THE NIRENBERG PROBLEM ON HALF SPHERES: 
A BUBBLING OFF ANALYSIS

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dedicated to the memory of Prof. Antonio Ambrosetti

Abstract

In this paper we perform a refined blow up analysis of finite energy approximated solutions to a Nirenberg type problem on half spheres. The later consists of prescribing, under minimal boundary conditions, the scalar curvature to be a given function. In particular we give a precise location of blow up points and blow up rates. Such an analysis shows that the blow up picture of the Nirenberg problem on half spheres is far more complicated that in the case of closed spheres. Indeed besides the combination of interior and boundary blow ups, there are non simple blow up points for subcritical solutions having zero or nonzero weak limit. The formation of such non simple blow ups is governed by a vortex problem, unveiling an unexpected connection with Euler equations in fluid dynamic and mean fields type equations in mathematical physics.

Key Words: Blow up analysis, Subcritical approximation, Critical Sobolev exponent, Non-simple blow up points, Vortex problems.

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1. Introduction and statement of main results

The Nirenberg problem in conformal geometry has a long history. Indeed this problem goes back to the following question raised by Louis Nirenberg in the academic year 1969-1970: Given a smooth function $K$ on the standard sphere $S^n$ endowed with its standard metric $g_0$, does there exist a metric $g$, in the conformal class of $g_0$, whose scalar curvature is given by the function $K$? On the two dimensional sphere $S^2$ this amounts to solve an elliptic PDE involving exponential nonlinearity while on spheres $S^n$ of dimension $n \geq 3$, using the transformation law of the scalar curvature under conformal change of metric, one sees that the Nirenberg problem is equivalent to solving the following nonlinear partial differential equation involving the critical Sobolev exponent:

\[(NP) \quad L_{g_0} u = Ku^{(n+2)/(n-2)}, \quad u > 0,\]

where $L_{g_0} := -\Delta_{g_0} + n(n-2)/4$ denotes the conformal Laplacian.

The Nirenberg problem has attracted a lot of attention in the last half century. See [3, 4, 7, 9, 11, 17, 19, 20, 21, 22, 23, 24, 28, 29, 33, 34, 42] and the references therein.

Due to Kazdan-Warner obstructions [29, 18] the above PDE is not always solvable and the corresponding Euler-Lagrange functional lacks compactness. One way to overcome the lack of compactness, which goes back to Yamabe [46] and R. Schoen [41] (see also [27]) consists of considering the following subcritical approximation

\[(NP_\epsilon) \quad L_{g_0} u = Ku^{((n+2)/(n-2)) - \epsilon}, \quad u > 0\]

whose Euler-Lagrange functional satisfies the Palais-Smale condition. One then studies the blow up behavior when $\epsilon$ goes to zero. It follows from such a refined analysis that blow ups occur at critical point with nonpositive Laplacian. Moreover it turns out that the blow up scenario depends strongly on the dimension and on the behavior of the function $K$ around its critical points. Indeed under the condition that $\Delta K \neq 0$ at the critical points of $K$, one has that, in dimension 3, solutions of the above approximation $(NP_\epsilon)$ could only develop one single bubble, see please [7, 20]. In dimension 4 there could be multiple blow ups but tuples of blow up points have to satisfy a balancing condition, see please [11, 34], while in dimensions $n \geq 5$ every tuple of distinct critical points of $K$ having negative Laplacian can be realized as the blow up set of a blowing up solution of the approximated PDE, see please [9, 37, 38]. Furthermore all blow up points of finite energy blowing up solutions are isolated simple in the sense that around every blow up point there is a ball which does not contain any other blow up point and the Dirichlet-Energy of the blow up solutions $u_\epsilon$ in a shrinking neighborhood around the blow up point tends as $\epsilon \to 0$ to the energy of one bubble concentrating at this point. See please [37, 38].

In this paper we consider a version of the Nirenberg problem on standard half spheres $(S^n_+, g)$. Namely we prescribe simultaneously the scalar curvature to be a positive function $K \in C^3(S^n_+)$ and the boundary mean curvature to be zero. This
amounts to solve the following boundary value problem

\[ \begin{align*}
(P) \quad & \begin{cases}
-\Delta_{g_0} u + \frac{n(n-2)}{4} u = Ku^{(n+2)/(n-2)}, & u > 0 \quad \text{in } S^n_+,
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial S^n_+.
\end{cases}
\end{align*} \]

This problem has been studied on half spheres of dimensions \( n = 3, 4 \). See the papers [12, 13, 14, 15, 26, 32, 36] and the references therein. Like the case of the Nirenberg problem on spheres, there are obstructions to the existence of solutions to \((P)\) and the corresponding variational problem is not compact. In order to recover compactness one considers the following subcritical approximation

\[ \begin{align*}
(P_{\varepsilon}) \quad & \begin{cases}
-\Delta_{g_0} u + \frac{n(n-2)}{4} u = Ku^{(n+2)/(n-2)-\varepsilon}, & u > 0 \quad \text{in } S^n_+,
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial S^n_+.
\end{cases}
\end{align*} \]

Regarding the behavior of a sequence of energy bounded solutions \( u_\varepsilon \) of \((P_{\varepsilon})\), it follows from the concentration compactness principle that either the \( \|u_\varepsilon\|_{L^\infty} \) remains uniformly bounded or \( u_\varepsilon \to 2n/(n-2) L^n \) (where \( L^n \) denotes the Lebesgue measure) converges in the sense of measures to a sum of Dirac masses, some of them are located in the interior and the others are on the boundary. Moreover it follows from the blow up analysis of \((P_{\varepsilon})\) that the interior points are critical points of \( K \) with nonpositive Laplacian and the boundary points are critical points of \( K_1 \) the restriction of \( K \) on the boundary and satisfying that \( \partial_\nu K \geq 0 \). See [13, 15, 26].

Furthermore, under the non degeneracy assumption that \( \Delta K \neq 0 \) at interior critical points of \( K \) and that \( \partial_\nu K \neq 0 \) at critical points of \( K_1 \), we have that in the dimension \( n = 3 \) multiple bubbling may occur but all blow up points are isolated simple, see [26, 32]. Moreover under additional condition on \( K_1 \) it has been proved in [14] that in dimension 4 all blow up points are isolated simple. More surprisingly and in contrast with the case of closed spheres, the Nirenberg problem on half spheres may have non simple blow up points for finite energy bubbling solutions of \((P_{\varepsilon})\) see [2].

In this paper we perform a systematic asymptotic analysis, as \( \varepsilon \) goes to zero, of finite energy blowing up solutions of \((P_{\varepsilon})\). Such an analysis is performed under the following non degeneracy conditions:

\[ \textbf{(H1)} \] The critical points \( y \)'s of \( K \) in \( S^n_+ \) are non degenerate and satisfy \( \Delta K(y) \neq 0 \).

\[ \textbf{(H2)} \] The critical points \( z \)'s of the restriction of \( K \) on the boundary \( K_1 := K|_{\partial S^n_+} \) are non degenerate and satisfy \( \partial_\nu K(z) \neq 0 \).

We first consider the case where the sequence of energy bounded solutions of \((P_{\varepsilon})\) has a zero weak limit \( u_\varepsilon \to 0 \). In this situation the description of the blow up picture is as follows:

**Theorem 1.1.** Let \( n \geq 5 \) and \( 0 < K \in C^3(S^n_+) \) be a positive function satisfying the assumptions \((H1), (H2)\) and let \( (u_\varepsilon) \) be a sequence of energy bounded solutions of \((P_{\varepsilon})\) with \( u_\varepsilon \to 0 \). Then \( u_\varepsilon \) blows up and decomposes as follows

\[ u_\varepsilon = \sum_{i:a_i, \nu} \frac{1}{K(a_i, \nu)^{(n-2)/4}} \delta_{a_i, \nu} + \sum_{i:a_i, \nu} \frac{1}{K(a_i, \nu)^{(n-2)/4}} \delta_{a_i, \nu} + v_\varepsilon, \]
where $\delta_{a,\lambda}$ is the standard bubble defined in (2.2) and $\|v_\lambda\| = a_\varepsilon(1)$ where $\|\cdot\|$ is defined by (2.1).

Furthermore, there hold:

(a) Each interior concentration point $a_{i,\varepsilon} \in \mathbb{S}^n_+$ converges to a critical point $y_i$ of $K$ with $\Delta K(y_i) < 0$, $\lambda_{i,\varepsilon}(a_{i,\varepsilon}, y_i)$ is uniformly bounded and $y_i$ is an isolated simple blow up point for the sequence $(u_\varepsilon)$. Moreover there exists a dimensional constant $\kappa_1(n) > 0$ (see (3.34) for the precise value) such that
$$
\lambda_{i,\varepsilon}^2 = -\kappa_1(n) \frac{\Delta K(y_i)}{K(y_i)} \frac{1}{\varepsilon (1 + o_\varepsilon(1))}.
$$

(b) Each boundary concentration point $a_{i,\varepsilon} \in \partial \mathbb{S}^n_+$ converges to a critical point $z_i$ of $K_1$ with $\partial_\nu K(z_i) > 0$ and there exists a dimensional constant $\kappa_2(n) > 0$ (see (3.34) for the precise value) such that
$$
\lambda_{i,\varepsilon} = \kappa_2(n) \frac{\partial_\nu K(z_i)}{K(z_i)} \frac{1}{\varepsilon (1 + o_\varepsilon(1))}.
$$

(c) For a boundary concentration point $a_{i,\varepsilon}$ converging to a critical point $z_i$ of $K_1$ there are two alternatives

(i) Either $d(a_{i,\varepsilon}, z_i)/\varepsilon$ is uniformly bounded and $z_i$ is an isolated simple blow up point for the sequence $(u_\varepsilon)$.

(ii) Or for a subsequence $d(a_{i,\varepsilon}, z_i)/\varepsilon \to \infty$, as $\varepsilon \to 0$. In this case $z_i$ is a non simple blow up point for the sequence $(u_\varepsilon)$ (in the sense that there exists at least another point $a_{j,\varepsilon}$ which converges to $z_i$).

(d) Let $z \in \partial \mathbb{S}^n_+$ be a non simple boundary blow up point and $\{a_{1,\varepsilon}, \ldots, a_{m,\varepsilon}\}$ be the set of points converging to $z$. Then for every $i = 1, \ldots, m$ we have that $\varepsilon^{(2-n)/n} d(a_{i,\varepsilon}, z_i)$ is uniformly bounded.

Denoting

$$
(1.3) \quad b_{i,\varepsilon} := \kappa_3(n) \frac{(\partial_\nu K(z))^{n-1}}{(K(z))^{n+1}} \frac{1}{(\varepsilon^{n+1})} (a_{i,\varepsilon} - < a_{i,\varepsilon}, z >, z) \in T_z \mathbb{S}^n_+ \quad \text{for} \quad 1 \leq i \leq m,
$$

where $\kappa_3(n) > 0$ is a dimensional constant (see (3.39) for the precise value), we have that $(b_{1,\varepsilon}, \ldots, b_{m,\varepsilon})$ converges to $(\tilde{b}_1, \ldots, \tilde{b}_m)$ which is a critical point of the following Kirchhoff-Routh type function:

$$
(1.4) \quad \mathcal{F}_{\varepsilon,m} : \mathbb{R}^m \to \mathbb{R}^m \quad \text{by} \quad \mathcal{F}_{\varepsilon,m}(\xi_1, \ldots, \xi_m) := \frac{1}{2} \sum_{i=1}^m D^2 K_1(z)(\xi_i, \xi_i) + \sum_{1 \leq i < j \leq m} \frac{1}{|\xi_i - \xi_j|^{n-2}}.
$$

where $\mathbb{R}^m (T_z \mathbb{S}^n_+) := \{(\xi_1, \ldots, \xi_m); \xi_i \neq \xi_j \in T_z(\mathbb{S}^n_+) \text{for} \ i \neq j\}$.

Remark 1.1. (i) We point out that given $z_1, \ldots, z_m \in \partial \mathbb{S}^n_+$ non degenerate critical points of $K_1 := K|_{\partial \mathbb{S}^n_+}$ satisfying $(\partial_\nu K_1(z_i)) > 0$ for each $i = 1, \ldots, m$ and $y_{m+1}, \ldots, y_{m+\ell}$ non degenerate critical points of $K$ with $\Delta K(y_i) < 0$ for each $i > m$, then there exists a sequence of solutions $u_\varepsilon$ of $(P_\varepsilon)$ which converges weakly to $0$ and blows up at $z_1, \ldots, z_m, y_{m+1}, \ldots, y_{m+\ell}$ and all these blow ups are isolated simple. See Theorem 1.1 in [2].
Since the function \( F_{z,m} \) does not have any critical point if \( z \) is a local maximum point, see Proposition 4.1 in [2], it follows that \( z \) in the situation of (d) cannot be local maximum of \( K_1 \). In other words if a local maximum point \( z \) is a blow up point then it has to be isolated simple.

The converse of statement (d) in the above theorem holds. Indeed given \( z \in \partial S^n_{\pm} \) a critical point of \( K_1 \) having \( \delta_s K(z) > 0 \) and \( m \in \mathbb{N} \). Then every non degenerate critical point of \( F_{z,m} \) gives rise to a solution of \( (P_e) \) building a cluster of \( m \) concentration points around \( z \). See Theorem 1.4 in [2].

In the next theorem we describe the blow up picture when the sequence of solutions has a non zero weak limit. We call such a behavior a blow up phenomenon with residual mass. Such a phenomenon does not occur in low dimension \( n \leq 4 \). We first describe the blow up scenario for half spheres of dimension greater than or equal to 7. Namely we prove:

**Theorem 1.2.** Let \( n \geq 7 \) and \( 0 < K \in C^3(S^n_{\pm}) \) be a positive function satisfying the assumptions (H1), (H2) and let \( (u_{\varepsilon}) \) be a sequence of energy bounded solutions of \( (P_e) \) converging weakly but not strongly \( u_{\varepsilon} \to \omega \neq 0 \). Then \( \omega \) is a solution of \( (P) \) and \( u_{\varepsilon} \) has to blow up and takes the following form

\[
  u_{\varepsilon} = \omega + \sum_{i \in a_{k,\varepsilon} \in \partial S^n_{\pm}} \frac{1}{K(a_{k,\varepsilon})((a_{k,\varepsilon} - 2)\varepsilon)} \delta_{a_{k,\varepsilon}, A_{k,\varepsilon}} + \sum_{i \in a_{k,\varepsilon} \in S^n_{\pm}} \frac{1}{K(a_{k,\varepsilon})((a_{k,\varepsilon} - 2)\varepsilon)} \delta_{a_{k,\varepsilon}, A_{k,\varepsilon}} + v_{\varepsilon},
\]

where \( \delta_{a,\lambda} \) is the standard bubble defined in \( (2.2) \) and \( \|v_{\varepsilon}\| = o_{\varepsilon}(1) \) where \( \|\cdot\| \) is defined by \( (2.1) \).

Furthermore the statements (a), (b), (c) and (d) of Theorem 1.1 hold.

**Remark 1.2.** The converse of Theorem 1.2 holds. See Theorems 1.2 and 1.7 in [2].

In the next theorem we single out the half sphere of dimension 5. Since in this case, unlike the closed case, blow up with residual mass may occur but involves exclusively boundary blow up points. Namely we prove

**Theorem 1.3.** Let \( n = 5 \) and \( 0 < K \in C^3(S^5_{\pm}) \) be a positive function satisfying the assumption (H1) and let \( (u_{\varepsilon}) \) be a sequence of energy bounded solutions of \( (P_e) \) converging weakly but non strongly \( u_{\varepsilon} \to \omega \neq 0 \). Then \( \omega \) is a solution of \( (P) \) and \( u_{\varepsilon} \) blows up in the following form

\[
  u_{\varepsilon} = \omega + \sum_{i \in a_{k,\varepsilon} \in \partial S^5_{\pm}} \frac{1}{K(a_{k,\varepsilon})^{3/2}} \delta_{a_{k,\varepsilon}, A_{k,\varepsilon}} + v_{\varepsilon},
\]

where \( \delta_{a,\lambda} \) is the standard bubble defined in \( (2.2) \) and \( \|v_{\varepsilon}\| = o_{\varepsilon}(1) \) where \( \|\cdot\| \) is defined by \( (2.1) \).

Furthermore Statements (b), (c) and (d) of Theorem 1.1 hold.

**Remark 1.3.**

i) The converse of Theorem 1.3 holds. See Theorems 1.2 and 1.7 in [2].

ii) On \( S^5_{\pm} \) we constructed in [2] blowing up solutions with residual mass involving only boundary blow up points. However the existence of blowing
up solutions with residual mass involving interior points remains an open problem even for the Nirenberg problem on the six dimensional sphere \( \mathbb{S}^6 \).

**Theorem 1.4.** Let \( n = 5 \) and \( 0 < K \in C^3(\mathbb{S}^5) \) be a positive function satisfying the assumption \((H1)\) or \( n \geq 7 \) and \( 0 < K \in C^3(\mathbb{S}^n) \) satisfy the assumptions \((H1)\) and \((H2)\). Let \( (\omega_k) \) be a sequence of energy bounded solutions of \((P)\). Then \( |\omega_k|_{\infty} \) is uniformly bounded (that is \( \omega_k \) cannot blow up).

**Remark 1.4.** We point out that the above results extend easily on compact riemannian manifolds with umbilic boundary. For more general manifolds there is a difficulty in the choice of a suitable modified bubble when the boundary is not umbilic. This technical point will the subject of a forthcoming paper.

