Operational verification of the existence of a spacetime manifold

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We argue that there exists an operational way to establish the objective reality of the notions of space and time. Specifically, we propose a theory-independent protocol for a gedanken-experiment, whose outcome is a signal establishing the observability of the spacetime manifold, without a priori assuming its existence. The experimental signal contains the information about the dimension and the topology of spacetime (with the currently achievable precision), and establishes its manifold structure, while respecting its underlying diffeomorphism symmetry. We also introduce and discuss appropriate criteria for the concept of emergence of spacetime, which any tentative theoretical model of physics must satisfy in order to claim that spacetime does emerge from some more fundamental concepts.

Keywords: space, time, spacetime manifold, spacetime emergence, dimension, topology, diffeomorphism symmetry

I. INTRODUCTION

Both in physics and philosophy, one often asks deep fundamental questions such as “What is time?”, “What is space?”, “Do they objectively exist or not?”, and similar. While a lot has been said and written about these questions throughout history of science and philosophy [1–8], there appear to be very few attempts to address these questions in terms of experimental evidence for the notions of time and space. On one hand, we all have intuitive feeling for both, since we heavily rely on them in daily life, and this intuition is partially based on some experimental evidence. On the other hand, conceptual analysis of the objective existence of space and time turned out to be quite a hard problem, due to the high level of symmetry properties of space and time, encoded in the principle of general relativity.

Namely, there is a long-standing common claim that the points on a spacetime manifold are physically unobservable, given the principle of general relativity, i.e., because we expect all physics to be diffeomorphism-invariant and background-independent. As far as the statement goes, one cannot distinguish between “this point” versus “that point” of spacetime itself, but only “the point where fields have this value” versus “the point where fields have that value”, along the lines of the paradigm of relational approach to physics (see for example [9]).

Loosely speaking, the basic argument for the unobservability of individual spacetime points goes as follows. If we choose one point of spacetime (by specifying its coordinates in some coordinate system), determine the values of all fields at that point, and then perform an “active diffeomorphism” (permutation of manifold points), we “move” all physics from that point to another point. After that, we can perform a “passive diffeomorphism” (choice of a different manifold chart), to undo the active one, i.e., we use the same set of numbers as coordinates for the new point in the new coordinate chart as we have used for the old point in old coordinates. Given that physics does not change throughout the process, we conclude that one cannot distinguish between the “old spacetime point” and the “new spacetime point”. Thus, individual spacetime points are unobservable.

While this is correct in itself, apparently there are proposals that go even further, and generalize this argument to deny the existence of the spacetime manifold altogether. The argument could possibly be paraphrased in the following form — if spacetime points are not observable, they do not objectively exist, so therefore the manifold itself does not objectively exist.

The purpose of this work is to challenge this generalization. Namely, our statement is the following: the fact that we cannot observe individual points does not imply that the spacetime manifold as a whole cannot be observed, or that it does not objectively exist in nature. In particular, a manifold has properties which are invariant with respect to diffeomorphisms — specifically, its dimension and its topology, and a priori both of these might in principle be observable. The main point of this work is to demonstrate that these properties of a manifold indeed are observable. As a consequence of this, we argue that the spacetime manifold has an objective physical reality, without violating

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either the diffeomorphism symmetry or background independence. Additionally, our argument for the existence of spacetime, presented below, relies on an operational, theory-independent experimental protocol, incorporated in a specific proposed gedanken-experiment. It is thus mainly based on experimental evidence, rather than on some kind of theoretical or metaphysical assumptions.

In order to further emphasise this last point, one can imagine a hypothetical scenario involving an artificial intelligence (AI) implemented within a memory of a computer, without having any a priori notion about time and space. One can further imagine that this AI is perfectly capable of performing complex mathematical analyses. In such a setup, the experimenter could perform our proposed protocol, and feed AI the resulting experimental data to analyse it. The outcome of the analysis performed by the AI would then be a conclusion that the experimenter “lives in a space and time” of dimension 4 and simply connected topology, despite the fact that AI itself does not have any intuitive or a priori notions of either space or time. This hypothetical scenario emphasises the independence of our proposed experimental protocol on any a priori notions of time and space that an experimenter may be subject to, because the AI would reach the same conclusion without such a priori notions.

The layout of the paper is as follows. Sections II and III discuss the details of the thought experiment in mechanics and field theory, respectively. The two cases have been separated into two sections merely for the pedagogical purpose of gradually introducing the relevant analysis and techniques, while in principle the analysis in the context of mechanics is merely a special case of the analysis in the context of field theory. Section IV contains a discussion of various related aspects of the analysis, as well as a discussion of several topics which put our results in a wider context. The Appendix A contains some important technical results needed for the analysis.

II. MECHANICS AND TIME

In order to have a clear understanding of the ideas proposed in this work, it is prudent to first discuss the toy-example of a (0 + 1)-dimensional spacetime manifold, i.e. the time manifold. Various key properties of the analysis can be discussed using simple mechanical systems, such as pendulums, while the proper (3+1)-dimensional spacetime manifold is postponed for Section III. Of course, throughout the text we uphold an initial assumption that the concepts of time and spacetime are not observable a priori.

A. System with one observable

Let us introduce the following gedanken-experiment. We are given a swinging pendulum, denoted $A$. Without any assumptions about Newtonian mechanics (since it relies on the a priori notion of time), our objective is to describe the motion of the pendulum, as precisely as we can. The measurement apparatus we have available is a camera that can take still photos of the pendulum, along with a ruler that can measure the distances on the photos.

Start by taking $N$ photos of the pendulum, completely randomly, and use a ruler to measure the signed distance $a_k$ of the pendulum from its vertical axis, for every photograph $k \in \{1, \ldots , N\}$. The distance is signed in the sense that if the pendulum is left of its axis we consider the distance to be negative, while if the pendulum is right of its axis, the sign of distance is positive (the choices of left and right are merely a convention and do not influence any conclusions). Moreover, we want to eliminate any information about the “time order” in which the photos might have been taken, so we remember only the following unordered set of measurements, describing what we can tell about the motion of pendulum $A$, 

$$A = \{a_1, a_2, \ldots , a_N\},$$

where

$$a_k \in S_A \equiv [a_{\min} , a_{\max}] \subset \mathbb{R}, \quad \forall k = 1, \ldots , N.$$  

Here $a_{\min}$ and $a_{\max}$ denote the left and right amplitudes of the pendulum, respectively. In usual circumstances one should obtain $a_{\min} = -a_{\max}$, but for our purposes this equality does not really matter. Also, we assume that no two measurements produce exactly equal values of $a$, so that $a_i \neq a_j$ for all measurements $i$ and $j$. This assumption is just for convenience, and we will discuss later the conceptual case in which the measurement results are repeating themselves, and always falling into a discrete set of outcomes.

Given the measurement dataset $A$, one can draw it as a one-dimensional scatter plot on the $a$-axis, which looks similar to the plot in FIG. If we take additional $N$ measurements (i.e., a total of $2N$), the scatter plot will look similar, only with twice as many points in the segment $S_A$. In the limit $N \rightarrow \infty$, the scatter plot will become dense,
and practically fill the whole segment $S_A$, given that we can measure distances $a_k$ with an arbitrarily small but still nonzero precision. The notion of arbitrary small distance is not in conflict with quantum mechanics, since we are measuring only the position of the pendulum, and not its momentum. With sufficiently many measurements, we can describe the motion of the pendulum using the whole set $S_A$ of possible positions it might be in, and we can call this set the configuration space of the pendulum.

However, our naive intuition suggests that such a description of pendulum’s motion is not completely satisfactory, since each photo demonstrates that the pendulum is at some particular distance $a$, while the configuration space of the pendulum describes the pendulum merely “in all positions”, suggesting that we are missing some extra information. We therefore ask for a more precise description of the pendulum’s motion. The precise formalization of this “intuition” will be given below, but at this stage let us appeal to the ideas of relationalism, and try to obtain the missing information by comparing the motion of the pendulum against some other physical system.

