QED in the worldline representation

Christian Schubert

Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo,
Edificio C-3, Ciudad Universitaria, C.P. 58040 Morelia, Michoacan, Mexico,
schubert@ifm.umich.mx

Abstract. Simultaneously with inventing the modern relativistic formalism of quantum electrodynamics, Feynman presented also a first-quantized representation of QED in terms of worldline path integrals. Although this alternative formulation has been studied over the years by many authors, only during the last fifteen years it has acquired some popularity as a computational tool. I will shortly review here three very different techniques which have been developed during the last few years for the evaluation of worldline path integrals, namely (i) the “string-inspired formalism”, based on the use of worldline Green functions, (ii) the numerical “worldline Monte Carlo formalism”, and (iii) the semiclassical “worldline instanton” approach.

Keywords: Quantum electrodynamics, perturbation theory, worldline, string inspired formalism

PACS: 11.15.Bt,11.15.Kt,11.25.Db,12.20.Ds

1. FEYNMAN’S WORLDLINE REPRESENTATION OF QED

In 1950 Feynman presented, in an appendix to one of his groundbreaking papers on the modern, manifestly relativistic formalism of perturbative QED [1], also a first-quantized formulation of scalar QED, “for its own interest as an alternative to the formulation of second quantization”. There he provides a simple rule for constructing the complete scalar QED S-matrix by representing virtual scalars and photons in terms of relativistic particle path integrals, and coupling them in all possible ways. Restricting ourselves, for simplicity, to the purely photonic part of the S-matrix (no external scalars), and moreover to the “quenched” contribution (only one virtual scalar), this “worldline representation” can be given most compactly in terms of the (quenched) effective action $\Gamma[A]$:

$$\Gamma_{\text{scalar}}[A] = \int d^4x \mathcal{L}_{\text{scalar}}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)\rightarrow x(T)} \mathcal{D}x(\tau) e^{-S[x(\tau)]}$$

(1)

Here $T$ denotes the proper-time of the scalar particle in the loop, $m$ its mass, and $\int_{x(T)\rightarrow x(0)} \mathcal{D}x(\tau)$ a path integral over all closed loops in spacetime with fixed periodicity in the proper-time. The worldline action $S[x(\tau)]$ has three parts,

$$S = S_0 + S_{\text{ext}} + S_{\text{int}}$$

(2)

(see fig. [1]). Of these, the kinetic term $S_0$ describes the free propagation of the scalar, $S_{\text{ext}}$ its interaction with the external field, and $S_{\text{int}}$ the corrections due to internal photon exchanges in the loop. The connection to a standard Feynman diagrammatic description is made simply by expanding out the two interaction exponentials.
$S_0 = \int_0^T d\tau \frac{\dot{x}^2}{4}$ (free propagation)

$S_{\text{ext}} = ie \int_0^T \dot{x}^\mu A_\mu(x(\tau))$ (external photons)

$S_{\text{int}} = -\frac{e^2}{8\pi^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(x(\tau_1) - x(\tau_2))^2}$ (internal photons)

FIGURE 1. Perturbative expansion of the worldline path integral.

The generalization of this representation to include multiple scalar loops and open scalar lines is straightforward. It yields a first-quantized representation of the full effective action, and thus, by Fourier transformation, of the S-matrix.

While for scalar QED this representation is essentially unique, when generalizing it to spinor QED one has a number of choices. First, worldline representations of spin half particles can be derived either from standard first-order Dirac theory, or from its second-order formulation (see [2] and refs. therein), based on the identity

$$(\not{\partial} + ieA)^2 = -(\not{\partial} + ieA)^2 - \frac{i}{2} e \sigma^{\mu\nu} F_{\mu\nu}$$

Contrary to the situation in second-quantized field theory, in the worldline formalism the second-order approach is the more standard one, and I will restrict myself to it in this review (see [3, 4] for the first-order approach). Using (3) one arrives at a worldline representation for the fermion QED effective action $\Gamma_{\text{spinor}}[A]$ which, at the one-loop level, differs from the scalar one (1) only by the addition of a global factor of $-\frac{1}{2}$, and the insertion of a spin factor $S[x,A]$ under the path integral [5],

$$S[x,A] = \text{tr}_\Gamma \mathcal{P} \exp \left[ \frac{i}{2} e \sigma^{\mu\nu} \int_0^T d\tau F_{\mu\nu}(x(\tau)) \right]$$

Here tr$_\Gamma$ denotes the Dirac trace and $\mathcal{P}$ path ordering.

A more modern way of writing the same spin factor is in terms of an additional Grassmann path integral [6, 7, 8, 9, 10],

\[ S[x,A] = \text{tr}_\Gamma \mathcal{P} \exp \left[ \frac{i}{2} e \sigma^{\mu\nu} \int_0^T d\tau F_{\mu\nu}(x(\tau)) \right] \]
Here the path integration is over the space of anticommuting functions antiperiodic in proper-time, $\psi^\mu(\tau_1) \psi^\nu(\tau_2) = - \psi^\nu(\tau_2) \psi^\mu(\tau_1)$, $\psi^\mu(T) = - \psi^\mu(0)$. The main advantage of introducing this second path integral is that, as it turns out, there is a “worldline” supersymmetry between the coordinate function $x(\tau)$ and the spin function $\psi(\tau)$ [8]. Although this supersymmetry is broken by the different periodicity conditions for $x$ and $\psi$, it still has a number of useful computational consequences. In particular, introducing a worldline superformalism allows one to combine the two path integrals, and to write down a formula for $\Gamma_{\text{spinor}}[A]$ which is completely analogous to (1), (2), including the internal photon corrections [11].

