TWIST FORMULAS FOR ONE-ROW COLORED $A_2$ WEBS AND $\mathfrak{sl}_3$ TAILS OF $(2, 2m)$-TORUS LINKS

WATARU YUASA

ABSTRACT. The $\mathfrak{sl}_3$ colored Jones polynomial $J^{\mathfrak{sl}_3}_\lambda(L)$ is obtained by coloring the link components with two-row Young diagram $\lambda$. Although it is difficult to compute $J^{\mathfrak{sl}_3}_\lambda(L)$ in general, we can calculate it by using Kuperberg’s $A_2$ skein relation. In this paper, we show some formulas for twisted two strands colored by one-row Young diagram in $A_2$ web space and compute $J^{\mathfrak{sl}_3}_{(m,0)}(T(2, 2m))$ for an oriented $(2, 2m)$-torus link. These explicit formulas derives the $\mathfrak{sl}_3$ tail of $T(2, 2m)$. They also gives explicit descriptions of the $\mathfrak{sl}_3$ false theta series with one-row coloring because the $\mathfrak{sl}_2$ tail of $T(2, 2m)$ is known as the false theta series.

1. INTRODUCTION

In this paper, we will make tools to calculate the quantum invariant of knots and links obtained from $\mathfrak{sl}_3$ by using the Kuperberg’s linear skein theory. Especially, we will prove a full twist formula which is useful to compute explicitly the $\mathfrak{sl}_3$ colored Jones polynomial. The quantum invariants of knots are obtained through a functor from the category of framed oriented tangles to the representation category of a quantum group of a simple Lie algebra. Thus, one can define an invariant $J^\mathfrak{g}_V(K)$ of a knot $K$ for each simple Lie algebra $\mathfrak{g}$ and its representation $V$, that is, there exist infinitely many quantum invariants of knots. However, it is difficult to compute $J^\mathfrak{g}_V(K)$ explicitly. The most simple case is $\mathfrak{g} = \mathfrak{sl}_2$ with the 2-dimensional irreducible representation $V = \mathbb{C}^2$. This quantum invariant reconstructs the Jones polynomial $J(K)$. We know that $J(K)$ can be calculated from a knot diagram of $K$ by using the Kauffman bracket skein relation, for example [Kau87]. More generally, one can compute the colored Jones polynomial $J^\mathfrak{g}_n(K)$ obtained from the $(n + 1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ by applying the Kauffman bracket skein relation to a knot diagram colored by the Jones-Wenzl projector. The Jones-Wenzl projector [Jon83, Wen87] corresponds to the project onto the $(n + 1)$-dimensional irreducible representation in $V^\otimes n$. In this case, $J_n(K)$ is explicitly calculated for many knots and there are many useful formulas which decompose a tangle diagram to the linear sum of the web basis by the skein relation. This method to calculate quantum invariants from knot diagrams is called the linear skein theory, see for example [Lic97]. The linear skein theory is constructed for some other $\mathfrak{g}$ than $\mathfrak{sl}_2$. In a calculational point of view, one of the most reasonable quantum invariants next after $\mathfrak{sl}_2$ is the $\mathfrak{sl}_3$ colored Jones polynomial $J^{\mathfrak{sl}_3}_{(m, n)}(K)$ which is obtained from the irreducible representation with the highest weight $(m, n)$ of $\mathfrak{sl}_3$. We use the linear skein theory for $\mathfrak{sl}_3$ constructed by Kuperberg [Kup94, Kup96], see Definition 2.1 of the $A_2$ skein relation. However, there are few non-trivial examples of explicit formulas of $J^{\mathfrak{sl}_3}_{(m, n)}(K)$. For example, Lawrence [Law03] calculated $J^{\mathfrak{sl}_3}_{(1, 0)}(K)$ for the trefoil knot, more generally, In [GMV13, GV17] for the $(2, 2m + 1)$- and $(4, 5)$-torus knots by using
a representation theoretical method. In [Yua17], the author calculated $J_{n}^{\text{sl}_3}_{(n,0)}(K)$ for the two-bridge links $K$ with one-row coloring $(n,0)$ by the Kuperberg’s linear skein theory. Important formulas used in [Yua17] is the following full twist formulas:

**Theorem 1.1** (anti-parallel full twist formula [Yua17]). Let $d = \min\{s, t\}$ and $\delta = |s - t|$. 

\[
\begin{array}{c}
\text{knotted disk} \\
\text{with anti-parallel orientation by using}
\end{array}
\]

\[
q^{\frac{1}{2}d(d+\delta)} - d \sum_{k=0}^{d} q^{k(k+\delta)+k} \left( q \right)_{d-k}^{-d-k} \delta \sum_{k_{0}\geq k_{1}\geq \cdots \geq k_{m}\geq 0} C(k_{0}, k_{0}-k_{m}) \sum_{k_{0}+k_{m}+(s-t)k_{0}} \left( s \right)_{q} \left( t \right)_{q}.
\]

From this theorem, the author obtained an $m$-full twist formula for anti-parallel two strands.

**Theorem 1.2** (anti-parallel $m$-full twist formula [Yua17]). Let $k = (k_{1}, \ldots, k_{m})$ be an $m$-tuple of integers, $k_{0} = d = \min\{s, t\}$, and $\delta = |s - t|$. 

\[
\begin{array}{c}
\text{colored Jones polynomial. We call it the}\n\end{array}
\]

\[
q^{\frac{1}{2}d(d+\delta)} - d \sum_{k=0}^{d} q^{k(k+\delta)+k} \left( q \right)_{d-k}^{-d-k} \delta \sum_{k_{0}\geq k_{1}\geq \cdots \geq k_{m}\geq 0} C(k_{0}, k_{0}-k_{m}) \sum_{k_{0}+k_{m}+(s-t)k_{0}} \left( s \right)_{q} \left( t \right)_{q}.
\]

where 

\[
C(k_{0}, k_{0}-k_{m}) = q^{m} \sum_{i=1}^{m} k_{i}(k_{i}+\delta)+2k_{i} q^{k_{0}-k_{m}} \left( q \right)_{k_{m}+\delta}^{-k_{m}} \left( k_{i}, k_{i}, \ldots, k_{i}, k_{m} \right)_{q},
\]

and $k'_{i+1} = k_{i} - k_{i+1}$ for $i = 0, 1, \ldots, m - 1$.

