BUSER-SARNAK INVARIANT AND PROJECTIVE NORMALITY OF ABELIAN VARIETIES

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Abstract. We show that a general $n$-dimensional polarized abelian variety $(A, L)$ of a given polarization type and satisfying $h^0(A, L) \geq \frac{8^n}{2} \cdot n^n$ is projectively normal. In the process, we also obtain a sharp lower bound for the volume of a purely one-dimensional complex analytic subvariety in a geodesic tubular neighborhood of a subtorus of a compact complex torus.

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1. Introduction and Statement of Results

Let $A$ be an abelian variety of dimension $n$, and let $L$ be an ample line bundle over $A$. Such a pair $(A, L)$ is called a polarized abelian variety. We are interested in studying the projective normality of $(A, L)$, which plays an important role in the theory of linear series associated to $(A, L)$. For each $r \geq 1$, we consider the multiplication map

\begin{equation}
\rho_r : \text{Sym}^r H^0(A, L) \to H^0(A, L^\otimes r)
\end{equation}

induced by $(\sigma_1, \ldots, \sigma_r) \to \sigma_1 \cdots \sigma_r$ for $\sigma_1, \ldots, \sigma_r \in H^0(A, L)$. Here $\text{Sym}^r H^0(A, L)$ denotes the r-fold symmetric tensor power of $H^0(A, L)$. Recall that $(A, L)$ (or simply $L$) is said to be projectively normal if $\rho_r$ is surjective for each $r \geq 1$. The projective normality of a polarized abelian variety $(A, L)$ is well-understood in the case when $L$ is not primitive, i.e., when there exists a line bundle $L'$ such that $L = L'^\otimes m$ for some integer $m \geq 2$ (cf. the references in \cite{Iy}). However, not much is known for the case when $L$ is primitive.

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In the primitive case, the main interest is to find conditions on the polarization type $d_1 | d_2 | \cdots | d_n$ of $(A, L)$ or on $h^0(A, L) := \dim_{\mathbb{C}} H^0(A, L)$ (note that $h^0(A, L) = d_1 \cdots d_n$) which will guarantee the projective normality of a general $(A, L)$ of a given polarization type. Along this line, J. Iyer [Iy] proved the following result:

**Theorem 1.1.** ([Iy, Theorem 1.2]) Let $(A, L)$ be a polarized simple abelian variety of dimension $n$. If $h^0(A, L) > 2^n n!$, then $L$ is projectively normal.

See also [FG] for related results in the lower dimensional cases when $n = 3, 4$. These works use the theory of theta functions and theta groups.

Our goal is to relate this problem to the Buser-Sarnak invariant $m(A, L)$ of the polarized abelian variety (cf. [L2, p. 291]). Since $A$ is a compact complex torus, one may write $A = \mathbb{C}^n / \Lambda$, where $\Lambda$ is a lattice in $\mathbb{C}^n$. It is well-known that there exists a unique translation-invariant flat Kähler form $\omega$ on $A$ such that $c_1(L) = [\omega] \in H^2(A, \mathbb{Z})$. The real part of $\omega$ gives rise to an inner product $\langle \ , \ \rangle$ on $\mathbb{C}^n$, and we denote by $\| \|$ the associated norm on $\mathbb{C}^n$. The Buser-Sarnak invariant is given by

$$m(A, L) := \min_{\lambda \in \Lambda \setminus \{0\}} \| \lambda \|^2.$$  

In other words, $m(A, L)$ is the square of the minimal length of a non-zero lattice vector in $\Lambda$ with respect to $\langle \ , \ \rangle$. The study of this invariant was initiated by Buser and Sarnak in [BS], where they studied it for principally polarized abelian varieties and Jacobians. In particular, they showed the existence of a principally polarized abelian variety $(A, L)$ with

$$m(A, L) \geq \frac{1}{\pi} \sqrt{2L^n}.$$  

In [Ba], Bauer generalized this to abelian varieties of arbitrary polarization type (cf. [L2, p. 292-293]).

The relevance of the invariant $m(A, L)$ in the study of algebro-geometric questions was first observed by Lazarsfeld [L1], where he obtained a lower bound for the Seshadri number of $(A, L)$ in terms of $m(A, L)$ (cf. [L2, p. 293]). In particular, $m(A, L)$ gives information on generation of jets by $H^0(A, L)$. Furthermore, Bauer used the existence of $(A, L)$ satisfying (1.3) together with Lazarsfeld’s above result to obtain the following result:
Theorem 1.2. ([Ba, Corollary 2]) Let \((A, L)\) be a general \(n\)-dimensional polarized abelian variety of a given polarization type. If \(h^0(A, L) \geq \frac{8^n}{2} \cdot \frac{n^n}{n!}\), then \(L\) is very ample.

