CONFIGURATIONS OF POINTS AND THE SYMPLECTIC BERRY-ROBBINS PROBLEM

JOSEPH MALKOUN

Abstract. We present a new problem on configurations of points, which is in some sense a new version of a similar problem by Atiyah and Sutcliffe, except it is related to the Lie group $Sp(n)$, instead of the Lie group $U(n)$. More precisely, we wish to construct a map from the manifold $C_n$ of $n$ points $x_i \in \mathbb{R}^3 \setminus \{0\}$, such that $x_i \neq x_j$ and $x_i \neq -x_j$ for all $1 \leq i < j \leq n$, into the flag manifold $F_n = Sp(n)/T$, where $T$ is the diagonal torus, such that replacing an $x_i$ with $-x_i$ corresponds to multiplying the $i$th column of the corresponding point in $F_n$ by $j$, and which is equivariant under the action of the permutation group on $n$ elements. Namely, permuting the $x_i$ corresponds to a similar permutation of the columns of the corresponding point in the flag manifold $F_n$. We present a candidate for such an equivariant map, which would be a genuine map provided a certain linear independence conjecture holds. We prove the linear independence conjecture for $n = 2$.

1. Introduction

We introduce the relevant manifolds.

$$C_n = \{(x_1, \ldots, x_n) \in (\mathbb{R}^3 \setminus \{0\})^n; x_i \neq x_j \text{ and } x_i \neq -x_j \text{ for all } 1 \leq i < j \leq n\}$$

We denote by $F_n$ the following flag manifold

$$F_n = Sp(n)/T$$

where $T$ is the diagonal $n$-dimensional torus in $Sp(n)$. The following finite group will be needed

$$W_n = (\mathbb{Z}/(2))^n \rtimes \Sigma_n$$

where $\Sigma_n$ is the permutation group on $n$ elements. We refer to it as the Weyl group, for it is the Weyl group of the Lie group $Sp(n)$. The element $(1, \ldots, -1, \ldots, 1) \in (\mathbb{Z}/(2))^n$, with a $-1$ in the $i$th position only, acts on $(x_1, \ldots, x_n)$ by replacing $x_i$ with $-x_i$, and leaving all other $x_j$ invariant. An element $\sigma \in \Sigma_n$ acts by permuting the $n$ points $x_j$.

On the other hand, the action of $W_n$ on $F_n$ can be described as follows: an element in $(\mathbb{Z}/(2))^n$ having $-1$ only in the $i$th position, multiplies the $i$th column of each point in $F_n$ by the quaternionic structure $j$ (leaving the other columns invariant), while a permutation $\sigma$ simply permutes the columns of each point $gT \in F_n$.

Having described the main players in our story, we ask the following question, which was asked by Berry and Robbins for the group $U(n)$ ([5]), and solved positively by Atiyah in [1], and then later generalized (and solved positively) to an arbitrary compact Lie group $G$ by Atiyah and Bielawski in [3]:

**Question:** Is there for each $n \geq 2$, a continuous map $f_n : C_n \to F_n$ which is intertwining for the corresponding actions of $W_n$?
Actually, as we wrote earlier, Atiyah and Bielawski have already posed and solved in [3] a more general problem for any compact Lie group \( G \) (in our case, \( G = Sp(n) \)). But their solution is non-elementary, as it relies on an analysis of the Nahm equations. Here we propose, similar to Atiyah ([1] and [2]), and Atiyah and Sutcliffe ([4]), a more elementary construction in the same spirit as those papers, but for the case \( G = Sp(n) \), instead of \( G = U(n) \).

2. The Main Construction

We first associate to each configuration \( x \in \mathbb{C}^n \), \( n \) polynomials \( p_1 \) to \( p_n \) of \( t \in \mathbb{C} \) of degree less than or equal to \( 2n - 1 \), each defined up to a factor only. Namely, we let \( p_i \) be a polynomial having as roots the stereographic projections of the normalizations of \(-x_i, -x_i + x_j \) and \(-x_i - x_j \) for all \( j \neq i \) (1 \( \leq j \leq n \)). Similarly, we introduce \( n \) other polynomials \( q_1 \) to \( q_n \), with \( q_i \) having as roots the antipodals of the roots of \( p_i \), namely the stereographic projections of \( x_i, x_i + x_j \) and \( x_i - x_j \), for \( j \neq i \).

