Affine Toda Solitons and Vertex Operators

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Abstract

Affine Toda theories with imaginary couplings associate with any simple Lie algebra \(\mathfrak{g}\) generalisations of Sine Gordon theory which are likewise integrable and possess soliton solutions. The solitons are "created" by exponentials of quantities \(\hat{F}^i(z)\) which lie in the untwisted affine Kac-Moody algebra \(\hat{\mathfrak{g}}\) and ad-diagonalise the principal Heisenberg subalgebra. When \(\mathfrak{g}\) is simply-laced and highest weight irreducible representations at level one are considered, \(\hat{F}^i(z)\) can be expressed as a vertex operator whose square vanishes. This nilpotency property is extended to all highest weight representations of all affine untwisted Kac-Moody algebras in the sense that the highest non-vanishing power becomes proportional to the level. As a consequence, the exponential series mentioned terminates and the soliton solutions have a relatively simple algebraic expression whose properties can be studied in a general way. This means that various physical properties of the soliton solutions can be directly related to the algebraic structure. For example, a classical version of Dorey’s fusing rule follows from the operator product expansion of two \(\hat{F}\)’s, at least when \(\mathfrak{g}\) is simply laced. This adds to the list of resemblances of the solitons with respect to the particles which are the quantum excitations of the fields.

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1. Introduction

This work continues that of our previous paper [1] in which we identified the manner whereby solutions describing any number of solitons fitted into the more general class of solutions of the affine Toda theories associated with an affine untwisted Kac-Moody algebra \( \hat{\mathfrak{g}} \). Substitution into the energy-momentum tensor confirmed that the energy was real and positive, being of the form appropriate to a set of moving, relativistic particles. This was despite the fact that the solutions were complex as a consequence of the coupling constant being imaginary in order to provide topological stability. Although the energy density was likewise complex, the reality of its integral was a consequence of the fact that it was a total derivative which was asymptotically real. This work therefore extended previous results of the last year [2] [3], [4], [5], putting them on a more systematic and general footing and unifying them with other developments. Related recent developments are due to Niedermaier [3] and Aratyn et al. [7].

The soliton solutions could be expressed simply in terms of rational functions of expectation values of the \( r \) “fields” \( \hat{F}^i(z) \), where \( r = \text{rank} \mathfrak{g} \), which diagonalise the ad-action of the principal Heisenberg subalgebra of \( \hat{\mathfrak{g}} \). The role of this Heisenberg subalgebra in other, non-relativistic, soliton theories was first established by the Kyoto school [8], but the affine Toda theory is of particular interest as it possesses relativistic symmetry in space-time with two dimensions and well illustrates [1] Zamolodchikov’s idea [10] that integrable theories result from particular breakings of a conformally invariant theory in which the conservation laws are relics of the conformal (or W-) symmetry. Thus it is possible to relate the structure found in affine Toda theory to the concepts of particle physics in an intriguing way and it is this which motivates our interest in precisely this theory. For example, mass formulae and coupling rules for the solitons emerge bearing remarkable similarities to those enjoyed by the quantum particle excitations of the original fields, as elucidated in recent years [11], [12], [13], [14], [15], [16]. This points to a duality symmetry in which the inversion of the coupling constant is accompanied by a replacement of the roots of \( \mathfrak{g} \) by the corresponding coroots. All this structure is very reminiscent of ideas entertained fifteen years ago in connection with magnetic monopole solitons arising in spontaneously broken gauge theories in four dimensional space time [17], [18], [19], [20]. The difference here is that the construction of the dual quantum field theory describing the solitons is now a more realistic possibility.

Our construction of the soliton solutions has the intuitively attractive feature that the soliton of species “\( i \)” is “created” by the “Kac-Moody” group element given by the exponential

\[
\exp Q \hat{F}^i(z),
\]

where \( \ln |Q| \) and \( \ln |z| \) relate to the coordinate and rapidity of the soliton respectively while the superscript “\( i \)” denoting the soliton species, labels one of the \( r \) orbits of the \( hr \) roots of \( \mathfrak{g} \) under the action of the Coxeter element of its Weyl group. It will be convenient to take as the representative element of each such orbit the unique one of the form

\[
\gamma(i) = c(i) \alpha_{(i)}
\]
where $\alpha_i$ is simple and $c(i) = \pm 1$ denotes its “colour”, black or white, as explained in [14] and [16]. That the exponentials (1.1) are meaningful as Kac-Moody group elements without any normal ordering is a consequence of the fact that $\hat{F}^i(z)$ is nilpotent: in an irreducible representation of $\hat{g}$ at level $x$, powers of $\hat{F}^i(z)$ higher than $2x/\gamma_i^2$ vanish while lower powers usually have finite matrix elements. (In saying this we have chosen to normalise the highest root of $g$ to have length $\sqrt{2}$). (The exceptions to this finiteness will have a physical interpretation to be discussed below).

The present paper supplies the proof of these properties. At the same time techniques are found for calculating the expectation values of the $F$’s needed to evaluate the soliton solutions explicitly. The techniques used have an intrinsic interest; the principal vertex operator construction is used together with an exploitation of the outer automorphisms of $\hat{g}$ corresponding to symmetries of the extended Dynkin diagram of $g$. By-products of interest to the physical interpretation of the soliton solutions concern (a) a classical analogue of Dorey’s fusing rule [14] applicable to soliton solutions rather than the quantum excitation particles, (b) insight into generalised “breather” solutions, and (c) expressions for the weight lattice coset of the topological charge when $g$ is simply laced.

In section 2 we present our notations and recapitulate the equations considered together with our solutions. In section 3 we establish properties of the outer automorphisms of $\hat{g}$ to be used while in section 4 we present the properties of the expectation values of the $F$’s with respect to the highest weight states of the fundamental representations of $\hat{g}$. These form a pseudo-unitary matrix which plays an important role in subsequent work.

Since the results sought are representation dependent we have to proceed by considering successively more elaborate irreducible representations of $\hat{g}$. The simplest possibility occurs when $g$ is simply laced and at level one, being known as the “basic” representation. As the $g$ weight of its highest weight state vanishes, that state can be thought of as the vacuum. In this representation, the $F$’s can be constructed explicitly in terms of the principal Heisenberg subalgebra via the principal vertex operator construction [21], [22] as explained in section 5. It is therefore possible to calculate the operator product expansion of two $F$’s. The result possesses double poles with $c$-number residues and simple poles whose residues are proportional to elements of the principal Heisenberg subalgebra or a third $F$. The $F$’s so occurring satisfy Dorey’s fusing rule and this is the algebraic origin of this rule as reported previously [1]. It is easy to check that the square of each $F$ vanishes and to “normal order” products of $F$’s in order to calculate matrix elements, rather as is done in evaluating dual string scattering amplitudes. Section 6 extends these results to the other irreducible representations of $\hat{g}$ at levels higher than one (when $g$ is simply laced) in which the vertex operator construction no longer applies.

Section 7 treats non-simply laced Lie algebras, denoted $g_\tau$, as they are obtained in a unique way by the familiar folding procedure from a simply laced $g$ and an outer automorphism $\hat{\tau}$ [24], [24]. The key result is that $\hat{\tau}$ preserves the principal $su(2)$ subalgebra of $g$. As a consequence, the principal Heisenberg subalgebra of $g_\tau$ is a subalgebra of that of $g$. This makes it easy to relate the $F$’s for $\hat{g}$ and $g_\tau$ and hence establish the final versions of the quoted nilpotency and operator product properties.
The physical interpretation of these results, (a), (b) and (c), cited four paragraphs above, are discussed in section 8. Sample calculations of soliton solutions are also given for $su(N)$, $so(8)$ and $G_2$.

2. The Affine Toda Equations and their Soliton Solutions

even though it is not difficult to generalise virtually all of our arguments to the latter. Before presenting the equations and the solutions of interest, we shall summarise our conventions for such algebras and their representations.

2.1 The Chevalley Basis and the Principal Gradation

The Chevalley basis of the affine untwisted Kac-Moody algebra $\hat{g}$ is generated by the elements

$$\{e_i, f_i, h_i\} \quad i = 0, 1 \ldots r.$$ 

It has been proven \[22\] that

\[
\begin{align*}
[h_i, h_j] &= 0 \\
[h_i, e_j] &= K_{ji} e_j \\
[h_i, f_j] &= -K_{ji} f_j \\
[e_i, f_j] &= \delta_{ij} h_i
\end{align*}
\]  

(2.1)

together with the Serre relations

\[
\begin{align*}
(ade_i)^{1-K_{ji}} e_j = 0 \\
(adf_i)^{1-K_{ji}} f_j = 0
\end{align*}
\]  

(2.2)

are sufficient to completely specify the algebra, given $K_{ji}$, the Cartan matrix of $\hat{g}$, satisfying the usual properties. This is the analogue of Serre’s theorem for a finite-dimensional Lie algebra. It is then convenient to add the element $d'$ with the defining properties

\[
[d', e_i] = e_i, \quad [d', h_i] = 0 \quad \text{and} \quad [d', f_i] = -f_i,
\]  

(2.3)

and to regard it, together with the $h_i$, as spanning the Cartan subalgebra which therefore has dimension $r + 2$. Notice that the adjoint action of $d'$ grades $\hat{g}$; this is generally known as the principal gradation in view of its connection with the principal $so(3)$ subalgebra of $g$ as seen below. Since the $e_i$ can be regarded as the step operators for the simple roots of $\hat{g}$, the grade of a step operator counted by $d'$ is simply the height of the associated root.

We define the positive integers $m_i$ and $n_i$ to be the lowest for which

\[
\sum_i K_{ji} m_i = 0 \quad \text{and} \quad \sum_j n_j K_{ji} = 0.
\]  

(2.4a)

It follows that

\[
n_i = 2m_i/a_i^2
\]  

(2.4b)
where \( a_i \) denotes the \( i \)th simple root of \( \hat{g} \) and the long roots are chosen to have length \( \sqrt{2} \). The Coxeter and dual Coxeter numbers of \( g \) are defined respectively as

\[
h = \sum_{i=0}^{r} n_i \quad \text{and} \quad \tilde{h} = \sum_{i=0}^{r} m_i.
\]

(2.5)

The following element of the Cartan subalgebra is central in that it commutes with all of \( \hat{g} \):

\[
x = \sum_{i=0}^{r} m_i h_i.
\]

(2.6)

We shall want to compare this presentation of \( \hat{g} \), which is particularly relevant to the affine Toda theory, to the more familiar one as a central extension of the loop algebra of \( g \):

\[
[\lambda^m \otimes \gamma_1, \lambda^n \otimes \gamma_2] = \lambda^{m+n} \otimes [\gamma_1, \gamma_2] + \delta_{m+n,0}(\gamma_1, \gamma_2)m x,
\]

(2.7)

where \( \gamma_1, \gamma_2 \in g \) and \((,\) is the Killing form on \( g \). When \( \hat{g} \) is viewed this way it is more natural to append an element \( d \) with the grading property

\[
[d, \lambda^m \otimes \gamma] = m\lambda^m \otimes \gamma.
\]

This is known as the homogeneous grading and \( d \) can be thought of as the Virasoro generator \(-L_0\).

