Reflection and transmission of elastic waves in non-local band-gap metamaterials: a comprehensive study via the relaxed micromorphic model

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May 31, 2016

Abstract

In this paper we propose to study wave propagation, transmission and reflection in band-gap mechanical metamaterials via the relaxed micromorphic model. To do so, guided by a suitable variational procedure, we start deriving the jump duality conditions to be imposed at surfaces of discontinuity of the material properties in non-dissipative, linear-elastic, isotropic, relaxed micromorphic media. Jump conditions to be imposed at surfaces of discontinuity embedded in Cauchy and Mindlin continua are also presented as a result of the application of a similar variational procedure. The introduced theoretical framework subsequently allows the transparent set-up of different types of micro-macro connections granting the description of both i) internal connexions at material discontinuity surfaces embedded in the considered continua and, as a particular case, ii) possible connections between different (Cauchy, Mindlin or relaxed micromorphic) continua. The established theoretical framework is general enough to be used for the description of a wealth of different physical situations and can be used as reference for further studies involving the need of suitably connecting different continua in view of (meta-)structural design. In the second part of the paper, we focus our attention on the case of an interface between a classical Cauchy continuum on one side and a relaxed micromorphic one on the other side in order to perform explicit numerical simulations of wave reflection and transmission. This particular choice is descriptive of a specific physical situation in which a classical material is connected to a phononic crystal. The reflective properties of this particular interface are numerically investigated for different types of possible micro-macro connections, so explicitly showing the effect of different boundary conditions on the phenomena of reflection and transmission. Finally, the case of the connection between a Cauchy continuum and a Mindlin one is also presented as a numerical study, so showing that band-gap description is not possible for such continua, in strong contrast with the relaxed micromorphic case.

Key words: micromorphic elasticity, dynamic problem, wave propagation, band-gap phenomena, interface, reflection, transmission.

AMS 2010 subject classification: 74A10 (stress), 74A30 (nonsimple materials), 74A35 (polar materials), 74A60 (micromechanical theories), 74B05 (classical linear elasticity), 74M25 (micromechanics), 74Q15 (effective constitutive equations), 74J05 (Linear waves), 74A50 (Structured surfaces and interfaces, coexistent phases)

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1 Introduction

1.1 Band gap metamaterials and the relaxed micromorphic model

Engineering metamaterials showing exotic behaviors with respect to both mechanical and electromagnetic wave propagation are recently attracting growing attention for their numerous possible astonishing applications [2, 16, 29, 30]. Actually, materials which are able to “stop” or “bend” the propagation of waves of light or sound with no energetic cost could suddenly disclose rapid and extremely innovative technological advancements.

In this paper, we focus our attention on those metamaterials which are able to “stop” wave propagation, i.e. metamaterials in which waves within precise frequency ranges cannot propagate. Such frequency intervals at which wave inhibition occurs are known as frequency band-gaps and their intrinsic characteristics (characteristic values of the gap frequency, extension of the band-gap, etc.) strongly depend on the metamaterial microstructure. Such unorthodox dynamical behavior can be related to two main different phenomena occurring at the micro-level:

- local resonance phenomena (Mie resonance): the micro-structural components, excited at particular frequencies, start oscillating independently of the matrix thus capturing the energy of the propagating wave which remains confined at the level of the microstructure. Macroscopic wave propagation thus results to be inhibited.

- micro-diffusion phenomena (Bragg scattering): when the propagating wave has wavelengths which are small enough to start interacting with the microstructure of the material, reflection and transmission phenomena occur at the micro-level that globally result in an inhibited macroscopic wave propagation.

Such resonance and micro-diffusion mechanisms (usually a mix of the two) are at the basis of both electromagnetic and elastic band-gaps (see e.g. [2]) and they are manifestly related to the particular microstructural topologies of the considered metamaterials. Indeed, it is well known (see e.g. [2, 30, 31]) that the characteristics of the microstructures strongly influence the macroscopic band gap behavior. In this paper, we will be concerned with mechanical waves, even if some of the used theoretical tools can be thought to be suitably generalized for the modeling of electromagnetic waves as well. Such generalizations could open new long-term research directions, for example in view of the modeling of so called “phoxonic crystals” which are simultaneously able to stop both electromagnetic and elastic wave propagation [2, 30].

In recent contributions [21, 22] we proposed a new generalized continuum model, which we called relaxed micromorphic which is able to account for the onset of microstructure-related frequency band-gaps while remaining in the macroscopic framework of continuum mechanics. Well posedness results have already been proved for this model. It turns out that the relaxed micromorphic model is the only macroscopic continuum model known to date which is simultaneously able to account for

- the onset and prediction of complete band-gaps in metamaterials

- the possibility of non-local effects via the introduction of higher order derivatives of the micro distortion tensor in the strain energy density.

Effective numerical homogenization methods [26] have been recently introduced which allow to account for frequency band gap at the homogenized level. Such methods make use of a “separation of scales hypothesis” which basically implies that the vibrations of the microstructural elements remain confined in the considered unit cells. Such hypothesis intrinsically leads, at the homogenized level, to generalized models, sometimes called internal variable models, in which no space derivatives of the introduced internal variable appear. In other words, non-local effects cannot be accounted for at the homogenized level.

Our relaxed micromorphic model allows to account for the possibility of non-local effects in band-gap metamaterials and contains the internal variable model as a degenerate limit case when suitably setting the characteristic length to be zero. Even if the internal variable model can be considered to be an acceptable tool for studying the behavior of a certain sub-class of band gap metamaterials, the fact of neglecting a priori any non-locality might be hazardous since microstructured materials are intrinsically expected to exhibit non-local behaviors when subjected to particular loading and/or boundary conditions [23].

The main interest of using macroscopic theories for the description of the behavior of materials with microstructures can be found in the fact that they feature the introduction of few parameters which are, in an
averaged sense, reminiscent of the presence of an underlying microstructure. If, on the one hand, this fact provides a drastic modeling simplification which is optimal to proceed towards (meta-)structural design, some drawbacks can be reported which are mainly related to the difficulty of directly relating the introduced macroscopic parameters to the specific characteristics of the microstructure (topology, microstructural mechanical properties, etc.). In order to account in detail for the effect of the underlying microstructures on the overall mechanical properties of the material at the homogenized level, enhanced homogenization techniques, including higher order terms in the performed expansions of the micro-fields, may be used [14, 15, 10, 36].

The aforementioned difficulty of explicitly relating macro-parameters to micro-properties is often seen as a limit for the effective application of enriched continuum models. As a matter of fact, it is the authors’ belief that such models are a necessary step if one wants to proceed towards the engineering design of metastructures, i.e., structures which are made up of metamaterials as building blocks. Of course, the proposed model will introduce a certain degree of simplification, but it is exactly this simplicity that makes possible to envision the next step which is that of proceeding towards the design of complex structures made of metamaterials.

To be more precise, the relaxed micromorphic model proposed here, is able to describe the onset of the first (and sometimes the second) band-gap which occurs at lower frequencies. In order to catch more complex behaviors the kinematics and the constitutive relations of the proposed model should be further enriched in a way that is not yet completely clear. Nevertheless, we do not see this fact as a limitation since we intend to use the unorthodox dynamical behavior of some metamaterials exhibiting band-gaps to fit, by inverse approach, the parameters of the relaxed micromorphic model following what has been done e.g. in [23]. This fitting, when successfully concluded for some specific metamaterials will allow the setting up of the design of metastructures by means of tools which are familiar to engineers, such as Finite Element codes. Of course, as classical Cauchy models show their limits for the description of the dynamical behavior of metamaterials, even at low frequencies, the relaxed micromorphic model will show its limit for higher frequencies, yet remaining accurate enough for accounting for some macroscopic manifestations of microstructure. To proceed in this direction, we intended to use the simplest possible model (linear, elastic) which is able to account for the wanted phenomena (band-gap onset and description). This allows for the introduction of few extra parameters that may be calibrated on the basis of suitable experimental or numerical “discrete” simulations. This is sufficient when remaining in the linear-elastic framework which is the target of the present paper. Generalized continuum models of the micromorphic type featuring the description of band-gaps when introducing non-linearities in the micro-inertia terms can also be found in the literature [13], but the interpretation of the introduced non-linearities would be more complex to be undertaken.

1.2 Boundary conditions and reflection and transmission in metamaterials

The problem of studying boundary conditions to be imposed at surfaces of discontinuity of the material properties in Cauchy continua is classically studied e.g. in [1] and comes back to the fact of assigning jumps of forces and/or jumps of displacements at the considered interfaces.

As far as generalized continua are concerned (micromorphic, micropolar, second gradient or also porous media), the setting up of correct boundary conditions becomes more delicate and can be successfully achieved using suitable variational procedures [21, 32, 6, 22, 20, 33, 19, 8, 18, 7, 6, 28].

In this paper, basing ourselves upon an appropriate variational procedure, we obtain the jump conditions that have to be imposed at internal surfaces of discontinuity of the material properties in relaxed micromorphic continua. The considered surfaces do not possess their own material properties (mass, inertia, etc.), but such generalization could be easily achieved using the methods presented in [27, 35]. As a result of the use of similar variational arguments, we also present the analogous jump conditions that have to be imposed at internal surfaces of discontinuity in Cauchy and Mindlin continua. On the basis of the introduced sets of jump conditions we are able to establish

- different types of (internal) micro-macro connections which are possible between two relaxed micromorphic media, as well as between two Cauchy or two Mindlin continua
- As a particular case of the preceding point, we are able to deduce different possible types of connections between different continua (Cauchy, relaxed micromorphic or Mindlin).

