SPECTRAL PROPERTIES OF ELLIPTIC OPERATOR WITH DOUBLE-CONTRAST COEFFICIENTS NEAR A HYPERPLANE

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ABSTRACT. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). Let $D^\varepsilon$ be a family of spherical shells depending on a parameter $\varepsilon > 0$, by $B^\varepsilon$ we denote the union of balls surrounded by these shells. It is supposed that the shells are distributed along the plane $\Gamma$ having non-empty intersection with $\Omega$, the number of shells tends to infinity, while their diameters tend to zero as $\varepsilon \to 0$. The paper deals with the asymptotic behaviour as $\varepsilon \to 0$ of the spectrum of the operator $A^\varepsilon = -\frac{1}{\varepsilon^2} \text{div}(a^\varepsilon \nabla)$ in $\Omega$ (subject to the Dirichlet boundary conditions on $\partial \Omega$). It is supposed that $a^\varepsilon \to 0$ on $D^\varepsilon$ and $b^\varepsilon \to \infty$ on $B^\varepsilon$ as $\varepsilon \to 0$. We prove that the spectrum of $A^\varepsilon$ converges as $\varepsilon \to 0$ to the spectrum of some operator $A$ acting in $L^2(\Omega) \oplus L^2(\Gamma)$. The form of this operator depends on the behaviour of $a^\varepsilon$ and $b^\varepsilon$ as $\varepsilon \to 0$. In particular, in some cases the limit operator may have nonempty essential spectrum. Also we study the same problem, when $\Omega$ is an infinite strip and $\Gamma$ is parallel to its boundary. We will show that $A^\varepsilon$ has gap in the spectrum when $\varepsilon$ is small enough.

INTRODUCTION

The problem we are going to study lies on the intersection of spectral theory and the homogenization theory for partial differential operators. We recall that one of the central problems of the homogenization theory is to study the asymptotic behaviour as $\varepsilon \to 0$ of the solution to the problem

$$-\text{div}(a^\varepsilon(x)\nabla u^\varepsilon) = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,$$

where $\varepsilon > 0$ is a small parameter, the real measurable function $a^\varepsilon(x)$ satisfies

$$a_-^\varepsilon \leq a^\varepsilon(x) \leq a_+^\varepsilon \quad (a_\pm^\varepsilon \text{ are positive constants})$$

and becomes highly oscillating as $\varepsilon \to 0$. The typical example is $a^\varepsilon(x) = a(xe^{-1})$, where $a(x)$ is a fixed periodic function. It is well-known (see, e.g., [5, 10]) that if

$$\inf_\varepsilon a_-^\varepsilon > 0, \quad \sup_\varepsilon a_+^\varepsilon < \infty \quad (0.1)$$

then the family $\{u^\varepsilon\}_\varepsilon$ is compact in $L_2(\Omega)$ and if $u^\varepsilon \to u$ as $\varepsilon = \varepsilon_k \to 0$ then $u(x)$ is a solution of the problem

$$-\text{div}(A(x)\nabla u) = f \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,$$

where $A(x)$ is some bounded and bounded away from zero function, which depends, in general on the subsequence $\varepsilon_k$. Note, that if $a^\varepsilon(x) = a(xe^{-1})$ than the whole sequence $u^\varepsilon$ converges, and in this case $A(x)$ is a constant. On the language of the operator theory one can say that the operator $A^\varepsilon = -\text{div}(a^\varepsilon(x)\nabla)$ strongly resolvent converges to the operator $A = -\text{div}(A(x)\nabla)$ as $\varepsilon = \varepsilon_k \to 0$. Thus if (0.1) holds then the limit operator has the same form as the initial one.

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If, on the contrary, conditions (0.1) are violated, for example there exist subsets $D^\varepsilon \subset \Omega$ such that
\[
\lim_{\varepsilon \to 0} \sup_{x \in D^\varepsilon} a^\varepsilon(x) = \infty \text{ or } \lim_{\varepsilon \to 0} \inf_{x \in D^\varepsilon} a^\varepsilon(x) = 0,
\]
then the limit operator may have more complicated form, which depends essentially on the structure of the domains $D^\varepsilon$. We refer to monograph [23], where various problems of this type were studied. In particular, the authors studied the case, when condition (0.2) holds on the union $D^\varepsilon$ of thin shells, distributed periodically, with period $\varepsilon$, in the domain $\Omega$. The authors studied the behaviour of linear evolution equations involving such operators $A^\varepsilon$. The semi-linear evolution equations were studied in papers [14, 26, 27]. Spectral properties of such operators were studied in [18].

In all papers mentioned above the case of bulk distribution of shells was considered. In the present work we are interesting in the case of surface distribution of shells, i.e. when the shells are located in a neighbourhood of some hyperplane. Below we briefly present our main results.

We deal with the operator
\[
A^\varepsilon = -\frac{1}{b^\varepsilon(x)} \text{div}(a^\varepsilon(x) \nabla)
\]  
acting in $L^2(\Omega)$ endowed with the weighted scalar product $\int_\Omega b^\varepsilon(x)u(x)v(x)dx$. Here $\Omega$ is a domain in $\mathbb{R}^n (n \geq 2)$, $\varepsilon > 0$ is small parameter, the functions $a^\varepsilon$ and $b^\varepsilon$ are bounded above and bounded away from zero uniformly in $\varepsilon$ everywhere except a small neighbourhood of some hyperplane $\Gamma$ having non-empty intersection with the domain $\Omega$. More precisely, we denote by $D^\varepsilon = \{D^\varepsilon_i\}$ the family of spherical thin shells distributed periodically along $\Gamma$, by $B^\varepsilon = \{B^\varepsilon_i\}$ we denote the union of balls surrounded by these shells (see Figure 1). When $\varepsilon \to 0$ the number of shells goes to infinity as $\varepsilon \to 0$, whereas their external radii goes to zero. We assume that $a^\varepsilon$ and $b^\varepsilon$ are periodic along $\Gamma$ and as $\varepsilon \to 0$
\[
da^\varepsilon \to 0 \text{ on } D^\varepsilon, \quad b^\varepsilon \to \infty \text{ on } B^\varepsilon.
\]  
Operators of the form (0.3) occur in various areas of physics. For example, the operator $A^\varepsilon$ describes vibrations of the body occupying the domain $\Omega$, the functions $a^\varepsilon$ and $b^\varepsilon$ are the stiffness and the
mass density, correspondingly. Conditions (0.4) means that the body contains many small heavy inclusions $B_i^\varepsilon$ surrounded by thin soft layers $D_i^\varepsilon$. The asymptotic behaviour of eigenvalues of a body with many small heavy inclusions was studied in a number of papers (see, e.g., [11, 12, 21, 22, 29] and references therein).

We start from the case when $\Omega$ is a bounded domain. In this paper additionally to (0.4) we suppose that the total mass of inclusions $B_i^\varepsilon$ is bounded above and bounded away from zero uniformly in $\varepsilon$, that is

$$0 < \beta_- \leq \int_{B_i^\varepsilon} b^\varepsilon \, dx \leq \beta_+ < \infty, \quad \beta_+ > 0.$$ 

It turns out that the form of the limit operator depend essentially on the quantity $a := \lim_{\varepsilon \to 0} \frac{\inf_{x} a(x)}{\text{diam} D_i^\varepsilon}$.

There are three qualitatively different cases.

1: $a > 0$. In this case we prove that the spectrum of $\sigma(\mathcal{A}^\varepsilon)$ converges in the Hausdorff sense to the spectrum of the operator $\mathcal{A}$ acting on the space $L_2(\Omega) \oplus L_2(\Gamma)$.

We will define this operator accurately in Section 1 here we only note that the resolvent equation $\mathcal{A}U - \lambda U = F$ (here $U = (u_1, u_2)$, $F = (f_1, f_2)$) can be formally written as follows:

$$\begin{cases}
-\Delta u_1 - \lambda u_1 = f_1 & \text{in } \Omega \setminus \Gamma, \\
(u_1)^+ = (u_1)^- & \text{on } \Gamma, \\
\left( \frac{\partial u_1}{\partial n} \right)^+ - \left( \frac{\partial u_1}{\partial n} \right)^- + p(u_1 - u_2) = 0 & \text{on } \Gamma, \\
q(u_2 - u_1) - \lambda u_2 = f_2 & \text{on } \Gamma, \\
u_1 = 0 & \text{on } \partial \Omega,
\end{cases}$$

(0.5)

where $n = -(0, 0, \ldots, 0, 1)$, by $+$ (resp. $-$) we denote the values of the function $u$ and its normal derivative on the upper (resp. lower) side of $\Gamma$. Here $p, q$ are some positive numbers which depend only on $a^\varepsilon, b^\varepsilon$ and are independent of $\Omega$.

The spectrum of $\mathcal{A}^\varepsilon$ consists of two sequences of eigenvalues $\{\lambda_k^-\}_{k \in \mathbb{N}}, \{\lambda_k^+\}_{k \in \mathbb{N}}$ and the point $q$ which is a point of the essential spectrum:

$$0 < \lambda_1^- \leq \lambda_2^- \leq \ldots \leq \lambda_k^- \leq \ldots \to q < \lambda_1^+ \leq \lambda_2^+ \leq \ldots \leq \lambda_k^+ \leq \ldots \to \infty.$$

2: $a = 0$. In this case $\sigma(\mathcal{A}^\varepsilon)$ converges to the set $\sigma(\mathcal{A}_0) \cup \{0\}$, where $\mathcal{A}_0$ is the Dirichlet Laplacian in $L_2(\Omega)$.

3: $a = \infty$. In this case $\sigma(\mathcal{A}^\varepsilon)$ converges to the spectrum of the spectral problem, which formally can be written as follows:

$$\begin{cases}
-\Delta u - \lambda u = 0 & \text{in } \Omega \setminus \Gamma, \\
(u)^+ = (u)^- & \text{on } \Gamma, \\
\left( \frac{\partial u}{\partial n} \right)^+ - \left( \frac{\partial u}{\partial n} \right)^- - \lambda \rho u = 0 & \text{on } \Gamma, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\rho$ is a positive constant.

In the last part of the work we consider the same problem for waveguide-like domain $\Omega$:

$$\Omega = \mathbb{R} \times (d_-, d_+), \quad \Gamma = \{x = (x^1, x^2): x^2 = 0\},$$

where $\pm d_+, d_- > 0$. We are interested in the case $a > 0$ only.

Due to the periodicity of $\mathcal{A}^\varepsilon$ its spectrum is a locally finite union of compact intervals (bands). In general the bands may overlap each other and the natural question arising here is whether the gaps open up in the spectrum (i.e. whether there is an open interval $(\alpha, \beta) \subset (0, \infty)$ such that $(\alpha, \beta) \cap \sigma(\mathcal{A}^\varepsilon) =$ }
periodically distributed on the plane $\Omega$ and the bulk distribution of shells, the spectrum of $\mathcal{A}$ has gap when $\varepsilon$ is small enough.

Our goal is to study whether the gaps will open up in case of our waveguide-like domain. We will prove that the spectrum of $\mathcal{A}$ converges in the Hausdorff sense to the spectrum of the operator $\mathcal{A}$ which is defined by the same expression as in the case of compact domain. Its spectrum is as follows:

if $q < \min \left\{ \frac{x^2}{\varepsilon^2}, \frac{x^2}{\varepsilon^2} \right\}$ then

$$\sigma(\mathcal{A}) = [a_1, q] \cup [a_2, \infty),$$

otherwise $\sigma(\mathcal{A}) = [a_1, \infty)$. Here $a_1, a_2$ are some positive numbers satisfying $0 < a_1 < q < a_2$. Thus if the waveguide is thin enough we have a gap in the spectrum of $\mathcal{A}$ when for small enough $\varepsilon$.

Periodic perturbations of the Laplacian in waveguide-like domains leading to opening of spectral gaps were also studied in [4, 6, 13, 24, 25, 30]. In all these papers (except [7]) spectral gaps appear because of a perturbation of the boundary of the waveguide (for example by making small holes periodically distributed along the waveguide [13] or by dividing the waveguide on two parts coupled by a periodic system of small windows [6]). In the recent paper [7] the authors considered small perturbations of the Laplace operator in a cylindrical domain by second-order differential operators with periodic coefficients; they gave sufficient conditions on this perturbation for gap opening. These conditions are not valid for the operators considered in the present work.

The paper is organized as follows. In Section 1 we set up the problem and formulate the main result (Theorems 1.1-1.2). They are proved in the next section: in Subsection 2.1 we prove the Hausdorff convergence of $\sigma(\mathcal{A})$ for the case $a < \infty$, Subsection 2.2 is devoted to the proof of Lemma 1.1 describing the spectrum of the operator $\mathcal{A}$, in Subsection 2.3 we study the case $a = \infty$. Finally in Section 3 we study the behaviour of $\sigma(\mathcal{A})$ for the case of the waveguide-like domain.

1. Setting of the problem and main result

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n \geq 2$). It is supposed that $0 \in \Omega$. We denote by $\Gamma$ the intersection of $\Omega$ with the hyperplane $\{x^n = 0\}$:

$$\Gamma = \left\{ x = (x^1, \ldots, x^n) \in \Omega : x^n = 0 \right\}. $$

Let $\varepsilon > 0$ be a small parameter. We denote by $x^\varepsilon_i$, $i = (i_1, \ldots, i_n) \in \mathbb{Z}^{n-1}$ the family of points periodically distributed on the plane $\{x^n = 0\}$:

$$x^\varepsilon_i = (\varepsilon i_1, \varepsilon i_2, \ldots, \varepsilon i_{n-1}, 0).$$

By $I^\varepsilon$ we denote the set of all multiindices $i \in \mathbb{Z}^{n-1}$ such that $x^\varepsilon_i \in \Gamma$.

We introduce the following sets (see Fig. 1):

$$D^\varepsilon_i = \left\{ x \in \mathbb{R}^n : r^\varepsilon - d^\varepsilon < |x - x^\varepsilon_i| < r^\varepsilon \right\},$$

$$B^\varepsilon_i = \left\{ x \in \mathbb{R}^n : |x - x^\varepsilon_i| < r^\varepsilon - d^\varepsilon \right\}.$$
We introduce the following piecewise constant functions:
\[
\alpha^\varepsilon(x) = \begin{cases} 
1, & x \in \Omega \setminus \bigcup_{i \in I^\varepsilon} D_i^+, \\
\alpha^\varepsilon, & x \in \bigcup_{i \in I^\varepsilon} D_i^+, 
\end{cases}
\]
\[
b^\varepsilon(x) = \begin{cases} 
1, & x \in \Omega \setminus \bigcup_{i \in I^\varepsilon} B_i^e, \\
b^\varepsilon, & x \in \bigcup_{i \in I^\varepsilon} B_i^e, 
\end{cases}
\]
(1.1)

where \(\alpha^\varepsilon, \beta^\varepsilon\) are positive constants satisfying
\[
\alpha^\varepsilon \to 0, \beta^\varepsilon \to \infty \text{ as } \varepsilon \to 0. \tag{1.2}
\]

Below (see (1.6)) we impose additional restrictions on \(\alpha^\varepsilon\) and \(\beta^\varepsilon\).

