Quantum exceptional group $G_2$
and its conjugacy classes

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Dedicated to the memory of Petr Kulish

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Abstract

We construct quantization of semisimple conjugacy classes of the exceptional group $G = G_2$ along with and by means of their exact representations in highest weight modules of the quantum group $U_q(g)$. With every point $t$ of a fixed maximal torus we associate a highest weight module $M_t$ over $U_q(g)$ and realize the quantized polynomial algebra of the class of $t$ by linear operators on $M_t$. Quantizations corresponding to points of the same orbit of the Weyl group are isomorphic.

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Introduction

Exceptional Lie groups occupy a special position in mathematics among simple groups and find important applications in theoretical physics in connection with string theories, supergravity, and grand unification, [1, 2]. At the same time their quantum analogs are the least studied compared to other quantum groups. In this paper we focus on the simplest exceptional group \( G = G_2 \) and construct quantization of its semisimple conjugacy classes via operator algebra realization.

To this end, we undertake a thorough analysis of certain modules over the quantum group \( U_q(g) \), where \( g \) is the Lie algebra of \( G \). In particular, we associate such module, \( M_\lambda \), with every point \( t \) of a fixed maximal torus of \( G \) and prove that the polynomial algebra of the class \( O_t \ni t \) can be quantized as a subalgebra in \( \text{End}(M_\lambda) \). Here \( \lambda \) is the highest weight of \( M_\lambda \), that depends on \( t \). The quantum polynomial algebra is simultaneously presented as a quotient of the locally finite part \( \mathbb{C}_q[G] \) of the adjoint module \( U_q(g) \) by the explicitly given ideal annihilating \( M_\lambda \).

The quantization is facilitated by the properties of the matrix \( Q \in U_q(g) \otimes U_q(g) \) expressed through the universal \( R \)-matrix \( R = R_{21}R \). It is known to commute with the coproduct \( \Delta(x) \) of all elements \( x \in U_q(g) \) and satisfy the reflection equation [3]. On specialization of the left tensor factor to the minimal representations of \( U_q(g) \) on \( V = \mathbb{C}^7 \), the entries of matrix \( Q \) generate \( \mathbb{C}_q[G] \subset U_q(g) \). Quantization is possible due to semisimplicity of the operator \( Q \) on \( V \otimes M_\lambda \), which is a consequence of direct sum decomposition of \( V \otimes M_\lambda \) into a sum of submodules of highest weight. It may be interpreted as quantization of the basic quantum homogeneous vector bundles over \( O_t \). We give exact criteria for such decomposition for each module \( M_\lambda \), which required a detailed study of singular vectors in \( V \otimes M_\lambda \).

The setup of the paper is as follows. Section 2 presents a classification of semisimple conjugacy classes of \( G_2 \) adopted to our purposes. It is followed by the basic information about the quantum group \( U_q(g) \) and its minimal representation on \( \mathbb{C}^7 \) in Section 3. The quantization theorem based on decomposition of \( V \otimes M_\lambda \) is stated in Section 3. The subsequent sections prove this decomposition for each type of \( M_\lambda \). In Section 5 we define generalized parabolic Verma modules and establish some their properties. We do regularization of singular vectors in \( V \otimes M_\lambda \) for a general Verma module \( M_\lambda \), in Section 6. The last three sections are devoted to regularization of singular vectors in tensor products and direct sum decomposition of \( V \otimes M_\lambda \) for all other types of \( M_\lambda \). Some useful formulas including the entries of matrix participating in the Shapovalov inverse form can be found in Appendix.
Throughout the paper we adopt the following general convention:

- For better readability of formulas, we denote a scalar inverse by the bar, e.g. \( \bar{q} = q^{-1} \).
- The notation \( a \simeq b \) means that \( a \) is proportional to \( b \) with a non-zero scalar factor. If the coefficient is a scalar function, we thereby assume that it never turns zero. The symbol \( \simeq \) also stands for isomorphism, which is always clear from the context and causes no confusion.
- Divisibility by a regular scalar function \( \phi \) is denoted by \( \phi \sqsubset \).

## 2 Semisimple conjugacy classes of \( G_2 \)

In this section we describe semisimple conjugacy classes of the complex algebraic group \( G = G_2 \). Let \( \mathfrak{g} \) denote the Lie algebra of \( G \) and with a fixed Cartan subalgebra \( \mathfrak{h} \). Its root system \( R \) is displayed on the figure below.

![Affine Dynkin diagram of \( G_2 \)](image)

The subset of positive roots \( R^+ \subset R \) contains the basis of simple roots \( \Pi = \{\alpha, \beta\} \). Fix an inner product on \( \mathfrak{h}^* \) as

\[
(\alpha, \alpha) = 2, \quad (\alpha, \beta) = -3, \quad (\beta, \beta) = 6.
\]

The half-sum of positive roots \( 5\alpha + 3\beta \) is denoted by \( \rho \).

To all \( \lambda \in \mathfrak{h}^* \) we assign its image \( h_\lambda \) under the isomorphism \( \mathfrak{h}^* \simeq \mathfrak{h} \) via the canonical form: \( \mu(h_\lambda) = (\lambda, \mu) \) for all \( \mu \in \mathfrak{h}^* \).

The group \( G \) has an exact representation in \( \mathbb{C}^7 \). The corresponding representation of the quantum group is given in Section 3.1.

The affine Dynkin diagram of \( \mathfrak{g} \) suggests the following stabilizers \( \mathfrak{t} \subset \mathfrak{g} \) of semisimple conjugacy classes labelled by the their root bases \( \Pi_\mathfrak{t} \subset R^+_\mathfrak{g} \):
\[ \begin{array}{|c|c|c|c|} \hline \mathfrak{t} & \Pi_k & \mathfrak{s} & \mathfrak{t}_k & \mathfrak{s}_k & \mathfrak{t}_{k,1} & \mathfrak{t}_{k,2} \\ \hline \mathfrak{h} & \emptyset & \{\alpha\}, & \{\alpha + \beta\}, & \{2\alpha + \beta\} \\ \mathfrak{e}_s & \{\beta\}, & \{3\alpha + \beta\}, & \{3\alpha + 2\beta\} \\ \mathfrak{e}_t & \{\alpha, 3\alpha + 2\beta\}, & \{\alpha + \beta, 3\alpha + \beta\}, & \{2\alpha + \beta, \beta\} \\ \mathfrak{e}_{t,1} & \{\beta, 3\alpha + 2\beta\} \\ \hline \end{array} \]

The subscripts indicate the lengths of roots. There are three Levi types with \(\#\Pi_k \leq 1\) and two pseudo-Levi types with \(\#\Pi_k = 2\).

Different although isomorphic \(\mathfrak{t}\) give rise to the same conjugacy class \(G/K\), where \(K\) is the subgroup with the Lie algebra \(\mathfrak{t}\). Still we make this distinction because we associate with them different representations of quantized \(G/K\).

Let \(T\) denote the maximal torus of \(G\) corresponding to \(\mathfrak{h}\) and fix \(t \in T\) such that \(\mathfrak{t}\) is the centralizer of \(t\). We parameterize \(T\) with a pair of complex coordinates \(x, y \in \mathbb{C}^\ast\). In the matrix realization that gives

\[ t = \text{diag}(xy, x, y, 1, \bar{x}, \bar{y}, \bar{x}\bar{y}) \in \text{End}(\mathbb{C}^7). \]

Regarding the roots as characters on \(T\), we have \(\alpha(t) = y, \beta(t) = xy^{-1}\).

Define \(T^\mathfrak{t} \subset T\) as the subset of points whose centralizer Lie algebra is \(\mathfrak{t}\). We will also use the notation \(T^{\Pi_k} = T^\mathfrak{t}\). We select the subset \(T^\mathfrak{t}_{\text{reg}} \subset T^\mathfrak{t}\) of regular points, whose minimal polynomial in the basic representation has maximal degree. The complementary subset in \(T^\mathfrak{t}\) is denoted by \(T^\mathfrak{t}_{\text{bord}}\) and called borderline, following [4]. Such points are present only for \(\mathfrak{t} = \mathfrak{h}\) and \(\mathfrak{t} = \mathfrak{t}_k\). They are a sort of “transitional” from Levi to pseudo-Levi type, hence the name.

The set \(T^\mathfrak{s} = T^\mathfrak{h}\) is determined by the conditions \(x \neq 1, y \neq 1, x \neq y, xy^2 \neq 1, x^2y \neq 1, xy \neq 1\). It is convenient to use the first three diagonal matrix entries for description of \(t\):

\[
\begin{align*}
T^\mathfrak{s}_{\text{reg}} &: (xy, x, y), \quad x \neq y, \quad x^iy^j \neq 1, \quad i, j = 0, 1, 2; \\
T^\mathfrak{s}_{\text{bord}} &: (−x, −1, x), \quad (−x, x, −1), \quad (−1, x, −\bar{x}), \quad x^4 \neq 1; \\
T^\alpha &: (x, x, 1), \quad T^{\alpha + \beta} &: (x, 1, x), \quad T^{2\alpha + \beta} &: (1, x, x^{-1}), \quad x^2 \neq 1; \\
T^\mathfrak{s}_\text{reg}^\beta &: (x^2, x, x^3), \quad T^{3\alpha + \beta} &: (\bar{x}, \bar{x}^2, x), \quad T^{3\alpha + 2\beta} &: (\bar{x}, x, \bar{x}^2), \quad x^3, x^4 \neq 1; \\
T^\mathfrak{s}_\text{bord}^\beta &: (−1, x, x), \quad T^{3\alpha + \beta} &: (\bar{x}, −1, x), \quad T^{3\alpha + 2\beta} &: (\bar{x}, x, −1), \quad x = e^{\pm \frac{\pi}{2}}; \\
T^{\alpha, 3\alpha + 2\beta} &: (−1, −1, 1), \quad T^{\alpha + \beta, 3\alpha + \beta} &: (−1, 1, −1), \quad T^{2\alpha + \beta, \beta} &: (1, −1, −1); \\
T^{\beta, 3\alpha + \beta} &: (\bar{x}, x, x), \quad x = e^{\pm \frac{2\pi}{3}}.
\end{align*}
\]
We have listed all possible \( \mathfrak{t} \subset \mathfrak{g} \), so that the sets

\[
T^s = T^\alpha \cup T^{\alpha+\beta} \cup T^{2\alpha+\beta}, \quad T^l = T^\beta \cup T^{3\alpha+\beta} \cup T^{3\alpha+2\beta},
\]

\[
T^{s,l} = T^{\alpha,3\alpha+2\beta} \cup T^{\alpha+\beta,3\alpha+\beta} \cup T^{2\alpha+\beta,\beta}, \quad T^{l,l} = T^{\beta,3\alpha+\beta},
\]

along with \( T^\varnothing \) exhaust all of \( T \). They consist of points whose conjugacy classes are isomorphic as homogeneous spaces.

Denote by \( \tilde{G} \) the group \( SL(7) \) and by \( \tilde{W} \) its Weyl group. The Weyl group of \( G \) is denoted by \( W \). It is elementary fact that the intersection of \( \tilde{G} \)-conjugacy class with \( G \) consists of a finite number of \( G \)-classes.

**Proposition 2.1.** The conjugacy class of each semisimple point \( t \in G \) is the intersection of its \( \tilde{G} \)-conjugacy class with \( G \).

