ON DISPERSION MANAGED NONLINEAR SCHRÖDINGER EQUATIONS WITH LUMPED AMPLIFICATION

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Abstract. We show the global well–posedness of the nonlinear Schrödinger equation with periodically varying coefficients and a small parameter $\varepsilon > 0$, which is used in optical–fiber communications. We also prove that the solutions converge to the solution for the Gabitov–Turitsyn or averaged equation as $\varepsilon$ tends to zero.

1. Introduction

We consider the nonlinear Schrödinger equation (NLS) with periodically varying coefficients
\[ i\partial_t u + d(t)\partial_x^2 u + c(t)|u|^2 u = 0 \] (1.1)
which describes the behavior of a signal transmitted on an optical–fiber cable. Here, $x$ denotes the (retarded) time, $t$ the position along the cable, and periodic functions $d(\cdot)$ and $c(\cdot)$ the dispersion and the fiber loss/amplification along the cable respectively.

The original evolution of optical pulses in a dispersion managed system with lumped amplification is described by the nonlinear Schrödinger equation
\[ i\partial_t E + d(t)\partial_x^2 E + |E|^2 E = ig(t)E. \]
The fiber loss and amplification coefficient along the cable is given by
\[ g(t) = -\frac{\Gamma}{2} + \Gamma \sum_{j=1}^{\infty} \delta(t - t_j), \]
where $\Gamma > 0$ is the fiber loss, $t_j$ corresponds to the location of amplifiers, and $\delta(\cdot)$ is the Dirac delta function. For more information on this equation, see, e.g., [2]. Taking
\[ E(x,t) = u(x,t) \exp \left( \int_0^t g(t')dt' \right), \]
we obtain equation (1.1) with $c(t) = \exp \left( 2\int_0^t g(t')dt' \right)$ provided that $g$ is a periodic function with mean zero.

The dispersion management with alternating sections of positive and negative dispersion in fibers was introduced in 1980, see [23]. It was successful to transfer the data at ultra–high speed over long distances, see, e.g., [1, 12, 14, 15, 22, 27]. For more information on the dispersion management, see [26] and references therein.

In the strong dispersion management regime, the dispersion is given by
\[ d(t) = d_{av} + \frac{1}{\varepsilon}d_0 \left( \frac{t}{\varepsilon} \right), \]

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where $d_0(\cdot)$ is the mean zero part of the dispersion which is a 2–periodic function satisfying
\[
d_0(t) = \begin{cases} 
1, & 0 \leq t < 1, \\
-1, & 1 \leq t < 2,
\end{cases}
\]
d_{av} \in \mathbb{R} the average dispersion over one period, and $\varepsilon > 0$ a small parameter. The fiber loss and amplification is defined to be
\[
c(t) = G \left( \frac{t}{\varepsilon} \right),
\]
where $G$ is also a 2–periodic function given by
\[
G(t) = \exp \left( 2 \int_0^t \left( -\frac{\Gamma}{2} + \Gamma \sum_{j \in \mathbb{Z}} \delta(t' - 2j) \right) dt' \right).
\]

The first main result of this paper is the well–posedness of the Cauchy problem
\[
\begin{cases}
  i \partial_t u + \left( d_{av} + \frac{1}{\varepsilon} d_0 \left( \frac{t}{\varepsilon} \right) \right) \partial_x^2 u + G \left( \frac{t}{\varepsilon} \right) |u|^2 u = 0, \\
u(x, 0) = u_0(x),
\end{cases}
\]
(1.2)

**Theorem 1.1** (Global well–posedness). Let $d_{av} \in \mathbb{R}$. For every $u_0 \in H^1(\mathbb{R})$, there exists a unique solution $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ of (1.2). Moreover, $u$ depends continuously on the initial datum in the following sense. For every $M > 0$, the map $u_0 \mapsto u(t)$ from $H^1(\mathbb{R})$ to $C([-M, M], H^1(\mathbb{R}))$ is locally Lipschitz continuous.

Now we change the variables $u = T_{D(t/\varepsilon)} v$ in (1.2) to obtain
\[
\begin{cases}
  i \partial_t v + d_{av} \partial_x^2 v + G \left( \frac{t}{\varepsilon} \right) T^{-1}_{D(t/\varepsilon)} \left( |T_{D(t/\varepsilon)} v|^2 T_{D(t/\varepsilon)} v \right) = 0, \\
v(x, 0) = u_0(x),
\end{cases}
\]
(1.3)

where $D(t) = \int_0^t d_0(t') dt'$ and $T_t$ is the solution operator for the free Schrödinger equation in dimension one. Note that since $d_0$ is a 2–periodic function with mean zero, $D$ is also 2–periodic and, therefore, the map $t \mapsto T_{D(t/\varepsilon)}$ is $2\varepsilon$–periodic.

For small $\varepsilon > 0$, that is, in the regime of strong dispersion management, equation (1.3) contains the fast oscillating terms $T_{D(t/\varepsilon)}$ and $G(t/\varepsilon)$ in the nonlinearity and hence Gabitov and Turitsyn suggested averaging the equation over one period, see [14, 15]. This yields the following “averaged” equation
\[
i \partial_t v + d_{av} \partial_x^2 v + \frac{1}{2} \int_0^2 G(\tau) T^{-1}_{D(\tau)} \left( |T_{D(\tau)} v|^2 T_{D(\tau)} v \right) d\tau = 0.
\]
We make the change of variables $D(\tau) = r$, then we have
\[
i \partial_t v + d_{av} \partial_x^2 v + \int_0^1 T^{-1}_r \left( |T_r v|^2 T_r v \right) \psi(r) dr = 0,
\]
(1.4)

where
\[
\psi(r) = e^{-\Gamma} \cosh \Gamma (r - 1).
\]

The well–posedness of the averaged equation (1.4) in $H^s(\mathbb{R})$ for all $s \geq 0$ is proved in [3] for a general dispersion profile. For more general nonlinearities including saturated nonlinearities, see [11].

The averaging procedure is rigorously justified in [28] when the fiber loss and amplification are not present. More precisely, it is shown that for $\varepsilon > 0$, the solutions of (1.3) and (1.4) with the same initial datum in $H^s(\mathbb{R})$ stay $\varepsilon$–close in $H^{s-3}(\mathbb{R})$ for a long time in $O(\varepsilon^{-1})$ when $s$ is sufficiently large. Note that the convergence is not shown in $H^s(\mathbb{R})$ where
the solutions exist. However, we prove that the solutions for (1.3) converge to the solution for (1.1) in $H^1(\mathbb{R})$, where the solutions exist, as $\varepsilon \to 0$.

**Theorem 1.2** (Averaging Theorem). Let $d_{av} \in \mathbb{R}$, $M > 0$ and $v \in C(\mathbb{R}, H^1(\mathbb{R}))$ be the solution of the averaged equation (1.1) with the initial datum $v_0 \in H^1(\mathbb{R})$. Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ and $\|u_0 - v_0\|_{H^1(\mathbb{R})} \leq \varepsilon$, then

$$\|v_\varepsilon - v\|_{C([-M,M], H^1(\mathbb{R}))} \leq C\varepsilon,$$

(1.5)

where $v_\varepsilon$ is the solution of (1.3) with the initial datum $u_0 \in H^1(\mathbb{R})$.

