Weyl-Invariant Light-Like Branes
and Black Hole Physics

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Abstract

We propose a new class of $p$-brane theories which are Weyl-conformally invariant for any $p$. For any odd world-volume dimension the latter describe intrinsically light-like branes, hence the name WILL-branes (Weyl-Invariant Light-Like branes). Next we discuss the dynamics of WILL-membranes (i.e., for $p = 2$) both as test branes in various external physically relevant $D = 4$ gravitational backgrounds, as well as within the framework of a coupled $D = 4$ Einstein-Maxwell-WILL-membrane system. In all cases we find that the WILL-membrane materializes the event horizon of the corresponding black hole solutions, thus providing an explicit dynamical realization of the membrane paradigm in black hole physics.

1 Introduction - Main Motivation

The consistent Lagrangian formulation of geometrically motivated field theories (gravity, strings, branes, etc.; for a background on string and brane theories, see refs.[1].) requires among other things reparametrization-covariant (generally-covariant) integration measure densities (volume-forms). The usual choice is the standard Riemannian integration measure given by $\sqrt{-g}$ with $g \equiv \det |g_{\mu\nu}|$, where $g_{\mu\nu}$ indicates the intrinsic Riemannian metric on the underlying manifold.

However, equally well-suited is the following alternative non-Riemannian integration measure density:

$$\Phi(\varphi) \equiv \frac{1}{D!} \varepsilon^{\mu_1 \ldots \mu_D} \varepsilon_{i_1 \ldots i_D} \partial_{\mu_1} \varphi^{i_1} \ldots \partial_{\mu_D} \varphi^{i_D} , \quad i = 1, \ldots, D$$

(1)

built in terms of $D$ auxiliary scalar fields independent of the intrinsic Riemannian metric.

In a series of papers [2] two of us have proposed new classes of models involving gravity, called two-measure theories, whose actions contain both standard Riemannian and alternative non-Riemannian integration measures:

$$S = \int d^D x \Phi(\varphi) L_1 + \int d^D x \sqrt{-g} L_2$$

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The scalar Lagrangians are of the following generic form:

\[ L_1 = e^{\frac{2\phi}{M_P}} \left[ -\frac{1}{\kappa} R(g, \Gamma) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (\text{Higgs}) + (\text{fermions}) \right] \]

and similarly for \( L_2 \) (with different choice of the normalization factors in front of each of the terms). Here \( R(g, \Gamma) \) is the scalar curvature in the first order formalism, \( \phi \) is the dilaton field, \( M_P \) denotes the Planck mass, etc. The auxiliary fields \( \phi^i \) are pure-gauge degrees of freedom except for the new dynamical “geometric” field \( \zeta(x) \equiv \frac{\Phi(\phi)}{\sqrt{-g}} \), whose dynamics is determined only through the matter fields locally (i.e., without gravitational interaction).

Two-measure theories address various basic problems in cosmology and particle physics, and provide plausible solutions for a broad array of issues, such as:

- Scale invariance and its dynamical breakdown; Spontaneous generation of dimensionfull fundamental scales;
- Cosmological constant problem;
- The problem of fermionic families;
- Applications in modern brane-world scenarios.

For a detailed exposition we refer to the series of papers [2, 3].

Subsequently, the idea of employing an alternative non-Riemannian integration measure was applied systematically to string, \( p \)-brane and \( Dp \)-brane models [4]. The main feature of these new classes of modified string/brane theories is the appearance of the pertinent string/brane tension as an additional dynamical degree of freedom beyond the usual string/brane physical degrees of freedom, instead of being introduced \textit{ad hoc} as a dimensionfull scale. In the next section we briefly recall the construction of the modified bosonic string model with a dynamical tension before proceeding to our main task. It is the construction of a novel class of \( p \)-brane theories which are Weyl-conformal invariant for any \( p \) and whose dynamics significantly differs both from the standard Nambu-Goto (or Dirac-Born-Infeld) branes as well as from their modified versions with dynamical string/brane tensions [4] mentioned above.

## 2 Strings and Branes with a Modified World-Sheet/World-Volume Integration Measure

The modified-measure bosonic string model is given by the following action:

\[
S = -\int d^2 \sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \frac{\varepsilon^{ab}}{2\sqrt{-\gamma}} F_{ab}(A) \right] + \int d^2 \sigma \sqrt{-\gamma} A_a J^a; \quad J^a = \frac{\varepsilon^{ab}}{\sqrt{-\gamma}} \partial_b u,
\]

with the notations:

\[
\Phi(\varphi) \equiv \frac{1}{2} \varepsilon_{ij} \varepsilon^{ab} \partial_a \varphi^i \partial_b \varphi^j, \quad F_{ab}(A) = \partial_a A_b - \partial_b A_a,
\]

\( \gamma_{ab} \) denotes the intrinsic Riemannian world-sheet metric with \( \gamma = \det ||\gamma_{ab}|| \) and \( G_{\mu\nu}(X) \) is the Riemannian metric of the embedding space-time \((a, b = 0, 1; i, j = 1, 2; \mu, \nu = 0, 1, \ldots, D - 1)\).