**Proof.** A semisimple conjugacy class of \( \tilde{G} \) is determined by the set of eigenvalues and their multiplicities. It is sufficient to check that \( \tilde{W}t \cap T = Wt \) for all \( t \in T \).

Fix \( t = (xy, x, y) \in T^\varnothing \) so that \( y \neq -1 \). One can assume that the multiplicity of \( y \) is 1, since otherwise \( x \neq -1 \) and one can flip \( x \rightarrow y \) by \( \sigma_\beta \). Present \( \tilde{W}t \cap T \) as a union \( Y \cup \bar{Y} \) of sets whose elements have either \( y \) or \( \bar{y} \) among their first three coordinates. It is disjoint for \( t \in T^\varnothing \).

One can check that \( |Y| = |\bar{Y}| = 6 \). Then \( Y \subset Wt \) since

\[
(xy, x, y) \sim (x, y, x) \sim (y, xy, \bar{x}) \sim (y, \bar{x}, xy) \sim (\bar{x}, y, \bar{y}) \sim (\bar{x}, \bar{y}, y).
\]

The set \( \bar{Y} \) is obtained from \( Y \) by inverting the coordinates. One has \( \bar{Y} \subset Wt \) as \( Y \ni (xy, x, y) \sim (x, xy, \bar{y}) \in \bar{Y} \).

Further,

\[
T^s \ni (x, 1, x)^{\sigma_{2\alpha+\beta}} (\bar{x}, \bar{1}, \bar{x})^{\sigma_{\alpha}} (\bar{x}, 1, \bar{x})^{\sigma_{2\alpha+\beta}} (x, x, 1)^{\sigma_{\alpha+\beta}} (1, \bar{x}, \bar{x})^{\sigma_{\beta}} (1, x, \bar{x}), \quad x^2 \neq 1,
\]

\[
T^l \ni (\bar{x}, x, x^2)^{\sigma_{\beta}} (\bar{x}, \bar{x}^2, x)^{\sigma_{\alpha}} (\bar{x}^2, \bar{x}, \bar{x})^{\sigma_{\beta+2\alpha}} (x^2, x, x)^{\sigma_{\alpha}} (x, x^2, \bar{x})^{\sigma_{\beta}} (x, \bar{x}, x^2), \quad x^2, x^3 \neq 1,
\]

\[
T^{l,s} \ni (1, -1, -1)^{\sigma_{\sigma}} (-1, 1, -1)^{\sigma_{\beta}} (-1, -1, 1),
\]

\[
T^{l,l} \ni (e^{\frac{2\pi}{3}}, e^{-\frac{2\pi}{3}}, e^{\frac{2\pi}{3}})^{\sigma_{\alpha}} (e^{-\frac{2\pi}{3}}, e^{\frac{2\pi}{3}}, e^{-\frac{2\pi}{3}}).
\]

This proves \( \tilde{W}t = Wt \) for all \( t \in T \).

**Corollary 2.2.** The ideal of a semisimple conjugacy class in \( \mathbb{C}[G] \) is generated by the entries of the minimal polynomial over the maximal ideal of the subalgebra of invariants.
Throughout the paper we assume that \( \hat{O}_t \subset \hat{G} \) and \( O_t \subset G \). Let \( F_1 \) and \( F_2 \) be \( G \)-submodules in \( \text{End}(\mathbb{C}^7) \) generating the ideals \( N(\hat{O}_t) \) and \( N(\hat{G}) \), respectively. Put \( f_i : \text{End}(\mathbb{C}^7) \to F_i \) be the corresponding maps and set \( f = f_1 \oplus f_2 \). By construction, \( f \) is \( G \)-equivariant. It is sufficient to prove that \( \ker df(t) = \ker df_1(t) \cap \ker df_2(t) \) has the same dimension as \( O_t \), cf. [5], Prop. 2.1. For the general linear group, \( \tilde{m}_t = \tilde{g} \oplus \tilde{t} = \ker df_1(t) \), and \( \tilde{m}_t = \text{Ad}_t(\tilde{m}_t) \). On the other hand, \( \ker df_2(t) = \text{Ad}_t(g) \), and \( \ker df = \ker df_1 \cap \ker df_2 = \tilde{m}_t \cap g = g \oplus t \) since \( t \in G \). This completes the proof. \( \Box \)

Recall from linear algebra that a semisimple \( \hat{G} \)-class is determined by the set of eigenvalues and their multiplicities. Eigenvalues are fixed by the minimal polynomial while the multiplicities by the character of the subalgebra of invariants under conjugation. The subalgebra of invariants when restricted to maximal torus \( \hat{T} \subset \hat{G} \) is generated by the functions \( t \mapsto \text{Tr}(tm) \), \( m = 1, \ldots, 7 \), and the character is evaluation at \( t \). The possible minimal polynomials of \( t \in T \) are listed here:

\[
(t - xy)(t - x)(t - y)(t - 1)(t - \bar{x})(t - \bar{y})(t - \bar{x}\bar{y}), \quad t \in T^\circ, \quad (x, y) \in \mathbb{C}_\text{reg},
\]

\[
(t^2 - x^2)(t^2 - 1)(t^2 - \bar{x}^2), \quad t \in T^\circ, \quad x^4 \neq 1, \quad \text{mlt}(-1) = 2,
\]

\[
(t - x)(t - 1)(t - \bar{x}), \quad t \in T^s, \quad x^2 \neq 1, \quad \text{mlt}(x^\pm 1) = 2,
\]

\[
(t - x^2)(t - x)(t - 1)(t - \bar{x})(t - \bar{x}^2), \quad t \in T^l, \quad x^3 \neq 1 \neq x^4, \quad \text{mlt}(x^\pm 1) = 2,
\]

\[
(t^4 - 1), \quad t \in T^b, \quad \text{mlt}(-1) = \text{mlt}(\pm i) = 2,
\]

\[
(t^2 - 1), \quad t \in T^{s,l}, \quad \text{mlt}(-1) = 4,
\]

\[
(t^3 - 1), \quad t \in T^{l,l}, \quad \text{mlt}(e^{\pm 2\pi i}) = 3.
\]

Remark that regular points, contrary to borderline, separate irreducible \( \mathfrak{k} \)-submodules in \( \mathbb{C}^7 \). The two bottom lines correspond to the two pseudo-\( \text{Levi} \) classes.

### 3 Quantized universal enveloping algebra

Throughout the paper we assume that \( \mathfrak{q} \in \mathbb{C} \) is not a root of unity. Denote by \( U_q(\mathfrak{g}_\pm) \) the \( \mathbb{C} \)-algebra generated by \( e_{\pm\alpha}, e_{\pm\beta} \) subject to the \( q \)-\text{Serre} relations

\[
e_{\pm\alpha}e_{\pm\beta} - [4]_q e_{\pm\alpha}^3 e_{\pm\beta} e_{\pm\alpha} + [3]_q (q^2 + q^{-2}) e_{\pm\alpha}^2 e_{\pm\beta}^2 e_{\pm\alpha}^2 - [4]_q e_{\pm\alpha}^2 e_{\pm\beta} e_{\pm\alpha}^3 + e_{\pm\beta}^4 e_{\pm\alpha}^4 = 0,
\]

\[
e_{\pm\beta}^2 e_{\pm\alpha} - (q^3 + q^{-3}) e_{\pm\beta} e_{\pm\alpha} e_{\pm\beta} + e_{\pm\alpha}^2 e_{\pm\beta}^2 = 0.
\]
Here and further on, \( [z]_q = \frac{q^z - q^{-z}}{q - q^{-1}} \) whenever \( q^\pm z \) make sense.

Denote by \( U_q(h) \) the commutative \( \mathbb{C} \)-algebra generated by \( \{ q^{\pm h_\alpha} \}_{\alpha \in \Pi} \), with \( q^{h_\alpha}q^{-h_\alpha} = 1 \). The quantum group \( U_q(g) \) is a \( \mathbb{C} \)-algebra generated by \( U_q(g_{\pm}) \) and \( U_q(h) \) subject to the relations [6]:

\[
q^{h_\alpha}e_{\pm \alpha}q^{h_\alpha} = q^{\pm 2}e_{\pm \alpha}, \quad q^{h_\alpha}e_{\pm \beta}q^{-h_\alpha} = q^{\mp 3}e_{\pm \beta},
\]

\[
q^{h_\beta}e_{\pm \alpha}q^{-h_\beta} = q^{\pm 3}e_{\pm \alpha}, \quad q^{h_\beta}e_{\pm \beta}q^{-h_\beta} = q^{\pm 6}e_{\pm \beta},
\]

\[
[e_\alpha, e_{-\alpha}] = \frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}}, \quad [e_\beta, e_{-\beta}] = \frac{q^{h_\beta} - q^{-h_\beta}}{q^3 - q^{-3}}.
\]

Remark that the vector space \( h \) is not contained in \( U_q(g) \), still it is convenient to keep reference to \( h \) for additive parametrization of monomials in \( U_q(h) \).

Set up the comultiplication on the generators as

\[
\Delta(e_\alpha) = e_\alpha \otimes 1 + q^{h_\alpha} \otimes e_\alpha, \quad \Delta(e_{-\alpha}) = 1 \otimes e_{-\alpha} + e_{-\alpha} \otimes q^{-h_\alpha},
\]

\[
\Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},
\]

for all \( \alpha \in \Pi \). It is opposite to that in [7].

We will use the notation \( f_\alpha = e_{-\alpha}, f_\beta = [3]_q e_{-\beta} \), so the only relation that is inhomogeneous in \( e_{-\alpha} \) translates to \([e_\beta, f_\beta] = \frac{q^{h_\beta} - q^{-h_\beta}}{q - q^{-1}} \).

The subalgebras in \( U_q(g) \) generated by \( U_q(g_{\pm}) \) over \( U_q(h) \) are quantized universal enveloping algebras of the Borel subalgebras \( b_{\pm} = h + g_{\pm} \subset g \) denoted further by \( U_q(b_{\pm}) \).

The Chevalley generators \( e_{\pm \alpha} \) can be supplemented with higher root vectors \( e_{\pm \beta} \) for all \( \beta \in \mathbb{R}^+ \). They participate in construction of a Poincaré-Birkhoff-Witt (PBW) basis in \( U_q(g) \) and universal \( R \)-matrix, [7].

The universal \( R \)-matrix is an element of a certain extension of \( U_q(g) \otimes U_q(g) \). Let \( \{ \xi_i \}_{i=1}^2 \) be an orthogonal basis in \( h^* \). The exact expression for \( R \) up to the flip is extracted from [7], Theorem 8.3.9:

\[
R = q^{\sum_{i=1}^2 h_{\xi_i} \otimes h_{\xi_i}} \prod_{\mu \in \mathbb{R}^+} \exp_q(1 - q_\mu^{-2})(e_{-\mu} \otimes e_\mu) \in U_q(b_{-}) \hat{\otimes} U_q(b_{+}), \tag{3.8}
\]

where \( \exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{q^k} \), \( q_\mu = q^{(\mu, \mu)} \), and the product is ordered in a certain way. Its reduction to the minimal representation can be found in [8, 9].