A third, even more subtle, way of implementing spin on the worldline is the "Polyakov spin factor", a purely geometric quantity depending only on the worldline itself [12, 13]. This aspect of spin has been studied by a number of authors, but usually using the first-order formalism; the form of the spin factor appropriate for the second-order formalism has been established only recently [14].

A vast amount of work has been done on these QED worldline representations and their generalizations to other background fields and couplings (see [15] for an extensive list of references). Nevertheless, most of the earlier work on this subject is concerned with formal aspects of relativistic particle Lagrangians, rather than with attempts at performing state-of-the-art calculations in quantum field theory (notable exceptions are [16, 17] and [18]). It is only during the last fifteen years that the usefulness of the first-quantized approach as an alternative to standard Feynman diagrammatic methods has been seriously investigated. This development was triggered by string theory, where first-quantized path integrals are the standard tool for perturbative calculations of scattering amplitudes, and by the fact that many amplitudes in quantum field theory can be represented as infinite string tension limits of the corresponding string amplitudes. Bern and Kosower [19] investigated this limit in detail for the case of the QCD $N$-gluon amplitudes, and found in this way a new set of rules for the construction of these amplitudes. Shortly later, Strassler’s work [20] showed that the same type of “string-inspired” representation of photon/gluon amplitudes can also be obtained more directly by evaluating the corresponding first-quantized path integrals using appropriate worldline Green functions. This approach was then generalized to some cases of multiloop amplitudes [21, 22, 23], as well as to QED amplitudes in a constant external field [24, 25] (see [15] for a review). Moreover, the success of this program led to the development of alternative techniques for the calculation of worldline path integrals. In this talk, I will restrict myself to the computational aspects of the worldline representation, and to the prototypical case, QED. I will discuss, in turn, three quite different methods which are now available for the computation of the scalar/spinor QED worldline path integrals, (i) the original “string-inspired” approach, (ii) the numerical “worldline Monte Carlo” method and (iii) the semiclassical “worldline instanton” technique.
2. THE “STRING-INSPIRED” APPROACH

The strategy in the “string-inspired approach” is simple. The path integral(s) will be manipulated into Gaussian form, after which they can be performed using worldline correlators with the appropriate periodicity properties. For the one-loop path integrals (1),(5) those are [26, 20]

\[
\langle y^\mu(\tau_1)y^\nu(\tau_2) \rangle = -G_B(\tau_1, \tau_2) \delta^{\mu\nu}
\]

\[
G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{1}{T}(\tau_1 - \tau_2)^2
\]

\[
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2) \rangle = G_F(\tau_1, \tau_2) \delta^{\mu\nu}
\]

\[
G_F(\tau_1, \tau_2) = \text{sign}(\tau_1 - \tau_2)
\]

(6)

Here \( y^\mu(\tau) \equiv x^\mu(\tau) - x_0^\mu \), where \( x_0^\mu \) denotes the loop center of mass, \( x_0^\mu \equiv \frac{1}{T} \int_0^T d\tau x^\mu(\tau) \).

There is a certain freedom in choosing the coordinate correlator \( G_B \) (see, e.g., [15]); the one given above is usually the most convenient one, but some of the references given below use the alternative \( G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - (\tau_1 + \tau_2) + \frac{2}{T} \tau_1 \tau_2 \).

Now, to get gaussian form,

- Expand all the interaction exponentials \( e^{-S_{\text{ext}}[x(\tau)]} \) etc.
- For the effective action: Taylor expand the external field at the loop center of mass,

\[
A^\mu(x(\tau)) = e^{y(\tau) \cdot \partial} A^\mu(x_0)
\]

(7)

- For the \( N \) photon amplitudes : Expand the field in \( N \) plane waves,

\[
A^\mu(x(\tau)) = \sum_{i=1}^N \epsilon_i^\mu e^{ik_i \cdot x(\tau)}
\]

(8)

- Exponentiate the denominator of the photon insertion terms,

\[
- \frac{e^2}{8\pi^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(x(\tau_1) - x(\tau_2))^2} =
\]

\[
- \frac{e^2}{2} \int_0^{\infty} \frac{d\bar{T}}{(4\pi\bar{T})^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) \exp \left[-\frac{(x(\tau_1) - x(\tau_2))^2}{4\bar{T}} \right]
\]

(9)

For example, for the four-photon amplitude in scalar QED this procedure yields the following integral representation:
\[
\Gamma[k_1, \epsilon_1; \ldots; k_4, \epsilon_4] = e^4 \int_0^\infty \frac{dT}{T} [4\pi T]^{-2} e^{-m^2 T} \prod_{i=1}^4 \int_0^T d\tau_i Q_4 e^{G_{Bij} k_i k_j}
\]

(10)

\[
Q_4 = Q_4^1 + Q_4^2 + Q_4^3 - Q_4^{22}
\]

(11)

\[
Q_4^1 = \dot{G}_{B12} \dot{G}_{B23} \dot{G}_{B34} \dot{G}_{B41} Z_4(1234) + 2 \text{ permutations}
\]

(12)

\[
Q_4^2 = \dot{G}_{B12} \dot{G}_{B21} Z_2(12) \left\{ G_{B3i} e_3 \cdot k_i \dot{G}_{B4j} e_4 \cdot k_j + \frac{1}{2} G_{B34} e_3 \cdot e_4 \left[ \dot{G}_{B3i} k_3 \cdot k_i - \dot{G}_{B4j} k_4 \cdot k_j \right] \right\} + 5 \text{ perm.}
\]

\[
Q_4^{22} = \dot{G}_{B12} \dot{G}_{B21} Z_2(12) \dot{G}_{B34} \dot{G}_{B43} Z_2(34) + 2 \text{ perm.}
\]

