On Infravacua and Superselelction Theory

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Abstract

In the DHR theory of superselection sectors, one usually considers states which are local excitations of some vacuum state. Here, we extend this analysis to local excitations of a class of “infravacuum” states appearing in models with massless particles. We show that the corresponding superselection structure, the statistics of superselection sectors and the energy-momentum spectrum are the same as with respect to the vacuum state. (The latter result is obtained with a novel method of expressing the shape of the spectrum in terms of properties of local charge transfer cocycles.) These findings provide evidence to the effect that infravacua are a natural starting point for the analysis of the superselection structure in theories with long-range forces.

1 Introduction

The description of the superselection structure of theories with long-range interactions such as QED is a long-standing problem in the algebraic approach to quantum field theory. As a characteristic feature, such theories contain charges which obey Gauss’ law and thus do not fit into the framework of superselection sectors as described by Doplicher, Haag and Roberts.

Focusing (mainly for definiteness) on QED, Gauss’ law has the following three consequences: First, since every charge is accompanied by an electromagnetic field, an operator $\psi_c$ which describes the creation of an electric charge in a gauge-invariant way in the “physical” Hilbert space, cannot be local. There is, however, the possibility of localising the charge and (the
Cauchy data of its field in a space-like cone $C$ (as a simple example of a region extending to space-like infinity). In this case, the charge cannot be detected by measurements in the space-like complement $C'$ of that cone.

Second, the configuration of the asymptotic electric flux at space-like infinity is a classical observable whose values label an uncountable number of superselection sectors in the physical Hilbert space. Among these is the vacuum sector, but also the sector generated from the vacuum by the operator $\psi_c$. The former is identified with the vacuum representation $\pi_0$ of the observables, the latter with the representation $\pi_0 \circ \gamma_C$, where $\gamma_C \equiv \text{Ad} \psi_c^*$. Obviously, the space-like direction of $C$ can be determined by measurements of the asymptotic flux distribution in the sector $\pi_0 \circ \gamma_C$. This implies that the representations $\pi_0$ and $\pi_0 \circ \gamma_C$, equal by construction when restricted to $C'$, will not be equivalent in restriction to the space-like complement $C'_1$ of some arbitrary space-like cone $C_1$.

Third, Gauss' law lies at the origin of what is called the infraparticle problem: electrons are infraparticles, i.e. they do not correspond to eigenvalues of the mass operator. The physical picture for this is that every electron is accompanied by a cloud of low-energy photons. Related to this is the emission of bremsstrahlung in collisions of charged particles. The creation of infinitely many photons in such a process manifests itself on the mathematical side in the fact that the representations of the incoming and the outgoing free photon fields cannot both be Fock representations. However, this does not preclude that both representations can be equivalent, as is indeed the case for certain representations constructed by Kraus, Polley and Reents [2]. Therefore these so-called KPR representations are interpreted as a background radiation field which is sufficiently chaotic in the sense that the addition of bremsstrahlung only amounts to a slight perturbation of them.

Motivated by this picture, D. Buchholz has pointed out in [3] that in front of such a background the space-like direction of the localisation cone $C$ of $\psi_c$ should not be detectable any more. In mathematical terms, the representations $\pi_I$ and $\pi_I \circ \gamma_C$ (where $\pi_I$ corresponds to the KPR-like background) are expected to satisfy the so-called BF (Buchholz-Fredenhagen) criterion, i.e. they should become unitarily equivalent in restriction to the space-like complement of any space-like cone $C_1$, in contrast to the situation prevailing in front of the vacuum.

Thus, trading the vacuum $\pi_0$ for a background $\pi_I$ is expected to lead to an important improvement of the localisability properties of the charged states and should therefore permit to carry through the BF variant of the
DHR analysis \[4, 5\] in this case too.

In this perspective, the aim of the present letter is to provide a consistency check of these ideas for the case of (massless) theories with DHR-like charges. We will show that in such theories, a DHR analysis can be based on a wide class of background representations \(\pi_I\) and that the main results of this analysis are invariant with respect to the choice of that background. More interestingly, also the sets of sectors satisfying Borchers’ criterion are invariant.

These results provide a justification for viewing such representations as a background and corroborate the hope that, in theories with long-range interactions, the superselection structure may indeed be described more easily in relation to such a background rather than to the vacuum.

We end this introduction by explaining the main assumptions and some notation. The starting point of the following analysis will be a quantum field theory given in its vacuum representation. More specifically, let \((\mathcal{H}_0, U_0, \Omega_0)\) be a vacuum Hilbert space, that is a Hilbert space \(\mathcal{H}_0\) which carries a strongly continuous unitary representation \(U_0\) of the translation group \(\mathbb{R}^{1+s}\) (of \((1+s)\)-dimensional Minkowski space, also denoted by \(\mathbb{R}^{1+s}\)) whose spectrum is contained in the forward lightcone \(V_+ \subset \mathbb{R}^{1+s}\) and such that the translation invariant subspace of \(\mathcal{H}_0\) is spanned by a single unit vector \(\Omega_0\).

Assume further that we are given a Haag-Kastler net

\[ \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \]

of von Neumann algebras on \(\mathcal{H}_0\), i.e. an isotonomous mapping from the bounded regions \(\mathcal{O} \subset \mathbb{R}^{1+s}\) to the von Neumann subalgebras of \(\mathcal{B}(\mathcal{H}_0)\) which satisfies locality (i.e. \(\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'\) if \(\mathcal{O}_1 \subset \mathcal{O}_2\), in standard notation), translation covariance w.r.t. \(\alpha_x := \text{Ad}U_0(x)\) (i.e. \(\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + x)\)) and weak additivity. The quasilocal algebra \(\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})\) is also denoted by \(\mathcal{A}\) and is assumed to act irreducibly. It will be of crucial importance later on that one can conclude from these assumptions that the net \(\mathcal{A}\) has Borchers’ property \(B\) and that the \(C^*\)-algebra \(\mathcal{A}\) is simple, cf. \(6\). (These conclusions do not, in fact, rely on the existence and/or uniqueness of the vacuum vector \(\Omega_0\).)

It is customary to introduce the following notations at this point: the identical representation \(\pi_0 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_0)\) is called the vacuum representation; it is the GNS representation of the vacuum state \(\omega_0 = (\Omega_0, \cdot, \Omega_0)\). The spectral family associated with the translations \(U_0\) is denoted by \(E_0\).
2 Energy Components

In this section, we will present certain folia of states, namely energy components of positive energy representations. This notion has been introduced by H.-J. Borchers and D. Buchholz [7] and discussed further by R. Wanzenberg [8]. We begin this section by recalling some details which are relevant in the present context.