Now, let us define accurately the operator \(\mathcal{A}^\varepsilon = -\frac{1}{\varepsilon^2} \text{div}(\sigma^\varepsilon \nabla)\). By \(\mathcal{H}^\varepsilon\) we denote the Hilbert space of functions from \(L_2(\Omega)\) endowed with a scalar product
\[
(u, v)_{\mathcal{H}^\varepsilon} = \int_\Omega b^\varepsilon(x)u(x)v(x)dx. \tag{1.3}
\]

By \(\eta^\varepsilon\) we denote the sesquilinear form in \(\mathcal{H}^\varepsilon\) defined by the formula
\[
\eta^\varepsilon[u, v] = \int_\Omega a^\varepsilon(x)\nabla u \cdot \nabla vdx \tag{1.4}
\]
with \(\text{dom}(\eta^\varepsilon) = H^1_0(\Omega)\). The form \(\eta^\varepsilon\) is densely defined, closed, positive and symmetric. Thus (see e.g. [17]) there exists the unique self-adjoint and positive operator \(\mathcal{A}^\varepsilon\) associated with the form \(\eta^\varepsilon\), i.e.
\[
(\mathcal{A}^\varepsilon u, v)_{\mathcal{H}^\varepsilon} = \eta^\varepsilon[u, v], \quad \forall u \in \text{dom}(\mathcal{A}^\varepsilon), \forall v \in \text{dom}(\eta^\varepsilon).
\]

The domain of \(\mathcal{A}^\varepsilon\) consists of functions \(u\) with the corresponding restriction belonging to the spaces \(H^2(D_i^\varepsilon), H^2(B_i^\varepsilon)\) (for any \(i \in I^\varepsilon\)), \(H^2(\Omega \setminus \bigcup_{i \in I^\varepsilon} (D_i^\varepsilon \cup B_i^e)) \cap H^1_0(\Omega)\), and satisfying the following conditions on \(\partial D_i^\varepsilon\):
\[
\begin{cases} 
(u)^+ = (u)^-, & \left(\frac{\partial u}{\partial n}\right)^+ = \alpha^\varepsilon \left(\frac{\partial u}{\partial n}\right)^-, \quad x \in \partial D_i^\varepsilon \setminus \partial B_i^e, \\
(u)^+ = (u)^-, & \alpha^\varepsilon \left(\frac{\partial u}{\partial n}\right)^+ = \left(\frac{\partial u}{\partial n}\right)^-, \quad x \in \partial D_i^\varepsilon \cap \partial B_i^e
\end{cases}
\]
(1.5)

where by + (resp. -) we denote the traces of the function \(u\) and its normal derivative taken from the exterior (resp. interior) side of either \(\partial D_i^\varepsilon \setminus \partial B_i^e\) or \(\partial D_i^\varepsilon \cap \partial B_i^e\).

The spectrum \(\sigma(\mathcal{A}^\varepsilon)\) of the operator \(\mathcal{A}^\varepsilon\) is purely discrete. Our goal is to describe the behaviour of \(\sigma(\mathcal{A}^\varepsilon)\) as \(\varepsilon \to 0\).

Additionally to (1.2) we suppose that the following conditions hold:
\[
\lim_{\varepsilon \to 0} \alpha^\varepsilon = a \in [0, \infty], \quad \lim_{\varepsilon \to 0} \frac{\alpha^\varepsilon}{(d^\varepsilon)^2} = \infty, \quad \lim_{\varepsilon \to 0} \varepsilon \beta^\varepsilon = b \in (0, \infty). \tag{1.6}
\]

The main attention in this work will be paid to the case \(a > 0\).

Remark 1.1. As we mentioned above the operator \(\mathcal{A}^\varepsilon\) describes vibrations of the body occupying domain \(\Omega\) and containing many small heavy inclusions surrounded by thin soft layers. The last condition in (1.6) implies that the total mass of heavy inclusions \(B_i^e\) is bounded above and bounded away from zero uniformly in \(\varepsilon\), namely
\[
\sum_{i \in I^\varepsilon} \int_{B_i^e} b^\varepsilon(x)dx = \varepsilon \beta^\varepsilon \sum_{i \in I^\varepsilon} e^{\varepsilon - 1} \sim b|\Gamma| \text{ as } \varepsilon \to 0.
\]
We start to the case \( a < \infty \). In order to formulate the result we need some additional notations. We set

\[
p = ar^{n-1} \omega_{n-1}, \quad q = \frac{an}{br}, \quad \rho = br^n \omega_n, \tag{1.7}
\]

where \( \omega_{n-1} \) is the volume of \((n-1)\)-dimensional unit sphere, \( \omega_n \) is the volume of \( n \)-dimensional unit ball.

By \( \mathcal{H} \) we denote the Hilbert space of functions from \( L_2(\Omega) \oplus L_2(\Gamma) \) endowed with the scalar product

\[
(U, V)_{\mathcal{H}} = \int_{\Omega} u_1(x)v_1(x)dx + \int_{\Gamma} \rho u_2(x)v_2(x)ds, \quad U = (u_1, u_2), \ V = (v_1, v_2) \tag{1.8}
\]

(here \( ds \) is an area form on \( \Gamma \)). By \( \eta \) we denote the sesquilinear form in \( \mathcal{H} \)

\[
\eta[U, V] = \int_{\Omega} \nabla u_1 \cdot \nabla u_2 dx + \int_{\Gamma} p(u_1 - u_2)(v_1 - v_2)ds \tag{1.9}
\]

with \( \text{dom}(\eta) = H^1_0(\Omega) \oplus L_2(\Gamma) \). This form is densely defined, closed, positive and symmetric. We denote by \( \mathcal{A} \) the self-adjoint operator associated with this form. Formally the equation \( \mathcal{A}U - \lambda U = F \) (where \( U = (u_1, u_2), \ F = (f_1, f_2) \) ) has the form (1.5).

If \( a = 0 \) then the operator \( \mathcal{A} \) is a direct sum of the Dirichlet Laplacian in \( L_2(\Omega) \) (we denote it \( \mathcal{A}_0 \)) and the null operator in \( L_2(\Gamma) \). As a result we have:

\[
\text{if } a = 0 \text{ then } \sigma_{\text{ess}}(\mathcal{A}) = \{0\}, \ \sigma_{\text{disc}}(\mathcal{A}) = \sigma(\mathcal{A}_0).
\]

The following statement describes the spectrum of the operator \( \mathcal{A} \) in the case \( a > 0 \).

**Lemma 1.1.** The spectrum of the operator \( \mathcal{A} \) has the form

\[
\sigma(\mathcal{A}) = \{q\} \cup \{\lambda_k^-, k = 1, 2, 3...\} \cup \{\lambda_k^+, k = 1, 2, 3...\}.
\]

The points \( \lambda_k^\pm, k = 1, 2, 3... \) belong to the discrete spectrum, \( q \) is a point of the essential spectrum and they are distributed as follows:

\[
0 < \lambda_1^- \leq \lambda_2^- \leq \ldots \leq \lambda_k^- \leq \ldots \rightarrow q < \lambda_1^+ \leq \lambda_2^+ \leq \ldots \leq \lambda_k^+ \leq \ldots \rightarrow \infty. \tag{1.10}
\]

We will prove this lemma in Subsection 2.2.

Now, we formulate the main result.

**Theorem 1.1.** Let \( a < \infty \). Then the spectrum \( \sigma(\mathcal{A}^\varepsilon) \) converges to the spectrum \( \sigma(\mathcal{A}) \) in the Hausdorff sense, i.e.

(A) if \( \lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon) \) and \( \lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda \) then \( \lambda \in \sigma(\mathcal{A}) \),

(B) for any \( \lambda \in \sigma(\mathcal{A}) \) there are \( \lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon) \) such that \( \lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda \).

We consider the case \( a = \infty \). Let \( \mathcal{H} \) be again the Hilbert space of functions from \( L_2(\Omega) \oplus L_2(\Gamma) \) endowed with the scalar product (1.8) (note, that the weight \( \rho \) is independent of \( a \)). We introduce in \( \mathcal{H} \) the sesquilinear form

\[
\widetilde{\eta}[U, V] = \int_{\Omega} \nabla u_1 \cdot \nabla u_2 dx, \ U = (u_1, u_2), \ V = (v_1, v_2)
\]

with \( \text{dom}(\widetilde{\eta}) = \{U = (u_1, u_2) \in H^1_0(\Omega) \oplus L_2(\Gamma) : \ u_1|_{\Gamma} = u_2\} \). The form \( \widetilde{\eta} \) is densely defined, closed, positive and symmetric. We denote by \( \mathcal{A} \) the self-adjoint operator associated with this form. Formally
the equation $\hat{\mathcal{A}} U - \lambda U = F$ (where $U = (u, u_1)$, $F = (f_1, f_2)$) has the form

$$
\begin{aligned}
-\Delta u - \lambda u &= f_1 & \text{in } \Omega \setminus \Gamma, \\
(u)^+ - (u)^- &= 0 & \text{on } \Gamma, \\
\left(\frac{\partial u}{\partial n}\right)^+ - \left(\frac{\partial u}{\partial n}\right)^- - \lambda pu &= \rho f_2 & \text{on } \Gamma, \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}
$$

It clear that $\hat{\mathcal{A}}$ has compact resolvent in view of the trace theorem and the Sobolev-Kondrashev embedding theorem. Therefore the spectrum of $\hat{\mathcal{A}}$ is purely discrete.

**Theorem 1.2.** Let $a = \infty$. Then the spectrum $\sigma(\mathcal{A}^\varepsilon)$ converges to the spectrum $\sigma(\hat{\mathcal{A}})$ in the Hausdorff sense.

2. Proof of the main results

The proof of Theorem 1.1 is presented in Subsection 2.1. Lemma 1.1 is proved in Subsection 2.2. Finally, the proof of Theorem 1.2 is given in Subsection 2.3.

2.1. Proof of Theorem 1.1

2.1.1. Let $\lambda^\varepsilon \in \sigma(\mathcal{A})$ and $\lambda^\varepsilon \to \lambda$. We have to show that $\lambda \in \sigma(\hat{\mathcal{A}})$.

We present the proof for the case $n \geq 3$ only. For the case $n = 2$ the proof needs some small modifications (for example in (2.10) the function $|x - x_i^\varepsilon|^{2-n}$ has to be replaced by $-\ln|x - x_i^\varepsilon|$).

In what follows by $C, C_1, \ldots$ we denote generic constants that do not depend on $\varepsilon$.

By $\langle u \rangle_B$ we denote the mean value of the function $u(x)$ over the domain $B$:

$$
\langle u \rangle_B = \frac{1}{|B|} \int_B u(x) \, dx
$$

Here by $|B|$ we denote the Lebesgue measure of the domain $B$. If $\Sigma \subset \mathbb{R}^n$ is a $(n-1)$-dimensional surface then the Euclidean metric in $\mathbb{R}^n$ induces on $\Sigma$ the Riemannian metrics and measure. We denote by $ds$ the density of this measure. Again by $\langle u \rangle_\Sigma$ we denote the mean value of the function $u$ over $\Sigma$, i.e $\langle u \rangle_\Sigma = \frac{1}{|\Sigma|} \int_\Sigma u \, ds$, where $|\Sigma| = \int_\Sigma ds$.

We introduce the following sets:

$$
\begin{aligned}
\Omega^\varepsilon &= \Omega \setminus \bigcup_{i \in I^\varepsilon} \left( D^\varepsilon_i \cup B^\varepsilon_i \right), \\
S^\varepsilon_{i,+} &= \left\{ x \in \mathbb{R}^n : |x - x_i^\varepsilon| = r^\varepsilon \right\}, \\
S^\varepsilon_{i,-} &= \left\{ x \in \mathbb{R}^n : |x - x_i^\varepsilon| = r^\varepsilon - d^\varepsilon \right\}, \\
Y^\varepsilon_i &= \left\{ x \in \mathbb{R}^n : |x|^2 - (x_i^\varepsilon)^2 < \frac{\varepsilon}{2}, \forall k \right\}, \\
\Gamma^\varepsilon_i &= Y^\varepsilon_i \cap \Gamma.
\end{aligned}
$$

We denote by

$$
0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_k^\varepsilon \leq \cdots \to \infty
$$

the sequence of eigenvalues of $\mathcal{A}^\varepsilon$ repeated according to their multiplicity. By $u^\varepsilon_1, u^\varepsilon_2, \ldots, u^\varepsilon_k, \ldots$ we denote a corresponding sequence of eigenfunctions normalized by the condition $(u^\varepsilon_k, u^\varepsilon_l)_{\mathcal{H}^\varepsilon} = \delta_{kl}$.

We denote by $n^\varepsilon$ the index corresponding to $\lambda^\varepsilon$ (i.e., $\lambda^\varepsilon = \lambda^\varepsilon_{n^\varepsilon}$). By $u^\varepsilon = u^\varepsilon_{n^\varepsilon} \in H_0^1(\Omega)$ we denote the corresponding eigenfunction. One has

$$
\|u^\varepsilon\|_{\mathcal{H}^\varepsilon} = 1, \quad n^\varepsilon [u^\varepsilon, u^\varepsilon] = \lambda^\varepsilon. \quad (2.1)
$$
By \((\ref{1.1}), \ (\ref{1.4}), \ (\ref{2.1})\) we obtain
\[
\|\nabla u^e\|_{L^2(\Omega^e)} + \sum_{i \in I^e} \|\nabla u^e\|_{L^2(B_i^e)}^2 \leq \lambda^e \leq C. \tag{2.2}
\]

In order to describe the behavior of \(u^e\) as \(\epsilon \to 0\) we need some additional operators. It is known (see, e.g., \([1]\)) that there exists an extension operator \(\pi^e : H^1(Y_0^e \setminus (D_0^e \cup B_0^e)) \to H^1(Y_0^e)\) satisfying the following properties: \(\forall u \in H^1(Y_0^e \setminus (D_0^e \cup B_0^e))\) one has
\[
[\pi^e u](x) = u(x), \quad \forall x \in Y_0^e \setminus (D_0^e \cup B_0^e),
\]
\[
\|\pi^e u\|_{H^1(Y_0^e)} \leq C\|u\|_{H^1(Y_0^e \setminus (D_0^e \cup B_0^e))},
\]
where the constant \(C\) is independent of \(u\).

