Some normed binomial difference sequence spaces related to the $\ell_p$ spaces

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Abstract
The aim of this paper is to introduce the normed binomial sequence spaces $b^\ell_p(\nabla)$ by combining the binomial transformation and difference operator, where $1 \leq p \leq \infty$. We prove that these spaces are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively. Furthermore, we compute Schauder bases and the $\alpha$-, $\beta$- and $\gamma$-duals of these sequence spaces.

Keywords: sequence space; matrix domain; Schauder basis; $\alpha$-, $\beta$- and $\gamma$-duals

1 Introduction and preliminaries
Let $w$ denote the space of all sequences. By $\ell_p$, $\ell_\infty$, $c$ and $c_0$, we denote the spaces of $p$-absolutely summable, bounded, convergent and null sequences, respectively, where $1 \leq p < \infty$. Let $Z$ be a sequence space, then Kizmaz [1] introduced the following difference sequence spaces:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the set of positive integers. Since then, many authors have studied further generalization of the difference sequence spaces [2–6]. Moreover, Altay and Polat [7], Başarir and Kara [8–12], Kara [13], Kara and İlkhan [14], Polat and Başar [15], and many others have studied new sequence spaces from a matrix point of view that represent difference operators.

For an infinite matrix $A = (a_{n,k})$ and $x = (x_k) \in w$, the $A$-transform of $x$ is defined by $Ax = ((Ax)_n)$ and is supposed to be convergent for all $n \in \mathbb{N}$, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k}x_k$.

For two sequence spaces $X, Y$ and an infinite matrix $A = (a_{n,k})$, the sequence space $X_A$ is defined by $X_A = \{x = (x_k) \in w : Ax \in X \}$, which is called the domain of matrix $A$ in the space $X$. By $(X : Y)$, we denote the class of all matrices such that $X \subseteq Y_A$.

The Euler means $E^r$ of order $r$ is defined by the matrix $E^r = (e^r_{n,k})$, where $0 < r < 1$ and

$$e^r_{n,k} = \begin{cases} \binom{n}{k} (1 - r)^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$
The Euler sequence spaces \( e_p^r \) and \( e_{\infty}^r \) were defined by Altay, Başar and Mursaleen [16] as follows:

\[
e_p^r = \left\{ x = (x_k) \in w : \sum_{n} \left| \sum_{k=0}^{n} \left( \frac{n}{k} \right) (1-r)^{n-k} r^k x_k \right|^p < \infty \right\}
\]

and

\[
e_{\infty}^r = \left\{ x = (x_k) \in w : \sup_{n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (1-r)^{n-k} r^k x_k \right| < \infty \right\}
\]

Altay and Polat [7] defined further generalization of the Euler sequence spaces \( e_0^r(\nabla) \), \( e_c^r(\nabla) \) and \( e_{\infty}^r(\nabla) \) by

\[
e_0^r(\nabla) = \left\{ x = (x_k) \in w : (\nabla x_k) \in e_0^r \right\},
\]

\[
e_c^r(\nabla) = \left\{ x = (x_k) \in w : (\nabla x_k) \in e_c^r \right\}
\]

and

\[
e_{\infty}^r(\nabla) = \left\{ x = (x_k) \in w : (\nabla x_k) \in e_{\infty}^r \right\}
\]

where \( \nabla x_k = x_k - x_{k-1} \) for each \( k \in \mathbb{N} \). Here any term with negative subscript is equal to naught. Many authors have used especially the Euler matrix for defining new sequence spaces, for instance, Kara and Başarır [17], Karakaya and Polat [18] and Polat and Başar [15].

Recently Bişgin [19, 20] defined another type of generalization of the Euler sequence spaces and introduced the binomial sequence spaces \( b_p^{rs} \), \( b_c^{rs} \), \( b_{\infty}^{rs} \) and \( b_{\infty}^{r}p \). Let \( r, s \in \mathbb{R} \) and \( r + s \neq 0 \). Then the binomial matrix \( B^{rs} = (b_n^{rs})_{n,k} \) is defined by

\[
b_n^{rs}_{n,k} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases}
\]

for all \( k, n \in \mathbb{N} \). For \( sr > 0 \) we have

(i) \( \| B^{rs} \| < \infty \),

(ii) \( \lim_{n \to \infty} b_n^{rs}_{n,k} = 0 \) for each \( k \in \mathbb{N} \),

(iii) \( \lim_{n \to \infty} \sum_{k} b_n^{rs}_{n,k} = 1 \).

