A NONSMOOTH VARIATIONAL APPROACH TO DIFFERENTIAL PROBLEMS. A CASE STUDY OF NONRESONANCE UNDER THE FIRST EIGENVALUE FOR A STRONGLY NONLINEAR ELLIPTIC PROBLEM

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Abstract. We adapt a technique of nonsmooth critical point theory developed by Degiovanni-Zani for a semilinear problem involving the Laplacian to the case of the $p$-Laplacian. We suppose only coercivity conditions on the potential and impose no growth condition of the nonlinearity. The coercivity is obtained using similar nonresonance conditions to [1] and to [12] in two different results and using some comparison functions and comparison spaces in a third one. it is also shown that neither of the three theorems implies the two others.

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1. Introduction

The new nonsmooth trends in critical point theory dealing with continuous functionals on metric spaces [1, 2, 3, 4, 5] brought important developments, in particular in the abstract part of the theory [6, 7, 8, 9, 10, 11]. The reason is that the definition of the weak slope they use to measure regularity relies on a qualitative topological property on level sets which avoid us the technicalities of using pseudo-gradients to construct deformations. Hence, it is particularly useful to prove abstract critical point theorems. While considering applications to boundary value problems would be, a priori, more delicate. This is natural because we would need some appropriate calculus and a new sense for weak solutions, weaker than the traditional one, strong enough to be useful and still with a practical relevance. Many early applications of the nonsmooth theory to several problems in partial differential equations and variational inequalities where the functionals involved are not locally Lipschitz continuous were given [12, 13, 14, 15, 16, 17, 18, 19, 20] but this passes in general through a great deal of technicality. Recently, a specific subdifferential calculus has been developed by Campa and degiovanni [19, 21] when the underlying space is normed and has been used successfully in applications where the Clarke’s subdifferential is in general inappropriate [21, 22, 23, 24, 25, 26, 27, 28, 29, 30].
In the present paper, we use a general approach that combines the basic ideas of variational methods with these new developments and study non-resonance under the first eigenvalue for a strongly nonlinear elliptic problem involving the \( p \)-Laplacian operator where no growth condition is supposed. The strategy followed can be described by the following scheme:

1. Give an acceptable definition of a weak solution to the problem to be treated by changing the traditional space \( C_0^\infty(\Omega) \) of test functions by another dense subspace of \( W^{1,p}_0(\Omega) \).
2. Show that a critical point is a weak solution in the sense of the previous point with the help of a minimum of assumptions.
3. Prove the effective existence of a critical point for the nonsmooth theory by one of the available abstract theorems for our specific nonlinear problem.

The first two points are general and may be used with different boundary value problems. We could trace their origin to the papers [31, 32]. But, they are more visible in [22] where the authors have successfully tried a nonsmooth variational approach for a coercive semilinear problem with the Laplacian operator. Nevertheless, these are general and may be used for different boundary value problems with other geometries as in the situation of linking results.

2. Situating the problem

We recall the definition of regularity in the sense of Degiovanni-Marzocchi [2] and Katriel [3] and the key properties of the subdifferential calculus developed for by Campa-Degiovanni [21, 19] we will use in the sequel.

Let \((E,d)\) be a metric space, \( f: E \to \mathbb{R} \) a function and denote the ball of center \( u \) and radius \( \sigma \) by \( B(u,\sigma) \) and the epigraph of \( f \) by

\[
\text{epi}(f) = \{(u,\lambda) \in E \times \mathbb{R}; f(u) \leq \lambda\}.
\]

Consider \( \text{epi}(f) \) as a subspace of the metric space \( E \times \mathbb{R} \) for the metric

\[
\text{dist}((u,\lambda),(u',\lambda')) = \left( \text{dist}(u,u')^2 + (\lambda' - \lambda)^2 \right)^{1/2}.
\]

**Definition 1.** A point \( u \in E \) such that \( f(u) \in \mathbb{R} \) is \( \delta \)-regular for a \( \delta > 0 \), if there exists \( \sigma > 0 \) and a continuous deformation \( \nu: (B_\sigma(u,f(u)) \cap \text{epi}(f)) \times [0,\sigma] \to E \) such that for all \((w,\lambda) \in B(u,f(u)) \cap \text{epi}(f) \) and all \( t \in [0,\sigma] \):

\[
\text{dist}(\nu((u,\mu),t),u) \leq t \quad \text{and} \quad f(\nu((w,t),t)) \leq \mu - \delta t.
\]

The point \( u \) is regular if there exists \( \delta > 0 \) such that \( u \) is \( \delta \)-regular and it is critical if it is not regular.

When \( f \) is continuous, this definition may be simplified to the following form.

**Definition 2.** Let \( f: E \to \mathbb{R} \) be a continuous function. A point \( u \in E \) is \( \delta \)-regular if there exists a continuous deformation \( \eta: [0,\sigma] \times B(u,\sigma) \to E \)
such that for all \((t, u) \in [0, \sigma] \times B(u, \sigma)\):
\[
\text{dist} (\eta(t, u), u) \leq t \quad \text{and} \quad f(u) - f(\eta(t, u)) \geq \delta t.
\]

The regularity of \(u\) is evaluated by a generalized notion of the norm of the derivative called by Degiovanni-Marzocchi and Katriel respectively weak slope and regularity constant of \(f\) at \(u\):
\[
|df|(u) = \sup \{\delta > 0; \ u \text{ is } \delta - \text{regular}\},
\]
which reduces to \(||f'(u)||\) when \(f\) is of class \(C^1\).

When \(u\) is a local minimum of \(f\), it is a critical point, i.e.,
\[
|df|(u) = 0.
\]

And when \(f\) satisfies the geometric conditions of the mountain pass theorem of Ambrosetti-Rabinowitz [33], and an appropriate form of the Palais-Smale condition, it has a critical point whose value is characterized by the usual inf max argument (cf. [2, 3]).

