Check of the Mass Bound Conjecture in de Sitter Space

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Abstract

Recently an interesting conjecture was put forward which states that any asymptotically de Sitter space whose mass exceeds that of exact de Sitter space contains a cosmological singularity. In order to test this mass bound conjecture, we present two solutions. One is the topological de Sitter solution and the other is its dilatonic deformation. Although the latter is not asymptotically de Sitter space, the two solutions have a cosmological horizon and a cosmological singularity. Using surface counterterm method we compute the quasilocal stress-energy tensor of gravitational field and the mass of the two solutions. It turns out that this conjecture holds within the two examples. Also we show that the thermodynamic quantities associated with the cosmological horizon of the two solutions obey the first law of thermodynamics. Furthermore, the nonconformal extension of dS/CFT correspondence is discussed.

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I. INTRODUCTION

It is well-known that to calculate the conserved charges including mass is a difficult task in an asymptotically de Sitter (dS) spacetime. This is due to the absence of the spatial infinity and the globally timelike Killing vector in such a spacetime. In a recent paper [1], a novel prescription was proposed for computing the boundary stress tensor and conserved charges of asymptotically dS spacetimes from data at early or late time infinity. This uses the surface counterterm method [2–4], which was developed in the AdS/CFT correspondence [5–7]. On the other hand, if one accepts the dS/CFT correspondence [8,9], the resulting quantities then correspond to the stress-energy tensor and corresponding conserved charges of the dual Euclidean conformal field theory (CFT).

Following this prescription, the authors of [1] calculated the masses of the 3,4,5-dimensional Schwarzschild-de Sitter black hole solutions, respectively. It is found that these masses are always less than those of dS spaces in corresponding dimensions. Furthermore, they argued that this result is consistent with the dS/CFT correspondence and the Bousso’s observation [10] on the asymptotically dS space that the entropy of dS space is an upper bound for the entropy of any asymptotically dS space. On the basis of this result, the authors of [1] put forward a conjecture (BBM conjecture): *Any asymptotically de Sitter space whose mass exceeds that of de Sitter contains a cosmological singularity.* Because a rigorous proof of this conjecture is not yet carried out, it is very interesting to check this conjecture with some examples. This is the main aim of this paper.

The organization of this paper is as follows. In Sec. II we introduce briefly the prescription to calculate the boundary stress-energy tensor and conserved charges of gravitational field in the asymptotically de Sitter space. We present the topological de Sitter solution and compute the boundary stress-energy tensor and mass of this solution in Sec. III. In Sec. IV, we check the BBM conjecture in a dilatonic deformation of the topological dS solution. We summarize our results in Sec. V with some discussions.

II. PRESCRIPTION

In this section we briefly review the surface counterterm method to compute the conserved charges in asymptotically de Sitter space. We consider an \((n+2)\)-dimensional Einstein action with a positive cosmological constant, \(\Lambda = n(n+1)/2l^2\),

\[
S = \frac{1}{16\pi G} \int_M d^{n+2}x \sqrt{-g} \left( R - \frac{n(n+1)}{l^2} \right) + \frac{1}{8\pi G} \int_{\partial M^+} d^{n+1}x \sqrt{h} K. \tag{1}
\]

Here the first term is the bulk action with \(n+2\)-dimensional Newtonian constant \(G\). The second is the Gibbons-Hawking surface term, which is necessary to have a well-defined Euler-Lagrange variation. \(M\) denotes the bulk manifold, \(\partial M^\pm\) are spatial boundaries at early and late times. \(g_{\mu \nu}\) is the bulk metric and \(h_{ij}\) and \(K\) are the induced metric and the trace of the extrinsic curvature of the boundaries. In dS space the spacelike boundaries \(I^\pm\) are Euclidean surfaces at early and late time infinities. The notation \(\int_{\partial M^-} d^{n+1}x\) indicates an integral over the late time boundary minus an integral over the early time boundary which are both Euclidean surfaces.
Some surface counterterms have been given in [1], which can render the action finite in 3,4,5-dimensional asymptotically dS spaces:

\[
S_{ct} = \frac{1}{8\pi G} \int_{\partial M^+} d^{n+1}h \mathcal{L}_{ct} + \frac{1}{8\pi G} \int_{\partial M^-} d^{n+1}x \sqrt{h} \mathcal{L}_{ct},
\]

where

\[
\mathcal{L}_{ct} = \frac{n}{l} - \frac{l}{2(n-1)} \mathcal{R}
\]

and \(\mathcal{R}\) is the intrinsic curvature of the induced metric. This is an extension of the surface counterterm in the asymptotically anti-de Sitter (AdS) space [2–4]. Decomposing the bulk spacetime in the ADM form as

