SU(2) low energy quark effective couplings in weak external magnetic field

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In this work corrections to the usual flavor SU(2) Nambu-Jona-Lasinio coupling due to a weak external magnetic field are calculated by considering quark polarization in a (dressed) gluon exchange mechanism for quark interactions. The quark field is split into two components, one that condenses and another one that is a background field for interacting quarks, being the former integrated out. The resulting determinant is expanded for relatively large quark mass and small magnetic field, \((eB_0/M^2) < 1\) by resolving magnetic field dependent low energy quark effective interactions. Besides the corrections for the NJL and vector NJL effective couplings, different \(B_0\)-dependent effective couplings that break isospin and chiral symmetry emerge.

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I. INTRODUCTION

Quark interactions involve a large variety of different effects and mechanisms. To identify their particular roles in observables and to establish a realistic hierarchy among all of them for each of the processes under their conditions within the complexity of Quantum Chromodynamics (QCD) is a difficult task. High energy density (temperature and baryon density) systems are known to be suitable to test quark (and gluon) dynamics, from relativistic (heavy) ion collisions (r.h.i.c.) to several dense stars. Magnetic fields are also expected to be sizeable in such systems \([1-3]\) and, actually, they are expected to produce a large variety of effects not only in such high energy systems but also in the vacuum by means of phenomena such as the magnetic catalysis and the inverse effect at finite temperatures, for example Refs. \([2, 4-9]\), to produce changes in the CP violation phase transition \([10]\), the emergence of superconducting vacuum \([11]\) or chiral asymmetry/imbalance and the chiral magnetic effect \([12-14]\) among others. In particular, it has been argued that finite temperature inverse magnetic catalysis may be traced back to the chirality imbalance \([15]\). In the core of dense stars (magnetars) and in the early Universe magnetic fields are expected to be sizeable \([1-3]\), favoring or not chiral imbalance or vector condensation.

Even before the establishment of quantum electrodynamics, vacuum fluctuations for the electromagnetic field had already been calculated with the Euler-Heisenberg action \([31]\). With QED several approaches have been employed to derive effective actions or Hamiltonians for higher order contributions of the electromagnetic field firstly in the absence and then in the presence of matter, few examples are given in Refs. \([32, 33]\). For several strongly interacting systems where magnetic fields are sizeable and relevant it becomes interesting to investigate the vacuum polarization effects in the presence of magnetic fields. In this work, effective quark-quark interactions are calculated in the presence of constant weak magnetic field from vacuum polarization effects. The one loop background field method for quarks, as employed in Refs. \([35, 36]\), will be considered in the presence of a constant weak magnetic field from vacuum polarization effects. The one loop background field method for quarks, as employed in Refs. \([35, 36]\), will be considered in the presence of a constant weak magnetic field from vacuum polarization effects.

The departure point of the present work is the global color model (GCM) obtained by considering quark exchange corrected by gluon interactions and its non-Abelian character, i.e. it can be a realistic gluon propagator. It is given by \([35-40]\):

\[
S_{\text{eff}}[\bar{\psi}, \psi] = \int \left[ \bar{\psi} (i\partial - M) \psi - \frac{g^2}{2} \int \frac{d^4 x}{M^2} \int \frac{d^4 y}{M^2} (R_{\mu\nu}^{-1}(x-y))(\bar{\psi}_a(x)j^\mu_{\nu a}(x)) \right],
\]

(1)

Where the color quark current is \(j^\mu_{\nu a} = \bar{\psi}_a \gamma_\mu \gamma_5 \psi\), the sums in color, flavor and Dirac indices are implicit, the kernel \((R_{\mu\nu}^{-1})\) is the gluon propagator. Non abelian gluon interactions can be considered to dress the gluon exchange...
by considering a non-perturbative (realistic) gluon propagator that, together with the quark-gluon vertex, will be assumed to provide the strength for Dynamical Chiral Symmetry Breaking (DChSB). Several different effects are known to contribute to the strength of the quark-quark interaction above \[47\]. Therefore this calculation presents, in this sense, a similar level of approximation to the rainbow ladder approximation for the Schwinger-Dyson equations that yield DChSB \[38–41, 44–46\]. The quark gluon vertex was shown to depend on \(B_0\) \[22\] and this will not be considered in the present work. This model will be coupled to the electromagnetic field via the quark minimal coupling. To investigate the flavor structure of the model, one performs a Fierz transformation from which a NJL coupling emerges in the local limit, besides other structures. This work is organized as follows. In the next Section the Fierz transformation of this GCM interaction coupled (minimally) to a constant magnetic field is presented and the quark field is integrated out in the presence of background quark. In the following Section the determinant is assumed to provide the strength for Dynamical Chiral Symmetry Breaking (DChSB). Several simple ratios between the effective couplings in the limit of large quark effective mass are obtained. In the final Section a summary and discussion are presented.

II. QUARK COMPONENTS AND LIGHT MESON FIELDS

The generating functional to be considered is the following:

\[
Z[\xi, \tilde{\xi}] = N \int D[\bar{\psi}, \psi] \exp \left( \mathcal{L} + \bar{\psi} J + \psi \right),
\]

where the Lagrangian density for the minimal electromagnetic coupling (for a background electromagnetic field) to the global color model can be written as:

\[
\mathcal{L} = \bar{\psi} (i\gamma \cdot D - m) \psi - \frac{g^2}{2} \int_y \bar{\psi}(x)\gamma_{\mu} \lambda^b \psi(x) R^{\mu
u}_{bc}(x - y) \bar{\psi}(y) \gamma_{\nu} \lambda^c \psi(y),
\]

Where \(a, b, \ldots = 1, \ldots (N_f^2 - 1)\) stand for color in the adjoint representation, and \(i, j, k = 0, \ldots (N_f^2 - 1)\) will be used for SU(2) flavor indices, the sums in color, flavor and Dirac indices are implicit, and the covariant quark derivative is:

\[
D^\mu = \partial^\mu - i e Q_{ij} A^i \delta_{ij}.
\]

In several gauges, the gluon kernel can be written in terms of the transversal and longitudinal components as:

\[
R^{\mu\nu}(x - y) = \delta_{ab} \left( (g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{k^2}) R_T(x - y) + \frac{\partial^\mu \partial^\nu}{k^2} R_L(x - y) \right).
\]

The infrared regime of the gluon propagator exhibits a non trivial behavior that is often parameterized in terms of an effective gluon mass \[47\]. This will be discussed further in Section IV.

