RELATIVISTIC $N$-BOSON SYSTEMS
BOUND BY OSCILLATOR PAIR POTENTIALS

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Abstract

We study the lowest energy $E$ of a relativistic system of $N$ identical bosons bound by harmonic-oscillator pair potentials in three spatial dimensions. In natural units $\hbar = c = 1$ the system has the semirelativistic “spinless-Salpeter” Hamiltonian

$$H = \sum_{i=1}^{N} \sqrt{m^2 + p_i^2} + \sum_{j>i=1}^{N} \gamma |r_i - r_j|^2, \quad \gamma > 0.$$

We derive the following energy bounds:

$$E(N) = \min_{r>0} \left[ N \left( m^2 + \frac{2(N-1)P^2}{N^2 r^2} \right)^{\frac{1}{2}} + \frac{N}{2} (N-1) \gamma r^2 \right], \quad N \geq 2,$$

where $P = 1.376$ yields a lower bound and $P = 3/2$ yields an upper bound for all $N \geq 2$. A sharper lower bound is given by the function $P(\mu)$, where $\mu = m(N/(\gamma(N-1)^2))^{\frac{1}{2}}$, which makes the formula for $E(2)$ exact: with this choice of $P$, the bounds coincide for all $N \geq 2$ in the Schrödinger limit $m \to \infty$.

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I. Introduction and Main Result

Many-body problems form essential links between quantum-theoretical models and real nuclear, atomic, or macroscopic systems. However, even for nonrelativistic quantum theory, there are very few many-body problems that have explicit analytic solutions; the harmonic oscillator and the attractive delta interaction are well-known exceptions. In relativistic quantum theories the situation is even worse, in spite of the fact that the phenomenon of particle creation allowed by quantum field theory would suggest that there is no such thing as a one-body problem in that theory. Therefore, it is of considerable interest to study model $N$-body systems within the framework of the semirelativistic “spinless-Salpeter” equation. For this problem there exists a well-defined nonrelativistic limit which yields a useful consistency check. Specifically, we investigate in this paper the relative energy $E$ of a system of $N$ identical bosons represented by a semirelativistic “spinless-Salpeter” Hamiltonian \[ H = \sum_{i=1}^{N} \sqrt{m^2 + \mathbf{p}_i^2} + \sum_{j>i=1}^{N} \gamma |\mathbf{r}_i - \mathbf{r}_j|^2, \] where $m$ is the boson mass, and $\gamma > 0$ is a coupling parameter, and we have chosen units in which $\hbar = c = 1$. The operators $\mathbf{p}_i$ are defined \[3,4\] in the momentum-space representation where they become multiplicative operators ($c$-variables). The present work is an extension to the case of $N$ bosons of our earlier study \[5\] in which we derived energy bounds for the corresponding 1-body problem. We may compare $H$ with the corresponding Schrödinger $N$-body problem with Hamiltonian \[ H_S = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{j>i=1}^{N} \gamma |\mathbf{r}_i - \mathbf{r}_j|^2. \] Given our goal of investigating the relative (that is, binding) energies, both of these Hamiltonians have the unwelcome feature that they include the kinetic energy of the center-of-mass motion. This is easy to remedy for $H_S$, but a correct form is not so immediate in the relativistic case $H$. The exact solution to the $N$-body harmonic-oscillator problem is periodically “rediscovered” but has been known at least since 1935 when Houston \[6\] solved it. Later, Post \[7\] studied the non-relativistic translation-invariant problem: the exact ground-state energy $E_S$ may be expressed \[8\] for $N \geq 2$ (in three dimensions) by the simple formula \[ \varepsilon = 3v \frac{\lambda}{2}, \quad \varepsilon = \frac{mE_S}{N-1}, \quad v = \frac{mN\gamma}{2}. \]
Thus $\varepsilon$ is exactly the bottom of the spectrum of the 1-body Hamiltonian $-\Delta + vr^2$. In this paper we shall prove the following statement.

