ALGEBRAIC SEMIGROUP ACTIONS I. C*-ALGEBRAS AND GROUPOIDS

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ABSTRACT. We provide a framework for studying the concrete C*-algebras associated with algebraic semigroup actions: Given such an action, we construct an inverse semigroup, and we introduce conditions for algebraic semigroup actions that characterize Hausdorffness, topological freeness, and minimality of the associated tight groupoid. We parameterize all closed invariant subspaces of the unit space of our groupoid, and characterize topological freeness of the associated reduction groupoids. We prove that our groupoids are purely infinite whenever they are minimal, and in the topologically free case, we prove that the concrete C*-algebra is always a (possibly exotic) groupoid C*-algebra in the sense that it sits between the full and essential C*-algebras of our groupoid. As an application, we obtain structural results for C*-algebras associated with, for instance, shifts over semigroups, actions coming from commutative algebra, and non-commutative rings.

1. Introduction

Algebraic group actions form an important class of dynamical systems with deep connections to commutative algebra and operator algebras; they have been studied intensely since their inception by Kitchens and Schmidt in the late 80s, see [40], [51], [26, Chapters 13&14] and the references therein. The theory of their “one-sided” or “irreversible” counterparts, algebraic semigroup actions, is much less developed. This work is the first in a series of papers in which we systematically study algebraic semigroup actions from the perspective of operator algebras and groupoids. Here, we consider such actions through the lens of C*-algebra theory: Each algebraic semigroup action gives rise to a concrete C*-algebra generated by the Koopman representation for the action together with the left regular representation of the group, and we develop a general framework for studying these C*-algebras.

An algebraic semigroup action is an action of a semigroup on a group by injective endomorphisms. Until now, the algebraic semigroup actions that have been investigated from the point of view of C*-algebra theory fall into two classes. The first class comes from ring theory: The multiplicative monoid of left regular elements of a ring acts by injective endomorphisms on the additive group of the ring, and the associated C*-algebra is the reduced ring C*-algebra of the ring. These were introduced and studied by Cuntz for the ring \( \mathbb{Z} \) [13], for integral domains by Cuntz and the second-named author in [15], and for general rings by the second-named author in [34]. The ring C*-algebras of integral domains that are not fields were proven to be UCT Kirchberg algebras in [15] (following the approach in [13] for the ring \( \mathbb{Z} \)). For general commutative rings, conditions for pure infiniteness and simplicity were established in [34], but the question of pure infiniteness and simplicity for non-commutative rings was left open. Closely related to these are the C*-algebras associated with actions of congruence monoids on rings of algebraic integers, which have been studied by the authors in the context of boundary quotients of semigroup C*-algebras, and were proven to be UCT Kirchberg algebras, see [8 § 8] and [9 § 3]. The second class comes from special actions of right LCM semigroups (i.e., semigroups in which all elements which have a right common multiple have a least right common multiple). Their study began with a note by Hirshberg on the concrete C*-algebras associated with algebraic \( \mathbb{N} \)-actions on amenable groups [23]. Cuntz and Vershik later proved that these C*-algebras are UCT Kirchberg algebras for exact, finite
The aforementioned actions do not include several fundamental example classes: For instance, actions on solenoids, shifts over semigroups, and algebraic dynamical systems. Results on pure infiniteness and simplicity were obtained in this setting, see [52, Theorem 3.26] and [53, Theorem 5.10]. The actions studied in [52] were generalized by the so-called algebraic dynamical systems introduced by Brownlowe, Larsen, and Stammeier in [7, Definition 2.1]. These are algebraic semigroup actions where the acting semigroup is required to be right LCM and the action is required to respect the order in the sense of [9, Definition 8.1]. These conditions are quite restrictive, e.g., for the examples coming from rings, they are tantamount to requiring that the ring is a principal ideal domain, yet this class is already broad enough to include many interesting examples which have been studied also in, e.g., [53] and [5]. These authors were primarily focused on universal C*-algebras attached to the actions and the relationship with (boundary quotients of) semigroup C*-algebras. In the case where these C*-algebras agree with the boundary quotients of the semi-direct product semigroup attached to the action, results on pure infiniteness and simplicity were obtained in [6, Theorem 8.12] and [5, Theorem 4.17].

Let us now explain the main construction of this paper (see §3) and our results on groupoids. Given any algebraic semigroup action \( \sigma: S \curvearrowright G \) (see Definition 2.1), we construct an inverse semigroup as follows: Let \( I_\sigma \) be the inverse semigroup of partial bijections of \( G \) generated by the endomorphisms implementing the action of \( S \) together with the translations for the action of \( G \) on itself (see Definition 3.5). There is a canonical faithful representation of this inverse semigroup by partial isometries on \( \ell^2(G) \), and the C*-algebra generated by this representation is precisely the C*-algebra \( \mathfrak{A}_\sigma \) generated by the Koopman representation of \( S \) and the left regular representation of \( G \). The action of \( S \) on \( G \) generates a distinguished family of subgroups of \( G \), which we call \( S \)-constructible subgroups. The collection of non-zero idempotents of \( I_\sigma \) is then equal to the collection of cosets for these subgroups. This collection of cosets defines a topology on \( G \), and the resulting completion is a compact, totally disconnected space \( \partial E \) that is canonically identified with the Gelfand spectrum of the commutative C*-algebra \( \mathfrak{D}_\sigma \) of \( \mathfrak{A}_\sigma \) generated by the projections coming from idempotents in \( I_\sigma \). The inverse semigroup \( I_\sigma \) acts on \( \partial E \) by partial homeomorphisms, and the associated transformation groupoid \( G_\sigma := I_\sigma \ltimes \partial E \) is the tight groupoid of \( I_\sigma \) in the sense of Exel [18, 19]. In §4 we characterize properties of \( G_\sigma \) in terms of the initial algebraic semigroup action: We introduce three conditions on the action \( \sigma: S \curvearrowright G \) and we prove that these characterize Hausdorffness, topological freeness, and minimality of \( G_\sigma \). We prove that if \( G_\sigma \) is minimal, then it is purely infinite, we parameterize all closed invariant subspaces of \( \partial E \) and we characterize topological freeness for the reduction groupoid to any closed invariant subset of \( \partial E \). We do not need to assume our groupoids are Hausdorff for these results.
We prove that there is always a canonical, surjective *-homomorphism \( C^*(G_\sigma) \to \mathfrak{A}_\sigma \) that is an isomorphism at the level of canonical commutative subalgebras. Since \( C^*(G_\sigma) \) is the universal C*-algebra for tight representations of \( I_\sigma \), the C*-algebra \( C^*(G_\sigma) \) provides a good universal model for \( \mathfrak{A}_\sigma \). The full and reduced C*-algebras \( C^*(I_\sigma) \) and \( C^*_r(I_\sigma) \) of \( I_\sigma \) are then the “Toeplitz-type” C*-algebras attached to the action \( \sigma : S \acts G \).

Recently, there have been several advances in the study of non-Hausdorff groupoids and their C*-algebras, see, e.g., [27], [45], [39], [12] and [30], and [47]. The essential C*-algebra captures—at the C*-algebraic level—minimality and pure infinitness, see, e.g., [27], [45], [39], [12] and [30], and [47]. The essential C*-algebra of an étale groupoid, as defined by Kwaśniewski and Meyer [30], captures—at the C*-algebraic level—minimality and pure infinitness, see, e.g., [27], [45], [39], [12] and [30], and [47].

Having established our general framework, we turn to example classes. In §7.1, we consider actions satisfying what we call the finite index property. This condition means that every constructible subgroup of \( G \), such actions include endomorphisms of (duals of) solenoids, examples from self-similar groups, and actions from torsion-free rings of finite rank. The question of structure for ring C*-algebras of non-commutative rings had been left open in [34]. For such actions, our groupoid is always minimal and purely infinite, so we obtain, in particular, that these ring C*-algebras are simple and purely infinite. We then turn to actions by left reversible monoids in §7.2. This covers several large classes of actions, including actions from commutative algebra, which are the semigroup versions of the actions from [51]. In this setting, too, our groupoids are always minimal and purely infinite. Even for the case of reduced ring C*-algebras of commutative rings, our results offer improvements on those from [34].

Finally, we apply the general K-theory formula from [38] to our Toeplitz-type C*-algebras (§8). This computes, in particular, the K-groups of C*-algebras associated with many classes of shifts over semigroups.

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## 2. Preliminaries

### 2.1. Algebraic semigroup actions

This work is centered around the following class of actions, which are the “one-sided” analogues of the algebraic group actions from [51].

**Definition 2.1.** An algebraic semigroup action is an action of a semigroup \( S \) on a discrete group \( G \) by injective group endomorphisms, i.e., a semigroup homomorphism \( \sigma : S \to \text{End}(G) \) such that \( \sigma_s \) is injective for all \( s \in S \).
We shall write $\sigma : S \curvearrowright G$ to denote an algebraic semigroup action. When we want to emphasize the acting semigroup, we call $\sigma : S \curvearrowright G$ an algebraic $S$-action.

The case where $G$ is Abelian is the most interesting from the point of view of topological dynamics:

**Remark 2.2.** When $G$ is Abelian, an algebraic $S$-action on $G$ is the same as having an action of $S$ on the compact Pontryagin dual $\hat{G}$ of $G$ by surjective group endomorphisms. More precisely, if $\sigma : S \curvearrowright G$ is an algebraic $S$-action, then we obtain a right action $\hat{\sigma} : S \curvearrowright \hat{G}$ (i.e., a semigroup homomorphism from $S$ to the semigroup of continuous maps $\hat{G} \to \hat{G}$) characterized by $\langle \hat{\sigma}(\chi)g, h \rangle = \langle \chi, \sigma_g(h) \rangle$ for all $s \in S$, $\chi \in \hat{G}$ and $g \in G$, where $\langle \cdot, \cdot \rangle : \hat{G} \times G \to \mathbb{T}$ is the duality pairing. Moreover, it is easy to see that, for all $s \in S$, $\hat{\sigma}_s$ is surjective if and only if $\sigma_s$ is injective.

**Standing assumptions:** Assume that $\sigma : S \curvearrowright G$ is an algebraic $S$-action. The identity element of $G$ will be denoted by $e$. We shall always assume that $S$ is a monoid with identity element denoted by 1 and that $G$ is non-trivial.

We will always assume that the action $\sigma : S \curvearrowright G$ is faithful, i.e., $\sigma$ is injective. This implies that $S$ is left cancellative.

We say $\sigma : S \curvearrowright G$ is automorphic if $\sigma_s \in \text{Aut}(G)$ for all $s \in S$; thus, $\sigma : S \curvearrowright G$ is non-automorphic if there exists $s \in S$ such that $\sigma_s G \leq G$ (this implies that $S$ is non-trivial). The action $\sigma : S \curvearrowright G$ is said to be an algebraic group action if $S$ is a group (such actions are automatically automorphic). We will primarily be interested in non-automorphic actions.

Let us now introduce the important concept of globalization.

**Definition 2.3.** We say that the algebraic semigroup action $\sigma : S \curvearrowright G$ has a globalization if $S$ can be embedded into a group $\mathcal{F}$ and there is a group $\mathcal{G}$ containing $G$ together with an algebraic group action $\bar{\sigma} : \mathcal{F} \curvearrowright \mathcal{G}$ such that $\bar{\sigma}_s|G = \sigma_s$ for all $s \in S$.

There are many large example classes of algebraic semigroup actions that admit a globalization.

**Example 2.4.** Suppose $S$ is left Ore, i.e., cancellative and right reversible ($Ss \cap St \neq \emptyset$ for all $s, t \in S$). Then $S$ acts on the group $\mathcal{G} := S^{-1}G := \lim_{\to} \{ G \overset{\sigma}{\to} G \}$ by automorphisms, so we obtain a globalization $\bar{\sigma} : \mathcal{F} \curvearrowright S^{-1}G$, where $\mathcal{F} = S^{-1}S$ is the enveloping group of $S$. More precisely, consider the inductive system $\mathcal{G}_p := G$ (for $p \in S$, where $p \leq q$ if $q = rp$ for some $r \in S$), with connecting map $\mathcal{G}_p \to \mathcal{G}_r$ given by $\sigma_r : G \to G$, $g \mapsto \sigma_r g$. Then the inductive limit $\mathcal{F} := \lim_{\to} \mathcal{G}_p$ can be constructed as $\mathcal{F} := (\bigsqcup_p \mathcal{G}_p) / \sim$, with $\mathcal{G}_p \ni g \sim \sigma_r(g) \in \mathcal{G}_r$, for all $p, r \in S$ and $g \in G$. Now define $\bar{\sigma}_s : \mathcal{F} \to \mathcal{G}$ as follows: Given $g \in \mathcal{G}_q$, find $q', s' \in S$ with $q' q = s' s$. Such elements exist because $S$ is right reversible. Now define $\bar{\sigma}_s([g]) := ([\sigma_{q'}(g)])$, where we view $\sigma_{q'}(g)$ as an element of $\mathcal{G}_s$. It is straightforward to check that $\bar{\sigma}_s$ is an automorphism of $\mathcal{F}$ which satisfies $\bar{\sigma}_s|G = \sigma_s$ for all $s \in S$, so that $\bar{\sigma}$ extends to the desired algebraic group action $\bar{\sigma} : \mathcal{F} \curvearrowright \mathcal{G}$.

Note that when $G$ is Abelian, the dual action of $\bar{\sigma}$ can be constructed explicitly from $X = \hat{G}$ and $\alpha = \hat{\sigma}$ (the underlying space will be given as a projective limit obtained from $X, \alpha$).

**Example 2.5.** Suppose $G \leq \mathbb{Q}^r$ is torsion-free and of finite rank $r \in \mathbb{Z}_{> 0}$. Then $S$ acts by automorphisms on $\mathbb{Q} \otimes ZG$, so we obtain a globalization by considering the action of the group generated by $S$ on $\mathbb{Q} \otimes ZG$.

**Example 2.6.** Let $\Sigma$ be a non-trivial group, and consider the semigroup shift $S \curvearrowright \bigoplus S \Sigma$. This action admits a globalization if and only if $S$ can be embedded into a group (say $\mathcal{F}$), in which case a globalization is given by the group shift $\mathcal{F} \curvearrowright \bigoplus \mathcal{F} \Sigma$.

Suppose that $S \subseteq \Sigma$ is an embedding of $S$ into a group $\mathcal{F}$. Let $\sigma : S \curvearrowright G$ be an algebraic semigroup action with $G$ Abelian. Consider $\mathbb{Z} \mathcal{F} \otimes_{\mathbb{Z} \Sigma} G$ with the natural $\mathcal{F}$-action and the map $G \to \mathbb{Z} \mathcal{F} \otimes_{\mathbb{Z} \Sigma} G$, $g \mapsto 1 \otimes g$. This is the universal enveloping action of $\sigma$ with respect to $S \subseteq \mathcal{F}$, in the sense that any enveloping
action of $\sigma$ with respect to $S \subseteq \mathcal{I}$ factors through it. As a consequence, $G \to \mathbb{Z} \mathcal{I} \otimes_{\mathbb{Z} S} G$ is injective if and only if there exists a globalization $\tilde{\sigma}: \mathcal{I} \rtimes \mathcal{G}$ of $\sigma$.

2.2. Étale groupoids and their C*-algebras. In this and the next subsection, we introduce some notation for groupoids and their C*-algebras that will be used throughout this paper.

Let $\mathcal{G}$ be a (not necessarily Hausdorff) locally compact étale groupoid such that the unit space $\mathcal{G}(0)$ is Hausdorff in the relative topology. We let $r$ and $s$ denote the range and source maps on $\mathcal{G}$. A subset $B \subseteq \mathcal{G}$ is said to be a bisection if the restrictions $r|_B$ and $s|_B$ are injective on $B$. Given an open bisection $U \subseteq \mathcal{G}$, we let $C_c(U)$ denote the set of continuous compactly supported complex-valued functions on $U$. Extension-by-zero gives an embedding $C_c(U) \subseteq \ell^\infty(\mathcal{G})$, and we let $C(\mathcal{G})$ be the linear space spanned by the subspaces $C_c(U)$ as $U$ runs through the open bisections of $\mathcal{G}$. Then $C(\mathcal{G})$ carries the natural structure of a *-algebra (see, e.g., [18, §3]). The (full) C*-algebra of $\mathcal{G}$, which we denote by $C^*(\mathcal{G})$, is the enveloping C*-algebra of $C(\mathcal{G})$. For each unit $x \in \mathcal{G}(0)$, there is a representation $\pi_x: C(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$, where $\mathcal{G}_x := s^{-1}(x)$, such that $\pi_x(f)\delta_\gamma = \sum_{\delta \in \mathcal{G}_x(\gamma)} f(\delta)\delta_\gamma$ for all $f \in C(\mathcal{G})$.

The reduced C*-algebra of $\mathcal{G}$ is the completion of $C(\mathcal{G})$ with respect to the norm $||f||_r := \sup_{x \in \mathcal{G}(0)} ||\pi_x(f)||$. We shall view each $\pi_x$ as a representation of $C^*_r(\mathcal{G})$, and let $\pi_r: C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ be the projection map.

The groupoids constructed in this paper will not always be Hausdorff, and we shall see that the essential C*-algebra, as defined by Kwaśniewski and Meyer, will provide the best model for our concrete C*-algebras. It agrees with the reduced C*-algebra in the Hausdorff case.

**Definition 2.7** ([30 §4]). The singular ideal of $C^*_r(\mathcal{G})$ is $J_{\text{sing}} := \{ a \in C^*_r(\mathcal{G}) : \{ x \in \mathcal{G}(0) : \pi_x(a)\delta_x \neq 0 \} \text{ is meager} \}$

The essential C*-algebra of $\mathcal{G}$ is the quotient $C^*_\text{ess}(\mathcal{G}) := C^*_r(\mathcal{G})/J_{\text{sing}}$.

2.3. Induced representations. We continue with the setup from the previous subsection. Fix $x \in \mathcal{G}(0)$. If $\pi$ is a representation of the group C*-algebra $C^*(\mathcal{G}_x^0)$ on a Hilbert space $H_\pi$, then the associated induced representation $\text{Ind } \pi$ of $C^*(\mathcal{G})$ can be explicitly defined as follows (cf. [12 §1]): Let $H_{\text{Ind } \pi} := \{ \xi: \mathcal{G}_x \to H_\pi : \xi(gh) = \pi(u_h)\pi(g) \text{ for all } g \in \mathcal{G}_x, h \in \mathcal{G}_x^0, \text{ and } \sum_{g \in \mathcal{G}_x/\mathcal{G}_x^0} ||\xi(g)||^2 < \infty \}$.

Then, $H_{\text{Ind } \pi}$ is a Hilbert space with the obvious linear structure and inner product $\langle \xi, \eta \rangle_{H_{\text{Ind } \pi}} := \sum_{g \in \mathcal{G}_x/\mathcal{G}_x^0} \xi(g)\overline{\eta(g)}$ for $\xi, \eta \in H_{\text{Ind } \pi}$, and $\text{Ind } \pi: C^*(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$ is defined by $((\text{Ind } \pi)(f)\xi)(g) = \sum_{h \in \mathcal{G}_x^0(g)} f(h)\xi(h^{-1}g)$ for all $f \in C(\mathcal{G})$ and $\xi \in H_{\text{Ind } \pi}$.

Let $N \subseteq \mathcal{G}_x^0$ be a subgroup. The quasi-regular representation $\lambda_{\mathcal{G}_x^0/N}$ of $C^*(\mathcal{G}_x^0)$ on $\ell^2(\mathcal{G}_x^0/N)$ is defined by $\lambda_{\mathcal{G}_x^0/N}(u_g)\delta_{hN} = \delta_{ghN}$ for all $g, h \in \mathcal{G}_x^0$, where $u_g$ is the canonical unitary in $C^*(\mathcal{G}_x^0)$ corresponding to $g$. We shall make use of the following observation in several places below.

**Proposition 2.8.** There is a unitary $W: H_{\text{Ind } \lambda_{\mathcal{G}_x^0/N}} \cong \ell^2(\mathcal{G}_x/N)$ such that $(W(\text{Ind } \lambda_{\mathcal{G}_x^0/N})(f)W^*\xi)(gN) = \sum_{h \in \mathcal{G}_x^0(g)} f(h)\xi(h^{-1}gN)$ for all $f \in C(\mathcal{G})$, $\xi \in \ell^2(\mathcal{G}_x/N)$, and $gN \in \mathcal{G}_x/N$. 

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Proof. For \( \xi \in H_{\text{Ind}} \), define \( W: G_x/N \to \mathbb{C} \) by \( (W\xi)(\gamma N) := \xi(\gamma)(N) \). Let \( R \) be a complete set of representatives for \( G_x/N \), so that \( G_x/N = \bigsqcup_{r \in R} \{h N : h N \in G_x^r/N \} \). Then,
\[
\sum_{\gamma N \in G_x/N} |(W \xi)(\gamma N)|^2 = \sum_{\gamma \in R} \sum_{h N \in G_x^r/N} |\xi(\gamma h)(N)|^2 = \sum_{\gamma \in R} \sum_{h N \in G_x^r/N} |(\lambda_{G_x^r/N}(h)^* \xi(\gamma))(r)|^2
\]
so \( W \) defines an isometry \( H_{\text{Ind}} \to L^2(G_x/N) \). For \( \eta \in L^2(G_x/N) \) and \( \gamma \in G_x \), define \( (V \eta)(\gamma) : G_x^\gamma/N \to \mathbb{C} \) by \( [(V \eta)(\gamma)](h N) := \eta(\gamma h) \). Then \( \sum_{g N \in G_x/N} |\eta(g N)|^2 \leq ||\eta(g N)||_{L^2(G_x/N)}^2 \), so \( (V \eta)(\gamma) \in L^2(G_x/N) \) and \( V \) defines a map \( L^2(G_x/N) \to H_{\text{Ind}} \). For \( \eta \in L^2(G_x/N) \) and \( \gamma \in G_x/N \), we have \( (V \eta)(\gamma N) = (V \eta)(\gamma)(N) = (V \eta)(\gamma) \), so that \( WV = I \), which shows that \( W \) is surjective. \( \square \)

Remark 2.9 (cf. [12, § 1]). The induced representation \( \text{Ind} \lambda_{G_x^r/N} \) coincides with \( \pi_x \circ \pi_r \).

3. C*-algebras and groupoids associated with algebraic semigroup actions

3.1. The concrete C*-algebra associated with an algebraic semigroup action. Each algebraic semigroup action \( \sigma: S \curvearrowright G \) naturally gives rise to a concrete C*-algebra acting on \( L^2(G) \): Define an isometric representation \( \kappa_\sigma: S \to \text{Isom}(L^2(G)) \) by \( \kappa_\sigma(s) \delta_h = \delta_{s \cdot h} \), where \( \{\delta_h : h \in G\} \) is the canonical orthonormal basis for \( L^2(G) \). We shall call \( \kappa_\sigma \) the Koopman representation associated with the action \( \sigma \).

Definition 3.1. Suppose \( \sigma: S \curvearrowright G \) is an algebraic semigroup action. We let
\[
\mathfrak{A}_\sigma := C^*\{\kappa_\sigma(s) : s \in S\} \cup \{\lambda(g) : g \in G\},
\]
where \( \lambda: G \to \mathcal{U}(L^2(G)) \) is the left regular representation of \( G \).

In the following, we will simply write \( \kappa \) for \( \kappa_\sigma \) if the algebraic semigroup action \( \sigma \) is understood.

Remark 3.2. The C*-algebra \( \mathfrak{A}_\sigma \) depends only on the image of \( \sigma \), so if we are only interested in \( \mathfrak{A}_\sigma \), then there is no loss in generality in assuming faithfulness of \( \sigma: S \curvearrowright G \).

Let \( P : = G \times S \) denote the semi-direct product with respect to \( \sigma \) taken in the category of monoids. We identify \( G \) and \( S \) via the embeddings \( g \mapsto (g,1) \in P \) and \( s \mapsto (e,s) \in P \) as submonoids of \( P \).

Remark 3.3. Define \( \lambda \times \kappa: P \to \text{Isom}(L^2(G)) \) by \( \lambda \times \kappa(g,s) := \lambda(g) \kappa(s) \). Then \( \mathfrak{A}_\sigma = C^*_{\lambda \times \kappa}(P) \) is the C*-algebra generated by the isometric representation \( \lambda \times \kappa \) of \( P \). We will see that in general there will not exist a canonical *-homomorphism from the C*-algebra \( C^*(P) \) to \( C^*_{\lambda \times \kappa}(P) \) (see § [3]), so we cannot appeal to the theory of semigroup C*-algebras to study \( \mathfrak{A}_\sigma \). Here, \( C^*(P) \) is the full semigroup C*-algebra of \( P \) as defined in [13, Definition 5.6.38].