B. System with two observables

For lack of a better idea, let us introduce another pendulum, denoted $B$, in addition to $A$. In a generic situation, the pendulum $B$ may have different length and other properties compared to $A$. We put the two pendulums side by side, and randomly take photos of the whole system. However, in order to eliminate any notion of “simultaneity” of measurements between $A$ and $B$, we cut each photo into two pieces, such that each pendulum is displayed separately on its half of the photo. As before, we now shuffle all photos to erase the order in which the photos were taken, as well as the relation which photo of $A$ is paired to which photo of $B$. We end up with the following unordered datasets for the two pendulums:

$$A = \{a_1, a_2, \ldots, a_N\}, \quad B = \{b_1, b_2, \ldots, b_N\}. \quad (3)$$

After sufficiently many measurements, we can establish the individual configuration spaces for each pendulum like before,

$$S_A \equiv [a_{\min}, a_{\max}] \subset \mathbb{R}, \quad S_B \equiv [b_{\min}, b_{\max}] \subset \mathbb{R}. \quad (4)$$

Again, we have an intuition that this is not a complete description of the motion of the two pendulums. But now we can formalize this intuition as follows. Given the individual configuration spaces $S_A$ and $S_B$, a priori one can say that the joint configuration space will be the set $S_A \times S_B$. One can visualize this experimental result within this set graphically, as follows. Pick a random permutation $\pi$ of $N$ elements, and construct the following $2 \times N$ matrix,

$$\left( \begin{array}{ccccc} a_1 & a_2 & \ldots & a_N \\ b_{\pi(1)} & b_{\pi(2)} & \ldots & b_{\pi(N)} \end{array} \right), \quad (5)$$

which describes an arbitrary “pairing” of each measurement $a_k$ to some measurement $b_{\pi(k)}$. These pairings define $N$ points on a scatter plot in the space $S_A \times S_B$, which will typically look like the one in FIG. 2. One can draw a similar scatter plot for every choice of the permutation function $\pi$. Since for $N$ measurements there are $N!$ possible permutations, there are also $N!$ scatter plots. But, as an experimental result of our thought experiment, among all those scatter plots, there will be one special plot corresponding to the permutation denoted as $\bar{\pi}$, which is visually very distinguishable from the rest, and looks like the one in FIG. 3. Looking at the plot, it is straightforward to interpret it as a set of points on a 1-dimensional curve, denoted $C$. The experimental fact that the joint configuration space $S_{A,B}$ is equal to the curve $C$ has one important consequence—it captures our intuition of a more precise description of the motion of pendulums $A$ and $B$. This is because the curve $C$ is equivalent to a functional dependence (a correlation) between observables $a$ and $b$, of the general form

$$F(a, b) = 0. \quad (6)$$

The equation (6) actually defines the curve $C$ as a set of all pairs $(a, b) \in S_A \times S_B$ which satisfy (6). At this point we can spell out the formalization of our intuition for the results of the thought experiment, in the form of the following conjecture.
Conjecture 1 (definition of special correlations). The joint configuration space $S_{A \cup B}$ is a strict subspace of $S_A \times S_B$, which is of measure zero compared to $S_A \times S_B$:

$$S_{A \cup B} \subset S_A \times S_B, \quad \frac{\text{meas}(S_{A \cup B})}{\text{meas}(S_A \times S_B)} = 0.$$  \hfill (7)

The measure over these sets is defined as an induced measure from the standard Euclidean metric over $\mathbb{R}^2$.

Note that, while visual distinguishability is appealing, it is a particular pattern-recognition trait of a human brain. In order to have a more formal definition of why this particular diagram is “special”, without appealing to our eyes and brains, we can resort to the statistical analysis described in detail in the Appendix A. This can be implemented as a computer algorithm operating on dataset [3], eliminating any reliance on visual inspection of scatter plots. Also, the formal statistical analysis has an advantage of being applicable to higher dimensions, i.e., beyond the 2- and 3-dimensional scatter plots, in contrast to visual inspection by a human. This property will become important in Section III.

The “special” permutation $\tilde{\pi}$, corresponding to the “special” plot above, has several crucial properties:

Conjecture 2 (properties of the special correlations).
• **Existence.** Given a completely random set of numbers $a_k$ and $b_k$, $k = 1, \ldots, N$, the permutation $\tilde{\pi}$ does not necessarily exist. This is actually one of the definitional properties for a sequence of numbers to be “random” — absence of any noticeable correlation. It should be stressed that the existence of $\tilde{\pi}$ is a specific property of the dataset (9), which fails to be random, as opposed to an arbitrary set of numbers not obtained by experimental measurements of the two pendulums. The existence of $\tilde{\pi}$ is therefore an experimental signal.

• **Self-reinforcement.** Suppose that, after taking $N$ measurements (3), we continue to take additional $M$ measurements,

$$A' = \{a_{N+1}, a_{N+2}, \ldots, a_{N+M}\}, \quad B' = \{b_{N+1}, b_{N+2}, \ldots, b_{N+M}\}. \quad (8)$$

We can then perform the same analysis independently over the three datasets — the original dataset $(A, B)$, the new dataset $(A', B')$, and their union dataset $(A \cup A', B \cup B')$, to arrive at the three “special” permutations $\tilde{\pi}$, $\tilde{\pi}'$ and $\tilde{\Pi}$, defined over $N$, $M$ and $N + M$ numbers respectively. Then, it is again an experimental result that the restriction of $\Pi$ to the first $N$ numbers will be equal to the original permutation $\tilde{\pi}$, while the restriction of $\Pi$ to the remaining $M$ numbers will be equal to the new permutation $\tilde{\pi}'$. Thus, we have a piecewise-defined equation

$$\tilde{\Pi}(k) = \begin{cases} \tilde{\pi}(k), & k = 1, \ldots, N, \\ \tilde{\pi}'(k), & k = N + 1, \ldots, N + M, \end{cases} \quad (9)$$

which holds for all possible choices of $N$ and $M$. Pictorially, this property means that once we draw scatter plots corresponding to $\tilde{\pi}$ and $\tilde{\pi}'$, the data points will remain nicely aligned along the same curve. In other words, the correlation which represents our experimental signal reinforces itself when one adds additional data.

• **Dimensionality.** In the limit $N \to \infty$, one can see that there exists a special permutation $\tilde{\pi}$ such that the data points are arranged in a dense way along the whole curve $C$. In other words, the actual joint configuration space $C = S_{A \cup B}$ for pendulums $A$ and $B$ is not only a subset of measure zero in the direct product $S_A \times S_B$ of two individual configuration spaces, but also a 1-dimensional subset. This is also an experimental result. Namely, it could have happened that $S_{A \cup B}$ is some patch in $S_A \times S_B$ of nonzero area (2-dimensional), or a discrete set of points (0-dimensional), or any combination thereof. But these alternative scenarios may happen only for non-experimental datasets, while for experimental datasets it always turns out that $S_{A \cup B}$ is 1-dimensional.

• **Topology.** The statistical analysis from Appendix A provides us not only with the dimension, but also with a convenient set of patches along the curve $C$, which define a basis of open sets, giving rise to an induced topology on $C$. In the example of the curve in the picture, the topology is that of a circle, but it can also be open curve in other examples, as we shall discuss in more detail below.

Once the existence and properties of the 1-dimensional curve $C$ have been established, one may be tempted to call it the “time manifold”. However, there are two problems associated with that. First, in our case of the two pendulums the curve $C$ is almost certainly self-intersecting, preventing it from being a manifold in the proper sense. Second, if we denote the length of the pendulums $A$ and $B$ as $l_A$ and $l_B$ respectively, it may happen that their square-roots are incommensurate,

$$\sqrt{\frac{l_A}{l_B}} \notin \mathbb{Q}. \quad (10)$$

Note that if we recall Newtonian mechanics, (10) in fact means that the periods of the two pendulums are incommensurate. Of course, in order to be consistent, we have to assume that we have no knowledge about either Newtonian mechanics nor about the concept of a “period”, so we refrain from this interpretation. Nevertheless, we are still allowed to measure the lengths $l_A$ and $l_B$ using a ruler, calculate their square roots, and discuss the validity of (10) in a given experiment. And if it happens that (10) holds, in the limit $N \to \infty$ the curve $C$ becomes a space-filling curve (passing through almost every point in the rectangle $S_A \times S_B$), invalidating the statement that it is 1-dimensional, in the sense of the analysis discussed in Appendix A.

