Gauge Miura and Bäcklund transformations for generalized $A_n$-KdV hierarchies

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Abstract
The construction of Miura and Bäcklund transformations for $A_n$ mKdV and KdV hierarchies are presented in terms of gauge transformations acting upon the zero curvature representation. As in the well known $sl(2)$ case, we derive and relate the equations of motion for the two hierarchies. Moreover, the Miura-gauge transformation is not unique, instead, it is shown to be connected to a set of generators labeled by the exponents of $A_n$. The construction of generalized gauge-Bäcklund transformation for the $A_n$-KdV hierarchy is obtained as a composition of Miura and Bäcklund-gauge transformations for $A_n$-mKdV hierarchy. The zero curvature representation provide a framework which is universal within all flows and generate systematically Bäcklund transformations for the entire hierarchy.

Keywords: solitons, integrable hierarchies, Miura transformation, Backlund transformation

1. Introduction
The recent studies of integrable models has revealed an increasing importance of its underlying algebraic structure, which is intimately connected to a series of peculiar properties, such as the existence of an infinite number of conservation laws, soliton solutions, Bäcklund transformation, etc. Structural connection has been known for some time, see for instance, [1–3]. Many of these properties can be derived from the zero curvature formulation in terms of two
dimensional gauge potentials, $A_t$ and $A_x$, lying in an affine algebra, $\hat{G}$, i.e.

$$\partial_t A_x - \partial_x A_t + [A_t, A_x] = 0. \quad (1.1)$$

A particular virtue of the zero curvature representation (1.1) is its gauge invariance. Conservation laws can be derived by abelianizing the two dimensional potentials by gauge transforming them into the Cartan subalgebra of $\hat{G}$ [4]. Soliton solutions in turn, can be constructed by gauge transforming a vacuum solution $A^\text{vac}_x$ and $A^\text{vac}_t$ into some nontrivial configuration (dressing) see for instance [5]. More recently, gauge transformation was shown to be a key ingredient in constructing Bäcklund transformations [6–9]. In all cases, gauge transformation is an essential ingredient.

In this paper we shall consider the multicomponent mKdV and KdV hierarchies within the generalized Drinfeld–Sokolov matrix hierarchy connected to the affine $G = A_n$ algebra discussed in [1]. Also important are its connection to several physically relevant nonlinear soliton equations, $W$ algebras and discrete matrix models [2, 3, 10, 11]. Bäcklund transformation interpolates between two soliton solutions of an integrable model [12]. More recently, it was employed to describe integrable defects in the sense that two field configurations can be connected by a defect at certain space location [13, 14]. Thus, the classification of integrable defects and the construction of the various types of Bäcklund transformations are intimately connected.

The construction of Bäcklund transformations can be formulated in terms of gauge transforming the two dimensional potentials $A_\mu$ from one field configuration, say $\phi$ to another, $\psi$, i.e.

$$U(\phi, \psi)A_\mu(\phi) = A_\mu(\psi)U(\phi, \psi) + \partial_\mu U(\phi, \psi). \quad (1.2)$$

A natural construction of Bäcklund-gauge transformations $U(\phi, \psi)$ in terms of a graded affine algebra was proposed in [7, 8] for the $A_1$ mKdV (sinh-Gordon) and in [9] for the generalized $A_n$ Toda hierarchies.

Moreover $U(\phi, \psi)$ was shown to be an universal object and provide a systematic construction of Bäcklund transformation extended to all evolution equations (flows) within the hierarchy since (1.2) is valid for all flows (i.e. $A_t \equiv A^{N}_t$).

Another application of gauge invariance of the zero curvature formulation is the construction of a Miura transformation connecting the mKdV and KdV hierarchies. Results in this direction were obtained for the $A_1$ loop algebra involving a single field in reference [15, 16]. In this paper we shall discuss the construction of the Miura transformation for the multicomponent $A_n$ affine algebra. We shall also discuss the interrelation between the Bäcklund transformation of the two systems. Several aspects of KdV hierarchies have been developed in the past with regards to spherically symmetric reductions of self-dual Yang–Mills theories in 4D, the realization of $W$-algebras, etc (e.g. see for instance [17–19] and references therein). In particular, in reference [8] a Miura-gauge transformation $S$ was proposed to map the $A_1$ two dimensional potentials $A^{\text{mKdV}}_\mu$ into $A^{\text{KdV}}_\mu$, i.e.

$$A^{\text{KdV}}_\mu = SA^{\text{mKdV}}_\mu S^{-1} + S\partial_\mu S^{-1}. \quad (1.3)$$

In this paper we follow the same line of reasoning of reference [8] to discuss the construction of Miura-gauge transformation from the $A_n$-mKdV to the $A_n$-KdV systems. An important consequence of our construction in (1.2) is that it is valid for all flows $t = t_N$ and henceforth (1.2) provide a systematic derivation of Bäcklund-gauge transformation for all flows of the $A_n$-KdV from the $A_n$-mKdV system as a composition of $U$ and $S$ (see equation (6.4)).
This paper is organized as follows. In sections 2 and 3 we discuss the algebraic structure and construct the general \(A_n\) equations of motion for the second flow of zero curvature representation for both mKdV and KdV systems. In section 4 we discuss the structure of the Miura-gauge transformation for \(A_1, A_2\) systems (and for \(A_3\) in the appendix B). In particular, we argue that there are more than one Miura transformation connecting both systems. In fact, for \(A_n\) there are \(n+1\) transformations labeled by the identity \(I\) and by the discrete set of constant generators \(E^{(k)} \in \hat{K}\), of grade \(k = 1, \ldots, n\). We construct explicitly the Miura transformations for \(A_1, A_2\) and \(A_3\) and propose a general construction for the \(A_n\) case. Section 5 contains the general structure between equations of motion of the two integrable hierarchies. In section 6 we present the construction of Bäcklund transformation for the \(A_n\)-KdV system as a combination of Miura and Bäcklund transformations for the mKdV system. Section 7 contains our conclusions.

2. mKdV equations

Here we shall discuss in detail the general construction of the \(A_n\)-mKdV flows. We shall use ‘\(n\)’ for the rank of the underlying \(\hat{G} = A_n\) affine algebra (which also equals the number of fields of the theory) and ‘\(N\)’, to label the flows (or time evolution according to \(t_N\)).

Consider the generic time evolution equations for the \(A_n\)-mKdV hierarchy which are classified according to the grading structure developed in the appendix A. Consider the decomposition of an affine Lie algebra \(\hat{G} = \sum G_a\) according to a grading operator \(Q\) such that \([Q, G_a] = aG_a\) and \([G_a, G_b] \subset G_{a+b}\). Let \(E \equiv E^{(1)}\) be a constant grade one generator responsible for a second decomposition of \(\hat{G}\) into Kernel of \(E\), \(\hat{K} = \{ x \in \hat{K}, [x, E] = 0 \}\) and its complement, \(\hat{M}\) i.e.

\[
\hat{G} = \hat{K} \oplus \hat{M}.
\] (2.1)

In particular, projecting into the zero grade subspace, \(G_0 = K \oplus M\). Define now the Lax operator as

\[
L = \partial_x + E + A_0,
\] (2.2)

where \(A_0 \in \hat{M}\) (and consequently \(A_0 \in G_0\)). In the case of \(\hat{G} = \hat{sl}(n+1)\) and principal gradation with subspaces given as in (A.5), \(E^{(1)}\) is given by

\[
E \equiv E^{(1)} = \sum_{k=1}^{n} E_{0k} + \lambda E_{-(\alpha_1+\cdots+\alpha_n)}
\] (2.3)

and \(A_0\) parameterized as

\[
A_0 = \sum_{k=1}^{n} v_k h_k = \begin{pmatrix}
  r_1 & 0 & \cdots & 0 \\
  0 & r_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & r_{n+1}
\end{pmatrix},
\] (2.4)

where \(r_i = v_i - v_{i-1}, i = 1, \ldots, n+1, v_0 = v_{n+1} = 0\). The relation \(v_i = \sum_{j=1}^{i} r_j\) follows from trace condition.

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We now propose the construction of time evolution equations in the zero curvature representation,
\[
[\partial_x + A_x, \partial_N + A_N] = \partial_t A_N - \partial_N A_x + [A_x, A_N] = 0,
\]
where \( A_x = E^{(1)} + A_0 \).

