Massive higher spin fields
in the frame-like multispinor formalism

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Abstract

In this paper, a gauge invariant description of massive higher spin bosonic and
fermionic particles in frame-like Lagrangian and unfolded formalism in (A)dS\textsubscript{4} is built. A complete set of gauge invariant object is also constructed and the Lagrangian is rewritten in terms of these objects. The unitarity of the theories is studied alongside with the partially massless limits. The calculations are carried out in the multispinor formalism, which simplifies them and is particularly convenient for the supersymmetry studies.

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# 1 Introduction

There are two well-known formalisms for the description of the massless higher spin fields: the metric one [11,14], which can be considered as a higher spin generalization of the metric formulation of gravity, and a so-called frame-like one [5,7] generalizing a frame formulation for gravity. Both formalisms are drastically based on the gauge invariance which guarantees the correct number of the physical degrees of freedom and almost completely fixes the possible forms of consistent interactions.

The metric formulation of the massive bosonic and fermionic fields was proposed long ago in [8,9]. It does not possess any gauge invariance; instead, it provides us with the set of the forms of consistent interactions. One of the possible routes for the investigation of the consistent interactions for the massive higher spin fields is to use their constraints which follow from the Lagrangian equations. It does not possess any gauge invariance; instead, it provides us with the set of the forms of consistent interactions.

The tensor formulation used in [13] is universal in a sense that it works in any space-time dimensions $d \geq 4$, but it appears technically quite involved. It becomes even more complicated in the case of the massive fermions and this is, at least, one of the reasons why such formalism has not been developed so far. In this work we restrict ourselves with the four-dimensional space-time. This allows us to use a multispinor formalism which greatly simplifies calculations especially when one has to deal with the mixed symmetry (spin-)tensors. So we managed not only reproduce the results of [13] (with a number of generalizations) but also developed an analogous formulation for the massive fermions. Note, that such formalism, where bosons and fermions appear on equal footing, is very well suited for the investigation of the supersymmetric models. Indeed, it has been already used in our recent investigations of different $N = 1$ supermultiplets in four dimensions [17,19].

In this paper, we develop two different but tightly connected formalisms - namely, the frame-like Lagrangian one and the unfolded one. Let us illustrate them on the simple case of the massless spin-$s$ field propagating over the $(A)dS_4$ background. In the frame-like multispinor formalism we use here (see Appendix A for notations and conventions), the massless spin-$s$ boson is described by the physical one-form $\Phi^{(s-1)}(s-1)$ and auxiliary one-forms $\Omega^{(s)}(s-2) + h.c.$ The free Lagrangian (which is a four-form in our formalism) looks as follows:

$$-i(-1)^s \mathcal{L} = s\Omega^{(s-2)}(s-1)E_{\gamma\delta}\Omega^{(s-2)}(s-1)\hat{\alpha}(s-1) - (s-2)\Omega^{(s-3)}(s-3)\hat{\alpha}(s-1) - (s-2)\Omega^{(s-3)}(s-3)\hat{\alpha}(s-1) - (s-2)\Omega^{(s-3)}(s-3)\hat{\alpha}(s-1)$$

$$+2\lambda^2\Phi^{(s-1)}(s-1)E^{(s-1)}(s-1)\hat{\beta}(s-1) - h.c.$$

This Lagrangian is invariant under the following gauge transformations:

$$\delta \Omega^{(s-2)}(s-1) = D\eta^{(s-2)}(s-1) + (s-2)e^{(s-2)}(s-1)\hat{\alpha}(s-1)$$

$$\delta \Phi^{(s-1)}(s-1) = D\xi^{(s-1)}(s-1) + (s-1)e^{(s-1)}(s-1)\hat{\alpha}(s-1).$$

(2)
It is easy to construct a gauge invariant two-form (an analogue of the torsion in gravity):

\[ R^{\alpha(s-1)\dot{\alpha}(s-1)} = D\Phi^{\alpha(s-1)\dot{\alpha}(s-1)} + (s - 1)e^\alpha_\dot{\alpha}\Omega^{\alpha(s-2)\dot{\alpha}(s)} + (s - 1)e_\alpha^{\dot{\alpha}}\Omega^{\alpha(s)\dot{\alpha}(s-2)}. \] (3)

The straightforward attempt to generalize the curvature by substituting the frame and the spin-connection with the physical and the auxiliary fields respectively, however, fails: the result is not invariant under the part of the transformation \( \zeta^{\alpha(s+1)\dot{\alpha}(s-3)} \), which has no analogue in the spin-2 case. To restore the full invariance, one has to introduce a so-called extra field \( \Sigma^{\alpha(s+1)\dot{\alpha}(s-3)} \) (with its complex conjugate), which does not enter the free Lagrangian (although it is required to build the gauge invariant interactions) and plays the role of the gauge field for this extra gauge transformations:

\[ \delta \Sigma^{\alpha(s+1)\dot{\alpha}(s-3)} = D\zeta^{\alpha(s+1)\dot{\alpha}(s-3)} + (s - 3)e_\beta^{\dot{\alpha}}\zeta^{\alpha(s+1)\beta\dot{\alpha}(s-4)} + (s + 1)\lambda^2 e^\alpha_\beta\eta^{\alpha(s)\beta\dot{\alpha}(s-3)}. \] (4)

With the use of this extra field, one builds the generalization of the Riemann tensor as:

\[ R^{\alpha(s)\dot{\alpha}(s-2)} = D\Omega^{\alpha(s)\dot{\alpha}(s-2)} + s\lambda^2 e^\alpha_\beta\Phi^{\alpha(s-1)\dot{\alpha}(s-2)} + (s - 2)e_\alpha^{\dot{\alpha}}\Sigma^{\alpha(s+1)\dot{\alpha}(s-3)}. \] (5)

It is possible to construct a gauge invariant object which contains the derivative of the extra field \( \Sigma^{\alpha(s+1)\dot{\alpha}(s-3)} \); however, the extra field \( \Sigma^{\alpha(s+2)\dot{\alpha}(s-4)} \) is needed for that. In the end, one arrives at the complete set of extra fields:

\[ \delta \Sigma^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} = D\zeta^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} + (s - 1 - m)e_\beta^{\dot{\alpha}}\zeta^{\alpha(s-1+m)\beta\dot{\alpha}(s-2-m)} + (s - 1 + m)\lambda^2 e^\alpha_\beta\zeta^{\alpha(s-2+m)\beta\dot{\alpha}(s-1-m)}. \] (6)

used to build the complete set of the gauge invariant curvatures:

\[ R^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} = D\Sigma^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} + (s - 1 + m)\lambda^2 e^\alpha_\beta\Sigma^{\alpha(s-2+m)\dot{\alpha}(s-1-m)} + (s - 1 - m)e_\alpha^{\dot{\alpha}}\Sigma^{\alpha(s+1)\dot{\alpha}(s-2-m)}. \] (7)

The index \( |m| \leq s - 1 \); in case of \( |m| = s - 1 \) the terms with zero coefficients \( (s - 1 \pm m) \) are omitted. This set of curvatures is closed, i.e. for each field \( \Sigma^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} \) there is a unique curvature \( R^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} = DW^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} + \ldots \). Note the differential relation

\[ DR^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} = -(s - 1 + m)\lambda^2 e^\alpha_\beta R^{\alpha(s-2+m)\dot{\alpha}(s-1-m)} + (s - 1 - m)e_\alpha^{\dot{\alpha}}R^{\alpha(s+1)\dot{\alpha}(s-2-m)}. \] (8)

As in the case of gravity, the Lagrangian can be rewritten in the manifestly gauge invariant form:

\[ i(-1)^{s+1}L = \sum_{m=1}^{s-1} \frac{(s - 2)!(s - 1)!}{(s - 1 - m)!(s + m - 1)!\lambda^{2m}} R^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} R^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} - h.c. \] (9)
The coefficients in the (9) are determined by the extra field decoupling condition
\[ \frac{\delta \mathcal{L}}{\delta \Sigma^\alpha(s-1+m)\hat{\alpha}(s-1-m)} = 0 \] (10)
up to the normalization factor.

We now turn on to describe of the unfolded formalism. It describes the particle via an infinite chain of first order equations closed on-shell. No Lagrangian is known that would imply the whole chain of the equations. It is remarkable that the unfolded formalism is the only one in which a complete non-linear theory has been constructed \[20–22\]. In \[13\] the unfolded formulation was constructed for the massive bosons; one of the aims of our paper is to build the unfolded description for the massive fermions.

We derive the unfolded equations now. We start with an analogue of the zero torsion condition:
\[ T^{\alpha(1)}\hat{\alpha}(s-1) = D\Phi^\alpha(s-1)\hat{\alpha}(s-1) + (s-1)e_\alpha \hat{\alpha} \Omega^\alpha(\hat{\alpha}(s-2) + h.c. = 0, \]
which holds on-shell and allows us to express \( \Omega^\alpha(\hat{\alpha}(s-2) \) in terms of \( D\Phi^\alpha(s-1)\hat{\alpha}(s-1) \) up to the gauge transformations. Its derivative can be expressed via \( R^\alpha(\hat{\alpha}(s-2) \) (with its complex conjugate) using the identity (5):
\[ DT^{\alpha(s-1)\hat{\alpha}(s-1)} = -(s-1)e_\alpha \hat{\alpha} R^\alpha(\hat{\alpha}(s-2) + h.c. = 0. \]

One can see that the condition \( e_\alpha \hat{\alpha} R^\alpha(\hat{\alpha}(s-2) + h.c. = 0 \) ensures there is one-to-one correspondence between the components of \( R^\alpha(\hat{\alpha}(s-2) \) and \( \Sigma^\alpha(s+1)\hat{\alpha}(s-3) \) up to the gauge invariance. So we can set
\[ 0 = R^\alpha(\hat{\alpha}(s-2) = s\lambda^2 e_\alpha \Phi^\alpha(s-1)\hat{\alpha}(s-1) + D\Omega^\alpha(\hat{\alpha}(s-2) + (s-2)e_\alpha \hat{\alpha} \Sigma^\alpha(s+1)\hat{\alpha}(s-3), \]
and solve it for the \( \Sigma^\alpha(s+1)\hat{\alpha}(s-3) \). By repeating the steps above, we arrive at the number of zero curvature conditions:
\[ R^\alpha(s+m-1)\hat{\alpha}(s-m-1) = 0, \quad |m| \leq s - 2 \] (14)

The case of the curvature \( R^\alpha(2s-2) \) is quite different. In the previous steps, we always had an extra field which could be chosen to set the curvature to zero. There is no extra field left when we obtain \( e_\alpha \hat{\alpha} R^\alpha(2s-2) = 0 \). Thus to write the most general consistent equation we have to introduce a first gauge invariant zero-form:
\[ R^\alpha(2s-2) = E_\alpha(2) W^\alpha(2s) \] (15)
It is the closest analogue of the Weyl tensor in the gravity; it parametrizes all the components which do not vanish on-shell. The derivative of the curvature \( R^\alpha(2s-2) \) can be expressed via other curvatures and thus vanish. This gives the condition
\[ E_\alpha(2) DW^\alpha(2s) = 0 \] (16)
for the zero-form \( W^\alpha(2s) \). Similarly to the previous steps, this means that its derivative can be uniquely expressed via the components of another field. In this case, the field is \( W^\alpha(2s+1)\hat{\alpha} \):
\[ 0 = DW^\alpha(2s) + e_\alpha \hat{\alpha} W^\alpha(2s+1)\hat{\alpha}. \] (17)
In turn, the equation for the $W^{(2s+1)\dot{a}}$ requires introduction of $W^{(2s+2)\dot{a}(2)}$ and so on. We obtain an infinite chain of zero-forms $W^{(2s+m)\dot{a}(m)} + h.c., m \geq 0$ (see Figure 1):

$$0 = DW^{(2s+m)\dot{a}(m)} + e_{\alpha\dot{a}} W^{\alpha(2s+m+1)\dot{a}(m+1)} + (2s + m)\lambda^2 e^{\alpha\dot{a}} W^{\alpha(2s+m-1)\dot{a}(m-1)}$$

(18)

One can see that each equality in the unfolded equations chain gives the parametrization of the derivatives of the previous field not vanishing on-shell by the next field (up to the $\lambda^2$ terms induced by the space-time curvature). Hence, the $i$-th pair of the one-forms (taking physical field as zeroth) represents all the $i$-th derivatives of the physical field which do not vanish on-shell. The same holds for the zero-forms - the $i$-th pair of zero-forms represents the $(s+i)$-th derivative. This is an explanation why the extra fields do not enter the free Lagrangian - it has second order in the derivatives, if one expresses all the fields via the physical one. Moreover, the fermionic Lagrangian has the first order and hence contains the physical field only.

In the next sections, we build the frame-like and unfolded formulation for massive higher spin particles. Our main interest here is the general massive case, however we also investigate all possible partially massless and/or infinite spin limits. The Sections 2 and 3 are devoted to bosons and fermions respectively. Each section is divided into five parts. In the first part, we construct the gauge invariant Lagrangian for the particle and study its unitarity. In the second part, we build the complete set of the gauge invariant curvatures, introducing all necessary extra fields. In the third part, we use the set of the curvatures to express the Lagrangian in the explicitly gauge invariant form. In the fourth part, we build the unfolded equations. And in the final fifth part, we discuss the applications of our work to the formalism developed in [23]. In Appendix A the notations, conventions and the facts about the multispinor formalism are presented. In Appendix B we list the general facts about the gauge invariant curvatures for the massive particles.
2 Bosonic case

2.1 The Lagrangian

To construct the gauge invariant description for the massive spin-$s$ boson, one has to introduce a complete set of components for the massless fields with spins from zero to $s$. As it was already mentioned in the introduction, one needs two fields to describe the massless boson of spin $s > 1$ — namely, the physical one-form $\Phi^{(k)}(k)$ and the auxiliary one-form $\Omega^{(k+1)}(k)$ with its complex conjugate. The cases with spins $s = 1$ and $s = 0$ are special: one needs one-form $A$ and zero-form $B^{(2)} + h.c.$ for spin-1 and zero-forms $\phi$, $\pi^{\alpha\bar{\alpha}}$ for spin-0. The complete Lagrangian is built as a sum of kinetic terms for all fields with all possible cross and mass terms added. The most general ansatz (up to the normalization choice) is:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

$$-i\mathcal{L}_0 = \sum_{k=1}^{s-1} (-1)^{k+1} [\Omega^{(k+1)}(k) E_{\gamma\beta} \Omega^{(k)}(k) \dot{\beta}(k) - (k-1) \Omega^{(k)}(k) E_{\gamma\beta} \Omega^{(k-1)}(k) \dot{\beta}(k-1)] + 2 \Omega^{(k+1)}(k) \dot{\beta}(k) h.c.$$

$$-i\mathcal{L}_1 = \sum_{k=2}^{s-1} (-1)^{k+1} \frac{2(k+1) \mu_k \Omega^{(k+1)}(k) E_{\alpha(2)}(k) \Phi^{(k)}(k) - h.c.}{(k-1)}$$

$$-i\mathcal{L}_2 = \sum_{k=1}^{s-1} (-1)^{k+1} [2 \beta_{k+1}(k+1) \Phi^{(k)}(k) E_{\alpha\beta} \Phi^{(k)}(k-1) \dot{\alpha}(k-1) - h.c.]$$

$$-24 \mu_1 \beta_2 E_{\alpha\bar{\alpha}} \Phi^{\alpha\bar{\alpha}} \phi + 24 \mu_1 \beta_2 E_{\phi^2}$$

Here all the terms are arranged into three sums $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ by the dimensionality of the coefficients. To simplify the calculations, a non-canonical normalization of the fields $B^{(2)}$, $\pi^{\alpha\bar{\alpha}}$ and $\phi$ is chosen. The same Lagrangian can also be used to describe the infinite spin particle by taking $s \rightarrow \infty$.

