SOLVING FOR ROOT SUBGROUP COORDINATES: THE SU(2) CASE

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Abstract. In [3] and [5] we showed that a loop in a simply connected compact Lie group \( \hat{K} \) has a unique triangular factorization if and only if the loop has a unique root subgroup factorization (relative to a choice of a reduced sequence of simple reflections in the affine Weyl group). In this paper we show that in the \( \hat{K} = SU(2) \) case, root subgroup coordinates are rational functions (with positive denominators) of the triangular factorization coordinates. We conjecture that in general they are algebraic functions.

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0. Introduction

In this paper, unless stated otherwise, we suppose that \( \hat{K} := SU(2) \) and \( \hat{G} := SL(2, \mathbb{C}) \).

A triangular factorization for \( g \in L\hat{K} \) is a factorization of the form

\[
    g(z) = l(z) \cdot m \cdot a \cdot u(z),
\]

where

\[
    l(z) = \begin{pmatrix} l_{11}(z) & l_{12}(z) \\ l_{21}(z) & l_{22}(z) \end{pmatrix} \in H^0(\Delta^*, G), \quad l(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\( l \) has appropriate boundary values on \( S^1 \) (depending on the smoothness properties of \( g \)), \( m = \begin{pmatrix} m_0 & 0 \\ 0 & m_0^{-1} \end{pmatrix} \), \( m_0 \in S^1 \), \( a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \), \( a_0 > 0 \),

\[
    u(z) = \begin{pmatrix} u_{11}(z) & u_{12}(z) \\ u_{21}(z) & u_{22}(z) \end{pmatrix} \in H^0(\Delta, G), \quad u(0) = \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix},
\]

and \( u \) has appropriate boundary values on \( S^1 \), where \( \Delta \ (\Delta^*) \) is the open unit disk centered at \( z = 0 \) (\( z = \infty \), respectively), and \( H^0(U) \) denotes holomorphic functions in a domain \( U \subset \mathbb{C} \).

The basic fact is that for \( g \in L\hat{K} \) having a triangular factorization, there is a second unique root subgroup factorization

\[
    g(z) = k_1(\eta)^*(z) \begin{pmatrix} e^{\chi(z)} & 0 \\ 0 & e^{-\chi(z)} \end{pmatrix} k_2(\zeta)(z), \quad |z| = 1,
\]

where

\[
    k_1(\eta)(z) = \lim_{n \to \infty} a(\eta_n) \begin{pmatrix} 1 & -\eta_n z^n \\ \eta_n z^{-n} & 1 \end{pmatrix} a(\eta_0) \begin{pmatrix} 1 & 0 \\ \eta_0 & 1 \end{pmatrix},
\]
\[ \chi(z) = \sum \chi_j z^j \] is a \( i\mathbb{R} \)-valued Fourier series (modulo \( 2\pi i\mathbb{Z} \)),

\[ k_2(\zeta)(z) = \lim_{n \to \infty} a(\zeta_n) \left( \frac{1}{-\zeta_n z^n} \frac{\zeta_n z^{-n}}{1} \right) a(\zeta_1) \left( \frac{1}{-\zeta_1 z} \frac{\zeta_1 z^{-1}}{1} \right), \]

\[ a(\cdot) = (1 + |\cdot|^2)^{-1/2}, \] and it is understood that if \( g \in C^\infty(S^1, K) \), then the coefficients are rapidly decreasing, and similarly for other function spaces; conversely a root subgroup factorization as in (0.2) implies that \( g \) has a triangular factorization (3).

It is a relatively simple matter to pass from a root subgroup factorization to a triangular factorization; we will recall how this is done in section 1. The main point of this paper is to explain how to directly find the root subgroup factors \( \eta \) and \( \zeta \) in terms of the triangular factorization. The proof in [3] for the existence of these factors uses (in part) an inverse function theoretic argument. The basic observation is that the Taylor series centered at \( z = 0 \) for the meromorphic functions \( l_{21}(z)/l_{11}(z) \) and \( u_{21}(z)/u_{22}(z) \) have coefficients which are related in a triangular way with the variables \( \eta \) and \( \zeta \), respectively. From this we can see that \( \eta(\zeta) \) is a rational function of the coefficients of \( l(u) \) and their conjugates, with positive denominators. Unfortunately we have not yet found closed form expressions for \( \eta \) and \( \zeta \).

