Supersymmetric Vertex Models with Domain Wall Boundary Conditions

Shao-You Zhao\textsuperscript{a} and Yao-Zhong Zhang\textsuperscript{a,b}

\textsuperscript{a} Department of Mathematics, University of Queensland, Brisbane, QLD 4072, Australia
\textsuperscript{b} Physikalisches Institut, Universität Bonn, D-53115 Bonn, Germany
E-mail: syz@maths.uq.edu.au, yzz@maths.uq.edu.au

Abstract

By means of the Drinfeld twists, we derive the determinant representations of the partition functions for the $gl(1|1)$ and $gl(2|1)$ supersymmetric vertex models with domain wall boundary conditions. In the homogenous limit, these determinants degenerate to simple functions.

Keywords: Drinfeld twist; Domain wall boundary condition; Supersymmetric vertex model.
1 Introduction

The six vertex model on a finite square lattice with the so-called domain wall (DW) boundary condition was first proposed by Korepin in [1], where recursion relations of the partition functions of the were derived. In [2, 3] it was found that the partition functions of the DW model can be represented as determinants. Taking the homogenous limit of the spectral parameters, Sogo found that the partition function satisfies the Toda differential equations [4]. Then by using the equations Korepin and Zinn-Justin obtained the bulk free energy of the system [5]. The determinant representations of DW partition functions are useful in solving some pure mathematical problems, such as the problem of alternating sign matrices [6]. By using the fusion method, Caradoc, Foda and Kitanine obtained the determinant expressions of the partition functions for the spin-$k/2$ vertex models with the DW boundary conditions [7]. In [8], Bleher and Fokin obtained the large $N$ asymptotics of the DW six-vertex model in the disordered phase.

Supersymmetric integrable models based on superalgebras are important for describing strongly correlated electronic systems of high $T_c$ superconductivity [9, 10, 11, 12, 13, 14, 15]. In this paper, we investigate the $gl(1|1)$ and $gl(2|1)$ supersymmetric vertex models on a $N \times N$ square lattice with DW boundary conditions. By using the approach of Drinfeld twists [16, 18, 19], we derive the determinant representations of the DW partition functions for the two systems. We find that the partition functions degenerate to simple functions in the homogenous limit.

The framework of the paper is as follows. In section 2, we obtain the determinant representation of the partition function of the $gl(1|1)$ vertex model with DW boundary condition. Then we derive the homogenous limit of the partition function. In section 3, we solve the $gl(2|1)$ supersymmetric vertex model with the DW boundary conditions, by deriving its DW partition functions exactly. In section 4, we present some discussions.

2 $gl(1|1)$ vertex model with DW boundary condition

In this section, we study the DW boundary condition for the $gl(1|1)$ vertex model on a $N \times N$ square lattice.
2.1 Description of the model

Let $V$ be the 2-dimensional irreducible $gl(1|1)$-module and $R \in \text{End}(V \otimes V)$ the $R$-matrix associated with this representation. In this paper, we choose the FB grading for $V$, i.e. $[1] = 1, [2] = 0$. The $R$-matrix satisfies the graded Yang-Baxter equation (GYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$  \hspace{1cm} (2.1)

where $R_{ij} \equiv R_{ij}(\lambda_i, \lambda_j)$ with spectral parameters specified by $\lambda_i$. Explicitly,

$$R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix} c_{12} & 0 & 0 & 0 \\ 0 & a_{12} & b_{12} & 0 \\ 0 & b_{12} & a_{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (2.2)

where

$$a_{12} = a(\lambda_1, \lambda_2) \equiv \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2 + \eta}, \quad b_{12} = b(\lambda_1, \lambda_2) \equiv \frac{\eta}{\lambda_1 - \lambda_2 + \eta},$$

$$c_{12} = c(\lambda_1, \lambda_2) \equiv \frac{\lambda_1 - \lambda_2 - \eta}{\lambda_1 - \lambda_2 + \eta}$$  \hspace{1cm} (2.3)

and $\eta$ is the crossing parameter which could be normalized to 1 in the rational cases we are considering.

In the following, we study the $gl(1|1)$ 6-vertex model on a 2-d $N \times N$ square lattice. At any site of the lattice, there may be a vacuum $\phi$ or a single fermion $\rho$. A vertex configuration in the lattice is constructed by two nearest particle states in a horizontal line and two nearest particle states in a vertical line. For the present model, there are altogether 6 possible weights corresponding to a vertex configuration. The possible configurations and their corresponding Boltzmann weights $w_i$ are given as follows:
Figure 1. Vertex configurations and their Boltzmann weights for the $gl(1|1)$ vertex model.

Figure 2. DW boundary condition for the $gl(1|1)$ vertex model.

One may check that these weights preserve the fermion numbers. If we assign horizontal lines parameters $\{\lambda_j\}$ and vertical lines parameters $\{z_k\}$, and let

$$w_1 = 1, \quad w_2 = c(\lambda, z), \quad w_3 = w_4 = a(\lambda, z), \quad w_5 = w_6 = b(\lambda, z),$$

then the Boltzmann weights correspond to the elements of the $gl(1|1)$ $R$-matrix ($2.2$).

We now propose the $gl(1|1)$ vertex model with DW boundary condition. Similar to [1, 3], if the ends of the square lattice satisfy the special boundary condition, that is states of all
left and top ends are vacuum $\phi$ while states of all right and bottom ends are single fermion $\rho$, we then call the boundary condition the DW boundary condition.