In our main theorems, Theorems 1.1 and 1.2, we prove the well–posedness of the Cauchy problem (1.2) and the validity of the averaging process in the strong dispersion, $\frac{1}{\varepsilon}d_0(\frac{x}{\varepsilon})$, while the well–posedness of the averaged equation is already proved in [3]. There are analogous results for the fast dispersion, $d_0(\frac{x}{\varepsilon^2})$, and the random dispersion, $\frac{1}{\varepsilon}d_0(\frac{x}{\varepsilon^3})$, with some centered stationary random process $a_0$, in [13] and [17], respectively.

We here remark some provable facts which are not dealt with in this paper. Related to standing wave solutions $v(x,t) = e^{i\omega t}f(x)$, $\omega \in \mathbb{R}$, of the averaged equation (1.1), a constrained minimization problem is well studied. The existence of minimizers can be found in [10, 19] when $d_{av} \geq 0$. One can easily show that every minimizer is a weak solution of the corresponding Euler–Lagrange equation. Each weak solution and its Fourier transform decay exponentially, which can be proven by modifying the proofs in [13, 17] a little. Particularly, every minimizer is smooth. Moreover, the set of ground states is orbitally stable, see [11, 18] as well as [8]. In the case $d_{av} > 0$, using the averaging theorem and the orbital stability, it is possible to obtain the stable soliton–like solution for (1.2).

The paper is organized as follows. In Section 2 we prove Theorem 1.1 the global well–posedness result in $H^1(\mathbb{R})$. We start by showing the local well–posedness in $H^1(\mathbb{R})$ and the global existence in $L^2(\mathbb{R})$. Although we do not have the energy conservation, we prove the existence of a global solution in $H^1(\mathbb{R})$ based on the mass conservation and the boundedness of the mixed norm of the solution for Strichartz admissible pairs. In Section 3 we prove Theorem 1.2 the averaging theorem. In Appendix A we gather basic properties of the free Schrödinger time evolution to prove Theorem 1.1.

### 2. Well–posedness

To begin with, let us introduce some notations. For $1 \leq p < \infty$, we use $L^p(\mathbb{R})$ to denote the Banach space of functions $f$ whose norm

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}}$$

is finite with the essential supremum instead when $p = \infty$. The space $L^2(\mathbb{R})$ is a Hilbert space with scalar product given by $(f,g) = \int_{\mathbb{R}} f(x)g(x) dx$. We use $L^p_t(J, L^q_x(I))$ to denote, for $1 \leq p, q < \infty$ and intervals $I, J \subset \mathbb{R}$, the Banach space of all functions $u$ with the mixed norm

$$\|u\|_{L^p_t(J, L^q_x(I))} := \left(\int_J \left(\int_I |u(x,t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}.$$

If $p = \infty$ or $q = \infty$, use the usual modification. For notational simplicity, we use $L^q(J, L^p)$ for $L^q(J, L^p_\mathbb{R})$. We say that $u \in L^p_{loc}(J, L^p)$ when $u \in L^p(\tilde{J}, L^p)$ for every bounded interval $\tilde{J} \subset J$. 

The Fourier transform on $\mathbb{R}$ is defined by
\[
\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx
\]
for $f \in \mathcal{S}(\mathbb{R})$, the Schwartz space of infinitely smooth, rapidly decreasing functions. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ is defined as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ for which
\[
\|f\|_{H^s} := \left( \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty.
\]

For a Banach space $X$ with norm $\| \cdot \|_X$ and an interval $J$, $C(J, X)$ is the space of all continuous functions $u : J \to X$. When $J$ is compact, it is a Banach space with norm
\[
\|u\|_{C(J, X)} = \sup_{t \in J} \|u(t)\|_X
\]
and $C^1(J, X)$ is the Banach space of all continuously differentiable functions $u : J \to X$.

Let $T_t$ denote the solution operator for the free Schrödinger equation in spatial dimension one. In terms of the Fourier transform, this is given by
\[
\hat{T_t}f(\xi) = e^{it\xi^2} \hat{f}(\xi) = e^{-it\xi^2} \hat{f}(\xi)
\]
for $f \in \mathcal{S}(\mathbb{R})$, thus, one can express
\[
T_t f(x) = e^{it\xi^2} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^2} \hat{f}(\xi) d\xi.
\]
It is a unitary operator on $L^2(\mathbb{R})$ and also on $H^1(\mathbb{R})$. Therefore, for every $t \in \mathbb{R}$,
\[
\|T_t f\|_{L^2} = \|f\|_{L^2} \quad \text{and} \quad \|T_t f\|_{H^1} = \|f\|_{H^1}.
\]
We use the notation $f \lesssim g$ when there exists a positive constant $C$ such that $f \leq Cg$.

Now we prove the well–posedness of the Cauchy problem (1.2). Since the proof does not rely on the factor $\varepsilon$ in (1.2), we only consider the case $\varepsilon = 1$,
\[
\begin{cases}
i\partial_t u + (d_\alpha v + d_0(t)) \partial_x^2 u + G(t)|u|^2 u = 0, \\
u(x, 0) = u_0(x),
\end{cases}
\tag{2.1}
\]
or, equivalently,
\[
\begin{cases}
i\partial_t v + d_\alpha \partial_x^2 v + G(t)T_{D(t)}^{-1} \left( |T_{D(t)}v|^2 T_{D(t)}v \right) = 0, \\
v(x, 0) = u_0(x),
\end{cases}
\tag{2.2}
\]
where $u = T_{D(t)}v$. We first prove the local existence of a unique solution for the integral equation of (2.2),
\[
v(t) = e^{id_\alpha \partial_x^2} u_0 + i \int_0^t e^{i(t-t')d_\alpha \partial_x^2} Q(v(t')) dt',
\tag{2.3}
\]
with $u_0 \in H^1(\mathbb{R})$, where
\[
Q(v)(t) = G(t)T_{D(t)}^{-1} \left( |T_{D(t)}v(t)|^2 T_{D(t)}v(t) \right).
\]
Here and below, we use $C$ to denote various constants.
Lemma 2.1. For every \( f, g \in H^1(\mathbb{R}) \), we have
\[
\|Q(f)(t)\|_{H^1} \lesssim \|G\|_{L^\infty} \|f\|_{H^1}^3
\]  
and
\[
\|Q(f)(t) - Q(g)(t)\|_{H^1} \lesssim \|G\|_{L^\infty} (\|f\|_{H^1}^2 + \|g\|_{H^1}^2) \|f - g\|_{H^1}.
\]  

Proof. We define, for every \( f_1, f_2, f_3 \in H^1(\mathbb{R}) \),
\[
Q(f_1, f_2, f_3)(t) := G(t) T_{D(t)}^{-1} \left( T_{D(t)} f_1 f_2 T_{D(t)} f_3 \right),
\]
which is multi-linear. Note that \( Q(f)(t) = Q(f, f, f)(t) \) for any \( f \in H^1(\mathbb{R}) \). Since
\[
\|fg\|_{H^1} \leq C \|f\|_{H^1} \|g\|_{H^1}
\]
and \( T_{D(t)} \) is unitary on \( H^1(\mathbb{R}) \), we obtain
\[
\|Q(f_1, f_2, f_3)(t)\|_{H^1} \lesssim \|G\|_{L^\infty} \|f_1\|_{H^1} \|f_2\|_{H^1} \|f_3\|_{H^1}
\]
which proves (2.4). Observing
\[
Q(f)(t) - Q(g)(t) = Q(f - g, f, f)(t) + Q(g, f - g, f)(t) + Q(g, g, f - g)(t),
\]
one can easily obtain (2.5) from (2.4). \( \square \)

