A Generalized Fluctuation-Dissipation Theorem for Nonlinear Response Functions

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Abstract

A nonlinear generalization of the Fluctuation-Dissipation Theorem (FDT) for the $n$-point Green functions and the amputated 1PI vertex functions at finite temperature is derived in the framework of the Closed Time Path formalism. We verify that this generalized FDT coincides with known results for $n = 2$ and 3. New explicit relations among the 4-point nonlinear response and correlation (fluctuation) functions are presented.
I. INTRODUCTION

Within the framework of non-equilibrium statistical mechanics a successful linear response theory has been established by Kubo [1] and other authors [2,3]. One of the focuses in linear response theory is the Fluctuation-Dissipation Theorem (FDT) [2] which describes the relation between the response and the correlation (fluctuation) functions. To seek for a nonlinear generalization of the FDT is an important and interesting issue [4–11] in nonlinear response theory which deals with functions of many time arguments. Earlier attempts [4–8] to derive a nonlinear generalization of the FDT were based on studies of the KMS condition [1,12], the conditions of time reversal invariance, and the exact algebra of transformations between different representations of the thermal Green functions (see Eq. (21) below). As stated in Ref. [10], these attempts failed since none of these relations by themselves provide a nonlinear generalization of the FDT, and their correct combination has not yet been found.

It was shown in Refs. [9,10] that the Closed Time Path (CTP) formalism suggested by Schwinger [13] and further elaborated by Keldysh [14] provides a suitable theoretical framework for nonlinear response theory near thermal equilibrium. By combining the three types of relations mentioned above, the authors of Ref. [10] arrived at a plausible nonlinear generalization of the FDT for the nonlinear response functions, without, however, giving an explicit proof of its general validity. Their suggested nonlinear generalization of the FDT (Eq. (5.60) in Ref. [10]) involves a parameter $\varepsilon = \pm 1$ which depends on the signature under time reversal of the physical quantity considered. For arbitrary field operators it is not clear which value of $\varepsilon$ should be taken, which makes it difficult to verify the correctness of the generalized FDT given in [10]. In this paper we will give a complete derivation of the generalized FDT for nonlinear response functions. Instead of time reversal we exploit the closely related “tilde conjugation” transformation. We find that the correct generalized FDT equations contain no sign parameter $\varepsilon$.

Several recent investigations [11,15–24] of $n$-point functions ($n \geq 3$) and their spectral representations have tried to clarify the analytical continuation from the imaginary time
formalism (ITF) \cite{25} to the real time formalism (RTF) \cite{13,14,23,26}, the RTF Feyman rules and their transformation between various RTF representations, and the calculation of the temperature dependent running coupling constant in QCD. The generalized FDT plays a crucial role in the determination of spectral representations for the retarded and advanced $n$-point Green functions in the CTP formalism \cite{11,20,21,24}. In Ref. \cite{11} the corresponding relations were derived for the retarded and advanced 3-point functions, by way of a rather tedious evaluation of the corresponding Lehmann spectral representation. As we will show, these same relations can be deduced very easily from the generalized FDT which, however, applies equally well to more general situations, $n \geq 4$. Due to the known intimate relation between transport coefficients and higher order $n$-point Green functions as provided by the Kubo formalism \cite{1,27–29}, we expect the generalized FDT to become a useful tool for the calculation of collective phenomena in, say, a quark-gluon plasma. As a non-trivial example for its application we give explicit expressions for the relations between the 4-point nonlinear response and correlation functions which may be useful for a RTF calculation of the shear viscosity in scalar field theories \cite{27–29}.

This paper is organized as follows: In Sec. II we review some important relations between various components of the $n$-point RTF Green functions in the CTP formalism and the general expressions for the nonlinear response functions which will be used in the following sections. In Sec. III we derive the generalized FDT for nonlinear response functions. By solving the generalized FDT equations, explicit relations among the correlation (fluctuation) and response functions are obtained for the cases of $n = 2, 3, 4$ and checked against known results. In Sec. IV we repeat the derivation for the amputated 1PI vertex functions. A short summary is given in Sec. V. In the appendix we deduce relations between the vertex functions in the physical representation and the $R/A$-functions by Aurenche and Becherrawy \cite{21}.

II. NONLINEAR RESPONSE THEORY IN THE REAL-TIME FORMALISM
A. $n$-point Green functions and their generating functional

In the literature there exist three equivalent representations of the RFT $n$-point Green functions and their generating functional \[10,30\]: the “closed time-path”, “single-time” and “physical” representations.

1. In the closed time-path (CTP) representation, the $n$-point Green functions $G_p(1, \ldots, n)$ (which may include disconnected parts, especially if the fields develop a vacuum condensate) and their generating functional $Z[J(x)]$ are defined as

\[
G_p(1, \ldots, n) \equiv (-i)^{n-1} \langle T_p[\hat{\phi}(1) \cdots \hat{\phi}(n)] \rangle = i(-1)^n \frac{\partial^n Z[J(x)]}{\partial J(1) \cdots \partial J(n)} \bigg|_{J=0}, \\
Z[J(x)] \equiv \langle T_p \left[ \exp \left( i \int_p d^4x J(x) \hat{\phi}(x) \right) \right] \rangle. \tag{1}
\]

Throughout this paper the field operator $\hat{\phi}$ will denote a bosonic (elementary or composite) field; the obvious extension to fermions is left to the reader. $J(x)$ is a classical external source, and $\langle \ldots \rangle$ the thermal expectation value. The numbers $1, \ldots, n$ stand for the coordinate arguments $x_1, \ldots, x_n$ in 4-dimensional Minkowski space, and $p$ denotes the integration contour (the closed time-path) which consists of a branch $C_1$ running from negative infinity to positive infinity and another branch $C_2$ running back from positive infinity to negative infinity. $T_p$ represents the time ordering operator along this time-path; it is the usual time ordering operator $T$ on the branch $C_1$ and the anti-chronological time ordering operator $\tilde{T}$ on the branch $C_2$.

2. Instead of the closed time-path which covers the real time axis twice one can use a single real time parameter if one distinguishes the external source $J_1$ on the branch $C_1$ from $J_2$ on the branch $C_2$, setting them equal (and to zero) only at the end of the calculation. Then Eqs. (1) and (2) can be rewritten as

\[
G_{a_1, \ldots, a_n}(1, \ldots, n) \equiv (-i)^{n-1} \langle T_p[\hat{\phi}_{a_1}(1) \cdots \hat{\phi}_{a_n}(n)] \rangle = i(-1)^{a_1+\ldots+a_n} \frac{\delta^n Z[J_1(x), J_2(x)]}{\delta J_{a_1}(1) \cdots \delta J_{a_n}(n)} \bigg|_{J_1=J_2=0}, \tag{3}
\]
\[ Z[J_1(x), J_2(x)] \equiv \langle T_p \left[ \exp \left( i \int_{-\infty}^{\infty} d^4x \left[ J_1(x) \hat{\phi}_1(x) - J_2(x) \hat{\phi}_2(x) \right] \right) \right] \rangle. \] (4)

Here the single time parameter varies from \(-\infty\) to \(\infty\), and \(a_1, a_2, \ldots, a_n = 1, 2\) indicate on which of the two branches, \(C_1\) or \(C_2\), the fields are located. The definition (3) gives, for example,

\[ G_{\overrightarrow{2.1\ldots n}}(1, \ldots, n) = (-i)^{n-1} \left\langle \left\langle \hat{T}[\hat{\phi}(1) \cdots \hat{\phi}(m)]T[\hat{\phi}(m+1) \cdots \hat{\phi}(n)] \right\rangle \right\rangle. \] (5)

3. The “physical” or \(r/a\) representation of the formalism is defined by setting

\[ J_a(x) = J_1(x) - J_2(x), \quad J_r(x) = \frac{1}{2} \left( J_1(x) + J_2(x) \right), \] (6a)

\[ \hat{\phi}_a(x) = \hat{\phi}_1(x) - \hat{\phi}_2(x), \quad \hat{\phi}_r(x) = \frac{1}{2} \left( \hat{\phi}_1(x) + \hat{\phi}_2(x) \right), \] (6b)

Eqs. (3) and (4) are then transformed into

\[ G_{\alpha_1 \ldots \alpha_n}(1, \ldots, n) \equiv (-i)^{n-1} 2^{n_r-1} \langle T_p[\hat{\phi}_{\alpha_1}(1) \cdots \hat{\phi}_{\alpha_n}(n)] \rangle \]

\[ = i (-1)^n 2^{n_r-1} \frac{\delta^n Z[J_a(x), J_r(x)]}{\delta J_{\alpha_1}(1) \cdots \delta J_{\alpha_n}(n)} \bigg|_{J_a = J_r = 0}, \] (7)

\[ Z[J_a(x), J_r(x)] \equiv \langle T_p \left[ \exp \left( i \int_{-\infty}^{\infty} d^4x \left[ J_a(x) \hat{\phi}_r(x) + J_r(x) \hat{\phi}_a(x) \right] \right) \right] \rangle. \] (8)

Here \(\alpha_1, \ldots, \alpha_n = a, r\), and \(n_r\) is the number of indices \(r\) among \((\alpha_1, \alpha_2, \ldots, \alpha_n)\). For \(\alpha_i = a\) one defines \(\bar{\alpha}_i = r\), and vice versa. One shows generally that [10,30]

\[ G_{aa \ldots a}(1, \ldots, n) = 0. \] (9)

Furthermore, the functions \(G_{raa \ldots a}, G_{ara \ldots a}, G_{aar \ldots a}, \ldots\), with only one index \(r\) can be expressed [10,30] as sums over expectation values of \(n-1\) nested commutators which are just the fully retarded \(n\)-point Green functions defined in Ref. [31]. \(G_{rr \ldots r}\) is a sum over expectation values of \(n-1\) nested anticommutators; it is the \(n\)-point correlation function, with no retarded or advanced relations among any of its time arguments. The remaining components \(G_{\alpha_1 \ldots \alpha_n}(1, \ldots, n)\) are sums over expectation values of combinations of \(n-1\) nested commutators and anticommutators, with both retarded and advanced relations between some of the \(n\) time arguments.
The generating functional for $n$-point connected Green functions in the closed time-path representation is defined as

$$ W[J(x)] = -i \ln Z[J(x)]. \quad (10) $$

In analogy to Eqs. (4) and (8), we can define its single-time representation $W[J_1(x), J_2(x)]$ and its physical representation $W[J_a(x), J_r(x)]$. Replacing in Eqs. (1, 3, 7) the generating functional $Z$ by $W$ we obtain the connected Green functions $G^c_p(1, 2, \ldots, n)$, $G^c_{a_1 \ldots a_n}(1, \ldots, n)$ and $G^c_{\alpha_1 \ldots \alpha_n}(1, \ldots, n)$ in the three different representations.

