ON HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES
FOR STRONGLY $\varphi_h$-CONVEX FUNCTIONS

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Abstract. In this paper, using functions whose derivatives absolute values are strongly $\varphi_h$-convex with modulus $c > 0$, we obtained new inequalities related to the right and left side of Hermite-Hadamard inequality by using new integral identities.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [4], [8, p. 137]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1.1}$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see (3-12) and the references cited therein.

Let $I$ be an interval in $\mathbb{R}$ and $h : (0, 1) \rightarrow (0, \infty)$ be a given function. A function $f : I \rightarrow \mathbb{R}$ is said to be $h$-convex if

$$f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in (0, 1)$ [23]. This notion unifies and generalizes the known classes of functions, s-convex functions, Gudunova-Levin functions and P-functions, which are obtained by putting in (1.2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$, and $h(t) = 1$, respectively. Many properties of them can be found, for instance, in [6], [7], [16], [18], [19], [21], [23].

Let us consider a function $\varphi : [a, b] \rightarrow [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the $\varphi$-convex functions in [17]:

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be $\varphi$-convex on $[a, b]$ if for every two points $x \in [a, b], y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)).$$

Obviously, if function $\varphi$ is the identity, then the classical convexity is obtained from the previous definition. Many properties of the $\varphi$-convex functions can be found, for instance, in [1], [2], [17], [20], [21].

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Moreover in [2], Cristescu have presented a version Hermite-Hadamard type inequality for the \( \phi \)-convex functions as follows:

**Theorem 1.** If a function \( f : [a, b] \rightarrow \mathbb{R} \) is \( \phi \)-convex for the continuous function \( \phi : [a, b] \rightarrow [a, b] \), then

\[
(1.3) \quad f \left( \frac{\phi(a) + \phi(b)}{2} \right) \leq \frac{1}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} f(x) dx \leq \frac{f(\phi(a)) + f(\phi(b))}{2}.
\]

Recall also that a function \( f : I \rightarrow \mathbb{R} \) is called strongly convex with modulus \( c > 0 \), if

\[
f(t \phi(x) + (1 - t) \phi(y)) \leq t f(x) + (1 - t) f(y) - ct(1 - t)(x - y)^2
\]

for all \( x, y \in I \) and \( t \in (0, 1) \). Strongly convex functions have been introduced by Polyak in [13] and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature see ([13]-[16]) and the references cited therein.

In [22], Sarikaya have introduced the following notion of the strongly \( \phi_h \)-convex functions with modulus \( c > 0 \), and give some properties of them:

A function \( f : D \rightarrow [0, \infty) \) is said to be strongly \( \phi_h \)-convex with modulus \( c > 0 \), if

\[
f(t \phi(x) + (1 - t) \phi(y)) \leq h(t) f(\phi(x)) + h(1 - t) f(\phi(y)) - ct(1 - t)(\phi(x) - \phi(y))^2
\]

for all \( x, y \in D \) and \( t \in (0, 1) \). In particular, if \( f \) satisfies (1.4) with \( h(t) = t^s \) (\( s \in (0, 1) \)), \( h(t) = \frac{1}{t} \), and \( h(t) = 1 \), then \( f \) is said to be strongly \( \phi \)-convex, strongly \( \phi_s \)-convex, strongly \( \phi \)-Gudunova-Levin function and strongly \( \phi \)-P-function, respectively. The notion of \( \phi_h \)-convex function corresponds to the case \( c \rightarrow 0 \).

In this article, using functions whose derivatives absolute values are strongly \( \phi_h \)-convex with modulus \( c > 0 \), we obtained new inequalities related to the right and left side of Hermite-Hadamard inequality by using new integral identities. In particular if \( \phi = 0 \) is taken as, our results obtained reduce to the Hermite-Hadamard type inequality for classical convex functions.

