Orbit-equivalent infinite permutation groups

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Abstract

Let $G, H$ be closed permutation groups on an infinite set $X$, with $H$ a subgroup of $G$. It is shown that if $G$ and $H$ are orbit-equivalent, that is, have the same orbits on the collection of finite subsets of $X$, and $G$ is primitive but not 2-transitive, then $G = H$.

Keywords: primitive permutation group, orbit-equivalent, set-homogeneous.

1 Introduction

We consider closed permutation groups acting on an infinite set $X$; that is, subgroups of $\text{Sym}(X)$ which are closed in $\text{Sym}(X)$ in the topology of pointwise convergence on $\text{Sym}(X)$ with respect to the discrete topology on $X$ (so the basic open sets are cosets of pointwise stabilisers of finite sets). It is easily checked that a closed permutation group on $X$ is precisely the automorphism group of a relational structure with domain $X$. Two permutation groups $G, H$ on the set $X$ are said to be orbit-equivalent if, for every positive integer $k$, $G$ and $H$ have the same orbits on the collection of unordered $k$-element subsets of $X$, denoted here by $X^{[k]}$. This generalises a definition for finite permutation groups. Observe that if $G, H$ are orbit-equivalent, then they are each orbit-equivalent to $\langle G, H \rangle$. Thus, to investigate such pairs, it suffices to consider $G, H$ with $H$ a subgroup of $G$. Easily, if $H \leq G$ and $G, H$ are orbit-equivalent, then $G$ is transitive (on $X$) if and only if $H$ is transitive, and also $G$ and $H$ preserve the same
systems of imprimitivity on $X$; so $G$ is primitive on $X$ (that is, preserves no proper non-trivial equivalence relation on $X$) if and only if $H$ is primitive.

Our main theorem is the following. Our particular interest is in the case when $X$ is countably infinite, but the proofs below do not use countability.

**Theorem 1.1.** Let $G, H$ be orbit-equivalent closed permutation groups on the infinite set $X$, with $H \leq G$, and suppose that $G$ is primitive but not 2-transitive. Then $H = G$.

We stress that if $H$ is a closed proper subgroup of $G \leq \text{Sym}(X)$, then for some $k > 0$, some $G$-orbit on $X^k$ (the set of $k$-tuples from $X$) breaks into more than one $H$-orbit. The assumption in the theorem that $G$ and $H$ are closed seems essential; indeed, any subgroup $H$ of $\text{Sym}(X)$ is orbit-equivalent to its closure, and, for example, the dense (and so orbit-equivalent) subgroups of $\text{Sym}(X)$ are exactly the subgroups of $\text{Sym}(X)$ which are $k$-transitive for all positive integers $k$, and these seem hopelessly unclassifiable.

This paper takes its motivation from two sources. First, there is an extended literature on primitive orbit-equivalent pairs of permutation groups on a finite set $X$; see for example [20, 11, 21]. Clearly, the symmetric and alternating groups $\text{Sym}_n$ and $\text{Alt}_n$, in their natural actions on $\{1, \ldots, n\}$, are orbit-equivalent for $n \geq 3$. Also, if $G$ is a permutation group on a finite set $X$ and has a regular orbit $U$ on the power set $\mathcal{P}(X)$, and $H$ is a proper subgroup of $G$, then $H$ is intransitive on $U$, and so $H$ is not orbit-equivalent to $G$. It is shown in [3] that if $X$ is finite then there are just finitely many primitive subgroups of $\text{Sym}(X)$ which do not contain $\text{Alt}(X)$ and have no regular orbit on $\mathcal{P}(X)$ (and so could have an orbit-equivalent proper subgroup). Such primitive groups $G$ (with no regular orbit on $X$) are classified by Seress in [18], who then classifies all pairs of finite primitive orbit-equivalent permutation groups $(H, G)$ with $H < G$. There is further work on the finite imprimitive case in [19].

The second source of motivation is more model-theoretic, namely the study of homogeneous structures. Recall that a countable (possibly finite) structure $M$ in a first order relational language is said to be homogeneous if every isomorphism between finite substructures of $M$ extends to an automorphism of $M$. A natural generalisation, originally considered by Fraïssé in [8], is to say that the countable structure $M$ is set-homogeneous if, whenever $U, V$ are isomorphic finite substructures of $M$, there is $g \in \text{Aut}(M)$ with $U^g = V$. Finite set-homogeneous graphs are classified by Ronse in [17], and a very short proof was given by Enomoto in [7] that every finite set-homogeneous graph is homogeneous. There is a classification of set-homogeneous digraphs (allowing two vertices to be linked by an arc in each direction) in [9], building on a corresponding classification of finite homogeneous digraphs by Lachlan [12]. Also, there are initial results on countably infinite set-homogeneous structures, in particular graphs and digraphs, in [6] and [9]. The latter paper poses the following related question: given a homogeneous structure $M$, when does $\text{Aut}(M)$ have a proper closed subgroup $H$ which acts set-homogeneously on $M$, that is, has the same orbits as $\text{Aut}(M)$ on the collection of unordered finite subsets of $M$? Equivalently, for which $M$ does $\text{Aut}(M)$ have a proper closed orbit-equivalent subgroup? (Here, and throughout the paper, we use the same symbol $M$ for a structure and for its domain.)
A countably infinite set $X$ in the empty language is homogeneous, and has automorphism group $\text{Sym}(X)$. By a theorem of Cameron [2], $\text{Sym}(X)$ has just four orbit-equivalent closed proper subgroups, namely $\text{Aut}(X, <)$, $\text{Aut}(X, B)$, $\text{Aut}(X, C)$, and $\text{Aut}(X, S)$. Here $<$ is a dense linear order without end points on $X$, $B$ is the (ternary) linear betweenness relation on $X$ induced from $<$, $C$ is the (also ternary) circular order on $X$ induced from $<$, and $S$ is the corresponding arity 4 separation relation. Observe that $\text{Aut}(X, S) = \langle \text{Aut}(X, B), \text{Aut}(X, C) \rangle$ and is 3-transitive but not 4-transitive. Our conjecture below would strengthen Theorem 1.1 by removing the ‘not 2-transitive’ assumption.

**Conjecture 1.2.** Let $G$ and $H$ be distinct orbit-equivalent primitive closed permutation groups on a countably infinite set $X$. Then $G$ and $H$ belong to the list $\text{Aut}(X, <)$, $\text{Aut}(X, B)$, $\text{Aut}(X, C)$, $\text{Aut}(X, S)$, $\text{Sym}(X)$ described above.