Finally, the amputated 1PI $n$-point vertex functions $\Gamma^{(n)}(1, \ldots, n)$ and their generating functional $\Gamma[\varphi(x)]$ in the closed time-path representation are defined as

$$ \Gamma^{(n)}(1, \ldots, n) \equiv (-1)^n \frac{\delta^n \Gamma[\varphi(x)]}{\delta \varphi(1) \cdots \delta \varphi(n)} \bigg|_{\varphi = \langle \hat{\phi} \rangle}, \quad (11) $$

$$ \Gamma[\varphi(x)] \equiv W[J(x)] - \int_p d^4 x J(x) \varphi(x), \quad (12) $$

with the classical field

$$ \varphi(x) = \frac{\delta W[J(x)]}{\delta J(x)}. \quad (13) $$

In the single-time representation these definitions are rewritten as

$$ \Gamma^{(n)}_{a_1 \ldots a_n}(1, \ldots, n) \equiv (-1)^n \frac{\delta^n \Gamma[\varphi_1(x), \varphi_2(x)]}{\delta \varphi_{a_1}(1) \cdots \delta \varphi_{a_n}(n)} \bigg|_{\varphi_1 = \langle \hat{\phi} \rangle, \varphi_2 = 0}, \quad (14) $$

$$ \Gamma[\varphi_1(x), \varphi_2(x)] \equiv W[J_1(x), J_2(x)] - \int_{-\infty}^{\infty} d^4 x \left( J_1(x) \varphi_1(x) - J_2(x) \varphi_2(x) \right), \quad (15) $$

with the classical fields on the two branches of the closed time-path

$$ \varphi_1(x) = \frac{\delta W[J_1(x), J_2(x)]}{\delta J_1(x)}, \quad \varphi_2(x) = -\frac{\delta W[J_1(x), J_2(x)]}{\delta J_2(x)}. \quad (16) $$

The corresponding definitions in the physical representation are

$$ \Gamma^{(n)}_{a_1 \ldots a_n}(1, \ldots, n) \equiv (-1)^n 2^{n-1} \frac{\delta^n \Gamma[\varphi_a(x), \varphi_r(x)]}{\delta \varphi_{a_1}(1) \cdots \delta \varphi_{a_n}(n)} \bigg|_{\varphi_a = 0, \varphi_r = \langle \hat{\phi} \rangle}, \quad (17) $$

$$ \Gamma[\varphi_a(x), \varphi_r(x)] \equiv W[J_a(x), J_r(x)] - \int_{-\infty}^{\infty} d^4 x \left( J_a(x) \varphi_r(x) + J_r(x) \varphi_a(x) \right), \quad (18) $$
where \( n_a \) is the number of indices \( a \) among \((\alpha_1, \ldots, \alpha_n)\), and

\[
\varphi_a(x) = \varphi_1(x) - \varphi_2(x) ,
\quad \varphi_r(x) = \frac{1}{2}(\varphi_1(x) + \varphi_2(x)).
\]  

(19)

Corresponding to Eq. (19) we have

\[
\Gamma^{(n)}_{rr, \ldots, r}(1, \ldots, n) = 0.
\]

(20)

The relation between the \( r/a \) vertex functions \( \Gamma^{(n)}_{\alpha_1 \ldots \alpha_n} \) and the \( R/A \) vertex functions defined in Ref. [20] can be found in Appendix A.

In Refs. [10,14,30] the following transformation law between the single-time and physical representations for the \( n \)-point Green functions (including disconnected parts) was established:

\[
G_{\alpha_1 \ldots \alpha_n}(1, \ldots, n) = 2^{1 - \frac{n}{2}}Q_{\alpha_1 a_1} \cdots Q_{\alpha_n a_n}.
\]

(21)

Here

\[
Q_{a1} = -Q_{a2} = Q_{r1} = Q_{r2} = \frac{1}{\sqrt{2}}
\]

(22)

are the four elements of the orthogonal Keldysh transformation for 2-point functions [14]. In the following, a summation over repeated indices is understood. In a similar way we derive

\[
G^{c}_{\alpha_1 \ldots \alpha_n}(1, \ldots, n) = 2^{1 - \frac{n}{2}}Q_{\alpha_1 a_1} \cdots Q_{\alpha_n a_n};
\]

(23)

\[
\Gamma^{(n)}_{\alpha_1 \ldots \alpha_n}(1, \ldots, n) = 2^{1 - \frac{n}{2}}\Gamma^{(n)}_{\alpha_1 \ldots \alpha_n}(1, \ldots, n)Q_{\alpha_1 a_1} \cdots Q_{\alpha_n a_n}.
\]

(24)

The transformation laws for the connected \( n \)-point Green functions and the amputated 1PI vertex functions are thus identical to those for the normal \( n \)-point Green functions (including disconnected parts), contrary to a remark made on page 24 in Ref. [10].

**B. Tilde conjugation and KMS condition**

The tilde conjugation operation is defined by reversing the time ordering in coordinate space [26]. For example, the tilde conjugate of Eq. (13) is
\[
\tilde{G}_{a_1 \ldots a_n}^{2 \ldots 2_{m \ldots n}}(1, \ldots, n) = (-i)^{n-1} \langle T[\hat{\phi}(1) \cdots \hat{\phi}(m)]T[\hat{\phi}(m+1) \cdots \hat{\phi}(n)] \rangle .
\]

In momentum space one can show that tilde conjugation is equivalent to complex conjugation:

\[
\begin{align*}
\tilde{G}_{a_1 \ldots a_n}(k_1, \ldots, k_n) &= (-1)^{n-1} G_{a_1 \ldots a_n}^*(k_1, \ldots, k_n), \tag{26a} \\
\tilde{G}_{\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n) &= (-1)^{n-1} G_{\alpha_1 \ldots \alpha_n}^*(k_1, \ldots, k_n). \tag{26b}
\end{align*}
\]

The factor \((-1)^{n-1}\) here results from the corresponding factor \((-i)^{n-1}\) in the definition (3). Energy-momentum conservation requires \(k_1 + k_2 + \ldots + k_n = 0\). The relations (26) carry over without modification for the connected Green functions \(G_{a_1 \ldots a_n}^c(k_1, \ldots, k_n)\) and \(G_{\alpha_1 \alpha_2 \ldots \alpha_n}^c(k_1, k_2, \ldots, k_n)\). For the amputated 1PI vertex functions we have

\[
\begin{align*}
\tilde{\Gamma}^{(n)}_{\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n) &= -\Gamma^{(n)*}_{\alpha_1 \ldots \alpha_n}(k_1, \ldots, k_n), \tag{27a} \\
\tilde{\Gamma}^{(n)}_{a_1 \ldots a_n}(k_1, \ldots, k_n) &= -\Gamma^{(n)*}_{a_1 \ldots a_n}(k_1, \ldots, k_n). \tag{27b}
\end{align*}
\]

Using the KMS condition for periodicity of the Green functions in imaginary time \([1,12]\) one derives in coordinate space

\[
\begin{align*}
\tilde{G}_{a_1 \ldots a_n}(1, \ldots, n) &= \exp \left( i\beta \sum_{\{i|a_i=2\}} \partial_{t_i} \right) G_{\bar{a}_1 \ldots \bar{a}_n}(1, \ldots, n), \tag{28a} \\
\tilde{\Gamma}^{(n)}_{a_1 \ldots a_n}(1, \ldots, n) &= \exp \left( i\beta \sum_{\{i|a_i=2\}} \partial_{t_i} \right) \Gamma^{(n)}_{\bar{a}_1 \ldots \bar{a}_n}(1, \ldots, n), \tag{28b}
\end{align*}
\]

where \(\beta\) is the inverse temperature, and \(\bar{a}_i = 2, 1\) for \(a_i = 1, 2\). The connected Green functions \(G_{a_1 \ldots a_n}^c(1, \ldots, n)\) satisfy the same relation Eq. (28a) as the disconnected ones. In momentum space the exponential prefactors become products of simple inverse Boltzmann factors \(e^{\beta k_i^0}\).

C. General expression for nonlinear response functions

Let us consider a thermal system with global equilibrium initial conditions at \(t_0 = -\infty\) which is driven out of equilibrium by a real time-dependent external source \(J(t)\) which
is switched on adiabatically. In the closed time-path representation we define the Green functions for a nonvanishing external source by

\[ D_p(1, \ldots, n) = i(-1)^n \frac{\delta^n Z[J(x)]}{\delta J(1) \cdots \delta J(n)}. \]  

(29)

As shown in Refs. [9,10], the observables in nonlinear response theory are the correlation functions \( D_r, D_{rr}, D_{rrr}, \ldots \) in the physical representation. These functions can be expressed as a Volterra series in powers of the external source \( J \), with \( G_{ra}, D_{raa}, D_{rra}, \ldots \) as their coefficients. Taking \( J_1(x) = J_2(x) = J(x) \), the nonlinear response can be expressed as

\[ D_r(1) = G_r(1) - \int d2 G_{ra}(1, 2)J(2) + \frac{1}{2!} \int d2 d3 G_{raa}(1, 2, 3)J(2)J(3) + \ldots, \]  

(30a)

\[ D_{rr}(1, 2) = G_{rr}(1, 2) - \int d3 G_{rra}(1, 2, 3)J(3) + \frac{1}{2!} \int d3 d4 G_{rraa}(1, 2, 3, 4)J(3)J(4) + \ldots, \]  

(30b)

\[ D_{rrr}(1, 2, 3) = G_{rrr}(1, 2, 3) - \int d4 G_{rrra}(1, 2, 3, 4)J(4) + \frac{1}{2!} \int d4 d5 G_{rrraa}(1, 2, 3, 4, 5)J(4)J(5) + \ldots. \]  

(30c)

Here \( di \) is a shorthand for \( d^4x_i \). Following Refs. [4–8], the correlation functions \( D_r, D_{rr}, D_{rrr}, \ldots \) and \( G_r, G_{rr}, G_{rrr}, \ldots \) are called fluctuation functions in the non-equilibrium and equilibrium state, respectively. The fully retarded Green functions \( G_{ra}, G_{raa}, G_{raaa}, \ldots \) correspond to linear, second-order, third-order, and higher order response functions of the averaged physical field \( \varphi \) to the external source \( J \). Similarly, \( G_{rra}, G_{rraa}, G_{rrra}, \ldots \) are the response functions of the fluctuations at different order.

III. NONLINEAR GENERALIZATION OF THE FLUCTUATION-DISSIPATION THEOREM FOR \( n \)-POINT GREEN FUNCTIONS

Eqs. (28a) show that not all \( 2^n \) components of the \( n \)-point Green functions \( G_{a_1 \ldots a_n} \) in the single-time representation are independent of each other. Taking also into account Eq. (21) with Eq. (9), one sees that there are at most \( 2^{n-1} - 1 \) independent components. We will soon see that the same is true for the physical representation of the \( n \)-point Green function, \( G_{a_1 \ldots a_n} \).
As a simple example consider the 2-point Green function. According to (9) \( G_{aa} = 0 \). The remaining three components are defined in coordinate space by

\[
G_{ra}(1, 2) = G^{\text{ret}}(1, 2) \equiv -i\theta(t_1 - t_2)\langle [\hat{\phi}(1), \hat{\phi}(2)] \rangle, \tag{31a}
\]

\[
G_{ar}(1, 2) = G^{\text{adv}}(1, 2) \equiv i\theta(t_2 - t_1)\langle [\hat{\phi}(1), \hat{\phi}(2)] \rangle, \tag{31b}
\]

\[
G_{rr}(1, 2) = G^{\text{cor}}(1, 2) \equiv -i\langle \{\hat{\phi}(1), \hat{\phi}(2)\} \rangle. \tag{31c}
\]

In momentum space the retarded and advanced 2-point functions, \( G^{\text{ret}} \) and \( G^{\text{adv}} \), are related by complex conjugation, while the correlation function \( G^{\text{cor}} \) is related to them by the well-known fluctuation-dissipation theorem [2], in our notation

\[
G_{rr}(k) = \left( 1 + 2n(k^0) \right) \left( G_{ra}(k) - G_{ar}(k) \right). \tag{32}
\]

Here \( n(k^0) = [e^{\beta k^0} - 1]^{-1} \) is the Bose-Einstein distribution function. The FDT thus plays an important role for the analytic structure of the thermal 2-point function.

We will now derive similar relations among the \( n \)-point \( (n \geq 3) \) retarded and advanced Green functions in the physical representation. Since, as discussed in the preceding section, these functions are related to the higher order fluctuations and nonlinear response functions, the relations among them contain the desired nonlinear generalization of the FDT.

