ON THE OPTIMIZATION PROBLEMS OF THE PRINCIPAL EIGENVALUES OF MEASURE DIFFERENTIAL EQUATIONS WITH INDEFINITE MEASURES

ZHUYAN WEN∗
School of Mathematical Sciences
Inner Mongolia University
Huhhot, 010021, China

MEIRONG ZHANG
Department of Mathematical Sciences
Tsinghua University
Beijing, 100084, China

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Abstract. In this paper, we will first establish the necessary and sufficient conditions for the existence of the principal eigenvalues of second-order measure differential equations with indefinite weighted measures subject to the Neumann boundary condition. Then we will show the principal eigenvalues are continuously dependent on the weighted measures when the weak∗ topology is considered for measures. As applications, we will finally solve several optimization problems on principal eigenvalues, including some isospectral problems.

1. Introduction. In this paper, we are concerned with the principal eigenvalues of the second-order ordinary differential equations (ODEs or ODE)

\[-y'' = \lambda w(x)y\] on \(I := [0, 1]\),

subject to the Neumann boundary condition

\[y'(0) = y'(1) = 0.\] (N)

Here \(w(x) \in L^1(I)\) are nonzero indefinite (integrable) weights. By a principal eigenvalue \(\lambda\) of problem (1)-(N), it means that the problem admits eigenfunctions \(\varphi(x) > 0\) on \(I\). For any weight \(w \neq 0\), \(\lambda = 0\) is a trivial principal eigenvalue with an eigenfunction \(\varphi(x) \equiv 1\). Henceforth, our concern is on nontrivial principal eigenvalues. For simplicity, the principal eigenvalues in this paper are always meant the positive principal eigenvalues. For the ODE case (1), the existence

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∗ Corresponding author: Zhuyan Wen.
of principal eigenvalues is clear [2, 5]. In fact, the necessary and sufficient conditions for (1)-(N) to admit a principal eigenvalue $\lambda_{\text{prin}} > 0$ are

$$\int_I w_+(x) \, dx > 0 \quad \text{and} \quad \int_I w(x) \, dx < 0. \quad (2)$$

Here $w_\pm(x) = \max\{\pm w(x), 0\}$ are the positive and negative parts of $w(x)$. See, for example, [22, Section 4]. The proofs in [22] are based on the Prüfer transformation and detailed analysis for the resulted argument functions. When $w$ satisfies (2), the principal eigenvalue $\lambda_{\text{prin}} = \lambda_{\text{prin}}(w)$ is unique. On higher dimensional domains, the existence of principal eigenvalues of partial differential equations can be found from [4, 28].

Principal eigenvalues $\lambda_{\text{prin}} = \lambda_{\text{prin}}(w)$ are important in ecology systems and population dynamics [5, 17, 18]. When the environment is spatially heterogeneous, $w(x)$ is the (indefinite) density of the distribution of resource on $I$ and the principal eigenvalue $\lambda_{\text{prin}}(w)$ stands for the surviving threshold for a single species. This leads to many interesting optimization problems on principal eigenvalues. Some have been solved [7, 15, 18], while many are still open [17]. In this paper, we consider the following related optimization problem. Given $0 < A < B$, we denote

$$W_{A,B} := \{ w \in L^1(I) : \int_I w_+(x) \, dx > 0, \int_I w(x) \, dx \leq -A, \text{and} \int_I |w(x)| \, dx \leq B \}. \quad (3)$$

It is not difficult to see that

$$\sup_{w \in W_{A,B}} \lambda_{\text{prin}}(w) = +\infty. \quad (4)$$

The optimization problem we are interested in is

$$\tilde{L}(A, B) := \inf_{w \in W_{A,B}} \lambda_{\text{prin}}(w), \quad (5)$$

which is well defined. Due to the non-compactness of $W_{A,B}$ (even in the weak topology of $L^1(I)$), we will see that the solution of problem (5) will lead to more general distributions of resources which have no densities (with respect to the Lebesgue measure). Moreover, principal eigenvalues and eigenfunctions have to be explained using the so-called measure differential equations (MDEs or MDE), which are a special class of generalized ordinary differential equations (GODEs) studied extensively in [24, 27].

Let us use $\mathcal{M}_0(I)$ to denote the space of (normalized, real, Radon) measures on $I$. The norm $\| \cdot \|$ of $\mathcal{M}_0(I)$ is the total variation of measures. The space $\mathcal{M}_0(I)$ is isometrically identical as the dual space of $C(I)$, the space of real continuous functions on $I$ with supremum norm $\| \cdot \|_{\infty}$. For more details on $\mathcal{M}_0(I)$, see §2.

The second-order linear MDE with a measure $\mu \in \mathcal{M}_0(I)$ is written as

$$-d u^{\bullet}(x) + u(x) \, d\mu(x) = 0, \quad x \in I, \quad (6)$$

following the notations from [23]. (Strong) solutions $u(x)$ of initial value problems of (6) are explained using the equivalent system of integral equations. Here $u^{\bullet}(x)$, $x \in I$ stands for the generalized right derivative of the solution $u(x)$. It is known that $u^{\bullet}(x)$ is a function of bounded variation, or a non-normalized Radon measure on $I$. With a measure $\nu \in \mathcal{M}_0(I)$ as a potential, eigenvalues of MDE

$$-d u^{\bullet}(x) + u(x) \, d\nu(x) = \tau u(x) \, dx, \quad x \in I, \quad (7)$$

with the Dirichlet boundary condition

$$u(0) = u(1) = 0, \quad (D)$$
or, with the Neumann boundary condition
\[ u^*(0) = u^*(1) = 0, \] (N)

have been established in [23]. The structure of these eigenvalues are the same as those of the ODEs with integrable potentials. For example, problem (7)-(N) possesses a sequence of eigenvalues \( \tau_0(\nu) < \tau_1(\nu) < \cdots \to +\infty \), where \( \tau_0(\nu) \) is called the zeroth Neumann eigenvalue.

Spectral theory of MDEs and some types of GODEs has been extensively studied in recent works like [11, 12, 13, 16, 26, 37]. In [37], by considering a nonzero non-negative measure \( \rho \in M_0(I) \) as a weight, for the (weighted) eigenvalue problem
\[ -du^*(x) = \lambda u(x) \, d\rho(x), \quad x \in I, \] (8)
with (D) or (N), some completely different features have been revealed. For example, the numbers of eigenvalues of problems (8)-(D) and (8)-(N) depend on measures \( \rho \) and may be finite. These results on second-order MDEs, together with the so-called completely continuous dependence of eigenvalues on measures, have been extended in [16] to some class of third-order symmetric MDEs. Motivated by the study on the Camassa-Holm equation [8, 9] and general strings, some spectral and inverse spectral problems on second-order GODEs with indefinite distributions as weights have been developed in [11, 12, 13] and are still under developing. In order to solve problems (4) and (5) and for the theoretical purpose, in this paper, by taking indefinite measures \( \rho \in M_0(I) \), we will give a relatively complete study on the (positive) principal eigenvalues of MDEs (8) with respect to the Neumann boundary condition (N).

According to the Jordan’s decomposition theorem [31, Theorem 3.3], any \( \rho \in M_0(I) \) can be uniquely decomposed as \( \rho = \rho_+ - \rho_- \). Here \( \rho_+ \) and \( \rho_- \) are the non-negative and non-positive part of \( \rho \), respectively. As functions of bounded variation, both \( \rho_\pm(x) \) are non-decreasing in \( x \in I \). Corresponding to (2) for indefinite integrable weights, let us introduce a subset
\[ M_{\text{prin}} := \{ \rho \in M_0(I) : \|\rho_+\| > 0 \text{ and } \int_I d\rho < 0 \} \]
of measures on \( I \). As before, by a principal eigenvalue \( \lambda \) of (8)-(N), we always refer \( \lambda \) to a positive principal eigenvalue.

The first part of this paper contains two main results. This first one gives the necessary and sufficient conditions for the existence of positive principal eigenvalues.