In order to work around these two issues, it is useful to enlarge our physical system yet again.

C. **Systems with three and more observables**

In addition to pendulums $A$ and $B$, let us introduce yet another pendulum, denoted $C$, and repeat the whole analysis, taking photos of all three pendulums side-by-side. This time we end up with the dataset

$$A = \{a_1, a_2, \ldots, a_N\}, \quad B = \{b_1, b_2, \ldots, b_N\}, \quad C = \{c_1, c_2, \ldots, c_N\}. \quad (11)$$
After sufficiently many measurements, we can establish the individual configuration spaces for each pendulum, as before,

\[ S_A \equiv [a_{\min}, a_{\max}] \subset \mathbb{R}, \quad S_B \equiv [b_{\min}, b_{\max}] \subset \mathbb{R}, \quad S_C \equiv [c_{\min}, c_{\max}] \subset \mathbb{R}. \] (12)

We can now expect that the joint configuration space will be a measure-zero subset of the 3-dimensional box \( S_A \times S_B \times S_C \). To see this, introduce two arbitrary permutations of \( N \) elements, \( \pi \) and \( \rho \), and construct the following \( 3 \times N \) matrix,

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_N \\
b_{\pi(1)} & b_{\pi(2)} & \cdots & b_{\pi(N)} \\
c_{\rho(1)} & c_{\rho(2)} & \cdots & c_{\rho(N)}
\end{bmatrix},
\] (13)

which describes an arbitrary ordering of our dataset into triplets \((a_k, b_{\pi(k)}, c_{\rho(k)})\), \(k = 1, \ldots, N\). We use these triplets as data points in the 3-dimensional scatter plot whose domain is the box \( S_A \times S_B \times S_C \). One can construct \((N!)^2\) such scatter plots, one for each choice of the permutations \( \pi \) and \( \rho \). A typical plot will have data points randomly distributed and filling up the whole box, as in FIG. 4. However, either by using visual inspection over \((N!)^2\) such diagrams, or by using the data-analysis approach from Appendix A, one can again establish the existence of the “special” diagram, see FIG. 5, corresponding to the “special” permutations \( \tilde{\pi} \) and \( \tilde{\rho} \).

There are now several new features to be discussed. To begin with, as an experimental result of our thought-experiment, we can establish that our dataset \((A, B, C)\) together with the permutations \( \tilde{\pi} \) and \( \tilde{\rho} \), satisfies all three previously discussed properties of existence, self-reinforcement and dimensionality. Therefore, the dataset again gives rise to a 1-dimensional curve \( C \), as an experimental result. In principle, the addition of the third pendulum could have rendered the data-points distributed along some 2-dimensional surface in the box rather than a 1-dimensional
In general, if we have a very big mechanical system of \( K \) observables \( a, b, c \),

\[
F_1(a, b, c) = 0, \quad F_2(a, b, c) = 0.
\]

In general, if we have a very big mechanical system of \( K \) observables \( a^{(1)}, \ldots, a^{(K)} \), we will find in total \( K - 1 \) correlation functions

\[
F_1(a^{(1)}, \ldots, a^{(K)}) = 0, \quad \ldots, \quad F_{K-1}(a^{(1)}, \ldots, a^{(K)}) = 0,
\]

whose set of solutions describes a 1-dimensional curve \( C \) in a \( K \)-dimensional configuration space.

Second, the 1-dimensional nature of the curve \( C \) is equivalent to the set of two correlation functions between three observables \( a, b, c \),

\[
F_1(a, b, c) = 0, \quad F_2(a, b, c) = 0.
\]

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\]

whose set of solutions describes a 1-dimensional curve \( C \) in a \( K \)-dimensional configuration space.

Next, given the curve \( C \), one can project it onto the \((a, b)\) plane, simply by ignoring the value of the observable \( c \), so that one recovers a curve defined by the points \((a_k, b_{\pi(k)})\) in the space \( S_A \times S_B \). Assuming that the datasets \( A \) and \( B \) are identical to those used in the previous, two-pendulum example, the “special” permutation \( \tilde{\pi} \) that we have found in the previous subsection will be identical to the one we found now. In other words, the presence or absence of the dataset \( C \) in the statistical analysis of the Appendix A will not change the permutation \( \tilde{\pi} \), although a priori this is mathematically possible. The fact that this does not happen in our thought experiment is a consequence of the nontrivial nature of the dataset \((A, B, C)\).

Since the permutation \( \tilde{\pi} \) is independent of the presence of the third dataset, we can use the third dataset to “resolve” the self-intersecting points of the projection of the curve \( C \) to the space \( S_A \times S_B \). Namely, the presence of the third observable establishes that we are actually observing a projection of two different points of the curve \( C \) onto the same point in the \( S_A \times S_B \) subspace. Specifically, if we have the following two points on the curve \( C \),

\[
\mathcal{P}_k \equiv (a_k, b_{\tilde{\pi}(k)}, c_{\tilde{\rho}(k)}), \quad \mathcal{P}_m \equiv (a_m, b_{\tilde{\pi}(m)}, c_{\tilde{\rho}(m)}),
\]

such that \( a_k \approx a_m, b_{\tilde{\pi}(k)} \approx b_{\tilde{\pi}(m)} \) but \( c_{\tilde{\rho}(k)} \neq c_{\tilde{\rho}(m)} \), we see that although the points \( \mathcal{P}_k \) and \( \mathcal{P}_m \) may belong to quite distant parts of the curve \( C \), they will both project to the “same point” in the subspace \( S_A \times S_B \), leading to the apparent intersection. This is again a general feature of mechanical systems — if our curve \( C \) happens to self-intersect anywhere, we can always enlarge the physical system by adding another observable which will “distinguish” between the two appearances of the apparent intersection point, resolving it into two distinct points.

Finally, let us just shortly note that the issue of space-filling curve \( C \) can be eliminated by extending the physical system with an observable \( x \) which has a noncompact domain \( S_X \), and can be considered to be monotonically increasing, such as entropy or whatever else is at hand. This means that our big configuration space will be noncompact, and the curve \( C \) will intersect every hypersurface \( x = \text{const} \) precisely once, sidestepping any issue of space-filling curves.

D. The time manifold

The thought experiment described above provides an operational protocol to establish the existence of a 1-dimensional, non-self-intersecting, non-space-filling curve \( C \) as a joint configuration space of a given number of observable mechanical degrees of freedom. The curve \( C \) has all the hallmarks of a manifold (it is a nonempty set with a well-defined topology and dimension), and in order to properly promote it into a manifold, all we need is a set of coordinate charts, i.e., an atlas. Assuming for simplicity that \( C \) has a topology of an open line, it is enough to consider a single coordinate chart across the whole curve (as opposed to a circle which requires at least two charts), called time chart:

\[
T : C \rightarrow \mathbb{R}, \quad (a, b, c, \ldots) \mapsto t.
\]

Being a chart, i.e., a homeomorphism, this map is invertible, giving rise to parametric equations of motion for the observables \( a, b, c, \ldots \),

\[
T^{-1} : \mathbb{R} \rightarrow C, \quad t \mapsto (a(t), b(t), c(t), \ldots).
\]

This map defines \( a(t), b(t), \) etc., as functions of the time coordinate \( t \in \mathbb{R} \), and they satisfy the correlation functions \( (13) \). Of course, one can introduce a different chart \( T' : C \rightarrow \mathbb{R} \), such that the composition \( \phi \equiv T' \circ T^{-1} : \mathbb{R} \rightarrow \mathbb{R} \) is a
homeomorphism of the real line. One can assume that \( \phi \) is a smooth function, in which case it is a diffeomorphism. It defines the time reparametrization transformation, as

\[
t' = \phi(t),
\]
which keeps the correlation functions (15) invariant. This becomes obvious if we use the composition notation \( F_k(a(t), b(t), \ldots) \equiv (F_k \circ T^{-1})(t) \), so that

\[
F_k \circ T^{-1} = 0
\]
implies

\[
F_k \circ T^{-1} \circ \phi^{-1} = 0,
\]
for all \( k = 1, 2, \ldots, K \), where \( \phi^{-1} \) is a coordinate transformation from \( t' \) to \( t \). Since the curve \( \mathcal{C} \) is in fact the joint configuration space of \( K \) observables, and it is defined by the equations \( F_k = 0 \), the invariance of the correlation functions establishes the invariance of the joint configuration space \( \mathcal{C} \) under the action of the 1-dimensional group of diffeomorphisms \( \text{Diff} (\mathbb{R}) \). This feature of \( \mathcal{C} \) is called the time reparametrization invariance, and the group \( \text{Diff} (\mathbb{R}) \) is a subgroup of the larger group of spacetime diffeomorphisms, \( \text{Diff} (\mathbb{R}^4) \), as we shall see in the next section.

