A SHARP CONVERGENCE THEOREM FOR THE MEAN CURVATURE FLOW IN SPHERES I

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ABSTRACT. In this paper, we prove a sharp convergence theorem for the mean curvature flow of arbitrary codimension in spheres which improves Baker's convergence theorem. In particular, we obtain a new differentiable sphere theorem for submanifolds in spheres.

1. INTRODUCTION

Let $M$ be an $n$-dimensional compact submanifold isometrically immersed in a Riemannian manifold $N^{n+p}$. Denoted by $F_0$ the isometric immersion. We consider the one-parameter family $F_t = F(\cdot, t)$ of immersions $F_t : M \to N^{n+p}$ with corresponding images $M_t = F_t(M)$ which satisfies

\[
\begin{aligned}
\frac{\partial}{\partial t} F(x, t) &= H(x, t), \\
F(x, 0) &= F_0(x),
\end{aligned}
\]

where $H(x, t)$ is the mean curvature vector of $M_t$. We call $F_t : M \to N^{n+p}$ the mean curvature flow with initial value $F_0$. In 1980's, Huisken proved the convergence theorems for the mean curvature flow of hypersurfaces under certain conditions in a series of papers. For the initial hypersurfaces satisfying the convexity condition, Huisken \[4, 5\] proved that the solution of the mean curvature flow converges to a point as the time approaches the finite maximal time. Motivated by the rigidity theorem for hypersurfaces with constant mean curvature in spheres due to Okumura \[14\], Huisken \[6\] proved the convergence theorem for the mean curvature flow of hypersurfaces under the curvature pinching condition for $|H|^2$ and the squared norm of the second fundamental form $|A|^2$ in the sphere $S^{n+1}$.

**Theorem A.** Let $F_0 : M^n \to S^{n+1}(\frac{1}{\sqrt{\bar{K}}})$ be a smooth compact hypersurface in a sphere with constant curvature $\bar{K}$. Assume $M$ satisfies

\[
|A|^2 < \begin{cases} 
\frac{4}{n} |H|^2 + \frac{4}{3} \bar{K}, & n = 2, \\
\frac{n}{4} |H|^2 + 2\bar{K}, & n \geq 3.
\end{cases}
\]

Then the mean curvature flow with the initial value $F_0$ either converges to a round point in finite time, or converges to a total geodesic sphere of $S^{n+1}(\frac{1}{\sqrt{\bar{K}}})$ as $t \to \infty$.

For the mean curvature flow of arbitrary codimension in the Euclidean space, Andrews and Baker \[11\] investigated the convergence problem. In \[2\], Baker studied the mean curvature flow of arbitrary codimension in spheres and obtained the following result.
Theorem B. Let $F_0 : M^n \to S^{n+p}(\frac{1}{\sqrt{K}})$ be a smooth compact submanifold in a sphere with constant curvature $\bar{K}$. Assume $M$ satisfies

\begin{equation}
|A|^2 \leq \begin{cases}
\frac{1}{n-1}H^2 + \frac{2(n-1)}{n-1} \bar{K}, & n = 2, 3, \\
\frac{1}{n-1}H^2 + 2\bar{K}, & n \geq 4.
\end{cases}
\end{equation}

Then the mean curvature flow with the initial value $F_0$ either converges to a round point in finite time, or converges to a total geodesic sphere of $S^{n+p}(\frac{1}{\sqrt{K}})$ as $t \to \infty$.

For $n \geq 4$, the compact submanifold $M^n \subset S^1(r) \times S^{n-1}(s) \subset S^{n+1}(1) \subset S^{n+p}(1)$ with $r^2 + s^2 = 1$ implies that the pinching conditions of Theorem A and Theorem B are sharp. In fact, the sharp pinching condition means the linear relationship of $|A|^2$, $|H|^2$ and $\bar{K}$ is sharp. If the relationship is not linear, the pinching condition can be improved. Afterwards, Lei-Xu [10] also obtained a sharp convergence theorem for mean curvature flow of high codimension in spheres. Set

\begin{equation}
\alpha(x) = n\bar{K} + \frac{n}{2(n-1)}x - \frac{n}{2(n-1)}\sqrt{x^2 + 4(n-1)\bar{K}x},
\end{equation}

they proved the following theorem.

Theorem C. Let $F_0 : M^n \to S^{n+p}(\frac{1}{\sqrt{K}})$ be an $n$-dimensional ($n \geq 6$) smooth compact submanifold in a sphere with constant curvature $\bar{K}$. Assume $M$ satisfies

\begin{equation}
|A|^2 < \gamma(n, |H|, \bar{K}).
\end{equation}

Then the mean curvature flow with the initial value $F_0$ either converges to a round point in finite time, or converges to a total geodesic sphere of $S^{n+p}(\frac{1}{\sqrt{K}})$ as $t \to \infty$.

Here $\gamma(n, |H|, \bar{K})$ is an explicit positive scalar defined by

\[\gamma(n, |H|, \bar{K}) = \min\{\alpha(|H|^2), \beta(|H|^2)\},\]

where

\[\beta(x) = \alpha(x_0) + \alpha'(x_0)(x - x_0) + \frac{1}{2} \alpha''(x_0)(x - x_0)^2;
\]

\[x_0 = \frac{2n + 2}{n - 4}\sqrt{n - 1}(\sqrt{n - 1} - \frac{n - 4}{2n + 2}) \bar{K}.
\]

The scalar $\gamma(n, |H|, \bar{K})$ in Theorem C satisfies the following: (i) $\gamma(n, |H|, \bar{K}) > 1$; (ii) $\gamma(n, |H|, \bar{K}) > \frac{1}{3}\sqrt{n - 1}\bar{K}$; (iii) $\gamma(n, |H|, \bar{K}) = \alpha(|H|^2)$ when $|H|^2 \geq x_0$.

Lawson-Simons [8] proved the topological sphere theorem in the unite sphere under the pinching condition $|A|^2 < 2\sqrt{n - 1}$. Applying the convergence results of Hamilton and Brendle for Ricci flow and the Lawson-Simons formula for the nonexistence of stable currents, Xu and Zhao [18] proved a differentiable sphere theorem in spheres.

Theorem D. Let $M$ be an $n$-dimensional ($n \geq 4$) oriented complete submanifold in the unit sphere $S^{n+q}$. Then

(i) if $n = 4, 5, 6$ and $\sup_M |A|^2 - 2\sqrt{n - 1} < 0$, then $M$ is diffeomorphic to $S^n$;

(ii) if $n \geq 7$ and $|A|^2 < 2\sqrt{2}$, then $M$ is diffeomorphic to $S^n$.

Motivated by the rigidity and topological sphere theorems for submanifolds in spheres, Lei-Xu proposed the conjecture that $\alpha(|H|^2)$ is the optimal pinching condition for mean curvature flow in spheres.
Conjecture 1. Let $M_0$ be an $n$-dimensional complete submanifold in the sphere $S^{n+p}(\frac{1}{\sqrt{K}})$. Suppose that $\sup_{M_0}(|h|^2 - \alpha(|H|^2)) < 0$. Then the mean curvature flow with initial value $M_0$ converges to a round point in finite time, or converges to a totally geodesic sphere as $t \to \infty$. In particular, if $|A|^2 < 2\sqrt{n-1}\bar{K}$, $M_0$ is diffeomorphic to $S^n$.

For mean curvature flow of submanifolds in hyperbolic spaces, Liu-Xu-Ye-Zhao [12] proved the convergence theorem. Recently, Lei-Xu [9] proved a convergence theorem of arbitrary codimension in hyperbolic spaces that the initial submanifold $M^n$ of dimension $n(\geq 6)$ under the optimal pinching condition $|A|^2 < \alpha(|H|^2)$. Note that initial submanifolds in the almost all convergence results implies the positive sectional curvatures. However, Lei-Xu’s convergence theorems imply that the Ricci curvatures of the initial submanifolds are positive, but don’t imply the positivity of the sectional curvatures. Therefore, their convergence theorems also imply the differentiable sphere theorems for submanifolds with positive Ricci curvatures. For other results and applications for the mean curvature flow, we refer the readers to [3, 5, 7, 11, 13, 17].

