The electroweak monopole–antimonopole pair in the standard model

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Abstract

We present the first numerical solution that corresponds to a pair of Cho–Maison monopoles and antimonopoles (MAPs) in the SU(2) × U(1) Weinberg–Salam (WS) theory. The monopoles are finitely separated, while each pole carries a magnetic charge \( \pm 4\pi/e \). The positive pole is situated in the upper hemisphere, whereas the negative pole is in the lower hemisphere. The Cho–Maison MAP is investigated for a range of Weinberg angles, 0.4675 \( \leq \tan \theta_W \leq 10 \), and Higgs self-coupling, 0 \( \leq \beta \leq 1.7704 \). The magnetic dipole moment (\( \mu_M \)) and pole separation (\( d_\perp \)) of the numerical solutions are calculated and analyzed. The total energy of the system, however, is infinite due to point singularities at the locations of monopoles.

Keywords: magnetic monopole, Cho-Maison monopole, Weinberg-Salam model

(Some figures may appear in colour only in the online journal)

1. Introduction

Magnetic monopoles have remained a topic of extensive study ever since Dirac [1] introduced the idea into Maxwell’s theory. This was generalized to non-abelian gauge theories in 1968 by Wu and Yang [2]. However, both Dirac and Wu–Yang monopoles possess infinite energy due to the presence of a point singularity at the origin. The first finite energy magnetic monopole solution is the ‘t Hooft–Polyakov monopole [3] in the SU(2) Yang–Mills–Higgs (YMH) theory, found independently by ‘t Hooft and Polyakov in 1974. The mass of their monopole was estimated to be around 137 \( m_W \), where \( m_W \) is the mass of the intermediate vector boson.

Since then, numerous solutions have been found in the SU(2) YMH theory. In the Bogomolny–Prasad–Sommerfield (BPS) limit of the vanishing Higgs potential, exact monopole [4] and multimonopole [5] (MAP) solutions exist. Other well-known cases are the axially symmetric MAP of Kleihaus and Kunz [6], and the MAP chain of Kleihaus et al [7]. These solutions possess finite energy and represent a chain of magnetic monopoles and antimonopoles lying in alternating order along the symmetrical axis.

In 1997, Cho and Maison [8] found a monopole solution in the standard Weinberg–Salam (WS) theory. The Cho–Maison monopole is an electroweak generalization of the Dirac monopole. It acquires a W-boson dressing and becomes a hybrid between the Dirac monopole and ‘t Hooft–Polyakov monopole. Moreover, its magnetic charge is twice as large because the period of the electromagnetic U(1) is 4\( \pi \) in the SU(2) × U(1) WS theory.

The importance of the Cho–Maison monopole comes from the following fact. As the electroweak generalization of the Dirac monopole, it must exist if the standard model is correct [8–10]. One might argue that the standard WS theory cannot accommodate magnetic monopoles because the second homotopy of quotient space, SU(2) × U(1)/U(1)_{em} is trivial. However, it is not the only monopole topology. As pointed out by Cho and Maison [8], the WS theory, with hypercharge U(1), can be viewed as a gauged CP\(^1\) model in which the normalized Higgs doublet plays the role of the CP\(^1\) field. This way, the SU(2) part of WS theory has exactly the same monopole topology as the Georgi–Glashow model [11]: that is, \( \pi_2(S^2) = \mathbb{Z} \). Originally, the Cho–Maison monopole solution was obtained by numerical integration, but a mathematically rigorous proof of existence was established later on [9].

The mass of the Cho–Maison monopole cannot be calculated due to the point singularity at the origin. However, it is premature to deny its existence simply because it has infinite energy. Classically, the electron has an infinite electric energy but a finite mass [10]. Additionally, it has been shown that the solution can be regularized. In [10], the mass of a
Ché-Maison monopole was estimated to be around 4 to 10 TeV. More recently, different methods have been used; it is reported that the new BPS bound for the Ché-Maison monopole may not be smaller than 2.98 TeV, more probably 3.75 TeV [12]. Another estimate puts the lower bound of the mass of the Ché-Maison monopole at 2.37 TeV [13].

