A splitting algorithm for fixed points of non-expansive mappings and equilibrium problems

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Abstract. We consider the problem of finding a fixed point of a nonexpansive mapping, which is also a solution of a pseudo-monotone equilibrium problem, where the bifunction in the equilibrium problem is the sum of two ones. We propose a splitting algorithm combining the gradient method for equilibrium problem and the Mann iteration scheme for fixed points of nonexpansive mappings. At each iteration of the algorithm, two strongly convex subprograms are required to solve separately, one for each of the component bifunctions. Our main result states that, under paramonotonicity property of the given bifunction, the algorithm converges to a solution without any Lipschitz type condition as well as Hölder continuity of the bifunctions involved.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with weak topology defined by the inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset and $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ a bifunction such that $f(x, y) < +\infty$ for every $x, y \in C$. The equilibrium problem defined by the Nikaido-Isoda-Fan inequality that we are going to deal with in this paper is given as

$$\text{Find } x \in C \text{ such that } f(x, y) \geq 0 \text{ for all } y \in C. \quad (EP)$$

In 1955, Nikaido and Isoda [25] first used this inequality in convex game models. Then in 1972 Ky Fan [12] called this inequality a minimax one and established existence theorems for $(EP)$. After the appearance of the paper by Blum and Oettli [6], this problem has been attracted much attention of researchers. In [3, 6, 22] it has been shown that some important problems
such as optimization, variational inequality, Kakutani fixed point, and Nash equilibria can be formulated in the form of \((EP)\). Many papers concerning the solution existence, stabilities as well as algorithms for \((EP)\) have been published (see e.g. [10, 16, 18, 20, 23, 24, 27, 28, 30] and the excellent survey paper [3]).

Recently the problem of finding a solution of an equilibrium problem which is also a fixed point of a nonexpansive mapping has been considered in some papers (see e.g. [31, 33, 34, 38] and the references therein). The existing methods combine algorithms for solving \((EP)\) such as the projection, extragradient, and proximal point methods with iterative schemes for finding fixed points of nonexpansive mappings. These methods require either computing the projection onto the feasible domain \(C\), or solving convex and/or strongly monotone regularized equilibrium subproblems (see e.g. [2, 3, 15, 19, 33, 34, 38]). However, in general, solving these subproblems is computational cost. In order to reduce the computational cost, several splitting algorithms have been developed for some classes of maximal monotone operator inclusion, variational inequality, and equilibrium problems (see e.g. [1, 8, 9, 11, 13, 14, 21, 26, 36]).

In this paper we propose splitting algorithms for finding a point in the intersection of the fixed point set of a finite number of nonexpansive mappings and the solution set of an equilibrium problem, where the bifunction is the sum of two bifunctions. The algorithm is a combination between the gradient method for equilibrium problem and the Mann iteration scheme for fixed point of nonexpansive mappings. The main features of the proposed algorithm are the followings:

- At each iteration, it requires solving two strongly convex programs, one for each component bifunction separably rather than for their sum;
- Evaluating each nonexpansive mapping can be done in parallel;
- Convergence of the proposed algorithms is ensured without any Lipschitz type or Hölder conditions that are required in some existing splitting algorithms for equilibrium problems (e.g. in [1, 13]).

The remaining part of the paper is organized as follows. The next section are preliminaries containing some lemmas that will be used in proving the convergence of our proposed algorithms. Section 3 is devoted to the formulation of our considered problem, the description of the proposed algorithm, and its convergence analysis. Section 4 shows some variants of the algorithm when applying to solve some special cases of the problem. The last section closes the paper with some conclusions.

2. Preliminaries

We recall the following well-known definition on monotonicity of bifunctions (see e.g. [5]).

**Definition 1.** A bifunction \(f : H \times H \to \mathbb{R} \cup \{+\infty\}\) is said to be
(i) strongly monotone on $C$ with modulus $\beta > 0$ (shortly $\beta$-strongly monotone) if
\[ f(x, y) + f(y, x) \leq -\beta \|y - x\|^2 \quad \forall x, y \in C; \]
(ii) monotone on $C$ if
\[ f(x, y) + f(y, x) \leq 0 \quad \forall x, y \in C; \]
(iii) strongly pseudo-monotone on $C$ with modulus $\beta > 0$ (shortly $\beta$-strongly pseudo-monotone) if
\[ f(x, y) \geq 0 \implies f(y, x) \leq -\beta \|y - x\|^2 \quad \forall x, y \in C; \]
(iv) pseudo-monotone on $C$ if
\[ f(x, y) \geq 0 \implies f(y, x) \leq 0 \quad \forall x, y \in C; \]
(v) paramonotone on $C$ with respect to a set $S$ if
\[ x^* \in S, x \in C \text{ and } f(x^*, x) = f(x, x^*) = 0 \implies x \in S. \]

Obviously, (i) $\implies$ (ii) $\implies$ (iv) and (i) $\implies$ (iii) $\implies$ (iv).

