Precise many-body calculations and hyperfine interaction effect on dynamic polarizabilities at the low-lying energy levels of $Y^{2+}$

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Abstract

The present work determines the precise values of magic wavelengths corresponding to the clock transitions $5^2S-4^2D$ of $Y^{2+}$ ion both at the levels of fine- and hyperfine-structures due to the external light beams having linear as well as circular polarization. To calculate the dynamic polarizabilities of the associated states of the transitions, we employ the sum-over-states technique, where dominating and correlation sensitive part of the sum is evaluated using a highly correlated relativistic coupled-cluster theory. The estimated magic wavelengths of the light beams have substantial importance to cool and trap the ion using a state-insensitive and blue-detuned trapping scheme. We also present the tune-out wavelengths which are useful in state-insensitive trapping and cooling. Noticeable contributions to the total polarizabilities were observed from the vector part due to the circularly polarized light beams which provide additional magic wavelengths compared to linearly polarized light beams. Considerable effects of hyperfine interaction on the values of polarizabilities and number of magic wavelengths divulge the importance of precise estimations of hyperfine structure splitting.
I. INTRODUCTION

Accurate information about the dynamic polarizabilities of the atomic states at the hyperfine levels can be significant for quantum experiments such as trapping and cooling [1–3], atom interferometry [4, 5], quantum registers [6], etc.. Also, the cooling and trapping of ions are absolutely necessary for error-free experiments of frequency standard, fundamental constants [7–9], quantum computer [10] and many other modern advanced technologies [11–14]. For instance, the preciseness of trapping and cooling parameters of ions mostly decide the fractional uncertainties of frequency standards which is demanded to be of the order of $10^{-18}$ or less [15–17]. It has been shown over the past two decades that singly [18], doubly [19] or multiply charged [7] ions can be competing candidates for the frequency standard in terms of the accuracy and frequency stability. Since the perturbation due to external field reduces with increasing ionization of an atomic system [20], doubly ionized systems can be better choices for many of the above experiments than singly ionized atoms in this regard. One of the ways to improve the accuracy is to make states or system, involved in the experiment, insensitive to external field [21, 22] and, motions of atoms [23] in the optical lattice.

The narrow line-width quadrupole transitions between ground states and long-lived metastable states for some moderate to heavy ions are being targeted from a long past as quantum experiments, such as atomic clock [24–28], quantum computer [29] etc.. Rubidium-like Sr$^+$ is a well-known ion for quantum technology and $^2S_{1/2} - ^2D_{3/2}$ quadrupole transition of it is particularly utilized in quantum computing [30, 31]. Here we are proposing the same narrow line-width transition of Y$^{2+}$, but now the ground state is $^2D_{3/2}$ instead of $^2S_{1/2}$. However, as indicated earlier, the advantage of using Y$^{2+}$ instead of Sr$^+$ is that the transition of Y$^{2+}$ is affected less by the perturbation of the electric field of the laser beam due to one unit of more positive charge. But certainly the effect of this perturbation can not be avoided for Y$^{2+}$ in general experimental circumstances. Nevertheless, with the advent of cryogenic methods, highly charged ions are also possible to keep cool for longer period [32] for experimental studies.

Further, recent experiment on the isotope shift of $^4D_{3/2} \rightarrow ^5P_{3/2}$ transition of Y$^{2+}$ ion at 294.6 nm [33] motivate us to propose $^4D_{3/2} \rightarrow ^5S_{1/2}$ transition (clock transition) for the above quantum experiments. These prodigious technological and conceptual advancements on quantum experiments cannot be worthwhile for precision measurements without the
proper choice of frequencies of the external field at which differential ac-Stark shift among the relevant energy levels of the atom or ion vanishes. The corresponding wavelengths of these frequencies are called magic wavelengths [34]. Also, the precise measurement [35] of the tune-out wavelengths of light, where the polarizability of an atomic state vanishes, are crucial for atom interferometry, and vector polarizability plays a significant role [36] in that due to its dependance on $m_F$. Urge of finding these magic zero wavelengths inspires the theorists for highly accurate calculations on the dynamic polarizabilities of the electronic states of atoms or ions involved in the experiments. Here we are interested in evaluating polarizabilities of the ground ($4^2D_{3/2}$), first excited ($4^2D_{5/2}$) and second excited ($5^2S_{1/2}$) states of $Y^{2+}$. Another advantage of using $Y^{2+}$ is the long lifetime of the first (244.08s) and second (10.76s) excited states [37], which can be advantageous for experiments. Beside of all these three states, the $5^2P_{1/2}$ states of $Y^{2+}$ are also important as the cooling transitions $4^2D_{3/2} \rightarrow 5^2P_{1/2}$ achievable near 232 nm [38–43]. Therefore, the precise knowledge of static and dynamic polarizabilities of the low lying states of $Y^{2+}$ are very much important for the above experiments and also relevant for measurements such as collinear laser spectroscopy [33] and black-body radiation shift [34, 44, 45], etc.

Precise knowledge of hyperfine levels or structure of ions is extremely important to experimentalists before studying the dynamic polarizabilities at those hyperfine multiplets. So far, there has been one recent reference [33] along with a very old experimental work [46] providing hyperfine constants (magnetic dipole) of $5s^2S_{1/2}$ and $5p^2P_{1/2}$ states only of $Y^{2+}$. The hyperfine splitting of the clock transition levels can induce interesting effects to the magic wavelengths. As an important example of the applications of quantum technologies, experiments on the hyperfine splittings are also known as one of the first applications of trapped ions [47]. Moreover, these constants are also important in the studies of the chemical composition of the sun and stars. $Y^{2+}$ has been studied theoretically as well as experimentally several times for that purpose [48–50].

We further calculate dynamic polarizabilities of the $4^2D_{3/2}$ and $5^2S_{1/2}$ states at hyperfine levels of $Y^{2+}$ and estimate the magic wavelengths for $5^2S_{1/2} - 4^2D_{3/2}$ clock transitions at those levels of experimental interests due to the external field of linearly as well as circularly polarized light. We also calculate the tune-out [51] wavelengths for which the ac-Stark-shifts for the above hyperfine states becomes zero. The circularly polarized light yields an extra component to the valence polarizability, “vector part”, compare to linearly polarized light.
This vector polarizability part arises from the induced dipole moment perpendicular to the polarization of the field. As a consequence, the circularly polarized light is advantageous for the quantum experiments [52, 53].

