Certifiable randomness from temporal correlations

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Abstract

Leggett-Garg inequalities (LGI) are constrains on certain combinations of temporal correlations obtained by measuring one and the same system at two different instants of time. The usual derivations of LGI assume macroscopic realism per se and noninvasive measurability. We derive these inequalities under a different set of assumptions, namely the assumptions of predictability and no signaling in time. As a novel implication of this derivation, we show that LGI can be used to certify randomness in a device independent way.

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For testing the existence of superposition of macroscopically distinct quantum states, Leggett and Garg [1] put forward the notion of macrorealism. This notion rests on the classical paradigm [2] that

(A1) A macroscopic system with two or more macroscopically distinct states available to it will at all times be in one or the other of these states. [Macroscopic realism per se (MR)]

(A2) It is possible, at least in principle, to determine which of these states the system is in, without any effect on the state itself or on its subsequent dynamics. [Noninvasive measurability (NIM)]

The conjunction of these two assumptions, namely the assumptions of MR and NIM, together with the postulate of Induction [3–6] give rise to Leggett-Garg inequalities (LGI).

Leggett-Garg inequalities have been a source of considerable research-interest [7–10]. However, there is still some controversy about the implication of its violation [7, 11–13].

In the present work, we give an alternative derivation of LGI by using different set of assumptions: the assumption of no signaling in time (NSIT) and predictability. The assumption of NSIT, described in [6, 14], says that a measurement does not change the outcome statistics of a later measurement, whereas predictability is the assumption that one can predict the outcomes of all possible measurements to be performed on a system [15]. This derivation, therefore, allows us to conclude that in a situation where NSIT is satisfied, the violation of LGI will imply the presence of “true” randomness. Apart from the theoretical significance, this also has practical implications in the generation of Device-Independent (DI) certified randomness.

From algorithmic information theory it is known that randomness cannot be certified by any mathematical procedure [16]. The generation of randomness, therefore, must be based on unpredictability of some physical phenomena, so that the randomness is guaranteed by the inherent uncertain nature of the physical theory. There is no such thing as true randomness in classical world as any classical phenomenon, can, in principle, be predicted. They appear random to us due to lack of our knowledge and control of all the relevant degrees of freedom. Measurement on a quantum particle, on the other hand, is postulated to give intrinsically random results. The quantum measurements, therefore, can be used to generate true randomness [17]. But, for the reliability of the randomness thus generated, one needs to trust the devices which prepare and measure the quantum states. Can randomness be certified in a Device-Independent way, i.e., can it be certified even without knowing the details of the devices used in its generation – is a topic of current research interest [18–21]. Interestingly, it can be certified in a DI way, provided, we consider the scenario which either involves different measurements on two correlated particles [15, 18–22] or which involve measurements on one and the same particle at different times. This latter is the scenario for a Leggett-Garg test and is the subject matter of the present manuscript.

But, before moving to the Leggett-Garg test under the said assumptions of predictability and NSIT, we, in the following, briefly describe the ontological framework of an operational theory (for details of this framework, we refer to [23, 24]), as this will subsequently be used in our derivation of LGI.

The goal of an operational theory is merely to specify the probabilities \( p(k|M, P, T) \) of different outcomes \( k \in K_M \) that may result from a measurement procedure \( M \in M \) given a particular preparation procedure \( P \in P \), and a particular transformation procedure \( T \in T \); where \( M \), \( P \) and \( T \) respectively denote the sets of measurement procedures, preparation procedures and transformation procedures; \( K_M \) denotes the set of measurement results for the measurement \( M \).

Whereas an operational theory does not tell anything

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about physical state of the system, in an ontological model of an operational theory, the primitives of description are the actual state of affairs of the system. A preparation procedure is assumed to prepare a system with certain properties and a measurement procedure is assumed to reveal something about those properties. A complete specification of the properties of a system is referred to as the ontic state of that system. In an ontological model for quantum theory, a particular preparation method \( P_\psi \) which prepares the quantum state \( |\psi\rangle \), actually puts the system into some ontic state \( \lambda \in \Lambda \), \( \Lambda \) denotes the ontic state space. An observer who knows the preparation \( P_\psi \) may nonetheless have incomplete knowledge of \( \lambda \). Thus, in general, an ontological model associates a probability distribution \( \mu(\lambda|P_\psi) \) with preparation \( P_\psi \) of \( |\psi\rangle \). \( \mu(\lambda|P_\psi) \) is called the epistemic state as it encodes observer’s epistemic ignorance about the state of the system. It must satisfy

\[
\int_\Lambda \mu(\lambda|P_\psi) d\lambda = 1 \quad \forall \; |\psi\rangle \quad \text{and} \quad P_\psi.
\]