The above formula plays an important role in the study of $\text{sl}_3$ tails of knots and links. The tail of a knot $K$ is a $q$-series which is a limit of the colored Jones polynomials $\{J_{n}(K; q)\}_{n}$. Independently, the existence of the tails for alternating knots was shown in [DL06, DL07] and [GL15], more generally, for adequate links in [Arm13]. In [GL15], Garoufalidis and Lê showed a more general stability, the existence of the tail is the zero-stability, for alternating knots. Some explicit descriptions of tails are known for $T(2, 2m + 1)$ in [AD11], for $T(2, 2m)$ in [Haj16], for a pretzel knot $P(2k + 1, 2, 2l + 1)$ in [EH17], for knots with small crossing numbers in [Ko16, Bo17], and [GL15]. Especially, the tail of $T(2, 2m + 1)$ is given by the theta series and one of $T(2, 2m)$ is the false theta series.

One can consider a tail for the $\text{sl}_3$ colored Jones polynomial. We call it the $\text{sl}_3$ tail. However, there are many problems about how do we define a limit for $\{J_{n}^{\text{sl}_3}_{(n,k,l)}(K)\}_{k,l}$ and the existence of the $\text{sl}_3$ tail etc. As a case study, the author gave an explicit formula of the $\text{sl}_3$ tail for the $(2, 2m)$-torus link $T_{2m}(2, 2m)$ with anti-parallel orientation by using Theorem 1.2 in [Yua18], see also Theorem 4.5. This $\text{sl}_3$ tail can be considered a special type of the $\text{sl}_3$ false theta series. In fact, Bringmann-Kaszian-Milas [BKM19] commented that the $\text{sl}_3$ tail of $T_{2m}(2, 2m)$ coincides with the diagonal part of the $\text{sl}_3$ false theta series defined through the study of vertex operator algebras in [BM15, BM17, CM14, CM17].
This paper is organized as follows. In section 2, we review the Kuperberg’s linear skein theory for $A_2$ and give a diagrammatic definition of the Jones-Wenzl projector for $A_2$. In section 3, we derive formulas which decompose an $m$-full twist of two-strands with one-row coloring into the linear sum of $A_2$ basis webs. We use the generating function for Young diagrams by involving the decompositions of the $A_2$ web with lattice paths in proofs of these formulas. As an application, we compute the $\mathfrak{sl}_3$ colored Jones polynomials and the $\mathfrak{sl}_3$ tails of oriented $(2, m)$-torus links with one-row coloring in section 4.

2. Definition and Properties of the $A_2$ Clasp

We use the following notation.

- $[n] = \frac{q^n - q^{-n}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ is a quantum integer for $n \in \mathbb{Z}_{\geq 0}$.
- $[n]_k = \frac{[n]}{[n-k]}$ for $0 \leq k \leq n$ and $[n]_k = 0$ for $k > n$.
- $(q)_n = \prod_{i=1}^{n} (1 - q^i)$ is a $q$-Pochhammer symbol.
- $(n)_k = \frac{(q)_n}{(q)_k(q)_{n-k}}$ for $0 \leq k \leq n$ and $(n)_k = 0$ for $k > n$.
- $(k_1, k_2, \ldots, k_m)_n = \frac{(q)_n}{(q)_{k_1}(q)_{k_2} \cdots (q)_{k_m}}$ for positive integers $k_i$'s such that $\sum_{i=1}^{m} k_i = n$.

Let us define $A_2$ web spaces based on [Kup96]. We consider a disk $D$ with signed marked points $(P, \epsilon)$ on its boundary where $P \subset \partial D$ is a finite set and $\epsilon: P \to \{+,-\}$ a map.

A tangled bipartite uni-trivalent graph on $D$ is an immersion of a directed graph into $D$ satisfying (1) – (4):

1. the valency of a vertex of underlying graph is 1 or 3,
2. all crossing points are transversal double points of two edges with under/over information,
3. the set of univalent vertices coincides with $P$,
4. a neighborhood of a vertex is one of the followings:

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
\quad
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
\quad
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
\quad
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
\end{array}
\]

Definition 2.1 ($A_2$ web space [Kup96]). Let $G(c; D)$ be the set of boundary fixing isotopy classes of tangled trivalent graphs on $D$. The $A_2$ web space $W(c; D)$ is the quotient of the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-vector space on $G(c; D)$ by the following $A_2$ skein relation:

\[
\begin{align*}
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
&= q^{\frac{1}{2}} \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture} - q^{-\frac{1}{2}} \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture},
\end{align*}
\begin{align*}
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
&= q^{-\frac{1}{2}} \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture} - q^{\frac{1}{2}} \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture},
\end{align*}
\begin{align*}
\begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}
&= \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture} + \begin{tikzpicture}[scale=0.5]
\draw (-1,0) -- (0,0) -- (1,0);
\draw (0,0) circle (1);
\end{tikzpicture}.
\end{align*}
\]
An element in $W(\epsilon; D)$ is called web and an element in $G(\epsilon; D)$ without crossings which has no internal 0-, 2-, 4-gons a basis web. Any web is described as the sum of basis webs.

The $A_2$ skein relation realize the Reidemeister moves (R1) – (R4), that is, we can show that webs represent diagrams in the left side and right side is the same web in $W(D; \epsilon)$.

We review a diagrammatic definition of an $A_2$ clasp introduced in [Kup96, OY97, Kim07] and its properties. The $A_2$ clasp gives a coloring of strands in a web by pairs of non-negative integers. It plays an important role as is the case with the Jones-Wenzl projector.

We construct a projector called the $A_2$ clasp of type $(n, m)$ in a special web space $TL_{A_2}(n - m, +n - m)$.

**Definition 2.2** (the Temperley-Lieb category for $A_2$). Let $D = [0, 1] \times [0, 1]$ and $n$ denotes a set of $n$ points on $I = [0, 1]$ dividing it into $n + 1$ equal parts. The Temperley-Lieb category $TL_{A_2}$ is a linear category over $\mathbb{C}(q^\pm)$ is defined as follows:

- an object is a word (finite sequence) over $\{+, -\}$,
- the tensor product of two words is defined by the product (concatenation),
- the space of morphisms $TL_{A_2}(\alpha, \beta)$ is the web space $W(\bar{\alpha} \sqcup \beta; D)$ where $\bar{\alpha}$ is the word consisting of opposite signs of $\alpha$.

We identify an object $\alpha$ with length $n$ as a map $n \to \{+, -\}$ by using the order on $n$. A map $\bar{\alpha} \sqcup \beta$ means that the domain $n \subset I$ of the map $\bar{\alpha}$ is identified with the top edge $[0, 1] \times \{0\}$ and $\beta$ identified with the bottom edge $[0, 1] \times \{1\}$.

