Blackbody Radiation Noise Broadening of Quantum Systems

Eric B. Norrgard,¹,² Stephen P. Eckel,¹ Christopher L. Holloway,³ and Eric L. Shirley¹

¹Sensor Science Division, National Institute of Standards and Technology, Gaithersburg, Maryland 20899, USA
²Joint Quantum Institute, University of Maryland, College Park, Maryland 20742, USA
³RF Technology Division, National Institute of Standards and Technology (NIST), Boulder, CO 80305

(Dated: 2021-11-13)

Precision measurements of quantum systems often seek to probe or must account for the interaction with blackbody radiation. Over the past several decades, much attention has been given to AC Stark shifts and stimulated state transfer. For a blackbody in thermodynamic equilibrium, these two effects are determined by the expectation value of photon number in each mode of the Planck spectrum. Here, we explore how the photon number variance of an equilibrium blackbody generally leads to a parametric broadening of the energy levels of quantum systems that is inversely proportional to the square-root of the blackbody volume. We consider the effect in two cases which are potentially highly sensitive to this broadening: Rydberg atoms and atomic clocks. We find that even in blackbody volumes as small as 1 cm⁻³, this effect is unlikely to contribute meaningfully to transition linewidths.

Precision spectroscopy of atomic and molecular systems is the basis of numerous metrological applications [1, 2] and tests of fundamental physics [3, 4] and symmetries [5, 6]. A correct determination of transition energies [1, 2] and tests of fundamental physics [3, 4] and symmetries is the basis of numerous metrological applications [1, 2].

For both systems, we find the BBR noise is too small to significantly contribute to line broadening under any currently feasible scenario.

Before deriving the BBR noise broadening, we summarize the relevant photon statistics and AC Stark interaction (see Appendix A for more details on photon statistics). For photons with angular frequency $\omega$, we will use $x = h\omega/k_B T$ to write the partition function $Z = (1 - e^{-x})^{-1}$. The $m$th moment of the photon number $n$ is

$$\langle \hat{n}^m \rangle = \sum_{n=0}^{\infty} n^m e^{-nx} = \frac{1}{Z} \frac{\partial^m}{\partial(-x)^m} Z(x). \quad (1)$$

The variance in photon number for each BBR mode is $\sigma_i^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = e^x/(e^x - 1)^2$. In this work, a hat over a symbol indicates an operator.

We consider an ideal blackbody of effective volume $V$ and temperature $T$ which surrounds a quantum system in state $i$. In this work, we define the effective length $\ell$ and volume $V$ of the cavity through the mode spacing. Physically, a blackbody cavity is created through a combination of a resistive wall material, light trapping geometry [14], and surface micro-structure [15, 16]. A plane wave incident on a surface with non-zero resistance incurs a phase shift according to the Fresnel equations. Effectively, these phase shifts increase the cavity size and decrease the mode spacing compared to that of an ideal conductor. Similarly, surface micro-structure can increase the physical length of each incident mode. For this work, we will not concern ourselves with these complications and instead compare cavities with equal effective size – that is, cavities with identical mode spacing – rather than cavities with equal physical dimensions.

The AC Stark interaction is characterized by a real and an imaginary component [17, 19]:

$$\mathcal{E}_i^{BBR} = \Delta E_i - \frac{i\hbar}{2} \Gamma_i^{BBR} = -\int_0^{\infty} d\omega \frac{\langle \hat{E}^2 \rangle}{2} \alpha_i^*(\omega), \quad (2)$$
where $\hat{E}$ is the electric field operator, $\alpha_i^s(\omega)$ is the scalar polarizability of level $i$

$$\alpha_i^s(\omega) = \sum_j \left[ \frac{|\mu_{ij}|^2}{\hbar} \left( \frac{1}{\omega_{ij} - \omega - \frac{i}{2}\Gamma_{ij}} + \frac{1}{\omega_{ij} + \omega + \frac{i}{2}\Gamma_{ij}} \right) \right].$$

Here, $\mu_{ij}$ and $\Gamma_{ij}$ are the $z$ component of the dipole matrix element and the partial decay rate, respectively, between state $i$ to state $j$. The imaginary part $\Gamma_{BBR}$ of Eq. (2) is associated with stimulated state transfer from level $i$ to other levels $j$ of the system:

$$\Gamma_{BBR}^\imath = \sum_j \frac{\mu_{ij}^2 \omega_{ij}^3}{\epsilon_0 \hbar c^3} \langle \hat{n}(\omega_{ij}) \rangle.$$  

The real part of Eq. (2) is a shift in the energy of level $i$, given by (10):

$$\Delta E_i = -P \int_0^\infty d\omega \left[ \frac{\hbar \omega^3 \langle \hat{n} \rangle}{2\epsilon_0 \pi^2 c^3} \alpha_i^s(\omega) \right]$$

where $P$ denotes the Cauchy principal value.

