Diffusive fluctuations for one-dimensional totally asymmetric interacting random dynamics

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Abstract

We study central limit theorems for a totally asymmetric, one-dimensional interacting random system. The models we work with are the Aldous-Diaconis-Hammersley process and the related stick model. The A-D-H process represents a particle configuration on the line, or a 1-dimensional interface on the plane which moves in one fixed direction through random local jumps. The stick model is the process of local slopes of the A-D-H process, and has a conserved quantity. The results describe the fluctuations of these systems around the deterministic evolution to which the random system converges under hydrodynamic scaling. We look at diffusive fluctuations, by which we mean fluctuations on the scale of the classical central limit theorem. In the scaling limit these fluctuations obey deterministic equations with random initial conditions given by the initial fluctuations. Of particular interest is the effect of macroscopic shocks, which play a dominant role because dynamical noise is suppressed on the scale we are working.

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1 Introduction

We study fluctuations in the scale of the classical central limit theorem for totally asymmetric interacting random systems in one space dimension. The model system for which we prove theorems is the Aldous-Diaconis-Hammersley process. To summarize this model in one sentence, it consists of point particles on the real line that jump to the left, at rate equal to the distance to the left neighbor, with new locations chosen uniformly at random between the jumper and its left neighbor. The idea for this process appeared in Hammersley’s classical paper [16], and Aldous and Diaconis [1] first defined it as an infinite system of interacting particles.

We consider the general nonequilibrium hydrodynamic limit situation where the limiting interface (or tagged particle, depending on one’s point of view) is governed by a Hamilton-Jacobi equation
\[ u_t + f(u_x) = 0. \]
The initial distributions can be fairly arbitrary, subject to a limit assumption on the fluctuations around the initial macroscopic profile and some moment bounds. In particular, we do not restrict to product initial distributions or particular types of initial macroscopic profiles.

The overall picture is this: the limiting fluctuation field \( \zeta(x,t) \) is governed by the linearization of the hydrodynamic equation: \( \zeta_t + f'(u_x)\zeta_x = 0 \) where \( u(x,t) \) is the deterministic limit around which the random interface fluctuates. This is a deterministic equation, and all the randomness is confined to the initial condition. The dynamics transports the initial fluctuations along the characteristics and shocks of the hydrodynamic equation. This picture of characteristics rigidly transporting fluctuations has been understood to some degree for quite a while, and has been proved in some special cases. What our paper furnishes are proofs in a general setting (but for the particular model). In addition we clarify some interesting details of this picture that are produced by the shocks of the hydrodynamic equation, such as the definition of the limiting fluctuation variable \( \zeta(x,t) \) at a shock location \((x,t)\).

The mathematical reason for the suppression of dynamical noise lies in two facts: (i) The most general evolution of Hammersley’s process can be realized as an envelope of an infinite family of simpler processes with deterministic initial conditions. This is the microscopic variational representation of the process. (ii) The results of Baik-Deift-Johansson [3] imply that these simpler processes have fluctuations of order \( n^{1/3} \) which are then swamped by the initial diffusive fluctuations of order \( n^{1/2} \).

It has been more common to use the exclusion process and its variants for mathematical theory of large scale behavior. For our purposes Hammersley’s process has one advantage over the exclusion process. The totally asymmetric versions of both processes can be conveniently coupled with simple growth models, and both processes possess particle-level variational formulations in this coupling. For Hammersley’s process this growth model is the increasing sequences model on a planar Poisson point process [1] [22]. For exclusion it is the last-passage percolation model with weakly
increasing paths on the two-dimensional square lattice\cite{24}. The advantage of Hammersley’s process comes from the fact that presently better probability estimates are available for the planar increasing sequences model than for the lattice last-passage model. In particular, Lemma 4.1(b) in our proof has not yet been proved for the exclusion model. This estimate for the increasing sequences model was proved by Baik, Deift and Johansson\cite{3} with Riemann-Hilbert techniques. Obtaining this estimate for exclusion is not simply a matter of repeating the argument, but has turned out to be a somewhat tricky problem (personal communication from J. Baik). But once this estimate for exclusion becomes available, we believe that the results of this paper can be repeated for totally asymmetric simple exclusion.

The reader can find comprehensive overviews of fluctuation results for interacting systems in\cite{15} and in Chapter 11 of\cite{19}. So we make only a few remarks here. Past work on the fluctuations of asymmetric systems has concentrated on the exclusion process. The proofs use couplings and monotonicity arguments and necessitate special initial distributions such as i.i.d. distributions or product measures with piecewise constant densities.

The deepest and most important work on the fluctuations of the asymmetric exclusion process is undoubtedly by Ferrari and Fontes. In a series of papers\cite{11},\cite{12},\cite{13} they study the fluctuations of the current and the tagged particle in equilibrium, and the fluctuations of a second class particle with shock initial conditions, given by a product measure with different densities to the left and right of the origin. In this situation the authors prove the basic feature of asymmetric fluctuations, namely the rigid transport along the characteristics. Our paper complements their work on some questions, by going into more general nonequilibrium profiles and initial distributions, and by giving more complete results on the convergence of the entire interface and the distribution-valued density fluctuation field. Our results cover tagged particles for Hammersley’s process, and thereby also the current for the stick process. The stick process is the process of increments for Hammersley’s process, and hence the displacements of Hammersley’s particles are the currents of the stick process.

Let us contrast our methods and results with those of symmetric, reversible processes. In the one-dimensional setting symmetric means that particles are equally likely to jump both left and right. The fluctuation theory of reversible interacting processes relies on methods of martingales and the Holley-Stroock theory of generalized Ornstein-Uhlenbeck processes. The limiting fluctuation fields of reversible processes are governed by equations driven by white noise. Both our methods and the qualitative results are different. We use no martingale theory. Instead our methods rely on sharp control of the paths of individual particles, and on the theory of shocks and characteristics of one-dimensional conservation laws and Hamilton-Jacobi equations. And we already highlighted the main difference, that for asymmetric systems no dynamical noise is visible on the diffusive scale.
Between symmetric and asymmetric systems are the weakly asymmetric systems where the asymmetry vanishes in the hydrodynamic limit. The central limit behavior of weakly asymmetric systems is qualitatively the same as that of symmetric systems, governed by a linear stochastic partial differential equation whose drift term is the linearization of the hydrodynamic equation. But the weakly asymmetric systems have an additional important feature proved by Bertini and Giacomin [4]: A small perturbation of a flat profile obeys, on larger space and time scales, a nonlinear stochastic equation of KPZ type. This raises the question whether such a result could be obtained for asymmetric systems at some suitable scaling.

As mentioned above, the shocks of the hydrodynamic equation turn out to have interesting effects on fluctuations. For example, a basic result one would expect is that the motion of a tagged particle converges to something related to Brownian motion. But we find that in the presence of shocks the fluctuation processes of a tagged particle are not tight in the Skorokhod space $D([0, \infty), \mathbb{R})$. We can still prove the tagged particle’s convergence to a function of Brownian motion uniformly on compact time intervals away from shocks, and even pointwise at the shocks. The limiting path has discontinuities at the shock times, and is not right-continuous (in time), but instead lower semicontinuous at the discontinuities.

We prove a distributional limit theorem for the entire interface in a weaker topology, as an element of $L^p_{\text{loc}}(\mathbb{R})$. This limiting process is a weak solution of the linearization of the hydrodynamic equation, as mentioned above. However, due to the shocks a particular version of the limiting process has to be chosen (from among the a.e. equal versions) to get a weak solution of the linearized equation. And this weak solution turns out to disagree with the pointwise distributional limit at the shocks.

Organization of the paper. Section 2 describes the particle process and the results. The last part of that section gives a rigorous construction of the process in terms of increasing sequences among Poisson points on the Euclidean plane. This is the variational coupling formulation basic for our approach. Section 3 develops properties of the characteristics and shocks of a one-dimensional conservation law. The approach here is based on the Hopf-Lax and Lax-Oleinik formulas, with the results of [21] as a starting point.

Sections 4 and 5 contain probability estimates needed for taking advantage of the variational coupling formulation. These are based on the known estimates for increasing sequences ([3], [18], [23]). The remaining sections go through the proofs of the theorems. Some technical measurability proofs are collected in an appendix at the end.

2 Results

We study the large scale behavior of the Aldous-Diaconis-Hammersley process, or Hammersley’s process for short. The state of this process is $z(t) = (z_i(t) : i \in \mathbb{Z})$ that re-
resents a countable collection of labeled point particles on \( \mathbb{R} \). The variable \( z_i(t) \in \mathbb{R} \) represents a countable collection of labeled point particles on \( \mathbb{R} \). The variable \( z_i(t) \in \mathbb{R} \) is the location of particle \( i \) at time \( t \). The particles are ordered, so that \( z_{i-1}(t) \leq z_i(t) \) for all \( i \in \mathbb{Z} \) and \( t \geq 0 \). All particles make jumps to the left, according to the following rule. Suppose the state at time \( t \) is \( z(t) = (z_i(t) : i \in \mathbb{Z}) \). To determine the next jump of particle \( i \), let \( \sigma \) be a random exponential waiting time with expectation \( (z_i(t) - z_{i-1}(t))^{-1} \). At time \( t + \sigma \) particle \( i \) jumps to its new location \( z_i(t + \sigma) \), chosen uniformly at random from the interval \( (z_{i-1}(t), z_i(t)) \). This type of event happens independently and simultaneously for all \( i \). Of course this description needs justification because infinitely many jumps happen in every positive time interval. In Section \( 2.4 \) below we give a rigorous construction of this infinite-particle dynamics in terms of increasing sequences on the plane.

Instead of thinking about a particle configuration, we can regard Hammersley’s process as a model for a 1-dimensional interface on the plane. The interface is represented by the height function \( z(t) \) defined on the integers, so that \( z_i(t) \) is the height of the interface above site \( i \). Through the jumps of the \( z_i \)'s the interface moves downward.

The stick process \( \eta(t) = (\eta_i(t) : i \in \mathbb{Z}) \) is the process of increment variables defined by

\[
\eta_i(t) = z_i(t) - z_{i-1}(t).
\]

The dynamics of \( \eta(\cdot) \) can be represented by the following generator \( \mathcal{L} \) which acts on bounded cylinder functions \( \psi \) on the product space \( [0, \infty)^\mathbb{Z} \):

\[
\mathcal{L}\psi(\eta) = \sum_{i \in \mathbb{Z}} \int_0^{\eta_i} [\psi(\eta^{u,i,i+1}) - \psi(\eta)]du
\]

where \( \eta^{u,i,i+1} \) represents the configuration after a piece of size \( u \) has been moved from site \( i \) to \( i+1 \): \( \eta^{u,i,i+1}_i = \eta_i - u \), \( \eta^{u,i,i+1}_{i+1} = \eta_{i+1} + u \), and \( \eta^{u,i,i+1}_j = \eta_j \) for \( j \neq i, i+1 \). This process can be rigorously defined on a certain subspace of the full product space \( [0, \infty)^\mathbb{Z} \), see [23] for details.

Let \( u_0 \) be a nondecreasing locally Lipschitz continuous function on \( \mathbb{R} \). It represents the initial macroscopic interface. The evolving macroscopic interface \( u(x,t), (x,t) \in \mathbb{R} \times [0, \infty) \), is the unique viscosity solution of the Hamilton-Jacobi equation

\[
u_t + f(u_x) = 0, \quad u(x,0) = u_0(x),
\]

with velocity function \( f(\rho) = \rho^2 \). Equivalently, \( u \) is defined for \( t > 0 \) by the Hopf-Lax formula

\[
u(x,t) = \inf_{y \geq x \leq t} \left\{ u_0(y) + tg \left( \frac{x-y}{t} \right) \right\}
\]

where \( g(x) = x^2/4 \) is the convex dual of \( f \). For a fixed \( t \) the partial \( x \)-derivative \( \rho(x,t) = u_x(x,t) \) exists for all but countably many \( x \). This function is the unique entropy solution of the Burgers equation

\[
u_t + f(\rho)x = 0, \quad \rho(x,0) = \rho_0(x),
\]
where \( \rho_0 = u'_0 \) (a.e. defined derivative). We cover some properties of these equations later in Section 3. See chapters 3, 10, 11 in [10] for basic theory.

Assume we have a sequence \( z^n(\cdot) \) of Hammersley’s processes, with random initial configurations \( \{z^n_i(0) : i \in \mathbb{Z}\} \), and \( n = 1, 2, 3, \ldots \) is the index of the sequence. The objective of our paper is to study the fluctuations of the random interface \( z^n_{[nt]}(nt) \) around the deterministic interface \( nu(x,t) \) in the diffusive, or central limit theorem, scale \( n^{1/2} \). The fluctuations are described by the stochastic process \( \zeta_n(x,t) \) defined for \( (x,t) \in \mathbb{R} \times [0, \infty) \) by

\[
\zeta_n(x,t) = n^{-1/2} \{ z^n_{[nt]}(nt) - nu(x,t) \}.
\]

Think of the initial process \( \{\zeta_n(y,0) : y \in \mathbb{R}\} \) as a random function with values in the Skorokhod space \( D(\mathbb{R}) \) of right-continuous functions on \( \mathbb{R} \) with left limits (RCLL functions). This space is metrized as follows. Let \( \Lambda \) be the collection of strictly increasing, bijective Lipschitz functions \( \lambda : \mathbb{R} \to \mathbb{R} \) such that

\[
\|\lambda\| = |\lambda(0)| + \sup_{x \neq y} \left| \log \frac{\lambda(x) - \lambda(y)}{x - y} \right| < \infty.
\]

For \( \alpha, \beta \in D(\mathbb{R}) \) and \( u > 0 \) let

\[
d(\alpha, \beta, \lambda, u) = \sup_{x \in \mathbb{R}} |\alpha \left( (x \wedge u) \vee (-u) \right) - \beta \left( (\lambda(x) \wedge u) \vee (-u) \right) | \wedge 1
\]

and then

\[
d_S(\alpha, \beta) = \inf_{\lambda \in \Lambda} \left[ \|\lambda\| + \int_0^\infty e^{-u} d(\alpha, \beta, \lambda, u) du \right].
\]

The metric \( d_S \) is complete and separable. Convergence \( d_S(\alpha_j, \alpha) \to 0 \) is equivalent to the existence of a sequence \( \lambda_j \in \Lambda \) such that \( \lambda_j \) converges to the identity function uniformly on compacts, and \( |\alpha_j - \alpha \circ \lambda_j| \to 0 \) uniformly on compacts. Let \( C(\mathbb{R}) \) denote the subspace of continuous functions.

Our basic hypothesis is weak convergence at time 0 to a continuous limit function:

\[
\text{There exists a } C(\mathbb{R})\text{-valued random function } \zeta_0 \text{ such that } \zeta_n(\cdot, 0) \to \zeta_0(\cdot) \text{ in distribution as } n \to \infty, \text{ on the space } D(\mathbb{R}).
\]

Assumption (7) is in fact equivalent to a stronger assumption, which is important for us so we clarify it right away. Let \( D_u(\mathbb{R}) \) be the space of RCLL functions endowed with the \( d_u \)-metric of uniform convergence on compact sets:

\[
d_u(\alpha, \beta) = \sum_{j=1}^\infty 2^{-j} \left\{ \sup_{-j \leq r \leq j} |\alpha(r) - \beta(r)| \wedge 1 \right\} \text{ for } \alpha, \beta \in D_u(\mathbb{R}).
\]

The metric \( d_u \) is much stronger than the Skorokhod metric \( d_S \), and in fact \( D_u(\mathbb{R}) \) is not even separable. But on \( C(\mathbb{R}) \) the two metrics induce the same topologies. Because the
jumps of $\zeta_n(\cdot,0)$ occur at deterministic locations and because the limit process $\zeta_0(\cdot)$ is continuous, it follows that $\zeta_n(\cdot,0)$ is measurable as a $D_u(\mathbb{R})$-valued random function, and assumption (7) is equivalent to this stronger assumption:

$$\zeta_n(\cdot,0) \rightarrow \zeta_0(\cdot) \text{ in distribution as } n \rightarrow \infty, \text{ on the space } D_u(\mathbb{R}).$$

(9)

We shall not go through the details of this point, and refer the reader to section 18 in [5].

Since the state space is large, we need a uniformity assumption. But only on one side since the dynamics is totally asymmetric.

There exists a fixed $b \in \mathbb{R}$ such that for every $\varepsilon > 0$ one can find $q$ and $n_0$ such that

$$\sup_{n \geq n_0} P \left\{ \sup_{k:k \leq nq} nk^{-2} \left( z^n_{[nt]}(0) - z^n_k(0) \right) \geq \varepsilon \right\} \leq \varepsilon.$$

(10)

Note that if (11) holds for some $b$, it holds for all $b$. It forces $u_0$ to satisfy

$$\lim_{y \rightarrow -\infty} |y|^{-2} u_0(y) = 0.$$

(11)

A consequence of assumption (3) is convergence in probability to the macroscopic interface $u_0$:

$$\lim_{n \rightarrow \infty} P \left( \sup_{y \in [a,b]} |n^{-1} z^n_{[nt]}(0) - u_0(y)| \geq \varepsilon \right) = 0.$$

(12)

This and (10) are sufficient for a hydrodynamic limit: $n^{-1} z^n_{[nt]}(nt) \rightarrow u(x,t)$ in probability as $n \rightarrow \infty$, uniformly over $(x,t)$ in compact sets. See [22].

Property (13) guarantees that there exists a nonempty compact set $I(x,t) \subseteq (-\infty, x]$ on which the infimum in (2) is achieved:

$$I(x,t) = \left\{ y \leq x : u(x,t) = u_0(y) + tg \left( \frac{x-y}{t} \right) \right\}.$$

For $t = 0$ it is convenient to have the convention $I(x,0) = \{x\}$. The minimal and maximal Hopf-Lax minimizers are

$$y^-(x,t) = \inf I(x,t) \quad \text{and} \quad y^+(x,t) = \sup I(x,t).$$

(13)

Define

$$\rho^\pm(x,t) = g' \left( \frac{x-y^\pm(x,t)}{t} \right) \quad \text{for } (x,t) \in \mathbb{R} \times (0, \infty).$$

(14)

It turns out that, for a fixed $t$, $y^-(x,t) = y^+(x,t)$ for all except at most countably many $x$. At all such points the function $\rho(x,t) = \rho^\pm(x,t)$ is defined and continuous, and is the $x$-derivative $\rho(x,t) = u_x(x,t)$ of the viscosity solution of (1). Definition (4)
is called the Lax-Oleinik formula. We say that \((x, t) \in \mathbb{R} \times (0, \infty)\) is the location of a shock if \(y^-(x, t) < y^+(x, t)\). We will not call \((x, 0)\) a shock even if the initial function \(u_0\) is nondifferentiable at \(x\).

Our first result shows that later fluctuations are close to a deterministic transformation of the initial fluctuations.

**Theorem 2.1** Suppose \(u_0\) is a locally Lipschitz continuous function. Assume \((9)\) and \((10)\).

(i) Let \(A \subseteq \mathbb{R} \times [0, \infty)\) be a compact set such that either (a) \(A\) is finite, or (b) there are no shocks in \(A\), in other words \(y^-(x, t) = y^+(x, t)\) for all \((x, t) \in A\). Then

\[
\lim_{n \to \infty} \sup_{(x, t) \in A} |\zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0)| = 0 \quad \text{in probability.} \tag{15}
\]

(ii) For all \(-\infty < a < b < \infty, \ 0 < \tau < \infty, \ \text{and} \ 1 \leq p < \infty,\)

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq \tau} \int_a^b \left|\zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0)\right|^p dx = 0 \quad \text{in probability.} \tag{16}
\]

From this theorem we deduce distributional limits for the interface and the stick profile.

### 2.1 Weak limits and the linearized equation

In assumption \((10)\) we assumed the existence of a \(C(\mathbb{R})\)-valued random function \(\zeta_0\). On the probability space of \(\zeta_0\) define random variables \(\zeta(x, t), \ (x, t) \in \mathbb{R} \times [0, \infty), \) by

\[
\zeta(x, t) = \inf_{y \in I(x, t)} \zeta_0(y). \tag{17}
\]

To formulate a process-level weak convergence result, we consider, for a fixed \(t\), the random function \(x \mapsto \zeta_n(x, t)\) as an element of the space \(L^p_{\text{loc}}(\mathbb{R})\) of functions that are locally in \(L^p\). By definition, a measurable function \(f\) on \(\mathbb{R}\) lies in \(L^p_{\text{loc}}(\mathbb{R})\) if for all \(0 < k < \infty,\)

\[
\|f\|_{L^p[-k, k]} \equiv \left(\int_{[-k, k]} |f(x)|^p \, dx\right)^{1/p} < \infty.
\]

\(L^p_{\text{loc}}(\mathbb{R})\) is a complete separable metric space under the metric

\[
d_p(f, g) = \sum_{k=1}^{\infty} 2^{-k} \left(\|f - g\|_{L^p[-k, k]} \wedge 1\right), \tag{18}
\]
and we endow $L^p_{\text{loc}}(\mathbb{R})$ with its Borel $\sigma$-algebra. We show that for a fixed $t$, $\zeta_n(\cdot, t)$ is measurable as an $L^p_{\text{loc}}(\mathbb{R})$-valued random element. And that the path $\zeta_n : t \mapsto \zeta_n(\cdot, t)$ is a measurable map from the underlying probability space into the Skorokhod space $D([0, \infty), L^p_{\text{loc}}(\mathbb{R}))$ of right-continuous $L^p_{\text{loc}}(\mathbb{R})$-valued paths with left limits at all time points $t$. Similarly the random variables $\zeta(x, t)$ defined in (17) specify an $L^p_{\text{loc}}(\mathbb{R})$-valued path $\zeta : t \mapsto \zeta(\cdot, t)$. We show that $\zeta$ is a random element of the space $C([0, \infty), L^p_{\text{loc}}(\mathbb{R}))$ of continuous paths.

**Theorem 2.2** Suppose $u_0$ is a locally Lipschitz continuous function. Assume (9) and (10).

(i) For any finitely many points $(x_i, t_i) \in \mathbb{R} \times [0, \infty)$, $1 \leq i \leq k$, we have the limit in distribution

$$
(\zeta_n(x_1, t_1), \ldots, \zeta_n(x_k, t_k)) \xrightarrow{d} (\zeta(x_1, t_1), \ldots, \zeta(x_k, t_k)) \quad \text{as } n \to \infty \quad (19)
$$

in the space $\mathbb{R}^k$.

(ii) The process $\zeta_n$ converges in distribution to the process $\zeta$ on the path space $D([0, \infty), L^p_{\text{loc}}(\mathbb{R}))$.

As one would expect, $\zeta(x, t)$ is a solution of the linearization of the Hamilton-Jacobi equation (9). For this we must choose the correct version of $\zeta$ in the a.e. sense. Let

$$
\tilde{\zeta}(x, t) = \frac{1}{2} \{ \zeta_0(y^-(x, t)) + \zeta_0(y^+(x, t)) \}.
$$

For a fixed $t$, $\tilde{\zeta}(x, t) = \zeta(x, t)$ at all $x$ except shock locations. $\tilde{\zeta}$ is a weak solution of the equation

$$
\tilde{\zeta}_t(x, t) + f'(\rho(x, t))\tilde{\zeta}_x(x, t) = 0, \quad \tilde{\zeta}(\cdot, 0) = \zeta_0(\cdot). \quad (20)
$$

This is a linear transport equation with a discontinuous coefficient. The appropriate definition of a weak solution is that, for all $\phi \in C^\infty_c(\mathbb{R} \times [0, \infty))$, $\zeta$ satisfies this integral criterion:

$$
\int_0^\infty \int_{\mathbb{R}} \phi_t(x, t)\tilde{\zeta}(x, t)dx \, dt + \int_0^\infty dt \int_{\mathbb{R}} \tilde{\zeta}(x, t)\phi'\bigl(\rho^+(\cdot, t)\bigr)\, dx(x)
$$

$$
+ \int_{\mathbb{R}} \zeta_0(x)\phi(x, 0)dx = 0. \quad (21)
$$

For each $t$, the $x$-integral in the second term is with respect to the signed measure $\mu = \mu(t)$ defined by

$$
\mu(a, b] = \phi(b, t)f'(\rho^+(b, t)) - \phi(a, t)f'(\rho^+(a, t)).
$$

For this to make sense we took the right-continuous version $\rho^+(\cdot, t)$ of $\rho(\cdot, t)$. The definition also requires that $\rho(\cdot, t)$ be locally of bounded variation, which is true by the Lax-Oleinik formula (14).

Equation (21) shows why the choice of $\tilde{\zeta}$ matters. Suppose $(r(t), t)$ is a shock location for $t_0 \leq t \leq t_1$. Then $f'(\rho(\cdot, t))$ jumps at $r(t)$ and the
measure \( \mu \) gives nonzero mass to the singleton \( \{r(t)\} \) for each \( t \). Clearly the value of the second term in (21) depends on which value \( \zeta(r(t),t) \) takes. It is a curious discord that the correct weak solution of (20) differs from the pointwise limit in (19) at the shocks. There is no dynamically generated noise in equation (20), as all the randomness is in the initial data \( \zeta_0 \). The equation expresses the point that on the diffusive scale the initial noise is transported along the characteristics, and the noise created by the dynamics is not visible because it is of lower order.

That \( \bar{\zeta} \) is a weak solution of (20) follows from this more general result. Given a convex, differentiable flux function \( f \), let \( \Theta(\lambda, \rho) \in [0, 1] \) for \( \lambda \neq \rho \) be defined by

\[
\frac{f(\lambda) - f(\rho)}{\lambda - \rho} = \Theta(\lambda, \rho)f'(\rho) + (1 - \Theta(\lambda, \rho))f'(\lambda).
\]  

(22)

Let \( \rho^\pm(x, t) \) be the functions defined by the Lax-Oleinik formula (14). Given a continuous function \( v_0 \), set for \( (x, t) \in \mathbb{R} \times (0, \infty) \) first

\[
\theta(x, t) = \Theta(\rho^-(x, t), \rho^+(x, t))
\]

and then

\[
v(x, t) = \theta(x, t)v_0(y^+(x, t)) + (1 - \theta(x, t))v_0(y^-(x, t)).
\]  

(23)

**Theorem 2.3** Suppose \( f \) is a convex flux function with convex conjugate \( g \), the minimizers \( y^\pm(x, t) \) are defined by (13), and \( \rho^\pm(x, t) \) are defined by the Lax-Oleinik formula (14). Let \( v_0 \) be an arbitrary continuous function on \( \mathbb{R} \), and define \( v \) by (23). Then \( v \) is a weak solution of the linear transport equation

\[
v_t + f'(\rho(x, t))v_x = 0, \quad v|_{t=0} = v_0,
\]  

(24)

in the sense of the integral criterion (21).

We would expect \( v \) to be the unique weak solution of (24) under some natural uniqueness criterion. Presently a uniqueness theory exists for continuous solutions of equations of this type. See Petrova and Popov [20] and their references.

For the special case \( f(\rho) = \rho^2 \) we get \( \Theta \equiv \frac{1}{2} \), which explains why we defined \( \bar{\zeta} \) as the \( \frac{1}{2}, \frac{1}{2} \) convex combination of \( \zeta_0(y^\pm(x, t)) \). Next some remarks on the hypotheses and results.

### 2.1.1 Remark

Above we chose to work with the \( x \)-right-continuous function \( \zeta_n(x, t) \) defined by (4). The reader may prefer to linearly interpolate between the point locations \( z^n_k \) to define an \( x \)-continuous random interface

\[
z_n(x, t) = (nx - \lfloor nx \rfloor) z^n_{\lfloor nx \rfloor + 1}(nt) + (\lfloor nx \rfloor + 1 - nx) z^n_{\lfloor nx \rfloor}(nt),
\]  

(25)
and then consider the $x$-continuous fluctuation process
\begin{equation}
\zeta_n^{(c)}(x,t) = n^{-1/2}\{z_n(x,t) - nu(x,t)\}.
\end{equation}

The results would be the same. In particular, assumption (1) is equivalent to $\zeta_n^{(c)}(\cdot,0) \rightarrow \zeta_0(\cdot)$ weakly in $C(\mathbb{R})$. Our estimates imply the following proposition, which shows that on the scale $n^{1/2}$ large microscopic variations in the index are not visible.