The general theoretical framework introduced in the present paper allows to deal with a wealth of different boundary conditions which may be of use for the description of different physical situations corresponding to
specific connections between metamaterials or between classical materials and metamaterials. In the second part of the paper, we decide to focus our attention on the case of the different possible connections between a Classical Cauchy medium and a relaxed micromorphic one, since, as shown in [23], it is of use for the simulation of experiments of reflection and transmissions at the interface between an aluminum plate and a phononic crystal of the type proposed e.g. by [17]. Notwithstanding the focus given to this particular case for the implementation of the proposed numerical simulations, the present paper is intended to be a reference for all subsequent works concerning the correct setting up of boundary conditions at internal interfaces in Cauchy, relaxed micromorphic and Mindlin continua, as well as for the proper description of connections at interfaces between different media.

It has been proven [12, 24] that the relaxed micromorphic continuum is a degenerate model, in the sense that only the tangential part of the micro-distortion tensor field $P$ can be assigned in order to have a well posed problem. This is not the case in standard Mindlin’s micromorphic model in which all the 9 components of the micro-distortion tensor $P$ must be assigned at the considered boundary.

Due to the complexity of the kinematics of micromorphic media (3+9=12 degrees of freedom) a great variety of connections can be envisaged at material surfaces of discontinuity in such continua. The fact of establishing in a clear fashion all the possible jump conditions that can be imposed in micromorphic media is a delicate point which is rarely treated with the due care in the literature. Indeed, as far as boundary conditions in standard micromorphic media are concerned, Mindlin [25] and Eringen [11] propose suitable jump conditions to be imposed at surfaces of discontinuities of the material properties. To the sake of completeness, after introducing the jump conditions that have to be used in relaxed micromorphic media, we also present the analogous conditions for Mindlin and Cauchy continua, so recovering classical results.

After having introduced the theoretical tools which are needed to deal with interfaces in Cauchy, Mindlin and relaxed micromorphic media, we focus on the study of reflection and transmission of waves at Cauchy/relaxed and Cauchy/Mindlin interfaces, respectively. We thus present numerical simulations showing the behavior of the reflection coefficient as function of the frequency of the traveling waves and for different types of boundary conditions (internal clamp with fixed microstructure and internal clamp with free microstructure). To the author’s knowledge, such studies of reflection and transmission properties in the spirit of micromorphic modeling are not found in the literature. Some results of a rather simplified 1D situation can be found in [3] where some expressions for reflection and transmission coefficients are presented, but finally not exploited in the proposed numerical simulations.

1.3 Organization of the paper

The paper is organized as follows

- In Section 2 we start recalling the equations of motion and jump duality conditions that can be imposed at surfaces of discontinuity of the material properties in classical Cauchy continua. This is useful to suitably introduce the generalizations that occur for discontinuity surfaces in relaxed micromorphic and Mindlin continua. In fact, the equations of motions and associated jump duality conditions are derived for both such generalized continua. At the end of the section we particularize our findings to the case of an interface between a Cauchy continuum and a relaxed micromorphic (or a Mindlin) medium.

- In Section 3 we explicitly set up a series of different connections which are possible at surfaces of discontinuity in Cauchy, relaxed micromorphic or Mindlin’s media. All the introduced connections are conceived in order to be compatible with the jump duality conditions presented in Section 2. If, as expected, the possible constraints in classical Cauchy media are the internal clamp, the free boundary and the fixed boundary, a great variety of more complex connections can be envisaged in micromorphic media, due to the richer kinematics allowing for microstructural motions. We present the complete list of all possible micro-macro connections both in relaxed micromorphic and Mindlin’s media, but we will focus our attention on two of them, namely the internal clamp with free microstructure and the internal clamp with fixed microstructure. These two constraints impose continuity of the displacement of the macroscopic matrix and allow particular kinematics at the level of the microstructure. Such two constraints will be used in the sequel to study wave reflection and transmission at interfaces between classical Cauchy and relaxed micromorphic (or Mindlin’s) media.
In section 4 we derive the principle of conservation of total energy $E$ for Cauchy, relaxed micromorphic and Mindlin’s media in the form $E_t + \text{Div} \, H = 0$, where $H$ is the energy flux vector. If the definition of the total energy $E = J + W$ is straightforward once the kinetic energy $J$ and potential energy $W$ are introduced for the considered continuum, the computation of the energy flux for generalized media is more elaborate. The explicit form of the energy fluxes is established in terms of micro and macro velocities and of stresses and hyper-stresses.

In section 5 we introduce what we will call plane wave ansatz in the remainder of the paper. This hypothesis consists of assuming that all unknown fields of displacement $u$ and micro-distortion tensor $P$ only depend on one scalar space variable $x_1$ which will also coincide with the direction of propagation of the considered waves. This hypothesis allows us to rewrite the governing equations of the considered continua in a simplified form. In particular, for relaxed and standard micromorphic media, we are able to obtain systems of uncoupled partial differential equations for longitudinal and for transverse waves, as well as for some other waves which are only connected to purely microstructural deformation modes and which we call “uncoupled waves”. The jump conditions to be imposed at surfaces of discontinuity of the material properties and the expressions of the energy fluxes are also particularized to the considered 1D case.

In section 6 we study plane wave propagation in semi-infinite Cauchy, relaxed micromorphic and standard Mindlin’s media. This allows to unveil the band gap behavior of the relaxed micromorphic media in opposition to Cauchy and Mindlin ones. To present the dispersion curves of the considered generalized models, we follow a procedure similar to the one presented in [11, 12] where dispersion curves for standard micromorphic media are provided. In addition to the previously presented results, we provide extra arguments concerning the asymptotic behavior of the dispersion curves which are the main feature of the relaxed micromorphic model allowing for band-gap description. This step concerning the study of bulk wave propagation is mandatory for the future determination of the constitutive parameters of our relaxed micromorphic model on real band-gap metamaterials. We identify specific cut-off frequencies and characteristic velocities which are related both to the macro and micro material properties of the considered generalized continua. We finally present the case of an internal variable model as a degenerate limit case of our relaxed model when setting $L_c = 0$. This model does not allow for non-local effects, but can be still thought to describe band-gap behaviors in particular metamaterials as the ones considered in [16, 26, 34]. Nevertheless, the fact that a clear singularity occurs (the solution for the case $L_c = 0$ is different from that which is found for very small but non-vanishing $L_c$) indicates that the case $L_c = 0$ could lead to imprecise results compared to the relaxed micromorphic model with small $L_c$.

In section 7 we study the phenomena of reflection and transmission at the considered interfaces, generalizing the classical definitions of reflection and transmission coefficients. We show that the reflective properties of the relaxed micromorphic and of Mindlin’s media are drastically different, especially in the vicinity of the frequencies for which band-gaps are likely to occur. We repeat this study for two different constraints (internal clamp with free microstructure and internal clamp with fixed microstructure) and show that the constraints which are imposed on the microstructure of the considered generalized media have a relevant effect on the global reflective properties of the interfaces which are investigated. In particular, we claim that by suitably varying the parameters of the relaxed micromorphic model, the constraint of internal clamp with free microstructure can provide a second band gap (additional to the one evidenced in the study of bulk propagation) which is completely due to the presence of the interface. Even if we do not explicitly present this case of “double band-gap” in the present paper, we mention this possibility that also allows the description of local resonances at the level of the microstructure (see [23]).

1.4 Notational agreement

In this paper, we denote by $\mathbb{R}^{3 \times 3}$ the set of real $3 \times 3$ second order tensors, written with capital letters. We denote respectively by $\cdot, :$ and $\langle \cdot, \cdot \rangle$ a simple and double contraction and the scalar product between two tensors of any suitable order. Everywhere we adopt the Einstein convention of sum over repeated indices if $1$For example, $(A \cdot v)_i = A_{ij}v_j$, $(A \cdot B)_{ik} = A_{ij}B_{jk}$, $A : B = A_{ij}B_{ji}$, $(C \cdot B)_{ijk} = C_{ijp}B_{pk}$, $(C : B)_i = C_{ijp}B_{pj}$, $\langle v, w \rangle = v \cdot w = v_iw_i$, $(A, B) = A_{ij}B_{ij}$ etc.
not differently specified. The standard Euclidean scalar product on $\mathbb{R}^{3\times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3\times 3}} = \text{tr}(X \cdot Y^T)$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{R}^{3\times 3}$. The identity tensor on $\mathbb{R}^{3\times 3}$ will be denoted by $\mathbf{I}$, so that $\text{tr}(X) = \langle X, \mathbf{I} \rangle$.

We consider a body which occupies a bounded open set $B_L$ of the three-dimensional Euclidean space $\mathbb{R}^3$ and assume that its boundary $\partial B_L$ is a smooth surface of class $C^2$. An elastic material fills the domain $B_L \subset \mathbb{R}^3$ and we refer the motion of the body to rectangular axes $Ox_i$. We denote by $\Sigma$ any material surface embedded in $B_L$. We also denote by $n$ the outward unit normal to $\partial B_L$, or to the surface $\Sigma$ embedded in $B_L$.

In the following, given any field $a$ defined on the surface $\Sigma$ we will also set

$$[a] := a^+ - a^-,$$

which defines a measure of the jump of $a$ through the material surface $\Sigma$, where

$$[\cdot]^+ := \lim_{x \to \Sigma^+} [\cdot], \quad [\cdot]^− := \lim_{x \to \Sigma} [\cdot],$$

and where we denoted by $B_L^-$ and $B_L^+$ the two subdomains which results form dividing the domain $B_L$ through the surface $\Sigma$.

