Insuring uninsurable income∗

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Abstract

In an exchange economy composed of risk-averse individuals facing shocks, these shocks may become uninsurable if they are private information. Efficient shock smoothing across the economy requires self-reporting of the shocks that each individual privately suffers. This efficient allocation of resources leads to growing inequality between individuals, and the problem remains largely unsolved. This paper proposes a solution. The proposed mechanism (i) promises within-period full insurance by postponing risks; (ii) does not lead to social ranking among the same age groups; and (iii) is sustainable under some conditions.

Key words: Incentives, optimal resource allocation, recursive contract, equality of same age group.

1 Introduction

In an exchange economy composed of risk-averse individuals facing shocks, these shocks may become uninsurable if they are private information. This problem has been studied extensively since the work of Green (1987). These shocks are uninsurable because efficient shock smoothing across the economy requires self reporting of shocks that each individual privately suffers. Proposed solutions are limited. Inducing truth-telling at the expense of equality leads to immiserization, where the average expected lifetime utility of individuals converge to negative infinity while driving inequality without bound (Green 1987, Thomas and Worrall 1990). If discount factor is approximately one, meaning individuals value the future and care little about the present, their contracts approach to the first best (Thomas and Worrall 1990, Carrasco et al. 2019). Atkeson and Lucas (1992) consider a scenario where discount factors represent resource prices, and social planners trade claims to current and future resources at these prices. They show that full information is required for efficient allocation.

This paper extends an alternative mechanism proposed by Marcet and Marimon (1992), which distributes risk across periods for the same individual. This approach could reduce the inequality observed in previous studies that allocate risk among individuals within each period. The mechanism proposed by Marcet and Marimon (1992) was an efficient growth mechanism combining state-contingent investment and transfers. This paper uses it for transfers in an exchange economy facing income shocks.

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The first key finding of this paper is that our mechanism can achieve both efficiency and avoidance of immiserization under certain conditions — specifically, when individuals’ degree of risk aversion is not excessively high relative to the disparity between their highest and lowest possible incomes. Furthermore, the mechanism preserves the equality of opportunity within the same age group. Both efficiency and equality of opportunity are achieved by shifting risks to future periods. However, if the degree of risk aversion is too high relative to the disparity in income levels, challenges may arise. In such cases, it may become necessary to temporarily adjust the optimal transfer rules. This disruption occurs when the promised lifetime utility cannot be defined, given the assumption that each individual’s utility function is bounded from above. Nevertheless, even when the sufficient condition is not met, the mechanism continues to maintain the optimal allocation under the given condition and avoids immiserization.

The second key finding pertains to the potential sustainability of this mechanism in a society with a continuum of individuals, where the population, consisting of different age group, remains constant over time. In such a society, each individual faces a probability of dying in the next period, while new individuals continuously enter. Since risks exist across periods, maintaining unchanged expected lifetime utility across periods incurs costs. To ensure sustainability, earlier entrants must bear these costs, resulting in average consumption increasing with age. However, this does not lead to the formation of social ranking within the same age group. Importantly, this mechanism can ensure higher expected lifetime utility compared to autarky for all generations, while sustaining a positive balance of payments over time. Thus, a society that achieves both efficiency and equality of opportunity is feasible.

The paper is structured as follows. Section 2 introduces the notation and terminology and proposes the mechanism for the inter-period risk transfers. Section 3 investigates the potential sustainability of the mechanism and examines the properties for individuals. Section 4 investigates the potential sustainability of the mechanism. Section 5 provides some numerical examples. Section 6 evaluates the mechanism’s efficiency again using Bellman equation. Section 7 summarizes the conclusions.

2 Model

Consider an infinitely existing risk-neutral government for a society with a continuum of individuals in a unit interval. Each individual is risk-averse and will be alive in the next period with probability \( \alpha \in (0, 1) \). In each period, newly born risk-averse individuals enter the society, holding the total population constant. Individuals differ only in their periods of birth and death. Each individual has a discount factor \( r \in (0, 1) \). Let \( \beta = \alpha r \) be the multiplication of \( r \) and \( \alpha \). A similar stationary population model is used in Fujiwara-Greve and Okuno-Fujiwara (2009) for their Prisoner’s Dilemma game. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \( \{e_t\}_{t \in \mathbb{N} \cup \{0\}} \) be a sequence of integrable random variables on \((\Omega, \mathcal{F}, P)\) denoting identically and independently distributed income shocks of each individual in a sequence of periods. These income shocks take values in \( E := \{e^1, e^2, \ldots, e^M\} \), where \( e^i \in \mathbb{R}_+ \) with \( e^i < e^j \) for \( i < j \in \{1, \ldots, M\} \), \( M \geq 2 \). The government considers a transfer mechanism \( \mathcal{M} \) that repeats from period zero to infinity, using a sequence of transfers (controls) \( \{\tau_t\}_{t=0}^\infty \), \( \tau_t \in \mathbb{R} \) and a sequence of utility control variables \( \{\lambda_t\}_{t=0}^\infty \), \( \lambda_t \in \mathbb{R}_{++} \) which will be specified later.
2.1 Pareto optimal transfer mechanism with a utility control variable $\lambda_0$

First, we derive an optimal transfer function which, under symmetric income information, ensures a level of utility corresponding to a utility control variable $\lambda_0 > 0$. Following [Marcet and Marimon (1992)], we will use this utility control variable $\lambda_0$ as a state variable under asymmetric income information. For this purpose, in this subsection only, we consider the case where individuals cannot hide their income from the government in each period.

In period 0, the government is considering an efficient transfer mechanism that increases individuals’ lifetime utility as much as possible, but the government cannot allow an infinite net subsidy. The government solves the following problem, having decided on an upper bound for the net subsidy.

\[
\begin{align*}
\max_{\{\tau_t\}_{t=0}^\infty} & \quad (1 - \beta) E_0 \left[ \sum_{t=0}^\infty \beta^t u(c_t) \right] \\
\text{subject to} & \quad (1 - \beta) E_0 \left[ \sum_{t=0}^\infty \beta^t (\tau_t) \right] \leq \text{Const},
\end{align*}
\]

where $E_0$ is the conditional expectation given $e_0$, $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ is the individual’s utility function, $c_t = e_t + \tau_t$ is the consumption in period $t$ starting from the individual’s birth period. The utility function $u$ is assumed to be $u' > 0$, $u'' < 0$, to satisfy the Inada conditions ($\lim_{c \rightarrow 0} u'(c) = +\infty$, $\lim_{c \rightarrow \infty} u'(c) = 0$) and to be bounded from above.

Without specifying the upper bound of the net subsidy, $\{\tau_t\}_{t=0}^\infty$ is determined by $\lambda_0$, and the same $\lambda_0$ determines the upper bound of the net subsidy as well. Thus, the problem (*) can be rewritten as the following problem, where the objective function is the Lagrangian of problem (*) with an undetermined multiplier $1/\lambda_0 > 0$.

\[
\max_{\{\tau_t\}_{t=0}^\infty} (1 - \beta) E_0 \left[ \sum_{t=0}^\infty \beta^t \lambda_0 u(c_t) - \tau_t \right].
\]

From the first order condition,

\[ u'(c_t) = \frac{1}{\lambda_0} \text{ for } t \in \mathbb{N} \cup \{0\} \]

and we obtain $c_t = (u')^{-1}(\lambda_0^{-1})$ and $\tau_t = (u')^{-1}(\lambda_0^{-1}) - e_t$ for $t \in \mathbb{N} \cup \{0\}$. This is the Pareto optimal transfer mechanism that gives the individual constant consumption in each period.