A configuration with the DW boundary condition is shown in figure 2. In this figure, $\lambda_i$ and $z_j$ are parameters associated with the horizontal and vertical lines, respectively. The partition function for this system is then given by

$$Z_N = \sum \prod_{i=1}^{6} w_i^{n_i},$$

(2.5)

where the summation is over all possible configurations, $n_i$ is the number of configurations with the Boltzmann weight $w_i$. By means of the R-matrix, the partition function may be rewritten as

$$Z_N = \prod_{i=0}^{N-1} \rho_{(N-i)} \prod_{j=0}^{N-1} \phi_{(N-j)} \prod_{k=1}^{N} \prod_{l=1}^{N} R_{kl}(\lambda_k, z_l) \prod_{j=1}^{N} \rho_{(j)} \prod_{i=1}^{N} \phi_{(i)}.$$  

(2.6)

Here $\rho_{(i)}$ ($\phi_{(i)}$) stands for the state $\rho$ ($\phi$) in $i$th horizontal (vertical) line.

Following Korepin, we use the transfer matrix to arrange the partition function. Define the monodromy matrix $T_k(\lambda_k)$ along the horizontal lines

$$T_k(\lambda_k) = R_{k,N}(\lambda_k, z_N)R_{k,N-1}(\lambda_k, z_{N-1}) \ldots R_{k,1}(\lambda_k, z_1) \equiv \begin{pmatrix} A(\lambda_k) & B(\lambda_k) \\ C(\lambda_k) & D(\lambda_k) \end{pmatrix}_{(k)}.$$  

(2.7)

therefore, in terms of $T_k(\lambda_k)$ the partition function is equal to

$$Z_N = \prod_{i=0}^{N-1} \langle 1|_{(N-i)} \prod_{j=0}^{N-1} \langle 0|_{(N-j)} \prod_{k=1}^{N} T_k(\lambda_k) \prod_{j=1}^{N} |1\rangle_{(j)} \prod_{i=1}^{N} |0\rangle_{(i)}$$

$$= \prod_{i=0}^{N-1} \langle 1|_{(N-i)} C(\lambda_1)C(\lambda_2) \ldots C(\lambda_N) \prod_{i=1}^{N} |0\rangle_{(i)}.$$  

(2.8)

Here we have introduced the notation

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(2.9)

to denote the particle states $\phi, \rho$, respectively.

### 2.2 Exact solution of the partition function

In this subsection, we will compute the partition function $[2.8]$ for the $gl(1|1)$ vertex model with DW boundary conditions by using the Drinfeld twist approach.
Let us remark that even though we believe that Drinfeld twists do exist for all algebras, so far only those related to $A$-type (super) algebras and to $XYZ$ model have been explicitly constructed [17, 18, 19, 20, 21].

Let $\sigma$ be any element of the permutation group $S_N$. We then define the following lower-triangular matrix [19, 20]

$$F_{1,\ldots,N} = \sum_{\sigma \in S_N} \sum_{\alpha = \alpha(1) \ldots \alpha(N)}^{*} \prod_{j=1}^{N} F_{\alpha(j)}^\sigma(c, \sigma, \alpha_\sigma) R_{1,\ldots,N}^\sigma,$$  

(2.10)

where $P_{\alpha}^{(k)}$ has the elements $(P_{\alpha}^{(k)})_{mn} = \delta_{am}\delta_{an}$ at the $k$th space with root indices $\alpha = 1, 2$, the sum $\sum^{*}$ is taken over all non-decreasing sequences of the labels $\alpha_\sigma(i):

$$\alpha_{\sigma(i+1)} \geq \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) > \sigma(i),$$

$$\alpha_{\sigma(i+1)} > \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i+1) < \sigma(i)$$  

(2.11)

and the c-number function $S(c, \sigma, \alpha_\sigma)$ is given by

$$S(c, \sigma, \alpha_\sigma) \equiv \exp \left\{ \frac{1}{2} \sum_{l<k=1}^{N} \left( 1 - (-1)^{\lceil \alpha_\sigma(k) \rceil} \right) \delta_{\alpha_\sigma(k), \alpha_\sigma(l)} \ln(1 + c_{\sigma(k)\sigma(l)}) \right\}. $$  

(2.12)

In (2.10), $R_{12\ldots,N}^\sigma$ is the $N$-fold $R$-matrix which can be decomposed in terms of elementary $R$-matrices (2.2) by the decomposition law

$$R_{1\ldots,N}^{\sigma'\sigma} = R_{\sigma'(1\ldots,N)}^{\sigma} R_{1\ldots,N}^{\sigma'}. $$  

(2.13)

We showed in [18] that the $F$-matrix is nondegenerate and satisfies the relation

$$F_{\sigma(1\ldots,N)}(z_{\sigma(1)}, \ldots, z_{\sigma(N)}) R_{1\ldots,N}^{\sigma}(z_1, \ldots, z_N) = F_{1\ldots,N}(z_1, \ldots, z_N). $$  

(2.14)

The nondegeneracy of the $F$-matrix means that its column vectors form a complete basis, which is called the $F$-basis. The nondegeneracy also ensures that the $F$-matrix is invertible. The inverse is given by [19]

$$F_{1\ldots,N}^{-1} = F_{1\ldots,N}^{*} \prod_{i<j} \Delta_{ij}^{-1} $$  

(2.15)

with

$$\Delta_{ij} = \text{diag} \left( (1 + c_{ij})(1 + c_{ji}), a_{ji}, a_{ij}, 1 \right) $$  

(2.16)
\[ F_{1\ldots N}^* = \sum_{\sigma \in S_N} \sum_{\alpha_{1}(\ldots, \alpha_{n}(N)} S(c, \sigma, \alpha_{\sigma}) R_{\sigma_{(1\ldots N)}}^{-1} \prod_{j=1}^{N} P_{\sigma_{(j)}}^{\alpha_{\sigma(j)}}, \]

(2.17)

Here the sum \( \sum^{**} \) is taken over all possible \( \alpha_{i} \) which satisfies the following non-increasing constraints:

\[
\begin{align*}
\alpha_{\sigma(i+1)} & \leq \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i + 1) < \sigma(i), \\
\alpha_{\sigma(i+1)} & < \alpha_{\sigma(i)}, \quad \text{if} \quad \sigma(i + 1) > \sigma(i). 
\end{align*}
\]