In order to have the right amount of the physical degrees of freedom, the Lagrangian has to possess all the symmetries of the initial massless Lagrangians. In this, the gauge transformations also have to be modified with cross and mass-like terms. The ansatz for the transformations consistent with the Lagrangian has the form:

$$\delta \Omega^{(k+1)}(k) = D\eta^{(k+1)}(k-1) + (k-1) \epsilon_{\alpha}^{\dot{\alpha}} \dot{\xi}^{(k+2)}(k-2) + (k+1) \beta_{k+1}^{\dot{\alpha}} \epsilon_{\alpha}^{\dot{\alpha}} \dot{\xi}^{(k)}(k)$$

$$+ \frac{(k+1)}{(k+2)} \mu_k \epsilon^{\alpha\dot{\alpha}} \eta^{(k)}(k-2) + \frac{(k+2)}{k} \mu_{k+1} \epsilon_{\alpha\dot{\alpha}} \eta^{(k+2)}(k),$$
\[ \delta \Phi^{\alpha(k)} = D \xi^{\alpha(k)} + ke_\alpha \eta^{\alpha(k-1)\dot{\alpha}(k+1)} + ke_\alpha \eta^{\alpha(k+1)\dot{\alpha}(k)} \]
\[ + \frac{\mu_k}{k(k-1)} k^2 e^{\alpha\dot{\alpha}} \xi^{\alpha(k-1)\dot{\alpha}(k-1)} + \mu_{k+1} e_{\alpha\dot{\alpha}} \xi^{\alpha(k+1)\dot{\alpha}(k+1)}, \]
\[ \delta \Omega^{\alpha(2)} = D \eta^{\alpha(2)} + 2\beta_2 e^{\alpha\dot{\alpha}} \xi^{\alpha\dot{\alpha}a} + 3\mu_2 e_{\alpha\dot{\alpha}} \eta^{\alpha(3)\dot{\alpha}}, \]
\[ \delta \Phi^{\alpha\dot{\alpha}} = D \xi^{\alpha\dot{\alpha}} + e^{\alpha\dot{\alpha}} \xi^{\alpha\dot{\alpha}a} + e_\alpha \dot{\alpha} \eta^{\alpha(2)\dot{\alpha}} + \frac{\mu_1}{2} e^{\alpha\dot{\alpha}} \xi + \mu_2 e_{\alpha\dot{\alpha}} \xi^{\alpha(2)\dot{\alpha}(2)}, \]
\[ \delta B^{\alpha(2)} = \eta^{\alpha(2)}, \quad \delta A = D \xi + \mu_1 e_{\alpha\dot{\alpha}} \xi^{\alpha\dot{\alpha}}, \]
\[ \delta \pi^{\alpha\dot{\alpha}} = \xi^{\alpha\dot{\alpha}}, \quad \delta \phi = \xi. \]

Here, in case of \( k = s - 1 \), the terms which contain the fields with more than \( 2s - 2 \) indices should be omitted. The gauge invariance condition leads to the following recurrent relations for \( \mu_k, \beta_k \):

\[
\frac{(k+2)}{k} \mu_{k+1}^2 = \frac{(k+1)}{(k-1)} \mu_k^2 - 2\beta_{k+1}(k+1) + 2\lambda^2(k+1),
\]
\[
3\mu_2^2 = \mu_1^2 - 4\beta_2 + 4\lambda^2,
\]
\[
(k-1)k\beta_k = (k+2)(k+1)\beta_{k+1}. \tag{24}
\]

One can see that the general solution of these relations depends on two free parameters. In case of the finite spin \( s \), the condition \( \mu_s = 0 \) reduces the number of free parameters to one. We use the ”mass” parameter

\[
M^2 = \frac{s(s-1)\mu_{s-1}^2}{2(s-2)}
\]
as the second one. We put the word ”mass” in quote since the translation generators of the (A)dS space do not commute, and the square of momentum \( P^2 \) is not thus a Casimir operator anymore. Hence, there is no straightforward generalization of the notion of the mass to the constant curvature space. However, in case of completely symmetric fields (which is the only type of fields we need in four dimensional case) we may propose a consistent definition for the massless limit. Namely, this is the limit where the main gauge field (i.e. that described by \( \Phi^{\alpha(s-1)\dot{\alpha}(s-1)}, \Omega^{\alpha(s)\dot{\alpha}(s-2)}, \Omega^{\alpha(s-2)\dot{\alpha}(s)} \)) decouples from all the Stueckelberg ones. This corresponds to the limit \( \mu_{s-1} \to 0 \) when the Lagrangian splits in two independent parts, one containing \( \Phi^{\alpha(s-1)\dot{\alpha}(s-1)}, \Omega^{\alpha(s)\dot{\alpha}(s-2)}, \Omega^{\alpha(s-2)\dot{\alpha}(s)} \), while the other one — the rest of the fields. As for the concrete normalization, we choose it so that this parameter coincides with the usual mass in the flat limit \( \lambda \to 0 \). The coefficients \( \mu_k, \beta_k \) parametrized by \( s \) and \( M \) have the following form:

\[
\mu_k^2 = \frac{(s-k)(s+k+1)(k-1)}{k(k+1)^2} [M^2 + (s+k)(s-k-1)\lambda^2],
\]
\[
\mu_1^2 = \frac{(s-1)(s+2)}{2} [M^2 + (s+1)(s-2)\lambda^2], \tag{25}
\]
\[
\beta_k = \frac{s(s+1)}{(k-1)k^2(k+1)} [M^2 + s(s-1)\lambda^2].
\]
In case of the infinite spin, the notions of spin is inapplicable; we choose the lowest coefficients $\beta_2$ and $\mu_1$ as the two free parameters:

$$\mu_k^2 = \left(\frac{k-1}{k+1}\right)\left[\mu_1^2 - \frac{6(k-1)(k+2)}{k(k+1)}\beta_2 + (k-1)(k+2)\lambda^2\right],$$
$$\beta_k = \frac{12\beta_2}{(k-1)k(k+1)}.$$ (26)

Now let us discuss the hermiticity of the Lagrangian. In case of the finite spin it implies that all $\mu_k^2$ are non-negative. For flat and $AdS$ spaces that leads to the condition $M^2 \geq 0$. In case of equality $M^2 = 0$ in $AdS$, the highest field decouples while in flat space the Lagrangian splits into $s + 1$ massless ones. In $dS$ space ($\lambda^2 < 0$) the hermiticity condition leads to the appearance of the so-called unitary forbidden region $M^2 < -s(s-1)\lambda^2$. At the boundary of this region the spin-0 component decouples and we obtain the first partially massless limit. Inside the forbidden region we obtain a number of other partially massless ones. Indeed, the Lagrangian splits into two independent parts at the values of mass corresponding to the condition $\mu_{k-1} = 0$:

$$M^2 = -(s+k-1)(s-k)\lambda^2.$$ 

In this case, one of the two parts contains the components with spins $k, s$, while the other contains components $0, s$, only one of the two parts is unitary. In $dS$, it is the part with components $k, s$ which is unitary; the lower fields entering the non-unitary part decouple. The resulting theory is hence unitary, even though the value of the mass lays in the unitary forbidden region.

In case of infinite spin, it is convenient to introduce a variable $y_k = k^2 + k - 2$. Then, the sign of $\mu_k^2$ is determined by a square trinomial on $y_k$:

$$\mu_k^2 \propto \mu_1^2(y_k + 2) - 6y_k\beta_2 + y_k(y_k + 2)\lambda^2.$$ 

It immediately follows that in $dS$ space no infinite spin particle can exist, since all the $\mu_k^2$, starting from sufficiently large $k$ are negative. Consider the flat case first. There exists a whole set of unitary solutions with the complete spectrum of helicities $0, \pm \infty$. The unitarity condition reads:

$$\mu_1^2 \geq 0, \quad \mu_1^2 > 6\beta_2.$$ (27)

Most of these solutions are tachyonic, while the solution $\mu_1^2 = 6\beta_2$ corresponds to the massless infinite spin field. Note, that it is this solution that can be obtained from the massive finite spin one if one takes the limit $M \to 0$, $s \to \infty$ so that $Ms = const$.

Besides, for $\mu_1^2 < 0$ the situation analogous to the partially massless limit in $dS$ is possible. Indeed, if we set

$$\mu_{s-1} = 0 \Rightarrow \mu_1^2(y_s + 2) = 6\beta_s y_s,$$

then the non-unitary part decouples so that the components $s, +\infty$ form an unitary theory.

The $AdS$ case is more complicated. Here we also have a whole set of solutions with the complete spectrum of helicities $0, \pm \infty$. The unitarity region is an infinite area with
piece-wise linear boundary (in coordinates \( \mu_1^2, \beta_2 \)):

\[
12\beta_2 \in [(k-1)k^2(k+1)\lambda^2; k(k+1)^2(k+2)\lambda^2], \quad k \in \mathbb{N},
\]

\[
\mu_1^2 > (k^2 + k - 2)[\frac{6\beta_2}{k^2 + k} - \lambda^2].
\]

(28)

Each segment of the boundary corresponds to the condition \( \mu_1^2 > 0 \). Once again, a set of partially massless limits is also possible. The solution for partially massless limit corresponding to the components \( s, +\infty \) can be written in a form very similar to that of the massive finite spin case:

\[
\begin{align*}
\beta_k &= \frac{s(s + 1)\hat{M}^2}{(k - 1)k^2(k + 1)}, \quad \hat{M}^2 < s(s + 1)\lambda^2, \\
\mu_k^2 &= \frac{(k - s)(k + s + 1)}{k(k + 1)}[k(k + 1)\lambda^2 - \hat{M}^2].
\end{align*}
\]

(29)

### 2.2 Gauge invariant curvatures

As it was already mentioned, one of the advantages of the frame-like formalism is the possibility to construct a complete set of gauge invariant objects, or curvatures. However, in contrast to the case of massless spin-2 particle, for the massless spin \( s > 2 \) particles one has to introduce the so-called extra fields, which do not enter the free Lagrangian. They, however, do transform under the gauge transformations and enter the curvatures as well as the interaction Lagrangian. In the massless case the complete set of fields is \( \Omega^{\alpha(s-1-m)\hat{a}(s-1-m)}, |m| \leq s - 1 \), where the field with \( m = 0 \) is the physical one, while the fields with \( m = \pm 1 \) are the auxiliary ones. Thus in the massive case we need the following set of one-forms \( \Omega^{\alpha(k+m)\hat{a}(k-m)}, |m| \leq k \leq s - 1 \). However, our Lagrangian contains zero-forms as well, and it appears that to construct the complete set of the gauge invariant objects one has to introduce the following set of zero-forms \( W^{\alpha(k+m)\hat{a}(k-m)}, m \leq k \leq s - 1 \), so that we have a one to one correspondence between the one-forms and the zero-forms (see Figure 2a). From here on in the section, the notations are unified: \( \Phi^{\alpha(k)\hat{a}(k)} \equiv \Omega^{\alpha(k)\hat{a}(k)}, B^{\alpha(2)} \equiv W^{\alpha(2)}, \pi^{\alpha\hat{a}} \equiv W^{\alpha\hat{a}}, \phi \equiv W \). The gauge transformation law of the physical and auxiliary fields has already been obtained. Then the most general ansatz for the extra fields gauge transformation, up to normalization choice, is (see Appendix B about the coefficients \( \alpha_{ij}^k \)):

\[
\begin{align*}
\delta \Omega^{\alpha(k+m)\hat{a}(k-m)} &= D\eta^{\alpha(k+m)\hat{a}(k-m)} + (k + m)(k - m)\alpha^{-\hat{m}}_{\hat{m},m}e^{\hat{m}\hat{a}}\eta^{\alpha(k+m-1)\hat{a}(k-m-1)}
+ \alpha_{\hat{a}\hat{m}}^{++}e^{\hat{a}\hat{a}}\eta^{\alpha(k+m+1)\hat{a}(k-m+1)} + (k + m)\alpha_{\hat{a}\hat{m}}^{-\hat{a}}e^{\hat{a}\hat{a}}\eta^{\alpha(k+m-1)\hat{a}(k-m-1)},
\delta \Omega^{2\alpha(k)} &= D\eta^{2\alpha(k)} + \alpha_{\hat{a}\hat{a}}^{++}e^{\hat{a}\hat{a}}\eta^{2\alpha(k+1)\hat{a}} + 2k\alpha_{\hat{a}\hat{a}}^{-\hat{a}}e^{\hat{a}\hat{a}}\eta^{2\alpha(k-1)\hat{a}},
\delta W^{\alpha(k+m)\hat{a}(k-m)} &= \eta^{\alpha(k+m)\hat{a}(k-m)},
\end{align*}
\]

(30)

where

\[
\begin{align*}
\alpha_{\hat{a}\hat{a}}^{++} &= \frac{(k + 2)}{k}\mu_{k+1}, \quad \alpha_{\hat{a}\hat{a}}^{--} = \beta_{k+1}, \quad \alpha_{\hat{a}\hat{a}}^{-\hat{a}} = \frac{\mu_k}{(k - 1)(k + 2)}
\end{align*}
\]

\[
\begin{align*}
\alpha_{\hat{a}\hat{a}}^{++} &= \mu_{k+1}, \quad \alpha_{\hat{a}\hat{a}}^{-\hat{a}} = 1, \quad \alpha_{\hat{a}\hat{a}}^{--} = \frac{\mu_k}{k(k - 1)}.
\end{align*}
\]
The expressions for the lower spin curvatures have different coefficients and thus have to be
squares of Skvortsov-Vasiliev formalism.