- The composition $GF \in G(\bar{\alpha} \sqcup \gamma; D)$ of $F \in G(\bar{\alpha} \sqcup \beta; D)$ and $G \in G(\bar{\beta} \sqcup \gamma; D)$ is given by gluing the top edge of $F$ and the bottom edge of $G$.
- The “tensor product” $F \otimes G \in G(\bar{\alpha_1} \bar{\alpha_2} \bar{\beta_1} \bar{\beta_2}; D)$ of $F \in G(\bar{\alpha_1} \sqcup \beta_1; D)$ and $G \in G(\alpha_2 \sqcup \beta_2; D)$ by gluing the right edge $\{1\} \times [0, 1]$ of $F$ and the left edge $\{0\} \times [0, 1]$ of $G$.

They define the composition and the tensor product on the space of morphisms by linearization. The diagrammatic description of them is in Figure 2.1.
TWIST FORMULAS AND $a_3$ TAILS OF (2, 2m)-TORUS LINKS

\[ \text{Figure 2.1. the composition and the tensor product in } TL^{A_2} \]

$TL^{A_2}$ is generated by identity morphisms $\text{id}_+ \in TL^{A_2}(+, +)$, $\text{id}_- \in TL^{A_2}(-, -)$ and the following morphisms.

\[ t^{++} = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array} ,
\text{ } t^{+-} = \begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array} ,
\text{ } t^{-+} = \begin{array}{c}
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} ,
\text{ } t^{--} = \begin{array}{c}
\text{Diagram 7} \\
\text{Diagram 8}
\end{array} . \]

Let us define one colored $A_2$ clasp $JW_{+m-n}^0$ before defining $JW_{+m-n}$. We do not describe the boundary of $D$ in what follows.

**Definition 2.3** (The $A_2$ clasp in $TL^{A_2}(+, +)$). The $A_2$ clasp $JW_{+m-n}^0$ described by a white box with $m \in \mathbb{Z}_{\geq 0}$ is defined as follows.

1. $0 = \emptyset$ (the empty diagram), $1 = \text{id}_+ \in TL^{A_2}(+, +)$
2. $m_{+1} = \frac{m}{n+1}$, $m_{-1} = \frac{m}{n+1}$, $m_{\text{min}} = \frac{m}{n+1}$, $m_{\text{max}} = \frac{m}{n+1}$, $m_{\text{mid}} = \frac{m}{n+1} \in TL^{A_2}(+, +)$

The $A_2$ clasp in $JW_{+m-n}^0 \in TL^{A_2}(-m, -m)$ is also defined in the same way.

We introduce the $A_2$ clasp $JW_{+m-n}$ in $TL^{A_2}(+m-n, +m-n)$ based on [OY97].

**Definition 2.4** (The $A_2$ clasp in $TL^{A_2}(+m-n, +m-n)$).

\[ JW_{+m-n} = m_{\min} \sum_{i=0}^{\min(m,n)} (-1)^i \frac{m}{i} \frac{n}{i} \frac{m+n+1}{i} m_{-i} n_{-i} \]

The $A_2$ clasp has the following well-known properties

**Proposition 2.5.** Let $m, n$ be non-negative integers.

1. $(JW_{+k-l}) (JW_{+m-n}) = (JW_{+m-n}) (JW_{+k-l}) = JW_{+m-n}$ for $0 \leq k \leq m$ and $0 \leq l \leq n$.
2. $F (JW_{+m-n}) = 0$ if $F \in \{t^{--}, t^{+-}, d_{+-}, d_{-+}\}$.
3. $(JW_{+m-n}) F = 0$ if $F \in \{t^{++}, t^{+-}, b^{++}, b^{+-}\}$.

In the above, we omit tensor components of identity morphisms in $F$ and $JW_{+k-l}$. 
Next, we construct a special web $JW^\alpha_\beta \in TL^2_2(\alpha, \beta)$ which has a similar properties to the $A_2$ clasp where $\alpha$ and $\beta$ are obtained by rearranging the order of $+^{m-n}$.

Let us introduce two basis webs called $H$-webs:

\[ H_{+}^{+} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}, \quad H_{+}^{-} = \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}. \]

Let $\alpha \in TL^2_2$ be a object obtained by rearranging the order of $+^{m-n}$. Then, one can construct a morphism from $+^{m-n}$ to $\alpha$ by composing $H$-webs. $\sigma(\alpha)$ denotes such a web minimizing the number of $H$-webs and it is uniquely determined. In the same way, one can construct the morphism from $\alpha$ to $+^{m-n}$ by pre-composing $H$-webs and denote it by $\tau(\alpha)$. We remark that $\sigma(\alpha)$ (resp. $\tau(\alpha)$) does not contain $H_{+}^{-}$ (resp. $H_{+}^{+}$).

**Definition 2.6.** Let $\alpha$ and $\beta$ be words obtained by rearranging $+^{m-n}$. We define the following webs:

\[
JW^\alpha_+ = \sigma(\alpha) \ (JW_{+}^{m-n}) \in TL^2_2(+^{m-n}, \alpha),
\]

\[
JW^\beta_+ = (JW_{+}^{m-n}) \tau(\beta) \in TL^2_2(\beta, +^{m-n}),
\]

\[
JW^\alpha_\beta = JW^\alpha_{+} JW^\beta_{+} \in TL^2_2(\beta, \alpha).
\]

Let us describe $JW^\alpha_\beta$ diagrammatically by $id_\beta \circ$ (the white box) $\circ id_\alpha$.

**Lemma 2.7.** $F \ (JW^\alpha_+ + ) = 0$ if $F \in \{t^+_-, t^-_+, d^+, d^-\}$ and $(JW^\beta_+) F = 0$ if $F \in \{t^+_+, t^-_-, b^+, b^-\}$.

**Proof.** We only show the first case and denote $\sigma(\alpha)$ by $\sigma$ for simplicity. We give a proof by induction on the number $h(\sigma)$ of $H_{+}^{+}$ contained in $\sigma$. If $h(\sigma) = 0$, it is clear since $JW^\alpha_{+} = JW^\alpha_{+}$. If $h(\sigma) = 1$, then $\sigma = id_{+} m \otimes H_{+}^{+} \otimes id_{-} m$ and one can prove easily. When $h(\sigma) = 2$, we choose $\alpha' \in TL^2_2$ such that $\sigma(\alpha) = (id \otimes H_{+}^{+} \otimes id) \sigma(\alpha')$. Then, $\sigma$ is one of the following webs:

\[
\text{type I: } \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}, \quad \text{type II: } \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array},
\]

\[
\text{type III: } \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}, \quad \text{type IV: } \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array},
\]

where $\sigma = \sigma(\alpha')$. Because of $h(\sigma') = k$, $JW^\alpha_{+} m$ satisfies the statement in Lemma 2.7 by the induction hypothesis. We only need to calculate a web obtained by multiplication of $F$ at the part we pictured in the above.

