bar{\xi}_L = 1\). In this case each \(\xi_i\) is equal to \(E_{st}\) for some \((s, t) \in \mathcal{I}_l\), since \(E_{st}^{(2)} = 0\).

Assume \(L = 2\) and \(\xi_2 < \xi_1\). Let \(\xi_1 = E_{st}\) and let \(\xi_2 = E_{ij}\). We show that

\[
(\ast) \quad \xi_1\xi_2 = \sum_{\xi_2 \leq x_i \leq \xi_1, c_i \xi_1 y_i, c_i \in A}
\]
where each \(x_i\) or \(y_i\) is in the form \(E_{ij}, (i, j) \in \mathcal{I}_1\). In view of the proof of Case 1, one needs only verify the case \(i < s < j < t\), in which we have by Lemma 8.1(5) that

\[
\xi_1 \xi_2 = E_{st}E_{ij} = -E_{ij}E_{st} + (q_j - q_j^{-1})E_{it}E_{sj}.
\]

By definition, \(E_{ij} \prec E_{it} \prec E_{sj} \prec E_{st}\), so the formula (*) follows.

Using the formula (*) and applying the induction on the minimal element in a product \(\xi_1 \cdots \xi_L\) as in Case 1, we obtain that \(\xi_1 \cdots \xi_L\) is an \(\mathcal{A}\)-linear combination of \(E^d\), as desired. 

Since there is a unique super-ring isomorphism \(\mathcal{V}^- \to (\mathcal{V}^+)^{opp}\) which carries \(F^{(N)}_{ij}\) into \(F^{(N)}_{ij}(i, j) \in \mathcal{I}\) and \(q \to q^{-1}\), we get

**Proposition 8.4.** (a) \(\mathcal{V}^-\) is generated as an \(\mathcal{A}\)-superalgebra by the elements \(F^{(N)}_{\alpha_i}(i \in [1, m+n], N \geq 0)\).

(b) \(\mathcal{V}^-\) is generated as an \(\mathcal{A}\)-module by the monomials

\[
F^d F_0^{(\psi)} (\psi \in \mathbb{N}^\mathcal{I}, d \in \{0, 1\}^\mathcal{I}).
\]

By [14, 2.14], \(\mathcal{V}^0\) is generated as an \(\mathcal{A}\)-module by the elements

\[
K^\delta_{\alpha_1} \cdots K^\delta_{\alpha_m+n} \left[ K_{\alpha_1}; 0 \right] \cdots \left[ K_{\alpha_m+n}; 0 \right],
\]

\(\delta_i \in \{0, 1\}, t_i \in \mathbb{N}\).

Clearly we have \(\mathcal{A}\)-superalgebra homomorphisms from \(\mathcal{V}^-, \mathcal{V}^0\), and \(\mathcal{V}^+\) to \(\mathcal{V}\) stabilizing the generators. Then we obtain an \(\mathcal{A}\)-linear map \(\pi: \mathcal{V}^- \otimes \mathcal{A} \mathcal{V}^0 \otimes \mathcal{A} \mathcal{V}^+ \to \mathcal{V}\).

It follows from the defining relations (h1) – (h6) of the \(\mathcal{A}\)-superalgebra \(\mathcal{V}\) that \(\pi\) is surjective.

**Proposition 8.5.** (a) \(\mathcal{V}\) is generated as an \(\mathcal{A}\)-superalgebra by the homogeneous elements

\[
E^{(N)}_{\alpha_i}, F^{(N)}_{\alpha_i}, K^\pm_{\alpha_j}, \left[ K_{\alpha_j}; 0 \right], i \in [1, m+n], j \in [1, m+n], N \geq 0, t \geq 0.
\]

(b) \(\mathcal{V}\) is generated as an \(\mathcal{A}\)-module by the elements

\[
F^d F_0^{(\psi)} \prod_{i=1}^{m+n} (K^\delta_{\alpha_i} \left[ K_{\alpha_i}; 0 \right]) E_0^{(\psi')} F^d',
\]

\(d, d' \in \{0, 1\}^\mathcal{I}, \psi, \psi' \in \mathbb{N}^\mathcal{I}, \delta_i \in \{0, 1\}, t_i \geq 0\).

**Proof.** (b) follows from the surjective map \(\pi\), Prop. 8.3(b) and Prop. 8.4(b).

By Prop. 8.3(a), Prop. 8.4(a) and the definition of \(\mathcal{V}\), \(\mathcal{V}\) is generated as an \(\mathcal{A}\)-superalgebra by the elements

\[
E^{(N)}_{\alpha_i}, F^{(N)}_{\alpha_i}, K^\pm_{\alpha_j}, \left[ K_{\alpha_j}; c \right], i \in [1, m+n], j \in [1, m+n], c \in \mathbb{Z}, t \in \mathbb{N}.
\]
To prove (a), one needs only show that, for any $c \in \mathbb{Z}$, $t \in \mathbb{N}$, the element $\left[K_{\alpha_j}^c, t\right]_t$, $1 \leq j \leq m + n$ is generated by the elements $K_{a_i}^{\pm 1} \left[K_{\alpha_j}^0, t\right]_t$, $i \in [1, m + n], t \geq 0$. This is given in [14, 2.17].

We now form the $\mathcal{A}'$-superalgebras $\mathcal{V}^+_{\mathcal{A}'}$, $\mathcal{V}^-_{\mathcal{A}'}$, and $\mathcal{V}_{\mathcal{A}'}$ by applying $- \otimes \mathcal{A}'$ to $\mathcal{V}^+$, $\mathcal{V}^-$, $\mathcal{V}_0$, and $\mathcal{V}$. Write $E_{ij}^{(1)} \otimes 1$ and $F_{ij}^{(1)} \otimes 1$ as $E_{ij}$ and $F_{ij}$ respectively.

**Proposition 8.6.** $\mathcal{V}_{\mathcal{A}'}$ is the $\mathcal{A}'$-superalgebra defined by the generators $E_{ij}, F_{ij}(i, j) \in \mathcal{I}$, $K_{a_i}^{\pm 1}(s \in [1, m + n])$, and the following relations:

(a1) $E_{ij}^2 = 0$, $(i, j) \in \mathcal{I}_1$, 

(a2) $E_{ij}E_{st} = (-1)^{E_{ij}E_{st}E_{ij}}E_{st}E_{ij}$ if $i < s < t < j$ or $s < t < i < j$, 

(a3) $E_{ta}E_{tb} = (-1)^{E_{ta}E_{tb}}E_{tb}E_{ta}$, $t < a < b$,

(a4) $E_{bt}E_{at} = (-1)^{E_{bt}E_{at}}E_{at}E_{bt}$, $a < b < t$, 

(a5) $E_{ij} = E_{ic}E_{cj} - q_{ij}E_{c}E_{ic}E_{cj}$, $i < c < j$, 

(b1) $F_{ij}^2 = 0$, $(i, j) \in \mathcal{I}_1$, 

(b2) $F_{ij}F_{st} = (-1)^{F_{ij}F_{st}F_{ij}}F_{st}F_{ij}$ if $i < s < t < j$ or $s < t < i < j$, 

(b3) $F_{ta}F_{tb} = (-1)^{F_{ta}F_{tb}}F_{tb}F_{ta}$, $t < a < b$, 

(b4) $F_{bt}F_{at} = (-1)^{F_{bt}F_{at}}F_{at}F_{bt}$, $a < b < t$, 

(b5) $F_{ij} = -q_{ij}F_{ic}F_{cj} + F_{cj}F_{ic}$, $i < c < j$, 

(c1) $K_{a_i}K_{a_j} = K_{a_j}K_{a_i}$, 

(c2) $K_{a_i}K_{a_i}^{-1} = 1$, 

(d1) $E_{a_i}F_{a_j} - (-1)^{\delta_{a_i}a_j}F_{a_j}E_{a_i} = \delta_{ij}\frac{K_{a_i} - K_{a_i}^{-1}}{q_i - q_i^{-1}}$, $i, j \in [1, m + n)$, 

(d2) $K_{a_i}E_{a_j} = q_{a_i}^{-a_j}E_{a_j}K_{a_i}$, $a_{ij} \in \bar{A}$, 

Proof. It is clear that all the formulas above follow immediately from the defining relations of $\mathcal{V}$. To complete the proof, we must show that, conversely, all the defining relations of $\mathcal{V}$ follow from above formulas. This can be verified by induction (see Sec.6).
Similarly one obtains that $V^+_A$ (resp. $V^-_A; V^0_A$) is the $A'$-superalgebra defined by the generators $E_{ij}((i, j) \in \mathcal{I})$ (resp. $F_{ij}((i, j) \in \mathcal{I}); K^{\pm 1}_{\alpha_i}(s \in [1, m+n])$ and relations (a1)-(a5)(resp. (b1)-(b5); (c1)-(c2)). Since the generators of $U^0_A$ also satisfy the relations (c1), (c2), there is the canonical epimorphism $f : V^0_A \rightarrow U^0_A$ such that

$$f(K_{\alpha_i}) = \begin{cases} K_i K_{i+1}^{-1}, & i \in [1, m+n) \\ K_{m+n}, & i = m+n. \end{cases}$$