**Theorem 1**

**Bounds on the ground-state energy eigenvalue** $E$ of the semirelativistic Hamiltonian (1.1) are provided by the formula

$$E = \min_{r>0} \left[ N \left( m^2 + \frac{2(N-1)P^2}{N^2} \right)^{\frac{1}{2}} + \frac{N}{2} (N-1)\gamma r^2 \right], \quad N \geq 2,$$

which yields an upper bound on $E$ when $P = 3/2$, and a lower bound on $E$ when $P = P(\mu)$, where $\mu = m/(\gamma(N-1)^2)^{\frac{1}{2}}$, a function that makes the approximation (1.4) exact in the case $N = 2$. The function $P(m)$ is monotone increasing with $m$, has bounds

$$1.376 < P(m) < \frac{3}{2}, \quad (1.5)$$

and has the limit

$$\lim_{m \to \infty} P(m) = \frac{3}{2}. \quad (1.6)$$

In the large-$m$ limit, the upper and lower bounds coalesce to the corresponding exact (nonrelativistic) Schrödinger energy $E_{NR} = E_S + Nm$.

The paper is primarily concerned with proving Theorem 1. The main technical difficulties are twofold: to keep the fundamental symmetries of translation invariance and boson permutation symmetry, and to find ways of “penetrating” the square-root operator of the Salpeter kinetic energy. Our policy is to work with Jacobi relative coordinates to guarantee translation invariance of the wave functions, and to accept the concomitant complications of permutation symmetry. We discuss the relative coordinates and some of their properties in Sec. II. We shall exploit the necessary permutation symmetry to relate the $N$-body energy to that of a scaled and reduced 2-body problem. The exact solution of the 1-body problem is discussed in Sec. III. It is well known that the 1-body Salpeter problem is equivalent to a Schrödinger problem with Hamiltonian $-\Delta + \sqrt{m^2 + r^2}$ [9,10]. We take the position in this paper that the lowest eigenvalue $e(m)$ of this problem, which is easy to find numerically, is at our disposal. In Fig. 1 we exhibit graphs of the functions $\{e(m), P(m)\}$. The extension of these results to the 2-body problem is treated in Sec. IV. The lower
bound discussed in Sec. V is rendered possible by an operator property introduced in Sec. II that allows us, in a sense, to remove certain annihilation operators from inside the square-root operator. For the $N$-body upper bound discussed in Sec. VI we use a Gaussian wave function and minimize the energy expectation with respect to a scale variable. The calculation is helped by special factoring properties of the Gaussian and by the use of Jensen’s inequality. The bounds corresponding to $P = \{1.376, 1.5\}$ are depicted in Fig. 2, and the convergence of the bounds $P = \{P(\mu), 3/2\}$ with increasing $m$ (where $\mu = m(N/(\gamma(N-1)^2))^{1/2}$) is shown in Fig. 3, for $2 \leq N \leq 8$.

II. Relative Coordinates

Jacobi relative coordinates may be defined with the aid of an orthogonal matrix $B$ relating the column vectors of the new $[\rho_i]$ and old $[r_i]$ coordinates according to

$$[\rho_i] = B[r_i].$$

The first row of $B$ defines a center-of-mass variable with every entry $1/\sqrt{N}$, the second row defines a pair distance $\rho_2 = (r_1 - r_2)/\sqrt{2}$, and the $k$th row, $k \geq 2$, has the first $k-1$ entries $B_{ki} = 1/\sqrt{k(k-1)}$, the $k$th entry $B_{kk} = -\sqrt{(k-1)/k}$, and the remaining entries zero. We define the corresponding momentum variables as

$$[\pi_i] = (B^{-1})^t[p_i] = B[p_i].$$

These coordinates have some nice properties which we shall need. Firstly, we have

$$k \sum_{i=2}^{k} \rho_i^2 = \sum_{j>i=1}^{k} (r_i - r_j)^2, \quad k = 2, 3, \ldots, N,$$  \hspace{1cm} (2.3)

and similarly for the momenta

$$k \sum_{i=2}^{k} \pi_i^2 = \sum_{j>i=1}^{k} (p_i - p_j)^2, \quad k = 2, 3, \ldots, N.$$  \hspace{1cm} (2.4)