Relating the two presentations of \( \hat{g} \) is straightforward. For \( i \neq 0 \)

\[
 e_i \leftrightarrow \lambda^0 \otimes E^{\alpha_i}, \quad f_i \leftrightarrow \lambda^0 \otimes E^{-\alpha_i}, \quad h_i \leftrightarrow \lambda^0 \otimes H^{\alpha_i},
\]

(2.8a)

in terms of the Chevalley generators of \( g \) while, for \( i = 0 \),

\[
e_0 \leftrightarrow \lambda^1 \otimes E^{-\psi}, \quad f_0 \leftrightarrow \lambda^{-1} \otimes E^{\psi}, \quad h_0 \leftrightarrow \lambda^0 \otimes (H^{-\psi} + x),
\]

(2.8b)

where \( \psi \equiv -\alpha_0 \) denotes the highest root of \( g \). Finally we can now state the relation between \( d \) and \( d' \) which count the homogeneous and principal grades respectively

\[
d' = hd + T_0^3.
\]

(2.9)

Here \( T_0^3 \) is defined to be \( \lambda^0 \otimes T^3 \) where \( T^3 \) is the generator of the principal or maximal \( so(3) \) subalgebra of \( g \) lying in the Cartan subalgebra of the latter. Its adjoint action on the step operators of \( g \) counts the height of the corresponding roots. The analogue of \( d' \) for the finite dimensional Lie algebra \( g \) must be obtained by somehow modding out \( d \). By (2.9), this is achieved by defining the element of the corresponding Lie group \( G, S = e^{2\pi it^3/h}. \)

Conjugation with respect to \( S \) then grades \( g \) according to powers of \( \omega = e^{\exp(2\pi i/h)} \). This structure played a crucial role in understanding the properties of the affine Toda particles arising as field quanta \([13], [14]\), and is explained further in section 2.4.
2.2 Highest Weight Representations

The construction of solutions to the affine Toda equations will utilise irreducible highest weight representations of $\hat{g}$, particularly the fundamental ones. The corresponding representation spaces are generated by the action of arbitrary products of the $f_i$ on the “highest weight” state $|\Lambda >$, say, which is annihilated by the $e_i$. The representation is precisely characterised by the action of $h_0, h_1 \ldots h_r$ on this state, which must be an eigenvector under this action. We write
\[ h_i |\Lambda > = \Lambda(h_i) |\Lambda > = \frac{2\Lambda \cdot a_i}{a_i^2} |\Lambda > . \tag{2.10} \]

The scalar product here is deduced from the invariant scalar product on $\hat{g}$. Unitarity requires the $r+1$ quantities $\Lambda(h_i)$ to be non-negative integers. It follows that the eigenvalue of $x$ take the same positive integral value on all states of the irreducible representation, which we denote $V(\Lambda)$. This is known as the level, and is again denoted $x$, by abuse of notation. The number of irreducible representations at each level is finite and at least one.

It is convenient to define the “fundamental” weights $\Lambda_0, \Lambda_1, \ldots \Lambda_r$ of $\hat{g}$ by $\Lambda_i(h_j) = \delta_{ij}$. Since the $r+1$ simple roots $a_i$ of $\hat{g}$ relate to the $r$ simple roots $\alpha_i$ of $g$ by
\[ a_0 = (-\psi, 1, 0), \quad a_i = (\alpha_i, 0, 0) \quad i = 1, 2, \ldots r, \tag{2.11} \]
given the Cartan Weyl basis of the Cartan subalgebra of $\hat{g}$, $(H_0, x, d)$, we find
\[ \Lambda_0 = (0, 1, ?) \quad \Lambda_i = (\lambda_i, m_i, ?) \quad i = 1, 2, \ldots r. \tag{2.12} \]
The entries denoted by the question mark are undetermined since the roots span a subspace of codimension 1 but are conventionally taken to vanish. The levels $m_i$ in (2.12) are the integers defined in (2.4). For more details, see [25] where the same notation is used.

Notice that the state $\otimes_j |\Lambda_j >^{p_j}$ of the representation $\otimes_j V(\Lambda_j)^{p_j}$, where the $p_j$ are positive integers, is automatically annihilated by the $f_i$ and so is a highest weight state. Thus
\[ \otimes_j |\Lambda_j >^{p_j} = | \sum_j p_j \Lambda_j > , \tag{2.13} \]

where the right hand side denotes the highest weight state of $V(\sum_j p_j \Lambda_j >$. Thus, in principle, all irreducible representations of $\hat{g}$ can be found by the decomposition of products of its fundamental representations. In fact it is possible to improve on this by considering products of only those fundamental representations with level one, ie $m_i = 1$. These include the vacuum representation, that with highest weight $\Lambda_0$. If $g$ is simply laced this condition means that $\lambda$ either vanishes or is minimal [25], [26].

2.3 Alternative basis for $\hat{g}$

The subspace of $\hat{g}$ with principal, $d'$ grade unity contains an element of special significance
\[ \hat{E}_1 = \sum_{i=0}^r \sqrt{m_i} e_i \tag{2.14} \]
As we see in section 3 it is invariant under all the diagram automorphisms of $\hat{g}$. It was found some time ago [27] that the integrability of the affine Toda equations owed itself to the existence of field dependent zero curvature potentials [28]. These can be lifted from the loop algebra of $g$ to $\hat{g}$ at the cost of introducing an extra, innocuous, field [29]. The constant, vacuum solutions to the affine Toda equations give rise to potentials depending linearly on $\hat{E}_1$, (2.14), and its conjugate, $\hat{E}_{-1} = \hat{E}_1^\dagger$. We see from (2.1) and (2.7) that

$$[\hat{E}_1, \hat{E}_{-1}] = x.$$ 

The affine Toda Hamiltonian is one member of an infinite hierarchy of integrable systems and consideration of this makes it natural to seek the subalgebra $C(\hat{E}_1)$ that commutes with $\hat{E}_1$, modulo the central term $x$. It is not difficult to show that can be expressed as a direct sum of graded subspaces

$$C(\hat{E}_1) = \bigoplus \sum_{M \in \mathbb{Z}} C_M(\hat{E}_1) \quad \text{where} \quad [d', C(\hat{E}_1)] = MC_M(\hat{E}_1).$$

The integers $M$ for which $\dim C_M(\hat{E}_1)$ is non-zero are called exponents of $\hat{g}$ of multiplicity $\dim C_M(\hat{E}_1)$. Tables of these exponents can be found in [22]. These exponents possess the symmetry $M \leftrightarrow -M$ and possess period $h$, the Coxeter number of $g$. Modulo $h$ the exponents of $\hat{g}$ equal the exponents of $g$. Since the only occasion in which the multiplicity exceed one (and equals two) is when $g = D_{2n}$ and the exponent equals $n - 1 \mod 2n - 2$, we shall permit ourselves the temporary simplifying assumption that $C_M(\hat{E}_1)$ is spanned by $\hat{E}_M$ when $M$ is an exponent. Using the invariant bilinear form $(,)$ on $\hat{g}$ it follows that

$$[\hat{E}_M, \hat{E}_N] = \frac{Mx}{h} (\hat{E}_M, \hat{E}_N) \delta_{M+N,0},$$

(2.15)

So the $\hat{E}_M$ generate what is known as the principal Heisenberg subalgebra of $\hat{g}$. We suppose $\hat{E}_M^\dagger = \hat{E}_{-M}$ and choose to normalise consistently with (2.14) so that

$$[\hat{E}_M, \hat{E}_N] = Mx \delta_{M+N,0}.$$ 

(2.15)

It is understood that $[M]$ denotes $M$ modulo $h$ and that $q([M])$ and $\gamma_i$ lie in the root space of $g$. These quantities arise naturally in the construction of the Cartan subalgebra of $g$ in apposition, a structure which is essentially the analogue for $g$ of the construction above for $\hat{g}$ as we briefly explain in the next subsection. (More details can be found in Chapter 14 of [22] and in appendix A of [1].)
If we define the generating function for the \( \hat{F}_N^i \) in terms of a formal complex parameter \( z \) as
\[
\hat{F}_N^i(z) = \sum_{N=-\infty}^{\infty} z^{-n} \hat{F}_N^i,
\]
we find from (2.12) that
\[
[\hat{E}_M, \hat{F}_N^i(z)] = \gamma_i \cdot q([M]) z^M \hat{F}_N^i(z).
\]
Thus the complete Heisenberg subalgebra of \( \hat{g} \) has been ad-diagonalised by the \( \hat{F}_N^i(z) \), which, as we mentioned in the introduction, create the solitons of affine Toda theory when exponentiated, (1.1).

2.4 A construction of the Alternative Basis for \( \hat{g} \)

The construction outlined above has a well established analogue for the finite-dimensional Lie algebra \( g \). Indeed this played a fundamental role in the construction of the local conservation laws of affine Toda theory and in understanding the mass and coupling properties of the particles which are the quantum excitations of the \( r \) fields [14], [15], [16]. The mathematics of the construction is due to Kostant [30]. The analogue of (2.14) in \( g \) is
\[
E^1 = \sum_{i=1}^{r} \sqrt{m_i} E^{\alpha_i} + E^{-\psi}.
\]
Now it commutes with its conjugate and, in fact, as \( E^1 \) is in “general position”, the subalgebra of \( g \) commuting with it possesses \( r \) dimensions and so is abelian. It can therefore be considered as an alternative Cartan subalgebra \( h' \), say, said to be “in apposition”. Conjugation by \( S = e^{2\pi i T^3/h} \) furnishes a linear map \( \sigma \) on \( h' \). \( \sigma \) can be shown to be an element of the Weyl group, of order \( h \), called the Coxeter element. This possesses \( r \) eigenvalues of the form \( e^{2\pi i \nu/h} \), where \( 1 \leq \nu \leq h - 1 \) is one of the exponents of \( g \). Accordingly, if \( h_1, h_2, \ldots, h_r \) denotes an orthonormal basis of \( h' \), we can express the \( S \)-graded elements of \( h' \) as
\[
S E^{\nu} S^{-1} = e^{2\pi i \nu/h} S, \quad E^{\nu} = q(\nu) \cdot h \quad \text{where} \quad \sigma(q(\nu)) = e^{2\pi i \nu/h} q(\nu),
\]
so that the \( q(\nu) \) are the eigenvectors of \( \sigma \). Because \( \sigma \) is unitary and real we can orthonormalise in the following sense
\[
q(\nu) \cdot q(\nu')^* = \delta_{\nu,\nu'}, \quad q(h - \nu) = q(\nu)^*.
\]
This is explained in more detail in [17]. There it is also explained how it is possible to define the step operators \( F^{\alpha} \) for the roots with respect to \( h' \) and that \( \sigma \) splits them into precisely \( r \) orbits each containing \( h \) roots. From the definition (2.20), we have the commutator
\[
[E^{\nu}, F^{\alpha}] = \alpha \cdot q(\nu) F^{\alpha}.
\]
The modulus of the structure constant here is proportional to the mass of the affine Toda
particles. The fact that this is also proportional to the right Perron Frobenius eigenvector
\( y_j(1) \) of the Cartan matrix of \( g \) follows from the important identity
\[
\gamma_j \cdot q(\nu) = 2i \sin \frac{\pi \nu}{h} y_j(\nu) e^{-i \delta_j B \pi i/\nu}.
\]
(2.23)
where \( \delta_j B = 1 \) if the vertex \( j \) is “black”, and zero otherwise. \( \gamma_j \) is defined in (1.2), being
the standard representative of the \( i \)-th orbit.