### 3.1 Minimal representation of \( U_q(g) \)

In this section we describe a representation of \( U_q(g) \) in the vector space \( \mathbb{C}^7 \). It is a deformation of the classical representation of \( g \) restricted from \( \mathfrak{so}(7) \). Our realization is close to [10].
Let $I$ denote the set of integers from $1, \ldots, 7$. Fix a basis $\{w_i\}_{i \in I} \subset \mathbb{C}^7 = V$ and let $e_{ij} \in \mathrm{End}(V)$ denote the standard matrix units, $e_{ij}w_k = \delta_{jk}w_i$, $i, j, k \in I$. One can check that the assignment

$$
q^h \alpha \mapsto qe_{11} + q^{-1}e_{22} + q^2e_{33} + e_{44} + q^{-2}e_{55} + qe_{66} + q^{-1}e_{77},
$$

$$
q^h \beta \mapsto e_{11} + q^3e_{22} + q^{-3}e_{33} + e_{44} + q^3e_{55} + q^{-3}e_{66} + e_{77},
$$

$$
e_\alpha \mapsto e_{12} + e_{34} + e_{45} + e_{67},
$$

$$
f_\alpha \mapsto e_{21} + [2]q^{43} + [2]q^{54} + e_{76},
$$

$$
e_\beta \mapsto e_{23} + e_{56},
$$

$$
f_\beta \mapsto e_{32} + e_{65}.
$$

is compatible with the defining relations and extends to a homomorphism $U_q(\mathfrak{g}) \to \mathrm{End}(V)$.

Up to scalar multiplies, the action of $U_q(\mathfrak{g}^-)$ can be depicted by the graph by

4 Quantum conjugacy classes

In this section we describe quantum semisimple conjugacy classes along the lines of [11] and [13]. The construction is based on certain facts from representation theory to be established in the subsequent sections.

We regard elements of $\mathfrak{h}^*$ as characters of $T$ and elements of $T$ as spectral points of $U_q(\mathfrak{h})$ via the correspondence $t: q^h \alpha \mapsto \alpha(t)$, for all $t \in T$. Fix $t \in T$ and its stabilizer subalgebra $\mathfrak{k}$. Choose the weight $\lambda$ from the condition $q^{2\lambda} = t_q^{2\rho - 2\rho} = t_q$ regarded as an equality in $T$ upon the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ via the inner product. Consider a 1-dimensional representation $\mathbb{C}_\lambda$ of $U_q(\mathfrak{h}^+)$ extending $t_q: U_q(\mathfrak{h}) \to \mathbb{C}$: it is implemented
by the assignment $q^{\pm h_{\alpha}} \mapsto q^{\pm (\lambda, \alpha)}$, $e_{\alpha} \mapsto 0$. Let $M_{\lambda} = U_q(\frak{g}) \otimes_{U_q(\frak{g}^+)} \mathbb{C}$ denote the Verma $U_q(\frak{g})$-module of highest weight $\lambda$ with the canonical generator $v_{\lambda} \in M_{\lambda}$.

Due to the special choice of $\lambda$, it satisfies the conditions $q^{2(\lambda + \rho, \alpha)} = q^{(\alpha, \alpha)}$ for all $\alpha \in \Pi$, and we denote by $\Theta^{\mathfrak{t}}$ the set of all such weights. There exist singular vectors $v_{\lambda - \alpha} \in M_{\lambda}$ annihilated by $e_{\alpha}$, $\forall \alpha \in \Pi$. Up to a scalar multiplier, they can be written explicitly as $f_{1i}v_{\lambda}$ with $(1, i) \in P(\alpha)$, where $f_{ij}$ are matrix elements of the (reduced) Shapovalov inverse. The vectors $v_{\lambda - \alpha}$ generate submodules $M_{\lambda - \alpha} \subset M_{\lambda}$. Let $M_{\lambda}^{\mathfrak{t}}$ denote the quotient module $M_{\lambda}/ \sum_{\alpha \in \Pi} M_{\lambda - \alpha}$. On transition to the $\mathbb{C}[[\mathfrak{h}]]$-extension via $q = e^{\mathfrak{h}}$, $M_{\lambda}^{\mathfrak{t}}$ is free over $\mathbb{C}[[\mathfrak{h}]]$, by Lemma 5.2.

Denote by $Q = (\pi \otimes \text{id})(\mathcal{R}_{21} \mathcal{R}) \in U_q(\frak{g})$ a matrix with entries in the quantum group. It commutes with the coproduct of all elements from $U_q(\frak{g})$ and plays a key role in the theory. Its entries generate a subalgebra $\mathbb{C}_q[G] \subset U_q(\frak{g})$, which is an equivariant quantization of $\mathbb{C}[G]$. So $Q$ is the matrix of ”coordinate” functions on $G$. It satisfies the so called ”reflection equation” rather then the RTT-relations of the Hopf dual to $U_q(\frak{g})$.

The operator $Q$ is scalar on every submodule of highest weight in $V \otimes M_{\lambda}^{\mathfrak{t}}$ as well as a quotient module. Its eigenvalues can be described as follows. In the classical limit, the module $V$ is completely reducible over $\mathfrak{t}$. Let $I^{\mathfrak{t}} \subset I$ denote the subset of indices of $\mathfrak{t}$-highest vectors $w_i$. Then the eigenvalues are $x_j = q^{2(\lambda + \rho, \nu_j) + (\nu_j, \nu_j) - (\nu_i, \nu_i) + 2(\rho, \nu_i)}$. It is proved in the subsequent sections that $Q$ is semisimple on $V \otimes M_{\lambda}^{\mathfrak{t}}$, so its minimal polynomial is

$$\prod_{j \in I^{\mathfrak{t}}} (Q - x_j) = 0,$$

where the prime means that only distinct eigenvalues count (coincidences occur only for borderline $t$). One can check that the set $\{x_j\}_{j \in I^{\mathfrak{t}}}$ is invariant under the shifted action of Weyl group $\sigma: \lambda \mapsto \sigma(\lambda + \rho) - \rho$.

Put $\text{Tr}_q(X) = \text{Tr}(\pi(q^{2\mathfrak{h}_{\rho}})X)$ for a matrix $X$ with arbitrary entries. Then the elements $\tau^m = \text{Tr}_q(Q^m)$, $m \in I$, generate the center of $\mathbb{C}_q[G]$. Note that $\tau^m$ are not independent, as the rank of $G$ is two.

Let $\chi_\lambda$ denote the central character of $\mathbb{C}_q[G]$ returning $\tau^m v_\lambda = \chi_\lambda(\tau^m) v_\lambda$,

$$\chi_\lambda(\tau^m) = \sum_{i=1}^{7} x_i^{\nu_i} \prod_{\alpha \in \mathbb{R}_+} \frac{q^{(\lambda + \rho + \nu_i, \alpha)} - q^{-(\lambda + \rho + \nu_i, \alpha)}}{q^{(\lambda + \rho, \alpha)} - q^{-(\lambda + \rho, \alpha)}}.$$

It is invariant under the shifted action of the Weyl group $W$ on $\frak{h}^*$.

**Theorem 4.1.** The image of $\mathbb{C}_q[G]$ in $\text{End}(M_{\lambda}^{\mathfrak{t}})$ is an equivariant quantization of the polynomial algebra $\mathbb{C}[O_{1^t}]$. It is the quotient of $\mathbb{C}_q[G]$ by the ideal generated by the entries of the
minimal polynomial over the kernel of \( \chi_\lambda \). The algebra \( \mathbb{C}_q[O_t] \) depends only on the \( W \)-orbit of the point \( t \in T \).

Proof. The proof is similar to [11]. Denote by \( S \) the quotient of \( \mathbb{C}_q[G] \) by the ideal generated by \( \ker \chi_\lambda \). Denote by \( T \) the extension of \( \text{End}(M^t_\lambda) \) over the Laurent series. Let \( U_h(\mathfrak{g}) \) be the \( \mathbb{C}[[h]] \)-extension of \( U_q(\mathfrak{g}) \) completed in \( h \)-adic topology. The map \( S \to T \) is \( U_h(\mathfrak{g}) \)-equivariant. The algebra \( S \) is a direct sum of \( U_h(\mathfrak{g}) \)-modules that are \( \mathbb{C}[[h]] \)-finite (as embedded in the locally finite part of \( \text{End}(M_\lambda) \)). Since \( T \) has no \( h \)-torsion, the image of \( S \) in \( T \) is \( \mathbb{C}[[h]] \)-free. The relations (4.9) and (4.10) go over to the relations of \( N(O_t) \), hence they generate the kernel of the map \( \mathbb{C}_q[O_t] \to T \), cf. [11] for details. This kernel is invariant under the shifted action of \( W \) on the weight \( \lambda \), therefore, \( \mathbb{C}_q[O_t] \) depends on the class of \( t \).

The key fact the proof based upon is semisimplicity of \( Q \). To this end, it is sufficient to show that highest weight submodules exhaust all of \( V \otimes M^t_\lambda \). We solve a stronger problem: we establish exact criteria when \( V \otimes M^t_\lambda \) splits into a direct sum of highest weight modules. The rest of the paper is devoted to this analysis.

5 Generalized parabolic Verma modules

Fix a weight \( \lambda \in \mathfrak{h}^* \) and consider the Verma module \( M_\lambda \). Let \( *M_\lambda \) denote the Verma module of lowest weight \(-\lambda \). There is a unique, up to a scalar multiple, \( U_q(\mathfrak{g}) \)-invariant form \( M_\lambda \otimes *M_\lambda \to \mathbb{C} \) (equivalent to the contravariant Shapovalov on \( M_\lambda \)), which is non-degenerate if and only if \( M_\lambda \) is irreducible. As that is the case for generic weight, [12], there is a unique lift \( \hat{F} \in U_q(\mathfrak{g}_+) \otimes U_q(\mathfrak{b}_-) \) of the inverse form, where the Borel subalgebra is extended over the ring of fractions of \( U_q(\mathfrak{b}) \). The matrix \( \hat{F} = (\pi \otimes \text{id})(\hat{F}) \in \text{End}(V) \otimes U_q(\mathfrak{b}_-) \) is expressed through a matrix \( F \in \text{End}(V) \otimes U_q(\mathfrak{g}_-) \) whose entries \( f_{ij} \) are presented in Section A.2.

Put \( \rho_i = (\rho, \nu_i) \), \( \tilde{\rho}_i = \rho_i + \frac{1}{2}\| \nu_i \|^2 \), \( i \in I \), and define

\[
\eta_{ij} = h_{\nu_i} - h_{\nu_j} + \rho_i - \rho_j - \frac{\| \nu_i - \nu_j \|^2}{2}, \quad A^i_j = \frac{q-q^{-1}}{1-q^{2\eta_{ij}}}, \quad i \neq j.
\]

For an ascending sequence of integers \( m_1, \ldots, m_k \), put \( f_{m_1, m_2 \ldots m_{k-1}, m_k} = f_{m_1, m_2} \cdots f_{m_{k-1}, m_k} \) and \( A^i_{m_1, \ldots, m_k} = A^i_{m_1} \cdots A^i_{m_k} \). Then

\[
\hat{f}_{ij} = \sum_{i < \tilde{m} < j} f_{i, \tilde{m}, j} A^i_{i, \tilde{m}} q^{\eta_{ij} - \tilde{\rho}_i + \tilde{\rho}_j}, \quad (5.11)
\]

where the summation is taken over all sequences \( \tilde{m} = (m_1, \ldots, m_k) \) such that \( i < m_1 \), \( m_k < j \), including \( \tilde{m} = \emptyset \). Finally, \( \hat{f}_{ii} = 1 \) for all \( i \) and \( \hat{f}_{ij} = 0 \) for \( i > j \).
We regard $U_q(\mathfrak{h})$ as the algebra of trigonometric polynomials on $\mathfrak{h}^*$. The linear isomorphism $U_q(\mathfrak{b}_-) \simeq U_q(\mathfrak{g}_-) \otimes U_q(\mathfrak{h})$ identifies elements of $U_q(\mathfrak{b}_-)$ with functions $\mathfrak{h}^* \to U_q(\mathfrak{g}_-)$. The singular vector $v_{\lambda-\alpha} \in M_{\lambda}$ can be constructed as follows. It is known to be unique, up to a scalar factor (follows from the basics properties of reduction algebras). Therefore it is proportional to \( \hat{f}_{ij}v_{\lambda} = \hat{f}_{ij}(\lambda)v_{\lambda} \) with \((i, j) \in P(\alpha)\), upon an appropriate regularization of \( \hat{f}_{ij}(\lambda) \) if needed.

