RANK ONE EISENSTEIN COHOMOLOGY
OF LOCAL SYSTEMS ON THE MODULI SPACE
OF ABELIAN VARIETIES

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Abstract. We give a formula for the Eisenstein cohomology of local
systems on the partial compactification of the moduli of principally po-
larized abelian varieties given by rank 1 degenerations.

1. Introduction

Let \( \mathcal{A}_g \) be the moduli stack of principally polarized abelian varieties over
a field \( k \). The universal abelian scheme \( \pi : \mathcal{X}_g \to \mathcal{A}_g \) over \( \mathcal{A}_g \) defines a
local system \( V = R^1 \pi_* \mathbb{Q}(1) \) of rank 2 on \( \mathcal{A}_g \), and for a prime \( \ell \) different
from the characteristic of \( k \) its \( \ell \)-adic variant \( V = R^1 \pi_* \mathbb{Q}_\ell(1) \) for the étale
topology. This local system is associated to the standard representation
of \( \text{GSp}(2g, \mathbb{Q}) \). To an irreducible representation of \( \text{Sp}(2g, \mathbb{Q}) \) with highest
weight \( \lambda = (\lambda_1, \ldots, \lambda_g) \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_g \) we can associate a local
system \( V_\lambda \) of weight \( \sum \lambda_i \) which occurs ‘for the first time’ in
\[ \otimes_{j=1}^{g-1} \text{Sym}^{\lambda_j - \lambda_{j+1}}(\wedge^j V) \otimes \wedge^g V^\lambda. \]
The cohomology of this local system is closely related to vector-valued Siegel
modular forms, cf. [4, 7, 6]. We are interested in the Euler characteristic
\[ e(\mathcal{A}_g, V_\lambda) := \sum (-1)^i [H^i(\mathcal{A}_g, V_\lambda)] \]
in a suitable \( K \)-group of mixed Hodge structures or of Galois representations
if one works with étale cohomology. Similarly, we can consider the analogue
for compactly supported cohomology
\[ e_c(\mathcal{A}_g, V_\lambda) := \sum (-1)^i [H^i_c(\mathcal{A}_g, V_\lambda)]. \]
There is a natural map \( H^*_c(\mathcal{A}_g, V_\lambda) \to H^*(\mathcal{A}_g, V_\lambda) \) and the image is called
the interior cohomology. We define the Eisenstein cohomology as the difference
\[ e_{\text{Eis}}(\mathcal{A}_g, V_\lambda) := e(\mathcal{A}_g, V_\lambda) - e_c(\mathcal{A}_g, V_\lambda). \]
We consider the partial compactification of rank \( \leq 1 \) degenerations. One
can obtain it by considering a compactification \( \bar{\mathcal{A}}_g \) of Faltings-Chai-type
together with the natural map \( q : \bar{\mathcal{A}}_g \to \mathcal{A}_g^* \) to the Satake compactification.
The Satake compactification has a stratification \( \mathcal{A}_g^* = \bigcup_{i=0}^g \mathcal{A}_i \). Then the

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partial compactification $A'_g$ is defined as $q^{-1}(A_g \cup A_{g-1})$ and can be viewed as the moduli space of semi-abelian varieties with torus part of rank $\leq 1$. It is independent of the choice of the compactification $\tilde{A}_g$.

Let $j : A_g \to \tilde{A}_g$ be the inclusion map. We can rewrite the Eisenstein cohomology for a local system $V_\lambda$ (both in Hodge cohomology and in étale cohomology) as

$$e_{Eis}(A_g, V_\lambda) = e(\tilde{A}_g, Rj_*V_\lambda) - e(\tilde{A}_g, Rj!V_\lambda),$$

where $j!V_\lambda$ is the extension by zero. This is a sum over strata

$$\sum_{j=0}^{g-1} e_c(q^{-1}(A_j), Rj_*V_\lambda - Rj!V_\lambda).$$

We define now the rank 1 part of the Eisenstein cohomology as the part that comes from the boundary component lying over $A_{g-1}$:

$$e_{Eis, 1}(A_g, V_\lambda) := e_c(q^{-1}(A_{g-1}), Rj_*V_\lambda - Rj!V_\lambda).$$

This contribution to the Eisenstein cohomology is independent of the compactification.

Let $L := h^2(\mathbb{P}^1)$ be the Lefschetz motive $h^2(\mathbb{P}^1)$ of rank 1 and weight 2. Our result is an explicit formula for the rank 1 part of the Eisenstein cohomology.

**Theorem 1.1.** The contribution $e_{Eis, 1}(A_g, V_\lambda)$ to the Eisenstein cohomology of $V_\lambda$ from the codimension 1 boundary is (both in Hodge cohomology and in étale cohomology) of the form

$$\sum_{k=1}^{g} (-1)^k e_c(A_{g-1}, V_{\lambda_1+1, \lambda_2+1, \ldots, \lambda_k+1, \lambda_{k+1}, \ldots, \lambda_g}) (1 - L^{\lambda_k+g+1-k}).$$

**Remark 1.2.** The fact that the result is formally the same for both Hodge and étale cohomology can be deduced by using the compactifications of the powers of the universal abelian variety as in [7] and by decomposing the direct image of the cohomology under the action of algebraic correspondences as done in [7], p. 238.

**Example 1.3.** For $g = 1$ the cohomology of $V_k$ can only be non-trivial for even $k$. In this case one gets for the Eisenstein cohomology $e_{Eis}(A_1, V_k) = e_{Eis, 1}(A_1, V_k)$ of $V_k$ the polynomial $1 - L^{l+1}$, in agreement with the Eichler-Shimura isomorphisms of [4]

$$e_c(A_1, V_k) = -S_{k+2} \oplus \bar{S}_{k+2} - 1$$

and

$$e(A_1, V_k) = -S_{k+2} \oplus \bar{S}_{k+2} - C(k + 1).$$

Here $S_k$ denotes the space of cusp forms of weight $k$ on $SL(2, \mathbb{Z})$. 
For $g = 2$, cohomology of $\mathbb{V}_{l,m}$ can only be non-trivial for $l \equiv m \pmod{2}$ and in this case one gets for $e_{\text{Eis},1}(\mathbb{V}_{l,m})$ the expression
\[ e_c(A_1, \mathbb{V}_{l+1})(1 - \mathbb{L}^{l+2}) - e_c(A_1, \mathbb{V}_m)(1 - \mathbb{L}^{l+2}), \]
and for $g = 3$ we get for $e_{\text{Eis},1}$ the expression
\[ e_c(A_2, \mathbb{V}_{l+1,m+1})(1 - \mathbb{L}^{l+3}) - e_c(A_2, \mathbb{V}_{l+1,n})(1 - \mathbb{L}^{l+2}) + e_c(A_2, \mathbb{V}_{m,n})(1 - \mathbb{L}^{l+3}). \]

For $g = 2$ we also prove a formula for the total Eisenstein cohomology $e_{\text{Eis}}(A_2, \mathbb{V}_{l,m}) = -s_l - m + 2(1 - \mathbb{L}^{l+m+3}) + s_{l+m+4}(1 - \mathbb{L}^{l+m+3}) - e_c(A_1, \mathbb{V}_m)(1 - \mathbb{L}^{l+2}) - (1 - \mathbb{L}^{m+1}) l \text{ even,}
\[ l \text{ odd,} \]
where $s_m$ denotes for $m > 2$ the dimension of the space of cusp forms of weight $m$ on $\text{SL}(2, \mathbb{Z})$ and $s_2 = -1$. From this formula one can deduce for regular $\lambda$ (i.e., $l > m > 0$) the formula for the Eisenstein cohomology that was announced in joint work with Carel Faber [6], see Corollary 9.2. In that paper the term Eisenstein cohomology refers only to the kernel of $H^i_c \to H^i$.