Here each of the integrals \( \int_0^T d\tau_i \) represents one of the four photon legs moving around the scalar loop. \( G_{Bij} \) stands for \( G_B(\tau_i, \tau_j) \) and \( \dot{G}_{Bij} \) for its derivative. Repeated indices are to be summed over \( 1, \ldots, N \). In writing down the integrand, a number of integrations by parts have already been performed which eliminated all second derivatives of the \( G_{Bij} \)'s that initially appear in the factor \( Q_4 \). Moreover, this factor has been decomposed into terms \( Q_4^m \) according to “cycle content”, indicated by the superscript, where a “cycle” is a factor of \( \dot{G}_{B_{i_1} j_1} \dot{G}_{B_{i_2} j_2} \dot{G}_{B_{i_3} j_3} \cdots \dot{G}_{B_{i_n} j_1} \). Such a ‘\( \tau \)-cycle’ always gets multiplied by a ‘Lorentz-cycle’ \( Z_n(i_1 i_2 \ldots i_n) \), which is the trace of the products of the field strength tensors associated to the corresponding legs. The various \( Q_4^m \)'s are individually gauge invariant. This representation can be generalized to an arbitrary number of photons, and is unique if one requests manifest permutation symmetry in the photon legs [19, 20, 27]. The generalization to higher loop photon amplitudes can be achieved either using (9), or by explicit sewing of pairs of external photons, or more directly using generalized worldline Green functions adapted to the multiloop graph topologies [21, 22, 28].

In all cases the arising multiple integrals are equivalent to standard Feynman parameter integrals. For example, in (10) and its \( N \)-point generalizations, the prefactor \( Q_N \) relates to a Feynman numerator and the exponential factor to the universal denominator of one-loop \( N \)-point integrals (see, e.g., [29]). However, the string-inspired representation has a number of interesting properties which are not usually manifest in standard Feynman parameter calculations:

1. The use of the cycle notation allows a more compact way of writing the \( N \) photon amplitudes than usual.
There is a simple “cycle replacement rule” which allows one to construct the integrand of the spinor loop $N$ photon amplitude from the scalar loop one \[19\], \[20\]. This implies that, in this formalism, the calculations of the same quantity in scalar and spinor QED are not independent; any spinor QED calculation yields the corresponding scalar QED result as a byproduct.

The integral (10) and its higher $N$ generalizations represent the whole amplitude, with no need to sum over permutations. This property is not very relevant at the one-loop level, but becomes interesting at higher loop orders. In the QED case, it generally allows one to combine into one integral all contributions from Feynman diagrams which can be identified by letting photon legs slide along scalar/electron loops or lines. As an example, we show in fig. 2 the "quenched" contributions to the three-loop photon propagator.

This property is particularly interesting in view of the fact that it is precisely this type of sums of diagrams which in QED generally leads to extensive cancellations between diagrams, and to final results which are substantially simpler than intermediate ones (see \[30\] and refs. therein). And for the two-loop QED $\beta$ function indeed a way was found for calculating the corresponding integral in a way which avoided splitting up the multiple parameter integral into sectors with a fixed ordering of the photon legs, and which led to dramatic simplifications \[11\]. However, so far no generalization of the method used there to higher loop orders has been found.

The string-inspired method provides a particularly convenient way of implementing constant external fields in QED calculations. Photon amplitudes or effective lagrangians in a constant field are obtained from the corresponding vacuum quantities simply by substituting the vacuum Green functions $G_{B,F}(\tau_1, \tau_2)$ by appropriate field-dependent Green functions $G_{B,F}(\tau_1, \tau_2; F)$, and by a change of the free worldline path integral determinants \[24\], \[25\] (see also \[31\]). In particular, the "cycle replacement rule" carries over to the constant field case.

The efficiency of the technique for QED in a constant field has, at the one-loop level, been demonstrated by recalculations of the photon splitting amplitude in a magnetic field \[32\], and of the one-loop vacuum polarization in a general constant field \[33\]. At the two-loop level it has been extensively applied to the QED effective Lagrangian in a constant field. This includes recalculations of the standard two-loop Euler-Heisenberg Lagrangians \[25\], \[34\], \[35\], closed-form expressions for the weak field expansion coefficients of the magnetic two-loop Euler-Heisenberg Lagrangian \[36\], and a generalization of these Lagrangians to the case of a self-dual Euclidean field \[37\]. I will show here only the last result, which is particularly nice: a Euclidean self-dual field fulfills
\[ F_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} F^{\alpha \beta}, \]
which implies that the square of the field strength tensor is proportional to the unit matrix, \( F_{\mu \nu} F^{\nu \lambda} = - f^2 \delta^\lambda_\mu. \) Remarkably, this simplifies matters so much that all parameter integrals can be done in closed form, leading to the following explicit formulas for the two-loop scalar and spinor QED effective Lagrangians in a constant self-dual field \([37]\),

\[
\begin{align*}
\mathcal{L}^{(2)}_{\text{scalar}}(\kappa) &= \alpha \frac{m^4}{(4\pi)^3} \frac{1}{\kappa^2} \left[ \frac{3}{2} \xi^2(\kappa) - \xi'(\kappa) \right] \\
\mathcal{L}^{(2)}_{\text{spinor}}(\kappa) &= -2\alpha \frac{m^4}{(4\pi)^3} \frac{1}{\kappa^2} \left[ 3\xi^2(\kappa) - \xi'(\kappa) \right]
\end{align*}
\]

Here \( \kappa = \frac{m^2}{2e_f} \) and

\[
\xi(x) \equiv -x \left( \frac{\Gamma'(x)}{\Gamma(x)} - \ln(x) + \frac{1}{2x} \right)
\]

The simplicity of these results made it possible to use this self-dual case for studying the generic properties of the weak and strong field expansions of Euler-Heisenberg Lagrangians \([38]\), as well as the corresponding maximally helicity violating components of the two-loop \(N\)-photon amplitudes in the low energy limit \([37]\).

