Ergodic Approach for Nonconvex Robust Optimization Problems

Pedro Pérez-Aros

Received: date / Accepted: date

Abstract In this work we show the consistency of an approach for solving robust optimization problems using sequences of sub-problems generated by ergodic measure preserving transformations.

The main result of this paper is that the minimizers and the optimal value of the sub-problems converge, in some sense, to the minimizers and the optimal value of the initial problem, respectively. Our result particularly implies the consistency of the scenario approach for nonconvex optimization problems. Finally, we show that our method can be used to solve infinite programming problems.

Keywords Stochastic optimization · scenario approach · robust optimization · epi-convergence · ergodic theorems.

Mathematics Subject Classification (2010) MSC 90C15 · 90C26 · 90C90 · 60B11

1 Introduction

Robust optimization (RO) corresponds to a field of optimization dedicated to the study of problems under uncertainty. In this class of models, the constraint set is given by the set of points, which satisfy all (or in the presence of measurability the almost all) possible cases. Roughly speaking, an RO problem corresponds to the following mathematical optimization model

$$\begin{align*}
\min & \quad g(x) \\
\text{s.t.} & \quad x \in M(\xi), \quad \text{almost surely } \xi \in \Xi,
\end{align*}$$

(1)

where $X$ is a Polish space, $(\Xi, \mathcal{A}, \mathbb{P})$ is a probability space, $M : \Xi \rightrightarrows X$ is a measurable multifunction with closed values and $g : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function. We refer to [4, 5, 17, 20, 22] and the references therein for more details and applications.

When the amount of possible scenarios $\xi \in \Xi$ is infinite in Problem (1), the computation of necessary and sufficient optimality conditions presents difficulties and require a more delicate
analysis than a simpler optimization problem. As far as we know, only works related to infinite programming deals directly with infinite-many constraints (see, e.g., [13,10] and the references therein). For that reason, it is necessary to solve an approximation of Problem (1). In this direction, the so-called *scenario approach* emerges as a possible solution. The *scenario approach* corresponds to a min-max approximation of the original robust optimization problem using a sequence of samples, it has used to provide a numerical solution to convex and nonconvex optimization problems (see, e.g., [8–10]). Furthermore, recently the consistency of this method was provided in [7] for convex optimization problems.

The intention of this work is to provide the consistency of the following method used to solve RO problems: Considers an *ergodic measure preserving transformation* $T : \Xi \rightarrow \Xi$, then systematically one solves the sequence of optimizations problems

$$\begin{align*}
\min g_n(x) \\
\text{s.t. } x \in M(T^n(\xi)); \quad i = 1, \ldots, n,
\end{align*}$$

(2)

where $g_n$ is a sequence of functions, which converge continuously to the objective function $g$. Here, the desired conclusion is that the optimal value and the minimizers of (2) converge, in some sense, to the optimal value and the minimizers of (1) for almost all possible choices of $\xi \in \Xi$. This conclusion is established in Corollary 3.1 which follows directly from our main result Theorem 3.1.

The key point in our results is to make a connection among the ideas of *scenario approach*, an *ergodic theorem for random lower semicontinuous functions* established in [1, Theorem 1.1], and the *theory of epi-convergence*. After that, and due to the enormous developments in the theory of epi-convergence (see, e.g., [2,19]), we can quickly establish some link between the minimizers and the optimal value of the robust optimization problem (1) and its corresponding approximation (2).

As a consequence of this method we obtain the consistency of the *scenario approach* for nonconvex optimization problems. More precisely, in this method one considers a drawing of independent and $\mathbb{P}$-distributed random function $\xi_1, \xi_2, \ldots$, and systematically solves the sequence of optimizations problems.

$$\begin{align*}
\min g_n(x) \\
\text{s.t. } x \in M(\xi_i); \quad i = 1, \ldots, n,
\end{align*}$$

(3)

again, the conclusion relies in showing that the optimal value and the minimizers of (3) converges to the solution of (1) for almost all possible sequences ($\xi_1, \xi_2, \ldots$).

It is worth mentioning that our method allows us to solve nonconvex optimization problems and to consider a perturbation of the objective function $g$ in (2) and (3), which is not guaranteed by the results of [7]. Here, it has not escape our notice that the perturbation over $g$ could be useful to ensure smoothness of the objective function in (2) and (3). On the other hand, the functions $g_n$ could be used to guarantee the existence and uniqueness of the numerical solutions of (2) and (3).

