An Algorithm for Best Generalised Rational Approximation of Continuous Functions

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Received: 5 November 2020 / Accepted: 16 December 2021 / Published online: 28 February 2022
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Abstract
In this paper we introduce an algorithm for solving variational inequality problems when the operator is pseudomonotone and point-to-set (therefore not relying on continuity assumptions). Our motivation is the development of a method for solving optimisation problems appearing in Chebyshev rational and generalised rational approximation problems, where the approximations are constructed as ratios of linear forms (linear combinations of basis functions). The coefficients of the linear forms are subject to optimisation and the basis functions are continuous functions. It is known that the objective functions in generalised rational approximation problems are quasiconvex. In this paper we prove a stronger result, the objective functions are pseudoconvex in the sense of Penot and Quang. Then we develop numerical methods, that are efficient for a wide range of pseudoconvex functions and test them on generalised rational approximation problems.

Keywords Chebyshev generalised rational approximation · Pseudoconvex functions · Point-to-set operators

Mathematics Subject Classification (2010) 90C25 · 90C26 · 90C90 · 90C47 · 65D15 · 65K10

1 Introduction
Consider the set of all real-valued polynomials with degree up to \( n \), denoted by \( \Pi_n \), and the continuous function \( f : \mathbb{R} \to \mathbb{R} \). We are interested in the problem of approximating the
function \( f \) by a rational function \( \frac{p}{q} \) where \( p \in \Pi_n \) and \( q \in \Pi_m \) for some given nonnegative numbers \( n \geq 0 \) and \( m \geq 0 \). In other words we want to solve the optimisation problem:

\[
\min_{p \in \Pi_n, q \in \Pi_m} \sup_{t \in I} \left| f(t) - \frac{p(t)}{q(t)} \right|.
\]

Problem (1) is also known as Chebyshev rational approximation problem. For \( t \in I \) where \( I \subset \mathbb{R} \) is any compact subset of \( \mathbb{R} \), we define the vector \( t_n = (1, t, t^2, \ldots, t^n) \in \mathbb{R}^{n+1} \). We represent the polynomial \( p \in \Pi_n \) by the vector of its coefficients \( a = (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1} \) as follows:

\[
p(t) = \langle a, t_n \rangle = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n.
\]

Now Problem (1) can be written as:

\[
\min_{(a, b) \in C} \Psi_f(a, b),
\]

where \( \Psi_f(a, b) = \sup_{t \in I} \left| f(t) - \frac{\langle a, t_n \rangle}{\langle b, t_m \rangle} \right| \) is a maximal deviation for the approximation of \( f \) on the interval \( I \), and

\[
C = \{(a, b) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} : \langle b, t_m \rangle \geq 1, \forall t \in I \}
\]

is a feasible set.

Rational approximation was a popular research topic in the 1950s–1960s \cite{1, 9, 19, 20} as a promising alternative to the free knot spline approximation. Rational approximation models combine simplicity and significant flexibility, two properties attractive for practical applications \cite{6, 17}.

The authors of the current paper started investigating the connections between optimisation and rational approximation in \cite{17}. Here we extend these investigations in several significant ways. We show that the problem under consideration is pseudoconvex. This is an important step forward, as this guarantees that any local minimiser is also global, and therefore opens up opportunities to develop new algorithms. Furthermore, the algorithm we propose in this paper applies to a general class of problems, which includes rational approximation.

In \cite{10} Cheney and Loeb demonstrated that some of the results in the area of Chebyshev rational approximation can be extended to approximation by a ratio of linear forms

\[
\frac{G(a, t)}{H(b, t)} = \frac{a_0 g_0(t) + a_1 g_1(t) + a_2 g_2(t) + \cdots + a_n g_n(t)}{b_0 h_0(t) + b_1 h_1(t) + b_2 h_2(t) + \cdots + b_m h_m(t)} = \frac{\langle a, g(t) \rangle}{\langle b, h(t) \rangle},
\]

where \( g_i(t), i = 0, \ldots, n \) and \( h_j(t), j = 0, \ldots, m \) are not limited to bases of polynomials, but can be any set of continuous functions. For example, \( g_i(t) = \sin^i(t) \), or \( h_j(t) = e^{jt} \). The authors call this type of approximation \textit{generalised rational approximation}. There are a number of ways to generalise rational approximations. In the current paper we use the same terminology as in \cite{10} and therefore we approximate continuous functions by the ratios of linear forms and the coefficients of these forms are subject to optimisation. It is still required for the linear form in the denominator to be positive.

These extensions of the results are possible due to the fact that the corresponding objective functions in the optimisation problems are quasiconvex. We will talk about this property in Section 2. In this paper we also prove a stronger result, the objective functions are pseudoconvex in the sense of Penot and Quang (which extends the notion of pseudoconvexity to the case of nonsmooth functions, see \cite{2, 12, 18}). Then we develop numerical methods, that are efficient for a wide range of pseudoconvex functions and test them on generalised rational approximation problems.
The generalised rational approximation problem can be formulated as the following optimisation problem:

\[
\min_{\mathbf{a}, \mathbf{b}} \sup_{t \in I} \left| f(t) - \langle \mathbf{a}, g(t) \rangle \langle \mathbf{b}, h(t) \rangle \right|. \tag{4}
\]

The decision variables are the coefficients of the linear forms \( \mathbf{a} = (a_0, a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n+1} \) and \( \mathbf{b} = (b_0, b_1, \ldots, b_m) \in \mathbb{R}^{m+1} \).

The functions \( g_i(t), i = 0, \ldots, n \) and \( h_j(t), j = 0, \ldots, m \) are called the basis functions, \( t \in I, I \subset \mathbb{R} \) any compact subset of \( \mathbb{R} \). We also define the deviation function \( \sigma_t(\mathbf{a}, \mathbf{b}) = f(t) - \langle \mathbf{a}, g(t) \rangle \langle \mathbf{b}, h(t) \rangle \) and absolute deviation function \( \Phi^f_{\mathbf{a}, \mathbf{b}}(t) = |\sigma_t(\mathbf{a}, \mathbf{b})| \) this problem can be written as:

\[
\min_{(\mathbf{a}, \mathbf{b}) \in C} \Psi^f(\mathbf{a}, \mathbf{b}), \tag{5}
\]

where \( \Psi^f(\mathbf{a}, \mathbf{b}) = \sup_{t \in I} \Phi^f_{\mathbf{a}, \mathbf{b}}(t) \) is a maximal deviation for the approximation of \( f \) on the interval \( I \), and

\[
C = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} : \langle \mathbf{b}, h(t) \rangle \geq 1, \forall t \in I\}
\]

is the feasible set.

To solve Problem (2), we develop a projection-type Algorithm inspired by the algorithms presented in [5, 7]. The presented algorithm specifically solves the Variational Inequality Problem (VIP), defined and studied in Section 4, which is well known to be equivalent to Problem (2). The presented algorithm solves the VIP for a class of multivalued operators satisfying that the dual and primal VIP share the same solution. All pseudomonotone operators satisfy this property. For more details we refer the reader to [5, 7, 13–15] and references therein.