**Proposition 2.1** Suppose $0 \leq \ell = \ell(n) \leq Cn^{1/3-\delta}$ for some $C < \infty$ and $\delta > 0$. Fix $-\infty < a < b < \infty$ and $\tau < \infty$. Under assumptions (9) and (10),
\begin{equation}
\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq \tau} n^{-1/2}\{z_{k+\ell}^n(nt) - z_k^n(nt)\} = 0 \quad \text{in probability.}
\end{equation}

Under the stronger assumptions (27) and (28) of the next section, the limit above holds a.s.

We sketch the proof of this proposition in the Appendix.

2.1.2 **Remark**

In both theorems part (i) is a sharper statement for a restricted set of space-time points, and part (ii) is a weaker statement without restriction on space-time points. Let us emphasize that the limits in (15) and (19) are valid for any finite collection of points, including shock locations. For the global results, (16) and part (ii) of Theorem 2.2, we integrate over space so that the values of the processes at shocks become immaterial because the shocks are a Lebesgue null set. The same effect could be achieved by integrating over time.

2.1.3 **Remark**

The uniform convergence in (13) cannot be extended to sets that contain shocks. To see why, suppose $(x,t)$ is a shock, and suppose there are points $x_k \nearrow x$ and $x'_\ell \searrow x$ such that $(x_k,t)$ and $(x'_\ell,t)$ lie in $A$ but are not shocks. Let $y_k = y^-(x_k,t)$ and $y'_\ell = y^+(x'_\ell,t)$ be the (unique) Hopf-Lax minimizers for these points. They satisfy $y_k \nearrow y^-(x,t)$ and $y'_\ell \searrow y^+(x,t)$. Consider the $x$-continuous version $\zeta_n^{(c)}(x,t)$ defined in Remark 2.1.1. If (13) were to hold for this set $A$, then with high probability $|\zeta_n^{(c)}(x_k,t) - \zeta_n^{(c)}(y_k,0)|$ and $|\zeta_n^{(c)}(x'_\ell,t) - \zeta_n^{(c)}(y'_\ell,0)|$ are small uniformly over $k$ and $\ell$. As we let $k,\ell \rightarrow \infty$ and use the continuity of $\zeta_n^{(c)}(\cdot,t)$, we conclude that the random variable $\zeta_n^{(c)}(x,t)$ is forced to simultaneously approximate $\zeta_n^{(c)}(y^-(x,t),0)$ and $\zeta_n^{(c)}(y^+(x,t),0)$. This is impossible (except in trivial cases) because in the shock case the points $y^-(x,t)$ and $y^+(x,t)$ are macroscopically separated, and the values $\zeta_n^{(c)}(y^\pm(x,t),0)$ can differ with probability 1 with suitable choice of initial distributions.
2.2 Starting in local equilibrium

Now we take the point of view of an observer on the initial interface, whose location is taken as the origin. Furthermore, we assume that at time zero this observer sees the interface to his left and right in local equilibrium, which is an assumption on the local slopes \( \eta_n^i(0) = z_n^i(0) - z_{n-1}^i(0) \). (If we want to think of the \( z_n^i \)'s as particles, we call the \( \eta_n^i \)'s interparticle distances.) Then we can strengthen the distributional limits to almost sure limits, and give the limiting objects concrete descriptions in terms of Brownian motion.

For the precise hypotheses, let \( \rho_0 \) be a nonnegative, locally bounded measurable function on \( \mathbb{R} \). It will be the macroscopic profile of the \( \eta_n^i(0) \) variables. Assume that for some real number \( b \) (and hence for all \( b \)),

\[
\lim_{r \to -\infty} |r|^{-1} \sup_{r \leq x \leq b} \rho_0(x) = 0. \tag{27}
\]

Define a locally Lipschitz function \( u_0 \) by

\[
u_0(0) = 0, \quad u_0(x) - u_0(y) = \int_y^x \rho_0(r)dr \quad \text{for all } y < x.
\]

Let \( u(x,t) \) and \( \rho(x,t) \) be again the relevant solutions of the macroscopic equations (I) and (II). The assumption on the initial interfaces \( z_n^i(0) \) is as follows.

For each \( n \), \( z_0^n(0) = 0 \) with probability 1, and the variables \( \{\eta_n^i(0) : i \in \mathbb{Z}\} \) are mutually independent, exponentially distributed with expectations

\[
E[\eta_n^i(0)] = nu_0(i/n) - nu_0((i - 1)/n) = n \int_{(i-1)/n}^{i/n} \rho_0(x)dx. \tag{28}
\]

Note that now \( -z_0^n(t) \) is the cumulative current from site 0 for the stick process \( \eta^n(\cdot) \), in other words the total stick length that has moved across the bond \((0,1)\) during time interval \((0,t)\).

Let \( B(\cdot) \) denote a two-sided standard Brownian motion. In other words, take two independent 1-dimensional standard Brownian motions \( B_1(s) \) and \( B_2(s) \) defined for \( 0 \leq s < \infty \), and set

\[
B(s) = \begin{cases} B_1(s), & s \geq 0 \\ -B_2(-s), & s < 0. \end{cases} \tag{29}
\]

The limiting processes are defined in terms of this Brownian motion by

\[
\zeta_0(y) = B \left( \int_0^y \rho_0^2(s)ds \right)
\]

and

\[
\zeta(x,t) = \inf_{y \in I(x,t)} B \left( \int_y^x \rho_0^2(s)ds \right) = \inf_{y \in I(x,t)} \zeta_0(y). \tag{30}
\]

Note that in the above definitions the integrals are signed, in other words for \( y < 0 \)

\[
\int_y^0 \rho_0^2(s)ds = -\int_0^y \rho_0^2(s)ds \leq 0.
\]
Theorem 2.4 Assume (27) and (28). Then we can construct the processes \( \{z^n(\cdot)\} \) on a common probability space with a two-sided Brownian motion \( B(\cdot) \) so that the following almost sure limits hold.

(i) Let \( A \subseteq \mathbb{R} \times [0, \infty) \) be a compact set such that either (a) \( A \) is finite, or (b) there are no shocks in \( A \), in other words \( y^- (x, t) = y^+ (x, t) \) for all \( (x, t) \in A \). Then

\[
\lim_{n \to \infty} \sup_{(x, t) \in A} |\zeta_n(x, t) - \zeta(x, t)| = 0 \quad \text{a.s.} \tag{31}
\]

(ii) For all \( -\infty < a < b < \infty \), \( 0 < \tau < \infty \), and \( 1 \leq p < \infty \),

\[
\lim_{n \to \infty} \sup_{0 \leq t \leq \tau} \int_a^b |\zeta_n(x, t) - \zeta(x, t)|^p \, dx = 0 \quad \text{a.s.} \tag{32}
\]

2.2.1 Remark

We assumed the initial increment variables \( \{\eta^n_i(0)\} \) exponentially distributed in assumption (28) just to be concrete. It is a natural choice because i.i.d. exponential distributions are invariant for the \( \eta(\cdot) \) process so we can call (28) “local equilibrium.” But the validity of Theorem 2.4 does not depend on this special choice at all. The reader can verify that the proof works as long as the initial distribution can be embedded in Brownian motion, and the moments are sufficiently bounded so that the probabilities in (10) and (12) are summable in \( n \). However, definition (30) of \( \zeta(x, t) \) would change with different choices of initial distributions. The \( \rho^2_0(s) \) inside the integral \( \int_0^y \rho^2_0(s) \, ds \) appears because the variance of an exponential random variable is the square of the mean.

2.2.2 Moving along a characteristic from the origin

If \( y^\pm(x, t) = 0 \), which means that \( (x, t) \) is a point on a genuine characteristic (not a shock) emanating from \((0, 0)\), then \( \zeta(x, t) = 0 \) and (31) gives \( \zeta_n(x, t) \to 0 \) a.s. This tells us that \( n^{-1/2} \) is the wrong normalization. We might expect the fluctuation to be of size \( n^{1/3} \) because the situation studied by Baik, Deift and Johansson [3] is of this type. Their initial condition corresponds to setting \( z_i(0) = 0 \) for \( i \leq 0 \) and \( z_i(0) = \infty \) for \( i > 0 \). And their result can be expressed as the weak limit of \( n^{-1/3} \{z_{[nx]}(nt) - nx^2/(4t)\} \) for \( x, t > 0 \). In this situation \( u_0(x) = \infty \cdot 1_{(0, \infty)}(x) \) and \( y^\pm(x, t) = 0 \) for all \( x, t > 0 \).

On the other hand, suppose \( y^-(x, t) \leq 0 \leq y^+(x, t) \) with at least one inequality strict. Then \( (x, t) \) lies on a characteristic from the origin that is a shock. Now \( \zeta(x, t) \neq 0 \) with positive probability, and with probability 1 if \( y^-(x, t) < 0 < y^+(x, t) \). (31) says that the current across a shock has fluctuations of order \( n^{1/2} \).
2.2.3 Rarefaction fan

This means that \( y^\pm(x,t) = \bar{y} \) for a nontrivial interval of \( x \)'s. The simplest way to produce this is to take two densities \( \lambda > \rho \) and the initial profile

\[
\rho_0(y) = \begin{cases} 
\rho, & y < \bar{y} \\
\lambda, & y > \bar{y}.
\end{cases}
\]

Then \( y^\pm(x,t) = \bar{y} \) for \((x,t) \in F\) where \( F \) denotes the “fan” (cut off at \( T < \infty \) to make it compact)

\[
F = \{ (x,t) : 0 \leq t \leq T, \ \bar{y} + 2\rho t \leq x \leq \bar{y} + 2\lambda t \}.
\]

As a corollary of (31) we get

\[
\lim_{n \to \infty} \sup_{(x,t),(x',t') \in F} |\zeta_n(x,t) - \zeta_n(x',t')| = 0.
\]

So \( n^{-1/2} \) is not the right normalization for fluctuations inside a rarefaction fan, and further work is called for.

2.2.4 Shock

A shock produces discontinuous fluctuations that jump across segments of the Brownian path that represents the initial fluctuations. Consider the simplest shock case, with initial profile

\[
\rho_0(y) = \begin{cases} 
\lambda, & y < 0 \\
\rho, & y > 0,
\end{cases}
\]

where still \( \lambda > \rho \). The convex flux \( f(\rho) = \rho^2 \) preserves a downward jump. (An upward jump is smoothed out into the rarefaction fan.) At later times \( t > 0 \) the shock is located at \( x = (\rho + \lambda)t \), and the profile is given by

\[
\rho(x,t) = \begin{cases} 
\lambda, & x < (\rho + \lambda)t \\
\rho, & x > (\rho + \lambda)t.
\end{cases}
\]

The Hopf-Lax minimizers are \( y^\pm(x,t) = x - 2\lambda t \) for \( x < (\rho + \lambda)t \), \( y^\pm(x,t) = x - 2\rho t \) for \( x > (\rho + \lambda)t \), and \( I(x,t) = \{ (\rho - \lambda)t, (\lambda - \rho)t \} \) for \( x = (\rho + \lambda)t \). At macroscopic time \( t \), the limiting fluctuation process is

\[
\zeta(x,t) = \begin{cases} 
B \left( \lambda^2 (x - 2\lambda t) \right), & x < (\rho + \lambda)t \\
\min \{ B \left( \lambda^2 t(\rho - \lambda) \right), B \left( \rho^2 t(\lambda - \rho) \right) \}, & x = (\rho + \lambda)t \\
B \left( \rho^2 (x - 2\rho t) \right), & x > (\rho + \lambda)t.
\end{cases}
\]

(33)

There is a jump in \( \zeta(\cdot,t) \) at the shock \( x = (\rho + \lambda)t \), and the path may be left- or right-continuous, depending on which choice makes it lower semicontinuous. The initial fluctuation in the range \( \{ B(s) : \lambda^2 t(\rho - \lambda) < s < \rho^2 t(\lambda - \rho) \} \) disappeared from (33). Ferrari and Fontes \([11]\) show that in asymmetric exclusion this becomes the fluctuation of a second class particle.
2.2.5 A tagged particle fails to be tight in the presence of shocks

A basic question is to ask about the fluctuations of the motion of a tagged particle. In other words, fix \( x \) and consider the process \( \zeta_n(x, t) = \frac{1}{n} \left[ z^n_{nx}(nt) - nu(x, t) \right] \) as \( t \) varies in \([0, T]\). If there are no shocks in \([x] \times [0, T]\), \([31]\) gives uniform convergence to a time-changed Brownian path.

But if \((x, \sigma)\) is a shock for some \( \sigma \in (0, T)\), it turns out that the sequence of processes \( \{\zeta_n(x, \cdot)\} \) is not even tight in the Skorokhod space \( D([0, T], \mathbb{R}) \). To see this, recall this condition for tightness: for every \( \varepsilon > 0 \) there must exist a \( \delta > 0 \) such that \( P(w'_n(\delta) > \varepsilon) < \varepsilon \) for all \( n \), where \( w'_n(\delta) \) is the following modulus of continuity: \( w'_n(\delta) = \inf_{\{t_i\}} w_n(\{t_i\}) \) where the infimum is over partitions \( \{t_i\} \) of \([0, T]\) such that \( t_i - t_{i-1} > \delta \) for all \( i \), and \( w_n(\{t_i\}) = \max_i \sup \{|\zeta_n(x, s) - \zeta_n(x, t)| : s, t \in [t_{i-1}, t_i]\} \).

(See \([3], \text{Chapter 3}\) or \([1], \text{Chapter 3}\).)

Now fix \( y_0 < y^-(x, \sigma) \), a constant \( \alpha > 0 \), the event
\[
A = \{\zeta_0(y^+(x, \sigma)) < \zeta_0(y) - \alpha \text{ for } y \in [y_0, y^-(x, \sigma)]\},
\]
and \( \beta = P(A) > 0 \). The probability \( P(A) \) is positive because \( y^-(x, \sigma) < y^+(x, \sigma) \) by the assumption that \((x, \sigma)\) is a shock. Let \( \varepsilon < (\alpha \wedge \beta)/8 \), and suppose there is a \( \delta > 0 \) such that \( P(w'_n(\delta) > \varepsilon) < \varepsilon \) for all \( n \). Fix \( \tau \in (\sigma, \delta/2) \) so that \((x, \tau)\) is not a shock and so that \( y(x, \tau) \in [y_0, y^-(x, \sigma)]\). This is possible because \( t \mapsto y^-(x, t) \) is right-continuous and nonincreasing. Let \( F_n \) be the event
\[
F_n = \{w'_n(\delta) < \varepsilon, |\zeta_n(x, t) - \zeta(x, t)| \leq \varepsilon \text{ for } t = \sigma, \tau\}.
\]
By \([31]\), \( P(F_n) \geq 1 - 2\varepsilon \) for large enough \( n \), and then \( P(A \cap F_n) \geq \beta/2 > 0 \). Fix a sample point \( \omega \in A \cap F_n \). Fix a partition \( \{t_i\} \) that achieves \( w_n(\{t_i\}) < \varepsilon \) for this \( \omega \). At this \( \omega \),
\[
\zeta_n(x, \sigma) \leq \zeta(x, \sigma) + \varepsilon \leq \zeta_0(y^+(x, \sigma)) + \varepsilon \leq \zeta_0(y(x, \tau)) - \alpha + \varepsilon = \zeta(x, \tau) - \alpha + \varepsilon \leq \zeta_n(x, \tau) - 3\alpha/4.
\]
This implies that \( \sigma \) and \( \tau \) cannot lie in the same partition interval \([t_{i-1}, t_i]\), so there must be at least one partition point in \((\sigma, \tau)\). Since \( \tau - \sigma < \delta/2 \), there must be a unique partition point \( t_k \in \{t_i\} \cap (\sigma, \tau) \). Then \( w_n(\{t_i\}) < \varepsilon \) forces \( |\zeta_n(x, t) - \zeta_n(x, \sigma)| < \varepsilon \) for \( t \in [t_{k-1}, t_k) \), while \( |\zeta_n(x, t) - \zeta_n(x, \tau)| < \varepsilon \) for \( t \in [t_k, t_{k+1}) \). Combining this with the earlier inequality gives
\[
\zeta_n(x, t_k^-) \leq \zeta_n(x, t_k) - \alpha/2,
\]
which by the continuity of \( u(x, t) \) implies
\[
z^n_{nx}(nt_k^-) \leq z^n_{nx}(nt_k) - n^{1/2}\alpha/2
\]
and contradicts the basic rule that the particle \( z^n_{nx}(\cdot) \) jumps leftward.
2.3 Fluctuations for the conserved quantity

We continue assuming that the process starts in local equilibrium according to assumptions (27) and (28). In this section we consider the fluctuations of the empirical density of the stick variables \( \{ \eta_i^n(nt) \} \). Total stick length is conserved by the dynamics, as each jump of particle \( z_i \) means that a random portion is subtracted from \( \eta_i \) and added on to \( \eta_{i+1} \). Under assumptions (27) and (28) the empirical measure \( n^{-1} \sum_i \eta_i^n(nt) \delta_{i/n} \) satisfies a hydrodynamic limit. Precisely, for any finite \( a < b \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=\lfloor na \rfloor+1}^{\lfloor nb \rfloor} \eta_i^n(nt) = \int_a^b \rho(x,t)dx \quad \text{a.s. (34)}
\]

See [22]. Actually only a limit in probability is proved in [22], but the result can be strengthened under assumption (28).

The next theorem is the fluctuation theorem for this hydrodynamic limit. The result is stated for the random distribution \( \xi_n(t) \) defined below. First set

\[
\rho_i^n(t) = n \int_{(i-1)/n}^{i/n} \rho(x,t)dx = nu(i/n,t) - nu((i-1)/n,t).
\]

Then, for compactly supported test functions \( \phi \), define

\[
\xi_n(t,\phi) = n^{-1/2} \sum_{i \in \mathbb{Z}} \phi(i/n)(\eta_i^n(nt) - \rho_i^n(t)).
\]

Define another random distribution \( \xi(t) \) by

\[
\xi(t,\phi) = -\int_{-\infty}^{\infty} \phi'(x)\zeta(x,t)dx,
\]

where \( \zeta(x,t) \) is defined by (30) in terms of the Brownian motion \( B(\cdot) \).

We want to put \( \xi_n(t) \) and \( \xi(t) \) into some reasonable metric space, and a workable choice turns out to be the space \( H^{-1}_0(\mathbb{R}) \) of distributions that are locally in \( H^{-1}(\mathbb{R}) \). To explain this we need some definitions. For the reader unfamiliar with this, Chapter 9 in [14] covers enough of the theory for following our paper. Let \( \mathcal{D}' \) be the space of distributions in Schwartz’s notation. Elements \( F \in \mathcal{D}' \) are linear functionals on the space \( C_c^\infty(\mathbb{R}) \) of compactly supported infinitely differentiable functions, and they are continuous in this sense: \( F(\phi_j) \to F(\phi) \) if all derivatives of \( \phi_j \) converge uniformly to the corresponding derivatives of \( \phi \), and all \( \phi_j \) and \( \phi \) are supported on a common compact set. Distributions can be multiplied by smooth functions: if \( \chi \) is a \( C^\infty \)-function then the distribution \( \chi F \) is defined by \( \chi F(\phi) = F(\chi \phi) \).

The Sobolev space \( H^1(\mathbb{R}) \) contains those \( L^2 \)-functions \( v \) that possess a weak derivative \( v' \) in \( L^2 \). It is a separable Hilbert space with (one possible) norm

\[
\|v\|_{H^1(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})} + \|v'\|_{L^2(\mathbb{R})}.
\]
$H^{-1}(\mathbb{R})$ is the dual space of $H^1(\mathbb{R})$, and itself a separable Hilbert space. A continuous linear functional on $H^1(\mathbb{R})$ also acts continuously on $C_c^\infty(\mathbb{R})$, and consequently the elements of $H^{-1}(\mathbb{R})$ are from the space $\mathcal{D}'$. Give $H^{-1}(\mathbb{R})$ the operator norm

$$
\|F\|_{H^{-1}(\mathbb{R})} = \sup\{|F(v)| : \|v\|_{H^1(\mathbb{R})} \leq 1\}.
$$

Now we can define the space of distributions locally in $H^{-1}$:

$$
H^{-1}_{\text{loc}}(\mathbb{R}) = \{F \in \mathcal{D}': \chi F \in H^{-1}(\mathbb{R}) \text{ for all } \chi \in C_c^\infty(\mathbb{R})\}.
$$

Fix once and for all an increasing sequence of $C_c^\infty(\mathbb{R})$ functions $\chi_k$ such that

$$
1_{[-k+1,k-1]} \leq \chi_k \leq 1_{(-k,k)}.
$$

Then a distribution $F$ lies in $H^{-1}_{\text{loc}}(\mathbb{R})$ iff $\chi_k F \in H^{-1}(\mathbb{R})$ for all $k$. We metrize $H^{-1}_{\text{loc}}(\mathbb{R})$ by

$$
R(F,G) = \sum_{k=1}^{\infty} 2^{-k} \left\{ 1 \wedge \|\chi_k F - \chi_k G\|_{H^{-1}(\mathbb{R})} \right\}.
$$

Under this metric $H^{-1}_{\text{loc}}(\mathbb{R})$ is a complete separable metric space.

We shall show that the process $t \mapsto \xi_n(t)$ is a random element of the Skorokhod space $D([0,\infty),H^{-1}_{\text{loc}}(\mathbb{R}))$, and that $t \mapsto \xi(t)$ is a random element of the space $C([0,\infty),H^{-1}_{\text{loc}}(\mathbb{R}))$ of continuous $H^{-1}_{\text{loc}}(\mathbb{R})$-valued paths.

### Theorem 2.5

Assume (27) and (28). Construct the processes $\{z^n(\cdot)\}$ on a common probability space with a two-sided Brownian motion $B(\cdot)$ so that the conclusions of Theorem 2.4 are valid. Fix a finite time horizon $\tau < \infty$. Then almost surely

$$
\lim_{n \to \infty} \sup_{t \in [0,\tau]} R(\xi_n(t),\xi(t)) = 0.
$$

In particular, $\xi_n(\cdot)$ converges almost surely to $\xi(\cdot)$ on the path space $D([0,\infty),H^{-1}_{\text{loc}}(\mathbb{R}))$.

Theorem 2.5 is a corollary of Theorem 2.4 and is valid under any hypotheses that make Theorem 2.4 true. See Remark 2.2.1.

To complement the theorem, we give alternative characterizations of the limiting distribution-valued process $\xi(\cdot)$. Spohn [28, page 260] argued that the limiting fluctuations of an asymmetric conservative system should be governed by the equation

$$
\partial_t \xi + \partial_x [f'(\rho)\xi] = 0.
$$

By definition (35), $\xi(t) = \partial_x \zeta(\cdot,t)$ in the distribution sense. Formally differentiating through (20) with respect to $x$ then gives exactly equation (39). Thus we can regard
\( \xi(\cdot) \) as a distribution solution to (39). Following (21), the correct interpretation of the distribution \( \partial_x [f'(\rho(\cdot, t))\xi(t)] \) is then, applied to a test function \( \psi \in C_c^\infty(\mathbb{R}) \),

\[
\partial_x [f'(\rho(\cdot, t))\xi(t)](\psi) = \int_\mathbb{R} \tilde{\xi}(x, t)d[f'(\rho(\cdot, t))\psi'](x).
\] (40)

We can also consider \( \xi \) as a Gaussian process indexed by time and compactly supported test functions. For this we briefly introduce forward characteristics \( w^\pm(a, t) \).

These are inverse functions of \( y^\pm(x, t) \) defined in (13), themselves defined by

\[
w^-(a, t) = \inf\{x : y^+(x, t) \geq a\} \quad \text{and} \quad w^+(a, t) = \sup\{x : y^+(x, t) \leq a\}.
\]

We discuss these characteristics in Section 3. For now, we note that for a fixed \( t \), \( w^-(a, t) = w^+(a, t) \) for all but countably many points \( a \in \mathbb{R} \). As functions of \( t \), \( w^\pm(a, \cdot) \) are the minimal and maximal Filippov solutions of the initial value problem

\[
\frac{dx}{dt} = f'(\rho(x, t)), \quad x(0) = a.
\] (41)

See [6] and [21] for more about this.

Ignoring the Lebesgue null set of shocks, we can write

\[
\xi(t, \phi) = -\int_\mathbb{R} \phi'(x)B \left( \int_0^{y^+(x, t)} \rho^2_0(r)dr \right) dx
\]

which shows that \( \xi = \{\xi(t, \phi) : t \in [0, \infty), \phi \in C_c^\infty(\mathbb{R})\} \) is a mean zero Gaussian process. Its distribution is determined by the correlations \( E[\xi(s, \psi)\xi(t, \phi)] \), which we will show in Section 3 to equal

\[
E[\xi(s, \psi)\xi(t, \phi)] = \int_\mathbb{R} \psi(w(r, s))\phi(w(r, t))\rho^2_0(r)dr.
\] (42)

Here we wrote \( w(r, t) \) for the a.e. defined function that agrees with both \( w^-(r, t) \) and \( w^+(r, t) \) at a.e. \( r \), for any fixed \( t \).

Correlations (42) show that \( \xi(t, \phi) \) can be equivalently described as follows. Fix a single two-sided Brownian motion \( W(\cdot) \). For \( t \in [0, \infty) \) and \( \phi \in C_c^\infty(\mathbb{R}) \), define the random variables \( \tilde{\xi}(t, \phi) \) by the Itô integrals

\[
\tilde{\xi}(t, \phi) = \int_\mathbb{R} \phi(w(r, t))\rho_0(r)dW(r).
\] (43)

The function \( \phi(w(r, t)) \) is supported on some compact interval \( a \leq r \leq b \) so there is no problem in defining the stochastic integral (13) as a function of the increments \( \{W(r) - W(a) : a \leq r \leq b\} \). The process \( \tilde{\xi} = \{\tilde{\xi}(t, \phi) : t \in [0, \infty), \phi \in C_c^\infty(\mathbb{R})\} \) has the correlations given in (12). From this we conclude that on the product space \( \mathbb{R}^{[0, \infty) \times C_c^\infty(\mathbb{R})} \) the distributions of \( \xi \) and \( \tilde{\xi} \) are identical.
2.4 Construction of the process and the variational coupling

The purpose of this section is mainly to establish the notation. For more explanation and justification of this construction we refer to [1], [22], [23], [26]. Consider a rate one, homogeneous Poisson point process on $\mathbb{R} \times (0, \infty)$. A sequence $(x_1, t_1), (x_2, t_2), \ldots, (x_m, t_m)$ of Poisson points is increasing if

$$x_1 < x_2 < \cdots < x_m \quad \text{and} \quad t_1 < t_2 < \cdots < t_m.$$

For $(a, s), (b, t) \in \mathbb{R} \times [0, \infty)$, let $L((a, s), (b, t))$ be the maximal number of Poisson points on an increasing sequence contained in $(a, b] \times (s, t]$. Abbreviate $L((b, t)) = L((0, 0), (b, t))$. Define an inverse to $L$ by

$$L^{-1}((a, s), m, \tau) = \inf \left\{ h > 0 : L((a, s), (a + h, s + \tau)) \geq m \right\}.$$

Again abbreviate $L^{-1}(m, \tau) = L^{-1}((0, 0), m, \tau)$. The well-known laws of large numbers are

$$\lim_{s \to \infty} \frac{1}{s} L(sb, st) = 2\sqrt{bt} \quad \text{and} \quad \lim_{s \to \infty} \frac{1}{s} L([sa], st) = \frac{a^2}{4t} \quad \text{a.s.}$$

Assume given a probability space $(\Omega, \mathcal{F}, P)$ on which are defined the homogeneous Poisson point process on $\mathbb{R} \times (0, \infty)$ and an initial configuration $(z_i(0) : i \in \mathbb{Z})$ for Hammersley’s process. The process $z(t) = (z_k(t) : k \in \mathbb{Z})$ is defined by

$$z_k(t) = \inf_{i : i \leq k} \left\{ z_i(0) + L^{-1}(\tau) \right\} \quad (44)$$

for all $k \in \mathbb{Z}$ and $t > 0$. Define the state space

$$\mathcal{Z} = \left\{ z = (z_i) \in \mathbb{R}^\mathbb{Z} : z_{i-1} \leq z_i \text{ for all } i, \text{ and } \lim_{i \to \infty} i^{-2} z_i = 0 \right\}.$$

If $(z_i(0)) \in \mathcal{Z}$ a.s., then the infimum in (44) is attained at some finite $i$ and $z(t) \in \mathcal{Z}$ for all $t$ a.s. Thus (44) defines a time-homogeneous Markov process $z(\cdot)$ with state space $\mathcal{Z}$.

