A Proof of “Goldbach’s Conjecture”

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Abstract: Presented is a proof of “Goldbach’s Conjecture” that is simple, direct, and concise and that is readily understandable by most mathematically or scientifically trained persons.

Keywords: Goldbach, mathematics, proof

1. INTRODUCTION
Goldbach's Conjecture states:

Every even number greater than two can be expressed as the sum of two primes.

2. STEP 1

General

All of the prime numbers other than 2 are odd. The sum of any two of those odd prime numbers is always an even number. Therefore, it only remains to show that the combinations* of all prime numbers other than 2, taken two at a time, summed in pairs, yields all of the even numbers greater than 4. That, along with that the even number 4 is the sum of the pair of prime numbers [2 + 2] will complete the proof. Goldbach's Conjecture has been verified up to $10^8$ by numerical calculations. 1

* [Here the combinations in pairs may include the same number twice.]

3. STEP 2

How combinations of primes summed in pairs might yield all even numbers.

The odd prime numbers comprise a string of odd numbers each greater than the prior by two except that there are various gaps [intervals of one or more non-primes] in the sequence. For example, all of the odd numbers < 200 are as in equation (1) with the non-primes in bold face [the number 1, which is divisible only by 1 and itself, is nevertheless defined as non-prime].

\[
\begin{align*}
1 &\quad 3 &\quad 5 &\quad 7 &\quad 9 &\quad 11 &\quad 13 &\quad 15 &\quad 17 &\quad 19 &\quad 21 \\
23 &\quad 25 &\quad 27 &\quad 29 &\quad 31 &\quad 33 &\quad 35 &\quad 37 &\quad 39 &\quad 41 &\quad 43 \\
45 &\quad 47 &\quad 49 &\quad 51 &\quad 53 &\quad 55 &\quad 57 &\quad 59 &\quad 61 &\quad 63 &\quad 65 \\
67 &\quad 69 &\quad 71 &\quad 73 &\quad 75 &\quad 77 &\quad 79 &\quad 81 &\quad 83 &\quad 85 &\quad 87 \\
89 &\quad 91 &\quad 93 &\quad 95 &\quad 97 &\quad 99 &\quad 101 &\quad 103 &\quad 105 &\quad 107 &\quad 109 \\
111 &\quad 113 &\quad 115 &\quad 117 &\quad 119 &\quad 121 &\quad 123 &\quad 125 &\quad 127 &\quad 129 &\quad 131 \\
133 &\quad 135 &\quad 137 &\quad 139 &\quad 141 &\quad 143 &\quad 145 &\quad 147 &\quad 149 &\quad 151 &\quad 153 \\
155 &\quad 157 &\quad 159 &\quad 161 &\quad 163 &\quad 165 &\quad 167 &\quad 169 &\quad 171 &\quad 173 &\quad 175 \\
177 &\quad 179 &\quad 181 &\quad 183 &\quad 185 &\quad 187 &\quad 189 &\quad 191 &\quad 193 &\quad 195 &\quad 197 \\
169 &\quad ... 
\end{align*}
\] (1)

Designating the set of all prime numbers to be $\{P_i, i = 1, 2, \infty \ldots \} = 3, 5, 7, not 9, 11, \ldots$, the first sub-set of the set of all combinations* of $P_i$ taken and summed in pairs is the sub-set
\{3 + P_i\}. That sub-set produces some, but not all, of the even numbers generated as the sum of two primes as in equation (2). Bold face indicates gaps in the even numbers sequence, even numbers that are the sum of two odd numbers one or both of which is non-prime and therefore do not satisfy the conjecture.