Recall the following standard terminology, for a permutation group $G$ on a set $X$, and an integer $k > 0$: $G$ is $k$-transitive if it is transitive on the ordered $k$-subsets of $X$; and $G$ is $k$-homogeneous if it is transitive on the unordered $k$-subsets of $X$. Also, if $U$ is a subset of $X$ then $G_{(U)}$ and $G_{(U)}$ denote respectively the setwise and pointwise stabilisers of $U$ in $G$, and if $x \in X$ then $G_x := \{ g \in G : x^g = x \}$.

The proof of Theorem 1.1 splits into two cases:

1. $G$ is primitive but not 2-homogeneous;
2. $G$ is 2-homogeneous (and so primitive) but is not 2-transitive.

Our main tool for both cases is the notion of local rigidity. We shall say that a permutation group $G$ acting on an infinite set $X$ acts locally rigidly if for all finite $U \subset X$, there is some finite $V \subset X$ such that $U \subseteq V$ and the setwise stabiliser $G_{(V)}$ of $V$ fixes $U$ pointwise. Likewise, a first order relational structure $M$ is locally rigid if, for every finite substructure $U$ of $M$, there is a finite substructure $V$ of $M$ containing $U$ such that every automorphism of $V$ fixes $U$ pointwise. Clearly, if a relational structure $M$ is locally rigid, then any subgroup of its automorphism group acts locally rigidly on $M$. Strengthening the notion of local rigidity, we shall later say that a countably infinite first order structure $M$ is cofinally rigid if, for every finite substructure $U$ of $M$, there is a finite substructure $V$ of $M$ with $U \subseteq V$ such that the automorphism group of $V$ is trivial. Here, ‘substructure’ is used in the standard model-theoretic sense, corresponding to the graph-theoretic notion of ‘induced subgraph’.

**Lemma 1.3.** Let $G, H$ be closed permutation groups on $X$, with $H \leq G$. If $G$ and $H$ are orbit-equivalent and $G$ acts locally rigidly, then $H = G$.

**Proof.** It suffices to show that $H$ has the same orbits as $G$ on $X^k$ for all $k$. So let $\overline{u}_1, \overline{u}_2 \in X^k$ be in the same orbit of $G$; that is, there is $g \in G$ such that $\overline{u}_1^g = \overline{u}_2$. Let $U_1, U_2 \subset X$ be enumerated by $\overline{u}_1, \overline{u}_2$ respectively. Since $G$ acts locally rigidly on $X$, there is finite $V_1 \subset X$ such that $U_1 \subseteq V_1$ and $G_{(V_1)} \leq G_{(U_1)}$. Let $V_2 := V_1^g$. Then $V_1, V_2$ are in the same orbit of $G$, so by orbit-equivalence there is some $h \in H$ such that $V_1^h = V_2$. Now $gh^{-1} \in G_{(V_1)}$, so in fact $gh^{-1} \in G_{(U_1)}$. Thus $\overline{u}_1^h = \overline{u}_2$ as required. \qed
In both cases (1) and (2) \((G\) primitive, and either not 2-homogeneous, or 2-homogeneous but not 2-transitive\) we shall show that \(G\) acts locally rigidly on \(X\). In fact, in the second case we show that \(G\) is a group of automorphisms of a **cofinally rigid** tournament. Our method to show the local rigidity of such actions stems from a similar result in \([6]\), which we adapt. Formally, we view a graph \(\Gamma\) as a relational structure \(\Gamma = (X, R)\), where \(R\) is a symmetric irreflexive binary relation on \(X\). Given a graph \(\Gamma\), if \(x,y\) are vertices we write \(x \sim y\) if \(x\) and \(y\) are adjacent, and let \(\Gamma(x) := \{v \in X : v \sim x\}\), the *neighbour set* of \(x\). We shall prove in Lemma 2.3 a strengthening of the following result.

**Lemma 1.4.** \([6]\) Let \(\Gamma\) be an infinite graph such that, for all distinct vertices \(x,y\) of \(\Gamma\), the sets \(\Gamma(x) \setminus \Gamma(y)\) and \(\Gamma(y) \setminus \Gamma(x)\) are both infinite. Then \(\Gamma\) is locally rigid.

We draw attention to a basic Ramsey-theoretic principle which is well-known, for example in model theory, and used below in both the primitive not 2-homogeneous case, and the 2-homogeneous not 2-transitive case.

**Definition 1.5.** Let \(L\) be a finite relational language, let \(M\) be a first order \(L\)-structure, \(A\) a finite subset of the domain of \(M\), and \(P_1, \ldots, P_r\) disjoint subsets of \(M \setminus A\), with \(P_i := \{p_{i,0}, \ldots, p_{i,n-1}\}\) for each \(i = 1, \ldots, r\). We say that \(P_1, \ldots, P_r\) are **mutually indiscernible** over \(A\) if the following holds for any positive integers \(e_1, \ldots, e_r < n\): for each \(j = 1, \ldots, r\), let \(\vec{p}_j, \vec{p}'_j\) be \(e_j\)-tuples from \(P_j\), each listed in increasing order (so if \(\vec{p}_j = (p_{j,1(1), \ldots, p_{j,1(e_j)})}\), then \(i(1) < \ldots < i(e_j)\)); then the map taking \(\vec{p}_j\) to \(\vec{p}'_j\) for each \(j\), extended by the identity on \(A\), is an isomorphism of \(L\)-structures.

**Lemma 1.6.** Let \(M, L, A\) be as in Definition 1.2 with \(M\) infinite, and let \(Q_1, \ldots, Q_r\) be countably infinite disjoint subsets of \(M \setminus A\). Let \(n\) be a positive integer. Then the following hold.

(i) There are subsets \(P_1 \subset Q_1, \ldots, P_r \subset Q_r\), each of size \(n\), such that \(P_1, \ldots, P_r\) are mutually indiscernible over \(A\) (with respect to some indexing of each \(P_j\)).

(ii) If every relation of \(L\) is of arity at most 2, and \(P_1, \ldots, P_r\) are as in (i), then for each \(i = 1, \ldots, r\), either some relation of \(L\) induces a total order on \(P_i\), or every permutation of \(P_i\), extended by the identity on \(S_i := A \cup \bigcup_{j \neq i} P_j\), is an automorphism of the induced \(L\)-structure on \(S := A \cup P_1 \cup \ldots \cup P_r\).