To make possible a more detailed investigation of the different flavor channels of quark interactions a Fierz transformation \[18, 45\] can be done next. Then for the quark interaction above:

\[
\Omega \equiv \frac{g^2}{2} \bar{\psi}(x)\gamma_{\mu} \lambda^b \psi(x) R^{\mu\nu}_{bc}(x - y) \bar{\psi}(y) \gamma_{\nu} \lambda^c \psi(y),
\]

the Fierz transformed \(\mathcal{F}(\Omega)\) color singlet expression is given by:

\[
\mathcal{F}(\Omega) = \alpha g^2 \left\{ \left[ j_S(x, y) j_S(y, x) + j_P^I(x, y) j_P^I(y, x) \right] R(x - y)
\right.
\]

\[
- \frac{1}{2} \left\{ \left[ j^{V,\mu}_I(x, y) j^{V,\mu}_I(y, x) - j^{V,\mu}_I(x, y) j^{V,\mu}_I(y, x) \right] \tilde{R}^{\mu\nu}(x - y) \right\},
\]

where \(\alpha = 8/9\) for SU(2) flavor, and the following bi-local quark bilinears \((j_S^q(x, y) = \bar{\psi}(x)\Gamma^q \psi(y))\) for operators \(\Gamma_q\) where \(q = s, p, v, a\) were defined:

\[
\begin{align*}
  j^S_S(x, y) &= \bar{\psi}(x)\psi(y), & j^P_S(x, y) &= \bar{\psi}(x)\sigma_i\gamma_5 \psi(y), \\
  j^{V,\mu}_S(x, y) &= \bar{\psi}(x)\gamma^\mu \sigma_i \psi(y), & j^{V,\mu}_I(x, y) &= \bar{\psi}(x) i\gamma_5 \gamma^\mu \sigma_i \psi(y).
\end{align*}
\]

In these expressions following kernels were used:

\[
R(x - y) \equiv R = 3 R_T(x - y) + R_L(x - y),
\]

\[
\tilde{R}^{\mu\nu}(x - y) \equiv \tilde{R}^{\mu\nu} = g^{\mu\nu} (R_T(x - y) + R_L(x - y)) + 2 \frac{\partial^\mu \partial^\nu}{k^2} (R_T(x - y) - R_L(x - y)).
\]
The long-wavelength or local limit of the scalar and pseudo-scalar interactions yields the NJL coupling with \( G \sim \frac{g^2}{\Lambda_{QCD}^2} \) or \( G \sim \frac{s^2}{M_Q^2} \) for massless and massive gluons [35, 49, 50].

The quark field will be split according to the Background Field Method (BFM) [51, 52]. One component is considered to be a (constituent quark) background field \( \psi_1 \), and the sea quark field \( \psi_2 \) will be integrated out. This splitting of the field can be made by means of the bilinears \( \bar{\psi} \Gamma \psi \) where \( \Gamma \) stands for Dirac, color or flavor operators, such that the resulting determinant corresponds basically to the one loop BFM results. The splitting can be written as [35, 36]:

\[
\bar{\psi} \Gamma^q \psi \rightarrow (\bar{\psi} \Gamma^q \psi)_2 + (\bar{\psi} \Gamma^q \psi)_1,
\]

where \((\bar{\psi} \psi)_2\) will be integrated out, being possible that it composes light mesons and the scalar condensate and the component \((\bar{\psi} \psi)_1\) stands for the background field that yields baryon constituent quarks. This separation preserves chiral symmetry, and it may not correspond to a simply mode separation of low and high energies which might be a very restrictive assumption since pions and constituent quarks might be composed by quarks with similar energy modes (fully or in part). Therefore it seems the criterium might not involve a separation scale and at the end, the shape of the results should be basically the same. The shift of bilinear may also be suitable for envisaging quark-anti-quark states which are the most important states for the very low energy QCD, i.e. below the nucleon mass shape of the results should be basically the same. The shift of bilinear may also be suitable for envisaging quark-anti-quark states which are the most important states for the very low energy QCD, i.e. below the nucleon mass scale.

The effective interaction \( \Omega \) is split accordingly and terms with mixed bilinear of \( \psi_1 \) and \( \psi_2 \) can be written such that the quadratic part of bilinear \( \psi_2 \psi_2 \) will be suitable to be integrated out. The interaction \( \Omega_2 \) deserves some more attention and it can be handled in two ways: (1) By resorting to a weak field approximation \( \Omega_2 \ll \Omega_1 \) which yields directly the one loop BFM that might receive corrections by a perturbative expansion which incorporates \( \Omega_2 \) [51]; (2) By making use of the auxiliary field method according to which a set of auxiliary fields is introduced by means of unitary functional integrals multiplying the generating functional [36, 38, 39, 53, 54]. Auxiliary fields (a.f.) allow to incorporate properly DChSB with the formation of the scalar quark condensate which endows quarks with a large effective mass. Therefore the use of the a.f. improves the one loop background field method as usually implemented. Auxiliary fields are introduced by multiplying the generating functional by the following normalized auxiliary fields:

\[
\int D[V_{\mu}] \int D[\bar{A}_\mu] e^{-\frac{i}{4} \int \bar{V}_{\mu} \Gamma_{\mu\nu} (\bar{V}_{\nu} - gj_{\nu}^{(2)}) + (\bar{A}_{\mu} - gj_{\mu}^{(2)})] e^{-\frac{i}{4} \int \bar{A}_{\mu} \Gamma_{\mu\nu} (\bar{A}_{\nu} - gj_{\mu}^{(2)})},
\]

In these expressions the bilocal a.f. are \( S(x, y), P(x, y), V_{\mu}(x, y) \) and \( \bar{A}_\mu(x, y) \) and they have been shifted by quark currents such as to cancel out the fourth order interactions \( \Omega_2 \). These shifts have unit Jacobian and they generate a coupling to quarks. The non locality of these auxiliary fields give rise to form factors which nevertheless can produce punctual meson fields by expanding in an infinite basis of local fields. Finally it is also possible to consider the long-wavelength limit by keeping only the lowest energy states and by simply considering the local limit for structureless light mesons [35]. The resulting effective action for quarks \( (\psi_1 \text{ and } \psi_2) \) interacting with auxiliary fields (quark-anti-quark mesons) is quadratic in \( \psi_2 \) requiring a typical Gaussian integration. The resulting determinant can be written, by considering the identity \( \det A = \exp \text{Tr} \ln(A) \), as:

\[
S_{\text{eff}} = i \text{Tr} \ln \left\{ S_0^{-1}(x-y) + \Xi(x-y) \right. \\
- \alpha g^2 R^{\mu\nu}(x-y)\gamma_{\mu} \sigma_{\nu} \left[ (\bar{\psi}(y)\gamma_{\mu} \sigma_{\nu} \psi(x))_1 + i\gamma_{\nu}(\bar{\psi}(y) i\gamma_{\mu} \sigma_{\nu} \psi(x))_1 \right] \\
+ 2\bar{R}(x-y)\alpha g^2 \left[ (\bar{\psi}(y)\psi(x))_1 + i\gamma_{\nu} \sigma_{\nu} \psi(x))_1 \right] \\
- \frac{1}{2} \int_{x,y} \left\{ R \left[ S^2 + P^2 \right] + \frac{1}{2} \bar{R}^{\mu\nu} [V_{\mu}V_{\nu} + \bar{A}_\mu \bar{A}_\nu] \right\}, \\
- \int_{x} \bar{\psi}_1(x) (i\gamma_{\mu} D_{\mu} - m) \psi_1(x) - \frac{g^2}{2} \int_{x,y} j_{\mu}^{a(1)}(x) R^{\mu\nu}(x-y) j_{\nu}^{b(1)}(y)
\]