Now we denote \(\tilde{u}_i(x) = u(x + x_i^e)\) and define the operator \(\Pi_i^e : H^1(\Omega^e) \to H^1(\Omega)\) by
\[
[\Pi_i^e u](x) = \begin{cases} 
  u(x), & x \in \Omega \setminus \bigcup_{i \in I^e} Y_i^e; \\
  [\pi^e \tilde{u}_i](x - x_i^e), & x \in Y_i^e. 
\end{cases}
\]

It is clear that
\[
[\Pi_i^e u](x) = u(x), \quad \forall x \in \Omega^e,
\]
\[
\|\Pi_i^e u\|_{H^1(\Omega^e)} \leq C\|u\|_{H^1(\Omega)}. \tag{2.3}
\]

Also we introduce the operator \(\Pi_2^e : L_2(\bigcup_{i \in I^e} B_i^e) \to L_2(\Gamma)\):
\[
\Pi_2^e u(x) = \begin{cases} 
  (u)_{B_i^e}, & x \in \Gamma_i^e; \\
  0, & x \in \Gamma \setminus \bigcup_{i \in I^e} \Gamma_i^e.
\end{cases}
\]

Using the Cauchy inequality and \((\ref{1.6})\) we obtain
\[
\|\Pi_2^e u\|_{L_2(\Gamma)} \leq \frac{\|\Pi^e_i\|_{B_i^e}}{|B_i^e|} \int_{B_i^e} |u(x)|^2 \, dx \leq C \sum_{i \in I^e} \int_{B_i^e} |x - x_i^e| \, dx \|u\|^2_{\Omega^e} \leq C\|u\|^2_{\Omega^e}. \tag{2.4}
\]

Using \((\ref{2.2})-(\ref{2.4})\) and the Sobolev-Kondrasheff embedding theorem we conclude that there is a subsequence (still denoted by \(\epsilon\)) and \(u_1 \in H^1(\Omega), u_2 \in L_2(\Gamma)\) such that
\[
\Pi^e_i u^e \rightharpoonup u_1 \quad \text{in} \quad H^1(\Omega), \tag{2.5}
\]
\[
\Pi^e_i u^e \rightharpoonup u_1 \quad \text{in} \quad L_2(\Omega), \tag{2.6}
\]
\[
\Pi_2^e u^e \rightharpoonup u_2 \quad \text{in} \quad L_2(\Gamma). \tag{2.7}
\]

It is clear that \(\Pi^e_i u^e|_{\partial \Omega} = 0\), whence it follows from \((\ref{2.5})\) and the trace theorem that \(u_1|_{\partial \Omega} = 0\) and
\[
\Pi^e_i u^e \rightharpoonup u_1 \quad \text{in} \quad L_2(\Gamma). \tag{2.8}
\]

**Case 1.** \(u_1 \neq 0\). We will show that in this case \(U = (u_1, u_2)\) is an eigenfunction of the operator \(\mathcal{A}\) corresponding to the eigenvalue \(\lambda\).

For all \(w \in H_0^1(\Omega)\) one has
\[
\int_{\Omega} a^e \nabla u^e \cdot \nabla w \, dx = \lambda^e \int_{\Omega} b^e u^e w \, dx. \tag{2.9}
\]

Our strategy of proof will be to plug into \((\ref{2.9})\) some specially chosen test-function \(w\) depending on \(\epsilon\) and then pass to the limit as \(\epsilon \to 0\) to obtain the equality \(\mathcal{A}U = \lambda U\) written in weak form.
For constructing this special test-function we introduce several additional functions. Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( \Phi(r) = 1 \) as \( r \leq 1 \) and \( \Phi(r) = 0 \) as \( r \geq 2 \). For \( i \in I^e \) we denote
\[
\varphi_i^\varepsilon(x) = \Phi \left( \frac{|x - x_i^\varepsilon| + \frac{\varepsilon}{2} - 2r^\varepsilon}{\frac{\varepsilon}{2} - r^\varepsilon} \right).
\]
It is clear that
\[
supp(\varphi_i^\varepsilon) \subset Y_i^\varepsilon, \quad \varphi_i^\varepsilon = 1 \text{ in } D_i^\varepsilon \cup B_i^\varepsilon, \quad supp(D^m \varphi_i^\varepsilon) \subset Y_i^\varepsilon \setminus D_i^\varepsilon \cup B_i^\varepsilon \text{ and } |D^m \varphi_i^\varepsilon| \leq \frac{C}{\varepsilon^m} (m \neq 0).
\]

By \( v_i^\varepsilon(x) \) we denote the following function:
\[
v_i^\varepsilon(x) = \begin{cases} 
1 - \frac{A^\varepsilon}{|x - x_i^\varepsilon|^{n-2}} + B^\varepsilon, & |x - x_i^\varepsilon| \leq r^\varepsilon - d^\varepsilon, \\
0, & r^\varepsilon - d^\varepsilon < |x - x_i^\varepsilon| < r^\varepsilon, \\
\end{cases} \quad (2.10)
\]
where
\[
A^\varepsilon = \left( \frac{1}{(r^\varepsilon - d^\varepsilon)^{n-2}} - \frac{1}{(r^\varepsilon)^{n-2}} \right)^{-1} \sim \frac{(r^\varepsilon)^{n-1}}{(n-2)d^\varepsilon}, \quad B^\varepsilon = -\frac{A^\varepsilon}{(r^\varepsilon)^{n-2}}. \quad (2.11)
\]
Here we use the fact that \( d^\varepsilon = o(\varepsilon) \).

It is easy to see that \( v_i^\varepsilon \) is a continuous and piecewise smooth function, \( supp(v_i^\varepsilon) \subset D_i^\varepsilon \cup B_i^\varepsilon \). Using (1.6), (1.7) one can easily check that \( v_i^\varepsilon \) satisfies the following properties:
\[
\int_{Y_i^\varepsilon} \alpha^\varepsilon |\nabla v_i^\varepsilon|^2 \, dx = \int_{D_i^\varepsilon} \alpha^\varepsilon |\nabla v_i^\varepsilon|^2 \, dx \sim p\varepsilon^{n-1} \text{ as } \varepsilon \to 0, \quad (2.12)
\]
\[
\int_{Y_i^\varepsilon} |v_i^\varepsilon|^2 b^\varepsilon \, dx = \int_{B_i^\varepsilon} \beta^\varepsilon \, dx + O(d^\varepsilon \varepsilon^{n-1}) \sim \rho\varepsilon^{n-1} \text{ as } \varepsilon \to 0. \quad (2.13)
\]

Finally taking arbitrary functions \( w_1 \in C_0^\infty(\Omega), w_2 \in C^\infty(\Gamma) \) we construct the following test-function:
\[
w^\varepsilon(x) = w_1(x) + \sum_{i \in I^e} (w_1(x_i^\varepsilon) - w_1(x)) \varphi_i^\varepsilon(x) + \sum_{i \in I^e} v_i^\varepsilon(x) (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)). \quad (2.14)
\]
It is clear that \( w^\varepsilon \in H_0^1(\Omega) \).

We plug \( w = w^\varepsilon(x) \) into (2.9). At first we study the integral staying at the left-hand-side. One has
\[
\int_{\Omega} \alpha^\varepsilon |\nabla u^\varepsilon|^2 \, dx = \int_{\Omega^e} \nabla u^\varepsilon \cdot \nabla w_1 \, dx + \sum_{i \in I^e} \int_{Y_i^\varepsilon \setminus D_i^\varepsilon \cup B_i^\varepsilon} \nabla u^\varepsilon \cdot \nabla (w_1(x_i^\varepsilon) - w_1(x)) \varphi_i^\varepsilon \, dx + \sum_{i \in I^e} (w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon)) \int_{D_i^\varepsilon} \alpha^\varepsilon \nabla u^\varepsilon \cdot \nabla v_i^\varepsilon \, dx. \quad (2.15)
\]
Due to (2.2), (2.3), (2.5) and since \( \lim_{\varepsilon \to 0} |\Omega \setminus \Omega^e| = 0 \) one has
\[
\int_{\Omega} \nabla u^\varepsilon \cdot \nabla w_1 \, dx = \int_{\Omega} \nabla (\Pi^e u^\varepsilon) \cdot \nabla w_1 \, dx - \int_{\Omega^e} \nabla (\Pi^e u^\varepsilon) \cdot \nabla w_1 \, dx \to \int_{\Omega} \nabla u_1 \cdot \nabla w_1 \, dx. \quad (2.16)
\]
Using (2.2) and the estimates
\[
\left| \nabla (w_1(x_i^\varepsilon) - w_1(x)) \varphi_i^\varepsilon \right| < C, \quad \sum_{i \in I^e} \varepsilon^{n-1} < C
\]
we conclude that the second integral in (2.15) vanishes as $\varepsilon \to 0$:

$$\left| \sum_{i \in I^\varepsilon} \int_{\overline{D^\varepsilon \cup B^\varepsilon_i}} \nabla u^\varepsilon \cdot \nabla \left( (w_1(x^\varepsilon_i) - w_1) \varphi^\varepsilon \right) \, dx \right|^2 \leq C \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \int_{i \in I^\varepsilon} Y_i^\varepsilon \leq C_1 \sum_{i \in I^\varepsilon} \varepsilon^n \leq C_2 \varepsilon. \quad (2.17)$$

Let us now study the third integral in (2.15). Integrating by parts and taking into account that $\Delta v_i^\varepsilon = 0$ in $D^\varepsilon_i$ we get:

$$\sum_{i \in I^\varepsilon} \left( w_2(x^\varepsilon_i) - w_1(x^\varepsilon_i) \right) \int_{D^\varepsilon_i} \alpha^\varepsilon \nabla u^\varepsilon \cdot \nabla v_i^\varepsilon \, dx =$$

$$= \sum_{i \in I^\varepsilon} \left( w_2(x^\varepsilon_i) - w_1(x^\varepsilon_i) \right) \alpha^\varepsilon \left( \int_{S^\varepsilon_i^+} \frac{\partial v_i^\varepsilon}{\partial |x - x^\varepsilon_i|} u^\varepsilon \, ds - \int_{S^\varepsilon_i^-} \frac{\partial v_i^\varepsilon}{\partial |x - x^\varepsilon_i|} u^\varepsilon \, ds \right) =$$

$$= \sum_{i \in I^\varepsilon} \left( w_1(x^\varepsilon_i) - w_2(x^\varepsilon_i) \right) \alpha^\varepsilon \frac{d}{d\varepsilon} \varepsilon^{n-1} \omega_{n-1} \varepsilon^{n-1} (\langle u^\varepsilon \rangle_{S^\varepsilon_i^+} - \langle u^\varepsilon \rangle_{S^\varepsilon_i^-}) \quad (2.18)$$

Recall, that by $\Omega_{n-1}$ we denote the volume of $(n - 1)$-dimensional unit sphere.

**Lemma 2.1.** One has the following estimates: $\forall u \in H^1(Y_i^\varepsilon)$

$$\left| \langle u \rangle_{S^\varepsilon_i^+} - \langle u \rangle_{Y_i^\varepsilon} \right|^2 \leq C \varepsilon^{2-n} \|\nabla u\|^2_{Y_i^\varepsilon}, \quad (2.19)$$

$$\left| \langle u \rangle_{S^\varepsilon_i^-} - \langle u \rangle_{B^\varepsilon_i} \right|^2 \leq C \varepsilon^{2-n} \|\nabla u\|^2_{B_i^\varepsilon}. \quad (2.20)$$

**Proof.** We denote $v = u - \langle u \rangle_{B_i^\varepsilon}$. One has the following standard trace inequality (see, e.g., [2]):

$$\|v\|^2_{L^2(S_i^\varepsilon)} \leq C \left( \varepsilon^{-1} \|v\|^2_{L^2(B_i^\varepsilon)} + \varepsilon \|\nabla v\|^2_{L^2(B_i^\varepsilon)} \right). \quad (2.21)$$

Then using (2.21), the Cauchy inequality and the Poincare inequality

$$\|u - \langle u \rangle_{B_i^\varepsilon}\|^2_{L^2(B_i^\varepsilon)} \leq C \varepsilon^2 \|\nabla u\|^2_{L^2(B_i^\varepsilon)}$$

we obtain:

$$\left| \langle u \rangle_{S^\varepsilon_i^-} - \langle u \rangle_{B_i^\varepsilon} \right|^2 = \left| \langle v \rangle_{S^\varepsilon_i^-} \right|^2 \leq \frac{1}{|S_i^\varepsilon^-|} \|v\|^2_{L^2(S_i^\varepsilon^-)} \leq$$

$$\leq C \left( \varepsilon^{-n} \|u - \langle u \rangle_{B_i^\varepsilon}\|^2_{L^2(B_i^\varepsilon)} + \varepsilon^{2-n} \|\nabla u\|^2_{L^2(B_i^\varepsilon)} \right) \leq C_1 \varepsilon^{2-n} \|\nabla u\|^2_{L^2(B_i^\varepsilon)}$$

and (2.20) is proved.

In the same way we prove the estimates

$$\left| \langle u \rangle_{S^\varepsilon_i^+} - \langle u \rangle_{Y_i^\varepsilon} \right|^2 \leq C \varepsilon^{2-n} \|\nabla u\|^2_{Y_i^\varepsilon},$$

$$\left| \langle u \rangle_{Y_i^\varepsilon} - \langle u \rangle_{Y_i^\varepsilon} \right|^2 \leq C \varepsilon^{2-n} \|\nabla u\|^2_{Y_i^\varepsilon}, \quad (2.22)$$

