Singularities in $K$-space and Multi-brane Solutions in Cubic String Field Theory

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Abstract

In a previous paper [arXiv:1111.2389], we studied the multi-brane solutions in cubic string field theory by focusing on the topological nature of the “winding number” $N$ which counts the number of branes. We found that $N$ can be non-trivial owing to the singularity from the zero-eigenvalue of $K$ of the $KBc$ algebra, and that solutions carrying integer $N$ and satisfying the EOM in the strong sense is possible only for $N = 0, \pm 1$. In this paper, we extend the construction of multi-brane solutions to $|N| \geq 2$. The solutions with $N = \pm 2$ is made possible by the fact that the correlator is invariant under a transformation exchanging $K$ with $1/K$ and hence $K = \infty$ eigenvalue plays the same role as $K = 0$. We further propose a method of constructing solutions with $|N| \geq 3$ by expressing the eigenvalue space of $K$ as a sum of intervals where the construction for $|N| \leq 2$ is applicable.

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1 Introduction

Construction of multi-brane classical solutions in cubic string field theory (CSFT) [1] has attracted much attention recently [2, 3, 4, 5, 6, 7, 8, 9]. Most of the analytical solutions in CSFT are written in pure-gauge form [10], \( \Psi = U Q B U^{-1} \), and let us consider

\[
\mathcal{N} = \frac{\pi^2}{3} \int (U Q B U^{-1})^3, \tag{1.1}
\]

which is related to the energy density \( \mathcal{E} \) of the solution by \( \mathcal{E} = \mathcal{N} / (2\pi^2) \). Since the energy density of the tachyon vacuum is \(-1/(2\pi^2)\), the solution represents the \( \mathcal{N} + 1 \) branes. An important property of \( \mathcal{N} \) is that it is a “topological quantity” invariant under a small deformation of \( U \). The problem of constructing multi-brane solutions is equivalent to understanding what kind of large deformation of \( U \) can change \( \mathcal{N} \).

In our previous paper [4], we examined \( \mathcal{N} \) by focusing on its similarity to its counterpart in the Chern-Simons (CS) theory in three dimensions:

\[
N_{\text{CS}} = \frac{1}{24\pi^2} \int_M \text{tr} \left( g d g^{-1} \right)^3. \tag{1.2}
\]

This is also a topological quantity and counts the winding number of the mapping \( g(x) \) from the manifold \( M \) to the gauge group. Then, a natural question is whether \( \mathcal{N} \) in CSFT also has an interpretation as a kind of “winding number” taking possibly integer values. For convenience, we call \( \mathcal{N} \) simply winding number in this paper.

Our argument in [4] is restricted to the pure-gauge solution of Okawa type [10] with \( U \) given by

\[
U = 1 - Bc \left( 1 - G(K) \right), \tag{1.3}
\]

and specified by the function \( G(K) \). Here, \((K, B, c)\) satisfies the \( KBc \) algebra [10]:

\[
[B, K] = 0, \quad \{ B, c \} = 1, \quad B^2 = c^2 = 0, \tag{1.4}
\]

and

\[
Q_B B = K, \quad Q_B K = 0, \quad Q_B c = cKc. \tag{1.5}
\]

For \( U \) of (1.3), \( \Psi = U Q_B U^{-1} \) is given by

\[
\Psi = c \frac{K}{G(K)} Bc \left( 1 - G(K) \right). \tag{1.6}
\]

In this paper, we assume that \( G(K) \) is a rational function of \( K \).

Let us recapitulate our results in [4]:

1 We are considering static and translationally invariant solutions, and have put both the space-time volume and the open string coupling constant equal to 1.

2 In this paper, we adopt for convenience the non-Hermitian convention for \( \Psi = U Q_B U^{-1} \). This can be converted to the Hermitian one used in [4] by a gauge transformation.
Corresponding to the fact that $N_{CS}$ (1.2) is written as the integration of an exact quantity, $N_{CS} = \int_M dH$, we showed that $\mathcal{N}$ has the following expression:

$$\mathcal{N} = \int Q_B A.$$  \hspace{1cm} (1.7)

In the case of $N_{CS}$, singularities of $H$ can make $N_{CS}$ non-vanishing; $N_{CS} = \int_{\partial M} H \neq 0$ with $\partial M$ being the set of singularities of $H$. Quite similarly, although the RHS of (1.7) vanishes naively, it can take non-trivial values owing to the singularities existent in $A$.

We found that the origin of the singularity of $A$, and therefore the origin of the winding number $\mathcal{N}$ is the zero eigenvalue of $K$.

In order to properly evaluate the RHS of (1.7), we have to introduce a regularization for the singularity at $K = 0$. Since the eigenvalue distribution of $K$ is restricted to real and non-negative, we adopted in [4] the $K_\varepsilon$-regularization of making the following replacement:

$$K \to K_\varepsilon = K + \varepsilon,$$  \hspace{1cm} (1.8)

with $\varepsilon$ being a positive infinitesimal. In calculating the RHS of (1.7) in the $K_\varepsilon$-regularization, we make the replacement (1.8) after evaluating $Q_B A$ by using (1.5). Then, we have

$$\int (Q_B A)_{K \to K_\varepsilon} \sim \varepsilon \times \frac{1}{\varepsilon} \neq 0,$$  \hspace{1cm} (1.9)

where $\varepsilon$ and $1/\varepsilon$ are from the violation of the BRST-exactness of the integrand due to the $K_\varepsilon$-regularization and from the singularity of $A$ at $K = 0$, respectively.

We evaluated $\mathcal{N}$ (1.7) for $G(K)$ of the form \[G(K) = \left(\frac{K}{1 + K}\right)^n, \quad (n = 0, \pm 1, \pm 2, \cdots).\] (1.10)

We found that $\mathcal{N}$ takes integer values, $\mathcal{N} = -n$, only when $n = 0, \pm 1$. For $|n| \geq 2$, $\mathcal{N}$ deviates from the integer $-n$; $\mathcal{N} = \mp 2 \pm 2\pi^2$ for $n = \pm 2$. For a generic $n$, we obtain

$$\mathcal{N} = -n + A(n),$$  \hspace{1cm} (1.11)

with

$$A(n) = \frac{\pi^2}{3} n(n^2 - 1) \text{Re}_1 F_1(2 + n, 4; 2\pi i),$$  \hspace{1cm} (1.12)

where $1 F_1(\alpha, \gamma; z)$ is the confluent hypergeometric function (see (A.8)). The result (1.11), whose derivation is outlined in Appendix A, has been obtained by expanding

\[3 \text{ Precisely speaking, these singularities mean those in the integrand of the function } B(\tau) \text{ (2.25) at } \tau = 0 \text{ or } \tau = 1.\]

\[4 \text{ A(n) is an odd function of } n. \text{ The same anomaly as (1.12) appears in [5].}\]
G(K) around K = 0; only the leading power n of G(K) determines N. This is validated by the observation mentioned above that only the singularity of A at K = 0 determines the value of N. The anomaly A(n) in N (i.e., the deviation of N from the integer −n) is inevitable except when G(K) is finite or has simple zero or simple pole at K = 0. (As we shall see later, the behavior of G(K) at K = ∞ also affects the winding number. Here and in [4], we are assuming that G(K = ∞) is finite and non-vanishing.)

Here, we comment on the relation between our result (1.11) and that in the pioneering works [2, 5]. In [2, 5], they claim that G(K) of (1.10) gives N = −n without the anomaly term for any n. However, this is owing to their choice of deforming the contour of z-integration along the pure-imaginary axis in the formula (B.3) to the left (Re z < 0) for avoiding the singularity at z = 0. This manifestly contradicts with our Kε-regularization (1.8) which corresponds to integrating along Re z = ε > 0 and multiplying the integrand of (B.3) by e−εs. The contour of [2, 5] rather corresponds to K → K − ε, which cannot work as a regularization. In this paper, we admit that the anomaly in N is unavoidable for |n| ≥ 2 and pursue new ways of constructing solutions without anomaly.

