We show that two-dimensional sigma models are related to certain perturbed conformal field theories. When the fields in the sigma model take values in a space $G/H$ for a group $G$ and a maximal subgroup $H$, we argue that the corresponding conformal field theory is the $k \to \infty$ limit of the coset model $(G/H)_k$, and the perturbation is related to the currents of $G$. Non-perturbative instanton contributions to the sigma model free energy are perturbative when $k$ is finite. We use this mapping to find the free energy for the $O(n)$ model at non-zero temperature. It also results in a new approach to the $CP^n$ model.

Sigma models are used frequently in particle physics and condensed-matter physics to describe Goldstone excitations and their interactions. When a field transforming under some symmetry group $G$ has an expectation value invariant under some subgroup $H$, the low-energy modes of the field take values in the manifold $G/H$. The $G/H$ sigma model is the field theory describing these low-energy modes. Even in two dimensions, where quantum effects restore the original symmetry group $G$ and the low-energy excitations are massive, sigma models are very useful. Two-dimensional sigma models have been the subject of a huge amount of study because they can be interesting toy models for gauge theories, because they often arise in experimentally-realizable condensed-matter systems, because this is the highest dimension in which they are naively renormalizable, and because of the powerful theoretical methods applicable.

One of the great breakthroughs in two-dimensional field theory was the realization that many known models (and even more previously-unknown theories) could be written as perturbed conformal field theories. One starts with the model at its critical point, which is described by a conformal field theory. In many conformal field theories, all the relevant operators are known. One can thus define a massive field theory by adding some relevant operator to the action. This defines the theory to all orders in perturbation theory, even if the action of the conformal field theory is not known. Not any such model is integrable (most are not) but if it is, one can apply a variety of techniques to find for example the exact $S$ matrix and the free energy. Literally dozens of infinite hierarchies of models have been solved over the last decade, the most famous single model being the Ising field theory at $T = T_c$ in a magnetic field.

Sigma models have stood somewhat apart from this line of development. Some exact $S$ matrices (for example, in the $O(n)$ model and the principal chiral model) have been known for quite some time. The energy in a magnetic field at zero temperature in these models can be computed, but further progress has been slow.

Usually one computes finite-temperature properties in an integrable model using the thermodynamic Bethe ansatz, but often it is not known how to categorize the solutions of the Bethe ansatz equations, a necessary step for the thermodynamics. There are a few cases where the computation is possible: the $O(3)$ model and its sausage deformation, the $O(4)$ model (equivalent to the $SU(2)$ principal chiral model) and the supersymmetric $CP^n$ models. All these models have an intriguing similarity: they can all be expressed as limits of certain perturbed conformal field theories. The purpose of this paper is to explain the general principle behind these results, and to extend it further.

We will make a general conjecture that these $G/H$ sigma models are equivalent to the $k \to \infty$ limit of a particular perturbation of the coset conformal field theory $(G/H)_k$. The utility of this result is threefold. First of all, it uncovers a nice general structure of $G/H$ sigma models. Moreover, it makes it possible to use the powerful methods of perturbed rational conformal field theory on sigma models. Finally, a great deal is known about integrable perturbed conformal field theory (much more than in sigma models), and therefore these results can be applied to sigma models. For example, technical complications had prevented the computation of the free energy of the $O(n)$ sigma models directly. We will show how this construction enables this computation for any $n$.

One of the interesting consequences of this reformulation is that non-perturbative instanton contributions to the free energy are perturbative when $k$ is finite. For example, in the appropriate perturbation of the coset models $(O(3)/O(2))_k$, there is a contribution to the free energy at order $k$ times any integer. When one takes $k \to \infty$ to obtain the $O(3)$ sigma model, this contribution turns into the instanton contribution, which is not polynomial in the perturbing parameter. This remains true even when a theta term is present. For example, this yields the result that the $SU(N)/SO(N)$ sigma models are integrable when $\theta = \pi$, and they flow to the $SU(N)_1$ conformal field theory.
We study symmetric spaces $G/H$, where $G$ and $H$ are Lie groups, and $H$ is a maximal subgroup of $G$. The $G/H$ sigma model has action

$$S = \int d^2 z \, g_{ij}(X) \partial_\mu X^i(z, \bar{z}) \partial^\mu X^j(z, \bar{z}), \quad (1)$$

where $z$ and $\bar{z}$ are coordinates for two-dimensional spacetime, and $X^i$ and $g_{ij}(X)$ are the coordinates and metric for the manifold $G/H$. Symmetric spaces have non-vanishing curvature, so $(1)$ defines an asymptotically-free massive field theory. When $G = \text{H} \times H$ and $H$ is a simple Lie group diagonally embedded in $G$, the resulting sigma model is called the principal chiral model. Another example is the $O(n)$ model, where $G = O(n)$ and $H = O(n-1)$. This space is an $n-1$ dimensional sphere: $O(n)$ is the rotational symmetry of the sphere, while $O(n-1)$ is the subgroup leaving a given point fixed.