The gauge transformations completely define the form of the curvatures. For $k \geq 2$ those curvatures are:

$$R^{\alpha(k+m)\dot{\alpha}(k-m)} = D\Omega^{\alpha(k+m)\dot{\alpha}(k-m)} + (k + m)(k - m)\alpha_{k,m}^{\alpha - \dot{\alpha}} + e^{\alpha \dot{\alpha}} \Omega^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} + (k + m)\alpha_{k,m}^{\alpha - \dot{\alpha}} \Omega^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} + (k - m)e^{\alpha \dot{\alpha}} \Omega^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} \Omega^{\alpha(k-m+1)\dot{\alpha}(k-m-1)},$$

$$R^{\alpha(2k)} = D\Omega^{\alpha(2k)} + \alpha_{k,m}^{\alpha + \dot{\alpha}} e^{\alpha \dot{\alpha}} \Omega^{\alpha(2k+1)\dot{\alpha}} + 2k\alpha_{k,k}^{\alpha} e^{\alpha \dot{\alpha}} \Omega^{\alpha(2k-1)\dot{\alpha}}$$

$$+ 4k(2k-1)\alpha_{k,k}^{\alpha} \alpha_{k,k-1}^{\alpha} W^{\alpha(2k-2)} - 2\alpha_{k,k}^{\alpha} E^{\alpha(2k)} W^{\alpha(2k+2)}$$

$$+ \frac{-\alpha_{k+1}^{\alpha} E^{\alpha(2k-1)\dot{\alpha}}}{k+1},$$

$$C^{\alpha(k+m)\dot{\alpha}(k-m)} = DW^{\alpha(k+m)\dot{\alpha}(k-m)} - \Omega^{\alpha(k+m)\dot{\alpha}(k-m)} + (k + m)(k - m)\alpha_{k,m}^{\alpha - \dot{\alpha}} e^{\alpha \dot{\alpha}} W^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} + (k - m)e^{\alpha \dot{\alpha}} W^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} + (k + m)\alpha_{k,m}^{\alpha - \dot{\alpha}} e^{\alpha \dot{\alpha}} W^{\alpha(k+m+1)\dot{\alpha}(k-m+1)}.$$
\[
R = D\Omega + \mu_1 e_{a\dot{\alpha}} \Omega^a\dot{\alpha} - 2\mu_1 E_{a(2)} W^{a(2)} - 2\mu_1 E_{\dot{a}(2)} W^{\dot{a}(2)},
\]
\[
C = DW - \Omega + \mu_1 e_{a\dot{\alpha}} W^{a\dot{\alpha}}.
\]

It is convenient to introduce auxiliary coefficients \(\alpha^{ij}_{k,m}\) which have one index only. The coefficients \(\alpha^{ij}_{k,m}\) can be expressed in terms of these auxiliary ones as follows:

\[
\begin{align*}
\alpha^{++}_{k,m} &= \frac{\alpha^{++}_k}{(k-m+1)(k-m+2)}, \\
\alpha^{--}_{k,m} &= \frac{\alpha^{--}_k}{(k+m)(k+m+1)}, \\
\alpha^{-+}_{k,m} &= \frac{\alpha^{-+}_m}{(k-m+1)(k-m+2)(k+m)(k+m+1)}.
\end{align*}
\]

These expressions are applicable both for the finite and the infinite spin cases. To express the auxiliary coefficients, we use the same parameter choice as in the previous subsection. Namely, spin \(s\) and "mass" parameter \(M\) in case of the finite spin:

\[
\begin{align*}
\alpha^{++}_{k-1,2} &= k(k-1)(s-k)(s+k+1)[M^2 + (s+k)(s-k-1)\lambda^2], \\
\alpha^{--}_k &= \frac{(s-k)(s+k+1)}{k(k-1)}[M^2 + (s+k)(s-k-1)\lambda^2], \\
\alpha^{-+}_m &= (s-m+1)(s+m)[M^2 + (s-m)(s+m-1)\lambda^2],
\end{align*}
\]

and the lowest coefficients \(\mu_1, \beta_2\) in case of the infinite spin:

\[
\begin{align*}
\alpha^{++}_{k-1,2} &= k(k-1)[\mu_1^2 k(k+1) - 6(k-1)(k+2)\beta_2 + (k-1)k(k+1)(k+2)\lambda^2], \\
\alpha^{--}_k &= \frac{1}{k(k-1)}[\mu_1^2 k(k+1) - 6(k-1)(k+2)\beta_2 + (k-1)k(k+1)(k+2)\lambda^2], \\
\alpha^{-+}_m &= [\mu_1^2 k(k-1)k(k-2)(k+1)\beta_2 + (k-2)(k-1)k(k+1)\lambda^2].
\end{align*}
\]

Note the useful relation \(\alpha^{--}_{m-1,2} \alpha^{++}_{m-2} = \alpha^{--}_m\). The relation is a general rule, i.e. it holds not only for the bosonic \(\alpha^{ij}_k\), but for their fermionic analogues as well.

Note that the hermiticity of the curvatures \(R^{a(2)\dot{a}(m)\dot{a}(k-m)}\) requires the coefficients \(\alpha^{++}_{k,m}, \alpha^{--}_{k,m}\) to be real. One can see that \(\alpha^{--}_m \propto \alpha^{--}_{m-1,2} \propto \mu_1^2, \alpha^{--}_{m+2} \propto \mu_2^2\). Hence, the hermiticity of the Lagrangian is equivalent to the hermiticity of the curvatures.

In case of the partially massless limit where the unitary part contains the components \(k, s\), all the lower spin fields (i.e. \(\Omega^{a(l+m)\dot{a}(l-m)}, W^{a(l+m)\dot{a}(l-m)}\) for \(l < k-1\)) completely decouple. Besides, all the zero-forms \(W^{a(l+m)\dot{a}(l-m)}\) with \(l \geq k-1, |m| \leq k-1\) also decouple. This leaves us with the set of one forms \(\Omega^{a(l+m)\dot{a}(l-m)}\) with \(l \geq k-1, |m| \leq k-1\) (which exactly correspond to the Skvortsov-Vasiliev formalism \[23\], see below) as well as the pairs of one-forms and zero-forms with \(l \geq k, l \geq |m| \geq k\) (see Figure 2b).

### 2.3 Lagrangian in terms of the curvatures

The existence of the complete set of gauge invariant curvatures allows us to rewrite the Lagrangian in the explicitly gauge invariant form. The most general ansatz for the Lagrangian
in terms of the curvatures is:
\[ -i\mathcal{L} = \sum_{k=0}^{s-1} \sum_{m=-k}^{k} (-1)^{k+1} a_{k,m} R^{\alpha(k+m)}_{\alpha(k-m)} \delta_{\alpha(k)}^{(k)} \]
\[ + \sum_{k=0}^{s-2} \sum_{m=-k}^{k} (-1)^{k+1} b_{k,m} R^{\alpha(k+m)}_{\alpha(k-m)} \delta_{\alpha(k)}^{(k)} \]
\[ + \sum_{k=1}^{s-1} \sum_{m=-k+1}^{k-1} (-1)^{k+1} c_{k,m} R^{\alpha(k+m)}_{\alpha(k-m)} \delta_{\alpha(k)}^{(k)} \]
\[ + \sum_{k=1}^{s-1} \sum_{m=-k+1}^{k-1} (-1)^{k+1} d_{k,m} R^{\alpha(k+m)}_{\alpha(k-m)} \delta_{\alpha(k)}^{(k)} \]
\[ - \sum_{k=1}^{s-1} \sum_{m=-k+1}^{k-1} (-1)^{k+1} e_{k,m} R^{\alpha(k+m)}_{\alpha(k-m)} \delta_{\alpha(k)}^{(k)} \]
\[ + \sum_{k=1}^{s-1} \sum_{m=-k+1}^{k-1} (-1)^{k+1} f_{k,m} R^{\alpha(k+m)}_{\alpha(k-m)} \delta_{\alpha(k)}^{(k)} \]
\[ \delta \mathcal{L} \]
\[ \frac{\delta \mathcal{L}}{\delta \Omega_{\alpha(k-1)}^{\alpha(k-s+1)}} = 0, \quad |m| \geq 2, \]
\[ \frac{\delta \mathcal{L}}{\delta W_{\alpha(k-1)}^{\alpha(k-1)}} = 0, \quad k \geq 2, \]  
(37)
where \( a_{k,m} = -a_{k,-m}, b_{k,m} = -b_{k,-m}, c_{k,m} = -c_{k,-m} \) (for the hermiticity of the Lagrangian). The most straightforward way to calculate the coefficients \( a_{k,m} - f_{k,m} \) is to substitute the curvatures with their expressions via fields and require the result to be equal to (19). It is much more convenient, however, to require the Lagrangian equations to match. Since the equations are gauge invariant, they can be expressed via the curvatures as well. Hence, the curvatures can be used during the whole process of calculation reducing the number of terms. The requirement of matching the equations is equivalent to the extra field decoupling conditions:
\[ \frac{\delta \mathcal{L}}{\delta \Omega_{\alpha(k-1)}^{\alpha(k-s+1)}} = 0, \quad |m| \geq 2, \]
\[ \frac{\delta \mathcal{L}}{\delta W_{\alpha(k-1)}^{\alpha(k-1)}} = 0, \quad k \geq 2, \]  
(37)
up to the normalization, which is fixed by the normalization of the equations for the physical and auxiliary fields:
\[ \frac{\delta \mathcal{L}}{\delta \Omega_{\alpha(k-1)}^{\alpha(k+1)}} = 2(-1)^{k+1} e^{\beta} g_{\alpha(k-1)}^{\beta \alpha(k+1)}, \]
\[
\frac{\delta \mathcal{L}}{\delta W^{\alpha(2)}} = -2\mu_1 E_{\alpha(2)} R, \quad \frac{\delta \mathcal{L}}{\delta W^{\alpha\alpha}} = -24\mu_1 \beta_2 e_{\alpha\dot{\alpha}} C,
\]
\[
\frac{\delta \mathcal{L}}{\delta \Omega^{\alpha(k)\alpha(k)}} = 2(-1)^{k+1} e_{\alpha} \dot{\gamma} R_{\alpha(k-1)\dot{\gamma}\dot{\alpha}(k)} + h.c., \quad \frac{\delta \mathcal{L}}{\delta \Omega^{\alpha\alpha}} = 2\mu_1 E_{\alpha(2)} C^{\alpha(2)} + h.c., \quad \frac{\delta \mathcal{L}}{\delta W} = -24\mu_1 \beta_2 e_{\alpha\dot{\alpha}} C^{\alpha\dot{\alpha}}.
\] (38)

Those conditions yield a system of linear equations for \(a_{k,m} - f_{k,m}\). However, there is an arbitrariness in the choice of \(a_{k,m} - f_{k,m}\). It stems from the fact that there exist terms quadratic in the curvatures that are equal to the total derivative of some object, which does not alter the equations of motion (see Appendix B):

\[
i(\mathcal{L} - \mathcal{L}_0) = \sum_{k=0}^{s-1} \sum_{m=1}^{k} (-1)^{k+1} p_{k,m} D(R^{\alpha(k+m)\dot{\alpha}(k-m)} C^{\alpha(k+m)\dot{\alpha}(k-m)} - h.c.)
+ \sum_{k=1}^{s-1} \sum_{m=0}^{k-1} (-1)^{k+1} q_{k,m} D(C^{\alpha(k+m)\dot{\alpha}(k-m)} e^{\alpha}_{\dot{\alpha}} C^{\alpha(k+m+1)\dot{\alpha}(k-m-1)} - h.c.)
+ \sum_{k=0}^{s-1} \sum_{m=1}^{k} (-1)^{k+1} r_{k,m} D(C^{\alpha(k+m)\dot{\alpha}(k-m)} e^{\alpha}_{\dot{\alpha}} C^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} - h.c.).
\] (39)

Hence, the parameters of the Lagrangian is determined up to the shifts with \(p_{k,m}, q_{k,m}, r_{k,m}\) (see their explicit expressions in Appendix B). By an appropriate choice of \(p_{k,m}, q_{k,m}, r_{k,m}\) one can set to zero all the \(b_{k,m}, c_{k,m}\) and \(d_{k,m}\) for \(m \geq 0\). It follows from the equations that all the \(d_{k,m}, e_{k,m}, f_{k,m}\), except \(e_{k,k}, f_{k,k}\) turn out to be zero as well. We obtain the following expressions for the remaining coefficients \(a^{(0)}_{k,m}, e^{(0)}_{k,k}, f^{(0)}_{k,k}\):

\[
a^{(0)}_{k,\pm m} = \pm \frac{(k-1)!(k+m+1)k!}{(k-m)!^2(k-m+1)! \prod_{i=1}^{m} \alpha^\mu_{i}}, \quad m > 0,
\]
\[
e^{(0)}_{k,k} = \frac{a^{++}_{k,k} \alpha^+_{k+1}}{k+1}, \quad f^{(0)}_{k,k} = -4a^{++}_{k,k} a_{k,k}, \quad k > 0,
\] (40)

For such choice of the coefficients, the structure of the Lagrangian simplifies to:

\[
-i\mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \sum_{m=0}^{k} a_{k,m} R^{\alpha(k+m)\dot{\alpha}(k-m)} R^{\alpha(k+m)\dot{\alpha}(k-m)}
+ \sum_{k=1}^{s-1} (-1)^{k+1} e_{k,k} C^{\alpha(2k)} E^{\beta}_{\alpha} C^{\alpha(2k-1)\beta} - h.c.
+ \sum_{k=0}^{s-2} (-1)^{k+1} f_{k,k} C^{\alpha(2k)} E^{\alpha(2)} C^{\alpha(2k+2)} - h.c.
\] (41)

Note that the structure of the expression is the same as in \[13\].