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on $B_L$ with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$, respectively will be denoted by $L^2(B_L)$. Moreover, we introduce the standard Sobolev spaces

$$\begin{align*}
\mathcal{H}^1(B_L) &= \{ u \in L^2(B_L) \mid \text{grad } u \in L^2(B_L) \}, \quad \|u\|_{\mathcal{H}^1(B_L)} := \|u\|_{L^2(B_L)} + \|\text{grad } u\|_{L^2(B_L)}, \\
\mathcal{H}^1(\text{curl}; B_L) &= \{ v \in L^2(B_L) \mid \text{curl } v \in L^2(B_L) \}, \quad \|v\|_{\mathcal{H}^1(\text{curl}; B_L)} := \|v\|_{L^2(B_L)} + \|\text{curl } v\|_{L^2(B_L)}, \\
\mathcal{H}^1(\text{div}; B_L) &= \{ v \in L^2(B_L) \mid \text{div } v \in L^2(B_L) \}, \quad \|v\|_{\mathcal{H}^1(\text{div}; B_L)} := \|v\|_{L^2(B_L)} + \|\text{div } v\|_{L^2(B_L)}.
\end{align*}$$

of functions $u$ or vector fields $v$, respectively.

For vector fields $v$ with components in $\mathcal{H}^1(B_L)$, i.e. $v = (v_1, v_2, v_3)^T, v_i \in \mathcal{H}^1(B_L)$, we define $\nabla v = ((\nabla v_1)^T, (\nabla v_2)^T, (\nabla v_3)^T)^T$, while for tensor fields $P$ with rows in $\mathcal{H}^1(\text{curl}; B_L)$, resp. $\mathcal{H}^1(\text{div}; B_L)$, i.e. $P = (P_1^T, P_2^T, P_3^T), P_i \in \mathcal{H}^1(\text{curl}; B_L)$ resp. $P_i \in \mathcal{H}^1(\text{div}; B_L)$ we define $\text{Curl } P = ((\text{curl } P_1)^T, (\text{curl } P_2)^T, (\text{curl } P_3)^T)^T$, $\text{Div } P = (\text{div } P_1, \text{div } P_2, \text{div } P_3)^T$. The corresponding Sobolev-spaces will be denoted by

$$\begin{align*}
\mathcal{H}^1(B_L), \quad \mathcal{H}^1(\text{Div}; B_L), \quad \mathcal{H}^1(\text{Curl}; B_L).
\end{align*}$$

## 2 Equations of motion and jump duality conditions

In the present section, we present, as the result of the application of a suitable variational principle, the bulk equations and associated jump duality conditions that have to be verified at internal interfaces in Cauchy, relaxed micromorphic and Mindlin continua, respectively. Such preliminary theoretical framework is needed for the subsequent introduction of

- different types of internal connections in Cauchy, relaxed micromorphic or Mindlin continua (internal surfaces embedded in such continua)
- as a particular case of the previous point, connections between different media (any combination of Cauchy, Mindlin and relaxed micromorphic).

Let us consider a fixed time $t_0 > 0$ and a bounded domain $B_L \subset \mathbb{R}^3$. We introduce the action functional of the considered system to be defined as

$$A = \int_0^{t_0} \int_{B_L} (J - W) \, dX \, dt,$$

where $J$ and $W$ are the kinetic and potential energies of the considered system.
As for the kinetic energy, we consider that it takes the following form for a Cauchy medium and a micromorphic (standard or relaxed) medium respectively

\[
J = \frac{1}{2} \rho \| u_{,t} \|^2, \quad J = \frac{1}{2} \rho \| u_{,t} \|^2 + \frac{1}{2} \eta \| P_{,t} \|^2,
\]

(4)

where \( u \) denotes the classical macroscopic displacement field and \( P \) is the micro-distortion tensor which accounts for independent micro-motions at lower scales. On the other hand, as it will be explained in the following, the strain energy density \( W \) takes a different form depending whether one considers a classical Cauchy medium, a relaxed medium or a standard micromorphic one.

In what follows, we will explicitly derive, both for classical Cauchy and micromorphic (standard and relaxed) media, the equations of motion in strong form as well as the jump duality conditions which have to be imposed at material discontinuity surfaces in such media and which are intrinsically compatible with the least action principle associated to the action functional \( J \). Starting from the derived duality conditions we will deduce different types of possible connections between Cauchy and generalized media which are automatically compatible with the associated bulk equations.

### 2.1 Classical Cauchy medium

In this Subsection, since it will be useful in the following, we recall that the strain energy density for the classical, linear-elastic, isotropic Cauchy medium takes the form

\[
W = \mu_{\text{macro}} \| \text{sym} \nabla u \|^2 + \frac{\lambda_{\text{macro}}}{2} (\text{tr} (\text{sym} \nabla u))^2,
\]

(5)

where \( \lambda_{\text{macro}} \) and \( \mu_{\text{macro}} \) are the classical Lamé parameters and \( u \) denotes the classical macroscopic displacement field.

The associated equations of motion in strong form, obtained by a classical least action principle take the usual form\(^2\)

\[
\rho u_{,tt} = \text{Div} \sigma, \quad \rho u_{i,tt} = \sigma_{ij,j},
\]

(6)

where

\[
\sigma = 2 \mu_{\text{macro}} \text{sym} \nabla u + \lambda_{\text{macro}} \text{tr} (\text{sym} \nabla u) \mathbf{1}, \quad \sigma_{ij} = \mu_{\text{macro}} (u_{i,j} + u_{j,i}) + \lambda_{\text{macro}} u_{k,kj} \delta_{ij}
\]

is the classical Cauchy stress tensor for isotropic materials.

#### 2.1.1 Jump duality conditions in classical Cauchy media

In Cauchy continua, only force-displacement duality conditions are possible, and take the form

\[
\| \langle f, \delta u \rangle \| = 0, \quad \| f_i \delta u_i \| = 0,
\]

(8)

with

\[
f = \sigma \cdot n, \quad f_i = \sigma_{ij} n_j.
\]

#### 2.2 Relaxed micromorphic medium

The strain energy density for the relaxed medium is given by

\[
W = \mu_c \| \text{sym} (\nabla u - P) \|^2 + \frac{\lambda_c}{2} (\text{tr} (\nabla u - P))^2 + \mu_c \| \text{skew} (\nabla u - P) \|^2
\]

\[ + \mu_{\text{micro}} \| \text{sym} P \|^2 + \frac{\lambda_{\text{micro}}}{2} (\text{tr} P)^2 + \frac{\mu_c L_c}{2} \| \text{Curl} P \|^2.
\]

\(^2\)Here and in the sequel we denote by the subscript \( t \) the partial derivative with respect to time of the considered field.

\(^3\)Here and in the sequel we equivalently write, for the sake of completeness, our equations both in compact and in index form.
Imposing the first variation of the action functional to be vanishing (i.e. \( \delta A = 0 \)), integrating by parts a suitable number of times and the considering arbitrary variations \( \delta u \) and \( \delta P \) of the basic kinematical fields, we obtain the strong form of the bulk equations of motion of considered system (see also [12, 21, 22, 24]) which read

\[
\begin{align*}
\rho u_{tt} &= \text{Div} \, \vec{\sigma} , \\
\eta P_{tt} &= \vec{\sigma} - s - \text{Curl} \, m , \\
\rho u_{i,t} &= \vec{\sigma}_{ij,j} , \\
\eta P_{ij,t} &= \vec{\sigma}_{ij} - s_{ij} - m_{ik,p} \epsilon_{jpk} ,
\end{align*}
\]

(11)

where \( \epsilon_{jpk} \) is the Levi-Civita alternator and

\[
\begin{align*}
\vec{\sigma} &= 2 \mu_e \text{sym} (\nabla u - P) + \lambda_c \text{tr} (\nabla u - P) \mathbb{1} + 2 \mu_e \text{skew} (\nabla u - P) , \\
s &= 2 \mu_{\text{micro}} \text{sym} P + \lambda_{\text{micro}} \text{tr} P \mathbb{1} , \\
m &= \mu_e L_c^2 \text{Curl} \, P ,
\end{align*}
\]

or equivalently, in index notation:

\[
\begin{align*}
\vec{\sigma}_{ij} &= \mu_e (u_{i,j} - P_{ij} + u_{j,i} - P_{ji}) + \lambda_c (u_{k,k} - P_{kk}) \delta_{ij} + \mu_e (u_{i,j} - P_{ij} - u_{j,i} + P_{ji}) \\
s_{ij} &= \mu_{\text{micro}} (P_{ij} + P_{ji}) + \lambda_{\text{micro}} P_{kk} \delta_{ij} , \\
m_{ik} &= \mu_e L_c^2 P_{ia,b} \epsilon_{kba}.
\end{align*}
\]

(12)

Since it is useful for further calculations, we explicitly note that the last term in the last equation [11] can be rewritten in terms of the basic kinematical fields as:

\[
(C \text{curl} \, m)_{ij} = m_{ik,p} \epsilon_{jpk} = \mu_e L_c^2 P_{ia,bp} \epsilon_{kba} \epsilon_{jpk} = \mu_e L_c^2 P_{ia,bp} (\delta_{kj}\delta_{ap} - \delta_{kp}\delta_{aj}) = \mu_e L_c^2 (C \text{curl} \, P)_{ij}.
\]

We remark that as soon as in the last balance equation [11] (which is in duality with \( \delta P_{ij} \)) one sets \( j = p = 1, 2, 3 \), then the relaxed term \( m_{ik,p} \epsilon_{jpk} = \mu_e L_c^2 (P_{ip,jp} - P_{ij,pp}) \) is identically vanishing. This means that in the considered relaxed model, there are three terms which are vanishing compared to the standard micromorphic one. This is equivalent to say that we are somehow lowering the order of the considered differential system, so that, as we will see, less boundary conditions will be necessary in order to have a well-posed problem.