Then Coro.3.11(4) implies that $f$ is an isomorphism. By the PBW theorem and [14, 2.21], $U_A$ has the following PBW basis

$$F^d_1 F^\psi_0 \prod_{i=1}^{m+n} (K^{\delta_i}_{\alpha_i} \left[ K_{\alpha_i}; 0 \right]) E^{\psi'}_0 E^d_1$$

$$(d, d' \in \{0, 1\}^{I_A}, \delta_i \in \{0, 1\}, \psi, \psi' \in \mathbb{N}^{I_A}, t_i \in \mathbb{N}).$$

Since the relations in Prop. 8.6 are also satisfied by the generators in $U_A$ of the same notion, we get a unique $A'$-superalgebra epimorphism $\rho : V_A \rightarrow U_A$ such that

$$\rho(E_{ij}) = E_{ij}, \quad \rho(F_{ij}) = F_{ij}, \quad \rho(K_{\alpha_i}) = K_{\alpha_i}.$$ 

By the definition of $U_A$, we obtain

$$\rho(V) = U_A, \quad \rho(V^\pm) = U^\pm_A, \quad \rho(V^0) = U^0_A.$$ 

By Prop. 8.5(b), $\rho$ carries a set of vectors that spans $V$ as an $A$-module to the PBW-type basis of $U_A$. It follows that $\rho$ is an $A'$-superalgebra isomorphism and the set vectors in Prop. 8.5(b) forms an $A$-basis of $V_A$ (hence an $A'$-basis of $V_A$). Then $\rho_V : V \rightarrow U_A$ is an $A$-superalgebra isomorphism. This implies that $U_A \cong U_A \otimes A$. 

By induction, one obtains

$$\Delta(E^{(N)}_{\alpha_i}) = \sum_{j=0}^{N} q_i^{j(N-j)} E^{(j)}_{\alpha_i} \otimes K^{N-j}_{\alpha_i} E^{(N-j)}_{\alpha_i},$$

$$\Delta(F^{(N)}_{\alpha_i}) = \sum_{j=0}^{N} q_i^{j(N-j)} F^{(j)}_{\alpha_i} K^{-j}_{\alpha_i} \otimes F^{(N-j)}_{\alpha_i}.$$ 

Then the $A$-superalgebra $U_A$ obtains a unique Hopf superalgebra structure from $U_A$. 

### 9 The relations with modular representations

Assume $\mathbb{F}$ is a field of characteristic $p > 2$. Let $G$ be the general linear $\mathbb{F}$-supergroup $GL(m, n)$. In this section we study the relations between quantum supergroups and modular representations of $G$. 

30
9.1 Kostant $\mathbb{Z}$-forms

Let $U(\mathfrak{g})_Q$ be the universal enveloping superalgebra of the Lie superalgebra $\mathfrak{g} = gl(m,n)$ over $Q$. Recall the maximal torus $H$ of $\mathfrak{g}$. Let $U(H)_Q \subseteq U(\mathfrak{g})_Q$ be its universal enveloping algebra. For each $h \in H$ and each $r \in \mathbb{N}$, set

\[
\binom{h}{r} = \frac{1}{r!} h(h-1) \cdots (h-r+1) \in U(H)_Q.
\]

As defined in [3], the Kostant $\mathbb{Z}$-form $U(\mathfrak{g})_\mathbb{Z}$ is a $\mathbb{Z}$-sub-superalgebra of $U(\mathfrak{g})_Q$ generated by

\[
e_{ij}^{(r)}(i,j) \in \mathcal{I}, r \geq 0, \left( e_{ss}^{(r)} \right)_r(s \in [1,m+n], r \geq 0).
\]

By [3, 3.1], $U(\mathfrak{g})_\mathbb{Z}$ is a free $\mathbb{Z}$-module with a basis consisting of all the monomials of the form

\[
\Pi_{(i,j) \in \mathcal{I}} f_{ij}^{d_{ij}} \Pi_{(i,j) \in \mathcal{I}_0} f_{ij}^{(a_{ij})} \Pi_{s=1}^{m+n} \left( e_{ss}^{(r)} \right)_r \Pi_{(i,j) \in \mathcal{I}} e_{ij}^{d_{ij}},
\]

$a_{ij}', a_{ij}, r_s \geq 0, d_{ij}', d_{ij} = 0, 1$, where the product is taken in any fixed order.

Recall the notation $h_{a_i}, i \in [1,m+n]$. Let $T$ be the maximal torus of $G$ such that, for each commutative superalgebra $A$, $T(A)$ is the subgroup of $G(A)$ consisting of all diagonal matrices. Choose a basis of $\phi_i, \phi_1, \cdots, \phi_{m+n}$ of $Y(T)$ defined by

\[
\phi_i(t) = \begin{cases} 
\text{diag}(1 \cdots 1, t, t^{-1}, 1, \cdots, 1) & \text{if } i \in [1,m+n) \setminus m \\
\text{diag}(1, \cdots, 1, t^{(i)}_{(i+1)}, 1, \cdots, 1) & i = m \\
\text{diag}(1, \cdots, 1, t^{(m+n-1)}_{(m+n)}, \cdots, t^{(m+n)}_{(m+n)}) & i = m+n,
\end{cases}
\]

for any $t \in G_m(A)$. Then under the isomorphism from $U_\mathbb{Z} \otimes \mathbb{F}$ into $\text{Dist}(G)$ provided by [3, Th. 3.2], we have $h_{a_i} = (d \phi_i)(1), i \in [1,m+n]$. By [9, II, 1.11], the set of all $\Pi_{i=1}^{m+n} (h_{a_i})^r_i, r_i \geq 0$ is a basis of $\text{Dist}(T_\mathbb{Z})$. By taking a natural basis of $Y(T)$, one gets another basis of $\text{Dist}(T_\mathbb{Z})$ (see [3]) consisting of elements $\Pi_{i=1}^{m+n} (e_i)^r_i, r_i \geq 0$. This gives us

**Lemma 9.1.** $U(\mathfrak{g})_\mathbb{Z}$ has a $\mathbb{Z}$-basis consisting of all the monomials

\[
\Pi_{(i,j) \in \mathcal{I}} f_{ij}^{d_{ij}'} \Pi_{(i,j) \in \mathcal{I}_0} f_{ij}^{(a_{ij})} \Pi_{s=1}^{m+n} \left( h_{a_i}^{r_s} \right)_r \Pi_{(i,j) \in \mathcal{I}} e_{ij}^{d_{ij}},
\]

$a_{ij}', a_{ij}, r_s \geq 0, d_{ij}', d_{ij} \in \{0,1\}$, where the product is taken in any fixed order.

Similarly one can describe the Kostant $\mathbb{Z}$-form $U(\mathfrak{g}_0)_\mathbb{Z}$ and its $\mathbb{Z}$-bases.

The closed $\mathbb{F}$-subgroups $G_{ev}, P$ of $G$ are defined in [3] as follows. For each commutative superalgebra $A$, let $P(A)$ (resp. $G_{ev}(A)$) be the group of all invertible
$(m+n) \times (m+n)$ matrices of the same form as the one in $G(A)$ with the additional condition $Y = 0$ (resp. $Y = 0, X = 0$). Then we have $\text{Lie}(P) = g^+$.

The Kostant $\mathbb{Z}$-form $U(g^+)_\mathbb{Z}$ is a free $\mathbb{Z}$-module with a basis being given by the set of all monomials of the form

$$\Pi_{(i,j) \in \mathcal{I}_0} f^{(a_{ij})}_{i,j} \Pi_{1 \leq i \leq m+n} \left( h_{\alpha_i} \right) \Pi_{(i,j) \in \mathcal{I}_0} e^{(a'_{ij})}_{i,j} \Pi_{(i,j) \in \mathcal{I}_1} d_{ij}$$

for all $a_{ij}, a'_{ij}, r_i \in \mathbb{N}$ and $d_{ij} \in \{0, 1\}$, where the product is taken in any fixed order. Then by [3, Th.3.2], we have

$$U(g)_\mathbb{F} = U(g)_\mathbb{Z} \otimes \mathbb{F} \cong \text{Dist}(G), \quad U(g^+)_\mathbb{F} = U(g^+)_\mathbb{Z} \otimes \mathbb{F} \cong \text{Dist}(P).$$

To establish Th. 9.15 later, we introduce two more $\mathbb{Z}$-subalgebras of $U(g)_\mathbb{Z}$. Let $U(g^-)_\mathbb{Z}$ be the $\mathbb{Z}$-subalgebra of $U(g)$ generated by the elements $f_{ij}, (i,j) \in \mathcal{I}_1$. Then it is easy to see that $U(g^-)_\mathbb{Z}$ is a free $\mathbb{Z}$-module with a basis consisting of elements $\Pi_{(i,j) \in \mathcal{I}_1} d_{ij}, d_{ij} \in \{0, 1\}$, where the product is taken in any fixed order. Let $U(g^+_\mathbb{Z})$ be the $\mathbb{Z}$-submodule of $U(g)_\mathbb{Z}$ spanned by the basis vectors

$$\Pi_{(i,j) \in \mathcal{I}_0} f^{(a_{ij})}_{i,j} \Pi_{1 \leq i \leq m+n} \left( h_{\alpha_i} \right) \Pi_{(i,j) \in \mathcal{I}_0} e^{(a'_{ij})}_{i,j} \Pi_{(i,j) \in \mathcal{I}_1} d_{ij}$$

for all $a_{ij}, a'_{ij}, r_i \in \mathbb{N}$ and $d_{ij} \in \{0, 1\}$ with $\sum_{(i,j) \in \mathcal{I}_1} d_{ij} > 0$. Then it is routine to verify that $U(g^+_\mathbb{Z})$ is a $\mathbb{Z}$-subalgebra of $U(g)_\mathbb{Z}$. Clearly we have

$$U(g^+)_\mathbb{Z} = U(g_0)_\mathbb{Z} \oplus U(g^+)_\mathbb{Z}. $$

Set

$$U(g_0)_\mathbb{F} = U(g_0)_\mathbb{Z} \otimes \mathbb{F}, \quad U(g^-)_\mathbb{F} = U(g^-)_\mathbb{Z} \otimes \mathbb{F}, U(g^+_\mathbb{F}) = U(g^+_\mathbb{Z}) \otimes \mathbb{Z} \mathbb{F}.$$ 

