Gravitational shock waves and vacuum fluctuations

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Abstract: We show that the vacuum expectation value of the stress-energy tensor of a scalar particle on the background of a spherical gravitational shock wave does not give a finite expression in second order perturbation theory, contrary to the case seen for the impulsive wave. No infrared divergences appear at this order. This result shows that there is a qualitative difference between the shock and impulsive wave solutions which is not exhibited in first order.
INTRODUCTION

Both from physical and mathematical points, the cosmic string solutions [1] of Einstein’s field equations are interesting [2]. An immediate question is whether these strings decay. Exact solutions describing such decays are given for impulsive waves by Nutku-Penrose [3] and Gleiser and Pullin [4], for shock waves by Nutku [5].

Once these solutions are found one may question whether they give rise to vacuum fluctuations. We have investigated these fluctuations in several papers. In first order perturbation theory, we found that we could not isolate a finite part for the vacuum expectation value, VEV, of the stress-energy tensor both for the impulsive [6] and shock wave solutions [7]. When the calculation is carried to second order for the impulsive wave case, a finite result is found [8] if a detour is taken to de Sitter space. The essential point in this calculation is the generation of an infrared divergence in second order perturbation theory which is regulated by an infrared mass. We go to de Sitter space and there cancel this mass by the cosmological constant. At the end we let both the infrared mass and the cosmological constant go to zero and obtain a finite result.

In this note we carry the calculation in the background of the shock wave metric to second order and investigate whether the same trick gives us a finite expression for the VEV of the stress-energy tensor for this case. If the calculation does not generate an infrared divergence, going to de Sitter space gives us a finite result only in this space which vanishes when we go back to Minkowski space [9].

In first order perturbation theory both the impulsive and the shock wave cases showed similar behaviour. The two solutions are essentially different, though. The shock wave solution has a dimensional constant which is lacking in the impulsive wave solution. Since in quantum field theory, models with dimensional and dimensionless constants belong to different classes, we thought similar distinction between these two models may exist. Relying on these motivations we planned to check whether there is a qualitative difference between the two solutions exhibited by their behaviour at higher orders.

We will show that the infrared divergences which may be cancelled via a detour in de Sitter space are absent in the shock wave calculation. Another point of difference is the importance attributed to the homogenous solutions in these two cases. The homogenous solutions give just the free Green's function for the impulsive case, whereas they result in a totally different contribution to the Green's function in the shock wave. This is an artefact of the presence of a dimensional coupling constant in the latter case. What may be more interesting is the fact that just the contribution of the first order calculation contributes to \( < T_{\mu\nu} > \) in de Sitter space. The higher order terms cancel out when the VEV of the stress-energy tensor is computed.

We give our calculation in the next section. We calculate the Green's function and the VEV of the stress-energy tensor exactly as we have done in references 6-9. These methods are more thoroughly described in [10], details are in [11]. We conclude with few remarks.
CALCULATION

We start with the metric
\[ ds^2 = 2P dudv + 2uP_\zeta d\zeta dv + 2uP_\zeta d\zeta dv - 2u^2 d\zeta d\zeta \]
whose properties are described in references 5 and 7. Here \( P = \frac{1}{|\nabla\zeta|} \), where \( \zeta \) is an arbitrary function of the argument \( \zeta + gv\Theta(v) \). \( g \) is the dimensional coupling constant and \( \Theta \) is the Heavyside unit step function. In our particular case we take \( h = (\zeta + gv\Theta(v))^{1+i\delta} \), where \( \delta << 1 \) and is the expansion parameter.

We expand the operator \( L \) which is equal to \( \sqrt{-g} \) times the d’Alembertian operator, in powers of \( \delta \).
\[ L = L_0 + \delta L_1 + \delta^2 L_2 + ... \]

If we use real variables, \( \zeta = \frac{x+iy}{\sqrt{2}} \), we get
\[ L_0 = 2u^2 \frac{\partial^2}{\partial u \partial v} + 2u \frac{\partial}{\partial v} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \]
\[ L_1 = -\frac{2u}{x^2 + y^2} \left( \frac{y}{\partial x} - x \frac{\partial}{\partial y} \right) \frac{\partial}{\partial u} - tan^{-1} \frac{y}{x} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \]
\[ L_2 = -\frac{1}{x^2 + y^2} \left( u \frac{\partial}{\partial u} + u^2 \frac{\partial^2}{\partial u^2} + 2tan^{-1} \frac{y}{x} \left( \frac{y}{\partial x} - x \frac{\partial}{\partial y} \right) u \frac{\partial}{\partial u} \right) \]
\[ + \frac{1}{2} \left( 1 - (tan^{-1}\frac{y}{x})^2 \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]

