Iterated stochastic measurements

Michel Bauer\textsuperscript{1}, Denis Bernard\textsuperscript{2,3} and Tristan Benoist\textsuperscript{2}

\textsuperscript{1} Institut de Physique Théorique\textsuperscript{4} de Saclay, CEA-Saclay, Saclay, France
\textsuperscript{2} Laboratoire de Physique Théorique de l’ENS, CNRS and Ecole Normale Supérieure de Paris, Paris, France

E-mail: michel.bauer@cea.fr, denis.bernard@ens.fr and tristan.benoist@ens.fr

Received 3 July 2012, in final form 13 September 2012
Published 27 November 2012
Online at stacks.iop.org/JPhysA/45/494020

Abstract

We describe a measurement device principle based on discrete iterations of Bayesian updating of system-state probability distributions. Although purely classical by nature, these measurements are accompanied with a progressive collapse of the system-state probability distribution during each complete system measurement. This measurement scheme finds applications in analysing repeated non-demolition indirect quantum measurements. We also analyse the continuous time limit of these processes, either in the Brownian diffusive limit or in the Poissonian jumpy limit. In the quantum mechanical framework, this continuous time limit leads to Belavkin’s equations which describe quantum systems under continuous measurements.

This article is part of ‘Lattice models and integrability’, a special issue of Journal of Physics A: Mathematical and Theoretical in honour of F Y Wu’s 80th birthday.

PACS numbers: 03.65.Ta, 05.40.—a, 02.50.Cw

(Some figures may appear in colour only in the online journal)

1. Introduction

Informal and formal similarities between Bayesian inference \cite{1} and quantum mechanics were noted quite some time ago; see, e.g., \cite{2}. The Bayesian inference may be seen as a way to update trial probability distributions by taking into account the partial information that one has gained on the system under study. An indirect quantum measurement consists of obtaining partial information on a quantum system by letting it interact with another quantum system, called a probe, and performing a direct Von Neumann measurement on this probe. Iterating the process of system–probe interaction and probe measurement increases the information on the system because of system–probe entanglements.
This has been experimentally implemented in electrodynamics in cavities [3], but also in superconductor circuits [4]. As shown by these experiments, repeating a large number of times (formally, infinitely many times) the indirect non-demolition measurements [5] reproduces macroscopic direct measurements with collapse of the system quantum wavefunction. Each collapse is stochastic and progressive, becoming sharper and sharper as the number of indirect measurements increases.

Controlling quantum systems [7] by repeating measurements is, in some way, as old as quantum mechanics, but it has recently been further developed aiming at quantum state manipulations and quantum information processing [8]. At a theoretical level, the concept of quantum trajectories [9, 10] emerges from the need to describe quantum jumps and randomness inherent to repeated measurements. In parallel, studies of open quantum systems [11] led to the theory of quantum feedback [12] and quantum continual measurements [13]. Belavkin’s equations [14] are stochastic nonlinear generalizations of the Schrödinger equation adapted to quantum systems under continual measurements.

Contact between experiments of the type described in [3] and classical stochastic processes was made in [6], showing in particular that the approach to the collapse is controlled by relevant relative entropy. The aim of this paper is to follow and complement the study performed in [6], by, in some way, reversing the logic. We start by forgetting quantum mechanics for a while and we study a random process obtained by discretely and randomly updating a system-state probability distribution using Bayes’ rules. Iterated stochastic measurements refer to this random recursive updating. We describe why and how this leads to a stochastic measurement principle allowing the measurement of the initial system-state probability distribution but which implements a random collapse of the system-state distribution at each individual complete system measurement. The initial system-state distribution is nevertheless reconstructed by repeating the complete system measurements. We point out a connection between De Fenetti’s theorem on exchangeable random variables, see, e.g., [15], and iterated stochastic measurements. We also show that these discrete measurement devices admit continuous formulations with continual updating. There are two limits: a Brownian diffusive limit in which the random data used to update the system-state distribution are coded into Brownian motions, this case was studied in [16], and a Poissonian jumpy limit in which these random data are coded in point processes. The construction of the continuous time process relies on deforming an a priori probability measure on the updating data. The key tool is Girsanov’s theorem. Then, we transport these results, in an almost automatic way, to quantum mechanics, and we show that quantum mechanical systems under repeated non-demolition indirect measurements admit a continuous time limit described by Belavkin’s equations (18) and (19). This completes results proved in [17] and makes contact with those described in [18].

2. Iterated indirect stochastic measurements

Let $S$ be the system under study and $A$ be a chosen countable set of system states $\alpha \in A$ that we shall call pointer states\(^5\). The model apparatus is going to measure the probability distribution $Q_0(\alpha)$, with $\sum_\alpha Q_0(\alpha) = 1$, for the system $S$ to be in one of the pointer states.

The model apparatus is made up of an infinite series of indirect partial measurements. Let $I$ denote the set of possible results of one partial measurement, which we assume to be finite or countable. For each system complete measurement, the output datum is thus an infinite sequence of data $(i_1, i_2, \ldots, i_k) \in I$, associated with the series of successive

---

5 According to the quantum terminology, but the concept of states is here more general as it simply refers to a complete list of labels characterizing the system behaviour.
partial measurements. The output data are random. The probability distribution \( Q_0(\alpha) \) is to be reconstructed from the sequences \((i_1, i_2, \ldots)\).

To be concrete, one may keep in mind that the indirect partial measurements arise from direct measurements on probes which have been coupled to the system. The model apparatus is then made up of an infinite set of in-going probes—which, for simplicity, are supposed to be all identical—passing through the system \( S \) and interacting with it one after the other. Measurements are carried out on the out-going probes.

Specifications of the model apparatus depend on the chosen set of pointer states. One of its manufacturing characteristics is a collection of probability distributions \( p(i|\alpha) \), \( \sum_i p(i|\alpha) = 1 \), for the output partial measurement to be \( i \in I \) conditioned on the system \( S \) being in the state \( \alpha \in A \). For simplicity, we shall assume a non-degeneracy hypothesis which amounts to supposing that all probability distributions \( p(\cdot|\alpha) \) are distinct, i.e. for any pair of distinct pointer states \( \alpha \) and \( \beta \), there exists \( i \in I \), such that \( p(i|\alpha) \neq p(i|\beta) \).

2.1. Discrete time description

In the model apparatus, a complete measurement is made up of an infinite series of partial measurements such that each output of these partial measurements provides a gain of information on the system. Our first aim is to decipher what information one is gaining from the \( n \)th first partial measurements. This will allow us to spell out the way the model apparatus is working as a measurement device.

- **Series of partial measurements and specification of the model apparatus.** Suppose that the first partial measurement gives result \( i_1 \in I \). Bayes’ law then tells us that the probability for the system \( S \) to be in the state \( \alpha \) conditioned on the first measurement \( i_1 \) is \( Q_1(\alpha|i_1) = Q_0(\alpha)p(i_1|\alpha)/\pi_0(i_1) \), with \( \pi_0(i) := \sum_\alpha Q_0(\alpha)p(i|\alpha) \), if \( Q_0(\alpha) \) is the initial probability for the system \( S \) to be in the state \( \alpha \) (this probability is yet unknown but shall be recovered from the series of partial measurements making a complete measurement). Let us now ask ourselves what is the probability to get \( i_2 \) as second output partial measurement? By the law of conditioned probabilities, \( \pi_1(i_2|i_1) = \sum_\alpha p(i_1, i_2|\alpha)Q_0(\alpha)/\pi_0(i_1) \) with \( p(i_1, i_2|\alpha) \) being the probability to measure \( i_1 \) and \( i_2 \) on the two first partial measurements conditioned on the system being in the state \( \alpha \). At this point, we need to make an assumption: we assume that the output partial measurements are independent and identically distributed (i.i.d.) provided that the system \( S \) is in one of the pointer state \( \alpha \in A \). This translates into the relation

\[
p(i_1, i_2|\alpha) = p(i_2|\alpha)p(i_1|\alpha),
\]

which implies that \( \pi_1(i_2|i_1) = \sum_\alpha p(i_2|\alpha)Q_1(\alpha|i_1) \). That is, the probability \( \pi_1(i_2|i_1) \) is identical to the probability to get \( i_2 \) as output partial measurement assuming that the system distribution is \( Q_1(\alpha|i_1) \).

Hence, as a defining characteristic property of our model apparatus, we assume that the output of the \( n \)th partial measurement is independent of those of the \((n-1)\)-first outputs provided the system \( S \) is in one of the pointer states \( \alpha \in A \):

\[
p(i_1, \ldots, i_{n-1}, i_n|\alpha) = p(i_n|\alpha)p(i_1, \ldots, i_{n-1}|\alpha) = \prod_{k=1}^n p(i_k|\alpha), \quad \forall \alpha.
\]  

This specifies our model apparatus. This specification is clearly attached to the chosen set of pointer states.

