Large-time behavior for spherically symmetric flow of viscous polytropic gas in exterior unbounded domain with large initial data

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Abstract

This paper deals with the spherically symmetric flow of compressible viscous and polytropic ideal fluid in unbounded domain exterior to a ball in $\mathbb{R}^n$ ($n \geq 2$). We show that the global solutions are convergent as time goes to infinity. The critical step is obtaining the point-wise bound of the specific volume $v(x, t)$ and the absolute temperature $\theta(x, t)$ from up and below both for $x$ and $t$. Note that the initial data can be arbitrarily large and, compared with [14], our method applies to the spatial dimension $n = 2$. The proof is based on the elementary energy methods.

1 Introduction

We study the asymptotic behavior of spherically symmetric solutions to a polytropic ideal model of a compressible viscous gas over an unbounded exterior domain $\Pi = \{ \xi \in \mathbb{R}^n : |\xi| > 1 \}$, where $n \geq 2$ denotes the spatial dimension. The motion of a viscous polytropic ideal gas which can be described by the equations in Eulerian coordinates (cf. [3])

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + R\nabla(\rho \theta) &= \mu \Delta \mathbf{u} + \nabla((\mu + \lambda) \text{div} \mathbf{u}), \\
c_v \rho(\theta_t + \mathbf{u} \cdot \nabla \theta) + R \rho \theta \text{div} \mathbf{u} &= \kappa \Delta \theta + \lambda (\text{div} \mathbf{u})^2 + 2\mu D : D, \\
\end{align*}
\]

Here, as usual, the unknown functions $\rho$, $\theta$ and $\mathbf{u} = (u_1, \ldots, u_n)$ symbol the density, the absolute temperature and the velocity, respectively. $R, c_v, \kappa$ are given positive constants; $\mu$ and $\lambda$ are the constant viscous coefficients satisfy $\mu > 0$, $2\mu + n\lambda > 0$; and $D = D(\mathbf{u})$ is the deformation tensor,

\[
D_{ij} = \frac{1}{2} (\partial_j u_i + \partial_i u_j) \quad \text{and} \quad D : D = \sum_{i,j=1}^{n} D^2_{ij}.
\]

We shall consider the equations (1.1) supplemented with the initial and boundary conditions

\[
\rho(\xi, 0) = \rho_0(\xi), \quad \mathbf{u}(\xi, 0) = \mathbf{u}_0(\xi), \quad \theta(\xi, 0) = \theta_0(\xi), \quad \xi \in \overline{\Pi},
\]

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and
\[ u(\xi, t)|_{\xi \in \partial \Omega} = u_0(\xi), \quad \partial \theta / \partial n(\xi, t)|_{\xi \in \partial \Omega} = \theta_0(\xi), \quad t \geq 0, \quad (1.3) \]
with \( \nu \) being the exterior normal vector.

If the initial functions \((\rho_0(\xi), u_0(\xi), \theta_0(\xi))\) are assumed to be spherically symmetric, i.e.,
\[ \rho_0(\xi) = \tilde{\rho}(r), \quad u_0(\xi) = \xi r \tilde{u}(r), \quad \theta_0(\xi) = \tilde{\theta}(r), \quad r = |\xi| \geq 1, \quad (1.4) \]
so does the corresponding solution \((\tilde{\rho}, \tilde{u}, \tilde{\theta})(r, t)\) because \((1.1)\) is rotationally invariant (cf. [117]), and thereby, the equations \((1.1)\) takes the form (ignore the “\(\hat{\cdot}\)“)
\[ \rho_t + \frac{(r^{n-1}\rho u)_r}{r^{n-1}} = 0, \]
\[ \rho(u_t + u \partial_r u) + R \partial_r \rho = \beta \left( \frac{(r^{n-1}u)_r}{r^{n-1}} \right)_r, \]
\[ c \rho (\theta_t + u \partial_\rho \theta) + R \rho \theta = \kappa \left( \frac{(r^{n-1}u)_r}{r^{n-1}} \right)^2 + 2\mu (\partial_r u)^2 + 2\mu \frac{n-1}{r^2} u^2, \quad \beta = 2\mu + \lambda > 0, \quad (1.5) \]
\[ u(1, t) = 0, \quad \partial_r \theta(1, t) = 0, \quad t \geq 0. \quad (1.7) \]
For our analysis convenience, it is desirable to convert the \((1.5)\) from the Euler coordinates \((r, t)\) into that in Lagrangian coordinates \((x, t)\). Define
\[ r(x, t) = r_0(x) + \int_0^t u(r(x, \tau), \tau)d\tau, \quad (1.8) \]
with
\[ \int_{r_0(x)}^{r(x, t)} y^{n-1} \rho_0(y)dy = x. \quad (1.9) \]
Using \((1.8), (1.9), (1.5)\), and the boundary condition \(u(1, t) = 0\), we check for \(t \geq 0\)
\[ \int_1^{r(x, t)} y^{n-1} \rho(y, t)dy = \int_1^{r_0(x)} y^{n-1} \rho_0(y)dy = x. \quad (1.10) \]
By this, \(r = 1\) iff \(x = 0\) and \(r \to \infty\) iff \(x \to \infty\), as long as \(\rho > 0\) for all \((y, t) \in [0, \infty) \times [0, \infty)\). Moreover, it is easy to see from \((1.8)\) and \((1.10)\) that
\[ \partial_\rho r(x, t) = u(r(x, t), t) \quad \text{and} \quad r^{n-1}(x, t)\rho(r(x, t), t)\partial_x r(x, t) = 1. \quad (1.11) \]
Introduce new functions
\[ \tilde{v}(x, t) = 1/\rho(r(x, t), t), \quad \tilde{u}(x, t) = u(r(x, t), t), \quad \tilde{\theta}(x, t) = \theta(r(x, t), t), \quad (1.12) \]
we express \((1.5)\) in terms of \((\tilde{v}, \tilde{u}, \tilde{\theta})\) (denoted still by \((v, u, \theta)\) below) in variables \((x, t)\)
\[ v_t = (r^{n-1}u)_x, \]
\[ u_t = r^{n-1}u_x, \]
\[ c\tilde{\theta}_t = \kappa \left( \frac{2(n-1)\tilde{\theta}}{v} x + (r^{n-1}u)\sigma - 2\mu(n-1)(r^{n-2}u^2)_x, \quad x \in \Omega, \quad t > 0, \quad (1.13) \]
where $\sigma = \beta (r^{n-1}u)_r / v - R \theta / v$, $\Omega = (0, +\infty)$, the initial functions
\[
v(x, 0) = v_0(x), 
\ u(x, 0) = u_0(x), 
\ \theta(x, 0) = \theta_0(x), \quad x \in \Omega,
\]
the boundary and the far field behavior
\[
u(0, t) = 0, \ \partial_y \theta(0, t) = 0, \ \lim_{x \to \infty} (v(x, t), u(x, t), \theta(x, t)) = (1, 0, 1), \quad t \geq 0.
\]

In view of (1.12), the (1.8) and (1.11) is reduced to
\[
r(x, t) = r_0(x) + \int_0^t u(x, s) ds, \quad r_t = u, \quad r^{n-1} - r_x = v.
\]
Integration of last term in (1.16) yields
\[
r^n(x, t) = 1 + n \int_0^x v(y, t) dy.
\]
Furthermore, it follows from [6, eq.(3.19)] that
\[
r(x, t) \geq r(0, t) = 1, \quad (x, t) \in \mathbb{R} \times [0, \infty).
\]

We first state the global existence in time of (generalized) solution to the initial-boundary-value problem (1.13)-(1.15).

**Theorem 1.1 (See [6])** Assume that the initial function $(v_0, u_0, \theta_0)$ in (1.14) are compatible with the boundary conditions (1.15), and satisfy
\[
v_0 - 1, \ u_0, \ \theta_0 - 1, \ r^{n-1} \partial_r v_0, \ r^{n-1} \partial_r u_0, \ r^{n-1} \partial_r \theta_0 \in L^2(\Omega),
\]
\[
\inf_{x \in \Omega} v_0(x) > 0 \quad \text{and} \quad \inf_{x \in \Omega} \theta_0(x) > 0.
\]

Then for any fixed $T > 0$, the problem (1.13)-(1.15) admits a unique global (large) generalized solution $(v, u, \theta)$ over $[0, T]$, with $v(x, t)$ and $\theta(x, t)$ having positive bounds from above and below (depending on $T$). Moreover,
\[
v - 1, \ u, \ \theta - 1 \in L^\infty (0, T; H^1(\Omega)), \ v_1, \ r^{n-1} u_x, \ r^{n-1} \theta_x \in L^\infty (0, T; L^2(\Omega)),
\]
\[
v_{x1}, \ u_t, \ \theta_t, \ r^{2(n-1)} u_{xx}, \ r^{2(n-1)} \theta_{xx} \in L^2 (0, T; L^2(\Omega)).
\]

After the existence of global solution, the study of asymptotic behavior for solutions comes out naturally. The first paper in this aspect is due to Matsumura-Nishida [13], where they prove that, with initial data having a small oscillatory around a non-vacuum equilibrium, the solution (with external force and may not be symmetric) converges to the corresponding stationary solution as time tends to infinity in exterior domain of $\mathbb{R}^3$. From then on, much progress have been made with smallness assumptions. See [4, 5, 9, 11, 15, 16], and the references cited therein.

In case of large initial data in bounded annulus domain, Itaya [17] showed global existence of the spherically symmetric solution to (1.11). Matsumura [12] considered the isothermal flow and proved that the stationary solution is time asymptotically stable (with large external force); also, an exponential convergence rate was obtained there.