whose combination gives (2.19). Lemma is proved. \qed
We introduce the operator $Q^\varepsilon : C^1(\Gamma) \to L_2(\Gamma)$ by the formula
\[
Q^\varepsilon w = \begin{cases} 
  w(x_i^\varepsilon), & x \in \Gamma_i^\varepsilon, \\
  0, & x \in \Gamma \setminus \bigcup_i \Gamma_i^\varepsilon.
\end{cases}
\tag{2.23}
\]
It is easy to see that
\[
\forall w \in C^1(\Gamma) : \lim_{\varepsilon \to 0} Q^\varepsilon w = w \text{ in } L_2(\Gamma).
\tag{2.24}
\]
Taking into account (2.7), (2.8), (2.24) we obtain from (2.18):
\[
\sum_{i \in I^\varepsilon} \left( w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon) \right) \int_{D_i^\varepsilon} \nabla u^\varepsilon \cdot \nabla v_i^\varepsilon \, dx = \sum_{i \in I^\varepsilon} \left( w_1(x_i^\varepsilon) - w_2(x_i^\varepsilon) \right) \frac{d\varepsilon}{d^\varepsilon} r_{n-1}^{\varepsilon} \omega_{n-1} \varepsilon^{n-1} \left( (\Pi_1^\varepsilon u^\varepsilon)_{Y_i^\varepsilon} - (u^\varepsilon)_{B_i^\varepsilon} \right) + \\
+ \delta(\varepsilon) = \frac{d\varepsilon}{d^\varepsilon} r_{n-1}^{\varepsilon} \omega_{n-1} \int_{\Gamma} \left( Q^\varepsilon (w_1 - w_2) \right) (\Pi_1^\varepsilon u^\varepsilon - \Pi_2^\varepsilon u^\varepsilon) \, ds + \delta(\varepsilon)
\tag{2.25}
\]
where the reminder $\delta(\varepsilon)$ vanishes as $\varepsilon \to 0$, namely applying (2.19) for $u = \Pi_1^\varepsilon u^\varepsilon$ and (2.19) for $u = u^\varepsilon$ and taking into account that $a^\varepsilon = O(d^\varepsilon)$ (since $a < \infty$) we obtain:
\[
|\delta(\varepsilon)|^2 \leq C \varepsilon^{n-1} \sum_{i \in I^\varepsilon} \left( |(u^\varepsilon)_{S_i^\varepsilon} - (\Pi_2^\varepsilon u^\varepsilon)_{Y_i^\varepsilon}|^2 + |(u^\varepsilon)_{S_i^\varepsilon} - (u^\varepsilon)_{B_i^\varepsilon}|^2 \right) \\
\leq C \varepsilon \sum_{i \in I^\varepsilon} \left( \|\nabla \Pi_1^\varepsilon u^\varepsilon\|^2_{L_2(Y_i^\varepsilon)} + \|\nabla u^\varepsilon\|^2_{L_2(B_i^\varepsilon)} \right) \leq C_1 \varepsilon.
\]
Thus we obtain, using (2.6), (2.7), (2.24),
\[
\sum_{i \in I^\varepsilon} \left( w_2(x_i^\varepsilon) - w_1(x_i^\varepsilon) \right) \int_{D_i^\varepsilon} \nabla u^\varepsilon \cdot \nabla v_i^\varepsilon \, dx = \frac{d\varepsilon}{d^\varepsilon} r_{n-1}^{\varepsilon} \omega_{n-1} \int_{\Gamma} \left( Q^\varepsilon (w_1 - w_2) \right) (\Pi_1^\varepsilon u^\varepsilon - \Pi_2^\varepsilon u^\varepsilon) \, ds + O(\varepsilon) \\
\to p \int_{\Gamma} (w_1 - w_2)(u_1 - u_2) \, ds \text{ as } \varepsilon \to 0.
\tag{2.26}
\]
It follows from (2.15)-(2.17), (2.26) that
\[
\lim_{\varepsilon \to 0} \int_{\Omega^\varepsilon} a^\varepsilon \nabla u^\varepsilon \cdot \nabla w^\varepsilon \, dx = \int_{\Omega} \nabla u_1 \cdot \nabla w_1 \, dx + \int_{\Gamma} p(u_1 - u_2)(w_1 - w_2) \, ds.
\tag{2.27}
\]
Now, let us consider the right-hand-side of (2.9). One has:
\[
\lambda^\varepsilon \int_{\Omega} b^\varepsilon u^\varepsilon w^\varepsilon \, dx = \lambda^\varepsilon \left( \int_{\Omega^\varepsilon} u^\varepsilon w_1 \, dx + \sum_{i \in I^\varepsilon} \int_{(D_i^\varepsilon \cup B_i^\varepsilon)} u^\varepsilon \left( w_1(x_i^\varepsilon) - w_1 \right) \varphi_i^\varepsilon \, dx + \\
+ \sum_{i \in I^\varepsilon} \int_{D_i^\varepsilon} u^\varepsilon w_2 \, dx + \sum_{i \in I^\varepsilon} \int_{B_i^\varepsilon} b^\varepsilon u^\varepsilon w_2(x_i^\varepsilon) \, dx \right).
\tag{2.28}
\]
It is clear that
\[
\int_{\Omega^\varepsilon} u^\varepsilon w_1 \, dx \to \int_{\Omega} u^\varepsilon w_1 \, dx \text{ as } \varepsilon \to 0
\tag{2.29}
\]
and the next two integrals in (2.28) vanishes since $|w^e| < C$:

$$\left| \sum_{i \in I} \int_{Y_i^\varepsilon \setminus D_i^\varepsilon \cup B_i^\varepsilon} u^e \left( w_1(x_i^e) - w_1 \right) \varphi_i^e dx + \sum_{i \in I} \int_{D_i^\varepsilon} u^e w^e dx \right|^2 \leq C \sum_{i \in I} |Y_i^\varepsilon \setminus B_i^\varepsilon| \sum_{i \in I} \|u^e\|^2_{L^2(Y_i^\varepsilon \setminus B_i^\varepsilon)} = O(\varepsilon) \text{ as } \varepsilon \to 0. \quad (2.30)$$

Finally we study the last integral in (2.28). Using the equality $|B_i^\varepsilon| = (r^e - d^e)^n \kappa_n \sim r^e \kappa_n e^n$ (recall, that by $\kappa_n$ we denote the volume of $n$-dimensional unit ball) we get:

$$\sum_{i \in I} \int_{B_i^\varepsilon} u^e w_2(x_i^e) dx = e \beta^e \int_{B_i^\varepsilon} (e^\varepsilon)_{B_i^\varepsilon} w_2(x_i^e) e^{\alpha_1-1} = e \beta^e (r^e - d^e)^n \int_\Gamma \Pi_1^e u^e \varphi_2 w_2 ds \to \rho \int_\Gamma u_2 w_2 ds. \quad (2.31)$$

It follows from (2.28)-(2.31) and lim $\lambda^e = \lambda$ that

$$\lim_{\varepsilon \to 0} \left( \lambda^e \int_\Omega u^e w^e dx \right) = \lambda \left( \int_\Omega u_1 w_1 dx + \rho \int_\Gamma u_2 w_2 ds \right). \quad (2.32)$$

Finally, combining (2.9), (2.27) and (2.32) we get

$$\int_\Omega \nabla u_1 \cdot \nabla w_1 dx + \int_\Gamma \rho(u_1 - u_2)(w_1 - w_2) ds = \lambda \left( \int_\Omega u_1 w_1 dx + \rho \int_\Gamma u_2 w_2 ds \right). \quad (2.33)$$

Since $C_0^\infty(\Omega) \oplus C^\infty(\Gamma)$ is densely embedded into $H^1_0(\Omega) \oplus L^2(\Gamma)$ then equality (2.33) is valid for an arbitrary $(w_1, w_2) \in \operatorname{dom}(\eta)$, and therefore, using (1.8) and (1.9),

$$U \in \operatorname{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A} U = \lambda U,$$

i.e. $\lambda$ is an eigenvalue of $\mathcal{A}$.

**Case 2.** $u_1 = 0$. We will show that in this case $\lambda = q$ (and therefore by Lemma 1.1 $\lambda \in \sigma(\mathcal{A})$).

We express the eigenfunction $u^e$ in the form

$$u^e = v^e - g^e + \delta^e$$

where

$$v^e = \sum_{i \in I} (u_i^e)_{B_i^\varepsilon} \varphi_i^e, \quad g^e = \sum_{k=1}^{n-1} (v_k^e, u_k^e)_{H^1_0} u_k^e, \quad (2.34)$$

where the function $v^e_i$ is again defined by (2.10). It is clear that $v^e, g^e \in \operatorname{dom}(\mathcal{A})$, supp$(v^e) \subset \bigcup_{i \in I} Y_i^\varepsilon$.

At first we obtain some estimates for the eigenfunction $u^e$.

For any $u \in H^1(Y_i^\varepsilon)$ one has the estimate (see [18, Lemma 4.3])

$$\|u\|^2_{Y_i^\varepsilon} \leq C \left( (d^e)^2 \|u\|^2_{L^2(D_i^\varepsilon)} + \varepsilon d^e \|\nabla u\|^2_{L^2(Y_i^\varepsilon)} + \varepsilon^{-1} d^e \|u\|^2_{L^2(Y_i^\varepsilon)} \right). \quad (2.35)$$
Recall that \((d^p)^2 = o(\alpha^p)\) (see (1.6)). Using this, (2.2) and taking into account that \(\sum_{i \in I^e} \|\nabla u^e\|_{D_i^e}^2 \leq (\alpha^p)^{-1} \eta^e [u^e, u^e] \leq C(\alpha^p)^{-1}\), we obtain from (2.35):

\[
\lim_{\varepsilon \to 0} \sum_{i \in I^e} \|u^e\|^2_{L^2(D_i^e)} = 0. \tag{2.36}
\]

Finally using the Poincaré inequality and (2.2) we obtain:

\[
\sum_{i \in I^e} \|u^e - \langle u^e\rangle_{B_i^e}\|^2_{L^2(B_i^e)} = O(\varepsilon^2) \text{ as } \varepsilon \to 0. \tag{2.37}
\]

Since \(e\beta^e = O(1)\) then (2.37) implies

\[
\sum_{i \in I^e} \beta^e ||u^e - \langle u^e\rangle_{B_i^e}||^2_{L^2(B_i^e)} = O(\varepsilon) \text{ as } \varepsilon \to 0. \tag{2.38}
\]

Using the fact that \(||u^e||_{L^2(\Omega)} \leq C||\Pi_{i}u^e||_{L^2(\Omega)} \to ||u_1||_{L^2(\Omega)} = 0\) and (2.1), (2.36), (2.38) we obtain

\[
1 = \sum_{i \in I^e} \|\langle u^e\rangle_{B_i^e}\|^2 |B_i^e| e^2 \rho^{e-1} + o(1) = \rho \sum_{i \in I^e} \|\langle u^e\rangle_{B_i^e}\|^2 e^{n-1} + o(1) \quad (\varepsilon \to 0). \tag{2.39}
\]

Using estimates (2.12), (2.13) for \(v_i^e\) and taking into account (2.39) we obtain the following estimates for \(\nu^e\):

\[
\eta^e[v^e, v^e] = \left( e^{1-n} \int_{\gamma_i^e} \alpha^e(x) \left| \nabla v_i^e \right|^2 dx \right) \sum_{i \in I^e} \|\langle u^e\rangle_{B_i^e}\|^2 e^{n-1} \sim \frac{D}{\rho^e} = q \quad (\varepsilon \to 0), \tag{2.40}
\]

\[
\|v^e\|^2_{H^0} = \left( e^{1-n} \int_{\gamma_i^e} b^e(x) |v_i^e|^2 dx \right) \sum_{i \in I^e} \|\langle u^e\rangle_{B_i^e}\|^2 e^{n-1} \sim \frac{D}{\rho^e} = 1 \quad (\varepsilon \to 0), \tag{2.41}
\]

\[
\sum_{i \in I^e} \|v_i^e||^2_{L^2(D_i^e)} \leq C|D_i^e| e^{1-n} \sum_{i \in I^e} \|\langle u^e\rangle_{B_i^e}\|^2 e^{n-1} \leq C_1 d^e, \tag{2.42}
\]

Here we also use the fact that \(\rho q = p\). Also we note that

\[
\nu^e = \langle u^e\rangle_{B_i^e} \text{ in } B_i^e, \quad \nu_i^e = 0 \text{ in } \Omega^e. \tag{2.43}
\]

Using the Bessel inequality we can estimate the function \(g^e\) as follows:

\[
\|g^e\|^2_{H^0} = \sum_{k=1}^{n^e-1} \left( \nu_k^e, u_k^e \right)_H^2 = \sum_{k=1}^{n^e-1} \left( \nu^e - u^e, u_k^e \right)_H^2 \leq \|\nu^e - u^e\|^2_{H^0},
\]

\[
\eta^e[g^e, g^e] = (\lambda_k^e)^2 \sum_{k=1}^{n^e-1} \left( \nu_k^e, u_k^e \right)_H^2 = (\lambda^e)^2 \sum_{k=1}^{n^e-1} \left( \nu^e - u^e, u_k^e \right)_H^2 \leq (\lambda^e)^2 \|\nu^e - u^e\|^2_{H^0}.
\]

In view of (2.36), (2.38), (2.42), (2.43) and the fact that \(u_1 = 0\) one has

\[
\|u^e - \nu^e\|^2_{H^0} = ||d||^2_{L^2(\Omega^e)} + \sum_{i \in I^e} \|u^e - \nu^e\|^2_{L^2(D_i^e)} + \sum_{i \in I^e} \beta^e ||u^e - \langle u^e\rangle_{B_i^e}||^2_{L^2(B_i^e)} \to 0 \text{ as } \varepsilon \to 0 \tag{2.44}
\]

and therefore

\[
\|g^e\|^2_{H^0} + \eta^e[g^e, g^e] \to 0 \text{ as } \varepsilon \to 0. \tag{2.45}
\]
Now let us estimate the remainder $\delta^\varepsilon$. We denote $\bar{v}^\varepsilon = v^\varepsilon - g^\varepsilon$. Since $\bar{v}^\varepsilon \in \{u_1^\varepsilon, \ldots, u_{n^\varepsilon - 1}^\varepsilon\}$ then by the min-max principle (see, e.g., [28])

$$\lambda^\varepsilon = \eta^\varepsilon [u^\varepsilon, u^\varepsilon] \leq \frac{\eta^\varepsilon [\bar{v}^\varepsilon, \bar{v}^\varepsilon]}{||\bar{v}^\varepsilon||^2_{L^2_\varepsilon}}.$$  

or equivalently, using $u^\varepsilon = \bar{v}^\varepsilon + \delta^\varepsilon$,

$$\eta^\varepsilon [\delta^\varepsilon, \delta^\varepsilon] \leq -2\eta^\varepsilon [\bar{v}^\varepsilon, \delta^\varepsilon] + \eta^\varepsilon [\bar{v}^\varepsilon, \bar{v}^\varepsilon] \left(||\bar{v}^\varepsilon||^2_{H^2_\varepsilon} - 1\right).$$  

(2.46)

In view of (2.40), (2.41), (2.45)

$$\eta^\varepsilon [\bar{v}^\varepsilon, \bar{v}^\varepsilon] \left(||\bar{v}^\varepsilon||^2_{H^2_\varepsilon} - 1\right) \to 0 \text{ as } \varepsilon \to 0. \tag{2.47}$$

Now let us estimate the first term on the right-hand-side of (2.46). One has

$$\eta^\varepsilon [\bar{v}^\varepsilon, \delta^\varepsilon] \to \eta^\varepsilon [v^\varepsilon, u^\varepsilon - v^\varepsilon] + \eta^\varepsilon [v^\varepsilon, g^\varepsilon] - \eta^\varepsilon [g^\varepsilon, \delta^\varepsilon].$$  

(2.48)

Integrating by parts and using (2.10), (2.11), (2.43) we get:

$$\eta^\varepsilon [v^\varepsilon, u^\varepsilon - v^\varepsilon] = \sum_{i \in I^\varepsilon} \int_{D_i^\varepsilon} a^\varepsilon \nabla v^\varepsilon_i \cdot \nabla (u^\varepsilon - v^\varepsilon) dx =$$

$$= a^\varepsilon \langle u^\varepsilon \rangle_{B_i^\varepsilon} \sum_{i \in I^\varepsilon} \left( \int_{\gamma_i^\varepsilon} \frac{\partial v^\varepsilon_i}{\partial |x - x_i^\varepsilon|} u^\varepsilon_i ds - \int_{\gamma_i^\varepsilon} \frac{\partial v^\varepsilon_i}{\partial |x - x_i^\varepsilon|} (u^\varepsilon - \langle u^\varepsilon \rangle_{B_i^\varepsilon}) ds \right) \sim$$

$$\sim \frac{a^\varepsilon}{\varepsilon} \delta^{n-1} \omega_{n-1} \sum_{i \in I^\varepsilon} \langle u^\varepsilon \rangle_{B_i^\varepsilon} \left( \langle u^\varepsilon \rangle_{S_i^{\varepsilon, +}} - \langle u^\varepsilon \rangle_{S_i^{\varepsilon, -}} \right) e^{n-1}.$$

