Members of Thin $\Pi^0_1$ Classes and Generic Degrees *

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August 12, 2020

Abstract
A $\Pi^0_1$ class $P$ is thin if every $\Pi^0_1$ subclass $Q$ of $P$ is the intersection of $P$ with some clopen set. In 1993, Cenzer, Downey, Jockusch and Shore initiated the study of Turing degrees of members of thin $\Pi^0_1$ classes, and proved that degrees containing no members of thin $\Pi^0_1$ classes can be recursively enumerable, and can be minimal degree below $0'$. In this paper, we work on this topic in terms of genericity, and prove that all 2-generic degrees contain no members of thin $\Pi^0_1$ classes. In contrast to this, we show that all 1-generic degrees below $0'$ contain members of thin $\Pi^0_1$ classes.

*Stephan is supported by Singapore Ministry of Education Academic Research Fund Tier 2 grant MOE2016-T2-1-019 / R146-000-234-112. Wu is supported by Singapore Ministry of Education Academic Research Fund Tier 2 grant MOE2016-T2-1-083 (M4020333); NTU Tier 1 grants RG32/16 (M4011672) and RG111/19 (M4012245).
1 Introduction

In this paper, we will continue the study of Turing degrees of members of thin $\Pi^0_1$ classes. Here a $\Pi^0_1$ class $P$ is thin if every $\Pi^0_1$ subclass $Q$ of $P$ is the intersection of $P$ with some clopen set. Historically, thin classes were first constructed by Martin and Pour-El in their paper [12], when they constructed an axiomatizable essentially undecidable theory such that any axiomatizable extension of it is a finite extension. The concept of thin $\Pi^0_1$ class was first raised explicitly by Downey in his PhD thesis [5]. In a thin $\Pi^0_1$ class, computable elements are all isolated, and hence, perfect thin $\Pi^0_1$ classes contain no computable element.

In [2], Cenzer, Downey, Jockusch and Shore considered the Cantor-Bendixson ranks of members of countable thin $\Pi^0_1$ classes, and then constructed an r.e. degree, and also a minimal degree below $0'$, whose elements are not members of any thin $\Pi^0_1$ class. We call such degrees thin-free degrees. Downey, Wu and Yang recently proved in [6] that the r.e. thin-free degrees are both dense and co-dense in r.e. degrees, providing another class of r.e. degrees which are both dense and co-dense in r.e. degrees. Recall that the first such a class is the set of of branching r.e. degrees, proved by Fejer [7] and Slaman [14]. Consequently, we call a Turing degree thin if it is not thin-free.

The construction of thin-free minimal degrees below $0'$ uses the $e$-splitting trees, which provides a framework for us to include and exclude infinitely paths of a given $\Pi^0_1$ class $P$, to show that $P$ is not a thin class. In the construction of a thin-free r.e. degree, they used an effective version of this idea, and constructed various intervals to capture the wanted infinitely paths. The construction of thin-free minimal degrees below $0''$ turns out to be much easier as it is based on Spector’s construction, instead of Sacks’ construction. Based on this observation, in his PhD thesis [16], Yuan proved the existence of a hyperimmune-free minimal thin-free degree below $0''$. We will use the same idea to show that nonrecursive sets below a 2-generic degree are thin-free. In particular, all 2-generic sets are of thin-free degree. In contrast to this, we show that all 1-generic degrees below $0'$ are not thin-free.

Our notation are standard. Most of the concepts and notation we use in paper can be found in books [4, 13, 15].
2 2-generic sets are thin-free

Recall that a set $A$ is 2-generic if it meets or avoids every $\Sigma^0_2$ subset of $2^{<\omega}$.

Jockusch observed [9] that for any 2-generic set $A$, for any partial recursive functional $\Phi$ with $\Phi^A$ total and nonrecursive, $A$ has an initial segment $\sigma$ such that the set $T = \{ \rho : \rho \subseteq \Phi^\tau, \text{ where } \tau \supseteq \sigma \}$ is a recursive extendible tree without isolated infinite paths.

