Spacelike Wilson Loops at Finite Temperature

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\textbf{ABSTRACT}

In the high temperature phase of Yang-Mills theories, large spatial Wilson loops show area law behaviour with a string tension that grows with increasing temperature. Within the framework of the commonly used string picture we use a large scale expansion, which allows us to determine the string tension from measurements of intermediate and symmetric Wilson loops.

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1. Introduction

Wilson loops are basic gauge invariant observables in lattice gauge theory. It has been proposed that the area/perimeter law behaviour of large spatial Wilson loops in the quark-gluon plasma provides evidence for magnetic confinement/deconfinement [1]. Although these claims are not based on rigorous evidence, large Wilson loops serve as observables in the appropriate physical situation: the time axis can be defined in a spatial direction which makes it possible to describe zero temperature QCD in a volume with one compactified dimension. Nevertheless, for simplicity we use here the standard notation common to finite temperature field theory.

Wilson loops can be studied by both analytical and numerical methods. For example, weak coupling perturbation theory can be used to calculate the short distance behaviour of the local potentials. On the other hand, their large scale behaviour is believed to be described by a model of a fluctuating thin tube that contains the chromo-electric flux from the quark to the anti-quark.

Here we recapitulate that the Coulomb part of a spatial Wilson loop is well described by infrared-finite perturbation theory [2]. The perturbative range ends where the string tension part starts to dominate. The larger the loop the better the string model should work. This yields an effective large distance expansion and allows us to determine the string tension from Monte Carlo measurements. Since most of the data are obtained for almost symmetric Wilson loops we use an expansion for large rectangular Wilson loops that is effective if both sides become large at a constant rate. In this way we avoid the problems of standard approaches, which determine the string tension from the local potentials. This would necessitate large asymmetric Wilson loops and, in turn, large lattices. Our approach also allows for the inclusion of symmetric Wilson loops and gives reliable results already for relatively small lattices. The string tension appears to be enhanced due to a pinching of the gluon string, which results from the compactification of one of the transverse dimensions.

2. Short and large distance expansions.

At short distances the Wilson loop is dominated by its Coulomb part which has been calculated by renormalised perturbation theory [2]. The comparison of this short-range perturbative contribution with the Monte Carlo data is shown in Fig. 1 for the gauge group SU(2) at a temperature of approximately $4T_c$. In
this instance the Coulomb potential, which has a logarithmic behaviour, predicts
that perturbation theory is valid up to a distance of around $T^{-1}$. Beyond this
perturbative region, the distance dependence of the local potentials

$$Y(S, L) = \ln W(S, L - 1) - \ln W(S, L),$$

where $W(S, L)$ is an $S \times L$ rectangular Wilson loop, is essentially described by the
linearly rising string tension term.

The standard method for calculating the non-perturbative contributions is to
consider the potential

$$V(S) = \lim_{L \to \infty} Y(S, L),$$

which needs rather large lattices in order to allow for Wilson loops that are them-
selves large and asymmetric enough to satisfy the limit above. The model of a
fluctuating thin flux tube extending from the quark to the anti-quark with rigid
internal degrees of freedom predicts a large distance expansion of the form [3,4]

$$V(S) = c_1 S + c_2 + \frac{c_3}{S} + O(S^{-2}).$$

The coefficient $c_3$ is universal [3] in the sense that it does not depend on the
particular string action and the gauge group under consideration, a fact that can
be traced back to the Mermin-Wagner theorem [5].

In practice, however, the lattices that can be simulated are not large enough
to restrict attention to $V(S)$ alone. Hence it is more appropriate to investigate
$Y(S, L)$ or even $W(S, L)$. If $L$ is large compared to $S$, but still finite, the appro-
priate modification to the expression above has been given in ref. [6]. In order to
allow for the inclusion of nearly symmetric Wilson loops, we apply a large scale
expansion where both $S$ and $L$ become large at a constant rate, i.e. with $S/L =
\text{const.}$

We start by considering the action of a non-critical string with frozen internal
degrees of freedom that is defined on a two dimensional space-time lattice of size
$L \times S$ with Dirichlet boundary conditions.