Let $v_1 : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a function expressing each individual’s lifetime utility provided by the transfer mechanism. Let $v_2 : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a function expressing the expected balance of transfers. Then, for given $\lambda_0 > 0$ and $e_0 \in E$, these values are written as follows.

\[
\begin{align*}
v_1(\lambda_0, e_0) & := (1 - \beta) E_0 \left[ \sum_{t=0}^\infty \beta^t u(c_t) \right] = u((u')^{-1}(\lambda_0^{-1})) =: \overline{v}_1(\lambda_0) \\
v_2(\lambda_0, e_0) & := (1 - \beta) E_0 \left[ \sum_{t=0}^\infty \beta^t (-\tau_t) \right].
\end{align*}
\]

Note that the function $v_1$ could be seen as a function of the value $\lambda_0$ whose inverse was the Lagrangian multiplier. We write it as $\overline{v}_1$. The function $\overline{v}_1$ is bijective and increasing.

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On the other hand, we see that $v_2(\lambda_0, e_0)$ depends on income. So we write its expectation as $\nu_2(\lambda_0)$.

\[ \nu_2(\lambda_0) := E[v_2(\lambda_0, e_t)]. \]

In Section 4 the government will adjust the value of $\lambda_0$ so that the expected balance of total transfers across generations is non-negative.

Let $\tau^*(\cdot | \lambda_0)$ be a time-invariant transfer function, which is the solution to the problem $[\Pi]$. That is, $\tau^*: \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$ is defined such that, for a reported income $x \in \mathbb{R}_+$ and a utility control variable $\lambda > 0$

\[ \tau^*(x|\lambda) := (u')^{-1}(\lambda^{-1}) - x. \]

We will use these notation in the following section for the case where income is private information. There we observe that a utility control variable $\lambda_t$ can play the role of a state variable and the role of an incentive device. Let $\mathcal{T}$ be the set of all admissible control functions there, and let $\tau: \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$ be a typical element of $\mathcal{T}$. Then the lifetime utility of individuals who are guaranteed utility level $\lambda_t$ at an initial period $t$ is given by

\[ J(\lambda_t, t, \tau) = (1 - \beta)E_t \left[ \sum_{k=t}^{\infty} \beta^{k-t}u(e_k + \tau(e_k|\lambda_t)) \right] \quad \text{for} \quad \tau \in \mathcal{T}, \tag{2} \]

where $E_t$ denotes the conditional expectation given $e_t$. We will see that $\tau^* \in \mathcal{T}$ could be an optimal control function for any initial period $t$ and for later specified $\{\lambda_k\}_{k=t}^{\infty}$ such that

\[ J(\lambda_t, t, \tau^*) = \sup_{\tau \in \mathcal{T}} (1 - \beta)E_t \left[ \sum_{k=t}^{\infty} \beta^{k-t}u(e_k + \tau(e_k|\lambda_t)) \right]. \tag{3} \]

2.2 Pareto optimal transfer mechanism with asymmetric information

If income is private information, the government has to ask individuals to declare their income. In general, a mechanism $\Gamma$ that arranges transfers $\{\tau_t\}$ needs to satisfy sequential incentive compatibility defined below. For this definition, let $z$ denote the current observable state, which can be a finite vector of state variables, and may include the past history of observables or be in a recursive form. The distribution of the tomorrow’s state $z'$ is determined by this $z$ and a vector of variables determining the current level of consumption $x$. We write $\pi(x)$ for the value of each individual’s current payoff and $v(z)$ for the present value of the mechanism $\Gamma$ at $z$.

**Definition 1.** (Sequential incentive compatibility, Marcet and Marimon 1992) The mechanism $\Gamma$ is said to be sequentially incentive compatible if for every state $z$

\[ v(z) = (1 - \beta)\pi(x) + \beta E_{(x,z)}[v(z')] \geq (1 - \beta)\pi(\tilde{x}) + \beta E_{(\tilde{x},z)}[v(z')], \]

where $E_{(x,z)}$ means that the conditional expectation given $(x, z)$.

We further evaluate a mechanism according to the following notion.

**Definition 2.** (Sequentially efficient mechanism, Marcet and Marimon 1992) The mechanism $\Gamma$ is said to be sequentially efficient mechanism if it is sequentially incentive com-
patible and it is not Pareto dominated by any other sequentely incentive compatible mechanism.

In addition to taking these things into account, since it is known to lead to an immis-
arization outcome, the government does not hope to manipulate the transfer function \( \tau^* \). The government takes an alternative approach. It is to vary the utility control variable \( \lambda \) in each period. This could be an incentive to tell the truth. For a given \( \lambda \) in a given period, the government promises to ensure that the lifetime utility level \( \overline{\tau}_1(\lambda) \), and the utility control variable for the next period is renewed taking into account the current utility control variable and the income reported in the current period. If an individual truthfully reports income \( e \) in a period \( t \), then the lifetime utility from that period should be

\[
\overline{\tau}_1(\lambda) = E\left[ (1 - \beta)u(e + \tau^*(e|\lambda)) + \beta \overline{\tau}_1(\lambda'(e|\lambda)) \right],
\]

where \( \lambda'(\cdot|\lambda) : E \to \mathbb{R}_{++} \) is the utility control variable for the next period. If \( \lambda' \) satisfies \([1]\), then \( E[\overline{\tau}_1(\lambda'(e|\lambda))] = \overline{\tau}_1(\lambda) \). Truth telling is then induced if, given the current utility control variable \( \lambda \), the renewed utility control variable \( \lambda' \) satisfies the following incentive constraints.

\[
(1 - \beta)u(e + \tau^*(e|\lambda)) + \beta \overline{\tau}_1(\lambda'(e|\lambda)) \\
\geq (1 - \beta)u(e + \tau^*(\hat{e}|\lambda)) + \beta \overline{\tau}_1(\lambda'(\hat{e}|\lambda)) \text{ for } e, \hat{e} \in E.
\]

Such an utility control variable \( \lambda' \) that satisfies both \([4]\) and \([5]\) could be adapted from the \( \lambda \) mechanism of [Marcet and Marimon]\\mbox{[1992]}, provided that there is a feasible \( \lambda' \). By a feasible \( \lambda' \) we mean that the updated utility control variable keeps the level of expected utility for the next period below the supremum of each individual’s utility function. That is, let \( S \) be the supremum of each individual’s utility function, \( S = \sup_{x \in \mathbb{R}_+} u(x) \). For given \( \lambda > 0 \), if \( \overline{\tau}_1(\lambda) + \lambda^{-1} \beta^{-1} (v_2(\lambda, eM) - \overline{\tau}_2(\lambda)) < S \), define \( \lambda' \) such that

\[
\lambda'(e|\lambda) = \overline{\tau}_1^{-1}\left( \overline{\tau}_1(\lambda) + \lambda^{-1} \beta^{-1} (v_2(\lambda, e) - \overline{\tau}_2(\lambda)) \right).
\]