**Remark 1.4.** Note that the result of [1] is compatible with Poincaré duality, which says that
\[ H^i(A_g, \mathbb{V}_\lambda)^\vee \cong H^{2d-i}_c(A_g, \mathbb{V}_\lambda^\vee(\nu^d)), \]
where $d = g(g + 1)/2$. Here $\mathbb{V}(\nu)$ means the twist of $\mathbb{V}$ by the multiplier, cf. Section 2.

The study of Eisenstein cohomology was initiated by Harder and carried on by his students Schwermer and Pink, cf. [10, 14, 15, 11], cf. also the work of Franke [8]. One may view the result here as an explicit formula for the general results of [11] for the symplectic group. For us the interest in Eisenstein cohomology arose in joint work [6] with Carel Faber where we tried to obtain information on Siegel modular forms by counting curves over finite fields. There Eisenstein cohomology contributes terms that one wants to remove, cf. [11, 6].

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2. The Group

Fix a positive integer $g$ and let $V_{\mathbb{Z}}$ be the standard symplectic lattice of rank $2g$ with generators $e_i$ and $f_i$ ($i = 1, \ldots, g$) with $\langle e_i, f_j \rangle = \delta_{ij}$, the Kronecker delta, for $i \leq j$. We let $G = \text{GSp}(2g)$ be the corresponding Chevalley group of symplectic similitudes of $V_{\mathbb{Z}}$. We shall write $V$ for $V_{\mathbb{Z}} \otimes \mathbb{Q}$.

An element $\gamma \in G$ can be written as $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $a, b, c, d$ integral $g \times g$-matrices. For such a $\gamma$ we write $ad^t - bc^t = \nu(\gamma) \text{Id}_g$.

Here $\nu : G \to \text{G}_m$ is called the multiplier representation. It satisfies
det = ν^\nu : G \rightarrow G_m. We let G act on the left on V by matrix multiplication. We denote by M the subgroup of G of elements that respect the two subspaces \langle e_i : i = 1, \ldots, g \rangle and \langle f_i : i = 1, \ldots, g \rangle. We can interpret the elements of M as matrices \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \text{ with } a \cdot d^\nu = ν 1_g. So M \cong GL(g) \times G_m. Furthermore, we let Q be the (maximal) parabolic subgroup of G that stabilizes the sublattice spanned by the vectors e_i (i = 1, \ldots, g). Then Q = M \rtimes U with $U = Hom(Sym^2(X), \mathbb{Z})$, the group of \mathbb{Z}-valued bilinear forms, or in terms of matrices, the group of matrices \begin{pmatrix} 1_g & b \\ 0 & 1_g \end{pmatrix} in G. The standard maximal torus T of G can be identified with $\mathbb{G}_m^{g+1}$ via 
\[(t_1, \ldots, t_g, x) \mapsto \text{diag}(t_1, \ldots, t_g, x/t_1, \ldots, x/t_g).\]
The character group may thus be identified with the lattice 
\[\{(a_1, \ldots, a_g, c) \in \mathbb{Z}^{g+1} : \sum a_i \equiv c \text{ (mod2)}\},\]
via $\chi(\text{diag}(t_1, \ldots, t_g, x/t_1, \ldots, x/t_g)) = x(c−\sum a_i)/2 \prod t_i^{a_i}$. We will sometimes view $t_i$ (i = 1, \ldots, g) and x as characters on T.

Let t be the complex Lie algebra of T. We have root systems $\Phi_G$ and $\Phi_M$ in $t^\vee$ and we choose compatible systems of positive roots $\Phi_G^+$ and $\Phi_M^+$. So
\[\Phi_M := \{(t_i/t_j)_{\pm} : 1 \leq i < j \leq g\},\]
and we set 
\[\Psi_M = \{(t_i t_j/x)_{\pm} : 1 \leq i < j \leq g\}\]
so that
\[\Phi_G = \Phi_M \cup \Psi_M \quad \text{ and } \quad \Phi_G^+ = \Phi_M^+ \cup \Psi_M^+\]
with 
\[\Phi_M^+ = \{(t_i/t_j) : 1 \leq i < j \leq g\} \quad \text{ and } \quad \Psi_M^+ = \{(t_i t_j/x) : 1 \leq i < j \leq g\}.\]
As usual $\rho = (1/2) \sum_{x \in \Phi_G^+} x$.

3. The Final Elements of the Weyl Group

The Weyl group $W_G$ of G is isomorphic to the semi-direct product $S_g \rtimes (\mathbb{Z}/2\mathbb{Z})^g$, where the symmetric group $S_g$ on g letters acts on $(\mathbb{Z}/2\mathbb{Z})^g$ by permuting the g factors. We interpret elements of $W_G$ as signed permutations. The Weyl group $W_M$ of M is isomorphic to the symmetric group $S_g$. They operate on the complex Lie algebra t and its dual $t^\vee$.

We define the set of Kostant representatives by 
\[W_M = \{w \in W_G : \Phi_M^+ \subset w(\Phi_G^+)\},\]
or equivalently as 
\[\{w \in W_G : \langle w(\rho) - \rho, u \rangle \geq 0 \quad \text{ for all } u \in \Phi_M^+\}.
Concretely, for a sign change $\epsilon : (x_1, \ldots, x_g) \mapsto (\epsilon_1 x_1, \ldots, \epsilon_g x_g)$ with $\epsilon_i \in \{-1\}$ there exists exactly one $\sigma \in S_g$ such that $\sigma \epsilon(\rho) - \rho$ is of the form...
(a_1, \ldots, a_g) \text{ with } a_1 \geq a_2 \geq \cdots \geq a_g \text{ and this is the Kostant representative in } W^M. \text{ Note that } \rho = (g, g-1, \ldots, 2, 1, 0). \text{ Recall that } W_G \text{ carries a length function } \ell.

Another way to describe the Weyl group (and the one we shall use in the following) is as the group of permutations

\[ W_g := \{ \sigma \in S_{2g} : \sigma(i) + \sigma(2g + 1 - i) = 2g + 1 \ (i = 1, \ldots, g) \}. \]

Then the Weyl group of \( M \) can be identified with the subgroup

\[ S_g = \{ \sigma \in W_g : \sigma \{1, 2, \ldots, g\} = \{1, 2, \ldots, g\} \}. \]

The length \( \ell(w) \) of an element \( w \in W_g \subset S_{2g} \) is then defined by

\[ \#\{i < j \leq g : w(i) > w(j)\} + \#\{i \leq j < g : w(i) + w(j) > 2g + 1\}. \]

**Lemma 3.1.** An element \( \sigma \in W_g \subset S_{2g} \) is a Kostant representative if and only if \( \sigma(i) < \sigma(j) \) for all \( 1 \leq i < j \leq g \).

For the proof we refer to [5], Lemma 1. In that paper the Kostant representatives are called final elements. We shall adopt that usage here too. The set of \( 2g \) final elements in the Weyl group is denoted by \( F_g \subset S_{2g} \).

**Lemma 3.2.** The set \( \{ \sigma \in F_g : \sigma^{-1}(k) \leq k \} \) (resp. the set \( \{ \sigma \in F_g : \sigma^{-1}(2g+1-k) \leq g \} \)) has cardinality \( 2^{g-1} \) and can be identified in a natural way with \( F_{g-1} \) compatibly with the length function \( \ell \).

