Modelling of TM Modes in Periodically-Shorted Cavities for Circuit QED

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Electromagnetic cavities are ubiquitous in superconducting quantum circuit research, employed to control a circuit’s electromagnetic environment, suppress radiative loss, and implement functionalities such as qubit readout and inter-qubit coupling. Here we consider the case of a rectangular cavity shorted by a periodic array of conducting cylinders. This is a potential enclosure geometry for large-scale quantum chips with many qubits. We develop simple, accurate models for the TM modes of the cavity, over a wide range of cylinder spacing and radii, using a plasma model and a coupled cavity array circuit model. We compare predictions with finite-element simulation and find good agreement. We investigate inter-qubit couplings mediated by such cavities for circuits at the 100-qubit scale, and discuss additional applications to circuit QED.

In this Letter, we construct accurate, closed-form, physically intuitive models for the frequencies and field distribution of the TM modes in a rectangular cavity shorted by a periodic array of conducting cylinders, and discuss the implications to enclosed circuit QED devices. This object is closely linked to a metallic photonic crystal [13, 14], the difference being that our cavity is finite in size. We are therefore looking for closed-form standing-wave solutions that depend on the cavity side-lengths, that should tend to the photonic crystal solutions when the side-lengths go to infinity. We build on a work which looked at small conducting cylinders [15], here deriving a key empirical result. We then develop a circuit model for larger cylinders, where we show the system behaves as a coupled cavity array. The results have primary applications to mode suppression in enclosures for large-scale quantum chips, and additional applications to circuit QED are also discussed.

We will consider the case of a perfect rectangular cavity, with side-lengths \( \ell_x \), \( \ell_y \) and height \( d \), filled with a dielectric of permittivity \( \epsilon_r \). The cavity is shorted by a periodic array of perfectly conducting cylinders, radius \( r \) and equal spacing \( a \) in \( \hat{x} \) and \( \hat{y} \) (figure 1). For simplicity, we treat only one dielectric layer within the cavity. However the models can accommodate many layers stacked along the \( \hat{z} \) axis (e.g. vacuum - substrate - vacuum) simply by an adjustment to \( \epsilon_r \), so long as the DC connections between cylinders and cavity are not broken; see appendix A.

As a primitive model, we use the periodicity of the cylinders to impose 2D boundary conditions on the maximum standing wave wavelength, \( \lambda = 2a \), leading to a fundamental TM mode frequency

\[
f_a = c/(a\sqrt{2\epsilon_r})
\]  

(1)

\( f_a \) offers a frequency for comparison with the results that follow. For the region \( r/a < 0.1 \), where we use a plasma model to fit the TM modes, \( f_a \) significantly overestimates the fundamental frequency.
FIG. 1: A cavity filled with a dielectric (constant $\varepsilon_r$) shorted by a square array of conducting cylinders, spacing $a$ and radius $r$. The cavity is bounded by perfect conductor and has no internal losses.

**Plasma model**

If $r \ll a, \lambda$, scattering off single cylinders will be negligible and we can use a bulk quantity to describe the effect of the array. The free electrons in the cylinders then behave like free electrons in the cavity and so the cavity behaves as if filled with an anisotropic plasma [16], with the associated frequency dependent permittivity

$$\varepsilon_p(f) = (1 - (f_v/f)^2) \hat{\varepsilon}$$

(2)

This result is applicable for E-M waves propagating in the x-y plane [17]. The plasma frequency $f_p$ of this metamaterial can be accurately expressed as [18, 19]

$$f_p = \frac{1}{\sqrt{2\pi\varepsilon_0\mu_0a}(\ln(f/(\Gamma)) - \Gamma)^{0.5}}$$

(3)

$$\Gamma = \ln(2\pi) - \pi/6 - \sum_{m=1}^{\infty}(\coth(n\pi) - 1)/n = 1.3105...$$

We emphasize the derivation is valid only for $d \gg r$ (i.e. field solutions independent of $z$). However, this condition will generally be satisfied when considering small cylinders. If we define $f_{TM}$ as the TM mode frequencies of the cavity without the cylinder array, and $f'_{TM}$ as the frequencies with the array present, then

$$f_{TM}' = \frac{f_{TM}}{\sqrt{f_p(f_{TM})}}$$

(4)