Let us finish this section with a remark that the existence of the curve \( \mathcal{C} \) in our dataset means that it is impossible to “erase” the information about coincident measurements of the observables. Recall that we were “cutting the photos” to display only individual pendulums, in order to erase the information about the pairing of experiment outcomes. The existence of the curve \( \mathcal{C} \) fully recovers that information from our dataset, which means that this information cannot be erased. In other words, the information about “conditional measurements” — measuring the position of the pendulum \( A \) under the condition that pendulum \( B \) has some given position — has its own objective existence, and is encoded in the correlations present in the dataset.

### III. FIELD THEORY AND SPACETIME

The generalization of the analysis given in the previous section, from a time manifold to a spacetime manifold, is completely straightforward. The only difference is that we need to use fields instead of mechanical systems in our thought experiment.

For the sake of concreteness, let us imagine that we have a fluid, flowing through some big container (a pipe or a river bed). Suppose we have an instrument, called “a probe”, which we can insert into the fluid to measure its various properties. The probe has compact spherical shape and is small enough not to perturb the properties and flow of the fluid, while it samples (i.e., performs a coincident measurement of) various observables. When immersed into the fluid and activated, the probe provides the following set of numbers:

- mass-density \( \rho^m \) of the fluid,
- pressure \( p \) of the fluid,
- temperature \( T \) of the fluid,
- charge-density \( \rho^e \) of the fluid,
- magnitude \( E \equiv \| \vec{E} \| \) of the electric field inside the fluid,
- magnitude \( B \equiv \| \vec{B} \| \) of the magnetic field inside the fluid, and
- the angle \( \theta \) between \( \vec{E} \) and \( \vec{B} \), specified as \( \theta \equiv \vec{E} \cdot \vec{B} / E B \).

In total, the probe measures the 7 observables

\[
\rho^m, \ p, \ T, \ \rho^e, \ E, \ B, \ \theta.
\]

We then perform \( N \) measurements, by using \( N \) probes. Each probe randomly activates once, performs the measurements, and transmits the measured values wirelessly to our computer. In order to erase any information about “where
and when the measurements took place, the computer does not keep track of anything but the unordered sets of $N$ measurements for each observable,

\[
\rho^m = \{\rho^m_1, \ldots, \rho^m_N\}, \\
p = \{p_1, \ldots, p_N\}, \\
\vdots \\
\theta = \{\theta_1, \ldots, \theta_N\}.
\]

(23)

Each of the 7 observables belongs to its domain, which is either a compact or a noncompact subset of $\mathbb{R}$. Denote them in turn as

\[
S_{\rho^m}, \quad S_p, \quad \ldots, \quad S_{\theta}.
\]

(24)

and introduce the total (kinematic) configuration space as the Cartesian product of all of these,

\[
S_{kin} \equiv S_{\rho^m} \times S_p \times \cdots \times S_{\theta}.
\]

(25)

Next, since we work with 7 observables, introduce 6 arbitrary permutations $\pi_1, \ldots, \pi_6$ of a set of $N$ elements, and construct a $7 \times N$ matrix as

\[
\begin{pmatrix}
\rho^m_1 & \rho^m_2 & \ldots & \rho^m_N \\
p_{\pi_1(1)} & p_{\pi_1(2)} & \ldots & p_{\pi_1(N)} \\
\vdots \\
\theta_{\pi_6(1)} & \theta_{\pi_6(2)} & \ldots & \theta_{\pi_6(N)}
\end{pmatrix}.
\]

(26)

Each column in this matrix represents one data point in a 7-dimensional scatter plot, which obviously cannot be drawn on paper but is well-defined nevertheless. Since the permutations $\pi_1, \ldots, \pi_6$ can be chosen completely arbitrary, there are in total $(N!)^6$ such matrices, each with its own 7-dimensional scatter plot. Again for obvious reasons, visual inspection of all these plots is not possible, but the statistical analysis from Appendix A should work just fine, provided enough computational power. As a result of the thought experiment, the statistical analysis will provide us with one “special” choice of permutations $\tilde{\pi}_1, \ldots, \tilde{\pi}_6$, for which all data points align themselves nicely along one hypersurface, denoted $\mathcal{M}$, in the big 7-dimensional configuration space $S_{kin}$. In the limit $N \to \infty$, it satisfies the properties of being a strict measure-zero subset of $S_{kin}$:

\[
\mathcal{M} \subset S_{kin}, \quad \frac{\text{meas}(\mathcal{M})}{\text{meas}(S_{kin})} = 0.
\]

(27)

The hypersurface $\mathcal{M}$ represents the joint configuration space for our physical system, and exhibits the usual properties of existence, self-reinforcement and dimensionality, all established as an experimental result of our thought experiment. The statistically obtained dimension of $\mathcal{M}$, without any a priori reason whatsoever, turns out to be

\[
\dim \mathcal{M} = 4.
\]

(28)

Also, as before, if the hypersurface $\mathcal{M}$ happens to self-intersect or fills up the whole space $S_{kin}$, we should simply add additional convenient observables to our dataset, and convince ourselves that the above inconvenient properties dissipate. Also as before, adding additional observables fails to change the overall dimension of $\mathcal{M}$, which persistently keeps being equal to 4, regardless of the number of sampled observables.

If we extend the number of our observables from 7 to $K$, denoted $a^{(1)}, \ldots, a^{(K)}$, we find a total of $K - 4$ correlation functions

\[
F_1(a^{(1)}, \ldots, a^{(K)}) = 0, \quad \ldots, \quad F_{K-4}(a^{(1)}, \ldots, a^{(K)}) = 0.
\]

(29)

whose set of solutions describes a 4-dimensional hypersurface $\mathcal{M}$ in a $K$-dimensional configuration space.

The only thing left to do at this point is to introduce a set of coordinate charts, i.e., an atlas,

\[
f : \mathcal{M} \to \mathbb{R}^4, \quad (a^{(1)}, \ldots, a^{(K)}) \mapsto x \equiv (t, x, y, z),
\]

(30)

and its inverse

\[
f^{-1} : \mathbb{R}^4 \to \mathcal{M}, \quad x \mapsto (a^{(1)}(x), \ldots, a^{(K)}(x)),
\]

(31)
which establish the parametric functions $a^{(k)}(x)$ for the observables, which in turn satisfy all correlation functions \( (29) \). The hypersurface $\mathcal{M}$, now established as a proper 4-dimensional manifold, is of course called \textit{spacetime}, while the parametric functions $a^{(k)}(x)$ are called \textit{fields living on spacetime}.

One can also introduce a different chart, \( f^\prime: \mathcal{M} \rightarrow \mathbb{R}^4 \), \( (a^{(1)}, \ldots, a^{(K)}) \mapsto x' \equiv (t', x', y', z') \), such that the composition $\phi = f^\prime \circ f^{-1}: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a homeomorphism in $\mathbb{R}^4$. If $\phi$ is smooth, it is a diffeomorphism. It defines a coordinate transformation
\[
x' = \phi(x) ,
\]
which keeps the correlation functions \( (29) \) invariant, similar to the 1-dimensional case from Section II. Since $\mathcal{M}$ is equivalent to the solution of the system \( (29) \), and since it is the joint configuration space for the observables, it is invariant under the action of the group of 4-dimensional diffeomorphisms, $\text{Diff}(\mathbb{R}^4)$. This feature of $\mathcal{M}$ is called (passive) \textit{diffeomorphism invariance of spacetime}, or \textit{general coordinate invariance}.

