Thermodynamic Black Hole with Modified Chaplygin Gas as a Heat Engine

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We assume that the negative cosmological constant as a thermodynamical pressure and the asymptotically anti-de Sitter (AdS) black hole thermodynamics with modified Chaplygin gas. We have written the mass of the black hole, volume, entropy and temperature due to the thermodynamic system. We find a new solution of Einstein’s field equations of AdS black hole with modified Chaplygin gas as a thermodynamic system. We also examine the weak, strong and dominant energy conditions for the source fluid of black hole. We also show that the thermodynamic black hole with Chaplygin gas can be considered as a heat engine and then we calculate work done and its efficiency by this system.

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I. INTRODUCTION

Thermodynamic properties of black hole have been studied for many years. In recent years there is considerable interest in the physics of asymptotically AdS black hole [1] due to AdS/CFT correspondence. Hawking et al [2] studied the thermodynamic properties of non-rotating uncharged Schwarzschild-AdS black hole. After that Chamblin et al [3, 4] investigated the first order phase transition in the non-rotating charged Reissner-Nordstrom-AdS black hole. When the charge and/or rotation of the AdS black hole are included, the behaviour of the AdS black hole is qualitatively similar to the Van der Walls fluid [5, 6]. The concepts of black holes from the viewpoint of chemistry, in terms of concepts such as Van der Waals fluids, reentrant phase transitions, and triple points have been studied in [7]. The Van der Waals black hole has been determined by Rajagopal et al [8]. Subsequently, the Van der Waals black hole in $d$-dimensions has been described by Delsate et al [9]. Also the polytropic black hole has been formulated by Setare et al [10]. Kubiznak et al [11] assumed that the cosmological constant $\Lambda < 0$ [12], which represents the thermodynamic pressure $p = \frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}$ (1)

and first law of black hole thermodynamics [8, 9]

$$\delta M = T\delta S + V\delta p + ....$$ (2)

with the black hole thermodynamics volume [8, 9]

$$V = \left(\frac{\partial M}{\partial p}\right)_{S,...}$$ (3)

where, $M$ is the mass, $S$ is the entropy and $T$ is the temperature of the black hole. From this, an equation of state $p = p(V, T)$ can be written for the black hole and comparing it with the corresponding fluid equation of state, we may construct the temperature, volume, pressure, etc.

II. CHAPLYGIN BLACK HOLE

Motivated by the works for Van der Waals fluid [8, 9] and polytropic gas [10] in AdS black hole, we assume the modified Chaplygin gas in AdS black hole whose equation of state is given by [13]

$$p = A\rho - \frac{B}{\rho^\alpha}$$ (4)

where $A, B, \alpha$ are constants. The modified Chaplygin gas is one of the candidate of dark energy which drives the acceleration of the Universe. We want to construct an asymptotically AdS black hole with Chaplygin gas whose thermodynamics coincide with the above equation of state. So we consider the static spherically symmetric black hole metric [8, 10]

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$ (5)

where

$$f \equiv f(r, \rho) = \frac{r^2}{l^2} - \frac{2M}{r} - g(r, \rho)$$ (6)

Here the unknown function $g(r, \rho)$ is to be determined.

Now assume the negative cosmological constant $\Lambda$, so the Einstein’s equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$ (7)

Here negative $\Lambda$ represents the vacuum pressure. The entropy, mass, volume and temperature of the black hole

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are related to the horizon radius $r_h$ such that

$$S = \frac{A}{4} = \pi r_h^2,$$

(8)

$$M = \frac{4\pi}{3} r_h^3 - \frac{1}{2} r_h g(r_h, \rho),$$

(9)

$$V = \frac{\partial M}{\partial \rho} = \frac{4\pi}{3} r_h^3 - \frac{1}{2} r_h \left( \frac{\partial g(r_h, \rho)}{\partial \rho} \right) \frac{d\rho}{d\rho},$$

(10)

$$T = \frac{1}{4\pi} \left[ \frac{\partial f(r, \rho)}{\partial r} \right]_{r=r_h} = 2r_h \rho - \frac{g(r_h, \rho)}{4\pi r_h} - \frac{1}{4\pi} \left[ \frac{\partial g(r, \rho)}{\partial r} \right]_{r=r_h}$$

(11)