**Theorem 1.1.** (i) Eigenvalue problem (8)-(N) admits a principal eigenvalue if and only if \( \rho \in M_{\text{prin}} \).

(ii) Let \( \rho \in M_{\text{prin}} \) be given. Then problem (8)-(N) has a unique principal eigenvalue, denoted by \( \lambda_{\text{prin}} = \lambda_{\text{prin}}(\rho) > 0 \). Moreover, \( \lambda_{\text{prin}} \) can be characterized by
\[ \lambda_{\text{prin}} = \min_{u \in U_\rho} \frac{\int_I (u'(x))^2 \, dx}{\int_I u^2(x) \, d\rho(x)}, \] (9)
and the minimum is attained and only attained by eigenfunctions associated with \( \lambda_{\text{prin}} \). Here
\[ U_\rho := \{ u \in W^{1,2}(I) : \int_I u^2 \, d\rho > 0 \} \subset W^{1,2}(I). \] (10)

Because we will establish in Proposition 1 the equivalence of weak and strong solutions of (8), so different from the approach by the Prüfer transformation in [22, 23, 37], the proof of Theorem 1.1 is variational.
In the measure space $\mathcal{M}_0(I)$, besides the topology induced by norm $\| \cdot \|$, it also possesses the weak* topology $w^*$ induced by the weak* convergence of measures. The next result asserts that $\lambda_{\text{prin}}(\rho)$ is continuous in $\rho \in \mathcal{M}_{\text{prin}}$ even when the weak* topology $w^*$ is considered for $\mathcal{M}_{\text{prin}}$.

**Theorem 1.2.** Assume that $\rho \in \mathcal{M}_{\text{prin}}$ and $\rho_n \in \mathcal{M}_{\text{prin}}$ for all $n \geq 1$. If $\rho_n \rightharpoonup \rho$ with respect to the weak* topology $w^*$, then $\lambda_{\text{prin}}(\rho_n) \to \lambda_{\text{prin}}(\rho)$.

Since the weak* topology $w^*$ is considered for $\mathcal{M}_{\text{prin}}$, Theorem 1.2 asserts that $\lambda_{\text{prin}}(\rho)$ is continuously dependent on $\rho$ in a very strong way. Such a continuous dependence can be called the complete continuity or the strong continuity. This type of continuity has been first proved in [23, 37] for eigenvalues of (7) in potential measures $\nu$ and of (8) in non-negative weight measures $\rho$. Different from the proofs in [23, 37], in this paper, Theorem 1.2 is obtained from the minimization characterization (9) for principal eigenvalues.

In the second part of this paper, we will apply Theorems 1.1 and 1.2 to solve several optimization problems on principal eigenvalues. Corresponding to (3) for indefinite weights is the following set of indefinite measures

$$
\mathcal{M}_{A,B} := \{ \rho \in \mathcal{M}_0(I) : \| \rho_+ \| > 0, \int_I d\rho \leq -A, \| \rho \| \leq B \},
$$

where $0 < A < B$ are given constants. By letting $\alpha_{A,B} := (B - A)/2 > 0$, $\beta_{A,B} := (B + A)/2 > 0$,

$$
\hat{\rho}_{A,B} := \alpha_{A,B} \delta_0 - \beta_{A,B} \delta_1, \quad \check{\rho}_{A,B} := -\beta_{A,B} \delta_0 + \alpha_{A,B} \delta_1,
$$

(11)

where $\delta_0$ and $\delta_1$ are the Dirac measures at 0 and 1, respectively. We will obtain the following results.

**Theorem 1.3.** There hold

$$
\mathbf{L}(A, B) := \inf_{\rho \in \mathcal{M}_{A,B}} \lambda_{\text{prin}}(\rho) \equiv \frac{4A}{B^2 - A^2}, \quad (12)
$$

$$
\mathbf{M}(A, B) := \sup_{\rho \in \mathcal{M}_{A,B}} \lambda_{\text{prin}}(\rho) = +\infty. \quad (13)
$$

Furthermore, the minimal value $\mathbf{L}(A, B)$ is attained and only attained by either $\hat{\rho}_{A,B}$ or $\check{\rho}_{A,B}$, the measures defined by (11).

By using the minimization characterization (9) for principal eigenvalues, problems (12) and (13) can be reduced to finitely dimensional extremal problems, whose solutions are elementary. After giving the proofs in §4.1, we will give more comments on the method used in this paper.

Next, let $0 < A < B$ be given. It is known from Theorem 1.3 that $\lambda_{\text{prin}}(\rho) \in [\mathbf{L}(A,B), +\infty)$ for any $\rho \in \mathcal{M}_{A,B}$. Conversely, for any $\gamma \in [\mathbf{L}(A,B), +\infty)$, by letting

$$
\mathcal{M}_\gamma := \{ \rho \in \mathcal{M}_{A,B} : \lambda_{\text{prin}}(\rho) = \gamma \},
$$

we will study the following minimization problem on measures

$$
\mathbf{T}(\gamma) := \min_{\rho \in \mathcal{M}_\gamma} \int_I d\rho. \quad (14)
$$

Problem (14) is an isospectral problem. Theorem 1.3 asserts that when $\gamma = \mathbf{L}(A,B)$, one has $\mathcal{M}_{\mathbf{L}(A,B)} = \{ \hat{\rho}_{A,B}, \check{\rho}_{A,B} \}$. Since $\int_I d\hat{\rho}_{A,B} = \int_I d\check{\rho}_{A,B} = -A$, we have
\( T(L(A, B)) = -A \). Hence problem (12) is the inverse of problem (14) with the choice of \( \gamma = L(A, B) \). For general \( \gamma \in [L(A, B), +\infty) \), let us introduce

\[
\alpha_\gamma := \frac{B}{2} + \frac{1}{\gamma} - \left( \frac{B^2}{4} + \frac{1}{\gamma^2} \right)^{1/2} > 0, \quad \beta_\gamma := \frac{B}{2} - \frac{1}{\gamma} + \left( \frac{B^2}{4} + \frac{1}{\gamma^2} \right)^{1/2} > 0, \quad (15)
\]

\( \hat{\rho}_\gamma := \alpha_\gamma \delta_0 - \beta_\gamma \delta_1, \quad \hat{\rho}_\gamma := -\beta_\gamma \delta_0 + \alpha_\gamma \delta_1. \quad (16) \)

The solution of minimization problem (14) is shown by

**Theorem 1.4.** There holds

\[
T(\gamma) = \frac{2}{\gamma} - \left( B^2 + \frac{1}{\gamma^2} \right)^{1/2} \quad \forall \gamma \in [L(A, B), +\infty). \quad (17)
\]

Moreover, the minimal value \( T(\gamma) \) is attained and only attained by either \( \hat{\rho}_\gamma \) or \( \hat{\rho}_\gamma \) defined in (16).

The rest of this paper is organized as follows. In §2, we record some preliminaries and prove some basic facts regarding measure theory and second-order MDEs. §3 is devoted to prove Theorem 1.1 and Theorem 1.2. In §4.1 and §4.2, Theorem 1.3 and Theorem 1.4 are proved. Finally, in §4.3, by using the complete continuity of \( \lambda_{\text{prin}}(\rho) \) in Theorem 1.2 and a smooth approximation theorem for general measures in [19], one can find that the solution of problem (5) is the same as (13), i.e. \( \tilde{L}(A, B) \equiv L(A, B) \). For details, see Theorem 4.5.