Then it is clear that $U(g^-)_\mathbb{F}$ is the universal enveloping superalgebra of $g_1$ (see 2.2). Let us observe that $U(g^+_\mathbb{F})$ is the two-sided ideal of $U(g)_\mathbb{F}$ generated by the elements $e_{ij}, (i,j) \in \mathcal{I}_1$, and which is easily seen to be nilpotent. Since $U(g^+_\mathbb{F}) = U(g_0)_\mathbb{F} \oplus U(g^+_\mathbb{F})$, it follows that each simple $\text{Dist}(P)$-module is a simple $\text{Dist}(G_{ev})$-module annihilated by $U(g^+_\mathbb{F})$.

In the following, we identify $\Lambda$ with $\mathbb{Z}^{m+n}$ by mapping each $\lambda \in \Lambda$ into

$$(\lambda(h_{\alpha_1}), \cdots, \lambda(h_{\alpha_{m+n}})) \in \mathbb{Z}^{m+n}. $$

Then each $z = (z_1, \cdots, z_{m+n}) \in \mathbb{Z}^{m+n}$ is equal to $\lambda = \sum_{i=1}^{m+n} \lambda_i e_i \in \Lambda$ such that $z_i = \lambda_i - (-1)^{\delta_{im}} \lambda_{i+1}$, $i \in [1, m+n)$, $z_{m+n} = \lambda_{m+n}$. We say that $z \in \mathbb{Z}^{m+n}$ is $p$-typical if the corresponding $\lambda \in \Lambda$ is $p$-typical (see Sec.2).

Let $M$ be a $\text{Dist}(G)$-module. For each $z = (z_1, \cdots, z_{m+n}) \in \mathbb{Z}^{m+n}$, define the $z$-weight space of $M$ by

$$M_z = \{ m \in M | (h_{\alpha_i})_m = (z_i)_m \text{ for all } i = 1, \cdots, m+n, r \geq 1 \}. $$
For \( i \in [1, m+n] \), \( r \geq 1 \), by the discussion preceding Lemma 9.1 one can write \( (e_i)_r \) as a \( \mathbb{Z} \)-linear combination of products \( \Pi_{i,n}(h^0_{ij}, \bar{s}) \) \( U \mathbb{Z} \). This implies that the \( z \)-weight space \( M_z \) is exactly the \( \lambda \)-weight space (see [3, 3.3])

\[
M_\lambda = \{ m \in M | \left( \frac{e_i}{r} \right) m = \left( \frac{\lambda_i}{r} \right) m \text{ for all } i = 1, \ldots, m+n, r \geq 1 \},
\]

where \( \lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i \) is the image of \( z \) under the bijective map from \( \mathbb{Z}^{m+n} \) into \( \Lambda \) given above.

Following [3, 13], set

\[
X^+(T) = \{ \sum_{i=1}^{m+n} \lambda_i \epsilon_i | \lambda_1 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \}
\]

and let

\[
X^+_p(T) := \{ \lambda \in X^+(T) | \lambda_i - \lambda_{i+1} < p \text{ for all } i \in [1, m+n] \setminus m \}.
\]

For each \( \lambda \in X^+(T) \), let \( L(\lambda)(\text{resp. } L_0(\lambda)) \) be a simple \( G \)-module (resp. \( G_{ev} \)-module) with highest weight \( \lambda \). One can view \( L_0(\lambda) \) as a Dist(\( P \))-module on which all \( e_{ij}, (i, j) \in \mathcal{I}_1 \) act trivially. Define the induced \( G \)-module (see [3, p. 11])

\[
\text{Ind}_p^G \lambda =: \text{Dist}(G) \otimes_{\text{Dist}(P)} L_0(\lambda).
\]

A Lie superalgebra \( L = L_0 \oplus L_1 \) is called a *restricted Lie superalgebra* if \( L_0 \) is a restricted Lie algebra and \( L_1 \) is a restricted \( \mathfrak{g}_0 \)-module under the adjoint action. Let \( [p]: x \rightarrow x^{[p]} \) be the \( p \)-map in \( L_0 \). The quotient superalgebra of \( U(L) \) by its \( \mathbb{Z}_2 \)-graded ideal generated by the elements \( x^p - x^{[p]}, x \in L_0 \) is called the *reduced enveloping superalgebra* of \( L \), and denoted by \( u(L) \) (see [2]).

Example: The Lie superalgebra \( \mathfrak{g} = \mathfrak{gl}(m, n) \) is a restricted Lie superalgebra with \( p \)-map the \( p \)th power in \( \mathfrak{g} \). The Lie subalgebras \( \mathfrak{g}^+, \mathfrak{g}_0 \) are its restricted subalgebras.

Let us note that, by a similar proof to that for [9, 7.10(1)], the subalgebra of Dist(\( G \)) generated by the elements \( e_{ij}, f_{ij}, h_{\alpha_s}(i, j) \in \mathcal{I}, s \in [1, m+n] \) is isomorphic to \( u(\mathfrak{g}) \), and the subalgebra generated by the elements \( e_{ij}, (i, j) \in \mathcal{I}, f_{ij}, (i, j) \in \mathcal{I}_0, h_{\alpha_s}, s \in [1, m+n] \) is isomorphic to \( u(\mathfrak{g}^+) \). Let \( G_1 \) be the first Frobenius kernel of \( G \) (see [13, Sec.3]). Then we get by [13, 3.1] that \( u(\mathfrak{g}) \cong \text{Dist}(G_1) \).

### 9.2 Induced modules

Recall that we write \( \lambda \otimes 1 \in H^* \) as \( \bar{\lambda} \) for each \( \lambda \in \Lambda \), there is no confusion to write \( \alpha \otimes 1(\alpha \in \Phi^+) \), \( \rho \otimes 1 \) also as \( \alpha, \rho \) respectively.

Since \( u(\mathfrak{g}^+) \) is a subalgebra of Dist(\( P \)), the simple Dist(\( P \))-module \( L_0(\lambda) \) becomes a \( u(\mathfrak{g}^+) \)-module, and hence a \( U(\mathfrak{g}^+) \)-module by the canonical epimorphism from \( U(\mathfrak{g}^+) \) onto \( u(\mathfrak{g}^+) \). Let \( \lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i \) and let \( v^+ \in L_0(\lambda) \) be a maximal vector, unique up to scalar. Then we have \( e_{ii}v^+ = \lambda_i v^+, i \in [1, m+n] \); hence, \( hv^+ = \bar{\lambda}(h)v^+ \) for any \( h \in H \).
Lemma 9.2. Let $\lambda \in \Lambda$.

(1) There is a $u(\mathfrak{g})$-module isomorphism $\phi_1 : u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L_0(\lambda) \rightarrow Ind_{P}^{G} \lambda$.

(2) There is a $U(\mathfrak{g})$-module isomorphism $\phi_2 : U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} L_0(\lambda) \rightarrow u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L_0(\lambda)$.

Proof. (1) It is easy to check that the map $\phi_1$ from $u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L_0(\lambda)$ into $Ind_{P}^{G} \lambda$ with

$$\phi_1(x \otimes m) = x \otimes m, \quad x \in u(\mathfrak{g}), \quad m \in L_0(\lambda)$$

is well-defined. Clearly $\phi_1$ is a $u(\mathfrak{g})$-module homomorphism. Moreover, $\phi_1$ is an epimorphism, since we have by Lemma 9.1 that Dist($G$) has a basis consisting of elements $\Pi_{(i,j) \in I \subseteq \Delta} f_{ij} u_k$, where the elements $u_k$ is a basis of Dist($P$), so that $Ind_{P}^{G} \lambda$ is spanned by the elements $\Pi_{(i,j) \in I \subseteq \Delta} f_{ij} \otimes m$, $m \in L_0(\lambda)$. According to [9, 2.14(1)], $L_0(\lambda)$ is finite dimensional, so that $\phi_1$ is an isomorphism since

$$\dim u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L_0(\lambda) = \dim Ind_{P}^{G} \lambda.$$

(2) Let $\pi$ be the canonical epimorphism from $U(\mathfrak{g})$ into $u(\mathfrak{g})$. Then $\pi(U(\mathfrak{g}^+)) = u(\mathfrak{g}^+)$. One verifies easily that the map

$$\phi_2 : U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^+)} L_0(\lambda) \rightarrow u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L_0(\lambda)$$

such that $\phi_2(x \otimes m) = \pi(x) \otimes m, \quad x \in U(\mathfrak{g}), m \in L_0(\lambda)$ is well-defined. By a similar argument as in (1), one proves that $\phi_2$ is a $U(\mathfrak{g})$-module isomorphism. \[\square\]

Recall the notion $K(\mu)(\mu \in H^*)$ in Sec. 2. If $\lambda \in X^+_p(T)$, then $L_0(\lambda)$ is a simple $u(\mathfrak{g}_0)$-module of highest weight $\lambda$, so we have by Lemma 9.2 that

$$u(\mathfrak{g}) \otimes_{u(\mathfrak{g}^+)} L_0(\lambda) \cong K(\lambda).$$

Recall the symmetric bilinear form on $H^*$. Define

$$\tilde{P}(\mu) = \Pi_{x \in \Phi^+_1} (\mu + \rho, x) \mu \in H^*,$$

so that $\tilde{P}(\lambda) = P(\lambda)1_{\mathbf{F}}$ for any $\lambda \in \Lambda$.