Lemma 2.1 (Jockusch) If $A$ is 2-generic and $\Phi^A$ is total and nonrecursive, then there is a $\sigma \subset A$ such that for all $\tau \supseteq \sigma$,

- for any $x$, $\tau$ has an extension $\rho \supseteq \tau$ with $\Phi^\rho(x) \downarrow$,
- there is a $\Phi$-splitting extension above $\tau$.

This is a property we need to show that a given $\Pi^0_1$ class is not thin.

Theorem 2.2 Any nonrecursive set Turing below a 2-generic set is thin-free. Thus, 2-generic sets are of thin-free degree.

Proof: Let $A$ be a 2-generic set, and assume that $\Phi^A$ is total and nonrecursive. Suppose that $\Phi^A$ lies on a primitive recursive tree $P$, where $\Phi$ is a \{0,1\}-valued partial recursive functional. We will show that $[P]$ is not thin.

Let $V$ be the set of strings $\tau$ such that:

- $\exists x \forall \rho \supseteq \tau[\Phi^\rho(x) \uparrow]$, or
- $\forall \rho, \pi \supseteq \tau \forall x[\Phi^\rho(x) \downarrow \text{ & } \Phi^\pi(x) \downarrow \Rightarrow \Phi^\rho(x) = \Phi^\pi(x)]$, or
- $\Phi^\tau \notin P$.

$V$ is $\Sigma^0_2$, and hence, $A$ either meets or avoids $V$, as $A$ is 2-generic. Note that $A$ cannot meet $V$, because $\Phi^A$ is assumed to be total, nonrecursive, and lies on $P$. So $A$ avoids $V$. This implies the existence of a string $\sigma \subset A$ such that for any $\tau \supseteq \sigma$,

1. for any $x$, $\tau$ has an extension $\rho \supseteq \tau$ with $\Phi^\rho(x) \downarrow$, and
2. $\tau$ has two extensions $\rho, \pi$ such that for some $x$, $\Phi^\rho(x) \downarrow$, $\Phi^\pi(x) \downarrow$, and $\Phi^\rho(x) \neq \Phi^\pi(x)$, and
3. $\Phi^\tau \in P$. 

3
For $\sigma$ above, consider the tree $T = \{ \rho \subseteq \Phi^\tau : \tau \supseteq \sigma \}$. We show that $T$ is extendible as follows. Suppose that for $\tau \supseteq \sigma$, $\Phi^\tau$ is a finite string on $T$. Then for any $x \geq |\Phi^\tau|$ (meaning that $\Phi^\tau(x) \uparrow$), by (1), there is some $\pi \supseteq \tau$ such that $\Phi^{\pi}(x) \downarrow$, which means that $\Phi^{\pi}$ extends $\Phi^\tau$. As $\Phi^{\pi}$ is on $T$, $\Phi^{\pi}$ is extendible on $T$.

Moreover, $T$ is recursive. Fix $l$. If $l \leq |\Phi^\sigma|$, then only one string of length $l$, i.e., $\Phi^\sigma \downarrow l$, is on $T$. If $l > |\Phi^\sigma|$, let $\tau$ be the first string extending $\sigma$ such that $|\Phi^\tau| \geq l$. Such a $\tau$ exists by (1) and can be found recursively by enumerating strings extending $\sigma$. Let $u(l) = \max\{ \varphi(x) : x < l \}$, where $\varphi(x)$ is the use of $\Phi^{\tau}(x)$. $u(l)$ is recursive. By enumerating all strings of length $u(l)$ extending $\sigma$, we get all strings on $T$ of length smaller than $l$.