The so-called *positive grade* time evolution according to \( t_N \) is constructed from \( A_N \) decomposed as
\[
A_N = D^{(N)} + D^{(N-1)} + \cdots + D^{(0)}, \quad D^{(j)} \in \mathcal{G}_j, \quad N \in \mathbb{Z}_+.
\]

The zero curvature representation (2.5) decomposes according to the graded structure into
\[
[E^{(1)}, D^{(N)}] = 0,
\]
\[
[E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_t D^{(N)} = 0,
\]
\[
\vdots
\]
\[
[A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_N A_0 = 0,
\]
and allows solving for \( D^{(j)} \) recursively starting from the highest grade equation (2.7). In particular, the last equation (2.9) is the only equation involving time derivatives acting on \( A_0 \) and can be regarded as the time evolution for the fields parametrizing \( A_0 \in \mathcal{M} \).

Solving (2.7) for \( N = 2 \) and \( n > 1 \), according to the grading structure given in (A.5) we find
\[
D^{(2)} = a_2 \left( \sum_{k=1}^{n-1} E_{\alpha_k+\alpha_{k+1}} + \lambda E_{-(\alpha_1+\cdots+\alpha_{n-1})} + \lambda E_{-(\alpha_2+\cdots+\alpha_n)} \right) \in \hat{K}. \quad (2.10)
\]
where \( a_2 \) is an arbitrary coefficient. The lower grade matrices \( D^{(i)}, i = 0, 1 \) can be solved recursively from the appropriate grade projections of the zero-curvature equation (2.5). For example when \( N = 2 \), the grade one element \( D^{(1)} \) is solved from (2.8) i.e.
\[
[E^{(1)}, D^{(1)}] + [A_0, D^{(2)}] + \partial_x D^{(2)} = 0
\]
leading to
\[
\partial_x a_2 = 0 \quad \text{and} \quad D^{(1)} = \sum_{k=1}^n (v_{k+1} - v_{k-1}) E_{\alpha_k} + (v_1 - v_n)\lambda E_{-(\alpha_1+\cdots+\alpha_n)}, \quad (2.12)
\]
where we shall set \( a_2 = 1 \).

The grade 1 projection of the zero-curvature equation (2.5) leads to
\[
[E^{(1)}, D^{(0)}] + [A_0, D^{(1)}] + \partial_x D^{(1)} = 0,
\]
with \( D^{(0)} = \sum_{a=1}^n d_a \eta_a \). It therefore follows,
\[
K_{\mu\nu} \partial_{\mu} = K_{\mu\nu} (v_{j+1} - v_{j-1}) + \partial_{\mu} (v_{j+1} - v_{j-1}), \quad (2.14)
\]
where \( K_{\mu\nu} = 2 \frac{\alpha_{\mu}\alpha_{\nu}}{\alpha_0^2} = 2 \delta_{\mu,\nu} - \delta_{\mu,\nu-1} - \delta_{\mu,\nu+1} \) is the Cartan matrix of \( \mathfrak{sl}(n+1) \).
Solving (2.14) for \(d_a, a = 1, \ldots, n\) and inserting it into the grade 0 projection of the zero-curvature equation (2.9) yields the equation of motion for \(t \equiv t_2\),

\[
\partial_t v_a = \partial_x d_a. \tag{2.15}
\]

After multiplying both sides of relation (2.15) by the Cartan matrix \(K\) and making use of relation (2.14) we obtain, [22]

\[
\partial_t \sum_{a=1}^{n} K_{ba} v_a = \partial_t \left( 2v_b - v_{b+1} - v_{b-1} \right)
= \partial_t \left[ \partial_x (v_{b+1} - v_{b-1}) + (2v_b - v_{b+1} - v_{b-1})(v_{b+1} - v_{b-1}) \right], \quad b = 1, \ldots, n. \tag{2.16}
\]

Recalling equation (2.16) can be rewritten as,

\[
\partial_t \left( r_{b+1} - r_b \right) = -\partial_x \left[ \partial_x (v_b + r_{b+1}) - (r_{b+1} - r_b) (r_b + r_{b+1}) \right], \quad b = 1, \ldots, n, \tag{2.17}
\]

that defines \(t_2\) flows of \(sl(n + 1)\) mKdV hierarchy. Inverting the Cartan matrix explicitly for the cases where \(n = 2, 3\) in (2.16) we find in terms of \(v_j\) variables, respectively,

\[
\partial_t v_1 = \frac{1}{3} \partial_x \left( -\partial_x v_1 + 2\partial_x v_2 + v_1^2 - 2v_1 v_2 \right),
\]

\[
\partial_t v_2 = \frac{1}{3} \partial_x \left( -2\partial_x v_1 + \partial_x v_2 + 2v_1^2 - v_2^2 - 2v_1 v_2 \right), \tag{2.18}
\]

and

\[
\partial_t v_1 = \frac{1}{2} \partial_x \left( -\partial_x v_1 + \partial_x v_2 + \partial_x v_3 + v_1^2 - v_2^2 - v_3^2 + v_1 v_2 + v_2 v_3 \right),
\]

\[
\partial_t v_2 = \partial_x \left( -\partial_x v_1 + \partial_x v_3 + v_1^2 - v_2^2 - v_3^2 - v_1 v_2 + v_2 v_3 \right),
\]

\[
\partial_t v_3 = \frac{1}{2} \partial_x \left( -\partial_x v_1 - \partial_x v_2 + \partial_x v_3 + v_1^2 + v_2^2 - v_3^2 - v_1 v_2 - v_2 v_3 \right). \tag{2.19}
\]

3. KdV equations

For the \(A_n\)-KdV hierarchy we shall employ the same algebraic structure namely, principal gradation, \(Q^{\text{pal}}\) and constant grade one element, \(E(1)\) (2.3) and propose the following Lax operator

\[
L = \partial_x + E^{(1)} + A^{(-1)} + \cdots + A^{(-n)},
\]

\[
A^{(k-n-1)} = J_{n+1-k} E_{-(\alpha_2 + \cdots + \alpha_k)} \in \tilde{M} \subset \mathfrak{g}_{k-n-1} \tag{3.1}
\]
for $k = 1, \ldots, n$, or in matrix form

$$A_x = E^{(1)} + A^{(-1)} + \cdots + A^{(-n)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ \vdots & & & & & \vdots \\ \lambda + J_n & J_{n-1} & \ldots & J_2 & J_1 & 0 \end{pmatrix}. \quad (3.2)$$

The zero curvature representation for flow $t = t_2$, (i.e. $N = 2$),

$$[\partial_x + E^{(1)} + A^{(-1)} + \cdots + A^{(-n)}, \quad \partial_t + D^{(2)} + D^{(1)} + \cdots + D^{(-n)}] = 0 \quad (3.3)$$

leads to the following graded equations,

$$[E^{(1)}, D^{(2)}] = 0, \quad (3.4)$$
$$[E^{(1)}, D^{(1)}] + \partial_x D^{(2)} = 0, \quad (3.5)$$
$$[E^{(1)}, D^{(0)}] + [A^{(-1)}, D^{(2)}] + \partial_x D^{(1)} = 0, \quad (3.6)$$
$$[E^{(1)}, D^{(-1)}] + [A^{(-1)}, D^{(1)}] + [A^{(-2)}, D^{(2)}] + \partial_x D^{(0)} = 0, \quad (3.7)$$

which involves the unknowns $D^{(2)}, D^{(1)}, D^{(0)}$ and $D^{(-1)}$. Moreover, the lower graded equations,

$$[A^{(-1)}, D^{(-n)}] + [A^{(-2)}, D^{(-n+1)}] + \cdots + [A^{(-n)}, D^{(-1)}] = 0, \quad (3.8)$$
$$[A^{(-2)}, D^{(-n)}] + [A^{(-3)}, D^{(-n+1)}] + \cdots + [A^{(-n)}, D^{(-2)}] = 0, \quad (3.9)$$
$$\vdots$$
$$[A^{(-n)}, D^{(-n+1)}] + [A^{(-n+1)}, D^{(-n)}] = 0, \quad (3.10)$$
$$[A^{(-n)}, D^{(-n)}] = 0. \quad (3.11)$$

involves the unknowns $D^{(-1)}, \ldots, D^{(-n)}$. Together they lead to the time evolution equations

$$\partial_{t_2} A^{(-1)} = [E^{(1)}, D^{(-2)}] + [A^{(-1)}, D^{(0)}] + [A^{(-2)}, D^{(1)}] + [A^{(-3)}, D^{(2)}] + \partial_x D^{(-1)}, \quad (3.12)$$
$$\partial_{t_2} A^{(-2)} = [E^{(1)}, D^{(-3)}] + [A^{(-1)}, D^{(-1)}] + \cdots + [A^{(-4)}, D^{(2)}] + \partial_x D^{(-2)}, \quad (3.13)$$
$$\vdots$$
$$\partial_{t_2} A^{(-n)} = [A^{(-1)}, D^{(-n+1)}] + [A^{(-2)}, D^{(-n+2)}] + \cdots + [A^{(-n)}, D^{(0)}] + \partial_x D^{(-n)}. \quad (3.14)$$
In order to obtain the time evolution equations (3.12)–(3.14) we need to solve equations (3.4)–(3.7) for $D^{(2)}$, $D^{(1)}$, $D^{(0)}$, $D^{(-1)}$, and (3.8)–(3.11) for $D^{(-2)}$, $D^{(-3)}$, . . . , $D^{(-n)}$. Let us start from (3.4) which coincides with (2.7) for $N = 2$ and yields $D^{(2)} = D^{(2)} \in \mathcal{K}$ given by (2.10).

