EQUIVALENCE OF KUO AND THOM QUANTITIES
FOR ANALYTIC FUNCTIONS

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Abstract. Sufficiency of jets is a very important notion introduced by René Thom in order to establish the structural stability theory. The criteria for some sufficiency of jets are known as the Kuo condition and Thom type inequality, which are defined using the Kuo quantity and Thom quantity. Therefore these quantities are meaningful. In this paper we show the equivalence of Kuo and Thom quantities. Then we apply this result to the relative conditions to a given closed set.

1. Introduction

Let \( f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be a \( C^r \) function germ. The \( r \)-jet of \( f \) at \( 0 \in \mathbb{R}^n \), \( j^r f(0) \), has a unique polynomial representative \( z \) of degree not exceeding \( r \). We do not distinguish the \( r \)-jet \( j^r f(0) \) and the polynomial representative \( z \) here.

Kuiper-Kuo condition. There is a positive number \( C > 0 \) such that
\[
\| \text{grad } z(x) \| \geq C \| x \|^{r-1}
\]
holds in some neighbourhood of \( 0 \in \mathbb{R}^n \).

The Kuiper-Kuo condition is well-known as a criterion for \( C^0 \)-sufficiency and \( V \)-sufficiency of \( z \) in \( C^r \) functions (N. Kuiper [9], T.-C. Kuo [7], J. Bochnak and S. Lojasiewicz [4]). See [2] for the definitions of \( C^0 \)-sufficiency and \( V \)-sufficiency of jet.

Let us recall the Kuo condition.

Kuo condition. There are positive numbers \( C, \alpha, \bar{w} > 0 \) such that
\[
\| \text{grad } f(x) \| \geq C \| x \|^{r-1} \quad \text{in } \mathcal{H}_r(f; \bar{w}) \cap \{ \| x \| < \alpha \},
\]
where \( \mathcal{H}_r(f; \bar{w}) := \{ x \in \mathbb{R}^n : |f(x)| \leq \bar{w} \| x \|^r \} \) is the horn-neighbourhood of \( f^{-1}(0) \) of degree \( r \) and width \( \bar{w} \) (T.-C. Kuo [8]).

The Kuo condition is a criterion for \( V \)-sufficiency of \( z \) in \( C^r \) functions.

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**Condition** ($\tilde{K}$). There is a positive number $C > 0$ such that
$$\|x\| \|\grad f(x)\| + |f(x)| \geq C\|x\|^r$$
holds in some neighbourhood of $0 \in \mathbb{R}^n$.

This condition is the Kuo condition in a different way. Therefore condition ($\tilde{K}$) is also a criterion for $V$-sufficiency of $z$ in $C^r$ functions.

On the other hand, R. Thom formulated the following condition as a sufficient condition for $z$ to be $C^0$-sufficient in $C^r$-functions.

**Thom type inequality.** There are positive numbers $K, \beta > 0$ such that
$$\sum_{i<j} |x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i}|^2 + |f(x)|^2 \geq K\|x\|^{2r} \text{ for } \|x\| < \beta.$$

It is shown in [1] that Thom type inequality condition is equivalent to the Kuiper-Kuo condition.

Throughout this paper, we denote by $\mathbb{N}$ the set of natural numbers in the sense of positive integers. Let $s \in \mathbb{N} \cup \{\infty, \omega\}$, and let $\mathcal{E}_{[s]}(n, p)$ denote the set of $C^s$ map-germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

Now we introduce the Kuo quantity $K_m$ and Thom quantity $T_m$. The Thom quantity is a generalisation of the left side of Thom type inequality, and the Kuo quantity is a generalisation of the left side of a condition equivalent to condition ($\tilde{K}$).

**Definition 1.1.** Let $f \in \mathcal{E}_{[s]}(n, p), n \geq p$, and let $m \in \mathbb{N}$. Let us define two functions of the variable $x$:

(1.1) $K_m(f, x) := \|x\|^m \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \right) \right|^m + \|f(x)\|^m$

(1.2) $T_m(f, x) := \sum_{1 \leq i_1 < \ldots < i_{p+1} \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p, \rho)}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x) \right) \right|^m + \|f(x)\|^m$

where $\rho(x) = \|x\|^2$. Note that $T_m(f, x) = \|f(x)\|^m$ in the case where $n = p$.