where \( \text{Tr} \) stands for traces of discrete internal indices and integration of spacetime coordinates, the inverse Fierz transformation was used for the \( \psi_1 \) interaction that is written in the last line, and where \( S_0^{-1} = (i\not{D} - m) \), being \( \not{D} = \gamma_{\mu}(\partial_{\mu} - ieQ_{ij} A_{\mu}) \). \( \Xi \) stands for the auxiliary fields coupling to sea quarks. Vector and axial auxiliary fields yield heavier excitations and may be neglected for the low energy regime. The bilocal a.f. can be expanded in a basis of local meson excitations. However, this work is concerned with the effects of weak magnetic field in the low energy regime of quark effective interactions and the local limit of these composite fields might be adopted because the only
leading effect of the a.f. is to produce the large quark effective mass due to DChSB. The quark coupling to the local scalar and pseudo-scalar fields, in the absence of the heavier vector states, can be written as:

\[ \Xi(x, y) = g \alpha F_{0,0}(x - y) R \left[ S \left( \frac{x + y}{2} \right) + P_1 \left( \frac{x + y}{2} \right) i \gamma_5 \sigma_i \right] \]  

where, due to the structureless mesons approximation, it will be considered \( z = (x + y)/2 = x \). Then it reduces to:

\[ \Xi(x, y) \simeq F[s(x) + p_i(x) \gamma_5 \sigma_i] \delta(x - y), \]  

where \( F \) is the pion decay constant that allows for the canonical definition of the pion field as \( \pi_i = F p_i \). The saddle point equations for expression (7) yield the usual gap equations, by denoting the auxiliary fields \( \phi_q = S(x, y), P_i(x, y), V_\mu(x, y) \) and the axial field \( \vec{A}_\mu(x, y) \) these equations are:

\[ \frac{\partial S_{eff}}{\partial \phi_q} = 0. \]  

These equations for the NJL model and GCM have been analyzed in many works, under external B or not, for the vacuum or at finite temperatures or quark densities, including in the complete form which corresponds to Dyson Schwinger equations in the rainbow ladder approximation. The only possible non trivial solution might exist for the scalar field since the ground state is scalar. It yields a correction to the quark mass, as the leading effect, and therefore only the limit of local auxiliary field \( S \) is needed from here on. The magnetic field is known to increase the leading quark-quark effective couplings and their dependence on a constant magnetic field will be shown.

At this point it is worth noticing that an estimate of the effect of the a.f. on the eventual quark-quark effective interactions can be obtained by expanding the quark determinant above in powers of the a.f. However it is seen that the effects of a.f. on the quark-quark effective interactions only will appear at least in the third order of the expansion to produce, for example, terms of the following form \( \phi_q (\bar{\psi} \Gamma_q \psi)^2 \). These terms are of higher order in \( S_0 \) and consequently smaller in the large quark mass limit. Alternatively, if these auxiliary fields are kept as a whole and afterwards eliminated for example being integrated out approximately when expanding the determinant in a steepest descent approximation, their contribution to the photon-quark interaction would be again of higher order. Therefore the a.f. can be neglected by keeping the non trivial value of the scalar field that endows quarks with an effective mass. Results will be precisely those from one loop BFM with the corrected quark effective mass.

The determinant can be rewritten as:

\[ I_{det} = Tr \ln \left( S_0^{-1} + \sum_q a_q \Gamma_q j_q \right) \]

\[ = \frac{1}{2} Tr \ln \left( S_0^{-1} + \sum_q a_q \Gamma_q j_q \right) \left( S_0^{-1} + \left( \sum_q \bar{a}_q \Gamma_q j_q \right)^* \right) \]  

where \( \tilde{S}_0^{-1} = (i \slashed{D} + M^*) \); for \( q = s, p, v, a \) and \( \bar{a}_s = -a_s, \bar{a}_v = -a_v \) and \( \bar{a}_p = a_p, \bar{a}_a = a_a \), and also it has been defined the following shorthand notation for the four channels \( q \):

\[ \sum_q a_q \Gamma_q j_q \]  

\[ = -\alpha g^2 \bar{R}^{\mu\nu}(x - y) \gamma_\mu \sigma_i \left[ (\bar{\psi}(y) \gamma_\nu \sigma_i \psi(x)) + i \gamma_5 (\bar{\psi}(y) i \gamma_5 \gamma_\nu \sigma_i \psi(x)) \right] \]  

\[ + 2 \alpha g^2 R(x - y) \left[ (\bar{\psi}(y) \psi(x)) + i \gamma_5 \sigma_i (\bar{\psi}(y) i \gamma_5 \sigma_i \psi(x)) \right]. \]  

By turning the (background) quark currents to zero this determinant yields the celebrated Euler Heisenberg effective action for the electromagnetic field \[31, 32, 48, 55\]. Below, a large quark mass expansion will be performed and the leading quark-quark effective couplings and their dependence on a constant magnetic field will be shown.
III. EXPANSION OF THE DETERMINANT AND EFFECTIVE COUPLINGS

The large quark mass expansion of the determinant will be performed next by neglecting all the quark derivative couplings [56]. A shorthand notation will be used below to improve the reading of the expressions, the gluon kernels will be written shortly: \( R \equiv R(x - y) \), \( R^{\mu \nu} \equiv R^{\mu \nu}(x - y) \) and so on. By neglecting terms such as \( Tr \ln(iS_0^{-1}) \) that becomes an irrelevant constant in the generating functional, the dynamical part of expression [12], by considering the anti-commutation relations of the Dirac gamma matrices, can then be written as:

\[
S_d \simeq Tr \sum_{n=1}^{\infty} d_n \left\{ \tilde{S}_2 \left[ \Delta_A + \xi + \xi_{ab} + \xi_{\text{der}} + I_{\text{crossed}} + 4(\alpha g^2)^2 R^2 \left[ j_S(x,y)js(y,x) + \gamma_5^{\mu} \sigma_i j^\mu_p(x,y)j_{\mu}^p(y,x) \right] \right] \right\}^n ,
\]

where the following terms have been defined:

\[
\Delta_A = -\epsilon^2 \hat{Q}^2 (A_\mu A^\mu + A^\mu A^\nu \sigma_{\mu \nu}) + \frac{i\epsilon}{2} Q \sigma_{\mu \nu} F^{\mu \nu},
\]

\[
\xi = -\epsilon^2 (2 \alpha g^2) R \hat{Q} \sigma_{\mu \nu} \left[ A_\mu, \gamma_5 \right] j^\nu_p(x,y) + c(\alpha g^2) R^{\mu \nu} \hat{Q} \sigma_{\mu \nu} \left[ \gamma_5, \gamma_5 \right] j^\nu_p(x,y),
\]

\[
\xi_{ab} = 4(\alpha g^2) R M^a j_S(x,y) - 2M^a (\alpha g^2) R^{\mu \nu} \sigma_{\mu \nu} j^\nu_p(x,y),
\]