Starting from the transformation (21) and using \( Q_{r\bar{a}i} = Q_{rai} \) and \( Q_{a\bar{a}i} = -Q_{aa_i} \), we get

\[
G_{a_1...a_n}(1, \ldots, n) \pm G_{\bar{a}_1...\bar{a}_n}(1, \ldots, n) = 2^{1-\frac{n}{2}} \sum_{\alpha_1, \ldots, \alpha_n = a,r} \left( 1 \pm (-1)^{n_a(\alpha_1, \ldots, \alpha_n)} \right) G_{\alpha_1...\alpha_n}(1, \ldots, n) Q_{\alpha_1a_1} \cdots Q_{\alpha_nan},
\]

where \( n_a(\alpha_1, \ldots, \alpha_n) \) counts the number of \( a \) indices among \( (\alpha_1, \ldots, \alpha_n) \). Eqs. (33) form \( 2^{n-1} \) independent equations. We now consider separately the contributions to the sum on the r.h.s. of (33) with fixed even and odd numbers of \( a \) indices \( n^e_a \) and \( n^o_a \), and define

\[
G_{n^e_a(\alpha_1...\alpha_n)}(1, \ldots, n) = 2^{\frac{n^e_a}{2}} \sum_{\alpha_1, \ldots, \alpha_n = a,r \atop n_a(\alpha_1, \ldots, \alpha_n) = n^e_a} G_{\alpha_1...\alpha_n}(1, \ldots, n) Q_{\alpha_1a_1} \cdots Q_{\alpha_nan}, \tag{34a}
\]

\[
G_{n^o_a(\alpha_1...\alpha_n)}(1, \ldots, n) = 2^{\frac{n^o_a}{2}} \sum_{\alpha_1, \ldots, \alpha_n = a,r \atop n_a(\alpha_1, \ldots, \alpha_n) = n^o_a} G_{\alpha_1...\alpha_n}(1, \ldots, n) Q_{\alpha_1a_1} \cdots Q_{\alpha_nan}. \tag{34b}
\]
Due to Eq. (33) neither \( n^a_s \) nor \( n^o_a \) can be larger than \( n - 1 \). Eq. (33) then gives

\[
\sum_{0 \leq n^s_s \leq n-1} G_{n^s_s, (a_1 \ldots a_n)}(1, \ldots, n) = 2^{n-2} \left[ G_{a_1 \ldots a_n}(1, \ldots, n) + G_{\bar{a}_1 \ldots \bar{a}_n}(1, \ldots, n) \right], \tag{35a}
\]

\[
\sum_{1 \leq n^o_a \leq n-1} G_{n^a_o, (a_1 \ldots a_n)}(1, \ldots, n) = 2^{n-2} \left[ G_{a_1 \ldots a_n}(1, \ldots, n) - G_{\bar{a}_1 \ldots \bar{a}_n}(1, \ldots, n) \right]. \tag{35b}
\]

With the help of the KMS condition (28a) and some further algebra we find

\[
\sum_{0 \leq n^s_s \leq n-1} \left( G_{n^s_s, (a_1 \ldots a_n)}(1, \ldots, n) + \tilde{G}_{n^s_s, (a_1 \ldots a_n)}(1, \ldots, n) \right) = \coth \left( \frac{A}{2} \right) \sum_{1 \leq n^o_a \leq n-1} \left( G_{n^a_o, (a_1 \ldots a_n)}(1, \ldots, n) + \tilde{G}_{n^a_o, (a_1 \ldots a_n)}(1, \ldots, n) \right), \tag{36a}
\]

\[
\sum_{0 \leq n^s_s \leq n-1} \left( G_{n^s_s, (a_1 \ldots a_n)}(1, \ldots, n) - \tilde{G}_{n^s_s, (a_1 \ldots a_n)}(1, \ldots, n) \right) = \tanh \left( \frac{A}{2} \right) \sum_{1 \leq n^o_a \leq n-1} \left( G_{n^a_o, (a_1 \ldots a_n)}(1, \ldots, n) - \tilde{G}_{n^a_o, (a_1 \ldots a_n)}(1, \ldots, n) \right), \tag{36b}
\]

where

\[
A = i\beta \sum_{\{i|a_i=2\}} \partial_{i_i}. \tag{37}
\]

Going to momentum space and using (26a) we can separate the real and imaginary parts of these equations. For odd values of \( n \) we obtain

\[
\text{Re} \left[ \sum_{0 \leq n^s_s \leq n-1} G_{n^s_s, (a_1 \ldots a_n)}(k_1, \ldots, k_n) \right] = \coth \left( \frac{\beta}{2} \sum_{\{i|a_i=2\}} k^0_i \right) \text{Re} \left[ \sum_{1 \leq n^o_a \leq n-2} G_{n^a_o, (a_1 \ldots a_n)}(k_1, \ldots, k_n) \right], \tag{38a}
\]

\[
\text{Im} \left[ \sum_{0 \leq n^s_s \leq n-1} G_{n^s_s, (a_1 \ldots a_n)}(k_1, \ldots, k_n) \right] = \tanh \left( \frac{\beta}{2} \sum_{\{i|a_i=2\}} k^0_i \right) \text{Im} \left[ \sum_{1 \leq n^o_a \leq n-2} G_{n^a_o, (a_1 \ldots a_n)}(k_1, \ldots, k_n) \right], \tag{38b}
\]

while for even \( n \) we find

\[
\text{Im} \left[ \sum_{0 \leq n^s_s \leq n-2} G_{n^s_s, (a_1 \ldots a_n)}(k_1, \ldots, k_n) \right] = \coth \left( \frac{\beta}{2} \sum_{\{i|a_i=2\}} k^0_i \right) \text{Im} \left[ \sum_{1 \leq n^o_a \leq n-1} G_{n^a_o, (a_1 \ldots a_n)}(k_1, \ldots, k_n) \right], \tag{39a}
\]
\[
\text{Re} \left[ \sum_{0 \leq n_e \leq n - 2} G_{n_e}(a_1, \ldots, a_n)(k_1, \ldots, k_n) \right] \\
= \tanh \left( \frac{\beta}{2} \sum_{\{i|a_i=2\}} k_i^0 \right) \text{Re} \left[ \sum_{1 \leq n_o \leq n - 1} G_{n_o}(a_1, \ldots, a_n)(k_1, \ldots, k_n) \right].
\]

Eqs. (38,39) are the nonlinear generalization of the FDT for \(n\)-point Green functions. Each of these two sets of equations contains \(2^{n-1}\) pairs of (real) relations among the components \(G_{a_1 \ldots a_n}\) in the physical representation. Taking also into account Eq. (9) we see that at most \(2^{n-1} - 1\) (complex) components are independent, as promised above.

We should point out that the nonlinear generalization of the FDT for the connected \(n\)-point Green functions is the same as Eqs. (38,39). This is because \(G\) and \(G^c\) have the same transformation properties (21) and (26) and satisfy the same KMS condition (28a). The following examples will therefore be written down only for the standard \(n\)-point functions which may contain disconnected contributions like, e.g., vacuum condensates.

### A. 2-point function

Combining Eqs. (3) and (21) for \(n = 2\) leads to

\[
G_{22} = \frac{1}{2}(G_{rr} - G_{ra} - G_{ar}) , \quad G_{21} = \frac{1}{2}(G_{rr} + G_{ra} - G_{ar}) , \quad (40a) \\
G_{12} = \frac{1}{2}(G_{rr} - G_{ra} + G_{ar}) , \quad G_{11} = \frac{1}{2}(G_{rr} + G_{ra} + G_{ar}) . \quad (40b)
\]

We can then write down the definitions (34) for the two relevant cases \(n^e_a = 0, n^o_a = 1\):

\[
G_{0,(22)} = G_{rr} , \quad G_{1,(22)} = -G_{ra} - G_{ar} ; \quad (41a) \\
G_{0,(21)} = G_{rr} , \quad G_{1,(21)} = G_{ra} - G_{ar} . \quad (41b)
\]

Substitution into (39) yields in momentum space

\[
\text{Im} \left[ G_{ra}(k_1, k_2) + G_{ar}(k_1, k_2) \right] = 0 , \quad (42a) \\
\text{Re} \ G_{rr}(k_1, k_2) = 0 ; \quad (42b) \\
\text{Im} \ G_{rr}(k_1, k_2) = \coth \frac{\beta k_0^1}{2} \text{Im} \left[ G_{ra}(k_1, k_2) - G_{ar}(k_1, k_2) \right] , \quad (42c) \\
\text{Re} \ G_{rr}(k_1, k_2) = \tanh \frac{\beta k_0^1}{2} \text{Re} \left[ G_{ra}(k_1, k_2) - G_{ar}(k_1, k_2) \right] . \quad (42d)
\]
The zeros in the first two equations follow from energy conservation, \(k_1^0 + k_2^0 = 0\), which leads to \(\coth[\beta(k_1^0 + k_2^0)/2] \to \infty\) and \(\tanh[\beta(k_1^0 + k_2^0)/2] \to 0\). Eqs. (12) are easily solved by

\[
G_{rr}(k) = \coth \frac{\beta k_0}{2} \left[ G_{ra}(k) - G_{ar}(k) \right], \quad G_{ra}(k) = G_{ar}^*(k). \quad (43)
\]

The first of these equations is the well-known fluctuation-dissipation theorem (32).

Substituting Eq. (43) into Eq. (40) and using the two-component column vectors introduced in Refs. [23,26] one can express Eq. (40) in terms of an outer product of these column vectors [23,11]

\[
G_{a_1 a_2}(k) = G_{ra}(k) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left( 1 + n(k^0) \right) - G_{ar}(k) \begin{pmatrix} n(k^0) \\ 1 + n(k^0) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (44)
\]

B. 3-point function

For \(n = 3\) Eqs. (3) and (21) give

\[
G_{222} = \frac{1}{4} (G_{rrr} - G_{rra} - G_{rar} - G_{arr} + G_{raa} + G_{ara} + G_{aar}), \quad (45a)
\]

\[
G_{211} = \frac{1}{4} (G_{rrr} + G_{rra} + G_{rar} - G_{arr} + G_{raa} - G_{ara} - G_{aar}), \quad (45b)
\]

\[
G_{121} = \frac{1}{4} (G_{rrr} + G_{rra} - G_{rar} + G_{arr} - G_{raa} + G_{ara} - G_{aar}), \quad (45c)
\]

\[
G_{112} = \frac{1}{4} (G_{rrr} - G_{rra} + G_{rar} + G_{arr} - G_{raa} - G_{ara} + G_{aar}), \quad (45d)
\]

\[
G_{221} = \frac{1}{4} (G_{rrr} + G_{rra} - G_{rar} - G_{arr} - G_{raa} + G_{ara} + G_{aar}), \quad (45e)
\]

\[
G_{212} = \frac{1}{4} (G_{rrr} - G_{rra} + G_{rar} - G_{arr} + G_{raa} - G_{ara} - G_{aar}), \quad (45f)
\]

\[
G_{122} = \frac{1}{4} (G_{rrr} - G_{rra} - G_{rar} + G_{arr} + G_{raa} - G_{ara} + G_{aar}), \quad (45g)
\]

\[
G_{111} = \frac{1}{4} (G_{rrr} + G_{rra} + G_{rar} + G_{arr} + G_{raa} + G_{ara} + G_{aar}). \quad (45h)
\]

The definitions (34) yield

\[
G_{0,(222)} = G_{0,(211)} = G_{0,(121)} = G_{0,(112)} = G_{rrr} ; \quad (46a)
\]

\[
G_{1,(222)} = -G_{rra} - G_{rar} - G_{arr} ; \quad G_{2,(222)} = G_{raa} + G_{ara} + G_{aar} ; \quad (46b)
\]

\[
G_{1,(211)} = G_{rra} + G_{rar} - G_{arr} ; \quad G_{2,(211)} = G_{raa} - G_{ara} - G_{aar} ; \quad (46c)
\]
\[ G_{1,(121)} = G_{rra} - G_{rar} + G_{arr}, \quad G_{2,(121)} = -G_{raa} + G_{ara} - G_{aar}; \]  
\[ G_{1,(112)} = -G_{rra} + G_{rar} + G_{arr}, \quad G_{2,(112)} = -G_{raa} - G_{ara} + G_{aar}. \]  
(46d)
(46e)