It follows immediately that if $\Psi$ is a translation-invariant wave function which is symmetric (or antisymmetric) under the permutation of the individual-particle indices, then it follows that

$$(\Psi, \rho_i^2 \Psi) = (\Psi, \rho_i^2 \Psi), \quad 2 \leq i \leq N,$$  \hspace{1cm} (2.5)
(\Psi, \pi_i^2 \Psi) = (\Psi, \pi_2^2 \Psi), \quad 2 \leq i \leq N. \quad (2.6)

These expectation symmetries might suggest that the wave function \( \Psi \) is symmetric under permutation of the relative coordinates; but this stronger property is not generally true; it is the case for Gaussian wave functions. Moreover, Gaussian boson wave functions of Jacobi relative coordinates uniquely [11, 12] have the further factoring property that

\[ \Phi(\rho_2, \rho_3, \ldots, \rho_N) = \phi(\rho_2)\theta(\rho_3, \ldots, \rho_N), \quad (2.7) \]

where \( \phi \) and \( \theta \) are also Gaussian.

III. The 1-Body Problem

We consider the 1-body problem with Hamiltonian

\[ H_1 = \sqrt{m^2 + p^2} + r^2 \to e(m), \quad (3.1) \]

where, for coupling \( \gamma = 1 \), \( e(m) \) is the lowest eigenvalue as a function of the mass \( m \). By transforming this problem into momentum space we obtain the equivalent problem

\[ \tilde{H}_1 = -\Delta + \sqrt{m^2 + r^2} \to e(m). \quad (3.2) \]

Since this Schrödinger problem is easy to solve numerically to arbitrary accuracy, we shall take the position that \( e(m) \) is “known” and at our disposal. We note that in the large-\( m \) (nonrelativistic or Schrödinger) limit, we have

\[ e(m) \simeq e_{\text{NR}}(m) = m + \frac{3}{(2m)^{\frac{1}{2}}}. \quad (3.3) \]

We now define, for a given value of \( m \), the (lowest) “kinetic potential” [13–15] \( \bar{h}(s) \) associated with the relativistic-kinetic-energy square-root operator \( \sqrt{m^2 + p^2} \) and the harmonic-oscillator potential \( r^2 \) by

\[ \bar{h}(s) = \inf_{\psi \in \mathcal{D}(H_1) \atop \|\psi\| = 1} (\psi, r^2 \psi), \quad (3.4) \]

where \( \psi(r) \) is a wave function in the domain \( \mathcal{D}(H_1) \) of \( H_1 \). That is to say, we find the minimum mean-value of the potential, subject to the constraint that the mean
kinetic energy is held constant at the value $s$. It follows that the eigenvalue may now be recovered from $\bar{h}(s)$ by a further minimization with respect to the kinetic energy $s$. Thus we have

$$e(m) = \min_{s > m} [s + \bar{h}(s)].$$

(3.5)

It may be difficult to find the kinetic potential $\bar{h}(s)$ exactly from (3.4). Instead we construct an effective kinetic potential $\bar{h}_{\text{eff}}(s)$ which, when substituted in (3.5), yields $e(m)$ exactly. We do this by changing the minimization variable from $s > m$ to $r > 0$ according to the following equations:

$$\bar{h}_{\text{eff}}(s) = r^2, \quad s = \sqrt{m^2 + \left(\frac{P(m)}{r}\right)^2}.$$  

(3.6)

Now, by rewriting (3.5) in terms of the minimization variable $r$ we obtain the defining relation for $P(m)$ as follows:

$$e(m) = \min_{r > 0} \left[ \sqrt{m^2 + \left(\frac{P(m)}{r}\right)^2} + r^2 \right].$$

(3.7)

