II_{∞} Factors and M-theory in Asymptotically Flat Space

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Abstract: I discuss a formulation of M-theory at null infinity, which is based on general principles of holographic space-time, and is manifestly covariant. The construction utilizes a certain Type II Von Neumann algebra, which provides a kinematic framework, alternative to Fock Space, for describing the scattering states of eleven dimensional asymptotically flat M-theory. The construction provides a greatly clarified statement of the connection between SUSY and holography. I make preliminary remarks about dynamical equations for the S-matrix, and compactifications.
1. Introduction

In a series of papers primarily devoted to cosmology[1], W. Fischler and the present author have developed a general framework for quantum gravity based on the holographic principle. The purpose of the present paper is to begin to make contact between that formalism and conventional formulations of M-theory. We will deal primarily with the simplest version of the theory in 11 asymptotically flat dimensions.

The basic ingredient of the holographic approach to quantum gravity is the operator algebra of a causal diamond\(^1\). This replaces the notion of the fields at a point in local field theory. The covariant entropy bound [2] bounds the entropy in a causal diamond by one quarter of the area in Planck units of its holographic screen. The holographic screen is the maximal area spacelike \(d - 2\) surface on the boundary of the diamond. Fischler and I hypothesized that the entropy referred to in the entropy bound was that of the maximally uncertain density matrix in the Hilbert space of states associated with the diamond. Thus the entropy is the logarithm of the dimension of this Hilbert space.

\(^1\)A causal diamond is the region bounded by the backward lightcone of a point \(P\) and the forward lightcone of a point \(Q\) in the causal past of \(P\).
The operator algebra of the diamond can be constructed from the quantization of classical variables specifying the orientation of its holoscreen. A small area or pixel on the holoscreen can be described by giving the direction of the null ray\(^2\) penetrating the screen and a \((d - 2)\) dimensional area element transverse to this null ray. Both are specified, up to conformal rescaling, by a solution of the Cartan-Penrose equation:

\[
\bar{\psi} \gamma^\mu \psi \gamma^\mu \psi = 0. \tag{1.1}
\]

The independent real components of this pure spinor are quantized by the formula

\[
[S_a(n), S_b(n)]_+ = \delta_{ab}. \tag{1.2}
\]

The dimension of the irreducible representation of this algebra specifies the area of the pixel, by the Bekenstein-Hawking rule.

The quantization condition breaks the conformal invariance of the CP equation, leaving over only a \((\text{local}) Z_2\). Using this, we can Klein transform the \textit{a priori} commuting variables associated with different pixels, so that the full operator algebra of the diamond is

\[
[S_a(n), S_b(m)]_+ = \delta_{ab} \delta_{mn}. \tag{1.3}
\]

This residual \(Z_2\) gauge invariance is identified with \((-1)^F\).

The proposed quantization rule says that the degrees of freedom associated with a pixel of the holographic screen are precisely the spin degrees of freedom of a massless superparticle, which may be viewed as the quantum degree of freedom which entered (exited) the diamond via that pixel. This connection between the holographic principle and supersymmetry is one of the most exciting features of our formalism.

Note that, although we will not discuss compactification of eleven dimensions in this paper, the way to accommodate it is to enlarge the algebra of pixel generators to included charges associated with Kaluza-Klein symmetries, their magnetic duals, and wrapped branes. These are precisely the features of compact spaces which are invariant under the various dualities of M-theory. We will propose a quasi-geometrical picture only for the non-compact dimensions of space-time (which are however taken to include de Sitter space in some cases).

\(^2\)There is an implicit choice of whether the future directed null ray is entering or leaving the diamond. In asymptotically flat space one should associate different variables with incoming and outgoing rays, which are related by the approximate S-matrix described below.
2. M-theory in Asymptotically Flat Space (AFM-theory) - Light Cone Gauge

At present, our most complete formulation of M-Theory in Asymptotically Flat Space-time is for those cases where Matrix Theory[3] is well defined. Matrix Theory is the DLCQ of M-Theory compactified on $T^d$ with $0 \leq d \leq 5$ and on K3 [4]. For $d \leq 3$ it is described in terms of a Lagrangian quantum field theory, while for $d = 4$ and K3 one needs the $(2,0)$ superconformal field theory in six dimensions, and for $d = 5$ the even more mysterious Little String Theory.

The S-matrix of M-Theory is supposed to be obtained as the $N \to \infty$ limit of the scattering matrix of these quantum theories. All of them have a moduli space on which one can define a scattering problem (note that the space on which the various field theories are defined is compact and does not admit scattering). Apart from amplitudes protected by non-renormalization theorems, one does not expect to get covariant results before taking the large $N$ limit.

It would clearly be useful to have a formulation in which $N$ was already infinite. One would then expect to have exact Lorentz invariance. Moreover, some of the problems associated with compactifying more dimensions appear to go away at $N = \infty$. In this paper I will present some ideas about formulating theories of asymptotically flat quantum gravity directly at null infinity, in a manifestly Lorentz invariant manner. For the present, I will restrict attention to the uncompactified eleven dimensional theory.

The first ideas, which evolved into this paper, were developed several years ago in an attempt to formulate Matrix Theory directly at $N = \infty$. The obvious conjecture is that the variables of Matrix Theory should be replaced by elements of an infinite dimensional associative algebra. The algebra should be a complex algebra with involution, and the basic variables are Hermitian with respect to the involution. The form of the Matrix Theory action requires that the algebra possess a well defined trace. This then defines an inner product via

\[
(A, B) = Tr A^\dagger B. \tag{2.1}
\]

Using this one can realize the algebra as an algebra of linear operators acting on an inner product space. Various theorems in the literature suggest[?] that it will always in fact be a Von Neumann algebra, that is, a weakly closed algebra of bounded operators in a Hilbert space. By a celebrated theorem of Von Neumann, a condition

3This attempt has run into various problems. I will present it elsewhere, if a solution to those problems is found.