2. Measures and measure differential equations.

2.1. Measures. Let us first recall from [6, 31] some preliminaries of (Radon) measures on the interval \( I = [0, 1] \). For a real function \( \rho : I \rightarrow \mathbb{R} \), we use \( \|\rho\| \) to denote the total variation of \( \rho \) on \( I \). Generally, \( \|\rho\| \) may take value \( +\infty \). Let us define

\[
\mathcal{M}(I) := \left\{ \rho : I \rightarrow \mathbb{R} : \rho(x+)=\rho(x) \quad \text{for all } x \in (0,1), \quad \|\rho\| < +\infty \right\},
\]

the space of functions of bounded variation on \( I \). Here \( \rho(x+) := \lim_{t \uparrow x} \rho(t) \) is the right limit at \( x \). Let

\[
\mathcal{M}_0(I) := \{ \rho \in \mathcal{M}(I) : \rho(0+) = 0 \}.
\]

The Riesz representation theorem asserts that \( (\mathcal{M}_0(I), \|\cdot\|) \) is isometrically identical as the dual space of \( (C(I), \|\cdot\|_\infty) \), the Banach space of continuous functions on \( I \) endowed with the supremum norm \( \|\cdot\|_\infty \). In fact, by using the Riemann-Stieltjes integrals, any \( \rho \in \mathcal{M}_0(I) \) defines a linear functional \( \rho^* \in (C(I), \|\cdot\|_\infty)^* \) by

\[
\rho^*(u) := \int_I u(x) d\rho(x) \quad \forall u \in C(I).
\]

For this reason, \( \mathcal{M}(I) \) and \( \mathcal{M}_0(I) \) are respectively called the space of non-normalized Radon measures on \( I \) and the space of (normalized Radon) measures on \( I \).

We use \( \delta_a, a \in I \) to denote the (unit) Dirac measure at \( a \). Clearly, \( \|\delta_a\| = 1 \) for any \( a \in I \) and \( \|\delta_a - \delta_b\| = 2 \) for all \( a, b \in I \) with \( a \neq b \). We say that \( \rho = \rho(x) \in \mathcal{M}_0(I) \) is a non-negative measure, if

\[
\int_I u(x) d\rho(x) \geq 0 \quad \forall u \in C(I) \text{ with } u(x) \geq 0.
\]

A measure \( \rho \) is non-negative if and only if \( \rho(x) \) is non-decreasing on \( I \); see [31]. For example, the Dirac measures \( \delta_a \) are non-negative. The Jordan’s decomposition theorem for measures is as follows.
Lemma 2.1. [31, Theorem 3.3] Any \( \rho \in M_0(I) \) can be written uniquely as \( \rho = \rho_+ - \rho_- \), where both \( \rho_+, \rho_- \in M_0(I) \) are non-negative measures.

We call \( \rho_+ \) the non-negative part and \( \rho_- \) the non-positive part of \( \rho \). One has then

\[
\begin{align*}
\int_I d\rho_+(x) &= \rho_+(1) - \rho_+(0) = \|\rho_+\|, \\
\int_I d\rho_-(x) &= \|\rho_+\| - \|\rho_-\|, \\
\|\rho\| &= \|\rho_+\| + \|\rho_-\|, \\
\int_I u(x) d\rho(x) &\leq \|\rho\| u_\infty \quad \forall u \in C(I).
\end{align*}
\]

See [6, 31].

For \( \rho \in M_0(I) \) and \( x \in I \), let \( \|\rho\|_{[0,x]} \) be the total variation of \( \rho \) on interval \([0, x]\). If we consider \( \rho(x) \) and \( \|\rho\|_{[0,x]} \) as functions of \( x \in I \), then for any \( x_0 \in I \), \( \rho(x) \) is (left/right) continuous at \( x = x_0 \) if and only if \( \|\rho\|_{[0,x]} \) is (left/right) continuous at \( x = x_0 \). Based on such an observation, we have the following conclusion.

Lemma 2.2. If \( \rho \in M_0(I) \) satisfies \( \|\rho_+\| > 0 \), then \( \int_I u^2(x) d\rho(x) > 0 \) for some \( u(x) \in C^\infty(I) \); that is, the set \( U_\rho \) defined in (10) is nonempty.

Proof. The condition \( \|\rho_+\| > 0 \) implies that \( \rho(x) \) cannot be a decreasing function in \( x \in I \). Thus there exist \( 0 \leq \xi < \eta \leq 1 \) such that \( \rho(\xi) < \rho(\eta) \). Without loss of generality, we assume that \( 0 < \xi < \eta < 1 \) and let \( h := \rho(\eta) - \rho(\xi) > 0 \). Since \( \rho(x) \) is right continuous at \( \xi \) and \( \eta \), there exists \( \delta > 0 \) small enough such that

\[
|\rho(\xi) - \rho(\xi + \delta)| \leq h/4 \quad \text{and} \quad \|\rho\|_{[\xi, \xi + \delta]} + \|\rho\|_{[\eta, \eta + \delta]} \leq h/4.
\]

Let us take a smooth function \( u \in C^\infty(I) \) such that

\[
0 \leq u(x) \leq 1 \quad \forall x \in I, \quad u(x) = 1 \quad \forall x \in [\xi + \delta, \eta],
\]

and with the support being contained in \([\xi, \xi + \delta]\). Consequently,

\[
\int_I u^2(x) d\rho(x) = \left( \int_{[\xi + \delta, \eta]} + \int_{[\xi, \xi + \delta]} \right) u^2(x) d\rho(x) \\
\geq (\rho(\eta) - \rho(\xi + \delta)) - \|\rho\|_{[\xi, \xi + \delta]} - \|\rho\|_{[\eta, \eta + \delta]} \\
\geq \frac{h}{2} > 0.
\]

This completes the proof. \( \square \)

2.2. Second-order linear MDEs. Given a measure \( \mu \in M_0(I) \), the second-order linear MDE with the measure \( \mu \) is written as

\[
du^*(x) + u(x) d\mu(x) = 0, \quad x \in I.
\]

See [23]. Here the notation \( u^*(x) \) stands for the generalized right derivative of \( u(x) \). With an initial value

\[
(u(0), u^*(0)) = (u_0, v_0) \in \mathbb{R}^2,
\]

the (strong) solution \( u(x) \) of MDE (18), together with \( v(x) := u^*(x) \), is explained using the following equivalent system of integral equations

\[
u(x) = u_0 + \int_{[0,x]} v(t) \, dt \quad \forall x \in I,
\]

\[
u(x) = \begin{cases} v_0, & \text{for } x = 0, \\
v_0 - \int_{[0,x]} u(t) \, d\mu(t), & \text{for } x \in (0, 1].
\end{cases}
\]

\[
\int u^2(x) d\rho(x) = \left( \int_{[\xi + \delta, \eta]} + \int_{[\xi, \xi + \delta]} \right) u^2(x) d\rho(x) \\
\geq (\rho(\eta) - \rho(\xi + \delta)) - \|\rho\|_{[\xi, \xi + \delta]} - \|\rho\|_{[\eta, \eta + \delta]} \\
\geq \frac{h}{2} > 0.
\]

This completes the proof. \( \square \)

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\]

\[
u(x) = \begin{cases} v_0, & \text{for } x = 0, \\
v_0 - \int_{[0,x]} u(t) \, d\mu(t), & \text{for } x \in (0, 1].
\end{cases}
\]
It is known that both $u(x) = u(x; \mu) = u(x; \mu, u_0, v_0)$ and $v(x) = u^\bullet(x; \mu) = u^\bullet(x; \mu, u_0, v_0)$ are uniquely defined on $I$. Moreover, $u(x) \in C(I)$ and $u^\bullet(x) \in M(I)$. See [24] or [23, Theorem 3.1]. By (19), one sees that

$$u^\bullet(x) = u'(x) \quad \ell\text{-a.e. } x \in [0,1].$$

(21)

Here $u'(x)$ is the classical right derivative of $u(x)$ and $\ell$ is the Lebesgue measure of $I$, i.e. $\ell(x) \equiv x$.

**Remark 1.** Due to the uniqueness of solutions of initial value problems, if a non-trivial solution $u(x)$ of MDE (18) has a zero $u(x_0) = 0$, then $u^\bullet(x_0) \neq 0$. Though $u^\bullet(x_0)$ is only the generalized right-derivative, it is possible to use integral equations (19) and (20) to show that $x_0$ is an isolated zero of $u(x)$. In fact, if $x_0 \in (0,1)$, $u(x)$ must change signs at two sides of $x_0$.

The next proposition shows that the Neumann eigenvalues and eigenfunctions can also be determined by weak solutions.