Proposition 2.2. Let \( d_{av} \in \mathbb{R} \). For every \( K > 0 \), there exist \( M_\pm > 0 \) such that for every initial datum \( u_0 \in H^1(\mathbb{R}) \) with \( \|u_0\|_{H^1} \leq K \), there is a unique solution \( v \in \mathcal{C}([-M_-, M_+], H^1) \) of (2.3). Moreover,
\[
\|v(t)\|_{H^1} \leq 2K \quad \text{for all} \quad t \in [-M_-, M_+].
\]

Corollary 2.3. Let \( d_{av} \in \mathbb{R} \). For any initial datum \( u_0 \in H^1(\mathbb{R}) \), there exist maximal life times \( T_\pm \in (0, \infty) \) such that there is a unique solution \( v \in \mathcal{C}([-T_-, T_+], H^1) \) of (2.3). Moreover, the blow–up alternative for solutions holds: if \( T_+ \) is finite, then
\[
\lim_{t \to T_+} \|v(t)\|_{H^1} = \infty
\]
and if \( T_- \) is finite, then
\[
\lim_{t \to T_-} \|v(t)\|_{H^1} = \infty.
\]

Proof of Proposition 2.2. Without loss of generality, we assume that \( t > 0 \).

To prove the existence of a solution, we use a fixed point argument. For each \( M > 0 \) and \( a > 0 \), let
\[
B_{M,a} = \{ v \in \mathcal{C}([0, M], H^1) : \|v\|_{\mathcal{C}([0, M], H^1)} \leq a \}
\]
be equipped with the distance
\[
d(v, w) = \|v - w\|_{\mathcal{C}([0, M], H^1)}.
\]
Let \( K > 0 \) and \( u_0 \in H^1(\mathbb{R}) \) with \( \|u_0\|_{H^1} \leq K \) be fixed. Define the map \( \Phi \) on \( B_{M,a} \) by
\[
\Phi(v)(t) = e^{itd_{av} \partial_x^2} u_0 + i \int_0^t e^{i(t-t')d_{av} \partial_x^2} Q(v)(t') dt'.
\]
It follows from Lemma 2.1 that if \( v(t), w(t) \in H^1(\mathbb{R}) \), then
\[
\|\Phi(v)(t)\|_{H^1} \leq \|u_0\|_{H^1} + \int_0^t \|Q(v)(t')\|_{H^1} dt' \leq \|u_0\|_{H^1} + C \int_0^M \|v(t')\|_{H^1}^3 dt'.
\]
Given the initial datum $0$

Proof of Corollary 2.3.

and

$\|\Phi(v)(t) - \Phi(w)(t)\|_{H^1} \leq \int_0^t \|Q(v)(t') - Q(w)(t')\|_{H^1} dt'$

$\leq C \int_0^M (\|v(t')\|_{H^1}^2 + \|w(t')\|_{H^1}^2) \|v(t') - w(t')\|_{H^1} dt'$

for every $0 \leq t \leq M$. Therefore, there is a positive constant $C$ such that for all $v, w \in B_{M,a}$,

$\|\Phi(v)\|_{C([0,M],H^1)} \leq K + CMa^3$

and

$d(\Phi(v), \Phi(w)) \leq CMa^2d(v, w)$.

Now set $a = 2K$ and choose $M_+ > 0$ satisfying

$CM_+(2K)^2 < \frac{1}{2}$,

then we obtain that $\Phi$ is a contraction from $B_{M+,2K}$ into itself. Thus, Banach’s contraction mapping theorem shows that there exists a unique solution $v$ of (2.3) in $B_{M+,2K}$ and, moreover,

$\|v\|_{C([0,M+],H^1)} \leq 2K$.

To show the uniqueness of a solution, let $v_1, v_2 \in C([0,M+],H^1)$ be solutions of (2.3). Then it follows from (2.5) that for every $t \in [0,M+]$

$\|v_1(t) - v_2(t)\|_{H^1} \leq C \left(\|v_1\|_{C([0,M+],H^1)}^2 + \|v_2\|_{C([0,M+],H^1)}^2\right) \int_0^t \|v_1(t') - v_2(t')\|_{H^1} dt'$

(2.8)

which implies $\|v_1 - v_2\|_{C([0,M+],H^1)} = 0$.

Proof of Corollary 2.3. Given the initial datum $0 \neq u_0 \in H^1(\mathbb{R})$, let us define the maximal life time $T_+$ by

$T_+ = \sup \{ M > 0 : \text{a unique solution of (2.3) exists in } C([0,M],H^1) \}$.

Then it immediately follows from Proposition 2.2 that $T_+ \in (0, \infty]$.

Note that if a solution exists in $C([0,M],H^1)$ for any $M > 0$, then it is a unique solution in $C([0,M],H^1)$, by the same argument in the proof of the uniqueness in Proposition 2.2. Thus, there exists a unique solution $v \in C([0,T_+],H^1)$ of (2.3). To prove the blow-up alternative, let $T_+ < \infty$. Suppose to the contrary that there exist a positive number $K$ and a sequence $\{t_j\}$ in $(0,T_+)$ such that $\|v(t_j)\|_{H^1} \leq K$ and $t_j \to T_+$ as $j \to \infty$. Then, by Proposition 2.2, there exists $M > 0$ such that a unique solution of (2.3) with the initial datum $v(t_j)$ exists in $C([t_j,t_j + M],H^1)$ for all $j$. Since we can choose $j^*$ such that $t_{j^*} + M > T_+$, this contradicts the definition of $T_+$. The case of $T_-$ can be done similarly.

Next, we show the continuous dependence of the solutions for (2.3) on the initial data to finish the local well-posedness. Indeed, the map $u_0 \mapsto v(t)$ is locally Lipschitz continuous on $H^1(\mathbb{R})$.

**Proposition 2.4.** Let $d_{av} \in \mathbb{R}$. For every $K > 0$, there exists a positive constant $C$ such that for all initial data $v_0, w_0 \in H^1(\mathbb{R})$ with $\|v_0\|_{H^1}, \|w_0\|_{H^1} \leq K$, we have

$\|v - w\|_{C([-M_,M+],H^1)} \leq e^{C \max(M_-,M_+)}\|v_0 - w_0\|_{H^1}$,

where $v$ and $w$ are the corresponding local solutions of (2.3) with initial data $v_0, w_0$ on the time interval $[-M_,M+]$ of existence, guaranteed by Proposition 2.2.
Proof. We consider positive $t$ only and fix $t \in (0, M_+]$. From Proposition 2.2 we know that
\[ \|v(t)\|_{H^1} \leq 2K \quad \text{and} \quad \|w(t)\|_{H^1} \leq 2K. \]
By a similar argument of (2.7), we obtain
\[
\|v(t) - w(t)\|_{H^1} \leq \|v_0 - w_0\|_{H^1} + \int_0^t \|Q(v)(t') - Q(w)(t')\|_{H^1} dt'
\leq \|v_0 - w_0\|_{H^1} + C \int_0^t \left( \|v(t')\|_{H^1}^2 + \|w(t')\|_{H^1}^2 \right) \|v(t') - w(t')\|_{H^1} dt'
\leq \|v_0 - w_0\|_{H^1} + CK^2 \int_0^t \|v(t') - w(t')\|_{H^1} dt'.
\]
Thus, it follows from Gronwall’s inequality that
\[ \|v(t) - w(t)\|_{H^1} \leq e^{CK^2t} \|v_0 - w_0\|_{H^1} \leq e^{CK^2M_+} \|v_0 - w_0\|_{H^1}, \]
which completes the proof. \hfill \Box