In the case of type I, the possibility of $F$ is $d_+$ or $d_-$. Then, the vanishing of $F (\sigma JW^\alpha_{+} m)$ is clear.

In the case of type II, it is similar to type I if $F = d_+$ or $d_-$. We only have to consider $F = t^+_+$, that is, show

\[
\sigma' \circ JW^\alpha_{+} + = 0.
\]
The construction of $\sigma$ and $\alpha' = \cdots + - + \cdots$ say that there exists $\sigma''$ such that

\[
\begin{array}{c}
\cdots \\
\sigma'' \\
\cdots
\end{array}
= 
\begin{array}{c}
\cdots \\
\sigma'' \\
\cdots
\end{array}
+ 
\begin{array}{c}
\cdots \\
\sigma'' \\
\cdots
\end{array}
\]

The first term of the RHS has $t_{++}$ and the second $d_{-+}$ on the top of $\sigma''$. Because of $h(\sigma'') = k - 1$ and the induction hypothesis, the composition of the RHS with $J\!W_{+m-n}$ vanish. One can prove type III and type IV in the same way.

**Lemma 2.8.** Let $\alpha$, $\beta$, and $\gamma$ in $\text{TL}^{A_2}$ be obtained by rearranging $+m-n$. Then, $J\!W_{\gamma}J\!W_{\beta} = J\!W_{\alpha^\ast}$.

**Proof.** It is easy to see by Lemma 2.7 because $\tau(\beta)$ is the reflection of $\sigma(\beta)$ at the horizontal line with the oppositely directed edges. \qed

Let us introduce two types of a web, “stair-step” and “triangle”. These webs also appear in [Kim06, Kim07, Yua17, FS20].

**Definition 2.9.** For positive integers $n$ and $m$, $n \begin{array}{c}1 \end{array} \begin{array}{c}n \end{array} m$ is defined by

\[
\begin{array}{c}
\begin{array}{c}1 \\
\vdots \\
m \\
\end{array} \\
\begin{array}{c}
n \\
m \\
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}1 \\
\vdots \\
m \\
\end{array} \\
\begin{array}{c}
n \\
m \\
\end{array}
\end{array}
\text{ and }

\begin{array}{c}
\begin{array}{c}
n \\
m \\
\end{array}
= 
\begin{array}{c}
\begin{array}{c}m-1 \\
\vdots \\
m \\
\end{array} \\
\begin{array}{c}
n \\
m \\
\end{array}
\end{array}
\text{ for } m > 1.
\]

Specifying a direction on an edge around the box determine the all directions of edges in the box.

**Definition 2.10.** For positive integer $n$, $n \begin{array}{c}1 \end{array} \begin{array}{c}n \end{array}$ is defined by $\begin{array}{c}1 \\
\vdots \\
n-1 \\
\end{array}$ and

\[
\begin{array}{c}
\begin{array}{c}1 \\
\vdots \\
n-1 \\
\end{array} \\
\begin{array}{c}
n \\
n-1 \\
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}n-1 \\
\vdots \\
n-1 \\
\end{array} \\
\begin{array}{c}
n \\
n-1 \\
\end{array}
\end{array}
\text{ for } n > 1. \text{ We obtain a web by specifying a direction of an edge around the triangle.}

We note a graphical description of properties of $A_2$ clasp and useful formulas.
Lemma 2.21.

\[ \begin{align*}
\gamma & = \alpha, \\
\beta & = \alpha, \\
\Lambda & = 0, \\
\Pi & = 0,
\end{align*} \]

\[ \begin{align*}
\sigma & = \sigma, \\
\Lambda & = \sigma,
\end{align*} \]

\[ \begin{align*}
\gamma & = (-q^{-\frac{1}{2}})^{mn}, \\
\delta & = (-q^{\frac{1}{2}})^{mn},
\end{align*} \]

\[ \begin{align*}
\theta & = q^{\frac{1}{2}}, \\
\xi & = q^{\frac{1}{2}},
\end{align*} \]

\[ \begin{align*}
\sigma & = \sigma, \\
\Lambda & = \sigma,
\end{align*} \]

\[ \begin{align*}
\gamma & = (-q^{-\frac{1}{2}})^{mn}, \\
\delta & = (-q^{\frac{1}{2}})^{mn},
\end{align*} \]

Proof. Graphical interpretation of the previous definitions and lemmas. One can prove formulas related to the braiding and the triangle by easy calculation, see [Yua17].

3. Twist formula

In this section, we show new twist formulas derived from a combinatorial method related to the generating function of Young diagrams. We firstly explain this method which is used in the proof of full twist formulas for the Kauffman bracket and Theorem 1.1 in [Yua17].

Let \( \{ \sigma_n(k, l) \mid 0 \leq k, l \leq n \} \) be a set of webs satisfying the following conditions:

1. \( \sigma_n(k, l) \) is basis web if \( k + l = n \),
2. \( \sigma_n(k, l) = X(k, l)\sigma_n(k + 1, l) + Y(k, l)\sigma_n(k, l + 1) \),
3. \( X(k, l)Y(k + 1, l) = qY(k, l)X(k, l + 1) \),

where \( X(k, l), Y(k, l) \in \mathbb{C}(q^{\frac{1}{2}}) \).

Proposition 3.1.

\[ \sigma_n(0, 0) = \sum_{k+l=n} \prod_{i=0}^{l-1} Y(0, j) \prod_{i=0}^{k-1} X(i, l) \binom{n}{k} q \sigma_n(k, l). \]

Proof. Let us label a lattice point \((k, l)\) in \( \mathbb{Z} \times \mathbb{Z} \) with \( \sigma(k, l) \), and a directed edge from \((k, l)\) to \((k + 1, l)\) with \( X(k, l) \), and a directed edge from \((k, l)\) to \((k, l + 1)\) with \( Y(k, l) \). We denote the label of an edge \( e \) by \( w(e) \). In the expansion of \( \sigma_n(0, 0) \), it is seen that the coefficient of \( \sigma_n(k, l) \), for \( k + l = n \), is

\[ \sum_{\gamma} \prod_{e \in \gamma} w(e) = \sum_{a_i \in \{X, Y\}} a_1a_2\cdots a_n, \]

where \( X \) (resp. \( Y \)) appears \( k \) (resp. \( l \)) times in \( a_1a_2\cdots a_n \) and \( \gamma \) is a path from \((0, 0)\) to \((k, l)\) which consists of the above directed edges. Although we describe \( X \) and \( Y \) without its coordinate in the above, it is uniquely determined. For example, \( a_1a_2a_3a_4 \cdots =
TWIST FORMULAS AND \(a_n\) TAILS OF \((2, 2m)\)-TORUS LINKS

\[XYXY \cdots = X(0, 0)Y(1, 0)Y(1, 1)X(1, 2)\cdots.\] By the condition (3), the RHS of (3.1) is
\[Y^tX^k \sum q^{\#\{i<j|a_i=X, a_j=Y\}} = Y^tX^k \left(\frac{n}{k}\right)_q.\]

In [Yua17], we used Young diagrams instead of sequences of \(X\) and \(Y\). However, it is intrinsically the same because a lattice path from \((0, 0)\) to \((k, l)\) is interpreted as a Young diagram. The formula of Theorem 1.1 is obtained from Proposition 3.1 by setting \(n = d = \min\{s, t\}\) and
\[\sigma_n(k, l) = \begin{cases} \begin{array}{c} \vdots \\ 1 \\ s \end{array} & \text{if } d = s, \\ \begin{array}{c} \vdots \\ 1 \\ s \\ t \end{array} & \text{if } d = t. \end{cases}\]

In a similar way, we will prove the following formula.