$$S_{\text{eff}}(\xi) = K \sum_{x_1=1}^{S-1} \sum_{x_2=1}^{L-1} \left( \frac{1}{2} \xi(x)(-\Delta)\xi(x) + \frac{1}{2} \lambda_1 \xi(x)(-\Delta)^2 \xi(x) \right. \right.$$

$$\left. \left. + \lambda_2 [\xi(x)(-\Delta)\xi(x)]^2 + \cdots \right), \right.$$

$$2$$
where \( \xi(x) \in \mathbb{R}^2 \), \( K \) and \( \lambda \) are unknown coupling constants, and \( \cdots \) represent even more infrared regular contributions. The Wilson loop becomes

\[
W(S, L) = \int_{-\infty}^{\infty} \prod_x d^2 \xi(x) \exp(-S_{\text{eff}}(\xi)).
\]

(5)

The expansion with respect to the couplings \( \lambda \) yields [3]

\[
- \ln W(S, L) = \ln \frac{K}{2\pi} (S - 1)(L - 1) + F(S, L)
\]

\[
+ \lambda_1 F_1(S, L) + \frac{\lambda_2}{K} F_2(S, L) + O(\lambda^2),
\]

(6)

where the coefficient functions are given by

\[
F(S, L) = \sum_{n_1=1}^{S-1} \sum_{n_2=1}^{L-1} \ln 4(\sin^2 \frac{\pi}{S}n_1 + \sin^2 \frac{\pi}{L}n_2),
\]

(7)

\[
F_1(S, L) = \sum_{n_1=1}^{S-1} \sum_{n_2=1}^{L-1} 4(\sin^2 \frac{\pi}{S}n_1 + \sin^2 \frac{\pi}{L}n_2),
\]

(8)

and

\[
F_2(S, L) = \sum_{n_1=0}^{S-1} \sum_{n_2=0}^{L-1} \left\{ \left[ \sum_{\mu=1,2} \partial_\mu(x)\partial_\mu(y)G(x, y) \right]^2 \right. \\
+ \left. 2 \sum_{\mu, \nu=1,2} (\partial_\mu(x)\partial_\nu(y)G(x, y))^2 \right\}_{x=y=(n_1, n_2)}.
\]

(9)

\( G(x, y) \) denotes the lattice Green function that is derived from the difference equation

\[
\sum_x (-\Delta)(z, x)G(x, y) = \delta(z, y)
\]

(10)

with Dirichlet boundary conditions imposed.

The behaviour of the coefficient functions for large \( S \), with \( \omega = S/L \) fixed, can be derived by an asymptotic expansion. The results are as follows:

\[
F_1(S, L) = SL \int_{-\pi}^{\pi} \frac{d^2 \vec{k}}{(2\pi)^2} \hat{k}^2 - 4(S + L) + 4 + O(S^{-n}), \quad \text{any } n > 0,
\]

(11)

\[
F_2(S, L) = d_0 SL + d_1(S + L) + d_2 + O(S^{-1}),
\]
and
\[ F(S, L) = SL \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^2} \ln \hat{k}^2 - (S + L) \frac{1}{2} \ln (3 + 2\sqrt{2}) + d_3 \]
\[ + \left\{ G_0(\omega) - \frac{1}{4} \ln (SL) \right\} \]
\[ + \frac{1}{S^2} \frac{\pi \omega}{16} \left( \frac{1}{4} + G_2(\omega) \right) + O(S^{-4}), \]
(12)

where \( \hat{k}^2 = \sum_{i=1,2} 4 \sin^2 k_i \) and \( d_0, d_1, d_2 \) and \( d_3 \) are constants that do not depend on the asymmetry \( \omega \). The functions \( G_0 \) and \( G_2 \) involved are given by
\[ G_0(\omega) = \frac{1}{4} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \left[ \text{Ei}(-y) - \frac{\exp(-y)}{y} \right] \bigg|_{y = \pi(n_1^2 \omega + n_2^2 \omega)} \]
(13)
\[ G_2(\omega) = \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \exp(-y) \left[ 1 + \frac{3}{2y} + \frac{3}{2y^2} - \frac{z}{3y} (2 + \frac{3}{y} + \frac{6}{y^2} + \frac{6}{y^3}) \right] \bigg|_{y = \pi(n_1^2 \omega + n_2^2 \omega)} \]
(14)