In this paper we work with a family of processes $\{z^n(\cdot)\}$. For each $n$ we assume the existence of some probability space $(\Omega, \mathcal{F}, P)$ that supports the initial configuration $z^n(0) = (z^n_i(0) : i \in \mathbb{Z})$ in addition to the space-time Poisson point process. On this probability space define the random variables

$$\Gamma^n_{m,i}(t) = \Gamma((z^n_i(0), 0), (a, b, s, \tau)). \quad (45)$$

Then, following (44), the processes $\{z^n(t)\}$ are defined by

$$z^n_k(t) = \inf_{i : i \leq k} \left\{ z^n_i(0) + \Gamma^n_{k-i}(t) \right\}.$$

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3 Characteristics and the Hopf-Lax formula

The proofs of this paper take advantage of the correspondence between the macroscopic and microscopic situations, and estimates on the probability that the microscopic situation deviates from the macroscopic one. First we study the macroscopic situation. Without any additional trouble, we can relax the regularity assumption on the initial interface $u_0$. We adopt this standing assumption for this section:

**Assumption 3.1** $u_0$ is a nondecreasing, left-continuous real-valued function on $\mathbb{R}$ that satisfies the left growth bound $[11]$.

We work throughout with the flux $f(\rho) = \rho^2$ with convex conjugate $g(x) = x^2/4$. Same results can be derived for any strictly convex, differentiable conjugate pair $(f, g)$. The growth bound $[11]$ would need to be tailored to the $g$ in question.

Under Assumption 3.1 the function $\Phi(y) = u_0(y) + tg((x - y)/t)$, minimized in the Hopf-Lax formula (2) over $y \in (-\infty, x]$, is lower semicontinuous and satisfies $\lim_{y \to -\infty} \Phi(y) = \infty$. Consequently the minimum in (2) is achieved at some point $y$, and the set of minimizers is compact. Define the function $u(x, t)$ by the initial condition $u(x, 0) = u_0(x)$ and by (2). Then for a fixed $t > 0$, $u(\cdot, t)$ is locally Lipschitz in $x$ (we check this below), and $x^{-2}u(x, t) \to 0$ as $x \to -\infty$. The Hopf-Lax formula can be iterated as a semigroup:

$$u(x, t) = \inf_{y : y \leq x} \left\{ u(y, s) + (t - s)g\left(\frac{x - y}{t - s}\right) \right\} \quad (46)$$

for all $0 < s < t$ and $x \in \mathbb{R}$. Define the set of minimizers in (46) by

$$I(x; s, t) = \left\{ y \leq x : u(x, t) = u(y, s) + (t - s)g\left(\frac{x - y}{t - s}\right) \right\}. \quad (47)$$

$I(x; s, t)$ is nonempty and compact. Define minimal and maximal minimizers by

$$y^- (x; s, t) = \inf I(x; s, t) \quad \text{and} \quad y^+ (x; s, t) = \sup I(x; s, t). \quad (48)$$

The following properties can be checked: If $x_1 < x_2$ then $y^+(x_1; s, t) \leq y^-(x_2; s, t)$, while if $t_1 < t_2$ then $y^+(x; s, t_2) \leq y^-(x; s, t_1)$. $y^\pm(x; s, t)$ is nondecreasing in $x$ and nonincreasing in $t$. $y^+$ is right- and $y^-$ left-continuous in $x$, while $y^+$ is left- and $y^-$ right-continuous in $t$. Consequently, for fixed $s < t$, $y^\pm(\cdot; s, t)$ have the same continuity points, they coincide on these continuity points, and $y^-(x; s, t) < y^+(x; s, t)$ iff $x$ is a discontinuity point. A similar statement holds for $y^\pm(x; s, \cdot)$ as a function of $t$, for fixed $x, s$.

Next define minimal and maximal forward characteristics by

$$w^- (a; s, t) = \sup \{ x : y^+(x; s, t) < a \} = \inf \{ x : y^+(x; s, t) \geq a \} \quad (49)$$
and
\[
w^+(a; s, t) = \sup\{x : y^+(x; s, t) \leq a\} = \inf\{x : y^+(x; s, t) > a\}. \tag{50}\]
The equalities between the alternative definitions follow from the properties of \(y^\pm(x; s, t)\). For \(w^\pm(a; s, t)\) we have these properties: nondecreasing in \(a\), nondecreasing in \(t\), \(w^+\) is right- and \(w^-\) left-continuous in \(a\), \(w^+(a_1; s, t) \leq w^-(a_2; s, t)\) for \(a_1 < a_2\). As above, for fixed \(s < t\), \(w^\pm(\cdot; s, t)\) have the same points of continuity, coincide on continuity points, and \(w^-(a; s, t) < w^+(a; s, t)\) iff \(a\) is a discontinuity point. Note the equivalence
\[
y^-(a; s, t) \leq a \leq y^+(a; s, t) \iff w^-(a; s, t) \leq x \leq w^+(a; s, t). \tag{51}\]
Note also that as a trivial consequence of the definitions, \(y^-(a; s, t) \leq y^+(a; s, t) \leq x\), and \(a \leq w^-(a; s, t) \leq w^+(a; s, t)\).

We adopt the following notational conventions. When the \(\pm\) functions coincide we write \(y^\pm(x; s, t) = y(x; s, t)\) and \(w^\pm(a; s, t) = w(a; s, t)\). When \(s = 0\) abbreviate \(y^\pm(x; 0, t) = y^\pm(x, t)\) and similarly \(y(x, t), w^\pm(a, t), w(a, t)\).

As mentioned earlier, \(u(x, t)\) is the unique viscosity solution of the Hamilton-Jacobi equation \(u_t + f(u_x) = 0\) with \(f(\rho) = \rho^2\) and initial data \(u|_{t=0} = u_0\). Set \(b(x) = g'(x) = x/2\), and define two functions \(\rho^\pm(x, t)\) by
\[
\rho^\pm(x, t) = b\left(\frac{x - y^\pm(x, t)}{t}\right) \quad \text{for } (x, t) \in \mathbb{R} \times (0, \infty). \tag{52}\]
For a fixed \(t > 0\), \(\rho^\pm\) give the one-sided \(x\)-derivatives of \(u\):
\[
\rho^\pm(x, t) = \lim_{\varepsilon \to 0^\pm} \frac{u(x + \varepsilon, t) - u(x, t)}{\varepsilon}.
\]
There is a function \(\rho\) such that \(\rho(x, t) = \rho^\pm(x, t)\) for all but countably many \(x\), because \(y^-(x, t) = y^+(x, t)\) for all but countably many \(x\) (for fixed \(t\) again). The a.e. defined function \(\rho(x, t)\) is the unique entropy solution of the Burgers equation \(\rho_t + f(\rho)_x = 0\) with initial condition given by the Radon measure \(d\rho_0(x)\). More precisely, we mean that \(\rho(x, t)\) is a weak solution in this integral sense: for all \(\phi \in C^\infty_c(\mathbb{R})\),
\[
\int_\mathbb{R} \phi(x)\rho(x, t)dx - \int_\mathbb{R} \phi(x)d\rho_0(x) = \int_0^t \int_\mathbb{R} \phi'(x)f(\rho(x, s))dxds. \tag{53}\]
Formula (52) is known as the Lax-Oleinik formula. See [10] for the textbook p.d.e. theory. The appendix in [22] develops a uniqueness theory for \(\rho(x, t)\) when the initial condition \(d\rho_0\) is a measure with singularities.

The next lemma collects some properties proved in Section 3 of Rezakhanlou [21]. In that paper the initial density profile \(\rho_0 = u'_0\) is assumed bounded and integrable, but the proofs work with at most minor modifications under our Assumption [3,4]. It

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is worthwhile to note that strict convexity and continuous differentiability of \( g(x) = x^2/4 \) are critical for many of the good properties of the characteristics utilized in this section. The reader can compare with [27] where the \( g \) function corresponding to the \( K \)-exclusion process is not known to possess these properties.

**Lemma 3.1** (a) Suppose \( 0 \leq t_1 < t_2 < t_3 \), \( y_2 \in I(x; t_2, t_3) \), and \( y_1 \in I(y_2; t_1, t_2) \). Then \( y_1 \in I(x; t_1, t_3) \) and

\[
\frac{x - y_1}{t_3 - t_1} = \frac{x - y_2}{t_3 - t_2} = \frac{y_2 - y_1}{t_2 - t_1}.
\]

In other words, the points \((y_1, t_1), (y_2, t_2)\), and \((x, t_3)\) lie on a line segment.

(b) For \( 0 \leq s < s_1 < t \),

\[
y^\pm(x; s_1, t) = \frac{s_1 - s}{t - s} x + \frac{t - s_1}{t - s} y^\pm(x; s, t).
\]

(c) For \( 0 < s < t \) and all \( a \in \mathbb{R} \), \( w^\pm(a; s, t) = w(a; s, t) \). For \( s = 0 \) and \( a \in \mathbb{R} \), \( w^\pm(a, t) = w(a, t) \) is guaranteed by

\[
\liminf_{\varepsilon \searrow 0} \frac{u_0(a + \varepsilon) - u_0(a)}{\varepsilon} \leq \limsup_{\varepsilon \searrow 0} \frac{u_0(a) - u_0(a - \varepsilon)}{\varepsilon}.
\]

(d) Suppose \( 0 \leq t_1 < t_2 < t_3 \). Then \( w^\pm(a; t_1, t_3) = w(w^\pm(a; t_1, t_2); t_2, t_3) \).

Statements (a) and (b) can be augmented as follows.

**Lemma 3.2** Let \( y_1 \in I(x_1, t_1) \). Let \( z(t) = (t/t_1)x_1 + (1 - (t/t_1))y_1 \), \( t \in [0, t_1] \), be the line segment from \((y_1, 0)\) to \((x_1, t_1)\). Then

(i) \( z(t) \in I(x_1; t_1) \) for each \( t \in [0, t_1] \), and

(ii) \( I(z(t); s, t) = \{z(s)\} \) for all \( 0 \leq s < t < t_1 \).

**Proof.** **Step 1:** we show that \( I(z(t); 0, t) = \{y_1\} \) for all \( 0 < t < t_1 \). The assumption \( y_1 \in I(x_1, t_1) \) implies that

\[
u_0(y_1) + t_1 g \left( \frac{x_1 - y_1}{t_1} \right) \leq u_0(y) + t_1 g \left( \frac{x_1 - y}{t_1} \right) \quad \text{for all } y \leq x_1.
\]

This rearranges to give

\[
u_0(y_1) - u_0(y) \leq g \left( \frac{x_1 - y}{t_1} \right) \left( \frac{x_1 - y_1}{t_1} \right) \quad \text{for all } y < y_1
\]

and

\[
u_0(y) - u_0(y_1) \leq g \left( \frac{x_1 - y_1}{t_1} \right) \left( \frac{x_1 - y_1}{t_1} \right) \quad \text{for all } y_1 < y \leq x
\]
Now let \( y \in (y_1, z(t)] \). By the definition of \( z(t) \),
\[
\frac{z(t) - y}{t} < \frac{x_1 - y}{t_1} < \frac{z(t) - y_1}{t_1} = \frac{x_1 - y_1}{t_1}.
\]

By the strict convexity of \( g(x) = x^2/4, \)
\[
g \left( \frac{x_1 - y_1}{t_1} \right) - g \left( \frac{x_1 - y}{t_1} \right) > g \left( \frac{z(t) - y_1}{t} \right) - g \left( \frac{z(t) - y}{t} \right).
\]

This combined with (56) gives
\[
u_0(y) - u_0(y_1) > g \left( \frac{z(t) - y_1}{t} \right) - g \left( \frac{z(t) - y}{t} \right)
\]
which implies that no \( y > y_1 \) can be in the minimizing set \( I(z(t);0,t) \). A similar argument that utilizes (53) rules out \( y < y_1 \), and Step 1 is complete.

**Step 2:** we show \( z(t) \in I(x_1; t_1) \).
\[
u(x_1, t_1) = u_0(y_1) + t_1 g \left( \frac{x_1 - y_1}{t_1} \right) = u_0(y_1) + tz \left( \frac{z(t) - y_1}{t} \right) + (t_1 - t)g \left( \frac{x_1 - z(t)}{t_1 - t} \right)
\]
which implies the conclusion. Above we used the line segment assumption in the form
\[
\frac{x_1 - y_1}{t_1} = \frac{z(t) - y_1}{t} = \frac{x_1 - z(t)}{t_1 - t}
\]
and then Step 1.

**Step 3:** It remains to show \( I(z(t);s,t) = \{z(s)\} \) for \( 0 < s < t < t_1 \). Now we know \( z(s) \in I(x_1; s, t_1) \) by Step 2, so we can simply repeat Step 1 for \( s > 0 \) in place of \( s = 0 \). 

We emphasize the conclusion of part (ii) of the last lemma: Along the line segment \( z(t), 0 \leq t < t_1 \), Hopf-Lax minimizers are unique.

**Lemma 3.3** (i) Let \( 0 < s < t \) and \( x_1 = w(x_0; s, t) \). Then
\[
y^-(x_1, t) \leq y^-(x_0, s) \leq y^+(x_0, s) \leq y^+(x_1, t).
\]
Conversely, if (57) holds and the middle inequality is strict, then \( x_1 = w(x_0; s, t) \).

(ii) Let \((x, t)\) and \((x_1, t_1)\) be arbitrary points in \( \mathbb{R} \times (0, \infty) \) with \( t \leq t_1 \). Suppose the open intervals \( J_{x,t} = (y^-(x, t), y^+(x, t)) \) and \( J_{x_1, t_1} = (y^-(x_1, t_1), y^+(x_1, t_1)) \) are
nonempty. Then one of two cases happens: either the intervals are disjoint, which happens if \( t = t_1 \) and \( x \neq x_1 \), or if \( t < t_1 \) and \( x_1 \neq w(x; t, t_1) \). Or \( J_{x,t} \subset J_{x_1,t_1} \), which happens if \( (x, t) = (x_1, t_1) \) or if \( t < t_1 \) and \( x_1 = w(x; t, t_1) \).

(iii) Let \( y \in I(x, t) \) and \( z(s) = \frac{s}{t} x + \left(1 - \frac{s}{t}\right) y \), \( s \in [0, t] \), be the line segment from \((y, 0)\) to \((x, t)\). Suppose \( 0 \leq s_1 < s_2 \leq t \) and \( w^-(x_1; s_1, s_2) \leq x_2 \leq w^+(x_1; s_1, s_2) \). Then the points \((x_1, s_1)\) and \((x_2, s_2)\) must be on the same side of the line segment \( z(\cdot) \). In other words, \( x_1 < z(s_1) \) implies \( x_2 \leq z(s_2) \), and \( x_1 > z(s_1) \) implies \( x_2 \geq z(s_2) \).

Proof. (i) By (57),
\[
y' = y^-(x_1; s, t) \leq x_0 \leq y^+(x_1; s, t) = y''.
\]
By Lemma 3.1(a), \( y' (y'; 0, s), y'' (y''; 0, s) \in I(x_1, t) \). By the monotonicity of \( y^\pm(\cdot, s) \),
\[
y^-(x_1; 0, t) \leq y'(y'; 0, s) \leq y^-(x_0; 0, s) \leq y^+(x_0; 0, s) \leq y''(y''; 0, s) \leq y^+(x_1; 0, t).
\]
For the converse part, if \( x_1 > w(x_0; s, t) \) then by monotonicity and the part already proved,
\[
y^-(x_1; 0, t) \geq y^+(w(x_0; s, t); 0, t) \geq y^+(x_0; 0, s).
\]
This contradicts (57) if the middle inequality of (57) is strict. Similarly rule out the case \( x_1 < w(x_0; s, t) \).

(ii) If \( t = t_1 \) then either \( x = x_1 \) or the intervals must be disjoint, because \( x < x_1 \) implies \( y^+(x, t) \leq y^-(x_1, t) \). Suppose \( t_1 > t \). If \( x_1 = w(x; t, t_1) \) then by part (i) \( (y^-(x, t), y^+(x, t)) \) is contained in \((y^-(x_1, t_1), y^+(x_1, t_1))\). On the other hand, if \( x_1 > w(x; t, t_1) \) then we have disjointness by the argument already used in part (i):
\[
y^-(x_1; 0, t_1) \geq y^+(w(x; t, t_1); 0, t_1) \geq y^+(x_0; 0, s).
\]

Similar for the remaining cases.

(iii) Let us show \( x_1 > z(s_1) \) implies \( x_2 \geq z(s_2) \). By Lemma 3.2 \( y^-(z(s_2); s_1, s_2) \leq z(s_1) < x_1 \), so
\[
x_2 \geq w^-(x_1; s_1, s_2) = \sup\{\xi : y^-(\xi; s_1, s_2) < x_1\} \geq z(s_2).
\]

Part (i) of the previous lemma has the following meaning. We say that \((x, t)\) is a shock if \( y^-(x, t) < y^+(x, t) \). By (54), this is the same as saying that the Lax-Oleinik solution \( \rho(\cdot, t) \) is discontinuous at \( x \). In fact this is the same as saying that \( \rho \) is continuous at \((x, t)\). For when the minimizer \( y(x, t) = y^\pm(x, t) \) is unique and \((x_j, t_j) \to (x, t)\), then any choice \( y_j \in I(x_j, t_j) \) satisfies \( y_j \to y(x, t) \). Inequalities (57) imply that once a shock is created, it moves along a forward characteristic and never disappears. Note though that shocks merge when characteristics merge, so the number of shocks may decrease.

Next we look at the continuity of characteristics and \( u(x, t) \).
Lemma 3.4  (a) Given $a < b$ and $T > 0$, there exists a constant $C$ such that $I(x, t) \subseteq [x - Ct^{1/2}, x]$ for all $x \in [a, b]$ and $t \in (0, T]$.

(b) Fix $0 \leq s < T < \infty$ and suppose the function $u(\cdot, s)$ on the right-hand side of (16) is locally Lipschitz in the $x$-variable. Fix $a < b$, and let $K$ be the Lipschitz constant of $u(\cdot, s)$ on the interval $[y^{-}(a; s, T), b]$. Then for all $x \in [a, b]$ and $t \in (s, T]$, $I(x; s, t) \subseteq [x - 4K(t - s), x]$.

(c) Assume $u_0$ is locally Lipschitz, in addition to assumption (11). Then $u(x, t)$ is locally Lipschitz on $\mathbb{R} \times [0, \infty)$. For any $a < b$ and $T < \infty$ there exists a constant $L = L(a, b, T)$ such that $I(x; s, t) \subseteq [x - L(t - s), x]$ for all $x \in [a, b]$ and $0 \leq s < t \leq T$.

(d) Under the original assumptions of left-continuity and (11), $u(x, t)$ is locally Lipschitz on $\mathbb{R} \times [0, \infty)$, and $\lim_{t \to 0} u(x, t) = u_0(x)$ for all $x \in \mathbb{R}$.

Proof. (a) Pick $\varepsilon > 0$ small enough so that $\sqrt{T \varepsilon} < 1/2$. Pick $M$ so that $|u_0(y)| \leq \varepsilon y^2$ for $y \leq M$. Let $(x, t) \in [a, b] \times (0, T]$. Any $y \in I(x, t)$ must satisfy $y \leq x$ and $u_0(x) \geq u_0(y) + (x - y)^2/(4t)$, from which follows

$$(x - y)^2 \leq 4(t(u_0(x) - u_0(y))) \leq 4t(C_1 + \varepsilon y^2)$$

where we picked $C_1 \geq 0$ so that $2|u_0| \leq C_1$ on $[M, b]$. Square roots and algebra give

$$x - 2\sqrt{C_1 t} \leq y + 2y\sqrt{t \varepsilon} \leq y + 2\sqrt{t \varepsilon}(|b| + |y^{-}(a, T)|),$$

from which the conclusion follows.

(b) Let $c = y^{-}(a; s, T)$. Since $y^\pm(x; s, t)$ is nonincreasing in $t$, $I(x; s, t) \subseteq [c, b]$ for all $(x, t) \in [a, b] \times (s, T]$. Any $y \in I(x; s, t)$ must satisfy $y \leq x$ and $u(x, s) \geq u(y, s) + (x - y)^2/(4(t - s))$, from which follows

$$(x - y)^2 \leq 4(t - s)(u(x, s) - u(y, s)) \leq 4(t - s)K(x - y).$$

(c) First step: to show that on a bounded interval $[a, b]$, $u(\cdot, t)$ is locally Lipschitz in the $x$-variable with Lipschitz constant independent of $t \in [0, T]$. Let $K$ be the Lipschitz constant of $u_0$ on $[y^{-}(a, T), b]$. Let $x_1 < x_2$ in $[a, b]$ and pick $y_1 \in I(x_1, t)$. Then $y_1 \in [y^{-}(x_1, t), x_1] \subseteq [y^{-}(a, T), b]$. Set $y_2 = y_1 + x_2 - x_1 \in [y_1, x_2] \subseteq [y^{-}(a, T), b]$. Then

$$0 \leq u(x_2, t) - u(x_1, t)$$

$$\leq u_0(y_2) + tg\left(\frac{x_2 - y_2}{t}\right) - u_0(y_1) - tg\left(\frac{x_1 - y_1}{t}\right)$$

$$= u_0(y_2) - u_0(y_1) \leq K|y_2 - y_1| = K|x_2 - x_1|.$$ 

Second step: to show the existence of a constant $C$ such that for all $x \in [a, b]$ and $0 \leq s < t \leq T$, $|u(x, t) - u(x, s)| \leq C|t - s|$. The two steps together imply that $u$ is locally Lipschitz.
For the second step, apply the first step to let $K$ be the common Lipschitz constant for the functions $\{u(\cdot, t) : 0 \leq t \leq T\}$ on the $x$-interval $y^-(a, T) \leq x \leq b$. Let $y = y^-(x; s, t) \in I(x; s, t)$. By part (b) of this lemma, $|x-y| \leq 4K(t-s)$. Furthermore, by Lemma $3.1$ (b),

$$y = y^-(x; s, t) = \frac{s}{t} x + \left(1 - \frac{s}{t}\right) y^-(x, t) \in [y^-(a, T), b].$$

Now we may reason as follows for $x \in [a, b]$ and $0 \leq s < t \leq T$:

$$0 \leq u(x, s) - u(x, t) = u(x, s) - u(y, s) - \frac{(x-y)^2}{4(t-s)} \leq u(x, s) - u(y, s) \leq K|x-y| \leq 4K^2(t-s).$$

We may take $C = 4K^2$ and the proof of Lipschitz continuity is complete.

By Lemma $3.1$ (b), $[y^-(a; s, T), b] \subseteq [y^-(a, T), b]$ for all $0 \leq s < T$, so a single Lipschitz constant $K$ works for all $s$ in part (b).

(d) For local Lipschitz continuity of $u$ on $\mathbb{R} \times (0, \infty)$ it suffices to show that $u(\cdot, s)$ is Lipschitz in $x$ for any fixed $s > 0$, for then we can apply part (c) to the solution obtained for $t \geq s$. Let $x_1 < x_2$ in $[a, b]$ and $y \in I(x_1, s)$.

$$0 \leq u(x_2, s) - u(x_1, s) \leq u_0(y) + sg((x_2 - y)/s) - u_0(y) - sg((x_1 - y)/s) = sg((x_2 - y)/s) - sg((x_1 - y)/s) = g'(\xi)(x_2 - x_1) \leq C(x_2 - x_1)$$

where we used the mean value theorem, and chose $C$ as an upper bound for $g'$ on the interval $[0, s^{-1}b - s^{-1}y^-(a, s)]$.

The limit $u(x, 0+) = u_0(x)$ follows from $u_0(y^+(x, t)) \leq u(x, t) \leq u_0(x)$, part (a), and left-continuity of $u_0$. 

Let $0 \leq t_0 < T$. A forward characteristic emanating from $(x_0, t_0)$ is any function $r(t)$, $t_0 \leq t \leq T$, that satisfies $r(t_0) = x_0$ and

$$w^-(r(s); s, t) \leq r(t) \leq w^+(r(s); s, t) \text{ for all } t_0 \leq s < t \leq T.$$  

(58)

Notice that this implies $r(t) = w(r(s); s, t)$ for all $0 < s < t$. Multiple forward characteristics can emanate only from a point $(a, 0)$ on the $t = 0$ line [and only if (54) fails]. For example, $w^+(a, t)$ are forward characteristics that emanate from $(a, 0)$.

**Lemma 3.5** (i) A forward characteristic $r(\cdot)$ is absolutely continuous on $[0, T]$ for any $T < \infty$, and locally Lipschitz continuous on $(0, \infty)$. If $u_0$ is locally Lipschitz, then $r(\cdot)$ is locally Lipschitz on $[0, \infty]$.

(ii) Let $A \subseteq \mathbb{R} \times [0, \infty)$ be any set of points such that no two points of $A$ have the same $t$-coordinate, and for every pair $(x_1, t_1), (x_2, t_2) \in A$, if $t_1 < t_2$ then $w^-(x_1; t_1, t_2) \leq x_2 \leq w^+(x_1; t_1, t_2)$. (If $t_1 > 0$ it follows we must have equality $x_2 = w(x_1; t_1, t_2)$.) Then there exists a forward characteristic $r(t)$, $t \geq 0$, such that all points of $A$ lie on $r(\cdot)$. 

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Proof. Part (i) is a consequence of Lemma 3.4.

For part (ii), let
\[ \tau = \inf\{ t : \text{there exists } x \in \mathbb{R} \text{ such that } (x, t) \in A \}. \]

Let \( \xi \) be such that \((\xi, \tau) \in A\) if such a point exists. If not, pick a sequence \((x_n, t_n) \in A\) such that \(t_n \rightarrow \tau\). By assumption \(x_n = w(x_{n+1}; t_{n+1}, t_n)\) for all \(n\) (note that now \(t_n > 0\) for all \(n\)) so that \(x_n \geq x_{n+1}\). Thus there is a limit \(x_n \rightarrow \xi\). This limit must be finite because \(x_1 = w(x_n; t_n, t_1)\) implies \(x_n \geq y^-(x_1; t_n, t_1) \geq y^-(x_1, t_1)\).

Let us first show that for all \((x, t) \in A\) with \(t > \tau\),
\[ w^-(\xi; \tau, t) \leq x \leq w^+(\xi; \tau, t). \] (59)

If \((\xi, \tau) \in A\) then (59) is part of the assumption. Otherwise, for large \(n\) so that \(t_n < t\), the assumption gives \(x = w(x_n; t_n, t)\) from which follows
\[ y^-(x; t_n, t) \leq x_n \leq y^+(x; t_n, t). \]

Let \(n \rightarrow \infty\) and apply Lemma 3.3(b) to get
\[ y^-(x; \tau, t) \leq \xi \leq y^+(x; \tau, t) \]
which implies (59).

Suppose first \(\tau > 0\). Pick any \(y \in I(\xi, \tau)\) and define \(r(\cdot)\) by
\[ r(t) = \begin{cases} (t/\tau)\xi + (1 -(t/\tau))y, & t \in [0, \tau] \\ w(\xi; \tau, t), & t \in (\tau, \infty). \end{cases} \]

Properties (58) are satisfied by Lemmas 3.1(d) and 3.2.

If \(\tau = 0\) set \(r(0) = \xi\) and for every \(s > 0\) set \(r(s) = w(x; t, s)\) where \((x, t)\) is an arbitrary point in \(A\) such that \(0 < t < s\). Such points exist since \(\tau = 0\). \(r(s)\) is well defined because if \(t_1 < t_2 < s\) and \((x_1, t_1), (x_2, t_2) \in A\), then \(w(x_2; t_2, s) = w(w(x_1; t_1, t_2); t_2, s) = w(x_1; t_1, s)\) by assumption and by Lemma 3.1(d). This same Lemma 3.1(d) implies that \(r(\cdot)\) satisfies the defining properties (58).