Related to the Kuo condition and Thom type inequality, we have shown the following result.

**Theorem 1.2.** ([1]) Let $r \in \mathbb{N}$. For $f \in \mathcal{E}_{[r]}(n, p), n \geq p$, the following conditions are equivalent.

1. There are positive numbers $C, \alpha > 0$ such that $K_2(f, x) \geq C\|x\|^{2r}$ for $\|x\| < \alpha$.
2. There are positive numbers $K, \beta > 0$ such that $T_2(f, x) \geq K\|x\|^{2r}$ for $\|x\| < \beta$.

The main purpose of this paper is to show the equivalence of the Kuo quantity and Thom quantity, which is a generalisation of the above result in a certain sense.

**Theorem 1.3.** (Main Theorem). Let $f \in \mathcal{E}_{[\omega]}(n, p), n \geq p$. Then for any $m \in \mathbb{N},$
$$K_m(f,.) \approx T_m(f,.).$$
Throughout this paper, we use the equivalence $\approx$ in the following sense:

Let $f, g : U \to \mathbb{R}$ be non-negative functions, where $U \subset \mathbb{R}^N$ is an open neighbourhood of $0 \in \mathbb{R}^N$. If there are real numbers $K > 0$, $\delta > 0$ with $B_\delta(0) \subset U$ such that $f(x) \leq Kg(x)$ for any $x \in B_\delta(0)$, where $B_\delta(0)$ is a closed ball in $\mathbb{R}^N$ of radius $\delta$ centred at $0 \in \mathbb{R}^N$, then we write $f \lesssim g$ (or $g \gtrsim f$). If $f \lesssim g$ and $f \gtrsim g$, we write $f \approx g$.

In the next section we mention the definitions of $C^0$-sufficiency and $V$-sufficiency of jets, and give the notion of the relative jet of a given closed subset $\Sigma$. We shall show our Main Theorem in \S 3 and apply the theorem to the relative conditions to a closed set $\Sigma$ in \S 4.

2. Preliminaries

2.1. Sufficiency of jets. Let $s \in \mathbb{N} \cup \{\infty, \omega\}$. Let us recall $E_{s|}(n, p)$, the set of $C^s$ map-germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. Let $j^r f(0)$ denote the r-jet ($r \in \mathbb{N}$) of $f$ at $0 \in \mathbb{R}^n$ for $f \in E_{s|}(n, p)$, $s \geq r$, and let $J^r(n, p)$ denote the set of r-jets in $E_{s|}(n, p)$.

We say that $f, g \in E_{s|}(n, p)$ are $C^0$-equivalent (resp. $SV$-equivalent), if there exists a local homeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \sigma$ (resp. $\sigma(f^{-1}(0)) = g^{-1}(0)$). In addition, we say that $f, g \in E_{s|}(n, p)$ are $V$-equivalent, if $f^{-1}(0)$ is homeomorphic to $g^{-1}(0)$ as germs at $0 \in \mathbb{R}^n$.

Let $w \in J^r(n, p)$. We call the r-jet $w$ $C^0$-sufficient, $SV$-sufficient and $V$-sufficient in $C^s$ mappings, $s \geq r$, if any two realisations $f, g \in E_{s|}(n, p)$ of $w$, namely $j^r f(0) = j^r g(0) = w$, are $C^0$-equivalent, $SV$-equivalent and $V$-equivalent, respectively.

Let us recall the Thom type inequality for $f \in E_{s|}(n, p)$, $n \geq p$ :

There are positive numbers $K, \alpha, \beta > 0$ such that $T_2(f, x) \geq K ||x||^\alpha$ for $||x|| < \beta$.

As mentioned in the Introduction, R. Thom considered this condition with $\alpha = 2r$ in the function case as a sufficient condition for $z = j^r(f)(0)$ to be $C^0$-sufficient in $C^r$ functions. On the other hand, he considered this condition in the mapping case as a sufficient condition for $SV$-sufficiency of jet.

