A REMARK ON SOLITON EQUATION OF MEAN CURVATURE FLOW

LI MA AND Y.YANG

Abstract. In this short note, we consider self-similar immersions $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$ of the Graphic Mean Curvature Flow of higher co-dimension. We show that the following is true: Let $F(x) = (x, f(x)), x \in \mathbb{R}^n$ be a graph solution to the soliton equation

$$\overline{H}(x) + F_{\perp}(x) = 0.$$ 

Assume $\sup_{\mathbb{R}^n} |Df(x)| \leq C_0 < +\infty$. Then there exists a unique smooth function $f_\infty : \mathbb{R}^n \rightarrow \mathbb{R}^k$ such that

$$f_\infty(x) = \lim_{\lambda \to \infty} f_\lambda(x)$$

and

$$f_\infty(rx) = rf_\infty(x)$$

for any real number $r \neq 0$, where

$$f_\lambda(x) = \lambda^{-1} f(\lambda x).$$

1. Introduction

Let $M^{n+k}$ be a Riemannian manifold of dimension $n + k$. Assume that $\Sigma^n$ be a Riemannian manifold of dimension $n$ without boundary. Let $F : \Sigma^n \rightarrow M^{n+k}$ be an isometric immersion. Denote $\nabla$ (respectively $D$) the covariant differentiation on $\Sigma$ (on $M$). Let $T\Sigma$ and $N\Sigma$ be the tangent bundle and normal bundle of $\Sigma$ in $M$ respectively. We define the second fundamental form of the immersion $\Sigma$ by

$$II : T\Sigma \times T\Sigma \rightarrow N\Sigma,$$

with

$$II(X, Y) = D_X Y - \nabla_X Y,$$

for tangential vector fields $X, Y$ on $\Sigma$. We define the mean curvature vector field (in short, MCV) by

$$\overline{H} = \text{tr}_\Sigma II.$$

In recent years, many people are interested in studying the evolution of the immersion $F : \Sigma^n \rightarrow M^{n+k}$ along its Mean Curvature Flow (in short, just say MCF). The MCF is defined as follows. Given an one-parameter family of sub-manifolds $\Sigma_t = F_t(\Sigma)$ with immersions $F_t : \Sigma \rightarrow M$. Let $\overline{H}(t)$ be the MCV of $\Sigma_t$. Then our MCF is the equation/system

$$\frac{\partial F(x, t)}{\partial t} = \overline{H}(x, t).$$

This flow has many very nice results if the codimension $k = 1$. See the work of G.Huisken [3] for a survey in this regard. Since there is very few result about MCF in higher codimension, we will study it in the target when $M^{n+k} = \mathbb{R}^{n+k}$, which is the standard Euclidian space.

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In this short note, we will consider a family of self-similar graphic immersions \( F(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k} \) of the Mean Curvature Flow (MCF):

\[
\frac{\partial}{\partial t} F(x, t) = \vec{H}(x, t), \quad \forall x \in \mathbb{R}^n, \quad \forall t \in (-\infty, 0).
\]

Write

\[ \Sigma_t = F(\mathbb{R}^n, t), \]

and

\[ F = (F^A), \quad 1 \leq A \leq n + k. \]

By definition, we call the family \( \Sigma_t \) self-similar if

\[ \Sigma_t = \sqrt{-t} \Sigma_{-1}, \quad \forall t < 0. \]

In this case, we can reduce the MCF into an elliptic system. In the other word, we have the following parametric elliptic equation for the family \( \Sigma_t \):

\[
\overline{H}(x) + F^\perp(x) = 0, \quad \forall x \in \Sigma_{-1} := \Sigma.
\]

We will call this system as the soliton equation of the MCF. Note that this equation is usually obtained from the monotonicity formula of G.Huisken [2] for blow-up. It is a hard and open problem to classify solutions of this equation.