**Theorem 3.2** (parallel full twist formula). Let \(d = \min\{s, t\}\) and \(\delta = |s - t|\).
\[\begin{align*}
&= q^{\frac{n}{2}} \sum_{l=0}^{\infty} q^{\frac{n}{2} - \frac{l}{2}} q^{-l(t+1)} (q)_l \left(\frac{s}{q}, q, l, q, t\right) \\
&= q^{-\frac{d(x+y)}{2} - \frac{d}{2}} \sum_{k=0}^{d} q^{k(k+\delta)+\frac{k}{2}} \left(q\right)^{d+\delta} (q)_k (q)_d (q)_{d-k} (q)_{d-k} (q)_{d-k}.
\end{align*}\]

To prove this formula by Proposition 3.1, we set
\[\begin{align*}
\sigma_d(k, l) &= \begin{cases} \begin{array}{c} \vdots \\ 1 \\ s \end{array} & \text{if } d = s, \\ \begin{array}{c} \vdots \\ 1 \\ s \end{array} & \text{if } d = t. \end{cases}
\end{align*}\]

We remark that it is easy to see that \(A_2\) clasps in the middle of \(\sigma_d(k, l)\) can be removable from the definition of \(A_2\) clasp and Lemma 2.7.

**Proposition 3.3.**
\[\sigma_d(k, l) = q^{\frac{n}{2}(d+\delta-l)} \sigma_d(k+1, l) + q^{-\frac{d}{2}} q^{\frac{k+d}{4}} q^{-\frac{n}{2}(d+\delta-l)} (1 - q^{d+\delta-l}) \sigma_d(k, l+1)\]

We can confirm that \(X(k, l) = q^{-\frac{n}{2}(d+\delta-l)}\) and \(Y(k, l) = q^{-\frac{d}{2}} q^{\frac{k+d}{4}} q^{-\frac{n}{2}(\alpha-l)} (1 - q^{\alpha-l})\) satisfy (2) of Proposition 3.1. Thus, we can prove Theorem 3.2 by applying Proposition 3.1. This equation is obtained as a sequel to a calculation of \(A_2\) webs. We will prove it in the case of \(d = s\), but the otherwise is proven in the same way.
Lemma 3.4. Let $n$ and $i$ are positive integers satisfying $i < n$.

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= q^{\frac{i}{2}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
+ q^{-\frac{i}{2}} q^{-\frac{n}{2}} q^\frac{i}{2} (1 - q)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array}
\end{array}
\]

Proof. We will make a red mark on a place where we use the skein relation and Lemma 2.11.

Lemma 3.5. Let $n$ be a positive integer.

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 3}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= q^{\frac{n}{2}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
+ q^{-\frac{n}{2}} q^{-\frac{n}{2}} (1 - q^n)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Diagram 4}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 5}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= q^{\frac{n}{2}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
+ q^{-\frac{n}{2}} q^{-\frac{n}{2}} (1 - q^n)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Diagram 6}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 7}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= q^{\frac{n}{2}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
+ q^{-\frac{n}{2}} q^{-\frac{n}{2}} (1 - q^n)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Diagram 8}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 9}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= q^{\frac{n}{2}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
+ q^{-\frac{n}{2}} q^{-\frac{n}{2}} (1 - q^n)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Diagram 10}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 11}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
= q^{\frac{n}{2}}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
+ q^{-\frac{n}{2}} q^{-\frac{n}{2}} (1 - q^n)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Diagram 12}
\end{array}
\end{array}
\end{array}
\]
Proof. By using Lemma 3.4, we obtain
\[
q^{2\frac{i}{2}} + q^{-\frac{i}{2}} q^{-\frac{n}{2}} (1 - q^i) = q^{2(i+1)} + q^{-\frac{i}{2}} q^{-\frac{n}{2}} (1 - q^{i+1}).
\]
We repeatedly apply the above equation to the RHS of Lemma 3.4 at \(i = 0\). □

Lemma 3.6. Let \(i\) and \(j\) be positive integers.
\[
q^{2\frac{i}{2}} + q^{-\frac{i}{2}} q^{-\frac{j}{2}} (1 - q^i) = q^{2\frac{j}{2}} + q^{-\frac{j}{2}} q^{-\frac{i}{2}} (1 - q^j).
\]
Proof. By Lemma 3.5,
\[
q^{2\frac{i}{2}} + q^{-\frac{i}{2}} q^{-\frac{j}{2}} (1 - q^i) = q^{2\frac{j}{2}} + q^{-\frac{j}{2}} q^{-\frac{i}{2}} (1 - q^j).
\]
As we remarked before Proposition 3.3, the definition of \(A_2\) clasp and its property says that
\[
q^{2\frac{i}{2}} = q^{2\frac{j}{2}} = (-q^{\frac{1}{2}})^2(i-1).
\] □
Proof of Proposition 3.3. Apply Lemma 3.6 to $\sigma_d(k, l)$. We have to confirm the web in the second term is equal to $\sigma_d(k, l + 1)$. It is deformed as follows.

In the first equation, we remove an $A_2$ clasp by using its definition and property of the triangle in Lemma 2.11. One can also apply the same deformation on the right side of the web and obtain the following equation:

\[
\begin{array}{c}
\text{(3.2)} \\
\end{array}
\]

Theorem 3.7 (parallel $m$-full twist formula). Let $k = (k_1, \ldots, k_m)$ be an $m$-tuple of integers, $k_0 = d = \min\{s, t\}$, and $\delta = |s - t|$. For a positive integer $m$, we denote a right-handed $m$ half twist by

\[
\begin{array}{c}
\text{m full twists} \\
\end{array}
\]

where

\[
D(k) = q^{\sum_{i=1}^{m} k_i(k_i + \delta) + \frac{d}{2}\left(k_0(k_0 - k_m) + \frac{d}{2}\right)} \sum_{k_0 \geq k_1 \geq \cdots \geq k_m \geq 0} D(k) \left( \frac{d}{k_0 + \delta}, \frac{d}{k_m + \delta}, \frac{d}{k_0 + \delta} \right)_{q}
\]

and $k'_i = k_i - k_{i+1}$ for $i = 0, 1, \ldots, m - 1$.