Conversely, the pointer states associated with this device are those system states for which the values of the output partial measurements are independent, i.e. conditioned on the system being in a pointer state, the output variables \( i_1, i_2, \ldots \) are independent and
identically distributed. If the system is initially in a pointer state \( \alpha \), i.e. its probability distribution is peaked, \( Q_0(\cdot) = \delta_\alpha \), the occurrence frequency \( v(i) \) of the value \( i \) in the output sequence \( (i_1, i_2, \ldots) \) is \( p(i|\alpha) \). As we shall see later, one may then identify the pointer states as the system states for which independent infinite series of partial measurements (i.e. independent complete measurements) provide identical occurrence frequencies \( v(\cdot) \), and this gives a way to calibrate the device and to determine the conditioned probabilities \( p(\cdot|\alpha) \).

If the system is not in a pointer state, its initial distribution \( Q_0(\alpha) \)—to be determined—is not peaked. Let \( Q_n(\alpha|i_1, \ldots, i_n) \) be the probability for the system to be in the state \( \alpha \) conditioned on the \( n \)-first output partial measurements \( i_1, i_2, \ldots, i_n \). From our hypothesis (1), the probability to get \( i \) as the \( n \)th output conditioned on the \( (n-1) \)th first outputs being \( (i_1, \ldots, i_{n-1}) \) is

\[
\pi_{n-1}(i|i_1, \ldots, i_{n-1}) = \sum_{\alpha} p(i|\alpha) Q_{n-1}(\alpha|i_1, \ldots, i_{n-1}).
\]  

By Bayes’ law, the probability for the system to be in the state \( \alpha \) conditioned on the \( n \)-first measurements \( i_1, i_2, \ldots, i_n \) is then recursively computed by

\[
Q_n(\alpha|i_1, \ldots, i_n) = \frac{p(i|\alpha) Q_{n-1}(\alpha|i_1, \ldots, i_{n-1})}{\pi_{n-1}(i_1, \ldots, i_{n-1})},
\]

where \( \pi_{n-1} \) is the probability to get \( i_n \) as the \( n \)th output. To simplify notations, we denote \( Q_n(\alpha|i_1, \ldots, i_n) \) by \( Q_n(\alpha) \) and \( \pi_{n-1}(i_1, \ldots, i_{n-1}) \) by \( \pi_{n-1}(i) \). Equation (3) can be solved explicitly:

\[
Q_n(\alpha) = Q_0(\alpha) \frac{\prod_i p(i|\alpha)^{N_i(i)}}{\sum_{\beta} Q_0(\beta) \prod_i p(i|\beta)^{N_i(i)}},
\]

with \( N_i(i) \) being the number of times the value \( i \) appears in the \( n \)th first outputs.

Let us point out an interesting reformulation of the above conditions on the outputs of the model apparatus. A sequence \( (i_1, i_2, \ldots) \) of random variables is called exchangeable if the distribution of \( (i_1, i_2, \ldots, i_n) \) is the same as the distribution of \( (i_{\sigma_1}, i_{\sigma_2}, \ldots, i_{\sigma_n}) \) for each \( n \) and each permutation \( \sigma \) of \([1, 2, \ldots, n]\). A remarkable theorem proposed by De Finetti (see e.g. [15] or the last two items of [19]) asserts that an infinite sequence \( (i_1, i_2, \ldots) \) of random variables is exchangeable if and only if there is a random variable \( A \) such that, conditionally on \( A \), \( (i_1, i_2, \ldots) \) is a sequence of independent identically distributed (i.i.d.) random variables. In our construction, the values taken by \( A \) are nothing but the pointer states and the measure on \( A \) is \( Q_0 \). So the hypotheses on the model apparatus can be rephrased as the fact that the order of partial measurements is immaterial.

More concretely, let \( \Omega \) be the data set of all complete measurements. This is made up of all infinite series \( \omega := (i_1, i_2, \ldots) \), \( i_k \in I_n \), of output partial measurements. We may endow \( \Omega \) with the filtration \( \mathcal{F}_n \) of \( \sigma \)-algebras generated by the sets \( B_{i_1, \ldots, i_n} := \{ \omega = (i_1, \ldots, i_n, \text{anything else}) \in \Omega \} \), i.e. \( \mathcal{F}_n \) codes for the knowledge of the \( n \)th first partial measurements. This filtered space is equipped with a probability measure recursively defined by \( \mathbb{P}[\omega = i | \mathcal{F}_{n-1}] = \pi_{n-1}(i) \). Note that, given \( Q_0(\alpha) \), this probability measure decomposes as a sum

\[
\mathbb{P} = \sum_{\alpha} Q_0(\alpha) \mathbb{P}_\alpha,
\]

where \( \mathbb{P}_\alpha \) will be the probability measure induced on \( \Omega \) if the system happened to be initially in the pointer state \( \alpha \), i.e. if \( Q_0(\cdot) \) is peaked at \( \alpha \). Under \( \mathbb{P}_\alpha \) the partial outputs are independent random variables so that

\[
\mathbb{P}_\alpha[B_{i_1, \ldots, i_n}] = \prod_{k=1}^{n} p(i_k|\alpha).
\]
Figure 1. A schematic view of iterated stochastic measurements: probes are sent one after the other to interact with the system for a while. After the interaction, a measurement is performed on each probe. The information gained is summarized in the occurrence frequencies, which allow one to identify the limiting state.

Let us then quote properties of the random probability distribution $Q_n(\cdot)$, which will be keys for specifying the model measurement device.

(i) Peaked distributions are stable under the recursion relation (3). That is, if $Q_0(\cdot) = \delta_{\gamma\omega}$, then $Q_n(\cdot) = \delta_{\gamma\omega}$ for any $n$.

(ii) Given $Q_0(\cdot)$ generic, the random variables $Q_n(\alpha)$ converge as $n$ tends to infinity almost surely and in $L^1$. The limiting distribution $Q_\infty(\cdot)$ is peaked at a random target pointer state:

$$Q_\infty(\cdot) = \delta_{\gamma\omega}$$

with a target pointer state $\gamma\omega$ depending on the event $\omega$. The probability for the target to be a given pointer state $\alpha$ is the initial probability distribution:

$$P[\gamma\omega = \alpha] = Q_0(\alpha).$$

(iii) The asymptotic occurrence frequencies $\nu(i) := \lim_n N_n(i)/n$, with $N_n(i)$ being the number of times the value $i$ appears in the $n$th first outputs, are those of the target pointer state:

$$\lim_{n \to \infty} N_n(i)/n = p(i|\gamma\omega).$$

(iv) The convergence is exponentially fast:

$$Q_n(\alpha) \simeq \exp(-nS(\gamma\omega|\alpha)), \quad \alpha \neq \gamma\omega,$n$$

for $n$ large enough, with $S(\gamma\omega|\alpha)$ being the relative entropy of $p(\cdot|\gamma\omega)$ relative to $p(\cdot|\alpha)$. These facts have been proved in [6]. They are based on the fact that the random variables $Q_n(\alpha)$ are bounded $\mathbb{P}$-martingales with respect to the filtration $\mathcal{F}_n$. That is, $\mathbb{E}[Q_n(\alpha)|\mathcal{F}_{n-1}] = Q_{n-1}(\alpha)$. A classical theorem of probability theory [19] says that a bounded martingale converges almost surely and in $L^1$, so that $Q_\infty(\alpha) := \lim_n Q_n(\alpha)$ exists and $Q_\infty(\alpha) = \mathbb{E}[Q_\infty(\cdot)|\mathcal{F}_n]$. More general results, involving for instance extra randomness on the partial measurements or relaxing the non-degeneracy hypothesis on the conditioned probability $p(\cdot|\alpha)$, have been obtained in [16].

**How to read-off a complete measurement and consequences.** Let us summarize how the model apparatus is (concretely) working and how data are analysed; see figure 1. For a given system measurement, the data are an infinite sequence $\omega = (i_1, i_2, \ldots)$ of output partial measurements. From its asymptotic behaviour, the apparatus computes the asymptotic frequencies $\nu(i)$ of occurrences of the values $i$ in the sequence $\omega$, and it compares it to one of the apparatus data-base distributions $p(i|\alpha)$. By the non-degeneracy hypothesis and the above convergence theorem [6], each of the asymptotic frequencies coincides with one of the data-base distributions, so that the comparison identifies uniquely the target...
pointer state and that the identified state is by definition the result of a complete system measurement. Since by the above theorem the distribution of the target pointer states is the initial distribution \( Q_0(\cdot) \), the histogram of repeated independent complete system measurements yields the initial distribution.