The Kuo condition mentioned in the Introduction is a criterion for $V$-sufficiency of $z = j^r(f)(0)$ in $C^r$ functions. This condition is generalised to the mapping case, as a criterion for $V$-sufficiency of $z = j^r(f)(0)$ in $C^r$ mappings : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n \geq p$. For the details, see T.-C. Kuo [9].

2.2. Relative jet to a given closed set. Throughout this paper, let $\Sigma$ be a germ of a given closed subset of $\mathbb{R}^n$ at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$. Then we denote by $d(x, \Sigma)$ the distance from a point $x \in \mathbb{R}^n$ to the subset $\Sigma$.

We consider on $E_{s|}(n, p)$ the following equivalence relation:

Two map-germs $f, g \in E_{s|}(n, p)$ are $r$-$\Sigma$-equivalent, denoted by $f \sim g$, if there exists a neighbourhood $U$ of $0$ in $\mathbb{R}^n$ such that the r-jet extensions of $f$ and $g$ satisfy $j^r f(\Sigma \cap U) = j^r g(\Sigma \cap U)$.

We denote by $j^r f(\Sigma; 0)$ the equivalence class of $f$, and by $J^r_\Sigma(n, p)$ the quotient set $E_{s|}(n, p)/\sim$. 
Proof of Theorem 1.3. It is obvious that 

It follows that 

where 

show the converse.

respectively.

Thom quantity 
the relative Kuiper-Kuo condition and relative Kuo condition (or condition 

$r$

relative jets to $\Sigma$, similarly to in the non-relative case. In [2] we gave criteria for

$\lambda$

$C$

$\delta$

$g$

Here we remark that the functions

$(3.1)$

$(C(t))^m K_m(f,\lambda(t)) > T_m(f,\lambda(t)).$

We may write $(3.1)$ as:

$(C(t))(h \circ \lambda(t)))^m > (g \circ \lambda(t))^m$.

Here we remark that the functions $g \circ \lambda, h \circ \lambda, u \circ \lambda, v \circ \lambda$ and $w \circ \lambda$ are real analytic on $[0, \delta)$ and satisfying the conditions

$g \circ \lambda(0) = h \circ \lambda(0) = u \circ \lambda(0) = v \circ \lambda(0) = w \circ \lambda(0) = 0$

and

$\lambda(t) \neq 0, \ C(t) > 0, \ h \circ \lambda(t) > 0, \ g \circ \lambda(t) \geq 0 \ \text{for} \ 0 < t < \delta$

By $(3.2)$, $C(t)(h \circ \lambda(t)) > u \circ \lambda(t), C(t)(h \circ \lambda(t)) > w \circ \lambda(t)$ and

$v \circ \lambda(t) = h \circ \lambda(t) - u \circ \lambda(t) \geq h \circ \lambda(t)(1 - C(t))$.  

3. PROOF OF MAIN THEOREM

In this section we show the equivalence between the Kuo quantity $K_m$ and the Thom quantity $T_m$, namely our main theorem (Theorem 1.3).

Let $\text{ord}(\gamma(t))$ denote the order of $\gamma$ in $t$ for a $C^\infty$ function $\gamma: [0, \delta) \to \mathbb{R}$.

Proof of Theorem 1.3. It is obvious that $K_m(f,.) \preceq T_m(f,.)$. Therefore we have to show the converse.

We first remark that if $x$ and $y$ are bigger than or equal to 0, we have

$$(x + y)^m \geq x^m + y^m \geq \frac{(x + y)^m}{2^m}.$$  

It follows that

$$K_m(f, x) \approx v^m(x) + u^m(x) \approx (h(x))^m$$

$$T_m(f, x) \approx w^m(x) + u^m(x) \approx (g(x))^m,$$

where $u(x) = \|f(x)\|,$

$$v(x) = \|x\| \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \right) \right|,$$

$$w(x) = \sum_{1 \leq i_1 < \ldots < i_{p+1} \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p, \rho)}{D(x_{i_1}, \ldots, x_{i_p+1})}(x) \right) \right|, \text{(where } \rho(x) = \|x\|^2),$$

$h(x) = v(x) + u(x)$ and $g(x) = w(x) + u(x)$.