Let $v^+ \in L_0(\lambda)$ be a maximal vector. According to [18, Th. 4.2], we have in $K(\lambda)$ that

$$\Pi_{(i,j) \in I \subseteq \Delta} f_{ij} \otimes v^+ = \tilde{P}(\lambda)v^+ = P(\lambda)v^+.$$

By [18, Prop. 3.1], we have

Lemma 9.3. For $\lambda \in \Lambda$, $K(\lambda)$ is simple if and only if $\lambda$ is $p$-typical.
9.3 Simple $G$-modules

In this subsection we determine the simplicity of the $\text{Dist}(G)$-module $\text{Ind}_P^G \lambda$.

**Theorem 9.4.** Assume $\mathbb{F}$ is algebraically closed. If $\lambda \in X^+(T)$ is $p$-typical, then

\[ \text{Ind}_P^G \lambda \cong L(\lambda). \]

**Proof.** We split the proof into two cases according to whether $\lambda \in X_p^+(T)$ or not.

Case 1. $\lambda \in X_p^+(T)$. By the discussion above we have $\text{Ind}_P^G \lambda \cong \mathcal{K}(\lambda)$. Since $\lambda$ is $p$-typical, we get by Lemma 9.3 that $\text{Ind}_P^G \lambda$ is a simple $U(\mathfrak{g})$-module, and hence a simple $u(\mathfrak{g})$-module. Recall that $u(\mathfrak{g}) \cong \text{Dist}(G_1)$. Then [13, 4.3] shows that $\text{Ind}_P^G \lambda \cong L(\lambda)$.

Case 2. $\lambda \notin X_p^+(T)$. Note that $\lambda$ can be written uniquely as $\lambda = \lambda' + p\lambda''$ with $\lambda' \in X_p^+(T)$ being $p$-typical and $\lambda'' \in X^+(T)$. By the Steinberg tensor product theorem([9, Coro. 3.17]), we have an isomorphism of $\text{Dist}(P)$-modules

\[ L_0(\lambda) \cong L_0(\lambda') \otimes L_0(\lambda'')^	ext{[1]} . \]

By [13, Th.4.4], The simple $G$-module $L(\lambda)$ is isomorphic to $L(\lambda') \otimes L_0(\lambda'')^	ext{[1]}$. Since $\lambda'$ is $p$-typical, we get $L(\lambda') \cong \text{Ind}_P^G \lambda'$ from Case 1. Clearly, the embedding of the $\text{Dist}(P)$-module $L_0(\lambda') \otimes L_0(\lambda'')^	ext{[1]}$ into $L(\lambda') \otimes L_0(\lambda'')^	ext{[1]}$ induces a nontrivial $\text{Dist}(G)$-module homomorphism $f$ from $\text{Ind}_P^G \lambda$ into $L(\lambda') \otimes L_0(\lambda'')^	ext{[1]}$. Then the simplicity of the latter implies that $f$ is surjective. Note that the codimension of $\text{Dist}(P)$ in $\text{Dist}(G)$ is $2^{|\mathcal{I}_1|} = 2^{nm}$. Hence, $f$ is isomorphic since

\[
\dim \text{Ind}_P^G \lambda = 2^{nm} \dim L_0(\lambda) \\
= 2^{nm} \dim L_0(\lambda') \dim L_0(\lambda'') \\
= \dim \text{Ind}_P^G \lambda' \dim L_0(\lambda'') \\
= \dim L(\lambda') \otimes L_0(\lambda'')^	ext{[1]},
\]

so we get $\text{Ind}_P^G \lambda \cong L(\lambda)$. \qed

Let $U(\mathfrak{g})_{\overline{\mathbb{F}}_p} = U(\mathfrak{g})_\mathbb{Z} \otimes _\mathbb{Z} \overline{\mathbb{F}}_p$ and let $\overline{\mathfrak{u}}$ be the sub-superring of $U(\mathfrak{g})_{\overline{\mathbb{F}}_p}$ generated by the elements $e_{ij}, f_{ij}((i,j) \in \mathcal{I}), e_{m+n,m+n}$. For any field $k$ of characteristic $p > 2$, set

\[ \overline{\mathfrak{u}}_k := \overline{\mathfrak{u}} \otimes _{\overline{\mathbb{F}}_p} k. \]

Recall in Sec.9.1 that $\overline{\mathfrak{u}}_k$ is the reduced enveloping algebra of the Lie superalgebra $\mathfrak{g} = \text{gl}(m,n)$ over $k$; that is, $\overline{\mathfrak{u}}_k = u(\mathfrak{g})$. The following result is due to [13, Prop. 3.4].

**Lemma 9.5.** Every simple $\overline{\mathfrak{u}}_k$-module contains a unique (up to scalar multiple) homogeneous element $v^+ \neq 0$ such that $e_{ij} v^+ = 0$ for any $(i,j) \in \mathcal{I}$. There exists $z = (z_1, \ldots, z_{m+n}) \in \mathbb{F}_{p^{m+n}}$ such that $e_{ii} v^+ = z_i v^+$ for all $1 \leq i \leq m+n$. Non-isomorphic modules yield distinct weights $z$. Thus there are totally $p^{m+n}$ isomorphism classes of simple $\overline{\mathfrak{u}}_k$-modules.
The element $v^+$ as in the lemma is called a maximal vector of weight $z$.

**Definition 9.6.** [4, 29.13] Let $F$ be a field and let $\mathfrak{A}$ (resp. $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$) be a $F$-algebra (resp. $F$-superalgebra) and $M$ a simple $\mathfrak{A}$-module. $M$ is called absolutely simple if $M \otimes_F \mathbb{L}$ is also a simple $\mathfrak{A} \otimes_F \mathbb{L}$-module for any extension field $\mathbb{L} \supseteq F$.

Let $k$ be field of characteristic $p > 2$, and let $M(z) = M_0 \oplus M_1$ be a simple $\mathfrak{u}_k$-module having a maximal vector $v^+$ of weight $z$. Let $|\mathfrak{u}_k|$ denote the associate $k$-algebra $\mathfrak{u}_k$ forgetting its $\mathbb{Z}_2$-structure. Then the uniqueness of the maximal vector $v^+$ implies that $M(z)$ contains the unique (up to scalar multiple) maximal vector $v^+$ even as a $|\mathfrak{u}_k|$-module. Let $f : M(z) \rightarrow M(z)$ be a $|\mathfrak{u}_k|$-module homomorphism. Then we must have $f(v^+) = cv^+$ for some $0 \neq c \in k$, so we get

$$\text{Hom}_{|\mathfrak{u}_k|}(M(z), M(z)) = k.$$ 

By [4, 29.13], $M(z)$ is a absolutely simple $|\mathfrak{u}_k|$-module. Therefore $M(z) \otimes_k \mathbb{L}$ is a simple $|\mathfrak{u}_l|$-module for any extension field $\mathbb{L}$. This implies that $M(z) \otimes_k \mathbb{L}$ is also simple as a $\mathfrak{u}_l$-module, so that $M(z)$ is absolute simple.

Now let $\overline{k}$ be an algebraic closure of $k$, and let $M(z)_{\overline{k}}$ be a simple $\mathfrak{u}_{\overline{k}}$-module having a unique maximal vector $v^+$ of weight $z$. Then we may identify $M(z)$ with the $\mathfrak{u}_k$-lattice $\mathfrak{u}_k \cdot v^+ \subseteq M(z)_{\overline{k}}$.

Thus, the representation theory of $\mathbb{F}_p$-superalgebra $\mathfrak{u}$ is completely determined by that of $\mathfrak{u}_{\mathbb{F}_p}$. Let $\lambda \in X^+_p(T)$. Then [13, Lemma 4.3] tells us that $M(\lambda)_{\mathbb{F}_p}$ is isomorphic to $L(\lambda)$ restricted to $\mathfrak{u}_{\mathbb{F}_p}$.

