QUADRATIC RESPONSE AND SPEED OF CONVERGENCE OF INvariant MEASURES IN THE ZERO-NOISE LIMIT

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Abstract. We study the stochastic stability in the zero-noise limit from a quantitative point of view.

We consider smooth expanding maps of the circle perturbed by additive noise. We show that in this case the zero-noise limit has a quadratic speed of convergence, as suggested by numerical experiments and heuristics published by Lin, in 2005 (see [25]). This is obtained by providing an explicit formula for the first and second term in the Taylor’s expansion of the response of the stationary measure to the small noise perturbation. These terms depend on important features of the dynamics and of the noise which is perturbing it, as its average and variance.

We also consider the zero-noise limit from a quantitative point of view for piecewise expanding maps showing estimates for the speed of convergence in this case.

1. Introduction. Deterministic dynamical systems are often used as models of physical and natural phenomena despite the ubiquitous presence in nature of small random perturbations or fluctuations. It is natural to study the robustness of the deterministic model to such random perturbations and which of the aspects of the deterministic dynamics are stable under small random perturbations. In this paper we consider the many important aspects of the statistical behavior of the system which are encoded in its invariant measures. We study hence quantitatively how these measures change when the system is perturbed by the addition of a small quantity of noise, in the so called zero-noise limit. More precisely, we study this limit and its speed of convergence from a quantitative point of view, also considering first and second order terms in this convergence. We will see that these terms depend

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on important features of the dynamics and of the noise which is perturbing it, as its average and variance.

Let \( S_0 := (X, T) \) be a discrete time deterministic dynamical system where \( X \) is a metric space and \( T : X \to X \) is a Borel measurable map. It is well known that \( (X, T) \) can have several invariant measures, let us consider one of these measures and denote it by \( \mu_0 \).

Suppose now we perturb the system at each iteration by the addition of a small quantity of noise whose amplitude is expressed by a certain parameter \( \delta \in [0, \delta) \) obtaining a family of random systems \( \{S_\delta\}_{\delta \in [0, \delta)} \) (these systems will formally be defined as suitable random dynamical systems, a precise definition will be given in Section 2.2.4). Suppose \( \{\mu_\delta\}_{\delta \in [0, \delta)} \) are stationary measures for \( S_\delta \). It is natural to investigate under which assumptions one may have

\[
\lim_{\delta \to 0} \mu_\delta = \mu_0.
\]

In this case the system and the measure \( \mu_0 \) are said to be \textit{statistically stable} under small noise perturbations (or in the zero-noise limit). An invariant measure of a deterministic model which is stable under the small random perturbations which are present in nature is a measure that can be observed in the real phenomenon behind the model. For this reason this zero-noise limit was proposed by A. N. Kolmogorov as a tool to select the \textit{physically meaningful} measures among the a priori many invariant measures of a deterministic system (see e.g. [13], [36]).

The statistical stability for the zero-noise limit (also called stochastic stability) was proved for several classes of systems, starting from uniformly hyperbolic ones to many interesting cases of non-uniform hyperbolic behavior ([23],[37], [11], [12], [10], [28], [33], [5], [27], [6], [3], [2], [4], [32], [9]).

The mere existence of the zero-noise limit gives a qualitative information on the behavior of the system under perturbation. In practice it can be useful to have quantitative information on the convergence of this limit, both on the speed of the convergence and on the “direction” of change of the invariant measure after the perturbation. In [25] several numerical experiments have been done to estimate the speed of convergence in the limit. From the experiments and the heuristic presented one could conjecture a quadratic speed in the case of smooth expanding maps and linear speed for the piecewise expanding and hyperbolic case. Other exponents can been conjectured in cases of weakly chaotic, non-uniformly hyperbolic systems.

In this paper we will consider these kinds of questions, investigating both quantitative estimates for the speed of the convergence and the direction of change of the invariant measure of the system under perturbation. This is strongly related to the linear response theory, although in this case we will not be only interested in the linear term in the response of the system to the perturbation, but also in the higher order terms, and in particular to the quadratic one.

The \textit{Linear Response} means to quantify the response of the system when submitted to a certain infinitesimal perturbation as a derivative. For example, if one is interested in the linear response of the stationary measure of the system, the derivative of the invariant measure of interest with respect to the perturbation will be considered.

More precisely, let \( \{S_\delta\}_{\delta \in [0, \delta)} \) as above be the family of systems arising by some small perturbation of the initial system \( S_0 \) with unique stationary measures \( \{\mu_\delta\}_{\delta \in [0, \delta)} \). The linear response of the invariant measure \( \mu_0 \) of \( S_0 \) under the given
perturbation is defined by the limit
\[ \hat{\mu} := \lim_{\delta \to 0} \frac{\mu_\delta - \mu_0}{\delta} \] (1)
where the meaning of this convergence can vary from system to system. In some
systems and for a given perturbation, one may get $L^1$-convergence for this limit;
in other systems or for other perturbations one may get convergence in weaker or
stronger topologies. The linear response to the perturbation hence represents the
first order term of the response of a system to the perturbation and in this case, a
linear response formula can be written:
\[ \mu_\delta = \mu_0 + \hat{\mu} \delta + o(\delta) \] (2)
which holds in some weaker or stronger sense. We remark that given an observable
function $c : X \to \mathbb{R}$, if the convergence in (1) is strong enough with respect to the
regularity of $c$, we get
\[ \lim_{t \to 0} \int c \, d\mu_t - \int c \, d\mu_0 = \int c \, d\hat{\mu} \] (3)
showing how the linear response controls the behavior of observable averages. For
instance the convergence in (3) holds when $c \in L^\infty$ and the convergence of the
linear response is in $L^1$.

Once the first order (the linear part) of the response of a system to a perturbation
is understood, it is natural to study further orders. The second order of the response
may then be related to the second derivative and to other natural questions, as
convexity aspects of the response of the system under perturbation, or the stability
of the first order response. If the Linear Response $\hat{\mu}$ represents the first order
term of the response (see (2)), the Quadratic Response $\ddot{\mu}$ will represent the second
order term of this response, analogous to the second derivative in usual Taylor's
expansion:
\[ \mu_\delta = \mu_0 + \hat{\mu} \delta + \frac{1}{2} \ddot{\mu} \delta^2 + o(\delta^2). \] (4)
We refer to [8] for a recent survey on linear response for deterministic systems
and perturbations and to the introduction of [20] for a very recent survey in the case
of response for random systems and higher order terms in the response of a system to
deterministic or random perturbations. Focusing on zero-noise limits, we point out
the paper [21], where among other results, linear and high order response are proven
for deterministic perturbations and zero-noise limits of uniformly hyperbolic systems
(see also [26] for some earlier examples of linear response in the zero-noise limit
for expanding maps and [7] for rigorous numerical methods for its approximation
including an example of zero-noise limit).

In the paper [20], a relatively simple and quite general approach to the first and
second order terms in the response of a system to perturbations is proposed and
applied to deterministic perturbations of deterministic systems and perturbations
of random systems (other approaches to higher order response can be found e.g. in
[22, 21, 29, 31, 30]). In Subsection 2.1 we recall the main general results of [20].
We then apply it to the zero-noise limit, providing precise quantitative information
on the convergence of the zero-noise limit and proving some of the statements sug-
gested by the numerical experiments and the heuristic exposed in [25], in particular
considering zero-noise limits of expanding and piecewise expanding maps.

In the literature, the general approach to this problem is often based on considering
the family of transfer operators $\{L_\delta\}_{\delta \in [0,\delta]}$ associated to the dynamical system
and its perturbations, remarking that invariant and stationary measures are fixed points of this family of operators. Quantitative perturbative statements about these operators and its spectral picture will hence give information on the perturbation of invariant measures. In this paper we use these tools to study the zero-noise limit from a quantitative point of view in two main cases: smooth expanding maps and piecewise expanding maps of the circle. In the following two subsections we enter in more details about our main results in these two cases.

**Smooth expanding maps, response and zero-noise limit.** We consider a smooth uniformly expanding map $T \in C^8(S^1 \to S^1)$, with its associated transfer operator $L_T : SM(S^1) \to SM(S^1)$, where $SM(S^1)$ is the space of finite Borel measures with sign on $S^1$, defined by

$$(L_T(\mu))(A) = \mu(T^{-1}(A))$$

for each signed measure $\mu \in SM(S^1)$. $L_T$ is also called the transfer operator associated to $T$ or pushforward map associated to $T$. We consider an i.i.d. random perturbation distributed according to a zero mean kernel $\rho \in BV([-1, 1])$. \forall \delta \in [0, \delta)$ we denote by $\rho_\delta$ the rescaling of $\rho$ with amplitude $\delta$ defined by

$$\rho_\delta(x) = \frac{1}{\delta} \rho\left(\frac{x}{\delta}\right).$$

The transfer operator associated to the randomly perturbed map is then defined as

$L_\delta = \rho_\delta * L_T$

where $*$ stands for the ordinary convolution operator on $S^1$. Note that we can extend this definition to $\delta = 0$ with $\rho_0 = \delta_0$ the Dirac mass. It can be proved that (see Section 2) each operator $L_\delta$ has a unique fixed point $h_\delta$ in the Sobolev space $W^{7, 1}(S^1)$ and hence $h_\delta$ is the stationary measure of the perturbed system.

