Entanglement, Particle Identity
and the GNS Construction: A Unifying Approach

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Abstract

A novel approach to entanglement, based on the Gelfand-Naimark-Segal (GNS) construction, is introduced. It considers states as well as algebras of observables on an equal footing. The conventional approach to the emergence of mixed from pure ones based on taking partial traces is replaced by the more general notion of the restriction of a state to a subalgebra. For bipartite systems of nonidentical particles, this approach reproduces the standard results. But it also very naturally overcomes the limitations of the usual treatment of systems of identical particles. This GNS approach seems very general and can be applied for example to systems obeying para- and braid- statistics including anyons.
INTRODUCTION

In spite of the numerous efforts to achieve a satisfactory understanding of entanglement for systems of identical particles, there is no general agreement on the appropriate generalization of concepts valid for non-identical constituents [1]. That is because many concepts are usually only discussed in the context of quantum systems for which the Hilbert space $\mathcal{H}$ is a simple tensor product with no additional structure. An example is the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of two non identical particles. In this case the partial trace $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$ for $|\psi\rangle \in \mathcal{H}$ to obtain the reduced density matrix has a good physical meaning: it corresponds to observations only on the subsystem $A$.

In contrast, the Hilbert space of a system of $N$ identical bosons (fermions) is given by the symmetric (antisymmetric) $N$-fold tensor product of the single-particle spaces. The consequence is that any multi-particle state contains intrinsic correlations between subsystems due to quantum indistinguishability. This, in turn, forces a departure from the straightforward application of entanglement-related concepts like singular value decomposition (SVD), Schmidt rank, or entanglement entropy.

In this context, Schliemann et al. [2] have introduced an analogue of the Schmidt rank, the ‘Slater rank’ to study entanglement in two-fermion systems, using a new version of the SVD adapted to deal with antisymmetric matrices. The extension of these ideas to the boson case was worked out in [3] and [4]. These approaches also have not found general acceptance.

The problems arising in the interpretation of the Slater rank and the von Neumann entropy of the reduced density matrix (obtained by partial tracing) for these systems have been analyzed in [5] [6].

Numerous other proposals for the treatment of identical particles have recently been put forward. But, as a closer look at the literature on the subject [1–20] reveals, it is apparent that there is no consensus yet as to what the proper formalism should be.

In this paper, we propose an approach to the study of entanglement based on the theory of operator algebras. The foundational results of Gelfand, Naimark and Segal on the representation theory of $C^*$-algebras, abbreviated as the GNS-construction [21] [23] are used in order to obtain a generalized notion of entanglement. In particular, the notion of partial trace is replaced by the much more general notion of restriction of a state to a subalgebra [24].
This will allow us to treat entanglement of identical and non-identical particles on an equal footing, without the need to resort to different separability criteria according to the case under study.

In order to display the usefulness of our approach, several explicit examples will be worked out. In particular we show that the GNS-construction gives zero for the von Neumann entropy of a fermionic or a bosonic state containing the least possible amount of correlations. We believe that this settles an issue that has caused a lot of confusion regarding the use of von Neumann entropy as a measure of entanglement for identical particles [3, 6, 8, 17, 25].

**THE BASIC IDEA**

**Preliminary Remarks**

A vector state of a quantum system is usually described by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$ (pure case). More generally, a state is a density matrix $\rho: \mathcal{H} \to \mathcal{H}$, a linear map satisfying $\text{Tr} \rho = 1$ (normalization), $\rho^\dagger = \rho$ (self-adjointness) and positivity $\rho \geq 0$. For pure states the additional condition $\rho^2 = \rho$ is required, which amounts to the assertion that $\rho$ is of the form $|\psi\rangle\langle\psi|$ for some normalized vector in $\mathcal{H}$.

Now, given that the expectation value of an observable $\mathcal{O}$ is defined by $\langle \mathcal{O} \rangle = \text{Tr}(\rho \mathcal{O})$, we can equivalently regard $\rho$ as a **linear functional** $\omega_\rho: \mathcal{A} \to \mathbb{C}$ on a unital ($C^*$-) algebra $\mathcal{A}$ of observables with unity $1_\mathcal{A}$ (we consider only unital algebras). That is, $\omega_\rho(\mathcal{O}) \in \mathbb{C}$, for $\mathcal{O} \in \mathcal{A}$. The normalization and positivity conditions above then take the form $\|\omega_\rho\| := \omega_\rho(1_\mathcal{A}) = 1$ and $\omega_\rho(\mathcal{O}^\dagger \mathcal{O}) \geq 0$ (for any $\mathcal{O} \in \mathcal{A}$). Such a positive linear functional of unit norm is called a **state** on the algebra $\mathcal{A}$.

As already mentioned, in the bipartite case $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the definition of $\rho_A$ involves a partial trace operation. It is therefore natural to ask for a characterization of this operation in terms of the notion of states on an algebra. For this purpose consider the subalgebra $\mathcal{A}_0$ consisting of all operators of the form $K \otimes 1_B$, for $K$ an observable on $\mathcal{H}_A$. Defining a new state $\omega_0: \mathcal{A}_0 \to \mathbb{C}$ as the **restriction** of $\omega_\rho$ to the subalgebra $\mathcal{A}_0$, one easily checks that for any observable $K$ of $\mathcal{H}_A$ the equality $\omega_0(K \otimes 1_B) \equiv \text{Tr}_A(\rho_K K)$ holds.

As we show below, an algebraic description of the quantum system where the basic objects are a $C^*$-algebra $\mathcal{A}$ and a state $\omega$ on the algebra provides the solution to some of...
the problems that appear when $\mathcal{H}$ does not have the form of a ‘simple tensor product’.

The GNS Construction

The basic idea of the GNS-construction is that given an algebra $\mathcal{A}$ of observables and a state $\omega$ on this algebra, we can construct the Hilbert space on which the algebra of observables acts on. The key steps are: (a) Using $\omega$, we can endow $\mathcal{A}$ itself with an inner product. So it becomes an ‘inner product’ space $\hat{\mathcal{A}}$. (b) This inner product may be degenerate in the sense that the norm of some non-null elements of $\hat{\mathcal{A}}$ may be zero. (c) If we remove these null vectors by taking the quotient of $\hat{\mathcal{A}}$ by the null space $\mathcal{N}$ of zero norm vectors to get $\hat{\mathcal{A}}/\mathcal{N}$, then we have a well-defined positive definite inner product on $\hat{\mathcal{A}}/\mathcal{N}$. Hence we get a well-defined Hilbert space (after completion). (d) The algebra of observables $\mathcal{A}$ acts naturally on this Hilbert space in a simple manner.