A key observation is that the space of polynomials of degree less than or equal to \( 2n - 1 \) has a natural quaternionic structure \( j \), this having to do with \( 2n - 1 \) being odd. Writing such a polynomial as

\[
p(t) = \sum_{i=0}^{2n-1} a_i t^i
\]

the quaternionic structure \( j \) maps \( p \) to \( jp \), defined by

\[
(jp)(t) = \sum_{i=0}^{2n-1} a'_i t^i
\]

where

\[
a'_i = (-1)^{i+1} \bar{a}_{2n-1-i}
\]

Moreover, since the roots of \( q_i \) are the antipodals of those of \( p_i \), it follows that \( q_i \) is a factor times \( jp_i \).

If we think of the space of polynomials above with its quaternionic structure as \( \mathbb{H}^n \), then the \( p_i \) define \( n \) column vectors \( v_i \) in \( \mathbb{H}^n \), each defined up to a \( \mathbb{C}^* \)-ambiguity.

We observe that \( x_i \mapsto -x_i \) has the effect of mapping \( v_i \mapsto jv_i \) (i.e. \( p_i \mapsto q_i \)). On the other hand, permuting the \( x_i \) corresponds to permuting the \( v_i \) in the same way.

Hence, assuming the \( v_i \) are always \( \mathbb{H} \)-linearly independent, for any \( x \in \mathbb{C}^n \), we would have defined a \( W_n \)-equivariant map from \( C_n \) into \( GL(n, \mathbb{H})/(\mathbb{C}^*)^n \). Following this map by an orthogonalization procedure which respects the action of \( W_n \), we finally get a smooth map from \( C_n \) into \( F_n \). But this is so provided the following conjecture is true:

**Conjecture 1:** given any \( x \in \mathbb{C}^n \), the \( 2n \) polynomials \( p_i \) and \( q_i \) (1 \( \leq i \leq n \)) are linearly independent over \( \mathbb{C} \).

In the following section, we define a natural determinant function, and then, we prove the conjecture above for \( n = 2 \).

3. A Determinant Function

Let \( t_i^\pm \) and \( t_{ij}^{\pm\pm} \in \mathbb{C}P^1 \) be the stereographic projections of the normalizations of the following vectors, respectively

\[
\pm x_i \ (1 \leq i \leq n) \text{ and } \pm x_i \pm x_j \ (1 \leq i, j \leq n \text{ and } i \neq j)
\]
We then choose lifts \( u_i^\pm = (u_i^\pm, v_i^\pm) \) and \( u_{ij}^{\pm\pm} = (u_{ij}^{\pm\pm}, v_{ij}^{\pm\pm}) \in \mathbb{C}^2 \setminus \{0\} \) of these roots under the Hopf map:

\[
h : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1
\]

We then form the polynomials \( p_i \) having as roots the chosen lifts \( (u_i^-, v_i^-) \) and \( (u_{ij}^{ij-}, v_{ij}^{ij-}) \) for all \( j \neq i \). Thus in particular, once the lifts are chosen, the polynomials \( p_i \) are determined uniquely, in other words, the factor of each \( p_i \) gets fixed. Similarly, form the polynomials \( q_i \) having as roots the chosen lifts \( (u_i^+, v_i^+) \) and \( (u_{ij}^{ij+}, v_{ij}^{ij+}) \) and \( (u_{ij}^{ij-}, v_{ij}^{ij-}) \) for all \( j \neq i \). We now form the complex 2n by 2n matrix

\[
M = (p_1, q_1, \ldots, p_n, q_n)
\]

having the coefficients of \( p_i \) and \( q_i \) as column vectors. One then defines the quantity

\[
P = \prod_i \det(u_i^-, u_i^+) \prod_{i<j} \left( \det(u_{ij}^{ij+}, u_{ij}^{ij-}) \det(u_{ij}^{ij-}, u_{ij}^{ij+}) \right)^2
\]

Then the determinant function

\[
D(x_1, \ldots, x_n) = \det(M)/P
\]

is independent of the choices of lifts, and thus well defined. Similar to the Atiyah-Sutcliffe determinant, \( D \) is actually invariant under the action of the Weyl group \( W_n \) on \( C_n \), and is also invariant under scaling, and rotations in \( \mathbb{R}^3 \). However, unlike the Atiyah-Sutcliffe determinant, it is always real-valued, because it is the determinant of a 2n by 2n complex matrix, which represents an n by n quaternionic matrix, and thus is always real (indeed, the complex conjugate of such a 2n by 2n complex matrix can be shown to be in the same conjugacy class as the complex matrix itself, so they must have equal determinants).