Notice that since $[E^{(1)}, D^{(1)}] \in \mathcal{M}$, equation (3.5) imply that $\partial_\alpha a_2 = 0$, $D^{(1)} = 0$ and henceforth $D^{(1)} = D^{(1)} \in \mathcal{K}$, i.e.

$$D^{(1)} = a_1 \left( E^{(0)}_{\alpha_1} + E^{(0)}_{\alpha_2} + \cdots + E^{(0)}_{\alpha_n} + E^{(1)}_{-(\alpha_1 + \cdots + \alpha_n)} \right).$$  

(3.15)

Equation (3.6) imply that

$$\partial_\alpha a_1 = 0, \quad [E^{(1)}, D^{(0)}] + [A^{(-1)}, D^{(2)}] = 0$$

(3.16)

and we shall take $a_1 = 0$ and $a_2 = 1$. Inserting $A^{(-1)} = J_1 E^{(0)}_{\alpha_1}$ and $D^{(0)} = \sum_{i=1}^{n} d_i (\mu_i \cdot H^{(0)})$ we find

$$D^{(0)} = -J_1 \left( \mu_{n-1} \cdot H^{(0)} \right),$$

(3.17)

where $\mu_{n-1}$ is the $(n - 1)$th fundamental weight of $A_n$, i.e.

$$\mu_{n-1} = \sum_{i=1}^{n-1} \left( \frac{2i}{n+1} \right) \alpha_i + \left( \frac{n-1}{n+1} \right) \alpha_n.$$  

(3.18)

where $\alpha_j, j = 1, \ldots, n$ are simple roots of $A_n$. Following the same philosophy, we find from (3.7) that

$$D^{(-1)} = \sum_{i=1}^{n-2} \frac{2i}{n+1} \partial_\alpha J_1 E^{(0)}_{\alpha_{i+1}} + \left( \frac{n-1}{n+1} \right) \partial_\alpha J_1 + J_2 \right) E^{(0)}_{\alpha_{i+1}}$$

$$+ \left( \frac{n-1}{n+1} \partial_\alpha J_1 + J_2 \right) E^{(0)}_{\alpha_1}.$$  

(3.19)

In order to solve equations (3.8)–(3.11) we propose a general solution for $D^{(-i)}, i = 1, \ldots, n$ by assigning to them only negative step operators (lower diagonal) of grade $-i$ as follows,

$$D^{(-i)} = b^{(-i)}_1 E^{(0)}_{\alpha_i},$$

$$D^{(-i-1)} = b^{(-i-1)}_1 E^{(0)}_{-(\alpha_i + \cdots + \alpha_{i-1})} + b^{(-i-1)}_2 E^{(0)}_{-(\alpha_i + \cdots + \alpha_{i+1})},$$

$$\vdots$$

$$D^{(-1)} = b^{(-1)}_1 E^{(0)}_{\alpha_1} + b^{(-1)}_2 E^{(0)}_{\alpha_2} + \cdots + b^{(-1)}_n E^{(0)}_{\alpha_n}$$

(3.20)

where $b^{(-i)}_k$ are coefficients to be determined by equations (3.12)–(3.14).

1 In general the subspaces defined by $E$ is such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M},$$

and each of the above equations gives rise to $\mathcal{K}$ and $\mathcal{M}$ components.
In order to illustrate our formulation consider the following explicit examples,

- \( G = \mathfrak{sl}(3), n = 2 \)

\[
D^{(0)} = -J_1 \left( \frac{2}{3} h_1^{(0)} + \frac{1}{3} h_2^{(0)} \right), \\
D^{(-1)} = (J_2 + \frac{2}{3} \partial_x J_1) E^{(0)}_{-\alpha_1} + (J_2 + \frac{1}{3} \partial_x J_1) E^{(0)}_{\alpha_2}.
\] (3.21)

Equations (3.12) and (3.13) give rise to

\[
D^{(-2)} = -\left( \frac{2}{3} \partial_x^2 J_1 + \partial_x J_2 \right) E^{(0)}_{-\alpha_1 - \alpha_2}
\] (3.22)

and respectively to the following equations of motion

\[
\partial_t J_1 = \partial_x^2 J_1 + 2 \partial_x J_2, \\
\partial_t J_2 = \frac{1}{3} \left( 2 \partial_x^3 J_1 + 3 \partial_x^2 J_2 - 2 J_1 \partial_x J_1 \right).
\] (3.23)

- \( G = \mathfrak{sl}(4), n = 3 \)

\[
D^{(0)} = -J_2 \left( \frac{1}{2} h_1^{(0)} + h_2^{(0)} + \frac{1}{2} h_3^{(0)} \right), \\
D^{(-1)} = \frac{1}{2} \partial_x J_1 E^{(0)}_{-\alpha_1} + (J_2 + \partial_x J_1) E^{(0)}_{\alpha_2} + (J_2 + \frac{1}{2} \partial_x J_1) E^{(0)}_{-\alpha_3}.
\] (3.25)

Likewise, (3.12) and (3.13) yields,

\[
D^{(-2)} = \left( -\frac{1}{2} \partial_x^2 J_1 + J_3 \right) E^{(0)}_{-\alpha_1 - \alpha_2} + \left( -\frac{3}{2} \partial_x^2 J_1 - \partial_x J_2 + J_3 \right) E^{(0)}_{-\alpha_2 - \alpha_3}
\] (3.26)

and the equations of motion

\[
\partial_t J_1 = 2 \left( \partial_x^3 J_1 + \partial_x J_2 \right), \\
\partial_t J_2 = -2 \partial_x^3 J_1 - \partial_x^2 J_2 + J_3 \partial_x J_1 + 2 \partial_x J_3, \\
\partial_t J_3 = \frac{1}{2} \left( \partial_x^4 J_1 - J_1 \partial_x^2 J_1 - 2 \partial_x^2 J_3 + J_2 \partial_x J_1 \right).
\] (3.27)

4. Miura transformation

In this section we consider \( G = A_4 \) and propose a Miura-gauge transformation \( S \) to connect the two gauge potentials \( A_{mKdV}^x \) and \( A_{KdV}^x \) as,

\[
A_{KdV}^x = S A_{mKdV}^x S^{-1} + S \partial_x S^{-1},
\] (4.1)
where

\[ A_{x}^{mKdV} = E^{(1)} + \sum_{i=1}^{n} v_{i}h_{i}, \quad A_{x}^{KdV} = E^{(1)} + \sum_{j=1}^{n} J_{n+1-j}E_{-(\alpha_{i}+\alpha_{i+1}+\cdots+\alpha_{n})}. \]  \hspace{1cm} (4.2)

The first thing to notice is that both potentials in (4.2) are defined according to the same Lie algebraic structure, i.e. principal gradation, \( Q_{ppal} \) and share the same constant, grade one semisimple element \( E^{(1)} \) given in (2.3) (see appendix A). The desired Miura-gauge transformation \( S \) is then constructed to preserve such structure. Let us recall the \( \mathfrak{sl}(2) \) case where the Miura transformation \( J = \epsilon \partial_{x}v - v^{2}, \epsilon = \pm 1 \) connects the two dimensional gauge potentials

\[ A_{x}^{mKdV} = \begin{pmatrix} v & 1 \\ \lambda & -v \end{pmatrix}, \quad A_{x}^{KdV} = \begin{pmatrix} 0 & 1 \\ \lambda + J & 0 \end{pmatrix}. \]  \hspace{1cm} (4.3)

In fact, in reference [8] we have constructed two solutions for the Miura-gauge transformation (4.1), namely,

\[ S_{\epsilon=1} = I + s^{(-1)} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, \quad S_{\epsilon=-1} = E^{(-1)} + s^{(-2)} = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & -\lambda^{-1}v \end{pmatrix}, \]  \hspace{1cm} (4.4)

where \( s^{(-i)} \in G_{-i} \) and \( E^{(-1)} = E_{-\alpha_{1}} + \lambda^{-1}E_{\alpha_{1}} = \begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} \) corresponding to \( \epsilon = +1 \) and \( \epsilon = -1 \) respectively.