For the whole of this section, let \((\mathcal{H}, \pi)\) be a positive energy representation of \((\mathcal{A}, \alpha)\). This means that there exists, on the Hilbert space \(\mathcal{H}\), a strongly continuous representation \(U_\pi\) of the translations such that 
\[
\pi \circ \alpha_x = \text{Ad} U_\pi(x) \circ \pi \quad \text{and} \quad \text{sp} U_\pi \subset \mathcal{V}_+.
\]
It is then known [9] that the operators \(U_\pi(x)\) can be chosen such that \(U_\pi(x) \in \pi(\mathcal{A})''\), and it is a remarkable fact [7] that there is a unique choice for which, on any subspace \(\mathcal{H}' \subset \mathcal{H}\) which is invariant under \(\pi(\mathcal{A})\), the spectrum \(\text{sp} U_\pi|_{\mathcal{H}'}\) has Lorentz-invariant lower boundary. From now on, \(U_\pi\) will always denote this canonical implementation of the translations and \(E_\pi\) the associated spectral family.

The notion of energy contents of \(\pi\)-normal states can now be introduced. For any compact set \(\Delta \subset \mathcal{V}_+\), denote by 
\[
\mathcal{S}_\pi(\Delta) := \left\{ \text{tr} \rho \pi(\cdot) \left| \rho \in \mathcal{I}_1(\mathcal{H}), \rho \geq 0, \text{tr} \rho = 1, \rho E_\pi(\Delta) = \rho \right\}
\]
the set of all \(\pi\)-normal states which have energy-momentum in \(\Delta\). Then 
\[
\mathcal{S}_\pi = \bigcup_\Delta \mathcal{S}_\pi(\Delta) = \mathcal{S}_\pi(\mathcal{V}_+),
\]
is just the folium of \(\pi\). Another set \(\tilde{\mathcal{S}}_\pi\) of states, called the energy component of \(\pi\), is defined by 
\[
\tilde{\mathcal{S}}_\pi := \overline{\bigcup_\Delta \mathcal{S}_\pi(\Delta)} = \mathcal{S}_\pi(\mathcal{V}_+),
\]
where \(\tilde{\mathcal{S}}_\pi(\Delta)\) denotes the set of all locally \(\pi\)-normal states in the weak closure of \(\mathcal{S}_\pi(\Delta)\). Of course, \(\mathcal{S}_\pi \subset \tilde{\mathcal{S}}_\pi\), and one can show that \(\tilde{\mathcal{S}}_\pi\) is a folium, too. Physically, it is interpreted as the set of states which can be reached from the folium of \(\pi\) by operations requiring only a finite amount of energy. Some important properties of the elements of \(\tilde{\mathcal{S}}_\pi\) are collected in the following Lemma, parts 2 and 3 of which are due to R. Wanzenberg [8].

\[\footnote{In the sequel, \(\Delta\) always stands for a compact subset of \(\mathcal{V}_+\). Moreover, we will use the notation \(\Delta_q := \mathcal{V}_+ \cap (q - \mathcal{V}_+)\) for the double cone in momentum space with apices 0 and \(q \in \mathcal{V}_+\).} \]
Lemma 2.1 Let \( \omega \in \tilde{\mathcal{S}}_\pi \) and denote by \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) its GNS triple.

1. Then \((\mathcal{H}_\omega, \pi_\omega)\) is a locally \( \pi \)-normal positive energy representation of \((A, \alpha)\).

2. Moreover, if \( \omega \in \tilde{\mathcal{S}}_\pi(\Delta_q) \) for some \( q \in \tilde{\mathcal{V}}_r \), then one has \( \Omega_\omega \in E_{\pi_\omega}(\Delta_q) \mathcal{H}_\omega \).

3. In the situation of 2, one has \( \tilde{\mathcal{S}}_{\pi_\omega}(\Delta) \subset \tilde{\mathcal{S}}_{\pi_\omega}(\Delta + \Delta_q - \Delta_q) \) for any \( \Delta \).

Proof: Using the positivity of the energy in the representation \( \pi \), part 1 follows from the fact that \( \omega \) is locally \( \pi \)-normal by arguments similar to those in [10]. (In [10], these arguments are only applied to states \( \omega \in \tilde{\mathcal{S}}_\pi(\Delta_q) \), but they carry over to norm limits of such states as well.)

For simplicity, part 2 will only be proved in the case that \( \pi_\omega \) is factorial. For the general case, see [8]. Let \( \Delta_q \) be given. Then the idea is to show that \( \Omega_\omega \) does not have momentum outside \( \Delta_q \). To this end, fix some \( p \in \tilde{\mathcal{V}}_r \) and choose a neighbourhood \( \mathcal{N}_p \subset \tilde{\mathcal{V}}_r \setminus \Delta_q \) of \( p \) and an open set \( \mathcal{N} \subset \text{sp}\mathcal{U}_{\pi_\omega} \) such that \( (\Delta_q + \mathcal{N} - \mathcal{N}) \cap \tilde{\mathcal{V}}_r = \emptyset \).

(Such a choice is always possible because the lower boundary of \( \text{sp}\mathcal{U}_{\pi_\omega} \) is Lorentz invariant.) Now choose a test function \( f \) satisfying \( \text{supp} \tilde{f} \subset \mathcal{N} - \mathcal{N} \) and take an arbitrary \( A \in A \). Then, \( A(f) := \int \text{d}x f(x) \alpha_x(A) \) is an element of \( A \) and satisfies \( \pi(A(f))E_{\pi}(\Delta_q) \mathcal{H}_\omega = \{0\} \). Since \( \omega \in \tilde{\mathcal{S}}_\pi(\Delta_q) \), this implies \( \omega(A(f)^*A(f)) = 0 \), hence \( \pi_\omega(A(f))\Omega_\omega = 0 \). This means that \( \Omega_\omega \) is orthogonal to

\[
\mathcal{D} := \text{span} \{ \pi_\omega(A(f))^* \Psi \mid \Psi \in \mathcal{H}_\omega, A \in A, \text{supp} \tilde{f} \subset \mathcal{N} - \mathcal{N} \).
\]