Sub-Set \{3 + P_i\} =

\begin{align*}
4 & \quad 6 & \quad 8 & \quad 10 & \quad 12 & \quad 14 & \quad 16 & \quad 18 & \quad 20 & \quad 22 & \quad 24 \\
26 & \quad 28 & \quad 30 & \quad 32 & \quad 34 & \quad 36 & \quad 38 & \quad 40 & \quad 42 & \quad 44 & \quad 46 \\
48 & \quad 50 & \quad 52 & \quad 54 & \quad 56 & \quad 58 & \quad 60 & \quad 62 & \quad 64 & \quad 66 & \quad 68 \\
70 & \quad 72 & \quad 74 & \quad 76 & \quad 78 & \quad 80 & \quad 82 & \quad 84 & \quad 86 & \quad 88 & \quad 90 \\
92 & \quad 94 & \quad 96 & \quad 98 & \quad 100 & \quad 102 & \quad 104 & \quad 106 & \quad 108 & \quad 110 & \quad 112 \\
114 & \quad 116 & \quad 118 & \quad 120 & \quad 122 & \quad 124 & \quad 126 & \quad 128 & \quad 130 & \quad 132 & \quad 134 \\
136 & \quad 138 & \quad 140 & \quad 142 & \quad 144 & \quad 146 & \quad 148 & \quad 150 & \quad 152 & \quad 154 & \quad 156 \\
158 & \quad 160 & \quad 162 & \quad 164 & \quad 166 & \quad 168 & \quad 170 & \quad 172 & \quad 174 & \quad 176 & \quad 178 \\
180 & \quad 182 & \quad 184 & \quad 186 & \quad 188 & \quad 190 & \quad 192 & \quad 194 & \quad 196 & \quad 198 & \quad 200
\end{align*}

The gaps in the sequence of even numbers generated by the sub-set \{3 + P_i\} are due to the gaps in the sequence of primes \{P_i\} per equation (1), above. The next sub-set, \{5 + P_i\}, fills in some of those gaps while leaving corresponding other ones, equation (3), again in bold face.

Sub-Set \{5 + P_i\} =

\begin{align*}
6 & \quad 8 & \quad 10 & \quad 12 & \quad 14 & \quad 16 & \quad 18 & \quad 20 & \quad 22 & \quad 24 & \quad 26 \\
28 & \quad 30 & \quad 32 & \quad 34 & \quad 36 & \quad 38 & \quad 40 & \quad 42 & \quad 44 & \quad 46 & \quad 48 \\
50 & \quad 52 & \quad 54 & \quad 56 & \quad 58 & \quad 60 & \quad 62 & \quad 64 & \quad 66 & \quad 68 & \quad 70 \\
72 & \quad 74 & \quad 76 & \quad 78 & \quad 80 & \quad 82 & \quad 84 & \quad 86 & \quad 88 & \quad 90 & \quad 92 \\
94 & \quad 96 & \quad 98 & \quad 100 & \quad 102 & \quad 104 & \quad 106 & \quad 108 & \quad 110 & \quad 112 & \quad 114 \\
116 & \quad 118 & \quad 120 & \quad 122 & \quad 124 & \quad 126 & \quad 128 & \quad 130 & \quad 132 & \quad 134 & \quad 136 \\
138 & \quad 140 & \quad 142 & \quad 144 & \quad 146 & \quad 148 & \quad 150 & \quad 152 & \quad 154 & \quad 156 & \quad 158 \\
160 & \quad 162 & \quad 164 & \quad 166 & \quad 168 & \quad 170 & \quad 172 & \quad 174 & \quad 176 & \quad 178 & \quad 180 \\
182 & \quad 184 & \quad 186 & \quad 188 & \quad 190 & \quad 192 & \quad 194 & \quad 196 & \quad 198 & \quad 200 & \quad \ldots
\end{align*}

Applying the two sub-sets together, however, the number of gaps in the sequence of all even numbers that are generated as the sum of two primes is reduced. Sub-set \{5 + P_i\} generates some even numbers that are the sum of two primes that \{3 + P_i\} does not. The combined effect is as in equation (4), for which if a number is not bold in one or both of equation 2 and 3 then it is not bold below and is not a gap.

