A NEW INVARIANT THAT'S A LOWER BOUND OF LS-CATEGORY

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Abstract. Let $X$ be a finite type simply connected CW-complex, $(AV, d)$ its Sullivan minimal model and let $k \geq 2$, the biggest integer such that $d = \sum_{i \geq k} d_i$ such that $d_i(V) \subseteq AV$. We introduce two new spectral sequences that generalizes the Milnor-Moore spectral sequence and its Ext-version [17]:

$$H^{p,q}(AV, d_k) \Rightarrow H^{p+q}(AV, d)$$

$$Ext^{p,q}_{(AV, d_k)}(\mathbb{Q}, (AV, d_k)) \Rightarrow Ext^{p+q}_{(AV, d)}(\mathbb{Q}, (AV, d)).$$

We prove that the last one is isomorphic to the Eilenberg-Moore spectral sequence:

$$Ext^{p,q}_{H_*^*(\Omega X, K)}(\mathbb{K}, H_*^*(\Omega X, \mathbb{K})) \Rightarrow Ext^{p+q}_{C_*^*(\Omega X, \mathbb{K})}(\mathbb{K}, C_*^*(\Omega X, \mathbb{K}))$$

and use it to introduce a new invariant $r(AV, d) = \sup \{ p \mid \mathcal{E}_p^{AV} \neq 0 \} \pm \infty$. Moreover, we show that this is is a lower bound of $\text{cat}(X_\mathbb{Q})$ and for an elliptic pure Sullivan minimal model, we have:

$$r(AV, d) = \dim V^{odd} + (\dim V^{even} - 1)(k - 2).$$

1. Introduction

LS-category was introduced since 1934 by L. Lusternik and L. Schnirelman in connection with variational problems [14]. They showed that for any closed manifold $M$, this category denoted $\text{cat}(M)$, is a lower bound of a number of critical points that any smooth function on $M$ must have. Later, it was shown that this is also right for a manifold with a boundary [18].

If $X$ is a topological space, $\text{cat}(X)$ is the least integer $n$ such that $X$ is covered by $n + 1$ open subset $U_i$, each contractible in $X$. It is an invariant of homotopy type (c.f. [10] or [3] Prop 27.2 for example). Though its definition seems easy, the calculation of this invariant is hard to compute. In [1], Y. Félix and S. Halperin developed a deep approach for computing the LS-category for rational spaces. They defined the rational category of a 1-connected space $X$ in terms of its Sullivan minimal model and showed that it coincides with LS-category of its rationalization $X_\mathbb{Q}$. Since then, much estimations of the rational LS-category were developed in connection with other numerical invariants. Later on, Y. Félix, S. Halperin and J.M. Lemaire [2] showed that for Poincaré duality spaces, the rational LS-category

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coincide with the rational Toomer invariant -denoted $e_Q(X)$- which is difficult to define, but at first sight, it may look easier to compute it (c.f. next section for details).

A version of this invariant was given in terms of the Milnor-Moore spectral sequence (cf. [1], Prop. 9.1). Later in [17] A. Murillo introduced what he called the $E\text{-}Xt$-version of the Milnor-Moore spectral sequence and used it to characterize Gorenstein algebras.

Before giving some necessary ingredients to introduce our main results, we point out the ulterior essential results in this area.

Let $(\Lambda V, d)$ be a commutative differential graded algebra (cdga for short) over a field $\mathbb{K}$ of characteristic zero. We suppose that $\dim(V) < \infty$, $V^0 = \mathbb{K}, V^1 = 0$ and denote $V^{even} = \bigoplus_k V^{2k}$ and $V^{odd} = \bigoplus_k V^{2k+1}$. If $d = \sum_{i \geq 2} d_i$ (where $d_i$ designate the part of $d$ such that $d_i(V) \subseteq \Lambda^i V$), $(\Lambda V, d)$ is said to be minimal. Recall that to any nilpotent space $X$ (in particular any simply connected space), D. Sullivan in [20] associated a minimal cdga $(\Lambda V, d)$ over the rationales, which is unique up to quasi-isomorphism. Later on, this is called the (Sullivan) minimal model for $X$.

In there famous article [1] where rational LS-category is defined in terms of Sullivan minimal models, Y. Félix and S. Halperin had shown that a coformal rationally elliptic space $Y$ (i.e. a space with minimal model $(\Lambda V, d_2)$) has $e(Y) = e_0(\Lambda V, d_2) = \dim(V^{odd})$. Longer after we showed in ([19], Prop 2.1 and Prop B) that for any rationally elliptic space $X$ with minimal model $(\Lambda V, d)$:

$$\dim H^* (\Lambda V, d_2) < \infty \iff \text{gldim}(H_*(\Omega X, \mathbb{Q})) < \infty \iff e_Q(X) = \text{depth}(H_*(\Omega X, \mathbb{Q})).$$

However, $\text{depth}(H_*(\Omega X, \mathbb{Q})) = \dim(V^{odd})$ ([5], Th 36.4), thus according to [2], we have

$$\text{cat}(X) = \dim(V^{odd}) \iff (\Lambda V, d_2) \text{ is also elliptic.}$$

This last result is generalized in one direction by L. Lechuga and A. Murillo in [12] as follow: Let $X$ an elliptic space with minimal model $(\Lambda V, d)$, such that $d = \sum_{i \geq k} d_i$ and $k \geq 2$. If $(\Lambda V, d_k)$ is also elliptic, then $\text{cat}(X) = \dim(V^{odd}) + (k-2) \dim(V^{even})$.

