Magnetic catalysis effect in the (2+1)-dimensional Gross–Neveu model with Zeeman interaction

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Magnetic catalysis of the chiral symmetry breaking and other magnetic properties of the (2+1)-dimensional Gross-Neveu model are studied taking into account the Zeeman interaction of spin-1/2 quasi-particles (electrons) with tilted (with respect to a system plane) external magnetic field $\vec{B} = \vec{B}_\perp + \vec{B}_\parallel$. The Zeeman interaction is proportional to magnetic moment $\mu_B$ of electrons. For simplicity, temperature and chemical potential are equal to zero throughout the paper. We compare in the framework of the model the above mentioned phenomena both at $\mu_B \neq 0$ and $\mu_B = 0$. It is shown that at $\mu_B \neq 0$ the magnetic catalysis effect is drastically changed in comparison with the $\mu_B = 0$ case. Namely, at $\mu_B \neq 0$ the chiral symmetry, being spontaneously broken by $\vec{B}$ at subcritical coupling constants, is always restored at $|\vec{B}| \to \infty$ (even at $\vec{B}_\parallel = 0$). Moreover, it is proved in this case that chiral symmetry can be restored simply by tilting $\vec{B}$ to a system plane, and in the region $\vec{B}_\perp \to 0$ the de Haas–van Alphen oscillations of the magnetization are observed. At supercritical values of coupling constant we have found two chirally non-invariant phases which respond differently on the action of $\vec{B}$. The first (at rather small values of $|\vec{B}|$) is a diamagnetic phase, in which there is an enhancement of chiral condensate, whereas the second is a paramagnetic chirally broken phase. Numerical estimates show that phase transitions described in the paper can be achieved at low enough laboratory magnetic fields.

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I. INTRODUCTION

It is well known that during last three decades a lot of attention is paid to the investigation of (2+1)-dimensional quantum field theories (QFT) under influence of different external conditions. In particular, the (2+1)-dimensional Gross-Neveu (GN) type models are among the most popular $^{[14]}$. There are several basic motivations for this interest. Since low dimensional theories have a rather simple structure, they can be used in order to develop our physical intuition for different physical phenomena taking place in real (3+1)-dimensional world (such as dynamical symmetry breaking $^{[1]}$, color superconductivity $^{[10]}$ etc). Another example of this kind is the spontaneous chiral symmetry breaking induced by external magnetic fields, i.e. the magnetic catalysis effect (see the recent reviews $^{[11,12]}$ and references therein). For the first time this effect was also studied in terms of (2+1)-dimensional GN models $^{[13]}$. In addition, low dimensional models are useful in elaborating new QFT methods like the large-N technique $^{[1,3]}$ and the optimized expansion method $^{[14,15]}$ etc.

However, a more fundamental reason for the study of these theories is also well known. Indeed, there are a lot of condensed matter systems which, firstly, have a (quasi-)planar structure and, secondly, their low-energy excitation spectrum is described adequately by relativistic Dirac-like equation rather than by Schrödinger one. Among these systems are the high-T$_c$ cuprate and iron superconductors $^{[10]}$, the one-atom thick layer of carbon atoms, or graphene, $^{[17,18]}$ etc. Thus, many properties of such condensed matter systems can be explained in the framework of various (2+1)-dimensional QFTs, including the GN-type models (see, e.g., $^{[13,21]}$ and references therein). Especially, it is necessary to note that the magnetic catalysis phenomenon supplies a very effective mechanism for a description of high temperature superconductivity $^{[10,22]}$, quantum Hall effect in graphene $^{[28]}$ and other condensed matter systems, which have a thin film structure and are exposed to an external magnetic field.

Since elementary excitations of an arbitrary condensed matter system are usually electrons (or quasi-particles) with $\pm 1/2$ spin projection on the direction of external magnetic field $\vec{B}$, there are two independent ways to introduce $\vec{B}$ into consideration in such systems, i) when $\vec{B}$ couples only to an orbital angular momentum of electrons (note, in planar systems only perpendicular component $\vec{B}_\perp$ of external magnetic field $\vec{B}$ contributes in this case), and ii) when, in addition, the Zeeman interaction of an electron spin (or its magnetic moment) with $\vec{B}$ is also taken into account. In early investigations of the magnetic catalysis effect in the framework of (2+1)-dimensional GN type models (see, e.g., $^{[13,19,22]}$ just the first way i) was used, where it was shown that chiral symmetry of the model is spontaneously broken down at arbitrary values of $\vec{B}$ and even at infinitesimal values of the coupling constant. Later, especially in connection with graphene physics, the Zeeman interaction was also taken into account (see, e.g., $^{[28,29]}$), but only in the case when $\vec{B}$ is perpendicular to the system plane. In contrast, in the recent paper $^{[27]}$ another limiting case for the direction of an external magnetic field was considered. There the influence of in-plane magnetic field $\vec{B}_\parallel$, i.e. parallel to the system plane, on the (2+1)-dimensional GN model was studied (in this case, since $\vec{B}_\parallel = 0$, the magnetic field couples to fermions only due to the Zeeman interaction). In particular, it was shown in $^{[27]}$ that at sufficiently
high values of $\vec{B}$, chiral symmetry of the model is restored. Naturally, one might wonder about the response of the chiral symmetry of $(2+1)$-dimensional GN type models upon the action of an external arbitrarily directed magnetic field, when $\vec{B}$ interacts both with orbital angular momentum and spin of electrons. (In this connection it is worth to note that in the framework of a planar system of free electrons a jumping behavior of the magnetization vs $|\vec{B}|$ was predicted in [30] at rather small (laboratory) values of $|\vec{B}|$. The effect was explained in [30] just due to the Zeeman interaction of spin with tilted magnetic field.)

So, in the present paper the magnetic catalysis effect, as well as other magnetic phenomena, is investigated in the leading order of the large-$N$ expansion technique in the framework of the $(2+1)$-dimensional GN model subjected to a tilted external magnetic field $\vec{B}$. The model is invariant with respect to the discrete chiral symmetry and describes the four-fermion interaction of quasi-particles (electrons) with $(\pm1/2)$-spin projections on the direction of $\vec{B}$. The Zeeman interaction of an electron magnetic moment with $\vec{B}$ is also taken into account. Temperature and chemical potential are equal to zero throughout the consideration. In particular, we show that at subcritical values of the coupling constant the chiral symmetry, being broken spontaneously at rather small values of $\vec{B}$, is restored (in contrast to the case when Zeeman interaction is ignored [13, 19, 22]) at sufficiently high values of external magnetic field. Moreover, we have found a jumping behavior of dynamical fermion mass vs $|\vec{B}|$, i.e. the evolution of the system vs $\vec{B}$ is accompanied by its passing through several different phases with broken chiral symmetry. It turns out that due to a presence of the Zeeman interaction one can observe in the model the de Haas – van Alphen oscillations of the magnetization as well as diamagnetic and paramagnetic phenomena. We hope that our results can be useful in explaining physical phenomena in thin organic films, graphene etc, i.e. in all condensed matter systems having a spatially (quasi-) two-dimensional structure.

The paper is organized as follows. In Sec. II the $(2+1)$-dimensional GN model is presented as well as its renormalized thermodynamic potential is obtained in different particular cases, i.e. with- and without Zeeman interaction, when external magnetic field is taken into account. In the next Sec. III the modification of the magnetic catalysis effect is investigated in the case of subcritical values of the coupling constant (Sec. III A). Here it is also shown (Sec. III B) that tilting of the magnetic field leads to oscillations of the magnetization. In Sec. IV A and IV B the case of supercritical values of the coupling constant is analyzed. It is established here that the phase portrait of the model contains two chirally non-invariant phases. One of them has a diamagnetic ground state, in another phase it is a paramagnetic one. It is also shown that at sufficiently high values of $|\vec{B}|$ and at $B_\parallel \to 0$ chiral symmetry of the model is restored. In Sec. IV C some numerical estimates in the context of condensed matter systems are performed, which show that phase transitions induced by Zeeman effect are really achieved at laboratory magnitudes of external magnetic fields. Finally, in Sec. V the summary of the paper is presented.