When the exterior domains becomes unbounded, Jiang [6] obtained the global in time solutions to (1.13)-(1.15). When it comes to the large-time behavior, some difficulties arise: for example, the useful representation (see [2]) for $v$, the specific volume, and the imbedding inequality $L^2 \to L^1$ do not valid any more for unbounded domains. However, when the space dimension $n \geq 3$, Jiang [6] proved a partial result on the asymptotic behavior, precisely, he shows that $\|u\|_{L^2(\Omega)}$ is zero stable as time tends into infinity, with $j \in [2, \infty)$ being an arbitrary integer. Nakamura-Nishibata [14] proved that the solutions (with external force) correspond to the stationary solution time asymptotically. But the proof in [14] still requires $n \geq 3$, which is
essential to conquer (with (3.16)) the difficulty caused by the unboundedness. Finally, we mention the progress for the one-dimensional (1D) case. By means of cut-off function, Jiang [7, 8] obtained the uniform (in $x$ and $t$) bounds for $v(x, t)$, but leaves that for absolute temperature $\theta(x, t)$ open. Recently, Li and the author [10] gives a complete description on the large-time behavior of solutions to 1D Cauchy problem (1.13).

This paper concerns the spherically symmetric solutions in unbounded domain exterior to $\mathbb{R}^n$ with spatial dimension $n \geq 2$. Our goal is to show the large-time behavior of solutions to the initial-boundary-value problem (1.13)-(1.15). The proof depends heavily on the bounds of the specific volume $v(x, t)$ and the absolute temperature $\theta(x, t)$. We adopt the idea in [2,7] and use a local representation to derive the bound for $v(x, t)$. To get the bound for the temperature, we multiply the equations (1.13) by $(\theta - 2)_+$, because the spatial domain always keeps bounded when $\theta(x, t)$ leaves far away from the equilibrium state at any fixed time.

Our main result in current paper lies in the following theorem.

**Theorem 1.2 (Large time behavior)** Assume that the initial data defined in (1.14) are compatible with boundary conditions (1.15) and satisfy (1.19)-(1.20). Let $(v, u, \theta)$ be the (unique) generalized solution to (1.13)-(1.15) described in Theorem 1.1 which satisfies (1.21). Then there exists a positive constant $C$ depending only on $\mu, \lambda, R, c_v, \kappa, n, \v$, and the initial data, such that

$$
\sup_{t \in [0, \infty)} \left( \| (v - 1, u, \theta - 1)(\cdot, t) \|_{L^2(\Omega)} + \| r^{n-1}(v_x, u_x, \theta_x)(\cdot, t) \|_{L^2(\Omega)} \right) \\
+ \int_0^\infty \left( \| r^{2(n-1)}u_x \|_{L^2(\Omega)} + \| r^{2(n-1)}\theta_x \|_{L^2(\Omega)}^2 \right) dt \leq C,
$$

(1.22)

and

$$
C^{-1} \leq v(x, t), \quad \theta(x, t) \leq C, \quad \forall \ (x, t) \in \Omega \times [0, \infty).
$$

(1.23)

Moreover, the following asymptotic behavior holds

$$
\lim_{t \to \infty} \| (v - 1, u, \theta - 1)(\cdot, t) \|_{C(\Omega)} = 0.
$$

(1.24)

A few remarks are in order:

**Remark 1.1** In comparison with [2,14], the (1.23) in Theorem 1.2 is valid for $n = 2$. Moreover, we do not need any type of smallness assumptions on the initial data.

**Remark 1.2** The same conclusion as Theorem 1.2 holds true if the boundary condition (1.15) is replaced by

$$
u(0, t) = 0, \quad \theta(0, t) = 1, \quad t \geq 0.
$$

Throughout this paper, $C(\Omega), L^p(\Omega)$ and $H^1(\Omega)$ denote the usual Sobolev spaces. See, for example, the definitions in [1]. The same letter $C$ symbolizes a positive generic constant which may rely on $\mu, \lambda, R, c_v, \kappa, n, \v$, and the initial data, but does not depend on the time $t$. Particularly, we use $C(\alpha)$ to emphasize that $C$ depends on $\alpha$.

The remainder sections are arranged as follows:

In section 2, some known Lemmas and facts are collected for our usage.

In sections 3, we use a local representation to get the uniform bound for $v(x, t)$ from above and below, and in section 4, the uniform upper bound for $\theta(x, t)$ is derived by means of elaborate energy computation.

We give the $L^2$-norm estimates for derivatives of the solutions in section 5, and complete the proof of Theorem 1.2 in the final section 6.
2 Preliminaries

The first lemma provides the basic energy estimate.

Lemma 2.1 The solution \((v, u, \theta)\) obtained in Theorem 1.1 satisfies for all \(t \geq 0\)

\[
\int_{\Omega} U(x, t) + \int_{0}^{t} \int_{\Omega} \left( \frac{vu^2}{r^2 \theta} + \frac{r^2(n-1)u_x^2}{v \theta} + \frac{(r^2(n-1)u_x^2)}{v \theta^2} \right) \leq C, \tag{2.1}
\]

where

\[
U(x, t) = \left( R(v - \ln v - 1) + \frac{1}{2} u^2 + c_v(\theta - \ln \theta - 1) \right)(x, t).
\]

Proof. Multiplying \((1.13)_1\) by \(R(1 - v - 1)\), \((1.13)_2\) by \(u\), \((1.13)_3\) by \((1 - \theta - 1)\), respectively, adding them together, we arrive at

\[
U_t + \left( \frac{\beta(r^{n-1}u_x^2)v}{v \theta} - 2\mu(n-1)\frac{(r^{n-2}u_x^2)}{\theta} + \kappa \frac{r^{2(n-1)}\theta^2_x}{v \theta^2} \right)
= \left\{ \frac{\beta(r^{n-1}u_xu^{n-1})}{v} + R\frac{r^{n-1}u_x(1 - \theta)}{v} + \kappa \frac{r^{2(n-1)}\theta_x(\theta - 1)}{v \theta} - 2\mu(n-1)(r^{n-2}u_x^2) \right\} x.
\tag{2.2}
\]

Utilizing \((1.16)\), a careful calculation (see [14, Lemma 3.1] for detail) shows there exists a positive constant \(C\) such that

\[
\beta(r^{n-1}u_x^2) - 2\mu(n-1)v(r^{n-2}u_x^2) \geq C \left( r^{-2}v^2u^2 + r^{2(n-1)}u_x^2 \right).
\tag{2.3}
\]

With \((2.3)\), integrating \((2.2)\), using integration by parts, Taylor theorem, initial conditions \((1.19)-(1.20)\), we obtain

\[
\int_{\Omega} U(x, t) + \int_{0}^{t} \int_{\Omega} \left( \frac{vu^2}{r^2 \theta} + \frac{r^2(n-1)u_x^2}{v \theta} + \frac{r^2(n-1)\theta^2_x}{v \theta^2} \right) \leq C.
\tag{2.4}
\]

Again using \((1.16)\) we compute

\[
(r^{n-1}u_x)x = r^{n-1}u_x + (n-1)r^{-1}vu,
\tag{2.5}
\]

which together with \((2.4)\) yields

\[
\int_{0}^{t} \int_{\Omega} \frac{(r^{n-1}u_x^2)}{v \theta} \leq C.
\]

This inequality plus \((2.1)\) yields \((2.1)\). \(\square\)

Having \((2.1)\) in hand, we use Jensen inequality and check that

\[
\int_{k}^{k+1} v - \ln \int_{k}^{k+1} v - 1, \quad \int_{k}^{k+1} \theta - \ln \int_{k}^{k+1} \theta - 1 \leq C, \quad k = 0, 1, 2, \ldots.
\]

This implies from mean value theorem

\[
0 < \alpha_1 \leq v(a_k(t), t) = \int_{k}^{k+1} v(x, t), \quad \theta(b_k(t), t) = \int_{k}^{k+1} \theta(x, t) \leq \alpha_2 < \infty, \tag{2.6}
\]

where \(\alpha_1, \alpha_2\) are two positive roots of the equation \(y - \ln y - 1 = C\).
3 Uniform bounds of \( v(x, t) \)

**Lemma 3.1** Let \((v, u, \theta)\) be a solution described in Theorem 1.1. Then it satisfies

\[
C^{-1} \leq v(x, t) \leq C, \quad (x, t) \in \Omega \times [0, +\infty). \tag{3.1}
\]

**Proof.** The strategy is to adopt some ideas in \([2, 7, 8]\) to localize the problem.

*Local representation for \( v(x, t) \):*

Define the cut-off function

\[
\varphi(x) = \begin{cases} 
1, & y \leq k; \\
 k + 1 - x, & k \leq y \leq k + 1; \\
0, & y \geq k + 1.
\end{cases}
\]

Utilizing (1.16), we multiply (1.13) by \( \varphi \) and compute

\[
(\varphi r^{1-n} u)_{t} + (n-1)\varphi r^{-n} u^{2} = (\sigma \varphi)_{x} - \sigma \varphi'. \tag{3.2}
\]

Let \( x \in I = (k-2, k) \cap \Omega \) with \( k \in \mathbb{N}_{+} \), integrating (3.2) over \((x, +\infty)\) and using (1.13) lead to

\[
-\partial_{t} \int_{x}^{\infty} \varphi r^{1-n} u - (n-1) \int_{x}^{\infty} \varphi r^{-n} u^{2} = \sigma + \int_{x}^{\infty} \varphi' \sigma \\
= \beta (\ln v)_{t} - R \frac{\theta}{v} - \int_{k}^{k+1} \sigma.
\]

Integration of it in time shows

\[
\int_{x}^{\infty} \varphi(r_{0}^{1-n} u_{0} - r^{1-n} u) - (n-1) \int_{0}^{t} \int_{x}^{\infty} \varphi r^{-n} u^{2} \\
= \beta \ln \frac{v(x, t)}{v_{0}} - R \int_{0}^{t} \frac{\theta}{v} - \int_{k}^{k+1} \int_{0}^{t} \sigma,
\]

which gives after taken the exponential

\[
\frac{1}{v(x, t)} \exp \left\{ R \int_{0}^{t} \frac{\theta}{v} \right\} = \frac{1}{B(x, t) Y(t)}, \quad x \in I, t \geq 0, \tag{3.3}
\]

where

\[
B(x, t) = v_{0} \exp \left\{ \frac{1}{\beta} \int_{x}^{\infty} \varphi(r_{0}^{1-n} u_{0} - r^{1-n} u) \right\}
\]

and

\[
Y(t) = \exp \left\{ \frac{1}{\beta} \left( \int_{k}^{k+1} \sigma - (n-1) \int_{0}^{t} \int_{x}^{\infty} \varphi r^{-n} u^{2} \right) \right\}.
\]