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt),
\]

one then has the induced metric \(h_{ij}\) on spacelike surfaces of fixed time. Denoting the future pointing unit normal to these surfaces by \(u^i\), the extrinsic curvature of these surfaces can be obtained using the formula:

\[
K_{ij} = -h_{ij}^{\mu} \nabla_\mu u^j.
\]

With these and the Brown-York prescription [14], one can get the Euclidean quasilocal stress-energy tensor of an asymptotically dS space

\[
T^x_{ij} = \frac{2}{\sqrt{h\delta h^{ij}}} \frac{\delta I}{\delta h^{ij}} = -\frac{1}{8\pi G} \left( K_{ij} - K h_{ij} - \frac{n}{l} h_{ij} - \frac{l}{n-1} G_{ij} \right),
\]

\[
T^-_{ij} = \frac{2}{\sqrt{h\delta h_{ij}}} \frac{\delta I}{\delta h_{ij}} = -\frac{1}{8\pi G} \left( -K_{ij} + K h_{ij} - \frac{n}{l} h_{ij} - \frac{l}{n-1} G_{ij} \right).
\]

Here \(I = S + S_{ct}\), and \(G_{ij}\) is the Einstein tensor of the induced surface. Since there exist two spacelike boundaries in dS space, the superscripts \(\pm\) in \(T_{ij}\) represent the quantity on the late or early time boundary. The difference in signs of the two stress-energy tensors in (3) arises because the extrinsic curvature \(K\) is defined with respect to a future pointing timelike normal, leading to sign changes between the early and late time boundaries [1]. For this reason as in [1] we will use \(T_{ij} = T^x_{ij}\) in what follows. This means that we calculate the conserved charges on the late time boundary \(I^+\).

Next let us decompose the induced metric \(h_{ij}\) in the form

\[
h_{ij} dx^i dx^j = N^2 \rho^2 + \sigma_{ab}(d\phi^a + N^a_\rho d\rho)(d\phi^b + N^b_\rho d\rho),
\]

where the notation \(\phi^a\) are angular variable parameterizing closed surfaces around the origin. Suppose \(\xi^i\) to be a Killing vector generating an isometry of the boundary geometry. Following [14],[4], one can define the conserved charge \(Q\) associated with the Killing vector \(\xi^i\) using the quasilocal stress-energy tensor \(T_{ij}\) as follows

\[1\]The surface counterterms in the asymptotically dS space have also been discussed in [11–13].
\[ Q = \oint_{\Sigma} d^n \phi \sqrt{\sigma n^i \xi^j T_{ij}}, \tag{8} \]

where \( n^i \) is the unit normal to the surface \( \Sigma \) with a fixed \( \rho \), and the coordinate \( \rho \) is obtained by analytic continuation of a timelike Killing vector.

Recall that an important obstacle to define the mass of gravitational field in the asymptotically dS space is the absence of a globally timelike Killing vector. However, there is a Killing vector which is timelike within the cosmological horizon of dS space in the static coordinates, while it is spacelike outside the cosmological horizon and then on \( I^+ \), future null infinity. Thus any spacetime which is asymptotically dS space will have such an asymptotic symmetry generator. Adapting the coordinates (7) so that “radial” normal \( n^i \) is proportional to the relevant (spacelike) boundary Killing vector \( \xi^i \), the authors of [1] proposed a mass formula for asymptotically dS spaces:

\[ M = \oint_{\Sigma} d^n \phi \sqrt{\sigma} N_\rho \epsilon, \quad \epsilon \equiv n^i n^j T_{ij}. \tag{9} \]

Here the Killing vector \( \xi^i \) is normalized as \( \xi^i = N_\rho n^i \). Similarly the angular momenta can be defined as

\[ P_a = \oint_{\Sigma} d^n \phi \sqrt{\sigma} J_a, \quad J_a = \sigma_{ab} n^i T^{bi}. \tag{10} \]

Using this prescription, the masses of 3, 4, 5-dimensional Schwarzschild-dS black hole solutions have been calculated in [1]. It was found that the mass of dS space is always larger than that of the black hole solution in the dS space in corresponding dimensions. This leads to the BBM conjecture. Now we wish to check this conjecture with the following two solutions.

