GRADIENT BASED BLOCK COORDINATE DESCENT ALGORITHMS
FOR JOINT APPROXIMATE DIAGNOLIZATION OF MATRICES

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Abstract. In this paper, we propose a gradient based block coordinate descent (BCD-G) framework to solve the joint approximate diagonalization of matrices defined on the product of the complex Stiefel manifold and the special linear group. Instead of the cyclic fashion, we choose the block for optimization in a way based on the Riemannian gradient. To update the first block variable in the complex Stiefel manifold, we use the well-known line search descent method. To update the second block variable in the special linear group, based on four different kinds of elementary rotations, we construct two classes: Jacobi-GLU and Jacobi-GLQ, and then get two BCD-G algorithms: BCD-GLU and BCD-GLQ. We establish the weak convergence and global convergence of these two algorithms using the Lojasiewicz gradient inequality under the assumption that the iterates are bounded. In particular, the problem we focus on in this paper includes as special cases the well-known joint approximate diagonalization of Hermitian (or complex symmetric) matrices by invertible transformations in blind source separation, and our algorithms specialize as the gradient based Jacobi-type algorithms. All the algorithms and convergence results in this paper also apply to the real case.

1. Introduction

1.1. Problem formulation. Let $1 \leq m \leq n$. For a matrix $X \in \mathbb{C}^{n \times m}$, we denote by $X^T$, $X^*$ and $X^H$ its transpose, conjugate and conjugate transpose, respectively. Let $\text{St}(m, n, \mathbb{C}) \overset{\text{def}}{=} \{X \in \mathbb{C}^{n \times m}, X^H X = I_m\}$ be the complex Stiefel manifold. Let $\text{GL}_m(\mathbb{C}) \overset{\text{def}}{=} \{X \in \mathbb{C}^{m \times m}, \det(X) \neq 0\}$ be the general linear group, and $\text{SL}_m(\mathbb{C}) \overset{\text{def}}{=} \{X \in \text{GL}_m(\mathbb{C}), \det(X) = 1\}$ be the special linear group.

Suppose that $\{A^{(\ell)}\}_{1 \leq \ell \leq L} \subseteq \mathbb{C}^{n \times n}$ is a set of complex matrices, and $\mu_\ell \in \mathbb{R}^+$ for $1 \leq \ell \leq L$. Denote $(\cdot)^T = (\cdot)^H$ or $(\cdot)^T$. In this paper, we mainly study the joint approximate diagonalization of matrices (JADM) problem on $\text{St}(m, n, \mathbb{C}) \times \text{SL}_m(\mathbb{C})$, which consists in minimizing the objective

$$f : \text{St}(m, n, \mathbb{C}) \times \text{SL}_m(\mathbb{C}) \to \mathbb{R}^+, \quad (U, X) \mapsto \sum_{\ell=1}^L \mu_\ell \|\text{offdiag}\{W^{(\ell)}\}\|^2,$$

where $W^{(\ell)} = \text{offdiag}(U^T A^{(\ell)} U) X$. Here $\text{offdiag}(Y)$ denotes the matrix obtained from $Y$ by removing the diagonal entries. We refer to Section 5 for the definitions of other matrices and vectors.

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where $W^{(\ell)} = (UX)^\dagger A^{(\ell)}(UX) \in \mathbb{C}^{m \times m}$ and $\text{offdiag}\{\cdot\}$ is the zero diagonal operator, which sets all the diagonal elements of a square matrix in $\mathbb{C}^{m \times m}$ to zero.

1.2. Equivalent representation. If we define the rectangular special linear set as

$$\text{RSL}(m, n, \mathbb{C}) \overset{\text{def}}{=} \{X \in \mathbb{C}^{n \times m}, X^H X \in \text{SL}_m(\mathbb{C})\},$$

then we have the following simple lemma.

**Lemma 1.1.** Let $1 \leq m \leq n$ and $Y \in \mathbb{C}^{n \times m}$. Then $Y \in \text{RSL}(m, n, \mathbb{C})$ if and only if there exist $U \in \text{St}(m, n, \mathbb{C})$ and $X \in \text{SL}_m(\mathbb{C})$, such that $Y = UX$.

By Lemma 1.1, problem (1) is equivalent to minimizing

$$f : \text{RSL}(m, n, \mathbb{C}) \rightarrow \mathbb{R}^+, \quad Y \mapsto \sum_{\ell=1}^L \mu_\ell \|\text{offdiag}\{W^{(\ell)}\}\|^2,$$

where $W^{(\ell)} = Y^\dagger A^{(\ell)}Y \in \mathbb{C}^{m \times m}$.

1.3. Examples. If we set $m = n$ and fix $U = I_m$, then the cost function (1) becomes

$$g : \text{SL}_m(\mathbb{C}) \rightarrow \mathbb{R}^+, \quad X \mapsto f(I_m, X) = \sum_{\ell=1}^L \mu_\ell \|\text{offdiag}\{W^{(\ell)}\}\|^2,$$

where $W^{(\ell)} = X^\dagger A^{(\ell)}X \in \mathbb{C}^{m \times m}$. A complex matrix $A \in \mathbb{C}^{m \times m}$ is called Hermitian if $A^H = A$. It is called complex symmetric if $A^T = A$. Then problem (3) has the following well-known special cases:

- **joint approximate diagonalization of Hermitian matrices** (JADM-H) [37, 32]: $(\cdot)^\dagger = (\cdot)^H$, $A^{(\ell)}$ is Hermitian and $\mu_\ell = 1$ for $1 \leq \ell \leq L$;
- **joint approximate diagonalization of complex symmetric matrices** (JADM-CS) [32]: $(\cdot)^\dagger = (\cdot)^T$, $A^{(\ell)}$ is complex symmetric and $\mu_\ell = 1$ for $1 \leq \ell \leq L$;
- **joint approximate diagonalization of real symmetric matrices** (JADM-RS) [4, 5]: on the real field $\mathbb{R}$, $(\cdot)^\dagger = (\cdot)^T$, $A^{(\ell)}$ is real symmetric and $\mu_\ell = 1$ for $1 \leq \ell \leq L$.

These three problems have been widely used in the **Blind source separation** and **Independent component analysis** [12, 15, 4]. Compared with the joint approximate diagonalization by orthogonal transformations [10, 11, 14, 22, 23, 24, 25, 40], the non-orthogonal joint diagonalizer in (3) does not require prewhitening and can suffer less from the noise. However, since $\text{SL}_m(\mathbb{C})$ is not compact, solving this problem is much more difficult.

To solve the JADM-RS problem, Jacobi-type algorithms were introduced based on the LU and QR decompositions in [5], and based on the Givens rotations, hyperbolic rotations and diagonal rotations in [38]. To solve the JADM-H problem, Jacobi-type algorithms were proposed based on the LU decomposition in [32, 33], and based on the QL decomposition in [37]. To solve the JADM-CS problem, a Jacobi-type algorithm was proposed based on the LU decomposition in [31, 32]. However, to our knowledge, there was no theoretical result about the convergence of these Jacobi-type algorithms in the literature.
1.4. **Block coordinate descent.** Suppose that \( \{ \mathcal{M}_i \}_{1 \leq i \leq d} \) is a set of smooth manifolds. To minimize the following smooth function

\[
f : \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_d \rightarrow \mathbb{R}^+,
\]

a popular approach is the *block coordinate descent* (BCD) \([7, 28, 29, 42, 43]\). In this method, only one block variable is updated at each iteration, while other block variables are fixed. Then problem \((4)\) is decomposed to a sequence of lower-dimensional optimization problems, similar as the subproblems in Jacobi-type algorithms. In BCD algorithm, there are different ways to choose blocks for optimization, including the *essentially cyclic, cyclic, random* fashions \([42, 43]\) and the so-called *maximum block improvement* (MBI) method \([13, 26]\).

1.5. **BCD-G algorithm on** \( \text{St}(m, n, \mathbb{C}) \times \text{SL}_m(\mathbb{C}) \). Suppose that

\[
f : \text{St}(m, n, \mathbb{C}) \times \text{SL}_m(\mathbb{C}) \rightarrow \mathbb{R}^+ (5)
\]

is an abstract smooth function, which includes the cost function \((1)\) as a special case. For \( \omega = (U, X) \in \text{St}(m, n, \mathbb{C}) \times \text{SL}_m(\mathbb{C}) \), we define

\[
f_{1, X} : U \mapsto f(U, X), \quad f_{2, U} : X \mapsto f(U, X),
\]

as the restricted functions. For simplicity, we denote the Riemannian gradient

\[
\text{grad } f_{t_k}(\omega_{k-1}) \quad \text{def} = \text{grad } f_{t_k}(U_{k-1}) \quad \text{and} \quad \text{grad } f_{t_k}(X_{k-1})
\]

and \( \omega_k \text{ def} = (U_k, X_k) \) for \( k \geq 0 \). Now we propose the following *gradient based block coordinate descent* (BCD-G) algorithm to minimize the cost function \((5)\).

| Algorithm 1: BCD-G algorithm |
|-------------------------------|
| 1: **Input:** A starting point \( \omega_0 \), \( 0 < \nu < \frac{\sqrt{2}}{2} \). |
| 2: **Output:** Sequence of iterates \( \omega_k \). |
| 3: **for** \( k = 1, 2, \cdots \), **do** |
| 4: Choose \( t_k = 1 \) or \( 2 \) such that |
| \[
\| \text{grad } f_{t_k}(\omega_{k-1}) \| \geq \nu \| \text{grad } f(\omega_{k-1}) \|; \] |
| 5: **if** \( t_k = 1 \) **then** |
| 6: Update \( U_k \) using a certain algorithm; |
| 7: Set \( X_k = X_{k-1} \); |
| 8: **else** |
| 9: Set \( U_k = U_{k-1} \); |
| 10: Update \( X_k \) using a certain algorithm. |
| 11: **end if** |
| 12: **end for** |

It is easy to see that, in Algorithm 1, we can always choose \( t_k = 1 \) or \( 2 \) such that the inequality \((7)\) is satisfied\(^2\). In this paper, to update \( U_k \), we choose the *line search descent* method \([3]\),

\(^1\) See [3, Sec. 3.6] for a detailed definition.

\(^2\) The inequality \((7)\) can be seen as a block coordinate analogue of \([18, \text{Eq. (3.3)}]\) and \([22, \text{Eq. (10)}]\).
which will be presented in Section 3. To update $X_k$, we will use the Jacobi-GLU or Jacobi-GLQ rotations, which will be formulated in Section 4. We call Algorithm 1 the BCD-GLU (or BCD-GLQ) algorithm if we choose the Jacobi-GLU (or Jacobi-GLQ) rotations. It will be seen that $f(\omega_k) \leq f(\omega_{k-1})$ always holds in Algorithm 1. In Algorithm 1 for cost function (1), we denote $M_0 \equiv f(\omega_0)$.

