Embedding functors and their arithmetic properties

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Abstract

In this article, we focus on how to embed a torus $T$ into a reductive group $G$ with respect to a given root datum $\Psi$ over a scheme $S$. This problem is also related to embedding an étale algebra with involution into a central simple algebra with involution (cf. [PR10]). We approach this problem by defining the embedding functor, which is representable and is a left homogeneous space over $S$ under the automorphism group of $G$. In order to fix a connected component of the embedding functor, we define an orientation $u$ of $\Psi$ with respect to $G$. We show that the oriented embedding functor is also representable and is a homogeneous space under the adjoint action of $G$. Over a local field, the orientation $u$ and the Tits index of $G$ determine the existence of an embedding of $T$ into $G$ with respect to the given root datum $\Psi$. We also use the techniques developed in Borovoi’s paper [Bo99] to prove that the local-global principle holds for oriented embedding functors in certain cases. Actually, the Brauer-Manin obstruction is the only obstruction to the local-global principle for the oriented embedding functor. Finally, we apply the results on oriented embedding functors to give an alternative proof of Prasad and Rapinchuk’s Theorem, and to improve Theorem 7.3 in [PR10].

Key words: torus, root datum, reductive group, central simple algebra, étale algebra, local-global principle, Tits index

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Introduction

Let $K$ be a field, $A$ be a central simple algebra over $K$ with involution $\tau$, and $E$ be an étale algebra over $K$ with involution $\sigma$. Suppose that $\tau|_K = \sigma|_K$. Let $k$ be the field of invariants $K^\tau$, which is a global field. Motivated by the weak commensurability and length-commensurability between locally symmetric spaces ([PR09]), Prasad and Rapinchuk discuss in [PR10] the local-global principle for embeddings of $(E, \sigma)$ into $(A, \tau)$ over $K$. This problem is also related to studying the condition under which the isomorphism classes of simple groups are determined by their isomorphism classes of maximal tori over a number field (cf. [Ga12] and [PR09] Thm 7.5).
Motivated by the work of Prasad and Rapinchuk, we consider the embedding problem of a twisted root datum. Loosely speaking, a twisted root datum is a torus equipped with some extra data related to the roots. In this article, we transform such embedding problem of algebras into a embedding problem of algebraic groups, in order to work in a more conceptual framework. Moreover, in this framework, our criteria can be applied not only to the classical groups but also to the exceptional groups. Instead of a global field, we work over an arbitrary scheme.

Let $S$ be a scheme and $G$ be a reductive group scheme over $S$. Given an $S$-torus $T$ and a twisted root datum $\Psi$ associated to $T$, we want to know when it is possible to embed $T$ in $G$ so that the corresponding twisted root datum $\Phi(G, T)$ is isomorphic to $\Psi$. To approach this problem, we first define the embedding functor $\mathcal{E}(G, \Psi)$. Roughly speaking, each point of the embedding functor is a closed immersion $f$ from $T$ to $G$ such that the twisted root datum $\Phi(G, f(T))$ is isomorphic to $\Psi$. For the formal definition, we refer to Section 1.1. Then our problem can be reformulated as: when is the set $\mathcal{E}(G, T)(S)$ nonempty?

We first prove that the embedding functor is a sheaf for the étale topology (in the sense of big étale site). To be more precise, the embedding functor is a homogeneous sheaf under the action of the automorphism group $\text{Aut}_{-gr}(G)$ over $S$, and it is a principal homogeneous space under the automorphism group $\text{Aut}(\Psi)$ over the scheme of maximal tori of $G$. Then by the result in [SGA3] Exp. X, 5.5, we conclude that the embedding functor $\mathcal{E}(G, \Psi)$ is representable.

However, the embedding functor $\mathcal{E}(G, \Psi)$ can be disconnected if $\text{Aut}_{-gr}(G)$ is. Therefore, instead of dealing with $\mathcal{E}(G, \Psi)$, we fix a particular connected component of $\mathcal{E}(G, \Psi)$ which will be called an oriented embedding functor. The way we fix a connected component is to fix an orientation of $\Psi$ with respect to $G$. An orientation between semisimple $S$-groups was previously defined by Petrov and Stavrova ([PS]). Here, we generalize it to an orientation between a twisted root datum $\Psi$ and a reductive $S$-group $G$, which is an element $u$ in $\text{Isom}_{\text{ext}}(\Psi, G)(S)$ (Section. 1.2.1). We show that the oriented embedding functor $\mathcal{E}(G, \Psi, u)$ is homogeneous under the adjoint action of $G$ over $S$ and is principal homogeneous under the action of the Weyl group $W(\Psi)$. Hence, $\mathcal{E}(G, \Psi, u)$ is also representable (ref. [SGA3] Exp. X, 5.5). Moreover, in Theorem 2.13 we show that over a local field $L$, the orientation together with the Tits index of the given group determine the existence of $L$-points of the oriented embedding functor.

The main application of embedding functors is to the embedding problem of Azumaya algebras with involutions. Let $\tilde{R}$ be a commutative ring, where 2 is invertible in $\tilde{R}$. Let $E$ be an étale algebra over $\tilde{R}$ with involution $\sigma$ and $A$ be an Azumaya algebra with involution $\tau$. Suppose $\sigma|_{\tilde{R}} = \tau|_{\tilde{R}}$. We ask when $(E, \sigma)$ can be embedded into $(A, \tau)$. The case where $\tilde{R}$ is a global field is discussed in Prasad and Rapinchuk’s paper ([PR10]).

Over a commutative ring $\tilde{R}$ in which 2 is an invertible element, we let $R$ be the ring of invariants $\tilde{R}^*$. If $\tilde{R}$ is equal to $R$, then $\sigma$ is said to be of the first
kind. If $\tilde{R}$ is a quadratic extension of $R$, then $\tau$ is said to be of the second kind. We consider the reductive group $G = U(A, \tau)$, the torus $T = U(E, \sigma)$ over $R$, and a twisted root datum $\Psi$ attached to $T$. There is a nice correspondence between the $R$-points of the embedding functor $\mathcal{E}(G, \Psi)$ and the embeddings $\iota : (E, \sigma) \rightarrow (A, \tau)$, namely:

**Theorem.** Keep all the notation defined above. The set of $k$-embeddings from $(E, \sigma)$ into $(A, \tau)$ is in one-to-one correspondence with the set of $R$-points of $\mathcal{E}(G, \Psi)$, except for $G$ of type $D_4$ or $A$ of degree 2 with $\tau$ orthogonal.

For $G$ of type $D_4$, we have a finer treatment and we refer to Proposition 1.17. Moreover, for the involution $\tau$ of the second kind, we prove that there is an orientation $u$ such that all $R$-points on the connected component $\mathcal{E}(G, \Psi, u)$ are in one-to-one correspondence with $R$-embeddings from $(E, \sigma)$ into $(A, \tau)$ (see Remark 1.16, Lemma 2.24).

The second part of this article is devoted to the arithmetic properties of the embedding functor. In particular, we want to know if the Hasse principle holds for the existence of $k$-points of the oriented embedding functor $\mathcal{E}(G, \Psi, u)$, when $k$ is a global field. Since $\mathcal{E}(G, \Psi, u)$ is a homogeneous space under the group $G$ whose stabilizer is a torus, we use the technique developed by Borovoi to solve this problem, cf. [Bo99]. Actually, in [Bo99], Borovoi proved that the Brauer-Manin obstruction to the Hasse principle is the only obstruction in this case. He also computed the obstruction using the Galois hypercohomology. We apply his result to show the following:

**Theorem.** Let $G, \Psi$ be as above, and $T$ be the torus determined by $\Psi$. Let $u \in \text{Isomext}(\Psi, G)(k)$ be an orientation. Suppose that $\Psi$ satisfies one of the following conditions:

1. all connected components of $\text{Dyn}(\Psi)(k_s)$ are of type $C$, where $k_s$ is a separable closure of $k$.
2. $T$ is anisotropic at some place $v \in \Omega_k$.

Then the local-global principle holds for the existence of a $k$-point of the oriented embedding functor $\mathcal{E}(G, \Psi, u)$. In particular, when $\Psi$ is generic, the local-global principle holds.

Finally, for the global field $k$ of characteristic different from 2, we combine these techniques and the correspondence established in Theorem 1.15 and Proposition 1.17 to give an alternative proof of Theorem A and Theorem 6.7 in [PR10]. Besides, we provide an example (2.23) to show that the Hasse principle fails in some cases when the involution $\tau$ is orthogonal and $A$ is $M_2m(D)$, where $D$ is a division algebra over $K$. The main reason for the failure is that the embedding functor $\mathcal{E}(G, \Psi)$ is disconnected in this case. Let $\mathcal{E}(G, \Psi) = X_1 \coprod X_2$. Then it may happen that the embedding functor has a $k_v$ point at each place $v$, but only $X_1$ has a $k_{v_1}$-point, at some place $v_1$, and only $X_2$ has a $k_{v_2}$ point, at another place $v_2$. This explains the failure of the Hasse principle.
0  Some general facts and notations

In this section, we briefly recall the notation and definitions which will be used later. We also state some well-known theorems which are necessary for the development of the main results about the embedding functor. Most of the material here can be found in [SGA3], and in the Appendix A of the book by Conrad, Gabber, and Prasad [CGP].

0.1 Notation and conventions

Let $S$ be a scheme and $S'$ an $S$-scheme. For an $S$-scheme $X$, we let $X_S$ be the scheme $X \times_S S'$ over $S'$. For a set $\Lambda$, we let $\Lambda_S$ denote the disjoint union of the schemes $S_i$, where $i \in \Lambda$ and each $S_i$ is isomorphic to $S$, i.e. $\Lambda_S = \bigsqcup_{i \in \Lambda} S_i$. We call $\Lambda_S$ the constant scheme over $S$ of type $\Lambda$. (ref. [SGA3], Exp. I, 1.8).

Let $\mathbf{Sch}/S$ be the category of all $S$-schemes. Throughout this article, the étale site of $S$ means the big étale site. Namely, we equip the category $\mathbf{Sch}/S$ with the following topology: for an $S$-scheme $U$, \{ $U_i \to U$ \}$_i$ is a covering of $U$ if for each $i$, $f_i$ is an étale morphism and $U = \bigcup_i f_i(U_i)$. For a detailed introduction to Grothendieck topology, we refer to the lecture notes by Brochard [Br].

0.2 Torsors and homogeneous spaces

Let $G$ be an $S$-group sheaf for étale topology. Let $F$ and $X$ be $S$-sheaves. Let $p : F \to X$ be a morphism between $S$-sheaves. Then $F$ is called a right (resp. left) $G$-sheaf over $X$ with respect to $p$ if $F$ is equipped with a $G$-action satisfying $p(fg) = p(f)$ (resp. $p(gf) = p(f)$ ) for all $(f, g) \in (F \times G)(S')$ and for all $S$-schemes $S'$. Note that $X \times G$ can be equipped with a right $G$-action as $(x, g)\sigma = (x, g\sigma)$. Let $p_X$ be the projection from $X \times G$ to $X$. Then $X \times G$ is a right $G$-sheaf over $X$ with respect to $p_X$. A $G$-sheaf $F$ over $X$ with respect to $p$ is trivial if it is isomorphic to $X \times G$ with respect to $p_X$ as $G$-sheaves over $X$. A right $G$-sheaf $F$ over $X$ with respect to $p$ is called a $G$-torsor over $X$ if there is an epimorphism of $S$-sheaves $\pi : Y \to X$ such that $Y \times F$ over $Y$ is a trivial $G$-sheaf.

**Proposition 0.1.** Let $G$ be an $S$-group sheaf, $X$ be a sheaf over $S$. Let $F$ be a $G$-sheaf over $X$ with respect to an $S$-sheaf morphism $p : F \to X$. Then $F$ is a torsor over $X$ if and only if $p$ is an epimorphism of $S$-sheaves and the morphism $i : F \times G \to F \times F$ defined as $i(x, h) = (x, xh)$ is invertible.

**Proof.** [DG70], Chap. III, §4, Corollary 1.7. \qed

Let $G$ be an $S$-group sheaf. Let $F$ and $X$ be $S$-sheaves. Let $p : F \to X$ be a morphism between $S$-sheaves. A $G$-sheaf $F$ over $X$ with respect to $p$ is called
a G-homogeneous space if $p$ is an epimorphism of $S$-sheaves and the morphism $i : \mathcal{F} \times G \to \mathcal{F} \times \mathcal{F}$ defined as $i(x, h) = (x, xh)$ is an epimorphism between sheaves.

0.3 Root data and twisted root datum

Let $\psi = (M, M^\vee, R, R^\vee)$ be a root datum (ref. [SGA3], Exp. XXI, 1.1.1). Let $\Delta \subseteq R$ be a system of simple roots of $R$. The root datum $\psi$ plus a system of simple roots $\Delta$ of $R$ is called a pinning root datum, and we denote it as $(M, M^\vee, R, R^\vee, \Delta)$.

The subgroup of the automorphism group of $M$ generated by the reflections $\{s_\alpha\}_{\alpha \in R}$ is called the Weyl group of $\psi$, and we denote it as $W(\psi)$.

For a finite subset $R$ (resp. $R^\vee$) of $M$ (resp. $M^\vee$), we let $\Gamma_0(R)$ (resp. $\Gamma_0(R^\vee)$) be the subgroup generated by $R$ (resp. $R^\vee$) and we let $\mathcal{V}(R)$ (resp. $\mathcal{V}(R^\vee)$) be the vector space defined by $\Gamma_0(R) \otimes \mathbb{Q}$ (resp. $\Gamma_0(R^\vee) \otimes \mathbb{Q}$).

A root datum is called reduced if for all $\alpha \in R$, we have $2\alpha \not\in R$. A root datum is called semisimple if $\text{rank}(\Gamma_0(R)) = \text{rank}(M)$. A root datum $(M, M^\vee, R, R^\vee)$ is called adjoint (resp. simply connected) if $M = \Gamma_0(R)$ (resp. $M^\vee = \Gamma_0(R^\vee)$).

We define the dual root datum of $\psi$ to be $(M^\vee, M, R^\vee, R)$, and denote it as $\psi^\vee$.

0.3.1 Radical and coradical of root data

Let

$$N = \{x \in M| \alpha^\vee(x) = 0, \forall \alpha^\vee \in R^\vee\}.$$ 

Then the dual of $N$ can be identified with $M^\vee/\mathcal{V}(R^\vee) \cap M^\vee$ (ref. [SGA3], Exp.XXI, 6.3.1).

Define the coradical of $\psi$ to be the root datum $(N, N^\vee, \emptyset, \emptyset)$ and denote it as $\text{corad}(\psi)$. We define the radical of $\psi$ to be $\text{corad}(\psi)^\vee$, and denote it as $\text{rad}(\psi)$.

0.3.2 Induced and coinduced root data

Given a root datum $\psi = (M, M^\vee, R, R^\vee)$, and a subgroup $N$ of $M$ which contains $\Gamma_0(R)$, let $i_N : N \to M$ be the natural inclusion, and $i_N^\vee : M^\vee \to N^\vee$ be the corresponding map on $M^\vee$. Let $R_N = R$ and $R_N^\vee = i_N^\vee(R^\vee)$. We define the root datum $\psi_N$ as $(N, N^\vee, R_N, R_N^\vee)$, which is called the induced root datum of $\psi$ with respect to $N$. If $N = \Gamma_0(R)$, then $\psi_N$ is an adjoint root datum, and we denote it as $\text{ad}(\psi)$. If $N = \mathcal{V}(R) \cap M$, then $\psi_N$ is a semisimple root datum, and we denote $\psi_N$ as $\text{ss}(\psi)$. We let $\text{der}(\psi) = \text{ss}(\psi^\vee)^\vee$, and $\text{sc}(\psi) = \text{ad}(\psi^\vee)^\vee$. 

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0.3.3 Morphisms between root data

Let \( \psi_1 = (M_1, M_1^\vee, R_1, R_1^\vee) \), \( \psi_2 = (M_2, M_2^\vee, R_2, R_2^\vee) \) be two root data. A module morphism \( f : M_1 \to M_2 \) is a morphism between \( \psi_1 \) and \( \psi_2 \) if \( f \) induces a bijection between \( R_1 \) and \( R_2 \) and the transpose map \( t^f : M_2^\vee \to M_1^\vee \) is a bijection between \( R_2^\vee \) and \( R_1^\vee \).

**Proposition 0.2.** Keep all the notations above. If \( f : M_1 \to M_2 \) is a morphism between \( \psi_1 \) and \( \psi_2 \), then \( t^f(\alpha^\vee) = \alpha^\vee \), and the map \( s_\alpha \to s_{f(\alpha)} \) for \( \alpha \in R_1 \) extends to an isomorphism between \( W(\psi_1) \) and \( W(\psi_2) \).

**Proof.** [SGA3], Exp. XXI, 6.1.1 and 6.2.2.

Let \( \text{Aut}(\psi) \) be the automorphism group of \( \psi \), and fix a system of simple roots \( \Delta \) of \( R \). Define the abstract group

\[ E_\Delta(\psi) = \{ u \in \text{Aut}(\psi) \mid u(\Delta) = \Delta \}. \]

Then we have the following:

**Proposition 0.3.** \( W(\psi) \) is a normal subgroup of \( \text{Aut}(\psi) \), and \( \text{Aut}(\psi) \) is a semi-direct product of \( W(\psi) \) by \( E_\Delta(\psi) \).

**Proof.** [SGA3], Exp. XXI, 6.7.1 and 6.7.2.

0.3.4 Twisted root data

Let \( T \) be an \( S \)-torus. Let \( M \) be the character group scheme associated to \( T \), i.e. \( M(S') = \text{Hom}_{S'-\text{gr}}(T_{S'}, \mathbb{G}_m, S') \). Let \( \Psi = (M, M^\vee, R, R^\vee) \) be a twisted root datum associated to \( T \) (ref. [SGA3], Exp. 22, Def. 1.9).

The root datum \( \Psi \) is *split* if \( T \) is split. A twisted root datum is called *reduced* if for all \( S \)-schemes \( S' \) and all \( \alpha \in R(S') \), we have \( 2\alpha \notin R(S') \).

Let \( \psi = (M, M^\vee, R, R^\vee) \) be a root datum. A twisted root datum \( \Psi \) is said to have type \( \psi \) at the point \( s \) of \( S \) if \( \Psi_s \simeq (M_s, M_s^\vee, R_s, R_s^\vee) \).

Let \( \Psi = (M, M^\vee, R, R^\vee) \) be a twisted root datum. Since at each \( s \in S \), there is an étale neighborhood such that \( T \) splits, we can define \( \text{ad}(\Psi), \text{sc}(\Psi), \text{ss}(\Psi), \text{der}(\Psi) \) étale locally, and by the functoriality of induced root data, define them over \( S \) by descent ( [SGA3], Exp. XXI, 6.5).

Let \( \Psi_1 = (M_1, M_1^\vee, R_1, R_1^\vee) \), \( \Psi_2 = (M_2, M_2^\vee, R_2, R_2^\vee) \) be two twisted root data. Let \( T_1, T_2 \) be the tori determined by \( \Psi_1 \) and \( \Psi_2 \) respectively. An \( S \)-group morphism \( f : T_2 \to T_1 \) is a morphism from \( \Psi_1 \) to \( \Psi_2 \) if \( f \) induces an isomorphism from \( R_1 \) to \( R_2 \) and an isomorphism from \( R_2^\vee \) to \( R_1^\vee \). We can also define the induced twisted root data by étale descent, and define \( \text{ad}(\Psi), \text{ss}(\Psi), \text{der}(\Psi) \) and \( \text{sc}(\Psi) \) as we have done for the root data.
0.3.5 Weyl groups, Isom, Isomext and Isomint for twisted root data

Let \( \Psi \) be a twisted root datum. Suppose \( \Psi \) is split and \( \Psi = (M_S, M^\vee_S, R_S, R^\vee_S) \). Let \( \psi \) be the root datum \((M, M^\vee, R, R^\vee)\). Let \( W \) be the Weyl group of \( \psi \), and define \( W(\Psi) = W_S \). Suppose that \( \Psi \) is not split. Then we can find an étale covering \( \{S_i \to S\} \) such that \( \Psi_{S_i} \) is split. By Proposition 0.3, the canonical isomorphism between \( (\Psi_{S_i})_{S_j} \) and \( (\Psi_{S_j})_{S_i} \) gives a canonical isomorphism between \( W(\Psi_{S_i})_{S_j} \) and \( W(\Psi_{S_j})_{S_i} \) and hence gives descent data for \( \{W(\Psi_{S_i})\}_i \), which allows us to define \( W(\Psi) \).

Let \( \text{Aut}(\Psi) \) be the automorphism functor of \( \Psi \). By descent, we can define the Weyl group \( W(\Psi) \). Then from Proposition 0.3, we can define the following exact sequence by étale descent:

\[ 1 \to W(\Psi) \to \text{Aut}(\Psi) \to \text{Autext}(\Psi) \to 1 \]

Then we have the following proposition:

**Proposition 0.4.** Keep all the notations above. The automorphism group \( \text{Aut}(\Psi) \) is representable by a twisted constant \( S \)-scheme, and \( W(\Psi) \) is normal in \( \text{Aut}(\Psi) \). Let \( \text{Autext}(\Psi) \) be the quotient group of \( \text{Aut}(\Psi) \) by \( W(\Psi) \). Then \( \text{Autext}(\Psi) \) is also representable by a twisted constant \( S \)-scheme.