Note that by the end of a complete measurement the system-state distribution has collapsed into one of the pointer states. The need for an infinite series of partial measurements reflects the need for a macroscopic apparatus to implement the collapse. If the system measurement is stopped after a finite number of partial measurements, the collapse is only partial, i.e. the probability distribution \( Q_n(\cdot) \) is still smeared around the target pointer state. The target pointer state may nevertheless be identified with high fidelity if the differences between the data-base probability distributions \( p(\cdot|\alpha) \) are bigger than the fluctuations of the frequencies \( \nu_n(\cdot) \) which generically scale like \( n^{-1/2} \).

2.2. Continuous time limit

We now describe continuous time limits of the previous model apparatus in which the partial measurements are performed continuously in time. There are different continuous time limits, depending on the behaviour of the data-base conditioned probability distributions of \( p(\cdot|\alpha) \): a Brownian diffusive limit, a Poissonian jumpy limit, or a mixture of both.

These limits may be understood by looking at properties of the counting process \( N_n(\cdot) := \sum_{i=1}^{\infty} I_{t_i=\nu} \), which is the number of times the value \( i \) appears in the \( n \)th first partial measurements. Recall that \( \pi_{m-1}(i) = \mathbb{E}[I_{t_m=\nu}|\mathcal{F}_{m-1}] \) is the probability to get \( i \) as the \( m \)th partial output conditioned on the \((m-1)\)th first partial outputs. We may tautologically decompose \( N_n(\cdot) \) as

\[
N_n(i) = X_n(i) + A_n(i), \quad \text{with} \quad A_n(i) := \sum_{m=0}^{n-1} \pi_m(i), \tag{5}
\]

where this equation serves as the definition of \( X_n(\cdot) \), i.e. \( X_n(i) := N_n(i) - A_n(i) \), with \( \sum_{i} X_n(i) = 0 \) as both \( N_n(\cdot) \) and \( A_n(\cdot) \) add up to \( n \). Then, by construction, \( \mathbb{E}[X_n(i)|\mathcal{F}_{n-1}] = X_{n-1}(i) \), so that the processes \( X_n(\cdot) \) are \( \mathbb{P} \)-martingales with respect to the filtration \( \mathcal{F}_n \). Equation \( (5) \) is the so-called Doob decomposition of \( N_n(\cdot) \) as the processes \( A_n(\cdot) \) are predictable, i.e. \( A_n(\cdot) \) is \( \mathcal{F}_{n-1} \)-measurable; see [19]. The martingale property in particular implies that \( \mathbb{E}[X_n(i)] = 0 \).

Recall now the recursion relation \( (3) \) that we may rewrite as

\[
Q_n(\alpha) - Q_{n-1}(\alpha) = Q_{n-1}(\alpha) \sum_i \frac{p(i|\alpha)}{\pi_{n-1}(i)} (\Delta I_{\nu=i})
\]

which holds true because \( \sum_i p(i|\alpha) = 1 \). By construction, \( \Delta I_{\nu=i} = X_n(i) - X_{n-1}(i) \) so that

\[
(\Delta Q)_n(\alpha) = Q_{n-1}(\alpha) \sum_i \frac{p(i|\alpha)}{\pi_{n-1}(i)} (\Delta X)_{n}(i), \tag{6}
\]

with \( (\Delta Q)_n(\alpha) := Q_n(\alpha) - Q_{n-1}(\alpha) \) and \( (\Delta X)_{n}(i) := X_n(i) - X_{n-1}(i) \). We thus have rewritten the recursion relation \( (3) \) as a discrete nonlinear difference equation for the probability distribution \( Q_n(\cdot) \) driven by discrete differences of the martingales \( X_n(\cdot) \). This will be the starting point of the continuous time limits.

Before going on let us point out a geometrical interpretation of \( Q_n(\alpha) \) which will be useful later. On the set of complete measurements, we have defined a global measure \( \mathbb{P} \) and a series of measures \( \mathbb{P}_\alpha \) associated with each of the pointer states with \( \mathbb{P} = \sum_\alpha Q_0(\alpha)\mathbb{P}_\alpha \). It
is a simple matter to check that \( P_\alpha \) is non-singular with respect to \( P \), so that there exists a Radon–Nikodym derivative of \( P_\alpha \) with respect to \( P \); see [19]. This derivative is \( Q_\infty(\alpha)/Q_0(\alpha) \). More concretely, for any \( \mathcal{F}_n \)-measurable integrable function \( X \),

\[
Q_0(\alpha)E_\alpha[X] = E[Q_\alpha(\alpha)X],
\]

with \( Q_\alpha(\alpha) = E[Q_\infty(\alpha)|\mathcal{F}_n] \), as may be checked directly.

We may tautologically refine this geometrical construction. Let us start from an arbitrary probability measure on \( \Omega \), and let us set \( Z_{n,\alpha} := P^0[B_{i_1,\ldots,i_n}] \), assuming that none of these probabilities vanish. Let \( Z_n \) and \( Z_{n,\alpha} \) be \( \mathcal{F}_n \)-measurable functions defined by

\[
Z_{\alpha}(i_1, \ldots, i_n) := \mathbb{I}_{i_1,\ldots,i_n} \prod_k p(i_k|\alpha), \quad Z_n := \sum_\alpha Q_\alpha(\alpha)Z_{\alpha}(\alpha),
\]

so that \( Q_\alpha(\alpha)/Q_0(\alpha) = Z_{\alpha}(\alpha)/Z_n \). It is easy to check that each \( Z_{\alpha}(\alpha) \) is a \( P^0 \)-martingale,

\[
E_0[Z_{\alpha}(\alpha)|\mathcal{F}_{n-1}] = Z_{n-1}(\alpha).
\]

As is clear from their definition, the \( Z_{\alpha}(\alpha) \) are the Radon–Nikodym derivatives of the measures \( P_\alpha \) with respect to \( P^0 \) on \( \mathcal{F}_n \)-measurable functions, that is

\[
E_\alpha[X] = E_0[Z_{\alpha}(\alpha)X],
\]

for any \( \mathcal{F}_n \)-measurable function \( X \). Of course, \( Z_n \) are the Radon–Nikodym derivatives of \( P \) with respect to \( P^0 \), i.e. \( E[X] = E_0[Z_nX] \) for any \( \mathcal{F}_n \)-measurable function \( X \). Choosing \( P^0 \) adequately helps in taking the continuous time limit, a fact that we shall illustrate below.

2.2.1. Brownian diffusive limit. The Brownian diffusive limit occurs when the conditioned probability \( p(i|\alpha) \) depends on an extra small parameter \( \delta \) such that

\[
p(i|\alpha) \approx_{\delta \to 0} p_0(i)(1 + \sqrt{\delta} \Gamma(i|\alpha) + \cdots),
\]

with all \( p_0(i) \) being non-vanishing and \( \alpha \)-independent. Since \( \sum_j p_0(i) = 1 \), the \( p_0(\cdot) \) define an \( \alpha \)-independent probability measure on \( I \). Note that \( \sum_j p(i|\alpha) \Gamma(j|\alpha) = 0 \) for all \( \alpha \) since \( \sum_j p(i|\alpha) = 1 \) for all \( \delta \). By the non-degeneracy assumption, the functions \( \Gamma(\cdot|\alpha) \) on \( I \) are all different.

The continuous time limit is then obtained by performing the scaling limit \( \delta \to 0 \), \( n \to \infty \) with \( t := n/\delta \) being fixed.

