Blackbody Radiation in $q$-deformed Statistics

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Abstract

More general canonical ensemble which gives rise the generalized statistics or $q$-deformed statistics can represent the realistic scenario than the ideal one, with proper parameter sets involved. We study the Planck’s law of blackbody radiation, Wein’s and Rayleigh-Jeans radiation formulae from the point of view of $q$-deformed statistics. We find that the blackbody energy spectrum curve for a given temperature $T$ corresponding to different $q$ values differs from each other: the location of the peak(i.e. $\nu_m$) of the energy distribution $u_\nu$ (corresponding to different $q$ ) shifted towards higher $\nu$ for higher $q$. From the $q$-deformed Wein’s displacement law, we find that $\lambda_m T$ varies from 0.0029 m K to 0.0017 m K as the deformation parameter $q$ varies from 1.0(undeformed) to 1.1(deformed).

Keywords: $q$-deformed statistics, blackbody radiation, Rayleigh-Jeans law, Wein’s displacement law.
I. INTRODUCTION

Statistical mechanics, the art of turning the microscopic laws of physics into a description of nature on a macroscopic scale, has been proved to be a very powerful tools in various domains over the last century. It has been successfully used not only in different branch of physics (e.g. condensed matter physics, high energy physics etc.), but in different areas e.g. share price dynamics, traffic control, hydroclimatic fluctuation etc. Results predicted by the statistical method are found to be in good agreement with the experiments. The important fact is that, without the detailed knowledge of each and every microstate of the concerned system, we can predict the macroscopic properties of the system. Things got easier as the connection has been build up between the statistical average of the microscopic properties and the macrostates.

Attempts have been made to generalize this connection mentioned above in recent years [1][2]. Recently, the techniques of the generalized statistics (popularly known as superstatistics or \( q \)-deformed (Tsallis) statistics) have been applied to a large class of complex systems e.g. hydrodynamic turbulence, defect turbulence, share price dynamics, random matrix theory, random networks, wind velocity fluctuations, hydroclimatic fluctuations, the statistics of train departure delays and models of the metastatic cascade in cancerous systems [3][9]. In this approach, the key parameter is the inverse temperature parameter \( \beta (= 1/k_BT) \) which exhibits fluctuations on a large time scale. One can model these type of complex systems by a kind of superposition of ordinary statistical mechanics with varying temperature parameters, which in short called superstatistics or deformed statistics. The stationary distributions of deformed/superstatistical systems differs from the usual Boltzmann-type statistical mechanics and can exhibit asymptotic power laws or other functional forms in the energy \( E \) [10][12].

By using the non-extensive statistical methods we can incorporate the fact of temperature fluctuations and the proceeding sections are devoted mainly for an attempt to give more insight on it [13][15]. This approach deals with the fluctuation(deformation) parameter \( q \) which corresponds to the degree of the temperature fluctuation effect to the concerned system. In this formalism we can treat our normal Boltzmann-Gibbs statistics as a special case of this generalized one, where temperature fluctuation effects are negligible, corresponds to \( q = 1.0 \) (un-deformed statistics). More deviation from \( q = 1 \) denotes a system with more
fluctuating temperature [16][18].

II. TEMPERATURE FLUCTUATION AND THE MODIFIED ENTROPY

The phenomena of temperature fluctuation can be interpreted physically as the deformation of an ideal canonical ensemble to a more realistic case. Ideal canonical ensemble is supposed to be the statistical ensemble that represents the possible states of a mechanical system in thermal equilibrium with a heat bath at a fixed temperature (say T). Consequently each and every cell (very small identical portions) of the system will be at temperature T. To make it more realistic we can think about a modified canonical system which is in thermal equilibrium with a heat bath at a fixed temperature T but there will be a small variation in temperature in different cells (say T − δT to T + δT) though the average temperature of the system will be T still.

A connection between the entropy (s) and the number of microstates (Ω) of a system can be derived using intuition mainly. We know the entropy is the measure of the number of microscopic configuration, i.e., the degree of randomness of a system. The only thing we can infer clearly is that, they both (s and Ω) will increase (or decrease) together. We can assume $s = f(Ω)$. Further we know, entropy is additive and the number of microstates is multiplicative. These will lead to the form $s = k_B \ln Ω$.