We now make this set of ideas more precise.

From the mathematical point of view, the algebra of observables is a $C^*$-algebra. This guarantees that one has enough structure to perform all the tasks in the list above.

A $C^*$-algebra is a (complete normed) algebra $(\mathcal{A}, \| \cdot \|)$, together with an antilinear *-involution $\alpha \mapsto \alpha^*$, such that the basic property $\| \alpha^* \alpha \| = \| \alpha \|^2$ is satisfied for all $\alpha$ in $\mathcal{A}$. The prototypical example of a $C^*$-algebra is the algebra $\mathcal{B} (\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$, with the involution given by the adjoint: $\alpha^* = \alpha^\dagger$. Here we will only be interested in unital algebras, that is, we assume the existence of a unit $1_\mathcal{A}$ for the algebra.

Given a state $\omega$ on a $C^*$-algebra $\mathcal{A}$, we can obtain a representation $\pi_\omega$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ as follows. Since $\mathcal{A}$ is an algebra, it is in particular a vector space. When elements $\alpha \in \mathcal{A}$ are regarded as elements of a vector space $\hat{\mathcal{A}}$ we write them as $|\alpha\rangle$. We then set $\langle \beta | \alpha \rangle = \omega (\beta^* \alpha)$. This is almost a scalar product, $\langle \alpha | \alpha \rangle \geq 0$, but there could be a null space $\mathcal{N}_\omega$ of zero norm vectors: $\mathcal{N}_\omega = \{ \alpha \in \mathcal{A} | \omega (\alpha^* \alpha) = 0 \}$. Schwarz inequality shows that $\mathcal{N}_\omega$ is a left ideal:

$$ a \mathcal{N}_\omega \subseteq \mathcal{N}_\omega, \quad \forall a \in \mathcal{A}. \quad (1) $$

It also shows that

$$ \langle a | \alpha \rangle = 0, \quad \forall a \in \mathcal{A}, \alpha \in \mathcal{N}_\omega. \quad (2) $$

We denote the elements of the quotient space $\mathcal{H}_\omega = \hat{\mathcal{A}}/\mathcal{N}_\omega$ by $[\alpha]$, where $[\alpha] = \alpha + \mathcal{N}_\omega$.
∀α ∈ A. It has a well-defined scalar product

\[ \langle [\alpha] | [\beta] \rangle = \omega(\alpha^* \beta) \]  

(3)

(it is independent of the choice of α from [α] because of (1)) and no nontrivial null vectors. Moreover we have a representation \( \pi_\omega \) of \( A \) on \( H_\omega : \pi_\omega(\alpha) [\beta] = [\alpha \beta] \). (To show this, use \( \alpha N_\omega \in N_\omega \)).

This representation \( \pi_\omega \) is in general reducible. We decompose \( H_\omega \) into a direct sum of irreducible spaces:

\[ H_\omega = \bigoplus_i H_i \]  

(6)

where \( \sum_i P_i \) is the corresponding orthogonal projections and define

\[ \mu_i = \| P_i [1_A] \| \quad (\mu_i > 0 \text{ always}) \quad \text{and} \quad [\chi_i] = (1/\mu_i) P_i [1_A]. \]  

(4)

Observe that

\[ \langle [\chi_i] | [\chi_j] \rangle = \delta_{ij}, \]  

(5)

and also that

\[ \omega(\alpha) = \langle [1_A] | \pi_\omega(\alpha) [1_A] \rangle, \quad [1_A] = \sum_i P_i [1_A]. \]  

(6)

One then obtains

\[ \omega(\alpha) = \text{Tr}_{H_\omega} (\rho_\omega \pi_\omega(\alpha)), \]  

(7)

where \( \rho_\omega \) is a density matrix on \( H_\omega \), given by

\[ \rho_\omega = \sum_i \mu_i^2 [\chi_i] \langle [\chi_i] |. \]  

(8)

This follows from

\[ \sum_{ij} \mu_j \mu_i \langle [\chi_i] | \pi_\omega(\alpha) | [\chi_j] \rangle = \sum_i \mu_i^2 \langle [\chi_i] | \pi_\omega(\alpha) | [\chi_i] \rangle, \]  

(9)

as \( \pi_\omega(\alpha) \) has zero matrix elements between different irreducible subspaces.

This is a crucial fact. Since \( [\chi_i] \langle [\chi_i] | \) is a pure state, it shows that \( \omega \) is pure if and only if the representation \( \pi_\omega : A \to \mathcal{B}(H_\omega) \) is irreducible, a well-known result. In particular, the von Neumann entropy of \( \omega \), \( S(\omega) = -\text{Tr}_{H_\omega} \rho_\omega \log \rho_\omega \), is zero if and only if \( H_\omega \) is irreducible. The latter is a property that depends on both the algebra \( A \) and the state \( \omega \).

Consider now a (unital) subalgebra \( A_0 \subset A \) of \( A \) and let \( \omega_0 \) denote the restriction to \( A_0 \) of a pure state \( \omega \) on \( A \) [24]. We can apply the GNS-construction to the pair \( (A_0, \omega_0) \) and use the von Neumann entropy of \( \omega_0 \) to study the entropy which arises from the restriction.
Example 1: $M_2(\mathbb{C})$

In order to illustrate the above GNS-construction, consider the algebra $\mathcal{A} = M_2(\mathbb{C})$ of $2 \times 2$ complex matrices. Denoting by $e_{ij}$ the $2 \times 2$ matrix with one on its $(i,j)$ entry and zero elsewhere, we can write any $\alpha \in \mathcal{A}$ as $\alpha = \sum_{i,j \in \{1,2\}} \alpha_{ij} e_{ij}$.