These predictions strongly indicate that it could be produced at the Large Hadron Collider (LHC) in the near future. Therefore, when discovered, it will become the first magnetically charged topological elementary particle. Secondly, the Ché-Maison monopole could induce the density perturbation in the early Universe due to its heavy mass. For the same reason, it could become the seed of the large-scale structures in the Universe and the source of the intergalactic magnetic field. Moreover, it could also generate primordial magnetic black holes, which offers a possible explanation for the origin of dark matter [11]. For these reasons, MoEDAL and ATLAS at the LHC are actively searching for the monopole [14, 15].

Plainly, if magnetic monopoles were detected in the lab, it would be through pair production and, therefore, the importance of studying a pair of MAPs is self-explanatory. In 1977, Nambu [16] predicted the existence of a pair of magnetic MAPs bounded by a $Z^0$ flux string in the WS theory. Monopoles in an Nambu MAP carry magnetic charge, $\pm 4\pi \sin^2 \theta_W/e$. The existence of Nambu MAPs was confirmed numerically by Teh et al [17] using an axially symmetric magnetic ansatz. They also confirmed that Nambu MAPs are actually electroweak sphalerons reported by Kleihaus et al [18].

In this work, we demonstrate that it is feasible to construct a finitely separated Ché-Maison MAP. This configuration, achieved through an axially symmetric magnetic ansatz, does not have a $Z^0$ flux string connecting the poles. The magnetic charge carried by each pole of the MAP solutions found in this study is $\pm 4\pi / e$, confirming that they are indeed Ché-Maison monopoles. Additionally, it is worth noting that Gervalle and Volkov [19] explored Ché-Maison multimonomole solutions, which are also axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different, since multimonopoles are axially symmetric, but it is important to note that multimonopoles and MAPs are fundamentally different. In one location.

The Ché-Maison MAP solutions were investigated at a physical Weinberg angle, $\tan \theta_W = 0.53557042$, while the Higgs self-coupling constant, $\beta$, runs from 0 to 1.7704 and at physical $\beta = 0.77818833$, while $\tan \theta_W$ is allowed to vary (0.4675 $\leq \tan \theta_W \leq 10$). The investigated quantities include the magnetic dipole moment ($\mu_m$) and pole separation ($d_o$). The total energy of the configuration is infinite due to point singularities at the location of monopoles.

2. MAP ansatz in standard model

The Lagrangian of the bosonic sector of SU(2) × U(1) WS theory is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - (D_\mu \phi^a)(D^\mu \phi^a) - \frac{\lambda}{2} (\phi^a - \frac{\mu_m^2}{\lambda})^2.$$  (1)

Here, $D_\mu$ is the covariant derivative of the SU(2) × U(1) group and is defined as

$$D_\mu = D_\mu - \frac{i}{2} g' B_\mu = \partial_\mu - \frac{i}{2} g A_\mu^a \sigma^a - \frac{i}{2} g B_\mu,$$  (2)

where $D_\mu$ is the covariant derivative of the SU(2) group only.

The SU(2) gauge coupling constant, potential and electromagnetic tensor are $g$, $A_\mu^a$ and $F_{\mu\nu}^a$. Their counterparts in the U(1) gauge field are denoted as $g'$, $B_\mu$ and $G_{\mu\nu}$. The term $\sigma^a$ is the Pauli matrices, while $\phi$ and $\lambda$ are the complex scalar Higgs doublet and Higgs field self-coupling constant. Higgs boson mass and $\mu_m$ are related through $m_H = \sqrt{2} \mu_m$. In addition, the Higgs field can be expressed as $\phi = H \xi / \sqrt{2}$, where $\xi$ is a column 2-vector that satisfies $\xi^\dagger \xi = 1$. The metric used in this paper is ($+-----$).