In fact, by inverting (3.7), we find the following expression for $P(m)$ in terms of the 1-body energy $e(m)$:

$$P(m) = \left( \frac{2 \left( e(m) + \sqrt{e^2(m) + 3m^2} \right)}{27} \right)^{\frac{1}{2}} \left( 2e(m) - \sqrt{e^2(m) + 3m^2} \right).$$

(3.8)

The graphs of $e(m) - m$ and $P(m)$ are shown in Fig. 1: both $e(m)$ and $P(m)$ are monotone increasing with $m$; $e(m) - m$, however, is monotone decreasing, in agreement, for large $m$, with the Feynman–Hellmann theorem for the corresponding nonrelativistic case. In the (ultrarelativistic) limit $m \to 0$ we have $\tilde{H}_1 \to -\Delta + r$, that is to say, the operator limit is the Schrödinger operator for the linear potential in three dimensions, with lowest energy $e(0) = 2.33810741$. In the (nonrelativistic) large-$m$ limit we have $H_1 \to m - (1/2m)\Delta + r^2$, that is to say, the Schrödinger harmonic oscillator with energy $e(m) \simeq m + 3/\sqrt{2m}$. By substituting these “outer” energies in (3.8), we obtain the bounds

$$1.376 < P(m) < \frac{3}{2}.$$  

(3.9)
It is clear from Eq. (3.7) that the expression for $e(m)$, as a function of $m$ and $P$, is monotone increasing in $P$. Thus, by substituting, respectively, the constants $P = 1.376$ and $P = 1.5$, we obtain from this formula lower and upper bounds on the 1-body energy $e(m)$. These bounds agree exactly with the bounds we obtained earlier [5,15] for this 1-body harmonic-oscillator problem.

For later application to the $N$-body problem, we now consider a more general 1-body problem with Hamiltonian

$$H = \beta \sqrt{m^2 + \lambda p^2} + \gamma r^2$$

(3.10)

and positive parameters $\{\beta, \gamma, \lambda\}$. We find by elementary scaling arguments that the eigenvalue $\varepsilon(m, \beta, \gamma, \lambda)$ corresponding to the operator $H$ may be expressed in terms of the energy function $e(m)$ by the explicit formula

$$\varepsilon(m, \beta, \gamma, \lambda) = (\beta^2 \gamma \lambda)^{\frac{1}{3}} e \left( m \left( \frac{\beta}{\gamma \lambda} \right)^{\frac{1}{3}} \right).$$

(3.11)

In terms of $P$, we therefore have

$$\varepsilon(m, \beta, \gamma, \lambda) = \min_{r > 0} \left[ \beta \left( m^2 + \lambda \left( \frac{P}{r} \right)^2 \right)^{\frac{1}{2}} + \gamma r^2 \right].$$

(3.12)

For each $\beta > 0$, $\gamma > 0$, $\lambda > 0$, this formula is therefore exact when

$$P = P(\mu), \quad \text{where} \quad \mu = m \left( \frac{\beta}{\gamma \lambda} \right)^{\frac{1}{3}},$$

(3.13)

it yields a lower bound when $P = 1.376$, and an upper bound when $P = 1.5$. As we shall see in the next section, the 2-body energy is obtained from (3.11) or (3.12) by simply setting $\lambda = 1$, $\beta = 2$. It is an extension of this reasoning that will allow us, in Sec. V, to obtain also the $N$-body, $N \geq 2$, lower energy bound by using suitable values for $\beta$, $\gamma$, and $\lambda$.

IV. The 2-Body Problem

For the case $N = 2$ we have explicitly

$$H = \sqrt{m^2 + p_1^2} + \sqrt{m^2 + p_2^2} + \gamma |r_1 - r_2|^2.$$

(4.1)
Let $\psi(\rho_2)$ be a normalized boson wave function. Then the lowest relative eigenvalue of the operator $H$ is the infimum of expectation values of the form $(\psi, H\psi)$. But the boson symmetry of $\psi(\rho_2)$ means that the two kinetic-energy terms in $(\psi, H\psi)$ must have the same value. Moreover, in terms of relative coordinates, the operator $p_2^2$ may be written

$$p_2^2 = \frac{(\pi_1 - \pi_2)^2}{2}. \quad (4.2)$$

Now, the operator $\pi_1$ would immediately annihilate $\psi(\rho_2)$ if it were not contained in the square root. We claim that, inside the expectation value, the operator $\pi_1$ may simply be removed; this may be seen as an immediate generalization of the following observation.