Here we have used the fact that the electric field of TM modes is oriented along $\hat{z}$. Solving for $f_{TM}'$, we find

$$f_{TM}' = \sqrt{f^2_{TM} + f^2_p}$$

(5)

This equation has been empirically used to fit FE simulations of cavities containing arrays of small conducting cylinders [15]. Here we see that it derives from the cylinders introducing a plasma-like contribution to the permittivity inside the cavity. We performed FE eigenmode simulations in ANSYS Electronics Desktop to verify this result over a range of $r/a$ (figure 2). We find that the agreement breaks down as $r$ approaches 0.1$a$. This is due to the mode frequencies approaching $f_a/\sqrt{2}$, where

Bragg scattering effects disturb the simple plasma picture [20]. We reiterate that equations (4) and (5) apply only to TM modes. The TE modes, where the electric field is perpendicular to the conducting cylinders, are unaffected by the cylinders if they are sufficiently small [13, 18]. Equation (2) implies the field distribution of the TM modes is unaffected by the cylinders. Apart from directly beside the cylinders (where the electric field must be 0), this is borne out in our finite-element (FE) simulations, in agreement with previous results [15]. Equation (5) has all TM modes shifted above the plasma frequency cutoff, below which no TM modes can form. We note that this cutoff frequency can also be approximated with a simple circuit model.

First, we impose $\ell_x, \ell_y \to \infty$ and so neglect the cavity edges. Then the cavity becomes a parallel plate capacitor discharging through the array of conducting cylinders. The magnetic field around each conducting cylinder while discharging can be approximated by [20]

$$H(R) = \frac{I}{2\pi R}(1 - \pi R^2/a^2)$$

(6)

where $I$ is the current through each cylinder, $R$ is the distance from the cylinder axis, leading to an inductance of each conducting cylinder of [20]

$$L_c \approx \frac{\mu_0d}{2\pi}(\ln(a/\ell) - \Gamma)^{0.5}$$

(7)

$$\Gamma = 1/2 + \ln(\sqrt{\pi}) = 1.072...$$

There are $\ell_x\ell_y$ conducting cylinders, and they add in parallel, thereby reducing the total inductance. The total
The isolated fundamental mode of each cavity is represented by $L_0$ and $C_0$. The magnetic coupling between cavities is given by mutual inductances $L_g$. The circuit includes boundary inductances $L_b$, that can be used to include asymmetry in the outermost cavity. The mutual inductance coupling coefficient $k$ (where $M = k\sqrt{L_1L_2}$) has been set to unity for simplicity, as differences can be absorbed in $L_0$ and $L_b$.

capacitance and inductance are then given by

$$C = \epsilon_0\epsilon_r \frac{\ell_x\ell_y}{d} \quad L = L_c \frac{a^2}{\ell_x\ell_y} \quad (8)$$

Thus charge oscillates back and forth through the conducting cylinder array with frequency

$$f = \frac{1}{\sqrt{2\pi\epsilon_0\epsilon_r\mu_0\ell_0}(\ln(\pi) - \Gamma)^{0.5}} \quad (9)$$

$$\Gamma = 1/2 + \ln(\sqrt{\pi}) = 1.072...$$

This frequency is the plasma frequency originally derived in [20], with capacitance and inductance the analogues of inverse electron density and electron mass. Its form is very close to equation (3), differing only in the value of the constant $\Gamma$.

Coupled cavity array circuit model

In order to develop a model for larger cylinders, we begin by remarking that at some size, scattering will effectively confine waves between cylinders [13, 22]. In this case, the array of cylinders will form a coupled cavity array [23]. The tight binding model has been used to model such coupled cavity arrays [23], and photonic crystals [14, 24]. Separately, circuit models have been used to model one dimensional coupled cavity arrays for linear accelerators [25, 26]. Here, we extend the circuit model treatment to two dimensional arrays, and show a better fit to FE simulations than the tight binding model for a single free parameter. This is expected, as the circuit model treats the magnetic coupling between unit cells (nearest, next-nearest etc.,) but cannot be extended to include diagonal couplings between cells; see appendix B for details.