Motivated by above theorems, we investigate the submanifold $M^n (n \geq 3, p = 1$ or $n \geq 4$) pinched by a sharp curvature pinching condition for the mean curvature flow of arbitrary codimension in the sphere $S^{n+p}(\frac{1}{\sqrt{K}})$. Putting

$$b(x) = (1 - \delta)(\frac{x}{n-1} + 2\bar{K}) + \delta \alpha(x), \quad x \geq 0,$$

where $\delta = \frac{\sqrt{(n+2)(n-2)} - 2}{2(n-2)}$, $n = 4, 5, 6$. The pinching condition $b(|H|^2)$ is obvious sharp and satisfies $b(|H|^2) > \frac{|H|^2}{n-1} + 2\bar{K}$. Then we prove the following sharp convergence theorem.

**Theorem 1.1.** Let $F_0 : M \to S^{n+p}(\frac{1}{\sqrt{K}})$ be an $n$-dimensional ($n \geq 4$) smooth compact submanifold immersed in the sphere. If $M$ satisfies

$$|A|^2 \leq \begin{cases} b(|H|^2), & n = 4, 5, 6, \\ \sqrt{\frac{|H|^2}{n-1} + 2\bar{K}}^2 + (2n-4)\bar{K}^2, & n \geq 7. \end{cases}$$

then the mean curvature flow with the initial value $F_0$ converges to a round point in finite time, or converges to a totally geodesic sphere of $S^{n+p}(\frac{1}{\sqrt{K}})$ as $t \to \infty$.

**Remark 1.2.** (i) Consider the compact submanifold $M^n = S^1(r) \times S^{n-1}(s) \subset S^{n+1}(1) \subset S^{n+p}(1)$ with $r^2 + s^2 = 1$. We have $|A|^2 - (\frac{|H|^2}{n-1} + 2)^2 - (2n-4) = \frac{(n-3)^2-1}{(n-1)^2} \cdot s^2$, which implies that the pinching condition is sharp.

(ii) Since $\sqrt{\frac{|H|^2}{n-1} + 2\bar{K}}^2 + (2n-4)\bar{K}^2 > \frac{|H|^2}{n-1} + 2\bar{K}$, this convergence theorem improves Theorem A and Theorem B. In fact, whatever the form of the pinching condition is, the relationship of $|A|^2$ and $|H|^2$, i.e., $\frac{1}{n-1}$ is optimal. So this pinching condition largely improved the proportion of $K$.

(iii) This convergence theorem implies that the Ricci curvatures of the initial submanifold is positive, but doesn’t imply the positivity of the sectional curvature.

(iv) For $n \geq 3, p = 1$, the mean curvature flow of hypersurfaces in the sphere
convergence under the sharp condition

$$|A|^2 \leq \sqrt{\left(\frac{|H|^2}{n-1} + 2 \tilde{K}\right)^2 + (2n - 4)\tilde{K}^2}.$$  

In particularity, the convergence theorem implies a sharp differentiable sphere theorem in the sphere. Noting that $\left(\frac{|H|^2}{n-1} + 2\right)^2 + 2n - 4 \geq 2n$, we obtain a new differentiable sphere theorem which improves Theorem D.

**Corollary 1.3.** Let $M$ be an $n$-dimensional ($n \geq 7$) smooth compact submanifold in the unit sphere $S^{n+q}$. If $|A|^2 \leq \sqrt{2n}$, then $M$ is diffeomorphic to $S^n$.

2. Preliminaries

Let $(M^n, g)$ be the $n$-dimensional ($n \geq 3, p = 1$ or $n \geq 4$) Riemannian submanifold isometrically immersed in a simply connected space form $\mathbb{R}^{n+d}(\tilde{K})$ with constant curvature $\tilde{K}$. Denote by $\nabla$ the Levi-Civita connection of the ambient space $\mathbb{R}^{n+d}$. We use the same symbol $\nabla$ to represent the connection of the tangent bundle $TM$ and the normal bundle $NM$. Denote by $(\cdot)^T$ and $(\cdot)^\perp$ the projections onto $TM$ and $NM$, respectively. For $u, v \in \Gamma(TM), \xi \in \Gamma(NM)$, the connection $\nabla$ is given by $\nabla_u v = (\nabla_u v)^T$ and $\nabla_u \xi = (\nabla_u \xi)^\perp$. The second fundamental form of $M$ is defined by

$$A(u, v) = (\nabla_u v)^\perp.$$  

Let $\{e_i | 1 \leq i \leq n\}$ be a local orthonormal frame for the tangent bundle and $\{\nu_\alpha | n + 1 \leq \alpha \leq n + d\}$ be a local orthonormal frame for the normal bundle. Let $\{\omega_i\}$ be the dual frame of $\{e_i\}$. With the local frame, the first and the second fundamental forms can be written as $g = \sum_i \omega^i \otimes \omega^i$ and $A = \sum_{i, j, \alpha} h_{ij, \alpha} \omega^i \otimes \omega^j \otimes \nu_\alpha = \sum_{i, j} h_{ij} \omega^i \otimes \omega^j$, respectively. The mean curvature vector is given by

$$H = \sum_\alpha H_\alpha \nu_\alpha, \quad H_\alpha = \sum_i h_{i, i, \alpha}.$$  

Let $\bar{A} = A - \frac{|H|^2}{n} g$ be the traceless second fundamental form, then we have $|\bar{A}|^2 = |A|^2 - \frac{|H|^2}{n}$ and $|\nabla \bar{A}|^2 = |\nabla A|^2 - \frac{|\nabla H|^2}{n}$. Denote by $\nabla^2_{ij} T = \nabla_i (\nabla_j T) - \nabla_{\nabla_i e_j} T$ the second order covariant derivative of tensors. Then the Laplacian of a tensor is defined by $\Delta T = \sum_i \nabla^2_{ii} T$.

We have the following evolution equations for the mean curvature flow.

**Lemma 2.1** ([1] [2]).

$$\nabla_{\partial_t} h_{ij} = \Delta h_{ij} + h_{ij} \cdot h_{pq} h_{pq} + h_{iq} \cdot h_{qp} h_{pj} + h_{j} \cdot h_{qp} h_{pi} - 2 h_{iq} \cdot h_{jp} h_{pq} + 2 \tilde{K} H g_{ij} - n \tilde{K} h_{ij},$$

$$\nabla_{\partial_t} H = \Delta H + H \cdot h_{pq} h_{pq} + n \tilde{K} H,$$

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2 |\nabla A|^2 + 2 R_1 + 4 \tilde{K} |H|^2 - 2 n \tilde{K} |A|^2,$$

$$\frac{\partial}{\partial t} |H|^2 = \Delta |H|^2 - 2 |\nabla H|^2 + 2 R_2 + 2 n \tilde{K} |H|^2,$$

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2 |\nabla \bar{A}|^2 + 2 R_1 - \frac{2}{n} R_2 - 2 n \tilde{K} |A|^2.$$
Proof. By direct computations, we get
\begin{equation}
R_1 = 2 \sum_{a, \beta} \left( \sum_{i, j} h_{i j a, h_{i j \beta}} \right)^2 + |Rm|^2,
\end{equation}
\begin{equation}
R_2 = \sum_{i, j} \left( \sum_{a} H_{a, h_{i j a}} \right)^2.
\end{equation}

We have the following following curvature estimates.

Lemma 2.2 (\[2\]).
\begin{equation}
|\nabla A|^2 \geq \frac{3}{n + 2} |\nabla H|^2,
\end{equation}
\begin{equation}
R_2 = |\tilde{A}|^2 |H|^2 + \frac{1}{n} |H|^4 - P_2 |H|^2, P_2 = \sum_{a > 1} \left( \hat{h}_{i j a} \right)^2,
\end{equation}
\begin{equation}
R_3 = \sum H^a h_{i j a}^a h_{j k i} h_{k i},
\end{equation}
\begin{equation}
R_1 - \frac{1}{n} R_2 \leq |\tilde{A}|^4 + \frac{1}{n} \tilde{A} |H|^2 + 2 P_2 |\tilde{A}|^2 - \frac{3}{2} P_2 - \frac{1}{n} P_2 |H|^2,
\end{equation}
\begin{equation}
R_3 - R_1 \geq \frac{|\tilde{A}^2| H^2}{2(n - 1)} - \frac{n}{2} (|\tilde{A}|^2 - P_2) - \max\{4, \frac{n + 2}{2}\} (|\tilde{A}|^2 - P_2) P_2 - \frac{3}{2} P_2.
\end{equation}

For convenience, we denote
\[a(x) = \sqrt{\left( \frac{x}{n - 1} + 2 \tilde{K} \right)^2 + (2n - 4) \tilde{K}^2}, \quad \hat{a}(x) = a(x) - \frac{x}{n}.\]

Moreover, we prove the following inequalities.