The idea is then to prove that this family of operators admits a linear, and even quadratic response when $\delta$ tends to 0. In particular, we prove and extend the numerical findings shown in [25], in which the author predicted a convergence of order $\delta^2$. We will precise the coefficients of the order two Taylor’s expansion of this zero-noise limit proving the following theorem:

**Theorem 1.1** (Quadratic response in the zero-noise limit for a smooth uniformly expanding map). Let $T \in C^8(S^1, S^1)$ be a smooth uniformly expanding map of the circle perturbed by additive noise distributed with a bounded variation density $\rho_\delta$ as described above. Let $h_\delta$ the stationary measure of the associated random dynamical system. Then the map $\delta \mapsto h_\delta$ has an order two Taylor’s expansion at $\delta = 0$, with

$$\left\| \frac{h_\delta - h_0}{\delta^2} - \frac{\sigma^2(\rho)}{2} (Id - L_T)^{-1} h_0'' \right\|_{W^{2, 1}} \to 0,$$

where $\sigma^2(\rho) = \int_{-1}^1 x^2 \rho(x) dx$ and $h_0''$ is the second derivative of the invariant measure of $T$.

We remark that in the above statement we require the noise distribution kernel $\rho$ only to be of bounded variation, allowing discontinuities, as it happen in the case of uniformly distributed noise.

Next Section 2 is essentially devoted to the proof of this result. We prove the theorem by the application of some general linear and quadratic response statements we recall in subsection 2.1. In subsection 2.2 we verify the several assumptions needed to apply those theorems, completing the proof at the end of Section 2.
Piecewise expanding maps, quantitative stability and zero-noise limit.

We have seen that for smooth expanding maps there is a quadratic speed of convergence in the zero-noise limit. This depend both on the smoothness and on the strong chaoticity of the system. When having less regularity in the map and in the invariant desities, the speed of convergence changes. In the second part of the paper we consider indeed piecewise expanding maps, allowing discontinuities. In this case we have systems still having strong chaoticity, and exponential decay of correlations, but the speed of convergence in the zero limit is of order 1 in some sense. We prove in fact the following

**Proposition 1.** Let $T : S^1 \to S^1$ be a topologically mixing piecewise expanding map having no periodic turning points (see Section 3 for the precise definitions). Suppose we perturb the associated dynamical system with noise of amplitude $\delta$ as above. Let $L_\delta$ be the associated transfer operators and let $h_\delta$ be a family of invariant measures for $L_\delta$. Then $h_\delta \in Lip[0,1]$ and there is $C \geq 0$ such that for each $\delta \in [0,\delta_0)$

$$||h_0 - h_\delta||_{L^1} \leq C\delta \log \delta.$$ 

Furthermore, there are examples of piecewise expanding maps (with periodic turning points) for which there is a constant $C'$ such that for each $\delta \in [0,\delta_0)$,

$$||h_0 - h_\delta||_{L^1} \geq C'\delta.$$ 

2. Quadratic response and the zero-noise limit of uniformly expanding maps.

In this section we consider the zero-noise limit of uniformly expanding maps on the circle. We get precise estimates on the speed of convergence of this limit, proving Theorem 1.1.

In this section we will consider maps $T : S^1 \to S^1$ satisfying the following assumptions

1. $T \in C^8(S^1)$,
2. $|T'(x)| \geq \alpha^{-1} > 1 \forall x \in S^1$.

To $T$ is associated a linear map $L_T : SM(S^1) \to SM(S^1)$ as defined in the introduction. We consider a perturbation of this transfer operator by adding to the deterministic dynamics generated by $T$ a random independent and identically distributed perturbation: the noise. In other words we consider a random dynamical system, corresponding to the stochastic process $(X_n)_{n \in \mathbb{N}}$ defined by

$$X_{n+1} = T(X_n) + \Omega_n \mod 1 \quad (6)$$

where $(\Omega_n)_{n \in \mathbb{N}}$ are i.i.d random variables distributed according to a zero average probability density $\rho_\delta$ (the noise kernel) where $\delta$ represent the “size” of the perturbation. We suppose that $\rho_\delta$ is obtained by rescaling a certain zero average probability density $\rho \in BV([-1,1])$, satisfying

$$\int_{-1}^1 \rho(z)dz = 1 \quad \int_{-1}^1 \rho(z)zdz = 0 \quad \int_{-1}^1 \rho(z)z^2dz := \sigma^2(\rho). \quad (7)$$

as follows

$$\rho_\delta(x) := \frac{1}{\delta}\rho\left(\frac{x}{\delta}\right). \quad (8)$$

For each $\delta \in (0,1]$. The (annealed) transfer operator associated to the perturbed random system is then defined by

$$L_\delta = \rho_\delta * L_T \quad (9)$$
where * is the convolution operator on $S^1$ (see e.g. \cite{34}, Section 5 or \cite{19}, Section 8 for more details on the definition of the annealed transfer operator).

Remark that for each $f \in L^1(S^1)$, one has for almost every $x \in S^1$:

$$(\rho(x) - \delta_0) \ast f(x) = \frac{1}{\delta} \int_{-\delta}^{\delta} \rho(y) f(x-y) dy - f(x) = \int_{-1}^{1} \rho(z) f(x-\delta z) dz - f(x) \quad (10)$$

where $\delta_0$ is the Dirac measure placed at 0.

To keep the notation compact we will denote $L_0 := L_T$. We remark that the invariant measures of the map $T$ are fixed points of $L_0$ and the stationary measures of the random system constructed by the addition of the noise are fixed points of $L_\delta$. We are interested in the properties of these measures and how they vary as $\delta$ goes to 0. Their characterization as fixed points of $L_\delta$ will be sufficient for our purposes. We recall that in the case we are considering (expanding maps) there can be a large set of invariant measures for the deterministic map $T$ but only one which is absolutely continuous with respect to the Lebesgue measure. On the other hand the stationary measures for $L_\delta$ will always be absolutely continuous. This is well known (see e.g. \cite{15}) but it can also be easily derived from the regularization estimates we prove in the following.

2.1. General linear response and quadratic response results. In this subsection we state some general results from \cite{20} about linear and quadratic response of fixed points of Markov operators under suitable perturbations. These results will be applied to our zero-noise limit and to the operators $L_\delta$ to get precise estimates on the speed of convergence of $h_\delta$ to $h_0$.

In the following we consider four normed vectors spaces of signed Borel measures on $S^1$. The spaces $(B_{ss}, \| \|_{ss}) \subseteq (B_s, \| \|_s) \subseteq (B_w, \| \|_w) \subseteq (B_{ww}, \| \|_{ww}) \subseteq BS(S^1)$ with norms satisfying

$$\| \|_{ww} \leq \| \|_w \leq \| \|_s \leq \| \|_{ss}.$$  

We will assume that the linear form $\mu \mapsto \mu(S^1)$ is continuous on $B_i$, for $i \in \{ss, s, w, ww\}$. Since we will consider Markov operators acting on these spaces, the following (closed) spaces $V_{ss} \subseteq V_s \subseteq V_w \subseteq V_{ww}$ of zero average measures will play an important role. We define $V_i$ as:

$$V_i := \{ \mu \in B_i \mid \mu(S^1) = 0 \}$$

where $i \in \{ss, s, w, ww\}$. Suppose hence we have a one parameter family of Markov operators $L_\delta$. The following is a general statement (see \cite{20}, Theorem 1) establishing linear response for suitable perturbations of such operators.

**Theorem 2.1 (Linear Response).** Suppose that the family of bounded Markov operators $L_\delta : B_i \to B_i$, where $i \in \{ss, s, w\}$ satisfy the following:

(LR1) (regularity bounds) for each $\delta \in [0, \bar{\delta})$ there is $h_\delta \in B_{ss}$, a probability measure such that $L_\delta h_\delta = h_\delta$. Furthermore, there is $M \geq 0$ such that for each $\delta \in [0, \bar{\delta})$

$$\|h_\delta\|_{ss} \leq M.$$

(LR2) (convergence to equilibrium for the unperturbed operator) There is a sequence $a_n \to 0$ such that for each $g \in V_{ss}$

$$\|L_0^\delta g\|_s \leq a_n \|g\|_{ss};$$

1A Markov operator is a linear operator $L$ preserving positive measures and such that for each positive measure $\mu$, it holds $[L(\mu)](X) = \mu(X)$. 

Remark that for each $g \in V_{ss}$, $g = \int g \rho d\mu + \sum_i a_n g_i$ for some positive measures $\mu$, $\rho$ and $g_i$ with $\|g_i\|_{ss} \leq a_n \|g\|_{ss},$
(LR3) (resolvent of the unperturbed operator) \((\text{Id} - L_0)^{-1} := \sum_{i=0}^{\infty} L_i^0\) is a bounded operator \(V_w \rightarrow V_w\).

(LR4) (small perturbation and derivative operator) There is \(K \geq 0\) such that 
\[
\|L_0 - L_\delta\|_{B_w \rightarrow B_w} \leq K\delta, \quad \text{and} \quad \|L_0 - L_\delta\|_{B_{ss} \rightarrow B_{ss}} \leq K\delta.
\]
There is \(Lh_0 \in V_w\) such that 
\[
\lim_{\delta \rightarrow 0} \left\| \frac{(L_\delta - L_0)}{\delta} h_0 - \ddot{L}h_0 \right\|_w = 0.
\]  
(11)

Then we have the following Linear Response formula
\[
\lim_{\delta \rightarrow 0} \left\| h_\delta - h_0 - (\text{Id} - L_0)^{-1} \dot{L}h_0 \right\|_w = 0.
\]  
(12)

The following is an abstract response result for the second derivative (see [20], Theorem 7).