**Proof.** Identify an element \( \sigma \) with its image \( g \)-tuple \( [\sigma(1), \ldots, \sigma(g)] \). Let \( A = \{ \sigma \in F_g : \sigma^{-1}(k) \leq k \} \). For each \( \sigma \in A \) we delete the entry equal to \( k \) from \( [\sigma(1), \ldots, \sigma(g)] \). We rename the entries by replacing an entry \( m \) by \( m-1 \) if \( k < m < 2g + 1 - k \) and by \( m-2 \) if \( m > 2g + 1 - k \). We thus find the \( 2^{g-1} \) elements of \( F_{g-1} \) as one easily checks. The other statements are proved in a similar way. \( \square \)

**Example 3.3.** \( g = 3 \) (For the third column see next two sections.)

\[
\begin{array}{|c|c|c|}
\hline
\ell(w) & (w(1), w(2), w(3)) & w(\lambda + \rho) - \rho \\
\hline
0 & 123 & (l, m, n) \\
1 & 124 & (l, m, -n - 2) \\
2 & 135 & (l, n - 1, -m - 3) \\
3 & 236 & (m - 1, n - 1, -l - 4) \\
3 & 145 & (l, -n - 3, -m - 3) \\
4 & 246 & (m - 1, -n - 3, -l - 4) \\
5 & 356 & (n - 2, -m - 4, -l - 4) \\
6 & 456 & (-n - 4, -m - 4, -l - 4) \\
\hline
\end{array}
\]

4. Representations and Vector Bundles

Let \( t \) be the complex Lie algebra of \( T \). The irreducible representations of \( G \) are parametrized by the characters of \( T \) that correspond to \( G \)-dominant (i.e., the scalar product with \( \Phi^+_G \) is non-negative) integral weights \( \lambda \in t^\vee \).
If \( \lambda \) is given by \((a_1, \ldots, a_g, c)\) this means that we have \( a_1 \geq a_2 \geq \ldots \geq a_g \). The standard representation corresponds to \((1, 0, \ldots, 0, 1)\). The irreducible representation associated to \( \lambda \) is denoted by \( V(\lambda) \).

We have

\[ V(\lambda)^\vee = V(\lambda) \otimes \nu^k \text{ with } k \text{ equal to } \lambda \text{ evaluated at } -1_{2g} \in t. \]

The irreducible representations of \( M \) are parametrized by characters \( \mu \) of \( T \) which are \( M \)-dominant (i.e., with non-negative scalar product with \( \Phi_M^+ \)) so that the representation \( W(\mu) \) corresponding to \( \mu \) has highest weight \( \mu \).

Let \( \mathcal{A}_g \) be the Deligne-Mumford stack of principally polarized abelian varieties of dimension \( g \) and let \( \pi : \mathcal{X}_g \to \mathcal{A}_g \) be the universal family. In the complex category we may identify \( \mathcal{A}_g(\mathbb{C}) \) with \( \text{Sp}(2g, \mathbb{Z})/\mathbb{D} \) with \( \mathbb{D} \) the space of Lagrangian subspaces \( W \subset V_\mathbb{C} \) on which \( -\langle v, \bar{v} \rangle > 0 \). The complex manifold \( \mathbb{D} \) is contained in its so-called compact dual \( \mathbb{D}^\vee = G/Q(\mathbb{C}) \) of Lagrangian subspaces \( W \subset V_\mathbb{C} \).

To each finite-dimensional complex representation \( r \) of \( Q(\mathbb{C}) \) on a vector space \( U \) one can associate a \( G \)-equivariant bundle \( E_r \) on \( \mathbb{D}^\vee \) defined as the quotient of \( G(\mathbb{C}) \times U \) under the equivalence relation

\[ (gq, r(q)^{-1}u) \sim (g, u) \text{ for all } g \in G(\mathbb{C}), q \in Q(\mathbb{C}). \]

Its restriction to \( \mathbb{D} \) descends to a vector bundle \( E_r \) on \( \mathcal{A}_g(\mathbb{C}) \). If \( r \) is the restriction to \( Q(\mathbb{C}) \) of a finite-dimensional complex representation \( \rho \) of \( G(\mathbb{C}) \) then \( E_r \) carries an integrable connection defined as follows. An element \( u \in U \), the fibre over the base point, defines a trivialization of \( E_r \) by \( \gamma \mapsto \rho(\gamma)u \); hence an integrable connection on \( E_r \) and it descends to \( \mathcal{A}_g(\mathbb{C}) \).

The vector bundle associated to the standard representation of \( G(\mathbb{C}) \) can be identified with the relative de Rham homology of \( \mathcal{X}_g \) and \( R^1\pi_*\mathbb{Q}(1) \) is the vector bundle associated to \( \nu^{-1} \otimes \) the standard representation. The integrable connection is the Gauss-Manin connection.

After the choice of a base point we can view \( G \) as the fundamental group (arithmetic fundamental group) of \( \mathcal{A}_g \). Therefore we can associate a local system (or a smooth \( \mathbb{Q}_l \)-sheaf) to each finite-dimensional representation of \( G \). To the standard representation it associates the local system \( R^1\pi_*\mathbb{Q}(1) \) (resp. \( \mathbb{Q}_l \)-sheaf \( R^1\pi_*\mathbb{Q}_l(1) \)) on \( \mathcal{X}_g \otimes \mathbb{Z}[1/l] \). The character \( \nu \) corresponds to \( \mathbb{Q}(1) \) (or \( \mathbb{Q}_l(1) \)). We have a non-degenerate alternating pairing

\[ R^1\pi_*\mathbb{Q} \times R^1\pi_*\mathbb{Q} \to \mathbb{Q}(-1) \]

The local system associated to such a \( \lambda \) is denoted by \( \nabla_\lambda \). If \( \lambda \) is given by \((a_1, \ldots, a_g, c)\) this means that we have \( a_1 \geq a_2 \geq \ldots \geq a_g \). The local system corresponding to the standard representation defined by \((a_1, \ldots, a_g, c) = (1, 0, \ldots, 0, 1)\) is \( R^1\pi_*\mathbb{Q}(1) \). If we do not specify \( c \) then we assume that \( c = \sum_{i=1}^{g} a_i \). Duality now says that we have a non-degenerate pairing

\[ \nabla_\lambda \times \nabla_\lambda \to \mathbb{Q}(-|\lambda|) \]

with \( |\lambda| = \sum \lambda_i \).
The irreducible representations of $M$ are parametrized by characters $\mu$ of $T$ which are $M$-dominant (i.e., with non-negative scalar product with $\Phi_M^+$). To such a character we can associate a locally free $O_{A_g}$-module (or vector bundle) $W_\mu$. For example, the Hodge bundle of the universal family corresponds to the representation $\gamma = (a, b, 0, d) \mapsto \nu(\gamma)^{-1} a$ acting on $\mathbb{C}^g$. Duality for $W_\mu$ says that $W_\mu^\vee = W_{-\sigma_1(\mu)}$ with $\sigma_1$ the longest element of $S_g$. Faltings showed that one can extend the vector bundles thus obtained to appropriate toroidal compactifications, cf. [7], Thm. 4.2.