First, we consider only nearest neighbour couplings. In this case $Z_{1D}$ is almost tridiagonal-toeplitz, and for the special cases $L_b = 0, L_g, 2L_g$ has simple closed-form solutions [28]. Taking the case $L_b = 0$ and using eq. (10):

$$f_{ij} = \frac{f_0}{\sqrt{1 + 4\beta(1 + \frac{1}{2}(\cos(\frac{i\pi}{n}) + \cos(\frac{j\pi}{m})))}} \quad (11)$$

$$f_0 = 1/\sqrt{L_0C_0} \quad \beta = L_g/L_0 \quad (1 \leq i \leq n) \quad (1 \leq j \leq m)$$

The minimum mode frequency for a lattice of such cavities then tends to $f_0/\sqrt{1 + 8\beta}$ as $n, m \to \infty$. This new cutoff frequency is determined by the frequency of each unit cell and the coupling between them. Unlike the low scattering case, the field distribution of the cavity modes is now significantly altered by the conducting cylinders.

We can find the expected fields (or currents) from the scattering case, the field distribution of the cavity modes having simple solutions [28]. Taking the case $L_b = 0$ and using eq. (10):

$$X_{ij}(a, b) = X_0 \sin\left(\frac{i(2a - 1)\pi}{2n}\right) \sin\left(\frac{j(2b - 1)\pi}{2m}\right) \quad (12)$$

(a,b index the unit cell; $i_j$ index the eigenvector). The lowest mode $(i,j = 1)$ is symmetric, and the highest mode $(i = n,j = m)$ is anti-symmetric, as we would expect of hybridized modes.

If $\beta$ is small, expanding (11) gives:

$$f_{ij} \approx f_0 - 2t(2 + \cos(k_ia) + \cos(k_ja)) \quad (13)$$

$$t = \beta f_0/2 \quad k_i = i\pi/na \quad k_j = j\pi/na$$
This is the tight-binding model dispersion for a square lattice, and so the circuit model tends to the tight binding model when $\beta \ll 1$.

We performed FE eigenmode simulations of a cavity containing conducting cylinders $r/a > 0.1$. Residuals and fit parameters to different models are shown in figure 5. The agreement with the next nearest neighbour circuit model is NRE < 1% for $r/a > 0.15$.

Simulated mode frequencies for some different lattice sizes are shown explicitly in figure 6. We find including the next nearest neighbour coupling term is important to accurately fit the degeneracy of these modes.

We note the emergence of a band-gap between the $n \times m$ and $m \times n + 1$ modes as $r/a$ increases (figure 6), in accordance with results from metallic photonic crystals [13, 14].

**Applications to enclosure design and circuit QED**

Superconducting circuits are a natural platform for experiments with periodically shorted cavities. The cryogenic temperatures mean superconducting materials can be used, greatly reducing internal losses, and the typical microwave frequency range of 5-10 GHz means effective mode suppression can be achieved with mm scale arrays.

That the fundamental mode frequency of the cavity ultimately depends on the cylinder radius and spacing lifts the constraints on the cavity side-lengths. Effective mode suppression can be achieved in arbitrarily large (in $\hat{x}$ and $\hat{y}$) superconducting circuit devices by enclosing them in such a cavity, and the models discussed above can be used to accurately fit the cavity mode structure.

