The study of models of space selenophysics using multi-parameter analysis and fractal geometry

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Abstract. In this work, using multi-parameter harmonic analysis and expansion of altimetry into spherical functions, models of the Moon’s physical surface (digital lunar selenocentric map - DLSM) are built, and a comparison of similarity of chosen local areas of the complex lunar structure is performed. The constructed DLSM have radius-vectors of the surface points in accordance with the space measurements taken. As result, the averaged fractal dimension of the selenocentric surfaces profiles models was found to 1.34 ± 0.05. The similarity and difference parameters were determined by the author’s method using fractal similarity coefficients. The analysis of macrosurfaces based on multi-parameter and fractal methods for the selenocentric models built in this work has not been conducted before.

1. Introduction

This work is dedicated to the creation of a method for analyzing planetary models in order to assess their reliability and similarity. Currently, there are highly accurate dynamic theories of celestial bodies, space altimetric measurements are taken with an accuracy up to 1 ms, but the construction and analysis of planetary models remain a complex and not fully solved problem. This also applies to researches in selenodesy, as the physical surface of the Moon and its connection with the celestial coordinate system are both non-linear multi-parameter systems. The altimetry for the construction of the models were produced by “Clementine” [1], “Kaguya” [2] and “LRO” [3] missions. One should emphasize that due to the fact that multiple space missions were used to construct the models under investigation, they have different coordinate systems [4], and the same applies to the position of origin [5]. Therefore, the data of space missions were transformed into a single dynamic selenocentric system [6]. For their analysis it is therefore necessary to use methods of statistical physics and fractal geometry [7]. Based on those methods, one may study similarity and structure of a model’s elements and conduct an analysis of dynamic parameters in various phase spaces. The classic methods for comparing topographic systems require a lot of measurements and calculations, and as the physical relief is of non-linear structure, the use of classic approaches to its analysis is almost impossible. The problem of developing new methods for reduction and studying selenodetic non-linear models is modern and relevant [8]. It also worth noting that while constructing various structural models of
selenographic physical surface, a great number of observations and methods are used, and thus arises a
necessity to estimate reliability of the data presented in such systems.

2. Method of robust estimations
The basic method used for studying the lunar macro-relief is numerical and analytical one which
implies expansion of a catalogue data in harmonic series of spherical functions [9]. For this purpose, it
is advisable to use methods of regression analysis [10]. Among statistical methods, regression analysis
is most often used for solving the tasks related to experimental data processing. Thus, regression
analysis allows constructing mathematical models based on various observations [11]. The simulation
implies introducing the parametric interrelationships between processes observed and parameters
corresponding to them. There are 4 aspects of regression analysis [12]:

I. Representation of observations data on the basis of mathematical methods;
II. Construction of models characterizing observed process;
III. Determining unknown parameters in the mathematical models using Least Square Method
(LSM);
IV. Finding and selecting the most reliable model based on the parameters being determined.

For solving the tasks of constructing macrofigures of the Moon, the method based on regression
simulation is used. For this purpose, the software package “Automated System of Transformation
Coordinate” (ASTC) has been developed.

The software modules included in ASTC provide solutions of normal and overdetermined systems
of linear algebraic equations. Solution of latter is implemented on the basis of LSM. Values of
unknowns and their errors, values of correlation matrix elements, internal and external quality
measures used for determining reliability and receiving recommendations for structuring a model can
take on the role of output [13]. In most cases the program contains one or another procedure of
searching for the most reliable structure of the model. There is a possibility of using step-by-step
regression analysis, which is applied for obtaining model by less number of observations n than the
number of coefficients p [14]. This is possible, since the terms are introduced in the model
consecutively and the calculation procedure can end earlier than excess solutions appear. In the work,
regression modeling was used to transformation altimetry data into a single coordinate reference
system.

3. The modelling of DLSM
For creating DLSM itself were used the harmonic expansion of the “Clementine”, “Kaguya” and
“LRO” missions’ altimetry data into spherical functions. The method and mathematical apparatus of
constructing DLSM are as follows. A model of the DLSM can be constructed using variations of
radius vectors expansion in spherical functions according to the following formula [15, 16]:

\[
h(\lambda, \beta) = \sum_{n=0}^{N} \sum_{m=0}^{n} (\tilde{C}_{nm} \cos m\lambda + \tilde{S}_{nm} \sin m\lambda) \cdot \tilde{P}_{nm}(\cos \beta) + \varepsilon,
\]

where \(h(\lambda, \beta)\) – altitude function dependent on longitude and latitude;
\(\lambda, \beta\) – longitude, latitude (known parameters);
\(\tilde{C}_{nm}, \tilde{S}_{nm}\) – harmonic normalized amplitudes;
\(\tilde{P}_{nm}\) – Legendre functions, normalized and associated;
\(\varepsilon\) – regression error, random.