Here is the list of differences w.r.t. the standard Nambu-Goto string (in the Polyakov-like formulation):

- \textit{Scale invariance and its dynamical breakdown}; Spontaneous generation of dimensionfull fundamental scales.
- \textit{Cosmological constant problem}.
- \textit{The problem of fermionic families}.
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- \textit{Scale invariance and its dynamical breakdown}; Spontaneous generation of dimensionfull fundamental scales.
- \textit{Cosmological constant problem}.
- \textit{The problem of fermionic families}.
- \textit{Applications in modern brane-world scenarios}.
• New non-Riemannian integration measure density $\Phi(\varphi)$ instead of $\sqrt{-\gamma}$;

• Dynamical string tension $T \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}$ instead of ad hoc dimensionfull constant;

• Auxiliary world-sheet gauge field $A_a$ in a would-be “topological” term $\int d^2 \sigma \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \varepsilon^{ab} F_{ab}(A)$;

• Optional natural coupling of auxiliary $A_a$ to external conserved world-sheet electric current $J^a$ (see last equality in (2) and Eq.(5) below).

The modified string model (2) is Weyl-conformally invariant similarly to the ordinary case. Here Weyl-conformal symmetry is given by Weyl rescaling of $\gamma_{ab}$ supplemented with a special diffeomorphism in $\varphi$-target space:

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \rho \gamma_{ab} \ , \ \varphi^i \longrightarrow \varphi'^i = \varphi^i(\varphi) \text{ with } \det \left| \frac{\partial \varphi'^i}{\partial \varphi^j} \right| = \rho .$$

(4)

The dynamical string tension appears as a canonically conjugated momentum w.r.t. $A_1$: $\pi_{A_1} \equiv \frac{\partial L}{\partial \dot{A}_1} = \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \equiv T$, i.e., $T$ has the meaning of a \textit{world-sheet electric field strength}, and the eqs. of motion w.r.t. auxiliary gauge field $A_a$ look exactly as $D = 2$ Maxwell eqs.:

$$\varepsilon^{ab} \partial_b T + J^a = 0 .$$

(5)

In particular, for $J^a = 0$ :

$$\varepsilon^{ab} \partial_b \left( \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \right) = 0 \ , \ \frac{\Phi(\varphi)}{\sqrt{-\gamma}} \equiv T = \text{const} ,$$

(6)

one gets a \textit{spontaneously induced} constant string tension. Furthermore, when the modified string couples to point-like charges on the world-sheet (i.e., $J^0 \sqrt{-\gamma} = \sum_i e_i \delta(\sigma - \sigma_i)$ in (5)) one obtains classical charge \textit{confinement}: $\sum_i e_i = 0$.

The above charge confinement mechanism has also been generalized in [4] to the case of coupling the modified string model with dynamical tension to non-Abelian world-sheet “color” charges. The latter is achieved as follows. Notice the following identity in 2D involving Abelian gauge field $A_a$:

$$\frac{\varepsilon^{ab}}{2\sqrt{-\gamma}} F_{ab}(A) = \sqrt{-\frac{1}{2} F_{ab}(A) F_{cd}(A) \gamma^{ac\gamma^{bd}}} .$$

(7)

Then the extension of the action (2) to the non-Abelian case is straightforward:

$$S = - \int d^2 \sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \sqrt{-\frac{1}{2} \text{Tr}(F_{ab}(A) F_{cd}(A)) \gamma^{ac\gamma^{bd}}} \right] + \int d^2 \sigma \text{ Tr} (A_a J^a)$$

(8)

with $F_{ab}(A) = \partial_a A_b - \partial_b A_a + i [A_a, A_b]$, sharing the same principal property – dynamical generation of string tension as an additional degree of freedom.

3 New Class of Weyl-Invariant $p$-Brane Theories

3.1 Weyl-Invariant Branes: Action and Equations of Motion

The identity (7) suggests how to construct \textbf{Weyl-invariant} $p$-brane models for any $p$. Namely, we propose the following novel $p$-brane actions:

$$S = - \int d^{p+1} \sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - \sqrt{F_{ab}(A) F_{cd}(A) \gamma^{ac\gamma^{bd}}} \right]$$

(9)

3
\[
\Phi(\varphi) \equiv \frac{1}{(p+1)!} \varepsilon_{i_1 \ldots i_{p+1}} \varepsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} \varphi^{i_1} \ldots \partial_{a_{p+1}} \varphi^{i_{p+1}}, \tag{10}
\]

where notations similar to those in (2) are used (here \(a, b = 0, 1, \ldots, p; i, j = 1, \ldots, p + 1\)).