We expand both the solutions and the eigenvalue in terms of \( \delta \), \( \phi = \phi_0 + \delta \phi_1 + \delta^2 \phi_2 + ... , \lambda = \lambda_0 + \delta \lambda_1 + \delta^2 \lambda_2 + ... \). We take \( \phi_1 = f \phi_0, \phi_2 = h \phi_0 \). Here \( f = f_0(z,y,u) + g f_1(z,y,u), h = h_0(z,y,u) + gh_1(z,y,u) \) where \( z = x + gv\Theta(v) \). \( \phi_0 \) is given by
\[ \phi_0 = \exp[i(k_1x + k_2y + Rv - \frac{K}{\pi^2})] \]
\[ u \sqrt{|R|}(2\pi)^2. \]

\( K, k_1, k_2, R \) are the separation constants which act as eigenfrequencies to be integrated over to find the Greens Function.

In second order in \( \delta \), we can reduce the differential equation to the system
\[ L_0 h_0 = I_0, \]
\[ L_0 h_1 + L_1 h_0 = I_1, \]
where
\[ L_0 = -2iR \frac{\partial}{\partial s} - 2i \left( k_1 \frac{\partial}{\partial z} + k_2 \frac{\partial}{\partial y} \right) - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2}, \]
\[ L_1 = -2u \frac{\partial}{\partial v} + \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \]
\[ L_2 = - \frac{1}{x^2 + y^2} \left( u \frac{\partial}{\partial u} + u^2 \frac{\partial^2}{\partial u^2} + 2tan^{-1} \frac{y}{x} \left( \frac{y}{\partial x} - x \frac{\partial}{\partial y} \right) u \frac{\partial}{\partial u} \right) \]
\[ + \frac{1}{2} \left( 1 - (tan^{-1}\frac{y}{x})^2 \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]
\[ \mathcal{L}_1 = -2 \frac{\partial^2}{\partial s \partial z} + \frac{iK}{R} \frac{\partial}{\partial z}. \]

Here \( s = \frac{1}{u} \). \( I_0 \) and \( I_1 \) are given as

\[
I_0 = \frac{1}{2} K - \frac{1}{z^2 + y^2} \left( 1 - \frac{3iKs}{2R} - \frac{K^2 s^2}{4R^2} \right) \left( 1 - 2i(k_1 y - k_2 z) \tan^{-1} \frac{y}{z} \right) + K \left( \frac{5}{2} - \frac{iKs}{2R} \right) \left( \tan^{-1} \frac{y}{z} \right)^2
\]

\[
-2K \left( \frac{\log(z^2 + y^2)}{2} - 1 \right) \left( k_1 y - k_2 z \right) + \tan^{-1} \frac{y}{z} \left( k_1 z + k_2 y \right)
\times \left( 1 - \frac{iKs}{R} \frac{k_1 y - k_2 z}{(z^2 + y^2)K} - \frac{iKs}{2R} \tan^{-1} \frac{y}{z} \right)
\]

\[
- \frac{i}{z^2 + y^2} \left( 1 - \frac{iKs}{2R} \left( (k_1 z + k_2 y) \log(z^2 + y^2) + 2(k_2 z - k_1 y) \tan^{-1} \frac{y}{z} \right) \right)
\]

\[
I_1 = \frac{2iyK}{(z^2 + y^2)R} \left( 1 - \frac{iKs}{4R} \tan^{-1} \frac{y}{z} - \frac{2k_1}{(z^2 + y^2)R} \left( 1 - \frac{iKs}{R} - \frac{K^2 s^2}{8R^2} \right) \right)
\]

\[
- \left( 2k_1 \tan^{-1} \frac{y}{z} - k_2 \log(z^2 + y^2) \right)
\times \left( \left( -\frac{iK^2 s}{8R^2} + \frac{5K}{4R} \right) \tan^{-1} \frac{y}{z} - \frac{i(k_1 y - k_2 z)}{(z^2 + y^2)R} \left( 1 - \frac{iKs}{R} - \frac{K^2 s^2}{8R^2} \right) \right)
\]