Note that a strongly pseudo-monotone bifunction may not be monotone. Paramonotone bifunctions have been used in e.g. [3, 31, 32]. Clearly in the case of optimization problem when $f(x, y) = \varphi(y) - \varphi(x)$, the bifunction $f$ is paramonotone on $C$ with respect to the solution set of the problem $\min_{x \in C} \varphi(x)$. Conditions for a bifunction $f$ to be paramonotone can be found in [17].

The following well known lemmas will be used for proving the convergence of the algorithm proposed in the next section.

Lemma 1. (see [35] Lemma 1) Let $\{\alpha_k\}$ and $\{\sigma_k\}$ be two sequences of nonnegative numbers such that $\alpha_{k+1} \leq \alpha_k + \sigma_k$ for all $k \in \mathbb{N}$, where $\sum_{k=1}^{\infty} \sigma_k < \infty$. Then the sequence $\{\alpha_k\}$ is convergent.

Lemma 2. Let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. Then for $x, y, z \in \mathcal{H}$ and $0 \leq \gamma \leq 1$, one has
\[ \|\gamma x + (1 - \gamma)y - z\|^2 = \gamma \|x - z\|^2 + (1 - \gamma)\|y - z\|^2 - \gamma(1 - \gamma)\|x - y\|^2. \]

Proof. By definition of the inner product and its reduced norm we have
\[
\begin{align*}
\|\gamma x + (1 - \gamma)y - z\|^2 &= \|\gamma(x - z) + (1 - \gamma)(y - z)\|^2 \\
&= \gamma^2\|x - z\|^2 + (1 - \gamma)^2\|y - z\|^2 + 2\gamma(1 - \gamma)\langle x - z, y - z \rangle \\
&= \gamma\|x - z\|^2 + (1 - \gamma)\|y - z\|^2 \\
&\quad - \gamma(1 - \gamma)\left(\|x - z\|^2 + \|y - z\|^2 - 2\langle x - z, y - z \rangle\right) \\
&= \gamma\|x - z\|^2 + (1 - \gamma)\|y - z\|^2 - \gamma(1 - \gamma)\|x - z\|\|y - z\| \\
&= \gamma\|x - z\|^2 + (1 - \gamma)\|y - z\|^2 - \gamma(1 - \gamma)\|x - y\|^2.
\end{align*}
\]

This proves the lemma. $\square$
3. Problem formulation, algorithm and its convergence

3.1. The problem and its special cases

Let $T : C \to C$ be a nonexpansive mapping, that is

$$
\|T(x) - T(y)\| \leq \|x - y\| \quad \forall x, y \in C.
$$

The set of all fixed points of the mapping $T$ is denoted by $\text{Fix}(T)$. Let $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction. In what follows we suppose that $f(x, y) = f_1(x, y) + f_2(x, y)$ and that $f_i(x, x) = 0$ ($i = 1, 2$) for every $x, y \in C$. The following assumptions will be used in the sequel.

(A1) For each $x \in C$, the functions $f_1(x, \cdot)$ and $f_2(x, \cdot)$ are convex, subdifferentiable on an open set containing $C$, while the function $f(\cdot, x)$ is weakly upper semicontinuous on $C$.

(A2) The bifunction $f$ is pseudo-monotone on $C$.

(A3) Either $\text{int} C \neq \emptyset$ or, for every $x \in C$, each function $f_i(x, \cdot)$ is continuous at a point of $C$.

The main problem we are considering in this paper is to find a fixed point of $T$ which is also an equilibrium point of $f$ on $C$. More formally, the problem is stated as follows.

Find $x^* \in C$ such that $x^* = T(x^*)$ and $f(x^*, y) \geq 0$ for all $y \in C$. \hspace{1cm} (P)

Let us mention some typical examples for Problem (P).