**Lemma 5.1.** Suppose that \( t \in T^t \) and \( \lambda = \frac{1}{2} \ln t_q \in \mathfrak{h}^* \) is fixed as in Section 4. For all \( \alpha \in \Pi_t \) choose a pair \((i, j) \in P(\alpha)\). Then \( \hat{f}_{ij}(\lambda) \in U_q(\mathfrak{g}_-) \) is a deformation of a classical root vector, \( f_\alpha \in \mathfrak{g}_- \).

**Proof.** The proof is based on the fact that in any presentation of \( \alpha \) as a sum of positive roots the summands do not belong to \( R^+ \), see e.g. [13], Lemma 2.2. \( \square \)

**Corollary 5.2.** The \( \mathbb{C}[[\hbar]] \)-extension of \( M^t_\lambda \) is \( \mathbb{C}[[\hbar]] \)-free.

**Proof.** The proof is similar to [5], Proposition 6.2. \( \square \)

## 5.1 Standard filtration of \( V \otimes M_\lambda \)

Define \( \mathcal{V}_j \subset V \otimes M_\lambda, j \in I \), to be the submodule generated by \( \{ w_i \otimes v_{\lambda} \}_{i=1}^j \). They form an ascending filtration \( \mathcal{V}_\bullet \) of \( V \otimes M_\lambda \), which we call standard. Its graded module \( \text{gr} \mathcal{V}_\bullet \) is the direct sum \( \oplus_{j=1}^j \text{gr} \mathcal{V}_j \), where \( \text{gr} \mathcal{V}_j = \mathcal{V}_j / \mathcal{V}_{j-1} \) is isomorphic to \( M_{\lambda + \nu_j} \) for all \( \lambda \) (the proof is similar to [14] for classical \( U(\mathfrak{g}) \)). It is generated by the image \( w^j_\lambda \) of \( w_j \otimes v_{\lambda} \) in \( \text{gr} \mathcal{V}_j \).

**Proposition 5.3.** Suppose \((i, j) \in P(\beta)\), and \( \psi \) is a Chevalley monomial of weight \( -\beta \). If \( \psi \not\approx \psi^j \), then \( w_i \otimes \psi v_{\lambda} \in \mathcal{V}_{j-1} \). Otherwise,

\[
\quad w_i \otimes \psi^j v_{\lambda} \simeq w_j \otimes v_{\lambda} \mod \mathcal{V}_{j-1}. \tag{5.12}
\]

**Proof.** The proof is similar to [13], Proposition 3.5. \( \square \)

Assuming \( \lambda \in \Theta^t \) denote by \( \mathcal{V}^t_j \) the image of \( \mathcal{V}_j \) under the projection \( V \otimes M_\lambda \to V \otimes M^t_\lambda \). Clearly the sequence \( \mathcal{V}_\bullet^t = (\mathcal{V}^t_j) \) forms an ascending filtration of \( V \otimes M^t_\lambda \). Denote by \( \bar{I}^t \) the complement of \( I^t \) in \( I \). Then \( j \in \bar{I}^t \) if and only if there is \( i < j \) such that \( \nu_i - \nu_j \in \Pi_t \).

**Proposition 5.4.** The graded module \( \text{gr} \mathcal{V}^t_\bullet \) of the filtration \( \mathcal{V}^t_\bullet \) is isomorphic to \( \oplus_{j \in \bar{I}^t} \text{gr} \mathcal{V}^t_j \).
Proof. Fix $j$ and put and $\beta = \nu_1 - \nu_j$. The module $\text{gr} V^j_{\ell}$ is a quotient of $V^j_{\ell}$ by the submodule $V^j_{\ell-1} + (V \otimes M) \cap V^j_{\ell}$, where $M = \sum_{\alpha \in \Pi_k} M_{\lambda - \alpha}$. By Proposition 5.3, its subspace of weight $\nu_j + \lambda$ is isomorphic to the quotient of $w_1 \otimes M_{\lambda - \beta}$ by $w_1 \otimes \mathcal{N}_\beta v_\lambda + w_1 \otimes M_{\lambda - \beta}$, where $\mathcal{N}_\beta \subset U_q(\mathfrak{g}_-)$ is spanned by Chevalley monomials of weight $-\beta$ that are not proportional to $\psi^j_\beta$. Then $M_{\lambda - \beta} = \mathcal{N}_\beta v_\lambda + M_{\lambda - \beta}$ if and only if $\psi^j_{\beta} v_\lambda \in M_{\lambda - \beta}$ mod $\mathcal{N}_\beta v_\lambda$, which is equivalent to $j \in I^\ell$. Otherwise $\text{gr} V^j_{\ell} \subset M_{\lambda - \beta}$ is spanned by the image of $w_1 \otimes \psi^j_{\beta} v_\lambda \simeq w_1 \otimes v_\lambda$, the generator of $\text{gr} V^j_{\ell}$. This proves that $V^j_{\ell} = V^j_{\ell-1}$ for each $j \in I^\ell$ and $V^j_{\ell}/V^j_{\ell-1} \neq \{0\}$ for $j \in I^\ell$.

Let $M_j \subset V \otimes M_\lambda$ denote the submodule of highest weight $\lambda + \nu_j$ and let $w_j$ be its highest weight generator. Furthermore, consider $M^j_{\ell}$ for $\lambda \in \Theta^\ell$ and let $\pi^j_{\ell}$ denote the projection $V \otimes M_\lambda \to V \otimes M^j_{\ell}$. Define $M^j_{\ell} = \pi^j_{\ell}(M_j)$ and $\mathcal{W}^j_{\ell} = \sum_{i=1}^7 M^j_{\ell}$. The ascending sequence $\mathcal{W}^j_{\ell} = (\mathcal{W}^j_{\ell})$, $j = 1, \ldots, 7$, of submodules is also invariant under the action of $\mathcal{Q}$, which is semisimple on $\mathcal{W}^j_{\ell}$. Semi-simplicity of $\mathcal{Q}$ is important for our studies, so the question is when $\mathcal{W}^j_{\ell} = V \otimes M^j_{\ell}$ or, more specifically, $\mathcal{W}^j_{\ell} = \oplus_{j \in \ell} M^j_{\ell}$. We answer this question by comparing $\mathcal{W}^j_{\ell}$ against $V^j_{\ell}$. First of all, observe that $\mathcal{W}^j_{\ell} \subset V^j_{\ell}$, by Proposition 5.3.

Proposition 5.5. Suppose that $\ell$ is Levi and fix $j \in I^\ell$. Then the following statements are equivalent: i) $V^j_{\ell} = \mathcal{W}^j_{\ell}$, ii) $V^j_{\ell} = \mathcal{W}^j_{\ell}$ all $i \leq j$, iii) projection $\varphi^j_{\ell}: V \otimes M^j_{\ell} \to \text{gr} V^j_{\ell}$ is an isomorphism on $M^j_{\ell}$ for all $i \leq j$, iv) $\mathcal{W}^j_{\ell} = \oplus_{i=1}^j M^j_{\ell}$.

Proof. It can be proved that, for Levi $\ell$, both $M^j_{\ell}$ and $\text{gr} V^j_{\ell}$ are parabolically induced from the same $U_q(\mathfrak{g})$-module. Hence the map $M^j_{\ell} \to \text{gr} V^j_{\ell}$ is epimorphism and isomorphism simultaneously unless it is zero.

The implication ii) $\Rightarrow$ i) is trivial. With $\mathcal{W}^j_{\ell} = V^j_{\ell}$, assume that ii) is violated and let $k > 1$ be the smallest such that $\mathcal{W}^j_{k} \neq V^j_{k}$. Comparison of weight subspaces gives $\dim \mathcal{W}^j_{k}[\lambda+\varepsilon_j] = \dim \mathcal{W}^j_{k}[\lambda+\varepsilon_j]+1$ for all $j \geq k$, so ii) $\Rightarrow$ ii). Assuming ii) we find that all maps $M^j_{\ell} \to \text{gr} V^j_{\ell}$ are surjective and therefore injective; hence iv). Conversely, iv) implies that all maps $\mathcal{W}^j_{\ell} \to \text{gr} V^j_{\ell}$ are surjective. Since, $\mathcal{W}^j_{\ell} = V^j_{\ell}$, induction on $i$ then proves ii). Furthermore, iv) implies that $M^j_{\ell} \to \text{gr} V^j_{\ell}$ are isomorphisms, and then $M^j_{\ell} \cap \mathcal{W}^j_{\ell-1} \subset M_i \cap \mathcal{W}^j_{\ell-1} = \{0\}$, which proves iii). Finally, induction on $i$ yields iii) $\Rightarrow$ ii).

A direct sum decomposition $V \otimes M^j_{\ell} = \oplus_{i} M^j_{\ell}$ implies that the operator $\mathcal{Q}$ is semisimple on $V \otimes M^j_{\ell}$. More generally, $\mathcal{Q}$ is semisimple if $V \otimes M_\lambda = W^j_{\ell}$. That is the case if all maps $\varphi^j_{\ell}: M^j_{\ell} \to \text{gr} V^j_{\ell}$ are onto, i.e. the generators of $M^j_{\ell}$ are not killed by $\varphi^j_{\ell}$.
6 Module structure of $V \otimes M_\lambda$

In this section, where work out exact criteria for decomposition of $C \otimes M_\lambda$ into a direct sum of submodules of highest weight. To that end, we undertake a detailed study of singular vectors $\hat{u}_j = \hat{F}(w_j \otimes v_\lambda) \in C \otimes M_\lambda$ as rational functions $\mathfrak{h}^* \rightarrow V \otimes U(\mathfrak{g}_-)$ upon the natural identification of $M_\lambda$ with $U_q(\mathfrak{g}_-)$ as a vector space. We end up with rescaled singular vectors $u_j, j \in I$, that are regular on $\mathfrak{h}^*$ and never turn zero.

6.1 Singular vectors in $V \otimes M_\lambda$

The vectors $\hat{u}_j = \hat{F}(w_j \otimes v_\lambda), j \in I$, are expanded as

$$\hat{u}_j = \sum_{i=1}^{j} w_i \otimes \hat{f}_{ij} v_\lambda \in V \otimes M_\lambda. \quad (6.13)$$

They are singular for all $\lambda$ where defined and generate submodules $M_j \subset C \otimes M_\lambda$ of highest weight $\lambda + \nu_j$. They have rational trigonometric dependence on $\lambda$ and may have zeros and poles. As they matter up to scalar factors, it is convenient to pass from $\hat{u}_j$ to $\check{u}_j = \check{A}_{1,\ldots,j-1}(\lambda) \hat{u}_j$, which are regular in $\lambda$. Then

$$\check{u}_j = \sum_{i=1}^{j} w_i \otimes \check{u}_{ij}, \quad \text{where} \quad \check{u}_{ij} = \check{A}_{1,\ldots,i-1}(\lambda) \check{f}_{ij} v_\lambda, \quad \check{f}_{ij} = \check{f}_{ij} \check{A}_{i,\ldots,j-1} \in U_q(\mathfrak{b}_-). \quad (6.14)$$

They generate submodules $M_j \subset V \otimes M_\lambda$ if do not turn zero, otherwise they needs rescaling. That is the subject of our further study.