In this context, our first goal is to estimate the rational Toomer invariant of any Gorenstein algebra (resp. LS category of any rationally elliptic space). For this we introduce the following spectral sequences, which we call the generalized (resp. $E\text{-}Xt$-version generalized) -Milnor-Moore spectral sequence of $(\Lambda V, d)$:

$$(1.0.1) \quad H^{p,q}(\Lambda V, d_k) \Longrightarrow H^{p+q}(\Lambda V, d)$$

$$(1.0.2) \quad E^{p,q}_{(\Lambda V, d_k)}(\mathbb{K}, (\Lambda V, d_k)) \Longrightarrow E^{p+q}_{(\Lambda V, d)}(\mathbb{K}, (\Lambda V, d)).$$

Here and in what follow, $k$ is the biggest integer $k \geq 2$ such that $d = \sum_{i \geq k} d_i$, with $d_i(V) \subseteq \Lambda^i V$ and $\mathbb{K}$ a field with $\text{char}(\mathbb{K}) \neq 2$.

Using the later one, we introduce a new invariant which we call an $E\text{-}Xt$-version of the Toomer invariant. Given a 1-connected commutative differential graded algebra
(A,d) over K and (\Lambda V, d) its minimal model (cf. section 2), if E_{\infty}^* designate the infty term of the generalized Ext-version Milnor-Moore spectral sequence, we set:

\[ r(A, d) := r(\Lambda V, d) := \text{Sup} \{ p \mid E_{\infty}^{p,*} \neq 0 \} \text{ or } \infty. \]

**Remark 1.0.1.** The filtration inducing the spectral sequence \[\mathcal{F}_p\] is

\[ \mathcal{F}_p = \{ f \in \text{Hom}_{\Lambda V}(\Lambda V \otimes \Gamma(sV), \Lambda V) \mid f(\Gamma(sV)) \subseteq \Lambda^p \Lambda V \}. \]

We deduce immediately the following:

1. If (\Lambda V, d) is a Sullivan minimal model, then the invariant r(\Lambda V, d) is the largest integer p such that some nontrivial class in E_{\infty}^{p,*} is represented by a cocycle in F_p. Equivalently it is the least integer p such that the projection \( \mathcal{A} \rightarrow \mathcal{A}/\mathcal{F}^{>p} \) induces an injection in cohomology, where \( \mathcal{A} = \text{Hom}(\Lambda V, d)(\Lambda V \otimes \Gamma(sV), D, (\Lambda V, d)). \)

2. Suppose \( \dim(V) < \infty \), so that (\Lambda V, d) is a Gorenstein algebra, i.e. \( E^{*}_{\infty} \Lambda V, d(\Lambda V, d) \) is one dimensional \([17]\). Denote by \( \Omega \) its generator. The projection \( \pi : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{F}^{>p} \) is then injective in cohomology if and only if \( H(\pi)(\Omega) \neq 0 \). Therefore

\[ r(\Lambda V, d) = \text{sup} \{ p \mid \Omega \text{ can be represented by a cocycle in } F_p \}. \]

Our first result in this paper is:

**Theorem 1.0.2.** Let \( X \) be a simply connected CW-complex of finite type and (\Lambda V, d) its Sullivan minimal model. If \( X \) is rationally elliptic, then \( \text{cat}(X, \mathbb{Q}) \geq r(\Lambda V, d) \).

Now let \( X \) be a simply connected CW-complex of finite type. Denote by \( A = (C_*(\Omega X), d) \) the chain complex on the loop space \( \Omega X \) and by \( (A \otimes B(A), d) \) the bar construction on \( (A, d) \) with coefficients in \( (A, d) \) \([5] \text{ p } 268\). Recall that \( A \otimes B(A) = \bigoplus_{n \geq 0} A \otimes T^n(sA) \) with \( A = \ker(\epsilon : A \rightarrow \mathbb{K}) \) and \( sA \) its suspension. The filtration \( F^q = \bigoplus_{k \geq q} A \otimes T^k(sA) \) on \( A \otimes B(A) \) is such that \( d(T^q(sA)) \subseteq \bigoplus_{k \geq q} A \otimes T^k(sA) \).

It is an \( A \)-semifree resolution of \( \mathbb{K} \) and gives rise to an spectral sequence with first term \( E^2_q = H(A, d) \otimes T^q(sA) \). By \([6] \text{ pro. } 20.11\) one can suppose this an \( H(A) \)-semifree resolution of \( \mathbb{K} \). We then filter \( \text{Hom}_A(A \otimes B(A), \mathbb{A}) \) by

\[ F^q = \{ f \in \text{Hom}_A(A \otimes B(A), A) \mid f(F^k) = 0, \forall k < q \}. \]

This induce the converging Eilenberg-Moore spectral sequence:

\[ E_2 = \text{Ext}^{p,q}_{H_*(\Omega X, \mathbb{K})}(\mathbb{K}, H_*(\Omega X, \mathbb{K})) \Rightarrow \text{Ext}^{p+q}_{C_*(\Omega X, \mathbb{K})}(\mathbb{K}, C_*(\Omega X, \mathbb{K})). \]

Our second result is:

**Theorem 1.0.3.** Let \( X \) be a simply connected CW-complex of finite type and (\Lambda V, d) its Sullivan minimal model. Then, the \( \text{Ext}-version generalized)-Milnor-Moore spectral sequence of (\Lambda V, d) is isomorphic to the Eilenberg-Moore spectral sequence.

Recall also that a graded differential algebra \( (A, d) \) is a Gorenstien algebra if \( \text{dim}(\text{Ext}^{*,*}_{(A,d)}(\mathbb{K}, (A, d))) = 1 \) \([17]\) and that \( (\Lambda V, d) \) is a Gorensten algebra if and only if \( \text{dim}(V) < \infty \) \([17]\). It follow immediately that there is a unique \( (p, q) \) such that \( E^{p,q}_{\infty}(\mathbb{Q}, (\Lambda V, d)) \neq 0 \) and it is one dimensional. By the use of \([1.0.2]\) we then have:
Proposition 1.0.4. With the notation above, if \((\Lambda V, d)\) is a Sullivan minimal algebra such that \(\dim(V) < \infty\), then \(r(\Lambda V, d) = r(\Lambda V, d_k)\).