Apart from the effective action in a constant field, the string-inspired method has also been extensively applied to the calculation of the full one-loop QED effective action in an arbitrary background field, using the derivative expansion \([39, 40, 41]\).

Moreover, at the one-loop level the method has been generalized to the finite temperature case, both to obtain representations of the \(N\)-photon amplitudes \([42, 43]\) and for the effective action in a general background \([44]\).

Finally, as should be clear from the above, work on the string-inspired technique in QED so far has been concerned mainly with purely photonic amplitudes. The extension to amplitudes involving also external scalars \([23]\) or fermions \([45, 46]\) is possible without difficulties, although its practical usefulness is difficult to judge at present due to a lack of state-of-the-art applications.

### 3. THE “WORLDLINE MONTE CARLO” APPROACH

During the last few years it was found that Feynman’s worldline representation \([1]\) and its various generalizations are also very amenable to a direct numerical evaluation using standard Monte Carlo techniques \([47, 48, 49]\). At the one-loop level, this approach has been shown to work well for QED effective actions in quite general background fields. This includes singular fields such as a magnetic field of the step-function type \([47]\). It also extends to the imaginary part of the effective action \([50]\), to be discussed in section 4 below. First steps towards a multiloop extension have been taken in \([51]\), although, as with all numerical approaches to quantum field theory, implementing the full renormalization program in such a formalism poses a formidable challenge.
The Monte Carlo approach appears to hold particular promise for the calculation of Casimir energies for arbitrary geometries [49, 52, 53]. Although this has so far been done only for scalar fields, not for the QED case, it shall be discussed here since it is interesting to see how Dirichlet boundary conditions can be implemented at the level of the path integral (see the recent [54]) for a treatment of boundary conditions in the “string-inspired” approach. For a (massless) scalar field in a background potential \( V(x) \), the gauge coupling term in Feynman’s path integral (1) has to be replaced by 
\[
- \int_0^T d\tau V(x(\tau)).
\]
Dirichlet boundary conditions on an (infinitely thin) surface \( \Sigma \) can be implemented by choosing 
\[
V(x) = \lambda \int_\Sigma d\sigma \, \delta^3(x - x_0)
\]  
(16)
with \( \lambda \to \infty \). Considering the case of two disjoint surfaces \( \Sigma = \Sigma_1 \cup \Sigma_2 \), it can then easily be shown [49] that the Casimir interaction energy between the surfaces is given by 1
\[
E_{\text{Casimir}} = \frac{1}{2} \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dT}{T^2} \int d^3x_0 \int \mathcal{D}x(\tau) e^{-\frac{1}{2} \int_0^T d\tau x^2} \Theta_\Sigma[x(\tau)]
\]  
(17)
where \( \Theta_\Sigma = 1 \) if the path \( x(\tau) \) intersects both \( \Sigma_1 \) and \( \Sigma_2 \), and \( \Theta_\Sigma = 0 \) otherwise. The trivial time component of the path integral has been integrated out. \( x_0 \) denotes the loop center of mass of the remaining integral over spatial loops; the energy density at a point \( x_0 \) is obtained by restricting the path integral to worldloops with that point as their common center of mass.

Figure 3 shows the Casimir energy density obtained by a numerical evaluation of (17) for the case of two infinite perpendicular plates at separation \( a \) [52].

![Figure 3](image)

**FIGURE 3.** Left panel: sketch of the perpendicular-plates configuration with (an artist’s view of) a typical worldline that intersects both plates. Right panel: Density plot of the effective action density \( \mathcal{L} \) for the perpendicular plates case; the plot shows \( \ln(2(4\pi a^2)^2 \mid \mathcal{L} \mid) \).

---

1 When comparing with [49] note that they use a different normalization of the free path integral.

2 Fig. 3 is reprinted from [52] with the permission of the authors and of the American Institute of Physics.
A particularly nice example is the case of a sphere of radius $R$ and an infinite plate at a distance $a$ [53]. Here an exact result for the Casimir interaction energy is known as a function of $a/R$ for $a/R \lesssim 0.1$ [55]. Fig. 4 shows the result of the worldline evaluation for this case. As can be seen from fig. 5 the result is in very good agreement with [55] for the whole parameter range.

FIGURE 4. Contour plot of the negative Casimir interaction energy density for a sphere of radius $R$ above an infinite plate; the sphere-plate separation has been chosen as $a = R$. The plot results from a pointwise evaluation of eq. (17) using worldlines with a common center of mass.

FIGURE 5. Casimir interaction energy of a sphere with radius $R$ and an infinite plate vs. the curvature parameter $a/R$. The energy is normalized with respect to the zeroth-order proximity force approximation (PFA) [53]. The numerical worldline result is compared to the exact result of [55] and the PFA estimate.

3 Figs. 4 and 5 are reprinted from [53] with the permission of the authors and of the American Physical Society (copyright by APS, http://link.aps.org/abstract/PRL/v96/e220401).
4. THE “WORLDLINE INSTANTON” APPROACH

We come now to a third approach which is more specialized than the two previous ones, since it applies only to the imaginary part of amplitudes or the effective action. Although the basic idea was presented by Affleck et al. in 1982 [18], it seems not to have been followed up on until very recently [56, 57, 58].