1. Equilibrium problem over the set of common fixed points of nonexpansive mappings. Let $T_i : C \to C$ be nonexpansive mappings. Consider the following problem

Find $x^* \in C$ such that

$$
x^* = T_i(x^*) \quad \text{for all } i = 1, \ldots, m \text{ and } f(x^*, y) \geq 0 \quad \text{for all } y \in C.
$$

This problem can be casted into Problem (P), thanks to the following lemma.

Lemma 3. \hspace{1cm} (see [4] Proposition 4.34). Let $\mu_i > 0$ ($i = 1, \ldots, m$), $\sum_{i=1}^m \mu_i = 1$ and $T(x) := \sum_{i=1}^m \mu_i T_i(x)$ for every $x \in C$. Then $T$ is nonexpansive on $C$ and its fixed point set coincides the intersection of the fixed point sets of $T_i$ ($i = 1, \ldots, m$).

2. Equilibrium problem over the intersection of closed convex sets. Consider the problem

Find $x \in C := \bigcap_{i=1}^m C_i$ such that $f(x, y) \geq 0$ for all $y \in C$. \hspace{1cm} (P_1)

where $C_i$ ($i = 1, \ldots, m$) are closed convex sets. In this case, we can take $T_i(x) := P_{C_i}(x)$ for each $i = 1, \ldots, m$ (i.e., $T_i$ is the projection map on $C_i$), and take $T(x) := \sum_{i=1}^m \mu_i T_i(x)$ with $0 < \mu_i < 1$ for every $i$, $\sum_{i=1}^m \mu_i = 1$. Then by Lemma 3 we have $\text{Fix}(T) \equiv C$, and therefore Problem (P_1) can be formulated in form of Problem (P).
3. Common solution of equilibrium problem and maximal monotone operator inclusion. Consider the problem

\[ \text{Find } x \in C \text{ such that } f(x, y) \geq 0 \text{ for all } y \in C \text{ and } 0 \in M_i(x) \text{ for all } i = 1, \ldots, m, \quad (P_2) \]

where \( M_i(i = 1, \ldots, m) \) is maximal monotone multi-valued operators on \( \mathcal{H} \). It is well-known (see e.g. [29]) that the operator \( T_i := (M_i + cI)^{-1} \) with \( c > 0 \) is defined everywhere, single-valued, nonexpansive on the whole space and its fixed point set coincides with the solution set of the inclusion \( 0 \in M_i(x) \). Thus Problem \( (P_2) \) can be reformulated as \( (P) \).

4. Split equilibrium problem. The split feasibility problem introduced in [7] is given as

\[ \text{Find } x \in U \text{ such that } Ax \in V, \quad (SFP) \]

where \( U \) and \( V \) are respectively nonempty closed convex subsets of Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), and \( A \) is a bounded linear operator from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \). Let us consider this problem with \( U \) being the solution set of equilibrium problem \( (EP) \) and the inclusion \( Ax \in V \) being represented by the solution set of the system of inequalities

\[ \langle a^i, x \rangle \leq b_i \quad (i = 1, \ldots, m). \]

In this setting, Problem \( (SFP) \) can be written as

\[ \text{Find } x \in C \text{ such that } f(x, y) \geq 0 \text{ for all } y \in C \text{ and } \langle a^i, x \rangle \leq b_i \quad (SEP) \]

Let \( H_i \) be the half space \( \{ x \in \mathcal{H} \mid \langle a^i, x \rangle - b_i \leq 0 \} \). Then Problem \( (SEP) \) can take the form of Problem \( (P) \) with \( T(x) := \sum_{i=1}^{m} \mu_i P_{H_i}(x) \), where \( P_{H_i} \) is the projection operator onto the half space \( H_i \), and \( \mu_i(i = 1, \ldots, m) \) are positive real numbers such that \( \sum_{i=1}^{m} \mu_i = 1 \).

3.2. The algorithm and its convergence analysis

The algorithm below is a combination between the gradient one for pseudo-monotone equilibrium problem \( (EP) \) and the Mann iterative scheme for finding fixed points of the nonexpansive mapping \( T \). The stepsize is computed as in the algorithm for equilibrium problem in [30].
Algorithm 1 A splitting algorithm for solving \((P)\).