When \(X\) is a normed space.

**Definition 3.** For any \(u \in X\) such that \(f(u) \in \mathbb{R}, v \in X\) and \(\varepsilon > 0\), we define \(f^0(u; v)\) as the infimum of \(r \in \mathbb{R}\) such that there exists \(\delta > 0\) and a continuous mapping
\[
\nu: (B_\delta(u, f(u)) \cap \text{epi}(f)) \times ]0, \delta[ \to B_\varepsilon(v)
\]
satisfying
\[
f(w + t\nu((w, \mu), t)) \leq \mu + rt
\]
for all \((w, \mu) \in B_\delta(u, f(u)) \cap \text{epi}(f)\) and \(t \in ]0, \delta[\).

Define also
\[
f^0(u; v) = \sup_{\varepsilon > 0} f^0(\varepsilon; u; v).
\]

The function \(f^0(u; \cdot)\) is convex, lower semicontinuous (l.s.c.) and positively homogeneous of degree 1 (cf. [21]).

**Definition 4.** For all \(u \in X\) such that \(f(x) \in \mathbb{R}\), we set
\[
\partial f(u) = \{\alpha \in X^*; \ \langle \alpha, v \rangle \leq f^0(u; v) \ \forall v \in X\}.
\]

The real \(f^0(u; v)\) is larger than the Clarke-Rockafellar generalized directional derivative. Hence, \(\partial f(u)\) contains the Clarke’s subdifferential of \(f\) at \(u\) which may be empty when \(\partial f(u)\) is not, as we can see for example for \(f(u) = u - \sqrt{|u|}\) at \(u = 0\) (there \(|df|(0) = 0\). Nevertheless they agree when \(f\) is locally Lipschitz. The notions \(f^0(u; v)\) and \(\partial f(u)\) have been introduced in [21], [19] and are more adapted to use with the weak slope of Degiovanni-Marzocchi.

**Proposition 1.** If \(u \in X\) is such that \(f(u) \in \mathbb{R}\), then
(a) \(|df|(u) < +\infty \iff \partial f(u) \neq \emptyset\).
(b) \(|df|(u) < +\infty \Rightarrow |df|(u) \geq \min\{||\alpha||; \ \alpha \in \partial f(u)\}.

**Remark 1.** Examples where (b) is strict exist.
Proposition 2. When \( u \in X \) is such that \( f(u) \in \mathbb{R} \) and \( g: X \to \mathbb{R} \) is Lipschitz continuous, then

\[
\partial(f + g)(u) \subset \partial(f)(u) + \partial(g)(u).
\]

If moreover, \( g \) is of class \( \mathcal{C}^1 \), we have equality and \( \partial g(u) = \{ g'(u) \} \).

To illustrate the abstract approach, we will study nonresonance under the first eigenvalue for a strongly nonlinear elliptic problem for the \( p \)-Laplacian operator as in [34]. The former reference contains improvements to some results in [31] adapting the contents of [32] to the case of the \( p \)-Laplacian. Both results might be adapted to a nonsmooth variational context. Degiovanni-Zani adapted their results of [32] in [22]. We will do the same, by analogy to [22] to the results of [34] extending [22] to the case where \( p \) may be different from 2.

The usual norm of \( W^{1,p}_0(\Omega) \) will be denoted \( ||.||_{1,p} \), the Lebesgue measure of a set \( A \) is \( |A| \) while \( B(x, \rho) = B_\rho(x) \) is the open ball of center \( x \) and radius \( \rho \). The notation \( m' \) stands for the conjugate of \( m \), i.e. \( \frac{1}{m} + \frac{1}{m'} = 1 \).

The problem to be treated is:

\[
(P) \quad \left\{ \begin{array}{ll}
-\Delta_p u &= f(x,u) + h \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{array} \right.
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( \Delta_p: W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega) \) is the \( p \)-Laplacian operator defined by

\[
-\Delta_p u \equiv \text{div} \left( |\nabla u|^{p-2}\nabla u \right), \quad 1 < p < \infty.
\]

The \( p \)-Laplacian is a degenerated quasilinear elliptic operator that reduces to the classical Laplacian if \( p = 2 \). The notation \( \langle ., . \rangle \) stands hereafter for the duality pairing between \( W^{-1,p'}(\Omega) \) and \( W^{1,p}_0(\Omega) \). While \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function and \( h \in W^{-1,p'}(\Omega) \).

Consider the energy functional \( \Phi: W^{1,p}_0(\Omega) \to \mathbb{R} \) associated to the problem

\[
\Phi(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u) \, dx - \langle h,u \rangle,
\]

where \( F(x,s) = \int_0^s f(x,t) \, dt \). We are interested in conditions to be imposed on the nonlinearity \( f \) in order that problem \((P)\) admits at least one solution \( u(x) \) for any given \( h(x) \). Such conditions are usually called nonresonance conditions.

When the nonlinearity satisfies a growth condition of the type:

\[
|f(x,s)| \leq a|s|^{q-1} + b(x) \quad \text{for all } s \in \mathbb{R}, \text{ and a.e. in } \Omega, \tag{2.1}
\]

with \( q < p^* \) where the Sobolev exponent \( p^* = \frac{Np}{N-p} \) when \( p < N \) and \( p^* = +\infty \) when \( p \geq N \) and \( b(x) \in L^{(p^*)'}(\Omega) \), the functional \( \Phi \) is well defined, of class \( \mathcal{C}^1 \), l.s.c. and its critical points are weak solutions of \((P)\) in the usual sense.
But when this growth condition is not satisfied, $\Phi$ is not necessarily of class $C^1$ on $W^{1,p}_0(\Omega)$ and may take infinite values.