### III. TOPOLOGICAL DE SITTER SOLUTION

We start with an \((n + 2)\)-dimensional topological black hole solution in AdS space

\[ ds_{TBA_{dS}}^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 \tilde{g}_{ab} dx^a dx^b. \tag{11} \]

where

\[ f(r) = k - \frac{2Gm}{r^{n-1}} + \frac{r^2}{l^2}, \quad k = 1, -1, 0. \tag{12} \]

\( \tilde{g}_{ab} dx^a dx^b \) is the line element of an \( n \)-dimensional hypersurface with constant curvature \( kn(n - 1) \) and volume \( V = \int d^n x \sqrt{\tilde{g}} \). \( l \) is the curvature radius of AdS space. \( m \) is a constant related with the ADM mass of the black hole [15]. It is believed that black holes in asymptotically flat spacetime should have a spherical horizon. When there is a negative cosmological constant in a spacetime, however, a black hole can have a non-spherical horizon. In this sense this black hole [14] is referred to as a topological black hole in AdS space. When \( m = 0 \), the solution (11) reduces to the AdS space. Replacing \( l^2 \) by \(-l^2\) in (11), one has a solution

\[ ds_{TBdS}^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 \tilde{g}_{ab} dx^a dx^b. \tag{13} \]
where

\[ f(r) = k - \frac{2Gm}{r^{n-1}} - \frac{r^2}{l^2}, \quad k = 1, -1, 0. \]  

(14)

Obviously this is a solution to the Einstein equations with a positive cosmological constant in \((n + 2)\) dimensions.

When \(k = 1\), it is just the Schwarzschild-de Sitter solution. The case \(m = 0\) reduces to the dS space with a cosmological horizon \(r_c = l\). When \(m\) increases, a black hole horizon occurs and increases with the size of \(m\), while the cosmological horizon shrinks. Finally the black hole horizon touches the cosmological horizon when

\[ m = \frac{1}{G(n + 1)} \left( \frac{n - 1}{n + 1} \right)^{(n-1)/2}. \]  

(15)

This is the Nariai black hole, the maximal black hole in dS space. The mass of the solution in this case has been calculated in [1]. We will discuss the cases \(k = 0\) and \(k = -1\), respectively.

(i) The case of \(k = 0\). In this case, \(\tilde{g}_{ab} dx^a dx^b\) is an \(n\)-dimensional Ricci flat hypersurface. Changing the sign in front of \(m\) in (14), one has

\[ ds^2_\text{dS} = -\left( \frac{2Gm}{r^{n-1}} - \frac{r^2}{l^2} \right) dt^2 + \left( \frac{2Gm}{r^{n-1}} - \frac{r^2}{l^2} \right)^{-1} dr^2 + r^2 dx_n^2, \]  

(16)

where \(dx_n^2\) denotes the Ricci flat hypersurface. It is easy to check that the metric (16) is still a solution to the Einstein equations with a positive cosmological constant in \((n + 2)\) dimensions. From this solution we see that there is a Ricci flat cosmological horizon at 

\[ r = r_c = \left( \frac{2Gml^2}{n(n+1)} \right)^{1/(n+1)}. \]  

Also there exists a cosmological singularity at \(r = 0\) for \(n \geq 2\) and \(m \neq 0\). Therefore, this solution is a good example to check the BBM conjecture. For this cosmological horizon, we have the Hawking temperature \(T_{HK}\) and entropy \(S\),

\[ T_{HK} = \frac{(n + 1)r_c}{4\pi l^2}, \]

\[ S = \frac{r_c^n V}{4G}. \]  

(17)

When \(m = 0\), the solution (16) goes to

\[ ds^2 = -\frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} dt^2 + r^2 dx_n^2, \]  

(18)

in which \(t(r)\) becomes a spacelike (timelike) coordinate. In fact, this is a pure dS space:

One can rewrite the metric (18) as follows,

\[ ds^2 = -d\tau^2 + e^{\pm 2\tau/l} dx_{n+1}^2, \]  

(19)

where \(dx_{n+1}^2\) is an \((n + 1)\)-dimensional Ricci-flat space. This is just the dS space in the planar coordinates.