Fix \( \Sigma = \Sigma_t \). Assume that \( F(x) = (x, f(x)) \). Let

\[ Q = (Q^A_\alpha), \quad n + 1 \leq \alpha \leq n + k \quad 1 \leq A \leq n + k \]

is the orthogonal projection onto \( N_p \Sigma \), where \( p \in \Sigma \). Then the second fundamental form of \( \Sigma \) can be written as

\[ \Pi^A_{ij} = Q^A_\alpha D^2_{ij} f^\alpha. \]

Hence, we have the expression for the mean curvature vector of \( \Sigma \) in \( \mathbb{R}^{n+k} \):

\[
\overline{H}^A = g^{ij} Q^A_\alpha D^2_{ij} f^\alpha.
\]

Our main result in this paper is the following

**Theorem 1.1.** Let \( F(x) = (x, f(x)), x \in \mathbb{R}^n \) be a graph solution to the soliton equation

\[
\overline{H}(x) + F^\perp(x) = 0.
\]

Assume \( \sup_{\mathbb{R}^n} |Df(x)| \leq C_0 < +\infty \). Then there exists a unique smooth function \( f_\infty : \mathbb{R}^n \rightarrow \mathbb{R}^k \) such that

\[ f_\infty(x) = \lim_{\lambda \rightarrow \infty} f_\lambda(x) \]

and

\[ f_\infty(rx) = rf_\infty(x) \]

for any real number \( r \neq 0 \), where

\[ f_\lambda(x) = \lambda^{-1} f(\lambda x). \]

We remark that the proof of this result given below is very simple. But it is based on a nice observation. We just use the divergence theorem with a nice test function. In the next section, we recall the form of divergence theorem for convenient of the readers. In the last section we give a proof of our Theorem.

We point out that we may consider \( F_\infty(x) = (x, f_\infty(x)) \) obtained above as a tangential minimal cone along the research direction done by L.Simon [5].
2. Preliminary

Given a vector field \( X : \Sigma \rightarrow TM \). Let \( X^T \) and \( X^N \) denote the projection of \( X \) onto \( T \Sigma \) and \( N \Sigma \) respectively. We define the divergence of \( X \) on \( \Sigma \) as

\[
\div_{\Sigma} X = \sum g^{ij} \langle D_i X, \frac{\partial}{\partial x^j} \rangle
\]

where \((g^{ij}) = g^{-1}_{ij}\), and \((g^{ij})\) is the induced metric tensor written in local coordinates \((x^i)\) on \( \Sigma \).

Note that, for any tangential vector field \( Y \) on \( \Sigma \),

\[
D_Y X = D_Y X^T + D_Y X^N.
\]

So

\[
\langle D_Y X, Y \rangle = \langle D_Y X^T, Y \rangle + \langle D_Y X^N, Y \rangle
\]

\[
= \langle \nabla_Y X^T, Y \rangle - \langle D_Y Y, X^N \rangle
\]

\[
= \langle \nabla_Y X^T, Y \rangle - \langle II(Y, Y), X \rangle.
\]

Hence

\[
\div_{\Sigma} X^T = \div_{\Sigma} X + \langle X, \Pi \rangle,
\]

and by the Stokes formula on \( \Sigma \), we have

\[
\int_{\Sigma} \div X^T = \int_{\partial \Sigma} \langle X, \nu \rangle d\sigma
\]

and

\[
\int_{\Sigma} \div_{\Sigma} X d\nu = - \int_{\Sigma} \langle \Pi, X \rangle d\nu + \int_{\partial \Sigma} \langle X, \nu \rangle d\sigma,
\]

where \( \nu \) is the exterior normal vector field to \( \Sigma \) on \( \partial \Sigma \).

3. Proof of Main Theorem

In the following, we take \( M^{n+k} = \mathbb{R}^{n+k} \) as the standard Euclidean space. We assume that the assumption of our Theorem 1.1 is true in this section.

Define the vector field

\[
X = (1 + |F|)^{-s} F
\]

where \( s \in \mathbb{R} \) to be determined.