It is readily checked that the map $\omega_\lambda : \mathcal{A} \to \mathbb{C}$ given by $\omega_\lambda(\alpha) = \lambda \alpha_{11} + (1 - \lambda) \alpha_{22}$ defines a state on the algebra, as long as $0 \leq \lambda \leq 1$. The vector space $\mathcal{H}_{\omega_\lambda}$ is generated by the four vectors $|e_{ij}\rangle$ ($i,j = 1,2$) and the null space $\mathcal{N}_{\omega_\lambda}$ is generated by those $\alpha$ such that $\omega_\lambda(\alpha^* \alpha) = 0$. Since

$$\alpha^* \alpha = \sum_{ijk} \bar{\alpha}_{ki} \alpha_{kj} |i\rangle \langle j|,$$

the explicit form of the null vector condition is:

$$\lambda(|\alpha_{11}|^2 + |\alpha_{21}|^2) + (1 - \lambda)(|\alpha_{12}|^2 + |\alpha_{22}|^2) = 0. \quad (11)$$

Case 1: $0 < \lambda < 1$

For $0 < \lambda < 1$ the only solution to this equation is $\alpha = 0$, implying that there are no null vectors: $\mathcal{N}_{\omega_\lambda} = \{0\}$. Therefore, in this case the GNS-space is given by $\mathcal{H}_{\omega_\lambda} \cong \mathbb{C}^4$. The matrices $e_{ij}$ act on this space as:

$$\pi_{\omega_\lambda}(e_{ij}) |[e_{kl}]\rangle = \delta_{jk} |[e_{il}]\rangle. \quad (12)$$

The matrix of $\pi_{\omega_\lambda}(e_{11})$ for instance is:

$$\pi_{\omega_\lambda}(e_{11}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (13)$$

when we order the basis as $e_{11}, e_{12}, e_{21}, e_{22}$.

This representation is clearly reducible, the subspaces $\mathcal{H}^{(l)}$ ($l = 1,2$) spanned by $\{|[e_{kl}]\rangle\}_{k=1,2}$ being invariant. Hence $\pi_\omega$ is the direct sum of two isomorphic irreducible representations, the corresponding decomposition of $\mathcal{H}_{\omega_\lambda}$ being

$$\mathcal{H}_{\omega_\lambda} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}. \quad (14)$$

Next we decompose $\omega_\lambda$ into pure states. The unity $1_\mathcal{A}$ of $\mathcal{A}$ is just $1_2$ so that

$$|[1_\mathcal{A}]\rangle = |[e_{11}]\rangle + |[e_{22}]\rangle \quad (15)$$
gives the decomposition of $| [I_A] \rangle$ into irreducible subspaces. The norms of the two components are $\sqrt{\lambda}$ and $\sqrt{1 - \lambda}$ by (3). So

$$
| [I_A] \rangle = \sqrt{\lambda} | [x_1] \rangle + \sqrt{1 - \lambda} | [x_2] \rangle, \quad \langle [x_i] | [x_j] \rangle = \delta_{ij},
$$

(16)

with $| [x_i] \rangle$ as in (4). It follows that

$$
\rho_{\omega \lambda} = \lambda | [x_1] \rangle \langle [x_1] | + (1 - \lambda) | [x_2] \rangle \langle [x_2] |,
$$

(17)

so that $\omega \lambda$ is not pure. It has von Neumann entropy

$$
S(\omega \lambda) = -\lambda \log \lambda - (1 - \lambda) \log(1 - \lambda).
$$

(18)

Case 2: $\lambda = 0$ or 1

If we choose $\lambda = 0$, from (11) we see that $\mathcal{N}_{\omega \lambda} \cong \mathbb{C}^2$, since it is spanned by elements of the form

$$
\alpha = \begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{21} & 0 \end{pmatrix},
$$

(19)

that is, by linear combinations of $| e_{11} \rangle$ and $| e_{21} \rangle$. Accordingly, the GNS-space $\mathcal{H}_{\omega \lambda} = \hat{A}/\mathcal{N}_{\omega \lambda} \cong \mathbb{C}^2$ is generated by $| [e_{12}] \rangle$ and $| [e_{22}] \rangle$. In this case the representation of $\mathcal{A}$ is irreducible and given by $2 \times 2$ matrices $\pi_{\omega \lambda} (e_{ij})$:

$$
\pi_{\omega \lambda} (e_{ij}) | [e_{k2}] \rangle = \delta_{jk} | [e_{i2}] \rangle.
$$

(20)

The state $\omega \lambda$ is pure with zero entropy. A similar situation is found for $\lambda = 1$.

Example 2: Bell State

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$ and consider the state vector

$$
| \psi \rangle = \frac{1}{\sqrt{2}} (| + \rangle \otimes | - \rangle - | - \rangle \otimes | + \rangle).
$$

(21)

Then $| \psi \rangle \langle \psi |$ can be thought of as a state $\omega$ on the algebra $\mathcal{A}$ of linear operators on $\mathcal{H}$. This algebra is isomorphic to the $4 \times 4$ matrix algebra $M_4(\mathbb{C})$ and is generated by elements of the form $\sigma_\mu \otimes \sigma_\nu (\mu, \nu = 0, 1, 2, 3)$, with $\sigma_0 = 1_2$ and $\{ \sigma_1, \sigma_2, \sigma_3 \}$ the Pauli matrices.

In this context, the entanglement of $| \psi \rangle$ is understood in terms of correlations between “local” measurements performed separately on subsystems $A$ and $B$. Measurements performed on $A$ correspond to the restriction of $\omega$ to the subalgebra $\mathcal{A}_A \subset \mathcal{A}$ generated by elements of the form $\sigma_\mu \otimes 1_2$. 

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We now study this case. We set $\omega_A = \omega |_{A_A}$. In order to construct the GNS-space $\mathcal{H}_{\omega_A}$, we first notice that $\omega_A((\sigma_\mu \otimes 1_2)^*(\sigma_\nu \otimes 1_2)) = \omega(\sigma_\mu^* \sigma_\nu \otimes 1_2) = \langle \psi | \sigma_\mu^* \sigma_\nu \otimes 1_2 | \psi \rangle = \delta_{\mu\nu}$, so that in this case there are no nontrivial null states, that is, $N_{\omega_A} = \{0\}$. We obtain $\mathcal{H}_{\omega_A} = \hat{A}_A/N_{\omega_A} \cong \mathbb{C}^4$, with basis vectors $[[\sigma_\mu]] \equiv [[\sigma_\nu \otimes 1_2]]$ and inner product $\langle [[\sigma_\mu]] | [[\sigma_\nu]] \rangle = \delta_{\mu\nu}$.