(2.18)

Working in the \( F \)-basis, the entries of the monodromy matrix (2.7) can be simplified to symmetric forms, e.g. the lower entry \( C(\lambda) \) becomes [19]

\[
\tilde{C}(\lambda) \equiv F_{12\ldots N} C(\lambda) F_{12\ldots N}^{-1} = \sum_{i=1}^{N} b(\lambda, z_i) E_{(i)}^{12} \otimes_{j \neq i} \text{diag} (2a(\lambda, z_j), 1),
\]

(2.19)

where \( E_{(l)}^{\alpha\beta} \) \( (\alpha, \beta = 1, 2; l = 1, 2, \ldots, N) \) are generators of the superalgebra \( gl(1|1) \) at the site \( l \). From (2.19), one can see that in the \( F \)-basis all compensating terms (polarization clouds) in the original expression of \( C(\lambda) \) in terms of local generators disappear from \( \tilde{C}(\lambda) \).

Applying the \( F \)-matrix and its inverse to the states \( |0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)} \) and \( \langle 1\rangle_{(N)} \otimes \cdots \otimes \langle 1\rangle_{(1)} \), we have

\[
F_{1\ldots N}|0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)} = |0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}
\]

\[
\langle 1\rangle_{(N)} \otimes \cdots \otimes \langle 1\rangle_{(1)} F_{1\ldots N}^{-1} = \langle 1\rangle_{(N)} \otimes \cdots \otimes \langle 1\rangle_{(1)} \prod_{i<j}(2a(z_i, z_j))^{-1}.
\]

(2.20)

Substituting (2.19) and (2.20) into the partition function (2.8), we obtain

\[
Z_{N} = \prod_{i=0}^{N-1} \langle 1\rangle_{(N-i)} C(\lambda_{1}) C(\lambda_{2}) \ldots C(\lambda_{N}) \prod_{i=1}^{N} |0\rangle_{(i)}
\]

\[
= \prod_{i=0}^{N-1} \langle 1\rangle_{(N-i)} F_{1\ldots N}^{-1} F_{1\ldots N} C(\lambda_{1}) \ldots C(\lambda_{N}) F_{1\ldots N}^{-1} F_{1\ldots N} \prod_{i=1}^{N} |0\rangle_{(i)}
\]

\[
= \prod_{i<j}(2a(z_i, z_j))^{-1} \prod_{i=0}^{N-1} \langle 1\rangle_{(N-i)} \tilde{C}(\lambda_{1}) \ldots \tilde{C}(\lambda_{N}) \prod_{i=1}^{N} |0\rangle_{(i)}
\]

7
\[
= \prod_{i<j} (2a(z_i, z_j))^{-1} \prod_{i=0}^{N-1} \langle 1 | (N-i) \rangle 
\times 2^{\frac{N(N-1)}{2}} \sum_{i_1 < \ldots < i_N} B_N(\lambda_1, \ldots, \lambda_N | z_{i_1}, \ldots, z_{i_N}) E_{(i_1)}^{12} \ldots E_{(i_n)}^{12} \prod_{i=1}^{N} |0\rangle_{(i)} 
= \prod_{i<j} a^{-1}(z_i, z_j) B_N(\lambda_1, \ldots, \lambda_N | z_1, \ldots, z_N),
\] (2.21)

where

\[
B_N(\lambda_1, \ldots, \lambda_N | z_1, \ldots, z_N) = \sum_{\sigma \in S_N} \text{sign}(\sigma) \prod_{k=1}^{N} b(\lambda_k, z_{\sigma(k)}) \prod_{l=k+1}^{N} a(\lambda_l, z_{\sigma(l)})
= \det \mathcal{B}(\{\lambda\}, \{z\})
\] (2.22)

with \(\mathcal{B}(\{\lambda\}, \{z\})\) being a \(N \times N\) matrix with elements

\[
(\mathcal{B}(\{\lambda\}, \{z\}))_{\alpha\beta} = b(\lambda_{\alpha}, z_{\beta}) \prod_{\gamma=1}^{\alpha-1} a(\lambda_{\gamma}, z_{\beta}).
\] (2.23)

Therefore the partition function of the \(gl(1|1)\) vertex model is given by the following determinant of the \(N \times N\) matrix,

\[
Z_N = \prod_{i<j} a^{-1}(z_i, z_j) \det \mathcal{B}(\{\lambda\}, \{z\}).
\] (2.24)

### 2.3 Homogenous limit of the partition function

In this subsection, we discuss the homogenous limit, i.e. when \(\lambda_1 = \lambda_2 = \ldots = \lambda_N\) and \(z_1 = z_2 = \ldots = z_N\), of the partition function (2.24).

For later convenience, we rewrite the inhomogeneous partition function more explicitly as

\[
Z_N = \prod_{i<j} \frac{z_i - z_j + \eta}{z_i - z_j} 
\times \begin{vmatrix}
  b(\lambda_1, z_1) & \cdots & b(\lambda_1, z_N) \\
  b(\lambda_2, z_1) a(\lambda_1, z_1) & \cdots & b(\lambda_2, z_N) a(\lambda_1, z_N) \\
  \vdots & \ddots & \vdots \\
  b(\lambda_N, z_1) \prod_{\gamma=1}^{N-1} a(\lambda_{\gamma}, z_1) & \cdots & b(\lambda_N, z_N) \prod_{\gamma=1}^{N-1} a(\lambda_{\gamma}, z_N)
\end{vmatrix}
\] (2.25)
It is easy to check that in the limit $\lambda_1 \to \lambda, \lambda_2 \to \lambda, \ldots, \lambda_N \to \lambda$, we have

$$Z_N = \prod_{i<j} \frac{z_i - z_j + \eta}{z_i - z_j} \prod_{i=1}^N \frac{\eta}{\lambda - z_i + \eta} \times \left| \begin{array}{cccccc}
1 & \cdots & 1 & \cdots & 1 \\
a(\lambda, z_1) & \cdots & a(\lambda, z_k) & \cdots & a(\lambda, z_N) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a^{N-1}(\lambda, z_1) & \cdots & a^{N-1}(\lambda, z_k) & \cdots & a^{N-1}(\lambda, z_N)
\end{array} \right|,$$