This reveals a surprising fact about the variables \( \eta \) and \( \zeta \): their natural domain is the set of all loops \( g \) in the formal completion of the complex loop group \( \hat{L}G \) having a triangular factorization. For example this immediately implies that (the components of) \( \eta \) and \( \zeta \) are well-defined random variables with respect to the invariant measures considered in [2].

In finite dimensions there is a Gaussian elimination algorithm for finding the LDU decomposition for a matrix. There does not exist an algorithm for finding the analogous triangular decomposition for a matrix valued loop - one has to invert a Toeplitz operator. What we are showing is that triangular factorization and root subgroup factorization are at roughly the same level of complexity.

0.1. Higher Rank and Generalizations. Suppose that \( \hat{K} \) is an arbitrary simply connected compact Lie group with simple Lie algebra \( \hat{\mathfrak{k}} \). Fix a triangular decomposition of the complexified Lie algebra

\[ \hat{\mathfrak{g}} := \hat{\mathfrak{k}}^c = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \]

which is compatible with \( \hat{\mathfrak{k}} \) in the sense that \( \mathfrak{t} \cap \mathfrak{h} \) is maximal abelian. In this case there is again a triangular factorization for \( g : S^1 \to K \), \( g(z) = l(z)mau(z) \), if and only if there is a root subgroup factorization, \( g(z) = k_1(\eta)^*(z)exp(\chi(z))k_2(\zeta)(z) \), where in the case \( \text{rank}(\mathfrak{k}) > 1 \), the details of the root subgroup factorization depend additionally on choices of reduced factorization of the longest element \( w_0 \) of the Weyl group, and a reduced sequence of simple reflections for the affine Weyl group; see [5].

Based on experiments with SU(3) and SU(4), it appears that in this more general context, the components of \( \eta \) and \( \zeta \) are algebraic functions of the triangular factorization coordinates. But solving for the components is far less straightforward.

The result in this paper does resurrect the hope that there might be a way to similarly solve for the root subgroup coordinates for homeomorphisms of a circle; see [4] (this is what has long motivated me to search for formulas similar to those in this paper). The formulas that we find also apply for root subgroup factorization
for loops in $G_0 = SU(1, 1)$, as in [1]. It is possible that these formulas might help clarify some of the complications for root subgroup factorization that arise in that noncompact context.

0.2. Notation. For a function $f : U \subset \bar{\Sigma} \rightarrow \mathcal{L}(\mathbb{C}^N)$, define $f^*(q) = f(R(q))^*$, where $(\cdot)^*$ is the Hermitian adjoint. If $f \in H^0(\Sigma)$ (i.e. a holomorphic function in some open neighborhood of $\Sigma$), then $f^* \in H^0(\Sigma^*)$. If $q \in S$, then $f^*(q) = f(q)^*$, the ordinary complex conjugate of $f(q)$.

1. FROM ROOT SUBGROUP FACTORIZATION TO A TRIANGULAR FACTORIZATION: THE SU(2) CASE

Suppose that $g : S^1 \rightarrow SU(2)$ has a root subgroup factorization as in (1.2). Recall from [3] that $k_1 = k_1(\eta)$ and $k_2 = k_2(\zeta)$ have triangular factorizations of the following special forms:

\[
(1.1) \quad k_1 = \begin{pmatrix} 1 & 0 \\ y^* & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix},
\]

and

\[
(1.2) \quad k_2 = \begin{pmatrix} 1 & x^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},
\]

where for example $y = \sum_{n \geq 0} y_n z^n$ and $x = \sum_{n \geq 1} x_n z^n$ are holomorphic functions in $\Delta$, with appropriate boundary behavior, depending the smoothness of $g$.