5. THE BGG COMPLEX

Let $\tilde{A}_g$ be a Faltings-Chai compactification of $A_g$ and let $j : A_g \to \tilde{A}_g$ be the natural inclusion and let $i : D \to \tilde{A}_g$ be the inclusion of the divisor at infinity. Recall that $D$ is a stack quotient of a Kuga-Satake variety, namely a compactified quotient of the universal abelian variety of dimension $g - 1$ by the group $GL(1, \mathbb{Z})$ which acts by $\{\pm 1\} \in \text{End}(X_\eta)$ on the generic fibre $X_\eta$.

According to Deligne the logarithmic de Rham complex w.r.t. the divisor $D$ represents $Rj_* \mathcal{O}$. This generalizes for our sheaves $V_\lambda$, where the role of the de Rham complex is played by the dual BGG complex which is obtained by applying ideas of Bernstein-Gelfand-Gelfand of [2] to our situation as worked out in [7].

The dual BGG complex for $\lambda$ is a direct summand of the de Rham complex for $V_\lambda^\vee$ and consists of a complex $K_\lambda^\bullet$ of vector bundles on $A_g$:

$$K_\lambda^q = \oplus_{w \in F_g, \ell(w) = q} W_{w*\lambda}^\vee,$$

where $w*\lambda = w(\lambda + \rho) - \rho$ and $W_\mu^\vee = W_{-\sigma_1(\mu)}$. This complex is a filtered resolution of $V_\lambda^\vee$ on $A_g$. The differentials of this complex are given by homogeneous differential operators on $\mathbb{D}^\vee$.

The vector bundles $W_\mu$ extend over the compactification $\tilde{A}_g$ and so do the differential operators, resulting in a complex $\tilde{K}_\lambda^\bullet$ on $\tilde{A}_g$. We shall denote the extensions of $W_\mu$ again by the same symbol $W_\mu$ (or by $\tilde{W}_\mu$ if confusion might arise). There is a variant of this complex $\tilde{K}_\lambda^\bullet \otimes O_{\tilde{A}_g}(-D)$ and the differentials extend also for this complex.

The filtration on the dual BGG complex induce decreasing filtrations on $\tilde{K}_\lambda^\bullet$ and $\tilde{K}_\lambda^\bullet \otimes O(-D)$ given by

$$F^p(\tilde{K}_\lambda^\bullet) = \oplus_{w \in F_g, \ell(w, \lambda) \geq p} \tilde{W}_{w*\lambda}^\vee,$$

where $f(w, \lambda) = (\sum \lambda_i + \sum \mu_i)/2$ with $\mu = -\sigma_1(w*\lambda)$, and in an analogous way for $\tilde{K}_\lambda^\bullet \otimes O(-D)$. A term $W_\mu$ belongs to $F^p$ if and only if $(\sum \lambda_i + \sum \mu_i)/2 \geq p$.

In [7], p. 233, it is shown that the (filtered) dual BGG complex $\tilde{K}_\lambda^\bullet$ is quasi-isomorphic to $Rj_* V_\lambda^\vee$, while $\tilde{K}_\lambda^\bullet \otimes O(-D)$ is quasi-isomorphic to $Rj_* V_\lambda^\vee$ and that the inclusion $\tilde{K}_\lambda^\bullet \otimes O(-D) \subset \tilde{K}_\lambda^\bullet$ corresponds to the natural map $Rj_* V_\lambda^\vee \to Rj_* V_\lambda^\vee$. 
We have an exact sequence of complexes

\[ 0 \to \bar{K}_\lambda^\bullet \otimes O(-D) \to \bar{K}_\lambda^\bullet \to \bar{K}_\lambda^\bullet | D \to 0. \]

Therefore we can calculate the Eisenstein cohomology \( e_{Eis}(A_g, V_\lambda^\vee) \) by using the complex \( \bar{K}_\lambda^\bullet | D = i^* \bar{K}_\lambda^\bullet \) obtained by restricting the dual BGG complex to \( D \).

### 6. A Calculation at the Boundary

The boundary stratum \( D \) in \( \tilde{A}_g \) is a stratified space itself via the map \( q \) to the Satake compactification. The part \( D' = D \cap A'_g \) of \( D \) lying in \( A'_g \) has an étale cover \( X_{g-1} \to D' \) with \( X_{g-1} \to \mathcal{A}_{g-1} \) the universal principally polarized abelian variety of relative dimension \( g - 1 \) and \( X_{g-1} \to \mathcal{A}_{g-1} \) factors through \( q : D' \to \mathcal{A}_{g-1}/\text{GL}(1, \mathbb{Z}) \). The vector bundles \( W_\mu \) extend canonically from \( D' \) to \( D \).

We shall calculate the rank 1 part of the Eisenstein cohomology by using the Leray spectral sequence for the complex

\[ \bigoplus_{w \in F_g} W_\lambda^\vee \]

by first restricting the factors \( W_\lambda^\vee \) of the complex \( \bigoplus_{w \in F_g} W_\lambda^\vee \) to \( D' \), extending these to a suitable Faltings-Chai compactification \( \bar{D}' \) and then tensoring this complex with \( O(-F) \) with \( F \) the divisor at infinity \( \bar{D}' - D' \) of \( D' \) and by calculating the cohomology using the Leray spectral sequence for the map \( q : D' \to \mathcal{A}_{g-1}/\text{GL}(1, \mathbb{Z}) \). (The tensoring with \( O(-F) \) is done to get the rank-1 part of the Eisenstein cohomology and because of this the choice of the compactification of \( D' \) does not matter.)

As we shall now work at the same time with \( \mathcal{A}_g \) and \( \mathcal{A}_{g-1} \) we will write \( W_\mu^{(g)} \) for \( W_\mu \) on \( \tilde{A}_g \) in order to avoid confusion. We first do a computation on \( X_{g-1} \to \mathcal{A}_{g-1} \) and later take into account the action of \( \text{GL}(1, \mathbb{Z}) \).

**Proposition 6.1.** For \( W_a = W_a^{(g)} \) with \( a = (a_1, \ldots, a_g) \) on \( \tilde{A}_g \) we have

\[
\sum_j (-1)^j R^j q_*(W_a^{(g)}|X_{g-1}) = \sum_{k=1}^g (-1)^{g-k} W_{(a_1, \ldots, a_{k-1}, a_{k+1}-1, \ldots, a_g)}^{(g-1)}.
\]

**Proof.** We consider the Hodge bundle \( \mathcal{E}_g \) on \( \tilde{A}_g \). Its pullback to \( X_{g-1} \) fits into the exact sequence

\[ 0 \to q^*\mathcal{E}_{g-1} \to \mathcal{E}_g \to O_{X_{g-1}} \to 0, \]

and we thus get

\[
Rq_*\mathcal{E}_g = \mathcal{E}_{g-1} \otimes Rq_*O_{X_{g-1}} + Rq_*O_{X_{g-1}} = (\mathcal{E}_{g-1} + 1) \otimes \bigoplus_{j=0}^{g-1} \wedge^j \mathcal{E}_{g-1}^\vee,
\]

where \( \wedge^j \mathcal{E}_{g-1}^\vee \) is the exterior product of \( j \)-fold of \( \mathcal{E}_{g-1}^\vee \).
since $R^iq_*O_{X_g-1} = \wedge^i R^1q_*O_{X_g-1}$. Note that

$$
\sum_{j=0}^{g-1} \wedge^j E_{g-1} = (-1)^k \sum_{k=0}^{g-1} W_{t_k}^{(g-1)}
$$

with $t_k$ denoting a vector $(0, \ldots, 0, -1, \ldots, -1)$ of length $g - 1$ with $g - 1 - k$ zeros. Since $W_{a}^{(g)}$ is made by applying a Schur functor to the Hodge bundle the exact sequence for $E_g$ implies

$$Rq_*W_a = \text{Res}^{g}_{g-1} W_{a}^{(g)} \otimes \sum_{k=0}^{g-1} (-1)^k W_{t_k}^{(g-1)}.$$