The first eigenvalue of the $p$-Laplacian characterized by the variational formulation
\[ \lambda_1 = \lambda_1(-\Delta_p) = \min \left\{ \int_{\Omega} |\nabla u|^p \, dx \mid u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\} \]
is known (cf. [35] for example) to be associated to a simple eigenfunction that does not change sign. We will denote by $\varphi_1$ the normalized eigenfunction such that $\varphi_1 > 0$ almost everywhere.

A procedure used to treat $(P)$ when the nonlinearity lies asymptotically on the left of $\lambda_1$ consists in supposing a “coercivity” condition on $F$ of the type:
\[ \limsup_{s \to \pm\infty} \frac{pF(x,s)}{|s|^p} < \lambda_1 \quad \text{for almost every } x \in \Omega \]
and minimizing $\Phi$ on $W^{1,p}_0(\Omega)$. The minimum being a weak solution of $(P)$ in an appropriate sense (cf. [31, 32, 22, 34]).

An other way, is to obtain a priori estimates on the solutions of some equations approximating $(P)$ and to show that their weak limit is indeed a weak solution.

Notice that with the help of the conditions (2.1) and (2.2), we know since the work of Hammerstein (1930) that $(P)$ admits a weak solution that minimizes the functional $\Phi$ on $W^{1,p}_0(\Omega)$.

The condition (2.2) does not imply a growth condition on $f$ unless $f(x,u)$ is convex in $u$ (see for example [36, 37]).

In [31], Anane and Gossez supposed only a one-sided growth condition with respect to the Sobolev (conjugate) exponent that do not suffice to guarantee the differentiability of $\Phi$, which may even take infinite values. Nevertheless, they showed that any minimum of $\Phi$ solves $(P)$ in a suitable sense.

Then, Degiovanni-Zani [32] in the case $p = 2$ and Chakrone in [34] for $1 < p < \infty$ supposed only that $f$ maps $L^\infty(\Omega)$ to $L^1(\Omega)$:
\[ \sup_{|s| \leq R} |f(.,s)| \in L^1_{loc}(\Omega), \quad \forall R > 0 \]
and a coercivity condition of the type (2.2). They proved that any minimum $u$ of $\Phi$, which is not of class $C^1$ on $W^{1,p}_0(\Omega)$ and may take infinite values too, is a weak solution of $(P)$ in the sense
\[ \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v \, dx = \int_{\Omega} f(x,u)v \, dx + \langle h, v \rangle, \]
for $v$ in a dense subspace of $W^{1,p}_0(\Omega)$. Recently, in 1998 Degiovanni and Zani adapted their technique of [32] for the Laplacian operator ($p = 2$) to a nonsmooth analytical context in [22]. We will do here the same with the results of [34] for the $p$-Laplacian ($1 < p < \infty$).
In the autonomous case \( f(x,s) = f(s) \), De Figueiredo and Gossez have proved the existence of solutions for any \( h \in L^\infty(\Omega) \) by a topological method. They supposed only a coercivity condition and established that
\[
\int_{\Omega} |\nabla u|^p - 2\nabla u \nabla v dx = \int_{\Omega} f(x,u)v dx + \langle h, v \rangle
\]
for all \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \cup \{ u \} \) but the solution obtained may not minimize \( \Phi \). Indeed, an example is given in [38] in the case \( p = 2 \) and another one is given in [34], where \( p \) may be different from 2.

Notice that in our case, the condition (2.3) implies no growth condition on \( f \) as it may be seen in the following example from [34].

**Example 1.** Consider the function
\[
f(x,s) = \begin{cases} 
    d(x) \left( \sin \left( \frac{\pi s}{2} \right) - \text{sign} (s) \right) \exp \left( \frac{2 \cos \left( \frac{\pi s}{2} \right)}{\pi} + \frac{|s| - 1}{2} \right) & \text{if } |s| \geq 1 \\
    d(x) \frac{s^2}{2} (10s^2 - 9) & \text{if } |s| \leq 1
\end{cases}
\]
where \( d(x) \in L^1_{\text{loc}}(\Omega) \) and \( d(x) \geq 0 \) almost everywhere in \( \Omega \), so that
\[
F(x,s) = \begin{cases} 
    -d(x) \exp \left( \frac{2 \cos \left( \frac{\pi s}{2} \right)}{\pi} \right) \exp \left( \frac{|s| - 1}{2} \right) & \text{if } |s| \geq 1 \\
    -d(x) \frac{s^2}{4} (-5s^2 + 9) & \text{if } |s| \leq 1
\end{cases}
\]
Then, \( F(x,s) \leq 0 \) for all \( s \in \mathbb{R} \) almost everywhere in \( \Omega \). So, \( \Phi \) is coercive. Nevertheless, as we can check easily, \( f \) satisfies no growth condition.

3. **Theoretical Approach**

We will show that when (2.3) is fulfilled, a critical point in the sense of Degiovanni-Marzocchi, is a weak solution of (P) in an acceptable sense.

**Definition 5.** For \( 1 \leq p \leq \infty \), the space \( L^p_0(\Omega) \) is defined by:
\[
L^p_0 = \{ v \in L^p_0; v(x) = 0 \text{ a.e. outside a compact subset of } \Omega \}. 
\]

Consider \( u \in W^{1,p}_0(\Omega) \), we set
\[
V_u = \{ v \in W^{1,p}_{\text{loc}}; u \in L^\infty(\{ x \in \Omega; v(x) \neq 0 \}) \}.
\]

**Proposition 3 (Brezis-Browder [39]).** If \( u \in W^{1,p}_0(\Omega) \), there exists a sequence \( (u_n) \subset W^{1,p}_0(\Omega) \) such that:
(i) \( (u_n) \subset W^{1,p}_0(\Omega) \cap L^\infty_0(\Omega) \).
(ii) \( |u_n(x)| \leq |u(x)| \) and \( u_n(x)u(x) \geq 0 \text{ a.e. in } \Omega \).
(iii) \( u_n \to u \text{ in } W^{1,p}_0(\Omega) \).