As in [3], given these triangular factorizations for $k_1$ and $k_2$, we can derive the triangular factorization for $g$ as follows:

\[
g = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}^* \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 a_2 e^{x_- + x_0 + x_+} & 0 \\ 0 & (a_1 a_2 e^{x_- + x_0 + x_+})^{-1} \end{pmatrix} \begin{pmatrix} 1 & X^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}
\]

\[
= \begin{pmatrix} \alpha_1^* & \gamma_1^* \\ \beta_1^* & \delta_1^* \end{pmatrix} \begin{pmatrix} e^{x_-} & 0 \\ 0 & e^{-x_-} \end{pmatrix} \begin{pmatrix} 1 & e^{-2x_-} \cdot Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 a_2 e^{x_0} & 0 \\ 0 & (a_1 a_2 e^{x_0})^{-1} \end{pmatrix} \begin{pmatrix} 1 & e^{2x_+} \cdot X^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{x_+} & 0 \\ 0 & e^{-x_+} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}
\]

where $Y = a_1^2 y$ and $X = a_2^2 x$.

The product of the middle three factors is upper triangular, and it is easy to find its triangular factorization. Thus $g = l(g)m(g)a(g)u(g)$, where

\[
l(g) = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1^* & \gamma_1^* \\ \beta_1^* & \delta_1^* \end{pmatrix} \begin{pmatrix} e^{x_-} & 0 \\ 0 & e^{x_+} \end{pmatrix} \begin{pmatrix} 1 & e^{-2x_-} \cdot Y + (a_1 a_2)^2 e^{2(x_0 + x_+)} \cdot X^* \\ 0 & 1 \end{pmatrix}
\]

\[
m(g) = \begin{pmatrix} e^{x_0} & 0 \\ 0 & e^{-x_0} \end{pmatrix}, \quad a(g) = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & (a_1 a_2)^{-1} \end{pmatrix}
\]

\[
u(g) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 1 & ((a_1 a_2)^{-2} e^{-2(x_- + x_0)} Y + e^{2x_+} \cdot X^*)^+ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{x_+} & 0 \\ 0 & e^{-x_+} \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}
\]

In particular

\[
\begin{pmatrix} l_{11} \\ l_{21} \end{pmatrix} = e^{x_-} \begin{pmatrix} \alpha_1^* \\ \beta_1^* \end{pmatrix}, \quad (a, b) = a_1(\alpha_1, \beta_1)
\]
\[ (u_{21}, u_{22}) = e^{-x} (\gamma_2, \delta_2) \] and \((c, d) = a_2^{-1}(\gamma_2, \delta_2)\) 

Therefore 

\[ \mathbb{P}(T_{\diamond 1}^*) = \mathbb{P}(a_1) = \mathbb{P}(a) \] 

and 

\[ \mathbb{P}(u_{21}, u_{22}) = \mathbb{P}(\gamma_2, \delta_2) = \mathbb{P}(c, d) \]

2. FINDING THE ROOT SUBGROUP FACTORS: THE SU(2) CASE

Suppose that \( g : S^1 \to SU(2) \) (in what follows we will suppress mention of the necessary degree of smoothness of \( g \); but everything works if \( g \in W^{1/2, L^2}(S^1, SU(2)) \)).

If \( A(g) \) is invertible, then \( g \) has a unique Birkhoff (or Riemann-Hilbert) factorization factorization 

\[ g = g_0 g^+ \]

\[ g \in H^0(\Delta^*, G), \ g_0 \in G, \ \text{and} \ g^+ \in H^0(\Delta, G), \] 

where 

\[ (g_0 g^+)^{-1} = [A(g)]^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

where \( A(g) \) is the (block) Toeplitz operator corresponding to \( g \) (see Section 1 of [4]).

If \( A(g) \) is invertible, and \( g_0 \) has a triangular factorization in the finite dimensional sense, then \( g \) has a triangular factorization \( g = lmau \). In this case we know that there also exists a root subgroup factorization 

\[ g = k_1(\eta)^* \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix} k_2(\zeta) \]

The key to solving for the \( \eta, \chi, \) and \( \zeta \) factors is the following

**Theorem 2.1.** (a) For \( k_1(\eta) \), 

\[ b/a = \beta_1/\alpha_1 = l_{21}^*/l_{11}^* \]

This meromorphic function in \( \Delta \) has Taylor series \( \sum_{n=0}^{\infty} \psi_n z^n \) where \( \psi_n \) is the sum of terms

\[ \psi_n = (-1)^s (\eta_{i_1} (\eta_{j_1}, -\eta_{i_1})) \cdots \eta_{i_s} (-\eta_{i_s}) \]

where \( j_s < i_s \) and \( j_s \leq i_{s-1} \) for \( s = 1, \ldots, r \), and \( \sum_{s=1}^{r+1} i_s - \sum_{s=1}^{r} j_s = n; \) in particular

\[ \xi_n = (-\eta_{i_0}) \prod_{s=1}^{n-1} (1 + |\eta_s|^2) + \text{polynomial}(\eta_s, \eta_{i_s}, s < n) \]