Through Lagrangian (1), three equations of motion can be obtained as follows,

$$D^\mu D_\mu \phi^a = \lambda (\phi^a \phi - \frac{\mu_m^2}{\lambda}) \phi,$$  (3)

$$D^\mu F_{\mu\nu}^a = \frac{i g}{2} [\phi^b \sigma^a (D_\nu \phi) - (D_\nu \phi)^b \sigma^a \phi],$$  (4)

$$\partial^\mu G_{\mu\nu} = \frac{i g'}{2} [\phi^b \sigma^a (D_\nu \phi) - (D_\nu \phi)^b \phi].$$  (5)

The magnetic ansatz used to obtain the Ché-Maison MAP is [17]:

$$g A_i^a = -\frac{1}{r} \psi_1 (r, \theta) \hat{n}_\theta^a \hat{n}_\theta^a \hat{n}_\phi^a \hat{n}_\phi^a \phi_i + \frac{1}{r} R_i (r, \theta) \hat{n}_\theta^a \hat{n}_\phi^a \phi_i + \frac{n}{r} \frac{1}{r} R_2 (r, \theta) \hat{n}_\theta^a \hat{n}_\phi^a \phi_i,$$

$$g' B_i = \frac{n}{r} \frac{1}{r} B_i (r, \theta) \hat{n}_\theta^a \phi_i, \quad g A_0^a = g' B_0 = 0,$$

$$\delta^a = \Phi_1 (r, \theta) \hat{n}_\theta^a + \Phi_2 (r, \theta) \hat{n}_\phi^a = H (r, \theta) \hat{\phi}^a,$$  (6)

where the Higgs unit vector, $\hat{\Phi}^a$, can be written as,

$$\hat{\Phi}^a = -\xi^a \sigma^a \xi,$$

$$= \cos (\alpha - \theta) \hat{n}_\theta^a + \sin (\alpha - \theta) \hat{n}_\phi^a + h_1 \hat{n}_\theta^a + h_2 \hat{n}_\phi^a,$$

$$\xi = \begin{pmatrix} \sin (\alpha (r, \theta) / 2) & e^{-i \omega (r, \theta)} \\ - \cos (\alpha (r, \theta) / 2) & 0 \end{pmatrix}.$$  (7)

The functions $\cos \alpha$ and $\sin \alpha$ are defined as

$$\cos \alpha = \frac{\Phi_1 \cos \theta - \Phi_2 \sin \theta}{\sqrt{\Phi_1^2 + \Phi_2^2}} = h_1 \cos \theta - h_2 \sin \theta,$$

$$\sin \alpha = \frac{\Phi_1 \sin \theta + \Phi_2 \cos \theta}{\sqrt{\Phi_1^2 + \Phi_2^2}} = h_1 \sin \theta + h_2 \cos \theta.$$  (8)

Moreover, $H (r, \theta) = \Phi = \sqrt{\Phi_1^2 + \Phi_2^2}$ is the Higgs modulus. The angle $\alpha (r, \theta) \to p \theta$ asymptotically [17], where $p$ is the parameter controlling the number of poles in the solution and is set to two for MAP solutions. Finally, the spatial spherical coordinate unit vectors are defined as

$$\hat{A}_\theta = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ \sin \theta \end{pmatrix}, \quad \hat{A}_\phi = \begin{pmatrix} - \sin \theta \cos \phi \\ - \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}.$$  (9)

$$\hat{A}_r = \begin{pmatrix} - \sin \phi \\ \cos \phi \\ 0 \end{pmatrix}.$$  (10)
\[ H_i = \sin \theta \cos \phi \delta_{i3} + \sin \theta \sin \phi \delta_{i2} + \cos \theta \delta_{i1}, \]
\[ \delta_i = \cos \theta \cos \phi \delta_{i1} + \cos \theta \sin \phi \delta_{i2} - \sin \theta \delta_{i3}, \]
\[ \phi_i = -\sin \phi \delta_{i1} + \cos \phi \delta_{i2}. \]  

(9)

Similarly, the unit vectors for the isospin coordinate system are given by
\[ \hat{n}_i^a = \sin \theta \cos n \phi \delta_i^a + \sin \theta \sin n \phi \delta_i^b + \cos \theta \delta_i^c, \]
\[ \hat{n}_i^b = \cos \theta \cos n \phi \delta_i^a + \cos \theta \sin n \phi \delta_i^b - \sin \theta \delta_i^c, \]
\[ \hat{n}_i^c = -\sin \phi \delta_i^a + \cos \phi \delta_i^b, \]  

(10)

where \( n \) is the \( \phi \)-winding number and is set to one in this research.