Here $\text{Res}^{g}_{g-1} W_a$ is the bundle obtained from restriction (branching) from $GL(g)$ to $GL(g - 1)$. A well-known formula from representation theory (cf. e.g. [9]) says that we thus get

$$\text{Res}^{g}_{g-1} W_{a}^{(g)} = \sum_{b} W_{b}^{(g-1)},$$

where the sum is over all (interlacing) $b = (b_1, \ldots, b_{g-1})$ with $a_1 \geq b_1 \geq a_2 \geq \cdots \geq b_{g-1} \geq a_g$. Moreover, we have that $W_{b}^{(g-1)} \otimes \sum_{k=0}^{g-1} (-1)^k W_{t_k}^{(g-1)}$ is a signed sum of $W_{b'}^{(g-1)}$'s with the sum running over all vectors $b' \in \mathbb{Z}^{g-1}$ obtained from subtracting a 1 from $k$ entries $b_i$ with $k$ between 0 and $g - 1$ and deleting those $b_i$ that does not satisfy the condition that $b_i' \geq b_i' + 1$. Carrying out the summation we see that most terms telescope away and what remains is the right hand side of the statement in the proposition. \square

For example, for $g = 2$ we get that $Rq_*W_{a,b,c}^{(3)}$ is equal to

$$\sum_{\alpha, \beta} W_{\alpha, \beta}^{(2)} - W_{\alpha, \beta-1}^{(2)} - W_{\alpha-1, \beta}^{(2)} + W_{\alpha-1, \beta-1}^{(2)}$$

where the sum is over all $(\alpha, \beta)$ with $a \geq \alpha \geq b \geq \beta \geq c$ and $W_2(r, s) = 0$ if $r < s$. What remains is $W_2(a, b) - W_2(a, c - 1) + W_2(b - 1, c - 1)$.

However, we still need to take the action of $GL(1, \mathbb{Z})$ into account. The non-trivial elements acts by $-1$ on the fibres of $q$. Therefore only the terms in the right hand side of Proposition 6.1 which are even (in the sense that $\sum_{i=1}^{k-1} a_i + \sum_{i=k+1}^{g} (a_i - 1) \equiv 0 \pmod{2}$) will contribute to

$$\sum_{j} (-1)^j R^j q_* (W_{a}^{(g)}|D').$$

Recall that we write $w \ast \lambda$ for the operation $\lambda \mapsto w(\lambda + \rho) - \rho$ of $W_g$ on the set of lambda’s. Recall also we have $W(w \ast \lambda)^\vee = W(-\sigma_1(w \ast \lambda))$ with $\sigma_1$ the longest element of $S_g$.

Our Eisenstein cohomology is now given by a complex which is a sum over $w \in F_g$ of terms $Rq_* W_{w \ast \lambda}^{\vee}$. By Prop. 6.1 the term $Rq_* W_{w \ast \lambda}^{\vee}$ yields a
complex
\[ \sum_{l=1}^{g} (-1)^l W^{(g-1)}_{\tau_l(-\sigma(w*\lambda))}, \]
where \( \tau_l \) applied to a vector \((a_1, \ldots, a_g)\) is the vector \((a_1, \ldots, a_{l-1}, a_{l+1} - 1, \ldots, a_g - 1)\). We thus get a double sum \( \sum_{w \in F_g} \sum_{l=1}^{g} X_{w,l} \) of terms \( X_{w,l} \) (that are (signed) sheaves of the form \( W_{\mu} \)) which we rewrite as a sum of two double sums
\[ \sum_{k=1}^{g} \sum_{w \in F_g, w^{-1}(k) \leq k} X_{w,k} + \sum_{k=1}^{g} \sum_{w \in F_g, w^{-1}(2g+1-k) \leq g} X_{w,k}. \]

By Lemma [3.2] the inner sum \( \sum_{w \in F_g, w^{-1}(k) \leq k} X_{w,k} \) in the first double sum is equal to
\[ \sum_{w \in F_g-1} (-1)^k (W^{(g-1)}_{\tau_k'\mu} \lambda)^\vee, \]
where \( \tau_k'(a_1, \ldots, a_g) = (a_1 + 1, \ldots, a_{k-1} + 1, a_{k+1}, \ldots, a_g) \). The double sum \( \sum_{k=1}^{g} \sum_{w \in F_g, w^{-1}(k) \leq k} X_{w,k} \) thus contributes the complex
\[ K'_\lambda := \sum_{k=1}^{g} (-1)^k K_{\tau_k'\mu}(\lambda). \]

The Hodge weight of these terms can be read off from Lemma [3.2] and Faltings’ results and thus contribute the cohomology of \( \sum_k (-1)^{k} V_{\tau_k'\mu}(\lambda) \). (See the diagram in Section [9] for an illustration in case \( g = 2 \).) In view of duality (Poincaré and Serre duality, see [7], p. 236) the remaining \( 2g-1 \) \( g \) terms of the sum \( \sum_{k=1}^{g} \sum_{w \in F_g, w^{-1}(2g+1-k) \leq g} X_{w,k} \) will contribute the dual terms:
\[ \sum_{k=1}^{g} (-1)^{k+1} K_{\tau_k'\mu}(\lambda) \otimes \nu^{\lambda_k + g+1-k} \]
and this contributes the cohomology of \( \sum_k (-1)^{k+1} V_{\tau_k'\mu}(\lambda) \) twisted by the power \( \nu^{g+1+\lambda_k-k} \) of \( \nu \). But we need the rank 1 part. To get this we consider the divisor \( \Delta \) at infinity of \( \tilde{\mathcal{A}}_{g-1} \) (where \( \tilde{\mathcal{A}}_{g-1} \) is defined as the closure of the zero-section of \( \mathcal{X}_{g-1} \to \mathcal{A}_{g-1} \)) and take \( \tilde{K}'_\lambda \otimes O(-\Delta) \) instead of \( K'_\lambda \). Here \( \tilde{K}'_\lambda \) is the extension over \( \tilde{\mathcal{A}}_{g-1} \) which we know to exist. Then our Eisenstein cohomology is of the form
\[ \sum_{k=1}^{g} (-1)^k e_c(\mathcal{A}_{g-1}, V_{\tau_k'\mu}(\lambda))(1 - \nu^{g+1+\lambda_k-k}). \]

In the next section we show that this is compatible with the action of the Hecke algebras.
7. THE ACTION OF THE HECKE OPERATORS

The Hecke algebra acts on the cohomology \( H^*(\mathcal{A}_g, \mathbb{V}_\lambda) \) and \( H_c^*(\mathcal{A}_g, \mathbb{V}_\lambda) \) as explained in [7]. It also acts in a compatible way on the dual BGG complex. These operators are defined by algebraic cycles and this guarantees that they respect the mixed Hodge structure on the Betti cohomology and the Galois structure on étale cohomology. Moreover, they are self-adjoint for Serre and Poincaré duality.

The compatible action of the Hecke operators on both \( H^*(\mathcal{A}_g, \mathbb{V}_\lambda) \) and \( H_c^*(\mathcal{A}_g, \mathbb{V}_\lambda) \) induces an action on the (total) Eisenstein cohomology. We can see this action by means of its action on the dual BGG complex and its restriction to the boundary \( D \). We now show that it factors through the action of the Hecke algebra for \( GSp(2g-2, \mathbb{Z}) \).