On the other hand, S. Halperin in \((\mathbb{R})\) associated to any simply connected space \(X\) with minimal model \((\Lambda V, d)\) its pure model \((\Lambda V, d_\sigma)\) and introduce a spectral sequence with \(E_2\)-term \(H(\Lambda V, d_\sigma)\) and converging to \(H(\Lambda V, d)\).

As a corollary we deduce that \(r(\Lambda V, d_\sigma) = r(\Lambda V, d_{\sigma, k})\). So, using this relation, we obtain an explicit formula of \(r(\Lambda V, d_\sigma)\):

Theorem 1.0.5. Let \((\Lambda V, d)\) any pure minimal Sullivan algebra. If \(\dim(V) < \infty\) then

\[ r(\Lambda V, d) = \dim(V^{\text{odd}}) + (k - 2)(\dim(V^{\text{even}}) - 1). \]

2. Preliminary

In this section we recall some notions we will use in other sections.

2.1. A Sullivan minimal model: Let \(\mathbb{K}\) a field of characteristic \(\neq 2\).

A Sullivan algebra is a free commutative differential graded algebra (\(cdga\) for short) \((\Lambda V, d)\) (where \(\Lambda V = \text{Exterior}(V^{\text{odd}}) \otimes \text{Symmetric}(V^{\text{even}})\)) generated by the graded \(\mathbb{K}\)-vector space \(V = \bigoplus_{i=0}^{\infty} V^i\) which has a well ordered basis \(\{x_\alpha\}\) such that \(dx_\alpha \in \Lambda V^{\leq \alpha}\). Such algebra is said minimal if \(\deg(x_\alpha) < \deg(x_\beta)\) implies \(\alpha < \beta\). If \(V^0 = V^1 = 0\) this is equivalent to saying that \(d(V) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^i V\).

A Sullivan model for a commutative differential graded algebra \((A, d)\) is a quasi-isomorphism (morphism inducing an isomorphism in cohomology) \((\Lambda V, d) \xrightarrow{\sim} (A, d)\) with source, a Sullivan algebra. If \(H^0(A) = \mathbb{K}, H^1(A) = 0\) and \(\dim(H^i(A, d)) < \infty\) for all \(i \geq 0\), then \((\mathbb{R})\), Th. 7.1), this minimal model exists. If \(X\) is a topological space any (minimal) model of the algebra \(C^*(X, \mathbb{K})\) is said a Sullivan (minimal) model of \(X\).

The uniqueness (up quasi-isomorphism) in the rational case was assured by D. Sullivan in \((\mathbb{R})\). Indeed he associated to any simply connected space \(X\) the \(cdga\) \(A(X)\) of polynomial differential forms and showed that its minimal model \((\Lambda V_X, d)\) satisfy:

\[ V_X^i \cong \text{Hom}_\mathbb{Z}(\pi_i(X, \mathbb{Q})); \quad \forall i \geq 2, \]

that is, in the case where \(X\) is a finite type CW-complex, the generators of \(V_X^i\) corresponds to those of \(\pi_*(X) \otimes \mathbb{Q}\).

If \(\text{char}(\mathbb{K}) = p > 2\), for any \(r\)-connected CW-complex \(X\) with \(\dim(X) < rp\), there exists a sequence \((\mathbb{R})\):

\[ (\Lambda V_X, d) \xrightarrow{\sim} C^*(L) \xrightarrow{\sim} B^\vee(C_*(\Omega X, \mathbb{K})) \xrightarrow{\sim} C^*(X, \mathbb{K}) \]

of quasi-isomorphisms, where \(C^*(L)\) is the Cartan-Chevalley-Eilenberg cochain complex and \(B^\vee(C_*(\Omega X, \mathbb{K}))\) is the dual of the bar construction of \(C_*(\Omega X, \mathbb{K})\). The uniqueness of \((\Lambda V_X, d)\) is also assured under this restrictive conditions on \(X\). As I know there is any relationship between \(V_X^i\) and \(\pi_*(X) \otimes \mathbb{K}\).
In both cases, one can associate to \((\Lambda^V, d)\) another cdga (called the pure model associated to \(X\)) denoted \((\Lambda^V, d_\sigma)\) with \(d_\sigma\) defined as follow:

\[
d_\sigma(V_X^{even}) = 0 \quad \text{and} \quad (d - d_\sigma)(V_X^{odd}) \subseteq \Lambda^V \otimes \Lambda^+ V_X^{odd}.
\]

This pure model was introduced by S. Halperin ([7]) as the \(E_1\)-term of the following spectral sequence:

\[
E_2^{p,q} = H^{p,q}(\Lambda^V, d_\sigma) \implies H^{p+q}(\Lambda^V, d)
\]

This spectral sequence is induced by the filtration \(F^p = (\Lambda^V)_{\geq p,*}\), where \((\Lambda^V)^{n+q-s} = (\Lambda^Q \otimes \Lambda^n P)^n\) (with \(Q = V^{even}\) and \(P = V^{odd}\)).

For completeness, we recall here the filtration inducing the following Ext-\textit{version} odd spectral sequence introduced in [10]:

\[
\mathcal{E}xt_{(\Lambda^V, d_\sigma)}^{p,q}(Q, (\Lambda^V, d_\sigma)) \implies \mathcal{E}xt_{(\Lambda^V, d_\sigma)}^{p+q}(Q, (\Lambda^V, d)).
\]

Recall first that if \((\Lambda^V \otimes \Gamma sV, D)\) designate the acyclic closure of \((\Lambda^V, d)\) then \((\Lambda^V \otimes \Gamma sV, D_\sigma)\), (with \(D_\sigma = d_\sigma\) on \(V\) and \(D_\sigma(sv) = v - s(d_\sigma v), \forall v \in V\)) is the one of \((\Lambda^V, d_\sigma)\).