Thus, the binomial matrix \( B^{rs} \) is regular for \( sr > 0 \). Unless stated otherwise, we assume that \( sr > 0 \). If we take \( s + r = 1 \), we obtain the Euler matrix \( E^r \). So the binomial matrix generalizes the Euler matrix. Bişgin [20] defined the following spaces of binomial sequences:

\[
b_p^{rs} = \left\{ x = (x_k) \in w : \sum_{n} \left| \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}
\]

and

\[
b_{\infty}^{rs} = \left\{ x = (x_k) \in w : \sup_{n} \left| \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}
\]
The main purpose of the present paper is to study the normed difference spaces $b^s_p(\nabla)$ and $b^\infty_s(\nabla)$ of the binomial sequence whose $B^s(\nabla)$-transforms are in the spaces $\ell_p$ and $\ell_\infty$, respectively. These new sequence spaces are the generalization of the sequence spaces defined in [7] and [20]. Also, we compute the bases and $\alpha$-, $\beta$- and $\gamma$-duals of these sequence spaces.

2 The binomial difference sequence spaces

In this section, we introduce the spaces $b^s_p(\nabla)$ and $b^\infty_s(\nabla)$ and prove that these sequence spaces are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively.

We first define the binomial difference sequence spaces $b^s_p(\nabla)$ and $b^\infty_s(\nabla)$ by

$$b^s_p(\nabla) = \{ x = (x_k) \in w : (\nabla x_k) \in b^s_p \}$$

and

$$b^\infty_s(\nabla) = \{ x = (x_k) \in w : (\nabla x_k) \in b^\infty_s \}.$$ 

Let us define the sequence $y = (y_n)$ as the $B^s(\nabla)$-transform of a sequence $x = (x_k)$, that is,

$$y_n = \left[ B^s(\nabla x_k) \right]_n = \frac{1}{(s + r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (\nabla x_k)$$

for each $n \in \mathbb{N}$. Then the binomial difference sequence spaces $b^s_p(\nabla)$ or $b^\infty_s(\nabla)$ can be redefined by all sequences whose $B^s(\nabla)$-transforms are in the space $\ell_p$ or $\ell_\infty$.

**Theorem 2.1** The sequence spaces $b^s_p(\nabla)$ and $b^\infty_s(\nabla)$ are complete linear metric spaces with the norm defined by

$$f_{b^s_p(\nabla)}(x) = \|x\|_p = \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}}$$

and

$$f_{b^\infty_s(\nabla)}(x) = \|x\|_\infty = \sup_{n \in \mathbb{N}} |y_n|,$$

where $1 \leq p < \infty$ and the sequence $y = (y_n)$ is defined by the $B^s(\nabla)$-transform of $x$.

**Proof** The proof of the linearity is a routine verification. It is obvious that $f_{b^s_p(\nabla)}(\alpha x) = |\alpha| f_{b^s_p(\nabla)}(x)$ and $f_{b^s_p(\nabla)}(x) = 0$ if and only if $x = \theta$ for all $x \in b^s_p(\nabla)$, where $\theta$ is the zero element in $b^s_p$ and $\alpha \in \mathbb{R}$. We consider $x, z \in b^s_p(\nabla)$, then we have

$$f_{b^s_p(\nabla)}(x + z) = \left( \sum_{n} \left( B^s_p(\nabla (x_k + z_k)) \right)_n \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{n} \left( B^s_p(\nabla x_k) \right)_n \right)^{\frac{1}{p}} + \left( \sum_{n} \left( B^s_p(\nabla z_k) \right)_n \right)^{\frac{1}{p}} = f_{b^s_p(\nabla)}(x) + f_{b^s_p(\nabla)}(z).$$
Hence $f_{b_p^r(\nabla)}$ is a norm on the space $b_p^r(\nabla)$.

Let $(x_m)$ be a Cauchy sequence in $b_p^r(\nabla)$, where $x_m = (x_{m,n})_{n=1}^\infty \in b_p^r(\nabla)$ for each $m \in \mathbb{N}$. For every $\varepsilon > 0$, there is a positive integer $m_0$ such that $f_{b_p^r(\nabla)}(x_{m,n} - x_l) < \varepsilon$ for $m, l \geq m_0$.