The correspondences \( T \to \mathcal{A}_g \times \mathcal{A}_g \) that define the Hecke operators extend in a natural way to \( \mathcal{A}_g' \times \mathcal{A}_g' \). The (pullback of the) restriction of such a \( T \) to \( \mathcal{X}_{g-1} \times \mathcal{X}_{g-1} \), a cover of \( D' \times D' \), is given by \( T' \to \mathcal{X}_{g-1} \times \mathcal{X}_{g-1} \) which lies over a component of a Hecke correspondence \( T'' \to \mathcal{A}_{g-1} \times \mathcal{A}_{g-1} \). In the next paragraph we indicate this for the complex case.

The map \( T' \to T'' \) is a universal family of abelian varieties. The action is on bundles that are pullbacks from \( \mathcal{A}_{g-1} \). We thus see that the action of \( T'/T'' \) is induced by an element of \( \mathbb{Z} \subset \text{End}(X_\eta) \) with \( X_\eta \), an abelian variety, the generic fibre of \( T' \) over \( T'' \) on the cohomology of the structure sheaf \( O_{X_\eta} \). This action is by scalars. The action of \( T'' \) belongs to the action of the Hecke algebra of \( GSp(2g-2, \mathbb{Z}) \). Hence the Hecke algebra of \( GSp(2g, \mathbb{Z}) \) on the Eisenstein cohomology factors through an action of the Hecke algebra of \( GSp(2g-2, \mathbb{Z}) \).

We work this out over \( \mathbb{C} \). A component of a Hecke correspondence is defined by an embedding \( \mathcal{H}_g \to \mathcal{H}_g \times \mathcal{H}_g \), with \( \mathcal{H}_g \) the Siegel upper half plane, and given by equations

\[
wcz + dw - az - b = 0, \tag{1}
\]

where \( w, z \) are in \( \mathcal{H}_g \) and \( (a, b; c, d) \) is an integral \( 2g \times 2g \)-matrix which lies in \( GSp(2g, \mathbb{Q}) \). This component can be extended to a correspondence for \( \mathcal{A}_g' \) that restricts to \( T' \to \mathcal{X}_{g-1} \times \mathcal{X}_{g-1} \) given by an embedding of \( \mathcal{H}_{g-1} \times \mathbb{C}^{g-1} \) into \( (\mathcal{H}_{g-1} \times \mathbb{C}^{g-1})^2 \). We consider the behavior at infinity given by

\[
\lim_{t \to \infty} \begin{pmatrix} z' \\ \xi t \end{pmatrix} = z' \in \mathcal{H}_{g-1}, \xi \in \mathbb{C}^{g-1}.
\]

In order that the component given by (1) does intersect our boundary component the matrix \( (a, b; c, d) \) must have the form

\[
\begin{pmatrix}
a' & 0 & b' & * \\
* & u & * & * \\
c' & 0 & d' & * \\
0 & 0 & 0 & u^{-1}
\end{pmatrix} \quad \left( \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} \in GSp(2g-2, \mathbb{Q}), u \in \mathbb{Q}^* \right).
\]
This correspondence in $X_{g-1} \times X_{g-1}$ lies over a component of a Hecke correspondence $T'' \to \mathcal{A}_{g-1} \times \mathcal{A}_{g-1}$ given by

$$w'c'z' + d'w' - a'z' - b' = 0 \quad (z', w' \in \mathcal{H}_{g-1}).$$

and in the fibres it is given by

$$\eta^i(c'z' + d') = u\zeta + (\alpha z' + \beta),$$

where $\eta$ is the analogue for $w$ of $\zeta$ for $z$.

8. An Example: $g = 1$

We consider the case of a local system $\mathbb{V}_k = \text{Sym}^k(\mathbb{V})$ for $k$ even. For $g = 1$ the BGG complex is $0 \to j_* \mathbb{V}_k \to W_{-k} \to W_{k+2} \to 0$ with $\mathbb{V}_k = \text{Sym}^k(R^1\pi_*\mathbb{Q}(1))$. Similarly, there is a complex $0 \to j'_* \mathbb{V}_k \to W_{-k}(-D) \to W_{k+2}(-D) \to 0$ with $D$ the divisor $\mathcal{A}_1 - \mathcal{A}_1$ at infinity. By the exact sequence

$$0 \to W_m(-D) \to W_m \to W_m|D \to 0$$

we get

$$e(j_* \mathbb{V}_k') - e(j'_* \mathbb{V}_k') = e(D, W_{-k}|D) - e(D, W_{k+2}|D).$$

The bundle $W_a$ is associated to a representation of $Q$, the parabolic subgroup. The action of the central multiplicative group is not trivial: $W_1$ is associated to the representation where

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

acts by multiplication by $1/d$ on $\mathbb{C}$.

In view of $\mathbb{V}_k' = \mathbb{V}_k \otimes \nu^k$ we see that $e_{\text{Eis}}(\mathbb{V}_k)$ is equals to $e(D, W_{-k}'|D) - e(D, W_{k+2}'|D)$, where the prime refers to $k$ times twisting. This implies that $W_c'|D$ is $\mathbb{C}((k+c)/2)$ on $D$ and that

$$e_{\text{Eis}}(\mathbb{V}_k) = L^0 - L^{k+1}$$

for $k \geq 0$ even.

Our answer for the Eisenstein cohomology is compatible with Poincaré duality. It has a part with Hodge weight $k+1$ and one with Hodge weight 0. The part of Hodge weight $k+1$ occurs for $k \geq 2$ in the exact sequence

$$0 \to H^0(W_{k+2} \otimes O(-D)) \to H^0(W_{k+2}) \to H^0(D, W_{k+2}|D) \to 0.$$ 

This can be identified with

$$0 \to S_{k+2} \to M_{k+2} \to \mathbb{C}(k+1) \to 0,$$

with $M_{k+2}$ (resp. $S_{k+2}$) the space of modular forms (resp. of cusp forms) of weight $k + 2$ on $\text{SL}(2, \mathbb{Z})$. The other part occurs in

$$0 \to H^0(D, W_{-k}|D) \to H^1(W_{-k} \otimes O(-D)) \to H^1(W_{-k}) \to 0.$$ 

All in all we get

$$e_c(\mathcal{A}_1, \mathbb{V}_k) = -S_{k+2} \oplus \check{S}_{k+2} - 1$$

and

$$e(\mathcal{A}_1, \mathbb{V}_k) = -S_{k+2} \oplus \check{S}_{k+2} - \mathbb{C}(k + 1).$$
We identify \( \mathbb{C} \) and \( \mathbb{C}(k+1) \) with the Eisenstein cohomology. Note that if we let \( W'_m = W_m \otimes O(-\Delta) \) with \( \Delta \) the divisor on \( D \) that is the fibre over the cusp \( \infty \) of \( \tilde{A}_1 \) we also have the identities

\[
e_\varepsilon(A_1, V_k) = -e(\tilde{A}_1, W'_{k+2}) + e(\tilde{A}_1, W'_k) \\
e(A_1, V_k) = -e(\tilde{A}_1, W_{k+2}) + e(\tilde{A}_1, W_{-k}).
\]