We now calculate the boundary stress-energy tensor and the mass of the solution (16). For \(r > r_c\), this solution can be rewritten as
\[ ds^2 = -f(r)^{-1}dr^2 + f(r)dt^2 + r^2dx_n^2, \quad f = \frac{r^2}{l^2} - \frac{2Gm}{r^{n-1}} > 0 \]  

(20)

in which \( t(r) \) becomes a spacelike (timelike) coordinate. Since \( dx_n^2 \) is a Ricci-flat space, therefore, for a hypersurface with a fixed \( r > r_c \), its induced metric, \( f(r)dt^2 + r^2dx_n^2 \), is also Ricci flat. Thus those counterterms involving the intrinsic curvature and Ricci tensor of the induced metric vanish, and the Lagrangian of the required surface counterterm for the solution (10) is

\[ \mathcal{L}_{ct} = \frac{n}{l}, \]  

(21)

and the boundary stress-energy tensor becomes

\[ T_{ij} = -\frac{1}{8\pi G}(K_{ij} - Kh_{ij} - \frac{n}{l}h_{ij}). \]  

(22)

Considering a surface \( \Sigma \) with fixed \( r > r_c \) in (20) and calculating its extrinsic curvature \( K_{ij} \), we obtain from (22)

\[ T_{tt} = \frac{nm}{8\pi lr^{n-1}} + \cdots, \]
\[ T_{ab} = -\frac{ml}{8\pi r^{n-1}}\delta_{ab} + \cdots, \]  

(23)

where the ellipses denote higher order terms, which have no contribution when we take the limit \( r \to \infty \) on the \( I^+ \).

Now we are in a position to calculate the mass of the solution (16). Substituting this boundary stress-energy tensor (23) into the mass formula (9), we find

\[ M = \frac{nmV}{8\pi}. \]  

(24)

When \( n = 1 \) and identifying the coordinate \( x \) in (16) with a circle with period \( 2\pi \), from (24) one has \( M = m/4 \), precisely reproducing the result in Ref. [1] for the mass of three dimensional Schwarzschild-dS solution. When \( m = 0 \), we have \( M = 0 \). This is consistent with the result obtained in [15] that the mass of the three-dimensional dS space vanishes in the planar coordinates. Our result (24) indicates that the mass of dS space vanishes (\( M_{dS} = 0 \)) in any dimension in the planar coordinates. When \( m \neq 0 \), we have \( M > M_{dS} = 0 \). According to the BBM conjecture [1], there should be a cosmological singularity. Indeed it is clear from (14) that there is a cosmological singularity at \( r = 0 \). As a result, we verify that the BBM conjecture holds in the solution (16). Furthermore, we can easily check that the mass \( M \) in (24), Hawking temperature \( T_{HK} \) and entropy \( S \) in (17) satisfy the first law of thermodynamics

\[ \text{Note the difference of notations used in this paper and in Ref. [1]: } m_{\text{here}} = 4m_{\text{there}}. \]

\[ \text{In the static coordinates, however, the three-dimensional dS space has a nonvanishing mass } M = 1/8G. \]  

[117]
Next we calculate the stress-energy tensor of the Euclidean CFT dual to the solution (16). As the asymptotically AdS case, the induced metric diverges when the boundary of dS space is approached. However, a surface metric on which the Euclidean CFT resides can be fixed, up to a conformal factor, from the bulk metric (20). For example, a simple surface metric can be obtained as follows,

\[ ds_{ECFT}^2 = \gamma_{ij} dx^i dx^j = \lim_{r \to \infty} \frac{l^2}{r^2} ds_{TdS}^2 = dt^2 + l^2 dx_n^2. \]  

Note that here \( t \) is a spacelike coordinate. The stress-energy tensor \( \tau_{ij} \) of the boundary Euclidean CFT can be obtained using the following relation [19]

\[ \sqrt{\gamma} \gamma^{ij} \tau_{jk} = \lim_{r \to \infty} \sqrt{h} h^{ij} T_{jk}. \]  

Substituting (23) into the above and using (26), one has

\[ \tau_{tt} = \frac{nm}{8 \pi l^n}, \quad \tau_{ab} = -\frac{m}{8 \pi l^{n-2}} \delta_{ab}. \]  

As expected, the trace of the stress-energy tensor vanishes.