The action of $\mathcal{A}_A$ on $\mathcal{H}_{\omega_A}$ is given, as explained above, by linear operators $\pi_{\omega_A}(\alpha)$:

$$\pi_{\omega_A}(\alpha) [[\beta]] = [[\alpha \beta]].$$

(22)

The RHS can be explicitly computed using the identity $\sigma_i \sigma_j = \delta_{ij} 1_2 + i \varepsilon_{ijk} \sigma_k$. One then finds that the GNS space splits into the sum of two invariant subspaces:

$$\mathcal{H}_{\omega_A} = \mathbb{C}^2 \oplus \mathbb{C}^2.$$

(23)

They are spanned by

$$\left\{ [[\sigma_+ \otimes 1_2]], \quad [(1/2)(1 - \sigma_3) \otimes 1_2]] \right\}$$

(24)

and

$$\left\{ [[(1/2)(1 + \sigma_3) \otimes 1_2]], \quad [[\sigma_- \otimes 1_2]] \right\},$$

(25)

where

$$\sigma_\pm = \sigma_1 \pm i \sigma_2.$$

(26)

The corresponding projections are

$$P_i = \frac{1}{2} \pi_{\omega_A}(1_A + (-1)^i \sigma_3 \otimes 1_2), \text{ with } i = 1, 2.$$

(27)

We obtain

$$\mu_i^2 = \|P_i[[1_A]]\|^2 = \frac{1}{2} \omega_A(1_A + (-1)^i \sigma_3 \otimes 1_2) = \frac{1}{2},$$

(28)

$$[[\chi_i]] = \frac{1}{\sqrt{2}} ([[\sigma_0]] + (-1)^i [[\sigma_3]]), \quad \text{with} \quad \langle [[\chi_i]] | [[\chi_j]] \rangle = \delta_{ij}.$$

(29)

Thus, the representation of $\omega_A$ as a density matrix on the GNS-space $\mathcal{H}_{\omega_A}$ is:

$$\rho_{\omega_A} = \frac{1}{2} [[\chi_1]] \langle [[\chi_1]] | + \frac{1}{2} [[\chi_2]] \langle [[\chi_2]].$$

(30)

The von Neumann entropy computed via the GNS-construction is therefore

$$S(\omega_A) = \log 2,$$

(31)

reproducing the standard result, as expected.
SYSTEMS OF IDENTICAL PARTICLES

Let $\mathcal{H}^{(1)} = \mathbb{C}^d$ be the Hilbert space of a one-particle system. The group $U(d) = \{g\}$ acts on $\mathbb{C}^d$ by the representation $U^{(1)}$ and the algebra of observables is given by a $\ast$-representation of the group algebra $\mathbb{C}U(d)$ on $\mathcal{H}^{(1)}$. Its elements are of the form

$$\tilde{\alpha} = \int_{U(d)} d\mu(g) \alpha(g) U^{(1)}(g), \quad (32)$$

where $\alpha$ is a complex function on $U(d)$ and $\mu$ the Haar measure \cite{26}.

The elements $\tilde{\alpha}$ span the matrix algebra $M_d(\mathbb{C})$. We can understand (32) in terms of matrix elements

$$\tilde{\alpha}_{ij} = \int_{U(d)} d\mu(g) \alpha(g) U^{(1)}(g)_{ij} \quad (33)$$

which are just the integrals of the functions $\alpha(g) U^{(1)}(g)_{ij}$.

Consider now a fermionic system with single-particle space $\mathcal{H}^{(1)}$. The Hilbert space of this system is the Fock space $\mathcal{F} = \bigoplus_{k=0}^{d} \mathcal{H}^{(k)}$, where $\mathcal{H}^{(k)} = \Lambda^k \mathcal{H}^{(1)}$ is the space of antisymmetric $k$-tensors in $\mathcal{H}^{(1)}$. Let $\{|e_1\rangle, |e_2\rangle, \ldots, |e_d\rangle\}$ denote an orthonormal basis for $\mathcal{H}^{(1)}$. Then, the set $\{|e_{i_1} \wedge \ldots \wedge e_{i_k}\rangle\}_{1 \leq i_1 < \ldots < i_k \leq d}$ provides an orthonormal basis for $\mathcal{H}^{(k)}$.

We can alternatively consider the canonical anticommutation relations (CAR) algebra $\{a_i, a_i^\dagger\} = \delta_{ij}$, and obtain all basis vectors by repeated application of creation operators to the vacuum vector $|\Omega\rangle$:

$$|e_{i_1} \wedge \ldots \wedge e_{i_k}\rangle = a_{i_1}^\dagger \ldots a_{i_k}^\dagger |\Omega\rangle. \quad (34)$$

A self-adjoint operator $A$ on $\mathcal{H}^{(1)}$ can be made to act on $\mathcal{H}^{(k)}$ in a way that preserves the antisymmetric character of the vectors by considering combinations of the form

$$A^{(k)} := (A \otimes 1_d \otimes \cdots \otimes 1_d) + (1_d \otimes A \otimes 1_d \otimes \cdots \otimes 1_d) + \cdots + (1_d \otimes \cdots \otimes 1_d \otimes A). \quad (35)$$

The map $A \rightarrow A^{(k)}$ is a Lie algebra homomorphism. We further comment on this important point below.

At the group level, we may consider exponentials of such operators, of the form $e^{tA}$. We then see that the operators of the form

$$\tilde{\alpha}^{(k)} = \int_{U(d)} d\mu(g) \alpha(g) U^{(1)}(g) \otimes \cdots \otimes U^{(1)}(g), \quad (36)$$

act properly on $\mathcal{H}^{(k)}$. 

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The map $\hat{\alpha} \to \hat{\alpha}^{(k)}$ is an isomorphism from $M_d(\mathbb{C})$ into $M_{dk}(\mathbb{C})$. This is also important as discussed below.

These constructions are most conveniently expressed in terms of a coproduct $\Delta$. In fact, an approach based on Hopf algebras (as explained in [26]), has the great advantage that para- and braid-statistics can be automatically included. In our present case, the construction of the observable algebra corresponds to the following simple choice for the coproduct: $\Delta(g) = g \otimes g$, ($g \in U(d)$), linearly extended to all of $\mathbb{C}U(d)$. This choice fixes the form of (36). At the Lie algebra level, it reduces to (35).