The rest of the paper is organized as follow: In Section 2 we summarize the main definitions and notions using in the presented manuscript. Posteriorly, we divide Section 3 into two subsections. The first one gives us a generalization of the *scenario approach* using *ergodic measure preserving transformation* instead of a sequence of independent and identically $\mathbb{P}$-distributed random functions. The second one aims to give a direct proof of the consistency of the *scenario approach* for nonconvex optimization problem using the results of Subsection 3.1. Finally, in Section 4 we show that our ergodic approach can be applied to problems related with infinite programming.
2 Notation and Preliminary

In the following, we consider that \((X, d)\) is a Polish space, that is to say, a complete separable metric space and \((\Xi, A, P)\) is a complete probability space. The Borel \(\sigma\)-algebra on \(X\) is denoted by \(\mathcal{B}(X)\), which we recall that is the smallest \(\sigma\)-algebra containing all open set of \(X\).

For a function \(f : X \to \mathbb{R} \cup \{+\infty\}\) and \(\alpha \in \mathbb{R}\) we define the \(\alpha\)-sublevel set as

\[
\text{lev}_{\leq \alpha} f := \{ x \in X : f(x) \leq \alpha \}.
\]

We say that \(f\) is lower semicontinuous (lsc) if for all \(\alpha \in \mathbb{R}\) the \(\alpha\)-sublevel set is closed. Following [2], let us define the \(\varepsilon\)-infimal value of \(f\), given for \(\varepsilon \geq 0\)

\[
v_\varepsilon(f) := \begin{cases} 
\inf_X f + \varepsilon, & \text{if } \inf_X > -\infty, \\
-\frac{1}{\varepsilon}, & \text{if } \inf_X = -\infty.
\end{cases}
\]

with the convention \(\frac{1}{0} = +\infty\), when \(\varepsilon = 0\) we simply write \(v(f)\). Furthermore, we define the \(\varepsilon\)-argmin of \(f\) by

\[
\varepsilon\text{-argmin}_X f := \{ x \in X : f(x) \leq v_\varepsilon(f) \},
\]

the especial case \(\varepsilon = 0\) is simply denoted by \(\text{argmin}_X f\) and we omit the symbol \(X\) when there is no confusion.

For a set \(A \subseteq X\), we define the indicator function of \(A\), given by,

\[
\delta_A(x) := \begin{cases} 
0, & \text{if } x \in A, \\
+\infty, & \text{if } x \notin A.
\end{cases}
\]

A function \(f : \Xi \times X \to \mathbb{R} \cup \{+\infty\}\) is called a random lower semicontinuous function (also called a normal integrand function) if

(i) the function \((\xi, x) \to f(\xi, x)\) is \(A \otimes \mathcal{B}(X)\)-measurable, and

(ii) for every \(\xi \in \Xi\) the function \(f_\xi := f(\xi, \cdot)\) is lsc.

Let us consider a set-valued map (also called a multifunction) \(M : \Xi \rightrightarrows X\). We said that \(M\) is measurable if for every open set \(U \subseteq X\) the set

\[
M^{-1}(U) := \{ x \in X : M(x) \cap U \neq \emptyset \} \in A.
\]

For more details about the theory of normal integrand and measurable multifunctions we refer to [3, 11, 14, 19].

Consider a sequence of sets \(S_n \subseteq X\), we set \(\liminf_{n \to \infty} S_n\) and \(\limsup_{n \to \infty} S_n\) as the inner-limit and the outer-limit, in the sense of Painlevé-Kuratowski, of the sequence \(S_n\), respectively, that is to say,

\[
\liminf_{n \to \infty} S_n := \left\{ x \in X : \limsup_{n \to \infty} d(x, S_n) = 0 \right\}
\]

\[
\limsup_{n \to \infty} S_n := \left\{ x \in X : \liminf_{n \to \infty} d(x, S_n) = 0 \right\},
\]

where \(d(x, S_n) := \inf \{ d(x, y) : y \in S_n \}\).

Now, let us recall some notations about convergence of functions.
**Definition 2.1** Let $f_n : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functions. The functions $f_n$ are said to epi-converge to $f$, denoted by $f_n \xrightarrow{e} f$, if for every $x \in X$

a) $\liminf_{n \to \infty} f_n(x_n) \geq f(x)$ for all $x_n \to x$.

b) $\limsup_{n \to \infty} f_n(x_n) \leq f(x)$ for some $x_n \to x$.

We refer to [2, 19] for more details about the theory of epigraphical convergence, and several properties about the convergence of infimal value and the argmin of the approximate sequence of functions with respect to the minimizers of the epigraphical limit.

Also, we will need the following notation, which is equivalent to uniform convergence over compact sets for continuous functions (see, e.g., [19]).