As we have seen, the presence of the conducting cylinder array significantly changes the mode structure of the cavity. It is therefore interesting to consider the form of cavity mode mediated inter-qubit couplings in these periodically shorted cavities. Qubits below cutoff (in the first band-gap) will interact with the modes around the band-edge. For $L_1, L_2 \gg a$, we can treat the cavity as a photonic crystal, and approximate these modes as being continuous. The interaction between a qubit and the continuous modes around a band-edge forms a bound state [29], with a spatially exponentially decaying envelope. Such bound states have already been experimentally measured in superconducting circuits [30, 31], using transmon [32] qubits embedded in dielectric photonic crystals (metallic photonic crystals distinguish themselves by having their fundamental is shown. The dotted line shows the fundamental decreasing towards a bound in accordance with equation (11). Inset shows the lowest $n^2 + 1$ modes, showing band-gaps below the first mode and between modes $n^2$ and $n^2 + 1$.

\[ H_I^{ij} = \hbar g^{ij} \sigma_+^{i} \sigma_-^{j} \int d\mathbf{k} \frac{E_k(x_i)E_k(x_j)}{\omega_k - \omega_e(k)} + \text{h.c.} \] (14)

For qubits in the band-gap of a two dimensional system with a quadratic dispersion around the band-edge, this results in: $J_{ij} \propto e^{-d_{ij}/L}/\sqrt{d_{ij}}$, where $d_{ij}$ is the qubit separation, $L$ is the effective bound state length.
FIG. 7: Simulated $J$ coupling between qubits $i$ and $j$, for different values of $r/a$. The model consists of a 20 mm $\times$ 20 mm cavity, containing a 10 $\times$ 10 grid of qubits and a 9 $\times$ 9 grid of shorting cylinders, both with 2 mm pitch. $\epsilon_r = 11.9$. Inset shows the positions of qubits $i$ (red) and $j$ (blue) in the grid. Both have frequency 5 GHz. In the absence of cylinders, the couplings do not have a simple spatial dependence. With cylinders present, couplings decay exponentially with separation, with the rate of decay increasing with the ratio $r/a$. For details of the simulation model see appendix C.

We simulated coupling strengths between 100 coaxial Transmon qubits [35] in a cavity containing a 9 by 9 array of cylinders (figure 7), with a FE simulation method described in [36]. We see exponential decay between qubit couplings, suggesting the 9 by 9 cylinder array is sufficiently large for a bound states interpretation to be appropriate. This exponential decay is dramatically different from typical cavity mediated couplings in the Jaynes-Cummings model [4], and is a useful property where local connectivity is desired. We note it can be understood intuitively in the plasma model picture, where waves below the plasma frequency decay away exponentially. Alternatively, by choosing qubit frequencies close to the band-edge, long range interactions can be engineered [33].

That the cylinders can form a coupled cavity array also raises interesting prospective applications. Current implementations of coupled cavity arrays in circuit QED have focused around using printed superconducting circuits to form the array [37, 38]. 3D coupled cavity arrays may benefit from the higher intrinsic quality factors typically achievable in 3D cavities [7, 39]. Moreover, by using the 3D modes to form the coupled cavity array the circuitry is simplified, freeing up more space on the substrate surface. If each array site of the coupled cavity array is coupled to a controllable qubit there are many applications, for example in generation of stabilized W-states via bath-engineering [40, 41], or the quantum simulation of phase transitions in the Bose-Hubbard model [38, 42, 43].

Such a periodically shorted cavity could be made using readily available technology: for example by contacting a substrate containing periodic superconducting TSVs [12, 44] to a metal cavity or metalised substrate, with superconducting bonds [45–47] or high quality pogo pins [9, 48] to ensure low loss DC contact between the TSVs and the cavity. Alternatively, a recess cavity can be machined in metal having cylinders protruding from it, circumventing TSV fabrication or substrate bonding [49].

In summary, we have developed accurate, analytic models for the effects of conducting cylinder arrays on TM cavity mode frequencies; and considered their applications for circuit QED. Arrays of small cylinders imbue the cavity with an anisotropic frequency-dependent permittivity, whereas arrays of larger cylinders form a coupled cavity array. In both cases, the cylinders set a cutoff frequency for the fundamental TM mode of the cavity that is independent of the cavity side-lengths. Such periodically shorted cavities are an attractive platform for quantum computing, bath-engineering, and simulation of many-body physics within circuit QED.
A. STACKED DIELECTRIC LAYERS

First we assume the materials in the stack have no magnetic properties. To determine the modified plasma frequency, we use the following first principles argument:

If we displace electrons an amount $\delta z$ in a cylinder, a charge $e\eta\delta z$ will form at the top of the cylinder and an opposite charge $-e\eta\delta z$ will form at the bottom (here $\eta$ is the electron density). There will then be a restoring force on the the electrons at the top. Assuming there is only a single dielectric layer in the cavity

$$F = eE$$  \hspace{1cm} (A.1)

$$E = -e\eta\delta z/\epsilon_0\epsilon_r$$  \hspace{1cm} (A.2)

and so

$$e^2\eta\delta z/\epsilon_0\epsilon_r = -m^*d^2\delta z/dt^2$$  \hspace{1cm} (A.3)

Here $m^*$ is the effective electron mass. This results in oscillations at frequency

$$\omega_p = \sqrt{(e^2\eta/m^*\epsilon_0\epsilon_r)}$$  \hspace{1cm} (A.4)

However if there are multiple dielectric layers, we must use the average force, found using the total work done on the electrons assuming they traverse the entire stack.

$$<F> = W/d$$  \hspace{1cm} (A.5)

$$W = e\sum_{i=1}^{l} E_i d_i$$  \hspace{1cm} (A.6)

$$E_i = -e\eta\delta z/\epsilon_0\epsilon_r$$  \hspace{1cm} (A.7)

In this case

$$<F> = eE'$$  \hspace{1cm} (A.9)

$$E' = -e\eta\delta z/\epsilon_0\epsilon'_r$$  \hspace{1cm} (A.10)

Where

$$\epsilon'_r = 1/\sum_{i=1}^{l} (d_i/d) \epsilon_i$$  \hspace{1cm} (A.11)

$$d = \sum_{i=1}^{l} d_i$$  \hspace{1cm} (A.12)

Resulting in a plasma frequency

$$\omega'_p = \sqrt{(e^2\eta/m^*\epsilon_0\epsilon'_r)}$$  \hspace{1cm} (A.13)

Therefore eq. (3) in the main text is modified simply by replacing $\epsilon_r$ with $\epsilon'_r$.

For the coupled cavity array circuit model, we again assume the materials in the stack have no magnetic properties. Then inductances in the circuit are unchanged, and only the capacitance $C_0$ changes. This capacitance corresponds to the fundamental mode of each unit cell, and is essentially a parallel plate capacitance between the
bottom and top of the cell. It is then just a matter of calculating the effective capacitance of the series of capacitors formed by the stack:

\[ C_0(\{\epsilon_1, d_1\}, \{\epsilon_2, d_2\}, ..., \{\epsilon_l, d_l\}) = \frac{1}{\sum_{i=1}^{l} \frac{1}{C_0(\epsilon_i, d_i)}} \]

(A.14)

\[ = C_0(\epsilon'_r, d) \]

(A.15)

where

\[ \epsilon'_r = \frac{1}{\sum_{i=1}^{l} \frac{d_i}{\epsilon_i}} \]

(A.16)

\[ d = \sum_{i=1}^{l} d_i \]

(A.17)

So the coupled cavity array filled with a stack of dielectrics behaves the same as the coupled cavity array filled with a single dielectric layer with dielectric constant \(\epsilon'_r\).

In summary, in both the plasma and the circuit model, the only effect of a dielectric stack is to replace the relative permittivity \(\epsilon_r\) wherever it appears with the effective relative permittivity \(\epsilon'_r\) in eq. (A.16).

**B. MESH ANALYSIS & MAPPING THE ARRAY CIRCUIT TO CHAIN CIRCUIT**

Using mesh analysis [1], the circuit in figure (3) of the main text can be represented by the matrix

\[
Z_{2D} = \begin{bmatrix}
Z_\alpha & Z_G & \cdots & 0 \\
Z_G & Z_\beta & Z_G & \\
& Z_G & Z_\beta & Z_G \\
& & \ddots & \ddots \\
& & & Z_G & Z_\beta & Z_G \\
0 & \cdots & & Z_G & Z_\beta & Z_G \\
\end{bmatrix}
\]