To understand the scaling limit of the counting processes \( N_n(i) \), let us look at its behaviour under \( P_\alpha \), i.e. for a system in the pointer state \( \alpha \) with the initial distribution \( Q_\alpha(\cdot) = \delta_{\alpha,\alpha} \). Then, by hypothesis, the output partial measurements are random independent variables, so that \( N_n(i) = \sum_{n=1}^n \mathbb{I}_{i_n=i} \) is the sum of \( n \) i.i.d. variables \( \xi_k(i) \) with value 1 (if the output of the \( k \)th partial measurement is \( i \)) with the probability \( p(i|\alpha) \) and zero (if the output of the \( k \)th partial measurement is different from \( i \)) with complementary probability. By the law of large numbers, the \( N_n(i) \) at large \( n \) become Gaussian processes with mean \( n p(i|\alpha) \) and covariance \( \min(n,m) \{ p(i|\alpha) \delta_{i,j} - p(i|\alpha) p(j|\alpha) \} \). Under these hypotheses, the probability \( \mathbb{P}_{n,m}(i) \) for the \( m \)th output partial measurement to be \( i \) is \( p(i|\alpha) \) for all \( m \), so that \( N_n(i) = n p(i|\alpha) \). Hence, under \( P_\alpha \) and for such peaked initial distribution, the law of the processes \( X_n(i) \) at large \( n \) is that of Gaussian processes with zero mean and covariance:

\[
E_n[X_n(i)X_n(j)] = \min(n,m) \{ p(i|\alpha) \delta_{i,j} - p(i|\alpha) p(j|\alpha) \}, \quad \text{for } Q_\alpha(\cdot) = \delta_{\alpha,\alpha}.
\]

After appropriate rescaling, this clearly admits a finite limit as \( \delta \to 0 \), which is \( \alpha \)-independent. Hence, under this hypothesis, \( X_n(i) \) admits a continuous time limit \( X(i) \), formally to be thought of as \( \lim_{\delta \to 0} \sqrt{\delta} X(\delta|\beta)(i) \). However, the previous equation is not enough to describe this limit under the law \( P \) and some care has to be taken; see [16].
So, let us define the scaling diffusive limit of the state distribution and the Doob martingales,

\[ Q_\delta(i) := \lim_{\delta \to 0} Q_{i/\delta}(\alpha), \quad X_t(i) := \lim_{\delta \to 0} \sqrt{\delta} X_t(i/\delta), \]

and of the counting processes,

\[ W_t(i) := \lim_{\delta \to 0} \sqrt{\delta} (N_{i/\delta}(i) - p_0(i)t/\delta). \]

These equalities have to be thought in law, but we shall still denote by \( \mathbb{P} = \sum_\alpha Q_\alpha(\alpha) \mathbb{P}_\alpha \) the probability measure for the continuous time processes. By construction, \( X_t(i) \) are \( \mathbb{P} \)-martingales.

The discrete difference equation (6) naïvely translates into the nonlinear stochastic equation for \( Q_\alpha(\alpha) \). Recall that, in the diffusive limit, \( p(i|\alpha) \simeq p_0(i)[1 + \sqrt{\delta} \Gamma(i|\alpha) + \cdots] \) as \( \delta \) tends to zero, so that \( \pi_{n-1}(i) \simeq p_0(i)[1 + \sqrt{\delta} (\Gamma(i)|_t) + \cdots] \) with \( (\Gamma(i)|_t) := \sum_\alpha \Gamma(i|\alpha) Q_\alpha(\alpha) \).

In the continuous time limit, equation (6) then becomes

\[ \frac{dQ_\alpha(\alpha)}{\alpha} = Q_\alpha(\alpha) \sum_t \left( \Gamma(i|\alpha) - \langle \Gamma(i)_t \rangle \right) dX_t(i) \tag{7} \]

with Itô’s convention. We used \( \sum_i X_t(i) = 0 \) and \( \sum_i p_0(i) \Gamma(i|\alpha) = 0 \) to take this limit. Remark that this equation preserves the normalization condition \( \sum_\alpha Q_\alpha(\alpha) = 1 \). This equation is that which governs the evolution of the system probability distribution under continuous Bayes’ updating in the diffusive limit. The random fields \( X_t(i) \) code for the information of the continuous time series of partial measurements. Not all of these fields are independent since \( \sum_i X_t(i) = 0 \). As we shall see, the main feature of the Brownian diffusive limit is that the \( X_t(i) \) are the Gaussian processes with zero mean and covariance:

\[ \mathbb{E}[X_t(i)X_t(j)] = \min(t,s) (p_0(i)\delta_{i,j} - p_0(i)p_0(j)), \tag{8} \]

Alternatively, the fields \( X_t(i) \) are zero mean Gaussian martingales with quadratic variation

\[ dX_t(i)dX_t(j) = (p_0(i)\delta_{i,j} - p_0(i)p_0(j)) \, dt, \]

which is of course compatible with the relation \( \sum_i X_t(i) = 0 \). Actually, the proofs of equation (7) and of the correctness of (8) are a bit tricky; see [16] for details. We shall here present an alternative less rigorous but quicker and simpler argument.

Let us now argue for equation (8). Recall the Doob decomposition of the counting process \( N_\alpha(i) = X_t(i) + A_\alpha(i) \). Its naïve scaling limit reads

\[ W_t(i) = X_t(i) + \int_0^t ds p_0(i) \langle \Gamma(i)_t \rangle, \]

whose infinitesimal differential form is

\[ dW_t(i) = dX_t(i) + p_0(i) \langle \Gamma(i)_t \rangle \, dt. \tag{9} \]

Contrary to the \( X_t(i) \), the \( W_t(i) \) are not \( \mathbb{P} \)-martingales but they are globally defined and independent of the initial distribution \( Q_\alpha(\alpha) \) because they are defined as limit of the counting process. We however know that, under \( \mathbb{P}_\alpha \), the \( W_t(i) \) are Gaussian processes with mean and covariance:

\[ \mathbb{E}[W_t(i)] = tp_0(i) \langle \Gamma(i)_t \rangle, \]
\[ \text{Cov}_{\alpha}[W_t(i)W_t(j)] = \min(t,s)(p_0(i)\delta_{i,j} - p_0(i)p_0(j)). \]

We would now like to use this information to read off properties of the martingales \( X_t(i) \). The key point consists in using Girsanov’s theorem [19]. Recall that \( Q_{\infty}(\alpha) \) may be thought of as the Radon–Nikodym derivative of \( \mathbb{P}_\alpha \) with respect to \( \mathbb{P} \) and that \( Q_\alpha(\alpha) = \mathbb{E}[Q_{\infty}(\alpha) | \mathcal{F}_t] \). Assume
for a while that the $X_t(i)$ are the Gaussian processes under $\mathbb{P}$ with zero mean and the quadratic variation $G(i, j)dt := \langle dX_t(i), dX_t(j) \rangle$ is to be determined. Girsanov’s theorem tells us that modifying the measure $\mathbb{P}$ by multiplying with the martingale $Q_t(\alpha)$ adds a supplementary drift in the stochastic differential equation (9), given by the logarithmic derivative of the martingales. In the present case, using equation (7), Girsanov’s theorem implies that
\[
\tilde{d}W_t(i) = \tilde{d}X_t(i) + p_0(i)(\Gamma(i)\tilde{\alpha})dt + \sum_j G(i, j)(\Gamma(j|\alpha) - \langle \Gamma(j) \rangle)\,dt,
\]
with $\tilde{X}_t(i)$ being Gaussian processes under $\mathbb{P}_\alpha$ with zero mean and identical quadratic variation $G(i, j)\,dt$. Comparing now with the known properties of $W_t(i)$ under $\mathbb{P}_\alpha$, mentioned above, we deduce that $G(i, j) = (p_0(i)\delta_{i,j} - p_0(i)p_0(j))$, as claimed, so that the previous equation reduces to
\[
\tilde{d}W_t(i) = \tilde{d}X_t(i) + p_0(i)\Gamma(i|\alpha)\,dt,
\]
under $\mathbb{P}_\alpha$, as required. This ends our argument for equations (7) and (8).

A way to rigorously construct processes with all the above properties is to deform a suitable a priori measure $\mathbb{P}^0$. Details have been given in [16]. In this paper, we shall illustrate this strategy in the Poissonian case.

Equation (7) may actually be integrated explicitly; see [16]. Furthermore, as bounded martingales the $Q_t(\alpha)$ again converge almost surely and in $L^1$. Under the non-degeneracy assumption that all $\Gamma(\cdot|\alpha)$ are different, the limit distribution is peaked, $Q_\infty(\cdot) = \delta_{i;\gamma_0}$, at a random target pointer state. The convergence is still exponential.

2.2.2. Poissonian jump limit. The Poissonian limit occurs when the conditioned probabilities $p(i|\alpha)$ vanish as a small parameter $\delta$ vanishes. Not all $p(i|\alpha)$ may vanish simultaneously as they sum up to 1. So let us single out one value $i^*$ for which $p(i^*|\alpha)$ tends to 1 as $\delta \to 0$ for all $\alpha$ and assume that all other $p(i|\alpha)$ vanish in this limit:
\[
p(i^*|\alpha) \approx_{\delta \to 0} 1, \quad p(i|\alpha) \approx_{\delta \to 0} \delta \theta(i|\alpha) \quad \text{for } i \neq i^*, \quad \forall \alpha.
\]
By consistency $p(i^*|\alpha) = 1 - \delta(\sum_{i \neq i^*} \theta(i|\alpha)) + o(\delta)$ and all $\theta(i|\alpha)$ are positive and assumed to be non-vanishing. In the limit $\delta \to 0$, the output of the partial measurements is most frequently $i^*$ with sporadic jumps to another value $i$ different from $i^*$.