**Definition 2.2** We say that a sequence of functions $f_n : X \to \mathbb{R} \cup \{+\infty\}$ converges continuously to $f$, if for every $x \in X$ and every $x_n \to x$

$$\lim_{n \to \infty} f_n(x_n) = f(x).$$

The following definition is an extension of the notation *eventually level-bounded* used in finite-dimension, which can be found in [19] Chapter 7.E. We extend this notation as following: We say that a sequence of functions $f_n$ is *eventually level-compact*, if for each $\alpha \in \mathbb{R}$ there exists $n_\alpha \in \mathbb{N}$ such that

$$\bigcup_{n \geq n_\alpha} \text{lev}_{\leq \alpha} f_n \text{ is relatively compact.}$$

Now, let us rewrite some of the previous notions in terms of optimization problems. Consider

$$\min c_n(x) \quad \text{s.t. } x \in C_n. \quad (P_n)$$

We define the $\varepsilon$-optimal value of each $(P_n)$ given by $v_\varepsilon(P_n) := v_\varepsilon(f_n)$, where $f_n := c_n + \delta_{C_n}$ and the

$$\varepsilon\text{-argmin}\,(P_n) := \varepsilon\text{-argmin}\,f_n.$$ 

Furthermore, we say that the family of optimization problems if *eventually level-compact*, if for each $\alpha \in \mathbb{R}$ there exists $n_\alpha \in \mathbb{N}$ such that

$$\bigcup_{n \geq n_\alpha} \{ x \in C_n : c_n(x) \leq \alpha \} \text{ is relatively compact},$$

which is terms of the functions $f_n$ is equivalent to say that the sequence $f_n$ is *eventually level-compact*.

The following lemma shows that the sum of an epi-convergence sequence and a continuously converge sequence still epi-convergences to the sum of limits.

**Lemma 2.1** Consider sequences $p_n, q_n : X \to \mathbb{R} \cup \{+\infty\}$ such that $p_n$ converges equicontinuously $p$ and $q_n$ epi-converges to $q$. Then, $p_n + q_n \xrightarrow{e} p + q$. 
Proof Consider $x \in X$ and a sequence $x_n \to x$, then
\[
\liminf_{n \to \infty} (p_n + q_n)(x_n) \geq \liminf_{n \to \infty} p_n(x_n) + \liminf_{n \to \infty} q_n(x_n) \geq p(x) + q(x).
\]
Now, by definition of epi-convergence we now that there exists $x_n \to x$ such that
\[
\limsup_{n \to \infty} q_n(x_n) \leq q(x).
\]
Furthermore, $\limsup_{n \to \infty} p_n(x_n) = p(x)$, and consequently
\[
\limsup_{n \to \infty} (p_n + q_n)(x_n) \leq \limsup_{n \to \infty} p_n(x_n) + \limsup_{n \to \infty} q_n(x_n) \leq p(x) + q(x).
\]

The following result corresponds to a slightly generalization of [2, Proposition 2.9] (see also [19, Proposition 7.30]), where only sequences $\varepsilon_n \to 0$ were considered.

**Proposition 2.1** Let $f_n \rightharpoonup f$ and $\varepsilon_n \geq 0$ be a sequence such that $\varepsilon = \limsup \varepsilon_n < +\infty$. Then,
\[
a) \limsup \varepsilon_n (f_n) \leq \varepsilon(f).
b) \limsup \varepsilon_n \cdot \text{argmin} f_n \subseteq \varepsilon \cdot \text{argmin} f.
\]

**Proof** First let us prove a), on the one hand if $\limsup \varepsilon_n (f_n) = -\infty$ the conclusion of is trivial, so we can assume that
\[
\limsup \varepsilon_n (f_n) > -\infty,
\]
and by passing to a subsequence, we also assume that $\inf X f_n > -\infty$ for all $n \in \mathbb{N}$. By definition of epi-convergence we have that for every $x \in X$ there exists $x_n \to x$ such that $f(x) \geq \limsup f_n(x_n)$, it implies that
\[
-\infty < \limsup \varepsilon_n (f_n) = \limsup \left( \inf_X f_n + \varepsilon_n \right) \leq \limsup \left( f_n(x_n) + \varepsilon_n \right) \leq f(x) + \varepsilon,
\]
which yields that $-\infty < \limsup \varepsilon_n (f_n) \leq \inf_X f(x) + \varepsilon = \varepsilon(f)$.