Likewise the \( \mathfrak{sl}(2) \) case (4.4), we now consider \( G = \mathfrak{sl}(3) \) where

\[ A_{x}^{mKdV} = \begin{pmatrix} v_{1} & 1 & 0 \\ 0 & -v_{1} + v_{2} & 1 \\ \lambda & 0 & -v_{2} \end{pmatrix}, \quad A_{x}^{KdV} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda + J_{1} & J_{2} & 0 \end{pmatrix} \]  \hspace{1cm} (4.5)

\( E \equiv E^{(1)} = E_{\alpha_{1}} + E_{\alpha_{2}} + \lambda E_{-\alpha_{1}-\alpha_{2}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix} \) and propose 3 solutions for \( S \) satisfying (4.1),

\[ S_{1} = I + s^{(-1)} + s^{(-2)} = \begin{pmatrix} 1 & 0 & 0 \\ v_{1} & 1 & 0 \\ v_{1}^{2} - \partial_{x}v_{1} & v_{2} & 1 \end{pmatrix} \]  \hspace{1cm} (4.6)

leading to the Miura transformation

\[ J_{1}^{(1)} = -\partial_{x}(v_{1} + v_{2}) + v_{1}^{2} - v_{1}v_{2} + v_{2}^{2}, \]
\[ J_{2}^{(1)} = v_{1}\partial_{x}(-2v_{1} + v_{2}) + \partial_{x}^{2}v_{1} + v_{1}^{2}v_{2} - v_{1}v_{2}^{2}. \]  \hspace{1cm} (4.7)

The second solution is given by

\[ S_{2} = E^{(-1)} + s^{(-2)} + s^{(-3)} = \begin{pmatrix} 0 & 0 & \lambda^{-1} \\ 1 & 0 & -\lambda^{-1}v_{2} \\ v_{1} - v_{2} & 1 & \lambda^{-1}(v_{2}^{2} + v_{2}) \end{pmatrix}, \]  \hspace{1cm} (4.8)
following general proposition for the
structure the two gauge potentials
Miura-gauge transformation (4.1) and notice a general pattern emerging that induces to the
transformation
\[
J_1^{(2)} = -\partial_x(v_1 - 2v_2) + v_1^2 - v_1v_2 + v_2^2,
J_2^{(2)} = -v_2\partial_x(v_1 + v_2) - \partial^2_xv_2 + v_1^2v_2 - v_1v_2^2.
\]
(4.9)
A third solution is given by
\[
S_3 = E^{(-2)} + s^{(-3)} + s^{(-4)} = \begin{pmatrix}
0 & \lambda^{-1} & 0 \\
0 & \lambda^{-1}(-v_1 + v_2) & \lambda^{-1} \\
1 & \lambda^{-1}A & -\lambda^{-1}v_1
\end{pmatrix},
\]
(4.10)
for
\[
E^{(-2)} = \lambda^{-1}E = \lambda^{-1}E_{(1)} + \lambda^{-1}E_{(2)} + E_{(1-1)} + E_{(-1-2)} = \begin{pmatrix}
0 & \lambda^{-1} & 0 \\
0 & 0 & \lambda^{-1} \\
1 & 0 & 0
\end{pmatrix},
\]
and
\[
A = \partial_x(v_1 - v_2) + (v_1 - v_2)^2 \text{ leading to the Miura transformation}
\]
\[
J_1^{(3)} = 2\partial_xv_1 - \partial_xv_2 + v_1^2 - v_1v_2 + v_2^2,
J_2^{(3)} = -(v_1 + v_2)\partial_xv_1 + 2(v_1 - v_2)\partial_xv_2 + \partial^2_xv(-v_1 + v_2) + v_1^2v_2 - v_1v_2^2.
\]
(4.11)
In the appendix B we develop the example for \(G = sl(4)\) by constructing 4 solutions for the
Miura-gauge transformation (4.1) and notice a general pattern emerging that induces to the
following general proposition for the \(sl(n+1)\) case,

- Given \(E^{(1)} = \sum_{j=1}^{n} E_{a_j} + \lambda E_{(a_1 + \cdots + a_n)}\), there are \(n\) generators of grade
  \(q = -1, -2, \ldots, -n\) commuting with \(E^{(1)}\), namely, \(\{E^{(-1)}, E^{(-2)}, \ldots, E^{(-n)}\} \in \hat{K}\)
  (see for instance appendix of reference [4]).
- Propose solution \(S_i = E^{(-i)} + s^{(-i-1)} + s^{(-i-2)} + \cdots + s^{(-i-n)}\), \(i = 0, 1, \ldots, n\) where
  \(E^{(0)} = I\) and \(S_i+n+1 = \lambda S_i\) such that the algebraic structure of gauge potentials (4.2) is
  preserved.2
- Each solution generates a Miura transformation \(J^{(0)}_a, a = 1, \ldots, n+1\).

Notice that the Miura transformation \(S\) acting on the zero curvature representation (2.5)
transforms both gauge potentials, i.e. \(A_{1KdV} \to A_{1KdV}^{KdV}\) and \(A_{mKdV} \to A_{mKdV}^{KdV}\), for all values of \(N\).
This implies that the gauge-Miura transformation \(S\) is universal within the hierarchy in the
sense that all flows (labeled by \(N\)) are transformed by the same \(S\).

5. Equations of motion

We now conjecture that the equations of motion of the two hierarchies are connected as in the
well known case of the \(A_1\) KdV and mKdV equations,
\[
4\partial_xJ - \partial^3_xJ - 6J\partial_xJ = \mathcal{P}_\epsilon(4\partial_xv - \partial_x(\partial^2_xv - 2v^3)) = 0, \quad \mathcal{P}_\epsilon = (\epsilon\partial_x - 2v), \quad \epsilon = \pm 1.
\]
(5.1)

2 It is clear that \(S\) contains a highest grade component commuting with \(E^{(1)}\) in order to preserve the common graded
structure the two gauge potentials \(A_{1KdV}^{KdV}\) and \(A_{mKdV}^{KdV}\).
In fact, we argue that the equations of motion of the generalized $A_n$ mKdV and KdV hierarchies are related by a matrix operator $\mathcal{P}$, as $[\text{KdV}(J_i)] = \mathcal{P}[\text{mKdV}(v_i)]$, or in components,

\[
\begin{pmatrix}
\frac{\partial J_1}{\partial t} \\
\frac{\partial J_2}{\partial t} \\
\vdots \\
\frac{\partial J_n}{\partial t}
\end{pmatrix} = \mathcal{P} \begin{pmatrix}
\frac{\partial v_1}{\partial t} \\
\frac{\partial v_2}{\partial t} \\
\vdots \\
\frac{\partial v_n}{\partial t}
\end{pmatrix},
\]

(5.2)

where the matrix operator $\mathcal{P}$ is denoted $\mathcal{P} = (P_{ij})$.

Explicitly we have considered the equations of motion (2.18) and (3.24) for $sl(3)$ written in the form (5.2) as

\[
\begin{pmatrix}
\frac{\partial J_1}{\partial t} \\
\frac{\partial J_2}{\partial t}
\end{pmatrix} = \mathcal{P} \begin{pmatrix}
\frac{\partial v_1}{\partial t} \\
\frac{\partial v_2}{\partial t}
\end{pmatrix},
\]

(5.3)

It is clear that there is a different $\mathcal{P}$ operator associated to each solution of the Miura-gauge transformation (4.1).