4Weakly closed means that if for a sequence of operators $A_n$, $A_n|\psi> \to A|\psi>$, for all states $|\psi>$, then $A$ is a member of the algebra.
equivalent to weak closure is that the algebra be equal to its own double commutant in the algebra of all bounded operators on Hilbert space. A Von Neumann algebra whose center is just the complex numbers is called a factor. Every Von Neumann algebra can be constructed as a direct sum (integral) of factors.

Murray and Von Neumann [5] classified factors according to the possible values that the trace takes on projectors i.e. operators satisfying $e_i^2 = e_i$. Type $I_N$ algebras are the algebras of all bounded operators in an $N$ dimensional Hilbert space ($N = \infty$ is included in the list). The trace of projectors is always a positive integer. In Type II algebras the trace of projectors takes on continuous positive values. $II_1$ algebras have a maximal value of the trace, that of the unit operator, which is normalized to 1. In Type $II_\infty$ algebras the trace of projectors is unbounded. Every such algebra is a direct product of a $I_\infty$ factor and a Type $II_1$ factor. In Type III factors the trace of projectors takes on only the values 0 and $\infty$. We will not have further occasion to discuss them here.

It is obvious that the factors of interest for us are of Type $II_\infty$. In fact I will suggest that it is a very particular $II_\infty$ factor. The trace of an element of the algebra will measure its longitudinal momentum (in Planck units). That is, for any operator $A$ in our algebra, find the projector $e_{p^+}$ of maximal trace ($\equiv p^+$) such that $A = e_{p^+} A e_{p^+}$. $P^+(A)$ is the function from the algebra to the nonnegative real numbers defined by this maximal trace.

One of the invariants that characterizes inequivalent factors of the same type, is the group of outer automorphisms of the algebra. An automorphism is an invertible mapping of the algebra into itself which preserves the algebraic operations (including Hermitian conjugation) and is continuous in the topology defined by weak operator convergence. If $U$ is any unitary element of the algebra, the mapping $A \rightarrow U^* AU$ is an inner automorphism. The group of outer automorphisms, $Out[A]$ is the factor group of the full automorphism group by the normal subgroup of inner automorphisms, (also called the unitary group of the algebra). Given any automorphism $\rho$ we can define a new trace by $Tr_\rho[A] = Tr[\rho(A)]$. For inner automorphisms this is just the old trace. For outer automorphisms it is in principle different, but Murray and Von Neumann proved the uniqueness of the trace up to a positive multiplicative factor. Thus $Tr_\rho[A] = e^{\lambda_\rho} Tr[A]$ where $\lambda_\rho$ is real. This mapping from the group $Out[A]$ to the group of positive reals, is a homomorphism, and the image subgroup is an invariant which can be used to distinguish factors.

There is a particular $II_\infty$ factor for which this group is the group of all positive real numbers, and the homomorphism is an isomorphism. Thus, there is a unique outer automorphism that multiplies the trace by any given positive number. For this factor, which is denoted $R_{0,1}$ in the classification of Araki and Woods, [6], every two projectors
with the same trace, are unitarily equivalent.

The factor \( R_{0,1} \) is the product of the algebra of all bounded operators (the unique \( I_\infty \) factor) with a special \( II_1 \) factor first constructed by Murray and Von Neumann. The construction is fairly easy for physicists to understand. One considers the \( n \) dimensional Clifford-Dirac algebra \( [\gamma^a, \gamma^b]_+ = 2\delta^{ab} \) in its irreducible Dirac spinor representation. Normalize the trace of the unit operator in this representation to 1. The trace of any projector is then a rational number between 0 and 1, and it is easy to see that as \( n \to \infty \) these numbers become dense in the interval. Embed this sequence of finite dimensional algebras in the algebra of all bounded operators in Hilbert space. Obviously, each member of the sequence can be realized as a subalgebra of the next member. Murray and Von Neumann show that an appropriately defined limit of this sequence is a Type \( II_1 \) factor, named \( R \). The factor \( R \) is in fact the only \( II_1 \) factor which is generated by such an increasing sequence of finite dimensional subalgebras. This property, with an appropriate definition of what it means to approximate any element by a finite dimensional matrix, is called \textit{hyperfiniteness}, and is shared by its daughter, the \( II_\infty \) factor \( R_{0,1} \).

A maximal abelian subalgebra of \( R_{0,1} \) is thus generated by finite linear combinations of orthogonal projectors, whose traces are arbitrary positive real numbers. If two such linear combinations have the same collection of traces then they are unitarily equivalent to each other. Now consider a 9-vector \( \mathbf{X} \) of elements of the commuting subalgebra. It is a limit of terms of the form \( \sum x_I e_I \), where \( e_I e_J = \delta_{IJ} e_I \) and \( \text{Tr} e_I = P_I^+ \). The subgroup of the unitary group that preserves the maximal abelian subalgebra, acts on these vectors by permuting the \( x_I \) with the same value of \( P_I^+ \).