• The pure-gauge solution with the Kε-regularization, Ψε = (UQBU−1)K→Kε, is no longer a pure-gauge, and breaks the EOM by apparently O(ε). We examined the EOM in the strong sense for Ψε:

\[
\text{EOM-test}[\Psi] = \int \Psi * (Q_B \Psi + \Psi * \Psi) = \varepsilon \times \mathcal{E}_\varepsilon, \tag{1.13}
\]

with \(\mathcal{E}_\varepsilon\) given by

\[
\mathcal{E}_\varepsilon[G(K)] = \int BcG(K_\varepsilon)c \frac{K_\varepsilon}{G(K_\varepsilon)}c \frac{K_\varepsilon}{G(K_\varepsilon)}. \tag{1.14}
\]

Since (1.13) is multiplied by \(\varepsilon\), it vanishes unless \(\mathcal{E}_\varepsilon\) contains the 1/ε singularity which again comes from the zero eigenvalue of K. The situation is quite similar to the case of the winding number N (1.7), and only the leading power of G(K) around K = 0 determines EOM-test[\Psi] (1.13). For G(K) of (1.10), we found that the EOM in the strong sense holds for n = 0, ±1, but it is violated for |n| ≥ 2. Concretely, we obtain (see Appendix A)

\[
\text{EOM-test}[\Psi] = B(n) \equiv \frac{n(n-1)}{\pi} \text{Im}_1 F_1(1 + n, 2, 2\pi i), \tag{1.15}
\]

which vanishes only when n = 0, ±1.

Summarizing, the singularity due to the zero eigenvalue of K is the origin of both the winding number N and the EOM-test. The winding number N deviates from integer and the EOM in

\[B(n) \text{ has the property } B(-n) = (1 + n)/(1 - n) B(n).\]
the strong sense is violated unless \( G(K) \) is finite or has simple pole or simple zero at \( K = 0 \). Namely, we can construct satisfactory (multi-)brane solutions only for the tachyon vacuum \((N = -1)\), a single brane \((N = 0)\) and two branes \((N = 1)\).

In this paper, we present a way to construct multi-brane solutions with \(|N| \geq 2\). The point is that the singularities from the \( K = \infty \) eigenvalue as well as that from \( K = 0 \) can be the origins of the winding number \( N \) \((1.7)\). This is owing to the invariance of the correlator under a transformation which replaces \( K \) with \( 1/K \). This remarkable property implies, in particular, that \( G(K) \) and \( G(1/K) \) give the same values of \( N \) and EOM-test. By using the singularities from both \( K = 0 \) and \( K = \infty \), we can construct solutions with \( N = \pm 2 \) and satisfying the EOM in the strong sense. The corresponding \( G(K) \) with \( N = 2 \), for example, should behave near \( K = 0 \) and \( K = \infty \) as \( G(K) \sim 1/K \) \((K \to 0)\) and \( G(K) \sim K \) \((K \to \infty)\), respectively. The singularity at \( K = \infty \) needs a regularization besides the \( K_\varepsilon \)-regularization \((1.8)\) for \( K = 0 \). We devise such a regularization from the \( K_\varepsilon \)-regularization by using the transformation \( K \mapsto 1/K \) mentioned above.

However, it is impossible to construct satisfactory solutions with \(|N| \geq 3\) since higher order zeros/poles of \( G(K) \) at \( K = 0 \) and \( K = \infty \) inevitably lead to the anomaly in \( N \) and the breaking of the EOM in the strong sense. As a possible resolution to this problem of constructing satisfactory solutions with \(|N| \geq 3\), we consider using \( G(K) \) which has zeros/poles in \( 0 < K < \infty \). Such \( G(K) \) is “dangerous” since its \( N \) and EOM-test contain quantities which do not have well-defined Schwinger parameter representations. We propose a way of defining \( N \) and EOM-test for such \( G(K) \) by giving them as a sum of well-defined ones obtained by expressing the eigenvalue space of \( K \), \([0, \infty]\), as a sum of intervals. This division into intervals is analogous to the division of a sphere \( S^2 \) into two hemispheres for defining the vector field of \( U(1) \) monopole. By this method, we can construct satisfactory multi-brane solutions carrying any integer \( N \).

The organization of the rest of this paper is as follows. In Sec. 2, we show that \( K = \infty \) is equivalent to \( K = 0 \), and can be the origin of the winding number. This leads us to the construction of satisfactory solutions with \( N = \pm 2 \). In Sec. 3, we propose a way of defining solution carrying \(|N| \geq 3\) by division of the eigenvalue space of \( K \) into intervals. We summarize the paper and discuss future problems in Sec. 4. In Appendix A we present derivations of eqs. \((1.11)\) and \((1.15)\). In Appendix B we give a proof of the invariance of the correlator under the transformation which exchanges \( K \) with \( 1/K \).

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6 In this paper, we mean by a “satisfactory solution” a one carrying an integer \( N \) and satisfying the EOM in the strong sense. It is of course a problem in what sense the EOM should hold and whether the EOM in the strong sense is sufficient. We discuss this matter in Sec. 4.
2 \( K = \infty \) as another origin of \( \mathcal{N} \)

The winding number \( \mathcal{N} \) (1.1) in CSFT has another expression (1.7), which is the integration of a BRST-exact quantity \( Q_BA \) and hence is apparently equal to zero. We saw in [4] that \( \mathcal{N} \) can take non-trivial values due to the singularity of \( A \) at \( K = 0 \). This allowed us to construct satisfactory multi-brane solutions only for \( \mathcal{N} = 0 \) and \( \pm 1 \). In this section, we show that another eigenvalue of \( K, K = \infty \), can play the same role as \( K = 0 \) and hence can make \( \mathcal{N} \) non-trivial. By using the singularities both at \( K = 0 \) and \( K = \infty \), we can extend the construction of satisfactory multi-brane solutions to the cases \( \mathcal{N} = \pm 2 \).

2.1 Equivalence of \( K = 0 \) and \( K = \infty \)

We wish to show that \( K = \infty \) is in a sense equivalent to \( K = 0 \). This is due to the invariance of the correlator on the infinite cylinder under the following transformation which replaces \( K \) with \( 1/K \) and keeps the \( KBc \) algebra (1.4) and (1.5):

\[
\begin{align*}
K &\mapsto \tilde{K} = \frac{1}{K}, & B &\mapsto \tilde{B} = \frac{B}{K^2}, & c &\mapsto \tilde{c} = cK^2Bc. \\
\end{align*}
\]

(2.1)

Namely, the following surprising equality holds for any \( W(K, B, c) \) with ghost number 3:

\[
\int W(K, B, c) = \int W(\tilde{K}, \tilde{B}, \tilde{c}).
\]

(2.2)

A proof of (2.2) is given in Appendix [3]. There, we show (2.2) by using the \((s, z)\)-integration expression of the correlator given in [2, 5], and making a suitable change of integration variables. We call the transformation (2.1) “inversion” in the rest of this paper.

The inversion (2.1) is a special case \((g(K) = 1/K)\) of a more general transformation \( M_g \) which keeps the \( KBc \) algebra and is specified by a function \( g(K) \) [12, 8, 13]:

\[
\begin{align*}
M_g(K) &= g(K), & M_g(B) &= \frac{g(K)}{K} B, & M_g(c) &= c\frac{K}{g(K)} Bc. \\
\end{align*}
\]

(2.3)

We call the transformation \( M_g \) (2.3) Erler-Masuda-Noumi-Takahashi (EMNT) transformation hereafter. Note that the correlator is not invariant under the EMNT transformation of an arbitrary \( g(K) \). This is because the correlator is not determined by the \( KBc \) algebra alone. The inversion (the tilde operation) (2.1) is \( M_{1/K} \). Note that \( Bc \) is invariant under any EMNT transformation, \( M_g(Bc) = Bc \).