In the absence of other broadening mechanisms, the interaction of a quantum system with the mean BBR field leads to a Lorentzian lineshape with full width at half maximum (FWHM)

$$\Gamma_i = \Gamma_{i}^{\text{sp}} + \Gamma_{i}^{\text{BBR}},$$

where $\Gamma_{i}^{\text{sp}}$ and $\Gamma_{i}^{\text{BBR}}$ are the rates of spontaneous decay and BBR stimulated depopulations for level $i$, respectively. Here, we seek to quantify the small, additional broadening of the quantum level induced by fluctuations in the BBR electric field. These fluctuations lead to RMS deviations $\sigma_{\Delta E_i}$ in the AC Stark shift $\Delta E_i$. These fluctuations occur with typical timescale of the blackbody coherence time $t_c = \hbar / 4k_B T$ ($t_c \approx 40$ fs at room temperature) $^{21}$. Therefore, the expected lineshape for a single measurement with duration $t_m \gg t_c$, is a Voigt profile with Lorentzian width $\gamma = \Gamma_i$ and Gaussian half width at half maximum (HWHM) $\sigma = \sigma_{\Delta E_i} \sqrt{t_c / t_m}$. The FWHM of a Voigt profile is approximately

$$\text{FWHM}_V \approx \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} + 8\sigma^2 \ln 2}.$$  

In the limit $\gamma \gg \sigma$, the Voigt FWHM becomes

$$\text{FWHM}_V \approx \gamma + 8 \ln 2 \frac{\sigma^2}{\gamma}.$$  

In order to calculate $\sigma_{\Delta E_i}$, we begin by considering each photon mode as contributing $\Delta \epsilon_i$ to the total shift, i.e. $\Delta E_i = \sum_{\text{modes}} \Delta \epsilon_i$. Assuming uncorrelated modes, $\sigma_{\Delta E_i}^2 = \sum_{\text{modes}} \sigma_{\Delta \epsilon_i}^2$; it suffices to find the contribution of each mode independently. The summation from $\text{modes}$ corresponds to $P \int V D_{\text{mode}}$ in the continuous limit, where $D_{\text{mode}}$ is the density of modes per unit volume per unit angular frequency $d\omega$, and

$$D_{\text{mode}} = D_{\text{FS}} = \omega^2 d\omega / (2\epsilon_0 V)^3$$

for free space. Combining this with Eq. (3) yields

$$\Delta \epsilon_i = -\frac{\hbar \omega (\langle \hat{n} \rangle)}{2\epsilon_0 V} \alpha_i^s(\omega),$$

and the variance of the shift is given by

$$\sigma_{\Delta \epsilon_i}^2 = \frac{\alpha_i^s(\omega)}{\sigma_{\Delta E_i}^2} \left[ \frac{\partial \Delta \epsilon_i}{\partial n} \right]^2 \sigma_n^2 = \left[ \frac{\hbar \omega}{2\epsilon_0 V} \alpha_i^s(\omega) \right]^2 \sigma_n^2.$$  

Finally we add the contributions of each mode in quadrature to arrive at the variance of the AC Stark shift due to BBR noise:

$$\sigma_{\Delta E}^2 = \sum_{\text{modes}} \sigma_{\Delta \epsilon_i}^2$$

$$= P \int_0^\infty V D_{\text{mode}} \frac{\hbar^2 \omega^3 \sigma_n^2}{4\epsilon_0^2 V^3} \alpha_i^s(\omega)^2$$

$$= P \int_0^\infty d\omega \left[ \frac{\hbar^2 \omega^4}{4\epsilon_0^2 \sigma_n^2 c^3} \frac{\hbar \omega / k_B T}{(\hbar \omega / k_B T - 1)^2} \right.$$

$$\times \left[ \sum_j \frac{1}{\hbar} \left( \frac{|\mu_{ij}|^2}{\omega_{ij} - \omega - \frac{i}{2}\Gamma_{ij}} + \frac{|\mu_{ij}|^2}{\omega_{ij} + \omega + \frac{i}{2}\Gamma_{ij}} \right) \right]^2.$$  