(2.26)

where and below $a^k(\lambda, z_i)$ stands for the $k$-th power of $a(\lambda, z_i)$. For the homogenous limit of the parameters $z_k$, we first compute the limit $z_2 \to z_1 \equiv z$. Taylor expanding the second column as $z_2 \to z$,

$$a^k(\lambda, z_2) = a^k(\lambda, z) + (a^k(\lambda, z))' (z_2 - z) + \mathcal{O}((z_2 - z)^2),$$

where and below, $X'$, $X''$ and $X^{(n)}$ stand for the 1st, 2nd and $n$-th order derivatives of $X$ with respect to the parameter $z$, respectively, and subtracting the first column from the second, (2.26) becomes

$$Z_N = (-\eta)^3 \prod_{j=3}^N \left( \frac{z - z_j + \eta}{z - z_j} \right)^2 \prod_{j>i=3} \frac{z_i - z_j + \eta}{z_i - z_j} \frac{\eta}{\lambda - z + \eta} \prod_{i=3}^N \frac{\eta}{\lambda - z_i + \eta} \times \left| \begin{array}{cccccc}
1 & 0 & 1 & \cdots & 1 \\
a(\lambda, z) & (a(\lambda, z))' & a(\lambda, z_3) & \cdots & a(\lambda, z_N) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a^{N-1}(\lambda, z) & (a^{N-1}(\lambda, z))' & a^{N-1}(\lambda, z_3) & \cdots & a^{N-1}(\lambda, z_N)
\end{array} \right|.
$$

(2.27)

Then Taylor expanding the third column as $z_3 \to z$,

$$a^k(\lambda, z_3) = a^k(\lambda, z) + (a^k(\lambda, z))' (z_3 - z) + \frac{1}{2} (a^k(\lambda, z))'' (z_3 - z)^2 + \mathcal{O}((z_3 - z)^3),$$

and subtracting multiples of previous columns, we obtain

$$Z_N = (-\eta)^3 \frac{1}{2} \prod_{j=4}^N \left( \frac{z - z_j + \eta}{z - z_j} \right)^3 \prod_{j>i=4} \frac{z_i - z_j + \eta}{z_i - z_j} \frac{\eta}{\lambda - z + \eta} \prod_{i=4}^N \frac{\eta}{\lambda - z_i + \eta} \times \left| \begin{array}{cccccc}
1 & 0 & 0 & 1 & \cdots & 1 \\
a(\lambda, z) & (a(\lambda, z))' & (a(\lambda, z))'' & a(\lambda, z_4) & \cdots & a(\lambda, z_N) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a^{N-1}(\lambda, z) & (a^{N-1}(\lambda, z))' & (a^{N-1}(\lambda, z))'' & a^{N-1}(\lambda, z_4) & \cdots & a^{N-1}(\lambda, z_N)
\end{array} \right|.
$$

(2.28)
Continuing with such process, we obtain, instead of the determinant of the \( N \times N \) matrix, the following determinant of the \( (N - 1) \times (N - 1) \) matrix for the partition function:

\[
Z_N = \frac{(-\eta)^{\sum_{i=1}^{N-1} i} b^N(\lambda, z)}{\prod_{i=2}^{N-1} i!} \times \left| \begin{array}{cccc}
(a(\lambda, z))^\prime & (a(\lambda, z))^\prime & \cdots & (a(\lambda, z))^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
(a^{(N-1)}(\lambda, z))^\prime & (a^{(N-1)}(\lambda, z))^\prime & \cdots & (a^{(N-1)}(\lambda, z_N))^{(N-1)}
\end{array} \right|.
\] (2.29)

To simplify the determinant in (2.29) further, we investigate its elements. Computing \((a^j(\lambda, z))^{(n)}\), we obtain

\[
(a^j(\lambda, z))^{(n)} = \sum_{k=1}^{n} f_k \prod_{l=0}^{k-1} (j - l) a^{j-k}(\lambda, z),
\] (2.30)

where \( f_k \) are functions of \( \{(a(\lambda, z))^{(l)}\} \) \((l = 1, \ldots, n)\) and are independent of \( j \) in (2.30). One can easily obtain \( f_k \) for the first and last terms,

\[
f_1 = (a(\lambda, z))^{(n)}, \quad f_n = ((a(\lambda, z))^{(n)})^n.
\] (2.31)

Thus, by means of the properties of determinants, the partition function of the \( gl(1|1) \) vertex model is simplified to the following function:

\[
Z_N = \frac{(-\eta)^{\sum_{i=1}^{N-1} i} b^N(\lambda, z)}{\prod_{i=2}^{N-1} i!} \prod_{i=2}^{N-1} i! \left[ (a(\lambda, z))^{(n)} \right]^{\sum_{i=1}^{N-1} i} = (b(\lambda, z))^{N^2}.
\] (2.32)