The following explains our name for \( \kappa \):

Remark 3.4. Suppose \( G \) is Abelian. Then in terms of the dual action \( \hat{\sigma}: S \curvearrowright \hat{X} = \hat{G} \), \( \kappa \) and \( \mathfrak{A}_\sigma \) can be understood as follows: If \( \mu \) denotes normalized Haar measure on \( X \), then \( \alpha \) is measure-preserving in the sense that \( \mu(W) = \mu(\alpha^{-1}(W)) \) for all Borel subsets \( W \subseteq X \). Hence we obtain an isometric representation of \( S \) on \( L^2(X,\mu) \) via \( L^2(X,\mu) \to L^2(X,\mu), f \mapsto f \circ \sigma_s \) (for \( s \in S \)), which is the analogue of the Koopman representation in the group case. Moreover, \( C(X) \) acts on \( L^2(X,\mu) \) by multiplication operators. This yields a unitary representation of \( G \) on \( L^2(X,\mu) \) via the canonical identification \( C^*(G) \cong C(X) \). These two representations of \( S \) and \( G \) together give rise to a representation of \( P \) which is unitarily equivalent to \( \lambda \times \kappa \) via the canonical unitary \( L^2(X,\mu) \cong L^2(G) \).

A priori, we have the following description of \( \mathfrak{A}_\sigma \):
\[
\mathfrak{A}_\sigma = \text{span}\{\kappa(s_1)^* \lambda(g_1) \kappa(t_1) \cdots \kappa(s_m)^* \lambda(g_m) \kappa(t_m) : g_i \in G, s_i, t_i \in S, m \in \mathbb{Z}_{>0}\}.
\]
We now set out to construct (a candidate for) a groupoid model for $\mathfrak{A}_\sigma$. For this, the language of inverse semigroups is very convenient.

### 3.2. The inverse semigroup associated with an algebraic semigroup action.

For background on inverse semigroups see [32], and for background on their C*-algebras, see [48] and [18, 19]. Let $\mathcal{I}_G$ denote the inverse semigroup of all partial bijections of $G$. We shall now view $\sigma_s$ as a partial bijection of $G$, so that $\sigma_s^{-1}$ makes sense as an element of $\mathcal{I}_G$; namely, $\sigma_s^{-1}$ is the partial bijection $\sigma_sG \to G$ given by $\sigma_s(g) \mapsto g$. For each $g \in G$, let $\tau_g \in \mathcal{I}_G$ denote the bijection given by $\tau_g(h) = gh$ for all $h \in G$.

**Definition 3.5.** We let $I_\sigma$ denote the inverse sub-semigroup of $\mathcal{I}_G$ generated by the endomorphisms $\sigma_s$ for $s \in S$ and the translations $\tau_g$ for $g \in G$. Explicitly, we have

$$I_\sigma = \{\sigma_s^{-1} \tau_{g_1} \sigma_{t_1} \cdots \sigma_s^{-1} \tau_{g_m} \sigma_{t_m} : s_i, t_i \in S, g_i \in G, m \in \mathbb{Z}_{>0}\}.$$

In the following, we will simply write $I$ for $I_\sigma$ if the algebraic semigroup action $\sigma$ is understood.

There is a canonical faithful representation by partial isometries $\Lambda: \mathcal{I}_G \to \text{Pls}(\ell^2(G))$ such that, for $\phi \in \mathcal{I}_G$ with domain $\text{dom}(\phi)$ and $h \in G$, $\Lambda_\phi \delta_h = \begin{cases} \delta_{\phi(h)} & \text{if } h \in \text{dom}(\phi), \\ 0 & \text{if } h \notin \text{dom}(\phi). \end{cases}$

From now on, we shall use $\Lambda$ to denote the restriction of the above representation to $I$. It is easy to see that this restriction extends the isometric representation $\kappa$, so that $\Lambda: I \to \text{Pls}(\mathfrak{A}_\sigma)$ is a representation of the inverse semigroup $I$ in $\mathfrak{A}_\sigma$. Now it follows immediately that $\mathfrak{A}_\sigma = \text{span}(\{\Lambda_\phi : \phi \in I\})$, so it is reasonable to expect that the structure of $\mathfrak{A}_\sigma$ is closely related to properties of the inverse semigroup $I$.

This motivates the analysis of the inverse semigroup $I$. In particular, we wish to compare the C*-algebras associated with $I$ with the C*-algebra $\mathfrak{A}_\sigma$.

In the following, let $\mathcal{E}_\sigma$ be the idempotent semilattice of $I$.

**Definition 3.6.** We let $\mathcal{C}_\sigma$ be the smallest family of subgroups of $G$ such that

(i) $G \in \mathcal{C}_\sigma$;

(ii) if $C \in \mathcal{C}_\sigma$, then $\sigma_sC \in \mathcal{C}_\sigma$ and $\sigma_s^{-1}C \in \mathcal{C}_\sigma$ for every $s \in S$.

Members of $\mathcal{C}_\sigma$ are called $S$-constructible subgroups. In the following, we will simply write $C$ for $\mathcal{C}_\sigma$ and $\mathcal{E}$ for $\mathcal{E}_\sigma$ if the algebraic semigroup action $\sigma$ is understood. Note that

$$C = \{\sigma_t^{-1} \sigma_{s_1} \cdots \sigma_{s_m} G : s_i, t_i \in S, m \in \mathbb{Z}_{>0}\},$$

where for a subset $C \subseteq G$, we put $\sigma_s^{-1}C := \{h \in G : \sigma_s(h) \in C\}$. Here, we are writing $\sigma_s^{-1}C$ for the set-theoretic inverse image of $C$ under $\sigma_s$, rather than the more cumbersome notation $\sigma_s^{-1}(C \cap \sigma_s G)$, which would be used when viewing $\sigma_s$ as a partial bijection.

**Remark 3.7.** If $\sigma: S \curvearrowright G$ is non-automorphic, then $C$ is non-trivial, i.e., $C \supseteq \{G\}$.

**Definition 3.8.** Let $I^c$ be the inverse sub-semigroup of $I$ generated by $\{\sigma_s : s \in S\}$, i.e.,

$$I^c = \{\sigma_s^{-1} \sigma_{t_1} \cdots \sigma_{t_m} : s_i, t_i \in S, m \in \mathbb{Z}_{>0}\}.$$

Note that $\mathcal{C}$ is the semilattice of idempotents of $I^c$.

**Proposition 3.9.** The family $\mathcal{C}$ of $S$-constructible subgroups satisfies the following properties:

(i) $\mathcal{C}$ is closed under taking finite intersections;

(ii) if $C \in \mathcal{C}$, then $\sigma_s^{-1}C$ and $\sigma_sC$ lie in $\mathcal{C}$;
(iii) if \( s \in S, \ g \in G, \ C \in \mathcal{C} \), then \( \sigma_s^{-1}(gC) = \emptyset \) or \( \sigma_s^{-1}(gC) = h\sigma_s^{-1}C \) for any \( h \in \sigma_s^{-1}(gC) \).

Proof. (i) follows because \( \mathcal{C} \) is the idempotent semilattice of \( I^e \), and (ii) follows immediately from the definition of \( \mathcal{C} \). Note that a direct proof of (i) can be given following the proof of [33, Lemma 3.3].

(iii): Let \( s \in S, \ g \in G, \text{ and } C \in \mathcal{C} \). If \( \sigma_s^{-1}(gC) \neq \emptyset \), then there exists \( h \in G \) with \( \sigma_s(h) \in gC \). We claim that \( \sigma_s^{-1}(gC) = h\sigma_s^{-1}C \). The containment “\( \supseteq \)" is clear. For “\( \subseteq \)”, let \( k \in \sigma_s^{-1}(gC) \), so that \( \sigma_s(k) \in gC \). Now \( \sigma_s(h^{-1}k) = \sigma_s(h)^{-1}\sigma_s(k) \in C \), so that \( h^{-1}k \in \sigma_s^{-1}C \), i.e., \( k \in h\sigma_s^{-1}C \). \( \square \)

**Corollary 3.10.** If \( \sigma : S \curvearrowright G \) is non-automorphic, then \( \mathcal{E} = \{gC : C \in \mathcal{C}, g \in G\} \cup \{\emptyset\} \).

Members of \( \mathcal{E} \) shall be called \( S \)-constructible cosets. We let \( \mathcal{E}^\times := \mathcal{E} \setminus \{\emptyset\} \).

Let us now develop a standard form for elements of \( I \).

**Proposition 3.11.**

(i) For \( \phi \in I \), we have \( \phi \in I^e \) if and only if \( \phi(e) = e \).

(ii) Every \( \phi \in I^\times \) is of the form \( \phi = \tau_h \varphi \tau_{g^{-1}} \) for some \( \varphi \in I^e \) and \( g, h \in G \). More precisely, if \( \text{dom}(\phi) = gC \) for some \( g \in G \) and \( C \in \mathcal{C} \), and if \( \phi(g) = h \), then \( \varphi := \tau_{h^{-1}} \phi \tau_g \in I^e \).

Proof. (i): By definition of \( I \), we know that \( \phi = \sigma_{s_1}^{-1} \tau_{g_1} \sigma_{t_1} \cdots \sigma_{s_m}^{-1} \tau_{g_m} \sigma_{t_m} \). We proceed inductively on \# \{\: g_i = e \:.\: \} . \) Note that \( \phi(e) = e \) implies that \( e \in \text{dom}(\phi) \), so that \( g_m = \tau_{g_m} \sigma_{t_m}(e) \in \text{dom}(\sigma_{s_m}^{-1}) \), i.e., \( g_m = \sigma_{s_m}(h_m) \) for some \( h_m \in G \). Hence

\[
\phi = \sigma_{s_1}^{-1} \tau_{g_1} \sigma_{t_1} \cdots \sigma_{s_m}^{-1} \tau_{g_m} \sigma_{t_m} = \cdots \sigma_{s_m}^{-1} \tau_{g_m} \sigma_{t_m} = \cdots \tau_{h_m} \sigma_{s_m}^{-1} \sigma_{t_m}
\]

In this way, we reduce \# \{\: g_i = e \:.\: \} , so that we arrive at \( \phi = \tau_g \varphi \) for some \( \varphi \in I^e \). But then \( \phi(e) = e \) implies that \( g = \tau_g \varphi(e) = \phi(e) = e \). Thus \( \varphi \in I^e \), as desired.

(ii): We know by Corollary 3.10 that \( \text{dom}(\phi) = gC \) for some \( g \in G \), \( C \in \mathcal{C} \). Let \( h \in G \) be such that \( \phi(g) = h \). Then \( \tau_{h^{-1}} \phi \tau_g (e) = e \). (i) implies that \( \varphi := \tau_{h^{-1}} \phi \tau_g \in I^e \). Hence, \( \phi = \tau_h \varphi \tau_{-g} \), as desired. \( \square \)

**Remark 3.12.** The elements in \( I^e \) are partial group automorphisms, i.e., they are group isomorphisms from their domains onto their ranges.

**Corollary 3.13.** The inverse semigroup \( I \) is 0-E-unitary if and only of \( I^e \) is \( E \)-unitary.

Proof. It is easy to see that \( I^e \) is \( E \)-unitary whenever \( I \) is 0-E-unitary. Assume \( I^e \) is \( E \)-unitary, and suppose \( \phi \in I \) and \( kD \in \mathcal{E}^\times \) are such that \( kD \subseteq \text{dom}(\phi) \) and \( \phi|_{kD} = id_{kD} \). Let \( gC = \text{dom}(\phi) \). Note that \( kD \subseteq gC \), so that \( k \in gC \) and \( D \subseteq \text{dom}(\varphi) \). By Proposition 3.11, we can write \( \phi = \tau_h \varphi \tau_{g^{-1}} \) for \( \varphi \in I^e \) with \( h = \phi(g) \) and \( \text{dom}(\varphi) = C \). Now \( \varphi|_{kD} = id_{kD} \equiv (1)

\[
\varphi(g^{-1}kD) = h^{-1}kD \quad \text{for all } D.
\]

Taking \( d = e \) in (1) gives \( \varphi(g^{-1}k) = h^{-1}k \). Since \( g^{-1}k \in D = \text{dom}(\varphi) \), we have \( \varphi(g^{-1}kD) = \varphi(g^{-1}k)\varphi(d) \) (see Remark 3.12), so (1) reduces to \( \varphi(d) = d \) for all \( D \). Since \( I^e \) is \( E \)-unitary, it follows that \( \varphi = id_C \), so that (1) implies \( g = h \). Finally, we see that \( \phi = \tau_g id_C \tau_{g^{-1}} = id_{gC} \).

**Remark 3.14.** If \( \sigma : S \curvearrowright G \) is non-automorphic, then \( \emptyset \in \mathcal{E} \), and we view \( \emptyset \) as the distinguished zero element of \( \mathcal{E} \). We shall always regard \( I^e \) as a semilattice without a distinguished zero element, even though \( I^e \) may contain the trivial subgroup \( \{e\} \).

3.3. The partial algebraic group action associated with a globalization. For the basics of partial group actions, see, for instance, [25] or [14] § 5.5. If \( \tilde{\sigma} : \mathcal{G} \curvearrowright \mathcal{G} \) is a globalization for \( \sigma : S \curvearrowright G \), then the monoid \( P = G \times S \) embeds into the group \( \Gamma := \mathcal{G} \times \mathcal{G} \). We get an affine action of \( \Gamma \) on \( \mathcal{G} \) by \( (g, s).x := g\tilde{\sigma}_s(x) \); we shall often identify \( \mathcal{G} \) and \( \mathcal{G} \) with their images in \( \Gamma \). Elements of the group \( \langle P \rangle \leq \Gamma \) are then of the form \( s_1^{-1}g_1t_1 \cdots s_m^{-1}g_mt_m \), where \( s_i, t_i \in S, \ g_i \in G, \text{ and } m \in \mathbb{Z}_{>0} \).
By restricting to the subgroup $G \leq \mathcal{G}$, we obtain a partial affine action $\Gamma \curvearrowright G$, where for $\gamma = gs \in \Gamma$ with $g \in \mathcal{G}$ and $s \in \mathcal{I}$, $\gamma$ acts as follows: The domain of $\gamma$ is

$$G_{\gamma^{-1}} := \tilde{\sigma}_s^{-1}(g^{-1}G) \cap G,$$

and the action of $\gamma$ is given by $G_{\gamma^{-1}} \to G_{\gamma x}, x \mapsto g\tilde{\sigma}_s(x)$.

**Remark 3.15.** Since the action of $g \in G$ is given by $t_g$ and the action of $s \in S$ is given by $\sigma_s$, we see that $\sigma_s^{-1}g_1\sigma_{t_1} \cdots \sigma_s^{-1}g_m\sigma_{t_m} \in I$ is a restriction of the partial bijection corresponding to $\gamma = s_1^{-1}g_1 \cdots s_m^{-1}g_m \in \langle G \times S \rangle$ and $g_1 \in G$, $s_1, t_1, \ldots, t_m \in S$.

For $\phi = t_h \varphi \varphi_{g^{-1}} \in I$, with $\varphi = \sigma_s^{-1}\sigma_{t_1} \cdots \sigma_s^{-1}\sigma_{t_m} \in I^e$, we have $\text{dom}(\phi) = g\sigma_1^{-1}\sigma_{s_m} \cdots \sigma_1^{-1}\sigma_1 G$. Moreover, for all $x \in \text{dom}(\phi)$, we have $\phi(x) = g\tilde{\sigma}_s(x)$, where $s\phi = s_1^{-1}t_1 \cdots s_m^{-1}t_m \in \langle S \rangle$ and $g\phi = h\tilde{\sigma}_s(x)^{-1} \in G$.

In light of the above remark, it is natural to ask for conditions that will ensure there is a well-defined map $I^\times \to \Gamma$ given by $\phi \mapsto g_\phi s\phi$.

Suppose $\sigma : S \cap G$ has a globalization $\tilde{\sigma} : \mathcal{I} \cap \mathcal{G}$. Consider the following condition:

(JF) $C \subseteq \text{fix}(\tilde{\sigma}_s) \implies s = e$, for all $C \in \mathcal{C}$, $s \in \langle S \rangle$.

This condition is a kind of joint faithfulness for the partial action of $\langle S \rangle \leq \mathcal{I}$ on $G$.

**Proposition 3.16.** Assume $\sigma : S \cap G$ has a globalization $\tilde{\sigma} : \mathcal{I} \cap \mathcal{G}$. Then $\tilde{\sigma} : \mathcal{I} \cap \mathcal{G}$ satisfies \textbf{(JF)} if and only if $I$ is strongly $0$-$E$-unitary. If these equivalent conditions hold, then there exists an idempotent pure partial homomorphism $g : I^\times \to \Gamma$ such that $g(\phi) = g\phi s\phi$, where $g\phi s\phi \in \Gamma$ is associated with $\phi$ as in Remark 3.15.

**Proof.** Assume \textbf{(JF)} is satisfied. Let $\phi \in I^\times$. By Proposition 3.11 we can write $\phi = t_h \varphi \varphi_{g^{-1}}$ for some $g, h \in G$ and $\varphi = \sigma_s^{-1}\sigma_{t_1} \cdots \sigma_s^{-1}\sigma_{t_m}$, where $s_1, t_1, \ldots, t_m \in S$ and $m \in \mathbb{Z}_{>0}$. We have $\text{dom}(\phi) = gC$, where $C = \sigma_1^{-1}\sigma_{s_m} \cdots \sigma_1^{-1}\sigma_1 G$, and $\phi(x) = g\tilde{\sigma}_s(x)$ for all $x \in gC$, where $s\phi = s_1^{-1}t_1 \cdots s_m^{-1}t_m$ and $g\phi = h\tilde{\sigma}_s(x)^{-1}$ (see Remark 3.15). Suppose there exists $k \in G$ and $t \in \langle S \rangle$ such that

$$g\tilde{\sigma}_s(x) = k\tilde{\sigma}_t(x)$$

for all $x \in gC$. Taking $x = g$ in (2) gives $g\tilde{\sigma}_s(g) = k\tilde{\sigma}_t(g)$. Plugging this into (2) gives $\tilde{\sigma}_s(c) = \tilde{\sigma}_t(c)$ for all $c \in C$, i.e., $\tilde{\sigma}_t^{-1}g\tilde{\sigma}_s(c) = c$ for all $c \in C$. Now $t^{-1}s\phi = e$ by \textbf{(JF)}, i.e., $t = s\phi$. Now $k = g\phi$ follows from (2). This shows that $g : I^\times \to \Gamma$ defined by $g(\phi) = g\phi s\phi$ is well-defined. An argument similar to the one above shows that $\phi \in \mathcal{E}^\times$ if and only if $(g\phi, s\phi) = (0, 1)$, and it is easy to see that $g$ is a partial homomorphism. Hence, $g$ is an idempotent pure partial homomorphism, so that $I$ is strongly $0$-$E$-unitary.

Now assume that $I$ is strongly $0$-$E$-unitary, so that there exists an idempotent pure partial homomorphism $g : I^\times \to \Lambda$ for some discrete group $\Lambda$. The restriction of $g$ to the copy of $G \times S$ in $I^\times$ is injective by [14, Lemma 5.5.7], which gives us an embedding $G \times S \hookrightarrow \Lambda$. Let $s = s_1^{-1}t_1 \cdots s_m^{-1}t_m \in \langle S \rangle$, and suppose $\tilde{\sigma}_s(x) = x$ for all $x \in C$, where $C \in \mathcal{C}$. This implies

$$\sigma_1^{-1}\sigma_{s_1} \cdots \sigma_1^{-1}\sigma_{s_m} \cdot t \cdot \text{id}_C = \text{id}_C$$

in $I$ (cf. Remark 3.15). Since both sides are nonzero, applying $g$ to (3) yields $s = s_1^{-1}t_1 \cdots s_m^{-1}t_m = e$. Hence, \textbf{(JF)} is satisfied.

Let us now discuss the special case when $S$ is left Ore.

**Example 3.17.** Assume $S$ is left Ore and that $G$ is Abelian. We shall write $G$ additively.

(i) let $t, u \in S$. Write $tu^{-1} = \alpha^{-1}\beta$ in $\langle S \rangle = S^{-1}S$, where $\alpha, \beta \in S$. Then $\sigma_t \sigma_u^{-1} = \sigma_\alpha \text{id}_G G \sigma_\beta$.

(ii) $I = \{\sigma_s^{-1} \text{id}_{D} \cdot t_g \sigma_t : g, h \in G, s, t \in S, D \in \mathcal{C}\}$. 


(iii) $\sigma : S \rightrightarrows A$ satisfies \([\text{IF}]\) if and only if $C \leq \ker (\sigma_s - \sigma_t) \implies s = t$ for all $C \in \mathcal{C}$ and $s, t \in S$.

To see (i), observe that we have $\alpha t = \beta u$, so $\sigma_s^{-1} \sigma_t \sigma_s^{-1} \sigma_t = \sigma_s^{-1} \sigma_t \sigma_s^{-1} \sigma_t = \sigma_s^{-1} \sigma_t \sigma_s^{-1} \sigma_t = \sigma_s^{-1} \text{id}_\sigma G \sigma_s$.

For (ii), it suffices to show that $\{\sigma_s^{-1} \text{id}_h + D \oplus g : g \in H, s, t \in S, D \in \mathcal{C}\}$ is closed, up to 0, under taking products. Let $s, t, u, v \in S, g, h, k \in \mathbb{N} \cup \{0\}$, $x, y \in G$, and $D, E \in \mathcal{C}$. Since $S$ is left Ore, we can find $\alpha, \beta \in S$ such that $tu^{-1} = \alpha^{-1} \beta$ in $\langle S \rangle = S^{-1} S$. By (i), we have $\sigma_s^{-1} \sigma_t^{-1} = \sigma_s^{-1} \text{id}_\sigma G \sigma_s$. Now we compute:

\[
\sigma_s^{-1} \text{id}_h + D \oplus g \sigma_s^{-1} \text{id}_E \oplus k \sigma_v = \sigma_s^{-1} \text{id}_h + D \oplus g \sigma_s^{-1} \text{id}_E \oplus k \sigma_v = \frac{\sigma_s^{-1} \text{id}_h + D \sigma_s^{-1} \sigma_t (g) \text{id}_E \sigma_t (k) \sigma_v}{\text{id}_G} = \frac{\sigma_s^{-1} \text{id}_h (\sigma_t (g) \sigma_c (k) \sigma_v)}{\text{id}_G} = \frac{\sigma_s^{-1} \text{id}_h (\sigma_t (g) \sigma_c (k) \sigma_v)}{\text{id}_G} = \frac{\sigma_s^{-1} \text{id}_h (\sigma_t (g) \sigma_c (k) \sigma_v)}{\text{id}_G}.
\]

It remains to observe that $\text{id}_h (\sigma_t (g) \sigma_c (k) \sigma_v)$ is zero or of the form $\text{id}_E + C$.

(iii) is true because, for $C \in \mathcal{C}$ and $s, t \in S$, we have $C \leq \ker (\sigma_s - \sigma_t)$ if and only if $C \leq \ker (\text{id} - \sigma_s - \sigma_t)$.

**Remark 3.18.** (i) in Example 3.17 is equivalent to the statement that if $S$ is left Ore and $G$ is Abelian, then

\[\mathfrak{A}_\sigma = \text{span}(\{\kappa (s)^* 1_{hD} \lambda (g) \kappa (t) : s, t \in S, g, h \in G, D \in \mathcal{C}\}).\]

**3.4. The unit space of the groupoid model.** Let us assume in this subsection that our action is non-automorphic, so that $\mathcal{E} = \{gC : C \in \mathcal{C}, g \in G\} \cup \{\emptyset\}$ is the idempotent semilattice of our inverse semigroup $I$. As before, we write $\mathcal{E}^\times := \mathcal{E} \setminus \{\emptyset\}$. Define $\widehat{\mathcal{E}}$ as the space of characters of $\mathcal{E}$, i.e., non-zero multiplicative maps $\mathcal{E} \to \{0, 1\}$ sending $0$ to $0$, equipped with the topology of point-wise convergence. A basis of open sets is given by

\[\mathcal{E}(gC; (g_i C_i)) := \left\{ \chi \in \widehat{\mathcal{E}} : \chi (gC) = 1; \chi (g_i C_i) = 0 \forall i \right\},\]

where $gC \in \mathcal{E}^\times$ and $\{g_i C_i\} \subseteq \mathcal{E}^\times$ is a finite subset. Without loss of generality we may assume $g_i C_i \subseteq gC$ for all $i$. There is a one-to-one correspondence between characters of $\mathcal{E}$ and filters on $\mathcal{E}$, i.e., subsets $\mathcal{F} \subseteq \mathcal{E}$ with the following properties: $\emptyset \notin \mathcal{F}$, $G \subseteq \mathcal{F}$, if $gC \in \mathcal{F}$ and $hD \in \mathcal{E}$ with $gC \subseteq hD$, then $hD \in \mathcal{F}$, and if $gC, hD \in \mathcal{F}$, then $gC \cap hD \in \mathcal{F}$. This one-to-one correspondence is implemented by the assignment $\widehat{\mathcal{E}} \ni \chi \mapsto \mathcal{F}(\chi) := \{gC \in \mathcal{E} : \chi (gC) = 1\}$.

**Definition 3.19.** Let $\widehat{\mathcal{E}}_{\text{max}}$ denote the characters $\chi$ of $\mathcal{E}$ for which $\mathcal{F}(\chi)$ is maximal with respect to inclusion.

In other words, $\chi \in \widehat{\mathcal{E}}$ belongs to $\widehat{\mathcal{E}}_{\text{max}}$ if and only if we cannot find $\chi' \in \widehat{\mathcal{E}}$ with $\mathcal{F}(\chi) \subseteq \mathcal{F}(\chi')$.

**Definition 3.20.** The boundary of $\widehat{\mathcal{E}}$ is given by $\partial \widehat{\mathcal{E}} := \overline{\widehat{\mathcal{E}}}_{\text{max}}$.