Remark that, for any $U_q(\mathfrak{g})$-module $Z$, a singular vector $u = \sum_{i \in I} w_i \otimes z_i \in V \otimes Z$, defines a $U_q(\mathfrak{g}_+)$-equivariant map $V^* \rightarrow Z$. Since the $U_q(\mathfrak{g}_+)$-module $V^*$ is cyclicly generated by $z_1$, we call it generating coefficient of $u$.

Lemma 6.1. Suppose that $Z$ is generated by highest weight vector $v_\lambda$. Suppose that $\check{f}_{mj} v_\lambda = 0$ for $m < j$. Then $\check{f}_{ij} v_\lambda = 0$ for all $i \leq m$.

Proof. Recall that for all $k \geq m$ and $\mu = \nu_k - \nu_{k+1}, e_\mu \check{f}_{kj} = \check{f}_{k+1,j} \check{A}_{k}^j \mod U_q(\mathfrak{g}_+)$, cf. [15]. Then $\check{f}_{mj} v_\lambda = 0$ implies $\check{A}_{m,\ldots,k-1}(\lambda) \check{f}_{kj} v_\lambda = 0$ for all $k \geq m + 1$. If follows from (5.11) that

$$\check{f}_{m-1,j} v_\lambda = a_m(\lambda) \check{f}_{m-1,m} \check{f}_{mj} v_\lambda + \sum_{k=m+1}^{j} a_k(\lambda) \check{f}_{m-1,k} (\check{A}_{m,\ldots,k-1}^j(\lambda) \check{f}_{kj}) v_\lambda,$$

where $a_i(\lambda)$ are non-vanishing numerical factors. This implies $\check{f}_{m-1,j} v_\lambda = 0$. Induction on $m$ proves the statement for all $i \leq m$. \hfill \Box
Corollary 6.2. The vector $\tilde{u}_j \in V \otimes \mathbb{Z}_\lambda$ turns zero i) only if $\tilde{A}_m^j(\lambda) = 0$, ii) if only if $\tilde{f}_{m,j}v_\lambda = 0$, for some $m < j$.

Proof. "Only if" in both statements follow from the equalities $\tilde{u}_{ij} = \tilde{A}^j_{i-1}v_\lambda$ and, respectively, $\tilde{u}_{1j} = \tilde{f}_{1j}v_\lambda$. "If" is due to Lemma 6.1.

Remark 6.3. Suppose that $Z$ is a family of $U_q(\mathfrak{g})$-modules of highest weight $\lambda$ ranging in an algebraic set $\Theta \subset \mathfrak{h}^*$. Then Lemma 6.1 admits an obvious modification if one replaces equality to zero with divisibility by some $\phi \in \mathbb{C}_q[\Theta]$. Assuming it indecomposable, Corollary 6.2 can be appropriately reworded if $\mathbb{C}_q[\Theta]$ is a unique factorization domain and $Z$ has no zero divisors. In what follows, we apply this modification to $\Theta = \Theta^\mathfrak{f}$ and $Z = M_\lambda^\mathfrak{f}$.

In the next statement we essentially assume that $\mathfrak{f} \neq \mathfrak{h}$.

Proposition 6.4. The submodule $M_j^\mathfrak{f} \subset V \otimes M_\lambda^\mathfrak{f}$ vanishes for all $j \in \tilde{R}$.

Proof. It is sufficient to consider the case $\Pi_\mathfrak{f} = \{\mu\}$. Choose $\lambda \in \Theta^\mathfrak{f}$ so that $\tilde{u}_j \neq 0$. By Proposition 6.8 below, such weights are dense in $\Theta^\mathfrak{f}$. Let $i \in I$ be such that $\nu_i - \nu_j = \mu$. As $\tilde{A}_i^j(\lambda) = 0$, for all $k \leq i$ one has

$$\tilde{f}_{k,j}v_\lambda \simeq \sum_{k < m < i} f_{k,m,i} \tilde{A}_{k,m,i}(\lambda) \tilde{f}_{ij}v_\lambda \in M_{\lambda-\mu} \subset M_\lambda.$$ 

Therefore $\tilde{u}_j$ is in the submodule $V \otimes M_{\lambda-\mu}$ and so does $u_j$ for all $\lambda \in \Theta^\mathfrak{f}$. 

6.2 Projection to $\text{gr} \mathcal{V}_\bullet$

It follows from Proposition 5.3 that the image of $\tilde{u}_j$ in $\text{gr} \mathcal{V}_j$, lies in $\text{gr} \mathcal{V}_j$, and thus $\tilde{u}_j = \hat{D}_j w_\lambda^j$ mod $\mathcal{V}_{j-1}$ with some $\hat{D}_j \in \mathbb{C}$. Obviously $\hat{D}_1 = 1$. For higher $j$ calculation gives

$$\hat{D}_2 \simeq \frac{[\xi_{12}]_q}{[\eta_{12}]_q}, \quad \hat{D}_3 \simeq \frac{[\xi_{13}]_q [\xi_{23}]_q}{[\eta_{13}]_q [\eta_{23}]_q}, \quad \hat{D}_4 \simeq \frac{[\xi_{14}]_q [\xi_{24}]_q [\xi_{34}]_q}{[\eta_{14}]_q [\eta_{24}]_q [\eta_{34}]_q}, \quad \hat{D}_5 \simeq \frac{[\xi_{15}]_q [\xi_{25}]_q [\xi_{35}]_q [\xi_{45}]_q}{[\eta_{15}]_q [\eta_{25}]_q [\eta_{35}]_q [\eta_{45}]_q},$$ 

$$\hat{D}_6 \simeq \frac{[\xi_{16}]_q [\xi_{36}]_q [\xi_{26}]_q [\xi_{56}]_q}{[\eta_{16}]_q [\eta_{36}]_q [\eta_{26}]_q [\eta_{56}]_q}, \quad \hat{D}_7 \simeq \frac{[\xi_{27}]_q [\xi_{37}]_q [\xi_{47}]_q [\xi_{57}]_q [\xi_{67}]_q}{[\eta_{27}]_q [\eta_{37}]_q [\eta_{47}]_q [\eta_{57}]_q [\eta_{67}]_q},$$

where the quantities

$$\xi_{ij} = h_i - h_j + \rho_i - \rho_j + \frac{1}{2}(||\nu_i||^2 - ||\nu_j||^2) \in \mathfrak{h} + \mathbb{C}$$ 

are related to eigenvalues of the operator $\mathcal{Q}$ by $q^{2\xi_{ij}} = x_i x_j$. For reader’s convenience, $\xi_{ij}$ are given in Section A.1.

Let $\Theta_j \subset \mathfrak{h}^*, j > 4$, denote the set of weights such that $q^{2\eta_{ij}(\lambda)} = -q^2$. 

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Lemma 6.5. The module $M_j$ is not contained in $V_{j-1}$ for a dense open subset $\Theta_j^\circ \subset \Theta_j$.

Proof. Observe that $q^{2\xi_{ij}} = -q^2 \neq 1$ and $q^{2\xi_{ij}} = q^8 \neq 1$. For all other $i < j$, the functions $q^{2\xi_{ij}}$ are not constant on $\Theta_j$. Therefore all $q^{2\xi_{ij}}$ with $i < j$ are distinct from 1, off an algebraic subset in $\Theta_j$. For such weights, $M_j$ cannot be in $V_j$, since the eigenvalue $x_j$ is distinct from the $\mathbb{Q}$-eigenvalues $\{x_i\}_{i=1}^j$ on $V_{j-1}$.

Corollary 6.6. For all $j > 4$, $\tilde{f}_{1j}$ identically vanishes on $\Theta_j$.

Proof. The product $\tilde{D}_j = \prod_{k=1}^{j-1} \tilde{A}_k \tilde{D}_j$ is polynomial in $q^{\pm h_\nu}$ and turns zero on $\Theta_j$ for $j > 4$ thanks to the factor $\tilde{A}_j$, and the equality $\frac{\eta_{ij}}{2} = \eta_{ij} - 1$. The tensor $\tilde{u}_j \in V \otimes M_\lambda$ is projected to $\tilde{C}(\lambda)w_\lambda^2 = 0 \mod V_{j-1}$. That is possible only in the following two cases: either $\tilde{u}_j$ turns zero or $M_j \subset V_{j-1}$. By Lemma 6.5, $\tilde{u}_j(\lambda) = 0$ on $\Theta_j^\circ$ and hence on $\Theta_j$. This yields $\tilde{u}_{1j} = \tilde{f}_{1j}v_\lambda = 0$ on $\Theta_j$ for the generating coefficient.

6.3 Regularization of singular vectors in $V \otimes M_\lambda$

In this section, we evaluate a scalar function $\delta_j \subset \tilde{u}_j$ and show that renormalized singular vectors $\delta_j^{-1} \tilde{u}_j$ do not turn zero at all $\lambda$.

Denote by $J$ the two-sided ideal in $U_q(\mathfrak{g}_-)$ generated by the relation $f_{ij}f_{ij} = q^4 f_{ij}f_{ij}$. The non-zero elements $f_{ij}$ modulo $J$ read

$$f_{12} = f_\alpha, \quad f_{23} = [3]_q f_\beta, \quad f_{34} = f_\alpha, \quad f_{45} = f_\alpha, \quad f_{56} = [3]_q f_\beta, \quad f_{67} = f_\alpha,$$

$$f_{24} = (q^3 - q^3) f_{ij} f_\alpha, \quad f_{35} = \frac{q^2 - 1}{[2]_q} f_\alpha^2, \quad f_{57} = (q^3 - q^3) f_{ij} f_\alpha,$$

$$f_{25} = (q^3 - q)(q^3 - q^3) f_{ij} f_\alpha^2.$$

Introduce $\tilde{g}_{ij}$ as polynomials in $y_1^{\pm 1}, \ldots, y_7^{\pm 1}$ with coefficients in $U_q(\mathfrak{g}_-)$ by similar formulas as $\tilde{f}_{ij}$ with all $\tilde{A}_k^i$ in (5.11) and (6.14) replaced by $\tilde{A}_k = \frac{1-y_k}{q-y_k}$.

Lemma 6.7. One has $\tilde{g}_{35} \simeq f_\alpha^2 \frac{2q^4 + q}{q+q}$. Furthermore, $\tilde{g}_{25} \simeq [3]_q f_\beta f_\alpha^2 \frac{2q^4 + q}{q+q} \mod J$.