**Lemma 1**

*Suppose $\Psi(x, y) = \psi(x)$, then*

$$\left[ 1 - \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right)^2 \right]^{\frac{1}{2}} \Psi = \left( 1 - \frac{\partial^2}{\partial x^2} \right)^{\frac{1}{2}} \Psi. \quad (4.3)$$

**Proof of Lemma 1**

If $\mathcal{F}$ indicates the 2-dimensional Fourier transform and our new variables are $\{p, q\}$, then we find $\mathcal{F}(\Psi)(p, q) = \tilde{\psi}(p)\delta(q)$, and, by definition, the Fourier transform of the left-hand side of (4.3) becomes

$$\left( 1 + (p - q)^2 \right)^{\frac{1}{2}} \tilde{\psi}(p)\delta(q) = \left( 1 + p^2 \right)^{\frac{1}{2}} \tilde{\psi}(p)\delta(q). \quad (4.4)$$

By transforming back to the variables $\{x, y\}$, we obtain the right-hand side of (4.3).

Applying the generalization of this lemma to our problem in three dimensions, we find, for $\psi = \psi(\rho_2)$,

$$(\psi, H\psi) = \left(\psi, \left( 2\sqrt{m^2 + \frac{1}{2}\pi_1^2 + 2\gamma\rho_2^2} \right) \psi \right). \quad (4.5)$$

By defining the pair-distance variable $r = r_1 - r_2 = \sqrt{2}\rho_2$, and the corresponding momentum as $p = -i\nabla r = \pi_2/\sqrt{2}$, we may rewrite (4.5) as

$$(\psi, H\psi) = \left(\psi, \left( 2\sqrt{m^2 + p^2 + \gamma r^2} \right) \psi \right). \quad (4.6)$$
By using a formal relative coordinate $r$, we have thus recovered the well-known [16] 2-body result: the minimum of the right-hand side of (4.6) is the bottom of the spectrum of $H$ which corresponds precisely to the energy of a 1-body problem with the kinetic-energy parameter $\beta = 2$. This result may also be expressed in terms of the 1-body energy function $e(m)$ by means of Eq. (3.11). Thus we have explicitly for $N = 2$

$$E = (4\gamma)^{\frac{1}{3}} e \left( m \left( \frac{2}{\gamma} \right)^{\frac{1}{3}} \right).$$  \hspace{1cm} (4.7)

In the next section we shall apply a similar reasoning to the $N$-body problem; however, for $N > 2$ we obtain, instead of the exact energy, a lower energy bound.

V. The Lower Bound

Suppose that $\Psi(\rho_2, \rho_3, \ldots, \rho_N)$ is a normalized translation-invariant $N$-boson wave function. Boson symmetry and, in particular, formula (2.3) allow us to write

$$E \leq (\Psi, H \Psi) = N \left( \Psi, (m^2 + p_N^2)^{\frac{1}{2}} \Psi \right) + \left( \frac{N}{2} \right) \gamma \left( \Psi, 2 \rho_N^2 \Psi \right).$$  \hspace{1cm} (5.1)

Now, from the definition of the relative coordinates, we have

$$p_N = \frac{1}{\sqrt{N}} \pi_1 - \sqrt{\frac{N - 1}{N}} \pi_N.$$  \hspace{1cm} (5.2)

Consequently, an application of an immediate generalization of Lemma 1 allows us to “remove” the operator $\pi_1$ from the square root of the kinetic-energy term and write

$$E \leq N \left( \Psi, \left( m^2 + \frac{N - 1}{N} \pi_N^2 \right)^{\frac{1}{2}} \Psi \right) + \left( \frac{N}{2} \right) \gamma \left( \Psi, 2 \rho_N^2 \Psi \right).$$  \hspace{1cm} (5.3)