Upon substituting the magnetic ansatz, equation (6), into the equations of motion, equations (3)-(5), the set of equations was reduced to seven coupled second-order partial differential equations, which correspond to the seven profile functions in the magnetic ansatz, equation (6), consistently \((\psi_1, \psi_2, R_1, R_2, B_1, \Phi_1, \Phi_2)\). These equations were further simplified with the following substitutions,
\[ x = m_w r, \quad \hat{H} = \frac{H}{H_0}, \quad \tan \theta_w = \frac{g_l}{g}, \quad \beta^2 = \frac{\lambda}{\beta^2}. \]  

(11)

Here, \( H_0 = \sqrt{2} \mu_{H}/\sqrt{\lambda} \) and \( m_w = g/H_0/2 \). The new radial coordinate, \( x \), is dimensionless, which is then compactified through \( \hat{x} = x/(x + 1) \). The rescaled Higgs field, \( \hat{H} \), approaches one asymptotically. Only two free parameters were left after the transformation, \( \tan \theta_w \) and \( \beta \). Both of these parameters can be expressed in terms of the mass of elementary particles, and by adopting \( m_\text{H} = 125.10 \text{GeV} \), \( m_\text{W} = 80.379 \text{GeV} \) and \( m_\text{Z} = 91.1876 \text{GeV} \), these parameters are calculated to be \( \beta = 0.77818833 \) and \( \tan \theta_w = 0.53557042 \).

The seven coupled equations are then subject to the following boundary conditions. Along the positive and negative \( z \)-axis, when \( \theta = 0 \) and \( \pi \),
\[ \partial_\theta \psi_1 = R_1 = \partial_\theta \Phi_1 = \Phi_2 = \partial_\theta B_2 = 0, \]  

(12)

where \( A = 1, 2 \). Asymptotically, when \( r \) approaches infinity,
\[ \psi_1(\infty, \theta) = 2, \quad R_1(\infty, \theta) = B_2(\infty, \theta) = 0, \quad \Phi_1(\infty, \theta) = \cos \theta, \quad \Phi_2(\infty, \theta) = \sin \theta, \]  

(13)

and finally, at the origin,
\[ \psi_1(0, \theta) = R_1(0, \theta) = 0, \quad B_2(0, \theta) = -2, \quad \Phi_1(0, \theta) \sin \theta + \Phi_2(0, \theta) \cos \theta = 0, \quad \partial_r (\Phi_1(r, \theta) \cos \theta - \Phi_2(r, \theta) \sin \theta)|_{r=0} = 0. \]  

(14)

Using the finite difference method (central difference approximation), the set of seven coupled partial differential equations was converted into a system of non-linear equations, which was then discretized onto a non-equidistant grid of \( M \times N \), where \( M = 70 \) and \( N = 60 \). The associated error with the numerical method employed is \( O(1/M^2) \) in the \( r \) direction and \( O(\pi^2/N^2) \) in the \( \theta \) direction. The region of integration covers all space which translates to \( 0 \leq \hat{x} \leq 1 \) and \( 0 \leq \theta \leq \pi \). Good initial guesses are needed for the numerical computation to converge.

### 3. Properties of Cho–Maison MAP

In the SU(2) \( \times \) U(1) WS model, the energy density of Cho–Maison MAP solutions is obtained from the energy-momentum tensor, \( T_{\mu\nu} \):
\[ \varepsilon = T_{00} = \frac{1}{4} G_{ij} G_{ij} + \frac{1}{4} F_{ij}^{\mu} F_{ij}^{\mu} + (D_i \phi)(D_i \phi) + \frac{\lambda}{2} \left( \phi^2 - \mu_\text{h}^2 \right)^2. \]  

(15)

The curve of weighted energy density \((\varepsilon_\text{w} = r^2 \sin \theta \cdot \varepsilon)\) near the \( z \)-axis was plotted to show that its value blows up and the singularities appear at the location of the monopoles.