Now let

\[
(\Lambda^V \otimes \Gamma sV)^r,s = \bigoplus_{a_1, a_2, a_3} (\Lambda^Q \otimes \Gamma^{a_1} sQ \otimes \Lambda^{a_2} P \otimes \Gamma^{a_3} sP)^s.
\]

The filtration on \((A, D) = (Hom_{(\Lambda^V, d)}[(\Lambda^V \otimes \Gamma sV, D), (\Lambda^V, d)], D)\) is then given by

\[
\mathcal{F}^p(A^n) = \bigoplus_{r,s} Hom_{(\Lambda^V, d)}[(\Lambda^V \otimes \Gamma sV)^r,s, (\mathcal{F}^{p+r+s}(\Lambda V))^{r+s}].
\]

Remark 2.1.1. Let \(\varphi \in \mathcal{F}^p(A^n)\), it is clear that \(\varphi(1) \in F^p = \Lambda^{2p} V\).

2.2. The evaluation map: Let \((A, d)\) be an augmented \(K\)-dga and choose an \((A, d)\)-semifree resolution ( [4]) \(\rho : (P, d) \xrightarrow{\bar{\rho}} (K, 0)\) of \(K\). Providing \(K\) with the \((A, d)\)-module structure induced by the augmentation we define a chain map:

\[
Hom_{(A, d)}[(P, d), (A, d)] \longrightarrow (A, d) \; \text{by} \; f \mapsto f(z), \; \text{where} \; z \in P \; \text{is a cycle representing 1.}
\]

Passing to homology we obtain the evaluation map of \((A, d)\):

\[
ev_{(A, d)} : \mathcal{E}xt_{(A, d)}(K, (A, d)) \longrightarrow H(A, d),
\]

where \(\mathcal{E}xt\) is the differential Ext of Eilenberg and Moore [15]. Note that this definition is independent of the choice of \(P\) and \(z\) and it is natural with respect to \((A, d)\).

The authors of [4] also defined the concept of a Gorenstein space over any field \(K\). It is a space \(X\) such that \(\dim(\mathcal{E}xt^*_{C^*}(K, X)) = 1\). Moreover, \(\dim H(X, K) < \infty\), then it satisfies Poincaré duality property over \(K\) and its fundamental class is closely related to the evaluation map [17].
2.3. **The Toomer invariant:** The Toomer invariant is defined by more than one way. Here we recall its definition in the context of minimal models. Let \((\Lambda V, d)\) any minimal cdga on a field \(K\). Let

\[
p_n : \Lambda V \rightarrow \frac{\Lambda V}{\Lambda \geq n+1 V}
\]
denote the projection onto the quotient differential graded algebra obtained by factoring out the differential graded ideal generated by monomials of length at least \(n + 1\). The "commutative" Toomer invariant \(e_{c,K}(\Lambda V, d)\) of \((\Lambda V, d)\) is the smallest integer \(n\) such that \(p_n\) induces an injection in cohomology or \(\infty\) if there is no such smallest \(n\).

In \cite{[9]}, S. Halperin and J.M. Lemaire defined for any simply connected finite type CW-complex \(X\) (and any field \(K\)) the invariant \(e_K(X)\) in terms of its free model \((\mathcal{T}(W), d)\) and showed that it coincides with the classical Toomer invariant. Consequently if \(\text{char}(K) \neq 2\) and \(X\) is an \(r\)-connected CW-complex with \(\dim(X) < rp\), \((r \geq 1)\) then using both its Sullivan minimal model \((\Lambda V, d)\) and its free minimal model \((\mathcal{T}(W), d)\), we deduce (cf \cite{[9]}, Th 3.3 ) that \(e_K(X) = e_K(\Lambda V, d) = e_{c,K}(\Lambda V, d)\).

For \(K = \mathbb{Q}\), we shall denote \(e_0(\Lambda V, d)\) instead of \(e_{c,Q}(\Lambda V, d)\).

**Remark 2.3.1.**

1. The definition given above for \(e_0(\Lambda V, d)\) is also expressed in terms of the Milnor-Moore spectral sequence (which coincides with \(1.0.1\), for \(k = 2\)) (cf. \cite{[1]} by:

\[
e_0(\Lambda V, d) = \sup \{p \mid E_\infty^{pq} \neq 0\} or \infty.
\]

2. In (\cite{[1]}, Lemma 10.1), Y. Félix and S. Halperin showed also that whenever \(H(\Lambda V, d)\) has Poincaré duality, then:

\[
e_0(\Lambda V, d) = \sup \{k / \omega\text{ can be represented by a cycle in }\Lambda \geq k V\}
\]

where \(\omega\) represents the fundamental class. This remains true when one replace \(\mathbb{Q}\) by any field \(K\).

### 3. Spectral sequences and the proof of the results:

In this section we will work in a field \(K\) of \(\text{char}(K) \neq 2\). Let \((\Lambda V, d)\) a cdga with \(\dim(V) < \infty\). Suppose that \(d = \sum i \geq k \ d_i\) and \(k \geq 2\). The filtrations that induce the spectral sequences \(1.0.1\) and \(1.0.2\) given in the introduction are defined respectively as follow:

\[
(3.0.1) \quad F^p = \Lambda \geq p V = \bigoplus_{i=p} \Lambda i V
\]

\[
(3.0.2) \quad F^p = \{f \in Hom_{AV}(\Lambda V \otimes \Gamma(sV), \Lambda V) \mid f(\Gamma(sV)) \subseteq \Lambda \geq p V\}
\]

