A Geometrical Anisotropic Model of Space-time Based on Finslerian Metric

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Abstract

In this work we study an anisotropic model of general relativity based on the framework of Finsler geometry. The observed anisotropy of the microwave background radiation is incorporated in the Finslerian structure of space-time. We also examine the electromagnetic (e.m.) field equations in our space. As a result a modified wave equation of e.m. waves yields.

1 Introduction

A Finslerian geometrical structure of models which can correspond to anisotropic structures of regions of spacetime (radius ≤ 10^8 light years) can be introduced. Our work was motivated by the observed anisotropy of the microwave cosmic radiation. This anisotropy is of dipole type, i.e. the intensity of the radiation is maximum at one direction and minimum at the opposite direction.

It is known that this anisotropy can be explained if we use Robertson-Walker metric and take into account the movement of our galaxy with respect to distant galaxies of the universe. A small anisotropy is expected, however, due to the anisotropic distribution of galaxies in space.

From the above mentioned results, it is reasonable to seek for a Lagrangian which expresses this anisotropy. As such, we choose:

\[ \mathcal{L} = \sqrt{a_{ij}y^i y^j + \varphi(x)\hat{k}_a y^a} \]  \hspace{1cm} (1)

The vector \( \hat{k}_a \) expresses the observed anisotropy of the microwave background radiation.

In §2 we give the necessary mathematical formalism, upon which we develop our theory.
In §3 we develop the geometric anisotropic structure of space-time based on
the tangent bundle. Some physical interpretations are given.

In §4 we study the changes that are imposed on the electromagnetic field as a
result of the anisotropic geometry. It is shown that the e.m. field tensor remains
unchanged in our approach. The wave equation of e.m. waves is modified
(\(\Box F A_i(x)\)) in such a way that it expresses anisotropy of the electromagnetic
field, i.e. in the generalized D’Alambertian there exist terms of anisotropy
which affect the conventional form of the wave equation.

2 Preliminaries

The framework in which we develop our present work is a Finsler tangent bundle.
For this we consider a smooth 4-dimensional pseudoriemannian manifold M,
\((TM, \pi, M)\) its tangent bundle and \(\tilde{T}M = TM \setminus \{0\}\), where 0 means the image
of the null cross-section of the projection \(\pi : TM \to M\). We also consider a
local system of coordinates \((x^i)\), \(i = 0, 1, 2, 3\) and \(U\) a chart of \(M\). Then the
couple \((x^i, y^a)\) is a local system of coordinates on \(\pi^{-1}(U)\) in \(TM\). A coordinate
transformation on the total space \(TM\) is given by

\[
\tilde{x}^i = \tilde{x}^i(x^0, \ldots, x^3), \quad \det \left| \frac{\partial \tilde{x}^i}{\partial x^j} \right| \neq 0, \quad \tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b} y^b, \quad x^a = \delta_i^a x^i \tag{2}
\]

By definition [5] a Finsler metric on \(M\) is a function \(F : TM \to \mathbb{R}\) having
the properties:

1. The restriction of \(F\) to \(\tilde{T}M\) is of the class \(C^\infty\) and \(F\) is only continuous
   on the image of the null cross section in the tangent bundle to \(M\).

2. The restriction of \(F\) to \(\tilde{T}M\) is positively homogeneous of degree 1 with
   respect to \((y^a)\).

\[
F(x, ky) = kF(x, y), \quad k \in \mathbb{R}_+^*
\]

3. The quadratic form on \(\mathbb{R}^n\) with the coefficients

\[
f_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{3}
\]

defined on \(\tilde{T}M\) is non degenerate \((\det(f_{ij}) \neq 0)\), with \(\text{rank}(f_{ij}) = 4\).

As it is known a non linear connection \(N\) on \(TM\) is a distribution on \(TM\),
supplementary to the vertical distribution \(V\) on \(TM\):

\[
T_{(x,y)}(TM) = N_{(x,y)} \oplus V_{(x,y)}
\]

In our case a non linear connection can be defined by

\[
N^a_j = \frac{\partial G^a}{\partial y^j} \tag{4}
\]
where \( G^a \) are defined from
\[
G^a = \frac{1}{4} f^{aj} \left( \frac{\partial^2 \mathcal{L}}{\partial y^j \partial x^k} y^k - \partial_j \mathcal{L} \right)
\] (5)

and the relation
\[
\frac{dy^a}{ds} + 2G^a(x, y) = 0
\] (6)
yields from the Euler-Lagrange equations:
\[
\frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial y^a} \right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0
\] (7)

The transformation rule of the non-linear connection coefficients is
\[
\tilde{N}^a_i = \frac{\partial \tilde{x}^a}{\partial x^b} \frac{\partial x^b}{\partial x^i} N^b_j(x, y) + \frac{\partial \tilde{x}^a}{\partial x^b} \frac{\partial^2 x^b}{\partial x^i \partial \tilde{x}^b} y^b
\] (8)

also
\[
\frac{\delta}{\delta \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\delta}{\delta x^j} \quad \frac{\partial}{\partial \tilde{y}^a} = \frac{\partial x^b}{\partial \tilde{x}^i} \frac{\partial}{\partial y^b}
\]
\[
d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j \quad \delta \tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b} \delta y^b
\]

A local basis of \( T_{(x,y)}(TM) \), \((\delta^i, \hat{\partial}_a)\) adapted to the horizontal distribution \( N \) is
\[
\delta^i = \partial_k - N^a_k(x, y) \hat{\partial}_a, \text{ where } \partial_k = \frac{\partial}{\partial x^k}, \hat{\partial}_a = \frac{\partial}{\partial y^a}
\] (9)

where \( N^a_k(x, y) \) are the coefficients of the non-linear Cartan connection \( N \) as we mentioned above.
The dual local basis is
\[
\{d^i = dx^i, \delta^a = \delta y^a = dy^a + N^a_j dx^j \}_{i,a=0,3} = \{\delta^3\}_{\beta=0,7}
\]

A d-connection on tangent bundle \( TM \) is a linear connection on \( TM \) which preserves by parallelism the horizontal distribution \( N \) and the vertical distribution \( V \) on \( TM \).