1.6. Contributions. The main contributions of this paper can be summarized as follows:

- To solve problem (1), we propose the BCD-G framework (Algorithm 1), which chooses the block for optimization in a way based on the Riemannian gradient. This is similar to the gradient based way of choosing index pairs in Jacobi-G algorithms [18, 22, 40].
- To update the first block variable $U$, we adopt the well-known line search descent method. To update the second block variable $X$, based on four kinds of elementary rotations, we construct two classes of Jacobi-type rotations (Jacobi-GLU and Jacobi-GLQ), and then get two BCD-G algorithms: BCD-GLU and BCD-GLQ.
- We establish the weak convergence\(^3\) and global convergence\(^4\) of BCD-GLU and BCD-GLQ algorithms using the Lojasiewicz gradient inequality under the assumption that the iterates are bounded, that is, there exists $M_\omega > 0$ such that

$$\|\omega_k\| \leq M_\omega$$ (8)

always holds.
- If we set $m = n$ and fix $U = I_n$, then BCD-GLU and BCD-GLQ algorithms specialize as the Jacobi-GLU and Jacobi-GLQ algorithms on $\text{SL}_m(\mathbb{C})$, which can be seen as non-orthogonal analogues of the Jacobi-G algorithm on orthogonal or unitary group [18, 22, 40].

Remark 1.2. This paper is based on the complex matrices, complex Stiefel manifold $\text{St}(m, n, \mathbb{C})$ and complex special linear group $\text{SL}_m(\mathbb{C})$. In fact, all the algorithms and convergence results described in this paper also apply to the real case.

1.7. Organization. The paper is organized as follows. In Section 2, we recall the basics of first order geometries on the Stiefel manifold $\text{St}(m, n, \mathbb{C})$ and special linear group $\text{SL}_m(\mathbb{C})$, as well as the Lojasiewicz gradient inequality. In Section 3, we show the details of how to use line search descent method to update the first block variable. In Section 4, using four kinds of elementary rotations, we construct two classes of Jacobi-type rotations to update the second block variable, which induce two BCD-G algorithms: BCD-GLU and BCD-GLQ. In Section 5 and Section 6, we present the details of these four kinds of elementary rotations. In Section 7, we prove our main results about the weak and global convergence of BCD-G algorithms. For more clarity, we also present our convergence results for special cases of BCD-GLU and BCD-GLQ algorithms, which are gradient based Jacobi-type algorithms to solve problem (3).

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\(^3\)Every accumulation point is a stationary point.
\(^4\)For any starting point, the iterates converge as a whole sequence.
2. Geometries on $\text{St}(m, n, \mathbb{C})$ and $\text{SL}_m(\mathbb{C})$

2.1. Notations. Let $1 \leq m \leq n$. For a complex matrix $X \in \mathbb{C}^{n \times m}$ and a complex number $z \in \mathbb{C}$, we write the real and imaginary parts as $X = X^R + iX^\Im$ and $z = \Re(z) + i\Im(z)$, respectively. For $X, Y \in \mathbb{C}^{n \times m}$, we introduce the following real-valued inner product

$$\langle X, Y \rangle \overset{\text{def}}{=} \langle X^R, Y^R \rangle + \langle X^\Im, Y^\Im \rangle = \Re \left( \text{tr} (X^H Y) \right),$$

which makes $\mathbb{C}^{n \times m}$ a real Euclidean space of dimension $2nm$. Let $h : \mathbb{C}^{n \times m} \to \mathbb{R}$ be a differentiable function and $X \in \mathbb{C}^{n \times m}$. We denote by $\frac{\partial h}{\partial X^R}, \frac{\partial h}{\partial X^\Im} \in \mathbb{R}^{n \times m}$ the matrix Euclidean derivatives of $h$ with respect to real and imaginary parts of $X$. The Wirtinger derivatives \[1, 9, 20\] are defined as

$$\frac{\partial h}{\partial X^R} := \frac{1}{2} \left( \frac{\partial h}{\partial X^R} + i \frac{\partial h}{\partial X^\Im} \right), \quad \frac{\partial h}{\partial X^\Im} := \frac{1}{2} \left( \frac{\partial h}{\partial X^R} - i \frac{\partial h}{\partial X^\Im} \right).$$

Then the Euclidean gradient of $h$ with respect to the inner product (9) becomes

$$\nabla h(X) = \frac{\partial h}{\partial X^R} + i \frac{\partial h}{\partial X^\Im} = 2 \frac{\partial h}{\partial X^R}.$$ \hfill (10)

For real matrices $X, Y \in \mathbb{R}^{n \times m}$, we see that (9) becomes the standard inner product, and (10) becomes the standard Euclidean gradient. We denote by $S_{n-1} \subseteq \mathbb{R}^n$ the unit sphere.

2.2. Riemannian gradient on $\text{St}(m, n, \mathbb{C})$. For $X \in \mathbb{C}^{m \times m}$, we denote

$$\text{sym} (X) = \frac{1}{2} \left( X + X^H \right), \quad \text{skew} (X) = \frac{1}{2} \left( X - X^H \right).$$

Let $T_U \text{St}(m, n, \mathbb{C})$ be the tangent space to $\text{St}(m, n, \mathbb{C})$ at a point $U$. Let $U_\perp \in \mathbb{C}^{n \times (n - m)}$ be an orthogonal complement of $U$, that is, $[U, U_\perp] \in \mathbb{C}^{n \times n}$ is a unitary matrix. By [30, Def. 6], we know that

$$T_U \text{St}(m, n, \mathbb{C}) = \{ Z \in \mathbb{C}^{n \times m}, Z = UA + U_\perp B, A \in \mathbb{C}^{m \times m}, A^H + A = 0, B \in \mathbb{C}^{(n-m) \times m} \}, \quad \text{(11)}$$

which is a $(2nm - m^2)$-dimensional vector space. The orthogonal projection of $\xi \in \mathbb{C}^{n \times m}$ to $T_U \text{St}(m, n, \mathbb{C})$ is

$$\text{Proj}_U \xi = (I_n - UU^H) \xi + U \text{skew}(U^H \xi) = \xi - U \text{sym} \left( U^H \xi \right).$$ \hfill (12)

We denote $\text{Proj}_U \xi \overset{\text{def}}{=} \xi - \text{Proj}_U \xi$. Note that $\text{St}(m, n, \mathbb{C})$ is an embedded submanifold of the Euclidean space $\mathbb{C}^{n \times m}$. By (12), we have the Riemannian gradient of $h$ at $U$ as:

$$\text{grad} h(U) = \text{Proj}_U \nabla h(U) = \nabla h(U) - U \text{sym} \left( U^H \nabla h(U) \right).$$ \hfill (13)

By [3, Ex. 5.4.2], the exponential map at $U$ is defined as

$$\text{Exp}_U : T_U \text{St}(m, n, \mathbb{C}) \to \text{St}(m, n, \mathbb{C}) \quad Z \mapsto [U, Z] \exp \left( \begin{bmatrix} U^H Z & -Z^H Z \\ I_m & U^H Z \end{bmatrix} \right) \left[ \exp \left( -U^H Z \right) I_{m \times m} \right].$$
2.3. Matrix groups. Let \( \text{sl}_m(\mathbb{C}) \overset{\text{def}}{=} \{ X \in \mathbb{C}^{m \times m}, \text{tr}(X) = 0 \} \). Then the tangent space to \( \text{SL}_m(\mathbb{C}) \) at a point \( X \in \text{SL}_m(\mathbb{C}) \) can be constructed [6, Eq. (3.7), (3.8)] by \( T_X \text{SL}_m(\mathbb{C}) = \{ X\Omega, \Omega \in \text{sl}_m(\mathbb{C}) \} \). Let \( T \) \( \text{SU}_m \subseteq \mathbb{C}^{m \times m} \) be the special unitary group. Let

\[
\text{su}_m(\mathbb{C}) \overset{\text{def}}{=} \{ X \in \mathbb{C}^{m \times m}, X^H = -X, \text{tr}(X) = 0 \}.
\]

Then the tangent space to \( \text{SU}_m \) at a point \( X \in \text{SU}_m \) can be constructed [6, Eq. (3.15)] by \( T_X \text{SU}_m = \{ X\Omega, \Omega \in \text{sl}_m(\mathbb{C}) \} \).

A matrix \( X \in \mathbb{C}^{m \times m} \) is said to be upper triangular if \( X_{ij} = 0 \) for \( i > j \). An upper triangular matrix \( X \) is said to be unipotent if it satisfies \( X_{ii} = 1 \) for \( 1 \leq i \leq m \). Let \( \text{UT}_m(\mathbb{C}) \subseteq \text{GL}_m(\mathbb{C}) \) be the upper triangular subgroup, that is

\[
\text{UT}_m(\mathbb{C}) \overset{\text{def}}{=} \{ X \in \text{GL}_m(\mathbb{C}), X \text{ is upper triangular} \}.
\]

Let \( \text{SUT}_m(\mathbb{C}) \subseteq \text{GL}_m(\mathbb{C}) \) be the upper unipotent subgroup, that is

\[
\text{SUT}_m(\mathbb{C}) \overset{\text{def}}{=} \{ X \in \text{GL}_m(\mathbb{C}), X \text{ is upper triangular and unipotent} \}.
\]

A matrix \( X \in \mathbb{C}^{m \times m} \) is said to be strictly upper triangular if \( X_{ij} = 0 \) for \( i \geq j \). Let \( \text{sut}_m(\mathbb{C}) \subseteq \mathbb{C}^{m \times m} \) be the set of strictly upper triangular matrices. Then the tangent space to \( \text{SUT}_m(\mathbb{C}) \) at a point \( X \in \text{SUT}_m(\mathbb{C}) \) can be constructed [6, Eq. (3.11)], [4, Sec. 6.4] by \( T_X \text{SUT}_m(\mathbb{C}) = \{ X\Omega, \Omega \in \text{sut}_m(\mathbb{C}) \} \). For the case \( n = 2 \),

\[
\text{SUT}_2(\mathbb{C}) = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, z \in \mathbb{C} \right\}, \quad \text{sut}_2(\mathbb{C}) = \left\{ \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}, z \in \mathbb{C} \right\}.
\]

Similary, we let \( \text{LT}_m(\mathbb{C}) \subseteq \mathbb{C}^{m \times m} \) be the lower triangular subgroup, \( \text{SLT}_m(\mathbb{C}) \subseteq \mathbb{C}^{m \times m} \) be the lower unipotent subgroup, and \( \text{slt}_m(\mathbb{C}) \subseteq \mathbb{C}^{m \times m} \) be the set of strictly lower triangular matrices. Then the tangent space to \( \text{SLT}_m(\mathbb{C}) \) at a point \( X \in \text{SLT}_m(\mathbb{C}) \) can be constructed by \( T_X \text{SLT}_m(\mathbb{C}) = \{ X\Omega, \Omega \in \text{slt}_m(\mathbb{C}) \} \).