Integrating (3.3) after multiplied by \( R\theta/\beta \) arrives at

\[
\exp \left\{ R \int_{0}^{t} \frac{\theta}{v} d\tau \right\} = 1 + \frac{R}{\beta} \int_{0}^{t} \frac{\theta(x, \tau)}{B(x, \tau) Y(\tau)} d\tau, \quad x \in I, \tag{3.4}
\]

which again with (3.3) yields

\[
v(x, t) = B(x, t) Y(t) + \frac{R}{\beta} \int_{0}^{t} \frac{\theta(x, \tau) B(x, t) Y(t)}{B(x, \tau) Y(\tau)} d\tau, \quad x \in I, t \geq 0. \tag{3.5}
\]

*Estimate for \( B(x, t) \) and \( Y(t) \):
Clearly, it follows from (1.18) and (2.1) that for \( x \in I \)
\[
\left| \int_x^\infty \varphi(r_0^{-n}u_0 - r^{-n}u)dy \right| \leq C\|u_0\|_{L^2(k-2,k+1)} + C\|u\|_{L^2(k-2,k+1)} \leq C,
\]
and therefrom,
\[
C^{-1} \leq B(x,t) \leq C. \tag{3.6}
\]

Next, we following in [7] to estimate \( Y(t) \). Making use of (2.6) and (1.18), one has for \( x \in [k,k+1] \) and \( 0 \leq s < t \)
\[
\left| \int_s^t \int_{b_k}^\infty \frac{\theta(x)}{\theta} \right| \leq \int_s^t \left( \int_k^{k+1} \frac{\alpha_2^2}{v\theta^2} \right)^{1/2} \left( \int_k^{k+1} v(y,\tau) \right)^{1/2} \\
\leq C\alpha_2^{1/2} \int_s^t \left( \int_k^{k+1} \frac{r^{-2(n-1)}\theta^2}{v\theta^2} \right)^{1/2} \leq C(\alpha_2) + (t-s)\ln 2.
\]

By this we use Jensen inequality to estimate
\[
\int_s^t \theta(x,\tau) = \int_s^t \exp\{\ln \theta(x,\tau)\} \\
\geq (t-s) \exp \left\{ \frac{1}{(t-s)} \int_s^t \ln \theta(x,\tau) \right\} \\
= (t-s) \exp \left\{ \frac{1}{(t-s)} \left[ \int_s^t \int_{b_k}^\infty \frac{\theta(x)}{\theta} + \int_s^t \ln \theta(b_k,\tau) \right] \right\} \\
\geq (t-s) \exp \left\{ -\frac{1}{(t-s)} \left[ \int_s^t \int_{b_k}^\infty \frac{\theta(x)}{\theta} + \ln \alpha_1 \right] \right\} \\
\geq \frac{\alpha_1}{2} (t-s) \exp \left\{ -C(\alpha_2) \right\},
\]
whence,
\[
-t \int_s^t \inf_{x \in [k,k+1]} \theta(\cdot,\tau) \leq \left\{ \begin{array}{ll}
0, & 0 \leq t-s \leq 1,
-C(t-s), & 1 \leq t-s.
\end{array} \right. \tag{3.7}
\]

By virtue of (2.1), (2.6), (3.7), Jensen inequality, we have
\[
\int_s^t \int_k^{k+1} \sigma - (n - 1) \int_0^t \int_x^\infty \varphi r^{-n} u^2 \\
\leq \int_s^t \int_k^{k+1} \sigma = \int_s^t \int_k^{k+1} \left( \frac{r^{-n-1}u_x}{v} - R\theta \right) \\
\leq C \int_s^t \int_k^{k+1} \frac{(r^{-n-1}u_x)^2}{v\theta} - \frac{R}{2} \int_s^t \int_k^{k+1} \frac{\theta}{v} \\
\leq C - \frac{R}{2} \int_s^t \inf_{\tau} \left( \int_k^{k+1} \frac{1}{v} \right)^{-1} \\
\leq C - C \int_s^t \inf_{\tau} \left( \int_k^{k+1} \frac{1}{v} \right) \leq C - C(t-s).
\]

Therefore,
\[
0 \leq Y(t)/Y(s) \leq C \exp\{-C(t-s)\}, \quad 0 \leq s < t. \tag{3.8}
\]

**Uniform bounds of \( v(x,t) \) from up and below:**

In terms of (3.6) and (3.8), we deduce from (3.5) that for \( x \in I \)
\[
v(x,t) \leq C + C \int_0^t \theta(x,s) \exp\{-C(t-s)\} ds. \tag{3.9}
\]
Observe from (1.18) and (2.6) that for \( x \in [k, k + 1] \)
\[
|\sqrt{\theta(x, t)} - \sqrt{\theta(b_k(t), t)}| \leq \int_k^{k+1} \frac{|\theta|}{\sqrt{\theta}} \leq \left( \int_k^{k+1} \frac{\theta_2}{v\theta^2} \right)^{1/2} \left( \int_k^{k+1} \frac{v}{\theta} \right)^{1/2}
\]
\[
\leq C\sqrt{\alpha_2} \max_{x \in [k, k+1]} \sqrt{v} \left( \int_k^{k+1} \frac{r^{2(n-1)}\theta_2}{v\theta^2} \right)^{1/2},
\]
which implies that
\[
\frac{\alpha_1}{2} - \alpha_2 f(t) \max_{x \in [k, k+1]} v(\cdot, t) \leq \theta(x, t) \leq 2\alpha_2 + 2\alpha_2 f(t) \max_{x \in [k, k+1]} v(\cdot, t), \quad (3.10)
\]
where
\[
f(t) = \int_\Omega \frac{r^{2(n-1)}\theta_2}{v\theta^2}.
\]
Inserting (3.10) into (3.9) yields
\[
v(x, t) \leq C + C \int_0^t f(\tau) \max_{x \in [k, k+1]} v(\cdot, \tau). \quad (3.12)
\]
Recall (2.1), exploiting Gronwall inequality to (3.12) concludes
\[
v(x, t) \leq C, \quad (x, t) \in [k, k + 1] \times [0, \infty). \quad (3.13)
\]
Integrate (3.5) over \([k, k + 1]\), use (2.6), we infer
\[
\alpha_1 \leq C \exp\{-Ct\} + C \int_0^t \frac{Y(t)}{Y(\tau)} d\tau.
\]
which, along with (3.6), (3.8), (3.10), (3.13), deduces from (3.5) that for \( x \in [k, k + 1] \)
\[
v(x, t) \geq C \int_0^t \theta(x, \tau) \frac{Y(t)}{Y(\tau)} d\tau
\]
\[
\geq C \int_0^t \frac{Y(t)}{Y(\tau)} d\tau - C \int_0^t f(s) \frac{Y(t)}{Y(\tau)} d\tau
\]
\[
\geq C - C \exp\{-Ct\} - C \left( \int_0^{t/2} + \int_{t/2}^t \right) f(\tau) \exp\{-C(t - \tau)\} d\tau
\]
\[
\geq C - C \exp\{-Ct/2\} - \int_{t/2}^t f(\tau) d\tau
\]
\[
\geq C,
\]
as long as \( t \geq T_0 \) for some large \( T_0 \). On the other hand, it satisfies from (6) eq.(4.9)] that
\[
v(x, t) \geq C(T_0), \quad (x, t) \in \Omega \times [0, T_0). \quad (3.15)
\]
Notice that the \( C \) is independent of \( k \), the proof ends up with (3.10), (3.11) and (3.15) if the integers \( k \) traverses \( \mathbb{N}_+ \). \( \square \)

**Corollary 3.2** Inequalities (1.17) and (3.1) ensure that
\[
C^{-1}(1 + x) \leq r^3(x, t) \leq C(1 + x), \quad (3.16)
\]
where the positive constant \( C \) independent of either \( x \) or \( t \).

**Remark 3.1** With the aid of (3.15), the validity of (3.1) has been proven by Jiang in (7) for \( n = 3 \).
4 Uniform bound for $\theta(x, t)$ from above

The following lemma plays a critical role in deriving the upper bound for $\theta$.

**Lemma 4.1** Let $(v, u, \theta)$ be the solution described in Theorem 1.1. Then it holds that

$$
\int_\Omega [(\theta - 1)^2 + u^4] (x, t) + \int_0^t \int_\Omega \left[(1 + \theta + u^2)(r^{n-1}u)_x^2 + r^{2(n-1)}\theta_x^2\right] \leq C,
$$

(4.1)

where the $C$ is independent of $t$.

**Proof.** First notice that the set

$$
\Omega_a(t) = \{x \in \Omega : \theta(x, t) > a > 1\}
$$

is uniformly bounded in time, that is, for any $t \in [0, \infty)$

$$
\text{meas} \Omega_a(t) \leq \int_{\Omega_a(t)} \leq C(a) \int_{\Omega_a(t)} c_v(\theta - \ln \theta - 1) \leq C(a),
$$

(4.2)

by (2.1). This, together with (2.6), yields

$$
\int_{\Omega_a(t)} \theta(x, t) \leq C(a).
$$

(4.3)

The proof is broken into several steps.