Generally an \( h-v \) metric on the tangent bundle \( (TM, \pi, M) \) is given by
\[
G = f_{ij}(x, y)dx^i \otimes dx^j + h_{ab} \delta y^a \otimes \delta y^b
\] (10)

We consider a metrical d-connection \( CT = (N^a_j, L^i_{jk}, C^i_{jk}) \) with the property
\[
f_{ij|k} = \delta_k f_{ij} - L^h_{ik} f_{jh} - L^h_{jk} f_{ih} = 0
\] (11)
\[
f_{ij|k} = \hat{\partial}_k f_{ij} - C^h_{ik} f_{jh} - C^h_{jk} f_{ih} = 0
\] (12)
where
\[ L_{jk}^l = \frac{1}{2} f^{ir} (\delta_j f_{rk} + \delta_k f_{jr} - \delta_r f_{jk}) \] (13)
\[ C_{jk}^l = \frac{1}{2} f^{ir} (\delta_j f_{rk} + \delta_k f_{jr} - \delta_r f_{jk}) \] (14)

The coordinate transformation of the objects \( L_{jk}^l \) and \( C_{jk}^l \) is:
\[
\tilde{L}_{jk}^l = \frac{\partial \tilde{x}^i}{\partial x^h} \frac{\partial x^l}{\partial x^k} L_{ir}^h (x, y) + \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k} (15)
\]
\[
\tilde{C}_{jk}^l = \frac{\partial \tilde{x}^i}{\partial x^h} \frac{\partial x^l}{\partial x^k} C_{ir}^h (x, y) (16)
\]

The Cartan torsion coefficients \( C_{ijk} \) are given by
\[ C_{ijk} = \frac{1}{2} \frac{\partial f_{ij}}{\partial x^k} \] (17)

while the Christoffel symbols of the first and second kind for the metric \( f_{ij} \) are respectively:
\[ \gamma_{ijk} = \frac{1}{2} \left( \frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k} \right) \] (18)
\[ \gamma_{ij}^l = \frac{1}{2} f^{lk} \left( \frac{\partial f_{kj}}{\partial x^i} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{ij}}{\partial x^k} \right) \] (19)

The torsions and curvatures which we use are given by \[ P_{ijk} \]:
\[ T_{kj}^i = 0 \quad S_{kj}^i = 0 \quad R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i \] (20)
\[ P_{jk}^i = \dot{\delta}_k N_j^i - L_{kj}^i \quad P_{jk}^i = f^{im} P_{mj}^i \quad P_{ijk} = C_{ijk} | y \] (21)

\[ R_{jkl}^i = \delta_i L_{jkl}^i + L_{ik}^h L_{jh}^l - L_{ih}^h L_{jk}^l + C_{ij}^s R_{kl}^s \] (22)
\[ S_{jik} = C_{iks} C_{jkh}^s - C_{khs} C_{jik}^s \] (23)
\[ P_{ijk} = C_{ijk} | y - C_{hjs} C_{rkh} | y \] (24)
\[ S_{ijk} = f^{ij} S_{jik} \] (25)
\[ P_{ijk} = f^{ij} P_{jik} \] (26)

3 The Geometrical Structure of the anisotropic model (based on \( TM \))

In the following, the lowering and raising of the indices of the objects \( \hat{k}_a, y^a \) and all related Riemannian tensors will be performed with the metric \( a_{ij} \). For the related Finslerian tensors we shall use the Finsler metric \( f_{ij} \).
The Lagrangian which gives the equation of geodesics in the case of (pseudo)-
Riemannian space-time is given by:

\[ L = \sqrt{a_{ij}y^i y^j}, \quad y^i = \frac{dx^i}{ds} \]  

(27)

or, equivalently, we may write for the line element:

\[ ds_R = \sqrt{a_{ij} dx^i dx^j} \]  

(28)

where \( a_{ij} \) is the Riemannian metric with signature \((-\,\,+,\,\,+\,)\). Because of the
observed anisotropy, we must insert an additional term to the Riemannian line
element \((28)\). This term must fulfill the following requirements:

(a) It must give absolute maximum contribution for direction of movement par-

   parallel to the anisotropy axis.

(b) It must give zero contribution for movement in direction perpen
dicular to

   the anisotropy axis, i.e. the new line element must coincide with the Rie-

   mannian one for direction vertical to the anisotropy axis.

(c) It must not be symmetric with respect to replacement \( y^a \rightarrow -y^a \). This

   requirement is necessary in order to express the anisotropy of dipole type of

   the Microwave Background Radiation (MBR). We need to have maximum

   (positive) contribution for direction that coincides with the direction of

   the anisotropy axis, and minimum (negative) contribution for the opposite
direction.