It is possible to affinise \( g \) to form \( \hat{g} \) following the construction (2.7) but using the
basis just discussed. The resulting quantities \( E^\mu_m \) and \( F^\alpha_m \) are graded with respect to \( d \),
the homogeneous grade, rather than \( d' \), the principal grade. Regarding \( z \) as a formal
parameter, the following formula succinctly converts between the grades
\[
\sum_{M \in \mathbb{Z}} z^{-M} \hat{E}_M = \sum_{m \in \mathbb{Z}} z^{-mh} z^{-T_0^3} \left( \sum_{\nu} E^\nu_m \right) z^{T_0^3}.
\]
(2.24)
This is equivalent to the result derived in the appendix of [29]. (See also [31].)

A similar expression applies to the \( \hat{F}^i(z) \)
\[
\hat{F}^i(z) = \sum_{M} z^{-M} \hat{F}^i_M = \sum_{m \in \mathbb{Z}} z^{-mh} z^{-T_0^3} F^\gamma_i z^{T_0^3} - x(T^3, F^\gamma_i) / h.
\]
(2.25)
The necessity for the last, central, term was explained in [1].

2.5 The Affine Toda Equations and their Soliton Solutions

The affine Toda field theory equations associated with \( \hat{g} \) comprise a set of \( r \) coupled,
differential equations satisfied by \( r \) scalar fields \( \phi \), taken to lie in the root space of \( g \), taking
the relativistically invariant form in two space-time dimensions
\[
\partial^2 \phi + \frac{4\mu^2}{\beta} \left( \sum_{i=1}^r m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \cdot \phi} - \frac{\psi}{\psi^2} e^{-\beta \psi \cdot \phi} \right) = 0.
\]
(2.26)
\( \mu \) denotes a real inverse length scale and \( \beta \) an imaginary coupling constant. The coefficients
are so arranged that \( \phi = 0 \) is a constant solution. The most general solution was found
in a formal sense in [3] but we shall be interested in what we claimed to be the solitonic
specialisation of this solution. These specialisations are given by
\[
e^{-\beta \lambda_j \cdot \phi} = \frac{\langle \Lambda_j | e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} | \Lambda_j \rangle}{\langle \Lambda_0 | e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} | \Lambda_0 \rangle^{m_j}},
\]
(2.27)
for \( j = 1, 2, \ldots r \). \( g(0) \) was a Kac-Moody group element arising as an integration constant.
The choice which described \( n \) solitons was a product of \( n \) factors of the form (1.1), one for
each soliton, with the moduli of the variables $z$ suitably ordered. The “time development” operators could be eliminated, using the identity valid for $j = 0, 1, 2, \ldots r$,

$$< \Lambda_j | e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_1 x^-} | \Lambda_j > = e^{\mu^2 m_j x^+ x^-} < \Lambda_j | g(t) | \Lambda_j >,$$

(2.28)

where $g(t)$ is obtained by multiplying each occurrence of $\hat{F}^i(z)$ by the factor

$$W(i, z, x^\mu) = \exp \left( -\mu \{ \gamma_i \cdot q(1) z x^+ - \gamma_i \cdot q(1)^* z^{-1} x^- \} \right), \quad (2.29a)$$

$$= \exp \left( \pm \mu |\gamma_i \cdot q(1)| \{ e^\eta x^+ - e^{-\eta} x^- \} \right), \quad (2.29b)$$

choosing

$$z = \mp e^{\eta} |\gamma_i \cdot q(1)| / \gamma_i \cdot q(1) \quad (2.30)$$

for the reasons explained in [[1]]. There $\eta$ was shown to be the rapidity of the soliton created by this $\hat{F}^i(z)$. The result (2.29) is a straightforward application of the commutation relations (2.15) and (2.18).

The justification of the specialisation was that it was possible to evaluate the energy and momentum of these solutions, finding the expressions appropriate to $n$ moving solitons, given the choice of the variables $z$ explained in [[1]], despite the complexity implied by the imaginary coupling constant $\beta$. The remainder of this paper will be devoted to the study of these formulae. It was crucial in the arguments of [[1]] that products of factors were bona-fide Kac-Moody group elements. This will be justified by developing the properties of the $\hat{F}^i(z)$, some of which are Lie algebraic and some of which, such as the nilpotency, are representation dependent.

3. Automorphisms of $\hat{\mathfrak{g}}$ and symmetries of its Dynkin diagram

Let $\Delta(\hat{\mathfrak{g}})$ denote the Dynkin diagram of the affine untwisted Kac-Moody algebra $\hat{\mathfrak{g}}$. It is well known that this is the same as the extended Dynkin diagram of the finite-dimensional Lie algebra $\mathfrak{g}$. $\Delta(\hat{\mathfrak{g}})$ tends to be more symmetrical than $\Delta(\mathfrak{g})$, the Dynkin diagram of $\mathfrak{g}$, and this will be important in what follows as it will mean that $\hat{\mathfrak{g}}$ has more outer automorphisms than $\mathfrak{g}$ has. There will be several ways in which these can be exploited to simplify calculations of matrix elements of the $F$’s and hence the soliton solutions themselves. As the discussion of this section is rather technical, the reader may wish, at a first reading, to pass immediately to section 4.

The symmetries of $\Delta(\hat{\mathfrak{g}})$ form a finite group denoted $Aut \Delta(\hat{\mathfrak{g}})$ whose elements can be denoted by permutations of the vertices of $\Delta(\hat{\mathfrak{g}})$ preserving its structure:

$$\tau : i \to \tau(i) \quad \tau \in Aut \Delta(\hat{\mathfrak{g}}). \quad (3.1)$$

We immediately have a representation of $aut \Delta(\hat{\mathfrak{g}})$ of dimension $r + 1$:

$$D_{ij}(\tau) = \delta_{i\tau(j)}, \quad (3.2)$$

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where $D$ is a real, orthogonal matrix. Its character is simply given by

$$
\chi(\tau) \equiv Tr(D(\tau)) = \text{No. of points of } \Delta(\hat{g}) \text{ fixed by } \tau. \tag{3.3}
$$

Given the character table of $Aut\Delta(\hat{g})$, (3.3) can be used to deduce the decomposition of $D$ into irreducible representations. Because of its defining property, $D(\tau)$ must commute with the Cartan matrix of $\hat{g}$, denoted $K$:

$$
[K, D(\tau)] = 0. \tag{3.4a}
$$

It also follows from the definitions (2.4) that

$$
m_{\tau(i)} = m_i \quad n_{\tau(i)} = n_i \tag{3.4b}
$$

There is a standard way of lifting any non-trivial $\tau \in Aut\Delta(\hat{g})$ to an automorphism of $\hat{g}$ which is outer. The key point is that, if $X_i$ denotes $e_i$, $f_i$ or $h_i$, the map $\hat{\tau}$ defined by

$$
X_i \rightarrow X_{\tau(i)} = X_j D_{ji}(\tau) \equiv \hat{\tau}(X_i) \tag{3.5}
$$

respects the defining relations (2.1) and (2.2) of $\hat{g}$ by virtue of (3.4) and hence can be extended to an automorphism of $\hat{g}$, by the reconstruction theorem. Notice that, by (2.6), (3.4) and (3.5), the level, $x$, is invariant

$$
\hat{\tau}(x) = x. \tag{3.6}
$$

$\hat{\tau}$ can readily extended to the enveloping algebra by imposing $\hat{\tau}(ab) = \hat{\tau}(a)\hat{\tau}(b)$ and similarly to states of representations of $\hat{g}$. If $|\Lambda >$ denotes a highest weight state, so that it is annihilated by all $r + 1$ $e_i$, then, by (3.5), so is $\tau(|\Lambda >)$. Furthermore, if it is one of the $r + 1$ fundamental highest weight states so that

$$
h_i|\Lambda_j > = \delta_{ij}|\Lambda_j >,
$$

we find, by (3.5) that

$$
\hat{\tau}|\Lambda_i > \sim |\Lambda_{\tau(i)} >. \tag{3.7}
$$

As long as $\tau$ is non-trivial there exists an $i$ for which $\tau(i)$ is distinct from it. Thus as $|\Lambda_i >$ and $|\Lambda_{\tau(i)} >$ are highest weight states of inequivalent irreducible representations of $\hat{g}$, we conclude that $\hat{\tau}$ is an outer automorphism of $\hat{g}$. In fact it is known that the group of outer automorphisms of $\hat{g}$, understood as the quotient of the group of all automorphisms by the invariant subgroup of inner automorphisms is a finite group isomorphic to $Aut\Delta(\hat{g})$.

Because of the way the soliton solutions are expressed in terms of the principal Heisenberg subalgebra $\{ \hat{E}_M \}$ and the $F$’s which ad-diagonalise it, we need to determine the action of $\hat{\tau}$ on these quantities. The first comment is that, by (3.5), $\hat{\tau}$ leaves invariant the subspaces of $\hat{g}$ with principal grade 1, −1 and 0 respectively. As these subspaces generate all
of \( \hat{\mathfrak{g}} \) it follows that \( \hat{\tau} \) always respects the principal grade, which is counted by the adjoint action of \( d' \). Furthermore, by its explicit form, (2.21), \( \hat{E}_1 \) is invariant:

\[
\hat{\tau}(\hat{E}_1) = \hat{E}_1 \quad \text{for all } \tau \in \text{Aut}(\hat{\mathfrak{g}}). \tag{3.8}
\]

Since the \( \hat{E}_M \) are characterised by their principal grade and their commutation relation with \( \hat{E}_1 \), we conclude that

\[
\hat{\tau}(\hat{E}_M) = \eta(\tau,M)\hat{E}_M, \quad M = \text{exponent of } \hat{\mathfrak{g}}, \tag{3.9}
\]

where \( \eta(\tau,M) \) denotes a two dimensional representation of \( \text{Aut}(\hat{\mathfrak{g}}) \) when \( g = D_{2k} \) and \( M = 2k - 1 \mod 4k - 2 \) and a one dimensional representation otherwise (that is, a phase). We shall need explicit formulae for \( \eta \) and shall calculate it in all cases of interest. Most often it simply equals unity, as in (3.8), but there are crucial cases when it does not.

To proceed further, we need to know more about the structure of \( \text{Aut}(\hat{\mathfrak{g}}) \) and this can be deduced by considering its action on \( g \) rather than \( \hat{g} \). That it has such an action follows from the comment that \( \Delta(\hat{\mathfrak{g}}) \) is also the extended Dynkin diagram of \( g \) defined by the \( r \) simple roots augmented by the negative of the highest root. Therefore

\[
\text{Aut}(\hat{\mathfrak{g}}) \subset \text{Aut}(\Phi(g)),
\]

where \( \Phi(g) \) denotes the root system of \( g \). The structure of \( \text{Aut}(\Phi(g)) \) is well understood \[26\]. If \( W(g) \) denotes the Weyl group of \( g \), it constitutes an invariant subgroup of \( \text{Aut}(\Phi(g)) \) such that

\[
\text{Aut}(\Phi(g))/W(g) \cong \text{Aut}(\Delta(g)),
\]

where \( \text{Aut}(\Delta(g)) \) itself can be thought of as the subgroup of \( \text{Aut}(\Delta(\hat{\mathfrak{g}})) \) respecting the vertex labelled 0. Let us define

\[
W_0(g) = \text{Aut}(\Delta(\hat{\mathfrak{g}})) \cap W(g), \tag{3.10}
\]

and analyse its structure. We see immediately that \( W_0 \) is likewise an invariant subgroup of \( \text{Aut}(\Delta(\hat{\mathfrak{g}})) \) and that,

\[
\text{Aut}(\Delta(\hat{\mathfrak{g}}))/W_0(g) \cong \text{Aut}(\Delta(g)). \tag{3.11}
\]

We can now deduce that any \( \tau \in \text{Aut}(\Delta(\hat{\mathfrak{g}})) \) has a unique decomposition

\[
\tau = \rho\tau_0, \quad \rho \in \text{Aut}(\Delta(g)), \quad \tau_0 \in W_0. \tag{3.12}
\]

By (3.11) there exists \( \tau_0 \in W_0 \) with \( \tau_0(0) = \tau(0) \). Hence \( \tau\tau_0^{-1} = \rho \in \text{Aut}(\Delta(g)) \). Furthermore, \( \tau_0 \) is unique, for, if not, and \( \tau_0(0) = \tau_0'(0) \), we would have \( \tau_0\tau_0'^{-1} \in W_0 \cap \text{Aut}(\Delta(g)) \). As this intersection contains only the unit element the result (3.12) follows.