Remark 2.5. If we define the energy $E(v(t))$ of the solution $v$ for (2.2) by
\[
E(v(t)) = \frac{d_{av}}{2} \|\partial_x v(t)\|_{L^2}^2 - \frac{G(t)}{4} \int_{\mathbb{R}} |T_{D(t)} v(t)|^4 dx,
\]
then, however, the energy is neither conserved nor decreasing. Indeed, its derivative is given by
\[
\frac{dE(v(t))}{dt} = -\frac{1}{4} G'(t) \int_{\mathbb{R}} |T_{D(t)} v(t)|^4 dx \quad (2.9)
\]
for all $t \in \mathbb{R} \setminus 2\mathbb{Z}$. Note that $E(v(t))$ is not differentiable nor continuous at $t \in 2\mathbb{Z}$.
If there is no fiber loss nor amplification, i.e., $G \equiv 1$, then $T_\pm = \infty$ by the conservation of energy and the blow-up alternative, which immediately gives the global well-posedness. However, as you see in (2.9), the energy is no longer conserved in our case.

Now we consider the integral form of the Cauchy problem (2.11)
\[ u(t) = U(0, t)u_0 + i \int_0^t U(t', t) \left( G(t') |u(t')|^2 u(t') \right) dt'. \quad (2.10) \]
Here and below, $U(0, t)$ is the solution operator for the linear Schrödinger equation associated with (2.1), i.e., for every $f \in L^2(\mathbb{R})$, $U(0, t)f$ solves the initial value problem
\[
\begin{cases}
i \partial_t w + (d_{av} + d_0(t)) \partial_x^2 w = 0, \\
w(x, 0) = f(x).
\end{cases}
\]
Next we define, for all $t_0, t \in \mathbb{R}$,
\[ U(t_0, t) := U(0, t)(U(0, t_0))^{-1} \quad (2.11) \]
on $L^2(\mathbb{R})$. Then $U(t_0, t)$ is unitary on $L^2(\mathbb{R})$ and also on $H^1(\mathbb{R})$, and therefore, for every $t, t_0 \in \mathbb{R}$,
\[
\|U(t_0, t)f\|_{L^2} = \|f\|_{L^2} \quad \text{and} \quad \|U(t_0, t)f\|_{H^1} = \|f\|_{H^1}.
\]
For more properties of $U(t_0, t)$, see Appendix A.

Note that, given $u_0 \in H^1(\mathbb{R})$, there exists a unique solution $v \in \mathcal{C}((-T_-, T_+), H^1)$ for equation (2.3) by Corollary 2.3. If we let $u = T_{D(t)} v$, then $u \in \mathcal{C}((-T_-, T_+), H^1)$ solves equation (2.10) and the blow-up alternative holds since $\{T_{D(t)} : t \in \mathbb{R} \}$ is a strongly continuous group of unitary operators on $H^1(\mathbb{R})$. Moreover, since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we
have \( u \in L^\infty((-T_-, T_+), L^2) \cap L^4_{\text{loc}}((-T_-, T_+), L^\infty) \) and, therefore, by the Riesz–Thorin interpolation theorem,

\[
u \in L^q_{\text{loc}}((-T_-, T_+), L^p)
\]

(2.12)

for every admissible pair \((p, q)\). Before we prove the global existence of a solution, we show the existence of a unique global solution of (2.10) with the initial datum \( u_0 \in L^2(\mathbb{R}) \) when \( d_{av} \neq \pm 1 \). As usual, we use the Strichartz estimates (Lemma A.2) to prove the existence of a local solution for (2.10), see [7][20] for example.

**Proposition 2.6.** Let \( d_{av} \neq \pm 1 \). For any \( u_0 \in L^2(\mathbb{R}) \), there exists a unique global solution \( u \in C(\mathbb{R}, L^2) \cap L^6_{\text{loc}}(\mathbb{R}, L^6) \) of (2.10). Moreover, the solution \( u \) satisfies

\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \text{for all } t \in \mathbb{R}.
\]

Furthermore, for every \( M > 0 \) and admissible pair \((p, q)\), there exists a positive constant \( C \) depending on \( d_{av} \) and \( \|u_0\|_{L^2} \) such that

\[
\|u\|_{L^6([-M, M], L^p)} \leq C.
\]

(2.13)

**Proof.** Let \( 0 \neq u_0 \in L^2(\mathbb{R}) \) be fixed. Without loss of generality, we consider positive \( t \) only. First, to prove the existence of a unique solution in \( C([0, 1], L^2) \cap L^6([0, 1], L^6) \) of (2.10), let us define the closed ball

\[
B_{M,a} := \{ u \in L^\infty([0, M], L^2) \cap L^6((0, M], L^6) : \|u\|_{L^\infty([0, M], L^2)} + \|u\|_{L^6([0, M], L^6)} \leq a \}
\]

equipped with the distance

\[
d(u, v) = \|u - v\|_{L^\infty([0, M], L^2)} + \|u - v\|_{L^6([0, M], L^6)}
\]

for each \( 0 < M \leq 1 \) and \( a > 0 \). Define the map \( \Phi \) on \( B_{M,a} \) by

\[
\Phi(u)(t) = U(0, t)u_0 + i \int_0^t U(t', t) \left( G(t') |u(t')|^2 u(t') \right) dt'.
\]

For appropriate values of \( M \) and \( a \), the map \( \Phi \) is a contraction on \( (B_{M,a}, d) \). Indeed, it follows from the Strichartz estimates (Lemma A.2) and the Cauchy–Schwarz inequality that

\[
\|\Phi(u)\|_{L^6([0, M], L^6)} \leq C\|u_0\|_{L^2} + C\|G(\cdot)|u|^2 u\|_{L^1([0, M], L^2)} \\
\leq C\|u_0\|_{L^2} + CM^{1/2}\|u\|_{L^6([0, M], L^6)}^3.
\]

(2.14)

On the other hand, using the unitarity of \( U(0, t) \) on \( L^2(\mathbb{R}) \) and the argument used in (2.14), we obtain

\[
\|\Phi(u)\|_{L^\infty([0, M], L^2)} \leq \|u_0\|_{L^2} + C\|G(\cdot)|u|^2 u\|_{L^1([0, M], L^2)} \\
\leq \|u_0\|_{L^2} + CM^{1/2}\|u\|_{L^6([0, M], L^6)}^3.
\]

Next, noting

\[
|z_1|^2 z_1 - |z_2|^2 z_2 | \leq C(|z_1|^2 + |z_2|^2)|z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{C},
\]

(2.15)

by the same arguments above, we get

\[
\|\Phi(u) - \Phi(v)\|_{L^6([0, M], L^6)} \leq C\|G(\cdot)(|u|^2 u - |v|^2 v)\|_{L^1([0, M], L^2)} \\
\leq C \int_0^M \|u(t)\|_{L^6}^2 + \|v(t)\|_{L^6}^2 \|u(t) - v(t)\|_{L^6} dt \\
\leq CM^{1/2}\|u\|_{L^6([0, M], L^6)}^3 + \|v\|_{L^6([0, M], L^6)}^3 \|u - v\|_{L^6([0, M], L^6)}
\]