The continuous time limit is obtained by performing the scaling limit $\delta \to 0$, $n \to \infty$ with $t = n/\delta$ being fixed.

To understand the continuous time limit of the counting processes $N_t(i)$, let us again look at the behaviour under $\mathbb{P}_\alpha$. As before, the output partial measurements are then random independent variables so that $N_t(i) = \sum_{n=1}^N \zeta_{n(i)}$ is the sum of $n$ i.i.d. variables $\epsilon_k(i)$ having value 1 with probability $p(i|\alpha)$ and zero with complementary probability. Let us first consider $i \neq i^*$ and compute $\log E_\alpha[e^{N_t(i)}] = n \log(1 - p(i|\alpha) + e^\epsilon p(i|\alpha))$. In the scaling limit with $n = t/\delta$ and $p(i|\alpha) \approx \delta \theta(i|\alpha)$, we obtain
\[
\lim_{\delta \to 0} \log E_\alpha[e^{N_{t/\delta}(i)}] = \theta(i|\alpha)(e^\epsilon - 1), \quad i \neq i^*.
\]
Similar computations, based on the general formula
\[
E_\alpha\left[ e^{\sum_{l=1}^t \sum_i \epsilon_l(i)(N_{\alpha l}(i) - N_{\alpha l-1}(i))} \right] = \prod_{l=1}^k \left( \sum_i e^{\epsilon_l(i)p(i|\alpha)} \right)^{B_{t/\delta} - B_{t/\delta - 1}}.
\]

6 This expression for the quadratic variation of the $X_t(i)$ is compatible with the relation $\sum_i X_t(i) = 0$, since $\sum_i G(i, j) = 0$ as it should.

7 We may generalize this by assuming that more than one conditioned probability remains finite as $\delta$ tends to zero. In these cases, the continuous time limit will be a mixture between the Brownian and the Poissonian limits.
for \( k \geq 1 \), arbitrary non-decreasing sequences of integers \( 0 = n_0 \leq n_1 \leq \cdots \leq n_k \) of length \( k \), and arbitrary (complex) \( \gamma_k(i) \), show that, in the limit \( \delta \to 0 \), the \( \mathbb{P}_\alpha \)-distributions of the counting processes \( N_{(i)}(t) \), \( i \neq i^* \), converge to those of independent Poisson point processes with intensities \( \theta(i)\alpha \) dt. Note that this statement is true under \( \mathbb{P}_\alpha \) but not under \( \mathbb{P} \). However, we can compute their \( \mathbb{P} \)-generating functions using the decomposition of the measure \( \mathbb{P} = \sum_{\alpha} Q_\alpha(\mathbb{P}_\alpha) \). For instance

\[
\mathbb{E}[e^{N_{(0)}(1)}] \to 0 \sum_{\alpha} Q_\alpha(\mathbb{P}_\alpha) e^{\theta(i)\alpha(1)} - 1, \quad i \neq i^*.
\]

The properties of \( N_n(i^*) \) and their limits are reconstructed using the sum rule, \( \sum_i N_n(i) = n \).

In particular, for small \( \delta \), \( N_{(i)}(t^*) \approx t/\delta \) up to order 1 random corrections.

So, let us define the scaling Poisson limits of the state distribution and of the Doob martingales \( X_n(i) \)

\[
Q_{(i)}(\alpha) := \lim_{\delta \to 0} Q_{(i)}(\alpha), \quad Y_t(i) := \lim_{\delta \to 0} X_{t}(i)
\]

and of the jump counting processes

\[
N_n(i) := \lim_{\delta \to 0} N_{(i)}(t^*), \quad \text{for } i \neq i^*,
\]

and \( M_i(i^*) := \lim_{\delta \to 0} \left( N_{(i)}(t^*) - i/\delta \right) \). Again, these equalities have to be thought in law, but we still denote by \( \mathbb{P} = \sum_{\alpha} Q_\alpha(\mathbb{P}_\alpha) \) the probability measure for the time continuous processes. By construction, the martingales \( Y_t(i) \) sum up to zero, i.e. \( \sum_i Y_t(i) = 0 \), and have zero mean, i.e. \( \mathbb{E}[Y_t(i)] = 0 \). Similarly, \( M_i(i^*) + \sum_{i \neq i^*} N_n(i) = 0 \).

Again, the naïve scaling limit of the difference equation (6) yields a stochastic equation for the system-state distribution. In the Poissonian limit, one has \( p(i|\alpha) \simeq \delta \theta(i)\alpha + \cdots \) for \( i \neq i^* \) as \( \delta \to 0 \), so that \( \pi_{n-1}(i) \simeq \delta \theta(i)\alpha \), \( \cdots \) with \( \theta(i)\alpha := \sum_i \theta(i)\alpha Q_\alpha(\alpha) \), for \( i \neq i^* \), whereas both \( p(i^*|\alpha) \) and \( \pi_{n-1}(i^*) \) approach 1 as \( \delta \) tends to zero. The continuous time limit of equation (6) is then

\[
dQ_t(\alpha) = Q_t(\alpha) \sum_{i \neq i^*} \left( \frac{\theta(i)\alpha}{\theta(i)\alpha} - 1 \right) dY_t(i), \quad (10)
\]

where we used \( dY_t(i^*) = -\sum_{i \neq i^*} dY_t(i) \) to deal with the term associated with \( i^* \) in equation (6). As we shall show just below, the \( Y_t(i) \) are related to the counting processes by

\[
dN_t(i) = dY_t(i) + \theta(i)\alpha dt, \quad i \neq i^*. \quad (11)
\]

We shall furthermore argue that the processes \( dN_t(i), i \neq i^* \), are point processes with intensities \( \theta(i)\alpha \) dt. This intensity is sample dependent—a point that we shall explain—but predictable. Equations (10) and (11) are those which govern the evolution of the system probability distribution under continuous Bayes’ updating in the Poissonian limit. The random counting processes \( N_n(i) \) code for the information on the continuous time series of partial measurements.

Let us now argue for equation (11). Consider again the Doob decomposition \( N_n(i) = X_n(i) + A_n(i) \). Because \( \pi_{(i)}(i^*) \simeq \delta \theta(i)\alpha \), for small \( \delta \), its naïve scaling reads

\[
N_n(i) = Y_t(i) + \int_0^t dx\theta(i)\alpha, \quad i \neq i^*.
\]

Its infinitesimal version is equation (11), as announced. Since \( p(i^*|\alpha) \simeq 1 - \delta \sigma(i^*|\alpha) \) with \( \sigma(i^*|\alpha) := \sum_i \sigma (i^*|\alpha) i^* \), the counting function \( N_n(i^*) \) slightly deviates from \( n \), and \( M_i(i^*) = Y_t(i^*) - \int_0^t dx \sigma(i^*) i^* \) with \( \sigma(i^*) i^* := \sum_i \sigma (i^*|\alpha) Q_\alpha(\alpha) \).

By the martingale property, \( \mathbb{E}[dN_t(i)|F_t] = \theta(i)\alpha dt, \quad i \neq i^* \).
That is, the number of jumps in the direction $i$ in the time interval $[t, t+dr]$ depends on the past of the process and is equal to $\langle \theta(i) \rangle_i$, $dr$ in mean. We may go a little further and compute the generating function of those jumps. Indeed, since the conditional measure $E[\cdot|\mathcal{F}_t]$ decomposes as a sum, i.e. $E[\cdot|\mathcal{F}_t] = \sum_\alpha Q_\alpha(\alpha)E_\alpha[\cdot|\mathcal{F}_t]$, and since under $E_\alpha[\cdot|\mathcal{F}_t]$ the $dN(t)$ are Poisson point processes with the intensity $\langle \theta(i)|\alpha \rangle dr$, we have
\[
\log E[e^{\sum_{\alpha}^{}dN(t)|\mathcal{F}_t}] = dr \langle \theta(i) \rangle_i (e^r - 1),
\]
with $\langle \theta(i) \rangle_i = \sum_\alpha Q_\alpha(\alpha)\theta(i|\alpha)$. That is, under $\mathbb{P}$, the $dN(t)$ are point processes with intensities $\langle \theta(i)|\alpha \rangle$, as announced. As above, a similar computation shows that the $dN(t)$ for fixed $i$, are independent variables for $i \neq j$ under $\mathbb{P}_\omega$ but not under $\mathbb{P}$. An alternative description of this limit is given in [17]; see also the forthcoming [21].