Where \(i_i\) is the current through mesh \(i\) and \(V_i\) is the voltage applied to mesh \(i\). At the mode frequencies, currents can oscillate in the absence of any excitations. Therefore modes exist at frequencies where an eigenvalue of \(Z_{2D}\) is 0. \(Z_{2D}\) can be expressed in block matrix form as a diagonal matrix. For a nearest neighbour coupling model it is tridiagonal, for a next nearest neighbour coupling model it is pentadiagonal, and so on. We will show the mapping for the tridiagonal and the pentadiagonal cases, from which it is clear the mapping will extend to arbitrary coupling distance. The tridiagonal matrix is
Where

\[
Z_{\alpha} = \begin{bmatrix}
Z_{\alpha_1} & -Z_g & \cdots & 0 \\
-Z_g & Z_{\alpha_2} & Z_b & \cdots \\
-Z_g & -Z_g & Z_{\alpha_2} & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
-Z_g & Z_{\alpha_2} & -Z_g & \cdots \\
-Z_g & -Z_g & Z_{\alpha_2} & \cdots \\
0 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

\[
Z_{\alpha_1} = Z_0 + 2Z_g + 2Z_b \\
Z_{\alpha_2} = Z_0 + 3Z_g + Z_b
\]

\[
Z_{\beta} = \begin{bmatrix}
Z_{\beta_1} & -Z_g & \cdots & 0 \\
-Z_g & Z_{\beta_2} & Z_b & \cdots \\
-Z_g & -Z_g & Z_{\beta_2} & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
-Z_g & Z_{\beta_2} & -Z_g & \cdots \\
-Z_g & -Z_g & Z_{\beta_2} & \cdots \\
0 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

\[
Z_{\beta_1} = Z_0 + 3Z_g + Z_b \\
Z_{\beta_2} = Z_0 + 4Z_g
\]

\[
Z_G = -Z_g I_{m \times m}
\]

and

\[
Z_0 = i\omega L_0 - i\frac{\omega}{C_0} \\
Z_g = i\omega L_g \\
Z_b = i\omega L_b
\]

It can be verified by expansion that \(Z_{2\Delta}\) can be written

\[
\begin{align*}
Z_{2\Delta_{n \times m}} &= Z_{1\Delta_{n}} \oplus Z_{1\Delta_{m}} - Z_0 I_{n \times m}
\end{align*}
\]

where \(\oplus\) is the Kronecker sum and

\[
Z_{1\Delta_{n}} = \begin{bmatrix}
Z_1 & -Z_g & \cdots & 0 \\
-Z_g & Z_2 & -Z_g & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
-Z_g & Z_2 & -Z_g & \cdots \\
0 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

\[
Z_1 = Z_0 + Z_g + Z_b \\
Z_2 = Z_0 + 2Z_g
\]

\(Z_{1\Delta_{n}}\) is exactly the impedance matrix of the circuit in figure (4) of the main text. The form of this mapping makes solving the eigenvalues much easier, since

\[
\lambda(Z_{2\Delta})_{ij} = \lambda(Z_{1\Delta})_i + \lambda(Z_{1\Delta})_j - Z_0 \tag{A.19}
\]
Finding the mode frequencies of the 2-D lattice is now simply a matter of solving the 1-D eigenvalues, inserting into eq. (A.19), and solving for \( \lambda(\mathbf{Z}_{2D})_{ij} = 0 \).

\( Z_{1D_n} \) is a tridiagonal matrix, for which simple closed-form solutions exist for \( L_b = 0, L_g, 2L_g \) \[2\], leading to

\[
\begin{align*}
 f_{ijL_b=0} &= \frac{f_0}{\sqrt{1 + 4\beta(1 + \frac{1}{2}\cos(\frac{i\pi}{n}) + \cos(\frac{j\pi}{m}))}} \quad (A.20) \\
 f_{ijL_b=L_g} &= \frac{f_0}{\sqrt{1 + 4\beta(1 + \frac{1}{2}\cos(\frac{i\pi}{n}) + \cos(\frac{j\pi}{m+1}))}} \quad (A.21) \\
 f_{ijL_b=2L_g} &= \frac{f_0}{\sqrt{1 + 4\beta(1 + \frac{1}{2}\cos(\frac{(i-1)\pi}{n}) + \cos(\frac{(j-1)\pi}{m}))}} \quad (A.22)
\end{align*}
\]

The predicted modes of a 10x10 array are shown in figure (2).