(2.49)

Then, using the Cauchy inequality, (2.2), (2.8), Lemma 2.1 and the fact that $u_1 = 0$, we obtain from (2.49):

$$\left| \eta^\varepsilon [v^\varepsilon, u^\varepsilon - v^\varepsilon] \right|^2 \leq C \left\{ \sum_{i \in I^\varepsilon} \left| \langle u^\varepsilon \rangle_{B_i^\varepsilon} \right|^2 e^{n-1} \right\} \left\{ \sum_{i \in I^\varepsilon} \left( \left| \langle u^\varepsilon \rangle_{S_i^{\varepsilon, +}} \right|^2 + \left| \langle u^\varepsilon \rangle_{S_i^{\varepsilon, -}} - \langle u^\varepsilon \rangle_{B_i^\varepsilon} \right|^2 \right) e^{n-1} \right\} \leq$$

$$\leq C_1 \sum_{i \in I^\varepsilon} \left( \left| \Pi_{i^\varepsilon} u^\varepsilon \right|^2_{L^2(\Gamma_i^\varepsilon)} + \varepsilon \left| \nabla \Pi_{i^\varepsilon} u^\varepsilon \right|^2_{L^2(\Gamma_i^\varepsilon)} + \varepsilon \left| \nabla u^\varepsilon \right|^2_{L^2(\Gamma_i^\varepsilon)} \right) \to 0 \text{ as } \varepsilon \to 0. \tag{2.50}$$

Further, in view of (2.40), (2.45)

$$\lim_{\varepsilon \to 0} \eta^\varepsilon [v^\varepsilon, g^\varepsilon] = 0. \tag{2.51}$$

And, finally using (2.1), (2.40), (2.45) we obtain:

$$\left| \eta^\varepsilon [g^\varepsilon, \delta^\varepsilon] \right| \leq \left| \eta^\varepsilon [g^\varepsilon, u^\varepsilon] \right| + \left| \eta^\varepsilon [g^\varepsilon, v^\varepsilon] \right| + \left| \eta^\varepsilon [g^\varepsilon, g^\varepsilon] \right| \to 0 \text{ as } \varepsilon \to 0. \tag{2.52}$$

It follows from (2.48), (2.50), (2.52) that

$$\lim_{\varepsilon \to 0} \eta^\varepsilon [\bar{v}^\varepsilon, \delta^\varepsilon] = 0. \tag{2.53}$$

Combining (2.46), (2.47), (2.53) we conclude that

$$\lim_{\varepsilon \to 0} \eta^\varepsilon [\delta^\varepsilon, \delta^\varepsilon] = 0. \tag{2.54}$$
Then it follows from (2.1), (2.40), (2.45), (2.54) that
\[ \lambda = \lim_{\varepsilon \to 0} \lambda^\varepsilon = \lim_{\varepsilon \to 0} \eta^\varepsilon[u^\varepsilon, u^\varepsilon] = \lim_{\varepsilon \to 0} \eta^\varepsilon[v^\varepsilon, v^\varepsilon] = q \]
and hence \( \lambda \in \sigma(A) \).

2.1.2. Let \( \lambda \in \sigma(A) \). We have to prove that there exist \( \lambda^\varepsilon \in \sigma(A^\varepsilon) \) such that \( \lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda \).

Proving this indirectly we assume the opposite. Then some subsequence (still denoted by \( \varepsilon \)) exists and a positive number \( \delta \) exists such that
\[ (\lambda - \delta, \lambda + \delta) \cap \sigma(A^\varepsilon) = \emptyset. \tag{2.55} \]
Since \( \lambda \in \sigma(A) \) there exists \( F = (f_1, f_2) \in H \), such that that \( F \notin \text{im}(A - \lambda I) \).

We introduce the function \( f^\varepsilon \in H^\varepsilon \) by the formula
\[ f^\varepsilon(x) = \begin{cases} f_1(x), & x \in \Omega^\varepsilon, \\ 0, & x \in \bigcup_{i \in I^\varepsilon} D_i^\varepsilon, \\ (f_2)_{i^\varepsilon}, & x \in B_i^\varepsilon. \end{cases} \]

One has:
\[ \|f^\varepsilon\|_{H^\varepsilon}^2 = \|f_1\|_{\Omega^\varepsilon}^2 + \sum_{i \in I^\varepsilon} \beta_i \|B_i^\varepsilon\| \|f_i\|_{D_i^\varepsilon}^2 \leq \|f_1\|_{L^2(\Omega)}^2 + C\|f_2\|_{L^2(\Gamma)}^2 \leq C_1 \|F\|_{H^*}^2. \]

In view of (2.55) \( \lambda \) is in the resolvent set of \( A^\varepsilon \) and therefore there exists a unique \( u^\varepsilon \in \text{dom}(A^\varepsilon) \) such that
\[ A^\varepsilon u^\varepsilon - \lambda u^\varepsilon = f^\varepsilon \]
and moreover the following estimate is valid:
\[ \|u^\varepsilon\|_{H^\varepsilon} \leq \delta^{-1} \|f^\varepsilon\|_{H^\varepsilon} \leq C_1, \tag{2.56} \]
\[ \eta^\varepsilon[u^\varepsilon] \leq \lambda \|u^\varepsilon\|_{H^\varepsilon} + (f^\varepsilon, u^\varepsilon)_{H^\varepsilon} \leq C_2. \tag{2.57} \]

Then it follows from (2.56)-(2.57) that there is a subsequence (still denoted by \( \varepsilon \)) and \( u_1 \in H^1_0(\Omega), u_2 \in L_2(\Gamma) \) such that
\[ \Pi_1^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } H_0^1(\Omega), \quad \Pi_2^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } L_2(\Omega), \quad \Pi_1^\varepsilon u^\varepsilon \rightharpoonup u_1 \text{ in } L_2(\Gamma), \quad \Pi_2^\varepsilon u^\varepsilon \rightharpoonup u_2 \text{ in } L_2(\Gamma). \]

For an arbitrary \( w \in H^1_0(\Omega) \) one has
\[ \int_{\Omega} b^\varepsilon \nabla u^\varepsilon \cdot \nabla w dx - \lambda \int_{\Omega} b^\varepsilon u^\varepsilon w dx = \int_{\Omega} b^\varepsilon f^\varepsilon w dx. \tag{2.58} \]

We plug into (2.58) the function \( w = w^\varepsilon(x) \) defined by formula (2.14) and pass to the limit as \( \varepsilon \to 0 \). In the same way as above we obtain that \( (u_1, u_2) \) satisfies the equality
\[ \int_{\Omega} \nabla u_1 \cdot \nabla w_1 dx + \int_{\Gamma} p(u_1 - u_2)(w_1 - w_2) ds - \lambda \left( \int_{\Omega} u_1 w_1 dx + \rho \int_{\Gamma} u_2 w_2 ds \right) = \int_{\Omega} f_1 w_1 dx + \rho \int_{\Gamma} f_2 w_2 ds. \tag{2.59} \]

which holds for an arbitrary \((w_1, w_2) \in C_0^\infty(\Omega) \oplus C_0^\infty(\Gamma) \) (and by the density arguments for an arbitrary \((w_1, w_2) \in \text{dom}(\eta)) \).
It follows from (2.59) that
\[ U = (u_1, u_2) \in \text{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}U - \lambda U = F, \]
We obtain a contradiction. Thus there is \( \lambda^c \in \sigma(\mathcal{A}^c) \) such that \( \lim_{\varepsilon \to 0} \lambda^c = \lambda. \)

Theorem 1.1 is proved.

2.2. Spectrum of operator \( \mathcal{A} \). The last subsection is devoted to the proof of Lemma 1.1.

At first we study the discrete spectrum of \( \mathcal{A} \). Let \( \lambda \neq q \) be the eigenvalue of \( \mathcal{A} \) corresponding to the eigenfunction \( U = (u_1, u_2) \in \mathcal{H} \). It means that
\[
\int_{\Omega} \nabla u_1 \cdot \nabla v_1 \, dx + p \int_{\Gamma} (u_1 - u_2)(v_1 - v_2) \, ds = \lambda \left( \int_{\Omega} u_1 v_1 \, dx + p q^{-1} \int_{\Gamma} u_2 v_2 \, ds \right), \quad \forall v = (v_1, v_2) \in \mathcal{H}.
\]
(2.60)

One can easily derive the following lemma.

**Lemma 2.2.** that if \( U = (u_1, u_2) \in \mathcal{H} \) satisfies (2.60) then \( u_1 \) satisfies
\[
\int_{\Omega} \nabla u_1 \cdot \nabla v_1 \, dx - \frac{\lambda p}{q - \lambda} \int_{\Gamma} u_1 v_1 \, ds = \lambda \int_{\Omega} u_1 v_1 \, dx, \quad \forall v_1 \in H_0^1(\Omega).
\]
(2.61)

Conversely if \( u_1 \in H_0^1(\Omega) \) satisfies (2.61) then \( U = (u_1, u_2) \), where \( u_2 = \frac{q u_1}{q - \lambda} \), satisfies (2.60).

Let \( \mu \in \mathbb{R} \). By \( \eta^\mu \) we denote the sesquilinear form in \( L_2(\Omega) \) defined as follows
\[
\eta^\mu[u, v] = \int_{\Omega} \nabla u \cdot \nabla v - \mu \int_{\Gamma} uv \, ds, \quad \text{dom}(\eta^\mu) = H_0^1(\Omega).
\]
We denote by \( \mathcal{A}^\mu \) the operator generated by this form. Formally the eigenvalue problem \( \mathcal{A}^\mu u = \lambda u \) can be written as
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega \setminus \Gamma, \\
(u^+) = (u^-) & \text{on } \Gamma, \\
\left( \frac{\partial u}{\partial n}^+\right)^- - \left( \frac{\partial u}{\partial n}^-\right)^- - \mu u = 0 & \text{on } \Gamma.
\end{cases}
\]
(2.62)

The spectrum of \( \mathcal{A}^\mu \) is purely discrete. We denote by
\[
0 < \lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_k(\mu) \leq \cdots \to \infty
\]
the sequence of eigenvalues of \( \mathcal{A}^\mu \) repeated according to their multiplicity. By \( \{u_k(\mu)\}_{k=1}^\infty \) we denote the corresponding set of eigenfunctions satisfying \( (u_k(\mu), u_l(\mu))_{L_2(\Omega)} = \delta_{kl}. \)

We denote by \( \sigma_\mu(\mathcal{A}) \) the set of eigenvalues of the operator \( \mathcal{A} \). It follows from Lemma 2.2 that
\[
\sigma_\mu(\mathcal{A}) = \left\{ \lambda \in \mathbb{R} : \lambda = \lambda_k(\mu) = \frac{q \mu}{\mu + p} \text{ for some } \mu \in \mathbb{R} \text{ and some } k \in \mathbb{N} \right\}
\]
(2.63)

In the next three lemmas we establish some properties of the spectrum of the operator \( \mathcal{A}^\mu \).

**Lemma 2.3.** For each fixed \( k \in \mathbb{N} \) the function \( \left\{ \begin{array}{ccl} \mathbb{R} & \to & \mathbb{R} \\ \mu & \mapsto & \lambda_k(\mu) \end{array} \right\} \) is continuous and monotonically decreasing.
Proof. One has the following min-max principle (see, e.g., [15]):

\[ \lambda_k(\mu) = \min_{\mathcal{H} \in \mathcal{L}_k} \max_{\|u\|_{L_2(\Omega)}^2} \frac{\eta^\mu[u]}{\|u\|_{L_2(\Omega)}^2}, \quad k = 1, 2, 3 \ldots \]  

(2.64)

where \( L_k \) is a set of all \( k \)-dimensional subspaces of \( H_0^1(\Omega) \). Then the monotonicity follows easily from (2.64) and the monotonicity (for fixed \( u \)) of the function \( \mu \mapsto \eta^\mu[u] \).

Now let us prove continuity. Let \([\mu_0, \mu_1] \subset \mathbb{R}\) be an arbitrary compact interval. We choose some \( \eta > 0 \) such that

\[ \mu_1 \leq \frac{1}{2} \]  

(2.65)

(if \( \mu_1 < 0 \) we can choose an arbitrary \( \eta > 0 \).) By the usual trace inequality (see, e.g., [2]) there exists \( C_\eta > 0 \) such that

\[ \forall u \in H^1(\Omega) : \quad p\|u\|^2_{L_2(\Gamma)} \leq \eta \|
abla u\|_{L_2(\Omega)}^2 + C_\eta \|u\|_{L_2(\Omega)}^2. \]  

(2.66)

Now, let \( \mu, \tilde{\mu} \in [\mu_0, \mu_1] \), \( \mu \leq \tilde{\mu} \). We denote for abbreviation:

\[ \alpha := 1 + \frac{\eta(\mu - \tilde{\mu})}{1 - \eta \tilde{\mu}}, \quad \beta := \frac{C_\eta(\tilde{\mu} - \mu)}{1 - \eta \tilde{\mu}}. \]  

(2.67)

In view of (2.65) \( \alpha \) and \( \beta \) are positive.

For each \( u \in H^1_0(\Omega) \setminus \{0\} \) we obtain, using (2.66),

\[ (1 - \alpha)\|
abla u\|^2_{L_2(\Omega)} + (\alpha \tilde{\mu} - \mu)p\|u\|^2_{L_2(\Gamma)} \leq \leq \leq (1 - \alpha + (\alpha \tilde{\mu} - \mu)\eta)\|
abla u\|^2_{L_2(\Omega)} + (\alpha \tilde{\mu} - \mu)C_\eta\|u\|^2_{L_2(\Omega)} \]  

(2.68)

and therefore

\[ \frac{\eta^\mu[u]}{\|u\|_{L_2(\Omega)}^2} \leq \alpha \frac{\eta^{\tilde{\mu}}[u]}{\|u\|_{L_2(\Omega)}^2} + \beta. \]  

(2.69)

It follows from (2.64) and (2.68) that for each fixed \( k \in \mathbb{N} \)

\[ \lambda_k(\mu) \leq \alpha \lambda_k(\tilde{\mu}) + \beta, \]  

Using (2.65), (2.66) and the monotonicity of \( \lambda_k(\cdot) \), we obtain:

\[ 0 \leq \lambda_k(\mu) - \lambda_k(\tilde{\mu}) \leq \frac{\eta \lambda_k(\mu_0) + C_\eta(\tilde{\mu} - \mu)}{1 - \eta \mu_1} \leq 2(\eta \lambda_k(\mu_0) + C_\eta)(\tilde{\mu} - \mu) \]  

(2.69)

which implies the desired continuity on the interval \([\mu_0, \mu_1]\). Since this interval was chosen arbitrarily then we prove the continuity on the whole axis. The lemma is proved.