### 2.2.1 Jump duality conditions for the relaxed medium

Together with the bulk governing equations, the least action principle simultaneously provides the duality jump conditions which can be imposed at surfaces of discontinuity of the material properties in relaxed media which reads

\[
[\langle t, \delta u \rangle] = 0, \quad [\langle \tau, \delta P \rangle] = 0,
\]

or, equivalently, in index notation

\[
[t_{ij} \delta u_i] = 0, \quad [\tau_{ij} \delta P_{ij}] = 0,
\]

(13)

with

\[
t = \vec{\sigma} \cdot n , \quad t_i = \vec{\sigma}_{ij} n_j , \quad \tau = -m \cdot \epsilon \cdot n , \quad \tau_{ij} = -m_{ik} \epsilon_{kj} n_h .
\]

(14)

It can be checked that, given the definition (14) of the double forces \( \tau_{ij} = -m_{ik} \epsilon_{kj} n_h = \mu_e L_c^2 (P_{ij,k} - P_{ik,j}) n_h \), three out of the nine components of the double force \( \tau \) are identically vanishing. For example, this check is immediate when choosing the normal oriented along the \( x_1 \) axis: \( n = (1, 0, 0) \), since it is straightforward that \( \tau_{11} = \tau_{21} = \tau_{31} = 0 \). This fact implies that only six of the nine duality jump conditions \( [\tau_{ij} \delta P_{ij}] = 0 \) are actually independent so that one can claim that the set of boundary conditions to be imposed in a relaxed model is actually underdetermined with respect to the standard micromorphic model. This result is equivalent to the one presented in [12, 24] in which it is said that only tangential boundary conditions on \( P \) must be imposed in order to prove existence and uniqueness for the relaxed problem.
2.3 Standard micromorphic model

The strain energy density for the standard Mindlin’s micromorphic model is

\[
W = \mu \| \text{sym} (\nabla u - P) \|^2 + \frac{\lambda}{2} (\text{tr} (\nabla u - P))^2 + \mu_c \| \text{skew} (\nabla u - P) \|^2 \\
+ \mu_{\text{micro}} \| \text{sym} P \|^2 + \frac{\lambda_{\text{micro}}}{2} (\text{tr} P)^2 + \frac{\mu_c L^2}{2} \| \nabla P \|^2.
\]  

The equations of motion obtained by the associated least action principle are

\[
\rho u_{tt} = \text{Div} \tilde{\sigma}, \quad \rho u_{i,tt} = \tilde{\sigma}_{ij,j},
\]

\[
\eta P_{tt} = \tilde{\sigma} - s + \text{Div} M, \quad \eta P_{ij,tt} = \tilde{\sigma}_{ij} - s_{ij} M_{ijk,k},
\]

where

\[
M = \mu_c L^2 g \nabla P, \quad M_{ijk} = \mu_c L^2 g P_{ij,k}.
\]

2.3.1 Jump duality conditions for the standard micromorphic medium

The jump duality conditions arising from the least action principle read

\[
[ \langle t, \delta u \rangle ] = 0, \quad [ \langle \tilde{\tau}, \delta P \rangle ] = 0,
\]

or, equivalently, in index notation

\[
[ t_i \delta u_i ] = 0, \quad [ \tilde{\tau}_{ij} \delta P_{ij} ] = 0,
\]

where the force \( t \) is the same as in the relaxed micromorphic case, while the double-forces \( \tilde{\tau} \) are defined now as

\[
\tilde{\tau}_{ij} = \mu_c L^2 P_{ij,k} n_k.
\]

2.4 Connections between a classical Cauchy medium and a relaxed micromorphic (or a standard Mindlin’s) medium

Once that the possible jump duality conditions are established at surfaces of discontinuity of the material properties of all the introduced continua (Cauchy, relaxed micromorphic, standard micromorphic), the subsequent step is to interconnect two different continua by introducing suitable connections which are compatible with such jump duality conditions. These connections can be envisaged by suitably exploiting the jump duality conditions introduced in Sections 2.1.1-2.3.1. In particular, connections between Cauchy/relaxed, relaxed/relaxed, Cauchy/standard-micromorphic, standard-micromorphic/standard-micromorphic, relaxed/standard-micromorphic media may be introduced as particular cases of the duality jump conditions previously presented. Although we present here the general theoretical framework for establishing all such connections, we postpone to further investigations the interesting problem of studying wave reflection and transmission in all these cases, limiting ourselves to treat here in detail the case of reflection and transmission of waves at a surface of discontinuity between a classical Cauchy medium and a relaxed micromorphic (or standard Mindlin) medium. We chose to focus our attention on this particular case essentially for two reasons

- In a classical Cauchy medium band gaps are not allowed, so that propagation occurs for any real frequency. This is a good feature if one wants to control the properties of the incident wave and to be sure to be able to send a wave at the considered interface for any real frequency value.

- The great majority of engineering materials can be modeled by means of classical Cauchy continuum theories, so that in view of possible applications (see [17, 23]) it is reasonable to start with a standard material on one side of the discontinuity.
Before studying wave reflection and transmission at a discontinuity surface between a Cauchy and a relaxed micromorphic medium, we need to explicitly set up the jump duality conditions that may be established at such interfaces.

It is possible to check that the duality jump conditions \( (8) \) and \( (13) \) for the Cauchy and relaxed micromorphic media respectively, allow us to conclude that the following relations must be verified at the interface between Cauchy and relaxed micromorphic media

\[
\langle f, \delta u^- \rangle - \langle t, \delta u^+ \rangle = 0,
\langle \tau, \delta P^+ \rangle = 0,
\]

or, equivalently, in index notation

\[
f_i \delta u_i^- - t_i \delta u_i^+ = 0, \quad \tau_{ij} \delta P_{ij}^+ = 0.
\] (18)

The case of an interface between a Cauchy and a Mindlin medium is formally equivalent to the case treated here, except that the double force \( \tau \) for a relaxed medium has 3 vanishing components out of the 9, while in the standard Mindlin’s medium it has all the 9 components which are non-vanishing. We will explain in more detail this fact in the following sections.

3 How to consider different connections between two generalized continua

If one thinks to classical structural mechanics, it is immediate to understand that the same structural elements can be interconnected using different constraints (for example, in beam theory, one can deal with clamps, pivots, rollers, etc.). The reasoning that we present in this sections is intended to establish analogous results for generalized continua, in view of the effective modeling of complex (meta-)structures.

In this spirit, we present some considerations concerning the possible choice of different boundary conditions to be imposed at surfaces of discontinuity between two different generalized media. We start by considering the case in which we have on the two sides the same type of medium (Cauchy/Cauchy, relaxed/relaxed, standard/standard). Such boundary conditions can be subsequently straightforwardly generalized to the case in which two different media are considered on the two sides. We start by presenting different connections between two Cauchy continua, since they are classical and the reader will have an immediate feeling of the physics which is involved. We will then generalize such constraints to the case of relaxed media and standard micromorphic media, trying to understand which are their intrinsic meanings with a particular attention to their possible physical interpretation.

In other words, in this section we come back to the duality conditions established in Section 2 and we list all the possible jump conditions that one can envisage and which satisfy these duality conditions. We are going to provide a list of all the possible sets of jump conditions which are intrinsically compatible with the strong form of the considered equations of motion due to the fact that both (PDEs and BCs) are deduced from the same variational principle.

We remind once again that at the end of the paper we will be devoted to the numerical study of wave reflection and transmission at an interface between a classical Cauchy continuum and a relaxed micromorphic continuum since this can be of use to describe experimental results of the type presented in [17]. Nevertheless, for the sake of completeness, we present here all the possible connections between different types of continua (Cauchy, relaxed micromorphic and standard micromorphic) in order to clearly establish the general theoretical framework for further investigations and generalizations.

3.1 Connections between two Cauchy media

In this subsection, we present some well known types of internal constraints between two Cauchy continua starting from the analysis of the duality conditions \( (8) \). As it is classically established, in order to verify the jump duality conditions \( (8) \) one can suitably impose displacements and/or forces, so giving rise to different types of constraints. In the case of classical Cauchy continua, when considering kinematical constraints, it is only possible to impose conditions on displacement and not on its space derivatives as instead happens for second gradient continua (or also for Euler-Bernoulli beams). It is for this reason that only three types of constraints
can be analyzed in the case of Cauchy continua, namely the internal clamp, the free boundary and the fixed boundary. When one can impose continuity of the higher derivatives of displacement (as in second gradient theory), then one can envisage more complex types of boundary conditions such as internal hinges and internal rollers (see e.g. [9]).

In the following Subsections, we will explicitly set up all the possible sets of boundary conditions which can be imposed at an interface between two classical Cauchy media. We will see that, independently of the type of constraint, we always end up with $3 + 3 = 6$ scalar conditions to be imposed at the aforementioned interface in order to have a well-posed problem.

We finally recall that in a classical Cauchy medium the constitutive expression of forces $f$ is given by Eq. (9) together with (7).

3.1.1 Internal clamp

When imposing continuity of the three components of displacement between the two sides of the considered discontinuity, one obtains what we call an internal clamp. This constraint is equivalent to a continuity constraint and can be used as a check of the used numerical code for testing that an incident wave continues undisturbed to propagate when reaching the interface at which the internal clamp is located. As we said, the conditions to be imposed are given by the continuity of displacement as follows

$$\|u_i\| = 0, \quad i = 1, 2, 3.$$  

Since the virtual displacements must be compatible with the imposed boundary conditions, also the virtual displacements must verify such jump conditions at the considered interface $[\delta u_1] = [\delta u_2] = [\delta u_3] = 0$. Such conditions on the virtual displacements, together with the duality conditions (8), imply for a generalized internal clamp also the following jump conditions on force must be satisfied

$$\|f_i\| = 0, \quad i = 1, 2, 3.$$  

3.1.2 Free boundary

We call free boundary that type of connection between the two Cauchy media corresponding to which the displacements on the two sides of the considered medium can be completely arbitrary on the two sides. This corresponds to the physical situation in which the two media are simply in contact, without any specific constraint. This implies that also the virtual displacements $\delta u^+$ and $\delta u^-$ on the two sides of the interface can be completely arbitrary. This means that, in order to have the duality jump conditions (8) to be satisfied, one necessarily has to have

$$f_i^+ = f_i^- = 0, \quad i = 1, 2, 3.$$  

In other words, when assigning arbitrary displacements on the two sides, one necessarily has to have vanishing forces on the two sides of the boundary in order to have a well posed problem.