(2.16)
and
\[ \|\Phi(u) - \Phi(v)\|_{L^\infty([0,M],L^2)} \leq C\|G(\cdot)(|u|^2u - |v|^2v)\|_{L^1([0,M],L^2)} \]
\[ \leq CM^{1/2} \left( \|u\|_{L^6([0,M],L^6)}^2 + \|v\|_{L^6([0,M],L^6)}^2 \right) \|u - v\|_{L^6([0,M],L^6)}. \]

Therefore, we have a positive constant \( C \) such that for all \( u, v \in B_{M,a} \),
\[ \|\Phi(u)\|_{L^\infty([0,M],L^2)} + \|\Phi(u)\|_{L^6([0,M],L^6)} \leq C\|u_0\|_{L^2} + CM^{1/2}a^3 \]
and
\[ d(\Phi(u), \Phi(v)) \leq CM^{1/2}a^2d(u, v). \]

Now set \( a = 2C\|u_0\|_{L^2} \) and choose \( 0 < M \leq 1 \) satisfying
\[ CM^{1/2}(2C\|u_0\|_{L^2})^2 < \frac{1}{2}, \]
then we obtain that \( \Phi \) is a contraction from \( B_{M,2C\|u_0\|_{L^2}} \) into itself. Thus, \( \Phi \) has a unique fixed point \( u \in B_{M,2C\|u_0\|_{L^2}} \) and
\[ \|u\|_{L^\infty([0,M],L^2)} + \|u\|_{L^6([0,M],L^6)} \leq 2C\|u_0\|_{L^2}. \tag{2.17} \]

Moreover, the Stricharz estimates guarantee that \( u \) is even in \( C([0,M],L^2) \cap L^6([0,M],L^6) \).

To prove the uniqueness in \( C([0,M],L^2) \cap L^6([0,M],L^6) \), it is enough to find a small \( \delta > 0 \) so that the uniqueness is guaranteed in \( C([0,\delta],L^2) \cap L^6([0,\delta],L^6) \). Suppose that \( u_1, u_2 \in C([0,\delta],L^2) \cap L^6([0,\delta],L^6) \) solve (2.10), for \( 0 < \delta \leq M \). It follows from (2.10) that
\[ \|u_1 - u_2\|_{L^6([0,\delta],L^6)} \leq C\delta^{1/2} \left( \|u_1\|_{L^6([0,\delta],L^6)}^2 + \|u_2\|_{L^6([0,\delta],L^6)}^2 \right) \|u_1 - u_2\|_{L^6([0,\delta],L^6)}. \]

Choosing \( \delta > 0 \) small enough, we obtain
\[ \|u_1 - u_2\|_{L^6([0,\delta],L^6)} \leq \frac{1}{2}\|u_1 - u_2\|_{L^6([0,\delta],L^6)}, \]
which implies that \( u_1 = u_2 \) on \( C([0,\delta],L^2) \cap L^6([0,\delta],L^6) \).

Let \( u \in C([0,M],L^2) \cap L^6([0,M],L^6) \) be a unique solution of (2.10). We now show that \( u \in L^p([0,M],L^p) \) for every admissible pair \((p,q)\). Applying the Strichartz estimates to (2.10) and the Hölder inequality with exponents \( \frac{2}{3} \) and \( \frac{3}{2} \) in \( x - \) integral and \( t - \) integral, we get
\[ \|u\|_{L^p([0,M],L^p)} \leq C\|u_0\|_{L^2} + C\|G(\cdot)|u|^2u\|_{L^{6/5}([0,M],L^{6/5})} \]
\[ \leq C\|u_0\|_{L^2} + C \left( \int_0^M \|u(t)\|_{L^2}^{6/5}\|u(t)\|_{L^6}^{12/5} dt \right)^{5/6} \]
\[ \leq C\|u_0\|_{L^2} + CM^{1/2}\|u\|_{L^\infty([0,M],L^2)}\|u\|_{L^6([0,M],L^6)} \]
\[ \leq C\|u_0\|_{L^2} + CM^{1/2}\|u_0\|_{L^2}^2, \]
where (2.17) is used. Moreover, since \( u \) satisfies the mass conservation, that is,
\[ \|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \text{for all } t \in [0,M], \]
one can iterate this argument to obtain a unique solution in \( C([0,1],L^2) \cap L^6([0,1],L^6) \) of (2.10). Iterating this process, we obtain a unique global solution and it satisfies (2.13). \( \blacksquare \)

Now we give the proof of Theorem 1.1.
Proof of Theorem 1.1 Let \( u_0 \in H^1(\mathbb{R}) \) be fixed and let us consider positive times only. We first consider the case \( d_{av} \neq \pm 1 \). Let \( u \in C([0, \infty), L^2) \cap L^6_{\text{loc}}([0, \infty), L^6) \) be the solution of (2.10), obtained in Proposition 2.6. On the other hand, Corollary 2.3 gives us the solution \( \tilde{u} \in C([0, T_+), H^1) \) for (2.10), where \( T_+ > 0 \) is the maximal life time. Moreover, we have \( \tilde{u} \in C([0, T_+), L^2) \cap L^6_{\text{loc}}([0, T_+), L^6) \) from (2.12). By the uniqueness of a solution in \( C([0, T_+), L^2) \cap L^6_{\text{loc}}([0, T_+), L^6) \), \( u = \tilde{u} \) and, therefore, \( u \) is also in \( C([0, T_+), H^1) \) and it remains to show \( T_+ = \infty \).

Suppose to the contrary that \( T_+ < \infty \) and choose \( n \in \mathbb{N}_0 \) such that \( n < T_+ \leq n + 1 \). It follows from (2.18) that
\[
\|u\|_{L^4([n, T_+], L^p)} < \infty
\] for every admissible pair \((p, q)\).

For now, we assume to have
\[
\frac{d}{dt} \| \partial_x u(t) \|_{L^2}^2 = -2G(t) \text{Im} \int_\mathbb{R} (u(t) \overline{\partial_x u(t)})^2 dx
\] (2.19)
for all \( t \in (n, T_+) \). Then, for every \( t \in (n, T_+) \),
\[
\frac{d}{dt} \| \partial_x u(t) \|_{L^2}^2 \leq 2 \| u(t) \|_{L^\infty}^2 \| \partial_x u(t) \|_{L^2}^2
\]
and, therefore, by Gronwall’s inequality and the Cauchy–Schwarz inequality, we have
\[
\| \partial_x u(t) \|_{L^2}^2 \leq \| \partial_x u(n) \|_{L^2}^2 \exp \left( 2 \int_n^t \| u(t') \|_{L^\infty}^2 dt' \right) \\
\leq \| \partial_x u(n) \|_{L^2}^2 \exp \left( 2 \| u \|_{L^4([n, T_+], L^\infty)}^2 \right).
\]
Thus, we use (2.18) and the mass conservation to obtain
\[
\sup_{n < t < T_+} \| u(t) \|_{H^1}^2 < \infty
\] (2.20)
which contradicts the blow–up alternative.