The function \( F \) results from the gaussian part of the measure Eq. (5). It contains the universal term \( G_0(\omega) - \frac{1}{4} \ln (SL) \) which does not depend on the particular bare string tension \( K \) and the long range couplings \( \lambda \). It is the direct generalisation of Lüsher’s universal \( (-\pi/12S) \) term in the case of \( V(S) \) to Wilson loops \( W(S, L) \). This particular contribution does not depend on the cutoff scheme, except for a (non-universal) constant. It has also been calculated in ref. [7] using the \( \zeta \)-function regularisation. Although the explicit form of their result looks rather different than the one above, it actually agrees with our result up to a constant, as it should. In our case the symmetry with respect to \( S \) and \( L \) is more obvious since \( G_0(\omega) = G_0(\frac{\omega}{2}) \). As the Wilson loop becomes more asymmetric, the expansion approaches the result given in [6].

To summarise, the belief that Eq. (4) provides an effective low energy model for the heavy quark potential of QCD implies that the appropriate ansatz for a large scale expansion of Wilson loops is
\[ -\ln W(S, L) = a(1)SL + a(2)(S + L) + a(3) \]
\[ + a(4)(-\frac{1}{4} \ln (SL) + G_0(\frac{S}{L})) + O(S^{-1}, L^{-1}). \]
(15)

Consistency then requires that the coefficient in front of the universal term satisfies \( a(4) = 1 \).
What happens if one of the dimensions in which the gluon string can fluctuate becomes finite, in particular if it has a toroidal symmetry? The assumption that the flux tube behaves as a string with frozen internal degrees of freedom implies that compactification should not have much influence on the spin wave part, except possibly for the case that the dimension shrinks to zero. On the other hand, the non-universal contributions will be sensitive. For instance, we would expect that the string tension increases due to a pinching of the string, respectively stronger "self-interference" of the wave function that describes the ground state of the string in the hamiltonian approach [3].

3. Simulations

The SU(2) data are generated using a standard heat bath method, combined with an $\omega = 2$ overrelaxation algorithm, discussed in [8].

We investigate couplings $\beta = 2.50, 2.80$ and $3.00$ on lattice sizes $16^3 \times 4$ and $24^3 \times 4$ (with $32^3 \times 4$ in the case of $\beta = 3.00$). These couplings correspond to $T/T_c \approx 2.0, 4.5$ and $8.0$ respectively (cf. ref. [9]). A typical MC run on the larger lattices consists of around $130,000$ thermalised iterations, with measurements carried out every 25th iteration. Since the spatial Wilson loops under consideration here have a typical integrated correlation time $\tau_{int}$ of around 3, we have effectively around $1700 \tau_{int}$ independent measurements in the case of the two largest couplings, while for $\beta = 2.50$ we have slightly less.

For SU(3) we use a pseudo-heat bath algorithm, combined with $(\omega = 2)$ pseudo-overrelaxation steps [10]. The run consists of around $63,000$ sweeps on a $16^3 \times 4$ lattice at $\beta = 6.65$ ($T/T_c \approx 6 - 8$), with measurements performed every 10th sweep. Discarding the first $9000$ iterations for thermalisation, we again find that $\tau_{int} \approx 3$.

In the MC runs we measure the (on-axis) spatial Wilson loops

$$W(S, L) = < \prod_{(x;i) \in C(S,L)} U(x;i) >, \quad (16)$$

where $C(S, L)$ are the links of an $S \times L$ rectangular path in the spatial directions and $U(x;i)$ are the matrices defined on those links. These can be combined to obtain local potentials defined in Eq. (1).

Comparing the results for the Wilson loops obtained on the different lattice sizes ($L_s = 16$ and 24), we find little finite-size dependence within the statistical
errors as long as periodicity effects are neglected (i.e. we consider Wilson loops with sides less than $L_s/2$). The error analysis is carried out using the standard jackknife procedure.