Similarly, the model may be such that the ontic state \( \lambda \) determines only the probability \( \xi(k|\lambda, M) \), of different outcomes \( k \) for the measurement method \( M \). However, in a deterministic model \( \xi(k|\lambda, M) \in \{0, 1\} \). The response functions \( \xi(k|\lambda, M) \in [0, 1] \), should satisfy

\[
\sum_{k \in \mathcal{K}_M} \xi(k|\lambda, M) = 1 \quad \forall \; \lambda, \; M.
\]

Thus, in the ontological model, the probability \( p(k|M, P) \) is specified as

\[
p(k|M, P) = \int_\Lambda \xi(k|M, \lambda)\mu(\lambda|P)d\lambda.
\]

As the model is required to reproduce the observed frequencies (quantum predictions) hence the following must also be satisfied

\[
\int_\Lambda \xi(\phi|M, \lambda)\mu(\lambda|P)d\lambda = |\langle \phi|\psi \rangle|^2.
\]

The transformation processes \( T \) are represented by stochastic maps from ontic states to ontic states. \( T(\lambda'|\lambda) \) represents the probability distribution over subsequent ontic states given that the earlier ontic state one started with was \( \lambda \).

In a standard Leggett-Garg test, we consider a macroscopic object which is described by a set of macro variable \( \{Q, Q', \ldots\} \) whose values are considered to be macroscopically distinct by some measure \([4]\). In a series of runs, the object is prepared in the same initial state, and each preparation defines a new origin of time. Let us consider the case where macro variable \( A \in \{Q, Q', \ldots\} \) is measured at time \( t_A(t_A > 0) \) and macro variable \( B \in \{Q, Q', \ldots\} \) at a later time \( t_B \) [25]. The correlation function \( C_{t_A t_B} = \langle Q_{t_A} Q_{t_B} \rangle \) for measurements at \( t_A \) and \( t_B \) is obtained from the joint probability \( P(A_{t_A} B_{t_B}|Q_{t_A} Q_{t_B}) \) of obtaining the results \( A_{t_A} \) and \( B_{t_B} \) from measurements of \( Q \) at time \( t_A \) and \( t_B \) (\( t_B > t_A \)) as

\[
C_{t_A t_B} = \sum_{A_{t_A} B_{t_B}} A_{t_A} B_{t_B} P(A_{t_A} B_{t_B}|Q_{t_A} Q_{t_B}).
\]

In the simplest case, the macro variable may obtain only two different values \( \pm 1 \). In such cases, macrorealism together with induction imply the LGI [4] of the Clauser-Horne-Shimony-Holt (CHSH) type [26] \((t_1 < t_2 < t_3 < t_4)\):

\[
f_4^{\text{LG}} = -2 \leq C_{t_1 t_2} + C_{t_2 t_3} + C_{t_3 t_4} - C_{t_4 t_1} \leq 2.
\]

or of the Wigner type [27]

\[
f_3^{\text{LG}} = -3 \leq C_{t_1 t_2} + C_{t_2 t_3} + C_{t_3 t_1} \leq 1
\]

In the ontological framework, the system’s state is described by an ontic variable \( \lambda \) and \( P(A_{t_A}, B_{t_B}|Q_{t_A}, Q_{t_B}, \lambda) \rightarrow \lambda' \) denotes the joint probability of obtaining outcome \( A_{t_A} \) of measurement \( Q_{t_A} \) performed at time \( t_A \) and outcome \( B_{t_B} \) of measurement \( Q_{t_B} \) performed at a later time \( t_B \); \( \lambda \rightarrow \lambda' \) denotes the change of the system’s ontic state conditioned that \( A_{t_A} \) outcome has been obtained in measurement \( Q_{t_A} \) at time \( t_A \). The ontological model then predicts for the observed probability as

\[
P(A_{t_A} B_{t_B}|Q_{t_A} Q_{t_B}) = \int_\lambda \int_{\lambda'} d\lambda d\lambda' \mu(\lambda|Q_{t_A}, A_{t_A}, \lambda) P(A_{t_A} B_{t_B}|Q_{t_A} Q_{t_B}, \lambda, \lambda' \rightarrow \lambda'),
\]