Then we get

$$|(B^r(\nabla x_{m,n} - x_k))|_p \leq \left( \sum_{n=0}^{\infty} (B^r(\nabla x_{m,n} - x_k))_n^p \right)^{\frac{1}{p}} < \varepsilon$$

for $m, l \geq m_0$ and each $k \in \mathbb{N}$. So $(B^r(\nabla x_{m,n}))_{m=1}^\infty$ is a Cauchy sequence in the set of real numbers $\mathbb{R}$. Since $\mathbb{R}$ is complete, we have $\lim_{m \to \infty} B^r(\nabla x_{m,n}) = B^r(\nabla x_k)$ for each $k \in \mathbb{N}$. We compute

$$\sum_{n=0}^{i} (B^r(\nabla x_{m,n} - x_k))_n \leq f_{b_p^r(\nabla)}(x_m - x_l) < \varepsilon$$

(2.2)

for $m > m_0$. We take $i$ and $l \to \infty$, then the inequality (2.2) implies that

$$f_{b_p^r(\nabla)}(x_m - x_l) \to 0.$$ We have

$$f_{b_p^r(\nabla)}(x) \leq f_{b_p^r(\nabla)}(x_m - x) + f_{b_p^r(\nabla)}(x_m) < \infty,$$

that is, $x \in b_p^r(\nabla)$. Thus, the space $b_p^r(\nabla)$ is complete. For the space $b_p^r(\nabla)$, the proof can be completed in a similar way. So, we omit the detail.

**Theorem 2.2** The sequence spaces $b_p^r(\nabla)$ and $b_p^r(\nabla)$ are linearly isomorphic to the spaces $\ell_p$ and $\ell_\infty$, respectively, where $1 \leq p < \infty$.

**Proof** Similarly, we only prove the theorem for the space $b_p^r(\nabla)$. To prove $b_p^r(\nabla) \cong \ell_p$, we must show the existence of a linear bijection between the spaces $b_p^r(\nabla)$ and $\ell_p$.

Consider $T : b_p^r(\nabla) \to \ell_p$ by $T(x) = B^r(\nabla x_k)$. The linearity of $T$ is obvious and $x = \theta$ whenever $T(x) = \theta$. Therefore, $T$ is injective.

Let $y = (y_n) \in \ell_p$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k} (s + r)^i \sum_{j=i}^{k} \binom{i}{j} r^{-j} (-s)^{j-i} y_i$$

(2.3)

for each $k \in \mathbb{N}$. Then we have

$$f_{b_p^r(\nabla)}(x) = \|B^r(\nabla x_k)\|_p$$

$$= \left( \sum_{n=1}^{\infty} \left( \frac{1}{(s + r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} k^p (\nabla x_k) \right)^p \right)^{\frac{1}{p}}$$

$$= \left( \sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}} = \|y\|_p < \infty,$$
which implies that $x \in b_p^\infty(\mathbb{V})$ and $T(x) = y$. Consequently, $T$ is surjective and is norm preserving. Thus, $b_p^\infty(\mathbb{V}) \cong \ell_p$. □

3 The Schauder basis and $\alpha$-, $\beta$- and $\gamma$-duals

For a normed space $(X, \| \cdot \|)$, a sequence $\{x_k : x_k \in X\}_{k \in \mathbb{N}}$ is called a Schauder basis [21] if for every $x \in X$, there is a unique scalar sequence $(\lambda_k)$ such that $\|x - \sum_{k=0}^{n} \lambda_k x_k\| \to 0$ as $n \to \infty$. Next, we shall give a Schauder basis for the sequence space $b_p^\infty(\mathbb{V})$.

We define the sequence $g^{(k)}(r, s) = \{g^{(k)}_i(r, s)\}_{i \in \mathbb{N}}$ by

$$g^{(k)}_i(r, s) = \begin{cases} 0 & \text{if } 0 \leq i < k, \\ (s + r)^k \sum_{j=0}^{i-k} \binom{i}{j} r^{-j}s^{i-j-k} & \text{if } i \geq k, \end{cases}$$

for each $k \in \mathbb{N}$.

**Theorem 3.1** The sequence $(g^{(k)}(r, s))_{k \in \mathbb{N}}$ is a Schauder basis for the binomial sequence spaces $b_p^\infty(\mathbb{V})$ and every $x = (x_i) \in b_p^\infty(\mathbb{V})$ has a unique representation by

$$x = \sum_k \lambda_k (r, s) g^{(k)}(r, s),$$

(3.1)

where $1 \leq p < \infty$ and $\lambda_k (r, s) = [B^{\infty}(\nabla x_i)]_k$ for each $k \in \mathbb{N}$.