The work of Affleck et al. concerned the imaginary part of the scalar QED effective Lagrangian in a constant electric field. This quantity has been of much interest ever since Schwinger, building on earlier work by Sauter, Heisenberg, Euler, and Weisskopf [59, 60, 61], showed that its existence implies the possibility of electron-positron pair creation in vacuum by the electric field [62]. For small production rates this rate is simply given by twice the imaginary part itself, \( P_{\text{production}} \approx 2\text{Im} \mathcal{L}[E] \). Moreover, Schwinger was able to explicitly calculate the imaginary part in terms of a sum of exponentials,

\[
\text{Im} \mathcal{L}_{\text{spinor}}(E) = \frac{m^4}{8\pi^3} \beta^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left[ -\frac{\pi n}{\beta} \right] \quad (18)
\]

(\( \beta = eE/m^2 \)). In this sum the \( n \)th term relates to the coherent production of \( n \) pairs by the field. The appearance of \( E \) in the denominator of the exponents indicates that the pair creation effect is nonperturbative in nature. It also suggests an interpretation as a tunnel effect where a virtual electron-positron pair separates out along the field lines and extracts a sufficient amount of energy from the field to turn real.

The corresponding formula for scalar QED differs only by a global factor and signs (due to the difference in statistics),

\[
\text{Im} \mathcal{L}_{\text{scalar}}(E) = -\frac{m^4}{16\pi^3} \beta^2 \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^n \frac{(-1)^n}{n^2} \exp \left[ -\frac{\pi n}{\beta} \right] \quad (19)
\]

The production rates are exponentially small for

\[
E \ll E_{\text{crit}} = \frac{m^2}{e} = 1.3 \times 10^{18} \text{ V/m} \quad (20)
\]

Until a few years ago producing an electric field close to this critical field strength \( E_{\text{crit}} \), and with a sufficient spatial extension, appeared far out of the reach of laboratory experiments. However, due to recent advances in laser technology it seems now conceivable that pair production could be seen in laser fields in the near future. The optical laser POLARIS, under construction at the Jena high-intensity laser facility, is projected to reach a maximal field strength of \( E_{\text{max}} \approx 2 \times 10^{14} \text{ V/m} \) [63], while the European X-ray free electron laser (XFEL), under construction at DESY, is expected to come even closer, \( E_{\text{max}} \approx 1.2 \times 10^{16} \text{ V/m} \) [64].

For realistic laser experiments it is usually far from clear whether the constant field approximation is justified; it would be preferable to have generalizations of Schwinger’s formula (18) to inhomogeneous and time-dependent fields. This is a subject which has
been pursued by many authors, and many results have been obtained over the years (see, e.g. \cite{65, 66, 67, 68, 69, 70, 71, 72, 73, 74}). However, considering this large volume of work there is a surprising lack of variety in the calculation methods used; apart from a few special field configurations, virtually all results of a more general nature have been obtained by WKB, or some variant of it. The formalism developed below is, while similar in spirit to WKB, technically quite different, and apparently more general.

Let us start with retracing Affleck et al.’s \cite{18} recalculation of the Schwinger formula for scalar QED, \cite{19}. At one loop, Feynman’s representation (1) reads

\[ \Gamma_{\text{scalar}}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int Dx e^{-\int_0^T d\tau \left( \frac{x^2}{2} + i e A \cdot \dot{x} \right)} \]  

(21)

Rescaling \( \tau = Tu \), this becomes

\[ \Gamma_{\text{scalar}}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int Dx e^{-\left( \frac{1}{T} \int_0^1 du \dot{x}^2 + i e \int_0^1 du A \cdot \dot{x} \right)} \]  

(22)

The \( T \) integral has a stationary point at

\[ T_c = \frac{\sqrt{\int du \dot{x}^2}}{m} \]  

(23)

This allows us to calculate its imaginary part using a stationary phase approximation, yielding

\[ \text{Im} \Gamma_{\text{scalar}} = \frac{1}{m} \text{Im} \int Dx \sqrt{\frac{2\pi}{T_c}} e^{-\left( m\sqrt{\int du \dot{x}^2 + i e \int_0^1 du A \cdot \dot{x} } \right)} \]  

(24)

with a new worldline action,

\[ S = m\sqrt{\int du \dot{x}^2 + ie \int_0^1 du A \cdot \dot{x}} \]  

(25)

We would like to calculate the remaining path integral using a stationary phase approximation, too. The action \( S \) is stationary if

\[ m \frac{\ddot{x}_\mu}{\sqrt{\int du \dot{x}^2}} = ie F_{\mu \nu} \dot{x}_\nu \]  

(26)

Contracting with \( \dot{x}^\mu \) yields \( x^2 = \text{constant} \equiv a^2 \), and thus

\[ m \ddot{x}_\mu = iea F_{\mu \nu} \dot{x}_\nu \]  

(27)
Thus the extremal action trajectory $x^{\text{cl}}(u)$, the “worldline instanton”, will be a periodic solution of the Lorentz force equation, with a parameter $a$ to be determined by the periodicity condition. Once the instanton is found, its worldline action immediately provides a semiclassical approximation for the imaginary part of the effective Lagrangian. Closer inspection shows that this approximation also corresponds to a weak field approximation 56:

$$\text{Im} L_{\text{scalar}}(E) \approx \lim_{E \to 0} e^{iS[x^{\text{cl}}]}$$

For a the case of a constant electric field, $\vec{E} = (0,0,E) = \text{const.}$, it is easy to see that the periodicity condition is solved by

$$a_n = \frac{m}{eE} 2n\pi, \quad n \in \mathbb{Z}^+$$

The worldline instantons are simply circles in the $t-z$ plane, with a winding number $n$:

$$x_n^{\text{cl}}(u) = \frac{m}{eE} (x_1, x_2, \cos(2n\pi u), \sin(2n\pi u))$$

$$S[x_n^{\text{cl}}] = n\pi \frac{m^2}{eE}$$

The evaluation of the worldline action on the $n$th instanton thus yields just the $n$th exponent in Schwinger’s formula (19). Moreover, Affleck et al. were able to compute also the prefactor, which here involves the determinant of fluctuations around the instanton path (see below). Thus, this approach provides a very simple and elegant rederivation of Schwinger’s result for scalar QED.