(b) For \( k_2(\zeta) \), 

\[ c/d = \gamma_2/\delta_2 = u_{21}/u_{22} = (g_+)^{21}/(g_+)^{22} \]

This meromorphic function in \( \Delta \) has Taylor series \( \sum_{n=1}^{\infty} \xi_n z^n \) where \( \xi_n \) is the sum of terms

\[ (-1)^s (\xi_{i_1} (\xi_{j_1}, -\xi_{i_1})) \cdots \xi_{i_s} (-\xi_{i_s}) \]

where \( j_s < i_s \) and \( j_s \leq i_{s-1} \) for \( s = 1, \ldots, r \), and \( \sum_{s=1}^{r+1} i_s - \sum_{s=1}^{r} j_s = n; \) in particular

\[ \xi_n = (-\xi_{i_0}) \prod_{s=1}^{n-1} (1 + |\xi_s|^2) + \text{polynomial}(\xi_s, \xi_{i_s}, s < n) \]
For example
\[ b/a = (-\eta_0) + (-\eta_1)(1 + |\eta_0|^2)z + ((-\eta_2)(1 + |\eta_0|^2)(1 + |\eta_1|^2) + ...) z^2 + ... \]
and
\[ c/d = (-\zeta_1)z + (-\zeta_2)(1 + |\zeta_1|^2)z^2 + \left((-\zeta_3)(1 + |\zeta_1|^2)(1 + |\zeta_2|^2) + (-\zeta_1^2 \zeta_2)(1 + |\zeta_1|^2)\right) z^3 \]

\[ + ((-\zeta_4)(1 + |\zeta_1|^2)(1 + |\zeta_2|^2)(1 + |\zeta_3|^2) + (1 + |\zeta_1|^2)(\zeta_2 \zeta_3(1 + |\zeta_2|^2) \right) z^4 + ... \]

For this latter sum, if \( n = 2 \), the terms in \( \xi_2 \) are \(-\zeta_2 \) and \( (1)(-\zeta_1)\xi_1(-\zeta_2) \).

**Remarks.**

(a) Note that we can solve for the \( \zeta \) variables using the coefficients of \( (g_+)_1/(g_+)_2 \). We only need the Riemann-Hilbert factorization, not the full triangular factorization, in order to find \( \zeta \).

(b) The finiteness of the formulas in the theorem contrasts sharply with the infinite formulas for the coefficients of the terms \( \gamma_2, \delta_2 \), and so on; see Lemma 1 below.

(c) It would be nice to find a quicker route to finding \( \chi \). It is essentially determined by either of the formulas

\[ Re(\chi_+) = -\frac{1}{2} \log|l_1|^2 + |l_2|^2) \]

where \( a_1 \) (\( a_2 \)) is determined by \( \eta \) (\( \zeta \), respectively); see 3. Note that for the equality of these two formulas, it is essential that the loop is unitary. For nonunitary loops these formulas diverge, and it is in this sense that we said in the introduction, whereas \( \eta \) and \( \zeta \) extend naturally to the top stratum of the formal completion of the complex loop group, \( \chi \) does not have a preferred extension even to ordinary complex loops; \( \chi \) is much more complicated from an analytic point of view.

**Proof.** We will prove part (b); part (a) is proven in the same way.

We first consider the equality of the meromorphic functions, \( b/a, \gamma_2/\delta_2, u_{21}/u_{22}, \) and \( (g_+)_1/(g_+)_2 \). The first two equalities follow from 1. As we noted before the statement of the theorem, a triangular factorization implies a Riemann-Hilbert factorization: If \( g = \text{Im} au \), then \( g = g_+ g_+^{-1} \), where

\[ g_+ = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \]

\[ g_0 = \begin{pmatrix} 1 & 0 \\ l_{21}(\infty) & 1 \end{pmatrix} \begin{pmatrix} a_1 a_2 e^{\chi_0} & 0 \\ 0 & (a_1 a_2 e^{\chi_0})^{-1} \end{pmatrix} \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix} \]

In particular \( (g_+)_1 = u_{21} \) and \( (g_+)_2 = u_{22} \). This obviously implies the third equality.