### 9.4 Lusztig’s finite dimensional Hopf superalgebras

We first fix an integer $l' \geq 1$. Let $\mathcal{B}$ be the quotient ring of $\mathcal{A}$ by the ideal generated by the $l'$th cyclotomic polynomial $\phi_l \in \mathbb{Z}[q]$. Let $l \geq 1$ be defined by

$$l = \begin{cases} l' & \text{if } l' \text{ is odd} \\ \frac{l'}{2} & \text{if } l' \text{ is even} \end{cases}$$

Define the $\mathcal{B}$-superalgebras $U^+_\mathcal{B}$, $U^-\mathcal{B}$, $U^0\mathcal{B}$, and $U_{\mathcal{B}}$ by applying $-\otimes_{\mathcal{A}} \mathcal{B}$ to $\mathcal{A}$-superalgebras $U^+_{\mathcal{A}}$, $U^-_{\mathcal{A}}$, $U^0_{\mathcal{A}}$, and $U_{\mathcal{A}}$. Let $\mathfrak{u}^+$, $\mathfrak{u}^-$, $\mathfrak{u}^0$, and $\mathfrak{u}$ be the $\mathcal{B}$-sub-superalgebras of $U^+_{\mathcal{B}}$, $U^-_{\mathcal{B}}$, $U^0_{\mathcal{B}}$, and $U_{\mathcal{B}}$ generated respectively by the elements

$$E^{(N)}_{ij}((i, j) \in \mathcal{I}_0, N \in [0, l]), \quad E^{(\sigma)}_{st}((s, t) \in \mathcal{I}_1, \sigma = 0, 1);$$

$$F^{(N)}_{ij}((i, j) \in \mathcal{I}_0, N \in [0, l]), \quad F^{(\sigma)}_{st}((s, t) \in \mathcal{I}_1, \sigma = 0, 1);$$

$$K^{\pm 1}_{\alpha \epsilon}, \left[ K^{\alpha \epsilon ; 0}_{t \epsilon} \right] (c \in [1, m + n], t \epsilon \in [0, l]);$$

and

$$E^{(N)}_{ij}, E^{(\sigma)}_{st}, F^{(N)}_{ij}, F^{(\sigma)}_{st}, \left[ K^{\alpha \epsilon ; 0}_{t \epsilon} \right], K^{\pm 1}_{\alpha \epsilon}.$$
Let $\mathcal{B}'$ be the quotient field of $\mathcal{B}$. We form the $\mathcal{B}'$-superalgebras $'u^+$, $'u^-$, $'u^0$, $'u$ and $U_{\mathcal{B}'}$ by applying $- \otimes_{\mathcal{B}} \mathcal{B}'$ to the $\mathcal{B}$-superalgebras $u^+$, $u^-$, $u^0$, $u$ and $U_{\mathcal{B}}$ respectively.

**Proposition 9.7.** (a) $u^+$ is generated as a $\mathcal{B}$-superalgebra by the elements

$$E_{\alpha_i}^{(N)}, E_{\alpha_m}^{(\sigma)}(i \in [1, m + n] \setminus m, N \in [0, l], \sigma = 0, 1)$$

and as a free $\mathcal{B}$-module by the basis $E_{\psi}^{(\sigma)}{E_{i}^{d}}, \psi \in [0, l]^{\mathcal{I}_0}, d \in \{0, 1\}^{\mathcal{I}_1}$.

(b) $u^-$ is generated as a $\mathcal{B}$-superalgebra by the elements

$$F_{\alpha_i}^{(N)}, F_{\alpha_m}^{(\sigma)}(i \in [1, m + n] \setminus m, N \in [0, l], \sigma = 0, 1)$$

and as a free $\mathcal{B}$-module by the basis $F_{\psi}^{(\sigma)}{F_{i}^{d}}, \psi \in [0, l]^{\mathcal{I}_0}, d \in \{0, 1\}^{\mathcal{I}_1}$.

(c) $u^0$ is generated as a free $\mathcal{B}$-module by the basis

$$\Pi_{i=1}^{m+n}(K_{\alpha_i}^{\delta_i} \left[ K_{\alpha_i}^{\delta_i} : 0 \right]), t_i \in [0, l], \delta_i \in \{0, 1\}.$$

(d) $u$ is generated as a $\mathcal{B}$-superalgebra by the elements $E_{\alpha_i}^{(N)}, E_{\alpha_m}^{(\sigma)}, F_{\alpha_i}^{(\sigma)}, F_{\alpha_m}^{(\sigma)}$ and

$$K_{\alpha_i}^{\pm 1} \left[ K_{\alpha_i}^{\delta_i} : 0 \right] (i \in [1, m + n] \setminus m, j \in [1, m + n], t, N \in [0, l], \sigma = 0, 1)$$

and as a free $\mathcal{B}$-module by the basis

$$F_{1}^{d}E_{0}^{(\psi)}n_{i=1}^{m+n}(K_{\alpha_i}^{\delta_i} \left[ K_{\alpha_i}^{\delta_i} : 0 \right])E_{1}^{d'}, \psi, \psi' \in [0, l]^{\mathcal{I}_0}, d, d' \in \{0, 1\}^{\mathcal{I}_1}, \delta_i \in \{0, 1\}, t_i \in [0, l].$$

(e) $'u^+$, $'u^-$, $'u^0$, $'u$ may be viewed as $\mathcal{B}'$-sub-superalgebras of $U_{\mathcal{B}'}$ having the following bases:

$$'u^+ : E_{0}^{\psi}{E_{1}^{d}}, \psi \in [0, l]^{\mathcal{I}_0}, d \in \{0, 1\}^{\mathcal{I}_1};$$

$$'u^- : F_{1}^{d}E_{0}^{(\psi)} : \psi \in [0, l]^{\mathcal{I}_0}, d \in \{0, 1\}^{\mathcal{I}_1};$$

$$'u^0 : \Pi_{i=1}^{m+n}K_{\alpha_i}^{N_i}, N_i \in [0, 2l);$$

$$'u : F_{1}^{d}F_{0}^{\psi}n_{i=1}^{m+n}K_{\alpha_i}^{N_i}E_{0}^{\psi}{E_{1}^{d'}}, \psi, \psi' \in [0, l]^{\mathcal{I}_0}, N_i \in [0, 2l], d, d' \in \{0, 1\}^{\mathcal{I}_1}. $$

**Proof.** (a) For any $(i, j) \in \mathcal{I}$, we have $E_{ij}^{(M)}E_{ij}^{(N)} = 0$ if $M \geq 1, N \geq 1$, and $M + N \geq l$. It then follows from Lemma 8.1(1) that $u^+$ is generated as a $\mathcal{B}$-superalgebra by the elements $E_{\alpha_i}^{(N)}, E_{\alpha_m}^{(\sigma)}(i \in [1, m + n] \setminus m, N \in [0, l], \sigma = 0, 1)$. In view of the proof of Lemma 8.3(b), we obtain that $u^+$ is spanned as a $\mathcal{B}$-module by the elements $E_{0}^{(\psi)}{E_{1}^{d}}, \psi \in [0, l]^{\mathcal{I}_0}, d \in \{0, 1\}^{\mathcal{I}_1}$. By the discussion following Prop. 8.6, $U_{\mathcal{B}'}^{+}$ is a free $\mathcal{A}$-module having a basis consisting of elements (see Prop. 8.3(b)) $E_{0}^{(\psi)}{E_{1}^{d}}, \psi \in N^{\mathcal{I}_0}, d \in \{0, 1\}^{\mathcal{I}_1}$, so that $U_{\mathcal{B}'}^{+}$ is a free $\mathcal{B}$-module having an analogous basis. It then follows that $u^+$ is a free $\mathcal{B}$-submodule.

(b), (c) and (d) can be proved similarly. (e) follows from (a)-(d). \(\square\)
In the following we assume $l = l'$ is odd. Use Lusztig's notion

$$K_{i,t} = K_{\alpha_i}^{-t} \left[ K_{\alpha_i}^{}; 0 \right]_t, i \in [1, m + n], t \geq 0.$$ 

By [14, Lemma 6.4], the elements $\Pi_{i=1}^{m+n} K_{\alpha_i}^{t_i} \Pi_{i=1}^{m+n} K_{i,t_i} (t_i \geq 0, \delta_i \in \{0, 1\})$ form a $B$-basis of $\tilde{U}_B^\prime$. 

Let $\tilde{U}_B, \tilde{u}$ (resp. $\tilde{U}_B', \tilde{u}'$) be the quotient of $B$-superalgebras (resp. $B'$-superalgebras) $U_B, u$ (resp. $U_B', u'$) by the two-sided ideal generated by the central elements $K_{\alpha_i}^1 - 1, \ldots, K_{\alpha_{m+n}}^1 - 1$. 

Then we get:

(a) The elements

$$F_1^d E_0^{(\psi)} \Pi_{i=1}^{m+n} K_{i,t_i} E_0^{(\psi')} E_1^{d'} (\psi, \psi' \in \mathbb{N}^{\mathbb{T}_0}, d, d' \in \{0, 1\}^{\mathbb{T}_1}, t_i \in \mathbb{N})$$

form a $B$-basis of $\tilde{U}_B$ and $B'$-basis of $\tilde{U}_B'$. 

(b) The elements

$$F_1^d E_0^{(\psi)} \Pi_{i=1}^{m+n} K_{i,t_i} E_0^{(\psi')} E_1^{d'} (\psi, \psi' \in [0, l]^{\mathbb{T}_0}, d, d' \in \{0, 1\}^{\mathbb{T}_1}, t_i \in [0, l])$$

form a $B$-basis of $\tilde{u}$ and $B'$-basis of $\tilde{u}'$. 