The even numbers [\leq 200] as generated by

Sub-Sets \{3 + P_i\} and \{5 + P_i\} combined =

\begin{align*}
4 & \quad 6 & \quad 8 & \quad 10 & \quad 12 & \quad 14 & \quad 16 & \quad 18 & \quad 20 & \quad 22 & \quad 24 \\
26 & \quad 28 & \quad 30 & \quad 32 & \quad 34 & \quad 36 & \quad 38 & \quad 40 & \quad 42 & \quad 44 & \quad 46 \\
48 & \quad 50 & \quad 52 & \quad 54 & \quad 56 & \quad 58 & \quad 60 & \quad 62 & \quad 64 & \quad 66 & \quad 68 \\
70 & \quad 72 & \quad 74 & \quad 76 & \quad 78 & \quad 80 & \quad 82 & \quad 84 & \quad 86 & \quad 88 & \quad 90 \\
92 & \quad 94 & \quad 96 & \quad 98 & \quad 100 & \quad 102 & \quad 104 & \quad 106 & \quad 108 & \quad 110 & \quad 112 \\
114 & \quad 116 & \quad 118 & \quad 120 & \quad 122 & \quad 124 & \quad 126 & \quad 128 & \quad 130 & \quad 132 & \quad 134 \\
136 & \quad 138 & \quad 140 & \quad 142 & \quad 144 & \quad 146 & \quad 148 & \quad 150 & \quad 152 & \quad 154 & \quad 156 \\
158 & \quad 160 & \quad 162 & \quad 164 & \quad 166 & \quad 168 & \quad 170 & \quad 172 & \quad 174 & \quad 176 & \quad 178 \\
180 & \quad 182 & \quad 184 & \quad 186 & \quad 188 & \quad 190 & \quad 192 & \quad 194 & \quad 196 & \quad 198 & \quad 200
\end{align*}

The sub-set \{5 + P_i\} supplies a missing even number wherever it encounters the beginning of a gap in the sequence of even numbers that were generated by the \{3 + P_i\} sub-set. That happens because each number in \{5 + P_i\} is 2 more [one odd number higher] than the corresponding number in \{3 + P_i\}. The effect is that any number in equation (2) that is at the beginning of a gap [bold face with non-bold face to its left] moves one position to the left in equation (3), moves to a position not in a gap [non-bold face]. The only exception is the initial gap, 4, which has already been addressed.

Because of this, were the sequence 3, 5, ... of the subsets \{3 + P_i\}, \{5 + P_i\}, ... to continue without any breaks, for example were it to proceed \{7 + P_i\}, \{9 + P_i\}, \{11 + P_i\}, ... then all of those sub-sets collectively would eventually fill in all of the gaps in the original sequence generated by \{3 + P_i\} and generate all of the even numbers as sums of two primes. That is, that would happen...
provided that each of the gaps is such that there are more odd numbers, 5, 7, 9, 11, ... (but not 3, which generates the original gaps), preceding the gap than there are even numbers in the gap.

However, the sequence of subsets has breaks such as the \( \{9 + P_i\} \) sub-set, which generates no even numbers as the sum of two primes because one of the two numbers summed is always the non-prime 9. There now remain two issues: what of the invalid sub-sets, ones that produce breaks in the sub-set sequence such as \( \{9 + P_i\} \), and are there always sufficient prime [not merely odd] numbers preceding each gap?

4. STEP 3

The invalid sub-sets such as sub-set \( \{9 + P_i\} \). The equations below present the even numbers generated by each of sub-sets \( \{7 + P_i\} \) through \( \{17 + P_i\} \) for which the index, \( I \), where \( I = 7, 9, 11, 13, 17, ... \), is prime (except 9) and for each presents the cumulative effect of all of the sub-sets through that point in eliminating gaps.

Sub-Set \( \{7 + P_i\} \) =

\[
\begin{align*}
8 & 10 12 14 16 18 20 22 24 26 28 \\
30 & 32 34 36 38 40 42 44 46 48 50 \\
52 & 54 56 58 60 62 64 66 68 70 72 \\
74 & 76 78 80 82 84 86 88 90 92 94 \\
96 & 98 100 102 104 106 108 110 112 114 116
\end{align*}
\]