The manuscript is organised as follows. In Section 2 we introduce notation and preliminaries. In Section 3 we introduce essential results related to the approximation problem. Section 4 is dedicated to developing an algorithm for solving a variational inequality which is equivalent to solve the approximation problem. In Section 5 we present a general algorithm to solve non-monotone variational inequalities for point-to-set operators. Section 6 shows some numerical experiments to demonstrate the behaviour of the presented algorithm. Some conclusion remarks are presented in Section 7.

## 2 Preliminary Results

This section is devoted to some classic notations, definitions and results we will use in the present work.

By \( \mathbb{R}^n \) we denote the \( n \) dimensional Euclidean space, \( |\cdot| \) is the absolute value function, \( \|\cdot\| \) the norm induced by the inner product \( \langle \cdot, \cdot \rangle \). The set \( \text{conv} \ D \) is the convex hull of the set \( D \). The orthogonal projection of a point \( x \in \mathbb{R}^n \), onto the convex, closed and nonempty set \( C \subset \mathbb{R}^n \), is defined by the unique point in \( C \), which is the solution of the minimal distance problem:

\[
P_C(x) = \arg\min_{y \in C} \|x - y\|.
\]

Given a point-to-set the operator \( T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \), the graph of \( T \) is denoted and defined by \( Gr(T) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in T(x)\} \).
Fact 1 [3, Proposition 4.8 and Theorem 3.14] Let $C \subseteq \mathbb{R}^n$ be closed, convex and nonempty. For all $x, y \in \mathbb{R}^n$ and all $z \in C$, the following holds:

1. \[ \| P_C(x) - P_C(y) \|^2 \leq \| x - y \|^2 - \| (x - P_C(x)) - (y - P_C(y)) \|^2. \]
2. \[ (x - P_C(x), z - P_C(x)) \leq 0. \]

One of the most useful tools in projection algorithms is the Fejér convergence, defined as follows:

**Definition 1** (Fejér convergence) Let $S$ be a nonempty subset of $\mathbb{R}^n$. A sequence $(x_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$ is said to be Fejér convergent to $S$, if and only if, for all $x \in S$ there exists $k_0 \in \mathbb{N}$ such that $\| x^{k+1} - x \| \leq \| x^k - x \|$ for all $k \geq k_0$.

For more details on Fejér convergent sequences we refer the reader to [4]. The main property that Fejér convergent sequences satisfy is as follows:

**Proposition 1** If $(x_k)_{k \in \mathbb{N}}$ is Fejér convergent to $S$, then it is bounded.

**Proof** By definition, taking $\bar{x} \in S$, we have $\| x^k - \bar{x} \| \leq \| x^n - \bar{x} \|$ for any $n \leq k$, then for all $k \in \mathbb{N}$, $\| x^k - \bar{x} \| \leq \| x^{k_0} - \bar{x} \|$. □

The concepts of monotonicity, pseudomonotonicity and quasimonotonicity are mentioned in this work. Now, we define these operators.

**Definition 2** A point-to-set operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is called:

(i) **Monotone**, if and only if, for all $(x, u), (y, v) \in Gr(T)$,
\[ \langle u - v, x - y \rangle \geq 0. \]

(ii) **Pseudomonotone**, if and only if, for all $(x, u), (y, v) \in Gr(T)$, the following implication holds:
\[ \langle u, y - x \rangle \geq 0 \implies \langle v, y - x \rangle \geq 0. \]

(iii) **Quasimonotone**, if and only if, for all $(x, u), (y, v) \in Gr(T)$, the following implication holds:
\[ \langle u, y - x \rangle > 0 \implies \langle v, y - x \rangle \geq 0. \]

It is clear that every monotone operator is pseudomonotone, and every pseudomonotone operator is quasimonotone.

Next we define the well known Clarke Subdifferential:

**Definition 3** (Clarke Subdifferential [11]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, the Clarke Subdifferential of the function $f$ in $x \in \mathbb{R}^n$ is defined by:
\[ \partial f (x) := \{ x^* \in \mathbb{R}^n : \langle x^*, d \rangle \leq f^0(x; d), \forall d \in \mathbb{R}^n \}, \]
where \[ f^0(x; d) = \liminf_{y \to x, t \downarrow 0} \frac{f(y + td) - f(x)}{t}. \]

In this paper we prove that the operator under consideration (the Clarke subdifferential of the objective function) is pseudomonotone. We will rely on the following known results.
Note that a function is said to be radially continuous when its restrictions to line segments is continuous. Every continuous function is radially continuous.

**Theorem 1** [2, Theorem 4.1] Let \( f \) be a lower semicontinuous function and radially continuous function. The following assertions are equivalent:

1. \( f \) is pseudoconvex.
2. \( f \) is quasiconvex and \( (0 \in \partial_C f(x^*) \implies f \) has a global minimum at \( x^* \).

**Proposition 2** [12, Proposition 2.2] Let \( f \) be a lower semicontinuous function and a radially continuous function. Then \( f \) is pseudoconvex if and only if the operator \( \partial_C f \) is pseudomonotone.

For completeness we present the definition of inner semicontinuous operator.

**Definition 4** (Inner semicontinuous - see [8]) The point-to-set operator \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \), is said to be inner semicontinuous at \( x \in \mathbb{R}^n \), if for any sequence \( (x_i)_{i \in \mathbb{N}} \subseteq \mathbb{R}^n \) convergent to \( x \) and any \( y \in T(x) \), there exists a sequence \( (y_i)_{i \in \mathbb{N}} \) such that \( y_i \in T(x_i) \) for all \( i \in \mathbb{N} \) and \( y_i \rightarrow y \).

The next definition will be used as part of the assumptions for the convergence of the algorithm.

**Definition 5** A point-to-set operator is said to be sequentially continuous if for any two sequences \( (x^k)_{k \in \mathbb{N}} \) and \( (y^k)_{k \in \mathbb{N}} \) such that \( \lim_{k \rightarrow \infty} x^k = \lim_{k \rightarrow \infty} y^k = x \) then, there exist \( u \in T(x) \) and \( u^k \in T(x^k) \) and \( v^k \in T(y^k) \) for all \( k \in \mathbb{N} \), satisfying that \( \lim_{k \rightarrow \infty} u^k = \lim_{k \rightarrow \infty} v^k = u \).

### 3 Approximation of Continuous Functions

This section is dedicated to analysing Problem (2). We start from the following lemma.

**Lemma 1** [17, Theorem 5] For any real function \( f : \mathbb{R} \rightarrow \mathbb{R} \) and a compact set \( I \subseteq \mathbb{R} \), the maximal deviation \( \Psi^f : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \rightarrow \mathbb{R} \) is a quasiconvex function.

Interestingly, this result is a direct corollary of [10, Lemma 2], but it was not elaborated by the authors and was left unnoticed for several decades. The most reasonable explanation for this is that most efficient techniques for quasiconvex optimisation were developed much later: around 20 years after [10].