- For $S_1$ (4.6) and Miura (4.7),

\[
\begin{align*}
P_{11}^{(1)} &= -\partial_x + 2v_1 - v_2, \\
P_{12}^{(1)} &= -\partial_x - v_1 + 2v_2, \\
P_{21}^{(1)} &= \partial_x^2 - 2v_1 \partial_x - 2\partial_x v_1 + \partial_x v_2 + 2v_1 v_2 - v_2^2, \\
P_{22}^{(1)} &= v_1 \partial_x - 2v_1 v_2 + v_1^2.
\end{align*}
\]

(5.4)

- Likewise for the Miura-gauge transformation $S_2$ (4.8) and Miura (4.9),

\[
\begin{align*}
P_{11}^{(2)} &= -\partial_x + 2v_1 - v_2, \\
P_{12}^{(2)} &= 2\partial_x - v_1 + 2v_2, \\
P_{21}^{(2)} &= -v_2 \partial_x + 2v_1 v_2 - v_2^2, \\
P_{22}^{(2)} &= -\partial_x^2 - v_2 \partial_x - \partial_x v_1 - \partial_x v_2 - 2v_1 v_2 + v_1^2.
\end{align*}
\]

(5.5)

- For $S_3$ given in (4.10) and Miura (4.11),

\[
\begin{align*}
P_{11}^{(3)} &= 2\partial_x + 2v_1 - v_2, \\
P_{12}^{(3)} &= -\partial_x - v_1 + 2v_2, \\
P_{21}^{(3)} &= -\partial_x^2 - v_1 \partial_x + v_2 \partial_x - \partial_x v_1 + 2\partial_x v_2 + 2v_1 v_2 - v_2^2, \\
P_{22}^{(3)} &= \partial_x^2 + 2v_1 \partial_x - 2v_2 \partial_x + \partial_x v_1 - 2\partial_x v_2 + v_2^2 - 2v_1 v_2.
\end{align*}
\]

(5.6)

For the $sl(4)$ case the same procedure can be employed for each associated Miura-gauge transformation given in appendix B.
6. Bäcklund transformation

In this section we generalize the results of reference [8] by constructing the Bäcklund transformation for the $A_n$-KdV hierarchy from the Miura and Bäcklund-gauge transformations constructed for the $A_n$-mKdV hierarchy. Consider the Bäcklund-gauge transformation for the $A_n$-KdV hierarchy proposed in [9],

$$U(\phi_i, \psi_i)A_{\mu}^{\text{mKdV}}(\phi_i) = A_{\mu}^{\text{mKdV}}(\psi_i)U(\phi_i, \psi_i) + \partial_{\mu}U(\phi_i, \psi_i),$$  \hspace{1cm} (6.1)

where $u_i = \partial_i \phi_i, v_i = \partial_i \psi_i$. Also, the Miura-gauge transformation can be written as,

$$A_{\mu}^{\text{KdV}}(J_i) = SA_{\mu}^{\text{mKdV}}(u_i)S^{-1} + S\partial_{\mu}S^{-1}.$$  \hspace{1cm} (6.2)

Let $K(I, J)$ be the generator of the Bäcklund-gauge transformation for the $A_n$-KdV hierarchy, i.e.

$$K(I_i, J_i)A_{\mu}^{\text{KdV}}(I_i) = A_{\mu}^{\text{KdV}}(J_i)K(I_i, J_i) + \partial_{\mu}K(I_i, J_i).$$  \hspace{1cm} (6.3)

Inserting (6.2) into (6.1), we find

$$K(I_i, J_i) = S(v_i)U(\phi_i, \psi_i)S^{-1}(u_i).$$  \hspace{1cm} (6.4)

Notice that $K(I_i, J_i)$ given by the rhs of (6.4) depend upon mKdV variables, $u_i$ and $v_i$. In reference [8] we have shown that the gauge-Bäcklund transformation $K(I_i, J_i)$ was entirely written in terms of KdV variables, $I$ and $J$ if we use two different Miura-gauge transformations, for the right $S(u) = S_1$ and for the left $S(v)^{-1} = S_2^{-1}$ multiplications (see (4.4)). Here we will follow the same principle by choosing different $S$ solutions for right and left multiplication in (6.4). Bäcklund transformation for the $A_n$ Toda theory was first proposed in [20] and in [9], it was generalized as a gauge transformation constructed and classified according to a graded affine Lie algebraic structure. Such construction was shown to be universal within the hierarchy, i.e. in the sense that it is same for all positive and negative flows.

Consider, as an illustration, the type I Bäcklund-gauge transformation for $sl(3)$ mKdV hierarchy (see for instance [9]),

$$U(\phi_i, \psi_i) = \begin{pmatrix} 1 & 0 & \sigma e^{-\phi_1 - \psi_1} \\ \sigma e^{\phi_1 + \psi_1 - \psi_2} & 1 & 0 \\ 0 & \sigma e^{\phi_1 + \phi_2 + \psi_2} & 1 \end{pmatrix},$$  \hspace{1cm} (6.5)

which yield from (6.1) for $A_1$ the following equations corresponding to the mKdV Bäcklund transformation,

$$u_1 - v_1 = \sigma (e^{\phi_1 + \phi_2 - \psi_2} - e^{-\phi_1 - \psi_1}), \quad u_2 - v_2 = \sigma (e^{-\phi_1 + \phi_2 + \psi_2} - e^{-\phi_1 - \psi_1}),$$  \hspace{1cm} (6.6)

where $u_i = \partial_i \phi_i, v_i = \partial_i \psi_i, i = 1, 2$ and take the two solutions $S_1(u_i)$ and $S_2(v_i)$ given by (4.6) and (4.10) to be inserted as right and left multiplications in (6.4) yielding the following matrix elements,

$$K_{11} = (-u_1 + \sigma e^{\phi_1 + \psi_1 - \psi_2})\lambda^{-1}$$  \hspace{1cm} (6.7)

\footnote{Observe that the relativistic Toda model correspond to the first negative flow [21].}
\[ K_{22} = (-u_2 - v_1 + v_2 + \sigma e^{-\phi_1 + \phi_2 + \psi_2})\lambda^{-1} \]  
(6.8)

\[ K_{33} = (-v_1 + \sigma e^{-\phi_2 - \psi_1})\lambda^{-1} \]  
(6.9)

\[ K_{12} = K_{21} = \lambda^{-1} \]  
(6.10)

\[ K_{31} = 1 + Y\lambda^{-1} \]  
(6.11)

\[ K_{13} = 0 \]  
(6.12)

\[ K_{21} = (\partial_x u_1 - u_1^2 + u_1 u_2 - u_1 \sigma e^{-\phi_1 + \phi_2 + \psi_2} - (v_1 - v_2)(-u_1 + \sigma e^{\phi_1 + \psi_1 - \psi_2}))\lambda^{-1} \]  
(6.13)

\[ K_{32} = (\partial_x v_1 - \partial_x v_2 + v_1^2 + v_2^2 - 2v_1 v_2 + u_2 v_1 - u_2 \sigma e^{-\phi_2 - \psi_1} - v_1 \sigma e^{-\phi_1 + \phi_2 + \psi_2})\lambda^{-1}, \]  
(6.14)

where

\[ Y = u_1 \sigma e^{-\phi_1 + \phi_2 + \psi_2} + (\partial_x u_1 - u_1^2 + u_1 u_2)(-v_1 + \sigma e^{-\phi_2 - \psi_1}) \]

\[ + (\partial_x v_1 - \partial_x v_2 + v_1^2 + v_2^2 - 2v_1 v_2)(-u_1 + \sigma e^{\phi_1 + \psi_1 - \psi_2}). \]  
(6.15)

We now show how to re-write the Bäcklund matrix \( K(I_i, J_i) \) in terms of KdV variables. Subtracting the diagonal terms and using (6.6),

\[ K_{11} - K_{22} = (-u_1 + v_1 + u_2 - v_2 + \sigma e^{\phi_1 + \psi_1 - \psi_2} - \sigma e^{-\phi_1 + \phi_2 + \psi_2})\lambda^{-1} = 0 \]  
(6.16)

\[ K_{22} - K_{33} = (-u_2 + v_2 + \sigma e^{-\phi_1 + \phi_2 + \psi_2} - \sigma e^{-\phi_1 - \psi_2})\lambda^{-1} = 0 \]  
(6.17)

and henceforth

\[ K_{11} = K_{22} = K_{33} \equiv \frac{1}{3}Q\lambda^{-1}. \]  
(6.18)

Acting with \( x \) derivative on \( Q \) and re-arranging terms,

\[ \partial_x Q = \lambda\partial_x (K_{11} + K_{22} + K_{33}) \]

\[ = \partial_x (-u_1 - u_2 - 2v_1 + v_2) + u_1 \sigma (e^{-\phi_1 + \psi_1 - \psi_2} - e^{-\phi_1 + \phi_2 + \psi_2}) \]

\[ + u_2 \sigma (e^{-\phi_1 + \phi_2 + \psi_2} - e^{-\phi_2 + \psi_1}) \]

\[ + v_1 \sigma (e^{\phi_1 + \psi_1 - \psi_2} - e^{-\phi_1 + \phi_2 + \psi_2}). \]

