IMPROVEMENTS OF UPPER CURVATURE BOUNDS

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Abstract. We show that any space with a positive upper curvature bound has in a small neighborhood of any point a closely related metric with a negative upper curvature bound.

1. INTRODUCTION

1.1. Main result. There is a significant difference between the existence of global non-positive upper curvature bounds and the existence of some upper curvature bound \( \kappa \in \mathbb{R} \). For instance, any complete non-positively curved space is aspherical. On the other hand, any simplicial complex carries a metric of curvature bounded from above by 1 in the sense of Alexandrov, [Ber83].

The main result of this note confirms the expectation that in local considerations the value of the upper curvature bound does not matter:

Theorem 1.1. For \( \kappa \in \mathbb{R} \) let the metric space \((X,d)\) be CAT(\(\kappa\)). Let \( O = B_r(x) \) be an open ball of radius \( r \) around \( x \) in \( X \). If \( \kappa > 0 \) assume \( r < \frac{\pi}{2\sqrt{\kappa}} \). Then there exists a complete CAT(-1) metric \( d' \) on \( O \), such that the identity map \((O,d) \to (O,d')\) is locally bilipschitz.

In particular, in many questions concerning only local topological properties of CAT(\(\kappa\)) spaces, like most of [Kle99], [LN19], [LN18], one may always assume \( \kappa \) to be \(-1\).

Remark 1.2. The metric \( d' \) provided by the proof of Theorem 1.1 has the following additional properties. The distance to the central point \( x \) in \((O,d')\) depends only on the distance to \( x \) in \((O,d)\). Moreover, the tangent spaces at any point \( y \in O \) with respect to both metrics \( d \) and \( d' \) are isometric. This condition implies that \((O,d')\) is geodesically complete if \((O,d)\) is locally geodesically complete, see [LN19].

Our construction follows [LS17] and defines the metric \( d' \) via a conformal change of the original metric \( d \) with a sufficiently convex function. The same approach shows that any CAT(0) space \((X,d)\) admits another CAT(0) metric which is locally negatively curved, see Theorem 5.2 below. This result also applies to everywhere branching Euclidean buildings, where some rigidity might have been expected, see [KL97].

2010 Mathematics Subject Classification. 53C20, 53C23, 58E20.

Key words and phrases. Non-positive curvature, conformal change, minimal disc, harmonic maps.
However, in Theorem 5.2 the local negative curvature bound needs to tend to zero at infinity. Indeed, no conformal change of the flat Euclidean plane results in a complete Riemannian manifold of curvature $\leq -1$, [MT02, Corollary 7.3]. Thus, the answer to the following question, very natural in view of [AB04] and our Theorems 1.1, 5.2, cannot be obtained by means of this paper:

**Question 1.3.** Given a CAT(0) space $X$, does there exist a CAT(-1) metric on $X$ defining the same topology?

### 1.2 Auxiliary results of independent interest.

It has been shown in [LS17] that conformal changes with sufficiently convex functions preserve non-positive curvature. The improvement of the curvature bound is derived from the following analog of the formula expressing the curvature of a Riemannian manifold after a conformal change.

**Theorem 1.4.** For $c, C, \kappa, \lambda \in \mathbb{R}$, let $X$ be a CAT($\kappa$) space and let $f : X \to [c, C]$ be a Lipschitz continuous $\lambda$-convex function. Further, let $Y = e^f \cdot X$ denote the conformally equivalent space.

- If $\kappa - 4\lambda \leq 0$ then $Y$ is CAT($\bar{\kappa}$) with $\bar{\kappa} = e^{-2C} \cdot (\kappa - 4\lambda)$.
- If $\kappa - 4\lambda > 0$ and $\lambda > 0$ then $Y$ is CAT($\bar{\kappa}$) with $\bar{\kappa} = e^{-2c} \cdot (\kappa - 4\lambda)$.

We refer to [LS17] and Subsection 4.1 below for the definition and basic properties of conformally changed spaces and recall that a function $f : X \to \mathbb{R}$ is called $\lambda$-convex if the function $t \to f \circ \gamma(t) - \frac{\lambda}{2} \cdot t^2$ is convex, for any unit speed geodesic $\gamma$ in $X$.

Besides the theory of minimal discs, Theorem 1.4 relies on a generalization of the classical observation that the restriction of a convex function to a harmonic map is subharmonic, [Ish79], [KS93], [Che95], [Fug05]. Here we derive the following natural extension to semi-convex functions as a direct consequence of the properties of gradient flows of such functions.

**Theorem 1.5.** Let $\Omega$ be a domain in a Euclidean space $\mathbb{R}^n$ and let $u : \Omega \to X$ be a harmonic map into a CAT($\kappa$) space $X$. Let $f : X \to \mathbb{R}$ be a Lipschitz continuous $\lambda$-convex function.

Then for the composition $u \circ f \in W^{1,2}_{\text{loc}}(\Omega)$ the distributional Laplacian $\Delta (f \circ u)$ is a signed locally finite measure which satisfies

$$\Delta (f \circ u) \geq \lambda \cdot e_u^2,$$

where $e_u^2 \in L^1(\Omega)$ is the energy density of $u$.

Any Sobolev function with vanishing energy density has a constant representative. As an immediate consequence we obtain:
Corollary 1.6. Let $\Omega$ be a domain in a Euclidean space $\mathbb{R}^n$ and let $u : \Omega \to X$ be a harmonic map into a $\text{CAT}(\kappa)$ space $X$. Let $f : X \to \mathbb{R}$ be Lipschitz and 1-convex. If the composition $f \circ u$ is constant then $u$ itself is a constant map.

1.3. Comments. The proof of Theorem 1.5 does not use any specific property of the domain $\Omega$ and applies without changes to any domain in a Riemannian manifold and to domains in admissible Riemannian polyhedra in the sense of [Fug05], [DM10], [BFH+16].