More general assumption can be made which will deform the fundamental connection between s and Ω as follows

$$s = f(Ω^q)$$

(1)

This assumption with $q > 0$ on the first hand will lead to the deformed/generalized statistics. The generalized entropy will take the following form.

$$s_q = k_B \ln_q Ω$$

(2)

where the generalized log function is defined as

$$\ln_q Ω = \frac{Ω^{1-q} - 1}{1 - q} = x, \text{ (say)}$$

(3)

and consequently the generalized exponential function becomes

$$e_q^x = [1 + (1 - q)x]^{1/(1-q)} \longrightarrow e^x \text{ as } q \to 1$$

(4)
Extremizing $S_q$ subject to suitable constraints yields more general canonical ensembles, where the probability to observe a microstate with energy $\epsilon_i$ is given by:

$$p_i = \frac{e^{-\beta\epsilon_i}}{z} = \frac{1}{z} \left[ 1 - (1 - q)\beta\epsilon_i \right]^{\frac{1}{1-q}}$$  \hspace{1cm} (5)

with partition function $z$ and inverse temperature parameter $\beta = \frac{1}{k_B T}$.

This generalized modification can be related to the temperature fluctuation of the system. One can show the connection by expanding the deformed/generalized exponential function as follows:

$$e^x_q = \left[ 1 + (1 - q)x \right]^{\frac{1}{1-q}}$$
$$= 1 + x + q\frac{x^2}{2!} + q(2q - 1)\frac{x^3}{3!} + q(2q - 1)(3q - 1)\frac{x^4}{4!} + \cdots$$  \hspace{1cm} (6)

where, $x = -\beta\epsilon_i = -\frac{\epsilon_i}{k_B T}$.

The $q$ factors in the expansion can be absorbed in $T$ and those will account for the temperature fluctuation of the system, whereas, $q = 1.0$ will lead us back to the normal Boltzmann-Gibbs statistics i.e., the case of zero or negligible temperature fluctuation.

\section*{III. EFFECTS OF $q$-DEFORMED STATISTICS IN BLACKBODY RADIATION}

In the last section we have seen that we can use the generalized statistical mechanics wherever the system is subjected to the temperature fluctuation. If the temperature fluctuation effect is not negligible enough to disclose itself, then definitely there will be some deviation from the ideal phenomena.

Ideal blackbody radiation (at constant temperature $T$) spectrum, which do not suffer any temperature fluctuation, unfortunately is not available in nature. If we want to use the blackbody radiation formula to determine something in our real world, e.g. the surface temperature of a star from it’s spectrum analysis, we can use $q$-deformed statistics and it’s impact on the Planck’s law of blackbody radiation. The generalized (or $q$-deformed) distribution function takes the following form for small deformation, i.e., small $|1 - q|$ (see Appendix, Eq.(29))

$$f_q = \frac{1}{(e^{-h\nu/k_B T})^{1-q} - 1} = \frac{1}{\left[ 1 - (1 - q)\frac{h\nu}{k_B T} \right]^{\frac{1}{1-q}} - 1}$$  \hspace{1cm} (7)
Number of photons in the frequency interval $\nu$ and $\nu + d\nu$ and volume $V$

$$dN = \frac{8\pi V}{c^3} \cdot \frac{\nu^2 d\nu}{\left[1 - (1 - q) \frac{h\nu}{k_B T}\right]^{\frac{q}{q-1}} - 1}$$  \hspace{1cm} (8)$$

The radiation energy corresponding to photons of frequency lying between $\nu$ and $\nu + d\nu$ is

$$dE (= h\nu \times dN) = \frac{8\pi hV}{c^3} \cdot \frac{\nu^3 d\nu}{\left[1 - (1 - q) \frac{h\nu}{k_B T}\right]^{\frac{q}{q-1}} - 1}$$  \hspace{1cm} (9)$$

The distribution of the Photon energy density per unit volume is given by (see Appendix, Eq.(29))

$$u_q(\nu)d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3 d\nu}{\left[1 - (1 - q) \frac{h\nu}{k_B T}\right]^{\frac{q}{q-1}} - 1}$$  \hspace{1cm} (10)$$

and this is the $q$-deformed blackbody radiation formula.