2. Main Results

In order to prove our main results, we establish an important integral identity as follows:

**Lemma 1.** Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \) and \( \phi : [a, b] \rightarrow [a, b] \). If \( f' \) is a Lebesgue integrable function, then the
following equality holds;
\[
\begin{align*}
\frac{1}{2} \left( f(\varphi(a)) + f(\varphi(b)) \right) - \frac{1}{(\varphi(b) - \varphi(a))} \int_0^{\varphi(b)} f'(x) \, dx \\
= \frac{1}{2} \left( \varphi(b) - \varphi(a) \right) \int_0^{2t - 1} \left[ f'(t \varphi(b) + (1 - t) \varphi(a)) + ct(1 - t)(\varphi(b) - \varphi(a))^2 \right] \, dt.
\end{align*}
\]

**Proof.** By integration by parts, we can state:
\[
I = \int_0^{2t - 1} \left[ f'(t \varphi(b) + (1 - t) \varphi(a)) + ct(1 - t)(\varphi(b) - \varphi(a))^2 \right] \, dt
\]
\[
= (2t - 1) \frac{f(t \varphi(b) + (1 - t) \varphi(a))}{(\varphi(b) - \varphi(a))} - \frac{2}{(\varphi(b) - \varphi(a))^2} \int_0^1 f(t \varphi(b) + (1 - t) \varphi(a)) \, dt
\]
\[
= \frac{f(\varphi(b)) + f(\varphi(a))}{(\varphi(b) - \varphi(a))} - \frac{2}{(\varphi(b) - \varphi(a))^2} \int_0^1 f(t \varphi(b) + (1 - t) \varphi(a)) \, dt.
\]

Using the change of the variable \( x = t \varphi(b) + (1 - t) \varphi(a) \) for \( t \in [0, 1] \), which gives
\[
I = \frac{f(\varphi(b)) + f(\varphi(a))}{(\varphi(b) - \varphi(a))} - \frac{2}{(\varphi(b) - \varphi(a))^2} \int_0^1 f(x) \, dx.
\]

Multiplying the both sides of (2.2) by \( \frac{\varphi(b)}{2} \), we obtain
\[
\frac{\varphi(b)}{2} \int_0^1 f(x) \, dx = \frac{f(\varphi(b)) + f(\varphi(a))}{(\varphi(b) - \varphi(a))} - \frac{1}{(\varphi(b) - \varphi(a))} \int_0^1 f(x) \, dx
\]
which is required. \( \square \)

**Theorem 2.** Let \( h : (0, 1) \to (0, \infty) \) be a given function. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \) and Lebesgue integrable function. If \( |f'| \) is strongly \( \varphi_h \)-convex with respect to \( c > 0 \) for the continuous function \( \varphi : [a, b] \to [a, b] \) and \( \varphi(a) < \varphi(b) \), then the following inequality holds:
\[
\begin{align*}
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_0^{\varphi(b)} f(x) \, dx \right| \\
\leq \left( \frac{\varphi(b) - \varphi(a)}{2} \right) \left( \sqrt{\int_0^1 |f'(\varphi(b))|^2 \, dt} + \sqrt{\int_0^1 |f'(\varphi(a))|^2 \, dt} \right) \int_0^1 |2t - 1| h(t) \, dt
\end{align*}
\]
for all \( t \in (0, 1) \).
Proof. From Lemma 1 and by using strongly $\varphi_h$-convexity functions with modulus $c > 0$ of $|f'|$, we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right| \leq \frac{(\varphi(b) - \varphi(a))}{2} \int_0^1 |2t - 1| \left[ |f'(t\varphi(b) + (1 - t)\varphi(a))| + ct(1 - t)(\varphi(b) - \varphi(a))^2 \right] \, dt$$

$$\leq \frac{(\varphi(b) - \varphi(a))}{2} \int_0^1 |2t - 1| \left[ |h(t)| f'(\varphi(b)) + h(1 - t) |f'(\varphi(a))| \right] \, dt$$