To whet the reader’s appetite for what follows, I will consider the Lagrangian of [3] where both the \( \mathbf{X} \) and \( \Theta \) variables are taken to be elements of the \( R_{0,1} \) algebra rather than an \( N \times N \) matrix algebra. We also drop the variable \( R \) representing the length of the compactified null circle since we are attempting to describe the Lorentz invariant theory with noncompact longitudinal direction. We will continue to call elements of the Von Neumann algebra, \textit{matrices}, in order to distinguish them from quantum mechanical operators. For the moment, we restrict attention to the maximal abelian subalgebra, dropping the commutator terms in the Lagrangian. A general configuration is given by

\[
\mathbf{X} = \sum X_I(t)e_I \tag{2.2}
\]

\[
\Theta = \sum \theta_I(t)e_I. \tag{2.3}
\]

In terms of the ordinary variables \( x_I, \theta_I \) the Lagrangian is
\[ \mathcal{L} = 1/2 \sum \dot{x}_I^2 p_I^+ + i \theta_I \dot{\theta}_I p_I^+. \]  

(2.4)

This is just the light cone Lagrangian for \( M \) copies of the eleven dimensional super-particle. When we quantize it we get the states of \( M \) supergravitons. Configurations with permutations of the indices, \( I \) are related by unitary equivalence in the algebra \( R_{[0,1]} \) and, as in matrix theory, this is to be treated as a gauge invariance. This \( S_M \) gauge invariance, and the anti-commutation relations of the \( \theta_I \) give us the correct statistics of the supergravitons. In other words, when the Matrix Theory Lagrangian is applied to the maximal abelian subalgebra of \( R_{0,1} \), then quantization of the theory leads to the Fock space of 11 dimensional SUGRA. In the next section, we will try to cast this new form of gravitational kinematics in a manifestly Lorentz invariant form.

3. M-theory at null infinity

The success of the AdS/CFT correspondence tempts us to construct a manifestly covariant formalism for AFM-theory on null-infinity. This might cause difficulties for massive particles. Null infinity is not a manifold and the asymptotic wave-functions of massive particles are concentrated near two of its singularities. However, at least in eleven dimensions, all stable finite energy states of M-theory are massless supergravitons, so a formulation on null-infinity does not run into a priori difficulties. We will return briefly to the question of massive particles below.

A formulation on null infinity cannot share the dynamical properties of AdS/CFT. This is most clearly seen in Ashtekar’s description of massless free field theory on null infinity[7]. The coordinates of null infinity in eleven dimensions are \((u, \Omega)\), where \( u \) is null and \( \Omega \) parametrizes a 9-sphere. This “manifold” does not have a metric, but only a conformal structure: the set of conformal rescalings of the round metric on the sphere. The coordinate \( u \) is also rescaled by the conformal factor. If \( g_{ab}(\Omega) \rightarrow \omega^2(\Omega) g_{ab}(\Omega) \), then \( u \rightarrow \omega u \).

The conformal group of the 9-sphere is \( SO(1,10) \) and this is interpreted as the Lorentz group of asymptotically flat space-time. The translation generators, in the Lorentz frame where the metric on the sphere is round, are the vector fields \( P_\mu = (1, \Omega) \partial_u \), where \( \Omega^2 = 1 \), parametrizes the 9-sphere.

It is important to understand that in eleven dimensions, the gravitational S-matrix for finite numbers of particles does not suffer from infra-red divergences. There is no need to consider classical gravitational radiation in the initial or final states, and one can use the stringent asymptotic condition[7], which only allows the vacuum as an asymptotically flat solution. Consequently there is no need to discuss the Bondi-
Metzner-Sachs group. The definition of asymptotically flat space-time used by relativists, allows classical gravitational radiation in all dimensions, and the invariance group of such a formalism would have to be the BMS group. This is puzzling to string theorists, who are used to computing a gravitational S-matrix with only Poincare invariance. The absence of IR divergences is the explanation of this puzzle, and we will adopt the string theory definition of asymptotically flat space, in which the classical background is forced to satisfy the stringent asymptotic condition that Ashtekar calls restriction to the vacuum sector.

Multiparticle states of massless particles can be described in terms of fields at null infinity, but there are no propagation equations. For example, a massless scalar field is completely specified by the commutation relation

$$[\phi(u, \Omega), \phi(v, \Theta)] = \epsilon(u - v)\delta^9(\Omega, \Theta),$$  

where we have used the usual Heaviside $\epsilon$ function and the invariant $\delta$ function on the sphere. Thus, the $u$ coordinate plays a role analogous to longitudinal position, rather than light front time. There is no analog of light front time at null infinity.

Dynamics at null infinity is instead encoded in the S-matrix. Indeed, so far we have only described future null infinity. Past null infinity is an identical copy of the same conformal “manifold”, and the scattering matrix is a mapping between the natural bases of states on $I_\pm$. The problem of dynamics thus reduces to finding a set of equations for determining the scattering matrix in terms of more elementary objects, the analogs of the Hamiltonian of the light front formalism.

Our discussion of this problem breaks into two parts, a long kinematical discussion of a Matrix Theory-like parametrization of the Hilbert spaces at $I_\pm$, and a short speculative subsection on dynamical equations for the S-matrix.

### 3.1 Kinematics

The alternative to Ashtekar’s Fock space description of kinematics at null infinity is based on the work of [1] and [3]. One of the primary purposes of the present paper is to make contact between the formalism of [1] and [8], and established theories of quantum gravity.

The fundamental geometrical object in Lorentzian space-time is a causal diamond. In the holographic proposal for the kinematic description of quantum space-time, each causal diamond is replaced by a Hilbert space which is the fundamental representation of the anti-commutation relations,

$$[S_a(n), S_b(m)]_+ = \delta_{ab}\delta_{mn}.$$  

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The logarithm of the dimension of this Hilbert space is the quantum version of \((\frac{1}{2} \times)\) the area of the largest \(d - 2\) surface encountered on the boundary of the causal diamond. This surface is called the holographic screen.

The idea behind this association is the Cartan-Penrose relation between pure spinors of \(SO(1,10)\) and null flags consisting of a null direction and a bit of area transverse to it. Consider a 10 dimensional null hypersurface in eleven dimensional space-time, and let \(n^\mu\) be the null direction pointing out of the surface at a given point. Both this null direction and the orientation of the spacelike 9 plane orthogonal to it are captured by a pure spinor satisfying

\[
n^\mu \gamma_\mu \psi = 0,
\]

the Cartan-Penrose equation. Indeed, \(n^\mu \propto \bar{\psi} \gamma^\mu \psi\), and the orientation of a transverse 9 plane is specified by the non-vanishing components of \(\bar{\psi} \gamma^{[\mu_1 \cdots \mu_k]} \psi\). The pure spinor has 16 independent real components \(S_a(\Omega)\). \(\Omega\) is a coordinate on the holographic screen and the notation indicates that we should think of the collection of spinor variables describing bits of the screen as sections of the spinor bundle over the screen. Note that in ordinary Lorentzian geometry, specifying all of these variables for every screen would over determine the conformal structure of the manifold. That is, there must be consistency conditions, relating the \(S_a(\Omega)\) variables for different screens.