Following [18, 19], characters of $\mathcal{E}$ which belong to $\partial \widehat{\mathcal{E}}$ are called *tight*, and we also call the corresponding filters tight. We briefly recall several notions from [18, 19, 21]. A cover (resp. outer cover) of a subset $\mathcal{F} \subseteq \mathcal{E}$ is a subset $\mathcal{C} \subseteq \mathcal{F}$ (resp. $\mathcal{C} \subseteq \mathcal{E}$) such that for each $gC \in \mathcal{F}^\times := \mathcal{F} \cap \mathcal{E}^\times$, there exists $hD \in \mathcal{C}$ with $hD \cap gC \neq \emptyset$. For $e \in I$, let $\text{fix} (e) := \{g \in \text{dom} (e) : e (g) = g\}$, and let $\mathcal{J}_\phi := \{hD \in \mathcal{E} : hD \subseteq \text{fix} (\phi)\} \cup \{\emptyset\}$. A subset $\mathcal{C} \subseteq \mathcal{E}$ is a *cover* of the constructible coset $gC$ if $\mathcal{C}$ is a cover of $\mathcal{J}_\phi$. It is shown in [18, 19] that $\chi \in \widehat{\mathcal{E}}$ belongs to $\partial \widehat{\mathcal{E}}$ if and only if for every $gC \in \mathcal{E}^\times$ with $\chi (gC) = 1$ and every cover $\mathcal{C}$ of $gC$, there exists $hD \in \mathcal{C}$ such that $\chi (hD) = 1$. To ease notation, let $\cup \mathcal{J}_\phi := \bigcup_{gC \in \mathcal{J}_\phi} gC$. 


Lemma 3.21. Let $\phi \in I^\times$. A finite collection $c = \{h_iD_i\} \subseteq E^\times$ is a (finite) outer cover for $J_\phi$ if and only if $\cup J_\phi \subseteq \bigcup_i h_iD_i$. In particular, $c$ is a cover of the constructible coset $gC$ if and only if $gC = \bigcup_i h_iD_i$.

Proof. The “if” direction is clear, so suppose $c = \{h_iD_i\}$ is a finite outer cover for $J_\phi$. Set $D := \bigcap_i D_i$. We have $D \in C$ because $C$ is closed under finite intersections. Let $gC \in J_\phi$. For every $k \in gC$, we have $k(C \cap D) \subseteq gC$, so that, since $c$ is a cover for $J_\phi$, there exists $i$ such that $h_iD_i \cap k(C \cap D) \neq \emptyset$. Since $k(C \cap D) \subseteq kD_i$, this implies that $kD_i \cap h_iD_i \neq \emptyset$, i.e., $k \in h_iD_i$. Hence, $gC \subseteq \bigcup_i h_iD_i$, so that $\cup J_\phi \subseteq \bigcup_i h_iD_i$.

If $c$ is a cover of $J_\phi$, then $c \subseteq J_\phi$, so that $\bigcup_i h_iD_i \subseteq \bigcup J_\phi$. Since $\cup J_gC = gC$, we see that $c$ is a cover of the constructible coset $gC$ if and only if $gC = \bigcup_i h_iD_i$. $\square$

Let us now give a characterization of tight characters in our situation.

Lemma 3.22. Let $\chi \in \hat{E}$. Then $\chi$ lies in $\partial \hat{E}$ if and only if for all $gC \in E^\times$ with $\chi(gC) = 1$ and all $D \in C$ with $D \subseteq C$, $|C : D| < \infty$, and $C = \bigcup_i k_iD$ for some $k_i \in G$, there exists $i$ such that $\chi(gk_iD) = 1$.

Proof. “$\Leftarrow$” follows from the characterization of tight characters in [18, 19] mentioned above because $\{gk_iD\}$ is a cover of $gC$. For “$\Rightarrow$”, suppose that $\{h_jD_j\}$ is a cover of $gC$. By [19, (4.4)], we may without loss of generality assume that $|C : D_j| < \infty$ for all $j$. Set $D := \bigcap_j D_j$. Then $|C : D| < \infty$, so that $C = \bigcup_i k_iD$ for some $k_i \in G$. By assumption, there exists $i$ such that $\chi(gk_iD) = 1$. Moreover, as $gC = \bigcup_j h_jD_j$ by Lemma 3.21, there exists $j$ such that $gk_iD \subseteq h_jD_j$. Hence $\chi(h_jD_j) = 1$, as desired. $\square$

Let us now relate characters and filters on $E$ with those on $C$. As above, a filter on $C$ is a subset $\mathcal{F} \subseteq C$ with the following properties: $G \in \mathcal{F}$, if $gC \in \mathcal{F}$ and $hD \in C$ with $gC \subseteq hD$, then $hD \in \mathcal{F}$, and if $gC, hD \in \mathcal{F}$, then $gC \cap hD \in \mathcal{F}$.

Definition 3.23. We call a filter $\mathcal{F}$ on $C$ finitely hereditary if whenever $C \in \mathcal{F}$ and $D \in C$ satisfy $D \subseteq C$ with $|C : D| < \infty$, then $D \in \mathcal{F}$.

Given $\chi \in \hat{E}$, define $\Pi(\chi) : C \to \{0, 1\}$ by

$$\Pi(\chi)(C) := \begin{cases} 1 & \text{if there exists } g \in G \text{ with } \chi(gC) = 1, \\ 0 & \text{if } \chi(gC) = 0 \text{ for all } g \in G. \end{cases}$$

Moreover, set $\mathcal{F}(\chi) := \mathcal{F}(\Pi(\chi)) := \{C \in C : \Pi(\chi)(C) = 1\}$.

Lemma 3.24. (i) For every $\chi \in \hat{E}$, $\mathcal{F}(\chi)$ is a filter on $C$.

(ii) $\Pi$ is a surjective map from $\hat{E}$ onto the space of characters of $C$. More precisely, for each filter $\mathcal{F}$ on $C$ there exists a character $\chi(\mathcal{F}) \in \hat{E}$ uniquely determined by $\mathcal{F}(\chi(\mathcal{F})) = \mathcal{F}$. Moreover, for every filter $\mathcal{F}$ on $C$, $\chi \in \hat{E}$ lies in $\Pi^{-1}(\mathcal{F})$ if and only if $\chi(gC') = 0$ for all $C' \in C$ with $C' \notin \mathcal{F}$, and for all $C \in \mathcal{F}$, there exists $gC \in G$ such that $\chi(gC'C) = 1$, and $(gC)$ satisfies the compatibility condition that $gC \in gD$ for all $C, D \in \mathcal{F}$ with $C \subseteq D$. (iii) Given $\chi \in \hat{E}$, we have $\chi \in \partial \hat{E}$ if and only if $\mathcal{F}(\chi)$ is a finitely hereditary filter.

(iv) The maximal filter on $C$ is given by $\mathcal{F}(\max) = \mathcal{C}$.

(v) $\Pi^{-1}(\mathcal{F}(\max))$ is a subset of $\hat{E}(\max)$ which is dense in $\partial \hat{E}$.

Proof. (i) is true by construction. For surjectivity of $\Pi$ in (ii), just observe that, given a filter $\mathcal{F}$ on $C$, $\chi(\mathcal{F})(C) := 1$ if $C \in \mathcal{F}$ and $\chi(\mathcal{F})(gC) := 0$ for all $gC \in E$ with $gC \notin \mathcal{F}$ defines a character of $E$ which satisfies $\mathcal{F}(\chi(\mathcal{F})) = \mathcal{F}$. The remaining claims in (ii) are easy to see.

(iii) follows from Lemma 3.22. (iv) is true because any two elements of $\mathcal{C}$ always have non-empty intersection, as they all contain $e \in G$. It remains to show (v). $\Pi^{-1}(\mathcal{F}(\max)) \subseteq \hat{E}(\max)$ is clear. Given a non-empty basic open set $\partial \hat{E}(gC; \{h_iD_i\})$, there exists $z \in gC \setminus \bigcup_i h_iD_i$. Hence $kD_i \cap h_iD_i = \emptyset$ for all $i$. Moreover,
by (ii), there exists $\chi \in \Pi^{-1}(\mathfrak{F}_{\max})$ with the property that $\chi(kC) = 1$ for all $C \in \mathfrak{F}_{\max}$. It follows that $\chi \in \partial \hat{E}(gC; \{h_iD_i\})$, as desired. \qed

**Remark 3.25.** The compatibility condition in Lemma 3.24 (ii) is equivalent to the condition that $(gC)$ is an element of the projective limit $\lim_{C \in \hat{E}} \{G/C\}$.

The following notation will be convenient.

**Definition 3.26.** Given $k \in G$, we denote by $\chi_k$ the character of $E$ which satisfies $\chi(kC) = 1$ for all $C \in C$.

Note that $\chi_k$ exists by Lemma 3.24 (ii).

**Remark 3.27.** It follows from Lemma 3.24 (v) that $\{\chi_k : k \in G\}$ is dense in $\partial \hat{E}$. Thus, $\partial \hat{E}$ is a completion of (a quotient of) $G$.

3.5. The groupoid associated with an algebraic semigroup action. We present (a candidate for) a groupoid model of $A$. Let $I \ast \hat{E} := \{(\phi, \chi) \in I \times \hat{E} : \chi(\text{dom}(\phi)) = 1\}$, and define an equivalence relation on $I \ast \hat{E}$ by

$$(\phi, \chi) \sim (\psi, \gamma) \text{ if } \chi = \gamma \text{ and there exists } gC \in \mathcal{E}^\times \text{ with } \phi|_{gC} = \psi|_{gC} \text{ and } \chi(gC) = 1.$$ 

We let $[\phi, \chi]$ denote the equivalence class of $(\phi, \chi)$. For each $gC \in \mathcal{E}$, let $\hat{E}(gC) := \{\chi \in \hat{E} : \chi(gC) = 1\}$. For $gC \in \mathcal{E}$ and a finite subset $f \subseteq \mathcal{E}$, put

$$\hat{E}(gC; f) := \hat{E}(gC) \setminus \bigcup_{hD \in f} \hat{E}(hD).$$

For $\chi \in \hat{E}(\text{dom}(\phi))$, define $\phi, \chi := \chi(\phi^{-1} \cup \phi) \in \hat{E}(\text{im}(\phi))$.

**Definition 3.28.** Let

$$I \rhd \hat{E} := I \ast \hat{E} / \sim = \{[\phi, \chi] : (\phi, \chi) \in I \ast \mathcal{E}\}$$

be the transformation groupoid attached to $I \rhd \hat{E}$ with range and source maps given by $r([\phi, \chi]) = \phi, \chi$ and $s([\phi, \chi]) = \chi$, respectively. Multiplication and inversion are given by $[\phi, \psi, \chi][\psi, \chi] = [\phi\psi, \chi]$ and $[\phi, \chi]^{-1} = [\phi^{-1}, \phi, \chi]$, respectively.

Now $I \rhd \hat{E}$ becomes an ample groupoid when equipped with the topology with basis of compact open bisections

$$[\phi, \hat{E}(gC; f)] := \{[\phi, \chi] : \chi \in \hat{E}(gC; f)\} \text{ for } gC \in \mathcal{E}, f \subseteq \mathcal{E} \text{ finite with } \hat{E}(gC; f) \subseteq \hat{E}(\text{dom}(\phi)).$$

The subspace $\partial \hat{E}$ is $I$-invariant by [18, Proposition 12.8.], so that we get an action $I \rhd \partial \hat{E}$. The basic open subsets of $\partial \hat{E}$ are of the form

$$\partial \hat{E}(gC; \{h_iD_i\}) := \partial \hat{E} \cap \hat{E}(gC; \{h_iD_i\}).$$

**Definition 3.29.** We set $G_\sigma := I \rhd \partial \hat{E}$.

In the terminology from [18], $G_\sigma$ is the tight groupoid of $I$. When $I$ is countable, the $C^*$-algebra $C^*(G_\sigma)$ is universal for tight representations of the inverse semigroup $I$ by [18, Theorem 13.3].

Since $\Lambda : I \to \mathfrak{A}_r$ is a representation, the universal property of $C^*(I)$ yields a (surjective) $^*$-homomorphism $\tilde{\rho} : C^*(I \rhd \hat{E}) \cong C^*(I) \to \mathfrak{A}_r$ such that $\tilde{\rho}(1_{[\phi, \partial \hat{E}(\text{dom}(\phi))]} = \Lambda_\phi$ for all $\phi \in I^\times$. 

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Proposition 3.30. Assume $I$ is countable and that $\sigma: S \curvearrowright G$ is non-automorphic. Then, representation $\Lambda: I \to \mathfrak{A}_\sigma$ is tight, so that $\tilde{\rho}: C^*(I \ltimes \hat{E}) \to \mathfrak{A}_\sigma$ factors through a representation $\rho: C^*(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma$. Moreover, the restriction of $\rho$ to $C(\partial \hat{E})$ is an isomorphism onto $\mathfrak{D}_\sigma := \overline{\text{span}}\{1_gC : g \in G, C \in \mathcal{C}\}$.

Proof. Since $\rho$ is unital, [20] Corollary 4.3 implies that $\Lambda$ is tight if and only if it is cover-to-join in the sense of [20] §3, i.e., for every $gC \in \mathcal{E}^\times$, we have

$$\Lambda(gC) \subseteq \bigvee_{hD \in \mathcal{E}} \Lambda(hD)$$

for all finite covers $\mathcal{E}$ of $gC$. Now suppose $\mathcal{E} = \{g_i C_i : i \in F\}$ is a finite cover of $gC$. By Lemma 3.21 we have $gC = \bigcup_{i \in F} g_i C_i$, and by [46] Lemma 4.1, there exists $j \in F$ with $|C : C_j| < \infty$. In $\mathfrak{D}_\sigma$, we have

$$\bigvee_{i \in F} 1_{g_i C_i} = \sum_{\emptyset \neq I \subseteq F} (-1)^{|I|-1} \prod_{i \in I} 1_{g_i C_i} = 1_{\bigcup_{i \in F} g_i C_i} = 1_{gC}.$$

Hence, $\bigvee_{i \in F} \Lambda(g_i C_i) = \Lambda(gC)$.

We now turn to the second claim. It is clear that the restriction of $\rho$ to $C(\partial \hat{E})$ has image equal to $\mathfrak{D}_\sigma$. It follows from [14] Proposition 5.6.21] that there is an inverse homomorphism, which implies injectivity. □

Remark 3.31. The $C^*$-algebra $C^*(\mathcal{G}_\sigma)$ provides a universal model for $\mathfrak{A}_\sigma$.

The $C^*$-algebras $C^*(I) = C^*(I \ltimes \hat{E})$ and $C^r(I) = C^r(I \ltimes \hat{E})$ can be regarded the full and reduced “Toeplitz-type” $C^*$-algebras associated with $\sigma: S \curvearrowright G$.

Let $\sigma: S \curvearrowright G$ be an algebraic semigroup action that admits a globalization $\tilde{\sigma}: \hat{\mathcal{S}} \curvearrowright \hat{\mathcal{G}}$. Assume [JF] is satisfied, and let $g: I^\times \to \Gamma = \mathcal{G} \ltimes \mathcal{S}$ be the idempotent pure partial homomorphism from Proposition 3.16. General results for inverse semigroups admitting idempotent pure partial homomorphisms to groups now give us a partial action of $\Gamma = \mathcal{G} \ltimes \mathcal{S}$ on $\hat{E}$ (cf. [35], [14] Chapter 5), or [35]): For each $\gamma \in \Gamma$, let

$$U_{\gamma^{-1}} := \{\chi \in \hat{E} : \chi(gC) = 1\}, \text{ where } gC = \text{dom}(\phi) \text{ for some } \phi \in I^\times \text{ with } g(\phi) = \gamma\}.$$

Then $\gamma \in \Gamma$ acts via the homeomorphism $U_{\gamma^{-1}} \to U_{\gamma}, \chi \mapsto \gamma.\chi$ defined by

$$(\gamma.\chi)(hD) := \chi(\gamma^{-1}(hD) \cap \text{dom}(\phi))$$

for any $\phi \in I^\times$ with $g(\phi) = \gamma$. We have isomorphisms $I \ltimes \hat{E} \cong \Gamma \ltimes \hat{E}$ and $I \ltimes \partial \hat{E} \cong \Gamma \ltimes \partial \hat{E}$.

4. Properties of the groupoid model

In this section, we study properties of the groupoid $\mathcal{G}_\sigma$ attached to the non-automorphic algebraic semigroup action $\sigma: S \curvearrowright G$. These groupoid properties then translate into properties of the $C^*$-algebras $C^*_r(\mathcal{G}_\sigma)$ and $C^*_{\text{sa}}(\mathcal{G}_\sigma)$.

4.1. Hausdorffness. Combining Lemma 3.21 with [21] Theorem 3.16, we obtain:

**Proposition 4.1.** The groupoid $\mathcal{G}_\sigma$ is Hausdorff if and only if the following condition is satisfied:

(H) For all $\phi \in I$, there exists constructible cosets $g_1 C_1, ..., g_n C_n \subseteq \text{fix}(\phi)$ such that $\cup J_\phi = \bigcup g_i C_i$.

**Remark 4.2.** Clearly, if $\text{fix}(\phi)$ is finite for all $\phi \in I \setminus \mathcal{E}$, then (H) is satisfied and $\mathcal{G}_\sigma$ is Hausdorff.

Assume that $\sigma: S \curvearrowright G$ has a globalization $\tilde{\sigma}: \hat{\mathcal{S}} \curvearrowright \hat{\mathcal{G}}$. If (JF) is satisfied, then $\mathcal{G}_\sigma \cong (\mathcal{G} \ltimes \mathcal{S}) \ltimes \partial \hat{E}$ is a partial transformation groupoid and hence Hausdorff. If $\hat{\mathcal{G}}$ is torsion-free and $\tilde{\sigma}$ is mixing, then $\text{fix}(\tilde{\sigma}_s) = \{e\}$ for all $1 \neq s \in \mathcal{G}$ (see [50]). Hence $\text{fix}(\tilde{\sigma}_s)$ is finite for all $1 \neq s \in \mathcal{G}$ and thus $\mathcal{G}_\sigma$ is Hausdorff by the observation above. If we know in addition that $\{e\} \notin \mathcal{C}$, then $I$ is strongly $0$-$E$-unitary.
4.2. Closed invariant subspaces of the boundary. Recall that we introduced the notion of filters on $\mathcal{E}$ in §3.4 and that there is a one-to-one correspondence between characters of $\mathcal{E}$ (i.e., elements of $\hat{\mathcal{E}}$) and filters given by $\hat{\mathcal{E}} \ni \chi \mapsto \mathcal{F}(\chi) := \{ gC \in \mathcal{E} : \chi(gC) = 1 \}$. This bijection restricts to a one-to-one correspondence between tight characters and tight filters. Moreover, we defined the map $\Pi$ from characters of $\mathcal{E}$ to characters of $\mathcal{C}$ and set $\mathcal{F}(\chi) := \mathcal{F}(\Pi(\chi)) := \{ C \in \mathcal{C} : \Pi(\chi)(C) = 1 \}$ for every $\chi \in \hat{\mathcal{E}}$. We observed that for all $\chi \in \hat{\mathcal{E}}$, $\chi$ is tight if and only if $\mathcal{F}(\chi)$ is finitely hereditary. Moreover, for every filter $\mathcal{F}$ on $\mathcal{C}$ there exists a uniquely determined character $\chi(\mathcal{F}) \in \hat{\mathcal{E}}$ with $\mathcal{F}(\chi(\mathcal{F})) = \mathcal{F}$. In particular, for a finitely hereditary filter $\mathcal{F}$ on $\mathcal{C}$, $\chi(\mathcal{F})$ is a tight character.

In order to describe closed invariant subspaces of $\partial \hat{\mathcal{E}}$, we need some terminology.

**Definition 4.3.** Let $\mathcal{F}_C$ be the set of finitely hereditary filters on $\mathcal{C}$. We define a partial action of $I^e$ on $\mathcal{F}_C$ as follows: Given $\mathcal{F} \in \mathcal{F}_C$ and $\varphi \in I^e$, $\varphi.\mathcal{F}$ is defined if there exists $C_\varphi \in \mathcal{F}$ such that $C_\varphi \subseteq \text{dom}(\varphi)$. In that case, $\varphi.\mathcal{F}$ is defined as the smallest element of $\mathcal{F}_C$ containing $\{ \varphi(C_\varphi \cap C) : C \in \mathcal{F} \} = \{ \varphi(C) : C \in \mathcal{F}, C \subseteq C_\varphi \}$. A subset $\mathcal{F} \subseteq \mathcal{F}_C$ is called $I^e$-invariant if for all $\mathcal{F} \in \mathcal{F}_C$ and $\varphi \in I^e$ such that $\varphi.\mathcal{F}$ is defined, we have $\varphi.\mathcal{F} \in \mathcal{F}$.

A subset $\mathcal{F} \subseteq \mathcal{F}_C$ is called $\subseteq$-closed if for all $\mathcal{E} \in \mathcal{F}_C$, $\mathcal{E} \subseteq \bigcup_{\mathcal{F} \in \mathcal{F}} \mathcal{F}$ implies $\mathcal{E} \in \mathcal{F}$.

Our main result concerning closed invariant subspaces of $\partial \hat{\mathcal{E}}$ reads as follows:

**Theorem 4.4.** There is a one-to-one correspondence between closed invariant subspaces of $\partial \hat{\mathcal{E}}$ and $I^e$-invariant, $\subseteq$-closed subsets of $\mathcal{F}_C$ sending $\mathcal{X} \subseteq \partial \hat{\mathcal{E}}$ to $\mathcal{F}(\mathcal{X}) := \{ \mathcal{F}(\chi) : \chi \in \mathcal{X} \}$. The inverse map sends $\mathcal{F} \subseteq \mathcal{F}_C$ to $\mathcal{F}^{-1}(\mathcal{F}) := \{ \chi \in \partial \hat{\mathcal{E}} : \mathcal{F}(\chi) \in \mathcal{F} \}$.

For the proof, we first show that closed, $G$-invariant subspaces of $\partial \hat{\mathcal{E}}$ are in one-to-one correspondence with $\subseteq$-closed subsets of $\mathcal{F}_C$.

**Lemma 4.5.** Given $\mathcal{X} \subseteq \partial \hat{\mathcal{E}}$, we have $\Pi^{-1}(\Pi(\mathcal{X})) \subseteq \{ t_k.\chi : k \in G, \chi \in \mathcal{X} \}$.

**Proof.** Suppose that $\chi, \omega \in \partial \hat{\mathcal{E}}$ satisfy $\Pi(\omega) = \Pi(\chi)$. Then we claim that $\omega \in \{ t_k.\chi : k \in G \}$. Indeed, take a basic open neighbourhood $\partial \hat{\mathcal{E}}(gC ; \{ h_iD_i \})$ of $\omega$. Then $\omega(gC) = 1$ and $\omega(h_iD_i) = 0$ for all $i$. Without loss of generality, we can assume that $D_i \not\subseteq \mathcal{F}(\omega)$ (otherwise replace $C$ by $C \cap D_i$). As $\Pi(\omega) = \Pi(\chi)$, this implies that there exists $k \in G$ with $\chi(kC) = 1$ whereas $\mathcal{F}(\omega)$ and $\mathcal{F}(\chi)$ are $\subseteq$-closed, as desired.

**Proposition 4.6.** Given $\mathcal{X} \subseteq \partial \hat{\mathcal{E}}$ and $\omega \in \partial \hat{\mathcal{E}}$, we have $\omega \in \{ t_k.\chi : k \in G, \chi \in \mathcal{X} \}$ if and only if $\mathcal{F}(\omega) \subseteq \bigcup_{\chi \in \mathcal{X}} \mathcal{F}(\chi)$.

**Proof.** “$\Rightarrow$”: We have $C \in \mathcal{F}(\omega)$ if and only if $\omega(gC) = 1$ for some $g \in G$. If $\omega \in \{ t_k.\chi : k \in G, \chi \in \mathcal{X} \}$, then there exists $\chi \in \mathcal{X}$ and $k \in G$ such that $t_k.\chi(gC) = 1$. Hence $C \in \mathcal{F}(t_k.\chi) = \mathcal{F}(\chi)$.

“$\Leftarrow$”: Without loss of generality, we may assume that $\mathcal{F}(\omega) = \mathcal{F}(\omega)$ because Lemma 4.6 allows us to replace $\omega$ by $\chi(\mathcal{F}(\omega))$ if necessary. Take a basic open neighbourhood $\partial \hat{\mathcal{E}}(gC ; \{ h_iD_i \})$ of $\omega$, with $h_iD_i \subseteq gC$ for all $i$. We may assume $g = e$. Hence $\omega(C) = 1$ and $\omega(h_iD_i) = 0$ for all $i$. We may also assume that $D_i \not\subseteq \mathcal{F}(\omega)$ for all $i$ (otherwise replace $C$ by $C \cap D_i$). Thus $[C : D_i] = \infty$ for all $i$. Since $\mathcal{F}(\omega) \subseteq \bigcup_{\chi \in \mathcal{X}} \mathcal{F}(\chi)$, we can find $\chi \in \mathcal{X}$ with $\chi(lC) = 1$ for some $l \in G$. Without loss of generality assume that $l = e$, i.e., $\chi(C) = 1$. For each $i$ with $D_i \subseteq \mathcal{F}(\chi)$ choose $k_i \in G$ so that $\chi(k_iD_i) = 1$. For each $i$, we have $h_iD_i = e$ implies $h_iD_i \neq h_iD_i$ for all $i$. Hence $(h_iD_i) \cap (h_iD_i) = \emptyset$ for all $i$, and thus $t_{h_iD_i}(h_iD_i) = 0$ for all $i$. We conclude that $t_{h_iD_i} \in \partial \hat{\mathcal{E}}(C ; \{ h_iD_i \})$, as desired.
Corollary 4.7. The map in Theorem 4.4 implements a one-to-one correspondence between closed, $G$-invariant subspaces of $\partial \hat{E}$ and $\subseteq$-closed subsets of $\tilde{S}_C$.