$$= \frac{(\varphi(b) - \varphi(a))}{2} \left[ |f'(\varphi(b))| + |f'(\varphi(a))| \right] \int_0^1 |2t - 1| h(t) \, dt$$

where using the fact that

$$\int_0^1 h(t) \, dt = \int_0^1 h(1 - t) \, dt$$

which completes the proof. \qed

The following inequalities are associated with the right side of Hermite-Hadamard type inequalities for strongly $\varphi$-convex, strongly $\varphi_s$-convex strongly $\varphi - P$-convex with respect to $c > 0$, respectively.

**Corollary 1.** Under the assumptions of Theorem 2 with $h(t) = t$, $t \in (0, 1)$, we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right| \leq (\varphi(b) - \varphi(a)) \left( \frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{8} \right).$$

**Corollary 2.** Under the assumptions of Theorem 2 with $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right| \leq \frac{(\varphi(b) - \varphi(a))}{2} \left( s + \frac{1}{2s+1} \right) \left( \frac{|f'(\varphi(b))| + |f'(\varphi(a))|}{(s+1)(s+2)} \right).$$
Corollary 3. Under the assumptions of Theorem 2 with \( h(t) = 1, \ t \in (0,1) \), we have

\[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \varphi(b) \int f(x) \, dx \right| \leq \frac{\varphi(b) - \varphi(a)}{2} \left( \frac{1}{2} \left| f'(\varphi(b)) \right| + \frac{1}{2} \left| f'(\varphi(a)) \right| \right).
\]

Remark 1. (a) In the case \( c \to 0 \) and \( \varphi(x) = x \) for all \( x \in [a,b] \), then inequality (2.6) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal and Agarwal in (3).

(b) In the case \( c \to 0 \) and \( \varphi(x) = x \) for all \( x \in [a,b] \), then inequality (2.6) gives the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in (3).

Theorem 3. Let \( h : (0,1) \to (0,\infty) \) be a given function. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a,b \in I \) with \( a < b \) and Lebesgue integrable function. If \( |f'|^q \) is strongly \( \varphi_h \)-convex with respect to \( c > 0 \) for the continuous function \( \varphi : [a,b] \to [a,b] \), \( \varphi(a) < \varphi(b) \), and

\[
A = c^q(\varphi(b) - \varphi(a))^qB(q + 1, q + 1) - \frac{c}{6}(\varphi(b) - \varphi(a))^2 > 0,
\]

then the following inequality holds;

\[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \varphi(b) \int f(x) \, dx \right| \leq \frac{(\varphi(b) - \varphi(a))}{2^q} \left( \frac{1}{p + 1} \left( \int_0^1 \left| f'(\varphi(b)) \right|^q + \left| f'(\varphi(a)) \right|^q \right) \left( \int_0^1 h(t) \, dt \right) + A \right)^{\frac{1}{q}}
\]

for all \( t \in (0,1) \), \( q \geq 1 \) where \( B \) is a beta function.

Proof. From Lemma 1 and by using Hölder’s integral inequality, we have

\[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{(\varphi(b) - \varphi(a))} \varphi(b) \int f(x) \, dx \right| \leq \frac{(\varphi(b) - \varphi(a))}{2^q} \left( \int_0^1 |2t - 1|^p \, dt \right)^{\frac{1}{p}}
\]

\[
\times \left( \int_0^1 [\left| f'(t\varphi(b) + (1 - t) \varphi(a)) \right| + ct(1 - t)(\varphi(b) - \varphi(a))^2]^q \, dt \right)^{\frac{1}{q}}.
\]

Since \( |f'|^q \) is strongly \( \varphi_h \)-convex on \([a,b]\) and using the following inequality