Since \( v_2(\lambda, e) - \overline{\tau}_2(\lambda) = (1 - \beta)(e - E[e_t]) \), if the reported income \( e \in E \) is above the average income, \( E[e_t] \), this will be rewarded with higher utility control variables for the next period. Conversely, if the reported income \( e \in E \) is below the average income, \( E[e_t] \), this will result in lower utility control variables for the next period. Note that \( u(e + \tau^*(e|\lambda)) = u((u')^{-1}(\lambda^{-1})) = \overline{\tau}_1(\lambda) \). From this we see that \( \lambda' \) defined in \([6]\) satisfies \([1]\):

\[
E[(1 - \beta)u(e + \tau^*(e|\lambda)) + \beta \overline{\tau}_1(\lambda'(e|\lambda))] \\
= E[\overline{\tau}_1(\lambda) + \lambda^{-1} (v_2(\lambda, e) - \overline{\tau}_2(\lambda))] = \overline{\tau}_1(\lambda).
\]

Furthermore, remind that the government’s objective function in problem \([1]\) with \( \lambda_0 = \lambda \):
is optimal at true income report \( e \). It follows that

\[
\lambda[(1 - \beta)u(e + \tau^*(e|\lambda)) + \beta\pi_1(\lambda)] + v_2(\lambda, e) - v_2(\lambda) \\
\geq \lambda[(1 - \beta)u(e + \tau^*(\hat{e}|\lambda)) + \beta\pi_1(\lambda)] + v_2(\lambda, \hat{e}) - v_2(\lambda) \quad \text{for } e, \hat{e} \in E.
\]

Then we have the following inequality and see that (5) holds.

\[
\lambda[(1 - \beta)u(e + \tau^*(e|\lambda)) + \beta\pi_1(\lambda)] + v_2(\lambda, e) - v_2(\lambda) \\
= \lambda[(1 - \beta)u(e + \tau^*(e|\lambda)) + \beta\pi_1(\lambda')] + v_2(\lambda, e) - v_2(\lambda) \\
\geq \lambda[(1 - \beta)u(e + \tau^*(\hat{e}|\lambda)) + \beta\pi_1(\lambda)] + v_2(\lambda, \hat{e}) - v_2(\lambda) \\
= \lambda[(1 - \beta)u(e + \tau^*(\hat{e}|\lambda)) + \beta\pi_1(\lambda')] \quad \text{for } e, \hat{e} \in E.
\]

On the other hand, if there is an income \( \hat{e} \in E \) for given \( \lambda \) such that \( \lambda' \) defined in (6) must promise the expected utility for the next period above the supremum of the individual’s utility function, then the government must find the closest utility control variable for which there is a certainty equivalent to the promised level of lifetime utility. That is, for given \( \lambda \), if there is \( \hat{e} \in E \) such that \( \pi_1(\lambda) + \lambda^{-1}\beta^{-1}(v_2(\lambda, \hat{e}) - v_2(\lambda)) \geq S \), for reported income \( e \in E \), define \( \lambda' \) such that

\[
\lambda'(e|\lambda) = \pi_1^{-1}(\pi_1(\lambda) + \lambda^{-1}\beta^{-1}(v_2(\lambda, \min\{e, \bar{\pi}_\lambda\}) - E[v_2(\lambda, \min\{e_t, \bar{\pi}_\lambda\})]), \tag{7}
\]

where \( \bar{\pi}_\lambda \) is defined such that

\[
\bar{\pi}_\lambda = \arg \max_{e' \in E_N} \left(v_2(\lambda, e') - E[v_2(\lambda, \min\{e_t, e'\})]\right)
\]

subject to

\[
\pi_1(\lambda) + \frac{1}{\lambda\beta}(v_2(\lambda, e') - E[v_2(\lambda, \min\{e_t, e'\})]) < S,
\]

where \( E_N \supset E \) is a set with \(|E_N| = N > M, N \in \mathbb{N} \), and for \( e \in E_N \) \( e^1 \leq e \leq e^M \).

This adjustment also changes the transfers in the current period \( t \). However, we see the recursive relation in (7) satisfies the following corresponding incentive constraint.

\[
(1 - \beta)u(e + \tau^*(\min\{e, \bar{\pi}_\lambda\}|\lambda)) + \beta\pi_1(\lambda'(e|\lambda)) \\
\geq (1 - \beta)u(e + \tau^*(\min\{e, \bar{\pi}_\lambda\}|\lambda)) + \beta\pi_1(\lambda'(e|\lambda)) \quad \text{for } e, \hat{e} \in E. \tag{8}
\]

This is because the transfer \( \tau^*(\min\{e, \bar{\pi}_\lambda\}|\lambda) \) in (8) is the solution to the problem (1) subject to \( \tau(e_t) \geq (u')^{-1}(\lambda^{-1}) - \bar{\pi}_\lambda \) for \( t \in \mathbb{N} \cup \{0\} \), where \( \lambda = \lambda_0 \). In other words, since the transfer to the government is constant for individuals whose income is above the threshold \( \bar{\pi}_\lambda \), they are indifferent between reporting and misreporting their true income.

With respect to the promise-keeping constraint, it holds because the expectation of the left-hand side of the inequality in (8) is greater than or equal to the promised level of lifetime utility \( \pi_1(\lambda) \):

\[
E[(1 - \beta)u(e + \tau^*(\min\{e, \bar{\pi}_\lambda\}|\lambda)) + \beta\pi_1(\lambda')] \geq \pi_1(\lambda).
\]

\(^1\) To avoid getting stuck on a pass where the same utility control variable \( \lambda \) goes on forever, if you find \( \bar{\pi}_\lambda = e^1 \) for given \( N \), choose \( N' \in \mathbb{N} \) with \( N' > N \), and use \( E_{N'} \) instead of \( E_N \) so as to obtain \( \bar{\pi}_\lambda \geq e^2 \).
An inter-period transfer mechanism $\mathcal{M}$ is represented by a sequence of utility control variables $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ which satisfies either (6) or (7) and the corresponding transfers: 

\[ \tau^*(e_t|\lambda_t) \text{ for } \lambda_t \text{ in (6)} \text{ and } \tau^*(\min\{e_t, \tau_{\lambda_t}\}|\lambda_t) \text{ for } \lambda_t \text{ in (7)} \] 

in each period $t$.

A sufficient condition for the sequence of utility control variables $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ to always satisfy the recursive condition (6) is, as we see in the following lemma, that the Arrow-Pratt measure of the absolute risk aversion of $u$ satisfies the recursive condition (6). Consequently, we see the desired result if condition (9) does not hold, there may be a case where an utility control variable $\lambda_t$ is defined in (6) instead of in (7), e.g. if the individuals’ utilities have constant absolute risk aversion, and if it is greater than $\beta/((1-\beta)(e^M - E[e_t]))$. Since $F$ decreases in $\lambda$ and $\lim_{\lambda \to 0} F(\lambda) = \infty$, there must be a threshold $\lambda$ below which $F(\lambda) \geq S$.

The inter-period transfer mechanism $\mathcal{M}$ is sequentially incentive compatible regardless of whether each $\lambda_t$ is defined in (6) or in (7). Furthermore, if $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ is always defined in (6), like the $\lambda$ mechanism of Marcet and Marimon (1992), $\mathcal{M}$ is sequentially efficient.