For even \( k \geq 4 \) we let \( S[k] \) denote the motive of cusp forms of weight \( k \) on \( SL(2, \mathbb{Z}) \) as constructed by Scholl, cf. \([13]\), see also \([3]\). For \( k = 2 \) we put \( S[2] = -\mathbb{L} - 1 \). In the category of Hodge structures we have for \( k \geq 2 \)

\[
S[k+2] = e(\tilde{A}_1, W'_{k+2}) - e(\tilde{A}_1, W_{-k}).
\]

Note that by Serre duality we have \( H^1(\tilde{A}_1, W_{-k}) \cong H^0(\tilde{A}_1, \Omega^1 \otimes W_k)^{\vee} = H^0(\tilde{A}_1, W'_{k+2})^\vee \) but if we take into account the action of the central \( \mathbb{G}_m \) we have to twist by \( \eta^{k+1} \). We now have for even \( k \geq 0 \)

\[
e_\varepsilon(A_1, V_k) = -S[k+2] - 1, \quad e(A_1, V_k) = -S[k+2] - \mathbb{L}^{k+1}.
\]

9. Another Example: \( g = 2 \)

We now look at the case \( g = 2 \) and consider the local system \( V_{l,m} \) with \( l \equiv m \) (mod2). Calculations with Carel Faber in \([6]\) led to the formulas for this case. We have a standard Faltings-Chai compactification \( \tilde{A}_2 \) in this case which coincides with Igusa’s blow-up of the Satake compactification and also with the moduli space \( \mathcal{M}_2 \) of stable curves of genus 2. We consider the full Eisenstein cohomology

\[
e_{\text{Eis}}(A_2, V_{l,m}) := e(A_2, Rj_* V_{l,m}) - e(A_2, Rj_{\vee} V_{l,m}).
\]

**Theorem 9.1.** The Eisenstein cohomology \( e_{\text{Eis}}(A_2, V_{l,m}) \) is given by

\[
-s_{l-m+2}(1 - \mathbb{L}^{l+m+3}) + s_{l+m+3}(\mathbb{L}^{m+1} - \mathbb{L}^{l+2}) + \\
\begin{cases}
e_\varepsilon(A_1, (V_m)(1 - \mathbb{L}^{l+2}) - (\mathbb{L}^{l+2} - \mathbb{L}^{l+m+3}) & l \text{ even}, \\
-e_\varepsilon(A_1, V_{l+1})(1 - \mathbb{L}^{m+1}) - (1 - \mathbb{L}^{m+1}) & l \text{ odd}.
\end{cases}
\]

Alternatively, the Eisenstein cohomology can be written as

\[
-(s_{l-m+2} + 1)(1 - \mathbb{L}^{l+m+3}) + s_{l+m+3}(\mathbb{L}^{m+1} - \mathbb{L}^{l+2}) + \\
\begin{cases}
-S[m+2](1 - \mathbb{L}^{l+2}) & l \text{ even}, \\
S[l+3](1 - \mathbb{L}^{m+1}) & l \text{ odd}.
\end{cases}
\]

We can check this for example for \( l = m = 0 \). We have

\[
e(Rj_* V_{0,0}) - e(Rj_{\vee} V_{0,0}) = 1 + \mathbb{L} - \mathbb{L}^2 - \mathbb{L}^3,
\]

while for the compactly supported Eisenstein cohomology (i.e., the kernel of \( H^*_c \to H^* \)) we find \( 1 + \mathbb{L} \), which fits. If \( V_{l,m} \) is a regular local system (i.e., \( l > m > 0 \)) then the Hodge weights of the terms are either \( > l + m + 3 \)...
or \(< l + m + 3\) and this determines whether the term belongs to compactly supported Eisenstein cohomology, cf., [12] Thm. 3.5. We thus get the result announced in [6].

**Corollary 9.2.** ([6]) The compactly supported Eisenstein cohomology for a regular local system \(\mathcal{V}_{l,m}\) is given by

\[
sl - m + 2 - S[l + 3] + 1 \quad l \text{ even}, \\
- S[l + 3] \quad l \text{ odd}.
\]

In [6] one finds numerical confirmation of these formulas. The BGG complex for \(j_*\mathcal{V}_{l,m}^\vee\) is:

\[
0 \to j_*\mathcal{V}_{l,m}^\vee \to W_{-m,-l} \to W_{m+2,-l} \to W_{l+3,1-m} \to W_{l+3,m+3} \to 0
\]

and for the compactly supported cohomology the similar complex is

\[
0 \to j!\mathcal{V}_{l,m}^\vee \to W_{-m,-l} \otimes O(-D) \to \ldots
\]

The extended complexes over \(\tilde{\mathcal{A}}_g\) are quasi-isomorphic to \(Rj_* (\mathcal{V}_\lambda)^\vee\) and \(Rj! (\mathcal{V}_\lambda)^\vee\), see [7] Prop. 5.4 and above. We have to twist \(l + m\) times if we work with \(Rj_* (\mathcal{V}_\lambda)\) and \(Rj! (\mathcal{V}_\lambda)\). We hope that the details of the first part of the proof in this case will illustrate the proof of the formula for \(e_{Eis,1}\) for the general case. The short exact sequence

\[
0 \to W_{\mu}(-D) \to W_{\mu} \to W_{\mu}|D| \to 0
\]

yields

\[
e(Rj_* \mathcal{V}_{l,m}) - e(j! \mathcal{V}_{l,m}) = e(W_{-m,-l}|D) - e(W_{m+2,-l}|D) + e(W_{l+3,1-m}|D) - e(W_{l+3,m+3}|D))
\]

if we take into account a \(l + m\) th twist. Therefore we first determine the Euler characteristic \(e(W_{a,b}|D)\). Note that \(D\) is the quotient by \(-1\) of the compactification \(\tilde{X}_1 \to \tilde{\mathcal{A}}_1\) of the universal elliptic curve \(X_1 \to \mathcal{A}_1\). We stratify \(\tilde{X}_1\) by the open part and the fibre \(F\) over the cusp \(\infty\) of \(\mathcal{A}_1\). The cohomology

\[
e(\tilde{\mathcal{A}}_1, \sum_j (-1)^j R^j q_*(W_{\mu}|X_1))
\]

can be calculated via the exact sequence for the Hodge bundle \(E_2 = W_{1,0}\)

\[
0 \to q^* E_1 \to E_2|X_1 \to O_{X_1} \to 0
\]

More generally, the pull back of \(W_{a,b}\) to \(X_1\) is \(\text{Sym}^{a-b} E_1 \otimes \det E_1^b\) with \(E_1\) the Hodge bundle. But we need to keep track of the twisting. The exact sequence above gives (using that \(R^i q_* O_{X_1} = \wedge^i E_1^\vee\))

\[
Rq_* W_{a,b}|X_1 = \sum_{a \geq \nu \geq b} W_\nu \otimes (1 - E_1^\vee)
\]

and carrying out the summation we find

\[
Rq_* W_{a,b}|X_1 = W_a - W_{b-1}.
\]
We now replace the $W_\mu$’s in our calculation by $W'_\mu := W_\mu(-F)$ and then calculate separately the contribution from the stratum $F$. We collect the terms. An element $w \in F_2 \subset S_4$ is identified with its images $[w(1)w(2)]$ of the elements 1 and 2.

| $w$ | $\ell(w)$ | $-\sigma_1(w * \lambda)$ | contribution |
|-----|-----------|----------------|-------------|
| 12  | 0         | $(-m,-l)$     | $W'_{-m} - W'_{-l-1}$ |
| 13  | 1         | $(m+2,-l)$    | $-W'_{m+2} + W'_{-l-1}(\nu^{m+1})$ |
| 24  | 2         | $(l+3,1-m)$   | $W'_{l+3} - W'_{-m}(\nu^{l+2})$ |
| 34  | 3         | $(l+3, m+3)$  | $-W'_{l+3}(\nu^{m+1}) + W'_{m+2}(\nu^{l+2})$ |

We can collect this into $-e_c(A_1, V_m)(1 - L^{l+2}) + e_c(A_1, V_{l+1})(1 - L^{m+1})$.