**Proof.** (Sketch) (i) Let \(Q_i := \{q_{i,j} : j \in \mathbb{N}\}\) for each \(i = 1, \ldots, r\). Let \(d\) be the maximum arity of a relation in \(L\). Colour each subset \(\{i_1, \ldots, i_d\}\) of \(\mathbb{N}\) in such a way that given natural numbers \(i_1 < \ldots < i_d\) and \(k_1 < \ldots < k_d\), the map

\[(q_{i_1,i_1}, \ldots, q_{i_1,i_d}, \ldots, q_{r,i_1}, \ldots, q_{r,i_d}) \mapsto (q_{i_1,k_1}, \ldots, q_{i_1,k_d}, \ldots, q_{r,k_1}, \ldots, q_{r,k_d})\]

is an isomorphism over \(A\) if and only if \(\{i_1, \ldots, i_d\}\) and \(\{k_1, \ldots, k_d\}\) have the same colour. By Ramsey’s Theorem, replacing \(\mathbb{N}\) by an infinite monochromatic subset if necessary, we may suppose that \(\mathbb{N}\) is monochromatic. Now let \(p_{i,j} := q_{i,(i-1)n+j}\) for each \(i = 1, \ldots, r\) and \(j = 0, \ldots, n-1\). Put \(P_i := \{p_{i,0}, \ldots, p_{i,n-1}\}\) for each \(i = 1, \ldots, r\). Then \(P_1, \ldots, P_r\) are mutually indiscernible over \(A\).

(ii) This is immediate from (i). \(\square\)
The case of Theorem 1.1 when $G$ is primitive but not 2-homogeneous is handled in Section 2, and the 2-homogeneous but not 2-transitive case is treated in Section 3. Section 4 consists of some further observations, about bounds in local rigidity, approaches to Conjecture 1.2, and regular orbits on the power set. We also observe that our proofs give a slight strengthening of Theorem 1.1, namely Theorem 4.1.

## 2 $G$ primitive but not 2-homogeneous

In this section we prove the following.

**Proposition 2.1.** Let $G$ be a primitive but not 2-homogeneous permutation group on an infinite set $X$. Then the action of $G$ on $X$ is locally rigid.

The proposition follows rapidly from the following two lemmas. The first uses an argument in [15, Proposition 4.4].

**Lemma 2.2.** Let $G$ be a primitive but not 2-homogeneous permutation group on an infinite set $X$. Then there is a $G$-invariant graph $\Gamma$ with vertex set $X$ such that for all distinct $x, y \in X$, the symmetric difference $\Gamma(x) \Delta \Gamma(y)$ is infinite.

**Proof.** Let $U$ be any $G$-orbit on the collection of 2-subsets of $X$. Then $U$ is the edge set of a $G$-invariant graph $\Gamma_0$ with vertex set $X$, and as $G$ is not 2-homogeneous, $\Gamma_0$ is not complete. For $x \in X$, write $\Gamma_0(x)$ for the neighbour set of $x$ in $\Gamma_0$. Define the equivalence relation $\equiv_0$ on $X$, putting $x \equiv_0 y$ if and only if $|\Gamma_0(x) \Delta \Gamma_0(y)|$ is finite. Then $\equiv_0$ is $G$-invariant, so by primitivity $\equiv_0$ is trivial or universal. The lemma holds if $\equiv_0$ is trivial, so we shall suppose that $\equiv_0$ is universal.

Recall that a graph is locally finite if all of its vertices have finite degree.

**Claim.** Either $\Gamma_0$ or its complement is locally finite.

**Proof of Claim.** Suppose not, and fix $x \in X$. Then both $\Gamma_0(x)$ and $X \setminus \Gamma_0(x)$ are infinite. If $y \in \Gamma_0(x)$ then (as $\equiv_0$ is universal) $\Gamma_0(y) \setminus \Gamma_0(x)$ is finite. Hence as $G_x$ has at most two orbits on $\Gamma_0(x)$ there is $k \in \mathbb{N}$ such that for all $y \in \Gamma_0(x)$, we have $|\Gamma_0(y) \setminus \Gamma_0(x)| \leq k$. Pick distinct $z_1, \ldots, z_{k+1} \in X \setminus (\{x\} \cup \Gamma_0(x))$. Then as $x \equiv_0 z_i$ for each $i$, each set $\Gamma_0(z_i) \cap \Gamma_0(x)$ is cofinite in $\Gamma_0(x)$. Hence there is $y \in \Gamma_0(x) \cap \bigcap_{i=1}^{k+1} \Gamma_0(z_i)$. Then $z_1, \ldots, z_{k+1} \in \Gamma_0(y) \setminus \Gamma_0(x)$, so $|\Gamma_0(y) \setminus \Gamma_0(x)| \geq k + 1$, which is a contradiction.

By the claim, replacing $\Gamma_0$ by its complement if necessary, we may suppose that $\Gamma_0$ is locally finite. By our original assumption that $\Gamma_0$ is not complete (or null), $\Gamma_0$ has an edge. By primitivity, $\Gamma_0$ is connected. Now let $\Gamma$ be the graph on $X$ whose edge set consists of the set of unordered pairs an even distance apart in $\Gamma_0$. Then $\Gamma$ is also $G$-invariant. Pick $v_0 \in X$, and choose a $\Gamma_0$-path $v_0 \sim v_1 \sim v_2 \sim \ldots$ so that the distance $d_0(v_0, v_i)$ between $v_0$ and $v_i$ in $\Gamma_0$ equals $i$ for each $i$ (this is certainly possible, for example by König’s Lemma). Then $v_{2i} \in \Gamma(v_0) \setminus \Gamma(v_1)$ for each $i > 0$. Thus $\Gamma(v_0)$ and $\Gamma(v_1)$ have infinite symmetric difference, and since $G$ is primitive, this holds for all pairs of distinct vertices in $\Gamma$. \[\square\]
In the next lemma, and later in the paper, if $A, B$ are sets we write $A \subset B$ if $B \setminus A$ is infinite and $A \setminus B$ is finite. The lemma below extends Lemma 1.4 since under the assumptions of that lemma, $x < y$ (as defined below) never holds. If $u, v, w$ are distinct vertices of the graph $\Gamma$, we say $w$ separates $u$ and $v$ if $w \in \Gamma(u) \cap \Gamma(v) \setminus \{u, v\}$, and call a collection of such vertices $w$ a separating set for $u$ and $v$.

**Lemma 2.3.** Let $\Gamma = (X, R)$ be an infinite graph, and suppose that $\Gamma(x) \cap \Gamma(y)$ is infinite for any distinct $x, y$. Write $x < y$ whenever $\Gamma(x) \supset \Gamma(y)$. Then the structure $\Gamma_\prec = (X, R, <)$ is locally rigid.

**Proof.** We slightly adapt the proof of Proposition 6.1 from [6]. So let $U = \{u_1, \ldots, u_n\}$ be a finite subset of $X$. We aim to find finite $V$ with $U \subset V \subset X$, such that $\text{Aut}(V, R, <)$ fixes $U$ pointwise.

Let $K$ be a positive integer. By Lemma 1.4 with respect to the language $L = \{R, <\}$, we can choose for each $i < j$ a subset $P_{ij}$ of $U$ with $|P_{ij}| = K$, such that the collection of all sets $P_{ij}$ is mutually indiscernible over $U$. Let $W = U \cup \bigcup_{i < j \leq n} (P_{ij} : 1 \leq i < j \leq n)$. Then each $P_{ij}$ carries a complete or null induced graph structure, and for each $x, y \in P_{ij}$ and $z \in W \setminus P_{ij}$, we have $x \sim z$ if and only if $y \sim z$.