A coset conformal field theory utilizes the affine Kac-Moody algebra $G_k$ defined by the operator product

$$J^A(z)J^B(w) = \frac{k}{(z-w)^2} + \frac{f^{ABC}f^{CD}(w)}{z-w} + \ldots, \quad (2)$$

where the $f^{ABC}$ are the structure constants of the ordinary Lie algebra for $G$ and $k$ is called the level; $k$ is a positive integer for a compact Lie group. A conformal field theory with current algebra $G_k$ is called a Wess-Zumino-Witten model, and is equivalent to the principal chiral model for $G$ plus an extra piece called the Wess-Zumino term $\xi$. The central charge (coefficient of the conformal anomaly) of the $G_k$ WZW model is $k \dim G/(k+h)$, where $f^{ACD}f^{BCD} = h\delta_{AB}/2$. For $G = SU(n)$, $h = n$, while for $G = SO(n)$, $h = n-2$ (for $n \geq 4$). The primary fields of the WZW model have scaling dimensions $x_j = 2C_j/(k+h)$, where $C_j$ is the quadratic Casimir defined by $T^A T^A = C_j I$, with the $T^A$ the generators of the Lie algebra of $G$ in the $j$th representation and $I$ the identity matrix. All the other scaling fields arise from the operator product of the $J^A(z)$ with the primary fields; it follows from $(2)$ that $J$ has dimension 1 and all fields have dimensions $x_j$ plus an integer.

Given a subgroup $H$ of $G$, a $(G/H)_k$ coset conformal field theory is defined from the generators of $G_k$ not in the subalgebra $H_l$ (where $k$ is the index of the embedding of $H$ into $G$). The central charge of this new conformal field theory is $c_G - c_H$. The energy-momentum tensor obeys the orthogonal decomposition $T_G = T_H + T_{G/H}$, so a field $\phi_G$ (some representation of $G_k$) decomposes into representations $\phi^a_H$ of $H_l$ as

$$\phi_G \cong \oplus_a \phi^a_G \otimes \phi^a_H. \quad (3)$$

The coefficients $\phi^a_G/H$ are the fields of the coset model $(G/H)_k$. A consequence of $G/H$ being a symmetric space is that the generators of $G$ not in $H$ form a real irreducible representation of $H_l$. Thus when the currents $J^A(z)$ are decomposed into representations of $H_l$, the resulting fields in the coset model form a real irreducible representation of $H_l$, which we denote as $J^a$, for $a = 1, \ldots, (\dim G - \dim H)$.

Obviously, the $G/H$ sigma model cannot be equivalent to a coset theory $(G/H)_k$, because the latter is massless while the former is not. A massive field theory is defined by perturbing $(G/H)_k$ by a relevant operator. We can now state our conjecture precisely.

**Conjecture** The sigma model for the symmetric space $G/H$ is equivalent to the $k \to \infty$ limit of the $(G/H)_k$ coset conformal field theory perturbed by the operator

$$\mathcal{O}_a = \sum_{a=1}^{\dim G - \dim H} J^a(z)J^a(\bar{z}). \quad (4)$$

Because the $J^a$ form a real irreducible representation of $H_l$, their dimension is independent of $a$.

This perturbed coset has the general properties of a sigma model. In the ultraviolet limit, the perturbation of $(G/H)_k$ goes away, and its central charge when $k \to \infty$ is $\dim G - \dim H$. In the ultraviolet limit of the sigma model, asymptotic freedom means that the manifold $G/H$ becomes flat (e.g. in the $O(n)/O(n-1)$ model, the radius of the sphere goes to infinity). The action $(4)$ reduces to $\dim G - \dim H$ free bosons, which also have the central charge $\dim G - \dim H$. Moreover, when $J^A$ is decomposed into representations of $H_l$, the resulting field $\phi^a_G$ has dimension going to zero as $k \to \infty$. Thus the field $J^a$ has dimension 1 in this limit, so the perturbation $\mathcal{O}_a$ is of dimension 2 and so is naively marginal. It is not exactly marginal – this is the phenomenon of dimensional transmutation common to sigma models.