Adapting the argument presented in Sec. IV for the 2-body case $N = 2$, we define a relative coordinate $r = \sqrt{2} \rho_N$, and the corresponding momentum $p = \pi_N / \sqrt{2}$. The expression for the upper bound to the lowest $N$-boson energy $E$ then becomes

$$E \leq N \left( \Psi, \left( m^2 + \frac{2(N - 1)}{N} p^2 \right)^{\frac{1}{2}} \Psi \right) + \left( \frac{N}{2} \right) \gamma \left( \Psi, r^2 \Psi \right).$$  \hspace{1cm} (5.4)

The inequality (rather than an equality) in (5.4) comes only from the choice of wave function. If we find the infimum of such expressions over all normalized translation-invariant $N$-boson wave functions, we would obtain the exact energy $E$; if we find
this minimum but without the constraint of boson symmetry, then the right-hand side of (5.4) will in general fall below $E$ but will in any case be bounded from below by the bottom of the spectrum of the 1-body semirelativistic Salpeter Hamiltonian

$$H = N \left( m^2 + \frac{2(N-1)}{N} p^2 \right) \frac{1}{2} + \left( \frac{N}{2} \right) \gamma r^2. \quad (5.5)$$

But this latter problem corresponds precisely to Eq. (3.7) if we make the parameter substitutions

$$\beta = N, \quad \lambda = \frac{2(N-1)}{N}, \quad \gamma \rightarrow \left( \frac{N}{2} \right) \gamma = \frac{N(N-1)}{2} \gamma. \quad (5.6)$$

Thus, in view of the $P$ representation (3.12), it is clear by choosing $P = P(\mu)$, where $\mu = m(N/(\gamma(N-1)^2))^{\frac{3}{2}} > 1.376$, that we have established the lower bound (1.4) of Theorem 1.

It is interesting to note that we can also substitute the $N$-body values (5.6) for the parameters $\beta$, $\gamma$, and $\lambda$ into the result (3.11) for the 1-body ground-state energy $\varepsilon(m, \beta, \gamma \lambda)$ in order to obtain the following explicit expression for the lower bound:

$$E \geq \left( N^2(N-1)^2 \gamma \right)^{\frac{1}{2}} \varepsilon \left( m, \frac{N}{(N-1)^2 \gamma} \right)^{\frac{1}{2}}. \quad (5.7)$$

This expression—which is equivalent to the lower bound (1.4) of Theorem 1—gives the exact energy and agrees with Eq. (4.7) when $N = 2$. Meanwhile, for all $N \geq 2$, in the nonrelativistic large-$m$ (Schrödinger) limit it yields the exact $N$-body energy

$$E_{\text{NR}} = Nm + 3 \left( \frac{\gamma}{2m} \right)^{\frac{1}{2}} N^{\frac{3}{2}}(N-1), \quad (5.8)$$

reproducing thus the old result of Houston and Post recalled in Eq. (1.3).

**VI. The Upper Bound**

For the upper bound we employ a Gaussian wave function of the form

$$\Phi(\rho_2, \rho_3, \ldots, \rho_N) = C \exp \left( -\alpha \sum_{i=2}^{N} \rho_i^2 \right), \quad \alpha > 0, \quad (6.1)$$

where $C$ is a normalization constant. The factoring property (2.7) of this function and the boson-symmetry reduction leading to (5.4) allows us to write

$$E \leq N \left( \phi, \left( m^2 + \frac{2(N-1)}{N} p^2 \right)^{\frac{1}{2}} \phi \right) + \left( \frac{N}{2} \right) \gamma (\phi, r^2 \phi), \quad (6.2)$$
where the function $\phi(r)$ is given by

$$
\phi(r) = \left( \frac{\alpha}{\pi} \right)^{\frac{3}{4}} \exp \left( -\frac{\alpha r^2}{2} \right).
$$