Since \( \pi_\omega \) is factorial and \( \mathcal{N} - \mathcal{N}_p \) is open, it follows by an argument explained in [4] (see, in particular, the proof of Prop. 2.2 therein) that the closure of \( \mathcal{D} \) equals \( E_{\pi_\omega}(\text{sp}\mathcal{U}_{\pi_\omega} + \mathcal{N}_p - \mathcal{N}) \mathcal{H}_\omega \). Thus, \( \Omega_\omega \in \mathcal{D}^\perp \) yields

\[
\{\Omega_\omega\}^\perp \supset \mathcal{D}^\perp = E_{\pi_\omega}(\text{sp}\mathcal{U}_{\pi_\omega} + \mathcal{N}_p - \mathcal{N}) \mathcal{H}_\omega \supset E_{\pi_\omega}(\mathcal{N}_p) \mathcal{H}_\omega,
\]

where the last inclusion holds because \( \text{sp}\mathcal{U}_{\pi_\omega} - \mathcal{N} \ni 0 \). From this, we get

\[
E_{\pi_\omega}(\mathcal{N}_p)\Omega_\omega = 0 \text{ or, as } p \in \tilde{\mathcal{V}}_r \setminus \Delta_q \text{ was arbitrary, } \Omega_\omega \in E_{\pi_\omega}(\Delta_q) \mathcal{H}_\omega.
\]

To prove part 3, let \( \Psi \in E_{\pi_\omega}(\Delta) \mathcal{H}_\omega \). Part 2 and the cyclicity of \( \Omega_\omega \) imply that there exists in \( A \) a sequence \( (A_n)_{n \in \mathbb{N}} \) of operators with energy-momentum support in \( \Delta - \Delta_q \) such that \( \Psi = \lim_{n \to \infty} \pi_\omega(A_n)\Omega_\omega \). From
\( \omega \in \hat{S}_\pi(\Delta_q) \) follows \( \omega(A_n^* \cdot A_n) \in \hat{S}_\pi(\Delta_q + (\Delta - \Delta_q)) = \hat{S}_\pi(\Delta + \Delta_q - \Delta_q) \), i.e.

\[
\langle \Psi, \pi_\omega(\cdot) \Psi \rangle = \lim_{n \to \infty} \omega(A_n^* \cdot A_n) \in \hat{S}_\pi(\Delta + \Delta_q - \Delta_q).
\]

Thus any vector state from \( S_\pi(\Delta) \) lies in \( \hat{S}_\pi(\Delta + \Delta_q - \Delta_q) \). One now gets the assertion by taking convex combinations, norm limits and locally normal \( w^* \)-limits.

The preceding lemma has shown that positivity of the energy is a property which “survives” the process of going from the representation \( \pi \) to the GNS representation of a state in \( \hat{S}_\pi \). Other properties survive as well, as for instance the compactness condition \( C^\# \) of Fredenhagen and Hertel \([11, 12]\) which can be formulated as follows:

**Definition:** Condition \( C^\# \) is said to be satisfied in the (positive energy) representation \( \pi \) if, for any \( \Delta \) and any bounded region \( O \subset \mathbb{R}^{1+s} \), the set

\[
S_\pi(\Delta)|_{A(O)} = \{ \omega|_{A(O)} \mid \omega \in S_\pi(\Delta)\}
\]

is contained in a \( \| \cdot \| \)-compact subset of \( A(O)^* \).

This condition controls the infrared properties of the model under consideration, cf. \([12]\). It has been established in the theory of a massive free particle (in any space-time dimension) and in the theory of a massless (scalar or vector) particle in at least 1 + 3 space-time dimensions \([13, 12]\) and is believed to hold in QED as well. In the present context, it will play a technical role in the proof of Prop. 3.2 since it allows (by part 1 of the following lemma) a simplification in the definition of \( \hat{S}_\pi(\Delta) \) given above.

**Lemma 2.2** Let \( C^\# \) be satisfied in the representation \( \pi \) and let \( \omega \) be a state in the weak closure of \( S_\pi(\Delta) \). Then

1. \( \omega \) is locally \( \pi \)-normal, i.e. \( \omega \in \hat{S}_\pi(\Delta) \);

2. \( C^\# \) is satisfied in the GNS representation of \( \omega \).

**Proof:** Both parts follow from the fact that, in restriction to \( A(O) \), any \( w^* \)-limit of \( S_\pi(\Delta) \) is, as a consequence of \( C^\# \), even a \( \| \cdot \| \)-limit of \( S_\pi(\Delta) \). Part 1 now follows directly. For part 2, we note that the above fact implies \( \hat{S}_\pi(\Delta)|_{A(O)} \subseteq \overline{S_\pi(\Delta)|_{A(O)}} \| \cdot \| \), which, in view of Lemma 2.1(3) yields the assertion.
3 Infravacuum Representations

Intuitively, an infrared cloud (in QED) can be split, given any energy threshold $\epsilon > 0$, into a finite number $N_\epsilon$ of “hard” photons and infinitely many “soft” photons whose total energy is less than $\epsilon$. As the creation of hard photons can be described by a quasilocal operation, this picture motivates the following mathematical notion.

**Definition:** An irreducible representation $(\mathcal{H}_I, \pi_I)$ of $\mathcal{A}$ is called an infravacuum representation if, for any $q \in \mathcal{V}_+$, the set $\tilde{S}_{\pi_0}(\Delta_q)$ contains some $\pi_I$-normal state.

With any arbitrarily small amount of energy, one can thus create from the vacuum representation some state in an infravacuum representation. In the terms of $\tilde{S}$, this means that the transition energy between the sectors $\pi_0$ and $\pi_I$ vanishes (cf. Prop. 3.2 below). As an example for infravacuum representations, we mention the KPR representations, i.e. a class of non-Fock representations (of the free asymptotic electromagnetic field) devised by Kraus, Polley and Reents [4], cf. also [14], so as to be stable (up to unitary equivalence) under the bremsstrahlung produced in typical collision processes of charged particles. We note that with the above definition, the vacuum $\pi_0$ itself is an infravacuum representation, which will be convenient in the sequel. However, the notion of infravacuum representations is tailored to theories with massless particles in the sense that in a purely massive theory, $\pi_0$ would be the only infravacuum representation.

The following lemma collects some basic properties of any infravacuum representation.

**Lemma 3.1** Let $\pi_I$ be an infravacuum representation. Then

1. for any $q \in \mathcal{V}_+$, $\tilde{S}_{\pi_0}(\Delta_q)$ contains a pure $\pi_I$-normal state;

2. $\pi_I$ is a locally normal positive energy representation of $(\mathcal{A}, \alpha)$;

3. for any bounded $\mathcal{O}$, the restriction of $\pi_I : \mathcal{A}(\mathcal{O}) \rightarrow \pi_I(\mathcal{A}(\mathcal{O}))$ to uniformly bounded subsets of $\mathcal{A}(\mathcal{O})$ is a homeomorphism with respect to the weak operator topologies.