(ii) The case of \( k = -1 \). In this case, changing the sign in front of \( m \) in (13), we have

\[ ds_{TdS}^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 \tilde{g}_{ab} dx^a dx^b, \]

where

\[ f(r) = -1 + \frac{2Gm}{r^{n-1}} - \frac{r^2}{l^2}. \]

Once again, when \( m > 0 \), this solution has a cosmological singularity at \( r = 0 \) and a cosmological horizon \( r_c \), which is a negative constant curvature hypersurface. In this sense, we refer to the solution (29) together with the solution (16) as topological dS solutions. The cosmological horizon of the solution (29) has the Hawking temperature \( T_{HK} \) and entropy \( S \)

\[ S = \frac{4 \pi l^n}{m}. \]

\[^4\]In the asymptotically AdS case, the induced metric also diverges when the boundary of AdS space is approached. But a well-defined surface metric on which the dual CFT resides can be determined, up to a conformal factor, from the bulk metric. The behavior of induced metric near the boundary, the normalizations of action and boundary stress-energy tensor, and the gravitational conformal anomaly have been analyzed in detail in [16]. For the asymptotically dS case, a similar analysis can be made as well, for example, see [18].

\[^5\]In general, there is a conformal anomaly for a CFT in even dimensions (for a review see [20]). In the case we are discussing, however, the spacetime background (23) is Ricci flat, therefore the conformal anomaly vanishes.
\[ T_{HK} = \frac{1}{4\pi r_c} \left( n - 1 + (n + 1) \frac{r_c^2}{l^2} \right), \]
\[ S = \frac{r_c^n V}{4G}. \]  \hspace{1cm} (31)

Since the \( \tilde{g}_{ab}dx^a dx^b \) is a negative constant curvature hypersurface, in this case, so the surface counterterm \((21)\) is not sufficient. The needed surface counterterms will depend on the spacetime dimension. We consider therefore the four- and five-dimensional cases below. In that case, the required surface counterterms are given in \((3)\), and corresponding boundary stress-energy tensor can be computed using \((6)\).

Repeating the steps as the case of \( k = 0 \) and using \((3)\), in four dimensions, we obtain the boundary stress-energy tensor
\[ T_{tt} = \frac{m}{4\pi rl} + \cdots, \]
\[ T_{ab} = -\tilde{g}_{ab} \frac{ml}{8\pi r} + \cdots. \]  \hspace{1cm} (32)

The mass of the solution is
\[ M_4 = \frac{mV}{4\pi}. \]  \hspace{1cm} (33)

And the stress-energy tensor of corresponding Euclidean CFT is
\[ \tau_{tt} = \frac{m}{4\pi l^2}, \]
\[ \tau_{ab} = -\tilde{g}_{ab} \frac{m}{8\pi}, \]  \hspace{1cm} (34)

which has a vanishing trace. The CFT resides on the three dimensional space with metric
\[ \gamma_{ij} dx^i dx^j = dt^2 + l^2 \tilde{g}_{ab} dx^a dx^b. \]  \hspace{1cm} (35)

In five dimensions, we find that it is quite different from the case of four dimensions. The boundary stress-energy tensor is
\[ T_{tt} = \frac{3l}{8\pi Gr^2} \left( \frac{1}{8} + \frac{Gm}{l^2} \right) + \cdots, \]
\[ T_{ab} = -\tilde{g}_{ab} \frac{l^3}{8\pi Gr^2} \left( \frac{1}{8} + \frac{Gm}{l^2} \right) + \cdots. \]  \hspace{1cm} (36)

The mass of the solution is
\[ M_5 = \frac{3V}{8\pi G} \left( \frac{l^2}{8} + Gm \right), \]  \hspace{1cm} (37)

and the stress-energy tensor of Euclidean CFT
\[ \tau_{tt} = \frac{3}{8\pi Gl} \left( \frac{1}{8} + \frac{Gm}{l^2} \right), \]
\[ \tau_{ab} = -\tilde{g}_{ab} \frac{l}{8\pi G} \left( \frac{1}{8} + \frac{Gm}{l^2} \right). \]  \hspace{1cm} (38)
The surface metric is of the form (35), but in four dimensions. Once again, the stress-energy tensor has a vanishing trace. From (37) we see that unlike the case (33) in four dimensions, the mass in five dimensions does not vanish even when \( m = 0 \). This is reminiscent of the difference between the four and five dimensional Schwarzschild-dS black hole solutions [1], there it is also found that there is a nonvanishing mass for the five dimensional pure dS space. But from (35) and (37) we see that the masses with \( m > 0 \) are always larger than those of dS solutions with \( m = 0 \). As a result, we confirm the BBM conjecture in the topological dS solution. Furthermore, these masses (35) and (37) also satisfy the first law (25) of thermodynamics.