The linear space \( V_u \) enjoys some nice properties. The next proposition that refines in some sense the former one is due to [32] when \( p = 2 \).
Proposition 4 (Chakrone [34]). The space $V_u$ is dense in $W^{1,p}_0(\Omega)$. And if we suppose that (2.3) holds, then

$$A_u = \{ \varphi \in W^{1,p}_0(\Omega); \ f(x,u)\varphi \in L^1(\Omega) \}$$

is a dense subspace of $W^{1,p}_0(\Omega)$ as $V_u \subset A_u$. More precisely, Brezis-Browder’s result holds true if we replace $W^{1,p}_0(\Omega) \cap L^\infty_0(\Omega)$ by $A_u$.

**Proof.** It suffices to show that $V_u$ is dense in $W^{1,p}_0(\Omega)$ and that $V_u \subset A_u$ when (2.3) holds.

1. **The density of $V_u$ in $W^{1,p}_0(\Omega)$:**

We have to show that for any $\varphi \in W^{1,p}_0(\Omega)$, there exists a sequence $(\varphi_n)_n \subset V_u$ satisfying (ii) and (iii). This is done in two steps. First, we show it is true for all $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty_0(\Omega)$. Then, using Proposition 3, we show it is true in $W^{1,p}_0(\Omega)$.

**First Step:** Suppose $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty_0(\Omega)$ and consider a sequence $(\Theta_n)_n \subset C^\infty_0(\mathbb{R})$ such that:

1. $\mathrm{supp} \, \Theta_n \subset [-n,n]$,
2. $\Theta \equiv 1$ on $[-n+1,n-1]$,
3. $0 \leq \theta_n \leq 1$ on $\mathbb{R}$ and
4. $|\Theta_n'(s)| \leq 2$.

The sequence we are looking for is obtained by setting

$$\varphi_n(x) = (\Theta \circ u)(x)\varphi(x) \quad \text{for a.e. } x \in \Omega.$$ 

Indeed, let’s check the following three points

(a) $\varphi_n \in V_u$,
(b) $|\varphi_n(x)| \leq |\varphi(x)|$ and $\varphi_n(x)\varphi(x) > 0$ a.e. in $\Omega$ and
(c) $\varphi_n \to \varphi$ in $W^{1,p}_0(\Omega)$.

For (a), since $\varphi \in L^\infty_0(\Omega)$, we have that $\varphi_n \in L^\infty_0(\Omega)$ and it’s clear by (4) that $\varphi_n \in W^{1,p}_0(\Omega)$. Finally, by (1), $u(x) \in [-n,n]$ for a.e. $x$ in $\{x \in \Omega; \varphi_n(x) \neq 0\}$.

The assumption (b) is a consequence of (3).

For (c), by (2), $\varphi_n(x) \to \varphi(x)$ a.e. in $\Omega$ and

$$\frac{\partial \varphi_n}{\partial x_i}(x) = \Theta'_n(u(x)) \frac{\partial u}{\partial x_i}(x) \varphi(x) + \Theta_n(u(x)) \frac{\partial \varphi}{\partial x_i} \to \frac{\partial \varphi}{\partial x_i} \text{ in } \Omega.$$ 

And by (4),

$$\left| \frac{\partial \varphi_n}{\partial x_i}(x) \right| \leq \left| \frac{\partial u}{\partial x_i}(x) \right| |\varphi(x)| + \left| \frac{\partial \varphi}{\partial x_i}(x) \right| \in L^p(\Omega).$$

Finally, by the dominated convergence theorem we get (c).

**Second Step:** Suppose that $\varphi \in W^{1,p}_0(\Omega)$. By Proposition 3, there is a sequence $(\psi_n)_n \subset W^{1,p}_0(\Omega)$ satisfying (i), (ii) and (iii).

For $k = 1,2,\ldots,$ there is $n_k \in \mathbb{N}$ such that $||\psi_{n_k} - \varphi||_{1,p} \leq 1/k$. Since $\psi_{n_k} \in W^{1,p}_0(\Omega) \cap L^\infty_0(\Omega)$, by the first step, there is $v_k \in V_u$ such that $|\varphi_k(x)| \leq |\psi_{n_k}(x)|$ and $\varphi_k(x)\psi_{n_k}(x) \geq 0$ almost everywhere in $\Omega$ and $||\varphi_k - \psi_{n_k}||_{1,p} \leq 1/k$. Since $V_u$ is dense in $W^{1,p}_0(\Omega)$, we get

$$\lim_{k \to \infty} \varphi_k = \varphi \quad \text{in } W^{1,p}_0(\Omega).$$ 

So, $\varphi_n \to \varphi$ in $W^{1,p}_0(\Omega)$ and

$$\lim_{k \to \infty} \frac{\partial \varphi_k}{\partial x_i}(x) \to \frac{\partial \varphi}{\partial x_i} \quad \text{in } L^p(\Omega).$$ 

Therefore, $\varphi_n \to \varphi$ in $W^{1,p}_0(\Omega)$, as required.
ψ_{n_k}||_{1,p} \leq 1/k$, so that $(\varphi_k)_k$ is the sequence we are seeking. Indeed, $|\varphi_k(x)| \leq |\psi_{n_k}(x)| \leq |\varphi(x)|$, $\varphi_k(x)\varphi(x) \geq 0$ a.e. in $\Omega$ and $||\varphi_k - \varphi||_{1,p} \leq ||\varphi_k - \psi_{n_k}||_{1,p} + ||\psi_{n_k} - \varphi||_{1,p} \leq 2/k$.