Proof: Choose, for some $q \in \mathcal{V}_+$, a state $\omega \in S_{\pi_I} \cap \tilde{S}_{\pi_0}(\Delta_q)$ and let $(\mathcal{H}, \pi, \Omega)$ be its GNS triple. Since $\omega \in S_{\pi_I}$, $\pi$ is quasi-equivalent to the irreducible representation $\pi_I$. This implies that $\pi(\mathcal{A})'$ is a type I factor and thus contains some minimal projection $P$. Obviously, $\pi|_{\mathcal{P}} \cong \pi_I$, which
implies assertion 2 by Lemma 2.1(1). Moreover, recalling \( \Omega \in E_\pi(\Delta_q) \mathcal{H} \), it also implies that the pure \( \pi_I \)-normal state \( \omega_P := (P\Omega, \pi(\cdot)P\Omega)/\|P\Omega\|^2 \) fulfills

\[
\omega_P \in \mathcal{S}_{\pi_I}(\Delta_q) \subset \mathcal{S}_\pi(\Delta_q) \subset \tilde{\mathcal{S}}_\pi(\Delta_q) \subset \tilde{\mathcal{S}}_{\pi_0}(2\Delta_q).
\]

Since \( q \) was arbitrary, this proves part 1. Finally, part 3 follows by applying Corollary 7.1.16. of \([13]\) to the maps \( \pi_I : \mathcal{A}(\mathcal{O}) \rightarrow \pi_I(\mathcal{A}(\mathcal{O})) \) which are isomorphisms of von Neumann algebras since \( \pi_I \) is faithful and locally normal.

Let us remark that one can even establish local unitary equivalence between \( \pi_0 \) and \( \pi_I \) under the additional assumption that \( \pi_I \) satisfies weak additivity\(^2\). The reasoning for this goes along the following line: According to the arguments collected nicely in \([4]\), there exist Reeh-Schlieder vectors in \( \mathcal{H}_0 \) and \( \mathcal{H}_I \), i.e. the von Neumann algebras \( \mathcal{A}(\mathcal{O}) \) and \( \pi_I(\mathcal{A}(\mathcal{O})) \) possess cyclic and separating vectors. From this, one obtains local unitary equivalence by applying Thm. 7.2.9. of \([13]\). We recall, however, that an attempt to establish unitary implementability of \( \pi_I \) on an algebra pertaining to an unbounded region (such as \( \mathcal{O}' \)) would fail because \( \pi_I \) need not be normal on such a region. Indeed, the interesting case will be the situation in which \( \pi_I \) does not fulfill the DHR criterion with respect to \( \pi_0 \).

To conclude this section about general properties of infravacuum representations, we want to compare the nets \( \tilde{\mathcal{S}}_{\pi_0} \) and \( \tilde{\mathcal{S}}_{\pi_I} \) of states. In the next proposition, we will have to make the additional assumption that the vacuum state is unique in the sense that, for some \( q \in \mathbf{V}_+ \), \( \omega_0 \) is the only vacuum state in \( \tilde{\mathcal{S}}_{\pi_0}(\Delta_q) \).

**Proposition 3.2** Let \( \pi_I \) be an infravacuum representation. Then we have, for any \( \Delta \) and any \( q \in \mathbf{V}_+ \):

\[
\tilde{\mathcal{S}}_{\pi_I}(\Delta) \subset \tilde{\mathcal{S}}_{\pi_0}(\Delta + \Delta_q - \Delta_q).
\]

If, moreover, the defining representation \( \pi_0 \) satisfies \( C_2 \) and if the vacuum state is unique in the sense explained above, then also the converse is true, namely

\[
\tilde{\mathcal{S}}_{\pi_0}(\Delta) \subset \tilde{\mathcal{S}}_{\pi_I}(\Delta + \Delta_q - \Delta_q).
\]

Proof: Using Lemma 3.1(1), the first statement follows immediately from Lemma 2.1(3). To prove the second statement, we will first use the

\(\text{\footnotesize \textsuperscript{2}If additivity and not merely weak additivity is assumed for } \pi_0 \text{, additivity and hence weak additivity follow for } \pi_I \text{ by local normality.}\)
two additional assumptions to show \( \omega_0 \in \tilde{S}_{\pi_I}(\Delta_q) \) for any \( \Delta_q \). This is done as follows. Take some \( \omega \in S_{\pi_I} \cap \tilde{S}_{\pi_0}(\Delta) \). It is easy to see that this implies \( \omega \in \tilde{S}_{\pi_I}(\Delta) \). Now consider the family \( (\omega_L)_{L>0} \) of states defined by \( \omega_L := \frac{1}{|V_L|} \int_{V_L} e^{1+s} \omega \circ \alpha_x \), where \( V_L := [-L,L] \cont R^{1+s} \). Since \( \omega_L \in \tilde{S}_{\pi_I}(\Delta) \), any \( \omega^* \)-limit \( \tilde{\omega} \) of this family is a \( \omega^* \)-limit of \( \tilde{S}_{\pi_I}(\Delta) \). Now \( \pi_I \) satisfies \( C_2 \) by Lemma 2.2(2), so Lemma 2.2(1) can be applied to \( \tilde{\omega} \) and yields \( \tilde{\omega} \in \tilde{S}_{\pi_I}(\Delta) \). In particular, \( \tilde{\omega} \) has positive energy. On the other hand, \( \tilde{\omega} \) is translation invariant by construction, i.e. it is a vacuum state.

Moreover, by the first part of the present proposition, \( \tilde{\omega} \in \tilde{S}_{\pi_0}(2\Delta) \). Now if \( q \) is sufficiently small, the uniqueness assumption yields \( \tilde{\omega} = \omega_0 \) whence \( \omega_0 \in \tilde{S}_{\pi_I}(\Delta) \). Trivially, this conclusion remains true for any \( q \). With this information, the second statement follows again from Lemma 2.1(3).

We remark, in addition, that the previous proposition does not, in general, entail that one of the sets \( \tilde{S}_{\pi_0}(\Delta) \) and \( \tilde{S}_{\pi_I}(\Delta) \) is contained in the other because the nets \( \Delta \mapsto \tilde{S}_{\pi}(\Delta) \) (\( \pi = \pi_0 \) or \( \pi_I \)) need not be regular from the outside (which would mean \( \tilde{S}_{\pi}(\Delta) = \tilde{S}_{\pi}(\Delta) := \bigcap_q \tilde{S}_{\pi}(\Delta + \Delta_q) \)), but the outer regularized nets \( \tilde{\tilde{S}}_{\pi} \) coincide. We note, as an aside, that the nets \( S_{\pi_0} \) and \( S_{\pi_I} \), in contrast, are regular from the outside, as follows from the continuity of the spectral families \( E_0 \) and \( E_{\pi_I} \).