After eliminating the exponentials from the mKdV Bäcklund transformations (6.6) and subsequent use of Miura transformations (4.7) and (4.11), i.e.

\[ I_1(u_1) = J_1^{(1)}(u_1) = -\partial_x u_1 - \partial_x u_2 + u_1^2 + u_2^2 - u_1 u_2 \]  
(6.19)

\[ J_1(v_1) = J_1^{(3)}(v_1) = 2\partial_x v_1 - \partial_x v_2 + v_1^2 + v_2^2 - v_1 v_2 \]  
(6.20)

we find,

\[ \partial_x Q = I_1 - J_1. \]  
(6.21)
The $K_{21}$ matrix element also can be written entirely in terms of KdV variables, e.g. replacing the $\partial_t u_1$ term using (6.7),

$$
\lambda K_{21} = -\frac{1}{3} \partial_t Q + u_1 \sigma (e^{\phi_1 + \phi_2} - e^{-\phi_1 - \phi_2}) - u_1 (u_1 - u_2 - v_1 + v_2)
$$

$$
= -\frac{1}{3} \partial_t Q,
$$

where we have used (6.6). Likewise, using the Miura transformation (6.20) in (6.14) and then replacing the remaining $\partial_t v_1$ using (6.9) we find,

$$
\lambda K_{32} = J_1 - \partial_t v_1 - v_1 v_2 + u_2 v_1 - u_2 \sigma e^{-\phi_2 - \phi_1} - v_1 \sigma e^{-\phi_1 + \phi_2 + \phi_3}
$$

$$
= J_1 + \frac{1}{3} \partial_t Q + v_1 \sigma (e^{-\phi_2 - \phi_1} - e^{-\phi_1 + \phi_2 + \phi_3}) + v_2 (u_2 - v_2) = J_1 + \frac{1}{3} \partial_t Q,
$$

where we have used (6.6) to eliminate the exponentials. Finally doing the same procedure for (6.11) we find

$$
K_{31} = 1 + \frac{Y}{\lambda},
$$

(6.22)

where $Y = \sigma^3 - \frac{Q^3}{2} + \frac{1}{2} Q J_1$ and

$$
K(I, J) = 
\begin{pmatrix}
\frac{1}{3\lambda} Q & \frac{1}{\lambda} (I - J_1) & 0 \\
\frac{1}{3\lambda} (I - J_1) & \frac{1}{3\lambda} Q & \frac{1}{\lambda} \\
1 + \frac{Y}{\lambda} & \frac{1}{3\lambda} (I + 2 J_1) & \frac{1}{3\lambda} Q
\end{pmatrix},
$$

(6.23)

It can be verified that $\det K = \frac{1}{\lambda} + \frac{Y}{\lambda}$.

We have verified that the very same argument follows if we instead of the pair $S_3(v_1)$ and $S_3^{-1}(u_1)$, use in (6.4) $S_1(v_1)$ and $S_2^{-1}(u_1)$. The resulting Bäcklund-gauge transformation $K$ for the KdV hierarchy has the same form as (6.23) but now written in terms of the corresponding Miura fields given in (4.7) and (4.9) and so is the resulting Bäcklund-gauge generator for the remaining pair $S_1, S_2$.

Let us now write down explicit Bäcklund transformation for the $A_2$-mKdV system (3.23) and (3.24) according to $t = t_2$. Employing the gauge-Bäcklund transformation (6.23) in (6.3) where gauge potentials $A_{KdV}^\mu$ are given by,

$$
A_{KdV} = 
\begin{pmatrix}
0 & 1 & 0 \\
\lambda + J_2 & 0 & 1 \\
J_1 & 0 & \lambda
\end{pmatrix},
$$

$$
A_{KdV}^\mu = 
\begin{pmatrix}
-\frac{2 J_1}{3} & 0 & 1 \\
\lambda + \frac{2}{3} \partial_x J_1 + J_2 & \frac{J_1}{3} & 0 \\
\frac{2}{3} \partial_x^2 J_1 - \partial_x J_2 & \lambda + \frac{1}{3} \partial_x J_1 + J_2 & \frac{J_1}{3}
\end{pmatrix}
$$

(6.24)

The nontrivial equations obtained from (6.3) for $A_{KdV}^\mu$ are

$$
I_2 - J_2 = \partial_x J_1 - \frac{1}{3} (I_1 - J_1) Q
$$

(6.25)
\[ \partial_t (I_1 - J_1) = -3I_2 + J_1Q - \frac{1}{9}Q^3 + 3\sigma^3 \]  
(6.26)
\[ \partial_t (I_1 + 2J_1) = -3J_2 + I_1Q - \frac{1}{9}Q^3 + 3\sigma^3 \]  
(6.27)

which are compatible in the sense that a linear combination of any two yields the third. Those correspond to matrix elements (3, 1), (2, 1) and (3, 2) respectively.

As for the \( A_{KdV} \) potential, the nontrivial equations corresponding to matrix diagonal elements are,

\[ \partial_t Q = 2\partial_t I_1 + 3I_2 - I_1Q + \frac{1}{3}Q(I_1 - J_1) + \frac{1}{9}Q^3 - 3\sigma^3, \]  
(6.28)
\[ \partial_t Q = \partial_t (I_1 - 2J_1) + 3(I_2 - J_2) + \frac{1}{3}(I_1 - J_1)Q, \]  
(6.29)
\[ \partial_t Q = -\partial_t J_1 - 3J_2 + J_1Q + \frac{1}{3}Q(I_1 - J_1) - \frac{1}{9}Q^3 + 3\sigma^3. \]  
(6.30)

Other nontrivial equations are,

\[ 3\partial_t (I_1 - J_1) = 6\partial_t^2 I_1 + 9\partial_t J_2 - 2I_1^2 + J_1^2 + I_1J_1 - (2\partial_t(I_1 - J_1) + 3(I_2 - J_2))Q, \]  
(6.31)
\[ 3\partial_t (I_1 + 2J_1) = 6\partial_t^2 I_1 + 9\partial_t J_2 + I_1^2 - 2J_1^2 + I_1J_1 + (\partial_t(I_1 - J_1) + 3(I_2 - J_2))Q, \]  
(6.32)
\[ \partial_t \left( -\frac{Q^3}{27} + \frac{1}{3}Q(I_1) \right) = \frac{1}{9} \left( I_1 \left( \partial_t(4I_1 - J_1) + 6I_2 - 3J_2 - 2I_1Q + \frac{Q^3}{9} - 3\sigma^3 \right) \right) \]
\[ + I_1 \left( \partial_t(2I_1 + J_1) + 3I_2 + 3J_2 + \frac{2Q^3}{9} - 6\sigma^3 \right) \]
\[ + Q\partial_t \left( -3I_2 + 3J_2 + 2\partial_t I_1 + 2\partial_t J_1 - J_1^2 Q \right) \]  
(6.33)

and correspond to matrix elements (2, 1), (3, 2) and (3, 1) respectively.

Subtracting (6.28)–(6.29) and (6.29)–(6.30) we eliminate \( \partial_t Q \) and recover (6.27) and (6.26) respectively. Moreover, substituting (6.25) in (6.29) we find

\[ \partial_t Q = \partial_t (I_1 - J_1) + 2(I_2 - J_2), \]  
(6.34)

which leads directly to the equations of motion (3.23) for \( I_1 \) and \( J_1 \) by acting with \( \partial_t \) in (6.34), i.e.

\[ \partial_t (I_1 - J_1) = \partial_t^2 (I_1 - J_1) + 2\partial_t (I_2 - J_2). \]  
(6.35)

Subtracting (6.32) and (6.31) we find

\[ 3\partial_t J_1 = -2\partial_t^2 (I_1 - J_1) - 3\partial_t (I_2 - J_2) + (I_1^2 - J_1^2) + (\partial_t(I_1 - J_1) + 2(I_2 - J_2))Q. \]  
(6.36)
Acting with $\partial_t$ on equation (6.25) and using the fact that $\partial_t Q = I_1 - J_1$,

$$3\partial_t(I_2 - J_2) = \partial_t(3\partial_t J_1 - Q\partial_t Q)$$

$$= \partial_t \left( Q(-(I_2 - J_2) + \partial_t J_1 - \frac{1}{3}(I_1 - J_1)Q) \right)$$

$$+ \partial_t \left(-2\partial_t^3(I_1 - J_1) - 3\partial_t(I_2 - J_2) + (I_1^2 - J_1^2)\right),$$

where we have used (6.36) and substituted $\partial_t Q$ using (6.29). After making use of (6.25) to eliminate the first term (proportional to $Q$) we end up with the equation of motion for $I_2$ and $J_2$ (3.24), i.e.

$$\partial_t(I_2 - J_2) = -\frac{1}{3}\partial_t \left( 2\partial_t^3(I_1 - J_1) + 3\partial_t(I_2 - J_2) - (I_1^2 - J_1^2)\right). \quad (6.37)$$

Notice that, since the Bäcklund transformation generator $K(I, J)$ in (6.23) is the same for all flows, the same procedure can be employed for higher flows $t = t_n$ generating therefore, the Bäcklund transformation for the entire hierarchy in a systematic manner. The same can be extended to a general $A_n$ KdV/mKdV integrable models.