We denote by \( \mathcal{A}^N \) and \( \mathcal{A}^D \) the operators acting in \( L_2(\Omega) \) and generated by the forms \( \eta^N \) and \( \eta^D \), correspondingly, which are defined as follows:

\[ \text{dom}(\eta^N) = H_0^1(\Omega), \quad \text{dom}(\eta^D) = \{ u \in H_0^1(\Omega) : u = 0 \text{ on } \Gamma \}, \quad \eta^N[u, v] = \eta^D[u, v] = \int_\Omega \nabla u \cdot \nabla v \, dx. \]

We denote by \( \lambda^N_k \) and \( \lambda^D_k \) the sequences of eigenvalues of \( \mathcal{A}^N \) and \( \mathcal{A}^D \) written in the increasing order and with account of their multiplicity.
Lemma 2.4. For each $k \in \mathbb{N}$

$$\lambda_k(\mu) \nearrow \lambda^D_k \text{ as } \mu \to -\infty,$$

(2.70)

Moreover, along a subsequence,

$$u_k(\mu) \to u^D_k \text{ in } H^1(\Omega) \text{ as } \mu \to -\infty$$

(2.71)

for each $k \in \mathbb{N}$, where $u^D_k$ is a normalized eigenfunction of $\mathcal{A}^D$ associated with $\lambda^D_k$.

Proof. We denote by $L^D_k$ the set of all $k$-dimensional subspaces of $\text{dom}(\eta^D)$. It is clear that $L^D_k \subset L_k$.

By the min-max principle we have for each $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$:

$$\lambda^D_k = \min \max_{H \in L^D_k, \mu \in \mathbb{R}} \frac{\eta^D[H]}{||H||_{L^2(\Omega)}} = \min \max_{H \in L^D_k, \mu \in \mathbb{R}} \frac{\eta^\mu[H]}{||H||^2_{L^2(\Omega)}} \geq \min \max_{H \in L^D_k, \mu \in \mathbb{R}} \frac{\eta^\mu[H]}{||H||^2_{L^2(\Omega)}} = \lambda_k(\mu).$$

(2.72)

We prove the assertion of the lemma by induction in $k$.

- $k = 1$. Using (2.72) we obtain

$$\lambda^D_1 \geq \lambda_1(\mu) = ||\nabla u_1(\mu)||^2_{L^2(\Omega)} - \mu p ||u_1||^2_{L^2(\Gamma)}$$

(2.73)

and therefore the set $\{u_1(\mu)\}_{\mu \in (-\infty, 0]}$ is bounded in $H^1(\Omega)$. Then using the compactness of embedding $H^1(\Omega) \subset L^2(\Omega)$ and of the trace map $H^1(\Omega) \to L^2(\Gamma)$ we conclude that there exist $u^D_1 \in H^1_0(\Omega)$ and the subsequence (for convenience still denoted by $\mu$) such that

$$u_1(\mu) \to u^D_1 \text{ in } H^1(\Omega), \quad u_1(\mu) \to u^D_1 \text{ in } L^2(\Omega), \quad u_1(\mu) \to u^D_1 \text{ in } L^2(\Gamma)$$

(2.74)

Moreover it follows from (2.73) that

$$||u^D_1||^2_{L^2(\Gamma)} \leq \frac{\lambda^D_1}{p|\mu|} \to 0 \text{ as } \mu \to -\infty.$$ 

and hence $u^D_1 = 0$ on $\Gamma$, i.e. $u^D_1 \in \text{dom}(\eta^D)$.

Using (2.73), (2.74) and the monotonicity of $\lambda_1(\mu)$ we obtain

$$||\nabla u^D_1||^2_{L^2(\Omega)} \leq \liminf_{\mu \to -\infty} ||\nabla u_1(\mu)||^2_{L^2(\Omega)} \leq \liminf_{\mu \to -\infty} \lambda_1(\mu) \leq \lambda^D_1.$$ 

(2.75)

Furthermore in view of (2.74) $||u^D_1||_{L^2(\Omega)} = 1$, whence

$$\lambda^D_1 = \min_{0 \neq u \in H^1_0(\Omega)} ||\nabla u||^2_{L^2(\Omega)} \leq ||\nabla u^D_1||^2_{L^2(\Omega)},$$

and therefore equality holds everywhere in (2.75). In particular,

$$\lambda^D_1 = \lim_{\mu \to -\infty} \lambda_1(\mu).$$

and

$$||\nabla u^D_1||^2_{L^2(\Omega)} = \lambda^D_1 = \min \{ ||\nabla u||^2_{L^2(\Omega)} \mid u \in H^1_0(\Omega), \quad ||u||_{L^2(\Omega)} = 1, \quad u|_{\Gamma} = 0 \},$$

whence $u_1$ is an eigenfunction associated with $\lambda^D_1$, and finally

$$||\nabla u_1(\mu)||^2_{L^2(\Omega)} \to ||\nabla u^D_1||^2_{L^2(\Omega)}$$

along another subsequence, which together with (2.74) gives

$$u_1(\mu) \to u^D_1 \text{ in } H^1(\Omega).$$
\* \( m - 1 \rightarrow m \). Let \( m \geq 2 \) and let the assertion of the lemma hold for \( k = 1, \ldots, m - 1 \). We prove it for \( k = m \). By the same arguments as used for \( k = 1 \) we conclude that along a subsequence (still denoted by \( \mu \)) there is \( u_m^D \in H_0^1(\Omega) \) such that

\[
    u_m(\mu) \rightarrow u_l^D \text{ in } H^1(\Omega), \quad u_m(\mu) \rightarrow u_l^D \text{ in } L_2(\Omega), \quad u_m^D|\Gamma = 0, \quad \text{(2.76)}
\]

and

\[
    \|\nabla u_m^D\|_{L_2(\Omega)}^2 \leq \liminf_{\mu \rightarrow -\infty} \|\nabla u_m(\mu)\|_{L_2(\Omega)}^2 \leq \lim_{\mu \rightarrow -\infty} \lambda_m(\mu) \leq \lambda_m^D. \quad \text{(2.77)}
\]

Furthermore, since \((u_1(\mu), u_j(\mu))_{L_2(\Omega)} = \delta_{ij} (\forall i, j)\), then, using \((2.76)\) and the induction premise we get

\[
    (u_i(\mu), u_j(\mu))_{L_2(\Omega)} \rightarrow (u_i^D, u_j^D)_{L_2(\Omega)} \text{ for } i, j = 1, \ldots, m.
\]

In particular,

\[
    \|u_m^D\|_{L_2(\Omega)} = 1, \quad (u_i^D, u_j^D)_{L_2(\Omega)} = \delta_{ij}, \quad i, j = 1, \ldots, m - 1.
\]

Thus

\[
    \lambda_m^D = \min \left\{ \frac{\|\nabla u\|_{L_2(\Omega)}^2}{\|u\|_{L_2(\Omega)}^2}, \quad 0 \neq u \in H_0^1(\Omega), \quad u|\Gamma = 0, \quad (u, u^D)_{L_2(\Omega)} = 0 \right\} \leq \|\nabla u_m^D\|_{L_2(\Omega)}^2.
\]

Combining this with \((2.77)\) gives the assertion of the lemma for \( k = m \).

\[\square\]

**Lemma 2.5.** For each \( k \in \mathbb{N} \)

\[
    \lambda_k(\mu) \searrow -\infty \text{ as } \mu \rightarrow \infty. \quad \text{(2.78)}
\]

**Proof.** Let \( m \in \mathbb{N} \). Let \( B_j, j = 1, \ldots, m \) be the open balls with a centres at \( z_j \in \Gamma \) and with the radius \( R \). It is supposed that \( R \) is small enough so that

\[
    B_j \subset \Omega, \quad j = 1, \ldots, m \text{ and } B_i \cap B_j = \emptyset, \quad i \neq j. \quad \text{(2.79)}
\]

Let \( v(x) \) be an arbitrary smooth function such that \( v(x) > 0 \) as \( |x| < R \) and \( v(x) = 0 \) as \( |x| \geq R \). We denote \( v_j(x) = v(x - z_j) \). Since \( \text{supp}(v_j) \subset B_j \) then \( v_j \in H_0^1(\Omega) \). We denote

\[
    U = \text{span}\{v_1, \ldots, v_m\}.
\]

It is clear that \( \dim(U) = m \) and then using \((2.64)\) we get

\[
    \lambda_m(\mu) \leq \max_{u \in U} \frac{\eta^D[u]}{\|u\|_{L_2(\Omega)}^2}. \quad \text{(2.80)}
\]

Let \( 0 \neq \bar{u} \in U \) maximize the quotient staying in the right-hand-side of \((2.80)\). It can be represented in the form \( \bar{u} = \sum_{j=1}^m \alpha_j v_j \), where \( \alpha_j \in \mathbb{R} \), \( \sum \alpha_j^2 > 0 \). Then we get

\[
    \lambda_m(\mu) \leq \frac{\|\nabla v\|_{L_2(\Omega)}^2 - \mu\|v\|_{L_2(\Gamma)}^2}{\|v\|_{L_2(\Omega)}^2} \leq A - \mu B, \quad A, B > 0. \quad \text{(2.81)}
\]

The assertion of lemma follows directly from \((2.81)\).

\[\square\]

Now, with Lemmas \([2.3, 2.5]\) we can easily establish the properties of the set staying in the right-hand-side of \((2.63)\).

**Lemma 2.6.** The set

\[
\{ \lambda \in \mathbb{R} : \lambda = \lambda_k(\mu) = \frac{q\mu}{\mu + p} \text{ for some } \mu \in \mathbb{R} \text{ and } k \in \mathbb{N} \}
\]

consists of two sequences \( \{\lambda_k^-\}_{k \in \mathbb{N}} \) and \( \{\lambda_k^+\}_{k \in \mathbb{N}} \) with the following properties:
(i) \(0 < \lambda_1^+ \leq \lambda_2^+ \leq \cdots \leq \lambda_k^+ \not\rightarrow q\) as \(k \rightarrow \infty\),

(ii) \(q < \lambda_1^+ \leq \lambda_2^+ \leq \cdots \leq \lambda_k^+ \not\rightarrow \infty\) as \(k \rightarrow \infty\),

(iii) \(\lambda_{k+k_0}^N \leq \lambda_k^+ \leq \lambda_{k+k_0}^D\) (\(k \in \mathbb{N}\)), where

\[ k_0 = \min \{k \in \mathbb{N} : \lambda_k^D \leq q < \lambda_{k+1}^D\} \quad \text{(if \(\lambda_1^D > q\) then we set \(k_0 = 0\)).} \]

**Proof.** Since \(0 < \lambda_1(0) \leq \lambda_2(0) \leq \cdots\) then in view of Lemmas 2.3 2.5 we conclude that for each \(k \in \mathbb{N}\) there is \(\mu_k \in (0, \infty)\), which is unique within \((0, \infty)\) and such that

\[ \lambda_k(\mu_k) = \frac{g\mu_k}{\mu + p}. \tag{2.82} \]

We set \(\lambda_k^- := \lambda_k(\mu_k)\).

It is clear that \(\lambda_k^- \in (0, q)\) and \(\lambda_k^-\) monotonically increases (see Fig. ). Therefore there exists \(\bar{q} \in (0, q]\) such that \(\lambda_k^- \rightarrow \bar{q}\) as \(k \rightarrow \infty\). The assumption \(\bar{q} < q\) implies that for each \(k \in \mathbb{N}\) \(\lambda_k(\mu_k) \leq \bar{q}\) and thus by (2.82)

\[ \mu_k \leq \frac{p\bar{q}}{q - \bar{q}} =: \bar{\mu} \text{ for all } k \in \mathbb{N}. \tag{2.83} \]

Choosing \(k' \in \mathbb{N}\) such that \(\lambda_{k'}(\bar{\mu}) > q = \frac{q\bar{\mu}}{\mu + 1}\), and taking into account that \(\lambda_{k'}(\mu)\) decreases while \(\frac{q\mu}{\mu + 1}\) increases, we conclude that \(\mu_k \leq \bar{\mu}\), which contradicts to (2.83). Thus \(\bar{q} = q\), which completes the proof of (i).

Furthermore, with \(k_0\) chosen such that \(\lambda_{k_0}^D \leq q < \lambda_{k_0+1}^D\), Lemmas 2.3 2.4 show that

- for \(k \in \{1, \ldots, k_0\}\) there is no solution \(\mu \in (-\infty, -p)\) of \(\lambda_k(\mu) = \frac{q\mu}{\mu + 1}\),
- for \(k \geq k_0 + 1\) (in this case we can write \(k = k_0 + k\) for some \(k \in \mathbb{N}\)) there is \(\bar{\mu}_k \in (-\infty, -1)\),

which is unique within \((0, -p)\), such that

\[ \lambda_{k_0+k}(\bar{\mu}_k) = \frac{q\bar{\mu}_k}{\bar{\mu}_k + p}. \]

We set \(\lambda_k^+ := \lambda_{k_0+k}(\bar{\mu}_k)\).

It is clear that \(\lambda_k^+ \in (q, \infty)\) and \(\lambda_k^+\) monotonically increases as \(k \rightarrow \infty\). Moreover, by Lemmas 2.3 2.4

\[ \lambda_{k_0+k}^D \leq \lambda_k^+ \leq \lambda_{k_0+k}^N \quad (k \in \mathbb{N}) \]

which implies \(\lambda_k^+ \not\rightarrow \infty\) (since \(\lambda_k^N \not\rightarrow \infty\)) and thus we get (ii) and (iii).

For \(\mu \in (-1, 0)\), \(\frac{q\mu}{\mu + 1} < 0 \leq \lambda_k^N = \lambda_k(0) \leq \lambda_k(\mu)\) for all \(k\), whence in \((-1, 0)\) there is no further solutions of \(\lambda_k(\mu) = \frac{q\mu}{\mu + 1}\). This completes the proof of the lemma. \(\Box\)

Finally we study the essential spectrum \(\sigma_{ess}(\mathcal{A})\) of the operator \(\mathcal{A}\).