For any subset $Y$ of $X$, define the equivalence relation $\equiv_Y$ on $Y$, where, for $x, y \in Y$, $x \equiv_Y y$ if and only if $(\Gamma(x) \cap \Gamma(y)) \cap Y \subseteq \{x, y\}$. Then $\equiv_Y$-classes always carry a complete or null (that is, independent set) induced subgraph structure. If $Z$ is an $\equiv_Y$-class, then for $z_1, z_2 \in Z$ and $y \in Y \setminus Z$, we have $y \sim z_1 \iff y \sim z_2$; in particular, $\text{Aut}(Y \cap Y(z))$ induces $\text{Sym}(Z)$. Observe that if $Y_1 \subset Y_2 \subseteq X$ and $x, y \in Y_1$, then $x \equiv_{Y_2} y$ implies $x \equiv_{Y_1} y$. We often identify such $Y$ with the induced subgraph $(Y, R \cap Y^2)$ of $\Gamma$ which it carries. Thus, $\equiv_Y$ is $\text{Aut}(Y, R)$-invariant.

Now, each $P_{ij}$ lies in an $\equiv_W$-class of $W$. Deleting some sets $P_{ij}$ if necessary (only where elements of distinct sets $P_{ij}$ are $\equiv_W$-equivalent, and retaining the assumption that any two distinct elements of $U$ are separated by some set of form $P_{ij}$), we may suppose: no two elements $x, y$ in distinct sets $P_{ij}, P_{kl}$ are $\equiv_W$-equivalent. Also, $\equiv_W$-classes contain at most one point of $U$; for if $u, u_j \in U$ with $i < j$ then there is a non-empty set $P_{kl}$ whose elements separate $u_i$ and $u_j$, so witness that $u_i \not\equiv_W u_j$. Let $m = \binom{n}{2}$, an upper bound on the number of distinct sets $P_{ij}$. Adjusting the $P_{ij}$ and hence $W$ further, we arrange the sizes of the $P_{ij}$ so that $|P_{ij}| \geq 2$ for each $i, j$ and distinct $\equiv_W$-classes of $W$ of size at least two all have different sizes, with size at most $m + 1 \in \mathbb{N}$. Now every $\equiv_W$-class of $W$ of size greater than 1 consists of a set $P_{ij}$, possibly together with an element of $U$. We will say that a set $Y \subseteq X$ is huge if $|Y| > m + 1$.

Any automorphism of $(W, R)$ will preserve $\equiv_W$, and will fix setwise each $\equiv_W$-class of size at least two (as these classes all have different sizes). Hence, if no element of $U$ is $\equiv_W$-equivalent to any element of any $P_{ij}$ (that is, elements of $U$ lie in $\equiv_W$-classes of size 1), then as the $P_{ij}$ separate the elements of $U$, any automorphism of $W$ will fix $U$ pointwise, as required. The concern is that some $\equiv_W$-class $C$ in $W$ of size
at least two might consist of a set \( P_{ij} \) together with some \( u \in U \), in which case there would be an automorphism of \((W, R)\) mapping \( u \) to some vertex in \( C \setminus \{u\} \).

So suppose \( u \in C \cap U \) as in the last paragraph. By the Pigeonhole Principle (retaining all the above reductions, so initially working with larger sets \( P_{ij} \)) we may suppose for all such \( C, u \) that either \( u \not\in c \) (that is, \( u \) and \( c \) are incomparable with respect to \( < \)) for all \( c \in C \setminus \{u\} \), or \( u < c \) for all \( c \in C \setminus \{u\} \), or \( c < u \) for all \( c \in C \setminus \{u\} \). For such \( u, c \) and \( C \), we add a finite set \( S_{cu} \) of additional vertices of \( \Gamma \) to \( W \) according to the following recipe.

If \( C \) is null, then for each \( c \in C \setminus \{u\} \) for which \( c \not\in u \), the set \( \Gamma(c) \setminus \Gamma(u) \) is infinite, and we choose \( S_{cu} \subset \Gamma(c) \setminus (\Gamma(u) \cup W) \). If \( C \) is complete, then for each \( c \in C \setminus \{u\} \) for which \( c \not\in u \), the set \( \Gamma(u) \setminus \Gamma(c) \) is infinite, and we choose \( S_{cu} \subset \Gamma(u) \setminus (\Gamma(c) \cup W) \). In other cases \((C \text{ null and } c > u \text{ for all } c \in C \setminus \{u\})\), or \( C \) complete and \( c < u \) for all \( c \in C \setminus \{u\} \) we do not add any corresponding set \( S_{cu} \). Each \( S_{cu} \) (for \( u \in U \) and \( c \in W \setminus U \) with \( c \approx_W u \)) is chosen to be huge, and these sets are chosen so that if \( (c, u) \neq (c', u') \) then \( S_{cu} \cap S_{c'u'} = \emptyset \). We may suppose, again by the Pigeonhole Principle, that for each such \( c, u \), either \( c \not\in x \) for all \( x \in S_{cu} \), or \( c < x \) for all \( x \in S_{cu} \), or \( x < c \) for all \( x \in S_{cu} \). By Lemma 1.6 with respect to \( L = \{R, <\} \), we may suppose that the collection of all such sets \( S_{cu} \) is mutually indiscernible over \( W \) (formally, before applying the lemma, the \( S_{cu} \) may be taken to be infinite). Let \( V \) be the union of all such sets \( S_{cu} \) and of \( W \). Observe that each \( S_{cu} \) is either complete or null, and for each \( x, y \in S_{cu} \) and \( z \in V \setminus S_{cu} \), we have \( x \sim z \) if and only if \( y \sim z \). In particular, any two elements of a set \( S_{cu} \) are \( \approx_V \)-equivalent. We arrange that all elements of \( V \setminus W \) lie in huge \( \approx_V \)-classes, and that distinct huge \( \approx_V \)-classes have different sizes, so each is fixed setwise by any automorphism of \((V, R)\).