Now, we focus on b). Consider a sequence of points $x_k \in \varepsilon_{n_k} \cdot \text{argmin}_X f_{n_k}$ such that $x_k \to x$. Then, by definition of epi-convergence and part a) we have that
\[
f(x) \leq \liminf f_{n_k}(x_{n_k}) \leq \liminf \varepsilon_{n_k}(f_{n_k}) \leq \varepsilon(f),
\]
which ends the proof of b).

### 3 Consistency of the approach to robust optimization problems

In this section we consider the following optimization problem
\[
\min g(x)
\text{ s.t. } x \in M_{\text{as}},
\text{(RO)}
\]
where $M_{\text{as}} := \{ x \in X : x \in M(\xi) \text{ a.s.} \}$, where $g : X \to \mathbb{R} \cup \{ +\infty \}$ is an lsc function and $M : \Xi \rightrightarrows X$ is a measurable multifunction with closed values. We study two approaches for solving (RO). The first one corresponds to an approach using _ergodic measure preserving transformation_, in this subsection we show the consistency of this method. Second, we show that our results implies the consistency of _scenario approach_ method.
3.1 Approach by measure preserving transformations

Now, we consider the following approach using ergodic measure preserving transformation. First, let us formally introduce this notions. Consider a (complete) probability space $(\Xi, \mathcal{A}, \mathbb{P})$ and a measurable function $T : \Xi \to \Xi$. We say that $T$ preserves measure if

$$\mathbb{P}(T^{-1}(A)) = \mathbb{P}(A), \text{ for all } A \in \mathcal{A}. \quad (4)$$

Furthermore, we say that $T$ is ergodic provided that for all $A \in \mathcal{A}$

$$A = T^{-1}(A) \Rightarrow \mathbb{P}(A) = 0, \text{ or } \mathbb{P}(A^c) = 0. \quad (5)$$

Consequently, we $T$ satisfies (1) and (5) we say that $T$ is an ergodic measure preserving transformation.

We consider a sequence of lsc functions $g_n$, which converge continuously to $g$, let us consider an ergodic measure preserving transformation $T : \Xi \to \Xi$. With this setting, we define the following family of optimization problems: For a point $\xi \in \Xi$ we define

$$\min_{x} g_n(x) \quad s.t. \quad x \in M(T^n(\xi)); \quad i = 1, \ldots, n, \quad (\text{EO}_n(\xi))$$

where $T^n$ denotes the $n$-times composition of $T$. In order to show clearer the link of epigraphical convergence and the relation between (RO) and (EO), let us define $f_n : \Xi \times X \to \mathbb{R} \cup \{+\infty\}$ as

$$f_n(\xi, x) := g_n(x) + \frac{1}{n} \sum_{k=1}^{n} \delta_{M(T^k(\xi))}(x), \quad f(x) := g(x) + \delta_{M^\infty}(x). \quad (6)$$

With this notation we can write the relationship between (RO) and (SO) in a functional formulation.

**Theorem 3.1** Under the above setting we have that $f_n(\xi, \cdot) \xrightarrow{\text{a.s.}} f$, $\mathbb{P}$-a.s. Consequently for any measurable sequence $\varepsilon_n : \Xi^\infty \to (0, +\infty)$ with $\varepsilon(\xi) := \limsup \varepsilon_n(\xi) < +\infty$, $\mathbb{P}$-a.s. we have that:

a) \( \limsup_{n \to \infty} v_{\varepsilon_n(\xi)}(f_n(\xi, \cdot)) \leq v_{\varepsilon(\xi)}(f), \mathbb{P}\text{-a.s.} \)

b) \( \limsup_{n \to \infty} \varepsilon_n(\xi) \cdot \arg\min f_n(\xi, \cdot) \subseteq \varepsilon(\xi) \cdot \arg\min f, \mathbb{P}\text{-a.s.} \)

**Proof** Let us consider the sequence of functions $p_n(x) := g_n(x)$ and $q_n(\xi, x) := \frac{1}{n} \sum_{k=1}^{n} \delta_{M(T^k(\xi))}(x)$. It is not difficult to see that for each $n \in \mathbb{N}$ the function $q_n$ is a random lsc function.

Then, by [13, Theorem 1.1], we have that

$$q_n(\xi, \cdot) \xrightarrow{\text{P}} E\xi(\delta_M(\cdot)) = \delta_{M^\infty}(\cdot), \mathbb{P}\text{-a.s.}$$

Now, define $\tilde{\Xi} := \{ \xi \in \Xi : q_n(\xi, \cdot) \xrightarrow{\text{P}} \delta_{M^\infty} \}$, it follows that $\mathbb{P}(\tilde{\Xi}) = 1$, then for all $\xi \in \tilde{\Xi}$ we apply Lemma 2.1 which implies that for all $\xi \in \tilde{\Xi}$, we have $p_n + q_n(\xi, \cdot) \xrightarrow{\text{a.s.}} f$, that is to say, $f_n(\xi, \cdot) \xrightarrow{\text{a.s.}} f$ for all $\xi \in \Xi$. 