The CP equation is conformally invariant, and invariant under rescaling of \(\psi\), as well as under rotations in the transverse plane. In classical Lorentzian geometry, the spinors are only sensitive to the causal structure. We think of the \(S_a(n)\) operators above as the quantization of the screen variables \(S_a(\Omega)\).

Each \(S_a(n)\) should be thought of as representing a particular pixel of the holographic screen, with a quantized area equal (in Planck units) to \(4\ln 256\) (the logarithm of the dimension of the irreducible representation of the Clifford algebra with 16 generators. Actually, since the algebra 3.2 is invariant under orthogonal transformations \(S_a(n) \to O_{nm} S_a(m)\), only one basis for the \(S_a(n)\) algebra should be associated with a pixel (a small area element on the holographic screen). A more invariant way to describe the pixelation of the screen geometry is to imagine that the algebra of continuous (or measurable) functions on the screen is replaced by a finite dimensional algebra, with a particular basis of operators corresponding (morally, as one says) to operators \(\int d\Omega S_a(\omega)f_n(\Omega)\), with \(f_n\) some basis of the finite function algebra.

For general space-times there will not be a canonical algebra of functions for a given causal diamond. Rather, there are many choices, related by the analog of general coordinate transformations. However, in asymptotically flat space-times it is reasonable to insist on causal diamonds which preserve the full symmetry of spatial rotations. A nested sequence of causal diamonds corresponds to a time-like observer, and it is
reasonable to insist on keeping the maximal symmetry of such an observer’s world line in the quantum theory. For M-theory in 11 dimensions, a possible procedure is to equate the function algebra of the holographic screen of a causal diamond, with the matrix algebra generated by a sequence of \( d_N \) dimensional representations of the \( SO(10) \) Dirac algebra, with \( d_N \to \infty \). The \( S_a(n) \) for a nested sequence of causal diamonds, converging to null infinity, should transform as sections of (appropriately defined) spinor bundles over this sequence of algebras.

When the space-time boundary conditions admit a TCP transformation, we expect to be able to choose TCP invariant causal diamonds. That is, we expect two descriptions of the Hilbert space \( S_a^+(n) = T_{a,n}^{b,m} S_b^- (m) T^{-1} \), related by an anti-unitary involution, \( T \). The matrix \( C \) implements the geometrical space inversion symmetry as well as charge conjugation. We may view these two collections of operators as being associated with the future and past null boundary of the causal diamond respectively.

As such, we expect them to also be related by the unitary time evolution operator

\[
S_a^+(n) = S_D^{+a} S_a^{-}(n) S_D,
\]

where \( S_D \) is the scattering matrix of the causal diamond. If we view an observer as a nested sequence of causal diamonds, the S-matrix of one diamond can be constructed by concatenating the evolution operators in individual diamonds, as in [1]. It is clear that these observer dependent quantities cannot be exactly gauge invariant observables of a theory of quantum gravity. However, in an asymptotically flat space-time, the S-matrix for the limit of large causal diamonds should be universal and gauge invariant.

Thus we want to imagine a limit in which our causal diamond becomes the interior of null infinity in asymptotically flat 11 dimensional space-time. In order to achieve that we first study the single particle states of supergravitons. These are described by the null momentum of the state, tensored with a spin index. The wave function has the form \( \Phi_A(p, \Omega) \), where \( A \) takes on 256 values. The null momentum (in an appropriate Lorentz frame) is \( p^\mu = p(1, \Omega) \), and \( \Omega^2 = 1 \) parametrizes a point on \( S^9 \). The spin space is a representation of a single copy of the 16 generator Clifford algebra, with generators \( S_a \). These are the independent components of a pure spinor satisfying \( p_\mu \gamma^\mu \psi = 0 \).

We can organize the operator algebra on the single particle Hilbert space in the following way: for any measurable section \( f^a(\Omega) \) of the spinor bundle over the 9-sphere introduce the operator \( S(f) \) by

\[
S(f) \Phi_A(p, \Omega) = f^a(\Omega)(S_a)_A^B \Phi_B(p, \Omega).
\]

These operators, combined with the \( SO(10) \) rotations of the 9-sphere, generate the full operator algebra for fixed value of \( p \). We can describe this in the following language.
The algebra of measurable functions on the sphere has an outer automorphism group $SO(10)$ which preserves the round metric on the sphere. The operators $S(f)$ are a linear map from sections of the spinor bundle to the algebra of operators on Hilbert space. If we write

$$S(f) = \int d\Omega f^a(\Omega) S_a(\Omega),$$

then, under $SO(10)$, $S_a(\Omega)$ transforms like a section of the spinor bundle on the 9-sphere.

So far we have worked at fixed $p$, and restricted attention to single particle states. We deal with these omissions simultaneously by combining Ashtekar’s insight that $p$ is analogous to longitudinal momentum on a light front, with the treatment of multi-particle states in Matrix Theory and the theory of Type II Von Neumann algebras. Define the Von-Neumann algebra

$$\mathcal{A} \equiv R_{[0,1]} \otimes \mathcal{M},$$

the tensor product of the hyperfinite type II$_\infty$ factor with the algebra of measurable functions on the 9-sphere. We define the $\mathcal{S}$, the spinor bundle over this algebra to be the tensor product of $R_{[0,1]}$ with the spinor bundle over the nine sphere.