where in the last line we take $D_{\text{mode}} = D_{\text{FS}}$. Note that the summation within the modulus in Eq. (11) is identical to that found in the Kramers-Heisenberg formula for differential scattering cross-section of light (e.g. Ref. $^{22}$ Section 8.7); that is, we can consider the BBR noise broadening of each level $i$ to be due to variance in BBR Rayleigh scattering rate due to fluctuating photon number in each mode. Also of note is the RMS fluctuation in the BBR shift $\sigma_{\Delta E_i}$ is proportional to $1/\sqrt{V}$. While the RMS electric field of a blackbody is independent of volume, smaller volumes contain fewer modes which contribute to the field, and thus are subject to larger field variations. This relationship between BBR fluctuations and volume was first noted by Einstein $^{23}$, and Eq. (11) can alternately be derived using standard thermodynamic relations (see Appendix B).

Often, the quantum observable of interest is not the AC Stark shift, but the differential AC Stark shift $\Delta \epsilon'_{ij} = \Delta E_j - \Delta E_i$, such as when measuring the transition frequency $\omega_{ij}$ between two states $i, j$. Conventionally, negative shifts imply a decrease in the observed transition frequency. Likewise, for the BBR noise line broadening, we define the differential shift per mode

$$\Delta \epsilon'_{ij} = \Delta \epsilon_j - \Delta \epsilon_i,$$

with $\alpha'_{ij}(\omega) = \alpha^s(\omega) - \alpha^s(\omega)$ the differential dynamic scalar polarizability.
where we have used the identities
differential polarizability
this effect is quite small. If we consider a "typical"
experimentally. That is unsurprising, as in most cases,
ening effect described by Eq. (11) has not been observed
value of \( \sigma \)

\[
\begin{align*}
\Delta E'_{ij} & = \frac{\hbar}{2 \varepsilon_0 \pi^2 c^3 15} \left( \frac{k_B T}{\hbar} \right)^4 \alpha_{ij}'(0), \tag{13} \\
\sigma_{\Delta E'_{ij}} & = \frac{\hbar^2}{4 \varepsilon_0^2 \pi^2 c^3 V} \left( \frac{k_B T}{\hbar} \right)^5 \left| \alpha_{ij}'(0) \right|^2, \tag{14}
\end{align*}
\]

where we have used the identities

\[
\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}, \quad \int_0^\infty dx \frac{x^4 e^x}{(e^x - 1)^2} = \frac{4\pi^4}{15}. \tag{15}
\]

To the best of our knowledge, the BBR noise broadening effect described by Eq. (11) has not been observed experimentally. That is unsurprising, as in most cases, this effect is quite small. If we consider a "typical" atomic transition to have frequency \( \omega_{ij} = 10^{15} \text{s}^{-1} \) and differential polarizability \( \alpha_{ij}' = 4 \pi \varepsilon_0 a_0^3 \) (where \( a_0 \approx 52.9 \text{pm} \) is the Bohr radius), then \( \sigma = \sigma_{\Delta E'_{ij}} \sqrt{t_c/t_m} \approx 3 \times 10^{-16} \sqrt{E/T m/V} \text{Hz m}^{3/2} \text{K}^{-5/2} \). For \( T = 300 \text{ K} \) and \( V = 1 \text{ m}^3 \), \( \sigma_{\Delta E'_{ij}} \sqrt{t_c/t_m} \approx \hbar \times 14 \text{ nHz} \).

In nearly all practical measurements, the fluctuation \( \sigma = \sigma_{\Delta E'_{ij}} \sqrt{t_c/t_m} \) averages down the numerically small value of \( \sigma_{\Delta E'_{ij}} \) by a considerable factor (e.g. for \( t_m = 1 \text{ s}, \sqrt{t_c/t_m} \approx 2 \times 10^{-7} \) at room temperature). We then see using Eq. (8) that the BBR noise broadening becomes a vanishingly small correction to the linewidth (\( \sigma \approx 3 \text{ Hz} \) in this example).

We detail below two special cases where BBR noise broadening might be most noticeable: atomic clocks and circular Rydberg atoms. However, we find in both cases the BBR noise is too small to be detected with modern experimental techniques.