We see that a term which satisfies the above conditions is \( k_a(x)y^a \), where
\( k_a(x) \) expresses this anisotropy axis. For constant direction of \( k_a(x) \) we may
consider \( k_a(x) = \varphi(x)\hat{k}_a \), where \( \hat{k}_a \) is the unit vector in the direction \( k_a(x) \).
Then \( \varphi(x) \) plays the role of “length” of the vector \( k_a(x), \varphi(x) \in \mathbb{R} \). Hence, we
have the Lagrangian

\[ \mathcal{L} = \sqrt{a_{ij}y^i y^j + \varphi(x)\hat{k}_a y^a} \]  

(29)

From \((28)\) we define the Finsler metric function \( F(x,y) = \mathcal{L} \). Setting \( y^a = dx^a \)
we have

\[ ds_F = \sqrt{a_{ij} dx^i dx^j + \varphi(x)\hat{k}_a dx^a} \]  

(30)

d\( s_F \) is the Finslerian line element and \( ds_R \) is the Riemannian one. We see
that the Finslerian line element is generated by an additional increment to the
Riemannian one due to the anisotropy axis. Now

\[ ds_F^2 = a_{ij} dx^i dx^j + 2\varphi(x)\hat{k}_a dx^a \sqrt{a_{ij} dx^i dx^j + \varphi^2(x)\hat{k}_a dx^a \hat{k}_b dx^b} \]  

(31)
In order for the Finslerian metric to be physically consistent with General Relativity theory, it must have the same signature with the Riemannian metric \((-, +, +, +)\). We have

\[
ds_R = c\, d\tau = c\, \gamma dt = \gamma dx^0
\]  
(32)

where \(\gamma = \sqrt{1 - (v/c)^2}\) and \(v\): 3-velocity in Riemannian space-time. From relations (32), (31) we obtain:

\[
ds_F^2 = a_{00} dx^0 dx^0 + 2a_{0\alpha} dx^0 dx^\alpha + a_{\alpha\beta} dx^\alpha dx^\beta + 2\varphi(x)\hat{k}_0 dx^0 ds_R \\
+ 2\varphi(x)\hat{k}_\alpha dx^\alpha ds_R + \varphi^2(x)\hat{k}_0 dx^0 dx^0 + 2\varphi^2(x)\hat{k}_0 \hat{k}_\alpha dx^0 dx^\alpha \\
+ \varphi^2(x)\hat{k}_\alpha dx^\alpha \hat{k}_\beta dx^\beta
\]

or

\[
ds_F^2 = \left(a_{00} + 2\varphi(x)\hat{k}_0 + \varphi^2\hat{k}_0\hat{k}_0\right) dx^0 dx^0 + \\
+ \left(a_{\alpha\beta} + \varphi^2(x)\hat{k}_\alpha \hat{k}_\beta\right) dx^\alpha dx^\beta + 2\varphi(x)\hat{k}_\alpha dx^\alpha dx^0 \\
+ 2a_{\alpha\alpha} dx^0 dx^\alpha + 2\varphi^2(x)\hat{k}_0 \hat{k}_\alpha dx^0 dx^\alpha
\]

(33)

where \(\alpha, \beta = 1, 2, 3\). From relation (33) it is evident that we must have

\[
(k_0(x))^2 + 2\gamma k_0(x) + a_{00} < 0
\]

(34)

\[
\delta^{\alpha\beta}(a_{\alpha\beta} + k_\alpha(x)k_\beta(x)) > 0
\]

(35)

for the signature to be preserved, where we have written \(\varphi(x)\hat{k}_i = k_i(x)\). Relation (34) admits negative values for

\[
-\gamma - \sqrt{\gamma^2 - a_{00}} < k_0(x) < -\gamma + \sqrt{\gamma^2 - a_{00}}
\]

(36)

while from (35) yields:

\[
(k_\alpha(x))^2 > -a_{\alpha\alpha}
\]

(37)

which is true for any \(k_\alpha(x)\) since \(a_{\alpha\alpha} > 0\).

Then for any physically acceptable vector, its 0 component \(k_0(x)\) must lie in the interval (36). Relation (34) is a restriction upon the anisotropy of space-time, i.e. the anisotropy vector can not take arbitrary values.

The equation of geodesics is given by:

\[
\frac{d^2x^l}{ds^2} + \Gamma_{ij}^l y^i y^j + \sigma a^m l (\partial_j \varphi k_m - \partial_m \varphi k_j) y^j = 0
\]

(38)

We observe that in the equation of geodesics we have an additional term, namely \(\sigma a^m l (\partial_j (\varphi k_m) - \partial_m (\varphi k_j)) y^j\) which expresses rotation of the anisotropy axis.
Now for the case of electromagnetic waves we must modify relation (38). This is because the world line of an e.m. wave is null. In geometrical optics the direction of propagation of a light ray is determined by the wave vector tangent to the ray. Let \( \frac{w_i}{\lambda} = \frac{dx^i}{d\lambda} \) be the four-dimensional wave vector, where \( \lambda \) is some parameter varying along the ray. We have:

\[
\frac{dw_i}{d\lambda} + \Gamma_{ij}^{(a)} w_i w_j + \sigma a^m \left( \partial_j \varphi_k m - \partial_m \varphi_k j \right) k^m = 0
\]  

(39)

One possible explanation of the anisotropy axis could be that it expresses the resultant of the spin densities of the angular momenta of galaxies in a restricted region of space \( k_a(x) \) spacelike). It is known that the mass is anisotropically distributed for regions of space with radius \( \leq 10^8 \) light years \([6]\). Then an important kind of anisotropy might result from the ordering of the angular momenta of galaxies. As we move to greater distances (radius \( \geq 10^8 \) l.y.) the resultant of the spin densities is approximately zero, as it is expected for an isotropic universe.

\[
k_a(x) = \sum_i \left( k_a(x) \approx 0 \right)
\]  

(40)

where \( k_a(x) \) is the spin density tensor of each rotating mass distribution.

The spin is defined through the spin density tensor \([2]\) from the relation

\[
S_{ab} = \sqrt{-g} \epsilon_{abc} k^c(x)
\]  

(41)

In the case that \( \varphi(x) \) expresses spin density, the function \( \varphi(x) \) is related to mass density (angular momenta depends upon angular velocity and mass distribution).

From equation (38) we see that for small variation of the resultant of the spin densities vector, the deviation from the Riemannian geodesics is very small, if not negligible.