**Proposition 1.** The following are equivalent:

(i) $u(x) \in W^{1,\infty}(I)$ is a solution of MDE (18) satisfying (N).

(ii) $u(x) \in W^{1,2}(I)$ is a weak solution of (18)-(N) in the sense

$$\int_I u'(x)\phi'(x) \, dx = \int_I u(x)\phi(x) \, d\mu(x) \quad \forall \phi \in W^{1,2}(I).$$

(22)

**Proof.** Assume first that $u \in W^{1,\infty}(I)$ is a solution of (18)-(N). Then $u$ and $v = u^\bullet$ fulfill (19) and (20) with $v_0 = 0$ and $u_0 = u(0) \in \mathbb{R}$. Since $u$ satisfies (N), it follows from $v_0 = 0$ and (20) that

$$v(1) = -\int_I u(t) \, d\mu(t) = 0.$$

(23)

According to (21), one has

$$u'(x) = -\int_{[0,x]} u(t) \, d\mu(t) \quad \text{for } \ell\text{-a.e. } x \in I.$$

Thus

$$\int_I u'(x)\phi'(x) \, dx = -\int_I \phi'(x) \left(\int_{[0,x]} u(t) \, d\mu(t)\right) \, dx$$

$$= -\int_I \left(\int_{[t,1]} \phi'(x) \, dx\right) u(t) \, d\mu(t) \quad \text{(by the Fubini’s theorem)}$$

$$= -\int_I \phi(1) u(t) \, d\mu(t) + \int_I \phi(t) u(t) \, d\mu(t).$$

This gives (22) by exploiting (23).

Conversely, assume that $u(x) \in W^{1,2}(I)$ satisfies (22). By letting $\phi(x) \equiv 1$ in (22), one has

$$\int_I u(t) \, d\mu(t) = 0.$$

(24)

For any $x \in I$, by taking in (22) the following test function

$$\phi_x(t) = \begin{cases} t, & \text{if } 0 \leq t \leq x, \\ x, & \text{if } x < t \leq 1, \end{cases}$$

...
we obtain
\[ u(x) - u(0) = \int_I u'(t)\phi_x'(t) \, dt = \int_{[0,x]} \phi_x(t)u(t) \, d\mu(t) \]
\[ = \int_{[0,x]} tu(t) \, d\mu(t) + \int_{(x,1]} xu(t) \, d\mu(t) \]
\[ = \int_{[0,x]} tu(t) \, d\mu(t) - \int_{[0,x]} xu(t) \, d\mu(t) \quad \text{(by using (24))} \]
\[ = \int_{[0,x]} (t-x)u(t) \, d\mu(t). \quad (25) \]

Meanwhile, let us define
\[ U(x) = \begin{cases} 0, & \text{if } x = 0, \\ -\int_{[0,x]} u(t) \, d\mu(t), & \text{if } x \in (0,1]. \end{cases} \quad (26) \]

It then follows from the Fubini’s theorem that
\[ \int_{[0,x]} U(t) \, dt = -\int_{[0,x]} \left( \int_{[0,t]} u(s) \, d\mu(s) \right) \, dt \]
\[ = \int_{[0,x]} (t-x)u(t) \, d\mu(t) \quad \forall x \in I. \quad (27) \]

From (25) and (27), one has \( u(x) - u(0) = \int_{[0,x]} U(t) \, dt. \) Thus \( U(x) = u^*(x). \) Now (25)-(26) corresponds to system (19)-(20) with the initial value \( (u(0), u^*(0)) = (u(0), 0) \). By the uniqueness of solutions of MDE (18), we know that \( u(x) \) is actually in \( W^{1,\infty}(I) \). Moreover, besides \( u^*(0) = 0 \), we have from (24) and (26) that \( u^*(1) = U(1) = 0 \), i.e. \( u(x) \) also satisfies \( (N) \).

3. Positive principal eigenvalues.

3.1. Existence of principal eigenvalues — Proof of Theorem 1.1. In this subsection we prove the existence of the principal eigenvalue of problem (8)-(N) and complete the proof of Theorem 1.1.

Let \( \rho \in \mathcal{M}_{\text{prin}} \) be fixed. By Lemma 2.2, \( U_{\rho} \subset W^{1,2}(I) \) is non-empty. Since \( \int_I d\rho < 0 \), any \( u \in U_{\rho} \) is nonconstant. Hence the Rayleigh form
\[ R(u) = R(u,\rho) := \frac{\int_I (u^'(x))^2 \, dx}{\int_I u^2(x) \, d\rho(x)}, \quad u \in U_{\rho}, \]
is well defined and positive. Define
\[ \lambda_{\text{prin}} := \inf_{u \in U_{\rho}} R(u). \quad (28) \]

We give the proof of Theorem 1.1 in a series of claims.

Claim 1. We claim that \( \lambda_{\text{prin}} > 0 \) and the infimum in (28) is attained by some \( \varphi \in U_{\rho} \).

Let us take any minimizing sequence \( u_n \in U_{\rho} \) so that \( \lim_{n \to \infty} R(u_n) = \lambda_{\text{prin}} \).

Due to the homogeneity of \( R(u) \), we can assume that \( \|u_n\|_\infty = 1 \) for all \( n \geq 1 \). Then for any \( \varepsilon > 0 \), there exists a sufficient large integer \( n_\varepsilon \) such that
\[ \|u_n^'\|^2_2 \leq (\lambda_{\text{prin}} + \varepsilon) \int_I u_n^2(x) \, d\rho(x) \leq (\lambda_{\text{prin}} + \varepsilon) \|\rho\| \quad \forall n > n_\varepsilon. \quad (29) \]
Here $\| \cdot \|_2 = \| \cdot \|_{L^2(I)}$. Hence the sequence $\{u_n\}_{n \geq 1}$ is bounded in $W^{1,2}(I)$. It then follows from the local weak compactness of $W^{1,2}(I)$ that there is a subsequence $\{u_{n_j}\}_{j \geq 1}$ such that

$$u_{n_j} \rightharpoonup \varphi \text{ weakly in } W^{1,2}(I) \text{ for some } \varphi \in W^{1,2}(I). \quad (30)$$

As the imbedding $W^{1,2}(I) \hookrightarrow C(I)$ is compact [1], passing to a subsequence if necessary, one has $u_{n_j} \to \varphi$ in $(C(I), \| \cdot \|_{\infty})$. It turns out that $\| \varphi \|_{\infty} = 1$ and $\int_I \varphi^2 \, d\rho \geq 0$. Moreover, it follows from (30) that

$$\| \varphi' \|_2^2 \leq \liminf_{j \to \infty} \| u'_{n_j} \|_2^2. \quad (31)$$

Since $\int_I d\rho < 0$, we infer from $\int_I \varphi^2 \, d\rho \geq 0$ that $\varphi$ cannot be a constant function. Thus $\| \varphi' \|_2 > 0$. Now (29) and (31) show that

$$\| \varphi' \|_2^2 \leq (\lambda_{\text{prin}} + \varepsilon) \int_I \varphi^2(x) \, d\rho(x) \leq (\lambda_{\text{prin}} + \varepsilon) \| \rho \| \quad \forall \varepsilon > 0.$$

By letting $\varepsilon \to 0$, we obtain $\int_I \varphi^2(x) \, d\rho(x) > 0$, i.e. $\varphi \in U_\rho$. Moreover,

$$\lambda_{\text{prin}} \geq \frac{\| \varphi' \|_2^2}{\int_I \varphi^2(x) \, d\rho(x)} = \mathcal{R}(\varphi) > 0.$$

Combining with (28), we obtain

$$\lambda_{\text{prin}} = \min_{u \in U_\rho} \mathcal{R}(u) = \mathcal{R}(\varphi) > 0. \quad (32)$$

**Claim 2.** The minimum $\lambda_{\text{prin}}$ in (32) is an eigenvalue of (8)-(N) with $\varphi(x)$ being an eigenfunction.