Before closing this introduction we describe our strategy of proof. To perform an asymptotic analysis of blowing up solutions \( u_\varepsilon \) the usual blow up analysis techniques are based on precise pointwise \( C^0 \)-estimates of \( u_\varepsilon \) and the extensive use of Pohozaev identities [22, 23, 27, 30, 33, 34, 38, 41]. Our method of analysis is different. Indeed it consists of testing the equation by vector fields which bring the parameters of the concentration to their critical positions. These vector fields correspond to the leading term of the gradient of the Euler-Lagrange functional with respect to these parameters combined with what we call the barycentric vector field which consists of pushing a cluster of nearby blow up points to their common center of mass. We then derive balancing conditions to be satisfied by these parameters. Through a careful analysis of these balancing conditions, we derive the information regarding the location of the concentration points and the speed of the concentration. Eventually we derive the nature of the blow up point (i.e isolated simple or not). We believe that our method, which avoids the use of pointwise estimates and Pohozaev identities might be useful to deal with non compact variational problems where non simple blow ups occur like in the singular mean field equation with quantized singularities [10, 31, 44, 45, 25]. Indeed the existence of non simple blow up points makes the task of establishing pointwise \( C^0 \)-estimates a daunting one.

The remainder of the paper is organized as follows: in Section 2 we set up the variational framework of the problems \((P)\) and \((P_\varepsilon)\), introduce the neighborhood at infinity and its parametrization. Section 3 is devoted to the analysis of finite energy blowing up solutions in the zero limit case. Indeed in this section, after giving a precise estimates of the infinite dimensional part, we prove various balancing conditions satisfied by the parameters of the concentration and provide the proof of Theorem 1.1. In Section 4 we deal with the non zero weak limit case and provide the proofs of Theorems 1.2 and 1.3. While Section 5 is devoted to the proof of Theorem 1.4. Finally we collect in the appendix some useful estimates and technical lemmas needed in the proof of various statements in this paper.

2. Parametrization of the neighborhood at infinity

In this section we consider \( \varepsilon \geq 0 \) and we will set up the general variational framework, recall the description of the lack of compactness that one derives from the
concentration’s compactness principle, introduce the *neighborhood at infinity* and its parametrization.

The space of variation is the Sobolev space $H^1(\mathbb{S}^n_\ast)$ endowed with the norm

\begin{equation}
(2.1) \quad \|u\|^2 := \int_{\mathbb{S}^n_\ast} |\nabla u|^2 + \frac{n(n-2)}{4} \int_{\mathbb{S}^n_\ast} u^2.
\end{equation}

Indeed, for $\epsilon \geq 0$, the problem $(\mathcal{P}_\epsilon)$ has a variational structure. Namely its solutions are in one to one correspondence with the positive critical points of the functional

\[ L_\epsilon(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p+1-\epsilon} \int_{\mathbb{S}^n_\ast} |\nabla u|^{p+1-\epsilon}, \quad u \in H^1(\mathbb{S}^n_\ast) \text{ with } p := \frac{n+2}{n-2}. \]

For $a \in \mathbb{S}^n_\ast$ and $\lambda > 0$ we define the *standard bubble* to be

\begin{equation}
(2.2) \quad \delta_{a,\lambda}(x) := c_0 \frac{\lambda^{(n-2)/2}}{(\lambda^2 + 1 + (1-\lambda^2)\cos d(a,x))^{(n-2)/2}},
\end{equation}

where $d$ is the geodesic distance on $\mathbb{S}^n_\ast$ and $c_0 := (n(n-2))^{(n-2)/4}$ is chosen such that

\[-\Delta \delta_{a,\lambda} + \frac{n(n-2)}{4} \delta_{a,\lambda} = \delta_{a,\lambda}^{(n+2)/(n-2)} \quad \text{in } \mathbb{S}^n_\ast.\]

For $a \in \mathbb{S}^n_\ast$, we define *projected bubble* $\varphi_{a,\lambda}$ to be the unique solution to

\[-\Delta \varphi_{a,\lambda} + \frac{n(n-2)}{4} \varphi_{a,\lambda} = \varphi_{a,\lambda}^{(n+2)/(n-2)} \quad \text{in } \mathbb{S}^n_\ast, \quad \frac{\partial \varphi_{a,\lambda}}{\partial \nu} = 0 \text{ on } \partial \mathbb{S}^n_\ast.\]

We point out that $\varphi_{a,\lambda} = \delta_{a,\lambda}$ if $a \in \partial \mathbb{S}^n_\ast$.

Note that, the stereographic projection induces an isometry $\iota : H^1(\mathbb{S}^n_\ast) \to D^{1,2}(\mathbb{R}^n_\ast)$ and the function $\iota \delta_{a,\lambda}$ is a solution of the Yamabe problem in $\mathbb{R}^n_\ast$. Precisely, for $a \in \partial \mathbb{S}^n_\ast$, using the stereographic projection $\pi_a$ (see [5] for the precise formulae), we have

\[-\Delta (\iota \delta_{a,\lambda}) = (\iota \delta_{a,\lambda})^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n_\ast \quad \text{and} \quad \iota \delta_{a,\lambda}(x) := c_0 \frac{\lambda^{(n-2)/2}}{(1 + \lambda^2|x|^2)^{(n-2)/2}}.\]

Note that, if we use $\pi_b$ (with $b \in \partial \mathbb{S}^n_\ast$ instead of $-a$) the image $\iota \delta_{a,\lambda}$ becomes $c_0 \mu^{(n-2)/2}/(1 + \mu^2|x-a|^2)^{(n-2)/2}$ where the new variables $\tilde{\mu}$ and $\mu$ depend strongly on $a$ and $\lambda$ (see [2, 6] for the precise relations). In the sequel, we denote the function $\iota \delta_{a,\lambda}$ also by $\delta_{a,\lambda}$.

Let $\omega$ be a solution of $(\mathcal{P}_0)$ and let $N_0(\omega)$ be the kernel of the associated quadratic form defined by:

\begin{equation}
(2.3) \quad Q_\omega(h) := \|h\|^2 - \frac{n+2}{n-2} \int_{\mathbb{S}^n_\ast} K \omega^4/(n-2) h^2 \quad \text{for } h \in H^1(\mathbb{S}^n_\ast).
\end{equation}

Let $m$ be the dimension of $N_0(\omega)$ and $(e_1, \cdots, e_m)$ be an orthonormal basis of $N_0(\omega)$. We set

\[ H_0(\omega) := \text{span}(\omega) \oplus \text{span}(e_1, \cdots, e_m). \]
Following M. Mayer, see Lemma 3.6 and Proposition 3.7 in [39], we parameterize a neighborhood of $\omega$ by

$$u_{\alpha,\beta} := \alpha(\omega + \sum_{i=1}^{m} \beta_i e_i + h(\beta)) \quad \text{with}$$

$$h(\beta) \in H_0(\omega); h(\beta) = O(\|\beta\|^2) \text{ and } \|h(\beta)\|_{C^2} \to 0,$$

where the function $u_{\alpha,\beta}$ satisfies

$$\langle \nabla I_0(u_{\alpha,\beta}), h \rangle := \langle u_{\alpha,\beta}, h \rangle - \int Ku_{\alpha,\beta}^0 h = 0 \quad \text{for each } h \in H_0(\omega)^\perp.$$

Next for $\omega$ a solution of $(P)$ whose Kernel is of dimension $m$, $q$, $\ell \in \mathbb{N}_0$ and $\tau$ a small positive constant, we define the so called neighborhood at infinity $V(\omega, q, \ell, \tau)$ as follows:

$$V(\omega, q, \ell, \tau) := \left\{ u \in H^1(\mathbb{R}^n_+) : \exists \lambda_1, \ldots, \lambda_N > \tau^{-1} \text{ with } e \ln \lambda_i \leq \tau; \exists \alpha_1, \ldots, \alpha_N \in \mathbb{S}^n_+,$$

$$\lambda_i d_i < \tau, \forall i \leq q, \text{ and } \lambda_i d_i > \tau^{-1} \forall i > q, \varepsilon_{ij} < \tau; \exists \alpha_0 \in (1-\tau, 1+\tau);$$

$$\beta \in \mathbb{R}^m \text{ with } \|\beta\| \leq c \tau \text{ such that } \|u - \sum_{i=1}^{N} K(\alpha_i) 2^{2/n} \varphi_{\alpha_i,\lambda_i} - u_{\alpha_0,\beta}\| < \tau \right\}$$

where $N := q + \ell$, $d_i := d(\alpha_i, \partial \mathbb{S}^n_+)$ and

$$\varepsilon_{ij} := \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{1}{2} \lambda_i \lambda_j (1 - \cos d(\alpha_i, \alpha_j)) \right)^{(2-n)/2}.$$

Following A. Bahri and J-M. Coron [8] we consider for $u \in V(\omega, q, \ell, \tau)$ and $N = q + \ell$ the following minimization problem

$$\min_{\alpha_i > 0, \beta \in \mathbb{R}^m, \lambda_i > 0, u \in \mathbb{C}_0^\tau, \forall i \leq q; \alpha_i \in \mathbb{S}^n_+, \forall i > q} \left\| u - \sum_{i=1}^{N} \alpha_i \varphi_{\alpha_i,\lambda_i} - u_{\alpha_0,\beta} \right\|.$$

We then have the following proposition whose proof is identical, up to minor modification to the one of Proposition 7 in [8]

**Proposition 2.1.** For any $q, \ell \in \mathbb{N}_0$ there exists $\tau_0 > 0$ such that if $\tau < \tau_0$ and $u \in V(\omega, q, \ell, \tau)$ the minimization problem (2.7) has, up to permutation of the indices, a unique solution.

Hence it follows from Proposition 2.1 that every $u \in V(\omega, q, \ell, \tau)$ can be written in a unique way as

$$u = \sum_{i=1}^{q} \alpha_i \delta_{a_i, \lambda_i} + \sum_{i=q+1}^{N} \alpha_i \varphi_{\alpha_i,\lambda_i} + u_{\alpha_0,\beta} + v,$$

where $a_i \in \partial \mathbb{S}^n_+, \forall i \leq q$ and $a_i \in \mathbb{S}^n_+ \text{ with } \lambda_i d_i > \tau^{-1}, \forall i \geq q + 1,$

$$\alpha_i^{4/(n-2)} K(\alpha_i) = 1 + o(1) \forall i \geq 1, \quad \alpha_0 = 1 + o(1),$$
and \( v \in H^1(S^1) \) satisfying
\[
(V_0) \quad ||v|| < \tau, \quad <v, \psi> = 0, \text{ for } \psi \in E^\perp_{\omega,a,\lambda}
\]

(2.10)

(2.11)

Finally by combining the analysis of the Palais-Smale sequences, performed in Lemma 3.3 of [40], whose proof goes along with the concentration compactness arguments developed in [43] [35] [16] [27], with the above parametrization of the neighborhood at infinity we derive that:

**Proposition 2.2.** Let \( u_\varepsilon \) be an energy bounded solution of \((P_\varepsilon)\) which converges weakly to 0. Then there exist \( q \) and \( \ell \) such that \( u_\varepsilon \) can be written as

\[
u_{\varepsilon} := \sum_{i \leq q} \alpha_i \varepsilon \delta_{a_i, \lambda_i} + \sum_{i = q+1}^{q+\ell} \alpha_i \varepsilon \varphi_{a_i, \lambda_i} + v_\varepsilon \in V(q, \ell, \tau) := V(0, q, \ell, \tau)
\]

with \( v_\varepsilon \in E^\perp_{a_i, \lambda_i} := E^\perp_{0, a_i, \lambda_i}\).

3. The zero weak limit case

In this section we deal with the case where the blowing up sequence \( u_\varepsilon \) of \((P_\varepsilon)\) has a bounded energy and converges weakly to zero. We notice that, in the sequel, for sake of simplicity of the presentation, we will cancel the index \( \varepsilon \) from the points \( a_{i, \varepsilon} \) and the speeds \( \lambda_{i, \varepsilon} \).

We remark that, in the first part of this section, we consider \( \varepsilon \geq 0 \). That is the sequence \( (u_\varepsilon) \) (in the case of \( \varepsilon = 0 \)) will be assumed to be a sequence \( (\omega_k) \) of solutions of \((P)\). However, in Proposition 3.11 and Lemma 3.12 we will assume that \( \varepsilon > 0 \).

3.1. Estimates of the infinite dimensional part. In this subsection we deal with \( v_\varepsilon \) the infinite dimensional part of the blowing up solutions \( u_\varepsilon \). In the next lemma we provide an accurate estimate of this part, which shows that it does not have any contribution to the blowing up phenomenon. Namely we prove

**Proposition 3.1.** Let \( v_\varepsilon \) be defined as in Proposition 2.2. Then it satisfies:

\[
||v_\varepsilon|| \leq \varepsilon R(\varepsilon, a, \lambda)
\]

where

\[
R(\varepsilon, a, \lambda) := \varepsilon + \sum_{i=1}^N \frac{||K(a_i)||}{\lambda_i} + \frac{1}{\lambda_i^2} \begin{cases} 
\sum e_{\varepsilon i j} (\ln e_{\varepsilon i j}^{-1})^{\frac{\alpha_{d_{i j}}}{\varepsilon}} + \sum_{i > q} \frac{\ln(\lambda_{d_i})}{(\lambda_{d_i})^{\frac{3}{2}}} & \text{if } n \geq 6, \\
\sum e_{\varepsilon i j} (\ln e_{\varepsilon i j}^{-1})^{3/5} + \sum_{i > q} \frac{1}{(\lambda_{d_i})^{3/5}} & \text{if } n = 5.
\end{cases}
\]

**Proof.** Observe that, for each \( u, h \in H^1(S^1) \), it holds that

\[
\langle \nabla I_\varepsilon(u), h \rangle = \langle u, h \rangle - \int K|u|^{p-\varepsilon-1}uh.
\]
Hence taking \( u = u_e \) and \( h = v_e \) and using the fact that \( v_e \in E_{a_i}^+ \), we derive that

\[
\|v_e\|^2 = \int K u_e^{p-\varepsilon} v_e = \int K \overline{u}_e^{p-\varepsilon} v_e + p \int K \overline{u}_e^{p-1-\varepsilon} v_e^2 + o(\|v_e\|^2)
\]

where \( \overline{u}_e := u_e - v_e \). Since \( \alpha_i^{4/(n-2)} K(a_i) = 1 + o(1) \) for each \( i \), easy computations imply that

\[
\|v_e\|^2 - p \int K \overline{u}_e^{p-1-\varepsilon} v_e^2 = \|v_e\|^2 - p \sum \int \delta_i^{p-1} (v_e)^2 + o(\|v_e\|^2) := Q_{a_i}(v_e) + o(\|v_e\|^2).
\]

We remark that \( Q_{a_i} \) is a positive definite quadratic form in the space \( E_{a_i}^+ \) (see Proposition 3.1 of [5]). Furthermore, the linear form is computed in Eq. (23) of [2] and we have

\[
\int_{E^+_a} K u_e^{p-\varepsilon} v_e = O \left( R(\varepsilon, a, \lambda) \|v_e\|^2 \right).
\]

Combining the previous estimates we obtain

\[
Q_{a_i}(v_e) = o(\|v_e\|^2) + O \left( \|v_e\| \left( R(\varepsilon, a, \lambda) \right) \right).
\]

The estimate of \( v_e \) follows from the fact that \( Q_{a_i} \) is a positive definite quadratic form. \( \blacksquare \)

3.2. **Balancing conditions for the parameters of the concentration.** We next provide various balancing conditions that have to be satisfied by the parameters of concentration. The following propositions are quoted from [2] (Propositions 3.1 and 3.2) by using the fact that \( \nabla I_e(u_e) = 0 \) since \( u_e \) is a solution of \( (P_e) \). Our first such a condition concerns the gluing parameter \( \alpha_i \) for \( i = 1, \ldots, N \). Namely we have:

**Proposition 3.2.** [2] For each \( 1 \leq i \leq N \), it holds:

\[
|1 - \lambda_i^\varepsilon(n-2)/2 \alpha_i^{4/(n-2)} K(a_i)| = O(R_{a_i}) \quad \text{where}
\]

\[
R_{a_i} := \varepsilon + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \sum_{j \neq i} \varepsilon_{ij} + R(\varepsilon, a, \lambda)^2 + \begin{cases} \|\nabla K(a_i)\|/\lambda_i & \text{if } i \leq q, \\ 1/(\lambda_i \varepsilon^{2q-2}) & \text{if } i \geq q + 1 \end{cases}.
\]

Our next balancing condition concerns the rate of the concentration \( \lambda_i \) for \( 1 \leq i \leq N \). Using Propositions 3.1, 3.2 of [2] and Proposition 3.2 above, we get:

**Proposition 3.3.** [2] For \( \varepsilon \) small enough, the following equations hold:

\[
\left\{ \begin{array}{l}
-c_2 \sum_{i \neq j \leq q} \alpha_j \lambda_i \frac{\partial E_{ij}}{\partial \lambda_i} - \frac{\alpha_i}{K(a_i)} \frac{c_3}{\lambda_i} \frac{\partial K}{\partial \varepsilon} (a_i) = c_5 K(a_i) \varepsilon \\
= O \left( \frac{1}{\lambda_i^2} + \sum_{j \neq q} \varepsilon_{ij} + R_{1,i} \right); & \text{if } i \leq q
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
-c_2 \sum_{j \neq i} \alpha_i \lambda_j \frac{\partial E_{ij}}{\partial \lambda_i} + c_2 \frac{n-2}{2} \sum_{j=q+1}^N \frac{\alpha_j}{(\lambda_i \lambda_j)^{n-2}} \\
+ \alpha_i \left( \frac{\Delta K(a_i)}{\lambda_i^2 K(a_i)} + 2c_5 \varepsilon \right) = O(R_{2,i}); & \text{if } i > q
\end{array} \right.
\]
Proposition 3.4. Namely we prove

upt. Namely we will prove that while interior blow ups are isolated simple (3.7). The existence of such blow up points non simple boundary our last balancing condition concerns the point of concentration and 1.