Introduce linear maps $S(\rho(f))$ for $f \in \mathcal{S}$, from $\mathcal{S}$ to the algebra of operators in Hilbert space. $\rho$ is any element of a group of outer automorphisms of $\mathcal{A}$, which we specify below.

We require $S$ to be invariant under inner automorphisms of $\mathcal{A}$. A general element of $\mathcal{A}$ can be written as a limit of a finite sum $r_i \chi_i$, where the $\chi_i$ are characteristic functions of disjoint subsets of $S^9$, and $r_i \in R_{[0,1]}$. A general inner automorphism has the form $\sum U_i \chi_i$, where the $\chi_i$ form a partition of unity on $S^9$. Thus, we can use inner automorphism invariance to diagonalize all the $r_i$.

Let us verify that this prescription generates the Fock space of eleven dimensional SUGRA. Invariance under unitary transformations means that we can write every element of the algebra as a limit of finite sums of the form $A = \sum e_k f_k(\Omega)$, where each $e_k$ is a projector in $R_{[0,1]}$, and $e_k e_l = \delta_{kl} e_k$. This description is redundant, as unitary transformations can permute projectors if they have the same trace. However, as in Matrix Theory, this is the gauge transformation of particle statistics. We will adopt the conventional treatment of this symmetry in first quantized theories: we work in a large Hilbert space on which this symmetry acts, and impose invariance under it as a condition on states. Let $p_k = \text{Tr} e_k$. In $R_{[0,1]}$ a projector is characterized up to inner automorphism by its trace, so

$$S(e_k \otimes f_k) = S(p_k, f_k).$$

Linearity of $S_a$ implies that the algebra of operators $S_a(p_k, \Omega_k)$ combined with their images under outer automorphisms, generate the entire operator algebra on the Hilbert
space. We could choose the $S_a(p_k, \Omega_k)$ to anti-commute for different values of $k$, but it is more convenient to do a Klein transformation so that they commute.

We conclude that the Hilbert space of our system is a direct sum of $K$ particle sectors, where $K$ is any positive integer. $K$ corresponds to the number of independent projectors in the tensor decomposition of an element $A \in \mathcal{A}$. The $K$ particle sector is the symmetrized tensor product of single particle sectors. Each single particle sector is characterized by a positive real number $p_k$. For each section of the spinor bundle on the sphere, $f^a(\Omega)$, we have an operator $S[f] = \sqrt{p_k} S_a f^a(\Omega)$, where $[S_a, S_b]_+ = \delta_{ab}$.

From a physicist’s point of view, the operators $S[f]$ are a complete set of operators in the single particle Hilbert space. That is, we will momentarily define operations on the $S[f]$ which change the values of $p_k$ and $\Omega$, by rotations and Lorentz boosts. The infinitesimal generators will act on $S[f]$ as linear differential operators and we will be able to write these operations as the result of commutation with bilinears in the $S[f]$. From the mathematical point of view, in which we think of the $S(f)$ as linear maps from the spinor bundle over $A$ to the quantum operator algebra, the full operator algebra is generated by composing these linear operators with outer automorphisms of $A$. The whole system is viewed as arising as a limit of a similar construction for finite dimensional algebras $A_N$, corresponding to finite causal diamonds. In the finite dimensional case, all the automorphisms will be inner, and the $S_a(n)$ really generate the operator algebra even in the strict mathematical sense.

The variables of our system are thus concisely characterized as operators $S(\psi)$ where $\psi$ is a section of the spinor bundle over the algebra $R_{0,1} \otimes L_1(S^9)$, and $S(\psi)$ is invariant under inner automorphism of the algebra. These operators have commutation relations

$$[S(\psi), S(\phi)]_+ = (\psi, \phi),$$

(3.9)

where the scalar product in the spinor bundle includes the scalar product on the algebra, defined by its trace. The map from the spinor bundle to operators is linear. To be more explicit, a general element of the spinor bundle is $\psi = \sum A_i \psi_i(\Omega)$, where $A_i$ is an element of $R_{0,1}$ and $\psi(\Omega)$ a section of the ordinary spinor bundle over the 9-sphere. $S(\psi)$ is defined to be invariant under the inner automorphisms $A_i \rightarrow U_i^\dagger A_i U_i$, where the $U_i$ are unitary elements of $R_{0,1}$, as well as under Hermitian conjugation in $R_{0,1}$.

The scalar product is defined as

$$(\psi, \phi) = \sum \text{Tr}(A_i B_i) \int d^9 \Omega \, \psi^a_i(\Omega) \phi^a_i(\Omega).$$

(3.10)

We have shown that the irreducible representation of this operator algebra is precisely the Fock space of eleven dimensional supergravitons.
I would like to emphasize the insight that the above construction provides, regarding the operator algebra $S_a(n)$ of a finite causal diamond. I have already emphasized that for some choice of the basis of labels $n$, we should think of the operator $S_a(n)$ as representing the information stored in a pixel of the holographic screen. Our current discussion emphasizes that this “local” operator algebra is the SUSY algebra of a massless superparticle. That is, the information content in a pixel can be encoded in the spin states of a massless supermultiplet. This is a much clearer statement of the relation between SUSY and holography than the one I presented in [8].

In eleven dimensions, SUSy kinematics forces us to consider a theory of gravitation. In lower dimensions, we can have massless supermultiplets which do not include the graviton, and the necessity for gravitation in this holographic theory of space-time, may become evident only at the dynamical level. The consistency conditions on the dynamics of overlapping causal diamonds[1] are discrete analogs of general coordinate invariance. At the moment, there is no direct proof that this requires us to use representations of the lower dimensional SUSY algebras with spin two, but if the formalism does have a correspondence limit with low energy effective field theory then this must be the case.