By Lemma 3.3(i), \(r'(t)\) exists at a.e. \(t\), and \(r\) is the integral of its derivative. Next, a formula for the derivative. Let
\[ h(x, t) = \begin{cases} f'(\rho(x, t)), & \text{if } y^-(x, t) = y^+(x, t) \\ \frac{f(\rho^+(x, t)) - f(\rho^-(x, t))}{\rho^+(x, t) + \rho^-(x, t)}, & \text{if } y^-(x, t) < y^+(x, t). \end{cases} \] (60)

**Theorem 3.1** For any forward characteristic \(r(\cdot)\), \(r'(t) = h(r(t), t)\) for Lebesgue a.e. \(t\).

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Proof. Let $t_1 > t_0 > 0$ and $x_1 = w(x_0; t_0, t_1)$. Let $r(t) = w(x_0; t_0, t)$ be the forward characteristic emanating from $(x_0, t_0)$.

Case 1. If $(x_0, t_0)$ and $(x_1, t_1)$ are not shocks, the proof on p. 136–137 of [2] shows that $(x_1 - x_0)/(t_1 - t_0) = f'(\rho(x_0, t_0)) = f'(\rho(x_1, t_1))$.

Case 2. Suppose $(x_0, t_0)$ is a shock. Abbreviate $\rho^\pm = \rho^\pm(x_0, t_0)$. Let $\varepsilon > 0$. Take $t_1$ close enough to $t_0$ and $a < x_0 < b$ close enough to $x_0$ so that, for $(x, t) \in [a, b] \times [t_0, t_1]$,

$$|\rho^\pm(x, t) - \rho^-| < \varepsilon \quad \text{if} \ x < r(t), \text{and}$$

$$|\rho^\pm(x, t) - \rho^+| < \varepsilon \quad \text{if} \ x > r(t).$$

This can be achieved because $\rho^\pm(x, t) = b((x - y^\pm(x, t))/t), b = g'$ is continuous, and $y^\pm(x, t) \to y^-(x, t_0)$ as $(x, t)$ approaches $(x_0, t_0)$ with $t \geq t_0$ and $x < r(t)$. Decrease $t_1$ further towards $t_0$ so that $a < x_0 \leq r(t) \leq x_1 < b$ for $t \in [t_0, t_1]$.

Now apply (53) to a test function $\phi \in C_\infty^c(\mathbb{R})$ compactly supported inside $(a, b)$ and such that $\phi \equiv 1$ on $[x_0, x_1], \phi' \geq 0$ on $(a, x_1]$ and $\phi' \leq 0$ on $[x_1, b)$. After a calculation this gives

$$\frac{x_1 - x_0}{t_1 - t_0} = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-} + O(\varepsilon)$$

provided the ratio $(b - a)/(t_1 - t_0)$ is kept bounded. Letting $t_1 \searrow t_0$ shows that the right derivative $r'(t_0^+)$ at $t_0$ is given by $h(r(t_0), t_0)$.

The only type of point $(x_0, t_0)$ not covered by Cases 1 and 2 is such that $(x_0, t_0)$ is not a shock but $(x_1, t_1)$ is a shock for every $t_1 > t_0$. There can be only one such point on any forward characteristic. These two calculations suffice to prove the theorem, because from absolute continuity we know $r'(t)$ exists almost everywhere. \[\square\]

From this theorem one concludes a result of Dafermos [3] and Rezakhanlou [21]: A forward characteristic is a Filippov solution of the initial value problem (41). We shall not discuss this point further as we make no use of it. But we will use the formula $(d/dt)w^\pm(q, t) = h(w^\pm(q, t), t)$.

At certain stages in the proofs we need to know that an expression such as

$$\sup_{x \in [a, b], t \in [0, T]} \left| \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \right|$$

is a measurable random variable. We want to argue that the supremum can be taken over a countable set $\{(x_k, t_k)\}$. Such an attempt will reveal as troublesome those points $(x, t)$ for which the Hopf-Lax formula (3) possesses more than two minimizers. [In other words, $I(x, t)$ does not coincide with $\{y^\pm(x, t)\}$, which is either a singleton or a two-point set.] We show here that such points are at most countable.

**Theorem 3.2** Let $T > 0$, and

$$\mathcal{U} = \{(x, t) \in \mathbb{R} \times (0, T] : I(x, t) \neq \{y^\pm(x, t)\}\}.$$
The set \( \mathcal{U} \) is countable.

**Proof.** If \((x,t) \in \mathcal{U}\), then necessarily \(y^+(x,t) - y^-(x,t) > 0\). Thus it suffices to prove the countability of the set
\[
\mathcal{U}_\alpha = \{(x,t) \in \mathcal{U} : y^+(x,t) - y^-(x,t) \geq \alpha\}
\]
for an arbitrary \(\alpha > 0\). The countability of \(\mathcal{U}_\alpha\) will be achieved by showing that on any bounded set \([a,b] \times (0,T]\), the points of \(\mathcal{U}_\alpha\) lie on a finite collection of forward characteristics, and each characteristic contains at most countably many \(\mathcal{U}_\alpha\)-points.

**Step 1.** Fix a forward characteristic \(r(\cdot)\). We show that \(r(\cdot)\) contains at most countably many points from \(\mathcal{U}\), so in particular, at most countably many points from \(\mathcal{U}_\alpha\). We do this by associating to each point \((r(t),t) \in \mathcal{U}\) a nonempty open interval \(J_t\) so that these \(J_t\)'s are pairwise disjoint.

Let \(t_1 > 0\) be such that \((r(t_1),t_1) \in \mathcal{U}\). This implies that the minimizing set \(I(r(t_1),t_1)\) contains at least three points \(y^-(r(t_1),t_1) < y_1 < y^+(r(t_1),t_1)\). Let \(\zeta = \{(z(t),t) : 0 \leq t \leq t_1\}\) be the line segment from \((y_1,0)\) to \((r(t_1),t_1)\). By Lemma 3.3, the curve \(\{(r(t),t) : 0 \leq t \leq t_1\}\) must lie entirely on one side of \(\zeta\). Also, no \((z(t),t) \in \zeta\) lies in \(\mathcal{U}\) for \(t \in (0,t_1)\). This is because by Lemma 3.3(ii) the minimizer set \(I(z(t),t)\) is the singleton \(\{y_1\}\), while every \(\mathcal{U}\)-point has multiple minimizers. Thus one of these cases holds:

- **Case I:** \(r(t) < z(t)\) for all \(t \in (0,t_1)\) such that \((r(t),t) \in \mathcal{U}\).
- **Case II:** \(r(t) > z(t)\) for all \(t \in (0,t_1)\) such that \((r(t),t) \in \mathcal{U}\).

In **Case I** set \(J_{t_1} = (y_1,y^+(r(t_1),t_1))\), and in **Case II** set \(J_{t_1} = (y^-(r(t_1),t_1),y_1)\). In the special case where \((r(t),t) \notin \mathcal{U}\) for all \(t \in (0,t_1)\) we may set \(J_{t_1}\) either one of the two alternatives.

For the pairwise disjointness it suffices to show that \(J_{t_1} \cap J_t = \emptyset\) for any \(t < t_1\) such that \((r(t),t),(r(t_1),t_1) \in \mathcal{U}\). In **Case I** \(r(t) < z(t)\), and Lemma 3.3(ii) implies
\[
y^-(r(t),t) < y^+(r(t),t) \leq y^-(z(t),t) = y_1.
\]
Consequently \(J_{t_1} \cap J_t = \emptyset\) because \(J_t \subseteq (y^-(r(t),t),y^+(r(t),t))\) and \(J_{t_1} = (y_1,y^+(r(t_1),t_1))\). Similarly in **Case II**.

**Step 2.** We show that in a bounded set \([a,b] \times (0,T]\), all the \(\mathcal{U}_\alpha\)-points lie on a finite collection \(\{r_j(\cdot) : 1 \leq j \leq M\}\) of forward characteristics.

Let
\[
\Lambda = \{U \subseteq \mathcal{U}_\alpha : U \subseteq [a,b] \times (0,T], \text{ and for any two distinct points } (x,t), (x',t') \in U, \text{ the intervals } (y^-(x,t),y^+(x,t)) \text{ and } (y^-(x',t'),y^+(x',t')) \text{ are disjoint}\}.
\]
Note that for each \((x, t) \in \mathcal{U}_\alpha\), the interval \((y^-(x, t), y^+(x, t))\) is a subinterval of \([y^-(a, T), b]\) of length at least \(\alpha\). Consequently no set \(U \in \Lambda\) contains more than \((b - y^-(a, T))/\alpha\) points. Let \(M = \max\{|U| : U \in \Lambda\}\) be the maximal number of points in any element of \(\Lambda\). Fix \(U_0 \in \Lambda\) that has \(M\) points,

\[
U_0 = \{(x_j, t_j) : 1 \leq j \leq M\}.
\]

For each \(j\), define the forward characteristic emanating from \((x_j, t_j)\) by \(r_j(t_j) = x_j\) and \(r_j(t) = w(x_j; t_j, t)\) for \(t \in (t_j, \infty)\). And let \(A_j\) be the open triangle with vertices \((x_j, t_j), (y^-(x_j, t_j), 0),\) and \((y^+(x_j, t_j), 0)\). By Lemma 3.1(b) the sides of this triangle are formed by the line segments \(\{(y^\pm(x_j; t, t_j), t) : 0 < t < t_j\}\). Hence

\[
A_j = \{(x, t) : 0 < t < t_j, y^-(x_j; t, t_j) < x < y^+(x_j; t, t_j)\}.
\]

As an intermediate conclusion we claim that each \(\mathcal{U}_\alpha\)-point in \([a, b] \times (0, T]\) lies either on the forward characteristic from some \((x_j, t_j)\), or in some \(A_j\). To justify this, note first that no \(\mathcal{U}_\alpha\)-point can lie on a side \(\{(y^\pm(x_j; t, t_j), t) : 0 < t < t_j\}\) of a triangle \(A_j\) because by Lemma 3.2(ii) such a point has a unique minimizer. Secondly, if a \(\mathcal{U}_\alpha\)-point \((x, t)\) lies outside the closure of the union of \(\{r_j(\cdot), A_j\}\), then by Lemma 3.3(i)–(ii) the open interval \((y^-(x, t), y^+(x, t))\) is disjoint from all \((y^-(x_j, t_j), y^+(x_j, t_j))\). Then we can add \((x, t)\) to \(U_0\), thereby contradicting the definition of \(M = |U_0|\) as the maximal size of an element of \(\Lambda\).

To complete Step 2 it remains to argue that we can extend the definition of \(r_j(\cdot)\) to \([0, t_j]\) so that all \(\mathcal{U}_\alpha\)-points in \(A_j\) lie on \(r_j(\cdot)\).

**Final claim.** There cannot exist two \(\mathcal{U}_\alpha\)-points \((x, t)\) and \((x', t')\) in \(A_j\) such that \(t' > t\) but \(x' \neq w(x; t, t')\).

Suppose such points did exist. But then \((y^-(x, t), y^+(x, t))\) and \((y^-(x', t'), y^+(x', t'))\) are disjoint. Furthermore, as subintervals of \((y^-(x_j, t_j), y^+(x_j, t_j))\) they are both disjoint from all the other intervals \((y^-(x_i, t_i), y^+(x_i, t_i)), i \neq j\). We can contradict the maximality of \(M = |U_0|\) by replacing \((x_j, t_j)\) with \((x, t)\) and \((x', t')\). This proves the final claim.

By the final claim, we can apply Lemma 3.2(ii) to get a single characteristic \(r_j(\cdot)\) that contains all \(\mathcal{U}_\alpha\)-points in \(A_j\) and those on the forward characteristic from \((x_j, t_j)\). This completes the proof of Step 2 and thereby the proof of Theorem 3.2. \(\blacksquare\)

Another technical result we need is that all the shocks in a compact set can be enclosed in an open set with small \(t\)-sections.

**Proposition 3.1** Fix \(-\infty < a < b < \infty, 0 < \tau < \infty, \) and \(\varepsilon > 0\). Then there exists an open set \(G \subseteq \mathbb{R} \times (0, \infty)\) such that \(G\) contains all the shocks in \([a, b] \times (0, \tau]\), and for each \(t \in (0, \tau]\), the \(t\)-section \(G_t = \{x : (x, t) \in G\}\) has \(1\)-dimensional Lebesgue measure \(|G_t| \leq \varepsilon\).
The following notion will be helpful for the proof: Say a shock \((x, t)\) is a new shock if there does not exist a shock \((x_0, t_0)\) such that \(t_0 < t\) and \(x = w(x_0; t_0, t)\). There can be at most countably many new shocks because if \((x, t)\) and \((x', t')\) are new shocks, the open intervals \((y^-(x, t), y^+(x, t))\) and \((y^-(x', t'), y^+(x', t'))\) must be disjoint by Lemma 3.3(ii).

**Proof of Proposition 3.4.** Fix a point \(c < y^-(a, \tau)\). Let \(\{(x_i, t_i) : i \geq 1\}\) be the (at most countably many) new shocks in \([a, b] \times (0, \tau]\). Let \(S\) be the set of all shocks in \([c, b+1] \times (0, \tau + 1]\). Define \(\Delta y(x, t) = y^+(x, t) - y^-(x, t)\), so that \(\Delta y(x, t) > 0\) iff \((x, t)\) is a shock. Let \(\varepsilon_1 < (\varepsilon/2) \cdot (b - y^-(c, \tau))^{-1}\). Write \(B(x, \delta)\) for the Euclidean ball in \(R^2\) centered at \(x\) with radius \(\delta\). Define the following subset of \(R^2\):

\[
H = \left\{ \left( x - \varepsilon_1 \Delta y(x, t), x + \varepsilon_1 \Delta y(x, t) \right) \times \{ t \} \right\} \cup \left\{ \bigcup_{i \geq 1} B((x_i, t_i), 2^{-i-2} \varepsilon) \right\}.
\]

We claim that every shock in \([a, b] \times (0, \tau]\) is an interior point of \(H\). This is clear for new shocks. If \((x_1, t_1)\) is a non-new shock, we can find a shock \((x_0, t_0)\) such that \(0 < t_0 < t\) and \(x_1 = w(x_0; t_0, t_1)\). Since we chose \(c < y^-(a, \tau) \leq y^-(x_1, t_1)\), the shock \((x_0, t_0)\) and the forward characteristic \(w(x_0; t_0, t)\) for \(t_0 \leq t \leq t_1\) lie in \(S\). Furthermore, since \(S\) contains the shocks in \([x_1, b+1] \times [t_1, \tau + 1]\), we can choose \(t_2 > t_1\) so that \(S\) contains the forward characteristic \(r(t) \equiv w(x_0; t_0, t)\) for \(t_0 \leq t \leq t_2\). Let \(h = \varepsilon_1 \Delta y(x_0, t_0) > 0\). By (67), \(\varepsilon_1 \Delta y(r(t), t) \geq h\) for \(t_0 \leq t \leq t_2\). Consequently \(H\) contains the set \(\bigcup_{t_0 < t < t_2} (r(t) - h, r(t) + h) \times \{t\}\). This latter contains an open neighborhood of \((x_1, t_1)\) because \(r(\cdot)\) is a Lipschitz curve by Lemma 3.3(i).

Let \(G\) be the interior of \(H\). Then for \(t \in (0, \tau]\)

\[
G_t \subseteq \left\{ \bigcup_{x:(x,t) \in S} (x - \varepsilon_1 \Delta y(x, t), x + \varepsilon_1 \Delta y(x, t)) \right\} \cup \left\{ \bigcup_{i \geq 1} (x_i - 2^{-i-2} \varepsilon, x_i + 2^{-i-2} \varepsilon) \right\},
\]

and consequently

\[
|G_t| \leq \sum_{c \leq x \leq b} 2 \varepsilon_1 \Delta y(x, t) + \sum_{i \geq 1} 2^{-i-1} \varepsilon \\
\leq 2 \varepsilon_1 (y^+(b, t) - y^-(c, t)) + \varepsilon/2 < \varepsilon.
\]

The inequalities above follow because, as \(x \in [c, b]\) ranges over the shock locations with time coordinate \(t\), the open intervals \((y^-(x, t), y^+(x, t))\) are disjoint subintervals of \((y^-(c, t), y^+(b, t))\), which itself is a subinterval of \((y^-(c, \tau), b)\). ■
4 Estimates for increasing sequences

We have the following bounds on $L$ and $\Gamma$.

**Lemma 4.1** Suppose $a$, $s$ and $h$ are positive real numbers.

(a) For $x \geq 2$, define

$$I(x) = 2x \cosh^{-1}(x/2) - 2\sqrt{x^2 - 4}.$$  

When $x > 0$ is small enough, there is a constant $C$ such that $I(2 + x) \geq Cx^{3/2}$. For any $C$, $I(x) \geq Cx$ for large enough $x$. For all real $b > 0$ and $m \geq 2b$,

$$P\{L(b,b) \geq m\} \leq \exp\left(-bI(m/b)\right).$$  \hspace{1cm} (61)

(b) There are fixed positive constants $B_0$, $B_1$, $d_0$, $C_0$ and $C_1$ such that if $a \geq B_0$ and $B_1a^{4/3} \leq hs \leq d_0a^2$, then

$$P\left\{\Gamma([a],s) > \frac{a^2}{4s} + h\right\} \leq C_0\exp\left\{-C_1s^3h^3/a^4\right\}.$$  

(c) There are finite positive constants $C_0$ and $C_1$ such that for all $0 < a \leq s$,

$$P\{\Gamma([a],s) > s\} \leq C_0\exp(-C_1s^2).$$

Part (a) was first proved by Kim \[18\]. Seppäläinen \[23\] proved that $I(x)$ is the correct rate function for the deviations in (61). Part (b) is a consequence of Lemma 7.1(iv) in Baik-Deift-Johansson \[3\]. [See Lemma 5.2 in \[26\] for the conversion of Baik-Deift-Johansson’s lemma into part (b) above.] Part (c) is a consequence of Lemma 2.2 in Johansson \[17\].

Next we use these inequalities to derive estimates tailored to our needs. Most technical complications arise from the need to treat small $t$ that vanish as $n \to \infty$, in order to get the $t$-uniformity of the theorems. When using Lemma 4.1, it is often useful to note that $L(a,b) \overset{d}{=} L\left(\sqrt{ab},\sqrt{ab}\right)$ (\(\overset{d}{=} \) means equality in distribution). This follows from the invariance of the homogeneous planar Poisson point process under the maps $(x,y) \mapsto (rx,r^{-1}y)$, $r > 0$.

**Lemma 4.2** (a) Let $\beta, \tau > 0$. Then there exists a constant $\alpha \in (0, \infty)$ such that

$$\sum_{n \geq 1} P\left\{\Gamma([\alpha nt^{1/2}], nt) \leq n\beta \right\} \text{ for some } t \in [n^{-2}(\log n)^2, \tau]\} < \infty.$$
(b) Let $b, \delta, \gamma, \tau > 0$. Assume these restrictions: $\gamma < 3/4$ and $\gamma(1 + \delta) < 1$. Then there exists a constant $\alpha \in (0, \infty)$ such that
\[
\sum_{n \geq 1} n P \left\{ \Gamma([a \gamma], nt) \leq b \alpha nt^{1/2} \text{ for some } t \in \left[ n^{-(1+\delta)}, \tau \right] \right\} < \infty.
\]

(c) Let $b, \delta, \gamma, \eta, \tau > 0$. Assume these restrictions: $1/2 < \gamma < 3/4$, $\gamma(1 + \delta) < 1$, and $\delta < (3 - 4\gamma - 2\eta)/(4\gamma - 2)$. Then there exists a constant $\alpha \in (0, \infty)$ such that
\[
\sum_{n \geq 1} n P \left\{ \Gamma([a \gamma], nt) \leq b n^{1/2+\eta} \text{ for some } t \in \left[ n^{-(1+\delta)}, \tau \right] \right\} < \infty.
\]

**Proof.** Part (a). Set $t_i = 4^i n^{-2}(\log n)^2$ for $i \geq 0$. Pick $K$ so that $t_{K-1} < \tau \leq t_K$. Then $K \leq C \log n$ for a constant $C$. Let $\alpha_0 = \alpha/2$ to account for the effect of replacing the integer $[a \gamma]$ by $a \gamma$. Suppose $\alpha_0 > 6\beta^{1/2}$. Pick $a_1$ so that $I(x) \geq a_1 x$ for $x \geq 3$. Increase $\alpha_0$ further so that $a_1 \alpha_0 \geq 2$. Use Lemma 4.1(a) to bound the probability:
\[
P \left\{ \Gamma([a \gamma], nt) \leq n \beta \text{ for some } t \in \left[ n^{-2}(\log n)^2, \tau \right] \right\}
\leq
P \left\{ L(n, nt) \geq \alpha_0 nt^{1/2} \text{ for some } t \in \left[ n^{-2}(\log n)^2, \tau \right] \right\}
\leq
\sum_{i=0}^{K-1} P \left\{ L(n, nt_{i+1}) \geq \alpha_0 nt_i^{1/2} \right\}
=\sum_{i=0}^{K-1} P \left\{ L \left( n(\beta t_{i+1})^{1/2}, n(\beta t_{i+1})^{1/2} \right) \geq \alpha_0 nt_i^{1/2} \right\}
\leq\sum_{i=0}^{K-1} \exp \left\{ -n(\beta t_{i+1})^{1/2} I \left( \alpha_0 (4\beta)^{-1/2} \right) \right\}
\leq K \exp \left\{ -a_1 \alpha_0 \log n \right\} \leq C n^{-2} \log n.
\]

This bound is summable over $n$.

Part (b). Follow a similar partition argument, with $t_i = 4^i n^{-(1+\delta)}$. Take $\alpha$ sufficiently large, $\alpha_0 = \alpha/2$, and use Lemma 4.1(a) to get the upper bound
\[
P \left\{ \Gamma([a \gamma], nt) \leq b \alpha nt^{1/2} \text{ for some } t \in \left[ n^{-(1+\delta)}, \tau \right] \right\}
\leq
\sum_{i=0}^{K-1} P \left\{ L \left( b \alpha t_{i+1}^{1/2}, nt_{i+1} \right) \geq \alpha_0 nt_i^{1/2} \right\}
\leq\sum_{i=0}^{K-1} \exp \left\{ -n(\beta a)^{1/2} t_{i+1}^{3/4} I \left( (1/2)(\alpha/b)^{1/2} 4^{-\gamma t_{i+1}^{3/4}} \right) \right\}
\leq\sum_{i=0}^{K-1} \exp \left\{ -(1/2) n a_1 \alpha t_i^{1/2} \right\} \leq C \log n \exp(-C n^{1-\gamma(1+\delta)}).
To get the inequality in part (b), multiply this bound by \( n \) and use the assumption \( \gamma(1 + \delta) < 1 \).

Part (c). With the same partition as in part (b),

\[
\sum_{i=0}^{K-1} P \left\{ L \left( b n^{1/2 + \eta}, n t_{i+1} \right) \geq a_0 n t_i^2 \right\}
\leq \sum_{i=0}^{K-1} \exp \left\{ -n^{3/4 + \eta/2} (b t_{i+1})^{1/2} I \left( (1/2) a_0 b^{-1/2} n^{1/4 - \eta/2} t_i^{\gamma - 1/2} \right) \right\}.
\]

Replace \( t_i \) by its lower bound \( n^{-(1+\delta)} \) inside \( I \) to get \( n^{1/4 - \eta/2} t_i^{\gamma - 1/2} \geq n^{1/4 - \eta/2 - (1+\delta)(\gamma - 1/2)} \).

This last exponent is positive by the assumption on \( \delta, \eta, \gamma \). Now proceed as above.

\[ \Box \]

**Lemma 4.3** Let \( r, \tau > 0 \) be positive constants. Then there exist finite positive constants \( C_0 \) and \( C_1 \) such that, for all \( n \geq 1 \),

\[
\sum_{m=1}^{[nr]} P \left\{ \Gamma(m, nt) \leq m^2 - \frac{m^{4/3} \log n}{4nt} \right\} \text{ for some } t \in [n^{-2}, \tau] \leq C_0 n^{5/3} \exp \left( -C_1 (\log n)^{3/2} \right).
\]

**Proof.** It suffices to consider \( m^{2/3} > \log n \), otherwise the probability is 0. As in the previous proof, partition the time interval \([n^{-2}, \tau]\) by \( n^{-2} = t_0 < t_1 < \cdots < t_{K-1} < \tau \leq t_K \), and bound the probability by Lemma 4.1(a):

\[
P \left\{ \Gamma(m, nt) \leq m^2 - \frac{m^{4/3} \log n}{4nt} \right\} \text{ for some } t \in [n^{-2}, \tau] \leq P \left\{ L \left( m^2(4nt)^{-1}(1 - m^{-2/3} \log n), nt \right) \geq m \right\} \text{ for some } t \in [n^{-2}, \tau] \leq \sum_{i=0}^{K-1} P \left\{ L \left( m^2(4nt_i)^{-1}(1 - m^{-2/3} \log n), nt_{i+1} \right) \geq m \right\} \leq \sum_{i=0}^{K-1} \exp (-b_i I(m/b_i)) ,
\]

where we wrote

\[
b_i = \frac{m}{2} \left( \frac{t_{i+1}}{t_i} \right)^{1/2} (1 - m^{-2/3} \log n)^{1/2}.
\]

We argue separately for two ranges of \( m \).

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\textbf{Case 1}: \( \log n < m^{2/3} \leq (1 + \delta) \log n \) for a small \( \delta \in (0, 1/4) \). Define the partition by \( t_i = 4^i t_0 \). Check that then \( m/b_i \geq \delta^{-1/2} \). Pick a constant \( a_0 \) so that \( I(x) \geq a_0 x \) for \( x \geq \delta^{-1/2} \). (This makes sense because \( \delta^{-1/2} > 2 \).) The size \( K \) of the partition satisfies \( K \leq C \log n \) for a constant \( C \) that depends on \( \tau \). Summing the bound \( \exp (-b_i I(m/b_i)) \leq \exp (-a_0 m) \) over \( m \) and \( i \) gives the following upper bound:

\[
\sum_{m: \log n < m^{2/3} \leq (1 + \delta) \log n} \sum_{i=0}^{K-1} e^{-a_0 m} \leq C_0 (\log n)^{5/2} \exp \left(-C_1 (\log n)^{3/2}\right)
\]

for suitable constants \( C_0, C_1 \).

\textbf{Case 2}: \( (1 + \delta)^{3/2} (\log n)^{3/2} \leq m \leq nr \). Set \( \theta = 1 + (1/2)m^{-2/3} \log n \), and define the partition by \( t_i = t_0 \theta^i \). Now \( K \leq C m^{2/3} \). Note that

\[
\frac{m}{b_i} = 2 \left( \frac{t_i}{t_{i+1}} \right)^{1/2} \left( 1 - \frac{\log n}{m^{2/3}} \right)^{-1/2} \geq 2 \left( \frac{t_i}{t_{i+1}} \right)^{1/2} \left( 1 + \frac{\log n}{2m^{2/3}} \right)
\]

\[
= 2 \left( 1 + \frac{\log n}{2m^{2/3}} \right)^{1/2} \geq 2 + \frac{\log n}{4m^{2/3}},
\]

where we used the inequalities \( (1 - h)^{-1/2} \geq 1 + h/2 \) and \( (1 + h)^{1/2} \geq 1 + h/4 \) that are valid for \( 0 \leq h < 1 \). Choose \( a_1 \) so that \( I(2 + x) \geq a_1 x^{3/2} \) for \( 0 < x < 1/4 \). Check that \( b_i \geq C m \) for a constant \( C = C(\delta) \). Then finally the bound becomes

\[
\sum_{i=0}^{K-1} \exp (-b_i I(m/b_i)) \leq K \exp \left\{ -C m \left( \frac{\log n}{4m^{2/3}} \right)^{3/2} \right\} \leq C m^{2/3} \exp \left(-C_1 (\log n)^{3/2}\right).
\]

Summing this over \( (1 + \delta)^{3/2} (\log n)^{3/2} \leq m \leq nr \), and combining with Case 1 above, gives the bound in the statement of the lemma. \( \blacksquare \)

\textbf{Lemma 4.4} Let \( \tau > 0 \). There exist finite positive constants \( M_0, n_0, C_0, \) and \( C_1 \) such that, for all \( n \geq n_0 \) and \( m \geq M_0 (\log n)^{3/2} \),

\[
P \left\{ \Gamma(m, nt) > \frac{m^2}{4nt} + \frac{m^{4/3} \log n}{4nt} \text{ for some } t \in [n^{-1}, \tau] \right\} \leq C_0 m^{2/3} \exp \left(-C_1 (\log n)^3\right).
\]

\textbf{Proof}. Fix \( n \geq 3, m \geq 1 \). Set

\[
\theta = \frac{2m^{2/3} + \log n}{2m^{2/3}} > 1.
\]
Consider a partition of \([n^{-1}, \tau]\), defined by \(t_0 = n^{-1}, t_i = t_0 + \theta^i\) for \(i \geq 1\), and choose \(K\) so that \(t_{K-1} < \tau \leq t_K\). Then \(K \leq Cm^{2/3}\) for a constant \(C = C(\tau)\).