The inclusion $V_u \subset A_u$:
Indeed, for $\varphi \in V_u$, set $E = \{x \in \Omega; \varphi(x) \neq 0\}$ so that

$$
|f(x,u)\varphi| = |f(x,u)\chi_E \varphi(x)| \\
\leq \max \{ |f(x,s)\varphi(x)|; |s| \leq ||u||_{L^\infty(E)} \}
$$

where $\chi_E$ is the characteristic function of the set $E$.

By (2.3), the last term lies to $L^1(\Omega)$, so that $\varphi \in A_u$.

So, we have that

$$
\Lambda u = \sup \left\{ \int_\Omega f(x,u)v \, dx; \ v \in A_u, ||v||_{1,p} \leq 1 \right\} = \\
\sup \left\{ \int_\Omega f(x,u)v \, dx; \ v \in V_u, ||v||_{1,p} \leq 1 \right\}.
$$

(3.1)

Definition 6. Consider $u \in W^{1,p}_{loc}(\Omega)$, we say that $f(x,u) \in W^{-1,p'}(\Omega)$ if

$$
\Lambda u < +\infty.
$$

Then, the mapping $T: V_u \rightarrow \mathbb{R}$ defined by

$$
T(\varphi) = \int_\Omega f(x,u)\varphi \, dx
$$

is linear, continuous and admits an extension $\tilde{T}$ to the whole space $W^{1,p}_0(\Omega)$. Henceforth, we will make the identification $f(x,u) = \tilde{T}$; this justifies the terminology of Definition 3.

Definition 7 (Weak solution). A point $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (P) if $f(x,u) \in W^{-1,p'}(\Omega)$ (in the sense of Definition 3) and (P) is satisfied in $W^{-1,p'}(\Omega)$.

In particular, we would have

$$
\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \int_\Omega f(x,u)v \, dx + \langle h, v \rangle, \quad \forall v \in A_u.
$$

According to the conventions of [4], adopted also in [22], we will consider that by definition

$$
\int_\Omega F(x,u)^+ \, dx = \int_\Omega F(x,u)^- \, dx = +\infty \quad \text{implies that} \quad \int_\Omega F(x,u) \, dx = +\infty.
$$

And since the definition of the functional $\Phi$ contains the term $\int_\Omega F(x,u) \, dx$, the above convention has to be considered for $-F$. 
Theorem 5. Let $u \in W_0^{1,p}(\Omega)$ such that $\Psi(u) = \int_{\Omega} F(x, u) \, dx \in \mathbb{R}$, then
(a) For any $v \in W_0^{1,p}(\Omega)$ such that $f(x, u)v \in L^1(\Omega)$, we have
$$\Psi^v(u; v) \leq \int_{\Omega} f(x, u)v \, dx.$$ 
This is true in particular for all $v \in V_u$.
(b) If $\partial \Psi(u) \neq \emptyset$, then $f(x, u) \in W^{-1,p}(\Omega)$ and $\partial \Psi(u) = \{f(x, u)\}$.

Proof. (a) We will prove that the assertion holds for all $v \in V_u$ in a first step and then, for all $v \in A_u$ in a second step.

1st step:
Consider $v \in V_u$, $\varepsilon > 0$, $R > 0$ and
$$r > \int_{\Omega} f(x, u)v \, dx.$$ 
Consider also a smooth function $\Theta_R: \mathbb{R} \to [0, 1]$ with support in $[-2R, 2R]$ such that $\Theta_R(s) = 1$ in $[-R, R]$ and $|\Theta'_R(s)| \leq 2/R$ in $\mathbb{R}$. For $R$ large enough, we have $||\Theta_R(u)v - v||_{1,p} < \varepsilon$ and
$$r > \int_{\Omega} f(x, u)\Theta(u)v \, dx.$$ 

Affirmation:
$$\lim_{w \to u} \lim_{t \to 0^+} \int_{\Omega} \frac{F(x, w + t\Theta_R(w)v) - F(x, w)}{t} \, dx = \int_{\Omega} f(x, u)\Theta_R(u)v \, dx.$$ 
Indeed, consider $(w_h)_h$ such that $w_h \to u$ in $W_0^{1,p}(\Omega)$ and $(t_h)_h$ such that $t_h \to 0^+$. Without loss of generality, we may suppose that $w_h(x) \to u(x)$ almost everywhere and that $t_h \leq 1$. It follows that
$$\lim_{t_h \to 0^+} \frac{F(x, w_h + t_h\Theta_R(w_h)v) - F(x, w_h)}{t_h} = f(x, u)\Theta_R(u)v \quad \text{a.e. in } \Omega.$$ 
On the other hand, for almost every $x$ in $\Omega$, there exists $\tau_h \in [0, t_h]$ such that
$$\frac{F(x, w_h + t_h\Theta_R(w_h)v) - F(x, w_h)}{t_h} = \frac{|f(x, w_h + t_h\Theta_R(w_h)v)|}{|v|}\Theta_R(w_h)$$ 
$$\leq \left(\sup_{|s| \leq T} |f(x, s)|\right)|v|$$
where $T = 2R + ||v||_{\infty}$. The affirmation follows then from (2.3) and Lebesgue’s dominated convergence theorem.
Therefore, there exists $\delta > 0$ such that for any $w \in W_0^{1,p}(\Omega)$ satisfying $||w - v||_{1,p} < \delta$ and $0 < t < \delta$ we have
$$||\Theta_R(w)v - v||_{1,p} < \varepsilon,$$ 
$$F(x, w + t\Theta_R(w)v) - F(x, w) \in L^1(\Omega),$$ 
and
$$\int_{\Omega} \frac{F(x, w + t\Theta_R(w)v) - F(x, w)}{t} \, dx < r.$$
Consider now the continuous function \( \nu : (B_\delta(u, \Psi(u)) \cap \text{epi} (\Psi)) \times [0, \delta) \to B_\varepsilon(v) \) defined by \( \nu((w, \mu), t) = \Theta_R(w)v. \)