We denote by \( A(a, b) = A^+(a, b) \cup A^-(a, b) \) the set of active values, where

\[
A^+(a, b) := \left\{ t \in I : \Psi^f(a, b) = \sigma^f_t(a, b) \right\}
\]

\[
A^-(a, b) := \left\{ t \in I : \Psi^f(a, b) = -\sigma^f_t(a, b) \right\},
\]

and \( \sigma^f_t(a, b) := f(t) - \frac{\langle a, g(t) \rangle}{\langle b, h(t) \rangle} \).
Theorem 2 The Clarke subdifferential of the function $\Psi^f$ can be computed as follows:

$$\partial \Psi^f(a, b) = \text{conv}\left\{ \nabla \sigma_t^f(a, b), -\nabla \sigma_t^f(a, b) : t \in A^+(a, b), l \in A^-(a, b) \right\}.$$  (6)

Proof See Theorem 10.31 in [21].

Proposition 3 Given a continuous function $f$, if $0 \in \partial \Psi^f(a, b)$ then $(a, b)$ is a global minimiser of $\Psi^f$.

Proof This proof is largely based on [10].

Define by

$$s_t(a, b) = \frac{\sigma_t^f(a, b)}{\Psi^f(a, b)},$$

so that $s_t(a, b) = 1$ (resp. $-1$) when $t \in A^-(a, b)$ (resp. $t \in A^+(a, b)$).

Because of these two respective equalities, we can rewrite the subdifferential as $\partial \Psi^f = \text{conv}\{s_t(a, b)\nabla \sigma_t^f(a, b), t \in A(a, b)\}$, where

$$\nabla \sigma_t^f(a, b) = \frac{1}{(a, b) h(t))} \left( -g_0(t)(a, h(t)), -g_1(t)(a, h(t)), \ldots, -g_n(t)(a, h(t)), h_0(t)(a, g(t)), h_1(t)(a, g(t)), \ldots, h_m(t)(a, g(t)) \right)$$

$$= \frac{1}{(a, b) h(t))} \left( (a, g(t)), (a, g(t))(a, b) h(t)) \right).$$

Suppose that the point $(a, b)$ is not the global optimiser. In this case, there is a direction $(a', b') \neq 0_{n+m+2}$ such that $\Psi^f(a - a', b - b') < \Psi^f(a, b)$, and $(a - a', b - b') \in C$. In particular for any $t \in A(a, b)$, $|\sigma_t^f(a - a', b - b')| < |\sigma_t^f(a, b)|$.

From there, we can find that:

$$s_t(a, b) \left( f - \frac{\langle a, g(t) \rangle}{(a, h(t))} \right) - s_t(a, b) \left( f - \frac{\langle a - a', g(t) \rangle}{(a - b', h(t))} \right) > 0$$

$$s_t(a, b) \left( \frac{\langle a - a', g(t) \rangle}{(a - b', h(t))} - \frac{\langle a, g(t) \rangle}{(a, h(t))} \right) > 0$$

$$s_t(a, b) \left( \langle a - a', g(t) \rangle - \frac{\langle a, g(t) \rangle}{(a, h(t))} \langle b - b', h(t) \rangle \right) > 0$$

$$s_t(a, b) \left( -\langle a', g(t) \rangle + \frac{\langle a, g(t) \rangle}{(b, h(t))} \langle b - b', h(t) \rangle \right) > 0$$

$$s_t(a, b) \left( -\langle b, h(t) \rangle \langle a', g(t) \rangle + \langle a, g(t) \rangle (b', h(t)) \right) > 0$$

$$s_t(a, b) \left( \frac{\langle a', g(t) \rangle}{(b, h(t))} - \langle b, h(t) \rangle \langle a', g(t) \rangle + \langle a, g(t) \rangle \langle b', h(t) \rangle \right) > 0$$

$$\left( \frac{\langle a', b' \rangle}{(b, h(t))}, s_t(a, b) \nabla \sigma_t^f(a, b) \right) > 0.$$
Since this product is positive for every \( t \in A(a, b) \), it is also positive for every element in \( \partial f \Psi(a, b) \). Indeed, any such element can be written as

\[
g = \sum_{t \in A(a, b)} \alpha_t s_t(a, b) \nabla \alpha_t \sigma_t(a, b), \quad \text{with} \quad \alpha_t \geq 0, \quad \sum_{t \in A(a, b)} \alpha_t = 1.
\]

Then

\[
\langle (a', b'), g \rangle = \sum_{t \in A(a, b)} \alpha_t \langle (a', b'), s_t(a, b) \nabla \sigma_t(a, b) \rangle \neq 0.
\]

Hence \( g \neq 0 \) and this implies that \( 0 \not\in \partial f \Psi(a, b) \).

\[\square\]

**Theorem 3** The function \( \Psi f \) is a pseudoconvex function in terms of \([2, 12, 18]\). Consequently the Clarke Subdifferential \( \partial \Psi f \) is a pseudomonotone operator.

**Proof** From Lemma 1, Proposition 3 and Theorem 1, it follows that \( \Psi f \) is pseudoconvex. The second part is due to Proposition 2.

This result is especially important, since the basis functions are not restricted to monomials, that is, we can consider the general problem of approximating a function by ratios of linear forms, as in (3).

In [7], the authors introduce an algorithm for solving variational inequalities, when the operator is pseudomonotone, subject to some continuity requirements. In the following example we show that the subdifferential of the function \( \Psi f \) is not necessarily inner semicontinuous, and therefore does not satisfy the requirements from [7].

**Example 1** Consider the constant function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(t) = 0 \), the polynomials \( p(t) = 2t^2 - 1 \) and \( q(t) = 1 \), and the compact set \( I = [-1, 1] \). Then \( a = (-1, 0, 2) \ b = (1) \), \( A^+ = \{0\} \) and \( A^- = \{-1, 1\} \). Therefore \( \partial \Psi f(a, b) = \operatorname{conv}\{(-1, -1, 1, -2), (-1, 0, 0, 1), (1, 1, 1, -2)\} \) corresponding to the active points \( t_0 = -1, t_1 = 0 \) and \( t_2 = 1 \). Now consider the sequence \((a_n, b_n) = (-1, 1/n, 2, 1)\) which converges to the point \((a, b) = (-1, 0, 2, 1)\). For all \( n \in \mathbb{N} \), \( A^+(a_n, b_n) = \emptyset \) and \( A^-(a_n, b_n) = \{1\} \). That is:

\[
\Psi f(a_n, b_n) = \sup_{t \in [-1, 1]} \left| -1 + \frac{t}{n} + 2t^2 \right| = 1 + \frac{1}{n}.
\]

Then for all \( n \in \mathbb{N} \) we have that \( \partial \Psi f(a_n, b_n) = (1, 1, 1, -1 - \frac{1}{n}) \), which proves that there is no sequence of elements of \( \partial \Psi f(a_n, b_n) \) converging to \( y = (-1, 0, 0, 1) \in \partial \Psi f(a, b) \). Therefore the operator \( \partial \Psi f \) is not inner semicontinuous.