Substituting these quantities into Eqs. (38) we get in momentum space

\[
\begin{align*}
\text{Re} \left[ G_{rra} + G_{rar} + G_{arr} \right] &= 0, \quad (47a) \\
\text{Im} \left[ G_{rrr} + G_{raa} + G_{ara} + G_{aar} \right] &= 0; \quad (47b) \\
\text{Re} \left[ G_{rrr} + G_{raa} - G_{ara} - G_{aar} \right] &= \coth \left( \frac{\beta k_0^0}{2} \right) \text{Re} \left[ G_{rra} + G_{rar} - G_{arr} \right], \quad (47c) \\
\text{Im} \left[ G_{rrr} + G_{raa} - G_{ara} - G_{aar} \right] &= \tanh \left( \frac{\beta k_0^0}{2} \right) \text{Im} \left[ G_{rra} + G_{rar} - G_{arr} \right]; \quad (47d) \\
\text{Re} \left[ G_{rrr} - G_{raa} + G_{ara} - G_{aar} \right] &= \coth \left( \frac{\beta k_0^0}{2} \right) \text{Re} \left[ G_{rra} - G_{rar} + G_{arr} \right], \quad (47e) \\
\text{Im} \left[ G_{rrr} - G_{raa} + G_{ara} - G_{aar} \right] &= \tanh \left( \frac{\beta k_0^0}{2} \right) \text{Im} \left[ G_{rra} - G_{rar} + G_{arr} \right]; \quad (47f) \\
\text{Re} \left[ G_{rrr} - G_{raa} - G_{ara} + G_{aar} \right] &= \coth \left( \frac{\beta k_0^0}{2} \right) \text{Re} \left[ -G_{rra} + G_{rar} + G_{arr} \right], \quad (47g) \\
\text{Im} \left[ G_{rrr} - G_{raa} - G_{ara} + G_{aar} \right] &= \tanh \left( \frac{\beta k_0^0}{2} \right) \text{Im} \left[ -G_{rra} + G_{rar} + G_{arr} \right]. \quad (47h)
\end{align*}
\]

Again the zeros in the first two equations result from energy conservation. To solve Eqs. (47), we choose \( G_{raa}, \ G_{ara} \) and \( G_{aar} \) as the \( 2^{n-1} - 1 = 3 \) independent physical components; in the notation of Ref. (11) these are the three retarded 3-point functions \( \Gamma_{R_i}, \ \Gamma_R, \) and \( \Gamma_{Ro}, \) respectively. We further introduce the shorthands (11) \( N_i = N(k_i^0) = \coth(\beta k_i^0/2) = 1 + 2n(k_i^0) = 1 + 2n_i \) which satisfy the identity

\[ N_1 N_2 + N_2 N_3 + N_3 N_1 = -1. \]  
(48)

Solving Eqs. (47) in terms of the three selected components we find

\[
\begin{align*}
G_{rar} &= N_1(G_{ara}^* - G_{aar}) + N_3(G_{ara}^* - G_{raa}), \quad (49a) \\
G_{arr} &= N_2(G_{raa}^* - G_{aar}) + N_3(G_{raa}^* - G_{ara}), \quad (49b) \\
G_{rra} &= N_1(G_{aar}^* - G_{ara}) + N_2(G_{aar}^* - G_{raa}), \quad (49c) \\
G_{rrr} &= G_{raa}^* + G_{ara}^* + G_{aar}^* + N_2 N_3(G_{raa} + G_{raa}^*) \
&\quad + N_1 N_3(G_{ara} + G_{ara}^*) + N_1 N_2(G_{aar} + G_{aar}^*). \quad (49d)
\end{align*}
\]
Noting the identities $G_{rar} = \Gamma_F$, $G_{arr} = \Gamma_{Fi}$, $G_{rra} = \Gamma_{Fo}$, and $G_{rrr} = \Gamma_E$ between the 3-point functions in $r/a$ notation and the 3-point functions used in [11], Eq. (49) are seen to agree with Eqs. (33) in [11]. Obviously, the algebra leading from (47) to (49) is much simpler than the method of derivation presented in [11].

Eq. (49d) expresses the fluctuation function $G_{rrr}$ in terms of the second order non-linear response functions ($G_{raa}$, $G_{ara}$, $G_{aar}$), i.e. in terms of the fully retarded vertex functions. Furthermore, Eqs. (49a-c) express the linear response functions for the fluctuation ($G_{rra}$, $G_{rar}$, $G_{arr}$) in terms of the same set of retarded Green functions. These four relations form the nonlinear generalization of the FDT for the case $n = 3$.

When substituting Eq. (49) into (45), Eq. (45) can be written in terms of outer products of column vectors as

\[
\begin{align*}
(G_{a_1 a_2 a_3}) &=
G_{raa} \left( \begin{array}{c} 1 \\ 1 \\ 1 + n_2 \\ 1 + n_3 \\ 1 + n_3 \\ 1 + n_1 \end{array} \right) - G_{raa}^* \left[ (1 + n_2)(1 + n_3) - n_2 n_3 \right] \left( \begin{array}{c} n_1 \\ 1 + n_1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \\
+ &G_{ara} \left( \begin{array}{c} n_1 \\ 1 + n_1 \\ 1 \\ 1 \\ 1 + n_3 \\ 1 + n_3 \end{array} \right) - G_{ara}^* \left[ (1 + n_1)(1 + n_3) - n_1 n_3 \right] \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} n_2 \\ 1 + n_2 \\ 1 \\ 1 \end{array} \right) \\
+ &G_{aar} \left( \begin{array}{c} n_1 \\ 1 + n_1 \\ 1 + n_2 \\ 1 + n_2 \\ 1 \end{array} \right) - G_{aar}^* \left[ (1 + n_1)(1 + n_2) - n_1 n_2 \right] \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} n_3 \\ 1 + n_3 \\ 1 \end{array} \right)
\end{align*}
\]

(50)

C. 4-point function

Following the same steps as for $n = 2, 3$, we get from Eqs. (9), (21), and (34)

\[
\begin{align*}
G_{0,(2222)} &= G_{0,(2221)} = G_{0,(2212)} = G_{0,(2122)} = G_{0,(1222)} \\
&= G_{0,(2211)} = G_{0,(2121)} = G_{0,(1221)} = G_{rrrr}; \\
G_{1,(2222)} &= -G_{rrra} - G_{rrar} - G_{rarr} - G_{arrr}, \\
G_{2,(2222)} &= G_{rraa} + G_{araa} + G_{raar} + G_{arar} + G_{aarr}, \\
G_{3,(2222)} &= -G_{raaa} - G_{raar} - G_{aara} - G_{aarr}; \\
G_{1,(2221)} &= G_{rrra} - G_{rrar} - G_{rarr} - G_{arrr},
\end{align*}
\]

(51a, 51b)
\[ G_{2,(2221)} = -G_{rraa} - G_{rara} + G_{raar} - G_{arra} + G_{arar} + G_{aarr}, \]
\[ G_{3,(2221)} = G_{raaa} + G_{araa} + G_{aara} - G_{aar}; \]  
(51c)
\[ G_{1,(2212)} = -G_{rrra} + G_{rrar} - G_{rarr} - G_{arrr}, \]
\[ G_{2,(2212)} = -G_{rraa} + G_{rara} - G_{raar} + G_{arra} - G_{arar} + G_{aarr}, \]
\[ G_{3,(2212)} = G_{raaa} + G_{araa} - G_{aara} + G_{aar}; \]  
(51d)
\[ G_{1,(2122)} = -G_{rrra} - G_{rrar} + G_{rarr} - G_{arrr}, \]
\[ G_{2,(2122)} = G_{rraa} + G_{rara} - G_{raar} - G_{arra} - G_{arar} - G_{aarr}, \]
\[ G_{3,(2122)} = -G_{raaa} - G_{araa} + G_{aara} + G_{aaar}; \]  
(51e)
\[ G_{1,(2211)} = G_{rrra} + G_{rrar} + G_{rarr} + G_{arrr}, \]
\[ G_{2,(2211)} = -G_{rraa} + G_{rara} + G_{raar} - G_{arra} - G_{arar} + G_{aarr}, \]
\[ G_{3,(2211)} = -G_{raaa} - G_{araa} - G_{aara} + G_{aaar}; \]  
(51f)
\[ G_{1,(2121)} = G_{rrra} + G_{rrar} - G_{rarr} + G_{arrr}, \]
\[ G_{2,(2121)} = -G_{rraa} - G_{rara} - G_{raar} - G_{arra} + G_{arar} + G_{aarr}, \]
\[ G_{3,(2121)} = -G_{raaa} + G_{araa} - G_{aara} + G_{aaar}; \]  
(51g)
\[ G_{1,(1222)} = G_{rrra} - G_{rrar} + G_{rarr} - G_{arrr}, \]
\[ G_{2,(1222)} = G_{rraa} - G_{rara} - G_{raar} + G_{arra} + G_{arar} - G_{aarr}, \]
\[ G_{3,(1222)} = -G_{raaa} - G_{araa} + G_{aara} + G_{aaar}; \]  
(51h)
\[ G_{1,(1221)} = G_{rrra} - G_{rrar} - G_{rarr} + G_{arrr}, \]
\[ G_{2,(1221)} = -G_{rraa} + G_{rara} - G_{raar} - G_{arra} + G_{arar} - G_{aarr}, \]
\[ G_{3,(1221)} = -G_{raaa} + G_{araa} - G_{aara} + G_{aaar}. \]  
(51i)

Substituting these quantities into Eq. (33), we get in momentum space

\[
\text{Im}[G_{1,(2222)} + G_{3,(2222)}] = 0, \quad (52a)
\]
\[
\text{Re}[G_{0,(2222)} + G_{2,(2222)}] = 0; \quad (52b)
\]
\[
\text{Im}[G_{0,(2221)} + G_{2,(2221)}] = -\coth\frac{\beta k_0^4}{2} \text{Im}[G_{1,(2221)} + G_{3,(2221)}], \quad (52c)
\]
\begin{align}
\text{Re}[G_{0,(2221)} + G_{2,(2221)}] &= -\tanh\frac{\beta k_4^0}{2} \text{Re}[G_{1,(2221)} + G_{3,(2221)}] ; \\
\text{Im}[G_{0,(2212)} + G_{2,(2212)}] &= -\coth\frac{\beta k_3^0}{2} \text{Im}[G_{1,(2212)} + G_{3,(2212)}] , \\
\text{Re}[G_{0,(2212)} + G_{2,(2212)}] &= -\tanh\frac{\beta k_3^0}{2} \text{Re}[G_{1,(2212)} + G_{3,(2212)}] ; \\
\text{Im}[G_{0,(2122)} + G_{2,(2122)}] &= -\coth\frac{\beta k_2^0}{2} \text{Im}[G_{1,(2122)} + G_{3,(2122)}] , \\
\text{Re}[G_{0,(2122)} + G_{2,(2122)}] &= -\tanh\frac{\beta k_2^0}{2} \text{Re}[G_{1,(2122)} + G_{3,(2122)}] ; \\
\text{Im}[G_{0,(1222)} + G_{2,(1222)}] &= -\coth\frac{\beta k_1^0}{2} \text{Im}[G_{1,(1222)} + G_{3,(1222)}] , \\
\text{Re}[G_{0,(1222)} + G_{2,(1222)}] &= -\tanh\frac{\beta k_1^0}{2} \text{Re}[G_{1,(1222)} + G_{3,(1222)}] ; \\
\text{Im}[G_{0,(2211)} + G_{2,(2211)}] &= \coth\frac{\beta (k_1^0 + k_2^0)}{2} \text{Im}[G_{1,(2211)} + G_{3,(2211)}] , \\
\text{Re}[G_{0,(2211)} + G_{2,(2211)}] &= \tanh\frac{\beta (k_1^0 + k_2^0)}{2} \text{Re}[G_{1,(2211)} + G_{3,(2211)}] ; \\
\text{Im}[G_{0,(2121)} + G_{2,(2121)}] &= \coth\frac{\beta (k_1^0 + k_2^0)}{2} \text{Im}[G_{1,(2121)} + G_{3,(2121)}] , \\
\text{Re}[G_{0,(2121)} + G_{2,(2121)}] &= \tanh\frac{\beta (k_1^0 + k_2^0)}{2} \text{Re}[G_{1,(2121)} + G_{3,(2121)}] ; \\
\text{Im}[G_{0,(1221)} + G_{2,(1221)}] &= \coth\frac{\beta (k_2^0 + k_3^0)}{2} \text{Im}[G_{1,(1221)} + G_{3,(1221)}] , \\
\text{Re}[G_{0,(1221)} + G_{2,(1221)}] &= \tanh\frac{\beta (k_2^0 + k_3^0)}{2} \text{Re}[G_{1,(1221)} + G_{3,(1221)}] .
\end{align}