We now want to describe how the super Poincare algebra of 11D SUGRA acts on the operator algebra. It is sufficient to define the action on the single particle operators $S(e_i \otimes f)$ and then extend it to the full operator algebra by linearity of $S(\psi)$. A general Lorentz transformation is the product of an $SO(10)$ rotation and a boost, $B(\Omega', \zeta)$ along some direction, $\Omega'$, with rapidity $\zeta$. The combined action is a general conformal transformation of the nine sphere. In particular, the conformal transformation corresponding to the boost is

$$\Omega \rightarrow \frac{(\Omega \cdot \Omega')\Omega' e^\zeta + (\Omega - (\Omega' \cdot \Omega)\Omega')}{\sqrt{e^{2\zeta} - 1}[\Omega' \cdot \Omega']^2 + 1}.$$  \hfill (3.11)

For any pair of directions $\Omega$ and $\Omega'$, let $K(\Omega', \Omega)$ be the counterclockwise rotation in the $\Omega, \Omega'$ plane, which takes $\Omega$ into $\Omega'$. If $\Lambda$ is a general element of the conformal group $SO(1,10)$, then

$$L(\Lambda, \Omega) \equiv K^{-1}(\Lambda(\Omega), \Omega)\Lambda,$$  \hfill (3.12)

is in the little group of the point $\Omega$. Thus it is the product of a boost

$$B(\Omega, \zeta(C, \Omega))$$

and an $SO(9)$ rotation, $R(C, \Omega)$ in the 9 plane perpendicular to $\Omega$ in $R^{10}$. We define the action of the Lorentz group of the spinor bundle over $S^9$ as follows. First we relate
the bases of the spinor spaces at two points by parallel transport via the rotation $K$

$$K(\Omega', \Omega)f_a(\Omega) = f_a(\Omega'),$$  \hspace{1cm} (3.13)

for any section $f_a$ of the bundle. Now for a general conformal transformation $\Lambda$ in $SO(1,10)$ we write

$$\Lambda = K(\Omega', \Omega)L(\Lambda, \Omega),$$  \hspace{1cm} (3.14)

so that the action of $\Lambda$ on the spinor bundle is induced by the action of the little group of a point. The latter is defined by

$$L(\Lambda, \Omega)f_a(\Omega) = e^{\frac{i}{2}\zeta(\Lambda, \Omega)}D_{ab}[R(\Lambda, \Omega)]f_b(\Omega),$$  \hspace{1cm} (3.15)

where $D_{ab}[R]$ is the usual sixteen dimensional spinor representation of $SO(9)$. With this transformation law, the Conformal Killing Spinor Equation,

$$D_{ab}m^q \alpha^b(\Omega) \equiv (\partial_m \delta_{ab} - \omega_{m}^{ab}\gamma_{ab})q^a(\Omega) = \frac{1}{9}e_{m}^A(\gamma A)^{ab}\mathcal{D}^{bc}q^a,$$  \hspace{1cm} (3.16)

is Lorentz covariant. $\mathcal{D}$ is the Dirac operator on $S^9$. $\alpha$ labels the 32 linearly independent solutions of this equation, which transform as a spinor under $SO(1,10)$.

We define the SUSY generators by

$$Q^\alpha \equiv S[q] \equiv \sum \int d^9 \Omega_i S_a^{(i)}(\Omega(\Omega_i)) q^a(\Omega(\Omega_i)),$$  \hspace{1cm} (3.17)

where the sum has $K$ terms in the $K$ supergraviton sector. These operators satisfy

$$[\bar{Q}^\alpha, Q^\beta]_+ = \sum p_i \bar{q}_i^0(\Omega(\Omega_i)) q^\beta(\Omega(\Omega_i)).$$  \hspace{1cm} (3.18)

The $32 \times 32$ matrix $\bar{q}_i^0(\Omega(\Omega_i)) q^\beta(\Omega(\Omega_i))$, on the right hand side of this equation can be expanded in anti-symmetric products of the 11 dimensional Dirac matrices $\Gamma^{\mu_1\ldots\mu_n}$, where $n = 0, 1, 2$ or 5. The transformation properties of sections of the spinor bundle under $SO(9)$ rotations perpendicular to $\Omega_i$ show that only the matrices $1, (\Omega(\Omega_i))_1 \Gamma^1$ and $(\Omega(\Omega_i))_1 \Gamma^6$ are allowed. The transformation under boosts in the $\Omega_i$ direction allows only the linear combination $1 + (\Omega(\Omega_i))_1 \Gamma^1$. Thus, the right hand side is just the momentum operator $P_\mu$ for positive energy incoming particles, dotted into $\Gamma^\mu$. Our Hilbert space carries a representation of the 11 dimensional super-translation algebra.

We now define the action of the Lorentz group on the operator algebra by

$$U(\Lambda)^\dagger S(e_i \otimes f)U(\Lambda) = S(\rho(\Lambda, \Omega)[e_i] \otimes f^{(\Lambda)}).$$  \hspace{1cm} (3.19)

Here $f^{(\Lambda)}$ is the transformed element of the spinor bundle over the sphere, which we defined above. $\rho(\Lambda, \Omega)$ is an element of the automorphism group of $R[0,1]$. Recall that $\lambda_p$ is defined by

$$\text{Tr} \rho[a] = e^{\lambda_p} \text{Tr} a,$$  \hspace{1cm} (3.20)
for every element of the algebra. Let $J(\Lambda, \Omega)$ be the Jacobian of the conformal
transformation $\Lambda$ on the nine sphere. If we choose
\[ e^{\lambda \rho(\Lambda, \Omega))} = J^{-1}e^{-\zeta(\Lambda, \Omega)}, \quad (3.21) \]
(recall that $\zeta$ is the rapidity of the boost in the little group) then the anti-commutation
relations
\[ [S(\psi), S(\phi)]_+ = (\psi, \phi), \quad (3.22) \]
are invariant under the action of the Lorentz group, as they must be if this action is
implemented by a unitary transformation.