For a positive integer $m$, we denote a right-handed $m$ half twist by

\[
\begin{array}{c}
\text{m crossings} \\
\end{array}
\]
Proof. By sliding the right-hand triangle,

\[
\begin{align*}
&= \left( (-1)^{d-k} q^{\frac{k}{2}} (d-k)^2 - \frac{8}{3} (d-k) q^{\frac{k+1}{2}} (d-k)(k+s-d) q^{\frac{k-1}{2}} (d-k)(k+t-d) \right)^2 \\
&\times \left( -q^{\frac{k}{2}} \right)^{2k(s-d)(d-k)} \left( -q^{\frac{k}{2}} \right)^{2(k+s-d)(d-k)}
\end{align*}
\]

and

\[
\begin{align*}
&= \left( -q^{\frac{k}{2}} \right)^{2k(s-d)(d-k)} \left( -q^{\frac{k}{2}} \right)^{2(k+s-d)(d-k)}
\end{align*}
\]

Let \( m \) be positive integers and \( \{k_i\} \) are non-negative integers satisfying \( k_i \geq k_{i+1} \) for \( 0 \leq i \leq m - 1 \). We set \( m_i = (m - i) \), \( k_0 = d \), \( t_i = k_i + (t-d) \), and \( s_i = k_i + (s-d) \).
Then, as the above calculation, we obtain

\[
\begin{align*}
&= q^{\frac{m}{3}(k_{i-1} - k_i)^2 - m_s(k_{i-1} - k_i)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + s - d)} q^{\frac{m - 1}{3}(k_{i-1} - k_i)(k_i + t - d)} \\
&\times q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + t - d)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + s - d)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + t - d)} \\
&= q^{\frac{m}{3}(k_{i-1} - k_i)^2 - m_s(k_{i-1} - k_i)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + s - d)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + t - d)} \\
&\times q^{\frac{m}{3}(k_{i-1} - k_i)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + s - d)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + t - d)} \\
&= q^{\frac{m}{3}(k_{i-1} - k_i)^2 - m_s(k_{i-1} - k_i)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + s - d)} q^{\frac{m}{3}(k_{i-1} - k_i)(k_i + t - d)}
\end{align*}
\]

Let \( \tau_0 \) be the parallel \( m \) full twist in the LHS of Theorem 3.2 and
for $i = 1, 2, \ldots, m - 1$. Then, we apply Theorem 3.2, the index of summation is $k_{i+1}$, to the left-most full twist in the box $2m_i$ of $\tau_i$ and the above deformation.

$$
\tau_i = q^{-\frac{1}{2}} k_i (k_i + \delta) - \frac{1}{2} \sum_{k_{i+1}=0}^{k_i} q^{k_{i+1}(k_{i+1} + \delta) + \frac{k_i+1}{2}} \left( \frac{q}{q} \right)^{k_i + \delta} \left( \frac{k_i}{k_{i+1}} \right) q
$$

$$
\times q^{\frac{m_{i+1} - m_i}{2} k_i (k_i + \delta) - m_{i+1} k_i} q^{\frac{m_{i+1} - m_i}{2} k_{i+1}(k_{i+1} + \delta) + m_{i+1} k_{i+1}}
$$

$$
= \sum_{k_{i+1}=0}^{k_i} q^{-\frac{m_{i+1} - m_i}{2} k_i (k_i + \delta) + \frac{m_{i+1} - m_i}{2} k_{i+1}(k_{i+1} + \delta) - \frac{1}{2} (k_i - k_{i+1}) q^{m_{i+1} - m_i + m_{i+1} k_{i+1}}
$$

$$
\times q^{k_i - k_{i+1}} k_{i+1} (k_{i+1} + \delta) \left( \frac{q}{q} \right)^{k_{i+1} + \delta} \left( \frac{k_{i+1}}{k_i} \right) q
$$

where the shaded box with $\tau_{i+1}$ means replacement of the box with $\tau_{i+1}$. We remark that $m_0 = m, t_0 = t, s_0 = s$, and $\min\{s_{i-1}, t_{i-1}\} = k_{i-1}$. By using a similar way to (3.2), we can confirm that
Consequently,

\[
\tau_0 = \sum_{k_0 \geq k_1 \geq \ldots \geq k_m \geq 0} \prod_{i=0}^{m-1} q^{-\frac{m}{2} k_i (k_i + \delta) + \frac{m+1}{2} k_{i+1} (k_{i+1} + \delta)} q^{-\frac{1}{2} (k_i - k_i + 1)} q^{-m k_i + m_{i+1} k_{i+1}}
\]

\[
\times q^{k_{i+1} (k_{i+1} + \delta) + k_i} \left( \frac{q}{(q)} k_{i+1} \right) \left( \frac{k_i}{(q)_{k_i+1}} \right) q \]

\[
= \sum_{k_0 \geq k_1 \geq \ldots \geq k_m \geq 0} q^{-\frac{m}{2} k_0 (k_0 + \delta)} q^{-\frac{1}{2} (k_0 - k_m)} q^{-m k_0} q^{\sum_{i=0}^{m-1} k_{i+1} (k_{i+1} + \delta) + k_i}
\]

\[
\times \left( \frac{q}{(q)_{k_0+\delta}} \right) \left( \frac{q}{(q)_{k_m+\delta}} \right) \left( \frac{q}{(q)_{k_0-k_1}} \frac{q}{(q)_{k_1-k_2}} \cdots \frac{q}{(q)_{k_{m-1}-k_m}} \frac{q}{(q)_{k_m}} \right)
\]

\[
= q^{-\frac{m}{2} k_0 (k_0 + \delta) - m k_0} \sum_{k_0 \geq k_1 \geq \ldots \geq k_m \geq 0} q^{\frac{1}{2} (k_0 - k_m)} q^{\sum_{i=1}^{m-1} k_i (k_i + \delta) + k_i}
\]

\[
\times \left( \frac{q}{(q)_{k_0+\delta}} \right) \left( \frac{q}{(q)_{k_m+\delta}} \right) \left( \frac{q}{(q)_{k_0-k_1}} \frac{q}{(q)_{k_1-k_2}} \cdots \frac{q}{(q)_{k_{m-1}-k_m}} \frac{q}{(q)_{k_m}} \right)
\]

\[
\square
\]