Since $\mathcal{M}$ is the joint configuration space for the observables, choosing a particular data point $k$ in spacetime (one of the measured points on $\mathcal{M}$) gives rise to a $K$-tuple of particular values of the observables. In our example, those are
\[
(\rho^m_k, p_{x_1(k)}, \ldots, p_{x_6(k)}) ,
\]
and they reconstruct the information about coincident measurements coming from each particular probe — the information we had tried to erase by ignoring any particular order of values in the dataset \( (23) \). As in the case of the 1-dimensional time manifold, this information is present in the correlations of the dataset itself, and \textit{cannot be erased}.

As a final point, note that the above operational reconstruction of spacetime is actually one precise implementation of the relationalism paradigm, defining the spacetime manifold using nothing but fields that supposedly live on it. The idea for a mental image “fields do not live on top of spacetime, but on top of each other” (pointed out for example by Rovelli in [9]) can be explicitly realized if we were to take 4 parametric functions $a^{(1)}(x), \ldots, a^{(4)}(x)$, conveniently chosen so that they uniquely specify a spacetime point $x$ in some given chart, i.e., such that one can solve those 4 parametric functions for the coordinates as functions of the observables,
\[
x^\mu(a^{(1)}, \ldots, a^{(4)}), \quad \mu = 0, \ldots, 3 ,
\]
and then use them to eliminate $x$ from the remaining $K - 4$ observables $a^{(5)}, \ldots, a^{(K)}$. In this way one arrives precisely at the $K - 4$ correlation functions \( (29) \), which actually define the manifold $\mathcal{M}$ using nothing but the information about fields. These correlation functions are the precise technical implementation of the statement that “fields live on fields”, in the sense that we can only observe coincidences among fields, as opposed to the the values of fields “at a given spacetime point”.

Nevertheless, in contrast to the paradigm of relationalism, the fact that we always have precisely $K - 4$ correlation functions (that is, for every choice of $K$) tells us that the set of solutions of those $K - 4$ correlation functions, namely the spacetime manifold $\mathcal{M}$, has its own objective reality, independent of the choice and the amount of the fields one uses to describe it, and despite the diffeomorphism symmetry of those fields. Thus, the paradigm of relationalism falls short of achieving its goal of completely eliminating the objective reality of the underlying spacetime manifold. The invariant properties of $\mathcal{M}$ such as dimension and topology are present as correlations in the experimental data, and there is no theoretical account of why they have the values that we observe in the experiment. The 4-dimensionality and simply connected topology are \textit{brute experimental facts}, and are \textit{independent} of the choice, properties and even the very number of the fields we use to measure them. Of course, one could attempt to construct a theoretical model which would be able to deduce these correlations among observables from some simpler set of principles. This would implement the concept of the \textit{emergence of spacetime}. So far, however, no such model has ever been successfully constructed.

**IV. CONCLUSIONS AND DISCUSSION**

In this work, we have argued that one can give an operational, model independent experimental protocol whose outcome would be the determination of the dimension and the topology of the time manifold (Section II) and the spacetime manifold (Section III). After introducing the thought experiment, we have studied the relevant properties (see Conjecture 2 from Section II) of the experimental data which give rise to a signal that reflects the objective reality of the underlying manifold, over which the observable fields are defined. As it turns out, there indeed exists such a signal, establishing the \textit{objective reality of the spacetime manifold}.

In what follows, we will discuss various aspects of the thought experiment that were not discussed in detail in previous sections, but nevertheless deserve to be mentioned and commented on.
A. Distinguishing space from spacetime

In Section II we have introduced the time manifold by looking at a mechanical system, while in Section III we have introduced the spacetime manifold, looking at fields instead. However, one may ask a natural question about the notion of a space manifold, and its differences from spacetime. To that end, let us discuss another illustrative example of the application of our gedanken-experiment.

Consider a room with a lamp and some furniture. The experimental apparatus used to perform measurements over this system consists of two cameras separated several centimeters apart. Each camera performs a measurement that provides the following data:

• the polar and azimuthal angles $\theta$ and $\varphi$ of the incident light ray,
• the frequency $\nu$ and intensity $I$ of the ray.

Since we have two cameras, in total there are eight observables per measurement. As always, we collect the measured data ignoring any order, and perform the analysis of the protocol, to reach the conclusion that there are five correlation functions between observables, which means that all measurements can be arranged on a 3-dimensional manifold. Intuitively, this camera setup corresponds to the stereoscopic eyesight, that gives one a perception of depth and thus the notion of a 3-dimensional space of the room. One should note that the notion of time is absent from this description, since the scene of the room is static. Hence the resulting 3-dimensional manifold deserves the name space manifold.

Nevertheless, one can also consider a room with a lamp and furniture, and additionally a cat moving around inside. If we now perform the same type of measurements of the system with identical cameras as before, we will find only four correlation functions between eight observables, which means that all measurements can be arranged on a 4-dimensional manifold, rather than a 3-dimensional one. It is obvious that the motion of the cat renders the dataset fundamentally more complicated, with less amount of correlation. In other words, the observed 4-dimensional manifold corresponds to spacetime, since in this case the scene of the room is not static anymore.

This example illustrates the dependence of the outcome of the experiment on the properties of the system being observed. It may happen that a system has a high level of symmetry (in the example above, the time-translation invariance), which introduces additional correlations into the dataset and lowers the dimension of the resulting manifold. This is the main way to distinguish space from spacetime — space is in fact spacetime with a certain global symmetry, which renders time unobservable. Similarly, the time manifold from the pendulum example in Section II is also spacetime with a global space-translation symmetry (the pendulum swings the same way regardless of its spatial position), which renders space unobservable.

B. Quantum-mechanical treatment

It is not completely obvious what would change regarding the outcomes of our thought experiment, if one were to take into account the effects of non-commuting observables, i.e., quantum effects. On one hand, the above analysis makes use of only mutually commuting observables (positions of several pendulums in mechanics, or values of different fields in field theory all mutually commute). This may suggest that our results should not be disturbed by the fact that there exist other observables, which fail to commute with the ones used in the experiment. In principle, one can extend the set of sampled observables up to the so-called complete set of compatible observables, without changing anything in the above analysis.

On the other hand, a very precise measurement of a coordinate of a given pendulum may uncontrollably perturb its momentum, so much that the pendulum fails to swing in the usual way, which would introduce a form of an intrinsic noise in the experimental data. In that case, subsequent measurements of the position of the pendulum may fail to be well correlated to the measurements of other observed pendulums, potentially hindering the predictions for the dimension and topology of the proper configuration space. Nevertheless, in a carefully controlled experiment, this noise can probably be kept below the expected experimental error. Additionally, care should be taken to distinguish the single-shot measurement of a single instance of a pendulum (which gives a random result, per QM), from the statistical measurement of an ensemble of identically prepared pendulums (which is probabilistically determined by QM). The uncertainty relations hold for the latter, while in our study we are interested in the former.

The proper quantum mechanical treatment is out of the scope of this paper, and we postpone it for future work. However, we would like to point the reader to a recent work [10], studying the notion of the so-called quantum switch [11]. There, a quantum mechanical observable was proposed which counts the number of relevant spacetime points, distinguishing 4-point optical realisations of the quantum switch from a 2-point gravitational switch (see also a related subsequent work [12]). The quantum mechanical measurement operator constructed in [10] represents
another example of a property of a spacetime manifold that is observable. Specifically, despite the fact that individual points of the manifold are not observable, they are distinguishable, in the sense that one can experimentally tell apart two different spacetime points. This property is of course also diffeomorphism-invariant, since diffeomorphisms are bijective maps and can never map two different points into a single point. Therefore, there should be a corresponding physical observable that distinguishes two (or more) different spacetime points. This further supports our current result, opening up a possibility for generalisations to the quantum realm.