The proof of Theorem 1.5 does not use the upper curvature bound assumption in an essential way either. It only relies on the existence and the contracting behaviour of gradient flows of semi-convex functions, which is valid in much greater generality. For instance these features are true in spaces with lower curvature bounds, [AKP16], [Pet07] and in some spaces of probability measures, [AGS05], [Oht09].

The importance of the result for spaces of curvature bounded above lies in a great variety of semi-convex functions. For instance, for every point $x$ in a $\text{CAT}(\kappa)$ space $X$ the function $f(y) = d^2(x, y)$ is 1-convex if $\kappa \leq 0$. If $\kappa > 0$ then the function $f$ is $\epsilon$-convex on the closed ball $\bar{B} = B_r(\kappa)$ for any $r < \frac{\pi}{2\sqrt{\kappa}}$ and some $\epsilon = \epsilon(r, \kappa) > 0$. This in conjunction with Theorem 1.4 leads to the proofs of Theorems 1.1, 5.2.

Moreover, on any $\text{CAT}(0)$ space $X$ the distance function $d : X \times X \to \mathbb{R}$ is convex. Similarly, for any closed ball $B$ of radius less than $\frac{\pi}{2\sqrt{\kappa}}$ in any $\text{CAT}(\kappa)$ space there exists a convex function $\psi : B \times B \to \mathbb{R}$, comparable (up to a bounded factor) with the distance $d(x, y)$, [Ken91], [Yok16]. This directly implies the uniqueness of solutions of the Dirichlet problem and the continuous dependence of harmonic maps on their traces. The existence of the harmonic maps with prescribed trace and their regularity involve some finer arguments but are heavily based on Theorem 1.5 for the functions $d$ and $\psi$, respectively, [KS93], [Ser95].

The proof of Theorem 1.4 provides also a local statement in the case $\kappa - 4\lambda > 0$ and $\lambda \leq 0$, see Theorem 5.1. Moreover, as the proof of Theorem 1.1 shows, one can localize the statement by writing the formulas in Theorem 1.4 using only local bounds and local semi-convexity of $f$.

1.4. Structure of the paper. In the preliminaries we recall basics of Sobolev maps and variation of length under gradient flows of semi-convex functions. This variation is applied in Section 3 to obtain a proof of Theorem 1.5. In Section 4, we recall some structural results about spaces with upper curvature bounds, minimal discs and conformal changes used in Section 5 to prove the main results of the paper.

1.5. Acknowledgments. We would like to thank Grigori Avramidi for posing a question which has lead to Theorem 1.1.

Both authors were partially supported by DFG grant SPP 2026.
2. Preliminaries

2.1. Notations and spaces with upper curvature bounds. We refer the reader to [BBI01], [BH99] and [AKP16] for basics on metric geometry and CAT(κ) spaces. Here we just agree on notation, some finer properties will be discussed in Subsection 4.2.

In this paper all CAT(κ) spaces will be complete length spaces by definition. By \( d \) we will denote distances in metric spaces. We will let \( D \subset \mathbb{R}^2 \) be the open Euclidean unit disc, \( \bar{D} \subset \mathbb{R}^2 \) the closed Euclidean unit disc and \( S^1 = \partial \bar{D} \subset \mathbb{R}^2 \) the unit circle.

For a Lipschitz function \( f : X \to \mathbb{R} \) on a metric space \( X \) we denote by \( |\nabla^- f| \in [0, \infty) \) the descending slope of \( f \) at \( p \in X \) defined by

\[
|\nabla^- f| = \max\{0, \limsup_{x \to p} \frac{f(p) - f(x)}{d(p,x)}\}.
\]

2.2. Semi-convex functions and their gradient flows. Let \( X \) be a CAT(κ) space. For any Lipschitz continuous \( \lambda \)-convex function \( f : X \to \mathbb{R} \) there exists the locally Lipschitz continuous gradient flow \( \Phi : [0, \infty) \times X \to X \) of \( f \), such that for any \( x \) the flow line \( t \to \Phi_t(x) \) is the gradient curve of the function \( f \) starting at \( x \).

As a reference one can use [OP17] or [Lyt05], see also [Pet07], [May98] and [AGS05] for a general theory of gradient flows in metric spaces.

From all properties of gradient flows we will only need the following distance estimate on the change of length under the gradient flow. In [Pet07, Lemma 2.2.1, Lemma 2.1.4] it is proven for Alexandrov spaces but the proof relies only on the first variation formula and is identical in our setting of CAT(κ) spaces.

**Corollary 2.1.** Let \( X \) be CAT(κ) and let \( f \) a Lipschitz continuous \( \lambda \)-convex function on \( X \) with gradient flow \( \Phi \). Let \( \gamma : [a,b] \to X \) be an absolutely continuous curve and let \( \rho : [a,b] \to [0,\infty) \) be Lipschitz.

Then \( \eta(s) := \Phi_{\rho(s)}(\gamma(s)) \) is an absolutely continuous curve and for almost all \( s \in [a,b] \) its velocity is bounded by

\[
|\eta'(s)|^2 \leq e^{-2\lambda \rho(s)} (|\gamma'(s)|^2 - 2(f \circ \gamma)'(s) \cdot \rho'(s) + |\nabla^- \gamma(s) f|^2 \cdot (\rho'(s))^2).
\]

2.3. Sobolev maps and energy. By now there exists a well established theory of Sobolev maps with values in metric spaces, [HKST15]. We will follow [LW17] and restrict our revision to the special case needed in this paper.

Let \( X \) be a complete metric space. Let \( \Omega \subset \mathbb{R}^n \) be a Lipschitz domain and denote by \( L^2(\Omega, X) \) the set of measurable and essentially separably valued maps \( u : \Omega \to X \) such that for some and thus every \( x \in \Omega \) the function \( u_x(z) := d(x, u(z)) \) belongs to \( L^2(\Omega) \).