From the equation of geodesics (38) we obtain for movement \( y^i \) perpendicular to \( k^i \):

\[
\frac{d^2 x^a}{ds^2} + \Gamma_{ij}^{(a)} y^i y^j + \sigma a^m \partial_j \varphi_k m y^j = 0
\]  

(42)

From (42) it is evident that although the contribution to the \( ds_R \) line element is zero for \( y^i \) vertical to \( k^i \), the equation of geodesics is different from the Riemannian case. In the case, however, where \( \partial_i \varphi(x) \) is parallel to \( k_i \), i.e. the increment of anisotropy takes place only along the anisotropy axis, then the equation of geodesics is identical with the geodesics of the Riemannian space-time.

Using the notation \( \beta = k_a y^a \), \( \sigma = \sqrt{a_{ij} y^i y^j} \), we calculate the metric tensor from (3):

\[
f_{ij} = \frac{F}{\sigma} a_{ij} + \frac{\varphi(x)}{2\sigma} \delta_{ij} (y_i \hat{k}_j) - \frac{\beta \varphi(x)}{\sigma^3} y_i y_j + \varphi^2(x) \hat{k}_i \hat{k}_j
\]  

(43)
where $S_{ij}$ is an operator and denotes symmetrization of the indices $i, j$, e.g.

$$S_{ij}(A_{ikjl}) = \frac{1}{2}(A_{ikjl} + A_{jkil}).$$

Accordingly we define the antisymmetric operator

$$A_{ij}(M_{ikjl}) = \frac{1}{2}(M_{ikjl} - M_{jkil}).$$

The inverse metric is

$$f_{ij} = \sigma a_{ij} - \frac{\sigma \varphi}{2F} \mathcal{S}_{ij}(y^i \dot{k}^j) + \frac{\varphi (\beta + m \sigma \varphi)}{F^3} y^i y^j$$

as it may be verified by direct calculation, where $m = \dot{\tilde{k}}_a \tilde{k}^a = 0, \pm 1$ according whether $\dot{\tilde{k}}_a$ is null, spacelike or timelike (in order to not loose generality, we do not identify $\dot{\tilde{k}}_a$ as spacelike). It must be noted, however, that if $y^a$ represents the velocity of a particle ($y^i$ timelike) then $\dot{\tilde{k}}_a$ is bound to be spacelike. This follows from the fact that one possible value of $y^a \dot{\tilde{k}}_a$ is zero.

The determinant of the metric is

$$f = \det(f_{ij}) = \left(\frac{F}{\sigma}\right)^5 \det(a_{ij})$$

The Cartan torsion coefficients which are given by (17), take the form:

$$C_{ijl} = \frac{3 \beta \varphi}{2 \sigma^3} y_i y_j y_l + \frac{3 \varphi}{\sigma} \mathcal{S}_{ijl}(a_{ij}) \frac{3 \varphi}{\sigma^3} \mathcal{S}_{ijl}(y_i y_j y_l) - \frac{3 \beta \varphi}{\sigma^3} \mathcal{S}_{ijl} a_{ij} y_l$$

We observe from (46) that an increment of the anisotropy, i.e. increment of $\varphi$, results in a change in the values of the components of the Cartan coefficients. This is expected since the condition

$$C_{ijk} = 0$$

is the condition for the Finsler metric to be Riemannian.

The finslerian Christoffel symbols of the first kind are given by (18)

$$\gamma_{ijl} = \frac{F^{(a)}}{\sigma} \Gamma_{ijl} + \Lambda_{ijl} + M_{ijl}$$

where

$$\Gamma_{ijl}^{(a)} = \frac{1}{2}(\partial_i a_{lj} + \partial_j a_{il} - \partial_l a_{ij})$$

are the Christoffel symbols corresponding to the metric $a_{ij}$.

$$\Lambda_{ijl} = \mathcal{G}_{ijl} \left[ \frac{3 \beta \varphi}{2 \sigma^3} y_i y_j \frac{\varphi}{\sigma^3} \mathcal{S}_{ijl} \dot{k}_j - \frac{\varphi \beta}{4 \sigma^3} a_{ij} \right] (49)$$

and

$$\Lambda_{ijl} = \mathcal{G}_{ijl} \left[ \frac{3 \beta \varphi}{2 \sigma^3} y_i y_j \frac{\varphi}{\sigma^3} \mathcal{S}_{ijl} \dot{k}_j - \frac{\varphi \beta}{4 \sigma^3} a_{ij} \right] (\partial_l a_{ab} y^a y^b)$$

(50)
and
\[ M_{ijl} = \mathcal{G}_{ij(t)} \left[ \left( \frac{\beta}{2\sigma} a_{ij} + \frac{1}{\sigma} y_i k_j - \frac{\beta}{\sigma^3} y_i y_j + 2\phi k_i k_j \right) \partial_l \phi \right]. \] (51)

The operator \( \mathcal{G} \) denotes an interchange of the indices in the form this interchange appears in the definition of the Christoffel symbols of a metric, e.g.
\[ \mathcal{G}_{ij(t)} A_{ij} = A_{ji} + A_{ij} - A_{ijl} \]
\[ \mathcal{G}_{ij(t)} \partial_l a_{ij} = 2 (a) \Gamma^{(a)}_{ij} \]

The Christoffel symbols of the second kind are found from (19):
\[ \gamma_{ij} = (a) \Gamma^{(a)}_{ij} + \left( \frac{\varphi (\beta + m \sigma \varphi)}{\sigma F^2} y^a y^l - \frac{2 \varphi}{F} \frac{S(k^l)}{a^l} \right) (a) \Gamma^{(a)}_{ij} + \frac{\sigma}{F} (A_{ij} + M_{ij}) \]
\[ + (A_{ij} + M_{ij}) \left( \frac{\varphi (\beta + m \sigma \varphi)}{F^3} y^a y^l - \frac{2 \sigma \varphi}{F^2} \frac{S(k^l)}{a^l} \right) \] (52)

where \( A_{ij} = A_{jik} a^{ik} \) and \( M_{ij} = M_{jik} a^{ik} \). In relation (52) it is seen that, besides the \( \Gamma^{(a)}_{ij} = 0 \) terms, the rest express the anisotropic deviation from the Riemannian Christoffel symbols. When \( \varphi = 0 \), i.e. absence of anisotropy, the Finsler Christoffel symbols coincide with the Riemannian ones. From the above relation, for \( \Gamma^{(a)}_{ij} = 0 \) we have \( \gamma_{ij} \neq 0 \). This shows the dependence of \( \gamma_{ij} \) from the anisotropy terms.