**Lemma 1.** The sequence of utility control variables $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ is always defined in (6) if

\[ \frac{u''((u')^{-1}(\lambda^{-1}))}{u'((u')^{-1}(\lambda^{-1}))} < \frac{\beta}{(1-\beta)(e^M - E[e_t])} \text{ for } \lambda > 0. \] (9)

**Proof.** It suffices to show the proposition that $F(\lambda) < S$ for all $\lambda > 0$.

Since the utility function $u$ is continuous, bounded from above and satisfies the Inada condition, we have

\[ \lim_{\lambda \to -\infty} \tau_1(\lambda) = u(\lim_{\lambda \to -\infty} (u')^{-1}(\lambda^{-1})) \leq S \]

\[ \lim_{\lambda \to -\infty} (u')^{-1}(\lambda^{-1}) = \infty. \]

So we see that

\[ \lim_{\lambda \to -\infty} F(\lambda) = \lim_{\lambda \to -\infty} \tau_1(\lambda) = S. \]

If (9) holds, then we have $F'(\lambda) > 0$ for $\lambda > 0$, which implies that $F$ is strictly increasing in $\lambda$. Consequently, we see the desired result $F(\lambda) < S$ for all $\lambda > 0$. □

If condition (9) does not hold, there may be a case where an utility control variable $\lambda_t$ in $\mathcal{M}$ is defined in (7) instead of in (6), e.g. if the individuals’ utilities have constant absolute risk aversion, and if it is greater than $\beta/((1-\beta)(e^M - E[e_t]))$. Since $F$ decreases in $\lambda$ and $\lim_{\lambda \to 0} F(\lambda) = \infty$, there must be a threshold $\lambda$ below which $F(\lambda) \geq S$.

The inter-period transfer mechanism $\mathcal{M}$ is sequentially incentive compatible regardless of whether each $\lambda_t$ is defined in (6) or in (7). Furthermore, if $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ is always defined in (6), like the $\lambda$ mechanism of Marcet and Marimon (1992), $\mathcal{M}$ is sequentially efficient.

**Proposition 1.** $\mathcal{M}$ is sequentially incentive compatible. Furthermore, if condition (9) holds, then $\mathcal{M}$ is sequentially efficient.

**Proof.** With respect to $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ defined throughout in (6), the propositional statement is shown by Marcet and Marimon (1992). For completeness, we report the entire proof.

We first show that $\mathcal{M}$ is sequentially incentive compatible. For $\lambda_t > 0$ with $F(\lambda_t) < S$

\[ \lambda_t\left[(1-\beta)u(e + \tau^*(e|\lambda_t)) + \beta\tau_1(\lambda_{t+1}(e|\lambda_t))\right] = \lambda_t\left[(1-\beta)u(e + \tau^*(e|\lambda_t)) + \beta\tau_1(\lambda_t)\right] + v_2(\lambda_t, e) - \tau_2(\lambda_t) \]

\[ \geq \lambda_t\left[(1-\beta)u(e + \tau^*(\tilde{e}|\lambda_t)) + \beta\tau_1(\lambda_t)\right] + v_2(\lambda_t, \tilde{e}) - \tau_2(\lambda_t) \]

\[ = \lambda_t\left[(1-\beta)u(e + \tau^*(\tilde{e}|\lambda_t)) + \beta\tau_1(\lambda_{t+1}(\tilde{e}|\lambda_t))\right] \text{ for } e, \tilde{e} \in E. \]

The last inequality follows from the optimality of $\lambda\tau_1(\lambda) + v_2(\lambda, e)$ in the problem (1) given $\lambda$. Hence, $\mathcal{M}$ is sequentially incentive compatible for $\lambda > 0$ defined in (6).
For $\lambda_t > 0$ with $F(\lambda_t) \geq S$
\[
\lambda_t \left[(1 - \beta)u(e + \tau^*(\min\{e, \tau_{\lambda_t}\} | \lambda_t)) + \beta \tau_1(\lambda_t)\right]
\geq \lambda_t \left[(1 - \beta)u(e + \tau^*(\min\{\bar{e}, \tau_{\lambda_t}\} | \lambda_t)) + \beta \tau_1(\lambda_t)\right]
\hspace{1cm} + v_2(\lambda_t, \min\{e, \tau_{\lambda_t}\}) - E[v_2(\lambda_t, \min\{e_t, \tau_{\lambda_t}\})]
\geq \lambda_t \left[(1 - \beta)u(e + \tau^*(\min\{\bar{e}, \tau_{\lambda_t}\} | \lambda_t)) + \beta \tau_1(\lambda_t)\right]
\hspace{1cm} + v_2(\lambda_t, \min\{\bar{e}, \tau_{\lambda_t}\}) - E[v_2(\lambda_t, \min\{e_t, \tau_{\lambda_t}\})]
\] = $\lambda_t \left[(1 - \beta)u(e + \tau^*(\min\{\bar{e}, \tau_{\lambda_t}\} | \lambda_t)) + \beta \tau_1(\lambda_t')\right]$ for $e, \bar{e} \in E$.

The last inequality follows from the fact that $\tau^*(\min\{e, \tau_{\lambda_t}\} | \lambda_t)$ corresponds to the solution of the problem (1) subject to $\tau_t \geq (u')^{-1}(\lambda_0^{-1}) - \tau_{\lambda_t}$ for $t \in \mathbb{N}$, where $\lambda_0 = \lambda_t$.

We proceed to show that if condition (9) holds, then $\mathcal{M}$ is Pareto optimal and not dominated by any other sequentially incentive-compatible mechanisms. By the first part of this proof, $\mathcal{M}$ is sequentially incentive compatible. By Lemma 1 since condition (9) holds, the sequence of utility control variables $\{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ in $\mathcal{M}$ always defined in (6). Hence $\tau^*$ in $\mathcal{M}$ corresponds to transfers that solve the problem (1) in every period. Therefore $\mathcal{M}$ is a Pareto optimal transfer mechanism.

It remains to prove that under condition (9) $\mathcal{M}$ is not Pareto dominated by any other sequentially incentive compatible mechanism. Suppose, contrary to our claim, that there exists a sequentially incentive compatible mechanism $\Gamma$ that Pareto dominates $\mathcal{M}$ for a given income state $e$. Let $(v^*_1, v^*_2)$ be the present value achieved by $\Gamma$. Set $\lambda_0 = \tau_1^{-1}(v^*_1)$. Since condition (9) holds for any $\lambda > 0$, this $\lambda_0$ satisfies $F(\lambda_0) < S$ as in the proof of Lemma 1. We may now use $(\lambda_0, e)$ as the initial condition for $\mathcal{M}$ under condition (9). Then, by construction, each individual has the same present value for both contracts. Since $\Gamma$ Pareto dominates $\mathcal{M}$, its Pareto dominance requires that $v^*_2 > v_2(\lambda_0, e)$. However, this contradicts the fact that solutions of $\mathcal{M}$ are Pareto optimal under condition (9).

Proposition 1 shows sequential efficiency of the mechanism $\mathcal{M}$ only for the case where condition (9) holds. However, we could relax this restriction by removing the assumption that the utility function is bounded from above. This assumption provides a sufficient condition to ensure Proposition 2, but it is not a necessary condition.