Lemma 2.3. For \(x \geq 0\), \(\hat{a}\) has the following properties.
\begin{itemize}
\item[(i)] \(\frac{4a(\hat{a})^2}{n} < 1\),
\item[(ii)] \(2x \hat{a}'' + \hat{a}' < \frac{2(n - 1)}{n(n + 2)}\),
\item[(iii)] \(\frac{n - 2}{n(n - 1)} \sqrt{x \hat{a} + \hat{a}' < \frac{x}{n} + n \tilde{K}}\),
\item[(iv)] \(n \tilde{K}(\hat{a} + x \hat{a}') - a(\hat{a} - x \hat{a}') \geq \frac{2n(n - 2)(n - 1)^2 \tilde{K}^4}{(x + \sqrt{2 + \sqrt{2n(n - 1)}}) n} \),
\item[(v)] \(2 \hat{a} - \frac{x}{n} + x \hat{a}' \leq 2 \sqrt{2n \tilde{K}} - \frac{n - 4 x}{n(n - 1)} - \frac{6 \sqrt{2n - 2} \tilde{K} x}{3x + 2 \sqrt{2n(n - 1)}} \),
\item[(vi)] \(\frac{x}{n - 1} (a + n \tilde{K}) - (\frac{x}{n - 1} + 2n \tilde{K}) (\hat{a} + a - n \tilde{K} - x \hat{a}') < - \frac{2 \tilde{K}^2}{n - 1} + 2n(n - 4) \tilde{K}^2 \).
\end{itemize}

Proof. By direct computations, we get
\[\hat{a} = \sqrt{\left( \frac{x}{n - 1} + 2 \tilde{K} \right)^2 + (2n - 4) \tilde{K}^2} - \frac{x}{n},\]
\[\hat{a}' = \frac{\frac{x}{n - 1} + 2 \tilde{K}}{(n - 1) \sqrt{\left( \frac{x}{n - 1} + 2 \tilde{K} \right)^2 + (2n - 4) \tilde{K}^2} - \frac{1}{n} < \frac{1}{n(n - 1)}},\]
\[ \dot{\alpha}'' = \frac{2(n-2)\bar{K}^2}{(n-1)^2 \left( \sqrt{\frac{x}{n-1} + 2\bar{K}} \right)^2 (2n-4)\bar{K}^2} \]

(i)
\[ \frac{4x(\dot{\alpha})^2}{\dot{\alpha}} < \frac{x}{n(n-1)\dot{\alpha}} < 1. \]

(ii)
\[ 2x\ddot{\alpha}'' < \frac{4(n-2)x\bar{K}^2}{(n-1)^2 \left( \frac{x}{n-1} + 2\bar{K} \right) \left( \frac{x}{n-1} + 2\bar{K} \right)^2 (2n-4)\bar{K}^2} < \frac{4(n-2)\bar{K}^2}{x^2/n-1 + 6\bar{K}x + 4n(n-1)^2\bar{K}^2 + (2n+8)(n-1)\bar{K}^2} < \frac{2(n-2)}{(n-1)(n+2\sqrt{6n+1})} < \frac{2n-5}{(n-1)(n+2)}. \]

\[ 2x\ddot{\alpha}'' + \dot{\alpha}' < \frac{2n-5}{(n-1)(n+2)} + \frac{1}{n(n-1)} = \frac{2(n-1)}{n(n+2)}. \]

(iii)
\[ \frac{n-2}{\sqrt{n(n-1)}} \sqrt{\frac{x}{n-1} + \dot{\alpha}} < \frac{x}{n} + n\bar{K} \]
\[ \Leftrightarrow \frac{(n-2)^2}{n(n-1)} \left( x(a - \frac{x}{n}) \right) < \left( \frac{2x}{n} - a + n\bar{K} \right)^2 \]
\[ \Leftrightarrow a \left( \frac{x}{n-1} + 2\bar{K} \right) < \left( \frac{x}{n-1} + 2\bar{K} \right)^2 + (n-2)\bar{K}^2 \]
\[ \Leftrightarrow a < \left( \frac{x}{n-1} + 2\bar{K} \right) + \frac{(n-2)\bar{K}^2}{\frac{x}{n-1} + 2\bar{K}}. \]

(iv)
\[ n\bar{K}(\dot{\alpha} + x\dot{\alpha}') - a(\dot{\alpha} - x\dot{a}') = \frac{2n\bar{K}}{n-1} \left( x^2 + 3(n-1)\bar{K}x + n(n-1)^2\bar{K}^2 - x - (n-1)\bar{K} \right) \]
\[ = \frac{2n(n-2)(n-1)(2x + n(n-1)\bar{K})\bar{K}^4}{(x^2 + 3(n-1)\bar{K}x + n(n-1)^2\bar{K}^2)a + (x + (n-1)\bar{K})(n-1)a^2} \geq \frac{2n(n-2)(n-1)^2\bar{K}^4}{(x + \sqrt{2} + \sqrt{2n(n-1)\bar{K}})^2}. \]
The last inequality above is equivalent to
\[ x^3 + \left( 4\sqrt{2 + \sqrt{2n + n - 5}} \right) (n - 1) \bar{K} x^2 \]
\[ + 2 \left( n\sqrt{2 + \sqrt{2n + n - n}} \right) (n - 1)^2 \bar{K}^2 x + \sqrt{2n(n - 1)^3} \bar{K}^3 \]
\[ \geq (x + \sqrt{2n(n - 1) \bar{K}}) (x^2 + 3(n - 1) \bar{K} x + n(n - 1)^2 \bar{K}^2) \]
\[ \geq (n - 1) a \cdot (x^2 + 3(n - 1) \bar{K} x + n(n - 1)^2 \bar{K}^2). \]

(v) set \( A = (n - 1) a = \sqrt{x^2 + 4(n - 1) \bar{K} x + 2n(n - 1)^2 \bar{K}^2} \).

\[ 2 \bar{a} - \frac{x}{n} + x \bar{a}' \]
\[ = \frac{3x^2 + 10(n - 1) \bar{K} x + 4n(n - 1)^2 \bar{K}^2}{A} - \frac{4x}{n} \]
\[ \leq 2\sqrt{2n \bar{K}} - \left( \frac{(n - 4)x}{n(n - 1)} - \frac{6(\sqrt{2n - 2})(n - 1) \bar{K} x}{3x + 2\sqrt{2n(n - 1) \bar{K}}}. \right) \]

The inequality above is equivalent to
\[ x^3 + 2\sqrt{2n(n - 1) \bar{K}} - \frac{3x^2 + 10(n - 1) \bar{K} x + 4n(n - 1)^2 \bar{K}^2}{A} \]
\[ = \frac{(3x + 2\sqrt{2n(n - 1) \bar{K}})^2 A^2 - (3x^2 + 10(n - 1) \bar{K} x + 4n(n - 1)^2 \bar{K}^2)^2}{A^2 (3x + 2\sqrt{2n(n - 1) \bar{K}})^2 + A (3x^2 + 10(n - 1) \bar{K} x + 4n(n - 1)^2 \bar{K}^2)^2} \]
\[ \geq \frac{12(\sqrt{2n - 2})(n - 1) \bar{K} x - A}{A (3x + 2\sqrt{2n(n - 1) \bar{K}}) + (3x^2 + 10(n - 1) \bar{K} x + 4n(n - 1)^2 \bar{K}^2)} \]
\[ \geq \frac{6(\sqrt{2n - 2})(n - 1) \bar{K} x}{3x + 2\sqrt{2n(n - 1) \bar{K}}}. \]

The last inequality above is equivalent to
\[ A (3x + 2\sqrt{2n(n - 1) \bar{K}}) \]
\[ \geq 3x^2 + 10(n - 1) \bar{K} x + 4n(n - 1)^2 \bar{K}^2. \]

(vi)
\[ \frac{x}{n - 1} (a + n \tilde{K}) - \left( \frac{x}{n - 1} + 2n \tilde{K} \right) \left( a + \tilde{a} - n \tilde{K} - x \bar{a}' \right) \]
\[ = \frac{x}{n - 1} \left( 2n \tilde{K} + x \bar{a}' - \bar{a} + 2n \tilde{K} (n \tilde{K} + x \bar{a}' - a - \bar{a}) \right) \]
\[ < \frac{x}{n - 1} \left( 2n \tilde{K} + x \bar{a}' - \left( \frac{x}{n - 1} + 2n \tilde{K} \right) \right) \]
\[ + 2n \tilde{K} \left( n \tilde{K} + x \bar{a}' - 2 \left( \frac{x}{n - 1} + 2n \tilde{K} \right) \right) \]
\[ < - \frac{2x \tilde{K}}{n - 1} + 2n(n - 4) \tilde{K}^2. \]
3. Preservation of curvature pinching

In this section, we prove the curvature pinching condition preserves along the mean curvature flow in spheres. First we consider the submanifold $M^n(n \geq 7)$ of arbitrary codimension in the sphere $S^{n+p}(1/K)$. For the mean curvature flow of hypersurfaces $M^n(n \geq 3)$ in spheres, the proof is similar. First, we prove the following inequality.