Proof. Proposition 4.6 implies that for any closed, $G$-invariant subspace $X$ of $\partial \hat{E}$, $\tilde{S}(X)$ is $\subseteq$-closed. Moreover, given a $\subseteq$-closed subset $\tilde{S}$ of $\tilde{S}_C$, $\tilde{S}^{-1}(\tilde{S})$ is clearly $G$-invariant. To see that it is also closed, take $\omega \in \tilde{S}^{-1}(\tilde{S})$. Proposition 4.6 implies that $\tilde{S}(\omega) \subseteq \bigcup_{\tilde{S} \in \tilde{S}_C} \tilde{S}$. But since $\tilde{S}$ is $\subseteq$-closed, this implies $\tilde{S}(\omega) \in \tilde{S}$ and thus $\omega \in \tilde{S}^{-1}(\tilde{S})$, as desired.

To see that these maps are inverse to each other, first note that $\tilde{S}(\tilde{S}^{-1}(\tilde{S})) = \tilde{S}$ because $\tilde{S}$ is surjective. Moreover, to show $\tilde{S}^{-1}(\tilde{S}(X)) = X$, it suffices to show $\subseteq$ as $\supseteq$ is clear. So take $\omega \in \partial \hat{E}$ with $\tilde{S}(\omega) \in \tilde{S}(X)$. Then $\tilde{S}(\omega) = \tilde{S}(\chi)$ for some $\chi \in X$. It follows that $\Pi(\omega) = \Pi(\chi)$ and thus $\omega \in \{t_k \cdot \chi : k \in G\}$ by Lemma 4.5. This shows $\subseteq$.

For the proof of Theorem 4.4 it remains to show that the one-to-one correspondence in Corollary 4.7 restricts to a one-to-one correspondence between closed invariant subspaces of $\partial \hat{E}$ and $I^\omega$-invariant, $\subseteq$-closed subsets of $\tilde{S}_C$. In other words, we have to show that the $I^\omega$-action is preserved. This is a consequence of the following observations:

Lemma 4.8. (i) For all $\varphi \in I^\omega$ and $\chi \in \partial \hat{E}$, if $\varphi \cdot \chi$ is defined, then $\varphi \cdot \tilde{S}(\chi)$ is defined.

(ii) For all $\varphi \in I^\omega$ and $\chi \in \partial \hat{E}$, if $\varphi \cdot \tilde{S}(\chi)$ is defined, then there exists $g \in G$ such that $\varphi \cdot (t_g \cdot \chi)$ is defined.

(iii) For all $\varphi \in I^\omega$ and $\chi \in \partial \hat{E}$ such that $\varphi \cdot \chi$ and $\varphi \cdot \tilde{S}(\chi)$ are defined, we have $\tilde{S}(\varphi \cdot \chi) = \varphi \cdot \tilde{S}(\chi)$.

Proof. (i) and (ii) follow from the observations that $\varphi \cdot \chi$ is defined if and only if there exist $g \in G$ and $C \in C$ with $\chi(gC) = 1$ and $gC \subseteq \text{dom}(\varphi)$, while the latter is equivalent to $g \in \text{dom}(\varphi)$ and $C \subseteq \text{dom}(\varphi)$ because $\varphi \in I^\omega$, and that $\varphi \cdot \tilde{S}(\chi)$ is defined if and only if there exist $C \in \tilde{S}(\chi)$ with $C \subseteq \text{dom}(\varphi)$.

For (iii), assume that $g_\varphi C_\varphi \in \tilde{E}^X$ satisfy $g_\varphi C_\varphi \subseteq \text{dom}(\varphi)$ and $\chi(g_\varphi C_\varphi) = 1$. Observe that $D \in \tilde{S}(\varphi \cdot \chi)$ if and only if there exists $h \in G$ such that $\varphi \cdot \chi(hD) = 1$, which is equivalent to existence of $h \in G$ such that $\chi(g_\varphi^{-1}(\varphi(g_\varphi)C_\varphi) \cap (hD)) = 1$. Furthermore, observe that $D \in \varphi \cdot \tilde{S}(\chi)$ if and only if there exists $C \in \tilde{S}(\chi)$ such that $\varphi(C_\varphi \cap C) \subseteq D$. Therefore, we need to show that there exists $h \in G$ such that $\chi(g_\varphi^{-1}(\varphi(g_\varphi)C_\varphi) \cap (hD)) = 1$ and if only if there exists $C \in \tilde{S}(\chi)$ such that $\varphi(C_\varphi \cap C) \subseteq D$.

Remark 4.9. Theorem 4.4 reduces the problem of computing all closed invariant subspaces of $\partial \hat{E}$ to the study of certain subsets of $\tilde{S}_C$, and the relevant subsets are singled out by two conditions involving the $I^\omega$-action and set-theoretical properties, but no topology. The question remains whether it is possible to compute closed invariant subspaces more concretely. In principle, $I^\omega$-invariant and $\subseteq$-closed subsets of $\tilde{S}_C$ are completely determined by the subset $\bigcup_{\tilde{S} \in \tilde{S}_C} \tilde{S}$ of $C$. However, in general, it seems to be a challenge to characterize which subsets of $C$ arise in this way.

4.3. Minimality. Let us now give several characterizations of minimality of our groupoid model.

Theorem 4.10. Let $\sigma : S \curvearrowright G$ be a non-automorphic algebraic semigroup action. The following conditions are each equivalent to minimality of $G_\sigma$:

(M1) For all $C, D \in C$, there exist $\phi_1, \ldots, \phi_n \in I$ such that $C \subseteq \bigcup_{i=1}^n \phi_i(\text{dom}(\phi_i) \cap D)$.

(M2) For all $D \in C$, there exist $\phi_1, \ldots, \phi_n \in I$ with $G = \bigcup_{i=1}^n \phi_i(\text{dom}(\phi_i) \cap D)$.

(M3) For all $C, D \in C$, there exist $\varphi \in I^\omega$ and $D' \in C$ with $D' \leq D \cap \text{dom}(\varphi)$ such that $\varphi(D') \leq C$ and $[C : \varphi(D')] < \infty$.

(M4) For all $D \in C$, there exist $\varphi \in I^\omega$ and $D' \in C$ with $D' \leq D \cap \text{dom}(\varphi)$ such that $[G : \varphi(D')] < \infty$.

(M5) $\tilde{S}_C$ contains no proper, $I^\omega$-invariant, $\subseteq$-closed subsets.
Proof. By Lemma [3.21] and [21] Theorem 5.5], minimality of \( G_\sigma \) implies condition (M1). The implications (M1) \( \Rightarrow \) (M2) and (M3) \( \Rightarrow \) (M4) are clear, and that (M5) implies minimality of \( G_\sigma \) follows from Theorem 4.4.

(M2) \( \Rightarrow \) (M3): Let \( C, D \in \mathcal{C} \). By assumption, there exist \( \phi_1, \ldots, \phi_n \in I \) with \( G = \bigcup_{i=1}^n \phi_i(\text{dom}(\phi_i) \cap D) \), where \( \text{dom}(\phi_i) \cap D \neq \emptyset \) for all \( i \). We have

\[
C = \bigcup_{i=1}^n C \cap \phi_i(\text{dom}(\phi_i) \cap D) = \bigcup_{i=1}^n \phi_i(\phi_i^{-1}(C \cap \text{im}(\phi_i)) \cap D).
\]

Without loss in generality, we may assume that for every \( i \), \( \phi_i(\phi_i^{-1}(C \cap \text{im}(\phi_i)) \cap D) \) is non-empty, so that there exists a constructible coset \( k_i D_i \subseteq \text{dom}(\phi_i) \cap D \) with \( \phi_i^{-1}(C \cap \text{im}(\phi_i)) \cap D = k_i D_i \). By Proposition 3.11 we can write \( \phi_i = t_i \phi_i \varphi_i^{-1} \) for some \( g_i, h_i \in G \) and \( \varphi_i \in I^e \) such that \( \text{dom}(\varphi_i) \subseteq \mathcal{C} \) and \( \text{dom}(\phi_i) = g_i \text{dom}(\varphi_i) \). Since \( k_i D_i \subseteq \text{dom}(\phi_i) \), we have \( k_i g_i c_i \) for some \( c_i \in \text{dom}(\varphi_i) \) and \( D_i \subseteq \text{dom}(\varphi_i) \), so that \( c_i D_i \subseteq \text{dom}(\varphi_i) \). By Remark 3.12 \( \varphi_i \) is a homomorphism on \( \text{dom}(\varphi_i) \), so that

\[
\phi_i(k_i D_i) = h_i \varphi_i(g_i^{-1} k_i D_i) = h_i \varphi_i(c_i D_i) = h_i \varphi_i(c_i \varphi_i(D_i)).
\]

Now \( C = \bigcup_{i=1}^n h_i \varphi_i(c_i \varphi_i(D_i)) \), so by [46 (4.3)], we may assume that \( \varphi_i(D_i) \) has finite index in \( C \) for all \( i \). Since \( e \in C \), there exists \( i \) such that \( e \in h_i \varphi_i(c_i \varphi_i(D_i)) \), i.e., \( h_i \varphi_i(c_i \varphi_i(D_i)) = \varphi_i(D_i) \). It remains to observe that \( D_i \subseteq D \) because \( k_i D_i \subseteq D \).

(M4) \( \Rightarrow \) (M5): Suppose \( \mathcal{F} \subseteq \mathcal{C} \) is a nonempty, \( I^e \)-invariant, \( \subseteq \)-closed subset. Let \( \mathcal{F} \in \mathcal{F} \). Given \( D \in \mathcal{C} \), there exists a constructible subgroup \( D' \subseteq D \) and \( \varphi \in I^e \) such that \( D' \subseteq \text{dom}(\varphi) \) and \( \varphi(D') \leq G \) has finite index. Since \( G \in \mathcal{F} \) and \( \mathcal{F} \) is finitely hereditary, it follows that \( \varphi(D') \in \mathcal{F} \). Thus, \( \varphi^{-1} \mathcal{F} \) is defined and contains \( D' \), so that \( D \subseteq \varphi^{-1} \mathcal{F} \) because \( \varphi^{-1} \mathcal{F} \) is a filter. Moreover, \( \varphi^{-1} \mathcal{F} \subseteq \mathcal{F} \) by \( I^e \)-invariance. Since \( D \) was arbitrary, it follows that \( \mathcal{F} \subseteq \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F} \). Since \( \mathcal{F} \) is \( \subseteq \)-closed, it follows that \( \mathcal{F} \in \mathcal{F} \) for all \( \mathcal{F} \in \mathcal{C} \).

4.4. Topological freeness. Following [30] § 2.4, we shall say that an étale groupoid \( G \) is topologically free if for every open bisection \( U \) of \( G \), \( U \subseteq \text{Iso}(G) \setminus G^{(0)} \) implies that \( \{ x \in G^{(0)} : G^x \cap U \neq \emptyset \} \) has empty interior. In general, \( G \) is topologically free whenever it is effective (i.e., \( \text{Iso}(G)^x = G^{(0)} \), where \( \text{Iso}(G) := \bigcup_{x \in G^{(0)}} G^x \) is the isotropy bundle of \( G \)). The converse holds if \( G \) is Hausdorff. Topological freeness is important because of its relationship to the intersection property, see [30] § 7 and [27] § 7.1.

We now turn to topological freeness for the groupoids \( I \rtimes \mathcal{X} \), where \( \mathcal{X} \subseteq \partial \mathcal{E} \) is a closed, invariant subset.

Definition 4.11. The algebraic semigroup action \( \sigma : S \rhd G \) is called exact if \( \bigcap_{C \in \mathcal{C}} C = \{ e \} \).

Remark 4.12. The group \( \bigcap_{C \in \mathcal{C}} C \) is the biggest subgroup \( G_C \) of \( G \) which is invariant under \( \sigma_s \) for all \( s \in S \) such that \( \sigma_s|_{G_C} \) is surjective for all \( s \in S \). Thus, \( \sigma : S \rhd G_C \) is an automorphic \( S \)-action, and exactness of \( \sigma : S \rhd G \) is equivalent to saying that there are no \( S \)-invariant subgroups \( H \leq G \) such that the associated \( S \)-action \( S \rhd H \) is automorphic.

Remark 4.13. Our definition of exactness for an algebraic semigroup action is a vast generalization of the notion of exactness for a single endomorphism given by Rohlin in [49].

When \( \sigma : S \rhd G \) is an algebraic dynamical system in the sense of [7], then it is exact if and only if \( \bigcap_{s \in S} \sigma_s G = \{ e \} \). This stronger condition is called “minimal” in [52]. Our condition (M1) is automatically satisfied for the actions in [52] 7 (see § 7.2 below), which could explain their choice of terminology (cf. [52] Remark 1.7]).

For \( \mathcal{F} \in \mathcal{C} \), put \( \mathcal{F} := \bigcap_{C \in \mathcal{C}} C \), and for \( H \leq G \), let \( \text{core}_G(H) := \bigcap_{g \in H} g H g^{-1} \). Our main result on topological freeness reads as follows:

Theorem 4.14. Suppose that \( \mathcal{X} \subseteq \partial \mathcal{E} \) is a closed invariant subspace, and let \( \mathcal{F} = \mathcal{F}(\mathcal{X}) \).

(i) If the groupoid \( I \rtimes \mathcal{X} \) obtained by restricting \( G_\sigma \) to \( \mathcal{X} \) is topologically free, then for all \( D \in \bigcup_{\mathcal{F} \in \mathcal{C}} \mathcal{F} \), we have \( \bigcap_{\mathcal{F} \in \mathcal{C}, D \in \mathcal{F}} \text{core}_G(C) = \{ e \} \).
We also deduce the following consequence, which is special to our situation and in general only holds for 
that I implies that with contains a basic open set of the form C \{gC_i\}.

For the proof, we need some preparations.

**Lemma 4.15.** Let gC_i, gC_i \in E^\times for 1 \leq i \leq n with gC \setminus \bigcup_i gC_i \neq \emptyset. Then there exists hD \in E^\times such that hD \subseteq gC \setminus \bigcup_i gC_i. Moreover, if \bigcup_i gC_i \subseteq gC, then for any constructible coset hD \subseteq gC \setminus \bigcup_i gC_i, we have \partial E(hD) \subseteq \partial E(gC; \{gC_i\}).

**Proof.** Put D := C \cap \bigcap_i C_i. Since D \leq C_i for all i and D \leq C, each of the cosets gC_i and gC_i can be written as a non-trivial (possibly infinite) disjoint union of D-cosets. Hence, gC \setminus \bigcup_i gC_i contains a D-coset, hD say.

Now suppose \bigcup_i gC_i \subseteq gC. If \chi \in \partial E(hD), i.e., \chi(hD) = 1, then \chi(gC) = 1 because hD \subseteq gC. If we had \chi(gC_i) = 1 for some i, then we would have \chi(hD \cap gC_i) = 1; this is impossible since hD and gC_i are disjoint. Thus, \partial E(hD) \subseteq \partial E(gC; \{gC_i\}).

\[ \square \]

**Lemma 4.16.** Let \mathfrak{X} and \mathfrak{F} be as in Theorem 4.14. The following are true:

(i) \bigcup_{\mathfrak{F} \in \mathfrak{F}} \{t_k \cdot \chi(\mathfrak{F}) : k \in G\} is dense in \mathfrak{X}. In particular, \{\chi_k : k \in G\} is dense in \partial \hat{E}.

(ii) Given \mathfrak{F} \in \mathfrak{F} and \phi \in I, if \phi \cdot \chi(\mathfrak{F}) = \chi(\mathfrak{F}), then \phi(\mathfrak{F}) = \mathfrak{F}.

**Proof.** (i) follows from Proposition 4.16. To see (ii), observe that for D \subseteq \text{dom}(\phi), we have \chi(\mathfrak{F})(D) = 1 if and only if \chi(\mathfrak{F})(\phi(D)) = 1. This implies \phi(\bigcap_{C \in \mathfrak{F}} C) \supseteq \bigcap_{C \in \mathfrak{F}} C. The reverse inclusion follows by replacing \phi by \phi^{-1}.

\[ \square \]

**Proof of Theorem 4.14.** (i): Suppose that there exists D \in \bigcup_{\mathfrak{F} \in \mathfrak{F}} \mathfrak{F} with \bigcap_{\mathfrak{F} \in \mathfrak{F}} \text{core}_G(\mathfrak{F}) \neq \{e\}. Take e \neq k \in \bigcap_{\mathfrak{F} \in \mathfrak{F}} \text{core}_G(\mathfrak{F}). For all \chi \in \mathfrak{X} with D \in \mathfrak{F}(\chi), we have \phi^{-1}gC = gC for every g \in G and \chi \in \mathfrak{F}(\chi), which implies t_k \cdot \chi = \chi. Hence, \{t_k \times (\mathfrak{X} \cap \partial \hat{E}(D))\} \subseteq \text{Iso}(I \times \mathfrak{X}) \setminus \mathfrak{X}. Since \mathfrak{X} \cap \partial \hat{E}(D) \neq \emptyset, this implies that I \times \mathfrak{X} is not topologically free.

(ii): Suppose that \{\phi, U\} \subseteq \text{Iso}(I \times \mathfrak{X}) for some \phi \in I and nonempty open set U \subseteq \mathfrak{X} \cap \partial \hat{E}(\text{dom}(\phi)). Then U contains a basic open set of the form \mathfrak{X} \cap \partial \hat{E}(gC; \{h_i D_i\}), and by Lemma 4.15 there exists hD \in E^\times such that \partial \hat{E}(hD) \subseteq \partial \hat{E}(gC; \{h_i D_i\}). Hence \phi \cdot \chi = \chi for all \chi \in \mathfrak{X} with \chi(hD) = 1. In particular, for all \mathfrak{F} \in \mathfrak{F} with D \in \mathfrak{F} and k \in hD, we have \phi \cdot t_k(\chi(\mathfrak{F})) = t_k(\chi(\mathfrak{F})). It follows that (t_{k^{-1}} \cdot \phi \cdot t_k(\chi(\mathfrak{F}))) = \chi(\mathfrak{F}) and thus (t_{k^{-1}} \cdot \phi \cdot t_k)(\chi(\mathfrak{F})) = \chi(\mathfrak{F}) by Lemma 4.16, i.e., \phi(k \cap \mathfrak{F}) = k \cap \mathfrak{F}. By assumption, \bigcap_{\mathfrak{F} \in \mathfrak{F}} \mathfrak{F} \cap \mathfrak{F} = \{e\}, which implies \phi(k) = k for all k \in hD, i.e., \phi|_{hD} = \text{id}_hD. Hence \mathfrak{X} \cap \partial \hat{E}(hD) = \{\phi\} \times (\mathfrak{X} \cap \partial \hat{E}(hD)) \subseteq [\phi, U] \cap \mathfrak{X} and thus [\phi, U] \cap \mathfrak{X} \neq \emptyset, as desired.

For the last claim, let us assume G is Abelian. (ii) implies that G_\sigma is topologically free if \sigma: S \rhd G is exact. For the converse, suppose there exists e \neq k \in \bigcap_{C \in \mathfrak{C}} C. Then, since G Abelian, t_k \cdot \chi = \chi for every \chi \in \partial \hat{E}, and [\mathfrak{F}, \partial \hat{E}] \subseteq \text{Iso}(G_\sigma) \setminus \partial \hat{E}, which implies G_\sigma is not topologically free.

\[ \square \]

Since G_\sigma is ample, we derive the following immediate consequence from [4, Corollary 3.15].

**Corollary 4.17.** If the conditions in part (ii) of Theorem 4.14 are satisfied for all closed, invariant subspaces \mathfrak{X} \subseteq \partial \hat{E} and G_\sigma is Hausdorff and inner exact, then C^*_\sigma(G_\sigma) has the ideal property.

We also deduce the following consequence, which is special to our situation and in general only holds for amenable groupoids. Put G_\epsilon := \bigcap_{C \in \mathfrak{C}} C.

**Corollary 4.18.** Consider the following statements:

(i) \sigma: S \rhd G is exact.

(ii) G_\sigma is topologically free.
(iii) $C(\partial \hat{E}) \subseteq C_{\text{ess}}^*(G_\sigma)$ has the ideal intersection property.

Then (i) \implies (ii) \implies (iii). If $G_c$ is amenable and $G_c = \bigcap_{C \in \mathcal{C}} \text{core}_G(C)$ (e.g., if $G$ is Abelian), then (iii) \implies (i).

**Proof.** The implication “(i) \implies (ii)” follows from part (ii) of Theorem 4.14 and “(ii) \implies (iii)” follows from [30, Theorem 7.29].

“(iii) \implies (i)”: Assume $G_c$ is amenable and $G_c = \bigcap_{C \in \mathcal{C}} \text{core}_G(C)$, and suppose that $G_c \neq \{e\}$. Then Remark 4.12 implies that $\{([\chi, [G_c, \chi]] : \chi \in \partial \hat{E})\}$, where $[G_c, \chi] := \{[t_g, \chi] : g \in G_c\}$, is an essentially confined amenable section of isotropy groups of $G_\sigma$, in the sense of [27, Definition 7.1]. Hence [27, Theorem 7.2] implies that $C(\partial \hat{E}) \subseteq C_{\text{ess}}^*(G_\sigma)$ does not have the ideal intersection property. \hfill $\square$

### 4.5. Amenability

We now consider amenability for our groupoids.

**Theorem 4.19.** Suppose $\sigma : S \curvearrowright A$ has a globalization $\tilde{\sigma} : \mathcal{I} \curvearrowright \mathcal{J}$ and that (JF) is satisfied. Assume (without loss of generality) that $\mathcal{I}$ is generated by $S$. If $G_\sigma$ is amenable, then $\mathcal{I}$ is amenable. The converse holds if $\mathcal{I}$ is amenable.

**Proof.** As explained in §3.5, our assumptions mean that we have the identification $G_\sigma \cong (\mathcal{J} \times \mathcal{I}) \rtimes \partial \hat{E}$. Our assumptions also give $\mathcal{I} \cong (G_\sigma)_{\chi_e}$, so amenability of $\mathcal{I}$ follows from amenability of $G_\sigma$ by [1, Proposition 5.1.1]. If $\mathcal{I}$ and $\mathcal{J}$ are amenable, then $\mathcal{I} \times \mathcal{J}$ is amenable, so that $(\mathcal{I} \times \mathcal{J}) \rtimes \partial \hat{E}$ is amenable.

**Remark 4.20.** In general, $C_r^*(G_\sigma)$ is nuclear if and only if $G_\sigma$ is amenable (see [1]). Assume we are in the setting of Theorem 4.19, so that $G_\sigma \cong (\mathcal{I} \times \mathcal{J}) \rtimes \partial \hat{E}$. If the groupoid $\mathcal{I} \times \mathcal{J}$ is exact, then $C_r^*(G_\sigma)$ is nuclear if and only if the canonical map $C^*(G_\sigma) \to C_r^*(G_\sigma)$ is an isomorphism by [11, Theorem 4.12].

### 4.6. Pure infiniteness

We now turn to pure infiniteness following [22, §4]. A subset $X \subseteq \partial \hat{E}$ is said to be properly infinite if there exist compact open bisections $U, V \subseteq G_\sigma$ such that $s(U) = s(V) = X$ and $r(U) \cup r(V) \subseteq X$; the groupoid $G_\sigma$ is said to be purely infinite if every compact open subset of $\partial \hat{E}$ is properly infinite.

Our main result on pure infiniteness is the following:

**Theorem 4.21.** If $G_\sigma$ is minimal, then $G_\sigma$ is purely infinite.

Combining this with our results above and general results for groupoid $C^*$-algebras, we obtain:

**Corollary 4.22.** Suppose $\sigma : S \curvearrowright G$ is a non-automorphic algebraic semigroup action. If $\sigma : S \curvearrowright G$ is exact and satisfies (M1), then $C_{\text{ess}}^*(G_\sigma)$ is simple and purely infinite.

**Proof.** By Theorem 4.14, $G_\sigma$ is topologically free, and by Theorem 4.10, $G_\sigma$ is minimal. The groupoid $G_\sigma$ is properly infinite by Theorem 4.21, which implies $G_\sigma$ is locally contacting. Now the result follows from [30, Theorem 7.26] (see [30, Remark 7.27]). \hfill $\square$

**Remark 4.23.** Condition (M1) implies that $\{e\} \notin C$. Moreover, if $\{e\} \in C$, then $\mathfrak{A}_\sigma \cong C_r^*(G_\sigma)$ contains the compact operators, so $C_r^*(G_\sigma)$ cannot be purely infinite in this case.

Before proceeding to the proof of Theorem 4.21, we need two lemmas.