To perform our analysis we make use of the following notation. We denote by (3.8)

where $R_{1,\lambda} := \sum_{j \neq k} e^{\frac{\mu}{x_{kj}}} \ln(e_{kj}^{-\frac{\mu}{x_{ija}}}) + R^2(\epsilon, a, \lambda)$, $R_{2,\lambda} := R_{1,\lambda} + \sum_{k>q} \ln(\lambda_k d_k)/(\lambda_k d_k)^n$ and

$c_2 := c_0^{p+1} \int_{\mathbb{R}^n} \frac{1}{x^2(|x|^n+2)^2} dx; \quad c_3 := (n-2) c_0^{p+1} \int_{\mathbb{R}^n} x_n(x_n^2 - 1) dx,$

c_4 := \frac{n-2}{2n} c_0^{p+1} \int_{\mathbb{R}^n} x_n(x_n^2 - 1) \left(1+|x|^2\right)^{n+1} dx; \quad c_5 := \frac{(n-2)^2}{4} c_0^{p+1} \int_{\mathbb{R}^n} \left(1+|x|^2\right)^{n+1} dx.$

Our last balancing condition concerns the point of concentration $a_i$ for $i = 1, \cdots, N$. Namely we prove

**Proposition 3.4.** For $\epsilon$ small enough, there hold:

(3.7)

(F_i) \left\{ \begin{array}{l}
- \frac{c_2}{2} \sum_{j \neq i} \frac{1}{\lambda_i} \frac{\partial e_{ij}}{\partial a_i} - \frac{a_i}{K(a_i) \lambda_i} \nabla K_i(a_i) = O\left(1/\lambda_i^5 + \sum_{j \neq i} e_{ij} + R_{\lambda_i}\right) \text{ for } i \leq q \\
\left| \nabla K(a_i) \right| \frac{1}{\lambda_i} \leq c \left(1/\lambda_i^5 + 1/(\lambda_i d_i)^{n-2} + \sum_{j \neq i} e_{ij} + R^2(\epsilon, a, \lambda) \right) \text{ for } i > q
\end{array} \right.

where

$R_{\lambda_i} := R_{1,\lambda} + \sum_{j \neq q, i \neq i} e_{ij}^q \lambda_j d(a_i, a_j)$, \quad $c_6 = \frac{n-2}{n} c_0^{p+1} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{n+1}} dx.$

3.3. **Refined blow up analysis.** In this section we analyze the nature of the blow ups. Namely we will prove that while interior blow ups are isolated simple and their rates of concentration are comparable, there might be non simple boundary blow up points. The existence of such non simple boundary blow up points at a boundary blow up point $z \in \partial \mathbb{S}^n_+$ is in one to one correspondence with the existence of critical points of some Kirchhoff-Routh type Hamiltonian $\mathcal{F}_z$. To perform our analysis we make use of the following notation. We denote by

(3.8) $I_b := \{i : a_i \in \partial \mathbb{S}^n_+\}; \quad I_{in} := \{i : a_i \in \mathbb{S}^n_+\}$; \quad $\mu_i := \begin{cases} \lambda_i & \text{if } i \in I_b \\ \lambda_i^2 & \text{if } i \in I_{in} \end{cases}$

Note that, in our case, $#I_b = q$ and $#I_{in} = \ell$. Furthermore, we order the $\mu_i$’s as

(3.9) $\mu_1 \leq \cdots \leq \mu_N$.

Let

(3.10) $I' := \{i : \lim_{\epsilon \to 0} \mu_i/\mu_1 = \infty\}; \quad I := \{1, \cdots, N\} \setminus I'$ where $N := q + \ell$.

We notice that, for $i, j \in I$, the speeds of concentration $\mu_i$ and $\mu_j$ are of the same order (i.e. the ratio is bounded above and below) and for each $k \notin I$ and each $i \in I$, we have that the ratio $\mu_k/\mu_i$ goes to $\infty$ as $\epsilon \to 0$. Finally, for a critical point $z$ of $K_1$ (resp. a critical point $y$ of $K$) we denote by

(3.11) $B_z := \{i \in I \cap I_b : \lim_{\epsilon \to 0} a_i = z\}; \quad B_y := \{i \in I \cap I_{in} : \lim_{\epsilon \to 0} a_i = y\}$.

Before studying the nature of the blow up we need the following Lemmata.
Lemma 3.5. For each solution \( u_{\nu} \) in \( V(q, \ell, \tau) \), it holds
\[
\sum_{k \neq i} e_{ik} + \sum_{k \in I_{in}} \left( \frac{|\nabla K(a_k)|}{\lambda_k} + \frac{1}{(\lambda_k d_k)^{n-2}} \right) + \varepsilon \leq c \frac{1}{\mu_1}.
\]
Furthermore \( R_{1,\delta} \) and \( R_{2,\delta} \) defined in Proposition 3.3 satisfy: \( R_{k,\delta} = o\left( \frac{1}{\mu_1^{(n-1)/(n-2)}} \right) \).

Proof. We first notice that
\[
-\lambda_i \frac{\partial e_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial e_{ij}}{\partial \lambda_j} \geq 0 \quad \text{for each } i \neq j \quad \text{and} \quad -\lambda_i \frac{\partial e_{ij}}{\partial \lambda_i} \geq c e_{ij} \quad \text{if } \lambda_i \geq c \lambda_j.
\]
Hence, summing \( 2^j(E_j) \) (defined in (3.6)) for \( i \in I_{in} \) and for \( i \in I_b \) respectively, we obtain
\[
\sum_{i \in I_{in}, k \neq i} e_{ik} + \sum_{i \in I_{in}} \frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon \leq c \left( R_{2,\delta} + \frac{1}{\mu_1} \right)
\]
\[
\sum_{i, k \in I_b, k \neq i} e_{ik} + \varepsilon \leq c \left( R_{1,\delta} + R_{2,\delta} + \frac{1}{\mu_1} \right).
\]
The result follows by using \( (F_i) \) for \( i \in I_{in} \) and the definitions of \( R_{1,\delta} \) and \( R_{2,\delta} \).

Lemma 3.6. For \( i \in I_{in} \), we define
\[
\Gamma_i := \lambda_i^2 \sum_{k \neq i} e_{ik} + \frac{H(a_i, a_i)}{\lambda_i^n} + \frac{|\nabla K(a_i)|}{\lambda_i} + \varepsilon + \sum_{k \neq j} e_{kj} + \frac{1}{(\lambda_j d_j)^{n-2}}.
\]
Let \( D_i' := \{ i \in I_{in} : \lim \Gamma_i = \infty \} \) and \( D_1 := I_{in} \setminus D_i' \).
(i) For each \( i \in D_1 \), there exists a critical point \( y_i \) of \( K \) such that \( \lambda_i |a_i - y_i| \leq C \).
(ii) If there exists an index \( i \in D_i' \), then it holds that
\[
\sum_{j \in I_{in}, j \neq i} \left( \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k \neq j} e_{kj} + \varepsilon + \frac{1}{(\lambda_j d_j)^{n-2}} \right) = o\left( \frac{1}{\mu_1^{(n-1)/(n-2)}} \right).
\]

Proof. Let \( i \in D_1 \), it follows that \( \Gamma_i \) is bounded which implies that \( |\nabla K(a_i)| \leq C/\lambda_i \) and therefore the first assertion follows. Concerning the second one, let \( i \in D_i' \), summing \( 2^j(E_j) + (F_j)/m \) for \( j \geq i \) and \( j \in I_{in} \) where \( m \) is a small constant, it holds that
\[
\sum_{j \in I_{in}, j \geq i} \left( \frac{|\nabla K(a_i)|}{\lambda_j} + \sum_{k \neq j} e_{kj} + \varepsilon + \frac{H(a_j, a_j)}{\lambda_j^n} \right) = O\left( \frac{1}{\lambda_i^2} + R_{2,\delta} \right).
\]
Since \( i \in D_i' \), it follows that \( 1/\lambda_i^2 \) is small with respect to the left hand side. Thus the proof follows from the estimate of \( R_{2,\delta} \) (see Lemma 3.5).

Lemma 3.7. (i) For each \( i \in I_{in} \) and each \( j \in I_b \), it holds that \( e_{ij} = o(1/\mu_1^{(n-1)/(n-2)}) \).
(ii) For each \( i, j \in I_{in} \), it holds that \( e_{ij} = o(1/\mu_1) \).
Proof. Claim (i) follows from Lemma 3.6. It remains to prove the second claim. Let \( i, j \in I_b \). Observe that, if \( i \) or \( j \) belongs to \( D'_1 \), then the result follows from Lemma 3.6. In the other case, that is \( i, j \in D_1 \), using again Lemma 3.6, there exist critical points \( y_i \) and \( y_j \) such that \( \lambda_k d(a_k, y_k) \leq c \) for \( k = i, j \). Two cases may occur:

(a) Either \( y_i \neq y_j \), and in this case we get \( d(a_i, a_j) \geq c \) and therefore the result follows easily,

(b) or \( y_i = y_j \). Since we have \( \lambda_k d(a_k, y_k) \leq c \) and \( \varepsilon_{ij} \) is small, it follows that \( \lambda_i / \lambda_j \to 0 \) or \( \infty \). Taking \( \lambda_i \leq \lambda_j \) and using the fact that \( \Gamma_j \) is bounded, it holds

\[
\varepsilon_{ij} \leq \frac{c}{\lambda_j} = \frac{\lambda_j^2}{\lambda_j^2 \lambda_j^2} = o\left( \frac{1}{\lambda_j} \right) = o\left( \frac{1}{\mu_1} \right).
\]

Hence the proof is completed. \( \blacksquare \)

3.3.1. Ruling out bubble towers. In this section we prove that the rate of concentration of boundary concentration points are comparable and the rate of concentration of interior points are also comparable. This fact rules out the phenomenon of bubble towers.

We start with a preliminary lemma:

Lemma 3.8. (1) Assume that there exist \( i_0 \neq j_0 \in I_b \) such that \( \lim_{\lambda_0} \varepsilon_{i_0 j_0} = \infty \). Then it holds that

\[
\varepsilon + \sum_{k \neq i_0} \varepsilon_{i_0 k} + \frac{1}{\lambda_{i_0}} = o\left( \frac{1}{\mu_1^{n-1}(n-2)} \right).
\]

(2) For each \( i, j \in I_b \) with \( j \neq i \), it holds

(a) \( \frac{1}{\lambda_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \leq c / \lambda_j^{(n-1)(n-2)} \) if \( c' \leq \frac{\lambda_i}{\lambda_j} \leq c \) and \( \lambda_i \varepsilon_{ij} \leq c \),

(b) \( \frac{1}{\lambda_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| = o\left( \frac{1}{\mu_1^{(n-1)(n-2)}} \right) \) in the other cases.

Proof. Summing \( 2'(E_i) \) for \( i \in I_b \) and \( i \geq i_0 \), it follows that

\[
\sum_{i \in I_b} \sum_{j \geq i_0} \varepsilon_{ik} + \varepsilon + O\left( \frac{1}{\lambda_{i_0}} \right) = O\left( \sum_{k \neq i_0} \varepsilon_{i_0 k} + R_{1, \lambda} \right).
\]

Hence assertion (1) follows from Lemmas 3.5, 3.7 and the fact that \( 1 / \lambda_{i_0} \) is small with respect to \( \varepsilon_{i_0 j_0} \) which exists in the left hand side.

Now we will focus on the second assertion. Observe that, in general, it holds

\[
\frac{1}{\lambda_i} \left| \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \leq c \lambda_j d(a_j, a_i) \varepsilon_{ij}^{(n-2)} \leq c \sqrt{\lambda_j / \lambda_i} \varepsilon_{ij}^{(n-1)(n-2)} \leq c \varepsilon_{ij}.
\]

Observe that, Claim (a) follows from the second inequality of (3.15). Concerning Claim (b), three cases may occur:

- if \( \lambda_i \varepsilon_{ij} \to \infty \) or \( \lambda_j \varepsilon_{ij} \to \infty \), then (b) follows from (3.15) and Claim (1),
- if \( (\lambda_i + \lambda_j) \varepsilon_{ij} \leq c \) and \( \lambda_i / \lambda_j \to \infty \), then (b) follows from the second inequality of (3.15).
Claim 3: On the other hand, using Lemma 3.5, we derive that
\[ \sqrt{\lambda_j / \lambda_i (\varepsilon_{ij})^{(n-1)/(n-2)}} \leq (\lambda_i / \lambda_j)^{\frac{n-1}{n-2}} (1 / \lambda_i)^{\frac{n-1}{n-2}} = o(1 / \mu_i^{(n-1)/(n-2)}). \]
Hence the proof of (b) is completed. \[ \square \]

Now, using the above lemmas, Equations \((E_i)\) and \((F_i)\) can be improved and we get
\[ (E'_i) : \left\{ \begin{array}{ll}
2c_5 \varepsilon + c_4 \Delta K(a_i) / \lambda_i^2 K(a_i) = o\left( \frac{1}{\mu_1} \right) & \text{for } i \in I_{in}, \\
-\frac{c_2}{2} \sum_{i \neq j \in I_b} \partial \varepsilon_{ij} / \partial a_i - \alpha_i \frac{\partial}{\partial \lambda_i} \left[ c_3 \frac{\partial K}{\partial \lambda_i} (a_i) - c_5 \varepsilon K(a_i) \right] = o\left( \frac{1}{\mu_1^{n-2}} \right) & \text{for } i \in I_b, \\
\end{array} \right. \]
\[ (3.17) \]
\[ (F'_i) : -\frac{c_2}{2} \sum_{j \in I_b, j \neq i} \alpha_j \frac{1}{\lambda_j} \partial \varepsilon_{ij} / \partial a_i - \alpha_i \frac{c_6}{\lambda_i} \nabla K(a_i) = o\left( \frac{1}{\mu_1^{(n-1)/(n-2)}} \right) & \text{for } i \in I_b. \]

Next we rule out bubble towers by proving that all concentration’s rates \(\mu_i\) are comparable. Namely we prove

**Proposition 3.9.** All the \(\mu_i\)'s are comparable. That is \(I = I_b \cup I_{in}\).

**Proof.** Recall that \(N = q + \ell\). Arguing by contradiction, we assume that \(N \notin I\). Thus we get that \(\lim \mu_N / \mu_1 = \infty\). We claim that

**Claim 1:** \(\varepsilon = o(1 / \mu_1)\). In fact two cases may occur: (a) Either \(N \in I_{in}\), and in this case the claim follows easily by using \((E'_N)\) or (b) \(N \in I_b\), in this case, \((E'_N)\) implies that
\[ \sum_{j \in I_b, j \neq N} \varepsilon_{ij} + \varepsilon + O\left( \frac{1}{\lambda_i} \right) = o\left( \frac{1}{\mu_1^{(n-1)/(n-2)}} \right) \]
and the claim follows in this case also. Hence the proof of Claim 1.

**Claim 2:** \(I \cap I_{in} = \emptyset\). Assume that there exists \(j \in I \cap I_{in}\). It follows that \(\mu_1\) and \(\lambda_j^2\) are of the same order. Hence using Claim 1 and \((E'_j)\) we derive that \(\Delta K(a_j) = o(1)\). On the other hand, using Lemma 3.3, we derive that \(a_j\) has to converge to a critical point of \(K\) which leads to a contradiction. Thus Claim 2 follows.

**Claim 3:** For each \(i \in I\), we have: \(a_i\) converges to a critical point \(z_i\) of \(K_1\) satisfying \(\partial K / \partial \nu(z_i) > 0\). Using Claim 2, it follows that \(I \subset I_b\). Let \(i \in I\). Using \((F'_i)\) and Lemma 3.3 we get \(|\nabla K_1(a_i)| / \lambda_i = O(1 / \mu_1)\) which implies that \(|\nabla K_1(a_i)| = o(1)\) since \(\lambda_i\) and \(\mu_1\) are of the same order. Thus \(a_i\) has to converge to a critical point \(z_i\) of \(K_1\). It remains to prove that \(\partial K(z_i) > 0\). First, we remark that
\[ (3.18) \quad \varepsilon_{kj} = o(1 / \mu_1) \quad \text{for each } k \in I \text{ and } j \in I_b \setminus I \text{ (that is } \lambda_j / \lambda_k \rightarrow \infty). \]
In fact, \((3.18)\) follows from Lemma 3.3 if \(\lambda_j \varepsilon_{kj} \rightarrow \infty\). In the other case, that is \(\lambda_j \varepsilon_{kj} \leq c\), it holds that \(\varepsilon_{kj} \leq c / \lambda_j = o(1 / \lambda_k)\) and \((3.18)\) follows in this case also. Secondly, using \((3.18)\), Claim 1 and \((E'_i)\), we get
\[ \sum_{k \in I, k \neq i} \alpha_k (c + o(1)) \varepsilon_{ik} - c \alpha_i \frac{\partial K}{\partial \nu}(z_i) \frac{1}{\lambda_i} = o\left( \frac{1}{\mu_1} \right) \]
which implies that \( \partial_i K(z_i) \) has to be positive. Thereby Claim 3 is completed.