Now we state our main result in this paper as follows:

Theorem 1.3. A general \(n\)-dimensional polarized abelian variety \((A, L)\) of a given polarization type and satisfying \(h^0(A, L) \geq \frac{8^n}{2} \cdot \frac{n^n}{n!}\) is projectively normal.

Using Stirling’s formula \((n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n})\), one easily sees that our bound in Theorem 1.3 improves Iyer’s bound in Theorem 1.1 substantially for large \(n\). Note that our bound in Theorem 1.3 for projective normality is the same as Bauer’s bound in Theorem 1.2 for very ampleness. To our knowledge, this is just a coincidence. Although the proofs of both theorems use Bauer’s generalization of (1.3), Theorem 1.2 itself is not used in the proof of Theorem 1.3. Finally it is worth comparing Theorem 1.3 with the result in [FG] and [Ru] that there is a polarization type \(d_1|d_2|\cdots|d_n\) with \(d_1 \cdots d_n = h^0(A, L) = \frac{4^n}{2}\) such that no abelian varieties of this polarization type is projectively normal.

We describe briefly our approach as follows. First we obtain an auxiliary result, which is a sharp lower bound for the volume of a purely one-dimensional complex analytic subvariety in a geodesic tubular neighborhood of a subtorus of a compact complex torus (see Proposition 2.3 for the precise statement). As a consequence, we obtain a lower bound of the Seshadri number of the line bundle \(p_1^*L \otimes p_2^*L\) along the diagonal of \(A \times A\) in terms of \(m(A, L)\) (see Proposition 3.2). Here \(p_i : A \times A \to A\) denotes the projection onto the \(i\)-th factor, \(i = 1, 2\). We believe that these two auxiliary results are of independent interest beside their application to the projective normality problem. Finally the proof of Theorem 1.3 involves the use of the second auxiliary result and applying Bauer’s result mentioned above in (1.3).

2. Volume of subvarieties near a complex subtorus

In this section, we are going to obtain a sharp lower bound for the volume of a purely 1-dimensional complex analytic subvariety in a tubular open neighborhood of a subtorus of a compact complex torus (see Proposition 2.3). This inequality is inspired by an analogous inequality in the hyperbolic setting proved in [HT]. The proof of the current
case is much simpler than the one in [HT], using a simple projection argument and Federer’s volume inequality for analytic subvarieties in a Euclidean ball in $\mathbb{C}^n$ (cf. e.g. [St] or [L2, p. 300]).

Let $T = \mathbb{C}^n/\Lambda$ be an $n$-dimensional compact complex torus associated to a lattice $\Lambda \subset \mathbb{C}^n$ and endowed with a flat translation-invariant Kähler form $\omega$. For simplicity, we call $(T, \omega)$ a polarized compact complex torus. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm on $\mathbb{C}^n$ associated to $\omega$ as in Section 1. Next we let $S$ be a $k$-dimensional compact complex subtorus of $T$, where $0 \leq k < n$. It is well-known that $S$ is the quotient of a $k$-dimensional linear subspace $F \sim \mathbb{C}^k$ of $\mathbb{C}^n$ by a sublattice $\Lambda_S \subset \Lambda$ of rank $2k$ and such that $\Lambda_S = \Lambda \cap F$. Let $F^\perp$ be the orthogonal complement of $F$ in $\mathbb{C}^n$ with respect to $\langle \cdot, \cdot \rangle$, and let $q_F: \mathbb{C}^n \to F$ and $q_{F^\perp}: \mathbb{C}^n \to F^\perp$ denote the associated unitary projection maps. Similar to (1.2), we define the relative Buser-Sarnak invariant $m(T, S, \omega)$ given by

$$m(T, S, \omega) := \min_{\lambda \in \Lambda \setminus \Lambda_S} \| q_{F^\perp}(\lambda) \|^2.$$  

In other words, $m(T, S, \omega)$ is the square of the minimal distance of a vector in $\Lambda \setminus \Lambda_S$ from the linear subspace $F$.

**Remark 2.1.** (i) The invariant $m(A, L)$ in (1.2) corresponds to the special case when $S = \{0\}$ and $[\omega] = c_1(L)$, i.e., $m(A, L) = m(A, \{0\}, \omega)$.
(ii) From the discreteness of $\Lambda$, the equality $\Lambda_S = \Lambda \cap F$ and the compactness of $S = F/\Lambda_S$, one easy checks that $m(T, S, \omega) > 0$ and its value is attained by some $\lambda \in \Lambda \setminus \Lambda_S$.