II. THE MODEL AND ITS THERMODYNAMIC POTENTIAL

We suppose that some physical system is localized in the spatially two-dimensional plane perpendicular to the \(\hat{z}\) coordinate axis of usual tree-dimensional space. Moreover, there is an external homogeneous and time independent magnetic field $\vec{B}$ tilted with respect to this plane. The corresponding $(3+1)$-dimensional vector potential $A_\mu$ is given by $A_{0,1} = 0$, $A_2 = B_2 x$, $A_3 = B_3 y$, i.e. the spatial components $B_{x,y,z}$ of an external magnetic field have the form $B_x = B_\parallel$, $B_y = 0$, $B_z = B_\perp$. We assume that the planar physical system consists of quasi-particles (electrons) with two spin projections, $\pm1/2$, on the direction of magnetic field $\vec{B}$. Moreover, it is also supposed that their low-energy dynamics is described by the following $(2+1)$-dimensional Gross-Neveu type Lagrangian\footnote{In the paper we use the natural units, $\hbar = k_B = c = 1$. However, for a more adequate application of our results to condensed matter physics it is necessary to modify slightly Lagrangian (1). Namely, one should use there the replacements $\nabla_{1,2} \to v_F \nabla_{1,2}$ as well as $G \to v_F G$, where $v_F$ is a Fermi velocity of quasi-particles (for example, in graphene $v_F \approx c/300$). For simplicity, in all the expressions of our paper we assume that $v_F = 1$. The case $v_F \neq 1$ is considered in the last section IV C, where some estimates are made in the context of condensed matter systems.}

\[
L = \sum_{k=1}^{2} \bar{\psi}_{ka} \left[ \gamma^0 \partial_t + \gamma^1 i \nabla_1 + \gamma^2 i \nabla_2 - \nu (-1)^{k} \gamma^0 \right] \psi_{ka} + \frac{G}{N} \left( \sum_{k=1}^{2} \bar{\psi}_{ka} \psi_{ka} \right)^2, \tag{1}
\]

where $\nabla_{1,2} = \partial_{1,2} + i e A_{1,2}$ and the summation over the repeated index $a = 1, \ldots, N$ of the internal $O(N)$ group is implied. For each fixed value of $k = 1, 2$ and $a = 1, \ldots, N$ the quantity $\psi_{ka}(x)$ in (1) means the Dirac fermion field, transforming over a reducible 4-component spinor representation of the $(2+1)$-dimensional Lorentz group. Moreover, all these Dirac fields $\psi_{ka}(x)$ form two fundamental multiplets, $\psi_1 a(x)$ and $\psi_2 a(x)$ ($a = 1, \ldots, N$), of the internal auxiliary $O(N)$ group, which is introduced here in order to make it possible to perform all the calculations in the framework of the nonperturbative large-$N$ expansion method. We suppose that spinor fields $\psi_1 a(x)$ and $\psi_2 a(x)$ ($a = 1, \ldots, N$) correspond to electrons with spin projections $1/2$ and $-1/2$ on the direction of an external magnetic field, respectively. In (1) the $\nu$-term is introduced in order to take into account the Zeeman interaction energy of electrons with external magnetic field $\vec{B}$. (To investigate the genuine role of a magnetic field in the phase structure...
of the model, we suppose throughout the paper that both electron number chemical potential and temperature are zero.) Hence, in our case \( \nu = g_S \mu_B |\vec{B}| / 2 \), where \( |\vec{B}| = \sqrt{B_x^2 + B_y^2} \), \( g_S \) is the spectroscopic Lande factor and \( \mu_B \) is an electron magnetic moment, i.e. the Bohr magneton. In what follows it is supposed that \( g_S = 2 \), however at the end of the paper we will also briefly discuss the influence of \( g_S > 2 \), as in \( [30] \), on some physical results. The algebra of the \( \gamma^\rho \)-matrices as well as their particular representations are given, e.g., in \( [24] \). The model (1) is invariant under the discrete chiral transformation, \( \psi_{ka} \rightarrow \gamma^5 \psi_{ka} \) (the particular realization of the \( \gamma^5 \)-matrix is also presented in \( [24] \)). Certainly, there is the \( O(N) \) invariance of the Lagrangian (1). Finally note that at \( N = 1 \) the quasi-particle spectrum of the model (1) is just the same as in the monolayer graphene \( [28] \), but at \( N > 1 \) one can interpret our results as occurring in the \( N \)-layered system.

In the following we use an auxiliary theory with the Lagrangian density

\[
\mathcal{L} = -\frac{N \sigma^2}{4G} + \sum_{k=1}^{2} \bar{\psi}_{ka} \left( \gamma^0 i\partial_t + \gamma^1 i\nabla_1 + \gamma^2 i\nabla_2 + \gamma^5 \mu k \gamma^0 - \sigma \right) \psi_{ka},
\]

where \( \mu_1 = \nu, \mu_2 = -\nu \) and from now on \( \nu = \mu_B |\vec{B}| \) (in this formula and below the summation over repeated indices is implied). Clearly, the Lagrangians \( [11] \) and \( [22] \) are equivalent, as can be seen by using the Euler-Lagrange equation of motion for scalar bosonic field \( \sigma(x) \) which takes the form

\[
\sigma(x) = -\frac{2G}{N} \sum_{k=1}^{2} \bar{\psi}_{ka} \psi_{ka}.
\]

One can easily see from \( [3] \) that the neutral field \( \sigma(x) \) is a real quantity, i.e. \( \langle \sigma(x) \rangle^\dagger = \sigma(x) \) (the superscript symbol \( \dagger \) denotes the Hermitian conjugation). Moreover, if \( \langle \sigma(x) \rangle \neq 0 \), then, after the shifting \( \sigma(x) \rightarrow \sigma(x) + \langle \sigma(x) \rangle \) in \( [22] \), it is clear that the discrete chiral symmetry of the model is spontaneously broken and fermions acquire dynamically the mass equal to \( \langle \sigma(x) \rangle \).

Let us now study the phase structure of the four-fermion model (1) by starting from the equivalent semi-bosonized Lagrangian \( [22] \). In the leading order of the large-\( N \) approximation, the effective action \( \mathcal{S}_{\text{eff}}(\sigma) \) of the considered model is expressed by means of the path integral over fermion fields

\[
\exp(i \mathcal{S}_{\text{eff}}(\sigma)) = \int \prod_{k=1}^{2} \prod_{a=1}^{N} [d\bar{\psi}_{ka}] [d\psi_{ka}] \exp\left( i \int \mathcal{L} d^3 x \right),
\]

where

\[
\mathcal{S}_{\text{eff}}(\sigma) = -\int d^3 x \frac{N \sigma^2}{4G} + \bar{\sigma} \mathcal{S}_{\text{eff}}.
\]

The fermion contribution to the effective action, i.e. the term \( \bar{\sigma} \mathcal{S}_{\text{eff}} \) in \( [11] \), is given by

\[
\exp(i \bar{\sigma} \mathcal{S}_{\text{eff}}) = \int \prod_{k=1}^{2} \prod_{a=1}^{N} [d\bar{\psi}_{ta}] [d\psi_{ta}] \exp\left( i \sum_{k=1}^{2} \int \bar{\psi}_{ka} \left( \gamma^0 i\partial_t + \gamma^1 i\nabla_1 + \gamma^2 i\nabla_2 + \gamma^5 \mu k \gamma^0 - \sigma \right) \psi_{ka} d^3 x \right).
\]

The ground state expectation value \( \langle \sigma(x) \rangle \) of the composite bosonic field is determined by the saddle point equation,

\[
\frac{\delta \mathcal{S}_{\text{eff}}}{\delta \sigma(x)} = 0.
\]

For simplicity, throughout the paper we suppose that the above mentioned ground state expectation value does not depend on space-time coordinates, i.e.