A diagonal matrix \( X \in \mathbb{C}^{m \times m} \) is said to be a diagonal rotation if the product of all the diagonal elements is equal to 1. Let \( \text{D}_m(\mathbb{C}) \subseteq \text{GL}_m(\mathbb{C}) \) be the set of diagonal rotation matrices. Let \( \text{d}_m(\mathbb{C}) \subseteq \mathbb{C}^{m \times m} \) be the set of diagonal traceless matrices. Then the tangent space to \( \text{D}_m(\mathbb{C}) \) at a point \( X \in \text{D}_m(\mathbb{C}) \) can be constructed by \( T_X \text{D}_m(\mathbb{C}) = \{ X\Omega, \Omega \in \text{d}_m(\mathbb{C}) \} \). For the case \( n = 2 \),

\[
\text{D}_2(\mathbb{C}) = \left\{ \begin{bmatrix} z & 0 \\ 0 & \pm i \end{bmatrix}, z \in \mathbb{C} \right\}, \quad \text{d}_2(\mathbb{C}) = \left\{ \begin{bmatrix} z & 0 \\ 0 & -z \end{bmatrix}, z \in \mathbb{C} \right\}.
\]

2.4. Riemannian gradient on \( \text{SL}_m(\mathbb{C}) \). For tangent vectors \( \xi, \eta \in T_X \text{SL}_m(\mathbb{C}) \), we use the left invariant [3], [4, Eq. (6.2)] Riemannian metric

\[
\langle \xi, \eta \rangle_X \overset{\text{def}}{=} \langle X^{-1}\xi, X^{-1}\eta \rangle = \Re \left( \text{tr}(\xi^H(XX^H)^{-1}\eta) \right).
\]

Let \( g : \text{SL}_m(\mathbb{C}) \rightarrow \mathbb{R}^+ \) be a differentiable cost function. Then the Riemannian gradient of \( g \) at \( X \in \text{SL}_m(\mathbb{C}) \) is the orthogonal projection [4, Lem. 6.2] of its Euclidian gradient \( \nabla g(X) \) to \( T_X \text{SL}_m(\mathbb{C}) \), that is,

\[
\text{grad}_g(X) = X \left( X^H \nabla g(X) - \frac{\text{tr}(X^H \nabla g(X))}{n} I_n \right). \quad (14)
\]
We denote \( \Lambda(X) \stackrel{\text{def}}{=} X^{-1} \text{grad} g(X) \in \mathfrak{s} \mathfrak{l}_m(\mathbb{C}) \) for \( X \in \mathbf{SL}_m(\mathbb{C}) \), which will be frequently used in this paper.

In what follows, we will use the following exponential map
\[
\text{Exp}_X : T_X \mathbf{SL}_m(\mathbb{C}) \to \mathbf{SL}_m(\mathbb{C}), \quad X \Omega \mapsto X \exp(\Omega^*) \exp(\Omega - \Omega^*),
\]
where \( \exp(\Omega) \) is the matrix exponential function \(^{\text{3, 6, 17}}\). For any \( \Delta \in T_X \mathbf{SL}_m(\mathbb{C}) \), we have the following relationship between \( \text{Exp}_X \) and the Riemannian gradient
\[
\langle \Delta, \text{grad} g(X) \rangle_X = \left. \left( \frac{d}{dt} g(\text{Exp}_X(t\Delta)) \right) \right|_{t=0}.
\]

2.5. **Lojasiewicz gradient inequality.** In this subsection, we present some results about the Lojasiewicz gradient inequality \(^{\text{21, 27, 2, 39}}\). These results were used in \(^{\text{24, 40}}\) to prove the global convergence of Jacobi-G algorithms on the orthogonal and unitary groups.

**Definition 2.1** \(^{\text{[36, Def. 2.1]}}\). Let \( M \subseteq \mathbb{R}^n \) be a Riemannian submanifold, and \( f : M \to \mathbb{R} \) be a differentiable function. The function \( f : M \to \mathbb{R} \) is said to satisfy a Lojasiewicz gradient inequality at \( x \in M \), if there exist \( \sigma > 0, \zeta \in (0, \frac{1}{2}] \) and a neighborhood \( U \) in \( M \) of \( x \) such that for all \( y \in U \), it follows that
\[
|f(y) - f(x)|^{1-\zeta} \leq \sigma \| \text{grad} f(y) \|.
\]

**Lemma 2.2** \(^{\text{[36, Prop. 2.2]}}\). Let \( M \subseteq \mathbb{R}^n \) be an analytic submanifold\(^{\text{5}}\) and \( f : M \to \mathbb{R} \) be a real analytic function. Then for any \( x \in M \), \( f \) satisfies a Lojasiewicz gradient inequality \(^{\text{(16)}}\) in the \( \delta \)-neighborhood of \( x \), for some \( \delta, \sigma > 0 \) and \( \zeta \in (0, \frac{1}{2}] \).

**Theorem 2.3** \(^{\text{[36, Thm. 2.3]}}\). Let \( M \subseteq \mathbb{R}^n \) be an analytic submanifold and \( \{x_k\}_{k \geq 1} \subseteq M \). Suppose that \( f \) is real analytic and, for large enough \( k \),
(i) there exists \( \sigma > 0 \) such that
\[
|f(x_{k+1}) - f(x_k)| \geq \sigma \| \text{grad} f(x_k) \| \| x_{k+1} - x_k \|;
\]
(ii) \( \text{grad} f(x_k) = 0 \) implies that \( x_{k+1} = x_k \).

Then any accumulation point \( x_* \) of \( \{x_k\}_{k \geq 1} \) must be the only limit point. If, in addition, for some \( \kappa > 0 \) and for large enough \( k \) it holds that
\[
\| x_{k+1} - x_k \| \geq \kappa \| \text{grad} f(x_k) \|,
\]
then the following convergence rates apply
\[
\| x_k - x_* \| \leq \begin{cases} Ce^{-ck}, & \text{if } \zeta = \frac{1}{2}; \\ Ck^{-\frac{1}{1-\zeta}}, & \text{if } 0 < \zeta < \frac{1}{2}, \end{cases}
\]
where \( \zeta \) is the parameter in \(^{\text{(16)}}\) at the limit point \( x_* \), and \( C > 0, c > 0 \) are some constants.

\(^{\text{5}}\)See \(^{\text{[19, Def. 2.7.1]}}\) or \(^{\text{[24, Def. 5.1]}}\) for a definition of an analytic submanifold.

\(^{\text{6}}\)The values of \( \delta, \sigma, \zeta \) depend on the specific point in question.
3. Line search descent method on $\text{St}(m,n,\mathbb{C})$

Let $f$ be the cost function (1). Let $\omega_{k-1} = (U_{k-1}, X_{k-1})$ and $p = f_{1}, X_{k-1}$ be the first restricted function. Denote $X = X_{k-1}$ for simplicity. Then $p$ can be expressed as

$$p : \text{St}(m,n,\mathbb{C}) \to \mathbb{R}^+, \quad U \mapsto \sum_{\ell=1}^{L} \mu_{\ell} \|\text{offdiag}\{W^{(\ell)}\}\|^2,$$

(19)

where $W^{(\ell)} = X^{\dagger}U^{\dagger}A^{(\ell)}UX$.

3.1. Riemannian gradient. We first present a lemma, which can be obtained by direct calculations. This result will help us to obtain the Riemannian gradient of $p$ in (19).

Lemma 3.1. Let $A \in \mathbb{C}^{n \times n}$ and the function $\tilde{p}$ be defined as

$$\tilde{p} : \mathbb{C}^{n \times m} \to \mathbb{R}^+, \quad Y \mapsto \|\text{offdiag}\{W\}\|^2,$$

(20)

where $W = Y^{\dagger}AY$. Let $V = AY = [v_1, \ldots, v_m]$ and $V^{'} = A^{\dagger}Y = [v_1^{\prime}, \ldots, v_m^{\prime}]$. Denote $Y = [y_1, \ldots, y_m]$. Then the Euclidean gradient is

$$\nabla \tilde{p}(Y) = 2 \left( \sum_{j \neq 1} v_j v_j^{\dagger} y_1, \ldots, \sum_{j \neq m} v_j v_j^{\dagger} y_m \right) + 2 \left( \sum_{j \neq 1} v_j^{\prime}(v_j^{\prime})^{\dagger} y_1, \ldots, \sum_{j \neq m} v_j^{\prime}(v_j^{\prime})^{\dagger} y_m \right).$$

(21)

In particular, it satisfies

$$YY^{\dagger} \nabla \tilde{p}(Y) = 2Y \Upsilon(W),$$

(22)

where

$$\Upsilon(W) \overset{\text{def}}{=} \begin{cases} W \text{ offdiag}\{W\}^{\dagger} + W^{\dagger} \text{ offdiag}\{W\}, & \text{if } (\cdot)^{\dagger} = (\cdot)^{\dagger}; \\ W^{\ast} \text{ offdiag}\{W\}^{\dagger} + W \text{ offdiag}\{W\}, & \text{if } (\cdot)^{\dagger} = (\cdot)^{\dagger}. \end{cases}$$

Lemma 3.2. Let $W^{(\ell)}$ and the function $p$ be as in (19). Then the Euclidean gradient satisfies

$$U^{\dagger} \nabla p(U) = 2 (X^{\dagger})^{-1} \sum_{\ell=1}^{L} \mu_{\ell} \Upsilon(W^{(\ell)}) X^{\dagger}.$$ 

(23)

Proof. By the product rule, we see that $\nabla p(U) = \nabla \tilde{p}(UX)X^{\dagger}$. Then, by (22), we have

$$XX^{\dagger}U^{\dagger} \nabla p(U) = U^{\dagger}UX(X^{\dagger})^{\dagger} \nabla \tilde{p}(UX)X^{\dagger} = 2U^{\dagger}UX \sum_{\ell=1}^{L} \Upsilon(W^{(\ell)}) X^{\dagger}.$$ 