\[
+ \left( k_1^2 - k_2^2 \right) \left( 2z \tan^{-1} \frac{y}{z} + y \log(z^2 + y^2) - 2y \right) - 2k_1 k_2 \left( z \log(z^2 + y^2) - 2y \tan^{-1} \frac{y}{z} - 2z \right)
\times \left( \left( \frac{3Ki}{8R} \tan^{-1} \frac{y}{z} - \frac{3}{4(z^2 + y^2)R} (k_1 y - k_2 z) (1 - \frac{iKs}{2R}) \right) - \frac{3i}{4(z^2 + y^2)R} \left( 1 - \frac{iKs}{2R} \right) \right)
\]

\[
\times \left( (k_1^2 - k_2^2) \left( z \log(z^2 + y^2) - 2y \tan^{-1} \frac{y}{z} \right) + 2k_1 k_2 \left( y \log(z^2 + y^2) + 2z \tan^{-1} \frac{y}{z} \right) \right)
\]

In these expressions we took the ‘mass-shell’ condition, which is imposed in the calculation of the Greens function; i.e. we set \( k_1^2 + k_2^2 \) equal to \( K \). One can check that after we perform the \( K \) and \( k_1, k_2 \) integrations the effect of these two expressions are exactly the same.

We see that, contrary to the impulsive wave calculation, in both \( I_0 \) and \( I_1 \), there are no terms that are independent of \( z \) and \( y \) except a single term which is proportional to \( K \). To be able to obtain terms in \( < T_{\mu \nu} > \) that diverge as the infrared parameter goes to zero, we need inverse powers of \( R \) which are not multiplied by \( K \) or \( k_1^2, k_2^2 \). Each inverse power of \( R \) means a higher infrared divergence, order going as \( m^2, 1, \log m^2, \frac{1}{m^2}, \frac{1}{m^3}, \) etc..., whereas each power of \( K, k_1^2, k_2^2 \) means one lower order in the same divergence. In free space the power of \( m \) is zero. There is no divergence.
In reference 4, we generated these divergences at second order and then cancelled them with the cosmological constant of the de Sitter solution. Our mechanism for obtaining these infrared divergences was as follows. We isolated the $-2iR \frac{\partial}{\partial s}$ in the operator $L_0$ from the others and equated it to the term which did not contain $z$ or $y$.

$$-2iR \frac{\partial}{\partial s} h'_0 = cs$$

where $c$ can be a function of $v$ but not that of $z$ and $y$. Then $h'_0 = \frac{i cs^2}{4R}$ which has an extra power of $1/R$ compared to the other terms. The second iteration gives us $h'_1 \propto s^3/R^2$. Such a term will induce $1/m^2$ factor in the expression for the Greens Function, $G_F$, and this infrared mass will be retained in $\langle T_{\mu\nu} \rangle$.

For the shock wave solution all the terms in $I_0$ and $I_1$ are either functions of $z$ and $y$ or are multiplied by $K$. We can not isolate a part of the operator $L_0$ and equate it to a single term on the RHS. Note that in the previous argument we may act in this way since the rest of the operators in $L_0$ will annihilate the resulting expression $h'_0$. If this terms is not annihilated by the other operators, there will be a mismatch in the powers in $s$ on both sides of the equation. To illustrate this let’s assume we had

$$-2iR \frac{\partial}{\partial s} h'_0 = f(z, y)s^2.$$  

Upon integration we get $h'_0 = \frac{if(z, y)s^3}{6R}$. Such a term will not be annihilated when rest of the terms in $L_0$ operate on it and we will generate a higher power of $s$ than the one we have started from, which has no match on RHS. Since $I_0$ and $I_1$ do not contain powers of $s$ higher than the quadratic, there is no way we can generate terms with the third power of $s$ in this way, also no way to generate $\frac{1}{R^2}$ which will multiply such a term. Similarly we can show that we can not generate a power of $\frac{1}{R^3}$ in a combination which does not already exist on RHS. On RHS only the combination $\frac{K^2}{R^2}$ and $\frac{K}{R^2}$ exist. $\frac{K^2}{R^2}$ gives exactly the singularity structure as the free case, and $\frac{K}{R^2}$ gives a logarithmic divergence which is cancelled in the $\langle T_{\mu\nu} \rangle$ calculation.