The tune-out wavelength of a particular sub-levels of hyperfine multiplets shifts due to circularly polarized beams compare to linearly polarized beam and the shift estimates fictitious magnetic field useful for trapping and cooling [36]. To calculate the polarizabilities of the above states we have employed a highly correlated relativistic many-body formalism, coupled- cluster theory. A brief discussion of which is given at the end of the Section II and the beginning of Section III, and a detail of this can be found in ref.[54, 55].

II. THEORY

Stark shift ($\Delta \xi_v$) of the $v$-th atomic energy level in the presence of external electric field ($E(\omega)$) can be obtained by second-order time-independent perturbation theory [34, 54]

$$\Delta \xi_v(\omega) = \sum_{i \neq v} \frac{1}{\omega_{vi}} |\langle \psi_v | - D \cdot E |\psi_i \rangle|^2 = -\frac{1}{2} \alpha_v(\omega) E^2,$$

where, $- D \cdot E$ is the interaction Hamiltonian and $\omega_{vi}$ is the resonance frequency associated with the electric dipole transition (here onward an electric dipole transition will be addressed as “$E1$ transition”) between the states $|\psi_v \rangle$ and $|\psi_i \rangle$. Here polarizability, $\alpha_v(\omega)$, is dependent on frequency ($\omega$) of the external light and can be decomposed into three components as [34],

$$\alpha_v(\omega) = \alpha^C(\omega) + \alpha^{VC}_v(\omega) + \alpha^V_v(\omega).$$

Here valence state independent quantity $\alpha^C(\omega)$ represents the contribution to the total polarizability due to the ionic core [34]. It can be expressed, in general, by the following sum-over-state form [54, 56, 57]

$$\alpha(\omega) = \frac{2}{3} \sum_{ap} \frac{|\langle \phi_a |d_{DF}||\phi_p \rangle \langle \phi_a |d_{corr}||\phi_p \rangle \omega_{pa}|}{(\omega_{pa})^2 - \omega^2}.$$

The subscripts $a$ and $p$ represent the core (fully degenerate) and virtual (empty) orbitals, respectively. $\langle \phi_a |d_{DF}||\phi_p \rangle$ and $\langle \phi_a |d_{corr}||\phi_p \rangle$ represent the reduced matrix element concerning the Dirac Fock (DF) and correlated atomic states, respectively. In our calculations of polarizability for ionic core, we have considered the latter matrix elements (see section III) at
second-order relativistic many-body perturbation theory (RMBPT(2)). \( \alpha^{V C}(\omega) \), the perturbation to the core polarizability in the presence of valence electron \[^58\] is almost frequency insensitive and can be assumed constant within the frequency range considered in this work.

The estimation of \( \alpha^V_v(\omega) \) is the most crucial part of the calculations of dynamic polarizability, and it has three components \[^56\]:

\[
\alpha^V_v(\omega) = \frac{mJ}{2J} \alpha^{(1)}_v(\omega) + \frac{3mJ - J(J + 1)}{J(2J - 1)} \alpha^{(2)}_v(\omega).
\]

Here, \( J \) and \( mJ \) are the total angular momentum and its magnetic component, respectively, for \( |\psi_v\rangle \). The polarization factor \( \sigma \), of the external field is zero for linearly polarized light and \( \pm 1 \) for circularly polarized light. \( \alpha^{(0)}_J(\omega) \), \( \alpha^{(1)}_J(\omega) \) and \( \alpha^{(2)}_J(\omega) \) represent scalar, vector, and tensor polarizabilities, respectively, of fine structure states with total angular momentum \( J \).

In the form of sum-over-states technique, these polarizabilities are expressed as \[^54, 56\]

\[
\alpha^{(0)}_J(\omega) = \frac{2}{3(2J + 1)} \sum_n d_{nv},
\]

\[
\alpha^{(1)}_J(\omega) = -\sqrt{\frac{6J}{(J + 1)(2J + 1)}} \sum_n (-1)^{J_n + J} \left\{ \begin{array}{ccc} J & 1 & J \\ 1 & J_n & 1 \end{array} \right\} \left( \frac{2\omega}{\omega_{nv}} \right) d_{nv},
\]

and

\[
\alpha^{(2)}_J(\omega) = 4\sqrt{\frac{5J(2J - 1)}{6(2J + 1)(2J + 3)}} \sum_n (-1)^{J_n + J} \left\{ \begin{array}{ccc} J & 1 & J_n \\ 1 & J & 2 \end{array} \right\} d_{nv},
\]

where the angular momentum independent factor \( d_{nv} = \{ |\langle \psi_v|d|\psi_n\rangle|^2 \omega_{nv} \}/(\omega_{nv}^2 - \omega^2) \) diverges for light at resonant frequency, \( \omega_{nv} \).

At the hyperfine energy level (usually indicated by the quantum number \( F = J + I \) with nuclear spin \( I \)), the derivation of the polarizability is similar. Here, the valence part of the polarizability can be calculated using the same form of the equation as shown in Eq 4, but the quantum number \( (J, m_J) \) will be replaced by \( (F, m_F) \). Also, the expression of the scalar part of polarizability is equal to \( \alpha^{(0)}_J \) as the second order scalar shift does not depend on \( F \) value \[^34\]. The vector and tensor polarizabilities of a hyperfine level have extra factors following angular momentum algebra and they are mathematically expressed as

\[
\alpha^{(1)}_F(\omega) = (-1)^{J + F + I + 1} \left\{ \begin{array}{ccc} F & J & I \\ J & F & 1 \end{array} \right\} \sqrt{\frac{F(2F + 1)(2J + 1)(J + 1)}{J(F + 1)}} \alpha^{(1)}_J(\omega)
\]
and
\[
\alpha_F^{(2)}(\omega) = (-1)^{J+F+I} \left\{ \begin{array}{ccc}
F & J & I \\
J & F & 2
\end{array} \right\} \sqrt{\frac{(F(2F-1)(2F+1))}{(2F+3)(F+1)}} \times \\
\sqrt{\frac{(2J+3)(2J+1)(J+1)}{J(2J-1)}} \alpha_I^{(2)}(\omega).
\tag{9}
\]