The following Lemma is a direct consequence of the continuity of the function \( \Psi f \).

**Lemma 2** Consider a converging sequence \((a_n, b_n)_{n \in \mathbb{N}} \subseteq C \), so that 

\[
\lim_{n \to \infty} (a_n, b_n) = (a, b) \in C.
\]

Taking for all \( n \in \mathbb{N} \) an active value \( t_n \in A^+(a_n, b_n) \cup A^-(a_n, b_n) \), for any converging subsequence \((t_{n_k})_{k \in \mathbb{N}} \subseteq (t_n)_{n \in \mathbb{N}}\), such that \( \lim_{k \to \infty} t_{n_k} = \tilde{t} \) we have that \( \tilde{t} \in A^+(a, b) \cup A^-(a, b) \).
Proof The existence of a convergent subsequence is due to the boundedness of the compact set $I$, from now on, suppose for simplicity that the sequence $(t_i)_{i \in \mathbb{N}}$ is convergent to a point $\bar{t} \in I$.

By continuity of $f$ we have

$$\lim_{i \to \infty} \left| f(t_i) - \frac{\langle a_i, g(t_i) \rangle}{\langle b_i, h(t_i) \rangle} \right| = \left| f(\bar{t}) - \frac{\langle a, g(\bar{t}) \rangle}{\langle b, h(\bar{t}) \rangle} \right|. \tag{7}$$

Take any $\hat{t} \in I$. For all $i \in \mathbb{N}$ we have

$$\left| f(\hat{t}) - \frac{\langle a_i, g(\hat{t}) \rangle}{\langle b_i, h(\hat{t}) \rangle} \right| = \left| f(t_i) - \frac{\langle a_i, g(t_i) \rangle}{\langle b_i, h(t_i) \rangle} \right|,$$

because all $t_i$ are active values of $\Psi f$ on its respective $(a_i, b_i)$. Taking limits when $i \to \infty$ we obtain, using the continuity of the absolute value function, that

$$\left| f(\hat{t}) - \frac{\langle a, g(\hat{t}) \rangle}{\langle b, h(\hat{t}) \rangle} \right| \leq \left| f(\bar{t}) - \frac{\langle a, g(\bar{t}) \rangle}{\langle b, h(\bar{t}) \rangle} \right|.$$

Since it is true for all $\hat{t} \in I$, it must hold that $\bar{t}$ is an active value for $(a, b)$. \qed

4 Variational Inequality and the Algorithm

In this section we develop an algorithm for solving the variational inequality problem defined below. The algorithm uses projections onto separating hyperplanes and onto the feasible set, and a feasible line search. Two different projection strategies are presented. We do not assume maximal monotonicity of the operator, which is a classical assumption in variational inequality problems. This algorithm applies for solving Problem (5). Algorithm 1 extends Algorithm F in [5].

Consider the pseudomonotone operator $T : \mathbb{R}^n \Rightarrow \mathbb{R}^n$ and the set $C \subset \mathbb{R}^n$, the variational inequality problem for $T$ and $C$, denoted by VIP($T, C$), is defined as:

$$\text{Find } x^* \in C : \exists u^* \in T(x^*) : \langle u^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{8}$$

Problem (8) is equivalent to the dual variational inequality problem (DVIP($T, C$)):

$$\text{Find } x^* \in C : \forall u \in T(x) : \langle u, x - x^* \rangle \geq 0, \forall x \in C. \tag{9}$$

This problem has been widely studied. For work on variational inequality problems with pseudomonotone operators, we refer the reader to [5, 7, 13–15] and references therein.

We denote the solution set of Problem (8) by $S^*$ and the solution set of Problem (9) by $S_0$. The equivalence of Problems (8) and (9) implies that $S^* = S_0$.

We present the version of Linesearch F in [5] suitable to our problem in Algorithm 1.

Notice that Algorithm 1 needs access to the whole set $T(x^k)$, where $x^k$ is the current iterate. In general the operator may not be easy to compute: for example, finding the Clarke subdifferential of a pseudoconvex function is difficult. However, in the case of Problem (5), we have an explicit formulation for the operator, given in Theorem 2.
Algorithm 1: Line search.

**Input:** \((a, b) \in C, \beta > 0\) and \(\delta \in (0, 1)\)

Set \(\alpha \leftarrow 1\) and \(\theta \in (0, 1)\).

for all \((u_a, u_b) \in \{\nabla \sigma^f_t(a, b), -\nabla \sigma^f_t(a, b) : t \in A^+(a, b), l \in A^-(a, b)\}\)

define \((z_a, z_b) = P_C((a, b) - \beta(u_a, u_b))\)

if for each \((u_a, u_b)\) we have

\[
\max_{(u_a^*, u_b^*) \in D_a} \langle (u_a^*, u_b^*), (a, b) - (z_a, z_b) \rangle < \delta \langle (u_a, u_b), (a, b) - (z_a, z_b) \rangle,
\]

where \(D_a := \{\nabla \sigma^f_t(a_\alpha, b_\alpha), -\nabla \sigma^f_t(a_\alpha, b_\alpha) : t \in A^+(a_\alpha, b_\alpha), l \in A^-(a_\alpha, b_\alpha)\}\)

and

\((a_\alpha, b_\alpha) := \alpha(z_a, z_b) + (1 - \alpha)(a, b), \) then

\[
\alpha \leftarrow \theta \alpha.
\]

else

Return \(\alpha\) and \((u_a, u_b)\)

end

**Output:** \((\alpha, (u_a, u_b))\)

This algorithm relies on the evaluation of an operator \(\mathcal{F}\) (we use \(\mathcal{F}\) for “feasible”, because the line search applies within the feasible set). We consider two variants for the operator \(\mathcal{F}\). Their main difference lies in the way to compute (10b):

\[
\mathcal{F}_1 \left((a_k, b_k)\right) = P_C \left(P_H \left((\bar{a}^k, \bar{b}^k), (u_a^k, u_b^k)\right) \left((a_k, b_k)\right)\right); \quad \text{(Variant 1)}
\]

\[
\mathcal{F}_2 \left((a_k, b_k)\right) = P_{C \cap H} \left((\bar{a}^k, \bar{b}^k), (u_a^k, u_b^k)\right) \left((a_k, b_k)\right); \quad \text{(Variant 2)}
\]

where

\[
H(x, u) := \left\{ y \in \mathbb{R}^{n+m+2} : \langle u, y - x \rangle \leq 0 \right\}. \quad \text{(11)}
\]

Variants 1, 2 are the adaptation of the Variants 1 and 2 from [5] and from the Algorithm presented in [7] to solve Problem (5). The main difference is the line search, because in the present version, we do not assume inner semicontinuity of the operator \(\partial \Psi^f\), as shown in Example 1. Another difference is that the operator \(\partial \Psi^f\) is point-to-set and not point-to-point as in [5].

4.1 Convergence

This section is dedicated to prove the convergence of Algorithm 2. We first show that Algorithm 1 (Line search) terminates.