With regard to the Riemannian geometry associated to $\omega$, one also easily sees that the geodesic distance function $d_T: T \times T \to \mathbb{R}$ of $T$ with respect to $\omega$ can be expressed in terms of $\| \cdot \|$ given by

$$d_T(x, y) = \inf \{ \| z - w \| \mid p(z) = x, p(w) = y \},$$

where $p: \mathbb{C}^n \to T$ denotes the covering projection map. For any given $r > 0$, we consider the open subset of $T$ given by

$$W_r := \{ x \in T, \ | \ d_T(x, S) < r \} \supset S,$$

where as usual,

$$d_T(x, S) := \inf_{y \in S} d_T(x, y) = \min \{ \| q_{F^\perp}(z) \| \mid p(z) = x \}$$

(note that the second equality in (2.4) follows from standard facts on inner product spaces, and as in Remark 2.1 the minimum value in the last expression in (2.4) is attained by some $z$). We simply call $W_r$ the
geodesic tubular neighborhood of $S$ in $T$ of radius $r$. Next we consider the biholomorphism $\tilde{\phi} : F \times F^\perp \to \mathbb{C}^n$ given by

$$
\tilde{\phi}(z_1, z_2) = z_1 + z_2 \quad \text{for} \quad (z_1, z_2) \in F \times F^\perp,
$$

(2.5)

It is easy to see that the covering projection map $p \circ \tilde{\phi} : F \times F^\perp \to T$ is equivariant under the action of $\Lambda_S$ on $F \times F^\perp$ given by $(z_1, z_2) \mapsto (z_1 + \lambda, z_2)$ for $(z_1, z_2) \in F \times F^\perp$ and $\lambda \in \Lambda_S$. It follows readily that $p \circ \tilde{\phi}$ descends to a well-defined covering projection map denoted by $\phi : S \times F^\perp \to T$ (in particular, $\phi$ is a local biholomorphism). Consider the flat translation-invariant Kähler form on $\mathbb{C}^n$ given by

$$
\omega_{\mathbb{C}^n} := \frac{-1}{2} \partial \bar{\partial} |z|^2, \quad z \in \mathbb{C}^n,
$$

(2.6)

which is easily seen to descend to the Kähler form $\omega$ on $T$. Consider also the flat Kähler form on $F^\perp$ given by $\omega_{F^\perp} := \omega_{\mathbb{C}^n}|_{F^\perp}$, and for any $r > 0$, let $B_{F^\perp}(r) := \{z \in F^\perp \mid \|z\| < r\}$ denote the associated open ball of radius $r$. Let $\omega_S := \omega|_S$. Note that $\phi|_{S \times \{0\}}$ is given by the identity map on $S$. It admits biholomorphic extensions as follows:

**Lemma 2.2.** For any real number $r$ satisfying $0 < r \leq \frac{\sqrt{m(T, S, \omega)}}{2}$, one has a biholomorphic isometry

$$
\phi_r : (S, \omega_S) \times (B_{F^\perp}(r), \omega_{F^\perp}|_{B_{F^\perp}(r)}) \to (W_r, \omega|_{W_r})
$$

(2.7)

given by $\phi_r := \phi|_{S \times B_{F^\perp}(r)}$.

**Proof.** First we fix a real number $r$ satisfying $0 < r \leq \frac{\sqrt{m(T, S, \omega)}}{2}$. From (2.3), (2.4) and the obvious identity $q_{F^\perp}(\tilde{\phi}(z_1, z_2)) = z_2$ for $(z_1, z_2) \in F \times F^\perp$, one easily sees that $\phi(S \times B_{F^\perp}(r)) \subset W_r$, and thus the map $\phi_r$ in (2.7) is well-defined. For each $x \in W_r$, it follows from the second equality in (2.4) that there exists $z \in \mathbb{C}^n$ such that $p(z) = x$ and $\|q_{F^\perp}(z)\| = d_T(x, S) < r$. Now, $q_{F^\perp}(z)$ descends to a point $x_S$ in $S$, and one easily sees that $\phi_r(x_S, q_{F^\perp}(z)) = x$ with $(x_S, q_{F^\perp}(z)) \in S \times B_{F^\perp}(r)$. Thus $\phi_r$ is surjective. Next we are going to prove by contradiction that $\phi_r$ is injective. Suppose $\phi_r$ is not injective. Then it implies readily that there exist two points $(z_1, z_2), (z_1', z_2') \in F \times B_{F^\perp}(r)$ such that

(i) either $z_1 - z_1' \notin \Lambda_S$ or $z_2 \neq z_2'$; and

(ii) $z_1 + z_2 - (z_1' + z_2') = \lambda$ for some $\lambda \in \Lambda$

(here (i) means that $(z_1, z_2), (z_1', z_2')$ descend to different points in $S \times B_{F^\perp}(r)$). In both cases in (i), one easily checks that $\lambda \in \Lambda \setminus \Lambda_S$. 
On the other hand, one also sees from (ii) that \( q_{F^\perp}(\lambda) = z_2 - z_2' \) and thus

\[
\|q_{F^\perp}(\lambda)\| \leq \|z_2\| + \|z_2'\| < r + r = 2r \leq \sqrt{m(T, S, \omega)},
\]