However, prior estimation of the energy shifts of these hyperfine levels due to hyperfine interaction is required here. These energy shifts or splittings can be calculated accurately from the precise knowledge of hyperfine-structure constants corresponding to the magnetic dipole and electric quadrupole moments, known as the hyperfine \(A\) and \(B\) constants. Followings are the definitions of the above constants in terms of reduced matrix elements \([60, 74]\) of electron-nucleus interaction Hamiltonian \([61]\).

\[A = \mu_N g_I \frac{\langle J||T^{(1)}||J \rangle}{\sqrt{J(J+1)(2J+1)}} = -\frac{g_I \kappa_v}{J(J+1)} \langle v|\frac{1}{r^2}|v \rangle \times 13074.7 \text{MHz}\]  
\tag{10}

and

\[B = 2eQ \sqrt{\frac{2J(2J-1)}{(2J+1)(2J+2)(2J+3)}} \langle J||T^{(2)}||J \rangle, = Q \frac{2J-1}{2J+2} \langle v|\frac{1}{r^3}|v \rangle \times 234.965 \text{MHz}\]  
\tag{11}

where \(\mu_N\) is the nuclear magneton, \(g_I\) is the nuclear \(g\)-factor and \(Q\) is the quadrupole moment of the nucleus. \(\kappa_v\) is the relativistic quantum number.

The energy splitting of the atomic energy level can be calculated using the following expression

\[E_{\text{hfs}} = \frac{AK}{2} + \frac{13K(K+1) - 4J(J+1)I(I+1)}{2I(2I-1)2J(2J-1)} B, \tag{12}\]

where \(K = F(F+1) - I(I+1) - J(J+1)\).

The various matrix elements present in the above expressions are calculated here in three theoretical approaches: DF, RMBPT(2) and relativistic coupled-cluster (RCC). Very brief, but adequate descriptions of the DF and RMBPT(2) theories are available in a write up by W. R. Johnson \([55]\). The RCC is a well known many-body method for the calculation of
electronic-structure properties [55, 62–64]. It has an ability of exhaustive consideration of electron correlations. There are many detailed write-ups on the RCC method in literature [65]; brief discussions of different correlations and their numerical contributions can be found in our recent works [63, 66–68]. The present version of the RCC theory includes single, double and a subset of partial triple excitations (RCCSD(T)) [69] to solve the energy eigenvalue equations.

III. RESULTS AND DISCUSSIONS

A. Dynamic Polarizabilities and Magic Wavelengths at spin-orbit levels

We use sum-over-state formalism to estimate the dynamic polarizabilities of $4^2D_{3/2}$ and $5^2S_{1/2}$ states along with the static polarizabilities of these states and $5^2P_{3/2}$ states. According to this formalism, the accurate calculations of a large number of $E1$ matrix elements of the contributing energy states are necessary. The summation index ($n$) in the equations of scalar, vector and tensor polarizabilities (Eq. (5)-(7)) refers to the single valence open-shell states with different values of principal quantum numbers and the relevant angular momentum quantum numbers which can give non-zero values to the $E1$ matrix elements associated with the targeted required state. In the present calculations, the principal quantum number is considered up to 25. The $E1$ matrix elements arising from all the states beyond this principal quantum number have very negligible contributions to the total valence polarizability of the clock states. Depending on the comparative strengths of $E1$ matrix elements in evaluating the valence polarizabilities, we consider three different levels of many-body theories concerning the effects of correlations. The most dominant contribution to a valence polarizability is appeared from the sum of the terms having matrix elements associated with first few low-lying states. Accordingly, the excited states involving $5-8s_{1/2}$, $5-8p_{1/2,3/2}$, $4-7d_{3/2,5/2}$ and $4-6f_{5/2,7/2}$ as valence orbitals are the most important states in the present sum-over-states formalism. The matrix elements associated with these states are calculated here using the correlation exhaustive RCCSD(T) method. Relatively less important $E1$-matrix elements are calculated by using RMBPT(2) [57] which includes core polarization corrections on top of the DF contributions. This part sums up the contribution from the next five single valence states for all the symmetries. The $E1$ matrix elements whose con-
tributions are less significant but cannot be neglected to the total valence polarizabilities, are associated with the atomic states with single valence orbitals 14-25s_{1/2}, 14-25p_{1/2,3/2}, 13-25d_{3/2,5/2} and 12-25f_{5/2,7/2}. The wavefunctions and corresponding E1 transitions of these states are calculated using the DF method.

In Table I, we tabulate static values of scalar and tensor polarizabilities of $\alpha_v$ along with the contributions from $\alpha^C$ and $\alpha^{VC}_v$ to the total polarizability. Our calculated values are compared with the corresponding theoretical estimations of Safranova and Safranova [70]. The latter calculations are the output of an all-order relativistic many-body perturbation method which is used to find the static valence polarizabilities of $4^2D_{3/2}^3$, $5^2S_{1/2}^2$ states. In that calculation, the random-phase approximation (RPA) is used to compute the core polarizability for the above states. This comparison indicates 4.7% deviation for $4^2D_{3/2}^3$ state, 4.9% deviation for $4^2D_{5/2}^5$ state and 3.6% deviation for $5^2S_{1/2}^2$ state in static polarizabilities. We also present the static polarizabilities for $5^2P_{1/2}^2$, $3^2P_{1/2}^3$ states for future experimental explorations of state-specific properties [49].

FIG. 1 shows the dependency of the total polarizabilities for the $5^2S_{1/2}^2$ and $4^2D_{3/2}^{3/2}$ states with increasing wavelength for a linearly as well as a circularly polarized light. Here we opted for the most important region of the electromagnetic spectrum from $\lambda = 77$ nm to $\lambda = 400$ nm as there is no significant magic wavelength which can be prescribed for laser trapping purpose outside of this region. The entire spectrum of the wavelength as considered in all the 4 figures span from the far ultraviolet to the starting zone of the visible region with increasing wavelength. The peaks in the polarizability curves representing the resonances associated with the transitions $5^2S_{1/2}^2 \rightarrow (5-7)^2P_{1/2}^2$ and $4^2D_{3/2}^{3/2} \rightarrow (5-7)^2P_{1/2}^2$, $(4-5)^2F_{3/2}^2$. It is very interesting to see there is no resonance line in between 100 to 400 nm for 4D5(5/2) with a linearly polarized light (fig 1(b)). We found a strong cancellation between scalar and tensor parts of the polarizability for this state. Same kind of nature of 3D5(5/2) was seen in the recent paper of dynamic polarizability of Sc$^{2+}$ [54]. The resonance line appeared for this state when we consider circular polarization of light.