Note that $X$ is invertible. The proof is complete. \qed

Now, by (23) and (13), we see that the Riemannian gradient of $p$ in (19) satisfies

$$U^{\dagger} \text{ grad } p(U) = (X^{\dagger})^{-1} \sum_{\ell=1}^{L} \Upsilon(W^{(\ell)}) X^{\dagger} - X \sum_{\ell=1}^{L} \Upsilon(W^{(\ell)}) X^{\dagger} X^{-1}.$$ 

(24)
3.2. **Line search descent method.** In this paper, we adopt the line search descent method \cite{2, 3, 34, 35} on \textit{St}(m, n, \mathbb{C}) to find \( U_k \) for the cost function (19). More precisely, we set

\[
U_k = \text{Exp}_{U_{k-1}}(t_{k-1}Z_{k-1}),
\]

where the search direction \( Z_{k-1} \) satisfies

\[
\langle \text{grad} \, p(U_{k-1}), Z_{k-1} \rangle U_{k-1} \leq -\delta_s \| \text{grad} \, p(U_{k-1}) \| \| Z_{k-1} \| \]

with \( \delta_s > 0 \), and the step size \( t_{k-1} \) satisfies the first Wolfe condition\footnote{It is also known as the Armijo condition in the literature.}

\[
p(U_k) \leq p(U_{k-1}) + \delta_w t_{k-1} \langle \text{grad} \, p(U_{k-1}), Z_{k-1} \rangle U_{k-1}
\]

with \( 0 < \delta_w < 1 \). We say that the shrinking gradient condition is satisfied, if there exists \( \kappa_p > 0 \) such that

\[
\| t_{k-1}Z_{k-1} \| \geq \kappa_p \| \text{grad} \, p(U_{k-1}) \|
\]

holds. It was shown \cite{34, 35} that we can always choose \( Z_{k-1} \) and \( t_{k-1} \) such that the conditions (26) and (27) are both satisfied. It is not difficult to see that there exists \( M_c > 0 \) such that

\[
\| \text{Exp}_U(Z_1) - \text{Exp}_U(Z_2) \| \leq M_c \| Z_1 - Z_2 \|,
\]

for any \( U \in \text{St}(m, n, \mathbb{C}) \) and \( Z_1, Z_2 \in T_U \text{St}(m, n, \mathbb{C}) \). Then the next result follows directly.

**Lemma 3.3.** If we set \( U_k \) as in (25) such that the conditions (26) and (27) are both satisfied, then we have

\[
p(U_{k-1}) - p(U_k) \geq \delta_s \| \text{grad} \, p(U_{k-1}) \| \| t_{k-1}Z_{k-1} \| \geq \sigma_p \| \text{grad} \, p(U_{k-1}) \| \| U_k - U_{k-1} \|,
\]

where \( \sigma_p = \frac{\delta_w}{M_c} \). Furthermore, if the condition (28) is always satisfied, then we have

\[
p(U_{k-1}) - p(U_k) \geq \eta_p \| \text{grad} \, p(U_{k-1}) \|^2.
\]

where \( \eta_p = \delta_s \delta_w \kappa_p \).

4. **Jacobi-GLU and Jacobi-GLQ rotations on \( SL_m(\mathbb{C}) \)**

4.1. **Elementary functions.** Let \((i, j)\) be a pair of indices with \( 1 \leq i < j \leq n \). Define a projection operator \( \mathcal{P}_{i,j} : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{2 \times 2} \) which extracts a submatrix of \( X \in \mathbb{C}^{m \times m} \) as follows:

\[
\mathcal{P}_{i,j}(X) = \begin{bmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj} \end{bmatrix}.
\]

Conversely, we introduce an operator

\[
\mathcal{E}_{i,j} : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^{m \times m}, \quad \Psi \mapsto \mathcal{E}_{i,j}(\Psi)
\]

where \( \mathcal{E}_{i,j}(\Psi) \) is the identity matrix \( I_m \) except that \( \mathcal{P}_{i,j}(\mathcal{E}_{i,j}(\Psi)) = \Psi \). Now we define the following four elementary rotations in \( SL_m(\mathbb{C}) \).

- \( G^{(i,j, \Psi)} \) def \( \mathcal{E}_{i,j}(\Psi) \): the plane rotation for a matrix \( \Psi \in SU_2 \).
- \( U^{(i,j, \Psi)} \) def \( \mathcal{E}_{i,j}(\Psi) \): the upper triangular rotation for a matrix \( \Psi \in SUT_2(\mathbb{C}) \).
- \( L^{(i,j, \Psi)} \) def \( \mathcal{E}_{i,j}(\Psi) \): the lower triangular rotation for a matrix \( \Psi \in SLT_2(\mathbb{C}) \).
- \( D^{(i,j, \Psi)} \) def \( \mathcal{E}_{i,j}(\Psi) \): the diagonal rotation for a matrix \( \Psi \in D_2(\mathbb{C}) \).
Remark 4.1. These four elementary rotations have all been used in the literature. The plane rotation \( G_{(i,j,\Psi)} \) was used very often in Jacobi-type algorithms for joint approximate diagonalization of matrices or tensors by orthogonal or non-orthogonal transformations \([15, 22, 40, 4, 5, 37]\). The upper triangular rotation \( U_{(i,j,\Psi)} \) and lower triangular rotation \( L_{(i,j,\Psi)} \) also appeared many times in the Jacobi-type algorithms on special linear group \( SL_m(\mathbb{C}) \) or \( SL_m(\mathbb{R}) \) \([4, 5, 31, 32, 33]\). In the real case, the diagonal rotation \( D_{(i,j,\Psi)} \) was used in \([38]\). In this paper, using these four elementary rotations, we will construct two classes (Jacobi-GLU and Jacobi-GLQ) in Section 4.3, such that a gradient inequality \((38)\) is always satisfied to establish the convergence.

In this paper, as in \([15, 40]\), we parameterize \( \Psi \in SU_2 \) as

\[
\Psi = \Psi(c, s_1, s_2) = \begin{bmatrix} c & -s_l \cr s_l & c \end{bmatrix} = \begin{bmatrix} c & -(s_1 + is_2) \cr s_1 - is_2 & c \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta e^{i\phi} \\
\sin \theta e^{-i\phi} & \cos \theta \end{bmatrix},
\]

(32)

where \((c, s_1, s_2) \in S_2 \) and \((\theta, \phi) \in \mathbb{R}^2\).

Let \( g : SL_m(\mathbb{C}) \to \mathbb{R}^+ \) be a smooth function. Let \( X \in SL_m(\mathbb{C}) \) and \( z = x + iy \). We define the following elementary functions:

\[
\phi_{(i,j)}(x, y) = \phi_{(i,j)}(\Psi) \overset{\text{def}}{=} g(XU_{(i,j,\Psi)}), \quad z \in \mathbb{C};
\]

(34)

\[
\psi_{(i,j)}(x, y) = \psi_{(i,j)}(\Psi) \overset{\text{def}}{=} g(XL_{(i,j,\Psi)}), \quad z \in \mathbb{C};
\]

(35)

\[
\rho_{(i,j)}(x, y) = \rho_{(i,j)}(\Psi) \overset{\text{def}}{=} g(XD_{(i,j,\Psi)}), \quad z \in \mathbb{C}_+.
\]

(36)

4.2. Derivatives of elementary functions. Denote \( \Lambda = \Lambda(X) \), which is defined as in Section 2.4. Let \( P_{i,j}^T : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{m \times m} \) be the adjoint operator of projection operator \( P_{i,j} \) defined in \((31)\), i.e.,

\[
P_{i,j}^T(\Psi) = \begin{bmatrix}
i & j \\
i & \Psi_{ii} & \Psi_{ij} \\
j & \Psi_{ji} & \Psi_{jj} \\
0 & 0 & 0
\end{bmatrix}
\]

for \( \Psi \in \mathbb{C}^{2 \times 2} \).
Lemma 4.2. The Riemannian gradients of the elementary functions defined in Section 4.1 at the identity matrix $I_2$ can be expressed as follows:

\[
\begin{align*}
\text{grad } h_{(i,j)}(I_2) &= \begin{bmatrix} \frac{1}{2} \Re (\Lambda_{ii} - \Lambda_{jj}) & \frac{1}{2} \Im (\Lambda_{ij}) - \frac{1}{2} \Im (\Lambda_{ji}) \\ \frac{1}{2} \Im (\Lambda_{ij}) + \frac{1}{2} \Im (\Lambda_{ji}) & -\frac{1}{2} \Re (\Lambda_{ii} - \Lambda_{jj}) \end{bmatrix}; \\
\varphi_{(i,j)}(I_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
\psi_{(i,j)}(I_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
\rho_{(i,j)}(I_2) &= \begin{bmatrix} \Re (\Lambda_{ii} - \Lambda_{jj}) \\ 0 \\ 0 \\ \Im (\Lambda_{ii} - \Lambda_{jj}) \end{bmatrix}.
\end{align*}
\]

Proof. We only prove the case of $h_{(i,j)}$. Other cases are similar. For any $\Delta \in su_2(\mathbb{C}) = T_{I_2}SU_2$, by equation (15), we have

\[
(\Delta, \text{grad } h_{(i,j)}(I_2)) I_2 = \left(\frac{d}{dt} h_{(i,j)}(X(\text{Exp} I_2(t\Delta))) \right)_{t=0} = \left(\frac{d}{dt} f(XG^{(i,j,\text{Exp} I_2(t\Delta)}) \right)_{t=0} = (X P_{i,j}^T(\Delta), \text{grad } f(X)) X = (\Delta, P_{i,j}(\Delta)) I_2.
\]

The proof is complete. \qed

Lemma 4.3. The partial derivatives of the elementary functions defined in Section 4.1 satisfy

\[
\begin{align*}
\varphi_{(i,j)}(I_2) &= \varphi_{(i,j)}(1,0,0) = [0, -\Re (\Lambda_{ij}), \Im (\Lambda_{ij})]^T; \\
\psi_{(i,j)}(I_2) &= \psi_{(i,j)}(0,0,0) = [\Re (\Lambda_{ij}), \Im (\Lambda_{ij})]^T; \\
\rho_{(i,j)}(I_2) &= \rho_{(i,j)}(0,0,0) = [\Re (\Lambda_{ii} - \Lambda_{jj}), \Im (\Lambda_{ii} - \Lambda_{jj})]^T.
\end{align*}
\]

4.3. Jacobi-G rotations. Let $\omega_{k-1} = (U_{k-1}, X_{k-1})$ be the $(k-1)$-th iterate produced by Algorithm 1, and $g : SL_m(\mathbb{C}) \to \mathbb{R}$ be the restricted function $f_{2U_{k-1}}$ defined as in (6). Let $(i_k, j_k)$ be a pair of indices with $1 \leq i_k < j_k \leq m$. We denote that

\[
h_k = h_{(i_k,j_k)}(X_{k-1}), \quad \varphi_k = \varphi_{(i_k,j_k)}(X_{k-1}), \quad \psi_k = \psi_{(i_k,j_k)}(X_{k-1}), \quad \rho_k = \rho_{(i_k,j_k)}(X_{k-1}).
\]

Any matrix $X \in SL_m(\mathbb{C})$ has the LU decomposition [16] $X = LU$ with $L \in LT_m(\mathbb{C})$ and $U \in UT_m(\mathbb{C})$, and the LQ decomposition $X = LQ$ with $L \in UT_m(\mathbb{C})$ and $Q \in SU_m$. Inspired by these two decompositions and a similar idea as in [18, 22, 40], in this section, we propose two types of Jacobi-G rotations on $SL_m(\mathbb{C})$. We call them the Jacobi-GLU (based on LU decomposition) and Jacobi-GLQ (based on LQ decomposition) rotations\(^8\), respectively.