The above action (9) is invariant under Weyl-conformal symmetry (the same as in the dynamical-tension string model (2)):

\[
\gamma_{ab} \rightarrow \gamma^\prime_{ab} = \rho \gamma_{ab}, \quad \varphi^i \rightarrow \varphi^i = \varphi^i(\varphi) \quad \text{with} \quad \det \left| \frac{\partial \varphi^i}{\partial \varphi^i} \right| = \rho. \tag{11}
\]

We notice the following significant differences of (9) w.r.t. the standard Nambu-Goto \(p\)-branes (in the Polyakov-like formulation):

- New non-Riemannian integration measure density \(\Phi(\varphi)\) instead of \(\sqrt{-\gamma}\), and no “cosmological-constant” term \((p-1)\sqrt{-\gamma}\);
- Variable brane tension \(\chi \equiv \frac{\Phi(\varphi)}{\sqrt{-\gamma}}\) which is Weyl-conformal gauge dependent: \(\chi \rightarrow \rho^\frac{1}{2}(1-p)\chi\);
- Auxiliary world-volume gauge field \(A_a\) in a “square-root” Maxwell (Yang-Mills) term\(^1\); the latter is straightforwardly generalized to the non-Abelian case \(-\sqrt{-\text{Tr} \left( F_{ab}(A)F_{cd}(A) \right) \gamma^{acbd}}\) similarly to (8);
- Natural optional couplings of the auxiliary gauge field \(A_a\) to external world-volume “color” charge currents \(j^a\);
- The action (9) is manifestly Weyl-conformal invariant for any \(p\); it describes intrinsically light-like \(p\)-branes for any even \(p\).

The eqs. of motion w.r.t. measure-building auxiliary scalars \(\varphi^i\) are:

\[
\frac{1}{2} \gamma^{cd} \left( \partial_c X \partial_d X - \sqrt{FF}\gamma \right) = M \left( = \text{const} \right), \tag{12}
\]

employing the short-hand notations:

\[
(\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \quad \sqrt{FF}\gamma \equiv \sqrt{F_{ab}F_{cd} \gamma^{acbd}}. \tag{13}
\]

The eqs. of motion w.r.t. \(\gamma^{ab}\) read:

\[
\frac{1}{2} \left( \partial_a X \partial_b X \right) + \frac{F_{ac} F_{db}}{\sqrt{FF}\gamma} = 0, \tag{14}
\]

and (upon taking the trace) imply \(M = 0\) in Eq.(12).

Next we have the following eqs. of motion w.r.t. auxiliary gauge field \(A_a\) and w.r.t. \(X^\mu\), respectively:

\[
\partial_b \left( \frac{F_{cd} \gamma^{acbd}}{\sqrt{FF}\gamma} \Phi(\varphi) \right) = 0, \tag{15}
\]

\[
\partial_a \left( \Phi(\varphi) \right) + \Phi(\varphi) \partial_a X^\mu \partial_b X^\nu \partial_c X^{\lambda} \Gamma_{\mu\nu\lambda} = 0, \tag{16}
\]

where \(\Gamma_{\mu\nu\lambda} = \frac{1}{2} G_{\mu\kappa} \left( \partial_b G_{\kappa\lambda} + \partial_\lambda G_{\kappa\nu} - \partial_\nu G_{\kappa\lambda} \right)\) is the affine connection corresponding to the external space-time metric \(G_{\mu\nu}\).

\(^1\)“Square-root” Maxwell (Yang-Mills) action in \(D = 4\) was originally introduced in the first ref.[5] and later generalized to “square-root” actions of higher-rank antisymmetric tensor gauge fields in \(D \geq 4\) in the second and third refs.[5].
### 3.2 Intrinsically Light-Like Branes

Let us consider the $\gamma^{ab}$-eqs. of motion (14). $F_{ab}$ is an anti-symmetric $(p + 1) \times (p + 1)$ matrix, therefore, $F_{ab}$ is not invertible in any odd $(p + 1)$ – it has at least one zero-eigenvalue vector $V^a$ ($F_{ab} V^b = 0$). Therefore, for any odd $(p + 1)$ the induced metric:

$$g_{ab} \equiv (\partial_a X \partial_b X) \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \quad (17)$$

on the world-volume of the Weyl-invariant brane (9) is singular as opposed to the ordinary Nambu-Goto brane (where the induced metric is proportional to the intrinsic Riemannian world-volume metric):

$$(\partial_a X \partial_b X) V^b = 0 \quad \text{i.e.} \quad (\partial_V X \partial_V X) = 0 \quad (\partial_\perp X \partial_V X) = 0 \quad (18)$$

where $\partial_V \equiv V^a \partial_a$ and $\partial_\perp$ are derivates along the tangent vectors in the complement of the tangent vector field $V^a$.

Thus, we arrive at the following important conclusion: every point on the world-surface of the Weyl-invariant $p$-brane (9) (for odd $(p + 1)$) moves with the speed of light in a time-evolution along the zero-eigenvalue vector-field $V^a$ of the world-volume electromagnetic field-strength $F_{ab}$. Therefore, we will name (9) (for odd $(p + 1)$) by the acronym WILL-brane (Weyl-Invariant Light-Like-brane) model.