To investigate the magnetic properties, the following gauge transformation was applied to the SU(2) gauge field, \( g \mathcal{A}_\mu \):
\[ g \mathcal{A}_\mu = g \mathcal{A}_\mu - \frac{1}{2} \frac{\partial_\mu \Phi}{\sin \theta}, \]  

(16)

where the unit vector, \( \hat{u}_\mu \), together with \( \hat{u}_\mu^a \) and \( \hat{u}_\nu^a \), are defined as
\[ \hat{u}_\mu^a = \frac{\sin \alpha}{2} \cos n \phi \delta_\mu^a + \frac{\sin \alpha}{2} \sin n \phi \delta_\mu^b + \cos \alpha \delta_\mu^c, \]
\[ \hat{u}_\nu^a = \frac{\cos \alpha}{2} \cos n \phi \delta_\nu^a + \frac{\cos \alpha}{2} \sin n \phi \delta_\nu^b - \sin \alpha \delta_\nu^c, \]
\[ \hat{u}_\nu^a = \hat{u}_\nu^a = -\sin \phi \delta_\nu^a + \cos \phi \delta_\nu^b. \]  

(17)

The transformed gauge potential has the following form:
\[ g \mathcal{A}_\mu = \frac{2n}{r} \left[ \psi_2 \sin \left( \frac{\alpha}{2} - \theta \right) - R_2 \cos \left( \frac{\alpha}{2} - \theta \right) \right] \hat{u}_\mu^\phi \hat{\phi}_i, \]
\[ - g \mathcal{A}_\mu = \frac{2n}{r} \sin \theta \hat{u}_\mu^\phi \hat{\phi}_i - \partial_\alpha \hat{u}_\nu^a. \]  

(18)

Note here, when \( a = 3 \), equation (18) becomes the ’t Hooft gauge potential,
\[ g \mathcal{A}_i^\phi = \frac{n}{r} \left( \psi_2 h_2 - R_2 h_1 - \frac{1 - \cos \alpha}{\sin \theta} \right) \hat{\phi}_i, \]
\[ = \frac{A_x}{r \sin \theta} \hat{\phi}_i. \]  

(19)

Through gauge transformation, equation (16), the physical fields can be expressed as
\[ \left( \begin{array}{c} \mathcal{A}_\mu^\text{em} \\ Z_i^\text{em} \end{array} \right) = \left( \begin{array}{cc} \cos \theta_w & \sin \theta_w \\ -\sin \theta_w & \cos \theta_w \end{array} \right) \left( \begin{array}{c} B_i \\ A_i^\phi \end{array} \right) \]
\[ = \frac{1}{\sqrt{g^2 + \xi^2}} \left( \begin{array}{c} g' \xi' \xi \left( \begin{array}{c} B_i \\ A_i^\phi \end{array} \right) \right) \]  

(20)

Note that the above definition is gauge independent [11].
Then, the real electromagnetic potential becomes

\[ \mathbf{A}^{\text{em}}_i = \frac{1}{2} \varepsilon_{ijk} \mathbf{F}^{\text{em}}_{jk} = -\frac{1}{2} \varepsilon_{ijk} (\partial_j \mathbf{A}^{\text{em}}_k - \partial_k \mathbf{A}^{\text{em}}_j) \]

where \( e \) is the unit electric charge. The 'em' magnetic field could then be calculated according to the mixing shown in equation (21),

\begin{align}
B^{\text{em}}_i &= -\frac{1}{e} \varepsilon_{ijk} \partial_j (e A^{\text{em}}_k) \\
&= -\frac{1}{e} \varepsilon_{ijk} \partial_j [\cos^2 \theta_W (g' B_i) + \sin^2 \theta_W (g A^3_i)] \\
&= -\frac{1}{e} \varepsilon_{ijk} \partial_j [\cos^2 \theta_W B_i + \sin^2 \theta_W A_i - \phi_k r \sin \theta] \\
&= -\frac{1}{e} \varepsilon_{ijk} \partial_j [\cos^2 \theta_W B_i + \sin^2 \theta_W A_i] \partial_k \phi.
\end{align}