Proof. All calculations will be done modulo $J$. First of all,

$$\tilde{g}_{35} \simeq f_{35} \tilde{A}_4 + f_{34} f_{45} = f_\alpha^2 \frac{q^2 - 1}{[2]_q} y_4 - \frac{1}{q-q} + 1 = f_\alpha^2 \frac{2q^4 + q}{q+q}.$$

Substitute $f_{ij}$ mod $J$ into $\tilde{g}_{25} \simeq f_{25} \tilde{A}_3 \tilde{A}_4 + f_{23} f_{35} \tilde{A}_4 + f_{24} f_{45} \tilde{A}_3 + f_{23} f_{34} f_{45}$ and get

$$\tilde{g}_{25} \simeq f_{ij} f_\alpha^2 \left( \frac{(q^3 - q)(q^3 - q^3)}{[2]_q^2} \tilde{A}_3 \tilde{A}_4 + [3]_q \frac{q^2 - 1}{[2]_q} \tilde{A}_4 + (q^3 - q^3) \tilde{A}_3 + [3]_q \right).$$

Computation of the coefficient in the brackets completes the proof.
Set \( \delta_j = \frac{q^{n_{ij} - 1} - q^{-n_{ij} + 1}}{q - q^{-1}} = \frac{n_{ij} - q^{-n_{ij} + 1}}{q - q^{-1}} \) for \( 5 \leq j \) and \( \delta_j = 1 \) for \( j = 1, 2, 3, 4 \). Corollary 6.6 assures that \( \tilde{f}_{ij}/\delta_j \) is a polynomial in \( q^{\pm h_\mu} \) via identification \( y_i = q^{2n_{ij}}, i < j \).

**Proposition 6.8.** For all \( j > 1 \), the vectors \( \tilde{f}_{ij}/\delta_j \) do not turn zero on \( h^* \).

**Proof.** Again, all calculations are done modulo \( J \). The statement is trivial for \( j = 2 \). For \( j = 3 \), it follows from the factorization \( \tilde{f}_{13} = \tilde{f}_{12}\tilde{f}_{23} \). We also have

\[
\tilde{f}_{14} \simeq f_{12}\tilde{f}_{24} \simeq f_\alpha([3]q\beta f_\alpha + (q^3 - q^3)\beta f_\alpha \bar{A}_3^4) = [3]qf_\alpha f_\beta f_\alpha q^{n_{15}} \neq 0,
\]

which proves the case \( j = 4 \).

For each \( j \geq 5 \) and all \( i < j \) we assign \( y_i = q^{2n_{ij}} \) and use Lemma 6.7: the key point is that \( \delta_j \) cancels the factor \( \bar{q}y_4 + q \) in all cases. Modulo \( J \), we have \( \tilde{f}_{15}(\lambda) \simeq f_{12}\bar{y}_{25} \) and \( \tilde{f}_{16}(\lambda) \simeq f_{12}\bar{y}_{25}\bar{f}_{56} \). This implies the statement for \( j = 5, 6 \). Finally,

\[
\tilde{f}_{17}(\lambda) \simeq f_{12}\bar{y}_{27} \simeq f_{12}(\bar{y}_{26}f_{67} + \bar{y}_{25}f_{57}\bar{A}_6) \simeq f_{12}(\bar{y}_{25}f_{56}f_{67} + \bar{y}_{25}f_{57}\bar{A}_6) = f_\alpha\bar{y}_{25}f_\beta f_\alpha[3]qy_6,
\]

which proves it for \( j = 7 \).

### 6.4 Decomposition of \( V \otimes M_\lambda \)

Denote by \( u_j \) the singular vectors \( \tilde{u}_j/\delta_j(\lambda) \) for all \( j \in I \). Then \( u_j = D_j w_j^\lambda \mod V_{j-1} \) with \( D_j \simeq \prod_{i=1}^{j-1} \phi_{ij} \), where \( \phi_{ij} = [\xi_{ij}]_q \) if \( i \neq j \) and \( \phi_{j'j} = [\xi_{j'j}^\lambda]_q \).

**Lemma 6.9.** The submodule \( M_j \) is contained in \( M_i \) with \( i < j \) if and only if \( \phi_{ij}(\lambda) = 0 \).

**Proof.** The eigenvalues \( q^{2(\rho,\nu_i)}x_j \) of the operator \( q^{2(\rho,\nu_i)}Q \) on the submodules \( M_j \) are

\[
q^{2(\lambda, 2\alpha + \beta) + 10}, q^{2(\lambda, \alpha + \beta) + 8}, q^{2(\lambda, \beta) + 2}, q^{-2}, q^{-2(\lambda, \beta) - 2}, q^{-2(\lambda, \alpha + \beta) - 8}, q^{-(\lambda, 2\alpha + \beta) - 10},
\]

counting from the left. If \( \phi_{ij}(\lambda) \) and hence \( D_j(\lambda) \) turns zero, then \( x_j = x_i \) and \( M_j \subset W_{j-1} \), by Proposition 5.5. Suppose that \( x_j \) is distinct from \( x_k \) if \( k < j \) and \( k \neq i \). As follows from (6.15) such weights are dense in the set of solution to \( \phi_{ij}(\lambda) = 0 \). Then \( M_j \) can lie only in \( M_i \) and hence it does for all such weights.

Conversely, let \( \lambda \) be so that \( M_j \subset M_i \). Then \( [\xi_{ij}]_q = 0 \) and hence \( \phi_{ij}(\lambda) = 0 \) if \( i \neq j' \). If \( i = j' \), we can assume that \( M_j \not\subset M_i \) for \( i \neq j' \) since \( \lambda \) is in the closure of such weights, cf. (6.15). Proposition 5.5 then suggests that \( D_j(\lambda) = 0 \), and the only vanishing factor can be \( \phi_{j'j} \). Then it is true for all \( \lambda \).
As an application of the obtained results, we describe direct sum decomposition of the module $V \otimes M_\lambda$. This corresponds to maximal conjugacy classes, with $\mathfrak{t} = \mathfrak{h}$. Fix $t = \text{diag}(t_i) \in T$ and choose $\lambda$ to fulfill the condition $t q^{-2 h_\rho} = q^{2 h_\lambda}$. This fixes the relation between the entries of $t$ and the $Q$-eigenvalues as $t_i = q^{2(\lambda + \rho, \nu_i)} = x_i q^{2h_\lambda - 2\rho_1}$.

**Proposition 6.10.** Suppose that $t \in T^h$ and parameterize it as in (2.1) and (2.2). Then $V \otimes M_\lambda = \oplus_{i=1}^7 M_i$ if and only if $q^{-2} \neq x, y, xy$ for regular $t$ and $q^{-4} \neq x^2$ for borderline $t$.

*Proof.* The sum $\sum_{i=1}^7 M_i$ exhausts all of $V \otimes M_\lambda$ if and only if it is direct, by Proposition 5.5 or, equivalently, if $D^h = \prod_{i=1}^7 D_i$ is not zero. Explicitly,

$$D^h \simeq \prod_{i,j < 4} (x_i - x_j) \prod_{i=1}^3 (x_i - q^{2m}) \simeq \prod_{i,j < 4} \prod_{i=1}^3 (t_i - t_j) \prod_{i=1}^3 (t_i - q^{-2}) \prod_{i=1}^3 (t_i - 1) \simeq \prod_{i=1}^3 (t_i - q^{-2})$$

for $t \in T^h$. This implies the stated conditions on $q$ guaranteeing $D^h \neq 0$ (mind that $q$ is not a root of unity).

Next we essentially assume that $\mathfrak{t} \neq \mathfrak{h}$ and put $\phi_j^\mathfrak{t} = \prod_{i \in \mathfrak{t}} \phi_{ij}$, $j \in I^\mathfrak{t}$.

**Lemma 6.11.** For every $j \in I^\mathfrak{t}$, the vector $\pi^\mathfrak{t}_V(u_j)$ vanishes once $\phi_j^\mathfrak{t}(\lambda) = 0$. If $\mathfrak{t}$ is of type $\mathfrak{k}_s$ or $\mathfrak{k}_l$, then $\pi^\mathfrak{t}_V(u_j)$ is divisible by $\phi_j^\mathfrak{t}$.

*Proof.* By Lemma 6.9, $u_j \in M_i$ once $\phi_j^\mathfrak{t}(\lambda) = 0$. On the other hand, $\pi^\mathfrak{t}_V(u_j) = 0$ if $\lambda \in \Theta^\mathfrak{t}$, by Proposition 6.4. Therefore $\pi^\mathfrak{t}_V(u_j)$ vanishes in $V \otimes M_\lambda^\mathfrak{t}$. If the semisimple part of $\mathfrak{t}$ has rank 1, trigonometric polynomials on $\Theta^\mathfrak{t}$ form a principal ideal domain as $\dim \Theta^\mathfrak{t} = 1$. Therefore $\pi^\mathfrak{t}_V(u_j)$ is divisible by $\phi_j^\mathfrak{t}(\lambda)$.

**7 Module structure of $V \otimes M_\lambda^\mathfrak{t}$**

Throughout this section $\nu \in \mathbb{R}^+$ is a short root and $\mathfrak{t} = \mathfrak{k}_s$ is the reductive subalgebra of maximal rank with the root system $\{ \pm \nu \}$. We aim to prove that the vectors $u_j^\mathfrak{t} = \frac{1}{\phi_j^\mathfrak{t}(\lambda)} \pi^\mathfrak{t}_V(u_j) \in V \otimes M_\lambda^\mathfrak{t}$ are regular functions of $\lambda$ and do not vanish at all weights. Their projections to $\text{gr} \, V^\mathfrak{t}$ are equal to $D_j^\mathfrak{t} w_\lambda^\mathfrak{t}$ with $D_j^\mathfrak{t} \simeq \prod_{i \in \mathfrak{t}} \phi_{ij}$. 
7.1 Regularization of singular vectors in $V \otimes M^k_\lambda$

Assuming $j \in I^k$, denote by $c^k_j$ the coefficient in the expansion $u^k_j \simeq w_j \otimes c^k_j v_\lambda + \ldots$, where the suppressed terms belong to $\sum_{i<j} w_i \otimes M^k_\lambda$. It is equal to $\prod_{i<j} [\eta_{ij}]^{d_j d_k}$ up to a non-vanishing factor.

| $\nu = \alpha$, $(\lambda, \alpha) = 0$, $\theta = (\lambda, \beta)$ | $\phi^k_j$ | $D^k_j$ | $c^k_j$ |
|---|---|---|---|
| $j \in I^k$ | | | |
| 1 | 1 | 1 | 1 |
| 3 | $[\theta+3]_q$ | $[\theta+3]_q$ | $[\theta]_q$ |
| 6 | $[\theta+4]_q[\theta+3]_q$ | $[\theta+9]_q[\theta+5]_q$ | $[\theta+4]_q[\theta+6]_q$ |
| $\nu = \alpha + \beta$, $q^{2(\lambda, \nu)+6} = 1$, $\theta = (\lambda, \alpha)$ | $\phi^k_j$ | $D^k_j$ | $c^k_j$ |
| $j \in I^k$ | | | |
| 1 | 1 | 1 | 1 |
| 2 | 1 | $[\theta+1]_q$ | $[\theta]_q$ |
| 5 | $[\theta+1]_q[\theta]_q$ | $[\theta+3]_q[\theta+2]_q$ | $[\theta+1]_q[2\theta]_q$ |
| $\nu = 2\alpha + \beta$, $q^{2(\lambda, \nu)+8} = 1$, $\theta = (\lambda, \alpha)$ | $\phi^k_j$ | $D^k_j$ | $c^k_j$ |
| $j \in I^k$ | | | |
| 1 | 1 | 1 | 1 |
| 2 | 1 | $[\theta+1]_q$ | $[\theta]_q$ |
| 3 | 1 | $[\theta]_q[2\theta+1]_q$ | $[\theta+1]_q[2\theta+4]_q$ |

Table 1: Type $\mathfrak{k} \simeq \mathfrak{g}$

**Proposition 7.1.** The vectors $u^k_j$ do not vanish at all weights.

*Proof.* We parameterize $\Theta^k \subset \mathfrak{h}^*$ with the complex variable $\theta$ as in Table 1.