The even numbers \([\leq 200]\) as generated by Sub-Sets \( \{3 + P_i\}, \{5 + P_i\} \) and \( \{7 + P_i\} \) combined =

\[
\begin{align*}
4 & 6 8 10 12 14 16 18 20 22 24 \\
26 & 28 30 32 34 36 38 40 42 44 46 \\
48 & 50 52 54 56 58 60 62 64 66 68 \\
70 & 72 74 76 78 80 82 84 86 88 90 \\
92 & 94 96 \underline{98} 100 102 104 106 108 110 112 \\
114 & 116 118 120 \underline{122} \underline{124} \underline{126} \underline{128} 130 132 134 \\
136 & 138 140 142 144 146 \underline{148} 150 152 154 156 \\
158 & 160 162 164 166 168 170 172 174 176 178 \\
180 & 182 184 186 188 \underline{190} \underline{192} 194 196 198 200 \\
\end{align*}
\]

Sub-Set \( \{11 + P_i\} \) =

\[
\begin{align*}
12 & 14 16 18 20 22 24 26 28 30 32 \\
34 & 36 38 40 42 44 46 48 50 52 54 \\
56 & 58 60 62 64 66 68 70 72 74 76 \\
78 & 80 82 84 86 88 90 92 94 96 98 \\
100 & 102 104 106 108 110 112 114 116 118 120 \\
122 & 124 126 128 130 132 134 136 138 140 142 \\
144 & 146 148 150 152 154 156 158 160 162 164 \\
166 & 168 170 172 174 176 178 180 182 184 186 \\
188 & 190 192 194 196 198 200 ...
\end{align*}
\]

The even numbers \([\leq 200]\) as generated by Sub-Sets \( \{3 + P_i\}, \{5 + P_i\}, \{7 + P_i\} \) and \( \{11 + P_i\} \) combined =
Sub-Set \{13 + \Pi\} =

\begin{align*}
&14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 \\
&36 & 38 & 40 & 42 & 44 & 46 & 48 & 50 & 52 & 54 & 56 \\
&58 & 60 & 62 & 64 & 66 & 68 & 70 & 72 & 74 & 76 & 78 \\
&80 & 82 & 84 & 86 & 88 & 90 & 92 & 94 & 96 & 98 & 100 \\
&102 & 104 & 106 & 108 & 110 & 112 & 114 & 116 & 118 & 120 & 122 \\
&124 & 126 & 128 & 130 & 132 & 134 & 136 & 138 & 140 & 142 & 144 \\
&146 & 148 & 150 & 152 & 154 & 156 & 158 & 160 & 162 & 164 & 166 \\
&168 & 170 & 172 & 174 & 176 & 178 & 180 & 182 & 184 & 186 & 188 \\
&190 & 192 & 194 & 196 & 198 & 200 & & & & & \\
\end{align*}

The even numbers \([\leq 200]\) as generated by Sub-Sets

\begin{align*}
\{3 + \Pi\}, \{5 + \Pi\}, \{7 + \Pi\}, \{11 + \Pi\} \text{ and } \{13 + \Pi\} \text{ combined} =
\begin{align*}
&4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 \\
&26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 \\
&48 & 50 & 52 & 54 & 56 & 58 & 60 & 62 & 64 & 66 & 68 \\
&70 & 72 & 74 & 76 & 78 & 80 & 82 & 84 & 86 & 88 & 90 \\
&92 & 94 & 96 & 98 & 100 & 102 & 104 & 106 & 108 & 110 & 112 \\
&114 & 116 & 118 & 120 & 122 & 124 & 126 & 128 & 130 & 132 & 134 \\
&136 & 138 & 140 & 142 & 144 & 146 & 148 & 150 & 152 & 154 & 156 \\
&158 & 160 & 162 & 164 & 166 & 168 & 170 & 172 & 174 & 176 & 178 \\
&180 & 182 & 184 & 186 & 188 & 190 & 192 & 194 & 196 & 198 & 200 \\
\end{align*}