From Euler-Lagrange equations we find for \( G^l \) (relation (3)):
\[ G^l = \frac{1}{2} (a) \Gamma^{(a)}_{ij} y^i y^j + \sigma a^{ml} y^j A_{jm} (\partial_j \varphi(x) \hat{k}_m) \] (53)

Using relation (4) we calculate the nonlinear connection coefficients:
\[ N^l_k = (a) \Gamma^{(a)}_{ik} y^i + \sigma a^{ml} A_{km} (\partial_k \varphi(x) \hat{k}_m) + \frac{1}{\sigma} a^{ml} y^j A_{jm} (\partial_j \varphi(x) \hat{k}_m) y_k \] (54)
or
\[ N^l_k = (a) N^l_j + \sigma a^{ml} A_{km} (\partial_k \varphi(x) \hat{k}_m) + \frac{1}{\sigma} a^{ml} y^j A_{jm} (\partial_j \varphi(x) \hat{k}_m) y_k \] (55)

Relation (55) clearly shows that the deviation from the Riemann non-linear connection is due to anisotropic terms. In the case of an irrotational anisotropic field, \( A_{km} (\partial_k \varphi(x) \hat{k}_m) = 0 \), the non-linear connection is identical with the Riemannian one.
The connection coefficients \( C_{ij}^l \) are given by (14):

\[
C_{ij} = \frac{\varphi}{2F} a_{ij} \hat{k}^l + \frac{\varphi}{F} S(\hat{k}^i \hat{k}^j) - \frac{\beta \varphi}{F \sigma^2} S(\hat{k}^i y_j) - \frac{\varphi(\beta + m \sigma \varphi)}{2F^2 \sigma} a_{ij} y^l - \\
- \frac{\varphi}{2F \sigma^2} y_i y_j \hat{k}^l - \frac{\varphi(\sigma - \beta \varphi)}{F^2 \sigma^2} y^l S(\hat{k}_i y_j) - \left( \frac{\varphi}{F} \right)^2 \hat{k}_i \hat{k}_j y^l + \\
+ \frac{\varphi(3 \beta + m \sigma \varphi)}{2F^2 \sigma^3} y_i y_j y^l
\]

Correspondingly, using (13) we get:

\[
L_{jk}^i = (a)_{ij} \Gamma_{ij}^l + \left( \frac{\varphi(\beta + m \sigma \varphi)}{\sigma F^2} y^a y^l - \frac{2\varphi}{F} S(y^a \hat{k}^l) \right) \Gamma_{ij}^l + \sigma (\Lambda_{ij}^l + \\
+ M_{ij}^l) + (\Lambda_{ija} + M_{ija}) \left( \frac{\varphi(\beta + m \sigma \varphi)}{F^3} y^a y^l - \frac{2\varphi}{F^3} S(y^a \hat{k}^l) \right) - \\
- \left( N_j^l C_{kl} + N_k^l C_{jl} - f^r N_j^l C_{jkl} \right)
\]

where \( N_j^l \) and \( C_{kl}^l \) are given explicitly by relations (54), (56). The curvature of the non linear connection is (21):
where \( R^i_{jk} \) is the Riemannian curvature of the metric \( a_{ij} \).

The torsion \( P^i_{jk} \) is given by (21):

\[
P^i_{jk} = \Gamma^i_{jk} + \frac{1}{\sigma} a^{mn} \left[ \mathcal{A}_m (\partial_\lambda \hat{\kappa}_n) y_j + a_{jl} \mathcal{A}_l (\partial_\lambda \hat{\kappa}_m) y^l \right] - L^i_{kj} \tag{59}
\]
then

\[ P_{ijk} = \frac{F^{(a)}}{\sigma} \Gamma^a_{jki} - \frac{\beta F}{2\sigma^2} a_{jk} \partial_i \varphi + \left[ \frac{F + m\sigma \varphi^2}{2\sigma^2} \partial_i \varphi y^i - \frac{\beta \varphi^2}{2\sigma} \partial_i \varphi \hat{k}^i \right] a_{jk} \hat{k}_i \]

\[ + \varphi^2 k_l \Gamma^l_{jk} \hat{k}_i + (\varphi^2 k_l + \frac{\varphi}{\sigma} y_l) \Gamma^l_{jk} \hat{k}_i - \frac{F}{2\sigma^2} y_j \hat{k}_k \partial_i \varphi + \]

\[ + \left( \frac{\sigma + 2\beta \varphi + m\sigma \varphi^2}{2\sigma^2} \right) y_j \partial_k \varphi \hat{k}_i \]

\[ - \left( \frac{\varphi^2}{2\sigma} (\partial_i \varphi \hat{k}^i) + \frac{\varphi}{2\sigma^2} (\partial_i \varphi y^i) \right) y_j \hat{k}_k \hat{k}_i + \left( \frac{\varphi}{\sigma} k_l \varphi - \frac{\beta \varphi}{\sigma^2} y_l \right) \Gamma^l_{jk} y_i + \]

\[ + \left( \frac{m\varphi}{2\sigma^2} \partial_i \varphi y^i - \frac{\beta \varphi}{2\sigma^2} \partial_i \varphi \hat{k}^i \right) a_{jk} y_i + \frac{(m\sigma^2 - \beta \varphi^2) y_j \partial_k \varphi y_i + \frac{\beta \varphi}{2\sigma^4} \partial_i \varphi y^i}{2\sigma^4} y_j \partial_k \varphi y_i + \]

\[ + \left( \frac{\beta \varphi}{2\sigma^4} \partial_i \varphi y^i - \frac{\varphi}{2\sigma^2} \partial_i \varphi \hat{k}^i \right) y_j \hat{k}_k y_i - \gamma_{kji} - \frac{g}{g_{jk}^i} (C_{jkl} N^l_{ij}) - f_{ik} L^h_{kj} \] (60)