3.1.3 Fixed boundary

We call fixed boundary that type of connection in which we set

$$u_i^+ = u_i^- = 0, \quad i = 1, 2, 3.$$  

Such connection corresponds to the physical situation in which the two Cauchy media are fixed on the two sides of the considered surface.

3.2 Connections between two relaxed micromorphic media

When considering micromorphic media (standard or relaxed) one can impose more kinematical boundary conditions than in the case of Cauchy continua. More precisely, one can act on the displacement field $u$ and also on the micro-distortion $P$. Clearly, more options are possible with respect to the case of classical Cauchy continua, so that we can introduce new types of constraints. In particular, we will show that, for any type of possible
connections between two relaxed micromorphic media, we always have \(3 + 3 + 6 + 6 = 18\) scalar conditions to be imposed at the considered interface.

We recall that for a relaxed micromorphic medium the constitutive expression for the force \(t\) and double-force \(\tau\) are given by Eq. \([14]\) together with \([12]\).

### 3.2.1 Micro/macro internal clamp

Generalizing what has been done previously, we impose continuity of the macro-displacement and we also consider continuity of the micro-distortion as follows

\[
[u_i] = 0, \quad [P_{ij}] = 0, \quad i = 1, 2, 3, \quad j = 2, 3,
\]

where we remember that the directions 2, 3 are the directions tangent to the considered surface. Imposing these \(3 + 6 = 9\) kinematical continuity conditions we are basically saying that there is no interruption either in the macroscopic matrix and in the tangent part of the micro-distortion tensor at the considered interface. Such conditions, together with the jump duality conditions \([13]\) also imply that the following conditions on forces and double-forces must be satisfied

\[
[t_i] = 0, \quad [\tau_{ij}] = 0, \quad i = 1, 2, 3, \quad j = 2, 3.
\]

As it has been pointed out before (see Section \(2.2\), in a relaxed model only 6 out of the 9 jump conditions on double forces are actually non-vanishing, so that we let here the subscript \(j\) take only the values 2 and 3. This choice is coherent to that of choosing the normal to the surface to take the particular form \(\mathbf{n} = (1, 0, 0)\) which is what we will do in all the remainder of this paper.

### 3.2.2 Free boundary

In this case the two media are simply in contact, with neither micro- nor macro-connection across the interface. In this case both micro-distortion and macro-displacement are arbitrary on both sides of the interface which implies the following 18 scalar conditions to be imposed at the interface

\[
t_i^+ = t_i^- = 0, \quad \tau_{ij}^+ = \tau_{ij}^- = 0, \quad i = 1, 2, 3, \quad j = 2, 3.
\]

Such conditions imply that all the micro and macro motions are left arbitrary.

### 3.2.3 Fixed boundary

We call fixed relaxed boundary that situation in which

\[
u_i^+ = u_i^- = 0, \quad P_{ij}^+ = P_{ij}^- = 0, \quad i = 1, 2, 3, \quad j = 2, 3.
\]

This case corresponds to the situation such that macroscopic matrix and the tangent part of the micro-distortion are blocked on both sides of the considered interface. Equivalently, this means that forces and the tangent part of double forces may take arbitrary values on the two sides of the interface.

### 3.2.4 Macro internal clamp with free microstructure

Another type of possible connection between two generalized media is the case in which the macroscopic matrix is continuous, while the microstructure is disconnected on the two sides. In formulas, this case can be formulated by saying that \(\delta P_{ij}\) are arbitrary on the two sides, while the macro-displacements verify the kinematical jump conditions

\[
[u_i] = 0, \quad i = 1, 2, 3.
\]

Such kinematical conditions, together with the duality conditions \([13]\) also imply that

\[
[t_i] = 0, \quad i = 1, 2, 3.
\]

The condition of free microstructure is finally given by the conditions

\[
\tau_{ij}^+ = \tau_{ij}^- = 0, \quad i = 1, 2, 3, \quad j = 2, 3
\]

which, in order to respect the duality conditions \([13]\), actually imply that the micro-distortions \(\delta P_{ij}\) are left arbitrary on the two sides.
3.2.5 Macro internal clamp with fixed microstructure

The macroscopic clamp is given by the continuity of displacement

\[ \| u_i \| = 0, \quad i = 1, 2, 3. \]

Such kinematical conditions, together with the duality conditions also imply that

\[ \| t_i \| = 0, \quad i = 1, 2, 3. \]

The condition of fixed microstructure is given by

\[ P^+_{ij} = P^-_{ij} = 0, \quad i = 1, 2, 3, \quad j = 2, 3, \]

that means that the double-force is left arbitrary on the two sides of the considered surface.

3.2.6 Micro internal clamp with free macrostructure

This type of constraint considers a connection between the two relaxed media which is only made through the microstructure. In other words, the microstructure is continuous across the interface, while the macroscopic matrix is free to move independently on the two sides. In this case the kinematical conditions to be imposed are

\[ \| P_{ij} \| = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

Such conditions together with the considered duality conditions imply

\[ \| \tau_{ij} \| = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

The condition of free macro-displacement is instead implied by imposing vanishing forces on the two sides as follows

\[ t^+_i = t^-_i = 0, \quad i = 1, 2, 3. \]

3.2.7 Micro internal clamp with fixed macrostructure

The condition of fixed macrostructure is given by

\[ u^+_i = u^-_i = 0, \quad i = 1, 2, 3. \]

In addition, we have continuity of the microstructure through the condition

\[ \| P_{ij} \| = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

The last jump condition also implies that

\[ \| \tau_{ij} \| = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

3.2.8 Fixed macrostructure and free microstructure

This connection is guaranteed by the relationships

\[ u^+_i = u^-_i = 0, \quad \tau^+_i = \tau^-_i = 0, \quad i = 1, 2, 3, \quad j = 2, 3, \]

which guarantees that the two sides of the interface remain fixed and that, on the other hand, the microstructure is able to deform arbitrarily on the two sides.

3.2.9 Free macrostructure and fixed microstructure

This is the converse situation with respect to the previous case, i.e. the macrostructure can move arbitrarily, while the microstructure cannot move on the two sides of the interface. Such connection is guaranteed by the conditions

\[ t^+_i = t^-_i = 0, \quad P^+_{ij} = P^-_{ij} = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]
3.3 Connections between two standard micromorphic media

The types of connections which are possible between two standard micromorphic media are formally the same as those presented for the relaxed micromorphic medium. The fundamental difference is that, in this last case, all the nine components of \( P \) (and hence all the nine components of the double-force \( \tilde{\tau} \)) are independent, so that one has to impose 24 instead of 18 scalar conditions at the interface. Hence, to rigorously define the possible constraints between two standard micromorphic media, it is sufficient to reformulate the jump conditions given in the previous Subsection by setting now \( \tau = \tilde{\tau} \) and \( j = 1, 2, 3 \), instead of \( j = 2, 3 \) as in the previous case. This means that, independently of the specific type of connection, one must always count \( 3 + 3 + 9 + 9 = 24 \) scalar conditions to be imposed at an interface between two standard micromorphic media.

We recall that for a standard micromorphic medium the constitutive expression for the force \( t \) is the same as in the relaxed micromorphic case (see Eq. (14) together with (12)) while the double-force \( \tilde{\tau} \) is redefined according to Eq. (17).

3.4 Connections between a Cauchy and a relaxed micromorphic medium

The study of the possible connections between a Cauchy and a relaxed micromorphic medium can be performed by suitably particularizing the results obtained in subsections 3.1 and 3.2. In particular, when considering connections between a Cauchy and a micromorphic medium (standard or relaxed) one can impose more kinematical boundary conditions than in the case of connections between Cauchy continua (see subsection 3.1), but less than in the case of connections between two micromorphic media (see subsection 3.2). More precisely, one can act on the displacement field \( u \) (on both sides of the interface) and also on the micro-deformation \( P \) (on the side of the interface occupied by the micromorphic continuum). We will show that for any type of possible connections between a Cauchy and a micromorphic medium, we always have \( 3 + 3 + 6 = 12 \) scalar conditions to be imposed at the interface. We recall that, in what follows, we consider the “−” region occupied by the Cauchy continuum and the “+” region occupied by the relaxed micromorphic continuum, so that, accordingly, we set

\[
\begin{align*}
\sigma_{ij}^- n_j^- & \quad \tau_{ij}^- = -m_{ik}^+ \epsilon_{kjh} n_h^+ ,
\end{align*}
\]

We recall once again that in a classical Cauchy medium the constitutive expression of forces \( f \) is given by Eq. (9) together with (7), while for a relaxed micromorphic medium the constitutive expression for the force \( t \) and double-force \( \tau \) are given by Eq. (14) together with (12).

3.4.1 Macro internal clamp with fixed microstructure

In the case in which we have a Cauchy continuum in contact with a relaxed micromorphic medium the conditions given in Subsections 3.2.1 and 3.2.5 collapse in the same set of condition which, in this particular case, become

\[
\begin{align*}
\| u_i \| = 0 , \quad t_i - f_i = 0 , \quad P_{ij}^- = 0 , \quad i = 1, 2, 3 , \quad j = 2, 3 .
\end{align*}
\]

\[\begin{array}{c}
\text{Cauchy} \\
\text{Relaxed}
\end{array}\]

\[\begin{array}{c}
[|u_{ij}|]=0 \\
P_{ij}^+=0
\end{array}\]

Figure 1: Macro internal clamp with fixed microstructure.
As we will see in the following, this will be one of the most representative connections when studying wave reflection and transmission at an interface between a Cauchy and a relaxed micromorphic continuum.