4. The case \( n = 2 \)

We consider here the case \( n = 2 \). We have two points \( x_1, x_2 \in \mathbb{R}^3 \) such that \( x_1 \neq x_2 \) and \( x_1 \neq -x_2 \). Using a rotation in \( \mathbb{R}^3 \), we can assume that they both lie on the xy-plane. We think of the xy-plane as the complex plane. Using a rotation in the xy-plane and scaling, we can further assume that \( x_1 = 1 \) and we then let \( z \) be the complex number representing \( x_2 \) in the xy-plane. Thus \( z \neq 1 \) and \( z \neq -1 \). We let

\[
A = \frac{z - 1}{|z - 1|},
B = -\frac{z + 1}{|z + 1|},
g = -\frac{z}{|z|}
\]

We then have

\[
-64D = \begin{vmatrix}
AB & 1 & ABg & 1 \\
AB - A - B & \bar{A} + \bar{B} - 1 & -Ag + Bg - AB & -\bar{A} + \bar{B} + \bar{g} \\
1 - A - B & \bar{A} - \bar{B} - g & A - B - g & -\bar{A} + \bar{B} - \bar{g} \\
1 & -\bar{A} & 1 & -\bar{A}\bar{B}\bar{g}
\end{vmatrix}
\]

We then multiply the second column by \(-AB\) and add it to the first column, and we multiply the second column by \(-ABg\) and add it to the third column, and finally subtract
the second column from the fourth one, and get, after expanding the determinant along the first row:

\[
64D = \begin{vmatrix} 2(AB - A - B) & -2Ag + ABg - AB & -2\bar{A} + 1 + \bar{g} \\ 0 & -2g + Bg - B + Ag + A & -2\bar{A}B + \bar{A}(1 - \bar{g}) + B(1 + \bar{g}) \\ 2 & \bar{A}B(1 - \bar{g}) \\ \end{vmatrix}
\]

Taking a 2 out from the first column, and using elementary column operations using the first column in order to make the entries in the (3, 2) and (3, 3) positions vanish, we get

\[
32D = \begin{vmatrix} AB - A - B & -2AB + A(1 - g) + B(1 + g) & 2\bar{g} - \bar{A}(1 + \bar{g}) + \bar{B}(1 - \bar{g}) \\ 0 & A(1 + g) - B(1 - g) - 2g & -2\bar{A}\bar{B} + \bar{A}(1 - \bar{g}) + \bar{B}(1 + \bar{g}) \\ 1 & 0 & 0 \\ \end{vmatrix}
\]

\[
= | -2AB + A(1 - g) + B(1 + g)|^2 + |A(1 + g) - B(1 - g) - 2g|^2
\]

\[
= 8 + 2|1 - g|^2 + 2|1 + g|^2 - 2B(1 - \bar{g}) - 2B(1 - g) - 2A(1 + \bar{g}) - 2\bar{A}(1 + g) + \ldots
\]

\[
\ldots + 2(\bar{A}B - \bar{A}B)(\bar{g} - g) - 2\bar{A}g(1 + g) - 2\bar{A}g(1 + \bar{g}) + 2B\bar{g}(1 - g) + 2B\bar{g}(1 - \bar{g})
\]

Using

\[
|1 + g|^2 + |1 - g|^2 = 4
\]

we get

\[
16D = 8 - 2A(1 + \bar{g}) - 2B(1 - \bar{g}) + A(1 + \bar{g})\bar{B}(1 - g) - 2A(1 + g) - 2\bar{B}(1 - g) + \bar{A}(1 + g)B(1 - \bar{g})
\]

If we let

\[
(w_1, w_2) = \frac{1}{2}(w_1\bar{w}_2 + w_2\bar{w}_1)
\]

\[
\det(w_1, w_2) = \frac{i}{2}(w_1\bar{w}_2 - w_2\bar{w}_1)
\]

we can then write

\[
4D = 2 - (A, 1 + g) - (B, 1 - g) - \Im(g) \det(A, B)
\]

where \(\Im(g)\) denotes the imaginary part of \(g\). Therefore, using the definitions of \(A, B\) and \(g\) in terms of \(z\), we get

\[
4D = 2 + \left( \frac{z - 1}{|z - 1|} \right)^2 \left( \frac{z}{|z|} \right) - 1 + \left( \frac{z + 1}{|z + 1|} \right)^2 \left( \frac{z}{|z|} + 1 \right) + 2 \left( \frac{3z}{|z||z - 1||z + 1|} \right)^2
\]