**Proof** Obviously, $B^{\infty}(\nabla g_i^{(k)}(r, s)) = e_k \in \ell_p$, where $e_k$ is the sequence with 1 in the $k$th place and zeros elsewhere for each $k \in \mathbb{N}$. This implies that $g^{(k)}(r, s) \in b_p^\infty(\mathbb{V})$ for each $k \in \mathbb{N}$.

For $x \in b_p^\infty(\mathbb{V})$ and $m \in \mathbb{N}$, we put

$$x^{(m)} = \sum_{k=0}^{m} \lambda_k(r, s) g^{(k)}(r, s).$$

By the linearity of $B^{\infty}(\mathbb{V})$, we have

$$B^{\infty}(\nabla x^{(m)}_i) = \sum_{k=0}^{m} \lambda_k(r, s) B^{\infty}(\nabla g^{(k)}_i(r, s)) = \sum_{k=0}^{m} \lambda_k(r, s) e_k$$

and

$$[B^{\infty}(\nabla (x_i - x^{(m)}_i))]_k = \begin{cases} 0 & \text{if } 0 \leq k \leq m, \\ [B^{\infty}(\nabla x_i)]_k & \text{if } k > m, \end{cases}$$

for each $k \in \mathbb{N}$.

For any given $\varepsilon > 0$, there is a positive integer $m_0$ such that

$$\sum_{k=m_0+1}^{\infty} \|B^{\infty}(\nabla x_i)\|_k^p < \left(\frac{\varepsilon}{2}\right)^p$$
for all \( k \geq m_0 \). Then we have

\[
\begin{align*}
f_{B^r_p(\nabla)}(\mathbf{x} - \mathbf{x}^{(m_0)}) &= \left( \sum_{k=m_1+1}^{\infty} \left| B^{r_+}(\nabla \mathbf{x}_i) \right|_k^p \right)^{\frac{1}{p}} \\
&\leq \left( \sum_{k=m_0+1}^{\infty} \left| B^{r_+}(\nabla \mathbf{x}_i) \right|_k^p \right)^{\frac{1}{p}} \\
&< \frac{\varepsilon}{2} < \varepsilon,
\end{align*}
\]

which implies that \( \mathbf{x} \in B^r_p(\nabla) \) is represented as (3.1).

To prove the uniqueness of this representation, we assume that

\[
\mathbf{x} = \sum_k \mu_k(r,s)g^{(k)}(r,s).
\]

Then we have

\[
[B^{r_+}(\nabla \mathbf{x}_i)]_k = \sum_k \mu_k(r,s)[B^{r_+}(\nabla g^{(k)}(r,s))]_k = \sum_k \mu_k(r,s)(e_k)_k = \mu_k(r,s),
\]

which is a contradiction with the assumption that \( \lambda_k(r,s) = [B^{r_+}(\nabla \mathbf{x}_i)]_k \) for each \( k \in \mathbb{N} \).

This shows the uniqueness of this representation. \( \square \)

**Corollary 3.2** The sequence space \( B^r_p(\nabla) \) is separable, where \( 1 \leq p < \infty \).

For the duality theory, the study of sequence spaces is more useful when we investigate them equipped with linear topologies. Köthe and Toeplitz [22] first computed duals whose elements can be represented as sequences and defined the \( \alpha \)-dual (or Köthe-Toeplitz dual).

For the sequence spaces \( X \) and \( Y \), define the multiplier space \( M(X, Y) \) by

\[
M(X, Y) = \{ u = (u_k) \in w : ux = (u_kx_k) \in Y \text{ for all } x = (x_k) \in X \}.
\]

Then the \( \alpha \)-, \( \beta \)- and \( \gamma \)-duals of a sequence space \( X \) are defined by

\[
X^\alpha = M(X, \ell_1), \quad X^\beta = M(X, c) \quad \text{and} \quad X^\gamma = M(X, \ell_\infty),
\]

respectively.