We now repeat the procedure for the next nearest coupling case, to demonstrate the effect of including further couplings. The resulting pentadiagonal matrix is

\[
\mathbf{Z}_{2D} = \begin{bmatrix}
\mathbf{Z}_\alpha & \mathbf{Z}_G & \mathbf{Z}_{G_2} & \cdots & 0 \\
\mathbf{Z}_G & \mathbf{Z}_\beta & \mathbf{Z}_G & \mathbf{Z}_{G_2} \\
\mathbf{Z}_{G_2} & \mathbf{Z}_G & \mathbf{Z}_\gamma & \mathbf{Z}_G & \mathbf{Z}_{G_2} \\
\mathbf{Z}_{G_2} & \mathbf{Z}_G & \mathbf{Z}_\gamma & \mathbf{Z}_G & \mathbf{Z}_{G_2} \\
0 & \cdots & \cdots & \cdots & \mathbf{Z}_{G_2} & \mathbf{Z}_G & \mathbf{Z}_\alpha \\
\end{bmatrix}
\]

Where

\[
\mathbf{Z}_\alpha = \begin{bmatrix}
\mathbf{Z}_{\alpha_1} & -Z_g & -Z_{g_2} & \cdots & 0 \\
-Z_g & \mathbf{Z}_{\alpha_2} & -Z_g & -Z_{g_2} \\
-Z_{g_2} & -Z_g & \mathbf{Z}_{\alpha_3} & -Z_g & -Z_{g_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & -Z_{g_2} & -Z_g & \mathbf{Z}_{\alpha_1} \\
\end{bmatrix}
\]

\[
\begin{align*}
\mathbf{Z}_{\alpha_1} &= Z_0 + 2Z_g + 2Z_{g_2} + 2Z_b \\
\mathbf{Z}_{\alpha_2} &= Z_0 + 3Z_g + 2Z_{g_2} + Z_b \\
\mathbf{Z}_{\alpha_3} &= Z_0 + 3Z_g + 3Z_{g_2} + Z_b
\end{align*}
\]

\[
\mathbf{Z}_\beta = \begin{bmatrix}
\mathbf{Z}_{\beta_1} & -Z_g & -Z_{g_2} & \cdots & 0 \\
-Z_g & \mathbf{Z}_{\beta_2} & -Z_g & -Z_{g_2} \\
-Z_{g_2} & -Z_g & \mathbf{Z}_{\beta_3} & -Z_g & -Z_{g_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & -Z_{g_2} & -Z_g & \mathbf{Z}_{\beta_1} \\
\end{bmatrix}
\]

\[
\begin{align*}
\mathbf{Z}_{\beta_1} &= Z_0 + 2Z_g + 2Z_{g_2} + 2Z_b \\
\mathbf{Z}_{\beta_2} &= Z_0 + 3Z_g + 2Z_{g_2} + Z_b \\
\mathbf{Z}_{\beta_3} &= Z_0 + 3Z_g + 3Z_{g_2} + Z_b
\end{align*}
\]
\[ Z_{\beta_1} = Z_0 + 3Z_g + 2Z_{g_2} + Z_b \]
\[ Z_{\beta_2} = Z_0 + 4Z_g + 2Z_{g_2} \]
\[ Z_{\beta_3} = Z_0 + 4Z_g + 3Z_{g_2} \]

\[
Z_\gamma = \begin{bmatrix}
Z_{\gamma_1} & -Z_g & -Z_{g_2} & \cdots & 0 \\
-Z_g & Z_{\gamma_2} & -Z_g & -Z_{g_2} & \cdots \\
-Z_{g_2} & -Z_g & Z_{\gamma_3} & -Z_g & -Z_{g_2} \\
0 & \cdots & \cdots & \cdots & \cdots \\
-Z_{g_2} & -Z_g & Z_{\gamma_2} & -Z_g & -Z_{g_1}
\end{bmatrix}
\]