Pedro Pérez-Aros
Now, by Proposition 2.1 we have that for all $\xi \in \hat{\Xi}$,

$$\limsup_{n \to \infty} v_{\varepsilon_n}(\xi; f_n(\xi, \cdot)) \leq v_{\varepsilon}(f),$$

and

$$\limsup_{n \to \infty} \varepsilon_n(\xi) \cdot \text{argmin } f_n(\xi, \cdot) \subseteq \varepsilon(\xi) \cdot \text{argmin } f,$$

which concludes the proof.

When there are additional assumptions over the feasibility and compactness of the optimization problems $(RO)$ and $(EO_n(\xi))$ we can establish a tighter conclusion. We translate the hypothesis into notation of the problems $(RO)$ and $(EO_n(\xi))$, respectively.

**Corollary 3.1** Let us assume that $(RO)$ is feasible, and the sequence of optimization problems $(EO_n(\xi))$ is eventually level compact $\mathbb{P}$-a.s. Then,

a) $\lim_{n \to \infty} v(EO_n(\xi)) = v(RO)$, $\mathbb{P}$-a.s.

b) For any measurable sequence $\varepsilon_n : \Xi \to (0, +\infty)$ with $\varepsilon_n(\xi) \to 0$, $\mathbb{P}$-a.s. we have that

$$\emptyset \neq \limsup_{n \to \infty} \varepsilon_n(\xi) \cdot \text{argmin } (EO_n(\xi)) \subseteq \text{argmin } (RO), \mathbb{P}$$.a.s.

c)

$$\bigcap_{\varepsilon > 0} \liminf_{n \to \infty} \varepsilon \cdot \text{argmin } (EO_n(\xi)) = \text{argmin } \bigcap_{\varepsilon > 0} \limsup_{n \to \infty} \varepsilon \cdot \text{argmin } (EO_n(\xi)), \mathbb{P}$.a.s.

**Proof** Consider the notation given in (6). Let us define $\alpha := \max\{\inf_X f + 1, 1\}$, we have that $\alpha < +\infty$ due to the feasibility of $(RO)$, consider a set $\hat{\Xi}$ of full measure such that for all $\xi \in \hat{\Xi}$

(i) $(f_n(\xi, \cdot))_{n \in \mathbb{N}}$ is eventually level compact,

(ii) $\varepsilon_n(\xi) \to 0$,

(iii) $\limsup_{n \to \infty} \varepsilon_n(\xi) \cdot \text{argmin } f_n(\xi, \cdot) \subseteq \text{argmin } f$.

Fix $\xi \in \hat{\Xi}$ and a sequence $x_k(\xi) \in \varepsilon_n(\xi) \cdot \text{argmin } f_n(\xi, \cdot)$, so there exists some $n_\xi \in \mathbb{N}$ such that for all $n \geq n_\xi$

$$\varepsilon_n(\xi) \leq 1, \text{ and } \bigcup_{n \geq n_\xi} \text{lev}_{\leq \alpha} f_n \text{ is relatively compact}.$$

This implies that the sequence $(x_k(\xi))_{k \geq n_\xi}$ belongs to a compact set, so it has an accumulation point. Consequently, we have that $\limsup_{n \to \infty} \varepsilon_n \cdot \text{argmin } f_n(\xi, \cdot) \neq \emptyset$, which proves (b).

Now, by [2] Theorem 2.11] we conclude that $\liminf_{n \to \infty} v(f_n(\xi, \cdot)) = v(f)$ for all $\xi \in \hat{\Xi}$, which concludes the proof of (a). Finally, using [2] Theorem 2.12] we get that (c) holds.
3.2 Approach by samples

In this section we consider \((\Xi_\infty, A_\infty, \mathbb{P}_\infty)\) as the denumerable product of the probability spaces \((\Xi, A, \mathbb{P})\).