\((u + v)^q \leq 2^{q-1}(u^q + v^q), \ u, v > 0, \ q > 1,\)
we get
\[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right|
\]
\[
\leq \frac{(\varphi(b) - \varphi(a))}{2^\frac{1}{p}} \left( \int_0^1 |2t - 1|^p \, dt \right)^{\frac{1}{p}}
\]
\[
\times \left( \int_0^1 \left[ |f'(t\varphi(b) + (1-t)\varphi(a))|^q + (ct(1-t)(\varphi(b) - \varphi(a))^2)^q \right] \, dt \right)^{\frac{1}{q}}
\]
\[
\leq \frac{(\varphi(b) - \varphi(a))}{2^\frac{1}{p}} \left( \int_0^1 |2t - 1|^p \, dt \right)^{\frac{1}{p}}
\]
\[
\times \left( \int_0^1 \left( h(t) |f'(\varphi(b))|^q + h(1-t) |f'(\varphi(a))|^q \right.
\]
\[
- c t(1-t)(\varphi(b) - \varphi(a))^2 + \left[ c t(1-t)(\varphi(b) - \varphi(a))^2 \right]^q \, dt \right)^{\frac{1}{q}}.
\]
Thus, with simple calculations we obtain
\[
\int_0^1 |2t - 1|^p \, dt = \frac{1}{p+1},
\]
\[
\int_0^1 t(1-t) \, dt = \frac{1}{6}, \quad \int_0^1 t^q(1-t)^q \, dt = B(q+1, q+1)
\]
and
\[
\int_0^1 h(t) \, dt = \int_0^1 h(1-t) \, dt.
\]
Therefore, we obtain
\[
\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right|
\]
\[
\leq \frac{(\varphi(b) - \varphi(a))}{2^\frac{1}{p}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( |f'(\varphi(b))|^q + |f'(\varphi(a))|^q \right)^{\frac{1}{q}} \left( \int_0^1 h(t) \, dt \right)^{\frac{1}{q}}
\]
\[
+ c^q(\varphi(b) - \varphi(a))^{2q} B(q+1, q+1) - \frac{c}{6} (\varphi(b) - \varphi(a))^2 \right)^{\frac{1}{q}}
\]
which completes the proof. \[\square\]
The following inequalities are associated with the right side of Hermite-Hadamard type inequalities for strongly $\varphi$-convex, strongly $\varphi_s$-convex strongly $\varphi - P$-convex with respect to $c > 0$, respectively.

**Corollary 4.** Under the assumptions of Theorem 3 with $h(t) = t$, $t \in (0,1)$, we have

\[
(2.7) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right| \\
\leq \frac{(\varphi(b) - \varphi(a))}{2^\frac{1}{q}} \left( \frac{1}{p+1} \right)^\frac{q}{p+1} \left( \frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{2} + A \right)^\frac{1}{q}. 
\]

**Corollary 5.** Under the assumptions of Theorem 3 with $h(t) = t^s$ $(s \in (0,1))$, $t \in (0,1)$, we have

\[
(2.8) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right| \\
\leq \frac{(\varphi(b) - \varphi(a))}{2^\frac{1}{q}} \left( \frac{1}{p+1} \right)^\frac{q}{p+1} \left( \frac{|f'(\varphi(b))|^q + |f'(\varphi(a))|^q}{s+1} + A \right)^\frac{1}{q}. 
\]

**Corollary 6.** Under the assumptions of Theorem 3 with $h(t) = 1$, $t \in (0,1)$, we have

\[
(2.9) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx \right| \\
\leq \frac{(\varphi(b) - \varphi(a))}{2^\frac{1}{q}} \left( \frac{1}{p+1} \right)^\frac{q}{p+1} \left( |f'(\varphi(b))|^q + |f'(\varphi(a))|^q + A \right)^\frac{1}{q}. 
\]

**Remark 2.** In the case $c \to 0$, inequalities (2.7), (2.8) and (2.9) reduce the right sides of Hermite-Hadamard type inequality for $\varphi$-convex, $\varphi_s$-convex and $\varphi - P$-convex functions, respectively.