In Table I we consider several current and proposed atomic frequency standards at \( T = 300 \text{ K} \). We calculate the RMS differential BBR shift deviation \( \sigma_{\Delta E'_{ij}} \) using Eq. (14) as well as the differential BBR Stark shift \( \Delta E'_{ij} \) using Eq. (13) as a check of consistency with the atomic transition data references [17, 27–29]. For the BBR shift of optical clock transitions, the static approximation is generally accurate to within a few percent [17]. Differential polarizabilities in Table I are given in their respective references in atomic polarizability units. These may be converted to SI units (Hz/(V/m)\(^2\)) by multiplying by a factor of \( 4\pi \varepsilon_0 a_0^3 / \hbar \), with \( a_0 \) the Bohr radius.

The results of Table I show that detection of BBR noise broadening in atomic frequency standards is not possible without several orders of magnitude improvement over current frequency sensitivity. Consider as examples the Sr and Yb systems, which now routinely achieve fractional uncertainty of \( \delta \omega / \omega \sim 10^{-18} [30, 52] \). Ignoring numerous technical noise sources (e.g. lattice phonon scattering in optical lattice clocks [33]), even in an exceptionally small BBR volume \( V = 10^{-6} \text{ m}^3 \) at \( T = 300 \text{ K} \), \( \sigma_{\Delta E'_{ij}} \) is only approximately 30 nHz in Sr and 10 nHz in Yb. For a measurement time \( t_m = 1 \text{ s} \), the BBR noise broadening is then \( \sigma \approx 6 \text{ nHz} \) for Sr and \( \sigma \approx 2 \text{ nHz} \) for Yb. Using Eq. (5), the observed FWHM would then
Rydberg atoms with principal quantum number \( n \) of Rb. The largest transition dipole matrix elements for transition matrix elements and energies for circular states make the BBR noise broadening in these systems even smaller than the Sr and Yb cases.

We next consider the possibility of observing BBR noise-limited linewidths in circular Rydberg states \(|nC\rangle \equiv |n, L = n - 1, J = n - 1/2\rangle\). Here we use the Alkali Rydberg Calculator python package [34] to calculate transition matrix elements and energies for circular states of Rb. The largest transition dipole matrix elements for Rydberg atoms with principal quantum number \( n \) are to states with \( n' = n \pm 1 \). For our circular Rydberg state calculations, we include electric dipole transitions to \( n' = n - 1, n + 1, n + 2, n + 3 \).

Rydberg transition wavelengths may be as large or larger than the length scale of its surroundings, and we must consider cavity effects on the mode density \( D_{\text{mode}} \). Figure 1 shows \( \sigma_{\Delta E} \), for the \(|52C\rangle\) state of Rb in an effective cubic volume \( V = \ell^3 \) for three key conceptual cases. The dashed colored lines depict \( \sigma_{\Delta E} \), calculated using \( D_{\text{mode}} = D_{\text{FS mode}} \); in this case \( \sigma_{\Delta E} \) is strictly proportional to \( \ell^{-3/2} \). The solid black line depicts an ideal blackbody (i.e. perfectly absorbing walls); the mode density for a blackbody \( D_{\text{BB mode}} \) is found by quantizing the cavity modes in the usual manner and assigning each mode a finesse \( F = 1/4 \) (see Appendix C for additional details on cavity effects on mode density). Finally, solid colored lines depict a cubic copper cavity. The mode density for copper \( D_{\text{Cu mode}} \) is complicated by the fact that the resistivity \( \rho \) of cryogenic copper may vary by two orders of magnitude depending on purity \[35\]; we assume residual-resistance ratios typical of oxygen-free high conductivity Cu: \( \rho(T = 300 \text{ K})/\rho(T = 77 \text{ K}) = 10 \) and \( \rho(T = 300 \text{ K})/\rho(T = 4 \text{ K}) = 100 \), with \( \rho(T = 300 \text{ K}) = 1.7 \times 10^{-8} \Omega \cdot \text{m} \).