We have that \( \Psi(w + t\nu((w, \mu), t)) = -\infty \) if and only if \( \Psi(w) = -\infty \) and \( \Psi(w + t\nu((w, \mu), t)) \leq \Psi(w) + rt \leq \mu + rt. \)

Therefore, \( \Psi_\varepsilon(w; v) \leq \mu. \) Since \( \varepsilon \) has been taken arbitrary, we have \( \Psi_\varepsilon(u; v) \leq \mu. \)

And since \( r \in \mathbb{R} \) has been also taken arbitrary, we get (a) for all \( v \in V. \)

2nd step:
Let \( v \in W^{1,p}_0 \) such that \( f(x,u)v \in L^1(\Omega), \) i.e \( u \in A_u. \) By Proposition 3, there exists \( (v_n)_n \subset V_u \) such that \( v_n \to v \) in \( W^{1,p}_0(\Omega). \) We can suppose that \( v_n \to v \) almost everywhere,

\[
|f(x,u)v_n| \leq |f(x,u)v| \text{ and } f(x,u)v_n \to f(x,u)v \quad \text{a.e. .}
\]

By the dominated convergence theorem, we have then

\[
\lim_h \int_\Omega f(x,u)v_n \, dx = \int_\Omega f(x,u)v \, dx.
\]

And since \( \Psi^0(u; \cdot) \) is l.s.c., the assertion (a) holds true for all \( v \in A_u. \)

(b) Let \( \alpha \in \partial \Psi(u). \) Suppose that \( v \in W^{1,p}_0(\Omega) \) and that \( f(x,u)v \in L^1(\Omega), \) then

\[
\langle \alpha, v \rangle \leq \Psi^0(u; v) \leq \int_\Omega f(x,u)v \, dx.
\]

As we can change in the above inequality \( v \) by \( -v, \) it follows that \( \langle \alpha, v \rangle = \int_\Omega f(x,u)v \, dx, \) so \( f(x,u) \in W^{-1,p'}(\Omega) \) and \( \alpha = f(x,u). \)

So, we get as an immediate consequence of Propositions 3 and 2 and Theorem 4, the following result.

**Theorem 6.** If \( u \in W^{1,p}_0(\Omega) \) is a critical point of \( \Phi \) in the sense of Degiovanni-Marzocchi such that \( \Psi(u) \in \mathbb{R}, \) then \( u \) is a weak solution of the problem (P).

4. Coercive problems

We will see now in an analogous approach to [22] some conditions due to [34] that guarantee the existence of a global minimum \( u \) of \( \Phi \) in \( W^{1,p}_0(\Omega) \) and such that \( \Psi(u) \in \mathbb{R} \) and hence are exactly what we need. Three results are obtained using similar nonresonance conditions respectively to Mawhin-Ward-Willem [41] and to Landesman-Lazer [42] in the two first ones and using some comparison functions and comparison spaces in the third. The three theorems are shown to be incomparable.

Set \( G(x,s) = F(x,s) - \lambda_1 |s|^p/p. \) The conditions will port on \( G(x,s)/|s|^\alpha \) for \( 1 \leq \alpha \leq p. \) Anane and Gossez introduced in [31] some comparison functions and comparison spaces. Other ones are used here.
Definition 8. A continuous even function $\varphi: \mathbb{R} \to \mathbb{R}^+$ is called a comparison function of order $\alpha$, where $1 \leq \alpha \leq p$ if

(i) $|s|^{p} \varphi(s) \to 0$ when $s \to +\infty$.

(ii) $\frac{\varphi(s)}{|s|} \to +\infty$ when $s \to +\infty$.

(iii) $\frac{\varphi(s)}{\varphi(t_n)} \to r^{\alpha}$ when $\frac{s_n}{t_n} \to r > 0$, $s_n \to \infty$ and $t_n \to \infty$.

(iv) For all $\beta > \alpha$, there exist $t_0, a$ and $b$ such that

$$\frac{\varphi(ts)}{\varphi(t)} \leq a|s|^\beta + b \quad \text{forall } t \geq t_0 \text{ and all } s \geq 0.$$

Example 2.

a) $\varphi(s) = |s|^\alpha$, $1 < \alpha < p$.

b) $\varphi(s) = \frac{1}{\log |s|}$, $1 < \alpha < p$.

c) $\varphi(s) = |s|^\alpha \log |s|$, $1 \leq \alpha < p$.

Definition 9. Let $1 \leq \alpha \leq p$. We denote by $X_\alpha$ the set of all measurable functions $\eta(x)$ on $\Omega$ satisfying:

(i) $\eta(x) \in L^1(\Omega)$ if $p = N$.

(ii) $\eta(x) \in L^q(\Omega)$ for some $q > 1$ if $p = N$.

(iii) $\eta(x) \in L^q(\Omega)$ for some $q > (p^*/\alpha)'$ if $p < N$.

The set $Y_\alpha$ is defined the same way as $X_\alpha$ except that it is required that $\eta(x) \in L^{(p^*/\alpha)'}(\Omega)$ if $p < N$.