### 3.3 Dual Formulation of WILL-Branes

The $A_a$-eqs. of motion (15) can be solved in terms of $(p - 2)$-form gauge potentials $\Lambda_{a_1...a_{p-2}}$ dual w.r.t. $A_a$. The respective field-strengths are related as follows:

$$F_{ab}(A) = -\frac{1}{\chi} \sqrt{-\gamma} \varepsilon_{a_1...a_{p-1}c_1...c_{p-1}} \cdots \gamma^c d \cdots \gamma^{c_{p-1}d_{p-1}} F_{d_1...d_{p-1}}(\Lambda) \gamma^{cd} (\partial_{a_1} X \partial_{d} X) , \quad (19)$$

$$\chi^2 = -\frac{2}{(p - 1)^2} \gamma^{a_1 b_1} \cdots \gamma^{a_{p-1} b_{p-1}} F_{a_1...a_{p-1}}(\Lambda) F_{b_1...b_{p-1}}(\Lambda) , \quad (20)$$

where $\chi \equiv \Phi^{(p)} / \sqrt{-\gamma}$ is the variable brane tension, and:

$$F_{a_1...a_{p-1}}(\Lambda) = (p - 1) \partial_{[a_1} \Lambda_{a_2...a_{p-1}]} \quad (21)$$

is the $(p - 1)$-form dual field-strength.

All eqs. of motion can be equivalently derived from the following dual WILL-brane action:

$$S_{\text{dual}} = -\frac{1}{2} \int d^{p+1} \sigma \chi(\gamma, \Lambda) \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \quad (22)$$

with $\chi(\gamma, \Lambda)$ given in (20) above.

### 4 Special case $p = 2$: WILL-Membrane

The WILL-membrane dual action (particular case of (22) for $p = 2$) reads:

$$S_{\text{dual}} = -\frac{1}{2} \int d^3 \sigma \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} (\partial_a X \partial_b X) , \quad (23)$$

$$\chi(\gamma, u) \equiv \sqrt{-2 \gamma^{cd} \partial_c u \partial_d u} \quad (24)$$
where $u$ is the dual “gauge” potential w.r.t. $A_a$:

$$F_{ab}(A) = -\frac{1}{2\chi(\gamma, u)}\sqrt{-\gamma} \varepsilon_{abc}\gamma^{cd}\partial_d u \gamma^{ef}(\partial_e X \partial_f X).$$

(25)

$S_{\text{dual}}$ is manifestly Weyl-invariant (under $\gamma_{ab} \rightarrow \rho \gamma_{ab}$).

The eqs. of motion w.r.t. $\gamma_{ab}$, $u$ (or $A_a$), and $X^\mu$ read accordingly:

$$\left(\partial_a X \partial_b X\right) + \frac{1}{2} \gamma^{cd} \left(\partial_c X \partial_d X\right) \left(\frac{\partial_a u \partial_b u}{\gamma^{ef} \partial_e u \partial_f u} - \gamma_{ab}\right) = 0,$$

(26)

$$\partial_a \left(\frac{\sqrt{-\gamma} \gamma^{ab} \partial_b u}{\chi(\gamma, u)}\right) \gamma^{cd} \left(\partial_c X \partial_d X\right) = 0, \quad (27)$$

$$\partial_a \left(\chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu\right) + \chi(\gamma, u) \sqrt{-\gamma} \gamma^{ab} \partial_b X^\nu \partial_b X^\lambda \Gamma^\mu_{\nu\lambda} = 0.$$  

(28)

The first eq. above shows that the induced metric $g_{ab} \equiv \left(\partial_a X \partial_b X\right)$ has zero-mode eigenvector $V^a = \gamma^{ab} \partial_b u$.

The invariance under world-volume reparametrizations allows to introduce the following standard (synchronous) gauge-fixing conditions:

$$\gamma^{0i} = 0 \quad (i = 1, 2) \, , \quad \gamma^{00} = -1.$$  

(29)

In what follows we will use the ansatz for the dual “gauge potential”:

$$u(\tau, \sigma^1, \sigma^2) = \frac{T_0}{\sqrt{2}} \tau,$$

(30)

where $T_0$ is an arbitrary integration constant with the dimension of membrane tension. In particular:

$$\chi \equiv \sqrt{-2\gamma^{ab} \partial_a u \partial_b u} = T_0$$

(31)

This means that we take $\tau \equiv \sigma^0$ to be evolution parameter along the zero-eigenvalue vector-field of the induced metric on the brane ($V^a = \gamma^{ab} \partial_b u = \text{const} \ (1, 0, 0)$).