which contradicts the definition of \( m(T, S, \omega) \) in (2.1). Thus, \( \phi_r \) is injective, and we have proved that \( \phi_r \) is a bihomorphism. Finally from the obvious identity \( \|z_1\|^2 + \|z_2\|^2 = \|z_1 + z_2\|^2 \) for \((z_1, z_2) \in F \times F^\perp\), and upon taking \( \frac{\sqrt{2}}{2} \partial \overline{\partial} \), one easily sees that \( \phi : (F, \omega_F) \times (F^\perp, \omega_{F^\perp}) \to (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \) is a biholomorphic isometry (cf. (2.6)). It follows readily that the induced covering projection map \( \phi : (S, \omega_S) \times (F^\perp, \omega_{F^\perp}) \to (T, \omega) \) is a local isometry. Upon restricting \( \phi \) to \( S \times B_{F^\perp}(r) \), one sees that the biholomorphism \( \phi_r \) is an isometry.

For each \( x \in S \) and each non-zero holomorphic tangent vector \( v \in T_{x,T} \) orthogonal to \( T_{x,S} \), it is easy to see that there exists a unique 1-dimensional totally geodesic (flat) complex submanifold \( \ell \) of \( W_{\sqrt{m(T, S, \omega)}\perp} \) passing through \( x \) and such that \( T_{x,T} \subseteq \mathbb{C}v \). We simply call such \( \ell \) an \( S \)-orthogonal line of \( W_{\sqrt{m(T, S, \omega)}\perp} \). For a complex analytic subvariety \( V \) in an open subset of \( T \), we simply denote by \( \text{Vol}(V) \) its volume with respect to the Kähler form \( \omega \), unless otherwise stated. It is easy to see that for each \( 0 < r \leq \sqrt{\frac{m(T, S, \omega)}{2}} \), the values of \( \text{Vol}(\ell \cap W_r) \) are the same for all the \( S \)-orthogonal lines \( \ell \) in \( W_{\sqrt{m(T, S, \omega)}\perp} \). As such, \( \text{Vol}(\ell \cap W_r) \) is an unambiguously defined number depending only on \( r \) only (cf. (2.9) below). Next we consider the blow-up \( \pi : \tilde{T} \to T \) of \( T \) along \( S \), and denote the associated exceptional divisor by \( E := \pi^{-1}(S) \). For a complex analytic subvariety \( V \) in an open subset of \( T \) such that \( V \) has no component lying in \( S \), we denote its strict transform with respect to \( \pi \) by \( \tilde{V} := \pi^{-1}(V \setminus S) \). As usual, for an \( \mathbb{R} \)-divisor \( \Gamma \) and a complex curve \( C \) in a complex manifold, we denote by \( \Gamma \cdot C \) the intersection number of \( \Gamma \) with \( C \). Our main result in this section is the following

**Proposition 2.3.** Let \((T, \omega)\) a polarized compact complex torus of dimension \( n \), and let \( S \) be a \( k \)-dimensional compact complex subtorus of \( T \), where \( 0 \leq k < n \). Let \( \pi : \tilde{T} \to T \) be the blow-up of \( T \) along \( S \) with the exceptional divisor \( E = \pi^{-1}(S) \) as above. Then for any real number \( r \) satisfying \( 0 < r \leq \frac{\sqrt{m(T, S, \omega)}}{2} \) and any purely 1-dimensional complex analytic subvariety \( V \) of the geodesic tubular neighborhood \( W_r \) of \( S \) such that \( V \) has no component lying in \( S \), one has

\[
\text{Vol}(V) \geq \pi r^2 \cdot (\tilde{V} \cdot E) = \text{Vol}(\ell \cap W_r) \cdot (\tilde{V} \cdot E).
\]
In particular, for each \( 0 < r \leq \sqrt{\frac{m(A, S, \omega)}{2}} \) and each non-negative value \( s \) of \( \widetilde{V} \cdot E \), the lower bound in (2.9) is attained by the volume of some (and hence any) \( V \) consisting of the intersection of \( W_r \) with the union of \( s \) copies of \( S \)-orthogonal lines counting multiplicity.

**Proof.** Let \( V \subset \widetilde{W}_r \) be as above. It is clear that Proposition 2.3 for the general case when \( V \) is reducible follows from the special case when \( V \) is irreducible, and that (2.9) holds trivially for the case when \( V \cap S = \emptyset \). As such, we will assume without loss of generality that

\[
(2.10) \quad V \text{ is irreducible, } V \cap S \neq \emptyset \quad \text{and} \quad V \not\subset S.
\]

Then \( \widetilde{V} \cap E \) consists of a finite number of distinct points \( y_1, \ldots, y_\kappa \) with intersection multiplicities \( m_1, \ldots, m_\kappa \) respectively, so that

\[
(2.11) \quad \widetilde{V} \cdot E = m_1 + \cdots + m_\kappa.
\]

By Lemma 2.2, we have a biholomorphic isometry

\[
(2.12) \quad (W_r, \omega)|_{W_r} \cong (S \times F^\perp(r), \eta_1^* \omega_S + \eta_2^* \omega_{F^\perp}).
\]