Proposition 4.4. In Algorithm 2 and Algorithm 3, we can always choose an index pair $(i_k, j_k)$ and an elementary function $\nu_k(x)$ such that the inequality (38) is satisfied.

\(^{8}\)The inequality (38) can be seen as a non-orthogonal analogue of [18, Eq. (3.3)] and [22, Eq. (10)].
Algorithm 2: Jacobi-GLU rotation

1: **Input:** A fixed positive constant $0 < \varepsilon < \sqrt{\frac{2}{3m(m-1)}}$.

2: **Output:** New iterate $X_k$.

3: Choose the index pair $(i_k, j_k)$ and $\nu_k$ such that

$$\|\partial \nu_k(I_2)\| \geq \varepsilon \|\Lambda(X_{k-1})\|,$$

(38)

where $\nu_k = \varphi_k, \psi_k$ or $\rho_k$;

4: Compute $\Psi_k^*$ that minimizes the function $\nu_k$;

5: Update $X_k = X_{k-1} V_k$, where $V_k = U^{(i_k, j_k, \Psi^*_{k})}, L^{(i_k, j_k, \Psi^*_{k})}$ or $D^{(i_k, j_k, \Psi^*_{k})}$.

Algorithm 3: Jacobi-GLQ rotation

1: **Input:** A fixed positive constant $0 < \varepsilon < \sqrt{\frac{3+\sqrt{5}}{3m(m-1)}}$.

2: **Output:** New iterate $X_k$.

3: Choose the index pair $(i_k, j_k)$ and $\nu_k$ satisfying (38), where $\nu_k = h_k, \psi_k$ or $\rho_k$;

4: Compute $\Psi_k^*$ that minimizes the function $\nu_k$;

5: Update $X_k = X_{k-1} V_k$, where $V_k = G^{(i_k, j_k, \Psi^*)}, L^{(i_k, j_k, \Psi^*)}$ or $D^{(i_k, j_k, \Psi^*)}$.

We need a simple lemma before the proof of Proposition 4.4.

**Lemma 4.5.** (i) Let $z_1, z_2 \in \mathbb{C}$. Then

$$|z_1 - z_2|^2 + |z_2|^2 \geq \frac{3 - \sqrt{5}}{2}(|z_1|^2 + |z_2|^2).$$

(ii) Let $\{z_i\}_{1 \leq i \leq m} \subseteq \mathbb{C}$ satisfy that $\sum_{1 \leq i \leq m} z_i = 0$. Then

$$\sum_{1 \leq i < j \leq m} |z_i - z_j|^2 = m \sum_{1 \leq i \leq m} |z_i|^2.$$

**Proof of Proposition 4.4.** We first prove the existence of such an index pair $(i_k, j_k)$ and elementary function $\nu_k$ in Algorithm 2. By Lemma 4.3, Lemma 4.5 and $\Lambda = \Lambda(X_{k-1}) \in \text{sl}_m(\mathbb{C})$, we get that

$$\sum_{1 \leq i_k < j_k \leq m} (\|\partial \varphi_k(I_2)\|^2 + \|\partial \psi_k(I_2)\|^2 + \|\partial \rho_k(I_2)\|^2)$$

$$= \sum_{1 \leq i_k < j_k \leq m} (|\Lambda_{i_k, j_k}|^2 + |\Lambda_{j_k, i_k}|^2) + m \sum_{1 \leq i_k \leq m} |\Lambda_{i_k, i_k}|^2 \geq \|\Lambda\|^2.$$

Therefore, there exist an index pair $(i_k, j_k)$ and $\nu_k(x)$ such that

$$\frac{3}{2} m(m-1) \|\partial \psi_k(I_2)\|^2 \geq \|\Lambda\|^2.$$
Next, we prove the existence in Algorithm 3. Similarly, we get that
\[
\sum_{1 \leq i_k < j_k \leq m} (\|\partial h_k(I_2)\|^2 + \|\partial \psi_k(I_2)\|^2 + \|\partial \rho_k(I_2)\|^2)
\]
\[
= \sum_{1 \leq i_k < j_k \leq m} (|\Lambda_{i_k,j_k}^* - \Lambda_{i_k,j_k}|^2 + |\Lambda_{j_k,i_k}^* - \Lambda_{j_k,i_k}|^2)
\]
\[
\geq 3 - \frac{\sqrt{5}}{2} \sum_{1 \leq i_k < j_k \leq m} (|\Lambda_{i_k,j_k}|^2 + |\Lambda_{j_k,i_k}|^2) + m \sum_{1 \leq i_k \leq m} |\Lambda_{i_k}|^2 \geq \frac{3 - \sqrt{5}}{2} \|\Lambda\|^2.
\]
Therefore, there exists an index pair \((i_k, j_k)\) and \(\nu_k(x)\) such that
\[
\frac{3}{3 + \sqrt{5}} m(m - 1) \|\partial v_k(I_2)\|^2 \geq \|\Lambda\|^2.
\]
The proof is complete. \(\square\)

5. Triangular and diagonal rotations for JADM problem

Let \(f\) be the cost function (1). Let \(\omega_{k-1} = (U_{k-1}, X_{k-1})\) and \(g: SL_m(\mathbb{C}) \to \mathbb{R}\) be the restricted function \(f_2U_{k-1}\) as in Section 4.3. Denote \(B^{(t)} = U_{k-1}^H A^{(t)} U_{k-1}\). Then \(g\) can be expressed as
\[
g : SL_m(\mathbb{C}) \to \mathbb{R}^+, \quad X \mapsto \sum_{t=1}^L \mu_t \|\text{offdiag}(W^{(t)})\|^2,
\]
where \(W^{(t)} = X^H B^{(t)} X\). In this section, we will first calculate the Riemannian gradient of \(g\) in (39), and the partial derivatives of elementary functions \(\varphi_k\), \(\psi_k\) and \(\rho_k\) in (37). Then, we will prove that conditions (71) and (72) are both satisfied in the triangular and diagonal rotations.

5.1. Riemannian gradient. Let \(W^{(t)}\) and \(g\) be as in (39). Then, by (22) and (14), we have
\[
\nabla g(X) = 2(X^H)^{-1} \sum_{t=1}^L \mu_t \Upsilon(W^{(t)}),
\]
\[
\text{grad} g(X) = 2X \sum_{t=1}^L \mu_t \left( \Upsilon(W^{(t)}) - \frac{\text{tr}(\Upsilon(W^{(t)}))}{n} I_n \right),
\]

Remark 5.1. In the real case, the Euclidean gradient in (40) was earlier derived in [4, Eq. (6.3)] and [8, Sec. 2.3]. In this paper, we extend it to problem (39) in the complex case, and calculate the Riemannian gradient (41) as well.

5.2. Elementary functions. Let \(W^{(t)} = X_{k-1}^H B^{(t)} X_{k-1}\). Let
\[
g = \begin{cases} 
1, & \text{if } (\cdot)^\dagger = (\cdot)^H; \\
-1, & \text{if } (\cdot)^\dagger = (\cdot)^T.
\end{cases}
\]
Denote \((i, j) = (i_k, j_k)\) for simplicity. Now we make the following notations:
Let the function \( \psi \) be as in (37). Then we can get the following results by direct calculations.

(iii) the elementary function \( \rho \)

\[
\alpha_1 \defeq \sum_{\ell=1}^L \sum_{p \neq j} \mu_{\ell} \left( |W_{jp}^{(\ell)}|^2 + |W_{pj}^{(\ell)}|^2 \right),
\]

\[
\alpha_2 \defeq \sum_{\ell=1}^L \sum_{p \neq j} \mu_{\ell} \left( W_{ip}^{(\ell,\Re)} W_{jp}^{(\ell,\Re)} + W_{ip}^{(\ell,\Im)} W_{jp}^{(\ell,\Im)} + W_{pi}^{(\ell,\Re)} W_{pj}^{(\ell,\Re)} + W_{pi}^{(\ell,\Im)} W_{pj}^{(\ell,\Im)} \right),
\]

\[
\alpha_3 \defeq \sum_{\ell=1}^L \sum_{p \neq j} \mu_{\ell} \left( g \left( W_{ip}^{(\ell,\Re)} W_{jp}^{(\ell,\Re)} - W_{ip}^{(\ell,\Im)} W_{jp}^{(\ell,\Im)} \right) + W_{pi}^{(\ell,\Re)} W_{pj}^{(\ell,\Im)} - W_{pi}^{(\ell,\Im)} W_{pj}^{(\ell,\Re)} \right).
\]

\[
\beta_1 \defeq \sum_{\ell=1}^L \sum_{p \neq i} \mu_{\ell} \left( |W_{i\ell}^{(\ell)}|^2 + |W_{i\ell}^{(\ell)}|^2 \right),
\]

\[
\beta_2 \defeq \sum_{\ell=1}^L \sum_{p \neq i} \mu_{\ell} \left( W_{ip}^{(\ell,\Re)} W_{ip}^{(\ell,\Re)} + W_{ip}^{(\ell,\Im)} W_{ip}^{(\ell,\Im)} + W_{pi}^{(\ell,\Re)} W_{pi}^{(\ell,\Re)} + W_{pi}^{(\ell,\Im)} W_{pi}^{(\ell,\Im)} \right),
\]

\[
\beta_3 \defeq \sum_{\ell=1}^L \sum_{p \neq i} \mu_{\ell} \left( g \left( W_{ip}^{(\ell,\Re)} W_{ip}^{(\ell,\Re)} - W_{ip}^{(\ell,\Im)} W_{ip}^{(\ell,\Im)} \right) + W_{pi}^{(\ell,\Re)} W_{pi}^{(\ell,\Im)} - W_{pi}^{(\ell,\Im)} W_{pi}^{(\ell,\Re)} \right).
\]

\[
\gamma_1 \defeq \sum_{\ell=1}^L \sum_{p \neq i,j} \mu_{\ell} \left( |W_{i\ell}^{(\ell)}|^2 + |W_{i\ell}^{(\ell)}|^2 \right), \quad \gamma_2 \defeq \sum_{\ell=1}^L \sum_{p \neq i,j} \mu_{\ell} \left( |W_{j\ell}^{(\ell)}|^2 + |W_{j\ell}^{(\ell)}|^2 \right).
\]

Then we can get the following results by direct calculations.