**Lemma 2.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0$ where $a, b \in I$ with $a < b$ and $\varphi : [a, b] \to [a, b]$. If $f'$ is Lebesgue integrable function, then the following equality holds:

\[
\frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx - f\left( \frac{\varphi(a) + \varphi(b)}{2} \right) \\
= (\varphi(b) - \varphi(a)) \left\{ \frac{1}{2} \int_0^1 t \left[ f'(t \varphi(a) + (1-t) \varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2 \right] dt \right\}. 
\]
Proof. By integration by parts, we can state:

\[
J_1 = \int_0^1 t \left[ f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2 \right] dt
\]

\[
= \left. \frac{f(t\varphi(a) + (1-t)\varphi(b))}{(\varphi(a) - \varphi(b))} \right|_0^1 + \frac{1}{2(\varphi(a) - \varphi(b))} \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt + c(\varphi(b) - \varphi(a))^2 \int_0^1 t(1-t) dt
\]

\[
= \frac{1}{2(\varphi(a) - \varphi(b))} f \left( \frac{\varphi(b) + \varphi(a)}{2} \right)
\]

(2.10) + \frac{1}{(\varphi(b) - \varphi(a))} \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt + \frac{5c}{3 \times 2^6} (\varphi(b) - \varphi(a))^2.

and similarly

\[
J_2 = \int_0^1 (t - 1) \left[ f'(t\varphi(a) + (1-t)\varphi(b)) + ct(1-t)(\varphi(b) - \varphi(a))^2 \right] dt
\]

\[
= (t - 1) \left. \frac{f(t\varphi(a) + (1-t)\varphi(b))}{(\varphi(a) - \varphi(b))} \right|_0^1 + \frac{1}{2(\varphi(a) - \varphi(b))} \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt - c(\varphi(b) - \varphi(a))^2 \int_0^1 1(1-t)^2 dt
\]

\[
= \frac{1}{2(\varphi(a) - \varphi(b))} f \left( \frac{\varphi(b) + \varphi(a)}{2} \right)
\]

(2.11) + \frac{1}{(\varphi(b) - \varphi(a))} \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt - \frac{5c}{3 \times 2^6} (\varphi(b) - \varphi(a))^2.

Adding (2.10) and (2.11) and rewriting, we easily deduce (2.12)

\[
J = J_1 + J_2 = \frac{1}{(\varphi(b) - \varphi(a))} \int_0^1 f(t\varphi(a) + (1-t)\varphi(b)) dt - \frac{1}{(\varphi(b) - \varphi(a))} f \left( \frac{\varphi(b) + \varphi(a)}{2} \right).
\]
Using the change of the variable \( x = t \varphi(a) + (1 - t) \varphi(b) \) for \( t \in [0, 1] \), and multiplying the both sides of (2.12) by \((\varphi(b) - \varphi(a))\), we obtain

\[
(\varphi(b) - \varphi(a)) \int f(x) \, dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \]

which is required. \( \square \)

**Theorem 4.** Let \( h : (0, 1) \to (0, \infty) \) be a given function. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \) and Lebesgue integrable. If \( |f'| \) is strongly \( \varphi_h \)-convex with respect to \( c > 0 \) for the continuous function \( \varphi : [a, b] \to [a, b] \) and \( \varphi(a) < \varphi(b) \), then the following inequality holds;

\[
\left| \frac{1}{(\varphi(b) - \varphi(a))} \int f(x) \, dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right| \\
\leq (\varphi(b) - \varphi(a)) \left( |f'(\varphi(a))| + |f'(\varphi(b))| \right) \left( \int_0^t |h(t) + h(1 - t)| \, dt \right)
\]

for all \( t \in (0, 1) \).