Physically, the existence of such a coproduct is very important, since it allows us to homomorphically represent the one-particle observable algebra on the $k$-particle sector. In a many particle system, the choice to perform observations of only one-particle observables corresponds to restricting the full-algebra of observables to the homomorphic image of the one-particle observable algebra obtained by using the coproduct.

Now, if for some reason we perform only partial one-particle observations (for instance, if at the one-particle level, we decide to measure -or have access to- only the spin degrees of freedom, or only the position), the one-particle observable algebra will have to be restricted accordingly and hence its homomorphic image at the $k$-particle level will be a subalgebra of the original algebra.

As we will see in what follows, the GNS approach covers all such cases whether or not particles are identical. In particular it merits to reemphasize that it covers observations of particles obeying para- and braid- statistics, including anyons.

Example 3: Two Fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$.

Keeping the same notation as above, put $d = 3$. We focus our attention on the two-fermion space $\mathcal{H}^{(2)} = \Lambda^2 \mathbb{C}^3 \subset \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$, with basis

$$\{|f^k\rangle := \varepsilon^{ijk}|e_i \wedge e_j\rangle\}_{1 \leq k \leq 3},$$

being an orthonormal basis for $\mathcal{H}^{(1)}$. The algebra $\mathcal{A}$ of observables for the two-fermion system is the matrix algebra generated by $|f^i\rangle \langle f^j|$ ($i, j = 1, 2, 3$). It is isomorphic to $M_3(\mathbb{C})$.

Now, $U(3)$ acts on $\mathcal{H}^{(1)}$ through the defining representation $(U^{(1)}(g) = g)$, so that one
particle observables are given - at the two fermion level - by the action of $CU(3)$ on $\mathcal{H}^{(2)}$. This action is given by the restriction of the operators $\hat{\alpha} = \int_{U(\mathfrak{g})} d\mu(g) \alpha(g) U^{(1)}(g) \otimes U^{(1)}(g)$ to the space of antisymmetric vectors. Let $3$ be the defining (or fundamental) $SU(3)$ representation on $\mathcal{H}^{(1)}$. Then the restriction can be obtained from the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$ of the $SU(3)$ representation. The $|f^i\rangle$ span this $\bar{3}$ representation.

**Choice 1 for $A_0$:**

Let $|\psi\rangle \in \mathcal{H}^{(2)}$ be any two-fermion state vector. If we take $A_0$ to be the full algebra of one-particle observables acting on $\mathcal{H}^{(2)}$, then $A_0 = \mathcal{A}$ and the GNS-representation corresponding to the pair $(A_0, \omega_\psi)$ is irreducible, the state remaining unchanged upon restriction. This is just the fact that the $\bar{3}$ representation of $SU(3)$ is irreducible and corresponds to the fact that, for $d = 3$, all two-fermion vector states $|\psi\rangle$ have Slater rank 1.

We hence get zero for the von Neumann entropy.

Notice, however, that the von Neumann entropy computed by partial trace is equal to 1 for all choices of $|\psi\rangle$ (cf. [6]), in disagreement with the GNS-approach.

**Choice 2 for $A_0$:**

The situation changes drastically if we make a different choice for the subalgebra $A_0$ of $\mathcal{A}$. Let us, for the sake of concreteness, choose $A_0$ to be given by those one-particle observables pertaining only to the one-particle states $|e_1\rangle$ and $|e_2\rangle$. In this case, $A_0$ will be the five dimensional algebra generated by $M^{ij} := |f^i\rangle\langle f^j| (i, j = 1, 2)$ and $1_\mathcal{A}$, the unit matrix.

As a physical illustration for the meaning of $M^{ij}$, let us think of $e_1, e_2, e_3$ as $u, d, s$ quarks. Observables acting just on $u$ and $d$ amounts to the isospin group $SU(2)_I$ acting just on $|f^1\rangle$ and $|f^2\rangle$ in the $\bar{3}$ representation. And the group algebra $CSU(2)_I$ is also generated by $M^{ij}$.

We now explicitly perform the GNS-construction for the particular choice

$$|\psi_\theta\rangle = \cos \theta |f^1\rangle + \sin \theta |f^3\rangle. \quad (38)$$

Let $\omega_\theta : \mathcal{A} \to \mathbb{C}$ denote the corresponding state:

$$\omega_\theta(\alpha) = \langle \psi_\theta | \alpha | \psi_\theta \rangle, \quad \forall \alpha \in \mathcal{A}, \quad (39)$$

and put

$$\omega_{\theta,0} = \omega_\theta |_{A_0}. \quad (40)$$

Now notice that $\omega_{\theta,0}(M^{12}M^{12}) = 0$ independently of $\theta$. The same holds true for $M^{22}$ so that both are null vectors: $|[M^{12}]\rangle = |[M^{22}]\rangle = 0$. 

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Case 1: $0 < \theta < \pi/2$

For $0 < \theta < \pi/2$, there are no more linearly independent null vectors. We can see this from
\[
\langle \psi_\theta | \alpha^* \alpha | \psi_\theta \rangle = 0 \Rightarrow \alpha | \psi_\theta \rangle = 0 \Rightarrow \alpha = \sum c_i M^{i2}, c_i \in C.
\] (41)
Therefore, the null space $\mathcal{N}_{\theta,0}$ is two-dimensional and the GNS-space $\mathcal{H}_\theta = \hat{\mathcal{A}}_0/\mathcal{N}_{\theta,0}$ is the three-dimensional space with basis $\{ ||M^{11}||, ||M^{21}||, ||E^3|| \}$, where $E^3 := 1_{\mathcal{A}} - M^{11} - M^{22}$.