Below in Fig. 1 we have plotted the blackbody radiation curves for different temperature without invoking temperature fluctuation i.e. we set $q = 1.0$. In Fig. 2 we have plotted

![Blackbody Radiation Curve for different temperature](image)

**FIG. 1: Spectrum of a Blackbody at different temperatures in undeformed statistics ($q = 1.0$).**

the $q$-deformed energy density $u_{\nu}$ (in units of $J m^{-3} s$ (obtained from Eq.[10]) against $\nu$ (in units of Tera Hertz(THz)) for different $q$ values (ranging from $q = 1$ to $q = 1.1$) and constant temperature $T \sim 6000K$ (the surface temperature of the sun-like star). Also we find that as $q$ increases, not only the location (i.e. $\nu_m$) of the peak of the energy distribution $u_{\nu}$ shifted
towards higher \( \nu \), the height of the peak also increases (indicates higher rate of radiation). In Table I we have shown the value of the frequency \( (\nu_m) \) at which \( u_q(\nu) \) is maximum for different \( q \) and also the maximum value of the \( q \)-generalized energy density.

| \( q \) | \( \nu_m \) (THz) | \( u_{q_{\nu}}(\nu)_{\text{max}} \) (J m\(^{-3}\) s) |
|-------|---------------|------------------|
| 1.0   | 352           | 171 \( \times \) \( 10^{-17} \) |
| 1.05  | 395           | 183 \( \times \) \( 10^{-17} \) |
| 1.10  | 446           | 198 \( \times \) \( 10^{-17} \) |

The radiation curves corresponding to different \( q \) values for a given \( T \) appears to be different radiation curves corresponding to different \( T \) values for \( q = 1 \) (undeformed case).

**A. Generalized (\( q \)-deformed) Wien’s Law and Rayleigh-Jeans Law**

We next study \( q \)-deformed blackbody radiation formula and its different limits.

**\( q \)-deformed Wein’s law:** In the limit of low temperature \( T \) or high frequency \( \nu \) (i.e. \( x = \frac{h\nu}{k_B T} \gg 1 \)), we find

\[
   u_q(\nu) = \frac{8\pi h}{c^3} \frac{\nu^3}{\left[1 - (1 - q) \frac{h\nu}{k_B T}\right]^\frac{q}{q-1}}
\]  

(11)
This is the generalized Wien’s law, which reduces the usual (undeformed) Wein’s law

\[ u(\nu) = \frac{8\pi\hbar}{c^3 \nu^3} e^{-\hbar\nu/k_BT} \] (12)

in the limit \( q \to 1 \).

**q-deformed Rayleigh-Jeans Law:** In the limit of high temperature \( T \) or low frequency \( \nu \) (i.e. \( x = \frac{\hbar\nu}{k_BT} \ll 1 \)), we find

\[ u_q(\nu) = \frac{8\pi(k_BT)^3}{\hbar^2 c^3} \cdot \frac{x^3}{qx + \frac{q}{2!}x^2 + \frac{q(2-q)}{3!}x^3 + \cdots} \] (13)

which is the generalized Rayleigh-Jeans law. This reduces to the usual (undeformed) Rayleigh-Jeans law

\[ u(\nu) = \frac{8\pi k_BT \nu^2}{c^3} \] (14)

in the limit \( q \to 1 \).

**FIG. 3:** Spectrum of a Blackbody along with the Wien’s Law and Rayleigh-Jeans Law at temperature 6000 K in q-deformed statistics.

In Fig. 3, we have shown the generalized Wien’s law and Rayleigh-Jeans Law for different amount of q-deformation using Eqs.(11) and (13) respectively along with the blackbody radiation curves obtained from Eq.(10).