Once more the zeros in the first two equations follow from energy conservation. For \( n = 4 \) there are at most \( 2^{4-1} - 1 = 7 \) independent components of \( G_{\alpha_1\alpha_2\alpha_3\alpha_4} \); we choose them as \( G_{raaa}, G_{araa}, G_{aara}, G_{ara}, G_{ara}, \) and \( G_{aarr} \). Introducing the notation

\[ N_{ij}^{(kl)} = \frac{N_i + N_j}{N_k + N_l} = \frac{1 + n_i + n_j}{1 + n_k + n_l}, \tag{53} \]

we can express the solution of the above equations as

\begin{align}
G_{rrrr} &= -N_2 N_3 N_4 G_{raaa} + \left(N_2 N_3 N_4 + N_2 + N_3 + N_4\right) G^*_{raraa} + N_1 N_3 N_4 G_{araa} \\
&\quad + N_2 \left(N_{(14)}^{(14)} + N_{(24)}^{(13)}\right) G^*_{araa} + N_1 N_2 N_4 G_{aara} + N_3 \left(N_{(34)}^{(12)} + N_2^2 N_{(23)}^{(14)}\right) G^*_{aara} \\
&\quad + N_1 N_2 N_3 G_{aar} + N_4 \left(N_{(24)}^{(13)} + N_2^2 N_{(34)}^{(12)}\right) G^*_{aar} + N_1 N_4 G_{arra} + N_2 N_3 N_{(23)}^{(14)} G^*_{arra} \\
&\quad + N_1 N_3 G_{arr} + N_2 N_4 N_{(24)}^{(13)} G^*_{arr} + N_1 N_2 G_{arr} + N_3 N_4 N_{(34)}^{(12)} G^*_{arr}. \tag{54a}
\end{align}
This solution expresses the fluctuation function $G_{rrrr}$ in terms of tensors constructed from outer products of 2-component column vectors:

$$G_{rrrr} = N_2 N_3 G_{raaa} - N_2 N_4 N^{(13)}(24) G^*_{araa} - N_3 N_4 N^{(12)}(34) G^*_{aara} - \left(N^{(13)}(24) + N_3 N^{(12)}(34)\right) G^*_{aaar}$$

$$-N_1 G_{arra} - N_2 N^{(13)}(24) G^*_{arar} - N_3 N^{(12)}(34) G^*_{arar}$$

$$G_{rrar} = N_2 N_4 G_{raaa} - N_2 N_3 N^{(14)}(25) G^*_{araa} - \left(N^{(12)}(34) + N_2 N^{(14)}(25)\right) G^*_{aara} - N_3 N_4 N^{(12)}(34) G^*_{aarr}$$

$$-N_2 N^{(14)}(23) G^*_{arr} - N_1 G_{arar} - N_4 N^{(12)}(34) G^*_{arar}$$

$$G_{rrar} = N_3 N_4 G_{raaa} - \left(N^{(14)}(23) + N_2 N^{(13)}(24)\right) G^*_{araa} - N_2 N_3 N^{(14)}(23) G^*_{aara} - N_2 N_4 N^{(13)}(24) G^*_{aarr}$$

$$-N_2 N^{(13)}(23) G^*_{arr} - N_4 N^{(13)}(24) G^*_{arar} - N_1 G_{arar}$$

$$G_{rrrr} = (1 + N_2 N_3 + N_2 N_4 + N_3 N_4) G^*_{raaa} - N_3 N_4 G_{araa} - N_2 N_4 G_{aara} - N_2 N_3 G_{aarr}$$

$$-N_4 G_{arra} - N_3 G_{arar} - N_2 G_{aarr}$$

$$G_{raar} = -N_4 G_{raaa} + N_3 N^{(14)}(23) G^*_{araa} + N_2 N^{(14)}(23) G^*_{aara} - N_1 G_{aarar} + N^{(14)}(23) G^*_{aarar}$$

$$G_{rara} = -N_3 G_{raaa} + N_4 N^{(13)}(24) G^*_{araa} - N_4 G_{aara} + N_2 N^{(13)}(24) G^*_{aara} + N^{(13)}(24) G^*_{aarar}$$

$$G_{rraa} = -N_2 G_{raaa} - N_1 G_{araa} + N_4 N^{(12)}(34) G^*_{araa} + N_3 N^{(12)}(34) G^*_{aara} + N^{(12)}(34) G^*_{aarr}$$

This solution expresses the fluctuation function $G_{rrrr}$, the linear response functions of the 3rd order fluctuations $G_{rrra}, G_{rrar}, G_{rara}, G_{rrrr}$, and 3 of the non-linear response functions of the 2nd order fluctuations ($G_{raar}, G_{rara}, G_{rraa}$) in terms of the fully retarded 4-point vertex functions (3rd order non-linear response functions $G_{raaa}, G_{araa}, G_{aara}, G_{aarr}$) and the remaining 2nd order response functions of the 2nd order fluctuations ($G_{arr}, G_{arar}, G_{arar}$). Although these equations still have the structure of a fluctuation-dissipation theorem, they are clearly much more involved than for the simpler cases $n = 2, 3$ since fluctuation and response functions of different order simultaneously.

Similar to Eq. (54) we can express the 4-point Green function in the single-time representation in terms of tensors constructed from outer products of 2-component column vectors:

$$\left(G_{a_1 a_2 a_3 a_4}\right) = -G_{raaa} \left(\begin{array}{c} 1 \\ n_2 \\ 1 + n_2 \\ 1 + n_3 \\ 1 + n_4 \end{array}\right) \left(\begin{array}{c} n_3 \\ 1 + n_3 \\ 1 + n_4 \\ n_4 \\ 1 + n_4 \end{array}\right)$$

$$-G^*_{raaa} \left(\begin{array}{c} n_1 \\ 1 + n_1 \\ 1 + n_1 \\ 1 + n_3 \\ 1 + n_3 \end{array}\right) \left(\begin{array}{c} n_1 \\ 1 + n_1 \\ 1 + n_1 \\ 1 + n_3 \\ 1 + n_3 \end{array}\right)$$

$$+ \frac{1}{2} G_{araa} \left(\begin{array}{c} n_1 \\ 1 + n_1 \\ 1 + n_1 \\ 1 + n_3 \\ 1 + n_3 \end{array}\right) \left(\begin{array}{c} n_3 \\ 1 + n_3 \\ 1 + n_4 \\ n_4 \\ 1 + n_4 \end{array}\right) + \left(\begin{array}{c} 1 + n_3 \\ 1 + n_3 \\ 1 + n_4 \\ n_4 \\ 1 + n_4 \end{array}\right)$$
\[-\frac{1}{2}G_{aara}^* \frac{1 + n_1}{n_2} \left( \frac{1}{1 + n_2} \right) \left( \frac{1}{1 + n_2} \right) \left( n_4 \right) + \frac{1 + n_4}{n_3} \left( n_4 \right) = \frac{1 + n_4}{n_3} \left( n_4 \right) \right] \]

\[\frac{1}{2}G_{aara} \left( \frac{n_1}{1 + n_1} \right) \left( \frac{1}{1 + n_2} \right) \left( \frac{1}{1 + n_2} \right) \left( n_4 \right) + \frac{1 + n_4}{n_3} \left( n_4 \right) = \frac{1 + n_4}{n_3} \left( n_4 \right) \right] \]

\[\frac{1}{2}G_{aara}^* \frac{1 + n_1}{n_3} \left( \frac{1}{1 + n_2} \right) \left( \frac{1}{1 + n_2} \right) \left( n_4 \right) + \frac{1 + n_3}{n_2} \left( n_4 \right) = \frac{1 + n_3}{n_2} \left( n_4 \right) \right] \]

\[\frac{1}{2}G_{aarr} \left( \frac{n_1}{1 + n_1} \right) \left( \frac{1}{1 + n_2} \right) \left( n_3 \right) \left( n_4 \right) + \frac{1 + n_4}{n_3} \left( n_4 \right) = \frac{1 + n_4}{n_3} \left( n_4 \right) \right] \]

\[\frac{1}{2}G_{aarr}^* \frac{1 + n_1}{n_4} \left( \frac{1}{1 + n_2} \right) \left( n_3 \right) \left( n_4 \right) + \frac{1 + n_3}{n_2} \left( n_4 \right) = \frac{1 + n_3}{n_2} \left( n_4 \right) \right] \]

\[\frac{1}{2}G_{aarr}^* \frac{1 + n_1}{n_4} \left( n_3 \right) \left( n_4 \right) + \frac{1 + n_4}{n_3} \left( n_4 \right) = \frac{1 + n_4}{n_3} \left( n_4 \right) \right] \]

In deriving this expression we have made frequent use of the following relations which hold in global thermal equilibrium:

\[n(-x) = -\left( 1 + n(x) \right), \quad (56a)\]

\[\frac{n(x_1)n(x_2)n(x_3)}{n(x_1 + x_2 + x_3)} = \left( 1 + n(x_1) \right) \left( 1 + n(x_2) \right) \left( 1 + n(x_3) \right) - n(x_1)n(x_2)n(x_3). \quad (56b)\]

They are a special case of a general relation proved in the appendix of Ref. [32].

**IV. GENERALIZED FLUCTUATION-DISSIPATION THEOREM FOR AMPUTATED 1PI n-POINT VERTEX FUNCTIONS**

One can derive in a similar way a nonlinear generalization of the FDT for the amputated 1PI n-point vertex functions. In analogy to (34) we define

\[\Gamma^{(n)}_{n^a_0(a_1\ldots a_n)}(1, \ldots, n) = 2^{\frac{n}{2}} \sum_{\text{all } a_i = a, r, n^a_0(a_1\ldots a_n)=n^a_0} \Gamma^{(n)}_{a_1\ldots a_n}(1, \ldots, n)Q_{a_1a_1} \cdots Q_{a_na_n}, \quad (57a)\]

\[\Gamma^{(n)}_{n^a_0(a_1\ldots a_n)}(1, \ldots, n) = 2^{\frac{n}{2}} \sum_{\text{all } a_i = a, r, n^a_0(a_1\ldots a_n)=n^a_0} \Gamma^{(n)}_{a_1\ldots a_n}(1, \ldots, n)Q_{a_1a_1} \cdots Q_{a_na_n}. \quad (57b)\]

Due to (20) \( n^a_0 \) must be \( \geq 2 \) and \( n^a_0 \geq 1 \). Using Eqs. (22) and (24) we then find instead of Eq. (55)
\[
\sum_{2 \leq n_s \leq n} \Gamma_{n_s, (a_1 \ldots a_n)}^{(n)}(1, \ldots, n) = 2^{n-2} \left[ \Gamma_{a_1 \ldots a_n}^{(n)}(1, \ldots, n) + \Gamma_{\bar{a}_1 \ldots \bar{a}_n}^{(n)}(1, \ldots, n) \right], \quad (58a)
\]
\[
\sum_{1 \leq n_s \leq n} \Gamma_{n_s, (a_1 \ldots a_n)}^{(n)}(1, \ldots, n) = 2^{n-2} \left[ \Gamma_{a_1 \ldots a_n}^{(n)}(1, \ldots, n) - \Gamma_{\bar{a}_1 \ldots \bar{a}_n}^{(n)}(1, \ldots, n) \right]. \quad (58b)
\]

With the help of the KMS condition (28b) and the tilde conjugation relation (27a) we finally find
\[
\text{Im} \left[ \sum_{2 \leq n_s \leq n} \Gamma_{n_s, (a_1 \ldots a_n)}^{(n)}(k_1, \ldots, k_n) \right] = \coth \left( \frac{\beta}{2} \sum_{\{i | a_i = 2\}} k^0_i \right) \text{Im} \left[ \sum_{1 \leq n_s \leq n} \Gamma_{n_s, (a_1 \ldots a_n)}^{(n)}(k_1, \ldots, k_n) \right], \quad (59a)
\]
\[
\text{Re} \left[ \sum_{2 \leq n_s \leq n} \Gamma_{n_s, (a_1 \ldots a_n)}^{(n)}(k_1, \ldots, k_n) \right] = \tanh \left( \frac{\beta}{2} \sum_{\{i | a_i = 2\}} k^0_i \right) \text{Re} \left[ \sum_{1 \leq n_s \leq n} \Gamma_{n_s, (a_1 \ldots a_n)}^{(n)}(k_1, \ldots, k_n) \right]. \quad (59b)
\]

These relations are the nonlinear generalization of the FDT for the amputated 1PI \(n\)-point vertex function. Contrary to the case for the \(n\)-point Green functions, we here obtain formally the same relations for even and odd values of \(n\). Eqs. (59) provide \(2^{n-1}\) pairs of (real) relations among the (complex) components of \(\Gamma_{a_1 \ldots a_n}^{(n)}\). Together with Eq. (20) this leaves at most \(2^{n-1} - 1\) independent complex components for the \(n\)-point vertex.