C. Extra dimensions of spacetime

Throughout Section [11], it was claimed that the correlations one ought to find in real experimental data will ultimately support the conclusion $D = \dim M = 4$. This claim is supported by everyday experience and virtually all experiments ever performed in the history of physics so far. These experiments roughly cover the scales from $10^{-20}$ m (the scale of the current LHC and LIGO experiments), up to $10^{26}$ m (the scale of the observable Universe). There is a further range of scales, from $10^{-20}$ m down to $10^{-35}$ m (the Planck scale), and maybe even beyond that. We currently have no experimental data from this range to either support the result $D = 4$ or disprove it. Eventual access to this data could in principle change the result for $D$. For example, in the context of string theory [13, 14], one imagines spacetime to have six additional spacelike dimensions, wrapped up into a small Calabi-Yau manifold. If we were to measure various fields at the scale smaller than the size of that Calabi-Yau manifold, our analysis of the data would yield $D = 10$, or 4 plus whatever number of compactified small extra dimensions exist at that scale.

In this sense, the dimension of the spacetime manifold may be a scale-dependent quantity, like running coupling constants in QFT, having different “effective” values at different scales. So far we have no data that would indicate anything other than $D = 4$, but in principle this may change. Related to this, one may ask if our analysis supports the “running” of $D$ to values smaller than 4 at smaller scales. Values larger than 4 are obviously possible, but smaller values are also possible in principle. Namely, with sufficient resolution, one may notice that what looks like a 4-manifold is (for example) in fact a 1-dimensional curve densely packed up in a space-filling fashion, like some finite iteration of the Peano curve. In this sense, a fundamentally 1-dimensional manifold can look at large scales as a 4-dimensional one. Something along these lines apparently happens in the context of Causal Dynamical Triangulations scenario [15, 16], giving rise to $D = 2$ near the Planck scale. That said, note that the CDT results discuss a different concept of a spacetime dimension, namely an effective dimension visible to a random walker. This is in general not equivalent to the notion of topological dimension that we discuss here.

The result $D = 4$ is also contingent on the choice of observables measured and used to conclude that $D = 4$. In principle, if we were to extend our set of observables, we could find additional large dimensions. For example, so far we can detect the presence of dark matter only through its gravitational interaction with regular matter. In addition to that, one could imagine that there are also direct (contact) interactions between dark and regular matter, but that they are obscure enough not to be easily visible (like neutrinos, which interact with other matter only through short-range weak interactions and even weaker gravity). Nevertheless, if we somehow manage to measure these additional observables dependent on dark matter, they may change our statistics even in the IR regime, i.e., at large scales. It may thus turn out that our analysis yields $D = 7$ or $D = 12$ or any number larger than 4. An example of this scenarios are braneworld cosmologies similar to Randall–Sundrum model [17, 18] and similar, where all “standard” observables we can measure happen to be nonzero only on a 4-dimensional submanifold of some target manifold of larger dimension, while the “dark matter” observables would be nonzero even in the bulk of the target manifold, ultimately giving rise to $D > 4$. So far we have not found any such observables, but this may also change in the future.

D. Uniqueness objections

Looking at the analysis of the dataset of our thought experiment, one can raise an objection that the analysis of the correlations between observables may fail to give a unique result. Namely, in generic circumstances, any finite dataset constructed by $N$ measurements of $K$ observables can be permuted into $(N!)^{K−1}$ possible arrangements, and for each of them one can evaluate the suitable critical parameter (see Appendix [A]) that singles out the one “special” permutation. Since the set of permutations is finite, we end up with a finite collections of the values of the critical parameter, and any such collection contains a minimal element. This minimal parameter corresponds to a “special” permutation, since it features the strongest correlation in the dataset, ultimately describing a spacetime manifold. Given this setup, it may turn out that this minimal parameter is not unique, in the sense that several different permutations of the dataset all have this same minimal value of the parameter. In such a case, there are
in principle several different possible arrangements of data featuring equally strong correlations, leading to several different possible candidate manifolds.

On one hand, it is not feasible to study this question numerically in practice, since the set of $(N!)^{K-1}$ of all possible permutations is incredibly huge, while permutations featuring the minimal parameter are likely to be a scarce subset of these. This limits us to theoretical arguments that generic datasets feature minimal critical parameter for only one permutation. The main argument provided so far is based on the self-reinforcement property, which is unlikely to hold for more than one permutation and its restrictions to arbitrary subsets of data.

On the other hand, it is easy to construct explicit examples which feature more than one “special” permutation, each corresponding to the same minimal value of the critical parameter. This is expected to happen if a physical system being measured features global symmetries. For example, pendulums are invariant with respect to a global left-right symmetry, in the sense that one can find two “special” permutations, which are mirror images of each other. While these formally give rise to two different time manifolds, these manifolds are represented by two isomorphic datasets, and therefore in fact represent the one and the same time manifold, despite the nonuniqueness. In this sense, nonuniqueness that stems from the existence of global symmetries is benign, and does not invalidate the analysis and conclusions of the thought experiment.

Also, one of the most often discussed examples is the “diagonal” permutation of the dataset, where the values for each observable are sorted in an ascending (or descending) fashion. This leads to a dataset where points are roughly aligned along a 1-dimensional curve connecting two opposite corners of the $K$-dimensional kinematic configuration space. Being 1-dimensional, this permutation is likely to have a very small value of the critical parameter, possibly the minimal one. Nevertheless, it is not hard to demonstrate that this permutation fails to satisfy the self-reinforcement property, since its restrictions to data subsets will fail to coincide with the full permutation. In other words, as one adds more data to the dataset, the resulting 1-dimensional curve “wiggles”, changing its shape for every extension of the dataset. Thus such a permutation has to be excluded from consideration, regardless of the very small value of the critical parameter. We conjecture that the self-reinforcement property will be violated for all permutations featuring small critical parameter, except for one — which can thus be uniquely recognized as “special”. However, any potential proof of this conjecture is not available at this time, and requires further study.

E. Emergence of spacetime

As we noted at the end of Section [11] in principle one can imagine a theoretical model which does not feature anything like a 4-manifold in its postulated structure. Further, such a model may give us predictions for the values of all possible observables, and we can use it to generate a dataset, study it using our methods, and end up with correlations among observables which give rise to a 4-manifold. If such a scenario happens, one says that spacetime emerges from the theoretical model, and that the model predicts the values of spacetime dimension and topology.

Despite many hopeful attempts (usually in the context of quantum gravity), there are no particular theoretical models that have managed to achieve this, even with a wrong prediction for the dimension and topology. The reason for this are two important criteria that such a model must satisfy in order to make a legitimate prediction:

(1) The information about the 4-manifold must not be present in the axiomatic structure of the model. Namely, if we include the 4-manifold structure as an input in the construction of the theory, it should come as no surprise that one can recover that information from the model later on, in various different forms. However, this can be tricky to test, since the information about a 4-manifold can be encoded in a non-obvious, cryptic fashion, and it may be hard to prove that some of the axioms of the model are indeed equivalent to the assumption that there exists a 4-manifold in the theory.

(2) One must demonstrate that the actual dimension of the would-be manifold is explicitly computable from the model. For example, one can try to evaluate a bunch of observables using the model, and then apply our analysis onto that data, in order to obtain $D = 4$. In this sense, the dimension of spacetime would be an explicit consequence of the dynamics of the observables in the model. Alternatively, one may use some other way to calculate the spacetime dimension, but this again needs to be a consequence of dynamics of the observables, giving rise to appropriate correlations in the dataset that would ultimately lead to a 4-manifold. However, it is not enough to handwavingly claim that the model “in principle” predicts these correlations, because they cannot be a generic feature of the model. The correlations must be explicitly calculated, or otherwise rigorously proved to exist, and in addition it must be also rigorously proved that there cannot be any further correlations beyond these, since any further correlations would lower the dimension below 4, falsifying the model.

For example, a typical spinfoam model of quantum gravity is constructed as a state sum over the values of the fields living on a 2-complex (the spin foam), providing one with a way to calculate (expectation) values of observables.
However, usually by construction, the 2-complex which is used is dual to a triangulation of a 4-manifold. Because of this property, the spinfoam model fails to satisfy the criterion (1) — the information about a 4-manifold is already integrated into the model as an axiom, rather than being a property of the dynamics of the observables. Moreover, if one manages to “fix” this by generalizing the 2-complex structure somehow, so that it fails to be dual to a manifold (for example, by arranging that each dual cell has different topological dimension), there is criterion (2) — one must use the model to explicitly demonstrate that the observables will always feature appropriate correlations (as a consequence of the dynamics of the model) so that they give rise to a 4-manifold, at least in some large-distance limit. No such model has ever been proposed, and no such calculation has ever been performed.