**Lemma 4.24.** Let $B \in C$, and let $\{k_iB_i\} \subseteq E^\times$ be a finite (possibly empty) collection. Then for all $gC \in E^\times$ with $gC \leq B$ and $1 < [B : C] < \infty$, we have

$$\partial \hat{E}(B; \{k_iB_i\} \cup \{gC\}) = \bigcup_{hC \in B/C, hC \neq gC} \partial \hat{E}(B \cap hC; \{k_iB_i\}).$$
Proof. “⊆”: Suppose \( \chi \in \partial \hat{E}(B; \{k_iB_i\} \cup \{gC\}) \). Then, \( \chi(B) = 1 \) and \( \chi(gC) = \chi(k_iB_i) = 0 \) for all \( i \). We have the finite decomposition \( B = gC \sqcup \bigcup_{h \in C \cap B, hC \neq gC} hC \), so \( \chi(hC) = 1 \) for some \( h \in C \cap B \) by Lemma 3.22. Since \( \chi(gC) = 0 \), \( hC \neq gC \), so that \( \chi \in \partial \hat{E}(B \cap hC; \{k_iB_i\}) \).

“⊇”: Suppose \( \chi \in \partial \hat{E}(B \cap hC; \{k_iB_i\}) \) for some \( h \in C \cap B \) with \( hC \neq gC \). Then \( \chi(k_iB_i) = 0 \) for all \( i \) and \( 1 = \chi(B \cap hC) = \chi(B) \chi(hC) \), so that \( \chi(B) = \chi(hC) = 1 \). Since \( gC \cap hC = \emptyset \), \( \chi(gC) = 0 \), so \( \chi \in \partial \hat{E}(B; \{k_iB_i\} \cup \{gC\}) \).

**Lemma 4.25.** Assume \( G_\sigma \) is minimal. If \( B \in C \) satisfies \( [B : C] < \infty \) for all constructible subgroups \( C \leq B \), then \( [G : B] < \infty \).

**Proof.** By Theorem 4.10, there exists a constructible subgroup \( C \leq B \) and a \( \varphi \in I^c \) such that \( C \leq \text{dom}(\varphi) \) and \( \varphi(C) \leq G \) has finite index. Since the partial isomorphism \( \varphi^{-1} \) maps \( \varphi(C) \cap B \) onto \( C' := C \cap \varphi^{-1}(B \cap \text{im}(\varphi)) \leq C \), we have an isomorphism \( \varphi(C)/(\varphi(C) \cap B) \cong C/C' \). By assumption, \( [B : C'] < \infty \), which implies \( [C : C'] < \infty \), so that \( \varphi(C) : \varphi(C) \cap B < \infty \). Now \( [G : \varphi(C) \cap B] = [G : \varphi(C)]/[\varphi(C) : \varphi(C) \cap B] < \infty \), so that \( [G : B] < \infty \).

We are now ready for the proof of our theorem.

**Proof of Theorem 4.21.** By Lemma 4.1, it suffices to prove that every nonempty basic open set \( \partial \hat{E}(kB; \{k_iB_i\}) \) is properly infinite. Suppose we are given \( \partial \hat{E}(kB; \{k_iB_i\}) \neq \emptyset \), where \( kB \in E^\times \) and \( \{k_iB_i\} \) is a finite (possibly empty) collection of constructible cosets. By conjugating by the homeomorphism \( t \) if necessary, we may assume \( k = e \). We may also assume \( k_iB_i \subseteq B \) for all \( i \).

First, let us suppose \( [B : C] < \infty \) for all \( C \in C \) with \( C \leq B \). By Lemma 4.24 we may assume \( \{k_iB_i\} = \emptyset \) in this case. By Lemma 4.25 \( [G : B] < \infty \). Since \( \sigma : S \cap G \) is non-automorphic, there exists \( s \in S \) such that \( \sigma_sG \leq G \). Put \( C := B \cap \sigma_{1}^{-1}B \). Then \( C \) is a constructible subgroup of \( B \) satisfying \( \sigma_sC \leq B \), and we have that \( C, \sigma_sC \leq G \) are of finite index. Since \( \sigma_sG / \sigma_sC = \sigma_s(G/C) \), we have \( [G : \sigma_sC] = [G : \sigma_sG] / [\sigma_sG : \sigma_sC] = [G : C] [G : \sigma_sG] / [G : \sigma_sC] \) and all indices are finite. Thus,

\[
[G : B][B : \sigma_sC] = [G : \sigma_sC] = [G : C][G : \sigma_sG] = [G : B][B : C][G : \sigma_sG],
\]

where all indices are finite, so that \( [B : \sigma_sC] = [B : C][G : \sigma_sG] \). Let \( g_1, \ldots, g_m \in B \) be a complete set of representatives for \( B/C \). Since \( [G : \sigma_sG] \geq 2 \), we can find \( h_1, \ldots, h_2m \in B \) such that the cosets \( h_j\sigma_sC \) are pairwise disjoint for \( 1 \leq j \leq 2m \). Then

\[
U := \bigcup_{j=1}^{m} \{t_{h_j}\sigma_s t_{g_{j-1}}\} \times \partial \hat{E}(g_{j}C) \quad \text{and} \quad V := \bigcup_{j=1}^{m} \{t_{h_{2m+j}}\sigma_s t_{g_{j-1}}\} \times \partial \hat{E}(g_{j}C).
\]

are compact open bisections. By Lemma 3.22

\[
s(U) = s(V) = \bigcup_{j=1}^{m} \partial \hat{E}(g_{j}C) = \partial \hat{E}(B).
\]

Moreover,

\[
r(U) = \bigcup_{j=1}^{m} \partial \hat{E}(h_j\sigma_sC) \quad \text{and} \quad r(V) = \bigcup_{j=1}^{m} \partial \hat{E}(h_{m+j}\sigma_sC)
\]

are disjoint subsets of \( \partial \hat{E}(B) \) by our choice of \( h_j \)'s. Thus, \( \partial \hat{E}(B) \) is properly infinite.

Now let us suppose there exists a constructible subgroup \( C' \leq B \) such that \( [B : C'] = \infty \). By Lemma 4.15 there exists a constructible coset \( hD \subseteq B \cup \bigcup_{1} k_iB_i \), so that \( \partial \hat{E}(hD) \subseteq \partial \hat{E}(B; \{k_iB_i\}) \). By replacing \( D \) with \( D \cap C' \), we may assume that \( D \leq C' \), so that \( [B : D] = \infty \). If \( [D : D'] < \infty \) for all constructible subgroups \( D' \leq D \), then \( [G : D] < \infty \) by Lemma 4.25 since the inclusion map \( B/D \to G/D \) is injective, this would then imply that \( [B : D] < \infty \), a contradiction. Thus, there exists a constructible subgroup \( D' \leq D \) such
that \([D : D'] = \infty\). By Theorem 4.10, there exists a constructible subgroup \(C \leq D'\) and \(\varphi \in I^e\) with \(C \leq \text{dom}(\varphi)\) such that \(\varphi(C) \leq B\) has finite index. Let \(g_1, \ldots, g_m\) be a complete set of representatives for \(B/\varphi(C)\). Choose \(h_1, \ldots, h_{2m} \in D\) such that the cosets \(h_kC\) are pairwise disjoint for \(1 \leq j \leq 2m\) (here, we are using that \([D : C] = \infty\)). Consider the compact open bisections

\[
U := \bigcup_{j=1}^{m} \{t_{hh_j}^{-1}g_j^{-1}\} \times \partial\hat{E}(g_j\varphi(C)) \quad \text{and} \quad V := \bigcup_{j=1}^{m} \{t_{hh_{m+j}}^{-1}g_j^{-1}\} \times \partial\hat{E}(g_j\varphi(C)).
\]

Using Lemma 3.22, we have

\[
s(U) = s(V) = \bigcup_{j=1}^{m} \partial\hat{E}(g_j\varphi(C)) = \partial\hat{E}(B) \supseteq \partial\hat{E}(B; \{k_iB_i\}).
\]

Moreover,

\[
r(U) = \bigcup_{j=1}^{m} \partial\hat{E}(hh_jC) \quad \text{and} \quad r(V) = \bigcup_{j=1}^{m} \partial\hat{E}(hh_{m+j}C)
\]

are disjoint by our choice of \(h_j\)'s and are contained in \(\partial\hat{E}(hD)\) because \(hh_kC \leq hD\) for all \(k\). Now put \(U' := (s|U)^{-1}(\partial\hat{E}(B; \{k_iB_i\}))\) and \(V' := (s|V)^{-1}(\partial\hat{E}(B; \{k_iB_i\}))\). Then \(U'\) and \(V'\) are compact open bisections satisfying \(s(U') = s(V') = \partial\hat{E}(B; \{k_iB_i\})\) and \(r(U') \sqcup r(V') \subseteq \partial\hat{E}(hD) \subseteq \partial\hat{E}(B; \{k_iB_i\})\). Thus, \(\partial\hat{E}(B; \{k_iB_i\})\) is properly infinite. □

5. Comparison of the groupoid model and the concrete C*-algebra

We now compare the C*-algebras of our groupoid with \(\mathfrak{A}_\sigma\). Throughout this section, \(\sigma : S \curvearrowright G\) will be a non-automorphic algebraic semigroup action with \(S\) and \(G\) countable.

5.1. Comparison with the essential groupoid C*-algebra. We shall first describe \(C^e_{\text{ess}}(G_\sigma)\) using induced representations. For this, we need some preliminary results. We let \(\mathcal{S}_e := (G_\sigma)^\chi_e\), where \(\chi_e\) is the character from Definition 3.26.

**Lemma 5.1.** Suppose we have \(g, h \in G\) and \(\phi \in I^e\) with \(g \in \text{dom}(\phi)\). Then \(\phi \cdot \chi_g = \chi_{\phi(g)}\), and \(\chi_h = \chi_g\) if and only if \(hG_e = gG_e\).

**Proof.** The first claim is easy to see. We have \(\chi_h = \chi_g\) if and only if \(\chi_h(kC) = \chi_g(kC)\) for all \(kC \in \mathcal{E}_\chi\), i.e., \(h \in kC\) if and only if \(g \in kC\) for all \(kC \in \mathcal{E}_\chi\). This in turn is equivalent to having \(hC = gC\) for all \(C \in \mathcal{E}_\chi\), i.e., \(g^{-1}h \in C\) for all \(C \in \mathcal{E}_\chi\).

**Lemma 5.2.** We have \(\mathcal{S}_e = \{[t_h\varphi, \chi_e] : h \in G_e, \varphi \in I^e\}\).

**Proof.** It follows from Lemma 5.1 that \(t_h\varphi \in \mathcal{S}_e\) for all \(h \in G_e\) and \(\varphi \in I^e\). Suppose \([\phi, \chi] \in \mathcal{S}_e\). Then \(\phi \cdot \chi_e = \chi_e\), so in particular \(e \in \text{dom}(\phi)\) which by Proposition 3.11 implies that \(\text{dom}(\phi)\) is a constructible subgroup and \(\phi = t_h\varphi\) for some \(h \in G\) and \(\varphi \in I^e\). Since \(\chi_e = \phi \cdot \chi_e = \chi_{\phi(e)} = \chi_h\), we have \(h \in G_e\) by Lemma 5.1.

Let us introduce some notation following [21]. For \(\phi \in I^X\), let \(F_\phi := \{\chi \in \partial\hat{E} : \chi(\text{dom}(\phi)) = 1, \phi \cdot \chi = \chi\}\) be the set of fixed characters of \(\phi\), and we let

\[
TF_\phi := \bigcup_{gC \in \mathcal{E}_\chi, gC \subseteq \text{fix}(\phi)} \partial\hat{E}(gC)
\]

be the set of trivially fixed characters of \(\phi\). It is straightforward to see that \(TF_\phi \subseteq F_\phi\).

**Lemma 5.3.** If \(h \in G_e\) and \(\varphi \in I^e\) with \(\partial_h\varphi, \chi_e \in \mathcal{S}_e \setminus \{\chi_e\}\), then \([t_h\varphi, \partial\hat{E}(\text{dom}(\varphi))] \cap \partial\hat{E} = \emptyset\).
Let $\rho$ be the left regular representation of the group $C^*$-algebra $C^*(\mathcal{J}_e)$, and let $\text{Ind} \lambda_{\mathcal{J}_e}$ be the representation of $C^*(\mathcal{G}_\sigma)$ on $\ell^2((\mathcal{G}_\sigma)_{\chi e})$ induced from $\lambda_{\mathcal{J}_e}$. Inspired by a recent result in the setting of semigroup $C^*$-algebras, we now prove the following:

**Proposition 5.4** (cf. [35 Proposition 2.5]). We have $C^*_{\text{ess}}(\mathcal{G}_\sigma) = (\text{Ind} \lambda_{\mathcal{J}_e})(C^*(\mathcal{G}))$.

**Proof.** First, we claim that if $(\chi_i)_i$ is a net in $\partial \mathcal{E}$ that converges to $\chi_k$ for some $k \in \mathcal{G}$, then $(\chi_i)_i$ does not converge to any point of $(\mathcal{G}_\sigma)_{\chi_k} \setminus \{\chi_k\}$. In the terminology from [30 § 7], this says that none of the points in $\{\chi_k : k \in \mathcal{G}\}$ are dangerous. Since $\mathcal{E}$ is a homomorphism of $\partial \mathcal{E}$ taking $\chi_k$ to $\chi_e$, it suffices to consider the case $k = e$. Suppose $[\phi, \chi_e] \in \mathcal{J}_e \setminus \{\chi_e\}$. By Lemma 5.2, there exists $h \in G_e$ and $\phi \in I_e$ such that $\phi = \epsilon h \phi$. We need to show that $(\chi_i)_i$ does not converge to $[\phi, \chi_e]$. But this follows immediately from Lemma 5.3.

Since $S$ and $G$ are assumed countable, $\mathcal{G}_\sigma$ can be covered by countably many open bisections. Thus, by [35 Proposition 1.12], we have $J_{\text{sing}} = \bigcap_{k \in G} \ker (\pi_{\chi_k})$. Since every $\pi_{\chi_k}$ is unitarily equivalent to $\pi_{\chi_e}$, it follows that $J_{\text{sing}} = \ker (\pi_{\chi_e})$. Since $\text{Ind} \lambda_{\mathcal{J}_e} = \pi_{\chi_e} \circ \pi_{\tau}$, we are done. □

Let $\pi_{\text{ess}} : C^*(\mathcal{G}_\sigma) \to C^*_{\text{ess}}(\mathcal{G}_\sigma)$ be the canonical projection map.

**Remark 5.5.** Assume $\mathcal{G}_\sigma$ is topologically free (e.g., $\sigma : S \rhd G$ is exact). Then, by [30 Proposition 5.8 & Theorem 7.29], $C^*_{\text{ess}}(\mathcal{G}_\sigma)$ enjoys the following co-universal property: If $\pi : C^*(\mathcal{G}_\sigma) \to B$ is a $*$-homomorphism to a $C^*$-algebra $B$ such that $\pi C(\partial \mathcal{E})$ is injective, then there exists a $*$-homomorphism $\pi(C^*(\mathcal{G}_\sigma)) \to C^*_{\text{ess}}(\mathcal{G}_\sigma)$ such that $\pi(a) \mapsto \pi_{\text{ess}}(a)$ for all $a \in C^*(\mathcal{G}_\sigma)$. In particular, if $\sigma : S \rhd G$ is exact, then there exists a $*$-homomorphism $\mathfrak{A}_\sigma \to C^*_{\text{ess}}(\mathcal{G}_\sigma)$ such that $\Lambda_{\phi} \mapsto \pi(v_{\phi})$ for all $\phi \in I$.

In order to compare $C^*_{\text{ess}}(\mathcal{G}_\sigma)$ with $\mathfrak{A}_\sigma$, we shall now also describe $\mathfrak{A}_\sigma$ using induced representations. Let $\rho : C^*(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma$ be the $*$-homomorphism from Proposition 3.30. Then $\rho$ is determined by $\rho(v_{\phi}) = \Lambda_{\phi}$, where for $\phi \in I$, we let $v_{\phi} := 1_{[\phi, \partial \mathcal{E}(\text{dom}(\phi))] \in C(\mathcal{G}_\sigma)}$.

Recall that if $\pi_1$ and $\pi_2$ are representations of $C^*(\mathcal{G}_\sigma)$, then $\pi_1$ is said to be weakly contained in $\pi_2$ (written $\pi_1 \preceq \pi_2$) if $\ker \pi_2 \subseteq \ker \pi_1$. Clearly, $\pi_1 \preceq \pi_2$ if and only if $\pi_1$ factors through $\pi_2$. For instance, $\rho \preceq \pi_{\text{ess}}$ if and only if there exists a $*$-homomorphism $C^*_{\text{ess}}(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma$ such that $v_{\phi} \mapsto \Lambda_{\phi}$.

Consider the subgroup $\mathcal{J}_e := \{[\phi, \chi_e] : \phi \in I_e\} \leq \mathcal{J}_e$. By Proposition 2.8, $\text{Ind} \lambda_{\mathcal{J}_e, \mathcal{J}_e}$ is (unitarily equivalent to) the representation on $\ell^2((\mathcal{G}_\sigma)_{\chi e})$ given by

$$\left(\text{Ind} \lambda_{\mathcal{J}_e, \mathcal{J}_e}(f)(\xi)[t_k, \chi_e]\mathcal{J}_e\right) = \sum_{[\psi, \chi] \in \mathcal{G}_\sigma^k} f([\psi, \chi])\xi([\psi^{-1}t_k, \chi_e]\mathcal{J}_e)$$

for all $f \in C(\mathcal{G}_\sigma)$ and all $\xi \in \ell^2((\mathcal{G}_\sigma)_{\chi e})$.

The essential observation is the following:

**Proposition 5.6.** The representation $\rho$ is unitarily equivalent to $\text{Ind} \lambda_{\mathcal{J}_e, \mathcal{J}_e}$. 

Proof. First, we will show that the map $G \to (G_\sigma)_{\chi_e}/\hat{\mathcal{J}}_e$ given by $g \mapsto [t_g, \chi_e]_{\hat{\mathcal{J}}_e}$ is bijective. Let $[\phi, \chi_e] \in (G_\sigma)_{\chi_e}$. Since $\chi_e(\text{dom}(\phi)) = 1$ (i.e., $e \in \text{dom}(\phi)$), it follows from Proposition 3.11 that we can write $\phi = t_h \check{\varphi}$ for some $h \in G$ and $\varphi \in I^e$. We have $[\phi, \chi_e]_{\hat{\mathcal{J}}_e} = [t_h, \chi_e]_{\hat{\mathcal{J}}_e}$, so our map is surjective. If $[t_g, \chi_e]_{\hat{\mathcal{J}}_e} = [t_h, \chi_e]_{\hat{\mathcal{J}}_e}$ for $g, h \in G$, then there exists $\varphi \in I^e$ such that $[t_g, \chi_e] = [t_h \varphi, \chi_e]$, so that there exists $C \in C$ with $t_g|C = (t_h \varphi)|C$. Evaluating at $e \in C$ gives $g = h$. Thus, our map is injective.

Second, we will show that the unitary $\ell^2(G) \cong \ell^2((G_\sigma)_{\chi_e}/\hat{\mathcal{J}}_e)$ associated with the above bijection intertwines $\rho$ and $\text{Ind} \lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e$. For $\phi \in I$, we have

$$
\left(\text{Ind} \lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e)(v_\phi)\delta_{[t_g, \chi_e]_{\hat{\mathcal{J}}_e}}\right)([t_k, \chi_e]_{\hat{\mathcal{J}}_e}) = \sum_{[\psi, \chi] \in G_\sigma^{\chi_e}} v_\phi([\psi, \chi])\delta_{[t_g, \chi_e]_{\hat{\mathcal{J}}_e}}([\psi^{-1}t_k, \chi_e]_{\hat{\mathcal{J}}_e}).
$$

In order for the sum to be non-zero, there must exist $[\psi, \chi] \in G_\sigma$ such that $\psi \cdot \chi = \chi_k$, $[\psi, \chi] \in [\phi, \partial \check{E}(\text{dom}(\phi))]$, and $[t_g, \chi_e]_{\hat{\mathcal{J}}_e} = [\psi^{-1}t_k, \chi_e]_{\hat{\mathcal{J}}_e}$. The condition $[\psi, \chi] \in [\phi, \partial \check{E}(\text{dom}(\phi))]$ means that $\chi(\text{dom}(\phi)) = 1$ and there exists $I^e \in E$ with $\psi|I^e = \phi|I^e$ and $\chi(I^e) = 1$. Since $\psi \cdot \chi = \chi_k$ is equivalent to $\chi = \psi^{-1} \cdot \chi_k = \chi_{g^{-1}(k)}$, $\chi(\text{dom}(\phi)) = 1$ implies $\psi^{-1}(k) \in \text{dom}(\phi)$. The condition $[t_g, \chi_e]_{\hat{\mathcal{J}}_e} = [\psi^{-1}t_k, \chi_e]_{\hat{\mathcal{J}}_e}$ means that there exists $\varphi \in I^e$ and $D \in C$ such that $t_g|D = \psi^{-1}t_k|D$. Evaluating this equation at $e$ gives $g = \psi^{-1}(k)$. Moreover, $\psi^{-1} \cdot \chi_k(I^e) = \chi(I^e) = 1$ implies that $g = \psi^{-1}(k)$ lies in $I^e \cap \text{dom}(\psi)$, so that $\psi|I^e = \phi|I^e$ implies $\psi(g) = \phi(g)$. Therefore, the sum is zero unless $g \in \text{dom}(\phi)$ and $k = \phi(g)$, in which case there is a single non-zero summand corresponding to $[\phi, \chi_g]$. Thus, $(\text{Ind} \lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e)(v_\phi)\delta_{[t_g, \chi_e]_{\hat{\mathcal{J}}_e}} = \delta_{[t_{\phi(g)}, \chi_e]_{\hat{\mathcal{J}}_e}}$ when $g \in \text{dom}(\phi)$ and $(\text{Ind} \lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e)(v_\phi)\delta_{[t_g, \chi_e]_{\hat{\mathcal{J}}_e}} = 0$ otherwise.

Corollary 5.7. If $\hat{\mathcal{J}}_e$ is amenable, then $\rho \preceq \text{Ind} \lambda_{\mathcal{J}_e}$. If moreover, $\sigma : S \curvearrowright G$ is exact, then

\begin{equation}
C_{\text{ess}}^* (G_\sigma) = (\text{Ind} \lambda_{\mathcal{J}_e})(C^* (G_\sigma)) \to A\sigma, \quad (\text{Ind} \lambda_{\mathcal{J}_e})(v_\phi) \mapsto \Lambda_\phi,
\end{equation}

is a $*$-isomorphism.

Proof. If $\hat{\mathcal{J}}_e$ is amenable, then $\lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e \preceq \lambda_{\mathcal{J}_e}$, which, because weak containment is preserved under induction (cf. [28] Lemma 2.1), implies $\text{Ind} \lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e \preceq \text{Ind} \lambda_{\mathcal{J}_e}$. Since $\rho \sim_\theta \text{Ind} \lambda_{\mathcal{J}_e}/\hat{\mathcal{J}}_e$ by Proposition 5.6, it follows that $\rho \preceq \text{Ind} \lambda_{\mathcal{J}_e}$.

If $\sigma : S \curvearrowright G$ is exact, then faithfulness of the map in (5) follows from Corollary 4.18 because our map is the identity on $C(\partial \check{E})$ by Proposition 3.30.

Combining Corollary 5.7 with Corollary 4.22, we obtain:

Corollary 5.8. Let $\sigma : S \curvearrowright G$ be a non-automorphic algebraic semigroup action with $S$ and $G$ countable. If $\sigma : S \curvearrowright G$ is exact, satisfies (M1), and $\hat{\mathcal{J}}_e$ is amenable, then $A\sigma$ is simple and purely infinite.

Remark 5.9. Surprisingly, amenability of $\hat{\mathcal{J}}_e$ is often also necessary for $\rho \preceq \text{Ind} \lambda_{\mathcal{J}_e}$, see §5.2 below.

Let us now make a couple of observations about the group $\hat{\mathcal{J}}_e$. By Lemma 5.2, we have:

Corollary 5.10. We have $\mathcal{J}_e = \hat{\mathcal{J}}_e$ if and only if $\sigma : S \curvearrowright G$ is exact.

Remark 5.11. If we have a globalization that satisfies (IF), then $\hat{\mathcal{J}}_e = \langle S \rangle$. Here $\langle S \rangle$ is the subgroup of $\mathcal{J}$ generated by $S$, where $\mathcal{J}$ is as in Definition 2.3.

If the universal group of $\{\sigma_s : s \in S\}$ is amenable, then $\hat{\mathcal{J}}_e$ is amenable. In particular, if the universal group of $S$ is amenable, then $\hat{\mathcal{J}}_e$ is amenable.

The group $\hat{\mathcal{J}}_e$ is canonically isomorphic to the maximal group homomorphic image of $I^e$.
5.2. Comparison with the reduced groupoid C*-algebra. When $I$ is strongly 0-E-unitary—which implies, in particular, that $C^*_r(\mathcal{G}_\sigma) = C^*_\text{ess}(\mathcal{G}_\sigma)$—we shall use recent results from [12] to characterize $\rho \preceq \text{Ind} \lambda_{\mathcal{J}_{\mathcal{E}}}$.