**Proposition 4** If \((a, b) \in C\) is not a solution of Problem (8), Algorithm 1 (Line search) terminates after finitely many iterations.

**Proof** Suppose that Algorithm 1 never stops. Then, for all \(\alpha \in \{1, \theta, \theta^2, \cdots\}\) and \((u_a, u_b) \in \partial \Psi^f(a, b)\) we have

\[
\langle (u_a^\alpha, u_b^\alpha), (a, b) - (z_a, z_b) \rangle < \delta \langle (u_a, u_b), (a, b) - (z_a, z_b) \rangle
\]
Algorithm 2: Main algorithm.

**Input:** \((\beta_k)_{k \in \mathbb{N}} \subset [\hat{\beta}, \tilde{\beta}]\) such that \(0 < \hat{\beta} \leq \beta < +\infty\) and \(\delta \in (0, 1)\).

**Initialisation:** Take \((a^0, b^0) \in C\) and set \(k \leftarrow 0\).

**Step 1:** Apply Algorithm 1 to compute
\[
(\alpha_k, (u^k_a, u^k_b)) = \text{Linesearch } F((a^k, b^k), \beta_k, \delta),
\]
Set \((z^k_a, z^k_b) = P_C ((a^k, b^k) - \beta_k (u^k_a, u^k_b))\) then we have
\[
(\ell(a^k, b^k), (z^k_a, z^k_b)) \geq \delta ((u^k_a, u^k_b), (a^k, b^k) - (z^k_a, z^k_b))
\]
with \((\tilde{a}^k, \tilde{b}^k) = \alpha_k (z^k_a, z^k_b) + (1 - \alpha_k)(a^k, b^k)\) and \((u^e_{a^k}, u^e_{b^k}) \in \partial \Psi f (\tilde{a}^k, \tilde{b}^k)\).

**Step 2 (Stopping Criterion):** if \((z^k_a, z^k_b) = (a^k, b^k)\) or \((a^k, b^k) = P_C ((a^k, b^k) - (v^k_a, v^k_b))\) with \((v^k_a, v^k_b) \in \partial \Psi f ((a^k, b^k))\), then stop.

**Step 3:** Let \((\bar{a}^k, \bar{b}^k) := \alpha_k (z^k_a, z^k_b) + (1 - \alpha_k)(a^k, b^k)\), \((10a)\) and \((a^{k+1}, b^{k+1}) := F ((a^k, b^k))\); \((10b)\)

**Step 4:** If \((a^{k+1}, b^{k+1}) = (a^k, b^k)\), then stop. Otherwise, set \(k \leftarrow k + 1\) and go to **Step 1**.

for all \((u^e_{a^k}, u^e_{b^k}) \in \partial \Psi f (a^k, b^k)\), with \((a^k, b^k) = \alpha (z^k_a, z^k_b) + (1 - \alpha)(a, b)\). Note that by the structure of the set \(\partial \Psi f (a, b)\) it is enough to verify the inequality over the points in \(D_\alpha\). Taking limits when \(\alpha \to 0\), we have that
\[
(a^a, b^a) = \alpha (z^a_a, z^a_b) + (1 - \alpha)(a, b) \to (a, b).
\]

By Lemma 2, there exists \((u^a_a, u^a_b) \in \partial \Psi f (a, b)\) such that
\[
\langle (u^a_a, u^a_b), (a, b) - (z^a_a, z^a_b) \rangle \leq \delta \langle (u^a_a, u^a_b), (a, b) - (z^a_a, z^a_b) \rangle.
\]
This implies that \((1 - \delta)(u^a_a, u^a_b), (a, b) - (z^a_a, z^a_b) \leq 0\).

Since \(\delta \in (0, 1)\) we have
\[
0 \geq \langle (u^a_a, u^a_b), (a, b) - (z^a_a, z^a_b) \rangle \\
\geq \| (a, b) - (z^a_a, z^a_b) \|^2 + \langle (z^a_a, z^a_b) - ((a, b) - (u^a_a, u^a_b)), (a, b) - (u^a_a, u^a_b) \rangle.
\]
Using now Fact 1 we have that
\[
\langle (z^a_a, z^a_b) - ((a, b) - (u^a_a, u^a_b)), (a, b) - (u^a_a, u^a_b) \rangle \geq 0,
\]
then
\[
0 \geq \langle (u^a_a, u^a_b), (a, b) - (z^a_a, z^a_b) \rangle \geq \| (a, b) - (z^a_a, z^a_b) \|^2 \geq 0,
\]
which implies that \(\| (a, b) - (z^a_a, z^a_b) \|^2 = 0\), then \((a, b) = (z^a_a, z^a_b)\) implying that \((a, b)\) is a solution of Problem (8).

**Proposition 5** \((a^k, b^k) \in S^*\) if and only if \((a^k, b^k) \in H ((\tilde{a}^k, \tilde{b}^k), (u^e_{a^k}, u^e_{b^k}))\).
Proof This results follows from Proposition 4.3 in [5], using $x^k = \bar{a}^k, \bar{b}^k$ and $\bar{u}^k = (u^a_\alpha^k, u^b_\alpha^k)$.

**Proposition 6** If Algorithm 2 stops at steps 2 or 4, then it stops at the solution.

Proof If the Algorithm 2 stops at the step 2, then using Fact 1 (2) the results follows. If the stops is in Step 4, then $(a^k, b^k) = (a^{k+1}, b^{k+1}) \in H ((\bar{a}^k, \bar{b}^k), (u^a_\alpha^k, u^b_\alpha^k))$ and using Proposition 5, the result follows.

**4.1.1 Convergence Analysis of $\mathcal{F}_1$**

Now we provide the convergence analysis of Variant $\mathcal{F}_1$. From now on, we assume that Algorithm 2 produces an infinite sequence $(a^k, b^k)_{k \in \mathbb{N}} \not\in S^*$. Due to Proposition 4.5(i) in [5] we have the Fejér convergence and consequently, the boundedness, by Proposition 1, of the sequence $(a^k, b^k)_{k \in \mathbb{N}}$. Also

$$\lim_{k \to \infty} \langle (u^a_\alpha^k, u^b_\alpha^k), (a^k, b^k) - (\bar{a}^k, \bar{b}^k) \rangle = 0. \quad (13)$$

The next theorem will be proved following the ideas of Theorem 4.6 in [5], adapted to our point-to-set case.

**Theorem 4** If the operator $\partial \Psi^f$ is sequentially continuous, the sequence $(a^k, b^k)_{k \in \mathbb{N}}$ converges to a point in $S^*$.

Proof For the sake of brevity, we will not reproduce the full proof given in [5]. Instead we will specify whenever the proofs are different due to the operator being point-to-set in our case, and point out that we consider all elements on the normal cone of $C$ as zero.