We will now sketch how to generalize this approach to general electric fields, as well as to the spinor QED case; see [56, 57, 58] for the details. Starting all over from Feynman’s representation of the one-loop effective action in scalar QED (1), for the general case we prefer not to eliminate the $T$ integral, but rather to first seek a stationary phase approximation for the path integral. The stationarity condition is again the Lorentz force equation,

$$\ddot{x}_\mu = 2ieF^\mu \nu(x) \dot{x}_\nu$$

We fix a point on the loop:

$$\int d^4x(\tau) e^{-S[x(\tau)]} = \int d^4x(0) \int d^4x(\tau) e^{-S[x(\tau)]}$$

Let us assume that we have found a worldline instanton $x^{\text{cl}}(\tau)$, a classical solution with $x(T) = x(0) = x(0)$. We expand around $x^{\text{cl}}$. 
\[ x_\mu(\tau) = x^{cl}_\mu(\tau) + \eta_\mu(\tau), \quad \eta_\mu(0) = \eta_\mu(T) = 0. \]  (33)

Obtain the operator of quadratic fluctuations (Jacobi or Hessian matrix) \( \Lambda_{\mu\nu} \)

\[ \Lambda_{\mu\nu} = -\frac{1}{2} \delta_{\mu\nu} \frac{d^2}{d\tau^2} - \frac{d}{d\tau} Q_{\nu\mu} + Q_{\mu\nu} \frac{d}{d\tau} + R_{\mu\nu}, \]  (34)

where

\[ Q_{\mu\nu} = \frac{\partial^2 L}{\partial x_\mu \partial \dot{x}_\nu}, \quad R_{\mu\nu} = \frac{\partial^2 L}{\partial x_\mu \partial x_\nu}. \]  (35)

Find the zero modes \( \eta_\nu^{(\lambda)}(\tau) \) of \( \Lambda_{\mu\nu} \),

\[ \Lambda_{\mu\nu} \eta_\nu^{(\lambda)} = 0 \]  (36)

with initial value conditions

\[ \eta_\nu^{(\lambda)}(0) = 0, \quad \dot{\eta}_\nu^{(\lambda)}(0) = \delta_{\nu\lambda}, \quad (\mu, \nu = 1, 2, 3, 4) \]  (37)

Evaluate them at \( \tau = T \). Then, the final result for the semiclassical approximation becomes [57]

\[ \int \mathcal{D}x(\tau) e^{-S[x(\tau)]} = \frac{e^{i\theta} e^{-S[x^{cl}]}}{(4\pi T)^2} \left| \frac{\det \left[ \eta_{\mu,\text{free}}^{(\lambda)}(T) \right]}{\det \left[ \eta_{\mu}^{(\lambda)}(T) \right]} \right| \]  (38)

Note that the calculation of this quite general fluctuation determinant has been reduced to finding the determinant of the 4 \( \times \) 4 matrix of zero modes. This remarkable simplification relies on a theorem by Levit and Smilansky [75]. Finally, at the very end one does \( \int dT \) using the stationary phase method again.

For a number of classical special cases of “planar” non-constant fields, such as the single-pulse time dependent field [68, 69], the single-bump space dependent field [67], the sinusoidal time-dependence [66, 69], and the sinusoidal space-dependence, the worldline instantons can be found explicitly in terms of special functions [56], leading to simple explicit formulas for the semiclassical exponent. For example, for the single-pulse field

\[ E(t) = E \text{sech}^2(\omega t) \]  (39)
the stationary action is [56]

\[ S_{\text{pulse}} = n \frac{m^2 \pi}{eE} \left( \frac{2}{1 + \sqrt{1 + \gamma^2}} \right) \]  

(40)

where \( n \) is the winding number and \( \gamma \equiv \frac{m0}{eE} \) the “adiabaticity parameter” [65]. For the analogous single-bump field

\[ E(x_3) = E \text{sech}^2(kx_3) \]  

(41)

one finds

\[ S_{\text{bump}} = n \frac{m^2 \pi}{eE} \left( \frac{2}{1 + \sqrt{1 - \tilde{\gamma}^2}} \right) \]  

(42)

where \( \tilde{\gamma} = mk/eE \). Note that \( S_{\text{pulse}} \) decreases with \( \gamma \), so that the pair production rate increases, while it is the other way round for \( S_{\text{bump}} \). We believe that this is just an instance of a quite general fact: **Inhomogeneity in time tends to shrink the size of the wordline instantons, leading to an increase in the pair production rate; spatial inhomogeneity increases the instanton size and decreases the pair production rate.**

The single-bump field also provides an excellent opportunity to test the validity of the semiclassical approximation, since for this case an exact integral representation has been obtained by Nikishov [67], suitable for numerical evaluation, and moreover there is a worldline Monte Carlo result [50]. As shown in figure 6 all three results are in close agreement over the whole range of the inhomogeneity parameter \( \tilde{\gamma} \).