Up to now, our arguments have been only in law. A rigorous construction\footnote{This is an alternative to [17] in that case.} of processes, living on a well-defined probability space, and having all the required properties, is to deform a suitable $a \text{ priori}$ measure $\mathbb{P}^0$. The hint that this is possible is the formula for $Q_\alpha(\alpha)$ obtained by taking the continuous time limit of equation (4). There are some cancellations of powers of $\delta$ between numerator and denominator yielding
\[
Q_\alpha(\alpha) = Q_0(\alpha) \frac{Z(\alpha)}{Z_\alpha}, \quad \text{with } Z_i := \sum_\beta Q_\beta(\beta)Z_i(\beta),
\]
where
\[
Z_i(\alpha) := \prod_{j \neq i^*}^{1\ldots\alpha} \theta(i|\alpha)^{N(j)} e^{-\sum_{\alpha}^{1\ldots\alpha} \theta(i|\alpha)^{-1}}. \quad (13)
\]
One recognizes $Z_i(\alpha)$ as the standard exponential Poisson martingale. So, let us start from an $a \text{ priori}$ probability measure $\mathbb{P}^0$ accommodating for independent Poisson processes $N_i(t)$, $i \neq i^*$, of intensity $dr$. Define $\mathbb{P}_\alpha := Z(\alpha)^{-1}\mathbb{P}^0$ on $\mathcal{F}_t$. Then, under $\mathbb{P}_\alpha$, the $N_i(t)$ are independent Poisson processes with the intensity $\theta(i|\alpha)dr$. Defining
\[
\mathbb{P} := \left( \sum_\alpha Q_0(\alpha)Z(\alpha) \right)^{-1}\mathbb{P}^0,
\]
it is plain that the $Q_\alpha(\alpha)$ are $\mathbb{P}$-martingales and the $N_i(t)$ have the law we were after. For instance, $Q_\alpha(\alpha) = Q_0(\alpha)Z(\alpha)/Z_\alpha$, we have $E[\cdot|\mathcal{F}_t] = \sum_\alpha Q_\alpha(\alpha)E_\alpha[\cdot|\mathcal{F}_t]$ so that $dN_\alpha := N_{\alpha+1}(t) - N_\alpha(t)$ is at most 1 and $\mathbb{P}[dN_i(t) = 1|\mathcal{F}_t] = dt \langle \theta(i) \rangle_i$.

3. Iterated indirect quantum measurements

Although purely probabilistic—involving classical probability only—the previous description of iterated stochastic measurements finds applications in the quantum world, in particular in the framework of repeated indirect non-demolition measurements [5]. Recall that an indirect quantum measurement consists in letting a quantum system interact with another quantum system, called the probe, and implementing a direct Von Neumann measurement on the probe. One then gains information on the system because the probe and the system have been entangled. Repeating the cycle of entanglement and measurement progressively increases the information on the system as in the model apparatus that we described above.

Let $S$ be the quantum system and $\mathcal{H}_s$ be its Hilbert space of states. Pick a basis of states $\{|i\rangle \}$ in $\mathcal{H}_s$, which are going to play the role of pointer states. Let $P$ be the probe and $\mathcal{H}_p$ be its Hilbert space. We assume that the probe–system interaction preserves the pointer states: a system initially prepared in one of the pointer states remains in this state after having interacted
with the probes. This requires a peculiar form for the unitary operator $U$ of the probe–system interaction:

$$
U = \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes U_{\alpha},
$$

with $U_{\alpha}$ being an unitary operator on $\mathcal{H}_{p}$. Alternatively, $U|\alpha\rangle \otimes |v\rangle = |\alpha\rangle \otimes U_{\alpha}|v\rangle$ for any $|v\rangle \in \mathcal{H}_{p}$, a property coding for the fact that the pointer states $|\alpha\rangle$ are preserved by this interaction.

We imagine sending identical copies of the probe, denoted $P_{1}, P_{2}, \ldots$, one after the other through the system and measuring an observable on each probe after the interaction. We assume that the in-going probes have all been prepared in the same state $|\psi\rangle \in \mathcal{H}_{p}$, and that the observables measured in the out-going channel are all identical with non-degenerate spectrum $I$. Let $\{|i\rangle\} \in \mathcal{H}_{p}$, $i \in I$, be the basis of eigenstates of the measured observable. We denote by $i_{k}$ the output of the measurement on the $k$th out-going probe. In analogy with previous section, we call the cycle entanglement and measurement on a probe a partial measurement. The results of repetitions of these cycles of partial measurements will be called a complete measurement.

The unitary operator $U$ codes for the probability of measuring a given value $i$ on the out-going probe. Suppose that the in-going probe has been prepared in the state $|\psi\rangle$ and the system $S$ in the state $|\alpha\rangle$. After interaction, the system–probe state is $|\alpha\rangle \otimes U_{\alpha}|\psi\rangle$ and the probability to measure the value $i$ of the probe observable is

$$
p(i|\alpha) := |\langle i|U_{\alpha}|\psi\rangle|^{2},
$$

by the rule of quantum mechanics. So $|\langle i|U_{\alpha}|\psi\rangle|^{2}$ is the probability to measure $i$ in the out-going channel conditioned on the system state being $|\alpha\rangle$. The analogy with the previous section should start to become clear.

### 3.1. Discrete time description

Let $\rho$ be the system density matrix. The system-state probability distribution is $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle$. The aim of this section is to describe how the system-state distribution and the density matrix evolve when the cycles of entanglement and measurement are repeated, and to make explicit contact with previous sections.

Assume that the system is initially prepared in a density matrix state $\rho_{0}$, and let us look at what happens during a cycle of entanglement and interaction. Recall that the probe is assumed to be prepared in the density matrix state $|\psi\rangle\langle\psi|$. After interaction, the joint system–probe density matrix is $U\rho_{0} \otimes |\psi\rangle\langle\psi|U^{\dagger}$. The observable, with spectrum $I$, is then measured on the probe. If $i_{1}$ is the output value of this measurement, the joint system–probe state is projected into $\rho_{1} \otimes |i_{1}\rangle\langle i_{1}|$ with

$$
\rho_{1} := \frac{1}{\pi_{0}(i_{1})} (i_{1}|U|\psi\rangle\rho_{0}\langle\psi|U^{\dagger}|i_{1}\rangle).
$$

This occurs with the probability $\pi_{0}(i_{1}) = \text{Tr} ( (i_{1}|U|\psi\rangle \rho_{0} \langle\psi|U^{\dagger}|i_{1}\rangle )$. Using the assumed property of $U$, equation (14), this can be rewritten as

$$
\pi_{0}(i) := \text{Tr} ( (i|U|\psi\rangle \rho_{0} \langle\psi|U^{\dagger}|i\rangle ) = \sum_{\alpha} p(i|\alpha)Q_{0}(\alpha).
$$

How this cycle is to be repeated is clear. Let $\rho_{n-1}$ be the system density matrix after the $n - 1$ first partial measurements—this density matrix depends on the random values of these measurements, so that $\rho_{n-1} = \rho_{n-1}(i_{1}, \ldots, i_{n-1})$, but we simplify the notation by not writing explicitly the values of the measurements. We let the system interact with the $n$th probe and
do a measurement on this probe. If $i_n$ is the output value of this $n$th partial measurement, the system state is projected into

$$\rho_n = \frac{1}{\pi_n - 1 (i_n)} |i_n\rangle\langle U| \rho_{n-1} \langle \psi | U^\dagger |i_n\rangle,$$

(15)

where again we simplified the notation by not writing the values of the partial measurements—$ho_n$ should have been written as $\rho_n(i_n|1, \ldots, i_{n-1})$ and similarly for $\pi_n - 1$. This projection occurs with the probability $\pi_n - 1 (i_n)$, with

$$\pi_n - 1 (i) := \text{Tr}(i|U| \rho_{n-1} \langle \psi | U^\dagger |i\rangle) = \sum_\alpha p(i|\alpha) Q_{n-1} (\alpha).$$

The diagonal matrix elements of the density matrix are the probabilities for the system being in a pointer state, i.e. $Q_n (\alpha) = \langle \alpha | \rho_n | \alpha \rangle$. From equation (15), we read that

$$Q_n (\alpha) = \frac{p(i|\alpha) Q_{n-1} (\alpha)}{\pi_n - 1 (i_n)}.$$

The two above equations exactly coincide with equations (2) and (3) defining iterated stochastic measurements. So everything we wrote in the previous sections applies. In particular, the collapse of the system probability distribution is a discrete implementation of the wavefunction collapse in the Von Neumann measurement. The quantum system observable measured by the iteration of cycles of entanglement and indirect measurement is that with the eigenstate basis $|\alpha\rangle$. The collapse happens only for an infinite sequence of partial measurements reflecting the fact that the iterated stochastic measurement apparatus is macroscopic only if an infinite sequence of partial measurements is implemented; see [6].