Now assume that all the thermodynamic parameters for black hole related with the parameters for modified Chaplygin gas. So the first law of thermodynamics yields (using integrability condition) \[10\]

$$S = \frac{\rho + p}{T} V$$

(12)

Using equations (4) and (8)-(11), the equation (12) reduces to

$$\frac{24\pi r^2 (A\rho - B\rho^{-\alpha}) - 3g(r, \rho) - 3r \frac{\partial g(r, \rho)}{\partial r}}{16\pi r^2 - 6 \frac{\partial g(r, \rho)}{\partial \rho} \left( A + \alpha B\rho^{-1-\alpha} \right)^{-1}} = 0$$

(13)

Since $g(r, \rho)$ is unknown function of $r$ and $\rho$, so without any loss of generality, we may assume the polynomial form of $g(r, \rho)$ as in the following form

$$g(r, \rho) = X(r) + Y(r)\rho + Z(r)\rho^{-\alpha}$$

(14)

where $X(r), Y(r)$ and $Z(r)$ are arbitrary functions of $r$. Now substituting the expression of $g(r, \rho)$ in equation (13), we obtain the following equation

$$F_0(r) + F_1(r)\rho + F_2(r)\rho^{-\alpha} + F_3(r)\rho^{-\alpha-1} + F_4(r)\rho^{-2\alpha-1} = 0$$

(15)

where

$$F_0(r) = -3A(X(r) + rX'(r)),$$

(16)

$$F_1(r) = 8A(A - 2)\pi r^2 + 3(A + 2)Y(r) - 3ArY'(r),$$

(17)

$$F_2(r) = -8B\pi(A + 2\alpha - A\alpha)r^2 - 3B(\alpha + 2)Y(r)$$

$$-3(A + 2\alpha + 2A\alpha)Z(r) - 3BarY'(r) - 3ArZ'(r),$$

(18)

$$F_3(r) = -3Ba(X(r) + rX'(r)),$$

(19)

$$F_4(r) = -8B\alpha r^2 + 3BaZ(r) - 3BarZ'(r)$$

(20)

From the identity equation (15), comparing the coefficients of powers of $\rho$ in both sides, we must get $F_i(r) = 0$, $i = 0, 1, 2, 3, 4$. Now set $F_0(r) = 0 = F_3(r)$, we get (from equations (16) and (19))

$$X(r) = \frac{X_0}{r}$$

(21)

where $X_0$ is an integration constant. Next put $F_1(r) = 0$ in equation (17), we have

$$Y(r) = \frac{8\pi}{3} Ar^2 - Y_0 r^{1+\alpha}$$

(22)

where $Y_0$ is an integration constant. Again put $F_4(r) = 0$ in equation (20), we must get

$$Z(r) = -\frac{8\pi}{3} Br^2 - Z_0 r$$

(23)

where $Z_0$ is another integration constant. Lastly, we set $F_2(r) = 0$, we obtain (from equation (18)) the relation between two parameters as

$$A = -\frac{\alpha}{1+\alpha}$$

(24)

For non-trivial solutions of $X, Y, Z$, we must have $\alpha \neq 0, A \neq 0, B \neq 0$. Putting the solutions of $X(r), Y(r)$ and $Z(r)$ in equation (14), we get the expression of $g(r, \rho)$ as in the following form:

$$g(r, \rho) = \frac{X_0}{r} + \left( \frac{8\pi}{3} Ar^2 - Y_0 r^{1+\alpha} \right) \rho$$

$$+ \left( -\frac{8\pi}{3} Br^2 - Z_0 r \right) \rho^{-\alpha}$$

(25)

Finally, putting the expression of $g(r, \rho)$ in equation (6), we obtain the solution of the function $f(r, \rho)$:

$$f(r, \rho) = -\frac{2M + X_0}{r} + Y_0 r^{1+\alpha} \rho + Z_0 r^{\alpha}$$

(26)

This is a new form of black hole solution which may be called Chaplygin black hole (after the names of Van der Waals black hole \[8\] and polytropic black hole \[10\]). Since the thermodynamic pressure $p = \frac{3}{8\pi r^2}$ depends on $l^2$ and compare this pressure with the fluid pressure (eq.(4)), we may obtain the expression of density $\rho$, which also depends on $l^2$ and $\rho$ can be written in terms of pressure $p$
if we choose pressure \( r \), the black hole solution depends of \( r \) and thermodynamic pressure \( p \) (which is obviously a constant). In particular, if we choose \( \alpha = -2/3 \), \( X_0 = Z_0 = 0 \) and \( Y_0 = \frac{8\pi r}{\rho} \), then from equations (25) and (27), we obtain

\[
f = \frac{r^2}{l^2} - \frac{2M}{r} \tag{28}
\]

which is a black hole solution with asymptotically AdS spacetime.