**Definition 2.2.** A map \( u \in L^2(\Omega, X) \) belongs to the Sobolev space \( W^{1,2}(\Omega, X) \) if there exists \( h \in L^2(\Omega) \) such that for every \( x \in \Omega \) the
composition $u_x$ is contained in the classical Sobolev space $W^{1,2}(\Omega)$ and its weak gradient satisfies $|\nabla u_x| \leq h$ almost everywhere on $\Omega$.

Each Sobolev map $u$ has an associated trace $\text{tr}(u) \in L^2(\partial \Omega, X)$, see [KS93]. If $u$ extends continuously to a map $\hat{u}$ on $\bar{\Omega}$, then $\text{tr}(u)$ is represented by the restriction $\hat{u}|_{\partial \Omega}$.

There are several natural definitions of energy for Sobolev maps, see [LW17, Section 4]. We will only use the Korevaar–Schoen energy. It can be defined in many different ways, for instance, using the approximate metric differentials [LW17, Proposition 4.6]. The expression we are going to use is the following one.

Any map $u \in W^{1,2}(\Omega, X)$ has a representative, also denoted by $u$, which is absolutely continuous on almost all curves in $\Omega$, [HKST15]. Then, for any vector $v \in \mathbb{R}^n$ the restriction of $u$ to almost any segment parallel to $v$ is absolutely continuous, hence has a well defined finite velocity at almost all times. Thus the function $m_u(x, v)$ which measures the velocity of the curve $t \to u(x + tv)$ at $t = 0$ is well-defined almost everywhere on $\Omega \times \mathbb{R}^n$. We mention that, for almost all $x \in \Omega$, the function $v \to m_u(x, v)$ is a semi-norm on $\mathbb{R}^n$, the approximate metric differential of $u$ at $x$, which in fact is Euclidean, if $X$ is CAT($\kappa$), [KS93], [LW17, Section 11].

The energy density of $u$ is defined as
$$e^2_u(z) = \frac{1}{\omega_n} \int_{S^{n-1}} |m_u(x, v)|^2 \, dv,$$
where $\omega_n$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The Korevaar–Schoen energy of $u$ is given by
$$E^2(u) := \int_{\Omega} e^2_u(z) \, dz.$$

The map $u \in W^{1,2}(\Omega, X)$ is called harmonic if for all $w \in W^{1,2}(\Omega, X)$ with the same trace as $u$ one has
$$E^2(u) \leq E^2(w).$$

If $X$ is CAT(0) or a CAT($\kappa$) space of sufficiently small diameter, then any harmonic map is locally Lipschitz continuous and uniquely determined by a prescribed trace [KS93], [Ser95], [Fug08], [BFH+16].

3. First variation of energy

For Sobolev maps in CAT($\kappa$) spaces we have the following analog of the classical first variation formula.

**Lemma 3.1.** Let $X$ be a CAT($\kappa$) space and $f$ a Lipschitz continuous $\lambda$-convex function on $X$ with gradient flow $\Phi : [0, \infty) \times X \to X$. 

Let $u \in W^{1,2}(\Omega, X)$ be given. For any Lipschitz continuous test function $\rho : \Omega \to [0, \infty)$ with compact support in $\Omega$, define a variation $u_t$ of $u$ by
\[
u_t(x) = \Phi(t \rho(x), u(x)).\]
Then $u_t \in W^{1,2}(\Omega, X)$ and the following inequality holds
\[
\begin{align*}
&\frac{d}{dt} \mid_{t=0} E^2(u_t) \leq -2 \int_\Omega \left( \lambda \cdot e^2_u(x) \cdot \rho(x) + \langle \nabla_x(f \circ u), \nabla \rho \rangle \right) dx \\
\end{align*}
\]
where the left-hand side is the upper Dini derivative.

**Proof.** We fix $t > 0$. As a composition of a Sobolev and a Lipschitz map, the map $u_t$ is contained in $W^{1,2}(\Omega, X)$.

Consider the curves $\gamma(s) = u(x + sv)$ and $\eta(s) = \Phi(t \cdot \rho(x + sv), \gamma(s))$.

By definition, for almost all $x, v \in \Omega \times S^{n-1}$, the velocities of $\gamma$ respectively $\eta$ at $s = 0$ are exactly $m_u(x, v)$ and $m_u(x, v)$, the values of the corresponding approximate metric differentials. Applying Corollary [2.1] we get the estimate, valid at all such $x, v$:
\[
m^2_{u_t}(x, v) \leq e^{-2\lambda t \cdot \rho(x)}(m^2_u(x, v)) - 2t \langle \nabla_x(f \circ u), v \rangle \cdot \langle \nabla \rho, v \rangle + t^2 |\nabla u(x)f|^2 \cdot \langle \nabla \rho, v \rangle^2
\]
Averaging over $S^{n-1}$ and using the equality
\[
\frac{1}{\omega_n} \int_{S^{n-1}} \langle w_1, v \rangle \cdot \langle w_2, v \rangle \, dv = \langle w_1, w_2 \rangle,
\]
valid for all $w_1, w_2 \in \mathbb{R}^n$, we obtain the estimate of the energy densities, valid pointwise almost everywhere on $\Omega$:
\[
e^2_{u_t}(x) \leq e^{-2\lambda t \cdot \rho(x)}(e^2_u(x)) - 2t \langle \nabla \rho, \nabla_x(f \circ u) \rangle + t^2 |\nabla u(x)f|^2 \cdot |\nabla \rho|^2 \leq \\
\leq (1 - 2 \lambda \rho(x) \cdot t)(e^2_u(x) - 2t \langle \nabla \rho, \nabla_x(f \circ u) \rangle) + C t^2,
\]
for some constant $C$ depending on $\lambda, \rho, f$ and $u$.