One can see that the expression contain singularities in case of partially massless limits (i.e. for \(\alpha_n^+ = \alpha_n^- = 0\)). In this case, our ansatz fails. However, we can return back
to the general solution and use the shifts \( p_{k,m}, q_{k,m}, r_{k,m} \) to remove the poles, so that the limit \( \alpha_n^+ \to 0 \) can be taken. We do this in the most straightforward way - we set all the singular coefficients \( a_{k,m}^{(0)}, m \geq n, e_{k,k}^{(0)} f_{k,k}^{(0)}, k > n \) to zero, while preserving zero values of \( b_{k,m}, c_{k,m} \) and \( d_{k,m} \) \((m \neq n-1)\). Then, the coefficients with \( k < n \) remain the same, except the coefficients \( e_{n-1,n-1}, f_{n-2,n-2}, f_{n-1,n-1} \), which become zero. The non-zero coefficients for \( k \geq n \) are:

\[
\pm a_{k,m} = \frac{(k-1)!(k+m+1)!}{(k-m)!2(k-m+1)! \prod_{i=1}^m \alpha_i^{-1}}, \quad 0 < m < n,
\]

\[
d_{k,n-1} = -(k-n+1)a_{k,n-1},
\]

\[
e_{k,n} = -(k-n+1)(k-n+2)a_{k,n-1},
\]

\[
e_{k,n} = -(k-n+1)(k-n)a_{k,n-1}.
\]

The Lagrangian has the structure:

\[
-i\mathcal{L} = -i\mathcal{L}^{(0,n-2)} + \sum_{k=n-1}^{s-1} \sum_{m=-n+1}^{n-1} (-1)^{k+1} a_{k,m} R^{\alpha(k+m)\beta(k-m)} R_{\alpha(k+m)\beta(k-m)}
\]

\[
+ \sum_{k=n}^{s-1} (-1)^{k+1} d_{k,n-1} [R^{\alpha(k+n-1)\beta(k-n+1)} e^\beta_{\dot{\alpha}} C_{\alpha(k+n-1)\beta(k-n)} - h.c.]
\]

\[
+ \sum_{k=n}^{s-1} (-1)^{k+1} e_{k,n} [C^{\alpha(k+n)\beta(k-n)} E_\alpha C_{\alpha(k+n-1)\beta(k-n)} - h.c.]
\]

\[
+ \sum_{k=n}^{s-1} (-1)^{k+1} e_{k,-n} [C^{\alpha(k+n)\beta(k-n)} E_{\dot{\alpha}} C_{\alpha(k+n-1)\beta(k-n)} - h.c.].
\]

Here \( \mathcal{L}^{(0,n-2)} \) contains all the terms with \( k \leq n-2 \).

One can see that the Lagrangian splits in two parts containing the fields with \( k \geq n-1 \) and \( k < n-1 \) respectively. This is an expected result for the partially massless limit. Note that the fields \( W^{\alpha(k+m)\beta(k-m)} \), \( k \geq n-1 \), \( |m| \leq n-1 \) also do not enter the Lagrangian for the components \( \overline{n,s} \).

### 2.4 Unfolded equations

Let us consider an unfolded formulation for massive spin-\( s \) boson. Using the explicit expressions for the curvatures given above, one can straightforwardly check that it is consistent to set to zero most of them, namely:

\[
0 = R^{\alpha(s-1+m)\beta(s-1-m)}, \quad |m| \neq s-1,
\]

\[
0 = R^{\alpha(k+m)\beta(k-m)}, \quad k < s-1
\]

\[
0 = C^{\alpha(k+m)\beta(k-m)}, \quad k < s-1.
\]

As for the remaining curvatures, to write consistent equations for them one has to introduce a first set of the gauge invariant zero-forms:

\[
0 = R^{\alpha(2s-2)} - 2E_\alpha^{(2)} W^{\alpha(2s)},
\]

\[
0 = C^{\alpha(s+m-1)\beta(s-m-1)} + e_{\alpha\dot{\alpha}} W^{\alpha(s+m)\beta(s-m)}.
\]

14
These equations connect the gauge sector with the infinite tail containing gauge-invariant zero-forms only. Indeed, the equations for these new zero-forms require introduction of additional zero-forms and so on. This procedure leads to the infinite set of the gauge invariant zero-forms $W^{\alpha(k+m)\dot{\alpha}(k-m)}$, $k \geq s, |m| \leq s$. Thus the complete set of one-forms and zero-forms for the massive spin-$s$ boson is equal to the sum of the one-forms and zero-forms necessary for the unfolded formulation for the massless fields with spins $0, s$ (see Figure 2a).

The main difference is that a part of zero-forms, namely, $W^{\alpha(k+m)\dot{\alpha}(k-m)}, |m| \leq k \leq s$ are not gauge invariant but play the role of the Stueckelberg fields. The equations for the tail are similar to their massless analogues; however, just as every other object (Lagrangian, gauge transformations, curvatures), they have to contain the cross-terms. The most general ansatz for the tail equations, up to the normalization choice, is:

$$
0 = DW_{\alpha(k+m)\dot{\alpha}(k-m)} + (k + m)(k - m)\beta^{\alpha}_{k,m} e^{\alpha\dot{\alpha}}W^{\alpha(k+m-1)\dot{\alpha}(k-m-1)} + e_{\beta\gamma}W^{\alpha(k+m)\beta\dot{\alpha}(k-m)\gamma} + (k + m)\beta^{\alpha}_{k,m} e^{\alpha\dot{\alpha}}W^{\alpha(k+m-1)\beta\dot{\alpha}(k-m-1)}
$$

where $\beta^{\pm}_{k,m} = \beta^{-\pm}_{k,-m}$ due to hermiticity. The equations must be consistent with each other as well as with the gauge sector equations. This leads to the unique possible choice of the coefficients:

$$
\begin{align*}
\beta^{\alpha}_{k,m} &= \frac{\beta^{\alpha}_{k,m}}{(k + m)(k + m + 1)}, \\
\beta^{-\alpha}_{k,m} &= \frac{\beta^{-\alpha}_{k,m}}{(k - m)(k + m + 1)}, \\
\beta^{\alpha}_{k,m} &= \frac{\alpha^{\alpha}_{k+1}}{(k + m)(k + m + 1)(k - m)(k + m + 1)}, \\
\beta^{-\alpha}_{m} &= \frac{\alpha^{-\alpha}_{m}}{(s - m)(s - m + 1)}, \quad 1 \leq m < s, \quad \beta^{\alpha}_{s} = \frac{\alpha^{\alpha}_{s}}{2}, \\
\beta^{\alpha}_{m} &= \frac{\alpha^{\alpha}_{m}}{(s - m - 1)(s - m)}, \quad 0 \leq m < s - 1, \quad \beta^{\alpha}_{s-1} = 2.
\end{align*}
$$

In case of the partially-massless limit given by $\alpha^{\alpha}_{m} = 0$, the curvatures $C^{\alpha(k+m)\dot{\alpha}(k-m)}, m < n$ decouple. This corresponds to the equality $\beta^{\alpha}_{n} = 0$ for the unfolded equations, which means that the fields $W^{\alpha(k+m)\dot{\alpha}(k-m)}, m < n$ decouple as well (see Figure 2b). Hence, only the components with $n = s$ remain, which is an expected result.

### 2.5 Application to the Skvortsov-Vasiliev formalism

In the paper of Skvortsov and Vasiliev [23], an approach for the description of partially massless particles was proposed, which use the one-forms only. Consider a partially massless limit. The Skvortsov-Vasiliev formalism corresponds to a partial gauge fixing, when all the zero-forms $W^{\alpha(k+m)\dot{\alpha}(k-m)}, k < s, |m| \geq n - 1$ are set to zero (see Figure 2b). Then, the one-form curvatures $C^{\alpha(k+m)\dot{\alpha}(k-m)}, |m| \geq n - 1$ become:

$$
C^{\alpha(k+m)\dot{\alpha}(k-m)} = -\Omega^{\alpha(k+m)\dot{\alpha}(k-m)}.
$$

(48)
Other one-form curvatures, namely \( C^{\alpha(k+m)\dot{\alpha}(k-m)} \), \(|m| < n - 1\), decouple. The curvatures \( R^{\alpha(k+m)\dot{\alpha}(k-m)} \), \(|m| < k\) do not change since they contain no zero-forms. However, it is convenient to introduce the modified \( \hat{R}^{\alpha(k+n-2)\dot{\alpha}(k-n+2)} \) curvature as:

\[
\hat{R}^{\alpha(k+n-2)\dot{\alpha}(k-n+2)} = R^{\alpha(k+n-2)\dot{\alpha}(k-n+2)} + (k - n + 2)e_\alpha^\dot{\alpha} C^{\alpha(k+n-1)\dot{\alpha}(k-n+1)}
\]

\[
= D \Omega^{\alpha(k+n-2)\dot{\alpha}(k-n+2)} + \alpha_{k,n-2}^{\alpha\dot{\alpha}} \Omega^{\alpha(k+n-1)\dot{\alpha}(k-n+3)}
\]

\[
+ (k + n - 2)(k - n + 2)\alpha_{k,n-2}^{\alpha\dot{\alpha}} \Omega^{\alpha(k+n-3)\dot{\alpha}(k-n+1)}
\]

\[
+ (k + n - 2)\alpha_{k,n-2}^{\alpha\dot{\alpha}} \Omega^{\alpha(k+n-3)\dot{\alpha}(k-n+3)}.
\]

(49)

Let \( \hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)} = R^{\alpha(k+m)\dot{\alpha}(k-m)} \), \(|m| < n - 2\) to make the notations uniform. Then, the new set of curvatures \( \hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)} \) does not contain \( \Omega^{\alpha(k+m)\dot{\alpha}(k-m)} \), \(|m| \geq n - 1\) at all. In this case, the Lagrangian (43) can be rewritten purely in terms of \( \hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)} \):

\[
- i \mathcal{L} = \sum_{k=n-1}^{s-1} \sum_{m=-n+1}^{n-1} (-1)^{k+1} a_{k,m} \hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)} \hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)}.
\]

(50)

Hence, all the 1-forms \( \Omega^{\alpha(k+m)\dot{\alpha}(k-m)} \), \(|m| \geq n - 1\) decouple.

Finally, we derive the gauge sector of unfolded equations via \( \hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)} \) dropping off all the decoupled curvatures:

\[
\hat{R}^{\alpha(k+m)\dot{\alpha}(k-m)} = 0, \quad k < s - 1,
\]

\[
\hat{R}^{\alpha(s-1+m)\dot{\alpha}(s-1-m)} = 0, \quad m < n - 2,
\]

\[
\hat{R}^{\alpha(s+n-3)\dot{\alpha}(s-n+1)} = 2E_{\alpha(2)} W^{\alpha(s+n-1)\dot{\alpha}(s-n+1)}.
\]

(51)

The last equation is the only link between gauge sector and infinite tail of gauge-invariant zero-forms \( W^{\alpha(k+m)\dot{\alpha}(k-m)} \), \( k \geq s \), \(|m| \geq n - 1\). The equations for the gauge-invariant forms remain the same. It is the expected result: since the derivation of the Skvortsov-Vasiliev-like description is reduced to the fixing the gauge, the gauge-invariant forms are left unaltered.

3 Fermionic case

3.1 The Lagrangian

The massless fermion with the spin \( s + \frac{1}{2} \) requires only the physical one-form \( \Psi^{\alpha(s)\dot{\alpha}(s-1)} \) with its hermitian conjugate. The Lagrangian for the massless spin-\( s + \frac{1}{2} \) fermion is:

\[
(-1)^s \mathcal{L}^{(s+\frac{1}{2})} = \Psi^{\alpha(s-1)\dot{\alpha}(s-1)} e_\beta^\alpha \hat{\partial} \psi^{\alpha(s-1)\dot{\alpha}(s-1)}
\]

\[
+ \frac{\lambda(s + 1)}{2} \Psi^{\alpha(s-1)\dot{\alpha}(s-1)} E_\beta^\gamma \psi^{\alpha(s-1)\dot{\alpha}(s-1)}
\]

\[
- \frac{\lambda(s - 1)}{2} \Psi^{\alpha(s-1)\dot{\alpha}(s-1)} E_\beta^\gamma \psi^{\alpha(s)\dot{\alpha}(s-2)} + h.c.
\]

(52)

The Lagrangian possesses the following gauge symmetries:

\[
\delta \psi^{\alpha(s)\dot{\alpha}(s-1)} = D \eta^{\alpha(s)\dot{\alpha}(s-1)} + s \lambda e_\alpha^\alpha \psi^{\alpha(s-1)\dot{\alpha}(s-1)} + (s - 1) e_\alpha^\alpha \psi^{\alpha(s+1)\dot{\alpha}(s-2)}.
\]

(53)
The Lagrangian for the massive fermion is built in the same way as for the boson [11,12]. One introduces the $s + 1$ massless fields for the components with spins $\frac{1}{2}, \frac{3}{2}, \ldots, s + \frac{1}{2}$. The spin-$\frac{1}{2}$ component is described by the fermionic zero-form $\psi^\alpha$ (with its conjugate), while the other components require the one-forms $\Psi^\alpha(k)\dot{\alpha}(k)$ (with their conjugates), $1 \leq k \leq s$, used to describe the massless spin-$k + \frac{1}{2}$ fields. The Lagrangian is built as a sum of the massless Lagrangians with all possible cross-terms and mass-like terms:

$$ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 $$

$$ \mathcal{L}_0 = \sum_{k=0}^{s-1} (-1)^{k+1} \Psi_{\alpha(k)\beta}(k) \epsilon^{\beta\dot{\beta}} \partial^\beta \Psi^\alpha(k)\dot{\alpha}(k) - \alpha_0^2 \psi_\alpha E^\alpha \partial_\alpha \psi^\alpha, \quad (55) $$

$$ \mathcal{L}_1 = \sum_{k=1}^{s-1} (-1)^{k+1} \alpha_k \Psi_{\alpha(k-1)\beta}(k) \epsilon^{\beta\dot{\beta}} \partial^\beta \Psi^\alpha(k-1)\dot{\alpha}(k) + \alpha_0^2 \psi_\alpha E^\alpha \partial_\alpha \psi^\alpha + h.c. $$

$$ + \sum_{k=1}^{s-1} (-1)^{k+1} \frac{\beta_{k+1}}{2} [(k + 2) \Psi_{\alpha(k)\beta}(k) \epsilon^{\beta\dot{\beta}} \partial^\beta \Psi^\alpha(k)\dot{\alpha}(k) $$

$$ - k \Psi_{\alpha(k+1)\beta}(k) \epsilon^{\beta\dot{\beta}} \partial^\beta \Psi^\alpha(k+1)\dot{\alpha}(k-1)] $$

$$ + \beta_1 \alpha_0^2 E \psi_\alpha \psi^\alpha + h.c. \quad (56) $$

The Lagrangian is split into $\mathcal{L}_0$ and $\mathcal{L}_1$ according to the order of the terms in derivatives. The field $\psi$ has a non-canonical normalization. The Lagrangian can be used to describe the infinite-spin particles as well. In this case one has to take the limit $s \to +\infty$ [15,16]. We require that the Lagrangian possesses all the gauge symmetries the massless components possess. The transformation laws of the fields then have to be modified with the cross-terms and mass-like terms:

$$ \delta \Psi^\alpha(k)\dot{\alpha}(k) = D\eta^\alpha(k)\dot{\alpha}(k) + \alpha_{k+1} e^{\alpha\dot{\alpha}} \eta^{(k+2)\dot{\alpha}(k+1)} + \beta_{k+1} (k + 1) e^{\alpha\dot{\alpha}} \eta^{(k)\dot{\alpha}(k+1)} $$

$$ + \frac{(k + 1) \alpha_{k}}{(k + 2)} e^{\alpha\dot{\alpha}} \eta^{(k)\dot{\alpha}(k-1)} + k e^{\alpha\dot{\alpha}} \eta^{(k+2)\dot{\alpha}(k-1)}, \quad (57) $$