As in the previous section, we consider a sequence of lsc functions \(g_n\), which converge continuously to \(g\), let us define the following family of optimization problems: For each \(\omega = (\xi_i)_{i=1}^{\infty} \in \Xi_\infty\) we set

\[
\min g_n(x) \\
\text{s.t. } x \in M(\xi_i); \ i = 1, \ldots, n.
\]

Let us define \(f_n : \Xi_\infty \times X \to \mathbb{R} \cup \{+\infty\}\) given by

\[
f_n(\omega, x) := g_n(x) + \frac{1}{n} \sum_{k=1}^{n} \delta_{M(\xi_k)}(x),
\]

\[
f(x) := g(x) + \delta_{M_\omega(x)}(x).
\]

The following results corresponds to the scenario approach version of Theorem 3.1.

**Theorem 3.2** Under the above setting we have that \(f_n(\omega, \cdot) \xrightarrow{n \to \infty} f, \mathbb{P}_\infty\text{-a.s.}\) Consequently for any measurable sequence \(\varepsilon_n : \Xi_\infty \to (0, +\infty)\) with \(\varepsilon_n(\omega) := \limsup_{n \to \infty} \varepsilon_n(\omega) < +\infty, \mathbb{P}_\infty\text{-a.s.}\) we have that:

a) \(\limsup_{n \to \infty} v_{\varepsilon_n}(f_n(\omega, \cdot)) \leq v_{\varepsilon(\omega)}(f), \mathbb{P}_\infty\text{-a.s.}\)

b) \(\limsup_{n \to \infty} \varepsilon_n(\omega) \cdot \argmin f_n(\omega, \cdot) \subseteq \varepsilon(\omega) \cdot \argmin f, \mathbb{P}_\infty\text{-a.s.}\)

Proof Consider the shift on \(\Xi_\infty\), that is, \(T : \Xi_\infty \to \Xi_\infty\) given by

\[
T(\xi_i)_{i=1}^{\infty} = (\xi_{i+1})_{i=1}^{\infty},
\]

by [12, Proposition 2.2] \(T\) is an ergodic measure preserving transformation (For more details we refer to [12, 23]). Furthermore, we extend the measurable multifunction \(M\) to \(\Xi_\infty\) just defining \(\tilde{M} : \Xi_\infty \rightrightarrows X\) by \(\tilde{M}(\omega) = M(\xi_1)\), where \(\omega = (\xi_i)_{i=1}^{\infty}\). Using this notation of (7) with this particular functions we get

\[
f_n(\omega, x) := g_n(x) + \frac{1}{n} \sum_{k=1}^{n} \delta_{\tilde{M}(T_k(\omega))}(x) = g_n(x) + \frac{1}{n} \sum_{k=1}^{n} \delta_{M(\xi_k)}(x),
\]

\[
f(x) := g(x) + \delta_{M_\omega(x)}(x) = g(x) + \delta_{M_\omega(x)}(x).
\]

Then, Theorem 3.1 gives us that for almost all \(\omega = (\xi_i)_{i=1}^{\infty} \in \Xi_\infty\)

i) \(f_n(\omega, \cdot) \to f, \mathbb{P}_\infty\text{-a.s.}\)

ii) \(\limsup_{n \to \infty} v_{\varepsilon_n}(f_n(\omega, \cdot)) \leq v_{\varepsilon(\omega)}(f), \mathbb{P}_\infty\text{-a.s.}\)

iii) \(\limsup_{n \to \infty} \varepsilon_n(\omega) \cdot \argmin f_n(\omega, \cdot) \subseteq \varepsilon(\omega) \cdot \argmin f, \mathbb{P}_\infty\text{-a.s.}\)

**Remark 3.1** It is worth mentioning that Theorem 3.2 can be proved using the same proof given in Theorem 3.1 copied step by step, but using [11, Theorem 2.3] instead of [15, Theorem 1.1].
Similarly to the previous subsection, we can get more precise estimations under some compactness assumptions. The proof of this result follows considering the representation of (7) given in (9) using the shift transformation defined in (8), and also it can follow mimicking the proof of Corollary step by step, and using Theorem 3.2 instead of Theorem 3.1.

**Corollary 3.2** Let us assume that (RO) is feasible, and the sequence optimization problem \((SO_n(\omega))\) is eventually level compact \(\mathbb{P}^\infty\text{-a.s.}\). Then,

a) \(\lim_{n \to \infty} v(SO_n(\omega)) = v(RO), \mathbb{P}^\infty\text{-a.s.}\)

b) For any measurable sequence \(\varepsilon_n : \Xi^\infty \to (0, +\infty)\) with \(\varepsilon_n \to 0, \mathbb{P}^\infty\text{-a.s.}\) we have that

\[\emptyset \neq \limsup_{n \to \infty} \varepsilon_n(\omega) \cdot \arg\min SO_n(\omega) \subseteq \arg\min (RO), \mathbb{P}^\infty\text{-a.s.}\]

c) \[\bigcap_{\varepsilon > 0} \liminf_{n \to \infty} \varepsilon \cdot \arg\min SO_n(\omega) = \arg\min X (RO) = \bigcap_{\varepsilon > 0} \limsup_{n \to \infty} \varepsilon \cdot \arg\min SO_n(\omega), \mathbb{P}^\infty\text{-a.s.}\]

**Remark 3.2** It has not escape our notice that in [18] the authors did not show the consistency of the scenario approach with perturbation over the objective function \(g\) as in \((SO_n(\omega))\). Furthermore, only linear objective function where considered in [18].