A. 2-point self-energy

Making use of Eqs. (21) and (24) in the definition (57), we find
\[
\Gamma_{1, (22)}^{(2)} = -\Gamma_{ra}^{(2)} - \Gamma_{ar}^{(2)}; \quad \Gamma_{2, (22)}^{(2)} = \Gamma_{aa}^{(2)}; \quad (60a)
\]
\[
\Gamma_{1, (21)}^{(2)} = \Gamma_{ra}^{(2)} - \Gamma_{ar}^{(2)}; \quad \Gamma_{2, (21)}^{(2)} = -\Gamma_{aa}^{(2)}. \quad (60b)
\]

Substitution into (59) yields
\[
\text{Im} \left[ \Gamma_{ra}^{(2)}(k_1, k_2) + \Gamma_{ar}^{(2)}(k_1, k_2) \right] = 0, \quad (61a)
\]
\[
\text{Re} \Gamma_{aa}^{(2)}(k_1, k_2) = 0; \quad (61b)
\]
\[
-\text{Im} \Gamma_{aa}^{(2)}(k_1, k_2) = \coth \frac{\beta k^0_1}{2} \text{Im} \left[ \Gamma_{ra}^{(2)}(k_1, k_2) - \Gamma_{ar}^{(2)}(k_1, k_2) \right], \quad (61c)
\]
−Re \( \Gamma^{(2)}_{rr}(k_1, k_2) = \tanh \frac{\beta k^0}{2} \text{Re} \left[ \Gamma^{(2)}_{ra}(k_1, k_2) - \Gamma^{(2)}_{ar}(k_1, k_2) \right] \). \tag{61d}

Replacing in these equations \( \Gamma^{(2)} \) by the 2-point Green function \( G \) and interchanging the indices \( r \) and \( a \) we recover Eq.(62). The solution of (61) is thus straightforward:

\[
\Gamma^{(2)}_{aa}(k) = \coth \frac{\beta k^0}{2} \left[ \Gamma^{(2)}_{ra}(k) - \Gamma^{(2)}_{ar}(k) \right], \quad \Gamma^{(2)}_{ar}(k) = \Gamma^{(2)}_{ra}^{\ast}(k) . \tag{62}
\]

This coincides with the results obtained in Refs. [10,11]. Here \( \Gamma^{(2)}_{ar} \) and \( \Gamma^{(2)}_{ra} \) are the retarded and advanced self-energies, \( \Sigma^{\text{ret}} \) and \( \Sigma^{\text{adv}} \), respectively.

Using the column vector notation one can write the 2-point vertex function in the single-time representation as

\[
\left( \Gamma^{(2)}_{a_1a_2}(k) \right) = \Gamma^{(2)}_{ar}(k) \begin{pmatrix} 1 & (1 + n(k^0)) \\ -1 & -n(k^0) \end{pmatrix} - \Gamma^{(2)}_{ra}(k) \begin{pmatrix} n(k^0) & 1 \\ -(1 + n(k^0)) & -1 \end{pmatrix} . \tag{63}
\]

This expression exemplifies the general substitution rule: *in order to obtain the vertex functions \( \Gamma \) from the corresponding Green functions \( G \), one interchanges all \( r \) and \( a \) indices and changes the sign in the lower components of all the column vectors.*

### B. Amputated 1PI 3-point vertex

Substituting the quantities

\[
\Gamma^{(3)}_{3,(222)} = \Gamma^{(3)}_{3,(211)} = \Gamma^{(3)}_{3,(121)} = \Gamma^{(3)}_{3,(112)} = -\Gamma^{(3)}_{aaa} ; \tag{64a}
\]

\[
\Gamma^{(3)}_{1,(222)} = -\Gamma^{(3)}_{rra} - \Gamma^{(3)}_{rar} - \Gamma^{(3)}_{arr} , \quad \Gamma^{(3)}_{2,(222)} = \Gamma^{(3)}_{raa} + \Gamma^{(3)}_{ara} + \Gamma^{(3)}_{aar} ; \tag{64b}
\]

\[
\Gamma^{(3)}_{1,(211)} = \Gamma^{(3)}_{rra} + \Gamma^{(3)}_{rar} - \Gamma^{(3)}_{arr} , \quad \Gamma^{(3)}_{2,(211)} = \Gamma^{(3)}_{raa} - \Gamma^{(3)}_{ara} - \Gamma^{(3)}_{aar} ; \tag{64c}
\]

\[
\Gamma^{(3)}_{1,(121)} = \Gamma^{(3)}_{rra} - \Gamma^{(3)}_{rar} + \Gamma^{(3)}_{arr} , \quad \Gamma^{(3)}_{2,(121)} = -\Gamma^{(3)}_{raa} + \Gamma^{(3)}_{ara} - \Gamma^{(3)}_{aar} ; \tag{64d}
\]

\[
\Gamma^{(3)}_{1,(112)} = -\Gamma^{(3)}_{rra} + \Gamma^{(3)}_{rar} + \Gamma^{(3)}_{arr} , \quad \Gamma^{(3)}_{2,(112)} = -\Gamma^{(3)}_{raa} - \Gamma^{(3)}_{ara} + \Gamma^{(3)}_{aar} . \tag{64e}
\]

into Eq. (64) we find

\[
\text{Im} \left[ \Gamma^{(3)}_{rra} + \Gamma^{(3)}_{rar} + \Gamma^{(3)}_{arr} + \Gamma^{(3)}_{aaa} \right] = 0 , \tag{65a}
\]

\[
\text{Re} \left[ \Gamma^{(3)}_{raa} + \Gamma^{(3)}_{ara} + \Gamma^{(3)}_{aar} \right] = 0 ; \tag{65b}
\]
Replacing $\Gamma^{(3)}_r$ by the 3-point Green function $G$ and interchanging the indices $r$ and $a$ we recover Eq. (17). Choosing as independent components the fully retarded vertices $\Gamma^{(3)}_{arr}, \Gamma^{(3)}_{rar}$ and $\Gamma^{(3)}_{rra}$, we thus obtain directly the solution

$$\Gamma^{(3)}_{ara} = N_1(\Gamma^{(3)*}_{rar} - \Gamma^{(3)*}_{rra}) + N_3(\Gamma^{(3)*}_{rar} - \Gamma^{(3)*}_{arr}),$$

$$\Gamma^{(3)}_{raa} = N_2(\Gamma^{(3)*}_{arr} - \Gamma^{(3)*}_{rra}) + N_3(\Gamma^{(3)*}_{arr} - \Gamma^{(3)*}_{rar}),$$

$$\Gamma^{(3)}_{aar} = N_1(\Gamma^{(3)*}_{rra} - \Gamma^{(3)*}_{rar}) + N_2(\Gamma^{(3)*}_{rra} - \Gamma^{(3)*}_{arr}),$$

$$\Gamma^{(3)}_{aaa} = \Gamma^{(3)*}_{arr} + \Gamma^{(3)*}_{rar} + \Gamma^{(3)*}_{rra} + N_2N_3(\Gamma^{(3)}_{arr} + \Gamma^{(3)}_{rar}) + N_1N_2(\Gamma^{(3)}_{rra} + \Gamma^{(3)*}_{rra}).$$

These relations are the nonlinear generalization of the FDT for the amputated 1PI 3-point vertex function.

In terms of outer products of column vectors the 3-point vertex function in the single-time representation can be expressed as

$$\begin{pmatrix} \Gamma^{(3)}_{a1a2a3} \end{pmatrix} = \Gamma^{(3)}_{arr} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_2 \\ -(1+n_2) \end{pmatrix} \begin{pmatrix} n_3 \\ -(1+n_3) \end{pmatrix}$$

$$- \Gamma^{(3)*}_{arr} \left[ (1+n_2)(1+n_3) - n_2n_3 \right] \begin{pmatrix} n_1 \\ -(1+n_1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$+ \Gamma^{(3)}_{rar} \begin{pmatrix} n_1 \\ -(1+n_1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_3 \\ -(1+n_3) \end{pmatrix}$$

$$- \Gamma^{(3)*}_{rar} \left[ (1+n_1)(1+n_3) - n_1n_3 \right] \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_2 \\ -(1+n_2) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
\begin{align*}
  &+ \Gamma_{rra}^{(3)} \left(\begin{array}{ccc}
  n_1 & n_2 & 1 \\
  1 & 1 + n_1 & 1 \\
  1 & 1 + n_2 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1 \\
 \end{array}\right) \\
  &= -\Gamma_{rra}^{(3)} \left[ (1 + n_1)(1 + n_2) - n_1 n_2 \right] \left(\begin{array}{ccc}
  1 & 1 & n_3 \\
  1 & 1 & -(1 + n_3) \\
 \end{array}\right),
\end{align*}

in accordance with the above substitution rule (cf. Eq. (54)).

C. Amputated 1PI 4-point vertex

Repeating the same technical steps for the 4-point function we find

\begin{align*}
  \Gamma_{aaaa}^{(4)} &= -N_2 N_3 N_4 \Gamma_{arrr}^{(4)} + \left( N_2 N_3 N_4 + N_2 + N_3 + N_4 \right) \Gamma_{arrr}^{(4)*} + N_1 N_3 N_4 \Gamma_{rrrr}^{(4)} \\
  &+ N_2 \left( N_{(23)}^{(14)} + N_{(24)}^{(14)} \right) \Gamma_{rrrr}^{(4)} \\
  &+ N_1 N_2 N_4 \Gamma_{rrar}^{(4)} \\
  &+ N_3 \left( N_{(34)}^{(12)} + N_{(23)}^{(14)} \right) \Gamma_{rrrr}^{(4)} \\
  &+ N_1 N_3 N_4 \Gamma_{rara}^{(4)} \\
  &+ N_2 N_4 N_{(24)}^{(14)} \Gamma_{rrrr}^{(4)} + N_1 N_2 \Gamma_{rraa}^{(4)} + N_3 N_4 N_{(34)}^{(12)} \Gamma_{rrrr}^{(4)*},
\end{align*}

(68a)

\begin{align*}
  \Gamma_{aara}^{(4)} &= -N_2 N_3 \Gamma_{arrr}^{(4)} - N_2 N_4 \left( N_{(24)}^{(14)} \right) \Gamma_{rrra}^{(4)*} \\
  &- N_3 N_4 N_{(34)}^{(12)} \Gamma_{rrrr}^{(4)*} \\
  &- N_1 \Gamma_{rrar}^{(4)} \\
  &- N_2 N_{(23)}^{(14)} \Gamma_{rrar}^{(4)*} \\
  &- N_1 \Gamma_{rraa}^{(4)*} \\
  &- N_3 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*},
\end{align*}

(68b)

\begin{align*}
  \Gamma_{aara}^{(4)} &= N_2 N_4 \Gamma_{arrr}^{(4)} - N_2 N_3 N_{(23)}^{(14)} \Gamma_{rrra}^{(4)*} - N_3 N_4 N_{(34)}^{(12)} \Gamma_{rrrr}^{(4)*} \\
  &- N_1 \Gamma_{rrar}^{(4)*} \\
  &- N_2 N_{(23)}^{(14)} \Gamma_{rrar}^{(4)} \\
  &- N_1 \Gamma_{rraa}^{(4)*} \\
  &- N_3 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*},
\end{align*}