We aim to show that every automorphism of \((V, R, <)\) must fix \( U \) pointwise, which will complete the proof of the lemma. As a first step, observe that every huge \( \approx_V \)-class \( S \) contains some set \( S_{cu} \). We claim that no huge \( \approx_V \)-class meets \( U \). For suppose \( S \) is a huge \( \approx_V \)-class, with \( a \in U \cap S \). There is \( u \in U \) and \( c \in W \setminus U \), and a \( \approx_W \)-class \( C \) with \( c, u \in C \), such that \( S \supset S_{cu} \). Clearly \( a = u \), since otherwise, as \( S_{cu} \) separates \( c \) from \( u \), \( a \) would separate \( c \) and \( u \) and lie in \( U \subset W \), contradicting that \( c \approx_W u \). Now if \( C \) is null then, by our rule for the process adding \( S_{cu} \), all vertices of \( S \setminus \{u\} \) are adjacent to \( c \); hence \( c \) separates \( u \) from other elements of \( S \), so \( u \not\in S \), a contradiction. Likewise, if \( C \) is complete, then all vertices of \( S \setminus \{u\} \) are non-adjacent to \( c \), so \( c \) separates \( u \) from the rest of \( S \), so \( u \not\in S \). This proves the claim.

**Claim.** Let \( g \in \text{Aut}(V, R, <) \). Then there is \( h \in \text{Aut}(V, R, <)_{(U)} \) such that \( gh \) fixes \( W \) setwise.

**Proof of Claim.** There are distinct (so different-sized) huge \( \approx_V \)-classes \( S_j \) (for \( j \in J \)), each fixed setwise by \( g \), such that \( V \setminus W \subseteq \bigcup_{j \in J} S_j \). We may assume that \( W \) is not fixed setwise by \( g \), as otherwise the claim is trivial. Hence, for some \( j \in J \), we have \((S_j \cap (V \setminus W))^g \neq S_j \cap (V \setminus W) \)

First, we show that \(|S_j \cap W| = 1\). There are \( u \in U \), and some \( \approx_W \)-class \( C \) containing distinct elements \( u, c \) of \( W \), such that \( S_j \supseteq S_{cu} \). We may suppose that \( C \) is null, and \( S_{cu} \subset \Gamma(c) \setminus \Gamma(u) \), as the other case where \( C \) is complete and \( S_{cu} \subset \Gamma(u) \setminus \Gamma(c) \) is similar. Now no element of \( W \setminus \{u, c\} \) could lie in \( S_j \), for otherwise it would separate
u from c in W contradicting that \( u \approx_W c \). Hence, \( S_j \cap W \subseteq \{ c \} \), so due to the existence of the element \( g \), we have \( S_j \cap W = \{ c \} \). In fact, \( S_j = S_{cu} \cup \{ c \} \): for if \( c' \in C \setminus \{ u, c \} \) then \( S_{cu} \neq \emptyset \) but \( c' \) separates elements of \( S_{cu} \) from \( c \) so elements of \( S_{cu} \) do not lie in \( S_j \); and if \( u' \in U \setminus \{ u \} \), then no set of form \( S_{du'} \) could be a subset of \( S_j \), for otherwise \( c \) (in \( W \)) would separate \( d \) from \( u' \) so the set \( S_{du'} \) would not have been added.

By our assumption, there is \( v \in S_{cu} \) such that \( v^g = c \). It is not possible that \( S_{cu} \) is totally ordered by \( < \); this follows easily from the facts that \( g \) induces an automorphism of \( (S_{cu} \cup \{ c \}, <) \), and the earlier assumption that either \( c < x \) for all \( x \in S_{cu} \), or \( x < c \) for all \( x \in S_{cu} \), or \( c \mid x \) for all \( x \in S_{cu} \). It follows by Lemma 1.6(ii) that any permutation of \( S_{cu} \), extended by the identity on the rest of \( V \), is an automorphism of \( (V, R, <) \).

In particular distinct elements of \( S_{cu} \) are \( < \)-incomparable, so as \( v^g = c \), \( S_j \) is an antichain with respect to \( < \). Now it could not happen that there is some \( t \in V \setminus S_j \) whose \( < \)-relation to \( c \) is different from its \( < \)-relation to all other elements of \( S_j \). For otherwise \( t^g \) would have a different \( < \)-relation to \( v \) and to all other elements of \( S_j \), contradicting the mutual indiscernibility in the construction of \( S_{cu} \).

The element \( h \) of the claim will be a product of elements of the form \( g' \), each acting on a different huge \( \approx_V \)-class.

To finish the proof of the lemma, let \( g \in \text{Aut}(V, R, <) \), and let \( h \) be as in the claim. We must show \( w^g = u \) for all \( u \in U \). Now by construction \( gh \) fixes \( W \) setwise, and we claim that \( gh \) fixes \( U \) setwise. Indeed, suppose for a contradiction that \( u \in U \) and \( w^gh \not\in U \). As the \( \approx_W \)-classes of \( W \) of size greater than one are all of different sizes, they are all fixed setwise by \( gh \). Hence, as all elements of \( W \setminus U \) lie in \( \approx_W \)-classes of size greater than one, \( w^gh \) and hence also \( u \) lie in some \( \approx_W \)-class \( C \) of size greater than one. Now, by the construction of \( V \) from \( U \), either \( u \) is the greatest or least element of \( C \) with respect to \( < \), or \( u \) and \( w^gh \) are separated by some huge set of form \( S_{u, w^gh} \). The first case is impossible as \( gh \) preserves \( < \). The second case is also impossible, since as the huge \( \approx_V \)-classes all have different sizes, they are fixed setwise by \( gh \). Thus, as claimed, \( gh \) induces an automorphism of \( (W, R, <) \) which fixes \( U \) setwise. Hence \( gh \) fixes \( U \) pointwise; for any two distinct elements of \( U \) are separated by an \( \approx_W \)-class of size greater than one, and all such classes have different sizes, so are fixed setwise by \( gh \). Thus, \( g \) fixes \( U \) pointwise. \( \square \)

Remark 2.4. Careful inspection of the above proof shows that if \( |U| = n \), then \( V \) may be chosen to have size at most \( O(n^8) \). For in constructing \( W \) from \( U \), if \( m = \binom{n}{2} \) we add at most \( m \) sets \( P_{ij} \), each of size at least 2 and all of different sizes, so \( |W| = n + k \) where \( k := |W \setminus U| \leq \binom{m+1}{2} - 1 \). Then in adding the sets \( S_{cu} \) to obtain \( V \), we add at most \( k \) such sets, each of size at least \( m + 2 \), and all of different sizes. Thus, \( |V \setminus W| \leq (m + 2) + (m + 3) + \ldots + (m + k + 1) = \frac{k}{2}(2m + k + 3) \). Thus, \( |V| \leq \frac{1}{2}(2n + k(2m + k + 5)) \). This is used in Theorem 4.1 below.

Proof of Proposition 2.7. By Lemma 2.2 there is a \( G \)-invariant graph \( \Gamma \) on \( X \) such that for all distinct \( x, y \in X \), the set \( \Gamma(x) \triangle \Gamma(y) \) is infinite. The partial order \( < \) defined in Lemma 2.3 is clearly also \( G \)-invariant. The proposition thus follows immediately from that lemma. \( \square \)
3 \(G\) 2-homogeneous but not 2-transitive

By Proposition 2.1, to complete the proof of Theorem 1.1 it suffices to prove the following.