3 The inter-period transfer mechanism for each individual

The government’s approach of using the mechanism $\mathcal{M}$ successfully avoids an immiserization outcome. That is, the promised lifetime utility of each individual in $\mathcal{M}$ converges almost everywhere to a random variable with a finite expectation. To see this, we reinterpret $\{\tau_1(\lambda_n)\}_{n \in \mathbb{N}}$ as a sequence of random variables on $(\Omega, \mathcal{F}, P)$. This is done recursively as follows. First we interpret $\tau_1(\lambda_1)$ as a random variable $\tau_1(\lambda_1(e_0(\cdot) | \lambda_0))$ and let $\mathcal{F}_1 \subset \mathcal{F}$ be the smallest $\sigma$-algebra induced by $\tau_1(\lambda_1)$. That is, $\mathcal{F}_1 = \{\tau_1^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$, where $\tau_1^{-1}(B) = \{\omega \in \Omega \mid \tau_1(\lambda_1(e_0(\omega) | \lambda_0)) \in B\}$. We see that $\tau_1(\lambda_1)$ is $\mathcal{F}_1$-measurable, $\tau_1(\lambda_1) : (\Omega, \mathcal{F}_1) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Second, we consider $\tau_1(\lambda_n)$ as a random variable $\tau_1(\lambda_n(e_{n-1}(\cdot) | \lambda_{n-1}))$ and define $\mathcal{F}_n \subset \mathcal{F}$ as the smallest $\sigma$-algebra induced by the product of the elements in $\{\tau_1(\lambda_i)\}_{i=1}^n$ such that
\[
\mathcal{F}_n = \left\{\left(\tau_1(\lambda_1) \times \cdots \times \tau_1(\lambda_n)\right)^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R}^n)\right\},
\]
where

\[(\overline{v}_1(\lambda_1) \times \ldots \times \overline{v}_1(\lambda_n))^{-1}(B) = \left\{ \omega \in \Omega \left| \left(\overline{v}_1(\lambda_1(e_0(\omega)|\lambda_0)), \ldots, \overline{v}_1(\lambda_n(e_{n-1}(\omega)|\lambda_{n-1}))\right) \in B \right\}.\]

We see that \(\overline{v}_1(\lambda_n)\) is \(\mathcal{F}_n\)-measurable, \(\overline{v}_1(\lambda_n) : (\Omega, \mathcal{F}_n) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))\).

**Proposition 2.** The sequence of promised lifetime utilities of individuals \(\{\overline{v}_1(\lambda_n)\}_{n \in \mathbb{N}}\) induced by \(\mathcal{M}\) converges to an integrable random variable.

**Proof.** We first show that \(\{\overline{v}_1(\lambda_n)\}_{n \in \mathbb{N}}\) is a martingale. Since the sequence \(\{\lambda_t\}_{t \in [0, n]} \in \mathcal{M}\) satisfies (6) or (7), we have

\[E[\overline{v}_1(\lambda_1)|\overline{v}_1(\lambda_{n-1})(\omega)] = \overline{v}_1(\lambda_{n-1})(\omega)\]

for given realised \(\overline{v}_1(\lambda_{n-1})(\omega)\). We thus have the following equalities: For all \(B \in \mathcal{B}(\mathbb{R})\) for all \(n \in \mathbb{N}\)

\[\int_B E[\overline{v}_1(\lambda_n)|\overline{v}_1(\lambda_{n-1})(\omega)]dP_{\overline{v}_1(\lambda_{n-1})} = \int_B \overline{v}_1(\lambda_{n-1})(\omega)dP_{\overline{v}_1(\lambda_{n-1})},\]

and

\[\int_B E[\overline{v}_1(\lambda_n)|\overline{v}_1(\lambda_{n-1})(\omega)]dP_{\overline{v}_1(\lambda_{n-1})} = \int_{\overline{v}_1(\lambda_{n-1})(\omega) \in B} \overline{v}_1(\lambda_n)dP = \int_{\overline{v}_1(\lambda_{n-1})(\omega) \in B} \overline{v}_1(\lambda_n)dP.\]

From the last equality we have for \(n \in \mathbb{N}\) \(E[\overline{v}_1(\lambda_n)|\mathcal{F}_{n-1}] = \overline{v}_1(\lambda_{n-1})\) a.e. \([P]\), and \(\{\overline{v}_1(\lambda_n)\}_{n \in \mathbb{N}}\) is a martingale as claimed.

We proceed to show that \(\sup_n \overline{v}_1(\lambda_n)^+ < \infty\). From (6) and (7) we have \(\overline{v}_1(\lambda_n) \leq S\) for \(n \in \mathbb{N}\). So we have \(\sup_n \overline{v}_1(\lambda_n)^+ < \infty\).

Since \(\{\overline{v}_1(\lambda_n)\}_{n \in \mathbb{N}}\) is a martingale and \(\sup_n E[\overline{v}_1(\lambda_n)^+] \leq \sup_n \overline{v}_1(\lambda_n)^+ < \infty\), from the submartingale convergence theorem, there is an integrable random variable \(v^\infty\) such that \(\overline{v}_1(\lambda_n) \to v^\infty\) almost everywhere. \(\square\)

As the proof of Proposition 2 shows, mechanism \(\mathcal{M}\) functions like a fair-value gamble on lifetime utilities across periods. The lifetime utility in each period is a random variable whose sequence forms a martingale. Consequently, the mechanism is fair in the sense that each individual’s expected lifetime utility remains unchanged from the time of the bet.

**4 The inter-period transfer mechanism for the government**

However, the convergence of individuals’ lifetime utilities to an integrable random variable with finite expectation does not yet ensure the sustainability of the \(\mathcal{M}\) mechanism. This is because the government predicts a declining trend in its budget for each individual in the long run, as we see in the following lemma. Let \(\tau_n\) be the average transfer to those who were born \(n\) periods ago. Then \(\tau_n\) is given by as follows.

\[\tau_n = E[u^{-1}(\overline{v}_1(\lambda_n))] - E[\epsilon_n] \quad \text{for } n \in \mathbb{N} \cup \{0\}.\]
Lemma 2. The average transfer for each age of individuals \(\{\tau_n\}_{n \in \mathbb{N} \cup \{0\}}\) is an increasing sequence.

Proof. We first consider the case \(n = 1\). For \(n = 1\), since \(E[\lambda_1(e_1|\lambda_0)] = \tau_1(\lambda_0)\) and \(u^{-1}\) is convex from Jensen’s inequality we have
\[
\tau_1 = E[u^{-1}(\lambda_1(e_0|\lambda_0))] - E[e_1] \\
\geq u^{-1}(\tau_1(\lambda_0)) - E[e_0] = \tau_0.
\]
For \(n > 1\) since \(u^{-1}\) is a convex function, from Jensen’s inequality we see the result.
\[
\tau_{n-1} = E[u^{-1}(\lambda_{n-1})] - E[e_{n-1}] \\
= E[u^{-1}(E[\lambda_n|\tau_1(\lambda_{n-1})])] - E[e_{n-1}] \\
\leq E[E[u^{-1}(\lambda_n)|\tau_1(\lambda_{n-1})]] - E[e_{n-1}] \\
= E[u^{-1}(\tau_1(\lambda_n))] - E[e_n] \\
= \tau_n \quad \text{for } n = 2, 3, \cdots.
\]

The proof of Lemma 2 shows that the increasing trend in \(\{\tau_n\}_{n \in \mathbb{N} \cup \{0\}}\) is due to the cost of keeping the expected lifetime utility in each period unchanged from \(E[\tau_1(\lambda_0)]\), although there are risks between periods.