We now turn to proving the formula for \( \xi_n \). Using \( \xi = \gamma_2/\delta_2 \), we need to show that for the product

\[ (2.2) \begin{pmatrix} \gamma_2(z) & -\gamma_2(z) \\ \gamma_2(z) & \delta_2(z) \end{pmatrix} = \lim_{n \to \infty} a(\zeta_n) \begin{pmatrix} 1 & \zeta_n z^{-n} \\ -\zeta_n z^n & 1 \end{pmatrix} \cdot a(\zeta) \begin{pmatrix} 1 & \zeta_1 z^{-1} \\ -\zeta_1 z^n & 1 \end{pmatrix}, \]
\[ \gamma_2/\delta_2 \] has the claimed form. Our strategy is completely straightforward: we will first recall the formulas for the coefficients of \( \gamma_2 \) and \( \delta_2 \), and we will then calculate the Taylor series for the quotient.

The following is from [3], which is reminiscent of the Pauli exclusion principle:

**Lemma 1.** For the product

\[
(2.3) = (\gamma_2(z) - \gamma_2(z)) = \lim_{n \to \infty} a(\zeta_n) \left( \frac{1}{-\zeta_n z^n} \right) \cdot a(\zeta_1) \left( \frac{1}{-\zeta_1 z} \right),
\]

where

\[
\gamma_2(z) = \sum_{n=1}^{\infty} \gamma_{2,n} z^n,
\]

\[
\gamma_{2,n} = \sum \left( -\zeta_{i_1} \zeta_{j_1} \cdots -\zeta_{i_r} \zeta_{j_r} \right),
\]

the sum over multiindices satisfying

\[
0 < i_1 < j_1 < \ldots < j_r < i_{r+1}, \quad \sum i_* - \sum j_* = n,
\]

and

\[
\delta_2(z) = 1 + \sum_{n=1}^{\infty} \delta_{2,n} z^n,
\]

\[
\delta_{2,n} = \sum \zeta_{j_1} (-\zeta_{i_1}) \cdots \zeta_{j_r} (-\zeta_{i_r}),
\]

the sum over multiindices satisfying

\[
0 < j_1 < i_1 < \ldots < i_r, \quad \sum (i_* - j_* ) = n
\]

To simplify notation, let \( \gamma := \gamma_2 \), and write \( \delta := \delta_2 + \delta^2 - \ldots \), and the nth coefficient of \( \gamma_2/\delta_2 \) equals

\[
\gamma_n - (\gamma \delta)_n + (\gamma \delta^2)_n - \ldots + (-1)^{n-1} (\gamma \delta^{n-1})_n
\]

Each of the terms in this sum, according to the Lemma, has an expression as an infinite sum. According to the statement of the theorem, all but finitely many of these terms cancel out.

To explain in a leisurely way how this comes about, first consider

\[
(2.4) = (\gamma \delta)_n = \sum_{k=1}^{n-1} \gamma_k \delta_{n-k}
\]

This is a sum of terms of the form

\[
\left( -\zeta_{i_1} \zeta_{j_1} \cdots -\zeta_{i_r} \zeta_{j_r} \zeta_{i_{r+1}} \right) \left( \zeta_{j'_1} (-\zeta_{i'_1}) \cdots \zeta_{j'_{r'}} (-\zeta_{i'_{r'}}) \right)
\]

where

\[
0 < i_1 < j_1 < \ldots < j_r < i_{r+1}, \quad \sum i_* - \sum j_* = k, \quad 0 < j'_1 < i'_1 < \ldots < i'_{r'}, \quad \sum i'_* - \sum j'_* = n-k
\]