(22)

Through Gauss’s law, the magnetic charge enclosed in a Gaussian surface, \( S \), with surface element, \( dS \), can be obtained with the following integral:

\[ Q_{M(S)} = \iint_S B^{\text{em}}_i dS = \iiint_V \partial_j B^{\text{em}}_j dV, \]

(23)

and for the magnetic charge enclosed in the upper hemisphere, the following Gaussian surface was defined:

\[ S_+ = H^2_x \cup D^2_{xy}, \]

(24)

where \( H^2_x \) is a half sphere above the \( xy \)-plane, and \( D^2_{xy} \) denotes a disk in the \( xy \)-plane centered at the origin; both \( H^2_x \) and \( D^2_{xy} \) have the same radius. Taking into account the correct orientation of the surface elements, the magnetic charge enclosed can be calculated as:

\[ Q_{M(S)} = \iint_{H^2_x} B^{\text{em}}_i r^2 \sin \theta \ d\theta \ d\phi \ \hat{\rho}^i - \iint_{D^2_{xy}} B^{\text{em}}_i r \ d\phi \ \hat{\chi}. \]

(25)

Upon applying the boundary conditions (12)–(14), the first integral vanishes, while the second one becomes:

\[ \iiint_{D_{xy}} B^{\text{em}}_i r \ d\phi \ \hat{\chi} = \frac{2 \pi}{e} \cos^2 \theta_W (2) + \sin^2 \theta_W (-2) \]

\[ = - \frac{4 \pi}{e}. \]

(26)

Therefore, the magnetic charge carried by the monopole in the upper space is \( Q_{M(S)} = 4 \pi / e \).

For the magnetic charge enclosed in the lower space, \( Q_{M(S)} \), the following calculation was applied:

\[ Q_{M(S)} = \iint_{H^2_x} B^{\text{em}}_i r^2 \sin \theta \ d\theta \ d\phi \ \hat{\rho}^i + \iint_{D^2_{xy}} B^{pm} r \ d\phi \ \hat{\chi}. \]

(27)

The Gaussian surface, \( S_- \), is now \( H^2_x \cup D^2_{xy} \), where \( H^2_x \) is a half sphere below the \( xy \)-plane. It is found that \( Q_{M(S)} = -4 \pi / e \). These values are characteristic of a Cho–Maison MAP.

Moreover, the magnetic dipole moment, \( \mu_m \), of a Cho–Maison MAP can be calculated according to the mixing shown in equation (21) and by considering at large \( r \), \( g' B_i = g A^3_i \),

\[ A_i \rightarrow \frac{1}{e} (g' B_i) = \frac{1}{e} \frac{n B_i}{r \sin \theta} \hat{\rho}. \]

(28)

Here, we perform an asymptotic expansion,

\[ n B_i = -\mu_m \frac{\sin^2 \theta}{r}, \]

(29)

and therefore, \( \mu_m = -n B_i / \sin^2 \theta \). The value of \( \mu_m \) (in units of \( 1 / (e \cdot m_W) \)) is evaluated at \( \theta = \pi / 2 \).

4. Results

Figure 1 shows a comparison of 3D Higgs modulus plots for three MAP configurations: (a) an SU(2) MAP [6], (b) a Nambu MAP (electroweak sphaleron) [17] and (c) a Cho–Maison MAP. In figure 1, the \( \rho \)-axis is defined as \( \rho = \sqrt{x^2 + y^2} \). Physical Higgs self-coupling, \( \beta = 0.77818833 \), was chosen for all three solutions and a physical Weinberg

Figure 1. The 3D Higgs modulus plots of (a) an MAP found in the SU(2) YM theory, (b) a Nambu MAP (electroweak sphaleron), and (c) a Cho–Maison MAP: all with \( \beta = 0.77818833 \). For (b) and (c), \( \tan \theta_W = 0.53557042 \).
angle, $\tan \theta_W = 0.53557042$, was used for the ones shown in figures 1(b) and (c). Evidently, in a Nambu MAP, the poles are connected through a $Z^0$ flux string, but in a Cho–Maison MAP, they are two separate entities, just like an SU(2) MAP.