Note that $U_\rho$ is an open subset of $W^{1,2}(I)$. It then follows from (32) that for any $\phi \in W^{1,2}(I)$, one has

$$\frac{d\mathcal{R}(\varphi + \varepsilon \phi)}{d\varepsilon} \bigg|_{\varepsilon = 0} = 0.$$

This gives the identity

$$\int_I \varphi'(x) \phi'(x) \, dx = \lambda_{\text{prin}} \int_I \varphi(x) \phi(x) \, d\rho(x) \quad \forall \phi \in W^{1,2}(I). \quad (33)$$

Applying Proposition 1 to (33), we know that $\lambda_{\text{prin}}$ is an eigenvalue of problem (8)-(N), while $\varphi(x)$ is an associated eigenfunction.

**Claim 3.** We show that $\lambda_{\text{prin}}$ is a positive principal eigenvalue of (8)-(N).

Recall that $\varphi \in W^{1,2}(I)$ is an eigenfunction associated with $\lambda_{\text{prin}}$. As a nontrivial solution of MDE, $\varphi(x)$ has only finitely many isolated zeros on $I$. Then $\psi(x) := |\varphi(x)| \in W^{1,2}(I)$ satisfies $\psi^2(x) \equiv \varphi^2(x)$ and $(\psi(x))' = \pm (\varphi(x))'$ for $\text{e.a.} \ x$. Therefore $(\psi'(x))^2 = (\varphi'(x))^2$ for $\text{e.a.} \ x$. Hence $\lambda_{\text{prin}} = \mathcal{R}(\psi)$ and $\psi$ is also an eigenfunction associated with $\lambda_{\text{prin}}$. Combining with Claim 2, we know that $|\varphi(x)| \equiv c \varphi(x)$ for some constant $c \neq 0$. This shows that $\varphi(x)$ does not change sign, say $\varphi(x) \geq 0$ for the case $c > 0$. Therefore $\varphi(x) > 0$ for all $x \in I$. See Remark 1. Hence $\lambda_{\text{prin}}$ is a positive principal eigenvalue of (8)-(N) with a positive principal eigenfunction $\varphi(x)$.

**Claim 4.** We claim that $\lambda_{\text{prin}}$ is the unique positive principal eigenvalue.
In (33), we obtain that \( \tilde{\tau} \) that \( \int \) the zeroth Neumann eigenvalue \( \tau \) eigenfunction. Accordingly, \( u \) and the equality holds only for \( \tau \). Let now \( \tilde{\tau} \) be an eigenfunction associated with \( \tilde{\tau} \) of Theorem 1.2. To emphasize their dependence of principal eigenvalue \( \lambda > 0 \) be any principle eigenvalue of (8)-(N) with potential measure \( \nu = -\tilde{\lambda} \rho \). Due to (34), one sees that

\[
\int_I (u'(x))^2 \, dx - \tilde{\lambda} \int_I u^2(x) \, d\rho(x) \geq 0 \quad \forall u \in W^{1,2}(I),
\]

and the equality holds only for \( u = c\tilde{\varphi} \). Then (28) and (35) taken together imply that \( \tilde{\lambda} = \lambda_{\text{prin}} \).

**Claim 5.** The necessity part of the theorem holds.

To see this, let \( \lambda > 0 \) be a principle eigenvalue of (8)-(N) and \( \tilde{\varphi}(x) \) be an eigenfunction associated with \( \lambda \). Since \( \tilde{\varphi}(x) > 0 \) for all \( x \in I \), then \( \tilde{\varphi} \) is the eigenfunction corresponding to the zeroth Neumann eigenvalue \( \tau = 0 \) of eigenvalue problem (7)-(N) with potential measure \( \nu = -\tilde{\lambda}\rho \). To make \( \varphi(x) \) respectively. To make \( \varphi(x) \) be uniquely determined, it is required to satisfy the following normalization conditions

\[
\int_I \varphi^2(x; \rho) \, d\rho(x) = 1, \quad \varphi(x; \rho) > 0 \text{ on } I.
\]

3.2. Complete continuity of principal eigenvalues in measures — Proof of Theorem 1.2.

To emphasize their dependence of principal eigenvalue \( \lambda_{\text{prin}} \) and principal eigenfunction \( \varphi(x) \) on \( \rho \), let us write \( \lambda_{\text{prin}} = \lambda_{\text{prin}}(\rho) \) and \( \varphi(x) = \varphi(x; \rho) \) respectively. To make \( \varphi(x; \rho) \) be uniquely determined, it is required to satisfy the following normalization conditions

\[
\int_I \varphi^2(x; \rho) \, d\rho(x) = 1, \quad \varphi(x; \rho) > 0 \text{ on } I.
\]

In the space \( M_0(I) \) of measures, \( \rho_n \rightharpoonup \rho \) in \( (M_0(I), w^*) \) if and only if, as \( n \to \infty \),

\[
\int_I u(x) \, d\rho_n(x) \to \int_I u(x) \, d\rho(x) \quad \forall u \in C(I).
\]

We give the proof of Theorem 1.2 in a series of claims.
Let now \( \rho, \rho_n \in M_{\text{prin}} \) be such that \( \rho_n \to \rho \) with respect to the weak* topology \( w^* \).

**Claim 1.** We have \( \limsup_{n \to \infty} \lambda_{\text{prin}}(\rho_n) \leq \lambda_{\text{prin}}(\rho) \). In particular, \( \{\lambda_{\text{prin}}(\rho_n)\}_{n \geq 1} \) is bounded.

Recall that \( \varphi(x) = \varphi(x; \rho) \) is the normalized eigenfunction associated with \( \lambda_{\text{prin}}(\rho) \), cf. (36). From the definition of \( \rho_n \to \rho \) in \( (M_0(I), w^*) \), we have

\[
\lim_{n \to \infty} \int_I \varphi^2(x) \, d\rho_n(x) = \int_I \varphi^2(x) \, d\rho(x) = 1.
\]

Hence, for any small \( \varepsilon > 0 \) there exists some \( n_\varepsilon \in \mathbb{N} \) such that

\[
\int_I \varphi^2(x) \, d\rho_n(x) \geq 1 - \varepsilon > 0 \quad \forall n > n_\varepsilon.
\]

According to the characterizations (9) for measures \( \rho \) and \( \rho_n \), we have \( \lambda_{\text{prin}}(\rho) = \int_I (\varphi'(x))^2 \, dx \) and

\[
\lambda_{\text{prin}}(\rho_n) \leq \frac{\int_I (\varphi'(x))^2 \, dx}{\int_I \varphi^2(x) \, d\rho_n(x)} \leq \frac{\lambda_{\text{prin}}(\rho)}{1 - \varepsilon} \quad \forall n > n_\varepsilon.
\]

Letting \( n \to \infty \) and considering the arbitrariness of \( \varepsilon > 0 \), we then obtain

\[
\limsup_{n \to \infty} \lambda_{\text{prin}}(\rho_n) \leq \lambda_{\text{prin}}(\rho).
\]

**Claim 2.** We claim the limit \( \lambda \) of any convergent subsequence of \( \{\lambda_{\text{prin}}(\rho_n)\}_{n \geq 1} \) is an eigenvalue of (8)-(N).