**Theorem 2.2** (Quadratic term in the response). Let \((L_\delta)_{\delta \in [0, \delta]} : B_i \rightarrow B_i, \; i \in \{ss, \ldots, ww\}\) be a family of Markov operators as in the previous theorem. Assume furthermore that:
- (QR1) The derivative operator \(\dot{L}\) can be defined as \(\dot{L} : B_w \rightarrow V_{ww}\), such that 
\[
\lim_{\delta \rightarrow 0} \left\| \frac{(L_\delta - L_0)}{\delta} h_0 - \ddot{L}h_0 \right\|_{w \rightarrow w} = 0.
\]  
(13)

- (QR2) There exists a “second derivative operator” at \(h_0\), i.e. \(\dddot{L}h_0 \in V_{ww}\) such that 
\[
\lim_{\delta \rightarrow 0} \left\| \frac{(L_\delta - L_0)h_0 - \delta \ddot{L}h_0}{\delta^2} - \dddot{L}h_0 \right\|_{w \rightarrow w} = 0.
\]  
(14)

- (QR3) The resolvent operator \((\text{Id} - L_0)^{-1}\) admits a bounded extension as an operator \(V_{ww} \rightarrow V_{ww}\).

Then one has the following: the map \(\delta \in [0, \delta] \mapsto h_\delta \in B_{ss}\) has an order two Taylor’s expansion at \(\delta = 0\), with
\[
\lim_{\delta \rightarrow 0} \left\| \frac{h_\delta - h_0 - \delta(\text{Id} - L_0)^{-1} \dot{L}h_0}{\delta^2} - (\text{Id} - L_0)^{-1} \left[ \dddot{L}h_0 + \dddot{L}(\text{Id} - L_0)^{-1} \dot{L}h_0 \right] \right\|_{w \rightarrow w} = 0.
\]  
(15)

Given the family of transfer operators \(L_\delta\) defined at 9, we will apply these response results using the sequence of stronger and weaker spaces

\[ W^{7,1}(S^1) \subset W^{5,1}(S^1) \subset W^{3,1}(S^1) \subset W^{1,1}(S^1) \]

where \(W^{k,1}\) stands for the Sobolev space of functions having the \(k^{th}\) derivative in \(L^1\) (see [1] for an introduction to these spaces).

In the following subsection we verify the assumptions needed to apply theorems 2.1 and 2.2. Theorem 1.1 will be proved at the end of the section.

### 2.2. Verifying the assumptions in the general response theorems.

In this subsection we verify the assumptions needed to apply theorems 2.1 and 2.2. First we verify the spectral gap and existence of the resolvent assumptions for the unperturbed system. This is somewhat well known for circle expanding maps. However for completeness we recall the main steps of this construction in subsection 2.2.1. In subsection 2.2.2 we prove a uniform Lasota Yorke inequality, to verify the assumption LR1. In subsection 2.2.3 we compute the first derivative operator associated to...
the small-noise perturbation, verifying assumptions LR4 and QR1. In subsection 2.2.4 we compute the second derivative operator, verifying assumption QR2.

2.2.1. Spectral gap and resolvent for the unperturbed operator (verifying LR2, LR3 and QR3). In this section we consider the tranfer operator $L_0$ of the unperturbed system acting on our Sobolev spaces and verify the convergence to equilibrium and the existence of the resolvent operator $(Id - L_0)^{-1}$ on the weak and weakest spaces $W^{3,1}(\mathbb{S}^1)$, $W^{1,1}(\mathbb{S}^1)$ as required in assumptions LR2, LR3 and QR3. Since we are considering the transfer operator associated to an expanding map of the circle, these results are nowadays not surprising (see e.g. [26]). The results follow from a standard construction, in which one can get information on the spectrum of $L_0$ acting on these spaces from the existence of a Lasota Yorke inequality on suitable functional spaces which are compactly embedded each other. As there are some variants of this construction, for completeness we briefly recall some precise statements which we can apply to our case.

The following theorem (see [20], Theorem 9 for a proof of the statement in this form or [26] for similar results) is a version of a classical tool to obtain spectral gap in systems satisfying a Lasota Yorke inequality. It allows one to estimate the contraction rate of zero average measures, and imply spectral gap when applied to Markov operators. Let us consider a Markov operator $L_0$ acting on two normed vector spaces of complex or signed measures $(B_s, \|\|_s)$, $(B_w, \|\|_w)$, $B_s \subseteq B_w$ with $\|\|_s \geq \|\|_w$. We furthermore assume that $\mu \mapsto \mu(X)$ is continuous in the $\|\|_s$ and $\|\|_w$ topologies, and let $V_i := \{ \mu \in B_i, \mu(X) = 0 \}, i \in \{w, s\}$.

**Theorem 2.3.** Suppose:

1. (Lasota Yorke inequality). There are $A \geq 0, B \geq 0$ and $\lambda < 1$ such that for each $g \in B_s$

   $$\|L_0^n g\|_s \leq A\lambda^n \|g\|_s + B \|g\|_w;$$

2. (Mixing) for each $g \in V_s$, it holds

   $$\lim_{n \to \infty} \|L_0^n g\|_w = 0;$$

3. (Compact inclusion) The image of the closed unit ball in $B_s$ under $L_0$ is relatively compact in $B_w$.

Under these assumptions, we have

a: $L_0$ admits a unique fixed point in $h \in B_s$, satisfying $h(X) = 1$.

b: There are $C > 0, \rho < 1$ such that for all $f \in V_s$ and $m \geq 0$

   $$\|L_0^m f\|_s \leq C\rho^m \|f\|_s.$$  \(16\)

c: The resolvent $(Id - L_0)^{-1}: V_s \to V_s$ is defined and continuous.

To apply this result to our case we recall the following (a proof can be found in [20], Lemma 29).

**Lemma 2.4.** A $C^{k+1}$ expanding map on $\mathbb{S}^1$ satisfies a Lasota-Yorke inequality on $W^{k,1}(\mathbb{S}^1)$: there is $\alpha < 1$, $A_k, B_k \geq 0$ such that for all $n \geq 1$

$$\begin{cases}
\|L_0^n f\|_{W^{k-1,1}} \leq A_k \|f\|_{W^{k-1,1}} \\
\|L_0^n f\|_{W^{k,1}} \leq \alpha^n \|f\|_{W^{k,1}} + B_k \|f\|_{W^{k-1,2}}.
\end{cases}$$
By the compact immersion of $W^{k,1}$ in $W^{s,1}$ for all $k > s$ (see [1], Section 5)) and
the well-known fact that an expanding map satisfies the mixing assumption, we can
apply Theorem 2.3 to our transfer operator of a $C^8$ expanding map and deduce the
following result.

**Proposition 2.** For each $k \in \{1, 2, \cdots, 7\}$ there are $C > 0, \rho < 1$ such that for
each $g \in V_k$ it holds
\[
\| L^k g \|_{W^{k,1}} \leq C \rho^k \| g \|_{W^{k,1}}.
\]
In particular, the resolvent $(Id - L)^{-1} = \sum_{i=0}^{\infty} L^i$ is a well-defined and bounded
operator on $V_k$.

This is enough to verify assumptions LR2, LR3 and QR3 in our case.

2.2.2. Uniform Lasota Yorke inequalities in the zero-noise limit (verifying LR1).

In order to prove the assumption LR1, we use Theorem 2.3 and show a uniform
Lasota-Yorke inequality.

**Lemma 2.5.** \( \forall k \geq 0, f \in W^{k,1}(S^1), \)
\[
\| L_\delta f \|_{W^{k,1}} \leq \| L_T f \|_{W^{k,1}}.
\]

**Proof.** We first prove the general statement: \( \forall f \in L^1(S^1), \| \rho_\delta \ast f \|_{L^1} \leq \| f \|_{L^1}. \)

\( \forall x \in S^1, \) we have
\[
\rho_\delta \ast f(x) = \int_{-\delta}^{\delta} \rho_\delta(y) f(x - y) dy.
\]

Hence
\[
\| \rho_\delta \ast f \|_{L^1} \leq \int_{S^1} \int_{-\delta}^{\delta} \rho_\delta(y) |f(x - y)| dy dx = \int_{-\delta}^{\delta} \rho_\delta(y) dy \cdot \| f \|_{L^1} = \| f \|_{L^1}.
\]

Using the fact that \( \forall i \geq 0, \)
\[
(L_\delta f)^{(i)} = (\rho_\delta \ast L_T f)^{(i)} = \rho_\delta \ast (L_T f)^{(i)},
\]
we then have
\[
\| (L_\delta f)^{(i)} \|_{L^1} \leq \| (L_T f)^{(i)} \|_{L^1}.
\]

Hence
\[
\| L_\delta f \|_{W^{k,1}} = \sum_{i=0}^{k} \| (L_\delta f)^{(i)} \|_{L^1} \leq \| L_T f \|_{W^{k,1}}.
\]

\[\Box\]

**Lemma 2.6.** For $k \geq 1$, the family $(L_\delta)$ verifies a uniform Lasota-Yorke inequality
on $W^{k,1}(S^1)$, which is: there is $\alpha < 1, C_k, D_k \geq 0$ such that:
\[\begin{align*}
\| L_\delta^k f \|_{W^{k-1,1}} & \leq C_k \| f \|_{W^{k-1,1}} \\
\| L_\delta^k f \|_{W^{k,1}} & \leq \alpha^k \| f \|_{W^{k,1}} + D_k \| f \|_{W^{k-1,1}}.
\end{align*}\]

**Proof.** We will prove this lemma by induction on $k \geq 1$. $L_T$ is a contraction on $L^1$:
using Lemma 2.5, it is also the case for $L_\delta$, proving the power-boundedness on $L^1$
(with $C_1 = 1$).