Visually, the pole separation, $d_z$, of the Cho–Maison MAP is significantly larger than that of the MAP found in the SU(2) YMH theory. The value of $d_z$ for the Cho–Maison MAP can be obtained numerically from the curve of $\Phi_0(x, 0)$ and $\Phi_1(x, \pi)$. This is because $\Phi_2(x, \theta)$ is zero along the $z$-axis in boundary conditions (12) and hence, the behavior of $|\Phi|$ when $\theta = 0$ or $\pi$ is solely determined by $\Phi_1$.

Figure 2 shows the curve of $\Phi_1(x, \pi)$ for the MAP solution shown in figure 1(c). The magnetic antimonopole is located at $\Phi_1(x, \pi) = 0$, which is labelled $C$ in figure 2. Meanwhile, the magnetic monopole is located at $\Phi_1(x, 0) = 0$.

Figure 2. The curve of $\Phi_1(x, \pi)$ versus $x$ for the Cho–Maison MAP shown in figure 1(c).

Figure 3. A plot of $\varepsilon_W$ versus $z$ at $\rho = 0.1077$ for a Cho–Maison MAP with $\beta = 0.77818833$ and $\tan \theta_W = 0.53557042$. The values of $\varepsilon_W$ clearly indicate that there are two point singularities at the location of monopoles.

Figure 3. A plot of $\varepsilon_W$ versus $z$ at $\rho = 0.1077$ for a Cho–Maison MAP with $\beta = 0.77818833$ and $\tan \theta_W = 0.53557042$. The values of $\varepsilon_W$ clearly indicate that there are two point singularities at the location of monopoles.

which shares the same value as that of point $C$ in figure 2. The pole separation is then defined as $d_z = 2 \times AC$.

The plot of $\varepsilon_W$ versus $z$ at $\rho = 0.1077$ is shown in figure 3 for the particular solution displayed in figure 1(c). The weighted energy density increases rapidly near the location of monopoles, which clearly indicates there are two point singularities. As a result, the total energy of the system is infinite.

By fixing the Weinberg angle at $\tan \theta_W = 0.53557042$, the Cho–Maison MAP configuration is investigated for a range of $\beta$ from 0 to 1.7704. It is found that $d_z$ varies with $\beta$ and the plot is shown in figure 4(a). The value of $d_z$ starts off as 9.7140 when $\beta = 0$, then monotonically decreases until $\beta_{\text{min}} = 1.25$, where the local minimum $d_z = 8.0920$ (green dot) is reached. For physical $\beta = 0.77818833$, $d_z$ is measured to be 8.2012 (red dot). Instead of reaching a constant value, $d_z$ increases after $\beta_{\text{min}} = 1.25$ until $\beta = 1.7704$, where no solution can be found for $\beta > \beta_c$. The existence of an upperbound in $\beta$ is unexpected as SU(2) MAPs do not possess such a feature. The corresponding value for $d_z$ when $\beta = \beta_c$ is 8.2760 and the behavior of $d_z$ near $\beta_c$ is shown in figure 4(b). It can be seen that the gradient of the curve drastically increases after $\beta = 1.765$.

The plot of $\mu_m$ versus $\beta$ for a Cho–Maison MAP is shown in figure 5(a). The shape of the curve is basically identical to figure 4(a), which is expected, considering the close relations between $\mu_m$ and $d_z$. The physical value is measured to be $\mu_m = 8.5514$ (red dot), while the minimum $\mu_m$ is 8.4350 around $\beta_{\text{min}} = 1.25$ (green dot). We also investigate the behavior of $\mu_m$ by fixing $\beta = 0.77818833$, while allowing $\tan \theta_W$ to vary between 0.4675 and 10, figure 5(b). It is found that $\mu_m \rightarrow 10.3178$ as $\tan \theta_W \rightarrow 0.4675$, where a lower bound is reached, below which no solutions can be found. In a
similar manner, $\mu_m$ decreases until $\tan \theta_W = 2.89$, then increases slightly with increasing $\tan \theta_W$ before converging to a limiting value. The minimum value found this way is much lower, $\mu_m = 2.8098$ (green dot). Selected data of the Cho–Maison MAP is tabulated in tables 1 and 2.