Here \( \eta_1 : S \times F^\perp(r) \to S \) and \( \eta_2 : S \times F^\perp(r) \to F^\perp(r) \) denote the projections onto the first and second factor respectively. Next we make an identification \( F^\perp \cong \mathbb{C}^{n-k} \) with Euclidean coordinates \( z_1, z_2, \ldots, z_{n-k} \) associated to an orthonormal basis of \( (F^\perp, \langle , \rangle)|_{F^\perp} \). Under this identification, we have

\[
(2.13) \quad F^\perp(r) = \{ z = (z_1, z_2, \ldots, z_{n-k}) \in \mathbb{C}^{n-k} \mid |z| < r \}, \quad \text{and} \quad \omega_{F^\perp} = \frac{-1}{2} \sum_{i=1}^{n-k} dz_i \wedge d \overline{z}_i.
\]

Here \( |z| = \sqrt{\sum_{i=1}^{n-k} |z_i|^2} \). Note that \( \eta_2 \) (and thus also \( \eta_2|_V \)) is a proper holomorphic mapping, and thus by the proper mapping theorem, \( V' := \eta_2(V) \) is a complex analytic subvariety of \( F^\perp(r) \). From (2.10), one easily sees that \( V' \) is irreducible and of pure dimension one, and \( \eta_2|_V : V \to V' \) is a \( \delta \)-sheeted branched covering for some \( \delta \in \mathbb{N} \). Note that \( 0 \in V' \) since \( V \cap S \neq \emptyset \), and we denote by \( \mu \) the multiplicity of \( V' \) at the origin \( 0 \in F^\perp(r) \). Let \([V]\) (resp. \([V']\)) denote the closed positive current defined by integration over \( V \) (resp. \( V' \)) in \( W_r \) (resp. \( F^\perp(r) \)). Then via the identifications in (2.13), it follows from Federer’s volume inequality for complex analytic subvarieties in a complex Euclidean ball (see e.g. [St] or [L2, p. 300]) that one has

\[
(2.14) \quad \int_{F^\perp(r)} [V'] \wedge \omega_{F^\perp} \geq \mu \cdot \pi r^2.
\]
Next we consider a linear projection map \( \psi : F^\perp \to \mathbb{C} \) from \( F^\perp \) onto some one-dimensional linear subspace (which we identify with \( \mathbb{C} \)). It follows readily from the definition of \( \mu \) that for a generic \( \psi \), \( \psi|_{V'} : V' \to \psi(V') \) is an \( \mu \)-sheeted branched covering. Furthermore, by considering the local description of the blow-up map \( \pi \) (cf. e.g. [GH, p. 603]), one easily sees that for each \( y_j \in \tilde{V} \cap E, 1 \leq j \leq \kappa \), there exists an open neighborhood \( U_j \) of \( y_j \) in \( \tilde{V} \) such that for a generic \( \psi \), the function \( \psi \circ \eta_2 \circ \pi|_{U_j} : U_j \to \mathbb{C} \) is an \( m_j \)-sheeted branched covering, shrinking \( U_j \) if necessary. Thus by considering the degree of the map \( \psi \circ \eta_2 \circ \pi|_{\tilde{V}} \) for a generic \( \psi \), one gets

\[
(2.15) \quad \delta \cdot \mu = m_1 + \cdots + m_\kappa.
\]

Under the identification in (2.12), we have

\[
(2.16) \quad \text{Vol}(V) = \int_{W_r} [V] \wedge \omega
= \int_{S \times F^\perp(r)} [V] \wedge (\eta_1^* \omega_S + \eta_2^* \omega_{F^\perp})
\geq \int_{S \times F^\perp(r)} [V] \wedge \eta_2^* \omega_{F^\perp} \quad (\text{since } \eta_1^* \omega_S \geq 0)
= \delta \int_{F^\perp(r)} [V'] \wedge \omega_{F^\perp}
\quad \text{(upon taking the direct image } \eta_2^*)
\geq \delta \cdot \mu \cdot \pi r^2 \quad \text{(by (2.14))}
= \pi r^2 \cdot (\tilde{V} \cdot E) \quad \text{(by (2.11) and (2.15))},
\]
which gives the first line of (2.9). Next we take an \( S \)-orthogonal line \( \ell \) of \( W_{\sqrt{\mu \pi r^2}} \). Then under the identifications in (2.12), (2.13) and upon making a unitary change of \( F^\perp \) if necessary, one easily sees that \( \ell \cap W_r \) can be given by \( \{x\} \times \{(z_1, 0, \cdots, 0) \in \mathbb{C}^{n-k} | |z_1| < r\} \) for some fixed point \( x \in S \), and it follows readily that

\[
(2.17) \quad \text{Vol}(\ell \cap W_r) = \int_{|z_1| < r} \frac{\sqrt{-1}}{2} dz_1 \wedge d\overline{z_1} = \pi r^2,
\]
which gives the second line of (2.9). Finally we remark that the last statement of Proposition 2.3 is a direct consequence of (2.9), and thus we have finished the proof of Proposition 2.3. \( \square \)
3. Seshadri number along the diagonal of $A \times A$