Now consider the leftmost path $C$ though $T$. Since $T$ is a recursive, extendible tree, $C$ is recursive. $C$ lies on $T$, so $C \supseteq \Phi^\sigma$. Let $\rho_0 = \sigma$, and for a given $\rho_i$, let $\rho_{i+1}$, $\tau_{i+1}$ and $x_{i+1}$ be the first triple $(\rho, \tau, x)$ such that $\rho, \tau \supseteq \rho_i$, $\Phi^\rho(x) \downarrow$, $\Phi^\tau(x) \downarrow$, $\Phi^\rho(x) \neq \Phi^\tau(x)$, and $\Phi^\rho \uparrow x \subset C \uparrow x$. Such a triple exists because of (2) and the choice of $C$. Furthermore, as $C$ is recursive, the list of $\tau_i$ for $i \geq 1$ is recursive. By (3), $T$ is a subtree of $P$, so for any $i \geq 1$, there is an infinite path through $P$ extending $\Phi^{\tau_i}$. Let $S$ be the collection of all initial segments of $C$ and $\Phi^{\tau_i}$ for all $i \geq 1$, and the strings on $P$ extending $\Phi^{\tau_i}$ for $i$ even. Then $S$ is a recursive subtree of $P$, which is not the intersection of $[P]$ with any clopen set.

![Figure 1: The construction of $S$.](image)

So for any 2-generic set $A$, for all partial recursive functionals $\Phi$, if $\Phi^A$ is total, not recursive and $\Phi^A \in [P]$, then $[P]$ is not thin, completing the proof.

\[\blacksquare\]
3 1-generic sets below 0' are not thin-free

In this section, we show that all 1-generic degrees below 0’ are thin. Let A be a 1-generic set reducible to 0’. To show a set A is not thin-free, it suffices to construct a recursive thin tree T containing a path C which is Turing equivalent to A.

By Shoenfield’s limit lemma, A admits a \( \Delta^0_2 \) approximation. As A is also 1-generic, A admits a \( \Delta^0_2 \) approximations with an extra property, the so-called \( \Sigma_1 \)-correctness. Here a recursive approximation \( \{\sigma_s : s \in \omega \} \) of A is \( \Sigma_1 \)-correct if for any infinite r.e. set S of natural numbers, there exists some \( s \in S \) such that \( \sigma_s \subset A \). Haught pointed out in [8] that this property is due to Shore, and used it to show that 1-generic degrees below 0’ are downwards closed. For completeness of the paper, we present a proof of Shore’s Lemma.

**Lemma 3.1 (Shore)** Any recursive approximation of a 1-generic set \( A < T 0' \) has a \( \Sigma_1 \)-correct approximation.

**Proof:** As \( A < T 0' \), by Shoenfield’s limit lemma, we can have a recursive approximation of A, \( \{\sigma_s \} \) say. For an infinite r.e. set \( S \subseteq \omega \), we define a set \( V = \{\sigma_s : s \in S \} \), which is r.e.. By 1-genericity, A either meets or avoids \( V \). A cannot avoid \( V \) because for any initial segment \( \sigma \) of A, as \( \{\sigma_s \} \) approximates A, there exists some \( s \in S \) such that for all \( t > s \), \( \sigma_t \supset \sigma_s \). Thus, A meets \( V \), i.e., there is a \( \sigma \subset A \) such that \( \sigma \in V \), which implies the existence of \( s \in S \) with \( \sigma_s \subset A \).

In addition, Haught also pointed out that A can actually have a \( \Sigma_1 \)-correct approximation \( \{\sigma_s \} \) satisfying that \( |\sigma_{s+1}| > |\sigma_s| \) for each \( s \in \omega \). To see this, for any \( \Sigma_1 \)-correct approximation \( \{\alpha_s \} \) of A, we define a function \( f : \omega \to \omega \) inductively by taking \( f(0) = 0 \) and \( f(s+1) = \mu t > f(s) (|\alpha_t| > |\alpha_{f(s)}|) \). Such a \( t \) exists because \( \alpha_s \) is a recursive approximation. Note that \( f \) is recursive and increasing. Let \( \sigma_s = \alpha_{f(s)} \), then \( \{\sigma_s \} \) is also a recursive approximation of A. For any infinite r.e. set \( S \), \( V = \{f(s) : s \in S \} \) is r.e. and infinite. Since \( \{\alpha_s \} \) is \( \Sigma_1 \)-correct, there is some \( f(s) \in V \) such that \( \alpha_{f(s)} \subset A \), which implies that \( \sigma_s = \alpha_{f(s)} \subset A \). Thus \( \{\sigma_s \} \) is also \( \Sigma_1 \)-correct.