Figure 2 considers Rb circular states for principal quantum number \( n \leq 80 \) in a cubic ideal blackbody. The black dashed lines depict the Lorentzian partial linewidth due to spontaneous decay \( \Gamma_i^\text{sp} \). Colored dashed lines depict the partial linewidth due to both spontaneous and BBR-stimulated decay \( \Gamma_i = \Gamma_i^\text{sp} + \Gamma_i^\text{BBR} \). Solid lines depict BBR noise Gaussian width \( \sigma_{\Delta E, \sqrt{\Gamma_i^\text{sp}}} \), where we have assumed a measurement time \( t_m = 1/\Gamma_i \). The magnitude of the BBR noise relative to the decay rate is increased in cryogenic environments, as \( \sigma_{\Delta E} \) decreases more slowly than \( \Gamma_i^\text{sp} + \Gamma_i^\text{BBR} \) with decreasing temperature. For effective volumes \( V = 1 \text{ m}^3, \sigma_{\Delta E} \)}
is smaller than \( \Gamma_i \) by roughly six orders of magnitude, even at \( T = 4 \text{ K} \) and \( n = 80 \). For blackbodies with \( V = 10^{-6} \text{ m}^3 \), \( \sigma_{\Delta E_i} \) is smaller than \( \Gamma_i \) by roughly two orders of magnitude at \( T = 4 \text{ K} \) and \( n = 80 \). We caution that for such large \( n \), typical relevant transition wavelengths are similar to or exceed \( \ell = 1 \text{ cm} \) for this case. The assumption of an ideal blackbody for high \( n \) and small \( V \) is likely invalid, with the BBR noise reduced from these estimates by one or more orders of magnitude as in Fig. 1. The observed FWHM for circular Rydberg states is therefore unlikely to exceed \( \Gamma_i \) by more than roughly \( 10^{-6} \times \Gamma_i \) in the most favorable cases.

In this work, we have assumed isotropic polarization of the BBR, and thus only considered the scalar polarizability. As each mode has two independent polarizations, polarization fluctuations in the BBR can lead to broadening which involves the vector and tensor polarizabilities as well, and could be considered in future work.

We have derived a parametric broadening, general to all quantum systems, which is due to interactions with fluctuations in a blackbody radiation field. This BBR noise broadening is most significant in applications which involve small effective BBR volumes, large transition dipole moments, and/or high frequency precision. However, our presented calculations for several atomic clock transitions and for circular Rydberg states of Rb show that BBR noise broadening is typically too small to be detected in these systems with modern sensitivity by at least several orders of magnitude. These calculations catalogue a novel quantum noise source as being well below relevance for many ongoing experiments, such as atomic frequency standards and quantum sensing experiments using circular Rydberg atoms. For future experiments with substantially improved frequency precision, this work also sets a benchmark for testing fundamental thermodynamics of photons.

**ACKNOWLEDGEMENT**

The authors thank Dazhen Gu, Andrew Ludlow, Kyle Beloy, Michael Moldover, Marianna Safronova, Clément Sayrin, Wes Tew, and Howard Yoon for insightful conversations, and thank Nikunjkumar Prajapati, Joe Rice, and Wes Tew for careful reading of the manuscript.

**Appendix A: Photon Statistics**

The partition function \( Z \) for a blackbody radiation mode with frequency \( \omega \) is

\[
Z = \sum_{n=0}^{\infty} e^{-n\hbar\omega/k_BT} = \frac{1}{1 - e^{-\hbar\omega/k_BT}}. \tag{A1}
\]

A standard trick to calculate the \( m \)th moment of the photon number \( n \) is

\[
\langle n^m \rangle = \sum_{n=0}^{\infty} n^m e^{-n\hbar\omega/k_BT} = \frac{1}{Z} \frac{\partial^m}{\partial(-x)^m} Z(x). \tag{A2}
\]

The first few moments of the photon number are listed in Table II.

**TABLE II. First four moments of the photon number \( n \) for blackbody radiation, with \( x = \hbar\omega/k_BT \).**

| \( \langle n^m \rangle \) | \( \frac{x^m}{\Gamma(x)} \) |
|-------------------------|-----------------|
| \( \langle n \rangle \)  | \( \frac{1}{x} \)  |
| \( \langle n^2 \rangle \) | \( \frac{e^x + 1}{x^2} \) |
| \( \langle n^3 \rangle \) | \( \frac{e^x + 4e^x + 1}{x^3} \) |
| \( \langle n^4 \rangle \) | \( \frac{e^{3x} + 11e^{3x} + 11e^x + 1}{x^4} \) |

**Appendix B: Thermodynamic Derivation**

Here we rederive the main result of the main text, Eq. [I], from thermodynamic principles. The total energy \( E_{\text{BBR}}(\omega) \) contained within the volume \( V \) of a blackbody cavity per unit angular frequency is

\[
E_{\text{BBR}}(\omega) = V \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_BT} - 1}. \tag{B1}
\]