4. Results

Considering directly the spatial Wilson loops, we perform correlated $\chi^2$ fits using the four parameter ansatz given in Eq. (15). Since its derivation and salient points were discussed in sect. 2, we concentrate here on the results. In Table 1 we list the results for SU(2) for the different couplings (temperatures), calculated on the $L_s = 24$ lattices, while the result for SU(3) is given in Table 2. As a consistency check, we notice that in all cases $a(4)$ is about 1. In general we obtain fairly large $\chi^2$ values for the correlated fits, which can be ascribed to the well-known fact that the covariance matrix, which takes into account the correlations between neighbouring loops, reduces the statistical errors in the data. A minimum value for the $L \times S$ lattice is necessary to ensure the validity of the string picture and the related large scale expansions as given in the previous sections. Since we are regarding an asymptotic expansion, this implies that the string tension contribution should dominate the universal (spin-wave) part of the potential. In practical terms this means that for the local potentials (which give rise the the potential through Eq. (2)), the following constraint must be satisfied

$$a(1)S > |a(4)(-\frac{1}{4} \ln \frac{L}{L-1} + G_0(S/L) - G_0(S/(L-1)))|.$$  (17)

The above condition implies that in all cases we must consider rectangular loops with sizes $[S,L] \geq [3,4]$. These lower critical distances are of the order of the perturbative horizon ($\approx T^{-1}$). This serves as an additional check that we are in the range where the (non-perturbative) string tension dominates the perturbative Coulomb part of the potential. We also note that, in the case of asymmetric loops, Eq. (17) reduces to the conditional relation (cf. also [11])

$$S^2 \geq \frac{\pi}{12\sigma}$$  (18)

where $\sigma \equiv a(1)$ is the bare string tension.

The maximum values of $S$ and $L$ are determined by the statistics of the run; already for $S, L \approx 7$ the statistical noise becomes so large that most of the data in the fits must be excluded as they become negative (Eq. (15)). Since this produces a
bias in the values of the averages, these data points must be excluded altogether. Also, since the statistics vary from one coupling to the next (and is moreover dependent on the value of the coupling and lattice volume for example), the range of Wilson loops included for the different runs must be dealt with separately. This explains the different values for $[S,L]$ in Table 1. In the case of SU(3), where $L_s = 16$, we have the added restriction that $S,L \leq 7$ to avoid periodicity effects.

Developing the large scale expansion one step further, the first correction to Eq. (15) will be proportional to $S^{-1}$ and $L^{-1}$. The simplest choice is to add a term of the form $a(5)(S^{-1} + L^{-1})$ to the fitting ansatz given in Eq. (15). This would effectively allow us to include slightly smaller loops in the fits.

We emphasize again that the major advantage of our approach lies in the fact that we may include symmetric Wilson loops in the fitting procedure and hence are not restricted to large asymmetric loops. The latter is necessary if one considers instead the potential $V(S)$, which is determined from the local potentials with $S$ fixed and $L \to \infty$. An estimate for $a(1)$ is then obtained by parametrising $V(S)$ as in Eq. (3). This is the method most often implemented in the literature to extract the string tension. However, for loops with $S \leq L$ this method becomes meaningless, while naively the condition $S >> L$ can hardly be met in numerical simulations, even with high statistics. We also note that the results obtained by using the ansatz proposed by Gao [6], which beyond strongly asymmetric Wilson loops also allows the inclusion of slightly symmetric ones, are consistent with our results, which were obtained using the general formula Eq. (15).

In Fig. 2 we compare the Monte Carlo data with the results of the fit for the same situation as in Fig. 1. This clearly illustrates the success of our method.

The physical string tension ($\sigma_{phys}$) is obtained by reintroducing the lattice spacing $a$ through

$$\sigma SL = \frac{\sigma}{a^2}(Sa)(La) = \sigma_{phys} SL.$$  \hspace{1cm} (19)

Since we are considering (finite temperature) gauge theories on an $L_s^3 \times L_0$ lattice we have

$$\sigma_{phys} \equiv \sigma_{phys}(T) = \sigma L_0^2 T^2$$  \hspace{1cm} (20)

with $T = (L_0 a)^{-1}$. From the equation above and the results for $\sigma = a(1)$ listed in Table 1 it is immediately clear that, although the bare string tension decreases for increasing temperature, the physical string tension increases as the contribution
from the $T^2$ -factor will dominate. This agrees with our intuitive picture that the string tension increases due to the pinching of the flux tube.