Initialization: Seek \(x^0 \in C\). Choose \(\gamma \in (0, 1)\) and a sequence \(\{\beta_k\}_{k \geq 0} \subset \mathbb{R}\) satisfying the following conditions
\[
\sum_{k=0}^{\infty} \beta_k = +\infty, \quad \sum_{k=0}^{\infty} \beta_k^2 < +\infty.
\]

Iteration \(k = 0, 1, \ldots\):
Take \(g_k^1 \in \partial_2 f_1(x^k, x^k), g_k^2 \in \partial_2 f_2(x^k, x^k)\).
Compute
\[
\eta_k := \max\{\beta_k, \|g_k^1\|, \|g_k^2\|\}, \quad \lambda_k := \frac{\beta_k}{\eta_k},
\]
\[
y^k := \arg\min\{\lambda_k f_1(x^k, y) + \frac{1}{2}\|y - x^k\|^2 \mid y \in C\},
\]
\[
z^k := \arg\min\{\lambda_k f_2(x^k, y) + \frac{1}{2}\|y - y^k\|^2 \mid y \in C\},
\]
\[
x^{k+1} := \gamma z^k + (1 - \gamma)T(x^k).
\] (1)

In order to prove the convergence of Algorithm 1, we need the auxiliary results in the following propositions. For that we denote by \(\Omega\) the solution set of Problem \((P)\) and assume that \(\Omega \neq \emptyset\).

Proposition 1. For each \(x^* \in \Omega\), the sequence \(\{\|x^k - x^*\|\}_{k \in \mathbb{N}}\) is convergent.