**Lemma 3.1.** For $x \geq 0$, the following inequality holds.

\[
a(\bar{a} - x\bar{a}') - n\bar{K}(\bar{a} + x\bar{a}') + P_2 \left(2\bar{a} - \frac{x}{n} + x\bar{a}'\right) - \frac{3}{2}P_2^2 < 0.
\]

**Proof.** Without loss of generality, we assume that $\bar{K} = 1$. Its discriminant satisfies

\[
\Delta = \left(2\bar{a} - \frac{x}{n} + x\bar{a}'\right) - \sqrt{6(n(\bar{a} + x\bar{a}') - a(\bar{a} - x\bar{a}')).}
\]

\[
= \frac{3x^2 + 10(n - 1)x + 4(n - 1)^2}{(n - 1)\sqrt{x^2 + 4(n - 1)x + 2n(n - 1)^2}} - \frac{4x}{n}
\]

\[
- \sqrt{\frac{12n}{n - 1} \left(\frac{x^2 + 3(n - 1)x + n(n - 1)^2}{\sqrt{x^2 + 4(n - 1)x + 2n(n - 1)^2}} - x - (n - 1)\right)}.
\]

From Lemma 2.3 (iv),(v), we need to prove the following inequality holds.

\[
\left(x + \sqrt{2 + \sqrt{2n(n-1)}}\right) \left(2\sqrt{2n} - \frac{(n - 4)x}{n(n-1)} - \frac{6(\sqrt{2n} - 2)x}{3x + 2\sqrt{2n(n-1)}}\right)
\]

\[
< 2\sqrt{3n(n-2)(n-1)}.
\]

\[
\left(x + \sqrt{2 + \sqrt{2n(n-1)}}\right) \left(2\sqrt{2n} - \frac{(n - 4)x}{n(n-1)} - \frac{6(\sqrt{2n} - 2)x}{3x + 2\sqrt{2n(n-1)}}\right)
\]

\[
< \left(x + \sqrt{2 + \sqrt{2n(n-1)}}\right) \left(2\sqrt{2n} - \frac{(n - 4)x}{n(n-1)}\right)
\]

\[
- 2(\sqrt{2n} - 2) x \cdot \frac{3\sqrt{2 + \sqrt{2n}}}{2\sqrt{2n}}
\]

\[
< 2\sqrt{3n(n-2)(n-1)}.
\]

The last inequality above is equivalent to

\[
\Leftrightarrow - \frac{n - 4}{n(n-1)}x^2 + \left(2\sqrt{2n} - \left(4 - \frac{4}{n} - 3\sqrt{2/n}\right)\sqrt{2 + \sqrt{2n}}\right) x
\]

\[
- 2\sqrt{n(n-1)} \left(\sqrt{3(n-2)} - \sqrt{4 + 2\sqrt{2n}}\right) < 0.
\]

Now we prove its discriminant is negative, i.e.,

\[
1 - \left(2 - \frac{2}{n} - \frac{3}{\sqrt{2n}}\right) \sqrt{\frac{2 + \sqrt{2n}}{2n}} < \sqrt{\frac{n - 4}{n\sqrt{n}} \left(\sqrt{3(n-2)} - \sqrt{4 + 2\sqrt{2n}}\right)}.
\]
(i) $n \geq 66$

\[
1 + \frac{4}{n-4} + \sqrt{\frac{4 + 2\sqrt{2n}}{n}} < \sqrt{3 \left(1 - \frac{2}{n}\right)}
\]

$\Leftrightarrow 1 < \frac{n-4}{n\sqrt{n}} \left(\sqrt{3(n-2)} - \sqrt{4 + 2\sqrt{2n}}\right)$.

(ii) $13 \leq n \leq 65$

Since

\[
1 - \left(2 - \frac{2}{n} - \frac{3}{\sqrt{2n}}\right) \sqrt{\frac{2 + \sqrt{2n}}{2n}} < 1 - \left(2 - \frac{2}{13} - \frac{3}{\sqrt{26}}\right) \sqrt{\frac{2 + \sqrt{130}}{130}}
\]

$< \sqrt{0.3555}$,

then we have

\[
0.3555 \left(1 + \frac{4}{n-4} + \sqrt{\frac{4 + 2\sqrt{2n}}{n}}\right) < \sqrt{3 \left(1 - \frac{2}{n}\right)}
\]

$\Leftrightarrow 0.3555 < \frac{n-4}{n\sqrt{n}} \left(\sqrt{3(n-2)} - \sqrt{4 + 2\sqrt{2n}}\right)$.

(iii) $9 \leq n \leq 12$

We can check that

\[
1 - \left(2 - \frac{2}{n} - \frac{3}{\sqrt{2n}}\right) \sqrt{\frac{2 + \sqrt{2n}}{2n}} < \sqrt{0.1814},
\]

then we have

\[
0.1814 < \frac{n-4}{n\sqrt{n}} \left(\sqrt{3(n-2)} - \sqrt{4 + 2\sqrt{2n}}\right).
\]

When $n = 8$, \(312 \) is

\[
8 - \frac{x}{14} - \frac{12x}{3x+56} - \frac{168}{x+7\sqrt{6}} < 0
\]

$\Leftrightarrow \frac{3}{14}x^3 - \left(8 - \frac{3\sqrt{6}}{2}\right)x^2 - 56(\sqrt{6} - 1)x + 56^2(3 - \sqrt{6}) > 0$.

We calculate the minimum is

\[
\min \left\{ \frac{3}{14}x^3 - \left(8 - \frac{3\sqrt{6}}{2}\right)x^2 - 56(\sqrt{6} - 1)x + 56^2(3 - \sqrt{6}) \right\}
\]

$= 86.697 > 0$, $x = 19.827$.

When $n = 7$, \(341 \) is

\[
\frac{x^2 + 20x + 336}{2\sqrt{x^2 + 24x + 504}} - \frac{4x}{7} - \sqrt{\frac{14}{\sqrt{x^2 + 24x + 504} - x - 6}} < 0.
\]
We calculate the maximum by numerical evaluation.

\[
\max \left\{ \frac{x^2 + 20x + 336}{2\sqrt{x^2 + 24x + 504}} - \frac{4x}{7} - \sqrt{\frac{14}{\sqrt{x^2 + 24x + 504}} - x - 6} \right\}
\]

\[= -0.264 < 0, \quad x = 20.399.\]

\[\square\]

We denote \(\dot{a}'(|H|^2)\) and \(\dot{a}''(|H|^2)\) by \(\dot{a}'\) and \(\dot{a}''\), respectively. Then the evolution equation of \(\dot{a}\) satisfies

\[\frac{\partial}{\partial t}\dot{a} = \Delta \dot{a} + 2\dot{a}' \cdot (-|\nabla H|^2 + R_2 + n\bar{K}|H|^2) - \dot{a}'' \cdot |\nabla |H|^2|^2.\]

Since \(M\) is compact, there exists a small positive number \(0 < \epsilon \ll \frac{1}{n^2}\), such that \(M\) satisfies

\[|\dot{A}|^2 < \dot{a} - \epsilon\omega, \quad \omega = \frac{|H|^2}{n-1} + 2n\bar{K} > a.\]

Then we prove that the pinching condition above is preserved along the flow.

**Proposition 3.2.** Let \(F_0 : M \to \mathbb{S}^{n+p}(\frac{1}{\sqrt{\bar{K}}})\) be a compact submanifold immersed in the sphere. Suppose there exists a small positive number \(\epsilon(\ll \frac{1}{n^2})\) such that \(|\dot{A}|^2 \leq a(|H|^2) - \epsilon\omega\), then this condition holds along the mean curvature flow for all time \(t \in [0, T]\) where \(T \leq \infty\).