Plots corresponding to the undeformed cases (\( q = 1 \)) are also shown in the same plot. We find that as \( q \) varies from 1.0 to 1.1, plots are getting well seperated at high and low frequency.
region, respectively. Also in addition, for the Planck’s blackbody radiation formula, the location of the peak (at which the radiation density is maximum) gets shifted towards high frequency value as \( q \) increases from 1.0 to 1.1.

**B. Generalized Stefan’s law of radiation**

By integrating the frequencies for all values, we find the energy emitted by the blackbody per unit time (i.e. power) per unit surface area as

\[
P_q = \int_0^\infty d\nu u_q(\nu) = \frac{8\pi k_B^4}{h^3 c^3} \cdot T^4 \cdot \int_0^\infty \frac{x^3 \, dx}{[1 - (1 - q)x]^\frac{q}{q-1} - 1} = \frac{4}{c} \cdot \sigma_q \cdot T^4	ag{15}
\]

This is the generalized or \( q \)-deformed form of the Stefan’s law of blackbody radiation. Here \( \sigma_q \) is the \( q \)-deformed Stefan’s constant. Note that \( \sigma_{q=1.05} \approx 7.6 \times 10^{-8} \text{ J sec}^{-1} \text{m}^{-2} \text{K}^{-4} \), \( \sigma_{q=1.1} \approx 1.1 \times 10^{-7} \text{ J sec}^{-1} \text{m}^{-2} \text{K}^{-4} \) in deformed scenario. In the undeformed scenario, it becomes \( \sigma_{q=1.0} \approx 5.7 \times 10^{-8} \text{ J sec}^{-1} \text{m}^{-2} \text{K}^{-4} \).

**C. Generalized Wien’s Displacement Law**

In the \( q \)-deformed formalism, the photon energy density per unit volume is given by

\[
u_q(\lambda) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{\left[1 - (1 - q)^{\frac{hc}{\lambda k_BT}}\right]^\frac{q}{q-1} - 1}	ag{16}
\]

Wien’s displacement law states that the black body radiation curve for different temperatures peaks at a wavelength inversely proportional to the temperature. Let \( \lambda_m \) be the wavelength at which \( u_q(\lambda) \) is maximum at temperature \( T \), i.e.

\[
\left[\frac{du_q(\lambda)}{d\lambda}\right]_{\lambda=\lambda_m} = 0
\]

Using Eqs.(16) and (17) we obtain

\[
\frac{8\pi hc}{\lambda_m^6} \left(\frac{1}{e_q^{-\frac{hc}{\lambda_m k_BT}}}ight)^{-q} - 1 \left\{\frac{hc}{\lambda_m k_BT} \left(\frac{e_q^{-\frac{hc}{\lambda_m k_BT}}}{e_q^{-\frac{hc}{\lambda_m k_BT}}}\right)^{-q} - 1\right\} = 0
\]

This equation(Eq.(18)) leads to

\[
y + \frac{1}{q} e^{-y} - \frac{1}{q} \left(e^{-y}\right)^{1-q} = 0 \quad \text{with} \quad y = \frac{hc}{\lambda_m k_BT}	ag{19}
\]
which is the $q$-deformed version of Wein’s displacement law. In the limit $q \to 1.0$ (no temperature fluctuation) it becomes

$$\frac{y}{5} + e^{-y} - 1 = 0$$ (20)

which is the normal (undeformed) version of Wein’s displacement law. Below in Fig. [4] we have shown the variation of $\lambda_m T$ with the deformation parameter $q$. Clearly, the value of $\lambda_m T$ is found to be decreasing with increasing $q$. This will result in a shift of the peak of the blackbody spectrum in the direction of decreasing wavelength $\lambda$ (increasing frequency $\nu$) for a fixed temperature. In Fig. [5] we have plotted the blackbody radiation energy density as a function of the wavelength corresponding to the fixed temperature 6000 K for different $q$. Shift in $\lambda_m$ clearly can be seen from the plot. In Table II, we have shown the solution of Eq. [19] and the corresponding $\lambda_m T$ for different $q$ values. We see that as $q$ increases the value of $\lambda_m$ (corresponding to the peak energy density) decreases, while the height of the peak increases.