We conclude that the elements of \( W_0 \) are labelled by the “tip points” of \( \Delta(\hat{\mathfrak{g}}) \), namely all those vertices symmetrically related to the vertex labelled 0. The fundamental \( g \)-weights associated to the tip points of \( \Delta(\hat{\mathfrak{g}}) \) other than 0 itself are all minimal \[26\], and hence (when 0 is included) are in one-to-one correspondence with the cosets of the weight lattice of \( g \) with respect to its root lattice, and hence with the centre, \( Z(g) \), of the simply connected
Lie group whose Lie algebra is $g$. This suggests that $W_0(g)$ and $Z(g)$ are related. In fact it has been checked that they are isomorphic [23]. We can make the isomorphism more explicit once we lift $\tau$ in $Aut\Delta(\hat{g})$ to a automorphism $\tilde{\tau}$ of the finite dimensional Lie algebra $g$.

The automorphism (3.5) of $\hat{g}$ worked at any level so, in particular, we may choose level zero in which case $\hat{g}$ is realised by the loop algebra

$$e_i = E^{\alpha_i}, \quad i = 1, \ldots, r \quad e_0 = \lambda E^{\alpha_0},$$

where we have written the step operators for the extended set of roots of $g$. With similar expressions for the $f_i$, we see that (3.5) now furnishes an automorphism of $g$ rather than $\hat{g}$. We denote this $\tilde{\tau}$ in order to distinguish it from $\hat{\tau}$. The difference is that now $\tau$ in $W_0$ furnishes an inner automorphism of $g$. We may as well put $\lambda$ equal to unity. We see that

$$\eta(\tau, N) = \eta(\tau, \nu), \quad (3.13)$$

where $N = nh(g) + \nu$, and $\nu$ is an exponent of $g$ and so taking values between 1 and $h - 1$. Furthermore, by the construction of $E_\nu$ in the appendix of [32],

$$\eta(\tau, \nu) = 1 \quad \tau \in W_0. \quad (3.14)$$

Furthermore $\eta$ only depends on $\tau$ through the coset (3.11). We can now apply $\tilde{\tau}$ to $S = exp(2\pi iT^3/h)$, where $T^3 = \rho^\nu \cdot H = \sum_{\alpha > 0} 2\alpha \cdot H/\alpha^2$ is the intersection of the principal $so(3)$ with its Cartan subalgebra. Taking scalar products with the simple roots of $g$ in turn enables us to deduce

$$\tau(\rho^\nu) - \rho^\nu = -\frac{2h\lambda_{\tau(0)}}{\alpha_{\tau(0)^2}},$$

from which we find

$$\tilde{\tau}(S) = S \exp(-4\pi i\lambda_{\tau(0)} \cdot H/\alpha_{\tau(0)^2}). \quad (3.15)$$

As the second factor lies in $Z(g)$ this establishes the isomorphism

$$W_0(g) \cong Z(g). \quad (3.16)$$

This establishes the fact that $W_0$ is abelian, and, in fact, it is maximally so in $\Delta(\hat{g})$ as any element therein, commuting with all elements of $W_0$, must lie in $W_0$. We shall see how (3.16) affords important information concerning the asymptotic behaviour of soliton solutions in section (8.1).

Applying (3.9) and (3.14) to the statement that $\hat{F}^i(z)$ ad-diagonalises the $\hat{E}_M$, we deduce that

$$\tilde{\tau}(\hat{F}^i(z)) = \epsilon(\tau, i)\hat{F}^i(z), \quad \tau \in W_0(g), \quad (3.17)$$
where $\epsilon(\tau, i)$ is one of the irreducible (and hence one-dimensional) representations of $W_0 \cong Z$. In sections 4 and 8 we shall present arguments to the effect that

$$\epsilon(\tau, i) = e^{-2\pi i \lambda_i \cdot \lambda_j} \quad \text{if} \quad \tau(0) = j,$$

(3.18)

when $g$ is simply laced and its roots are chosen to have length $\sqrt{2}$. It is understood that $\lambda_0$ denotes zero.

4. Fundamental Expectation Values

4.1 The Matrix $F$

The preceding work will help us solve two apparently unrelated problems at the same time. The first is the evaluation of expectation values of the $r \hat{F}^i(z)$’s with respect to the highest weight states $|\Lambda_j>$ of the $r + 1$ fundamental representations:

$$F_{ji} = < \Lambda_j | \hat{F}^i(z) | \Lambda_j >$$

as these play a crucial role in the expressions for soliton solutions. We shall find the coefficient of proportionality between $F_{ji}$ and $F_{0i}$ which is the “vacuum expectation value” of $\hat{F}^i(z)$. The second problem is the determination of the action of the automorphism $\hat{\tau}$ on the $\hat{F}^i(z)$. When $\tau \in \text{Aut}(g)$, this will be useful for constructing the $\hat{F}^i(z)$’s for the nonsimply-laced Lie algebras via the folding of a simply-laced Lie algebra [23], [24].

Notice that (4.1) is independent of $z$ as only the zero (principal) mode of $\hat{F}^i(z)$ contributes. This means that we can equally well write

$$\hat{F}^i_0 = \sum_{j=0}^{r} h_j F_{ji}.$$

(4.2)

The matrix $F$ is not square but there is a natural way of defining an extra column with the label 0 in terms of the level $x$:

$$\hat{F}^0(z) = x \quad \text{so} \quad F_{i0} = m_i.$$

(4.3)

Taking the $|\Lambda_j>$ expectation value of the equation

$$[\hat{E}_1, [\hat{E}_{-1}, \hat{F}_0]] = \begin{cases} |q(1) \cdot \gamma_i|^2 \hat{F}^i_0 & i \neq 0 \\ 0 & i = 0 \end{cases}$$

(4.4)

and substituting (2.16) and (4.2) yields

$$\sum_k C_{jk} F_{ki} = \begin{cases} |q(1) \cdot \gamma_i|^2 F_{ji} & i \neq 0 \\ 0 & i = 0 \end{cases}$$

(4.5)
where

$$C_{jk} = m_j K_{jk} \quad (4.6)$$

and $K$ is the Cartan matrix of $\hat{g}$. We see that the "$i$"th column of the matrix $F$ furnishes an eigenvector of $C$ corresponding to the eigenvalue $|q(1) \cdot \gamma_i|^2$ or zero. Notice that apart from the eigenvalue zero these eigenvalues are proportional to the squared masses of the Toda particles. The matrix $C$, (4.6), is not symmetric, but it is symmetrisable, that is, equivalent to its transpose. This means that there is a second scalar product with respect to which $C$ is symmetric and its eigenvectors corresponding to distinct eigenvalues orthogonal. Unfortunately the eigenspaces of $C$ are degenerate in general and the reason is simply the symmetry of $\hat{g}$ which implies that $D(\tau)$ commutes with $C$, by (3.4). That is, each eigenspace of $C$ carries a representation of $\text{Aut}_\Delta(\hat{g})$ contained in $D$, (3.2). Because $\hat{\tau}$, (3.5), respects the principal grades of $\hat{g}$ we can take the zero mode of (3.17) and insert (4.2) to find

$$\sum_k D_{jk}(\tau) F_{ki} = \epsilon(\tau, i) F_{ji}, \quad \tau \in W_0(\mathfrak{g}), \quad (4.7)$$

where we define $\epsilon(\tau, 0) = 1$. Thus, the columns of $F$ are simultaneous eigenvectors of $C$ and $D(\tau)$. Because $\text{Aut}\Delta(\hat{g})$ is, by definition, the symmetry group of $C$, and because, by the result of section 3, $W_0$ is its maximal abelian subgroup, we can consider $C$ and $W_0$ as furnishing a complete set of commuting observables whose simultaneous eigenvectors are the columns of $F$. These are therefore orthogonal with respect to the second scalar product. Taking them also to be normalised, the matrix $F$ is unitary with respect to this product, and hence invertible. Considering now any $\tau \in \text{aut}\Delta(\hat{g})$, we find

$$\hat{\tau}(\hat{F}_0^i) = \sum_j \hat{F}_j^i d_{ji}(\tau) \quad \text{where} \quad d(\tau) = F^{-1} D(\tau) F. \quad (4.8)$$

Thus $d$ is the matrix representing the action of the automorphism on the $\hat{F}_0^i$. It enjoys the following properties

$$d^\dagger(\tau) d(\tau) = 1, \quad (4.9a)$$

$$d_{i0}(\tau) = d_{0i}(\tau) = \delta_{i0}, \quad (4.9b)$$

$$d_{ij}(\tau) \text{ is diagonal,} \quad \tau \in W_0. \quad (4.9c)$$

Unitarity, (4.9a), follows using the $\text{Aut}\Delta(\hat{g})$ invariance of the second scalar product. (4.9b) follows from unitarity and (3.6) while (4.9c) expresses (3.17). When $\tau \in \text{Aut}\Delta(\mathfrak{g})$, $D(\tau)$ also enjoys property (4.9b). In fact we have verified case by case that, with a suitable choice of labelling of the superscripts of the $\hat{F}_i^j(z)$,

$$D(\tau) = d(\tau), \quad \tau \in \text{Aut}\Delta(\mathfrak{g}). \quad (4.10)$$

Unfortunately we do not know a simple proof.

4.2 The Phases $\epsilon$

We are now in a position to be more specific about the phases $\epsilon(\tau, i)$ in (4.7) and (3.17). Since $W_0 \cong Z(\mathfrak{g})$ is abelian, its irreducible representations are all one dimensional.
Evidently, by (4.8), the phases $\epsilon(\tau, i)$ constitute the $r + 1$ irreps of $W_0$ occurring in the decomposition of $D$, (3.2). There is a subset of these that we know precisely, namely those occurring in the representation of $W_0$ defined by $D$ acting on the tip points. The reason is that when $W_0$ acts on these tip points there are no fixed points so that it follows that this is the regular representation of $W_0$ and that each irreducible representation occurs precisely once. When $g = G_2$ or $F_4$, $W_0$ is trivial but when $g = B_r$ or $C_r$, $W_0 = Z_2$ and there are of course two irreps. It is more interesting when $g$ is simply laced. Then the tip points correspond to minimal weights (understanding $\lambda_0 = 0$) and, if we choose the labelling appropriately, we have

$$
\epsilon(\tau_i, j) = e^{-2\pi i \lambda_i \cdot \lambda_j} \quad \text{where} \quad \tau_i(0) = i.
$$

(4.11)

Later on, in section 8, we shall present an argument for the extension of (4.11) whereby the label $j$ runs over all $r + 1$ values, not just the $|W_0|$ values corresponding to minimal weights.