The variance of \( E_{\text{BBR}}(\omega) \) is given by

\[
\sigma_{E_{\text{BBR}}(\omega)}^2 = k_BT^2 \left( \frac{\partial E_{\text{BBR}}(\omega)}{\partial T} \right) = V \frac{\hbar^2 \omega^4}{\pi^2 c^3} \left( e^{\hbar\omega/k_BT} - 1 \right)^2 \tag{B2}
\]

Because the spectral energy density \( U(\omega) = \varepsilon_0 \mathcal{E}^2 = E_{\text{BBR}}(\omega)/V \), the variance of the spectral energy density is given by

\[
\sigma_{U(\omega)}^2 = \frac{k^2 \omega^4}{V \pi^2 c^3} \sigma_{E(\omega)}^2. \tag{B3}
\]

Note that the fluctuations in \( U(\omega) \) are inversely proportional to the volume of the cavity.

Since

\[
\Delta E_i = - \int_0^\infty d\omega \frac{U(\omega)}{2\varepsilon_0} \alpha_i^2(\omega), \tag{B4}
\]
then

\[ \sigma^2 E_i = P \int_{0}^{\infty} d\omega \left| \frac{\alpha^s(\omega)}{2\varepsilon_0} \right|^2 \sigma_i^2(\omega) \]

(B5)

which matches Eq. (11) of the main text.

Appendix C: Treating a Blackbody as an Optical Cavity

Here we derive the characteristic cavity parameters for a blackbody cavity. For a cavity mode of length \( \ell \), the free spectral range is

\[ \Delta f_{\text{FSR}} = \frac{c}{2\ell}. \]  

(C1)

If a photon is emitted into a mode of length \( \ell \), the characteristic length traveled is precisely \( \ell \), since blackbodies are by definition perfect absorbers. Therefore, the spectral width of a blackbody cavity mode (Lorentzian HWHM) is

\[ \Delta f_{\text{BB}} = \frac{c}{\ell}. \]

(C2)

The finesse of a cavity is defined as \( \mathcal{F} = \Delta f_{\text{FSR}}/2\Delta f_{\text{BB}} \). For an ideal blackbody, apparently all modes have a finesse of \( \mathcal{F} = 1/4 \).

The resonant frequencies of a cubic cavity with side \( \ell \) are \( f_q = q\Delta f_{\text{FSR}} \). Here, \( q_x^2 = q_y^2 + q_z^2 \), and \( q_x, q_y, q_z \) are the number of nodes in the \( x, y, z \) dimension plus 1, excluding the boundaries. Allowed modes have \( q_x, q_y, q_z \geq 0 \), with at least one of \( q_x, q_y, q_z \geq 1 \). The quality factor of mode \( q \) is then \( Q = f_q/2\Delta f_{\text{BB}} = q/4 \).

Because each mode of the cavity has a spectral width \( \Delta f_{\text{BB}} \), the spectral density of mode \( q \) is

\[ d_q(f) = \frac{1}{\pi} \frac{f_{\text{BB}}}{(f_{\text{BB}})^2 + (f_q - f)^2} df, \]

(C3)

where the prefactor of 2 accounts for two possible polarizations per mode. A related quantity is the "mode density"

\[ D_{\text{mode}}(f) = \sum_q d_q(f)/V, \]

(C4)

which is the density of modes per unit frequency per unit volume. In the limit of \( q \gg 1 \), the density of modes is approximately a continuous function

\[ D_{\text{FS}}(f) = \frac{8\pi f^2}{c^3} df. \]

(C5)

For real materials, the finesse will generally have a frequency dependence. One can estimate the quality factor by the relation [36]

\[ Q = \frac{\text{Energy stored in cavity}}{\text{Energy lost per round trip}} = \frac{3V}{2S\delta}, \]

(C6)

where \( S \) is the surface area of the cavity (\( S = 6\ell^2 \) for a cube) and \( \delta \) is the skin depth given by [36]

\[ \delta = \sqrt{\frac{\varepsilon_0 \rho c^2}{\pi f}}. \]

(C7)

Here \( \rho \) is the material’s resistivity and \( \varepsilon_0 \) is the permittivity of free space. The resistivity \( \rho \) is also a frequency dependent parameter, but may be approximated by its DC value for sufficiently low frequency. Under this approximation, we find the finesse of a cubic cavity with walls of resistivity \( \rho \) is

\[ \mathcal{F} = \frac{1}{8\sqrt{\varepsilon_0 \rho f/\pi}}. \]

(C8)
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