**Lemma 5.2.** Let the function \( g \) be as in (39). Then

(i) the elementary function \( \varphi_k \) in (37)

\[
\varphi_k(x, y) - \varphi_k(0, 0) = \alpha_1 x^2 + 2\alpha_2 x + \alpha_1 y^2 + 2\alpha_3 y,
\]

\[
\varphi_k(x_k^*, y_k^*) - \varphi_k(0, 0) = -\frac{1}{\alpha_1} \left( \alpha_2^2 + \alpha_3^2 \right),
\]

\[
\partial \varphi_k(0, 0) = 2[\alpha_2, \alpha_3]^T.
\]

(ii) the elementary function \( \psi_k \) in (37)

\[
\psi_k(x, y) - \psi_k(0, 0) = \beta_1 x^2 + 2\beta_2 x + \beta_1 y^2 + 2\beta_3 y,
\]

\[
\psi_k(x_k^*, y_k^*) - \psi_k(0, 0) = -\frac{1}{\beta_1} \left( \beta_2^2 + \beta_3^2 \right),
\]

\[
\partial \psi_k(0, 0) = 2[\beta_2, \beta_3]^T.
\]

(iii) the elementary function \( \rho_k \) in (37)

\[
\rho_k(x, y) - \rho_k(1, 0) = \gamma_1 (x^2 + y^2) + \gamma_2 \frac{1}{x^2 + y^2} - \gamma_1 - \gamma_2,
\]

\[
\rho_k(x_k^*, y_k^*) - \rho_k(1, 0) = -\left( \sqrt{\gamma_1} - \sqrt{\gamma_2} \right)^2,
\]

\[
\partial \rho_k(1, 0) = 2[\gamma_1 - \gamma_2, 0]^T.
\]

**Remark 5.3.** In the complex case, the solution \( x_k^* = x_k^* + iy_k^* \) in (43) was earlier derived in [41, Eq. (8)]. In the real case, the solution \( x_k^* \) in (43) was earlier derived in [5, Eq. (7)].
Remark 5.4. In Algorithm 1 for cost function (1), when \( \nu_k = \varphi_k \), we see that \( x_k^* = 0 \) if \( \alpha_1 \neq 0 \) and \( \alpha_2 = 0 \). It is not possible that \( \alpha_1 = 0 \) and \( \alpha_2 \neq 0 \). If \( \alpha_1 = \alpha_2 = 0 \), we set \( x_k^* = 0 \). In the case of \( \nu_k = \psi_k \), we make the similar settings for the value of \( y_k^* \).

Remark 5.5. Let \( 0 < \zeta < \frac{1}{4} \) be a small positive constant. In Algorithm 1 for cost function (1), if \( \nu_k = \rho_k \), we always set \( y_k^* = 0 \). Moreover, we determine \( x_k^* \) based on the following rules.

- If \( \gamma_1 = \gamma_2 = 0 \), we set \( x_k^* = 0 \).
- Let \( \omega = \frac{\alpha}{\gamma_1} \). If \( \omega \in [0, \zeta) \), we set \( x_k^* = \frac{1}{4} \). If \( \omega \in (\frac{1}{\zeta}, +\infty] \), we set \( x_k^* = 2 \).
- Otherwise, if \( \omega \in [\zeta, \frac{1}{\zeta}] \), we set \( x_k^* = \sqrt{\omega} \), which is the minimum point.

5.3. Condition for global convergence. In Algorithm 1 for cost function (1), since \( f \) is always decreasing, we have \( \gamma_1 + \gamma_2 \leq M_0 = f(\omega_0) \).

Lemma 5.6. In Algorithm 1 for cost function (1), there exists \( \iota_\rho > 0 \) such that

\[
g(X_{k-1}) - g(X_k) \geq \iota_\rho \| \Lambda(X_{k-1}) \| \| \Psi_k^* - I_2 \|, \tag{51}
\]

whenever \( \nu_k = \rho_k \).

Proof. We now prove the inequality (51) by Lemma 5.2(iii) in three different cases shown in Remark 5.5.

- If \( \gamma_1 = \gamma_2 = 0 \), it is clear that the inequality (51) is satisfied for any \( \iota_\rho > 0 \).
- If \( \omega \in [0, \zeta) \), we get that

\[
\rho_k(1,0) - \rho_k(x_k^*, y_k^*) = \frac{3}{2} \gamma_1(1 - 4\omega) = \frac{3(1 - 4\omega)}{8(1 - \omega)} |\partial \rho_k(1,0)| \geq \frac{3(1 - 4\omega)\varepsilon}{8(1 - \omega)} \| \Lambda(X_{k-1}) \|
\]

\[
\geq \frac{3(1 - 4\omega)\varepsilon \sqrt{5}}{4 \sqrt{5} (1 - \omega)} \| \Lambda(X_{k-1}) \| \geq \frac{3(1 - 4\omega)\varepsilon}{4 \sqrt{5} (1 - \omega)} \| \Lambda(X_{k-1}) \| \| \Psi_k^* - I_2 \|
\]

\[
\geq \frac{3(1 - 4\omega)\varepsilon}{4 \sqrt{5} \| \Lambda(X_{k-1}) \| \| \Psi_k^* - I_2 \|}.
\]

- If \( \omega \in (\frac{1}{\zeta}, +\infty] \), we similarly get that

\[
\rho_k(1,0) - \rho_k(x_k^*, y_k^*) = \frac{3}{2} \gamma_2(1 - \frac{4}{\omega}) = \frac{3(1 - \frac{4}{\omega})}{8(1 - \frac{1}{\omega})} |\partial \rho_k(1,0)| \geq \frac{3(1 - \frac{4}{\omega})\varepsilon}{8(1 - \frac{1}{\omega})} \| \Lambda(X_{k-1}) \|
\]

\[
\geq \frac{3(1 - \frac{4}{\omega})\varepsilon \sqrt{5}}{4 \sqrt{5} (1 - \frac{1}{\omega})} \| \Lambda(X_{k-1}) \| \geq \frac{3(1 - \frac{4}{\omega})\varepsilon}{4 \sqrt{5} (1 - \frac{1}{\omega})} \| \Lambda(X_{k-1}) \| \| \Psi_k^* - I_2 \|
\]

\[
\geq \frac{3(1 - \frac{4}{\omega})\varepsilon}{4 \sqrt{5} \| \Lambda(X_{k-1}) \| \| \Psi_k^* - I_2 \|}.
\]

- If \( \omega \in [\zeta, \frac{1}{\zeta}] \), it is easy to verify that

\[
\frac{3\sqrt{\gamma_1 \gamma_2}}{2} \geq \frac{3\sqrt{\gamma_1}}{2} \left( \sqrt{1 - \frac{4}{\omega}} + \sqrt{1 - \omega} \right).
\]

(52)
Then, we get that
\[ \rho_k(1,0) - \rho_k(x_k^*, y_k^*) = 2 (\sqrt{\gamma_1} - \sqrt{\gamma_2})^2 = \frac{1}{2} |\partial \rho_k(1,0)| \frac{|\sqrt{\gamma_1} - \sqrt{\gamma_2}|}{\gamma_1 + \gamma_2} \]
\[ \geq \frac{\varepsilon}{4} \frac{|\Lambda(X_{k-1})|}{\gamma_1 \gamma_2} \left( \frac{1}{\sqrt{\gamma_1} + \sqrt{\gamma_2}} \right)^{1/2} \]
\[ \geq \frac{\varepsilon}{4} \frac{|\Lambda(X_{k-1})|}{(\gamma_1 \gamma_2)^{1/2}} \left( \frac{1}{(1 + x_k^*)^2} \right) \geq \frac{\varepsilon}{4} \frac{|\Lambda(X_{k-1})|}{(X_{k-1})} \left| \Psi_k^* - I_2 \right|. \]

Now we set \( \nu_k = \min \left( \frac{3(1-k)}{4\sqrt{5}}, \frac{\varepsilon}{4} \right). \) The proof is complete.

Note that \( \|X_k - X_{k-1}\| \leq ||\Psi_k^* - I_2|| \|X_{k-1}\| \) and \( \|\text{grad} g(X_{k-1})\| \leq ||\Lambda(X_{k-1})|| \|X_{k-1}\|. \)

Let \( \nu_k \) be as in (51) and \( \sigma_p = \frac{\nu_k}{\|\Psi_k^* - I_2\|} > 0. \) Then the next result follows directly from Lemma 5.6.

**Corollary 5.7.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then
\[ g(X_{k-1}) - g(X_k) \geq \sigma_p \|\text{grad} g(X_{k-1})\| \|X_k - X_{k-1}\|, \]  
whenever \( \nu_k = \rho_k. \)

**Lemma 5.8.** In Algorithm 1 for cost function (1), there exists \( \nu_k > 0 \) such that
\[ g(X_{k-1}) - g(X_k) \geq \nu_k \|\Lambda(X_{k-1})\| \|\Psi_k^* - I_2\|, \]  
whenever \( \nu_k = \varphi_k \) or \( \psi_k. \)

**Proof.** We prove that the inequality (54) is satisfied in two cases.

- If \( \nu_k = \varphi_k, \) by Lemma 5.2(i), we see that
  \[ g(X_{k-1}) - g(X_k) = \varphi_k(0,0) - \varphi_k(x_k^*, y_k^*) = 2 \left( \frac{(\alpha_2)^2}{\alpha_1} + \frac{(\alpha_3)^2}{\alpha_1} \right) \]
  \[ = \frac{1}{2} \|\partial \varphi_k(0,0)\| \|\sigma^1\| \|\lambda_k^+ \| \geq \frac{\varepsilon}{2} \|\Lambda(X_{k-1})\| \|\Psi_k^* - I_2\|. \]

- If \( \nu_k = \psi_k, \) by Lemma 5.2(ii), we see that
  \[ g(X_{k-1}) - g(X_k) = \psi_k(0,0) - \psi_k(x_k^*, y_k^*) = 2 \left( \frac{(\beta_2)^2}{\beta_1} + \frac{(\beta_3)^2}{\beta_1} \right) \]
  \[ = \frac{1}{2} \|\partial \psi_k(0,0)\| \|\sigma^1\| \|\lambda_k^+ \| \geq \frac{\varepsilon}{2} \|\Lambda(X_{k-1})\| \|\Psi_k^* - I_2\|. \]

Now we set \( \nu_k = \frac{\varepsilon}{2}. \) The proof is complete.