### 3.4.2 Free boundary

The conditions of free boundary between a Cauchy and a relaxed micromorphic medium can be deduced particularizing the conditions given in Subsection [3.2.2](#) and read

\[ t_i = f_i = 0, \quad \tau_{ij} = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

Such conditions imply that all the micro and macro motions are left arbitrary.

### 3.4.3 Fixed boundary

The conditions for a fixed boundary between a Cauchy and a relaxed micromorphic medium can be obtained from the conditions presented in Subsections [3.2.3](#) and [3.2.7](#) which, in the considered particular case, collapse in the following set of conditions

\[ u_i^+ = u_i^- = 0, \quad P_{ij}^+ = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

This case corresponds to the situation where the macrostructure is blocked on both sides of the considered interface, while the microstructure is blocked on the side of the relaxed micromorphic medium.

### 3.4.4 Macro internal clamp with free microstructure

Another type of possible connection between two generalized media is the case in which the macroscopic matrix is continuous, while the microstructure of the relaxed medium is free to move at the interface. This type of connection can be obtained suitably particularizing the jump conditions given in Subsection [3.2.4](#) which now become

\[ \begin{align*}
\lbrack u_i \rbrack &= 0, \\
t_i - f_i &= 0, \\
\tau_{ij} &= 0, \quad i = 1, 2, 3, \quad j = 2, 3.
\end{align*} \]

Figure 2: Macro internal clamp with free microstructure.

Also for this case, we will see that it is actually one of the more interesting connections when studying wave reflection and transmission at an interface between a Cauchy continuum and a relaxed micromorphic one.

### 3.4.5 Free macrostructure and fixed microstructure

Such connection is a particular case of the more general conditions [3.2.6](#) and [3.2.9](#) which in this case collapse in the same set of conditions, namely

\[ \begin{align*}
t_i = f_i &= 0, \\
P_{ij}^+ &= 0, \quad i = 1, 2, 3, \quad j = 2, 3.
\end{align*} \]
3.4.6 Fixed macrostructure and free microstructure

This connection is a particular case of the conditions given in Subsection 3.2.8 and is provided by the relationships

\[ u^+_i = u^-_i = 0, \quad \tau_{ij} = 0, \quad i = 1, 2, 3, \quad j = 2, 3. \]

This connection guarantees that the two sides of the interface remain fixed and that, on the other hand, the microstructure is able to deform arbitrarily on the side occupied by the relaxed medium.

4 Conservation of total energy

It is known that if one considers conservative mechanical systems, like in the present paper, then conservation of total energy must be verified in the form

\[ E_t + \text{div} \, H = 0, \quad (19) \]

where \( E = J + W \) is the total energy of the system and \( H \) is the energy flux vector. It is clear that the explicit expressions for the total energy and for the energy flux are different depending on whether one considers a classical Cauchy model, a relaxed micromorphic model or a standard micromorphic model. If the expression of the total energy \( E \) is straightforward for the three mentioned cases (it suffices to look at the given expressions of \( J \) and \( W \)), the explicit expression of the energy flux \( H \) can be more complicated to be obtained. For this reason we specify the expression of the energy fluxes for the three cases in the following Subsections and we propose their detailed deduction.

The explicit derivation of the energy fluxes for the considered continua is a fundamental step towards the subsequent definition of the reflection and transmission coefficients at the interface between two (meta-)materials.

4.1 Classical Cauchy continuum

In classical Cauchy continua the flux vector \( H \) can be written as

\[ H = -\sigma \cdot u_t, \quad H_k = -\sigma_{ik} u_{kt}, \quad (20) \]

where the Cauchy stress tensor \( \sigma \) has been defined in (7) in terms of the displacement field.

Indeed, recalling expressions (4) and (5) for the kinetic and potential energy densities respectively, together with definition (7) for the stress tensor, we can write

\[ E_t = J_t + W_t = \rho \langle u_t, u_t \rangle + \langle (2\mu_{\text{macro}} \text{sym} \nabla u + \lambda_{\text{macro}} \text{tr}(\text{sym} \nabla u) \mathbb{I}) \cdot \text{sym} \nabla (u_t) \rangle 
= \rho \langle u_t, u_t \rangle + \langle \sigma \cdot \text{sym} \nabla (u_t) \rangle. \]

Using the equations of motion (6) for replacing the quantity \( \rho u_{tt} \), using the fact that \( \langle \sigma \cdot \text{sym} \nabla (u_t) \rangle = \text{Div} (\sigma \cdot u_t) - \text{Div} \sigma \cdot u_t \) and simplifying we have:

\[ E_t = \text{Div} \sigma \cdot u_t + \text{Div} (\sigma \cdot u_t) - \text{Div} \sigma \cdot u_t = \text{Div} (\sigma \cdot u_t). \]

This last expression for the time derivative of the total energy \( E \) yields expression (20) for the energy flux \( H \).

4.2 Relaxed micromorphic model

As for the relaxed micromorphic continuum, the energy flux vector \( H \) is defined as

\[ H = -\tilde{\sigma}^T \cdot u_t - (m^T \cdot P_{\epsilon}) : \epsilon, \quad H_k = -u_{ik} \tilde{\sigma}_{ik} - m_{ik} P_{ij,k} \epsilon_{jkh}, \quad (21) \]

where the stress tensor \( \tilde{\sigma} \) and the hyper-stress tensor \( m \) have been defined in (12) in terms of the basic kinematical fields.
To prove that in relaxed micromorphic media the energy flux takes the form (21), we start noticing that, using eqs. (4) and (10), the time derivative of the total energy $E$ can be computed as

$$
E_t = \rho \langle u, u_t \rangle + \eta \langle P_t, P_{tt} \rangle + \langle 2 \mu_c \text{sym} (\nabla u - P), \text{sym} (\nabla u - P) \rangle
+ \lambda_s \text{tr} (\nabla u - P)^2 + \rho \lambda_s \text{tr} (\nabla u - P), \nabla u_t - P_t \rangle
+ \langle 2 \mu_c \text{skew} (\nabla u - P), \text{skew} (\nabla u_t - P_t) \rangle
+ 2 \mu_{\text{micro}} \text{sym} P, \text{sym} P_t \rangle.
$$

or equivalently, using definitions (12) for $\bar{\sigma}, s,$ and $m$.

$$
E_t = \rho \langle u, u_t \rangle + \eta \langle P_t, P_{tt} \rangle + \langle 2 \mu_c \text{sym} (\nabla u - P), \text{sym} P_t \rangle
+ \lambda_s \text{tr} (\nabla u - P) \mathbf{1} + \langle 2 \mu_c \text{skew} (\nabla u - P), \nabla u_t - P_t \rangle
+ \langle 2 \mu_{\text{micro}} \text{sym} P, \text{sym} P_t \rangle.
$$

Using now the equations of motion (11) to replace the quantities $\rho u_{tt}$ and $\eta P_{tt}$, recalling that $\langle m, \text{Curl} P_t \rangle = \text{Div} \left( (m^T \cdot P_t) : \epsilon \right)$, manipulating and simplifying, it can be recognized that

$$
E_t = \langle u, \text{Div} \sigma \rangle + \langle P_t, \bar{\sigma} - s - \text{Curl} m \rangle + \text{Div} \langle u_t, \bar{\sigma} \rangle - \langle u_t, \text{Div}\bar{\sigma} \rangle - \langle \bar{\sigma} - s, P_t \rangle + \langle m, \text{Curl} P_t \rangle
$$

$$
= \text{Div} \left( \bar{\sigma}^T \cdot u_t \right) - \langle \text{Curl} m, P_t \rangle + \text{Div} \left( (m^T \cdot P_t) : \epsilon \right) + \langle \text{Curl} m, P_t \rangle
$$

$$
= \text{Div} \left( \bar{\sigma}^T \cdot u_t + (m^T \cdot P_t) : \epsilon \right).
$$

### 4.3 Standard micromorphic model

Concerning the standard micromorphic model, the energy flux $H$ can be introduced as

$$
H = -\bar{\sigma}^T \cdot u_t - P_t^T : M, \quad H_k = -u_{it} \bar{\sigma}_{ik} - P_{i,t} M_{ijk}.
$$

Analogously to what done for the relaxed micromorphic model we can compute the first time derivative of the total energy as follows

$$
E_t = \rho \langle u, u_{tt} \rangle + \eta \langle P_{tt}, P_{tt} \rangle + \langle \bar{\sigma}, \nabla u_t \rangle - \langle \bar{\sigma} - s, P_t \rangle + \langle M, \nabla P_t \rangle
$$

$$
= \langle u_t, \text{Div}\bar{\sigma} \rangle + \langle P_t, \bar{\sigma} - s + \text{Div} M \rangle + \text{Div} \left( \bar{\sigma}^T \cdot u_t \right) - \langle u_t, \text{Div}\bar{\sigma} \rangle - \langle \bar{\sigma} - s, P_t \rangle + \langle M, \nabla P_t \rangle
$$

$$
= \text{Div} \left( \bar{\sigma}^T \cdot u_t \right) + \langle \text{Div} M, P_t \rangle + \text{Div} \left( P_t^T : M \right) - \langle \text{Div} M, P_t \rangle = \text{Div} \left( u_t \cdot \bar{\sigma} + P_t^T : M \right),
$$

where the equations of motion have been used to replace the quantities to replace the quantities $\rho u_{tt}$ and $\eta P_{tt}$, together with the identity $\langle M, \nabla P_t \rangle = \text{Div} \left( P_t^T : M \right) - \langle \text{Div} M, P_t \rangle$. The obtained expression for the time derivative of the total energy $E_t$ proves the expression (21) for the energy flux in a standard micromorphic medium.