Let $C^*_r(\mathcal{J}_{\mathcal{E}})$ denote the completion of the complex group algebra $C^*_r(\mathcal{J}_{\mathcal{E}})$ with respect to the norm $\| \cdot \|$ defined in [12 Definition 2.1], and denote by $\lambda^e_{\mathcal{J}_{\mathcal{E}}}$ the canonical projection $C^*(\mathcal{J}_{\mathcal{E}}) \to C^*_r(\mathcal{J}_{\mathcal{E}})$.

The following is a consequence of Proposition 5.6.

**Proposition 5.12.** We have $\rho \preceq \pi_r$ if and only if $\lambda_{\mathcal{J}_{\mathcal{E}}/\mathcal{J}_{\mathcal{E}}} \preceq \lambda^e_{\mathcal{J}_{\mathcal{E}}}$. 

**Proof.** By Proposition 5.6 $\rho$ is unitarily equivalent to $\text{Ind} \lambda_{\mathcal{J}_{\mathcal{E}}/\mathcal{J}_{\mathcal{E}}}$, so Proposition 2.2 implies that $\rho \preceq \pi_r$ if and only if $\lambda_{\mathcal{J}_{\mathcal{E}}/\mathcal{J}_{\mathcal{E}}} \preceq \lambda^e_{\mathcal{J}_{\mathcal{E}}}$.

**Remark 5.13.** Suppose $\rho \preceq \pi_r$, and let $\hat{\rho} : C^*_r(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma$ be the $*$-homomorphism such that $\hat{\rho}(v_\phi) = \Lambda_\phi$ for all $\phi \in I$. If $\mathcal{G}_\sigma$ is Hausdorff and $\sigma : S \curvearrowright G$ is exact, then Corollary 4.18 implies that $\hat{\rho}$ is an isomorphism.

**Remark 5.14.** If $\| \cdot \|_{\mathcal{E}}$ coincides with the reduced norm $\| \cdot \|_r$ on $C^*_r(\mathcal{J}_{\mathcal{E}})$, then $\lambda^e_{\mathcal{J}_{\mathcal{E}}} = \lambda_{\mathcal{J}_{\mathcal{E}}}$. We always have $\lambda_{\mathcal{J}_{\mathcal{E}}} \preceq \lambda^e_{\mathcal{J}_{\mathcal{E}}}$, and $\lambda_{\mathcal{J}_{\mathcal{E}}/\mathcal{J}_{\mathcal{E}}} \preceq \lambda_{\mathcal{J}_{\mathcal{E}}}$ if and only if $\mathcal{J}_{\mathcal{E}}$ is amenable, so amenability of $\mathcal{J}_{\mathcal{E}}$ implies $\lambda_{\mathcal{J}_{\mathcal{E}}/\mathcal{J}_{\mathcal{E}}} \preceq \lambda_{\mathcal{J}_{\mathcal{E}}}$.

If $I$ is strongly 0-E-unitary, then $\mathcal{G}_\sigma$ is a partial transformation groupoid (see, e.g., [14 Lemma 5.5.22]), so $\| \cdot \|_{\mathcal{E}}$ and $\| \cdot \|_r$ coincide by [12 Corollary 4.15]. In general, it is not clear when these norms coincide.

We will conclude this section by explaining how $\mathfrak{A}_\sigma$ is, in many cases, an exotic groupoid C*-algebra.

**Lemma 5.15.** If $\rho : C^*(\mathcal{G}_\sigma) \to \mathfrak{A}_\sigma$ is an isomorphism, then $G$ is amenable.

**Proof.** The map $G \times \partial E \to \mathcal{G}_\sigma$ given by $(g, \chi) \mapsto [t_g, \chi]$ identifies the transformation groupoid $G \times \partial E$ with the closed subgroupoid $\{ [t_g, \chi] \in \mathcal{G}_\sigma : g \in G, \chi \in \partial E \}$ of $\mathcal{G}_\sigma$. We thus have a canonical embedding $C^*(\partial E) \times G \cong C^*(G \times \partial E) \hookrightarrow C^*(\mathcal{G}_\sigma)$ which, in particular, sends the canonical unitary $u_g$ in $C^*(\partial E) \times G$ corresponding to $g \in G$ to $v_g$. The composition of these canonical embeddings with $\rho$ coincides with the left regular representation $\lambda_G$ of $C^*(G)$. Thus, if $\rho$ is an isomorphism, then $\lambda_G$ must also be an isomorphism, in which case $G$ is amenable.

**Corollary 5.16.** Suppose $\sigma : S \curvearrowright G$ is exact and $I$ is strongly 0-E-unitary. If both $G$ and $\mathcal{J}_{\mathcal{E}}$ are non-amenable, then $\mathfrak{A}_\sigma$ is an exotic groupoid C*-algebra, in the sense that it sits properly between the full and reduced C*-algebra of $\mathcal{G}_\sigma$.

**Proof.** By Remark 5.5 we have a canonical projection map $\mathfrak{A}_\sigma \to C^*_r(\mathcal{G}_\sigma)$; it is not injective if $\mathcal{J}_{\mathcal{E}}$ is non-amenable by Proposition 5.12 and Remark 5.14 if $G$ is non-amenable, then $\rho$ is not injective by Lemma 5.15. Our claim follows.

6. Comparisons with the boundary quotient of the semigroup C*-algebra

It is natural to compare our C*-algebras $C^*_r(\mathcal{G}_\sigma)$, $C^*_r(\mathcal{J}_{\mathcal{E}})$, and $\mathfrak{A}_\sigma$ to the boundary quotient of the semigroup C*-algebra $C^*_r(P)$, where $P = G \rtimes S$. For background on semigroup C*-algebras and their boundary quotients, see [39 § 2] and [14 Chapter 5].

6.1. Comparison of groupoids. Let $I_t$ be the left inverse hull of $P$, $E$ the semilattice of idempotents of $I_t$, and $I_t \rtimes \partial E$ the associated boundary groupoid, so that $C^*_r(I_t \rtimes \partial E)$ is the boundary quotient of the semigroup C*-algebra $C^*_r(P)$. Let $\mathcal{J}_S$ denote the semilattice of constructible right ideals of $S$. A straightforward computation shows

$$(g_1, s_1)^{-1}(g_2, s_2) \cdots (g_{n-1}, s_{n-1})^{-1}(g_n, s_n) = (s_1^{-1} t_{g_1} t_{g_2} t_{s_2} \cdots s_{n-1}^{-1} t_{g_{n-1}} t_{g_n} t_{s_n}) \times (s_1^{-1} s_2 \cdots s_{n-1}^{-1} s_n)$$
for all $g_1, \ldots, g_n \in G$ and $s_1, \ldots, s_n \in S$. Here we identify $G \times S$ with its canonical copy inside $I_t$. If $S$ is left reversible (i.e., $sS \cap tS \neq \emptyset$ for all $s, t \in S$), then $\emptyset \notin \mathcal{JS}$ by [14 Lemma 5.6.43], and the projection onto the $G$-component defines an inverse semigroup homomorphism $I_t \rightarrow I$, $\Phi \mapsto \Phi_G$. It is straightforward to see that this map is surjective. By restricting the map $I_t \rightarrow I$, we obtain a surjective semilattice homomorphism $E \rightarrow \mathcal{E}$ and hence a continuous embedding $\hat{\mathcal{E}} \hookrightarrow \hat{E}$, $\chi \mapsto \hat{\chi}$.

For the remainder of this subsection, we assume that $S$ is left reversible.

**Lemma 6.1.** The map $\hat{\mathcal{E}} \hookrightarrow \hat{E}$, $\chi \mapsto \hat{\chi}$ restricts to a bijection $\mathcal{E}_{\max} \cong \hat{\mathcal{E}}_{\max}$.

**Proof.** Recall that $\chi \in \hat{\mathcal{E}}$ is maximal if and only if whenever $\chi(gC) = 0$ for some $gC \in \mathcal{E}$, there exists $hD \in \mathcal{E}$ with $\chi(hD) = 1$ and $gC \cap hD = \emptyset$. Moreover, every element of $E$ is of the form $gC \times X$ for some $gC \in \mathcal{E}$ and $X \in \mathcal{JS}$. Since $S$ is left reversible, we have $(gC \times X) \cap (hD \times Y) = \emptyset$ if and only if $gC \cap hD = \emptyset$.

Now suppose that $\chi \in \mathcal{E}_{\max}$. Then $\hat{\chi}(gC \times X) = 1$ if and only if $\chi(gC) = 1$. Assume that $\hat{\chi}(gC \times X) = 0$. Then $\chi(gC) = 0$. Hence there exists $hD \in \mathcal{E}$ with $\chi(hD) = 1$ and $gC \cap hD = \emptyset$. It follows that, for every $Y$ such that $hD \times Y \in E$, we have $\hat{\chi}(hD \times Y) = 1$. At the same time, $(gC \times X) \cap (hD \times Y) = \emptyset$. This shows that $\hat{\chi}$ is maximal.

Now take $\omega \in \mathcal{E}_{\max}$. Define $\chi \in \hat{\mathcal{E}}$ by $\chi(gC) = 1$ if $\omega(gC \times X) = 1$ for some $X \in \mathcal{JS}$. Then $\chi \in \mathcal{E}_{\max}$. We claim that $\hat{\chi} = \omega$. Indeed, $\hat{\chi}(gC) = 1$ if and only if $\omega(gC \times X) = 1$ for some $X$ and if only if $\omega(gC \times Z) = 1$ for all $Z$ such that $gC \times Z \in E$. The last equivalence follows from maximality of $\omega$. □

**Corollary 6.2.** The map $\hat{\mathcal{E}} \hookrightarrow \hat{E}$, $\chi \mapsto \hat{\chi}$ restricts to a homeomorphism $\partial \hat{\mathcal{E}} \cong \partial \hat{E}$.

Now we see that there is a canonical surjection $I_t \times \partial \hat{E} \twoheadrightarrow \mathcal{G}_\sigma$ given by $[\Phi, \hat{\chi}] \mapsto [\Phi_G, \chi]$.

**Lemma 6.3.** If the left inverse hull $I_t(S)$ of $S$ is $E$-unitary, then the surjection $I_t \rightarrow I$, $\Phi \mapsto \Phi_G$, is an isomorphism if and only if the surjection $I_t \times \partial \hat{E} \twoheadrightarrow \mathcal{G}_\sigma$, $[\Phi, \hat{\chi}] \mapsto [\Phi_G, \chi]$, is an isomorphism.

**Proof.** “$\Rightarrow$” is clear. For “$\Leftarrow$”, take $\Phi$ and assume that $\Phi_G \in \mathcal{E}$, say $\Phi_G = \text{id}_{gC}$ and $\chi(gC) = 1$, so that $[\Phi_G, \chi] \in \mathcal{G}_\sigma^{(1)}$. Thus if $I_t \times \partial \hat{E} \twoheadrightarrow \mathcal{G}_\sigma$ is an isomorphism, we deduce $[\Phi, \hat{\chi}] = \chi$ and thus $\Phi_{gC \times X} \circ \text{id}_{gC \times X}$ for some $X \in \mathcal{JS}$. Therefore, $\Phi = \Phi_G \times \Phi_S$ with $\Phi_G \in \mathcal{E}$ and $\Phi_S|_{X} = \text{id}_{X}$. As $I_t(S)$ is $E$-unitary, the latter implies that $\Phi_S \in E(S)$, and hence $\Phi \in E$, as desired. □

6.2. **Comparison of C*-algebras.** Let us now compare the C*-algebras. For $\Phi \in I_t$, let $w_\Phi$ denote the corresponding partial isometry in $\mathcal{C}(I_t \times \partial \hat{E})$.

**Proposition 6.4.** The following are equivalent:

(i) $S$ is left reversible;

(ii) there exists a *-homomorphism $\vartheta: C^*(I_t \times \partial \hat{E}) \rightarrow \mathfrak{A}_\sigma$ such that $\vartheta(w_\Phi) = \Lambda_{\Phi_G}$ for all $\Phi \in I_t$;

(iii) there exists a *-homomorphism $\pi: C^*(I_t \times \partial \hat{E}) \rightarrow C^*_r(\mathcal{G}_\sigma)$ such that $\pi(w_\Phi) = v_{\Phi_G}$ for all $\Phi \in I_t$.

**Proof.** For all $s, t \in S$, we have $\sigma_s G \cap \sigma_t G = \emptyset$, so that $\Lambda_{\sigma_s} \Lambda_{\sigma_t}^{*} \Lambda_{\sigma_t}^{*} \neq 0$ and $v_{\sigma_s} v_{\sigma_t}^{*} v_{\sigma_t} v_{\sigma_t}^{*} \neq 0$. Thus, left reversibility of $S$ is necessary for existence in both cases.

Now assume $S$ is left reversible. To prove (ii), it suffices to show that the representation $I_t \rightarrow \mathfrak{A}_\sigma$, $\Phi \mapsto \Lambda_{\Phi_G}$ is a tight representation of $I_t$ in the sense of [18]; since (the restriction to $E$) is unital, it suffices by [20 Corollary 4.3] to prove that this representation is cover-to-join in the sense of [20 § 3]. Let $gC \times X \in E^x$ and suppose $c \subseteq E$ is a finite cover of $gC \times X$. Then for every $hD \times Y \subseteq gC \times X$, there
exists \( kB \times Z \in \mathfrak{c} \) such that \((kB \times Z) \cap (hD \times Y) \neq \emptyset \). Let \( \mathfrak{c}_G := \{kB : kB \times Z \in \mathfrak{c} \text{ for some } Z \in \mathcal{J}_S \} \). It is easy to see that \( \mathfrak{c}_G \) is a (finite) cover of \( gG \). We have

\[
\bigvee_{hD,Y \in \mathfrak{c}} \Lambda_{\Phi_\mathfrak{c},hD \times Y} = \bigvee_{hDE \in \mathfrak{c}_G} \Lambda_{\text{id}_hD} = \Lambda_{\text{id}_{gG}},
\]

where the last equality uses that \( \phi \mapsto \Lambda_\phi \) is a cover-to-join representation of \( I \) in \( \mathfrak{A}_\sigma \) (see the proof of Proposition 5.30).

The proof of (iii) is essentially the same, using that \( I_1 \to C^*_r(G_\sigma) \), \( \Phi \mapsto \Psi_\Phi \) is a tight representation.

**Remark 6.5.** A special case of the equivalence of (i) \( \iff \) (ii) in Proposition 6.4 was observed in [5, Proposition 4.3] using different methods.

For the remainder of this section, we assume that \( S \) is left reversible. Let \( \vartheta : C^*(I_1 \ltimes \partial \tilde{E}) \to \mathfrak{A}_\sigma \) be the \(*\)-homomorphism from part (ii) of Proposition 6.4.

We now compare \( \mathfrak{A}_e \) and \( C^*_r(I_1 \ltimes \partial \tilde{E}) \). Let \( \pi_1 : C^*(I_1 \ltimes \partial \tilde{E}) \to C^*_r(I_1 \ltimes \partial \tilde{E}) \) be the canonical projection map, and put \( \hat{T}_e := \{[\Phi, \tilde{\chi}_e] : \Phi_G \in I^* \} \leq \mathcal{T}_e := (I_1 \ltimes \partial \tilde{E})_{\tilde{\chi}_e} \).

**Proposition 6.6.** The representation \( \vartheta \) is unitarily equivalent to \( \text{Ind} \lambda_{\mathcal{T}_e/\hat{T}_e} \).

**Proof.** The projection \( I_1 \ltimes \partial \tilde{E} \to G_\sigma \) descends to a bijection \((I_1 \ltimes \partial \tilde{E})_{\tilde{\chi}_e}/\hat{T}_e \cong (G_\sigma)_{\chi_e}/\mathcal{T}_e \). Composing this with the bijection \((G_\sigma)_{\chi_e}/\mathcal{T}_e \cong G \) from the proof of Proposition 5.6, we get that the map \( G \to (I_1 \ltimes \partial \tilde{E})_{\tilde{\chi}_e}/\hat{T}_e \) given by \( g \mapsto [g, \tilde{\chi}_e]_e \) is a bijection. Similarly to the proof of Proposition 5.6, it now follows from Proposition 2.8 that the unitary \( \ell^2(G) \cong \ell^2((I_1 \ltimes \partial \tilde{E})_{\tilde{\chi}_e}/\hat{T}_e) \) induced by the above bijection intertwines \( \vartheta \) and \( \text{Ind} \lambda_{\mathcal{T}_e/\hat{T}_e} \).

Let \( C^*_e(\mathcal{T}_e) \) denote the completion of the complex group algebra \( \mathbb{C}\mathcal{T}_e \) with respect to the norm \( || \cdot ||_e \) defined in [12, Definition 2.1], and denote by \( \lambda_{\mathcal{T}_e} \) the canonical surjection \( C^*(\mathcal{T}_e) \to C^*_e(\mathcal{T}_e) \). By [12, Proposition 2.2], we have:

**Corollary 6.7.** We have \( \vartheta \leq \pi_1 \) if and only if \( \lambda_{\mathcal{T}_e/\hat{T}_e} \leq \lambda_{\mathcal{T}_e}^e \).

**Remark 6.8.** If \( \hat{T}_e \) is amenable, so that \( \lambda_{\mathcal{T}_e/\hat{T}_e} \leq \lambda_{\mathcal{T}_e} \), then \( \lambda_{\mathcal{T}_e/\hat{T}_e} \leq \lambda_{\mathcal{T}_e} \leq \lambda_{\mathcal{T}_e}^e \).

If \( P \) embeds into a group, so that \( I_1 \ltimes \partial \tilde{E} \) is a partial transformation groupoid by [14, § 5.7], then it follows from [12, Corollary 4.15] that \( || \cdot ||_e = || \cdot ||_r \).

We have \( \hat{T}_e = \mathcal{T}_e \) if \( \sigma : S \curvearrowright G \) is exact.

We lastly compare \( C^*_\text{ess}(G_\sigma) \) and \( C^*_\text{ess}(I_1 \ltimes \partial \tilde{E}) \). For this, we also assume that \( S \) and \( G \) are countable, so that we can invoke Proposition 5.4.

The surjection \( I_1 \ltimes \partial \tilde{E} \to G_\sigma \) induces a projection \( \mathcal{T}_e \to \mathcal{T}_e \). Let \( \mathcal{N} \leq \mathcal{T}_e \) be the kernel of this map. Explicitly, \( \mathcal{N} = \{[\Phi, \tilde{\chi}_e] \in \mathcal{T}_e : \text{there exists } C \in \mathfrak{C} \text{ with } \Phi_G|_C = \text{id}_C \} \).

**Proposition 6.9.** We have \( C^*_\text{ess}(I_1 \ltimes \partial \tilde{E}) = (\text{Ind} \lambda_{\mathcal{T}_e})(C^*(I_1 \ltimes \partial \tilde{E})) \).

**Proof.** Similarly to the proof of Proposition 5.4 (using that \( S \) and \( G \) are countable), it suffices to show that the point \( \tilde{\chi}_e \) is not dangerous. Since \( \chi_e \) is not dangerous in \( G_\sigma \), it is enough to show that for any given \([\Phi, \tilde{\chi}_e] \in \mathcal{N} \setminus \{\tilde{\chi}_e\} \), there is no net in \( \partial \tilde{E} \) that converges to \( \tilde{\chi}_e \) and \([\Phi, \tilde{\chi}_e] \). So suppose \([\Phi, \tilde{\chi}_e] \in \mathcal{N} \setminus \{\tilde{\chi}_e\} \). Then, \([\Phi, \partial \tilde{E}(\text{dom}(\Phi))] \cap \partial \tilde{E} = [\Phi, TF_\Phi] \), where \( TF_\Phi := \bigcup_{\varepsilon \in E,e \subseteq \text{fix}(\Phi)} \partial \tilde{E}(\varepsilon) \). Thus, it suffices to show
that \( \text{fix}(\Phi) \) does not contain any member of \( E^\times \). Suppose \( hD \times X \in E^\times \) with \( hD \times X \subseteq \text{fix}(\Phi) \). Then \( hD \subseteq \text{fix}(\Phi_G) \) and \( X \subseteq \text{fix}(\Phi_S) \). Since \([\Phi, \tilde{\chi}_e] \) lies in \( \mathcal{N} \), there exists \( C \in \mathcal{C} \) such that \( \Phi_G|_C = \text{id}_C \). In particular, this implies that \( \Phi_G \in \mathcal{I}^C \), so that \( \text{dom}(\Phi_G) \) is a subgroup and \( \Phi_G \) is a homomorphism on its domain (see Remark 3.12). It follows that \( \Phi_G(hd) = \Phi_G(h)\Phi_G(d) \) for all \( d \in D \). Since \( \Phi_G(h) = h \), this implies \( D \subseteq \text{fix}(\Phi_G) \). Now we have \( D \times Y \subseteq \text{fix}(\Phi) \), and \( D \times Y = (h,1)^{-1}(hD,Y) \in E^\times \). Since \( \tilde{\chi}_e(D \times Y) = 1 \), this implies \([\Phi, \tilde{\chi}_e] = \tilde{\chi}_e \), which is a contradiction.

Let \( \tilde{\pi} : C^*(I_t \ltimes \partial \hat{E}) \to C^*_\text{ess}(\mathcal{G}_\sigma) \) be the composition of \( \pi \) from Proposition 6.4 with the quotient map \( C^*_\text{ess}(\mathcal{G}_\sigma) \to C^*_\text{ess}(\mathcal{G}_\sigma) \).

**Proposition 6.10.** The representation \( \tilde{\pi} \) is unitarily equivalent to \( \text{Ind} \lambda_{\mathcal{F}_e/N} \).

**Proof.** The projection \( I_t \ltimes \partial \hat{E} \to \mathcal{G}_\sigma \) induces a bijection \((I_t \ltimes \partial \hat{E})^\times \subseteq (\mathcal{G}_\sigma)^\times \). By Proposition 5.6, the representation \( \text{Ind} \lambda_{\mathcal{F}_e/N} \) is unitarily equivalent to the canonical representation of \( C^*(I_t \ltimes \partial \hat{E}) \) on \( L^2((I_t \ltimes \partial \hat{E})^\times \subseteq (\mathcal{G}_\sigma)^\times) \), and by Proposition 5.4, \( C^\ast(\mathcal{G}_\sigma) = (\text{Ind} \lambda_{\mathcal{F}_e})(C^*(\mathcal{G}_\sigma)) \subseteq B(L^2((\mathcal{G}_\sigma)^\times)) \).

It remains to check that the unitary \( L^2((I_t \ltimes \partial \hat{E})^\times \subseteq (\mathcal{G}_\sigma)^\times) \) associated with the above bijection implements a unitary equivalence between \( \tilde{\pi} \) and \( \text{Ind} \lambda_{\mathcal{F}_e/N} \). This is similar to the proof of Proposition 5.6.

**Corollary 6.11.**

(i) If \( \mathcal{N} \) is amenable, then there is a \(*\)-homomorphism \( C^\ast(\mathcal{G}_\sigma) \) such that \( w_\phi \mapsto v_{\Phi_G} \).

(ii) If \( C^\ast(\mathcal{G}_\sigma) \) is Hausdorff, then \( \pi \) is non-injective if and only if \( \lambda_{\mathcal{F}_e/N} \) is non-injective.

**Proof.** (i): If \( \mathcal{N} \) is amenable, then \( \lambda_{\mathcal{F}_e/N} \) is amenable, so that \( \lambda_{\mathcal{F}_e/N} \) is non-injective (see [28, Lemma 2.1]). By Proposition 6.10, \( \tilde{\pi} \) is unitarily equivalent to \( \lambda_{\mathcal{F}_e/N} \), and by Proposition 6.9, \( C^\ast(\mathcal{G}_\sigma) = (\text{Ind} \lambda_{\mathcal{F}_e})(C^*(I_t \ltimes \partial \hat{E})) \), so the result follows.

(ii): This is a direct application of [12, Proposition 2.2].

**Remark 6.12.** Hausdorffness of \( I_t \ltimes \partial \hat{E} \) is characterized in [39, § 4].

**Lemma 6.13.** Assume \( \mathcal{F} \) is left reversible. Suppose \( \Phi \in \langle S \rangle \subseteq \mathcal{F} \) is such that \( \Phi|_{C \times X} \neq \text{id}_{C \times X} \) for all \( \emptyset \neq C \times X \subseteq E \) with \( C \in \mathcal{C} \), but \( \Phi_G = \text{id} \). Then \( \tilde{\pi}'(w_\phi - w_{\Phi^{-1}\phi}) = 0 \) but \( w_\phi - w_{\Phi^{-1}\phi} \neq 0 \).