Consider a subsequence $(i_k)_{k \in \mathbb{N}}$ such that all subsequences involved in the algorithm are convergent, i.e., $(a^{i_k}, b^{i_k}) \to (\hat{a}, \hat{b}), (u^{a, i_k}_\alpha, u^{b, i_k}_\alpha) \to (\bar{u}_a, \bar{u}_b)$ and $(u^{i_k}_a, u^{i_k}_b) \to (\tilde{u}_a, \tilde{u}_b)$. It is possible because all sequences are bounded.

Using (13) and the same criteria used in [5], we have that

$$\lim_{k \to \infty} \alpha_{i_k} \|(a^{i_k}, b^{i_k}) - (z^{i_k}_a, z^{i_k}_b)\| = 0, \quad (14)$$

and here as in [5], we consider two cases: $\alpha_{i_k} \to \hat{\alpha} > 0$ or $\alpha_{i_k} \to 0$.

**Case 1:** $\lim_{k \in \mathbb{N}} \alpha_{i_k} = \hat{\alpha} > 0$. Due to Lemma 2 we have that,

$$\lim_{k \to \infty} (u^{a, i_k}_\alpha, u^{b, i_k}_\alpha) = (\tilde{u}_a, \tilde{u}_b) \in \partial \Psi^f (\hat{a}, \hat{b}).$$

By the continuity of the projection mapping and (14), $(\hat{a}, \hat{b}) = PC \left( (\hat{a}, \hat{b}) - (\tilde{u}_a, \tilde{u}_b) \right)$, which implies that $(\hat{a}, \hat{b}) \in S^*$.

**Case 2:** $\lim_{k \to \infty} \alpha_{i_k} = 0$. Defining $\bar{a}_{i_k} = \frac{a^{i_k}_\alpha}{\theta}$, then $\bar{a}_{i_k} \to 0$, let $(\bar{a}^{i_k}, \bar{b}^{i_k}) = (\bar{a}_{i_k}, \bar{b}_{i_k}) + (1 - \bar{a}_{i_k})(a^{i_k}, b^{i_k})$ then $(\bar{a}_{i_k}, \bar{b}_{i_k}) \to (\tilde{a}, \tilde{b})$. From Algorithm 1, we have that for all $(\bar{v}^{i_k}_a, \bar{v}^{i_k}_b) \in \partial \Psi^f (\tilde{a}^{i_k}, \tilde{b}^{i_k})$ and all $(u^{i_k}_a, u^{i_k}_b) \in \partial \Psi^f (a^{i_k}, b^{i_k})$ and $k \in \mathbb{N}$ we have

$$\langle (\bar{v}^{i_k}_a, \bar{v}^{i_k}_b), (a^{i_k}, b^{i_k}) - (z^{i_k}_a, z^{i_k}_b) \rangle < \delta \langle (u^{i_k}_a, u^{i_k}_b), (a^{i_k}, b^{i_k}) - (z^{i_k}_a, z^{i_k}_b) \rangle. \quad (15)$$
Since $\alpha_{ik}$ converges to zero and following the same idea as in the proof of Lemma 2, there exist sequences $(\tilde{v}_{ik}^a, \tilde{v}_{ik}^b) \in \partial \psi^f(\tilde{a}_{ik}, \tilde{b}_{ik})$ and $(\tilde{u}_{ik}^a, \tilde{u}_{ik}^b) \in \partial \psi^f(a_{ik}, b_{ik})$ such that $\lim_{k \to \infty} \|(\tilde{v}_{ik}^a, \tilde{v}_{ik}^b) - (\tilde{u}_{ik}^a, \tilde{u}_{ik}^b)\| = 0$. As $\partial \psi^f$ is sequentially continuous, Taking a convergent subsequence, we have $(\tilde{v}_{ik}^a, \tilde{v}_{ik}^b) \to (\tilde{u}_a, \tilde{u}_b)$ and $(\tilde{u}_{ik}^a, \tilde{u}_{ik}^b) \to (\tilde{u}_a, \tilde{u}_b)$ as well, also $(a_{ik}, b_{ik}) \to (\hat{a}, \hat{b})$ and $(z_{ik}^a, z_{ik}^b) \to (\hat{z}_a, \hat{z}_b) = \text{PC}((\hat{a}, \hat{b}) - (\tilde{u}_a, \tilde{u}_b))$. Then passing to the limits on (15) we obtain

$$\langle (\tilde{u}_a, \tilde{u}_b), (\hat{a}, \hat{b}) - (\hat{z}_a, \hat{z}_b) \rangle \leq \delta \langle (\tilde{u}_a, \tilde{u}_b), (\hat{a}, \hat{b}) - (\hat{z}_a, \hat{z}_b) \rangle.$$  

The above equation is the same as in (12), the proof continues as in Proposition 4, getting that the limits of $(a_{ik}, b_{ik})$ is a solution of the problem.

Now as the sequence is Fejér convergent to the solution set, following Proposition 4.5 (i) in [5], we get the convergence to the solution set of the whole sequence.

\[ \square \]

### 4.1.2 Analysis Convergence of $F_2$

For the case of $F_2$ all proofs are similar to the proofs in [5]. We therefore provide a brief explanation.

We suppose that Algorithm 2 generates an infinite sequence. We provide a sketch of the proofs, because they are very similar to presented on Section 4.2 of [5].

**Proposition 7** The sequence $(a^k, b^k)_{k \in \mathbb{N}}$ is Fejér convergent to the solutions set $S^*$.

**Proof** Take $(a^*, b^*) \in S^*$. Following the ideas of Proposition 4.7 in [5], we obtain that

$$\|(a^{k+1}, b^{k+1}) - (a^*, b^*)\|^2 \leq \|(a^k, b^k) - (a^*, b^*)\|^2 - \|(a^{k+1}, b^{k+1}) - (a^k, b^k)\|^2.$$  

(16)

This implies that the sequence is Fejér convergent to $S^*$ and also, we have that $\|(a^{k+1}, b^{k+1}) - (a^k, b^k)\| \to 0.$

Since $(a^k, b^k) \in C, (a^k, b^k) \notin H((\tilde{a}^k, \tilde{b}^k), (u_{\alpha k}^a, u_{\alpha k}^b))$ and

$$C \cap H((\tilde{a}^k, \tilde{b}^k), (u_{\alpha k}^a, u_{\alpha k}^b)) \neq \emptyset,$$

we can use Lemma 2.5 in [5] to obtain the following result:

$$\begin{aligned}
(a^{k+1}, b^{k+1}) &= P_{C \cap H((\tilde{a}^k, \tilde{b}^k), (u_{\alpha k}^a, u_{\alpha k}^b))}(P_H((\tilde{a}^k, \tilde{b}^k), (u_{\alpha k}^a, u_{\alpha k}^b))(a^k, b^k)).
\end{aligned}$$  

(17)

Using now the conclusion given by (16), $\|(a^{k+1}, b^{k+1}) - (a^k, b^k)\| \to 0$, we use the formulae to project onto a hyperplane to obtain

$$\lim_{k \to \infty} \left( u_{\alpha k}^a, u_{\alpha k}^b, (a^k, b^k) - (\tilde{a}^k, \tilde{b}^k) \right) = 0.$$  

(18)

Joining the results (17) and (18) and, as was done with the result (13) to prove the convergence of Variant 1; the convergence to the solution set of the sequence generated by Variant 2 follows in the same way.
5 General Algorithm for Non-monotone Variational Inequality

Algorithm 2 and Algorithm 1 (Line search) can be applied not just for the Rational Approximation problem. In fact, consider \( T : \mathbb{R}^n \Rightarrow \mathbb{R}^n \), and the convex and closed set \( C \subseteq \mathbb{R}^n \). With the following conditions, Algorithm G (G for “general”) described below, is convergent.