Another example would be bosonic string theory, which is formulated by explicitly assuming the existence of a $D$-manifold, albeit with an unspecified value for $D$. Then one uses the dynamics of the model to evaluate 1-loop beta-functions, and from the requirement that these vanish, one obtains a consistency requirement $D = 26$. The explicit assumption of a $D$-dimensional manifold is an input to the model, and thus already violates criterion (1). Criterion (2) comes very close to being satisfied, but unfortunately it is contingent on the particular choice of $\zeta$-function regularization, making use of the famous “identity”

$$\zeta(-1) \equiv \sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots = -\frac{1}{12},$$

obtained by analytic continuation of the Riemann $\zeta$-function, i.e., postulated as an axiom of the model. This ultimately renders the value of $D$ to be a part of the definition of the theory, completing the argument that criterion (1) is indeed violated.

One can naturally expect to find many other proposals for spacetime emergence from various theoretical models throughout the literature. However, in order to take any of these proposals seriously, one must first demonstrate that criteria (1) and (2) are fulfilled, which is highly nontrivial, and likely has not been done for any of the existing proposals.

Let us note that one might also consider another type of emergence, which stems from raw experimental data, as opposed to emergence from a theoretical model, which was discussed so far in this Subsection. In fact, the approach taken in this work represents an example of this second type of spacetime emergence. In that sense, the spacetime emergence from raw experimental data is precisely a synonym for the notion of the verification of the existence of spacetime, as used in the title. It is important to emphasise that the term “emergence” can thus have quite different meanings, depending on the context, and one should take care in its use.

Acknowledgments

The authors wish to thank Jovan Janjić for help in implementing numerical algorithms in Mathematica, and to Igor Salom, Časlav Brukner, Marcus Huber, Reinhard Werner and Klaus Fredenhagen for fruitful discussions.

NP acknowledges Fundação para a Ciência e Tecnologia (FCT), Instituto de Telecomunicações Research Unit, ref. UIDB/50008/2020, UIDP/50008/2020 and PEst-OE/EEI/LA0008/2013 and LASIGE Research Unit, ref. UIDB/00408/2020 and ref. UIDP/00408/2020. The authors also acknowledge the European Regional Development Fund (FEDER), through the Competitiveness and Internationalization Operational Programme (COMPETE 2020), under the project QuantumPrime reference: PTDC/EEI-TEL/8017/2020 and QuRUNNER, QUESTS action of Instituto de Telecomunicações and the QuantaGENOMICS project, through the EU H2020 QuantERA II Programme, Grant Agreement No 101017733, CERN/FIS-PAR/0023/2019, as well as the FCT Estímulo ao Emprego Científico grant no. CEECIND/04594/2017/CP1393/CT000.

MV was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, and by the Science Fund of the Republic of Serbia, grant 7745968, “Quantum Gravity from Higher Gauge Theory 2021” — QGHLG-2021. The contents of this publication are the sole responsibility of the authors and can in no way be taken to reflect the views of the Science Fund of the Republic of Serbia.

Appendix A: Statistical analysis technique

Suppose we are given a dataset of $N \to \infty$ points as $K$-tuples in some compact volume of a $K$-dimensional Euclidean space (the noncompact case can be reduced to the compact case by cutting it into countably infinitely many compact pieces, and studying each piece one by one). The volume of the box is known to be $V_K$. Our task is to verify whether, and to what extent, these $N$ points are “nicely” aligned on some $D$-dimensional hypersurface in the big $K$-dimensional space. Obviously, it is assumed that $D < K$, and the notion of “nice” alignment will be defined rigorously below.
1. The critical parameter

We proceed as follows. Pick any datapoint \( x \), and construct around it a \( K \)-dimensional cube, such that the point \( x \) is in the cube’s center. Let the length of each edge of the cube be specified as

\[
\epsilon = \kappa \sqrt[1/D]{\frac{V_K}{N}},
\]

i.e., such that the volume of the cube is

\[
V_{\text{cube}} \equiv \epsilon^K = \frac{V_K}{N}.
\]

Suppose first that datapoints are scattered randomly throughout the box. If we construct the same cube around every datapoint, the total volume of all cubes can be represented as:

\[
V_{\text{total}} = \sum_{n=1}^{N} (-1)^{n+1} n! \sum_{i_1 \neq \ldots \neq i_n} V_{i_1 \ldots i_n}.
\]

Here, \( V_{i_1 \ldots i_n} \) is the volume of the joint intersection of \( n \) cubes labeled by indices \( i_1, \ldots, i_n \), and the second sum is evaluated over all nonrepeating values of these indices, which take values from the set \( \{1, \ldots, N\} \). Given that all individual cubes are of the same volume, the term for \( n = 1 \) is

\[
V_{n=1} = \sum_{i=1}^{N} V_i = NV_{\text{cube}} = V_K.
\]

Up to an overall minus sign, the remaining terms in \( \text{(A3)} \) are collected into a total overlap volume

\[
V_{\text{overlap}} \equiv -\sum_{n=2}^{N} (-1)^{n+1} n! \sum_{i_2 \neq \ldots \neq i_n} V_{i_1 \ldots i_n},
\]

so that we have

\[
V_{\text{total}} = V_K - V_{\text{overlap}} = \alpha V_K.
\]

The coefficient \( \alpha \in [0, 1] \) is another measure of this overlap:

\[
\alpha = 1 - \frac{V_{\text{overlap}}}{V_K}.
\]

In the main text, we call \( \alpha \) the critical parameter.

Specifically, if the datapoints are very uniformly distributed throughout the box, their corresponding cubes will have very little overlap volume, \( V_{\text{overlap}} \approx 0 \), so that the volume covered by all cubes will be approximately equal to the volume of the box, \( V_K \), and \( \alpha \approx 1 \). On the other hand, one can suppose that the datapoints are all aligned along some \( D \)-dimensional hypersurface \( \mathcal{M} \), whose total \( D \)-dimensional volume is \( V_D \), while its \( K \)-dimensional volume is of course of measure zero. Now, if all \( N \) datapoints are aligned on \( \mathcal{M} \), their cubes will overlap quite a lot, giving \( \alpha \approx 0 \) in the limit \( N \to \infty \).

One can estimate how close \( \alpha \) is to zero in the following way. Construct a cube around every point lying on \( \mathcal{M} \). The intersection between \( \mathcal{M} \) and the cube will be again a \( D \)-dimensional hypersurface, while the cubes will “protrude out” of \( \mathcal{M} \) into \( K - D \) orthogonal directions. We end up with a “thick hypersurface”, having thickness \( \epsilon^{K-D} \). Since by assumption all \( N \) points are lying on \( \mathcal{M} \), one can approximate the value of \( V_{\text{total}} \) of this thick hypersurface as

\[
V_{\text{total}} \approx \epsilon^{K-D} V_D = \left( \frac{V_K}{N} \right)^{K-D/D} V_D.
\]

This gives us the following estimate for \( \alpha \):

\[
\alpha(N) \equiv \frac{V_{\text{total}}}{V_K} \approx \frac{V_D}{(N V_K)^D} N^{D-1} = \text{const} \cdot N^{D-1}.
\]
Since $D < K$, we see that in the limit $N \to \infty$ we have $\alpha \to 0$. Pictorially, the “thickness” $\epsilon^{K-D}$ of the hypersurface gradually shrinks as $N$ grows, due to (A1), so that in the limit $N \to \infty$ we have $\epsilon \to 0$, and our thick hypersurface becomes asymptotically infinitely thin. This is all illustrated for the case $K = 2$ in the diagrams given in FIG. 6. On the other hand, if we take a random permutation of the second coordinates, the resulting diagrams look as the ones from FIG. 7. We can observe that, as $N$ grows, the area covered by the gray squares becomes smaller in the first case, while it remains approximately constant in the second (permuted) case. In the limit $N \to \infty$ the gray area drops to zero in the first case, while it remains constant in the second case.