4.3 Sample Calculations

The symmetry group of $\Delta(su(N))$ is the dihedral group $D_N$ while $W_0$ is its cyclic subgroup $Z_N$. Labelling the vertices of $\Delta(su(N))$ in the obvious consecutive manner $0, 1, 2, \ldots, N - 1$, and considering $W_0$ to be generated by $\tau_1(i) = i + 1$, we find

$$
F_{mn} = e^{2\pi imn/N}.
$$

(4.12)

Then

$$
d_{mn}(\tau_1) = \delta_{mn} e^{-2\pi im/N},
$$

so that, if $\tau_j(i) = i + j = \tau_1^j(i)$,

$$
d_{mn}(\tau_j) = \delta_{mi} e^{-2\pi imj/N}
$$

in agreement with (4.11). Furthermore

$$
c_{mn} \equiv (F^{-1}CF)_{mn} = \delta_{mn} \left(2\sin\frac{\pi m}{N}\right)^2.
$$

These yield the appropriate squared masses.

The Lie algebra $D_4$ provides an interesting example as $\text{Aut}\Delta(\hat{D}_4) \cong S_4$ while $W_0 \equiv Z_2 \otimes Z_2$ so, by (3.11), $\text{Aut}\Delta(D_4) \cong S_3$. The eigenvalues of the matrix (4.6)

$$
C = \begin{pmatrix}
2 & 0 & 0 & 0 & -1 \\
0 & 2 & 0 & 0 & -1 \\
0 & 0 & 2 & 0 & -1 \\
0 & 0 & 0 & 2 & -1 \\
-2 & -2 & -2 & -2 & 4
\end{pmatrix},
$$

exhibits an unusually large degeneracy, being 0, 2, 2, 2 and 6. Labelling the rows and columns of this matrix 0, 1, 2, 3 and 4 we have that the three non-trivial elements of the group $W_0$ take
the following form when written in permutation notation:- (01)(23), (02)(31) and (03)(12). Acting on the tip points 0,1,2 and 3 of $\Delta(\hat{D}_4)$ each irreducible representation occurs precisely once whereas the action on the central vertex 4 yields the scalar representation. We then find that the matrix $F$ diagonalising $C$ and $W_0$ takes the form

$$F = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 \\
2 & 0 & 0 & 0 & -4
\end{pmatrix}. \quad (4.13)$$

Notice the order chosen for the columns 1,2 and 3 ensuring the entry 1 on the diagonal given that the first line consists of unit entries. With this choice, (4.11) again holds.

5. Construction of $\hat{F}^i(z)$ for Simply-Laced $g$ at Unit Level

5.1 The Principal Vertex Operator Construction

Our goal is to evaluate expectation values of products of $F$’s with respect to the fundamental highest weight states $|\Lambda_j>$, and hence explicit soliton solutions. We have seen that for a single $F$ the desired result is the solution to a problem involving finite matrices of dimension $r + 1$, whatever $g$. The key to the more general problem is the observation that sufficiently large powers of $\hat{F}^i(z)$ always vanish. Since the critical power involves the level and so is representation dependent, our strategy will be to build up from the irreducible representations with the simplest structure. These occur when $g$ is simply laced and at level 1. The Frenkel-Kac-Segal vertex operator construction \cite{[33]} is familiar in this situation and there is a variant (with historical priority) which expresses the $\hat{F}^i(z)$ in terms of the principal Heisenberg subalgebra, the $\hat{E}_M$. The highest weights $\Lambda_j$ of these irreducible level one representations, denoted $\rho_j$, correspond to the tip points of $\Delta(\check{g})$, namely those related to the point 0 by symmetries of $\Delta(\check{g})$ which can be taken to be unique elements of $W_0$. $\Lambda_0$ itself defines “basic” representation whose highest weight state is what physicists would call the vacuum.

Denoting \cite{[21]}

$$\rho_j(\hat{F}^i(z)) = F_{ji}e^{\exp\left(\sum_{N>0} \frac{\gamma_i \cdot q([N])z^N \hat{E}_{-N}}{N}\right)} e^{\exp\left(\sum_{N>0} \frac{-\gamma_i \cdot q([N])^*z^{-N} \hat{E}_N}{N}\right)}, \quad (5.1)$$

where the sums extend over the positive affine exponents, we can verify the correct commutation relations (2.18) with the $\hat{E}_N$ as well as the correct expectation value in the highest weight state $|\Lambda_j>$ by virtue of (4.1). We also use the fact that $|\Lambda_j>$ is the ground state of the Fock space built with the principal Heisenberg subalgebra:

$$\hat{E}_N|\Lambda_j> = 0 \quad N > 0, \quad (5.2)$$

and that this Fock space carries the irreducible representation $\rho_j$. 

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5.2 Operator Product Expansion of Two $F$’s

Using the familiar normal ordering procedures of string theory, whereby the $\hat{E}_N$ with positive suffices are moved to the right of those with negative suffices, we shall establish the crucial formula

$$\rho(\hat{F}^i(z))\rho(\hat{F}^j(\zeta)) = X_{i,j}(z,\zeta) : \rho(\hat{F}^i(z))\rho(\hat{F}^j(\zeta)) :$$

where

$$X_{i,j}(z,\zeta) = X_{j,i}(\zeta,z) = \prod_{p=1}^{h} (z - \omega^{-p}\zeta)^{\sigma^p(\gamma_i)\cdot\gamma_j}.$$ (5.3b)

Notice that the powers $\sigma^p(\gamma_i)\cdot\gamma_j$ in (5.3b) can only take the values 0, ±1 and ±2 and we shall study the consequences of this later.

The proof of (5.3) employs the standard techniques which immediately yield (5.3a) with

$$X_{i,j}(z,\zeta) = \exp \left( -\sum_{N>0} \frac{\gamma_i \cdot q([N])^* \gamma_j \cdot q([N]) (\zeta^N)}{N} \right),$$

in which the sum converges when $|z| > |\zeta|$. The sum over the positive affine exponents $N = \nu + nh$, where $\nu = [N]$ is one of the $r$ exponents of $g$, can be written as a double sum over $n$ and $\nu$. Using the identity for a subset of the logarithmic series

$$\sum_{n=0}^{\infty} \frac{y^{nh+v}}{nh+v} = -\frac{1}{h} \sum_{p=1}^{h} \omega^{\nu_p} \ln(1 - y\omega^{-p}),$$

where $\omega$ is the primitive $h$’th root of unity, this becomes

$$X_{i,j}(z,\zeta) = \exp \left( \sum_{p=1}^{h} \ln(1 - \omega^{-p}\zeta^z) \left\{ \frac{1}{h} \sum_{\nu} \omega^{\nu} \gamma_i \cdot q(\nu)^* \gamma_j \cdot q(\nu) \right\} \right)$$

$$= \prod_{p=1}^{h} \left( 1 - \omega^{-p}\zeta^z \right) \sum_{\nu} \sigma^p\gamma_i\cdot q(\nu)^*\gamma_j\cdot q(\nu)/h,$$

where the sum over $\nu$ extends over the $r$ exponents of $g$. Because the $r$ eigenvectors of the Coxeter element $\sigma$ satisfy the orthogonality property

$$q(\nu)^* \cdot q(\mu) = h\delta_{\mu,\nu},$$

we have the completeness relation

$$\sum_{\nu} q(\nu)q(\nu)^* = hI.$$
Inserting this in our previous expression for $X_{i,j}(z,ζ)$ yields the desired result (5.3), on realising that the sum of powers in (5.3b) vanishes as $∑_p σ^p = 0$. The symmetry of the factor $X$ cited in (5.3b) is very important but not quite trivial to verify. Use is made of the fact just mentioned as well as the identity $∑_h pσ^p = h\gamma_j · σ^{−(1+c(i))/2}λ_i ∈ h\mathbb{Z}$, where $c(i)$ denotes the “colour” of the vertex $i$ in the notation of [34] and identity (2.7a) there is used.

5.3 Singularities of the OPE and Dorey’s Fusing Rule

The fact that $X_{i,j}(z,ζ)$ is symmetric (5.3b) means that we can calculate commutators of the modes of the $F$’s in the familiar way, using deformations of contour integrals. Here we shall be content with studying the singularities of (5.3) which are simply double and simple poles. First note that by (5.3)

$$\hat{F}^i(z) \hat{F}^j(z) = 0$$

(5.5)

as the contribution of $p = h$ in the product vanishes and the other factors are regular. This is nilpotency property appropriate to unit level for simply laced algebras. The corresponding results at higher levels will be deduced from this.

Double poles can only occur in (5.3) at points $z = ω^{-p}ζ$ when the corresponding factor is raised to the power $−2$. this requires

$$σ^pγ_i + γ_j = 0$$

which is only possible when $−γ_i$ and $γ_j$ lie on the same Coxeter orbit, that is, $i$ and $j$ are conjugate. The precise value of $p$ in (5.6) was calculated in [34]. The coefficient of the double pole is then a c-number while the coefficient of the associated simple pole lies in the principal Heisenberg subalgebra.

The remaining simple poles in (5.3b) can only occur at points $z = ω^{-p}ζ$ and then only if the power of the responsible factor equals $−1$, that is if $σ^pγ_i + γ_j$ is a root. As each root has a unique expression $σ^qγ_k$, we have the condition

$$σ^pγ_i + γ_j = σ^qγ_k.$$  

(5.6)

To find the residue of this pole we collect the coefficient of $\hat{E}_{−N}$ in the normal ordered product of (5.3b) and find

$$((ω^{-p}ζ)^Nγ_i · q([N]) + ζ^Nγ_j q([N]) = (ω^{-q}ζ)^Nγ_k · q([N])$$

using (5.6) and the fact that $q$ is an eigenfunction of $σ$. A similar calculation applies to the coefficient of $\hat{E}_N$ and the conclusion is that the residue is proportional to $\hat{F}^k(ω^{-q}ζ)$. This is the basis of our claim that the poles of the OPE of two $F$’s are controlled by Dorey’s fusing rule which therefore has a purely Kac-Moody-Lie algebraic origin. Later we shall see that this statement applies to all representations and that it has a consequence for the “fusing” of soliton solutions.
5.4 Arbitrary products of $F$’s

Equation (5.3) for the product of two $F$’s can be extended to an arbitrary product, again in the manner familiar from string theory:

$$\rho(\hat{F}^{i(1)}(z_1)) \ldots \rho(\hat{F}^{i(k)}(z_k)) \ldots \rho(\hat{F}^{i(n)}(z_n)) = \prod_{1 \leq p < q \leq n} X_{i(p),i(q)}(z_p,z_q) \times: \rho(\hat{F}^{i(1)}(z_1)) \ldots \rho(\hat{F}^{i(k)}(z_k)) \ldots \rho(\hat{F}^{i(k)}(z_n)): \quad (5.7)$$

Initially the moduli of the arguments $z_k$ should decrease to the right for convergence but as the right hand side of (5.7) is meromorphic its domain of definition can be extended by analytic continuation. Because the same factors $X$ occur as before, (5.3), we can conclude that the singularities arising are of the same type, that is, either due to the occurrence of conjugate pairs or to the possibility of producing a third soliton by Dorey’s fusing rule. This conclusion is important as it provides the basis for seeing that the products of Kac-Moody group elements occurring in the soliton solutions are well defined and regular except at the special singular points just mentioned.