We now treat the codimension 2 boundary contribution. It is a priori clear that the outcome will be a polynomial in $L$ as we are calculating over a toric curve; in fact one can also deduce that the exponents of $L$ that occur are in $\{l + m + 3, l + 2, m + 1, 0\}$.

Let $F$ be the fibre of $q$ over $\tilde{A}_0$. Note that $F$ has dimension 1. The neighborhood of $F$ in the toroidal compactification $\tilde{A}_2$ is a toric variety obtained by glueing infinitely many copies of affine 3-space $A^3$ and dividing through an action of $GL(2, \mathbb{Z})$; more precisely it is an orbifold constructed as follows. The cone of symmetric real positive definite $2 \times 2$ matrices has a natural cone decomposition invariant under the action of $GL(2, \mathbb{Z})$. The group $GL(2, \mathbb{Z})$ acts on the cone of positive definite $2 \times 2$ matrices by

$$C = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \mapsto A^tCA$$

The cone decomposition consists of the orbit under $GL(2, \mathbb{Z})$ of the cone spanned by the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Associated to this we have the toroidal variety $T$ with $GL(2, \mathbb{Z})$-action. The coordinate axes in the $A^3$’s are glued to a union of $\mathbb{P}^1$’s. The 1-skeleton $T^1$ of the quotient of this toric variety under $GL(2, \mathbb{Z})$ can be identified with $F$. In general in level $n \geq 3$ we have $(1/4)n^3 \prod_p (1 - p^{-2})$ copies of $\mathbb{P}^1$ meeting three at each one of the $(1/6)n^3 \prod_p (1 - p^{-2})$ points. For $n = 3$ this looks like a tetrahedron, a cube for $n = 4$ etc. and in general as the polyhedral decomposition of the Riemann surface $\Gamma(n) \backslash \mathcal{H}_1$ with $\Gamma(n)$ the full level $n$ congruence subgroup.

The associated quadratic form $\alpha x^2 + 2\beta xy + \gamma y^2$ determines a point $z = (\beta + \sqrt{\beta^2 - 4\alpha \gamma})/2\alpha$ in the upper half plane (this factors through scaling by positive reals). The action of $\text{diag}(1, -1)$ is given by $\beta \mapsto -\beta$ and induces $z \mapsto -\bar{z}$ on the upper half plane.
In order to calculate the complex $\bar{K}_\lambda |F$ we restrict the Hodge bundle $E$ to $F$. Note that the fundamental domain for the action of $\Gamma[n] \subset \text{GL}(2,\mathbb{Z})$ is a cone over a fundamental domain for the modular curve of level $n$.

The restriction of the Hodge bundle to a $\mathbb{P}^1$ in $F$ is of the form $O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}$. The pullback of $E$ to $T^1$ is flat vector bundle of rank 2 determined by the standard representation of $\text{GL}(2)$. So the bundle $W(a,b)$ is associated to the irreducible representation of type $(a,b)$ of $\text{GL}(2)$. This and the toroidal construction makes it possible to express the cohomology in terms of group cohomology.

**Proposition 9.3.** Let

$$U = W_{-m,-l} - W_{m+2,-l} + W_{l+3,1-m} - W_{l+3,m+3}.$$ 

Then the Euler characteristic $e_c(F,U|F)$ equals

$$-s_{l-m+2}(L^0 - L^{l+m+3}) + s_{l+m+4}(L^{m+1} - L^{l+2}) + \begin{cases} -L^{l+2} + L^{l+m+3} & l \text{ even} \\ -1 + L^{m+1} & l \text{ odd.} \end{cases}$$

**Proof.** The Euler characteristic $e_c(F,W(a,b))$ can be expressed in group cohomology. We calculate it by using the stratification of $F$ by the open part and the cusp. We then get the Euler characteristic of compactly supported cohomology of $\text{GL}(2,\mathbb{Z})$ with values in $V_{a-b}$. This gives for the four cases $(-m,-l), \ldots, (l+3,m+3)$ the contributions $(-s_{l-m+2}-1)L^x, (s_{l+m+4}-1)L^y, (s_{l+m+4}+1)L^{l+m+3-y}$ and $(s_{l-m+2}+1)L^{l+m+3-x}$ for appropriate $x$ and $y$ which turn out to be 0 and $m+1$. We have to add the contribution from the cusp. This gives $L^{l+2} - L^{l+m+3}$ for $l$ odd and $1 - L^{m+1}$ for $l$ even. For this just look at the action of $\text{diag}(1,-1)$. If $\ell$ (and hence $m$) is even only the first two terms ($W_{-m,-l}$ and $-W_{m+2,-l}$) contribute because of the sign, while for the odd case the two other terms contribute. \qed

**Remark 9.4.** The same method suffices to prove the analogues for the moduli space $A_2[n]$ of abelian surfaces with a level $n$ structure.

**References**

[1] J. Bergström, C. Faber, G. van der Geer: Siegel modular forms of genus 2 and level 2: cohomological computations and conjectures. In preparation.

[2] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand: Differential operators on the base affine space and a study of $g$-modules. In: Lie groups and their representations. Ed. I.M. Gelfand. Adam Hilger, London, 1975.

[3] K. Consani, C. Faber: On the cusp form motives in genus 1 and level 1. In: Moduli Spaces and Arithmetic Geometry, Advanced Studies in Pure Mathematics, 2006.

[4] P. Deligne: *Formes modulaires et représentations $\ell$-adiques*, Séminaire Bourbaki 1968-1969, exp. 355.

[5] T. Ekedahl, G. van der Geer: Cycle Classes of the E-O Stratification on the Moduli of Abelian Varieties. arXiv:math/0412272 To appear in: Algebra, Arithmetic and Geometry- Manin Festschrift. Birkhäuser Verlag.

[6] C. Faber, G. van der Geer: *Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes*, I, II. C.R. Acad. Sci. Paris, Sér. I, 338 (2004), 381–384, 467–470.
[7] G. Faltings, C-L. Chai: Degeneration of abelian varieties. Ergeb. Math. Grenzgeb. (3) 22, Springer, Berlin, 1990.

[8] J. Franke: Harmonic analysis in weighted $L_2$-spaces. Ann. Sci. École Norm. Sup. 31 (1998), p. 181–279.

[9] W. Fulton, J. Harris: Representation Theory. A First Course, Springer-Verlag, New York, 1991.

[10] G. Harder: Eisensteinkohomologie und die Konstruktion gemischter Motive. Springer Lecture Notes in Mathematics 1562. Springer 1993.

[11] R. Pink: On $\ell$-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. Math. Ann. 292 (1992), 197–240.

[12] C. Peters: Lowest weights in Cohomology of Variations of Hodge structure. arXiv: 0708.0130v2.

[13] A.J. Scholl: Motives for modular forms. Invent. Math. 100 (1990), p. 419–430.

[14] J. Schwermer: On Euler products and residual Eisenstein cohomology classes of Siegel modular varieties. Forum Math. 7 (1995), 1-28.

[15] J. Schwermer: On arithmetic quotients of the Siegel upper half space of degree two. Comp. Math. 58 (1986), 233–258.

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