Now we examine the weak, strong and dominant energy conditions for the source fluid. The energy momentum tensor for the anisotropic source fluid is given by [10]

\[
T^\mu_\nu = \varrho e_0^\mu e_0^\nu + \sum_{i=1}^{3} p_i e_i^\mu e_i^\nu \tag{29}
\]

where \( \varrho \) is the energy density, \( p_i \) (\( i = 1, 2, 3 \)) are the pressures for the source fluid and \( e_i^\mu \) are the components of the vielbein. Now for the black hole metric (5), using the Einstein’s equation (7), we obtain the field equations [8] [10] (assume that the gravitational constant \( G = 1 \))

\[
\varrho = -p_1 = 1 - \frac{f - rf'}{8\pi r^2} + p
\]

\[
= \frac{1}{8\pi r^2} \left[ 1 + \frac{2}{\alpha} Y_0 r^{1+\frac{2}{\alpha}} \rho - 2Z_0 r^{\rho - \alpha} \right] + \frac{3}{8\pi l^2} \tag{30}
\]

and

\[
p_2 = p_3 = \frac{rf'' + 2f'}{16\pi r} - p
\]

\[
= \frac{1}{8\pi r^2} \left[ \frac{1}{\alpha} \left( 1 + \frac{2}{\alpha} \right) Y_0 r^{1+\frac{2}{\alpha}} \rho + Z_0 r^{\rho - \alpha} \right] - \frac{3}{8\pi l^2} \tag{31}
\]

Now it is easy to check that the weak energy condition: \( \varrho \geq 0 \), \( \varrho + p_i \geq 0 \) (\( i = 1, 2, 3 \)) may be satisfied for \( Y_0 \geq 0 \), \( Z_0 \leq 0 \) and \( \alpha > 0 \). The strong energy condition: \( \varrho + \sum p_i \geq 0 \), \( \varrho + p_i \geq 0 \) may be satisfied for \( Y_0 \geq 0 \), \( Z_0 = 0 \) and \( \alpha \geq -2/3 \). The dominant energy condition: \( \varrho \geq |p_i| \) (\( i = 1, 2, 3 \)) may be satisfied for \( Y_0 \geq 0 \), \( Z_0 \leq 0 \) and \( 0 < \alpha < 2 \). So all the energy conditions will be satisfied at a time if \( Y_0 \geq 0 \), \( Z_0 = 0 \) and \( \alpha > 0 \). In other cases, the above energy conditions may be violated. If we assume the above conditions of the parameters, we may checked that the energy conditions are satisfied on the horizon. For Van Der Waals black hole [8, 10], some of the energy conditions are violated but for polytropic black hole [10], all the energy conditions are satisfied. In our Chaplygin black hole, some of the energy conditions are satisfied for some restrictions of the parameters involved.

III. CLASSICAL HEAT ENGINE

In thermodynamics and engineering, a heat engine is a system that converts heat or thermal energy and chemical energy to mechanical energy, which can then be used to do mechanical work. That means a heat engine is a physical system that takes heat from hot reservoir and part of it converts into the works while the remaining is transferred to cold reservoir. In 2014, Johnson [16] has introduce the holographic heat engine for black hole, where the cosmological constant was considered a thermodynamic variable. Based on the holographic heat engine for black hole proposal, Johnson [17, 18] has studied the Gauss-Bonnet black holes, Born-Infeld AdS black holes and holographic heat engines beyond large \( N \) and the exact efficiency formula. Heat engines for dilatonic Born-Infeld black holes have been analyzed in [20]. Zhang et al [21] have studied the \( f(R) \) black holes as heat engines. The thermodynamic efficiency in charge rotating and dyonic black holes has been studied in [22]. Till now, several authors have studied the heat engine mechanism for various types of black holes [23] [31] [39] [44] [47]. Recently Setare et al [45] have discussed polytropic black hole as a heat engine. Motivated by their work, here we'll study the classical heat engine for our Chaplygin black hole.