**Proof.** To see our claim, first note that \( (\text{Ind} \lambda_{\mathcal{F}_e})(w_\phi)(\delta_{[\text{id} \tilde{\chi}_e]}) = \delta_{[\text{id} \tilde{\chi}_e]} \) while \( (\text{Ind} \lambda_{\mathcal{F}_e})(w_{\Phi^{-1}\phi})(\delta_{[\text{id} \tilde{\chi}_e]}) = \delta_{[\text{id} \tilde{\chi}_e]} \). Now \([\Phi, \tilde{\chi}_e] = \tilde{\chi}_e \) if and only if \( \Phi|_{C \times X} = \text{id}_{C \times X} \) for some \( \emptyset \neq C \times X \subseteq E \) with \( C \in \mathcal{C} \). At the same time, \( \Phi_G = \text{id} \) implies \( \Phi_G = \Phi^{-1}\Phi_G \). Hence

\[
\tilde{\pi}'(w_\phi) = (\text{Ind} \lambda_{\mathcal{F}_e})(v_{\Phi_G}) = (\text{Ind} \lambda_{\mathcal{F}_e})(v_{\Phi^{-1}\phi}) = (\text{Ind} \lambda_{\mathcal{F}_e})(v_{\Phi^{-1}\phi})(\text{Ind} \lambda_{\mathcal{F}_e})(v_{\Phi_G}) = \tilde{\pi}'(w_{\Phi^{-1}\phi}).
\]

**Remark 6.14.** Assume \( \mathcal{F} \) is left reversible. If \( I_t \) is \( 0 \)-E-unitary, then our condition above for non-injectivity of \( \tilde{\pi}' \) is satisfied whenever there exists \( \Phi \notin E \) with \( \Phi_G \in \mathcal{E} \) (i.e., \( \Phi_G \in \mathcal{E} \)), i.e., \( I_t \rightarrow I \) is not injective. In other words, if \( I_t \) is \( 0 \)-E-unitary, and if \( \tilde{\pi}' \) is an isomorphism, then the map \( I_t \rightarrow I \) from above must be an isomorphism.

Also note that, again if \( I_t \) is \( 0 \)-E-unitary, then our condition above for non-injectivity of \( \tilde{\pi}' \) is satisfied whenever \( I \) is not \( 0 \)-E-unitary.

**Proposition 6.15.** Assume \( \mathcal{N} \) is amenable. Then the \(*\)-homomorphism \( \tilde{\pi}' : C^\ast(\mathcal{G}_\sigma) \to C^\ast(\mathcal{G}_\sigma) \) from Corollary 6.17 is an isomorphism if and only if \( \mathcal{N} \) is trivial.
Proof. If \( \mathcal{N} \) is trivial, then \( \text{Ind} \lambda_{\mathcal{N}}/\mathcal{N} \equiv \text{Ind} \lambda_{\mathcal{E}} \), and the result follows from Proposition \[6.10\] . If \( \mathcal{N} \) is non-trivial, then \( \hat{\pi}' \) is not injective by Lemma \[6.13\] .

7. Examples

7.1. Algebraic semigroup actions with the finite index property.

Definition 7.1. We say that \( \sigma : S \curvearrowright G \) has the finite index property if

\[(\text{FI}) \quad \#G/\sigma_sG < \infty \quad \text{for all } s \in S.\]

Proposition 7.2. If \( \sigma : S \curvearrowright G \) satisfies \((\text{FI})\), then every member of \( C \) is a finite index subgroup of \( G \).

Proof. We proceed by induction. The induction start is provided by \((\text{FI})\). For the induction step, suppose \( C \in \mathcal{C} \) with \( \#G/C < \infty \). Now let \( s,t \in S \). Since \( \sigma_sG/\sigma_sC = \sigma_s(G/C) \), we see that \( [\sigma_sG : \sigma_sC] \) is finite by the induction hypothesis. Hence, since \( [G : \sigma_sG] \) is also finite by the induction hypothesis, \( [G : \sigma_sC] \) is finite (see, e.g., [24, Chapter I, Theorem 4.5]). Moreover, we have \( \sigma_t(G/\sigma_t^{-1}C) = \sigma_tG/((\sigma_tG) \cap C) \), and the latter is finite because we have an embedding \( \sigma_tG/((\sigma_tG) \cap C) \hookrightarrow G/C \).

We immediately obtain the following:

Corollary 7.3. If \( \sigma : S \curvearrowright G \) satisfies \((\text{FI})\), then \( \tilde{G} := \varprojlim_{C \in \mathcal{C}} G/C \) is compact. Moreover, every character in \( \partial \tilde{E} \) is maximal, and \( G \) coincides with \( \partial \tilde{E} \) (cf. Lemma \[3.24\]).

Corollary 7.4. If \( \sigma : S \curvearrowright G \) satisfies \((\text{FI})\), then \( G_\sigma \) is minimal.

Proof. It is clear that \((\text{FI})\) implies \((\text{M4})\) from Theorem \[4.10\] .

Proposition 7.5. Assume \( G \) that \( g^m = h^m \) implies \( h = g \) for all \( g,h \in G \) and all \( m \in \mathbb{Z}_{>0} \) and that \( \sigma : S \curvearrowright A \) satisfies \((\text{FI})\). Then \( I \) is 0-E-unitary. If \( \sigma : S \curvearrowright G \) admits a globalization \( \tilde{\sigma} : \mathcal{I} \curvearrowright \mathcal{I} \), then it satisfies \((\text{JF})\).

Proof. By Corollary \[3.13\] , it suffices to prove that \( I^e \) is E-unitary. Suppose \( \varphi \in I^e \) is such that \( \varphi |_C = \text{id}_C \) for some \( C \in \mathcal{C} \). By Proposition \[7.2\] , there exists \( m \in \mathbb{Z}_{>0} \) such that \( G^m \subseteq C \). Now we have \( \varphi(g^m) = \varphi(g) = g \) for all \( g \in G \), which implies \( \varphi(g) = g \) for all \( g \in G \) by our assumption. The proof that \((\text{JF})\) is satisfied is similar.

Note that if \( G \) is Abelian and torsion free, then \( g^m = h^m \) implies \( h = g \) for all \( g,h \in G \) and all \( m \in \mathbb{Z}_{>0} \). If \( I \) is 0-E-unitary, then \( G_\sigma \) is Hausdorff by [21, Corollary 3.17].

Example 7.6 (Algebraic semigroup actions on tori and solenoids). Let \( G \) be a torsion-free Abelian group of finite rank \( r \in \mathbb{Z}_{>0} \), so that we can view \( G \) as a subgroup of \( \mathcal{I} := \mathbb{Q}^r \). Note that these assumptions on \( G \) are equivalent to the dual group \( \hat{G} \) being a solenoid. Given an algebraic semigroup action \( \sigma : S \curvearrowright G \), every \( \sigma_s \) extends naturally to an automorphism \( \tilde{\sigma}_s \) of \( \mathcal{I} \), so that we obtain a natural globalization by considering the action of the subgroup \( \mathcal{I} \) of \( \text{Aut}(\mathcal{I}) \) generated by \( \tilde{\sigma}_s, s \in S \). Moreover, \((\text{JF})\) is satisfied by Proposition \[7.5\] so that \( G_\sigma \cong (\mathcal{I} \times \mathcal{I}) \ltimes \partial \tilde{E} \). In particular, our groupoid is Hausdorff. By [22, Exercise 92.5], \((\text{FI})\) is satisfied, so \( G_\sigma \) is minimal by Corollary \[7.4\] and hence purely infinite by Theorem \[4.21\].

Remark 7.7. Example \[7.6\] provides many algebraic semigroup actions which have globalizations even though acting semigroup is not left Ore. For instance, it is not difficult to find faithful exact actions of free monoids on tori. Such actions are necessarily very far from respecting the order in the sense of [8, Definition 8.1].

Remark 7.8. For a non-automorphic algebraic semigroup action \( \sigma : S \curvearrowright \mathbb{Z}^r \), property ID for the dual action in the sense of [3, 43] implies exactness for \( \sigma \).
Proposition 7.12. For every $\sigma$, the principal constructible subgroups are principal ideals of the ring. Consider the following condition on $\sigma$:

Example 7.13. Remark 7.13.

Example 7.10. Example 7.10 (Algebraic semigroup actions from self-similar actions of groups). We briefly explain how to obtain examples of algebraic semigroup actions from self-similar group actions. We refer the reader to [14] and [33] for background on self-similar actions. Let $d \in \mathbb{Z}_{>1}$ and $X := \{0, \ldots, d-1\}$. Let $X^*$ denote the free monoid on $X$, and for each $n \in \mathbb{N}$, let $X^n \subseteq X^*$ be the set of words of length $n$. Suppose $G \curvearrowright X^*$ is a faithful self-similar action of a non-trivial group $G$ on $X^*$ as in [33, §3] (cf. [14]). For each $\mu \in X^*$, let $G_\mu \leq G$ be the stabilizer subgroup of $\mu$, and let $\phi_\mu : G_\mu \to G$ be the homomorphism $\phi_\mu(g) := g|_\mu$, where $g|_\mu$ is the section of $g$ at $\mu$. Assume that $\phi_\mu$ is an isomorphism for all $x \in X$. By [33, Lemma 3.10], this is equivalent to assuming $\phi_\mu$ is an isomorphism for all $\mu \in X^*$. For each $\mu \in X^*$, put $\sigma_\mu := \phi_\mu^{-1} : G \to G_\mu$. For $\mu, \nu \in X^*$, we have $\sigma_\mu \circ \sigma_\nu = \sigma_{\mu \nu}$, so that $S = \{\sigma_\mu : \mu \in X^*\}$ is a submonoid of $\text{End}(G)$. Since $G \curvearrowright X^*$ is faithful, $\bigcap_{\mu \in X^*} G_\mu = \{g \in G : g(\mu) = \mu \text{ for all } \mu \in X^*\} = \{1\}$, so $S \curvearrowright G$ is exact. Since $[G : G_\mu] \leq d^{\|\mu\|}$ for all $\mu \in X^*$, $S \curvearrowright G$ satisfies (FI). Thus, our groupoid $G_\sigma$ is topologically free, minimal, and purely infinite.

Example 7.11 (Ring C*-algebras of non-commutative rings). Let $\mathcal{R}$ be a unital (not necessary commutative) ring with $1 \neq 0$, and let $\mathcal{R}^\times$ be the set of left regular elements of $\mathcal{R}$, i.e., $\mathcal{R}^\times := \{a \in \mathcal{R} : ax = ay \text{ implies } x = y \text{ for all } x, y \in \mathcal{R}\}$. Then $\mathcal{R}^\times$ acts on the additive group of $\mathcal{R}$ by injective endomorphisms. Since $\mathcal{R}$ is unital, the algebraic semigroup action $\mathcal{R}^\times \curvearrowright \mathcal{R}$ is faithful. The concrete C*-algebra associated with $\mathcal{R}^\times \curvearrowright \mathcal{R}$ is called the reduced ring C*-algebra of $\mathcal{R}$, see [33], and is denoted by $\mathcal{A}_r[\mathcal{R}]$. Assume that that additive group of $\mathcal{R}$ torsion-free and of finite rank. Examples of such rings include integral group rings of finite groups and $\mathcal{R}^\times$ or $\mathcal{M}_n(\mathcal{R})$, where $\mathcal{R}$ an order in a central simple algebra over an algebraic number field. It is straightforward to check that $\mathcal{R}^\times \curvearrowright \mathcal{R}$ is exact and satisfies (FI) and that $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{R})^\times \curvearrowright \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{R}$ is a globalization for $\mathcal{R}^\times \curvearrowright \mathcal{R}$ that satisfies (IF). By Theorem 4.19 and Corollary 5.16, we see that the following are equivalent:

(i) $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{R})^\times$ is amenable;
(ii) $\mathcal{A}_r[\mathcal{R}]$ is nuclear;
(iii) $\mathcal{A}_r[\mathcal{R}]$ is simple.

If the above equivalent conditions are satisfied, then $\mathcal{A}_r[\mathcal{R}]$ is a UCT Kirchberg algebra by Corollary 5.8.

7.2. Algebraic semigroup actions by left reversible monoids. Recall that the monoid $S$ is said to be left reversible if $sS \cap tS \neq \emptyset$ for all $s, t \in S$. We shall now demonstrate that our conditions from §4 are especially easy to check for actions by left reversible monoids.

7.2.1. General results for actions satisfying (PC). We shall call $S$-constructible subgroups of the form $\sigma_1 G$ principal constructible subgroups. This terminology comes from the ring-theoretic examples where the principal constructible subgroups are principal ideals of the ring. Consider the following condition on $\sigma : S \curvearrowright G$:

(PC) For every $C \in \mathcal{C}$, there exists $s \in S$ such that $\sigma(s) \leq C$.

This condition means that the family of principal constructible subgroups is co-final in $\mathcal{C}$.

Proposition 7.12. If $S$ is left reversible, then $\sigma : S \curvearrowright G$ satisfies (PC).

Proof. Let $C = \sigma_1^{-1} \cdots \sigma_n^{-1} s_n G \subseteq C$. As $S$ is left reversible, there exists $s \in S$ such that $sS \subseteq s_1^{-1} t_1 \cdots s_n^{-1} t_n S$ (see [14, Lemma 5.6.43.]). Hence $\sigma(s) \leq C$.

Remark 7.13. If $\sigma : S \curvearrowright G$ satisfies (PC), then $\bigcap_{C \in \mathcal{C}} C = \bigcap_{s \in S} \sigma(s) G$, so that $\sigma : S \curvearrowright G$ is exact if and only if $\bigcap_{s \in S} \sigma(s) G = \{e\}$.
If $G$ is Abelian, then $\bigcap_{s \in S} \sigma_s G = \{e\}$ if and only if $\bigcup_{s \in S} \text{fix}(\sigma_s)$ is dense in $\hat{G}$.

**Lemma 7.14.** If $\sigma : S \rtimes G$ satisfies (PC) (e.g., if $S$ is left reversible), then $\sigma : S \rtimes G$ satisfies (M2).

**Proof.** Let $C \in C$. Since $\sigma : S \rtimes G$ satisfies (PC), there exists $s \in S$ such that $\sigma_s G \leq C$. Now $G = \sigma_s^{-1} C$, so (M2) holds.

**Corollary 7.15.** Assume $G$ is Abelian, and $S$ is cancellative and right reversible (i.e., left Ore). If $\sigma : S \rtimes G$ is faithful and satisfies (PC), then $\langle S \rangle \rtimes S^{-1}G$ satisfies (JF).

**Proof.** For convenience, let us write $G$ additively. Since $S$ is left Ore, by (iii) in Example 3.17, it suffices to show that $C \leq \ker(\sigma_s - \sigma_t) \implies s = t$ for all $C \in C$ and $s, t \in S$. Suppose we have $C \in C$ and $s, t \in S$ with $C \leq \ker(\sigma_s - \sigma_t)$. By assumption, $S \rtimes G$ satisfies (PC), so there exists $r \in S$ such that $\sigma_r G \leq C$, so we have $\sigma_s g = \sigma_t g$ for all $g \in G$. By faithfulness, it follows that $sr = tr$, and hence $s = t$ by right cancellation.

**Lemma 7.16.** Assume that $\sigma : S \rtimes G$ has a globalization $\tilde{\sigma} : \mathcal{I} \rtimes \mathcal{G}$ and that (PC) holds. Then (JF) is satisfied if and only if $\tilde{\sigma}_{|G} = \text{id}$ implies $g = 1$ for all $g \in \mathcal{G} \rtimes \mathcal{I}$.

**Proof.** Assume $\tilde{\sigma}_{|G} = \text{id}$ implies $g = 1$ for all $g \in \mathcal{G} \rtimes \mathcal{I}$. Suppose $\tilde{\sigma}_{|C} = \text{id}_C$ for some $C \in C$. Then by (PC), we can find $s \in S$ with $\sigma_s G \leq C$. Now we have that $\tilde{\sigma}_s \circ \tilde{\sigma}_s$ and $\tilde{\sigma}_s$ agree on $G$. Therefore our assumption implies $gs = s$ and thus $g = 1$.

By Corollary 4.22 and Corollary 5.16, we have:

**Theorem 7.17.** Assume $\sigma : S \rtimes G$ is a non-automorphic algebraic semigroup action with $S$ reversible. Then the groupoid $G_\sigma \cong (\langle S \rangle \rtimes S^{-1}G) \rtimes \partial \mathcal{E}$ is Hausdorff, minimal, and purely infinite. Moreover, if additionally $\sigma : S \rtimes G$ is exact, then $G_\sigma$ topologically free, and

1. $C^*_r(G_\sigma)$ is simple and purely infinite;
2. The map $\mathfrak{A}_\sigma \to C^*_r(G_\sigma)$, $\Lambda_\phi \mapsto \nu_\phi$ from Remark 5.5 is an isomorphism if and only if $\langle S \rangle$ is amenable. In particular, $\mathfrak{A}_\sigma$ is simple if and only if $\langle S \rangle$ is amenable.

7.2.2. **Algebraic semigroup actions from commutative algebra à la Schmidt.** Inspired by the rich class of algebraic group actions arising from considerations in commutative algebra, see, e.g., [51], we now turn to examples of algebraic semigroup actions arising from modules over commutative rings.

Let $R$ be an infinite, commutative, unital ring with $0 \neq 1$, and let $M$ be a non-zero $R$-module. For $a \in R$, let $\alpha_a^M \in \text{End}_Z(M)$ be the associated endomorphism of $M$. We often omit the superscript and simply write $\alpha_a$ when the module in question is clear from context. Let $\mathcal{R}(M) := \{a \in R : \alpha_a \text{ is injective}\}$ be the commutative monoid of $M$-regular elements. For the $R$-module $R^\times := \mathcal{R}(R)$ is the multiplicative monoid of non-zero-divisors in $R$. Let $S$ be a non-trivial submonoid of $R^\times \cap \mathcal{R}(M)$; since $R^\times$ is cancellative, so is $S$. Consider the action $\alpha : S \rtimes M$ given by $a \mapsto \alpha_a$. For $M = R$ and $S = R^\times$, the concrete $C^*$-algebra $\mathfrak{A}_\alpha$ associated with $\alpha : R^\times \rtimes R$ is the reduced ring $C^*$-algebra of $R$ in the sense of [31, Definition 7], which is denoted by $\mathfrak{A}_\alpha[R]$.

**Remark 7.18.** In this setting, the canonical globalization has a particularly nice form. Let $S^{-1}R$ and $S^{-1}M$ denote the localizations of $R$ and $M$, respectively, at $S$. The canonical map $R \to S^{-1}R$ is injective since $S \subseteq R^\times$, and the canonical map $M \to S^{-1}M$ is injective since $S \subseteq \mathcal{R}(M)$. Let $\langle S \rangle$ denote the subgroup of $(S^{-1}R)^\times$ generated by $S$. The canonical algebraic group action $\langle S \rangle \rtimes S^{-1}M$ is a globalization of $S \rtimes M$. By Corollary 7.15, $\langle S \rangle \rtimes S^{-1}M$ satisfies (JF) if and only if $\langle S \rangle \rtimes M$ is faithful.

For $M = R$ and $S = R^\times$, the ring $Q(R) := (R^\times)^{-1}R$ is the total quotient ring of $R$. 29
If \( \alpha : S \curvearrowright M \) is faithful, then since \( S \) is Abelian, we are in the setting of Theorem \ref{thm:faithful-UCT-Kirchberg-algebra} and \( \mathcal{A}_\alpha \) will be a UCT Kirchberg algebra whenever \( \alpha : S \curvearrowright M \) is non-automorphic and exact (nuclearity comes from Theorem \ref{thm:nuclearity-of-UCT}). Thus, we set out to establish criteria for these conditions to be satisfied. We start with the case \( M = R \).

**Remark 7.19.** If \( M = R \), then \( S \curvearrowright R \) is always faithful, and \( S \curvearrowright R \) is automorphic if and only if \( S \subseteq R^\times \).

**Proposition 7.20.** Let \( R \) be a Noetherian integral domain and \( S \subseteq R^\times \) a submonoid. Then \( S \curvearrowright R \) is exact if and only if \( S \) contains a non-unit.

**Proof.** If \( S \subseteq R^\times \), then the action \( S \curvearrowright R \) is by automorphisms and is thus not exact. Suppose there exists \( a \in S \setminus R^\times \), so that \( (a) \subseteq S \). Since \( R \) is Noetherian by assumption, \cite[Corollary 10.18]{kn:Reams80} says that \( \bigcap_{n=1}^{\infty} (a)^n = \{0\} \). Since each \( (a)^n = (a^n) = a^n R \) lies in \( \mathcal{C}_{S \curvearrowright R} \) for every \( n \), exactness follows. \( \square \)

Applying Theorem \ref{thm:faithful-UCT-Kirchberg-algebra} and Theorem \ref{thm:nuclearity-of-UCT} with the above observations gives:

**Corollary 7.21.**
(i) If \( \bigcap_{a \in R^\times} aR = \{0\} \), then the reduced ring \( C^\ast \)-algebra \( \mathcal{A}_\alpha[R] \) associated with \( R^\times \curvearrowright R \) is a UCT Kirchberg algebra.

(ii) If \( R \) is a Noetherian integral domain, and \( S \subseteq R^\times \) is a submonoid containing a non-unit, then the \( C^\ast \)-algebra \( \mathcal{A}_\alpha \) associated with \( \alpha : S \curvearrowright R \) is UCT Kirchberg algebra.

**Remark 7.22.** Corollary \ref{cor:faithful-action} (i) applies to a larger class of rings than those treated by the results in \cite[§ 5.3]{kn:Reams12}; indeed, in order to apply the results of \cite{kn:Reams12} to the reduced ring \( C^\ast \)-algebra \( \mathcal{A}_\alpha[R] \), one needs \( \bigcap_{a \in R^\times} aR = \{0\} \) and also condition (***) from \cite[§ 5.3]{kn:Reams12}.

If \( R \) is an integral domain that is not a field, then \( \bigcap_{a \in R^\times} aR = \{0\} \) (see the proof of \cite[Corollary 9]{kn:Reams12}), so Corollary \ref{cor:faithful-action} (i) applies to all integral domains that are not fields. This special case is also covered by \cite[Corollaries 8 & 9]{kn:Reams12} or \cite[Corollary 8.4]{kn:Reams12}.

Let \( R \) be the ring of integers in an algebraic number field. The boundary quotients \( \partial C^\ast \alpha(R \times R_{m,\Gamma}) \) associated with the action of a congruence monoid \( R_{m,\Gamma} \) on \( R \) are UCT Kirchberg algebras by \cite[§ 8]{kn:Reams12} and \cite[Theorem 3.1]{kn:Reams12}. Corollary \ref{cor:faithful-action} (ii) generalizes and explains this.

Let us now turn to more general modules. Let \( \text{Ann}(M) \subseteq R \) denote the annihilator ideal of \( M \). It is easy to see that \( S \curvearrowright M \) is faithful if and only if \( (S - S) \cap \text{Ann}(M) = \{0\} \), where \( S - S := \{a - b : a, b \in S\} \).

**Remark 7.23 (Prime actions).** Let \( p \) be a prime ideal of \( R \), and consider \( M = R/p \) as an \( R \)-module. Then \( \mathcal{R}_R(R/p) = R \setminus p \), so \( S \curvearrowright R/p \) acts by injective endomorphisms if and only if \( S \subseteq R \setminus p \). Such actions are called prime actions. The action \( S \curvearrowright R/p \) is non-automorphic if and only if \( S \) contains an elements \( a \) such that \( a + p \in (R/p)^\times \) is a non-unit. In particular, this means that \( p \) cannot be a maximal ideal. If \( R/p \) is Noetherian (e.g., if \( R \) is Noetherian), then by Proposition \ref{prop:faithful-action}, \( S \curvearrowright R/p \) is exact if and only if there exists \( a \in S \) with \( a + p \neq R \) (i.e., \( a \) is a non-unit modulo \( p \)).

A prime \( p \) of \( R \) is said to be associated with \( M \) if \( p = \text{ann}_R(x) \) for some \( x \in M \). Let \( \text{Asc}(M) \) denote the (possibly empty) set of primes associated with \( M \). If \( p \) is a prime of \( R \), then \( p \in \text{Asc}(R) \) if and only if there is an embedding of \( R \)-modules \( R/p \hookrightarrow M \). Note that \( \text{Ann}(M) \subseteq p \) for every \( p \in \text{Asc}(M) \).

**Lemma 7.24.** Assume \( \text{Asc}(M) \) is nonempty and that \( S \subseteq R \setminus \bigcup_{p \in \text{Asc}(M)} p \). Then \( S \curvearrowright M \) is faithful if and only if there exists a prime \( p \in \text{Asc}(M) \) such that the canonical prime action \( S \curvearrowright R/p \) is faithful.

**Proof.** If \( S \curvearrowright M \) is not faithful, then there exist \( a, b \in S \) such that \( a \neq b \) and \( a.x = b.x \) for every \( x \in M \), i.e., \( a - b \in \text{Ann}(M) \). Since \( \text{Ann}(M) \subseteq p \) for every \( p \in \text{Asc}(M) \), we see that \( S \rightarrow R/p \) is not injective for every \( p \in \text{Asc}(M) \). Hence, \( S \curvearrowright R/p \) is not faithful for every \( p \in \text{Asc}(M) \).
For every \( p \in \text{Asc}(M) \), we have an embedding of \( R \)-modules \( R/p \hookrightarrow M \). Hence, if \( S \vartriangleleft R/p \) is faithful for some \( p \in \text{Asc}(M) \), then \( S \vartriangleleft M \) must be faithful. \( \square \)

If \( R \) is Noetherian, then \( \bigcup_{p \in \text{Asc}(M)} p = R \setminus \mathcal{R}(M) \) (see, e.g., [41] Theorem 6.1, p.38). If \( R \) is Noetherian and \( M \) is finitely generated, then \( \text{Asc}(M) \) is finite and non-empty (see, e.g., [17] Theorem 3.1).