### 5 Plane wave ansatz: simplification of the governing equations and boundary conditions

In order to proceed towards the numerical exploitation of some of the theoretical results presented in the first part of the paper and in view of suitable applications to cases of real interest (17, 23), we particularize to the case of plane waves the previously derived equilibrium equations and associated jump conditions for the relaxed micromorphic, the standard micromorphic and the Cauchy continua. In what follows we call “plane wave ansatz” the hypothesis according to which the unknown fields $(u$ and $P$) are supposed to depend only on one component $x_1$ of the space variable $X = (x_1, x_2, x_3)$ which is also supposed to be the direction of propagation of the considered waves. We will also refer to such simplified framework as “1D case” if no confusion can arise.

The simplified partial differential equations obtained by means of this hypothesis will be solved using different types of possible jump conditions in order to study reflection and transmission of plane waves at a Cauchy-relaxed interface and at a Cauchy-standard-micromorphic interface. Since it will be needed to determine the reflection and transmission coefficients, we also derive the particularized 1D form of the energy flux $H$ in all such media.
5.1 Classical Cauchy medium

We briefly recall that in the case of plane waves (the displacement field \( u = (u_1, u_2, u_3) \) is supposed to depend only on the first component \( x_1 \) of the space variable \( X = (x_1, x_2, x_3) \), the equations of motion in strong form take the form,

\[
\begin{align*}
    u_{1,t} &= \frac{\lambda_{\text{macro}} + 2\mu_{\text{macro}}}{\rho} u_{1,11}, \\
    u_{2,t} &= \frac{\mu_{\text{macro}}}{\rho} u_{2,11}, \\
    u_{3,t} &= \frac{\mu_{\text{macro}}}{\rho} u_{3,11}.
\end{align*}
\]

(23)

It can be noticed that the first equation only involves the longitudinal displacement, while the last two equations involve transverse displacements in the 2 and 3 directions.

5.1.1 Jump conditions in the 1D case

Considering the unit normal to be directed along the \( x_1 \) axis, the jump duality conditions simplify into

\[
\begin{align*}
    [f_1 \delta u_1] &= 0, \\
    [f_2 \delta u_2] &= 0, \\
    [f_3 \delta u_3] &= 0
\end{align*}
\]

with

\[
\begin{align*}
    f_1 &= (\lambda_{\text{macro}} + 2\mu_{\text{macro}}) u_{1,1}, \\
    f_2 &= \mu_{\text{macro}} u_{2,1}, \\
    f_3 &= \mu_{\text{macro}} u_{3,1}.
\end{align*}
\]

5.1.2 Energy flux for the Cauchy model in the 1D case

The first component of the energy flux vector given in Eq. (20), simplifies in the 1D case into

\[
H_1 = -u_{1,t} \left[ (\lambda_{\text{macro}} + 2\mu_{\text{macro}}) u_{1,1} \right] - u_{2,t} \left[ \mu_{\text{macro}} u_{2,1} \right] - u_{3,t} \left[ \mu_{\text{macro}} u_{3,1} \right].
\]

(24)

5.2 Relaxed micromorphic continuum

We first introduce the decomposition of \( P \) such that \( P_{ij} = P^D_{ij} + P^S \delta_{ij} \) and consider the new variables

\[
P^S = \frac{1}{3} P_{kk}, \quad P^D_{ij} = P_{ij} - P^S \delta_{ij} = (\text{dev } P)_{ij}.
\]

(25)

We also denote the symmetric and anti-symmetric part of \( P \) as

\[
P_{(ij)} = \frac{P_{ij} + P_{ji}}{2} = (\text{sym } P)_{ij}, \quad P_{[ij]} = \frac{P_{ij} - P_{ji}}{2} = (\text{skew } P)_{ij}
\]

(26)

and introduce the variable

\[
P^V = P_{22} - P_{33}.
\]

(27)

Conversely, we explicitly remark that the following relationships hold which relate the components of the micro-distortion tensor \( P \) to some of the new introduced variables:

\[
\begin{align*}
    P_{11} &= P^S + P^D_{11}, \\
    P_{22} &= \frac{1}{2} \left( P^V + 2P^S - P^D_{11} \right), \\
    P_{33} &= \frac{1}{2} \left( 2P^S - P^V - P^D_{11} \right), \\
    P_{23} &= P_{(23)} + P_{[23]}, \\
    P_{32} &= P_{(23)} - P_{[23]}, \\
    P_{1\alpha} &= P_{(1\alpha)} + P_{[1\alpha]}, \quad \alpha = 2, 3, \\
    P_{\alpha 1} &= P_{(1\alpha)} - P_{[1\alpha]}, \quad \alpha = 2, 3.
\end{align*}
\]

(28)

As done for Cauchy continua, also for relaxed micromorphic media we limit ourselves to the case of plane waves. In other words, we suppose that the space dependence of all the introduced kinematical fields \( u_i \) and \( P_{ij} \) is limited only to the component \( x_1 \) of \( X \) which we also will suppose to be the direction of propagation of the considered plane wave. Since it will be useful in the following, let us collect some of the new variables of our problem as

\[
\begin{align*}
    v_1 := (u_1, P^D_{11}, P^S), \\
    v_2 := (u_2, P_{(12)}, P_{[12]}), \\
    v_3 := (u_3, P_{(13)}, P_{[13]}), \\
    v_4 := P_{(23)}, \\
    v_5 := P_{[23]}, \\
    v_6 := P^V.
\end{align*}
\]

(29)

and, for having a homogeneous notation, let us set

\[
\begin{align*}
    v_4 := P_{(23)}, \\
    v_5 := P_{[23]}, \\
    v_6 := P^V.
\end{align*}
\]

(30)
Using the plane wave ansatz and with the proposed new choice of variables we are able to rewrite the governing equations \([11]\) for a relaxed micromorphic continuum as different uncoupled sets of equations (see \([21, 22]\) for extended calculations), namely:

\[
\begin{align*}
v_{1,t,t} &= A_1^R \cdot v''_1 + B_1^R \cdot v'_1 + C_1^R \cdot v_1, \\
v_{\alpha,t,t} &= A_\alpha^R \cdot v''_\alpha + B_\alpha^R \cdot v'_\alpha + C_\alpha^R \cdot v_\alpha, \quad \alpha = 2, 3
\end{align*}
\]

where \((\cdot)'\) denoted the derivative of the quantity \((\cdot)\) with respect to \(x_1\) and we set

\[
\begin{align*}
A_1^R &= \begin{pmatrix}
\lambda_e + 2\mu_e & 0 & 0 \\
0 & \frac{\mu_e L^2}{3\eta} - \frac{2\mu_e L^2}{3\eta} & 0 \\
0 & 0 & \frac{\mu_e L^2}{3\eta} - \frac{2\mu_e L^2}{3\eta}
\end{pmatrix}, & A_\alpha^R &= \begin{pmatrix}
\mu_e + \mu_{micro} & 0 & 0 \\
0 & \frac{\mu_e L^2}{2\eta} & \frac{\mu_e L^2}{2\eta} \\
0 & 0 & \frac{\mu_e L^2}{2\eta}
\end{pmatrix},

B_1^R &= \begin{pmatrix}
\frac{3\lambda_e + 2\mu_e}{\eta} & 0 & 0 \\
0 & \frac{2\mu_e}{\eta} - \frac{3\lambda_e + 2\mu_e}{\eta} & 0 \\
0 & 0 & \frac{3\lambda_e + 2\mu_e}{\eta}
\end{pmatrix}, & B_\alpha^R &= \begin{pmatrix}
\mu_e & 0 & 0 \\
0 & \frac{2\mu_e}{\eta} & \frac{2\mu_e}{\eta} \\
0 & -\frac{\mu_e}{\eta} & 0
\end{pmatrix}, \quad \alpha = 2, 3,

C_1^R &= C_\alpha^R = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
(3\lambda_e + 2\mu_e) + (3\lambda_{micro} + 2\mu_{micro}) & 0 & 0
\end{pmatrix}, & C_4^R &= C_6^R = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
2(\mu_e + \mu_{micro}) & 0 & 0
\end{pmatrix}, & C_5^R &= -\frac{2\mu_e}{\eta}.
\end{align*}
\]

These 12 scalar partial differential equations can be used to study plane wave propagation in the relaxed micromorphic media (as done in \([21, 22]\)). We remark again that, when studying bulk propagation of waves in unbounded relaxed micromorphic media by means of equations \((31)\), there are six different problems which can be studied separately for the variables \(v_1, v_2, v_3, v_4, v_5, v_6\) respectively. More precisely, the bulk differential equations \((31)\) associated to each of such six variables are completely uncoupled (see also \([21, 22]\)). As we will see, on the other hand, the problem becomes partially coupled when considering the bulk equations together with the jump conditions at a Cauchy/relaxed interface.