**Proof.** From Lemma 2 and by using strongly \( \varphi_h \)-convexity functions with modulus \( c > 0 \) of \( |f'| \), we have

\[
\left| \frac{1}{(\varphi(b) - \varphi(a))} \int f(x) \, dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right|
\leq (\varphi(b) - \varphi(a)) \left\{ \int_0^t |h(t) f'(\varphi(a))| + h(1 - t) |f'(\varphi(b))| \, dt \right\}
\leq (\varphi(b) - \varphi(a)) \left( |f'(\varphi(a))| + |f'(\varphi(b))| \right) \left( \int_0^t |h(t) + h(1 - t)| \, dt \right)
\]

where using the fact that

\[
\int_0^t th(t) \, dt = \int_0^1 (1 - t) h(1 - t) \, dt
\]
and
\[
\int_0^1 th(1-t)dt = \int_0^1 (1-t) h(t)dt
\]
which completes the proof. □

The following inequalities are associated the left side of Hermite-Hadamard type inequalities for strongly \( \varphi \)-convex, strongly \( \varphi_s \)-convex, strongly \( \varphi-P \)-convex with respect to \( c > 0 \), respectively.

**Corollary 7.** Under the assumptions of Theorem 4 with \( h(t) = t \), \( t \in (0,1) \), we have
\[
\left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right|
\leq \left( \varphi(b) - \varphi(a) \right) \frac{|f'(\varphi(a))| + |f'(\varphi(b))|}{8}.
\]

**Corollary 8.** Under the assumptions of Theorem 4 with \( h(t) = t^s \) (\( s \in (0,1) \)), \( t \in (0,1) \), we have
\[
\left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right|
\leq \left( \varphi(b) - \varphi(a) \right) \left( 1 + \frac{s + 3}{2s + 2} \right) \frac{|f'(\varphi(a))| + |f'(\varphi(b))|}{(s + 1)(s + 2)}.
\]

**Corollary 9.** Under the assumptions of Theorem 4 with \( h(t) = 1 \), \( t \in (0,1) \), we have
\[
\left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right|
\leq \left( \varphi(b) - \varphi(a) \right) \frac{|f'(\varphi(a))| + |f'(\varphi(b))|}{4}.
\]

**Remark 3.** (a) In the case \( c \to 0 \) and \( \varphi(x) = x \) for all \( x \in [a,b] \), then inequality (2.13) gives the right sides of Hermite-Hadamard inequality proved by Kirmaci in ([10]).
(b) In the case \( c \to 0 \) and \( \varphi(x) = x \) for all \( x \in [a,b] \), then inequality (2.14) coincide with the right sides of Hermite-Hadamard inequality proved by Dragomir and Agarwal in ([13]).

**Theorem 5.** Let \( h : (0,1) \to (0,\infty) \) be a given function. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) where \( a, b \in I \) with \( a < b \) and Lebesgue integrable
function. If $|f|^q$ is strongly $\varphi_h$-convex with respect to $c > 0$ for the continuous function $\varphi : [a, b] \rightarrow [a, b]$, $\varphi(a) < \varphi(b)$, and

$$G = c(\varphi(b) - \varphi(a))^{2q} B_{1/2} (q + 1, q + 1) - \frac{c}{12}(\varphi(b) - \varphi(a))^2 > 0,$$

then the following inequality holds;

$$\left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right|$$

$$\leq \frac{(\varphi(b) - \varphi(a)) \left( \frac{1}{p + 1} \right)^{\frac{1}{q}}}{2^q} \left\{ \left( \int_0^1 (h(t) |f'(\varphi(a))|^q + h(1 - t) |f'(\varphi(b))|^q) \, dt + G \right)^{\frac{1}{q}} \right. $$

$$+ \left. \left( \int_{\frac{1}{2}}^1 (h(t) |f'(\varphi(a))|^q + h(1 - t) |f'(\varphi(b))|^q) \, dt + G \right)^{\frac{1}{q}} \right\}$$

for all $t \in (0, 1)$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ where $B_{r}(., .)$ is incomplete beta function.