The effect of the inversion (2.1) on \( \Psi \) of (1.6) is simply to replace \( G(K) \) with \( G(1/K) \). This fact together with the invariance (2.2) implies that \( G(K) \) and \( G(1/K) \) should give the same \( \mathcal{N} \). In this sense, \( K = \infty \) is “equivalent” to \( K = 0 \), and can be the origin of the winding number.
and that of the breaking of the EOM in the strong sense. Recall that the formula (1.11) counts the winding number from the singularity at \( K = 0 \) alone. Therefore, the contribution of the singularity at \( K = \infty \) to \( \mathcal{N} \) is given by applying (1.11) to \( G(1/K) \). Suppose that the leading behaviors of \( G(K) \) near \( K = 0 \) and \( K = \infty \) are

\[
G(K) \sim \begin{cases} 
K^{n_0} & (K \to 0) \\
(1/K)^{n_\infty} & (K \to \infty)
\end{cases},
\]

with \( n_0 \) and \( n_\infty \) being integers. Then, the total winding number is given by

\[
\mathcal{N} = -n_0 - n_\infty + A(n_0) + A(n_\infty).
\]

### 2.2 Regularization for both \( K = 0 \) and \( K = \infty \)

For precisely defining \( \mathcal{N} \) and EOM-test, we need a regularization for \( K = \infty \) as well as for \( K = 0 \). We introduce the regularization for \( K = \infty \) as the map of the \( K_\varepsilon \)-regularization (1.8) by the inversion (2.1). Consider \((K, B, c) \) and \((\tilde{K}, \tilde{B}, \tilde{c}) \) related by (2.1). The regularization for \( K = \infty \) is naturally defined as the operation on \((K, B, c) \) induced by the regularization for \( \tilde{K} = 0; (\tilde{K}, \tilde{B}, \tilde{c}) \rightarrow (\tilde{K} + \eta, \tilde{B}, \tilde{c}) \) with \( \eta \) being positive infinitesimal:

\[
K = \frac{1}{K} \rightarrow \frac{1}{K + \eta} = \frac{K}{1 + \eta K}, \\
B = \frac{\tilde{B}}{K^2} \rightarrow \frac{\tilde{B}}{(K + \eta)^2} = \frac{B}{(1 + \eta K)^2}, \\
c = \tilde{c} K^2 \tilde{B} \tilde{c} \rightarrow \tilde{c} (\tilde{K} + \eta)^2 \tilde{B} \tilde{c} = c (1 + \eta K)^2 Bc.
\]

The regularization for \( K \) is simply \( 1/K \rightarrow 1/K + \eta \), and we have to change \( B \) and \( c \) as well. Let us introduce, in addition, the regularization for \( K = 0 \). There are two ways for this; one is to replace \( K \) on the RHS of (2.6) by \( K_\varepsilon = K + \varepsilon \), and the other is to start with \((K_\varepsilon, B, c) \) in (2.6). The regularized \((K, B, c) \) in each case is then given by

\[
K_{\varepsilon\eta} = \frac{K_\varepsilon}{1 + \eta K_\varepsilon}, \quad B_{\varepsilon\eta} = \frac{B}{(1 + \eta K_\varepsilon)^2}, \quad c_{\varepsilon\eta} = c (1 + \eta K_\varepsilon)^2 Bc,
\]

and

\[
K_{\varepsilon\eta}' = \frac{K}{1 + \eta K}, \quad B_{\varepsilon\eta}' = \frac{B}{(1 + \eta K)^2}, \quad c_{\varepsilon\eta}' = c (1 + \eta K)^2 Bc.
\]

These two way of regularization are interchanged by the inversion (2.1). Namely, the regularization by (2.7), \((K, B, c) \rightarrow (K_{\varepsilon\eta}, B_{\varepsilon\eta}, c_{\varepsilon\eta}) \), is expressed in terms the tilded variables of (2.1) by

\[
(\tilde{K}, \tilde{B}, \tilde{c}) \rightarrow \left( \frac{\tilde{K}}{1 + \varepsilon \tilde{K}} + \eta, \frac{\tilde{B}}{(1 + \varepsilon \tilde{K})^2}, \tilde{c} (1 + \varepsilon \tilde{K})^2 \tilde{B} \tilde{c} \right).
\]
In the rest of this paper, we mainly use the regularization by (2.7).

An important property of the regularized \((K, B, c)\), (2.7) and (2.8), is that they satisfy the (anti-)commutation relations of (1.4):

\[
[B_{\xi\eta}, K_{\xi\eta}] = 0, \quad \{B_{\xi\eta}, c_{\xi\eta}\} = 1, \quad B_{\xi\eta}^2 = c_{\xi\eta}^2 = 0, \quad (2.10)
\]

and the same equations for (2.8).\(^7\) Owing to this property, the present regularization is consistently and unambiguously defined. Namely, the regularized quantities are independent of whether we rewrite/simplify them by using the (anti-)commutation relation (1.4) before or after the replacement \((K, B, c) \rightarrow (K_{\xi\eta}, B_{\xi\eta}, c_{\xi\eta})\). On the other hand, the BRST algebra (1.5) is broken by the regularized \((K, B, c)\), (2.7) and (2.8). This is necessary for making non-trivial the winding number (1.7) given as the integration of \(Q_bA\) (see (1.9)).

The regularized version of the identity (2.2) reads

\[
\int \mathcal{W}(K_{\xi\eta}, B_{\xi\eta}, c_{\xi\eta}) = \int \mathcal{W}(\tilde{K}_{\xi\eta}, \tilde{B}_{\xi\eta}, \tilde{c}_{\xi\eta}),
\]

where \(\tilde{K}_{\xi\eta}\), for example, is the inversion of \(K_{\xi\eta}\). Explicitly, we have

\[
\tilde{K}_{\xi\eta} = \frac{1}{K'_{\eta\xi}}, \quad \tilde{B}_{\xi\eta} = \frac{B'_{\eta\xi}}{(K'_{\eta\xi})^2}, \quad \tilde{c}_{\xi\eta} = c'_{\eta\xi}(K'_{\eta\xi})^2 B'_{\eta\xi} c'_{\eta\xi},
\]

which also satisfy the (anti-)commutation relations (1.4).

### 2.3 EOM in the strong sense

Let us consider \(\Psi (1.6)\) in the regularization of (2.7):

\[
\Psi_{\xi\eta} = \Psi\big|_{(K,B,c) \rightarrow (K_{\xi\eta}, B_{\xi\eta}, c_{\xi\eta})} = c \frac{K_{\xi}^2}{K_{\xi\eta}G(K_{\xi\eta})} Bc(1 - G(K_{\xi\eta})).
\]

Due to the regularization, \(\Psi_{\xi\eta}\) no longer satisfies the EOM exactly, and the breaking consists of two terms which are apparently of \(O(\varepsilon)\) and \(O(\eta)\), respectively. Correspondingly, the EOM in the strong sense is given by

\[
\text{EOM-test}[\Psi] = \int \Psi_{\xi\eta} * (Q_b \Psi_{\xi\eta} + \Psi_{\xi\eta} * \Psi_{\xi\eta}) = \varepsilon \times \mathcal{E}_{\xi\eta} + \eta \times \mathcal{F}_{\xi\eta},
\]