The claim follows now directly by integration over $\Omega$. \hfill \square

Now we can easily derive:

**Proof of Theorem 1.5** Applying Lemma 3.1 and the definition of harmonicity, we see that the right hand side of (3.1) must be non-negative for any non-negative Lipschitz continuous function $\rho$ with compact support in $\Omega$. Thus the distributional Laplacian $\Delta(f \circ u)$ satisfies
\[
\Delta(f \circ u)(\rho) = - \int_\Omega \langle \nabla_x(f \circ u), \nabla \rho \rangle \, dx \geq \lambda \int_\Omega e^2_u \cdot \rho.
\]
By the representation theorem of Riesz in distirbution theory, this is sufficient to draw the conclusion. \hfill \square
4. Preparations

4.1. Length spaces and their conformal changes. The length of a rectifiable curve \( \gamma \) in a metric space \( X \) is denoted by \( \ell(\gamma) \). A metric space \( X \) is a length space if the distance between any two points is equal to the greatest lower bound for lengths of curves connecting the respective points. A curve \( c : [a, b] \to X \) will be called geodesic if it is an isometric embedding. The space \( X \) itself will be called geodesic if any two points in \( X \) are joined by a geodesic.

We refer to [LS17] for more details on what follows here. Let \( X \) be a length space and \( f : X \to (0, \infty) \) be a continuous function. We define the \( f \)-length of a rectifiable curve \( \gamma : [a, b] \to X \) by

\[
\ell_f(\gamma) = \int_a^b f(\gamma(t)) \cdot |\dot{\gamma}(t)| \, dt,
\]

where \( |\dot{\gamma}(t)| \) denotes the velocity of the curve \( \gamma \) at time \( t \). The conformally changed metric \( d_f \) on the space \( X \) is defined by

\[
d_f(x, y) = \inf \{ \ell_f(\gamma) : \gamma \text{ Lipschitz curve from } x \text{ to } y \}.
\]

The space \( f \cdot X := (X, d_f) \) is a length space called the metric space conformally equivalent to \( X \) with conformal factor \( f \).

The identity map \( \text{id}_f : X \to f \cdot X \) is a locally bilipschitz homeomorphism. If \( f \) is bounded from below by a positive constant and \( X \) is complete then \( f \cdot X \) is complete as well.

We will need the following observation:

**Lemma 4.1.** Let \( X \) be a length space and assume \( X = B_r(x) \) for some \( x \in X \) and \( r > 0 \). Let \( \xi : [0, r) \to (0, \infty) \) be continuous and consider the function \( f(y) := \xi(d(x, y)) \) on \( X \). Then, in the conformally changed space \( f \cdot X \), the distance function to \( x \) can be computed as:

\[
d_f(x, y) = \int_0^{d(x,y)} \xi(t) \, dt.
\]

**Proof.** Consider concentric metric spheres \( S_s(x) \) of radii \( s \leq d(x, y) \) around \( x \). Then for any \( 0 \leq s_0 < s_1 \leq d(x, y) \) and any point \( z \in S_{s_1}(x) \) we can estimate the \( d_f \)-distance from \( z \) to \( S_s(X) \) as

\[
|s_1 - s_0| \cdot \min_{s_0 \leq s \leq s_1} \xi(s) \leq d_f(z, S_s(x)) \leq |s_1 - s_0| \cdot \max_{s_0 \leq s \leq s_1} \xi(s).
\]

The proof of the lemma follows by writing the integral \( \int_0^{d(x,y)} \xi(t) \, dt \) as a limit of Riemann sums as on the right and left hand sides of the above inequality.

\[\square\]

4.2. Local-to-global in \( \text{CAT}(\kappa) \) spaces. The basic local-to-global theorem about \( \text{CAT}(\kappa) \) spaces is the theorem of Cartan–Hadamard,
saying that a complete length space, which is locally CAT(κ) with κ ≤ 0, is a CAT(κ) space if and only if it is simply connected.

In order to describe related local-to-global statements for all κ, we recall that the injectivity radius of a local CAT(κ) space X is the supremum of all r > 0, such that any pair of points x and y in X at distance less than r are connected by a unique geodesic and that this geodesic depends continuously on the endpoints.

Combining [Bal04, 6.10] and [AKP16, 8.11.3] we obtain the following.

**Lemma 4.2.** Let X be a complete length space which is locally CAT(κ). Let r > 0 be such that r < $\frac{\pi}{\sqrt{\kappa}}$ for κ > 0.

If the injectivity radius of X is larger than r then any ball $\bar{B}_r(x)$ is a convex CAT(κ) subset of X.

The next local-to-global result is also well-known.

**Lemma 4.3.** Let X be a complete length space which is locally CAT(κ). Let Λ > 0 be such that Λ ≤ $\frac{\pi}{\sqrt{\kappa}}$ if κ > 0.

Assume that for any closed curve Γ of length less than 2Λ there exists a homotopy $\Gamma_t, t \in [0, 1]$ from $\Gamma = \Gamma_0$ to a constant curve $\Gamma_1$ such that the length of $\Gamma_t$ is less than 2Λ for all t.

Then any closed ball of radius less than $\frac{\Lambda}{2}$ in X is convex and CAT(κ).

**Proof.** Let Γ be a closed curve of length less than 2Λ. Then our assumptions allow to apply [AKP16, 8.13.4], to conclude that Γ is majorized by a CAT(κ) space. It follows that the injectivity radius of X is at least $\frac{\Lambda}{2}$. Hence Lemma 4.2 completes the proof. □

**Corollary 4.5** below will slightly strengthen the next result.

**Proposition 4.4.** Let Z be a compact geodesic space homeomorphic to $\bar{D}$. If Z is locally CAT(1) and has Hausdorff area $H^2(Z)$ less than 2π, then Z is CAT(1).