\[
\langle \sigma(x) \rangle = M,
\]

where \( M \) is a constant quantity. In fact, it is a coordinate of the global minimum point of the thermodynamic potential (TDP) \( \Omega(M; \nu, B_{\perp}) \). In the leading order of the large-\( N \) expansion the TDP is defined by the following expression:

\[
\int d^3 x \Omega(M; \nu, B_{\perp}) = -\frac{1}{N} \mathcal{S}_{\text{eff}}(\sigma(x)) \bigg|_{\sigma(x) = M},
\]

which gives

\[
\int d^3 x \Omega(M; \nu, B_{\perp}) = \int d^3 x \frac{M^2}{4G} + \frac{i}{N} \ln \left( \prod_{k=1}^{2} \prod_{b=1}^{N} [d\bar{\psi}_{tb}] [d\psi_{tb}] \exp \left( i \int \sum_{k=1}^{2} \bar{\psi}_{ka} D_k \psi_{ka} d^3 x \right) \right),
\]

where \( D_k = \gamma^0 i\partial_t + \gamma^1 i\nabla_1 + \gamma^2 i\nabla_2 + \mu_k \gamma^0 - M \).

\[2\] In our consideration the Zeeman \( \nu \)-term in (1) is introduced phenomenologically, so the quantity \( \mu_B \) might be considered as a free model parameter. However, at the end of the paper, when doing some numerical estimates in Sec. IV C, we suppose that \( \mu_B \) is equal to the Bohr magneton. Recall, there is an interesting possibility of a dynamical generation of an electron magnetic moment \( [31] \). This approach, however, is outside of the scope of the present paper.
A. The particular case $B_\perp = 0$, $\nu \neq 0$

In order to find a convenient expression for the TDP in this case it is necessary to evaluate the Gaussian path integral (8) at $B_\perp = 0$ (see, e.g., the paper [24], where more general path integrals with difermion condensates were calculated). As a result, we obtain the following expression for the TDP of the model (1) at zero temperature:

$$\Omega(M; \nu) = \frac{M^2}{4G} + 2i \int \frac{d^3p}{(2\pi)^3} \ln \left[ (p_0^2 - (E + \nu)^2)(p_0^2 - (E - \nu)^2) \right], \quad (9)$$

where $E = \sqrt{M^2 + |\vec{p}|^2}$ and $|\vec{p}| = \sqrt{|p_1|^2 + |p_2|^2}$. It is clear from (9) that without loss of generality one can suppose that $\nu \geq 0$ and $M \geq 0$. Using in the expression (9) a rather general formula

$$\int_{-\infty}^{\infty} dp_0 \ln (p_0 - A) = i\pi |A| \quad (10)$$

(obtained rigorously, e.g., in Appendix B of [33] and true up to an infinite term independent on the real quantity $A$), it is possible to reduce it to the following one:

$$\Omega(M; \nu) \equiv \Omega^{un}(M; \nu) = \frac{M^2}{4G} - 2 \int \frac{d^2p}{(2\pi)^2} \left( |E + \nu| + |E - \nu| \right). \quad (11)$$

The integral term in (11) is an ultraviolet divergent one, hence to obtain any information from this expression we have to renormalize it. First of all, let us regularize the TDP (11) by cutting the integration region, i.e. we suppose that $|p_1| < \Lambda$, $|p_2| < \Lambda$ in (11). As a result we have the following regularized expression (which is finite at finite values of $\Lambda$):

$$\Omega^{reg}(M; \nu) = \frac{M^2}{4G} - \frac{2}{\pi^2} \int_0^{\Lambda} dp_1 \int_0^{\Lambda} dp_2 \left( |E + \nu| + |E - \nu| \right). \quad (12)$$

Let us use in (12) the following asymptotic expansion at $|\vec{p}| \to \infty$, i.e. at $|\vec{p}| \gg \nu$,

$$|E + \nu| + |E - \nu| = 2|\vec{p}| + \frac{M^2}{|\vec{p}|} + O(1/|\vec{p}|^3). \quad (13)$$

(Note, the leading asymptotic terms in (13) do not depend on $\nu$.) Then, upon integration there term-by-term, it is possible to find

$$\Omega^{reg}(M; \nu) = M^2 \left( \frac{1}{4G} - \frac{4\Lambda \ln(1 + \sqrt{2})}{\pi^2} \right) - \frac{4\Lambda^3(\sqrt{2} + \ln(1 + \sqrt{2}))}{3\pi^2} + O(\Lambda^0), \quad (14)$$

where $O(\Lambda^0)$ denotes an expression which is finite in the limit $\Lambda \to \infty$. Second, we suppose that the bare coupling constant $G$ depends on the cutoff parameter $\Lambda$ in such a way that in the limit $\Lambda \to \infty$ one obtains a finite expression in the square brackets of (14). Clearly, to fulfill this requirement it is sufficient to require that

$$\frac{1}{4G} = \frac{G}{4G(\Lambda_0)} = \frac{4\Lambda \ln(1 + \sqrt{2})}{\pi^2} + \frac{1}{\pi g} = \frac{1}{4G_c} + \frac{1}{\pi g}, \quad (15)$$

where $g$ is a finite and $\Lambda$-independent model parameter with dimensionality of inverse mass and $G_c = \frac{\pi^2}{16\Lambda \ln(1 + \sqrt{2})}$. Moreover, since bare coupling $G$ does not depend on a normalization point, the same property is also valid for $g$. Hence, taking into account in (12) and (13) the relation (15) and ignoring there an infinite $\Lambda$-independent constant, one obtains the following renormalized, i.e. finite, expression for the TDP

$$\Omega^{ren}(M; \nu) = \lim_{\Lambda \to \infty} \left\{ \Omega^{reg}(M; \nu) \bigg|_{G=G(\Lambda)} + \frac{4\Lambda^3(\sqrt{2} + \ln(1 + \sqrt{2}))}{3\pi^2} \right\}. \quad (16)$$

It should also be mentioned that the TDP (16) is a renormalization group invariant quantity.

Suppose that $\nu \equiv \mu|\vec{B}| = 0$. Then the $O(\Lambda^0)$ term in (14) can be calculated explicitly. As a result, we have for the TDP in this particular case the expression:

$$V(M) \equiv \Omega^{ren}(M; \nu) \bigg|_{\nu=0} = \frac{M^2}{\pi g} + \frac{2M^3}{3\pi}. \quad (17)$$

It follows from (17) that at $g > 0$, i.e. at $G < G_c$, (15), the global minimum point (GMP) of the TDP is arranged at $M = 0$, so the chiral symmetry is not broken. However, at $g < 0$, i.e. at $G > G_c$, the GMP lies at the point $M_0 = -1/g$, and there is a spontaneous chiral symmetry breaking.
Now, let us obtain an alternative expression for the renormalized TDP (16) at ν ≠ 0. For this purpose one can rewrite the unrenormalized TDP Ω\text{un}(M; ν) (11) in the following way

\[ Ω\text{un}(M; ν) = \frac{M^2}{4G} - 2 \int \frac{d^2p}{(2\pi)^2} (2E) - 2 \int \frac{d^2p}{(2\pi)^2} \left( |E + ν| + |E - ν| - 2E \right). \]  

Since the leading terms of the asymptotic expansion (13) do not depend on ν, it is clear that the last integral in (18) is a convergent one. Other terms in (18) form the unrenormalized TDP of the particular case with ν = 0, which is reduced after renormalization procedure to the expression (17). Hence, after renormalization we obtain from (18) the following finite expression (evidently, it coincides with renormalized TDP (16)):

\[ Ω\text{ren}(M; ν) = V(M) - 2 \int \frac{d^2p}{(2\pi)^2} \left( |E + ν| + |E - ν| - 2E \right), \]  

where \( V(M) \) is presented in (17). The integral terms in (19) can be explicitly calculated. As a result, we have

\[ Ω\text{ren}(M; ν) = V(M) - \frac{1}{3\pi} \theta(ν - M)(ν - M)^2(2M + ν). \]  