The \( h \)-covariant derivative of the \( C_{ijk} \) coefficients is

\[ \nabla^h C_{ijk} = \delta_l C_{ijk} - L^h_{il} C_{hjk} - L^h_{jl} C_{ikh} - L^h_{kl} C_{ijh} \] (61)

From relations (21), (24), (46), (56), (57), (59), and (61) we can calculate the \( P_{ijkl} \) curvature.
Taking into account relations:

\[
\delta_l L_{jk}^i = \delta_l f^{ir} \left( \gamma_{jkr} - \sum_{jk}(C_j k h N_r^h) \right) + f^{ir} \left( \delta_l g_{jkr} - \left[ (\delta_l N_j^h) C_{rkh} + N_j^h (\delta_l C_{rkh}) - (\delta_l N_j^h) C_{jkh} - N_j^h (\delta_l C_{jkh}) \right] \right)
\]

(62)

\[
\delta_l g_{jkr} = \left( \frac{1}{\alpha} \delta_l F - \frac{F}{\alpha^2} \delta_l \sigma \right) \Gamma_{jk}^r + \frac{F}{\alpha} \delta_l \Gamma_{jk}^r + \delta_l \Lambda_{jkr} + \delta_l M_{jkr}
\]

(63)

\[
\delta_l N_j^i = \frac{\partial \Gamma_j^i}{\partial x^k} A_{jk} + \left( \frac{1}{2\alpha} \frac{\partial a_{mn}}{\partial x^k} y^m y^n a_{ki} + \frac{\partial a_{ki}}{\partial x^k} \right) A(\partial_j \phi \hat{k}_k) +
\]

\[
+ \sigma a_{mi} \left[ A_{jm} \left( \partial^2 \phi \hat{k}_m \right) \right] + \frac{1}{\alpha} a_{mi} \left[ A_{rm} \left( \partial^2 \phi \hat{k}_m \right) \right] y^r y_j +
\]

\[
+ \left( \frac{1}{\sigma} \frac{\partial a_{mi}}{\partial x^k} y^m y^n a_{mi} \right) A \left( \partial_r \phi \hat{k}_m \right) y^r y_j -
\]

\[
- \frac{\beta}{2\sigma} \partial^2 \phi A_{jk} (\partial_j \phi \hat{k}_k) - \sigma a_{ji} \Gamma_{jk}^{i} \left( \partial_k \phi \hat{k}_i \right) + \frac{m}{4} \partial^2 \phi \partial_k \phi y_j -
\]

\[
- \frac{1}{4} \left( \partial_i \phi \hat{k}_a \right) \partial_j \phi \hat{k}_k y_j - \frac{1}{2} \partial_i \phi \hat{k}_i \left[ a_{ji} A(\partial_k \phi \hat{k}_i) y_j + y^i A(\partial_k \phi \hat{k}_i) \right] -
\]

\[
- \left( \frac{\beta}{2\sigma} \right)^2 \partial^2 \phi \partial_j \phi y_k - \frac{\beta}{2\sigma} \left[ a_{mj} \Gamma_{mn}^m \partial^2 \phi - \partial^2 \phi \Gamma_{ja}^{i} \right] -
\]

\[
- \frac{\beta}{2\sigma^2} \partial_{a} \phi A(\partial_i \phi \hat{k}_a) y_j y_k + \frac{\beta}{4\sigma^2} (\partial_{a} \phi y^a) \partial^i \phi \hat{k}_i y_j -
\]

\[
- \frac{1}{\sigma} y^a \Omega_{kba} y_j A(\partial^a \phi \hat{k}_i) - \frac{1}{2\sigma^2} \partial^a \phi \hat{k}_i \left( \partial_{a} \phi \hat{k}_i \right) + \Gamma_{jk}^{i} \partial_{a} \phi \hat{k}_i -
\]

\[
+ \left( \partial_{a} \phi \hat{k}_i \right) a_{ji} \Gamma_{kli}^{a} y^a \hat{k}_i + \frac{\beta}{4\sigma^2} (\partial_{a} \phi y^a) \partial_j \phi \hat{k}_i y_k +
\]

\[
+ \frac{m}{4\sigma^2} (\partial_{a} \phi y^a) \partial_j \phi \hat{k}_i y_k - \Gamma_{kji}^{a} \Gamma_{jli}^{a} y^b -
\]

\[
- \frac{1}{2\sigma^2} (\partial_{a} \phi y^a) \partial_j \phi \hat{k}_i y_k - \Gamma_{kji}^{a} \Gamma_{jli}^{a} y^b -
\]

(64)

\[
\delta_l \beta = -\hat{k}_h N_j^h
\]

(65)

\[
\delta_l \sigma = \frac{1}{\sigma^2} \partial_a \phi \hat{k}_i \gamma^a y^i - N_j^h \left( \frac{1}{\sigma} a_{ih} y^i \right)
\]

(66)

\[
\delta_l F = \frac{1}{\sigma^2} \partial_a \phi \hat{k}_i \gamma^a y^i + \partial \phi \beta - N_j^h \left( \frac{1}{\sigma} a_{ih} y^i + \phi(x) \hat{k}_h \right)
\]