Sub-Set \{17 + \Pi\} =

\begin{align*}
&18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 \\
&40 & 42 & 44 & 46 & 48 & 50 & 52 & 54 & 56 & 58 & 60 \\
&62 & 64 & 66 & 68 & 70 & 72 & 74 & 76 & 78 & 80 & 82 \\
&84 & 86 & 88 & 90 & 92 & 94 & 96 & 98 & 100 & 102 & 104 \\
&106 & 108 & 110 & 112 & 114 & 116 & 118 & 120 & 122 & 124 & 126 \\
&128 & 130 & 132 & 134 & 136 & 138 & 140 & 142 & 144 & 146 & 148 \\
&150 & 152 & 154 & 156 & 158 & 160 & 162 & 164 & 166 & 168 & 170 \\
&172 & 174 & 176 & 178 & 180 & 182 & 184 & 186 & 188 & 190 & 192 \\
&194 & 196 & 198 & 200 & & & & & & & \\
\end{align*}

The even numbers \([\leq 200]\) as generated by Sub-Sets

\begin{align*}
\{3 + \Pi\}, \{5 + \Pi\}, \{7 + \Pi\}, \{11 + \Pi\}, \{13 + \Pi\} \text{ and } \{17 + \Pi\} \text{ combined} =
\begin{align*}
&4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & 24 \\
&26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 & 44 & 46 \\
&48 & 50 & 52 & 54 & 56 & 58 & 60 & 62 & 64 & 66 & 68 \\
&70 & 72 & 74 & 76 & 78 & 80 & 82 & 84 & 86 & 88 & 90 \\
&92 & 94 & 96 & 98 & 100 & 102 & 104 & 106 & 108 & 110 & 112 \\
&114 & 116 & 118 & 120 & 122 & 124 & 126 & 128 & 130 & 132 & 134 \\
&136 & 138 & 140 & 142 & 144 & 146 & 148 & 150 & 152 & 154 & 156 \\
&158 & 160 & 162 & 164 & 166 & 168 & 170 & 172 & 174 & 176 & 178 \\
&180 & 182 & 184 & 186 & 188 & 190 & 192 & 194 & 196 & 198 & 200 \\
\end{align*}

Each of the individual single sub-set tables is identical to its predecessor except that in each successive table the number in each position is increased by the amount that the index has increased. The effect is that the numbers in the sequence of tables move continuously to the left and upward in...
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the tables while the table positions of unsatisfactory even numbers [ones in bold face because they are in the positions where there is no number in \( P_i \), where the corresponding number in the sequence of all odd numbers is a non-prime] remain unmoved.

The original sequence, that of sub-set \( \{3 + P_i\} \), sets the problem. It exhibits many even numbers as a sum of two primes, that is as satisfactory numbers, and it also exhibits many gaps, uninterrupted sequences of unsatisfactory even numbers. The length of a gap, \( G \), is its number of unsatisfactory numbers in uninterrupted sequence. In general, then, what is required to insure that every even number is generated as the sum of two primes? What is required to clear all of the gaps?

As the numbers in the individual single sub-set tables move to the left and upward with the increases in the sub-set index, \( I \), whenever a number from a gap moves onto the position of a prime [a non-bold face position] the number is expressed as a sum of two primes, a satisfactory even number, and the gap is reduced to that extent. Not more than \( G \) such events would be needed to clear the gap. Each valid sub-set that acts after the original \( \{3 + P_i\} \) is a candidate to provide such an event for each gap and whether it succeeds or not it moves the gap nearer to the beginning of the table.

From this point of view, clearing a gap of length \( G \) requires that the original odd number sequence, equation (1), exhibit at least \( G \) primes ahead of the gap. This requirement is conservative because sometimes the same position in a table clears more than one element of a gap, that is one or more of the \( G \) primes ahead of the gap in the table may sometimes act more than once on the same gap. [For example, for the \( G = 3 \) gap \( 94, 96, 98 \) per sub-set \( \{3 + P_i\} \), equation (2), it turns out that the prime 92 actually clears two elements in the gap, the 94 and 96 [see equations (3), (4), (5), and (6)]. And, for another example, equation (8) versus equation (10) where 122 and 126 are cleared simultaneously.] That type of help in clearing gaps is ignored in the present analysis so as to arrive at a worst case.