Recall that \(\Gamma(sV)\) is the divided power algebra of \(sV\) and the differential \(D\) on \(\Gamma(sV) \otimes \Lambda V\) is a \(\Gamma\)-derivation (i.e. \(D(\gamma^p(sv)) = D(sv)\gamma^{p-1}(sv), \quad p \geq 1, \quad sv \in (sV)^{\text{even}}, \text{ and } D(sv) = v + s(dv)\)) which restrict to \(d\) in \(\Lambda V\). With this differential, \((\Lambda V \otimes \Gamma(sV), D)\) is a dga called an acyclic closure of \((\Lambda V, d)\) hence it is \((\Lambda V, d)\)-semifree. Therefore the projection \((\Lambda V \otimes \Gamma(sV), D) \rightarrow K\) is a semifree resolution of \(K\).
Recall also that de differential $D$ of $Hom_{(AV,d)}((AV \otimes \Gamma(sV),D),(AV,d))$ is defined as follow: $D(f) = d \circ f + (-1)^{|f|} f \circ D$.

Let us denote $A = AV$ (resp. $Hom_{AV}(AV \otimes \Gamma(sV),AV)$, $G^p = F^p$ (resp. $G^p = F^p$) and $\delta = d$ (resp. $\delta = D$) the differential of $A$.

The filtrations 3.0.1 and 3.0.2 verify the following lemma and then they define the tow spectral sequences below:

**Lemma 3.0.2.** (i): $(G^p)_{p \geq 0}$ is decreasing.

(ii): $G^0(A) = A$.

(iii): $\delta(G^p(A)) \subseteq G^p(A)$.

**Proof.** (i) and (ii) are immediate. The propriety (iii) follows from (a): the definition of $D$ on $Hom_{AV}(AV \otimes \Gamma(sV),D),(AV,d))$, (b): $d$ is minimal and (c): $D(\gamma^p(sv)) = D(sv)\gamma^{p-1}(sv) = (v + s(sv))\gamma^{p-1}(sv)$.

3.1. **Determination of the first terms of the tow spectral sequences.** The two filtrations are bounded, so they induce convergent spectral sequences. We calculate here, there first terms.

Beginning with the filtration 3.0.1 one can check easily the following:

$$E^p_0 = F^p/F^{p+1} \cong \bigcap\limits_{k_0} E^p_k \cong \Delta^p V.$$

and

$$d_0 = d_1 = \ldots = d_{k-2} = 0.$$

The first non-zero differential is $d_{k-1} : F^p_{k-1} \rightarrow F^{p+(k-1)}_{k-1}$ which coincides with the $k^{th}$ term $d_k$ in the differential $d$ of $(AV,d)$, and then: $(E_{k-1}, d_{k-1}) = (AV, d_k)$. So the first term in the induced spectral sequence 1.0.1 is

$$E_k = H(AV,d_k).$$

For the second spectral sequence 1.0.2 its general term is:

$$E^p_r = \frac{\{f \in F^p, \ D(f) \in F_p^{r+r}\}}{\{f \in F^{p+r}, \ D(f) \in F^{p+r+r} \}}.$$

we first proves the following important lemma:

**Lemma 3.1.1.** Let $f \in F^p$, then for any $p \geq 0$,

1. $D(f) \in F^{p+2} \iff f \in Ker(D_2)$.

2. $f - D(g) \in F^{p+1} \iff f - D_2(g) \in F^{p+1}.$

**Proof.** Using the relations $D(f) = d \circ f + (-1)^{|f|} f \circ D$, $D(\gamma^p(sv)) = D(sv)\gamma^{p-1}(sv) = (v + s(sv))\gamma^{p-1}(sv)$ and $D(sv) = v + s(sv)$, we have successively:

1. $D(f) \in F^{p+2} \iff D(f)(\Gamma(sV)) \cap \Delta^{p+1} V = \{0\} \iff D_2(f)(\Gamma(sV)) \cap \Delta^{p+1} V = \{0\} \iff f \in Ker(D_2)$.

2. $f - D(g) \in F^{p+1} \iff (f - D(g))(\Gamma(sV)) \cap \Delta^p V = \{0\} \iff (f - D_2(g)) \cap \Delta^p V = \{0\}.$
Using this lemma, we have successively:

\[(\mathcal{E}_0^p, d_0) = (\mathcal{F}^p / \mathcal{F}^{p+1}, 0)\]
\[(\mathcal{E}_1^p, d_1) = (\mathcal{E}_0^p, d_1) = (\mathcal{F}^p / \mathcal{F}^{p+1}, d_1)\]

where \(d_1 : \mathcal{F}^p / \mathcal{F}^{p+1} \to \mathcal{F}^{p+1} / \mathcal{F}^{p+2}\) such that \(\forall f \in \mathcal{F}^p, \forall g \in \mathcal{F}^{p+1}:\)

\[
\begin{align*}
\{ d_1(f) = 0 &\Rightarrow D(f) \in \mathcal{F}^{p+2} \iff f \in \text{Ker}(D_2) \Rightarrow D_2(f) = 0 \\
\hat{f} - d_1(g) = 0 &\Rightarrow f - D(g) \in \mathcal{F}^{p+1} \iff f - D_2(g) \in \mathcal{F}^{p+1} \Rightarrow \hat{f} = D_2(g)
\end{align*}
\]

As a consequence:

If \(k = 2\), \(\mathcal{E}_2^* = \mathcal{E}_{x}^*(\mathbb{Q}, (\Lambda V, d_2))\), which is exactly the first term of the Ext-Milnor-Moore spectral sequence introduced by A. Murillo in \([17]\).

If \(k > 2\), the differential \(D_2\) is reduced to \(D_0\), where \(D_0(f) = (-1)^{|f| + 1} f \circ D_0\), \(D_0(\gamma^p(vg)) = D_0(vg)\gamma^{p-1}(sv) = v\gamma^{p-1}(sv)\). Then \(\mathcal{E}_2^* = \mathcal{E}_{x}^*(\mathbb{Q}, (\Lambda V, 0))\).