**Proof.** From Lemma 2 and by using the Hölder’s inequality, we have

$$\left| \frac{1}{(\varphi(b) - \varphi(a))} \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right|$$

$$\leq (\varphi(b) - \varphi(a))$$

$$\times \left\{ \left( \int_0^{\frac{1}{2}} t^p \, dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{2}} [f'(t\varphi(a) + (1 - t) \varphi(b)) + ct(1 - t)(\varphi(b) - \varphi(a))^2]^q \, dt \right)^{\frac{1}{q}} \right.$$ 

$$+ \left. \left( \int_{\frac{1}{2}}^1 (t - 1)^p \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{2}}^1 [f'(t\varphi(a) + (1 - t) \varphi(b)) + ct(1 - t)(\varphi(b) - \varphi(a))^2]^q \, dt \right)^{\frac{1}{q}} \right\}.$$

Since $|f'|^q$ is strongly $\varphi_h$-convex on $[a, b]$ and using the following inequality

$$(u + v)^q \leq 2^{q-1}(u^q + v^q), \quad u, v > 0, \quad q > 1,$$
we get
\[ \left| \frac{1}{(\varphi(b) - \varphi(a))} \int f(x) \, dx - f \left( \frac{\varphi(b) + \varphi(a)}{2} \right) \right| \]
\[ \leq 2^{1 - \frac{1}{q}} (\varphi(b) - \varphi(a)) \left\{ \left( \frac{1}{p+1} \right) \left( \frac{1}{2} \right)^{\frac{1}{q}} \int_{0}^{1} t^{p} \, dt \right\}^{\frac{1}{q}} \]
\[ + \left( \int_{0}^{1} (1 - t)^{p} \, dt \right)^{\frac{1}{q}} \left( \frac{1}{2} \right) \left[ f'(t\varphi(a) + (1 - t) \varphi(b))^{q} + (ct(1 - t)(\varphi(b) - \varphi(a))^{2})^{q} \right] dt \left( \frac{1}{2} \right) \}
\[ \leq \frac{(\varphi(b) - \varphi(a))}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \]
\[ \times \left\{ \left( \frac{1}{p+1} \right) \left( \frac{1}{2} \right)^{\frac{1}{q}} \int_{0}^{1} (h(t) |f'(\varphi(a))|^{q} + h(1 - t) |f'(\varphi(b))|^{q}) \right. \]
\[ - ct(1 - t)(\varphi(b) - \varphi(a))^{2} + (ct(1 - t)(\varphi(b) - \varphi(a))^{2})^{q} \right] dt \left( \frac{1}{2} \right) \}
\[ = \frac{(\varphi(b) - \varphi(a))}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{q}} \]
\[ \times \left\{ \left( \frac{1}{p+1} \right) \left( \frac{1}{2} \right)^{\frac{1}{q}} \int_{0}^{1} (h(t) |f'(\varphi(a))|^{q} + h(1 - t) |f'(\varphi(b))|^{q}) \right. \]
\[ - \frac{c}{12} (\varphi(b) - \varphi(a))^{2} + c(\varphi(b) - \varphi(a))^{2q} B_{\frac{1}{2}} (q + 1, q + 1) \right) \left( \frac{1}{2} \right) \}
\[ + \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ f'(t\varphi(a) + (1 - t) \varphi(b))^{q} + (ct(1 - t)(\varphi(b) - \varphi(a))^{2})^{q} \right] dt \left( \frac{1}{2} \right) \}
\[ - \frac{c}{12} (\varphi(b) - \varphi(a))^{2} + c(\varphi(b) - \varphi(a))^{2q} B_{\frac{1}{2}} (q + 1, q + 1) \right) \left( \frac{1}{2} \right) \}
\[ . \]
Thus, with simple calculations we obtain
\[ \int_{0}^{1} t^{p} \, dt = \int_{\frac{1}{2}}^{1} (1 - t)^{p} \, dt = \frac{1}{2^{p+1} (p+1)}. \]
\[ \int_0^{1/2} t(1-t)dt = \int_{1/2}^1 t(1-t)dt = \frac{1}{12}, \]

\[ \int_0^{1/2} t^q(1-t)^q dt = \int_{1/2}^1 t^q(1-t)^q dt = B_{1/2}(q+1, q+1). \]