**Lemma 2.7.** \(\sigma_{ess}(\mathcal{A}) = \{q\}\).

**Proof.** It follows from Lemma 2.6 that \(q \in \sigma_{ess}(\mathcal{A})\). Now let us prove that if \(\lambda \neq q\) then \(\lambda \not\in \sigma_{ess}(\mathcal{A})\)

We denote by \(\mathcal{B}\) the following bounded operator in \(\mathcal{H}\):

\[ \mathcal{B}U = (0, -qu_1), \quad U = (u_1, u_2) \in \mathcal{H}. \]

We set

\[ \mathcal{A} := \mathcal{A} - \mathcal{B}. \]

One has:

\[ (\mathcal{A} - iI)^{-1} - (\mathcal{A} - iI)^{-1} = -(\mathcal{A} - iI)^{-1} \mathcal{B}(\mathcal{A} - iI)^{-1}. \tag{2.84} \]
In view of the embedding theorem the operator staying in the right-hand-side of (2.84) is compact and therefore (see, e.g., [8])

\[ \sigma_{ess}(\mathcal{A}) = \sigma_{ess}(\mathcal{\bar{A}}). \]

Suppose that \( \lambda \neq q \) and let us prove that \( \lambda \notin \sigma_{ess}(\mathcal{\bar{A}}) \). We assume the opposite. Then there exists a bounded non-compact sequence \( U^k = (u^k_1, u^k_2) \in \text{dom}(\mathcal{A}), \ k \in \mathbb{N} \) such that

\[ \mathcal{\bar{A}}U^k - \lambda U^k \to 0 \quad \text{as} \quad k \to \infty. \quad (2.85) \]

It follows (2.85) and the definition the operator \( \mathcal{\bar{A}} \) that

\[ u^k_2 \to 0 \text{ as } k \to \infty \]

and furthermore

\[ \|\nabla u^k_1\|_{L^2(\Omega)}^2 + p|u^k_1|^2_{L^2(\Gamma)} - \lambda |u^k_1|^2_{L^2(\Omega)} \to 0 \text{ as } k \to \infty. \]

Therefore \( u^k_1 \) is bounded in \( H^1(\Omega) \) uniformly in \( k \). By the embedding theorem the sequence \( u^k_1 \) is compact in \( L_2(\Omega) \). We obtain a contradiction. The lemma is proved. \( \square \)

The assertion of Lemma 2.8 follows directly from (2.63) and Lemmas 2.6, 2.7.

2.3. **Proof of Theorem 1.2**. Let \( \lambda^e \in \sigma(\mathcal{A}) \) and \( \lambda^e \to \lambda \). We have to show that \( \lambda \in \sigma(\mathcal{\bar{A}}) \). Again by \( u^e \) we denote the corresponding to \( \lambda^e \) eigenfunction satisfying (2.1). In the same way as in the proof of Theorem 1.1 we conclude that there exists \( (u_1, u_2) \in H^1_0(\Omega) \oplus L^2(\Gamma) \) such that (2.5), (2.8) hold.

For an arbitrary \( w \in H^1_0(\Omega) \) one has the equality (2.9). Let \( w_0 \) be an arbitrary function from \( C^\infty_0(\Omega) \). We plug into (2.9) the function \( w = w^e \) defined by formula (2.14) with \( w_1 = w_2 := w_0 \). Since \( w^e = w_0(x^e_1) = \text{const} \) in \( D^e_i \) then the integral \( \int_{\bigcup_i D^e_i} \alpha^e \nabla u^e \cdot \nabla w^e dx \) is equal to zero. Then now the left-and right-hand sides of (2.9) do not contain the terms with \( \alpha^e \). Thus we can pass to the limit in (2.9) as \( \varepsilon \to 0 \) and in the same way as in the proof of Theorem 1.1 we conclude that \( (u_1, u_2) \) satisfies (2.88) with \( w_1 = w_2 := w_0 \), i.e.

\[ \int_{\Omega} \nabla u_1 \nabla w_0 dx = \lambda \int_{\Omega} u_1 w_0 dx + \lambda \int_{\Gamma} \rho u_2 w_0 ds. \quad (2.86) \]

Let us prove that \( u_1|_{\Gamma} = u_2 \). In order to do this we need an additional estimate.

**Lemma 2.8.** One has the following inequality: \( \forall u \in H^1(D^e_i) \)

\[ \left| \langle u \rangle_{S^e_i}^+ - \langle u \rangle_{S^e_i}^- \right|^2 \leq C d^e \varepsilon^{1-\eta} \|\nabla u\|_{L^2(D^e_i)}^2. \quad (2.87) \]

**Proof.** We introduce in \( D^e_i \) the spherical coordinates \((R, \Theta)\), where \( R \in (r^e - d^e, r^e) \) is a distance to \( x^e_i \), \( \Theta \) are the angle cooridnates. By \( S_{n-1} \) we denote the \((n-1)\)-dimensional unit sphere, by \( d\Theta \) we denote the Riemannian measure on \( S_{n-1} \). One has

\[ u(r^e, \Theta) - u(r^e - d^e, \Theta) = \int_{r^e - d^e}^{r^e} \frac{\partial u}{\partial R}(R, \Theta) dR. \]
We integrate this equality over $S_{n-1}$ (with respect to $\Theta$), divide by $|S_{n-1}|$ and square. Using the Cauchy inequality we obtain

\[
\left| \langle u \rangle_{S_{n-k}^+} - \langle u \rangle_{S_{n-k}^-} \right|^2 = \left| \frac{1}{|S_{n-1}|} \int_{S_{n-1}} \int_{r^2-d^2} \frac{\partial u}{\partial R} (R, \Theta) dR d\Theta \right|^2 \leq C \left( \int_{S_{n-1}} \int_{r^2-d^2} \frac{\partial u}{\partial R} (R, \Theta) ^2 dR d\Theta \right) \cdot \left( \int_{r^2-d^2} \frac{dR}{R^{n-1}} \right) \leq C_1 \left( \frac{1}{(r^2-d^2)^{n-2}} - \frac{1}{(r^2)^{n-2}} \right) \|
abla u\|_{L^2(D_0')}^2 \leq C_d^2 d^{2(n-1)} \|
abla u\|_{L^2(D_0')}^2.
\]

The lemma is proved. \qed 

Let $w$ be an arbitrary function from $C^1(\Gamma)$, the operator $Q^\varepsilon$ be defined by (2.23). One has, using (2.24):

\[
\int_{\Gamma} (u_1 - u_2) w ds = \lim_{\varepsilon \to 0} \int_{\Gamma} (\Pi_1^\varepsilon u^\varepsilon - \Pi_2^\varepsilon u^\varepsilon)(Q^\varepsilon w) ds = \lim_{\varepsilon \to 0} \sum_{i \in I^\varepsilon} w(x_i^\varepsilon) \left( (\langle \Pi_1^\varepsilon u^\varepsilon \rangle_{r_i} - \langle u^\varepsilon \rangle_{B_i^\varepsilon} \right) |\Gamma_i^\varepsilon | (2.88)
\]

Using (2.19), (2.20), (2.87), the inequality $\sum_{i \in I^\varepsilon} e_i^{n-1} \leq C$ and taking into account (1.1), (1.4) we obtain from (2.88):

\[
\int_{\Gamma} (u_1 - u_2) w ds \leq C \lim_{\varepsilon \to 0} \sum_{i \in I^\varepsilon} e_i^{n-1} \left| (\langle \Pi_1^\varepsilon u^\varepsilon \rangle_{r_i} - \langle u^\varepsilon \rangle_{B_i^\varepsilon} \right|^2 \leq C \lim_{\varepsilon \to 0} \sum_{i \in I^\varepsilon} e_i^{n-1} \left| (\langle \Pi_1^\varepsilon u^\varepsilon \rangle_{r_i} - \langle u^\varepsilon \rangle_{B_i^\varepsilon} \right|^2 \leq C_1 \lim_{\varepsilon \to 0} \sum_{i \in I^\varepsilon} e_i^{n-1} \left| (\langle \Pi_1^\varepsilon u^\varepsilon \rangle_{r_i} - \langle u^\varepsilon \rangle_{B_i^\varepsilon} \right|^2 \leq C_2 \lim_{\varepsilon \to 0} \left( e \|\nabla \Pi_1^\varepsilon u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + d^2 \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + c_1 \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \right) \leq C_3 \lim_{\varepsilon \to 0} \left( \varepsilon + \frac{d}{\alpha^2} \right) \eta^2 \|u^\varepsilon - u\| = 0
\]

Thus $\int_{\Gamma} (u_1 - u_2) w ds$ for all $w \in C^1(\Omega)$, whence $u_1|_\Gamma = u_2$. (2.89)

It follows from (2.86), (2.89) that $U = (u_1, u_2) \in \text{dom}(\eta)$ and $\tilde{A}U = \lambda U$. (2.90)

Finally we prove that $u_1 \neq 0$. One has

\[
1 = \|u^\varepsilon\|_{L^2(\Omega)}^2 = \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \sum_{i \in I^\varepsilon} \|u^\varepsilon\|_{L^2(D_i^\varepsilon)}^2 + \sum_{i \in I^\varepsilon} \beta_i^\varepsilon \|u^\varepsilon\|_{L^2(B_i^\varepsilon)}^2.
\]

One has, using (2.6) and taking in mind that $u^\varepsilon = \Pi_1^\varepsilon u^\varepsilon$ in $\Omega^\varepsilon$, $\lim_{\varepsilon \to 0} \Omega \setminus \Omega^\varepsilon = 0$:

\[
\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq \lim_{\varepsilon \to 0} \|u_1\|_{L^2(\Omega)} + \|u^\varepsilon - u_1\|_{L^2(\Omega)} = \|u_1\|_{L^2(\Omega)}.
\]
In the same way as in the proof of Theorem 1.1 (see (2.36)) we get:

$$\lim_{\varepsilon \to 0} \sum_{i \in I^\varepsilon} \|u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2 = 0$$

(2.93)

(the proof of (2.36) is based on inequality (2.35) and the condition \((d^\varepsilon)^2 = o(\varepsilon^2))\). Finally, using the Poincaré inequality, (2.19), (2.20), (2.87) and the Cauchy inequality, we get

$$\sum_{i \in I^\varepsilon} \beta^\varepsilon \|u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2 \leq C \sum_{i \in I^\varepsilon} \left( \varepsilon^{-1} \|u_i^\varepsilon - (d_i^\varepsilon)_{B_i^\varepsilon}\|_{L^2(B_i^\varepsilon)}^2 + \varepsilon^{n-1} \|u_i^\varepsilon\|_{S_i^\varepsilon}^2 + \varepsilon^{n-1} \|u_i^\varepsilon\|_{S_i^\varepsilon}^2 \right) +$$

$$+ \varepsilon^{n-1} \|u_i^\varepsilon\|_{S_i^\varepsilon}^2 + \varepsilon^{n-1} \|u_i^\varepsilon\|_{S_i^\varepsilon}^2 \right) \leq$$

$$\leq C_1 \left( \varepsilon \|\nabla u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2 + d_i^\varepsilon \|\nabla u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2 + \varepsilon \|\nabla u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2 \right) + C_1 \|u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2$$

(2.94)

Passing to the limit in (2.94) and taking into account that \(d^\varepsilon = o(\varepsilon^2)\) (since \(a = \infty\)), (1.1), (1.4), (2.1), (2.2), (2.8) we obtain

$$\lim_{\varepsilon \to 0} \sum_{i \in I^\varepsilon} \beta^\varepsilon \|u_i^\varepsilon\|_{L^2(B_i^\varepsilon)}^2 \leq C \|u_1\|_{L^2(\Gamma)}^2$$

(2.95)

It follows from (2.91), (2.93), (2.95) that \(u_1 \neq 0\). Therefore in view of (2.90) \(\lambda\) is the eigenvalue of \(\widehat{\mathcal{A}}\).

The second property of the Hausdorff convergence is proved in the same way as in Theorem 1.1. Theorem 1.2 is proved.

3. Spectrum of the waveguide

In this section we consider the unbounded waveguide type domain \(\Omega \subset \mathbb{R}^2\):

$$\Omega = \mathbb{R} \times (d_-, d_+), \quad \pm d_+ > 0.$$

In this case

$$\Gamma = \{ x = (x^1, x^2) : x^2 = 0 \}.$$

We again suppose that conditions (1.6) holds, moreover in this Section we are interested in the case \(a > 0\) only.

In the same way as before we introduce the numbers \(p, q, \rho\), Hilbert spaces \(\mathcal{H}^p\) and \(\mathcal{H}\), the sesquilinear forms \(\eta^p\) and \(\eta\), and the operators \(\mathcal{A}^p\) and \(\mathcal{A}\).

Since now \(\Omega\) is a non-compact domain its spectrum has another structure comparing with a compact case. To describe it we need some additional notations.

For fixed \(\mu \in \mathbb{R}\) we denote by \(\alpha(\mu)\) the first eigenvalue of the problem

$$-u'' = \lambda u \text{ in } (d_-, d_+), \quad (3.1)$$

$$u(d_-) = u(d_+), \quad (3.2)$$

$$u'(d_-) = u'(0), \quad u'(d_+) = \mu u(d_+). \quad (3.3)$$

It is easy to calculate that the function the function \(\mu \mapsto \alpha_1(\mu)\) is continuous, monotonically decreasing and moreover \(\alpha_1(\mu) \to \min \left\{ \left( \frac{\pi}{d_-} \right)^2, \left( \frac{\pi}{d_+} \right)^2 \right\} \) and \(\alpha_1(\mu) \to -\infty\) as \(\mu \to -\infty\). Using this we conclude that there exists one and only one point \(\alpha_1\) satisfying

$$\exists \mu_1 > -p : \alpha_1 = \alpha(\mu_1) = \frac{q\mu_1}{p + \mu_1}.$$
and if $q < \min \left( \frac{\pi^2}{\alpha^2}, \frac{\pi^2}{\beta^2} \right)$ then there exists one and only one point $\alpha_2$ satisfying
$$
\exists \alpha_2 : \quad \alpha_2 = \alpha(\mu_2) = \frac{q\mu_2}{p + \mu_2}.
$$
Moreover
$$
0 < \alpha_1 < q < \alpha_2.
$$

**Lemma 3.1.** One has
$$
\sigma(\mathcal{A}) = \mathcal{D} := \begin{cases}
[a_1, q] \cup [a_2, \infty) & \text{if } q < \min \left( \frac{\pi^2}{\alpha^2}, \frac{\pi^2}{\beta^2} \right), \\
[a, \infty) & \text{otherwise}.
\end{cases}
$$

**Proof.** We denote
$$
\Omega_L = \{ x = (x^1, x^2) \in \mathbb{R}^2 : \ x^1 \in (-L, L), \ x^2 \in (d_-, d_+) \}, \quad \Gamma_L = \Gamma \cap \Omega_L.
$$

By $\mathcal{H}_L$ we denote the Hilbert space of functions from $L_2(\Omega_L) \oplus L_2(\Gamma_L)$ and the scalar product defined by (1.8) with $\Omega_L$ and $\Gamma_L$ instead of $\Omega$ and $\Gamma$.

We denote by $\eta_L^\#$ the sesquilinear form in $\mathcal{H}_L$ which is defined by (1.9) (with $\Omega_L$ and $\Gamma_L$ instead of $\Omega$ and $\Gamma$) and the definitional domain
$$
\text{dom}(\eta_L^\#) = \{(u_1, u_2) \in H^1(\Omega_L) \oplus L^2(\Gamma_L) : \ u_1(-L, \cdot) = u_1(L, \cdot), \ u_1(\cdot, d_-) = u_1(\cdot, d_+) = 0 \}.
$$

By $\mathcal{A}_L^\#$ we denote the operator generated by this form.