\[
\xi_{\text{der}} = -i\epsilon^2 (\alpha g^2) R^{\mu \nu} \hat{Q} \sigma_{\mu \nu} \left[ \gamma_5, \gamma_5 \right] j^\nu_p(x,y),
\]

\[
I_{\text{crossed}} = 2(\alpha g^2) R \hat{Q} \sigma_{\mu \nu} \left[ \gamma_5, \gamma_5 \right] j^\nu_p(x,y) j^\nu_p(x,y) + 2(\alpha g^2) R^{\mu \nu} \hat{Q} \sigma_{\mu \nu} \left[ \gamma_5, \gamma_5 \right] j^\nu_p(x,y) j_{\nu}^p(x,y) + 2(\alpha g^2) R^{\mu \nu} \hat{Q} \sigma_{\mu \nu} \left[ \gamma_5, \gamma_5 \right] j^\nu_p(x,y) j^\nu_p(x,y) + 2(\alpha g^2) R^{\mu \nu} \hat{Q} \sigma_{\mu \nu} \left[ \gamma_5, \gamma_5 \right] j^\nu_p(x,y) j_{\nu}^p(x,y).
\]

The terms \( \Delta_A \) and \( \xi \) contain magnetic field dependent terms, \( \xi_{ab} \) presents the symmetry breaking terms since they appear to be proportional to the Lagrangian quark mass \( M \). However if DChSB is considered for the auxilar scalar field and the corresponding gap equation, this Lagrangian mass is corrected to an effective mass \( M^* \). The other terms above are: \( \xi_{\text{der}} \) with the derivative terms that, with an integration by parts, may produce constant magnetic field contribution when multiplied by \( \xi \), whereas \( I_{\text{crossed}} \) contains mixing interactions with different quark currents and they produce non zero terms in the expansion only in higher orders. In this work only the lowest order terms will be investigated, up to the second order in the expansion. Third order of the expansion will have additional factors \( \tilde{S}_2 \), being each of them \( O(1/M^{*2}) \) smaller than the second order ones. In expression [14] the following parameters were defined:

\[
d_n = \frac{1}{2} (-1)^{n+1},
\]

\[
\tilde{S}_2 = 1/(-\partial^2 - M^{*2}).
\]

A. First order terms

In the long-wavelength or local limit of the expressions bellow the effective couplings can be resolved to yield effective coupling constants. In the zero order derivative expansion for the first order expansion the following effective couplings appear:

\[
I_{\text{eff.1}} = \Delta M^* \bar{\psi} \psi + g_4 \left[ (\bar{\psi} \gamma_5 \psi)^2 + (\bar{\psi} \gamma_5 \gamma_5 \gamma_5 \psi)^2 \right] - g_{4v} \left[ (\bar{\psi} \gamma_5 \gamma_5 \gamma_5 \psi)^2 + (\bar{\psi} \gamma_5 \gamma_5 \gamma_5 \gamma_5 \psi)^2 \right].
\]

These couplings have already been found in Refs. [36, 37] and, for these expressions, traces of Dirac and Pauli matrices are taken. The effective coupling constants were defined in the following way:

\[
\Delta M^* = -i2(\alpha g^2) Tr \left( \tilde{S}_2 R M^* \right),
\]

\[
g_4(1, \delta_{ij}) = -i2(\alpha g^2)^2 Tr \left( \tilde{S}_2 R^2 (1, \sigma_i \sigma_j) \right),
\]

\[
g_{4v}^{\nu \sigma} \delta_{ij} = -\frac{1}{2} (\alpha g^2)^2 Tr \left( \tilde{S}_2 \hat{R}^{\nu \sigma} \hat{R}^{\mu} \gamma_5 \gamma_5 \gamma_5 \gamma_5 \sigma_i \sigma_j \right),
\]

\[
(23)
\]
where, by performing the trace in Dirac indices, the following kernel can be defined:

\[
\tilde{R}_\mu^\nu \tilde{R}_{\rho\nu} = \tilde{R}_{2 \mu\nu} = g_{\mu\nu}(R_T + R_L) + 8 \frac{k_{\mu}k_{\nu}}{k^2} R_T(R_T - R_L).
\]  

(24)

The expression for the effective mass \(23\) might be ultraviolet divergent or finite depending on the gluon propagator behavior. However, the effective couplings constants \(g_L\) and \(g_{L\nu}\) are finite unless the quark and gluon kernels present an unusual momentum dependence. For gluon propagators written in terms of an effective gluon mass these expressions should also be infrared finite.

**B. Second order quark terms up to \(\mathcal{O}(\bar{\psi}\Gamma_\psi)\)**

The second order non derivative couplings that depend on the magnetic field will be exhibited below. Those terms containing one derivative of quark currents \((\xi_{der})\) that multiply the vector \(A^\mu\) either can yield non trivial contributions to (non derivative) effective quark couplings if an integration by parts is performed, producing quark couplings to the strength tensor \(F^{\mu\nu}\), or it may disappear. The terms that produce non zero contributions are shown below \((I_4 = I_2 + I_4 + I_{4,\xi} + I_{4,der} + I_{cross,B})\). The two possible orders of combining structures for each of the term in the expansion will be written as a big anti-commutator in most of the terms. Although all the calculations will be performed for the Landau gauge for a constant magnetic field, \(A_\mu = -B_0(0,0,x,0)\), the electromagnetic field will be carried almost until the last expressions. The following terms:

\[
I_2 = i e^2(\alpha^2)^2 M^* \text{Tr} \tilde{Q}^2 \left\{ \tilde{S}_2(A_\mu A^\mu) , \tilde{S}_2 R^2 \right\} j_\sigma(x,y),
\]

\[
I_4 = -i e^2(\alpha^2)^2 \text{Tr} \left\{ \tilde{S}_2(A_\mu A^\mu + A^\mu A^\nu \sigma_{\mu\nu}) , \tilde{S}_2 R^2 \right\} \left( \tilde{Q}^2 j_\sigma(x,y) j_\sigma(y,x) + \gamma_5^2 \tilde{Q}^2 \sigma_\sigma j^\sigma_p(x,y) j^\sigma_p(y,x) \right)
\]

\[
- \frac{i}{4} e^2(\alpha^2)^2 \text{Tr} \left\{ \tilde{S}_2(A_\mu A^\mu + A^\mu A^\nu \sigma_{\mu\nu}) , \tilde{S}_2 R^2 \right\} j_\sigma(x,y) j_\sigma(y,x)
\]

\[
+ i \frac{e^2(\alpha^2)^2}{8} \text{Tr} \left\{ \tilde{S}_2 F^\mu_{\nu 1} , \tilde{S}_2 \tilde{R}^\mu_{\nu 1} \bar{\sigma}_{\sigma_{\mu\nu}} \gamma_5 \gamma_5 \tilde{Q}^2 \sigma_\sigma \right\} (j_\nu^\sigma(x,y) j_\nu^\sigma(y,x) - \gamma_5^2 j_\nu^\sigma(x,y) j_\nu^\sigma(y,x)),
\]