7. Discussion and further developments

The gauge invariance of zero curvature representation was explored in order to map the $A_n$-mKdV hierarchy into its counterpart, the $A_n$-KdV hierarchy. Such map is known as generalized Miura-gauge transformation and is generated by a gauge transformation denoted by $S$. We have shown by developing explicit examples, that $S$ has the virtue of preserving the algebraic structure of the Lax operators (i.e. two dimensional gauge potentials $A_{mKdV}^\mu$ and $A_{KdV}^\mu$). An interesting discovery is that $S$ is not unique, as it was already been suggested from the well known $sl(2)$ example (5.1) in which the Miura transformation is two-fold degenerated, parameterized by $\epsilon = \pm 1$. The $A_n$-Miura-gauge transformation present an $n + 1$ degeneracy and was shown to be classified according to the elements of the Kernel of $E^{(1)}, \tilde{K}$, supplemented with the identity element $I$. The role of the subgroup of transformations generated by such subset of generators is still not entirely clear and is currently under investigation.

The Bäcklund transformation for the KdV hierarchy, in turn was inherited from the gauge-Bäcklund transformation for the mKdV hierarchy by left–right multiplication by Miura-gauge generators as in (6.4), i.e. $K(J, I) = S(I) \hat{U}(\phi, \psi) S^{-1}(J)$.

A surprising feature of the construction, is a nontrivial change of mKdV to KdV variables $(\phi, \psi) \rightarrow (J, I)$ which was shown to be possible if the left and right Miura transformations appearing in (6.4) correspond to different degenerate solutions associated to $\tilde{K}$. Such fact was already realized for the $sl(2)$ case in [8] and was explicitly verified for several combinations of Miura solutions for $sl(3)$ example.

As a matter of fact, the resulting gauge Bäcklund transformation was entirely written in terms of KdV variables and appears as an universal object within the hierarchy, in the sense that it is the same for all flows. As a byproduct, it generates, in a systematic manner, the Bäcklund transformation for all flows. Moreover, this method provides a classification of integrable defects as proposed in [14] and may also be extended to other generalized type II Bäcklund transformations from mKdV to KdV hierarchies [9]. The framework employed in this paper may be also extended to Lie algebras other than $A_n$, whose Dynkin diagram may connect more than two nearest neighbors, e.g. $B_4, E_6, E_7, E_8$ in the line of reference [23] or to non-simply
laced algebras. Application to twisted algebras may also provide interesting examples in the lines of reference [24].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Affine algebraic structure

Consider an affine Kac–Moody algebra \( \hat{G} \) defined by

\[
[H^{(l)}_i, H^{(k)}_j] = \kappa l \delta_{i+k,0} \delta_{i,j},
\]

\[
[H^{(l)}_i, E^{(k)}_\alpha] = \alpha (E^{(l+k)}_\alpha)
\]

\[
[E^{(l)}_\beta, E^{(k)}_\alpha] = \begin{cases} \epsilon(\alpha, \beta)E^{(l+k)}_{\alpha+\beta}, & \alpha + \beta = \text{root}, \\ \frac{2}{\alpha^2} \alpha \cdot H^{(l+k)} + \kappa l \delta_{i+k,0}, & \alpha + \beta = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

\( i, j = 1, \ldots \text{rank } G, l, k \in \mathbb{Z} \). Let \( Q \) be a grading operator and consider a decomposition of the affine algebra \( \hat{G} \) into grades subspaces, \( G_a \) such that,

\[
\hat{G} = \sum_{a \in \mathbb{Z}} G_a, \quad [Q, G_a] = a G_a, \quad [G_a, G_b] \in G_{a+b}.
\]

In this paper we shall discuss \( \hat{G} = \hat{sl}(n+1) \) endowed with the principal gradation in which

\[
Q^{\text{pal}} = (n+1)d + \sum_{a=1}^n \mu_a \cdot h,
\]

where \( d \) is the derivation operator, i.e.

\[
d, T^{(l)}_i = lT^{(l)}_i, \quad T^{(l)}_i = h^{(l)}_i \text{ or } E^{(l)}_\alpha
\]

and

\[
[\mu_a \cdot h^{(l)}_i, E^{(k)}_\alpha] = (\mu_a \cdot \alpha)E^{(l+k)}_\alpha.
\]

Here \( \mu_a \) and \( \alpha_a \) are the fundamental weights and simple roots respectively, \( \mu_a \cdot \alpha_b = \delta_{a,b} \), \( a, b = 1, \ldots, n \), and have normalized all roots of \( \hat{sl}(n+1) \) such that \( \alpha^2 = 2 \). The operator \( Q \) in (A.2) induces the following graded subspaces,

\[
G_{(a+1)} = \{ h^{(l)}_1, \ldots, h^{(l)}_n \}.
\]
In fact there are precisely 4 solutions for \( S \) satisfying (4.1) acting on 

\[
\hat{G}_{l(\ell+1)+1} = \{ E_{a_1}^{(0)}, \ldots, E_{a_n}^{(0)}, E_{-a_1-\cdots-a_n}^{(\ell+1)} \},
\]

\[
\hat{G}_{l(\ell+1)+2} = \{ E_{a_1+a_2}^{(0)}, E_{a_2+a_3}^{(0)}, \ldots, E_{a_{n-1}+a_n}^{(0)}, E_{-a_{n-1}-\cdots-a_n}^{(\ell+1)}, E_{-a_2-\cdots-a_n}^{(\ell+1)} \},
\]

\[
\vdots
\]

\[
\hat{G}_{l(\ell+1)+n} = \{ E_{-a_1}^{(\ell+1)}, \ldots, E_{-a_n}^{(\ell+1)}, E_{a_1+\cdots+a_n}^{(0)} \}.
\]

where \( h^{(0)}_l = \alpha_i \cdot H^{(0)} \).

Let \( \hat{K} \) denote the Kernel of \( E \) composed by all elements within each subspace of grade \( q \) commuting with \( E \). In particular denote them by \( E^{(\cdot q)} \in \hat{K} \), \( q = 1, \ldots, n \) mod \( (n+1) \), (see appendix of reference [4]) e.g.

\[
E^{(-n)} = \sum_{k=1}^n E_{-\alpha_k}^{(-1)} + E_{-\alpha_1-\cdots-\alpha_n}^{(0)},
\]

\[
E^{(-n+1)} = \sum_{k=1}^{n-1} E_{-\alpha_k+\alpha_{k+1}}^{(-1)} + E_{-\alpha_1-\cdots-\alpha_{n-1}}^{(0)} + E_{-\alpha_2-\cdots-\alpha_n}^{(0)},
\]

\[
\vdots
\]

\[
E^{(-1)} = \sum_{k=1}^n E_{-\alpha_k}^{(0)} + E_{-\alpha_1-\cdots-\alpha_n}^{(-1)}.
\]

**Appendix B. Miura transformation for \( sl(4) \)—example**

We now consider \( \hat{G} = sl(4) \) where \( \hat{E} \equiv \hat{E}^{(1)} = E_{a_1} + E_{a_2} + E_{a_3} + \lambda E_{-a_1-a_2-a_3} \) and propose four solutions for \( S \) satisfying (4.1) acting on

\[
A^{\text{mKdV}}_x = \begin{pmatrix}
  v_1 & 1 & 0 & 0 \\
  0 & -v_1 + v_2 & 1 & 0 \\
  \lambda & 0 & -v_2 + v_3 & 0 \\
\end{pmatrix} \quad A^{\text{KaV}}_x = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  \lambda + J_1 & J_2 & J_3 & 0 \\
\end{pmatrix}.
\]