We give the following properties:

\[
\begin{align*}
sup \sum_{n \in \mathbb{N}} |a_{n,k}|^q < \infty, \quad & (3.2) \\
sup \sum_{k \in \mathbb{N}} |a_{n,k}| < \infty, \quad & (3.3) \\
sup \sum_{n,k \in \mathbb{N}} |a_{n,k}| < \infty, \quad & (3.4) \\
lim_{n \to \infty} a_{n,k} = a_k \quad \text{for each } k \in \mathbb{N}, \quad & (3.5)
\end{align*}
\]
We define the set $U_r$ by

$$\sup_{k \in \Gamma} \sum_{i \in K} a_{n,k}^q < \infty,$$  \hspace{1cm} (3.6)

and

$$\lim_{n \to \infty} \left| a_{n,k} \right| = \sum_{i \in K} \lim_{n \to \infty} a_{n,k}^i \left| \right. < \infty.$$  \hspace{1cm} (3.7)

where $\Gamma$ is the collection of all finite subsets of $\mathbb{N}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$.

**Lemma 3.3** ([23]) Let $A = (a_{n,k})$ be an infinite matrix. Then the following statements hold:

(i) $A \in (\ell_1 : \ell_1)$ if and only if (3.3) holds.

(ii) $A \in (\ell_1 : c)$ if and only if (3.4) and (3.5) hold.

(iii) $A \in (\ell_1 : \ell_{\infty})$ if and only if (3.4) holds.

(iv) $A \in (\ell_p : \ell_1)$ if and only if (3.6) holds with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$.

(v) $A \in (\ell_p : c)$ if and only if (3.2) and (3.5) hold with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

(vi) $A \in (\ell_p : \ell_{\infty})$ if and only if (3.2) holds with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

(vii) $A \in (\ell_{\infty} : \ell_{\infty})$ if and only if (3.5) and (3.7) hold with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$.

(viii) $A \in (\ell_{\infty} : \ell_{\infty})$ if and only if (3.2) holds with $q = 1$.

**Theorem 3.4** We define the set $U_{1}^{s}$ and $U_{2}^{s}$ by

$$U_{1}^{s} = \left\{ u = (u_k) \in w : \sup_{i \in \mathbb{N}} \sum_{k \in K} (s + r)^i \sum_{j=i}^{k} i \left( \right)^r j (-s)^j u_i \left( \right) < \infty \right\}$$

and

$$U_{2}^{s} = \left\{ u = (u_k) \in w : \sup_{K \in \Gamma} \sum_{i \in K} (s + r)^i \sum_{j=i}^{k} i \left( \right)^r j (-s)^j u_i \left( \right) < \infty \right\}.$$ 

Then $[b_{1}^{s}(\nabla)]^{a} = U_{1}^{s}$ and $[b_{p}^{s}(\nabla)]^{a} = U_{2}^{s}$, where $1 < p \leq \infty$.

**Proof** Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we have

$$u_k x_k = \sum_{i=0}^{k} (s + r)^i \sum_{j=i}^{k} i \left( \right)^r j (-s)^j u_k y_i = (G_{\epsilon}^{s})_k$$

for each $k \in \mathbb{N}$, where $G_{\epsilon}^{s} = (g_{k,m}^{s})$ is defined by

$$g_{k,m}^{s} = \begin{cases} (s + r)^i \sum_{j=i}^{k} i \left( \right)^r j (-s)^j u_k & \text{if } 0 \leq i \leq k, \\ 0 & \text{if } i > k. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in \ell_1$ whenever $x \in b_{1}^{s}(\nabla)$ or $b_{p}^{s}(\nabla)$ if and only if $G_{\epsilon}^{s} y \in \ell_1$ whenever $y \in \ell_1$ or $\ell_p$, which implies that $u = (u_k) \in [b_{1}^{s}(\nabla)]^{a}$ or $[b_{p}^{s}(\nabla)]^{a}$ if and only if $G_{\epsilon}^{s} \in (\ell_1 : \ell_1)$ and $G_{\epsilon}^{s} \in (\ell_p : \ell_1)$ by parts (i) and (iv) of Lemma 3.3, we obtain $u = (u_k) \in [b_{1}^{s}(\nabla)]^{a}$ if and only if

$$\sup_{i \in \mathbb{N}} \sum_{k \in K} (s + r)^i \sum_{j=i}^{k} i \left( \right)^r j (-s)^j u_k < \infty$$
and \( u = (u_k) \in [b_r^p(\mathbb{V})]^a \) if and only if
\[
\sup_{K \in \Gamma} \sum_i \left| \sum_{k \in K} (s + r)^i \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_k \right|^q < \infty.
\]