For principal chiral models, the conjecture is already known to be true \[13\], and is reminiscent of an earlier description in terms of an infinite number of fermion flavors \[10\]. Since $\dim G - \dim H = \dim H$ here, the perturbation $\mathcal{O}_a$ of the coset $H_k \times H_l/H_{2k}$ is in the adjoint of $H_{2k}$. The usual coset notation for such an operator is $(1,1;\text{adjoint})$. This means that the corresponding $\phi_G$ is a descendant of the identity primary field in the $H_k$ conformal field theories (i.e. $J^A$ operating on the vacuum), and the $\phi_H$ in its decomposition are in the adjoint of $H_{2k}$. Such an operator is often called the “thermal” operator (because when $k = 1$ and $H = SU(2)$, $\mathcal{O}_a$ is the thermal operator in the Ising model). The particles in the perturbed coset models are kinks whose exact $S$ matrices were conjectured in \[13\] \[14\]. For finite $k$, the kinks form representations of the quantum-group algebra $U_q(H)$ with $q = -\exp(i\pi/(k+h))$. As $k \to \infty$, $q \to -1$ and the quantum-group algebra reverts to the ordinary Lie algebra of $H$. For example, for $SU(4)$, this means that particles are in the 4, the 6 and the 10 representations, giving 14 particles all together. Once an “intertwiner” is used to change basis, the $S$ matrices in
the $k \to \infty$ limit are those conjectured for the $H \times H/H$ sigma models in [3]. The exact free energy for the coset models was found in [19].

In the “$O(3)$” sigma model, the fields take values on the sphere, which is the symmetric space $O(3)/O(2) \cong SU(2)/U(1)$. The curvature (or equivalently, the radius) of the sphere determines the mass scale of the model. In this case, the conjecture above was put forth in [1]. There it was phrased as taking the $k \to \infty$ limit of the $Z_k$ parafermion theories perturbed by the operator $\psi_1 \tilde{\psi}_1 + b.c.$ Parafermions are a generalization of fermions which instead pick up $Z_k$ phases when taken around one another; $\psi_i(z)$ and $\tilde{\psi}_i(\bar{z})$ are the parafermions, where $i$ runs from 1 to $k−1$. The $Z_k$ parafermion models can be described by the coset $SU(2)k/U(1)$, and the operator $J^1$ here is indeed the parafermion $\psi_1$, while $J^2 = \psi_1^4$ [21]. As opposed to the principal chiral models, the particles here are in the vector representation of $O(3)$. Thus our result provides a natural explanation and generalization of the conjecture of [3]. One interesting thing about this model is that a topological theta term can be added to the action of the coset model [7]. The Bethe ansatz of the RSOS model is equivalent to $SU(2)k/O(2)$ [23]. The Boltzmann weights of the RSOS model are precisely the $S$ matrix of the perturbed $O(n)k/O(n−1)k$ coset (up to an overall function which makes $S1S = 1$; this factor was worked out in [23]), and the problem of finding the free energy is closely related (an analogous computation and references are in [3]). We then can take the $k \to \infty$ limit to obtain the free energy of the sigma model.

We first discuss $n$ even, where $O(n)$ is simply laced. The function $c_0(\theta)$ is defined so that the filling fraction of particles at rapidity $\theta$ and temperature $T = 1/(1 + \exp(c_0(\theta)))$ (the filling fraction is the density of particles divided by the density of states). Equivalently, $Tc_0(\theta)$ is the energy it takes to create a particle of energy $m \cos \theta$ over the Fermi sea; the mass $m$ of a particle is related to the sigma model coupling (the radius of the $n−1$ dimensional sphere) in [3]. We also define a set of “magnon energies” $\epsilon^{(n)}_r(\theta)$, where $r = 1..k−1$ and $a = 1..n/2$. The functionals $A_n^{(a)}(\theta)$ are defined as

$$A_n^{(a)}(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{n-2}{2\cosh((n-2)(\theta - \theta')/2)} \ln(1 + e^{\epsilon^{(n)}_r(\theta')})$$

while $\tilde{A}_n^{(a)}(\theta)$ is defined with $\epsilon \to -\epsilon$. The matrix $I_{ab}$ is $2 - C_{ab}$, where $C_{ab}$ is the Cartan matrix for $O(n)$, while the matrix $I_{rs} = \delta_{r,s-1} + \delta_{r,s+1}$. Finally, with $M_{ab}(x) = 2\cosh(\pi x/(n-2))\delta_{ab} - I_{ab}$ and $M^{-1}$ its matrix inverse, the function $f_a(\theta)$ is defined as the Fourier transform

$$f_a(\theta) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{i\theta x} (M^{-1})_{1a}(x)$$