Due to the boundedness of \( \{\lambda_{\text{prin}}(\rho_n)\}_{n \geq 1} \), without loss of generality, let us simply assume that \( \lambda_{\text{prin}}(\rho_n) \to \lambda \in \mathbb{R} \) as \( n \to \infty \). By multiplying some positive factors, we can consider the corresponding eigenfunctions \( \varphi(\cdot, \rho_n) \) satisfying \( \|\varphi(\cdot, \rho_n)\|_\infty = 1 \). It then follows from (9) that

\[
\int_I \varphi^2(x; \rho_n) \, dx = \lambda_{\text{prin}}(\rho_n) \int_I \varphi^2(x) \, d\rho_n(x) \leq \lambda_{\text{prin}}(\rho_n) \|\rho_n\|.
\]

This implies that \( \{\varphi(x; \rho_n)\}_{n \geq 1} \) is bounded in \( W^{1,2}(I) \), whence it is weakly compact in \( W^{1,2}(I) \). Without loss of generality, we assume that the sequence \( \{\varphi(x; \rho_n)\}_{n \geq 1} \) itself is weakly convergent to some \( \varphi \in W^{1,2}(I) \). Since the imbedding \( W^{1,2}(I) \to C(I) \) is compact, passing to a subsequence, we can assume that \( \varphi(x; \rho_n) \to \varphi(x) \) in \( (C(I), \|\cdot\|_\infty) \). As each \( \varphi(x; \rho_n) \) fulfills identity (33) with measure \( \rho_n \) and spectral parameter \( \lambda_{\text{prin}}(\rho_n) \), by letting \( n \to \infty \), we then conclude that \( \varphi(x) \) fulfills identity (33) with measure \( \rho \) and the spectral parameter \( \lambda \). Consequently, \( \lambda \) is an eigenvalue of (8)-(N) with the measure \( \rho \), and \( \varphi(x) \) is an eigenfunction associated with \( \lambda \).

**Claim 3.** We claim that the eigenvalue \( \lambda \) of Claim 2 must be the principal eigenvalue \( \lambda_{\text{prin}}(\rho) \).

Since \( \int_I \varphi^2(x; \rho_n) \, d\rho_n(x) > 0 \) for all \( n \geq 1 \), we have \( \int_I \varphi^2(x) \, d\rho(x) \geq 0 \). Combining with \( \int_I d\rho < 0 \), we must have \( \|\varphi\|_2 > 0 \). Owing to identity (33), we thus conclude that \( \varphi \in U_\rho \) and \( \lambda > 0 \). Moreover, \( \varphi(x) \geq 0 \) on \( I \) because all \( \varphi(x; \rho_n) \) are positive. According to Remark 1, \( \varphi(x) > 0 \) on \( I \). It then follows that \( \lambda > 0 \) is a principal eigenvalue of (8)-(N) with measure \( \rho \). Thus \( \lambda = \lambda_{\text{prin}}(\rho) \) because of the uniqueness of the positive principal eigenvalue of (8)-(N).

Finally, it follows from Claim 2 and Claim 3 that, as \( n \to \infty \), \( \{\lambda_{\text{prin}}(\rho_n)\}_{n \geq 1} \) itself is convergent to \( \lambda_{\text{prin}}(\rho) \). The proof of Theorem 1.2 is now complete. \( \square \)
Lemma 4.1. Assume that

\[ \text{Let us first consider optimization problem (12).} \]

4. Optimization problems on principal eigenvalues and on measures.

4.1. Optimizing principal eigenvalues with indefinite measures — Proof of Theorem 1.3. Let us first consider optimization problem (12).

Remark 2. One can see from the proof above that, whenever \( \rho_n \to \rho \) in \( (\mathcal{M}_{\text{prin}}, w^*) \), the normalized eigenfunction sequence \( \{ \varphi(x; \rho_n) \}_{n \geq 1} \) is weakly convergent to \( \varphi(x; \rho) \) in \( W^{1,2}(I) \) and strongly in \( C(I) \).

4. Optimization problems on principal eigenvalues and on measures.

4.1. Optimizing principal eigenvalues with indefinite measures — Proof of Theorem 1.3. Let us first consider optimization problem (12).

Lemma 4.1. Assume that \( 0 \leq a < b \leq 1 \) and \( \beta > \alpha > 0 \). Then

\[ \lambda_{\text{prin}}(\alpha \delta_a - \beta \delta_b) = \frac{1}{|b-a|} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) =: F(a, b, \alpha, \beta). \quad (37) \]

Moreover, for the case of \( a < b \),

\[ \varphi(x; \alpha \delta_a - \beta \delta_b) = \begin{cases} 1, & x \in [0, a), \\ 1 - \frac{\beta - \alpha}{\beta(b-a)}(x - a), & x \in [a, b), \\ \frac{\alpha}{\beta}, & x \in [b, 1], \end{cases} \quad (38) \]

is an eigenfunction associated with \( \lambda_{\text{prin}}(\alpha \delta_a - \beta \delta_b) \). For the case of \( a > b \),

\[ \varphi(x; \alpha \delta_a - \beta \delta_b) = \begin{cases} 1, & x \in [0, b), \\ 1 + \frac{\beta - \alpha}{\alpha(a-b)}(x - b), & x \in [b, a), \\ \frac{\beta}{\alpha}, & x \in [a, 1], \end{cases} \quad (39) \]

is an eigenfunction associated with \( \lambda_{\text{prin}}(\alpha \delta_a - \beta \delta_b) \).

In particular, for those measures as in (11), one has from (37) that

\[ \lambda_{\text{prin}}(\tilde{\rho}_{A,B}) = \lambda_{\text{prin}}(\tilde{\rho}_{A,B}) = \frac{4A}{B^2 - A^2}. \quad (40) \]

Proof. We first consider the case of \( a < b \). Let \( u(x) := \varphi(x; \alpha \delta_a - \beta \delta_b) > 0 \) be an eigenfunction associated with \( \lambda := \lambda_{\text{prin}}(\alpha \delta_a - \beta \delta_b) \). Due to linearity of equation (8), we can assume that \( u(0) = 1 \). In view of (20), one has

\[ u^*(x) = \begin{cases} 0, & x \in [0, a), \\ -\alpha u(a), & x \in [a, b), \\ -\lambda(\alpha u(a) - \beta u(b)), & x \in [b, 1]. \end{cases} \]

Since \( u(x) = 1 + \int_0^x u^*(t) \, dt \), one has

\[ u(x) = \begin{cases} 1, & x \in [0, a), \\ 1 - \alpha u(a)(x - a), & x \in [a, b), \\ 1 - \lambda u(a)(x - a) + \lambda \beta u(b)(x - b), & x \in [b, 1]. \end{cases} \quad (41) \]

Therefore \( u(a) = 1 \) and \( u(b) = 1 - \alpha \lambda(b-a) \). As \( u^*(1) = 0 \), i.e., \( \alpha u(a) = \beta u(b) \), then we have \( \alpha = \beta \) \( (1 - \lambda \alpha(b-a)) \). This proves (37). As a consequence, the conclusion (38) follows from (41) at once.

For the case of \( a > b \), conclusions (37) and (39) can be verified via a similar argument. \( \square \)

Lemma 4.2. Given \( \rho \in \mathcal{M}_{\text{prin}}, \) let us define \( \alpha = \|\rho_+\| \) and \( \beta = \|\rho_-\| \). Then there exist \( a \in I \) and \( b \in I \) satisfying \( a \neq b \) such that

\[ \lambda_{\text{prin}}(\rho) \geq \lambda_{\text{prin}}(\alpha \delta_a - \beta \delta_b). \quad (42) \]
Proof. Denote \( \varphi(x) = \varphi(x; \rho) \), an eigenfunction associated with \( \lambda_{\text{prin}}(\rho) \). Since \( \varphi(x) \) is not constant and positive, there exist \( a, b \in I \) such that \( a \neq b \) and \( \varphi^2(a) = \max_{x \in I} \varphi^2(x) \) and \( \varphi^2(b) = \min_{x \in I} \varphi^2(x) \). Accordingly,

\[
\int_I \varphi^2 \, d\rho = \int_I \varphi^2 \, d\rho_+ - \int_I \varphi^2 \, d\rho_- \leq \varphi^2(a) - \varphi^2(b) \beta \tag{43}
\]

by recalling \( \alpha = \|\rho_+\| \) and \( \beta = \|\rho_-\| \). Moreover, the equality holds if and only if

\[
\int_I \varphi^2 \, d\rho_+ = \alpha \varphi^2(a) \quad \text{and} \quad \int_I \varphi^2 \, d\rho_- = \beta \varphi^2(b).
\]

See (44). It is clear that the measure \( \alpha \delta_a - \beta \delta_b \) is in \( M_{\text{prin}} \). Then (9) and (43) taken together yield

\[
\lambda_{\text{prin}}(\rho) \geq \frac{\int_I \varphi^2 \, dx}{\int_I \varphi^2 \, d(\alpha \delta_a - \beta \delta_b)} \geq \lambda_{\text{prin}}(\alpha \delta_a - \beta \delta_b). \tag{46}
\]