**Proof.** Let \(U = |\dot{A}|^2 - \dot{a} + \epsilon\omega\). By Lemmas 2.12.2 and (3.6), we have

\[
\left(\frac{\partial}{\partial t} - \Delta\right) U = -2|\nabla \dot{A}|^2 + 2(\dot{a}' - \frac{\epsilon}{n-1})|\nabla H|^2 + \dot{a}'' |\nabla |H|^2|^2
\]

\[+ 2R_1 - \frac{2}{n}R_2 - 2n\bar{K} |\dot{A}|^2 - 2(\dot{a}' - \frac{\epsilon}{n-1}) \cdot (R_2 + n\bar{K}|H|^2)
\]

\[\leq 2\left(-\frac{2(n-1)}{n(n+2)} + \dot{a}' - \frac{\epsilon}{n-1} + 2|H|^2 \dot{a}''\right)|\nabla H|^2
\]

\[+ 2|\dot{A}|^2 (|\dot{A}|^2 + \frac{1}{n}|H|^2 - n\bar{K}) + 2P_2 (2|\dot{A}|^2 - \frac{1}{n}|H|^2 - \frac{3}{2}P_2)
\]

\[+ 2(\dot{a}' - \frac{\epsilon}{n-1}) \cdot |H|^2 |\dot{A}|^2 + \frac{1}{n}|H|^2 + n\bar{K} - P_2).\]

From Lemma 2.3 (ii), the coefficient of \(|\nabla H|^2\) is negative.

Replacing \(|\dot{A}|^2\) by \(U + \dot{a} - \epsilon\omega\), the above formula becomes

\[
\left(\frac{\partial}{\partial t} - \Delta\right) U \leq 2U \left(2\dot{a} + \frac{1}{n}|H|^2 - n\bar{K} - \dot{a}'|H|^2 + 2P_2 + \epsilon\left(\frac{|H|^2}{n-1} - 2\omega\right)\right)
\]

\[+ 2U^2 + 2\left(a(\dot{a} - |H|^2 \dot{a}') - n\bar{K}(\dot{a} + |H|^2 \dot{a}')\right)
\]

\[+ 2P_2 \left(2\dot{a} - \frac{1}{n}|H|^2 + |H|^2 \dot{a}' - \epsilon\left(\frac{|H|^2}{n-1} + 2\omega\right) - \frac{3}{2}P_2\right)
\]

\[+ 2\epsilon \left(|H|^2 \cdot (a + n\bar{K}) - \omega (a + \dot{a} - n\bar{K} - \dot{a}'|H|^2)\right)
\]

\[+ 2\epsilon^2 \omega (\omega - \frac{|H|^2}{n-1}).\]
From Lemma 2.3 (vi) and Lemma 3.1, we have
\[ a(\dot{a} - |H|^2 \dot{a}' - n\bar{K}(\ddot{a} + |H|^2 \dot{a}')) + P_2 \left( 2\dot{a} - \frac{1}{n} |H|^2 + |H|^2 \ddot{a}' - \frac{3}{2} P_2 \right) + \epsilon \left( \frac{|H|^2}{n-1} \cdot (a + n\bar{K}) - \omega (a + \dot{a} - n\bar{K} - \ddot{a}'|H|^2) + 2n\bar{K}\omega \right) \]
\[ < a(\dot{a} - |H|^2 \dot{a}') - n\bar{K}(\ddot{a} + |H|^2 \dot{a}')) + P_2 \left( 2\dot{a} - \frac{1}{n} |H|^2 + |H|^2 \ddot{a}' - \frac{3}{2} P_2 \right) + 2\epsilon \bar{K} \left( n\bar{K}(n - 4 + 2n\epsilon) - \frac{|H|^2}{n-1}(1 - n\epsilon) \right) \]
\[ < 0. \]

Then the assertion follows from the maximum principle. \(\square\)

4. Convergence theorem in the sphere

In this section, we prove the convergence theorem for the mean curvature flow of submanifolds \(M^n(n \geq 7)\) in \(\mathbb{S}^{n+p}(\frac{1}{\sqrt{n-1}})\). First, we derive an estimate for the traceless second fundamental form, which guarantees that \(M\) becomes spherical along the mean curvature flow.

**Proposition 4.1.** There exist constants \(C < \infty\) and \(\sigma > 0\) both depending only on the initial surface such that for all time \(t \in [0, T)\) where \(T \leq \infty\), we have the estimate
\[ |\dot{A}|^2 \leq C(|H|^2 + \bar{K})^{1-\sigma} e^{-2\sigma t}. \]
(4.1)

To obtain Theorem 4.1, we need to find the upper bound of
\[ f_\sigma := \frac{|\dot{A}|^2}{\dot{a}^{1-\sigma}} \]
with the help of the Stampacchia iteration as in [2]. Here, we get the evolution equation of \(f_\sigma\) with two parameters.

**Lemma 4.2.** For every \(\sigma \in (0, 1)\), we have the evolution equation
\[ \frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + 2|\nabla f_\sigma| |\nabla H| - \frac{2\epsilon f_\sigma |\nabla H|^2}{n|A|^2} + 6\sigma|A|^2 f_\sigma - 2(n\sigma + \epsilon)\bar{K} f_\sigma. \]
(4.2)

**Proof.** From Lemma 2.1, we have
\[ \frac{\partial}{\partial t} f_\sigma = f_\sigma \left( \frac{\partial}{\partial t} \frac{|\dot{A}|^2}{|A|^2} - (1 - \sigma) \frac{\partial}{\partial t} \frac{\dot{a}}{a} \right). \]
(4.3)
By direct computation, we have

\[
\Delta f_\sigma = f_\sigma \left( \frac{\Delta |\dot{A}|^2}{|\dot{A}|^2} - \frac{(1 - \sigma) \Delta \dot{a}}{\dot{a}} \right) - \frac{2(1 - \sigma)}{\dot{a}} (\nabla \dot{a}, \nabla f_\sigma) + \frac{\sigma(1 - \sigma)}{\dot{a}^2} f_\sigma |\nabla \dot{a}|^2.
\]

Combining (4.3) and (4.4) and from Lemma 2.11.2.3, we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) f_\sigma = \frac{2(1 - \sigma)}{\dot{a}} (\nabla \dot{a}, \nabla f_\sigma) - \frac{\sigma(1 - \sigma)}{\dot{a}^2} f_\sigma |\nabla \dot{a}|^2
+ f_\sigma \left( \frac{1 - \sigma}{\dot{a}} (2\dot{a}^2 |\nabla H|^2 + \dot{a}'|\nabla |H|^2|) - \frac{2|\nabla \dot{A}|^2}{|\dot{A}|^2} \right)
+ 2f_\sigma \left( \frac{R_1 - \frac{1}{n} R_2 - nK|\dot{A}|^2}{|\dot{A}|^2} - \frac{(1 - \sigma)\dot{a}'}{\dot{a}} (R_2 + nK|H|^2) \right)
\leq \frac{4\dot{a}'|H|}{\dot{a}} |\nabla H| |\nabla f_\sigma| + 2f_\sigma \frac{2(n - 1)}{n(n + 2)} \frac{|\dot{A}|^2 - \dot{a}}{|\dot{a}|} |\nabla H|^2
+ 2\sigma f_\sigma \left( \frac{R_1 - \frac{1}{n} R_2 - nK}{|\dot{A}|^2} - \frac{\dot{a}'}{\dot{a}} (R_2 + nK) \right)
\leq 2 \frac{|\nabla f_\sigma| |\nabla H|}{|\dot{A}|} + \frac{2\epsilon f_\sigma |\nabla H|^2}{n|\dot{A}|^2} + 2\sigma f_\sigma \left( \frac{3|\dot{A}|^2 - nK}{|\dot{A}|^2} \right)
+ 2(1 - \sigma) f_\sigma \left( \frac{R_1 - \frac{1}{n} R_2 - nK}{|\dot{A}|^2} - \frac{\dot{a}'}{\dot{a}} (R_2 + nK) \right).
\]

The last term in the bracket satisfies

\[
(4.5) \quad \frac{R_1 - \frac{1}{n} R_2}{|\dot{A}|^2} - nK - \frac{\dot{a}'}{\dot{a}} (R_2 + nK|H|^2)
\leq \frac{|\dot{A}|^2 + 2P_2 - nK}{|\dot{A}|^2} - \frac{1}{|\dot{A}|^2} \left( \frac{3}{2} P_2^2 + \frac{|H|^2}{n} P_2 \right) - \frac{\dot{a}'|H|^2}{\dot{a}} (|\dot{A}|^2 - P_2 + nK)
\leq \frac{|\dot{A}|^2}{\dot{a}} (\dot{a} - \dot{a}'|H|^2) - \frac{nK}{\dot{a}} (\dot{a} + \dot{a}'|H|^2) + \frac{P_2}{\dot{a}} (2\dot{a} + \dot{a}'|H|^2 - \frac{|H|^2}{n} - \frac{3}{2} P_2)
\leq \frac{1}{\dot{a}} \left( a(\dot{a} - \dot{a}'|H|^2) - nK(\dot{a} + \dot{a}'|H|^2) + P_2 (2\dot{a} + \dot{a}'|H|^2 - \frac{|H|^2}{n} - \frac{3}{2} P_2) \right)
- \frac{\epsilon \omega}{\dot{a}} (\dot{a} - \dot{a}'|H|^2).
\]