Bound the probability in question from above by
\[
\sum_{i=0}^{K-1} P\left\{ \Gamma(m, nt_i) > \frac{m^2}{4nt_i+1} + \frac{m^{4/3}\log n}{4nt_i+1} \right\}.
\]

To apply Lemma 4.1(b) to each of these probabilities, identify \(a = m, s = nt_i\), and \(h = m^{4/3}\log n(4nt_i+1)^{-1} - m^2(4n)^{-1}(t_i^{-1} - t_{i+1}^{-1})\). Check that there exist \(n_0\) and \(M_0\) such that, if \(n \geq n_0\) and \(m \geq M_0(\log n)^{3/2}\), then the hypotheses of Lemma 4.1(b) are met. Observe that \(hs = (8\theta)^{-1}m^{4/3}\log n\). Applying the estimate in Lemma 4.1(b) to each term of the sum above gives an upper bound of
\[
C_0K\exp\left( -C_1(sh)^{3/4}a^{-4} \right) \leq C_0 m^{2/3}\exp(-C_1(\log n)^3),
\]
where the constants \(C_i\) are no longer the original ones from Lemma 4.1(b).

The range of \(m\)'s not covered by the last lemma are taken care of by the next statement.

**Lemma 4.5** Let \(\tau, \alpha, M\) be positive constants. Then there exist finite positive constants \(n_0, C_0\) and \(C_1\) such that, for all \(n \geq n_0\),
\[
P\left\{ \Gamma(m, nt) > n^{\alpha} \text{ for some } t \in [n^{-1}, \tau] \text{ and } 1 \leq m \leq M(\log n)^{3/2} \right\} \leq C_0 \exp(-C_1n^{\alpha}).
\]

**Proof.** Since \(\Gamma(m, nt)\) is nonincreasing in \(t\) and nondecreasing in \(m\), the probability is bounded by
\[
P\left\{ \Gamma([M(\log n)^{2/3}], 1) > n^{\alpha} \right\}
= P\left\{ \Gamma([M(\log n)^{2/3}], n^{\alpha/2}) > n^{\alpha/2} \right\} \leq C_0 \exp(-C_1n^{\alpha}).
\]
We used the equality in distribution \(L(a, b) \overset{d}{=} L((ab)^{1/2}, (ab)^{1/2})\) and Lemma 4.1(c).

**5 Estimates for the microscopic variational formula**

Recall that for the \(n\)th process the variational coupling equality, appropriately scaled, reads
\[
z^n_{[nx]}(nt) = \inf_{i: i \leq [nx]} \{z^n_i(0) + \Gamma^{n,i}_{[nx]-1}(nt)\}. \tag{62}
\]
Let \(i_n(x,t)\) denote the minimal \(i\) at which the infimum is attained in (62):

\[
i_n(x,t) = \inf \{ i : z^n_{[nx]}(nt) = z^n_i(0) + \Gamma^{n,i}_{[nx]-i}(nt) \}. \tag{63}
\]

It is proved in [22] that under assumption (10) \(i_n(x,t)\) is almost surely finite, and it is a nonincreasing function of \(t\). For \(t = 0\) we interpret \(\Gamma^{n,i}_{[nx]-i}(0)\), and consequently on the event \(\{ [nx] \in (\alpha, \beta) \}\), and also from \([na] \geq [nx] \geq [nb]\). The technical key to benefiting from (62) lies in estimating how far the \(n^{-1}\)-scaled random minimizing indices of (62) lie from the set \(I(x,t)\) of macroscopic minimizers. We start with a crude bound, and successively refine it.

**Lemma 5.1** For \(c < a < b\) and \(\tau > 0\) define the events

\[
G_n = \{ \text{for some } x \in [a,b] \text{ and } t \in (0,\tau), (62) \text{ is minimized by some } i \leq cn \}. \tag{64}
\]

For fixed \(a < b\) and \(\tau\) we can choose \(c < 0\) such that this holds:

(i) Under the uniformity assumption (10), \(\lim_{n \to \infty} P(G_n) = 0\).

(ii) Under assumptions (27) and (28), \(\sum_{n=1}^{\infty} P(G_n) < \infty\).

**Proof.** Since \(i_n(x,t)\) is nonincreasing in \(t\), we can express (64) as

\[
G_n = \{ \text{for some } x \in [a,b] \text{ and } t = \tau, (62) \text{ is minimized by some } i \leq cn \}.
\]

Let \(C_1 > 0\), and pick \(C_0 > 0\) so that \(I(x) \geq C_1 x\) for \(x \geq C_0\) [recall Lemma 4.1(a)]. Pick \(\varepsilon > 0\) small enough so that \(C_0^2 \tau \varepsilon < 1\). And then pick \(c < 0\) so that \(c < a (1 - C_0 \sqrt{\tau \varepsilon})^{-1}\) and \(c < q\) for the \(q\) that satisfies assumption (10) for \(\varepsilon\). Abbreviate \(Y_{n,i} = z^n_{[nb]}(0) - z^n_{[na]}(0)\). Under these conditions \(i \leq nc\) implies \(\varepsilon i^2 \leq ([na] - i)^2/(C_0^2 \tau)\), and consequently on the event

\[
A_n = \{ i^{-2} n Y_{n,i} \leq \varepsilon \text{ for all } i \leq nc \}
\]

we have

\[
([na] - i) \cdot (n \tau Y_{n,i})^{-1/2} \geq C_0 \quad \text{for all } i \leq nc. \tag{65}
\]

Now we can estimate:

\[
P(G_n) \leq P \left\{ \text{for some } i \leq cn, \ L \left( (z^n_i(0),0), (z^n_{[nb]}(0), n \tau) \right) \geq [na] - i \right\} \nonumber
\]

\[
\leq P(A_n^c) + \sum_{i \leq nc} E \left\{ 1_{A_n} \cdot \exp \left\{ -(n \tau Y_{n,i})^{-1/2} I \left( ([na] - i) (n \tau Y_{n,i})^{-1/2} \right) \right\} \right\} \nonumber
\]

\[
\leq P(A_n^c) + \sum_{i \leq nc} \exp \left[ -C_1 ([na] - i) \right].
\]

The first inequality above comes from \(z^n_i(0) + \Gamma^{n,i}_{[nx]-i}(n \tau) \leq z^n_{[nx]}(0)\) which must follow if \(i\) is to be a minimizer in (62), and also from \([na] \leq [nx] \leq [nb]\). The second inequality comes from (64), and the last from (63) and \(I(x) \geq C_1 x\).
Assumption (10) implies that \( P(A_n^c) \to 0 \), so part (i) of the lemma is proved. To get the summability \( \sum P(G_n) < \infty \) required in part (ii), we need to check that assumptions (27) and (28) imply \( \sum P(A_n^c) < \infty \).

Let \( \rho^*(r) = \sup_{r \leq x \leq b} \rho_0(x) \). Write

\[
P(A_n^c) = P\left( \sum_{i=j+1}^{[nb]} \eta_i^n(0) > \frac{\varepsilon j^2}{n} \text{ for some } j \leq nc \right).
\]

Since the variables \( \{\eta_i^n(0) : j < i \leq [nb]\} \) are independent exponentials with means bounded by \( \rho^*(j/n) \), they are stochastically dominated by \( \{\rho^*(j/n)X_i : j < i \leq [nb]\} \) where the \( X_i \)'s are i.i.d. exponential variables with common mean \( EX_i = 1 \). Recall that for \( s > 1 \) we have the large deviation bound

\[
P\left( \sum_{i=1}^{m} X_i \geq ms \right) \leq \exp\left( -m\kappa(s) \right)
\]

with the rate function \( \kappa(s) = s - 1 - \log s \).

Thus

\[
P(A_n^c) \leq \sum_{j \leq nc} P\left( \sum_{i=j+1}^{[nb]} X_i > \frac{\varepsilon j^2}{n\rho^*(j/n)} \right) \leq \sum_{j \leq nc} \exp\left\{ -([nb] - j)\kappa(s_{n,j}) \right\},
\]

where

\[
s_{n,j} = \frac{\varepsilon j^2}{n([nb] - j)\rho^*(j/n)}.
\]

By assumption (27) we can guarantee \( s_{n,j} \geq M \) for an arbitrarily large \( M \), for all \( n \) and \( j \), by taking \( c < 0 \) large enough negative. Then \( \sum P(A_n^c) < \infty \) follows, and the lemma is proved.

**Lemma 5.2** Let \( a < b \), \( \alpha > 0 \), \( \tau > 0 \), \( \gamma \in (1/2, 3/4) \). Suppose \( \delta > 0 \) satisfies \( \gamma(1 + \delta) < 1 \) and \( \delta < (3 - 4\gamma)/(4\gamma - 2) \). Define the events

\[H_{n,0} = \left\{ \text{for some } x \in [a,b] \text{ and } t \in [n^{-1+\delta}, \tau], \right.\]

\[
\left. \text{[62] is minimized by some } i < [nx] - \alpha nt^\gamma \right\}
\]

and

\[H_{n,1} = \left\{ \text{for some } x \in [a,b] \text{ and } t \in (0, n^{-1+\delta}], \right.\]

\[
\left. \text{[62] is minimized by some } i < [nx] - \alpha n^{(1-\delta)/2} \right\}.
\]

If \( \alpha \) is chosen large enough, the following is true:

(i) Under assumptions (9) and (10), \( \lim_{n \to \infty} P(H_{n,0} \cup H_{n,1}) = 0 \).

(ii) Under assumptions (27) and (28), \( \sum_{n=1}^{\infty} P(H_{n,0} \cup H_{n,1}) < \infty \).
Proof. To see that $H_{n,0}$ is a measurable event, let $T$ be a countable dense subset of $[n^{-(1+\delta)}, \tau]$ that contains $\tau$. Then almost surely

$$H_{n,0} = \bigcup_{k=[na]} \bigcup_{t \in T} \bigcup_{i<k} \{z_{k}^{n}(nt) = z_{i}^{n}(0) + \Gamma_{k-i}^{n,i}(nt)\}.$$ 

To see why it suffices to consider only $t \in T$ in the union above, note that as functions of $t$, $z_{k}^{n}(nt)$ and $\Gamma_{k-i}^{n,i}(nt)$ are right-continuous jump processes whose jumps do not accumulate with probability 1. So almost surely, for any $t$ there exists a (random) $\varepsilon > 0$ such that these processes do not jump in $(t, t+\varepsilon)$. A similar argument works for $H_{n,1}$ also.

We prove statements (i) and (ii) first for $H_{n,0}$. The challenge here is in the small values of $t$ that vanish as $n \to \infty$. The proof will be achieved in two rounds. First we rule out minimizers $i \leq nx - \alpha nt^{1/2}$, and then in the second step we rule out $i \leq nx - \alpha nt^{\gamma}$. By conditioning on the event $G_{n}$ of Lemma 5.1, it suffices to consider $i \geq nc$.

Suppose some $i \in [nc, nx - \alpha nt^{1/2}]$ minimizes (62) for some $x \in [a, b]$ and $t \in [n^{-(1+\delta)}, \tau]$. Since $[nx]$ is among the indices over which the infimum is taken in (62), it must follow that

$$z_{i}^{n}(0) + \Gamma_{[nx]-i}^{n,i}(nt) \leq z_{[nx]}^{n}(0) \leq z_{[nb]}^{n}(0).$$

Bound the left-hand side from below:

$$z_{i}^{n}(0) + \Gamma_{[nx]-i}^{n,i}(nt) \geq z_{[nc]}^{n}(0) + \Gamma_{[nx]-i}^{n,[nc]}(nt) \geq z_{[nc]}^{n}(0) + \Gamma_{[\alpha nt^{1/2}]}^{n,[nc]}(nt).$$

The consequence is that for some $t \in [n^{-(1+\delta)}, \tau]$,

$$\Gamma_{[\alpha nt^{1/2}]}^{n,[nc]}(nt) \leq z_{[nb]}^{n}(0) - z_{[nc]}^{n}(0).$$

Let $\beta = u_{0}(b) - u_{0}(c) + 1$, and $G_{n}$ be the event in Lemma 5.1. Then the previous reasoning gives

$$P \left\{ \text{for some } x \in [a, b] \text{ and } t \in [n^{-(1+\delta)}, \tau], \right. \left. \text{(62) is minimized by some } i \leq nx - \alpha nt^{1/2} \right\} \leq P(G_{n}) + P \left\{ \Gamma \left([\alpha nt^{1/2}], nt \right) \leq n \beta \text{ for some } t \in [n^{-(1+\delta)}, \tau] \right\} + P \left\{ z_{[nb]}^{n}(0) - z_{[nc]}^{n}(0) > n \beta \right\}.$$

Note that $\gamma(1+\delta) < 1$ forces $\delta < 1$, and then $n^{-(1+\delta)} > n^{-2}(\log n)^{2}$ for large $n$. Thus Lemma 4.2(a) applies, and we can conclude that the second probability after
the inequality above is summable over \( n \geq 1 \) if \( \alpha \) is chosen large enough. The last probability converges to zero in Case (i), and is summable over \( n \) in Case (ii) of the lemma.

Now condition on the event that all minimizers satisfy \( i \geq nx - \alpha nt^{1/2} \), for \((x, t)\) in the range under consideration. Under this condition,

\[
H_{n,0} \implies \text{for some } x \in [a, b], \ t \in [n^{-(1+\delta)}, \tau], \ i \in [nx - \alpha nt^{1/2}, nx - \alpha nt^2];
\]

\[
z^n_\delta(0) + \Gamma_{[nx]}^{\alpha n,i}(nt) \leq z^n_{[nx]}(0)
\]

\[
\implies \text{for some } x \in [a, b], \ t \in [n^{-(1+\delta)}, \tau],
\]

\[
\Gamma_{[\alpha nt^2]}^{n,[nx-\alpha nt^{1/2}]}(nt) \leq z^n_{[nx]}(0) - z^n_{[nx-\alpha nt^{1/2}]}(0).
\]

By the definition of \( \zeta_n \), we can write

\[
z^n_{[nx]}(0) - z^n_{[nx-\alpha nt^{1/2}]}(0)
\]

\[
= n \left( u_0(x) - u_0(x - \alpha t^{1/2}) \right) + n^{1/2} \left( \zeta_n(x, 0) - \zeta_n(x - \alpha t^{1/2}, 0) \right)
\]

\[
\leq C \alpha nt^{1/2} + 2n^{1/2} \cdot \sup_{x \in [c, b]} \zeta_n(x, 0),
\]

where \([c, b]\) is an interval that contains \([x - \alpha t^{1/2}, x]\) for all \((x, t)\) under consideration, and \(C\) is the Lipschitz constant for \(u_0\) on the interval \([c, b]\). By assumption \( \delta < (3 - 4\gamma)/(4\gamma - 2) \), so we can pick a small \(\eta > 0\) so that \( \delta < (3 - 4\gamma - 2\eta)/(4\gamma - 2) \).

Now summarize everything in this upper bound:

\[
P(H_{n,0}) \leq P(G_n) + P \left\{ \Gamma \left( \alpha nt^{1/2}, nt \right) \leq n\beta \text{ for some } t \in [n^{-(1+\delta)}, \tau] \right\}
\]

\[
+ P \left\{ z^n_{[nt]}(0) - z^n_{[nc]}(0) > n\beta \right\}
\]

\[
+ P \left\{ \Gamma_{[\alpha nt^2]}^{n,[nx-\alpha nt^{1/2}]}(nt) \leq C \alpha nt^{1/2} + 2n^{1/2+\eta} \text{ for some } x \in [a, b], \ t \in [n^{-(1+\delta)}, \tau] \right\}
\]

\[
+ P \left\{ \sup_{x \in [c, b]} \zeta_n(x, 0) > n\eta \right\}
\]

\[
\equiv P(G_n) + p_{n,1} + p_{n,2} + p_{n,3} + p_{n,4}.
\]

By Lemma 4.2(a)–(c), \( \sum p_{n,j} < \infty \) for \( j = 1, 3 \). Note that for \( p_{n,3} \) we have to sum over the superscript \([nx - \alpha nt^{1/2}]\) as \( x \) varies over \([a, b]\). This gives \( O(n) \) terms, which is why the probabilities in Lemma 4.2(b)–(c) are multiplied by \( n \). Under assumption (9), \( \lim_{n \to \infty} p_{n,k} = 0 \) for \( k = 2, 4 \). Under assumption (8) and local boundedness of \( \rho_0 \), \( \sum p_{n,k} < \infty \) for \( k = 2, 4 \). This proves the lemma for the event \( H_{n,0} \).

Repeat the first step for \( H_{n,1} \). Notice that \( \Gamma_{[\alpha nt^2]}^{n,[nx-\alpha nt^{1/2}]}(nt) \) is nonincreasing in \( t \), so we can replace \( t \) by its upper bound \( n^{-(1+\delta)} \). Then we get

\[
P(H_{n,1}) \leq P(G_n) + P \left\{ \Gamma \left( \alpha nt^{1-\delta), nt^2} \leq n\beta \right\}
\]

\[
+ P \left\{ z^n_{[nt]}(0) - z^n_{[nc]}(0) > n\beta \right\}.
\]
These probabilities are handled as above. Note that the next to last probability is the special case $t = n^{-1+\delta}$ of the event in Lemma 4.2(a).

As usual, the distance between a point $x$ and a set $A$ is denoted by $\text{dist}(x, A) = \inf\{|x - y| : y \in A\}$.

**Lemma 5.3** Let $A \subseteq \mathbb{R} \times [0, \infty)$ be a compact set. Assume that $A$ satisfies either assumption (a) or (b):
(a) $A$ is a finite set; or
(b) there are no shocks in $A$, in other words $y^-(x, t) = y^+(x, t)$ for all $(x, t) \in A$.
For $\delta > 0$, define the events $H_n = H_n(\delta)$ by

$$H_n = \{\text{for some } (x, t) \in A, \text{ (62) is minimized by some } i \text{ such that dist}(n^{-1}i, I(x, t)) > \delta\}.$$  

(i) Under assumptions (9) and (10), $\lim_{n \to \infty} P(H_n) = 0$.
(ii) Under assumptions (27) and (28), $\sum_{n=1}^{\infty} P(H_n) < \infty$.

**Proof.** Measurability of $H_n$ is obvious for a finite $A$. We prove the measurability of $H_n$ for the other case in the appendix.

Fix finite $a < b$ and $\tau > 0$ so that $A \subseteq [a, b] \times [0, \tau]$. For small enough $\sigma > 0$, $I(x, t) \subseteq [x - \delta/2, x]$ for all $x \in [a, b]$ and $t \in (0, \sigma]$ by Lemma 3.4, so Lemma 5.2 gives the conclusion for $0 < t \leq \sigma$. Thus for the proof we can assume that $A \subseteq [a, b] \times [\sigma, \tau]$ where $0 < \sigma < \tau$. The important point here is bounding $t$ away from 0 because the estimation gets harder if $t \to 0$ as $n \to \infty$.

Choose $c < 0$, $c < y^-(a, \tau) \leq a$ so that Lemma 5.1 is satisfied. By that Lemma we only need to consider minimizers in the range $[nc, nb]$. Let

$$I(x, t)^{(\delta)} = \{q : |q - y| < \delta \text{ for some } y \in I(x, t)\}$$

be the $\delta$-neighborhood of $I(x, t)$. Set

$$\varepsilon = \frac{1}{5} \cdot \inf\{u_0(y) + tg((x - y)/t) - u(x, t) : (x, t) \in A, y \in [c, x] \setminus I(x, t)^{(\delta)}\}.$$  

We claim that $\varepsilon$ is a positive quantity if $A$ satisfies one of the two assumptions (a) or (b) in the statement of the lemma. This is clear if $A$ is finite. Suppose next that $y^\pm(x, t) = y(x, t)$ for all $(x, t) \in A$, but $\varepsilon = 0$. Pick a sequence $(x_j, t_j)$ in $A$ and $y_j \in [c, x_j] \setminus I(x_j, t_j)^{(\delta)}$ so that

$$u_0(y_j) + t_j g((x_j - y_j)/t_j) - u(x_j, t_j) \to 0 \quad \text{as } j \to \infty.$$  

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Pass to a convergent subsequence \((x_j, t_j, y_j) \to (\bar{x}, \bar{t}, \bar{y})\) with \((\bar{x}, \bar{t}) \in A\). By continuity,
\[
u_0(\bar{y}) + \bar{t} g((\bar{x} - \bar{y})/\bar{t}) - u(\bar{x}, \bar{t}) = 0
\]
which implies that \(\bar{y}\) must be the Hopf-Lax minimizer for \((\bar{x}, \bar{t})\), in other words \(\bar{y} = y(\bar{x}, \bar{t})\). On the other hand, we also have \(y(x_j, t_j) \to y(\bar{x}, \bar{t})\) and \(|y_{j} - y(\bar{x}, t_{j})| \geq \delta\), so in the \(j \to \infty\) limit \(|\bar{y} - y(\bar{x}, \bar{t})| \geq \delta\). This contradiction shows that \(\varepsilon > 0\).

Note that we cannot make this argument in case \(y^-(\bar{x}, \bar{t}) \neq y^+(\bar{x}, \bar{t})\), because then it is perfectly possible that \(y_j \to \bar{y} = y^-(\bar{x}, \bar{t})\) while \(y(x_j, t_j) \to y^+(\bar{x}, \bar{t})\) without contradicting \(|\bar{y} - y^+(\bar{x}, \bar{t})| \geq \delta\). This is the step where the proof of a uniform limit fails for a compact set with shocks. Of course, Remark 2.1.3 already showed that we cannot hope to prove a uniform limit for such a set.

For each \(x \in [a, b], t \in [\sigma, \tau]\), choose finitely many points \(a_k = a_k(x, t)\) and \(b_k = b_k(x, t), 1 \leq k \leq K = K(x, t)\), so that
\[
[c, x] \setminus I(x, t)^{(\delta)} = \bigcup_{k=1}^{K} [a_k, b_k]
\]
and
\[
\left|tg\left(\frac{x-b_k}{t}\right) - tg\left(\frac{x-a_k}{t}\right)\right| \leq \varepsilon
\]
for each \(k = 1, \ldots, K\). To do this, pick \(\delta_1 \in (0, \delta)\) so that \(|tg(q/t) - tg(r/t)| < \varepsilon\) for all \(q, r \in [0, b - c], t \in [\sigma, \tau]\), such that \(|q - r| \leq \delta_1\). Then pick a partition \(c = y_0 < y_1 < \cdots < y_m = b\) with mesh \(\max(y_{i+1} - y_i) < \delta_1\). Every connected component of \(I(x, t)^{(\delta)}\) is an open interval of length at least \(2\delta\), so each \(\{[y_i, y_{i+1}] \cap [c, x]\} \setminus I(x, t)^{(\delta)}\) is either empty or a closed interval. Let \([a_k, b_k] : 1 \leq k \leq K\) be the collection of the nonempty ones among the intervals \(\{([y_i, y_{i+1}] \cap [c, x]) \setminus I(x, t)^{(\delta)} : 0 \leq i \leq m - 1\}\).

For each \((x, t) \in A\), choose a point \(y_{x,t} \in I(x, t)\). Reason as follows:

- for some \((x, t) \in A\), \((\bar{y}, \bar{t})\) is minimized by some \(i\) such that \(n^{-1}i \in [c, x] \setminus I(x, t)^{(\delta)}\)
  \(\implies\) for some \((x, t) \in A\), \((\bar{y}, \bar{t})\) is minimized by some \(i\) such that \(n^{-1}i \in [a_k, b_k]\) for some \(1 \leq k \leq K\)
  \(\implies\) for some \((x, t) \in A\) and \(1 \leq k \leq K\),
    \[
    z^n_{[na_k]}(0) + \Gamma^n_{[nx]-[nb_k]}(nt) \leq z^n_{[ny_{x,t}]}(0) + \Gamma^n_{[nx]-[ny_{x,t}]}(nt)
    \]
  \(\implies\) for some \((x, t) \in A\) and \(1 \leq k \leq K\),
    - either \(z^n_{[na_k]}(0) < nu_0(a_k) - n\varepsilon\),
    - or \(\Gamma^n_{[nx]-[nb_k]}(nt) < ntg((x - b_k)/t) - n\varepsilon\),
    - or \(z^n_{[ny_{x,t}]}(0) > nu_0(y_{x,t}) + n\varepsilon\),

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or \( \Gamma_{n}^{[n_{y},t]}(nt) > ntg((x - y_{x,t})/t) + n\varepsilon \)

\[ \Rightarrow \text{ for some } y \in [c, b], \left| z_{n_{y}}^{0} - nu_{0}(y) \right| > n\varepsilon, \text{ or for some } x \in [a, b], t \in [\sigma, \tau], y \in [c, x], \left| \Gamma_{n}^{[n_{y},t]}(nt) - ntg((x - y)/t) \right| \geq n\varepsilon. \]

The next to last implication above followed from the choice of \( \varepsilon \), because

\[ u_{0}(a_{k}) + tg \left( \frac{x - b_{k}}{t} \right) \geq u_{0}(a_{k}) + tg \left( \frac{x - a_{k}}{t} \right) - \varepsilon \]

\[ \geq u_{0}(y_{x,t}) + tg \left( \frac{x - y_{x,t}}{t} \right) + 4\varepsilon. \]

The entire argument can be summarized in this bound:

\[ P(H_{n}) \leq P(G_{n}) + P(H_{n,0}) + P(H_{n,1}) \]

\[ + P \left( \left| z_{n_{y}}^{0} - nu_{0}(y) \right| > n\varepsilon \text{ for some } y \in [c, b] \right) \]

\[ + P \left( \left| \Gamma_{n}^{[n_{y},t]}(nt) - ntg((x - y)/t) \right| \geq n\varepsilon \text{ for some } x \in [a, b], y \in [c, x], t \in [\sigma, \tau] \right). \]

Apply the assumptions and previous lemmas to treat the terms on the right-hand side above. The probabilities of \( \Gamma_{n}^{[n_{x},[n_{y}]}(nt) \) are handled by Lemmas 1.3 and 1.4.

Assumption (3) of weak convergence of \( n^{-1/2} \{ z_{n_{y}}^{0} - nu_{0}(y) \} \) to a \( y \)-continuous process in the topology of uniform convergence on compact sets of \( y \)'s guarantees that

\[ \lim_{n \to \infty} P \left( \left| z_{n_{y}}^{0} - nu_{0}(y) \right| > n\varepsilon \text{ for some } y \in [c, b] \right) = 0. \]

Under assumptions (27) and (29) use elementary large deviation estimates after a partitioning: if \( c = b_{0} < b_{1} < \cdots < b_{k} = b \) is a fine enough partition, monotonicity of both \( z_{n_{y}}^{0} \) and \( nu_{0}(y) \), and the Lipschitz continuity of \( u_{0}(y) \), give

\[ P \left( \left| z_{n_{y}}^{0} - nu_{0}(y) \right| > n\varepsilon \text{ for some } y \in [c, b] \right) \]

\[ \leq \sum_{j=0}^{k} P \left( \left| z_{n_{y}}^{0} - nu_{0}(b_{j}) \right| > n\varepsilon/2 \right). \]

These probabilities are summable over \( n \), by large deviation bounds for exponential random variables. 