The ansatz for $u$ (30) together with the gauge choice for $\gamma_{ab}$ (29) brings the eqs. of motion w.r.t. $\gamma_{ab}$, $u$ (or $A_a$) and $X^\mu$ in the following form (recall $\left(\partial_a X \partial_b X\right) \equiv \partial_a X^\mu \partial_b X^\nu \Gamma^\mu_{\nu\lambda}$):

$$\left(\partial_a X \partial_b X\right) = 0 \, , \quad \left(\partial_a X \partial_b X\right) = 0,$$

(32)

$$\left(\partial_a X \partial_b X\right) - \frac{1}{2} \gamma^{ij} \gamma^{kl} \left(\partial_i X \partial_j X\right) = 0,$$

(33)

(Eqs.(33) look exactly like the classical (Virasoro) constraints for an Euclidean string theory with world-sheet parameters $(\sigma^1, \sigma^2)$):

$$\partial_0 \left(\gamma_{(2)} \gamma^{kl} \left(\partial_k X \partial_l X\right)\right) = 0,$$

(34)

where $\gamma_{(2)} \equiv \det \|\gamma_{ij}\|$ (the above equation is the only remnant from the $A_a$-eqs. of motion (15));

$$\Box^{(3)} X^\mu + \left(-\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl} \partial_k X^\nu \partial_l X^\lambda\right) \Gamma^\mu_{\nu\lambda} = 0,$$

(35)

where:

$$\Box^{(3)} \equiv -\frac{1}{\sqrt{\gamma_{(2)}}} \partial_0 \left(\sqrt{\gamma_{(2)}^0} \partial_0\right) + \frac{1}{\sqrt{\gamma_{(2)}}} \partial_i \left(\sqrt{\gamma_{(2)}^{ij}} \partial_j\right).$$

(36)
5 WILL-Membrane Solutions in Various Gravitational Backgrounds

5.1 Example: WILL-Membrane in a PP-Wave Background

As a simplest non-trivial example let us consider in (23) external space-time metric $G_{\mu\nu}$ for plane-polarized gravitational wave (pp-wave) background:

$$(ds)^2 = -dx^+dx^- - F(x^+, x^I) (dx^+)^2 + dx^I dx^I,$$

and employ in (32)–(36) the following natural ansatz for $X^\mu$ (here $\sigma_0 \equiv \tau$; $I = 1, \ldots, D - 2$):

$$X^- = \tau, \quad X^+ = X^+(\tau, \sigma^1, \sigma^2), \quad X^I = X^I(\sigma^1, \sigma^2).$$

The non-zero affine connection symbols for the pp-wave metric (37) are: $\Gamma^{++} = \partial_+ F$, $\Gamma^{I+} = \partial_I F$, $\Gamma^{II} = \frac{1}{2} \partial_I F$.

It is straightforward to show that the solution does not depend on the form of the pp-wave front $F(x^+, x^I)$ and reads:

$$X^+ = X^+_0 = \text{const}, \quad \gamma_{ij} = \tau-\text{independent};$$

$$\partial_i X^I \partial_j X^I + \frac{1}{2} \gamma_{ij} \partial_k X^I \partial_l X^I = 0, \quad \partial_i \left( \sqrt{\gamma} \gamma^{ij} \partial_j X^I \right) = 0,$$

where the latter eqs. describe a string embedded in the transverse $(D-2)$-dimensional flat Euclidean space.

5.2 Example: WILL-Membrane in a Schwarzschild Black Hole

Let us consider spherically-symmetric static gravitational background:

$$(ds)^2 = -A(r)(dt)^2 + B(r)(dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta) (d\phi)^2].$$

For the Schwarzschild black hole we have $A(r) = B^{-1}(r) = 1 - \frac{2GM}{r}$.

We find the following solution to the eqs. of motion (and constraints) (32)–(36). Using the ansatz:

$$X^0 \equiv t = \tau, \quad X^1 \equiv r = r(\tau, \sigma^1, \sigma^2), \quad X^2 \equiv \theta = \theta(\sigma^1, \sigma^2), \quad X^3 \equiv \phi = \phi(\sigma^1, \sigma^2),$$

$$\gamma_{ij} = a(\tau) \tilde{\gamma}_{ij}(\sigma^1, \sigma^2),$$

with $\tilde{\gamma}_{ij}$ being some standard reference 2D metric on the membrane surface ($i, j = 1, 2$), we obtain from Eqs.(32) taking into account (41):

$$\frac{\partial}{\partial \tau} r = \pm A(r), \quad \frac{\partial}{\partial \sigma^i} r = 0.$$  (44)

From Eq.(34) we get $\frac{\partial}{\partial \tau} r = 0$ which upon combining with (44) gives:

$$r = r_0 \equiv 2GM = \text{const}, \quad \text{i.e.} \quad A(r_0) = 0.$$  (45)

For the rest of embedding coordinates and the intrinsic WILL-membrane metric (upon assuming the membrane surface to be of spherical topology) we obtain:

$$\theta = \sigma^1, \quad \phi = \sigma^2, \quad ||\gamma_{ij}|| = c_0 e^{\mp \tau/r_0} \left( \begin{array}{cc} 1 & 0 \\ 0 & \sin^2(\sigma^1) \end{array} \right),$$

where $c_0$ is an arbitrary integration constant.