**Claim 4:** For each critical point \( z \) of \( K_1 \), it holds \#\( B_z \neq 1 \) where \( B_z := \{ i \in I \cap I_b : \lim a_i = z \} \).

Assume that there exists \( z \) such that \#\( B_z = 1 \). Let \( B_z = \{ i \} \). It follows that \( \alpha_{ij} = o(1/\mu_1) \) for each \( j \in I_b \) with \( i \neq j \) (in fact, for \( j \in I \), we derive that \( |a_i - a_j| \geq c \) and for \( j \in I_b \setminus I \), it follows from (3.13)). Thus (\( E' \)) leads to a contradiction (by using Claim 1) and therefore Claim 4 follows.

In the sequel, let \( z \) be such that \#\( B_z \geq 2 \) and let \( i_0, i_1 \in B_z \) be such that \( |a_{i_0} - a_{i_1}| := \min\{ |a_i - a_j| : i, j \in B_z \text{ with } i \neq j \} \). We introduce the following sets:

\[ A'_z := \{ j \in B_z : \lim |a_j - a_{i_0}|/|a_{i_0} - a_{i_1}| = \infty \} \quad \text{and} \quad A_z := B_z \setminus A'_z. \]

We remark that \( A_z \) contains at least \( i_0 \) and \( i_1 \).

**Claim 5:** Let \( z \) be such that \#\( B_z \geq 2 \), it holds that (i): \( \sum_{k \neq i \in A_z} \alpha_{ik} = (c + o(1))/\lambda_i \) for each \( i \in A_z \). Furthermore, (ii): for each \( i \neq j \in A_z \), it holds \( \lambda_i^{(n-3)/(n-2)} d(a_i, a_j) \) is bounded above and below.

We notice that, for each \( i \neq j \in A_z \), it holds that \( d(a_i, a_j) \) and \( d(a_{i_0}, a_{i_1}) \) are of the same order. Furthermore, \( \lambda_i \) and \( \lambda_j \) are of the same order. Hence, \( \alpha_{ij} = (\lambda_i^1 d(a_i, a_j)^2/c) \) which implies that all the \( \alpha_{ij} \), for each \( i \neq j \in A_z \), are of the same order. Thus (ii) follows immediately from (i). Concerning (i), it follows from Claim 1, (\( E' \)) and the fact that \( \alpha_{ij} = o(1/\mu_1) \) for each \( i \in A_z \) and \( j \notin A_z \) (the last information is immediately for \( j \in I \backslash B_z \)). It follows from (3.13) for \( j \notin I \).

For \( j \in A'_z \), we have \( d(a_i, a_j)/d(a_{i_0}, a_{i_1}) = \infty \) and the \( \alpha_k \)'s are of the same order which implies that \( \alpha_{ij} = o(1/\mu_1) \). Hence Claim 5 is complete.

To conclude the proof of Proposition 3.9, we need to introduce the barycenter of the points \( a_i \)'s for \( i \in A_z \). Let \( b \in \mathbb{R}^{n+1} \) be such that \( \sum_{i \in A_z} (b - a_i) = 0 \) and we define \( \overline{a} := b/|b| \). This point is the barycenter of the points \( a_i \)'s and it satisfies

\[ \overline{a} \in \partial S^n_+ ; \quad a_i - \langle a_i, \overline{a} \rangle \overline{a} \in T_{\overline{a}}(\partial S^n_+) \quad \forall \ i \in A_z \quad \text{and} \quad \sum_{i \in A_z} a_i - \langle a_i, \overline{a} \rangle \overline{a} = 0. \]

In addition it is easy to get that \( \lambda_i |\overline{a} - \langle a_i, \overline{a} \rangle a_i| \leq c \lambda_i d(a_i, \overline{a}) \leq c \lambda_i^{1/(n-2)} \) (by using Claim 5).

Now, multiplying (\( F' \)) by \( \alpha_i \lambda_i (\overline{a} - \langle a_i, \overline{a} \rangle a_i) \) (noting that this quantity belongs to the tangent space of \( \partial S^n_+ \) at the point \( a_i \)) and summing for \( i \in A_z \), it holds that

\[ -c_0 \sum_{i \in A_z} \dfrac{\partial^2 K(a_i)}{K(a_i)} (\overline{a} - \langle a_i, \overline{a} \rangle a_i) - c_2 \sum_{i, j \in A_z ; j \neq i} a_i a_j \dfrac{\partial \alpha_{ij}}{\partial a_i} (\overline{a} - \langle a_i, \overline{a} \rangle a_i) = o\left( \dfrac{1}{\mu_1} \right). \]

Observe that, Proposition 3.2 implies that \( \alpha_i^2 = K(a_i)^{(2-n)/2} + O(\ln(\lambda_i)/\mu_1) \). Using Lemma 5.4 we get that

\[ \sum_{i \in A_z} \dfrac{\alpha_i^2}{K(a_i)} \nabla K(a_i)(\overline{a} - \langle a_i, \overline{a} \rangle a_i) \]

\[ = \sum_{i \in A_z} \dfrac{1}{K(a_i)^{n/2}} \nabla K(a_i)(\overline{a} - \langle a_i, \overline{a} \rangle a_i) + O\left( \dfrac{\ln \lambda_i}{\mu_1} d(a_i, \overline{a}) \right). \]
we will consider the case where

\[ (3.21) \quad - \sum_{i \in A_x} \frac{1}{K_1(\bar{a})^{n/2}} \nabla K_1(\bar{a})(a_i - \langle a_i, \bar{a} \rangle \bar{a}) + O(d(\bar{a}, \bar{x})^2) + o\left(\frac{1}{\mu_1}\right) = o\left(\frac{1}{\mu_1}\right) \]

by using (3.19). Finally, using Lemma 6.3 there holds

\[ (3.22) \quad \frac{\partial \epsilon_{ij}}{\partial a_i}(\bar{a} - \langle a_i, \bar{a} \rangle a_i) + \frac{\partial \epsilon_{ij}}{\partial a_j}(\bar{a} - \langle a_j, \bar{a} \rangle a_j) \geq c \epsilon_{ij}. \]

Hence (3.20)-(3.22) contradict Claim 5. Therefore the proof of Proposition 3.9 is completed. \(\blacksquare\)

**Remark 3.10.** We point out that for \(\varepsilon = 0\), each sequence of solutions \(\omega_k\) of \((P_0)\) cannot be in \(V(q, \ell, \tau)\) for \(k\) large. Indeed the assumption that \(N \notin I\), made in the beginning of the proof, is only used in the proof of Claim 1.

3.3.2. **Location of blow up points and the speeds of concentration.** In the sequel we will consider the case where \(\varepsilon > 0\). In the next proposition we characterize the location of blow up points and provide the rate of the concentration parameters \(\lambda_i\). Namely we prove

**Proposition 3.11.** Assume that \(\varepsilon > 0\).

(a) Every interior concentration point \(a_i\) converges to a critical point \(y_i\) of \(K\) with \(\Delta K(y_i) < 0\), \(\lambda_i d(a_i, y_i)\) is uniformly bounded and \(y_i\) is an isolated simple blow up point. Moreover the concentration speed satisfies:

\[ -c_4 \frac{\Delta K(y_i)}{\lambda_i^2 K(y_i)} = 2c_5 \varepsilon (1 + o(1)), \]

where \(c_4\) and \(c_5\) are dimensional constants defined in Proposition 3.3.

(b) Every boundary concentration point \(a_j\) converges to a critical point \(z_j\) of \(K_1\) with \(\partial_i K(z_j) > 0\). Furthermore, it holds:

\[ c_3 \frac{\partial_i K(z_j)}{K(z_j)} \frac{1}{\lambda_j} = c_5 \varepsilon (1 + o(1)), \]

where \(c_3\) and \(c_5\) are dimensional constants defined in Proposition 3.3.

**Proof.** We start by proving Claim (a). By Proposition 3.9 we have that \(I = I_b \cup I_{in}\). Hence, for each \(i \in I_{in}\), it follows that \(\mu_i := \lambda_i^2\) and \(\mu_1\) are of the same order. Hence using (ii) of Lemma 3.6 and \((E'_i)\), we derive that the set \(D'_i\) has to be the empty set (indeed: if there exists \(j \in D'_i\) then \(a_j\) will converge to a critical point \(y\) of \(K\) and therefore \(|\Delta K(a_j)| > c > 0\). Thus (ii) of Lemma 3.6 and \((E'_i)\) are not compatible). Thus, from (i) of Lemma 3.6 it follows that \(a_i\) converges to a critical point \(y_i\) of \(K\) and we have \(\lambda_i d(a_i, y_i) \leq c\). The sign of \(\Delta K(y_i)\) and the behavior of the \(\lambda_i\) follow from \((E'_i)\). Now assume that there exist \(i \neq j \in I_{in}\) such that \(y_i = y_j\). Since \(\lambda_i d(a_i, y_k) \leq c\) for \(k = i, j\) and \(\lambda_i/\lambda_j\) is bounded from below and above, we derive that \(\lambda_i \lambda_j d(a_i, a_j)^2\) is bounded and this is not compatible with the fact that \(\epsilon_{ij}\) is small. Hence the proof of the assertion (a) is completed. Concerning the second assertion, let \(i \neq j \in I_b\), since \(\lambda_i\) and \(\lambda_j\) are of the same order
we derive that $\lambda_k d(a_i, a_j) \to \infty$ as $\varepsilon \to 0$ for $k = i, j$ (by using the smallness of $e_{ij}$).

Using Lemma 3.5, it holds

$$\frac{1}{\lambda_i} \frac{\partial e_{ij}}{\partial a_i} \leq c \lambda d(a_i, a_j) e_{ij}^{n/(n-2)} \leq \frac{c}{\lambda_d(a_i, a_j)} e_{ij} = o(e_{ij}) = o(1/\mu_1).$$

Now using $(P'_i)$ and the fact that $\lambda_i$ and $\mu_1$ are of the same order, we derive that $|\nabla K_1(a_i)| = o(1)$ for each $i \in I_B$ and therefore, $a_i$ has to converge to a critical point $z_i$ of $K_1$. To conclude the sign of $\partial K/\partial \nu(z_i)$, we need the following claim

**Claim A:** $e_{ij} = o(1/\mu_1)$ for each $i \neq j \in I_B$.

Arguing by contradiction, we assume that, there exists $i \in I_B$ such that $\sum_{j \neq i, j \in I_B} e_{ij} \geq c/\lambda_i$. We remark that, in this case, the critical point $z_i$ (satisfying $a_i \to z_i$) has to satisfy $\#B_z \geq 2$. In fact, if not, i.e. $\#B_z = 1$ we derive that $|a_i - a_j| \geq c$ for each $j \neq i$ and therefore $e_{ij} = O(1/(\lambda_i \lambda_j)^{(n-2)/2})$. Furthermore, we notice that Claim 5 of Proposition 3.9 holds true (in fact, the proof relies on $(i)$ and this information is assumed in our case). Thus, arguing as in the end of the proof of Proposition 3.9, we derive a contradiction which implies the proof of our claim.

To achieve the proof of the proposition, we use Claim A and $(E'_i)$ for $i \in I_B$.

### 3.3.3. Type of boundary blow up points.

In this subsection we study more carefully the boundary blow up points giving a precise characterization of isolated simple blow up points and the non simple blow up points.

**Lemma 3.12.** Assume that all the $\mu_i$’s are of the same order and let $i_0 \neq i_1 \in I_B$ be such that $d(a_{i_0}, a_{i_1}) := \min\{d(a_i, a_j) : i \neq j \in I_B\}$. Then we have that:

(i) Either there exists a constant $c$ such that $d(a_{i_0}, a_{i_1}) \geq c > 0$ and in this case we have that $\#B_z \leq 1$ for each critical point $z$ of $K_1$,

(ii) $d(a_{i_0}, a_{i_1}) \to 0$ as $\varepsilon \to 0$ and in this case, there exists at least one critical point $z$ of $K_1$ with $\#B_z \geq 2$ and there exist positive constants $c, c', \bar{c}$ such that the following estimates hold:

(a) $\lambda_i^{-(n-2)/n} d(a_i, a_j) \geq c$ for each $i \neq j \in I_B$.

(b) $\lambda_i^{-(n-2)/n} d(a_i, z) \leq \bar{c}$ for each $z$ such that $\#B_z \geq 2$ and for each $i \in B_z$.

(c) $c \leq \lambda_i^{(n-2)/n} d(a_i, a_j) \leq c'$ for each $z$ such that $\#B_z \geq 2$ and for each $i \neq j \in B_z$.

**Proof.** The first assertion (i) is immediate. Hence we will focus on the second assertion (ii).

Since $\lambda_i$ and $\lambda_j$ are of same order for each $i, j \in I_B$, we derive that $e_{ij} \leq c d(a_i, a_j)$ for each $i, j \in I_B$. Now, let $z$ be a critical point of $K_1$ such that $\#B_z \geq 2$ and let $k_0 \neq k_1 \in B_z$. Observe that, for each $i \in B_z$ and for each $j \in I_{k_0} \cup (I_B \setminus B_z)$, it holds that $e_{ij} \leq c/(\lambda_i \lambda_j)^{(n-2)/2} \leq c/\lambda_i^2$. Furthermore, for $j \in I_B$, it holds that $\lambda_j d(a_i, a_j) e_{ij}^{n/(n+1)(n-2)} \leq c e_{ij}^{n/(n-2)}$. Thus $(F_i)$ (defined in (3.7)) implies

$(3.23)$ $(F''_i) \quad \frac{c_2}{2} \sum_{j \in B_i \setminus I \#i} \alpha_j \frac{\partial e_{ij}}{\partial a_j} - c \alpha_i \frac{\partial}{\partial K(a_i)} \nabla K_1(a_i) = O\left(\frac{1}{\lambda_i} + \lambda_i e_{0i}^{\beta} \ln(e_{0i}^{\alpha_i})\right)$.

Now, let $j_0$ be defined by: $|\nabla K_1(a_{j_0})| := \max\{|\nabla K_1(a_i)| : i \in B_z\}$. 


Claim 1: There exists $M$ such that: 
\[ \frac{d(a_{i_0}, z)}{\lambda_{i_0}} \leq \frac{M}{(\lambda_{i_0} d(a_{i_0}, a_{i_0}))^{\alpha-1}}. \]

Arguing by contradiction, we assume that such a $M$ does not exist. Since the $\lambda_j$’s are of the same order and $z$ is nondegenerate, it follows that: 
\[ e^{(n-1)/(n-2)}_{i_0 i_1} = o(|\nabla K_1(a_{i_0})|/\lambda_{i_0}). \]

Hence (3.23) (with $i = j_0$) implies:
\[ \frac{|\nabla K_1(a_{i_0})|}{\lambda_{i_0}} + \sum_{j \in B_z, j \neq j_0} O(e^{(n-1)/(n-2)}_{j_0 j}) = O(\frac{1}{\lambda_{i_0}^2} + e^{n/(n-2)}_{i_0 i_1} \ln(e^{-1}_{i_0 i_1})). \]

Since $e_{ij} \leq c e_{i_0 i_1}$ for each $i, j \in I_0$ and $e^{(n-1)/(n-2)}_{i_0 i_1} = o(|\nabla K_1(a_{i_0})|/\lambda_{i_0})$, then it follows that $\lambda_{i_0} |\nabla K_1(a_{i_0})|$ is bounded. Therefore, we get that $\lambda_{i_0} d(a_{i_0}, z)$ is bounded. In addition, by the definition of $j_0$, we get that $d(a_{i_0}, z) \leq c d(a_{i_0}, z)$ for each $i \in B_z$. Thus, since the $\lambda_j$’s are of the same order, we derive that $\lambda_j d(a_{i_0}, z)$ is bounded for each $i \in B_z$. This implies that $\lambda_i \lambda_j d(a_{i_0}, a_{i_0})^2$ is bounded which contradicts the smallness of $\varepsilon_{ij}$. Hence Claim 1 follows.

Note that, Claim 1 implies that
\[ (3.24) \]
\[ \exists M \text{ such that } \forall z \text{ (with } \#B_z \geq 2), \forall i \in B_z \text{ it holds } \frac{d(a_{i_0}, z)}{\lambda_i} \leq \frac{M}{(\lambda_{i_0} d(a_{i_0}, a_{i_0}))^{\alpha-1}}. \]

Claim 2: There exists $c > 0$ such that:
\[ \frac{d(a_{i_0}, a_{i_1})}{\lambda_{i_0}} \geq \frac{c}{(\lambda_{i_0} d(a_{i_0}, a_{i_0}))^{\alpha-1}}. \]

Recall that $d(a_{i_0}, a_{i_1}) := \min\{d(a_{i_0}, a_{i_1}) : i \neq j \in I_0\}$. Let $z_0$ be such that $a_{i_0} \rightarrow z_0$ and let
\[ A'_{z_0} := \{ i \in B_{z_0} : \lim d(a_{i_0}, a_{i_1})/d(a_{i_0}, a_{i_1}) = \infty \} \quad \& \quad A_{z_0} := B_{z_0} \setminus A'_{z_0}. \]

It follows that
- $\forall i \neq j \in A_{z_0}, d(a_{i_0}, a_{i_1})$ and $d(a_{i_0}, a_{i_1})$ are of the same order.
- $\forall i \in A_{z_0}$ and $\forall j \in A'_{z_0}$, it holds: $\lim d(a_{i_0}, a_{i_1})/d(a_{i_0}, a_{i_1}) = \infty$.