For a comparison function $\varphi$ of order $\alpha$, $1 \leq \alpha \leq p$, we denote by

$$G^\pm_\varphi(x) = \limsup_{s \to \pm \infty} \frac{G(x, s)}{\varphi(s)} \text{ for almost every } x \in \Omega.$$

If $\varphi(s) = |s|^\alpha$, we write only

$$G^\pm_\varphi(x) = G^\pm_\alpha(x)$$

Consider now a slightly stronger condition than (2.3),

$$\sup_{|s| \leq R} |f(., s)| \in L^1(\Omega), \quad \forall R > 0$$

We have then the following coercivity results for the problem (P):

Theorem 7. Suppose $(f_0)$,

(G1) $G^\pm_\alpha \leq 0$ a.e. in $\Omega$,

(G'1) $|\{x \in \Omega; G^+_\alpha < 0\}| \neq 0$ and $|\{x \in \Omega; G^-_\alpha < 0\}| \neq 0$.

Then, $\Phi$ achieves its minimum in a point $u$ in $W^{1,p}_0(\Omega)$ and $\Psi(u) \in \mathbb{R}$. And hence $u$ is a weak solution of (P).
Theorem 8. Suppose \((f_0), \quad (G_3) \quad G_1^+ \leq \eta \text{ a.e. uniformly in } \Omega \text{ for some } \eta \in Y_1, \text{ and} \quad \int_\Omega G_1^+(x)\varphi_1(x) \leq \langle h, \varphi_1 \rangle - \int_\Omega G_1^-(x)\varphi_1(x).

Then, the conclusion of Theorem 9 holds.

Theorem 9. Suppose that \(\varphi\) is a comparison function of order \(\alpha, 1 \leq \alpha \leq p\). Suppose also that \((f_0)\) holds and \((G_2) \quad G_2^+ \leq \eta \text{ a.e. uniformly in } \Omega \text{ for some } \eta \in X_\alpha, \text{ and} \quad \int_\Omega G_2^+(x)(\varphi_1(x))^\alpha < 0 \text{ and } \int_\Omega G_2^-(x)(\varphi_1(x))^\alpha < 0.

Then, the conclusion of Theorem 9 holds.

The proofs of the three theorems use the same technique. To give the best idea on the role played by comparison spaces and functions, we will prove Theorem 9. We begin first by some properties of \(X_\alpha\) and \(\varphi\) in the following lemma.

Lemma 10. Consider \(\eta_1(x) \in X_\alpha, u \in W_0^{1,p}(\Omega)\) and a sequence \((u_n)_n \subset W_0^{1,p}(\Omega)\) then
(a) \(\eta_1(x)\varphi(u(x)) \in L^1(\Omega)\).
(b) If \(u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega)\), then \(\eta_1(x)\varphi(u_n) \rightharpoonup \eta_1(x)\varphi(u) \text{ in } L^1(\Omega)\).
(c) If \(||u_n||_{1,p} \to +\infty\) and \(v_n = u_n/||u_n||_{1,p} \rightharpoonup v \text{ in } W_0^{1,p}(\Omega)\) and almost everywhere in \(\Omega\) then \(\eta_1(x)\varphi(u_n)/||u_n||_{1,p} \to 0 \text{ in } L^1(\Omega)\).

Proof of Lemma 10. Consider only the case \(p < N\) (When \(p \geq N\), the proof is immediate.)

Consider \(q > (p^*/\alpha')\) such that \(\eta_1(x) \in L^q(\Omega)\), there is \(\beta > \alpha\) such that \(q > (p^*/\beta') > (p^*/\alpha')\). Set \(q_1 = \beta q'\) and \(g : \Omega \times R \to R\) defined by \(g(x,s) = \eta_1(x)\varphi(x)\). Then, \(g\) is a Carathéodory function and maps \(L^1(\Omega)\) into \(L^1(\Omega)\), which follows immediately from the property (iv) of \(\varphi\). Since \(q_1 < p^*\), we have \(W_0^{1,p}\) embeds compactly in \(L^{q_1}(\Omega)\), hence we get (a), (b).

The property (c) is a consequence of the properties (i) and (iv) of \(\varphi\).

Proof of Theorem 9. The functional \(\Phi\) is well defined and takes its values in \(R \cup \{+\infty\}\). Indeed, let \(\varepsilon > 0\), by \((f_0)\) and \((G_2)\), there is \(d_\varepsilon(x) \in L^1(\Omega)\) such that
\[
F(x,u) \leq \frac{\lambda_1}{p} ||s||_p + (\eta(x) + \varepsilon)\varphi(s) + d_\varepsilon(x) \text{ a. e. in } \Omega \text{ and } \forall s \in R.
\]

For \(u \in W_0^{1,p}(\Omega)\), we have
\[
\Phi(u) \geq \frac{1}{p} \int_\Omega \left( ||\nabla u||_p^p - \lambda_1 ||u||_p^p \right) - \int_\Omega (\eta_\varepsilon(x)\varphi(u(x)) - d_\varepsilon(x)) - \langle h, u \rangle
\]
where \(\eta_\varepsilon(x) = \varepsilon + \eta(x) \in X_\alpha\). By the property (a) of Lemma 10, we get the result.

The functional \(\Phi\) is w.l.s.c., it suffices to use (4.1), the property (b) of Lemma 10 and Fatou’s lemma. \(\Phi\) is also coercive. Suppose by contradiction that there exists \(A \in R, (u_n)_n \subset W_0^{1,p}(\Omega)\) such that \(||u_n||_{1,p} \to \infty\) and
\[ \Phi(u_n) \leq A \text{ for all } n \in \mathbb{N}. \] Set \( v_n = u_n / ||u_n||_{1,p} \). We can suppose that \( v_n \rightharpoonup v \) in \( W_0^{1,p}(\Omega) \), \( v_n \rightarrow v \) in \( L^p(\Omega) \) and almost everywhere in \( \Omega \).