**Proof.** Otherwise Z contains an isometric embedding $\Gamma$ of a round circle $S^1_2$ of length 2l < 2π in Z, [Bal04, 6.9]. The closed Jordan domain $Z_1$ cut out of Z by $\Gamma$ is convex, hence locally CAT(1) and its Hausdorff area is also less than 2π.

Doubling $Z_1$, thus gluing two copies of it along $\Gamma$, we obtain a space homeomorphic to the 2-dimensional sphere, which has Hausdorff area less than 4π and which is locally CAT(1), by Reshetnyak’s gluing theorem, [AKP16, Theorem 8.9.1].

But this contradicts the Gauss-Bonnet formula, [Res93 (8.15)]. □

As a consequence we deduce the following analog of [LW18, Proposition 12.1] for κ \neq 0.

**Corollary 4.5.** Let $\tilde{Z}$ be a length space homeomorphic to the open disc $D$. For κ \in $\mathbb{R}$ let $\tilde{Z}$ be locally CAT(κ) and assume that for κ > 0 the area $H^2(\tilde{Z})$ is at most $\frac{2\pi}{\kappa}$. Then the completion Z of $\tilde{Z}$ is CAT(κ).
Proof. We exhaust the space \( \hat{Z} \) by compact closed discs \( Z_n \) with boundary being a geodesic polygon as in the proof of [LW18, Proposition 12.1]. As in [LW18, Section 11.2], we readily see that these subsets \( Z_n \) are locally \( \text{CAT}(\kappa) \) in their intrinsic metrics.

Moreover, a limiting argument as in [LW18, Proposition 12.1] shows that it suffices to verify that the closed discs \( Z_n \) are globally \( \text{CAT}(\kappa) \).

For \( \kappa \leq 0 \), this statement follows directly by the theorem of Cartan–Hadamard. For \( \kappa > 0 \), we may rescale the space and assume \( \kappa = 1 \).

Any open non-empty subset of \( Z_0 \) has positive \( \mathcal{H}^2 \)-area, [LN19, Theorem 1.2], thus \( \mathcal{H}^2(Z_n) < 2\pi \), for any \( n \). The global \( \text{CAT}(1) \) property of \( Z_n \) is exactly Proposition 4.4. \( \square \)

4.3. Recognizing \( \text{CAT}(\kappa) \) spaces. For us it will be important that \( \text{CAT}(\kappa) \) spaces can be recognized without referring to geodesic triangles. By a Jordan curve in a metric space \( X \) we denote a subset homeomorphic to a circle.

We say that a metric space \( Y \) majorizes a rectifiable Jordan curve \( \Gamma \) in a metric space \( X \) if there exists a 1-Lipschitz map \( P : Y \to X \) which sends a Jordan curve \( \Gamma' \subset Y \) bijectively in an arc length preserving way onto \( \Gamma \). The following is proved in [LS17, Proposition 2] for \( \text{CAT}(0) \) spaces. Along the same lines we deduce:

**Proposition 4.6.** Let \( \kappa \in \mathbb{R} \) and \( \Lambda > 0 \) be such that \( \Lambda \leq \frac{\pi}{\sqrt{k}} \) if \( \kappa > 0 \).

Let \( X \) be a complete length metric space.

If any Jordan curve \( \Gamma \) in \( X \) of length less than \( 2\Lambda \) is majorized by some \( \text{CAT}(\kappa) \) space \( Y_\Gamma \), then any closed ball \( B \) in \( X \) of any radius \( r < \frac{\Lambda}{2} \) is convex in \( X \) and \( \text{CAT}(\kappa) \).

Moreover, if \( \kappa > 0 \) and \( \Lambda = \frac{\pi}{\sqrt{k}} \) then \( X \) is \( \text{CAT}(\kappa) \).

**Proof.** The argument in [LS17, Proposition 2] shows that any pair of points in \( X \) at distance less than \( \Lambda \) is connected by a unique geodesic in \( X \). Moreover, such geodesics depend continuously on their endpoints.

As in [LS17, Proposition 2] the assumption implies that any triangle in \( X \) of perimeter less than \( 2\Lambda \) is not thicker than its comparison triangle in the constant curvature surface.

For \( \kappa > 0 \) and \( \Lambda = \frac{\pi}{\sqrt{k}} \) this implies by definition, that \( X \) is \( \text{CAT}(\kappa) \).

For general \( \Lambda \), the condition implies that \( X \) is locally \( \text{CAT}(\kappa) \) and the statement follows from Lemma 4.2. \( \square \)

4.4. Surfaces. In the case of flat domains the curvature of conformally changed metrics has been investigated in detail by Yuri Reshetnyak, see [Res93] and the references therein. In this case it is even possible to relax the continuity and positivity assumptions on conformal factors.

We say that a function \( f : U \to [0, \infty) \) on a domain \( U \subset \mathbb{R}^2 \) is \( \kappa \)-log-subharmonic, if \( f \) is upper semi-continuous, contained in \( L^1_{\text{loc}} \) and
satisfies weakly
\[ \Delta \log f + \frac{\kappa}{2} f^2 \geq 0. \]

For a \( \kappa \)-log-subharmonic function \( f \) one can use formulas (4.1) and (4.2) to define the conformally changed metric on \( U \). Indeed, we have the following result due to Reshetnyak, see Theorem 7.1.1 in [Res93], see also [Mes01, Theorem 6.1] and [LW18, Theorem 8.1, Section 17].

**Theorem 4.7.** Let \( U \subset \mathbb{R}^2 \) be a domain and \( f \) a \( \kappa \)-log-subharmonic function on \( U \). Then \( f \cdot U \) is locally CAT(\( \kappa \)) and \( \text{id}_f : U \to f \cdot U \) is a homeomorphism.

The next computational lemma will provide control on double conformal changes.