B. The particular case ν = 0, B⊥ ≠ 0

Here we briefly discuss how the nonzero perpendicular magnetic field \( B_⊥ \) influences the phase structure of the initial model (1) in the simplified case of μ₂ = 0, i.e., when Zeeman interaction is not taken into account (ν = 0). The renormalized TDP of the GN model with single \( O(N) \) fundamental multiplet of four-component Dirac spinor fields was obtained and investigated in this case in, e.g., [4, 13, 19, 22]. The generalization to the case of GN model with two \( O(N) \) multiplets, as in the model under consideration, is trivial. So we have

\[ Ω\text{ren}(M; B_⊥) = \frac{M^2}{πg} + \frac{MeB_⊥}{π} - \frac{(2eB_⊥)^{3/2}}{π} \zeta \left( -\frac{1}{2}, \frac{M^2}{2eB_⊥} \right), \]  

where \( ζ(s, x) \) is the generalized Riemann zeta-function. (Since \( ζ(-1/2, x) = -2x^{3/2}/3 + O(\sqrt{x}) \) at \( x \to \infty \), it is clear that the TDP (21) coincides with \( V(M) \) (17) at \( B_⊥ = 0 \).) It is evident that

\[ \frac{∂Ω\text{ren}(M; B_⊥)}{∂M} = \frac{2M}{πg} + \frac{eB_⊥}{π} - \frac{M(2eB_⊥)^{1/2}}{π} ζ \left( 1/2, \frac{M^2}{2eB_⊥} \right), \]  

where we have used the relation \( ∂ζ(s, x)/∂x = -sζ(s + 1, x) \). Since \( ζ(1/2, x) = x^{-1/2} + O(x^0) \) at \( x \to 0 \), we see from (22) that

\[ \frac{∂Ω\text{ren}(M; B_⊥)}{∂M} \bigg|_{M \to 0^+} = -\frac{eB_⊥}{π} < 0, \]  

which means that the TDP (21) can never have a global minimum point at \( M = 0 \), if \( B_⊥ ≠ 0 \). Hence, if the model (1) is subjected to the external (arbitrarily small) perpendicular magnetic field \( B_⊥ \) and, in addition, the Zeeman interaction of electrons with this magnetic field is not taken into account (ν = 0), then at arbitrary (even infinitesimal) positive values of the bare coupling constant \( G \) the chiral symmetry breaking occurs in the model. It is the so-called magnetic catalysis effect [13]. In particular, it means that at \( g > 0 \), i.e., at \( G < G_c \), the spontaneous chiral symmetry breaking is induced by external magnetic field. At \( g < 0 \), i.e., at \( G > G_c \), the chiral symmetry is broken even at \( B = 0 \) due to a rather strong self-interaction coupling constant \( G \), however an external magnetic field enhances chiral symmetry breaking in this case. Moreover, using in (22) the above asymptotic expansion of the \( ζ(1/2, x) \) function at \( x \to 0 \), it is possible to show that at \( g > 0 \) the solution \( M_0(B_⊥) \) of the gap equation \( ∂Ω\text{ren}(M; B_⊥)/∂M = 0 \) has the following behavior at \( eg^2B_⊥ → 0 \) [1, 52]:

\[ M_0(B_⊥) = eB_⊥ g/2 + o(egB_⊥). \]  

Note, the gap \( M_0(B_⊥) ) \sim √eB_⊥ \) at \( eg^2B_⊥ → ∞ \) both for negative and positive values of the coupling \( g \) [4].

3 Without loss of generality one can suppose that \( eB_⊥ ≥ 0 \).
C. The TDP in the general case \( \nu \neq 0, B_\perp \neq 0 \)

The TDP of the GN model with single \( O(N) \) multiplet of Dirac spinors and at nonzero values of a chemical potential and \( B_\perp \) was obtained, e.g., in [4, 32]. Taking into account the fact that in our case each of two \( O(N) \) multiplets has its own chemical potential \( \mu_k = \pm \nu \), one can easily generalize the results of [4, 32] and find the following expression for the renormalized TDP of the GN model (1):

\[
\Omega^{\text{ren}}(M; \nu, B_\perp) = \Omega^{\text{ren}}(M; B_\perp) - \frac{eB_\perp}{\pi} \sum_{n=0}^{\infty} s_n \theta(\nu - \varepsilon_n)(\nu - \varepsilon_n),
\]

where \( s_n = 2 - \delta_{0n}, \varepsilon_n = \sqrt{M^2 + 2n\varepsilon_B} \), and the TDP \( \Omega^{\text{ren}}(M; B_\perp) \) is presented in (21). Note that due to a presence of the \( \theta(x) \)-functions, the summation over \( n \) in (25) is performed really from 0 to \( \text{Int}[\nu^2 - M^2]/2eB_\perp] \), where \( \text{Int}(x) \) is the integer part of \( x \). It follows from (24) and (22) that the global minimum point \( M_0(B_\perp, \nu) \) (or the gap) of the TDP (25) obeys the gap equation

\[
\frac{\partial \Omega^{\text{ren}}(M; \nu, B_\perp)}{\partial M} = \frac{2M}{\pi g} + \frac{eB_\perp}{\pi} - \frac{M(2eB_\perp)^{1/2}}{\pi} = \left( \frac{1}{2}, \frac{M^2}{2eB_\perp} \right) + \frac{eB_\perp}{\pi} \sum_{n=0}^{\infty} s_n \theta(\nu - \varepsilon_n) \frac{M}{\varepsilon_n} = 0.
\]

Recall, in the framework of the model (1) the gap \( M_0(B_\perp, \nu) \) is also a dynamical mass of quasi-particles (electrons).

III. SOME PROPERTIES OF THE MODEL AT \( g > 0 \)

Let us first investigate how the external magnetic field influences the phase structure and magnetization of the model at \( g > 0 \), i.e. in the case of subcritical values of the bare coupling constant, \( G < G_c \). So, it is necessary to study the properties of the global minimum point of the TDP (25) at \( g > 0 \). Recall, there \( \nu \equiv \mu_B|\vec{B}| \), where \( |\vec{B}| = \sqrt{B_\perp^2 + B_\parallel^2} \).

A. Magnetic catalysis effect

Suppose for a moment that \( \nu \) does not depend on \( |\vec{B}| \). In such a situation the parameter \( \nu \) is not the Zeeman splitting energy, but rather a usual chemical potential. In this case, i.e. when there are two independent external parameters \( B_\perp \) and \( \nu \), the \( (egB_\perp, \nu) \)-phase structure of our model (1) is the same as that of the GN model considered in [32, 3]. Then, as it follows from [32], for each fixed \( B_\perp \) there exists a critical value \( \nu_c(B_\perp) \) of the chemical potential \( \nu \) such that at \( \nu < \nu_c(B_\perp) \) (at \( \nu > \nu_c(B_\perp) \)) a chiral symmetry broken phase (chirally symmetric phase) is realized in the \( (egB_\perp, \nu) \)-phase portrait. On the critical curve \( \nu = \nu_c(B_\perp) \) the first order phase transitions take place in the model. Moreover, it was established in [32] that \( \nu_c(B_\perp) = M_0(B_\perp) + o(egB_\perp) \) at \( B_\perp \to 0 \) and \( \nu_c(B_\perp) \sim \sqrt{e\varepsilon_B} \) at \( B_\perp \to \infty \) \( (M_0(B_\perp) \) is the gap (23)). Hence, and it is the most important thing for our further consideration, there is a straight line \( \lambda \) in the \( (egB_\perp, \nu) \)-plane, tangent to a critical curve \( \nu = \nu_c(B_\perp) \) at the point \( B_\perp = 0 \), such that the whole \( (egB_\perp, \nu) \)-region above \( \lambda \) belongs to a symmetric phase of the model. It is clear from (24) that

\[
\lambda = \{(egB_\perp, \nu) : \nu = egB_\perp/2\}.
\]

Moreover, any straight line \( \nu = k_\perp egB_\perp \) with \( k < 1/2 \) crosses the region of the \( (egB_\perp, \nu) \)-plane, corresponding to a chiral symmetry broken phase.