Writing \(z = re^{i\theta}\), and after simplification, we get

\[
4D = 2 + \left( \frac{1 + r}{|r|} \right) \left( 1 - \cos(\theta) \right) \left( \frac{1 + r}{|r|} \right) \left( 1 + \cos(\theta) \right) \left( \frac{1 + r}{|r|} \right) + 2 \left( \frac{1 + r}{|r|} \right) \left( 1 + \cos(\theta) \right) \left( \frac{1 + r}{|r|} \right)
\]

Using \(1 + r \geq |z + 1|\) and \(1 + r \geq |z - 1|\), and that \(1 + \cos(\theta)\) and \(1 - \cos(\theta)\) are both nonnegative,

\[
4D \geq 4 + 2r(1 - \cos(\theta))(1 + \cos(\theta)) \left( \frac{1}{|z - 1||z + 1|} \right)
\]

Thus

\[
D \geq 1 + \frac{r \sin^2(\theta)}{2|z - 1||z + 1|}
\]

This proves the inequality \(D \geq 1\), which in turns implies the linear independence conjecture, for \(n = 2\). Moreover, it is not too difficult to see that equality \(D = 1\) occurs if and only if \(\sin(\theta) = 0\), or, in other words, if the two points \(x_1\) and \(x_2\) lie on the same line through the origin.
5. A Conjecture

Similar to conjecture 2 in [4], we make the following conjecture

**Conjecture 2:** for any \( n > 2 \) and for any \( \mathbf{x} \in C_n \), we have \( D(\mathbf{x}) \geq 1 \).

The author wrote a small code in Python in order to test this conjecture numerically. For instance, for \( n = 4 \), by generating pseudo-randomly 5 configurations, with points in \( \mathbb{R}^3 \) inputted row-wise:

```python
array([[ 2.47281778, 3.62052364, -2.70504408],
       [ 2.27770781, 2.09191822, -3.6166916 ],
       [-3.07973444, -3.72377639, -1.38146312],
       [ 2.76502302, 0.7349035 , -4.5999035 ],],

       [[-2.72416902, 1.13834182, 2.58450461],
       [-3.79314958, -1.33356991, 2.87890805],
       [ 3.40334318, 4.23018182, 2.12190586],
       [ 2.9544125 , 0.60701485, 0.27360666]),

       [[-3.67385531, -1.9883244 , -0.7827265 ],
       [-0.0506483 , -0.21253557, -3.70488325],
       [ 1.95411965, 4.9861575 , 1.24062531],
       [-0.52216459, -3.65415534, 4.34877093]],

       [[-0.58277887, -4.78615108, 3.7967915 ],
       [ 4.67841397, -2.47607831, -1.36649441],
       [ 3.2036798 , 2.2235935 , -1.21034345],
       [ 4.46273948, -0.52665001, -0.20335324]]),

       [[-0.27182305, -0.19298397, -3.77511865],
       [ 3.28036712, 4.60665408, -3.38594041],
       [-0.67987703, 3.68537834, -1.46815721],
       [ 3.59217276, -4.65889793, -2.65232372]])
```

We get the following corresponding values for \( D \):

```python
array([10.30510163 -6.59500902e-15j, 17.85736143 +1.19825483e-16j, 30.10463298 +1.51630999e-14j, 16.66601689 -7.57016637e-16j, 37.59270816 -1.28861893e-14j])
```

Note that in Python, the complex number \( i \) is denoted by \( j \). We know that \( D \) is real, so that the imaginary parts appearing should not be there. However, they only appear because of rounding errors and, as one can see, they are very small.

The author did some numerical testing for conjecture 2 for \( n \leq 10 \). The code will be made available on the author’s website (www.malkoun.org).

6. Conclusion

We have developed an \( Sp(n) \) version of the Atiyah and Sutcliffe problem, and made two conjectures: conjecture 1, or the linear independence conjecture, and conjecture 2, namely that \( D \) is always greater or equal to 1. Conjecture 2, if true, would imply conjecture 1. Conjecture 1 is needed for our map \( f_n \) defined on \( C_n \) to take values in \( F_n \). We then
automatically get that \( f_n \) is equivariant for the actions of the Weyl group \( W_n \) of \( Sp(n) \), so that \( f_n \) would then be a solution of the generalized Berry-Robbins problem, in the sense of Atiyah and Bielawski, for the group \( Sp(n) \).

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Department of Mathematics and Statistics, Notre Dame University-Louaize, Lebanon
E-mail address: joseph.malkoun@ndu.edu.lb