Let \(g(t, \lambda_0)\) be the expected balance of payments in period \(t\) for the government for given initial utility control variable \(\lambda_0\). Since each individual’s probability of being alive is \(\alpha \in (0, 1)\), and from period one the next generation of continuum of individuals on the \((1 - \alpha)\) interval is born in each period, \(g(t, \lambda_0)\) is given by
\[
g(t, \lambda_0) = \alpha^t(-\tau_t) + \alpha^{t-1}(1 - \alpha)(-\tau_{t-1}) + \cdots + \alpha(1 - \alpha)(-\tau_1) + (1 - \alpha)(-\tau_0) \\
= \sum_{k=1}^{t} \alpha^k(\tau_{k-1} - \tau_k) - \tau_0.
\]
For the \(M\) mechanism to be sustainable, the balance of payments in each period must be greater than or equal to be zero. As we see in the following proposition, with the approximation of the second order Taylor polynomial, if incomes have a symmetric distribution, a sufficient condition for the balance of payments in period \(t\) to be greater than or equal to be zero is that (i) the absolute risk aversion of \(u\) is decreasing in consumption, (ii) the absolute risk aversion of the absolute risk aversion of \(u\) is less than the absolute risk aversion of \(u^{-1}\) and (iii) \(\lambda_0\) is set so that the inequality \((\tau_1 - \tau_0)\alpha/(1 - \alpha) \leq -\tau_0\) is satisfied.

Proposition 3. Assume that (i) incomes have a symmetric distribution, (ii) \(-u''/u'\) is a decreasing function, (iii) \((-u''/u')\circ u^{-1}\) is a concave function or equivalently,
\[
\frac{(-u''/u')''(u^{-1}(y))}{(-u''/u')'(u^{-1}(y))} \leq \frac{(u^{-1})''(y)}{(u^{-1})'(y)},
\]
and (iv) \(\alpha(\tau_1 - \tau_0)/(1 - \alpha) \leq -\tau_0\). Then \(g(t, \lambda_0) \geq 0\) for \(t \in \mathbb{N}\) with the approximation of the second order Taylor polynomial.
The proof of Proposition 3 uses the following lemma.

**Lemma 3.** Under the assumptions (i), (ii), (iii) in Proposition 3, the second order polynomial approximation of $\tau_n - \tau_{n-1}$ is a decreasing sequence.

**Proof.** First we consider the case where $\{\lambda_n\}_{n \in \mathbb{N} \cup \{0\}}$ is defined in (6). Using the third order Taylor polynomial, for a given utility control variable $\lambda_{n-1} > 0$ and a given income $e \in E$, the corresponding transfer is written as

$$u^{-1}(\tau_1(\lambda_n | \lambda_{n-1})) = u^{-1}(\tau_1(\lambda_{n-1})) + \frac{1}{u'(u^{-1}(\tau_1(\lambda_{n-1}))))} \Delta_{n-1} - \frac{1}{2!} \frac{u''(u^{-1}(\tau_1(\lambda_{n-1}))))}{2!} \Delta_{n-1}^2 + \frac{1}{3!} \left( \frac{u'''(u^{-1}(\tau_1(\lambda_{n-1}))))}{u'(u^{-1}(\tau_1(\lambda_{n-1}))))^3 + \frac{3u''(u^{-1}(\tau_1(\lambda_{n-1}))))}{u'(u^{-1}(\tau_1(\lambda_{n-1}))))^2} \right) \Delta_{n-1}^3$$

where $\Delta_{n-1} = (1 - \beta)(e - E[e_n])/(\beta \lambda_{n-1})$ and $h_3 : \mathbb{R} \to \mathbb{R}$ is a function such that $\lim_{\Delta_{n-1} \to 0} h_3(\tau_1(\lambda_n | \lambda_{n-1})) = 0$. Taking the expectation of (10), $\tau_n - \tau_{n-1}$ is written as

$$\tau_n - \tau_{n-1} = E \left[ \frac{1}{2!} \left( \frac{-u''(u^{-1}(\tau_1(\lambda_{n-1}))))}{u'(u^{-1}(\tau_1(\lambda_{n-1}))))} \right) \left( \frac{1 - \beta}{\beta} \right)^2 \text{Var}[e_n] \right] + o(\Delta_{n-1}^3).$$

Note that since incomes follow a symmetric distribution, the third term on the right-hand side of the equation (10) will cancel out when the expectation is taken. Since $(-u''/u') \circ u^{-1}$ is a concave function, from Jensen’s inequality we have

$$E \left[ \frac{-u''}{u'} \left( u^{-1}(\tau_1(\lambda_{n-1})) \right) \right] = E \left[ \frac{-u''}{u'} \left( u^{-1}(E[\tau_1(\lambda_n) | \tau_1(\lambda_{n-1}))]) \right) \right] \geq E \left[ \frac{-u''}{u'} \left( u^{-1}(\tau_1(\lambda_{n-1})) \right) \tau_1(\lambda_{n-1}) \right] = E \left[ \frac{-u''}{u'} \left( u^{-1}(\tau_1(\lambda_{n-1})) \right) \right].$$

It follows that the third order polynomial approximation of $\{\tau_n - \tau_{n-1}\}_{n \in \mathbb{N}}$ is a decreasing sequence.

We can now proceed analogously to the proof of the case where $\lambda_{n-1}$ is defined in (7). We change $\Delta_{n-1}$ in (10) to

$$\Delta_{n-1} = (1 - \beta) \left( \min\{e, \tau_{\lambda_{n-1}}\} - E[\min\{e, \tau_{\lambda_{n-1}}\}] \right) / \beta \lambda_{n-1}.$$

By the asymmetric distribution of conditional income $e \leq \tau_{\lambda_{n-1}}$, the third term on the right-hand side of (10) remains and $\tau_n - \tau_{n-1}$ is given by

$$\tau_n - \tau_{n-1} = E \left[ \frac{1}{2!} \left( \frac{-u''(u^{-1}(\tau_1(\lambda_{n-1}))))}{u'(u^{-1}(\tau_1(\lambda_{n-1}))))} \right) \left( \frac{1 - \beta}{\beta} \right)^2 \text{Var}[\min\{e, \tau_{\lambda_{n-1}}\}] \right] + o(\Delta_{n-1}^3).$$

By a similar argument, the second order polynomial approximation of $\{\tau_n - \tau_{n-1}\}_{n \in \mathbb{N}}$ is a decreasing sequence, and the proof is complete.

We prove Proposition 3 below.

**Proof.** From Lemma 3, with the second order polynomial approximation we have

$$\tau_n - \tau_{n-1} \leq \tau_1 - \tau_0 \quad \text{for} \ n \in \mathbb{N}.$$
Hence the following inequality holds.

\[ g(t, \lambda_0) = \sum_{k=1}^{t} \alpha^k (\tau_{k-1} - \tau_k) - \tau_0 \]
\[ \geq \alpha \cdot \frac{1 - \alpha^t}{1 - \alpha} (\tau_0 - \tau_1) - \tau_0. \]

The last term is positive if

\[ -\tau_0 \geq \frac{\alpha(1 - \alpha^t)}{1 - \alpha} (\tau_1 - \tau_0) \]

and we see a sufficient condition for \( g(t, \lambda_0) \geq 0 \) for \( t \in \mathbb{N} \) is \( \alpha(\tau_1 - \tau_0)/(1 - \alpha) \leq -\tau_0 \). \( \square \)

An intuitive sufficient condition for \( \lambda_0 \) to satisfy the initial condition \( \alpha(\tau_1 - \tau_0)/(1 - \alpha) \leq -\tau_0 \) specified in Proposition 3 is given by the third order polynomial approximation as follows.