Suppose now that $K_m(f,.) \preceq T_m(f,.)$ does not hold. Then by the curve selection lemma, there is a $C^\infty$ curve $\bar{\lambda} = (\lambda, C) : [0, \delta) \to \mathbb{R}^n \times \mathbb{R}$ with $\bar{\lambda}(0) = (0, 0)$ and $\bar{\lambda}(t) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^*$, for $t \neq 0$, such that

$$(3.1) \quad (C(t))^m K_m(f,\lambda(t)) > T_m(f,\lambda(t)).$$

We may write $(3.1)$ as:

$$(3.2) \quad (C(t)(h \circ \lambda(t)))^m > (g \circ \lambda(t))^m.$$  

Here we remark that the functions $g \circ \lambda, h \circ \lambda, u \circ \lambda, v \circ \lambda$ and $w \circ \lambda$ are real analytic on $[0, \delta)$ and satisfying the conditions

$g \circ \lambda(0) = h \circ \lambda(0) = u \circ \lambda(0) = v \circ \lambda(0) = w \circ \lambda(0) = 0$

and

$\lambda(t) \neq 0, \ C(t) > 0, \ h \circ \lambda(t) > 0, \ g \circ \lambda(t) \geq 0 \ \text{for} \ 0 < t < \delta$

By $(3.2)$, $C(t)(h \circ \lambda(t)) > u \circ \lambda(t), C(t)(h \circ \lambda(t)) > w \circ \lambda(t)$ and

$v \circ \lambda(t) = h \circ \lambda(t) - u \circ \lambda(t) \geq h \circ \lambda(t)(1 - C(t)).$
Then we have

\[
\begin{cases}
ord(C) + \ord(h \circ \lambda) \leq \ord(u \circ \lambda) \\
ord(C) + \ord(h \circ \lambda) \leq \ord(w \circ \lambda) \\
ord(v \circ \lambda) \leq \ord(h \circ \lambda).
\end{cases}
\] (3.3)

Note that we are not considering the second inequality in the case where \( n = p \).

Let \( \lambda \) be written as follows:

\[
\lambda_i(t) = a_1^{(i)} t^{\varepsilon_1(i)} + a_2^{(i)} t^{\varepsilon_2(i)} + \ldots
\]

where \( 1 \leq \varepsilon_1(i) < \varepsilon_2(i) < \ldots \) and 

\[
\begin{cases}
a_1^{(i)} \neq 0 & \text{if } \lambda_i(t) \neq 0 \\
\varepsilon_1(i) & \text{if } \lambda_i(t) \equiv 0 \quad (1 \leq i \leq n),
\end{cases}
\]

where \( C(t) = u_1 t^{b_1} + u_2 t^{b_2} + \ldots \) where \( 1 \leq b_1 < b_2 < \ldots \) and \( u_1 \neq 0 \).

Since condition (3.1) is invariant under rotation, we can assume that \( \varepsilon_1(1) < \varepsilon_1(i) \) for \( i \neq 1 \).

Let \( f_j(\lambda(t)) = d_1^{(j)} t^{q_1^{(j)}} + d_2^{(j)} t^{q_2^{(j)}} + \ldots \), where \( 1 \leq q_1^{(j)} < q_2^{(j)} < \ldots \) (\( 1 \leq j \leq p \)).