Consider now the case where \(k = 3\):

The general term:

\[
\mathcal{E}_3^p = \frac{\{ f \in \mathcal{F}^p, D(f) \in \mathcal{F}^{p+3}\}}{\{ f \in \mathcal{F}^{p+1}, D(f) \in \mathcal{F}^{p+3}\} + \mathcal{F}^p \cap D(\mathcal{F}^{p-2})}
\]

Notice that:

\(f \in \mathcal{F}^p\) and \(D(f) \in \mathcal{F}^{p+3} \iff f \in \mathcal{F}^p \cap \text{Ker}(D_3)\)

therefore the natural projection \(\gamma : \mathcal{F}^p \cap \text{Ker}(D_3) \to \mathcal{E}_3^p\) is well defined and we have:

**Lemma 3.1.2.** The induced morphism of graded \(\mathbb{Q}\)-vector spaces:

\(\gamma : \mathcal{F}^p \cap \text{Ker}(D_3) \to \mathcal{E}_3^p\)

is surjective with Kernel: \(\text{Ker}(\gamma) = \mathcal{F}^p \cap \text{Im}(D_3) + \mathcal{F}^{p+1} \cap \text{Ker}(D_3)\).

**Proof.** \(\gamma\) is clearly a surjective morphism of graded \(\mathbb{Q}\) vector-spaces. Let \(f \in \text{Ker}(\gamma) = \{ \varphi \in \mathcal{F}^{p+1}, D(\varphi) \in \mathcal{F}^{p+3}\} + \mathcal{F}^p \cap D(\mathcal{F}^{p-2})\). Write \(f = g + D(h) = D_3(h) + (g + D_{\geq 4}(h))\), where \(g \in \mathcal{F}^{p+1}, D(g) \in \mathcal{F}^{p+3}\) and \(h \in \mathcal{F}^{p-2}\). As \(D_3(h) \in \mathcal{F}^p\) and because \(D_3(f) = 0\) we have \(g + D_{\geq 4}(h) \in \mathcal{F}^{p+1} \cap \text{Ker}(D_3)\). \(\square\)

As a consequence of this Lemma, we have the isomorphisms of graded \(\mathbb{Q}\)-vector spaces:

\[
\mathcal{E}_3^p \cong \frac{\mathcal{F}^p \cap \text{Ker}(D_3)}{\mathcal{F}^p \cap \text{Im}(D_3) + \mathcal{F}^{p+1} \cap \text{Ker}(D_3)} \cong \frac{\mathcal{F}^p \cap \text{Ker}(D_3)}{\mathcal{F}^p \cap \text{Im}(D_3)} \cong \frac{\mathcal{F}^p \cap \text{Ker}(D_3)}{\mathcal{F}^p \cap \text{Im}(D_3)}.
\]

That is:

\[
\mathcal{E}_3^p \cong \frac{H(\mathcal{F}^p, D_3|\mathcal{F}^p)}{H(\mathcal{F}^{p+1}, D_3|\mathcal{F}^{p+1})}.
\]

Hence we deduce the isomorphism of graded algebras:

\[
\mathcal{E}_3^p \cong \oplus_{p \geq 0} \mathcal{E}_{x}^*(\mathbb{Q}, (\Lambda V, d_3))
\]

The same arguments used for \(k = 3\) can be applied term by term to conclude that for any \(k \geq 3\), the first term of the second spectral sequence \(3.0.2\) is:

\[
\mathcal{E}_k^* \cong \mathcal{E}_{x}^*(\mathbb{Q}, (\Lambda V, d_k))
\]
3.2. **Proof of the main results**

In this paragraph we give the proof of our main results Theorem 1.0.2, Theorem 1.0.3 and Theorem 1.0.5.

Recall that the chain map

\[ \text{ev} : \text{Hom}_{(AV,d)}((\Lambda V \otimes \Gamma(sV), D), (\Lambda V, d)) \to (\Lambda V, d) \]

that induce \( ev_{(AV,d)} \) is compatible with filtrations [3.0.1] and [3.0.2] so it is a morphism of filtered cochain complexes.

**Proof of theorem 1.0.2**

*Proof.* We denote as in the introduction \((\Lambda V, d)\) the Sullivan minimal model of \(X\). Since \( \dim(V) < \infty \), \((\Lambda V, d)\) is a Gorenstein graded algebra and there exists a unique \((p,q) \in \mathbb{N} \times \mathbb{N} \) such that

\[ \mathcal{E}xt^q_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)) = E^q_{1.0.2}(\mathbb{Q}, (\Lambda V, d)), \]

with a unique generator (as a \(\mathbb{Q}\)-vector space). So the \(\mathcal{E}_\infty\) term of \([1.0.2]\) is exactly \(\mathcal{E}xt^q_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d))\) and \(r(\Lambda V, d) = p\).

Now the ellipticity of \((\Lambda V, d)\) implies that it is a Poincaré duality space and so \(\dim H^{p+q}(\Lambda V, d) = 1\) with \(N = p + q\) its formal dimension. Consequently the \(\mathcal{E}_\infty\) term of the convergent spectral sequence \([1.0.2]\) is isomorphic to \(H^{p+q}(\Lambda V, d)\).

We deduce then the following commutative diagram (due to the compatibility of filtrations with \(ev_{(AV,d)}\) and the convergence of spectra sequences):

\[
\begin{array}{ccc}
\mathcal{E}xt^q_{\infty} & \xrightarrow{\cong} & \mathcal{E}xt^q_{(AV,d)}(\mathbb{Q}, (AV, d)) \\
\text{ev}_\infty \downarrow & & \downarrow \text{ev}_{(AV,d)} \\
\mathcal{E}xt^q_{\infty} & \cong & H^{p+q}(AV, d)
\end{array}
\]

Recall that \((AV, d)\) is elliptic \(\iff\) \(ev_{(AV,d)} \neq 0\) (cf. [12]), so \(ev_\infty \neq 0\) and then \(E^{p,q}_\infty \neq 0\).