Thus, we have \([b_r^p(\mathbb{V})]^a = U_1 \) and \([b_r^p(\mathbb{V})]^a = U_2 \), where \( 1 < p \leq \infty \). \( \Box \)

Now, we define the sets \( U_3^a \), \( U_4^a \), \( U_5^a \), \( U_6^a \) and \( U_7^a \) by
\[
U_3^a = \left\{ u = (u_k) \in w : \lim_{n \to \infty} (s + r)^n \sum_{i=0}^n \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_i \text{ exists for each } k \in \mathbb{N} \right\},
\]
\[
U_4^a = \left\{ u = (u_k) \in w : \sup_{n \in \mathbb{N}} \left| (s + r)^n \sum_{i=0}^n \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_i \right| < \infty \right\},
\]
\[
U_5^a = \left\{ u = (u_k) \in w : \lim_{n \to \infty} \sum_k \left| (s + r)^n \sum_{i=0}^n \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_i \right| \right\},
\]
\[
U_6^a = \left\{ u = (u_k) \in w : \sup_{n \in \mathbb{N}} \left| (s + r)^n \sum_{i=0}^n \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_i \right|^q < \infty \right\}, \quad 1 < q < \infty,
\]
and
\[
U_7^a = \left\{ u = (u_k) \in w : \sup_{n \in \mathbb{N}} \left| (s + r)^n \sum_{i=0}^n \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_i \right| < \infty \right\}.
\]

Theorem 3.5 We have the following relations:

(i) \([b_r^p(\mathbb{V})]^\beta = U_3^a \cap U_4^a \),
(ii) \([b_r^p(\mathbb{V})]^\beta = U_3^a \cap U_5^a \), where \( 1 < p < \infty \),
(iii) \([b_r^p(\mathbb{V})]^\beta = U_3^a \cap U_6^a \),
(iv) \([b_r^p(\mathbb{V})]^\gamma = U_4^a \),
(v) \([b_r^p(\mathbb{V})]^\gamma = U_6^a \), where \( 1 < p < \infty \),
(vi) \([b_r^p(\mathbb{V})]^\gamma = U_7^a \).

Proof Let \( u = (u_k) \in w \) and \( x = (x_k) \) be defined by (2.3), then we consider the following equation:
\[
\sum_{k=0}^n u_k x_k = \sum_{k=0}^n u_k \left[ \sum_{i=0}^k (s + r)^i \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} y_i \right] = \sum_{k=0}^n \left[ (s + r)^k \sum_{i=0}^n \sum_{j=0}^k \left( \begin{array}{c} j \\ i \end{array} \right) r^j (-s)^{i-j} u_i \right] y_k = (U^a \gamma)_n,
\]
where \( U^{rs} = (u_{rk}^{rs}) \) is defined by

\[
u_{nk}^{rs} = \begin{cases} 
(s + r)^{k} \sum_{i=0}^{n} \binom{i}{k} r^{-i} (-s)^{i-k} u_i & \text{if } 0 \leq k \leq n, \\
0 & \text{if } k > n.
\end{cases}
\]

Therefore, we deduce that \( u x = (u_{kk} x_k) \in c \) whenever \( x \in b_{rs}^{(r)}(V) \) if and only if \( U^{rs} y \in c \) whenever \( y \in \ell_1 \), which implies that \( u = (u_{kk}) \in \left[ b_{rs}^{(r)}(V) \right]^{0} \) if and only if \( U^{rs} \in (\ell_1 : c) \). By Lemma 3.3(ii), we obtain \( \left[ b_{rs}^{(r)}(V) \right]^{0} = U_{rs}^{(r)} \cap U_{rs}^{(s)} \). Using Lemma 3.3(i) and (iii)-(viii) instead of (ii), the proof can be completed in a similar way. So, we omit the details. \( \square \)

4 Conclusion

By considering the definitions of the binomial matrix \( B^{rs} = (b_{kk}^{rs}) \) and the difference operator, we introduce the sequence spaces \( b_{rs}^{p} (V) \) and \( b_{rs}^{\infty} (V) \). These spaces are the natural continuations of \([1, 7, 20]\). Our results are the generalizations of the matrix domain of the Euler matrix of order \( r \). In order to give fully inform the reader on related topics with applications and a possible line of further investigation, the e-book \([24]\) is added to the list of references.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

JM came up with the main ideas and drafted the manuscript. MS revised the paper. All authors read and approved the final manuscript.

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