**Affirmation**

\[ v = \varphi_1 \text{ or } v = -\varphi_1. \]

Indeed, it suffices to divide (4.1) by \( \frac{1}{p}||u_n||^p_{1,p} \). Tending \( n \rightarrow \infty \) and using the property (iv) of \( \varphi \), we obtain \( 0 \leq \lambda \int |v|^p \), and hence

\[ ||v||^p_{1,p} \leq \lambda \int |v|^p \leq ||v||^p_{1,p}. \]

The affirmation is then a consequence of the variational characterization of \((\lambda_1, \phi_1)\). Suppose that \( v = -\varphi \) (we get the same conclusion if \( v = \varphi_1 \)).

Since \( \frac{1}{p} \int |\nabla u_n|^p - \frac{A}{p} \int |u_n|^p \geq 0 \), then \( A \geq \Phi(u_n) \geq -\int G(x,u_n) - \langle h,u_n \rangle \). Divide by \( \varphi(||u_n||) \), when \( n \rightarrow \infty \), since \( ||u_n||/\varphi(||u_n||) \rightarrow 0 \) (property (ii) of \( \varphi \)) when \( n \) goes to \( \infty \), we get

\[ \liminf_n \int_\Omega \frac{G(x,||u_n|||v_n|)}{\varphi(||u_n||)} \geq 0. \]

By (4.3), the property (iv) of \( \varphi \) and Fatou's lemma, we obtain then

\[ 0 \leq \int_\Omega \limsup_n \frac{G(x,||u_n|||v_n|)}{\varphi(||u_n||)}. \]

Since \( \varphi(|s|) = \varphi(s) \), and by the property (iii) of \( \varphi \) we have for almost every \( x \in \Omega \) that

\[ \limsup_n \frac{G(x,||u_n|||v_n|)}{\varphi(||u_n||)} = \limsup_n \frac{G(x,||u_n|||v_n|) \varphi(||u_n|||v_n|)}{\varphi(||u_n|||v_n|) \varphi(||u_n||)} \leq G_{\varphi}^{
eg}(x).(\varphi_1(x))^\alpha. \]

It follows that \( 0 \leq \int_\Omega G_{\varphi}^{
eg}(x).(\varphi_1(x))^\alpha, \) a contradiction with the hypothesis \((G'_2)\).

In the following three examples, we can see that neither of the three theorems implies the two others.

**Example 3.** Let \( 1 \leq \alpha \leq p \), \( \varphi \) a comparison function of order \( \alpha \) and \( \eta(x) \in X_\alpha \). Set \( F(x,s) = \frac{\lambda_1}{p}|s|^p + \eta(x)\varphi(x) \). It is clear that under the condition \( \int_\Omega \eta(x)(\varphi(x))^\alpha < 0 \), Theorem 3 applies but Theorem 2 do not because \( \Gamma_{\varphi}(x) = 0 \) almost everywhere in \( \Omega \). If moreover, we suppose \( \{|x \in \Omega; \eta(x) > 0\}| \neq 0 \) then Theorem 3 do not apply either because the property \((G_3)\) is not satisfied.

**Example 4.** Set \( F(x,s) = \frac{\lambda_1}{p}|s|^p + \eta(x)|s| \) where \( \eta(x) \in Y_1 \). We can check easily that Theorem 2 applies under the condition \((G'_4)\) which is equivalent to

\[ \int_\Omega \eta(x)\varphi_1(x) < -|\langle h, \varphi_1 \rangle|. \]

But the two other theorems do not apply.
Example 5. Let $a(x)$ a functional in $C_0^\infty(\Omega)$ such that $a(x) \leq 0$ for all $x \in \Omega$, $\{x \in \Omega; a(x) = 0\} \neq \emptyset$ and $\{x \in \Omega; a(x) < 0\} \neq \emptyset$. Set for some comparison function $\varphi$ of order $\alpha$, $1 \leq \alpha \leq p$

$$F(x,s) = \left(\frac{\lambda_1}{p} + a(x)\right) + |s|^p + (\varphi(s)|s|^p)^{1/2}.$$ 

Then, $(G_1)$ and $(G'_1)$ are satisfied, while $(G_2)$ and $(G_3)$ are not because $G^+_\varphi = G^-_\varphi = G^+_1 = G^-_1 = +\infty$ in the set of positive measure $\{x \in \Omega; a(x) = 0\}$.

Remark 2. 1) The comparison functions are useless in the autonomous case $f(x,s) = f(x)$. Indeed we have

(1.a) $(G_1), (G'_1)$ imply $(G_2), (G'_2)$ and $G^+_\varphi = -\infty$.
(1.b) $(G_2), (G'_2)$ imply $(G_3), (G'_3)$ for any $h \in W^{-1,p'}(\Omega)$ and $G^+_1 = -\infty$.
(1.c) The condition

$$(4.3) \quad \lim_{s \to \pm \infty} G(s)/|s|^\beta = -\infty$$

implies $(G_3), (G'_3)$ for any $h \in W^{-1,p'}(\Omega)$ and $G^+_1 = -\infty$.

2) Always, in the autonomous case, we can check easily that the condition $(2.2)$ implies the condition $(4.3)$. The converse is false. Indeed, for $f(s) = \lambda_1 |s|^{p-2}s - \beta |s|^{\beta-2}s$ where $1 < \beta < p$ so that $F(s) = \lambda_1 |s|^p/p - |s|^\beta$ and $G(s) = -|s|^\beta$. We have that the function $G$ satisfies $(4.3)$ but $\limsup_{s \to \pm \infty} p.F(s)/|s|^\beta = \lambda_1$.

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