Then

\[
\frac{df_j \circ \lambda}{dt}(t) = q_1^{(j)} d_1^{(j)} t^{q_1^{(j)} - 1} + q_2^{(j)} d_2^{(j)} t^{q_2^{(j)} - 1} + \ldots \quad (1 \leq j \leq p).
\]

It follows from (3.3) that

\[
q_1^{(j)} \geq \ord(C) + \ord(h \circ \lambda) \quad \text{for all } j \in \{1, \ldots, p\}.
\]

By (3.3) again, we have

\[
\varepsilon_1(1) + \ord\left( \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(\lambda(t)) \right) \right| \right) \leq \ord(h \circ \lambda).
\] (3.5)

Therefore there is a \( p \)-tuple of integers \( (k_1, \ldots, k_p) \) with \( 1 \leq k_1 < \ldots < k_p \leq n \) such that

\[
\begin{cases}
\ord(|\det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(\lambda(t)) \right)|) \leq \ord(|\det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(\lambda(t)) \right)|)
\end{cases}
\]

for any \( (i_1, \ldots, i_p) \), and

\[
\ord(|\det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})}(\lambda(t)) \right)|) \leq \ord(h \circ \lambda) - \varepsilon_1(1).
\] (3.6)

We continue the proof of the converse, dividing it into two cases. We first consider the case where \( n > p \). Then we have the following.

Claim: \( k_1 > 1 \).

Proof: Since 

\[
\frac{df_j \circ \lambda}{dt}(t) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\lambda(t)) \frac{d\lambda_i}{dt}(t), \quad (1 \leq j \leq p),
\]

we have

\[
\begin{pmatrix}
\frac{df_1 \circ \lambda}{dt}(t) \\
\vdots \\
\frac{df_p \circ \lambda}{dt}(t)
\end{pmatrix} = \frac{d\lambda_1}{dt}(t) \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(\lambda(t)) \\
\vdots \\
\frac{\partial f_1}{\partial x_n}(\lambda(t))
\end{pmatrix} + \ldots + \frac{d\lambda_n}{dt}(t) \begin{pmatrix}
\frac{\partial f_p}{\partial x_1}(\lambda(t)) \\
\vdots \\
\frac{\partial f_p}{\partial x_n}(\lambda(t))
\end{pmatrix}.
\] (3.7)

Here we remark that, by (3.4)
Then the determinant of the matrix we have

\[ \text{ord} \left( \frac{\lambda_j(t)}{\lambda_i(t)} \right) = q_j(\lambda) - \varepsilon_1(1) \geq \text{ord}(C) + \text{ord}(h \circ \lambda) - \varepsilon_1(1) \quad (1 \leq j \leq p), \]

and

\[ \text{ord} \left( \frac{\lambda_i(t)}{\lambda_i(t)} \right) \geq 1 \quad (2 \leq i \leq n). \]

Assume, by contradiction, that \( k_1 = 1 \) in (3.6). For simplicity, set

\[ A(t) = \left( \begin{array}{cccc} \frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_p}(\lambda(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(\lambda(t)) & \frac{\partial f_p}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_p}{\partial x_p}(\lambda(t)) \end{array} \right). \]

Then the determinant of the matrix \( A(t) \) is the summation of determinants of the following matrices:

\[ \text{ord} \left( \frac{\lambda_i(t)}{\lambda_i(t)} \right) \left( \begin{array}{cccc} \frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_p}(\lambda(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(\lambda(t)) & \frac{\partial f_p}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_p}{\partial x_p}(\lambda(t)) \end{array} \right) \quad \text{for } i \in \{2, \ldots, n\}. \]

By (3.8) the order of the determinant of the matrix (3.10) is bigger than or equal to \( \text{ord}(C) + \text{ord}(h \circ \lambda) - \varepsilon_1(1) \), and by (3.9) the order of the determinant of the matrix (3.11) is bigger than the order of the determinant of the matrix (3.10). Therefore we have

\[ \text{ord}(|\det A(t)|) \geq \text{ord}(C) + \text{ord}(h \circ \lambda) - \varepsilon_1(1) > \text{ord}(h \circ \lambda) - \varepsilon_1(1) \]

which contradicts (3.6). This completes the proof of the claim. \( \square \)

It follows from the Claim that there is a \( p \)-tuple \( (k_1, \ldots, k_p) \) with \( 1 < k_1 < \cdots < k_p \leq n \) such that condition (3.6) holds. Then

\[ \text{ord} \left( \left| \det \left( \frac{D(f_1, \ldots, f_p, \rho)}{D(x_1, x_{k_1}, \ldots, x_{k_p})}(\lambda(t)) \right) \right| \right) \leq \text{ord}(\lambda) + \text{ord} \left( \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{k_1}, \ldots, x_{k_p})}(\lambda(t)) \right) \right| \right) \]

\[ \leq \varepsilon_1(1) + \text{ord}(h \circ \lambda) - \varepsilon_1(1) = \text{ord}(h \circ \lambda). \]