We consider right circularly polarized light ($\sigma = +1$ in Eq. 4) in the present calculations of polarizabilities at the various $m_J$ levels of $4^2D_{3/2}^{3/2}$ states. However, the orientation of polarization of light ($\sigma$) and the sign of $m_J$ value of the state decide the resultant sign of the vector part of polarization. The crossing points of the distributions of the dynamic polarizabilities of $5^2S_{1/2}^2$ and $4^2D_{3/2}^{3/2}$ states show a number of magic wavelengths for the
transitions between these states. These magic wavelengths are tabulated in TABLE II with associated polarizability values at the corresponding wavelengths. This table also highlights the impact of the vector part of polarizability as one changes the impinging light beam with polarization from linear to circular. For a linearly polarized light, polarizability of an energy state is independent of the sign of its \( m_J \) component. We find quite a few magic wavelengths with high polarizabilities at the mid-UV region and these can be potential wavelengths for \( Y^{2+} \) clock experiment with the best possible precision. Due to the presence of vector part in the polarizability, the circularly polarized light provides relatively more number of magic wavelengths for the clock transitions and many of them correspond to large polarizability values. Nevertheless, moderate changes on both magic wavelengths and corresponding polarizability values have been observed when we compare the data in TABLE II for linearly and circularly polarized light in the mid-UV range. This facilitates external control on slight tuning of the magic wavelengths. Since the wavelengths of the transitions \( 5^2S_{\frac{7}{2}} - 4^2D_{\frac{5}{2}} \) are 1339.2 nm and 1483.0 nm, respectively, all the magic wavelengths presented in the table support blue-detuned trapping scheme.

B. Dynamic Polarizabilities and Magic Wavelengths at Hyperfine levels

In some situations, instead of considering the electronic fine-structure atomic states as discussed till now, experimentalists may prefer to consider hyperfine states during trapping and cooling processes. Therefore, it may be physically more meaningful to estimate the magic wavelengths for the transitions between the different hyperfine levels of the clock states. However, before doing any estimation of magic wavelength associated with these hyperfine levels, one needs to compute the hyperfine-structure constants of the various fine-structure states of \( Y^{2+} \) such that hyperfine interaction corrected \( \omega_{nv} \) frequencies are used in the polarizability expressions (Eq(6),(7),(8), and (9))

Hyperfine-structure constant \( A \) values for nine low-lying states of \( Y^{2+} \) are calculated using the RCCSD(T) method and are presented in Table III. We compare the present RCCSD(T) values with the available hyperfine \( A \) values in the literature and we find good agreement between them [33]. In order to calculate these constants, we choose the most abundant isotope of Y with mass number 89, nuclear spin \( (I) = \frac{1}{2} \) and nuclear magnetic moment (\( \mu \)) = \(-0.1374154 \) \( \mu_N \). Also, the nuclear charge distribution of this ion is assumed to
have the Fermi type form\textsuperscript{[71]}. As the nuclear quadrupole moment is zero for a spin-half nucleus, the values of hyperfine-splitting as expressed in Eq.\textsuperscript{12} are calculated using hyperfine $A$ constants only. The splitting values are tabulated at the last column of Table \textsuperscript{III}. High correlation effects to the hyperfine-structure constants can be understood from this table by comparing the obtained from the DF and RCCSD(T) methods. The relative impact of correlation is exceptionally high for the $4^2F$ states among all the states due to strong effect of core polarization. This kind of large correlation contributions have been observed already for the $4^2F$ states in many other systems such as Sc$^{2+}$\textsuperscript{[54]}, W$^{5+}$\textsuperscript{[63]} and Ga$^{2+}$\textsuperscript{[74]}.

FIG 2. represents the variation profiles of dynamic polarizabilities for the hyperfine levels within the same spectral region which is considered for the fine-structure levels as shown in FIG 1. For the circularly polarized light, the polarizability profiles of $4^2D_\frac{3}{2}$ ($m_J = \pm \frac{3}{2}$), $4^2D_\frac{5}{2}$ ($m_J = \pm \frac{5}{2}$) and $5^2S_\frac{1}{2}$ ($m_J = \pm \frac{1}{2}$) states (as shown in Fig. 1) are same as the profiles of $4^2D_\frac{3}{2}$ ($F = 2, m_F = \pm 2$), $4^2D_\frac{5}{2}$ ($F = 3, m_F = \pm 3$) and $5^2S_\frac{1}{2}$ ($F = 1, m_F = \pm 1$) states, respectively. This is because of the unit value of the multiplication factors in Eq (8) and Eq (9) which relate $\alpha_i^{(i)}(\omega)$ with $\alpha_i^{(i)}(\omega)$ for these states with $i=1,2$. Also the polarizability values are same for $4^2D_\frac{3}{2}$ ($F = 2, m_F = 0$) and $4^2D_\frac{5}{2}$ ($F = 1, m_F = 0$) states.

In Table \textsuperscript{IV}, we tabulate the magic wavelengths at the hyperfine levels for linearly as well as circularly polarized light. Just like Table II, we get few more magic wavelengths for circularly polarized light compared to linearly polarized light due to the presence of vector component in the former. This table in comparison with TABLE II, highlights small but noticeable changes both in the magic wavelengths and corresponding polarizabilities with more degrees of freedom in the choices of $(F, M_F)$ combinations for each $(J, m_J)$.