Since $\alpha_0 E^3 = 0$ if $\alpha_0 \in \mathcal{A}_0$, we immediately recognize that, in terms of irreducibles, $\mathcal{H}_\theta = C^2 \oplus C^1$. Call $P_1$ and $P_2$ the corresponding projections. After noting that $[M^{11} + M^{22}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we obtain
\[
P_1 ||1_{\mathcal{A}}|| = ||M^{11}||, \quad P_2 ||1_{\mathcal{A}}|| = ||E^3||
\] (42)
The corresponding 'weights' $|\mu_i|^2 = ||P_i ||1_{\mathcal{A}}|||^2$ are computed using the inner product of $\mathcal{H}_\theta$. We obtain
\[
|\mu_1|^2 = \cos^2 \theta, \quad |\mu_2|^2 = \sin^2 \theta.
\] (43)
Hence,
\[
\omega_{\theta,0} = \cos^2 \theta \left( \frac{1}{\cos^2 \theta} ||M^{11}|| \langle |M^{11}| \rangle \right) + \sin^2 \theta \left( \frac{1}{\sin^2 \theta} ||E^3|| \langle |E^3| \rangle \right).
\] (44)
The result for the entropy as a function of $\theta$ is therefore
\[
S(\theta) = - \cos^2 \theta \log \cos^2 \theta - \sin^2 \theta \log \sin^2 \theta.
\] (45)

Case 2: $\theta = 0$

It is readily checked that at $\theta = 0$ additional null states appear as compared to Case 1. Thus, for $\theta = 0$ the null vectors are spanned by
\[
|M^{12}>, \quad |M^{22}>, \quad |E^3>.\]
(46)
The GNS space $\mathcal{H}_0 = \hat{\mathcal{A}}_0/\mathcal{N}_{0,0}$ is two-dimensional and irreducible. It is spanned by
\[
||M^{11}||, \quad ||M^{21}||.
\] (47)
Since $\pi_\omega(\mathcal{A}_0)$ acts nontrivially on this space, and the smallest nontrivial representation of $\mathcal{A}_0$ is its two-dimensional IRR, this representation is irreducible. Hence $\omega_{0,0}$ is pure with zero entropy. For completeness we note that the projector to $\mathcal{H}_0$ is $\pi_\omega(M^{11} + M^{22})$. 

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Case 3: $\theta = \frac{\pi}{2}$

For $\theta = \frac{\pi}{2}$ instead, all of $|M^{ij}\rangle$ are null vectors. So $H_{\pi/2}$ is one-dimensional and spanned by $|[E^3]\rangle$. Clearly $\omega_{\pi/2}$ is pure with zero entropy.

The decomposition of $H_{\theta}$ as a direct sum of irreducible subspaces as we change the value of $\theta$ is, therefore, as follows:

$$H_{\theta} \cong \begin{cases} 
\mathbb{C}^2, & \theta = 0 \\
\mathbb{C}^3 \cong \mathbb{C}^2 \oplus \mathbb{C}, & \theta \in (0, \pi/2) \\
\mathbb{C}, & \theta = \pi/2. 
\end{cases}$$

This result should be contrasted against the fact that the $\bar{3}$ representation, when regarded as a representation space for $SU(2)$ acting on $|[M^{11}]\rangle$, $|[M^{22}]\rangle$ and $|[E^3]\rangle$ splits as $2 \oplus 1$.

Example 4: Two Fermions, $\mathcal{H}^{(1)} = \mathbb{C}^4$

Consider, in the spirit of [12], a one-particle space describing fermions with two degrees of freedom which we call external (e.g. ‘left’ and ‘right’) and two degrees of freedom which we call internal e.g. ‘spin 1/2’). Here it is convenient to use a description in terms of fermionic creation/annihilation operators $a^{(f)}_\sigma, b^{(f)}_\sigma$, with $a$ standing for ‘left’, $b$ for ‘right’ and $\sigma = 1, 2$ for spin up and down, respectively. A basis for $\mathcal{H}^{(2)}$ is then given by the vectors $a^{\dagger}_1a^{\dagger}_2|\Omega\rangle$, $b^{\dagger}_1b^{\dagger}_2|\Omega\rangle$ and $a^{\dagger}_\sigma b^{\dagger}_{\sigma'}|\Omega\rangle$, with $\sigma, \sigma' \in \{1, 2\}$.

Again we consider a $\theta$-dependent state vector, this time given by

$$|\psi_\theta\rangle = \left( \cos \theta a^{\dagger}_1b^{\dagger}_2 + \sin \theta a^{\dagger}_2b^{\dagger}_1 \right) |\Omega\rangle.$$  \hspace{1cm} (49)

At the two-particle level, the full observable algebra $\mathcal{A}$ is the matrix algebra $M_6(\mathbb{C})$. From this algebra we pick the subalgebra $\mathcal{A}_0$ of one-particle observables corresponding to measurements at the left location. This is the six-dimensional algebra generated by

$$1_\mathcal{A}, \quad T_1 := \frac{i}{2}(a^{\dagger}_1a_2 + a^{\dagger}_2a_1), \quad T_2 := -\frac{i}{2}(a^{\dagger}_1a_2 - a^{\dagger}_2a_1), \quad T_3 := \frac{i}{2}(a^{\dagger}_1a_1 - a^{\dagger}_2a_2),$$

$$n_{12} := (a^{\dagger}_1a_1a^{\dagger}_2a_2), \quad N_a := (a^{\dagger}_1a_1 + a^{\dagger}_2a_2).$$  \hspace{1cm} (50)

Case 1: $0 < \theta < \frac{\pi}{2}$

For $0 < \theta < \pi/2$ we readily find that a basis of null vectors of $\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ are $|n_{12}\rangle$ and $|(1_\mathcal{A} - N_a)\rangle$. The GNS Hilbert space $H_\theta = \hat{\mathcal{A}}_0/N_{\theta,0}$ is hence four-dimensional and spanned by the vectors $|[1_\mathcal{A}]\rangle$ and $\{|[T_i]\rangle\}_{i=1,2,3}$.  

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Let $\pi_\theta$ be the GNS representation of $\mathcal{A}_0$ on $\mathcal{H}_\theta$. We can evidently find the decomposition of $\mathcal{H}_\theta$ into irreducible subspaces under $\pi_\theta$ by computing the Casimir operator and the highest weight vectors of the Lie algebra $\mathfrak{su}(2)$ given by the representation $T_i \mapsto \pi_\theta(T_i)$. We find $\mathcal{H}_\theta = \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\mathcal{H}_1$ spanned by $| [T_1 + iT_2] \rangle = | [a_1^\dagger a_2] \rangle$ and $| [a_2^\dagger a_2] \rangle$ and $\mathcal{H}_2$ spanned by $| [a_1^\dagger a_1] \rangle$ and $| [T_1 - iT_2] \rangle = | [a_2^\dagger a_1] \rangle$. The two representations are isomorphic.