To finish this case, it remains to show (2.11). We use the twisted solution
\[
w(t) = (U(n, t))^{-1} u(t).
\]
Since \( u \) solves (2.10), \( w \) solves
\[
w(t) = u(n) + i \int_n^t (U(n, t'))^{-1} G(t') |u(t')|^2 u(t') dt'
\]
and, therefore, \( w \) is differentiable on \((n, T_+)\) and
\[
\dot{w}(t) = \partial_t w(t) = i(U(n, t))^{-1} G(t) |u(t)|^2 u(t)
\]
which is in \( H^1(\mathbb{R}) \). Using this, one sees that
\[
\frac{d}{dt} \| \partial_x u(t) \|_{L^2}^2 = \frac{d}{dt} \| \partial_x w(t) \|_{L^2}^2 = 2 \text{Re} \langle \partial_x w(t), \partial_x \dot{w}(t) \rangle \\
= -2G(t) \text{Im} \langle \partial_x u(t), \partial_x (|u(t)|^2 u(t)) \rangle,
\]
which yields (2.19).

Next, we assume that \( d_{av} = 1 \) and solve (2.1) recursively. First, we find a solution in \( C([0, 1], H^1) \) of (2.1), i.e.,
\[
\begin{cases}
  i \partial_t u + 2 \partial_x^2 u + G(t) |u|^2 u = 0 & \text{for } t \in (0, 1), \\
  u = u_0 & \text{for } t = 0.
\end{cases}
\]
It is well-known that there exists a unique solution \( u \in C([0, 1], H^1) \), for example, see [9]. Denote \( u(1) \) by \( u_1 \) and solve the ordinary differential equation with the initial datum \( u_1 \)

\[
\begin{align*}
&\left\{ \begin{array}{ll}
    i\partial_t u + G(t)|u|^2 u = 0 & \text{for } t \in (1, 2) \\
    u = u_1 & \text{for } t = 1
\end{array} \right.
\end{align*}
\] (2.21)

of which solution is given by

\[ u(t) = u_1 \exp \left( i|u_1| \int_1^t G(t') dt' \right). \] (2.22)

Here, we used that \( |u(t)| = |u_1| \) for all \( t \in (1, 2) \). Indeed, multiplying the ordinary equation in (2.21) by \( \overline{u} \) and taking the imaginary part of the resulting identity, we obtain \( \partial_t |u(t)|^2 = 0 \) for all \( t \in (1, 2) \). Noting that (2.22) has the limit at \( t = 2 \), we continuously extend \( u \) to \([1, 2] \). Repeating this process completes the proof. The case \( \epsilon = 1 \) is done similarly.

**Remark 2.7.** To get (2.20), one can use a formal calculation as follows:

\[
\frac{d}{dt} \|\partial_x u(t)\|^2_{L^2} = -2\text{Re}(\partial_x u, \partial_x \partial_t u) = 2\text{Im} \left( (d_{av} + d_0(t)) \langle \partial_x u, \partial_x \partial_x^2 u \rangle + G(t) \langle \partial_x u, \partial_x |u|^2 u \rangle \right).
\]

However, the scalar product \( \langle \partial_x u, \partial_x \partial_x^2 u \rangle \) in \( L^2(\mathbb{R}) \) may not be defined for \( u \in H^1(\mathbb{R}) \) since \( \partial_x u \in L^2(\mathbb{R}) \) and \( \partial_x \partial_x^2 u = \partial_3^3 u \in H^{-2}(\mathbb{R}) \). Thus, as in [11], we used the twisting argument, see also, [4] [24].

### 3. Averaging theorem

In this section, to prove the averaging theorem (Theorem 1.2), we compare the solutions of equations (1.3) and (1.4)

\[ i\partial_t v + d_{av} \partial_x^2 v + Q_\epsilon(v) = 0, \] (3.1)

and

\[ i\partial_t v + d_{av} \partial_x^2 v + \langle Q \rangle(v) = 0 \] (3.2)

with close initial data \( u_0, v_0 \in H^1(\mathbb{R}) \), where the nonlinearities are given by

\[ Q_\epsilon(v) := G \left( \frac{t}{\epsilon} \right) T_{D(t/\epsilon)}^{-1} \left( |T_{D(t/\epsilon)}v|^2 T_D(t/\epsilon) v \right) \]

and

\[ \langle Q \rangle(v) := \int_0^1 T_{r}^{-1} \left( |T_r v|^2 T_r v \right) \psi(r) dr, \]

respectively. Recall that

\[ \psi(r) = e^{-r} \cosh \Gamma(r - 1). \]

We, first, show a lemma which is inspired by the proof of Theorem 4.1 in [28].

**Lemma 3.1.** For every \( M > 0 \), if \( v \in C([M, M], H^1) \), then

\[ \int_0^t e^{i d_{av}(t-t') \partial_x^2} Q_\epsilon(v(t')) dt' \to \int_0^t e^{i d_{av}(t-t') \partial_x^2} \langle Q \rangle(v(t')) dt' \] (3.3)

in \( C([-M, M], H^1) \) as \( \epsilon \to 0 \).

**Proof.** We consider positive times only. Let \( R_\epsilon(v) := Q_\epsilon(v) - \langle Q \rangle(v) \). Then, by the Plancherel identity, its Fourier transform in \( x \) can be expressed by

\[ \hat{R_\epsilon(v)}(\xi, t) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \delta(\xi_1 - \xi_2 + \xi_3 - \xi) A_\epsilon(\xi_1, \xi_2, \xi_3, \xi, t) \hat{v}(\xi_1, t) \hat{v}(\xi_2, t) \hat{v}(\xi_3, t) d\xi_1 d\xi_2 d\xi_3, \]

where

\[ A_\epsilon(\xi_1, \xi_2, \xi_3, \xi, t) = \frac{1}{2\pi} \int_{\mathbb{R}} \delta(\xi_1 - \xi_2 + \xi_3 - \xi) G \left( \frac{t}{\epsilon} \right) T_{D(t/\epsilon)}^{-1} \left( |T_{D(t/\epsilon)}v|^2 T_D(t/\epsilon) v \right) d\xi. \]
for all \( \xi \in \mathbb{R} \) and \( t \in [0, M] \), where
\[
A_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t) := G(t/\varepsilon)e^{-iD(t/\varepsilon)(\xi_1^2 - \xi_2^2 + \xi_3^2)} - \int_0^1 e^{-ir(\xi_1^2 - \xi_2^2 + \xi_3^2)} \psi(r)dr.
\]

Now, we define \( B_\varepsilon : \mathbb{R}^4 \times [0, M] \to \mathbb{C} \) by
\[
B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t) := \int_0^t A_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t')dt',
\]
then
\[
B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t) = \int_0^t A_1(\xi_1, \xi_2, \xi_3, \xi, t')dt' = \varepsilon \int_0^{t/\varepsilon} A_1(\xi_1, \xi_2, \xi_3, \xi, t')dt',
\]
and therefore,
\[
|B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t)| \leq \varepsilon \int_0^2 |A_1(\xi_1, \xi_2, \xi_3, \xi, t')|dt' \leq 4\varepsilon
\]
since \( A_1 \) is a 2-periodic function in \( t \) with mean zero and bounded by 2. Thus,
\[
\|B_\varepsilon\|_{L^\infty([0, M])} \leq 4\varepsilon.
\] (3.4)