Let \( \nu_k \) be as in (54) and \( \sigma_p = \frac{\nu_k}{\|\Psi_k^* - I_2\|} > 0. \) Similar as for Corollary 5.7, the next result follows directly from Lemma 5.8.

**Corollary 5.9.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then
\[ g(X_{k-1}) - g(X_k) \geq \sigma_\varphi \|\text{grad} g(X_{k-1})\| \|X_k - X_{k-1}\|, \]  
whenever \( \nu_k = \varphi_k \) or \( \psi_k. \)
5.4. Condition for weak convergence.

**Lemma 5.10.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then there exists $\kappa_\rho > 0$ such that

$$\|\Psi_k^* - I_2\| \geq \kappa_\rho \|\Lambda(X_{k-1})\|,$$

(56)

whenever $\nu_k = \rho_k, \varphi_k$ or $\psi_k$.

**Proof.** If $\nu_k = \varphi_k$, we have

$$\|\Psi_k^* - I_2\|^2 = \frac{\alpha_2^2 + \alpha_3^2}{\alpha_1^2} \geq \frac{\|\partial \varphi_k(0,0)\|^2}{4M_2^2} \geq \frac{\varepsilon^2}{4M_2^2} \|\Lambda(X_{k-1})\|^2.$$  

The case $\nu_k = \psi_k$ is similar. Now we prove the $\nu_k = \rho_k$ case. If $\varpi \in [0, \varsigma)$, we have

$$\|\Psi_k^* - I_2\|^2 \geq \frac{5}{4} \geq \frac{5}{4} \frac{1}{\gamma_1^2 + \gamma_2^2} \|\partial \varphi_k(1,0)\|^2 \geq \frac{\varepsilon^2}{416M_2^2} \|\Lambda(X_{k-1})\|^2.$$  

The case $\varpi \in [\frac{1}{\varsigma}, +\infty)$ is similar. If $\varpi \in [\varsigma, \frac{1}{\varsigma}]$, we have

$$\|\Psi_k^* - I_2\|^2 \geq (1 - x_k^*)^2 \geq \frac{1}{M_2^2 M_3^2} \gamma_2^2 (1 - \varpi)^2 = \frac{1}{4M_2^2 M_3^2} \|\partial \varphi_k(1,0)\|^2 \geq \frac{\varepsilon^2}{4M^2 M_2^2} \|\Lambda(X_{k-1})\|^2.$$  

We only need to set $\kappa_\rho^2 = \min(\frac{\varepsilon^2}{4M_2^2 M_3^2}, \frac{\varepsilon^2}{416M_2^2})$. The proof is complete. □

By Lemma 5.6, Lemma 5.8 and Lemma 5.10, we can easily get the following results by setting $\eta_\rho = \sigma_\rho \kappa_\rho$ and $\eta_\varphi = \sigma_\varphi \kappa_\rho$.

**Corollary 5.11.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then

$$g(X_{k-1}) - g(X_k) \geq \eta_\rho \|\text{grad } g(X_{k-1})\|^2$$

(57)

whenever $\nu_k = \rho_k$.

**Corollary 5.12.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then

$$g(X_{k-1}) - g(X_k) \geq \eta_\varphi \|\text{grad } g(X_{k-1})\|^2$$

(58)

whenever $\nu_k = \varphi_k$ or $\psi_k$.

6. Plane rotations for JADM problem

Let $g$ be as in (39). Let $W^{(\ell)}$ and $g$ be as in Section 5.2. Denote $(i, j) = (i_k, j_k)$ for simpliciy. Define

$$\Gamma(i, j; X_{k-1}) \defeq \frac{\theta}{2} \sum_{\ell=1}^{L} \mu_{\ell} \Re \left(z_{i,j}(W^{(\ell)}) z^H_{i,j}(W^{(\ell)}) \right),$$

where

$$z_{i,j}(W) \defeq \begin{cases} [W_{jj} - W_{ii} & W_{ij} + W_{ji} & -i(W_{ij} - W_{ji})]^T, & \text{if } (\cdot)^\dagger = (\cdot)^H; \\ [W_{ij} + W_{ji} & W_{ii} - W_{jj} & i(W_{ii} + W_{jj})]^T, & \text{if } (\cdot)^\dagger = (\cdot)^T. \end{cases}$$
Let
\[ c_0 = \begin{cases} \frac{1}{2} \sum_{\ell=1}^L \mu_{\ell} \left| W_{j\ell}^{(\ell)} - W_{i\ell}^{(\ell)} \right|^2, & \text{if } (\cdot)^{\dagger} = (\cdot)^H; \\ \frac{1}{2} \sum_{\ell=1}^L \mu_{\ell} \left| W_{i\ell}^{(\ell)} + W_{j\ell}^{(\ell)} \right|^2, & \text{if } (\cdot)^{\dagger} = (\cdot)^T. \end{cases} \] (59)

6.1. Elementary function. As in [40, Eq. (4.4)], we let
\[ r = r(c, s_1, s_2) \overset{\text{def}}{=} [2c^2 - 1, -2cs_1, -2cs_2]^T = [\cos 2\theta, -\sin 2\theta \cos \phi, -\sin 2\theta \sin \phi]^T. \] (60)
Then we can get the following result\(^9\) by direct calculations.

Lemma 6.1. In Algorithm 1 for cost function (1), the elementary function \( h_k \) satisfies
\[ h_k(c, s_1, s_2) - h_k(1, 0, 0) = -r^T \Gamma^{(i,j,X_{k-1})} r - c_0. \] (61)

Now we denote \( \Gamma = \Gamma^{(i,j,X_{k-1})} \) for simplicity. It follows by (60) and (61) that
\[ h_k(c, s_1, s_2) - h_k(1, 0, 0) = - (q(\theta, \phi) - c_0), \] (62)
where
\[ q(\theta, \phi) \overset{\text{def}}{=} \frac{1}{2} \left( \Gamma_{11} - \Gamma_{22} \cos^2 \phi - \Gamma_{33} \sin^2 \phi - \Gamma_{23} \sin(2\phi) \right) \cos(4\theta) \]
\[ - \left( \Gamma_{12} \cos \phi + \Gamma_{13} \sin \phi \right) \sin(4\theta) + \frac{1}{2} \left( \Gamma_{11} + \Gamma_{22} \cos^2 \phi + \Gamma_{33} \sin^2 \phi + \Gamma_{23} \sin(2\phi) \right). \] (63)

Note that, by Lemma 6.1 and (60), we have
\[ \partial h_k(J_2) = -4 \big[ 0 \Gamma_{12} \Gamma_{13} \big]^T. \] (64)

Remark 6.2. By equation (62), we see that \( h_k(\theta + \frac{\pi}{4}, \phi) = h_k(\theta, \phi) \) for any \( \theta, \phi \in \mathbb{R} \). Therefore, we can always set \( \theta_\ast \in [-\frac{\pi}{4}, \frac{\pi}{4}] \).

Remark 6.3. In Algorithm 1 for cost function (1), we set a positive constant \( \epsilon > 0 \). If \( \nu_k = h_k \), we find the eigenvector \( u \) of \( \Gamma \) corresponding to the largest eigenvalue. Define two vectors \( v_{i,j} \overset{\text{def}}{=} [\Gamma_{12} \Gamma_{13}]^T \in \mathbb{R}^2 \) and \( w_{i,j} \overset{\text{def}}{=} [u_2 u_3]^T \in \mathbb{R}^2 \).

- If it holds that
  \[ |\langle v_{i,j}, w_{i,j} \rangle| \geq \epsilon \|v_{i,j}\| \|w_{i,j}\|, \] (65)
  then we find \( \phi_\ast \) and \( \theta_\ast \) by setting \( r = u \), and \( \Psi_\ast^k = \Psi(\theta_\ast, \phi_\ast) \);
- Otherwise, we set \( \cos \phi_\ast \sin \phi_\ast \) and then determine the best \( \theta_\ast \) based on \( \phi_\ast \).

6.2. Condition for global convergence. We first present a lemma, which will help us to prove Lemma 6.5.

Lemma 6.4. Let \( \alpha, \beta \in \mathbb{R} \) be two constants. For \( \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \), we define
\[ p(\theta) \overset{\text{def}}{=} \alpha \cos(4\theta) + \beta \sin(4\theta). \]
If \( p(\theta_\ast) = \max p(\theta) \), we have
\[ p(\theta_\ast) - p(0) \geq 2\sqrt{2}\|\phi_\ast\| \sin(\theta_\ast/2). \]

\(^{9}\)In the \((\cdot)^{\dagger} = (\cdot)^H\) case, this expression was first formulated in [11].
Lemma 6.5. Let the function \( q(\theta, \phi) \) be as in (63). Suppose that \( \phi_* \) and \( \theta_* \) are determined as in Remark 6.3. Then we have
\[
q(\theta_*, \phi_*) - q(0, 0) \geq 2\sqrt{2}\varepsilon|\sin(\frac{\theta_*}{2})|\|v_{i,j}\|.
\]
Proof. By Remark 6.3, we see that
\[
|\langle v_{i,j}, [\cos \phi_* \sin \phi_*]^T \rangle| \geq \epsilon\|v_{i,j}\|	ag{66}
\]
always holds. By Lemma 6.4 and (66), we get that
\[
q(\theta_*, \phi_*) - q(0, 0) = q(\theta_*, \phi_*) - q(0, \phi_*) \geq 2\sqrt{2}\varepsilon|\sin(\frac{\theta_*}{2})|\Gamma_{12}\cos \phi_* + \Gamma_{13}\sin \phi_*
\]
\[
\geq 2\sqrt{2}\epsilon|\sin(\frac{\theta_*}{2})|\|v_{i,j}\|.
\]
The proof is complete. \( \Box \)

Lemma 6.6. In Algorithm 1 for cost function (1), there exists \( \iota_h > 0 \) such that
\[
g(X_{k-1}) - g(X_k) \geq \iota_h\|\Lambda(X_{k-1})\|\|\Psi^*_k - I_2\|,
\]
whenever \( \nu_k = h_k \). Proof. We only prove the \( (\cdot)^\dagger = (\cdot)^H \) case. The other case is similar. By Lemma 6.5 and (64), we get that
\[
h_k(0, 0) - h_k(\theta_*, \phi_*) \geq 2\sqrt{2}\varepsilon|\sin(\frac{\theta_*}{2})|\|v_{i,j}\| = \epsilon\frac{2\sqrt{2}}{4}|\sin(\frac{\theta_*}{2})|\|\partial h_k(I_2)\|
\]
\[
\geq \epsilon\frac{\varepsilon}{4}\|G^{(i,j), \Psi^*_k} - I_m\|\|\Lambda(X_{k-1})\|.
\]
We can set \( \iota_h = \frac{\epsilon\varepsilon}{4} \). The proof is complete. \( \Box \)

Let \( \iota_h \) be as in (67) and \( \sigma_h = \frac{\iota_h}{M^2} > 0 \). Similar as for Corollary 5.7, the next result follows directly from Lemma 6.6.