Taking the expectation value of (5.7), with respect to the highest weight state, $|\Lambda_j>$, of the irrep $\rho$ and using (5.2) yields the matrix element

$$<\Lambda_j|\hat{F}^{i(1)}(z_1) \ldots \hat{F}^{i(k)}(z_k) \ldots \hat{F}^{i(n)}(z_n)|\Lambda_j> = \prod_{1 \leq p < q \leq n} X_{i(p),i(q)}(z_p,z_q) \prod_{p=1}^{n} F_{ji}(p). \quad (5.8)$$

6. $g$ Simply-Laced and Higher Levels of $\hat{g}$

6.1 Nilpotency and OPE Properties of $\hat{F}^{i}(z)$

When $g$ is simply laced, the irreducible level 1 representations of $\hat{g}$ are labelled by their $g$ weights which are respectively 0 and the fundamental minimal weights $\lambda_j$

$$|\Lambda_0> = |1,0> \quad \text{and} \quad |\Lambda_j> = |1,\lambda_j>.$$

Each $\lambda_j$ defines a coset of the weight lattice of $g$ with respect to its root lattice. At higher levels $x > 1$, we expect all $\hat{g}$ irreps with $g$-weight in the same coset as $\lambda_j$ to occur in the decomposition of the representation:

$$D^{(1,\lambda_j)} \otimes \underbrace{D^{(1,0)} \otimes \ldots D^{(1,0)}}_{x-1 \text{ times}} \quad (6.1)$$

In general, it is not easy to perform the decomposition of (6.1) into irreducibles but that is not necessary to establish the nilpotency and operator product expansion properties of the $\hat{F}^{i}(z)$ that we are seeking.

Corresponding to (6.1) the construction of $\hat{F}^{i}(z)$ at level two in terms of the level one principal vertex operator construction is

$$\hat{\mathcal{F}}^{i}(z) = \hat{F}^{i}(z) \otimes 1 + 1 \otimes \hat{F}^{i}(z),$$
and we see by (5.5), that
\[ \hat{F}^i(z)^2 = 2 \hat{F}^i(z) \otimes \hat{F}^i(z). \]

So
\[ \hat{F}^i(z)^3 = 0 \quad \text{at level 2}. \]

Repeating the construction at level \( x \),
\[ \hat{F}^i(z) = \hat{F}^i(z) \otimes 1 \otimes \ldots 1 + 1 \otimes \hat{F}^i(z) \otimes \ldots 1 + \ldots 1 \otimes \hat{F}^i(z), \quad (6.2) \]
so
\[ \hat{F}^i(z)^x = x! \hat{F}^i(z) \otimes \hat{F}^i(z) \otimes \ldots \hat{F}^i(z), \quad (6.3) \]
we find the previously announced nilpotency property
\[ \hat{F}^i(z)^{x+1} = 0. \quad (6.4) \]

We can also apply this style of argument to the operator product expansion. Considering level 2 for ease of writing, we have
\[ \hat{F}^i(z) \hat{F}^j(\zeta) = \hat{F}^i(z) \hat{F}^j(\zeta) \otimes 1 + 1 \otimes \hat{F}^i(z) \hat{F}^j(\zeta) + \hat{F}^i(z) \otimes \hat{F}^j(\zeta) + \hat{F}^j(\zeta) \otimes \hat{F}^i(z). \]

Only the first two terms possess the singularities at \( z \) in terms of \( \zeta \). For example, the residue of the pole at \( z = \omega^{-p}\zeta \) is proportional to
\[ \hat{F}^k(\omega^{-q}\zeta) \otimes 1 + 1 \otimes \hat{F}^k(\omega^{-q}\zeta) = \hat{F}^k(\omega^{-q}\zeta). \quad (6.5) \]

So Dorey’s fusing rule still applies. The residue of the double pole has doubled and so is proportional to the level.

We conclude that at any level, \( e^{Q\hat{F}^i(z)} \) is well defined in the sense of having finite matrix elements between any pair of states in the representation space. Thus, despite its superficial resemblance to a vertex operator, it requires no normal ordering and so constitutes a bona-fide Kac-Moody group element. As we have emphasised, the key point is that the exponential series terminates, by (6.4).

6.2 Fundamental Expectation Values of \( e^{Q\hat{F}^i(z)} \)

In order to evaluate the solution (2.27) for a single soliton of species \( i \), we seek the fundamental expectation values
\[ < \Lambda_j | e^{-\mu E_1 x^+} e^{Q\hat{F}^i(z)} e^{-\mu E_{-1} x^-} | \Lambda_j >, \quad j = 0, 1, 2, \ldots r. \quad (6.6) \]

If \( \lambda_j \) is minimal, so \( m_j = 1 \), then, by (2.28), (2.29), (4.1) and (6.4), this reduces to
\[ < \Lambda_j | e^{-\mu E_1 x^+} (1 + Q\hat{F}^i(z)) e^{-\mu E_{-1} x^-} | \Lambda_j > = e^{\mu^2 x^+ x^-} (1 + F_{ji} QW(i)). \quad (6.7) \]
This suffices when \( g \) is of \( A \)-type but for the \( D \) and \( E \)-type simply laced algebras we may consider those \( \Lambda_k \) corresponding to vertices \( k \) of \( \Delta(\hat{g}) \) adjacent to a tip point \( j \) (with \( m_j = 1 \)). Then \( m_k = 2 \) and it is not difficult to show that

\[
|\Lambda_k| = \frac{1}{\sqrt{2}} \left( |\Lambda_j > \otimes f_j|\Lambda_j > - f_j|\Lambda_j > \otimes |\Lambda_j > \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( |\Lambda_j > \otimes \hat{E}_{-1}|\Lambda_j > - \hat{E}_{-1}|\Lambda_j > \otimes |\Lambda_j > \right) . \tag{6.8}
\]

The expectation value of \( \hat{F}^i(z) \) with respect to \( |\Lambda_k > \) is still given by (4.1) whereas that of its square is found, using the nilpotency properties (6.4), (6.3) and (6.8) to be given by

\[
< \Lambda_k| e^{-\mu \hat{E}_1 x^+} \hat{F}^i(z)^2 e^{-\mu \hat{E}_{-1} x^-} |\Lambda_k > = 2 < \Lambda_j| e^{-\mu \hat{E}_1 x^+} \hat{F}^i(z) e^{-\mu \hat{E}_{-1} x^-} |\Lambda_j > < \Lambda_j| e^{-\mu \hat{E}_1 x^+} \hat{F}^i(z) e^{-\mu \hat{E}_{-1} x^-} \hat{E}_{-1}|\Lambda_j > .
\]

The matrix elements occurring here can be deduced by differentiating (6.7) with respect to \( x^+, x^- \) or both and the result after substitution is simply

\[
2 e^{2 \mu^2 x^+ x^-} F^2_{ji} W(i)^2 .
\]

Thus, in this case, the expectation value (6.6) is given by

\[
e^{2 \mu^2 x^+ x^-} (1 + F_{ki} Q W(i) + F^2_{ji} Q^2 W(i)^2) . \tag{6.9}
\]

Notice that the dependence upon \( z \) is wholly contained in the factor \( W(i) \) and that the result is clearly finite. Obviously this line of argument could be extended. The results so far are sufficient for the single soliton solutions of affine \( A_r \) and \( D_4 \), as we see in section 8.4.

7. \( g \) Non-Simply-Laced

We now seek to extend the construction of the \( \hat{F}^i(z) \) and the verification of their properties to those untwisted affine Kac-Moody algebras which are not simply laced. We shall exploit the existence of a natural embedding of each such algebra within a simply laced one (\( g \) say) as the fixed subalgebra (\( g_\tau \)) of an element \( \tau \) of \( Aut \Delta(g) \) lifted to \( g \) and so “outer”.

Before studying the effect on the \( \hat{E}_M \) and \( \hat{F}^i(z) \) we shall recap the basic ideas. We shall call \( \tau \) “direct” if within each orbit of points given by its action on \( \Delta(g) \) no two vertices are linked. The Dynkin diagram of \( g_\tau \), \( \Delta(g_\tau) \), is obtained by “folding” \( \Delta(g) \), that is, by identifying the vertices on each separate orbit of \( \tau \). The set of vertices contained in such an orbit containing vertex \( i \) is denoted \( < i > \). This will also be used to label the vertices of \( \Delta(g_\tau) \). As \( \tau \) preserves the point 0 the same folding procedure relates the
extended Dynkin diagrams of \( g \) and \( \tilde{g}_\tau \). There is a precise correspondence between each of these direct reductions and the list of simply laced Lie algebras. This can be seen from the diagrams and is as follows:

\[
A_{2r-1} \rightarrow C_r, \quad E_6 \rightarrow F_4, \quad D_{r+1} \rightarrow B_r \quad \text{and} \quad D_4 \rightarrow G_2. \tag{7.1}
\]

Thinking either of \( g \) or \( \tilde{g} \), we define

\[
X_<i> = \sum_{i \in <i>} X_i \tag{7.2}
\]

where, as in (3.5), \( X_i \) denotes \( e_i, f_i \) or \( h_i \). It is then easy to check the commutation relations

\[
[h_<i>, h_<j>] = 0 \tag{7.3a}
\]

\[
[h_<j>, e_<i>] = K_<i><j>e_<i>, \tag{7.3b}
\]

\[
[h_<j>, f_<i>] = -K_<i><j>f_<i>, \tag{7.3c}
\]

\[
[e_<i>, f_<j>] = \delta_{ij}h_<i>, \tag{7.3d}
\]

where

\[
K_<i><j> = \frac{1}{|<i>|} \sum_{i \in <i>, j \in <j>} K_{ij} = \sum_{j \in <j>} K_{ij}. \tag{7.4}
\]

The Serre relation (2.2) can also be checked. Once we have checked that (7.4) indeed defines a Cartan matrix, we shall conclude that it the Cartan matrix of \( g_\tau \) since (7.3) constitute the defining relations.

Using the direct property, we see \( K_<i><i> = 2 \) while, from the second expression (7.4) we see that when \( <i> \neq <j> \), \( K_<i><j> = 0 \), \(-1, -2\), or \(-3\), since the number of points in \( <j> \), \( |<j>| \), can only equal 1, 2 or 3, according to the examples above.