**FIGURE 6.** Plot of the imaginary part of the effective action for the field \( E(x) = E \text{sech}^2(kx) \) as a function of the inhomogeneity parameter \( \tilde{\gamma} \), normalized by the weak field limit of the “locally constant field approximation” [57]. The dotted line shows the result obtained from the worldline instanton approximation. The dashed line is the same ratio using a numerical integration of Nikishov’s exact expression [67]. The circles represent the worldline Monte Carlo results of [50], evaluated for \( eE/m^2 = 1 \).
Note that the imaginary part vanishes for $\tilde{\gamma} > 1$, which in the instanton approach simply means that instanton solutions cease to exist. Although mathematically this absence of instanton solutions allows one to conclude the vanishing of the imaginary part only in the semiclassical approximation, this example of the single-bump case supports the conjecture that it might, in fact, signal the complete absence of pair production.

Although the method works very well for these classical cases, these are rather special configurations, and known to be amenable also to WKB methods. The existence of closed-form expressions for the worldline instanton can be expected only for a very restricted classes of fields. Much more interesting is the fact that the worldline instanton equations (27) and the zero mode equations (36) are ordinary differential equations, which leads us to expect that they can be solved numerically for more general classes of fields than have been treated by WKB. Dunne and Wang [58] have very recently applied this numerical approach to a class of electric fields which depend nontrivially on two spatial coordinates, parametrized by

$$ A_4(\vec{x}) = -i \frac{E}{k} f(\vec{x}) $$

(\(\gamma = mk/eE\)). Fig. 7 shows their results for two examples,

$$ f(\vec{x}) = \frac{k(x_1 + x_2)}{1 + k^2(x_1^2 + x_2^2)} $$

$$ f(\vec{x}) = k(x_1 + x_2) e^{-k^2(x_1^2 + x_2^2)} $$

(44)

![Graph showing pair creation rates for two nonplanar cases.](image)

**FIGURE 7.** Pair creation rates for two nonplanar cases.

Note that again Im$\Gamma$ vanishes for large inhomogeneities, i.e. for fields of insufficient extent.

4 Reproduced from [58] with the permission of the authors.
Finally, making the transition from scalar to spinor QED in the instanton approach is straightforward if we use Feynman’s original implementation of spin in the worldline path integral. Up to the global normalization, it amounts to multiplying with the spin factor \( S[x^{el},A] \) evaluated on the instanton trajectory. For the classical cases discussed above, and more generally for all “planar” fields the path ordering in the spin factor has no effect. Surprisingly, one obtains simply \[ S[x^{el},A] = 4 \cos \left[ eT \int_0^1 du E(x(u)) \right] = 4(-1)^n \] (45)

with \( n \) the winding number. Thus for planar fields one finds the same simple relation between \( \text{Im} \mathcal{L}_{\text{scalar}}(E) \) and \( \text{Im} \mathcal{L}_{\text{spinor}}(E) \) as in the constant \( E \) case, eqs.(18),(19). It is an interesting open question whether this property might even extend to general electric fields.

5. SUMMARY

The purpose of this short review was to show that the worldline approach in its various versions is turning into an efficient alternative to second-quantized methods for an increasing range of problems in QED. Unfortunately, despite of the restriction to QED it was not possible here to give due credit to all relevant work. Among other things, it was not possible here to discuss the worldline variational approach of [76, 77], nor the nonperturbative propagator calculations of [78].

Let me conclude with summarizing the main advantages which one can hope to achieve using the worldline formalism: (i) Compact parameter integral representations for arbitrary QED multiloop amplitudes (ii) Easy implementation of constant external fields (iii) Reliable numerical (Monte Carlo) results for one loop effective actions and Casimir energies (iv) Reliable numerical results for pair creation rates in arbitrary electric fields.