### 6 Proof of Theorem 2.1

#### 6.1 Proof of Theorem 2.1(i)

**Lemma 6.1** Suppose \( X \) is a measurable function defined on some measurable space \((\Omega, F)\), and \( C \) is a compact subset of \( \mathbb{R} \). Then there exists a measurable function \( Y \) such that, for all \( \omega \in \Omega \), \( Y(\omega) \in C \) and \( \text{dist}(X(\omega), C) = |X(\omega) - Y(\omega)| \).
Proof. The function \( g(x) = \inf\{y \in C : \text{dist}(x, C) = |x - y|\} \) is nondecreasing, hence Borel measurable. Set \( Y(\omega) = g(X(\omega)) \).

Recall the definition (13) of \( i_n(x, t) \). By Lemma 5.1, we may choose a random \( y_n(x, t) \in I(x, t) \) such that

\[
|n^{-1}i_n(x, t) - y_n(x, t)| = \text{dist}\left(n^{-1}i_n(x, t), I(x, t)\right).
\]

To prove the limit (13) in Theorem 2.1, we bound \( \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \) from below and from above, uniformly over \((x, t) \in A\), with four separate arguments for different ranges of \( t \). Let \([a, b] \times [0, \tau]\) be a compact rectangle that contains \( A\).

6.1.1 Lower Bound, Case 1

Consider \( t \in (0, n^{-(1+\delta)}) \) for a small \( \delta > 0 \). By Lemma 5.3(i) we may condition on the event \( H^c_{n, 1} \), and thereby assume that \( i_n(x, t) \geq nx - \alpha n^{(1-\delta)/2} \) for all \((x, t) \in [a, b] \times (0, n^{-(1+\delta)})\). Since the \( \Gamma \)-term is always nonnegative,

\[
z^n_{[nx]}(nt) \geq z^n_{i_n(x,t)}(0) \geq z^n_{[nx-\alpha n^{(1-\delta)/2}]}(0).
\]

Furthermore, by Lemma 5.4 and by the local Lipschitz property of \( u_0 \), there exists a constant \( C \) such that

\[
u_0(y) \geq u_0(x) - Ct \geq u_0(x) - Cn^{-(1+\delta)}
\]

for all \( x \in [a, b], t \in (0, n^{-(1+\delta)}) \), and \( y \in I(x, t) \). Monotonicity in time and space give \( u(x, t) \leq u_0(x) \), and \( z^n_{[ny]}(0) \leq z^n_{[nx]}(0) \) whenever \( y \in I(x, t) \). We get the following lower bound, valid on the event \( H^c_{n, 1} \) for \( y \in I(x, t) \):

\[
z^n_{[nx]}(nt) - nu(x, t) - \{z^n_{[ny]}(0) - nu_0(y)\} \\
\geq -\{z^n_{[nx]}(0) - z^n_{[nx-\alpha n^{(1-\delta)/2}]}(0)\} - Cn^{-\delta}.
\]

Add and subtract the term \( nu_0(x) - nu_0(x - \alpha n^{-(1+\delta)/2}) \), which is of order \( O(n^{1/2-\delta/2}) \) uniformly over \( x \in [a, b] \) by the local Lipschitz property of \( u_0 \). Multiply through by \( n^{-1/2} \) and uniformize over \((x, t)\):

\[
\inf\{\zeta_n(x, t) - \zeta_n(y, 0) : (x, t) \in [a, b] \times (0, n^{-(1+\delta)}), y \in I(x, t)\} \\
\geq -\sup\{\zeta_n(x, 0) - \zeta_n(y, 0) : x \in [a, b], |x - y| \leq 2an^{-(1+\delta)/2}\} - Cn^{-\delta/2}.
\]

This bound is valid on the event \( H^c_{n, 1} \), hence by Lemma 5.3 with probability \( 1 - \varepsilon \) if \( n \) is large enough. The lower bound converges to 0 in probability by assumption (4). The constant \( \alpha \) was replaced by \( 2\alpha \) to account for the effects of integer parts.
6.1.2 Lower Bound, Case 2

Now \( t \in [n^{-(1+\delta)}, \tau] \).

\[
\begin{align*}
&z^n_{[nx]}(nt) - nu(x, t) \\
&= z^n_{i_n(x,t)}(0) + \Gamma_{[nx]-i_n(x,t)}(nt) - nu(x, t) \\
&= \left\{ z^n_{ny_n(x,t)}(0) - nu_0(y_n(x,t)) \right\} + \left\{ \Gamma_{[nx]-i_n(x,t)}(nt) - (nx - i_n(x,t))^2 / 4tn \right\} \\
&\quad + \left\{ z^n_{i_n(x,t)}(0) - z^n_{ny_n(x,t)}(0) - nu_0(n^{-1}i_n(x,t)) + nu_0(y_n(x,t)) \right\} \\
&\quad + n \left\{ \Phi(x, n^{-1}i_n(x,t)) - \Phi(x, y_n(x,t)) \right\}.
\end{align*}
\]

Above we used the notation

\[
\Phi(x, y) = u_0(y) + t g \left( \frac{x - y}{t} \right)
\]

for the function minimized in the Hopf-Lax formula (2). Since \( y_n(x,t) \in I(x,t) \) minimizes \( \Phi(x, \cdot) \), the term \( \Phi(x, n^{-1}i_n(x,t)) - \Phi(x, y_n(x,t)) \) is nonnegative and can be discarded. Recalling the definition (3) of \( \zeta_n \), we get

\[
\zeta_n(x,t) = n^{-1/2} \left\{ z^n_{[nx]}(nt) - nu(x, t) \right\} \\
\geq \inf_{y \in I(x,t)} \zeta_n(y, 0) + n^{-1/2} \left\{ \Gamma_{[nx]-i_n(x,t)}(nt) - (nx - i_n(x,t))^2 / 4tn \right\} \\
+ n^{-1/2} \left\{ z^n_{i_n(x,t)}(0) - nu_0(n^{-1}i_n(x,t)) \right\} - n^{-1/2} \left\{ z^n_{ny_n(x,t)}(0) - nu_0(y_n(x,t)) \right\}.
\]

Recall the definitions of the events \( H_{n,0} \) and \( H_{n}(\delta) \) in Lemmas 5.2 and 5.3. By Lemma 5.3, \( \lim_{n \to \infty} P(H_{n}(\delta)) = 0 \) for any fixed \( \delta > 0 \). Then it is possible to find a sequence \( \delta_n \searrow 0 \) such that \( \lim_{n \to \infty} P(H_{n}(\delta_n)) = 0 \). Now condition on the event \( H_{n,0}^c \cap H_{n}(\delta_n)^c \), the complement of these events. Then for all \( (x, t) \in A \) such that \( t \in [n^{-(1+\delta)}, \tau] \),

\[
[nx] - i_n(x,t) \leq \alpha nt \gamma \text{ and } |n^{-1}i_n(x,t) - y_n(x,t)| \leq \delta_n.
\]

Consequently we get, on the event \( H_{n,0}^c \cap H_{n}(\delta_n)^c \), for all \( (x, t) \in A \) such that \( t \in [n^{-(1+\delta)}, \tau] \),

\[
\zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y, 0) \geq R_{n,1} + R_{n,2}
\]

where we abbreviated

\[
R_{n,1} = \inf \left\{ n^{-1/2} \left( \Gamma_{[m]}^n(nt) - m^2 / (4tn) \right) : |nc| \leq i \leq |nb|, 0 \leq m \leq \alpha nt \gamma, t \in [n^{-(1+\delta)}, \tau] \right\}
\]

\[
R_{n,2} = \inf \left\{ n^{-1/2} \left( \Gamma_{[m]}^n(nt) - m^2 / (4tn) \right) : |nc| \leq i \leq |nb|, m > \alpha nt \gamma, t \in [n^{-(1+\delta)}, \tau] \right\}
\]

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and
\[ R_{n,2} = \inf \{ \zeta_n(r,0) - \zeta_n(s,0) : |r-s| \leq \delta_n \text{ and } r, s \in [c,d] \}. \]
In the definition of \( R_{n,1} \) and \( R_{n,2} \) we picked \( c < d \) depending on \( \alpha, \gamma, \) and \( \delta_n \) in (67) to ensure that \( y_n(x,t) \) and \( n^{-1}i_n(x,t) \in [c,d] \) for all \((x,t) \in A \) and for all \( n \).

**Lemma 6.2** For any \( \varepsilon > 0 \), \( \sum_{n=1}^{\infty} P(R_{n,1} \leq -\varepsilon) < \infty \).

*Proof.* The case \( m = 0 \) in the definition of \( R_{n,1} \) can be ignored. For \( 1 \leq m \leq \alpha nt^7 \) and \( t \geq n^{-(1+\delta)} \),
\[ m^{4/3} \log n \over 4nt \leq Cn^{1/3-(1+\delta)(4\gamma/3-1)} \log n. \]
This is less than \( \varepsilon n^{1/2} \) for large \( n \), if we choose \( \gamma \) close enough to \( 3/4 \) and \( \delta \) small enough. This can be done while satisfying the hypotheses of Lemma 5.2. Consequently the estimate in Lemma 4.3 is valid for the entire range of \( m \)-values in the definition of \( R_{n,1} \), and gives
\[ \sum_{n=1}^{\infty} P(R_{n,1} \leq -\varepsilon) \leq C \sum_{n=1}^{\infty} n \sum_{m=1}^{[nr]} P \left\{ \Gamma(m,nt) \leq n^{2/4nt} - m^{4/3} \log n \right\} \leq \infty. \]
The factor \( Cn \) came from summing over \([na] \leq i \leq [nb] \) as required by the definition of \( R_{n,1} \), and \( r \) was chosen sufficiently large so that \( nr \geq \alpha nt^7 \) for all \( t \) in the range. \( \blacksquare \)

**Lemma 6.3** \( \lim_{n \to \infty} |R_{n,2}| = 0 \) in probability.

*Proof.* Recall the definition of \( D_u(R) \) as the space of RCLL functions with the locally uniform metric. For a fixed \( \beta > 0 \) define the continuous function \( \phi_\beta \) on \( D_u(R) \) by
\[ \phi_\beta(f) = \sup \{ |f(r) - f(s)| : |r-s| \leq \beta \text{ and } r, s \in [c,d] \}. \]
For large enough \( n \) so that \( \delta_n < \beta, |R_{n,2}| \leq \phi_\beta(\zeta_n(\cdot,0)) \). The set \( \{ f \in D_u(R) : \phi_\beta(f) \geq \varepsilon \} \) is closed, so by the weak convergence \( \zeta_n(\cdot,0) \to \zeta_0 \) on \( D_u(R) \),
\[ \limsup_{n \to \infty} P(|R_{n,2}| \geq \varepsilon) \leq \limsup_{n \to \infty} P(\phi_\beta(\zeta_n(\cdot,0)) \geq \varepsilon) \leq P(\phi_\beta(\zeta_0) \geq \varepsilon). \]
Since \( \zeta_0 \) has continuous paths by assumption (3), the events \( \phi_\beta(\zeta_0) \geq \varepsilon \) decrease to the null event as \( \beta \searrow 0 \) and \( \varepsilon, c, d \) are held fixed. \( \blacksquare \)

We now have for **Case 2**
\[ \lim_{n \to \infty} P \left( \inf_{(x,t) \in A, t \in [n^{-(1+\delta)},\varepsilon]} \{ \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \} \leq -\varepsilon \right) \]
\[ \leq \lim_{n \to \infty} \{ P(H_{n,1}) + P(H_n(\delta_n)) + P(R_{n,1} \leq -\varepsilon/2) + P(R_{n,2} \leq -\varepsilon/2) \} = 0 \]
for any $\varepsilon > 0$. **Cases 1 and 2** together give

$$\lim_{n \to \infty} P \left( \inf_{(x,t) \in A} \left\{ \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right\} \leq -\varepsilon \right) = 0.$$ 

This completes the proof of the lower bound. Next we bound $\zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0)$ from above.

### 6.1.3 Upper Bound, Case 1

Consider $0 < t \leq n^{-1}$. Let $C$ be a finite constant such that $u(x,t) \geq u_0(x) - Ct$ for all $a \leq x \leq b$, $0 \leq t \leq \tau$. Since $z_{nt}^n(0) \leq z_{nt}^n(0)$, we can write

$$z_{nt}^n = z_{nt}^n(0) - nu(x,t) - \{z_{nt}^n(0) - nu_0(y)\} \leq \frac{\inf_{y \in I(x,t)} \{z_{nt}^n(0) - nu_0(y)\} + Ctn}{\sqrt{n}}$$

Estimate this uniformly over $a \leq x \leq b$, $0 < t \leq n^{-1}$. By Lemma 3.4, there is a constant $\gamma$ such that $y \in I(x,t)$ implies $|x-y| \leq \gamma t \leq \gamma n^{-1}$, for all $(x,t)$ in this range.

Dividing by $\sqrt{n}$ above gives, for $n$ large enough to have $x - \gamma n^{-1} \geq a - 1$,

$$\sup_{a \leq x \leq b, 0 < t \leq n^{-1}, y \in I(x,t)} \{z_{nt}^n(0) - nu_0(y)\} \leq \sup_{a \leq x \leq b, 0 < t \leq n^{-1}, y \in I(x,t)} \{z_{nt}^n(0) - nu_0(y)\} + Cn^{-1/2}.$$

The last quantity converges to 0 in probability by assumption (3).

### 6.1.4 Upper Bound, Case 2

Lastly consider $n^{-1} \leq t \leq \tau$. Use the fact that $u(x,t) = u_0(y) + (x-y)^2/(4t)$ for any $y \in I(x,t)$.

$$z_{nt}^n = z_{nt}^n(0) - nu(x,t) = \inf_{y \leq x} \{z_{nt}^n(0) + \Gamma_{nt}^n(0) - nu(x,t)\} \leq \inf_{y \in I(x,t)} \{z_{nt}^n(0) - nu_0(y)\} + R_{n,3} + C,$$

where

$$R_{n,3} = \sup_{x \in [a,b], t \in [n^{-1},\tau]} \sup_{y \in I(x,t)} \left\{ \Gamma_{nt}^n - \frac{\Gamma_{nt}^n}{4nt} \right\},$$

and the constant $C$ accounts for replacing $(x-y)^2/(4t)$ with $(\lfloor nx \rfloor - \lfloor ny \rfloor)^2/(4nt)$. Consequently

$$\sup_{(x,t) \in A, n^{-1} \leq t \leq \tau} \left\{ \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right\} \leq n^{-1/2} R_{n,3} + C n^{-1/2}.$$

The required upper bound follows by taking $\alpha \in (1/3, 1/2)$ in the next lemma.
Lemma 6.4 For any $\alpha > 1/3$, $\sum_{n=1}^{\infty} P(R_{n,3} > n^{\alpha}) < \infty$.

Proof. $R_{n,3} > n^{\alpha}$ implies that

for some $y \in [c,b]$, $t \in [n^{-1}, \tau]$, and $1 \leq m \leq n\gamma t + 1$, $\Gamma_{m,\lfloor ny \rfloor}(nt) > \frac{m^2}{4nt} + n^{\alpha}$. (69)

To see this, choose $\gamma$ according to Lemma 3.4 so that for $y \in I(x,t)$, $m = \lfloor nx \rfloor - \lfloor ny \rfloor \leq n\gamma t + 1$. Choose $c \leq a - \gamma \tau$ so that the range of possible $y$-values is contained in $[c,b]$.

The case $m = 0$ is empty because $\Gamma_{0,\lfloor ny \rfloor}(nt) \equiv 0$.

By the assumption $\alpha > 1/3$, the inequality $n^{\alpha} \geq m^{4/3}(4tn)^{-1} \log n$ is valid for the range $1 \leq m \leq n\gamma t + 1$, for large enough $n$. Pick $r > 0$ so that $n\gamma \tau + 1 \leq nr$, and let $M_0$ be the constant that appeared in Lemma 4.4. We can then assert that the event in (69) is contained in the union of

$$\bigcup_{n\gamma \leq j \leq na \atop M_0(\log n)^{3/2} \leq m \leq nr} \left\{ \Gamma_{m,\lfloor nt \rfloor}^{n,j} \geq m^{4/3} \log n + 4nt \right\}$$

and

$$\bigcup_{n\leq j \leq na \atop M_0(\log n)^{3/2}} \left\{ \Gamma_{m,\lfloor nt \rfloor}^{n,j} > n^{\alpha} \text{ for some } t \in [n^{-1}, \tau] \right\}.$$

The conclusion now follows from the estimates in Lemmas 4.4 and 4.5, because for any fixed $(n,j)$, $\Gamma_{m,\lfloor nt \rfloor}^{n,j}$ has the same distribution as $\Gamma(m,nt)$. □

Combining Cases 1 and 2, we have bounded

$$\sup_{(x,t) \in A} \left\{ \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right\}$$

above by a random variable that vanishes in probability as $n \to \infty$. Together with the lower bound, this completes the proof of part (i) of Theorem 2.1.

6.2 Proof of Theorem 2.1(ii)

First a lemma whose proof is partly a repetition of the above argument.

Lemma 6.5 Let $-\infty < a < b < \infty$ and $0 < \tau < \infty$, and set

$$M_n = \sup_{x \in [a,b], t \in [0,\tau]} \left| \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right|.$$

Then for every $\varepsilon > 0$ there exists a finite constant $C$ such that $P(M_n \leq C) \geq 1 - \varepsilon$ for all $n$. 

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Proof. We prove the measurability of $M_n$ in the Appendix. Let $A_0 = [a, b] \times [0, \tau]$. If $A_0$ were an admissible compact set for the proof of Theorem 2.1(i) just completed, there would be nothing more to prove. But $A_0$ might have shocks. The only step where this makes a difference in the above proof is Section 6.1.2 Lower Bound, Case 2 because this case appealed to Lemma 5.3. Without Lemma 5.3 the argument still gives a lower bound. Condition on the event $G_n^c$ so that $i_n(x, t) \geq nc$ for $(x, t) \in A_0$. Then inequality (66) gives

$$
\inf_{(x, t) \in A_0} \left\{ \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \right\} \geq -2 \cdot \sup_{y \in [c, b]} |\zeta_n(y, 0)| \geq \sup_{1 \leq m \leq n(b-c), t \in \mathbb{N}^{(1+n)}, \tau} \left\{ \frac{\Gamma_{m, c}^{n}(nt) - \frac{m^2}{4nt}}{\zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0)} \right\}.
$$

(70)

This bound is valid on the event $G_n^c$, hence with probability $1 - \varepsilon$ for large enough $n$. Combined with the other cases proved in Section 6.1 it proves the lemma.

Fix $-\infty < a < b < \infty$ and $\tau < \infty$. The goal is to prove the limit in probability (16). Let $\varepsilon > 0$. Let

$$
Y_n = \sup_{(x, t) \in [a, b] \times [0, \tau]} \left| \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \right|^p.
$$

By Lemma 6.5 we can find a constant $C \in (0, \infty)$ such that $P(Y_n > C) \leq \varepsilon$ for all $n$. Define the event $D_n = \{ Y_n \leq C \}$. By Proposition 3.1 we can find an open set $G \subset \mathbb{R} \times (0, \infty)$ such that $G$ contains all the shocks in $[a, b] \times [0, \tau]$, and its $t$-section has 1-dimensional Lebesgue measure $|G_t| < \varepsilon/(3C)$ for all $t$. (Recall that by definition there are no shocks on the $t = 0$ line.)

Let $A = [a, b] \times [0, \tau] \setminus G$. $A$ is a compact set with no shocks, so by Theorem 2.1(i)

$$
X_n \equiv \sup_{(x, t) \in A} \left| \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \right|^p \to 0 \quad \text{in probability.} \quad (71)
$$

Let $A_t = \{ x : (x, t) \in A \}$ be the $t$-section of $A$. On the event $D_n$ we can now bound

$$
\sup_{0 \leq t \leq \tau} \int_a^b \left| \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \right|^p \, dx \leq \sup_{0 \leq t \leq \tau} \left\{ \int_{A_t} X_n \, dx + \int_{[a, b] \setminus A_t} Y_n \, dx \right\} \leq (b - a)X_n + Y_n\varepsilon/(3C) \leq (b - a)X_n + \varepsilon/3.
$$

Thus by (71)

$$
\limsup_{n \to \infty} P\left\{ \sup_{0 \leq t \leq \tau} \int_a^b \left| \zeta_n(x, t) - \inf_{y \in I(x, t)} \zeta_n(y, 0) \right|^p \, dx \geq \varepsilon \right\} \leq P(D_n^c) \leq \varepsilon.
$$

This proves (16).
7 Proof of the weak limit and the linearized equation

7.1 Proof of Theorem 2.2

For part (i), take $A = \{(x_1,t_1), \ldots, (x_k,t_k)\}$ in (15). Note that the mapping

$$h \mapsto \left( \inf_{y \in I(x_1,t_1)} h(y), \ldots, \inf_{y \in I(x_k,t_k)} h(y) \right)$$

from $D_u(R)$ into $R^k$ is continuous. Then use the assumption (1) of weak convergence at time zero, and the continuous mapping theorem [5, p. 30].

Part (ii) goes by the same general principle. Let us abbreviate

$$\sigma_n(x,t) = \inf_{y \in I(x,t)} \zeta_n(y,0).$$

We need to check that $\zeta_n$ defines a random element of $D ([0,\infty), L^p_{1,\text{loc}}(R))$ and that $\sigma_n$ and $\zeta$ define random elements of $C ([0,\infty), L^p_{2,\text{loc}}(R))$.

For $f \in D_u(R)$ let $Gf(x,t) = \inf_{y \in I(x,t)} f(y)$.

**Lemma 7.1** $G$ is a continuous map from $D_u(R)$ into $C ([0,\infty), L^p_{1,\text{loc}}(R))$, when we interpret $Gf$ as the path $t \mapsto Gf(\cdot,t) \in L^p_{1,\text{loc}}(R)$.

**Proof.** For $0 \leq t \leq T$ and $a \leq x \leq b$, $I(x,t) \subseteq [y^-(a,T),b]$, and so $Gf$ is locally bounded as a function of $(x,t)$. Consequently for a fixed $t$, $Gf(\cdot,t)$ is in $L^p_{1,\text{loc}}(R)$.

Secondly, we need to argue that as $s \to t$, $Gf(\cdot,s) \to Gf(\cdot,t)$ in $L^p_{1,\text{loc}}(R)$. By the local boundedness and dominated convergence, we need only show

$$Gf(x,s) \to Gf(x,t) \text{ for a.e. } x.$$ 

Recall from Section 3 that if $y_1 \in I(x,s_1)$ and $y_2 \in I(x,s_2)$ for $s_1 < s_2$, then $y_2 \leq y_1$. Consider first $s \not\to t$. Fix $x$ so that $(x,t)$ is not a shock. Then for any choice $y_s \in I(x,s)$, $y_s \searrow y(x,t)$ is the unique Hopf-Lax minimizer for $(x,t)$. By right-continuity $f(y_s) \to f(y(x,t)) = Gf(x,t)$, and since we can let $f(y_s)$ be arbitrarily close to $Gf(x,s)$, we have (73).

Now suppose $s \searrow t$. $f$ has at most countably many discontinuities, so we still have a.e. $x$ if we exclude all $x$ such that $(x,t)$ is a shock, and all points $x = w^\pm(y,t)$ for discontinuities $\ddot{y}$ of $f$. Suppose $x$ is not one of the excluded points. Then $(x,t)$ has a unique minimizer $y(x,t)$, and the previous paragraph shows again $f(y_s) \to f(y(x,t))$ if $f$ is continuous at $y(x,t)$. But suppose $\ddot{y} = y(x,t)$ is a discontinuity for $f$. Then it must be that $w^-(\ddot{y},t) < x < w^+(\ddot{y},t)$. [Justification: (72) forces $w^-(\ddot{y},t) \leq x \leq w^+(\ddot{y},t)$, but $x \in \{w^\pm(\ddot{y},t)\}$ cannot happen because $x$ is not among the excluded points.] Since
forward characteristics are continuous, it follows that for \( s > t \) but close enough to \( t \), \( w^-(\bar{y}, s) < x < w^+(\bar{y}, s) \). This implies \( y^+(x, s) = \bar{y} \) which in turn says \( Gf(x, s) = Gf(x, t) \). Again (3) checks.

We have now shown that the function \( t \mapsto Gf(\cdot, t) \in L^p_{\text{loc}}(\mathbb{R}) \) is continuous. Finally, we check that the map \( G : D_u(\mathbb{R}) \to C([0, \infty), L^p_{\text{loc}}(\mathbb{R})) \) is continuous. This is a consequence of having the locally uniform topology on \( D_u(\mathbb{R}) \). For by the observation made in the beginning of the proof,

\[
\sup_{0 \leq t \leq T} \int_a^b |Gf(x, t) - Gg(x, t)|^p \, dx \leq \sup_{y^- (a, T) \leq x \leq b} |f(x) - g(x)|.
\]

By definitions (17) and (72), the processes \( \zeta \) and \( \sigma_n \) are obtained by applying the mapping \( G \) to the \( D_u(\mathbb{R}) \)-valued random functions \( \zeta_0 \) and \( \zeta_n(\cdot, 0) \). This checks that \( \sigma_n \) and \( \zeta \) define random elements of \( C([0, \infty), L^p_{\text{loc}}(\mathbb{R})) \). Also, by assumption (9) and the continuous mapping theorem, \( \sigma_n \rightarrow \zeta \) in the space \( C([0, \infty), L^p_{\text{loc}}(\mathbb{R})) \).

Now consider \( \zeta_n \) defined by (4). It is jointly measurable in \( (x, t, \omega) \), where \( \omega \) is a sample point of the underlying probability space \( \Omega \) (see the Appendix). It is also locally bounded in \( (x, t) \) so local \( L^p \)-integrability is not a problem. Fix \( t \). First we argue that

\[
\omega \mapsto \zeta_n(\cdot, t; \omega) \quad \text{is a measurable map from } \Omega \text{ into } L^p_{\text{loc}}(\mathbb{R}),
\]

(74)

where \( L^p_{\text{loc}}(\mathbb{R}) \) is endowed with its Borel \( \sigma \)-algebra defined by the metric \( d_p \) of (15). By Fubini’s theorem, for any \( f \in L^p_{\text{loc}}(\mathbb{R}) \), \( d_p(f, \zeta_n(\cdot, t; \omega)) \) is a measurable function of \( \omega \). Hence for any open \( d_p \)-ball \( B(f, r) \) in \( L^p_{\text{loc}}(\mathbb{R}) \), the inverse image \( \{ \omega : \zeta_n(\cdot, t; \omega) \in B(f, r) \} \) is measurable. By the separability of \( L^p_{\text{loc}}(\mathbb{R}) \), (74) follows.

By convention interacting systems are constructed to be right-continuous in time, so \( t \mapsto \zeta_n(x, t; \omega) \) is right-continuous. By local boundedness and dominated convergence, the map \( t \mapsto \zeta_n(\cdot, t; \omega) \) from \([0, \infty)\) into \( L^p_{\text{loc}}(\mathbb{R}) \) is right-continuous. The measurability of \( \zeta_n(\cdot, \cdot; \omega) \) as a \( D([0, \infty), L^p_{\text{loc}}(\mathbb{R})) \)-valued random element follows because in \( D \)-space measurability is equivalent to measurability of the time-coordinate projections \( \zeta_n(\cdot, t; \omega) \).

The weak convergence \( \zeta_n \xrightarrow{d} \zeta \) now follows readily. For any finite time-horizon \( T \), sup\( _{0 \leq t \leq T} d_p(\zeta_n(t), \sigma_n(t)) \rightarrow 0 \) in probability by (14). Uniform in time is stronger than the Skorokhod topology. So it follows that, if we let \( d_D \) denote the Skorokhod metric on \( D([0, \infty), L^p_{\text{loc}}(\mathbb{R})) \), \( d_D(\zeta_n, \sigma_n) \rightarrow 0 \) in probability also. This together with \( \sigma_n \xrightarrow{d} \zeta \) implies \( \zeta_n \xrightarrow{d} \zeta \). We have proved Theorem 2.3.