It is known that correlation corrections are the main source of theoretical uncertainties in the RCC calculations for hyperfine structure estimations apart from basis set. The latter one for these calculations can be controlled by comparing expectation values of $1/r$ with respect to the DF wavefunctions based on our present optimized Gaussian-type-orbital (GTO) basis function and precise numerical program, GRASP 92 code\textsuperscript{[72]}. In order to analyze theoretical uncertainties arising from correlation, two sets of atomic states of Y$^{2+}$ are classified. Atomic states: $5^2S_{\frac{1}{2}}$, $5^2P_{\frac{1}{2}, \frac{3}{2}}$, $4^2D_{\frac{3}{2}}$ and $5^2D_{\frac{3}{2}}$ are the first set (say, Class-I), where correlation contributions to the energy eigenstates are less than 50%. Each states of other set (Class II), containing $4^2D_{\frac{5}{2}}$, $5^2D_{\frac{5}{2}}$ and $4^2F_{\frac{7}{2}, \frac{5}{2}}$, has correlation correction more than 50%. The uncertainty spread can be roughly estimated by comparing the correla-
tion contributions to the presented RCC hyperfine values with another similar method like SDpT (linearized coupled-cluster with single, double and partial triple) \[70, 73\], which is also correlation exhaustive many-body method. However, for Y\(^{2+}\), no such SDpT hyperfine values are available in literature to our knowledge. Therefore, we make comparison of the same correlation to the hyperfine values of Sr\(^{+}\), another Rb-isoelectronic element, and SDpT calculations were performed on it by Safronova et al. \[73\]. In general, the correlation effect in Sr\(^{+}\) will be comparatively stronger than Y\(^{2+}\) due to less ionization of alkaline earth isoelectronic sequence \[74, 76\] and the uncertainty spread in the RCC calculations of Sr\(^{+}\) in comparison to the SDpT calculations will set upper limit for the uncertainty in correlation calculations of Y\(^{2+}\). We find a maximum discrepancy of 1.8\% among the two calculations of Sr\(^{+}\) for the states belong to the Class I. Our experience of calculating hyperfine values for isoelectronic sequence of alkali-metal-like atoms \[74, 75\] says that the same discrepancy should not be more than ±2\% for these states of Y\(^{2+}\). However in the case of the Class II states, for which the correlation corrections are very high but decreases rapidly from singly to doubly ionized systems along an isoelectronic sequence of an alkali-metal-atom, a different strategy is considered due to non availability of SDpT results for \(4^2F_\frac{5}{2}\) and \(4^2F_\frac{7}{2}\) states of Sr\(^{+}\). Here we compare the correlation corrections in the hyperfine \(A\) values of \(4^2D_\frac{7}{2}\) state as calculated by our RCC method with the same using SDpT approach by Safronova for Sr\(^{+}\). We find a discrepancy of around 5\% in these correlation values. For Y\(^{2+}\), though this discrepancy is expected to be lower than 5\%, we even assume it to be 5\% invoking an uncertainty of around ±7\% in the total hyperfine value (RCC) of the \(4^2D_\frac{7}{2}\) state of Y\(^{2+}\). We extend the same uncertainties to its F-multiplets. Therefore, assuming the quality of the DF wavefunctions near to the nuclear region, an approximate contribution from the Breit interactions \[76\], and rough approximation in the spread of correlation uncertainties as mentioned above, we estimate the theoretical uncertainties in the RCC hyperfine \(A\) values of around ±3\% for the Class I states and ±8\% for the Class II states of Y\(^{2+}\).

To estimate the uncertainty of our calculated magic wavelengths of Y\(^{2+}\), we reevaluate the magic wavelengths by replacing only our calculated RCCSD(T) dipole matrix elements by the corresponding available dipole matrix elements in the literature \[70\] which were calculated using an all-order RMBPT calculations. The transition matrix elements which we find in their work \[70\] are \(5^2S-5^2P\), \(4^2D-n^2P\) with \(n = 5, 6, 7\) and \(4^2D-m^2F\) with \(m = 4, 5, 6\). The maximum difference between these reevaluated magic wavelengths and the
corresponding magic wavelengths presented in Table II and Table IV is considered as the uncertainty in the latter values which is estimated around ±1%.

C. Tune-out wavelengths of \(4^2D_{3/2} \frac{3}{2} \) and \(5^2S_{1/2} \) states

We also report a few tune-out wavelengths in Table V at the hyperfine levels of the ground state and low metastable states. Here, the beam is considered as linearly polarized light. This means the scalar and tensor component of the polarizability cancels each other at this particular wavelength. In case of circularly polarized beam, the vector component of the polarizability contributing at the ground state \( (4^2D_{3/2} \text{ with } F = 2, m_F = \pm 2) \) at the tune-out wavelengths of 230.83 nm and 215.32 nm are 414.14 a.u. and 25.84 a.u., respectively. Whereas the values are 53.86 a.u. and 14.98 a.u. correspond to tune-out wavelength 244.36 nm and 204.41 nm (for \( F = 1, m_F = \pm 1 \)) respectively. These values can be used to calculate fictitious magnetic field induced by the circular polarization of light. In an optical lattice, trapped atoms are released in these wavelengths of light so that lattice potential vanishes. The prior knowledge of the zero Stark-shift wavelengths for a particular atomic state can be advantageous for an accurate trap-insensitive experimental measurement [5, 51, 77, 78].

IV. CONCLUSION

In this present work, we have determined the magic wavelengths corresponding to the \(5^2S_{1/2} - 4^2D_{3/2} \) clock transitions of \(^{89}\text{Y}^{2+}\) for two different polarizations of the projected light beam. These wavelengths span between the vacuum ultraviolet and the near ultraviolet region of the electromagnetic spectrum. The data are presented both at the spin-orbit and hyperfine levels of atomic states and this gives an understanding of how much the hyperfine interaction can affect the magic trapping conditions viable for the fine-structure states. Indeed we have found slight modifications in the magic wavelength values after imposing hyperfine splitting on the fine-structure clock states. But most importantly and for obvious reason, the number of magic wavelengths are increased. Many of these magic wavelengths can have potential applications for trap related experimental explorations. Irrespective of the nature of the polarization of light, our calculations show that the polarizabilities of the magic wavelengths between 200 nm and 300 nm are higher than others and thus, they are
TABLE I. Total static polarizabilities $\alpha_v(0)$ are presented in a.u. along with contributions from scalar $\alpha_v^{(0)}(0)$, core $\alpha_v^C(0)$, valence-core $\alpha_{VC}^{(0)}$, tensor $\alpha_v^{(2)}(0)$ parts. "Other" refers to the data obtained from the work of Safronova and Safronova [70].