In this section, we let $(A = \mathbb{C}^n/\Lambda, L)$ be a polarized abelian variety of dimension $n$, and let the associated objects $\omega, \langle \cdot, \cdot \rangle, \| \cdot \|$ and $m(A, L)$ be as defined in Section 1. Next we consider the Cartesian product $A \times A$, and we denote by $p_i : A \times A \to A$ the projection map onto the $i$-th factor. It is easy to see that $p_1^* L \otimes p_2^* L$ is an ample line bundle over the $2n$-dimensional (product) abelian variety $A \times A$, and the associated translation-invariant flat Kähler form on $A \times A$ is given by $\omega_{A \times A} := p_1^* \omega + p_2^* \omega$. In particular, one has

$$[\omega_{A \times A}] = c_1(p_1^* L \otimes p_2^* L) \in H^2(A \times A, \mathbb{Z}).$$

Furthermore, it is easy to see that the diagonal of $A \times A$ given by

$$D := \{(x, y) \in A \times A \mid x = y\}$$

is an $n$-dimensional abelian subvariety of $A \times A$. Let $m(A \times A, D, \omega_{A \times A})$ be the relative Buser-Sarnak invariant as given in (2.1).

**Lemma 3.1.** We have

$$m(A \times A, D, \omega_{A \times A}) = \frac{m(A, L)}{2}. \tag{3.3}$$

**Proof.** First we write $A \times A = (\mathbb{C}^n \times \mathbb{C}^n)/(\Lambda \times \Lambda)$, and we denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^n \times \mathbb{C}^n}$ and $\| \cdot \|_{\mathbb{C}^n \times \mathbb{C}^n}$ the inner product and norm on $\mathbb{C}^n \times \mathbb{C}^n$ associated to $\omega_{A \times A}$. It is easy to see that as a compact complex subtorus of $A \times A$, $D$ is isomorphic to the quotient $F/\Lambda_D$, where $F := \{(z, z) \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^n \times \mathbb{C}^n$ and $\Lambda_D := \{(\lambda, \lambda) \mid \lambda \in \Lambda\} \subset \Lambda \times \Lambda$. Denote by $F^\perp$ the orthogonal complement of $F$ in $\mathbb{C}^n \times \mathbb{C}^n$ with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}^n \times \mathbb{C}^n}$, and let $q_{F^\perp} : \mathbb{C}^n \times \mathbb{C}^n \to F^\perp$ be the corresponding unitary projection map. Then for any $(\lambda_1, \lambda_2) \in \Lambda \times \Lambda$, one easily checks that $q_{F^\perp}(\lambda_1, \lambda_2) = (\frac{\lambda_1 - \lambda_2}{2}, \frac{\lambda_2 - \lambda_1}{2})$, and thus

$$\|q_{F^\perp}(\lambda_1, \lambda_2)\|_{\mathbb{C}^n \times \mathbb{C}^n}^2 = \frac{1}{2} \left(\|\lambda_1 - \lambda_2\|^2 + \|\lambda_2 - \lambda_1\|^2\right) = \|\lambda_1 - \lambda_2\|^2. \tag{3.4}$$

Together with the obvious equality $\{(\lambda_1 - \lambda_2)(\lambda_1, \lambda_2) \in (\Lambda \times \Lambda) \setminus \Lambda_D\} = \Lambda \setminus \{0\}$ (and upon writing $\lambda = \lambda_1 - \lambda_2$, one gets

$$\inf_{(\lambda_1, \lambda_2) \in (\Lambda \times \Lambda) \setminus \Lambda_D} \|q_{F^\perp}(\lambda_1, \lambda_2)\|_{\mathbb{C}^n \times \mathbb{C}^n}^2 = \frac{1}{2} \inf_{\lambda \in \Lambda \setminus \{0\}} \|\lambda\|^2, \tag{3.5}$$

which, upon recalling (1.2) and (2.1), gives (3.3) immediately. \qed

Next we let $\pi : \widehat{A \times A} \to A \times A$ be the blow-up of $A \times A$ along $D$ with the associated exceptional divisor given by $E := \pi^{-1}(D)$. We
consider the line bundle $p_1^*L \otimes p_2^*L$ over $A \times A$, and denote its pull-back to $\widetilde{A \times A}$ by

\begin{equation}
\mathcal{L} := \pi^*(p_1^*L \otimes p_2^*L).
\end{equation}

Then the Seshadri number $\epsilon(p_1^*L \otimes p_2^*L, D)$ of $p_1^*L \otimes p_2^*L$ along $D$ is defined by

\begin{equation}
\epsilon(p_1^*L \otimes p_2^*L, D) := \sup\{ \epsilon \in \mathbb{R} \mid \mathcal{L} - \epsilon E \text{ is nef on } \widetilde{A \times A} \}
\end{equation}