**Step 1.** Multiplied by $(\theta - 2)_+$ with $(\theta - 2)_+ = \max\{0, \theta - 2\}$, it yields from (4.1) that

$$
\frac{c_v}{2} \int_\Omega (\theta - 2)_+^2(x, t) + \kappa \int_0^t \int_\Omega \frac{r^{2(n-1)}|\partial_x(\theta - 2)_+|^2}{v}
$$

$$
= \frac{c_v}{2} \int_\Omega (\theta_0 - 2)_+^2 + 2\mu(n - 1) \int_0^t \int_\Omega r^{n-2}u^2\partial_x(\theta - 2)_+
$$

$$
+ \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x^2}{v}(\theta - 2)_+ - \int_0^t \int_\Omega R \frac{\theta}{v}(r^{n-1}u)_x(\theta - 2)_+.
$$

(4.4)

If multiply (4.2) by $2u(\theta - 2)_+$, we discover

$$
2 \int_0^t \int_\Omega u_tu(\theta - 2)_+ + 2\beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x^2}{v}(\theta - 2)_+
$$

$$
= -2\beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v}r^{n-1}u(\theta - 2)_+ - 2 \int_0^t \int_\Omega R \frac{(\theta)}{x}r^{n-1}u(\theta - 2)_+.
$$

(4.5)

Putting (4.4) and (4.5) together receives

$$
\frac{c_v}{2} \int_\Omega (\theta - 2)_+^2 + \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x^2}{v}(\theta - 2)_+ + \kappa \int_0^t \int_\Omega \frac{r^{2(n-1)}|\partial_x(\theta - 2)_+|^2}{v}
$$

$$
= \frac{c_v}{2} \int_\Omega (\theta_0 - 2)_+^2 + 2\mu(n - 1) \int_0^t \int_\Omega r^{n-2}u^2\partial_x(\theta - 2)_+
$$

$$
- 2\beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v}r^{n-1}u(\theta - 2)_+ + 2 \int_0^t \int_\Omega R \frac{\theta}{v}r^{n-1}u(\theta - 2)_+
$$

$$
+ \int_0^t \int_\Omega R \frac{\theta}{v}(r^{n-1}u)_x(\theta - 2)_+ - 2 \int_0^t \int_\Omega u_tu(\theta - 2)_+ + \sum_{i=1}^5 I_i.
$$

(4.6)
We estimate \( I_i \) \((i = 1 \sim 5)\) as follows: By Cauchy-Schwarz inequality and (1.18), it has

\[
I_1 = 2\mu(n - 1) \int_0^t \int_{\Omega} r^{n-2} u^2 \partial_x (\theta - 2)_+ \leq \kappa \int_0^t \int_{\Omega} \frac{r^{2(n-1)}|\partial_x (\theta - 2)_+|^2}{v} + C \int_0^t \int_{\Omega} u^4
\]

and

\[
I_2 = -2\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x r^{n-1} u \partial_x (\theta - 2)_+}{v} \leq \kappa \int_0^t \int_{\Omega} \frac{r^{2(n-1)}|\partial_x (\theta - 2)_+|^2}{v} + C \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 u^2.
\]

The third term

\[
I_3 = 2 \int_0^t \int_{\Omega} R \frac{\theta}{v} r^{n-1} u \partial_x (\theta - 2)_+ = 2 \int_0^t \int_{\Omega} R \frac{\theta - 2}{v} r^{n-1} u \partial_x (\theta - 2)_+ + 4 \int_0^t \int_{\Omega} R \frac{r^{n-1} u}{v} \partial_x (\theta - 2)_+ = 2 \int_0^t \int_{\Omega} R \frac{\theta - 2}{v} r^{n-1} u \partial_x (\theta - 2)_+ \leq \kappa \int_0^t \int_{\Omega} \frac{r^{2(n-1)}|\partial_x (\theta - 2)_+|^2}{v} + C \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 u^2 \tag{4.7}
\]

For one hand,

\[
I_3 = -4 \int_0^t \int_{\Omega} R \left(\frac{1}{v} \right)_x r^{n-1} u (\theta - 2)_+ = 4 \int_0^t \int_{\Omega} R \frac{1-v}{v} (r^{n-1}u)_x (\theta - 2)_+ + 4 \int_0^t \int_{\Omega} R \frac{1-v}{v} r^{n-1} u \partial_x (\theta - 2)_+ \leq \beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} + C(\beta) \int_0^t \int_{\Omega} (v - 1)^2 (\theta - 2)_+^2 + C(\kappa) \int_0^t \int_{\Omega} (v - 1) u^2 (\theta - 1)_+ \tag{4.8}
\]

where the compensated term \((\theta(x,t) - 1) \geq 1\) in \(\Omega_2(t)\).

Thanks to (2.1, 3.1, 4.18), and the fact \(\theta^{-1}(x,t) \leq 1/2\) in \(\Omega_2(t)\), it satisfies

\[
\left| \int_0^t \int_{\Omega_2(t)} \frac{r^{n-2} u^2 \theta_x}{v \theta^2} \right| \leq C \int_0^t \int_{\Omega_2(t)} \left( \frac{r^{2(n-1)} \theta^2}{v \theta^2} + u^4 \right) \leq C + C \int_0^t \max_{x \in \Omega_2(t)} u^4 \tag{4.9}
\]

From (4.18), we compute

\[
I_3^2 = -4 \int_0^t \int_{\Omega} R \frac{(r^{n-1}u)_x}{v} (\theta - 2)_+ = 4 \int_0^t \int_{\Omega} c \theta_x - \kappa \left( \frac{r^{2(n-1)} \theta_x}{v} \right)_x - \beta \frac{(r^{n-1}u)_x^2}{v} + 2\mu(n - 1) (r^{n-2} u^2)_x \left( 1 - \frac{2}{\theta} \right) + 4\kappa \int_0^t \int_{\Omega} \theta_t \left( \frac{1 - 2}{\theta} \right) + 8 \int_0^T \int_{\Omega_2(t)} \left( \frac{r^{2(n-1)} \theta_x^2}{v \theta^2} + \beta \frac{(r^{n-1}u)_x^2}{v \theta^2} \right) \tag{4.10}
\]

\[
- 4\beta \int_0^t \int_{\Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v} - 16\mu(n - 1) \int_0^T \int_{\Omega_2(t)} \frac{r^{n-2} u^2 \theta_x}{\theta^2} \leq C + C \int_0^t \max_{x \in \Omega_2(t)} u^4 - 4\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v}.
\]
where in the last inequality is valid because of (2.1), (1.9), and the following two inequalities
\[
\int_0^t \int_\Omega \theta_t \left( 1 - \frac{2}{\theta} \right) + \leq \int_0^t \int_\Omega \theta_t \left( 1 - \frac{2}{\theta} \right) + \\
= \int_\Omega \left( \theta - 2 \ln \theta - 2(1 - \ln 2) \right)_+ - \int_\Omega \left( \theta_0 - 2 \ln \theta_0 - 2(1 - \ln 2) \right)_+ \\
\leq C
\]
and
\[
- \int_0^t \int_{\Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v} = \int_0^t \int_{\Omega_2(t)} (r^{n-1}u)_x^2 - \int_0^t \int_\Omega (r^{n-1}u)_x^2 \\
\leq 2 \int_0^t \int_{\Omega_2(t)} (r^{n-1}u)_x^2 - \int_0^t \int_\Omega (r^{n-1}u)_x^2 \\
\leq C - \int_0^t \int_{\Omega_2(t)} (r^{n-1}u)_x^2.
\]
Substituting (4.8) and (4.10) into (4.7) and utilizing Cauchy-Schwarz inequality guarantee that
\[
I_3 \leq C + \frac{\kappa}{8} \int_0^t \int_\Omega \frac{r^{2(n-1)}|\partial_x(\theta - 2)_+|^2}{v} - 3\beta \int_0^t \int_{\Omega_2(t)} (r^{n-1}u)_x^2 \\
+ C \int_0^t \left[ \max_{x \in \Omega} (\theta - 2)_+^2 + \max_{x \in \Omega_2(t)} (u^4 + (\theta - 1)^2) \right],
\]
where we have used the inequality \( \int_\Omega (u^2 + (v - 1)^2) \leq C \) which is valid due to (2.1) and (3.1).

Young inequality and (1.3) imply that
\[
I_4 = \int_0^t \int_\Omega R \frac{\theta}{v} (r^{n-1}u)_x (\theta - 2)_+ = \varepsilon \int_0^t \int_\Omega \theta (r^{n-1}u)_x^2 + C(\varepsilon) \int_0^t \int_{\Omega_2(t)} \theta (\theta - 2)_+^2 \\
\leq \varepsilon \int_0^t \int_\Omega \theta (r^{n-1}u)_x^2 + C(\varepsilon) \int_0^t \max_{x \in \Omega_2(t)} (\theta - 2)_+^2,
\]
with small positive constant \( \varepsilon \) will be determined later.

Integration by parts and (1.13) lead to
\[
I_5 = -2 \int_0^t \int_\Omega u_t u (\theta - 2)_+ \\
\leq \int_\Omega u_0^2 (\theta_0 - 2)_+ + \int_0^t \int_{\Omega_2(t)} u^2 \partial_x \theta \\
\leq C + \frac{\kappa}{\kappa_v} \int_0^t \int_{\Omega_2(t)} u^2 \left( \frac{r^{2(n-1)}}{v} \right)_x + c_v^{-1} \int_0^t \int_{\Omega_2(t)} u^2 \tilde{R},
\]
with
\[
\tilde{R} = \beta \left( \frac{(r^{n-1}u)_x^2}{v} - \frac{R(\theta - 2)_+}{v} (r^{n-1}u)_x - \frac{2R}{v} (r^{n-1}u)_x - 2\mu(n - 1)(r^{n-2}u)_x \right).
\]

Define
\[
\text{sgn}_u s = \begin{cases} 
1, & s > \eta, \\
\frac{s}{s/\eta}, & 0 \leq s \leq \eta, \\
0, & s \leq 0,
\end{cases}
\]
we use Lebesgue dominated convergence theorem to estimate
\[
\int_0^t \int_{\Omega_2(t)} u^2 \left( \frac{r^{2(n-1)} \partial_x}{v} \right)_x
= \lim_{\eta \to 0^+} \int_0^t \int_{\Omega} u^2 \text{sgn}_\eta (\theta - 2) \left( \frac{r^{2(n-1)} \partial_x}{v} \right)_x
\]
\[
= - \lim_{\eta \to 0^+} \int_0^t \int_{\Omega} \left[ 2u u_x \text{sgn}_\eta (\theta - 2) + u^2 \text{sgn}'_\eta (\theta - 2) \right] \frac{r^{2(n-1)} \partial_x}{v}
\]
\[
\leq - \lim_{\eta \to 0^+} \int_0^t \int_{\Omega_2(t)} 2u u_x \text{sgn}_\eta (\theta - 2) \frac{r^{2(n-1)} \partial_x}{v}
\]
\[
\leq \frac{c_v}{8} \int_0^t \int_{\Omega_2(t)} \frac{r^{2(n-1)} \partial_x^2}{v} + C \int_0^t \int_{\Omega_2(t)} r^{2(n-1)} u^2 u_x^2
\]
\[
\leq \frac{c_v}{8} \int_0^t \int_{\Omega_2(t)} \frac{r^{2(n-1)} \partial_x^2}{v} + C \int_0^t \int_{\Omega_2(t)} u^2 (r^{n-1} u)_x^2 + C \int_0^t \max_{x \in \Omega_2(t)} u^4,
\]
where the last inequality owes to (2.5), (1.18), and (3.1).