### 4 Application to infinite programming problems

In this part of the work, we use the result of Section 3.1 to show that a sequence of sub-problems can be used to give an approach for infinite programming problems.

Consider the following problem of infinite programming (semi-infinite programming, if \(S\) is a subset of \(\mathbb{R}^n\))

\[
\begin{align*}
\min g(x) \\
\text{s.t. } x \in M(s), \forall s \in S,
\end{align*}
\]

where \(S\) is a topological space, and \(M : S \rightrightarrows X\) is an outer-semicontinuous set-valued map, that is to say, for every net \(s_n \to s\) and every sequence \(x_n \in M(s_n)\) with \(x_n \to x\) we have \(x \in M(s)\). We denote by \(\mathcal{A}\) any \(\sigma\)-algebra, which contains all open subsets on \(T\), and consider \(\mu : \mathcal{A} \to \mathbb{R}\) a strictly positive finite measure, that is to say, \(\mu(S) < +\infty\) and

\[\mu(U) > 0, \text{ for every open set } U \subseteq S,\]

let us consider an ergodic measure preserving transformation \(T : S \to S\). With this framework, we define the sequence of optimization problems

\[
\begin{align*}
\min g_n(x) \\
\text{s.t. } x \in M(T^k(s)); k = 1, \ldots, n,
\end{align*}
\]

where \(g_n\) converge continuously to \(g\). As a simple application of Theorem 3.1 we get the following result.

**Theorem 4.1** Let us assume that \((S, \mathcal{A}, \mu)\) is complete, \((IP)\) is feasible, and the sequence optimization problems \((IP_n(s))\) is eventually level compact \(\mu\text{-a.e.}\). Then,
a) \( \lim_{n \to \infty} v(\text{IP}_n(s)) = v(\text{IP}), \mu\text{-a.e.} \)

b) For any measurable sequence \( \varepsilon_n : \Xi \to (0, +\infty) \) with \( \varepsilon_n(s) \to 0, \mu\text{-a.e.} \) we have that

\[ \emptyset \neq \limsup_{n \to \infty} \varepsilon_n(s) - \arg\min \{ \text{IP}_n(s) \} \subseteq \arg\min \{ \text{IP} \}, \mu\text{-a.e.}. \]

c) \[ \bigcap_{\varepsilon > 0} \liminf_{n \to \infty} \varepsilon - \arg\min \{ \text{IP}_n(s) \} = \arg\min \{ \text{IP} \}, \mu\text{-a.e.} \]

**Proof** First, by the outer-semicontinuous of \( M \) we have that the optimization problem (IP) is equivalent to

\[ \min g(x) \]

\[ \text{s.t. } x \in M(s), \mu\text{-a.e.} \quad (10) \]

Indeed, let \( x \in M(s) \) for almost all \( s \in S \). Then, the set \( D_x := \{ s \in S : x \in M(s) \} \) is dense due to the fact that \( \mu \) is a strictly positive measure. Consequently for every \( s \in S \setminus D \) there exist \( s_\nu \to s \), so by the outer-semicontinuous of \( M \) we get that \( x \in M(s) \), and consequently \( x \in M(s) \) for all \( s \in S \).

Next, consider the probability measure \( \mathbb{P}(\cdot) = \mu(S)^{-1}\mu(\cdot) \). Then, applying Corollary 3.1 to (10) we get that a), b) and c) hold with (10), and by the equivalency with (IP) we conclude the proof.

Now, let us illustrate the above result with the following example.