\[ Z_{\gamma_1} = Z_0 + 3Z_g + 3Z_{g_2} + Z_b \]
\[ Z_{\gamma_2} = Z_0 + 4Z_g + 3Z_{g_2} \]
\[ Z_{\gamma_3} = Z_0 + 4Z_g + 4Z_{g_2} \]

\[
Z_G = -Z_g I_{m \times m}
\]
\[ Z_{G_2} = -Z_{g_2} I_{m \times m} \]

It can again be verified by expansion that \( Z_{2D} \) can be written

\[
Z_{2D_{n \times m}} = Z_{1D_n} \oplus Z_{1D_m} - Z_0 I_{n \times m}
\]

where \( \oplus \) is the Kronecker sum and

\[
Z_{1D_n} = \begin{bmatrix}
Z_1 & -Z_g & -Z_{g_2} & \cdots & 0 \\
-Z_g & Z_2 & -Z_g & -Z_{g_2} & \cdots \\
-Z_{g_2} & -Z_g & Z_3 & -Z_g & -Z_{g_2} \\
0 & \cdots & \cdots & \cdots & \cdots \\
-Z_{g_2} & -Z_g & Z_2 & -Z_g & -Z_1
\end{bmatrix}
\]

\[ Z_1 = Z_0 + Z_g + Z_{g_2} + Z_b \]
\[ Z_2 = Z_0 + 2Z_g + Z_{g_2} \]
\[ Z_3 = Z_0 + 2Z_g + 2Z_{g_2} \]

Simple closed form solutions to this pentadiagonal matrix do not exist [3]. However the mapping still greatly reduces the size of the matrices to be solved by analytic or numerical methods.
FIG. 2: Showing the 100 mode frequencies of the 10 by 10 coupled cavity array, for the three different boundary conditions with simple closed form solutions. $\beta = 0.05$ in all cases. Increasing the boundary inductance lowers the frequency of all modes.

FIG. 3: a) A top down view of the simulation model, showing the grid of 100 qubits, as well as the grid of perfectly conducting cylinders. The substrate is 0.5 mm thick silicon, which directly contacts the perfectly conducting cavity on all sides. Qubits are separated by a 2 mm pitch, meaning 100 can be inserted on a 20 mm $\times$ 20 mm chip. Also highlighted is the subset of qubits for which couplings are shown in the main text. b) A zoom in of some of the qubits in the simulation. The shorting cylinders are visible, as well as cylindrical recesses above each qubit, to prevent them shorting to the cavity.

C. SIMULATION MODEL FOR INTER-QUBIT COUPLINGS

The model is shown in figure (A.3). A cylindrical recess filled with vacuum is made above each qubit to prevent it shorting to the cavity. At the location of each Josephson-Junction is a lumped port. Using the Driven Terminal
solution type, the 12x12 impedance matrix \( Z \) between these ports is simulated at 5 GHz. We use the simulation method laid out in [4] because it doesn’t require frequency sweeps of the impedance matrix, nor any fitting of frequency sweeps to a circuit model. This becomes intractable when couplings are very small, as a prohibitively small step size must be used in the simulated frequency sweep.

Along with the \( Z \) matrix, junction inductances must be supplied. A value of 14 nH is given for the junction inductance. This is found by placing a lumped LRC at one of the junction locations and finding the inductance value that results in a qubit mode frequency of 5 GHz, using the Eigenmode solution type.

A trade-off of this approach is that when calculating the coupling \( J_{ij} \) between qubits \( i \) and \( j \), all other qubits have their junctions behave as opens, and so these qubits contribute to the coupling predominantly like a capacitance network. Therefore couplings mediated between \( i \) and \( j \) by other qubit modes are neglected. These modes can in principle be re-inserted in post-processing back into the \( Z \) matrix, but this becomes intractable as the \( Z \) matrix grows in size.

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