Then, the gauge invariance requirement yields the following relations for the coefficients $\alpha_k, \beta_k$:

$$ k \beta_k = \beta_{k+1} (k + 2), $$

$$ \alpha_{k+1}^2 = \alpha_k^2 + \lambda^2 (2k + 3) - \beta_{k+1}^2 (2k + 3). \quad (58) $$

One can see that the coefficients $\alpha_k, \beta_k$ are defined up to two free parameters. In the case of the finite spin $s + \frac{1}{2}$ an additional condition $\alpha_s = 0$ reduces the number of the free parameters to one. Similarly to the bosonic case, we choose the “mass” parameter as this free parameter. It has to be proportional to the $\alpha_{s-1}$ and tends to the usual mass in the flat-space limit $\lambda^2 \to 0$. This leads to:

$$ M^2 = \frac{s^2 \alpha_{s-1}^2}{(2s - 1)}. $$
This gives the following expressions for the coefficients $\alpha_k, \beta_k$:

$$\alpha_k^2 = \frac{(s-k)(s+k+2)}{(k+1)^2}[M^2 + (s+k+1)(s-k-1)\lambda^2],$$
$$\beta_k^2 = \frac{(s+1)^2}{k^2(k+1)^2}[M^2 + s^2\lambda^2].$$

Such parametrization is not applicable in the case of the infinite spin. As in the bosonic case, we choose the lower coefficients as the free parameters - namely, $\beta_1$ and $\alpha_0$:

$$\alpha_k^2 = \frac{1}{(k+1)^2}[\alpha_0^2(k+1)^2 - 4\beta_1^2k(k+2) + k(k+1)^2(2k+3)\lambda^2],$$
$$\beta_k^2 = \frac{4\beta_1^2}{k^2(k+1)^2}.$$  \hfill (60)

Let us study the hermiticity of the Lagrangian now; consider the finite spin $s + \frac{1}{2}$ first. The hermiticity condition is that all $\alpha_k^2, \beta_k^2$ must be non-negative. That requires $M^2 \geq 0$ in case of flat space and AdS. The case of $M^2 = 0$ corresponds to the decoupling of the highest field; moreover, the Lagrangian breaks component-wise into $s + 1$ pieces in the flat space case. In $dS$ the hermiticity condition leads to the unitary forbidden region $M^2 < -s^2\lambda^2$. Similarly to the bosonic case, inside the forbidden region there exists a number of partially massless limits

$$M^2 = -\lambda^2(s + k - 1)(s - k + 1),$$

however, none of them are unitary, since the fermionic Lagrangian contains $\beta_k$ and they are all imaginary in these cases.

Consider the infinite spin case now. We introduce the variable $y_k = k^2+2k$ for simplicity. Then the sign of the coefficients $\alpha_k^2$ is determined by the square trinomial:

$$\alpha_k^2 \propto \alpha_0^2 + \alpha_0^2 + \lambda^2 - 4\beta_1^2y_k + y_k^2\lambda^2.$$

The sign of $\beta_k^2$ is determined by that of $\beta_1^2$. It follows immediately that the infinite-spin representations are completely impossible in the $dS$, since $\alpha_k^2 \propto y_k^2\lambda^2 < 0$ for sufficiently large values of $k$. Consider the flat space case. The hermiticity condition reads:

$$\alpha_0^2 \geq 4\beta_1^2 \geq 0.$$  \hfill (61)

In this, the case $\alpha_0^2 = 4\beta_1^2$ corresponds to the massless infinite spin particle which can be obtained from the massive finite spin one by the limit $s \to \infty$, $M \to 0$, $Ms = const$. No unitary partially massless limits exist in the flat space.

In the $AdS$ case, similarly to the bosonic case, the unitarity region in the $\alpha_0^2, \beta_1^2$ parameter space is an infinite region with piece-wise linear boundary:

$$2\beta_1^2 \in [(k-1)k(k+1)(k+2)\lambda^2, k(k+1)(k+2)(k+3)\lambda^2], \quad k \in \mathbb{N},$$
$$\alpha_0^2 > k(k+2)[\frac{4\beta_1^2}{(k+1)^2} - \lambda^2].$$  \hfill (62)
Moreover, there exists a number of partially massless limits with the spectrum \( k + 1/2, +\infty \) for which solutions can also be written in a form similar to the massive finite spin one:

\[
\beta_k^2 = \frac{(s + 1)^2 \hat{M}^2}{k^2(k + 1)^2}, \quad \hat{M}^2 < (s + 1)^2 \lambda^2,
\]

\[
\alpha_k^2 = \frac{(k - s)(k + s + 2)}{(k + 1)^2} [(k + 1)^2 \lambda^2 - \hat{M}^2].
\] (63)

### 3.2 Gauge invariant curvatures

The construction of the complete set of the gauge invariant objects is similar to the bosonic case. One needs the complete set of the one-forms \( \Omega^{\alpha(k+m+1)\dot{\alpha}(k-m)} \) and zero-forms \( W^{\alpha(k+m+1)\dot{\alpha}(k-m)} \), \( k \leq s - 1, m \geq 0 \) with their conjugates (see Figure 3a). In what follows the notations are unified, i.e. we take \( \Psi^{\alpha(k+1)\dot{\alpha}(k)} \equiv \Omega^{\alpha(k+1)\dot{\alpha}(k)} \), \( \psi^\alpha \equiv W^\alpha \). The ansatz for the gauge transformations is similar to the bosonic one due to the multispinor formalism. Namely, the gauge transformations \( (m \geq 0) \) have the form:

\[
\delta \Omega^{\alpha(k+m+1)\dot{\alpha}(k-m)} = D\eta^{\alpha(k+m+1)\dot{\alpha}(k-m)} + (k + m + 1)\alpha^{++}_{k,m} e^\alpha e^{\dot{\alpha}} \eta^{\alpha(k+m+1)\dot{\alpha}(k-m+1)} + (k - m)e^\alpha e^{\dot{\alpha}} \eta^{\alpha(k+m+2)\dot{\alpha}(k-m-1)} + \alpha^{++}_{k,m} e^\alpha e^{\dot{\alpha}} \eta^{\alpha(k+m+2)\dot{\alpha}(k-m+1)} + (k + m + 1)(k - m)\alpha^{--}_{k,m} e^\alpha e^{\dot{\alpha}} \eta^{\alpha(k+m)\dot{\alpha}(k-m-1)},
\]

\[
\delta W^{\alpha(k+m+1)\dot{\alpha}(k-m)} = \eta^{\alpha(k+m+1)\dot{\alpha}(k-m)}.
\] (64)

Here

\[
\alpha^{++}_{k,0} = \alpha_{k+1}, \quad \alpha^{--}_{k,0} = \beta_{k+1}, \quad \alpha_{k,0} = \frac{1}{k(k + 2)} \alpha_k.
\]

The corresponding expressions for the gauge invariant curvatures are:

\[
R^{\alpha(k+m+1)\dot{\alpha}(k-m)} = D\Omega^{\alpha(k+m)\dot{\alpha}(k-m)} + (k + m + 1)(k - m)\alpha^{--}_{k,m} e^\alpha e^{\dot{\alpha}} \Omega^{\alpha(k+m)\dot{\alpha}(k-m-1)}
\]

Figure 3: a) Left figure — massive fermion with spin \( s = \frac{11}{2} \). b) Right figure — partially massless case with \( n = 3 \).
We use the same parameter choice for these coefficients, as for the \( \alpha \). In the infinite spin case, their values are:

\[
\begin{align*}
\alpha_{k,m}^{++} &= \frac{\alpha_k^{++}}{(k - m + 1)(k + m + 2)}, \\
\alpha_{k,m}^{--} &= \frac{\alpha_k^{--}}{(k + m + 1)(k + m + 2)}, \\
\alpha_{k,m}^{--} &= \frac{\alpha_m^{--}}{(k - m + 1)(k - m + 2)(k + m + 1)(k + m + 2)}, \\
\alpha_{k,m}^{++} &= \frac{\alpha_0^{++}}{(k + 1)(k + 2)}.
\end{align*}
\]

We use the same parameter choice for these coefficients, as for the \( \alpha_k, \beta_k \). The values of the \( \alpha_k^{++}, \alpha_k^{--}, \alpha_m^{--} \) in case of finite spin are:

\[
\begin{align*}
\alpha_{k-1}^{++} &= k^2(s - k)(s + k + 2)[M^2 + (s + k + 1)(s - k - 1)\lambda^2], \\
\alpha_k^{--} &= \frac{1}{k^2}(s - k)(s + k + 2)[M^2 + (s + k + 1)(s - k - 1)\lambda^2], \\
\alpha_m^{--} &= (s - m + 1)(s + m + 1)[M^2 + (s - m)(s + m)\lambda^2], \\
\alpha_0^{++} &= (s + 1)^2[M^2 + s^2\lambda^2].
\end{align*}
\]

In the infinite spin case, their values are:

\[
\begin{align*}
\alpha_{k-1}^{++} &= k^2[\alpha_0^2(k + 1)^2 - 4\beta_1^2k(k + 2) + k(k + 1)^2(k + 2)\lambda^2], \\
\alpha_k^{--} &= \frac{1}{k^2}[\alpha_0^2(k + 1)^2 - 4\beta_1^2k(k + 2) + k(k + 1)^2(k + 2)\lambda^2], \\
\alpha_k^{++} &= [\alpha_0^2(k + 1)^2 - 4\beta_1^2k(k + 2) + k(k + 1)^2(k + 2)\lambda^2], \\
\alpha_0^{--} &= 4\beta_1^2.
\end{align*}
\]

Let us outline the useful relation \( \alpha_{m-1}^{--}\alpha_{m-2}^{++} = \alpha_m^{--} \) once again.

The hermiticity of the curvatures is given by the same expressions as for the Lagrangian. We have already seen that there the partially massless limits are non unitary due to the presence of \( \beta_k \) in the finite spin case. In curvatures, the coefficient \( \alpha_{k,0}^{++} \) plays the role of \( \beta_k \).
In case of the partially massless limit with the components \( k + 1/2, s + 1/2 \) all the lower spin fields (i.e. \( \Omega^{\alpha(l+m+1)}(l-m) \), \( W^{\alpha(l+m+1)}(l-m) \) for \( l < k-1 \)) completely decouple. Besides, all the zero-forms \( W^{\alpha(l+m+1)}(l-m) \) with \( l \geq k-1, m \leq k-1 \) and their conjugates also decouple. This leaves us with the set of one-forms \( \Omega^{\alpha(l+m+1)}(l-m) \) with \( l \geq k-1, m \leq k-1 \) and their conjugates (which form an analogue of the Skvortsov-Vasiliev formalism for the fermions, see below) as well as the pairs of one-forms and zero-forms with \( l \geq k, l \geq m \geq k \) (see Figure 3b).

### 3.3 Lagrangian in terms of curvatures

The ansatz for the Lagrangian expressed in the terms of the curvatures is similar to the bosonic case:

\[
\mathcal{L} = \sum_{k=0}^{s-1} \sum_{m=-k}^{k} (-1)^{k+1} a_{k,m} R^{\alpha(k+m+1)}(k-m) R_{\alpha(k+m+1)}(k-m) \\
+ \sum_{k=0}^{s-2} \sum_{m=-k-1}^{k} (-1)^{k+1} b_{k,m} R^{\alpha(k+m+1)}(k-m) c^{\alpha \dot{\alpha}} C_{\alpha(k+m+2)}(k-m+1) \\
+ \sum_{k=0}^{s-1} \sum_{m=-k-1}^{k} (-1)^{k+1} c_{k,m} R^{\alpha(k+m+1)}(k-m) e_{\alpha \dot{\alpha}} C_{\alpha(k+m)}(k-m-1) \\
+ \sum_{k=0}^{s-1} \sum_{m=-k-1}^{k} (-1)^{k+1} d_{k,m} R^{\alpha(k+m+1)}(k-m) e^{\beta \dot{\beta}} C_{\alpha(k+m+1)}(k-m-1) \\
+ \sum_{k=0}^{s-1} \sum_{m=-k-1}^{k} (-1)^{k+1} d_{k,-m} R^{\alpha(k+m)}(k-m) e_{\alpha \dot{\alpha}} C_{\alpha(k+m+1)}(k-m-1) \\
+ \sum_{k=0}^{s-1} \sum_{m=-k}^{k} (-1)^{k+1} e_{k,m} C^{\alpha(k+m+1)}(k-m) E^{\beta \dot{\alpha}} C_{\alpha(k+m)}(k-m) \\
+ \sum_{k=0}^{s-1} \sum_{m=-k}^{k} (-1)^{k+1} e_{k,-m} C^{\alpha(k+m+1)}(k-m) E^{\beta \dot{\alpha}} C_{\alpha(k+m+1)}(k-m) \\
+ \sum_{k=0}^{s-2} \sum_{m=-k-1}^{k} (-1)^{k+1} f_{k,m} C^{\alpha(k+m+1)}(k-m) E^{(2) \alpha \beta} C_{\alpha(k+m+3)}(k-m) \\
+ \sum_{k=0}^{s-2} \sum_{m=-k-1}^{k} (-1)^{k+1} f_{k,-m} C^{\alpha(k+m+1)}(k-m) E^{(2) \alpha \beta} C_{\alpha(k+m+1)}(k-m+3). \tag{69}
\]

Here \( a_{k,m} = a_{k,-m-1}, b_{k,m} = b_{k,-m-1}, c_{k,m} = c_{k,-m-1} \) due to the hermiticity.