This proves (42). \( \square \)

Now we can give the proof of Theorem 1.3. At first we note that the left-hand side \( \lambda_{\text{prin}}(\rho) \) of (42) is an infinitely dimensional functional on \( M_{A,B} \), while the right-hand side is a 4-dimensional elementary function \( F(a, b, \alpha, \beta) \) defined on

\[
D_{A,B} := \{(a, b, \alpha, \beta) : \beta > \alpha > 0, \alpha - \beta \leq -A, \alpha + \beta \leq B \text{ and } 0 \leq a \neq b \leq 1\}.
\]

It is elementary to verify that \( F(a, b, \alpha, \beta) \) takes its minimum

\[
\inf_{(a, b, \alpha, \beta) \in D_{A,B}} F(a, b, \alpha, \beta) = \frac{4A}{B^2-A^2}. \tag{47}
\]

at and only at

\[
(a, b) = (0, 1) \text{ or } (1, 0), \quad \alpha = (B - A)/2, \quad \beta = (A + B)/2. \tag{48}
\]

On the other hand, one has

\[
\sup_{(a, b, \alpha, \beta) \in D_{A,B}} F(a, b, \alpha, \beta) = +\infty. \tag{49}
\]

By (42) and (49), one has trivially

\[
M(A, B) = \sup_{\rho \in M_{A,B}} \lambda_{\text{prin}}(\rho) \geq \sup_{(a, b, \alpha, \beta) \in D_{A,B}} F(a, b, \alpha, \beta) = +\infty.
\]

Hence \( M(A, B) = +\infty \), the desired result (13).

Next we consider the infimum value \( L(A, B) \). According to (42) and (47), one has

\[
L(A, B) = \inf_{\rho \in M_{A,B}} \lambda_{\text{prin}}(\rho) \geq \inf_{(a, b, \alpha, \beta) \in D_{A,B}} F(a, b, \alpha, \beta) = \frac{4A}{B^2-A^2}. \tag{50}
\]

Moreover, corresponding to (48), one has measures \( \hat{\rho}_{A,B}, \tilde{\rho}_{A,B} \in M_{A,B} \). Immediately, it follows from (40) and (50) that

\[
L(A, B) \geq \frac{4A}{B^2-A^2} = \lambda_{\text{prin}}(\hat{\rho}_{A,B}) = \lambda_{\text{prin}}(\tilde{\rho}_{A,B}).
\]

Combining with (50), we obtain result (12). Moreover, \( L(A, B) \) is attained by both \( \hat{\rho}_{A,B} \) and \( \tilde{\rho}_{A,B} \).

It remains to show that \( L(A, B) \) is only attained by \( \hat{\rho}_{A,B} \) and \( \tilde{\rho}_{A,B} \). To this end, let us assume \( \rho \in M_{A,B} \) attains \( L(A, B) \). We have to show either \( \rho = \hat{\rho}_{A,B} \) or \( \rho = \tilde{\rho}_{A,B} \).
Similarly, for the case (43) implies that point of \( \phi \rho \). Thus (51) and (52) show that either \( \lambda_{\text{prin}}(A,B) \) or \( \lambda_{\text{prin}}(\hat{\rho}_{A,B}) \). For the case \( (a,b) = (0,1) \), one has from (38)

\[
\varphi(x) = 1 - \frac{2A}{2 + A} x, \quad x \in I,
\]

an eigenfunction associated with \( \lambda_{\text{prin}}(\hat{\rho}_{A,B}) \). Clearly, \( x = 0 \) is the unique maximal point of \( \varphi(x) \) and \( x = 1 \) is the unique minimal point of \( \varphi(x) \). Then the identity (43) implies that

\[
\rho_+ = \alpha_{A,B} \delta_0 \quad \text{and} \quad \rho_- = \beta_{A,B} \delta_1. \tag{51}
\]

Similarly, for the case \( (a,b) = (1,0) \), one can obtain

\[
\rho_+ = \alpha_{A,B} \delta_1 \quad \text{and} \quad \rho_- = \beta_{A,B} \delta_0. \tag{52}
\]

Thus (51) and (52) show that either \( \rho = \hat{\rho}_{A,B} \) or \( \rho = \hat{\rho}_{A,B} \). \qed

**Remark 3.** Optimization problems like (12) and (13) for (principal) eigenvalues are infinitely dimensional variational problems. Some typical approaches to these problems include the Pontryagin’s maximum principle [7, 17], and the Lyapunov-type inequalities [25, 34]. By using the Lagrangian multiplier method, a variational method has been developed in [35, 36] to deal some optimization problems of eigenvalues. However, in the present proof of Theorem 1.1, we have adopted a direct method, as done in [20, 21]. In fact, based on the minimization characterization (9) for \( \lambda_{\text{prin}}(\rho) \), an MDE explanation (45) to the estimates of the Rayleigh form can reduce problems (12) and (13) to finitely dimensional optimization problems. Owing to this advantage, our approach is relatively elementary.

### 4.2. Minimizing measures with given principal eigenvalue — Proof of Theorem 1.4

This subsection is devoted to study the minimization problem (14). Using the complete continuity in Theorem 1.2, we first show the minimal value \( T(\gamma) \) defined in (14) can be attained by some measures from \( \mathcal{M}_{A,B} \).

**Lemma 4.3.** Given \( \gamma \in [L(A,B), +\infty) \), there exists some measure \( \rho_\gamma = \rho_{A,B,\gamma} \in \mathcal{M}_\gamma \) such that \( \int_I d\rho_\gamma = T(\gamma) \).

**Proof.** Take a minimizing sequence \( \rho_n \in \mathcal{M}_\gamma \) such that \( \int_I d\rho_n \to T(\gamma) \) as \( n \to \infty \). Since \( \mathcal{M}_\gamma \subset \mathcal{M}_{A,B} \), then \( \|\rho_n\| \leq B\) for all \( n \geq 1 \). According to the Banach-Alaoglu theorem [10, Theorem 3.1], there is a subsequence such that \( \rho_{n_k} \to \rho_\gamma \) in \( (\mathcal{M}_0(I), w^\star) \) for some \( \rho_\gamma \in \mathcal{M}_0(I) \). One has then \( \int_I d\rho_\gamma = T(\gamma) \). Moreover, it follows from Theorem 1.2 that \( \lambda_{\text{prin}}(\rho_\gamma) = \gamma \), and it is not difficult to verify that \( \rho_\gamma \in \mathcal{M}_\gamma \). \qed

A crucial observation on \( \rho_\gamma \) is that, when the measure \( \rho \) is taken as \( \rho_\gamma \), (42) will become an equality.