IV. DILATONIC SOLUTION

The second example is a dilatonic deformation of the topological dS solution with a Ricci flat cosmological horizon (16). Consider the following action of a dilaton gravity theory,

\[
S = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \left( R - \frac{1}{2}(\partial \phi)^2 + V_0 e^{-a\phi} \right),
\]

where \( a \) and \( V_0 \) are assumed to be two positive constants. This action is an effective one for some gauged supergravity theories [21,22]. In [24] (see also [25,26] for the case in four dimensions), a class of domain wall black hole solutions has been found,

\[
ds^2_{DB} = -f(r)dt^2 + f(r)^{-1}dr^2 + R^2 dx_n^2,
\]

\[R(r) = r^N, \quad \phi(r) = \phi_0 + \sqrt{2nN(1 - N)\ln r},\]

\[f(r) = \frac{V_0 e^{-a\phi_0} r^{2N}}{nN(N(n + 2) - 1) - mr^{1-nN}}.\]

where \( \phi_0 \) and \( m \) are two integration constants. Also the notation \( dx_n^2 \) denotes the line element of an \( n \)-dimensional Ricci-flat space and the parameter \( N \) obeys the relation

\[a = \frac{\sqrt{2nN(1 - N)}}{nN}.\]

From (11) and (10) one can see that \( N \) must satisfy \( 1/(n + 2) < N \leq 1 \). For \( N = 1 \), the solution (40) precisely recovers the topological black hole with a Ricci flat horizon in AdS space [13]. For a general \( N \), the solution (40) is neither asymptotically AdS, nor asymptotically flat. When \( m = 0 \), the solution describes a domain wall spacetime where a domain wall/QFT (quantum field theory) correspondence [21], including the AdS/CFT correspondence in the horospherical coordinates as a special case, arises: a certain gauged

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6In four dimensions, in general there is a gravitational conformal anomaly [20,3,12] proportional to \((\mathcal{R}_{ij}\mathcal{R}^{ij} - \mathcal{R}^2/3)\), where \( \mathcal{R} \) and \( \mathcal{R}_{ij} \) are curvature scalar and Ricci tensor of surface metric, respectively. For the spacetime background (83), however, it is easy check that this conformal anomaly vanishes. This explains why the stress-energy tensor of dual CFT has a vanishing trace.
supergravity on the domain wall spacetime is dual to a QFT residing on the domain wall. For details, see [21, 22, 24]. It has been shown in [22] that one can also get a well-defined boundary stress-energy tensor by adding an appropriate surface counterterm for a class of solutions like (40), even though those solutions are not asymptotically AdS. The quasilocal boundary stress-energy tensor of the gravitational field and therefore the stress-energy tensor of corresponding QFT for the solution (40) have been acquired in [24]. It turns out that the integration constant $m$ in (40) is proportional to the mass $M$ of the black hole [24]:

$$M = \frac{nN}{\sqrt{2nN(1-N)}} \frac{mV}{16\pi G},$$

(42)

where $V$ is the volume of the Ricci flat space $dx_n^2$ in (40).

We now turn to the case $V_0 < 0$ in the action (39). In this case, the action still can be viewed as an effective truncation of a certain gauged supergravity, for example, see [23]. For convenience, we make a replacement: $V_0 \to -V_0$ in (39) so that we still have $V_0 > 0$ in the following. Then it is easy to check the new action has still a solution like (40), but with a new function $f$:

$$f(r) = \frac{mr^{1-nN}}{\sqrt{2nN(1-N)}} - \frac{V_0 e^{-a\phi_0} r^{2N}}{nN(N(n+2)-1)},$$

(43)

and others keep unchanged. Here we have changed the sign in front of the integration constant $m$ in (40). For the case of $N = 1$, the new solution (43) reduces to the $k = 0$ topological dS solution considered above. For a general $N < 1$, this solution is not asymptotically de Sitter. But it has a cosmological horizon $r_c$

$$r_c = \left( \frac{nNm(N(n+2)-1)}{V_0 e^{-a\phi_0} \sqrt{2nN(1-N)}} \right)^{1/(n+2)(N-1)}.$$