The case \( B_\parallel = 0 \), i.e. \( B_\perp = |\vec{B}| \). Now, as it was intended from the very beginning, we suppose that \( \vec{B} \) and \( \nu \) are dependent quantities and, furthermore, the external magnetic field \( \vec{B} \) is perpendicular to a system plane, i.e. \( B_\perp = |\vec{B}| \) and \( \nu = \mu_B B_\perp \). Hence, in the case under consideration only the points of the straight line \( \nu = \mu_B B_\perp \equiv k_\perp egB_\perp \) of the above mentioned \( (egB_\perp, \nu) \)-plane are relevant to a real physical situation (evidently, \( k = \mu_B/(eg) \)). So, if \( k > 1/2 \), i.e. at sufficiently small values of \( g \), then the straight line \( \nu = \mu_B B_\perp \) as a whole is above the line \( \lambda \) and, spontaneous chiral symmetry breaking is forbidden in the system. However, if the coupling constant \( g \) is greater than \( g_c = 2\mu_B/e \), we have \( k < 1/2 \) and the line \( \nu = \mu_B B_\perp \) is below \( \lambda \). Obviously, in this case the straight line \( \nu = \mu_B B_\perp \) crosses the region of the \( (egB_\perp, \nu) \)-plane with chiral symmetry breaking. Hence, at \( g > g_c \) chiral symmetry might be broken only for some finite interval of \( B_\perp \)-values. It means that the magnetic catalysis effect at \( B_\parallel = 0 \) and \( \mu_B \neq 0 \), i.e. when the Zeeman interaction of electrons with magnetic field is taken into account, is qualitatively

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4 In [32] the GN model with one \( O(N) \) multiplet of fermion fields was investigated at \( G < G_c \) and in the presence of \( B_\perp \neq 0 \) and nonzero chemical potential \( \nu \). Since the TDP (25) of our model (1) coincides, up to a factor 2, with the TDP of [32], it can be concluded that both models have an identical \( (egB_\perp, \nu) \)-phase structure, if \( B_\perp \) and \( \nu \) are independent.
different from the case with $B_\parallel = 0$ and $\mu_B = 0$ (see the section IVB). Indeed, i) at $\mu_B = 0$ the external (arbitrary small) magnetic field $B_\perp$ induces spontaneous chiral symmetry breaking at arbitrary values of $g > 0$ (see the section IVB), whereas at $\mu_B \neq 0$ chiral symmetry might be broken by $B_\perp$ only at $g > g_c > 0$. ii) If $g > g_c$, then at $\mu_B \neq 0$ the chiral symmetry is allowed to be spontaneously broken only for rather small values of $B_\perp$, i.e. at $B_\perp < B_{\perp c}$, where $0 < B_{\perp c} < \infty$. The symmetry is restored at sufficiently high values of $B_\perp > B_{\perp c}$. In contrast, if the Zeeman interaction is neglected, we have $B_{\perp c} = \infty$ for arbitrary $g > 0$.

To illustrate these circumstances we made some numerical investigations of the TDP (25) at $B_\perp = |\vec{B}|$. For example, we have found that at $g = 2.5 g_c$, $g = 3.5 g_c$ and $g = 5 g_c$ the corresponding critical values $B_{\perp c}$ of the perpendicular magnetic field at which there is a restoration of the chiral symmetry are the following, $e g^2 B_{\perp c} \approx 0.059$, $e g^2 B_{\perp c} \approx 0.518$ and $e g^2 B_{\perp c} \approx 2.04$. Moreover, the behavior of the dynamical electron mass (or the gap) $M_0(B_\perp, \nu)$ vs $B_\perp$ in the particular case $g = 5 g_c$ is presented in Fig. 1. It is clear from this figure that the gap is an increasing function vs $B_\perp$ up to a critical value $B_{\perp c}$, where it vanishes sharply, i.e. the first order phase transition occurs.

The case $B_\perp \neq |\vec{B}|$. Now let us consider the general case when $B_\parallel \neq 0$, i.e. $B_\perp \neq |\vec{B}|$. In this case the mass gap $M_0(B_\perp, \nu)$ is really a function of two independent quantities, $B_\perp$ and $|\vec{B}|$, with an additional evident physical constraint $B_\perp \leq |\vec{B}|$. Investigating properties of the global minimum point of the TDP (25), depending on $B_\perp$ and $|\vec{B}|$, it is possible to obtain a corresponding phase portrait of the model. For a typical value of the parameter $g = 5 g_c$ the phase structure of the model is presented in Fig. 2.

It is clear from the figure that at arbitrary small and perpendicular external magnetic field $\vec{B}$, such that $|\vec{B}| < B_{\perp c}$ (see the previous paragraphs), the system is in the chiral symmetry broken phase 2. Then, the chiral symmetry can be restored by two qualitatively different ways. First, one may increase the strength of $\vec{B}$, or, second, it is possible simply to tilt $\vec{B}$ with respect to a system plane. In the last case, not too high deflection angle $\phi$ of the magnetic field is needed ($\phi \approx 45^\circ$, where $\phi$ is the angle between $\vec{B}$ and the normal to the system plane) in order to restore the symmetry.

### B. Oscillations of the magnetization

Now, let us consider the magnetization $m(|\vec{B}|, B_\perp)$ of the system under influence of an external tilted magnetic field at $g > 0$. At fixed angle $\phi$ between $\vec{B}$ and the normal to the system plane, we define the magnetization by the following relation

$$m(|\vec{B}|, B_\perp) \equiv -\left.\frac{d\Omega^{ren}(M; \nu, B_\perp)}{d|\vec{B}|}\right|_{M=M_0(B_\perp, \nu)},$$

(28)
in the paramagnetic ground state. However, for a sufficiently small values of parameters $e$ and $B$, the situation can be changed, if $\epsilon_B > 0$. For example, in Fig. 3 a graph (see, e.g., Fig. 4), then the magnetization is positive for all physical values of $B$.

where $M_0(B, \nu)$ is the mass gap. Certainly, one should take into account that at fixed $\phi$ the perpendicular component $B_{\perp}$ of the magnetic field is proportional to $|\vec{B}|$, i.e. $B_{\perp} = |\vec{B}| \cos \phi$, so in (28) we have

$$\frac{d}{d|\vec{B}|} = \frac{\partial}{\partial |\vec{B}|} + \frac{B_{\perp}}{|\vec{B}|} \frac{\partial}{\partial B_{\perp}}.$$ (29)

Taking this relation into account, it is possible to obtain

$$m(|\vec{B}|, B_{\perp}) = \frac{B_{\perp}}{|\vec{B}|^2} \frac{\partial}{\partial B_{\perp}} \left[ e\frac{\partial}{\partial |\vec{B}|} \left( M(B, B_{\perp}) \right) \right] \bigg|_{M=M_0(B_{\perp}, \nu)} + \frac{B_{\perp}}{|\vec{B}|} \sum_{n=0}^{\infty} s_n \theta(\nu - \epsilon_n) \left( 2\nu - \frac{\epsilon_n^2 + eB_{\perp}}{\epsilon_n} \right),$$ (30)

where the notations of the expression (25) are used. In the chirally broken phase 2 of Fig. 2 the mass gap $M_0(B_{\perp}, \nu)$ is greater than $\nu$, so the series in (30) does not contribute to the magnetization. As a result, the magnetization $m(|\vec{B}|, B_{\perp})$ is a rather smooth function both over $|\vec{B}|$ and $B_{\perp}$ in the phase 2. However, in the chirally symmetric phase 1 of Fig. 2 we have $M_0(B_{\perp}, \nu) = 0$ and, as a result, a jumping and discontinuous (or oscillating) behavior of the magnetization, which is originated just due to contributions coming from the series terms in (30). Indeed, at $M_0(B_{\perp}, \nu) = 0$ we have for the magnetization in the chirally symmetric phase 1:

$$m(|\vec{B}|, B_{\perp}) \bigg|_{\text{phase 1}} = \frac{eB_{\perp}}{|\vec{B}|} \left[ \frac{3}{\sqrt{2eB_{\perp}}} \left( \zeta(-1/2) + 2\mu_B \right) + \frac{2eB_{\perp}}{\sqrt{2eB_{\perp}}} \sum_{n=1}^{\infty} \theta(\nu - \zeta(-1/2)) \left( 2\nu - \frac{3}{2} \sqrt{2eB_{\perp}} \right) \right],$$ (31)

where $\zeta(-1/2) \approx -0.208$. The plot of the function (31) $m(|\vec{B}|, B_{\perp})$ vs $B_{\perp}$ is presented in Figs 3 and 4 in two particular cases $g = 5g_c$ and $g = 0.5g_c$, correspondingly, at fixed value of $|\vec{B}|$ such that $e|\vec{B}|^2 = 1$. It is clear from these figures that in the region of small values of $B_{\perp}$ the quantity (31) is a highly oscillating function.