**Proof.** Note that we can find an étale covering \( \{S_i \to S\}_i \) such that \( \Psi_S \) is split (\cite{SGA3}, Exp. X, 4.5). By Proposition 0.3, \( \text{Aut}(\Psi_S) \) and \( \text{Autext}(\Psi_S) \) are constant group schemes over \( S_i \). By \cite{SGA3}, Exp. X, 5.5, \( \{S_i \to S\}_i \) gives an effective descent datum, so \( \text{Aut}(\Psi_S) \) and \( \text{Autext}(\Psi_S) \) are representable.

Let \( \Psi_1, \Psi_2 \) be two twisted root data. Suppose \( \Psi_2 \) is a twisted form of \( \Psi_1 \). Let \( \text{Isom}(\Psi_1, \Psi_2) \) be the isomorphism functor between \( \Psi_1 \) and \( \Psi_2 \). Then \( \text{Isom}(\Psi_1, \Psi_2) \) is a right principal homogeneous space of \( \text{Aut}(\Psi_1) \) and a left principle homogeneous of \( \text{Aut}(\Psi_2) \). Since \( \text{Aut}(\Psi_2) \) is representable, \( \text{Isom}(\Psi_1, \Psi_2) \) is also representable.

Define \( \text{Isomext}(\Psi_1, \Psi_2) = W(\Psi_2) \setminus \text{Isom}(\Psi_1, \Psi_2) \).

Note that for \( f \in \text{Isom}(\Psi_1, \Psi_2)(S) \), \( f^{-1} \circ W(\Psi_2) \circ f = W(\Psi_1) \) by Proposition 0.2. Therefore we have a natural isomorphism from \( \text{Isomext}(\Psi_1, \Psi_2) \) to \( \text{Isom}(\Psi_1, \Psi_2)/W(\Psi_1) \). Then \( \text{Isomext}(\Psi_1, \Psi_2) \) is a left \( \text{Autext}(\Psi_2) \)-principal homogeneous space and a right \( \text{Autext}(\Psi_1) \)-principal homogeneous space. An orientation of \( \Psi_1 \) with respect to \( \Psi_2 \) is an \( S \)-point of \( \text{Isomext}(\Psi_1, \Psi_2) \).

Suppose that there is \( u \in \text{Isomext}(\Psi_1, \Psi_2)(S) \). Then we can regard \( S \) as an \( \text{Isomext}(\Psi_1, \Psi_2) \)-scheme through \( u \) and define

\[ \text{Isomint}_u(\Psi_1, \Psi_2) := S \times_{\text{Isomext}(\Psi_1, \Psi_2)} \text{Isom}(\Psi_1, \Psi_2). \]
0.4 Reductive groups

An S-group scheme $G$ is called reductive (resp. semi-simple) if it is affine and smooth over $S$, and all the geometrical fibers are connected and reductive (resp. semisimple) (ref. [SGA3], Exp. XIX, Def. 2.7).

Let $G$ be a reductive S-group scheme and suppose that $T$ is a maximal torus in $G$. We let $\Phi(G,T)$ be the twisted root datum of $G$ with respect to $T$ (ref. [SGA3] Exp. XXII, 1.10).

For a point $s \in S$, let $\kappa(s)$ be the residue field of $s$ and $\overline{\kappa(s)}$ be the algebraic closure of $\kappa(s)$. Let $s = \text{Spec}(\overline{\kappa(s)})$. The type of $G$ at $s$ is the type of $\Phi(G_s, T_0)$, where $T_0$ is a maximal torus of $G_s$ (ref. [SGA3], Exp. 22, Def. 2.6.1, 2.7). Note that the type of $G$ is locally constant over $S$ (ref. [SGA3], Exp. 22, Prop. 2.8).

A reductive S-group $G$ is split if there is a maximal torus $T$ of $G$ and a root datum $(M, M^\vee, R, R^\vee)$ such that $\Phi(G,T) \simeq (M_S, M_S^\vee, R_S, R_S^\vee)$ and satisfying the following:

1. $S$ is nonempty and each root $\alpha \in S$ (resp. $\alpha^\vee \in R^\vee$) can be identified as a constant map from $S$ to $M$ (resp. $M^\vee$).
2. Let $g = \text{Lie}(G/S)$ and $t = \text{Lie}(T/S)$. Under the adjoint action of $T$, $g = t \oplus \bigoplus_{\alpha \in R} g^\alpha$, where the $g^\alpha$’s are free $\mathcal{O}_S$-modules.

In this case, we say that $G$ is split relatively to $T$ (ref. [SGA3], Exp. XXII, 1.13 and 2.7).

Let us endow $S$ with the étale topology. Let $S'$ be an $S$-scheme, and $G$ be an $S$-group scheme. Let $\text{Aut}_{S-\text{gr}}(G)$ be the sheaf of group automorphisms of $G$. Then we can define the group homomorphism

$$ad : G \rightarrow \text{Aut}_{S-\text{gr}}(G)$$

which maps an element $g$ of $G(S')$ to an automorphism of $G$ defined by the conjugation by $g$. Let $\text{Centr}(G)$ be the center of $G$. Then the image sheaf of $ad$ is isomorphic to $G/\text{Centr}(G)$ and $ad(G)$ is normal in $\text{Aut}_{S-\text{gr}}(G)$. So we have the exact sequence of S-group sheaves:

$$1 \rightarrow \text{ad}(G) \rightarrow \text{Aut}_{S-\text{gr}}(G) \rightarrow \text{Aut}_{\text{ext}}(G) \rightarrow 1.$$

Theorem 0.5. Let $S$ be a scheme and $G$ be a reductive $S$-group scheme. For the exact sequence of S-sheaves:

$$1 \rightarrow \text{ad}(G) \rightarrow \text{Aut}_{S-\text{gr}}(G) \rightarrow \text{Aut}_{\text{ext}}(G) \rightarrow 1,$$

we have the following:

(i) $\text{Aut}_{S-\text{gr}}(G)$ is represented by a separated, smooth $S$-scheme.
(ii) $\text{Aut}_{\text{ext}}(G)$ is represented by a twisted finitely generated constant scheme.
(iii) Suppose that $G$ splits relatively to $T$, and $\Phi(G,T) \simeq (M_S, M_S^\vee, R_S, R_S^\vee)$.

Let $(\psi, \Delta) = (M, M^\vee, R, R^\vee, \Delta)$ be a pinning root datum. Then there is a monomorphism between sheaves $a : E_\Delta(\psi)_S \rightarrow \text{Aut}_{S-\text{gr}}(G)$ such that

$$p \circ a : E_\Delta(\psi)_S \rightarrow \text{Aut}_{\text{ext}}(G)$$
is an isomorphism.

Proof. [SGA3], Exp. XXIV, Theorem 1.3.

For a subgroup scheme $H$ of $G$, we let $\text{Aut}_{S, gr}(G, H)$ be the subsheaf of $\text{Aut}_{S, gr}(G)$ which normalizes $H$, i.e. $\text{Aut}_{S, gr}(G, H) = \text{Norm}_{\text{Aut}_{S, gr}(G)}(H)$ (cf. [SGA3] Exp. VI, Def. 6.1 (iii)). We let $\text{Aut}_{ext}(G, H)$ be the quotient sheaf $\text{Aut}_{S, gr}(G, H)/\text{Norm}_{\text{ad}(G)}(H)$.

0.5 Dynkin diagrams

For each reductive $S$-group $G$, we can associate a Dynkin diagram scheme $\text{Dyn}(G)$ to $G$ (ref. [SGA3], Exp. XXIV, 3.2 and 3.3). Moreover we have the following:

**Proposition 0.6.** If $G$ is semisimple (resp. adjoint or simply connected), then the morphism

$$\text{Aut}_{ext}(G) \to \text{Aut}_{\text{Dyn}}(\text{Dyn}(G))$$

is a monomorphism (resp. isomorphism).

Proof. [SGA3], Exp. XXIV, 3.6.

Given a twisted root datum $\Psi$ over $S$, we can also define the Dynkin scheme of $\Psi$ in a similar way and denote it by $\text{Dyn}(\Psi)$. We also have a natural morphism from $\text{Aut}_{ext}(\Psi)$ to $\text{Aut}_{\text{Dyn}}(\text{Dyn}(\Psi))$, which will be a monomorphism (resp. isomorphism) if $\Psi$ is reduced semisimple (resp. reduced adjoint or reduced simply connected).

For a root datum $\psi$, we can associate to each connected component of its Dynkin diagram $\text{Dyn}(\psi)$ a type according to the classification of Dynkin diagrams (ref. [SGA3] Exp. XXI, 7.4.6). Let $T$ be the set of all types of Dynkin diagram. Similarly, for each Dynkin scheme $D$ over $S$, we can associate the scheme of connected components $D_0$ to $D$ (ref. [SGA3] Exp. XXIV, 5.2). We can also define a morphism

$$a : D_0 \to T_S.$$

Let $v \in T$. If $D_0 = a^{-1}(v)$, then we say $D$ is isotypical of type $v$. If the Dynkin scheme $\text{Dyn}(\Psi)$ is connected at each fiber over $S$ and is of constant type $v$, then we say that $\Psi$ is simple of type $v$.

0.6 Parabolic subgroups

Let $S$ be a scheme and $G$ be a reductive $S$-group. A subgroup scheme $P$ of $G$ over $S$ is called parabolic if

1. $P$ is smooth over $S$.
2. For each $s \in S$, the quotient $G_s/P_s$ is proper.
Let us keep the notations in Section 0.4. Let $\mathcal{E} = \{G, T, R, \Delta, \{X_\alpha\}_{\alpha \in \Delta}\}$ be a pinning of $G$ and $P$ be a parabolic subgroup. The pinning $E$ is said to be adapted to $P$ if $P$ contains $T$ and $\text{Lie}(P/S) = t \oplus \bigoplus_{\alpha \in R'} g^\alpha$, where $R'$ is a subset of $R$ which contains all the positive roots. In this case, we denote $\Delta(P) = \Delta \cap -R'$.

Let $\text{Of}(\text{Dyn}(G))$ be the functor defined as the following: for each $S$-scheme $S'$, $\text{Of}(\text{Dyn}(G))(S')$ is the set of all subschemes of $\text{Dyn}(G)_{S'}$ which are open and closed. Then $\text{Of}(\text{Dyn}(G))$ is a twisted finite constant scheme. Let $\text{Par}(G)$ be the functor defined by $\text{Par}(G)(S')$ is the set of all parabolic subgroups of $G'_{S'}$, for each $S$-scheme $S'$. One can define a morphism

$$t : \text{Par}(G) \to \text{Of}(\text{Dyn}(G))$$

satisfying

1. $t$ is functorial in $G$.
2. If $\mathcal{E}$ is a pinning of $G$ adapted to the parabolic subgroup $P$, then $t(P) = \Delta(P)_{S}$.

For a parabolic subgroup $P$ of $G$, we call $t(P)$ the type of $P$.

**Proposition 0.7.** Let $S, G$ be as above. Let $P$ be a parabolic subgroup of $G$. Let $t'$ be a section of $\text{Of}(\text{Dyn}(G))$ over $S$ and $t' \supseteq t(P)$. Then there is a unique parabolic subgroup $P'$ of $G$ which contains $P$ and the type of $P'$ is $t'$.

**Proof.** [SGA3] Exp. XXVI, Lemme 3.8. \[\square\]

## 1 Embedding functors

Let $S$ be a scheme and $G$ be a reductive group over $S$. Let $T$ be an $S$-torus and $\Psi$ be a root datum associated to $T$. We would like to know if we can embed $T$ in $G$ as a maximal torus such that the twisted root datum $\Phi(G, T)$ is isomorphic to $\Psi$. To answer this question, we first define the embedding functor $\mathcal{E}(G, \Psi)$. The embedding functor is representable and is a left $\text{Aut}_{S-\text{gr}}(G)$-homogeneous space. Briefly speaking, each $S$-point of $\mathcal{E}(G, \Psi)$ corresponds to an embedding from $T$ to $G$ with respect to $\Psi$.

In the second part, we first define an orientation $v$ of $\Psi$ with respect to $G$. Once we can fix an orientation, we can fix a connected component of $\mathcal{E}(G, \Psi)$, which is called an oriented embedding functor. The oriented embedding functor $\mathcal{E}(G, \Psi, v)$ is also representable and is a left $G$-homogeneous space.

In the end of this section, we show that the embedding functor has an interpretation in the embedding problem of Azumaya algebras with involution. Moreover, we show that there is a one-to-one correspondence between the $k$-points of the embedding functor and the $k$-embeddings from an étale $k$-algebra with involution into an Azumaya algebra with involution.
1.1 Embedding functors

Let $S$ be a scheme, $G$ be a reductive $S$-group scheme. Let $T$ be an $S$-torus. Let $\mathcal{M}$ be the character group scheme associated to $T$, and $\Psi = (\mathcal{M}, \mathcal{M}', \mathcal{R}, \mathcal{R}')$ be a root datum associated to $T$. We define the embedding functor by:

\[
\mathcal{E}(G, \Psi)(S') = \left\{ f : T_{S'} \hookrightarrow G_{S'} \middle| \begin{array}{l}
 f \text{ is both a closed immersion and a group homomorphism which induces an isomorphism } \\
 f^{\Psi} : \Psi_{S'} \xrightarrow{\Phi} \Phi(G_{S'}, f(T_{S'})) \text{ such that } \\
 f^{\Psi}(\alpha) = \alpha \circ f^{-1}|_{f(T_{S'})} \text{ for all } \alpha \in \mathcal{M}(S''), \\
 \text{for each } S'-\text{scheme } S'.
\end{array} \right\}
\]

for $S'$ a scheme over $S$. In this article, we always assume that at each geometric point $\overline{s} \in S$, the root datum $\Psi_{\overline{s}}$ is isomorphic to the root datum of $G_{\overline{s}}$. Therefore, $\mathcal{E}(G, \Psi)$ is not empty in our case.

The embedding functor $\mathcal{E}(G, \Psi)$ is naturally equipped with a left $\text{Aut}_{S-gr}(G)$-action defined as compositions of functions. Namely define

\[
l : \text{Aut}_{S-gr}(G) \times \mathcal{E}(G, \Psi) \to \mathcal{E}(G, \Psi)
\]

as $l(\sigma, f) = \sigma \circ f$ for all $\sigma \in \text{Aut}_{S-gr}(G)(S')$, $f \in \mathcal{E}(G, \Psi)(S')$ and $S'$ an $S$-scheme.

Since $\text{Aut}(\Psi) \subseteq \text{Aut}_{S-gr}(\mathcal{M})$ and $\text{Aut}_{S-gr}(T) = \text{Aut}_{S-gr}(\mathcal{M})^{op}$, we can regard $\text{Aut}(\Psi)$ as a subgroup of $\text{Aut}_{S-gr}(T)$ through the inverse map between $\text{Aut}_{S-gr}(\mathcal{M})$ and $\text{Aut}_{S-gr}(\mathcal{M})^{op}$. We define a right $\text{Aut}(\Psi)$-action on $\mathcal{E}(G, \Psi)$ as a composition of an automorphism of $T$ followed by a closed embedding from $\mathcal{E}(G, \Psi)$.

Now, let $\mathcal{T}$ be the scheme of maximal tori of $G$ (cf. [SGA3] XII, 1.10). We think about the morphism $\pi : \mathcal{E}(G, \Psi) \to \mathcal{T}$ defined as $\pi(f) = f(T_{S'})$, where $f \in \mathcal{E}(G, \Psi)(S')$, and $S'$ is a scheme over $S$. Then we have the following:

**Theorem 1.1.** In the sense of the étale topology, $\mathcal{E}(G, \Psi)$ is a homogeneous space over $S$ under the left $\text{Aut}_{S-gr}(G)$-action, and a torsor over $\mathcal{T}$ under the right $\text{Aut}(\Psi)$-action. Moreover, $\mathcal{E}(G, \Psi)$ is representable by an $S$-scheme.

**Proof.** We divide the argument into the following three parts:

**Claim.** $\mathcal{E}(G, \Psi)$ is a sheaf for the étale topology.

**Proof.** Let $\{S_i \to S\}$ be an étale covering. Since $\mathcal{E}(G, \Psi)$ is a subfunctor of $\text{Hom}_{S-gr}(T, G)$ and $\text{Hom}_{S-gr}(T, G)$ is a sheaf, we only need to prove that for $f \in \text{Hom}_{S-gr}(T, G)(S)$ if $f_{S_i} \in \mathcal{E}(G, \Psi)(S_i)$, then $f \in \mathcal{E}(G, \Psi)(S)$.

We note that to verify that $f$ is a closed immersion, it is enough to verify it étale locally. Hence $f$ is a closed immersion since $f_{S_i}$ is. Since $f(T)$ is locally a maximal torus in $G$, $f(T)$ is a maximal torus. Finally, $f^{\Psi}$ is an isomorphism étale locally, so $f^{\Psi}$ is an isomorphism. We conclude that $\mathcal{E}(G, \Psi)$ is a sheaf. 

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Claim. \( \mathcal{E}(G, \Psi) \) is homogeneous under the left \( \text{Aut}_{S-\text{gr}}(G) \)-action, which is defined as composition of functions.

Proof. Let \( S' \) be a scheme over \( S \), and \( f_1, f_2 \) be two elements in \( \mathcal{E}(G, \Psi)(S') \). Let \( F_i = f_i(T_{S'}) \), \( i = 1, 2 \) respectively. Then there exists an étale neighborhood \( U \) of \( S' \) where \( F_1 \) and \( F_2 \) are conjugated (ref. [SGA3] Exp. 12, Theorem 1.7), so we can assume \( F_{1U} = F_{2U} \). Moreover, we can even assume \( G_U \) is split relatively to \( F_{1U} \) (ref. [SGA3] Exp. 22, 2.3). By abuse of notation, we still use \( f_2 \circ f_1^{-1} \) to denote the morphism from \( F_{1U} \) to \( F_{2U} \). Then by the definition of the \( \mathcal{E}(G, \Psi) \)-functor, we know that \( f_2 \circ f_1^{-1} \) induces an automorphism on \( \Phi(G_U, f(F_{1U})) \). According to Theorem 0.3, we can find \( \sigma \), which is an automorphism of \( G_U \), such that \( \sigma \circ f_2 = f_1 \), which proves the claim. \( \square \)

Claim. \( \mathcal{E}(G, \Psi) \) is a right \( \text{Aut}(\Psi) \)-torsor over \( \mathcal{T} \) for étale topology.

Proof. We first prove that \( \pi : \mathcal{E}(G, \Psi) \to \mathcal{T} \) is surjective as an \( S \)-sheaf morphism for the étale topology. For an \( S \)-scheme \( S' \) and an element \( F \) in \( \mathcal{T}(S') \), which means that \( F \) is a maximal torus in \( G_{s'} \), for each \( s' \in S' \), we can find an étale open neighborhood \( U' \to S' \) such that \( \Psi_{U'} \) splits and \( G_{U'} \) splits relatively to \( F_{U'} \). Therefore, \( \Psi_{U'} \) and \( \Phi(G_{s'}, F) \) are isomorphic as we assume that both of them are with the same type at each geometric point. Hence, there is \( f \in \mathcal{E}(G, \Psi)(U') \) such that \( \pi(f) = F \times U' \).

Next, let us show that \( \mathcal{E}(G, \Psi) \times \text{Aut}(\Psi) \simeq \mathcal{E}(G, \Psi) \times \mathcal{E}(G, \Psi) \) as \( \text{Aut}(\Psi) \)-space. By identifying \( \text{Aut}(\Psi) \) with a subgroup of \( \text{Aut}(T) \), we regard \( \sigma \in \text{Aut}(\Psi)(S') \) as an element of \( \text{Aut}_{S-\text{gr}}(T)(S') \). Define

\[ m : \mathcal{E}(G, \Psi) \times \text{Aut}(\Psi) \to \mathcal{E}(G, \Psi) \times \mathcal{E}(G, \Psi) \]

as \( m(f, \sigma) = (f, f \circ \sigma) \) for all \( S' \) a scheme over \( S \).

Given \( (f_1, f_2) \in (\mathcal{E}(G, \Psi) \times \mathcal{E}(G, \Psi))(S') \), we let \( F = f_1(T_{S'}) = f_2(T_{S'}) \) and \( \Phi = \Phi(G_{s'}, F) \). Then both \( f_1^\Psi, f_2^\Psi \) induce isomorphisms from \( \Psi_{S'} \) to \( \Phi \), so \( (f_1^\Psi)^{-1} \circ f_2^\Psi \) is an automorphism of \( \Psi_{S'} \). So we can define

\[ i : \mathcal{E}(G, \Psi) \times \text{Aut}(\Psi) \to \mathcal{E}(G, \Psi) \times \text{Aut}(\Psi) \]

as \( i(f_1, f_2) = (f_1, f_1^{-1} \circ f_2) \). Then we have

\[ i \circ m(f, \sigma) = i(f, f \circ \sigma) = (f, f^{-1} \circ f \circ \sigma) = (f, \sigma); \]
\[ m \circ i(f_1, f_2) = m(f_1, f_1^{-1} \circ f_2) = (f_1, f_2). \]

Therefore \( i \) is the inverse map of \( m \) and the claim follows from Proposition 0.1. \( \square \)

Now we want to show that \( \mathcal{E}(G, \Psi) \) is a scheme. As we have mentioned in Proposition 0.2 the group scheme \( \text{Aut}(\Psi) \) is étale locally constant. Therefore, the \( \text{Aut}(\Psi) \)-torsor \( \mathcal{E}(G, \Psi) \) is representable by [SGA3] Exp. X, 5.5. \( \square \)
For a maximal torus $X$ of $G$, we let $X^{ad}$ be the corresponding torus in $\text{ad}(G)$. Note that $X/\text{Centr}(G) \simeq X^{ad}$ (ref. 
\cite{SGA3}, Exp. 24, Prop. 2.1). For $f \in \mathcal{E}(G, \Psi)(S')$, we define the stabilizer of $f$ under the $\text{Aut}_{S'}(G_S)$ as:

$$\text{Stab}(f)(S'') = \{ x \in \text{Aut}_{S'}(G_S)| x \circ f_{S''} = f_{S''} \}$$

**Proposition 1.2.** Let $f \in \mathcal{E}(G, \Psi)(S')$ and $X = f(T_S')$. Then $\text{Stab}(f)$ is isomorphic to $X^{ad}$.