**Theorem 3.2** A 1-generic degree \( a < 0' \) is not thin-free.

**Proof:** Let \( \{\sigma_s : s \in \omega \} \) be a \( \Sigma_1 \)-correct approximation of A such that \( |\sigma_{s+1}| > |\sigma_s| \) for each s. We will construct a recursive tree T, such that \( [T] \) is thin and there is a path C in \( [T] \) with \( C \equiv_T A \).

Before we provide the construction of T, we first consider the set \( S \) of all initial segments of \( \sigma_s \), \( s \in \omega \). S is a tree since it is closed under initial
segment, and \( A \) is an infinite path through \( S \). It is clear that \( A \) is the only path on \( S \). On the other hand, \( S \) is r.e., as a string \( \tau \in S \) if and only if \( \sigma_s \supseteq \tau \) for some \( s \). As \( \{ \sigma_s : s \in \omega \} \) is a \( \Delta^0_2 \) approximation of \( A \), some strings on \( S \) may not be extendible.

We want the tree \( T \) we are constructing to be recursive and extendible, and the construction of \( T \) “follows” the enumeration of \( S \). With this in mind, we need an extra symbol \( B \), standing for “blank”, such that \( T \) is a subtree of \( \{0, 1, B\}^{<\omega} \) and all strings up to some length, \( l(s) \) say, are defined on \( T \) at each stage \( s \). Here \( l \) is a recursive function. It is clear that \( T \) defined in this way can be coded into a binary tree recursively.

For a finite string \( \tau \in \{0, 1, B\}^{<\omega} \) (or an infinite sequence \( C \in \{0, 1, B\}^{\omega} \), respectively), we let \( \tau^d \) (or \( C^d \)) denote the string (or a finite string or an infinite subsequence, respectively) obtained by deleting all \( B \) from \( \tau \) (or \( C \)) while keeping the appearance of 0’s and 1’s the same order. For example, \( (0010BB10B01)^d = 00101001 \).

Construction of \( T \):

**Stage 0** Let \( \emptyset \) be the root of \( T \), and \( l(0) = 0 \).

**Stage \( s + 1 \)** \( l(s) \), and all strings \( \tau \) on \( T \) of length \( l(s) \) are already defined by the end of stage \( s \). For \( \rho \), a string on \( T \) of length \( l \) with \( \rho^d \subseteq \sigma_s \), let \( l(s + 1) = l(s) + m + 1 \), where \( m = |\sigma_s| - |\rho^d| \). Now there are \( m + 1 \) steps for \( i = 0, 1, \ldots, m \), and at step \( i \), for strings \( \tau \) on \( T \) of length \( l(s) + i \), if \( \tau^d \subseteq \sigma_s \), put \( \tau \upharpoonright 0 \) and \( \tau \upharpoonright 1 \) into \( T \), else put \( \tau \upharpoonright B \) into \( T \).

Note that for all \( \tau \)'s above, \( \tau^d \upharpoonright \lceil |\tau^d| \rceil - 1 \) are already on \( S \) by stage \( s \), while none of \( \tau^d \) is on \( S \) yet, and at stage \( s + 1 \), with \( \sigma_s \) just being put on \( S \), for the string \( \rho \) of length \( l(s) \) on \( T \) and \( \rho^d \subseteq \sigma_s \), let \( \pi \) be the string such that \( \sigma_s = \rho^d \pi \) and \( m = |\pi| \). Then (1) we put, if \( m \geq 1 \), \( \rho \upharpoonright (\pi \upharpoonright i)^\upharpoonright 0 \) and \( \rho \upharpoonright (\pi \upharpoonright i)^\upharpoonright 1 \) for \( 0 \leq i \leq m - 1 \), and \( \rho \upharpoonright \pi^\upharpoonright 0 \) and \( \rho \upharpoonright \pi^\upharpoonright 1 \) into \( T \) in order. In this manner, all strings ending with 0 or 1 on \( T \) longer than \( l(s) \) are defined. (2) we extend all other strings on \( T \) by \( B \) up to length \( l(s) + m + 1 \). This completes the construction of \( T \) up to length \( l(s) + m + 1 \).