| State       | $\alpha_v^{(0)}(0)$ | $\alpha_v^C(0)$ | $\alpha_{VC}^{(0)}$ | $\alpha_v^{(2)}(0)$ | $\alpha_v(0)$ |
|-------------|---------------------|-----------------|---------------------|---------------------|--------------|
| $4^2D_{\frac{3}{2}}$ | 6.894   | 6.742(26)  | 4.277  | 4.048  | -0.331 | -0.313 | -3.48 | -3.45(2) | 7.36  | 7.03(4) |
| $4^2D_{\frac{5}{2}}$ | 6.933   | 6.815(32)  | 4.277  | 4.048  | -0.358 | -0.341 | -4.86 | -4.81(3) | 5.99  | 5.71(4) |
| $5^2S_{\frac{1}{2}}$ | 42.07   | 40.64(17)  | 4.277  | 4.048  | -0.17  | -0.17  | 0    | 0      | 46.1  | 44.5(2) |
| $5^2P_{\frac{1}{2}}$ | 8.7500  | 4.277      | -0.0017 | 13.0257 |
| $5^2P_{\frac{3}{2}}$ | 12.0243 | 4.277      | -0.0011 | 5.6092  | 21.9098 |
FIG. 1. The variations of polarizabilities for the states $5^2S_{1/2}$ and $4^2D_{3/2}$ with wavelength to extract magic wavelengths for the transitions between them. (a) and (b) are for linearly polarized light. (c) and (d) are for circularly polarized light.
FIG. 2. Representation of Fig 1, but now by including the effect of hyperfine splittings in the fine-structure states: \(5^2S_{\frac{1}{2}}\) and \(4^2D_{\frac{3}{2}, \frac{5}{2}}\). (a) and (b) are for linearly polarized light; and (c) and (d) are for circularly polarized light.
TABLE II: Magic Wavelengths (in nm) with corresponding polarizabilities (in a.u.) for $5^2S_{1/2}$ to $4^2D_{3/2}$ or $4^2D_{5/2}$ transitions due to linearly and circularly polarized ($\sigma=\pm 1$) light. Notation used in the second row is used for the transition $5^2S_{J,m_J} - 4^2D_{J',m_{J'}}$.

| Linearly Polarized | Circularly Polarized |
|--------------------|----------------------|
| $(\frac{3}{2}, \frac{3}{2})$ - $(J', m_{J'})$ | $(\frac{1}{2}, \frac{3}{2})$ - $(J', -m_{J'})$ |
| $(\frac{1}{2}, \frac{1}{2})$ - $(J', -m_{J'})$ | $(\frac{1}{2}, -\frac{1}{2})$ - $(J', -m_{J'})$ |
| $(\frac{1}{2}, -\frac{1}{2})$ - $(J', -m_{J'})$ | $(\frac{1}{2}, -\frac{1}{2})$ - $(J', -m_{J'})$ |

| $J', m_{J'}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ |
|--------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $(\frac{3}{2}, \frac{3}{2})$ | 293.84 | - | 294.85 | 32.99 | 294.83 | 24.58 | 286.02 | 36.85 |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |
| 293.75 | - | 294.83 | 24.58 | 286.02 | 36.85 | 286.00 | - | - |

Continued on next page
TABLE II – continued from previous page

| Linearly Polarized | Circularly Polarized |
|--------------------|---------------------|
| \((\frac{1}{2}, \frac{1}{2}) - (J', m_J')\) | \((\frac{1}{2}, \frac{1}{2}) - (J', m_J')\) |
| \((J', m_J')\) | \(\lambda_{\text{magic}}\) | \(\alpha_{\text{magic}}\) | \(\lambda_{\text{magic}}\) | \(\alpha_{\text{magic}}\) | \(\lambda_{\text{magic}}\) | \(\alpha_{\text{magic}}\) | \(\lambda_{\text{magic}}\) | \(\alpha_{\text{magic}}\) |
| 229.71 | -69.82 | 222.72 | -54.69 | 232.28 | -69.59 | 224.60 | -64.94 | 232.91 | -81.69 |
| 99.54 | 0.98 | 99.38 | 0.97 | 100.78 | 0.66 | 99.38 | 1.06 | 100.25 | 0.87 |
| 85.40 | 4.65 | 85.73 | 3.08 | 85.99 | 4.64 | 85.04 | 3.62 | 85.89 | 4.92 |
| 81.02 | 6.28 | 81.08 | 6.01 | 81.00 | 6.07 | 81.08 | 6.46 | 81.00 | 6.52 |
| \((\frac{5}{2}, \frac{1}{2})\) | 293.87 | 28.44 | 294.85 | 30.87 | 294.83 | 25.50 | 286.02 | 33.97 | 286.02 | 27.61 |
| 233.99 | -78.51 | 234.05 | -72.89 | 233.95 | -72.67 | 234.03 | -84.39 | 233.93 | -84.11 |
| 225.48 | -62.52 | 222.50 | -54.40 | 226.83 | -60.48 | 224.26 | -64.35 | 227.71 | -70.62 |
| 99.54 | 0.98 | 99.47 | 0.95 | 99.66 | 0.91 | 99.47 | 1.03 | 99.66 | 1.00 |
| 85.39 | 4.46 | 85.79 | 3.43 | 85.88 | 3.94 | 85.37 | 4.13 | 85.62 | 4.52 |
| 81.02 | 6.28 | 81.03 | 6.05 | 81.02 | 6.06 | 81.03 | 6.50 | 81.02 | 6.51 |

TABLE III. Hyperfine A constants and Hyperfine splitting \((E_{hfs})\) in the unit of MHz.

| State | Hyperfine A | \(F_1 \rightarrow F_2\) | \(E_{hfs}\) |
|-------|-------------|----------------|-----------|
|       | DF          | RCC Other[33] | MCDF      | Exp. | RCC Other[46] |
| 4^2D_{3/2} | 79.55 | 102.74 | - | - | 2→1 | 205.47 | - |
| 4^2D_{5/2} | 33.20 | 14.25 | - | - | 3→2 | 42.74 | - |
| 4^2F_{5/2} | 0.48 | 0.86 | - | - | 3→2 | 2.59 | - |
| 4^2F_{7/2} | -0.27 | -0.88 | - | - | 4→3 | 3.52 | - |
| 5^2S_{1/2} | 1441.92 | 1793.19 | 1780 | 1803(5) | 1→0 | 1793.19 | 1920±150 |
| 5^2P_{1/2} | 284.44 | 371.47 | 352 | 391(5) | 1→0 | 371.47 | - |
| 5^2P_{3/2} | 49.57 | 73.86 | - | - | 2→1 | 147.72 | 128 |
| 5^2D_{3/2} | 17.12 | 21.63 | - | - | 3→2 | 43.25 | - |
| 5^2D_{5/2} | 7.19 | 8.10 | - | - | 3→2 | 24.29 | - |
TABLE IV: Magic Wavelengths (in nm) with corresponding polarizabilities (in a.u.) for $^5S_{\frac{1}{2}}$ - $^2D_{\frac{3}{2},\frac{5}{2}}$ transitions for linearly and circularly polarized light ($\sigma=\pm 1$). Notation used in the second row is for the transition $^5S_{j,m_J}(F,m_F) - ^2D_{j',m_{J'}}(F',m_{F'})$.