Proof. To simplify the notations, for each \(k \geq 0\) let
\[
h_k^1(x) := \lambda_k f_1(x^k, x) + \frac{1}{2}\|x - x^k\|^2,
\]
\[
h_k^2(x) := \lambda_k f_2(x^k, x) + \frac{1}{2}\|x - y^k\|^2.
\]
By Assumption (A1), the function \(h_k^1\) is strongly convex with modulus 1 and sub-differentiable, which implies
\[
h_k^1(y^k) + \langle u_k^1, x - y^k \rangle + \frac{1}{2}\|x - y^k\|^2 \leq h_k^1(x) \quad \forall x \in C \quad (2)
\]
for any \(u_k^1 \in \partial h_k^1(y^k)\). As defined in Algorithm 1, \(y^k\) is a minimizer of \(h_k^1(\cdot)\) over \(C\). Therefore, by Assumption (A3) and the optimality condition for convex programming, we have
\[
0 \in \partial h_k^1(y^k) + N_C(y^k),
\]
which implies that there exists \(u_k^1 \in -\partial h_k^1(y^k)\) such that \(\langle u_k^1, x - y^k \rangle \geq 0\) for all \(x \in C\). Hence, for each \(x \in C\), it follows from (2) that
\[
h_k^1(y^k) + \frac{1}{2}\|x - y^k\|^2 \leq h_k^1(x),
\]
i.e.,
\[
\lambda_k f_1(x^k, y^k) + \frac{1}{2}\|y^k - x^k\|^2 + \frac{1}{2}\|x - y^k\|^2 \leq \lambda_k f_1(x^k, x) + \frac{1}{2}\|x - x^k\|^2,
\]
or equivalently,
\[ ||y^k - x||^2 \leq ||x^k - x||^2 + 2\lambda_k (f_1(x^k, x) - f_1(x^k, y^k)) - ||y^k - x^k||^2. \]  
By the same argument on \( h^k_x(\cdot) \) and \( z^k \), we have
\[ ||z^k - x||^2 \leq ||y^k - x||^2 + 2\lambda_k (f_2(x^k, x) - f_2(x^k, z^k)) - ||z^k - y^k||^2. \]
Combining (3) and (4) yields
\[ ||z^k - x||^2 \leq ||x^k - x||^2 + 2\lambda_k f(x^k, x) - ||y^k - x^k||^2 - ||z^k - y^k||^2 - 2\lambda_k (f_1(x^k, y^k) + f_2(x^k, z^k)). \]
Since \( g^k_1 \in \partial f_1(x^k; x^k) \) and \( f_1(x^k, x^k) = 0 \), we have
\[ f_1(x^k, y^k) = f_1(x^k, y^k) - f_1(x^k, x^k) \geq \langle g^k_1, y^k - x^k \rangle, \]
which implies
\[ -2\lambda_k f_1(x^k, y^k) \leq -2\lambda_k \langle g^k_1, y^k - x^k \rangle. \]
By Cauchy-Schwarz inequality and the fact that \( ||g_k^1|| \leq \eta_k \), from (6) we have
\[ -2\lambda_k f_1(x^k, y^k) \leq \frac{2\beta_k}{\eta_k} \|y^k - x^k\| = 2\beta_k \|y^k - x^k\|. \]
By the same argument, we obtain
\[ -2\lambda_k f_2(x^k, z^k) \leq 2\beta_k \|z^k - x^k\|. \]
Replacing (7) and (8) to (5) we get
\[ ||z^k - x||^2 \leq ||x^k - x||^2 + 2\lambda_k f(x^k, x) - ||y^k - x^k||^2 - 2\lambda_k f(x^k, y^k) + 2\beta_k \|y^k - x^k\| + 2\beta_k \|z^k - x^k\| \]
\[ \leq ||x^k - x||^2 + 2\lambda_k f(x^k, x) + 2\beta_k^2 \]
Taking \( x = x^* \in \Omega \subseteq C \) in (9) we get
\[ ||z^k - x^*||^2 \leq ||x^k - x^*||^2 + 2\lambda_k f(x^k, x^*) + 2\beta_k^2. \]
Furthermore, since \( x^{k+1} = \gamma z^k + (1 - \gamma)T(x^k) \) as defined in Algorithm 1 we have
\[ ||x^{k+1} - x^*||^2 = ||\gamma z^k + (1 - \gamma)T(x^k) - x^*||^2 \]
\[ = \gamma ||z^k - x^*||^2 + (1 - \gamma) ||T(x^k) - T(x^*)||^2 - \gamma (1 - \gamma) ||z^k - T(x^k)||^2 \]
\[ \leq \gamma ||z^k - x^*||^2 + (1 - \gamma) ||x^k - x^*||^2 - \gamma (1 - \gamma) ||z^k - T(x^k)||^2 \]
\[ \leq \gamma (||x^k - x^*||^2 + 2\lambda_k f(x^k, x^*) + 2\beta_k^2) + (1 - \gamma) ||x^k - x^*||^2 - \gamma (1 - \gamma) ||z^k - T(x^k)||^2 \]
\[ = ||x^k - x^*||^2 + 2\gamma \lambda_k f(x^k, x^*) + 2\gamma \beta_k^2 - \gamma (1 - \gamma) ||z^k - T(x^k)||^2. \]
Here, the second equality follows from Lemma 2 and the fact that $T(x^*) = x^*$, the first inequality is due to the non-expansiveness of the mapping $T$, the second inequality is a consequence of (10), while the last equality is trivial. Now we note that $f(x^*, x^k) \geq 0$ since $x^*$ belongs to the solution set of $(P)$. This implies that $f(x^k, x^*) \leq 0$ by pseudo-monotonicity of the bifunction $f$ on $C$. From (11), by the negativity of $f(x^k, x^*)$ and due to $\gamma \in (0, 1)$, we obtain

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\gamma\beta_k^2. \quad (12)$$

Since $\gamma > 0$ and $\sum_{k=1}^{\infty} \beta_k^2 < \infty$, in virtue of Lemma 1 the inequality (12) implies that the sequence $\{\|x^k - x^*\|\}_{k \in \mathbb{N}}$ is convergent. This closes the proof of the proposition.

Proposition 2. Any weakly cluster point of $\{x^k\}_{k \in \mathbb{N}}$ is a fixed point of $T$.

**Proof.** In the following we will show that $\|T(x^k) - x^k\| \to 0$ as $k \to \infty$. The proposition follows immediately from this claim.

Indeed, by taking $x = x^k$ in (9) and note that $f(x^k, x^k) = 0$, we obtain

$$\|z^k - x^k\|^2 \leq 2\beta_k^2,$$

which implies

$$\lim_{k \to \infty} \|z^k - x^k\| = 0, \quad (13)$$

since $\beta_k \to 0$ as $k \to \infty$. On the other hand, let $x^* \in \Omega$ be fixed, then $f(x^k, x^*) \leq 0$. Therefore, from (11) we have

$$\gamma(1 - \gamma)\|T(x^k) - z^k\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\gamma\lambda_k f(x^k, x^*) + 2\gamma\beta_k^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\gamma\beta_k^2,$$

which implies

$$\lim_{k \to \infty} \|T(x^k) - z^k\| = 0, \quad (14)$$

since $\{\|x^k - x^*\|\}$ is convergent, $\gamma \in (0, 1)$, and $\beta_k \to 0$ as $k \to \infty$. To the end, by (13) and (14), we obtain

$$\|T(x^k) - x^k\| \leq \|T(x^k) - z^k\| + \|z^k - x^k\| \to 0 \text{ as } k \to \infty.$$

This closes the proof of the proposition.