Therefore, using the above obtained results, we have

\[
\left| \frac{1}{(\varphi(b) - \varphi(a))} \int f(x) dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right| \\
\leq \frac{(\varphi(b) - \varphi(a))}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^{1/2} \left( h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q \right) dt + G \right)^{1/2} \\
+ \left[ \int_{1/2}^1 \left( h(t) |f'(\varphi(a))|^q + h(1-t) |f'(\varphi(b))|^q \right) dt + G \right]^{1/2},
\]

which completes the proof. \(\square\)

The following inequalities are associated the left side of Hermite-Hadamard type inequalities for strongly \(\varphi\)-convex, strongly \(\varphi_s\)-convex strongly \(\varphi - P\)-convex with respect to \(c > 0\), respectively.

**Corollary 10.** Under the assumptions of Theorem \(\square\) with \(h(t) = t\), \(t \in (0, 1)\), we have

\[
\left| \frac{1}{(\varphi(b) - \varphi(a))} \int f(x) dx - f \left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right| \\
\leq (\varphi(b) - \varphi(a)) \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^{1/2} \left( |f'(\varphi(a))|^q + |f'(\varphi(b))|^q \right) dt + G \right)^{1/2} \\
\times \left\{ \left( \frac{3 |f'(\varphi(a))|^q + |f'(\varphi(b))|^q}{8} + G \right)^{1/2} + \left( \frac{3 |f'(\varphi(a))|^q}{8} + G \right)^{1/2} \right\}.
\]
Corollary 11. Under the assumptions of Theorem 5 with $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, we have

\[
\left| \frac{1}{\varphi(b) - \varphi(a)} \int_a^b f(x) \, dx - f\left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right| \leq \frac{\varphi(b) - \varphi(a)}{2^{\frac{1}{q}}} \left( \frac{1}{p + 1} \right)^{\frac{1}{q}} \left( \frac{1}{2^{s+1}(s + 1)} |f'(\varphi(a))|^q + \frac{1}{s + 1} \left( 1 - \frac{1}{2^{s+1}} \right) |f'(\varphi(b))|^q + G \right)^{\frac{1}{q}}.
\]

(2.17)

\[
\times \left\{ \left( \frac{1}{2^{s+1}(s + 1)} |f'(\varphi(a))|^q + \frac{1}{s + 1} \left( 1 - \frac{1}{2^{s+1}} \right) |f'(\varphi(b))|^q + G \right)^{\frac{1}{q}} \right\}.
\]

Corollary 12. Under the assumptions of Theorem 5 with $h(t) = 1$, $t \in (0, 1)$, we have

\[
\left| \frac{1}{\varphi(b) - \varphi(a)} \int_a^b f(x) \, dx - f\left( \frac{\varphi(a) + \varphi(b)}{2} \right) \right| \leq \frac{\varphi(b) - \varphi(a)}{2^{\frac{1}{q} - 1}} \left( \frac{1}{p + 1} \right)^{\frac{1}{q}} \left( \frac{1}{2^{s+1}(s + 1)} |f'(\varphi(a))|^q + |f'(\varphi(b))|^q \right)^{\frac{1}{q}} + G \right)^{\frac{1}{q}}.
\]

(2.18)

Remark 4. In the case $c \to 0$, inequalities (2.16), (2.17) and (2.18) reduce the right sides of Hermite-Hadamard type inequality for $\varphi$-convex, $\varphi_s$-convex and $\varphi-P$-convex functions, respectively.

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