(44)

And thus it has associated Hawking temperature $T_{HK}$ and entropy $S$ as

$$T_{HK} = \frac{V_0 e^{-a\phi_0} r_c^{2N-1}}{4\pi nN},$$

$$S = \frac{r_c^{nN} V}{4G}.$$

(45)

Furthermore, we note that there exists a cosmological singularity at $r = 0$ in the solution (43). Although the solution is not asymptotically de Sitter, we find that one can get a well-defined quasilocal stress-energy tensor of gravitational field for the solution (43) by adding an appropriate surface counterterm to the bulk action. The surface counterterm is given by

$$\mathcal{L}_{ct} = \frac{n}{l_{eff}} \sqrt{\frac{N(n+1)}{N(n+2)-1}}, \quad \frac{1}{l_{eff}} = \sqrt{\frac{V_0 e^{-a\phi}}{n(n+1)}},$$

(46)

And then the boundary stress-energy tensor is
\[ T_{ij} = -\frac{1}{8\pi G} \left( K_{ij} - Kh_{ij} - \frac{n}{l_{\text{eff}}} \left( \frac{N(n+1)}{N(n+2)} - 1 \right) h_{ij} \right). \] (47)

Similarly we obtain the boundary stress-energy tensor of gravitational field on the surface \( \Sigma \) with a fixed \( r > r_c \),

\[ T_{tt} = \frac{nNmc^{1/2}r^{-(n-1)N}}{16\pi G \sqrt{2nN(1-N)}} + \cdots, \]
\[ T_{ab} = -\delta_{ab} \frac{(2N-1)mr^{-(n-1)N}}{16\pi Gc^{1/2} \sqrt{2nN(1-N)}} + \cdots, \] (48)

where the ellipses represent higher order terms which have no contribution when we take the limit \( r \to \infty \) on \( I^+ \). The constant \( c \) is

\[ c = \frac{V_0e^{-\phi_0}}{nN(N(n+2)-1)}. \] (49)

Using the mass formula (9) and the boundary stress-energy tensor (48), we obtain the mass of the solution (43)

\[ M = \frac{nNmV}{16\pi G \sqrt{2nN(1-N)}}. \] (50)

We find that the mass (50) of the dilatonic deformation (43) has the same form as the domain-wall black hole solution (44). Furthermore, we can see that the mass in (50), the Hawking temperature \( T_{\text{HK}} \) and entropy in (45) satisfy the first law (23) of thermodynamics. From (50) one has \( M_{\text{vac}} = 0 \), for the vacuum state \( (m = 0) \) in the solution (43). Thus \( M > M_{\text{vac}} = 0 \), showing that the BBM conjecture is also satisfied with the dilatonic deformation of the topological de Sitter solution, even though the solution is not asymptotically de Sitter.

Since the solution (43) is not asymptotically de Sitter, we do not expect that the dual is a Euclidean CFT. Instead we expect that there is a Euclidean QFT dual to the solution (43). This correspondence is an analog of the domain wall/QFT correspondence in the spacetime with a cosmological horizon. In the correspondence, we can calculate the stress-energy tensor of the QFT dual to the solution (43). As the asymptotically dS case, the surface metric \( \gamma_{ij} \) of the spacetime, on which the QFT resides, can be determined, up to a conformal factor, as follows,

\[ ds^2_{\text{EQFT}} = \gamma_{ij}dx^i dx^j = \lim_{r \to \infty} \frac{1}{r^{2N}} ds^2 = c dt^2 + dx^2_n, \] (51)

where \( t \) is a spacelike coordinate and \( c \) is given in (49). Using (48), we obtain

\[ \tau_{tt} = \frac{nNmc^{1/2}}{16\pi G \sqrt{2nN(1-N)}}, \]
\[ \tau_{ab} = -\delta_{ab} \frac{(2N-1)m}{16\pi Gc^{1/2} \sqrt{2nN(1-N)}}. \] (52)
As expected, its trace does not vanish unless $N = 1$. In the case of $N = 1$, the solution (43) reduces to the $k = 0$ topological dS solution, to which one has a Euclidean CFT dual. Note that for the $N = 1$ case, those ill-defined expressions can be remedied by redefining the integration constant $m$: for example, one can absorb the factor $\sqrt{1 - N}$ into the $m$.