Since the $\lambda_j$’s are of the same order, the second assertion implies that:
\[ (3.25) \]
\[ \frac{1}{\lambda_i} |\frac{\partial \varepsilon_{ij}}{\partial a_{i_1}}| \leq \frac{1}{\lambda_i (\lambda_{i_0})^{(n-2)/2}} \frac{1}{d(a_{i_0}, a_{i_1})^{\alpha-1}} = o(e^{(n-1)/(n-2)}_{i_0 i_1}) \quad \forall i \in A_{z_0}, \forall j \in A'_{z_0}. \]

Let $\bar{a}$ be defined as
\[ (3.26) \]
\[ \bar{a} := \frac{b}{|b|} \quad \text{where} \quad b \in \mathbb{R}^{n+1} \text{ satisfying } \sum_{j \in A_{z_0}} (b - a_j) = 0. \]

Note that (3.19) holds with $z = z_0$. Multiplying $(F''_j)$ (defined in (3.23)) by $\alpha_i(\bar{a} - \langle a_i, \bar{a} \rangle a_i)$ and summing over $i \in A_{z_0}$, we obtain (by using (3.25))
\[ -\frac{c_2}{2} \sum_{j \in A_{z_0}, j \neq i} \alpha_i(\bar{a} - \langle a_i, \bar{a} \rangle a_i) - c_6 \frac{\alpha_i^2}{K(a_i)} \nabla K_1(a_i) \bar{a} - \langle a_i, \bar{a} \rangle a_i) \]
\[ = O\left(\frac{d(\bar{a}, a_i)}{\lambda_i} + o \left( \lambda_j d(\bar{a}, a_i) e^{(n-1)/(n-2)}_{i_0 i_1} \right) \right). \]
Observe that Lemma 6.3 gives the estimate of the first term. Now, using \((3.4)\) and Proposition 3.11, we derive that
\[
\alpha_i^2 = \frac{1}{K(a_j)^{n-2}/2} + O(\varepsilon/\ln \varepsilon).
\]
Furthermore, using Claim \((i)\) of Lemma 6.4 \((\text{with } h = \overline{u})\) and \((3.19), (3.27)\) implies \((3.28)\)
\[
\sum_{i \neq j \in A_0} \varepsilon_{ij} \leq c \sum_{i \in A_0} d(\overline{u}, a_i)^2 + c \sum_{i \in A_0} \frac{d(\overline{u}, a_i)}{\lambda_i} + \sum_{i \in A_0} o\left(\lambda_i d(\overline{u}, a_i) e_{ij} + d(a_i, z) d(\overline{u}, a_i)\right).
\]
Note that \((3.24)\) implies that \(d(a_i, z) d(a_{i_0}, a_{i_1}) \leq c e_{ij} \). In addition, since \(\lambda_i d(a_{i_0}, a_{i_1})\) is very large and \(d(\overline{u}, a_i) \leq cd(a_{i_0}, a_{i_1})\) for each \(i \in A_0\), we get that \(d(\overline{u}, a_i)/\lambda_i = o(d(a_{i_0}, a_{i_1})^2)\). Furthermore, using the fact that \(\lambda_i d(\overline{u}, a_i) e_{ij} / \lambda_i^1/(n-2) \leq c\) for each \(i \in A_0\), then \((3.28)\) implies that \(e_{ij} \leq cd(a_{i_0}, a_{i_1})^2\) which implies Claim 2.

Now, we are in position to prove Assertion \((ii)\) of the lemma. In fact: Claim 2 implies that \(\lambda_i^{(n-2)/n} d(a_{i_0}, a_{i_1}) \geq c\) and therefore, since all the \(\lambda_j\)'s are of the same order and \(d(a_{i_0}, a_{i_1}) = \min[d(a_i, a_j) : i \neq j \in I_b]\), we get that
\[
\lambda_i^{(n-2)/n} d(a_i, a_j) \geq c \quad \text{for each } i \neq j \in I_b.
\]
Hence the proof of Assertion \((a)\) \((\text{in particular, the first inequality of the assertion } (c)\) follows.

Furthermore, let \(z\) be such that \#\(B_z\) \(\geq 2\), and let \(i \in B_z\), using \((3.24)\), we get
\[
(3.30) \quad \lambda_i^{2(n-2)/n} d(a_i, z) = \lambda_i^{2(n-2)/n} d(a_i, z) \leq \frac{MA_j^{(2n-2)/n}}{\lambda_i (\lambda_i d(a_{i_0}, a_{i_1}))^{n-1}} \leq \frac{C}{\lambda_i^{(n-2)/n} d(a_{i_0}, a_{i_1})^{n-1}} \leq c
\]
(by using \((3.29)\)) which gives the proof of Assertion \((b)\).

Now, for \(z\) such that \#\(B_z\) \(\geq 2\) and for \(i \neq j \in B_z\), using the fact that \(|a_i - a_j| \leq |a_i - z| + |a_j - z|\) and applying \((3.30)\), we derive the second inequality of Assertion \((c)\) which completes the proof of Assertion \((c)\).

Note that, once Assertion \((c)\) is proved, we conclude that \(A'_j = \emptyset\) for each \(z\) such that \#\(B_z\) \(\geq 2\). The proof of the lemma is thereby completed.

Note that from \((3.29)\), it follows that (we can be more precise if \(d(a_i, a_j) \geq c\)
\[
(3.31) \quad \varepsilon_{ij} \leq \frac{c}{\lambda_i^{2(n-2)/n}} \quad \text{for each } i \neq j \in I_b.
\]
Furthermore, Assertion \((c)\) of Lemma 3.12 implies that, for each \(z\) such that \#\(B_z\) \(\geq 2\), all the \(d(a_i, a_j)\)'s, for \(i \neq j \in B_z\), are of the same order. Hence \(B_z = A_z\) for each \(z\).

**Lemma 3.13.** Let \(z\) be such that \(B_z = \{i\}\). Then it holds that \(\lambda_i d(a_i, z) \leq c\).

**Proof.** We need to use Equation \((F_i)\) \((\text{introduced in } (3.7))\) but in this equation, it appears one term in \(R_{a_i}\) which we cannot control. This term is \(\sum_{k \neq j} \varepsilon_{kj} n/(n-2) \ln(\varepsilon_{kj}^{-1})\) \((\text{by using } (3.31))\), it is enough to get \(\gamma > n/(n-2)\) instead of \(n/(n-2)\). For this reason, we will repeat the proof of the equation \((F_i)\) in our special case, that is \(|a_j - a_i| \geq c\) for each \(j \neq i\). In fact, we will follow the proof of the third assertion of
Proposition 3.1 of [2] (by taking \( \omega = 0 \)). Multiplying the equation satisfied by \( u_e \) by \( (1/\lambda_i)(\partial \delta_i / \partial x_i) \) we get

\[
(3.32) \quad \langle u_e, \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} \rangle = \int_{\mathbb{R}^n} K_{w^{-p}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} \quad \text{with} \quad p := \frac{n + 2}{n - 2}.
\]

Since \( v_e \) satisfies (2.10) and \( a_i \in \partial \mathbb{D}_r^a \), the first term is \( \sum_{j \neq i} O(\epsilon_{ij}) \) which is \( o(1/\lambda_i^3) \) (by using the fact that \( d(a_i, a_j) > c > 0 \) for each \( j \neq i \)). Furthermore, the other term is estimated as

\[
(3.33) \quad \int_{\mathbb{R}^n} K_{w^{-p}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} = \int_{\mathbb{R}^n} K_{w^{-p}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} + (p - \epsilon) \int_{\mathbb{R}^n} K_{w^{-p-1-e}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} + O(||v_e||^2)
\]

where \( \overline{u} := u_e - v_e \). Furthermore, using the behavior of \( \lambda_i \) (given in Claim (b) of Proposition 3.11) and (3.31), we derive that \( ||v_e||^2 = O(1/\lambda_i^2) \). In addition, observe that

\[
\left| \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} - \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} \right| \leq c_{\mu^{-p-1}} - (\alpha_i \delta_i)^{1/2} \leq c \sum_{j \neq i} \left\{ \left( \delta_i \delta_j \right)^{1/2} \right\}
\]

Hence, following the proof of Eq (23) of [2], it follows that the second term of (3.33) is \( O(R(e, a, \lambda)||v_e||) \) and therefore it is also \( O(1/\lambda_i^2) \). For the first integral of the right hand side of (3.33), using the behavior of \( \lambda_i \) (given in Proposition 3.11), it holds that

\[
\int K_{w^{-p}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} = \int K_{w^{-p}} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} + \sum_{j \neq i} O(\epsilon_{ij}) = \alpha_i^{1-q} \frac{c}{\lambda_i} \frac{\partial K}{\partial x^2}(a_i) + O(1/\lambda_i^2)
\]

Thus, (3.32) becomes

\[
\nabla K_1(a_i) = O(1/\lambda_i^2)
\]

which implies the result (since the critical points of \( K_1 \) are non-degenerate).}

### 3.4. Proof of Theorem 3.11

The assertions (a) and (b) follow from Proposition 3.11 by choosing

\[
(3.34) \quad \kappa_1(n) := c_4/(2c_5) \quad \text{and} \quad \kappa_2(n) := c_4/c_5
\]

where the constants \( c_3, c_4 \) and \( c_5 \) are defined in Proposition 3.3.

It remains to see the last one. Let \( z \) be a critical point of \( K_1 \) such that \( \#B_z \geq 2 \) and let \( i_0 \) and \( i_1 \) be such that \( d(a_{i_0}, a_{i_1}) := \min[d(a_i, a_j) : i \neq j \in B_z]. \) Let \( b_i \) be defined by (1.5). From Assertion (c) of Lemma 3.12 and the behavior of the \( \lambda_i \)’s (given in Claim (b) of the theorem), we derive that \( 0 < c < |b_i - b_j| \leq c' \) for each \( i \neq j \). In addition, using the behavior of \( \lambda_i \) and using Assertion (b) of Lemma 3.12 we derive that \( |b_i| \leq C \) for each \( i \in B_z \). Hence, it follows that

\[
(3.35) \quad |b_i| \leq C \quad \text{and} \quad |b_i - b_j| \geq c \quad \forall i \neq j \in B_z.
\]
Furthermore observe that: \(2(1 - \cos(d(a_i, a_j))) = |a_i - a_j|^2\) (seen as two points of \(\mathbb{R}^{n+1}\)) and we notice that, if \(d(a, b)\) is small, it holds that \(d(a, b) = |a - b|(1 + o(1))\). Therefore, from the definition of \(\varepsilon_{ij}\) (see (2.6)) and the fact that \(\lambda_i\) and \(\lambda_j\) are of the same order, it follows that

\[
\frac{\partial \varepsilon_{ij}}{\partial a_i} = \frac{n - 2}{4} \lambda_i \lambda_j (a_j - a_i) \varepsilon_{ij}^{(n-2)/2} = \frac{2^{n-2}(n-2)}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{a_j - a_i}{|a_j - a_i|^n} + O(\lambda_i \varepsilon_{ij} n^{1/4}) \tag{3.36}
\]

In addition, using (3.3), it holds that

\[
\frac{\alpha_j}{\alpha_i} = \frac{(K_1(a_j))^{(n-2)/4}}{(K_1(a_i))} + O(\varepsilon |\varepsilon|)
\]

(3.37) Then, using Lemma 6.4 and (3.36), (3.37), (F’’), implies

\[
- \frac{c_0}{2} \sum_{j \in B_i; j \neq i} \frac{2^{n-2}(n-2)}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{1}{|a_j - a_i|^n} (a_j - a_i, e - \langle e, a_i \rangle a_i)
\]

\[
-c_6 \frac{1}{K_1(z)} D^2 K_1(z)(a_i - \langle a_i, z \rangle, e - \langle e, z \rangle) + O(d(a_i, z)^2) = o_{\varepsilon}(e^{(n-2)/n}).
\]

Note that it is easy to see that: \(|a_i - z|^2 = 2(1 - \langle a_i, z \rangle)\) and therefore, it holds that \(\langle a_i, z \rangle = 1 + O(\varepsilon^{2(n-2)/n})\) (by using (b) of Lemma 3.12 and the fact that \(d(a_i, z) = |a_i - z|(1 + o(1))\)) which implies that \(b_i\) (defined by (1.3)) satisfies \(b_i = \varepsilon^{2(n-2)/n}(a_i - z) + O(\varepsilon^{(n-2)/n})\). Thus, using the behavior of the \(\lambda_i\)'s (see Claim (b)) and the change of variables (1.3), and multiplying (3.38) by \(\varepsilon^{(n-2)/n}\), it follows that

\[
- \sum_{j \in B_i; j \neq i} \frac{n - 2}{|b_j - b_i|^n} (b_j - b_i, e - \langle e, z \rangle) - D^2 K_1(z)(b_i, e - \langle e, z \rangle) = o_{\varepsilon}(1),
\]

by choosing \(\kappa_3(n)\) equal to

\[
\kappa_3(n) := 2^{(n-3)/n}(c_2/c_6)^{1/n}(c_5/c_3)^{(n-2)/n}.
\]

Finally, using (3.35), we derive that, for each \(i \in B_i\), \(b_i\) converges to \(\overline{b}_i\) satisfying

\[
\overline{b}_i \in T_a(\partial \mathcal{B}_\delta) \, , \, |\overline{b}_i| \leq c \, , \, |\overline{b}_j - \overline{b}_i| \geq c \, \forall \, i \neq j \, \text{ and}
\]

\[
\sum_{j \in B_i; j \neq i} \frac{n - 2}{|\overline{b}_j - \overline{b}_i|^n} (\overline{b}_j - \overline{b}_i, e - \langle e, z \rangle) + D^2 K_1(z)(\overline{b}_i, e - \langle e, z \rangle) = 0
\]

which means that \((\overline{b}_1, \cdots, \overline{b}_m)\) is a critical point of \(\mathcal{P}_{\varepsilon,m}\) (where \(m := |B_z|\)). Thus the proof of the theorem is completed.

4. The non zero weak limit case

Proposition 4.1. Let \(u_\varepsilon\) be an energy bounded solution of \((\mathcal{P}_\varepsilon)\) which blows up. We assume that there exists a positive solution \(\omega\) of \((\mathcal{P}_0)\) such that \(u_\varepsilon \to \omega\) (but
$u_e \to \omega$. Then there exist $q$ and $\ell$ such that $u_e$ can be written as

\[(4.1) \quad u_e := u_{\alpha, \beta} + \sum_{i=1}^{q} \alpha_i \delta_i + \sum_{i=q+1}^{q+\ell} \alpha_i \varphi_j + v_e \in V(\omega, q, \ell, \tau) \quad \text{with} \quad v_e \in E_{\omega, a, \lambda}^L.\]

In this section, we will adapt the program done in Section 3 and we will present the contribution of $\omega$ in the expansions proved in the previous section. In the sequel, we will collect some properties which are satisfied by the parameters and the solution $u_e$. Our first result concerns the function $v_e$ to show that it does not have any contribution in the phenomenon.