Then, using Lemma 2.4, we know that there is a $B_1 \geq 0$ such that
\[
\| L_\delta f \|_{W^{1,1}} \leq \| L_T f \|_{W^{1,1}} \leq \alpha \| f \|_{W^{1,1}} + B_1 \| f \|_{1}.
\]
Applying this inequality to $L_\delta^2 f = L_\delta (L_\delta f)$ gives us
\[
\| L_\delta^2 f \|_{W^{1,1}} \leq \alpha \| L_\delta f \|_{W^{1,1}} + B_1 \| L_\delta f \|_1 \\
\leq \alpha^2 \| f \|_{W^{1,1}} + B_1 \| f \|_1 + B_1 \| L_\delta f \|_1.
\]
We can then iterate:
\[
\| L_\delta^n f \|_{W^{1,1}} \leq \alpha^n \| f \|_{W^{1,1}} + B_1 \sum_{i=0}^{n-1} \alpha^i \| L_\delta^{n-1-i} f \|_1 \\
\leq \alpha^n \| f \|_{W^{1,1}} + \frac{B_1 C_1}{1 - \alpha} \| f \|_1
\]
giving us the property for $k = 1$, with $C_1 = 1$ and $D_1 = \frac{B_1 C_1}{1 - \alpha}$.

The induction is then analogous to the base case. Using the induction hypothesis on $k-1$, more precisely the Lasota-Yorke inequality, we have the power-boundedness of $L_\delta$ on $W^{k-1,1}$:
\[
\| L_\delta^n f \|_{W^{k-1,1}} \leq (1 + D_{k-1}) \| f \|_{W^{k-1,1}}.
\]
Hence $C_k = 1 + D_{k-1}$. We can then use again Lemma 2.4 to have the first inequality
\[
\| L_\delta f \|_{W^{k,1}} \leq \alpha^k \| f \|_{W^{k,1}} + B_k \| f \|_{W^{k-1,1}},
\]
which we can iterate to
\[
\| L_\delta^n f \|_{W^{k,1}} \leq \alpha^{kn} \| f \|_{W^{k,1}} + B_k \sum_{i=0}^{n-1} \alpha^{ki} \| L_\delta^{n-1-i} f \|_{W^{k-1,1}}.
\]
We can finally use the power-boundedness result we just proved to conclude:
\[
\| L_\delta^n f \|_{W^{k,1}} \leq \alpha^{kn} \| f \|_{W^{k,1}} + \frac{B_k C_k}{1 - \alpha} \| f \|_{W^{k-1,1}}.
\]
Hence the result for $k$, with $C_k = (1 + D_{k-1})$ and $D_k = \frac{B_k C_k}{1 - \alpha}$.

We can then extend this inequality to our spaces, $W^{2k+1,1} (S^1)$.

**Corollary 1.** For $k \geq 2$, the family $(L_\delta)$ verifies a uniform Lasota-Yorke inequality on $W^{k,1} (S^1) \subset W^{k-2,1} (S^1)$: there is $\alpha < 1$, $E_k, F_k \geq 0$ such that for each $n \geq 0$ and $f \in W^{k,1}$
\[
\| L_\delta^n f \|_{W^{k,1}} \leq \alpha^{\frac{k-1}{2} n} E_k \| f \|_{W^{k,1}} + F_k \| f \|_{W^{k-2,1}}.
\]

**Proof.** By using Lemma 2.6, we have that $\forall n, p \geq 0$,
\[
\| L_\delta^{n+p} f \|_{W^{k,1}} \leq \alpha^{kn} \| L_\delta^n f \|_{W^{k,1}} + D_k \| L_\delta^p f \|_{W^{k-1,1}} \\
\leq \alpha^{kn} C_{k+1} \| f \|_{W^{k,1}} + D_k \left( \alpha^{(k-1)p} \| f \|_{W^{k-1,1}} + D_{k-1} \| f \|_{W^{k-2,1}} \right) \\
\leq \left( \alpha^{kn} C_{k+1} + \alpha^{(k-1)p} D_k \right) \| f \|_{W^{k,1}} + D_k D_{k-1} \| f \|_{W^{k-2,1}}.
\]
So in the case $p = n$ (an even exponent), we have
\[
\| L_\delta^n f \|_{W^{k,1}} \leq \alpha^{(k-1)n} \left( \alpha^n C_{k+1} + D_k \| f \|_{W^{k,1}} + D_k D_{k-1} \| f \|_{W^{k-2,1}} \right) \\
\leq \alpha^{\frac{k-1}{2} n} \left( C_{k+1} + D_k \right) \| f \|_{W^{k,1}} + D_k D_{k-1} \| f \|_{W^{k-2,1}}.
\]
And in the case $p = n+1$ (an odd exponent), we have
\[
\| L_\delta^{n+1} f \|_{W^{k,1}} \leq \alpha^{\frac{k-1}{2} (2n+1)} \left( \alpha^{n+\frac{k-1}{2}} C_{k+1} + \alpha^{\frac{k-1}{2}} D_k \right) \| f \|_{W^{k,1}} + D_k D_{k-1} \| f \|_{W^{k-2,1}} \\
\leq \alpha^{\frac{k-1}{2} (2n+1)} \left( \alpha^{n+\frac{k-1}{2}} C_{k+1} + D_k \right) \| f \|_{W^{k,1}} + D_k D_{k-1} \| f \|_{W^{k-2,1}}.
\]
Taking the maximum of the two constants that differ finishes the proof: because \( \alpha < 1 \), we can take \( E_k = C_k + D_k \) and \( F_k = D_k D_{k-1} \).

Using the compact embedding of \( W^{7,1}(S^1) \) into \( W^{5,1}(S^1) \) (by Rellich-Kondrachov embedding theorem, see [1], Section 5) and the Lasota-Yorke inequality we just proved, one can easily deduce assumption LR1 (an example of such reasoning can be found in [15]).

2.2.3. First derivative operator (verifying LR4 and QR1). In this subsection we prove LR4 and QR1. These assumptions concern the first derivative operator. We will first prove that this first derivative operator is zero.

**Lemma 2.7.** Let \((\rho_\delta)_\delta\) be the family of random kernels defined at (8). There exists \( C \geq 0 \) such that for all \( f \in W^{2,1}(S^1) \), the following inequality holds

\[
\left\| \frac{\rho_\delta - \delta_0}{\delta} * f \right\|_{L^1} \leq \delta \| f \|_{W^{2,1}} C.
\]

**Proof.** Let us use the following Taylor expansion for \( f \in W^{2,1}(S^1) \): for each \( x, z \) and small \( \delta \)

\[
f(x - \delta z) = f(x) - \delta zf'(x) - \int_{x-\delta z}^{x} (x - \delta z - t)f''(t)dt.
\]

Using (10), we have

\[
((\rho_\delta - \delta_0) * f)(x) = -\int_{-1}^{1} \int_{x-\delta z}^{x} \rho(z)(x - \delta z - t)f''(t)dtdz
\]

By using the substitution \( y = \frac{z-x}{\delta} \) in the last integral, we can re-write the result as

\[
((\rho_\delta - \delta_0) * f)(x) = \delta^2 \int_{-1}^{1} \int_{0}^{z} \rho(z)(y-z)f''(x - \delta y)dydz
\]

In particular,

\[
\left\| \frac{\rho_\delta - \delta_0}{\delta} * f \right\|_{L^1} \leq \delta \int_{S^1} \int_{-1}^{1} |z-y| |f''(x - \delta y)|dydzdx
\]

\[
= \delta \int_{-1}^{1} \int_{0}^{z} \rho(z)|z-y| \left( \int_{S^1} |f''(x - \delta y)|dx \right) dydz
\]

\[
\left\| \frac{\rho_\delta - \delta_0}{\delta} * f \right\|_{L^1} \leq \delta \| f \|_{W^{2,1}} \int_{-1}^{1} \int_{0}^{z} \rho(z)|z-y|dydz
\]

\[
= C
\]

We use this lemma to prove QR1 with a zero first derivative operator.

**Proposition 3.** Let \((L_\delta)_\delta\) be the family of operators defined at (9). The following limit, defining the first derivative operator holds

\[
\lim_{\delta \to 0} \left\| \frac{L_\delta - L_0}{\delta} \right\|_{W^{3,1} \to W^{1,1}} = 0.
\]
Proof. Let us consider \( f \in W^{3,1}(\mathbb{S}^1) \), we get
\[
\left\| \frac{L_\delta - L_0}{\delta} f \right\|_{W^{1,1}} = \left\| \frac{\rho_\delta - \rho_0}{\delta} \ast (L_T f) \right\|_{W^{1,1}} = \left\| \frac{\rho_\delta - \rho_0}{\delta} \ast (L_T f) \right\|_{L^1} + \left\| \frac{\rho_\delta - \rho_0}{\delta} \ast (L_T f)' \right\|_{L^1}
\leq \delta C \left( \|L_T f\|_{W^{2,1}} + \|(L_T f)'\|_{W^{2,1}} \right)
\leq 2\delta C \|L_T f\|_{W^{3,1}}
\leq 2\delta C \|L_T\|_{W^{3,1} \to W^{3,1}} \|f\|_{W^{3,1}}.
\]
The operator norm is then bounded by \( 2\delta C \|L_T\| \), which tends to 0 when \( \delta \) does.