4 The DHR Criterion

In the spirit of the theory of superselection sectors by Doplicher, Haag and Roberts \[1\], it is natural to consider, for any infravacuum representation \( \pi_I \), the set of representations

\[
\text{DHR}(\pi_I) := \left\{ \pi \mid \pi \mid A(O') \cong \pi_I \mid A(O') \right\} \forall O \in K
\]

(where \( K \) denotes the set of all double cones in \( R^{1+s} \)). The well-known DHR analysis can be carried through under one additional assumption, namely that of Haag duality. Thus, we will restrict our attention in the sequel to those infravacuum representations which have this property, i.e.

\[
\pi_I(A(O'))' = \pi_I(A(O)), \quad \forall O \in K.
\]

We mention as an aside that Haag duality has been established for KPR representations (cf. Sect. 3) if the net of observables fulfils an additional condition corresponding to an algebraic version of Gauss' law \[14, 16\].
Instead of DHR(\(\pi_I\)), one can now study the set
\[
\Delta_{\pi_I,t} := \bigcup_{\mathcal{O} \in \mathcal{K}} \Delta_{\pi_I,t}(\mathcal{O})
\]
of transportable localised endomorphisms of the C*-algebra \(\pi_I(A)\), where \(\Delta_{\pi_I,t}(\mathcal{O})\) denotes the subset of all such endomorphisms which act trivially on the subalgebra \(\pi_I(A(\mathcal{O}^\prime))\). Transportability means that, for any \(x \in \mathbb{R}^{1+s}\) the endomorphisms \(\rho_I\) and \(\rho_{I,x}\) are unitarily equivalent in \(\mathcal{H}_{\pi_I}\), where \(\rho_{I,x}\) is given by the action of the translations
\[
\Delta_{\pi_I,t}(\mathcal{O}) \longrightarrow \Delta_{\pi_I,t}(\mathcal{O} + x) : \rho_I \mapsto \rho_{I,x} := \text{Ad}U_{\pi_I}(x) \circ \rho_I \circ \text{Ad}U_{\pi_I}(x)^*.
\]
We recall that \(\Delta_{\pi_I,t}\) can be viewed as the set of objects of a tensor C*-category with subobjects and finite direct sums [17], also denoted by \(\Delta_{\pi_I,t}\), whose morphisms \(T \in I_{\pi_I}(\sigma_I, \rho_I)\) from \(\sigma_I\) to \(\rho_I\) are the intertwining operators \(T \in \mathcal{B}(\mathcal{H}_{\pi_I})\) from \(\sigma_I \circ \pi_I\) to \(\rho_I \circ \pi_I\). It is a crucial consequence of Haag duality that the intertwiners are again local observables:
\[
I_{\pi_I}(\sigma_I, \rho_I) \subset \pi_I(A(\mathcal{O})) \quad \text{if} \quad \sigma_I, \rho_I \in \Delta_{\pi_I,t}(\mathcal{O}).
\]
One eventually has to pass from \(\Delta_{\pi_I,t}\) to the full subcategory \(\Delta_{\pi_I,f}\) of objects with finite statistical dimension, which is a tensor C*-category with conjugates. Its importance resides in the fact that it is possible to recover from it (by a deep result of Doplicher and Roberts [18]) a compact gauge group and a field net with normal commutation relations whose gauge invariant part contains all finite-dimensional DHR sectors of \(\pi_I(A)\).

It is important to show that the above-mentioned notions do not, in fact, depend on the choice of the infravacuum representation \(\pi_I\). Indeed, we obtain the following slightly more general result:

**Proposition 4.1** Let \(\pi_I\) be an irreducible, locally normal representation of \(A\) and assume that \(\pi_0\) and \(\pi_I\) satisfy Haag duality. Then, a bijective functor
\[
F : \Delta_{\pi_0,t} \longrightarrow \Delta_{\pi_I,t}
\]
is defined by the actions (on objects resp. on morphisms)
\[
F(\rho_0) := \pi_I \circ \pi_0^{-1} \circ \rho_0 \circ \pi_0 \circ \pi_I^{-1}, \quad \rho_0 \in \Delta_{\pi_0,t},
\]
\[
F(T) := \pi_I \circ \pi_0^{-1}(T), \quad T \in I_{\pi_0}.
\]
This functor restricts to an isomorphism $F : \Delta_{\pi_0,f} \to \Delta_{\pi_I,f}$ of the corresponding tensor $C^*$-categories with conjugates.

Proof: As the algebra $\mathcal{A}$ is simple, any representation is faithful, so $\pi_I \circ \pi_0^{-1} : \pi_0(\mathcal{A}) \to \pi_I(\mathcal{A})$ is bijective. Disregarding for a moment the question of transportability, this means that $F$, as defined above, is well-defined on the objects $\rho_0$ and remembering $I_{\pi_0} \subset \pi_0(\mathcal{A})$, also on the morphisms. Obviously, $F$ is bijective. One checks that $F(\rho_0)$ is localised in $\mathcal{O}$ iff $\rho_0$ is, which implies that $F(\rho_0)$ is transportable iff $\rho_0$ is. Since the statistical dimension of an object is a purely algebraic notion [18], $F$ clearly restricts to the subcategories of finite statistical dimension. As any bijective functor, $F$ preserves all algebraic structures involved, such as subobjects, direct sums, conjugates and the symmetries.

Obviously, what lies at the heart of this proof is that $\pi_I \circ \pi_0^{-1}$ is a net isomorphism. The additional property of this isomorphism of being bicontinuous (w.r.t. the weak operator topologies) in restriction to norm-bounded subsets of local algebras (cf. Lemma 3.1.3) has not been exploited yet but will play an essential role in Section 6.

We should emphasise that the above Proposition and its proof remain valid if $\pi_0$ also is replaced by an irreducible, locally normal representation of $\mathcal{A}$ which satisfies Haag duality. In particular, the spectral properties of this representation do not enter into the argument.

As straightforward as the foregoing proposition may be, as important is its interpretation: it shows that replacing the vacuum $\pi_0$ by an infravacuum $\pi_I$ does not affect the superselection structure of the theory. Not only the charges of the theory remain the same, but also their fusion structure, their statistics, their localisation and their transportability properties. In more pictorial terms, an infravacuum is suited as well as the "empty" vacuum as a background in front of which DHR theory can take place.