Note that, \( \nabla |F| = \frac{F}{|F|} \) and \( \div_{\Sigma} F = n \). So

\[
\div_{\Sigma} X = \langle \nabla (1 + |F|)^{-s}, F \rangle + (1 + |F|)^{-s} \div_{\Sigma} F
\]

\[
= -s(1 + |F|)^{-s-1} \frac{|F|^2}{|F|} + n(1 + |F|)^{-s}.
\]

Locally, we may assume that \( \Sigma \) is a graph of the form \( (x, f(x)) \in B_R(0) \times \mathbb{R}^k \), where \( B_R(0) \) is the ball of radius \( R \) centered at 0. Let \( \Sigma_R = \Sigma \cap (B_R(0) \times \mathbb{R}^k) \). By the divergence theorem we have

\[
\int_{\Sigma_R} \div_{\Sigma} X = \int_{\Sigma_R} \langle \Pi, X \rangle - \int_{\partial \Sigma_R} \langle X, \nu \rangle
\]
By direct computation, we have that
\[
\int_{\Sigma_R} \text{div}_\Sigma X = -s \int_{\Sigma_R} \frac{(1 + |F|)^{-s-1}}{|F|} |F^\top|^2 + n \int_{\Sigma_R} (1 + |F|)^{-s} \\
= -\int_{\Sigma_R} (1 + |F|)^{-s} |F^\top|^2 - \int_{\partial \Sigma_R} (1 + |F|)^{-s} \langle F, \nu \rangle \\
= -\int_{\Sigma_R} (1 + |F|)^{-s} |F^\top|^2 - \int_{\partial \Sigma_R} (1 + |F|)^{-s} \langle F, \nu \rangle.
\]
Hence, we have
\[
\int_{\Sigma_R} (1 + |F|)^{-s} |F^\top|^2 = s \int_{\Sigma_R} \frac{(1 + |F|)^{-s-1}}{|F|} |F^\top|^2 - n \int_{\Sigma_R} (1 + |F|)^{-s} - \int_{\partial \Sigma_R} (1 + |F|)^{-s} \langle F, \nu \rangle.
\]
Since \(|F^\top| \leq |F| \leq 1 + |F|\), we have
\[
\int_{\Sigma_R} \frac{(1 + |F|)^{-s-1}}{|F|} |F^\top|^2 \leq \int_{\Sigma_R} (1 + |F|)^{-s}.
\]
Clearly we have
\[
\left| \int_{\partial \Sigma_R} (1 + |F|)^{-s} \langle F, \nu \rangle \right| \leq \int_{\partial \Sigma_R} (1 + |F|)^{-s}.
\]
Combining these two inequalities together we get
\[
\int_{\Sigma_R} (1 + |F|)^{-s} |F^\top|^2 \leq (s - n) \int_{\Sigma_R} (1 + |F|)^{-s} + \int_{\partial \Sigma_R} (1 + |F|)^{1-s}.
\]
Choosing \(s = n\) yields (*):
\[
\int_{\Sigma_R} (1 + |F|)^{-n} |F^\top|^2 \leq \int_{\partial \Sigma_R} (1 + |F|)^{1-n}.
\]
By our assumption we have that \(\exists C > 0\) such that for \(F(x) = (x, f(x))\) on \(\Sigma = \mathbb{R}^n\), we have
\[
\det(I + (df)^\top df) \leq C
\]
on \(\Sigma\). Since
\[
g_{ij} = \delta_{ij} + D_i f^\alpha \cdot D_j f^\alpha,
\]
we know that
\[
I \leq (g_{ij}) \leq CI.
\]
Hence
\[
(1 + |x|) \leq (1 + |F(x)|) \leq C(1 + |x|).
\]
Therefore we get from (*) the key estimate (K):
\[
\int_{B_R(0)} (1 + |x|)^{-n} |\Pi|^2 dx \leq C \int_{\partial B_R(0)} (1 + |x|)^{1-n} \leq C.
\]
We now go to the proof of our Theorem.

**Proof.** Note that the mean curvature flow for the graph of \(f\) can be read as
\[
\frac{\partial f^\alpha}{\partial t} = g^{ij} D_i f^\alpha \cdot D_j f^\alpha, \quad \alpha = 1, \ldots, k.
\]
The important fact about this equation is that it is invariant under the transformation
\[
f(x) \rightarrow \frac{1}{\lambda} f(\lambda x), \forall \lambda > 0.
\]
Compute
\[
\frac{d}{d\lambda} f_{\lambda}(x) = -\lambda^{-2}f(\lambda x) + \lambda^{-1}Df(\lambda x) \cdot x \\
= \lambda^{-2}Df(\lambda x) \cdot \lambda x - f(\lambda x) \\
= \lambda^{-2}(\langle Df(\lambda x), -1 \rangle, (\lambda x, f(\lambda x))) \\
= \lambda^{-2}(\langle Df(\lambda x), -1 \rangle, F(\lambda x)) \\
= \lambda^{-2}(\langle Df(\lambda x), -1 \rangle, F(\lambda x)^\perp).
\]