**Lemma 4.8.** Let \( c, C, \kappa, \lambda \in \mathbb{R} \). Let \( U \subset \mathbb{R}^2 \) be a domain and \( \varphi \) a \( \kappa \)-log-subharmonic function on \( U \). Suppose that \( \psi : U \to [c, C] \) is a continuous function which satisfies \( \Delta \psi \geq \mu \cdot \varphi^2 \) weakly. Then the product \( e^{\psi} \cdot \varphi \) is \( \bar{\kappa} \)-log-subharmonic, with

- \( \bar{\kappa} = e^{-2c} \cdot (\kappa - 2\mu) \) if \( \kappa - 2\mu \leq 0 \);
- \( \bar{\kappa} = e^{-2c} \cdot (\kappa - 2\mu) \) if \( \kappa - 2\mu \geq 0 \).

**Proof.** Suppose \( \kappa - 2\mu \leq 0 \). Then \(- (\kappa - 2\mu) \geq - \bar{\kappa} \cdot (e^{\psi})^2 \) and the claim follows from
\[
\Delta \log(e^{\psi} \varphi) = \Delta \psi + \Delta \log \varphi \geq \mu \cdot \varphi^2 - \frac{\kappa}{2} \varphi^2 = -\frac{1}{2}(\kappa - 2\mu) \varphi^2
\]

The case \( \kappa - 2\mu \geq 0 \) is analogous. \( \square \)

Combining Theorem 4.7 and Lemma 4.8 leads to:

**Lemma 4.9.** Let \( c, C, \kappa, \mu \in \mathbb{R} \) and set

- \( \bar{\kappa} = e^{-2c} \cdot (\kappa - 2\mu) \) if \( \kappa - 2\mu \leq 0 \);
- \( \bar{\kappa} = e^{-2c} \cdot (\kappa - 2\mu) \) if \( \kappa - 2\mu \geq 0 \).

Suppose that \( \varphi \) is a \( \kappa \)-log-subharmonic function on \( D \) and let \( \varphi \cdot D \) be the conformally changed disc. Let \( Z \) denote the completion of \( \varphi \cdot D \). If \( \bar{\kappa} > 0 \), assume in addition \( \mathcal{H}^2(\varphi \cdot D) \leq e^{-2c} \cdot \frac{2\pi}{\bar{\kappa}} \). Finally, let \( \psi : Z \to [c, C] \) be a continuous function on \( Z \) such that the restriction of \( \psi \) to \( D \) satisfies \( \Delta \psi \geq \mu \cdot \varphi^2 \) weakly. Then \( e^{\psi} \cdot Z \) is CAT(\( \bar{\kappa} \)).

**Proof.** The proof of Lemma 4 of [LS17] implies \( e^{\psi} \cdot Z \) is the completion of the length space \( e^{\psi \cdot (\varphi \cdot D)} \). Moreover, it shows that \( (e^{\psi \cdot \varphi}) \cdot D \) is isometric to \( e^{\psi} \cdot (\varphi \cdot D) \). Hence Lemma 4.8 together with Theorem 4.7 imply that \( (e^{\psi \cdot \varphi}) \cdot D \) is locally CAT(\( \bar{\kappa} \)).

The claim then follows from Corollary 4.5. \( \square \)
4.5. **Minimal discs.** A general solution of the classical Plateau’s problem has been provided in [LW17] for proper metric spaces and in [GW17] for complete CAT(0) spaces. We need to discuss an appropriate extension to non-proper CAT(κ) spaces with κ > 0.

**Lemma 4.10.** Let X be a CAT(κ) space and let Γ be a Jordan curve in X of finite length l. If κ > 0 assume in addition, that $l < \frac{2\pi}{\sqrt{\kappa}}$. Then there exists a closed ball $B = \bar{B}_r(x)$ which contains Γ. Moreover, if κ > 0 we can choose $r < \frac{\pi}{2}\sqrt{\kappa}$.

The space $\Lambda(\Gamma, B)$ of all maps $v \in W^{1,2}(D, B)$ such that $tr(v)$ is a weakly monotone parametrization of Γ is non-empty and contains a map $u_0$ of smallest energy $E^2(u_0) < \frac{1}{2} \cdot l^2$ in $\Lambda(\Gamma, B)$.

Moreover, any such map $u_0$ has a unique representative which extends continuously to $\bar{D}$.

**Proof.** Without loss of generality, we may assume $\kappa = 1$. The existence of the required ball $B$ is a consequence of Reshetnyak’s majorization theorem [AKP16, 8.12.4]. Moreover, putting together the majorization theorem, the isoperimetric inequality in the hemisphere and the fact that convex subdomains of the hemisphere allow for conformal parametrizations, we conclude that there exists a map $u \in \Lambda(\Gamma, B)$ with $E^2(u) < \frac{1}{2} \cdot l^2$. In particular, $\Lambda(\Gamma, B)$ is non-empty.

If $B$ is compact, the existence of an energy minimizer in $\Lambda(\Gamma, B)$ is proved in [LW17]. The same classical argument, extended in [LW17] to proper metric spaces also works in the present non-compact case as follows.

As in the classical case, we can precompose with Moebius maps and restrict to the subspace $\Lambda_0(\Gamma)$ of all maps in $\Lambda(\Gamma)$ whose trace sends three fixed points in $S^1$ to three prescribed points in Γ, [LW17 Section 7]. Take an energy minimizing sequence $u_n \in \Lambda_0(\Gamma)$. For any $u_n$, we find a unique harmonic map $v_n \in \Lambda_0(\Gamma)$ with the same trace as $u_n$, [Ser95], [Fug08, Theorem 4], since $r < \frac{\pi}{2}$. In particular, $E(v_n) \leq E(u_n)$ and $v_n$ is an energy-minimizing sequence as well.