(6.3)

Since the kinetic-energy operator is a \textit{concave} function of the square $p^2$ of the momentum, we can use Jensen’s inequality [17] to move the expectation value $\langle p^2 \rangle$ inside the square root and thus estimate the mean value of this operator from above and write

$$
E \leq N \left( m^2 + \frac{2(N-1)}{N} \langle \phi, p^2 \phi \rangle \right)^{\frac{1}{2}} + \left( \frac{N}{2} \right) \gamma \langle \phi, r^2 \phi \rangle.
$$

(6.4)

We shall minimize this upper bound with respect to the scale variable $\alpha > 0$. We parametrize the basic kinetic-energy and potential-energy expectation values in terms of a variable $r > 0$ by the following relations:

$$
\langle \phi, r^2 \phi \rangle = \frac{3}{2\alpha} := r^2, \quad \langle \phi, p^2 \phi \rangle = \frac{3\alpha}{2} = \left( \frac{P}{r} \right)^2, \quad P := \frac{3}{2}.
$$

(6.5)

By substituting these expressions in Eq. (6.4) and minimizing over the variable $r$, we establish the upper bound (1.4) of Theorem 1.

VII. Summary and Conclusion

This paper is devoted to the investigation of the ground-state eigenvalue of the semirelativistic (“spinless-Salpeter”) Hamiltonian (1.1) which governs the dynamics of a system of $N$ identical bosons that experience pair interactions described by a harmonic-oscillator potential with coupling strength $\gamma$. For a fixed coupling $\gamma = 1$, we have represented the exact ground-state energy eigenvalue of the corresponding 1-body problem, regarded as a function $e(m)$ of the boson mass $m$, by a monotone rising function $P(m)$, which is bounded by $1.376 < P(0) \leq P(m) \leq P(\infty) = 1.5$. Our bounds (1.4) on the energy of the $N$-body problem are expressed in terms of a formula which has this function $P$ as a parameter.

In Fig. 2 we have plotted the energy bounds corresponding to fixed lower and upper limiting values of $P(m)$, namely, $P = \{1.376, 1.5\}$. In Fig. 3 we have kept the same upper energy bound, obtained with the help of a Gaussian trial wave function and corresponding to $P = 1.5$, but added the best lower energy bound of this type, using a “running” $P = P(\mu), \mu = m(N/(\gamma(N-1)^2))^\frac{1}{4}$. The lower energy bound
of Fig. 3 is identical to the exact energy for the case $N = 2$. For higher $N > 2$, Fig. 3 shows the approach of both upper and lower bounds to the well-known exact nonrelativistic solution (1.3) in the large-$m$ limit.

A key ingredient in this analysis is the use of relative coordinates: only in such a framework could the upper and lower energy bounds be made to converge in the Schrödinger limit. This study of the semirelativistic harmonic-oscillator problem is a first step towards energy bounds valid for more general central pair interactions.

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Figure 1. The monotone energy function $e(m)$ of the 1-body problem defined by (3.1), and the monotone function $P(m)$ used in our standard representation (3.8) for $e(m)$; the function $P(m)$ is bounded by $P(0) = 1.376 \leq P(m) \leq P(\infty) = 3/2$. 
Figure 2. Upper (full lines) and lower (dashed lines) bounds to the lowest energy $E(m)$ of the $N$-boson relativistic harmonic-oscillator problem for $N = 2, 3, \ldots, 8$ obtained by employing the constant values $P = 1.376$ and $P = 1.5$, respectively, in Eq. (1.4) of Theorem 1.
Figure 3. Upper (full lines) and lower (dashed lines) bounds to the lowest energy $E(m)$ of the $N$-boson relativistic harmonic-oscillator problem for $N = 2, 3, \ldots, 8$ obtained by employing the values $P = P(\mu), \quad \mu = m(N/((\gamma(N - 1)^2))^{\frac{1}{3}}$, and $P = 1.5$, respectively, in Eq. (1.4) of Theorem 1. For $N = 2$, the lower bound is exact.