\(^7\) More generally, \(K_{(g,h)} = g(K)\), \(B_{(g,h)} = h(K)B\), \(c_{(g,h)} = c\frac{K}{h(K)} Bc\), defined by \(g(K)\) and \(h(K)\) satisfy (1.4). They break the BRST algebra (1.5) unless \(g(K) = h(K)\), namely, unless they are a EMNT-transform of \((K, B, c)\).
where $\mathcal{E}_{\varepsilon\eta}$ and $\mathcal{F}_{\varepsilon\eta}$ are expressed in terms of $K_{\varepsilon\eta}$ (2.7) and $K_\varepsilon = K + \varepsilon$ as

\[
\mathcal{E}_{\varepsilon\eta}[G(K)] = \int BcG(K_{\varepsilon\eta})c \frac{K_\varepsilon^2}{K_{\varepsilon\eta}G(K_{\varepsilon\eta})} G(K_{\varepsilon\eta})c \frac{K_\varepsilon^2}{K_{\varepsilon\eta}G(K_{\varepsilon\eta})},
\]

(2.15)

\[
\mathcal{F}_{\varepsilon\eta}[G(K)] = \int BcG(K_{\varepsilon\eta})cK_\varepsilon^2 \left[ \frac{1}{K_{\varepsilon\eta}G(K_{\varepsilon\eta})} + c \right] K_\varepsilon^2 [G(K_{\varepsilon\eta}), c] \frac{K_\varepsilon^2}{K_{\varepsilon\eta}G(K_{\varepsilon\eta})},
\]

(2.16)

The EOM-test can be non-trivial if $\mathcal{E}_{\varepsilon\eta}$ is of $O(1/\varepsilon)$ and/or $\mathcal{F}_{\varepsilon\eta}$ is of $O(1/\eta)$.

In [4], we considered only the EOM breaking by $\varepsilon$, namely, $\varepsilon \times \mathcal{E}_\varepsilon = \mathcal{E}_{\varepsilon,\eta=0}$ given by (1.14). We may simplify (2.14) by putting $\eta = 0$ ($\varepsilon = 0$) in $\mathcal{E}_{\varepsilon\eta}$ (in $\mathcal{F}_{\varepsilon\eta}$) to consider $\varepsilon \times \mathcal{E}_\varepsilon + \eta \times \mathcal{F}_{\varepsilon=0,\eta}$. Then, we can easily show by using the property (2.2) that $\mathcal{E}_\varepsilon$ and $\mathcal{F}_{\varepsilon=0,\eta}$ have a simple relationship:

\[
\mathcal{F}_{\varepsilon=0,\eta}[G(K)] = \mathcal{E}_\eta[G(1/K)],
\]

(2.17)

where $\mathcal{E}_\eta$ is (1.14) with $\varepsilon$ replaced with $\eta$. Therefore, the EOM-test is expressed only in terms of $\mathcal{E}_\varepsilon$:

\[
\text{EOM-test}[\Psi] = \lim_{\varepsilon \to +0} \varepsilon \times \left\{ \mathcal{E}_\varepsilon[G(K)] + \mathcal{E}_\varepsilon[G(1/K)] \right\}.
\]

(2.18)

This is also understood without explicit calculations by applying the invariance (2.2) to EOM-test (2.14), and using the facts that the inversion commutes with $Q_B$, $[M_{1/K}, Q_B] = 0$, and that the inversion of $\Psi_{\varepsilon\eta}$ (2.13) is given by

\[
\widehat{\Psi}_{\varepsilon\eta} = \Psi|_{(K,B,c) \to (\widehat{K}_{\varepsilon\eta}, \widehat{B}_{\varepsilon\eta}, \widehat{c}_{\varepsilon\eta})} = \left( K + \eta(1 + \varepsilon K) \right)^2 Bc \left( 1 - G(1/K'_{\eta\varepsilon}) \right),
\]

(2.19)

where $(\widehat{K}_{\varepsilon\eta}, \widehat{B}_{\varepsilon\eta}, \widehat{c}_{\varepsilon\eta})$ and $K'_{\eta\varepsilon}$ are defined in (2.14) and (2.8), respectively. Eq. (2.19) should be compared with (2.13).

Eq. (2.18) implies that, for $\Psi$ specified by $G(K)$ with the behavior (2.4), its EOM in the strong sense is given by

\[
\text{EOM-test}[\Psi] = B(n_0) + B(n_\infty),
\]

(2.20)

where $B(n)$ is the function appearing in (1.15).

### 2.4 Solutions with $\mathcal{N} = \pm 2$

Now from our results (2.5) and (2.20) including contributions from both the eigenvalues $K = 0$ and $K = \infty$, we see that there exist satisfactory solutions with integer $\mathcal{N}$ and satisfying the EOM in the strong sense only when both $n_0$ and $n_\infty$ are either of 0 and $\pm 1$. This implies that we can extend the construction of satisfactory solutions to the cases $\mathcal{N} = \pm 2$ (i.e., three branes and “$(-1)$ brane”). The corresponding $G(K)$ are, for example,

\[
G(K) = \frac{(1 + K)^2}{K} \quad (\mathcal{N} = 2), \quad G(K) = \frac{K}{(1 + K)^2} \quad (\mathcal{N} = -2).
\]

(2.21)
We can also construct satisfactory solutions with $N = \pm 1$ using $K = \infty$. They are, for example,

$$G(K) = 1 + K \quad (N = 1), \quad G(K) = \frac{1}{1 + K} \quad (N = -1). \tag{2.22}$$

Note that $G(K) = K$ has $(n_0, n_\infty) = (1, -1)$ and represents the perturbative vacuum with $N = 0$.

Eqs. (2.22) shows that satisfactory solutions with $|N| \geq 3$ are impossible. One might think that solutions with $|N| \geq 3$ can be constructed by adopting $G(K)$ which has zeros/poles in $0 < K < \infty$ in addition to $K = 0$ and/or $K = \infty$. However, recall that we rely on the Schwinger parameter representation of quantities containing $G(K)$ or $1/G(K)$ in evaluating $N$. For a quantity having zeros/poles at finite $K > 0$, its Schwinger parameter representation does not exist; for example, $1/(K - a)$ cannot be expressed as $\int_0^\infty dt e^{-(K-a)t}$ for $K < a$. We will return to this problem in the next section.

### 2.5 Direct evaluation of $N$

In the above arguments, we treated the singularity at $K = \infty$ by mapping it to $K = 0$ through the inversion (2.1). In this subsection, we present the evaluation of $N$ by treating $K = \infty$ directly in the regularization (2.7). Here, we take

$$G(K) = 1 + K, \tag{2.23}$$

as an example.

Let us summarize the formulas for $N$ (1.7) in the regularization (2.7). The winding number of a pure-gauge solution $\Psi = U Q_B U^{-1}$ is given by

$$N = \int (Q_B A)_{(K,B,c)} \rightarrow (K_{\epsilon\eta}, B_{\epsilon\eta}, c_{\epsilon\eta}) = \pi^2 \int_0^1 d\tau B(\tau), \tag{2.24}$$

in terms of the function $B(\tau)$ which vanishes without regularization:

$$B(\tau) = \int \left[ Q_B \left( \Psi_\tau * \frac{d\Psi_\tau}{d\tau} \right) \right]_{(K,B,c)} \rightarrow (K_{\epsilon\eta}, B_{\epsilon\eta}, c_{\epsilon\eta}) = \int \left( \frac{d\Psi_\tau}{d\tau} * \Psi_\tau * \Psi_\tau \right)_{(K,B,c)} \rightarrow (K_{\epsilon\eta}, B_{\epsilon\eta}, c_{\epsilon\eta}). \tag{2.25}$$

Here, $\Psi_\tau$ with a parameter $\tau$ ($0 \leq \tau \leq 1$) is given by $\Psi_\tau = U_\tau Q_B U_\tau^{-1}$ in terms of $U_\tau = 1 - Bc (1 - G_\tau(K))$ which interpolates $U_{\tau=0} = 1$ and $U_{\tau=1} = U$ (hence $G_{\tau=0}(K) = 1$ and $G_{\tau=1}(K) = G(K)$). Explicitly, $B(\tau)$ is given by

$$B(\tau) = \int BcG_\tau c \frac{K_\epsilon^2}{K_{\epsilon\eta}} \left[ \frac{1}{G_\tau} \frac{dG_\tau}{d\tau} \right] K_{\epsilon\eta}^2 \left[ \frac{1}{G_\tau} \right] K_{\epsilon\eta}^2 \left[ \frac{1}{G_\tau} \right] K_{\epsilon\eta}^2, \tag{2.26}$$
with \( G_\tau = G_\tau(K_{\eta \eta}) \).