| Linearly Polarized | Circularly Polarized |
|-------------------|----------------------|
|                   | (1,1)-(F',m_{F'})    | (1,-1)-(F',-m_{F'}) | (1,1)-(F',m_{F'})    | (1,1)-(F',-m_{F'}) |
| [J'](F',m_{F'})   | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ |
| $\frac{3}{2}$     | 293.77               | 9.57                   | 294.83               | 22.16                   | 294.77               | -3.11                 | 286.02               | 24.91                   | 286.00               | -5.11                 |
|                   | -                   | -                      | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
|                   | -                   | -                      | 244.18               | -97.23                 | 247.36               | -107.53               | 244.12               | -116.35               | 246.54               | -126.90              |
|                   | -                   | -                      | 226.14               | -59.47                 | 244.79               | -98.92                | 232.54               | -82.32                | 244.85               | -119.02              |
|                   | -                   | -                      | 101.08               | 0.59                   | -                   | -                     | 101.09               | 0.68                  | -                   | -                     |
|                   | 98.96               | 1.10                   | 98.96                | 1.06                   | 98.96                | 1.06                  | 98.96                | 1.15                  | 98.96                | 1.15                 |
|                   | 85.39               | 4.52                   | 85.31                | 3.35                   | 85.35                | 5.79                  | 85.50                | 3.36                  | 85.61                | 5.22                 |
|                   | 80.63               | 6.59                   | 80.52                | 6.45                   | 80.63                | 6.36                  | 80.50                | 6.93                  | 80.08                | 7.32                 |
| $\frac{3}{2}$     | 293.85               | 24.2                   | 294.88               | 30.28                   | 294.85               | 17.66                 | 286.03               | 32.76                 | 286.00               | 18.86                 |
|                   | 244.19              | -106.87                | 244.30               | -97.55                 | 243.93               | -96.51                | 244.27               | -116.84               | 243.82               | -115                 |
|                   | 231.91              | -74.45                 | 226.53               | -60.02                 | 227.81               | -62.45                | 226.97               | -69.28                | 237.54               | -93.76                |
|                   | 98.96               | 1.10                   | 98.97                | 1.06                   | 98.96                | 1.06                  | 99.41                | 1.05                  | 98.96                | 1.14                 |
|                   | 85.38               | 4.06                   | 85.30                | 3.53                   | 85.34                | 4.71                  | 85.51                | 3.44                  | 85.56                | 4.28                 |
|                   | 80.63               | 6.59                   | 80.63                | 6.36                   | 80.63                | 6.36                  | 81.06                | 6.48                  | 80.63                | 6.83                 |
| $\frac{1}{2}$     | 293.87               | 29.07                   | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
|                   | 244.27              | -107.13                | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
|                   | 230.62              | -71.63                | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
|                   | 98.96               | 1.10                   | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
|                   | 85.36               | 3.92                   | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
|                   | 80.63               | 6.59                   | -                   | -                      | -                   | -                     | -                   | -                      | -                   | -                     |
| $\frac{3}{2}$     | 293.81               | 14.45                   | 294.86               | 24.87                   | 294.82               | 3.82                  | 286.03               | 27.89                 | 286.02               | 2.88                 |
|                   | 243.55              | -104.76                | 244.24               | -97.36                 | -                   | -                     | 244.20               | -116.56               | -                   | -                     |
|                   | 237.45              | -86.66                | 226.26               | -59.63                 | -                   | -                     | 226.74               | -68.71                | -                   | -                     |
|                   | 98.96               | 1.10                   | 98.97                | 1.06                   | 98.96                | 1.06                  | 98.97                | 1.14                  | 98.96                | 1.15                 |

Continued on next page
| $[J'](F', m_{F'})$ | Linearly Polarized | Circularly Polarized |
|-----------------|-------------------|---------------------|
| $(1, 1)-(F', m_{F'})$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ |
| $(1, -1)-(F', m_{F'})$ | 85.39 | 4.36 | 85.34 | 3.40 | 85.37 | 5.38 | 85.50 | 3.36 | 85.60 | 4.91 |
| $(1, -1)-(F', m_{F'})$ | 80.63 | 6.59 | 84.20 | 3.78 | - | - | 84.66 | 3.61 | - | - |
| $(1, -1)-(F', m_{F'})$ | - | - | 82.05 | 5.34 | - | - | 81.71 | 6.00 | - | - |
| $(1, 1)-(F', m_{F'})$ | - | - | 80.66 | 6.34 | 80.63 | 6.36 | 80.66 | 6.80 | 80.63 | 6.82 |
| $[^{3/2}]$, (3, 3) | 293.75 | 6.26 | 294.83 | 19.71 | 294.77 | -7.21 | 286.02 | 22.18 | 286.00 | -9.64 |
| $[^{3/2}]$, (3, 3) | - | - | 238.33 | -81.99 | 246.62 | -105.03 | 238.26 | -96.02 | 245.47 | -121.85 |
| $[^{3/2}]$, (3, 3) | - | - | 229.28 | -64.40 | 239.15 | -83.97 | 230.70 | -76.69 | 239.23 | -99.04 |
| $[^{3/2}]$, (3, 3) | - | - | 113.16 | -0.82 | - | - | 112.17 | -0.95 | - | - |
| $[^{3/2}]$, (3, 3) | - | - | 100.53 | 0.72 | 100.40 | 0.75 | 100.53 | 0.81 | 100.40 | 0.83 |
| $[^{3/2}]$, (3, 3) | 99.54 | 0.98 | 99.30 | 0.99 | - | - | 99.30 | 1.07 | - | - |
| $[^{3/2}]$, (3, 3) | 85.40 | 5.04 | 98.63 | 1.13 | 86.15 | 5.57 | 98.63 | 1.22 | 86.22 | 5.42 |
| $[^{3/2}]$, (3, 3) | 81.02 | 6.28 | 88.78 | 3.67 | 80.99 | 6.08 | 88.86 | 3.69 | 80.99 | 6.53 |
| $[^{3/2}]$, (3, 2) | 293.81 | 18.59 | 294.83 | 26.81 | 294.81 | 10.07 | 286.02 | 29.79 | 286.00 | 9.99 |
| $[^{3/2}]$, (3, 2) | 233.86 | -78.21 | 234.20 | -73.18 | 234.21 | -94.54 | 234.15 | -84.70 | 242.84 | -111.49 |
| $[^{3/2}]$, (3, 2) | 230.63 | -71.59 | 224.20 | -56.72 | 240.77 | -87.79 | 226.17 | -67.66 | 240.82 | -103.99 |
| $[^{3/2}]$, (3, 2) | 98.89 | 0.98 | 99.37 | 0.97 | 99.62 | 0.92 | 98.72 | 1.06 | 99.62 | 0.86 |
| $[^{3/2}]$, (3, 2) | 84.92 | 4.71 | 85.35 | 3.37 | 85.37 | 5.06 | 85.02 | 3.28 | 85.15 | 5.55 |
| $[^{3/2}]$, (3, 2) | 80.59 | 6.28 | 81.09 | 6.00 | 80.57 | 6.41 | 80.67 | 6.45 | 80.57 | 6.52 |
| $[^{3/2}]$, (3, 1) | 293.86 | 25.97 | 294.87 | 29.59 | 294.85 | 17.04 | 286.03 | 33.37 | 286.02 | 22.76 |
| $[^{3/2}]$, (3, 1) | 233.98 | -78.47 | 234.13 | -73.03 | 233.82 | -72.39 | 234.07 | -84.46 | 233.87 | -83.94 |
| $[^{3/2}]$, (3, 1) | 227.00 | -64.97 | 222.65 | -54.59 | 230.73 | -66.81 | 224.38 | -64.53 | 230.15 | -75.42 |
| $[^{3/2}]$, (3, 1) | 99.54 | 0.98 | 99.41 | 0.96 | 100.09 | 0.81 | 99.44 | 1.04 | 99.80 | 0.97 |
| $[^{3/2}]$, (3, 1) | 85.39 | 4.54 | 85.36 | 3.59 | 85.38 | 5.66 | 85.52 | 3.63 | 85.59 | 4.83 |
| $[^{3/2}]$, (3, 1) | 81.02 | 6.28 | 81.06 | 6.03 | 81.01 | 6.07 | 81.04 | 6.49 | 81.01 | 6.52 |