That is, the WILL-membrane with spherical topology (and with exponentially blowing-up/deflating internal metric) “sits” on (materializes) the event horizon of the Schwarzschild black hole.
5.3 Example: WILL-Membrane in a Reissner-Nordström Black Hole

Now we need to extend the WILL-brane model (9) via a coupling to external space-time electromagnetic field $A_\mu$. The natural Weyl-conformal invariant candidate action reads (for $p = 2$):

$$ S = - \int d^3 \sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \sqrt{F_{ab}F_{cd}\gamma^{ac}\gamma^{bd}} \right] - q \int d^3 \sigma \varepsilon^{abc} A_\mu \partial_a X^\mu F_{bc} . \quad (47) $$

The last Chern-Simmons-like term is a special case of a class of Chern-Simmons-like couplings of extended objects to external electromagnetic fields proposed in ref.[6].

In the dual formulation we get accordingly:

$$ S_{\text{dual}} = - \frac{1}{2} \int d^3 \sigma \chi(\gamma, u, A) \sqrt{\gamma^{ab} (\partial_a X \partial_b X)} , \quad (48) $$

with a variable brane tension:

$$ \chi(\gamma, u, A) \equiv \sqrt{-2 \gamma^{cd} (\partial_c u - q A_c) (\partial_d u - q A_d)} , \quad A_a \equiv A_\mu \partial_a X^\mu . \quad (49) $$

Here $u$ is the dual “gauge” potential w.r.t. $A_a$ and the corresponding field-strength and dual field-strength are related as:

$$ F_{ab}(A) = - \frac{1}{2 \chi(\gamma, u, A)} \sqrt{-\gamma^{ef} \gamma^{cd} (\partial_d u - q A_d) \gamma^{ef} (\partial_e u - q A_e)} \chi(\gamma, u, A) \gamma^{cd} (\partial_c X \partial_d X) . \quad (50) $$

The extended WILL-membrane model in the dual formulation (48) is likewise manifestly Weyl-invariant (under $\gamma_{ab} \rightarrow \rho \gamma_{ab}$).

The eqs. of motion w.r.t. $\gamma^{ab}$, $u$ (or $A_a$), and $X^\mu$ read accordingly:

$$ (\partial_a X \partial_b X) + \frac{1}{2} \gamma^{cd} (\partial_c X \partial_d X) \left( \frac{(\partial_a u - q A_a) (\partial_b u - q A_b)}{\gamma^{ef} (\partial_e u - q A_e) (\partial_f u - q A_f)} - \gamma^{ab} \right) = 0 ; \quad (51) $$

$$ \partial_a \left( \sqrt{-\gamma^{ab} \partial_b X} \chi(\gamma, u, A) \gamma^{cd} (\partial_c X \partial_d X) \right) = 0 ; \quad (52) $$

$$ \partial_a \left( \chi(\gamma, u, A) \sqrt{-\gamma^{ab} \partial_b X^\mu} \right) + \chi(\gamma, u, A) \sqrt{-\gamma^{ab} \partial_a X^\mu \partial_b X^\lambda \Gamma_{\nu\lambda}^{\nu\lambda}} - q \varepsilon^{abc} F_{bc} \partial_a X^\nu \left( \partial_{\lambda} A_{\nu} - \partial_{\nu} A_{\lambda} \right) G^\lambda_{\mu} = 0 . \quad (53) $$

Using the same (synchronous) gauge choice (29) and ansatz for the dual “gauge potential” (30), as well as considering static external space-time electric field ($A_0 = Q/\sqrt{4\pi r}$ – relevant case for Reissner-Nordström blackholes, see next Section), the eqs. of motion w.r.t. $\gamma^{ab}$, $u$ (or $A_a$) and $X^\mu$ acquire the form (recall $\partial_a X \partial_b X \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$):

$$ (\partial_0 X \partial_0 X) = 0 , \quad (\partial_0 X \partial_i X) = 0 , \quad (54) $$

$$ (\partial_i X \partial_j X) - \frac{1}{2} \gamma_{ij} \gamma^{kl} (\partial_k X \partial_l X) = 0 , \quad (55) $$