This contradicts (3.3), and it follows that \( K_m(f, \cdot) \prec T_m(f, \cdot) \).
We next consider the case where \( n = p \). Using a similar argument to the proof of the above Claim, we get the same contradiction for

\[
A(t) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_n}(\lambda(t)) \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_n}{\partial x_1}(\lambda(t)) & \frac{\partial f_n}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_n}{\partial x_n}(\lambda(t))
\end{pmatrix}.
\]

Therefore it follows that \( K_m(f,.) \lesssim T_n(f,.) \), and this completes the proof. \( \Box \)

Remark 3.1. The proof of Theorem 3.3 uses essentially the curve selection lemma. Therefore it is not difficult to see that the results are still valid if we suppose only that \( f \) is an arc-analytic and differentiable subanalytic map-germ; see [10], [5] and [3] for the notions and properties of subanalytic and arc-analytic functions.

Example 3.2. Let \( f = (f_1, f_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) be a polynomial mapping defined by \( f_1(x, y) = x - y^2 \), \( f_2(x, y) = x^2 \). Then we have \( f_1(x, y)^2 + f_2(x, y)^2 = (x - y^2)^2 + x^4 \), \( \det \left( \frac{D(f_1, f_2)}{D(x, y)}((x, y)) \right) = 4xy \). Therefore we have

\[
T_2(f, (x, y)) = (x - y^2)^2 + x^4, \quad K_2(f, (x, y)) = 16(x^2 + y^2)x^2y^2 + (x - y^2)^2 + x^4.
\]

To show that \( T_2(f, (x, y)) \approx K_2(f, (x, y)) \), we consider two cases.

In the case where \( |x - y^2| \leq \frac{1}{2}y^2 \), we have \( x \geq \frac{1}{2}y^2 \). Therefore \( 64x^4 \geq 16x^2y^4 \) and since for any constant \( C > 65 \), \( 16x^4y^4 = o((C - 65)x^4) \) we get

\[
CT_2(f, (x, y)) \geq K_2(f, (x, y))
\]
in a small neighbourhood of \((0, 0) \in \mathbb{R}^2\).

In the case where \( |x - y^2| \geq \frac{1}{2}y^2 \) we can see that

\[
(x - y^2)^2 + x^4 \geq \frac{1}{4}y^4 + x^4 \geq 16x^2y^4 + 16x^4y^4 = 16(x^2 + y^2)x^2y^2
\]
in a small neighbourhood of \((0, 0) \in \mathbb{R}^2\).

Thus, for any constant \( C > 65 \), we have \( T_2(f, (x, y)) \leq K_2(f, (x, y)) \leq CT_2(f, (x, y)) \) in a small neighbourhood of \((0, 0) \in \mathbb{R}^2\), it follows that \( T_2(f, (x, y)) \approx K_2(f, (x, y)) \).

4. Application to the relative case

We now introduce some notion for a \( C^r \)-map germ \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) in order to extend to the relative case the previous equivalence defined in the non-relative case.

Let \( \Sigma \) be a germ at \( 0 \in \mathbb{R}^n \) of closed set such that \( 0 \in \Sigma \). Given a map \( g \in \mathcal{E}_{\chi}(n, p) \) with \( j^r g(\Sigma; 0) = j^r f(\Sigma; 0) \). Let \( f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) denote the \( C^r \) mapping defined by

\[
f_t(x) = f(x) + t(g(x) - f(x)) \quad \text{for} \quad t \in [0, 1].
\]