If \( i_{r+1} < j'_1 \), then this product will exactly cancel with a term in the corresponding sum for \( \gamma_n \); it is in some sense obeying a Pauli exclusion principle. The only term in the sum for \( \gamma_n \) that is not canceled is \( -\zeta_{i_1} \); this is the only term which cannot be broken into two terms as in the sum for \( \gamma \delta \). If \( j'_1 \leq i_{r+1} \), then we keep this term; however, we will see than many of these terms are canceled by subsequent terms appearing in the sum \( (2.4) \).
Consider \((\gamma \delta^s)_n\). This is a sum of terms of the form
\[(2.5) \left( \prod_{i=1}^{s} (\zeta_{i+1} - \zeta_{i+1}) \right) \times \left( \prod_{i=1}^{s} (\zeta_{i+1} - \zeta_{i+1}) \right) \times \ldots \times \left( \prod_{i=1}^{s} (\zeta_{i+1} - \zeta_{i+1}) \right)
\]
where each of the factors separated by \(\times\) (the first factor comes from \(\gamma\), and the other \(s\) factors come from \(\delta^s\)) satisfy the appropriate constraints in the Lemma (in particular the sum of the \(i\) indices minus the sum of the \(j\) indices equals \(n\)).

At one extreme, it may happen that all of the indices in (2.5) are increasing, i.e. \(i_{r+1} < j_1^1\) and \(i_{r+1}^{s'} < j_1^{s'+1}\) for \(s' = 1, \ldots, s - 1\). By removing some of the \(\times\), we see that this product will have occurred in all of the preceding terms \((\gamma \delta^s)_n\), \(s' = 0, \ldots, s - 1\) (which occur with alternating signs). Similarly if \(r > 0\) or \(r^j > 1\) for some \(j\), then we can insert \(\times\) and this more finely factored product will occur in some subsequent terms \((\gamma \delta^s)_n\), for \(s < s'\); the largest such \(s'\) is \(S = r + r^1 + \ldots + r^s\), in which for each factor of \(\delta\), the corresponding factor
\[
\zeta_{j_1} (\zeta_{i_1}) \ldots \zeta_{j_r} (\zeta_{i_r})
\]
is irreducible in the sense that it cannot be split into a product of two similar factors, i.e. \(r' = 1\). The number of times the product (2.5) occurs in one of the terms in (2.4) depends on the number of ways we can insert \(\times\). Taking into account the signs that occur in (2.4), the coefficient of this product in (2.4) is
\[
\sum_{j=0}^{S} (-1)^j \binom{S}{j} = 0
\]
Thus this product completely cancels out in the sum (2.4).

At the opposite extreme, the term (2.4) may have the same form as in the statement of the theorem, i.e. \(r = 0\), and \(r^j = 1\), \(j = 1, \ldots, s\). In this case this term occurs in exactly one of the terms \((\gamma \delta^s)_n\). There is no cancelation.

In between these two extremes, we are considering a product for which, in the expression (2.5), there is a positive number of instances when \(j_1^{s''} \leq i_{r^{s''-1}}^{s''-1}\) for some \(s'' = 1, \ldots, s - 1\). As in the first extreme case, we can possibly remove some of the \(\times\) to see that this term occurs in earlier terms \((\gamma \delta^s)_n\) \((s' < s)\), and we can possibly insert some \(\times\) to see that it occurs in some later terms \((\gamma \delta^s)_n\) \((s < s')\). For definiteness we can suppose that \(s\) is as large as possible, i.e. that \(r = 0\) and each \(r^j = 1\), so that it is just a question of removing \(\times\). If \(s_0\) is the smallest \(s'\) such that the product occurs in \(\gamma \delta^s\), then \(s_0 < s\) (because we are not in the second extreme case) and the coefficient of this product in (2.4) is
\[
\sum_{j=s_0}^{s} (-1)^j \binom{s-s_0}{j} = 0
\]
Thus this product completely cancels out.

If we multiply out \((-\zeta_n) \prod (1 + \zeta_{i})\), then we see that each of the terms does occur in the sum in part (b). This proves the last claim in part (b).

\[\square\]

2.1. Solving for \(\eta\) and \(\zeta\).

**Corollary 1.** (a) \(\eta_i\) is a rational function of the coefficients \(\psi_i\), and in turn the coefficients \(\psi\) are polynomials in the coefficients of \(l_{21}\) and \(l_{22}\).
(b) \( \zeta_k \) is a rational function of the coefficients \( \xi_k \), and in turn the coefficients \( \xi \) are polynomials in the coefficients of \( u_{21} \) and \( u_{22} \).

Unfortunately (based on Maple calculations) it appears hopeless to find a closed form expression for the \( \zeta \) variables in terms of the \( \xi \) variables.