What does \( T \) look like? Consider all the strings on \( T \) of length \( s + 1 \), i.e., those strings on the \((s + 1)\)-st level of \( T \). There are exactly \( s + 2 \) strings on the \((s + 1)\)-st level, and among them, two strings end with 0 and 1 and share the common initial segment of length \( s \), and the other strings end with \( B \). Let \( \tau_i \) for \( 0 \leq i \leq s + 1 \) be the strings on the \((s + 1)\)-st level of \( T \), then \( \tau^d_i \) are incompatible with each other, and only one of them, say \( \tau^d_j \), is extendible on \( S \) (equivalently, \( \tau^d_j \subset A \)). So above the \((s + 1)\)-st level of \( T \), there are only finitely many paths above \( \tau_i \) on \( T \) for each \( i \neq j \), and infinitely many paths
on $T$ extending $\tau_j$. Since $A$ is the unique path on $S$, we know that there is only one path $C$ on $T$ which contains infinitely many 0 and 1, with $(C)^d = A$. For other paths $D$, there is some $n$ such that for all $x > n$, $D(x) = B$, and $(D)^d$ is a string not in $S$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{A diagram showing the relation between $T$ and $C$.}
\end{figure}

$A \leq_T C$ by $(C)^d = A$. $C \leq_T A$ because for any $x$, there is some $s_x$ such that $l(s_x) \leq x \leq l(s_x + 1)$. Then find the least $s > s_x$ such that $\sigma_s \subset A$. Such an $s$ exists since $\{\sigma_s\}$ is $\Sigma_1$-correct. For this $s$, find $\tau$ on $T$ of length $l(s)$ with $\tau^d \subset A$. Then $\tau$ is an initial segment of $C$, and $\tau(x) = C(x)$.

We now show that $T$ has a neat "splitting" property.

**Lemma 3.3** For any $n$, there is some $m > n$, such that for strings on the $n$-th level of $T$, all but one string do not split above level $m$ on $T$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{An illustration of construction of $T$.}
\end{figure}
Proof: For any \( n \), let \( \tau_i \) for \( 0 \leq i \leq n \) be the strings on the \( n \)-th level of \( T \), and \( k = \max_i \{|\tau_i^d|\} \). For this \( k \), by \( \Sigma_1 \)-correctness, there is a stage \( t \geq k \) such that \( \sigma_t \subset A \). Then there is some stage \( s \) such that for all stages \( s' > s \), \( \sigma_{s'} \supset \sigma_t \).

Let \( m = l(s + 1) - 1 \) and \( \rho \) be the string on the \( m \)-th level of \( T \) such that \( \rho^d = \sigma_s \). Then \( \rho^d \supset \sigma_t \), and after stage \( s \), all strings on the \( m \)-th level of \( T \) except \( \rho \) can only be extended by \( B \), and thus if \( \tau_i \not\subset \rho \), \( \tau_i \) does not split above level \( m \) on \( T \).

Thus, all strings on \( T \) are extendible, and \([T]\) contains infinitely many paths.

The \( \Sigma_1 \)-correctness of \( \{\sigma_s\} \) guarantees that \( T \) is thin. Let \( U \) be a recursive subtree of \( T \). There are two cases.

(1) \( C \not\in [U] \). If so, then there is some \( n \) such that \( C \upharpoonright n \not\in U \). As \( C^d = A \), \( (C \upharpoonright n)^d \subset A \), and thus there are only finitely many paths on \( U \). Suppose that up to length \( m \), strings on \( U \) do not split, let \( N \) be the union of cones above the strings on \( U \) of length \( m \), then \( N \) is clopen and \([U]\) is the intersection of \([T]\) with \( N \).