**Proposition 4.2.** Let the remainder term $v_e$ be defined as in Proposition 2. Then there holds:

$$||v_e|| \leq cR(\varepsilon, a, \lambda) + \sum \xi(\lambda_i) \quad \text{where} \quad \xi(\lambda_i) := \begin{cases} 1/\lambda_i^{(n-2)/2} & \text{if } n \leq 5 \\ \ln(\lambda_i)^2/\lambda_i^2 & \text{if } n = 6 \end{cases}$$

and where $R(\varepsilon, a, \lambda)$ is defined in Proposition 3.1

**Proof.** Recall that $\omega$ is a solution of $(\mathcal{P}_0)$ (not necessary non degenerate), thus we decompose $H^1(S^n_+)$ as

\[(4.2) \quad H^1(\mathbb{S}^n_+) := N_-(\omega) \oplus H_0(\omega) \oplus N_+(\omega) \quad \text{with} \quad H_0(\omega) := \text{span}[\omega] \oplus N_0(\omega)\]

where $N_-(\omega)$, $N_0(\omega)$ and $N_+(\omega)$ are respectively the space of negativity, of nullity and of positivity of the quadratic form $Q_\omega$ (defined by (2.3)) in $\text{span}[\omega]$. Note that these spaces are orthogonal spaces with respect to $\langle ., . \rangle$ and the associated bilinear form $B_\omega(., .)$ (defined by (2.3)). Furthermore, the sequence of the eigenvalues (denoted by $(\sigma_i)$) corresponding to $Q_\omega$ satisfies $\sigma_i \nrightarrow 0$. Therefore, there exists a constant $c > 0$ such that

\[(4.3) \quad Q_\omega(h) \leq -c||h||^2 \quad \forall \ h \in N_-(\omega) ; \quad Q_\omega(h) \geq c||h||^2 \quad \forall \ h \in N_+(\omega).\]

Using (4.2), $v_e$ can be decomposed as follows

\[(4.4) \quad v_e := v_e^- + v_e^0 + v_e^+ \quad \text{where} \quad v_e^0 \in H_0(\omega) ; \quad v_e^- \in N_-(\omega) \quad \text{and} \quad v_e^+ \in N_+(\omega).\]

Since $v_e \in E_{\omega, a, \lambda}^L$, using Lemma 6.7 we get that $v_e^0 = o(||v_e||)$. For the other parts, note that $v_e^-$ and $v_e^+$ are not necessarily in $E_{\omega, a, \lambda}^L$ but they are in $H_0(\omega)$. Now we will focus on estimating $v_e^-$. Taking $u = u_e$ and $h = v_e^-$ in (3.1) and using (6.14), we derive that

\[(4.5) \quad \sum \alpha_j \langle \varphi_j, v_e^- \rangle + (u_{\alpha, \beta}, v_e^-) + ||v_e^-||^2 = \int K\overline{u}_e^{p-\varepsilon}v_e^-\]

where $\overline{u}_e := u_e - v_e$. Using (6.13), (2.9) and the fact that $v_e^-$ is in a finite dimensional space (which implies that $||v_e^-|| \leq c||v_e||$), we derive that

$$\int K\overline{u}_e^{p-1-\varepsilon}v_e^- = \sum_{i=1}^{N} \int K(\alpha_i \varphi_i)^{p-1-\varepsilon}v_e^- + \int K\alpha_{i, \beta}^{p-1-\varepsilon}v_e^- + o(||v_e|| ||v_e^-||)$$
(4.6) \[ \sum_{i=1}^{N} O\left(|v_{i}^{e}|_{\infty} \int \delta_{i}^{p-1} |v_{e}^{i}|\right) + \int K \omega^{p-1}(v_{e}^{0} + v_{e}^{+} + v_{e}^{-}) v_{e}^{-} + o(|v_{e}||v_{e}^{-}|) \]

where we have used the orthogonality of \( v_{e}^{-} \), \( v_{e}^{0} \) and \( v_{e}^{+} \) with respect to \( \int K \omega^{p-1} \cdots \). Thus (4.5) and (4.6) imply that

(4.7) \[ -Q_{\omega}(v_{e}^{-}) + o(||v_{e}^{e}|||v_{e}^{-}||) = \sum_{j} \alpha_j \langle \varphi_j, v_{e}^{-} \rangle + \langle u_{\alpha, \beta}, v_{e}^{-} \rangle - \int K \mu_{e}^{p-1} v_{e}^{-} := \ell(v_{e}^{-}). \]

Observe that, using \( ||v_{e}^{-}||_{\infty} \leq c ||v_{e}^{-}|| \), it holds that

(4.8) \[ (\varphi_j, v_{e}^{-}) \leq c \int \delta_{i}^{p} |v_{e}^{-}| \leq c ||v_{e}^{-}||_{\infty} \int \delta_{i}^{0} \leq c \frac{||v_{e}^{-}||}{\lambda_{i}^{(n-2)/2}} ; \quad \int \delta_{i} |v_{e}^{-}| \leq c \frac{||v_{e}^{-}||}{\lambda_{i}^{(n-2)/2}}. \]

(4.9) \[ \int K \mu_{e}^{p-1} v_{e}^{-} = \int K u_{\alpha, \beta} v_{e}^{-} + \sum_{i} O\left( \int u_{\alpha, \beta} \delta_{i}^{p-1} |v_{e}^{-}| + \int \delta_{i}^{p-1} |v_{e}^{-}| \right) \]

Therefore, combining (4.3), (4.7), (4.9) with (2.5), we get

(4.10) \[ c ||v_{e}^{-}||^{2} \leq -Q_{\omega}(v_{e}^{-}) \leq c ||v_{e}^{-}|| \left( \varepsilon + \sum_{i} \frac{1}{\lambda_{i}^{(n-2)/2}} + o(||v_{e}||) \right). \]

It remains to estimate the \( v_{e}^{+} \)-part. Recall that \( v_{e}^{+} \in N_{e}(\omega) \) and it is not necessarily in \( E_{\alpha i}^{e} \). Note that Eq (4.5) holds with \( v_{e}^{+} \) instead of \( v_{e}^{-} \). Observe that, using (6.13) and (2.9), it holds

(4.11) \[ \int K \mu_{e}^{p-1} v_{e}^{+} = \sum_{i=1}^{N} \int K(\alpha_{i} \varphi_{i})^{p-1} v_{e}^{+} + \sum_{i=1}^{N} \int K \omega^{p-1}(v_{e}^{0} + v_{e}^{+}) v_{e}^{+} + o(||v_{e}|||v_{e}^{+}||) \]

by using the fact that \( ||v_{e}^{-}||_{\infty} \leq c ||v_{e}^{-}|| \) and \( ||v_{e}^{0}||_{\infty} \leq c ||v_{e}^{0}|| \) and the orthogonality of \( v_{e}^{-} \) (respectively \( v_{e}^{0} \)) and \( v_{e}^{+} \) with respect to \( \int K \omega^{p-1} \cdots \). Hence we obtain

(4.12) \[ Q_{\omega, \alpha, \beta}(v_{e}^{+}) + o(||v_{e}|||v_{e}^{+}||) = -\sum_{j} \alpha_j \langle \varphi_j, v_{e}^{+} \rangle - \langle u_{\alpha, \beta}, v_{e}^{+} \rangle + \int K \mu_{e}^{p-1} v_{e}^{+} := -\ell(v_{e}^{+}). \]

Using Lemma (5.7) we derive that \( Q_{\omega, \alpha, \beta}(v_{e}^{+}) \geq c ||v_{e}^{+}||^{2} + o(||v_{e}||^{2}) \), hence it remains to estimate the linear part \( \ell(v_{e}^{+}) \). In fact, using (6.13), we have

(4.13) \[ \int K \mu_{e}^{p-1} v_{e}^{+} = \sum_{\alpha i} K(\alpha_{i} \varphi_{i})^{p-1} v_{e}^{+} + \sum_{\alpha i} K u_{\alpha, \beta} v_{e}^{+} \]

Concerning the remainder terms, using Lemma 6.6 it follows that

\begin{equation}
\int (\delta_i \delta_j)^{p/2} |v^+_e|^2 \leq ||v^+_e|| \left( \int (\delta_i \delta_j)^{n/2} \right)^{(n+2)/(2n)} \leq c ||v^+_e|| \ln^{n+2}(\epsilon_i^{-1}),
\end{equation}

\begin{equation}
\int \delta_i^{p/2} |v^+_e|^2 \leq ||v^+_e|| \left( \int \delta_i^{n/2} \right)^{(n+2)/(2n)} \leq c ||v^+_e|| \ln^{n+2}(\epsilon_i^{-1}).
\end{equation}

For the other integrals, we recall that \( v^+_e \in N_+^2(\omega) \) and therefore it follows that \( v^+_e \in H_0(\omega)^+ \). Hence using (2.5) we derive that

\begin{equation}
- (u_{a,\beta}, v^+_e) + \int_{S^n_i} K\alpha_{\alpha,\beta} v^+_e = - (u_{a,\beta}, v^+_e) + \int_{S^n_i} K\alpha_{\alpha,\beta} v^+_e + O(\epsilon ||v^+_e||) = O(\epsilon ||v^+_e||).
\end{equation}

Furthermore, using Lemma 6.2 it holds that

\begin{equation}
\int K(\alpha_i \delta_i)^{p-\epsilon} v^+_e = \frac{\alpha_i^p K(\alpha_i)}{\lambda_i^{(n-2)/2}} \int S^n_i \delta_i^{p} v^+_e + O \left( \int S^n_i (K(x) - K(a_i)) \delta_i^{p} v^+_e \right)
\end{equation}

\begin{equation}
= \frac{\alpha_i^p K(\alpha_i)}{\lambda_i^{(n-2)/2}} \int S^n_i \delta_i \ln \left( 2 + (\lambda_i^2 - 1)(1 - \cos \delta(a_i, x)) ||v^+_e|| + \epsilon ||v^+_e|| \right)
\end{equation}

Thus, for \( i \leq q \), (that is \( a_i \in \partial S^n_i^+ \)), we have \( \varphi_i = \delta_i \) and therefore we get

\begin{equation}
- \alpha_i (\delta_i, v^+_e) + \int K(\alpha_i \delta_i)^{p-\epsilon} v^+_e = \alpha_i \left( \frac{\alpha_i^p K(\alpha_i)}{\lambda_i^{(n-2)/2}} - 1 \right) (\delta_i, v^+_e) + O(\epsilon ||v^+_e||) \left( \frac{\delta_i \ln(2 + \lambda_i^2 - 1)(1 - \cos \delta(a_i, x)) ||v^+_e|| + \epsilon ||v^+_e||}{\lambda_i^2} \right).
\end{equation}

However, for \( i \geq q + 1 \) (that is \( a_i \in S^n_i^+ \)), we have

\begin{equation}
\int K\varphi_i^{p-\epsilon} v^+_e = \int K\delta_i^{p-\epsilon} v^+_e + O \left( \int \delta_i^{p-1} ||\varphi_i - \delta_i|| ||v^+_e|| \right).
\end{equation}

The first term is estimated in (4.20) and the second one satisfies

\begin{equation}
\int \delta_i^{p-1} ||\varphi_i - \delta_i|| ||v^+_e|| \leq c ||v^+_e|| \left( \int \delta_i^{8n/(n-4)} ||\varphi_i - \delta_i||^{2n/(n+2)} \right)^{(n+2)/(2n)}.
\end{equation}
The estimate of the last integral depends on the dimension \( n \). In fact, for \( n \geq 6 \), we have \( 2n/(n+2) \geq n/(n-2) \) and using the fact that \(|\varphi_i - \delta_i| \leq c \min(\delta_i, 1/(\lambda_i d_i^2)^{(n-2)/2}) \) (see Lemma \([6,1]\)), we derive that
\[
\int_{S^*} \delta_{i}^{2n/(n+2)} |\varphi_i - \delta_i|^{2n/(n+2)} \leq c |\varphi_i - \delta_i|_{C^0}^{2n/(n-2)} \int_{B(a_i,d_i)} \delta_{i}^{2n/(n+2)} + \int_{S^* \setminus B(a_i,d_i)} \delta_{i}^{2n/(n-2)} \leq c \frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^n}
\]
and for \( n \leq 5 \), we have \( 8n/(n^2-4) > n/(n-2) \) and therefore it holds
\[
\int_{S^*} \delta_{i}^{8n/(n^2-4)} |\varphi_i - \delta_i|^{2n/(n+2)} \leq c |\varphi_i - \delta_i|_{C^0}^{2n/(n-2)} \int_{B(a_i,d_i)} \delta_{i}^{8n/(n^2-4)} \leq c \frac{1}{(\lambda_i d_i)^{2(n-2)/(n+2)}}.
\]

Finally, using the fact that \( v_\xi \in E_{w,a_\lambda} \), as in the computations of (4.8), we derive that
\[
\langle \varphi_i, v_\xi^+ \rangle = \langle \varphi_i, v_\xi \rangle - \langle \varphi_i, v_\xi^0 \rangle - \langle \varphi_i, v_\xi^- \rangle = O\left( \frac{\|v_\xi\|}{\lambda_i^{(n-2)/2}} \right) \quad \forall 1 \leq i \leq q + \ell.
\]

Thus, using (4.3), (4.12) becomes
\[
(4.22) \quad c\|v_\xi^+\|^2 \leq Q_{w,a_\lambda}(v_\xi^+) \leq o(\|v_\xi\|^2) + O\left( \|v_\xi\| \left( R(\epsilon, a_\lambda) + \sum \xi(a_i) \right) \right).
\]

Thus using (4.4), (4.10) and (4.22) we derive that
\[
\|v_\xi\|^2 = \|v_\xi^+\|^2 + \|v_\xi^-\|^2 + o(\|v_\xi\|^2) \leq c\|v_\xi\| \left( R(\epsilon, a_\lambda) + \sum \xi(a_i) \right)
\]
which completes the proof. \( \blacksquare \)

4.1. **Balancing conditions for blow ups with residual mass.** In this subsection we prove various balancing conditions for the parameters of concentration in the case where the blowing up solution \( u_\epsilon \) of (\( \mathcal{P}_\epsilon \)) decomposes into a solution of the problem (\( \mathcal{P} \)) plus a sum of interior and or boundary bubbles. The following propositions are quoted from \([2]\) (Propositions 3.1 and 3.2) by taking \( u_{a,\beta} \) instead of \( \omega \). Note that in our case we have \( \nabla I_\epsilon(u_\epsilon) = 0 \).

We start by estimating the gluing parameters \( a_i \). Namely we have:

**Proposition 4.3.** \([2]\) For each \( 1 \leq i \leq N \), it holds:
\[
|1 - \lambda_i^{2n/(n-2)} a_i^{2n/(n-2)} K(a_i) = O\left( R_{a_i} + \frac{1}{\lambda_i^{(n-2)/2}} + \sum \frac{\ln \lambda_k}{\lambda_k^{n/2}} \right)
\]
where \( R_{a_i} \) is defined in Proposition \([3,2]\).

Next we provide balancing conditions involving the rate of the concentration \( \lambda_i \) and the mutual interaction of bubbles \( \epsilon_{ij} \).

**Proposition 4.4.** For \( \epsilon \) small enough, the following equations hold:
\[
-\frac{c_2}{2} \sum_{j \neq i, j \leq q} \alpha_j \lambda_i \partial \epsilon_{ij} \partial \lambda_i - \frac{\alpha_i}{K(a_i)} \left[ c_3 \frac{\partial K}{\lambda_i} \partial \nu (a_i) - c_5 K(a_i) \epsilon \right] + c_2 \frac{n-2}{4} (1 + o(1)) \frac{\omega(a_i)}{\lambda_i^{(n-2)/2}} = O\left( \frac{1}{\lambda_i^2} + \sum_{j \leq q} \epsilon_{ij} + R_{1,\lambda} + \sum \frac{\ln \lambda_k}{\lambda_k^{n/2}} \right) \quad (i \leq q)
\]
\[-c_2 \sum_{j \neq i} \alpha_j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + c_2 \frac{n-2}{2} \sum_{j=q+1}^{p} \alpha_j \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} + \alpha_i \left( c_4 \frac{\Delta K(a_i)}{\lambda_i^2 K(a_i)} + 2 c_5 \varepsilon \right) + c_2 \frac{n-2}{2} (1 + o(1)) \frac{\omega(a_i)}{\lambda_i^{n-2}/2} = O(R_{2,1} + \sum \frac{\ln \lambda_k}{\lambda_k^{n/2}}) \quad (\text{for } i > q)\]

where \(R_{1,1}, R_{2,1}, c_3, c_2 \text{ and } c_5 \) are defined in Proposition 3.3.

Finally we provide the following balancing conditions involving the points of concentration \(a_i\).

**Proposition 4.5.** Let \(\varepsilon\) be small enough. For each \(i \leq q\), it holds:

\[-\frac{c_2}{2} \sum_{j \leq q, j \neq i} \alpha_j \lambda_j \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{\alpha_i}{\lambda_i} \frac{c_6}{K(a_i)} \varepsilon K_1(a_i) = O\left( \frac{1}{\lambda_i^2} + \sum_{j>q} \varepsilon_{ij} + R_{a_i} + \sum \frac{\ln \lambda_k}{\lambda_k^{n/2}} \right).\]

Furthermore, for each \(i \geq q+1\), it holds

\[
\frac{|\nabla K(a_i)|}{\lambda_i} \leq c \left( \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^{n/2}} + \sum_{j \neq i} \varepsilon_{ij} + R(\varepsilon, a, \lambda) + \sum \frac{\ln \lambda_k}{\lambda_k^{n/2}} \right)
\]

where \(R_{a_i}\) and \(c_6\) are defined in Proposition 3.4 and \(R(\varepsilon, a, \lambda)\) is defined in Proposition 3.7.

### 4.2. Proofs of Theorems 1.2, 1.3

**Proof of Theorem 1.2** Recall that the proof of Theorem 1.1 relies on the Eqs \((E_i), (F_i)\) and (3.4) which follow from Propositions 3.2, 3.3 and 3.4. When \(\omega \neq 0\), the counterpart of these propositions are Propositions 4.3, 4.4 and 4.5. In the new propositions, the new terms are \(O(1/\lambda_i^{n-2}/2)\) which is \(O(1/\lambda_i^{n/2})\) if \(n \geq 7\) and therefore it is small with respect to the other principal terms. Hence, the Eqs \((E_i), (F_i)\) and (3.4) are not changed. Thus the proof can be repeated exactly by the same way.

**Proof of Theorem 1.3** We notice that, in the proof of Theorem 1.1 for \(a_i \in \mathbb{S}_n^0\), the term \(\frac{\Delta K(a_i)}{\lambda_i^2}\) is a principal term. However, for the dimension 5, when \(\omega \neq 0\), another term appears in Proposition 4.4 which is \(\frac{\omega(a_i)}{\lambda_i^{n/2}}\). This term will dominate the previous one. Hence, we will have a change in the behavior analysis.