2. The asymptotic behavior

Let us now formalize the procedure for the statistical analysis. Given the dataset of $N$ points and the volume $V_K$ of the big space, a computer can actually evaluate $V_{total}$ by explicit numerical evaluation of the overlap $V_{overlap}^{(n)}$ between the cube constructed around each point $n$ and all surrounding cubes. This means that $\alpha(N)$ can be explicitly evaluated, for each $N$. Provided that we have evaluated it for a number of different values of $N$, we can discuss its asymptotic structure in the limit $N \gg 1$ (which is formally equivalent to $N \to \infty$), as follows:

$$\alpha(N) = c_K + \sum_{n=0}^{K-1} c_n \frac{1}{(N)^{K-n}} + \mathcal{O} \left( \frac{1}{(N)^{K+1}} \right) + R(N).$$  \hspace{1cm} (A10)

Here we have expanded $\alpha(N)$ into an asymptotic power series, from the order $1/\sqrt[3]{N}$ up to the order $\mathcal{O}(1/\sqrt[3]{N})^{K+1}$, while the remainder-term $R(N)$ captures any behavior of $\alpha(N)$ which does not have the form of this power series —
terms like $\log N$, or $N^N$, or various other types of asymptotics.

Now we define that the datapoints are "nicely" aligned along a $D$-dimensional hypersurface if and only if in the limit $N \gg 1$ the asymptotics has the following form:

$$R(N) = 0, \quad c_K = 0, \quad c_n = \begin{cases} c_D & n = D, \\ 0 & n \neq D, \end{cases} \quad \forall n = 0, \ldots, K - 1.$$  \hspace{1cm} (A11)

This is actually a statement that $\alpha(N)$ has the asymptotics of the form \((A9)\), while the single nonzero coefficient $c_D$ is proportional to the $D$-dimensional volume of the hypersurface,

$$c_D \equiv \frac{V_D}{(\sqrt{N})^D}.$$  \hspace{1cm} (A12)

Moreover, in this case one can solve \((A9)\) for $D$, which gives us the statistically defined dimension of the hypersurface,

$$D = K \left[ 1 + \lim_{N \to \infty} \frac{\log \alpha(N)}{\log N} \right].$$  \hspace{1cm} (A13)

Thus, if the asymptotics of $\alpha(N)$ is "nice" in the above sense, the quantity $D$ will be a positive integer smaller than $K$ (since $\log \alpha(N)$ is negative), and it can be explicitly computed, asymptotically for ever larger $N$.

Finally, if we have a bunch of available datasets (for example, $(N!)^6$, as we do in the main text), we can calculate the asymptotic form \((A10)\) for each. Among all these, the one dataset that converges most efficiently toward \((A9)\) is called "special", and the corresponding value of $D$ calculated from \((A13)\) is called its dimension. It defines a hypersurface $\mathcal{M}$ as a strict subspace of the box of volume $V_K$, and is moreover of measure zero compared to the box, since

$$\frac{\text{meas}(\mathcal{M})}{\text{meas}(\text{box})} = \lim_{N \to \infty} \frac{V_{\text{total}}}{V_K} = \lim_{N \to \infty} \alpha(N) = 0,$$  \hspace{1cm} (A14)

according to the asymptotics \((A9)\).

At the end, we note that we have implicitly also used the self-reinforcement property of the dataset, in the sense that the limit $N \to \infty$ actually exists, i.e. that instead of the dataset of $N$ points we can use the dataset of $N + M$ points, such that the asymptotic properties \((A11)\) are maintained as we pass from one dataset to another. This is necessary for the limit $N \to \infty$ to actually make sense, in this context.

In light of the asymptotic formulas \((A10)\) and \((A11)\), the three properties from the main text, namely (a) existence, (b) self-reinforcement and (c) dimensionality mean in turn that (a) $c_K = R(N) = 0$, (b) the limit $N \to \infty$ is well-defined, and (c) only one of the coefficients $c_n$ is different from zero. Finally, the fourth property, namely (d) topology, can be determined from the overlapping $K$-dimensional cubes constructed around the datapoints. Specifically, in the limit $N \to \infty$ the total overlap volume $V_{\text{overlap}}$ must remain finite, precisely because $V_{\text{total}}$ falls to zero. This means that, despite the fact that each $K$-dimensional cube shrinks to zero, its overlap with neighboring cubes will remain finite. This property enables us to define a basis of open sets on our would-be manifold $\mathcal{M}$ as the $D$-dimensional projections of all $K$-dimensional cubes (of volume $V_{\text{cube}}$), giving rise to a topology on $\mathcal{M}$.

### 3. Alternative technique for evaluating the critical parameter

The algorithm for calculating $\alpha(N)$, while conceptually clear, turns out to be a bit inefficient for large datasets, due to the complexity of calculating the overlap volumes of neighboring cubes. The algorithm starts to choke already for $K = 2$ and $N = 100$ on a typical desktop computer. To remedy this, instead of using \((A7)\) to calculate the parameter $\alpha$, one should instead split the whole box into a convenient grid of cells, and then count the cells which contain at least one datapoint. The total volume of the nonempty cells should then be similar to the total volume of the overlapping cubes from the original approach.

To implement this idea, we first choose the number of cells along each axis of the box to be

$$\left[ \frac{\sqrt[3]{N}}{K} \right],$$  \hspace{1cm} (A15)

so that the total number of cells is

$$\left[ \frac{\sqrt[3]{N}}{K} \right]^K \leq N.$$  \hspace{1cm} (A16)
This is the biggest integer smaller than $N$ such that the $K$-dimensional box can be divided into that many cells. The size of each cell in the $n$-th direction $x_n$ is then given as
\[ \tilde{\epsilon}_n = \frac{x_n^{\text{max}} - x_n^{\text{min}}}{\sqrt[1/K]{N}} \tag{A17} \]
so that the total volume of each cell is
\[ \tilde{V}_{\text{cell}} = \prod_{n=1}^{K} \tilde{\epsilon}_n = \frac{V_K}{\sqrt[1/K]{N}} \geq \frac{V_K}{N} \equiv V_{\text{cube}} \tag{A18} \]
see (A2). Given this arrangement, we can define a new parameter $\tilde{\alpha}$, in analogy to (A6), as
\[ \tilde{V}_{\text{total}} \equiv p \tilde{V}_{\text{cell}} = \tilde{\alpha} V_K \tag{A19} \]
where $p$ is the number of cells which contain at least one datapoint. We thus arrive at a new parameter,
\[ \tilde{\alpha} = \frac{p \tilde{V}_{\text{cell}}}{V_K} = \frac{p}{\sqrt[1/K]{N}} \tag{A20} \]
Note that we have used the “floor” function in (A15) in order to avoid underestimating the volume (A19), and thus avoid underestimating $\tilde{\alpha}$ and consequently $D$.

Compared to (A7), parameter $\tilde{\alpha}$ can be calculated much more efficiently, since it boils down to counting the number of nonempty cells $p$, which is way faster than evaluating all the overlapping volumes of cubes. Indeed, for the case $K = 2$ and $N = 10^4$, the algorithm takes around 10 s to evaluate $\tilde{\alpha}$ on the same hardware as before.

Intuitively, in the limit $N \to \infty$ one expects that $\tilde{\alpha} \approx \alpha$, at least in the case of “nice” alignment of datapoints (as defined by (A11)). This establishes that in practical simulations we can calculate the statistical dimension $D$ of the dataset (see (A13)) using $\tilde{\alpha}$ instead of $\alpha$, which is numerically much more efficient. Indeed, this is also confirmed by explicit numerical calculations on several different example datasets.

On the other hand, using the grid-like construction above obscures the notion of basis of open sets, since cells in the grid never overlap, and therefore one cannot use this construction to introduce the topology of the manifold. In this sense, while the grid-like construction of $\tilde{\alpha}$ is numerically more efficient, the overlapping-cubes construction of $\alpha$ gives us the information about topology and is thus conceptually more useful.

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