Table 1. The table of $\mu_m$ and $d_c$ for selected $\beta$ of Cho–Maison MAP solutions at a physical Weinberg angle $\tan \theta_W = 0.53557042$ (numbers in the table are rounded).

| $\beta$  | 0      | 0.2    | 0.4    | 0.6    | 0.7782 | 0.9    | 1      | 1.25   | 1.4    | 1.6    | 1.7704 |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\mu_m$  | 10.2122| 9.1227 | 8.8754 | 8.6719 | 8.5514 | 8.4983 | 8.4684 | 8.4350 | 8.4385 | 8.4715 | 8.5958 |
| $d_c$    | 9.7140 | 8.7356 | 8.5170 | 8.3198 | 8.2012 | 8.1496 | 8.1208 | 8.0920 | 8.0994 | 8.1398 | 8.2760 |

Table 2. The table of $\mu_m$ and $d_c$ for selected $\tan \theta_W$ of Cho–Maison MAP solutions at a physical Higgs self-coupling constant $\beta = 0.77818833$ (numbers in the table are rounded).

| $\tan \theta_W$ | 0.4675 | 0.5356 | 0.6    | 0.8    | 1      | 2      | 2.89   | 4      | 6      | 8      | 10     |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\mu_m$         | 10.3178| 8.5514 | 7.5528 | 5.6489 | 4.6219 | 3.0113 | 2.8098 | 2.9240 | 3.0831 | 3.1513 | 3.3520 |
| $d_c$           | 9.9384 | 8.2012 | 7.2484 | 5.4582 | 4.4912 | 2.8624 | 2.6120 | 2.7568 | 2.9584 | 3.0228 | 3.0470 |

5. Conclusion

In conclusion, we have found numerical solutions in the SU(2) × U(1) WS theory corresponding to a pair of Cho–Maison MAPs. Poles of this MAP configuration are not connected through a $Z^0$ flux string and each pole carries a magnetic charge, $\pm 4\pi/e$. The Cho–Maison MAP resides in the topological trivial sector, indicating its possible existence in nature, but its life-time or the interactions between constituents of this MAP are yet to be answered.

When $\tan \theta_W$ is fixed at 0.535 570 42, an upperbound for the Cho–Maison MAP exists at $\beta = 1.7704$, beyond which no solutions can be found. This is a feature that MAP solutions found in the SU(2) YMH theory do not possess. When $\beta$ is fixed at 0.778188 33, a lower bound exists at $\tan \theta_W = 0.4675$, where both $\mu_m$ and $d_c$ reach their maximum values of 10.3178 and 9.9384. However, as solutions in the SU(2) × U(1) WS theory are controlled by two parameters, $\beta$ and $\tan \theta_W$, finding global extrema must be done on a two-dimensional curve, like $d_c(\beta, \tan \theta_W)$. Lastly, we were able to accurately measure the physical values of $\mu_m$ and $d_c$ when $\beta = 0.77818833$ and $\tan \theta_W = 0.53557042$. The magnetic dipole moment of a Cho–Maison MAP is $\mu_m = 8.5514$ in units of $1/(e \cdot m_H)$ and the pole separation is $d_c = 8.2012$ in units of $1/m_H$.

Although the numerical method employed in this research is a direct generalization of a well-known procedure used in [6, 18], the solutions obtained in this study are unique and new. Indeed, the originality of this work lies in the actual construction of a novel numerical solution, which is a finitely separated Cho–Maison MAP.

Finally, due to the presence of point singularities at the locations of monopoles, the total energy of the system is infinite. This also prohibits us from determining if the solutions are bound states. However, it is possible to regularize the solutions by introducing a non-trivial U(1) hypercharge permeability in the form of a dimensionless function, $\epsilon(\phi) = (H/H_0)^\delta$ [10, 12]. With this, the total energy of the Cho–Maison MAP can be evaluated. This will be reported in a future work. Additionally, electric charges can be introduced into the system, forming a pair of dyons and antidyons.

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