5 Cocycles and Their Spectral Properties

We have seen in the previous section that the DHR superselection structure of a theory does not depend on the infravacuum with respect to which it is defined. In elementary particle physics, physically meaningful (DHR) representations should, however, also fulfil what is called Borchers’ criterion, i.e. have positive energy. In the next two sections, we want to convince ourselves that this additional property too is independent of the infravacuum representation chosen.
It is well-known [1] that Borchers’ criterion is fulfilled for all $\rho_0 \in \Delta_{\pi_0,f}$ since the subcategory of finite-dimensional objects is closed under conjugates. However, it is important to note that the proof of this theorem, in relying on the additivity of the energy, makes crucial use of the fact that $\pi_0$ is a vacuum representation and thus fails when $\pi_0$ is replaced by $\pi_I$. Yet, any $\rho_I \in \Delta_{\pi_I,f}$ actually does fulfil Borchers’ criterion, as follows from Prop. 6.3 below.

Leaving DHR theory for a moment, we first discuss a mathematical notion whose relevance will soon become clear. Thus, let $(\mathcal{H}_I, U_I)$ be a pair consisting of a Hilbert space $\mathcal{H}_I$ and a strongly continuous unitary representation $U_I$ of $\mathbb{R}^{1+s}$ whose spectral family will be denoted by $E_I$. Note that no condition is imposed on its spectrum at this stage.

**Definition:** A strongly continuous function $\Gamma : \mathbb{R}^{1+s} \to U(\mathcal{H}_I)$ is called a cocycle over $(\mathcal{H}_I, U_I)$ if it fulfils the cocycle equation

$$\Gamma(x) \ AdU_I(x)(\Gamma(y)) = \Gamma(x + y).$$

A cocycle is called $C$-spectral, where $C$ is some closed subset of $\mathbb{R}^{1+s}$, if it complies with the following condition: for any compact $\Delta \subset \mathbb{R}^{1+s}$, one has $\text{sp}(\Gamma(\cdot)E_I(\Delta)) \subset C - \Delta$.

Here, the spectrum $\text{sp}A$ of a uniformly bounded, strongly continuous operator-valued function $A : \mathbb{R}^{1+s} \to \mathcal{B}(\mathcal{H}_I)$ is, by definition, the support of its Fourier transform in the sense of operator-valued distributions. In particular, one has $\int dx f(x)A(x) = 0$ for any test function $f$ with $\text{supp} f \cap \text{sp}A = \emptyset$. Of course, if $A = U : \mathbb{R}^{1+s} \to U(\mathcal{H}_I)$ is a group homomorphism, $\text{sp}A = \text{sp}U$ coincides with the common spectrum of the generators of the group.

The notion of spectral cocycles does not seem to have appeared in the literature. The idea of applying it to the above-mentioned problem and the main part of Prop. 5.2 are due to D. Buchholz.

The following basic properties of the spectrum will be important in the sequel:

**Lemma 5.1** Let $A, A_1$ and $A_2$ be uniformly bounded, strongly continuous operator-valued function on $\mathbb{R}^{1+s}$. Then, one has

1. $\text{sp}A^* = -\text{sp}A$, where $A^*(x) = (A(x))^*$;
2. $\text{sp}(A_1 + A_2) \subset \text{sp}A_1 \cup \text{sp}A_2$;
3. \( \text{sp}(A_1A_2) \subset \text{sp}A_1 + \text{sp}A_2 \) if \( \text{sp}A_1 \) or \( \text{sp}A_2 \) is bounded.

Proof: Parts 1 and 2 are elementary, whereas part 3 can be obtained, by considering matrix elements, from the corresponding property of scalar-valued functions which in turn is the contents of §§1,2 and Théorème II of [19].

Proposition 5.2 A bijective correspondence between strongly continuous unitary representations \( V : \mathbb{R}^{1+s} \rightarrow \mathcal{U}(\mathcal{H}_I) \) and cocycles \( \Gamma \) is given by

\[
\Gamma(x) = V(x)U_I(x)^*.
\]

Moreover, \( \text{sp}V \subset C \) iff \( \Gamma \) is \( C \)-spectral.

Proof: The first assertion is obvious. As to the second one, let \( V \) satisfy \( \text{sp}V \subset C \). Making use of the previous Lemma, one then has, for any compact \( \Delta \subset \mathbb{R}^{1+s} \),

\[
\text{sp}(\Gamma(\cdot)E_I(\Delta)) = \text{sp}(V(\cdot)U_I^*(\cdot)E_I(\Delta)) \subset \text{sp}V - \text{sp}(E_I(\Delta)U_I(\cdot)) \subset C - \Delta
\]

which shows that \( \Gamma \) is \( C \)-spectral. To prove the converse, let \( \Gamma \) be \( C \)-spectral. Let \( \Delta \subset \mathbb{R}^{1+s} \) be compact and choose a cover of \( \Delta \) by a finite number of pairwise disjoint Borel sets \( (\Delta_j)_{j=1,...,N} \). Then, again by Lemma 5.1, one gets

\[
\text{sp}\left(V(\cdot)E_I(\Delta)\right) \subset \text{sp}\left(V(\cdot) \sum_j E_I(\Delta_j)\right) = \text{sp}\left(\sum_j \left(\Gamma(\cdot)E_I(\Delta_j) E_I(\Delta_j)U_I(\cdot)\right)\right)
\]

\[
\subset \bigcup_j \left(\text{sp}(\Gamma(\cdot)E_I(\Delta_j)) + \text{sp}(E_I(\Delta_j)U_I(\cdot))\right)
\]

\[
\subset \bigcup_j \left(C - \Delta_j + \Delta_j\right) = C + \bigcup_j \left(\Delta_j - \Delta_j\right).
\]

Since the cover may be chosen such that the maximal diameter of the sets \( \Delta_j \) is arbitrarily small, this implies \( \text{sp}(V(\cdot)E_I(\Delta)) \subset C \) for any compact \( \Delta \subset \mathbb{R}^{1+s} \), which, in view of \( s-\lim_{\Delta \uparrow \mathbb{R}^{1+s}} E_I(\Delta) = 1_{\mathcal{H}_I} \), finally yields \( \text{sp}V(\cdot) \subset C \).