REFERENCES

1. R. P. Feynman, Phys. Rev. 80, 440 (1950).
2. A. Morgan, Phys. Lett. B 351, 249 (1995), hep-ph/9502230.
3. A. A. Migdal, Nucl. Phys. B 265, 594 (1986).
4. C. D. Fosco, J. Sánchez-Guillén, and R. A. Vázquez, Phys. Rev. D 69, 105022 (2004), hep-th/0310191; Phys. Rev. D 73, 045010 (2006), hep-th/0511026.
5. R. P. Feynman, Phys. Rev. 84, 108 (1951).
6. E. S. Fradkin, Nucl. Phys. B 76, 588 (1966).
7. F. A. Berezin, and M. S. Marinov, Ann. Phys. 104, 336 (1977).
8. L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P. S. Howe, Phys. Lett. B 64, 435 (1976).
9. A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cim. A 35, 377 (1976).
10. L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. B 118, 76 (1977).
11. M.G. Schmidt, and C. Schubert, Phys. Rev. D 53, 2150 (1996), hep-th/9410100.
12. A. Strominger, Phys. Lett. B 101, 271 (1981).
13. A. M. Polyakov, Mod. Phys. Lett. A 3, 335 (1988).
14. H. Gies, and J. Hämmerling, Phys. Rev. D 72, 065018 (2005), hep-th/0505072.
15. C. Schubert, Phys. Rep. 355, 73 (2001), hep-th/0101036.
16. M. B. Halpern, and W. Siegel, Phys. Rev. D 16, 2486 (1977).
17. M.B. Halpern, A. Jevicki, and P. Senjanoivc, Phys. Rev. D 16, 2476 (1977).
18. I.K. Affleck, O. Alvarez, and N.S. Manton, Nucl. Phys. B 197, 509 (1982).
19. Z. Bern, and D.A. Kosower, Nucl. Phys. B 379, 451 (1992).
20. M.J. Strassler, Nucl. Phys. B 385, 145 (1992).
21. M.G. Schmidt, and C. Schubert, Phys. Lett. B 331, 69 (1994), hep-th/9403158
22. K. Roland, and H.-T. Sato, Nucl. Phys. B 480, 99 (1996), hep-th/9604152. Nucl. Phys. B 515, 488 (1998), hep-th/9709019.
23. K. Daikouji, M. Shino, and Y. Sumino, Phys. Rev. D 53, 4598 (1996), hep-ph/9508377.
24. R. Shaisultanov, Phys. Lett. B 378, 354 (1996), hep-ph/9512142.
25. M. Reuter, M.G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) 259, 313 (1997), hep-th/9610191.
26. A. M. Polyakov, Gauge fields and strings, Harwood Academic Publishers, 1987.
27. K. Daikouji, M. Shino, and Y. Sumino, Phys. Rev. D 53, 4598 (1996), hep-ph/9508377.
28. C. Schubert, Eur. Phys. J. C 5, 693 (1998), hep-th/9710067.
29. P. Dai, and W. Siegel, YITP-SB-06-29, hep-th/0608062.
30. C. Itzykson, and J. Zuber, Quantum field theory, McGraw-Hill, 1985.
31. D.J. Broadhurst, R. Delbourgo, and D. Kreimer, Phys. Lett. B 366, 421 (1996), hep-ph/9509296.
32. D.G.C. McKeon, and T.N. Sherry, Mod. Phys. Lett. A 9, 2167 (1994).
33. S.L. Adler, and C. Schubert, Phys. Rev. Lett. 77, 1695 (1996), hep-th/9605035.
34. D. Fliegner, M. Reuter, M.G. Schmidt, and C. Schubert, Theor. Math. Phys. 113, 1442 (1997), hep-th/9704194.
35. B. Körs, and M.G. Schmidt, Eur. Phys. J. C 6, 175 (1999), hep-th/9803144.
36. G.V. Dunne, A. Huet, D. Rivera, and C. Schubert, JHEP 11, 013 (2006), hep-th/9211076.
37. G.V. Dunne, and C. Schubert, JHEP 0208, 0053 (2002), hep-th/9906156.
38. G.V. Dunne, and C. Schubert, JHEP 0206, 0042 (2002), hep-th/0205005.
39. M.G. Schmidt, and C. Schubert, Phys. Lett. B 318, 407 (2000), hep-ph/0001288.
40. D. Cangemi, E. D'Hoker, and G. Dunne, Phys. Rev. D 51, 2513 (1995), hep-th/9409113.
41. V.P. Gusynin, and I.A. Shovkovy, Can. J. Phys. 74, 282 (1996), hep-ph/9509383; J. Math. Phys. 40, 5406 (1999), hep-th/9804143.
42. D.G.C. McKeon, and A. Rebhan, Phys. Rev. D 47, 5487 (1993), hep-th/9309053.
43. I.A. Shovkovy, Phys. Lett. B 441, 313 (1998), hep-th/9806156.
44. D.G.C. McKeon, and A. Rebhan, Phys. Rev. D 48, 289 (1993).
45. A.I. Karanikas, and C.N. Ktorides, JHEP 0208, 033 (2002), hep-th/0205007.
46. H. Gies, and K. Langfeld, Nucl. Phys. B 613, 353 (2001), hep-ph/0102185.
47. H. Gies, and K. Langfeld, Int. J. Mod. Phys. A 17, 966 (2002), hep-ph/0112198.
48. H. Gies, and K. Langfeld, Int. J. Mod. Phys. A 18, 1499 (2003).
49. H. Gies, and K. Langfeld, and L. Moyaerts, JHEP 0306, 018 (2003), hep-th/0303264.
65. L.V. Keldysh, *JETP* **20**, 1307 (1965).
66. E. Brezin, and C. Itzykson, *Phys. Rev. D* **2**, 1191 (1970).
67. A.I. Nikishov, *Nucl. Phys. B* **21**, 346 (1970).
68. N.B. Narozhnyi, and A.I. Nikishov, *Yad. Fiz.* **11**, 1072 (1970).
69. V.S. Popov, *JETP* **34**, 709 (1971); V.S. Popov, and M.S. Marinov, *Yad. Fiz.* **16**, 809 (1972).
70. M. Stone, *Phys. Rev. D* **14**, 3568 (1976).
71. W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields*, Springer, 1985.
72. A.B. Balantekin, J.E. Seger, and S.H. Fricke, *IJMP A* **6**, 695 (1991).
73. G.V. Dunne, and T. Hall, *Phys. Rev. D* **58**, 105022 (1998), hep-th/9807031.
74. S.P. Kim, and D. Page, *Phys. Rev. D* **73**, 065020 (2006), hep-th/0301132, hep-th/0701047.
75. S. Levit, and U. Smilansky, *Ann. Phys.* **103**, 198 (1977).
76. R. Rosenfelder, and A.W. Schreiber, *Phys. Rev. D* **53**, 3337 (1996), nucl-th/9504002; *Eur. Phys. J. C* **37**, 161 (2004), hep-th/0406062.
77. A. Alexandrou, R. Rosenfelder, and A.W. Schreiber, *Phys. Rev. D* **62**, 085009 (2000), hep-th/0003253.
78. C. Savkli, J. Tjon, and F. Gross, *Phys. Rev. C* **60**, 055210 (1999), hep-ph/9906211; Erratum-ibid. **61**, 069901 (2000); *Phys. Rev. C* **62**, 116006 (2000), hep-ph/9907445.