The case $\nu = \alpha$ follows from the fact that the null sets of $D^k_j$ and $c^k_j$ do not intersect, cf. Table 1.

In the case of $\nu = \alpha + \beta$, the statement is trivial for $j = 1$ and immediate for $u^k_2 \simeq -q^{-1} w_1 \otimes f_\alpha v_\lambda + w_2 \otimes [\theta]_q v_\lambda$. Let us prove it for $j = 5$. Choose a basis $f_\beta f_\alpha^3 v_\lambda$, $f_\alpha f_\beta f_\alpha^3 v_\lambda$, $f_\alpha^2 f_\beta f_\alpha v_\lambda$ in the weight subspace $M^k_\lambda[\lambda - \beta - 3\alpha]$. The coefficient in $\tilde{f}_{15}/(\delta_5 \phi^k_5)$ corresponding to $f_\beta f_\alpha^3$ is equal to $[\eta_{25}]_q = [\theta + 1]_q$, up to an invertible multiplier. Hence $u^k_5$ does not turn zero unless $[\theta + 1]_q = 0$. However, $D^k_5 \neq 0$ at such $\lambda$. Therefore $u^k_5 \neq 0$ at all weights from $\Theta^k$. 
Finally, consider the case $\nu = 2\alpha + \beta$. The projection $M_\lambda \to M^\xi_\lambda$ is an isomorphism on subspaces of weights $\lambda - \mu$ with $\mu < 2\alpha + \beta$. So are the weights of the generating coefficients $u^\xi_j = \pi^\xi_V(u_j)$. They do not turn zero as $u_j \neq 0$ at all weights. This completes the proof. \hfill \square

7.2 Decomposition of $V \otimes M^\xi_\lambda$

Fix $t \in T^\nu$ and let $\mathfrak{t}$ be the stabilizer of $t$. Choose $\lambda \in \Theta^\xi$ from the equality $q^{2\lambda + 2\nu} = tq^\nu$. We use $x \in \mathbb{C}^*$, $x^2 \neq 1$, to parameterise the spectrum $\{x^{\pm 1}, 1\}$ of $t$ as (2.3).

**Proposition 7.2.** The module $V \otimes M^\xi_\lambda$ splits into the direct sum $\oplus_{j \in \mathfrak{t}} M^\xi_j$ if and only if $q \neq x^{\pm 1}$.

**Proof.** Table 1 gives $D^\xi = \prod_{j \in \mathfrak{t}} D^\xi_j \simeq (q - x)(q - x^{-1})$. The sum $\sum_{j \in \mathfrak{t}} M^\xi_j$ exhausts all of $V \otimes M^\xi_\lambda$ if and only if $D^\xi$ does not vanish. It also implies that the eigenvalues $qx^{\pm 1}, q^2$ of $q^{10}Q$ are pairwise distinct, hence the sum is direct. \hfill \square

8 Module structure of $V \otimes M^\xi_\lambda$

In this section $\nu \in \mathbb{R}^+$ is one of the three long roots, and $\mathfrak{t} \simeq \mathfrak{t}_l \subset \mathfrak{g}$ is the reductive subalgebra of maximal rank with the root system $\{\pm \nu\}$. Now $\Theta^\xi$ the set of weights $\lambda$ that satisfy the condition $q^{2(\lambda + \rho, \nu) - 6} = 1$. We parameterize it with the complex variable $\theta = (\lambda, \alpha)$. There are two pairs $(l, k) \in P(\nu)$, so $\# \mathfrak{I}_l = 2$ and $\# \mathfrak{I}_t = 5$.

8.1 Regularization of singular vectors for quasi-Levi $\mathfrak{t}$

The case of quasi-Levi $\mathfrak{t}_l$ turns out to be simpler, so we consider it first. For $\nu = 3\alpha + 2\beta$, we have $I^\xi = \{1, 2, 3, 4, 5\}$, $q^{2(\lambda, 3\alpha + 2\beta) + 12} = 1$, and $\phi^\xi_j = 1$ for all $j \in I^\xi$. For $\nu = 3\alpha + \beta$, we have $I^\xi = \{1, 2, 3, 4, 6\}$, $q^{2(\lambda, 3\alpha + \beta) + 6} = 1$, and $\phi^\xi_j = 1$ for $j \in I^\xi$ apart from $\phi^\xi_6 = [3\theta]_q$, where $\theta = (\lambda, \alpha)$.

For each $\mu < \nu$, projection $M_\lambda[\lambda - \mu] \to M^\xi_\lambda[\lambda - \mu]$ is an isomorphism. Therefore $\pi^\xi_V(u_j) = u^\xi_j$ does not turn zero unless maybe $\nu = 3\alpha + \beta$, $j = 6$. In the latter case $\mathbb{C}^\xi_6 = [3\theta - 3]_q[2\theta - 1]_q[\theta - 2]_q[2\theta]_q$ and $e^\xi_6 = [2\theta + 1]_q[\theta - 1]_q[2\theta]_q[3\theta + 3]_q$, so $u^\xi_6$ may vanish only at $q^{12\theta} = \pm 1, q^2$. However, $u^\xi_6 \simeq [2\theta + 1]_q\bar{f}_{36}\nu_\lambda$, and one can easily check that

$$\bar{f}_{36} = \frac{[f_\beta, f_\alpha]_q, f_\alpha]_q^2}{[2]_q^2} A_4 A_5 + f_\alpha[f_\alpha, f_\beta]_q A_5 + f^2_\alpha f_\beta (q + q^{-4\theta - 1})[3]_q \neq 0$$

at all weights. Hence $u^\xi_6 \neq 0$ at all weights.
8.2 Regularization of singular vectors for Levi \( \mathfrak{t} \)

| \( j \in I^\mathfrak{t} \) | \( \phi_j^\mathfrak{t} \) | \( \bar{D}_j/(\phi_j^\mathfrak{t} \delta_j) \) | \( \bar{c}_j/(\phi_j^\mathfrak{t} \delta_j) \) | \( D_j^{\mathfrak{t}} \) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | \([\theta+1]_q\) | \([\theta]_q\) | \([\theta+1]_q\) |
| 4 | \([\theta+2]_q\) | \([2\theta+6]_q[\theta+5]_q\) | \([\theta+4]_q[\theta+5]_q[\theta+3]_q\) | \([\theta+5]_q[\theta+5]_q\) |
| 5 | \([\theta+1]_q\) | \([3\theta+6]_q[\theta+5]_q[\theta]_q\) | \([\theta+4]_q[\theta+3]_q[\theta]_q\) | \([\theta+5]_q[\theta+5]_q[\theta+4]_q\) |
| 7 | \([3\theta+6]_q[\theta+1]_q\) | \([2\theta+9]_q[\theta+9]_q[2\theta+4]_q[\theta+4]_q\) | \([2\theta+3]_q[\theta+9]_q[\theta+4]_q[\theta+4]_q[\theta+3]_q[\theta]_q\) | \([\theta+5]_q[\theta+3]_q[\theta+4]_q[\theta+4]_q\) |

Table 2: \( \nu = \beta, (\lambda, \beta) = 0, (\lambda, \alpha) = \theta \)

From this table we conclude that \( u_j/\phi_j^\mathfrak{t} \) may be divisible by following factors:

1. \([\theta + 3]_q \sqcup \overline{A}_4^4\), for \( j = 4 \),
2. \([\theta + 2]_q \sqcup \overline{A}_5^2, [\theta]_q \sqcup \overline{A}_3^3\), for \( j = 5 \),
3. \([2\theta + 4]_q \sqcup \overline{A}_4^7, [\theta + 3]_q \sqcup \overline{A}_5^7\), for \( j = 7 \).

Introduce \( \psi_j^\mathfrak{t} \) for \( j \in I^\mathfrak{t} \) as

\[
\psi_1^\mathfrak{t} = 1, \quad \psi_2^\mathfrak{t} = 1, \quad \psi_4^\mathfrak{t} = [\theta + 3]_q, \quad \psi_5^\mathfrak{t} = [\theta + 2]_q, \quad \psi_7^\mathfrak{t} = [\theta + 2]_q[\theta + 3]_q.
\]

Proposition 8.1. For all \( j \in I^\mathfrak{t} \), the singular vector \( \pi^\mathfrak{t}_V(u_j)/\phi_j^\mathfrak{t} \) is divisible by \( \psi_j^\mathfrak{t} \), and \( u_j^\mathfrak{t} = \pi^\mathfrak{t}_V(u_4)/(\phi_4^\mathfrak{t} \psi_4^\mathfrak{t}) \neq 0 \) at all \( \lambda \).

Proof. There is nothing to prove for \( j = 1, 2 \), so we assume \( j = 4, 5, 7 \). We should only check divisibility, because non-zero property then follows from factorization of \( \bar{D}_j \). One can check that

\[
\bar{f}_{14}v_\lambda \simeq (f_{13}f_\alpha - f_\alpha f_\beta f_\alpha q^3)[\theta + 3]_q[\theta + 2]_q, \quad (8.16)
\]

\[
\bar{f}_{25}v_\lambda \simeq (f_{24}f_\alpha q^2 - q^{-2} q^{-2} q^{-2}) \left( f_\beta f_\alpha \frac{1}{2} (q[3]_q + \frac{1}{q - q^{-1}}) \right) \delta_5[\theta + 2]_q v_\lambda, \quad (8.17)
\]

\[
\bar{f}_{47}v_\lambda \simeq (f_{16}f_\alpha q^3 - f_\alpha f_\beta f_\alpha)[\theta + 3]_q[\theta + 2]_q, \quad (8.18)
\]

in the module \( M_\lambda^\mathfrak{t} \). Now the proof for \( j = 4 \) readily follows from (8.16).

Furthermore, (8.17) implies that \( \bar{f}_{15} \) is not divisible by \([\theta]_q \) and divisible by \([\theta + 2]_q \), by Lemma 6.1. Therefore, \( \bar{f}_{15}/(\delta_5[\theta + 1]_q[\theta + 2]_q) \) is regular and never turns zero. This proves the case \( j = 5 \).
Equality (8.18) implies that $\tilde{f}_{i7}$ are divisible by $[\theta + 2]_q[\theta + 3]_q$ for all $i \leq 4$. Since $[\theta + 2]_q \cong A_2 \cong \phi^t$, we have $[\theta + 2]_q^2 \cong A_2 f_{27} v_\lambda \simeq e_{\beta} f_{27} v_\lambda$ and need to prove $[\theta + 2]_q^2 \cong \tilde{f}_{27}$.

It is easy to check that the operator $M$ is divisible by $[\theta + 2]_q^2$ as required. 

Define $D^t_j$ from the equality $\tilde{\phi}_j^t(u_j^t) = D_j^t w_j^\lambda$ and put $D^t = \prod_{j \in I_t} D^t_j$. It follows that $D^t \simeq D_j/(\tilde{\phi}_j^t \phi_j^t).