6 Borchers’ Criterion and Infravacua

Let us now return to DHR theory. If \( \pi_I \) is an infravacuum representation of \( \mathcal{A} \), the role of what was denoted by \( (\mathcal{H}_I, U_I) \) in the previous section is of
course taken over by \((\mathcal{H}_{\pi_I}, U_{\pi_I})\). For any object \(\rho_I \in \Delta_{\pi_I,t}\), the following set \(Z^{\rho_I}\) of cocycles is of interest:

\[
Z^{\rho_I} := \left\{ \Gamma^{\rho_I} \bigg| \Gamma^{\rho_I} \text{ is a cocycle over } (\mathcal{H}_{\pi_I}, U_{\pi_I}) \text{ and } \Gamma^{\rho_I}(x) \in I_{\pi_I}(\rho_I, x, \rho_I) \right\}.
\]

Calling an object \(\rho_I\) covariant if there exists a strongly continuous unitary representation \(V_{\rho_I} : \mathbb{R}^{1+\delta} \to U(\mathcal{H}_{\pi_I})\) which implements the translations in the representation \(\rho_I \circ \pi_I\), we easily obtain the following result by relating \(V_{\rho_I}\) and \(\Gamma^{\rho_I}\) as in Prop. 5.2:

**Lemma 6.1** \(Z^{\rho_I}\) is nonempty iff \(\rho_I \in \Delta_{\pi_I,t}\) is covariant.

In the rest of this section, we will study the behaviour of the cocycles introduced above under the bifunctor \(F : \Delta_{\pi_0,t} \to \Delta_{\pi_I,t}\) described in Prop. 4.1. We now have cocycles \(\Gamma^{\rho_0} \in Z^{\rho_0}\) and \(\Gamma^{\rho_I} \in Z^{\rho_I}\) which take on values in the groups of local and unitary elements of \(\pi_0(A)\) resp. \(\pi_I(A)\). As a consequence, not only they are in the domain of \(F\) (which equals \(\pi_I \circ \pi_0^{-1}\) on intertwiners) resp. \(F^{-1}\), but we even have because of the continuity properties of \(F\) established in Lemma 3.1(3) (whose proof only relies on local normality — cf. also the remark after Proposition 4.1):

**Lemma 6.2** Let \(\rho_0 \in \Delta_{\pi_0,t}\), \(\rho_I := F(\rho_0)\). Then the functor \(F\) maps \(Z^{\rho_0}\) onto \(Z^{\rho_I}\).

In other words, \(\rho_0\) is covariant iff \(\rho_I\) is\(^5\). Moreover, an elementary calculation shows that, in this case, implementations \(V_{\rho_0}\) resp. \(V_{\rho_I}\) of the translations for \(\rho_0 \circ \pi_0\) resp. \(\rho_I \circ \pi_I\) can be obtained from each other by the formula

\[
V_{\rho_I}(x) = \pi_I \circ \pi_0^{-1} \left( V_{\rho_0}(x) U_{\pi_0}(x)^* \right) U_{\pi_I}(x).
\] (1)

We will now investigate in what extent the functor \(F\) also respects spectral properties. The inclusions obtained in Prop. 3.2 are taken as a starting point, but we will comment on more general situations at the end of this section.

---

\(^4\)Such a representation will be called an implementation and denoted with the symbol \(V_{\rho_I}\) in order to avoid confusion with the unique canonical implementation \(U_{\pi}\) introduced in Sec. 2 for the positive energy representations \(\pi\) of \(\mathcal{A}\).

\(^5\)We will stick to the notation \(\rho_I := F(\rho_0)\) in the sequel.
Proposition 6.3 Let $\pi_I$ be an irreducible, locally normal positive energy representation of $A$ and assume that $\pi_0$ and $\pi_I$ satisfy Haag duality. Denote by

$$F : \Delta_{\pi_0,t} \to \Delta_{\pi_I,t}$$

the bifunctor of Prop. 4.1. If both inclusions of Prop. 3.2 are valid, then one obtains for any covariant object $\rho_0 \in \Delta_{\pi_0,t}$:

1. $F$ maps the $C$-spectral cocycles in $Z^{\rho_0}$ onto the $C$-spectral ones in $Z^{\rho_I}$.
2. If $V_{\rho_0}$ and $V_{\rho_I}$ are related by (3) then $\text{sp} V_{\rho_0} = \text{sp} V_{\rho_I}$.
3. $\rho_0$ has positive energy iff $\rho_I$ has.
4. If $\rho_0$ and $\rho_I$ have positive energy and are finite direct sums of irreducibles, then the respective canonical implementations $U_{\rho_0 \otimes \pi_0}$ and $U_{\rho_I \otimes \pi_I}$ of the translations are related by (4); in particular, their spectra coincide:

$$\text{sp} U_{\rho_0 \otimes \pi_0} = \text{sp} U_{\rho_I \otimes \pi_I}.$$

Proof: 1. Let $\Gamma^{\rho_0} \in Z^{\rho_0}$ be $C$-spectral and let $\Gamma^{\rho_I}(x) := F(\Gamma^{\rho_0}(x))$. It will be shown in the next lemma that this entails, for any compact set $\Delta$,

$$\text{sp} \left( \Gamma^{\rho_I}(\cdot) E_I(\Delta) \right) \subset \bigcap_{\Delta_0} \{ C - \Delta_0 \mid \tilde{S}_{\pi_I}(\Delta) \subset \tilde{S}_{\pi_0}(\Delta_0) \}.$$

As a consequence of the first inclusion of Prop. 3.2, $\Delta_0$ can be chosen to be any neighbourhood of $\Delta$. This implies that the right hand side reduces to $C - \Delta$, i.e. $\Gamma^{\rho_I}$ is $C$-spectral. The converse is proved along the same lines, now using the second inclusion of Prop. 3.2.

2. Due to (4), part 1 applies to the cocycles $\Gamma^{\rho_0}(x) := V_{\rho_0}(x)U_0(x)^*$ and $\Gamma^{\rho_I}(x) := V_{\rho_I}(x)U_I(x)^*$, Using Prop. 5.2, this yields for any closed $C \subset \mathbb{R}^{1+s}$:

$$\text{sp} V_{\rho_0} \subset C \iff \text{sp} V_{\rho_I} \subset C,$$

which immediately gives the assertion. From this, part 3 follows directly.

As to part 4, we first consider the special case when $\rho_0$ (and therefore also $\rho_I$) is irreducible. Then all implementations of the translations coincide up to a one-dimensional one, i.e. a phase factor. Together with part 2, this yields the following equivalences:

$$V_{\rho_0} = U_{\rho_0 \otimes \pi_0} \Leftrightarrow \text{sp} V_{\rho_0} \text{ has L"ub} \Leftrightarrow \text{sp} V_{\rho_I} \text{ has L"ub} \Leftrightarrow V_{\rho_I} = U_{\rho_I \otimes \pi_I}.$$
(where “Lilb” means “Lorentz-invariant lower boundary”). Applying part 2 again gives all of the assertion in this case. In the general case of reducible \( \rho_0 \), we prove \( V_{\rho_0} = U_{\rho_0 \circ \pi_0} \iff V_{\rho_I} = U_{\rho_I \circ \pi_I} \) as follows: if \( W_j \in I_{\pi_0}(\rho_{0,j}, \rho_0) \) denote intertwiners which perform the decomposition of \( \rho_0 \) into irreducibles \( \rho_{0,j} \), then \( F(W_j) \) decompose \( \rho_I \) into irreducibles \( \rho_{I,j} = F(\rho_{0,j}) \). The implementations \( V_{\rho_0} \) and \( V_{\rho_I} \) decompose into subrepresentations \( W_j \ast V_{\rho_0}(x)W_j \) and \( F(W_j^*) V_{\rho_I}(x) F(W_j) \) on the corresponding subspaces and these subrepresentations can be seen to be related by \( (\mathbb{I}) \) iff \( V_{\rho_0} \) and \( V_{\rho_I} \) are. Since the latter are, by definition, canonical iff all their subrepresentations are, the assertion follows from the special case discussed first. ■