**Corollary 1.** The condition \( \alpha(\tau_1 - \tau_0)/(1 - \alpha) \leq -\tau_0 \) in Proposition 3 holds with the second order polynomial approximation if the initial utility control variable \( \lambda_0 \) satisfies

\[ \lambda_0(\tau_1(\lambda_E) - \tau_1(\lambda_0)) \geq \left[ \frac{(1 - \alpha r)^2}{\alpha r^2(1 - \alpha)} \right] Var[\epsilon_t], \tag{11} \]

where \( \lambda_E = 1/(u'(E[\epsilon_t])) \).

**Proof.** Let us first examine the case where \( \lambda_1 \) is defined in (6). Applying Tayler theorem to \( u^{-1}(\tau_1(\lambda_1)) \), \( \alpha(\tau_1 - \tau_0)/(1 - \alpha) \) is expressed as

\[ \frac{\alpha(\tau_1 - \tau_0)}{1 - \alpha} = \frac{(u^{-1})''(\tau_1(\lambda_0))}{2!} (1 - \beta) \left( \frac{\alpha}{\beta \lambda_0} \right)^2 \frac{\alpha}{1 - \alpha} \text{Var}[\epsilon_0] + o(\Delta_0^3). \tag{12} \]

On the other hand, applying Tayler theorem to \( u^{-1}(\tau_1(\lambda_E)) \), we have

\[ -\tau_0 = -u^{-1}(\tau_1(\lambda_0)) + u^{-1}(\tau_1(\lambda_E)) \]
\[ = \left( \frac{(u^{-1})'(\tau_1(\lambda_0))}{1!} \right) \Delta_E + \left( \frac{(u^{-1})''(\tau_1(\lambda_0))}{2!} \right) \Delta_E^2 + o(\Delta_E^3). \tag{13} \]

where \( \Delta_E = \tau_1(\lambda_E) - \tau_1(\lambda_0) \). Since the first term on the right-hand side of (13) is positive, it is sufficient to show the second term on the right-hand side of (13) is equal to or greater than the first term on the righthand side of (12). However, from the corollary assumption (11), we have

\[ \Delta_E^2 \geq \frac{(1 - \beta)^2}{\lambda_0^2 \beta^2} \frac{\alpha}{1 - \alpha} \text{Var}[\epsilon_t], \]

which is the desired conclusion that the condition \( \alpha(\tau_1 - \tau_0)/(1 - \alpha) \leq -\tau_0 \) holds with the second order polynomial approximation. For the case where \( \lambda_1 \) is defined in (7), we have

\[ \frac{\alpha(\tau_1 - \tau_0)}{1 - \alpha} = \frac{(u^{-1})''(\tau_1(\lambda_0))}{2!} (1 - \beta) \left( \frac{\alpha}{\beta \lambda_0} \right)^2 \frac{\alpha}{1 - \alpha} \text{Var}[\min\{\epsilon_0, \tau_1\}] + o(\Delta_0^3). \]

Since \( \text{Var}[\min\{\epsilon_0, \tau_1\}] \leq \text{Var}[\epsilon_0] \), (11) is sufficient to be \( \alpha(\tau_1 - \tau_0)/(1 - \alpha) \leq -\tau_0 \). \( \square \)
Notes: This figure shows a graph of the slope of a utility function. The horizontal axis is consumption, and the vertical axis is utility. The curve is the utility function. The length \( \lambda_0(v_1(\lambda E) - v_1(\lambda_0)) \) is a linear approximation of the length between the expected income and the consumption guaranteed in the mechanism \( M \) with \( \lambda_0 \).

The left-hand side of the inequality (11) is a linear approximation of the difference between the consumption level that ensures the expected income and the consumption level in the mechanism \( M \) with the utility control variable \( \lambda_0 \). The inequality (11) means that this length is greater than the square root of the weighted income variance. Figure 1 shows which part of the graph of a utility function corresponds to \( \lambda_0(v_1(\lambda E) - v_1(\lambda_0)) \).

The lifetime utility \( v_1(\lambda_0) \) in the mechanism \( M \) could be set higher than that in the state of autarky if the risk premium of the state of autarky is higher than the term in the right-hand side of the inequality (11).

A higher discount factor and higher probability of being alive will relax the sufficient condition of \( \lambda_0 \). If \((u^{-1})'''\) is positive, the condition (11) may be unnecessarily strict.

5 Numerical examples

According to Corollary 1, in the following case, the mechanism \( M \) is sustainable and the lifetime utilities of all generations are higher than that in the state of autarky. Suppose that the utility function is given by

\[
u(c) = \frac{1}{1 + \gamma} (c + 1)^{1+\gamma},\]

where \( \gamma = -3 \). The income set \( E \) is \( E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). We will refer to this income set as \( E_s \). The income shocks follow the distribution \( P(e_t = 0) = Pr(e_t = 9) = 0.025, P(e_t = 1) = P(e_t = 8) = 0.045781, P(e_t = 2) = P(e_t = 7) = 0.092762, P(e_t = 3) = P(e_t = 6) = 0.148524, P(e_t = 4) = Pr(e_t = 5) = 0.187933 \). We will refer to this distribution as \( P_s \). Suppose that the discount factor and the probability of being alive are \( r = \alpha = 0.93 \). Then Proposition 3 assumptions are satisfied for \( \lambda_0 \in [\lambda_{CE}, 133] \), where \( \lambda_{CE} = 49.608 = 1/u'(E[u(e_t)]) \). Note that \( \lambda_E = 166.375 \), the risk premium is 1.826, and \( \sqrt{(1 - \alpha r)^2/(\alpha r^2(1 - \alpha)) Var[e_t]} \) is 1.147.

Figure 2 shows a series of sample means of lifetime utilities of 1000 individuals and corresponding transfer balances for 100 periods for the above case. The initial utility control variable is set \( \lambda_0 = 50 \).

As Proposition 2 implies, the series of sample means of lifetime utilities does not approach negative infinity, which has been called immiserization. However, as Lemma
Figure 2: A series of sample means of lifetime utility and corresponding transfer balance
Notes: This figure shows a series of sample means of lifetime utilities and corresponding transfer balances for 100 periods, where 1000 individuals are assumed to have the utility \((c + 1)^{1+\gamma}/(1 + \gamma)\) with \(\gamma = -3\). The parameters are set so that the initial utility control variable \(\lambda_0 = 50\), the discount factor \(r = 0.93\), the probability of being alive \(\alpha = 0.93\). The income set \(E\) is \(E_s\). The random numbers are generated using the discrete probability distribution \(P_s\).

The series of sample means of transfer balances \(\{\tau_n\}_{n=0}^{100}\) is decreasing.

Figure 3: A series of sample means of lifetime utility and corresponding transfer balance
Notes: This figure shows a series of sample means of lifetime utilities and corresponding transfer balances for 100 periods, where 1000 individuals are assumed to have the utility \((c + 1)^{1+\gamma}/(1 + \gamma)\) with \(\gamma = -3\). The parameters are set so that the initial utility control variable \(\lambda_0 = 50\), the discount factor \(r = 0.93\), the probability of being alive \(\alpha = 0.93\). The income set \(E\) is \(E_s\). The random numbers are generated using the discrete probability distribution \(P_s\).