**Proposition 3.1.** Let \(G\) be a 2-homogeneous but not 2-transitive permutation group on an infinite set \(X\). Then the action of \(G\) on \(X\) is locally rigid.

Recall that a tournament is a directed loopless digraph \((T, \to)\) such that for any distinct vertices \(x, y\), exactly one of \(x \to y\) or \(y \to x\) holds. A group which is 2-homogeneous but not 2-transitive has just one orbit on unordered 2-sets, but two orbits on ordered pairs of distinct elements. Each of these orbits is the arc set of a \(G\)-invariant tournament with vertex set \(X\). Thus, to prove Proposition 3.1, we develop analogues of the methods of Section 2, but for tournaments.

Let \(\to\) denote the arc relation in a tournament \(T = (X, \to)\), and let \(G = \text{Aut}(T)\). For \(x \in X\), we let \(\Gamma^+(x) := \{y \in X : x \to y\}\), the set of outneighbours of \(x\). For \(x, y, z \in X\), we say that \(z\) separates \(x, y\) if \(x \to z \to y\) or \(y \to z \to x\). Furthermore \(Z \subseteq X \) separates \(x, y\) if each \(z \in Z\) separates \(x, y\). We write \(x \to Z\) if \(x \to z\) for each \(z \in Z\).

**Proposition 3.2.** Let \(T = (X, \to)\) be an infinite tournament such that for any distinct \(x, y \in X\), the sets \(\Gamma^+(x) \setminus \Gamma^+(y)\) and \(\Gamma^+(y) \setminus \Gamma^+(x)\) are both infinite. Then \(T\) is cofinally rigid.

We first isolate an easy lemma, used to prove Proposition 3.2 in case it has other uses. It may be known.

Let \(T = (X, \to)\) be a tournament. We will say that \(A \subseteq X\) is a nice set if \(A \neq \emptyset\) and for all \(a_1, a_2 \in A\) and \(v \in X \setminus A\), we have \(a_1 \to v\) if and only if \(a_2 \to v\). (That is, all vertices in a nice set are related in the same way to vertices outside the nice set; equivalently, no vertex outside a nice set separates a pair of vertices inside the nice set.) Note that vacuously any singleton is a nice set, and \(X\) is nice. Furthermore, we will say that \(A \subseteq X\) is a good set, if \(A\) is totally ordered by \(\to\) and is nice. We consider the maximal good subsets of \(X\), that is, good sets \(A\) such that there is no good set \(A' \subseteq X\) with \(A' \supset A\).

**Lemma 3.3.** If \(T = (X, \to)\) is a tournament, then the maximal good subsets of \(X\) form a partition of \(X\).

**Proof.** We claim that if \(A\) is good and \(B \neq A\) is maximal good (where \(A, B \subseteq X\)), then either \(A \subseteq B\) or \(A \cap B = \emptyset\). To see this, let \(d \in A \cap B\), and let \(C = A \cup B\). We show that \(C\) is good, which ensures \(B = C\).

Let \(c_1, c_2 \in C\), \(v \in X \setminus C\). Now \(c_1 \to v\) if and only if \(d \to v\) if and only if \(c_2 \to v\). This holds because \(A\) and \(B\) are both nice and \(d \in A \cap B\). Hence \(C\) is nice. If \(C\) is not totally ordered, then there is some 3-cycle \(c_1 \to c_2 \to c_3 \to c_1\) in \(C\). Since \(A\) and \(B\) are both totally ordered, we must have at least one of these points in \(A \setminus B\) and one in \(B \setminus A\). Suppose \(c_1 \in A \setminus B\) and \(c_2 \in B \setminus A\) (the other case is similar). Then if \(c_3 \in A\), then \(c_2\) separates \(c_1, c_3\), contradicting the fact that \(A\) is nice. Otherwise
$c_3 \in B$, then similarly $c_1$ separates $c_2, c_3$, contradicting the fact that $B$ is nice. Hence $C$ is totally ordered. Now $B \subseteq C$, and $C$ is good, so $A \subseteq B = C$ by maximality of $B$.

The lemma follows immediately from the claim (using Zorn’s Lemma if $X$ is infinite), since each singleton in $X$ is a good set.

Proof of Proposition 3.3. Let $U = \{u_1, \ldots, u_n\} \subset X$. For any distinct $i, j \in \{1, \ldots, n\}$, the set $\Gamma^+(u_i) \setminus \Gamma^+(u_j) = \{v \in X : u_i \rightarrow v \rightarrow u_j\}$ is infinite. Hence by Ramsey’s Theorem, there is $U_{ij} \subset \Gamma^+(u_i) \setminus (\Gamma^+(u_j) \cup \{u_j\})$ with $|U_{ij}| = \aleph_0$, such that $U_{ij}$ is totally ordered by $\rightarrow$. Note that the sets $U_{ij}, U_{ji}$ both separate $u_i, u_j$ (since $u_i \rightarrow U_{ij} \rightarrow u_j$, and $u_j \rightarrow U_{ji} \rightarrow u_i$). We may choose the $U_{ij}$ so that if $(i, j) \neq (k, l)$ then $U_{ij} \cap U_{kl} = \emptyset$.

Claim 1. Let $N$ be any positive integer. Then there are finite subsets $V_{ij}$ of $U_{ij}$ (for all distinct integers $i, j$ with $1 \leq i, j \leq n$) of size $N$ such that the following holds, where $T'$ is the induced subtournament of $T$ with vertex set $U \cup \bigcup_{i \neq j} V_{ij}$: for any distinct $i, j \in \{1, \ldots, n\}$, and for each $x, y \in U_{ij}$ and $v \in T' \setminus U_{ij}$, $x \rightarrow v$ if and only if $y \rightarrow v$.

Proof of Claim 1. This is an immediate application of Lemma 1.6.

Provided we initially choose $N$ large enough, we may cut the $V_{ij}$ down further, and so suppose that each set $V_{ij}$ has size exactly $2^r$ for some $r \geq 2$, and that distinct sets $V_{ij}$ and $V_{kl}$ have distinct sizes. Observe (for use in Theorem 4.1) that $T'$ has $n + \sum_{i=2}^{n} 2^i$ vertices where $m = 2^{n\choose 2}$, that is, it has $n + 2^{n^2-n+2} - 2$ vertices. We claim that $T'$ is rigid, which suffices to prove the lemma. Let $V$ denote the vertex set of $T$ (a union of $U$ and the sets $V_{ij}$).

The sets $V_{ij}$ are clearly all good, though possibly not maximal good. Hence, by Lemma 3.3 if $B \cap V_{ij} \neq \emptyset$ and $B$ is maximal good, then $V_{ij} \subseteq B$.