### 3.2. Continuous time limit

The aim of this section is to take the continuous time limit of the discrete recurrence equation (15) for the quantum density matrix using the results of the previous section. Doing this we will make contact with the so-called Belavkin’s equations [14], describing continuous time measurements in quantum mechanics, which are nonlinear stochastic Schrödinger equations [20].

The small parameter $\delta$ is the time duration of the system–probe interaction, so that the unitary operator is $U = \exp(-i \delta H)$, with $H$ being the system–probe Hamiltonian$^9$. As is well known, the dynamics of a quantum system under continuous measurements is frozen by continuous wave packet reductions, a fact named the quantum Zeno effect. To avoid it, we have to rescale the system–probe interaction and at the same time we decrease the interaction time duration. So we assume the following form of the Hamiltonian $H$:

$$H = H_s \otimes 1 + 1 \otimes H_p + \frac{1}{\sqrt{\delta}} H_I,$$

(16)

where $H_s$ is the system Hamiltonian, $H_p$ is the probe Hamiltonian and $H_I$ is the interaction Hamiltonian.

For the pointer state to be stable under the action of $U = e^{-i \delta H}$, equation (14), we should assume that $H_s$ is diagonal in the pointer basis, $H_s = \sum_\alpha |\alpha\rangle E_\alpha \langle \alpha|$, for some energies $E_\alpha$—this is linked to the non-demolition character of the measurement—and that

$$H_I = \sum_\alpha |\alpha\rangle \langle \alpha| \otimes H_a,$$

$^9$ We use the notation $i$, without a dot, to code for the square root of $-1$. 

13
with $H_\alpha$ acting on $\mathcal{H}_\rho$ but $\alpha$ dependent. The conditioned probabilities $p(i|\alpha)$ are then

$$p(i|\alpha) = |\langle i|\psi \rangle|^2 - \sqrt{\delta} \langle i|H_\alpha|\psi \rangle + \cdots,$$

so that the Brownian diffusive limit occurs when $\langle i|\psi \rangle \neq 0$ and the Poissonian jumpy case to $\langle i|\psi \rangle = 0$.

In both cases, the continuous time limit is then obtained by performing the scaling limit $\delta \to 0$, $n \to \infty$ with $t := n/\delta$ fixed as above.

It is useful to recast the quantum recursion relation (15) into a difference equation. This simplifies matter when taking the continuous time limit. Let us write $\rho_n = \rho_0 + \sum_i \rho_n(i_n) \mathbb{1}_{i_n=i}$ with $\rho_n(i_n)$ being the same as defined in equation (15). Recall that this scaling limit consists in $\delta t \to 0$.

It is then a simple matter to naively take the continuous time limit of the difference $\langle \mathcal{H}_\alpha \rangle$ to the assumed condition $\langle \mathcal{H}_\alpha \rangle = 0$. This leads to the Doob decomposition of the difference $\rho_n - \rho_{n-1}$ as

$$\rho_n - \rho_{n-1} = (D\rho)_{n-1} + (\Delta\rho)_n,$$

with $(D\rho)_{n-1} := \mathbb{E}[\rho_n|\mathcal{F}_{n-1}] - \rho_{n-1}$, which is $\mathcal{F}_{n-1}$-measurable, and $(\Delta\rho)_n := \rho_n - \mathbb{E}[\rho_n|\mathcal{F}_{n-1}]$, which satisfies $\mathbb{E}[\rho_n|\mathcal{F}_{n-1}] = 0$. Explicitly,

$$(D\rho)_{n-1} = \sum_i \langle \mathcal{H}_\alpha \rangle_{n-1} \langle \mathcal{H}_\alpha \rangle_i - \rho_{n-1},$$

$$(\Delta\rho)_n = \sum_i \langle \mathcal{H}_\alpha \rangle_{n-1} \langle \mathcal{H}_\alpha \rangle_i - \rho_{n-1},$$

where we used $\mathbb{1}_{i_n=i} - \mathbb{1}_{i_n=i_{n-1}} = X_n(i) - X_{n-1}(i)$, as in previous section. In the continuous time limit, the first term $(D\rho)_{n-1}$ is going to converge towards the drift term and the second one $(\Delta\rho)_n$ to the noisy source of the stochastic differential equation.

3.2.1. Brownian diffusive limit. The Brownian diffusive limit occurs when $\langle i|\psi \rangle \neq 0$ for all $i$. Then, $p(i|\alpha) \simeq p_0(i) (1 + \sqrt{\delta} \Gamma(i|\alpha) + \cdots)$ for $\delta$ small with

$$p_0(i) = |\langle i|\psi \rangle|^2, \quad \Gamma(i|\alpha) = 2\text{Im} \left( \frac{\langle i|H_\alpha|\psi \rangle}{\langle i|\psi \rangle} \right).$$

This is the situation we encountered in the previous section on the classical diffusive limit so that we can borrow all results obtained there.

It is then a simple matter to naively take the continuous time limit of the difference equations (17). This limit exists only if $\langle \mathcal{H}_\alpha |\psi \rangle = 0$, which is equivalent to

$$\langle \mathcal{H}_\alpha |\psi \rangle = 0 \quad \text{for all } \alpha,$$

a criterion which we assume to hold true. Recall that this scaling limit consists in $\delta \to 0$ with $t = n\delta$ fixed. Let us first expand the term $(D\rho)_{n-1}$ in powers of $\sqrt{\delta}$. The term of order $\sqrt{\delta}$ vanishes due to the condition $\langle \mathcal{H}_\alpha |\psi \rangle = 0$, and for the term of order $\delta$, we obtain $(D\rho)_{n|\delta} \simeq \delta L_d(\rho) \delta$, with the Linbladian

$$L_d(\rho) := -i[H_\alpha, \rho] + \sum_i \rho_0(i) \left( \hat{C}_i \rho C_i^\dagger - \frac{1}{2} \{C_i^\dagger C_i, \rho\} \right),$$

where we defined the operators $C_i$ acting on $\mathcal{H}_\alpha$ by $C_i := -\frac{i}{\langle i|\psi \rangle} \langle i|\mathcal{H}_\alpha|\psi \rangle$, or equivalently

$$C_i := -t \sum_{\alpha} \langle \alpha| \langle i|\mathcal{H}_\alpha|\psi \rangle (\langle i|\psi \rangle \rangle, \langle \alpha|, \langle \alpha|,$$

using the decomposition of $H_\alpha$ on pointer states. Remark that $\sum_i \rho_0(i) C_i = 0$ thanks to the assumed condition $\langle \mathcal{H}_\alpha |\psi \rangle = 0$. Similarly, expanding the term $(\Delta\rho)_n$ using
\[ \pi_{n-1}(i) \simeq p_0(i)[1 + \delta \text{Tr}((C_i + C_i^\dagger)\rho_i)] + \cdots, \]

we obtain \( \lim_{\delta \to 0} (\Delta \rho)_{i\beta} = \sum_j D_j(\rho_i) \, dX_j(i) \), with

\[ D_j(\rho_i) := C_j \rho_i + \rho_i C_j^\dagger - \rho_i \text{Tr}[(C_j + C_j^\dagger)\rho_i]. \]

Note that computing these limits only uses the decomposition (16) of the Hamiltonian \( H \) and not the existence of a preferred pointer state basis\(^{10}\). Gathering shows that the Brownian limit of the difference equation (15) is

\[ d\rho_t = L_\rho(\rho_t) \, dt + \sum_j D_j(\rho_t) \, dX_j(j), \quad (18) \]

where the \( X_j(j) \) are the Gaussian centred processes, with the quadratic variation

\[ dX_j(i) \, dX_k(j) = (p_0(i) \delta_{i,j} - p_0(i)p_0(j)) \, dt, \]

defined in equation (8) and in the discussion around this equation. This is an example of the diffusive Belavkin’s equation [14, 18]. It is important to recall that \( \sum_i p_0(i)C_i = 0 \) since, without this condition, but with \( \sum_i X_i(i) = 0 \) as we do have, equation (18) would not be positivity preserving [18]. Contact with previous sections can be made explicit by recalling that the state probability distribution is \( Q_i(\alpha) = \langle \alpha | \rho_i | \alpha \rangle \) and by noting that \( \text{Tr}[(C_i + C_i^\dagger)\rho_i] = \langle \Gamma^i(i) \rangle \). We only took a naïve limit of the difference equation (15); to mathematically prove the diffusive Belavkin’s equation for the system density matrix in the scaling limit would require more delicate arguments.