3. \( \eta \) and \( \zeta \) as Functions on the Formal Completion

The formal completion of the loop group \( L\mathcal{G} \) is defined by

\[
L\mathcal{G} = G(\mathbb{C}[[z^{-1}]]) \times G(\mathbb{C}[z,z^{-1}]) \times G(\mathbb{C}[[z]]),
\]

where \( \mathbb{C}(z) \) is the field of formal Laurent series \( \sum a_n z^n \), \( a_n = 0 \) for \( n < 0 \). There is a generalized Birkhoff decomposition

\[
L\mathcal{G} = \bigcup_{\lambda \in \text{Hom}(\mathbb{S}^1, T)} \sum_{\lambda}^{L\mathcal{G}}, \quad \sum_{\lambda}^{L\mathcal{G}} = G(\mathbb{C}[[z^{-1}]]) \times \lambda \cdot G(\mathbb{C}[[z]])
\]

where \( \mathbb{C}[[z]] \) denotes formal power series in \( z \). This decomposition reduces to the usual Birkhoff decomposition for the smooth loop group \( L\mathcal{G} \).

For many purposes one is primarily interested in the top stratum corresponding to \( \lambda = 1 \) (which is open and dense). For \( g \) in the top stratum, there is a unique formal Riemann-Hilbert factorization

\[
g = g_- \cdot g_+ \cdot g_0
\]

where \( g_- \in G(\mathbb{C}[[z^{-1}]]) \), \( g_-(-\infty) = 1 \) \( g_0 \in G \), and \( g_+ \in G(\mathbb{C}[[z]]) \), \( g_+(0) = 1 \). There are bijective correspondences

\[
\{g_- \in G(\mathbb{C}[[z^{-1}]]) : g_-(-\infty) = 1\} \leftrightarrow g(\mathbb{C}[[z^{-1}]]) : g_- \leftrightarrow \theta_-(\partial g_-)g_-^{-1}
\]

and

\[
\{g_+ \in G(\mathbb{C}[[z]]) : g_+(0) = 1\} \leftrightarrow g(\mathbb{C}[[z]]) : g_+ \leftrightarrow \theta_+(\partial g_+)g_+^{-1}
\]

The factors \( g_+, g_0, g_- (\theta_- g_0 \theta_+ \) are referred to as Riemann-Hilbert coordinates (linear Riemann-Hilbert coordinates, respectively) for \( g \in L\mathcal{G} \).

If \( g_0 \) additionally has a triangular factorization, then \( g \) has a unique formal triangular factorization

\[
g = l \cdot m \cdot a \cdot u
\]

where \( m \in T \) (the diagonal torus in \( SU(2) \)), \( a \in A = \exp(\mathbb{R}(1 \ 0 \\ 0 \ -1)) \), \( l \in \mathcal{N}^- \), the (profinite nilpotent) group consisting of formal power series in \( z^{-1} \),

\[
l = \left(1 + \sum_{j=1}^{\infty} A_j z^{-j} \sum_{j=1}^{\infty} B_j z^{-j} \right) / \left(1 + \sum_{j=1}^{\infty} D_j z^{-j} \right),
\]

with \( \text{det}(l) = 1 \) (as a formal power series in \( z^{-1} \), and \( u \in \mathcal{N}^+ \), the group consisting of formal power series in \( z \),

\[
u = \left(1 + \sum_{j=1}^{\infty} a_j z^j \sum_{j=1}^{\infty} b_j z^j \right) / \left(1 + \sum_{j=1}^{\infty} d_j z^j \right),
\]

with \( \text{det}(u) = 1 \).

Exactly as in the previous subsection, we can say that \( \eta \) and \( \zeta \) extend in a natural way to functions on the set of \( g \in L\mathcal{G} \) having a triangular factorization.
Corollary 2. In reference to the formal completion, (a) $\eta_i$ is a rational function of the coefficients of $l_{21}$ and $l_{22}$, which are in turn polynomials in the coefficients of $\theta_-$. (b) $\zeta_k$ is a rational function of the coefficients of $u_{21}$ and $u_{22}$, which are in turn polynomials in the coefficients of $\theta_+$. This has important measure-theoretic implications, which we will pursue elsewhere.

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