(68c)

\begin{align*}
  \Gamma_{araa}^{(4)} &= -N_3 N_4 \Gamma_{arrr}^{(4)} - \left( N_{(23)}^{(14)} + N_{(24)}^{(14)} \right) \Gamma_{rrra}^{(4)*} \\
  &+ N_2 N_3 N_{(23)}^{(14)} \Gamma_{rrrr}^{(4)*} \\
  &- N_1 \Gamma_{rrar}^{(4)*} \\
  &+ N_2 N_{(24)}^{(14)} \Gamma_{rrar}^{(4)} \\
  &+ N_1 \Gamma_{rraa}^{(4)*} \\
  &+ N_3 N_{(23)}^{(14)} \Gamma_{rrra}^{(4)*},
\end{align*}

(68d)

\begin{align*}
  \Gamma_{raaa}^{(4)} &= \left( 1 + N_2 N_3 + N_2 N_4 + N_3 N_4 \right) \Gamma_{arrr}^{(4)*} \\
  &- N_3 N_4 \Gamma_{rrra}^{(4)*} \\
  &- N_2 N_4 \Gamma_{rrrr}^{(4)*} \\
  &- N_2 N_3 \Gamma_{rrrr}^{(4)} \\
  &+ N_4 \Gamma_{rrar}^{(4)*} \\
  &- N_3 \Gamma_{rrra}^{(4)} \\
  &- N_2 \Gamma_{rraa}^{(4)}
\end{align*}

(68e)

\begin{align*}
  \Gamma_{arr}^{(4)} &= -N_4 \Gamma_{arr}^{(4)} + N_3 N_{(23)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &+ N_2 N_{(23)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &- N_1 \Gamma_{rrra}^{(4)*} \\
  &+ N_4 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &+ N_2 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &- N_1 \Gamma_{rraa}^{(4)*} \\
  &+ N_3 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*},
\end{align*}

(68f)

\begin{align*}
  \Gamma_{ara}^{(4)} &= -N_3 \Gamma_{arr}^{(4)} + N_4 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &+ N_2 N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &- N_1 \Gamma_{rrra}^{(4)*} \\
  &+ N_3 N_{(23)}^{(14)} \Gamma_{rrra}^{(4)*} \\
  &+ N_{(24)}^{(14)} \Gamma_{rrra}^{(4)*},
\end{align*}

(68g)

\begin{align*}
  \Gamma_{rara}^{(4)} &= -N_2 \Gamma_{arr}^{(4)} - N_1 \Gamma_{arr}^{(4)*} \\
  &+ N_4 N_{(34)}^{(12)} \Gamma_{rrra}^{(4)*} \\
  &+ N_3 N_{(34)}^{(12)} \Gamma_{rrra}^{(4)*} \\
  &+ N_{(34)}^{(12)} \Gamma_{rrra}^{(4)*},
\end{align*}

(68h)

These relations are the nonlinear generalization of the FDT for the amputated 1PI 4-point vertex function. They can again be obtained by substituting in (54) \( G \) by \( \Gamma^{(4)} \) and inter-
changing \( r \) with \( a \). The corresponding column vector representation is obtained by applying the above-mentioned substitution rule to Eq. (55).

V. CONCLUSIONS

Using the transformation between the single-time and physical representations of real-time thermal \( n \)-point functions, the tilde conjugation relation and the KMS condition, we have derived the nonlinear generalization of the fluctuation-dissipation theorem for real-time \( n \)-point Green functions and 1PI amputated vertex functions at finite temperature. This generalized FDT is formally identical for disconnected and connected \( n \)-point Green functions. The FDT for the amputated 1PI vertex functions is obtained from that for the Green functions by interchanging the indices \( r \) and \( a \).

The generalized FDT for nonlinear response functions provides model-independent relations between the various components of the RTF thermal \( n \)-point functions which can be used as consistency checks for approximations. The results derived in this paper can be used for elementary or composite bosonic fields with arbitrary interactions.

For the cases \( n = 2 \) and \( 3 \) the results from the generalized FDT were shown to reproduce known relationships. The new results for the case \( n = 4 \) are expected to be useful for the derivation of spectral representations for the 4-point function and in a calculation of transport coefficients like shear viscosity in scalar field theories. An experimental verification of these relations may be possible in applications to the recent developments of picosecond pulse techniques and multichannel data acquisition in nonlinear systems [33].

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APPENDIX A: RELATION BETWEEN $r/a$- AND $R/A$-FUNCTIONS

In this appendix we derive the relation between the $r/a$ vertex functions used in the text and the $R/A$-functions introduced by Aurenche and Becherrawy in Ref. [20]. As shown in [20], the propagator $G$ in the single-time representation can be diagonalized with two matrices $U$ and $V$,

$$ G = U \hat{G} V , $$

(A1)

where

$$ \hat{G} = \begin{pmatrix} G_{RR}, & G_{RA} \\ G_{AR}, & G_{AA} \end{pmatrix} = \begin{pmatrix} G_{\text{ret}}, & 0 \\ 0, & G_{\text{adv}} \end{pmatrix} , $$

(A2)

and we choose

$$ U = \begin{pmatrix} U_{1R}, & U_{1A} \\ U_{2R}, & U_{2A} \end{pmatrix} = \begin{pmatrix} 1, & n \\ 1, & 1 + n \end{pmatrix} , $$

(A3a)

$$ V = \begin{pmatrix} V_{R1}, & V_{R2} \\ V_{A1}, & V_{A2} \end{pmatrix} = \begin{pmatrix} 1 + n, & n \\ -1, & -1 \end{pmatrix} , $$

(A3b)

as in Ref. [23].

In the CTP formalism the relation between propagator $G$ and the advanced, retarded and correlation functions is [10]

$$ G = Q^\dagger \bar{G} Q , $$

(A4)

where

$$ \bar{G} = \begin{pmatrix} G_{aa}, & G_{ar} \\ G_{ra}, & G_{rr} \end{pmatrix} = \begin{pmatrix} 0, & G_{\text{adv}} \\ G_{\text{ret}}, & G_{\text{cor}} \end{pmatrix} , $$

(A5a)

$$ Q = \begin{pmatrix} Q_{a1}, & Q_{a2} \\ Q_{r1}, & Q_{r2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, & -1 \\ 1, & 1 \end{pmatrix} , $$

(A5b)

$$ Q^\dagger = Q^{-1} = \begin{pmatrix} Q_{1a}^\dagger, & Q_{1r}^\dagger \\ Q_{2a}^\dagger, & Q_{2r}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, & 1 \\ -1, & 1 \end{pmatrix} . $$

(A5c)
Combining Eqs. (A1) and (A4) leads to

$$\hat{G} = U^{-1}Q^\dagger GQV^{-1}. \quad (A6)$$

1. Self energy function

Substituting the Schwinger-Dyson equation

$$\hat{G}^{-1} = \hat{G}_0^{-1} - \hat{\Sigma}, \quad (A7a)$$

$$\bar{G}^{-1} = \bar{G}_0^{-1} - \bar{\Sigma} \quad (A7b)$$

into Eq. (A6) we deduce

$$\hat{\Sigma} = VQ^\dagger\bar{\Sigma}QU, \quad (A8)$$

where

$$\hat{\Sigma} = \begin{pmatrix} \Sigma_{RR}, \Sigma_{RA} \\ \Sigma_{AR}, \Sigma_{AA} \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \Gamma^{(2)}_{aa}, \Gamma^{(2)}_{ar} \\ \Gamma^{(2)}_{ra}, \Gamma^{(2)}_{rr} \end{pmatrix}. \quad (A9)$$

The explicit matrix form of Eq. (A8) is

$$\begin{pmatrix} \Sigma_{RR}, \Sigma_{RA} \\ \Sigma_{AR}, \Sigma_{AA} \end{pmatrix} =$$

$$\begin{pmatrix} \Gamma^{(2)}_{ar} + (1 + 2n)\Gamma^{(2)}_{rr}, -\frac{1}{2}\Gamma^{(2)}_{aa} + \frac{1}{2}(1 + 2n)(\Gamma^{(2)}_{ar} - \Gamma^{(2)}_{ra}) + \frac{1}{2}(1 + 2n)^2\Gamma^{(2)}_{rr} \\ -2\Gamma^{(2)}_{rr}, \Gamma^{(2)}_{ra} - (1 + 2n)\Gamma^{(2)}_{rr} \end{pmatrix}. \quad (A10)$$

Inserting the FDTs (62) and Eq. (20) into the above expression we get following relations:

$$\Sigma_{RR} = \Gamma^{(2)}_{ar}, \quad \Sigma_{AA} = \Gamma^{(2)}_{ra}, \quad \Sigma_{RA} = \Sigma_{AR} = 0, \quad \Sigma_{RR} = \Sigma_{AA}^*. \quad (A11)$$

The first two of these equations agree with Eqs. (4.3) and (4.7) in Ref. [20].

Please note that $\Sigma_{RR}$ and $\Sigma_{AA}$ do not vanish, in contrast to the multi-point cases (c.f. Eqs. (A16a) and (A20a)).
2. n-point vertex function \((n \geq 3)\)

As shown in Ref. [20], the thermal distribution functions which occur in the finite-temperature propagators via the matrices \(U(k^0)\) and \(V(k^0)\) can be fully absorbed into the vertex functions. To this end one combines the matrices \(U\) and \(V\) in the propagators \((A1)\) with the vertex to which \(G\) attaches; \(U(k^0)\) is associated with an outgoing line while \(V(k^0)\) is associated to an incoming line with momentum \(k\). This defines the so-called \(R/A\) vertex functions [20]. Thus the \(n\)-point \(R/A\) vertex function with all incoming momenta \(k_1, k_2, \ldots, k_n\) can be expressed as

\[
\hat{\Gamma}^{(n)}_{\Lambda_1 \ldots \Lambda_n}(k_1 \ldots k_n) = V_{\Lambda_1 a_1}(k_1^0) \ldots V_{\Lambda_n a_n}(k_n^0) \Gamma^{(n)}_{a_1 \ldots a_n}(k_1 \ldots k_n),
\]

where \(\Lambda_1, \ldots, \Lambda_n = R, A\). \(\hat{\Gamma}^{(n)}_{\Lambda_1 \ldots \Lambda_n}\) incorporates all temperature dependence of the finite-temperature retarded and advanced fields [14]. Inserting Eq. \((24)\) into the above equation and defining

\[
P_{\Lambda, \alpha_i}(k_i^0) = \sqrt{2} V_{\Lambda, a_j}(k_i^0) Q_{a_j \alpha_i}^{\dagger},
\]

we obtain the relation between the \(r/a\)-functions used in the text and the \(R/A\)-functions of Aurenche and Becherrawy [20]:

\[
\hat{\Gamma}^{(n)}_{\Lambda_1 \ldots \Lambda_n}(k_1 \ldots k_n) = 2^{1-n} P_{\Lambda_1 \alpha_1}(k_1^0) \ldots P_{\Lambda_n \alpha_n}(k_n^0) \Gamma^{(n)}_{\alpha_1 \ldots \alpha_n}(k_1 \ldots k_n),
\]

where \(\alpha_1, \ldots, \alpha_n = r, a\). The explicit form of Eq. \((A13)\) reads

\[
P_{Ra}(k_i^0) = 1, \quad P_{Rr}(k_i^0) = N(k_i^0) = 1 + 2n(k_i^0),
\]

\[
P_{Aa}(k_i^0) = 0, \quad P_{Ar}(k_i^0) = -2.
\]

Clearly the \(R/A\) vertex functions \(\hat{\Gamma}^{(n)}_{\Lambda_1 \ldots \Lambda_n}\) are linear combinations of the \(r/a\) vertex functions \(\Gamma^{(n)}_{\alpha_1 \ldots \alpha_n}\). We will now show that the \(R/A\) vertex functions have simple relations with the independent components of the \(r/a\) vertex functions after inserting the generalized FDTs.
a. 3-point vertex function