The idea of the proof is as follows. First observe that automorphisms of the subtournament $(V, \rightarrow)$ of $T$ preserve the family of maximal good sets. We aim to show that by our construction of $V$, all non-singleton maximal good sets in $V$ have different sizes, so in fact each is fixed setwise, and hence pointwise, by any automorphism. We then show that if some automorphism $\alpha$ of $(V, \rightarrow)$ fixes pointwise all non-singleton maximal good subsets of $V$, then $\alpha$ fixes $V$ pointwise.

Claim 2. If $A$ is a good subset of $V$, then $|A \cap U| \leq 1$.

Proof of Claim 2. Suppose $u_1, u_2 \in A \cap U$, with $u_1 \neq u_2$. We have $u_1 \rightarrow V_{12} \rightarrow u_2$. Since $A$ is good, we must have $V_{12} \subset A$: otherwise any $y \in V_{12} \setminus A$ separates $u_1, u_2$, contradicting the fact that $A$ is nice. Similarly, we have $u_2 \rightarrow V_{21} \rightarrow u_1$, and we must have $V_{21} \subset A$. But then we have $\{u_1, u_2\} \cup V_{12} \cup V_{21} \subset A$, and $u_1 \rightarrow V_{12} \rightarrow u_2 \rightarrow V_{21} \rightarrow u_1$. But then $A$ is not totally ordered by $\rightarrow$, which contradicts the fact that $A$ is good.

Thus, maximal good sets are unions of sets $V_{ij}$ with at most one element of $U$ added (this includes the case of a singleton point of $U$). Then by our choice of the sizes of the $V_{ij}$ in the construction, any two non-singleton maximal good sets have different sizes. (For let the $V_{ij}$ have sizes $n_1, \ldots, n_t$, say. These were chosen as distinct powers of 2, and so all numbers of the form $n_1 + \ldots + n_i$, or $n_1 + \ldots + n_i + 1$ are
distinct.) Hence any automorphism of $V$ fixes each non-singleton maximal good set setwise, and hence also pointwise since each is totally ordered and so rigid. Thus any automorphism fixes all elements of $V \setminus U$ pointwise, and so also fixes $U$ pointwise; indeed, for each pair of elements of $U$ there is some $Z \subset V \setminus U$ separating the pair, and so no automorphism can move points of $U$.

**Corollary 3.4.** Let $T$ be an infinite tournament with 2-homogeneous automorphism group. Then $T$ is cofinally rigid.

**Proof.** By Ramsey’s Theorem, there is a subtournament of $T$ of the form $\{x_i : i \in \mathbb{N}\}$ with $x_i \rightarrow x_j$ if and only if $i < j$ (or possibly with all arcs reversed). Clearly, if $i < j$, then $|\Gamma^+(x_j) \triangle \Gamma^+(x_i)| \geq j - i - 1$. By 2-homogeneity of $G$ (and therefore $\text{Aut}(T)$), there is $d \in \mathbb{N} \cup \{\aleph_0\}$ such that if $x \neq y$ then $|\Gamma^+(x) \triangle \Gamma^+(y)| = d$. Hence, $d \geq n$ for each $n \in \mathbb{N}$, so $d = \aleph_0$.

We may suppose that $T$ is not totally ordered by $\rightarrow$, since finite total orders are rigid. By Proposition 3.2, the proof of the corollary now reduces to the following claim.

Claim. For all distinct $x, y \in X$, the sets $\Gamma^+(x) \setminus \Gamma^+(y)$ and $\Gamma^+(y) \setminus \Gamma^+(x)$ are both infinite.

**Proof of Claim.** Suppose that for some $u, v \in X$ with $u \neq v$, the set $\Gamma^+(u) \setminus \Gamma^+(v)$ is infinite, but $\Gamma^+(v) \setminus \Gamma^+(u)$ is finite. Now, using 2-homogeneity, define an order relation $<$ on $X$, such that $x < y$ if and only if $\Gamma^+(x) \setminus \Gamma^+(y)$ is infinite. This is a $G$-invariant partial order on $X$, containing comparable pairs. By 2-homogeneity, it follows that $<$ is a total order, and it or its reverse agrees with $\rightarrow$. This contradicts the above assumption. 

**Proof of Proposition 3.4.** As noted above, there is a $G$-invariant tournament $T$ with vertex set $X$, whose arc set is a $G$-orbit on $X^{[2]}$. The proposition now follows immediately from 2-homogeneity and Corollary 3.4.

**Proof of Proposition 1.1.** This is immediate from Lemma 1.3 and Propositions 2.1 and 3.1.

### 4 Further remarks

The proof of Theorem 1.1 yields that there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $l \in \mathbb{N}$, if $H \leq G$ are closed permutation groups on an infinite set $X$ with $G$ primitive but not 2-transitive, and $G$ and $H$ have the same orbits on $X^{[n]}$ for all $n \leq f(l)$, then $G$ and $H$ have the same orbits on $X^m$ for all $m \leq l$. An upper bound for $f$ is given by the cardinality of $V$ in terms of $|U|$ in the definition in the Introduction of a group $G$ acting locally rigidly. By the proofs of Propositions 2.1 and 3.1, we obtain the following slight strengthening of Theorem 1.1 probably far from best possible. Observe that with $m$ and $k$ as in Remark 2.2, \( \frac{1}{2} (2n + k(2m + k + 5)) \leq n + 2^{n^2 - n + 2} - 2 \) for all $n > 1$, so the bound in the proof of Proposition 3.1 dominates.
Theorem 4.1. Let $G, H$ be closed permutation groups on the infinite set $X$, with $G$ primitive but not 2-transitive on $X$, and with $H \leq G$. Let $n \in \mathbb{N}$, and suppose that $G$ and $H$ have the same orbits on the set $X^l$ for each $l \leq n + 2^{n^2-n+2} - 2$. Then $G$ and $H$ have the same orbits on $X^m$ for each $m \leq n$.

Theorem 4.1 requires the assumption of primitivity. For example, $\text{Aut}(\mathbb{Q}, \prec)\text{Wr}C_2$ is orbit equivalent to $\text{Sym}(\mathbb{Q})\text{Wr}C_2$ (in the natural imprimitive action). However, a proof of Conjecture 1.2 should yield a lot of information about the imprimitive case.

A proof of Conjecture 1.2, at least if via local rigidity, would appear to require arguments considerably more involved than those of this paper. As an example, suppose that $G$ is 2-primitive (that is, 2-transitive and with primitive point stabilisers) but not 3-homogeneous on the infinite set $X$. We conjecture that $G$ acts locally rigidly. There is a $G$-invariant 3-hypergraph $\Gamma$ on $X$, and we would like to show that $\Gamma$ (possibly expanded by some other $G$-invariant relations) is locally rigid. Given $x \in X$, there is an induced graph $\Gamma_x$ on $X \setminus \{x\}$ on which $G_x$ acts primitively. However, it is not clear that local rigidity of $\Gamma_x$ transfers to local rigidity of $\Gamma$, or that a straightforward induction on the degree of transitivity of $G$ can be made to work. There may also be an approach to local rigidity of hypergraphs using [14, Lemma 2.5] and related results.