To conclude, by remark 2.3.1.(2) we note that \(e_0(\Lambda V, d)\) is the same for the Milnor-Moore spectral sequence and its generalisation \([1.0.1]\) and since \((AV, d)\) is elliptic, \(e_0(\Lambda V, d) = \text{cat}_2(X)\) \((\mathbb{E})\).

\(\square\)

**Proof of theorem 1.0.3**

*Proof.* First, as in the introduction, denote \(A = C_\ast(\Omega X, d)\), by \(B(A)\) its reduced bar construction and by \(\Omega(A) = T(s^{-1}A^{-})\), the dual of \(B(A)\).

The proof reposes on the following isomorphisms (used in the proof of Theorem 2.1 in \([4]\)):

\[
\begin{align*}
(\text{Hom}_A(A \otimes B(A), A), d) & \xrightarrow{\cong} (\text{End}_{A \otimes B(A)}(A \otimes B(A), [d, ])) \\
& \xrightarrow{\cong} (\text{End}_{(\Omega(A) \otimes A^\vee)}(\Omega(A) \otimes A^\vee), [d^\vee, ])) \\
& \xrightarrow{\cong} (\text{Hom}_{(\Omega(A))}(\Omega(A) \otimes A^\vee, \Omega(A)), d)
\end{align*}
\]

(Recall that \(\varphi_A(f) = (f \otimes id) \circ (id \otimes \Delta_A)\) and the same is for \(\varphi_{(\Omega(A))}\)).

We denote \(B(A)^{\geq q} = \bigoplus_{k \geq q} T^k(s^{-1}A)\) and \((\Omega(A))^{\geq q} = \bigoplus_{k \geq q} T^k(s^{-1}A^\vee)\).

Consider also on \((\text{Hom}_{(\Omega(A))}(\Omega(A) \otimes A^\vee, \Omega(A)), d)\) the filtration given by

\[ \mathcal{F}^q = \{ f \mid f(\Omega(A) \otimes A^\vee) \subseteq \Omega(A)^{\geq q} \}. \]

We have successively for any \(f \in \mathcal{F}^q\):

\[ \varphi_A(f)(A \otimes B(A)) \subseteq A \otimes B(A)^{\geq q}, \]
\[(\vee \circ \varphi_A(f))(\Omega(A) \otimes A^\vee) = (\varphi_A(f))^\vee(\Omega(A) \otimes A^\vee) \subseteq \Omega(A)^{\geq q} \otimes A^\vee.\]

Put \(h = (\varphi_A(f))^\vee\). It is clear that
\[h(\Omega(A) \otimes A^\vee) \subseteq \Omega(A)^{\geq q} \otimes A^\vee \iff \varphi_A^{-1}(h)(\Omega(A) \otimes A^\vee) \subseteq \Omega(A)^{\geq q}.\]

It results that the composite isomorphism of graded complexes:
\[
(Hom_A(A \otimes B(A), A), d) \xrightarrow{\cong} (Hom_{A}(\Omega(A) \otimes A^\vee, \Omega(A), A), d)
\]
is filtration preserving.

Consider now, the Adams-Hilton model \((UL_W, d)\) of \(X\) ([5], Rem. §26 (b)), that is \(UL_W\) is quasi-isomorphic with \(A\). It follows an induced quasi-isomorphism \((\Omega(A), d) \xrightarrow{\cong} (\Omega(UL_W), d)\) preserving wordlength filtrations. On the other hand, the dual of the quasi-isomorphism \(C_*(L_W) \xrightarrow{\cong} B(UL_W)\) ([5], Prop. 22.7) is the preserving filtrations quasi-isomorphism \((\Omega(UL_W), d) \xrightarrow{\cong} (C^*(L_W), d)\). Now \((C^*(L_W), d)\) is a cochain model of \(A_{PL}(X)\), then there is a quasi-isomorphism \((C^*(L_W), d) \xrightarrow{\cong} (AV, d)\). Also since \(X\) is simply connected and of finite type, \(L_W\) is a connected Lie algebra of finite type and then \((C^*(L_W), d)\) is itself a Sullivan algebra. Any morphism of Sullivan algebras is automatically filtration preserving, so the last one is.

Finally remark that the filtration of \(\Omega(A) \otimes A^\vee\) by the \(\Omega(A)\)-submodules \(\Omega(A) \otimes A^\vee_{\leq q}\) exhibits \(\Omega(A) \otimes A^\vee\) as an \(\Omega(A)\)-semifree resolution of \(K\).

Combining this results, we deduce that the tow spectral sequences defined by the filtered differential complexes \((Hom_A(A \otimes B(A), A), d)\) and \((Hom_{AV}(AV, AV, d)\) are isomorphic from the second term. It results that the spectral sequence [10,2] is isomorphic to the Eilenberg-Moore one.

\[\square\]

**Proof of theorem 1.0.5**

**Proof.** As it is noted in the introduction, it suffice to calculate the invariant \(r(AV, d)\) for a minimal model whose \(\text{dim}(V) < \infty\) and the differential being pure and homogeneous of a certain degree \(k\). Since \(\text{dim}(V) < \infty\), \((AV, d)\) is a Gorenstein algebra, i.e. \(\text{dim}_Q(\text{Ext}^*_AV(d), Q, (AV, d))) = 1\). On the other hand since the differential is pure, there exists \([10] h : (AV \otimes \Gamma_V, D) \to (AV, d)\), a \((AV, d)\)-morphism with \([h(1)] = ev_{AV}(d)\) represents the top cohomology class on \((AV, d)\) [11].