Conversely, because our Hilbert space is defined as the irreducible representation of
the anti-commutation relations, there is a unitary action, unique up to a multiple of the
identity, which implements the Lorentz group. Thus we have described the full action
of the super-Poincare algebra on our Hilbert space. This is not a big surprise, since we
have already identified this space as the Fock space of supergravitons. Nonetheless it
is interesting to see the role of the automorphism group of the algebra in the explicit
construction. I believe that these formulae for the action of the super-Poincare algebra
will be useful in the attempt to construct the S-matrix. I now turn to a brief discussion
of that, as yet unrealized, program.

3.2 Dynamics

The prehistory of string theory was the search for an alternative method for construct-
ing a scattering matrix consistent with unitarity, Lorentz invariance, and causality.
Field theory gave such a construction, but left an enormous amount of ambiguity. To-
day we recognize that ambiguity as the existence of many possible fixed points of the
renormalization group. That is to say, the ambiguity is connected to the high energy
behavior of the theory.

This is very explicit in the S-matrix theorist’s derivation of unitarity. One starts
with an assumed spectrum of particles and makes an asymptotic expansion of the
S-matrix
\[ S = 1 + i \sum_{n=1}^{\infty} T_n. \quad (3.23) \]
The unitarity condition becomes
\[ T_n - T_n^\dagger = \sum_{k=1}^{n-1} T_k T_{n-k}. \quad (3.24) \]
Assuming appropriate analyticity conditions (presumed to follow from causality) for
S-matrix elements, one then claims that this relation determines the amplitudes in
terms of $T_1$. It explicitly computes discontinuities across cuts in terms of low order amplitudes. Dispersion relations (Cauchy’s theorem) should then enable us to compute the full amplitude. $T_1$ itself is Hermitian and, if there are a finite number of particles, analyticity shows that it can be written in terms of an integral over a local, Lorentz scalar, Lagrangian. The problem is that the Cauchy integrals do not converge until we take some momentum derivatives, which leads to polynomial ambiguities in higher order amplitudes. These are what field theorists call renormalization counterterms, and the procedure suffers precisely the ambiguities of the local field theory with the same tree level Lagrangian.

String theory was born as an attempt to find a different and more unique solution by positing an infinite number of stable particles (at zeroth order) so that $T_1$ (the Veneziano-Virasoro-Shapiro- Koba-Nielsen amplitudes) could have “better” high energy behavior, corresponding to a series of Regge poles. We all know the story of how this inadvertently led to the construction of a theory of quantum gravity, and most of us also know that tree level string theory does not give a correct description of the high energy behavior of quantum gravity amplitudes. Rather, it is believed that the generic regime of large kinematic invariants is dominated by the production and decay of black holes[9].

A possible route to the construction of the scattering matrix for 11D SUGRA then, would be to follow the old S-matrix program, using the added input of supersymmetry and insights about black hole dominance of high energy interactions. A starting point might be the ideas of ’t Hooft[10]. More particularly, consider an $N$ particle scattering amplitude as a function of the following three variables: the center of mass energy $s$, the subenergy $s_{n-1}$ of a cluster of $n-1$ of the particles, and the impact parameter $b$ between the $n$th particle and the cluster. In the limit $s \sim s_{n-1} \gg M_P$ and $b \gg R_S(s_{n-1})$, (the Schwarzschild radius of the cluster), the following approximation to the $n$ particle S-matrix suggests itself: $S_n = S_{n-1} e^{i\delta}$. The first factor is the exact $n-1$ particle S-matrix, while $e^{i\delta}$ is the scattering amplitude of a single particle in the classical field of a black hole of mass $s_{n-1}$ (in the Lorentz frame where the total spatial momentum of the cluster vanishes). Perhaps this information about high energy behavior will help to determine the amplitudes.\footnote{We know other things about inclusive cross sections for black hole production and decay in certain kinematical regimes, which might be useful.}

The form of the SUSY generators in our formalism, suggests an entirely different approach to finding the scattering matrix. Note that, although our formula has non-vanishing values for all components of these generators, the contribution of any single particle state has only 16 non-vanishing components. This is the familiar fact that the
supergraviton representation of 11D SUSY is BPS. Note further that if we write the algebra corresponding to outgoing rather than incoming particles, then we find “the other half” of the components of each particle’s supergenerators.

This is reminiscent of Matrix Theory, in light cone frame, where half of the SUSY generators are kinematical. The other half have a non-trivial, but fairly simple, construction in terms of kinematical particle variables, which encodes the entire dynamics of the theory. By analogy one should seek a formula which expresses the incoming components of the SUSY generators as simple functions of the outgoing particle variables $S(f)$. Since incoming and outgoing variables are related by the S-matrix, this formula would, be a constraint on the S-matrix; perhaps enough of a constraint to determine it.

Finally, recall that his general approach to holographic space-time presented in [1] contains a consistency condition which is very hard to satisfy. Namely, a quantum space-time is defined in terms of Hilbert spaces and time evolution operators for causal diamonds, plus a system of overlap conditions designed to guarantee that two observers who share a piece of their respective causal diamonds, describe that piece in a consistent manner. This condition is very hard to implement and so far the only successful example corresponds to a particular space-time, the FRW universe with $p = \rho$. We conjectured that this consistency condition would completely determine the dynamics of possible theories of quantum gravity.

In asymptotically flat space, one would like to translate the consistency conditions on local causal diamonds, into constraints on the S matrix. Recall that the S matrix is the common limit of an infinite set of sequences of local S-matrices for particular observers. So far, I have not been able to translate the dynamical compatibility conditions into equations for the S-matrix, but this should be possible.

It should be clear to the reader that these are just ideas for ideas. The problem of finding a non-perturbative formulation of string/M-theory for general asymptotically flat space-times is unsolved, and would appear to be the most important unsolved formal problem in the subject.