(67)
where (22), (57), (56) we may calculate the curvature explicitly from (58).
The S-curvature (23) is:

\[ S_{jikh} = \frac{\varphi^2(m^2 - \beta^2)}{2F^2\sigma^2} A(a_{ij}a_{kh}) + \frac{\varphi^2}{2F}\left(A(\hat{a}_{ki}\hat{h}_j)\hat{k}_h + A(a_{ih}\hat{k}_i)\hat{k}_k\right) + \]
\[ + \frac{\beta\varphi^2}{2F\sigma^3}\left(\hat{a}_{ki}\hat{k}_j y_h + A(\hat{a}_{i j k}\hat{h}_k) y_i\right) + \frac{\varphi^2}{2F\sigma^3}\left(\hat{k}_h y_k A(\hat{k}_i y_j) + \hat{k}_k y_h A(\hat{k}_j y_i)\right) + \]
\[ + \frac{\varphi^2(m^2 - \beta^2)}{4F\sigma^5}\left(A(a_{ih}y_k) y_j + A(a_{j k}y_h) y_i\right) \] (72)

\[ S^r_{ikh} = \frac{(m^2 - \beta^2)}{2F^2\sigma^2} A(\delta_{ik} a_{hi}) + \frac{\varphi^2}{2F^{2}}\left(\hat{k}_i A(\delta_{ik} \hat{k}_j) + \hat{k}_j A(\delta_{ik} \hat{k}_i)\right) + \]
\[ + \frac{\beta\varphi^2}{2F^2\sigma^2}\left(\delta_{ik} S(\hat{k}_j y_h) - \delta_{ik} S(\hat{k}_i y_k)\right) + \frac{\beta\varphi^2}{2F^2\sigma^2}\hat{k}_i A(\hat{k}_j y_k) + \]
\[ + \frac{(m^2 - \beta^2)\varphi^2}{2F^2\sigma^4} y_i A(\delta_{ik} y_h) + \frac{\varphi^2}{2F^2\sigma^2}\left(\hat{k}_i y_i A(\hat{k}_j y_k) + y_i (\hat{k}_j A(\hat{k}_i y_h))\right) + \]
\[ + \frac{\varphi^2(m^2 - \beta^2)\varphi^2}{2F^3\sigma^2} y_i A(\hat{k}_j y_k) + \frac{\beta\varphi^2 - \frac{2\beta^2 m^2\varphi^2}{2F^3\sigma^4}}{2\sigma^3}\frac{y_i A(\hat{k}_j y_h)}{A(k_h y_k)} + \]
\[ + \frac{(m^2 - \beta^2)\varphi^2}{2F^3\sigma^4} y_i A(\hat{k}_j y_k) \] (73)

\[ S_{ih} = -\frac{3(m^2 - \beta^2)\varphi^2}{4F^2\sigma^2} a_{ih} - \frac{\varphi^2}{4F}\hat{k}_i \hat{k}_h + \]
\[ + \frac{\beta\varphi^2}{2F^2\sigma^2} S(\hat{k}_i y_h) + \frac{3m^2\varphi^2 - 4\beta^2\varphi^2}{4F^2\sigma^4} y_i y_h \] (74)

\[ S = \frac{5(\beta^2 - m^2)\varphi^2}{2\sigma F^3} \] (75)

From a physical point of view the S-curvature can be considered as a curvature parameter of anisotropy as it is evident from relation (73). In the absence of anisotropy \( \varphi = 0 \), we have \( S = 0 \). In other words, \( S \) represents the measure of anisotropy of matter [8].
4 Anisotropic Electromagnetic Field Equations in vacuum

The electromagnetic field tensor in special relativity is
\[ F_{ij} = \partial_j A_i(x) - \partial_i A_j(x). \]
A generalization in our approach yields
\[ \tilde{F}_{ij} = A_i|\langle j \rangle (x) - A_j|\langle i \rangle (x) = \delta_{j\langle i} A_{\langle i} - (\partial_{\langle i} - N_{\langle i}^j \partial_\langle i}) A_{\rangle i} \]  
(76)
or
\[ \tilde{F}_{ij} = \partial_j A_i(x) - \partial_i A_j(x) = F_{ij} \]  
(77)
since \( \partial_\langle i \partial_\langle i = 0 \). Therefore the electromagnetic field tensor remains invariant as in the usual electromagnetic theory of a Riemannian space-time.

The first pair of Maxwell equations is
\[ \partial_l F_{ik} + \partial_k F_{il} + \partial_i F_{kl} = 0 \]  
(78)
A generalization of the partial derivatives in our case is to replace them with the \( h \)-covariant derivative of the bundle:
\[ \partial_h F_{ik} \rightarrow F_{ik}|\langle l \rangle \]  
(79)
We have
\[ F_{ik}|\langle l \rangle = \delta_i F_{lk} - L^h_{ik} F_{th} - L^h_{ik} F_{lh} \]  
(80)
\[ F_{li}|\langle k \rangle = \delta_k F_{li} - L^h_{kl} F_{hi} - L^h_{kl} F_{hi} \]  
(81)
\[ F_{kl}|\langle i \rangle = \delta_i F_{kl} - L^h_{ik} F_{lh} - L^h_{ik} F_{kh} \]  
(82)
using relations
\[ \delta_i F_{ij} = (\partial_i - N^j_{\langle i} \partial_\langle i}) F_{ij}, \quad \partial_h F_{ij} = 0 \]
and summing \([84], [85], [86]\), yields: \( L^i_{jk} \)
\[ F_{ik}|\langle l \rangle + F_{li}|\langle k \rangle + F_{kl}|\langle i \rangle = \partial_l F_{ik} + \partial_k F_{li} + \partial_i F_{kl} = 0 \]  
(83)
where we took into account the symmetric properties of \( L^i_{jk} \) and \( F_{ij} = -F_{ji} \). It is seen that the first pair of Maxwell equations remains unchanged.