where \( \mu(\lambda) \) and \( \rho(\lambda'|Q_{t_A}, A_{t_A}, \lambda) \) respectively denote the distribution of the ontic variables prior to the measurement \( Q_{t_A} \) and distribution of the ontic variables after obtaining the result \( A_{t_A} \) in the measurement of \( Q_{t_A} \). A crucial step in the derivation of LGI is to establish the following factorizability relation which follows from the assumptions of macrorealism and induction [10, 28, 29]:

\[
P(A_{t_A} B_{t_B}|Q_{t_A} Q_{t_B}, \lambda, \lambda' \rightarrow \lambda') = P(A_{t_A}|Q_{t_A} Q_{t_B}, \lambda) P(B_{t_B}|Q_{t_B}, \lambda).
\]

It is noteworthy that, in contrast to macrorealism, quantum mechanics predicts the outcome probability as:

\[
P(A_{t_A} B_{t_B}|Q_{t_A} Q_{t_B}) = \text{Tr}[\hat{\rho}(t_A)|\hat{Q}_A]|\text{Tr}[\hat{\rho}(t_A, t_B)|\hat{Q}_B]|,
\]

where, \( \hat{\rho}(t_A) \) is the quantum state of the system at time \( t_A \), \( \hat{Q}_A \) and \( \hat{Q}_B \) are the measurement operators for outcomes \( A \) and \( B \), and \( \hat{\rho}(t_A, t_B) \) (the reduced) quantum state at time \( t_B \) given that at time \( t_A \) result \( A \) was obtained.

For a two-level system undergoing coherent oscillations between the states with \( Q = \pm 1 \), the optimal quantum violation of the inequality (1) is known to be \( 2\sqrt{2} \), whereas it is \( \frac{\lambda}{2} \) for the inequality (2) [30].
We now proceed to show that the assumptions of predictability and no signaling in time also lead to the factorizability condition (4) and thus to the derivation of LGI.

NSIT [6, 14, 31] is said to be satisfied if a measurement does not change the outcome statistics of a later measurement; i.e., \( P(B_{i_B}|Q_{t_B}) = P(B_{i_B}|Q_{t_A}Q_{t_B}) \). Though macrorealism implies both LGI as well as NSIT, the assumption of NSIT, alone, does not imply LGI. However, it together with the assumption of predictability imply LGI. A model is said to be predictable if \( P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B}) \in \{0,1\} \) for measurements at any time and for all measurement outcomes. As \( P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B}) \in \{0,1\} \) hence conditioning on further variables cannot alter it, i.e., \( P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B},\lambda,\lambda'\rightarrow\lambda') = P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B}) \) - no ontic variable further specify the probabilities. Now according to Baye’s theorem, \( P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B}) = P(A_{i_A}|B_{i_B}Q_{t_A}Q_{t_B})P(B_{i_B}|Q_{t_A}Q_{t_B}) \). Again predictability implies \( P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B}) = P(A_{i_A}|Q_{t_A}Q_{t_B}) \). Assuming NSIT, we get \( P(B_{i_B}|Q_{t_A}Q_{t_B}) = P(B_{i_B}|Q_{t_B}) \) and due to Induction, which says that measurement statistics at an earlier time should not depend on the what would be measured at a later time, we also have \( P(A_{i_A}|Q_{t_A}Q_{t_B}) = P(A_{i_A}|Q_{t_A}) \). We, thus, have \( P(A_{i_A}B_{i_B}|Q_{t_A}Q_{t_B},\lambda\rightarrow\lambda') = P(A_{i_A}|Q_{t_A})P(B_{i_B}|Q_{t_B}) \). The factorizability condition (4), then, follows from conditioning the probabilities in the RHS on \( \lambda \).

The above derivation of LGI implies that either both or at least one of the underlying assumptions is violated whenever LGI is violated. Imagine now a situation where LGI is violated but NSIT is satisfied. It would be worth mentioning here that NSIT is experimentally testable. In such situations, we can say that the model and hence the corresponding phenomena are not predictable. Using the said situation, in the following, we show that temporal correlations are useful for DI randomness certification.