4. APPLICATION TO THE sl3 TAIL

In this section, we discuss the sl3 tail of (2, 2m)-torus link with one-row colorings. A tail of a link \( L \) is a limit of the sl\( \frac{3}{2} \) colored Jones polynomials \( \{ J_n^{sl\frac{3}{2}} (L) \}_n \). The existence of the tail is proven for adequate links in [Arm13]. The tail of (2, 2m) torus link is given by the false theta series and its explicit formula was obtained in [Haj16]. The tail or such stability of the \( q \) colored Jones polynomials of torus knots in a case where \( q \) is the rank 2 simple Lie algebra was shown by Garoufalidis and Vuong [GV17]. In the same paper, They gave explicit formulas of the tail for (2, 2m + 1)- and (4, 5)-torus knots in the case of \( q = sl3 \). The author also gave two explicit formulas of the sl\( \frac{3}{2} \) tail for the (2, 2m)-torus link with anti-parallel orientation in [Yua18] and it gave Andrews-Gordon type identities for a “diagonal summand” of the sl\( \frac{3}{2} \) theta series, see also Section 9 in [BKM19].

In the following, we give an explicit formula of one-row colored sl\( \frac{3}{2} \) Jones polynomial \( J^{sl\frac{3}{2}} (T_{2m} (2, 2m)) \) for the (2, 2m)-torus link with parallel orientation and its sl\( \frac{3}{2} \) tail. We use the following normalization of sl\( \frac{3}{2} \) colored Jones polynomial for an oriented framed link. Let \( D = D_1 \cup D_2 \cup \ldots \cup D_p \) be a link diagram of a framed link \( L \) with \( p \) components through the blackboard framing. A coloring \( \omega_i \) of \( D_i \) is given by an irreducible representations of sl\( \frac{3}{2} \) with the highest weight \( (m_i, n_i) \) for \( p = 1, 2, \ldots, p \). One can identify the coloring \( \omega_i \) and \( (m_i, n_i) \) \( \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \). Let \( D(\omega) \) be the \( A_2 \) web obtained by replacing \( D_i \) with \( JW_{\omega_i} \) where the orientation of the strand colored by \( m \) coincides with the orientation of \( D_i \) for all \( i = 1, 2, \ldots, p \). See Figure 4.1

**Definition 4.1.** \( J^{sl\frac{3}{2}} (L) = D(\omega) \in \mathbb{C}(q^{\frac{1}{2}}) \).
Remark 4.2. The above normalization of the \( sl_3 \) colored Jones polynomial of a link is different from in [Yua18].

Lemma 4.3. Let \( s \) and \( t \) be positive integers and \( d = \min\{s, t\} \). For any integer \( 0 \leq k \leq d \),

\[
\frac{\Delta(s, 0) \Delta(t, 0)}{\Delta(d - k, 0)}
\]

where \( \Delta \) is an oriented circle.

Proof. One can calculate the above web by using the following formulas (see, for example, [Yua17]).

\[
\Delta(n, 0) = \frac{(n+1)[n+2]}{n-k+1} \Delta(n-k, 0)
\]

\[
\Delta(n, 0) = \frac{(n+1)[n+2]}{n-k+1} \Delta(n-k, 0)
\]

\[
q^{\sum_{i=1}^{m} k_i (k_i+\delta)} q^{d-k_m} \frac{(q)^{d+\delta}}{(q)^{d-k_m} (q)_{k_m-1-k_m}} \Delta(s, 0) \Delta(t, 0)
\]

\[
\frac{1 - q^2}{1 - q^{d-k_m+1}} \Delta(s, 0) \Delta(t, 0)
\]

where \( d = \min\{s, t\} \) and \( \delta = |s - t| \).

Proof. One can easily obtain the above formula from Theorem 3.7, Lemma 4.3, and

\[
\Delta(n, 0) = \frac{(n+1)[n+2]}{2} = q^{-n} (1 - q^{n+1}) (1 - q^{n+2})
\]

\[
(1 - q)(1 - q^2)
\]
Let us consider a family of formal power series \( \{ f_n(q) \in \mathbb{Z}[[q]] \mid n \in \mathbb{Z}_{\geq 0} \} \) and \( F(q) \in \mathbb{Z}[[q]] \). Then, we define \( \lim_{n \to \infty} f_n(q) = F(q) \) if \( f_n(q) = F(q) \) in \( \mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]] \) for any \( n \).

We review an explicit formula of the \( \mathfrak{sl}_3 \) tail of the \((2, 2m)\)-torus link with anti-parallel orientation \( T_{\pm}(2, 2m) \).

**Theorem 4.5** (The \( \mathfrak{sl}_3 \) colored Jones polynomial of \( T_{\pm}(2, 2m) \) [Yua18]). For any one-row colorings \( \omega_1 = (s, 0) \) and \( \omega_2 = (t, 0) \),

\[
\begin{align*}
&f_{\omega_1}^{\mathfrak{sl}_3}(T_{\pm}(2, 2m)) = q^{-2\pi i d(d+1)-2mmd} \\
&\quad \times \sum_{d\geq k_1 \geq \cdots \geq k_m \geq 0} q^{\sum_{i=1}^{m} k_i (k_i+\delta) + 2k_i} q^{2d-2km} \frac{(q)_{d+\delta}}{(q)_{km+\delta}} \frac{(q)_{d-k} (q)_{k_1-k_2} \cdots (q)_{k_{m-1}-k_m} (q)_{km}} \\
&\quad \times \frac{(1-q)(1-q^2)}{(1-q^{d-km+1})(1-q^{d-km+2})} \Delta(s, 0) \Delta(t, 0)
\end{align*}
\]

where \( d = \min\{s, t\} \) and \( \delta = |s-t| \).

**Theorem 4.6** (The \( \mathfrak{sl}_3 \) tail of \( T_{\pm}(2, 2m) \) [Yua18]). Let \( \omega_n \) be a special coloring of \( T_{\pm}(2, 2m) \) such that \( s = t = n \). Then,

\[
\lim_{n \to \infty} f_n(q) = \frac{1}{(1-q)(1-q^2)} \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{-2km} q^{\sum_{i=1}^{m} k_i^2 + 2k_i} \frac{(q)_{k_1-k_2}(q)_{k_2-k_3} \cdots (q)_{k_{m-1}-k_m} (q)_{km}}
\]

where \( f_n(q) = q^{\frac{2\pi i d}{n}} q^2 \mathcal{J}_{\omega_n}^{\mathfrak{sl}_3}(T_{\pm}(2, 2m)) \).