### 3 \( gl(2|1) \) vertex model with DW boundary condition

#### 3.1 Description of the model

Let \( R \in \text{End}(V \otimes V) \) be the \( R \)-matrix associated with the 3-dimensional irreducible \( gl(2|1) \) module. Choosing the FFB grading for \( V \), i.e. \([1] = [2] = 1, [3] = 0\), then the \( R \)-matrix reads

\[
R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix}
c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{12} & 0 & -b_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & a_{12} & 0 & 0 & 0 & b_{12} & 0 \\
0 & -b_{12} & 0 & a_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{12} & 0 & b_{12} & 0 \\
0 & 0 & b_{12} & 0 & 0 & 0 & a_{12} & 0 \\
0 & 0 & 0 & 0 & b_{12} & 0 & a_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\] (3.1)
which satisfies the GYBE (2.1). Here $a_{12}, b_{12}$ and $c_{12}$ are the same as those given in the previous section. The basis vectors $|0\rangle, |1\rangle$ and $|2\rangle$ of the 3-d $gl(2|1)$ representation are given by

$$|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (3.2)$$

Now we consider the $gl(2|1)$ 15-vertex model on a $N \times N$ square lattice. Excluding the double occupancy, there are three possible electronic states, i.e. up spin $\uparrow$, down spin $\downarrow$ and vacuum $\phi$, at each site of the lattice. For this model, the configuration of the vertex is decided by the electronic states around it. Corresponding to the $gl(2|1)$ invariance, there are altogether 15 possible configurations, which preserve the fermion numbers and spins, with non-zero Boltzmann weights. They are shown by using figures as follows.
The Boltzmann weights associated with the vertices are given by the elements of the $R$-matrix (3.1)

$$w_1 = 1, \quad w_2 = w_3 = c(\lambda, z), \quad w_4 = \ldots = w_9 = a(\lambda, z),$$

$$w_{10} = \ldots = w_{13} = b(\lambda, z), \quad w_{14} = w_{15} = -b(\lambda, z). \quad (3.3)$$

The $gl(2|1)$ supersymmetric vertex model with DW boundary condition is defined as follows. At the left and top ends, all electrons are in vacuum states while at the right and bottom ends, there are $P$ down spin states (corresponding to the 1st-$P$th lines) and $N - P$ up spin states (corresponding to the $(P+1)$th -$N$th lines). The boundary condition is shown in figure 4. The DW partition function of the $gl(2|1)$ vertex model is given by

$$Z_N = \sum \prod_{i=1}^{15} w_i^{n_i}$$

$$= \prod_{i=0}^{P-1} \frac{\phi_i^{(N-i)}}{\phi_i^{(P-i)}} \prod_{j=0}^{P-1} \frac{\phi_i^{(N-j)}}{\phi_i^{(P-j)}} \prod_{k=0}^{N-1} \frac{\phi_i^{(N-k)}}{\phi_i^{(P-k)}} \prod_{l=0}^{N} R_{kl}(\lambda_k, z_l) \prod_{j=1}^{P} \frac{\phi_i^{(j)}}{\phi_i^{(j+1)}} \prod_{i=1}^{N} \frac{\phi_i^{(i)}}{\phi_i^{(i+1)}}$$

$$= \prod_{i=0}^{P-1} \frac{\phi_i^{(N-i)}}{\phi_i^{(P-i)}} \prod_{j=0}^{P-1} \frac{\phi_i^{(N-j)}}{\phi_i^{(P-j)}} \prod_{k=0}^{N-1} \frac{\phi_i^{(N-k)}}{\phi_i^{(P-k)}} \prod_{l=1}^{N} T_{kl}(\lambda_k) \prod_{j=1}^{P} \frac{\phi_i^{(j)}}{\phi_i^{(j+1)}} \prod_{i=1}^{N} \frac{\phi_i^{(i)}}{\phi_i^{(i+1)}}$$

Figure 3. Vertex configurations of the $gl(2|1)$ vertex model and their Boltzmann weights.

Figure 4. DW boundary condition for the $gl(2|1)$ vertex model.
\[
\prod_{i=0}^{P-1} \prod_{j=0}^{P-1} C_2(\lambda_1) \cdots C_2(\lambda_P) C_1(\lambda_{P+1}) \cdots C_1(\lambda_N) \prod_{i=1}^{N} \phi(i),
\]

(3.4)

where \( n_i \) are the number of configurations with the weights \( w_i \), and \( T_k(\lambda_k) \) is the monodromy matrix along the horizontal lines and is defined by

\[
T_k(\lambda_k) = R_{k,N}(\lambda_k, z_N) R_{k,N-1}(\lambda_k, z_{N-1}) \cdots R_{k,1}(\lambda_k, z_1)
\]

\[
\equiv \begin{pmatrix} A_{11}(\lambda_k) & A_{12}(\lambda_k) & B_1(\lambda_k) \\ A_{21}(\lambda_k) & A_{22}(\lambda_k) & B_2(\lambda_k) \\ C_1(\lambda_k) & C_2(\lambda_k) & D(\lambda_k) \end{pmatrix}^{(k)}.
\]

(3.5)

Similar to the \( gl(1|1) \) vertex model case, the partition function (3.4) can be computed by using the approach of Drinfeld twists, as can be seen from the next subsection.

### 3.2 Exact solution of the partition function

We now compute the partition function (3.4) using the Drinfeld twist method. The \( F \)-matrix for this case is still defined by (2.10), except that now \( \alpha = 1, 2, 3 \). The inverse of the \( F \)-matrix is given by

\[
F_{1 \cdots N}^{-1} = F_{1 \cdots N}^* \prod_{i<j} \Delta_{ij}^{-1}
\]

(3.6)

with

\[
\Delta_{ij} = \text{diag} (4a_{ij}a_{ji}, a_{ji}, a_{ij}, 4a_{ij}a_{ji}, a_{ji}, a_{ij}, a_{ij}, 1).
\]

(3.7)