Let $k$ be a commutative ring and let $\eta$ be an invertible element in $k$. Set

$$U_{\eta,k} = U_A \otimes_A k, \quad U(\mathfrak{g}_0)_{\eta,k} = U_q(\mathfrak{g}_0)_A \otimes_A k,$$

where $k$ is regarded as an $A$-algebra with $q$ acting as multiplication by $\eta$. Similar notation are defined for the $A$-subalgebras $(N_{\pm 1})_A, U_q(\mathfrak{g}_0)_A(N_i)_A$ of $U_A$. 

Denote by $\tilde{U}_{\eta,k}$ (resp. $\tilde{U}(\mathfrak{g}_0)_{\eta,k}$) the quotient superring of $U_{\eta,k}$ (resp. $U(\mathfrak{g}_0)_{\eta,k}$) by its two-sided ideal generated by the central elements $K_{\alpha_i}^1 - 1, i \in [1, m + n]$. 

**Proposition 9.8.** There is an isomorphism of superalgebras $\phi: U(\mathfrak{g})_Q \rightarrow \tilde{U}_{1,Q}$ such that

$$\phi(e_{ij}) = E_{ij}, \phi(f_{ij}) = F_{ij}, (i, j) \in I, \phi(h_{\alpha_s}) = \left[ K_{\alpha_s}; 0 \right]_1, s \in [1, m + n].$$

In particular, we have $\phi(U(\mathfrak{g})_Z) = \tilde{U}_{1,Z}$ and $\phi(U(\mathfrak{g}_0)_Z) = \tilde{U}(\mathfrak{g}_0)_{1,Z}$. 

**Proof.** We must show that $\phi$ preserves all the relations 2.2(a1)-(a8). Note that by Remark 2.2(3), the relations (a8) are preserved. In view of [14, 6.7(a)], we need only verify the case $i = m, j = m + 1$ in the relation 2.2(a2). By a proof similar to that of Lemma 7.2, we have that the relation 2.2(a2) for $e_{m+1,m+2}$ is preserved; an analogous argument applies to 2.2(a2) for $f_{m+1,m+2}$. Then Lemma 9.1 and the PBW type basis of $\tilde{U}_{1,Q}$ given by the statement (a) above with $l = l' = 1$ ensures that $\phi$ is an isomorphism. 

□
9.5 Representations of the Hopf superalgebra $\tilde{u}$

In this subsection, we let $U_q$ be the quantum supergroup $U_q(\mathfrak{g})$ over $\mathbb{C}(q)$ (see 2.2).

Let

$$z \in \mathbb{Z}_+^{m+n} = \{(z_1, \ldots, z_m, n) \in \mathbb{Z}^{m+n} | z_i \geq 0, \text{ for all } i \neq m, m + n\}.$$  

For each $U_q(\text{resp. } U_q(\mathfrak{g}_0))$-module $M$. The $z$-weight space of $M$ is defined to be

$$M_z = \{x \in M | K_{\alpha_i} x = q_i^{z_i} x, i = 1, \ldots, m + n\}.$$  

Let $L_0(z)$ be a simple $U_q(\mathfrak{g}_0)$-module of highest weight $z \in \mathbb{Z}_+^{m+n}$. Then $L_0(z)$ is finite dimensional by [16, Th. 4.12], and so is the induced $U_q$-module

$$K(z) = U_q \otimes_{U_q(\mathfrak{g}_0)} L_0(z).$$  

Let $M(z)$ be a simple $U_q$-module of highest weight $z$ (see Th. 5.3).

**Lemma 9.9.** $M(z)$ is a homomorphic image of $K(z)$.

**Proof.** Let $v \in M(z)$ be a maximal vector. We claim that $U_q(\mathfrak{g}_0)\mathcal{N}_1 v \subseteq M(z)$ is a simple $U_q(\mathfrak{g}_0)\mathcal{N}_1$-submodule. Suppose on the contrary that $U_q(\mathfrak{g}_0)\mathcal{N}_1 v$ were not simple. Then it contains a proper simple $U(\mathfrak{g}_0)\mathcal{N}_1$-submodule $M'$. By the discussion following Lemma 7.1, $M'$ must be a simple $U_q(\mathfrak{g}_0)$-module annihilated by $U_q(\mathfrak{g}_0)\mathcal{N}_1^+$, so that $M'$ contains a unique maximal vector $v^+$ which by definition is a nonzero weight vector with $E_{\alpha_i} v^+ = 0$ for all $i \in [1, m + n] \setminus m$. Since $v^+$ is annihilated by $E_{\alpha_m} \in U_q(\mathfrak{g}_0)\mathcal{N}_1^+$, $v^+$ is also a maximal vector for the $U_q$-module $M(z)$. By Th. 5.3, we get $v^+ = cv$ for some nonzero $c \in \mathbb{C}(q)$, and hence $M' = U_q(\mathfrak{g}_0)\mathcal{N}_1 v$, a contradiction.

Thus, we have a $U_q(\mathfrak{g}_0)\mathcal{N}_1$-module isomorphism

$$L_0(z) \cong U_q(\mathfrak{g}_0)\mathcal{N}_1 v \subseteq M(z).$$

This induces a $U_q$-module homomorphism from $K(z)$ into $M(z)$ that must be surjective since the latter is simple. \qed

Assume $l = l'$ is an odd integer $\geq 3$, and let $\eta$ be a primitive $l$th root of unity. In what follows, we identify $B'$ with the subfield $Q(\eta)$ of $\mathbb{C}$ by identifying $q$ with $\eta$. Then we may view $U_{B'}$ as a Hopf $B'$-sub-superalgebra of $U_{\eta, \mathbb{C}}$ such that $U_{B'} \otimes_{B'} \mathbb{C} \cong U_{\eta, \mathbb{C}}$.

Set

$$\eta_i = \begin{cases} \eta, & \text{if } i \leq m \\ \eta^{-1}, & \text{if } m + 1 \leq i \leq m + n. \end{cases}$$

Let $V = V_0 \oplus V_1$ be a $U_{\eta, \mathbb{C}}$-module (resp. $U_{B'}$-module) of type 1. For each $z = (z_1, \ldots, z_{m+n}) \in \mathbb{Z}^{m+n}$, we define the $z$-weight space

$$V_z = \{x \in V | K_{\alpha_i} x = \eta_i^{z_i} x, \left[ K_{\alpha_i} ; 0 \right] \frac{1}{l} x = \left[ \frac{z_i}{l} \right] \eta \} \quad \text{for } 1 \leq i \leq m + n\},$$

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where \( \begin{bmatrix} z_i \\ l \end{bmatrix} \) denotes \( \begin{bmatrix} z_i \\ l \end{bmatrix} \otimes 1 \in U_q, \mathbb{C} \). Since the parities of \( K_i \) and \( \begin{bmatrix} K_{\alpha_i} \\ l \end{bmatrix} \) are 0, \( V_z \) is \( \mathbb{Z}_2 \)-graded. It is routine to show that \( \sum_z V_z \) is a \( U_q, \mathbb{C} \)-submodule(resp. \( U_{\mathbb{B}^*} \)-submodule) of \( V \) (cf.\cite{15, 5.2}). Also by \cite{15, 3.3(b)}, the sum \( \sum_z V_z \) is direct.

Let \( z \in \mathbb{Z}^{m+n} \). A homogeneous nonzero element \( v^+ \in V_z \) is said to be a maximal vector if

\[
E_{ij} v^+ = 0 \quad \text{for all} \quad (i, j) \in \mathcal{I} \quad \text{and} \quad E_{ij}^{(0)} v^+ = 0 \quad \text{for all} \quad (i, j) \in \mathcal{I}_0.
\]

\( V \) is a highest weight module if it is generated by a maximal vector \( v^+ \) as a \( U_q, \mathbb{C} \)-module. For such a module, we have \( V = \bigoplus_{z' \leq z} V_{z'} \) and \( V_z = \mathbb{C} v^+ \), from which it follows that any proper submodule \( N = N_0 \oplus N_1 \) is equal to \( \sum_{z' \leq z} N \cap V_{z'} \), and hence is contained in \( \sum_{z' < z} V_{z'} \), so that \( V \) has a unique simple quotient.

**Definition 9.10.** A \( U_q, \mathbb{C} \)-module(resp. \( U_{\mathbb{B}^*} \)-module) \( V = V_0 \oplus V_1 \) of type \( \mathbf{1} \) is called integral if \( V = \sum_{z \in \mathbb{Z}^{m+n}} V_z \).

Note: For \( i \in \{m, m+n\} \), since \( \begin{bmatrix} K_{\alpha_i} \\ l \end{bmatrix} \) does not have the property \cite{15, 5.1(b)}, not every finite dimensional simple \( U_q, \mathbb{C} \)-module(resp. \( U_{\mathbb{B}^*} \)-module) is integral.