We determine the coefficients \( a_{k,m} - f_{k,m} \) in the same way as for the boson, i.e. we require that the extra fields decouple and the equations of motion derived from (69) and (54) match. The equation of motion obtained from (54) are expressed via curvatures as:

\[
\frac{\delta \mathcal{L}}{\delta \Omega^{\alpha(k+1)}(k)} = (-1)^{k+1} e_{\alpha} \dot{R}_{\alpha(k+1)}(k). 
\]
\[ \frac{\delta L}{\delta Q^\alpha} = e_\alpha R_{\dot{\alpha}} \]
\[ \frac{\delta L}{\delta W^\alpha} = -\alpha_0^2 E_\alpha C_{\dot{\alpha}}. \]

As in the bosonic case, there is an arbitrariness in the choice for the coefficients \( a_{k,m} - f_{k,m} \). It comes from the fact that the Lagrangian (69) is defined up to the total derivatives:

\[
(\mathcal{L} - \mathcal{L}_0) = \sum_{k=0}^{s-1} \sum_{m=0}^{\infty} (-1)^{k+1} p_{k,m} D(R^{\alpha(k+m+1)\dot{\alpha}(k-m)} C_{\alpha(k+m+1)\dot{\alpha}(k-m)} + h.c.)
+ \sum_{k=0}^{s-1} \sum_{m=0}^{\infty} (-1)^{k+1} q_{k,m} D(C^{\alpha(k+m+1)\dot{\alpha}(k-m)} e_\alpha \dot{C}_{\alpha(k+m+2)\dot{\alpha}(k-m)} + h.c.)
+ \sum_{k=0}^{s-1} \sum_{m=0}^{\infty} (-1)^{k+1} r_{k,m} D(C^{\alpha(k+m+1)\dot{\alpha}(k-m)} e^{\alpha\dot{\alpha}} C_{\alpha(k+m+2)\dot{\alpha}(k-m+1)} + h.c.)
\]

(71)

Those terms, however, lead to the shifts of the coefficients \( a_{k,m} - f_{k,m} \), just like in the bosonic case (see Appendix B for the explicit expressions of these shifts). Similarly to the bosonic case, we choose the coefficients \( p_{k,m}, q_{k,m}, r_{k,m} \) so that all the \( b_{k,m}, c_{k,m}, d_{k,m} \) and \( e_{k,m}, f_{k,m} \) for \( k \neq m \) equal to zero. The expressions for the nonzero coefficients read:

\[
a_{k,m}^{(0)} = \frac{(k + m + 2)!k!}{4(k - m)!2(k - m + 1)!\prod_{i=0}^{m} \alpha_i^{+}},
\]
\[
e_{k,k}^{(0)} = \frac{(2k + 2)!k!\alpha_{k+1}^{-}}{2(2k + 3)\prod_{i=0}^{m} \alpha_i^{+}},
\]
\[
f_{k,k}^{(0)} = -\frac{\alpha_k^{++}(2k)!k!}{\prod_{i=0}^{m} \alpha_i^{+}}.
\]

(72)

A general solution \( a_{k,m} - f_{k,m} \) can be obtained from the special solution (72) by substituting the arbitrary \( p_{k,m}, q_{k,m}, r_{k,m} \) in (104).

We use the same procedure to eliminate the singularities in the coefficients \( a_{k,m} - f_{k,m} \) in the case of partially massless limit given by \( \alpha_{n-1}^{-} = \alpha_{n-1}^{--} = \alpha_{n-2}^{++} = 0 \). Let us recall the steps. First, we obtain the general solution \( a_{k,m} - f_{k,m} \) for arbitrary parameters \( s, M (\alpha_0, \beta_1) \). Then we choose \( p_{k,m}, q_{k,m}, r_{k,m} \) in a way that allows to take the limit \( \alpha_n^{+} = 0 \). The easiest way is to zero out all the "bad" coefficients preserving most \( b_{k,m}, c_{k,m}, d_{k,m} \) and \( e_{k,m}, f_{k,m} \) zero. The exact expressions for the non-zero \( p_{k,m}, q_{k,m} \) are (all \( r_{k,m} \) are zero) are given in Appendix B. The expressions for the coefficients with \( k < n \) remain the same, except the vanishing coefficients \( e_{n-1,n-1}, f_{n-2,n-2}, f_{n-1,n-1} \). The expressions for the non-zero coefficients for \( k \geq n \) are:

\[
a_{k,m} = \frac{(k + m + 2)!k!}{4(k - m)!2(k - m + 1)!\prod_{i=0}^{m} \alpha_i^{+}}, \quad m < n,
\]
\[
d_{k,n-1} = -\frac{(k + n + 1)!k!}{2(n - k)!(k - n + 1)!(k - n + 2)!\prod_{i=0}^{n-1} \alpha_i^{+}}.
\]
\[ e_{k,n} = \frac{(k + n + 1)!k!^2}{4(k - n)!(k - n + 1)!^2 \prod_{i=0}^{n-1} \alpha_i^{-}}, \quad (73) \]
\[ e_{k,-n-1} = \frac{(k + n + 1)!k!^2}{4(k - n+1)!(k - n + 2)! \prod_{i=0}^{n-1} \alpha_i^{-}}. \]

The Lagrangian is thus split into two parts, one containing the coefficients with \( k \geq n - 1 \) and the other containing the ones with \( k < n - 1 \):

\[ -i\mathcal{L} = -i\mathcal{L}^{(0,n-2)} - i\mathcal{L}^{(n,n-2)} \]
\[ + \sum_{k=n-1}^{s-1} \sum_{m=-n}^{n-1} (-1)^{k+1} a_{k,m} R_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)} R_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)} \]
\[ + \sum_{k=n}^{s-1} (-1)^{k+1} d_{k,n-1} \left[ R_\alpha^{(k+n)} \dot{a}_\alpha^{(k-n+1)} e^{\beta}_\alpha C_\alpha^{(k+n)} \dot{a}^{(k-n)} - h.c. \right] \]
\[ + \sum_{k=n}^{s-1} (-1)^{k+1} e_{k,n} \left[ C_\alpha^{(k+n+1)} \dot{a}_\alpha^{(k-n)} E^{\beta}_\alpha C_\alpha^{(k+n+1)} \dot{a}^{(k-n)} - h.c. \right] \]
\[ + \sum_{k=n-1}^{s-1} (-1)^{k+1} e_{k,-n-1} \left[ C_\alpha^{(k+n+1)} \dot{a}_\alpha^{(k-n)} E^{\beta}_\alpha C_\alpha^{(k+n+1)} \dot{a}^{(k-n)} - h.c. \right]. \quad (74) \]

One can see that the part with the higher components does not contain the curvatures which contain the fields \( W_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)}, \quad m \leq n - 1 \). That means that the part with the higher coefficients does not contain the components \( \frac{1}{2}, \frac{n - 1}{2} \), i.e. the components \( n + \frac{1}{2}, s + \frac{1}{2} \) decouple in this case as well. In case of \( n = 0 \) the expression is:

\[ -i\mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} d_{k,-1} \left[ R_\alpha^{(k)} \dot{a}_\alpha^{(k+1)} e^{\beta}_\alpha C_\alpha^{(k)} \dot{a}^{(k)} - h.c. \right] \]
\[ + \sum_{k=0}^{s-1} (-1)^{k+1} e_{k,0} \left[ C_\alpha^{(k+1)} \dot{a}_\alpha^{(k)} E^{\beta}_\alpha C_\alpha^{(k+1)} \dot{a}^{(k)} - h.c. \right] \]
\[ + \sum_{k=0}^{s-1} (-1)^{k+1} e_{k,-1} \left[ C_\alpha^{(k+1)} \dot{a}_\alpha^{(k)} E^{\beta}_\alpha C_\alpha^{(k+1)} \dot{a}^{(k)} - h.c. \right]. \quad (75) \]

This case correspond to unitary region boundary, when the curvatures contain the fields \( \Omega_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)}, W_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)} \) with \( m \geq 0 \) or with \( m < 0 \) only. It does not correspond to any partially massless limit.

### 3.4 Unfolded equations

We derive the unfolded equations chain in the same way as in the bosonic case. We start by setting to zero most of the gauge invariant curvatures:

\[ 0 = R_\alpha^{(s+m)} \dot{a}_\alpha^{(s-m-1)}, \quad m \neq s - 1, -s, \]
\[ 0 = R_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)}, \quad k < s - 1, \]
\[ 0 = C_\alpha^{(k+m+1)} \dot{a}_\alpha^{(k-m)}, \quad k < s - 1. \quad (76) \]
To construct the consistent equations for the remaining gauge invariant curvatures $R^\alpha_{(2s-1)}$ and $C^\alpha_{(s+m)}\hat{\alpha}(s-m-1)$ one has to introduce a first set of the gauge invariant zero-forms:

$$0 = R^\alpha_{(2s-1)} - 2E_\alpha(2)W^\alpha_{(2s+1)},$$
$$0 = C^\alpha_{(s+m)}\hat{\alpha}(s-m-1) + e_\alpha\hat{\alpha}W^\alpha_{(s+m+1)}\hat{\alpha}(s-m).$$

(77)

The first equation starts the zero-form chain for the highest-spin massless component while the other extend the zero-form chains for the components with spins $\frac{1}{2}, s - \frac{3}{2}$ (see Figure 3a). The most general ansatz (up to the normalization) for the infinite tail containing the gauge-invariant zero-forms is:

$$0 = DW^{\alpha(k+m+1)}\hat{\alpha}(k-m) + (k + m + 1)(k - m)\beta^{++}_{k,m}e^\alpha\hat{\alpha}W^{\alpha(k+m+1)}\hat{\alpha}(k-m-1)$$
$$+ e_\beta W^{\alpha(k+m+1)}\beta^{+-}(k-m)\hat{\beta} + (k - m)\beta^{++}_{k,m}e^\beta W^{\alpha(k+m+1)}\beta^{++}(k-m-1)$$
$$+ (k + m + 1)\beta^{+-}_{k,m}e^\alpha W^{\alpha(k+m+1)}\beta^{+-}(k-m)$$

(78)

Here $\beta^{+-}_{k,m} = \beta^{+-}_{k,-m-1}$ due to hermiticity. The equations (77) and (78) must agree with each other; the consistency requirement yields the following expression for the $\beta^{\pm}_{k,m}$:

$$\beta^{++}_{k,m} = \frac{\beta^{++}_{m}}{(k + m + 2)(k + m + 1)},$$
$$\beta^{+-}_{k,m} = \frac{\beta^{+-}_{m}}{(k - m + 1)(k - m)},$$
$$\beta^{--}_{k,m} = \frac{\alpha^{--}_{k+1}}{(k + m + 1)(k + m + 2)(k - m)(k - m + 1)},$$
$$\beta^{+-}_{m} = \frac{\alpha^{+-}_{m}}{(s - m)(s - m + 1)}, \quad 1 \leq m < s,$$
$$\beta^{--}_{s} = \frac{\alpha^{--}_{s}}{2}, \quad \beta^{+-}_{0} = \alpha^{+-}_{0},$$
$$\beta^{--}_{m} = (s - m - 1)(s - m), \quad 1 \leq m < s - 1, \quad \beta^{++}_{s-1} = 2.$$  

(79)

As in the bosonic case in the partially massless limit $\alpha^{--}_{n} = 0$ curvatures $C^{\alpha(k+m+1)}\hat{\alpha}(k-m)$, $m < n$ and fields $W^{\alpha(k+m+1)}\hat{\alpha}(k-m)$, $m < n$ decouple, as a result all equations, containing $W^{\alpha(k+m+1)}\hat{\alpha}(k-m)$, $m < n$ completely decouple (see Figure 3b).

3.5 Skvortsov-Vasiliev formalism for fermions

The Skvortsov-Vasiliev formalism [23] can be extended to the fermionic partially massless particles. Consider the $n + 1/2, s + 1/2$-partially massless limit. We start with the partial fixing of the gauge by setting $W^{\alpha(k+m+1)}\hat{\alpha}(k-m) = 0$, $m < n - 1$. Then, we introduce the modified 2-curvature $\hat{R}^{\alpha(k+n-1)}\hat{\alpha}(k-n+2)$ as:

$$\hat{R}^{\alpha(k+n-1)}\hat{\alpha}(k-n+2) = R^{\alpha(k+n-1)}\hat{\alpha}(k-n+2) + (k - n + 2)e^\alpha\hat{\alpha}C^{\alpha(k+n)}\hat{\alpha}(k-n+1)$$
$$= D\Omega^{\alpha(k+n-1)}\hat{\alpha}(k-n+2) + \alpha^{++}_{k,n-2}e^\alpha\hat{\alpha}\Omega^{\alpha(k+n-1)\hat{\alpha}(k-n+3)}$$
$$+(k + n - 1)(k + n - 2)e^\alpha\hat{\alpha}\Omega^{\alpha(k+n+1)\hat{\alpha}(k-n+1)}$$
$$+(k + n - 1)\alpha^{+-}_{k,n-2}e^\alpha\hat{\alpha}\Omega^{\alpha(k+n+3)\hat{\alpha}(k-n+3)}.$$  

(80)

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For simplicity we unify the notation defining $\hat{R}^{\alpha(k+m+1)\dot{\alpha}(k-m)} = R^{\alpha(k+m+1)\dot{\alpha}(k-m)}$, $m < n - 2$. Then, we reformulate the theory in terms of the modified curvatures $\hat{R}^{\alpha(k+m+1)\dot{\alpha}(k-m)}$, i.e. without zero-forms and the 1-forms $\Omega^{\alpha(k+m+1)\dot{\alpha}(k-m)}$, $|m| \geq n - 1$. Then we can express the Lagrangian via the modified curvatures as follows:

$$-i\mathcal{L} = -i\mathcal{L}^{(0,n-2)} + \sum_{k=n-1}^{s} \sum_{m=-n+1}^{n-1} (-1)^{k+1} a_{k,m} \hat{R}^{\alpha(k+m+1)\dot{\alpha}(k-m)} \hat{R}_{\alpha(k+m+1)\dot{\alpha}(k-m)}.$$  (81)

The derivation of the unfolded equations is straightforward. For this, we rewrite the gauge sector of unfolded equations via $\hat{R}$ dropping all the decouples curvatures:

$$\hat{R}^{\alpha(k+m+1)\dot{\alpha}(k-m)} = 0, \quad k < s - 1,$$
$$\hat{R}^{\alpha(s+m)\dot{\alpha}(s-1-m)} = 0, \quad m < n - 2,$$
$$\hat{R}^{\alpha(s+n-2)\dot{\alpha}(s-n+1)} = 2E^{(2)}_{\alpha} W^{\alpha(s+n)\dot{\alpha}(s-n+1)}.$$  (82)

The last equation is the only link between the gauge sector and the sector of the gauge invariant zero-forms $W^{\alpha(k+m+1)\dot{\alpha}(k-m)}$, $k > s$, $m \geq n - 1$ and their conjugates. In this, the equations for these gauge invariant zero-forms remains to be the same.