Corollary 6.7. In Algorithm 1 for cost function (1), if the condition (8) is always satisfied, then
\[
g(X_{k-1}) - g(X_k) \geq \sigma_h\|\mathrm{grad} g(X_{k-1})\|\|X_k - X_{k-1}\|,
\]
whenever \( \nu_k = h_k \).

6.3. Condition for weak convergence. If the condition (8) is satisfied, it is easy to see that there exists \( M_\Gamma > 0 \) such that \( \|\Gamma^{(i,j), X_{k-1}}\| \leq M_\Gamma \) always holds.

Lemma 6.8. In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then there exists \( \kappa_h > 0 \) such that
\[
\|\Psi^*_k - I_2\| \geq \kappa_h\|\Lambda(X_{k-1})\|,
\]
whenever \( \nu_k = h_k \).
Proof. If $v_{i,j}$ and $w_{i,j}$ satisfy the condition (65), then the inequality (69) can be proved by a similar method as for [40, Lem. 7.2]. Otherwise, if we set $[\cos \phi_* \sin \phi_*]^T = \frac{v_{i,j}}{\|v_{i,j}\|}$ and find $\theta_*$ based on $\phi_*$, then

$$|\sin(4\theta_*)| = \frac{|\Gamma_{12} \cos \phi_* + \Gamma_{13} \sin \phi_*|}{\sqrt{(\Gamma_{12} \cos \phi_* + \Gamma_{13} \sin \phi_*)^2 + \frac{1}{4} (\Gamma_{11} - \Gamma_{22} \cos^2 \phi_* - \Gamma_{33} \sin^2 \phi_* - \Gamma_{23} \sin(2\phi_*))^2}}$$

$$\geq \frac{\sqrt{\Gamma_{12}^2 + \Gamma_{13}^2}}{2\sqrt{5M_\Gamma}} = \frac{\|\partial h_k(I_2)\|}{8\sqrt{5M_\Gamma}} \geq \frac{\varepsilon}{8\sqrt{5M_\Gamma}} \|\Lambda(X_{k-1})\|.$$

Note that

$$\|\Psi^*_k - I_2\| = 2\sqrt{2} |\sin(\theta_*)| \geq \frac{\sqrt{2}}{4} |\sin(4\theta_*)|.$$ 

We only need to set $\kappa_h = \frac{\sqrt{2}}{32\sqrt{5M_\Gamma}}$ in this case. The proof is complete. \[\Box\]

Then, by Lemma 6.6 and Lemma 6.8, we can easily get the following result by setting $\eta_h = \sigma_h \kappa_h$.

**Corollary 6.9.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then

$$g(X_{k-1}) - g(X_k) \geq \eta_h \|\text{grad } g(X_{k-1})\|^2$$

whenever $\nu_k = h_k$.

7. **Convergence analysis of BCD-G algorithm**

7.1. **Weak convergence.** Since the special linear group $\text{SL}_m(\mathbb{C})$ is not compact, the iterates $\{\omega_k\}_{k \geq 1}$ produced by Algorithm 1 may have no accumulation point. However, if there exists an accumulation point, we have the following result about its weak convergence, which can be proved easily by condition (7) and the fact that $f(\omega) \geq 0$.

**Lemma 7.1.** In Algorithm 1 for cost function (5), if there exists $\eta > 0$ such that

$$f(\omega_{k-1}) - f(\omega_k) \geq \eta \|\text{grad } f_{i_k}(\omega_{k-1})\|^2$$

always holds, then $\lim_{k \to \infty} \text{grad } f(\omega_{k-1}) = 0$. In particular, if $\omega_*$ is an accumulation point of the iterates $\{\omega_k\}_{k \geq 1}$, then $\omega_*$ is a stationary point of $f$.

Now we prove the following result about the weak convergence of Algorithm 1 for cost function (1).

**Theorem 7.2.** In Algorithm 1 for cost function (1), if the conditions (8) and (28) are always satisfied, and $\omega_*$ is an accumulation point of the iterates $\{\omega_k\}_{k \geq 1}$, then $\omega_*$ is a stationary point of the cost function (1).

**Proof.** By Equation (30), Corollary 5.11, Corollary 5.12 and Corollary 6.9, we see that the condition (71) is always satisfied in Algorithm 1 for cost function (1), if we set $\eta = \min(\eta_p, \eta_\rho, \eta_\varphi, \eta_h)$. Then the proof is complete by Lemma 7.1. \[\Box\]
7.2. Global convergence. The following result about the global convergence of Algorithm 1 is a direct consequence of Theorem 2.3 and condition (7).

**Proposition 7.3.** Suppose that the function $f$ in (5) is real analytic and the sequence $\{\omega_k\}_{k \geq 1}$ produced by Algorithm 1 satisfies that, for large enough $k$,

(i) there exists $\sigma > 0$ such that

$$f_{t_k}(\omega_{k-1}) - f_{t_k}(\omega_k) \geq \sigma \| \nabla f_{t_k}(\omega_{k-1}) \| \| \omega_k - \omega_{k-1} \| $$ (72)

(ii) $\nabla f_{t_k}(\omega_{k-1}) = 0$ implies that $\omega_k = \omega_{k-1}$.

Then any accumulation point $\omega^*$ of the sequence $\{\omega_k\}_{k \geq 1}$ must be the only limit point.

Now we prove the following result about the global convergence of Algorithm 1 for cost function (1).

**Theorem 7.4.** In Algorithm 1 for cost function (1), if the condition (8) is satisfied, then the iterates $\{\omega_k\}_{k \geq 1}$ produced by Algorithm 1 converge to a stationary point $\omega^*$.

**Proof.** By Lemma 3.3, Corollary 5.7, Corollary 5.9 and Corollary 6.7, we see that the condition (72) is always satisfied in Algorithm 1 for cost function (1), if we set $\sigma = \min(\sigma_p, \sigma_\rho, \sigma_\varphi, \sigma_h)$. Then the proof is complete by Proposition 7.3. □

7.3. Special case: Jacobi-G algorithms on $\text{SL}_m(\mathbb{C})$. Let $n = m$ in the cost function (5) and fix $U_k = U_0$ for $k \geq 1$ in Algorithm 1. In other words, we keep the first block variable unchanged and only update the second block variable $X_k$ in $\text{SL}_m(\mathbb{C})$ using Jacobi-GLU and Jacobi-GLQ rotations in Section 4.3. Then we get the *Jacobi-GLU* and *Jacobi-GLQ* algorithms\textsuperscript{10} to minimize the restricted function

$$g : \text{SL}_m(\mathbb{C}) \to \mathbb{R}^+, \quad X \mapsto f(U_0, X).$$ (73)

---

**Algorithm 4: Jacobi-GLU**

1: **Input:** A starting point $X_0$, a positive constant $0 < \varepsilon < \sqrt{\frac{2}{3m(m-1)}}$.

2: **Output:** Sequence of iterates $\{X_k\}_{k \geq 1}$.

3: **for** $k = 1, 2, \cdots$ **do**

4: Choose the index pair $(i_k, j_k)$ and $\nu_k$ such that

$$\| \partial \nu_k(I_2) \| \geq \varepsilon \| \Lambda(X_{k-1}) \|,$$ (74)

where $\nu_k = \varphi_k, \psi_k$ or $\rho_k$;

5: Compute $\Psi_*^k$ that minimizes the function $\nu_k$;

6: Update $X_k = X_{k-1}V_k$, where $V_k = U^{(i_k, j_k, \Psi_*^k)}, L^{(i_k, j_k, \Psi_*^k)}$ or $D^{(i_k, j_k, \Psi_*^k)}$.

7: **end for**

\textsuperscript{10}These algorithms are based on the similar ideas as the Jacobi-G algorithm in [18, 22, 40].
Algorithm 5: Jacobi-GLQ

1: Input: A starting point $X_0$, a positive constant $0 < \varepsilon < \sqrt{\frac{3+\sqrt{5}}{3m(m-1)}}$.
2: Output: Sequence of iterates $\{X_k\}_{k \geq 1}$.
3: for $k = 1, 2, \cdots$ do
4:   Choose the index pair $(i_k, j_k)$ and $\nu_k$ satisfying (74), where $\nu_k = h_k, \psi_k$ or $\rho_k$;
5:   Compute $\Psi^*_k$ that minimizes the function $\nu_k$;
6:   Update $X_k = X_{k-1}V_k$, where $V_k = G^{(i_k,j_k,\Psi^*_k)}, L^{(i_k,j_k,\Psi^*_k)}$ or $D^{(i_k,j_k,\Psi^*_k)}$.
7: end for

7.4. Convergence analysis of Jacobi-G algorithms. By similar methods as for Theorem 7.2 and Theorem 7.4, we can obtain the following result.

**Theorem 7.5.** In Algorithm 4 and Algorithm 5 for cost function (3), if there exists $M_X > 0$ such that

$$\|X_k\| \leq M_X$$

always holds for $k \geq 1$, then the iterates $\{X_k\}_{k \geq 1}$ converge to a stationary point.

**Remark 7.6.** In Algorithm 4 and Algorithm 5, a more natural way of choosing the index pair $(i_k, j_k)$ is according to a cyclic ordering. In fact, this cyclic way has been often used in the literature [32, 37, 41]. In this case, we call them the Jacobi-CLU (based on LU decomposition) and Jacobi-CLQ (based on LQ decomposition) algorithms, respectively. To our knowledge, there was no theoretical result about the convergence of Jacobi-CLU and Jacobi-CLQ algorithms in the literature. In this paper, we propose Jacobi-GLU and Jacobi-GLQ algorithms, and establish their convergence. It may be also interesting to study how to establish the convergence of Jacobi-CLU and Jacobi-CLQ algorithms.

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