Let $\lambda$ be an eigenvalue of $\mathcal{A}_L^\#$, $U$ be the corresponding eigenfunction such that $\|U\|_{\mathcal{H}_L} = 1$. We extend $U$ to the whole $\Omega$ by periodicity and set
$$
U_N(x) = \frac{1}{\sqrt{N}} \Phi \left( N^{-1}|x_1| \right),
$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is smooth function such that $\Phi(r) = 1$ as $r \leq 1$ and $\Phi(r) = 0$ as $r \geq 2$. It is easy to show that
$$
\| \mathcal{A}_L U_N - \lambda U_N \|_{\mathcal{H}} \to 0, \quad N \to \infty,
$$

$$
0 < C_1 \leq \|U_N\|_{\mathcal{H}} \leq C_2
$$

(the constants $C_1, C_2$ are independent of $N$ but depend on $L$) and therefore (see, e.g., [15]) $\lambda \in \sigma(\mathcal{A})$.

Thus we have proved that
$$
\forall \lambda > 0 : \quad \sigma(\mathcal{A}_L^\#) \subset \sigma(\mathcal{A})
$$

But via direct calculations it is easy to show that
$$
\bigcup_{L=1}^{\infty} \sigma(\mathcal{A}_L^\#) = \left\{ [a_1, q] \cup [a_2, \infty) \quad \text{if } q < \min \left( \frac{\pi^2}{\alpha^2}, \frac{\pi^2}{\beta^2} \right), \right. \left. [a, \infty) \quad \text{otherwise} \right\},
$$

and thus $\mathcal{D} \subset \sigma(\mathcal{A}_\Omega)$.

Now, let us prove the reverse enclosure. Let $\lambda \in \mathbb{R} \setminus \mathcal{D}$. We have to prove that $\lambda$ belongs to the resolvent set of $\mathcal{A}$.

Let us fix an arbitrary $F = (f_1, f_2) \in \mathcal{H}$, i.e. $\forall F \in \mathcal{H}$ there is $U \in \text{dom}(\mathcal{A})$ such that $\mathcal{A}U - \lambda U = F$.

We denote by $\eta_L$ the sesquilinear form which is defined by (1.9) (with $\Omega_L$ and $\Gamma_L$ instead of $\Omega$ and $\Gamma$) and the definitional domain $\text{dom}(\eta_L) = H^1_0(\Omega_L) \oplus L^2(\Omega_L)$. Let $\mathcal{A}_L$ be the operator acting in $\mathcal{H}_L$ and generated by this form.

One can easily calculate that $\forall \lambda > 0$ $\sigma(\mathcal{A}_L^\#) \subset \mathcal{D}$ and then $\forall F_L \in \mathcal{H}_L$ there is $U_L = (u_{1L}, u_{2L}) \in \text{dom}(\mathcal{A}_L)$ such that $\mathcal{A}_L U_L - \lambda U_L = F_L$.
We set $F_L := F|_{\Omega_L}$. One has
\begin{equation}
\|U_L\|_{H_{02L}} \leq \text{dist}(\lambda, D)\|F_L\|_{H_L} \leq C, \tag{3.6}
\end{equation}
and as a consequence
\begin{equation}
\|\nabla u_{1L}\|_{H_L} \leq C. \tag{3.7}
\end{equation}

We extend $U_L$ by 0 to $\Omega \setminus \Omega_L$ using the same notation for the extended function. Obviously $u_1 \in H^1_0(\Omega)$, $u_2 \in L^2(\Omega)$. It follows from (3.6)-(3.7) that there exists a subsequence (still denote by $L$) and $u_1 \in H^1(\Omega)$ and $u_2 \in L^2(\Gamma)$ such that
\begin{equation}
u_{1L} \to u_1 \text{ in } H^1(\Omega), \quad u_{2L} \to u_2 \text{ in } L^2(\Gamma) \text{ as } L \to \infty. \tag{3.8}
\end{equation}

Let $(w_1, w_2) \in C^\infty(\Omega) \oplus L^2(\Gamma)$. When $L$ is large enough then supp$(w_1) \subset \Omega_L$ and therefore one can write:
\begin{align*}
\int_\Omega \nabla u_{1L} \cdot \nabla w_1 dx + \int_\Gamma p(u_{1L} - u_{2L})(w_1 - w_2)ds - \lambda \left( \int_\Omega u_{1L} w_1 dx + pq^{-1} \int_\Gamma u_{2L} w_2 ds \right) = \\
= \int_\Omega f_1 w_1 dx + pq^{-1} \int_\Gamma f_2 w_2 ds. \tag{3.9}
\end{align*}

Using (3.8) we pass to the limit in (3.9) and obtain that $(u_1, u_2)$ satisfies (2.59), i.e.
\begin{equation}
\mathcal{A}U - \lambda U = F. \tag{3.10}
\end{equation}

Thus $\lambda$ belongs to the resolvent set of $\mathcal{A}$. The lemma is proved.

The main result of this section is similar to Theorem 1.1.

**Theorem 3.1.** The spectrum $\sigma(\mathcal{A}^\varepsilon)$ converges to the spectrum $\sigma(\mathcal{A})$ in the Hausdorff sense.

**Proof.** The proof of the property (B) of the Hausdorff convergence repeats word-by-word the proof in cases of compact domain $\Omega$. Therefore we focus on the proof of property (A): let $\lambda^\varepsilon \in \sigma(\mathcal{A})$, $\lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda$ and we have to prove that $\lambda \in \sigma(\mathcal{A})$.

We denote
\begin{align*}
\Omega = \{ x = (x^1, x^2) \in \mathbb{R}^2 : x^1 \in (0, 1), \ x^2 \in (d_-, d_+) \}, \quad \Gamma = \Gamma \cap \Omega.
\end{align*}

It is clear that $\forall i \in \mathbb{Z} : a^\varepsilon(x_1 + i, x_2) = a^\varepsilon(x_1, x_2)$, $b^\varepsilon(x_1 + i, x_2) = b^\varepsilon(x_1, x_2)$ provided $\varepsilon^{-1} \in \mathbb{N}$, i.e. $\mathcal{A}^\varepsilon$ is a periodic operator with a period cell $\Omega$.

We will study the subsequence $\lambda^{\varepsilon_k}$, where $\varepsilon_k = k^{-1}$, $k = 1, 2, 3, \ldots$. For convenience we will use the notation $\varepsilon$ keeping in mind $\varepsilon_k$. To describe the spectrum of $\mathcal{A}^\varepsilon$ for fixed $\varepsilon$ we introduce some additional operators on the period cell.

Let $\varphi \in [0, 2\pi]$. By $\mathcal{H}^\varepsilon$ we denote the space of functions from $L^2(\Omega)$ and the scalar product defined by (1.3) with $\Omega$ instead of $\Omega$. In the space $\mathcal{H}^\varepsilon$ we consider the sesquilinear form $\eta^{\varphi, \varepsilon}$ defined by (1.4) with $\Omega$ instead of $\Omega$ and the definitional domain
\begin{equation}
\text{dom}(\eta^{\varphi, \varepsilon}) = \left\{ u \in H^1(\Omega) : u(0, \cdot) = e^{i\varphi} u(1, \cdot), \ u(\cdot, d_-) = u(\cdot, d_+) = 0 \right\}. \tag{1.5}
\end{equation}

By $\mathcal{A}^{\varphi, \varepsilon}$ we denote the operator generated by this form. The spectrum of $\eta^{\varphi, \varepsilon}$ is purely discrete. We denote by
\begin{align*}
0 < \lambda_{1}^{\varphi, \varepsilon} \leq \lambda_{2}^{\varphi, \varepsilon} \leq \cdots \leq \lambda_{k}^{\varphi, \varepsilon} \leq \cdots \to \infty,
\end{align*}
the sequence of eigenvalues of $\mathcal{A}^{\varphi, \varepsilon}$ repeated according to their multiplicity.
It is well-known (see, e.g. [9], [19]) that the analysis of the spectrum of $A^\varepsilon$ reduces to the analysis of the spectra of the operators $A^{\varphi,\varepsilon}$. Namely, one has

$$\sigma(A^\varepsilon) = \bigcup_{k=1}^{\infty} P_k^\varepsilon, \text{ where } P_k^\varepsilon = \bigcup_{\varphi \in [0,2\pi)} \Lambda_k^{\varphi,\varepsilon}. \quad (3.10)$$

The sets $P_k^\varepsilon$ are compact intervals called bands.

We also introduce the operator $A^\varphi$ as the operator acting in

$$H = \{(u_1, u_2) \in L_2(\Omega) \oplus L_2(\Gamma), \text{ the scalar product is defined by } (1.8) \text{ with } \Omega, \Gamma \text{ instead of } \Omega, \Gamma \}$$

and generated by the sesquilinear form $\eta^\varphi$ which is defined by (1.4) (with $\Omega, \Gamma$ instead of $\Omega, \Gamma$) and definitional domain $\text{dom}(\eta^\varphi) = \text{dom}(\eta^{\varphi,\varepsilon}) \oplus L_2(\Gamma)$.

**Lemma 3.2.** The spectrum of $A^\varphi$ has the form

$$\sigma(A^\varphi) = \{q\} \cup \{\lambda_{k}^{\varphi,\pm}, k = 1, 2, 3...\} \cup \{\lambda_{k}^{\varphi,\pm}, k = 1, 2, 3...\}.$$ 

The points $\lambda_{k}^{\varphi,\pm}, k = 1, 2, 3...$ belong to the discrete spectrum, $q$ is a point of the essential spectrum and they are distributed as follows:

$$\alpha_1 \leq \lambda_{1}^{-} \leq \lambda_{2}^{-} \leq \ldots \leq \lambda_{k}^{-} \leq \ldots \rightarrow k \rightarrow \infty q < \lambda_{1}^{+} \leq \lambda_{2}^{+} \leq \ldots \leq \lambda_{k}^{+} \leq \ldots \rightarrow k \rightarrow \infty.$$ 

Moreover if $q < \min\left\{\frac{\pi^2}{\varepsilon^2}, \frac{\pi^2}{d^2}\right\}$ then

$$\alpha_2 < \lambda_{1}^{+}.$$ 

The proof of this lemma is similar to the proof of Lemma 1.1. 

In view of (3.10) there exists $\varphi^\varepsilon \in [0, 2\pi]$ such that $\lambda^\varepsilon \in \sigma(A^{\varphi^\varepsilon,\varepsilon})$. We extract a subsequence (still denoted by $\varepsilon$) such that

$$\varphi^\varepsilon \rightarrow \varphi \in [0, 2\pi] \text{ as } \varepsilon \rightarrow 0.$$ 

Let $u^\varepsilon$ be the eigenfunction of $A^{\varphi^\varepsilon,\varepsilon}$ corresponding to $\lambda^\varepsilon$ and normalized by the condition $\|u^\varepsilon\|_H^\varepsilon = 1$.

In the same way as in the proof of Theorem 1.1 we conclude that there exists a subsequence (still denote by $\varepsilon$), $u_1 \in H^1(\Omega)$ and $u_2 \in L_2(\Gamma)$ such that

$$\Pi_1^\varepsilon u^\varepsilon \rightarrow u_1 \text{ in } H^1(\Omega), \quad \Pi_2^\varepsilon u^\varepsilon \rightarrow u_1 \text{ in } L_2(\Omega), \quad \Pi_2^\varepsilon u^\varepsilon \rightarrow u_2 \text{ in } L_2(\Gamma), \quad \Pi_1^\varepsilon u^\varepsilon \rightarrow u_1 \text{ in } L_2(\Gamma). \quad (3.11)$$

(the operators $\Pi_1^\varepsilon$ and $\Pi_2^\varepsilon$ were defined in Subsection 2.1). It follows from (3.11) that $u_1^\varepsilon \in \text{dom}(\eta^{\varphi^\varepsilon,\varepsilon})$. 

If $u_1 = 0$ then $\lambda = q$. The proof repeats word-by-word the proof of this fact in Theorem 1.1.

Now, let $u_1 \neq 0$. For an arbitrary $w \in \text{dom}(\eta^{\varphi^\varepsilon,\varepsilon})$ we have

$$\int_{\Omega} a^\varepsilon \nabla u^\varepsilon \cdot \nabla \tilde{w} dx = \int_{\Omega} b^\varepsilon u^\varepsilon \tilde{w} dx. \quad (3.12)$$

Let $w_1, \ w_2$ be an arbitrary functions from $C^\infty(\Omega)$ and $C^\infty(\Gamma)$, correspondingly, moreover

$$w_1(0, \cdot) = e^{\varphi^\varepsilon} w_1(1, \cdot), \quad w'_1(0, \cdot) = e^{\varphi^\varepsilon} w'_1(1, \cdot), \quad w_1(\cdot, d_+) = w_1(\cdot, d_+) = 0.$$ 

Using these functions we construct the function $w^\varepsilon$ by the formula (2.14). It is clear that $w^\varepsilon \in \text{dom}(A^{\varphi^\varepsilon,\varepsilon})$. Finally we set

$$\hat{w}^\varepsilon(x) = w^\varepsilon(x) \left((e^{(\varphi^\varepsilon - \varphi)} - 1)x^1 + 1\right), \quad x = (x^1, x^2).$$

It is easy to see that $\hat{w}^\varepsilon \in \text{dom}(\eta^{\varphi^\varepsilon,\varepsilon})$ and $\int_{\Omega} \alpha^\varepsilon(x)|\hat{w}^\varepsilon|^2 dx \rightarrow 0, \quad \varepsilon \rightarrow 0$. 


where the reminder $\delta(\varepsilon)$ is vanishingly small:

$$|\delta(\varepsilon)|^2 \leq \int_{\Omega} a^\varepsilon \nabla u^\varepsilon \cdot \nabla (\hat{w}^\varepsilon - w^\varepsilon) dx \leq \lambda^\varepsilon \eta^\varepsilon [\hat{w}^\varepsilon - w^\varepsilon] \to 0. \quad (3.14)$$

Then passing to the limit in (3.13) as $\varepsilon \to 0$ in the same as in the proof of Theorem 1.1 we obtain:

$$\int_{\Omega} -u_1 \Delta \hat{w}^\varepsilon dx + \int_{\Gamma} p(u_1 - u_2)(\hat{w}^\varepsilon - w^\varepsilon) ds = \lambda \left( \int_{\Omega} u_1 \hat{w}^\varepsilon dx + p q^{-1} \int_{\Gamma} u_2 \hat{w}^\varepsilon ds \right).$$

Again integrating by parts we conclude that $u_1, u_2$ satisfies equality (2.33) (with $\Omega, \Gamma$ instead of $\Omega, \Gamma$). Using the density arguments we conclude that (2.33) holds for an arbitrary $(w_1, w_2) \in \text{dom}(\eta^\varepsilon)$ which means

$$\mathbb{A}^\varepsilon U = \lambda U, \quad U = (u_1, u_2).$$

Since $u_1 \neq 0$ then $\lambda \in \sigma(\mathbb{A}^\varepsilon)$. Then in view of Lemma 2.5, $\lambda \in [\alpha_1, q] \cup [\alpha_2, \infty)$.

Theorem (3.1) is proved. □

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