4 Conclusion

In the paper, the gauge invariant description of the massive higher spin bosons and fermions was built in (A)dS$_4$. In both cases, we begin with the construction of the gauge invariant Lagrangian and investigate an unitarity of models obtained, including all possible partially massless and/or infinite spin limits. For both bosons and fermions, a complete set of the gauge invariant curvatures was constructed (introducing all necessary extra fields) and the Lagrangian was expressed via these curvatures. At last, the complete set of the unfolded equations was constructed. Also, the connection with the Skvortsov-Vasiliev formalism was discussed; it was shown that such formalism can be obtained by the partial gauge fixing to get rid of the zero-forms that do not decouple in the partially massless limit. As a byproduct, we obtain the unfolded equations for the Skvortsov-Vasiliev formalism. Moreover, we have shown that the analogous formalism exists for the partially massless fermions as well. The calculations were carried out in the multispinor formalism, which enabled us to simplify the formulae at the cost of the restriction to $d = 4$. Multispinor formalism also treats the bosons and fermions in a similar way, which make the work particularly useful for the supersymmetry studies. The results were already used in the studies of the different higher spin $N = 1$ supermultiplets in $d = 4$.

A Notations and conventions

We work in the frame-like multispinor formalism. It means that all objects are forms with multispinors as their local indices, i.e. $\Phi^{\alpha(k)\dot{\alpha}(k)}$. World indices are omitted everywhere; all expressions are completely antisymmetric on them. We use the condensed index notations.
Namely, if the expression is symmetric on upper/low indices $\alpha_1\alpha_2\cdots\alpha_k$, these indices are denoted with the same letter with the number of indices in parentheses. For example:

$$R^{\alpha_1\alpha_2\cdots\alpha_s} = R^\mu(s).$$

Symmetrization over the set of $n$ indices is defined as the sum of all the $n!$ expressions obtained from the initial one by all the possible permutations of these indices, with the normalization factor $1/n!$. According to the definition of the symmetrization, multiple symmetrization over the same set of indices is equivalent to the unique symmetrization. For example:

$$R^{\alpha(s)} R_{\alpha(s)} a^\alpha a_\alpha = \frac{s-1}{s} R^{\alpha(s-1)\beta} R_{\alpha(s)} a^\alpha a_\beta + \frac{1}{s} R^{\alpha(s)} R_{\alpha(s)} a^\beta a_\beta,$$

$$R^{\alpha(s-1)\beta} R_{\alpha(s)} a^\alpha a_\beta = R^{\alpha(s-1)\beta} R_{\alpha(s-1)\gamma\alpha\beta} a^\gamma a_\beta.$$

The indices are contracted according to the Einstein rule, with the respect to the symmetrization. For example:

$$\epsilon_\alpha A^\alpha B^{(2)} = \frac{1}{3} (\epsilon_\beta A^\beta B^{(2)} + 2 \epsilon_\beta A^\beta B^{(2)}).$$

In four-dimensional space-time, using the Lorenz algebra isomorphism $\mathfrak{so}(3,1) \sim \mathfrak{sl}(2,\mathbb{C})$ one can replace vector index by two spinor indices with values $i = 1, 2$ \[24\]: $T^\mu \sim T^{\alpha\dot{\alpha}}$. The spinor indices are raised and lowered with the antisymmetric tensors $\epsilon_{\alpha\beta}$ ($\epsilon_{\dot{\alpha}\dot{\beta}}$):

$$\epsilon_{\alpha\beta} \xi^\beta = -\xi_\alpha, \quad \epsilon_{\alpha\beta} \xi_\beta = \xi^\alpha,$$

the same is true for dotted indices. Hence, all the symmetric multispinors are automatically traceless. Under the Hermitian conjugation, dotted and undotted indices are transformed one into another. For example:

$$\left(A^{(2)}\right)\dagger = A^{\beta\dot{\alpha}}(2).$$

The mixed symmetry tensor $\Phi^{\mu(k),\nu(l)}$ which corresponds to the two-row Young tableaux $Y(k,l)$ \[25\] in multispinor formalism is described by a pair of multispinors $\Phi^{\alpha(k+l)\dot{\alpha}(k-l)}$, $\Phi^{\alpha(k-l)\dot{\alpha}(k+l)}$. If the tensor $\Phi^{\mu(k),\nu(l)}$ is real then:

$$\left(\Phi^{\alpha(k+l)\dot{\alpha}(k-l)}\right)\dagger = \Phi^{\alpha(k-l)\dot{\alpha}(k+l)}.$$  

Similarly, the mixed symmetry spin-tensor $\Psi^{\mu(k),\nu(l)}$ which corresponds to the Young tableaux $Y(k+1/2,l+1/2)$ is described by a pair of multispinors $\Psi^{\alpha(k+l+1)\dot{\alpha}(k-l)}$, $\Psi^{\alpha(k-l+1)\dot{\alpha}(k+l)}$. If the spin-tensor $\Psi^{\mu(k),\nu(l)}$ is Majorana one then

$$\left(\Psi^{\alpha(k+l+1)\dot{\alpha}(k-l)}\right)\dagger = \Psi^{\alpha(k-l+1)\dot{\alpha}(k+l+1)}.$$  

The fermionic fields are grassmanian, i.e. they anticommute.

The $AdS_4$ space is described by the background Lorentz connections $\omega^{(2)}$, $\omega^{\dot{\alpha}(2)}$, which enter implicitly through the Lorentz covariant derivative $D$, and the background frame $e^{\alpha\dot{\alpha}}$. We also use the basis elements for the two-, thee- and four-forms

$$e^\alpha \sim e^{\alpha\dot{\alpha}}, \quad E^{ab} \sim E^{\alpha(2)}, \quad E^{\dot{\alpha}c} \sim E^{\alpha\dot{\alpha}}, \quad E^{abc} \sim E, \quad E^{abcd} \sim E,$$  

26
defined as follows:

\[ e^{\alpha \hat{\alpha}} \wedge e^{\beta \hat{\beta}} = \varepsilon^{\alpha \beta} E^{\hat{\alpha} \hat{\beta}} + \varepsilon^{\hat{\alpha} \hat{\beta}} E^{\alpha \beta}, \]

\[ E^{\alpha(2)} \wedge e^{\beta \hat{\beta}} = \varepsilon^{\alpha \beta} E^{\alpha \hat{\beta}}, \]

\[ E^{\alpha \hat{\alpha}} \wedge e^{\beta \hat{\beta}} = \varepsilon^{\alpha \beta} \varepsilon^{\hat{\alpha} \hat{\beta}} E. \]  

(87)

The hermitian conjugation rules for the basis forms are:

\[ (e^{\alpha \hat{\alpha}})\dagger = e^{\alpha \hat{\alpha}}, \quad (E^{\alpha(2)})\dagger = E^{\hat{\alpha}(2)}, \quad (E^{\alpha \hat{\alpha}})\dagger = -E^{\alpha \hat{\alpha}}, \quad (E)\dagger = -E. \]  

(88)

The Lorentz covariant derivative is normalized so that

\[ D \Lambda D\Phi^{\alpha(k)\hat{\alpha}(l)} = -2\lambda^2 [(k + m) E^{\alpha \beta} \Phi^{\alpha(k-1)\beta} + (k - m) E^{\alpha \hat{\beta}} \Phi^{\alpha(k-1)\hat{\beta}}]. \]  

(89)

The parameter \( \lambda^2 \) is proportional to the curvature of the space-time. The AdS space has \( \lambda^2 > 0 \), while the dS space has \( \lambda^2 < 0 \). The case of \( \lambda^2 = 0 \) corresponds to the flat Minkowski space.

In the main text all the wedge product signs \( \wedge \) are omitted.

## B Relations for gauge invariant curvatures

Firstly, consider the following problem. A set of objects \( B^{\alpha(k+m)\hat{\alpha}(k-m)} \) is given. Each object has the form:

\[ B^{\alpha(k+m)\hat{\alpha}(k-m)} = DA^{\alpha(k+m)\hat{\alpha}(k-m)} + (k + m)(k - m) \alpha^{++}_{k,m} e^{\alpha \hat{\alpha}} A^{\alpha(k+m-1)\hat{\alpha}(k-m-1)} \]

\[ + \alpha^{-+}_{k,m} e^{\alpha \hat{\alpha}} A^{\alpha(k+m+1)\hat{\alpha}(k-m+1)} + (k + m) \alpha^{--}_{k,m} e^{\alpha \hat{\alpha}} A^{\alpha(k+m-1)\hat{\alpha}(k-m+1)} \]

\[ + (k - m) \alpha^{+-}_{k,m} e^{\alpha \hat{\alpha}} A^{\alpha(k+m+1)\hat{\alpha}(k-m-1)}, \]  

(90)

and for each \( k, m \) the following relation holds:

\[ 0 = DB^{\alpha(k+m)\hat{\alpha}(k-m)} + (k + m)(k - m) \beta^{++}_{k,m} e^{\alpha \hat{\alpha}} B^{\alpha(k+m-1)\hat{\alpha}(k-m-1)} \]

\[ + \beta^{-+}_{k,m} e^{\alpha \hat{\alpha}} B^{\alpha(k+m+1)\hat{\alpha}(k-m+1)} + (k + m) \beta^{--}_{k,m} e^{\alpha \hat{\alpha}} B^{\alpha(k+m-1)\hat{\alpha}(k-m+1)} \]

\[ + (k - m) \beta^{+-}_{k,m} e^{\alpha \hat{\alpha}} B^{\alpha(k+m+1)\hat{\alpha}(k-m-1)}. \]  

(91)

Then one has to determine the coefficients \( \alpha^{ij}_{k,m}, \beta^{ij}_{k,m} \). Such problem arise three times for bosons and, similarly, three times for fermions. Namely, the calculation of the right coefficients in expressions for 2-curvatures, the derivation of the linear relations for 2-curvatures and the derivation of the unfolded equations can be reduced to the problem stated above, with additional restrictions on the coefficients (for example, the normalization choice \( \alpha^{+ -}_{k,m} = 1 \) or the hermiticity condition \( \alpha^{+ -}_{k,m} = \alpha^{- +}_{k,m} \)). It is thus important to solve this problem once in the general case. It immediately follows that \( \alpha^{ij}_{k,m} = \beta^{ij}_{k,m} \). Then, the following recurrent relations for \( \alpha^{ij}_{k,m} \) hold:

\[ (k - m)[\alpha^{++}_{k,m} \alpha^{++}_{k+1,m} + \alpha^{-+}_{k,m+1} \alpha^{+ -}_{k,m}] + 2\lambda^2 = (k - m + 2)[\alpha^{--}_{k+1,m} \alpha^{++}_{k,m} + \alpha^{+-}_{k,m-1} \alpha^{+ -}_{k,m}], \]  

(92)
\[(k + m)[α^{-}_{k,m}α^{++}_{k-1,m} + α^{−+}_{k,m}α^{−+}_{k,m-1}] + 2λ^2 = (k + m + 2)[α^{-}_{k+1,m}α^{++}_{k,m} + α^{−+}_{k,m+1}α^{++}_{k,m}],\]

\[(k + m + 2)α^{++}_{k+1,m}α^{++}_{k,m} = (k + m)α^{++}_{k,m+1}α^{++}_{k,m-1},\]

\[(k - m)α^{−+}_{k+1,m}α^{−+}_{k,m} = (k - m)α^{−+}_{k,m+1}α^{−+}_{k,m},\]

\[(k + m)α^{-}_{k,m}α^{-}_{k-1,m} = (k + m + 2)α^{-}_{k,m+1}α^{-}_{k,m},\]

The coefficients $α^{ij}_{k,m}$ satisfy those relations iff:

\[
α^{++}_{k,m}α^{−+}_{k,m+1} = \frac{A_m}{(k - m)(k - m + 1)(k + m + 1)(k + m + 2)},
\]

\[
α^{−−}_{k+1,m}α^{++}_{k,m} = \frac{A_{k+1}}{(k - m + 1)(k - m + 2)(k + m + 1)(k + m + 2)},
\]

\[
A_m = C_1 + C_2(m + 1)m + (m + 1)^2m^2λ^2.
\]

Note that the fermionic coefficients $α^{ij}_{k,m}$ are redefined as $α^{ij}_{k-1/2,m-1/2}$ in the main text. The normalization of the coefficients $α^{ij}_{k,m}$ and the constants $C_1, C_2$ are determined from the additional restrictions. Also note that the given expressions are not applicable in case of $m = ±k$. In particular, this explains why the expressions for $R^{α(2k)}$ significantly differ from the general case $R^{α(k+m)α(k−m)}$.