(see e.g. [L2, Remark 5.4.3] for the general definition and [D] for its origin). Here as usual, an $\mathbb{R}$-divisor $\Gamma$ on an algebraic manifold $M$ is said to be nef if $\Gamma \cdot C \geq 0$ for any algebraic curve $C \subset M$. Our main result in this section is the following

**Proposition 3.2.** Let $(A, L)$ be a polarized abelian variety of dimension $n$, and let $\mathcal{L}$ be as in (3.6). Then $\mathcal{L} - \alpha E$ is nef on $\widetilde{A \times A}$ for all $0 \leq \alpha \leq \frac{\pi}{8} \cdot m(A, L)$. In particular, we have

\begin{equation}
\epsilon(p_1^*L \otimes p_2^*L, D) \geq \frac{\pi}{8} \cdot m(A, L).
\end{equation}

**Proof.** First it is easy to see from (3.6) that $\mathcal{L}$ is nef, and thus the proposition holds for the case when $\alpha = 0$. Now we fix a number $\alpha$ satisfying $0 < \alpha \leq \frac{\pi}{8} \cdot m(A, L)$. Then it is easy to see from Lemma 3.1 that $\alpha = \pi r^2$ for some $r$ satisfying $0 < r \leq \frac{\sqrt{m(A \times A, \omega_{A \times A})}}{2}$. For each such $r$, we let $W_r$ be the geodesic tubular neighborhood of $D$ in $A \times A$ of radius $r$ as defined in (2.3) (with $T$ and $S$ there given by $A \times A$ and $D$ respectively). Let $C$ be an algebraic curve in $\widetilde{A \times A}$. First we consider the case when $C$ is irreducible and $C \not\subset E$, so that $\pi(C) \not\subset D$ and $C$ coincides with the strict transform of $\pi(C)$ with respect to the blow-up map $\pi$ (i.e., $C = \pi(C)$ in terms of the notations in Section 2). Then by (3.1), (3.6) and upon taking the direct image $\pi_*$, we get

\begin{equation}
\mathcal{L} \cdot C = \int_{\widetilde{A \times A}} [C] \wedge \pi^*\omega_{A \times A}
= \int_{A \times A} [\pi(C)] \wedge \omega_{A \times A}
\geq \int_{W_r} [\pi(C)] \wedge \omega_{A \times A}
\geq \pi r^2 \cdot (E \cdot C) \quad \text{(by Proposition 2.3)},
= \alpha \cdot (E \cdot C).
\end{equation}
In other words, we have
\[(3.10) \quad (L - \alpha E) \cdot C \geq 0.\]

Next we consider the case when \(C\) is irreducible and \(C \subset E\). By considering translation-invariant vector fields on \(D\) and \(A \times A\), one easily sees that the normal bundle \(N_{D|(A \times A)}\) is holomorphically trivial over \(D\). It follows readily that the line bundle \([E]|_E\) is isomorphic to \(\sigma^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1)\), where \(\sigma : D \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}\) denotes the projection onto the second factor. Hence \(E \cdot C \leq 0\) for any irreducible curve \(C \subset E\). Together with the nefness of \(L\), it follows readily that (3.10) also holds for the irreducible case when \(C \subset E\). Finally one easily sees that (3.10) for the case when \(C\) is reducible follows readily from the case when \(C\) is irreducible. Thus we have finished the proof of the nefness of \(L - \alpha D\) for all \(0 \leq \alpha \leq \frac{\pi}{8} \cdot m(A, L)\), which also leads to (3.8) readily.

\[\Box\]

4. Projective normality

In this section, we are going to give the proof of Theorem 1.3 and we follow the notation in Section 3. First we have

**Proposition 4.1.** Let \((A, L), n, E\) and \(L\) be as in Proposition 3.2. If \(L \otimes \mathcal{O}(-nE)\) is nef and big, then \(L\) is projectively normal.