We now establish the convergence result in the following theorem.

**Theorem 1.** Suppose that $f$ is paramonotone on $C$ with respect to the solution set $\text{Sol}(C, f)$ of problem $(EP)$. Then under the assumptions (A1), (A2), (A3), the sequence $\{x^k\}_{k \in \mathbb{N}}$ generated by Algorithm 1 converges weakly to a solution of $(P)$, provided that $(P)$ admits a solution.

**Proof.** Let $x^*$ be in the solution set $\Omega$ of $(P)$. As obtained in the proof of Proposition 1 from (11) and the negativity of $f(x^k, x^*)$ we have

$$0 \leq -2\gamma\lambda_k f(x^k, x^*) \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\gamma\beta_k^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\gamma\beta_k^2$$
for every $k \in \mathbb{N}$, which implies that

$$0 \leq -2\gamma \sum_{k=0}^{\infty} \lambda_k f(x^k, x^*) \leq \|x^0 - x^*\|^2 + 2 \sum_{k=0}^{\infty} \beta_k^2 < +\infty,$$

(15)

since $\sum_{k=0}^{\infty} \beta_k^2 < +\infty$. On the other hand, note that the sequences $\{g_1^k\}_{k \in \mathbb{N}}$ and $\{g_2^k\}_{k \in \mathbb{N}}$ are bounded by Proposition 4.1 [38]. This fact, together with the construction of $\{\beta_k\}_{k \in \mathbb{N}}$, implies that there exists $M > 0$ such that $\|g_1^k\| \leq M$, $\|g_2^k\| \leq M$, $\beta_k \leq M$ for all $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$ we have

$$\eta_k = \max\{\beta_k, \|g_1^k\|, \|g_2^k\|\} \leq M,$$

which implies

$$\lambda_k = \frac{\beta_k}{\eta_k} \geq \frac{\beta_k}{M}.$$

Since $\sum_{k=0}^{\infty} \beta_k = +\infty$, it follows that

$$\sum_{k=0}^{\infty} \lambda_k = +\infty.$$

(16)

The combination of (15) and (16) implies that

$$\limsup_{k \to \infty} f(x^k, x^*) = 0.$$

Let $\{x^{k_j}\}_{j \in \mathbb{N}}$ be a subsequence of $\{x^k\}_{k \in \mathbb{N}}$ such that

$$\lim_{j \to +\infty} f(x^{k_j}, x^*) = \limsup f(x^k, x^*) = 0.$$

By Proposition 1, the sequence $\{\|x^k - x^*\|\}_{k \in \mathbb{N}}$ is convergent. It follows that the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded, and hence its subsequence $\{x^{k_j}\}_{j \in \mathbb{N}}$ is also bounded. We may therefore assume that $\{x^{k_j}\}_{j \in \mathbb{N}}$ weakly converges to some $\bar{x} \in C$. Since $f(\cdot, x^*)$ is weakly upper semicontinuous, we have

$$f(\bar{x}, x^*) \geq \lim_{j \to +\infty} f(x^{k_j}, x^*) = 0,$$

(17)

and as a consequence, $f(x^*, \bar{x}) \leq 0$ by pseudo-monotonicity of the bifunction $f$. On the other hand, $f(x^*, \bar{x}) \geq 0$ since $x^*$ belongs to the solution set $\Omega$ of $(P)$. Therefore we obtain

$$f(x^*, \bar{x}) = 0.$$

(18)

This implies $f(\bar{x}, x^*) \leq 0$ by pseudo-monotonicity of $f$. Together with (17), it follows that

$$f(\bar{x}, x^*) = 0.$$

(19)