To finish verifying assumption LR4: we can remark that \( \forall k \geq 0 \),
\[
\left\| (\rho_\delta - \rho_0) \ast f \right\|_{W^{1,1}} = \sum_{i=0}^{k} \left\| (\rho_\delta - \rho_0) \ast f \right\|_{L^1}^{(i)} \leq \delta^2 C \sum_{i=0}^{k} \left\| f^{(i)} \right\|_{W^{2,1}} \leq \delta^2 C (k+1) \|f\|_{W^{k+2,1}}.
\]
So \( \forall k \geq 0 \),
\[
\left\| (L_\delta - L_0) f \right\|_{W^{k,1}} \leq \delta C (k+1) A_{k+3} \|f\|_{W^{k+2,1}}
\]
with \( A_{k+3} \) the constant from Lemma 2.4. We then have LR4, with the result for \( k \in \{3,5\} \):
\[
\left\| L_\delta - L_0 \right\|_{W^{k+2,1} \to W^{k,1}} \leq \delta C (k+1) A_{k+3}.
\]

2.2.4. Second derivative operator (verifying QR2). In this subsection we prove assumption QR2, computing the second derivative operator and showing its relation with the variance of \( \rho \).

**Lemma 2.8.** Let \( (\rho_\delta) \) be the family of random kernels described in 8. Then there exists \( C > 0 \) such that for all \( f \in W^{3,1}(\mathbb{S}^1) \), the following inequality holds
\[
\left\| \frac{\rho_\delta - \rho_0}{\delta^2} \ast f - \frac{\sigma^2(\rho)}{2} f'' \right\|_{L^1} \leq \delta C \|f\|_{W^{3,1}}.
\]

**Proof.** We can extend for \( f \in W^{3,1}(\mathbb{S}^1) \) the Taylor expansion (17):
\[
f(x - \delta z) = f(x) - \delta z f'(x) + \frac{\delta^2 z^2}{2} f''(x) - \int_{x-\delta z}^{x} \frac{(x-\delta z-t)^2}{2} f^{(3)}(t) dt.
\]
We then have
\[
((\rho_\delta - \rho_0) \ast f)(x) = \delta^2 \frac{f''(x)}{2} \sigma^2(\rho) - \int_{-1}^{1} \int_{x-\delta z}^{x} \rho(z) \frac{(x-\delta z-t)^2}{2} f^{(3)}(t) dt dz.
\]
By using the substitution \( y = \frac{z-t}{\delta} \) in the last integral, we can re-write the result as
\[
((\rho_\delta - \rho_0) \ast f)(x) = \delta^2 \frac{f''(x)}{2} \sigma^2(\rho) + \frac{\delta^3}{2} \int_{-1}^{0} \rho(z)(z-y)^2 f^{(3)}(x-\delta y) dy dz.
\]
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\[
\left( \frac{\rho \delta - \delta_0}{\delta^2} \ast f \right)(x) = \frac{f''(x)}{2} \sigma^2(\rho) + \frac{\delta}{2} \int_{-1}^{1} \int_{z}^{0} \rho(z)(z - y)^2 f^{(3)}(x - \delta y) dy dz
\]

i.e.

\[
\int_{-1}^{1} \int_{z}^{0} \rho(z)(z - y)^2 f^{(3)}(x - \delta y) dy dz = R_1(x, \delta) + R_2(x, \delta)
\]

Once again we can use Fubini theorem to exchange the last integrals:

\[
\int_{-1}^{1} \int_{z}^{0} \rho(z)(z - y)^2 f^{(3)}(x - \delta y) dy dz = R_1(x, \delta) + R_2(x, \delta)
\]

with

\[
R_1(x, \delta) = \int_{-1}^{0} \int_{z}^{0} \rho(z)(z - y)^2 f^{(3)}(x - \delta y) dy dz
\]

\[
= \int_{-1}^{0} f^{(3)}(x - \delta y) \left( \int_{z}^{y} \rho(z)(z - y)^2 dz \right) dy
\]

and

\[
R_2(x, \delta) = \int_{0}^{1} \int_{z}^{0} \rho(z)(z - y)^2 f^{(3)}(x - \delta y) dy dz
\]

\[
= \int_{0}^{1} f^{(3)}(x - \delta y) \left( \int_{y}^{0} \rho(z)(z - y)^2 dz \right) dy.
\]

So

\[
\left( \frac{\rho \delta - \delta_0}{\delta^2} \ast f \right)(x) = \frac{f''(x)}{2} \sigma^2(\rho) + \frac{\delta}{2} \int_{-1}^{1} f^{(3)}(x - \delta y) \Omega(y) dy
\]

with

\[
\Omega(y) = \begin{cases} 
\frac{1}{y} \rho(z)(z - y)^2 dz & \text{if } y \geq 0 \\
\int_{-1}^{y} \rho(z)(z - y)^2 dz & \text{if } y < 0
\end{cases}
\]

We can then conclude that

\[
\left\| \frac{\rho \delta - \delta_0}{\delta^2} \ast f - \frac{\sigma^2(\rho)}{2} f'' \right\|_{L^1} \leq \frac{\delta}{2} \int_{-1}^{1} f^{(3)}(x - \delta y) \Omega(y) dx
\]

\[
\leq \frac{\delta}{2} \int_{-1}^{1} \Omega(y) |dy| \cdot \|f\|_{W^{3,1}}.
\]

\[\square\]

As in Subsection 2.2.3, we can apply this lemma to our problem, obtaining the following.

**Proposition 4.** Suppose \( T \) is a \( C^5 \) expanding map on the circle \( S^1 \). Let \( h_0 \in S^1 \), its invariant probability density and let \( L_\delta \) be the family of operators defined in (9) then the following holds

\[
\left\| \frac{L_\delta - L_0}{\delta^2} h_0 - \frac{\sigma^2(\rho)}{2} h_0'' \right\|_{W^{1,1}} \underset{\delta \to 0}{\longrightarrow} 0.
\]

**Proof.** Remark that because \( h_0 \) is the invariant probability measure of \( T \) and the property of derivation of a convolution product,

\[
\frac{(L_\delta - L_0)h_0}{\delta^2} = \frac{\rho \delta - \delta_0}{\delta^2} \ast h_0 \quad \text{and} \quad \left( \frac{(L_\delta - L_0)h_0}{\delta^2} \right)' = \frac{\rho \delta - \delta_0}{\delta^2} \ast h_0'.
\]
being a $C^5$ expanding map imply that $h_0$ is $C^4$ (see e.g. [24], Theorem 1): we can then apply our lemma to both $h_0$ and $h'_0$, giving us

$$\left\| \frac{(L_\delta - L_0)h_0}{\delta^2} - \frac{\sigma^2(\rho)}{2} h_0'' \right\|_{L^1} \leq \delta C \|h_0\|_{W^{3,1}}$$

and

$$\left\| \frac{(L_\delta - L_0)h_0}{\delta^2} - \frac{\sigma^2(\rho)}{2} h_0'' \right\|_{L^1} \leq \delta C \|h'_0\|_{W^{3,1}}.$$

The result then follows.

We have then verified the assumption QR2. Since all the assumptions are verified, we can hence apply Theorem 2.2 to the family of perturbed operators $L_\delta$, proving Theorem 1.1.

**Proof of Theorem 1.1.** We apply Theorem 2.2, with the spaces $B_i = W^{1,1}(S^1)$, with $i \in \{ss,s,w,ww\} = \{7,5,3,1\}$. We showed that our family of operator verifies the assumptions of both Theorem 2.1 and 2.2 in the previous subsections: assumptions LR2, LR3 and QR3 in subsection 2.2.1, LR1 in subsection 2.2.2, LR4 and QR1 in subsection 2.2.3, and QR2 in subsection 2.2.4.

3. **Quantitative zero-noise limit of piecewise expanding maps.** In this section we prove that for a certain family of piecewise expanding maps, the invariant densities in the zero-noise limit have a speed of convergence “of order at least about 1”, as stated more precisely in Proposition 1, confirming the numerical findings of [25]. In this paper, the author shows numerically one example of piecewise expanding map having a discontinuous invariant density, where the speed of convergence is of order 1. This is due to the presence of discontinuities in the map and in the corresponding invariant densities. We remark that the smooth expanding maps considered in the previous section can be considered as particular examples of piecewise expanding maps, hence in this family, speed of convergence of order 2 is possible. The proof of Proposition 1 is composed of three parts: in section 3.1.1 we introduce the concept of Uniform Family of Operators and state their link with the speed of convergence to equilibrium. We then show that the family of perturbations we consider in the small noise limit is uniform in this sense. Finally, in section 3.3 we show a lower bound on the speed of convergence, based on the approximation of a discontinuity by Lipschitz functions.

3.1. **Upper bounds: Convergence to equilibrium and stability.** In this section we provide the upper bounds sufficient to prove Proposition 1. We start by defining the class of maps we mean to consider.

**Definition 3.1.** A map $T : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be piecewise $C^2$ if there exists a finite set of points $d_1 = 0 < d_2 < \ldots < d_n = 1$ such that for each $1 \leq i < n$, $T_i := T(d_i,d_{i+1})$ extends to a $C^2$ function on the closure. Its expanding constant is defined as $\lambda_T = \inf_{i,x \in [d_i,d_{i+1}]} |T'(x)|$.

A piecewise $C^2$ map is called piecewise expanding if there is an integer $k > 0$ such that $\lambda_{T^k} > 1$, where $T^k$ is the $k$th iterate of $T$.