**Proof.** Let $\sigma \in \text{Aut}_{S'}(G_S)$. Then $\sigma \in \text{Stab}(f)(S'')$ if and only if $\sigma|_X$ is the identity map on $X$, which means $\text{Stab}(f) = \text{Aut}_{S'}(G, \text{id}_X)$. Since $\text{Aut}_{S'}(G, \text{id}_X) = X^{ad}$ (ref. \cite{SGA3}, Exp. 24, Prop. 2.11), $\text{Stab}(f) = X^{ad}$. □

### 1.2 Oriented embedding functors

#### 1.2.1 The definition of an orientation

Let $\Psi = (\mathcal{M}, \mathcal{M}', \mathcal{R}, \mathcal{R}')$ be a twisted reduced root datum over $S$, and $G$ be a reductive group over $S$. Suppose that $\Psi$ and $G$ have the same type at each $s \in S$. From Theorem 1.1, we know that $\mathcal{E}(G, \Psi)$ is a homogeneous space under the action of $\text{Aut}_{S'}(G)$. However, $\text{Aut}_{S'}(G)$ may be disconnected, so we would like to fix an extra datum “$v$” to make our embedding functor together with “$v$” to be a homogeneous space under the adjoint action of $G$. The “$v$” will be called an orientation of $\Psi$ with respect to $G$.

First, we suppose that $G$ has a maximal torus $T$. Let $\Phi(G, T)$ be the twisted root datum of $G$ with respect to $T$. For an $S$-scheme $S'$, and for $\sigma \in \text{Aut}_{S'}(G, T)(S')$, $\sigma$ induces an automorphism on $\Phi(G, T)$, and induces a left action on $f \in \text{Isom}(\Psi, \Phi(G, T))(S')$ which is defined as

$$(\sigma \cdot f)(x) = f(x) \circ \sigma^{-1},$$

for all $x \in \mathcal{M}_{S'}(S'')$, where $S''$ is an $S'$-scheme.

Let $T'$ be another maximal torus of $G$, and $\text{Transt}_G(T, T')$ be the strict transporter from $T$ to $T'$ (cf. \cite{SGA3} Exp. VI B, Def. 6.1 (ii)). Then we have a natural morphism (for the convention, we refer to \cite{Gir} Chap III, Def. 1.3.1.):

$$\text{Transt}_G(T, T') \xrightarrow{\text{Norm}_{G}(T)} \text{Isom}(\Psi, \Phi(G, T)) \rightarrow \text{Isom}(\Psi, \Phi(G, T')).$$

Since $\text{Transt}_G(T, T')$ is a right principal homogeneous space under $\text{Norm}_{G}(T)$ and $\text{Norm}_{G}(T)$ acts on the left of $\text{Isomext}(\Psi, \Phi(G, T))$ trivially, we have the following canonical morphism:

$$\text{Isomext}(\Psi, \Phi(G, T)) \simeq \text{Transt}_G(T, T') \xrightarrow{\text{Norm}_{G}(T)} \text{Isomext}(\Psi, \Phi(G, T)) \simeq \text{Isomext}(\Psi, \Phi(G, T')).$$

Therefore, for $G$ with a maximal torus $T$, we can define

$$\text{Isomext}(\Psi, G) := \text{Isomext}(\Psi, \Phi(G, T)).$$
In general, since $G$ has a maximal torus étale locally, we can find an étale covering $\{S_i \to S\}_i$ such that $G_{S_i}$ has a maximal torus, and we can define $\text{Isom}_{\text{ext}}(\Psi, G)$ by the descent data of $\text{Isom}_{\text{ext}}(\Psi_{S_i}, G_{S_i})$.

An orientation of $\Psi$ with respect to $G$ is an $S$-point of $\text{Isom}_{\text{ext}}(\Psi, G)$. A twisted root datum $\Psi$ together with an orientation $v \in \text{Isom}_{\text{ext}}(\Psi, G)(S)$ is called an oriented root datum and we denote it as $(\Psi, v)$.

One can also define the functor $\text{Isom}_{\text{ext}}(G, \Psi)$ in the same way. Suppose that $G$ is with a maximal torus $T$. Then there is a natural isomorphism $\iota$ between $\text{Isom}(\Psi, \Phi(G, T))$ and $\text{Isom}(\Phi(G, T), \Psi)$ which sends $u$ to $u^{-1}$. This isomorphism also induces an isomorphism between $\text{Isom}_{\text{ext}}(\Psi, \Phi(G, T))$ and $\text{Isom}_{\text{ext}}(\Phi(G, T), \Psi)$. Let $T'$ be another maximal torus of $G$. We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Trans}^G_G(T, T') & \xrightarrow{\text{Norm}_{\text{ad}}(T)} & \text{Isom}(\Psi, \Phi(G, T)) \\
\downarrow & & \downarrow \\
\text{Isom}(\Psi, \Phi(G, T')) & \xrightarrow{\text{Norm}_{\text{ad}}(T)} & \text{Isom}(\Phi(G, T), \Psi)
\end{array}
\]

Therefore, the morphism $\iota$ defines an isomorphism between $\text{Isom}_{\text{ext}}(\Psi, G)$ and $\text{Isom}_{\text{ext}}(G, \Psi)$ and we can define $\iota$ for an arbitrary reductive group $G$ by descent.

**Remark 1.3.** Actually, in our case, there is no difference between the transporter $\text{Trans}^G_G(T, T')$ and strict transporter $\text{Transt}^G_G(T, T')$ since both $T$ and $T'$ are maximal tori.

**Proposition 1.4.** Let $G'$ be another reductive group over $S$. Suppose that $G$ and $\Psi$ have the same type at each fibre over $S$. Then we will have the following map:

\[\text{Isom}_{\text{ext}}(\Psi, G') \times \text{Isom}_{\text{ext}}(G, \Psi) \to \text{Isom}_{\text{ext}}(G, G').\]

**Proof.** To see this, we first suppose that both $G$ and $G'$ have maximal tori. Let $T$ and $T'$ be the maximal tori of $G$ and $G'$ respectively. Then the natural map from $\text{Isom}(\Psi, \Phi(G', T')) \times \text{Isom}(\Phi(G, T), \Psi)$ to $\text{Isom}(\Phi(G, T), \Phi(G', T'))$, induces the map

\[\text{Isom}_{\text{ext}}(\Psi, \Phi(G', T')) \times \text{Isom}_{\text{ext}}(\Phi(G, T), \Psi) \to \text{Isom}_{\text{ext}}(\Phi(G, T), \Phi(G', T')).\]

We now want to show $\text{Isom}_{\text{ext}}(\Phi(G, T), \Phi(G', T')) \simeq \text{Isom}_{\text{ext}}(G, G')$. Note that we have natural morphisms from $\text{Isoms}_{\text{gr}}(G, T; G', T')/\text{Norm}_{\text{ad}(G)}(\text{ad}(T))$ to $\text{Isom}_{\text{ext}}(\Phi(G, T), \Phi(G', T'))$. By [SGA3] Exp.XXIV, 2.2,

\[\text{Isoms}_{\text{gr}}(G, T; G', T')/\text{Norm}_{\text{ad}(G)}(\text{ad}(T)) \supseteq \text{Isom}_{\text{ext}}(G, G').\]

So we have a map from

\[\iota_1 : \text{Isom}_{\text{ext}}(G, G') \to \text{Isom}_{\text{ext}}(\Phi(G, T), \Phi(G', T')).\]
Note that $\text{Isomext}(G; G')$ and $\text{Isomext}(\Phi(G, T), \Phi(G', T'))$ are principal homogeneous spaces under $\text{Autext}(G)$ and $\text{Autext}(\Phi(G, T))$ respectively.

By [SGA3] Exp. XXIV, 2.1, we have $\text{Autext}(G, T) \simeq \text{Autext}(G)$. By Theorem 0.5 and Proposition 0.3, the natural map between $\text{Autext}(G, T)$ and $\text{Autext}(\Phi(G, T))$ is an isomorphism on each geometric fiber, so $\text{Autext}(G) \simeq \text{Autext}(G, T) \simeq \text{Autext}(\Phi(G, T))$.

Under these identifications, the map $\iota_1$ is a morphism between $\text{Autext}(G)$-principal homogeneous spaces. So it is an isomorphism.

For general reductive groups $G$ and $G'$, we can define this map by descent.

**Remark 1.5.** For a semisimple group $G$, there is also a definition of an orientation of $G$ (ref. [PS]). Let $G_{qs}$ be a quasi-split form of $G$, and $T'$ be a maximal torus of $G_{qs}$. If we replace $\Psi$ above by $\Phi(G_{qs}, T')$, then an orientation of $\Psi$ with respect to $G$ is called an orientation of $G$ in [PS] §2.

### 1.2.2 Oriented embedding functors

Given an oriented twisted root datum $(\Psi, v)$ of $\Psi$ with respect to $G$, we define the oriented embedding functor as:

$$\mathcal{E}(G, \Psi, v)(S') = \left\{ f : T_{S'} \hookrightarrow G_{S'} \mid f \in \mathcal{E}(G, \Psi)(S'), \text{and the image of } f^\Psi \text{ in } \text{Isomext}(\Psi, G)(S') \text{ is } v \right\}$$

With all the notation defined above, we have the following result similar to Theorem 1.1:

**Theorem 1.6.** Suppose that $G$ is reductive. Then in the sense of the étale topology, $\mathcal{E}(G, \Psi, v)$ is a left homogeneous space under the adjoint action of $G$, and a torsor over $T$ under the right $W(\Psi)$-action. Moreover, $\mathcal{E}(G, \Psi, v)$ is representable by an affine $S$-scheme.

**Proof.** Since $\mathcal{E}(G, \Psi)$ and $\text{Isomext}(\Psi, G)$ are sheaves, $\mathcal{E}(G, \Psi, v)$ is an $S$-sheaf.

Let $S'$ be an $S$-scheme, $f_1, f_2 \in \mathcal{E}(G, \Psi, v)(S')$ and $T_i = f_i(T_{S'})$, for $i = 1, 2$ respectively. There is an étale neighborhood $U$ of $S'$ such that $G$ splits relatively to $T_i$'s and hence there is $g \in \text{Transt}(T_1, T_2)(U)$ (ref. [SGA3], Exp. XXIV, 1.5). By the definition of $\mathcal{E}(G, \Psi, v)$, we know that $f_1^\Psi$ and $f_2^\Psi$ have the same image in $\text{Isomext}(\Psi, G)(S')$, so $g \cdot f_1^\Psi = f_2^\Psi$. Since $G_U$ splits relatively to $T_{2,U}$, we can find $n \in \text{Norm}_{G_U}(T)(U)$ such that $n \cdot g \cdot f_1^\Psi = f_2^\Psi$, which proves that $\mathcal{E}(G, \Psi, v)$ is a homogeneous space under the adjoint action of $G$.

Next, we show that $\pi : \mathcal{E}(G, \Psi, v) \to T$ is surjective as a morphism of sheaves. As we have seen in the proof of Theorem 1.1, $\pi : \mathcal{E}(G, \Psi) \to T$ is surjective, so for an $S$-scheme $S'$ and $X \in T(S')$, there is an étale covering...
{S_1' \to S'} such that for each i, there is f_i \in \mathfrak{E}(G, \Psi)(S_i') with \pi(f) = X_{S_i}. Moreover, we can assume G_{S_i} is split relatively to X_{S_i}. Then \text{Aut}_{S-gr}(G, T)(S_i') is mapped surjectively to \text{Aut}_{ext}(G)(S_i') (ref. \text{[SGA3]}, Exp. XXIV, 2.1), which allows us to find \sigma_i \in \text{Aut}_{S-gr}(G, T)(S_i') such that \sigma_i \circ f_i \in \mathfrak{E}(G, \Psi, v)(S_i'). Therefore, \pi : \mathfrak{E}(G, \Psi, v) \to T is surjective.

Finally, we want to prove that \mathfrak{E}(G, \Psi, v) is a right W(\Psi)-torsor over T. We identify W(\Psi) with a subgroup of \text{Aut}_{S-gr}(T). So for w \in W(\Psi)(S'), we can regard it as an element in \text{Aut}_{S-gr}(T)(S'). By the definition of \text{Isom}_{ext}(\Psi, G), W(\Psi) acts trivially on \text{Isom}_{ext}(\Psi, G). Therefore, we can consider the map

\[ m_w : \mathfrak{E}(G, \Psi, v) \times W(\Psi) \to \mathfrak{E}(G, \Psi, v) \times \mathfrak{E}(G, \Psi, v) \]

defined as \( m_w(f, w) = (f, f \circ w) \), for \( f \in \mathfrak{E}(G, \Psi, v)(S') \), \( w \in W(\Psi)(S') \), where \( S' \) is an S-scheme.

On the other hand, given \( f_1, f_2 \in \mathfrak{E}(G, \Psi, v)(S') \) with \( f_1(T) = f_2(T), \ f_1^\Psi \) and \( f_2^\Psi \) have the same image in \( \text{Isom}_{ext}(\Psi, \mathfrak{E}(G, f_1(T))) \), so \( f_1^{-1} \circ f_2 \) defines an element in \( W(\Psi)(S') \).

Then we can define the map

\[ i_v : \mathfrak{E}(G, \Psi, v) \times \mathfrak{E}(G, \Psi, v) \to \mathfrak{E}(G, \Psi, v) \times W(T) \]

as \( i_v(f_1, f_2) = (f_1, f_1^{-1} \circ f_2) \) for \( (f_1, f_2) \in \mathfrak{E}(G, \Psi, v)(S') \times \mathfrak{E}(G, \Psi, v)(S') \), \( S' \) an S-scheme. As what we have verified in the proof of Theorem 1.1, we have that \( i_v, m_v \) are the inverse maps of each other. Again, by Proposition 1.1, we conclude that \( \mathfrak{E}(G, \Psi, v) \) is a \( W(\Psi) \)-torsor over \( T \) and by \text{[SGA3]} Exp. X, 5.5, \( \mathfrak{E}(G, \Psi, v) \) is representable. Since \( T \) is affine and \( W(\Psi) \) is finite, \( \mathfrak{E}(G, \Psi, v) \) is represented by an affine S-scheme. \( \square \)

For a reductive group G, we let \text{der}(G) be the derived group of G and \text{ss}(G) be the semisimple group associated to G. Let \text{sc}(G) be the simply connected group associated to \text{der}(G). The following corollary allows us to reduce the oriented embedding problem of reductive groups to that of semisimple simply connected groups, which is useful for arithmetic purposes.

**Corollary 1.7.** Let \( v \in \text{Isom}_{ext}(\Psi, G)(S) \). Then \( v \) induces an orientation \( v_{der} \in \text{Isom}_{ext}(\text{der}(\Psi), \text{der}(G))(S) \). Moreover, we have a natural isomorphism

\[ \mathfrak{E}(G, \Psi, v) \to \mathfrak{E}(\text{der}(G), \text{der}(\Psi), v_{der}). \]

One can also replace \text{der}(\Psi) and \text{der}(G) by \text{ad}(\Psi) and \text{ad}(G), \text{ss}(\Psi) and \text{ss}(G), \text{sc}(\Psi) and \text{sc}(G) respectively.

**Proof.** The key point lies in the functoriality of the induced and coinduced operation on the root data and the one-to-one correspondence between the maximal tori of G and the maximal tori of \text{der}(G) (resp. \text{ad}(G), \text{sc}(G)), which gives us a natural isomorphism from \text{Isom}_{ext}(\Psi, G) to \text{Isom}_{ext}(\text{der}(\Psi), \text{der}(G)) (resp. \text{Isom}_{ext}(\text{ad}(\Psi), \text{ad}(G)), \text{Isom}_{ext}(\text{ss}(\Psi), \text{ss}(G)), \text{Isom}_{ext}(\text{sc}(\Psi), \text{sc}(G))).
Hence, we only prove the case for $\text{der}(\Psi)$ and $\text{der}(G)$ in detail, all the other cases can be proved similarly.

Suppose that $G$ has a maximal torus $T$. Let $T' = T \cap \text{der}(G)$. Then $T'$ is a maximal torus of $\text{der}(G)$ and $\text{der}(\Phi(G, T)) = \Phi(\text{der}(G), T')$. Moreover, the scheme of maximal tori of $G$ is isomorphic to the scheme of maximal tori of $\text{der}(G)$ (ref. [SGA3], Exp. XXII, 6.2.7, 6.2.8). Therefore, there is a natural morphism $i_{\text{der}}$ from $\text{Isom}(\Psi, \Phi(G, T))$ to $\text{Isom}(\text{der}(\Psi), \Phi(\text{der}(G), T'))$.

Moreover, by Proposition 0.6, the natural morphism from $\Psi$ to $\text{der}(\Psi)$ induces an isomorphism from $W(\Psi)$ to $W(\text{der}(\Psi))$. Therefore, there is a natural morphism $\tilde{i}_{\text{der}}$ from $\text{Isom}(\Psi, \Phi(G, T))$ to $\text{Isom}(\text{der}(\Psi), \Phi(\text{der}(G), T'))$ by descent. Therefore, given $v \in \text{Isom}(\Psi, G)(S)$, we can have $v_{\text{der}} \in \text{Isom}(\text{der}(\Psi), \text{der}(G))$ induced by $v$.

Since $W(\Psi)$ is isomorphic to $W(\text{der}(\Psi))$, by Theorem 1.6, both $\mathcal{E}(G, \Psi, v)$ and $\mathcal{E}(\text{der}(G), \text{der}(\Psi), v_{\text{der}})$ are $W(\Psi)$-torsors over the scheme of maximal tori. Thus, the natural morphism

$$\mathcal{E}(G, \Psi, v) \cong \mathcal{E}(\text{der}(G), \text{der}(\Psi), v_{\text{der}})$$

is an isomorphism. \hfill \Box

### 1.3 Examples–Embedding functors and embedding problems of Azumaya algebras with involution

In this section, we want to show the relations between the embedding functors and embedding problems of Azumaya algebras with involution. For the background of Azumaya algebras, we refer to the book by Knus [KN] Chap. III, §5 and the paper [KPS90].

Let $K$ be a commutative ring and suppose that 2 is invertible in $K$. Let $A$ be an Azumaya algebra over $K$ of degree $n$ equipped with an involution $\tau$. Let $k = K^\tau$, which are the elements in $K$ fixed by $\tau$. If $k = K$, then $\tau$ is said to be of the first kind. If $K$ is an étale quadratic extension over $k$, then $\tau$ is said to be of the second kind. Let $E$ be a commutative étale algebra over $K$ of rank $n$ equipped with an involution $\sigma$. Assume $\sigma | K = \tau | K$.

Let $U(E, \sigma)$ and $U(A, \tau)$ be two algebraic $k$-groups defined as follows: for any commutative $k$-algebra $C$,

$$U(E, \sigma)(C) = \{ x \in E \otimes_k C | x\sigma(x) = 1 \},$$

and

$$U(A, \tau)(C) = \{ x \in A \otimes_k C | x\tau(x) = 1 \}.$$

Let $T = U(E, \sigma)^0$, the identity component of $U(E, \sigma)$, and $G = U(A, \tau)^0$. Since 2 is invertible in $K$, $G$ is smooth at each fiber.

Then we associate a root datum $\Psi$ to $T$. The idea is to associate a "split form" $(A_0, \tau_0)$ (resp. $(E_0, \sigma_0)$) to each $(A, \tau)$ (resp. $(E, \sigma)$). From the split
form \((A_0, \tau_0)\), we get a group \(G_0\) with a split maximal torus \(T_0\). Let \(\Phi(G_0, T_0)\) be the root datum of \(G_0\) with respect to \(T_0\). Then we use the isomorphism between \(\text{Aut}(E_0, \sigma_0)\) and \(\text{Aut}(\Psi_0)\) to associate a twisted root datum \(\Psi\) to \((E, \sigma)\). This allows us to transfer a \(k\)-embedding from \((E, \sigma)\) to \((A, \tau)\) to a \(k\)-point of the embedding functor \(\mathcal{E}(G, \Psi)\). Moreover, we will show that the \(k\)-points of \(\mathcal{E}(G, \Psi)\) are in one-to-one correspondence with the \(k\)-embeddings from \((E, \sigma)\) to \((A, \tau)\). To simplify things, we always assume that \(A\) and \(E\) have constant rank over \(K\).

### 1.3.1 The root datum associated to \(T\)

**Notations for the case where the involution \(\tau\) is of the second kind**

If \(\tau\) is an involution of the second kind, then we let \(A_0 = M_{n,k} \times M_{n,k}^\text{op}\), where \(M_{n,k}\) stands for the \(n \times n\)-matrix algebra defined over \(k\), and let \(E_0 = k^n \times k^n\), which is viewed as an étale algebra over \(k \times k\). In this case, let \(\tau_0\) be the exchange involution of \(A_0\) defined by \(\tau_0(M, N) = (N, M)\). Let \(\iota_0 : E_0 \to A_0\) be defined as \(\iota_0(x_1, ..., x_n, y_1, ..., y_n) = (\text{diag}(x_1, ..., x_n), \text{diag}(y_1, ..., y_n))\), where \(\text{diag}(x_1, ..., x_n)\) stands for the diagonal matrix with the \((i, i)\)-th entry \(x_i\). Clearly it is a \(k \times k\)-homomorphism and the image of \(\iota_0\) is invariant under \(\tau_0\). Let \(\sigma_0\) be the exchange involution on \(E_0\) induced by \(\tau_0\). Let \(T_0 = U(E_0, \sigma_0)\) and \(G_0 = U(A_0, \tau_0)\) and \(f_0 : T_0 \to G_0\) be the embedding induced by \(\iota_0\). Let \(\Psi_0\) be the root datum associated to \(T_0\) defined as

\[
\Psi_0(C) = \Phi(G_0, f_0(T_0))(C) \circ f_0
\]

for any \(k\)-algebra \(C\).