A1) \( T \) is sequentially continuous.
A2) \( T \) is bounded on bounded sets.
A3) Problem (9) and (8) are equivalent. That is, \( S^* = S_0 \).

By closed we mean that the graph of \( T \) is closed. A2 is a classical assumption. A3 is weaker than pseudomonotone, see examples in [7].

Following, we present the general version for the Linesearch 1 and Algorithm 2.

**Algorithm 3: LineSearch G:** General linesearch.

**Input:** \( x \in C, \beta > 0 \) and \( \delta \in (0, 1) \)

Set \( \alpha \leftarrow 1 \) and \( \theta \in (0, 1) \).

**for all** \( u \in T(x) \), Define \( z = P_C(x - \beta u) \).

**if for each** \( u \in T(x) \)

\[
\max_{u^\alpha \in T(x_\alpha)} \langle u^\alpha, x - z \rangle < \delta \langle u, x - z \rangle,
\]

**then**

where \( x_\alpha := \alpha z + (1 - \alpha)x, \alpha \leftarrow \theta \alpha \)

**else**

| Stop and choose \( u^\alpha \in T(x_\alpha) \) such that: \( \langle u^\alpha, x - z \rangle \geq \langle u, x - z \rangle \)

**end**

**Output:** \( (\alpha, u^\alpha) \)

Note that Line search G is the same as Algorithm 1 but for a general operator \( T \).

The Algorithm 1 (Line search) is the particular case of the Linesearch G applied to the Problem (5).

Two variants can be considered for computing (19b).

\[
\mathcal{F}_1(x^k) = P_C \left( P_H(\tilde{x}^k, u^{\alpha k})(x^k) \right);
\]

\[
\mathcal{F}_2(x^k) = P_{C \cap H(\tilde{x}^k, u^{\alpha k})}(x^k);
\]

where \( u^{\alpha k} \in T(\tilde{x}^k) \) and

\[
H(x, u) := \{ y \in \mathbb{R}^n : \langle u, y - x \rangle \leq 0 \}.
\]

The differences between the Algorithm G and Algorithm F in [5] and the algorithm in [7] are listed below:

- We consider point-to-set operators instead of point-to-point. The operator is not necessarily continuous as in [7]. This is a more general case.
Algorithm 4: Algorithm G.

Input: \((\beta_k)_{k \in \mathbb{N}} \subset [\hat{\beta}, \tilde{\beta}]\) such that \(0 < \hat{\beta} \leq \tilde{\beta} < +\infty\) and \(\delta \in (0, 1)\).

Initialisation: Take \((x^0) \in C\) and set \(k \leftarrow 0\).

Step 1: Apply Algorithm 3 to compute 
\[(\alpha_k, u^{\alpha_k}) = \text{Linesearch } G(x^k, \beta_k, \delta),\]

Set \(z^k = P_C(x^k - \beta_k u^k)\) then we have 
\[\langle u^{\alpha_k}, x^k - z^k \rangle \geq \delta \langle u^k, x^k - z^k \rangle \]

with \(u^{\alpha_k} \in T(\tilde{x}^k)\) and \(\tilde{x}^k = \alpha_k z^k + (1 - \alpha_k)x^k\).

Step 2 (Stopping Criterion): if \(z^k = x^k\) or \(x^k = P_C(x^k - v^k)\) with \(v^k \in T(x^k)\), then stop Step 3: Set 
\[\tilde{x}^k := \alpha_k z^k + (1 - \alpha_k)x^k, \quad (19a)\]

and 
\[x^{k+1} := \mathcal{F}(x^k); \quad (19b)\]

Step 4: If \(x^{k+1} = x^k\), then stop. Otherwise, set \(k \leftarrow k + 1\) and go to Step 1.

- The Linesearch G is different to Linesearch F of [5]. The difference consists of the search in the whole set \(T(x)\), this is less efficient than using only one element in \(T(x)\), but this allows us to avoid the restriction to point-to-point operators and continuity as in [7].

6 Numerical Experiments

In this section we present the results of numerical experiments to demonstrate the behaviour of Algorithm 2 to approximate continuous functions. We applied the algorithm on the operator \(T = \partial \Psi f\) for various functions \(f\). We consider two cases: the “classical approximation” (using ratios of polynomials) and “non-classical approximation” (using ratios of non-polynomial linear forms). In the classical case, we examine two instances: when the solution is known and when it is unknown.

For the non-classical approximation, we use rational functions composed by linear combinations of basis functions of the form \(h^i(t)\). That is, the numerator and denominator have the form: 
\[p(t) = \sum_{i=0}^{n} a_i h^i(t)\]

where \(a_i \in \mathbb{R}\) and \(n \in \mathbb{N}\). We consider two different functions for \(h(t)\): \(\sin(t)\) and \(e^t\). In all the cases we approximate functions over the interval \(I = [-1, 1]\).

We use MATLAB version R2016a on a PC with Intel(R) Core(TM) i7-7500U CPU 2.60-2.90 GHz and Windows 10 Home. For the calculation of the projection step we use the built-in Quadratic Programming (quadprog) tool.

In all experiments we report our results in tabular and plot forms. In all tables, \(\Psi_{i,j}^f\) with \(i = 1, 2\) denotes the objective function (defined in (1)) value evaluated on the output of Variant \(i\) of the algorithm, \(iter\) denotes the number of iterations required by the algorithm. \(n, m\) are the degrees of the numerator and denominator functions respectively (that is, the number of terms in the linear forms, plus 1). In the pictures, we plot the absolute deviation.
Table 1 Algorithm 2 for unknown solution

| $f(t)$ | $(n, m)$ | Iter | $\Psi^1_1(a, b)$ | $CPU_1$ | Iter | $\Psi^1_2(a, b)$ | $CPU_2$ |
|--------|----------|------|-----------------|--------|------|-----------------|--------|
| $|t|$   | (2, 2)   | 23964| 0.053           | 3601.55| 24191| 0.051           | 4702.56|
| $|t|$   | (3, 3)   | 36899| 0.046           | 4339.27| 36983| 0.046           | 6185.06|
| $|t|$   | (4, 3)   | 21327| 0.045           | 3281.7 | 21218| 0.046           | 3175.61|
| $\sin(t)$ | (2, 2)   | 237  | 0.0091          | 46.5   | 169  | 0.0086          | 49     |
| $\sin(t)$ | (3, 3)   | 14724| 0.048           | 2301.64| 14696| 0.048           | 3180.88|
| $\sqrt{|t|}$ | (4, 4)   | 10577| 0.1320          | 815.438| 10131| 0.1354          | 780.078|

function $\Phi(t) = t \rightarrow |\sigma^f_i(a, b)|$, with $g$ and $h$ as in (3). We denote by $r_i$, $i = 1, 2$, the approximation obtained using variants 1 and 2 of the algorithm respectively, namely the function $t \rightarrow r_i = \frac{g(t)}{h(t)}$, with $a_i$ and $b_i$ the output of Variant $i$. In all pictures the approximation function obtained with Variant 1 and Variant 2 are shown with circles and dashed lines ($\circ$), and $\times$ with dotted lines respectively. The solid and continuous lines is the graph of the function $f$ to be approximated.