Figure 3: A series of sample means of balance of payments \(\{g(t, \lambda_0)\}_{n=0}^{1000}\)
Notes: This figure shows a series of sample means of transfer balances for a society of 100 individuals for 1000 periods. As in Figure 2, each individual’s utility is \((c + 1)^{1+\gamma}/(1 + \gamma)\) with \(\gamma = -3\), and the parameters are the initial utility control variable \(\lambda_0 = 50\), the discount factor \(r = 0.93\), the probability of being alive \(\alpha = 0.93\). The income set \(E\) is \(E_s\). The random numbers are generated using the discrete probability distribution \(P_s\).
Immiserization observed in previous studies (e.g. Green 1987, Thomas and Worrall 1990) is due to the approach to allocate risks within each period across the population of individuals. To ensure truthful reporting, only individuals with the highest incomes keep their lifetime utility level in the next period, and lifetime utilities of other individuals in the next period are discounted according to reported incomes. As a result, the average contract value of individuals becomes a decreasing series.

However, the mechanism $\mathcal{M}$ simply shifts risks to future periods. A similar limitation is that the proposed mechanism leads to a decreasing series of the average transfer balances for each generation, which could make the government’s balance of payments negative infinity. However, this could be solved by setting the initial utility control variable appropriately, so that average transfers are negative for the first few periods from birth. If this sense of intergenerational cooperation is properly organised, efficient income allocations could be sustainable even under asymmetric information about income shocks.

The property of the mechanism $\mathcal{M}$ contrasts with the numerical results of Marcet and Marimon (1992), where the lifetime utility of agent 1 (manager) is higher in the state of autarky than in the state of the incentive contract with agent 2 (investor). This is probably because, first, Marcet and Marimon (1992) is a growth model with the assumption that capital and the manager’s consumption, increases when the manager chooses the optimal level of investment. Second, since the transfers in the numerical example for incentive contracts in Marcet and Marimon (1992) are not competitive, the external financing contracts end up reducing agent 1’s allocation.

6 Checking the mechanism optimality for each generation

Lastly, we study the efficiency of the mechanism $\mathcal{M}$ using a Bellman equation. Let $V(\lambda_t, t)$ be the lifetime utility of each individual whose realised utility control variable $\lambda_t$ for their period $t$ in $\mathcal{M}$. For $\lambda_t \in \{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}}$ it is written as

$$V(\lambda_t, t) = E_t\left[u(e_t + \tau^*(\min\{e_t, \tau_{\lambda_t}\})|\lambda_t)\right],$$

where $\min\{e_t, \tau_{\lambda_t}\} = e_t$ for $e_t \in E$ if $\lambda_t$ is defined in (6). We study whether the function $V : \mathbb{R}^+ \times \mathbb{N} \cup \{0\} \rightarrow \mathbb{R} \cup \{-\infty\}$ is the value function of the Bellman equation:

$$V(\lambda_t, t) = \sup_{\tau \in \Lambda(\lambda_t, t)} (1 - \beta)u(e_t + \tau(\epsilon_t|\lambda_t)) + \beta V(\lambda_{t+1}, t+1),$$

where $\Lambda : \mathbb{R}^+ \times \mathbb{N} \cup \{0\} \rightarrow \mathbb{U}$ is an available control correspondence defined such that

$$\Lambda(\lambda_t, t) = \arg\max_{\tau} \left[\lambda_t u(e_t + \tau) - \tau \text{ subject to } \tau \geq (u')^{-1}(\lambda_t^{-1}) - \tau_{\lambda_t}\right],$$

$\mathbb{U}$ is the set of control parameters, and $\lambda_{t+1}$ is given by the function describing the change of the state variable $f : \mathbb{R}^+ \times \mathbb{U} \times \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}^+$, defined such that

$$f(\lambda_t, \tau(\cdot|\lambda_t), t) = \begin{cases} \lambda_{t+1} \text{ given by (6)} & \text{if } F(\lambda_t) < S \\ \lambda_{t+1} \text{ given by (7)} & \text{otherwise.} \end{cases}$$

The available controls given by $\Lambda$ are chosen according to the government’s policy to maximise the social welfare for each generation in each period. We observe the following
Proposition 4. Suppose the sequence of utility control variables \( \{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}} \) satisfies (9). Then \( V \) fulfills the Bellman equation and it is the value function of the dynamic optimisation problem, where the control function \( \tau^* \circ \min \{\cdot, \cdot\} \) in \( V \) maximises \( J \).

A sufficient condition for \( V \) to be the value function of the Bellman equation is given by Wiszniewska-Matyszkiel (2011) which is to satisfy the Bellman equation and the following terminal condition.

Terminal condition (Wiszniewska-Matyszkiel 2011)

(i) For every \( \lambda_t \in \{\lambda_t\}_{t \in \mathbb{N} \cup \{0\}} \) in \( M \)

\[
\lim_{t \to \infty} V(\lambda_t, t) \beta^t \leq 0
\]

and

(ii) for every \( \lambda_t \in \{\lambda\}_{t \in \mathbb{N} \cup \{0\}} \) in \( M \), if \( \lim_{t \to \infty} V(\lambda_t, t) \beta^t < 0 \), then

\[
J(\lambda_t, t, \hat{\tau}) = -\infty
\]

for every \( \hat{\tau} : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lambda_t \) in \( M \) corresponding to \( \hat{\tau} \) and \( t \).

Proof. We first observe that \( V \) defined in (14) satisfies the Bellman equation in (15).

\[
V(\lambda_t, t) = E_t \left[ u(e_t + \tau^*(\min\{e_t, \bar{e}_{\lambda_t}\} | \lambda_t)) \right]
\]

\[
= (1 - \beta)u(e_t + \tau^*(\min\{e_t, \bar{e}_{\lambda_t}\} | \lambda_t)) + \beta E_t \left[ v_1(\lambda_{t+1}) | v_1(\lambda_t) \right]
\]

\[
= (1 - \beta)u(e_t + \tau^*(\min\{e_t, \bar{e}_{\lambda_t}\} | \lambda_t)) + \beta V(\lambda_{t+1}, t + 1). \tag{16}
\]

But \( \tau^* \circ \min \{\cdot, \bar{e}_{\lambda_t}\} \) is the solution to the problem

\[
\sup_{\tau \in \Lambda(\lambda_t, t)} (1 - \beta)u(e_t + \tau(e_t | \lambda_t)).
\]

It follows that (16) is written as

\[
V(\lambda_t, 1) = \sup_{\tau \in \Lambda(\lambda_t, t)} (1 - \beta)u(e_t + \tau(e_t | \lambda_t)) + \beta V(\lambda_{t+1}, t + 1),
\]

and see the desired result.

The task is now to check the terminal condition. Since \( u \) is bounded from above, we have

\[
\lim_{t \to \infty} V(\lambda_t, t) \beta^t = 0.
\]

Thus, \( V \) satisfies Terminal condition (i) and from Theorem 1 of Wiszniewska-Matyszkiel (2011), we see \( V \) defined in (14) is the value function of the Bellman equation in (15), and the control function \( \tau^* \circ \min \{\cdot, \cdot\} \) in \( V \) maximises \( J \). \qed
7 Conclusion

This paper presents an alternative solution to avoid the prediction of the previous work: if the incomes of infinitely lived individuals are unobservable, efficient allocations are achieved only at the cost of invoking permanent inequality, leading to an immiserization of society. The proposed inter-period transfer mechanism achieves within-period full insurance by postponing risks. It does not trade off equal opportunities to become wealthy. It could be sustained by intergenerational cooperation. The result sheds light on efficient resource allocation for sustainable societies with equal opportunities.

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