Now let us discuss the relations between curvatures. In case of the free field, each curvature is linear on the fields and has the form:

\[
R^A = DW^A + F^A(W^B), \quad F^A(W^B) = \sum_{B ∈ B(A)} f^A_B W^B,
\]

The exterior derivative of the curvature $R^A$ hence can be expressed in terms of curvatures which contain the derivatives of the fields $W^B, B ∈ B(A)$:

\[
DR^A = \sum_{B ∈ B(A)} (-1)^{deg f^A_B} f^A_B R(B) + G^A(W^B).
\]

Here $G^A(W^B)$ does not contain the exterior derivatives of the fields. The factor $(-1)^{deg f^A_B}$ is due to anticommutativity of the exterior product and the exterior derivative. The gauge invariance implies $G^A(W^B) ≡ 0$, which means that the derivative of the curvature is expressed via other curvatures. Indeed, one can check by straightforward calculation that:

\[
0 = DR^α(k+m)α(−)α(k−m) + (k + m)(k − m)α^{++}_{k,m}ε^{α}\hat{α} R^{α(k+m−1)α(k−m−1)} + α^{−+}_{k,m}ε^{α}\hat{α} R^{α(k+m+1)α(k−m+1)} + (k + m)α^{-}_{k,m}ε^{α}\hat{α} R^{α(k+m−1)α(k−m+1)} + (k - m)α^{−−}_{k,m}ε^{α}\hat{α} R^{α(k+m+1)α(k−m−1)}
\]

\[
0 = DR^α(2k) + ε^{α}\hat{α} R^{α(2k−1)α} + 2kα^{−+}_{k,k} ε^{α}\hat{α} R^{α(2k−2)α} + 4k(2k - 1)α^{−+}_{k,k} α^{−−}_{k,k} E^{α(2k−2)} + α^{−−}_{k,k} E^{α(2k−2)} + α^{−−}_{k,k} E^{α(2k−2)},
\]

\[
0 = DC^α(k+m)α(k−m) + R^{α(k+m)α(k−m)} + (k + m)(k − m)α^{−−}_{k,m}ε^{α}\hat{α} R^{α(k+m−1)α(k−m−1)} + α^{−+}_{k,m}ε^{α}\hat{α} R^{α(k+m+1)α(k−m+1)} + (k + m)α^{−−}_{k,m}ε^{α}\hat{α} R^{α(k+m+1)α(k−m−1)} + (k - m)α^{−−}_{k,m}ε^{α}\hat{α} R^{α(k+m−1)α(k−m+1)}.
\]
Here, in the last equality one has to omit the terms with the factor $(k \mp m)$ in case of $m = \pm k$. For the lowest bosonic curvatures, the expressions are slightly different:

\[
0 = DR^{(2)} + 2\beta e^\alpha \dot{a} R^{(2)} \dot{a} + 3\mu_2 e_{\alpha \dot{a}} R^{(3)} \dot{a} + \mu_1^1 2 E^\alpha \beta C^{\alpha \beta} + 2\beta_2 \mu_1 E^{(2)} \gamma C - 6\mu_2 E_{(2)}^{\alpha} C^{(4)},
\]
\[
0 = DC^{\alpha \dot{a}} + (k + m)\alpha_{(k-m)} C^{\alpha \dot{a}}\gamma_{(k-m)} + 3\mu_2 e_{\alpha \dot{a}} C^{(3)} \dot{a},
\]
\[
0 = DR + \mu_1 e_{\alpha \dot{a}} R^{(2)} \dot{a} + 2\mu_1 E_{(2)}^{\alpha} C^{(2)} + 2\mu_1 E_{(2)}^{\dot{a}} \gamma^{(2)},
\]
\[
0 = DC + R + \mu_1 e_{\alpha \dot{a}} C^{(2)}.
\]

It is possible therefore to obtain the following relations for the derivative of the product of the two curvatures:

\[
-D(R^{(k+m)}(k-m) \alpha_{(k-m)} C_{(k+m)}) = R^{(k-m)}(k-m) \alpha_{(k-m)} R_{(k+m)} C_{(k-m)}
\]
\[
+R_{(k-m)}(k-m) \alpha_{(k-m)} e_{\alpha \dot{a}} C^{(2)}(k-m) + 2\mu_1 e_{\alpha \dot{a}} C^{(3)} \dot{a},
\]
\[
-(k+m)(k-m) \alpha_{(k-m)} e_{\alpha \dot{a}} C^{(2)}(k-m) + (k+m)(k-m) \alpha_{(k-m)} e_{\alpha \dot{a}} C^{(3)} \dot{a},
\]
\[
+R_{(k-m)}(k-m) \alpha_{(k-m)} e_{\alpha \dot{a}} C^{(2)}(k-m) + 2\mu_1 e_{\alpha \dot{a}} C^{(3)} \dot{a},
\]
\[
0 = DC + R + \mu_1 e_{\alpha \dot{a}} C^{(2)}.
\]
\[-D(C^{\alpha(k+m)}\dot{\alpha}(k-m)e^{\alpha\dot{\alpha}}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1)) =
R^{\alpha(k+m)}\dot{\alpha}(k-m)e^{\alpha\dot{\alpha}}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1) - R^{\alpha(k+m+1)}\dot{\alpha}(k-m)e^{\alpha\dot{\alpha}}C_{\alpha(k+m)}\dot{\alpha}(k-m)\]
\[-\alpha_{k,m}^{++}[C^{\alpha(k+m)}\dot{\alpha}(k-m+1)E^{\alpha\beta}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1)\]
\[+C^{\alpha(k+m+1)}\dot{\alpha}(k-m)E^{\alpha\beta}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1)\]
\[-(k + m)\alpha_{k,m}^{--}C^{\alpha(k+m-1)}\dot{\alpha}(k-m)E^{\alpha(2)}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1)\]
\[+(k + m + 2)\alpha_{k+1,m}^{--}C^{\alpha(k+m)}\dot{\alpha}(k-m)E^{\alpha(2)}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+2)\]
\[+(k - m)\alpha_{k,m}^{++}C^{\alpha(k+m+1)}\dot{\alpha}(k-m+1)E^{\alpha(2)}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1)\]
\[+(k + m + 2)\alpha_{k+1,m}^{--}C^{\alpha(k+m)}\dot{\alpha}(k-m)E^{\alpha\beta}C_{\alpha(k+m-1)}\dot{\alpha}(k-m)\]
\[-(k + m + 2)(k - m)\alpha_{k+1,m}^{--}C^{\alpha(k+m)}\dot{\alpha}(k-m)E^{\alpha\beta}C_{\alpha(k+m+1)}\dot{\alpha}(k-m+1). \tag{101}\]

Those relations determine the arbitrariness of the coefficients in the Lagrangian. Namely, the explicit expressions for their shifts are:

\[
\pm a_{k,\pm m} = \pm a_{k,\pm m}^{(0)} + p_{k,m},
\]
\[
\pm b_{k,\pm m} = \pm b_{k,\pm m}^{(0)} + p_{k,m}\alpha_{k,m}^{++} - p_{k+1,m}(k + m + 1)(k - m + 1)\alpha_{k+1,m}^{--} + r_{k,m},
\]
\[
\mp c_{k+1,\pm m} = \mp c_{k+1,\pm m}^{(0)} + p_{k,m}\alpha_{k,m}^{++} - p_{k+1,m}(k + m + 1)(k - m + 1)\alpha_{k+1,m}^{--} + r_{k,m},
\]
\[
d_{k,m} = d_{k,m}^{(0)} - (k - m)p_{k,m} + (k + m + 1)\alpha_{k,m+1}^{++}p_{k,m+1} + q_{k,m},
\]
\[
d_{k,-1,m} = d_{k,-1,m}^{(0)} - (k - m)p_{k,m} + (k + m + 1)\alpha_{k,m}^{--}p_{k,m} + q_{k,m},
\]
\[
e_{k,m} = e_{k,m}^{(0)} + (k + m)\alpha_{k,m+1}^{--}q_{k,m} + (k - m + 2)q_{k,m},
\]
\[\quad -(k - m + 2)(k + m)\alpha_{k+1,m}^{--}r_{k,m} + \alpha_{k-1,m}^{++}r_{k,m},
\]
\[
e_{k,0} = e_{k,0}^{(0)} + k\alpha_{k,1}^{--}q_{k,0} - k(k + 2)\alpha_{k+1,0}^{--}r_{k,0} + \alpha_{k-1,0}^{++}r_{k-1,0},
\]
\[
e_{k,-m} = e_{k,-m}^{(0)} + (k + m + 2)\alpha_{k,m+1}^{++}q_{k,m} + (k - m)q_{k,m-1},
\]
\[\quad +(k + m + 2)(k - m)\alpha_{k+1,m}^{--}r_{k,m} - \alpha_{k-1,m}^{++}r_{k-1,m},
\]
\[
e_{1,0} = e_{1,0}^{(0)} + 4\beta_{2}q_{1,0} - 3\alpha_{2,0}^{--}r_{1,0},
\]
\[
e_{k,k} = e_{k,k}^{(0)} + \frac{\alpha_{k-1}^{--}\alpha_{k}^{--}}{k + 1}p_{k,k} + 2q_{k,k-1} - 4k\alpha_{k+1,k}^{--}r_{k,k},
\]
\[
e_{1,1} = e_{1,1}^{(0)} + \mu_{1}^{2}p_{1,1} + 2q_{1,0} - 4\alpha_{2,1}^{--}r_{1,1},
\]
\[
f_{k,m} = f_{k,m}^{(0)} - (k + m + 1)(k - m + 2)\alpha_{k+1,m}^{--}q_{k+1,m} + \alpha_{k,m+1}^{++}q_{k,m},
\]
\[\quad -(k + m + 1)\alpha_{k,m+1}^{++}r_{k,m+1} + (k - m + 2)r_{k,m},
\]
\[
f_{k,-m} = f_{k,-m}^{(0)} + (k + m + 2)(k - m + 1)\alpha_{k+1,m}^{--}q_{k+1,m} - \alpha_{k,m-1}^{++}q_{k,m-1},
\]
\[\quad -(k + m + 2)\alpha_{k+1,m}^{++}r_{k,m} + (k - m + 1)r_{k,m-1},
\]
\[
f_{k,k} = f_{k,k}^{(0)} - 2\alpha_{k+k}^{++}p_{k,k} - 4(k + 1)(2k + 1)\alpha_{k+1,k+1}^{--}\alpha_{k,k+1}^{--}p_{k,k+1},
\]
\[\quad -2(2k + 1)\alpha_{k+1,k}^{--}q_{k+1,k} + 2r_{k,k},
\]
\[
f_{0,0} = f_{0,0}^{(0)} - \mu_{1}q_{1,0} - 2\beta_{2}\mu_{1}p_{1,1} + 2\mu_{1}p_{0,0} + 2r_{0,0}.
\]
The values of the non-zero shift parameters $p_{k,m}, q_{k,m}$ (all $r_{k,m} = 0$) which gives the solution for the partially massless bosonic cases:

$$p_{k,m} = -\frac{(k-1)!(k+m+1)!k!}{(k-m)!^2(k-m+1)!\prod_{i=1}^{m}\alpha_i^{-}}, \quad m \geq n,$$

$$q_{k,n-1} = -\frac{(k-1)!(k+n+1)!k!}{(k-n)!(k-n+1)!(k-n+2)!\prod_{i=1}^{n-1}\alpha_i^{--}}, \quad \text{for } n \geq 2.$$ (103)

The explicit expressions for the shifts of the Lagrangian parameters for the fermionic case:

$$a_{k,\pm m} = a_{k,m}^{(0)} + p_{k,m};$$
$$b_{k,\pm m} = b_{k,m}^{(0)} + p_{k,m}\alpha^{++} + p_{k+1,m}(k + m + 2)(k - m + 1)\alpha^{--}_{k+1,m} + r_{k,m};$$
$$c_{k+1,\pm m} = c_{k,m}^{(0)} - p_{k,m}\alpha^{++} + p_{k+1,m}(k + m + 2)(k - m + 1)\alpha^{--}_{k+1,m} - r_{k,m};$$
$$d_{k,\pm m} = d_{k,m}^{(0)} - (k - m)p_{k,m} + (k + m + 2)\alpha^{++}_{k,m+1}p_{k,m+1} + q_{k,m};$$
$$d_{k,-1} = d_{k,-1}^{0} + 2q_{k,-1};$$
$$d_{k,-m} = d_{k,-m}^{0} + (k + m + 2)p_{k,m-2} - (k + m)\alpha^{--}_{k,m-1}p_{k,m-1} + q_{k,m-2};$$
$$e_{k,\pm m} = e_{k,m}^{(0)} + (k + m + 1)\alpha^{--}_{k,m+1}q_{k,m} + (k - m + 2)q_{k,m-1},$$
$$-\alpha^{--}_{k+1,m}r_{k,m} + \alpha^{--}_{k+1,m}r_{k,m-1},$$
$$e_{k,-m} = e_{k,-m}^{(0)} - (k + m + 2)\alpha^{--}_{k,m}q_{k,m-1} - (k - m + 1)q_{k,m-2},$$
$$+(k + m + 3)(k + m)\alpha^{--}_{k+1,m-1}r_{k,m-1} - \alpha^{--}_{k+1,m-1}r_{k,m-1},$$
$$e_{k,k} = e_{k,k}^{(0)} + \frac{2\alpha^{++}_{k-1}\alpha^{--}_{k}}{2k + 3}p_{k,k} + 2q_{k,k-1} - 2(2k + 1)\alpha^{--}_{k+1,k}r_{k,k};$$
$$f_{k,\pm m} = f_{k,m}^{(0)} - (k + m + 2)(k - m + 2)\alpha^{--}_{k,m+1}q_{k,m+1} + \alpha^{++}_{k,m+1}q_{k,m},$$
$$-\alpha^{--}_{k+1,m+1}r_{k,m+1} + (k - m + 2)r_{k,m};$$
$$f_{k,-m} = f_{k,-m}^{(0)} + (k + m + 1)(k - m + 3)\alpha^{--}_{k,m}q_{k,m+1} - \alpha^{++}_{k,m}q_{k,m-1},$$
$$+(k + m)\alpha^{--}_{k,m}r_{k,m} - (k - m + 1)r_{k,m-1};$$
$$f_{k,k} = f_{k,k}^{(0)} - 2\alpha^{++}_{k,k}p_{k,k} - 4(2k + 3)\alpha^{--}_{k+1,k+1}p_{k+1,k+1},$$
$$-4(2k + 4)\alpha^{--}_{k+1,k+1}p_{k+1,k+1} + 2r_{k,k}.$$ (104)

The values of the non-zero shift parameters $p_{k,m}, q_{k,m}$ (all $r_{k,m} = 0$) which gives the solution for the partially massless fermionic cases:

$$p_{k,m} = -\frac{(k + m + 2)!k!}{4(k-m)!^2(k-m+1)!\prod_{i=0}^{m}\alpha_i^{--}}, \quad m \geq n,$$

$$q_{k,n-1} = -\frac{(k + n + 1)!k!}{4(k-n)!(k-n+1)!(k-n+2)!\prod_{i=0}^{n-1}\alpha_i^{--}}, \quad \text{for } n \geq 2.$$ (105)

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