Then an extension of the results of [19] yields the following integral equations, valid for even $n > 4$ and $k \geq 2$:

$$c_0(\theta) = \frac{m}{T} \cosh \theta - \frac{n/2}{a=1} \int_{-\infty}^{\infty} d\theta' f_a(\theta - \theta') \ln(1 + e^{-\epsilon^{(n)}_r(\theta')})$$

$$\epsilon^{(a)}_r = -\tilde{A}_0 \delta_{r,1} a_{a,1} - \sum_{s=1}^{k-1} I_{rs} A^{(a)}_s + \sum_{b=1}^{n/2} I_{ab} A^{(b)}_r$$

The free energy per unit length is then given by

$$F = -\frac{mT}{2\pi} \int_{-\infty}^{\infty} d\theta \cosh \theta \ln(1 + e^{-c_0(\theta)})$$

When $n$ is odd so $O(n)$ is not simply laced, the structure is more complicated. Nevertheless, the conjecture still is valid and the free energy follows from [20].
function $M_{ab}$ is given by their equation (B.10), while the second equation in (3) is replaced by their (B.4a) with its left-hand side replaced by $-A_0\delta_{1,1}\delta_{a,1}$ (note also that the range of $r$ depends on the value of $(a)$).

These equations are straightforward to solve numerically. The free energy as $m \to 0$ gives the correct value, proportional to the central charge $c = k(n - 1)/(2k + n - 4)/(2(k + n - 4)(k + n - 2))$ of the $O(n)_k/O(n-1)_k$ conformal field theory. The equations remain well-defined as $k \to \infty$; an infinite number of magnons is a generic characteristic of models with Lie algebra symmetries (as opposed to quantum-group structure).

We think the above arguments are convincing for integrable models, but other cases remain mostly unexplored. The $CP^{n-1}$ sigma model, which has $G = SU(n)$ and $H = SU(n-1) \times U(1)$, is particularly interesting. This is believed to be not integrable except for $n=2$, where $SU(2)/U(1) \approx O(3)/O(2)$. The (not conclusive) evidence against integrability is that no local conserved charges have been found [24], and that anomalies appear in the non-local conservation laws [24]. Our conjecture may provide a useful way of exploring the model’s properties. The $SU(n)_k/SU(n-1)_k \times U(1)$ coset model is dual to the “$W^{(k)}$” minimal model $SU(k)_{n-1} \times SU(1)/SU(k)_{n}$. In the latter, the perturbing operator $\sigma_\tau$ is denoted $(k; k; 1) + (k, k; 1)$. For $k=2$, this model is the $\Phi^{15}$ perturbation of the nth minimal model. Both this $\Phi^{15}$ and the $k=3$ case [25] are integrable, but the counting argument used to prove integrability for $k=2,3$ does not yield a conserved current for $k > 3$. However, (at least to first order in perturbation theory), all these models have a nonlocal symmetry generated by the chiral part of the $W^{(k)}_{-1}(1, 1; \text{adjoint})$ operator ($\Phi^{15}$ for $k = 2$).

There are a number of prospective uses of our conjecture in the $CP^{n-1}$ model. One could use the truncated conformal scaling approach [25] to find the low-lying energy levels of the theory; a signal of integrability is that the levels can cross as the strength of the perturbation is varied. Also, the conjecture implies the existence of nonlocal conserved quantities in the sigma model, by taking the $k \to \infty$ limit of those in the perturbed coset model. These do not seem to be anomalous like the ones discussed in [24], so even if $CP^{n-1}$ is not integrable, it still should have an interesting symmetry structure.

We have found a broadly-applicable and useful feature of $G/H$ sigma models, a feature which we have conjectured to be completely general. In fact, we believe it is even true when $G/H$ is not a symmetric space; the complication is that there are multiple coupling constants in the sigma model, and multiple perturbations of the coset model. Moreover, simple extensions of this conjecture allow for topological or Wess-Zumino terms in the sigma model action, and also to supersymmetric sigma models.

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