By the first version of (7.4) we see that

\[
\frac{K_<i><j>}{K_<j><i>} = \frac{|<j>|}{|<i>|}. \tag{7.5}
\]

In an obvious notation, this gives the ratio of squared root lengths \( \alpha^2_<i>/\alpha^2_<j> \). Since \( \tau \in Aut\Delta(g) \), \( |<0>| = 1 \) and hence

\[
\frac{\alpha^2_<i>}{\alpha^2_<0>} = \frac{1}{|<i>|}. \tag{7.5}
\]

By the definitions (2.4) together with the normalisation \( m_0 = n_0 = 1 \), we deduce

\[
m_<i> = m_i, \quad n_<i> = |<i>| n_i \quad \text{and} \quad h(g_\tau) = h(g). \tag{7.6}
\]
The preservation of the Coxeter number $h$ in (7.6) is very striking, but in fact the structure goes deeper. The principal $so(3)$ subalgebra of $\mathfrak{g}$ plays a crucial role in the construction of the principal Heisenberg subalgebra of $\hat{\mathfrak{g}}$. We now show that it respected by $\tau$ and hence survives as the principal $so(3)$ subalgebra of $\mathfrak{g}_\tau$,

$$so(3) \subset \mathfrak{g}_\tau \subset \mathfrak{g}.$$  \hspace{1cm} (7.7)

The reason is simply that the Weyl vector of $\mathfrak{g}$ is $Aut\Delta(\mathfrak{g})$ invariant. Also the step operator for the lowest root of both $\mathfrak{g}$ and $\mathfrak{g}_\tau$ is identified. This together with (7.7), again implies the equality of the Coxeter numbers.

Turning to $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}_\tau$ we see that the principal Heisenberg subalgebra of the latter is a subset of that of $\hat{\mathfrak{g}}$. It follows that the exponents of $\mathfrak{g}_\tau$ are a subset of the exponents of $\mathfrak{g}$ (with the same true automatically of the affine exponents). Comparing with (3.9) and (3.13), we see that when $\tau \in Aut\Delta(\mathfrak{g})$ is direct and nontrivial, $\eta(\nu, \tau) = 1$ if and only if $\nu$ is an exponent of $\mathfrak{g}_\tau$. Looking at the examples (7.1), we note from tables that in all cases the common Coxeter number is even. Furthermore, with the exception of $D_{2k}$, it is always precisely the even exponents of $\mathfrak{g}$ which are deleted in order to obtain those of $\mathfrak{g}_\tau$. Hence for $\tau$ nontrivial and direct $\in Aut\Delta(\mathfrak{g})$

$$\hat{\tau}(\hat{E}_M) = (-1)^M \hat{E}_M \hspace{1cm} (7.8)$$

with the exception of $\mathfrak{g} = D_{2k}$. Since in this latter case, two exponents equal $2k - 1$, the principal Heisenberg subalgebra at this principal grade, $M = 2k - 1, \text{mod} \ 4k - 2$, must be two dimensional. As only one of these two exponents survive in the reduction to $B_{2k-1}$, we must be able to choose a basis in the two-dimensional space so that

$$\hat{\tau}(\hat{E}_M) = \hat{E}_M, \quad \hat{\tau}(\hat{E}'_M) = -\hat{E}'_M. \hspace{1cm} (7.9)$$

$D_4$ is exceptional in that $Aut\Delta(D_r)$ is larger when $r = 4$, being $S_3$ the permutation group of the three tip vertices, which we label 1, 2 and 3. $D_4$ has exponents 1, 3, 3' and 5, while the reductions $\mathfrak{g}_{(12)} = B_3$ and $\mathfrak{g}_{(123)} = G_2$ have exponents 1, 3, 5 and 1, 3 respectively. Choosing $\tau$ to be the permutation (12), we can define the basis at $M = 6n + 3$ to be as in (7.9). Since $S_3$ is represented in this two-dimensional subspace and since the permutation $\sigma = (123)$ which is responsible for the reduction to $G_2$ has no invariant subspace, this two dimensional representation has to be the two dimensional irreducible representation of $S_3$. Hence we calculate

$$\hat{\sigma}(\hat{E}_M) = \frac{1}{2} \hat{E}_M - \frac{\sqrt{3}}{2} \hat{E}'_M, \hspace{1cm} (7.10a)$$

$$\hat{\sigma}(\hat{E}_M)' = \frac{\sqrt{3}}{2} \hat{E}_M - \frac{1}{2} \hat{E}'_M. \hspace{1cm} (7.10b)$$

This completes what we have to say concerning the actions of the outer automorphisms of affine untwisted Kac-Moody algebras on their principal Heisenberg subalgebras.

Given the choice of labelling whereby (4.10) holds, we can elevate it to include all grades so that

$$\hat{\tau}(\hat{F}^i(z)) = \hat{F}^j(z) D_{ji}(\tau) = \hat{F}^{\tau(i)}(z), \quad \tau \in Aut\Delta(\mathfrak{g}) \text{ and direct.} \hspace{1cm} (7.11)$$
We now check this when \( \tau \) has order two. On the basis of preceding arguments we expect that

\[ \hat{\tau}(\hat{F}_i(z)) = \hat{F}^{\tau(i)}(z_\tau) \]

and need only prove that \( z_\tau = z \). Acting on the \( \hat{E}_M \hat{F}_i(z) \) commutator (2.18) with \( \hat{\tau} \),

\[ (-1)^{M+1}[\hat{E}_M, \hat{F}^{\tau(i)}(z_\tau)] = z^M \gamma_i \cdot q(\mu) \hat{F}^{\tau(i)}(z_\tau). \]

Now \( \gamma_{\tau(i)} \cdot q(\mu) = (-1)^{M+1} \gamma_i \cdot q(\mu) \), using (2.23) and the facts that \( x_{\tau(i)}(\mu) = (-1)^{\mu+1} x_i(\mu) \), as shown in FO and \( \delta_{\tau(i)B} = \delta_{iB} \) as \( \tau \) is direct. Therefore

\[ [\hat{E}_M, \hat{F}^{\tau(i)}(z_\tau)] = z^M \gamma_{\tau(i)} \cdot q(\mu) \hat{F}^{\tau(i)}(z_\tau). \]

Hence \( z^M = z^M_\tau \) for all exponents \( M \) of \( \hat{g} \). As this includes \( M = 1 \) the result (7.11) follows. It then follows by the preceding discussion that

\[ \hat{F}^{<i>}(z) = \sum_{i \in <i>} \hat{F}_i(z), \tag{7.12} \]

where, again, \( <i> \) labels an orbit of \( \tau \). If \( \tau \) is direct, we know that no two points \( i \) and \( j \), say, of \( <i> \) are linked in \( \Delta(g) \) and that, as a consequence, \( \gamma_i \pm \gamma_j \) cannot be a root. Hence, by the fusing rule,

\[ [\hat{F}^i(z), \hat{F}^j(z)] = 0, \tag{7.13} \]

and we can repeat the arguments of section 6 to deduce that the highest non-vanishing power of \( \hat{F}^{<i>}(z) \) is \( |<i>| x \), remembering that the level, \( x \), is preserved (3.6). By (7.5), this power equals \( x\alpha_{<0>}^2/\alpha_{<i>}^2 \) as claimed earlier. As this and lower powers have finite matrix elements, we have completed our proof of the stated properties of the powers of \( \hat{F}_i(z) \).

8. Soliton Solutions and Their Interpretation

8.1 Topological quantum number and the phase \( \epsilon \)

The spatial jump of any soliton solution is conserved throughout time:

\[ \frac{\partial}{\partial t} \Delta \phi = 0 \quad \text{where} \quad \Delta \phi \equiv \phi(t, x = \infty) - \phi(t, x = -\infty) \tag{8.1} \]

This is because asymptotically the soliton solution must take values in \( \frac{2\pi i}{\beta} \Lambda_W(g) \), where for the purposes of this section we have assumed that \( g \) is simply laced with roots chosen to have length \( \sqrt{2} \). The expression (8.1) is called the topological quantum number but is known to be difficult to calculate, even for single soliton solutions. The reason is that it depends discontinuously on the phase of the parameter \( Q \) in (1.1) where \( \ln|Q| \) signifies the spatial coordinate of the soliton. However we shall now see that, modulo \( 2\pi i\Lambda_R(g)/\beta \), it is independent of this phase, depending only on the label \( i \) in (1.1) in a way that we shall determine.
To do this we shall consider the quantities \( \exp(-\beta \lambda_j \cdot \Delta \phi) \) which are easily calculable from the general solution. Actually it is sufficient to consider only those values of \( j \) for which the fundamental weight \( \lambda_j \) is minimal. As the level of the corresponding representation with highest weight state \( | \Lambda_j > \) is unity, the exponential (1.1) terminates after the second term in the solution. So

\[
e^{-\beta \lambda_j \cdot \phi(x)} = \frac{1 + Q < \Lambda_j | \hat{F}_0^i | \Lambda_j > W(i, z, x^\mu)}{1 + Q < \Lambda_0 | \hat{F}_0^i | \Lambda_0 > W(i, z, x^\mu)}
\]

remembering (2.29). So, if the minus sign is taken in (2.30), so that a soliton rather than an anti-soliton is considered,

\[
e^{-\beta \lambda_j \cdot \Delta \phi} = \frac{< \Lambda_j | \hat{F}_0^i | \Lambda_j >}{< \Lambda_0 | \hat{F}_0^i | \Lambda_0 >}.
\]

The anti-soliton, coming from the alternative sign choice produces the inverse factor. Now consider the unique element of \( W_0 \) defined by \( \tau_j(0) = j \). Putting \( i = 0 \) in (3.7) and using our freedom to choose a phase to be unity, we have

\[
\hat{\tau}_j | \Lambda_0 > = | \Lambda_j >
\]

which on insertion into the previous expression gives

\[
e^{-\beta \lambda_j \cdot \Delta \phi} = \frac{< \Lambda_0 | \hat{\tau}_j^{-1} \hat{F}_0^i \hat{\tau}_j | \Lambda_0 >}{< \Lambda_0 | \hat{F}_0^i rlo >} = \epsilon (\tau_j, i),
\]

using (3.17). Thus the phase \( \epsilon \) acquires a direct significance. At least when \( g \) is simply laced, explicit calculations have led us to conjecture that

\[
\epsilon(\tau_j, i) = e^{-2\pi i \lambda_i \cdot \lambda_j}
\]

(8.3)

Considering a solution describing several solitons of species \( i(1), i(2) \ldots i(n) \), we find, by similar arguments that

\[
e^{-\beta \lambda_j \cdot \Delta \phi} = \prod_{k=1}^{n} \epsilon (\tau_j, i(k)).
\]

(8.4)

We shall find below that this taken together with the possibility of solitons fusing gives support for (8.3).

8.2 The “Fusing” of Two solitons

We saw that

\[
g = \exp(Q_1 \hat{F}^{i(1)}(z_1)) \exp(Q_2 \hat{F}^{i(2)}(z_2))
\]

(8.5)
constitutes a well-defined Kac-Moody group element unless \(z_1\) and \(z_2\) are related in the exceptional ways discussed in section (5.3). Here we discuss the interpretation of the singularity given by

\[
z_1 = \omega^{-p} z_2 \quad \text{where } p \text{ is such that } \sigma^p \gamma_{i(1)} + \gamma_{i(2)} = \sigma^q \gamma_k, \tag{8.6}
\]

for some \(q\) and \(k\). As the group element \(g\) “creates” two solitons of species \(i(1)\) and \(i(2)\) whose rapidities, \(\ln|z_1|\) and \(\ln|z_2|\) coincide at the singular point (8.6), we may suspect that some sort of bound state may be formed. We shall confirm this and that the bound state is a third soliton, of species \(k\), given by Dorey’s fusing rule.