**Proposition 7.25.** Let \( R \) be a Noetherian ring, \( M \) a finitely generated \( R \)-module, and \( S \subseteq R \setminus \bigcup_{p \in \text{Asc}(M)} p \) a submonoid. The following are equivalent:

(i) there exists \( a \in S \) such that \( a^n \vartriangleleft M \) is exact;

(ii) \( S \vartriangleleft M \) is exact;

(iii) \( S \vartriangleleft R/p \) is exact for every \( p \in \text{Asc}(M) \);

Proof. (i)\(\Rightarrow\)(ii) is obvious. (ii)\(\Rightarrow\)(iii): Assume \( S \vartriangleleft M \) is exact. Let \( p \in \text{Asc}(M) \), so that there is an \( R \)-module embedding \( i: R/p \hookrightarrow M \). By Proposition 7.12, \( S \vartriangleleft M \) and every \( S \vartriangleleft R/p \) satisfy \([PC]\).

Since \( \bigcap_{s \in S} s \text{Asc}(R/p) \leq \bigcap_{s \in S} s \text{Asc}(M) \), exactness of \( S \vartriangleleft R/p \) follows from exactness of \( S \vartriangleleft M \) (cf. Remark 7.13).

(iii)\(\Rightarrow\)(i): Assume \( S \vartriangleleft R/p \) is exact for every \( p \in \text{Asc}(M) \). By Remark 7.23, this is equivalent to the following statement: For each \( p \in \text{Asc}(M) \), there exists \( a_p \in S \) such that \( (a_p) + p \subseteq R \). Let \( a := \prod_{p \in \text{Asc}(M)} a_p \in S \). Since \( R \) is Noetherian and \( M \) finitely generated, by [2] Theorem 10.17, we have

\[
\bigcap_{n=0}^{\infty} (a)^n M = \{x \in M : \text{there exists } r \in R \text{ such that } (1 - ar)x = 0\}.
\]

Suppose \( x \in \bigcap_{n=0}^{\infty} a^n M \) is non-zero. Since \( a^n M \subseteq (a)^n M \), we see from (6) that there exists \( r \in R \) such that \((1 - ar)x = 0\), i.e., \( 1 - ar \) is not \( M \)-regular and thus lies in \( \bigcup_{p \in \text{Asc}(M)} p = R \setminus \mathcal{R}(M) \). Hence, \( 1 - ar \in p \) for some \( p \in \text{Asc}(M) \), i.e., \( a \) is invertible modulo \( p \), so that \( (a) + p \subseteq R \). But we know that \( (a) + p \subseteq (a_p) + p \subseteq R \), so this is a contradiction. Hence, \( x = 0 \). \( \square \)

**Example 7.26** (A result of Krzy˙zewski). Fix \( n \geq 1 \). Let \( A \in M_n(\mathbb{Z})^X \) be a matrix with non-zero determinant, and let \( \sigma_A \) be the associated endomorphism of \( \mathbb{Z}^n \). Let \( \chi_A(u) \) be the characteristic polynomial of \( A \). The main result in [29] says that \( \sigma_A: \mathbb{N} \vartriangleleft \mathbb{Z}^n \) is exact if and only if \( \chi_A(u) \) not divisible by any polynomial with constant term \( \pm 1 \) (i.e., \( \chi_A(u) \) has no unimodular factors). We demonstrate that this characterization follows from Proposition 7.25.

View \( \mathbb{Z}^n \) as a \( \mathbb{Z}[u] \)-module via \( f(u).n = f(A)n \), and similarly view \( \mathbb{Q}^n \) as a \( \mathbb{Q}[u] \)-module. Let \( m_A(u) \) be the minimal polynomial of \( A \), i.e., \( m_A(u) \) is the unique monic generator of the ideal \( \text{Ann}_{\mathbb{Q}[u]}(\mathbb{Q}^n) \subseteq \mathbb{Q}[u] \). Since \( A \) has integer entries, \( m_A(u) \in \mathbb{Z}[u] \). Let \( m_A(u) = \prod_{i=1}^{r} f_i(u)^{k_i} \) be the factorization of \( m_A(u) \) into powers of irreducible elements, which are defined up to multiplication by \( \pm 1 \) (here, we are using that \( \mathbb{Z}[u] \) is a UFD with unit group \( \{\pm 1\} \)).

The canonical \( \mathbb{N} \)-action \( \mathbb{N} \vartriangleleft \mathbb{Z}[u]/(f_i(u)) \) is exact if and only if \( u + (f_i(u)) \) is a non-unit, which is equivalent to \( f_i(0) \notin \{\pm 1\} \). Thus, in order to deduce Krzy˙zewski’s characterization of exactness from Proposition 7.25, it suffices to prove the following lemma.

**Lemma 7.27.** We have \( \text{Asc}_{\mathbb{Z}[u]}(\mathbb{Z}^n) = \{ (f_i(u)) : 1 \leq i \leq r \} \).

Proof. If \( p \in \text{Asc}_{\mathbb{Z}[u]}(\mathbb{Z}^n) \), then \( \mathbb{Z}[u]/p \) embeds as a \( \mathbb{Z}[u] \)-submodule of \( \mathbb{Z}^n \); in particular, \( \mathbb{Z}[u]/p \) is torsion-free as an additive group, so that \( p \cap \mathbb{Z} = (0) \). Hence, taking localizations with respect to \( \mathbb{Z}^X \) and applying [17] Theorem 3.1(c) gives us

\[
\text{Asc}_{\mathbb{Q}[u]}(\mathbb{Q}^n) = \{ p_Q : p \in \text{Asc}_{\mathbb{Z}[u]}(\mathbb{Z}^n) \},
\]

where \( p_Q \) denotes the prime of \( \mathbb{Q}[u] \) generated by \( p \). Let \( p \in \text{Asc}_{\mathbb{Q}[u]}(\mathbb{Q}^n) \). Since \( \mathbb{Q}[u] \) is a PID, we can write \( p = (p(u))_Q \) for some irreducible polynomial \( p(u) \in \mathbb{Z}[u] \). Here, we write \( (p(u))_Q \) for the ideal of
Q[u] generated by p(u). Since (p(u))Q ⊇ AnnQ[u](Q^n) = (m_A(u))Q, we have p(u) | m_A(u) in Q[u], so that (p(u))Q = (f_i(u))Q for some 1 ≤ i ≤ r. Thus, we have AscQ[u](Q^n) ⊆ {(f_i(u))Q : 1 ≤ i ≤ r}.

Fix 1 ≤ i ≤ r, and let x ∈ (m_A(u)/f_i(u))Q^n \ {0} (such an x exists by definition of m_A). Then f_i(u)x = 0, so (f_i(u))Q ⊆ AnnQ[u](x). Since 1.x = 0, AnnQ[u](x) is a proper ideal of Q[u]; since (f_i(u))Q is a prime ideal and Q[u] has Krull dimension 1, we must have (f_i(u))Q = AnnQ[u](x). Hence, (f_i(u))Q ∈ AscQ[u](Q^n).

Therefore, using (7), we have {pQ : p ∈ AscZ[u](Z^n)} = {(f_i(u))Q : 1 ≤ i ≤ r}. It remains to observe that if f(u) ∈ Z[u] is a monic polynomial, then (f(u))Q ∩ Z[u] = fZ[u] by Gauss’s lemma. □

**Example 7.28 (Algebraic N^d-actions).** Fix d ∈ Z_{>0}, and let R^+_d := Z[u_1, ..., u_d] be the polynomial ring with integer coefficients in the d commuting variables u_1, ..., u_d. For n = (n_1, ..., n_d) ∈ N^d, we let u^n := u_1^{n_1} ··· u_d^{n_d}. Given f ∈ R^+_d, we can write f = ∑_{n ∈ N^d} f_n u^n, where f_n ∈ Z is zero for all but finitely many n. Given any algebraic N^d-action N^d ∩ M, where M is an Abelian group, M naturally becomes a module over R^+_d via f.x := ∑_n f_n u^n.x.

**Proposition 7.29.** Let M be a finitely generated module over Z[u_1, ..., u_d], and assume that u_i ∉ p for every 1 ≤ i ≤ d and every p ∈ Asc(M), so that we get an algebraic N^d-action N^d ∩ M. Then

(i) N^d ∩ M is faithful if and only if there exists p ∈ Asc(M) such that

\[ \{u^n - u^m : n, m ∈ N^d\} \cap p = \{0\}; \]

(ii) N^d ∩ M is exact if for all p ∈ Asc(M) we can find f_1, ..., f_m ∈ Z[u_1, ..., u_d] and z ∈ Z^d with z_j = 0 for some 1 ≤ j ≤ d such that p = (f_1, ..., f_m) and gcd(f_1(z), ..., f_m(z)) ≠ 1.

**Proof.** The ring R^+_d is Noetherian, see, e.g., [2, Theorem 7.5].

(i): By Lemma 7.24, N^d ∩ M is faithful if and only if there exists a prime p ∈ Asc(M) such that N^d ∩ R^+_d/p is faithful, which happens if and only if \{u^n - u^m : n, m ∈ Z^d\} \cap p = \{0\}.

(ii): By Proposition 7.25, N^d ∩ M is exact if and only if there exists a prime p ∈ Asc(M) such that N^d ∩ R^+_d/p contains a non-unit for every p ∈ Asc(M). Let p ∈ Asc(M). Suppose u^n + p is a unit of R^+_d/p for every n ∈ N^d; this is equivalent to \{(u^n) + p = R^+_d\} for every n ∈ N^d. We have (u^n) + p = R^+_d if and only if there exists q^n, g_1^n, ..., g_n^n ∈ R^+_d such that u^n q^n + ∑_{i=1}^n f_i g_i^n = 1. Taking n = e_j in this equation and then evaluating both sides at the point z gives ∑_{i=1}^n f_i(z) g_i^n(z) = 1, so that gcd(f_1(z), ..., f_m(z)) = 1, which contradicts our assumption. □

### 7.3. Shifts over semigroups

Throughout this section, let S be a left cancellative monoid with S ≠ 1, and let Σ any nontrivial group.

**Definition 7.30.** The full S-shift over Σ is the algebraic S-action

\[ \sigma : S ∩ Σ, \quad \sigma_s(x)_t := \begin{cases} x_{s^{-1}t} & \text{if } t ∈ sS, \\ 0 & \text{if } t \notin sS. \end{cases} \]

For t ∈ S and x ∈ Σ, we let xe_t ∈ Σ be the element defined by (xe_t)_t = x and (xe_t)_s = e for s ≠ t. Then \(\sigma_s(xe_t)_t = xe_{st}\), and we can write every element of a = (a_t)_t ∈ Σ as a = ∏_{t∈S} a_{te}. Thus, we see that the algebraic S-action S ∩ Σ is faithful because the left translation action S ∩ S is faithful.

For a ∈ Σ, let \(\text{supp}(a) := \{s ∈ S : a_s ≠ e\}\). Given X ⊆ S, we identify Σ with the subgroup \(\{a ∈ Σ : \text{supp}(a) ⊆ X\}\).
Proposition 7.31. We have $\mathcal{C}_S \cap \mathcal{S} \Sigma = \{ \bigoplus X \Sigma : X \in \mathcal{J}_S \}$. Moreover, the map $\mathcal{J}_S \to \mathcal{C}_S \cap \mathcal{S} \Sigma$ given by $X \mapsto \bigoplus X \Sigma$ is an isomorphism of semilattices.

Proof. Given $a = \prod_{t \in S} a_t \epsilon_t \in \bigoplus \Sigma$, $X \subseteq S$, and $s \in S$, we have $\text{supp}(a) \subseteq X$ if and only if $\text{supp}(\sigma_s(a)) \subseteq sX$, and we have $\text{supp}(a) \subseteq s^{-1}X$ if and only if $\text{supp}(\sigma_s(a)) \subseteq X$. Hence, $\sigma_s(\bigoplus X \Sigma) = \bigoplus sX \Sigma$ and $\sigma_s^{-1}(\bigoplus X \Sigma) = \bigoplus s^{-1}X \Sigma$, which is enough for the first claim.

For the second claim, note that $\bigoplus X \Sigma \cap \bigoplus Y \Sigma = \bigoplus X \cap Y \Sigma$ for all $X, Y \in \mathcal{J}_S$.

We immediately obtain the following:

Corollary 7.32. \( (i) \) $S \cap \bigoplus \Sigma$ is exact; \( (ii) \) $S \cap \bigoplus \Sigma$ satisfies [M2] if and only if $S$ is left reversible; \( (iii) \) $S \cap \bigoplus \Sigma$ satisfies [FL] if and only if $\#\Sigma < \infty$ and $\#(S \setminus sS) < \infty$ for all $s \in S$.

Proof. (i): Suppose we have $t \in \bigcap_{s \in S} sS$. Then there exists $s \in S$ such that $t = t^2 s$, which by left cancellation implies $1 = ts$. Since $S$ is left cancellative, it follows that $s \in S^*$. But $\bigcap_{s \in S} sS$ is a proper right ideal of $S$ because $S$ contains a non-invertible element, so have a contradiction. It follows that $\bigcap_{X \in \mathcal{J}_S} X \subseteq \bigcap_{s \in S} sS = \emptyset$. Thus, $S \cap \bigoplus \Sigma$ is exact by Proposition 7.31.

(ii): If $S$ is left reversible, then [M2] holds by Lemma 7.1. Conversely, if $S$ is not left reversible, $\emptyset \in \mathcal{J}_S$ by [L] Lemma 5.6.43., so that $\{ e \}$ is constructible by Proposition 7.31. This implies that [M2] does not hold.

(iii): For $s \in S$, $\bigoplus \Sigma / \bigoplus sS \Sigma \cong \bigoplus S \Sigma$, which is finite if and only if $\# \Sigma < \infty$ and $\#(S \setminus sS) < \infty$.

Lemma 7.33. Assume $S$ is left reversible.

\( (i) \) If $S$ is right cancellative, then $S \cap \bigoplus \Sigma$ satisfies [H].

\( (ii) \) If $S$ embeds in a group $\mathcal{J}$, then the globalization $\tilde{\sigma} : \mathcal{J} \cap \bigoplus \Sigma$ satisfies [H].

Proof. (i): Suppose we are given a constructible subgroup $\bigoplus X \Sigma$ and $s \in S$ with $\bigoplus X \Sigma \subseteq \text{fix}(\sigma_s)$. Then $x \epsilon_s = \tilde{\sigma}_s(x \epsilon_t) = x \epsilon_t$ for all $x \in \Sigma$ and all $t \in X$, so that $st = t$ for all $t \in X$. Since $S$ is left reversible, $X \neq \emptyset$. Thus, since $S$ is right cancellative, we have $s = 1$. It follows that [H] is satisfied. The proof of (ii) is similar.

Example 7.34. Let $S$ be a countable, left reversible submonoid of a group $\mathcal{J}$ such that $S \neq S^*$, and let $\Sigma$ be a countable non-trivial group. Assume $(S) = \mathcal{J}$ and that $\mathcal{J}$ and $\Sigma$ are both non-amenable. Then, the concrete C*-algebra $\mathfrak{A}_\sigma$ associated with $S \cap \bigoplus \Sigma$ is an exotic groupoid C*-algebra by Corollary 5.16.

7.4. Non-simple examples. Let us now use Theorem 4.4 to describe a class of algebraic semigroup actions whose C*-algebras have exactly one proper, nonzero ideal. Throughout this section, let us assume that the trivial subgroup is constructible, i.e., $\{ e \} \in \mathcal{C}$. This implies that the dense invariant subset $\mathcal{U} := \{ x_k : k \in G \}$ of $\partial \hat{\mathcal{E}}$ is open, so that $\mathcal{Z} := \partial \hat{\mathcal{E}} \setminus \mathcal{U}$ is a non-empty, proper, closed invariant subset of $\partial \hat{\mathcal{E}}$. Consider the following condition on $\sigma : S \cap G$:

(WPC) For every $C \in \mathcal{C}$ with $C \neq \{ e \}$, there exists $s \in S$ such that $\sigma_s C \leq C$.

Proposition 7.35. Assume $\sigma : S \cap G$ satisfies (WPC) and that $\{ e \} \in \mathcal{C}$. Then

\( (i) \) $\mathcal{Z}$ is the only nonempty, proper, closed invariant subset of $\partial \hat{\mathcal{E}}$;

\( (ii) \) the reduction groupoid $I \ltimes \mathcal{Z}$ is purely infinite.

Proof. (i): By Theorem 4.4, this is equivalent to $\mathfrak{F}_C \setminus \{ C \}$ being the only non-empty, proper, $I^\sigma$-invariant, $\subseteq$-closed subset of $\mathfrak{F}_C$. Suppose $\emptyset \neq \mathfrak{F} \subseteq \mathfrak{F}_C$ is an $I^\sigma$-invariant and $\subseteq$-closed subset. Since $\mathfrak{F}$ is $\subseteq$-closed and proper, $\mathfrak{F} \subseteq \mathfrak{F}_C \setminus \{ C \}$. Let $C \in \mathcal{C} \setminus \{ e \}$, and choose any $\mathfrak{F} \in \mathfrak{F}$. By (WPC), there exists $s \in S$ such that $\sigma_s G \leq C$. Since $G \in \mathfrak{F}$, we have $\sigma_s G \in \sigma_s \mathfrak{F}$, so that $C \in \sigma_s \mathfrak{F}$ because $\sigma_s \mathfrak{F}$ is a filter. By
\(I^e\)-invariance of \(\mathcal{F}\), we have \(\sigma_s \cdot \mathcal{F} \subseteq \mathcal{F}\), so \(C \subseteq \bigcup_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F}\). Thus, \(C \setminus \{e\} \subseteq \bigcup_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F}\). Since \(\mathcal{F}\) is \(\subseteq\)-closed, it follows that \(\mathcal{F}\) contains every member of \(\mathcal{F}_C \setminus \{C\}\).

(ii): The basic (compact) open subsets of \(\mathcal{F}\) are of the form \(\mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\})\), where \(kB \in \mathcal{E}^\times\) and \(\{k_iB_i\} \subseteq \mathcal{E}^\times\) is a finite subset. Moreover, each compact open subset of \(\mathcal{F}\) can be written as a finite disjoint union of such sets (cf. \cite{36} Lemma 4.1), so to prove \(I \times \mathcal{F}\) is purely infinite it suffices to prove that every non-empty compact open subset of the form \(\mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\})\) is properly infinite.

If \(kB = \{k\}\), then \(\partial \mathcal{E}(kB) = \{\chi_k\}\), so \(\mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\}) = \emptyset\) in this case. If \(k_iB_i = \{k_i\}\), then \(\partial \mathcal{E}(k_iB_j) = \{\chi_{k_i}\}\), so that \(\mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\}) = \mathcal{F} \cap \partial \mathcal{E}(kB;\{k_i\} : i \neq j)\). Thus, we may even assume that \(e \notin \{B\} \cup \{B_i\}\). In this case, the proof of Theorem 4.21 goes through and gives that \(\partial \mathcal{E}(kB;\{k_iB_i\})\) is properly infinite as a subset of the groupoid \(\mathcal{G}_\sigma\), and it follows that \(\mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\})\) is properly infinite as a subset of \(I \times \mathcal{F}\).

**Lemma 7.36.** If \(\sigma : S \curvearrowright G\) satisfies \([\text{WPC}]\) and \(\bigcap_{s \in S} \sigma_s G = \{e\}\), then \(I \times \mathcal{F}\) is topologically free.

**Proof.** By Theorem 4.14, it suffices to show that for all \(D \in \bigcup_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F}\), we have \(\bigcap_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\}) = \emptyset\). Fix \(t \in S\) and \(D \in \bigcup_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F}\). By \([\text{WPC}]\), there exists \(s \in S\) such that \(\sigma_s G \subseteq D \cap \sigma_t G\). Since \(\sigma_s \mathcal{F}\) is a filter, we have \(D, \sigma_t G \subseteq \sigma_s \mathcal{F}\). Since \(\sigma_s \mathcal{F} \neq \mathcal{F}\), it follows that \(\bigcap_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\}) \subseteq \sigma_t G\). Hence, \(\bigcap_{\mathcal{F} \subseteq \mathcal{F}} \mathcal{F} \cap \partial \mathcal{E}(kB;\{k_iB_i\}) \subseteq \bigcap_{s \in S} \sigma_s G = \emptyset\).

**Corollary 7.37.** If \(\sigma : S \curvearrowright G\) satisfies \([\text{WPC}]\) and \(\bigcap_{s \in S} \sigma_s G = \{e\}\), then \(C^*_G(I \times \mathcal{F})\) is simple and purely infinite. If moreover \(S, G, G\) are countable and \(\mathcal{G}_\sigma\) is Hausdorff and inner exact, then \(C^*_G(I \times \mathcal{F})\) is the unique nonzero, proper ideal of \(C^*_G(\mathcal{G}_\sigma)\) and \(C^*_G(\mathcal{G}_\sigma)/C^*_G(I \times \mathcal{F})\) is simple and purely infinite.

**Remark 7.38.** Since \(\{e\} \in \mathcal{C}\), the group \(\mathcal{F}_e\) is trivial, and the canonical map \(\mathcal{C}_{\text{ess}}(\mathcal{G}_\sigma) \to \mathcal{A}_\sigma\) from Corollary 7.37 is a *-isomorphism. Thus, in the setting of Corollary 7.37 we have \(\mathcal{C}_{\text{ess}}(\mathcal{G}_\sigma) \cong \mathcal{A}_\sigma\). Moreover, under this isomorphism, \(C^*_G(I \times \mathcal{F}) = C^*_G(I \times \mathcal{F}) \cong \mathcal{K}(\ell^2(G))\).

**Remark 7.39.** Assume \(\sigma : S \curvearrowright A\) satisfies the following condition:

\[(9) \quad \sigma^{-1}_{s_1} \sigma_{t_1} \cdots \sigma_{s_m} \sigma_{t_m} G \neq \{e\} \implies s_1^{-1} t_1 \cdots s_m^{-1} t_m S \neq \emptyset \]

for all \(s_i, t_i \in S\) and \(m \in \mathbb{Z}_{>0}\). Arguing as in Proposition 7.12, we see \(\sigma : S \curvearrowright A\) satisfies \([\text{WPC}]\).

**Example 7.40.** Let \(\sigma : S \curvearrowright \bigoplus_S \Sigma\) be the full S-shift over \(\Sigma\) as in Definition 7.30. If \(S\) is not left reversible, then \(\{e\}\) is constructible. Using Remark 7.39, it is not hard to see that \(\sigma : S \curvearrowright \bigoplus_S \Sigma\) satisfies \([\text{WPC}]\). We have seen in the proof of Corollary 7.32 that \(\bigcap_{s \in S} \sigma_s \bigoplus_S \Sigma = \emptyset\).

8. **K-theory of the Toeplitz-type C*-algebra**

The (reduced) Toeplitz-type C*-algebra attached to \(\sigma : S \curvearrowright G\) is the reduced C*-algebra \(C^*_G(I)\) of the inverse semigroup \(I\). For such C*-algebras, there are powerful tools for computing K-theory. Given \(C \in \mathcal{C}\), we consider the group \(I_C := \{\phi \in I : \text{dom}(\phi) = \text{im}(\phi) = C\}\). Applying \cite{38} Theorem 1.1], we obtain the following K-theory formula:

**Theorem 8.1.** Assume that \(S\) and \(G\) are countable and that \(I\) admits an idempotent pure partial homomorphism to a group that satisfies the Baum-Connes conjecture with coefficients. Then,

\[(10) \quad K_*(C^*_G(I)) \cong \bigoplus_{[C] \in I^e \setminus \mathcal{C}} K_*(C^*_G(I_C))\]

where \(I^e \setminus \mathcal{C}\) is the set of orbits for the action of \(I^e\) on \(\mathcal{C}\).

**Definition 8.2.** We say that \(\sigma : S \curvearrowright G\) has the infinite index property if

\[(\Pi) \quad \#C/D = \infty \quad \text{for all } C, D \in \mathcal{C} \text{ with } D \subseteq C.\]
If $\sigma : S \curvearrowright G$ satisfies (II), then every character on $E$ is tight by Lemma 3.22 so that $\partial E = \tilde{E}$ and $C^*_\lambda(G) = C^*_\lambda(I)$. Theorem 8.1 then computes the K-theory of $C^*_\lambda(G)$.

Example 8.3. Assume $S$ embeds into a countable group $\mathcal{J}$. Let $S \curvearrowright \bigoplus_S \Sigma$ be the full $S$-shift over a non-trivial group $\Sigma$ such that $(\bigoplus_S \Sigma) \rtimes \mathcal{J}$ satisfies the Baum–Connes conjecture with coefficients. If either $\# \Sigma = \infty$ or $|(X \setminus Y)| = \infty$ for all $X, Y \in \mathcal{J}_S^\Sigma$ with $Y \subsetneq X$, then $S \curvearrowright \bigoplus_S \Sigma$ satisfies (II) by Proposition 7.31, so that Theorem 8.1 gives us

$$K_*(C^*_\lambda(G)) \cong \bigoplus_{[X] \in \mathcal{J} \setminus \mathcal{J}_S^\Sigma} K_*(C^*_\lambda((\bigoplus_{X \in \Sigma} \Sigma) \rtimes \mathcal{J}_X)),$$

where $\mathcal{J}_X = \{ \gamma \in \mathcal{J} : \gamma X = X \}$. If $\# \Sigma < \infty$, then $K_*(C^*_\lambda((\bigoplus_{X \in \Sigma} \Sigma) \rtimes \mathcal{J}_X))$ can be explicitly computed using [37] Theorem 1.1.

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