In this case, we let \(i_{T_0} : \mathbb{G}_{m,k} \to T_0\) (resp. \(i_{G_0} : \mathbb{G}_{m,K} \to G_0\)) denote the embedding defined by the \(k \times k\)-structure morphism of \(E_0\) (resp. \(A_0\)), and let \(i_{T} : R_{K/k}^{(1)}(\mathbb{G}_{m,K}) \to T\) (resp. \(i_{G} : R_{K/k}^{(1)}(\mathbb{G}_{m,K}) \to G\)) denote the embedding defined by the \(K\)-structure morphism of \(E\) (resp. \(A\)).

An isomorphism between \((E_0, \sigma_0, k \times k)\) and \((E, \sigma, K)\) is a \(k\)-isomorphism between \(E_0\) and \(E\) commuting with the involutions, and sends \(k \times k\) to \(K\). Let \(\mathfrak{X} = \text{Isom}((E_0, \sigma_0, k \times k), (E, \sigma, K))\) be the isomorphism functor between \((E_0, \sigma_0, k \times k)\) and \((E, \sigma, K)\). Note that \(\mathfrak{X}(\mathfrak{F})\) is not empty for each geometric point \(\mathfrak{F}\) of \(\text{Spec}(k)\), if and only if \(\text{rank}_k E^+ = \text{rank}_K E\). Throughout this article, we assume that \(\mathfrak{X}\) is non-empty. Then \(\mathfrak{X}\) is a right \(\text{Aut}(E_0, \sigma_0, k \times k)\)-torsor.

**Notations for the case where the involution \(\tau\) is of the first kind**

For \(\tau\) an involution of the first kind, we let \(A_0 = M_{n,k}\), and \(E_0 = k^n\). Let \(\iota_0 : E_0 \to A_0\) be defined as \(\iota(x_1, ..., x_n) = \text{diag}(x_1, ..., x_n)\).

If \(\tau\) is an orthogonal involution and \(n\) is odd, we let \(n = 2m + 1\) and \(B = (b_{ij})_{0 \leq i, j \leq 2m}\), where

\[
b_{i,j} = \begin{cases} 
1, & \text{if } i=j=0, \\
1, & \text{if } i=j \pm m, \text{ with } i,j \geq 1 \\
0, & \text{otherwise.}
\end{cases}
\]
For $\tau$ an orthogonal involution and $n$ even, we let $n = 2m$ and $B = (b_{i,j})_{1\leq i,j \leq 2m}$, where

$$b_{i,j} = \begin{cases} 
1, & \text{if } i = j \pm m, \text{ with } i,j \geq 1 \\
0, & \text{otherwise}.
\end{cases}$$

For $\tau$ a symplectic involution, we let $n = 2m$ and $B = (b_{i,j})_{1\leq i,j \leq 2m}$, where

$$b_{i,j} = \begin{cases} 
1, & j = i + m, \\
-1, & j = i - m, \\
0, & \text{otherwise}.
\end{cases}$$

Let $\tau_0$ be the involution on $A_0$ defined by $\tau_0(M) = BM'B^{-1}$, and let $\sigma_0$ be the involution on $E_0$ induced by $\tau_0$. Let $T_0 = U(E_0, \sigma_0)^o$ and $G_0 = U(A_0, \tau_0)^o$. Let $f_0 : T_0 \rightarrow G_0$ be the embedding induced by $\iota_0$.

Let $\Psi_0$ be the root datum associated to $T_0$ defined as

$$\Psi_0(C) = \Phi(G_0, f_0(T_0))(C) \circ f_0$$

for any $k$-algebra $C$. For $\tau$ of the first kind, let $X = \text{Isom}((E_0, \sigma_0), (E, \sigma))$.

Note that $X(\overline{s})$ is non-empty for each geometric point $\overline{s}$ of $\text{Spec}(k)$ if and only if $\text{rank}_KE^\sigma = \lfloor \frac{1}{2} \text{rank}_KE \rfloor$. Throughout this article, we assume that $X$ is non-empty.

**The definition of the twisted root datum $\Psi$**

**Definition 1.8.**

1. Suppose that $\tau$ is of the first kind. A $k$-embedding $\iota : (E, \sigma) \rightarrow (A, \tau)$ is an injective $k$-homomorphism commuting with the involutions.
2. Suppose $\tau$ is of the second kind. A $k$-embedding $\iota : (E, \sigma) \rightarrow (A, \tau)$ is an injective $k$-homomorphism commuting with the involutions and sends $K$ to $K'$.
3. Let $\iota$ be a $k$-embedding from $(E, \sigma)$ to $(A, \tau)$. Define the isomorphisms between $(E_0, A_0, \iota_0)$ and $(E, A, \iota)$ to be pairs $(\alpha, \beta)$, where $\alpha$ is an isomorphism between $(E_0, \sigma_0)$ and $(E, \sigma)$, $\beta$ is an isomorphism between $(A_0, \tau_0)$ and $(A, \tau)$, and $\alpha, \beta$ satisfy $\iota \circ \alpha = \beta \circ \iota_0$. Let $\text{Isom}((E_0, A_0, \iota_0), (E, A, \iota))$ be the isomorphism functor between $(E_0, A_0, \iota_0)$ and $(E, A, \iota)$.
4. A morphism $f : T \rightarrow G$ is called an embedding if it is a closed immersion and a group homomorphism. Let $f : T \rightarrow G$ be an embedding. Define the isomorphisms between $(G_0, T_0, f_0)$ and $(G, T, f)$ to be pairs $(h, g)$, where $h$ is an isomorphism from $T_0$ to $T$ and $g$ is an isomorphism from $G_0$ to $G$, and $h, g$ satisfy $f \circ h = g \circ f_0$. Let $\text{Isom}((G_0, T_0, f_0), (G, T, f))$ be the isomorphism functor between $(G_0, T_0, f_0)$ and $(G, T, f)$.

**Remark 1.9.** Suppose that $\tau$ is of the second kind, and $\iota$ is an embedding from $(E, \sigma)$ to $(A, \tau)$. For a $k$-algebra $C$ and $(\alpha, \beta) \in \text{Isom}((E_0, A_0, \iota_0), (E, A, \iota))(C)$, $\alpha$ will automatically be in $\text{Isom}((E_0, \sigma_0, k \times k), (E, \sigma, K))(C)$, because $\beta$ sends the center of $A_0$ to the center of $A$ and $\iota \circ \alpha = \beta \circ \iota_0$.
Remark 1.10. Let f be a k-point of $\mathcal{E}(G, \Psi)$. Since f induces an isomorphism between $\Psi$ and $\Phi(G, f(T))$, f induces an isomorphism between $\text{rad}(\Psi)$ and $\text{rad}(\Phi(G, f(T)))$. As $i_T(R_{K/k}^{(1)}(G_m, K))$ (resp. $i_G(R_{K/k}^{(1)}(G_m, K))$) is the torus associated to $\text{rad}(\Psi)$ (resp. $\text{rad}(\Phi(G, f(T)))$), f maps $i_T(R_{K/k}^{(1)}(G_m, K))$ to $i_G(R_{K/k}^{(1)}(G_m, K))$.

The following lemma enables us to attach a twisted root datum to the torus T.

Lemma 1.11. Let $S = \text{Spec}(k)$. Then we have the following:

1. The canonical homomorphism from $\text{Aut}(E_0, A_0, \iota_0)$ to $\text{Aut}_{s-\text{gr}}(G_0, f_0(T_0))$ is an isomorphism except for $G_0$ of type $D_4$ or A of degree 2 with $\tau$ orthogonal.

2. For the involution $\tau_0$ of the first kind, there is a canonical monomorphism $j_{E_0}$ from $\text{Aut}(E_0, \sigma_0)$ to $\text{Aut}(\Psi_0)$. In particular, if $\Psi_0$ is not of type $D_4$, then the homomorphism $j_{E_0}$ is an isomorphism.

3. For the involution $\tau_0$ of the second kind, there is a canonical isomorphism from $\text{Aut}(E_0, \sigma_0, k \times k)$ to $\text{Aut}(\Psi)$.

Proof. To verify that the canonical homomorphism $j_{A_0}$ from $\text{Aut}(E_0, A_0, \iota_0)$ to $\text{Aut}_{s-\text{gr}}(G_0, f_0(T_0))$ is an isomorphism, it suffices to verify that the natural morphism from $\text{Aut}(A_0, \tau_0)$ to $\text{Aut}_{s-\text{gr}}(G_0)$ is an isomorphism, since the automorphism preserves $\iota_0(E_0)$ if and only if it preserves $f_0(T_0)$. To see that the natural morphism from $\text{Aut}(A_0, \tau_0)$ to $\text{Aut}_{s-\text{gr}}(G_0)$ is an isomorphism, we check it case by case. For $G_0$ of type $A_n$, let $\sigma$ be the automorphism of $(A_0, \tau_0)$ which maps $(M, N) \in A_0$ to $(N', M')$, where $N'$ denotes the transpose of N. Then $\text{Aut}(A_0, \tau_0)$ is $\text{PGL}_{n+1} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by $\sigma$.

Note that $\sigma$ induces the outer automorphism of $G_0 = \text{GL}_{n+1}$, and we have $\text{Aut}_{s-\text{gr}}(G_0) = \text{Aut}_{s-\text{gr}}(\text{GL}_{n+1}) = \text{PGL}_{n+1} \rtimes \mathbb{Z}/2\mathbb{Z}$. For $G_0$ of type $B_n$, we have

$\text{Aut}(A_0, \tau_0) = \text{PGO}(A_0, \tau_0) \cong G_0$

(cf. [KMRT98], Thm. 12.15 and Prop. 12.4). Since $G_0$ is adjoint of type $B_n$ in this case, we have $G_0 = \text{Aut}_{s-\text{gr}}(G_0)$ and hence

$\text{Aut}(A_0, \tau_0) \cong G_0 = \text{Aut}_{s-\text{gr}}(G_0)$.

Similar calculation can be done for $G_0$ of type $C_n$ or $G_0$ of type $D_n$ with $n \geq 2$ and $n \neq 4$, and we refer to [KMRT98], Theorem 26.14 and Theorem 26.15.

To prove (2), we first note that there is a natural isomorphism $j_{E_0}$ from $\text{Aut}(E_0, \sigma_0)$ to $\text{Aut}_{s-\text{gr}}(T_0)$. To see that the image of $j_{E_0}$ is contained in $\text{Aut}(\Psi)$, we verify it case by case. For example, for $\sigma_0$ orthogonal and $E$ of degree $2m$, the automorphism group $\text{Aut}(E_0, \sigma_0)$ is isomorphic to the constant group scheme $((\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m)_S$. We can check that the corresponding action of $((\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m)_S$ on $T_0$ actually preserves the root datum $\Psi_0$. Moreover, by [Bon] Plan. IV, we know that $((\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m)_S$ is exactly the automorphism
group of $\Psi_0$ for $m \neq 4$ and is a subgroup of $\text{Aut}(\Psi_0)$ for $m = 4$. From this, we conclude that $j_{E_0}$ maps $\text{Aut}(E_0, \sigma_0)$ isomorphically to $\text{Aut}(\Psi)$ for $\Psi_0$ not of type $D_4$. One can check the other cases in the same way, which allows us to conclude the statement (2). One can prove (3) in the same way.

Since $T_0$ is a maximal torus of $G_0$, we have the following exact sequence:

\[ 0 \longrightarrow \text{ad}(T_0) \longrightarrow \text{Aut}_{S-\text{gr}}(G_0, f_0(T_0)) \longrightarrow \text{Aut}(\Psi_0) \longrightarrow 0 \]

by Proposition [1.3] and [SGA3] Exp. XXIV, Proposition 2.6. Therefore, we can summarize the above lemma as the following diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{ad}(T_0) & \longrightarrow & \text{Aut}(A_0, E_0, \iota_0) & \longrightarrow & \text{Aut}(E_0, \sigma) & \longrightarrow & 0, \\
& & & \downarrow \alpha & & \downarrow \beta & & & \\
0 & \longrightarrow & \text{ad}(T_0) & \longrightarrow & \text{Aut}_{S-\text{gr}}(G_0, f_0(T_0)) & \longrightarrow & \text{Aut}(\Psi_0) & \longrightarrow & 0
\end{array}
\]

where $\alpha$, $\beta$ are isomorphisms if $\tau_0$ is of the first kind and $G_0$ is not of type $D_4$. For the involution $\tau_0$ of the second kind, we just replace $\text{Aut}(E_0, \sigma)$ by $\text{Aut}(E_0, \sigma, k \times k)$ and $\alpha$, $\beta$ are isomorphisms. Now for the involution $\tau_0$ of the first kind, we define the twisted root datum $\Psi$ related to $T$ as

\[ \Psi := \mathfrak{X} \wedge \text{Aut}(E_0, \sigma_0) \Psi_0. \]

For the involution $\tau_0$ of the second kind, we define the twisted root datum $\Psi$ related to $T$ as

\[ \Psi := \mathfrak{X} \wedge \text{Aut}(E_0, \sigma_0, k \times k) \Psi_0. \]

**Remark 1.12.** If we regard $\Psi_0$ as a set of combinatorial data satisfying the axioms of root data, the canonical morphism $\beta$ between $\text{Aut}(E_0, \sigma_0)$ and $\text{Aut}(\Psi_0)$ is defined over any arbitrary base. However, for the involution $\tau_0$ of the first kind, the group $G_0$ is not reductive over arbitrary base. Hence, we ask $2$ to be invertible over the base so that $\Psi_0$ can be regarded as a root datum of $G_0$.

**Remark 1.13.** We have a canonical morphism from $\mathfrak{X}$ to $\text{Isom}(\Psi_0, \Psi)$ which is an isomorphism except $\Psi_0$ is of type $D_4$ by Lemma [1.11]. The natural morphism from $\text{Isom}((A_0, \tau_0), (A, \tau))$ to $\text{Isom}(G_0, G)$ over $k$ is a canonical monomorphism which is an isomorphism except $G$ is of type $D_4$, since $\text{Aut}(A_0, \tau_0) = \text{Aut}(G_0)$ except for $G_0$ of type $D_4$ ([KMRT98] Chap. IV, §23 and §26).

**Remark 1.14.** For $A$ of degree 2 with $\tau$ orthogonal, the corresponding split group $G_0$ is actually the one dimensional split torus. Therefore, we have $G_0$ acts trivially on itself but non trivially on $A_0$. However, the conjugation by

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

induces a nontrivial isomorphism of $G_0$. Hence the natural morphism
Theorem 1.15. Keep all the notations defined above. Then:

1. The set of $k$-embeddings from $(E, \sigma)$ to $(A, \tau)$ is in one-to-one correspondence with the set of $k$-points of $\mathcal{E}(G, \Psi)$, except for $G$ of type $D_4$ or $A_4$ of degree 2 with $\tau$ orthogonal.

2. If $\tau$ is of the second kind, the set of $K$-algebra embeddings from $(E, \sigma)$ into $(A, \tau)$ is in one-to-one correspondence with the set of $k$-points $f$ of $\mathcal{E}(G, \Psi)$ which satisfy $f \circ i_T = i_G$.

Proof. The crucial ingredient of the proof is Lemma 1.11. We prove (1) first. Let $\iota$ be a $k$-embedding from $(E, \sigma)$ to $(A, \tau)$. Clearly, $\iota$ induces an embedding $f : T \to G$. To see that $f$ is a $k$ point of $\mathcal{E}(G, \Psi)$, we need to verify that $f$ induces an isomorphism between $\Psi$ and $\Phi(G, f(T))$.

Let $\mathcal{Y} = \text{Isom}((E_0, A_0, \iota_0), (E, A, i))$. By Lemma 1.11, we have

$$\mathcal{Y} \simeq \text{Isom}((G_0, f_0(T_0)), (G, f(T))).$$

This allows us to define an isomorphism from $\mathcal{Y} \wedge \text{Aut}((E_0, A_0, \iota_0)) (G_0, f_0(T_0))$ to $(G, f(T))$, which induces an isomorphism from $\mathcal{Y} \wedge \text{Aut}((E_0, A_0, \iota_0)) \Phi(G_0, f_0(T_0))$ to $\Phi(G, f(T))$.

Given a $k$-algebra $C$ and $(\alpha, \beta) \in \mathcal{Y}(C)$, we have a natural map from $\mathcal{Y}$ to $\mathcal{X}$ which maps $(\alpha, \beta)$ to $\alpha$. By the definition of $\Psi_0$, $f_0$ induces an isomorphism between $\Phi(G_0, f_0(T_0))$ and $\Psi_0$. Therefore we have the following natural map

$$\mathcal{Y} \wedge \text{Aut}(G_0, f_0(T_0)) \Phi(G_0, f_0(T_0)) \simeq \mathcal{Y} \wedge \text{Aut}(G_0, f_0(T_0)) \Psi_0 \to \mathcal{X} \wedge \text{Aut}(\Psi_0) \Psi_0 = \Psi.$$
so we have a canonical morphism from $\mathcal{A}$ to $\text{Isom}((E_0, \sigma_0), (E, \sigma))$, and hence a canonical map from $\mathcal{A} \wedge (E_0, \sigma_0)$ to $(E, \sigma)$. Similarly, by Remark 1.13 we have a canonical map from $\mathcal{A}$ to $\text{Isom}((A_0, \tau_0), (A, \tau))$, and hence a canonical map from $\mathcal{A} \wedge (A_0, \tau_0)$ to $(A, \tau)$. Therefore, we get a $k$-embedding $\iota : (E, \sigma) \to (A, \tau)$ from the map $\mathcal{A} \wedge (E_0, \sigma_0)$ to $\mathcal{A} \wedge (A_0, \tau_0)$ induced by $i_0$, and we denote $\iota$ as $J_G(f)$.

Clearly, $I_A J_G(f) = f$ since we construct $\iota$ from $\mathcal{A}$ and $\mathcal{A} \wedge (G_0, T_0, f_0)$ is canonically isomorphic to $(G, T, f)$. On the other hand, we have

$$J_G \circ I_A(\iota) = \iota$$

because of the canonical isomorphism from $\text{Isom}((E_0, A_0, \iota_0), (E, A, \iota))$ to $\text{Isom}((G_0, T_0, f_0), (G, T, f))$, where $f$ is induced by $\iota$. Hence, the first assertion follows.

We prove (2) now. Clearly, if $\iota : (E, \sigma) \to (A, \tau)$ is a $K$-embedding, then the corresponding $k$-embedding $f$ will be a $k$-point of $\mathcal{E}(G, \Psi)$ and $f$ satisfies $f \circ i_T = i_G$.

Now suppose $f \in \mathcal{E}(G, \Psi)(k)$ and $f \circ i_T = i_G$. Then we need to verify that the map $J_G(f)$ from $\mathcal{A} \wedge (E_0, \sigma_0)$ to $\mathcal{A} \wedge (A_0, \tau_0)$ is a $K$-morphism. From the construction of $J_G(f)$, it is enough to prove that the two maps from $\text{Isom}((G_0, T_0, f_0), (G, T, f))(R)$ to $\text{Isom}(\mathbb{G}_{m,k}, \mathbb{G}_{m,K})(R)$, which map $(h, g)$ to $i_T^{-1} \circ h \circ i_{0,T}$ and $i_G^{-1} \circ g \circ i_{0,G}$ respectively, coincide. However, it is a direct consequence from the fact that $f \circ i_T = i_G$, since

$$i_T^{-1} \circ h \circ i_{0,T} = i_G^{-1} \circ f \circ h \circ i_{0,T}$$

$$= i_G^{-1} \circ g \circ f_0 \circ i_{0,T}$$

$$= i_G^{-1} \circ g \circ i_{0,G}$$

Therefore, $J_G(f)$ is a $K$-algebra morphism. □

**Remark 1.16.** Let $\tau$ be of the second kind. Suppose that $\mathcal{E}(G, \Psi)(k)$ is nonempty and fix $f \in \mathcal{E}(G, \Psi)(k)$. If $f \circ i_T \neq i_G$, then $f \circ \sigma$ will satisfy $(f \circ \sigma) \circ i_T = i_G$ since $\sigma$ acts on $\mathbb{R}_{K/k}^{(1)}(\mathbb{G}_{m,K})$ as $-1$. Therefore, the existence of a $k$-embedding will imply the existence of a $K$-embedding. Moreover, we will see that the condition $f \circ i_T = i_G$ gives a particular orientation $u \in \text{Isom}(\Psi, G)(k)$.

Now we want to consider the case where $G_0$ is of type $D_4$. Note that since there is a natural monomorphism from $\text{Aut}(A_0, E_0, \iota_0)$ to $\text{Aut}_{k-\text{gr}}(G_0, f_0(T_0))$, we can still get a $k$-point of the embedding functor $\mathcal{E}(G, \Psi)$ from a $k$-embedding $\iota : (E, \sigma) \to (A, \tau)$. The problem is that given a $k$-point $f$ of the embedding functor $\mathcal{E}(G, \Psi)$, we can not get a $k$-embedding from $f$ as we have done in the proof of Theorem 1.15 because the canonical map from $\text{Aut}(A_0, E_0, \iota_0)$ to $\text{Aut}_{k-\text{gr}}(G_0, f_0(T_0))$ is not an isomorphism.