The term in the big bracket of the last inequality is non-positive under our pinching assumption. We complete the lemma with the following inequality.

\[
- \frac{\epsilon \omega}{\dot{a}} (\dot{a} - \dot{a}'|H|^2) \leq - \frac{2\epsilon \omega K}{\dot{a}} \leq -2\epsilon K.
\]

\[\square\]
Since the term \( \sigma |A|^2 f_\sigma \) in the evolution equation is positive, we cannot use the ordinary maximum principle. As in [2, 6], we need the negative gradient terms to proceed the iteration. Applying the Simons identity [16], we get
\[
\frac{1}{2} \Delta |\dot{A}|^2 = \langle \dot{A}, \nabla^2 H \rangle + |\nabla \dot{A}|^2 + nK|\dot{A}|^2 - R_1 + R_3.
\]

Now we have the following estimate.

**Lemma 4.3.** Let \( F : M \to S^{n+p}(\frac{1}{\sqrt{K}}) \) be a compact submanifold immersed in the sphere with constant curvature \( \tilde{K} \). If \( F \) satisfies pinching condition (3.7), then
\[
n\tilde{K}|\dot{A}|^2 - R_1 + R_3 \geq \frac{n}{2} |\dot{A}|^2(\epsilon |A|^2 - \sqrt{2nK}).
\]

**Proof.** From Lemma 2.2, we have
\[
n\tilde{K}|\dot{A}|^2 - R_1 + R_3 \geq \frac{n}{2} |\dot{A}|^2 \left( \frac{|H|^2}{n(n-1)} + 2\tilde{K} \right) - \frac{n}{2} |\dot{A}|^4
\]
\[
= \frac{n}{2} |\dot{A}|^2 \left( \frac{|H|^2}{n-1} + 2\tilde{K} - |A|^2 \right)
\]
\[
\geq \frac{n}{2} |\dot{A}|^2 \left( \frac{|H|^2}{n-1} + 2\tilde{K} - a + \epsilon \omega \right)
\]
\[
\geq \frac{n}{2} |\dot{A}|^2 \left( \epsilon \omega - (\sqrt{2n-2}) \tilde{K} \right).
\]

Then we can get the required Poincaré inequality.

**Lemma 4.4.** For every \( p \geq 4 \) and \( \eta > 0 \) we have the estimate
\[
\int_{M_t} |A|^2 f_\sigma d\mu_t \leq \int_{M_t} \left( \frac{2(p-1)}{n\epsilon} f_\sigma^{p-2} |\nabla f_\sigma|^2 + \frac{2(p-1)\eta + 5}{n\epsilon} \cdot \frac{f_\sigma^p |\nabla H|^2}{|A|^2} \right) d\mu_t
\]
\[
+ \frac{\sqrt{2n}}{\epsilon} \int_{M_t} \tilde{K} f_\sigma^p d\mu_t.
\]

**Proof.** From Lemma 4.3, we have
\[
\Delta f_\sigma \geq \frac{\Delta |\dot{A}|^2}{a^{1-\sigma}} - (1-\sigma) \frac{f_\sigma \Delta a}{a} - \frac{2(1-\sigma)\dot{a}'}{a} \langle |\nabla H|^2, \nabla f_\sigma \rangle
\]
\[
\geq \frac{1}{a^{1-\sigma}} \left( 2 \langle \dot{A}, \nabla^2 H \rangle + n|\dot{A}|^2(\epsilon |A|^2 - \sqrt{2nK}) \right)
\]
\[
- \frac{2|\nabla f_\sigma||\nabla H|}{|A|} - (1-\sigma) \frac{f_\sigma \Delta a}{a}.
\]
\[
nf_\sigma(\epsilon |A|^2 - \sqrt{2nK}) \leq \Delta f_\sigma - \frac{2\langle \dot{A}, \nabla^2 H \rangle}{a^{1-\sigma}} + (1-\sigma) \frac{f_\sigma \Delta a}{a}
\]
\[
+ \frac{2|\nabla f_\sigma||\nabla H|}{|A|}.
\]
Lemma 4.5. For any $p \geq \frac{3}{2\varepsilon} + 1$ and $\sigma \leq \frac{\varepsilon \sqrt{2}}{2\sqrt{p-1}}$, there exist a constant $C$ depending only on the initial surface such that for all $t \in [0, T)$ where $T \leq \infty$, we have

\begin{equation}
\left( \int_{M_t} f_\sigma^p \mathrm{d}u \right)^{\frac{1}{p}} \leq C e^{-2\sigma \bar{K} t}.
\end{equation}
Proof. For $t \geq t_0$, form Lemma 4.2, we have

\[
\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \, d\mu_t \leq \int_{M_t} p f_\sigma^{p-1} \frac{\partial}{\partial t} f_\sigma \, d\mu_t \\
\leq -p(p-1) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu_t \\
+2p \int_{M_t} f_\sigma^{p-1} \frac{|\nabla f_\sigma| |\nabla H|}{|A|} \, d\mu_t -2pe \int_{M_t} f_\sigma^p |\nabla H|^2 \, d\mu_t \\
+6\sigma \int_{M_t} |A|^2 f_\sigma^p \, d\mu_t -2p(n\sigma + \epsilon) \bar{K} \int_{M_t} f_\sigma^p \, d\mu_t.
\]

In view of the pinching condition we can estimate

\[
2p \int_{M_t} f_\sigma^{p-1} \frac{|\nabla f_\sigma| |\nabla H|}{|A|} \, d\mu_t \\
\leq \frac{p}{\mu} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu_t + p\mu \int_{M_t} f_\sigma^p |\nabla H|^2 \, d\mu_t.
\]

Substituting (4.17) to (4.16), letting $\mu = \frac{2}{p-1}$ and $p \geq \frac{2}{\epsilon} + 1$ we obtain

\[
\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \, d\mu_t \leq -\frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu_t \\
-pec \int_{M_t} f_\sigma^{p-1} \frac{1}{|A|^2(1-\sigma)} |\nabla A|^2 \, d\mu_t \\
+6\sigma \int_{M_t} |A|^2 f_\sigma^p \, d\mu_t -2p(n\sigma + \epsilon) \bar{K} \int_{M_t} f_\sigma^p \, d\mu_t.
\]

This together with Lemma 4.4 implies

\[
\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \, d\mu_t \\
\leq -p(p-1) \left( \frac{1}{2} - \frac{12\sigma}{n\epsilon} \right) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu_t \\
-p \left( \epsilon - \frac{6\sigma(2(p-1)\eta + 5)}{n\epsilon} \right) \int_{M_t} f_\sigma^{p-1} \frac{1}{(|H| + \beta \bar{K})^{1-\sigma}} |\nabla A|^2 \, d\mu_t \\
-2 \left( \epsilon - \frac{3\sigma \sqrt{2n}}{\epsilon} \right) \bar{K} \int_{M_t} f_\sigma^p \, d\mu_t -2n\sigma \bar{K} \int_{M_t} f_\sigma^p \, d\mu_t.
\]

Now we pick $\eta = \frac{28\sigma}{n\epsilon}$, $p \geq \frac{n^3}{32\pi} + 1$ and let

\[
\sigma \leq \min \left\{ \frac{\epsilon^2}{3\sqrt{2n}}, \frac{n^2 \epsilon}{24\sqrt{p-1}}, \frac{n\epsilon \sqrt{\epsilon}}{24\sqrt{p-1}} \right\} = \frac{n\epsilon \sqrt{\epsilon}}{24\sqrt{p-1}}
\]
such that

\[
6\sigma(2(p-1)\eta + 5) \leq \frac{n}{2} \epsilon^2 + \frac{n}{2} \epsilon^2 = n\epsilon^2.
\]

Then (4.19) reduces to

\[
\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \, d\mu_t \leq -2\sigma \bar{K} \int_{M_t} f_\sigma^p \, d\mu_t,
\]
and this implies

\[(4.21) \int_{M_t} f^2_\sigma \tilde{\mu}_t \leq e^{-2\sigma \bar{K}_t} \int_{M_{t_0}} f^2_\sigma \tilde{\mu}_t.\]

Then we can proceed by a Stampacchia iteration procedure as in [2, 6] to bound $f_\sigma$ in $L^\infty$ and complete the proof of Proposition 4.1.