We can find the components of $| [1_A] \rangle$ into $\mathcal{H}_i$ by writing

$$| [1_A] \rangle = | [N_1] \rangle = | [a_1^\dagger a_1 + a_2^\dagger a_2] \rangle. \quad (51)$$

Hence

$$P_1| [1_A] \rangle = | [a_2^\dagger a_2] \rangle, \quad P_2| [1_A] \rangle = | [a_1^\dagger a_1] \rangle. \quad (52)$$

As for their normalization, using (49),

$$\| P_1| [1_A] \rangle \|^2 = \sin^2 \theta, \quad \| P_2| [1_A] \rangle \|^2 = \cos^2 \theta. \quad (53)$$

Hence the restriction $\omega_{\theta,0}$ of $\omega_\theta = | \psi_\theta \rangle \langle \psi_\theta |$ to $\mathcal{A}_0$ can be written in terms of pure states as

$$\omega_{\theta,0} = \sin^2 \theta | \chi_1 \rangle \langle \chi_1 | + \cos^2 \theta | \chi_2 \rangle \langle \chi_2 |, \quad \text{with} \quad | \chi_1 \rangle = \frac{1}{\sin \theta} | [a_2^\dagger a_2] \rangle, \quad | \chi_2 \rangle = \frac{1}{\cos \theta} | [a_1^\dagger a_1] \rangle, \quad \langle \chi_i | \chi_j \rangle = \delta_{ij}. \quad (54)$$

This gives the following result for entropy:

$$S(\theta) = - \cos^2 \theta \log \cos^2 \theta - \sin^2 \theta \log \sin^2 \theta. \quad (56)$$

**Case 2: $\theta = 0, \frac{\pi}{2}$**

Consider first $\theta = 0$. In this case,

$$| \psi_0 \rangle = a_1^\dagger b_2^\dagger | \Omega \rangle \quad (57)$$

The null vectors $| \alpha \rangle$ are obtained by solving $\alpha | \psi_0 \rangle = 0$ for $\alpha \in \mathcal{A}_0$. That shows that

$$\mathcal{N}_{0,0} = \text{Span} \left\{ | n_{12} \rangle, | [1_A - a_1^\dagger a_1] \rangle, | [a_2^\dagger a_2] \rangle, | [a_1^\dagger a_1] \rangle \right\}. \quad (58)$$

The quotient space $\hat{\mathcal{A}}_0 / \mathcal{N}_{0,0}$ is $\mathbb{C}^2$ and is isomorphic to $\mathcal{H}_2$ above:

$$\hat{\mathcal{A}}_0 / \mathcal{N}_{0,0} = \text{Span} \left\{ | [a_1^\dagger a_1] \rangle, | [a_2^\dagger a_2] \rangle \right\} = \mathbb{C}^2. \quad (59)$$
For $\theta = \frac{\pi}{2}$, when $|\psi_{\pi/2}\rangle = a_2^\dagger b_1^\dagger |0\rangle$, we find instead a $\mathbb{C}^2$ isomorphic to $\mathcal{H}_1$ above:

$$\mathcal{N}_{\pi/2,0} = \text{Span} \left\{ ||n_{12}||, ||1_A - a^\dagger_2 a_2||, ||a^\dagger_1 a_1||, ||a_2^\dagger a_1|| \right\}, \quad (60)$$

$$\hat{A}_0/\mathcal{N}_{\pi/2,0} = \text{Span} \left\{ ||a^\dagger_1 a_2||, ||a_2^\dagger a_2|| \right\} = \mathbb{C}^2 \quad (61)$$

The GNS representations on both these $\mathbb{C}^2$’s are irreducible. Hence $\omega_{0,0}$ and $\omega_{\pi/2,0}$ are pure states with zero entropy.

The decomposition of $\mathcal{H}_\theta$ into irreducible subspaces, as a function of $\theta$, is the following:

$$\mathcal{H}_\theta \cong \begin{cases} \mathbb{C}^2, & \theta = 0, \pi/2 \\ \mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2, & \theta \in (0, \pi/2). \end{cases} \quad (62)$$

The significant aspect of this example is the fact that for the values of $\theta$ for which the Slater rank of $|\psi_{\theta}\rangle$ is one, namely $\theta = 0$ and $\frac{\pi}{2}$, we obtain exactly zero for the entropy. In previous treatments of entanglement for identical particles, the minimum value for the von Neumann entropy of the reduced density matrix (obtained by partial trace) has been found to be $\log 2$ (cf. [1] and references therein). This has been a source of embarrassment: it seems to suggest that different entanglement criteria have to be adopted, depending on whether one is dealing with non-identical particles, or with bosons, or fermions.

We have shown here that, by replacing the notion of partial trace by the more general one of restriction to a subalgebra, all cases can be treated on an equal footing.

**Example 5: Two Bosons, $\mathcal{H}^{(1)} = \mathbb{C}^3$.**

Here we consider the bosonic analogue of Example 3. Consider the one-particle space $\mathcal{H}^{(1)} = \mathbb{C}^3$ with an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$. The two-boson space $\mathcal{H}^{(2)}$ is the space of symmetrized vectors in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. It corresponds to the six-dimensional space obtained from the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$ of $SU(3)$. An orthonormal basis for $\mathcal{H}^{(2)}$ is given by vectors $\{|e_i \vee e_j\rangle\}_{i,j \in \{1,2,3\}}$, where

$$|e_i \vee e_j\rangle \equiv \begin{cases} \frac{1}{\sqrt{2}}(|e_i \rangle \otimes |e_j\rangle + |e_j \rangle \otimes |e_i\rangle), & i \neq j, \\ |e_i \rangle \otimes |e_i\rangle, & i = j. \end{cases} \quad (63)$$

The algebra $\mathcal{A}$ of observables for the two boson system is thus isomorphic to $M_6(\mathbb{C})$.