We now construct integral simple modules for \( U_q, \mathbb{C} \). Assume \( z \in \mathbb{Z}^{m+n} \). Let \( M(z) \) be a simple \( U_q \)-module of highest weight \( z \), and let \( v^+ \in M(z) \) be a maximal vector. Denote by \( M_A(z) \) the \( U_A \)-invariant \( \mathcal{A} \)-lattice \( U_A v^+ \) of \( M(z) \). Set

\[
M_{\eta, \mathbb{C}}(z) = M_A(z) \otimes_\mathcal{A} \mathbb{C},
\]

where \( \mathbb{C} \) is regarded as an \( \mathcal{A} \)-algebra by letting \( q \) act as multiplication by \( \eta \). Then \( M_{\eta, \mathbb{C}}(z) \) is a highest weight \( U_q, \mathbb{C} \)-module. Let \( L_{\eta, \mathbb{C}}(z) \) be the unique simple quotient of \( M_{\eta, \mathbb{C}}(z) \). Then clearly \( L_{\eta, \mathbb{C}}(z) \) is integral. It follows from Lemma 9.9 that \( L_{\eta, \mathbb{C}}(z) \) is finite dimensional if \( z \in \mathbb{Z}_+^{m+n} \).

In view of the proof for \cite{15, 6.4}, we get

**Proposition 9.11.** The map \( z \mapsto L_{\eta, \mathbb{C}}(z) \) defines a bijection between \( \mathbb{Z}_+^{m+n} \) and the set of isomorphism classes of integral simple \( U_q, \mathbb{C} \)-modules of type \( \mathbf{1} \), of finite dimension over \( \mathbb{C} \).

Assume \( z \in \mathbb{Z}_+^{m+n} \). By Lemma 9.9, there is an epimorphism of \( U_q \)-modules \( f: K(z) \longrightarrow M(z) \). Since \( \mathcal{L}_0(z) \subseteq K(z) \) is a simple \( U_q(\mathfrak{g}_0) \)-submodule annihilated by \( U_q(\mathfrak{g}_0)N_{\mathcal{L}_0}^+ \), \( f|_{\mathcal{L}_0(z)} \) is a \( U_q(\mathfrak{g}_0) \)-module isomorphism onto its image, denoted also \( \mathcal{L}_0(z) \), which is annihilated also by \( U_q(\mathfrak{g}_0)N_{\mathcal{L}_0}^+ \). Then we have \( M(z) = N_{\mathcal{L}_0} \mathcal{L}_0(z) \), so that

\[
M_A(z) = (N_{\mathcal{L}_0})_A U_q(\mathfrak{g}_0)_A v^+,
\]

where \( v^+ \) is a unique maximal vector in \( \mathcal{L}_0(z) \). Denote the image of \( v^+ \) in \( L_{\eta, \mathbb{C}}(z) \) also by \( v^+ \). It then follows that \( L_{\eta, \mathbb{C}}(z) \) has as a \( U(\mathfrak{g}_0)_\eta, \mathbb{C} \)-submodule \( U(\mathfrak{g}_0)_\eta, \mathbb{C} v^+ \) annihilated by \( (U(\mathfrak{g}_0)N_{\mathcal{L}_0}^+)_{\eta, \mathbb{C}} \) and

\[
L_{\eta, \mathbb{C}}(z) = (N_{\mathcal{L}_0})_{\eta, \mathbb{C}} U(\mathfrak{g}_0)_{\eta, \mathbb{C}} v^+.
\]
In the light of the proof of Lemma 9.9, we see that \( U(\mathfrak{g}_0)_{\eta,C}v^+ \subseteq L_{\eta,C}(z) \) is a simple \( U(\mathfrak{g}_0)_{\eta,C} \)-submodule.

Let \( k \) be any intermediate field between \( B' = Q(\eta) \) and \( C \). Set \( \tilde{u}_k = \tilde{u} \otimes_{B'} k \).

We now study the finite dimensional \( \tilde{u}_k \)-modules. Since \( K_{\alpha_i} = 1(i \in [1, m+n]) \) in \( \tilde{u} \), each \( \tilde{u}_k \)-module \( M = M_0 \oplus M_1 \) has a decomposition

\[
M_0 = \oplus_z (M_0)_z, \quad M_1 = \oplus_{z'} (M_1)_{z'},
\]

where \( z, z' \in [0, l]^{m+n} \) and

\[
(M_j)_z = \{ x \in M_j | K_{\alpha_i}x = \eta_i^{z_i}x, 1 \leq i \leq m+n \}, \quad \bar{j} \in \mathbb{Z}_2.
\]

Let

\[
M^0 = \{ x \in h(M)|E_{ij}x = 0, \quad \text{for all} \quad (i, j) \in I \}.
\]

A nonzero vector \( v \in M^0 \cap M_z \) is called a maximal vector of weight \( z \). Then applying verbatim [14, 5.10, 5.11] we get

**Proposition 9.12.** Each simple \( \tilde{u}_k \)-module \( M = M_0 \oplus M_1 \) contains a unique (up to scalar multiple) maximal vector of weight \( z \in [0, l]^{m+n} \). The correspondence \( M \mapsto z \) defines a bijection between the set of isomorphism classes of simple \( \tilde{u}_k \)-modules and the set \([0, l]^{m+n}\).

Using the fact that each simple \( \tilde{u}_k \)-module \( M \) contains a unique maximal vector, together with a similar discussion as following Definition 9.6, we see that \( M \) is absolutely simple. So we may restrict our attention to just the case \( k = C \). It follows from the description of the bases of superalgebras \( \tilde{U}_B' \) and \( \tilde{u} \) in Sec. 9.4 that \( \tilde{u}_C \) can be viewed as a sub-superalgebra of \( \tilde{U}_{\eta,C} \).

Set

\[
\mathbb{Z}_l^{m+n} =: \{(z_1, \ldots, z_{m+n}) \in \mathbb{Z}_l^{m+n} | 0 \leq z_i \leq l - 1 \quad \text{for all} \quad i \neq m, m+n \}.
\]

The following lemma can be proved by a similar argument as that for [15, 7.1].

**Lemma 9.13.** Assume \( z \in \mathbb{Z}_l^{m+n} \) and let \( x \) be a maximal vector of \( L_{\eta,C}(z) \). Then

(a) \( F_{\alpha_i}^{(l)}x = 0 \) for \( i \in [1, m+n] \setminus m \).

(b) Let \( \nabla = \{ y \in h(L_{\eta,C}(z)) | E_{ii}y = 0 \quad \text{for all} \quad i \neq m, m+n \} \). Then \( \nabla = C \cdot x \).

(c) Then restriction of \( L_{\eta,C}(z) \) to \( \tilde{u}_C \) is a simple \( \tilde{u}_C \)-module.

(d) \( L_{\eta,C}(z) = \tilde{u}_C \cdot x \).

It then follows that each simple \( \tilde{u}_C \)-module can be lifted to an integral simple \( U_{\eta,C} \)-module of type \( \underline{1} \).
9.6 The extended Lusztig conjecture

In this subsection assume \( l = l' \) is an odd prime \( p \), and assume \( \mathbb{F} \) is an algebraically closed field of characteristic \( p > 2 \). Consider the ring homomorphism \( \mathcal{B} \longrightarrow \mathbb{F}_p \) which maps \( z \in \mathbb{Z} \) into \( z \mod p \in \mathbb{F}_p \) and \( q \) into 1, and let \( \mathfrak{m} \) be its kernel. Then applying a similar argument as that for [14, Th. 6.8], we get

**Proposition 9.14.** There are isomorphisms of Hopf superalgebras:

\[
\tilde{U}_\mathcal{B}/\mathfrak{m}\tilde{U}_\mathcal{B} \cong U(\mathfrak{g})_{\mathbb{F}_p}, \quad \tilde{\mathfrak{u}}/\mathfrak{m}\tilde{\mathfrak{u}} \cong \tilde{\mathfrak{u}}.
\]

Assume \( \eta \) is a primitive \( p \)th root of unity. By [9, Ch.H], \( \mathbb{Z}[\eta] \) is the ring of all algebraic integers in \( \mathbb{Q}(\eta) \) and \( 1 - \eta \) generates the unique maximal ideal \((1 - \eta)\) in \( \mathbb{Z}[\eta] \). Let \( \mathcal{R} \) denote the localization of \( \mathbb{Z}[\eta] \) at \((1 - \eta)\). Then \( \mathcal{R} \) is a discrete valuation ring with residue field \( \mathbb{F}_p \). Regard the field \( \mathbb{F} \) as a \( \mathcal{R} \)-algebra via the embedding of the residue field of \( \mathcal{R} \) into \( \mathbb{F} \). We can identify \( U_{\eta, \mathcal{R}} \otimes_{\mathcal{R}} \mathbb{F} \) with \( U_{1, \mathbb{F}} \) (see Sec. 9.4).

Assume \( z \in \mathbb{Z}_+^{m+n} \). Let \( v^+ \) be a maximal vector of the simple \( U_{\eta, C} \)-module \( L_{\eta, C}(z) \). Then \( L_{\eta, \mathcal{R}}(z) =: U_{\eta, \mathcal{R}}v^+ \) is a \( U_{\eta, \mathcal{R}} \)-invariant \( \mathcal{R} \)-lattice in \( L_{\eta, C}(z) \). Now

\[
L_{\eta, C}(z)_\mathbb{F} = L_{\eta, \mathcal{R}}(z) \otimes_{\mathcal{R}} \mathbb{F}
\]

has a natural structure as a \( U_{1, \mathbb{F}} \)-module. Since each \( K_{\alpha_i} \) acts on \( L_{\eta, C}(z)_\mathbb{F} \) as the identity, \( L_{\eta, C}(z)_\mathbb{F} \) is a \( \tilde{U}_{1, \mathbb{F}} \)-module. Recall the notation \( U(\mathfrak{g})_\mathbb{F} \) in 9.1. By Prop. 9.8, we have

\[
\tilde{U}_{1, \mathbb{F}} = \tilde{U}_{1, \mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} \cong U(\mathfrak{g})_\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F} = U(\mathfrak{g})_\mathbb{F},
\]

so that \( L_{\eta, C}(z)_\mathbb{F} \) is a \( U(\mathfrak{g})_\mathbb{F} \)-module and hence a \( \text{Dist}(G) \)-module. It then follows from [3, Coro. 3.5] that \( L_{\eta, C}(z)_\mathbb{F} \) is a \( G \)-module.