In order to estimate the increase of the string tension with temperature, we make the following simple ansatz to describe the high temperature dependence of the physical string tension

$$\sigma_{\text{phys}}(T) = \sigma_0 + cT^\alpha$$

(21)

where $\sigma_0$ is the physical string tension at $T = 0$ [12]. If we explicitly fix $\sigma_0$ to be the "average" value for the string tension at zero temperature [9], we have three couplings (temperatures) and two parameters. A simple two parameter fit to the ansatz above then gives a value for the exponent $\alpha = 1.6(2)$. We note that this result does not include the systematic uncertainties which arise in the determination of the ratio $T/T_c$ for the different couplings under consideration.

For SU(3) we cannot repeat this analysis since we have only one coupling value. However, as in the case for SU(2), the string tension does increase with temperature in the deconfined phase: the coupling corresponds to a temperature of $6 - 8T_c$, which leads to a physical string tension $\sigma_{\text{phys}} = 24.8T_c^2 - 44.0T_c^2$, which should be compared to the one at zero temperature $\sigma_0 = 3.2(2)T_c^2$ [9].

Table 1. The results of the fit Eq. (15) for SU(2). The range of Wilson loops included in the fits is also given. In all cases $L \leq 8$. The number of degrees of freedom is denoted by $df$.

| $\beta$ | $[S, L]$ | $a(1)$ | $a(2)$ | $a(3)$ | $a(4)$ | $df$ |
|---------|---------|--------|--------|--------|--------|------|
| 2.50    | [3,4] - [5,6] | 0.074(5) | 0.445(34) | -0.360(34) | 0.80(20) | 7    |
| 2.80    | [3,4] - [6,7] | 0.033(2) | 0.444(7) | -0.360(7) | 1.02(4) | 12   |
| 3.00    | [3,5] - [7,7] | 0.024(2) | 0.404(3) | -0.309(3) | 0.98(2) | 13   |
Table 2. The result of the fit Eq. (15) for SU(3). Here $L \leq 7$.

| $\beta$ | $[S, L]$ | $a(1)$   | $a(2)$   | $a(3)$   | $a(4)$   | $df$ |
|---------|----------|----------|----------|----------|----------|------|
| 6.65    | [3,4] - [5,6] | 0.043(4) | 0.527(20) | -1.54(3) | 1.22(14) | 6    |

5. Conclusions

In the high temperature phase of gauge theories the short distance behaviour of spatial Wilson loops is governed by its perturbative Coulomb part, which for the local potentials implies logarithmic distance dependence. Our approach is based on a more symmetric large scale expansion, which allows us to use symmetric as well as asymmetric Wilson loops of intermediate sizes for a quantitative determination of the string tension. Thus we are able to circumvent the problem of standard methods that rely on large, asymmetric Wilson loops. The method used here is generally valid for gauge theories and can therefore also be applied to full QCD.

On larger scales Wilson loops are well described by the model of a thin oscillating string with rigid internal degrees of freedom. The (physical) string tension $\sigma$ is enhanced by the compactification of one of the dimensions orthogonal to the string (the temperature direction). In contrast to earlier claims in the literature, we find that the string tension increases as a function of the temperature for both SU(2) and SU(3) gauge theories. Our results for SU(2) suggest that the string tension has a power-like behaviour as a function of the temperature.

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    See also updated results collected in [9].

Figure captions

Fig. 1. The local potentials $Y(S,6)$ for SU(2) at temperature $T \simeq 4T_c$, calculated
on a $4 \times 16^3$ lattice. The Monte Carlo data (○) are compared with the weak
coupling results (△) to subleading order [2]. The difference of these results
(∇) is a linearly increasing function.

Fig. 2. A comparison of the Monte Carlo data for the local potentials $Y(S,6)$ (○) with
the result from the fit Eq. (2) (△), again for SU(2) at temperature $T \simeq 4T_c$. 