The second pair of Maxwell equations in vacuum is
\[ \partial_k F^{ik} = 0 \]  
(84)
As before we consider:
\[ \partial_k F^{ik} \rightarrow F^{ik}_{|k} \]  
(85)
Then \( \delta_k F_{ik} \) gives:

\[
F_{ik} = \delta_k F_{ik} + L_{hk}^i F_{hk} + L_{hi}^k F_{ih} \tag{86}
\]

From the antisymmetry of \( F_{ik} \) and the symmetry of \( L_{jk}^i \) the second and third terms of \( \delta_k F_{ik} \) vanish. Therefore

\[
F_{ik} = \partial_k F_{ik} = 0 \tag{87}
\]

From relations (87) and (83) it is evident that the generalization of the electromagnetic field does not change under the presence of the anisotropic model of gravity. It is expected, however, in analogy with the case of general relativity, that the wave equation should change its form.

We denote the generalized D’Alambertian by \( \Box_F \). In our approach the D’Alambertian is defined by

\[
\Box_F = \frac{1}{\sqrt{-f}} \left( \delta_i \left( \sqrt{-f} f^{ij} \delta_j \right) \right) \tag{88}
\]

or, equivalently we have:

\[
\Box_F = \delta_i f^{ij} \delta_j + f^{ij} \delta_i \delta_j + \frac{1}{2f} (\delta_i f) f^{ij} \delta_j \tag{89}
\]

From relation (11) it follows that

\[
\delta_i f^{ij} = - \left( L_{jk}^i f^{jk} + L_{ik}^j f^{ik} \right) \tag{90}
\]

The following relations hold good:

\[
\delta_i f = \partial_i f - N_i^i \dot{\partial}_i f \quad \partial_i f = \frac{\partial f}{\partial f_{ab}} \frac{\partial f_{ab}}{\partial x^i} \quad \frac{\partial f}{\partial f_{ab}} = f f^{ab} \tag{91}
\]

\[
\dot{\partial}_i f = \frac{\partial f}{\partial f_{ab}} \dot{\partial}_i f_{ab} \quad \text{then} \quad \dot{\partial}_i f = 2 f f^{ab} C_{abl} \tag{92}
\]

\[
\delta_i f = f f^{ab} (\partial_i f_{ab} - 2 N_i^i C_{abl}) \quad \partial_i f_{ab} = \gamma_{aib} + \gamma_{bia} \tag{93}
\]

\[
\partial_i f_{ab} - 2 N_i^i C_{abl} = (\gamma_{aib} - N_i^i C_{ahl}) + (\gamma_{bia} - N_i^i C_{bl}) \tag{94}
\]

\[
L_{jk}^i = (\gamma_{jk}^i - N_i^i C_{jkl}) - N_i^j C_{lk} + f^{ir} N_i^l C_{jkl} \tag{95}
\]

From relations (83) and (91)-(95) we have

\[
\Box_F = f^{ij} (\delta_i \delta_j - L_{ij}^k \delta_k) \tag{96}
\]

For the electromagnetic potential \( A_i(x) \) relation (90) yields

\[
\Box_F A_i(x) = f^{ij} (\partial_i \partial_j - L_{ij}^k \partial_k) A_i(x) = 0 \tag{97}
\]
The Lorentz gauge is in special relativity

\[ \partial_a A^a = 0 \] (98)

In a procedure similar to the one used for the D’Alambertian, we find the generalization of Lorentz condition:

\[ \partial_a A^a = 0 \rightarrow \frac{1}{\sqrt{-f}} \left( \delta_a \sqrt{-f} A^a \right) = 0 \] (99)

Again using relations (89) and (91)–(95), we obtain equivalently:

\[ \partial_a A^a + L_{ab} A^b = 0 \] (100)

Relation (100) is equivalent to \( A^a_{[a} = 0 \)

From relations (97), (100) it is evident that the wave equation, as well as the Lorentz condition, are modified in such a way as to include anisotropic terms. However the transformation rule of the \( L^i_{jk} \) connection yields that the Lorentz condition, and the D’Alambertian are the same for any observer.

It may be possible that relation (97) is connected with the observed anisotropy of the electromagnetic propagation over cosmological distances [7].

Finally we give the equation of motion of a charged particle, subject to the anisotropic geometrical framework we developed and to the electromagnetic field (we consider \( \sigma = 1 \)):

\[ mc \left( \frac{d^2 x^l}{ds^2} + \Gamma^{(a)}_{ij} y^i y^j + a^{lm} (\partial_j \varphi \kappa_m - \partial_m \varphi \kappa_j) y^j \right) = \frac{q}{c} F^l_{lj} y^j \]

\[ \frac{d^2 x^l}{ds^2} + \Gamma^{(a)}_{ij} y^i y^j + \left( a^{lm} (\partial_j \varphi \kappa_m - \partial_m \varphi \kappa_j) - \frac{q}{mc^2} F^l_{lj} \right) y^j = 0 \] (101)

It is interesting to note that equation (101) is produced by a Lagrangian of the form

\[ \mathcal{L} = mc \left( \sqrt{a_{ij} y^i y^j} + \varphi(x) \kappa_a y^a \right) + \frac{q}{c} A_a y^a \] (102)

Therefore, one may use the Lagrangian (102) as a metric function and produce the equation of motion of a charged particle, subject to an e.m. field and the anisotropic geometrical model, as a geodesic of the space generated by (102).

5 Conclusion

The observed anisotropy of the microwave cosmic radiation, represented by a vector \( k_a(x) \), can be incorporated in the framework of Finsler geometry. The equations of geodesics are generalized in a Finsler anisotropic space-time. The calculation of a curvature parameter of anisotropy is performed explicitly by the
contraction of the $S_{ijkl}$ curvature. Also, the Maxwell equations are unaffected from the passage to the anisotropic geometry. The Lorentz condition, as well as the generalized D’Alambertian, are shown to be invariant under coordinate transformations. In our case, however, the generalized wave equation includes the anisotropic vector through the $L_{jk}^i$ coefficients and the metric tensor $f_{ij}$.

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