Recall the identification of \( \mathbb{Z}_+^{m+n} \) with \( \Lambda \) in Sec. 9.1. Under this identification we have

\[
\mathbb{Z}_+^{m+n} = X^+(T) \quad \text{and} \quad \mathbb{Z}_p^{m+n} = X^+_p(T),
\]

where

\[
\mathbb{Z}_p^{m+n} = \{(z_1, \ldots, z_{m+n}) \in \mathbb{Z}_+^{m+n} | 0 \leq z_i < p, \text{for all } \ i \neq m, m+n\}.
\]

By the proposition above, each simple \( \tilde{\mathfrak{u}} \)-module corresponds to a simple \( \mathfrak{u} \)-module \( \mathcal{M} \) and has dimension \( \leq \dim \mathcal{M} \). We now extend the Lusztig’s conjecture in [14, 0.3] to the super case as follows.

**Conjecture:** If \( p \) is sufficiently large and \( z \in \mathbb{Z}_p^{m+n} \), then the inequality above is an equality and \( \tilde{\mathfrak{u}} \) and \( \mathfrak{u} \) have identical representation theories.

The conjecture is supported by the following theorem.

**Theorem 9.15.** Let \( \mathbb{F} \) be an algebraically closed field of characteristic \( p > 2 \). Assume Lusztig’s conjecture in [14, 0.3]. If \( z \in \mathbb{Z}_p^{m+n} \) is \( p \)-typical, then \( L_{\eta, C}(z)_\mathbb{F} \) is simple as a \( \text{GL}(m, n) \)-module.
Proof. Let $v^+$ be a maximal vector of $L_{\eta,\mathbb{C}}(z)$. From the discussion following Prop. 9.11, we have

$$L_{\eta,\mathbb{R}}(z) = U_{\eta,\mathbb{R}}v^+ = (\mathcal{N}_-\eta,\mathbb{R}) U(g_0)_{\eta,\mathbb{R}}v^+,$$

where $U(g_0)_{\eta,\mathbb{R}}v^+$ is a $U(g_0)_{\eta,\mathbb{R}}$-invariant lattice of the simple $U(g_0)_{\eta,\mathbb{C}}$-module

$$U(g_0)_{\eta,\mathbb{C}}v^+ \subseteq L_{\eta,\mathbb{C}}(z)$$

which is annihilated by $(U(g_0)\mathcal{N}_1^+)^{-1,\eta,\mathbb{C}}$. Recall the notation $U(g_{-1})_{\mathbb{F}}$, $U(g_0)_{\mathbb{F}}$ and $U(g^+)_{\mathbb{F}}$. Note that

$$(\mathcal{N}_-\eta,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{F} = U(g_{-1})_{\mathbb{F}}, \quad U(g_0)_{\eta,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{F} = U(g_0)_{\mathbb{F}}$$

and

$$(U(g_0)\mathcal{N}_1^+)_{\eta,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{F} = U(g^+)_{\mathbb{F}}.$$

This gives us

$$L_{\eta,\mathbb{C}}(z)_{\mathbb{F}} = U(g_{-1})_{\mathbb{F}}U(g_0)_{\mathbb{F}}v^+,$$

with $U(g_0)_{\mathbb{F}}v^+$ being annihilated by $U(g^+)_{\mathbb{F}}$.

Since $z \in \mathbb{Z}_{p+1}^{m+1}$, the Lusztig conjecture in [14, 0.3] says that $U(g_0)_{\mathbb{F}}v^+$ is a simple $U(g_0)_{\mathbb{F}}$-module of highest weight $z$. By the discussion in 9.1, we can now express the all weights as a subset of $\Lambda$ using the identification of $\mathbb{Z}_{p+1}^{m+1}$ with $X_1^+(T)$. Then we have

$$U(g_0)_{\mathbb{F}}v^+ \cong L_0(\lambda).$$

Since $U(g_0)_{\mathbb{F}}v^+$ is annihilated by $U(g^+)_{\mathbb{F}}$, it is a Dist($P$)-module. Therefore $L_{\eta,\mathbb{C}}(z)_{\mathbb{F}}$ is a homomorphic image of the Dist($G$)-module $\text{Ind}_P^G\lambda$. Since $z$ is $p$-typical, and hence $\lambda$ is $p$-typical, we have by Th. 9.4 that $\text{Ind}_P^G\lambda \cong L(\lambda)$, so that $L_{\eta,\mathbb{C}}(z)_{\mathbb{F}} \cong L(\lambda)$, as desired. \qed

## 10 Lusztig’s tensor product theorem

The purpose of this section is to establish the tensor product theorem for the quantum supergroup $U_{\eta,\mathbb{C}}$. Assume $l$ is an odd number $\geq 3$ and $\eta$ is a primitive $l$th root of unity.

Applying a similar argument as that for [15, Lemma 7.2], one obtains

**Lemma 10.1.** Let $\mathfrak{A}_\eta$ be the subalgebra of $U_{\eta,\mathbb{C}}$ generated by $E^{(l)}_{\alpha_1}, F^{(l)}_{\alpha_1}, [K_{\alpha_i,m},c]$,

$$\left[\begin{array}{c}K_{\alpha_i,m},c \\ l\end{array}\right], i \in [1,m+n] \setminus m, c \in \mathbb{Z}.$$

Assume $z \in \mathbb{Z}_{p+1}^{m+n}$ with $z = lz'$ for some $z' \in \mathbb{Z}_{p+1}^{m+n}$. Let $x$ be a maximal vector of $L_{\eta,\mathbb{C}}(z)$. Then

(a) $E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_j} - 1$ act as 0 on $L_{\eta,\mathbb{C}}(z)$ ($i \in [1,m+n], j \in [1,m+n]$).

(b) Let $\nabla = \{y \in h(L_{\eta,\mathbb{C}}(z))|E^{(l)}_{\alpha_i}y = 0 \quad \text{for all} \quad i \neq m, m+n\}$. Then $\nabla = \mathbb{C}.x$.

(c) $L_{\eta,\mathbb{C}}(z) = \mathfrak{A}_\eta \cdot x$. 

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Let \( z \in \mathbb{Z}_{m+n}^+ \). We can write uniquely \( z = z' + lz'' \), where \( z' \in \mathbb{Z}_{m+n}^+ \) and \( z'' \in \mathbb{Z}_{m+n}^+ \). Then applying a similar proof as that of [15, 7.4], we get

**Theorem 10.2.** The \( U_{\eta,C}(z) \) and \( L_{\eta,C}(z') \otimes L_{\eta,C}(lz'') \) are isomorphic.

Let \( U(\mathfrak{g}_0) \) be the universal enveloping algebra of the Lie algebra \( \mathfrak{g}_0 \) over \( \mathbb{C} \). Then we have

**Proposition 10.3.** For any \( z = (z_1, \cdots, z_{m+n}) \in \mathbb{Z}_{m+n}^+ \), \( L_{\eta,C}(lz) \) is a simple \( U(\mathfrak{g}_0) \)-module with highest weight \( z \).

**Proof.** Let \( I \) be the two-sided ideal of \( U_{\eta,C} \) generated by \( E_{\alpha_i}, F_{\alpha_i}, K_{\alpha_i} - 1, i \in [1, m + n], j \in [1, m + n] \). In view of the proof of [15, 7.5], we have a unique superalgebra epimorphism \( \phi : U(\mathfrak{g}_0) \rightarrow U_{\eta,C}/I \) such that

\[
\phi(e_{\alpha_i}) = \overline{E_{\alpha_i}}^{(l)}, \quad i \in [1, m + n] \setminus m,
\]
\[
\phi(f_{\alpha_i}) = \overline{F_{\alpha_i}}^{(l)},
\]
\[
\phi(h_{\alpha_i}) = \overline{[K_{\alpha_i}; 0]}^{(l)}, i \in [1, m + n].
\]

Since \( \overline{E_{\alpha_i}}^{(l)}, \overline{F_{\alpha_i}}^{(l)}, \overline{[K_{\alpha_i}; 0]}^{(l)}, \overline{[K_{\alpha_i+n}; 0]}^{(l)} \) generate \( U_{\eta,C}/I \) as an algebra, \( \phi \) is surjective. By Lemma 10.1(a), \( L_{\eta,C}(lz) \) is a simple \( U_{\eta,C}/I \)-module and hence the pull back along \( \phi \) is a simple \( U(\mathfrak{g}_0) \)-module. Now let \( x \) be a maximal vector of \( L_{\eta,C}(lz) \). Then by [15, 3.2(a)], we have \( h_{\alpha_i} \cdot x = z_i x \) for all \( i \in [1, m + n] \).

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