Definition 4.1. A condition \((*)\) on a \( C^r \) map \( f \) is called \( \Sigma-r\)-compatible in the direction \( g \), if \( f_t \) satisfies condition \((*)\) for any \( t \in [0, 1] \). If condition \((*)\) is \( \Sigma-r\)-compatible in any direction \( g \in \mathcal{E}_{\chi}(n, p) \) with \( j^r g(\Sigma; 0) = j^r f(\Sigma; 0) \), we simply say condition \((*)\) is \( \Sigma-r\)-compatible.
Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be a \( C^1 \) map-germ, \( \Sigma \subset \mathbb{R}^n \) be a germ of a closed set such that \( 0 \in \Sigma \) and \( r \in \mathbb{N} \). For \( m \in \mathbb{N} \), we introduce the following conditions:

\[
I^T_r(m) : \exists c, \delta > 0 \text{ such that } T_m(f, x) \geq c(d(x, \Sigma))^{rm} \text{ for } \|x\| < \delta,
\]

\[
I^K_r(m) : \exists c, \delta > 0 \text{ such that } K_m(f, x) \geq c(d(x, \Sigma))^{rm} \text{ for } \|x\| < \delta.
\]

**Theorem 4.3.** Let \( r \in \mathbb{N} \), and let \( f \in \mathcal{E}_E(n, p) \), \( n \geq p \). Suppose that \( j^r f(\Sigma, 0) \) has a \( C^\omega \) realisation. Then for any \( m \in \mathbb{N} \),

\[
I^T_r(m) \text{ holds if and only if } I^K_r(m) \text{ holds.}
\]

**Proof.** Let \( g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be a \( C^\omega \) realisation of \( j^r f(\Sigma, 0) \). From Theorem \ref{thm:1.3}, conditions \( I^T_r(m) \) and \( I^K_r(m) \) are equivalent for \( g \). Therefore it suffices to show that conditions \( I^T_r(m) \) and \( I^K_r(m) \) are \( r \)-compatible. Let \( f_t = f + th \) with \( h = g - f \). Then \( \|h\| = o(d(., \Sigma)^r) \), \( \|f_t\| \geq \|f\| - \|h\| \) and the expansion of the determinants give

\[
T_m(f_t, x) = T_m(f, x) + o(d(x, \Sigma))^{rm} \text{ and } K_m(f_t, x) = K_m(f, x) + o(d(x, \Sigma))^{rm}.
\]

Thus the \( r \)-compatibilities of \( I^T_r(m) \) and \( I^K_r(m) \) follow. \( \square \)

**Remark 4.4.** As pointed out in \( \cite{2} \), any \( r \)-jet, \( r \in \mathbb{N} \), has a unique polynomial realisation of degree not exceeding \( r \) in the non-relative case, but some \( r \)-jets do not have even a \( C^\omega \) realisation in the general relative case. Therefore, in the above theorem, the assumption that \( j^r f(\Sigma, 0) \) has a \( C^\omega \) realisation makes sense.

**Corollary 4.5.** Let \( \Sigma \) be a germ at \( 0 \) of a closed set. Let \( r \in \mathbb{N} \), and let \( f \in \mathcal{E}_E(n, p) \), \( n \geq p \). Suppose that \( j^r f(\Sigma, 0) \) has a \( C^\omega \) realisation. Then the following conditions are equivalent:

1. There exists \( m \in \mathbb{N} \) such that \( I^T_r(m) \) holds
2. For all \( m \in \mathbb{N} \), \( I^T_r(m) \) holds
3. There exists \( m \in \mathbb{N} \) such that \( I^K_r(m) \) holds
4. For all \( m \in \mathbb{N} \), \( I^K_r(m) \) holds

**Remark 4.6.** 1) It follows from the proof of Theorem \ref{thm:1.3} that the equivalence between conditions \( T_m \) and \( K_m \) holds for any \( C^1 \) map \( f \) in a category where the analytic curve selection lemma is valid.

2) For \( X_1, \ldots, X_l \geq 0 \) and a positive integer \( m \in \mathbb{N} \), we have

\[
(X_1 + \ldots + X_l)^m \approx X_1^m + \ldots + X_l^m.
\]

Therefore we see that \( K_1 \approx T_1 \) if and only if for any \( m \in \mathbb{N} \), \( K_m \approx T_m \).
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