![Fig. 1: Function $g$ of Example 4.1](image-url)
Example 4.1 Let us consider the following optimization problem

\[
\begin{align*}
\min & \quad g(x, y) \\
\text{subject to} \quad & \alpha x + \beta y \leq 6 \quad \text{for all } (\alpha, \beta) \in S^1,
\end{align*}
\]

where \( g(x, y) = x^4 - 3x^3 - 51x^2 - 37x + y^4 + 2y^3 - 79y^2 + 220y + 90 + xy \) (see Figure 1), and \( S^1 \) is the 2-dimensional unit sphere, that is to say, \( S^1 := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1 \} \). It is not difficult to see that the above problem is noting more than

\[
\begin{align*}
\min & \quad g(x, y) \\
\text{subject to} \quad & x^2 + y^2 \leq 36.
\end{align*}
\]

For solving this problem, we use a irrational rotation \( T : S^1 \to S^1 \), that is, \( T(\xi) = \xi \cdot e^{2\pi \theta i} \) with \( \theta \in [0, 1) \setminus \mathbb{Q} \). Here the multiplication is in the sense of complex numbers. Therefore, we have to solve numerically the following optimization problems

\[
\begin{align*}
\min & \quad g(x, y) \\
\text{subject to} \quad & x \in M(T^k(\xi)); \quad \text{for } k = 1, \ldots, n, \quad (\text{Ex}_n)
\end{align*}
\]

where \( M(\xi) := \{(x, y) \in \mathbb{R}^2 : \langle \xi, (x, y) \rangle \leq 6 \} \).

First, we have that the global minimum of \( g \) is attained at \((x_u, y_u) = (6.3442, -7.6398)\) and the minimum is \( g(x_u, y_u) = -5490.9 \); on the other hand the optimal value of \( \text{(Ex)} \) is attained at \((x_e, y_e) = (3.2004, -5.0752)\) with value \( g(x_e, y_e) = -3519.1 \). In Table 1 we can compare different numerical solutions of Problem \( \text{(Ex)} \).

Table 1: Numerical solution of \( \text{(Ex}_n) \).

| \( \xi \)   | \( n \) | \( t(\text{Ex}_n) \) | \( x \)   | \( y \)   | \( n \) | \( t(\text{Ex}_n) \) | \( x \)   | \( y \)   |
|----------|-------|----------------|--------|--------|-------|----------------|--------|--------|
| \( e^{i\pi/10} \) | 5     | -3862.4        | 1.7623 | -6.7866| 15    | -3634.7        | 3.3542 | -5.1543|
| \( e^{i\pi/6} \) | 5     | -4257.4        | 2.6753 | -9.9991| 15    | -3519.3        | 3.0438 | -5.1746|
| \( e^{i\pi/3} \) | 5     | -5406.3        | 5.7143 | -7.2882| 15    | -3651.6        | 4.3877 | -4.5573|
| \( e^{i4\pi/3} \) | 5     | -4258.3        | 4.3254 | -5.5167| 15    | -3633.0        | 3.1903 | -5.2573|
| \( e^{i7\pi/4} \) | 5     | -5251.2        | 4.9946 | -7.5690| 15    | -3748.5        | 3.9081 | -4.9747|
| \( e^{i5\pi/10} \) | 30    | -3519.4        | 3.3562 | -4.9778| 40    | -3519.4        | 3.3562 | -4.9778|
| \( e^{i\pi/6} \) | 30    | -3519.3        | 3.0438 | -5.1746| 40    | -3519.3        | 3.1230 | -5.1243|
| \( e^{i\pi/3} \) | 30    | -3542.3        | 3.1288 | -5.1560| 40    | -3542.3        | 3.0304 | -5.2028|
| \( e^{i4\pi/3} \) | 30    | -3534.0        | 3.6424 | -4.8220| 40    | -3534.0        | 3.5479 | -4.7711|
| \( e^{i7\pi/4} \) | 30    | -3524.0        | 3.6397 | -4.8089| 40    | -3524.0        | 3.6397 | -4.8089|
| \( e^{i5\pi/10} \) | 70    | -3519.4        | 3.3562 | -4.9778| 100   | -3.5194        | 3.3562 | -4.9778|
| \( e^{i\pi/6} \) | 70    | -3515.3        | 3.1230 | -5.1242| 100   | -3.5193        | 3.1232 | -5.1242|
| \( e^{i\pi/3} \) | 70    | -3532.1        | 3.0304 | -5.2028| 100   | -3.5219        | 2.9328 | -5.2494|
| \( e^{i4\pi/3} \) | 70    | -3529.4        | 3.4538 | -4.9319| 100   | -3.5268        | 3.3599 | -4.9866|
| \( e^{i7\pi/4} \) | 70    | -3523.6        | 3.5467 | -4.8555| 100   | -3.5228        | 3.4538 | -4.9220|

References

1. Z. Artstein and R. J.-B. Wets. Consistency of minimizers and the SLLN for stochastic programs. J. Convex Anal., 2(1-2):1–17, 1995.