(B.1)

Let

\[
S_1 = I + s^{(-1)} + s^{(-2)} + s^{(-3)}
\]

\[
= \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  v_1 & 0 & 0 & 0 \\
  -v_1^2 + v_2^2 & 1 & 0 & 0 \\
  (-v_1^2 - 3v_2^2 - 3v_3^2) & -v_1 v_2 & v_2^2 & 1 \\
\end{pmatrix}
\]

(B.2)

\(^4\)In fact there are precisely \( n \) generators commuting with \( E \) of grade given by the exponents modulo the Coxeter number \( h \) which in the \( \Lambda_n \) case are \( q = 1, \ldots, n \) modulo \( n + 1 \).
leading to the Miura transformation

\[ J_1^{(1)} = -\partial_t(v_1 + v_2 + v_3) + v_1^2 + v_2^2 + v_3^2 - v_1 v_2 - v_2 v_3, \]
\[ J_2^{(1)} = \partial_t^2(2v_1 + v_2) + (-4v_1 + v_2)(\partial_t v_1) + 2(v_1 - v_2)(\partial_t v_2) \]
\[ + v_2(\partial_t v_3) + v_1 v_2^2 - v_1 v_2^2 + v_2^2 v_3 - v_2 v_3^2, \]
\[ J_3^{(1)} = -\partial_t^3 v_1 + v_1 \partial_t^2(2v_1 - v_2) + \partial_t(2v_1 - v_2)(\partial_t v_1) \]
\[ + (v_2^2 + v_3^2 - 2v_1 v_2 - v_2 v_3)(\partial_t v_1) \]
\[ + (-v_1^2 + v_1 v_2)(\partial_t v_2) + (v_1^3 - v_1 v_2)(\partial_t v_3) + v_1^2 v_2 v_3 \]
\[ - v_1 v_2^2 v_3 + v_1 v_2 v_3^2 - v_1^2 v_3^2. \]

The second solution is given by

\[ S_2 = E^{(-1)} + s^{(-2)} + s^{(-3)} + s^{(-4)} \]
\[ = \begin{pmatrix}
0 & 0 & 0 & \lambda^{-1} \\
1 & 0 & 0 & -\lambda^{-1} v_3 \\
v_1 - v_3 & 1 & 0 & \lambda^{-1}(\partial_t v_3 + v_3^2)
\end{pmatrix}, \]

(B.3)

for \( E^{(-1)} = E^{(-1)} = E_{-a_1} + E_{-a_2} + E_{-a_3} + \lambda^{-1}E_{a_1+a_2+a_3} \), which leads to the Miura transformation

\[ J_1^{(2)} = -\partial_t(v_1 + v_2 - 3v_3) + v_1^2 + v_2^2 + v_3^2 - v_1 v_2 - v_2 v_3, \]
\[ J_2^{(2)} = \partial_t^2(v_1 - 3v_3) - 2v_1(\partial_t v_1) + (v_1 - v_3)(\partial_t v_2) - 2v_3(\partial_t v_3) \]
\[ + v_1^2 v_2 - v_1 v_2^2 + v_2^2 v_3 - v_2 v_3^2, \]
\[ J_3^{(2)} = \partial_t^3 v_3 + v_3 \partial_t^2(v_1 + v_3) + \partial_t(v_1 + v_2)(\partial_t v_3) + (v_1^3 - 2v_1 v_2)(\partial_t v_1) + v_1 v_2(\partial_t v_2) \]
\[ + (-v_1^2 - v_2^2 + v_1 v_2 + v_2 v_3)(\partial_t v_3) + v_1^2 v_2 v_3 - v_1 v_2^2 v_3 + v_1 v_2 v_3^2 - v_1^2 v_3^2. \]

(B.4)

A third solution is given by

\[ S_3 = E^{(-2)} + s^{(-3)} + s^{(-4)} + s^{(-5)} = \begin{pmatrix}
0 & 0 & \lambda^{-1} & 0 \\
0 & 0 & \lambda^{-1}(v_1 - v_2) & \lambda^{-1} \\
v_1 - v_2 & 1 & \lambda^{-1}s_{33} & -\lambda^{-1}s_{22} \\
v_1 - v_2 & 1 & \lambda^{-1}s_{43} & \lambda^{-1}s_{44}
\end{pmatrix}, \]

(B.5)

for \( E^{(-2)} = \lambda^{-1}E = \lambda^{-1}E_{a_1+a_3} + \lambda^{-1}E_{a_2+a_3} + E_{-a_1-a_2} + E_{-a_2-a_3} \) and

\[ s_{33} = \partial_t(v_2 - v_3) + (v_2 - v_3)^2, \]
\[ s_{43} = \partial_t^2(-v_2 + v_3) - 3(v_2 - v_3)\partial_t(v_2 - v_3) - (v_2 - v_3)^2, \]

leading to the Miura transformation
\[ J_1^{(3)} = \partial_x(3v_1 - v_2 - v_3) + v_1^2 + v_2^2 + v_3^2 - v_1v_2 - v_2v_3, \]
\[ J_2^{(3)} = \partial_x^2(-v_2 + 2v_3) - v_2(\partial_x v_1) + 2(-v_2 + v_3)(\partial_x v_2) + (3v_2 - 4v_3)(\partial_x v_3) + v_1^2v_2 - v_1v_2^2 + v_2^2v_3 - v_2v_3^2, \]
\[ J_3^{(3)} = -\partial_x^3(v_2 - v_3) + (v_2 - v_3)\partial_x^2(v_2 - 2v_3) + \partial_x(v_1 - 2v_3)\partial_x(v_2 - v_3) + (v_3^2 - 2v_2v_3)(\partial_x v_1) + (-v_1^2 + v_1v_2 - v_2v_3)(\partial_x v_2) + (v_1^2 - v_1v_2 + 2v_2v_3)(\partial_x v_3) + v_1^2v_2v_3 - v_1v_2^2v_3 + v_1v_2v_3^2 - v_1^2v_3^2. \]

The fourth solution is
\[ S_4 = E^{(-3)} + s^{(-3)} + s^{(-5)} + s^{(-6)} = \begin{pmatrix}
0 & \lambda^{-1} & 0 \\
0 & \lambda^{-1}(v_2 - v_1) & \lambda^{-1} \\
0 & \lambda^{-1}t_{32} & -\lambda^{-1}(v_3 - v_1) \\
1 & \lambda^{-1}t_{42} & \lambda^{-1}t_{43} \\
\end{pmatrix} \]

for \( E^{(-3)} = \lambda^{-1}E_{01} + \lambda^{-1}E_{02} + \lambda^{-1}E_{03} + E_{-01-02-03} \) and
\[ t_{32} = \partial_x(v_1 - v_2) + (v_1 - v_2)^2, \]
\[ t_{42} = \partial_x^2(-v_1 + v_2) - 3(v_1 - v_2)\partial_x(v_1 - v_2) + (v_1 - v_2)^3, \]
\[ t_{43} = \partial_x(2v_1 - v_2 - v_3) + v_1^2 + v_2^2 + v_3^2 - v_1v_2 - v_1v_3 - v_2v_3, \]
leading to the Miura transformation
\[ J_1^{(4)} = \partial_x(3v_1 - v_2 - v_3) + v_1^2 + v_2^2 + v_3^2 - v_1v_2 - v_2v_3, \]
\[ J_2^{(4)} = \partial_x^2(-3v_1 + 2v_2 + v_3) + 2(-v_1 + v_3)(\partial_x v_1) + (3v_1 - 4v_2 + v_3) \times (\partial_x v_2) + 2(v_2 - v_3)(\partial_x v_3) + v_1^2v_2 - v_1v_2^2 + v_2^2v_3 - v_2v_3^2, \]
\[ J_3^{(4)} = \partial_x^3(v_1 - v_2) + (v_1 - v_2)\partial_x^2(v_1 - 2v_2 + v_3) - \partial_x(v_1 - v_2)\partial_x(2v_2 - v_3) + (-v_1^2 + v_1v_2 + 2v_1v_2 - 2v_2v_3) + v_1v_3)(\partial_x v_2) + (v_1^2 - v_1v_2 - 2v_1v_3 + 2v_2v_3)(\partial_x v_3) + v_1^2v_2v_3 - v_1v_2^2v_3 + v_1v_2v_3^2 - v_1^2v_3^2. \]

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