We order all the \(\lambda_i\)’s, for \(i = 1, \ldots, N\): \(\lambda_1 \leq \cdots \leq \lambda_N\) and as in the previous case, we denote by

\[
I_b := \{ i : a_i \in \mathbb{S}_n^0 \} \quad \text{and} \quad I_{in} := \{ i : a_i \in \mathbb{S}_n^0 \},
\]

\[
I' := \{ i : \lambda_i/\lambda_1 \to \infty \} \quad \text{and} \quad I := \{ 1, \ldots, N \} \setminus I'.
\]

Multiplying the equation of Proposition 4.4 by \(2^i \times M\) if \(i > q\) (with \(M\) a large constant to dominate the \(O(\varepsilon_{ij})\) which appears in Proposition 4.4 for \(i \leq q\) and \(2^i\) if \(i \leq q\), and summing over \(i\), we obtain:

\[
(4.23) \quad \varepsilon + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i>q} \frac{1}{(\lambda_i d_i)^3} = O\left( \frac{1}{\lambda_1} \right).
\]
This implies that $R_{1,i}$ and $R_{2,i}$ defined in Proposition 4.4 and $R_{a_i}$ defined in Proposition 4.5 satisfy
\[(4.24) \quad |R_{1,i}| + |R_{2,i}| \leq c \ln(\lambda_1)/\lambda_1^{5/3} \quad \text{and} \quad |R_{a_i}| \leq c/\lambda_1^{4/3} \quad \text{for each } i.\]

**Lemma 4.6.** (1) Assume that $I_{in} \neq \emptyset$, then
\[
\varepsilon + \sum_{i \in I_{in}, k \neq i} \varepsilon_{ik} + \sum_{i \in I_{in}} \frac{1}{(\lambda_i d_i)^3} \sum_{i \in I_{in}} c/\lambda_i^{3/2} = O\left(R_{2,i} + \sum_{i \in I_{in}} \frac{\ln \lambda_1}{\lambda_i^{5/2}}\right) = O\left(\frac{1}{\lambda_1^{5/3}}\right).
\]

This implies that $I \cap I_{in} = \emptyset$.
(2) For each $k \in I_b$, it holds:
\[
\varepsilon + \sum_{i \in I_b, i \neq k} \varepsilon_{ik} = O\left(\frac{1}{\lambda_k} + \frac{\ln \lambda_1}{\lambda_1^{5/3}}\right).
\]
(3) For each $i \in I_b$ and $k \in I_b$, it holds:
\[
(1/\lambda_i) |\partial \varepsilon_{ik}/\partial a_i| = o(1/\lambda_1).
\]

**Proof.** Multiplying the second equality in Proposition 4.4 by $2^i$ and summing over $i \in I_{in}$ the first claim follows. Using Claim (1) the second one follows by multiplying the first equality in Proposition 4.4 by $2^i$ and summing over $i \geq k$ and $i \in I_b$.

Concerning the last one, if $i, k \in I$ then, we derive using (4.23) that
\[
(1/\lambda_i) |\partial \varepsilon_{ik}/\partial a_i| \leq c \varepsilon_{ik}^{(n-1)/(n-2)} \leq c/\lambda_1^{(n-1)/(n-2)}.
\]

In the other case, using Claim (2), we obtain $(1/\lambda_i) |\partial \varepsilon_{ik}/\partial a_i| \leq c \varepsilon_{ik} = o(1/\lambda_1)$. \(\square\)

Next we claim that:

**Claim A:** $I = I_b$ and $I_{in} = \emptyset$.
To prove this claim, we follow the proof of Proposition 3.9. In fact, arguing by contradiction, assume that either $I \neq I_b$ or $I_{in} \neq \emptyset$. Then, from Lemma 4.6 it follows that $\varepsilon = o(1/\lambda_1)$ (which gives Claim 1). Claim 2 follows immediately from (1) of Lemma 4.6. Claim 3 follows from Proposition 4.5 and (3) of Lemma 4.6. Claim 4 and (i) of Claim 5 follow from Proposition 4.4 and Lemma 4.6. (ii) of Claim 5 follows from (i) and the fact that all the $\varepsilon_{ij}$’s, for $i, j \in A_z$, are of the same order. Finally, the sequel of the proof follows exactly in the same way which completes the proof of Claim A.

Now, we are in the same situation with Theorem 1.1 but with $\ell = 0$ and the sequel of the proof can be repeated exactly in the same way. \(\square\)

**5. Proof of Theorem 1.4**

Arguing by contradiction we assume that there exists a sequence $(u_k)$ of energy bounded solutions of $(P_0)$ which blows up. Thus two cases may occur: (1) either $(u_k)$ converges weakly to 0, (2) or $(u_k)$ converges weakly to a positive solution $\omega$ of $(P_0)$. In both cases, we will run to a contradiction.

We start by considering the first case, that is when $(u_k)$ converges weakly to 0.
In this case, \((a_k)\) has to blow up and therefore it will enter in some \(V(q,\ell,\tau)\). Arguing as in the proof of Theorem 1.1 and using the same notation. Observe that Propositions 3.1, 3.2, 3.3, 3.4 hold true with \(\varepsilon = 0\). Furthermore, Lemmas 3.5, 3.6, 3.7 and 3.8 hold with \(\varepsilon = 0\). Hence, in this case, for each \(i = 1, \ldots, q + \ell\), Eqs \((E'_i)\) and \((F'_i)\) (defined in (3.16) and (3.17)) become

\[
\begin{align*}
(E'_i): \quad & \frac{\Delta K(a_i)}{\lambda_i^2 K(a_i)} = o\left(\frac{1}{\mu_1}\right) \quad \text{for } i \in I_n, \\
& -\frac{c_2}{2} \sum_{i \neq j \in I_b} \alpha_j j_i \partial K_j - \frac{\alpha_i}{\lambda_i} \frac{c_3}{K(a_i) \lambda_i} \partial K(a_i) = o\left(\frac{1}{\mu_1^{(n-1)/(n-2)}}\right) \quad \text{for } i \in I_b.
\end{align*}
\]

\[
\begin{align*}
(F'_i): \quad & -\frac{c_2}{2} \sum_{i \neq j \in I_b} \alpha_j j_i \partial K_j - \frac{\alpha_i}{\lambda_i} \frac{c_6}{\lambda_i} \nabla K_1(a_i) = o\left(\frac{1}{\mu_1^{(n-1)/(n-2)}}\right) \quad \text{for } i \in I_b.
\end{align*}
\]

Now we claim that:

**Claim 1:** For each \(i \in I_n\), it holds that \(\mu_i/\mu_1 \to \infty\) as \(k \to +\infty\), (that is \(I \cap I_n = \emptyset\)). In fact, arguing by contradiction, assume that there exists \(i \in I \cap I_n\). Thus, using Lemma 3.6 we get that \(a_i\) has to converge to a critical point \(y \in \mathbb{S}^n_d\) of \(K\) and therefore \(|\Delta K(a_i)| \geq c > 0\) for \(k\) large. Thus Eq \((E'_i)\) gives a contradiction and therefore the claim follows.

Now summing \(2^i(E'_i)\) for \(i \in I_b \setminus I\) we derive that

\[
\sum_{i \in I_b \setminus I : i \neq j \in I_b} \varepsilon_{ik} = o\left(\frac{1}{\mu_1}\right).
\]

From Eq \((E'_i)\) and Lemma 3.8 we derive that \(a_1\) has to converge to a critical point \(z_1\) in \(\partial \mathbb{S}^n_\tau\). Using Eq \((E'_i)\) and (5.3) we get that

\[
\sum_{i \neq j \in I_b} \varepsilon_{ij} = \frac{c \left(1 + o(1)\right)}{\lambda_1}.
\]

Hence, let \(j\) be such that \(d(a_1, a_j) := \min\{d(a_1, a_k) : k \in I \cap I_b\}\), then it holds that \(d(a_j, a_1) \to 0\) which implies that \#\(B_{z_1}\) \(\geq 2\) where \(B_{z_1} := \{i \in I \cap I_b : a_i \to z_1\}\).

We introduce the following sets:

\[
A'_{z_1} := \{k \in B_{z_1} : \text{lim} d(a_k, a_1)/d(a_j, a_1) = \infty\} \quad \text{and} \quad A_{z_1} := B_{z_1} \setminus A'_{z_1}.
\]

**Claim 2:** For each \(i \neq k \in A_{z_1}\), it holds \(\lambda_i^{(n-3)/(n-2)}d(a_i, a_k)\) is bounded above and below.

We notice that, for each \(i \neq k \in A_{z_1}\), it holds that \(d(a_i, a_k)\) and \(d(a_j, a_1)\) are of the same order. Furthermore, \(\lambda_i\) and \(\lambda_k\) are of the same order. Hence, \(\varepsilon_{ik} = (\lambda_i\lambda_k d(a_i, a_k)^{2(n-2)/c + o(1)})\) and all the \(\varepsilon_{ik}\), for each \(i \neq k \in A_{z_1}\), are of the same order. Thus Claim 2 follows immediately from (5.4).

To conclude the proof of the theorem, as the proof of Proposition 3.9, we need to multiply \((F'_i)\) by \(\alpha_i\lambda_i(\bar{\sigma} - \langle a_1, \bar{\sigma}\rangle a_i)\) and summing for \(i \in A_{z_1}\) where \(\bar{\sigma}\) is the barycenter...
of the points $a_i$'s for $i \in A_\omega$. Observe that Claim 2 implies that $\lambda_i (a_1 - \langle a_i, \bar{a} \rangle a_i) \leq c_i d(a_i, \bar{a}) \leq c_{i, \omega} \lambda_i^{1/(n-2)}$. It holds that

$$
(5.5) \quad \frac{1}{K_1(a_i)} \sum_{j \in A_i} \partial \bar{a}_i \langle \bar{a} - \langle a_i, \bar{a} \rangle a_i \rangle = \sum_{j \in A_i} O(|a_i| \lambda_i) = O\left( \frac{1}{\lambda_i} \right),
$$

Observe that, (2.9) implies $a_i^2 = K(a_i)(2-n)/2 + o(1)$. Furthermore, using Lemmas 6.3 and 6.4 there hold

$$
(5.6) \quad \frac{1}{K_1(a_i^2)} \sum_{j \in A_i} \partial \bar{a}_i \langle \bar{a} - \langle a_i, \bar{a} \rangle a_i \rangle = \sum_{j \in A_i} O(|a_i| \lambda_i^2) = O\left( \frac{1}{\lambda_i} \right),
$$

(5.7) \quad \frac{\partial \bar{a}_i (\bar{a} - \langle a_i, \bar{a} \rangle a_i)}{\partial a_i} + \frac{\partial \bar{a}_j (\bar{a} - \langle a_j, \bar{a} \rangle a_j)}{\partial a_j} \geq O_i e_i j.

Eqs (5.6) and (5.7) imply that Eq (5.5) cannot occur. Hence the first case (that is $(u_k)$ converges weakly to $0$) cannot occur.

Now we will focus on the second case: that is $(u_k)$ converges weakly to $\omega \neq 0$.

The proof follows the previous one but we need to use Section 4 instead of Section 3. Observe that when $\omega \neq 0$, comparing with Propositions 3.1, 3.4 the new propositions contain other terms due to the presence of $\omega$.

For $n \geq 7$, these terms are small with respect to the other principal terms (which exist in Propositions 3.1, 3.4). Hence the new propositions seem as there exist no changes and therefore the previous proof work exactly in the same way.

However, for $n = 5$, the new term $\omega(a_i)/\lambda_i^{3/2}$ dominates $\Delta K(a_i)/\lambda_i^2$ and therefore the principal terms in Proposition 4.4 will be different comparing with Proposition 4.1.

Recall that Section 4 holds for $\epsilon = 0$. Using the same notation than the proof of Theorem 1.3 we derive that (4.23) and (4.24) hold with $\epsilon = 0$. Now, from the first assertion of Lemma 4.6 we derive that $a_1 \in \partial \bar{a}_i$. Moreover, using Proposition 4.5 and Lemma 4.6 we obtain that $a_1$ converges to a critical point $z$ of $K_1$. In addition, using Proposition 4.4 we get that

$$
(5.8) \quad \sum_{k \in I} e_{1k} = \frac{c}{\lambda_1} \partial \bar{a}_i K(z)(1 + o(1))
$$

which implies that $\partial \bar{a}_i K(z)$ has to be positive (if not the previous equality cannot occur). Observe that (5.8) implies that there exists at least one index $j \in I$ such that $d(a_j, a_1) \to 0$ as $k \to \infty$.

As before, let $B_\gamma := \{ j \in I : a_j \to z \}$. It holds that $\# B_\gamma \geq 2$. Furthermore, following the end of the proof of Proposition 3.9 we deduce that (3.20) holds. Finally observe that (3.21) and (3.22) imply that (3.20) cannot occur. Thus the proof of our theorem is complete.

6. Appendix

In this section we collect some technical Lemmas used in this paper.
Lemma 6.1. For $a \in \partial \mathbb{S}^n_+$, we have $\partial \delta_{a,\lambda}/\partial \nu = 0$ and therefore $\varphi_{a,\lambda} = \delta_{a,\lambda}$. For $a \notin \partial \mathbb{S}^n_+$, we have

\( i \) $\delta_{a,\lambda} \leq \varphi_{a,\lambda} \leq 2\delta_{a,\lambda}$; \( |\lambda \partial \varphi_{a,\lambda}/\partial \lambda| \leq c\delta_{a,\lambda}$; \( |(1/\lambda)\partial \varphi_{a,\lambda}/\partial a^k| \leq c\delta_{a,\lambda} \)

where $a^k$ denotes the $k$-th component of $a$.

\( ii \) $\varphi_{a,\lambda} = \delta_{a,\lambda} + c_0 \frac{H(a, \cdot)}{\lambda^{(n-2)/2}} + f_{a,\lambda}$

where

\[ |f_{a,\lambda}|_\infty \leq \frac{c}{(\lambda d_a)^{n/2}} \frac{H(a, \cdot)}{\lambda^{n/2}} \leq \frac{c}{\lambda^{n/2} d_a^n} \text{ and } |\frac{\partial f_{a,\lambda}}{\partial \lambda}|_\infty \leq \frac{c}{\lambda^{n/2} d_a^n} \]

where $d_a := d(a, \partial \mathbb{S}^n_+)$. Furthermore, it holds that $H(a, \cdot)/\lambda^{(n-2)/2} \leq c\delta_{a,\lambda}$.

Lemma 6.2. Let $a \in \mathbb{S}^n_+$ and $\lambda > 0$ be large.

\( i \) Assume that $e \ln \lambda$ is small enough, then it holds

\[
\delta_{a,\lambda}^{-e}(x) = c_0^{-e} \lambda^{-e(n-2)/2} \left( 1 + \frac{n-2}{2} e \ln(2 + (\lambda^2 - 1)(1 - \cos d(a, x))) \right) + O \left( e^2 \ln(2 + (\lambda^2 - 1)(1 - \cos d(a, x))) \right) \text{ for each } y \in \mathbb{S}^n_+.
\]

\( ii \) For each $\gamma > 0$ and each $\beta \in [0, n/(n-2))$, it holds

\[ 0 < \int_{\mathbb{S}^n_+} \delta_{a,\lambda}^{n+1-\beta}(x) \ln^\gamma \left( 2 + (\lambda^2 - 1)(1 - \cos d(a, x)) \right) dx = O \left( \frac{1}{\lambda^{\beta(n-2)/2}} \right). \]

Lemma 6.3. Let $i, j \in I_b$ be such that $\lambda_i$ and $\lambda_j$ are of the same order and $|a_k - h| \to 0$ for $k = i, j$ for some $h \in \partial \mathbb{S}^n_+$. Then we have

\[ e_{ij} := \frac{\partial E}{\partial a_i}(h - \langle a, h \rangle a_i) + \frac{\partial E}{\partial a_j}(h - \langle a, h \rangle a_j) \geq c e_{ij}. \]

Lemma 6.4. (1) Let $a, h \in \partial \mathbb{S}^n_+$ be close to a critical point $z$ of $K_1$. Then it holds that

\[
\frac{1}{K_1(a)^{n/2}} \nabla K_1(a)(h - \langle a, h \rangle a) = -\frac{1}{K_1(a)^{n/2}} \nabla K_1(a)(h - \langle a, h \rangle h) + O(|a - h|^2).
\]

(2) Let $e \in \partial \mathbb{S}^n_+$ be such that $|e - z| \geq c > 0$.

\[
\frac{1}{K_1(a)} \nabla K_1(a)(e - \langle e, a \rangle a) = \frac{1}{K_1(z)} \frac{D^2 K_1(z)(a - \langle a, z \rangle z, e - \langle e, z \rangle z)}{1} + O(|a - z|^2).
\]

Proof. Let

\[ \beta(t) := \frac{h + t(a - h)}{|h + t(a - h)|}, \quad g(t) := \frac{2/(n-2)}{K_1(\beta(t))^{n-2}/2} \text{ for } t \in [0, 1]. \]

It is easy to get that

\[ \beta'(t) = \frac{1}{|h + t(a - h)|} \left( a - h - \langle \beta(t), a - h \rangle \beta(t) \right), \quad \langle \beta(t), a - h \rangle = O(|a - h|^2), \]

where $a^k$ denotes the $k$-th component of $a$. 

(since we have \(|a - h|^2 = 2(1 - (a,h))\) and therefore it holds that \(|\beta'(t)| = |a - h|(1 + o(1))\) uniformly in \(t \in [0, 1]\). Furthermore, easy computations imply that \(|\beta''(t)| = O(|a - h|^2)\) uniformly in \(t \in [0, 1]\). On the other hand, we have
\[
g'(t) = -\frac{1}{K_1(\beta(t))^{2n/2}} \nabla K_1(\beta(t)) \cdot (\beta'(t))
\]
and, since \(a\) and \(h\) are close to a critical point \(z\) of \(K_1\), we derive that
\[
g''(t) = o(|\beta'(t)|^2) - \frac{1}{K_1(\beta(t))^{2n/2}} D^2 K_1(\beta(t)) \cdot (\beta'(t), \beta'(t)) + o(|\beta''(t)|) = O(|a - h|^2)
\]
(uniformly in \(t \in [0, 1]\)). Now,
\[
\frac{1}{K_1(a)^{2n/2}} \nabla K_1(a) \cdot (a - \langle a, h \rangle a) + \frac{1}{K_1(h)^{2n/2}} \nabla K_1(h) \cdot (a - \langle a, h \rangle h)
\]
\[
= g'(1) - g'(0) = \int_0^1 g''(t) \, dt
\]
which implies the first claim of the lemma.
To prove the second claim, since \(z\) is a critical point of \(K_1\), it follows that \(1/K_1(a) = 1/K_1(z) + O(|a - z|^2)\). Furthermore, let \(f(a) := \nabla K_1(a) \cdot (e - \langle e, a \rangle a)\). It follows that: \(f(a) = f(\langle z \rangle) + f'(\langle z \rangle)(a - \langle a, z \rangle z) + O(|a - z|^2)\) which completes the proof. 