We will consider piecewise expanding maps which are mixing for their physical invariant measure, to ensure this we will define the following (see [35], assumption E3)
Definition 3.2. We say that a piecewise expanding map $T$ is topologically mixing if for each interval $J \subset S^1$ there is $n \geq 1$ such that $T^n(J) = S^1$.

We refer to [35], section 3 for explanations on how this property can be verified in concrete examples. In this subsection we will consider maps which are piecewise expanding and topologically mixing.

Definition 3.3. A turning point of a map $T$ is a point where the derivative of the map is not well defined.

3.1.1. Uniform family of operators, exponential convergence to equilibrium and quantitative statistical stability. In this subsection we present a general quantitative result relating the stability of the invariant measure of an uniform family of operators and the speed of convergence to equilibrium.

Let $L$ be a Markov operator acting on two vector subspaces of signed measures on $S^1$, $L : (B_s, \| \|_s) \rightarrow (B_s, \| \|_s)$ and $L : (B_w, \| \|_w) \rightarrow (B_w, \| \|_w)$, endowed with two norms, $\| \|_s$ on $B_s$, and $\| \|_w$ on $B_w$, such that $\| \|_s \geq \| \|_w$. As before we will assume that the linear form $\mu \mapsto \mu(S^1)$ is continuous on $B_i$, for $i \in \{s, w\}$.

Suppose that, $B_s \subseteq B_w \subseteq BS(S^1)$, where again $BS(S^1)$ denotes the space of Borel finite signed measures on $S^1$. Let us consider again the space of zero average measures

$$V_s = \{ f \in B_s, f(S^1) = 0 \}.$$  \hfill (19)

This space is preserved by any Markov operator.

We say that $L$ has convergence to equilibrium with at least speed $\Phi$ and with respect to the norms $\| \|_s$ and $\| \|_w$, if for each $f \in V_s$ it holds

$$\| L^n f \|_w \leq \Phi(n) \| f \|_s,$$  \hfill (20)

where $\Phi(n) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 3.4. A one parameter family of transfer operators $\{L_\delta\}_{\delta \in [0, \bar{\delta}]}$ is said to be an uniform family of operators with respect to the weak space $(B_w, \| \|_w)$ and the strong space $(B_s, \| \|_s)$ if it satisfies

UF1 There is $M > 0$ such that for for each $\delta \in [0, \bar{\delta})$ and each $h_\delta \in B_s$ being a probability measure fixed under the operator $L_\delta$, it holds

$$\| h_\delta \|_s \leq M;$$

UF2 $L_\delta$ approximates $L_0$ when $\delta$ is small in the following sense: there is $C \in \mathbb{R}^+$ such that:

$$\|(L_0 - L_\delta)h_\delta\|_w \leq \delta C;$$

UF3 $L_0$ has exponential convergence to equilibrium with respect to the norms $\| \|_s$ and $\| \|_w$: there exists $0 < \rho_2 < 1$ and $C_2 > 0$ such that $\forall \ f \in V_s$ it holds

$$\|L_0^n f\|_w \leq \rho_2^n C_2 \| f \|_s;$$

UF4 The iterates of the operators are uniformly bounded for the weak norm: there exists $M_2 > 0$ such that

$$\forall \delta \in [0, 1), n \geq 1, g \in B_s \text{ it holds } \| L_\delta^n g \|_w \leq M_2 \| g \|_w.$$
Under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when $L_0$ is perturbed to $L_\delta$, for small values of $\delta$. Moreover, the modulus of continuity can be estimated. The following result was indeed proved in [18] (Proposition 17).

**Proposition 5.** Suppose $\{L_\delta\}_{\delta \in [0,1)}$ is an uniform family of operators as in Definition 3.4, where $h_0$ is the unique fixed point of $L_0$ in $B_w$ and $h_\delta$ is a fixed point of $L_\delta$. Then, there exists $C \geq 0$ and $\delta_0 \in (0,1)$ such that for all $\delta \in [0,\delta_0)$, it holds

$$||h_\delta - h_0||_{w} = C\delta \log \delta.$$ 

It is worth to remark that such a statement can be generalized to other speed of convergence to equilibrium, obtaining for example Hölder bounds to the statistical stability of systems having a power law speed of convergence to equilibrium (see [17],[16]).

In the next section, we will prove that our small noise perturbation gives us a uniform family of operator. We can then apply Proposition 5 to our family to prove an upper bound on the speed of convergence of invariant densities. Note that it does not give us a purely linear upper bound $O(\delta)$; however a convergence of the order $\delta \log \delta$ (up to a multiplicative constant) would still give an exponent 1 if extracted as a power law behavior:

$$\lim_{\delta \to 0} \frac{\log ||h_\delta - h_0||_{L^1}}{-\log(\delta)} = 1.$$ 

### 3.2. Proof that the small noise perturbation gives a uniform family of operators.

In this subsection we verify that the family of operators we consider in the small noise perturbation of our piecewise expanding maps are a uniform family of operators, considering $BV(S^1)$ and $L^1(S^1)$ as a strong and weak space.

#### 3.2.1. $UF3$ and $UF4$. Assumption $UF4$ is immediate, as transfer operators are contractions on $L^1$. As showed earlier, we have that for all $f \in L^1$, $||\rho_\delta * f||_{L^1} \leq ||f||_{L^1}$. $L_T$ being a contraction on $L^1$, we then have

$$||L_\delta f||_{L^1} \leq ||L_T f||_{L^1} \leq ||f||_{L^1}.$$ 

$L_\delta$ is then also a contraction on $L^1$, hence the result for all $n$: $||L^n_\delta f||_{L^1} \leq ||f||_{L^1}$.$^{[a]}$

Assumption $UF3$ is verified with the spaces $BV$ and $L^1$ since it is well known that for topologically mixing piecewise expanding maps, there is a unique absolutely continuous invariant measure and spectral gap on $BV$ (see [35], section 3 and [15], Section 9).

#### 3.2.2. $UF2$. We first prove a similar result, but only for smooth functions. The calculations are basically the same as the ones we had for the derivative operator in the smooth expanding maps case.

**Lemma 3.5.** *There exists a $C > 0$ such that for all $f \in C^\infty$ and $\delta \in [0,1)$,*

$$||\rho_\delta * f - f||_{L^1} \leq C ||f'||_{L^1} \delta.$$ 

Proof.
\[
\|\rho_\delta * f - f\|_{L^1} = \int_{S^1} \frac{1}{\delta} \left| \int_{-\delta}^{\delta} \rho \left( \frac{y}{\delta} \right) [f(x - y) - f(x)]dy \right| dx \\
\leq \int_{-\delta}^{\delta} \frac{1}{\delta} \rho \left( \frac{y}{\delta} \right) \left( \int_{S^1} \left| \int_{0}^{y} f'(x - t)dt \right| dx \right) dy \\
\leq \|f'\|_{L^1} \cdot \int_{-\delta}^{\delta} \frac{1}{\delta} \rho \left( \frac{y}{\delta} \right) |y|dy \\
\|\rho_\delta * f - f\|_{L^1} \leq \|f'\|_{L^1} \cdot \delta \cdot \int_{-1}^{1} \rho(z)|z|dz = C
\]

To extend the result for all BV functions, we will use the following lemma, whose proof can be found in [14]. Not all bounded variation functions can be approximated by a smooth function with a small error in the BV norm; however we can still approximate BV functions by smooth ones in a weaker sense.

**Lemma 3.6.** For all \( f \in BV \), there exists a sequence \( (f_n) \in (C^\infty \cap BV)^\mathbb{N} \) such that

\[
\begin{align*}
\|f - f_n\|_{L^1} &\rightarrow 0 \\
\text{Var}(f_n) &\rightarrow \text{Var}(f)
\end{align*}
\]

We can then extend our result from Lemma 3.5.

**Proposition 6.** There exists a \( C > 0 \) such that for all \( f \in BV \) and \( \delta \in [0, 1) \),

\[
\|\rho_\delta * f - f\|_{L^1} \leq C \text{Var}(f) \delta.
\]

**Proof.** Lemma 3.5 gives us the result for all \( f \in C^\infty \). Indeed, for them \( \text{Var}(f) = \|f'\|_{L^1} \). Now let \( g \) be a BV function, and \( \epsilon > 0 \) be arbitrarily small. Let us prove

\[
\|\rho_\delta * g - g\|_{L^1} \leq C \text{Var}(g) \delta + \epsilon.
\]

with \( C \) the same constant from Lemma 3.5.

Using Lemma 3.6, there is a \( f \in C^\infty \) such that

\[
\begin{align*}
\|f - g\|_{L^1} &\leq \frac{\epsilon}{3} \\
\text{Var}(f) &\leq \text{Var}(g) + \frac{\epsilon}{3}\delta
\end{align*}
\]

We then have

\[
\|\rho_\delta * g - g\|_{L^1} \leq \|\rho_\delta * (g - f)\|_{L^1} + \|g - f\|_{L^1} + \|\rho_\delta * f - f\|_{L^1} \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + C\|f'\|_{L^1} \delta \\
= \frac{2\epsilon}{3} + C \text{Var}(f) \delta \\
\|\rho_\delta * g - g\|_{L^1} \leq \epsilon + C \text{Var}(g) \delta
\]

We then have **UF2** assuming **UF1**: indeed, because \( L_T h_\delta \in BV \), we can use the result from Proposition 6 as:

\[
\|(L_\delta - L_0)h_\delta\|_{L^1} = \|\rho_\delta * L_T h_\delta - L_T h_\delta\|_{L^1}
\]
Let $\delta = (L_0 - L)\delta \leq C \Var(L_T h_\delta)\delta$
\begin{align*}
\leq C \|L_T\|_{BV \to BV} h_\delta \|_{BV} \delta
\end{align*}
with $M$ the constant of UF1.