\section*{7.2 Proof of Theorem 2.3}
Lemma 7.2 Let $F, G$ be right-continuous functions, $F$ locally BV and $G$ nondecreasing. Let $H^-$ the left-continuous inverse of $G$ defined by

$$H^-(y) = \sup\{x : G(x) < y\} = \inf\{x : G(x) \geq y\}.$$ 

Then for all continuous functions $\varphi$ for which the integrals exist,

$$\int \varphi(H^-(y))dF(y) = \int \varphi(x)d(F \circ G)(x).$$

Proof. It suffices to take $\varphi = 1_{[a,b]}$, the indicator function of a left-open right-closed interval. Check that $\{y : a < H^-(y) \leq b\} = (G(a), G(b)]$. Then

$$\int 1_{[a,b]}(H^-(y))dF(y) = \int 1_{(G(a),G(b)]}(y)dF(y) = F(G(b)) - F(G(a))$$

$$= \int 1_{[a,b]}d(F \circ G).$$

This lemma will be applied below to the pair $G(a) = w^+(a,t)$, $H^-(b) = y^-(b,t)$.

Fix a test function $\phi \in C_c^\infty(R \times [0, \infty))$. Let $(A,B) \times [0,T)$ contain the support of $\phi$. Let $\Phi(x,t) = \int_{-\infty}^x \phi(y,t)dy$. By Theorem 3.1, for any $q \in R$ we have the formula

$$-\Phi(q,0) = \int_0^T \frac{d}{dt}\Phi(w^+(q,t),t)dt$$

$$= \int_0^T \Phi_t(w^+(q,t),t)dt + \int_0^T \phi(w^+(q,t),t)h(w^+(q,t),t)dt. \quad (75)$$

Now we calculate, beginning with the leftmost term of (75), with $v$ in place of $\zeta$. Note that $v(x,t) = v_0(y^-(x,t))$ a.e. so in this first integral these two are interchangeable.

$$\int_0^T dt \int v(x,t)\phi_t(x,t)dx = \int_0^T dt \int v_0(y^-(x,t))d[\Phi_t(\cdot,t)](x)$$

$$= \int_0^T dt \int v_0(q)d[\Phi_t(w^+(\cdot,t),t)](q).$$

There is a fixed compact interval $[a,b]$ on which the Lebesgue-Stieltjes measure $d[\Phi_t(w^+(\cdot,t),t)](q)$ is supported for all $t \in [0,T]$. Let $a = q_0 < q_1 < \cdots < q_m = b$ be a partition of this interval with mesh $\Delta = \max(q_i - q_{i-1})$. We can choose the partitions so that $w(q_i,t) = w^+(q_i,t)$ for all $i$, because by Lemma 3.1(c) we only need to pick the $q_i$’s outside a certain Lebesgue null set. The integrand $v_0$ is continuous by assumption, hence the $q$-integral can be written as a limit, and the last line above equals

$$= \int_0^T dt \lim_{\Delta \to 0} \sum_i v_0(q_i)\{\Phi_t(w(q_i,t),t) - \Phi_t(w(q_{i-1},t),t)\}. $$
The function inside the \( t \)-integral is bounded by a constant, uniformly over \( t \in [0,T] \) and over partitions of \([a,b]\). Hence we can take the limit outside, apply (75), and then put the limit back inside, to get

\[
\lim_{\Delta \to 0} \sum_i v_0(q_i) \int_0^T \{ \Phi_i(w(q_i,t),t) - \Phi_i(w(q_{i-1},t),t) \} dt
\]

\[
= \lim_{\Delta \to 0} \left\{ - \sum_i v_0(q_i) \int_0^T \{ \phi(w(q_i,t),t)h(w(q_i,t),t)
- \phi(w(q_{i-1},t),t)h(w(q_{i-1},t),t) \} dt - \sum_i v_0(q_i)[\Phi(q_i,0) - \Phi(q_{i-1},0)] \right\}
\]

\[
= - \int_0^T \lim_{\Delta \to 0} \sum_i v_0(q_i)\{ \phi(w(q_i,t),t)h(w(q_i,t),t)
- \phi(w(q_{i-1},t),t)h(w(q_{i-1},t),t) \} dt - \int v_0(q)\phi(q,0)dq. \tag{76}
\]

At this point we replace \( h(w(q_i,t),t) \) by \( f'(\rho^+(w(q_i,t),t)) \) and write \( R_\Delta \) for the error term. Then the last line above equals

\[
= - \int_0^T \lim_{\Delta \to 0} \sum_i v_0(q_i)\{ \phi(w(q_i,t),t)f'(\rho^+(w(q_i,t),t))
- \phi(w(q_{i-1},t),t)f'(\rho^+(w(q_{i-1},t),t)) \} dt - \int v_0(q)\phi(q,0)dq + \lim_{\Delta \to 0} R_\Delta.
\]

Ignoring the term \( \lim_{\Delta \to 0} R_\Delta \) for the moment, take the \( \Delta \to 0 \) limit in the first sum to get again a Lebesgue-Stieltjes integral. After another application of Lemma 7.2, we get this intermediate equation:

\[
\int_0^T dt \int v(x,t)\phi_t(x,t)dx
= - \int_0^T dt \int v_0(q)d[\phi(w^+(:,t),t)f'(\rho^+(w^+(:,t),t))]\phi(q)\phi(q,0)dq
- \lim_{\Delta \to 0} R_\Delta
\]

\[
= - \int_0^T dt \int v_0(\rho^-(-,t))d[\phi^-(-,t)f'(\rho^+(-,t))]\phi(\rho^-(-,t),0)dq
- \lim_{\Delta \to 0} R_\Delta. \tag{77}
\]

It remains to take care of \( \lim_{\Delta \to 0} R_\Delta \). Notice that on line (74), at the stage where \( R_\Delta \) was introduced, the summation can be restricted to \( i \) such that \( w(q_{i-1},t) < w(q_i,t) \) because otherwise \( w(q_{i-1},t) = w(q_i,t) \) and the expression in braces \( \{\} \) equals zero. Thus we can write \( R_\Delta \) as follows, and sum by parts:

\[
R_\Delta = \int_0^T dt \sum_{i:w(q_{i-1},t)<w(q_i,t)} v_0(q_i) [\phi(w(q_i,t),t) \{ h(w(q_i,t),t) - f'(\rho^+(w(q_i,t),t)) \}]
\]

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-\phi(w(q_{i-1}, t), t) \{ h(w(q_{i-1}, t), t) - f'(\rho^+(w(q_{i-1}, t), t)) \} \\
= \int_0^T dt \sum_i \phi(w(q_i, t), t) \{ h(w(q_i, t), t) - f'(\rho^+(w(q_i, t), t)) \} \\
\times [v_0(q_i)1\{w(q_{i-1}, t) < w(q_i, t)\} - v_0(q_{i+1})1\{w(q_i, t) < w(q_{i+1}, t)\}].

Now note that the last sum can be restricted to \(i\) such that \((w(q_i, t), t)\) is a shock because \(h(x, t) - f'(\rho^+(x, t)) = 0\) unless \((x, t)\) is a shock. Supposing that \((x, t)\) is a shock, observe that if \(y^-(x, t) \leq q_i < q_{i+1} \leq y^+(x, t)\) then \(w(q_i, t) = w(q_{i+1}, t) = x.\) In general \(w^+(y^+(x, t), t)\) could be strictly larger than \(x\), but then \(w^-(y^+(x, t), t) \leq w^+(y^+(x, t), t),\) which we have prevented by assuming \(w^-(q_i, t) = w^+(q_i, t).\) Consequently, for the shock \((x, t),\)

\[
\sum_{i: w(q_i, t) = x} [v_0(q_i)1\{w(q_{i-1}, t) < w(q_i, t)\} - v_0(q_{i+1})1\{w(q_i, t) < w(q_{i+1}, t)\}]
\]

\[= v_0 \left( \min\{q_i : q_i > y^-(x, t)\} \right) - v_0 \left( \min\{q_i : q_i > y^+(x, t)\} \right).\]

To express this in a single function, write

\[
L_\Delta(x, t) = 1 \left( [x, t) \text{ is a shock, and } x \in \{w(q_i, t) : 0 \leq i \leq m\} \right) \\
\times \left[ v_0 \left( \min\{q_i : q_i > y^+(x, t)\} \right) - v_0 \left( \min\{q_i : q_i \geq y^-(x, t)\} \right) \right].
\]

The subscript \(\Delta\) expresses the dependence of \(L_\Delta\) on the partition. For any shock \((x, t),\) some \(q_i\) lies in \((y^-(x, t), y^+(x, t))\) when \(\Delta\) is small enough, and then \(x = w(q_i, 0).\) By the continuity of \(v_0\) we have the convergence

\[
\lim_{\Delta \to 0} L_\Delta(x, t) = v_0(y^+(x, t)) - v_0(y^-(x, t)), \quad (78)
\]

which happens boundedly and at all \((x, t).\) Now write

\[
R_\Delta = \int_0^T dt \sum_{a \leq x \leq b} \phi(x, t) \{ f'(\rho^+(x, t), t) - h(x, t) \} L_\Delta(x, t)
\]

\[
= \int_0^T dt \int \theta(x, t) L_\Delta(x, t) d[\phi(\cdot, t) f'(\rho^+(\cdot, t))] (x)
\]

where we recognized that

\[
f'(\rho^+(x, t), t) - h(x, t) = \theta(x, t) [f'(\rho^+(x, t), t) - f'(\rho^-(x, t), t)].
\]

The \(x\)-integral lives entirely on the countable set of shocks because \(L_\Delta\) vanishes elsewhere. This is why we can slip the continuous \(\phi(x, t)\) factor into the integrator. Taking the limit gives

\[
\lim_{\Delta \to 0} R_\Delta
\]

\[
= \int_0^T dt \int \theta(x, t) \left[ v_0(y^+(x, t)) - v_0(y^-(x, t)) \right] d[\phi(\cdot, t) f'(\rho^+(\cdot, t))] (x).
\]

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Substituting this on line (77) above completes the proof that \( v(x, t) \) satisfies (21). We have proved Theorem 2.3.

8 Proof of Theorem 2.4

We begin by realizing the initial configurations \( (z^n_i(0) : i \in \mathbb{Z}) \) with Skorokhod’s representation. Let \((\Omega, \mathcal{F}, P)\) be a probability space on which are defined a two-sided Brownian motion \( B(\cdot) \), and independently of it a space-time Poisson point process for constructing the Hammersley dynamics. Recall that \( B(\cdot) \) is defined by \( B(s) = B_1(s) \) for \( s \geq 0 \) and \( B(s) = -B_2(-s) \) for \( s < 0 \), where \( B_1(\cdot), B_2(\cdot) \) are two independent standard 1-dimensional Brownian motions defined on \([0, \infty)\). For each \( n \), define a two-sided Brownian motion \( B_n(\cdot) \) by \( B_n(s) = n^{1/2}B((s/n) \) for \( s < 0 \), where \( B_1(\cdot), B_2(\cdot) \) are two independent standard 1-dimensional Brownian motions defined on \([0, \infty)\).

Fix \( n \). Construct the Skorokhod representation for the independent mean zero random variables \( \eta^n_i(0) - E[\eta^n_i(0)] \) whose distribution is defined in assumption (28). The usual construction (see e.g. Section 7.6 in [8]) is applied to \( B_1 \) for \( i > 0 \) and to \( B_2 \) for \( i \leq 0 \). This gives random variables

\[
\cdots \leq T_{n,-2} \leq T_{n,-1} \leq 0 = T_{n,0} \leq T_{n,1} \leq T_{n,2} \leq \cdots
\]

such that the variables \( \{\tau_{n,i} = T_{n,i} - T_{n,i-1} : i \in \mathbb{Z}\} \) are mutually independent, we have the equality in distribution of the processes

\[
\{B_n(T_{n,i}) - B_n(T_{n,i-1}) : i \in \mathbb{Z}\} \overset{d}{=} \{\eta^n_i(0) - E[\eta^n_i(0)] : i \in \mathbb{Z}\}
\]

and for each \( i \)

\[
E[\tau_{n,i}] = \text{Var}[\eta^n_i(0)] = E[\eta^n_i(0)]^2 = \left( n \int_{(i-1)/n}^{i/n} \rho_0(s)ds \right)^2. \tag{79}
\]

Note that the assumption of exponentially distributed \( \eta^n_i(0) \) was used here.

Now we take this construction as the definition of the initial interface:

\[
z^n_i(0) = nu_0(i/n) + B_n(T_{n,i}) = nu_0(i/n) + n^{1/2}B(n^{-1}T_{n,i}). \tag{80}
\]

The initial process \( \zeta_n(y, 0) \) defined by (3) is now given by

\[
\zeta_n(y, 0) = B(n^{-1}T_{n,[ny]}) + n^{1/2} (u_0([ny]/n) - u_0(y)). \tag{81}
\]

Lemma 8.1 For any \(-\infty < a < b < \infty\),

\[
\lim_{n \to \infty} \sup_{y \in [a, b]} \left| \frac{T_{n,[ny]}}{n} - \int_0^y \rho_0(s)ds \right| = 0 \quad \text{almost surely.}
\]

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Proof. Suppose 0 ≤ a < b. The other cases are handled with similar arguments. Let us first check

\[ \lim_{n \to \infty} \sup_{y \in [a, b]} \left| \frac{1}{n} ET_{n, \lfloor ny \rfloor} - \int_0^y \rho_0^2(s) ds \right| = 0. \]  

(82)

By (79),

\[ \frac{1}{n} ET_{n, \lfloor ny \rfloor} = \frac{1}{n} \sum_{i=1}^{\lfloor ny \rfloor} \left( n \int_{(i-1)/n}^{i/n} \rho_0(s) ds \right)^2 \]

\[ = \int_0^{\lfloor ny \rfloor / n} \sum_{i=1}^{\lfloor ny \rfloor / n} \left( n \int_{(i-1)/n}^{i/n} \rho_0(s) ds \right)^2 \mathbf{1}_{\left[ \frac{i-1}{n}, \frac{i}{n} \right)}(r) dr. \]

The integrand is bounded by the assumption \( \rho_0 \in L^\infty_{\text{loc}}(\mathbb{R}) \), and converges to \( \rho_0^2(r) \) at every Lebesgue point \( r \) of \( \rho_0 \). So the required convergence in (82) holds for each fixed \( y \). To get uniformity over \( y \),

\[ \sup_{y \in [a, b]} \left| \frac{1}{n} ET_{n, \lfloor ny \rfloor} - \int_0^y \rho_0^2(s) ds \right| \]

\[ \leq \sup_{y \in [a, b]} \left| \int_0^{\lfloor ny \rfloor / n} \left\{ \sum_{i=1}^{\lfloor ny \rfloor / n} \left( n \int_{(i-1)/n}^{i/n} \rho_0(s) ds \right)^2 \mathbf{1}_{\left[ \frac{i-1}{n}, \frac{i}{n} \right)}(r) - \rho_0^2(r) \right\} dr \right| + \frac{C}{n} \]

\[ \leq \sup_{y \in [a, b]} \int_0^{\lfloor ny \rfloor / n} \left| \sum_{i=1}^{\lfloor nb \rfloor / n} \left( n \int_{(i-1)/n}^{i/n} \rho_0(s) ds \right)^2 \mathbf{1}_{\left[ \frac{i-1}{n}, \frac{i}{n} \right)}(r) - \rho_0^2(r) \right| dr + \frac{C}{n} \]

(note that the integrand no longer depends on \( y \))

\[ \leq \int_0^b \left| \sum_{i=1}^{\lfloor nb \rfloor / n} \left( n \int_{(i-1)/n}^{i/n} \rho_0(s) ds \right)^2 \mathbf{1}_{\left[ \frac{i-1}{n}, \frac{i}{n} \right)}(r) - \rho_0^2(r) \right| dr + \frac{C}{n}. \]

The error \( C/n \) accounts for the effect of switching between \( y \) and \( \lfloor ny \rfloor / n \) (\( b \) and \( \lfloor nb \rfloor / n \)) as the upper limit of integration. Now (82) follows by dominated convergence.

Next we show that, for a fixed \( y \),

\[ \lim_{n \to \infty} \left| n^{-1} T_{n, \lfloor ny \rfloor} - n^{-1} ET_{n, \lfloor ny \rfloor} \right| = 0 \quad \text{a.s.} \]  

(83)

The moments of the waiting times \( \tau_{n,i} \) satisfy

\[ E[(\tau_{n,i})^k] \leq C_k E \left[ \left( \eta_i^n(0) - \eta_i^n(0) \right)^{2k} \right] \leq C_k' < \infty \]

for all \( 0 \leq i \leq nb \), for constants \( C_k, C_k' \). The first inequality follows from the Burkholder-Davis-Gundy inequalities, and the second from the local boundedness of
Thus

\[
P \left( \left| T_{n,[ny]} - ET_{n,[ny]} \right| \geq n\varepsilon \right) = P \left( \left| \sum_{i=1}^{[ny]} (\tau_{n,i} - E\tau_{n,i}) \right| \geq n\varepsilon \right)
\]

\[
\leq \frac{1}{n^4\varepsilon^4} E \left[ \left| \sum_{i=1}^{[ny]} (\tau_{n,i} - E\tau_{n,i}) \right|^4 \right] \leq \frac{C}{n^2\varepsilon^4},
\]

and now Borel-Cantelli gives (83). Finally, given \( \varepsilon > 0 \), pick \( \delta > 0 \) so that

\[
\int_x^y \rho_0^2(s) ds < \varepsilon \quad \text{for any } a \leq y < x < y + \delta \leq b.
\]

Pick a partition \( a = a_0 < a_1 < \cdots < a_m = b \) such that \( a_{k+1} - a_k < \delta \). Apply (83) for the values \( y = a_k \), and between the partition points estimate by

\[
\inf_{[na_k] \leq i \leq [na_{k+1}]} \frac{T_{n,i} - ET_{n,i}}{n} \geq \frac{T_{n,[na_k]} - ET_{n,[na_k]}}{n} + \frac{ET_{n,[na_k]} - ET_{n,[na_{k+1}]}}{n},
\]

to get, by (82) and (83),

\[
\liminf_{n \to \infty} \inf_{[na] \leq i \leq [nb]} \frac{T_{n,i} - ET_{n,i}}{n} \geq - \max_{0 \leq k \leq m} \int_{a_k}^{a_{k+1}} \rho_0^2(s) ds \geq -\varepsilon.
\]

Repeat the argument for the upper bound, and let \( \varepsilon \searrow 0 \) to get

\[
\lim_{n \to \infty} \sup_{[na] \leq i \leq [nb]} \left| n^{-1}T_{n,i} - n^{-1}ET_{n,i} \right| = 0 \quad \text{a.s.}
\]

Combine this with (82) to get the conclusion of the lemma.

By definition (80) and the path-continuity of Brownian motion, Lemma 8.1 is sufficient for proving that

\[
\lim_{n \to \infty} \sup_{y \in [a,b]} \left| \zeta_n(y,0) - B \left( \int_0^y \rho_0^2(s) ds \right) \right| = 0 \quad \text{almost surely.} \quad (84)
\]

Thus to prove limit (31) in Theorem 2.4 it suffices to show

\[
\lim_{n \to \infty} \sup_{(x,t) \in A} \left| \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right| = 0 \quad \text{almost surely.} \quad (85)
\]

In other words, we need to strengthen (15) to a.s. convergence. The proof follows the case-by-case reasoning in Section 8 for (15). The error terms \( R_{n,j}, \ j = 1, 2, 3, \) are the same as there. We check that in each case the stronger assumptions (27) and (28) give almost sure convergence.
Lower Bound, Case 1. By the argument for the case \( t \in (0, n^{-(1+\delta)}] \) in Section 6.1.1,
\[
\sum_{n \geq 1} P \left( \inf_{(x,t) \in A, t \in [0, n^{-(1+\delta)}]} \left[ z_{[nx]}^n(nt) - nu(x,t) \right] \right.
- \left. \inf_{y \in I(x,t)} \{ z_{[ny]}^n(0) - nu_0(y) \} \right) \leq -2\varepsilon n^{1/2}
\]
\[
\leq \sum_{n \geq 1} P(H_{n,1}) + \sum_{n \geq 1} P \left( \sup_{x \in [a,b]} \left\{ z_{[nx]}^n(0) - z_{[nx-\alpha_n(1-\delta)/2]}^n(0) \right\} \geq \varepsilon n^{1/2} \right).
\]
\( \sum P(H_{n,1}) < \infty \) follows from Lemma 5.2(ii). To show that the last sum is finite, write
\[
z_{[nx]}^n(0) - z_{[nx-\alpha_n(1-\delta)/2]}^n(0) = \sum_{i=[nx-\alpha_n(1-\delta)/2]+1}^{[nx]} \eta_i^n(0)
\]
in terms of increment, or stick variables. By the local boundedness of \( \rho_0 \) and by assumption (28), the \( \eta_i^n(0) \) are stochastically dominated by i.i.d. exponential variables of finite mean. Now apply standard large deviation estimates.

Lower Bound, Case 2. Now \( t \in [n^{-(1+\delta)}, \tau] \). The argument for this case in Section 6.1.2 showed that on the event \( H_{n,0}^c \cap H_n(\delta_n)^c \),
\[
\inf_{(x,t) \in A, t \in [n^{-(1+\delta)}, \tau]} \left\{ \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right\} \geq R_{n,1} + R_{n,2}.
\]
By Lemma 5.3(ii), \( \sum P(H_n(\delta)) < \infty \) for any fixed \( \delta > 0 \). Then it is possible to find a sequence \( \delta_n \to 0 \) such that \( \sum P(H_n(\delta_n)) < \infty \). For example, for \( j > 0 \) find \( n_0(j) \to \infty \) such that \( \sum_{n \geq n_0(j)} P(H_n(j^{-1}) < 2^{-j}. And then set \( \delta_n = j^{-1} \) for \( n_0(j) \leq n < n_0(j+1) \).

Use also Lemmas 5.2 and 5.3 to see that
\[
\sum_{n=1}^{\infty} \{P(H_{n,0}) + P(H_n(\delta_n)) + P(R_{n,1} \leq -\varepsilon)\} < \infty.
\]

It remains to show \( \lim \inf_{n \to \infty} R_{n,2} \geq 0 \) a.s. Recall the definition of the function \( \phi_\beta \) in (68). We get the desired conclusion by showing that \( \phi_{\delta_n}(\zeta_n(\cdot,0)) \to 0 \) a.s. By (81),
\[
\phi_{\delta_n}(\zeta_n(\cdot,0)) = \sup \left\{ \left[ B \left( n^{-1} T_{n,[nr]} \right) - B \left( n^{-1} T_{n,[nq]} \right) \right] : \right.
\]
\[
|r - q| \leq \delta_n \text{ and } r, q \in [c,d] \} + O(n^{-1/2}).
\]

Let \( \varepsilon \in (0,1) \). Choose \( \beta > 0 \) so that \( \left| \int_q^r \rho_0^2(s)ds \right| < \varepsilon \) for \( q, r \in [c,d] \) such that \( |q - r| \leq \beta \). Let \( U_m \) be the event such that
\[
\left| n^{-1} T_{n,[nq]} - \int_0^{nq} \rho_0^2(s)ds \right| \leq \varepsilon \quad \text{for all } q \in [c,d] \text{ and } n \geq m.
\]

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Let \([g, h]\) be a compact interval such that \(\int_0^q \rho_0^2(s)ds \in [g + 1, h - 1]\) for all \(q \in [c, d]\). Then on the event \(U_m\) we also have \(n^{-1}T_{n,[nq]} \in [g, h]\) for all \(q \in [c, d]\), and
\[
\left| n^{-1}T_{n,[nq]} - n^{-1}T_{n,[nr]} \right| \leq 3\varepsilon
\]
for all \(q, r \in [c, d]\) such that \(|q - r| \leq \beta\), again for all \(n \geq m\). Since \(\delta_n < \beta\) for large \(n\), we have on the event \(U_m\)
\[
\limsup_{n \to \infty} \phi_{\delta_n}(\zeta_n(\cdot, 0)) \leq \sup_{v, s \in [g, h], |v - s| \leq 3\varepsilon} |B(v) - B(s)|.
\]
By Lemma 8.1 \(\lim_{m \to \infty} P(U_m) = 1\), so (87) holds almost surely. Letting \(\varepsilon \searrow 0\) turns (87) into \(\lim_{n \to \infty} \phi_{\delta_n}(\zeta_n(\cdot, 0)) = 0\) by the path-continuity of \(B(\cdot)\).

This completes the proof of the lower bound: we now have
\[
\liminf_{n \to \infty} \inf_{(x,t) \in A} \left\{ \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right\} \geq 0 \quad \text{almost surely.} \quad (88)
\]

**Upper Bound, Case 1.** This is handled by large deviation estimates as was done in **Lower Bound, Case 1** above.

**Upper Bound, Case 2.** Lemma 6.4 already gives almost sure convergence for the error \(R_{n,3}\).

This completes the proof of part (i) of Theorem 2.4.

Following Lemma 6.3, one can show that \(M \equiv \sup_{(x,t) \in A_0} |\zeta_n(x,t)|\) is a.s. finite for an arbitrary compact set \(A_0\). This furnishes the a.s. bound needed to turn the proof of part (ii) of Theorem 2.1 (Section 5.2) into a proof for part (ii) of Theorem 2.4.

9 The distribution-valued processes

First we discuss the general setting of Theorem 2.5. An element \(F \in \mathcal{D}'\) is determined by its actions on \(C_c^\infty(K)\)'s for compact sets \(K \subseteq \mathbb{R}\), so \(R(F, G)\) is clearly a metric on \(H_{\mathrm{loc}}^{-1}(\mathbb{R})\). The separability and completeness of this metric follow from the separability and completeness of \(H^{-1}(\mathbb{R})\).

Let us first check that \(\xi_n(t) \in H_{\mathrm{loc}}^{-1}(\mathbb{R})\), or in other words that \(\chi\xi_n(t) \in H^{-1}(\mathbb{R})\) for any \(\chi \in C_c^\infty(\mathbb{R})\). Let \(\varphi \in H^1(\mathbb{R})\). Set
\[
\bar{u}_n(x, t) = n^{1/2} \left( u(x, t) - u([nx]/n, t) \right).
\]
By a summation by parts,
\[
\chi\xi_n(t, \varphi) = \xi_n(t, \chi\varphi)
\]
Use \((\chi\varphi)' = \varphi' + \varphi'\chi\) and the Schwarz inequality. Let \([a, b]\) contain the support of \(\chi\). By the local Lipschitz property of \(u\) (Lemma 3.4(c)), \(\bar{u}_n = O(n^{-1/2})\) on any compact set. We get

\[
|\chi\xi_n(t, \varphi)| \leq \|\varphi\|_{L^2(\mathbb{R})} \left( \int |\chi'(x)|^2 \{\zeta_n(x, t) + \bar{u}_n(x, t)\}^2 \, dx \right)^{1/2} + \|\varphi'\|_{L^2(\mathbb{R})} \left( \int |\chi(x)|^2 \{\zeta_n(x, t) + \bar{u}_n(x, t)\}^2 \, dx \right)^{1/2} \\
\leq C \cdot \|\varphi\|_{H^1(\mathbb{R})} \left\{ \left( \int_a^b \zeta_n(x, t)^2 \, dx \right)^{1/2} + \frac{1}{\sqrt{n}} \right\}.
\]

This verifies that \(\chi\xi_n(t) \in H^{-1}(\mathbb{R})\), because for any fixed \(\omega\), the process \(\zeta_n(x, t)\) is locally bounded in \((x, t)\). In other words, \(\xi_n(t) \in H^{-1}_{\text{loc}}(\mathbb{R})\).