Working in the \( F \)-basis, the lower entries \( C_1 \) and \( C_2 \) of the monodromy matrix (3.5) are simplified to symmetry forms, that is they can be written as

\[
\tilde{C}_2(\lambda) = \sum_{i=1}^{N} b(\lambda, z_i) E_{(i)}^{2,3} \otimes_{j \neq i} \text{diag} (a(\lambda, z_j), 2a(\lambda, z_j), 1)_{(j)},
\]

(3.8)

\[
\tilde{C}_1(\lambda) = \sum_{i=1}^{N} b(\lambda, z_i) E_{(i)}^{1,3} \otimes_{j \neq i} \text{diag} (2a(\lambda, z_j), a(\lambda, z_j)(a(z_i, z_j))^{-1}, 1)_{(j)}
\]

\[
+ \sum_{i \neq j=1}^{N} \frac{a(\lambda, z_j) b(\lambda, z_j) b(z_i, z_j)}{a(z_i, z_j)} E_{(i)}^{1,2} E_{(j)}^{2,3} \otimes_{k \neq i,j} (2a(\lambda, z_k), a(\lambda, z_k)a^{-1}(z_i, z_k), 1)_{(k)}.
\]

(3.9)
Here $E^\alpha^\beta_{(l)} (\alpha, \beta = 1, 2, 3; l = 1, 2, \ldots, N)$ are the generators of $gl(2|1)$ on the $l$th vertical line.

In the following we will identify $|0\rangle, |1\rangle$ and $|2\rangle$ with the vacuum $\phi$, spin-up $\uparrow$ and spin-down $\downarrow$ states, respectively. Note that under the action of $F$ the state $|0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}$ is invariant, that is,

$$F_{1 \ldots N} |0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)} = |0\rangle_{(1)} \otimes \cdots \otimes |0\rangle_{(N)}. \quad (3.10)$$

Thus substituting (3.9)-(3.10) into the partition function (3.4), we obtain

$$Z_N = \prod_{i=0}^{P-1} \langle 1 |_{(N-i)} \prod_{j=0}^{P-1} \langle 2 |_{(P-j)} C_2(\lambda_1) \cdots C_2(\lambda_P) C_{1}(\lambda_{P+1}) \cdots C_1(\lambda_N) \prod_{i=1}^{N} | 0 \rangle_{(i)}$$

$$= \prod_{i=0}^{P-1} \langle 1 |_{(N-i)} \prod_{j=0}^{P-1} \langle 2 |_{(P-j)} F_{1 \ldots N}^{-1} \tilde{C}_2(\lambda_1) \cdots \tilde{C}_2(\lambda_P) \tilde{C}_1(\lambda_{P+1}) \cdots \tilde{C}_1(\lambda_N) \prod_{i=1}^{N} | 0 \rangle_{(i)}$$

$$= \sum_{i_1 < \ldots < i_P} \sum_{i_{P+1} < \ldots < i_N} \prod_{l=0}^{P-1} \langle 1 |_{(N-l)} \prod_{j=0}^{P-1} \langle 2 |_{(P-j)} F_{1 \ldots N}^{-1} \prod_{j=1}^{P} E_{ij}^{23} \prod_{l=P+1}^{N} E_{(i_j)}^{13} \prod_{l=0}^{N} | 0 \rangle_{(l)}$$

$$\times 2^{P(P+1)+\frac{(N-P)(N-P+1)}{2}} \prod_{k=1}^{P} \prod_{l=P+1}^{N} a(\lambda_k, z_{i_l})$$

$$\times \det B_P(\{\lambda_1, \ldots, \lambda_P\}, \{z_{i_1}, \ldots, z_{i_P}\})$$

$$\times \det B_{N-P}(\{\lambda_{P+1}, \ldots, \lambda_N\}, \{z_{i_{P+1}}, \ldots, z_{i_N}\})$$

$$\equiv \sum_{i_1 < \ldots < i_P} \sum_{i_{P+1} < \ldots < i_N} \prod_{l=0}^{P-1} \langle 1 |_{(N-l)} \prod_{j=0}^{P-1} \langle 2 |_{(P-j)} (2a(z_{i_k}, z_{i_l}))^{-1} \prod_{l=k+1}^{N} (2a(z_{i_k}, z_{i_l}))^{-1} G(\{z\})$$

$$\times 2^{P(P+1)+\frac{(N-P)(N-P+1)}{2}} \prod_{k=1}^{P} \prod_{l=P+1}^{N} a(\lambda_k, z_{i_l})$$

$$\times \det B_P(\{\lambda_1, \ldots, \lambda_P\}, \{z_{i_1}, \ldots, z_{i_P}\})$$

$$\times \det B_{N-P}(\{\lambda_{P+1}, \ldots, \lambda_N\}, \{z_{i_{P+1}}, \ldots, z_{i_N}\}), \quad (3.11)$$

where $\{i_1, \ldots, i_P\} \cap \{i_{P+1}, \ldots, i_N\} = \emptyset$ and $B_M(\{\lambda\}, \{z\})$ is a $M \times M$ matrix with elements

$$(B_M(\{\lambda\}, \{z\}))_{\alpha\beta} = b(\lambda_{\alpha}, z_{\beta}) \prod_{\gamma=1}^{a-1} a(\lambda_{\gamma}, z_{\beta}). \quad (3.12)$$

In (3.11), the function $G(\{z\})$ is defined by

$$G(\{z\}) \equiv \prod_{l>k}^{P} (2a(z_{i_l}, z_{i_k})) \prod_{l>k=P+1} (2a(z_{i_l}, z_{i_k})) \langle 1 |_{(N)} \cdots \langle 1 |_{(P+1)} \langle 2 |_{(P)} \cdots \langle 2 |_{(1)}$$

$$\times F_{1 \ldots N}^{-1} E_{(i_1)}^{23} \cdots E_{(i_p)}^{23} E_{(i_{P+1})}^{13} \cdots E_{(i_N)}^{13} | 0 \rangle_{(1)} \cdots | 0 \rangle_{(N)}. \quad (3.13)$$
Using the matrix $F^{-1}_{1...N}$ defined by (2.15)-(2.18), one may simplify $G(\{z\})$ as follows.