To fix the problem, we first observe that $\text{ad}(G_0)$ (resp. $W(\Psi_0)$) is in the image of the canonical morphism from $\text{Aut}(A_0, \tau_0)$ (resp. $\text{Aut}(E_0, \sigma_0)$)
to \( \text{Aut}_{\text{gr}}(G_0) \) (resp. \( \text{Aut}(\Psi_0) \)). So instead to associate a split form \((A_0, \tau_0)\) to \((A, \tau)\), we consider all "quasi-split" forms of \((A, \tau)\). Note that \( \text{Aut}(A_0, \tau_0) / \text{ad}(G_0) \) is the constant group scheme \((\mathbb{Z}/2\mathbb{Z})_k \) and we can find a section from \((\mathbb{Z}/2\mathbb{Z})_k \) to \( \text{Aut}(A_0, E_0, \iota_0) \). For example we can send 1 in \( \mathbb{Z}/2\mathbb{Z} \) to the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

Let us fix the section from \((\mathbb{Z}/2\mathbb{Z})_k \) to \( \text{Aut}(A_0, E_0, \iota_0) \) as above. Let

\[
\text{Isomext}(A_0, \tau_0; A, \tau) := \text{Isom}(A_0, \tau_0; A, \tau) / \text{ad}(G_0).
\]

Then for each \((A, \tau)\) we can associate a quasi-split form

\[
(A_q, \tau_q) = \text{Isomext}(A_0, \tau_0; A, \tau) \wedge (\mathbb{Z}/2\mathbb{Z})_k (A_0, \tau_0).
\]

Moreover, since the section has image in \( \text{Aut}(A_0, E_0, \iota_0) \), we get an étale algebra with involution \((E_q, \sigma_q)\) and an embedding \( \iota_q : (E_q, \sigma_q) \to (A_q, \tau_q) \) from the datum \((A_0, E_0, \iota_0) \). Let \( G_q = U(A_q, \tau_q) \) and \( T_q = U(E_q, \sigma_q) \) and \( f_q : T_q \to G_q \) be the morphism induced by \( \iota_q \).

From our construction, the group \( G \) in an inner form of \( G_q \), so we can always fix an orientation \( v \) in \( \text{Isomext}(G_q, G)(k) \). Then we have the following result:

**Proposition 1.17.** Let \( u \) be a \( k \)-point of \( \text{Isomext}(\Psi, G) \). Then each \( k \)-point of the oriented embedding functor \( \mathcal{E}(G, \Psi, u) \) corresponds to a \( k \)-embedding \( v \) from \((E, \sigma)\) to \((A, \tau)\).

**Proof.** The way to prove it is exactly the same as in Theorem 1.15. The only different thing is that we stay in the inner case. First we fix an orientation \( v \) in \( \text{Isomext}(G_q, G)(k) \). Let \( u_q \) be the orientation in \( \text{Isomext}(\Psi_q, G_q)(k) \) which comes from the map \( f_q \). Then there is an orientation \( v' = u^{-1} \circ v \circ u_q \) in \( \text{Isomext}(\Psi_q, \Psi)(k) \) by Proposition 1.4.

For \( f \in \mathcal{E}(G, \Psi, u)(k) \), we consider the \( \text{Norm}_{\text{ad}(G_q)}(\text{ad}(f_q(T_q))) \)-torsor \( \text{Isomint}_{\nu}(G_q, f_q(T_q); G, f(T)) \). Clearly, we have a canonical morphism from \( 3' \) to \( \text{Isomint}_{\nu}(\Psi_q, \Psi) \) which is an \( \text{ad}(G_q) \)-torsor; and a canonical morphism from \( 3' \) to \( \text{Isomint}_{\nu}(\Psi_q, \Psi) \) which is a \( \text{W}(\Psi_q) \)-torsor. Note that \( \text{ad}(G_q) \) (resp. \( \text{W}(\Psi_q) \)) are in the image of the canonical morphism from \( \text{Aut}(A_q, \tau_q) \) (resp. \( \text{Aut}(E_q, \sigma_q) \)) to \( \text{Aut}_{\text{gr}}(G_q) \) (resp. \( \text{Aut}(\Psi_q) \)) as they do in the split case. So we get a \( k \)-embedding \( \iota : (E, \sigma) \to (A, \tau) \) from the map \( 3' \wedge \text{W}(\Psi_q) (E_q, \sigma_q) \) to \( 3' \wedge \text{ad}(G_q) (A_q, \tau_q) \) induced by \( \iota_q \). \( \square \)
2 Arithmetic properties of embedding functors

In this section, we focus on the arithmetic properties of the embedding functor. The main arithmetic technique which we use here has been developed by Borovoi [Bo99].

In the first part, we recall the main result in [Bo99]. In the second part, we give a criterion for an oriented embedding functor to satisfy the local-global principle. Besides, over a local field \( L \), we use the Tits index to give a necessary and sufficient condition for an oriented embedding functor to have an \( L \)-point.

For a field \( k \) with characteristic different from 2, embedding an étale algebra over \( k \) into a central simple algebra over \( k \) commuting with involutions is equivalent to finding a \( k \)-point of the corresponding embedding functor. In Section 2.4, we use the arithmetic properties of oriented embedding functors to give an alternative proof of Theorem A, Theorem 6.7 and Theorem 7.3 in the work of Prasad and Rapinchuk [PR10].

Throughout this section, we let \( k \) be a global field and \( k^s \) be a separable closure. Let \( \Gamma \) be the absolute Galois group of \( k \), and \( \Omega_k \) be the set of all places of \( k \).

We start this section with some general facts of the local-global principle of homogeneous spaces established in Borovoi’s papers.

2.1 The local-global principle for homogeneous spaces

First, we let \( k \) to be a number field. For a \( k \)-linear algebraic group \( G \), we let \( G^o \) to be the connected component containing the neutral element of \( G \). Let \( G^a \) be the unipotent radical of \( G^o \); \( G^{\text{red}} = G^o/G^a \); \( G^{ss} \) be the derived group of \( G^{\text{red}} \); \( G^{\text{tor}} = G^{\text{red}}/G^{ss} \). Let \( G^{ssu} = \ker[G^o \to G^{\text{tor}}] \). If \( G/G^{ssu} \) is abelian, we let \( G^{\text{mult}} = G/G^{ssu} \) which is a multiplicative group.

Let \( X \) be a left homogeneous space under a connected linear algebraic group \( G \) over \( k \). Let \( x \in X(k^s) \) and \( \overline{\Pi} = \text{Stab}_{G_{ks}}(x) \) be the stabilizer of \( x \).

Throughout this section, we will assume that \( G^{ss} \) is simply connected, and \( \overline{\Pi}/\overline{\Pi}^{ssu} \) is abelian.

Since \( \Gamma \) has a natural action on \( G(k^s) \), we can define \( \mathfrak{G} \) to be the semidirect product \( G(k^s) \rtimes \Gamma \). We have a \( \mathfrak{G} \)-action on \( X(k^s) \) defined as \((g, \sigma)x = g \cdot \sigma(x)\). Let \( \mathcal{H} = \text{Stab}_{\mathfrak{G}}(x) \). Then we have the following exact sequence:

\[
(*) \quad 1 \to \overline{\Pi}(k^s) \xrightarrow{i} \mathcal{H} \xrightarrow{p} \Gamma \to 1,
\]

where \( i(h) = (\overline{h}, 1) \), and \( p \) is the projection to \( \Gamma \).

Since \( \overline{\Pi}^{\text{mult}} \) is commutative, we can define a \( \Gamma \)-action on \( \overline{\Pi}^{\text{mult}} \) by conjugation. To be precise, for \( \sigma \in \Gamma \), choose \((g_\sigma, \sigma) \in p^{-1}(\sigma)\). Then since \( g_\sigma \cdot \sigma(x) = x \), we have \( \text{int}(g_\sigma)^\sigma \overline{\Pi}^{\text{mult}} = \overline{\Pi}^{\text{mult}} \). Note that, the above definition doesn’t depend on the lifting of \( \sigma \) in \( \mathcal{H} \) because \( \overline{\Pi}^{\text{mult}} \) is commutative. Hence
we get a $\Gamma$-action on $\overline{\text{H}^{\text{mult}}}$.

Since the point $x$ is defined over some finite extension $L$ of $k$,

$\overline{\text{H}^{\text{mult}}}$ is defined over $L$. Moreover, for each $\sigma \in \text{Gal}(k^s/L)$,

we can choose $g_\sigma = 1$. Hence, there is a $k$-form $\text{H}^m$ of $\overline{\text{H}^{\text{mult}}}$ defined by this $\Gamma$-action (cf. [BS64] 2.12, [Se] Chap. V, §4, n° 20, and [FSS98] 1.15). One can verify that the isomorphism class of $\text{H}^m$ is independent of the choice of the geometric point $x$. Therefore, given $G$ and $X$, the isomorphism class of $\text{H}^m$ is well-defined ([Bo99], 1.2).

Let $j : \overline{\text{H}} \to G_{k^s}$ be the natural inclusion. Then $j$ induces a group morphism from $\overline{\text{H}^{\text{mult}}}$ to $G_{\text{tor}}^{k^s}$, which descends to a group morphism $j : \text{H}^m \to G_{\text{tor}}$ over $k$.

Consider the complex $0 \to \overline{\text{H}^{\text{mult}}} \to G_{\text{tor}}^{k^s} \to \text{H}^m \to 0$, where $\text{H}^m$ is in degree $-1$ and $G_{\text{tor}}$ is in degree 0. Let $H^1(k, \text{H}^m \to G_{\text{tor}})$ be the first Galois hypercohomology group of the above complex, and $\text{III}^1(k, \text{H}^m \to G_{\text{tor}})$ be the kernel of the localization map $H^1(k, \text{H}^m \to G_{\text{tor}}) \to \prod_{v \in \Omega_k} H^1(k_v, \text{H}^m \to G_{\text{tor}})$.

For $\sigma \in \Gamma$, let $(g_\sigma, \sigma) \in \mathfrak{H}$. Let $u_{\sigma, \tau} = g_{\sigma \tau}(g_{\sigma}^{-1})^{-1}$, $\hat{u}_{\sigma, \tau}$ be the image of $u_{\sigma, \tau}$ in $\overline{\text{H}^{\text{mult}}}(k^s)$, and $\hat{g}_{\sigma, \tau}$ be the image of $g_{\sigma}$ in $G_{\text{tor}}^{k^s}$. Then $(\hat{u}, \hat{g})$ is a hypercycle, and we let $\eta(X) = \text{Cl}(\hat{u}, \hat{g}) \in H^1(k, \text{H}^m \to G_{\text{tor}})$. Note that $\eta(X)$ is well-defined (see [Bo99], 1.4).

We will make use of the following two theorems later.

**Theorem 2.1.** Let $k_\nu$ be a nonarchimedean local field of characteristic 0. Let $G$, $X$ be as above. If $\eta(X) = 0$, then $X$ has a $k_\nu$-point.

**Proof.** [Bo99], Thm. 2.1. □

**Theorem 2.2.** Let $k$ be a number field, and let $G$, $X$ be as above. Assume that $X(k_\nu)$ is nonempty for every place $\nu$ of $k$ and $\eta(X) = 0$. Then $X$ has a rational point.

**Proof.** [Bo99], Thm. 2.2. □

**Remark 2.3.** Note that if $X$ has a $k_\nu$-point at all places $\nu \in \Omega_k$, then $\eta(X)$ lies in $\text{III}^1(k, \text{H}^m \to G_{\text{tor}})$.

**Remark 2.4.** In fact, up to a sign, $\eta(X)$ is the Brauer-Manin obstruction of $X$ (ref. [Bo99], Thm. 4.5).

Now, we let $k$ be a global function field, for example, $k = F_q(t)$, where $F_q$ is a finite field with $q$ elements. One may ask if Theorem 2.1 holds over $k$. Indeed, we have similar results when $G$ is a connected reductive group over $k$ and $X$ is a $G$-homogeneous space for the étale topology. Since $X$ is a $G$-homogeneous space for the étale topology, the set $X(k^s)$ is nonempty. Let $x \in X(k^s)$ and $\overline{H} = \text{Stab}_{G_{k^s}}(x)$. Suppose that $\overline{H}$ is connected reductive. Then we can define an $k$-torus $\text{H}^m$, which is an $k$-form of $\overline{\text{H}^{\text{mult}}}$ as above. Let $\mathfrak{H}$ and $\eta(X)$ be defined as above. Then we have the following:
**Proposition 2.5.** Let \( k \) be a global function field. Let \( G \) be a connected reductive group over \( k \) and \( X \) be a \( G \)-homogeneous space for the étale topology. Let \( x \in X(k^s) \). Define \( H \) as above. Suppose that \( G^{ss} \) is simply connected and \( H \) is a torus. If \( \eta(X) = 0 \), then \( X \) has a \( k \)-point. The same result also holds over \( k_v \) for \( v \in \Omega_k \).

**Proof.** The key point of the proof is that \( H^1(k, G^{ss}) = 0 \), for \( G^{ss} \) semisimple simply connected ([H75], Satz A and [Th08], Thm. A). For \( \sigma \in \Gamma \), let \( (g_{\sigma}, \sigma) \in H \). Let \( u_{\sigma, \tau} = g_{\sigma \tau}(g_{\sigma} \sigma \tau)^{-1} \). As above, we have

\[
\eta(X) = \text{Cl}(\hat{u}, \hat{g}) \in H^1(k, H^m \to G^{tor}).
\]

Since \( H \) is a torus, we have \( \overline{H}^{\text{mult}} = H \) and \( H^m \) is a \( k \)-form of \( H \). Suppose that \( \eta(X) = 0 \). Then we have \( a_{\sigma} \in H^m(k^s) \) and \( s \in G(k^s) \) such that

\[
(\hat{u}_{\sigma, \tau}, \hat{g}_{\sigma}) = (-\partial(a_{\sigma}), j(a_{\sigma}) \partial s),
\]

i.e. \( u_{\sigma, \tau} = a_{\sigma \tau}(a_{\sigma} \sigma a_{\tau})^{-1} \) and \( g_{\sigma} = s^{-1} \cdot j(a_{\sigma}) \cdot \sigma s \) (mod \( G^{ss} \)). After replacing \( g_{\sigma} \) by \( a_{\sigma}^{-1} g_{\sigma} \), we can assume \( u_{\sigma, \tau} = 0 \). We also replace \( x \) by \( s \cdot x \), then we get \( g_{\sigma} \in G^{ss}(k^s) \) and \( u_{\sigma, \tau} = 0 \). Therefore, \( (g_{\sigma}) \) is a cocycle of \( \Gamma \) with values in \( G^{ss} \). Since \( H^1(k, G^{ss}) = 0 \) when \( G^{ss} \) is semisimple simply connected, there is \( t \in G^{ss}(k^s) \) such that \( s_{\sigma} = t^{-1} \sigma t \). Then \( t \cdot x \) is a \( k \)-point of \( X \). \( \square \)

**Remark 2.6.** For \( k \) with positive characteristic, we ask \( G \) to be reductive because we want to ensure that \( G^{red} \), \( G^{ss} \) and \( G^{tor} \) are properly defined. Otherwise, it may happen that the \( k \)-unipotent radical of \( G \) is trivial but \( G \) is not reductive (see [CGP], Example 1.1.3).

**Remark 2.7.** The above proposition is also true over a totally imaginary number field \( k \), because in this case, \( H^1(k, G^{ss}) = 0 \) by Kneser’s Theorem ([K] Chap IV, Thm. 1 and Chap. V, Thm. 1).

### 2.2 Local-Global principle for oriented embedding functors

Let \( k \) be a global field. Unless otherwise specified, \( G \) is a reductive \( k \)-group, \( \Psi \) is a twisted root datum, and \( T \) is the \( k \)-torus determined by \( \Psi \). In the following, we always assume that \( G \) and \( \Psi \) have the same type.

Let \( \text{sc}(T) \) be the torus determined by the simply connected root datum \( \text{sc}(\Psi) \). We will show that the only obstruction to the local-global principal for the oriented functor \( \mathcal{E}(G, \Psi, u) \) lies in the Shafarevich group \( \Pi^2(k, \text{sc}(T)) \). Moreover, the local-global principle holds for the oriented embedding functor \( \mathcal{E}(G, \Psi, u) \) if \( \text{Dyn}(\Psi) \) is of type \( C \) or \( T \) is anisotropic at some place \( v \).

Note that since \( G \) is reductive, we have \( G^{ss} = \text{der}(G) \). A direct application of Theorem 2.2 is the following:
Proposition 2.8. Let $u$ be an orientation of $\Psi$ with respect to $G$. Then the only obstruction for $\mathcal{E}(G, \Psi, u)$ to satisfy the local-global principle lies in the group $\text{III}^2(k, \text{sc}(T))$. In particular, if $\text{gcd}$ has no outer automorphisms and $\text{III}^2(k, \text{sc}(T))$ vanishes, then $\mathcal{E}(G, \Psi)$ satisfies the local-global principle.

Before proving the above proposition, we prove the following lemma:

Lemma 2.9. Suppose furthermore that $G$ is a semisimple simply connected group over $k$, and $\Psi$ is a twisted simply connected root datum. Let $u \in \text{Isomext}(\Psi, G)(k)$. As we have shown in Theorem 1.6, the oriented embedding functor $\mathcal{E}(G, \Psi, u)$ is a left homogeneous space under the adjoint $G$-action.

Then under this $G$-action, the corresponding $\text{H}^m$ (which is defined in Section 2.1) is isomorphic to $T$.

Proof. Given $f \in \mathcal{E}(G, \Psi, u)(k^*)$, the stabilizer of $f$ in $G(k)$ is $f(T(k^*))$. Since $f(T(k^*))$ is a torus, we have $f(T(k^*))^{\text{mult}} = f(T(k^*))$. For $\sigma \in \Gamma$, $\sigma$ acts on $f$ as $\sigma f = \sigma \circ f \circ \sigma^{-1}$. Let $\mathfrak{S} = G(k^*) \rtimes \Gamma$, and $\mathfrak{H} = \text{Stab}_{\mathfrak{S}}(f)$. For $g \in G(k^*)$, we let $\text{int}(g)$ denote the conjugation action of $g$ on $G$. Then for $(g_\sigma, \sigma) \in \mathfrak{H}$, we have $\text{int}(g_\sigma) \circ \sigma f = f$, which means $g_\sigma \cdot \sigma f(t) \cdot g^{-1}_\sigma = f(t)$ for all $t \in T(k^*)$.

Therefore, we have

$$g_\sigma \cdot \sigma(f(t)) \cdot g^{-1}_\sigma = f(\sigma(t)),$$

which means $f$ is a $k$-isomorphism between $T$ and $f(T(k^*))^m$. Therefore, the $\text{H}^m$ defined in Section 2.1 is isomorphic to $T$. \hfill $\Box$

Now, we are ready to prove Proposition 2.8.

Proof of Proposition 2.8. By Corollary 1.7, it suffices to prove this proposition for $\mathcal{E}(\text{sc}(G), \text{sc}(\Psi), u_{\text{sc}})$. Since $\mathcal{E}(\text{sc}(G), \text{sc}(\Psi), u_{\text{sc}})$ is a homogeneous space under $\text{sc}(G)$, by Lemma 2.9, we know that the $\text{H}^m$ corresponding to this $\text{sc}(G)$-action is isomorphic to $\text{sc}(T)$.

Since $\text{sc}(G)^{\text{tor}}$ is trivial, by Theorem 2.2 and Proposition 2.5, the only obstruction for $\mathcal{E}(\text{sc}(G), \text{sc}(\Psi), u_{\text{sc}})$ to satisfy the local-global principle lies in the group $\text{III}^2(k, \text{sc}(T))$. The rest of the proposition then follows. \hfill $\Box$

Let $k_v$ be a nonarchimedean local field. Then by combining previous results, we get the following corollary:

Corollary 2.10. If the group $H^2(k_v, \text{sc}(T))$ is trivial, then the oriented embedding functor $\mathcal{E}(G, \Psi, u)$ has a $k_v$-point.

Proof. By Corollary 1.7, it is enough to prove that $\mathcal{E}(\text{sc}(G), \text{sc}(\Psi), u_{\text{sc}})(k_v)$ is nonempty. By Lemma 2.9 the group $\text{H}^m$ for the $\text{sc}(G)$-homogeneous space $\mathcal{E}(\text{sc}(G), \text{sc}(\Psi), u_{\text{sc}})$ is isomorphic to $\text{sc}(T)$. Since $\text{sc}(G)^{\text{tor}}$ is trivial, we have

$$H^1(k_v, H^m \to \text{sc}(G)^{\text{tor}}) = H^2(k_v, \text{sc}(T)).$$

By Theorem 2.2 and Proposition 2.5 the set $\mathcal{E}(\text{sc}(G), \text{sc}(\Psi), u_{\text{sc}})(k_v)$ is nonempty if $H^2(k_v, \text{sc}(T))$ is trivial. \hfill $\Box$
For a twisted root datum \( \Psi \), the Galois group \( \Gamma \) has a natural action on \( \Psi_{k^s} \). Therefore, we have a group homomorphism from \( \Gamma \) to \( \text{Aut}(\Psi)(k^s) \). Recall that \( \Psi \) is said to be \textit{generic}, if the image of \( \Gamma \) in \( \text{Aut}(\Psi)(k^s) \) contains the Weyl group \( W(\Psi)(k^s) \).

**Theorem 2.11.** Let \( G \) be a reductive group over \( k \), \( \Psi \) be a twisted root datum over \( k \), and \( T \) be the torus determined by \( \Psi \). Let \( u \in \text{Isomext}(\Psi,G)(k) \).