We cannot even prove the conjecture under the assumptions that $X$ is countable and $G$ is locally compact (that is, there is some finite $F \subset X$ such that all orbits of $G(F)$ on $X$ are finite). Even the case when $G$ is countable is open. A first class to consider would be that of primitive groups with finite point stabiliser, for which Smith [22] gives a useful-looking version of the O’Nan-Scott Theorem.

However, as evidence for the conjecture, we observe that an obvious place to look for a counterexample, suggested by the family of closed supergroups of $\text{Aut}(\mathbb{Q}, \prec)$ listed in Conjecture 1.2, fails. Indeed, let $(T, \prec)$ be any of the countable 2-homogeneous trees (that is, semilinar orders) classified by Drost in [5]. There is a family of interesting primitive closed permutation groups associated with $\text{Aut}(T, \prec)$, namely the primitive Jordan permutation groups with primitive Jordan sets classified in [1]: we have in mind $\text{Aut}(T, \prec)$, the automorphism group of the ternary general betweenness relation on $T$ induced from $\prec$, the automorphism group of the corresponding countable $C$-structure, a structure whose elements are a dense set of maximal chains in $(T, \prec)$, and the automorphism group of the corresponding $D$-relation (a quaternary relation on the set of ‘directions’ of the betweenness relation). It can be checked that each of these groups acts locally rigidly. We omit the details.

In [4] a permutation group $G$ on $X$ is defined to be orbit-closed if there is no $H \leq \text{Sym}(X)$ which properly contains $G$ and is orbit-equivalent to $G$. Such $G$ will be a closed permutation group, and Conjecture 1.2 asserts that if $X$ is countably infinite then the only primitive closed permutation groups which are not orbit-closed are the proper subgroups of $\text{Sym}(X)$ listed in that conjecture. In [4] the authors define $G \leq \text{Sym}(X)$ to be a relation group if there is a collection $R$ of finite subsets of $X$ such that

$$G = \{g \in \text{Sym}(X) : \forall a \in \mathcal{P}(X)(a \in R \leftrightarrow a^g \in R)\}.$$
Clearly any relation group is orbit-closed. Also, by [4, Corollary 4.3], any finite primitive orbit-closed group is a relation group. We do not know whether this holds without finiteness, and in particular cannot answer the following question, to which Siemons drew our attention.

**Question 4.2.** Is $\text{Aut}(\mathbb{Q}, <)$ the only primitive but not 2-transitive closed permutation group of countable degree which is not a relation group?

As a small example, let $\Gamma_3$ be the universal homogeneous 2-edge-coloured graph with edges coloured randomly red or green; that is, the unique countably infinite homogeneous 2-edge-coloured graph such that for any three finite disjoint sets $U, V, W$ of vertices, there is a vertex $x$ not adjacent to any vertex in $U$, adjacent by a red edge to each element of $V$ and by a green edge to each element of $W$. At first sight, $G = \text{Aut}(\Gamma_3)$ is not a relation group, but in fact it is a relation group; for we may take $R$ to consist of the 2-sets joined by a red edge and the 3-sets which carry a green triangle.

Our remarks in the Introduction suggest a further question. Again, for convenience, we shall consider actions on a countably infinite set $X$. A subset $Y$ of $X$ is a moiety of $X$ if $|Y| = |X \setminus Y|$. 

**Question 4.3.** Which primitive closed permutation groups $G$ on a countably infinite set $X$ have a regular orbit on moieties?

To say that $G$ has a regular orbit on moieties of $X$ is the same as to say, in the language of [13], that any first order structure $M$ on $X$ with $G = \text{Aut}(M)$ has distinguishing number 2. Some results on this are obtained in [13]. For example, if $M$ is a homogeneous structure such that the collection of finite structures which embed in it is a ‘free amalgamation class’, and $\text{Aut}(M)$ is primitive but for some $k$ is not $k$-transitive, then $\text{Aut}(M)$ has a regular orbit on moieties. In particular, this holds for the random graph, as follows already from [10] Theorem 3.1. On the other hand, as noted in [13] it is easily seen that $\text{Aut}(\mathbb{Q}, <)$ has no regular orbit on moieties; for if $A$ is a moiety of $\mathbb{Q}$ whose setwise stabiliser is trivial, then $A$ is dense and codense in $\mathbb{Q}$, but the structure $(\mathbb{Q}, <, P)$, where $P$ is a unary predicate naming a dense codense set, is homogeneous so admits $2^{\aleph_0}$ automorphisms.

This suggests the following strengthening of orbit-equivalence. Let us say that permutation groups $G, H$ on the countably infinite set $X$ are strongly orbit-equivalent if they have the same orbits on the power set $\mathcal{P}(X)$ of $X$ (not just on finite subsets of $X$). The following conjecture is implied by Conjecture 1.2, for it is easily seen that the five closed groups containing $\text{Aut}(\mathbb{Q}, <)$ all have different orbits on $\mathcal{P}(\mathbb{Q})$. For example, $\text{Aut}(\mathbb{Q}, <)$ has an orbit consisting of increasing subsets of order type $\omega$ with rational supremum, but this family of sets is not invariant under the automorphism groups of the induced circular order or linear betweenness relation.

**Conjecture 4.4.** Let $G, H$ be strongly orbit-equivalent closed permutation groups on the countably infinite set $X$. Then $H = G$.

Again, the assumption that the groups are closed is necessary. Stoller ([23], see also [16]) gives an example of a proper subgroup $H$ of $G = \text{Sym}(\mathbb{N})$ which is strongly
orbit-equivalent to $G$; namely, let $H$ consist of those permutations $g$ of $\mathbb{N}$ such that there are two partitions, dependent on $g$, of $\mathbb{N}$ into finitely many sets $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ (so $\mathbb{N} = A_1 \cup \ldots \cup A_k = B_1 \cup \ldots \cup B_k$, each partitions) such that for each $i = 1, \ldots, k$, the element $g$ induces an order isomorphism $(A_i, <) \to (B_i, <)$.

Finally, we mention a conjectural strengthening of Lemmas 1.4 and 2.3. It is a special case of a much stronger conjecture in [6].

**Conjecture 4.5.** Let $\Gamma$ be an infinite graph such that for any distinct vertices $x, y$ the set $\Gamma(x) \Delta \Gamma(y)$ is infinite. Then $\Gamma$ is locally rigid.

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