\[
I_{4,\xi} = i e^2(\alpha^2)^2 \text{Tr} \tilde{Q} \sigma_i \tilde{Q} \sigma_j \left\{ \tilde{S}_2 R [A , \gamma_5] \tilde{S}_2 [A , \gamma_5] \right\} j_\sigma^i(x,y) j_\sigma^j(y,x),
\]

\[
+ \frac{i}{4} e^2(\alpha^2)^2 \text{Tr} \tilde{Q} \sigma_i \tilde{Q} \sigma_j \left[ \tilde{S}_2 [\gamma_\mu , A] \tilde{R}^{\mu\nu} \tilde{S}_2 [\gamma_\rho , A] \tilde{R}^{\sigma\rho} \right] j_\nu^i(x,y) j_\nu^j(y,x)
\]

\[
+ \frac{i}{4} e^2(\alpha^2)^2 \text{Tr} \tilde{Q} \sigma_i \tilde{Q} \sigma_j \left( \tilde{S}_2 [\gamma_\mu \gamma_5 , A] \tilde{R}^{\mu\nu} \tilde{S}_2 [\gamma_\rho \gamma_5 , A] \tilde{R}^{\sigma\rho} \right) j_\nu^i(x,y) j_\nu^j(y,x)
\]

\[
+ i 2 e(\alpha^2)^2 \text{Tr} M^* \left( \tilde{S}_2 \tilde{Q} \sigma_i [\gamma_\mu , A] \tilde{R}^{\mu\nu} \tilde{S}_2 R \right) j_\nu^i(x,y) j_\sigma(x,y)
\]

\[
+ i 4 e(\alpha^2)^2 \text{Tr} M^* \tilde{S}_2 R^2 j_\sigma(x,y) j_\sigma(y,x)
\]

\[
+ i (\alpha^2)^2 \text{Tr} \sigma_i \sigma_j \gamma_{\mu} \gamma_{\rho} M^* \tilde{S}_2 \tilde{R}^{\mu\nu} M^* \tilde{R}^{\sigma\rho} j_\nu^i(x,y) j_\nu^j(y,x),
\]

\[
I_{4,der} = -i^2 e(\alpha^2)^2 \text{Tr} \left\{ \tilde{S}_2 R \tilde{Q} \sigma_i [A , \gamma_5] , \tilde{S}_2 R \tilde{Q} \sigma_j [\gamma_\mu , \gamma_\rho] \right\} j_\sigma^i(x,y) \delta^\rho(j_\sigma^j(x,y))
\]

\[
+ i^2 2 e(\alpha^2)^2 \text{Tr} \tilde{S}_2 \tilde{R}^{\mu\nu} \tilde{Q} \sigma_i (\gamma_\mu , A) , \tilde{S}_2 \tilde{R}^{\mu\nu} (x-y) \sigma_\sigma (\gamma_\rho , \gamma_{\mu2}) j_\nu^i(x,y) \delta^\rho(j_\nu^j(y,x))
\]

\[
+ i^2 2 e(\alpha^2)^2 \text{Tr} \tilde{S}_2 \tilde{R}^{\mu\nu} \tilde{Q} \sigma_i (\gamma_\mu , A) , \tilde{S}_2 \tilde{R}^{\mu\nu} (x-y) \sigma_\gamma_5 (\gamma_\rho , \gamma_{\mu2}) j_\nu^i(x,y) \delta^\rho(j_\nu^j(y,x))
\]

\[
+ i^2 2 e(\alpha^2)^2 \text{Tr} \tilde{S}_2 \tilde{R}^{\mu\nu} \tilde{Q} \sigma_i (\gamma_\mu \gamma_5 , A) , \tilde{S}_2 \tilde{R}^{\mu\nu} (x-y) \sigma_\gamma_5 (\gamma_\rho , \gamma_{\mu2}) j_\nu^i(x,y) \delta^\rho(j_\nu^j(y,x))
\]

\[
+ i^4 4 e(\alpha^2)^2 \text{Tr} \tilde{S}_2 \tilde{R}^{\mu\nu} \tilde{Q} \sigma_i (\gamma_\mu \gamma_5 , A) , \tilde{S}_2 \tilde{R}^{\mu\nu} \tilde{Q} \sigma_j (\gamma_\rho \gamma_5 , A) , j_\nu^i(x,y) \delta^\rho(j_\nu^j(y,x)),
\]

\[
I_{cross,B} = -i^2 e(\alpha^2)^2 e^2 \text{Tr} \left\{ \tilde{S}_2 \left( \tilde{Q}^2 (A_{\mu\nu} A_{\mu\nu} + A_{\mu\nu} A_{\mu\nu} \sigma_{\mu\nu}) + \frac{i e}{2} \tilde{Q} \sigma_{\mu\nu} F^{\mu\nu} \right) \right\}
\]

\[
\times \tilde{S}_2 \tilde{R}^{\mu\nu} \tilde{R}^{\sigma\rho} (\gamma_{\mu} \gamma_{\rho} \sigma_\sigma + \gamma_{\rho} \gamma_{\mu} \sigma_\sigma) \gamma_5 j_\nu^i(x,y) j_\nu^j(y,x).
\]

(25)
The following traces of isospin and Dirac indices ($T_F$ and $T_D$) will be used in the next steps:

$$
T_F(\sigma_i \sigma_j) = 2\delta_{ij},
$$

(26)

$$
T_F(\bar{Q} \sigma_i \sigma_j) = \frac{1}{3} \delta_{ij} + i \epsilon_{ij3},
$$

(27)

$$
T_F(\bar{Q}^2 \sigma_i \sigma_j) = \frac{5}{9} \delta_{ij} + \frac{i}{3} \epsilon_{ij3},
$$

(28)

$$
T_D(\gamma^\mu \gamma^\nu) = 4g_{\mu\nu},
$$

(29)

$$
T_D(\sigma_{\mu\nu} \sigma_{\rho\sigma}) = 4(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}),
$$

(30)

$$
T_D(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\lambda) = -4i\epsilon^{\alpha\beta\delta\lambda}.
$$

(31)

By resolving the effective coupling constants in the long-wavelength limit, several of the terms above disappear. Besides that, only the momentum derivatives of internal lines will be considered. The non zero contributions of these expressions can be written as:

$$
L_{4q} = \Delta_B M^* \bar{\psi} \psi + g_{4,B} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} \gamma_5 \psi)^2 + \left( \frac{3g_{4,B}}{5} i \epsilon_{ij3} + g_{ps,B} c_i \delta_{ij} + \left( \frac{3}{5} g_{4,v,B} + g_{4,v,B} - F + 3g_{4,v,B} \right) \right) \right]
$$

\begin{align*}
&+ \left[ \delta_{ij} \left( g_{4,v,B} + g_{4,v,B} - F + g_{4,v,B} \right) + i \epsilon_{ij3} \left( \frac{3}{5} g_{4,v,B} + g_{4,v,B} - F + 3g_{4,v,B} \right) \right]