The horizon radius \( r_h \) can be found from the equation \( Y_0 r_h^{2+\frac{2}{\alpha}} \rho + Z_0 r_h^{\rho - \alpha} - X_0 - 2M = 0 \), which depends on \( X_0, Y_0, Z_0, \alpha, \rho \). From equations (8) and (10), we obtain the volume

\[
V = \frac{Y_0 (\frac{2}{\alpha})^{1+\frac{2}{\alpha}} - \frac{\alpha Z_0 (1+\alpha)(2M + X_0)}{2(\pi A + \rho \alpha - \alpha - 1)}}{2A + B\alpha - \rho \alpha - \alpha - 1} \tag{32}
\]

Also from equation (11), we get the temperature

\[
T = \left(1 + \frac{1}{A} \right) \frac{2M + X_0}{2S} - \frac{Z_0(p - \rho)}{2\pi AB} \tag{33}
\]

which can be written as

\[
S = \frac{\pi(1 + A)(2M + X_0)}{2\pi AT + Z_0\rho - \alpha - 1} \tag{34}
\]

So the relation between \( V \) and \( T \) is obtained as in the form

\[
V = \frac{Y_0 (\frac{1+\alpha A(2M + X_0)}{2A + B\alpha - \rho \alpha - \alpha - 1})^{1+\frac{2}{\alpha}} - \alpha Z_0 (1+\alpha A)(2M + X_0)}{2A + B\alpha - \rho \alpha - \alpha - 1} \tag{35}
\]

To describe the thermodynamic behavior of the Chaplygin gas in presence of variable pressure (i.e., variable cosmological constant), one can identify mass \( M \) from being the energy \( U \) to being the enthalpy [46], i.e., the enthalpy function is defined by \( H = M = U + pV \). From the first law of thermodynamics, we get

\[
dH = dM = TD + Vdp \tag{36}
\]
By integration the above equation, the enthalpy function can be written in the form:

\[ H = -\frac{1}{2} X_0 + \frac{Y_0 \rho}{2} \left( \frac{S}{\pi} \right)^{\frac{1}{1+a}} + \frac{Z_0 S}{2\pi \rho^{\alpha}} \]  

(37)

The Gibb’s free energy is given by

\[ G = H - TS = -\frac{1}{2} X_0 - \frac{Y_0 \rho}{2A} \left( \frac{S}{\pi} \right)^{\frac{1}{1+a}} \]

(38)

Also the free energy is given by

\[ F = G - pV = -\frac{1}{2} X_0 - \frac{Y_0 \rho}{2A} \left( \frac{S}{\pi} \right)^{\frac{1}{1+a}} - \frac{pY_0}{2} \left[ \left( \frac{S}{\pi} \right)^{\frac{1}{1+a}} - \frac{\alpha Z_0}{\pi} \rho^{-\alpha-1} \right] \frac{S \pi}{2(A + B \rho^{\alpha-1})} \]

(39)

David Kubiznak and Robert B. Mann have showed the critical behaviour of charged AdS black holes. Following this, we will study the critical behavior of the Chaplygin black hole. Critical point is a point of inflection which can be found from the following conditions:

\[ \left( \frac{\partial p}{\partial r} \right)_{c.r} = 0, \quad \left( \frac{\partial^2 p}{\partial r^2} \right)_{c.r} = 0 \]

(40)

At the critical point \( r_{c.r} \), the critical pressure \( p_{c.r} \) and critical temperature \( T_{c.r} \) will be

\[ p_{c.r} = A \left( \frac{Y_0}{\alpha Z_0} \right)^{-\frac{1}{1+a}} r_{c.r}^{-\frac{1}{1+a}} - B \left( \frac{Y_0}{\alpha Z_0} \right)^{-\frac{1}{1+a}} r_{c.r}^{-2} \]

(41)

and

\[ T_{c.r} = \left[ \frac{2M + X_0}{2\pi \alpha} + \frac{Z_0}{2\pi A} \left( \frac{Y_0}{\alpha Z_0} \right)^{-\frac{1}{1+a}} \right] r_{c.r}^{-2} \]