By the lemma of Courant–Lebesgue, the traces $tr(v_n)$ are uniformly continuous, [LW17 Section 7]. By [Ser95], [Fug08, Theorem 2], the maps $v_n$ have unique representatives, which continuously extends to $\bar{D}$. Moreover, these representatives converge uniformly, once their traces converge uniformly. Thus, using the semi-continuity of energy, we obtain an energy minimizer in $\Lambda_0(\Gamma)$ by taking a uniform limit of a subsequence of the maps $v_n$.

We call a continuous map $u_0 : \bar{D} \to B$ provided by the above result a **minimal filling** of Γ in B and are going to summarize its properties:

**Theorem 4.11.** Let Γ be a Jordan curve of length l in a CAT(κ) space X, with $l < \frac{2\pi}{\sqrt{\kappa}}$ if κ > 0. As in Lemma 4.10 let $B = B_r(x)$ be a closed ball which contains Γ and let $u : \bar{D} \to B$ be a minimal filling of Γ.
Then the following hold true:

1. $u$ is harmonic and $E^2(u) < \frac{1}{2}l^2$.
2. There exists a function $\varphi \in L^2(D)$, the conformal factor of $u$, such that the approximate metric differential satisfies $m_u(z, v) = \varphi(z) \cdot \|v\|$ for almost all $z \in D$ and all $v \in \mathbb{R}^2$.
3. The conformal factor $\varphi$ can be chosen to be $\kappa$-log-subharmonic.
4. The completion $Z$ of $\varphi \cdot D$ is a $\text{CAT}(\kappa)$ space.
5. The space $Z$ is homeomorphic to $\bar{D}$, the map $u : \varphi \cdot D \to X$ is $1$-Lipschitz and extends to a majorization $v : Z \to X$ of $\Gamma$.

Proof. (1) follows by definition of a minimal filling and Lemma 4.10. (2) is verified in [LW17, Theorem 11.3]. (3) is verified in [Mes01]. In order to verify (4), we use (3) and Theorem 4.7 to see that $\varphi \cdot D$ is locally $\text{CAT}(\kappa)$. The area of $\varphi \cdot D$ can be computed as $H^2(\varphi \cdot D) = \int_D \varphi^2 < \frac{1}{2}l^2$. In particular, if $\kappa > 0$, we have $H^2(\varphi \cdot D) < \frac{2\pi}{\kappa}$. Thus, (4) is a consequence of Corollary 4.5. Finally, (5) is verified in [LS17, Theorem 9] for $\kappa = 0$, hence also for $\kappa \leq 0$. The proof applies without changes to the case $\kappa > 0$. □

5. Main result

5.1. Local control of curvature under conformal changes. Along the lines of [LS17], we are going to prove the following local version of Theorem 1.4.

**Theorem 5.1.** For $c, C, \kappa, \lambda \in \mathbb{R}$ there exists some $\rho_0 = \rho_0(c, C, \kappa, \lambda) > 0$ with the following property.

Let $X$ be a $\text{CAT}(\kappa)$ space and let $f : X \to [c, C]$ be a Lipschitz continuous $\lambda$-convex function. Further, let $Y = e^f \cdot X$ denote the conformally equivalent space. Then any closed ball of radius at most $\rho_0$ in $Y$ is $\text{CAT}(\bar{\kappa})$, where

- $\bar{\kappa} = e^{-2C} \cdot (\kappa - 4\lambda)$ if $\kappa - 4\lambda \leq 0$;
- $\bar{\kappa} = e^{-2C} \cdot (\kappa - 4\lambda)$ if $\kappa - 4\lambda \geq 0$.

**Proof.** Set $Y = e^f \cdot X$. Since $f$ is bounded, the identity map from $Y$ to $X$ is a bilipschitz homeomorphism. Thus any complete subset of $X$ is also complete in $Y$. Rescaling $Y$ by the factor $e^{-c}$, thus subtracting the constant $c$ from $f$ we may assume that $c = 0$.

Choose a positive constant $\Lambda < e^{-C} \cdot \frac{2\pi}{\sqrt{\kappa}}$.

We claim that any Jordan curve $\Gamma$ in $Y$ of length $< \Lambda$ is majorized by a $\text{CAT}(\bar{\kappa})$ space. Fix $\Gamma$ and denote by $\hat{\Gamma}$ the curve $\Gamma$ considered in $X$. Since the identity map $Y \to X$ is $1$-Lipschitz, the length of $\hat{\Gamma}$ in $X$ is at most $\Lambda < \frac{2\pi}{\sqrt{\kappa}}$.

By Lemma 4.10, we obtain a minimal filling $u$ of $\hat{\Gamma}$ in $X$, whose properties are described in Theorem 4.11. Denote by $\varphi$ the conformal
factor of $u$. By Theorem 4.11, $u: \varphi \cdot D \to X$ extends to a majorization $v: Z \to X$, where the completion $Z$ of $\varphi \cdot D$ is CAT($\kappa$). Moreover, the area of $Z$ is less than $\frac{1}{16} \Lambda^2 < e^{-2C} \cdot \frac{2\pi}{\kappa}$.

Due to Theorem 1.5 and the conformality of $u$, the composition $f \circ u$ fulfills $\Delta(f \circ u) \geq 2 \lambda \cdot \varphi^2$ weakly. Hence, Lemma 4.9 ensures that $e^{(f \circ u)} \cdot Z$ is CAT($\bar{\kappa}$). Moreover, the majorization $v: e^{(f \circ u)} \cdot Z \to e^f \cdot X = Y$ of $\Gamma$.

Thus, any Jordan curve $\Gamma$ of length less than $\Lambda$ in $Y$ is majorized by a CAT($\bar{\kappa}$) space. We finish the proof by setting $\rho_0 = \frac{1}{4}$ and applying Lemma 4.3.}

□

5.2. Global versions. Now we can turn to the main theorems.

Proof of Theorem 1.4. Due to Theorem 5.1, the space $Y$ is a complete length space, which is locally CAT($\bar{\kappa}$). It remains to globalize the statement.