As \( G_\tau(K) \) for \( G(K) \) of (2.23), we adopt \( G_\tau(K) = 1 + \tau K \). For the present \( G(K) \) we are allowed to put \( \varepsilon = 0 \) since the integrand of (2.26) is regular at \( K = 0 \). In this simplification, \( K^2/\varepsilon \) in (2.26) is reduced to \( \varepsilon/(1 + \eta K) \), and we have

\[
G_\tau(K_{\varepsilon=0,\eta}) = 1 + \frac{\tau K}{1 + \eta K}.
\] (2.27)

Let us first consider evaluating \( B(\tau) \) (2.26) by using the \((s, z)\)-integration formula (B.3) for the correlator. Adding a large semi-circle in \( \text{Re} z < 0 \) to closed the contour of the \( z \)-integration, the poles in the \( z \)-plane are located at three points; \( z = -1/(\tau + \eta) \) and \(-1/(\tau + \eta) \pm 2\pi i/s\). Then, carrying out the \( s \)-integration, we obtain the desired result:

\[
\pi^2 B(\tau) = \frac{3\eta \tau^2}{(\tau + \eta)^4} \rightarrow \delta(\tau).
\] (2.28)

Next, we consider another way of calculating \( B(\tau) \). Note that \( B(\tau) \) is given as a sum of terms of the form \( \int BcF_1(K)cF_2(K)cF_3(K)cF_4(K) \). We carry out the integrations over the Schwinger parameters \( t_i \) for \( F_i(K) \) without using the \((s, z)\)-trick of inserting \( 1 = \int_0^\infty ds \delta(s - \sum_i t_i) \). In this case, there arises a delicate problem. In order to get the same result as (2.28), we need a special care for \( F_i(K) \) \((i = 1, 2, 3) \) which are sandwiched between two \( c \)'s; we must use the partial fraction decomposed form without any \( K \) in the denominators for each \( F_i(K) \) \((i = 1, 2, 3) \). For example, for \( F_1(K) = G_\tau(K_{0,\eta}) \) of (2.27), we must use, instead of \( c G_\tau(K_{0,\eta})c = c\tau K/(1+\eta K)c \), the following expression:

\[
c G_\tau(K_{0,\eta})c = -\frac{\tau}{\eta} \frac{1}{1 + \eta K}.
\]

The two \( c G_\tau(K_{0,\eta})c \)'s should be the same thing if we can use \( c^2 = 0 \), which correspond to \( t_1 = 0 \). The problem arises when other \( t_i \)'s are also infinitesimal and the ratio \( t_1/t_i \) is finite.

To explain the problem in more detail, let us consider a simpler example:

\[
\int Bc \frac{1}{1 + K} cK^p cK^q c \frac{1}{a + K} = \int_0^\infty dt_1 e^{-t_1} \int_0^\infty dt_4 e^{-a t_4} f(t_1, t_4),
\] (2.29)

and

\[
\int Bc \frac{-K}{1 + K} cK^p cK^q c \frac{1}{a + K} = \int_0^\infty dt_1 e^{-t_1} \int_0^\infty dt_4 e^{-a t_4} \frac{\partial}{\partial t_4} f(t_1, t_4).
\] (2.30)

Carrying out the integration by parts with respect to \( t_1 \) in (2.30), it is reduced to (2.29) if we can discard the surface term. However, the surface term at \( t_1 = 0 \),

\[
\lim_{t_1 \rightarrow 0} \int_0^\infty dt_4 e^{-a t_4} f(t_1, t_4),
\] (2.31)

is a subtle quantity. We certainly have \( f(t_1 \rightarrow 0, t_4 > 0) \rightarrow 0 \). However, the scaling property \( f(\lambda t_1, \lambda t_4) = \lambda^{3-p-q} f(t_1, t_4) \) implies that the expansion of \( f(t_1, t_4) \) in powers of \( t_1 \) starts with
This series expansion does not make sense for \(p + q \geq 3\) since the \(t_4\)-integration is divergent at \(t_4 = 0\).

If we use the \((s, z)\)-integration formula \((B.3)\), there is no problem of \(c^2 \neq 0\). This is because, in the derivation of \((B.3)\) by inserting \(1 = \int_0^{\infty} ds \delta(s - \sum t_i)\), the quantity \(t_i / \sum_{j=1}^4 t_j\) in the original expression is replaced with \(t_i / s\), and the \(t_i\)-integrations are carried out with \(s\) kept fixed. Therefore, there appear no subtle ratios \(t_i / t_j\). That \(c^2 = 0\) is ensured in \((B.3)\) is also seen from the fact \(G(F_1, F_2, F_3, F_4)\) \((B.4)\) vanishes when at least one of \(F_{1,2,3}\) is a constant.

3 A proposal for solutions with \(|\mathcal{N}| \geq 3\)

We found in the previous section that satisfactory multi-brane solutions with \(|\mathcal{N}| \geq 3\) cannot be constructed by using only the singularities at \(K = 0\) and \(\infty\). One might be tempted to adopt \(G(K)\) which has zeros/poles in \(0 < K < \infty\). However, such \(G(K)\) is problematic since quantities containing \(G(K)\) or \(1/G(K)\) do not have well-defined Schwinger parameter representations, as we stated in Sec. 2.4. In this section, we propose a way of constructing satisfactory solutions with \(|\mathcal{N}| \geq 3\) by employing such apparently “dangerous” \(G(K)\).

To explain our proposal, let us consider the following \(G(K)\):

\[
G(K) = \frac{(K + 1)^3}{K(K - a)} \quad (a > 0).
\] (3.1)

Our idea of defining the winding number and the EOM-test for \(\Psi\) specified by this \(G(K)\) is as follows. First, we divide the original eigenvalue space of \(K\), \([0, \infty]\), into two intervals, \([0, a]\) and \([a, \infty]\). Then, we expand each of the two intervals to \([0, \infty]\) by a linear fractional transformation with real coefficients, \(K \mapsto (aK + \beta)/(\gamma K + \delta)\), which is a real-to-real and one-to-one mapping. For example, we take the transformations \(K \mapsto g_1(K) = aK/(1 + K)\) and \(K \mapsto g_2(K) = a + K\) for the first and the second intervals, respectively.