(42)

with the condition

\[ 2M + X_0 = \frac{Y_0(\alpha + 1)}{\alpha} \left( \frac{Y_0}{\alpha Z_0} \right)^{-\frac{1}{1+a}} \]

(43)

Now assume, \( T_H \) and \( T_C \) are the temperatures of the hot and cold reservoirs respectively and they consist of two isothermal processes with two adiabatic processes. The heat engine flow is shown figure 1. So the heat flow for the upper isotherm process from 1 to 2 is given by

\[ Q_H = T_H \Delta S_{1\to2} = T_H(S_2 - S_1) \]

(44)

and the exhausted heat from the lower isothermal process is given by

\[ Q_C = T_C \Delta S_{3\to4} = T_C(S_3 - S_4) \]

(45)

Here \( S_i \)'s are related to \( V_i \)'s satisfying

\[ V_i = \frac{Y_0 \left( \frac{S_i}{\pi} \right)^{\frac{1}{1+a}} - \frac{\alpha Z_0}{\pi} \rho_i^{-\alpha-1} S_i}{2(A + B \rho_i^{\alpha-1})}, \quad i = 1, 2, 3, 4. \]

(46)

where \( \rho_i \) can be calculated from the relation \( p_i = A\rho_i - B\rho_i^{\alpha}, \quad i = 1, 2, 3, 4 \). The \( p-V \) diagram shows the Carnot heat engine which forms a closed path in figure 2.

The work done by the heat engine is

\[ W = Q_H - Q_C \]

(47)

The efficiency of a heat engine relates how much useful work is output for a given amount of heat energy input and it is defined by

\[ \eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H} \]

(48)

We know that the Carnot cycle has the maximum efficiency. Also we mention that the Stirling cycle consists of two isothermal processes plus two isochores processes. So we have the maximum efficiency as

\[ \eta_{\text{max}} = 1 - \frac{T_C}{T_H} \]

(49)

which is the maximum one of all the possible cycles between the given higher temperature \( T_H \) and lower one \( T_C \). The specific heat of the thermodynamical system is

\[ C = T \left( \frac{\partial S}{\partial T} \right) \]

\[ = - \frac{2AS^2T}{(1 + A)(2M + X_0)} \times \left( 1 - \frac{Z_0 \alpha \rho^{-\alpha-1}}{2\pi A(A + B \alpha \rho^{-\alpha-1})} \frac{\partial p}{\partial T} \right) \]

(50)

If volume \( V \) is constant (i.e., \( S \) is constant), the we can obtain

\[ \left( \frac{\partial p}{\partial T} \right)_V = \frac{2\pi A(A + B \alpha \rho^{-\alpha-1})}{Z_0 \alpha \rho^{-\alpha-1}} \]

(51)

and hence consequently for constant volume, the specific heat \( C_V = 0 \). For constant pressure, \( \left( \frac{\partial p}{\partial T} \right)_P = 0 \), so we may obtain the specific heat for constant pressure as

\[ C_p = -\frac{2AS^2T}{(1 + A)(2M + X_0)} \]

(52)

which is not equal to zero. So we have a new engine, described in figure 3, which involves two isobars and two isochores/adiabats. The heat flows show along the top and bottom. The work done along the isobars is given by

\[ W = \Delta p_{1\to4} \Delta V_{1\to2} = (p_1 - p_4)(V_2 - V_1) \]

\[ = (p_1 - p_4) \left\{ \frac{Y_0}{2} \left\{ \left( \frac{S_1}{\pi} \right)^{\frac{1}{1+a}} - \left( \frac{S_2}{\pi} \right)^{\frac{1}{1+a}} \right\} - \frac{\alpha Z_0}{2\pi} \left\{ \frac{\rho_1^{\alpha-1} S_2}{(A + B \rho_1^{\alpha-1})} - \frac{\rho_2^{\alpha-1} S_1}{(A + B \rho_1^{\alpha-1})} \right\} \right\} \]