Another effect may further increase the above requirement. It can happen that a number in a gap could move onto the position of a prime in a sub-set having a non-prime index; or rather, the number in the gap could move more than one number to the left at one time because of the index skipping an intervening invalid sub-set [see equations (5) and (7) between which an invalid sub-set is skipped and likewise equations (9) and (11)]. However viewed, such an event could "waste" a prime, result in failure to benefit from it by the conversion of a number from unsatisfactory to satisfactory, meaning that there may need to be more than \( G \) available primes preceding the gap. [For example, the 98 in sub-set \( \{7 + P_i\} \), equation (5), is in position to move to the left onto a position [non-bold face] that would clear it; however, the next sub-set is \( \{9 + P_i\} \), an invalid sub-set that is skipped over because its non-prime index prevents clearing any elements. The next position after that is in bold face again and unable to clear an element. The net effect is that the potentially clearing position is missed, "wasted".]

How many more available primes might be needed to account for and offset this effect? A conservative estimate of the amount that \( G \) should be increased would be to multiply \( G \) by the ratio of the number of non-primes preceding the gap to the number of primes preceding the gap, that is increase \( G \) to \( G \cdot [1 + \text{that ratio}] \). That is equivalent to assuming that every invalid sub-set wastes a prime. Sometimes the "wasted prime" position is occupied by a bold face number or sometimes the number in position to move there is already cleared so that, either way nothing is really lost. [For example, the 142 in sub-set \( \{7 + P_i\} \), equation (5).] An adjustment to take account of that help in clearing gaps is ignored in the present analysis so as to arrive at a worst case.

5. Step 4

The required number of available primes preceding the gap now becomes as follows.

Where: \( n \) = the first number of a gap, the gap's beginning. \( \pi(n) \) = the number of primes that are \( \leq n \).

\[ R(n) = \text{the required number of primes ahead of the gap that begins at number "} n \text{" needed to clear that gap.} \]

Then:

\[
R(n) = G \cdot \left[ 1 + \frac{n-\pi(n)}{\pi(n)} \right] = G \cdot \left[ \frac{n}{\pi(n)} \right]
\]
6. STEP 5

Why there are always sufficient primes preceding each gap.

The subject of the distribution of primes has been studied in depth. Chapter 4, "How are the Prime Numbers Distributed?" of Reference [1] summarizes the history and results of those studies and presents a number of related proofs. The results fall into two categories for the present purposes: section I of the chapter, which treats the development of the Prime Number Theorem, that is expressions for the number of primes in designated intervals, and section II of the chapter, which treats gaps between primes. That referenced work is the source of the following data.

The Prime Number Theorem, the most fundamental theorem of prime numbers is as follows.

\[\pi(n) = \text{the prime counting function}\]

- \(n = \text{the number of primes} \leq n, \text{an integer}\)
- \(\approx n/\ln(n) \text{ the approximation improving as } n \text{ increases}\)

That approximation is low by \(5.78\%\) for \(n = 10^8\) improving to low by \(2.79\%\) for \(n = 10^{16}\). A better approximation is given by a function called the logarithmic interval as follows.

\[\pi(n) \approx \text{Li}(n) = \int_2^n \frac{dx}{\ln(x)}\]  

The logarithmic interval approximation to \(\pi(n)\) is high by only \(0.013\%\) for \(n = 10^8\) and by only \(0.000,000,5\%\) for \(n = 10^{16}\).

Even more accurate is the Riemann function, too involved to be worth specifying here, which for \(n=10^8\) differs from the correct value by only \(0.0017\%\) and for \(n = 10^{16}\) by \(0.000,000,1\%\).