**Lemma 6.4** Let \( \Gamma_{\rho_0} \) be \( C \)-spectral. Then we have for any compact \( \Delta \):

\[
\text{sp} \left( \Gamma_{\rho_I}(\cdot)E_I(\Delta) \right) \subset \bigcap_{\Delta_0} \{ C - \Delta_0 \mid \tilde{S}_{\pi_I}(\Delta) \subset \tilde{S}_{\pi_0}(\Delta_0) \}.
\]

Proof: Let \( \Delta_0 \) be such that \( \tilde{S}_{\pi_I}(\Delta) \subset \tilde{S}_{\pi_0}(\Delta_0) \) and let \( f \) be a test function satisfying \( \text{supp} \tilde{f} \cap (C - \Delta_0) = \emptyset \). Setting \( \Gamma_{\rho_0}(f) := \int dx f(x) \Gamma_{\rho_0}(x) \), we obtain \( \Gamma_{\rho_0}(f)E_0(\Delta_0) = \int dx f(x) \Gamma_{\rho_0}(x) E_0(\Delta_0) = 0 \) since \( \Gamma_{\rho_0} \) is \( C \)-spectral. This means \( \omega(\pi_0^{-1}(\Gamma_{\rho_0}(f)^* \Gamma_{\rho_0}(f))) = 0 \) for any vector state \( \omega \in \mathcal{S}_{\pi_0}(\Delta_0) \), hence (by weak continuity) for any \( \omega \in \tilde{S}_{\pi_0}(\Delta_0) \) and in particular for any \( \omega \in \tilde{S}_{\pi_I}(\Delta) \). We thus have \( \pi_I \circ \pi_0^{-1}(\Gamma_{\rho_0}(f)) E_I(\Delta) = 0 \) and hence by local normality \( \int dx f(x)(\pi_I \circ \pi_0^{-1}(\Gamma_{\rho_0}(x))) E_I(\Delta) = 0 \). (For this argument, one has to realize that \( f \) can be approximated in \( L^1(\mathbb{R}^{1+s}) \) by test functions with compact support.) As this holds for any \( f \) with \( \text{supp} \tilde{f} \cap (C - \Delta_0) = \emptyset \), it follows that \( \text{sp}(\pi_I \circ \pi_0^{-1}(\Gamma_{\rho_0}(\cdot))) E_I(\Delta) \subset C - \Delta_0 \). This yields the assertion. ■

The inclusions of Prop. 3.2 played a crucial role in the previous proposition. However, partial results remain valid if less information about the relation between \( \tilde{S}_{\pi_I} \) and \( \tilde{S}_{\pi_0} \) is available. For instance, let us merely assume (instead of these inclusions) that there exists a compact neighbourhood \( N \) of \( 0 \) such that (cf. Lemma 2.1(3))

\[
\tilde{S}_{\pi_I}(\Delta) \subset \tilde{S}_{\pi_0}(\Delta + N) \quad \text{for all compact sets } \Delta.
\]

In this case, we obtain with the same arguments as above that the cocycle \( \Gamma_{\rho_I} := F(\Gamma_{\rho_0}(\cdot)) \) is \((C - N)\)-spectral if \( \Gamma_{\rho_0} \) is \( C \)-spectral. In terms of \( V_{\rho_0} \) and \( V_{\rho_I} \), this means

\[
\text{sp} V_{\rho_I} \subset \text{sp} V_{\rho_0} - N.
\]
In general, all detailed information on the shape of \( \text{sp}V_{\rho_I} \) (such as the size of possible mass gaps) is lost, but what can easily be seen is that \( \rho_I \) has positive energy if \( \rho_0 \) has. The canonical implementations of the translations need, in general, not be related by (1) any more. A similar reasoning applies if the roles of \( \pi_0 \) and \( \pi_I \) are exchanged.

7 Conclusion and Outlook

We have seen that, as far as DHR theory is concerned, the role of the vacuum representation \( \pi_0 \) can be taken over by any locally normal representation \( \pi_I \) satisfying Haag duality. Moreover, we have given sufficient conditions on \( \pi_I \) which ensure that the class of representations fulfilling Borchers’ criterion is independent of \( \pi_I \). These representations may be interpreted as background fields whose interaction with the charged particles of the model is weak.

Natural candidates for states describing such a background were elements of the energy component of the vacuum. The latter notion, introduced in [7], deserves some more interest on its own, and one might ask in the spirit of Section 3 under which circumstances other properties such as, e.g., Haag duality carry over from \( \pi_0 \) to \( \pi_I \).

It seems thus that for most purposes, the characteristic feature of a vacuum representation, namely the existence of a translation invariant vector in the Hilbert space of that representation, is not an essential property. However, we recall that this property plays a crucial role for proving that the energy-momentum spectra are additive under the fusion of covariant sectors:

\[
\text{sp}U_{\rho_1 \circ \rho_2 \circ \pi_0} \supset \text{sp}U_{\rho_1 \circ \pi_0} + \text{sp}U_{\rho_2 \circ \pi_0}
\]

Trivially, using the results of Prop. 6.3, this yields

\[
\text{sp}U_{F(\rho_1 \circ \rho_2) \circ \pi_I} \supset \text{sp}U_{F(\rho_1) \circ \pi_I} + \text{sp}U_{F(\rho_2) \circ \pi_I},
\]

but it would of course be interesting to know under which assumptions the latter inclusion can be derived without relying on the vacuum sector.

Finally, returning to Buchholz’ proposal of reconciling QED with superselection theory, several interesting questions arise. First, it has to be clarified which parts of the present work (in particular of Sections 4 and 6) can be generalised from DHR-like localised charges to charges localised in space-like cones. Second, it will be important to check in specific models whether there exist backgrounds which permit better localisation properties.
of charges of electric type than the vacuum does. Candidates for such models might be the one proposed by Herdegen [20] or a simple model recently introduced by Buchholz et al. in [21].

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