Suppose that $|\vec{B}|$ is fixed. Since all terms of the series in (31) are positive quantities, one can conclude that in the region of sufficiently small $B_{\perp}$ both the expression in the square brackets of (31) and the magnetization as a whole are also positive quantities. Hence, at small values of $B_{\perp}$ the ground state of the model is a paramagnetic one. The situation can be changed, if $B_{\perp}$ approaches $|\vec{B}|$. In this case, depending on the relation between dimensionless parameters $e$ and $\mu_B/g$, one can obtain quite different magnetic properties of the ground state. Really, if $\mu_B/g \geq e$ (see, e.g., Fig. 4), then the magnetization is positive for all physical values of $B_{\perp}, 0 \leq B_{\perp} \leq |\vec{B}|$, and the system is in the paramagnetic ground state. However, for a sufficiently small values of $\mu_B/g \ll e$ there is an interval of rather large values of $B_{\perp}$, where both the expression in the square brackets of (31) and the magnetization $m(|\vec{B}|, B_{\perp})$ are negative quantities, so we have in this case a diamagnetic ground state of the system. For example, in Fig. 3 a graph of the magnetization $m(|\vec{B}|, B_{\perp})$ vs $B_{\perp}$ is drown at fixed $|\vec{B}|$ and at $\mu_B/g = 0.1e$. Clearly, in this case the system is in the paramagnetic state if $e|\vec{B}|^2 B_{\perp} < 0.051$, and it is a diamagnetic one at $e|\vec{B}|^2 B_{\perp} > 0.051$.

Clearly, in the range of $B_{\perp}$ values, which is near $|\vec{B}|$, the series in (31) consists of a finite number of nonzero terms (the greater the value of $B_{\perp}$, the less the number of nonzero terms there). So, the behavior of the magnetization vs
$B_\perp$ can be easily determined in this case. However, it is difficult to get any information about the low $B_\perp$ behavior of the magnetization just from the expression (31), since at $B_\perp \to 0$ an infinite number of nonzero terms appears in the sum of (31). The problem is well known in solid state physics \[34, 35\] as well as in relativistic condensed matter systems \[36, 37\], where the magnetic oscillations were investigated. Of course, the corresponding techniques can be used in the present model as well.

To study the $B_\perp \to 0$ asymptotic behavior of the magnetization we apply in (31) the well-known Poisson summation formula \[35\]

$$\sum_{n=0}^{\infty} \alpha_n \Phi(n) = 2 \sum_{k=0}^{\infty} \alpha_k \int_0^\infty \Phi(x) \cos(2\pi k x) dx,$$

(32)

where $\alpha_k = 2 - \delta_{0k}$. Then, after rather tedious calculations it is possible to find the following asymptotic behavior of the magnetization (31) at $B_\perp \to 0$ and arbitrary fixed $|\vec{B}|$ (recall, $\nu = \mu_B |\vec{B}|$):

$$m(|\vec{B}|, B_\perp) = \frac{\mu_B \nu^2}{\pi} + \frac{\mu_B e B_\perp}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{k} \sin \left( \frac{\pi k}{e B_\perp} \nu^2 \right) + o(e B_\perp).$$

(33)

Remark, the leading asymptotic term in this expression, i.e. the first term in the right hand side of (33), is the magnetization corresponding to the TDP (25) with zero $B_\perp$ component of an external magnetic field. Moreover, an infinite series in (33) is no more than Fourier expansion of the periodic function $f(x)$, where $x = \nu^2 / (2e B_\perp)$. Its period is equal to unity and in the interval $0 < x < 1$ it looks like $f(x) = \pi/2 - \pi x$.

Note, in condensed matter systems, both nonrelativistic \[34, 35\] and relativistic \[36–38\], magnetic oscillations usually occur in the presence of chemical potential $\mu$, i.e. in the systems with $\mu = 0$ magnetic oscillations are absent as a rule. However, as it follows from our consideration (see also a more earlier paper \[30\]), in systems with planar structure magnetic oscillations can be induced even at $\mu = 0$ by tilting the external magnetic field with respect to a system plane.

IV. PHASE STRUCTURE OF THE MODEL AT $g < 0$

In the present section we study the influence of an external magnetic field on the properties of the initial model (1) at $g < 0$, i.e. at supercritical values of the bare coupling constant, $G > G_c$. Recall, when the Zeeman interaction is not taken into account the chiral symmetry breaking, induced originally in this case by a rather strong coupling, is enhanced additionally by external magnetic field (see, e.g., in \[13, 19, 22\]). It means that dynamical mass of electrons is an increasing function vs $B_\perp$ throughout the interval $0 < B_\perp < \infty$ (in this case $B_\parallel$ does not influence the properties of the model). It turns out that Zeeman interaction drastically changes properties of the model.

A. The particular case, $|g| = \mu_B / e$.

The case of perpendicular magnetic field. First, let us suppose that external magnetic field $\vec{B}$ is directed normally to a system plane, i.e. $B_\perp = |\vec{B}|$ and $B_\parallel = 0$. For simplicity, we fix the value of $g$ by the relation $|g| = \mu_B / e$. Investigating in this case the TDP (25) as well as the gap equation (26), we have found the behavior of the mass gap $M_0(B_\perp, \nu)$ vs $B_\perp$ (it is the curve 1 in Fig. 5). It turns out that up to a some critical value $B_{\perp c_1}$ (such that $eg^2B_{\perp c_1} \approx 0.81$) the enhancement scenario is realized, i.e. the mass gap is an increasing function vs $B_\perp$. Moreover, in this chirally broken phase the gap $M_0(B_\perp, \nu)$ takes rather large values, such that $M_0(B_\perp, \nu) > \nu$. Consequently, the contribution to the magnetization $m(|\vec{B}|, B_\perp)$ coming from the Zeeman interaction vanishes, i.e. all terms of the series in (30) are zero. As a result, the magnetization in this phase is completely determined by an interaction of $\vec{B}$ with orbital angular momentum. Due to this reason $m(|\vec{B}|, B_\perp)$ is negative at $0 < B_\perp < B_{\perp c_1}$ (see Fig. 5, where the curve 2 corresponds to a magnetization), and the ground state of this phase is a diamagnetic one.

Then, in the critical point $B_\perp = B_{\perp c_1}$ the mass gap $M_0(B_{\perp c_1}, \nu)$ jumps to a significantly smaller nonzero value, and there is a phase transition of the first order to another chirally broken phase. Further increasing of $B_\perp$ leads to a restoration of the chiral symmetry at $B_\perp = B_{\perp c_2}$, where $eg^2B_{\perp c_2} \approx 0.94$. It is a second order phase transition, since in this point the mass gap $M_0(B_{\perp c_2}, \nu)$ continuously turns into zero (see Fig. 5). Note also that both in the second chirally broken phase (at $B_{\perp c_1} < B_\perp < B_{\perp c_2}$) and in the chirally symmetric one (at $B_{\perp c_2} < B_\perp < \infty$) the magnetization of the system is positive, i.e. the ground states of these phases are paramagnetic (see Fig. 5).

The case of tilted magnetic field. Now, a few words about a response of the system with $g < 0$ upon an arbitrarily directed external magnetic field, i.e. when $B_\perp \neq |\vec{B}|$. Numerical investigations of the TDP (25), where for simplicity we put $|g| = \mu_B / e$, bring us to the phase portrait of the model presented in Fig. 6. There the number 1 corresponds to a chirally symmetric paramagnetic phase, whereas notations 2 and 3 are used for two different chirally
broken phases. The first of them, i.e. the phase 2, is a diamagnetic with \( m(|\vec{B}|, B_\perp) < 0 \), however the second one, i.e. the phase 3, is a phase with paramagnetic ground state, since in this region \( m(|\vec{B}|, B_\perp) > 0 \). Note, at \( g < 0 \) one can also observe the oscillations of the magnetization only in the chirally symmetric phase 1 when \( B_\perp \to 0 \).