Continued on next page
| $[J'](F', m_{F'})$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $[J'](F', m_{F'})$ | $\lambda_{\text{magic}}$ | $\alpha_{\text{magic}}$ | $(1, 1)-(F', m_{F'})$ | $(1, 1)-(F', m_{F'})$ | $(1, -1)-(F', m_{F'})$ | $(1, -1)-(F', m_{F'})$ | $(1, 1)-(F', m_{F'})$ | $(1, 1)-(F', m_{F'})$ |
|-------------------|------------------|------------------|-------------------|------------------|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $(1, 1)-(F', m_{F'})$ | 230.46 | -71.22 | - | - | - | - | - | - | - | - | - |
| $(1, 1)-(F', m_{F'})$ | 221.19 | -56.04 | - | - | - | - | - | - | - | - | - |
| $(1, 1)-(F', m_{F'})$ | 98.90 | 1.12 | - | - | - | - | - | - | - | - | - |
| $(1, 1)-(F', m_{F'})$ | 85.38 | 3.62 | - | - | - | - | - | - | - | - | - |
| $(1, 1)-(F', m_{F'})$ | 80.59 | 6.62 | - | - | - | - | - | - | - | - | - |
| $(\frac{5}{2}), (2, 2)$ | 293.79 | 8.72 | 294.86 | 21.25 | 294.81 | -3.87 | 286.02 | 23.85 | 286.01 | -5.86 | - | - |
| $(\frac{7}{2}), (2, 2)$ | - | - | 237.43 | -79.84 | 245.50 | -101.19 | 237.35 | -93.14 | 244.83 | -118.97 | - | - |
| $(\frac{7}{2}), (2, 2)$ | - | - | 227.89 | -62.16 | 239.63 | -85.05 | 229.69 | -74.52 | 239.69 | -100.44 | - | - |
| $(\frac{5}{2}), (2, 1)$ | 293.85 | 22.44 | 294.87 | 29.59 | 294.85 | 17.04 | 286.03 | 32.76 | 286.02 | 17.92 | - | - |
| $(\frac{7}{2}), (2, 1)$ | 230.42 | -71.14 | 234.13 | -73.03 | 233.82 | -72.39 | 234.10 | -84.55 | 233.79 | -83.73 | - | - |
| $(\frac{7}{2}), (2, 1)$ | 224.08 | -60.28 | 222.65 | -54.59 | 230.73 | -66.81 | 224.49 | -64.73 | 231.16 | -77.72 | - | - |
| $(\frac{5}{2}), (2, 1)$ | 98.90 | 1.12 | 98.76 | 0.96 | 99.61 | 0.81 | 98.76 | 1.05 | 99.61 | 0.90 | 98.90 | 1.12 |
| $(\frac{7}{2}), (2, 1)$ | 85.38 | 3.83 | 84.85 | 3.62 | 84.87 | 5.73 | 85.03 | 3.44 | 85.13 | 5.13 | 85.38 | 3.83 |
| $(\frac{5}{2}), (2, 1)$ | 80.59 | 6.62 | 80.63 | 6.03 | 80.58 | 6.07 | 80.63 | 6.48 | 80.58 | 6.52 | 80.59 | 6.62 |
TABLE V. Tune-out wavelengths (nm) of Y$^{2+}$ at the hyperfine levels $(F,m_F)$. $(F = |I + J|)$

| $5^2S_{1/2}$ | $4^2D_{5/2}$ | $4^2D_{3/2}$ |
|--------------|--------------|--------------|
| (1,1) (2,2) (2,1) (2/1, 0) (1,1) (3,3) (3,2) (3,1) (3/2,0) (2,2) (2,1) | 293.73 230.83 244.46 244.47 244.36 99.54 234.14 234.15 230.59 230.50 230.59 | 108.68 215.32 186.85 180.59 204.41 91.04 193.06 180.46 175.24 220.54 181.86 |
| 98.96 98.96 98.96 98.96 99.54 99.54 98.90 98.89 98.97 | 90.01 88.02 88.02 89.38 89.19 88.45 87.74 90.06 88.16 | 80.61 80.62 80.62 80.62 81.00 81.01 80.58 80.57 80.58 |

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