Consider the state $\omega_{\theta,\phi} : \mathcal{A} \rightarrow \mathbb{C}$ corresponding to the vector

$$|\psi_{\theta,\phi}\rangle = \sin \theta \cos \phi |e_1 \vee e_2\rangle + \sin \theta \sin \phi |e_1 \vee e_3\rangle + \cos \theta |e_3 \vee e_3\rangle. \quad (64)$$
We are interested in the restriction $\omega(\theta, \phi)$ to the subalgebra $\mathcal{A}_0$ of one-particle observables pertaining only to the one-particle vectors $|e_1\rangle$ and $|e_2\rangle$. Proceeding in the same way as in the previous examples, we recognize that the 6 representation, when regarded as a representation space for $SU(2)$ acting nontrivially on $|e_1\rangle$ and $|e_2\rangle$, splits as $6 = 3 \oplus 2 \oplus 1$. The basis vectors for these three invariant subspaces are given below:

3: $|1\rangle = |e_1 \vee e_1\rangle$, $|0\rangle = |e_1 \vee e_2\rangle$, $| -1\rangle = |e_2 \vee e_2\rangle$,

2: $|1/2\rangle = |e_1 \vee e_3\rangle$, $| -1/2\rangle = |e_2 \vee e_3\rangle$,

1: $|\tilde{0}\rangle = |e_3 \vee e_3\rangle$.  \hspace{1cm} (65)

The one-particle observables on $\mathcal{H}^{(2)}$ are obtained from the operators $|e_i\rangle\langle e_j|$ (with $i, j = 1, 2$), as well as from the unit operator on $\mathcal{H}^{(1)}$, by means of the coproduct. Thus, the subalgebra $\mathcal{A}_0$ is generated by operators of the form $|u\rangle\langle v|$, with both $|u\rangle$ and $|v\rangle$ belonging to the same irreducible component of $\mathcal{H}^{(2)}$. (Note that the image of unity on $\mathcal{H}^{(1)}$ under the coproduct $\Delta$ is $1_{\mathcal{A}}$. Hence by taking combinations of images of the above $\mathcal{H}^{(1)}$-observables under $\Delta$, we see that $\mathcal{A}_0$ contains $|\tilde{0}\rangle\langle \tilde{0}|$). In other words, $\mathcal{A}_0$ is given by block-diagonal matrices, with each block corresponding to one of the irreducible components in the decomposition $6 = 3 \oplus 2 \oplus 1$. The dimension of $\mathcal{A}_0$ is therefore $3^2 + 2^2 + 1^2 = 14$.

The construction of the GNS-representation corresponding to each particular value of the parameters $\theta$ and $\phi$ is performed following the same procedure as in Example 3. Let us introduce the notation $B_{u,v} \equiv |u\rangle\langle v|$, for any pair $|u\rangle, |v\rangle$ in (66). Then, from (64) we see that as long as the $(\theta, \phi)$-coefficients are all different from zero, those elements of $\mathcal{A}_0$ of the form $B_{j,\pm 1}$ ($j = 0, \pm 1$) and $B_{\sigma, -1/2}$ ($\sigma = \pm 1/2$) generate the null vectors. That these generate all the null vectors follows from the fact that (64) contains one basis element for every irreducible component, so that no further linear relation can arise that lead to null vectors. So in this case we have

$$\mathcal{H}_{(\theta, \phi)} := \hat{\mathcal{A}}_0 / \mathcal{N}_{(\theta, \phi),0} = \mathbb{C}^6, \quad \mathcal{N}_{(\theta, \phi),0} = \text{Null space}.$$ \hspace{1cm} (66)

In terms of irreducible subspaces, one can readily see that $\mathbb{C}^6$ decomposes according to $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}^1$.

In general, we can read off the decomposition of $\mathcal{H}_{(\theta, \phi)}$ into irreducible subspaces from (64), depending on which of its coefficients vanish. For example, if only the first one vanishes,

$$\mathcal{H}_{(\theta, \phi)} = \mathbb{C}^2 \oplus \mathbb{C}^1.$$ \hspace{1cm} (67)

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The entropy equation (69) as a function of $x$ and $y$, the coordinates of a plane representing the $(\theta, \phi)$-sphere through stereographic projection. Darker regions correspond to lower values of the entropy. Five of the six vanishing points of the entropy can be seen on the picture (black spots). The sixth one, corresponding to the north-pole of the sphere, lies ‘at infinity’ in this representation.

It is interesting to consider the entropy as a function of $(\theta, \phi)$. For the case in which all $(\theta, \phi)$-coefficients are non-zero, we have:

$$|[1_A]| = |[B_{1,1}]| + |[B_{1/2,1/2}]| + |[B_{0,0}]|,$$  \hspace{1cm} (68)

from which the entropy is readily computed as before. The result is:

$$S(\theta, \phi) = -\sin^2 \theta [\cos^2 \phi \log(\sin \theta \cos \phi)^2 + \sin^2 \phi \log(\sin \theta \sin \phi)^2] - \cos^2 \theta \log(\cos \theta)^2. \hspace{1cm} (69)$$

The analytic formulae for entropy when one or more of the coefficients in (66) vanish can be obtained from (69) by taking suitable limits on $\theta$ and $\phi$.

We can see that the entropy vanishes whenever $|\psi(\theta, \phi)\rangle$ lies in a single irreducible component. This happens precisely at those points of the two-sphere generated by the parameters $(\theta, \phi)$ that correspond to the coordinate axes. There are therefore six points where the entropy vanishes exactly. This is depicted in Figure 1 where the $(\theta, \phi)$-sphere has been mapped to the $x$-$y$ plane through a stereographic projection. The figure shows the entropy as a function of the coordinates of that plane.
CONCLUSIONS

We have presented a new approach to the study of quantum entanglement based on restriction of states to subalgebras. The GNS-construction allows us to obtain a representation space for the subalgebra such that its decomposition into irreducible subspaces can be used to study quantum correlations. We showed that, when applied to bipartite systems for which the Hilbert space is a ‘simple’ tensor product, our method reproduces the standard results on entanglement. We furthermore showed, with explicit examples, how the formalism can be applied to systems of identical particles. Our results demonstrate in a clear fashion that the von Neumann entropy indeed remains a suitable entanglement measure, when understood in terms of states on algebras of observables.

The formalism used for the treatment of identical particles, using coproducts to identify algebras of subsystems can be easily generalized to more sophisticated situations such as those of particles obeying para- and braid statistics. Our results can hence be extended to the study of entanglement of such particles. It can even be extended to study for instance a \( k \)-particle subsystem in an \( N \) particle Hilbert space.

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