This mathematical expression was also used for the processing of other astronomical observations
[17, 18].

All space observational data were reduced to a single coordinate reference system [19]. As a result,
DLSM was created covering the entire lunar sphere. DLSM has the dynamic selenocentric coordinate
system [20].

4. Analysis of DLSM using the fractal method
The structure of an object under consideration may be presented as an ordered set $A(N^2)$, where $N^2$ is the number of elements $a_{ij}$ of the set $a_{ij} \in A(N^2)$, where $i, j = 1 \ldots N$.

All the numerical values of the series used to convert the reference frame were taken from [8], except for the last two arguments, which were taken from [9].

Unfortunately, there is ambiguity in the values of some arguments. The contribution of these quantities leads to a discrepancy by an amount not exceeding 2 arc seconds.

Figure 1. Hasse diagram.

A partial order in a finite set is defined by a Hasse diagram (figure 1). Elements of a set have some properties $H_\xi(a)$ (size, colour, volume, shape, etc.) inherent only to the elements of this set $\forall a_{ij} (a_{ij} \in A_{\xi}(H_\xi(a)))$. If there are more than 1 ($\xi > 1$) common properties, a set could be described by several fractal properties.

Let us represent the set $A(N^2)$ as

$$A(N^2) = Q(1)(n^2) \cup Q(2)(n^2) \cup \cdots \cup Q(\alpha)(n^2),$$

where $\alpha$ and $n$ are integers.

There are upper and lower borders of the $A(N^2)$ set according to the properties $H_\xi(a)$:

$$AG_\xi = \sup A(N^2),$$

$$g_\xi = \inf A(N^2),$$

while $G_\xi \in A(N^2)$ and $g_\xi \in A(N^2)$.

The fractal property $D_\xi$ of the $A(N^2)$ set according to the $H_\xi(a)$ property is defined by an angle dependence coefficient of $\log \Gamma_\xi(n^2)$ on $\log s_\xi n^2$, where $\Gamma_\xi(n^2)$ is the number of discontiguous cubes’ surfaces covering the $Q(k)(n^2)$ set:

$$D_\xi = \sum_\gamma \frac{\log \Gamma_\xi(n^2_{\gamma+1}) - \log \Gamma_\xi(n^2_\gamma)}{\text{abs}(\log S_\xi(n^2_{\gamma+1})) - \text{abs}(\log S_\xi(n^2_\gamma))} \cdot \left(\frac{\alpha_{\gamma+1} - \alpha_\gamma}{N - 1}\right).$$

The self-similarity coefficient $K_\xi$ is defined as

$$K_\xi = \frac{D_\xi^\theta}{D_\xi},$$

where $D_\xi^\theta$ is a fractal dimension of a self-similar set.
\begin{align}
D^0_\xi &= \frac{\log \Gamma_\xi(N^2) - \log \Gamma_\xi(1)}{abs(\log S_\xi(N^2)) - abs(\log S_\xi(1))}.
\end{align}

Figure 2 presents the distribution of the self-similarity coefficient on the DLSM divided into square areas with a resolution of $15^\circ \times 15^\circ$ in $\lambda_i$ and $\beta_i$ selenography coordinates.

![Figure 2](image)

**Figure 2.** The distribution of the self-similarity coefficient on the DLSM for different areas of the lunar surface.

An analysis of figure 2 shows that the distribution of the self-similarity coefficient over different regions of the lunar surface varies from 0.8 to 1, which indicates the fractality of the lunar surface and at the same time a change in its structure from region to region.

5. **Summary and conclusions**

Based on the results of the work we may draw the following conclusions. The analysis of the modern ways to solve the questions of lunar selenography on the basis of data obtained during the space missions is conducted. In particular, “Clementine” (USA), “KAGUYA” (JAPAN), and “LRO” (USA) lunar missions are analyzed. The need of multiple processing of selenographic data sets due to the continuous modernization of using methods are needed in accurate selenocentric reference nets is being developed, has been found. This direction has become particularly important after the appearance of dynamic reference systems based on space measurements [21]. For future space missions the databases of global altitude data are being made as well, which have to increase the accuracy of the lunar reference system and perform the lunar images investigation for structural analysis [22].

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