\[
G(\{z\}) = \prod_{l>k=1}^{P} (2a(z_{ik}, z_{il})) \prod_{l>k=P+1}^{N} (2a(z_{ik}, z_{il})) \\
\times \langle 0 \rangle_{(N)} \cdots \langle 0 \rangle_{(1)} E^{31}_{(N)} \cdots E^{31}_{(P+1)} E^{32}_{(P)} \cdots E^{32}_{(1)} \sum_{\sigma \in S_2} S(c, \sigma, \alpha_\sigma) \\
\times R^{-1}_{\sigma(1...N)} P_{(\sigma(1))} \cdots P_{(\sigma(N))} \prod_{i<j} \Delta^{-1}_{ij} E^{23}_{(i_1)} \cdots E^{23}_{(i_{P})} E^{13}_{(i_{P+1})} \cdots E^{13}_{(i_{N})} |0\rangle_{(1)} \cdots |0\rangle_{(N)} \\
= \prod_{k=1}^{P} \prod_{l=P+1}^{N} a^{-1}(z_{ik}, z_{il}) \prod_{k>l=1}^{P} (2a(z_{ik}, z_{il}))^{-1} \prod_{k>l=P+1}^{N} (2a(z_{ik}, z_{il}))^{-1} \\
\times \langle 0 \rangle_{(N)} \cdots \langle 0 \rangle_{(1)} E^{31}_{(N)} \cdots E^{31}_{(P+1)} E^{32}_{(P)} \cdots E^{32}_{(1)} \sum_{\sigma \in S_N} S(c, \sigma, \alpha_\sigma) \\
\times R^{-1}_{\sigma(1...N)} E^{23}_{(i_1)} \cdots E^{23}_{(i_P)} E^{13}_{(i_{P+1})} \cdots E^{13}_{(i_{N})} |0\rangle_{(1)} \cdots |0\rangle_{(N)} \\
= \prod_{k=1}^{P} \prod_{l=P+1}^{N} a^{-1}(z_{ik}, z_{il}) \prod_{k>l=1}^{P} (2a(z_{ik}, z_{il}))^{-1} \prod_{k>l=P+1}^{N} (2a(z_{ik}, z_{il}))^{-1} \\
\times \sum_{\sigma \in S_N} (-1)^{\text{sign}(\sigma)} S(c, \sigma, \alpha_\sigma) \left( R^{-1}_{\sigma(1...N)} \right)^{\alpha_1...\alpha_P \alpha_{P+1}...\alpha_N}_{2...21...1}, \quad (3.14)
\]

where $\alpha = 1, 2$, the subscribes of $\alpha$ are indices of space, $\text{sign}(\sigma) = 1$ if $\sigma$ is odd and $\text{sign}(\sigma) = 0$ if $\sigma$ is even.

The procedure of computing the homogenous limit is similar to that for the $gl(1|1)$ vertex model. Here we only give the results. In the homogenous limit, i.e. when $\lambda_1 = \ldots = \lambda_N \equiv \lambda$ and $z_1 = \ldots = z_N \equiv z$, the partition function (3.11) becomes

\[
Z_N = (a(\lambda, z))^{P(N-P)}(b(\lambda, z))^{P^2}(b(\lambda, z))^{(N-P)^2} \\
\times \lim_{z_1, \ldots, z_N \to z} \sum_{i_1 < \ldots < i_P} \sum_{i_{P+1} < \ldots < i_N} G(\{z\}) \quad (3.15)
\]

By using the decomposition law (2.13), the $R$-matrix $R^\sigma$ in (3.14) can be decomposed to elementary $R$-matrices. In homogenous limit, the elements of the elementary $R$-matrix will tend to

\[
R(z, z)^{12}_{21} = R(z, z)^{21}_{12} = -1, \quad R(z, z)^{11}_{11} = R(z, z)^{22}_{22} = -1, \quad R(z, z)^{12}_{12} = R(z, z)^{21}_{21} = 0. \quad (3.16)
\]

Therefore for the last factor in (3.15) involving $G(\{z\})$, one may easily obtain

\[
\lim_{z_1, \ldots, z_N \to z} \sum_{i_1 < \ldots < i_P} \sum_{i_{P+1} < \ldots < i_N} G(\{z\})
\]
\[ \lim_{z_1, \ldots, z_N \to z} \sum_{i_1 < \ldots < i_P} \sum_{i_{P+1} < \ldots < i_N} \prod_{k=1}^{P} \prod_{l=P+1}^{N} a^{-1}(z_{i_k}, z_l) \]

\[ = C^P_N, \quad (3.17) \]

where \( C^P_N \) is the combinatorial number.

In summary, in the homogeneous limit the DW partition function \( Z_N \) is

\[ Z_N = C^P_N (a(\lambda, z))^{P(N-P)} (b(\lambda, z))^{P^2+(N-P)^2}. \quad (3.18) \]

It is easily seen that when \( P = 0 \) or \( P = N \), the DW partition function of the \( gl(2|1) \) vertex model reduces to that of the \( gl(1|1) \) vertex model, confirming a statement made in [24] on DW partition functions of vertex models based on \( A_n \)-type algebras. However, when \( P \neq 0, N \), one obtains new partition functions of the true \( gl(2|1) \) supersymmetric vertex model with DW boundary conditions.

## 4 Conclusion and discussion

In this paper, we have proposed the \( gl(1|1) \) and \( gl(2|1) \) supersymmetric vertex models with the so-called DW boundary conditions. The DW partition functions of the models have been computed by means of the approach of the Drinfeld twists. We have found that in the homogeneous limit, the partition functions degenerate to simple functions. For the 2-d square \( gl(2|1) \) lattice model, we note here that the definition of the DW boundary condition is not unique. For the other cases, by using the same procedure, one may find their partition functions are similar with those in this paper.