(these constraints are the same as in the absence of coupling to space-time gauge field (32)–(33));

$$ \partial_0 \left( \sqrt{\gamma(2)} \gamma^{kl} (\partial_k X \partial_l X) \right) = 0 , \quad (56) $$
(once again the same equation as in the absence of coupling to space-time gauge field (34));

\[ \Box^{(3)} X^\mu + \left( -\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{k\ell} \partial_k X^\nu \partial_\ell X^\lambda \right) \Gamma^\mu_{\nu\lambda} - q \frac{\gamma^{kl} (\partial_k X \partial_\ell X)}{\sqrt{\chi}} \partial_0 X^\nu (\partial_\lambda A_\nu - \partial_\nu A_\lambda) G^{\lambda \mu} = 0 , \]

where \( \chi \equiv T_0 - \sqrt{2qA_0} \) (the variable brane tension), \( A_1 = \ldots = A_{D-1} = 0 \), and:

\[ \Box^{(3)} \equiv - \frac{1}{\chi \sqrt{\gamma^{(2)}}} \partial_0 \left( \chi \sqrt{\gamma^{(2)}} \partial_0 \right) + \frac{1}{\chi \sqrt{\gamma^{(2)}}} \partial_0 \left( \chi \sqrt{\gamma^{(2)}} \gamma^{ij} \partial_j \right). \]

Now, let us solve Eqs.(54)–(58) in Reissner-Nordstr"om background:

\[ (ds)^2 = -A(r)(dt)^2 + A^{-1}(dr)^2 + r^2((d\theta)^2 + \sin^2(\theta) (d\phi)^2) \]

\[ A(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}. \]

Employing the same ansatz (42) as in the case of Schwarzschild background, the solution for Reissner-Nordstr"om background reads:

\[ X^0 \equiv t = \tau , \quad \theta = \sigma^1 , \quad \phi = \sigma^2 \]

\[ r(\tau, \sigma^1, \sigma^2) = r_{\text{horizon}} = \text{const} \]

where \( A(r_{\text{horizon}}) = 0 \);

\[ \|\gamma_{ij}\| = \left[ c_0 e^{\frac{r}{\chi} (\frac{2}{\chi} A)_{r=r_{\text{horizon}}} + \frac{qQ}{2\pi (\chi \frac{2}{\chi} A)_{r=r_{\text{horizon}}}}} \left( \begin{array}{cc} 1 & 0 \\ 0 & \sin^2(\sigma^1) \end{array} \right) \right] \]

where \( c_0 \) is an arbitrary integration constant (recall \( \chi \equiv T_0 - \sqrt{2qA_0} \)).

In particular, taking \( c_0 = 0 \) one obtains the usual time-independent internal spherical metric on the brane surface. Thus, similar to the Schwarzschild case, the WILL-membrane with spherical topology “sits” on (materializes) the event horizon of the Reissner-Nordström black hole.

### 6 Coupled Einstein-Maxwell-WILL-Membrane System

We can extend the results from the previous section to the case of the full coupled Einstein-Maxwell-WILL-membrane system, i.e., taking into account the back-reaction of the WILL-membrane serving as a material and electrically charged source for gravity and electromagnetism. The pertinent action reads:

\[ S = \int d^4x \sqrt{-G} \left[ \frac{R}{16\pi G_N} - \frac{1}{4} F_{\mu\nu}(A) F_{\kappa\lambda}(A) G^{\mu\kappa} G^{\nu\lambda} \right] + S_{\text{WILL-brane}} , \]

where \( F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu \), and:

\[ S_{\text{WILL-brane}} = - \int d^3\sigma \Phi(\varphi) \left[ \frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \sqrt{F_{ab} F_{cd} \gamma^{ac} \gamma^{bd}} \right] - q \int d^3\sigma \varepsilon^{abc} A_\mu \partial_a X^\mu F_{bc}. \]

Eqs. of motion for the WILL-membrane subsystem are the same as above, namely Eqs.(54)–(58). The rest of the eqs. of motion are:

\[ R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} \right) = 8\pi G_N \left( T^{(EM)}_{\mu\nu} + T^{(brane)}_{\mu\nu} \right) , \]
\[ \partial \nu \left( \sqrt{\gamma} G^{\mu \nu} F_{\nu \lambda} \right) + j^\mu = 0 , \]  

where:

\[ T^{(EM)}_{\mu \nu} \equiv F_{\mu \nu} F_{\nu \lambda} G^{\nu \lambda} - G_{\mu \nu} \frac{1}{4} F_{\rho \kappa} F_{\sigma \lambda} G^{\rho \sigma} G^{\nu \lambda} , \]

\[ T^{(brane)}_{\mu \nu} \equiv - G_{\mu \kappa} G_{\nu \lambda} \int d^3 \sigma \delta^{(4)}(x - X(\sigma)) \chi \sqrt{-G} \chi_{ab} \partial_a X^\kappa \partial_b X^\lambda , \]

(recall \( \chi \equiv \sqrt{-2\gamma_{cd} (\partial_c u - qA_c) (\partial_d u - qA_d)} , \ A_a \equiv A_{\mu} \partial_a X^\mu ) , \]

\[ j^\mu \equiv q \int d^3 \sigma \delta^{(4)}(x - X(\sigma)) \epsilon^{abc} F_{bc} \partial_a X^\mu . \]