Here we have used the fact that
\[
(Df(\lambda x), -1)^\perp \Sigma.
\]

So
\[
\frac{d}{d\lambda} f_{\lambda}(x) = \lambda^{-2}\langle (-Df(\lambda x), 1), H \rangle.
\]

Hence
\[
\left| \frac{d}{d\lambda} f_{\lambda}(x) \right| \leq C\lambda^{-2}|H|.
\]

So, for \(x \in S^{n-1}\), we have
\[
|f_{\lambda}(x) - f_{\mu}(x)| \leq C \int_{\lambda}^{\mu} \frac{\|P(\lambda x)\|^2}{\sigma^2} d\sigma \\
\leq C\int_{\lambda}^{\mu} \frac{1}{\sigma^3} d\sigma)(\int_{\lambda}^{\mu} \frac{\|P(\sigma x)^2\sigma d\sigma}{\sigma}) \\
\leq C|\mu^{-2} - \lambda^{-2}| \int_{\lambda}^{\mu} \frac{P(\sigma x)^2}{\sigma} d\sigma.
\]

Notice that, for \(\mu \geq \lambda > 1\),
\[
\int_{S^{n-1}} dx \int_{\lambda}^{\mu} \frac{P(\sigma x)^2}{\sigma} d\sigma \leq \int_{0}^{\infty} \int_{S^{n-1}} \frac{P(\sigma x)^2}{(1 + \sigma)^n} \sigma^{n-1} dxd\sigma \leq C.
\]

The last inequality follows from the inequality (K). Therefore, we have the estimate (**):
\[
\int_{S^{n-1}} |f_{\lambda}(x) - f_{\mu}(x)|^2 dx \leq C|\mu^{-2} - \lambda^{-2}|.
\]

This implies that \((f_{\lambda})\) is a Cauchy sequence in \(L^2(S^{n-1})\). Let \(f_{\infty}\) be its unique limit. Since \(\sup_{\mathbb{R}^n} |Df_{\lambda}| = \sup_{\mathbb{R}^n} |Df| \leq C_0\), the Arzela-Ascoli theorem tells us that \((f_{\lambda})\) is compact in \(C^\alpha(S^{n-1}), \forall \alpha \in (0, 1)\). Therefore
\[
f_{\infty}(x) = \lim f_{\lambda}(x) \quad \text{uniformly on } S^{n-1},
\]
and
\[
f_{\infty}(rx) = rf_{\infty}(x), \quad \forall 0 \neq r \in \mathbb{R}.
\]

This finishes the proof of Theorem 1.1

In the following, we pose a question about the stability of self-similar solutions of (MCF). Let \(f_0: \mathbb{R}^n \to \mathbb{R}^k\) be a smooth function with uniformly bounded (Lipschitz) gradient. Assume
\[
\lim_{\lambda \to \infty} f_{0\lambda} = f_{0}^\infty, \quad \text{uniformly on } S^{n-1}.
\]
Assume \( f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^k \) such that \( F(x, t) = (x, f(x, t)) \) is a solution of (MCF) with the initial data \( F(x, 0) = (x, f_0(x)) \). We ask if there is a smooth mapping \( \hat{f} : \mathbb{R}^n \to \mathbb{R}^k \) such that \( \hat{f}(\cdot, s) \to \hat{f}(\cdot) \) uniformly on compact subsets of \( \mathbb{R}^n \) as \( s \to \infty \). Here \( \hat{f} \) is defined by

\[
\hat{f}(x, s) = t^{-\frac{1}{2}} f(\sqrt{t} x, t), \quad s = \frac{1}{2} \log t, \quad 0 \leq s < \infty \quad \text{with} \quad t \geq 1.
\]

A related stability result is done by one of us in [4].

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Department of Mathematics, Tsinghua University, Beijing,100084,China

E-mail address: lma@math.tsinghua.edu.cn