&\times \left[ (\bar{\psi} \gamma_\mu \gamma_\rho \psi)(\bar{\psi} \gamma_\mu \gamma_\rho \psi) + (\bar{\psi} \gamma_\mu \gamma_5 \psi)(\bar{\psi} \gamma_\mu \gamma_5 \psi) \right]

&+ g_{s,sb}(\bar{\psi} \psi)^2 + \left( \bar{\psi} \gamma_5 \psi \right)^2,
\end{align*}

(32)

where the following notation was adopted in the terms depending on the coefficients $c_i$ with operators $\Gamma_i$:

$$
c_i \left( \bar{\psi} \Gamma_i \psi \right)^2 = c_1 \left( \bar{\psi} \Gamma_1 \psi \right)^2 + c_2 \left( \bar{\psi} \Gamma_2 \psi \right)^2 + c_3 \left( \bar{\psi} \Gamma_3 \psi \right)^2,
$$

(33)

being defined the following isospin coefficients:

$$
c_1 = -\frac{4}{9}, \quad c_2 = \frac{4}{9}, \quad c_3 = \frac{5}{9},
$$

(34)

These coefficients are responsible for the pseudo-scalar, vector and axial quark-anti-quark states (pion, rho, $A_1$) couplings to the magnetic field. In the first and second lines of expression (32) there are effective couplings dependent on the magnetic field and in the last line those due to the explicit symmetry breaking discussed in Ref. [37]. The couplings $g_{4,B}$ and mainly $g_{ps,B}$ are responsible for extra contributions to the axial current and then they allow for chiral separation effect. Chiral and isospin breaking terms also appear in the vector channel. The effective coupling constants are defined as:

$$
\Delta_B M^* = -i 2e^2 (a g^2) \left[ \bar{Q}^2 \bar{S}_2 (A_\mu A^\mu) M^* \bar{S}_2 \right],
$$

(35)

$$
g_{4,B}(\delta_{ij} + \frac{3}{5} i \epsilon_{ij3} \sigma_3) = -i 2e^2 (a g^2)^2 \left[ \bar{Q}^2 \bar{S}_2 A^\mu A_\mu \bar{S}_2 R^2 \left( 1 \right) : \gamma_5 \sigma_i \sigma_j : \right],
$$

(36)

$$
g_{4,v,B}(\delta_{ij} + \frac{3}{5} i \epsilon_{ij3} \sigma_3) g^{\sigma_1 \sigma} = -\frac{i}{2} e^2 (a g^2)^2 \left[ \bar{Q}^2 \sigma_i \sigma_j \gamma_{\rho_1} \gamma_{\rho_2} \bar{S}_2 A^\mu A_\mu \bar{S}_2 R_{\mu \sigma_1} \bar{R}^{\sigma_2} \right],
$$

(37)

$$
g_{ps,B} c_i \delta_{ij} = -i 2e^2 (a g^2)^2 \left[ \bar{Q} \sigma_i \sigma_j \bar{S}_2 \left[ A, \gamma_\rho, \gamma_5 \right] R \left[ \bar{S}_2 \left[ A, \gamma_\rho, \gamma_5 \right] R \right] \right],
$$

(38)

$$
g_{4,v,B} c_i \delta_{ij} g^{\sigma_2} = -\frac{i}{4} e^2 (a g^2)^2 \left[ \bar{Q} \sigma_i \sigma_j \bar{S}_2 \left[ \gamma_\mu, A \right] \bar{R}^{\mu \nu} \bar{S}_2 \left[ \gamma_\rho, A \right] \bar{R}^{\rho \sigma} \right],
$$

(39)

$$
g_{4,v,B} (\delta_{ij} + \frac{3}{4} i \epsilon_{ij3} \sigma_3) g^{\sigma_1 \sigma} = \frac{i}{2} e^2 (a g^2)^2 \left[ \bar{Q} \sigma_i \sigma_j \bar{S}_2 \left[ \gamma_\mu, A \right] \bar{R}^{\mu \nu} \bar{R}^{\rho \sigma} \sigma_{\mu \nu} \gamma_\rho \gamma_\sigma \right],
$$

(40)

$$
g_{4,v,B} (\delta_{ij} + \frac{3}{4} i \epsilon_{ij3} \sigma_3) g^{\sigma_1 \sigma} = -i e^2 (a g^2)^2 \left[ \bar{Q} \sigma_i \sigma_j (\partial_\rho A_\rho) \left[ \gamma_\mu, \gamma_\rho \right] \left[ \gamma_\rho, \gamma_\mu \right] \bar{S}_2 \bar{R}^{\mu \nu} \bar{S}_2 \bar{R}_{\mu \nu} \right],
$$

(41)

$$
g_{s,sb} = -i 4 (a g^2)^2 \left[ \bar{S}_2 \bar{R}^{\mu \nu} \bar{S}_2 \bar{R}^{\rho \sigma} \right],
$$

(42)

$$
g_{s,sb} \delta_{ij} g^{\sigma_2} = -i (a g^2)^2 \left[ \sigma_i \sigma_j \gamma_\mu \gamma_\rho M^* \bar{S}_2 \bar{R}^{\mu \nu} \bar{S}_2 \bar{R}^{\rho \sigma} \right].
$$

(43)
By performing the traces in discrete indices, always by neglecting the quark derivative couplings, and by taking $x = -i \frac{\partial}{\partial q_x}$ the above expressions can be written as:

$$\Delta_B M^* = -i \frac{40}{9} (eB_0)^2 (\alpha g^2) N_c Tr' \left[ M^* \tilde{S}_2 x^2 \tilde{S}_2 R \right],$$  \hspace{1cm} (44)

$$g_{4,B} = -i \frac{40}{9} (eB_0)^2 (\alpha g^2)^2 N_c Tr' \left[ \tilde{S}_2 x^2 \tilde{S}_2 R^2 \right],$$  \hspace{1cm} (45)

$$g_{4v,B} g^{\sigma_1} = -i \frac{40}{9} (eB_0)^2 (\alpha g^2) N_c Tr' \left[ \tilde{S}_2 x^2 \tilde{S}_2 R^2 \sigma_1 \right],$$  \hspace{1cm} (46)

$$g_{ps,B} c_j = -i c_j (eB_0)^2 (\alpha g^2)^2 N_c Tr' \left[ (\tilde{S}_2 x R)^2 \right],$$  \hspace{1cm} (47)

$$g_{4v,B} g^{\mu \nu} c_j = -i c_j (eB_0)^2 (\alpha g^2)^2 N_c Tr' \left[ \tilde{S}_2 x R^{\mu \nu} \tilde{S}_2 x R^{\rho \sigma} \right] \left( 4g_{\mu \nu} g_{\rho \sigma} - 2g_{\mu \rho} g_{\nu \sigma} \right),$$  \hspace{1cm} (48)