### 3.2.2. Poissonian jumpy limit

The Poissonian limit occurs when \( \langle i | \psi \rangle = 0 \). This cannot happen for all \( i \) as \( \langle i | \psi \rangle \) forms an orthonormal basis of \( \mathcal{H}_\rho \) and \( | \psi \rangle \) is non-zero. So, we assume, for simplicity, that one element of this basis is \( | \psi \rangle \), say \( | i^* \rangle = | \psi \rangle \), and all others are orthogonal to \( | \psi \rangle \), i.e. \( \langle i | \psi \rangle = 0 \) for all \( i \neq i^* \). Then, \( p_0(i^*|\alpha) \simeq_{\delta \to 0} 1 \) and \( p_0(i|\alpha) \simeq_{\delta \to 0} \delta \theta(i|\alpha) \), for \( i \neq i^* \), with

\[ \theta(i|\alpha) = |\langle i | H_\alpha | \psi \rangle|^2. \]

This is the situation that we encountered in the previous section on the classical Poisson jumpy limit, so that we can borrow all results obtained there.

As in the diffusive case, it is a simple matter to naïvely take the continuous time limit of the difference equation (17). This only uses the decomposition of the Hamiltonian \( H \) but the limit exists only if \( \langle \psi | H | \psi \rangle = 0 \), and we assume this to be true. Expanding the first term \( (D\rho)_{n-1} \) to second order in \( \sqrt{\delta} \), we obtain \( (D\rho)_{i\beta} \simeq_{\delta \to 0} L_\rho(\rho_t) \delta \), with the Linbladian

\[ L_\rho(\rho_t) := -i[H_\rho, \rho_t] + \sum_{\alpha \neq \beta} \left( D_\alpha \rho_t D_\beta^\dagger - \frac{1}{2} [D_\alpha D_\beta^\dagger, \rho_t] \right), \]

where we defined the operators \( D_\alpha := -t \langle i | H_\alpha | \psi \rangle \) acting on \( \mathcal{H}_\rho \):

\[ D_\alpha := -t \sum_\alpha | \alpha \rangle \langle i | H_\alpha | \psi \rangle | \alpha \rangle. \]

To compute the limit of the second term \( (\Delta \rho)_n \), we note that, to leading order in \( \delta \), \( \langle i | U | \psi \rangle \rho \langle \psi | U^\dagger \rangle \simeq \delta D_\alpha D_\alpha^\dagger \) and \( \pi_{n-1}(i) \simeq \delta \text{Tr}[D_i \rho_i D_i^\dagger] \) for \( i \neq i^* \), and we obtain \( \lim_{\delta \to 0} (\Delta \rho)_{i\beta} = \sum_{\alpha \neq \beta} \tilde{D}_\alpha(\rho_t) \, dY_\beta(i) \), with

\[ \tilde{D}_\alpha(\rho_t) := \frac{D_\alpha \rho_t D_\alpha^\dagger}{\text{Tr}[D_i \rho_i D_i^\dagger]} = \rho_t, \]

\(^{10}\) The existence of the pointer state basis was however used in determining the statistical properties of the fields \( X_i(i) \).
where the last term $-\rho_t$ comes from using $dY_t(i^*) = -\sum_{i \neq i^*} dY_t(i)$ when computing the contribution of the $i^*$-term in $(\Delta \rho)_n$, as in the previous section. Gathering shows that the Poissonian limit of the difference equation (15) is

$$d\rho_t = L_p(\rho_t) \, dt + \sum_{i \neq i^*} D_i(\rho_t) \, dY_t(i), \quad (19)$$

where the $Y_t(j)$ are the Poisson-like compensated martingales defined in equation (11). That is,

$$dN_t(i) = dY_t(i) + \text{Tr}[D_i \rho_t D_i^\dagger] \, dt, \quad i \neq i^*,$$

where the $dN_t(i)$ are the point processes with the intensities $\langle \theta(i) \rangle_t \, dt$ defined in the previous section; see, e.g., equations (12) and (13). Note that using the decomposition of the interaction Hamiltonian $H_I$ on the pointer state basis, i.e., $H_I = \sum_\alpha |\alpha\rangle \langle \alpha| \otimes H_\alpha$, we have

$$\text{Tr}[D_i \rho_t D_i^\dagger] = \sum_\alpha \theta(i|\alpha) Q_i(\alpha) = \langle \theta(i) \rangle_t,$$

so that the previous equation indeed coincides with equation (11). Equation (19) is an example of a jumpy Belavkin’s equation [14]. Let us finally point out that the stochastic processes (18) and (19) are not the most general because we assumed that they preserve the pointer state basis\textsuperscript{11} so that the operators $H_s, C_i$ or $D_j$ are diagonal in the pointer basis. This is of course related to the non-demolition property. Equations (18) and (19) are also peculiar examples of a more general class of models for continuous quantum measurements whose long time behaviour leads to purification of mixed states; see, e.g., [22, 18].

**Acknowledgments**

This work was in part supported by ANR contracts ANR-2010-BLANC-0414.01 and ANR-2010-BLANC-0414.02.

**References**

[1] Box G E P and Tiao G C 1992 *Bayesian Inference in Statistical Analysis* (New York: Wiley)
[2] Caves C M 1986 Quantum mechanics of measurements distributed in time. A path-integral formulation *Phys. Rev.* D 33 1643
[3] Guerlin C *et al* 2007 Progressive field-state collapse and quantum non-demolition photon counting *Nature* 448 889
[4] Devoret M and Martinis J M 2004 Implementing qubits with superconducting integrated circuits *Quantum Inf. Process.* 3 163
[5] Thorne K S *et al* 1978 Quantum nondemolition measurements of harmonic oscillators *Phys. Rev. Lett.* 40 667
[6] Unruh W G 1978 Analysis of quantum-nondemolition measurement *Phys. Rev.* D 18 1764
[7] Grangier P, Levenson J A and Poizat J P 1998 Quantum non-demolition measurements in optics *Nature* 396 537
[8] Bauer M and Bernard D 2011 Convergence of repeated quantum nondemolition measurements and wavefunction collapse *Phys. Rev. A* 84 0444103
[9] Wiseman H and Milburn G 2010 *Quantum Measurement and Control* (Cambridge: Cambridge University Press)
[10] Clerk A A *et al* 2010 Introduction to quantum noise, measurement, and amplification *Rev. Mod. Phys.* 82 1155
[11] Dalibard J, Castin Y and Molner K 1992 Wave-function approach to dissipative processes in quantum optics *Phys. Rev. Lett.* 68 580 (arXiv:0805.4002)
[12] Breuer H P and Petruccione F 2006 *The Theory of Open Quantum Systems* (Oxford: Oxford University Press)

\textsuperscript{11} We made this assumption when computing the probability distributions of the processes $X_t(i)$ and $Y_t(i)$ but not when formally taking the continuous time limit of the discrete equation (15).
Wiseman H M 1994 Quantum theory and continuous feedback Phys. Rev. A 49 2133
Barchielli A 1986 Measurement theory and stochastic differential equations in quantum mechanics Phys. Rev. A 34 1642
Belavkin V P 1989 A new wave equation for a continuous nondemolition measurement Phys. Lett. A 140 355
Belavkin V P 1990 A posteriori Schrödinger equation for continuous non demolition measurement J. Math. Phys. 31 2930
Belavkin V P 1992 Quantum continual measurements and a posteriori collapse on CCR Commun. Math. Phys. 146 611
Hewitt E and Savage J L 1955 Symmetric measures on Cartesian products Trans. Am. Math. Soc. 80 470
Bauer M, Benoist T and Bernard D 2012 Repeated quantum non-demolition measurements: convergence and continuous-time limit to appear Ann. H Poincaré at press (doi:10.1007/S00023-01-0204-x)
Pellegrini C 2008 Existence, uniqueness and approximation for stochastic Schrödinger equation: the diffusive case Ann. Prob. 36 2332
Pellegrini C 2010 Existence, uniqueness and approximation of the jump-type stochastic Schrödinger equation for two-level systems Stoch. Process. Appl. 120 1722
Barchielli A and Gregoratti M 2009 Quantum Trajectories and Measurements in Continuous Time: The Diffusive Case (Lecture Notes in Physics vol 782) (Berlin: Springer)
Jacod J and Protter Ph 2003 L’essentiel en Théorie des Probabilités (Paris: Cassini)
Kallenberg O 2000 Foundations of Modern Probability 2nd edn (Berlin: Springer)
Klemke A 2008 Probability Theory: A Comprehensive Course (Universitext) (Berlin: Springer)
Gisin N 1984 Quantum measurements and stochastic processes Phys. Rev. Lett. 52 1657
Diosi L 1988 Quantum-stochastic processes as models for state vector reduction J. Phys. A: Math. Gen. 21 2885
Pellegrini C and Benoist T 2012 in preparation
Barchielli A and Paganoni A M 2003 On the asymptotic behaviour of some stochastic differential equations for quantum states Infinite Dimens. Anal. Quantum Probab. Relat. Top. 6 223