Next we check that \(\xi_n(t)\) is measurable as a random element of \(H^{-1}_{\text{loc}}(\mathbb{R})\). For fixed \(\chi\) and \(\varphi\), the function \(\omega \mapsto \xi_n(t, \chi\varphi; \omega)\) is measurable. For a fixed \(F \in H^{-1}_{\text{loc}}(\mathbb{R})\), the metric \(R(F, \xi_n(t; \omega))\) is a measurable function of \(\omega\), because the supremum in

\[
\|\chi_k F - \chi_k \xi_n(t)\|_{H^{-1}(\mathbb{R})} = \sup_{\|\varphi\|_{H^1(\mathbb{R})} \leq 1} |F(\chi_k \varphi) - \xi_n(t, \chi_k \varphi)|
\]

can be restricted to countably many \(\varphi\)'s due to the separability of \(H^1(\mathbb{R})\). Now the Borel measurability of the \(H^{-1}_{\text{loc}}(\mathbb{R})\)-valued function \(\omega \mapsto \xi_n(t; \omega)\) follows from the separability of \(H^{-1}_{\text{loc}}(\mathbb{R})\).

Right-continuity of the path \(t \mapsto \xi_n(t; \omega) \in H^{-1}_{\text{loc}}(\mathbb{R})\) follows from the right-continuity and local boundedness of the process \(\zeta_n\), via a calculation that resembles the one performed above.

To summarize, \(\xi_n(\cdot)\) is a random element of the Skorokhod space \(D([0, \infty), H^{-1}_{\text{loc}}(\mathbb{R}))\). We leave it to the reader to show that \(\xi(\cdot)\) is a random element of \(C([0, \infty), H^{-1}_{\text{loc}}(\mathbb{R}))\).

We move to the main point, to prove the strong law \(\sup_{0 \leq t \leq T} R(\xi_n(t), \xi(t)) \to 0\) a.s. Note that in the definition (35) of \(R(F, G)\) the quantities \(\|\chi_k F - \chi_k G\|_{H^{-1}(\mathbb{R})}\) are nondecreasing in \(k\). Consequently, for any \(k\),

\[
R(F, G) \leq \|\chi_k F - \chi_k G\|_{H^{-1}(\mathbb{R})} + 2^{-k}.
\]

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So it suffices to show that for any $k$, almost surely
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq \tau} \sup_{\|\varphi\|_{H^1(\mathbb{R})} \leq 1} |\xi_n(t, \chi_k \varphi) - \xi(t, \chi_k \varphi)| = 0.
\]
Following the earlier calculation and by the definition (35), we get
\[
|\xi_n(t, \chi_k \varphi) - \xi(t, \chi_k \varphi)| \\
\leq \left| - \int (\chi_k \varphi)'(x) \zeta_n(x, t) dx + \int (\chi_k \varphi)'(x) \zeta(x, t) dx \right| + \left| \int (\chi_k \varphi)'(x) \bar{u}_n(x, t) dx \right| \\
\leq C_k \cdot \|\varphi\|_{H^1(\mathbb{R})} \left\{ \left( \int_a^b |\zeta_n(x, t) - \zeta(x, t)|^2 dx \right)^{1/2} + \frac{1}{\sqrt{n}} \right\}. 
\]
The constant $C_k$ is determined by $\chi_k$. Consequently
\[
\sup_{0 \leq t \leq \tau} \sup_{\|\varphi\|_{H^1(\mathbb{R})} \leq 1} |\xi_n(t, \chi_k \varphi) - \xi(t, \chi_k \varphi)| \\
\leq C_k \cdot \left\{ \left( \sup_{0 \leq t \leq \tau} \int_a^b |\zeta_n(x, t) - \zeta(x, t)|^2 dx \right)^{1/2} + \frac{1}{\sqrt{n}} \right\} 
\]
which converges a.s. to 0 by Theorem 2.4(ii). This completes the proof of Theorem 2.5.

As the final item of this section, we verify the correlation formula (42), which requires us to show that, for $\phi, \psi \in C_c^\infty(\mathbb{R})$,
\[
E \left[ \int \int \phi'(x) \psi'(z) B \left( \int_0^{y^+(x,t)} \rho_0^2(r) dr \right) B \left( \int_0^{y^+(z,s)} \rho_0^2(r) dr \right) dx \right] dz \\
= \int \phi(w^+(r,t)) \psi(w^+(r,s)) \rho_0^2(r) dr.
\]
Recall that in the above integrals $y^\pm(x, t)$ are interchangeable because they differ only on Lebesgue null sets, and the same for $w^\pm(r, t)$. In the sequel we manipulate Lebesgue-Stieltjes integrals of the form $\int_a^b f \ dG$. Then we always use the right-continuous version of $G$, with the measure defined by $\mu(a, b) = G(b) - G(a)$, and in case jumps make a difference, the integral is taken over the set $(a, b]$.

We shall make use of the following integration by parts formula. Its proof follows from standard integration by parts [14, Theorem 3.36] and Lemma 2.2. Let $f \in L^1(\mathbb{R})$ and $\varphi$ be compactly supported and differentiable. Then
\[
\int_a^b \varphi'(x) \int_{-\infty}^{y^+(x,t)} f(r) dr \ dx = \varphi(b) \int_{-\infty}^{y^+(b,t)} f(r) dr - \varphi(a) \int_{-\infty}^{y^+(a,t)} f(r) dr \\
- \int_{y^+(a,t)}^{y^+(b,t)} \varphi(w^+(r,t)) f(r) dr. \tag{89}
\]
To begin, recall that \(B(\cdot)\) is a two-sided Brownian motion with independent halves, so if \(rq < 0\) then \(E[B(r)B(q)] = 0\). Assume \(s \leq t\) without loss of generality. Then

\[
E \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} \phi'(x)\psi'(z)B \left( \int_0^{y_+^+(x,t)} \rho_0^2(r)dr \right) B \left( \int_0^{y_+^+(z,s)} \rho_0^2(r)dr \right) dx \, dz \right]
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi'(x)\psi'(z) \left \{ \int_0^{y_+^+(x,t) \wedge y_+^+(z,s)} \rho_0^2(r)dr \right \} 1\{y_+^+(x,t) \wedge y_+^+(z,s) > 0\} dx \, dz
\]

\[
+ \int_{\mathbb{R}} \int_{\mathbb{R}} \phi'(x)\psi'(z) \left \{ \int_0^{y_+^+(x,t) \vee y_+^+(z,s)} \rho_0^2(r)dr \right \} 1\{y_+^+(x,t) \vee y_+^+(z,s) < 0\} dx \, dz
\]

\[
\equiv A_+ + A_-,
\]

where the last line defines the abbreviations \(A_\pm\). We do the calculation for \(A_+\) and leave the similar steps for \(A_-\) to the reader.

\[
A_+ = \int_{w^+(0,t)}^{\infty} dx \int_{w^+(0,s)}^{\infty} dz \, \phi'(x)\psi'(z) \left \{ \int_0^{y_+^+(x,t) \wedge y_+^+(z,s)} \rho_0^2(r)dr \right \}
\]

\[
= \int_{w^+(0,s)}^{\infty} dz \, \psi'(z) \left \{ \int_0^{y_+^+(z,s)} \rho_0^2(r)dr \right \} \int_{w(z,s,t)}^{\infty} dx \, \phi'(x)
\]

\[
+ \int_{w^+(0,s)}^{\infty} dz \, \psi'(z) \int_{w^+(0,t)}^{w(z,s,t)} dx \, \phi'(x) \int_0^{y_+^+(x,t)} \rho_0^2(r)dr
\]

\[
= -\int_{w^+(0,s)}^{\infty} dz \, \psi'(z)\phi(w(z,s,t)) \int_0^{y_+^+(z,s)} \rho_0^2(r)dr
\]

\[
+ \int_{w^+(0,s)}^{\infty} dz \, \psi'(z)\phi(w(z,s,t)) \int_0^{y_+^+(w(z,s,t),t)} \rho_0^2(r)dr
\]

\[
- \int_{w^+(0,s)}^{\infty} dz \, \psi'(z)\phi(w^+(0,t)) \int_0^{y_+^+(w^+(0,t),t)} \rho_0^2(r)dr
\]

\[
- \int_{w^+(0,s)}^{\infty} dz \, \psi'(z) \int_{y_+^+(w^+(0,t),t)}^{y_+^+(w^+(z,s,t),t)} \phi(w^+(r,t))\rho_0^2(r)dr,
\]

where the last equality came from applying \((89)\). Observe that

\[
w(z,s,t) = w^+(r,t)\] for \(y_+^+(z,s) < r < y_+^+(w(z,s,t),t)\), and

\[
w^+(0,t) = w^+(r,t)\] for \(0 < r < y_+^+(w^+(0,t),t)\).

Then the terms above add up to give

\[
A_+ = -\int_{w^+(0,s)}^{\infty} dz \, \psi'(z) \int_0^{y_+^+(z,s)} \phi(w^+(r,t))\rho_0^2(r)dr
\]

which after an application of \((89)\) is

\[
= \psi(w^+(0,s)) \int_0^{y_+^+(w^+(0,s),s)} \phi(w^+(r,t))\rho_0^2(r)dr
\]

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\[
+ \int_{y^+(w^+(0,s),s)}^{\infty} \psi(w^+(r,s)) \phi(w^+(r,t)) \rho_{0}(r)dr.
\]

Observe that \(w^+(0,s) = w^+(r,s)\) for \(0 < r < y^+(w^+(0,s),s)\), to turn this into

\[
A_+ = \int_{0}^{\infty} \psi(w^+(r,s)) \phi(w^+(r,t)) \rho_{0}(r)dr.
\]

Similar arguments show that \(A_-\) is the complementary integral \(\int_{-\infty}^{0}\). Equation (42) is proved.

10 Appendix: Some technical issues

The assumptions in this section are the same as those of Theorem 2.1. We first check the measurability of certain functions and sets. When convenient we add the sample point \(\omega\) as an argument to random quantities. At the end of this section we indicate how Proposition 2.1 follows from the estimates of Sections 4 and 5.

**Proposition 10.1** For each \(n\), the function \(\zeta_n(x,t,\omega) - \inf_{y \in I(x,t)} \zeta_n(y,0,\omega)\) is jointly measurable in \((x,t,\omega)\).

**Proof.** The term \(\zeta_n(x,t,\omega) = n^{-1/2} \{ z^n_{[nx]}(nt,\omega) - nu(x,t) \}\) needs no special argument, as for each \(k\) the variable \(z^n_k(t,\omega)\) is right-continuous in \(t\) and hence progressively measurable in \((t,\omega)\).

Fix \(n\) and consider the term \(\sigma(x,t,\omega) = \inf_{y \in I(x,t)} \zeta_n(y,0,\omega)\). For integers \(m\) and \((x,t,y) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}\) let

\[
h_m(x,t,y) = \begin{cases} 
0 & \text{if } y \in \bigcup_{q \in I(x,t)} [q,q+1/m], \\
\infty & \text{otherwise}.
\end{cases}
\]

Check that \(h_m\) is lower semicontinuous, and hence Borel measurable in \((x,t,y)\). Consequently

\[
\sigma^{(m)}(x,t,\omega) = \inf_{y \in \mathbb{Q}} \{ \zeta_n(y,0,\omega) + h_m(x,t,y) \}
\]

is measurable in \((x,t,\omega)\). [The infimum is over rational \(y\).]

It remains to check \(\sigma^{(m)}(x,t,\omega) \to \sigma(x,t,\omega)\) as \(m \to \infty\). We leave this to the reader. \(\blacksquare\)

In several places in the paper we needed the measurability of the function

\[
Z(\omega) = \sup_{(x,t) \in A} \left| \zeta_n(x,t,\omega) - \inf_{y \in I(x,t)} \zeta_n(y,0,\omega) \right|
\]

where \(A\) is one of three types of compact sets: (i) \(A = [a,b] \times [0,\tau]\), (ii) \(A\) has no shocks, or (iii) \(A\) is finite. Finite is of course trivial. We give the proof for type (i) and leave the (simpler) type (ii) to the reader.
Proposition 10.2 Let $A = [a, b] \times [0, \tau]$ in the definition of $Z(\omega)$. Then there exists a countable set $S \subseteq [a, b] \times (0, \tau]$ such that

$$Z = \sup_{(x,t) \in S} \left| \zeta_n(x,t) - \inf_{y \in I(x,t)} \zeta_n(y,0) \right|. \quad (90)$$

Consequently, by Proposition 10.4, $Z$ is a measurable function on the probability space of the process $z^{n}(\cdot)$.

**Proof.** Let $S$ be a countable subset of $[a, b] \times (0, \tau]$ that satisfies these requirements:

(i) $S$ contains a subset $S' \subseteq S$ such that (a) each $(x,t) \in S'$ has $y^+(x,t) = y(x,t)$, in other words, points in $S'$ are not shocks; and (b) $S'$ is dense in $[a, b] \times (0, \tau]$, and dense in the boundary line segments $\{a\} \times (0, \tau], \{b\} \times (0, \tau]$, and $[a, b] \times \{\tau\}$.

(ii) $S$ contains all the shocks on the boundary line segments, and on the vertical segments $(k/n) \times (0, \tau]$ where $k$ ranges over integers such that $k/n \in [a, b]$. Also, $S$ contains the points $(x, \tau)$ for $x = a$, $x = k/n$ for any integer $k$ such that $k/n \in [a, b]$, and for $x = b$.

(iii) For each integer $\ell$, let $V_\ell$ be the set of points $(b, t)$, $0 < t \leq \tau$, such that $(b, t)$ is not a shock and $y(b,t) = \ell/n$. (We already included shocks $(b, t)$ in $S$ in step (ii).) Include in $S$ a dense countable subset of $V_\ell$ so that each point of $V_\ell$ can be approached from above by a point of $S$.

(iv) $S$ contains all the $U$-points in $[a, b] \times (0, \tau]$.

Requirements (i)–(ii) can be satisfied because there are no more than countably many shocks on any horizontal or vertical line segment. Requirement (iv) can be satisfied by Theorem 3.2.

Suppose now that $(x, t)$ is an arbitrary point of $[a, b] \times (0, \tau]$ outside $S$. Since $U$-points are in $S$, it follows that $I(x, t) = \{y^+(x,t)\}$. We first show that we can find a sequence $(x_j, t_j)$ in $S$ such that $y^+(x, t_j) = y_j$ and

$$\lim_{j \to \infty} |\zeta_n(x_j, t_j) - \zeta_n(y_j,0)| = |\zeta_n(x, t) - \zeta_n(y^+(x,t), 0)|. \quad (91)$$

Start with the case $x < b$. Find $(x_j, t_j) \in S$ so that $I(x_j, t_j) = \{y_j\}$, $x_j \searrow x$, $t_j \searrow t$, and so that $x_j > w(x; t, t_j)$ [possible because $w(x; t, t_j) \searrow x$ as $t_j \searrow t$]. If $t = \tau$ we can choose $t_j = \tau$ for all $j$. Notice that since $x_j$ approaches $x$ from the right, $[nx_j] = \lfloor nx \rfloor$ for large enough $j$, and since the jump processes are right-continuous in time, $z_{[nx_j]}^n(nt_j) = z_{[nx]}^n(nt)$ for large enough $j$. From $x_j > w(x; t, t_j)$ we have $y_j \searrow y^+(x,t)$, so $z_{[ny_j]}^n(0) = z_{[ny^+(x,t)]}^n(0)$ for large $j$. Since $u(x,t)$ is continuous, we get (91) for $x < b$.

Now let $(x, t) = (b, t) \notin S$. Then $(b, t)$ cannot be a shock so $y^+(b,t) = y(b,t)$. Pick $(b, t_j) \in S$ so that $t_j \searrow t$. Exactly as above, $\zeta_n(b, t_j) \to \zeta_n(b,t)$ as $j \to \infty$. Now $y_j = y(b, t_j)$ satisfies $y_j \not\searrow y(b,t)$. We still have $\zeta_n(y_j, 0) \to \zeta_n(y(b,t), 0)$, except possibly in
the case where \( ny(b, t) \) is an integer, when it may happen that \([ny] = [ny(b, t)] - 1\) for all large enough \( j \). But by part (iii) of the definition of \( S \), then we can choose \((b, t_j) \in S\) so that \( y_j = y(b, t) \).

We have checked (91) for all \((x, t) \in [a, b] \times (0, \tau)\). Repeating (92) with \( y^- (x, t) \) in place of \( y^+ (x, t) \) is trickier, so we exclude all cases where the \( \zeta_n(y^- (x, t), 0) \) alternative is irrelevant for the value of \( Z \). First, for the \( \zeta_n(y^- (x, t), 0) \) alternative to matter we must have

\[
\zeta_n(y^-(x, t), 0) = \min\{\zeta_n(y^-(x, t), 0), \zeta_n(y^+(x, t), 0)\} < \zeta_n(y^+(x, t), 0). \tag{92}
\]

Secondly, the quantity \(|\zeta_n(x, t) - \zeta_n(y^-(x, t), 0)|\) will not influence the value of \( Z \) unless at least

\[
|\zeta_n(x, t) - \zeta_n(y^-(x, t), 0)| > \lim_{j \to \infty} |\zeta_n(x_j, t_j) - \zeta_n(y_j, 0)|
\]

where \((x_j, t_j)\) is the sequence from \( S \) that appears in (91). This implies

\[
|\zeta_n(x, t) - \zeta_n(y^-(x, t), 0)| > |\zeta_n(x, t) - \zeta_n(y^+(x, t), 0)|
\]

which together with (92) implies that

\[
\zeta_n(x, t) - \zeta_n(y^-(x, t), 0) > 0.
\]

Thus to show that \(|\zeta_n(x, t) - \zeta_n(y^-(x, t), 0)|\) is no larger than the supremum over \( S \) in (90), it suffices to find a sequence \((x_j, t_j)\) in \( S \) such that \( y^+(x, t_j) = y_j \) and

\[
\limsup_{j \to \infty} \{|\zeta_n(x_j, t_j) - \zeta_n(y_j, 0)|\} \geq \zeta_n(x, t) - \zeta_n(y^-(x, t), 0). \tag{93}
\]

Consider first \( x \in \{a, \{k/n\}\} \). Then we choose \((x_j, t_j) = (x, t_j)\) so that \( t_j \searrow t \). (If \( t = \tau \) we would not be able to approximate \( t \) from above; this is why the point \((x, \tau)\) was included explicitly in \( S \)). Now \( y_j = y(x_j, t_j) \) satisfies \( y_j \searrow y^-(x, t) \). Depending on whether \( ny^-(x, t) \) is an integer or not and whether \( y_j < y^-(x, t) \) or equal, \( z^n_{[ny]}(0) \) converges to \( z^n_{[ny^-(x, t)]}(0) \) or to \( z^n_{[ny^-(x, t)]}(0) \). In either case \( \lim_{j \to \infty} \zeta_n(y_j, 0) \leq \zeta_n(y^-(x, t), 0) \). Right \( t \)-continuity of the random dynamics and the continuity of \( u \) give \( \zeta_n(x, t_j) \to \zeta_n(x, t) \). Thus (93) holds in this case.

It remains to consider \( x \notin \{a, \{k/n\}\} \) in (93). Now pick \( t_j \searrow t \) and \( x_j \nearrow x \). (Again if \( t = \tau \) we can take \( t_j = \tau \).) This forces \( y_j \searrow y^-(x, t) \). Since \( nx \) is not an integer, \( [nx] = [nx] \) for large enough \( j \), and the right \( t \)-continuity of the random dynamics together with the continuity of \( u \) gives \( \zeta_n(x, t_j) \to \zeta_n(x, t) \). The argument of the last paragraph again gives \( \lim_{j \to \infty} \zeta_n(y_j, 0) \leq \zeta_n(y^-(x, t), 0) \) and (93) holds.

As the last measurability issue we show that the event \( H_n \) in Lemma 5.3 is a measurable subset of the underlying probability space.
Proposition 10.3 Fix \( n \) and \( \delta > 0 \). Let \( A \) be a compact subset of \( \mathbb{R} \times [0, \infty) \). Let

\[
H = \{ \omega : z^n_{[nt]}(nt, \omega) = z^n_i(0, \omega) + \Gamma^{n,i}_{[nt] - i}(nt, \omega) \text{ for some } (x, t) \in A \\
\text{and some } i \text{ such that dist}(i/n, I(x, t)) > \delta \}
\]

Then \( H \) is a measurable event on the probability space of the process \( z^n(\cdot) \).

Proof. Fix a left-closed right-open bounded rectangle \( [a, b) \times [0, \tau) \) that contains \( A \), and such that \( an, bn \) and \( \tau n \) are integers. Let \( p \) be a positive integer. For integers \( v, w \) such that \( 2^p na + 1 \leq v \leq 2^p nb \) and \( 1 \leq w \leq 2^p n \tau \), let

\[
K_{v,w}^p = \left[ \frac{v - 1}{2^p n}, \frac{v}{2^p n} \right) \times \left[ \frac{w - 1}{2^p n}, \frac{w}{2^p n} \right)
\]

be a tiling of \( [a, b) \times [0, \tau) \) with small rectangles. The size goes by multiples of \( 2^{-p} \) so that if \( p' > p \) then each \( K_{v,w}^{p'} \) lies inside a unique \( K_{v,w}^p \). \( \overline{K}_{v,w}^p \) is the closure. Let

\[
I_{v,w}^p = \bigcup_{(x,t) \in \overline{K}_{v,w}^p} I(x, t).
\]

For integers \( L < 0 \) put

\[
J_{v,w}^L = \{ i \in \mathbb{Z} : Ln \leq i \leq \lfloor 2^{-p} v \rfloor, \text{ dist}(i/n, I_{v,w}^p) > \delta \}.
\]

Define the measurable event

\[
V_{v,w}^L = \bigcup_{i \in J_{v,w}^L} \{ \omega : z^n_{\lfloor 2^{-p} v \rfloor}(2^{-p} w, \omega) = z^n_i(0, \omega) + \Gamma^{n,i}_{\lfloor 2^{-p} v \rfloor - i}(2^{-p} w, \omega) \}.
\]

Let \( \mathcal{I}^p \) be the set of indices \( (v, w) \) such that \( K_{v,w}^p \) intersects \( A \). Let

\[
U_p^L = \bigcup_{(v,w) \in \mathcal{I}^p} V_{v,w}^L.
\]

Our goal is now to show that

\[
H = \bigcup_{L<0} \bigcup_{m \geq 1} \bigcap_{p \geq m} U_p^L \quad \text{a.s.} \tag{94}
\]

The set on the right-hand side is evidently measurable, and we conclude that so is \( H \).

Fix \( L \) and suppose \( \omega \in U_p^L \) for all large enough \( p \). Then it is possible to choose a subsequence of \( p \)'s and \( (v_p, w_p) \in \mathcal{I}^p \) such that \( \omega \in V_{v_p,w_p}^L \) and the squares \( K_{v_p,w_p}^p \) are nested decreasing. Since \( 2^{-p} v \leq nb \), there are in general only finitely many choices for the index \( i \in J_{v_p,w_p}^L \). Thus by passing to an even further subsequence we may assume that there is a fixed \( i \) that satisfies \( i \in J_{v_p,w_p}^L \) and

\[
z^n_{\lfloor 2^{-p} v_p \rfloor}(2^{-p} w_p) = z^n_i(0) + \Gamma^{n,i}_{\lfloor 2^{-p} v_p \rfloor - i}(2^{-p} w_p) \tag{95}
\]
for all the \( p \) in the subsequence. Since the squares \( K^p_{v_p,w_p} \) are nested, there is a fixed \( k \) such that \( K^p_{v_p,w_p} \subseteq [k/n, (k+1)/n) \times (0, \tau) \). Now the treatment splits into two cases.

**Case 1.** Suppose \( 2^{-p}n^{-1}v < (k+1)/n \) for some \( p \) in the relevant subsequence. Then \( 2^{-p}n^{-1}v \) is bounded away from \((k+1)/n\) for all large enough \( p \) because the nesting of the \( K^p_{v_p,w_p} \)'s forces \( 2^{-p}n^{-1}v \) to be nonincreasing. Pass to the \( p \to \infty \) limit along the relevant subsequence. By the nesting and compactness, there exists a point \( (x,t) \in A \) such that \( 2^{-p}n^{-1}v \searrow x \) and \( 2^{-p}n^{-1}w \searrow t \). Since the convergence comes from the right, we can pass to the limit in (95) to get

\[
 z^n_{[nx]}(nt) = z^i_n(0) + \Gamma^{n,i}_{[nx]-i}(nt). \tag{96}\]

Note also that \( (x,t) \in K^p_{v_p,w_p} \) and \( i \in J^L_{v_p,w_p} \) imply \( \text{dist}(i/n, I(x,t)) > \delta \). Thus (96) says that \( \omega \in H \).

**Case 2.** Suppose \( 2^{-p}n^{-1}v = (k+1)/n \) for all \( p \) in the relevant subsequence. Then after passing to the limit \( p \to \infty \) we have \( x = (k+1)/n \). Again (95) gives (96) with the consequence \( \omega \in H \).

Conversely, we now show that \( H \) lies a.s. in the event on the right-hand side of (94). \( H \) is a.s. the union of the sets

\[
 H_L = \{ \omega \in H : i_n(x,t,\omega) \geq L \text{ for } (x,t) \in [a,b] \times [0, \tau] \} \]

over \( L < 0 \) [recall that \( n \) is fixed now], so it suffices to consider \( \omega \in H_L \) for a fixed \( L \).

Fix \( (x,t) \in A \) and \( i \in [Ln, [nx]] \) such that \( \text{dist}(i/n, I(x,t)) > \delta \) and (96) holds.

For each \( p \) let \( (v_p,w_p) \) be the index such that \( (x,t) \in K^p_{v_p,w_p} \). Pick \( \beta > 0 \) so that \( \text{dist}(i/n, I(x,t)) > \delta + \beta \). For all \( (x',t') \) close enough to \( (x,t) \), \( I(x',t') \) is contained in the \( \beta \)-neighborhood around \( I(x,t) \). Thus for large enough \( p \), \( P(v_p,w_p) \) lies in this \( \beta \)-neighborhood, and consequently \( i \in J^L_{v_p,w_p} \). Increase \( p \) so that \( [nx] = [2^{-p}v] \) and so that neither \( z^n_{[nx]}(\cdot) \) nor \( \Gamma^{n,i}_{[nx]-i}(\cdot) \) jumps in the time interval \((nt, 2^{-p}w_p]\). Then for these large enough \( p \)'s, (96) implies (93), which says that \( \omega \in V^L_{v_p,w_p} \subseteq U^L_p \). This completes the proof. 

Finally, we indicate briefly how to deduce Proposition 2.1 from the estimates. Fix \( \mu \in (2/3, 1) \). The task is to show that \( \lim_{n \to \infty} Z_{r,n} = 0 \) in probability for \( r = 1, 2 \) where

\[
 Z_{1,n} = \sup_{an < k \leq nb, 0 \leq t \leq n^{-\mu}} n^{-1/2} \{ z^n_{k+\ell}(nt) - z^n_k(nt) \}
\]

and

\[
 Z_{2,n} = \sup_{an < k \leq nb, n^{-\mu} \leq t \leq \tau} n^{-1/2} \{ z^n_{k+\ell}(nt) - z^n_k(nt) \}.
\]

Let \( i(k) \) be a minimizer for \( z^n_k(nt) \) in the variational formula (32). Bound \( Z_{1,n} \) above by

\[
 \sup_{an < k \leq nb, 0 \leq t \leq n^{-\mu}} n^{-1/2} \{ z^n_{k+\ell}(nt) - z^n_{i(k)}(nt) \},
\]

...
use Lemma 5.2 to bound $i(k)$ from below, and appeal to assumption (9).

Bound $Z_{2,n}$ above by

$$\sup_{an \leq k \leq nb, n^- \leq t \leq r} n^{-1/2} \left\{ \Gamma_{k+i(k)}^n(nt) - \Gamma_{k-i(k)}^n(nt) \right\} \leq n^{-1/2} \left[ \sup_{an \leq k \leq nb, n^- \leq t \leq r} \Gamma_{k+i(k)}^n(nt) \cdot 1\{k-i(k) \leq M_0(\log n)^{3/2}\} 
+ \left( \Gamma_{k+i(k)}^n(nt) - \Gamma_{k-i(k)}^n(nt) \right) \cdot 1\{k-i(k) > M_0(\log n)^{3/2}\} \right],$$

where $M_0$ is the constant appearing in Lemma 4.4. Now apply Lemmas 4.3 and 4.4.

Under the stronger assumptions of local equilibrium these estimations can be made summable in $n$ and a.s. convergence follows by Borel-Cantelli.

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