Suppose that \( \Psi \) satisfies one of the following conditions:

1. all connected components of \( \text{Dyn}(\Psi)(k^s) \) are of type \( C \).
2. \( T \) is anisotropic at one place \( v \in \Omega_k \).

Then the local-global principle holds for the existence of a \( k \)-point of the oriented embedding functor \( \mathfrak{E}(G,\Psi,u) \). In particular, when \( \Psi \) is generic, the local-global principle holds.

**Proof.** If \( \Psi \) satisfies one of the above conditions, then \( \text{sc}(\Psi) \) also satisfies one of them. Therefore, we can assume that \( \Psi \) and \( G \) are semisimple simply connected.

By Proposition 2.8, the local-global principle holds for the existence of \( k \)-points of the oriented embedding functor \( \mathfrak{E}(G,\Psi,u) \) if \( \Pi^2(k,T) \) vanishes. Therefore, it is enough to prove \( \Pi^2(k,T) = 0 \) for \( \Psi \) satisfying either condition.

Suppose that \( \Psi \) satisfies condition (1). Let \( \Psi_0 \) be the split simple simply connected root datum of type \( C_n \) (ref. [Bou] Plan. III). Let \( E_0 \) be the étale algebra \( k^n \times k^n \) and \( \sigma_0 \) be the involution which exchanges the two copies of \( k^n \). When \( \Psi \) is simple simply connected of type \( C_n \), \( \Psi \) corresponds to some twisted form \((E,\sigma)\) of \((E_0,\sigma_0)\) by Lemma 1.11, and the torus \( T \) determined by \( \Psi \) is \( U(E_0,\sigma) = R_{E/E_0}(\mathbb{G}_m) \).

Consider the exact sequence:

\[
1 \longrightarrow R_{E/E_0}(\mathbb{G}_m) \longrightarrow R_{E/E_0}(\mathbb{G}_m) \longrightarrow \mathbb{G}_m \longrightarrow 1.
\]

By Hilbert Theorem 90, we have

\[
0 \longrightarrow H^2(E^\sigma, R_{E/E_0}(\mathbb{G}_m)) \longrightarrow H^2(E^\sigma, R_{E/E_0}(\mathbb{G}_m)) \).
\]

By Shapiro’s Lemma, \( \Pi^2(E^\sigma, R_{E/E_0}(\mathbb{G}_m)) = \Pi^2(E,\mathbb{G}_m) \). By Brauer-Hasse-Noether Theorem, \( \Pi^2(E,\mathbb{G}_m) = 0 \). Therefore, we have

\[
\Pi^2(k,T) = \Pi^2(E^\sigma, R_{E/E_0}(\mathbb{G}_m)) = 0.
\]

For \( \Psi \) not simple, since we can decompose \( \Psi_{k^s} \) into a product of isotypic root data (for [SGA3] Exp. XXI, 6.4.1 and 7.1.6), we can also decompose \( \Psi \) into a product of isotypic twisted root data \( \Psi_i \) by descent. By the same reasoning as in [SGA3] Exp. XXIV, 5.8 or in [CGP] Theorem A.5.14, we know that there exists some étale algebra \( F_i \) over \( k \) such that \( \Psi_{i,F_i} \) is a product of copies of the twisted simple root datum \( \Psi_{i,0} \) over \( F_i \), and the automorphism group
Aut(F_i/k) acts on Ψ_i by permuting Ψ_i,0’s. So we have Ψ_i = R_{F_i/k}(Ψ_i,0) and the torus T will take the form \( \prod R_{F_i/k}(T_i,0) \), where T_i,0 is the torus determined by the twisted root datum Ψ_i,0. Then we know that Ψ_i,0 is a twisted root datum defined by an étale algebra with involution over \( F_i \) (Section 1.3.1). As in the above discussion, we will have \( T_i,0 = U(E_i, \sigma_i) \) where \( E_i \) is an étale algebra over \( F_i \). By Shapiro’s Lemma,

\[
\text{III}^2(k, R_{F_i/k}(T_i,0)) = \text{III}^2(F_i, T_i,0) = \text{III}^2(E_i^{\sigma_i}, R_{E_i/E_i^{\sigma_i}}(G_{m_i})) = 0.
\]

By Proposition 2.8 the theorem holds when \( \Psi \) satisfies the first condition.

Now, suppose that \( T \) is anisotropic at some place \( v \in \Omega_k \). Then by Kneser’s Theorem (ref. [San81] Lemma 1.9), we have \( \text{III}^2(k, T) = 0 \).

To complete the proof, we will show that if \( \Psi \) is generic, then \( T \) is anisotropic at some place \( v \). Suppose that \( \Psi \) is generic. Let \( L \) be a finite Galois extension of \( k \) which splits \( T \). Then there exists an element \( \sigma \in \text{Gal}(L/k) \) such that \( \sigma \) acts on \( \Psi_L \) as the Coxeter element \( \omega \in \mathcal{W}(\Psi)(L) \). Let \( M \) be the character group of \( T_L \). Then the set \( M^\sigma = 0 \), by Theorem 1 in [Bou] Chap. V, §6, and hence \( M^\sigma = 0 \). By Čeboarev Density Theorem, there exists a place \( v \) such that \( \sigma \) generates the Frobenius map at \( v \), so \( T \) is anisotropic at \( v \).

**Remark 2.12.** Note that for \( \Psi \) generic without type \( C_n \) (for \( n \geq 1 \)), there is an even stronger result by Klyachko (cf. [Kl89] I.5) saying that \( H^1(k, M) = 0 \), where \( M \) is the character group of \( \text{sc}(T) \).

### 2.3 Oriented embedding functors over local fields

Let \( G \) be a reductive group over a local field \( L \), and \( \Psi \) be a twisted root datum over \( L \). Suppose that \( G \) and \( \Psi \) have the same type and \( \text{Isomext}(\Psi, G)(L) \) is not empty. Let \( u \in \text{Isomext}(\Psi, G)(L) \). In the following, we are going to show that the existence of an \( L \)-point of the oriented embedding functor is actually determined by the Tits indices of \( \Psi \) and \( G \). Note that the existence of an orientation \( u \) is important here, because it gives a map between the Dynkin schemes \( \text{Dyn}(G) \) and \( \text{Dyn}(\Psi) \), which allows us to compare the Tits indices of \( G \) and \( \Psi \). An orientation also allows us to replace the reductive group \( G \) by the adjoint group \( \text{ad}(G) \) or simply connected group \( \text{sc}(G) \) as we have shown in Corollary 1.7.

#### 2.3.1 Tits indices

We recall briefly the definition of the Tits index. For a detailed introduction on Tits indices, we refer to Tits’s paper [T66]. For the Tits indices of reductive groups over connected semilocal rings, one can refer to [SGA3] Exp. XXVI, §§5, §6, and §7. One can also look at Petrov and Stavrova’s paper [PS] §5. For the Tits indices of a twisted root datum, we refer to Gille’s paper [Gi] §7.

Let \( S \) be the spectrum of a semilocal ring. Let \( G \) be an \( S \)-reductive group. For each \( G \), there exists a minimal parabolic subgroup \( P_{\text{min}} \) of \( G \). Let \( t_{\text{min}} \) be
the type of $P_{\text{min}}$. Note that given $G$, the type $t_{\text{min}}$ is well defined (ref. [SGA3] Exp. XXVI, 5.7). Moreover, if $S$ is connected, then we call the type $t_{\text{min}}$ the Tits index of $G$, and denote it by $\Delta^\circ(G)$.

For a reduced root datum $\psi = (M, M^\vee, R, R^\vee)$, a parabolic subset $P$ is a closed subset of $R$ which contains a system of simple roots. For a reduced twisted root datum $\Psi = (M, M^\vee, R, R^\vee)$ over $S$, a parabolic subsheaf (fpqc) $\mathcal{P}$ is a subsheaf of $R$ which is locally isomorphic to a parabolic subset. Let $\text{Par}(\Psi)$ be the functor such that for each $S$-scheme $S'$, $\text{Par}(\Psi)(S')$ is the set of all the parabolic subsheaves (fpqc) of $\Psi$ over $S'$. Similarly, we can define a type map $t_{\psi}$ from $\text{Par}(\Psi)$ to $\text{Dyn}(\Psi)$.

Let $t_{\text{min}}$ be the type of a minimal parabolic subsheaf of $\Psi$ ([Gi], Prop. 7.1). If $S$ is connected, then we call $t_{\text{min}}$ the Tits index of $\Psi$, and denote it by $\Delta^\circ(\Psi)$. Note that $\Delta^\circ(\Psi)$ only depend on the roots, so they are invariant under the operations $sc, ad, ...$.

### 2.3.2 A criterion for the existence of points of the oriented embedding functor over a local field $L$

Let $L$ be a local field of arbitrary characteristic. We have the following criterion for the existence of an $L$-point of the oriented embedding functor:

**Theorem 2.13.** Let $G$ be a reductive group over a local field $L$, and $\Psi$ be a twisted root datum over $L$. Suppose that $G$ and $\Psi$ have the same type and $\text{Isom}_{\text{ext}}(\Psi, G)(L)$ is not empty. Let $u \in \text{Isom}_{\text{ext}}(\Psi, G)(L)$. Then $\mathcal{E}(G, \Psi, u)(L) \neq \emptyset$ if and only if $u(\Delta^\circ(\Psi)) \supseteq \Delta^\circ(G)$.

**Proof.** First, we suppose that $\mathcal{E}(G, \Psi, u)(L) \neq \emptyset$ and let $f \in \mathcal{E}(G, \Psi, u)(L)$. Since $\Psi \simeq \Phi(G, f(T))$, from Prop. 7.2 in [Gi], $u(\Delta^\circ(\Psi)) \supseteq \Delta^\circ(G)$.

Now, suppose $u(\Delta^\circ(\Psi)) \supseteq \Delta^\circ(G)$, and we want to show that $\mathcal{E}(G, \Psi, u)(L)$ is nonempty. Again, by Corollary 1.7, we only need to consider the problem for $sc(G)$ and $sc(\Psi)$. Therefore, we can assume $G$ is simply connected, and $\Psi$ is reduced simply connected.

Let $T$ be the torus determined by $\Psi$ and $I = \Delta^\circ(\Psi)$. We start with the case where $T$ is anisotropic, i.e. $I = \text{Dyn}(\Psi)(L)$.

**Case 1.** $L$ is non-archimedean. Since $T$ is anisotropic, by Tate-Nakayama Theorem, we have $H^2(L, T) = 0$ (cf. [K] 3.2, Thm. 5). Since $G$ and $\Psi$ are simply connected, by Corollary 2.10, the oriented embedding functor $\mathcal{E}(G, \Psi, u)$ has an $L$-point.

**Case 2.** $L = \mathbb{R}$. In this case, we consider the oriented embedding functor $\mathcal{E}(\text{ad}(G), \text{ad}(\Psi), u_{\text{ad}})$. Let $\sigma$ be the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Let $\text{ad}(T)$ be the torus associated to the root datum $\text{ad}(\Psi)$. Since $T$ is anisotropic, the torus $\text{ad}(T)$ is also anisotropic and $\text{ad}(T) \simeq (\mathbb{R}^{(1)}_{\mathbb{R}/\mathbb{C}}(\mathbb{G}_m))^r$.

Suppose that $\text{ad}(G)$ is anisotropic and pick a maximal torus $S$ of $\text{ad}(G)$. Since $\text{ad}(\Psi)$ and $\text{ad}(G)$ have the same type, there is a $\mathbb{C}$-point $f$ of $\mathcal{E}(\text{ad}(G), \text{ad}(\Psi), u_{\text{ad}})$ which maps $\text{ad}(T)$ to $S$. Since $\sigma$ acts on the character group of the anisotropic
torus by -1, σ commutes with f. Therefore, f is an \( \mathbb{R} \)-point of the oriented embedding functor \( \mathcal{E}(\text{ad}(G), \text{ad}(\Psi), u_{\text{ad}}) \).

Suppose that \( \text{ad}(G) \) is not anisotropic. Then we can find an anisotropic form \( \tilde{G} \) of \( \text{ad}(G) \) by [Ge91], Corollary 7. Since \( \tilde{G} \) has the same type with \( \text{ad}(\Psi) \), by the above argument, we have an \( \mathbb{R} \)-point \( f \) of \( \mathcal{E}(\tilde{G}, \text{ad}(\Psi)) \). Then \( f \) defines an \( \mathbb{R} \)-point \( \tilde{u} \) of \( \text{Isomext}(\text{ad}(\Psi), \tilde{G}) \). The orientation \( u_{\text{ad}} \) together with \( \tilde{u} \) gives an orientation \( u_{\text{ad}} \circ \tilde{u}^{-1} \in \text{Isomext}(\tilde{G}, \text{ad}(G))(\mathbb{R}) \). Hence \( \text{ad}(G) \) is an inner form of \( \tilde{G} \). However, the natural inclusion from \( H^1(\mathbb{R}, f(\text{ad}(T))) \) to \( H^1(\mathbb{R}, \tilde{G}) \) is surjective ([Ge91] Thm. 3), so \( \text{ad}(G) \) has an anisotropic torus \( S \). Let \( h \) belong to \( \mathcal{E}(\text{ad}(G), \text{ad}(\Psi), u_{\text{ad}})(\mathbb{C}) \) and suppose that \( h \) maps \( \text{ad}(T) \) to \( S \). Again, since \( \sigma \) acts on the character group of the anisotropic torus by -1, \( \sigma \) commutes with \( h \) and \( h \) is an \( \mathbb{R} \)-point of the oriented embedding functor \( \mathcal{E}(\text{ad}(G), \text{ad}(\Psi), u_{\text{ad}}) \). By Corollary 7.7 the oriented embedding functor \( \mathcal{E}(G, \Psi, u) \) has an \( \mathbb{R} \)-point.

Therefore, the proposition is true when \( T \) is anisotropic.

Now, suppose that \( T \) is arbitrary. Since \( u(I) \supseteq \Delta^0(G) \), we can find a parabolic subgroup \( P_1 \) of \( G \) such that the type of \( P \) is \( u(I) \) by Proposition 7.5. Let \( L_1 \) be a Levi subgroup of \( P_1 \) and \( T' \) be a maximal torus of \( L_1 \). Let \( \Psi' = \Phi(G, T') \), and \( \Psi'_1 = \Phi(L_1, T') \). Let \( \mathcal{P}_1 \) be the subsheaf of roots of \( \Psi' \) which is determined by \( P_1 \). Note that \( u \) corresponds to an element in \( \text{Isomext}(\Psi, \Psi')(L) \), which we still denote as \( u \).

Let \( \Psi = (\mathcal{M}, \mathcal{M}', \mathcal{R}, \mathcal{R}') \). Let \( \mathcal{P} \) be a minimal parabolic subsheaf of \( \mathcal{R} \). Then by definition, type \( \mathcal{P} = I \). Let \( \mathcal{R}_1 \) be the subsheaf of \( \mathcal{P} \) defined by the property: for any \( L \)-scheme \( X \),

\[
x \in \mathcal{R}_1(X), \text{ if and only if both } x \text{ and } -x \text{ are in } \mathcal{P}_1(X).
\]

Let \( \Psi_1 \) be the root system given by \((\mathcal{M}, \mathcal{M}', \mathcal{R}_1, \mathcal{R}'_1)\). Define

\[
Q = \text{Isom}_{\Psi_1}(\Psi, \mathcal{P}; \Psi', \mathcal{P}_1) = \text{Isom}(\Psi, \mathcal{P}; \Psi', \mathcal{P}_1) \cap \text{Isom}_{\Psi_1}(\Psi, \Psi').
\]

Note that \( Q \) is a right \( W(\Psi_1) \)-torsor over \( \text{Spec}(L) \) (for the étale topology), so \( Q \) is representable. By the definition of \( Q \), each \( h \in Q(X) \) will send the sheaf \( \mathcal{R}_1 \) to the sheaf of roots of \( L_1 \), because \( L_1 \) is the unique Levi subgroup of \( P_1 \) which contains \( T' \). Therefore, we have a natural map

\[
i_1 : Q \to \text{Isom}(\Psi_1, \Psi'_1).
\]

Let \( L^* \) be a separable closure of \( L \). Let \( x \in Q(L^*) \). By the definition of \( Q \), the image of \( x \) in \( \text{Isomext}(\Psi, \Psi')(L^*) \) is \( u \). Moreover, since \( Q \) is a right \( W(\Psi_1) \)-torsor and \( W(\Psi_1) \) acts trivially on \( \text{Isomext}(\Psi_1, \Psi'_1) \), \( i_1(x) \) defines an \( L \)-point of \( \text{Isomext}(\Psi_1, \Psi'_1) \) and hence an \( L \)-point of \( \text{Isomext}(\Psi_1, L_1) \). We denote it by \( u_1 \). Note that the definition of \( u_1 \) is independent of the choice of \( T' \).

Now we consider the functor \( \mathcal{E}(L_1, \Psi_1, u_1) \). We claim that if \( \mathcal{E}(L_1, \Psi_1, u_1) \) has an \( L \)-point, then \( \mathcal{E}(G, \Psi, u) \) has an \( L \)-point.

Suppose that \( \mathcal{E}(L_1, \Psi_1, u_1) \) has an \( L \)-point. Let \( f \in \mathcal{E}(L_1, \Psi_1, u_1)(L) \). Then we replace the torus \( T' \) above by \( f(T) \). By the definition of \( Q \) and \( u_1 \), we have a natural morphism

\[
j : Q \to \text{Isom}_{u_1}(\Psi_1, \Psi'_1).
\]
Since both of them are $W(\Psi)$-torsors, $j$ is an isomorphism. As $E(L, \Psi, u)$ has an $L$-point, $\text{Isom}^\text{int}(\Psi, \Psi')(L)$ is not empty, so $Q$ has an $L$-point as well, which means $\text{Isom}^\text{int}(\Psi, \Psi')(L) \neq \emptyset$. Hence, $E(G, \Psi, u)$ has an $L$-point.

Now, by Corollary [11.7] it is enough to prove that $E(\text{der}(L), \text{der}(\Psi), u)$ has an $L$-point. Note that $\text{der}(\Psi)$ is reduced simply connected as $\Psi$ is (ref. [SGA3], Exp. XXI, 6.5.11). Since the torus $\text{der}(T)$ determined by $\text{der}(\Psi)$ is anisotropic, it follows that $E(\text{der}(L), \text{der}(\Psi), u)$ has an $L$-point as we have seen above. This finishes the proof.

Example 2.14. The above theorem does not hold over arbitrary fields. Here is an example. Let $K = \mathbb{Q}(\sqrt{-1})$ and $\sigma$ be the conjugation on $K$, and $k = \mathbb{Q}$. Let $T$ be the torus $T = R_{E/\mathbb{Q}}(\mathbb{G}_m)$. Since $T$ is of dimension 1, there is only one semisimple simply connected root datum with respect to $T$. Let $\Psi$ be this root datum. Let $v_1, v_2$ be two places of $\mathbb{Q}$ of the form $4n + 1$. Then $\Psi$ splits at $v_1$ and $v_2$. Let $D$ be a quaternion algebra over $\mathbb{Q}$ corresponding to $1/2$ in $\mathbb{Q}/\mathbb{Z} \simeq H^2(\mathbb{Q}_{v_i}, \mathbb{G}_m)$ for $i = 1, 2$, and corresponding to 0 in the other places. Note that such a quaternion exists by Brauer-Hasse-Noether’s Theorem. Let $G$ be $\text{SL}_1(D)$. Since $G$ has no outer form, there is an orientation $u$ between $\Psi$ and $G$. Since both $\Psi$ and $G$ are anisotropic over $\mathbb{Q}$, we have $u(\Delta(\Psi)) \supseteq \Delta(\Psi)$.

However, at place $v_1$ and $v_2$, the root datum $\Psi$ splits but $G$ is anisotropic, so $E(G, \Psi, u)(\mathbb{Q}_{v_i}) = \emptyset$. Therefore, $E(G, \Psi, u)(\mathbb{Q}) = \emptyset$ and Theorem 2.13 does not hold over $\mathbb{Q}$.

2.4 Applications-the problems to embed an étale algebra in a central simple algebra with respect to involutions

Let $K$ be a field, $(E, \sigma)$ be an étale $K$-algebra with the involution $\sigma$, and $(A, \tau)$ be a central simple algebra over $K$ with the involution $\tau$. Assume $\sigma |_{K} = \tau |_{K}$. Let $k = K^\sigma$. From now on, we assume that $k$ is a global field of characteristic different from 2. Let $\Omega_k$ be the set of all places of $k$. Fix a separable closure $k^s$ of $k$. Let $G = \text{Gal}(k^s/k)$ and $G_v = \text{Gal}(k^s_v/k_v)$ for $v \in \Omega_k$. Let $T = U(E, \sigma)^0$, and $G = U(A, \tau)^0$. Note that by the definition of $U(E, \sigma)^0$, $T = R_{E/\mathbb{Q}}(R_{E_0/\mathbb{Q}}(\mathbb{G}_m))$. We keep all the notation defined in section 1.3.

In the paper [PR10], Prasad and Rapinchuk consider the local-global principle for the $K$-embedding from $(E, \sigma)$ into $(A, \tau)$. As we have mentioned in Theorem 1.13, the local-global principle for the existence of $k$-embeddings from $(E, \sigma)$ into $(A, \tau)$ is equivalent to the local-global principle for the existence of $k$-points of $E(G, \Psi)$. Here, we will reduce the original problem to the existence of $k$-points of oriented embedding functors, and prove that the local-global principle holds in certain cases by computing the Shafarevich group $\text{III}(k, \text{sc}(T))$.  