V. CONCLUSION AND DISCUSSION

The dS space is the unique maximally symmetric curved spacetime. It enjoys the same degree of symmetry as Minkowski space. So it has been most studied by quantum field theorists (see [27] and references therein). On the other hand, the recent astronomical data of supernova [28–30] together with the need of the inflation model in the cosmology of early universe indicate that our universe approaches dS geometries in both the far past and the far future [31–33]. Moreover, it has been proposed recently that there is a dual between quantum gravity on a dS space and a Euclidean CFT residing on a boundary of the dS space [8,9], very like the AdS/CFT correspondence. Therefore both the dS space itself and the realistic universe motivate us to well understand the dS space (including asymptotically dS spaces). As a first step, one has to compute some conserved charges like mass and angular momentum associated with asymptotically dS spaces. However, as stated in INTRODUCTION, it is not an easy matter to obtain those conserved charges in asymptotically dS spaces because of the absence of spatial infinity and globally timelike Killing vector. In Ref. [1] a novel prescription has been proposed to calculate those conserved charges from data at early or late time infinity. And it has been found that the masses of pure dS spaces are always larger than those of Schwarzschild-dS black holes in the corresponding dimensions. The interesting result is consistent with the observation [10] that the entropy of pure dS space is an upper bound of any asymptotically dS spaces if one accepts the dS/CFT correspondence, because generically field theories should have entropies increasing with energy. The interesting result also leads to the BBM conjecture. If the BBM conjecture is correct, no doubt it is a very important feature of asymptotically dS spaces.

In order to check the BBM conjecture, we have presented two solutions, the topological dS solution and its dilatonic deformation. Both have a cosmological horizon and a cosmological singularity. Using surface counterterm method, we have calculated the boundary quasilocal stress-energy tensor of gravitational field and obtained the masses for both solutions. The resulting masses are always positive, while the masses of dS space and its dilatonic deformation vanish in the planar coordinates. Although the mass (37) of dS space in five dimensions does not vanish for the case $k = -1$, in all cases we considered in this paper, we have verified the BBM conjecture. Thus we have provided evidence in favor of the BBM conjecture.

Even though the dilatonic deformation of the topological dS solution is not asymptotically de Sitter, we expect that there exists a dual Euclidean QFT. This correspondence is considered as an analog of the domain wall/QFT correspondence [21]: quantum gravity on the background (13) is dual to a certain Euclidean QFT residing on the space (51). Thus we can view this correspondence as a Euclidean version of the domain wall/CFT correspondence. The Euclidean domain wall/QFT correspondence includes the dS/CFT correspondence in the planar coordinates as a special case, in the same way as that the AdS/CFT correspondence in the horospherical coordinates comes out as a special case in the domain
wall/QFT correspondence [21]. This Euclidean domain wall/QFT correspondence is a non-conformal extension of the dS/CFT correspondence. According to this correspondence, we have obtained the stress-energy tensor of corresponding Euclidean QFTs. Also we have calculated some thermodynamic quantities associated with the cosmological horizon of these solutions, and verified that they all obey the first law of thermodynamics.

If the BBM conjecture indeed holds, then an interesting question is what its implications are. The BBM conjecture says that any asymptotically de Sitter space whose mass exceeds that of pure de Sitter space contains a cosmological singularity. From another point of view to see it, it gives us an upper bound of mass for any asymptotically dS space without a cosmological singularity. Together with the dS/CFT correspondence and the Bousso’s observation about the maximal entropy bound for any asymptotically dS space, the BBM conjecture seems to imply that the maximal entropy bound must be violated in any asymptotically dS space with a cosmological singularity. Thus, the BBM conjecture further implies that some energy conditions must be also violated for asymptotically dS space with a cosmological singularity. Therefore the BBM conjecture might be quite useful to well understand the creation and fate of our realistic universe. In addition, there might exist CFTs dual to some asymptotically dS spaces with cosmological singularity. For such CFTs, there is no maximal entropy bound. Some energy conditions might be violated for these exotic CFTs. Therefore a well understanding of the BBM conjecture will provide help to establish the dS/CFT correspondence. However, the understanding so far gained to the conjecture is obviously incomplete. It is to be expected to reveal the deep implications of the conjecture in a near future.

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