(53)
The net inflow of heat in upper isobar is given by
\[ Q_H = \int_{T_1}^{T_2} C_p(p_1, T)dT \] (54)
which can be expressed as
\[ Q_H = \frac{(A + 1)(2M + X_0)}{2A} \times \left[ \frac{4\pi A Z_0 \rho^{-\alpha}(T_2 - T_1)}{(2\pi AT_1 + Z_0 \rho^{-\alpha}_1)(2\pi AT_2 + Z_0 \rho^{-\alpha}_2)} \right. \\
\left. + \log \left( \frac{2\pi AT_1 + Z_0 \rho^{-\alpha}_1}{2\pi AT_2 + Z_0 \rho^{-\alpha}_2} \right) \right] \] (55)
or in the other form:
\[ Q_H = \frac{(1 + A)(2M + X_0)}{2A} \log \frac{S_2}{S_1} + \frac{Z_0 \rho^{-\alpha}_1}{2\pi A} (S_1 - S_2) \] (56)
Finally we can demonstrate the performance of the heat engine by a thermal efficiency \(\eta\) and found in the following form:
\[ \eta = \left( 1 - \frac{p_1}{p_2} \right) \times \left\{ \frac{Y_0 \rho_1}{2} \left( \frac{\rho_2^{-\alpha} S_2}{(A + B \rho_2^{-\alpha})} - \frac{\rho_1^{-\alpha} S_1}{(A + B \rho_1^{-\alpha})} \right) \right\} \times \\
\frac{(1 + A)(2M + X_0)}{2A} \log \frac{S_2}{S_1} + \frac{Z_0 \rho^{-\alpha}_1}{2\pi A} (S_1 - S_2)^{-1} \] (57)
which crucially depends on the modified Chaplygin gas parameters \(\alpha, A\) and \(B\).

IV. DISCUSSIONS

We have assumed the negative cosmological constant as a thermodynamical pressure and the asymptotically anti-de Sitter (AdS) black hole thermodynamic parameters which are identical with the modified Chaplygin gas, which obeys the integrability condition of the thermodynamical system. We have written the mass of the black hole, volume, entropy and temperature due to the thermodynamic system. We found the solutions of Einstein’s field equations of AdS black hole for modified Chaplygin gas. The new form of solution for black hole may be called Chaplygin black hole (after the names of Van der Waals black hole [8] and polytropic black hole [10]). For \(A = 0\), the above Chaplygin black hole solution may be reduced to the polytropic black hole solution for negative \(\Lambda\). If we set \(A = 2, Y_0 = \frac{1}{27}\) and \(Z_0 = 0\), the Chaplygin black hole may be reduced to the asymptotically AdS black hole for negative \(\Lambda\). Also if we set \(A = -2, Y_0 = \frac{1}{3}\) and \(X_0 = Z_0 = 0\), the Chaplygin black hole may be reduced to the Schwarzschild black hole. We have also examined the weak, strong and dominant energy conditions for the source fluid of the Chaplygin black hole. For \(Y_0 \geq 0, Z_0 \leq 0\) and \(\alpha > 0\), the weak energy condition is satisfied, for \(Y_0 \leq 0, Z_0 \leq 0\) and \(\alpha \leq -2/3\), the dominant energy condition is satisfied and for \(Y_0 \geq 0, Z_0 = 0\) and \(0 < \alpha \leq 2\), the strong energy condition is satisfied but for other cases, the all energy conditions are violated.

We have described the classical heat engine for Chaplygin black hole. Using the horizon radius \(r_h\), we have found the relations between volume \(V\), temperature \(T\), entropy \(S\) and pressure \(p\) (or density \(\rho\)). Using the first law of thermodynamics, we have found the enthalpy function in terms of the entropy \(S\). The Gibb’s free energy and free energy have been evaluated. The critical pressure and critical temperature have been found at the critical point of the system. We have found the heat flows from upper and lower isotherms process. Also we have calculated the work done by the heat engine and its efficiency. We have found the maximum efficiency by the Carnot cycle. For static black holes, Johnson [16] has investigated that the Carnot and Stirling cycles are coincided. For constant volume, we found that specific heat \(C_V = 0\). On the other hand, for constant pressure, we have found that specific heat \(C_p \neq 0\). So we have considered another cycle which consists of two isobars and two isochores. We have calculated the net inflow of heat in upper isobar and efficiency of the heat engine for this cycle.

FIG. 1: The figure represents the heat engine flows [16].

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FIG. 2: The figure represents the Carnot Engine.

FIG. 3: The figure represents other Engine.

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