Notice from (1.16) that
\[
(r^{n-2} u^2)_x = u r^{-1} \left[ 2 (r^{n-1} u)_x - n r^{-1} v u \right].
\]

This, along with (4.2), (1.18) and (3.1), brings to
\[
\int_0^t \int_{\Omega_2(t)} u^2 (r^{n-2} u^2)_x = \int_0^t \int_{\Omega_2(t)} \left( 2 \frac{u^3 (r^{n-1} u)_x}{r} - \frac{n u^4 v}{r^2} \right)
\]
\[
\leq C \int_0^t \int_{\Omega_2(t)} u^2 (r^{n-1} u)_x^2 + \int_0^t \max_{x \in \Omega_2(t)} u^4,
\]
and therefore,
\[
\int_0^t \int_{\Omega_2(t)} u^2 \tilde{R} \leq C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 + C \int_0^t \int_{\Omega} (\theta - 2)_x^2 u^2
\]
\[
+ C \beta \int_0^t \max_{x \in \Omega_2(t)} u^4 (\cdot, t) + C \beta v \int_0^t \int_{\Omega} \left[ \max_{x \in \Omega_2(t)} u^4 (\cdot, t) + \max_{x \in \Omega_2(t)} u^4 \right] + \beta \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x^2}{v}.
\]
Inequalities (4.12) and (4.13) guarantee that (4.11) satisfies
\[
I_5 \leq C + \frac{\kappa}{8} \int_0^t \int_{\Omega} r^{2(n-1)} |\partial_x (\theta - 2)_x |^2 + C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2
\]
\[
+ C \int_0^t \int_{\Omega} \left[ \max_{x \in \Omega} (\theta - 2)_x^2 + \max_{x \in \Omega_2(t)} u^4 \right] + \beta \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x^2}{v}.
\]

On account of the estimates for $I_i$ ($i = 1 \sim 5$) above, it follows from (3.1) and (4.6) that
\[
\int_{\Omega} (\theta - 2)_x^2 + \int_0^t \int_{\Omega} \left[ (r^{n-1} u)_x^2 + (\theta - 2)_x^2 + r^{2(n-1)} |\partial_x (\theta - 2)_x |^2 \right]
\]
\[
\leq C + C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 + \varepsilon \int_0^t \int_{\Omega} \theta (r^{n-1} u)_x^2
\]
\[
+ C \int_0^t \int_{\Omega} \left[ \max_{x \in \Omega} (\theta - 2)_x^2 + \max_{x \in \Omega_2(t)} u^4 + \max_{x \in \Omega_2(t)} (\theta - 1)_x^2 \right].
\]
we obtain

By (2.1) and (1.18), one has

Noting from (2.1) and (2.5) that

In view of (2.1), (3.1), and (4.2), we deduce

and that

Multiply (1.13) by \( u^3 \) and use

we select \( \varepsilon \) in (4.14) so small such that

\[
\int_0^t \int_\Omega (r^{n-1} u)_x^2 = \int_0^t \left( \int_{\Omega \setminus \Omega_2(t)} + \int_{\Omega \setminus \Omega_2(t)} \right) (r^{n-1} u)_x^2 \\
\leq C \int_0^t \int_{\Omega \setminus \Omega_2(t)} \left( \theta - 2 \right)_+ (r^{n-1} u)_x^2 + C \int_0^t \int_{\Omega \setminus \Omega_2(t)} (r^{n-1} u)_x^2 \\
\leq C \int_0^t \int_{\Omega \setminus \Omega_2(t)} \left( \theta - 2 \right)_+ (r^{n-1} u)_x^2 + C,
\]

and that

we obtain

\[
\begin{align*}
\frac{1}{4} \int_\Omega u^4(x,t) + & 3 \beta \int_0^t \int_\Omega u^2 (r^{n-1} u)^2_x \\
= & \int_\Omega u_0^4 + 2 \beta (n-1) \int_0^t \int_\Omega \frac{(r^{n-1} u)_x u^3}{r} - R \int_0^t \int_\Omega \left( \theta \right)_+ r^{n-1} u^3.
\end{align*}
\]

Step 2. Multiply (1.13) by \( u^3 \) and use

\[
(r^{n-1} u^3)_x = 3 u^2 (r^{n-1} u)_x - \frac{2(n-1)}{4} uu^3,
\]

we obtain

\[
\begin{align*}
1 & \int_\Omega (r^{n-1} u)_x^2 \leq \int_0^t \int_\Omega \left( \max \left( \theta - 2 \right)_+ + \max u^4 + \max \left( \theta - 1 \right)_+ \right) .
\end{align*}
\]

(4.15)

(4.16)

By (2.1) and (1.18), one has

\[
\begin{align*}
2 \beta (n-1) \int_0^t \int_\Omega \frac{(r^{n-1} u)_x u^3}{r} \leq & \varepsilon \int_0^t \int_\Omega (r^{n-1} u)_x^2 + C(\varepsilon) \int_0^t \max u^4(\cdot,t).
\end{align*}
\]

(4.17)

(4.18)

In view of (2.1), (3.1), and (1.2), we deduce

\[
\begin{align*}
\varepsilon \int_0^t \int_{\Omega \setminus \Omega_2(t)} [(r^{n-1} u)_x^2 + 2 (\theta - 1)^2 u^4] + C \int_0^t \int_{\Omega \setminus \Omega_2(t)} [(\theta - 1)^2 u^4 + v u^2] \\
& + \varepsilon \int_0^t \int_{\Omega_2(t)} [u^2 (r^{n-1} u)_x^2 + (\theta - 1)^2] + C \int_0^t \int_{\Omega_2(t)} [(\theta - 1)^2 u^2 + u^6] \\
& \leq \varepsilon \int_0^t \int_{\Omega_2(t)} (1 + u^2)(r^{n-1} u)_x^2 + C \int_0^t \left[ \max \left( \theta - 1 \right)_+ \right] + C.
\end{align*}
\]

(4.19)
Similar argument, combining with (4.3), runs
\[
\int_0^t \int_\Omega \left[ 3u^2 (r^{n-1} u)_x \frac{1-v}{v} - 2(n-1) \frac{1-v}{r} u^3 \right]
\leq \varepsilon \int_0^t \int_\Omega (r^{n-1} u)_x^2 + C \int_0^t \int_\Omega (1-v)^2 u^4 + C \int_0^t \left( \int_{\Omega \setminus \Omega_1(t)} + \int_{\Omega_1(t)} \right) \frac{|1-v|u^3}{r}
\leq \varepsilon \int_0^t \int_\Omega (r^{n-1} u)_x^2 + C \int_0^t \int_\Omega (1-v)^2 u^4 + C \int_0^t \int_\Omega (1-v)^2 u^4 + C \int_0^t \int_\Omega \frac{v|u|^2}{r^2}
\leq \varepsilon \int_0^t \int_\Omega (r^{n-1} u)_x^2 + C \int_0^t \max u^4 + C.
\]

With the help of (4.19) and (4.20), we use (4.16) to estimate
\[
-R \int_0^t \int_\Omega \left( \frac{\theta}{v} \right)_x (r^{n-1} u^3)
= R \int_0^t \int_\Omega \left( \frac{\theta - 1}{v} + \frac{1-v}{v} \right) (r^{n-1} u^3)_x
= R \int_0^t \int_\Omega \left[ 3u^2 (r^{n-1} u)_x \frac{\theta - 1}{v} - 2(n-1) \frac{\theta - 1}{r} u^3 \right]
+ R \int_0^t \int_\Omega \left[ 3u^2 (r^{n-1} u)_x \frac{1-v}{v} - 2(n-1) \frac{1-v}{r} u^3 \right]
\leq \varepsilon \int_0^t \int_\Omega \left( 1+u^2 \right) (r^{n-1} u)_x^2 + C \int_0^t \left[ \max (\theta - 1)^2 + \max x \in \Omega \right] + C.
\]

Inequalities (4.18) and (4.21) ensure that (4.17) satisfies
\[
\int_\Omega u^4 + \int_0^t \int_\Omega u^2 (r^{n-1} u)_x^2
\leq C + C \varepsilon \int_0^t \int_\Omega (r^{n-1} u)_x^2 + C \int_0^t \left( \max u^4 + \max x \in \Omega \right) (\theta - 1)^2 + C \int_0^t \max x \in \Omega \left( [\theta - 3/2]^2 + u^4 \right).
\]

Multiplying (4.22) by a large constant, adding the resulting expression up to (4.15), choosing \( \varepsilon \) sufficiently small, we conclude
\[
\int \left[ (\theta - 2)_+^2 + u^4 \right] + \int_0^t \int_\Omega \left[ (\theta + u^2)(r^{n-1} u)_x^2 + r^2(n-1)\theta_x^2 \right]
\leq C + C \int_0^t \left[ \max (\theta - 2)_+^2 + \max x \in \Omega \right] (\theta - 1)^2 + \max u^4
\leq C + C \int_0^t \max x \in \Omega \left[ [\theta - 3/2]^2 + u^4 \right].
\]