Substituting equation (9) into equation (8) the following is obtained.

\[R(n) = G \cdot \left[\frac{n}{\pi(n)}\right]\]

= \(G \cdot n \cdot \left[\frac{\ln(n)}{n}\right]\)

= \(G \cdot \ln(n)\)

The issue is, of course, how does \(R(n)\), the number of preceding primes required, compare with \(\pi(n)\), the number of preceding primes actually available? That is as follows.

\[\frac{\pi(n)}{R(n)} = \frac{n/\ln(n)}{G \cdot \ln(n)} = \frac{n/G \cdot (\ln(n))^2}{}\]

Fig. 1, below lists some values for that function using values for \(G\) extrapolated from the percent deviation of the logarithmic interval from the exact count of \(\pi(n)\) presented above. From hundreds to thousands to even far greater multiples of the required number of primes preceding the gaps are actually available for clearing them.

| \(n\) | \(G\) | \(\pi(n)/R(n)\) | \(G/n \%)\) |
|------|------|----------------|-------------|
| \(10^8\) | \(10^{3.0}\) | 295 | 0.001 |
| \(10^9\) | \(10^{3.6}\) | 585 | 0.0004 |
| \(10^{10}\) | \(10^{4.3}\) | 945 | 0.0002 |
| \(10^{11}\) | \(10^{4.9}\) | 1962 | 0.00008 |
| \(10^{12}\) | \(10^{5.5}\) | 4142 | 0.00003 |
| \(10^{13}\) | \(10^{6.1}\) | 8865 | 0.00001 |
| \(10^{14}\) | \(10^{6.8}\) | 15251 | 0.000006 |
| \(10^{15}\) | \(10^{7.4}\) | 33372 | 0.00003 |
| \(10^{16}\) | \(10^{8.0}\) | 73676 | 0.000001 |

Figure 1
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The formulations of various accuracies for evaluating $\pi(n)$ cited on the previous page: $n/Ln(n)$, the logarithmic interval and the Riemann function, cannot be used to find individual prime numbers. Precise, not limited accuracy, is required for that. However, the formulations do indicate that the prime numbers are well distributed over the range of all numbers, that dense concentrations and large gaps cannot occur.

The function of equation (9) is smooth as shown in Fig. 2, below, and while the function only approximates the actual values of $\pi(n)$, that approximation is fairly close over most of the range. The same can be said, and more emphatically, of the more accurate functions cited for which the approximation is better. Gaps that are large relative to the location in the sequence of numbers, that is other than small values of $G/\pi(n)$ simply cannot occur. The precise condition, $\pi(n) \geq R(n) = G \cdot Ln(n)$, is greatly exceeded.

For example, at $n = 10^8$, the deviation of $\pi(n)$ from the exact number of primes up to that $n$ is $0.0017\%$, which is a number range of $1,700$ out of $100,000,000$. The largest gap in that neighborhood could not reasonably exceed twice that $3,400$, and certainly not $10$ times it. The number of primes available up to that point is $5,761,455$. That is far more than needed to eliminate the effect of the gap. [The largest gap that has ever been found is a sequence of $653$ non-primes following the prime $11,000,001,446,613,353$ at which the number of preceding primes available is more than $279,238,341,033,925$ (the value for $10^{16}$).]

7. IN SUMMATION

a. - All of the prime numbers other than 2 are odd, 2 being the only even prime number. Further, the even number $4 = 2 + 2$.

b. - The sum of any two of the odd prime numbers is always an even number.

c. - All combinations (the combinations in pairs may include the same number twice) of the odd numbers $\geq 3$ [whether prime or not] summed in pairs produces all of the even numbers $\geq 6$.

d. - While just the prime odd numbers in sequence is a sequence with gaps as compared to that of all of the odd numbers; nevertheless, all combinations of the odd prime numbers $\geq 3$ summed in pairs produces all of the even numbers provided that there are enough primes preceding the gaps.

e. - That requirement is that $\pi(n) \geq R(n) = G \cdot Ln(n)$ where $n$ is the first number in the gap, $\pi(n)$ is the number of primes less than or equal to $n$, $R(n)$ is the number of preceding primes needed to assure clearance of the gap, and $G$ is the number of sequential non-primes in the gap. This requirement is comprehensively satisfied by all of the prime numbers and gaps because of the sufficiently smooth nature of $\pi(n)$.

Which proves the conjecture.

REFERENCES

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A graduate of the United States Military Academy at West Point and post-graduate of Stanford University Roger Ellman left the military and pursued his primary interest in the origin of the universe and its cosmological physics. That led to the study of gravitation, various astronomical anomalies and philosophy. He is the author of a number of scientific papers on physics and astrophysics and several books.