Denote by \(\omega\) the cycle representing this top cohomology class and let \(x_1, x_2, \ldots, x_n\) and \(y_1, y_2, \ldots, y_m\) the generators of \(V^{even}\) and \(V^{odd}\) respectively. First, note that \(\omega\) is a cycle whose length is \(p = m + n(k - 2)\) [11]. Now any element, \([\varphi]\) in \(\text{Ext}^*_AV(d))Q, (AV, d) = Im(ev_{AV, d})\) (since they have the same dimension) is such that \(\varphi(1) = \omega + d(\omega')\).

For any \(x_i\) \((1 \leq i \leq n)\) and \(y_j\) \((1 \leq j \leq m)\), we have \(\varphi(x_i) = x_i\varphi(1)\) and \(\varphi(y_j) = y_j\varphi(1)\).

We discuss then tow cases:

1. \(\text{case: } d(\omega') \in \Lambda^{\geq p+1}V\).

   Since \(d\) is pure, we have \(D(sx_i) = x_i\) and \(D(sy_j) = y_j + s(dy_j)\) and so, \(d(\varphi(x_i)) = \varphi(D(sx_i)) = \varphi(x_i) = x_i(\omega + d(\omega')) \in \Lambda^{\geq p+1}V\). This implies that \(\varphi(sx_i) \in \Lambda^{\geq (p+1)-k-2}V\).

   Also, \(d(\varphi(sy_j)) = \varphi(D(sy_j)) = \varphi(y_j + s(dy_j)) = y_j(\omega + d(\omega')) + \varphi(s(dy_j))\).
But since
\[ dy_j = \sum_{j_1, \ldots, j_k} x_{j_1} x_{j_2} \cdots x_{j_k} \in \Lambda^k V \]
(with \( j_1 < j_2 < \cdots < j_k \)), it is clear that
\[ s(dy_j) = \frac{1}{k} \sum_{j_1, \ldots, j_k} \sum_{l=1}^{l=k} x_{j_1} x_{j_2} \cdots x_{j_{l-1}} (sx_{j_l}) x_{j_{l+1}} \cdots x_{j_k}. \]

Therefore
\[ \varphi(s(dy_j)) = \frac{1}{k} \sum_{j_1, \ldots, j_k} \sum_{l=1}^{l=k} x_{j_1} x_{j_2} \cdots x_{j_{l-1}} x_{j_{l+1}} \cdots x_{j_k} \varphi(sx_{j_l}). \]

We deduce that \( d(\varphi(sy_j)) \in \Lambda^{p+1} V \) and then \( \varphi(sy_j) \in \Lambda^{p-k+2} V \) also. Consequently, for such \( \varphi \), \( \varphi(\Gamma(sV)) \subseteq \Lambda^{p-k+2} V \).

(2) case: \( d(\omega') \in \Lambda^q V \), with \( q < p \), the same argument gives (for such \( \varphi \)) \( \varphi(\Gamma(sV)) \subseteq \Lambda^{p-k+2} \). This case is then dominated (by definition of the invariant \( r(\Lambda(V, d)) \)) by the first one.

We conclude then that: \( r(\Lambda(V, d)) = p - k + 2 = m + (n-1)(k-2) \).

\( \square \)

3.3. Final remark.

(1) With the same notations as in the last proof, we remark that for any \( [\varphi] \) in \( \mathcal{E}xt^*_r(\mathbb{Q}, (\Lambda V, d)) \) (the differential being pure and homogeneous of degree \( k \)), \( \varphi(1) \in \Lambda^p V \), while \( \varphi(\Gamma^+(sV)) \subseteq \Lambda^{p-k+2} V \). Now for a minimal Sullivan algebra \( (\Lambda V, d) \), with \( d : V \to \Lambda^2 V \) and \( \text{dim}(V) < \infty \), using the convergence of \( 2.4.2 \) for the unduced model \( (\Lambda V, d_k) \) and remark 2.1.1, we obtain a generator \( \varrho \) for \( \mathcal{E}xt^*_r(\mathbb{Q}, (\Lambda V, d_k)) \) such that \( \varrho(1) \in \Lambda^p V \). Also, using the convergence of \( 1.0.2 \) we obtain a generator \([\vartheta]\) of \( \mathcal{E}xt^*_r(\mathbb{Q}, (\Lambda V, d)) \), such that \( \vartheta(1) \in \Lambda^p V \). As a consequence, if in addition, we suppose \( (\Lambda V, d) \) elliptic, we deduce that \( e_0(\Lambda V, d) \geq p = m + n(k - 2) \), so we recover the result of S. Ghorbal and B. Jessup (6, cor.3 and 13, Rem 2.4).

On the other hand, the determination of \( \varrho(\Gamma^+(sV)) \) and \( \vartheta(\Gamma^+(sV)) \), depends on the calculation of the images \( \varrho(D(sx_i)) \) and \( \vartheta(D(sx_i)) \) (resp \( \vartheta(D(sy_i)) \) and \( \vartheta(D(sy_j)) \)) which seems more difficult. The following question is natural.

\textit{Question:} For a minimal Sullivan algebra \( (\Lambda V, d) \), with \( d : V \to \Lambda^2 V \) and \( \text{dim}(V) < \infty \), is \( r(\Lambda V, d) \geq m + (n-1)(k-2) \)?

(2) Let \( K \) a field of \( \text{char}(K) = \rho \neq 2 \) and \( X \) an \( r \)-connected \( K \)-elliptic, finite CW-complex, such that \( \text{dim}(X) < r \rho \), let \( (\Lambda V, d) \) its minimal model (cf. Preliminary). By the same argument as in the rational case, we have
\[ e_K(X) \geq \text{dim}(V^{\text{odd}}) + (k-2)(\text{dim}(V^{\text{even}}) - 1). \]
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