We have demonstrated, by working out the DW \( gl(1|1) \) and \( gl(2|1) \) supersymmetric vertex models as examples, that by means of the Drinfeld twist method, one can actually derive directly, rather than conjecture a formula and then verify it as usually done, the determinant representations of DW partition functions.

It is widely known that determinant representations of partition functions are closely related to some pure mathematical problems, such as algebraic combinations and tilings of the Aztec diamond [5, 22]. In our further work, we will study the mathematical problems arising from the present models. The results in this paper will also be useful for simplifying the correlation functions of the supersymmetric \( t-J \) model obtained in [23] and for further studying physical properties of the model.
Acknowledgements: SYZ was supported by the UQ Postdoctoral Research Fellowship. SYZ would like to thank W.-L. Yang and K.-J. Shi for helpful discussions. YZZ was supported by Australian Research Council and also by Max-Planck-Institut für Mathematik and Alexander von Humboldt-Stiftung. YZZ would like to thank the Max-Planck-Institute für Mathematik, where part of this work was done, and Physikalisches Institut der Universität Bonn, especially Günter von Gehlen, for warm hospitality.

References

[1] V.E. Korepin, Calculation of the norms of the Bethe wave functions, Commun. Math. Phys. 86, (1982) 391-418.

[2] A.G. Izergin, Partition function of the 6-vertex model in a finite volume, Sov. Phys. Dokl. 32 (1987), 878-879.

[3] A.G. Izergin, D.A. Coker and V.E. Korepin, Determinant formula for the six-vertex model, J. Phys. A25, (1992) 4315-4334.

[4] K. Sogo, Time-dependent orthogonal polynomials and theory of soliton applications to matrix model, vertex model and level statistics, J. Phys. Soc. Japan 62 (1993) 1887-1894.

[5] V.E. Korepin and P. Zinn-Justin, Thermodynamic limit of the six-vertex model with domain wall boundary conditions, J. Phys. A33, (2000) 7053-7066.

[6] G. Kuperburg, Another proof of the alternating sign matrix conjecture, Int. Math. Res. Not. 3 (1996) 139-150.

[7] A. Caradoc, O. Foda and N. Kitanine, Higher spin vertex models with domain wall boundary conditions. J. Stat. Mech (JSTAT), (2006) P03012.

[8] P.M. Bleher and V.V. Fokin, Exact solution of the six-vertex model with domain wall boundary conditions: Disordered phase, math-ph/0510033

[9] P.P. Kulish and E.K. Sklyanin, On the solutions of the Yang-Baxter equation, J. Soviet Math. 19, (1982) 1596-1620.

[10] F.H.L. Essler and V.E.Korepin, Higher conservation laws and algebraic Bethe ansatz for the supersymmetric t-J model. Phys. Rev. B46, (1992) 9147-9162;
[11] F.H.L. Essler, V.E. Korepin and K. Schoutens, Exact solution of an electronic model of supercondutivity, Int. J. Mod. Phys. B23, (1994) 3205-3242.

[12] A.J. Bracken, M.D. Gould, J.R. Links and Y.-Z. Zhang, A new supersymmetric and exactly solvable model of correlated electrons, Phys. Rev. Lett. 74, (1995) 2768-2771.

[13] M.P. Pfannmller and H. Frahm, Algebraic Bethe ansatz for gl(2,1) invariant 36-vertex models, Nucl. Phys. B479, (1996) 575-593.

[14] P.B. Ramos and M.J. Martins, One parameter family of an integrable spl(2|1) vertex model: Algebraic Bethe ansatz approach and ground state structure, Nucl. Phys. B474, (1996) 678-714.

[15] K.E. Hibberd, M.D. Gould and J.R. Links, Algebraic Bethe ansatz for the supersymmetric U model, Phys. Rev. B54, (1996) 8430-8437.

[16] J.M. Maillet and I. Sanchez de Santos, Drinfel’d twists and algebraic Bethe ansatz, q-alg/9612012.

[17] T.-D. Albert, H. Boos, R. Flume and K. Ruhlig, Resolution of the nested hierarchy for the rational sl(n) models, J. Phys. A33, (2000) 4963-4980.

[18] W.-L. Yang, Y.-Z. Zhang and S.-Y. Zhao, Drinfeld twists and algebraic Bethe ansatz of the supersymmetric t-J model. J. High Energy Phys. (JHEP) 12, (2004) 038.

[19] W.-L. Yang, Y.-Z. Zhang and S.-Y. Zhao, Drinfeld twists and algebraic Bethe ansatz of the quantum supersymmetric model associated with U_q(gl(m|n)), Commun. Math. Phys. 264, (2006) 87-114.

[20] S.-Y. Zhao, W.-L. Yang and Y.-Z. Zhang, Drinfeld twists and symmetric Bethe vectors of supersymmetric Fermion models. J. Stat. Mech. (JSTAT), (2005) P04005.

[21] T.-D. Albert, H. Boos, R. Flume, R.H. Poghossian and K. Ruhlig, An F-twisted XYZ model, Lett. Math. Phys. 53, (2000) 201-214.

[22] P. Zinn-Justin, Six-vertex model with domain wall boundary conditions and one-matrix model, Phys. Rev. E62, (2000) 3411-3418.
[23] S.-Y. Zhao, W.-L. Yang and Y.-Z. Zhang, Determinant representation of correlation functions for the supersymmetric $t$-$J$ Model, Commun. Math. Phys., 268 (2006), 505-541.

[24] A. Dow and O. Foda, On the domain wall partition functions of level-1 affine $so(n)$ vertex models, J. Stat. Mech. (JSTAT), (2006) P05010.