For simplicity, first consider the group element (8.5) evaluated at unit level so that \(\hat{F}^i(z)^2 = 0\), by (5.5). Also using (5.3), we find

\[
g = 1 + Q_1 \hat{F}^{i(1)}(z_1) + Q_2 \hat{F}^{i(2)}(z_2) + Q_1 Q_2 X_{i(1), i(2)}(z_1, z_2) : \hat{F}^{i(1)}(z_1) \hat{F}^{i(2)}(z_2) :. \tag{8.7}
\]

As the singular point (8.6) is approached, the factor \(X\) in the last term acquires a pole while everything else remain finite if the constants \(Q_i\) remain fixed. Alternatively, we can suppose \(Q_1\) and \(Q_2\) tend to zero in such a way that \(Q_1 Q_2 X_{i(1), i(2)}\) remains finite. Then the limit of (8.7) is simply

\[
g = 1 + Q' \hat{F}^k(\omega^{-q} \zeta) = \exp Q' \hat{F}^k(\omega^{-q} \zeta),
\]

again using (5.5) and letting \(\zeta = z_2\) and \(Q'\) be a new constant. Thus two solitons have “fused” to give a third with the selection rule given precisely by Dorey’s fusing rule, originally formulated for the particle excitations of the theory and now seen to extend to the solitons. It is not difficult to show that the above limiting behaviour of the product of two Kac-Moody group elements equalling a third is independent of the level.

This result provides strong support for the validity of (8.3). Recall Braden’s result [35] that if species \(i(1)\) and \(i(2)\) fuse to form \(k\), then if the corresponding fundamental representations are considered, \(D_{\lambda}^k\) occurs in the the Clebsch Gordon series of \(D_{\lambda}^{i(1)} \otimes D_{\lambda}^{i(2)}\). This implies that

\[
e^{-2\pi i \lambda_{i(1)} \cdot \lambda_j} e^{-2\pi i \lambda_{i(2)} \cdot \lambda_j} = e^{-2\pi i \lambda_k \cdot \lambda_j}.
\]

It follows that if the phases \(\epsilon\) for two solitons are given by (8.3) then so is that of any third soliton obtained by fusion, by (8.4).

### 8.3 Breather Solutions

We have just seen how an analytic continuation of the parameters describing the coordinates and rapidities of a solution describing two or more solitons can lead to a new solution describing a lesser number of solitons. This phenomenon, involving “fusing”, does not occur in the simplest affine Toda theory, namely that with \(g = su(2)\), Sine Gordon theory, but there is a related phenomenon that does. This is the occurrence of the
“breather” solution which can be regarded as the bound state of soliton and antisoliton, oscillating about their common centre of mass. Classically it possesses a continuous mass spectrum extending from zero to the soliton-antisoliton threshold. In the quantum theory, this continuous spectrum is quantised, with the number of states depending on the coupling constant $\beta$. The lowest mass such state (if it exists) is identified with the quantum excitation of the original Sine-Gordon field.

We shall now see, by finding the corresponding Kac Moody “group element”, that such solutions can be formed of any soliton antisoliton pair in affine Toda theory, and that the energy and momentum can be evaluated by the techniques of our previous paper.[1]

If $\delta_i$ denotes the phase of the complex number $\gamma_i \cdot q(1)$, and $z_i = \epsilon_i e^{i\delta_i} e^{\eta_i}, \quad \epsilon_i = \pm 1,$

it was shown [1] that, when $\eta_i$ is real, the contribution to $\sqrt{2}$ times the light cone components $P^\pm$ of the momenta due to a factor $\exp Q_i \hat{F}_i(z_i)$ in the group element $g(0)$, (2.27), was $M_i e^{\pm \eta_i}$. $M_i$ is the mass of the soliton of species $i$ and was calculated explicitly while $\eta_i$ can be interpreted as its rapidity. In fact this argument holds good even if $e^{\eta_i}$ is complex, providing that its real part is positive. Note that the contribution to the momentum is independent of both the complex number $Q_i$ and the sign $\epsilon_i$. Consider now a group element

$$g(0) = \exp Q_1 \hat{F}_1(e^{i\delta_1} e^{\eta_1}) \exp Q_2 \hat{F}_2(e^{i\delta_2} e^{\eta_2}).$$

If $\eta_1$ and $\eta_2$ are real, this produces a two soliton solution with momentum

$$\sqrt{2} P^\pm = M_1 e^{\pm \eta_1} + M_2 e^{\pm \eta_2}.$$  

However it is possible for this expression to remain real and positive for a complex choice of rapidities $\eta_1$ and $\eta_2$, providing $M_1 = M_2 = M$, say. Such a choice is given by

$$\eta_1 = \eta_2^* = \eta + i \theta, \quad -\pi/2 < \theta < \pi/2,$$

when

$$\sqrt{2} P^\pm = 2 M e^{\pm \eta \cos \theta}.$$ 

Since the familiar $su(2)$, Sine-Gordon, breather arises this way, (8.8) and (8.9) can be interpreted as giving rise to a generalised breather solution with mass $2M \cos \theta$ and rapidity $\eta$.

Usually the equal mass condition is satisfied by choosing species 1 and 2 to be antiparticles of each other but in $D_4$ there is another possibility, as, by triality, the $S_3$ symmetry of $\Delta(D_4)$, there exist three solitons species of equal mass, as is seen below in section 8.4. This raises the possibility of breather solutions with non-zero topological charge which, at present, we are unable to exclude. We anticipate that it will be important to evaluate the higher conserved charges for such solutions.

8.4 Sample Soliton Solutions
The affine $su(N)$ soliton solutions are particularly easy to evaluate as all $N$ fundamental representations of $\hat{s}u(N)$ have level one. The single soliton solution of species $I$ is given by

$$e^{-\beta \lambda_J \cdot \phi} = \frac{1 + Qe^{2\pi i J/N}W(I)}{1 + QW(I)}, \quad J = 1, 2, \ldots N - 1.$$  

This follows from (2.27), (4.12) and (6.7). If we consider two solitons of species $I$ and $J$, we find, using additionally (5.8), that $e^{-\beta \lambda_J \cdot \phi}$ equals

$$\frac{1 + e^{2\pi i I(N)W_1} + e^{2\pi i I(2)J/N}W_2 + X_{I(1)I(2)}(z_1, z_2)e^{2\pi i (I(1)+I(2))J/N}W_1W_2}{1 + W_1 + W_2 + X_{I(1)I(2)}(z_1, z_2)W_1W_2},$$

where $J = 1, 2, \ldots N - 1$ and $W_q = Q(q)W(I(q))$ for short.

We can also immediately write down the $s\hat{o}(8)$ solitons. Using, in addition, (4.13) and (6.9), we have for species $i = 1, 2$ and 3, corresponding to three of the four tip points of $\Delta(s\hat{o}(8))$,

$$e^{-\beta \lambda_J \cdot \phi} = \begin{cases} 1 & j = i \\ \frac{1-QW(i)}{1+QW(i)} & j \neq i, 4 \\ \frac{1+Q^2W(i)^2}{(1+QW(i))^2} & j = 4 \end{cases},$$

while, for species 4,

$$e^{-\beta \lambda_J \cdot \phi} = \begin{cases} 1 & j = 1, 2, 3 \\ \frac{1-4QW(4)+Q^2W(4)^2}{(1+QW(4))^2} & j = 4 \end{cases}.$$

Using the additional results of section 7, concerning non simply laced algebras, we can construct the two species of affine $G_2$ soliton solution. By (7.12), these are created by

$$\hat{F}^{<1>} = \hat{F}^{<2>} = \hat{F}^{<3>} = \hat{F}^{1} + \hat{F}^{2} + \hat{F}^{3} \quad \text{and} \quad \hat{F}^{<4>} = \hat{F}^{4},$$

in terms of the quantities for $\hat{D}_4$. We have retained the convention for labeling the vertices of $\Delta(\hat{D}_4)$ 0, 1, 2, 3 and 4. It follows from their defining properties that the fundamental highest weight states of $\hat{G}_2$ are

$$|\Lambda_0 > = |\Lambda_0 >, \quad |\Lambda_1 > = |\Lambda_1 > \quad \text{or} \quad |\Lambda_2 > \quad \text{or} \quad |\Lambda_3 > .$$

So the $\hat{F}^{<4>}$ solution for affine $G_2$ is given by the same expression as the affine $D_4$ solution above. For the other soliton solution, we have, for $j = 0$ or 1

$$< \Lambda_{<j}|\hat{F}^{<1>}(z)|\Lambda_{<j}> = F_{j1} + F_{j2} + F_{j3},$$

$$< \Lambda_{<j}|\hat{F}^{<1>}(z)^2|\Lambda_{<j}> = 2X_{12}(z, z)(F_{j1}F_{j2} + F_{j2}F_{j3} + F_{j3}F_{j1}),$$

$$< \Lambda_{<j}|\hat{F}^{<1>}(z)^3|\Lambda_{<j}> = 6X_{12}(z, z)^3F_{j1}F_{j2}F_{j3},$$

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where the $\hat{D}_4$ matrix $F_{ji}$ (4.13) enters the right hand side and (5.7) has been used. As $\hbar = 6$ for $D_4$ or $G_2$, we can evaluate $X_{12}(z,z) = 1/3$ using (5.3b). Inserting these numerical values, (2.27) yields

$$e^{-\beta\lambda_{<1>}}\phi = \frac{1 - W - \frac{4}{3}W^2 + \frac{1}{27}W^3}{1 + 3W + W^2 + \frac{1}{27}W^3},$$

where $W = QW_{<1>}$. The other component of this solution is more difficult to evaluate since powers of $\hat{F}$ up to six survive. Combining the previous methods, we find

$$e^{-\beta\lambda_{<4>}}\phi = \frac{1 + 3W^2 - \frac{16}{27}W^3 + \frac{1}{3}W^4 + \frac{1}{27}W^6}{(1 + 3W + W^2 + \frac{1}{27}W^3)^2}.$$

All the single soliton solutions for the affine untwisted Toda theories have recently been worked out explicitly using Hirota’s method [7]. The results so obtained for the enumeration of soliton species and for the mass formulae seem to agree with the results of the general arguments of [1] rather than with [4] whose results were incomplete. It is interesting to see that although the general features of our approach eventually emerge, this Hirota method appears comparatively cumbersome.

9. Discussion

We have succeeded in our main aim of establishing the conditions under which products of Kac-Moody group elements of the form (1.1) make sense. The key result is the proof of the vanishing, in a representation of level $x$, of all powers of $\hat{F}^i(z)$ exceeding $2x/\gamma_i^2$. In the course of the work some new structural features have emerged as well as new questions such as the general proof of (8.3).

The extension of Dorey’s fusing rule to solitons, (at least when $g$ is simply laced), strengthens the evidence for a duality symmetry between the particles which are the quantum excitations of the elementary fields and the solitons. An intriguing difference concerns the fact that the solitons possess an internal degree of freedom due to the different values of the topological charge possible for a soliton of a given species. It is therefore urgent to clarify the structure of this spectrum but old questions persist such as the dependence of the topological charge (8.1) upon the phase of the constant $Q$ in (1.1). But we have been able to show that, modulo $2\pi i\Lambda_R(g)/\beta$, it is independent of $Q$. Although we have seen how the breathers of familiar type fit naturally within our formalism, we have not been able to exclude breathers of a new kind.

There are many ways in which we have good reason to believe our arguments can be extended. One concerns the affine Toda theories associated with twisted affine Kac-Moody algebras (rather than just the untwisted ones we have considered). Another concerns the non-Abelian Toda theories corresponding to non-principal embeddings of $SO(3)$ in $g$.

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