As it is clear from Figs 5 and 6 the presence of the Zeeman interaction significantly changes the behavior of the chiral symmetry under influence of an external both perpendicular and tilted magnetic field at \( g < 0 \). Indeed, at \( \mu_B \neq 0 \) the enhancement of a chiral condensation in this case takes place only at sufficiently small values of \(|\vec{B}|\), i.e. in the phase 2 of Fig. 6 (it means that fixing the tilting angle of the magnetic field we obtain the growth of the mass gap \( M_0(B_\perp, \nu) \) at increasing \(|\vec{B}|\)). Further increasing of \(|\vec{B}|\) leads ultimately to a chiral symmetry restoration.

**B. Phase structure in the general case**

Clearly, for other relations between \(|g|\) and \( \mu_B \), i.e. at \(|g| \neq \mu_B/e\), the \((eg^2|\vec{B}|, eg^2B_\perp)-phase portrait of the model might be quite different from Fig. 6. To imagine the phase structure of the model for an arbitrary, but fixed, relation between \(|g|\) and \( \mu_B \) it is very convenient to use for its description the new dimensionless parameters, \( x = \mu_B/|\vec{B}||g| \) and \( y = eg^2B_\perp \).

Assuming for a moment that \( x \) and \( y \) are fully independent quantities, it is possible to investigate the behavior of the global minimum point of the TDP (25) as a function of \( x \) and \( y \) and then to obtain the \((x, y)-phase portrait of the model depicted in Figs 7 and/\ or 8. (The line L of these figures should be ignored in this case. Note also that in Fig. 8 the phase portrait is depicted for a more extended region of the parameter \( y \).) There one can see only three different phases which were already presented in Fig. 6. So we use the same notations for them, 1, 2 and 3. In reality, there is a constraint between \( x \) and \( y \) which is due to the physical requirement \( B_\perp \leq |\vec{B}| \). In terms of \( x \) and \( y \) it looks like \( y \leq cx \), where \( c = e|g|/\mu_B \), i.e. not the whole \((x, y)\)-planes of Figs 7 and 8 can be considered as a phase diagram, but only those areas which are below the line L. The points of the line L correspond to a perpendicular external magnetic field, i.e. we have \( B_\perp = |\vec{B}| \) on the line L. Clearly, if the quantity \( c = e|g|/\mu_B \) varies, then the line L of Figs 7 and 8 changes its slope and, as a result, the allowed physical region which is below L is also changed. However, the positions and forms of the critical curves in Figs 7, 8 are not changed at different values of the parameter \( c \).

It is easily seen from Fig. 8 that inside the interval \( 3 < y < 11 \) the critical curve \( l \) of the phase diagram can be approximated by a straight line with a slope coefficient \( c^* \approx 28 \). Extrapolating this behavior of the curve \( l \) to

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5 Strictly speaking, only \( x \) is a new parameter, since \( y \) was already used in Fig. 6 etc.
region with higher $y$-values, one can conclude that a typical phase portrait of the initial model corresponding to the weak coupling $|g|$, such that $c = |g|/\mu_B < c^*$, is presented in Fig. 7 (it is the region just below the line L). In this case the line L certainly crosses critical curve $l$ of a phase portrait, i.e. it passes through several different phases, including the chirally symmetric phase $1$. As a result, one can see that at $c < c^*$ the chiral symmetry is always restored at $|\vec{B}| \rightarrow \infty$ irrespective of the magnetic field directions (even at a perpendicular magnetic field). In particular, the case $c = 1$ was considered in details in the previous section IV A, and Fig. 7 at $c = 1$ coincides with the phase diagram of Fig. 6.

In contrast, if $c > c^*$ then a typical phase portrait of the model is depicted in Fig. 8 (it is a region which is also below and/or to the right of the line L). Clearly, in this case the line L does not cross any of the critical curves of the phase diagram, and at arbitrary values of a perpendicular magnetic field the chirality cannot be restored, since we move along the line L when $B_{\perp} = |\vec{B}|$ increases. However, if $|\vec{B}| \rightarrow \infty$ at fixed $|\vec{B}|$ it is also possible to restore the symmetry by tilting the magnetic field away from the normal direction. In particular, if the parameter $x$ lies, e.g., in the interval $0.7 < x < 1.4$ (see Fig. 8), then a number of phase transitions can occur in the system that are also caused only by the inclination of an external magnetic field.

### C. Numerical estimates in the context of condensed matter physics

Now let us estimate the order of magnitude of the magnetic field at which the phase transitions of Figs 6, 7, 8 might take place in (2+1)-dimensional condensed matter systems. To this end it is necessary to take into account in the Lagrangian (1) the Fermi velocity of quasi-particles $v_F \neq 1$ (see footnote 1 of the present paper). Using the same calculational technique as in Sec. II of the present paper and/or, e.g., in [24, 30, 39], it is possible to obtain the thermodynamic potential $\Omega_{v_F}$ for the case $v_F \neq 1$. Indeed, there is a very simple connection between $\Omega_{v_F}$ and the new renormalized TDP [24] corresponding to $v_F = 1$. Namely, one should perform in (24) the replacements $eB_{\perp} \rightarrow ev_F^2B_{\perp}$, $g \rightarrow g/v_F$ (note, the Zeeman term $\mu_B|\vec{B}|$ remains unchanged in this case) and then multiply the obtained expression by the factor $1/v_F^2$.

Suppose that $g < 0$ (recall, we still fix the spectroscopic Lande factor $g_S$ by the relation $g_S = 2$, as in graphene). Then, in the particular case of $\vec{B} = 0$ the TDP $\Omega_{v_F}$ thus obtained from the TDP $V(M)$ [17] of the case $v_F = 1$ has already the global minimum at the point $M_{0_F} \equiv -v_F/g$ (it is the mass gap of the system). Since in all numerical calculations of the case $v_F = 1$ an arbitrary dimensionless quantity is converted into a dimensionless one by multiplying it with an appropriate powers of $|g|$, in the case $v_F \neq 1$ the powers of $|g|/v_F$ should be used instead. So, at $v_F \neq 1$ the analogs of the $(x, y)$-phase diagrams of Figs 7, 8 are the same as the figures, but with the new $x_F$, $y_F$, where $x_F = x/v_F \equiv \mu_B|\vec{B}|g/v_F$, and $y_F = y$. (In the following, when referring to Figs 7, 8 in the case $v_F \neq 1$, we imply that instead of $x$ and $y$ the new parameters $x_F$ and $y_F$ should be used in these figures.) The line L, below which
the physical region is arranged, has the form
\[ y_F = c_F x_F, \]
where \( c_F \equiv v_F = e|g|v_F/\mu_B = c_F^2/(\mu_B M_0^2) \). It is clear from Figs 7, 8 that at \( B_{\perp} = 0 \) and \( v_F \neq 1 \) the phase transition of the first order occurs at in-plane magnetic field \(|B_0|\) corresponding to \( x_F = 1 \), i.e. \(|B_0| = v_F/(|g|\mu_B) = M_0/\mu_B \). Since the value of the mass gap \( M_0 \) in condensed matter systems is typically of the order of 1-10 meV, one can easily obtain that the magnitude of the critical magnetic field \(|B_0|\) is of order of 14-140 Teslas, correspondingly.\(^{\text{[6]}}\) It is clear from Figs 7, 8 that at \( B_{\perp} \neq 0 \) the magnitudes of \(|B_0|\) at which one can observe phase transitions, are even less and might be as small as 0.7\(|B_0|\).