After this preparation, we define \(\mathcal{N}\) and EOM-test for \(\Psi\) specified by \(G(K)\) \((3.1)\) by the sum of those for \(\mathcal{M}_{g_{1,2}}(\Psi)\), the EMNT-transform \((2.3)\) of \(\Psi\) associated with the maps \(g_{1,2}(K)\):

\[
\mathcal{O}[^\Psi] = \mathcal{O}[\mathcal{M}_{g_1}(\Psi)] + \mathcal{O}[\mathcal{M}_{g_2}(\Psi)], \quad (\mathcal{O} = \mathcal{N}, \text{ EOM-test}). \tag{3.2}
\]

Of course, each of the two terms on the RHS of \((3.2)\) should be evaluated by using the regularization of Sec. 2.2. Recall that, for \(\Psi\) of the form \((1.6)\), \(\mathcal{M}_{g}(\Psi)\) is again given by \((1.6)\) with \(G(K)\) replaced with \(G(g(K))\). Now, since \(G(g_{1,2}(K))\) has no zeros/poles in \(0 < K < \infty\), we can apply the arguments of Sec. 2 to \(G(g_{1,2}(K))\). The RHS of \((3.2)\) is determined only by \((n_0, n_{\infty})\) of \(G(g_{1,2}(K))\) (see eq. \((2.4)\)); \((n_0, n_{\infty}) = (-1, -1)\) for both \(G(g_{1,2}(K))\). This implies that \(\Psi\) of \(G(K)\) \((3.1)\) is a satisfactory solution representing five branes with \(\mathcal{N} = 4\) and EOM-test = 0.
Generalization of the above example is manifest. Suppose that \( G(K) \) has zeros/poles at \( K = a_i \ (i = 0, 1, \cdots, m + 1) \) with \( a_0 = 0 \) and \( a_{m+1} = \infty \), and the leading behavior of \( G(K) \) near \( K = a_i \) is \( G(K) \sim (K - a_i)^{n_i} \ (i = 0, \cdots, m) \). For this \( G(K) \), we divide the eigenvalue space \([0, \infty)\) of \( K \) into \( m + 1 \) intervals \([a_{i-1}, a_i]\) \((i = 1, \cdots, m + 1)\), and expand each of the intervals to \([0, \infty]\) by the map \( K \mapsto g_i(K) \) with \( g_i(K) \) given, for example, by

\[
g_i(K) = a_{i-1} + \frac{(a_i - a_{i-1})K}{1 + K} \quad (i = 1, \cdots, m), \quad g_{m+1}(K) = a_m + K. \tag{3.3}
\]

Then, we define \( \mathcal{O}[\Psi] = \mathcal{N}[\Psi] \) and EOM-test[\( \Psi \)] for \( \Psi \) specified by the present \( G(K) \) by

\[
\mathcal{O}[\Psi] = \sum_{i=1}^{m+1} \mathcal{O}[\mathcal{M}_{g_i}(\Psi)]. \tag{3.4}
\]

Concretely, we have

\[
\mathcal{N}[\Psi] = -n_0 - n_{\infty} + A(n_0) + A(n_{\infty}) + 2 \sum_{i=1}^{m} [-n_i + A(n_i)], \tag{3.5}
\]

\[
\text{EOM-test}[\Psi] = B(n_0) + B(n_{\infty}) + 2 \sum_{i=1}^{m} B(n_i), \tag{3.6}
\]

which is a generalization of (2.5) and (2.20). Note that the contribution from \( a_i \ (i = 1, \cdots, m) \) is multiplied by 2. This is because \( a_i \) is a boundary of the two intervals \([a_{i-1}, a_i]\) and \([a_i, a_{i+1}]\).

From our results (3.5) and (3.6), we find that satisfactory multi-brane solutions can be constructed for any integer \( N \) by using \( G(K) \) which has simple zeros/poles in \( 0 < K < \infty \) in addition to those at \( K = 0 \) and \( \infty \).

Let us examine the consistency of our definition (3.4). First, consider \( G(K) \) which has no zeros/poles in \( 0 < K < \infty \) and hence needs no division into intervals. For this \( G(K) \), the LHS of (3.4) is already given in Sec. 2, and it agrees with the RHS defined by arbitrarily chosen \( a_i \). This is because the present linear fractional transformations \( g_i(K) \) never produce new zeros/poles from \( G(K) \). Second, the functions \( g_i(K) \) are not uniquely determined by \( a_i \) alone. For example, we could have chosen \( g_1(K) = a/(1 + \alpha K) \) and \( g_2(K) = a + \beta K \) \((\alpha, \beta > 0)\) for \( G(K) \) of (3.1). However, \( \mathcal{N} \) and EOM-test defined by (3.4) are not affected by the arbitrariness of \( g_i(K) \) since \( \mathcal{O}[\mathcal{M}_{g_i}(\Psi)] \) is determined only by \( (n_{i-1}, n_i) \). Finally, we considered here only \( \mathcal{N} \) and EOM-test as \( \mathcal{O} \) in (3.4). It is unknown whether possible other observables associated with \( \Psi \) are also “topological” ones determined only by \( n_i \). This is an important question to be clarified.

In the above arguments, we have implicitly assumed that \( G(K) \) has no zeros/poles with \( \text{Im} K \neq 0 \) and \( \text{Re} K > 0 \). Such \( G(K) \) is also dangerous and needs division into intervals. For example, if \( G(K) \) has zeros/poles at \( K = a \pm ib \ (a, b > 0) \), we need two intervals \([0, a]\) and \([a, \infty]\). However, these complex zeros/poles cannot contribute to \( \mathcal{N} \) and EOM-test defined by (3.4) since \( G(K) \) has no zeros/poles at \( K = a \).
4 Conclusions

In this paper, we presented a new way of constructing satisfactory multi-brane solutions in CSFT, namely, solutions carrying integer winding number $\mathcal{N}$ and satisfying the EOM in the strong sense. We found in the previous paper [4] that the origin of the non-zero value of the topological quantity $\mathcal{N}$ is the singularity at $K = 0$, and that stronger singularity leads to non-integer $\mathcal{N}$ and the breaking of the EOM in the strong sense. Therefore, only satisfactory solutions with $\mathcal{N} = 0, \pm 1$ were possible. Our new construction consists of two steps, and, in both steps, the EMNT transformation (2.3) which keeps the $KBc$ algebra plays important roles. The first step is based on our finding that the correlator is invariant under the inversion, the EMNT transformation which interchanges $K$ with $1/K$. This property implies that the singularities at $K = \infty$ as well as at $K = 0$ can be the origin of $\mathcal{N}$, and allows us to extend the construction of satisfactory multi-brane solutions to the cases $\mathcal{N} = \pm 2$. In our second step, we further extended the construction to the cases $|\mathcal{N}| \geq 3$. For this, we took $G(K)$ which has zeros/poles in $0 < K < \infty$ besides at $K = 0, \infty$ and hence does not give well-defined $\mathcal{N}$. We proposed a way of making $\mathcal{N}$ and EOM-test for this $G(K)$ well-defined by dividing the eigenvalue space of $K$, $[0, \infty]$, into a sum of safe intervals, and mapping each interval to $[0, \infty]$ by suitable EMNT transformations.

By our construction presented in this paper, satisfactory solutions with any integer $\mathcal{N}$ is now possible. Among them, the solutions with $|\mathcal{N}| \leq 2$ which are based on singularities at $K = 0$ and/or $K = \infty$ are firm. On the other hand, for our proposal of the solutions with $|\mathcal{N}| \geq 3$, further study is necessary to conform that it is a fully consistent one. In particular, it is a problem whether all the physical quantities related to the solution can be well-definedly given by (3.4). For this, it would be necessary that all the observables in CSFT are “topological” ones.

Besides the problem of solutions with $|\mathcal{N}| \geq 3$, there are many questions to be clarified concerning our construction. For example, there have been analyses of multi-brane solutions by studying the gauge invariant observables [5, 9] and the boundary states [3, 8]. It is interesting to examine how the $K = \infty$ eigenvalue can affect these analyses. Further study of the general property of EMNT transformation (2.3) is also necessary. In particular, we wish to know whether the EMNT transformation has a simple CFT interpretation, and whether there exists more general type of invariance of the correlator under the EMNT transformation.