**Remark 6.5.** Let \(m = 3\) and \(z\) be a critical point of \(K_1\). Assume that the matrix associated to \(D^2 K_1(z)\) has a positive eigenvalue \(\sigma > 0\). Then the function \(F_{z,3}\) has a critical point of the form \((\overline{b}, 0, -\overline{b})\).
In fact, each critical point of \(F_{z,3}\) has to satisfy the following system
\[
\begin{align*}
D^2 K_1(z)(x_1, \ldots) &- (n - 2) \frac{x_1 - x_3}{|x_1 - x_3|^n} - (n - 2) \frac{x_1 - x_2}{|x_1 - x_2|^n} = 0 \\
D^2 K_1(z)(x_2, \ldots) &- (n - 2) \frac{x_2 - x_3}{|x_2 - x_3|^n} - (n - 2) \frac{x_2 - x_1}{|x_2 - x_1|^n} = 0 \\
D^2 K_1(z)(x_3, \ldots) &- (n - 2) \frac{x_3 - x_1}{|x_3 - x_1|^n} - (n - 2) \frac{x_3 - x_2}{|x_3 - x_2|^n} = 0
\end{align*}
\]
Taking \((x_1, x_2, x_3) = (\overline{b}, 0, -\overline{b})\) then the second equation is satisfied. The first and the third equations give the same one which is:
\[
D^2 K_1(z)(\overline{b}, \ldots) - (n - 2) \frac{\overline{b}}{|\overline{b}|^n} - (n - 2) \frac{\overline{b}}{|2\overline{b}|^n} = D^2 K_1(z)(\overline{b}, \ldots) - (n - 2) \frac{\overline{b}}{|\overline{b}|^n}(1 + \frac{1}{2^{n-1}}) = 0.
\]
This equation has a solution \(\overline{b} = \gamma e_\sigma\) where \(e_\sigma\) is an eigenvector (with norm 1) associated to the eigenvalue \(\sigma > 0\) and \(\gamma > 0\) satisfies
\[
\gamma^n = (n - 2)(1 + \frac{1}{2^{n-1}})(1 - \sigma)
\]
This remark implies that we cannot prove that \(\lambda_1^{(n-2)/n} |a_i - z| \geq c\) for each \(i\) (see Lemma 3.1) since it is possible that for one index \(j\) we have \(\lambda_j^{(n-2)/n} |a_j - z| \to 0\). But this can occur for at most one index.
Lemma 6.6. For each $k \neq j$, it holds
\[
\int (\delta_k \delta_j) \frac{n}{n+2} \leq c e^{\frac{n}{n+2}} \ln e_{k,j}^{-1} \quad \& \quad \int \delta_k \delta_j \frac{n}{n+2} \leq c e^{\frac{n}{n+2}} \ln e_{k,j}^{-1} + c \frac{\ln \lambda_k}{\lambda_k^{n/2}} + c \frac{\ln \lambda_j}{\lambda_j^{n/2}}.
\]

Proof. The first assertion is extracted from (5) (Estimate 2 page 4). For the second one, let $B_i := B(a_i, 1)$, we have
\[
\int \delta_k \delta_j \frac{n}{n+2} 
\leq \int_{[|\xi| \leq a_i] \cap B_k} \delta_k (\delta_j \delta_k) \frac{n}{n+2} + \int_{[|\xi| \leq a_i] \cap B_k^c} \delta_k^{n+2} + \int_{[|\xi| \leq a_i] \cap B_j^c} (\delta_k \delta_j) \frac{n}{n+2} \delta_j + \int_{|\xi| \leq a_i} \delta_j^{n+2} 
\leq \left( \int (\delta_j \delta_k) \frac{n}{n+2} \right)^{\frac{2}{n+2}} \left( \int B_k \delta_k^{n+2} \right)^{\frac{n+2}{n+2}} + \left( \int B_j (\delta_k \delta_j) \frac{n}{n+2} \delta_j \right)^{\frac{2}{n+2}} + \frac{c}{\lambda_k^{(n+2)/2}} + \frac{c}{\lambda_j^{(n+2)/2}} 
\leq c \left( e^{\frac{n}{n+2}} \ln e_{k,j}^{-1} \right)^{\frac{2}{n+2}} \left( \int \frac{\ln \lambda_k}{\lambda_k^{n/2}} \right)^{\frac{n+2}{n+2}} + \frac{c}{\lambda_k^{n/2}} + \frac{c}{\lambda_j^{n/2}}. 
\]
Hence the proof is completed.

Lemma 6.7. Assume that the $e_{ij}$'s are small enough. Let $v \in E_{\omega,\alpha,\lambda}^\perp$ (defined in (2.11)). Written $v$ as
\[
v := v_0 + v_0 + v_+ \quad \text{with} \quad v_- \in N_- (\omega) \quad v_0 \in H_0 (\omega) \quad v_+ \in N_+ (\omega)
\]
where $H_0 (\omega) := \text{span}(\omega) \oplus N_0 (\omega)$ (defined in (4.2)). Then there exists a positive constant $c$ such that
\[
||v_0|| = o(||v||),
\]
\[
Q_{\omega,\alpha,\lambda} (v_-) = Q_\omega (v_-) + o(||v_-||^2) \leq -c ||v_-||^2,
\]
\[
Q_{\omega,\alpha,\lambda} (v_+) \geq c ||v_+||^2 + o(||v||^2),
\]
where $Q_\omega$ is defined in (2.5) and $Q_{\omega,\alpha,\lambda}$ is defined by
\[
Q_{\omega,\alpha,\lambda} (v) = ||v||^2 - p \int_{\mathbb{S}^{n-1}} \delta_i^{p-1} v^2 - p \int_{\mathbb{S}^{n-1}} K \omega^{p-1} v^2.
\]
This implies that the quadratic form $Q_{\omega,\alpha,\lambda}$ is a non degenerate one in the space $E_{\omega,\alpha,\lambda}^\perp$.

Proof. First note that the spaces $H_0 (\omega)$, $N_- (\omega)$ and $N_+ (\omega)$ are orthogonal spaces with respect to $(\cdot, \cdot)$ and the associated bilinear form $B_\omega (\cdot, \cdot) := \int_{\mathbb{S}^{n-1}} K \omega^{p-1} (\cdot, \cdot)$.

We start by proving (6.4). Since $v_0 \in H_0 (\omega)$, it follows that $v_0 = \gamma_0 \omega + \sum \gamma_i e_i$ where $(e_1, \ldots, e_m)$ is an orthonormal basis of $N_0 (\omega)$. Using the fact that $v \in E_{\omega,\alpha,\lambda}^\perp$ (which implies that $v \perp u_{\alpha, \beta}$ and $v \perp du_{\alpha, \beta} / \partial \beta_i$ for each $i$), it follows that
\[
\gamma_0 = \langle v_0, \omega \rangle = \langle v, \omega \rangle = (1/\alpha) \langle v, u_{\alpha, \beta} \rangle - (1/\alpha) \sum \beta_i \langle v, e_i \rangle = o(||v||),
\]
\[ \gamma_i = \langle v_0, e_i \rangle = \langle v, e_i \rangle = \langle v, e_i + (\partial h(\beta)/\partial \beta_i) \rangle - \langle v, \partial h(\beta)/\partial \beta_i \rangle = o(\|v\|) \quad \forall \ 1 \leq i \leq m \]

(by using the smallness of \( \beta \) and \( h(\beta) \) in the \( C^1 \) sense with respect to \( \beta \)). This ends the proof of (6.4).

Concerning (6.5), we have

\[ Q_{\omega,d,l}(v_-) = \|v_-\|^2 - p \sum \int_{\mathbb{S}^n} \delta_i^{p-1} v_i^2 - p \int K \omega^{p-1} v_-^2 = Q_\omega(v_-) - p \sum \int \delta_i^{p-1} v_i^2. \]

Observe that, since \( v_- \) belongs to a fixed finite dimensional space, we derive that \( \|v_-\|_{\infty} \leq c\|v_-\| \) and therefore

\[ \int \delta_i^{p-1} v_i^2 \leq \|v_-\|_{\infty}^2 \int \delta_i^{p-1} = o(\|v_-\|^2) \quad \text{for each } i. \]

Hence the proof of (6.5) follows by using (4.3).

It remains to prove (6.6). Note that, using Proposition 3.1 of ([5]), there exists a constant \( \zeta_1 > 0 \) such that

\[ (6.8) \quad \|h\|^2 - \frac{n+2}{n-2} \sum_{i=1}^N \int_{\mathbb{S}^n} \delta_i^{4} h^2 \geq \zeta_1 \|h\|^2 \quad \text{for each } h \in E_{a,A}^+. \]

where \( E_{a,A}^+ \) is introduced in Proposition 2.2.

In addition, the sequence of the eigenvalues (denoted by \( (\sigma_i) \)) corresponding to \( Q_\omega \) (defined by (2.3)) satisfies \( \sigma_i \nearrow 1 \). Let \( N_k(\omega) \) be the eigenspace associated to the eigenvalue \( \sigma_k \). These spaces are orthogonal with respect to \( \langle , \rangle \) and the bilinear form \( B_\omega \). Let \( \sigma_{k_0} := \min \{\sigma_i : \sigma_i > 0\} \). Hence it is easy to see that \( N_+(\omega) = \oplus_{k \geq k_0} N_k(\omega) \). Furthermore, it holds

\[ (6.9) \quad Q_\omega(h) \geq \sigma_{k_1} \|h\|^2 \quad \text{for each } h \in \oplus_{j \leq k} N_j(\omega). \]

Let \( k_1 \) be such that \( \sigma_{k_1} \geq 1 - \zeta_1/4 \). We decompose \( N_+(\omega) \) as follows:

\[ N_+(\omega) := (\oplus_{k_0 \leq k \leq k_1} N_k(\omega)) \oplus (\oplus_{k > k_1} N_k(\omega)) := N^{0,1}_+(\omega) \oplus N^1_+(\omega). \]

Note that \( N^{0,1}_+(\omega) \) is a fixed finite dimensional space. Now, since \( v_+ \in N_+(\omega) \), it holds that

\[ (6.10) \quad v_+ := v_0^+ + v_1^+ \quad \text{where } v_0^+ \in N^{0,1}_+(\omega) \text{ and } v_1^+ \in N^1_+(\omega). \]

Hence, using the orthogonality of the spaces \( N^{0,1}_+(\omega) \) and \( N^1_+(\omega) \), it follows that

\[ Q_{\omega,d,l}(v_+) = \|v_0^+\|^2 + \|v_1^+\|^2 - p \sum_{i=1}^N \int_{\mathbb{S}^n} \delta_i^{4} \{ (v_0^+)^2 + (v_1^+)^2 + 2v_0^+ v_1^+ \} - p \int_{\mathbb{S}^n} K \omega^{1/2} \{ (v_0^+)^2 + (v_1^+)^2 \}. \]

Observe that

\[ \|v_0^+\|^2 - \frac{n+2}{n-2} \int_{\mathbb{S}^n} K \omega^{1/2} (v_0^+)^2 = Q_\omega(v_0^+) \geq \sigma_{k_0} \|v_0^+\|^2 \quad \text{(by using (6.9))} \]

\[ \int_{\mathbb{S}^n} \delta_i^{4} (v_0^+)^2 \leq \|v_0^+\|_{\infty}^2 \int_{\mathbb{S}^n} \delta_i^{4} = o(\|v_0^+\|^2) \]

\[ \int_{\mathbb{S}^n} \delta_i^{4} (v_1^+)^2 \leq \|v_1^+\|_{\infty}^2 \int_{\mathbb{S}^n} \delta_i^{4} = o(\|v_1^+\|^2) \]

\[ \int_{\mathbb{S}^n} \delta_i^{4} (v_1^+)^2 \leq \|v_1^+\|_{\infty}^2 \int_{\mathbb{S}^n} \delta_i^{4} = o(\|v_1^+\|^2) \]
\[ \int_{\mathbb{S}^n} \delta \frac{\partial}{\partial x_i} \left| v^+_0 \right| \left| v^+_1 \right| \leq \left| v^+_0 \right|_\infty \int_{\mathbb{S}^n} \delta \frac{\partial}{\partial x_i} \left| v^+_1 \right| = o\left(\left| v^+_0 \right| \left| v^+_1 \right| \right) = o\left(\left| v^+_1 \right|^2 + \left| v^+_0 \right|^2 \right) \]

\[-p \int_{\mathbb{S}^n} K \omega \frac{\partial}{\partial x_i} (v^+_1)^2 = Q_\omega (v^+_1) - \left| v^+_1 \right|^2 \geq (\sigma_{k_1} - 1)\left| v^+_1 \right|^2 \geq - (\xi_1/4)\left| v^+_1 \right|^2 \]

where, for the last formula, we have used (6.9) and the choose of \( k_1 \). Combining these estimates, we get

(6.11)

\[ Q_{\omega, a, i}(v^+) \geq \left| v^+_1 \right|^2 - p \sum_{i=1}^{N} \int_{\mathbb{S}^n} \delta \frac{\partial}{\partial x_i} (v^+_1)^2 \sigma_{k_0} \left| v^+_0 \right|^2 - \frac{\xi_1}{4} \left| v^+_1 \right|^2 + o\left(\left| v^+_1 \right|^2 + \left| v^+_0 \right|^2 \right) \]

Note that the function \( v^+_1 \) is not necessarily in \( E^+_{a, i} \). For this raison we write \( v^+_1 := \sum t_i \psi_i + \tilde{v}^+_1 \in E^+_{a, i} \), where the \( \psi_i \)'s are the functions \( \varphi_j \)'s and their derivatives with respect to \( \lambda_j \) and \( a_i^k \). Let \( \psi_i \in \{ \varphi_i, \lambda \partial \varphi_i / \partial \lambda_i, (1/\lambda_i) \partial \varphi_i / \partial a_i^k \} \), it follows that

\[ t_i + o\left( \sum |t_k| \right) = \langle \tilde{v}^+_1, \psi_i \rangle = \langle v, \psi_i \rangle - \langle v_0, \psi_i \rangle - \langle v_-, \psi_i \rangle - \langle \tilde{v}^+_1, \psi_i \rangle = O\left( \int \delta \frac{\partial}{\partial x_i} (|v_0| + |v_-| + \left| v^+_0 \right|) \right) = o\left(\left| v^+_0 \right| + \left| v_- \right| + \left| v^+_0 \right| \right) = o\left(\left| v \right| \right) \]

by using the fact that these functions are in fixed finite dimensional spaces. Thus we derive that \( \left| v^+_1 \right|^2 = \left| \tilde{v}^+_1 \right|^2 + o\left(\left| v \right|^2 \right) \) and therefore, using (6.3), we get

\[ \left| v^+_1 \right|^2 - p \sum_{i=1}^{N} \int_{\mathbb{S}^n} \delta \frac{\partial}{\partial x_i} (v^+_1)^2 = \left| \tilde{v}^+_1 \right|^2 - p \sum_{i=1}^{N} \int_{\mathbb{S}^n} \delta \frac{\partial}{\partial x_i} (\tilde{v}^+_1)^2 + o\left(\left| \tilde{v}^+_1 \right|^2 + \left| v \right|^2 \right) \]

(6.12)

\[ \geq \frac{1}{2} \xi_1 \left| v^+_1 \right|^2 + o\left(\left| v \right|^2 \right). \]

Combining (6.10), (6.11) and (6.12), we get

\[ Q_{\omega, a, i}(v^+) \geq \frac{1}{4} \xi_1 \left| v^+_1 \right|^2 + \sigma_{k_0} \left| v^+_0 \right|^2 + o\left(\left| v^+_1 \right|^2 + \left| v^+_0 \right|^2 \right) + o\left(\left| v \right|^2 \right) \geq c \left| v^+_1 \right|^2 + o\left(\left| v \right|^2 \right). \]

Thus the result follows. \( \blacksquare \)

In the following lemma, we collect some formulae whose proof follows immediately by some standard calculus computations

**Lemma 6.8.** Let \( t_i > 0 \) and \( a, b \in \mathbb{R} \), there hold

(6.13)

\[ \left| \sum t_i \right|^\gamma - \sum t_i^\gamma \leq c \left\{ \begin{array}{ll} \sum_{i \neq j} (t_{i, j})^{\gamma/2} & \text{if } 0 < \gamma \leq 2 \\ \sum_{i \neq j} t_i^{-1/2} t_j & \text{if } \gamma > 2 \end{array} \right. \]

(6.14)

\[ \left| a + b \right|^\gamma - \left| a \right|^\gamma - \gamma \left| a \right|^{\gamma - 2} a b \leq c \left\{ \begin{array}{ll} \left| b \right|^\gamma + \left| a \right|^{\gamma - 2} b^2 & \text{if } \gamma > 2 \\ \left| b \right|^\gamma & \text{if } 1 < \gamma \leq 2. \end{array} \right. \]
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