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2.4.1 Symplectic involutions

For $\tau$ symplectic, $\Psi$ and $G$ are semisimple simply connected of type $C_n$, which is the first case in Theorem 2.11, so we just restate the result as the following:

**Proposition 2.15.** If $\tau$ is symplectic, then the local-global principle holds for the existence of $K$-embeddings of $(E, \sigma)$ into $(A, \tau)$.

2.4.2 Orthogonal involutions

Throughout this subsection, for an étale algebra $F$ over $K$, we let $M_F$ be the character group of the torus $R_{F/K}(\mathbb{G}_m)$ and let $J_F$ be the character group of the torus $R_{1}^{\times} F/K(\mathbb{G}_m)$. Note that $K = k$ when $\tau$ is an orthogonal involution.

The case where the degree of $A$ is odd Let us consider the case where $\tau$ is orthogonal, and $A = M_{2n+1}(K)$. In this case, the corresponding group $G$ is adjoint of type $B_n$, so there is no outer automorphisms. By Theorem 1.15 and Proposition 2.8, to prove the local-global principle for the $K$-embeddings here, it suffices to prove that $\text{III}^2(K, \text{sc}(T))$ vanishes. Note that in this case, $E = K \times E'$ and $\sigma$ acts trivially on the component $K$, so $T = R_{E'/K}(R_{E'/(E')^\sigma}(\mathbb{G}_m))$.

Let $E' = \prod_{i=1}^{r} F_i$, where $F_i$ is a field over $K$ for all $i$. Let $d = (d_1, \ldots, d_r)$ be an element in $E'^\sigma$ such that $E' = E'[x]/(x^2 - d) = \prod_{i=1}^{r} F_i[x]/(x^2 - d_i).$ Let $E_i = F_i[x]/(x^2 - d_i)$ for all $i$ and $E_{i,v}$ (resp. $F_{i,v}$) be $E_i \otimes_K K_v$ (resp. $F_i \otimes_K K_v$) for all $v \in \Omega_K$.

**Theorem 2.16.** Suppose $\tau$ is orthogonal, and $A = M_{2n+1}(K)$. If there is a place $v \in \Omega_K$ such that the following condition holds:

$$\text{for all } i, d_i \in F_i^{x^2} \text{ if and only if } d_i \in (F_{i,v})^{x^2}.$$  

Then the local-global principle for the existence of $K$-embeddings from $(E, \sigma)$ into $(A, \tau)$ holds.

We start with some calculations:

**Lemma 2.17.** $\text{sc}(T) = R_{E'/K}(\mathbb{G}_m)/R_{E'^{\sigma}/K}(\mathbb{G}_m)$.

**Proof.** Consider the exact sequence over $E'^\sigma$:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow R_{E'/K}(\mathbb{G}_m) \longrightarrow R_{E'^{\sigma}/K}(\mathbb{G}_m) \longrightarrow 1,$$

where the map from $R_{E'/K}(\mathbb{G}_m)$ to $R_{E'^{\sigma}/K}(\mathbb{G}_m)$ sends $x$ in $R_{E'/K}(\mathbb{G}_m)(R)$ to $x/\sigma(x)$, for any $K$-algebra $R$. Let us take the Weil restriction of the above sequence over $K$. Then we get the exact sequence:

\[34\]
Let $M$ (resp. $P$) be the character group of $T$ (resp. $\text{sc}(T)$).

First, suppose that $(E', \sigma)$ is split. Then $\text{Aut}(E', \sigma) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ and there is a basis $\{e_i\}_i$ of $M$ such that the $S_n$-part of $\text{Aut}(E', \sigma)$ acts on $\{e_i\}_i$ by permuting the indices and $(\mathbb{Z}/2\mathbb{Z})^n$ acts on $\{e_i\}_i$ by change the sign of $e_i$. In this case, $P = M + \frac{1}{2}(e_1 + \ldots + e_n)$. We choose a basis $\{\varepsilon_i, \epsilon_i\}_{i=1}^n$ (resp. $\{h_i\}_{i=1}^n$) of $M_{E'}$ (resp. $M_{E'}^\sigma$) on which $\text{Aut}(E', \sigma)$ acts as the following: $S_n$ permutes the indices $i$ and $(\mathbb{Z}/2\mathbb{Z})^n$ exchanges $\varepsilon_i, \epsilon_i$ (resp. $(\mathbb{Z}/2\mathbb{Z})^n$ acts trivially on $h_i$).

The we have the following exact sequence corresponding to (1):

$$0 \to M \overset{i}{\to} M_{E'} \overset{j}{\to} M_{E'}^\sigma \to 0,$$

where $i$ maps $e_i$ to $\varepsilon_i - \epsilon_i$ and $j$ maps $\varepsilon_i, \epsilon_i$ to $h_i$. Consider the map $\tau$ from $M$ to $J_{E'}$ induced by $i$. Then $\tau(e_1 + \ldots + e_n) = 2(\overline{\varepsilon_1} + \ldots + \overline{\varepsilon_n})$, where $\overline{\varepsilon_i}$ is the image of $\varepsilon_1$ in $J_{E'}$. Hence $\tau$ induces a map from $P$ to $J_{E'}$ and we have the following exact sequence:

$$0 \to P \to J_{E'} \to J_{E'}^\sigma \to 0.$$

Since all the maps constructed are equivariant under $\text{Aut}(E', \sigma)$, we conclude $\text{sc}(T) = R_{E'/(\mathbb{G}_m)}(1)/R_{E'^\sigma/(\mathbb{G}_m)}(1)$.

Now we use the above lemma to compute $\Pi^2(K, \text{sc}(T))$.

**Proof of Theorem 2.16.** Keep all the notations in Lemma 2.17. By the Poitou-Tate duality (ref. [NSW] Chap. VIII, Thm. 8.6.9), we have

$$\Pi^2(K, \text{sc}(T)) \simeq \Pi^1(K, P)^*.$$

Hence, it is enough to show $\Pi^1(K, P) = 0$.

From the exact sequence (2) in the proof of Lemma 2.17 we derive the commutative exact diagram:

$$\begin{array}{cccc}
0 & \to & J_{E'}^G & \to & J_{E'}^G \\
| & & | & & |
0 & \to & M_{E'}^G & \to & M_{E'}^G \\
| & & | & & |
0 & \to & P_{E'}^G & \to & J_{E'}^G \\
| & & | & & |
0 & \to & H^1(K, P) & \to & H^1(K, J_{E'}) \\
| & & | & & |
H^1(K, \mathbb{Z}) = 0 & \to & H^1(K, J_{E'}) & \to & \cdots
\end{array}$$

Hence, it is enough to show $\Pi^1(K, P) = 0$.
Again, $\mathbb{III}^1(K, J_E) = \mathbb{III}^2(K, R_{E/K}(G_m))^*$. By Hilbert Theorem 90, we have that $H^2(K, R_{E/K}(G_m))$ injects into $H^2(K, R_{E/K}(G_m))$. However, $\mathbb{III}^2(K, R_{E/K}(G_m))$ vanishes, so does $\mathbb{III}^1(K, J_E)$. Let $x \in \mathbb{III}^1(K, P)$. Since $\mathbb{III}^1(K, J_E) = 0$, we have $y \in J_{E/v}^G$ mapped to $x$.

Let $I = \{1, 2, \ldots, r\}$. Let $I_1$ be the subset of $I$ such that $i \in I_1$ if and only if $d_i \in F_i^{<2}$. Let $I_2 = I \setminus I_1$. Note that $M_{E/v}^G = \bigoplus_{i=1}^r M_{E_i}^G$ and $M_{E/v}^G = \bigoplus_{i=1}^r M_{E_i}^G$. For $i \in I_1$, $E_i \cong F_i \times F_i$, so $M_{E_i}^G \cong M_{F_i}^G \bigoplus M_{F_i}^G$ and $M_{E_i}^G$ is mapped surjectively onto $M_{F_i}^G$.

Let $\gamma_i$ be a basis of $M_{F_i}^G$. For $i \in I_2$, we have the following observation:

**Lemma 2.18.** For $i \in I_2$ and $y \in M_{E_i}^G$, $y$ is in the image of $M_{E_i}^G$ if and only if the coefficient of $\gamma_i$ in $y$ is even.

**Proof of Lemma 2.18.** Since $E_i$ is a field over $F_i$ with degree 2 for $i \in I_2$, the module $M_{E_i}^G$ is of rank 1 and is generated by $\sum_{\epsilon_j, \epsilon_j \in M_{E_i}} (\epsilon_j + \epsilon_j)$. Since the element

$$\sum_{\epsilon_j, \epsilon_j \in M_{E_i}} (\epsilon_j + \epsilon_j)$$

is mapped to $2\gamma_i$ in $M_{F_i}^G$, the lemma then follows. \(\square\)

We return to the proof of Theorem 2.16. Since $M_{E/v}^G$ is mapped surjectively onto $J_{E/v}^G$, $J_{E/v}^G$ is generated by $\gamma_i$’s. Let $\gamma_i$ be the image of $\gamma_i$ in $J_{E/v}^G$. Let $y = \sum_{i=1}^r a_i\gamma_i$. If for all $i \in I_2$, the $a_i$’s have the same parity, then we can find $z = \sum_{i=1}^r b_i\gamma_i$, which is a lifting of $y$ in $M_{E/v}^G$, such that $b_i$ is even for any $i \in I_2$. Then by Lemma 2.18, $z$ is in the image of $M_{E/v}^G$ and hence $y$ is in the image of $J_{E/v}^G$. So it is enough to prove that for all $i \in I_2$, the $a_i$’s have the same parity.

Now, let $v$ be a place of $K$ such that for all $i$, $d_i \in F_i^{<2}$ if and only if $d_i \in (F_i/v)^{<2}$. Since $x$ is in $\mathbb{III}^1(K, P)$, $y$ is in the image of $J_{E/v}^G$. For each $i \in I_2$, since $d_i$ is not a square in $F_i/v$, there is some $h_{j(i)} \in M_{F_i}$ such that there exists $\tau_{j(i)} \in G_v$ which exchanges $\epsilon_{j(i)}$ and $\epsilon_{j(i)}$. Therefore, for all $i \in I_2$, the coefficients of $h_{j(i)}$’s in the expression of $y$ have the same parity. Since the coefficient of $h_{j(i)}$ in $y$ is $a_i$, we know that all $a_i$’s have the same parity for $i \in I_2$. By Lemma 2.18, $y$ is in the image of $J_{E/v}^G$, which means $\mathbb{III}^1(K, P) = 0$. \(\square\)

**Remark 2.19.** A special case of the above theorem is when there is a place $v$ such that $sc(T)$ is anisotropic over $K_v$. We now show that $sc(T)$ is anisotropic over $K_v$ implies all $d_i \not\in (F_i/v)^{<2}$. To see this, we note that in our case here, $sc(T)$ is anisotropic if and only if $T$ is anisotropic. If there is $d_i \in (F_i/v)^{<2}$, then $M_{E_i,v/K_v} = M_{F_i,v/K_v} \oplus M_{F_i,v/K_v}$. Let $\alpha$ be a nontrivial element in $M_{F_i,v/K_v}$. Then $(\alpha, -\alpha) \in M_{E_i,v/K_v}$ and it is in the image of $M$, which means $M_{E_i,v/K_v}$ is nontrivial and contradicts the condition that $T$ is anisotropic over $K_v$. Therefore, $d_i \not\in (F_i/v)^{<2}$ for all $i$. 36
The case where the degree of $A$ is even

Throughout this paragraph, we let $A$ be $M_{2n}(K)$, or $M_n(D)$ with orthogonal involution $\tau$, where $D$ is a quaternion division algebra over $K$. In this case, the corresponding group $G$ is semisimple of type $D_n$, and $\text{Isom}_{\text{ext}}(\Psi, G)$ satisfies the local-global principle.

For $A$ satisfying one of the conditions in Theorem 2.20, we first show that $\mathcal{E}(G, \Psi)(K_v)$ is nonempty implies that $\mathcal{E}(G, \Psi, u)(K_v)$ is nonempty for any orientation $u$. (see Lemma 2.21) Then we prove that the local global principle holds for the oriented embedding functor $\mathcal{E}(G, \Psi, u)$. By Theorem 2.15 and Proposition 1.17, we get the local-global principle for the existence of $K$-embeddings from $(E, \sigma)$ into $(A, \tau)$.

We first fix some notations. Let $E^\sigma = \prod_{i=1}^r F_i$, where the $F_i$’s are fields over $K$. Let $d = (d_1, \ldots, d_r)$ be in $E^\sigma$ and $E = E^\sigma[x]/(x^2 - d) = \prod_{i=1}^r F_i[x]/(x^2 - d_i)$.

Let $E_v = F_i[x]/(x^2 - d_i)$, and $E_{i,v}$ (resp. $F_i[x]/(x^2 - d_i)$) be $E_i \otimes_K K_v$ (resp. $F_i \otimes_K K_v$) for all $v \in \Omega_K$.

**Theorem 2.20.** Suppose that $A$ is equal to one of the following:

- (1) $M_{2n}(K)$, $n > 1$.
- (2) $M_{2n+1}(D)$, where $D$ is a quaternion division algebra over $K$.
- (3) $M_{2m}(D)$, where $D$ is a quaternion division algebra over $K$, and at each place $v \in \Omega_K$, if $A$ is not split and the discriminant splits, then $E_v$ is not split over $E_v^\sigma$, i.e. $E_v \neq E_v^\sigma \times E_v^\sigma$.

If there is a place $v \in \Omega_K$ such that for all $i$, $d_i \in F_i^{x^2}$ if and only if $d_i \in (F_i^\sigma)^{x^2}$, then the local-global principle for the $K$-embedding of $(E, \sigma)$ into $(A, \tau)$ holds.

First we prove the following lemma:

**Lemma 2.21.** For $A$ satisfying one of the three conditions in Theorem 2.20, the existence of a $K_v$-point of $\mathcal{E}(G, \Psi)$ implies the existence of a $K_v$-point of $\mathcal{E}(G, \Psi, u)$ for any $u \in \text{Isom}_{\text{ext}}(\Psi, G)(K_v)$.

**Proof.** Suppose that there is a $K_v$-point $f$ of $\mathcal{E}(G, \Psi)$. By Theorem 2.13, there is an orientation $u'$ induced by $f$ such that $u'(\Delta^0(G_{K_v})) \supseteq \Delta^0(G_{K_v})$.

According to the list of all possible Tits indices (ref. [166]), if $A$ satisfies (1) or (2) in Theorem 2.20, then the Tits index of $G_{K_v}$ will be symmetric under $\text{Autext}(G_{K_v})$. Therefore, for any $u \in \text{Isom}_{\text{ext}}(\Psi, G)(K_v)$, we have that $u(\Delta^0(\Psi_{K_v}))$ contains $\Delta^0(G_{K_v})$, and again, by Theorem 2.13, we have $\mathcal{E}(G, \Psi, u)(K_v) \neq \emptyset$.

Now assume that $A$ satisfies (3) in Theorem 2.20. If over $K_v$, $A$ is not split and the discriminant splits, then $G$ is a non-split inner form over $K_v$. In this case, the possible Tits indices of $G$ are symmetric except the following case:

![Diagram](image-url)
Suppose that $\Delta^o(G_{K_v})$ takes the above nonsymmetric form. We will show that condition (3) in Theorem 2.20 forces $\Delta^o(\Psi_{K_v})$ to be symmetric under $\text{Aut}(\Psi_{K_v})$ in this case.

Consider the Dynkin diagram of $\Psi$:

```
1  2  3  4  ...  2m  2m-1
```

Suppose that $I = \Delta^o(\Psi_{K_v})$ is not symmetric under $\text{Aut}(\Psi)$. Without loss of generality, we suppose that the vertex $2m$ is not in $I$. Let $I'$ be the Dynkin subdiagram with vertices $1, ..., 2m-1$ which is of type $A_{2m-1}$. So $I \subseteq I'$.

Since there is $f \in \mathfrak{E}(G, \Psi)(K_v)$, $G_{K_v}$ has a parabolic subgroup $P_1$ with the type $I$ and $P_1$ contains $f(T_{K_v})$ by Proposition [17]. Let $G_0$, $\Psi_0$ be the split form of $G$ and $\Psi$ respectively (Section. 1.3.1), and let $T_0$ be the split torus determined by $\Psi_0$. Let $P_{0,1}$ be a parabolic subgroup of $G_{0, \Psi}$ with type $I$ and contains $T_{0, \Psi}$.

Let $P_1$ (resp. $P_{0,1}$) be the subsheaf of the sheaf of roots of $\Psi$ (resp. $\Psi_0$) determined by $P_1$ (resp. $P_{0,1}$). Define $W_{0,1} = W(\Psi_{0,1})$ and $W_1 = W(\Psi_1)$ as we have done in the proof of Theorem 2.13 Define $W_{0,1}' = W(\Psi_{0,1}')$ in the same way.

Let $\Psi_0$ be $(M_0, M_0', R_0, R_0')$, and $\{e_i\}_{i=1}^{2m}$ be a basis of $M_0$ such that $R_0$ is the set $\{\pm e_i \pm e_j\}_{i<j}$, where the vertex $i$ corresponds to $e_i - e_{i+1}$ for $i = 1, .., 2m-1$, and the vertex $2m$ corresponds to $e_{2m-1} + e_{2m}$. Let $S_n$ be the permutation group of $n$ elements. Then we have

$$\text{Aut}(\Psi_{0,K_v})(K_v) = (\mathbb{Z}/2\mathbb{Z})^{2m} \rtimes S_{2m},$$

where $S_{2m}$ acts on $R_0$ by permuting the indices of $\{e_i\}_{i=1}^{2m}$, and $(\mathbb{Z}/2\mathbb{Z})^{2m}$ acts on $R_0$ by exchanging the sign of $e_i$’s ( [Bou], Plan. IV). Under this basis, $W_{0,1}'$ is just the permutation group of the set $\{e_i\}_{i=1}^{2m}$. Therefore, the natural inclusion $\nu_W : W_{0,1}' \rightarrow \text{Aut}(\Psi_{0,K_v})$ sends $w \in W_{0,1}' \simeq S_{2m}$ to $(1, w) \in (\mathbb{Z}/2\mathbb{Z})^{2m} \rtimes S_{2m}$.

Since $G_{K_v}$ is an inner form of $G_{0,K_v}$, there is an orientation

$$\mu \in \text{Isomext}(\Psi_{0,K_v}, \Phi(G_{K_v}, f(T_{K_v}))) (K_v).$$

The orientation $\mu$ together with $u^{-1}$ gives an orientation $\nu \in \text{Isomext}(\Psi_{0,K_v}, \Psi_{K_v})(K_v)$.

We then define

$$Q = \text{Isomint}_u(\Psi_{0,K_v}, \mathfrak{P}_{0,1}; \Psi_{K_v}, \mathfrak{P}_1)$$

as we have done in the proof of Proposition 2.13. Since $W_{0,1} \subseteq W_{0,1}'$, we can regard $W_{0,1}$ as a subgroup of $\{1\} \rtimes S_{2m} \subseteq \text{Aut}(\Psi_0)$ through $\nu_W$. Since $\Psi_{K_v} = Q \wedge \Psi_{0,K_v}$ by Remark [13] $(E_v, \sigma) \simeq Q \wedge (E_{0,v}, \sigma_0)$. Therefore, $E_v \simeq E_v' \times E_v'$ with $\sigma$ acts on $E_v$ as the exchange of the two copies of $E_v'$, which contradicts to the assumption (3) in Theorem 2.20. Therefore, $I$ is symmetric under $\text{Aut}(\Psi_{K_v})$ and we conclude that $u(\Delta^o(\Psi_{K_v})) \supseteq \Delta^o(G_{K_v})$ for any orientation $u$. Again, by Theorem 2.13, we have $\mathfrak{E}(G, \Psi, u)(K_v) \neq \emptyset$, for any $u \in \text{Isomext}(\Psi, G)(K_v)$.
Next, we prove that the Autext(G)-torsor Isomext(Ψ, G) satisfies the local-global principle. Namely,

**Lemma 2.22.** Let G (resp. Ψ) be the corresponding semisimple group (resp. root datum) defined by A (resp. E). If the Autext(G)-torsor Isomext(Ψ, G) has a K-v-point at each place v ∈ ΩK, then Isomext(Ψ, G) has a K-point.

**Proof.** If A is not equal to M_8(K) or A = M_4(D), then Autext(G) is (Z/2Z)_K, so the local-global principle for Isomext(Ψ, G) holds in this case. For G an inner form, the outer automorphism group Autext(G) is the symmetric group S_3. Therefore, to prove the local-global principal for the S_3-torsor Isomext(Ψ, G), we only need to prove III^1(K, S_3) = 0. Consider the exact sequence:

\[(1) \ 0 \to \mathbb{Z}/3\mathbb{Z} \to S_3 \to \mathbb{Z}/2\mathbb{Z} \to 0.\]

From the above exact sequence, we get the following exact sequence

\[0 \to \mathbb{Z}/3\mathbb{Z} \to S_3 \to \mathbb{Z}/2\mathbb{Z} \to H^1(K, \mathbb{Z}/3\mathbb{Z}) \to H^1(K, S_3) \to H^1(K, \mathbb{Z}/2\mathbb{Z}).\]

Since the map from S_3 ro Z/2Z is surjective, we have

\[0 \to H^1(K, \mathbb{Z}/3\mathbb{Z}) \to H^1(K, S_3) \to H^1(K, \mathbb{Z}/2\mathbb{Z}).\]

However, the group III^1(K, Z/2Z) = 0, so the set III^1(K, S_3) is in the image of H^1(K, Z/3Z).