**Step 3.** It remains to estimate the terms on the right hand side of (4.23). Utilizing (4.3) and (4.18) we compute
\[
(\theta(x,t) - 3/2)_+^2 = -2 \int_x^\infty \theta(x,t) \partial_y \theta(x,t) + \theta(x,t) \partial_y \theta(x,t)
\leq C \int_{\Omega_{3/2}(t)} (\theta - 3/2)_+ \left[ \theta_x \right]
\leq C \int_{\Omega_{3/2}} \frac{\theta_x^2}{\theta} + \int \frac{\theta_x^2}{\theta} \left[ \theta - 3/2 \right] + C \int_{\Omega_{3/2}} \max (\theta(t), \theta(t) - 3/2)_+.
\]
which satisfies if $\delta_1$ is chosen small

$$\max_{x \in \Omega} (\theta'(-3/2)_+^2) \leq \frac{\sqrt{\delta_1}}{1 - C \sqrt{\delta_1}} \int_{\Omega} r^{2(n-1)} \theta_x^2 + \frac{C}{\delta_1^{3/2} (1 - C \sqrt{\delta_1})} \int_{\Omega} r^{2(n-1)} \theta_x^2.$$

This combine with (2.1) lead to

$$\int_0^t \max_{x \in \Omega} (\theta - 3/2)_+^2 \leq \sqrt{\delta_1} \int_0^t \int_{\Omega} r^{2(n-1)} \theta_x^2 + C(\delta_1). \quad (4.24)$$

Next, by (3.1), (1.18) and (4.3), one has

$$u^4(x,t) = 4 \int_0^x u^3 u_y = \frac{C}{\delta_2} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + \delta_2 \int_{\Omega} \frac{u^6 \theta}{r^{2(n-1)}}$$

$$\leq \frac{C}{\delta_2} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + \delta_2 \int_{\Omega \backslash \Omega(t)} \frac{u^6 \theta}{r^{2(n-1)}}$$

$$\leq \frac{C}{\delta_2} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + C \delta_2 \max_{x \in \Omega} (\theta - 1) \leq C \delta_2 \max_{x \in \Omega} u^6(\cdot, t).$$

Thus, for small $\delta_2$ one has

$$\max_{x \in \Omega} u^6(\cdot, t) \leq \frac{C}{\delta_2 (1 - C \delta_2)} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + \frac{C \delta_2}{(1 - C \delta_2)} \max_{x \in \Omega} u^6(\cdot, t). \quad (4.25)$$

On the other hand,

$$\max_{x \in \Omega} u^6(\cdot, t) = 6 \int_0^x u^5 u_y = 6 \int_0^x u^5 (r^{n-1} u_x - (n-1) r^{-1} u u_x)$$

$$\leq 6 \int_0^x \frac{u^5 (r^{n-1} u_x)}{r^{n-1}}$$

$$\leq \frac{C}{\sqrt{\delta_2}} \int_{\Omega} u^2 (r^{n-1} u_x)^2 + \sqrt{\delta_2} \int_{\Omega} u^6$$

$$\leq \frac{C}{\sqrt{\delta_2}} \int_{\Omega} u^2 (r^{n-1} u_x)^2 + C \sqrt{\delta_2} \max_{x \in \Omega} u^6(\cdot, t).$$

Insert it back into (4.25) and utilize (2.1) give that

$$\int_0^t \max_{x \in \Omega} u^4(\cdot, t) \leq C(\delta_2) + C \delta_2 \int_0^t \int_{\Omega} u^2 (r^{n-1} u_x)^2. \quad (4.26)$$

In terms of (4.20) and (4.24), choosing $\delta_1$ and $\delta_2$ so small such that (4.23) satisfies

$$\int_{\Omega} \left[ (\theta - 2)_+^2 + u^4 \right] + \int_0^t \int_{\Omega} \left[ (1 + \theta + u^2)(r^{n-1} u_x)^2 + r^{2(n-1)} \theta_x^2 \right] \leq C,$$

where the $C$ does not rely on $t$. By this and (2.1) implies

$$\int_{\Omega} (\theta - 1)^2 = \left( \int_{\Omega \backslash \Omega_3(t)} + \int_{\Omega_3(t)} \right) (\theta - 1)^2 \leq C + C \int_{\Omega_3(t)} (\theta - 2)_+^2 \leq C.$$

The last two inequalities give birth to (4.1), the required. \[\Box\]

**Lemma 4.2** Let $(v, u, \theta)$ be the solution described in Theorem 1.1. There is some $C$ independent of $t$ such that

$$\int_{\Omega} v_x^2(x,t) + \int_0^t \int_{\Omega} (1 + \theta) v_x^2 \leq C. \quad (4.27)$$
Proof. By (1.13)\textsubscript{1}, rewriting (1.13)\textsubscript{2} as the form
\[ \beta \left( \frac{v_x}{v} \right)_t = R \left( \frac{\theta}{v} \right)_x + r^{1-n} u_t, \]
which yields after multiplied by \( v_x/v \),
\[
\frac{\beta}{2} \int_\Omega \frac{v_x^2}{v^2} (x, t) + R \int_0^t \int_\Omega \frac{\theta v_x}{v^3} = \frac{\beta}{2} \int_\Omega \frac{v_x^2}{v^2} (x, 0) + R \int_0^t \int_\Omega \frac{v_x \theta_x}{v^2} + \int_0^t \int_\Omega \frac{r^{1-n} u_t v_x}{v}. \tag{4.28}
\]
Cauchy-Schwarz inequality, (2.11), (1.18), and (3.1) guarantee that
\[
R \int_0^t \int_\Omega \frac{v_x \theta_x}{v^2} \leq \frac{R}{2} \int_0^t \int_\Omega \frac{\theta v_x^2}{v^3} + C \int_0^t \int_\Omega \frac{\theta^2}{v^2} \leq \frac{R}{2} \int_0^t \int_\Omega \frac{\theta v_x^2}{v^3} + C \int_0^t \int_\Omega \frac{r^{2(n-1)} \theta_x^2}{v^3} + C \int_0^t \int_\Omega \frac{r^{2(n-1)} \theta_x^2}{v^3} \leq \frac{R}{2} \int_0^t \int_\Omega \frac{\theta v_x^2}{v^3} + C.
\]
Thanks to (1.16), (1.19), and (2.1), it gives
\[
\int_0^t \int_\Omega \frac{r^{1-n} u_t v_x}{v} = \int_\Omega r^{1-n} u \left( \frac{v_x}{v} \right)_t - \int_\Omega r^{1-n} u \left( \frac{v_x}{v} \right) + (n-1) \int_0^t \int_\Omega r^{-n} u^2 \frac{v_x}{v} \leq C + \frac{\beta}{4} \int_\Omega \frac{v_x^2}{v} - \int_0^t \int_\Omega \frac{r^{1-n} u \left( \frac{v_x}{v} \right)_t}{v^2} + (n-1) \int_0^t \int_\Omega r^{-n} u^2 \frac{v_x}{v}. \tag{4.29}
\]
Therefore, (4.28) satisfies
\[
\frac{\beta}{4} \int_\Omega \frac{v_x^2}{v} + \frac{R}{2} \int_0^t \int_\Omega \frac{\theta v_x^2}{v^3} \leq C - \int_0^t \int_\Omega r^{1-n} u \left( \frac{v_x}{v} \right)_t + (n-1) \int_0^t \int_\Omega r^{-n} u^2 \frac{v_x}{v}. \tag{4.29}
\]
Making use of (1.13)\textsubscript{1}, (1.18), (4.1), and (3.1), and
\[
(r^{1-n} u)_x = r^{2(1-n)}(r^{1-n} u)_x + 2(1-n)r^{1-2n} u,
\]
we estimate
\[
- \int_0^t \int_\Omega r^{1-n} u \left( \frac{v_x}{v} \right)_t = \int_0^t \int_\Omega \frac{(r^{1-n} u)_x}{v} \left( \frac{r^{1-n} u}{v} \right) = \int_0^t \int_\Omega r^{2(1-n)} \left( \frac{r^{1-n} u}{v} \right)_x^2 + 2(1-n) \int_0^t \int_\Omega r^{-1-n} u (r^{1-n} u)_x \leq C \int_0^t \int_\Omega (1 + \theta) (r^{1-n} u)_x^2 + C \int_0^t \int_\Omega \frac{nu^2}{r^{2(1-n)}} \leq C. \tag{4.30}
\]
Remember that (3.1) and (3.10), one has
\[
\int_0^t \int_\Omega \frac{\theta v_x^2}{v^3} \leq C(\alpha_1, \alpha_2) \left( \int_0^t f(t) \int_\Omega \frac{v_x^2}{v^3} + \int_0^t \int_\Omega \frac{v_x^2}{v^3} \right), \tag{4.31}
\]
where \( f(t) \) is taken from (3.11). By this, (3.10), and Cauchy-Schwarz inequality, we have
\[
\int_0^t \int_\Omega r^{-n} u^2 \frac{v_x}{v} \leq C \int_0^t \max u^4 \int_\Omega r^{-2n} + \frac{R}{4} \left( \int_0^t f(t) \int_\Omega \frac{v_x^2}{v^2} + \int_0^t \int_\Omega \frac{\theta v_x^2}{v^2} \right) \leq C + \frac{R}{4} \left( \int_0^t f(t) \int_\Omega \frac{v_x^2}{v^2} + \int_0^t \int_\Omega \frac{\theta v_x^2}{v^2} \right), \tag{4.32}
\]
where the last inequality comes from (3.10), (1.20) and (4.1). Substituting (4.32) and (4.30) into (4.29) arrives at
\[ \int_{\Omega} |v_{x}|^2 + \int_{0}^{t} \int_{\Omega} \frac{\theta v_{x}^2}{v^3} \leq C + C \int_{0}^{t} f(t) \int_{\Omega} \frac{v_{x}^2}{v^2}. \]
Gronwall inequality concludes that
\[ \int_{\Omega} v_{x}^2 + \int_{0}^{t} \int_{\Omega} \theta v_{x}^2 \leq C, \]
which together with (2.1) and (3.1) deduce from (4.31) that
\[ \int_{0}^{t} \int_{\Omega} v_{x}^2 \leq C. \]
The proof is complete. ☐

**Lemma 4.3** It holds that
\[ \int_{\Omega} u_{x}^2(x, t) + \int_{0}^{t} \int_{\Omega} r^{2(n-1)} u_{xx}^2 \leq C \left( 1 + \max_{\Omega \times [0, t]} \theta \right), \tag{4.33} \]
where the C is independent of t.