**Lemma 4.4.** Let \( \rho_\gamma \) be as in Lemma 4.3 and \( \varphi(x; \rho_\gamma) \) be the eigenfunction associated with \( \lambda_{\text{prin}}(\rho_\gamma) \). If \( a_\gamma \) and \( b_\gamma \) are the maximal and minimal points of \( \varphi(x; \rho_\gamma) \), \( \alpha_\gamma := \|\rho_\gamma\|_+ \) and \( \beta_\gamma := \|\rho_\gamma\|_- \), then

\[
\lambda_{\text{prin}}(\alpha_\gamma \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma}) = \lambda_{\text{prin}}(\rho_\gamma) = \gamma. \tag{53}
\]
Proof. In view of Lemma 4.1 and Lemma 4.2, we have
\[ \frac{1}{|a_\gamma-b_\gamma|} \left( \frac{1}{\alpha_\gamma} - \frac{1}{\beta_\gamma} \right) = \lambda_{\text{prin}} (\alpha_\gamma \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma}) \leq \lambda_{\text{prin}} (\rho_\gamma) = \gamma. \]
It remains to show this is in fact an equality. Otherwise, let us assume the strict inequality
\[ \frac{1}{|a_\gamma-b_\gamma|} \left( \frac{1}{\alpha_\gamma} - \frac{1}{\beta_\gamma} \right) < \gamma. \]
Then there exists some 0 < \( \alpha < \alpha_\gamma \) such that
\[ \lambda_{\text{prin}} (\alpha \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma}) = \frac{1}{|a_\gamma-b_\gamma|} \left( \frac{1}{\alpha} - \frac{1}{\beta_\gamma} \right) = \gamma. \]
Thus \( \alpha \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma} \in \mathcal{M}_{\gamma} \). However,
\[ \int_I \text{d} (\alpha \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma}) = \alpha - \beta_\gamma < \alpha_\gamma - \beta_\gamma = \int_I \text{d} \rho_\gamma. \]
This contradicts with the assumption that \( \rho_\gamma \) is a minimizer. \( \square \)

In accordance with the identity (53), one sees from the proof of Lemma 4.2 that
\[ \int_I \varphi^2 (x; \rho_\gamma) \text{d} (\rho_\gamma) = \varphi^2 (a, \rho_\gamma) \alpha_\gamma, \quad (54) \]
and
\[ \int_I \varphi^2 (x; \rho_\gamma) \text{d} (\rho_\gamma) = \varphi^2 (b, \rho_\gamma) \beta_\gamma. \quad (55) \]
In particular, \( \varphi (x; \rho_\gamma) \) is an eigenfunction associated with \( \lambda_{\text{prin}} (\alpha_\gamma \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma}) = \gamma \). Explicitly, \( \varphi (x; \rho_\gamma) \) is given by either (38) or (39).

Based on these observations, we are now ready for the proof of Theorem 1.4.

Step 1. By keeping the notations as same as in Lemma 4.4, we claim that \( |a_\gamma - b_\gamma| = 1 \).

If this is false, one has from Lemmas 4.1 and 4.4 that
\[ \frac{1}{\alpha_\gamma} - \frac{1}{\beta_\gamma} = \lambda_{\text{prin}} (\alpha_\gamma \delta_0 - \beta_\gamma \delta_1) < \lambda_{\text{prin}} (\alpha_\gamma \delta_{a_\gamma} - \beta_\gamma \delta_{b_\gamma}) = \gamma. \]
Arguing as in the proof of Lemma 4.4, one can derive a contradiction. Thus \( |a_\gamma - b_\gamma| = 1 \).

Step 2. One has from Step 1 that \( (a_\gamma, b_\gamma) \) is either \( (0, 1) \) or \( (1, 0) \). For the former case, it follows from (38) and (39) that
\[ \varphi (x; \rho_\gamma) = 1 - (\beta_\gamma - \alpha_\gamma) x / \alpha_\gamma, \]
and then we can infer from identities (54) and (55) that \( \rho_\gamma = \alpha_\gamma \delta_0 - \beta_\gamma \delta_1 \).

Analogously, for the latter case, we have
\[ \varphi (x; \rho_\gamma) = 1 + (\beta_\gamma - \alpha_\gamma) x / \alpha_\gamma, \]
whence \( \rho_\gamma = -\beta_\gamma \delta_0 + \alpha_\gamma \delta_1 \).

Step 3. To obtain the explicit expression of \( T(\gamma) \), we need to derive the explicit forms of \( \alpha_\gamma \) and \( \beta_\gamma \).

Since \( \rho_\gamma \in \mathcal{M}_{A,B} \) and \( \lambda_{\text{prin}} (\rho_\gamma) = \gamma \), we find from Step 2 that the pair \( (\alpha_\gamma, \beta_\gamma) \) must be the solution of the programming problem
\[ \min (\alpha_\gamma - \beta_\gamma) \]
with constraints
\[
\begin{align*}
\alpha_\gamma - \beta_\gamma & \leq -A, \\
\alpha_\gamma + \beta_\gamma & \leq B, \\
\frac{1}{\alpha_\gamma} - \frac{1}{\beta_\gamma} & = \gamma.
\end{align*}
\]
A direct calculation shows that the problem has the unique solution \((\alpha_\gamma, \beta_\gamma) = (\bar{\alpha}_\gamma, \bar{\beta}_\gamma)\), where \(\bar{\alpha}_\gamma\) and \(\bar{\beta}_\gamma\) are defined in (15). Then \(T(\gamma) = \bar{\alpha}_\gamma - \bar{\beta}_\gamma\) is given in (17).

Finally, we have from Step 2 that either \(\rho_\gamma = \hat{\rho}_\gamma\) or \(\rho_\gamma = \hat{\rho}_\gamma\). This implies that the minimizers of \(T(\gamma)\) must be the measures defined in (16). \(\square\)

4.3. A relation between optimization problems. Let us go back to the optimization problem (5) with integrable weights.

**Theorem 4.5.** There holds

\[
\tilde{L}(A, B) = \inf_{w \in \mathcal{W}_{A,B}} \lambda_{\text{prin}}(w) \equiv L(A, B). \tag{56}
\]

Moreover, the infimum \(\tilde{L}(A, B)\) cannot be attained by any weight \(w \in \mathcal{W}_{A,B}\).

**Proof.** For \(w(x) \in \mathcal{W}_{A,B}\), it easy to verify that the absolutely continuous measure

\[
\rho_w(x) := \int_{[0,x]} w(t) \, dt, \quad x \in I, \tag{57}
\]

is in \(\mathcal{M}_{A,B}\). Hence

\[
\tilde{L}(A, B) = \inf_{w \in \mathcal{W}_{A,B}} \lambda_{\text{prin}}(w) \geq \inf_{\rho \in \mathcal{M}_{A,B}} \lambda_{\text{prin}}(\rho_w) = \inf_{\rho \in \mathcal{M}_{A,B}} \lambda_{\text{prin}}(\rho) = \tilde{L}(A, B).
\]

For the converse, one has from Theorem 1.3 that

\[
L(A, B) = \lambda_{\text{prin}}(\hat{\rho}_{A,B}) = \lambda_{\text{prin}}(\alpha_{A,B} \delta_0 - \beta_{A,B} \delta_1).
\]

For any \(n \geq 2\), define the weight

\[
w_n(x) := \begin{cases} n\alpha_{A,B}, & x \in [0, 1/n], \\ 0, & x \in (1/n, 1-1/n), \\ -n\beta_{A,B}, & x \in [1-1/n, 1]. \end{cases}
\]

It is easy to verify that \(w_n \in \mathcal{W}_{A,B}\) for all \(n \geq 2\), and, moreover, as measures, \(\rho_{w_n} \rightharpoonup \hat{\rho}_{A,B}\) with respect to the weak* topology \(w^*\). By the complete continuity of \(\lambda_{\text{prin}}(\rho)\) in Theorem 1.2, we obtain

\[
\tilde{L}(A, B) \geq \lim_{n \to \infty} \lambda_{\text{prin}}(w_n) = \lim_{n \to \infty} \lambda_{\text{prin}}(\rho_{w_n}) = \lambda_{\text{prin}}(\hat{\rho}_{A,B}) = L(A, B).
\]

These yield equality (56). \(\square\)

Relations like (56) are suggested by smooth approximations of Radon measures. In fact, it is known from [19, Lemma 2.6 and Lemma 2.9] that measures in \(\mathcal{M}_0(I)\) can be weakly* approximated by absolutely continuous measures (57) induced by \(w \in L^1(I)\). It can be proved that \(\mathcal{W}_{A,B}\) is dense in \(\mathcal{M}_{A,B}\) with respect to \(w^*\). Combining with the complete continuity \(\lambda_{\text{prin}}(\rho)\) of Theorem 1.2, one has

\[
\inf_{w \in \mathcal{W}_{A,B}} \lambda_{\text{prin}}(w) = \inf_{\rho \in \mathcal{M}_{A,B}} \lambda_{\text{prin}}(\rho).
\]

This gives another explanation to equality (56).
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E-mail address: zy-wen13@tsinghua.org.cn
E-mail address: zhangmr@tsinghua.edu.cn