3.2.3. UF1. To prove the strong boundedness of the family of BV functions \{h_\delta\}, we can use a uniform L-Y inequality on $L_\delta$.

Remark 1. In Section 2, the unperturbed operator $L_0$ verified a Lasota-Yorke inequality of type
\begin{align*}
\|L_0^n f\|_s \leq \alpha^n \|f\|_s + C \|f\|_w,
\end{align*}
where the proof is based on iterating the case $n = 1$: we proved the uniform Lasota-Yorke inequality using $\|L_\delta f\| \leq \|L_T f\|$. In the general case of a piecewise expanding map, we have (see [15])
\begin{align*}
\|L_0^n f\|_s \leq \alpha^n A \|f\|_s + B \|f\|_w.
\end{align*}
Proving a uniform Lasota-Yorke inequality is then more complex.

We use the result stated in [12], which was proved in [11].

Definition 3.7. A transition probability is a linear positive (sub)-Markovian operator $Q : L^1 \to L^1$ such that $\|Q\|_1 \leq 1$. $Q^*$ denote its dual operator on $L^\infty$. A transition probability can be represented via a (sub)-Markov transition kernel on $[0,1]$ into itself:
\begin{align*}
Q^* h(x) = \int h(y) Q(x,dy) \quad \text{and} \quad Q h(y) = \left( \frac{d}{dm}\int h(x)Q(x,.)m(dx) \right)(y).
\end{align*}
If $Q(x,.) \ll m$ for each $x$, we note $q(x,y) = \frac{d}{dm}Q(x,)(y)$.

Proposition 7. Let $T$ be a piecewise expanding map with no periodic turning point. Suppose it is perturbed by a family of transition probabilities \{Q_\delta\} (i.e. $L_\delta = Q_\delta L_T$) verifying the following assumptions:
\begin{align}
\text{(Small perturbation)} \quad d(Q_\delta) := \sup \{ \|Q_\delta f - f\|_{L^1} : \|f\|_{BV} \leq 1 \} \to 0. \tag{21}
\end{align}
\begin{align}
\text{(Locality)} \quad \forall x \in S^1, \forall A \text{ measurable s.t. } \text{dist}(x,A) > \delta, \quad Q_\delta(x,A) = 0 \tag{22}
\end{align}
\begin{align}
\text{(Regularity)} \quad \text{There is a constant } C \geq 0 \text{ s.t. } \forall f \in BV, \quad \Var(Q_\delta f) \leq \Var(f) + C \|f\|_{L^1}. \tag{23}
\end{align}
where $Q_\delta f$ represent the density of $A \to \int Q_\delta(x,A)f(x)dx$ with respect to the Lebesgue measure.

Then there exists constants $C \geq 0, \delta_0 > 0, \alpha < 1$ and $N \in \mathbb{N}$ such that
\begin{align*}
\Var(L_\delta^N f) \leq \alpha \Var(f) + C \|f\|_1
\end{align*}
$\forall \delta \leq \delta_0$ and $f \in BV$.

In the case of an additive noise distributed according the kernel $\rho_\delta$, the Markov kernels have densities $q_\delta$ with respect to the Lebesgue measure, defined as (with the subtraction on $S^1$)
\begin{align*}
Q_\delta(x,A) = \int_A q_\delta(x,y)dy \quad \text{with} \quad q_\delta(x,y) := \rho_\delta(y - x).
\end{align*}
Then $Q_\delta f = \rho_\delta * f$. The small perturbation assumption is a simple application of Proposition 6, as it gives us that $d(Q_\delta) \leq C\delta \to 0$. The locality assumption
is verified as the support of $\rho_\delta$ is included in the interval $[-\delta, +\delta]$. The regularity assumption is easily verified by our noise kernel via the following lemma.

**Lemma 3.8.** $\forall \delta$ and $f \in BV$, 

$$\text{Var}(\rho_\delta * f) \leq \text{Var}(f)$$ 

**Proof.** One equivalent definition of Var is the following:

$$\text{Var}(f) = \sup \left\{ \int_{\mathbb{S}^1} \phi'(x) f(x) dx \mid \phi \in C^1 \text{ s.t. } \|\phi\|_\infty \leq 1 \right\}$$

Let $\phi \in C^1$. We then have

$$\int_{\mathbb{S}^1} \phi' \cdot (\rho_\delta * f) dx = \int_{\mathbb{S}^1} \int_{-\delta}^{\delta} \frac{1}{\delta} \rho \left( \frac{y}{\delta} \right) \phi'(x) f(x - y) dy dx$$

$$= \int_{-\delta}^{\delta} \frac{1}{\delta} \rho \left( \frac{y}{\delta} \right) \left( \int_{\mathbb{S}^1} \phi'(x) f(x - y) dx \right) dy$$

$$= \int_{-\delta}^{\delta} \frac{1}{\delta} \rho \left( \frac{y}{\delta} \right) \left( \int_{\mathbb{S}^1} \phi'(\tilde{x} + y) f(\tilde{x}) d\tilde{x} \right) dy$$

$$\leq \int_{-\delta}^{\delta} \frac{1}{\delta} \rho \left( \frac{y}{\delta} \right) \text{Var}(f) dy$$

$$\int_{\mathbb{S}^1} \phi' \cdot (\rho_\delta * f) dx \leq \text{Var}(f) dy$$

Hence $\text{Var}(\rho_\delta * f) \leq \text{Var}(f)$. $\square$

Because our noise verifies all the assumptions, we can apply Proposition 7. Using the contracting property of $L_\delta$ on $L^1$, we easily deduce the following.

**Proposition 8.** Let $T$ be a piecewise expanding map with no periodic turning point. Then there exists constants $C, \delta, \alpha < 1$ and $N \in \mathbb{N}$ such that

$$\|L_\delta^N f\|_{BV} \leq \alpha \|f\|_{BV} + C \|f\|_{L^1}$$

$\forall \delta \leq \delta$ and $f \in BV$.

This can give us an uniform L-Y inequality.

**Proposition 9.** Under the same assumptions as before, we have that $\forall p \in \mathbb{N}$, $\delta \leq \delta$, $0 \leq k < N$, $f \in BV$,

$$\|L_\delta^{pN+k} f\|_{BV} \leq \alpha^p \|L_T\|_{BV \rightarrow BV}^k \|f\|_{BV} + \frac{C}{1 - \alpha} \|f\|_{L^1}$$

which then leads to

$$\|L_\delta^{N} f\|_{BV} \leq \alpha^n A \|f\|_{BV} + B \|f\|_{L^1} \quad \forall n \in \mathbb{N}.$$

**Proof.** The previous proposition gives us

$$\|L_\delta^{N} f\|_{BV} \leq \alpha \|f\|_{BV} + C \|f\|_{L^1}.$$

Using the same type of induction as in the proof of Lemma 2.6, we have the following result $\forall p \in \mathbb{N}$:

$$\|L_\delta^{pN} f\|_{BV} \leq \alpha^p \|f\|_{BV} + C \sum_{i=0}^{p-1} \alpha^i \|L_\delta^{p-1-i} f\|_{L^1} \leq \alpha^p \|f\|_{BV} + \frac{C}{1 - \alpha} \|f\|_{L^1}.$$
Note that using the $L^1$ contraction of the additive noise proved in Lemma 2.5, and the result of Lemma 3.8, we have that for all $f \in BV$, $\|L_\delta f\|_{BV} \leq \|L_T f\|_{BV}$. Then, $\forall k < N$, $\|L_\delta^k f\|_{BV} \leq \|L_T\|_{BV \to BV}^k \|f\|_{BV}$. We can then conclude that

$$\|L_\delta^{pN+k} f\|_{BV} \leq \alpha^p \|L_T^k f\|_{BV} + \frac{C}{1-\alpha} \|L_\delta^k f\|_{L^1} \leq \alpha^p \|L_T\|_{BV \to BV}^k \|f\|_{BV} + \frac{C}{1-\alpha} \|f\|_{L^1}. $$

Having proven a uniform Lasota-Yorke inequality, we can conclude that our family of operators verifies also assumption $UF1$. We then have proved that the dynamics resulting from a piecewise expanding maps of the circle with no periodic turning point perturbed by an additive noise have an upper bound on their modulus of continuity. More explicitly,

$$\|h_\delta - h_0\|_{L^1} = O(\delta \log \delta).$$

### 3.3. Lower bounds: Approximation of a discontinuity by Lipschitz functions.

Until now, we only proved an upper bound on the modulus of continuity. Here, we show examples of piecewise expanding map of the circle for which the speed of approximation in the zero-noise limit is in fact of order 1, providing the lower bound sufficient to prove the results summarized in Proposition 1. Let us consider the following map

$$T : x \mapsto \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq \frac{1}{2} \\
  2(1-x) & \frac{1}{2} \leq x \leq 1
\end{cases}. \quad (24)$$

It is a piecewise expanding map as defined in Definition 3.1, as $T^2$ verifies

$$T^2(x) = \begin{cases} 
  1 - 2x & 0 \leq x \leq \frac{1}{2} \\
  2(2x-1) & \frac{1}{2} \leq x \leq \frac{3}{4} \\
  2(1-x) + \frac{1}{2} & \frac{3}{4} \leq x \leq 1
\end{cases}. \quad (25)$$

One gets easily that $T$ has the following invariant density, which is discontinuous:

$$h_0 : x \mapsto \begin{cases} 
  \frac{2}{3} & 0 \leq x \leq \frac{1}{2} \\
  \frac{1}{3} & \frac{1}{2} < x \leq 1
\end{cases}. \quad (26)$$

This example has already been studied in [25], where the author numerically found linear speed of convergence in the zero-noise limit. Note that $T$ admits $\{0, \frac{1}{2}, 1\}$ as periodic turning points, we cannot apply the upper bound result proven in the previous section.