| $q$  | $\frac{y}{5}$ | $\lambda_m T$ (m K) |
|------|---------------|---------------------|
| 0.95 | 4.126         | $3.5 \times 10^{-3}$|
| 1.0  | 4.965         | $2.9 \times 10^{-3}$|
| 1.05 | 6.156         | $2.3 \times 10^{-3}$|
| 1.10 | 8.143         | $1.7 \times 10^{-3}$|
IV. CONCLUSION

We study the Planck’s law of blackbody radiation and its high frequency limit (Wein’s law) and low frequency limit (Rayleigh-Jeans law) from the point of view q-deformed (Tsallis). The radiation curves corresponding to different q values for a given T appears to be different radiation curves corresponding to different T values for q = 1 (undeformed case). The location i.e. νm of the peak of the energy distribution uν (corresponding to different q) shifted towards higher ν. In the case of Wein’s law, we find that λmT varies from 0.0029 m K to 0.0017 m K as the deformation parameter q varies from 1.0 to 1.1.

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APPENDIX

Indicial properties of q-deformed exponential function for small deformation

From Eq.(4), keeping only first order in (1 − q),

\[ e_q^a \cdot e_q^b = [1 + (1 - q)a]^{1 - q/1} \cdot [1 + (1 - q)b]^{1 - q/1} \]
\[ = [1 + (1 - q)(a + b) + (1 - q)^2 ab]^{1 - q/1} \]
\[ \approx e_q^{a+b} \]

(21)

Similarly, neglecting higher order terms we get (for small |1 − q|),

\[ (e_q^a)^b = [1 + (1 - q)a]^{b/1 - q} \]
\[ = \left[ 1 + (1 - q)ab + \frac{b(b - 1)}{2!} (1 - q)^2 a^2 + \cdots \right]^{1/1 - q} \]
\[ \approx e_q^{ab} \]

(22)

The energy density of the oscillators having frequency in the range ν to ν + dν

\[ u_q(\nu) = \frac{g(\nu)}{V} \langle E(\nu) \rangle = \frac{8\pi\nu^2}{c^3} \langle E(\nu) \rangle \]

(23)
Now the average energy per oscillator is defined as (in $q$-deformed scenario) \[1, 19, 20\]

$$\langle E(\nu) \rangle = \frac{\sum_i \epsilon_i p_i^q}{\sum_i p_i^q}$$ \hfill (24)

where the factor $\sum_i p_i^q$ stands for the normalization. Writing $x = \frac{\hbar \nu}{k_B T}$ we get

$$\langle E(\nu) \rangle = \hbar \nu \frac{\sum_{n=0}^{\infty} n \left( e^{-nx} \right)^q}{\sum_{n=0}^{\infty} \left( e^{-nx} \right)^q}$$ \hfill (25)

Now in the small deformation approximation (i.e., small $|1 - q|$) using Eq. (22)

$$\left( e^{-nx} \right)^q \approx \left( e^{-x} \right)^{qn}$$ \hfill (26)

Putting this back to the Eq. (25) we get

$$\langle E(\nu) \rangle = \hbar \nu \frac{\sum_{n=0}^{\infty} n \left( e^{-x} \right)^{qn}}{\sum_{n=0}^{\infty} \left( e^{-x} \right)^{qn}} = \frac{\hbar \nu}{\left( e^{-x} \right)^{-q} - 1}$$ \hfill (27)

where we used the fact that,

$$\sum_{0}^{\infty} ny^n = \frac{y}{(1 - y)^2} \quad \text{and} \quad \sum_{0}^{\infty} y^n = \frac{1}{1 - y}$$ \hfill (28)

This will give the following compact form for the $q$-deformed version of the energy density

$$u_q(\nu) = \frac{8\pi \hbar \nu^3}{c^3} \frac{1}{\left( e^{-x} \right)^{-q} - 1}$$ \hfill (29)

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