At this point we note that we can find solutions of equations 9 and 10 even if $I_0$ and $I_1$ are set to zero. These are the homogenous solutions of the problem which give a non trivial contribution for the shock wave calculation. Since $I_0$ and $I_1$ are independent of $v$ we can assume a powers series expansion in $v$ for a chosen order in $g$. For the sake of illustration we take a solution in third order in $g$ and write the expansion as

$$f^{(1,3)}_H = g^3(v^3 f^{(1,3)}_{1H} + v^2 f^{(1,3)}_{2H} + vf^{(1,3)}_{3H} + f^{(1,3)}_{4H}).$$

Here $f^{(1,3)}_{1H}(s, z, y)$ has dimension zero, $f^{(1,3)}_{2H}(s, z, y)$ has dimension minus one, etc.. Inverse powers of $v$ are excluded by the regularity at $v = 0$. One can show that taking powers of $v$ higher than that of $g$ do not give results that differ from the free case. A similar expansion in the impulsive case would go as $f^{(1,3)}_H = (v^3 R^3 f^{(1,3)}_{1H} + v^2 R^2 f^{(1,3)}_{2H} + ...) $ when $f^{(1,3)}_{1H}$ etc.,
have the same dimensions as above, since the only free dimensional parameters are v and R. This gives the free result.

Keeping track of powers of v we get a system of four equations. We note that the first of these equations

\[ \mathcal{L}_0 f^{(1,3)}_{1,H} = 0 \]

has a solution for any function \( F = F(\frac{z}{R}(k_1 \pm ik_2) - (z \pm iy)) \). We can also show that the singularity behaviour of the Greens Function is independent of the form of \( F \). At the end we get, for the worse infrared poles the expressions

\[
G^{(1,H)}_H = 2\pi c_1 \left( \frac{v^3 \Theta(v) + v'^3 \Theta(v')}{(u - u')(v - v')} \right),
\]

\[
G^{(2,H)}_H = 2\pi c_2 \left( \frac{uv^2 \Theta(v) + u'v'^2 \Theta(v')}{(u - u')^2 \log(2m^2(u - u')(v - v'))} \right),
\]

\[
G^{(3,H)}_H = 2\pi c_3 \left( \frac{u^2 v \Theta(v) + u'^2 v' \Theta(v')}{(u - u')^4 m^2} \right),
\]

\[
G^{(4,H)}_H = 2\pi c_4 \left( \frac{u^3 \Theta(v) + u'^3 \Theta(v')}{(u - u')^6 m^4} \right),
\]

Here \( c_i \) are functions of \( x \) and \( y \), depending on the form for \( F \) used. \( m \) is the infrared mass. If we use linear function, then \( c_i \) is proportional to \( y \) or \( x \).

Upon symmetric differentiation the terms with \( \frac{1}{m^2} \) and \( \frac{1}{m^4} \) vanish. We may find a finite contribution only if we go to the de Sitter space, i.e. multiply by the factor \((1 + \Lambda u v)(1 + \Lambda u' v')\). In this case we get

\[
< T_{vv} > = g^3 \left( f_1 \Lambda v \Theta(v) + f_2 \Lambda^2 u v^2 \Theta(v) \right)
\]

which goes to zero with \( \Lambda \) when we go back to Minkowski background. In this expression \( f_1 \) are regular functions of \( x, y \). One can also show that any terms with less diverging powers of \( m^{-2} \) in \( G^{(4,H)}_H \) do not give a finite contribution even in de Sitter background.

CONCLUSION

Here we tried to show that two qualitative differences exist between the shock and the impulsive wave solutions proposed by the same group /3,5/. In the shock wave solution the infrared divergences which may be used to tame the ultraviolet divergences to result in finite contributions to \( < T_{vv} > \) are absent in second order perturbation theory. We can not find finite contributions to \( < T_{vv} > \) in Minkowski space. If we go to de Sitter space, though, we get a finite contribution which is proportional to \( \Theta \) function, which is the signature of a shock wave solution.
The homogenous solutions, in the shock wave, give contributions to the Greens function expression which are different from the free case. These solutions also give a finite contribution to $< T_{vv} >$ in de Sitter space. The presence of these nontrivial solutions is only due to the dimensional coupling constant. The presence of $g$ in the expansion makes it necessary to have an extra power of $\frac{1}{R}$ in the solution which results in a nontrivial term in $G_F$.

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