Applying Proposition 4.1, we have the following gradient estimation.

**Proposition 4.6** ([2] Theorem 5.8). For every $\eta > 0$, there exists a constant $C_\eta$ depending only on $\eta$ such that for all time $t$, there holds

$$|\nabla H|^2 \leq (\eta |H|^4 + C_\eta) e^{-\sigma t}.$$ 

To estimate the diameter of $M_t$, we need the lower bound of the Ricci curvature along the mean curvature flow. By Proposition 2 of [15], the Ricci curvature of $M$ satisfies

$$\text{Ric}_M \geq n - 1 - \frac{1}{n} |\tilde{A}|^2 - \frac{1}{n} \sqrt{n(n-1)} |H||\tilde{A}|.$$ 

From the pinching condition $|\tilde{A}|^2 < \tilde{\alpha} - \epsilon \omega$ and Lemma 2.3 (iii), we obtain $\text{Ric}_M \geq \frac{n-1}{n} \epsilon \omega > \frac{\epsilon}{n} |H|^2$. Combining the gradient estimation and the well-known Myers theorem we can get the following lemma easily.

**Lemma 4.7** ([10] Lemma 6.2). Suppose that $M$ is an $n$-dimensional submanifold in $S^{n+p}(\frac{1}{\sqrt{\lambda}})$ satisfying $|\tilde{A}|^2 < \tilde{\alpha} - \epsilon \omega$ and $|\nabla H| < 2\eta^2 \max_M |H|^2$, where $0 < \eta < \epsilon/2$. Then we have $\min_M |H|^2 > 1 - \eta$ and $\text{diam } M \leq \frac{2}{\eta} \max_M |H|^{-1}$. 

Now we can complete the proof of the theorem 1.1 for $n \geq 7$.

**Proof of theorem 1.1**. From Lemma 4.7, we know $\min_M |H|^2 \to 1$ and the diameter of $M_t$ is bounded along the mean curvature flow. By similar arguments as in [2], we obtain the mean curvature flow with initial value $F_0$ either converges to a round point in finite time, or converges to a total geodesic sphere of $S^{n+p}(\frac{1}{\sqrt{\lambda}})$ as $t \to \infty$. 

5. **Another sharp convergence theorem**

Putting

\[(5.1) \quad b(x) = (1 - \delta) \left( \frac{x}{n-1} + 2\bar{K} \right) + \delta a(x), \quad \tilde{b}(x) = b(x) - \frac{x}{n},\]

where $\delta = \frac{\sqrt{2n-5} - 7}{2(2n-5)},$ $4 \leq n \leq 12$, and $\delta = \frac{2n - 5}{n},$ $n \geq 13$. Then we prove the following sharp convergence theorem.

**Theorem 5.1**. Let $F_0 : M \to S^{n+p}(\frac{1}{\sqrt{\lambda}})$ be an $n$-dimensional ($n \geq 4$) smooth compact submanifold immersed in the sphere. If $M$ satisfies

\[(5.2) |\tilde{A}|^2 \leq b(|H|^2),\]

then the mean curvature flow with the initial value $F_0$ converges to a round point in finite time, or converges to a total geodesic sphere of $S^{n+p}(\frac{1}{\sqrt{\lambda}})$ as $t \to \infty$. 

Lemma 5.2. For $x \geq 0$, we have the following properties.

(i) $\frac{4x(\hat{b})^2}{b} < 1$,

(ii) $2xb'' + \hat{b} < \frac{2(n-1)}{2(n+1)}$, for $\delta \leq \frac{2(n-5)}{n^2-4}$,

(iii) $\frac{n-2}{\sqrt{n(n-1)}} \sqrt{xb} + \hat{b} < \frac{x}{n} + n\bar{K}$,

(iv) $\hat{b} \cdot (b - n\bar{K}) - xb' \cdot (b + n\bar{K}) < -2(1 - \delta)(n - 2)\bar{K}^2$,

(v) $\frac{n-2}{n-1}(b + n\bar{K}) - \left(\frac{n+2}{n-1} + 2n\bar{K}\right) \left(\hat{b} + b - n\bar{K} - xb'\right)$

$\leq -\frac{2\bar{K}x}{n-1} + 2n(n-4)\bar{K}^2$,

(vi) $2\hat{b} - \frac{x}{n} + xb' < 2(\delta(n - 2) + 2)\bar{K}$.

Proof. By direct computations, we get

$\hat{b} = \delta n\bar{K} + 2(1 - \delta)\bar{K} + \frac{\delta n^2 - 2\delta n + 2}{2(n-1)n} x - \frac{\delta(n-2)}{2(n-1)} \sqrt{x^2 + 4(n-1)\bar{K}x}$,

$\hat{b}' = \frac{\delta n^2 - 2\delta n + 2}{2(n-1)n} - \frac{\delta(n-2)}{2(n-1)} \frac{x + 2(n-1)\bar{K}}{\sqrt{x^2 + 4(n-1)\bar{K}x}} \leq \frac{n}{n(n-1)},$

$\hat{b}'' = \frac{2\delta(n-1)(n-2)\bar{K}^2}{(x^2 + 4(n-1)\bar{K}x)^{3/2}}$.

(i)

$\frac{4x(\hat{b})^2}{b} < \frac{x}{n(n-1)} < 1$.

(ii)

$2xb'' + \hat{b} = \delta n^2 - 2\delta n + 2 \quad \frac{\delta(n-2)(x^2 + 3(n-1)\bar{K}x^2)}{2(n-1)} \frac{2(n-1)\bar{K}x^3}{(x^2 + 4(n-1)\bar{K}x)^{3/2}}$

$\leq \frac{\delta n^2 - 2\delta n + 2}{2(n-1)n} \leq \frac{2(n-1)}{n(n+2)}$, where $\delta \leq \frac{2(2n-5)}{n^2-4}$.

(iii)

$\frac{n-2}{\sqrt{n(n-1)}} \sqrt{xb} + \hat{b} < \frac{n-2}{\sqrt{n(n-1)}} \sqrt{x\alpha} + \hat{a} = \frac{x}{n} + n\bar{K}$.

(iv)

$a(\hat{b} - xb')$

$= a \left( \delta n\bar{K} + 2(1 - \delta)\bar{K} - \frac{\delta(n-2)\bar{K}x}{\sqrt{x^2 + 4(n-1)\bar{K}x}} \right)$

$= (\delta n\bar{K} + 2(1 - \delta)\bar{K})^2 - (\delta n\bar{K} + 2(1 - \delta)\bar{K}) \frac{\delta(n-2)(x^2 + 3(n-1)\bar{K}x)}{(n-1)\sqrt{x^2 + 4(n-1)\bar{K}x}}$

$+ \frac{\delta^2(n-2)^2 + (\delta n + 2(1 - \delta))^2}{2(n-1)} \bar{K}x.$
\[ nK(b + x\dot{b}') = nK \left( \delta nK + 2(1 - \delta)\bar{K} + \frac{\delta n^2 - 2\delta n + 2}{(n - 1)n} x - \frac{\delta(n - 2)(x^2 + 3(n - 1)\bar{K}x)}{(n - 1)\sqrt{x^2 + 4(n - 1)\bar{K}x}} \right). \]

\[ a(b - x\dot{b'}) - nK(b + x\dot{b}') \]
\[ = (\delta - 1)(n - 2) \left( \delta nK + 2(1 - \delta)\bar{K} - \frac{\delta(n - 2)(x^2 + 3(n - 1)\bar{K}x)}{(n - 1)\sqrt{x^2 + 4(n - 1)\bar{K}x}} \right) \bar{K} \]
\[ + \frac{\delta(\delta - 1)(n - 2)^2}{n - 1} \bar{K}x \]
\[ = (\delta - 1)(n - 2)\bar{K} \left( \delta nK + 2(1 - \delta)\bar{K} + \frac{\delta(n - 2)}{n - 1} \left( x - \frac{x^2 + 3(n - 1)\bar{K}x}{\sqrt{x^2 + 4(n - 1)\bar{K}x}} \right) \right) \]
\[ < - (1 - \delta)(n - 2)\bar{K} \left( \delta nK + 2(1 - \delta)\bar{K} + \frac{\delta(n - 2)}{n - 1} \left( - (n - 1)\bar{K} \right) \right) \]
\[ = - 2(1 - \delta)(n - 2)\bar{K}^2. \]