Recall that G is the absolute Galois group of K. Note that H^1(K, Z/3Z) = Hom_{gr}(G, Z/3Z), where Hom_{gr}(G, Z/3Z) is the set of continuous homomorphisms from G to Z/3Z. Suppose α ∈ Hom_{gr}(K, Z/3Z) is mapped into III^1(K, S_3). Then since the symmetric group S_3 is surjective to the group Z/2Z for each place v ∈ ΩK, we have

\[0 \to H^1(K_v, \mathbb{Z}/3\mathbb{Z}) \to H^1(K_v, S_3) \to H^1(K_v, \mathbb{Z}/2\mathbb{Z}).\]

Therefore, the homomorphism α is in III^1(K, Z/3Z). Now we claim that III^1(K, Z/3Z) is trivial, i.e. for each α ∈ Hom_{gr}(K, Z/3Z), if α is in III^1(K, Z/3Z), then α is the trivial homomorphism. Suppose that α is not the trivial homomorphism. Let H be the kernel of α. Let L = (K^*)^H. Then L is a Galois extension of K with Galois group Z/3Z and we can regard it as a Z/3Z-torsor. Since the homomorphism α is in III^1(K, Z/3Z), L_v is split completely over K_v for each place v ∈ ΩK. This contradicts Chebotarev’s density Theorem! Therefore, α is the trivial homomorphism and III^1(K, Z/3Z) is trivial. Since III^1(K, S_n) is in the image of III^1(K, Z/3Z), III^1(K, S_n) is also trivial.

For A = M_4(D) and G an outer form, let L be the splitting field of the discriminant of A. We choose a splitting z from Z/2Z to S_3 and we twist the sequence (1) by z. Since z acts on Z/3Z as −1, we have the exact sequence:

\[(2) \ 0 \to R^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}) \to z(S_3) \to \mathbb{Z}/2\mathbb{Z} \to 0,\]

\[\text{39}\]
where we regard $\mathbb{Z}/3\mathbb{Z}$ as a constant group scheme. Note that $z$ is invariant under the twisting because $\mathbb{Z}/2\mathbb{Z}$ is commutative. Therefore, the sequence still splits. Consider the exact sequence derived from (2):

$$0 \to R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z})(K) \to z(S_3)(K) \to (\mathbb{Z}/2\mathbb{Z})(K) \to H^1(K, R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z})) \to H^1(K, z(S_3)) \to H^1(K, R_{L/K}^{(1)}(\mathbb{Z}/2\mathbb{Z})).$$

Since the sequence (2) splits, $z(S_3)(K)$ is mapped onto $(\mathbb{Z}/2\mathbb{Z})(K)$. Hence we have the exact sequence

$$0 \to H^1(K, R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z})) \to H^1(K, z(S_3)) \to H^1(K, R_{L/K}^{(1)}(\mathbb{Z}/2\mathbb{Z})).$$

Since $\text{III}^1(K, \mathbb{Z}/2\mathbb{Z}) = 0$, the set $\text{III}^1(K, z(S_3)(K))$ is in the image of $H^1(K, R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z}))$. Again, because the exact sequence (2) splits, for each place $v \in \Omega_K$, we have

$$0 \to H^1(K_v, R_{L_v/K_v}^{(1)}(\mathbb{Z}/3\mathbb{Z})) \to H^1(K_v, z(S_3)) \to H^1(K_v, R_{L_v/K_v}^{(1)}(\mathbb{Z}/2\mathbb{Z})).$$

Therefore, $\text{III}^1(K, z(S_3)(K))$ is in the image of $\text{III}^1(K, R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z}))$. Now, we only need to prove $\text{III}^1(K, R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z})) = 0$. By Shapiro’s Lemma, we have $\text{III}^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z})) = \text{III}^1(L, \mathbb{Z}/3\mathbb{Z})$. As we have proved above, the group $\text{III}^1(L, \mathbb{Z}/3\mathbb{Z}) = 0$, so $\text{III}^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z})) = 0$. Consider the following exact sequence

$$0 \to R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z}) \to R_{L/K}(\mathbb{Z}/3\mathbb{Z}) \xrightarrow{\text{Nr}} \mathbb{Z}/3\mathbb{Z} \to 0.$$  

In our case, the norm map $\text{Nr}$ from $R_{L/K}(\mathbb{Z}/3\mathbb{Z})(K) = \mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ is just the multiplication by 2. Hence the norm map $\text{Nr}$ is a surjective map from $R_{L/K}(\mathbb{Z}/3\mathbb{Z})(K)$ to $(\mathbb{Z}/3\mathbb{Z})(K)$. Therefore, the map from $H^1(K, R_{L/K}^{(1)}(\mathbb{Z}/3\mathbb{Z}))$ to $H^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z}))$ is injective. Hence, $\text{III}^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z}))$ is also trivial. Therefore, in both cases, the local-global principle for $\text{Isomext}(\Psi, G)$ holds.

Now we have all the ingredients to prove Theorem 2.20.

**Proof of Theorem 2.20.** Suppose that $(E \otimes K_v, \sigma \otimes \text{id}_{K_v})$ can be embedded into $(A \otimes K_v, \tau \otimes \text{id}_{K_v})$ over $K_v$ for each $v \in \Omega_K$, i.e., $E(G, \Psi)(K_v) \neq \emptyset$ for each $v \in \Omega_K$. Then we have $\text{Isomext}(\Psi, G)(K_v) \neq \emptyset$ for each place $v$. By Lemma 2.22, we can fix an orientation $u$.

By Lemma 2.21, the oriented embedding functor $\mathcal{E}(G, \Psi, u)$ has a $K_v$-point for each $v \in \Omega_K$. By Proposition 2.8, the only obstruction for $\mathcal{E}(G, \Psi, u)$ to satisfy the local-global principle lies in $\text{III}^2(K, \text{sc}(T))$. As the proof of Theorem 2.16 shows, $\text{III}^2(K, \text{sc}(T))$ vanishes if there is a place $v \in \Omega_K$ such that for all $i, d_i \in F_i^{x^2}$ and only if $d_i \in (F_i \otimes_K K_v)^{x^2}$. Therefore, the oriented embedding functor $E(G, \Psi, u)$ satisfies the local-global principle in this case. By Theorem 1.15 and Proposition 1.17, the local-global principle for the existence of $K$-embeddings from $(E, \sigma)$ into $(A, \tau)$ holds. 


In the following, we provide an example when the local-global principle for the embedding functor fails.

**Example 2.23.** Let \( K \) be \( \mathbb{Q}(\sqrt{-1}) \), and \( F = K[x]/(x^2 - 3) \). Let \( E' = F \times F \times F \) and \( E = E' \times E' \). Let \( \sigma \) be the \( K \)-automorphism of \( E \) which exchanges the two copies of \( E' \). Then \( \sigma \) is an involution and \( E^{\sigma} \cong E' \). With the notations defined in Section 1.3, we know that the right \( \text{Aut}(E_0, \sigma_0) \)-torsor \( \text{Isom}((E_0, \sigma_0), (E, \sigma)) \) defines a class in \( H^1(K, S_6) \), where \( S_6 \) is contained in the Weyl group of \( \Psi_0 \) (Plan. IV). Let \( \Psi \) be the corresponding root datum. Since \( \Psi \) comes from a class of \( H^1(K, S_6), \Psi \) is an inner form of \( \Psi_0 \).

Let us fix four places of \( K \) such that \( F \) is not split over \( K_v \). For example, we can take a place \( v \) which corresponds to a prime number of the form \( 7 + 12l \), where \( l \) is a positive integer. By Gauss reciprocity, \( x^2 - 3 \) is not split at \( v \). Let \( v_1, \ldots, v_4 \) be the four places mentioned above. At these places, the corresponding Tits index of \( \Psi \) is the following:

\[
\begin{array}{cccc}
\oplus & \rightarrow & \oplus & \rightarrow \\
\oplus & \rightarrow & \oplus & \rightarrow \\
\oplus & \rightarrow & \oplus & \rightarrow \\
\oplus & \rightarrow & \oplus & \rightarrow \\
\end{array}
\]

Consider the following central isogeny:

\[
1 \rightarrow \mu_2 \times \mu_2 \rightarrow \text{Spin}_6 \rightarrow \text{PSO}_6 \rightarrow 1.
\]

Since we have no real places, by [San81] Corollary 4.5, we have

\[
H^1(K, \text{PSO}_6) \cong H^2(K, \mu_2 \times \mu_2).
\]

Also at each finite place \( v \), we have

\[
H^1(K_v, \text{PSO}_6) \cong H^2(K_v, \mu_2 \times \mu_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

(ref. [K] Chap. IV, Thm. 1 and Thm. 2). Let \( [\xi] \) be the class in \( H^1(K_{v_i}, \text{PSO}_6) \) corresponding to \( (1, 0) \) in \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), for \( i = 1, 2 \). Let \( [\xi] \) be the class in \( H^1(K_{v_i}, \text{PSO}_6) \) corresponding to \( (0, 1) \) in \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), for \( i = 3, 4 \). For the other places \( v \in \Omega_K \setminus \{v_1, v_2, v_3, v_4\} \), we let \( [\xi_v] \) in \( H^1(K_v, \text{PSO}_6) \) correspond to \( (0, 0) \) in \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). By Brauer-Hasse-Noether Theorem, we know that there exists a class \( [\xi] \) in \( H^1(K, \text{PSO}_6) \) such that the image of \( [\xi] \) in \( H^1(K_v, \text{PSO}_6) \) is \( [\xi_v] \) for each \( v \in \Omega_K \).

Choose a cocycle \( \zeta \) which represents the class \( [\xi] \). Let \( G \) be the \( K \)-form of \( G_0 \) twisted by \( \zeta \). Since \( G \) and \( \Psi \) are inner forms of \( G_0 \) and \( \Psi_0 \) respectively, we can fix an orientation \( u \) of \( \Psi \) with respect to \( G \). Without loss of generality, we can choose the orientation \( u \) such that \( u(\Delta^\circ(\Psi_{K_{v}})) \supseteq (\Delta^\circ(G_{K_{v}})) \). Note that there is no orientation \( u' \) such that \( u'(\Delta^\circ(\Psi_{K_{v}})) \) contains \( \Delta^\circ(G_{K_{v}}) \) for both \( v = v_1 \) and \( v = v_3 \).

For each place \( v \in \Omega_K \setminus \{v_3, v_4\} \), we have \( u(\Delta^\circ(\Psi_{K_{v}})) \supseteq \Delta^\circ(G_{K_{v}}) \), so by Theorem 2.13 there is a \( K_v \) point of \( E(G, \Psi, u) \). On the one hand, for the place \( v \in \{v_3, v_4\} \), by Theorem 2.13 \( E(G, \Psi, u)(K_v) \) is empty. Therefore, the embedding functor \( E(G, \Psi, u) \) has no \( K \)-points. For the same reason,
we conclude $E(G, \Psi, u')(K) = \emptyset$ for the other orientation $u'$. Hence $E(G, \Psi)$ has no $K$-point. However, at each place $v$, we can always find an orientation $u_v \in \text{Isom}_{\text{ext}}(\Psi, G)(K_v)$ such that $u_v(\Delta^0(\Psi_{K_v})) \supseteq \Delta^0(G_{K_v})$, so the embedding functor $E(G, \Psi)$ has a $K_v$-point for each place $v$. Therefore, the local-global principle fails in this case.

2.4.3 Involutions of the second kind

In this section, $A$ is of degree $n$ over $K$ and $\tau$ is of the second kind. The corresponding reductive group $G$ is of type $A_{n-1}$. In this case, $K$ and $k$ are no longer the same.

Recall that $i_T : R^{(1)}_{K/k}(\mathbb{G}_m, K) \to T$ (resp. $i_G : R^{(1)}_{K/k}(\mathbb{G}_m, K) \to G$) denote the embedding defined by the $K$-structure morphism of $E$ (resp. $A$). We first interpret the $K$-morphism condition into an orientation. Namely, we show that the following are equivalent:

1. A $k$-embedding $f$ is a $K$-embedding.
2. $f \circ i_T = i_G$.
3. $f$ is a $k$-point of $E(G, \Psi, u)$ for some particular orientation $u$

Using the following lemma, we can concretely define the orientation $u$ mentioned above.

**Lemma 2.24.** Let $Z = R^{(1)}_{K/k}(\mathbb{G}_m, K)$ and $\Psi_Z = \Phi(Z, Z)$. Then

1. The natural homomorphism from $\text{Aut}(\Psi)$ to $\text{Aut}(\text{rad}(\Psi))$ induces an isomorphism from $\text{Aut}_{\text{ext}}(\Psi)$ to $\text{Aut}(\Psi_Z)$.
2. $\text{Isom}_{\text{ext}}(\Psi, G)$ is a trivial $\text{Aut}(\Psi_Z)$-torsor.

**Proof.** Let $j_T$ be the homomorphism from the character group of $T$ to the character group of $Z$ induced by $i_T$. Then $j_T$ induces an isomorphism between $\text{rad}(\Psi)$ and $\Psi_Z$ and we have a canonical way to identify $\text{Aut}(\text{rad}(\Psi))$ and $\text{Aut}(\Psi_Z)$. Consider the natural morphism from $\text{Aut}(\Psi)$ to $\text{Aut}(\text{rad}(\Psi))$. Since the Weyl group acts trivially on $\text{Aut}(\text{rad}(\Psi))$, we have a natural morphism $\eta$ from $\text{Aut}_{\text{ext}}(\Psi)$ to $\text{Aut}(\text{rad}(\Psi))$. Note that since $Z$ is a torus of dimension one, $\text{Aut}(\Psi_Z) \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, $\text{Aut}(\text{rad}(\Psi)) \simeq \mathbb{Z}/2\mathbb{Z}$.

To prove $\eta$ is an isomorphism, we only need to check it over $k^s$, so we can assume that $\Psi$ is split and of type $(M, M', R, R')$.

We first prove the injectivity of $\eta$. By the definition of $\Psi$, we can find a basis $\{e_i\}_{i=1, \ldots, n}$ of $M$ such that $\Delta = \{e_i - e_{i+1}\}_{i=1, \ldots, n-1}$ is a system of simple roots of $R$ (ref. [Bou], Plan. I). By Proposition 0.3, $\text{Aut}(\Psi) \simeq E_\Delta(\Psi)$. Let $h \in E_\Delta(\Psi)$ and suppose that $h$ acts on $\text{rad}(\Psi)$ trivially. We claim that $h$ acts on $\Psi$ trivially.

To see this, we note that $h$ induces an isomorphism on the Dynkin diagram, so $h$ can only act on $\Delta$ trivially or exchange $e_i - e_{i+1}$ with $e_{n-i} - e_{n-i+1}$.

Let $h(e_1) = \sum_{i=1}^n a_i e_i$.

Suppose that $h$ exchanges $e_i - e_{i+1}$ with $e_{n-i} - e_{n-i+1}$. Since
\[ \alpha^\vee(e_1) = h(\alpha)^\vee(h(e_1)) \text{ for all } \alpha^\vee \in R^\vee, \]

we have

\[ a_{n-1} = a_n + 1, \]
\[ a_i = a_{n-1}, \ i = 1, \ldots, n - 1. \]

Besides, \( h \) acts on \( \text{rad}(\Psi) \) trivially, so \( e_1 - h(e_1) = \sum_{i=1}^{n} b_i(e_i - e_{i+1}) \). By summing up the coefficients, we have \( \sum_{i=1}^{n} a_i = 1 \) and hence \( n a_{n-1} = 2 \). Since \( a_i \)'s are integers, the only possibility is \( n = 2 \) and \( a_1 = 1 \). In this case, \( h(e_1) = e_1 \) and \( h(e_1 - e_2) = e_1 - e_2 \), so \( h \) is identity.

Now suppose that \( h \) acts on \( \Delta \) trivially. Then by the same reasoning, we have

\[ a_1 = a_2 + 1, \]
\[ a_2 = a_i, \ i = 2, \ldots, n. \]

Besides, \( h \) acts on \( \text{rad}(\Psi) \) trivially, so \( \sum_{i=1}^{n} a_i = 1 \). Therefore, \( a_1 = 1 \) and \( a_i = 0 \) for \( i \neq 1 \), which means \( h \) is the identity.

This proves that \( \eta \) is injective.

On the other hand, since the \(-1\) map on \( \Psi \) induces the \(-1\) map on \( \text{rad}(\Psi) \) and \( \text{Aut}(\text{rad}(\Psi)) \simeq \mathbb{Z}/2\mathbb{Z} \), we have \( \eta \) is surjective and hence an isomorphism.

To prove (2), we choose a maximal torus \( T' \) of \( G \). Note that since \( K \) is in the center of \( A \), \( i_G(Z) \) is in the center of \( G \) and hence \( i_G(Z) \) is in \( T' \). Let \( j_G \) be the map between character groups of \( T' \) and \( Z \) induced by \( i_G \). Let \( \Psi_G = \Phi(G, T') \). Then we have a natural morphism from \( \text{Isom}(\Psi, \Psi_G) \) to \( \text{Isom}(\text{rad}(\Psi), \text{rad}(\Psi_G)) \). Since the Weyl group \( W(\Psi) \) acts trivially on \( \text{rad}(\Psi) \), the above morphism induces an morphism from \( \text{Isomext}(\Psi, \Psi_G) \) to \( \text{Isom}(\text{rad}(\Psi), \text{rad}(\Psi_G)) \), which is an isomorphism. Through \( j_T \) and \( j_G \), we have an isomorphism from \( \text{Isom}(\text{rad}(\Psi), \text{rad}(\Psi_G)) \) to \( \text{Aut}(\Psi_Z) \), which sends \( f^\Psi \) to \( j_G \circ f^\Psi \circ j_T^{-1} \). Therefore, we have

\[ \zeta : \text{Isomext}(\Psi, \Psi_G) \to \text{Aut}(\Psi_Z). \]

Since \( \text{Auttext}(\Psi) \simeq \text{Aut}(\Psi_Z) \) and \( \zeta \) is compatible with the \( \text{Aut}(\Psi_Z) \)-action, \( \zeta \) is an isomorphism between \( \text{Aut}(\Psi_Z) \)-principal homogeneous spaces. Since there is a canonical isomorphism from \( \text{Isomext}(\Psi, \Psi_G) \) to \( \text{Isomext}(\Psi, G) \), the result then follows.

With the notations defined in the above lemma, we let \( u \in \text{Isomext}(\Psi, G)(k) \) be \( \zeta^{-1}(1) \). Then for a \( k \)-embedding \( f \), we see that \( f \circ i_T = i_G \) if and only if \( f \) is a \( k \)-point of \( \mathcal{E}(G, \Psi, u) \). Hence again, we can reduce the embedding problem to the existence of rational points of \( f \in \mathcal{E}(G, \Psi, u) \) and reformulate Prasad-Rapinchuk’s Theorem as the following ([PR10], Thm. 4.1):
Theorem 2.25. Suppose that $\tau$ is an involution of the second type. If $E$ is a field, then the local-global principle for the $K$-embeddings from $(E, \sigma)$ to $(A, \tau)$ holds.

Proof. By Lemma 2.24, we can fix an orientation $u$ such that $f$ is a $k$-point of $\mathcal{E}(G, \Psi, u)$ if and only if $f$ is a $K$-embedding. By Remark 1.16, $\mathcal{E}(G, \Psi, u)(k_v)$ is nonempty if and only if $\mathcal{E}(G, \Psi)(k_v)$ is nonempty. Hence, it suffices to show that the local-global principal for $\mathcal{E}(G, \Psi, u)$ holds. By Theorem 2.2, we only need to show that $X^2(k, \text{sc}(T))$ vanishes. Consider the exact sequence

$$
0 \longrightarrow \text{sc}(T) \longrightarrow R_{E^\sigma/k}(R_{E/E^\sigma}^1(G_m)) \longrightarrow R_{K/k}(G_m) \longrightarrow 0,
$$

where we derive the long exact sequence:

$$
\cdots \longrightarrow H^2(k, \text{sc}(T)) \longrightarrow H^2(k, R_{E^\sigma/k}(R_{E/E^\sigma}^1(G_m))) \longrightarrow H^1(k, R_{K/k}^1(G_m)) \longrightarrow H^1(k, \text{sc}(T)) \longrightarrow H^3(k, \text{sc}(T)) \longrightarrow \cdots.
$$

Since $\text{III}^2(k, R_{E^\sigma/k}(R_{E/E^\sigma}^1(G_m))) = \text{III}^2(E^\sigma, R_{E/E^\sigma}^1(G_m)) = 0$, we know that $\text{III}^2(k, \text{sc}(T))$ is in the image of $H^1(k, R_{K/k}^1(G_m)) = k^\times/Nr_{K/k}(K^\times)$, where $N_{K/k}$ denotes the norm map from $K$ to $k$. Let $x$ be an element of $k^\times$ and suppose that $x$ is mapped to $\text{III}^2(k, \text{sc}(T))$. At each place $v \in \Omega_k$, let $x_v$ be the image of $x$ in $k_v$. Since $x$ is mapped to $\text{III}^2(k, \text{sc}(T))$, $x$ belongs to $k^\times \cap N_{E^\sigma/k}(I_{E^\sigma})N_{K/k}(I_K)$, where $I_{E^\sigma}$ and $I_K$ are idèle groups of $E^\sigma$ and $K$ respectively. By Hasse multiform principle (ref. [PIR] Prop. 6.11), $x$ belongs to $N_{E^\sigma/k}^1(I_{E^\sigma})N_{K/k}(K)$, so $x$ is in the image of $H^1(k, R_{E^\sigma/k}(R_{E/E^\sigma}^1(G_m)))$. Hence $x$ is mapped to 0 in $\text{III}^2(k, \text{sc}(T))$, which implies $\text{III}^2(k, \text{sc}(T)) = 0$. The theorem then follows.

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