We obtain an explicit formula of the \( \mathfrak{sl}_3 \) tail of \( T_{\pm}(2, 2m) \) from Theorem 4.4.

**Theorem 4.7** (The \( \mathfrak{sl}_3 \) tail of \( T_{\pm}(2, 2m) \)). Let \( \omega_n \) be a special coloring of \( T_{\pm}(2, 2m) \) such that \( s = t = n \). Then,

\[
\lim_{n \to \infty} f_n(q) = \frac{1}{(1-q^2)(1-q^3)} \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{-km} q^{\sum_{i=1}^{m} k_i^2 + k_i} \frac{(q)_{k_1-k_2}(q)_{k_2-k_3} \cdots (q)_{k_{m-1}-k_m} (q)_{km}}
\]

where \( f_n(q) = q^{\frac{2\pi i d}{n} + mn} q^n \mathcal{J}_{\omega_n}^{\mathfrak{sl}_3}(T_{\pm}(2, 2m)) \).

**Proof.** It is easy to see that \( q^{k^2+k_1} \frac{(q)_{n-k_1}}{(q)_{n-k_1}} = q^{k^2+k_1} \in \mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]] \) and \( q^{k^2} \frac{(q)_{n}}{(q)_{km}} = q^{k^2} \in \mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]] \). \( \square \)

**Acknowledgment.** This work was supported by Grant-in-Aid for JSPS Fellows Grant Number 19J00252 and Grant-in-Aid for Early-Career Scientists Grant Number 19K14528.

**REFERENCES**

[AD11] Cody Armond and Oliver T. Dasbach, Rogers–ramanujan type identities and the head and tail of the colored Jones polynomial, arXiv:1106.3948 (2011).

[Arm13] Cody Armond, The head and tail conjecture for alternating knots, Algebr. Geom. Topol. 13 (2013), no. 5, 2809–2826. MR 3116304

[BKM19] Kathrin Bringmann, Jonas Kaszian, and Antun Milas, Higher depth quantum modular forms, multiple Eichler integrals, and \( \mathfrak{sl}_3 \) false theta functions, Res. Math. Sci. 6 (2019), no. 2, Paper No. 20, 41. MR 3914943

[BM15] Kathrin Bringmann and Antun Milas, \( \mathcal{W} \)-algebras, false theta functions and quantum modular forms, i, Int. Math. Res. Not. IMRN (2015), no. 21, 11351–11387. MR 3456046
[BM17] ______, W-algebras, higher rank false theta functions, and quantum dimensions, Selecta Math. (N.S.) 23 (2017), no. 2, 1249–1278. MR 3624911

[BO17] Paul Beirne and Robert Osburn, q-series and tails of colored Jones polynomials, Indag. Math. (N.S.) 28 (2017), no. 1, 247–260. MR 3597046

[CM14] Thomas Creutzig and Antun Milas, False theta functions and the Verlinde formula, Adv. Math. 262 (2014), 520–545. MR 3228436

[CM17] ______, Higher rank partial and false theta functions and representation theory, Adv. Math. 314 (2017), 203–227. MR 3658716

[DL06] Oliver T. Dasbach and Xiao-Song Lin, On the head and the tail of the colored Jones polynomial, Compos. Math. 142 (2006), no. 5, 1332–1342. MR 2264669

[DL07] ______, A volumish theorem for the Jones polynomial of alternating knots, Pacific J. Math. 231 (2007), no. 2, 279–291. MR 2346497

[EH17] Mohamed Elhamdadi and Mustafa Hajij, Pretzel knots and q-series, Osaka J. Math. 54 (2017), no. 2, 363–381. MR 3657236

[FS20] Charles Frohman and Adam S. Sikora, SU(3)-skein algebras and web on surfaces, arXiv:2002.08151 (2020).

[GL15] Stavros Garoufalidis and Thang T. Q. Lê, Nahm sums, stability and the colored Jones polynomial, Res. Math. Sci. 2 (2015), Art. 1, 55. MR 3375651

[GMV13] Stavros Garoufalidis, Hugh Morton, and Thao Vuong, The SL_3 colored Jones polynomial of the trefoil, Proc. Amer. Math. Soc. 141 (2013), no. 6, 2209–2220. MR 3034446

[GV17] Stavros Garoufalidis and Thao Vuong, A stability conjecture for the colored Jones polynomial, Topology Proc. 49 (2017), 211–249. MR 3570390

[Haj16] Mustafa Hajij, The tail of a quantum spin network, Ramanujan J. 40 (2016), no. 1, 135–176. MR 3485997

[Jon83] V. F. R. Jones, Index for subfactors, Invent. Math. 72 (1983), no. 1, 1–25. MR 699057

[Kim06] Dongseok Kim, Trihedron coefficients for U_q(sl(3, C)), J. Knot Theory Ramifications 15 (2006), no. 4, 453–469. MR 2221529

[Kim07] ______, Jones-Wenzl idempotents for rank 2 simple Lie algebras, Osaka J. Math. 44 (2007), no. 3, 691–722. MR 2360947

[KO16] Adam Keilthy and Robert Osburn, Rogers-Ramanujan type identities for alternating knots, J. Number Theory 161 (2016), 255–280. MR 3435728

[Kup94] Greg Kuperberg, The quantum G_2 link invariant, Internat. J. Math. 5 (1994), no. 1, 61–85. MR 1265145

[Kup96] ______, Spiders for rank 2 Lie algebras, Comm. Math. Phys. 180 (1996), no. 1, 109–151. MR 1403861

[Law03] Ruth Lawrence, The PSU(3) invariant of the Poincare homology sphere, Proceedings of the Pacific Institute for the Mathematical Sciences Workshop “Invariants of Three-Manifolds” (Calgary, AB, 1999), vol. 127, 2003, pp. 153–168. MR 1953324

[Lie97] W. B. R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978

[OY97] Tomotada Ohtsuki and Shuji Yamada, Quantum SU(3) invariant of 3-manifolds via linear skein theory, J. Knot Theory Ramifications 6 (1997), no. 3, 373–404. MR 1457194

[Wen87] Hans Wenzl, On sequences of projections, C. R. Math. Rep. Acad. Sci. Canada 9 (1987), no. 1, 5–9. MR 873400

[Yua17] Wataru Yuasa, The sl_3 colored Jones polynomials for 2-bridge links, J. Knot Theory Ramifications 26 (2017), no. 7, 1750038, 37. MR 3660093

[Yua18] ______, A q-series identity via the sl_3 colored Jones polynomials for the (2, 2m)-torus link, Proc. Amer. Math. Soc. 146 (2018), no. 7, 3153–3166. MR 3787374

Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

E-mail address: wyuasa@kurims.kyoto-u.ac.jp