Certifiable randomness from LGI: Consider the standard Leggett-Gerg function, where ±1-valued observables \( Q_{t_A} \) and \( Q_{t_B} \) are measured on a single system at two different times \( t_A \) and \( t_B \) respectively, where \( t_A < t_B \). The joint probability distribution \( P(1|Q_{t_A}Q_{t_B}), \) of getting results 1 at \( t_A \) and 1 at \( t_B \) in such a scenario is calculated either by repeating the experiment many times or by employing an array of many identical systems. The other joint probabilities involved in Leggett-Gerg inequalities (1) or (2) are calculated similarly to observe their violations. These probabilities are also analyzed to see whether NSIT is obeyed. In fact, in Ref. [6], it has been shown that there exists probability distribution which violates LGI but satisfy NSIT. As we now know that such distribution cannot be predictable and therefore some randomness is associated with it. The associated randomness can be quantified by min-entropy [32] which is a statistical measure of the amount of randomness that a particular distribution contains. For a distribution \( X \), it is as defined as

\[
H_\infty(X) = \log_2 \max_{x: \text{Prob}(X=x)} \frac{1}{\text{Prob}(X=x)}
\]

Thus, to obtain the minimum amount of randomness associated with the violations of LGI (represented by \( f_{LG}^\delta = f_{MR}^\delta + \epsilon \), where \( f_{MR}^\delta \) is the macrorealistic bound of \( f_{LG}^\delta; \delta = 3 \) for inequality (2) and for (1) \( \delta = 4; \epsilon > 0 \), we need to first solve the following optimization problem:

\[
P_{NSIT}(Q_{t_A}, Q_{t_B}) = \max_{i,j} P(Q_{t_A} = i, Q_{t_B} = j) \\
\text{subject to} \quad f_{LG}^\delta = f_{MR}^\delta + \epsilon \\
P(Q_{t_A} = i, Q_{t_B} = j) \geq 0 \\
\sum_{i,j} P(Q_{t_A} = i, Q_{t_B} = j) = 1 \\
P(Q_{t_A}, Q_{t_B}) \text{ satisfy NSIT}. \quad (6)
\]

Having the optimized solution \( P_{NSIT}^*(Q_{t_A}, Q_{t_B}) \), the minimum randomness is calculated as \( H_\infty(Q_{t_A}, Q_{t_B}) = -\log_2 P_{NSIT}(Q_{t_A}, Q_{t_B}) \). We have considered the Leggett-Gerg function \( f_{LG}^\delta \) and \( f_{LG}^\delta \), one after another, in the optimization problem (6) and have numerically calculated the minimum amounts of randomness with different values of \( \epsilon \). We plot our findings in Fig.1 and Fig.2 [33].

Concluding remarks: Randomness is a valuable resource for various important tasks ranging from cryptographic applications to numerical simulations such as

![FIG. 1. (Color on-line) Certifiable randomness associated with Leggett-Gerg function \( f_{LG}^\delta \).](image1)

![FIG. 2. (Color on-line) Certifiable randomness associated with Leggett-Gerg function \( f_{LG}^\delta \).](image2)
Monte Carlo method (a useful technique which finds application in computational Physics, Statistical Physics, Physical Chemistry, Computational Biology, Computer Graphics, Finance and many other areas). For various such tasks, the genuineness of the used randomness is of primary concern. Thus, device independent certification and generation of randomness is very important from a practical point of view. Motivated by the work of Pironio and coworkers [18], many interesting results have been obtained, in recent times, in the field of DI certification and generation of randomness. All such methods use nonlocal correlations among spatially separated parties (which is guaranteed by Bell type inequalities violation [20, 35, 36]) to certify randomness. In this work, we have shown that temporal nonlocal correlations, i.e., correlations which violate Leggett-Gerg inequality, can also be used to certify randomness. This work provides an important information theoretic application of LGI which can be implemented in laboratory with the present day's technology. From the perspective of experimental implementation the LGI-based DI randomness certification seems more feasible than its spatial analogue as it does not require entanglement [34]. Moreover, various successful experimental tests of LGI violation also give rise to the possibility towards further experiments with more macroscopicity involved. The present work also shows potential usefulness of such non-classical macroscopic systems.

This work is significant from an yet another perspective. Though Quantum Theory postulates to have random measurement outcomes, it does not deny for a finer theory where the measurement outcomes are only apparently random. In fact, there exists ontological models which, in principle, can predict the outcomes of each individual measurement on a single particle [35, 37]. However, for really predicting the outcomes of a measurement, such a theory needs perfect knowledge of some variables. But, these variables are not accessible to the present day’s technology. Our analysis shows that even with some future technology (to which these variables are accessible and controllable), one cannot predict the outcomes of measurements performed on a single particle at two different times if the two-time correlations thus obtained violate LGI and satisfy NSIT.

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