Finally, let us discuss in what sense the EOM should be satisfied by the solutions $\Psi$, namely, for what class of test states $\Phi$ the condition

$$\int \Phi \ast (Q_B \Psi_{\epsilon \eta} + \Psi_{\epsilon \eta} \ast \Psi_{\epsilon \eta}) = 0,$$

should hold. In this paper, we tried to construct solutions carrying an integer $\mathcal{N}$ and satisfying
the EOM in the strong sense, i.e., (4.1) with $\Phi = \Psi_{\varepsilon \eta}$, by calling them “satisfactory solutions”. If the EOM in the strong sense holds, the winding number $N$ is directory related to the energy density. However, it was shown, within the framework of the $K_{\varepsilon}$-regularization, that (4.1) is violated for $\Psi$ corresponding to $G(K)$ singular at $K = 0$ when we take as the test state $\Phi$ the Fock vacuum $[5]$. This is the case also in our $(K_{\varepsilon \eta}, B_{\varepsilon \eta}, c_{\varepsilon \eta})$-regularization (2.7) introduced here since $\eta$ has no effect on the singularity at $K = 0$. Indeed, the Fock states are “natural states” for the perturbative vacuum, namely, they are states representing the excitations around a single D-brane. However, is it necessary to demand (4.1) against $\Phi$ in the Fock space which have no special meanings for multi-brane solutions? Furthermore, it seems impossible to demand (4.1) for an arbitrary $\Phi$; for any given $\Psi$ we could make (4.1) non-vanishing by taking a test state $\Phi$ which carries strong enough singularities at $K = 0$ and/or $K = \infty$.

To consider for what class of $\Phi$ the EOM condition (4.1) should be satisfied, let us recall the process of obtaining the action of the fluctuation $\Phi$ around a multi-brane solution $\Psi$:

$$S[\Psi_{\varepsilon \eta} + \Phi] = S[\Psi_{\varepsilon \eta}] + \int \left( \frac{1}{2} \Phi \ast Q_{\Psi} \Phi + \frac{1}{3} \Phi^3 \right),$$

(4.2)

where $Q_{\Psi}$ is the BRST operator around the solution $\Psi$, and we have used the EOM condition (4.1) to drop the term linear in $\Phi$. Then, obviously (4.1) must hold against $\Phi$ representing the open string excitations specified by $Q_{\Psi}$. It is our future problem to carry out this analysis and clarify the relationship to the EOM in the strong sense. On the other hand, the Fock states would be far beyond the space of finite mass excitations. We do not know whether we have to demand (4.1) for such $\Phi$ outside the space of the perturbative fluctuation modes around $\Psi$.

The analysis of the fluctuation modes around the solutions is itself an important problem. For a solution with $N \geq 1$, we must show that there is the $(N + 1)^2$ degeneracy of the open string excitations on $N + 1$ branes. For solutions with $N \leq -2$, we must clarify whether such “ghost branes” can really exist, and if so, what their physical meanings are, through the analysis of the fluctuations.

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8 For $\Psi$ corresponding to $G(K) = 1 + K$, the EOM condition (4.1) holds against the Fock vacuum in our regularization for $K = \infty$. This is consistent with the result of [4] in a different regularization. However, we think this fact not so important for our construction of multi-brane solutions since the singularities both at $K = 0$ and $K = \infty$ are necessary.

9 See [15] for an analysis of lump solutions by demanding the EOM only against the space of fluctuations around the solution.
A Derivation of eqs. (1.11) and (1.15)

In this appendix, we present the derivation of eq. (1.11) for the winding number $N$ and eq. (1.15) for the EOM in the strong sense by taking into account only the zeros/poles of $G(K)$ at $K = 0$.

First, let us consider $N$ (2.24) given in terms of $B(\tau)$ (2.25). Since only the singularity at $K = 0$ is of significance, we may start with (2.26) with $\eta = 0$, which has the following convenient expression:

$$B(\tau) = \int Q_B \left\{ Bc G_\tau(K_\varepsilon) cK_\varepsilon \left[ (d/d\tau)G_\tau(K_\varepsilon) \right] \frac{K_\varepsilon}{G_\tau(K_\varepsilon)} \right\}$$

$$+ \varepsilon \int c G_\tau(K_\varepsilon) cK_\varepsilon \left[ (d/d\tau)G_\tau(K_\varepsilon) \right] \frac{K_\varepsilon}{G_\tau(K_\varepsilon)} .$$

(A.1)

We can drop the first term since it is the integration of the BRST-transform of a quantity which is perfectly regular for $\varepsilon > 0$. The second term is multiplied by $\varepsilon$ and therefore can be non-trivial only if the integration gives a $O(1/\varepsilon)$ quantity. Therefore, we have only to take the leading term of $G(K)$ near $K = 0$, which, as we will see, gives the desired $O(1/\varepsilon)$ term. Here, we consider the case that $G(K)$ has a pole at $K = 0$:

$$G(K) \sim K^{-m} \quad (K \sim 0, \ m \geq 1).$$

(A.2)

The treatment of the case $m < 0$ is quite similar.

We take the parameter $\tau$ in such a way that $\Psi_\tau$ (0 \leq \tau < 1) and $\Psi_{\tau=1}$ represent the perturbative vacuum and the non-trivial one $\Psi$, respectively. Then, $G_\tau(K)$ must have no zeros/poles at $K = 0$ for $\tau < 1$. Therefore, as the leading term of $G_\tau(K)$, we can take without loss of generality

$$G_\tau(K) \sim (1 - \tau + K)^{-m} .$$

(A.3)

Substituting this into the second term of (A.1), $B(\tau)$ is now given by

$$B(\tau) = -m\varepsilon \left\{ \Delta\tau \int c \frac{1}{(K_\varepsilon + \Delta\tau)^m} cK_\varepsilon(K_\varepsilon + \Delta\tau)^m + \int c \frac{1}{(K_\varepsilon + \Delta\tau)^m} cKcK_\varepsilon(K_\varepsilon + \Delta\tau)^{m-1} \right\} ,$$

(A.4)
with $\Delta \tau = 1 - \tau$. We can evaluate this $B(\tau)$ by using the $(s, z)$-integration formula (B.3) with $F_4 = 1$. First, we carry out the $z$-integration by adding a large semi-circle in Re $z < 0$ to close the contour and evaluating residues at $z = -\Delta \tau - \varepsilon$ and $z = -\Delta \tau - \varepsilon \pm 2\pi i/s$. Then, the $s$-integration is elementary and gives

$$\pi^2 B(\tau) = \frac{m}{2} \sum_{k=0}^{m-1} \left\{ d_1(\tau) + \frac{1}{2} d_2(\tau) \sum_{k=0}^{m-1} \left( \frac{m}{m-k-1} \right) p_{k,1}^{(\pm)} \right. \left. + \frac{1}{2} d_3(\tau) \sum_{k=0}^{m-1} \left( \frac{m}{m-k-1} - \frac{m-1}{m-k-1} \right) p_{k,0}^{(\pm)} - \sum_{k=0}^{m} \left( \frac{m+1}{m-k} \right) p_{k,0}^{(\pm)} \right. \right.$$ \left. \left. + d_4(\tau) \sum_{k=0}^{m} \left( \frac{m}{m-k} \right) p_{k,-1}^{(\pm)} - \sum_{k=0}^{m-1} \left( \frac{m-1}{m-k-1} \right) p_{k,-1}^{(\pm)} \right\}, \tag{A.5} \right.$$ 

where $p_{k,\ell}^{(\pm)}$ and $d_{1,2,3,4}(\tau)$ are defined by

$$p_{k,\ell}^{(\pm)} = \frac{(\pm 2\pi i)^{k+\ell}}{k!}, \tag{A.6}$$ 

and

$$d_1(\tau) = \frac{2}{\varepsilon(w+1)^3}, \quad d_2(\tau) = \frac{1}{\varepsilon(w+1)^2}, \quad d_3(\tau) = \frac{2w}{\varepsilon(w+1)^3}, \quad d_4(\tau) = \frac{3w^2}{\varepsilon(w+1)^4}, \tag{A.7}$$

with $w = \Delta \tau / \varepsilon$. The four functions in (A.7) are all reduced in the limit $\varepsilon \to +0$ to the delta function $\delta(1 - \tau)$ satisfying $\delta(1 - \tau) = 0$ for $\tau < 1$ and $\int_0^1 d\tau \delta(1 - \tau) = 1$. Using this fact and the definition of the confluent hypergeometric series,

$$\binom{1}{F_1}(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{\gamma(\gamma - 1) \cdots (\gamma - k + 1)} \frac{z^k}{k!}, \tag{A.8}$$

we find that

$$\pi^2 B(\tau) \to \frac{m + A(-m)}{\varepsilon \to 0} \delta(1 - \tau), \tag{A.9}$$

with $A(n)$ given by (1.12).