(2) \( C \in [U] \). Consider \( V = \{s : \tau \in T \setminus U, \tau^d \in S \text{ for some stage } s\} \). Then by the \( \Sigma_1 \)-correctness, \( V \) is finite. Otherwise, there is some \( s \in V \) such that \( \sigma_s \subset A \), which means that there is some \( \tau \) in \( T \setminus U \) such that \( \tau^d = \sigma_s \) is enumerated into \( S \) at stage \( s \). By the construction, \( \tau^d \subset A \), so \( \tau \subset C \). However, \( C \in [U] \) implies that for all \( n \geq 0 \), \( C \upharpoonright n \in U \), a contradiction.

Since \( V \) is finite, there are finitely many paths on \( T \) but not on \( U \). Thus, there is a clopen set \( N \) such that \([T] \setminus [U] = [T] \cap N \), and hence \([U] = [T] \setminus N = [T] \cap \overline{N} \). The complement of \( N \) is what we need.

This shows that among these infinitely paths, exact one path is Turing equivalent to \( A \). Thus, we obtain a path \( C \) in a thin class \([T]\) with \( C \equiv_T A \), and \( A \) is not of thin-free degree. \( \square \)

References

[1] D. Cenzer, \( \Pi_1^0 \) classes in computability theory, in Handbook of Computability Theory, Elsevier, Amsterdam, 1999, 37-85.

[2] D. Cenzer, R. Downey, C. G. Jockusch, Jr., and R. A. Shore, Countable thin \( \Pi_1^0 \) classes, Ann. Pure Appl. Logic 59 (1993), 79-139.
[3] C. T. Chong and R. Downey, *Minimal degrees recursive in 1-generic degrees*, Ann. Pure Appl. Logic **48** (1990), 215-225.

[4] S. B. Cooper, *Computability theory*, Chapman and Hall/CRC, 2017.

[5] R. Downey, *Abstract Dependence, Recursion Theory and the Lattice of Recursively Enumerable Filters*, PhD Thesis, Monash University, Clayton, Victoria, Australia, 1982.

[6] R. Downey, G. Wu, and Y. Yang, *Degrees containing members of thin $\Pi^0_1$ classes are dense and co-dense*, J. Math. Log. **18** (2018), 1-47.

[7] P. Fejer, *The density of the nonbranching degrees*, Ann. Pure Appl. Logic **24** (1983), 113-130.

[8] C. A. Haught, *The degrees below a 1-generic degree $<\mathcal{O}'$*, J. Symb. Log. **51** (1986), 770-777.

[9] C. G. Jockusch, Jr., *Degrees of generic sets*, in *Recursion Theory: its generalizations and applications* (F. R. Drake and S. S. Wainer, eds.), London Mathematical Society Lecture Notes Series, vol. 45, Cambridge University Press, Cambridge, 1980, 110-139.

[10] C. G. Jockusch, Jr. and R. I. Soare, *Degrees of members of $\Pi^0_1$ classes*, Pacific J. Math. **40** (1972), 605-616.

[11] C. G. Jockusch, Jr. and R. I. Soare, *$\Pi^0_1$ classes and degrees of theories*, Trans. Amer. Math. Soc. **173** (1972), 33-56.

[12] D. A. Martin and M. B. Pour-El, *Axiomatizable theories with few axiomatizable extensions*, J. Symb. Log. **35** (1970), 205-209.

[13] P. G. Odifreddi, *Classical Recursion Theory*, vol II, North-Holland, Amsterdam, 1999.

[14] T. A. Slaman, *The density of infima in the recursively enumerable degrees*, Ann. Pure Appl. Logic **52** (1991), 155-179.

[15] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin, Heidelberg, 1987.

[16] B. Yuan, *Computability Theory and Degree Structures*, PhD Thesis, Nanyang Technological University, 2020.