Using the same argument as in (2.7), for every \( v_1, v_2 \in \mathcal{C}([0, M], H^1(\mathbb{R})) \), we see
\[
\left\| \int_0^t e^{i\partial_x v(t')}R_\varepsilon(v_1)(t') - Q_\varepsilon(v_2)(t') dt' \right\|_{L^\infty([0, M], H^1)} \leq CM\|v_1 - v_2\|_{L^\infty([0, M], H^1)}.
\] (3.5)

The estimate (3.5) also holds when \( Q_\varepsilon \) is replaced by \( \langle Q \rangle \). Therefore, by a density argument, it suffices to prove (3.3) for \( v \in \mathcal{C}^1([0, M], S(\mathbb{R})) \) only. Since
\[
\frac{\partial}{\partial t}B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t) = A_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t)
\]
for almost every \( t \in [0, M] \), by the integration by parts, we obtain
\[
\int_0^t e^{i\partial_x v(t')\xi^2}R_\varepsilon(v)(\xi, t')dt' = \hat{I}_1(v)(\xi, t) - \int_0^t e^{i\partial_x v(t')\xi^2} \left( \hat{I}_2(v)(\xi, t') - i\partial_v \hat{I}_3(v)(\xi, t') \right) dt',
\]
where
\[
\hat{I}_1(v)(\xi, t) = \int_{\mathbb{R}^3} \delta(\xi_1 - \xi_2 + \xi_3 - \xi)B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t)\hat{\varphi}(\xi_1, t)\hat{\varphi}(\xi_2, t)\hat{\varphi}(\xi_3, t)d\xi_1d\xi_2d\xi_3,
\]
\[
\hat{I}_2(v)(\xi, t) = \int_{\mathbb{R}^3} \delta(\xi_1 - \xi_2 + \xi_3 - \xi)B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t)\partial_t(\hat{\varphi}(\xi_1, t)\hat{\varphi}(\xi_2, t)\hat{\varphi}(\xi_3, t))d\xi_1d\xi_2d\xi_3,
\]
and
\[
\hat{I}_3(v)(\xi, t) = \int_{\mathbb{R}^3} \delta(\xi_1 - \xi_2 + \xi_3 - \xi)B_\varepsilon(\xi_1, \xi_2, \xi_3, \xi, t)\xi^2\partial_t\hat{\varphi}(\xi_1, t)\hat{\varphi}(\xi_2, t)\hat{\varphi}(\xi_3, t)d\xi_1d\xi_2d\xi_3.
\]

Then, fix \( t \in [0, M] \), we have
\[
\left\| \int_0^t e^{i\partial_x v(t')\xi^2}R_\varepsilon(v)(\cdot, t')dt' \right\|_{H^1} \leq \left( \int_{\mathbb{R}^1} (1 + \xi^2) \left\| \int_0^t e^{i\partial_x v(t')\xi^2}R_\varepsilon(v)(\xi, t')dt' \right\|^2 d\xi \right)^{1/2}
\]
\[
\leq \| I_1(v)(\cdot, t) \|_{H^1} + \int_0^t \left( \| I_2(v)(\cdot, t') \|_{H^1} + \| I_3(v)(\cdot, t') \|_{H^1} \right) dt',
\] (3.6)

where we use Minkowski’s inequality. First, to get a bound of \( I_1(v) \), note
\[
|I_1(v)(\xi, t)| \leq \|B_\varepsilon\|_{L^\infty([0, M])} \left\| \int_{\mathbb{R}^2} \hat{\varphi}(\xi_1, t)\hat{\varphi}(\xi_2, t)\hat{\varphi}(\xi - \xi_1 + \xi_2, t)d\xi_1d\xi_2 \right\|.
\]
for all $\xi$ and $t$. If we define $I(f_1, f_2, f_3)$ by its Fourier transform

$$\hat{I}(f_1, f_2, f_3)(\xi) = \int_{\mathbb{R}^d} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi - \xi_1 + \xi_2) d\xi_1 d\xi_2$$

for all $f_1, f_2, f_3 \in \mathcal{S}(\mathbb{R})$, then by a straightforward calculation, we obtain

$$\|I(f_1, f_2, f_3)\|_{H^1} \leq C\|f_1\|_{H^1} \|f_2\|_{H^1} \|f_3\|_{H^1}.$$  

This together with (3.1), we have

$$\|I_1(v)(\cdot, t)\|_{H^1} \leq \|B_e\|_{L^\infty(\mathbb{R}^d \times [0, M])} \|I(v(t), v(t), v(t))\|_{H^1} \lesssim \varepsilon \|v(t)\|_{H^1}^2$$

for all $t$. By a similar argument, we have

$$\|I_2(v)(\cdot, t)\|_{H^1} \lesssim \varepsilon \|\partial_t v(t)\|_{H^1} \|v(t)\|_{H^1}^2$$

and

$$\|I_3(v)(\cdot, t)\|_{H^1} \lesssim \varepsilon \left(\|\partial_t^2 v(t)\|_{H^1} \|v(t)\|_{H^1}^2 + \|v(t)\|_{H^1} \|v(t)\|_{H^1}\right)$$

for all $t$. Substituting the last three inequalities into (3.6) completes the proof.

Now we are ready to give

**Proof of Theorem 1.2.** Fix $M > 0$ and consider positive times only. Let

$$K = 2 \sup_{t \in [0, M]} \|v(t)\|_{H^1}$$

and let $0 < \varepsilon \leq \frac{K}{2}$ for now. Then we have $\|u_0\|_{H^1} \leq K$ since $\|u_0 - v_0\|_{H^1} \leq \varepsilon$ and $\|v_0\|_{H^1} \leq \frac{K}{2}$. Then, it follows from Proposition 2.2 that there exists $M_+ = M_+(K)$, independent of $\varepsilon$, such that $v_\varepsilon \in \mathcal{C}([0, M_+], H^1(\mathbb{R}))$ and

$$\sup_{0 < \varepsilon \leq \frac{K}{2}} \sup_{t \in [0, M_+]} \|v_\varepsilon(t)\|_{H^1} \leq 2K. \quad (3.7)$$

We now prove that (1.5) holds on $[0, M_+]$, i.e., there exists $C > 0$ such that

$$\|v_\varepsilon - v\|_{\mathcal{C}([0, M_+], H^1(\mathbb{R}))} \leq C\varepsilon.$$

By Duhamel’s formula, we have

$$v_\varepsilon(t) - v(t) = e^{idav(t)^2\partial_x^2}(u_0 - v_0) + i\mathcal{I}_1(t) + i\mathcal{I}_2(t)$$

for all $0 \leq t \leq M_+$, where

$$\mathcal{I}_1(t) = \int_0^t e^{i d a v(t - t')^2 \partial_x^2} \left[Q_\varepsilon(v_\varepsilon)(t') - Q_\varepsilon(v)(t')\right] dt'$$

and

$$\mathcal{I}_2(t) = \int_0^t e^{i d a v(t - t')^2 \partial_x^2} \left[Q_\varepsilon(v)(t') - \langle Q\rangle(v)(t')\right] dt'.$$

It follows from Lemma 3.1 that there exists a constant $C > 0$ such that

$$\sup_{t \in [0, M_+]} \|\mathcal{I}_2(t)\|_{H^1} \leq C\varepsilon. \quad (3.8)$$