If \( v_F = 1/300 \) and \( g_S = 2 \), as in graphene, then the slope factor \( c_F \) of the line L is approximately equal to \( 10^3 \) at \( M_0F = 10 \) meV, whereas it is of order of \( 10^4 \) at \( M_0F = 1 \) meV, i.e. \( c_F \gg c^* \approx 28 \). Hence, just the phase diagram of Fig. 8 refers to graphene-like planar systems. In this case (recall, we assume that the critical line \( l \) of the figure can be extrapolated to the region of high \( y \)-values by a straight line with the slope coefficient \( c^* \approx 28 \)) chiral symmetry cannot be restored by an arbitrary strong external perpendicular magnetic field, and the enhancement effect is realized at \( B_{\perp} \lesssim |\vec{B}| \). However, tilting the magnetic field away from a normal of the plane, it is possible to restore the symmetry, if \(|\vec{B}| > 0.7|B_0|\). The angle \( \varphi_0 \) between \( \vec{B} \) and the plane of the system, at which the restoration of the symmetry occurs, can be estimated numerically. For example, at \(|\vec{B}| = 1.5|B_0|\) (i.e. at \( x_F = 1.5 \)) the \( B_{\perp}\)-component, at which the chiral symmetry is restored, corresponds to the value \( y \approx 30 \) (the point \((1.5, 30)\) lies on the critical curve \( l \) of Fig. 8). Hence, \( \sin \varphi_0 = B_{\perp}/|\vec{B}| \approx 30/(1.5c_F) \). It means that at \( M_0F = 10 \) meV and \( M_0F = 1 \) meV we have \( \sin \varphi_0 \approx 0.02 \) and \( \sin \varphi_0 \approx 0.002 \), correspondingly, i.e. the restoration of the chiral symmetry occurs at very weak \( B_{\perp}\)-components of the magnetic field.

It was noted in the paper \(^{\text{[30]}}\), where the properties of the planar system of free electrons were investigated, that in the gapless semiconductors Fermi velocity \( v_F \) may vary in the interval \( v_F \in (1/3000, 1/300) \), whereas in our numerical estimates we use the following relations (see, e.g., in \(^{\text{[39]}}\)): \( \mu_B = e/(2m_e) \), where \( m_e \) is the electron rest mass, \( m_e \approx 0.5 \) MeV; 1 Tesla \( \approx 700 \) eV; \( c \approx 1/\sqrt{137} \), as in graphene.

V. SUMMARY AND CONCLUSIONS

In the present paper we investigate (at zero temperature and chemical potential) the response of the \((2+1)\)-dimensional GN model (1) upon the action of external magnetic field \( \vec{B} \). The model describes a four-fermion self-interaction of quasi-particles (electrons) with spin 1/2. In addition, it describes the interaction of \( \vec{B} \) both with orbital angular momentum of electrons and with their spin. The last is known as the Zeeman interaction, and it is proportional to electron magnetic moment \( \mu_B \) which is a free model parameter in our consideration. So at \( \mu_B = 0 \) the properties of the model were considered, e.g., in \(^{\text{[13, 19, 22]}}\), where in particular it was established that an external perpendicular magnetic field \( \vec{B}_{\perp} \) induces spontaneous chiral symmetry breaking at \( G < G_c \), or it enhances chiral condensation at \( G > G_c \). (Such an ability of an external magnetic field is called the magnetic catalysis effect.) Moreover, in this case the system responds diamagnetically on the influence of external magnetic field, i.e. its magnetization is negative. In addition, there are no magnetic oscillations of any physical quantity if the Zeeman interaction is not taken into account.

In the paper we study the modifications that appear both in the magnetic catalysis effect and in the magnetization phenomena of the system when Zeeman interaction is taken into consideration, i.e. at \( \mu_B \neq 0 \). To this end, we have obtained in the leading order of the large-\( N \) expansion technique the renormalized thermodynamic potential

\[ F = 10 \]
The behavior of the global minimum point of this quantity with respect to $M$ defines the phase structure of the model, whereas its derivative with respect to $|\vec{B}|$ gives us the magnetization. Note also that the renormalized TDP depends no more on the bare coupling $G$. Instead, it appears the dependence of the TDP on the new finite parameter $\nu$, which is connected with $G$ by the relation \[ \nu = \mu B|\vec{B}|. \] The main results of our investigations are the following.

i) We have found that at $\mu_B \neq 0$ and $g > 0$ there is a critical coupling constant $g_c = 2\mu_B/e$ such that at $g > g_c$, an arbitrary rather weak external magnetic field $\vec{B}$ induces spontaneous chiral symmetry breaking provided that there is not too great a deviation of $\vec{B}$ from a vertical as well as that $|\vec{B}| < B_c(g)$, where $0 < B_c(g) < \infty$ (see Fig. 2). At $0 < g < g_c$, chiral symmetry cannot be broken by an external magnetic field. (In contrast, at $\mu_B = 0$ and any values of $g > 0$ the chiral symmetry breaking is induced by arbitrary external magnetic field $\vec{B}$ such that $\vec{B}_\perp \neq 0$.)

ii) Suppose that $\mu_B \neq 0$, $g > g_c > 0$ and chiral symmetry is broken, i.e. $\vec{B}$ has a rather large $B_\perp$ component. Then chiral symmetry can be restored simply by tilting magnetic field to a system plane, i.e. without any increase of its modulus $|\vec{B}|$.

iii) We have shown that at $\mu_B \neq 0$, $g > 0$ and arbitrary fixed $|\vec{B}| \neq 0$ one can observe oscillations of the magnetization in the region of small values of $B_\perp$ (see Figs 3 and 4). Note, de Haas – van Alphen magnetic oscillation phenomenon is rather typical for condensed matter physics as well as for dense relativistic matter. It occurs usually at nonzero chemical potential. In contrast, in our (2+1)-dimensional system this phenomenon is induced (at zero chemical potential) by tilting an external magnetic field only.

iv) If $\mu_B \neq 0$ and $g < 0$, then the phase structure and magnetic properties of the model are much richer than in the case of $\mu_B = 0$, $g < 0$. Indeed, it is clear from Figs 6, 7 and 8 that at non-vanishing Zeeman interaction the phase portrait of the model contains at least two chirally nonsymmetric phases, denoted as 2 and 3. In the phase 2, which is a diamagnetic one, the enhancement of the chiral symmetry is occurred, whereas in the paramagnetic phase 3 it is absent. Moreover, if in addition the parameter $c \equiv g/\mu B < c^* \approx 28$, then at sufficiently high values of $|\vec{B}|$ (even at a perpendicular magnetic field) the restoration of the chiral symmetry is occurred in the model. In contrast, at $\mu_B = 0$ and $g < 0$ only the diamagnetic phase 2 with enhancement of the chiral symmetry breaking is realized in the model at arbitrary values and directions of $\vec{B}$, such that $B_\perp > 0$.

v) Assuming that the critical line $l$ of Fig. 8 can be extrapolated to the region $y \equiv eg^2B_\perp > 11$ by a straight line with a slope coefficient $c^* \approx 28$, we see that at $g < 0$ and $c \equiv g/\mu B > c^*$ the line L of Fig. 8 does not cross any of the critical curves of the figure. So, in this case at an arbitrary perpendicular magnetic field chiral symmetry cannot be restored. However, tilting the magnetic field away from a normal position, it is possible to restore the symmetry. As our numerical estimates show (see in Sec. IV C), just this situation is typical for graphene-like planar systems.

vi) Look again at Fig. 8, where $c > c^*$, and fix $|\vec{B}|$, e.g., in the interval such that $0.8 < x \equiv \mu B|\vec{B}|/|g| < 1$. Then at $B_\perp \lesssim |\vec{B}|$ the chirally broken phase 2 is realized in the model. It is clear from the figure that decreasing the value of $B_\perp$ (or simply tilting $\vec{B}$), it is possible in this case to restore the chiral symmetry. However, a further reduction of $B_\perp$ leads to a transition of the system to the chirally broken phases 3 and 2. In contrast, if $|\vec{B}|$ is fixed in the interval corresponding to $1 < x < 1.3$, then after a chiral symmetry restoration, taking place for certain intermediate values of $B_\perp$, there should appear the chiral symmetry broken phase 3 and then, finally, again the symmetric phase of the model. Hence, at some fixed values of $|\vec{B}|$ a series of phase transitions, following one after another, can be caused in the system simply by changing the inclination of an external magnetic field.

While our work was in the preparation, we learned about the paper, where the same model with a tilted magnetic field has been investigated. However, in quite other properties of the model, such as quantum Hall effect etc, were studied at nonzero temperature and chemical potential.

\[ \Omega_{\text{TDP}}(M; \nu, B_\perp) \] where $\nu = \mu B|\vec{B}|$. [25]
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