Assume first that $\lambda > 0$ and consider the gradient flow $\Phi_t$ of the function $f$ on the space $X$. Let $\Gamma$ denote a rectifiable closed curve in $Y$. Considering $\Gamma$ as a curve in $X$, we apply the gradient flow $\Phi_t$ to $\Gamma$ and obtain closed curves $\Gamma_t$ in $X$. The value of $f$ (and hence of $e^f$) does not increase along flow lines of $\Phi$. Since $\Phi_t$ contracts length in $X$, at least by a factor of $e^{-\lambda t}$, Lemma 5.1 we deduce the following two consequences. Firstly, the $e^f$-length of $\Gamma_t$ (thus the length of $\Gamma_t$ in $Y$) is non-increasing in $t$. Secondly, for any $\Gamma$ as above, any $\epsilon > 0$ and any sufficiently large $t$, the length of $\Gamma_t$ in $Y$ is less than $\epsilon$.

Taking $\epsilon$ to be smaller than $\rho_0$ in Theorem 5.1 and applying the globalization Lemma 4.3, we deduce that $Y$ is CAT($\bar{\kappa}$).

It remains to deal with the case $\lambda \leq 0$ and $\kappa - 4\lambda \leq 0$. But then $\kappa \leq 0$, hence $X$ is simply connected. Since $X$ is homeomorphic to $Y$, we deduce from the theorem of Cartan–Hadamard that $Y$ is CAT($\bar{\kappa}$). □

5.3. Conclusions. We can now easily prove:

Theorem 5.2. Let $x$ be a point in a CAT(0) space $X$. Define the function $f: X \to \mathbb{R}$ by $f(y) := \frac{1}{2}d^2(x, y)$. Then the space $Y = e^f \cdot X$ is CAT($\kappa$). Moreover, for any $R > 0$, the closed ball $\bar{B}_R(x)$ around $x$ in $Y$ is CAT($\kappa$) for some $\kappa = \kappa(R) < 0$.

Proof. The function $f$ is 1-convex and Lipschitz continuous on bounded balls. Thus, $Y$ is CAT(0) by [LS17].

By Lemma 4.1, the closed ball $\bar{B}_R(x)$ in $Y$ has the form $e^f \cdot B$, where $B$ is the closed ball $\bar{B}_r(x)$ in $X$ and $r(R)$ is such that

$$\int_0^r e^{\frac{1}{2}t^2} \, dt = R .$$

From Theorem 1.4 we deduce that $e^f \cdot B$ is CAT($\kappa$) with

$$\kappa = -4 \cdot e^{-r^2} .$$
Finally we can provide

**Proof of Theorem 1.1.** Clearly, we may assume $\kappa > 0$ and, by rescaling, even $\kappa = 1$.

Thus let $X$ be a CAT(1) space and let $O = B_r(x)$ be an open ball in $X$ with $r < \frac{\pi}{2}$.

We can replace $X$ by the closed ball $\bar{B}_r(x)$. In order to simplify the calculation we proceed as follows. First we improve the curvature bound on the closed ball to 0. In a second step, we change the metric on the open ball, to make it complete and simultaneously decrease the upper curvature bound to $-1$.

There exists $A > 0$ depending only on $r$, such that the function $g(y) = A \cdot d^2(x, y)$ is 1-convex on $X$. Due to Theorem 1.4, the space $Z = e^g \cdot X$ is a CAT(0) space. Moreover, by Lemma 4.1, the subset $e^g \cdot O \subset Z$ is an open ball in $Z$ around the point $x$. Replacing the space $X$ by $Z$ we have reduced our task to the case $\kappa = 0$. In this case the function $g(y) = \frac{1}{2}d^2(x, y)$ is 1-convex on $X$.

Now consider the function $h : [0, \frac{r^2}{2}) \to \mathbb{R}$ given by

$$ h(t) = -\log\left(\frac{\frac{r^2}{2} - t}{2}\right). $$

Then the function $h$ is convex and $\lim_{t \to \frac{r^2}{2}} h(t) = \infty$. Moreover,

$$ h'(t) = e^{h(t)}. $$

Consider the locally Lipschitz continuous function $f(y) := h(g(y))$ on $O = B_r(x)$ and the space $Y = e^f \cdot O$.

For an arbitrary point $y \in O$, we choose a small closed ball $U$ around $y$, such that for all $z \in U$ holds

$$ h'(g(z)) \geq \frac{1}{2}h'(g(y)) = \frac{1}{2}e^{h(g(y))} \quad \text{and} \quad h(g(z)) \leq 2h(g(y)). $$

Due to the convexity of $h$ and the 1-convexity of $g$, the restriction of $f$ to any geodesic $\gamma$ in $O$ is at least $\lambda$-convex, where $\lambda$ denotes the minimum of $h'$ on the image $g(\gamma) \subset [0, \frac{r^2}{2})$.

Hence for any such ball $U$, the space $e^f \cdot U$ is CAT(-1), by Theorem 1.4. This shows that the space $Y$ is locally CAT(-1).

For any $s < r$ we deduce from Lemma 4.1, that the subset $e^f \cdot \bar{B}_s(x) \subset e^f \cdot O$ coincides with the closed ball in $Y$ around $x$ of radius $R(s) = \int_0^s h\left(\frac{t^2}{2}\right) dt = -\int_0^s \log\left(\frac{\frac{r^2}{2} - t^2}{2}\right) dt$

Moreover, this ball is CAT(0) by Theorem 1.4. Since $R(s)$ converges to infinity as $s$ converges to $r$, we deduce that $Y$ is CAT(0). In particular, it is complete, simply connected and geodesic. Since we have already seen that $Y$ is locally CAT(-1), this finishes the proof. □
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