**Proof.** By [Iy, Proposition 2.1], one knows that the surjectivity of the multiplication maps \(\rho_r\) in (1.1) for all \(r \geq 1\) will follow from the surjectivity of \(\rho_2\) (i.e., the case when \(r = 2\)). Thus to prove that \(L\) is projectively normal, it suffices to show that the multiplication map
\[(4.1) \quad \rho : H^0(A, L) \otimes H^0(A, L) \rightarrow H^0(A, L^{\otimes 2})\]
(as given in (1.1)) is surjective. We are going to reduce this to the question of vanishing of a certain cohomology group on \(\widetilde{A \times A}\) following the standard approach in [BEL, Section 3]. Here \(\pi : \widetilde{A \times A} \rightarrow A \times A\) is the blow-up of \(A \times A\) along the diagonal \(D\) as in Section 3. Consider the short exact sequence on \(A \times A\) given by
\[(4.2) \quad 0 \rightarrow p_1^*L \otimes p_2^*L \otimes \mathcal{I} \rightarrow p_1^*L \otimes p_2^*L \rightarrow p_1^*L \otimes p_2^*L|_D \rightarrow 0,\]
where \(\mathcal{I}\) denotes the ideal sheaf of \(D\). Note that \(p_1^*L \otimes p_2^*L|_D \cong L^{\otimes 2}\) under the natural isomorphism \(D \cong A\), and one has \(H^0(A \times A, p_1^*L \otimes p_2^*L) \cong H^0(A, L) \otimes H^0(A, L)\) by the K"unneth formula. Together with the long exact sequence associated to (4.2), one easily sees that \(\rho\) is...
surjective if $H^1(A \times A, p_1^* L \otimes p_2^* L \otimes I) = 0$. But one also easily checks that
\begin{equation}
H^1(A \times A, p_1^* L \otimes p_2^* L \otimes I) = H^1(\tilde{A} \times A, L \otimes O(-nE)),
\end{equation}
where the last line follows from the isomorphism $K_{\tilde{A} \times A} = \pi^* K_{A \times A} + O((n-1)E) = O((n-1)E)$. Finally if $L \otimes O(-nE)$ is nef and big, then it follows from Kawamata-Viehweg vanishing theorem that $H^1(\tilde{A} \times A, K_{\tilde{A} \times A} \otimes L \otimes O(-nE)) = 0$, which together with (4.3), imply that $\rho$ is surjective.

\begin{lemma}
Let $(A, L)$, $n$, $E$ and $L$ be as in Proposition 3.2. If $L \otimes O(-nE)$ is nef and $L^n > (2n)^n$, then $L \otimes O(-nE)$ is big.
\end{lemma}

\begin{proof}
Note that
\begin{equation}
L^{2n} = (p_1^* L \otimes p_2^* L)^{2n} = \frac{(2n)!}{n! \cdot n!} L^n \cdot L^n.
\end{equation}
Recall that we have the identification $E = D \times \mathbb{P}^{n-1}$ from the proof of Proposition 3.2. Denoting by $\sigma : E \to \mathbb{P}^{n-1}$ and $\eta : E \to D = A$ the projections, we have $O(E)|_E = \sigma^* O_{\mathbb{P}^{n-1}}(-1)$ and $L|_E = \eta^*(L \otimes L)$. From these, a straight-forward calculation gives
\begin{equation}
(L \otimes O(-nE))^{2n} = \frac{(2n)!}{n! \cdot n!} \cdot L^n \cdot (L^n - (2n)^n).
\end{equation}
Together with the well-known fact that a nef line bundle is big if and only if its top self-intersection number is positive, one obtains the lemma readily.
\end{proof}

Finally we complete the proof of our main result as follows:

\begin{proof}[Proof of Theorem 1.3]
Let $A_{(d_1, \ldots, d_n)}$ denote the moduli space of $n$-dimensional polarized abelian varieties $(A, L)$ of a given polarization type $d_1|d_2| \cdots |d_n$ and satisfying
\begin{equation}
d_1 \cdots d_n \geq \frac{8^n}{2} \cdot \frac{n^n}{n!},
\end{equation}
and recall that
\begin{equation}
h^0(A, L) = d_1 \cdots d_n = \frac{L^n}{n!} \quad \text{for all } (A, L) \in A_{(d_1, \ldots, d_n)}.
\end{equation}
By [Ba, Theorem 1], there exists some $(A_o, L_o) \in A_{(d_1, \ldots, d_n)}$ such that
\begin{equation}
m(A_o, L_o) = \frac{1}{\pi} \sqrt{2L_o^n}.
\end{equation}

Let $\mathcal{L}_o$ be the line bundle over the blow-up $\widetilde{A_o \times A_o}$ of $A_o \times A_o$ along the diagonal (with exceptional divisor $E_o$) as in Proposition 3.2. From (4.5), (4.6) and (4.7), one easily checks that $n \leq \frac{7}{8} \cdot m(A_o, L_o)$. Thus it follows from Proposition 3.2 that $\mathcal{L}_o \otimes \mathcal{O}(-nE_o)$ is nef. One also easily checks from (4.5) and (4.6) that $L_o^n_{\mathbb{P}} > (2n)^n$, and thus by Lemma 4.2, the nef line bundle $\mathcal{L}_o \otimes \mathcal{O}(-nE_o)$ is also big. Then it follows from Proposition 4.1 that $(A_o, L_o)$ is projectively normal. Finally it is easy to see that the existence of a projective normal $(A_o, L_o)$ implies readily that a general $(A, L)$ in $\mathcal{A}_{(d_1, \ldots, d_n)}$ is projectively normal. \hfill \Box

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