$$g_{4v,B-F} = i \frac{8}{3} (eB_0) (\alpha g^2)^2 N_c Tr' \left[ \tilde{S}_2 \tilde{S}_2 R^{xy} \right],$$  \hspace{1cm} (49)

$$g_{4v,B} = -i 16 (eB_0) (\alpha g^2)^2 N_c Tr' \left[ T_{xy} \left[ 8 \tilde{S}_2 (R_T - R_L) \tilde{S}_2 (R_T - R_L) + \tilde{S}_2 (R_T - R_L) \tilde{S}_2 (R_T + R_L) \right] \right],$$  \hspace{1cm} (50)

$$g_{s,SB} = -i 32 (\alpha g^2)^2 N_c Tr' \left[ M^* \tilde{S}_2 R \right],$$  \hspace{1cm} (51)

$$g_{v,SB} g^{\mu \nu} = -i 8 (\alpha g^2)^2 N_c Tr' \left[ M^* \tilde{S}_2 R^{\mu \nu} \tilde{S}_2 R^{\rho \sigma} \right],$$  \hspace{1cm} (52)

where $T_{xy} = \frac{k_x k_y}{M}$ in expression (50) and $Tr'$ stands for the trace/integral in internal momenta. Due to the structure of $\tilde{R}^{xy}_y$ in expression (24) the coupling $g_{4v,B-F}$ is non-zero only for a non-zero transversal component of the gluon propagator, i.e. if $R_T = 0$ it yields $g_{4v,B-F} = 0$.

![FIG. 1: In these diagrams, the wavy line with a full dot is a (dressed) non perturbative gluon propagator, and the short bold lines insertions for the vector potential whereas the full triangle is for the magnetic field insertion ($F^\mu\nu$). Diagram (a) shows the effective mass due to the magnetic field contribution, whereas diagrams (b1,b2,b3) represent all the quark-quark effective interactions shown above.](image)

In Figure 1, the diagrams corresponding to the one loop terms presented above are shown. The wavy line with a full dot is a (dressed) non perturbative gluon propagator, the short thick lines insertions stand for the vector potential whereas the full triangle insertion stands for the magnetic field insertions ($F^\mu\nu$). Diagram (a) shows the contribution to the effective mass due to the magnetic field, whereas diagrams (b1,b2,b3) represent the quark-quark effective interactions shown above.

The leading effective mass dependence on the magnetic field, shown in Fig. (1a), is of the order of $(eB_0)^2/M^*^3$ instead of the leading correction obtained from the gap equation $\sqrt{eB_0}$ [5]. The leading coupling constants in the expressions above are $g_{4v,B}$ and $g_{4v,B-F}$ that are linearly proportional to the magnetic field $\partial_\mu A_\mu$, $eB_0/M^*$. The corresponding diagram is shown in Fig. (1b3). By extracting $1/M^*^2$ from $\tilde{S}_2$ in the limit of large quark effective mass, it produces a quantity proportional to the dipole moment coupling itself $eB_0/(2M^*)$. In spite of the absence of a tensor current for the dipolar coupling in the leading effective action, the magnetic field couples directly to the vector/axial currents being a dipolar interaction. All the other couplings - in expressions (35)-(39) - have two insertions $A_\mu A^\mu$ introducing a larger (and suppressing) momentum dependence in internal lines, with corresponding factor $(eB_0)^2/(M^*)^4$. They are smaller in the limit of large quark effective mass.
IV. RATIOS BETWEEN EFFECTIVE COUPLING CONSTANTS

There are few ambiguities in performing numerical estimates of the effective coupling constants found above. The first reason is that the gluon propagator with its infrared behavior is not really well known and results depend strongly on it. Also, one has to choose a way of performing the momenta/energy traces, for example in Euclidean or Minkowski spaces, and this might yield different numerical results. Furthermore, other effects in the gluon sector, such as the $B_0$ dependence of the quark gluon coupling or the gluon propagator itself, might be expected to yield $B$-dependence of the same order of magnitude as the quark condensate (or quark effective mass) from the gap equation \[56\]. Due to these reasons numerical estimates will not be presented. Nevertheless, below few solutions for the gap equation are presented with the intention to justify the approximations done, i.e. to consider the quark effective mass from DChSB as the leading effect of the quark gluon coupling or the gluon propagator itself, might be expected to yield $B$-dependence at least of the same order of magnitude as the quark condensate (or quark effective mass) from the gap equation \[56\].

To solve the gap equation \[10\] a longitudinal (confining) effective gluon propagator was chosen of the form of:

\[g^2 R_{\mu\nu}^{\text{eff}}(k) = K_F/(k^2 + M^2)^2 g_{\mu\nu},\]

for $K_F = 8\pi^3 M^2/9$ \[43, 59\]. This effective propagator incorporates the large strength of the running coupling constant and, to some extent, some of the relatively important issues of the ultraviolet and infrared behavior of the gluon confining propagator \[43\]. With this gluon propagator the gap equation, as well as all the expressions for the effective quark masses and effective coupling constants, are finite, i.e. free of ultraviolet and infrared divergences. For example consider $M_q = M \approx 378$ MeV, that is of the order of the values discussed in Ref. \[43\] in spite of being relatively smaller than the usual theoretical and lattice findings \[31, 17\]. For a current quark mass $m = 10$ MeV, the gap equation \[10\] is non zero only for the scalar auxiliary field $s$ as defined, and it yields, for $B_0 = 0$, $s_0 \approx 210$ MeV for which $M^* \approx 220$ MeV. For a weak magnetic field $z = \frac{eB_0}{M^*} = 0.1$ the gap equation yields $M^* (z = 0.1) = 227$ MeV. By considering $M_G = 511$ MeV, which is closer to the values obtained in lattice QCD it yields $M^* (z = 0) = 300$ MeV and for weak magnetic field $M^* (z = 0.1) = 309$ MeV.

The effective coupling constants presented above can exhibit simple relations in the limit of large effective masses. For some of these effective coupling constants, this is achieved in specific limits of the gluon kernels. In the limit of very large quark effective mass, i.e. for $S_2 \rightarrow 1/M^*^4$, some of these ratios are independent of the chosen component for the gluon propagator (for $R_T(x-y)$ or $R_L(x-y)$), i.e.:

\[\frac{\Delta M^*_B}{\Delta M^*} \approx \frac{5}{9} \frac{(eB_0)^2}{M^*^4}, \quad \frac{g_{4,B}}{g_4} \approx \frac{5}{9} \frac{(eB_0)^2}{M^*^4}, \quad \frac{g_{ps,B}}{g_{4,B}} \approx \frac{9}{5}, \quad \frac{g_{4v,B^2}}{g_{4v,B}} \approx \frac{27}{40}\]  

(53)