*Proof.* Multiplying (1.13) by \(-u_{xx}\), a straight calculation shows
\[
\begin{align*}
\frac{1}{2} \partial_{t} u_{x}^2 + \beta r^{2(n-1)} u_{xx}^2 & = (u_{x} u_{t})_{x} + \beta u_{xx} \left( r^{2(n-1)} \frac{v_{x} u_{x}}{v^2} + (n-1) \frac{u v}{r^2} - 2(n-1) r^{-2} u_{x} \right) \\
& + R u_{xx} r^{-1} \left( \frac{\theta x}{v} - \frac{\theta v_{x}}{v^2} \right).
\end{align*}
\]
Applying Cauchy-Schwarz inequality, we integrate it to find
\[
\begin{align*}
\frac{1}{2} \int_{\Omega} u_{x}^2(x, t) + \beta \int_{0}^{t} \int_{\Omega} \frac{r^{2(n-1)} u_{xx}^2}{v} & \leq \frac{1}{2} \int_{\Omega} u_{0x}^2 + \beta \int_{0}^{t} \int_{\Omega} \frac{r^{2(n-1)} u_{xx}^2}{v} \\
& + C \int_{0}^{t} \int_{\Omega} \left[ r^{2(n-1)} u_{x}^2 + \frac{u^2}{r^{2(n+1)}} + \frac{u_{x}^2}{r^2} + \theta_x^2 + \theta_x^2 v_{x}^2 \right]. \tag{4.34}
\end{align*}
\]
It follows from (3.1), (1.1), (1.26), (1.18) and (2.1) that
\[
\begin{align*}
C \int_{0}^{t} \int_{\Omega} \left( \frac{u^2}{r^{2(n+1)}} + \frac{u_{x}^2}{r^2} + \theta_x^2 + \theta_x^2 v_{x}^2 \right) & \leq C \left( 1 + \max_{\Omega \times [0, t]} \theta \right) \int_{0}^{t} \int_{\Omega} \left( \frac{u^2}{r^{2} \theta} + \frac{r^{2(n-1)} u_{x}^2}{v \theta} + \frac{r^{2(n-1)} \theta_x^2 + \theta_x^2 v_{x}^2}{r^2} \right) \\
& \leq C \left( 1 + \max_{\Omega \times [0, t]} \theta \right).
\end{align*}
\]
Since \(H^1 \hookrightarrow L^{\infty}\), we use (1.27) and (2.1) to get
\[
\begin{align*}
C \int_{0}^{t} \int_{\Omega} r^{2(n-1)} u_{x}^2 u_{xx}^2 & \leq C \int_{0}^{t} \left\| r^{n-1} u_{x} \right\|^2_{L^{\infty}} \int_{\Omega} v_{x}^2 \\
& \leq \frac{\beta}{4} \int_{0}^{t} \int_{\Omega} r^{2(n-1)} u_{xx}^2 + C \int_{0}^{t} \int_{\Omega} r^{2(n-1)} u_{x}^2 \tag{4.35} \\
& \leq \frac{\beta}{4} \int_{0}^{t} \int_{\Omega} r^{2(n-1)} u_{xx}^2 + C \max_{\Omega \times [0, t]} \theta.
\end{align*}
\]
With the last two inequalities in hand, we conclude the desired (4.33) from (4.34). ☐
Lemma 4.4 It holds that

\[
\int_{\Omega} \theta_x^2(x,t) + \int_0^t \int_{\Omega} r^{2(n-1)} \theta_{xx}^2 \leq C \left( 1 + \max_{\Omega \times [0,t]} \theta^2 \right), \tag{4.36}
\]

where the C is independent of t.

Proof. Multiplying (1.13) by \(-\theta_{xx}\) brings to

\[
c_v \frac{\partial}{\partial t} \theta_x^2 + \kappa \frac{r^{2(n-1)} \theta_{xx}^2}{v}
= (c_v \theta_x \theta_t) + \kappa \theta_{xx} \left( r^{2(n-1)} \frac{\nu \theta_x}{v^2} - 2(n-1) r^{n-2} \theta_x \right)
+ \theta_{xx} \left( R \frac{\theta}{v} (r^{n-1} u)_x + 2 \mu (n-1) (r^{n-2} u^2)_x - \beta (r^{n-1} u)^2_x \right).
\]

By Cauchy-Schwarz inequality, (3.1), (2.5), as well as

\[
\text{Lemma 4.4}
\]

we see that

\[
(r^{n-2} u^2)_x = 2r^{-1} u (r^{n-1} u)_x - n r^{n-2} u^2_v,
\]

we see that

\[
\frac{c_v}{2} \int_{\Omega} \theta_x^2 + \kappa \int_{0}^{t} \int_{\Omega} r^{2(n-1)} \theta_{xx}^2 \frac{v}{v}
\leq \frac{c_v}{2} \int_{0}^{t} \int_{\Omega} \theta_x^2 + \kappa \frac{r^{2(n-1)} \theta_{xx}^2}{v} + C \int_{0}^{t} \int_{\Omega} r^{2(n-1)} \nu \theta_x^2
+ C \int_{0}^{t} \int_{\Omega} \left[ \theta_x^2 + (\theta^2 + r^{-2} u^2) (r^{n-1} u)_x^2 + r^{2(n-1)} u^4 + r^{2(n-1)} u^4_x \right].
\]

Owing to (4.27) and (4.1), a similar argument as (4.35) shows

\[
C \int_{0}^{t} \int_{\Omega} r^{2(n-1)} \nu \theta_x^2 \leq \kappa \frac{r^{2(n-1)} \theta_x^2}{4} + C \int_{0}^{t} \int_{\Omega} r^{2(n-1)} \theta_x^2
\leq \kappa \frac{r^{2(n-1)} \theta_x^2}{4} + C.
\]

In terms of (1.18), (3.16), (4.26), and (4.1), it satisfies

\[
\int_{0}^{t} \int_{\Omega} \left( \theta_x^2 + (\theta^2 + r^{-2} u^2) (r^{n-1} u)_x^2 + r^{2(n-1)} u^4 \right)
\leq C \left( 1 + \max_{\Omega \times [0,t]} \theta \right) \int_{0}^{t} \int_{\Omega} \left[ r^{2(n-1)} \theta_x^2 + (\theta + u^2) (r^{n-1} u)_x^2 \right] + C \int_{0}^{t} \max_{\Omega \times [0,t]} u^4 \int_{\Omega} r^{2(n-1)}
\leq C \left( 1 + \max_{\Omega \times [0,t]} \theta \right).
\]

Finally, from (4.33) and (4.35) we obtain

\[
\int_{0}^{t} \int_{\Omega} r^{2(n-1)} u^4_x \leq C \int_{0}^{t} \| r^{n-1} u_x \|_{L^\infty}^2 \int_{\Omega} u^2_x
\leq C \max_{\Omega \times [0,t]} \theta \int_{0}^{t} \| r^{n-1} u_x \|_{L^\infty}^2 \leq C \left( 1 + \max_{\Omega \times [0,t]} \theta^2 \right).
\]

Insert the last three inequalities guarantee that (4.36) receives the (4.36). □

Corollary 4.5 (Bound of \( u \) and \( \theta \)) There exists some constant C such that

\[
|u(x,t)| + \theta(x,t) \leq C, \quad (x,t) \in \Omega \times [0, \infty). \tag{4.39}
\]
Proof. In view of (4.1) and (4.36), we use Sobolev inequality to get
\[ \| \theta - 1 \|_{L^\infty(\Omega)} \leq C \| \theta_1 - 1 \|_{L^2(\Omega)} \| \theta_x \|_{L^2(\Omega)} \leq C \left( 1 + \max_{\Omega \times [0,t]} \theta \right), \]
which means for some \( C \) independent of \( t \), such that
\[ \theta(x,t) \leq C, \quad (x,t) \in \overline{\Omega} \times [0,t]. \tag{4.40} \]
Once (4.40) is obtained, it follows from (2.1) and (4.33) that
\[ |u(x,t)| \leq C, \quad (x,t) \in \Omega \times [0,t]. \]
The proof is done. \( \Box \)

5 Estimates for derivatives

The lemmas in this section concern the derivatives estimates, which are needed to show the large-time behavior of solutions.

Lemma 5.1 Let \((v,u,\theta)\) be the solution obtained in Theorem 1.1. Then there is some \( C \) independent of \( t \), such that
\[ \int_\Omega r^{2(n-1)}v_x^2(x,t) + \int_0^t \int_\Omega (1 + \theta)r^{2(n-1)}v_x^2 \leq C. \tag{5.1} \]

Proof. Utilizing (1.16) and (1.13), the (1.13) takes the form
\[ \beta \left( r^{n-1}v_x \right)_t + Rr^{n-1}v_x = Rr^{n-1}v_x + u_t - \beta (n-1)r^{n-1}v_x. \]

Multiplied by \( r^{n-1}v_x/v \), it gives
\[ \frac{\beta}{2} \int_\Omega \left| r^{n-1}v_x \right|^2 + R \int_0^t \int_\Omega r^{2(n-1)}v_x^2 \]
\[ \leq R \int_0^t \int_\Omega r^{2(n-1)}\frac{v_x^2}{v} + C \int_0^t \int_\Omega r^{2(n-1)}\frac{\theta_x^2}{v} \]
\[ + \int_0^t \int_\Omega u r^{n-1}v_x - \beta (n-1) \int_0^t \int_\Omega r^{2(n-1)}v_x^2. \tag{5.2} \]

It yields from (1.40), (2.1), (4.1) and (4.39) that
\[ \int_0^t \int_\Omega u r^{n-1}v_x \]
\[ = \int_\Omega ur^{n-1}\frac{v_x}{v}(x,t) - \int_\Omega ur^{n-1}\frac{v_x}{v}(x,0) \]
\[ - (n-1) \int_0^t \int_\Omega u^2 r^{n-2}v_x \]
\[ \leq C + \frac{\beta}{4} \int_\Omega \left| r^{n-1}v_x \right|^2 + \frac{R}{8} \int_0^t \int_\Omega r^{2(n-1)}v_x^2. \tag{5.3} \]

By (1.40), (3.1) and (4.39), Sobolev inequality gives
\[ \| r^{-1}u \|^2_{L^\infty} \leq C \left( \| ur^{-1} \|^2_{L^\infty(\Omega)} + \| (ur^{-1})_x \|^2_{L^2(\Omega)} \right) \]
\[ \leq C \int_\Omega \frac{vu}{r^2\theta} + C \frac{r^{2(n-1)}u^2}{v\theta} =: g(t). \]
This combine with Cauchy-Schwarz inequality conclude that

\[- \beta (n-1) \int_0^t \int_\Omega r^{2(n-1)}v_x^2 \leq \varepsilon \int_0^t \int_\Omega r^{2(n-1)}v_x^2 + C(\varepsilon) \int_0^t \| r^{-1}u \|_{L^\infty}^2 \int_\Omega r^{2(n-1)}v_x^2 \]

\[
\leq \varepsilon C(\alpha_1, \alpha_2) \int_0^t \int_\Omega r^{2(n-1)}v_x^2 + C(\varepsilon) \int_0^t \int_\Omega \| f(t) + g(t) \| \int_\Omega r^{2(n-1)}v_x^2.
\]

where in the last inequality we have used (4.31).