(v)
\[ \frac{x}{n - 1} (b + n\bar{K}) - \left( \frac{x}{n - 1} + 2n\bar{K} \right) (b - n\bar{K} + \hat{b} - x\dot{b}') \]
\[ = \frac{2nx\bar{K}}{n - 1} - 2n\bar{K}(a - n\bar{K}) \]
\[ = - \left( \frac{x}{n - 1} + 2n\bar{K} \right) \left( \delta n\bar{K} + 2(1 - \delta)\bar{K} - \frac{\delta(n - 2)\bar{K}x}{\sqrt{x^2 + 4(n - 1)\bar{K}x}} \right) \]
\[ = - \frac{2\bar{K}x}{n - 1} + 2(n - 4)\bar{K}^2 + \frac{\delta(n - 2)\bar{K}x}{n - 1} \left( \frac{x}{\sqrt{x^2 + 4(n - 1)\bar{K}x}} - 1 \right) \]
\[ + \frac{\delta n(n - 2)\bar{K}}{n - 1} \left( \frac{x^2 + 6(n - 1)\bar{K}x}{\sqrt{x^2 + 4(n - 1)\bar{K}x}} - x - 4(n - 1)\bar{K} \right) \]
\[ < - \frac{2\bar{K}x}{n - 1} + 2(n - 4)\bar{K}^2. \]
(vi)
\[
2\dot{b} - \frac{x}{n} + xb' = 2(\delta(n - 2) + 2)K + \frac{3\delta n^2 - 2\delta n + 2}{2(n - 1)n} x - \frac{x}{n}
- 2 \frac{\delta(n - 2)}{2(n - 1)} \sqrt{x^2 + 4(n - 1)Kx} - \frac{\delta(n - 2)}{2(n - 1)} \frac{x^2 + 2(n - 1)Kx}{\sqrt{x^2 + 4(n - 1)Kx}}
\]
\[
= 2(\delta(n - 2) + 2)K - \frac{n - 4}{(n - 1)n} x
+ \frac{3\delta(n - 2)x}{2(n - 1)} - \frac{\delta(n - 2)}{2(n - 1)} \frac{3x^2 + 10(n - 1)Kx}{\sqrt{x^2 + 4(n - 1)Kx}}
\leq 2(\delta(n - 2) + 2)K.
\]

\[\square\]

There exists a small positive number \(0 < \epsilon \ll \frac{1}{n^2}\), such that \(M_0\) satisfies

\[
|\dot{A}|^2 < \dot{b} - \epsilon \omega,
\]

where

\[
\omega = \frac{|H|^2}{n - 1} + 2nK.
\]

In the following we prove that the pinching condition above is preserved along the flow.

**Proposition 5.3.** Let \(F_0 : M \to \mathbb{S}^{n+p}(\frac{1}{\sqrt{K}})\) be a compact submanifold immersed in the sphere. Suppose there exists a small positive number \(\epsilon \ll \frac{1}{n^2}\) such that

\[
|\dot{A}|^2 \leq b(|H|^2) - \epsilon \omega,
\]

then this condition holds along the mean curvature flow for all time \(t \in [0, T)\) where \(T \leq \infty\).

**Proof.** As the proof of Proposition 5.2 we need the following inequality holds for \(x \geq 0\).

\[
b(\dot{b} - xb') - nK(\dot{b} + xb') + P_2 \left(2\dot{b} - \frac{x}{n} + xb'\right) - \frac{3}{2} P_2^2 < 0.
\]

From Lemma 5.2 we have

\[
b(\dot{b} - xb') - nK(\dot{b} + xb') + P_2 \left(2\dot{b} - \frac{x}{n} + xb'\right) - \frac{3}{2} P_2^2
\]

\[
< - 2(1 - \delta)(n - 2)K^2 + 2(\delta(n - 2) + 2)K P_2 - \frac{3}{2} P_2^2.
\]

Its discriminant is

\[
\Delta = \left(\delta(n - 2) + 2\right)^2 - 3(1 - \delta)(n - 2)
= \delta^2(n - 2)^2 + 7\delta(n - 2) + 4 - 3(n - 2)
\leq 0.
\]

\[\square\]

We complete the proof of Theorem 5.1 with the following estimate.
Lemma 5.4. Let $F : M \to \mathbb{S}^{n+p}(\frac{1}{\sqrt{\bar{K}}})$ be a compact submanifold immersed in the sphere with constant curvature $\bar{K}$. If $F$ satisfies pinching condition (5.3), then there exists a strictly positive constant $\epsilon$ such that

$$n\bar{K}|\hat{A}|^2 - R_1 + R_3 \geq \frac{n}{2}|\hat{A}|^2(\epsilon|\hat{A}|^2 - 4\bar{K}).$$

Proof.

From Lemma 2.2 we have

$$n\bar{K}|\hat{A}|^2 - R_1 + R_3 \geq \frac{n}{2}|\hat{A}|^2\left(\frac{|H|^2}{n(n-1)} + 2\bar{K} - 2\bar{K}|\hat{A}|^2\right).$$

From Lemma 2.2 we have

$$b \leq \frac{|H|^2}{n-1} + (1 - \delta)2\bar{K} + \delta n\bar{K}.$$

From Lemma 2.2 we have

$$n\bar{K}|\hat{A}|^2 - R_1 + R_3 \geq \frac{n}{2}|\hat{A}|^2\left(\frac{|H|^2}{n(n-1)} + 2\bar{K} - 2\bar{K}|\hat{A}|^2\right) \geq \frac{n}{2}|\hat{A}|^2\left(\epsilon\omega - \delta(n-2)\bar{K}\right).$$

References

[1] B. Andrews and C. Baker, Mean curvature flow of pinched submanifolds to sphere, J. Differential Geom. 85(2010),357-395.
[2] C. Baker, The mean curvature flow of submanifolds of high codimension, arXiv:1104.4409.
[3] C. Baker and H. Nguyen, Codimension two surfaces pinched by normal curvature evolving by mean curvature flow, Ann. Inst. H. Poincaré Anal. Non Linéaire 34(2017), 1599-1610.
[4] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. 20(1984), 237-266.
[5] G. Huisken, Contracting convex hypersurface in Riemannian manifolds by their mean curvature, Invent. Math. 84(1986), 463-480.
[6] G. Huisken, Deforming hypersurfaces of the sphere by their mean curvature, Math. Z. 195(1987), 205-219.
[7] G. Huisken, The volume preserving mean curvature flow, J. Reine Angew. Math. 382(1987), 35-48.
[8] H. B. Lawson and J. Simons, On stable currents and their application to global problems in real and complex geometry, Ann. of Math. 98(1973), 427-450.
[9] L. Lei and H. W. Xu, An optimal convergence theorem for mean curvature flow of arbitrary codimension in hyperbolic spaces, arXiv:1503.06747.
[10] L. Lei and H. W. Xu, Mean curvature flow of arbitrary codimension in spheres and sharp differentiable sphere theorems, arXiv:1506.06321.
[11] L. Lei and H. W. Xu, New developments in mean curvature flow of arbitrary codimension inspired by Yau rigidity theory, Proceedings of the Seventh International Congress of Chinese Mathematicians. Vol. I, 327-348, Adv. Lect. Math. (ALM), 43, 2019.
[12] K. F. Liu, H. W. Xu, F. Ye and E. T. Zhao, Mean curvature flow of higher codimension in hyperbolic spaces, Comm. Anal. Geom. 21(2012), 651-669.
[13] K. F. Liu, H. W. Xu, F. Ye and E. T. Zhao, The extension and convergence of mean curvature flow in higher codimension, Trans. Amer. Math. Soc. 370(2018), 2231-2262.
[14] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, Am. J. Math. 96(1974), 207-213.
[15] K. Shiohama and H. W. Xu, *The topological sphere theorem for complete submanifolds*, Compositio Math., 107(1997), 221-232.

[16] J. Simons, *Minimal varieties in Riemannian submanifolds*, Ann. Math. 88(1968), 62-105.

[17] K. Smoczyk, *Mean curvature flow in higher codimension: introduction and survey*, Global differential geometry, 231-274, Springer Proc. Math., 17, Springer, Heidelberg, 2012.

[18] H. W. Xu and E. T. Zhao, *Topological and differentiable sphere theorems for complete submanifolds*, Comm. Anal. Geom., 17(2009), 565-585.

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