For other effective coupling constants, still in the limit of very large effective quark mass, $M^*$, it is possible to obtain simple relations by considering particular relative contributions of the longitudinal and transversal components of the gluon propagator. In the following it will be considered that any of the two components present an effective gluon mass. The ratios will be computed in the limit of large masses $R_T/L \sim 1/M^2_0$ for $n = 2, 4$, by keeping $M_G > M^*$. If it is assumed $R_L = 0$ then the following ratios are obtained:

\[(g_{4v,B}/g_{4v,B})^T \sim \frac{3}{4}, \quad (g_{4v,B}/g_{4v,B})^T \sim 3 \frac{M^*^2}{20 \ eB_0}, \quad (g_{4v,B}/g_{4v,B})^L \sim 80 \frac{eB_0}{9 \ M^*^2},\]  

(54)

whereas for $R_T = 0$ it yields:

\[(g_{4v,B}/g_{4v,B})^L \sim \frac{1}{4}, \quad (g_{4v,B}/g_{4v,B})^L \sim 0, \quad (g_{4v,B}/g_{4v,B})^L \sim 80 \frac{eB_0}{63 \ M^*^2}.\]  

(55)

All the coupling constants of the order of $B_0$ or $B_0^2$ are smaller than the NJL coupling, from expression \[23\], since a large effective quark mass expansion has been done: i.e. $\frac{(eB_0)}{M^*^2} < 1$ or $\frac{(eB_0)}{M^*^2} << 1$. However, by increasing the magnetic field strength this expansion still may be reliable up to some limit by computing higher orders terms ($n$-th order expansion). This produces further quark-quark effective interactions dependent on $B_0^{n-j}$ where $j = 0, 1, 2...n$. Consequently the complete account of the Landau orbits that could be done for the quark kernel \[2, 58\] emerges as a series in powers of the magnetic field in agreement with \[60\].
V. SUMMARY AND CONCLUSIONS

By departing from a (dressed) one gluon exchange mechanism for the quark-quark interaction, different leading quark-quark effective interactions due to polarization were derived in the presence of a weak magnetic field, i.e. $eB_0 \ll M^2$. The relevant assumption for the GCM is that the gluon propagator is dressed by non perturbative effects due to the non Abelian character of gluon interactions. The one loop BFM method was applied with a correction due to the auxiliary field method. However only the leading effect of the auxiliary fields was considered, that is the correction to the quark effective mass. The one-loop quark effective action in the presence of the background field was expanded for large quark effective mass and weak magnetic field up to the second order in quark bilinears and to leading order in the magnetic field. The (leading) first and second order $B_0$-dependent terms provided corrections to the background quark mass and effective interactions such as the usual NJL and vector NJL ones, besides new chiral and isospin symmetry breaking terms. They correspond to the different couplings of the magnetic field to pseudo-scalar, vector and axial isospin triplets states. The set of $B_0$-dependent interactions from expressions \[^{32}\] is given by:

\[
\mathcal{L}_{q4} = \Delta_B M^* \bar{\psi} \psi + g_{4,B} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right] + \left( \frac{3g_{4,B}}{5} \epsilon_{ij3} + g_{ps,B} c_i \delta_{ij} \right) (\bar{\psi} \sigma_i \gamma_5 \psi)(\bar{\psi} \sigma_j \gamma_5 \psi) \\
+ \left[ \delta_{ij} \left( g_{4v,B} + g_{4v,B2} \epsilon_i + \frac{g_{4v,B-F}}{3} + g_{4v,2,B} \right) + i\epsilon_{ij3} \left( \frac{3}{5} g_{4v,B} + g_{4v,B-F} + 3g_{4v,2,B} \right) \right] \\
\times \left[ (\bar{\psi} \sigma_i \gamma_\mu \psi)(\bar{\psi} \sigma_j \gamma^\mu \psi) + (\bar{\psi} i \sigma_i \gamma_\mu \gamma_5 \psi)(\bar{\psi} i \sigma_j \gamma^\mu \gamma_5 \psi) \right] \\
+ g_{s,ab}(\bar{\psi} \psi)^2 + g_{v,ab}(\bar{\psi} i \gamma_5 \psi)^2, \\
\]  

The mass correction $\Delta_B M$ is positive as expected from the usual magnetic catalysis analysis from the NJL-type gap equation. However this effective mass contribution for weak field is of the order of $(eB_0)^2/M^4$ whereas the leading contribution from the gap equation for weak $B$-field is of the order of $\sqrt{eB_0}$. The gap equation for the auxiliary field was found to depend on the magnetic field as usual for NJL or GCM -type models. Almost all the effective coupling constants are of the order of $(eB_0)^2/M^4$ except two of them, $g_{4v,2,B}$ and $g_{4v,B-F}$ are $\mathcal{O}(eB_0/M^2)$, corresponding therefore to dipolar couplings in spite of the absence of the tensor current. These two effective couplings are the leading ones being that $g_{4v,B-F}$ is non zero only if the gluon propagator has a transversal component. There are overall corrections to the NJL and vector-NJL coupling constants respectively given by: $g_{4,B}$ and $g_{4v,B}$, $g_{4v,B-F}$ and $g_{4v,2,B}$. The effective coupling constant $g_{4,B}$ enhances the strength of the quark scalar interaction. This might be seen as an increase of the strength of quark interactions that produce dynamical chiral symmetry breaking. Although this may suggest that DChSB can be obtained for zero NJL coupling constant ($g_4 \to 0$) when $g_{4,B} \neq 0$, this might be misleading in the sense that in the present development both effective couplings have the same physical origin, namely the quark polarization with a quark-gluon coupling $g^2$. The physical content of magnetic catalysis would be clearer in this sense by considering a different mechanism for one of the two effective interactions ($g_4$ or $g_{4,B}$). The effective coupling $g_{4v,2,B}$ is also positive and therefore it might contribute to the vector condensation in the vacuum \[^{11}\]. However some new couplings appear signaling the emergence of pseudo-scalar and vector/axial multiplets with different interaction with the magnetic field, i.e. different electric charge ($+, -, 0$). These couplings therefore break chiral and isospin symmetries. In particular the effective couplings $g_{4,B}$ and $g_{ps,B}$ yield pions interactions with the magnetic field. The vector couplings to the magnetic field are $g_{4v,B}$, $g_{4v,B2}$, $g_{4v,B-F}$ and $g_{4v,2,B}$ providing the different couplings in the vector and axial channels therefore related to the $\rho$ and $A_1$ triplets. The two couplings due to the explicit symmetry breaking, $g_{s,ab}$ and $g_{v,ab}$, have already been investigated in Ref. \[^{37}\]. The analytical ratios exhibited in Section \[^{14}\] are very specific to the limit in which the large quark effective mass is smaller than an effective gluon mass that is expected to be present in a non perturbative gluon propagator \[^{17}\]. Other limits could be considered and will be presented elsewhere. The main sources of possible improvements are the simplified momentum dependence of the internal lines and the inclusion of auxiliary fields which however will produce numerically smaller contributions. Higher order interactions of the expansion of the quark determinant considered in this manuscript yield corrections for stronger magnetic fields with increasing powers of $eB_0$ for the effective coupling constants. Alternatively, the whole summation over the Landau levels for internal quark lines (quark kernel) can be considered for arbitrary values of the magnetic field. Pion and quark $B_0$-dependent effective interactions will be investigated elsewhere.
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