6.1 Rational Approximation with Polynomial

In this subsection we consider the Problem (4) where $p \in \Pi_n$ and $q \in \Pi_m$ are polynomials of degree $n$ and $m$ respectively (Tables 1, 2 and 3).

6.1.1 Unknown Solution

In these examples, we used as stopping criterion $\|\partial \Psi^f_i(a^k, b^k)\| \leq 10^{-2}$.

6.1.2 Known Solution

In this subsection we show the results of our experiments (see Table 2) in the case when we know the solution sets. Here again we approximate the function $f$ with ratios of polynomials. We denote by $d_i(x, S^*)$ the distance to the solution set of the last point returned by the algorithm for the variant $i$. We stop the algorithm when the function value $\Psi^f_i$ at the current point be less or equal to 0.005 (Figs. 1, 2, 3, 4 and 5).

Table 2 Algorithm 2 for known solution

| $f(t)$ | $(n, m)$ | Iter | $d_1(x, S^*)$ | $CPU_1$ | Iter | $d_2(x, S^*)$ | $CPU_2$ |
|--------|----------|------|--------------|--------|------|--------------|--------|
| $1$    | (1, 1)   | 214  | 0.0022       | 9.5625 | 265  | 0.0016       | 7.60938|
| $1$    | (2, 2)   | 150  | 0.0022       | 7.84375| 166  | 0.0015       | 7.3125 |
| $\frac{1}{t+1}$ | (1, 2)   | 75   | 0.0204       | 12.5625| 251  | 0.0820       | 37.0313|
| $\frac{1}{t+1}$ | (2, 3)   | 68   | 0.0407       | 14.8906| 42   | 0.1140       | 9.125 |
| $\frac{1}{t+1}$ | (1, 1)   | 3412 | 0.1941       | 581.984| 3411 | 0.2005       | 715.672|
| $\frac{1}{t+1}$ | (2, 2)   | 3032 | 0.3230       | 560.281| 3012 | 0.2564       | 651.813|
| $\frac{1}{t+1}$ | (2, 2)   | 3099 | 0.4820       | 586.438| 3111 | 0.4896       | 505.109|
| $\frac{1}{t+1}$ | (3, 2)   | 1865 | 0.8104       | 284.484| 2437 | 0.9057       | 428.359|
Table 3  Algorithm 2 with non-polynomial

| h(t) | (n, m) | Iter₁ | CPU₁   | Iter₂ | CPU₂   |
|------|--------|-------|--------|-------|--------|
| e^t  | (3, 2) | 65    | 13.7031| 65    | 14.2969|
| e^t  | (3, 3) | 249   | 45.25  | 249   | 48.6719|
| e^t  | (5, 4) | 528   | 113.75 | 528   | 115.54 |
| e^t  | (10, 8)| 1022  | 258.359| 1022  | 259.063|
| sin(t)| (3, 2)| 107   | 25.7031| 126   | 29.1094|
| sin(t)| (3, 3)| 39    | 8.65625| 39    | 9.23438|
| sin(t)| (5, 4)| 27    | 6.65625| 27    | 6.70313|
| sin(t)| (10, 8)| 29   | 7.54688| 29    | 7.4375 |

\[ f(t) = |t|, n = m = 2 \]

\[ f(t) = |t|, n = m = 3 \]

**Fig. 1** Absolute deviation (\( \Phi_f^i \)), \( i = 1, 2 \), for different values of \( m \) and \( n \)

\[ f(t) = |t|, n = m = 3 \]

**Fig. 2** Left: \( f \) and \( r_i, i = 1, 2 \). Right: Function \( \Phi_f^i, i = 1, 2 \)
6.2 Non-polynomial Rational Approximation

In this section we consider different rational functions to approximate continuous functions. We denote by CPU the CPU time. By $h(t)$, we denote the basis functions in the linear forms, i.e., $p(t) = \langle a, H(t) \rangle = \sum_{k=0}^{n} a_k h_k(t)$. In all cases we approximate the continuous function $f(t) = \frac{\sin t - \cos t}{t + 3}$. We stop the algorithm when $\Psi_i(a, b) \leq 0.05$. The purpose of these examples is to compare both variants in the number of iterations and the CPU time.

6.3 Discussion

Our experiments show that the proposed algorithm is able to find an approximate solution in a reasonable time in all cases. For the classical case of rational approximation, our algorithm is not as efficient as specialised methods (see eg. [16]), but on the other hand it is applicable to a wider class of functions.
Our stopping criterion is based on the value of the objective function (when it is known that the optimal value is 0), and on the norm of the subgradient of the objective function at the iterate.

Many classical algorithms rely on equioscillation properties instead. As can noted from the graphs the equioscillation is not attained in all cases. A possible explanation for this is that the objective function of the problem becomes almost flat near the solution. Therefore it is possible to achieve a good approximation without equioscillation. Similarly the experiments on the cases when the solution is known show that it is possible to achieve a near-optimal approximation (as measured by the objective function value) at a point whose Euclidean distance to the solution set is moderately large.

7 Conclusion

In this paper we proposed an algorithm for solving variational inequality problems when the operator is pseudomonotone and point-to-set (therefore not relying on continuity assumptions). The motivation for developing this algorithm is to approximate continuous functions by ratios of linear forms, which generalise rational functions to the case when the bases are not necessarily polynomial. We show that this problem is pseudoconvex. We propose two variants of our algorithm, which differ in how we project iterates onto the feasible set. Finally we report the results of numerical experiments on the approximation of functions by rational functions and by ratios of non-polynomial linear forms to demonstrate that our algorithm can be applied in practice.

In all cases our algorithm is able to find a good approximation in a reasonable time. In the case of classical rational approximation, our algorithm is unsurprisingly not as efficient as specialised algorithms, but on the other hand it is much more general and can be applied in contexts when the classical algorithms are not available, for example when the linear forms are not polynomials. Furthermore, our algorithm is also applicable to other optimisation problems with pseudoconvex objective functions.

This work can be extended further. In future research we will consider the problem of approximate multivariate functions. The pseudoconvexity of the approximation problem
opens up some opportunities to exploit this property to develop more efficient algorithms in the future.

Acknowledgements  This research was supported by the Australian Research Council (ARC), Solving hard Chebyshev approximation problems through nonsmooth analysis (Discovery Project DP180100602).

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