Next, let us calculate EOM-test (1.13). Since $\mathcal{E}_\varepsilon$ (1.14) is also multiplied by $\varepsilon$, we have only to take the leading term (A.2) of $G(\mathcal{K})$ as in the case of the second term of (A.1):

$$\mathcal{E}_\varepsilon = \int Bc \frac{1}{K_{\varepsilon}^m} c K_{\varepsilon}^{m+1} c \frac{1}{K_{\varepsilon}^m} c K_{\varepsilon}^{m+1}. \tag{A.10}$$

Again using the $(s, z)$-integration formula (B.3), we obtain $\mathcal{E}_\varepsilon = B(-m)/\varepsilon$ with $B(n)$ given by (1.15).
B Proof of eq. (2.2)

In this appendix, we present a proof of the marvelous property (2.2) of the correlator. Since any \( \int W(K, B, c) \) is reduced to the sum of terms of the form \( \int BcF_1(K)cF_2(K)cF_3(K)cF_4(K) \) by using (1.4), it is sufficient to show (2.2) for the latter. Then, the RHS of (2.2) is rewritten as follows:

\[
\int \tilde{B}cF_1(\tilde{K})\tilde{c}F_2(\tilde{K})\tilde{c}F_3(\tilde{K})\tilde{c}F_4(\tilde{K}) = \int BcF_1(\tilde{K})cK^2BcF_2(\tilde{K})cK^2BcF_3(\tilde{K})cK^2BcF_4(\tilde{K})
\]

\[= \int BcF_1(\tilde{K})cK^2 \left[ F_2(\tilde{K}), c \right] K^2 \left[ F_3(\tilde{K}), c \right] K^2 F_4(\tilde{K}), \tag{B.1} \]

where we have used

\[
Bcf_1(K)cF_2(K)Bc = [f_1(K), c] f_2(K)Bc, \tag{B.2}
\]

valid for any \( f_{1,2}(K) \).

Our proof is based on the \((s, z)\)-integration formula of the correlator \([2, 5]\), which reads

\[
\int BcF_1(K)cF_2(K)cF_3(K)cF_4(K) = \int_0^\infty ds \frac{s^2}{(2\pi)^3 i} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{sz} G(F_1, F_2, F_3, F_4), \tag{B.3}
\]

where \( G \) is given in our convention by

\[
G(F_1, F_2, F_3, F_4) = \left[ (\Delta_s F_1) F_2 F_3' + F_1' F_2 (\Delta_s F_3) + (F_1 (\Delta_s F_2) F_3)' - \Delta_s (F_1 F_2') F_3 - \Delta_s (F_1 F_2) F_3' - F_1 \Delta_s (F_2' F_3) - F_1 \Delta_s (F_2 F_3') + \Delta_s (F_1 F_2' F_3) \right] F_4, \tag{B.4}
\]

with \( F_i = F_i(z) \), \( F_i' = (d/dz) F_i(z) \) and

\[
(\Delta_s F_i)(z) \equiv F_i \left( z - \frac{2\pi i}{s} \right) - F_i \left( z + \frac{2\pi i}{s} \right). \tag{B.5}\]

Note that we are allowed to extend the range of the \( s \)-integration in (B.3) to \((-\infty, \infty)\) since \( (B.3) \) has been obtained by inserting \( \int_0^\infty ds \delta(s - \sum_i t_i) = 1 \) with \( t_i \geq 0 \) being the Schwinger parameter for \( F_i(K) \). In the rest of this proof, we use (B.3) with the extended \( s \)-integration region.

It is sufficient to show (2.2) for \( F_i(K) = e^{-t_i K} \) \((i = 1, 2, 3, 4)\). Then, (B.1) is given as the \((s, z)\)-integration (B.3) with \( G \) replaced by

\[
G(e^{-t_1/z}, z^2 e^{-t_2/z}, z^2 e^{-t_3/z}, z^2 e^{-t_4/z}) - G(e^{-t_1/z}, z^2 e^{-t_2/z}, z^2, z^2 e^{-(t_3+t_4)/z})
\]

\[- G(e^{-t_1/z}, z^2, z^2 e^{-(t_2+t_3)/z}, z^2 e^{-t_4/z}) + G(e^{-t_1/z}, z^2, z^2 e^{-t_2/z}, z^2 e^{-(t_3+t_4)/z}). \tag{B.6}\]

In order to relate the \((s, z)\)-integration for (B.1) to that for the LHS of (2.2), let us first make the following change of integration variables in the former:

\[
z \rightarrow \frac{1}{z}, \quad s \rightarrow z^2 s. \tag{B.7}\]
Note that this keeps $e^{sz}$ invariant. Then, we obtain

$$\text{(B.1)} = \frac{1}{(2\pi)^3 i} \int_{-\infty}^{\infty} ds \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{(s-\sum t_i)z} (G_+ + G_-),$$

where $G_\pm$ is defined by

$$G_\pm = \pm s_\pm^2 \left[ -t_3 (e^{\pm(2\pi i/s_\pm)t_1} - 1) - t_1 (e^{\pm(2\pi i/s_\pm)t_3} - 1) 
- (t_1 + t_2 + t_3) (e^{\pm(2\pi i/s_\pm)t_2} - 1) + (t_2 + t_3) (e^{\pm(2\pi i/s_\pm)(t_1+t_2)} - 1) 
+ (t_1 + t_2) (e^{\pm(2\pi i/s_\pm)(t_2+t_3)} - 1) - t_2 (e^{\pm(2\pi i/s_\pm)(t_1+t_2+t_3)} - 1) \right] \pm 4\pi^2 t_1 t_2 t_3, \quad \text{(B.8)}$$

with $s_\pm = s \mp (2\pi i/z)$. Let us further make a change of variables $s \rightarrow s \mp (2\pi i/z)$ in the part of integration (B.8) multiplied by $G_\pm$. This shift of $s$ is allowed since we have $G_\pm = O(1/s^2)$ as $s \rightarrow \infty$ owing to the last term of (B.9). Then, we finally obtain the following expression of (B.1):

$$\int_{-\infty}^{\infty} ds \frac{s^2}{(2\pi)^3 i} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{(s-\sum t_i)z} \sum_{\pm} (\pm) \left[ -t_3 e^{\pm(2\pi i/s)t_1} - t_1 e^{\pm(2\pi i/s)t_3} - t_2 e^{\pm(2\pi i/s)(t_1+t_2+t_3)} 
- (t_1 + t_2 + t_3) e^{\pm(2\pi i/s)t_2} + (t_2 + t_3) e^{\pm(2\pi i/s)(t_1+t_2)} + (t_1 + t_2) e^{\pm(2\pi i/s)(t_2+t_3)} \right]. \quad \text{(B.10)}$$

We find that this is nothing but the LHS of (2.2) with $F_i(K) = e^{-t_i K}$. This ends a proof of (2.2).

Eq. (2.2) can of course be confirmed by taking a concrete $\mathcal{W}(K, B, c)$ and calculating its both hand sides by using the $(s, z)$-integration formula (B.3). If we carry out the calculation without using the $(s, z)$-trick, the same care as we described in Sec. 2.5 is necessary.

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