If we choose \( \varepsilon \) sufficiently small such that \( \varepsilon C(\alpha_1, \alpha_2) \leq R/8 \), substitute (5.3) and (5.4) into (5.2), use (1.11), (3.1), (4.27) and (4.39) and Gronwall inequality, it provides that

\[
\int_\Omega r^{2(n-1)}v_x^2 + \int_0^t \int_\Omega r^{2(n-1)}\theta v_x^2 \leq C.
\]

This, again with (4.31), yields the desired (5.1). \( \Box \)

**Lemma 5.2** Let \((v, u, \theta)\) be the solution obtained in Theorem 1.7. It holds that

\[
\int_\Omega (r^{n-1}u_x^2(x,t)) + \int_0^t \int_\Omega u_t^2 \leq C,
\]

where the \( C \) is independent of \( t \).

**Proof.** Multiplying (1.13) by \( u_t \) gives rise to

\[
\int_0^t \int_\Omega u_t^2 = - \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x + R \int_0^t \int_\Omega \frac{\theta v_x}{v^2} \frac{\theta_x}{v} r^{n-1}u_t
\]

\[
\leq - \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x + \frac{1}{2} \int_0^t \int_\Omega u_t^2 + C \int_0^t \int_\Omega r^{2(n-1)}(\theta_x^2 + \theta^2 v_x^2)
\]

\[
\leq - \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x + \frac{1}{2} \int_0^t \int_\Omega u_t^2 + C,
\]

where in the last inequality we have used (4.11), (4.39) and (5.1).

In terms of (2.1), (1.15), (1.37), (3.1), (4.27) and (4.39), a straight computation shows

\[
- \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x
\]

\[
- \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} [(r^{n-1}u)_{xt} - (n-1)(r^{n-2}u^2)_x]
\]

\[
\beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} (x,0) - \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)^2_x}{v}(x,t)
\]

\[
- \frac{\beta}{2} \int_0^t \int_\Omega \frac{(r^{n-1}u)_x^2}{v^2} + \beta(n-1) \int_0^t \int_\Omega \frac{(r^{n-1}u)_x}{v} (r^{n-2}u^2)_x
\]

\[
\leq C - \frac{\beta}{2} \int_0^t \int_\Omega \frac{(r^{n-1}u)^2_x}{v}(x,t) + C \int_0^t \int_\Omega (r^{n-1}u)_x^2.
\]

By this and (3.1), we conclude from (5.6) that

\[
\int_0^t \int_\Omega u_t^2 + \int_\Omega (r^{n-1}u)_x^2 \leq C + C \int_0^t \| (r^{n-1}u)_x \|_{L^\infty}^2 \int_\Omega (r^{n-1}u)_x^2.
\](5.7)
Observe from (1.18), (2.3), (1.21), (4.33), (2.1) and (4.39) that
\[ \int_0^t \|(r^{n-1}u)_x\|_{L^2(\Omega)}^2 \leq C \int_0^t \left( \|(r^{n-1}u)_{xx}\|_{L^2(\Omega)} + \|(r^{n-1}u)_x\|_{L^2(\Omega)}^2 \right) \]
\[ \leq C \int_0^t \int_{\Omega} \left( r^{2(n-1)}u_{xx}^2 + v_x^2 + \frac{r^{2(n-1)}u_x^2}{v\theta} + \frac{vu_x^2}{r^2\theta} \right) \]
\[ \leq C, \] (5.8)
we apply Gronwall inequality to (5.7) to receive the (5.5).

Lemma 5.3 Let \((v, u, \theta)\) be the solution obtained in Theorem 1.1. Then it holds that
\[ \int_\Omega r^{2(n-1)}\theta^2(x, t) + \int_0^t \int_\Omega \theta^2 \leq C, \]
(5.9)
where the \(C\) is independent of \(t\).

Proof. Multiplied by \(\theta_t\), it gives from (1.13) that
\[ c_v \int_0^t \int_{\Omega} \theta_t^2 + \frac{\kappa}{2} \int_\Omega \frac{r^{2(n-1)}\theta_x^2}{v} \]
\[ = \frac{\kappa}{2} \int_\Omega \frac{r^{2(n-1)}\theta_x^2}{v}(x, 0) + \frac{\kappa}{2} \int_0^t \int_{\Omega} \left( 2(n-1)\frac{r^{2(n-1)}u_x\theta_x}{v} - \frac{r^{2(n-1)}\theta_x^2}{v^2} \right) \]
\[ + \beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x\theta_t}{v} - R \int_0^t \int_{\Omega} \frac{\theta}{v}(r^{n-1}u)_x\theta_t - 2(n-1)\mu \int_0^t \int_{\Omega} \frac{(r^{n-2}u_x^2)}{x}\theta_t, \]
which implies from Cauchy-Schwarz inequality, (3.1), (1.1), (4.39), (1.18), (1.13), (2.1), (4.37), and (5.5) that
\[ \int_0^t \int_{\Omega} \theta_t^2 + \int_\Omega r^{2(n-1)}\theta_x^2 \leq C + C \int_0^t \|(r^{n-1}u)_x\|_{L^\infty(\Omega)}^2 \left( 1 + \int_\Omega r^{2(n-1)}\theta_x^2 \right). \]

Using Gronwall inequality and (5.8) to the above inequality completes the proof. □

6 Proof of Theorem 1.2

This final section is devoted to proving Theorem 1.2.

Firstly, the (1.22) follows from inequality (2.1), Lemma 3.1, Lemma 4.1, Lemmas 5.1-5.3 and equations (1.13).

Next to prove (1.24), for this we verify
\[ \int_0^\infty \left| \frac{d}{dt} \|v_x\|^2_{L^2(\Omega)} \right| + \left| \frac{d}{dt} \|u_x\|^2_{L^2(\Omega)} \right| + \left| \frac{d}{dt} \|\theta_x\|^2_{L^2(\Omega)} \right| dt \leq C. \]
(6.1)

In fact, by (1.18), Lemmas 4.3-4.4, Lemmas 5.2-5.3, we compute
\[ \int_0^\infty \left| \frac{d}{dt} \|v_x\|^2_{L^2(\Omega)} \right| + \left| \frac{d}{dt} \|\theta_x\|^2_{L^2(\Omega)} \right| dt \]
\[ = 2 \int_0^\infty \left( \int_{\Omega} u_x u_{xx} + \int_{\Omega} u_x u_{xt} \right) dt \]
\[ \leq \int_0^\infty \left( \|u_x\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Omega)} + \|\theta_x\|^2_{L^2(\Omega)} + \|\theta_t\|^2_{L^2(\Omega)} \right) dt \]
\[ \leq C. \]
(6.2)
By (1.13), (4.27) and (5.8), one has
\[
\int_0^\infty \left| \frac{d}{dt} \|v_x\|_{L^2(\Omega)}^2 \right| dt = 2 \int_0^\infty \left| \int_{\Omega} v_x (r^{n-1} u) \right| dt \\
\leq C \int_0^\infty \left( \|v_x\|_{L^2(\Omega)}^2 + \|(r^{n-1} u)_{xx}\|_{L^2(\Omega)}^2 \right) dt
\]
which together with (6.1) conclude that
\[
\lim_{t \to \infty} \|(v_x, u_x, \theta_x)(\cdot, t)\|_{L^2(\Omega)} = 0.
\]
Combination of (6.2) with (6.3) generates (6.1). On the other hand, it satisfies from (4.27), (4.1), (2.1), (3.1), (4.39) and (1.18) that
\[
\int_0^\infty \left( \|v_x\|_{L^2(\Omega)}^2 + \|u_x\|_{L^2(\Omega)}^2 + \|\theta_x\|_{L^2(\Omega)}^2 \right) dt \leq C,
\]
which together with (6.1) conclude that
\[
\lim_{t \to \infty} \|(v_x, u_x, \theta_x)(\cdot, t)\|_{L^2(\Omega)} = 0.
\]
Therefore, the desired (1.24) is a direct consequence of (6.4), (2.1), (3.1) and (4.1).

It is only left to check (1.23). Thanks to (5.1) and (4.39), it suffice to show \(\theta(x, t)\) has a positive bound from below. For one hand, (1.24) implies there is some large time point \(T_1\) such that
\[
\theta(x, t) \geq 1/2, \quad \forall \ (x, t) \in \overline{\Omega} \times [T_1, \infty).
\]
For another hand, it satisfies from [6, eq.(4.9)] that
\[
\theta(x, t) \geq C(T_1), \quad (x, t) \in \overline{\Omega} \times [0, T_1].
\]
Combination of (6.5) with (6.6) proves the (1.23), and thus, the Theorem 1.2 is completed. \(\square\)

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