Substituting the FDTs (66) and Eq. (20) into Eq. (A14) for \( n=3 \) we deduce

\[
\begin{align*}
\hat{\Gamma}_{AAA}^{(3)} &= \hat{\Gamma}_{RRR}^{(3)} = 0, \quad (A16a) \\
\hat{\Gamma}_{RAA}^{(3)} &= \Gamma_{arr}^{(3)}, \quad (A16b) \\
\hat{\Gamma}_{ARA}^{(3)} &= \Gamma_{rar}^{(3)}, \quad (A16c) \\
\hat{\Gamma}_{AAR}^{(3)} &= \Gamma_{rra}^{(3)}, \quad (A16d) \\
\hat{\Gamma}_{ARR}^{(3)} &= -\frac{1}{2}(N_2 + N_3)\Gamma_{arr}^{(3)*}, \quad (A16e) \\
\hat{\Gamma}_{RAR}^{(3)} &= -\frac{1}{2}(N_1 + N_3)\Gamma_{rar}^{(3)*}, \quad (A16f) \\
\hat{\Gamma}_{RRA}^{(3)} &= -\frac{1}{2}(N_1 + N_2)\Gamma_{rra}^{(3)*}. \quad (A16g)
\end{align*}
\]

These relations can be also obtained in the following simpler manner. Noticing

\[
V(k_0^0)(1) = (0), \quad V(k_0^0)(n(k_i^0)) = (0), \quad (A17)
\]

and substituting Eq. (67) into Eq. (A12) we have

\[
\begin{align*}
\left(\hat{\Gamma}_{A_1 A_2 A_3}^{(3)}\right) &= \Gamma_{arr}^{(3)}\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) - \frac{1}{2}(N_2 + N_3)\Gamma_{arr}^{(3)*}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) \\
&+ \Gamma_{rar}^{(3)}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) - \frac{1}{2}(N_1 + N_3)\Gamma_{rar}^{(3)*}\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right) \\
&+ \Gamma_{rra}^{(3)}\left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right) - \frac{1}{2}(N_1 + N_2)\Gamma_{rra}^{(3)*}\left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right). \quad (A18)
\end{align*}
\]

Eqs. (A16) can now be directly read off. It should be noted that the FDTs play an important role in getting Eqs. (A16).

Using \( \hat{\Gamma}_{AAA}^{(3)}, \hat{\Gamma}_{ARA}^{(3)} \) and \( \hat{\Gamma}_{AAR}^{(3)} \) as independent components, Eqs. (A16) can be used to rewrite Eq. (57) as

\[
\left(\Gamma_{a_1 a_2 a_3}^{(3)}\right) = \hat{\Gamma}_{RAA}^{(3)}\left(\begin{array}{ccc} 1 & n_2 & n_3 \\ -1 & -(1+n_2) & -(1+n_3) \end{array}\right) - \hat{\Gamma}_{RAA}^{(3)*}\left(\begin{array}{ccc} 1 + n_2(1+n_3) - n_2 n_3 & n_1 \\ (1+n_1) - n_1 & -(1+n_1) \end{array}\right). \quad (A19)
\]
\[
+ \hat{\Gamma}_{ARA}^{(3)} \left( \frac{n_1}{-(1 + n_1)} \right) \left( \frac{1}{-1} \right) \left( \frac{n_3}{-(1 + n_3)} \right) \\
- \hat{\Gamma}_{ARA}^{(3)*} \left( \frac{1 + n_1}{(1 + n_2) - n_2} \right) \left( \frac{1}{-(1 + n_2)} \right) \left( \frac{n_2}{-1} \right) \\
+ \hat{\Gamma}_{AAR}^{(3)} \left( \frac{n_1}{-(1 + n_1)} \right) \left( \frac{n_2}{-(1 + n_2)} \right) \left( \frac{1}{-1} \right) \\
- \hat{\Gamma}_{AAR}^{(3)*} \left( \frac{1 + n_1}{(1 + n_3) - n_3} \right) \left( \frac{1}{-(1 + n_3)} \right) \left( \frac{n_3}{-1} \right) .
\] (A19)

\[b. \quad 4\text{-point vertex function}\]

Similar to the 3-point vertex function, using Eqs. (A12) or (A14) we deduce

\[
\hat{\Gamma}_{AAAA}^{(4)} = \hat{\Gamma}_{RRRR}^{(4)} = 0 ,
\] (A20a)

\[
\hat{\Gamma}_{RRRA}^{(4)} = \frac{1 + n_1}{n_4} (1 + n_2)(1 + n_3) \Gamma_{rrra}^{(4)*} ,
\] (A20b)

\[
\hat{\Gamma}_{RRAR}^{(4)} = \frac{1 + n_1}{n_3} (1 + n_2)(1 + n_4) \Gamma_{rrar}^{(4)*} ,
\] (A20c)

\[
\hat{\Gamma}_{RARR}^{(4)} = \frac{1 + n_2}{n_2} (1 + n_3)(1 + n_4) \Gamma_{rarr}^{(4)*} ,
\] (A20d)

\[
\hat{\Gamma}_{ARRR}^{(4)} = \frac{1 + n_2}{n_1} (1 + n_3)(1 + n_4) \Gamma_{arrr}^{(4)*} ,
\] (A20e)

\[
\hat{\Gamma}_{RRAA}^{(4)} = \frac{1}{2} N_{(34)}^{(12)} \left( \Gamma_{rraa}^{(4)*} + N_3 \Gamma_{rrra}^{(4)*} + N_4 \Gamma_{rrar}^{(4)*} \right) ,
\] (A20f)

\[
\hat{\Gamma}_{RAAA}^{(4)} = \frac{1}{2} N_{(24)}^{(13)} \left( \Gamma_{rara}^{(4)*} + N_2 \Gamma_{rrra}^{(4)*} + N_4 \Gamma_{rrar}^{(4)*} \right) ,
\] (A20g)

\[
\hat{\Gamma}_{RAAR}^{(4)} = \frac{1}{2} N_{(23)}^{(14)} \left( \Gamma_{raa}^{(4)*} + N_2 \Gamma_{rrra}^{(4)*} + N_3 \Gamma_{rrar}^{(4)*} \right) ,
\] (A20h)

\[
\hat{\Gamma}_{AARR}^{(4)} = \frac{1}{2} \left( \Gamma_{rraa}^{(4)*} + N_3 \Gamma_{rrra}^{(4)*} + N_4 \Gamma_{rrra}^{(4)*} \right) ,
\] (A20i)

\[
\hat{\Gamma}_{ARAA}^{(4)} = \frac{1}{2} \left( \Gamma_{rara}^{(4)*} + N_2 \Gamma_{rrra}^{(4)*} + N_4 \Gamma_{rrar}^{(4)*} \right) ,
\] (A20j)

\[
\hat{\Gamma}_{RAAA}^{(4)} = \frac{1}{2} \left( \Gamma_{raa}^{(4)*} + N_2 \Gamma_{rrra}^{(4)*} + N_3 \Gamma_{rrar}^{(4)*} \right) ,
\] (A20k)

\[
\hat{\Gamma}_{AAAR}^{(4)} = -\Gamma_{rraa}^{(4)} ,
\] (A20l)

\[
\hat{\Gamma}_{AARA}^{(4)} = -\Gamma_{rrra}^{(4)} ,
\] (A20m)

\[
\hat{\Gamma}_{ARAA}^{(4)} = -\Gamma_{rrra}^{(4)} ,
\] (A20n)

\[
\hat{\Gamma}_{RAAA}^{(4)} = -\Gamma_{arrr}^{(4)} .
\] (A20o)
for 4-point vertex function.

Using \( \hat{\Gamma}^{(4)}_{RAAA}, \hat{\Gamma}^{(4)*}_{ARAA}, \hat{\Gamma}^{(4)}_{AARA}, \hat{\Gamma}^{(4)}_{AAAR}, \hat{\Gamma}^{(4)}_{RAAR}, \hat{\Gamma}^{(4)}_{RARA}, \hat{\Gamma}^{(4)}_{RRAA} \) as independent components, similar to Eq. \((A19)\) we can express the 4-point vertex function in the single-time representation in terms of column vectors as \([32]\):

\[
\Gamma^{(4)}_{a_1a_2a_3a_4} = \hat{\Gamma}^{(4)}_{RAAA} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_2 \\ -n_2 \end{pmatrix} \begin{pmatrix} n_3 \\ -n_3 \end{pmatrix} \begin{pmatrix} n_4 \\ -n_4 \end{pmatrix} \\
+ \hat{\Gamma}^{(4)*}_{ARAA} \frac{(1 + n_2)(1 + n_3)(1 + n_4) - n_2n_3n_4}{(1 + n_1) - n_1} \begin{pmatrix} n_1 \\ -(1 + n_1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
+ \hat{\Gamma}^{(4)}_{AARA} \begin{pmatrix} n_1 \\ -(1 + n_1) \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} n_3 \\ -(1 + n_3) \end{pmatrix} \begin{pmatrix} n_4 \\ -(1 + n_4) \end{pmatrix} \\
+ \hat{\Gamma}^{(4)*}_{ARAR} \frac{(1 + n_1)(1 + n_3)(1 + n_4) - n_1n_3n_4}{(1 + n_2) - n_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_2 \\ -(1 + n_2) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
+ \hat{\Gamma}^{(4)}_{AAAR} \begin{pmatrix} n_1 \\ -(1 + n_1) \end{pmatrix} \begin{pmatrix} n_2 \\ -(1 + n_2) \end{pmatrix} \begin{pmatrix} n_3 \\ -(1 + n_3) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
+ \hat{\Gamma}^{(4)*}_{AARA} \frac{(1 + n_1)(1 + n_2)(1 + n_4) - n_1n_2n_4}{(1 + n_3) - n_3} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_4 \\ -(1 + n_4) \end{pmatrix} \\
+ \hat{\Gamma}^{(4)}_{RAAR} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_2 \\ -(1 + n_2) \end{pmatrix} \begin{pmatrix} n_3 \\ -(1 + n_3) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
+ \hat{\Gamma}^{(4)*}_{RARA} \frac{(1 + n_2)(1 + n_3) - n_2n_3}{(1 + n_1) - n_1} \begin{pmatrix} n_1 \\ -(1 + n_1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_4 \\ -(1 + n_4) \end{pmatrix} \\
+ \hat{\Gamma}^{(4)}_{ARRA} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} n_2 \\ -(1 + n_2) \end{pmatrix} \begin{pmatrix} n_3 \\ -(1 + n_3) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
+ \hat{\Gamma}^{(4)*}_{RRAR} \frac{(1 + n_3)(1 + n_4) - n_3n_4}{(1 + n_1) - n_1} \begin{pmatrix} n_1 \\ -(1 + n_1) \end{pmatrix} \begin{pmatrix} n_2 \\ -(1 + n_2) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{A21}\]

Please note the much more symmetric structure of this equation compared to Eq. \((B5)\). A generalization of Eqs. \((A19)\) and \((X21)\) to \(n \geq 5\) was recently given in \([32]\). We emphasize once more that these representations (in particular the vanishing of \(\hat{\Gamma}^{(n)}_{R...R}\)) rely heavily on the validity of the generalized FDTs (or, equivalently, of the KMS condition). Similarly
simple expressions can thus not be expected to hold outside of thermal equilibrium.

From Eqs. (A10) and (A20) we can verify the important symmetry relation

\[ \hat{\Gamma}^{(n)}_{\Lambda_1...\Lambda_n}(k_1...k_n) = (-1)^{n+1} \frac{\prod_{i|\Lambda_i=R} \left(1 + n(k_i^0)\right)}{\prod_{i|\Lambda_i=A} n(k_i^0)} \hat{\Gamma}^{(n)*}_{\bar{\Lambda}_1...\bar{\Lambda}_n}(k_1...k_n), \tag{A22} \]

where \( \bar{\Lambda}_i = A \) for \( \Lambda_i = R \) and vice versa. A similar relation was given in Ref. [21].
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