8.3 Decomposition of $V \otimes M^t_\lambda$

Fix $t \in T^\nu$ with the stabilizer $\mathfrak{t}$ and determine $\lambda \in \Theta^t$ from the equality $q^{2h_{\lambda} + 2h_{\rho}} = tq^{\lambda}.\nu$. We use $x \in \mathbb{C}^*$, $x^2, x^3 \neq 1$, to parameterise the spectrum $\{x^{\pm 2}, x^\pm 1, 1\}$ of $t$ as in (2.4) and (2.5), for the regular and borderline cases. Then the spectrum of the operator $q^{2h_{\nu}} Q$ on the module $V \otimes M^t_\lambda$ is $\{x^{\pm 2}, q^3 x^{\pm 1}, q^{-2}\}.

Proposition 8.2. The module $V \otimes M^t_\lambda$ splits into direct sum $\oplus_{j \in I_t} M^t_j$ if and only if a) $q^3 \neq x^{\pm 1}, \frac{x^3 - x^2}{x - x^2} \neq 0$, for $\nu = \beta$, b) $q^3 \neq x^{\pm 1}, x^{\pm 3}$ for $\nu \neq \beta$.

Proof. Assuming $t \in T^\nu$, we get $D^t \simeq (x - q^3)^2(x - q)(x - q^3)^3 \frac{x^3 - q^3}{x - q}$, for $\nu = \beta$ and $D^t \simeq (x - q^3)^2(x - q^3)^3(x^3 - q^3)(\bar{x}^3 - q^3)$, for $\nu \neq \beta$. The sum $\sum_{j \in I_t} M^t_j$ exhausts all of $V \otimes M^t_\lambda$ if and only if $D^t \neq 0$.

Now suppose that $D^t \neq 0$. Then the sum $\sum_{j \in I_t} M^t_j$ is direct for $\nu = \beta$ by Proposition 5.5. For $\nu \neq \beta$, that is obvious if the $Q$-eigenvalues are pairwise distinct (which is violated for a finite number of $q$). Still it is true in all cases. We give a sketch of the proof based on character analysis. One can check that $M^t_j$ are quotients of $\tilde{M}^t_j$, where $\tilde{M}^t_j = M_{\lambda_{\nu}} / (M_{\lambda_{\nu} - \ell_j})$ with $\ell_j = 2(\nu, 2) + 1 \in \{1, 2\}$. It is easy to see that $\sum_{j \in I_t} \text{ch} \tilde{M}^t_j = \text{ch} V \times \frac{e^{\lambda_+ / (1 - e^{-\rho_+})}}{\prod_{a+ \in T^+ / (1 - e^{-\rho_+})}} \text{ch} (V \otimes M^t_\lambda)$. Now the map $\oplus_{j \in I_t} \tilde{M}^t_j \rightarrow V \otimes M^t_\lambda$ is injective because it is surjective. This also implies $\tilde{M}^t_j \simeq M^t_j$ for all $j \in I_t$.

9 Decomposition of $V \otimes M^t_\lambda$ for pseudo-Levi $\mathfrak{t}$

Let $\mu$ and $\nu$ denote, respectively, the minimal and maximal weight in $\Pi^t$ and put $m \subset \mathfrak{t}$ to be the reductive subalgebra of maximal rank such that $\Pi_m = \{\mu\}$. For $\lambda \in \Theta_m \subset \Theta_t$, the homomorphism $M_\lambda \rightarrow M^t_\lambda$ factors through the projection $M^m_\lambda \rightarrow M^t_\lambda$. Its restriction
\[ M^m_\lambda[\lambda - \xi] \rightarrow M^m_\lambda[\lambda - \xi] \] is an isomorphism for \( \xi < \nu \). It follows from here that the map \( \sum_{\nu < \mu} w_i \otimes M^m_\lambda[\lambda - \nu] \rightarrow \sum_{\nu < \mu} w_i \otimes M^m_\lambda[\lambda - \nu] \) sends non-vanishing singular vectors \( u^m_j \) with \( j \in I^t \subset I^m \) over to non-zero singular vectors, \( u^t_j \in V \otimes M^t_\lambda \). One can check that \( D^t_j(\lambda) = D^m_j(\lambda) \neq 0 \), for all \( j \in I^t \).

| \( \Pi_t \)         | \( t \)    | \( q^{2(\lambda,\alpha)} \) | \( q^{2(\lambda,\beta)} \) | \( I^t \) |
|---------------------|------------|-----------------------------|-----------------------------|---------|
| \( \{ \beta,3\alpha + \beta \} \) | \( (e^{\pm \frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}) \) | \( e^{\pm \frac{2\pi i}{3}} q^{-2} \) | \( 1 \) | \( \{1,2,4\} \) |
| \( \{ \beta,3\alpha + \beta \} \) | \( (e^{-\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}) \) | \( e^{\frac{2\pi i}{3}} q^{-2} \) | \( 1 \) | \( \{1,2,4\} \) |
| \( \{ \alpha,3\alpha + 2\beta \} \) | \( (-1,-1,1) \) | \( 1 \) | \( -q^{-6} \) | \( \{1,3\} \) |
| \( \{ \alpha + \beta,3\alpha + \beta \} \) | \( (-1,1,-1) \) | \( -1 \) | \( -q^{-6} \) | \( \{1,2\} \) |
| \( \{ \beta,2\alpha + \beta \} \) | \( (1,-1,-1) \) | \( -q^{-4} \) | \( 1 \) | \( \{1,2\} \) |

Table 3: Pseudo-parabolic type

**Proposition 9.1.** For all pseudo-Levi \( \mathfrak{k} \in \mathfrak{g} \), \( V \otimes M^t_\lambda = \oplus_{j \in I^t} M^t_j \).

**Proof.** Since \( D^t \neq 0 \), the sum \( \sum_{j \in I^t} M^t_j \) gives all \( V \otimes M^t_\lambda \). It is direct as the \( Q \)-eigenvalues are distinct. Indeed, for \( \mathfrak{k} = \mathfrak{k}_{L,t} \) we have

\[
q^{2\xi_{12}} = e^{\mp \frac{2\pi i}{3}}, \quad q^{2\xi_{14}} = e^{\pm \frac{2\pi i}{3}} q^8, \quad q^{2\xi_{24}} = e^{\mp \frac{2\pi i}{3}} q^6,
\]

where the upper sign corresponds to the first row in Table 3. For the three \( \mathfrak{k}_{s,t} \)-points we have

\[
q^{2\xi_{12}} = -q^2, \quad q^{2\xi_{12}} = -q^2, \quad q^{2\xi_{12}} = -q^{-2},
\]

respectively, from the top downward. \( \square \)
## A Appendix

### A.1 Formulas for $\eta_{ij}$ and $\xi_{ij}$

Below we present the explicit expressions for $\eta_{ij}$ and $\xi_{ij}$, $i \leq j$, arranging them into matrices.

\[
(\eta_{ij}) = \begin{pmatrix}
0 & h_{\alpha} & h_{\alpha + \beta} + 3 & h_{2\alpha + \beta} + 4 & h_{3\alpha + \beta} + 3 & h_{3\alpha + 2\beta} + 6 & h_{4\alpha + 2\beta} + 6 \\
0 & h_{\beta} & h_{\alpha + \beta} + 3 & h_{2\alpha + \beta} + 4 & h_{2\alpha + 2\beta} + 4 & h_{3\alpha + 2\beta} + 6,
0 & h_{\alpha} & h_{2\alpha} - 2 & h_{2\alpha + \beta} + 4 & h_{3\alpha + \beta} + 3 & h_{2\alpha + \beta} + 4 \\
0 & h_{\alpha} & h_{\alpha + \beta} + 3 & h_{3\alpha + \beta} + 3 & 0 & h_{\alpha} \\
0 & h_{\beta} & h_{\alpha + \beta} + 3 & h_{3\alpha + \beta} + 3 & 0 & h_{\alpha} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
(\xi_{ij}) = \begin{pmatrix}
0 & h_{\alpha} + 1 & h_{\alpha + \beta} + 4 & h_{2\alpha + \beta} + 6 & h_{3\alpha + \beta} + 6 & h_{3\alpha + 2\beta} + 9 & h_{4\alpha + 2\beta} + 10 \\
0 & h_{\beta} + 3 & h_{\alpha + \beta} + 5 & h_{2\alpha + \beta} + 5 & h_{2\alpha + 2\beta} + 8 & h_{3\alpha + 2\beta} + 9, & h_{3\alpha + \beta} + 9 \\
0 & h_{\alpha} + 2 & h_{2\alpha} + 2 & h_{2\alpha + \beta} + 5 & h_{3\alpha + \beta} + 6 & h_{2\alpha + \beta} + 4 & h_{3\alpha + \beta} + 4 \\
0 & h_{\alpha} & h_{\alpha + \beta} + 3 & h_{3\alpha + \beta} + 3 & 0 & h_{\alpha + 1} & 0
\end{pmatrix}
\]

### A.2 Entries of the matrix $F$

Here we present explicit expressions of the entries $f_{ij}$, $i < j$, participating in the reduced Shapovalov inverse form.

\[
f_{12} = f_{\alpha}, \quad f_{23} = [3]_q f_{\beta}, \quad f_{34} = f_{\alpha}, \quad f_{45} = f_{\alpha}, \quad f_{56} = [3]_q f_{\beta}, \quad f_{67} = f_{\alpha},
\]

\[
f_{13} = [f_{\beta}, f_{\alpha}]_{q^3}, \quad f_{24} = [f_{\alpha}, f_{\beta}]_{q^3}, \quad f_{35} = \frac{q^2}{[2]_q} [f_{\alpha}, f_{\alpha}]_{q^3}, \quad f_{46} = [f_{\beta}, f_{\alpha}]_{q^3}, \quad f_{57} = [f_{\alpha}, f_{\beta}]_{q^3},
\]

\[
f_{14} = \frac{q [f_{\alpha}, [f_{\beta}, f_{\alpha}]_{q^3}]}{[2]_q}, \quad f_{25} = \frac{[f_{\alpha}, [f_{\alpha}, f_{\alpha}]_{q^3}]}{[2]_q}, \quad f_{36} = \frac{[f_{\beta}, f_{\alpha}, f_{\alpha}]_{q^3}}{[2]_q}, \quad f_{47} = \frac{[f_{\alpha}, f_{\beta}]_{q^3}}{[2]_q},
\]

\[
f_{15} = \frac{q^2 [f_{\alpha}, [f_{\alpha}, f_{\beta}, f_{\alpha}]_{q^3}]}{[2]_q}, \quad f_{26} = \frac{q^3 [f_{\beta}, [f_{\alpha}, f_{\alpha}, f_{\alpha}]_{q^3}]}{[2]_q}, \quad f_{37} = \frac{q^2 [f_{\alpha}, [f_{\beta}, f_{\alpha}, f_{\alpha}]_{q^3}]}{[2]_q},
\]

\[
f_{16} = \frac{q^2 [f_{\beta}, [f_{\alpha}, [f_{\beta}, f_{\alpha}]_{q^3}]]_{q^3}}{[2]_q}, \quad f_{27} = \frac{q^2 [f_{\alpha}, [f_{\beta}, f_{\alpha}, f_{\alpha}]_{q^3}]}{[2]_q}, \quad f_{38} = \frac{q^2 [f_{\alpha}, f_{\beta}]_{q^3}}{[2]_q},
\]

\[
f_{17} = \frac{q^2 [f_{\beta}, [f_{\alpha}, [f_{\beta}, f_{\alpha}, f_{\alpha}]_{q^3}]]_{q^3}}{[2]_q}.
\]
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