In this section we prove the following proposition

**Proposition 10.** Let $T$ be the map defined in (25) and $h_0$ be its invariant density, as in (26). Let $L_\delta$ be the annealed transfer operator of the system with noise as defined at (9) with $\rho \in BV[-1,1]$ and let $h_\delta \in L^1$ be an invariant density for $L_\delta$. Then there exists a constant $C \in \mathbb{R}$ such that

$$\|h_\delta - h_0\|_{L^1} \geq C\delta.$$

We prove the proposition by showing in section 3.3.1 that $h_\delta$ is Lipschitz and providing an estimate for its Lipschitz constant, showing that $h_\delta$ is $C'$-Lipschitz for some constant $C'$. Then in section 3.3.2 we prove that there is a $C''$ such that $\|f - h_0\|_{L^1} \geq C''/\alpha$ for any function $f$ which is $\alpha$-Lipschitz, completing the proof.
3.3.1. Estimating the Lipschitz constant of $h_\delta$. In this section we prove that under the assumptions of Proposition 10 for any $\delta > 0$, the invariant density $h_\delta$ of the perturbed system is $C'_0$-Lipschitz. This will be proved in Proposition 12.

Before the main proposition we need a technical lemma.

**Lemma 3.9.** For $f \in BV$, $h \geq 0$, we have
\[
\int |f(x+h) - f(x)| \, dx \leq \text{Var}(f) \cdot h.
\]

**Proof.** We first prove it for $f \in C^\infty \cap BV$:
\[
\int |f(x+h) - f(x)| \, dx = \int \left| \int f'(y) \chi_{x \leq y \leq x+h} \, dy \right| \, dx \\
\leq \int |f'(y)| \, dy \cdot h \\
= \text{Var}(f) \cdot h.
\]

Then for all $f \in BV$: let us set $\epsilon > 0$. Using Lemma 3.6, we can have $g \in C^\infty \cap BV$ such that $\|g - f\|_{L^1} \leq \epsilon$ and $\text{Var}(g) \leq \text{Var}(f) + \epsilon$. We then have
\[
\int |f(x+h) - f(x)| \, dx \\
\leq \int |f(x+h) - g(x+h)| + |g(x) - f(x)| + |g(x+h) - g(x)| \, dx \\
\leq 2\epsilon + \text{Var}(g)h \\
\leq (2 + h)\epsilon + \text{Var}(f)h.
\]

We have the inequality for all $\epsilon > 0$, so we have our result. \( \square \)

This lemma easily give us a Lipschitz constant for the convolution product.

**Proposition 11.** For all $f \in L^\infty$, the function $\rho_\delta * f$ is $\frac{\text{Var}(\rho)}{\delta} \cdot \|f\|_\infty$-Lipschitz.

**Proof.** We use the first lemma to write that, for all $x \in S^1$, $h \geq 0$,
\[
|\rho_\delta * f(x+h) - \rho_\delta * f(x)| \leq \int |f(y)| \cdot |\rho_\delta(x-y+h) - \rho_\delta(x-y)| \, dy \\
\leq \|f\|_\infty \text{Var}(\rho_\delta) h.
\]

Since
\[
\text{Var}(\rho_\delta) = \frac{\text{Var}(\rho)}{\delta}
\]
we get the conclusion. \( \square \)

We now want to use this result to bound the Lipschitz constant of $h_\delta$, the invariant density of the perturbed system.

**Proposition 12.** There is a $C' > 0$ such that for all $\delta > 0$, the invariant density of the perturbed system $h_\delta$ is $C'_0$-Lipschitz.

**Proof.** By definition, $h_\delta = L_\delta h_\delta = \rho_\delta * L_T h_\delta$. Proposition 11 gives us that $h_\delta$ is $\frac{\text{Var}(\rho)}{\delta} \|L_T h_\delta\|_\infty$-Lipschitz. Another well known result is the existence of a constant $A > 0$ such that for all $f \in BV(S^1)$, $\|f\|_\infty \leq A\|f\|_{BV}$. Hence:
\[
\|L_T h_\delta\|_\infty \leq A\|L_T h_\delta\|_{BV} \\
\leq AB\|h_\delta\|_{BV} \quad \text{because } L_T \text{ is bounded on } BV \\
\leq ABM \quad \text{by property UF1 proven earlier.}
\]
We then have our result, as all the constants are independent from $\delta$.

3.3.2. Approximation of a discontinuity. We prove here the lower bound on the approximation of $h_0$ by $a$-Lipschitz functions, with $a > 0$ fixed.

Recall that $h_0$ is defined as (Figure 1)

\[ h_0 : x \mapsto \begin{cases} \frac{2}{3} & 0 \leq x \leq 0.5 \\ \frac{4}{3} & 0.5 < x \leq 1 \end{cases}. \]

The intuitive “best approximation” function that is $a$-Lip would then be the linear path,

\[ f_a : x \mapsto \begin{cases} \frac{2}{3} & x \leq 0.5 \frac{1}{3a} \\ 1 + ax - \frac{a}{2} & 0.5 \frac{1}{3a} \leq x \leq 0.5 + \frac{1}{3a} \\ \frac{4}{3} & x \geq 0.5 + \frac{1}{3a} \end{cases}. \]

We now prove that this is the best approximation in $L^1$, in the sense of the following proposition.

**Figure 1.** Lipschitz approximation of a discontinuity, graphical representation of $h_0$ and $f_a (a = 3)$.

Proposition 13. Let $f$ be a real-valued $a$-Lipschitz function of $[0, 1]$. The following inequality holds:

\[ \|f - h_0\|_{L^1} \geq \|f_a - h_0\|_{L^1} = \frac{1}{9a}. \]

**Proof.** The first step is to only consider the difference in the neighborhood of the discontinuity where $f_a \neq h_0$: for all real-valued $f$,

\[ \|f - h_0\|_{L^1([0, 1])} \geq \|f - h_0\|_{L^1(\{0.5 - \frac{1}{3a}, 0.5 + \frac{1}{3a}\})}. \]

We can then simplify our problem by only considering functions with values on the interval $[\frac{2}{3}, \frac{4}{3}]$. Indeed, for every real-valued $a$-Lip function $f$, if we denote by $\hat{f}$ the function defined by $\hat{f} : x \mapsto \min(\max(f(x), \frac{2}{3}), \frac{4}{3})$, the latter is a better approximation of the discontinuity (in the sense $\|\hat{f} - h_0\|_{L^1} \geq \|f - h_0\|_{L^1}$) while also being $a$-Lipschitz.
By a linear change of coordinates, one can see that proving the result on the window $[0.5 - \frac{1}{3a}, 0.5 + \frac{1}{3a}] \times [\frac{2}{3}, \frac{4}{3}]$ for \(a\)-Lip functions is equivalent to proving it on $[0, 1] \times [0, 1]$ for 1-Lip functions, with \(h_0\) and \(f_a\) now being (Figure 2)

\[
h_0 : x \mapsto \begin{cases} 
0 & 0 \leq x \leq 0.5 \\
1 & 0.5 < x \leq 1 
\end{cases}
\]

and \(f_a : x \mapsto x\).

**Figure 2.** Lipschitz approximation of a discontinuity, rescaling of the problem.

Let \(f\) be a 1-Lip function of $[0, 1]$, with values in $[0, 1]$.

\[
\|f - h_0\|_{L^1} = \int_0^{0.5} f(x) \, dx + \int_{0.5}^1 1 - f(x) \, dx.
\]

Using the 1-Lip property, we have that for all $x > 0.5$,

\[
f(x) - f(0.5) \leq |f(x) - f(0.5)| \leq x - 0.5 \quad \text{i.e.} \quad -f(x) \geq 0.5 - x - f(0.5).
\]

Hence

\[
\|f - h_0\|_{L^1} \geq \int_0^{0.5} f(x) \, dx + \int_{0.5}^1 1 - x \, dx + \frac{1}{2}(0.5 - f(0.5)) \\
= \int_0^{0.5} f(x) + 0.5 - f(0.5) \, dx + \int_{0.5}^1 1 - x \, dx.
\]

We can re-use the 1-Lip property on $x < 0.5$ to obtain

\[
f(x) - f(0.5) + 0.5 \geq x
\]

and conclude

\[
\|f - h_0\|_{L^1} \geq \int_0^{0.5} x \, dx + \int_{0.5}^1 1 - x \, dx = \|f_a - h_0\|_{L^1}.
\]

\(\square\)

We are now ready to prove Proposition 10.
Proof of Proposition 10. We showed that there is a \( c' > 0 \) such that the invariant density of \( L_\delta \) is \( \frac{c'}{\delta} \)-Lipschitz. We can apply the last proposition to state the following lower bound on the modulus of continuity:

\[
\| h_\delta - h_0 \|_{L^1} \geq \frac{\delta}{9c'} = C\delta.
\]

Note that this lower bound result could easily be applied to all piecewise expanding maps with a discontinuity in their unperturbed invariant density, with a different constant for each map.

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