To bound $\mathcal{I}_1$, we use Minkowski’s inequality and Lemma 2.1, then we obtain

$$\|\mathcal{I}_1(t)\|_{H^1} \leq \int_0^t \|Q_\varepsilon(v_\varepsilon)(t') - Q_\varepsilon(v)(t')\|_{H^1} dt'$$

$$\lesssim \int_0^t (\|v_\varepsilon(t')\|_{H^1}^2 \|v(t')\|_{H^1}^2) \|v_\varepsilon(t') - v(t')\|_{H^1} dt'. \quad (3.9)$$
for all $0 \leq t \leq M_+$. Since $\|u_0 - v_0\|_{H^1} \leq \varepsilon$, it follows from (3.8) and (3.9) that, for all $0 \leq t \leq M_+$, there exist positive constants $C_1$, depending only on $K$, and $C_2$ such that
\[
\|v_\varepsilon(t) - v(t)\|_{H^1} \leq \|u_0 - v_0\|_{H^1} + \|\mathcal{I}_1(t)\|_{H^1} + \|\mathcal{I}_2(t)\|_{H^1} \\
\leq C_2\varepsilon + C_1 \int_0^t \|v_\varepsilon(t') - v(t')\|_{H^1} dt'.
\]
Thus, by Gronwall’s inequality, we obtain
\[
\sup_{t \in [0, M_+]} \|v_\varepsilon(t) - v(t)\|_{H^1} \leq C_2\varepsilon e^{-C_1 M_+}.
\] (3.10)

If $M_+ \geq M$, the proof is complete and now we assume that $M_+ < M$. It follows from (3.10) and $v_\varepsilon - v \in \mathcal{C}([0, M_+], H^1(\mathbb{R}))$ that
\[
\|v_\varepsilon(M_+) - v(M_+)\|_{H^1} \leq C_2\varepsilon e^{C_1 M_+}.
\]

Now choose $\varepsilon_1 > 0$ such that $C_2\varepsilon_1 e^{C_1 M_+} \leq \frac{K}{2}$ and $\varepsilon_1 \leq \frac{K}{2}$. Let $0 < \varepsilon \leq \varepsilon_1$. Then
\[
\|v_\varepsilon(M_+)\|_{H^1} \leq \frac{K}{2} + C_2\varepsilon e^{C_1 M_+} \leq K.
\]
Applying Proposition 2.2 to (2.3) with the initial datum $v_\varepsilon(M_+)$, which satisfies
\[
\sup_{0 < \varepsilon \leq \varepsilon_1} \|v_\varepsilon(M_+)\|_{H^1} \leq K,
\]
we have
\[
\sup_{0 < \varepsilon \leq \varepsilon_1} \sup_{t \in [M_+, 2M_+]} \|v_\varepsilon(t)\|_{H^1} \leq 2K.
\]
Combining this and (3.7), we have
\[
\sup_{0 < \varepsilon \leq \varepsilon_1} \sup_{t \in [0, 2M_+]} \|v_\varepsilon(t)\|_{H^1} \leq 2K.
\]
Repeat the above argument to obtain
\[
\sup_{t \in [0, 2M_+]} \|v_\varepsilon(t) - v(t)\|_{H^1} \leq C_2\varepsilon e^{2C_1 M_+}.
\]

Iterating this argument, we can choose $\varepsilon_0 > 0$ such that $C_2\varepsilon_0 \exp \left( (\lfloor \frac{M}{M_+} \rfloor + 1) C_1 M_+ \right) \leq \frac{K}{2}$ and $\varepsilon_0 \leq \frac{K}{2}$ to get (1.5) with $C = C_2 \exp \left( (\lfloor \frac{M}{M_+} \rfloor + 2) C_1 M_+ \right)$, where $\lfloor a \rfloor = \max\{n \in \mathbb{Z} : n \leq a\}$. This completes the proof. □

**Appendix A. Linear Propagator**

For the reader’s convenience, the properties of the linear propagator $U(t_0, t)$ defined in (2.11) are referred in this section, which can be also found in [3, 5].

**Lemma A.1.** Let $d_{av} \neq \pm 1$. Then there exists a constant $C$ depending only on $d_{av}$ such that
\[
\|U(t_0, t)f\|_{L^\infty} \leq C|t - t_0|^{-1/2}\|f\|_{L^1}
\]
for all $f \in L^1(\mathbb{R})$ and all distinct $t, t_0$ with $|t| = |t_0|$.

**Proof.** Using the kernel of the solution operator for the free Schrödinger equation
\[
(T_t f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-x^2y^2/4t} f(y) dy, \quad t \neq 0,
\]
for Schwartz functions \( f \), we obtain that
\[
\|U(t_0, t)f\|_{L^\infty} \leq \frac{\|f\|_{L^1}}{(4\pi)^{1/2}(\int_{t_0}^t \langle t' \rangle dt')}^{1/2}, \quad t \neq t_0.
\]
Using this and the fact that for all \( t \) and \( t_0 \) with \( [t] = [t_0] \),
\[
\left| \int_{t_0}^t \langle t' \rangle dt' \right| = \begin{cases} |d_{av} + 1|t - t_0|, & [t_0] \text{ even}, \\ |d_{av} - 1|t - t_0|, & [t_0] \text{ odd}, \end{cases}
\]
we complete the proof.

Using Lemma A.1 and the unitarity of \( U(t_0, t) \) on \( L^2(\mathbb{R}) \), via the well–known arguments, we obtain the one–dimensional Strichartz estimates on each time interval \([n, n+1]\), for every \( n \in \mathbb{Z} \). For the classical Strichartz estimates, see, e.g., \([23, 16, 21]\). As usual, we say that a pair of exponents \((p, q)\) is admissible if \( 2 \leq p \leq \infty \) and
\[
\frac{1}{p} + \frac{2}{q} = \frac{1}{2}.
\]
and, also, for every \( p \geq 1 \), denote by \( p' \) the Hölder conjugate, i.e.,
\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

**Lemma A.2.** Assume \( d_{av} \neq \pm 1 \). Let \((p, q)\) and \((p_0, q_0)\) be admissible pairs and \( t_0 \in [n, n+1]\) for some \( n \in \mathbb{Z} \).

(i) If \( f \in L^2(\mathbb{R}) \), then the function \( t \mapsto U(t_0, t)f \) on \([n, n+1]\) belongs to \( L^q([n, n+1], L^p) \cap C([n, n+1], L^2) \). Moreover, there exists a constant \( C \) depending only on \( p \) and \( d_{av} \) such that for all \( f \in L^2(\mathbb{R}) \)
\[
\|U(t_0, t)f\|_{L^q([n, n+1], L^p)} \leq C\|f\|_{L^2}.
\]

(ii) Let \( I \) be an interval contained in \([n, n+1]\) and \( t_0 \in I \). If \( F \in L^{q_0}(I, L^{p_0}) \), then the function
\[
t \mapsto \int_{t_0}^t U(t', t)F(t')dt'
\]
on \( I \) belongs to \( L^q(I, L^p) \cap C(I, L^2) \). Moreover, there exists a constant \( C \) depending only on \( p, p_0, \) and \( d_{av} \) such that for all \( F \in L^{q_0}(I, L^{p_0}) \)
\[
\left( \int_I \left\| \int_{t_0}^t U(t', t)F(\cdot, t')dt' \right\|_{L^p}^q dt \right)^{1/q} \leq C \left( \int_I \|F(\cdot, t)\|_{L^{p_0}}^{q_0} dt \right)^{1/q_0}.
\]

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