Following the same steps as in the previous section we obtain the following spherically symmetric stationary solution. For the Einstein subsystem we find a solution:

\[ (ds)^2 = -A(r) (dt)^2 + A^{-1}(dr)^2 + r^2 [(d\theta)^2 + \sin^2(\theta)(d\phi)^2] , \]

consisting of two different black holes with a common event horizon:

- Schwarzschild black hole inside the horizon:
  \[ A(r) \equiv A_-(r) = 1 - \frac{2GM_1}{r} , \text{ for } r < r_0 \equiv r_{horizon} = 2GM_1 . \]  

- Reissner-Norström black hole outside the horizon:
  \[ A(r) \equiv A_+(r) = 1 - \frac{2GM_2}{r} + \frac{GQ^2}{r^2} , \text{ for } r > r_0 \equiv r_{horizon} , \]

where \( Q^2 = 8\pi q^2 r^4_{horizon} \equiv 128\pi q^2 G^4 M_1^4 \). For the Maxwell subsystem we get \( A_1 = \ldots = A_{D-1} = 0 \) everywhere and:

- Coulomb field outside horizon:
  \[ A_0 = \frac{\sqrt{2} q r_{horizon}^2}{r} , \text{ for } r \geq r_0 \equiv r_{horizon} . \]

- No electric field inside horizon:
  \[ A_0 = \sqrt{2} q r_{horizon} = \text{ const} , \text{ for } r \leq r_0 \equiv r_{horizon} . \]

The WILL-membrane again “sits” on (materializes) the common event horizon of the pertinent black holes:

\[ X^0 \equiv t = \tau , \ \theta = \sigma^1 , \ \phi = \sigma^2 , \ r(\tau, \sigma^1, \sigma^2) = r_{horizon} = \text{ const} \]

In addition there is an important matching condition for the metric components along the WILL-membrane:

\[ \frac{\partial}{\partial r} A_+ \bigg|_{r=r_{horizon}} - \frac{\partial}{\partial r} A_- \bigg|_{r=r_{horizon}} = -16\pi G \chi , \]
which yields the following relations between the parameters of the black holes and the \textit{WILL}-membrane ($q$ being its surface charge density):

$$M_2 = M_1 + 32\pi q^2 G^3 M_1^3$$  \hspace{1cm} (78)

and for the brane tension $\chi$:

$$\chi \equiv T_0 - 2q^2 r_{\text{horizon}} = q^2 G M_1^2$$ \hspace{1cm} i.e. \hspace{0.3cm} $T_0 = 5q^2 G M_1$  \hspace{1cm} (79)

The matching condition (77) corresponds to the statically soldering conditions in the light-like thin shell dynamics in general relativity [7]. On the other hand we should stress that unlike the latter phenomenological models of thin shell dynamics (\textit{i.e.}, where the membranes are introduced \textit{ad hoc}), the present \textit{WILL}-brane models provide a systematic description of light-like branes from first principles starting with concise Weyl-conformal invariant actions (9), (64)–(65). As a consequence, these actions also yield additional information impossible to obtain within the phenomenological approach, such as the requirement that the light-like brane must sit on the event horizon of the pertinent black hole.

\section{Conclusions and Outlook}

In the present work we have demonstrated that employing alternative non-Riemannian world-sheet/world-volume integration measure significantly affects string and $p$-brane dynamics:

- Acceptable dynamics in the novel class of string/brane models (Eqs.(2) and (9)) \textit{naturally} requires the introduction of auxiliary world-sheet/world-volume gauge fields.

- By employing square-root Yang-Mills actions for the auxiliary world-sheet/world-volume gauge fields one achieves manifest \textit{Weyl-conformal symmetry} in the new class of $p$-brane theories for \textit{any} $p$.

- The string/brane tension is \textit{not} a constant dimensionful scale given \textit{ad hoc}, but rather it appears as an \textit{additional dynamical degree of freedom} beyond the ordinary string/brane degrees of freedom.

- The novel class of Weyl-invariant $p$-brane theories describes intrinsically \textit{light-like} $p$-branes for \textit{any} even $p$ (\textit{WILL}-branes).

- When put in a gravitational black hole background, the \textit{WILL}-membrane ($p = 2$) sits on ("materializes") the event horizon.

- The coupled Einstein-Maxwell-\textit{WILL}-membrane system (64) possesses self-consistent solution where the \textit{WILL}-membrane serves as a material and electrically charged source for gravity and electromagnetism, and it "sits" on (materializes) the common event horizon for a Schwarzschild (in the interior) and Reissner-Nordström (in the exterior) black holes. Thus our model (64) provides an explicit dynamical realization of the so called "membrane paradigm" in the physics of black holes [8].

One can think of various physically interesting directions of further research on the novel class of Weyl-conformal invariant $p$-branes such as: quantization (Weyl-conformal anomaly and critical dimensions); supersymmetric extension; possible relevance for the open string dynamics (similar to
the Dirichlet- \((D_p)\)-branes); \textit{WILL}-brane dynamics in more complicated gravitational black hole backgrounds (e.g., Kerr-Newman) etc.

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