Since $\bar{x} \in C, x^* \in \bar{\Omega} \subset Sol(C, f)$, and $f$ is paramonotone on $C$ with respect to $Sol(C, f)$, from (18) and (19) we have $\bar{x} \in Sol(C, f)$. Furthermore, since $\bar{x}$ is a weakly cluster point of $\{x^k\}_{k \in \mathbb{N}}$, by Proposition 2, $\bar{x}$ is a fixed point of $T$. Hence $\bar{x} \in Sol(C, f) \cap Fix(T) = \Omega$. It therefore follows from Proposition 1 that the sequence $\{\|x^k - \bar{x}\|\}_{k \in \mathbb{N}}$ converges. Note that $\{x^{k_j}\}_{j \in \mathbb{N}}$ weakly converges to $\bar{x}$, we can conclude that the whole sequence $\{x^k\}_{k \in \mathbb{N}}$ weakly converges to $\bar{x}$, which is a solution to $(P)$. □
Remark 1. When $H$ is a finite dimensional space, Assumption (A3) can be omitted (see e.g. [37] page 70).

4. Applications

In this section, we apply Algorithm 1 to some special cases of Problem (P) mentioned in Section 3.1. For equilibrium problem over the intersection of closed convex sets ($P_1$), by taking $T_i(x) := P_{C_i}(x)$ for each $i = 1, \ldots, m$, the computation of $x^{k+1}$ in (1) takes the form

$$x^{k+1} := \gamma z^k + (1 - \gamma) \sum_{i=1}^{m} \mu_i P_{C_i}(x^k).$$

So in this case, we obtain a splitting algorithm for Problem ($P_1$), where optimization problems are solved separately for each function $f_1$ and $f_2$, while the projection is computed in parallel onto each convex set $C_i$ rather than onto their intersection.

Similarly, applying Algorithm 1 to find a common solution of equilibrium problem and maximal monotone operator inclusion (Problem ($P_2$)), the iterate $x^{k+1}$ is computed separately for each resolvent operator by taking

$$x^{k+1} := \gamma z^k + (1 - \gamma) \sum_{i=1}^{m} \mu_i (M_i + cI)^{-1}(x^k).$$

To illustrate the proposed algorithm for split equilibrium problem ($SEP$), let us consider a game with $n$-players. Each player $i = 1, \ldots, n$ can take an individual action, which is represented by $x_i \in \mathbb{R}$. All players together can take a collective action $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Each player $i$ uses a payoff function $f_i$ which depends on actions of other players. The Nikaido-Isoda function of the game is defined as

$$f(x, y) := \sum_{i=1}^{n} (f_i(x) - f_i(x[y_i])),$$

where the vector $x[y_i]$ is obtained from $x$ by replacing component $x_i$ by $y_i$. Let $C_i \subset \mathbb{R}$ be the strategy set of player $i$, then the strategy set of the game is $C := C_1 \times \ldots \times C_n$. As usual, a point $x^* \in C$ is said to be a Nash equilibrium point of the game if

$$f_i(x^*) = \max_{y_i \in C_i} f_i(x^*[y_i]) \quad \forall i = 1, \ldots, n.$$  

It is well known that $x^*$ is an equilibrium point if and only if $f(x^*, y) \geq 0$ for all $y \in C$. A concrete practical equilibrium model, where the bifunction is a paramonotone one being the sum of two monotone functions can be found in [28]. In some practical games such as jointly constrained Nash-Cournot equilibrium models, the equilibrium points are required to satisfy additional constraints given by

$$\langle a^j, x \rangle \leq b_j \quad (j = 1, \ldots, m).$$
Such the game has exactly the form of Problem (SEP). For this problem, the computation of iterate point $x^{k+1}$ in (I) of Algorithm I takes the form

$$x^{k+1} = \mu z^k + (1 - \mu) \sum_{j=1}^{m} P_{H_j}(x^k),$$

where $H_j$ is the half space defined by the inequality $\langle a^j, x \rangle - b^j \leq 0$, and therefore the projection $P_{H_j}$ has a closed form.

5. Conclusion

We have proposed a splitting algorithm for finding a point in the intersection of solution set of a pseudo-monotone equilibrium problem and the fixed point set of a nonexpansive mapping. The bifunction involved in the equilibrium problem is the sum of the two ones. Exploiting this special structure, the proposed splitting algorithm requires solving two strongly convex subprograms separately for each component bifunction. Combining with the Mann iteration scheme, the algorithm converges under the paramonotonicity property of the involved bifunction. Some variants of the algorithm devoted to some special cases of the considered problem have been shown.

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