4. Massive particles and compactification

Two different issues arise when trying to generalize these considerations to situations with fewer non-compact dimensions and/or less SUSY. The first is the necessity of describing stable massive particles in a formalism based on null infinity. The second is the appearance of non-gravitational multiplets in systems with less than maximal SUSY.
For massive BPS states the problems of working at null infinity seem tractable. Indeed, our formalism is not really local at null infinity since we work at fixed longitudinal momentum. It is easy to describe massive BPS states in terms of an extended momentum space including the central charges. Already at the level of finite causal diamonds, we simply replace the pixel algebra by the SUSY algebra with appropriate central charges. However, we know of examples of stable massive states in string theory which are not BPS.\(^6\)

It should be noted that in general, the masses of even BPS particles is not determined by symmetries. The masses will appear in the pixel algebra, as part of the kinematics, and will have to be determined by the dynamical equations of the theory.

A prescription for incorporating K theory charges into the operator algebra of a holographic screen is also the key to understanding compactification in this formalism. The K theory charges of states are the only topological features of the internal manifold that are preserved in string theory, since ordinary topology is not invariant under duality transformations. Thus, except in certain limits, one should not be able to think of the compact dimensions of space-time in terms of ordinary geometrical notions. By contrast, our formalism implies a rather direct relation between geometrical notions in the non-compact dimensions and the structure of the quantum theory. The quantum formalism for finite causal diamonds has a causal structure, which determines that of the Lorentzian geometry that emerges in the large area limit. The conformal factor of that Lorentzian geometry is directly related to the size of Hilbert spaces. By contrast, duality invariant information about the geometry of the compact space is incorporated in the pixel algebra of the non-compact space.

As an aside, we should note that de Sitter space should be considered non-compact in the sense in which the phrase is used here. The global dS manifold can be foliated by compact spatial sections, but the holographic formalism describes only the static observer’s horizon volume. The observer’s cosmological horizon converges to null infinity in the large radius limit, and the interesting physics of the system is concerned with the way in which this limit is approached. The global manifold is really only a trick (the thermofield doubling trick) for discussing the thermal physics of a single horizon volume.

\(^6\)The simplest is the spinor in \(SO(32)\) heterotic string theory, and the K-theory classification of D branes gives rise to other examples. It is a reasonable hypothesis that all stable massive particles in asymptotically flat string theory, carry a K theory charge, which will be a torsion element for non-BPS particles.
5. Conclusions

I have introduced the von Neumann algebra $R_{0,1}$ in an attempt to construct a non-perturbative Lorentz invariant formulation of the quantum theory underlying 11 dimensional supergravity. The manifestly covariant kinematics on null infinity uses $R_{0,1}$ to incorporate the Matrix Theory description of multi-particle states. So far however it is only a kinematics.

One of the most significant results in this paper is the deeper understanding we have achieved of the relation between SUSY and holographic screens. The Cartan-Penrose equation gives us a way to relate the variables describing a pixel on a holographic screen to a pure spinor. The commutation relations of the pure spinor variables are identical to those of the reduced SUSY algebra for a massless superparticle with fixed momentum. In eleven dimensions this implies that the quantum states of a holographic pixel are precisely the spin states of the SUGRA multiplet.

I have attempted to make contact between a holographic formulation of quantum space-time, and extant descriptions of certain space-times in terms of string/M-theory. The holographic formulation is maximally local: its key ingredient is the operator algebra of a causal diamond, which should be thought of as a quantization of Cartan-Penrose variables belonging to the (dual space of the) spinor bundle of the diamond’s holographic screen. The geometrical information encoded in these variables are the orientation of a pixel on the screen, as well as its area. An observer following a time-like trajectory is modeled as a nested sequence of causal diamonds. A quantum space-time is a topological spatial lattice with such a quantum observer attached to each point, together with overlap/consistency conditions that enforce agreement between the joint observations of different observers. It is very hard to find solutions of these conditions, so perhaps they are the only dynamical information the formalism needs. Indeed, the single known solution[1] automatically describes the dynamical evolution of a flat $p = \rho$ FRW cosmology.

The local formulation is gauge dependent and is formulated in generic (unitary) gauge. That it must be so follows from the principle of general covariance, but also from the more profound holographic principle. Indeed the holographic principle (in the form advocated by Fischler and the present author) implies that a finite area causal diamond has a finite set of observables and therefore cannot make infinitely precise measurements of itself. This means that local physics is intrinsically ambiguous. The claim is that this ambiguity is the quantum origin of general coordinate invariance. Gauge invariant formulations of gravitational systems exist only when space-time has an infinite asymptotic boundary. In this paper I tried to show how the conventional S-matrix description of 11D asymptotically flat space-time could be obtained as a limit
of the local formulation of holographic space-time. The basic idea was to take the function algebras of a sequence of causal diamonds to be an increasing sequence of representations of the $SO(10)$ Clifford-Dirac algebra, converging to the algebra $R_{0,1} \otimes \mathcal{M}(S^9)$: the tensor product of the hyperfinite $II_\infty$ factor, and the algebra of measurable functions on the nine sphere. I showed that if the quantum algebra of observables was the space of linear functionals $S(a)$ on the spinor bundle of this algebra, invariant under inner automorphisms, then the obvious limit of the finite anti-commutation relations gave us precisely the Fock space of 11D SUGRA.

Although I did not go into detail, I also presented the basic idea for incorporating compactification in this framework. The nine sphere is replaced by a lower dimensional sphere, and the algebra incorporates charges encoding the quantum numbers of finite energy wrapped branes on the compact manifold. Questions remain about torsion elements in the K-theory classification of D-branes. In general, at the kinematic level, it seems that one will have to include the masses of particles as parameters, and hope that they will be determined by the dynamical equations.

These equations themselves remain a mystery. I presented several directions of research for determining them. Perhaps, in the future, someone will pay attention to these ideas and figure them out. I wish her well.

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