### 5.2.1 Jump duality conditions in the 1D case

Considering that the fields \(u_i\) and \(P_{ij}\) only depend on the variable \(x_1\) and supposing that the unit normal vector is directed along the \(x_1\) axis, i.e. \(n = (1, 0, 0)\) which is also the direction of propagation of waves, one can check that the jump duality conditions \([13]\) simplify into

\[
\begin{align*}
[\tau_i \delta u_i] &= 0, & \quad \alpha = 1, 2, 3, & \quad \beta = 1, 2, 3, \quad \iota = 1, 2, 3
\end{align*}
\]

or equivalently, in index notation

\[
\begin{align*}
[t_i \delta u_i] &= 0, & \quad \tau_{ij} \delta P_{ij} &= 0, & \quad i = 1, 2, 3, & \quad j = 1, 2, 3,
\end{align*}
\]

with

\[
\begin{align*}
t_1 &= \begin{pmatrix}
\lambda_e + 2\mu_e \\
0 \\
0
\end{pmatrix} \cdot v'_1 + \begin{pmatrix}
0 \\
-2\mu_e \\
(3\lambda_e + 2\mu_e)
\end{pmatrix} \cdot v_1 \\
t_\alpha &= \begin{pmatrix}
\mu_e + \mu_{micro} \\
0 \\
0
\end{pmatrix} \cdot v'_\alpha + \begin{pmatrix}
0 \\
-2\mu_e \\
2\mu_e
\end{pmatrix} \cdot v_\alpha, & \quad \alpha = 2, 3
\end{align*}
\]
and

\[
\begin{align*}
\tau_{11} &= 0, & \tau_{12} &= \frac{\mu_e L_c^2}{\mu_e L_c^2} \cdot v_1', & \tau_{13} &= \frac{\mu_e L_c^2}{\mu_e L_c^2} \cdot v_3', \\
\tau_{21} &= 0, & \tau_{22} &= \frac{\mu_e L_c^2}{2} \cdot v_1' + \mu_e L_c^2 \cdot \frac{v_4' + v_5'}{2}, & \tau_{23} &= \mu_e L_c^2 \cdot (v_4' + v_5'), \\
\tau_{31} &= 0, & \tau_{32} &= \mu_e L_c^2 \cdot (v_4' - v_5'), & \tau_{33} &= \frac{\mu_e L_c^2}{2} \cdot v_1' - \mu_e L_c^2 \cdot v_6'.
\end{align*}
\]

We can remark once again that, since three components of the relaxed double force are vanishing, 3 out of the 12 jump conditions (33) are automatically satisfied. This means that only 9 out of the 12 jump conditions (34) are independent. This is coherent with what has been underlined in [23] where it is proven that only tangential conditions on the micro-distortion tensor \( P \) can be imposed in a relaxed model.

Moreover, it can be remarked that, due to the expression of double forces given in Eqs. (36), the jump conditions (33) actually produce a partial coupling of the considered problem. More precisely, if the bulk equations for the introduced unknown variables \( v_1, v_2, v_3, v_4, v_5, v_6 \) are completely uncoupled, a coupling between the variables \( v_1, v_6 \) and \( v_4, v_5 \) intervenes through the jump conditions on double forces. This means that the study of reflection and transmission of waves at surfaces of discontinuity of the material properties cannot be studied without accounting for such coupling phenomena.

### 5.2.2 Energy flux for the relaxed model in the 1D case

When considering conservation of total energy, it can be checked that the first component of the energy flux (21) can be rewritten in terms of the new variables as

\[
H_1 = H_1^1 + H_1^2 + H_1^3 + H_1^4 + H_1^5 + H_1^6,
\]

with

\[
\begin{align*}
H_1^1 &= v_{1,t} \cdot \begin{bmatrix}
-(\lambda_e + 2\mu_e) & 0 & 0 \\
0 & \frac{-\mu_e L_c^2}{2} & \mu_e L_c^2 \\
0 & \mu_e L_c^2 & -2\mu_e L_c^2
\end{bmatrix} \cdot v_1' + \begin{bmatrix}
0 & 2\mu_e & 3(\lambda_e + 2\mu_e)
\end{bmatrix} \cdot v_1, \\
H_1^2 &= v_{2,t} \cdot \begin{bmatrix}
-(\lambda_e + 2\mu_e) & 0 & 0 \\
0 & \frac{-\mu_e L_c^2}{2} & \mu_e L_c^2 \\
0 & \mu_e L_c^2 & -2\mu_e L_c^2
\end{bmatrix} \cdot v_2' + \begin{bmatrix}
0 & 2\mu_e & -2\mu_e
\end{bmatrix} \cdot v_2, \\
H_1^3 &= v_{3,t} \cdot \begin{bmatrix}
-(\lambda_e + 2\mu_e) & 0 & 0 \\
0 & \frac{-\mu_e L_c^2}{2} & \mu_e L_c^2 \\
0 & \mu_e L_c^2 & -2\mu_e L_c^2
\end{bmatrix} \cdot v_3' + \begin{bmatrix}
0 & 2\mu_e & -2\mu_e
\end{bmatrix} \cdot v_3, \\
H_1^4 &= -2\mu_e L_c^2 \cdot (v_4)'_1 v_{4,t}, & H_1^5 &= -2\mu_e L_c^2 \cdot (v_5)'_1 v_{5,t}, & H_1^6 &= -\frac{\mu_e L_c^2}{2} \cdot (v_6)'_1 v_{6,t}.
\end{align*}
\]

### 5.3 Standard micromorphic

The governing equations (11) can be rewritten in terms of the new variables as

\[
\begin{align*}
v_{1,tt} &= A_1^C \cdot v_{1''} + B_1^C \cdot v_1' + C_1^C \cdot v_1, & v_{1,tt} &= A_1^C \cdot v_{1''} + B_1^C \cdot v_1' + C_1^C \cdot v_1, & \alpha = 2, 3
\end{align*}
\]

\[
\begin{align*}
v_{4,tt} &= A_4^C \cdot v_{4''} + C_4^C \cdot v_4, & v_{5,tt} &= A_5^C \cdot v_{5''} + C_5^C \cdot v_5, & v_{6,tt} &= A_6^C \cdot v_{6''} + C_6^C \cdot v_6.
\end{align*}
\]
where

\[
A_1^C = \begin{pmatrix}
\frac{\lambda_e + 2\mu_e}{\rho} & 0 & 0 \\
0 & \frac{\mu_e L_g^2}{\eta} & 0 \\
0 & 0 & \frac{\mu_e L_g^2}{\eta}
\end{pmatrix},
\quad A_\alpha^C = \begin{pmatrix}
\frac{\mu_e + \mu}{\rho} & 0 & 0 \\
0 & \frac{\mu_e L_g^2}{\eta} & 0 \\
0 & 0 & \frac{\mu_e L_g^2}{\eta}
\end{pmatrix},
\]

\[
B_1^C = B_1^R, \quad B_\alpha^C = B_\alpha^R, \quad C_1^C = C_1^R, \quad C_\alpha^C = C_\alpha^R, \quad \alpha = 2, 3,
\]

\[
A_4^C = A_5^C = A_6^C = \frac{\mu_e L_g^2}{\eta}, \quad C_4^C = C_6^C = C_4^R, \quad C_5^C = C_5^R.
\]

5.3.1 Jump duality conditions in the 1D case

Considering that the fields \(u_i\) and \(P_{ij}\) depend only on the variable \(x_1\) and that the unit normal vector is directed along the \(x_1\) axis, one can check that the jump duality conditions simplify into

\[
\lbrack (t, \delta u_i) \rbrack = 0, \quad \lbrack (\tilde{\tau}, \delta P) \rbrack = 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3, \quad \text{(40)}
\]

or equivalently, in index notation

\[
\lbrack t_i \delta u_i \rbrack = 0, \quad \lbrack \tilde{\tau}_{ij} \delta P_{ij} \rbrack = 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3, \quad \text{(41)}
\]

where the components of the force \(t\) are the same compared to the previous case (see Eqs. [35]), while the components of the double force are given by

\[
\tilde{\tau}_{11} = \begin{pmatrix} 0 \\ \mu_e L_g^2 \\ \mu_e L_g^2 \end{pmatrix} \cdot v'_i, \quad \tilde{\tau}_{12} = \begin{pmatrix} 0 \\ \mu_e L_g^2 \\ \mu_e L_g^2 \end{pmatrix} \cdot v'_j, \quad \tilde{\tau}_{13} = \begin{pmatrix} 0 \\ \mu_e L_g^2 \\ \mu_e L_g^2 \end{pmatrix} \cdot v'_3,
\]

\[
\tilde{\tau}_{21} = \begin{pmatrix} 0 \\ \mu_e L_g^2 \\ -\mu_e L_g^2 \end{pmatrix} \cdot v'_2, \quad \tilde{\tau}_{22} = \begin{pmatrix} 0 \\ -\mu_e L_g^2 \\ \mu_e L_g^2 \end{pmatrix} \cdot v'_1 + \frac{\mu_e L_g^2}{2} v'_6, \quad \tilde{\tau}_{23} = \mu_e L_g^2 (v'_1 + v'_5), \quad \text{(42)}
\]

\[
\tilde{\tau}_{31} = \begin{pmatrix} 0 \\ \mu_e L_g^2 \\ -\mu_e L_g^2 \end{pmatrix} \cdot v'_3, \quad \tilde{\tau}_{32} = \mu_e L_g^2 (v'_4 - v'_5), \quad \tilde{\tau}_{33} = \begin{pmatrix} 0 \\ -\mu_e L_g^2 \\ \mu_e L_g^2 \end{pmatrix} \cdot v'_1 - \frac{\mu_e L_g^2}{2} v'_6.
\]

We notice that in the case of standard micromorphic medium, all the nine components of the double force are no vanishing, which is equivalent to say that the set of boundary conditions is not under-determined as in the case of the relaxed model. Moreover, a coupling between the variables \(v_1, v_6\) and \(v_4, v_5\) is introduced also in this case through the jump conditions.

5.3.2 Energy flux for the standard micromorphic medium in the 1D case

When considering conservation of total energy, it can be checked that the first component of the energy flux may be rewritten in terms of the new variables as

\[
\bar{H}_1 = \bar{H}_1^1 + \bar{H}_1^2 + \bar{H}_1^3 + \bar{H}_1^4 + \bar{H}_1^5 + \bar{H}_1^6,
\]

with
Cauchy media simply replacing the expressions (44) in the equations of motion (23) and simplifying, so obtaining

\[ \tilde{H}_1^1 = v_{1,t} \cdot \begin{pmatrix} - (\lambda_c + 2\mu_c) & 0 & 0 \\ 0 & -3\mu_c L_g^2 & 0 \\ 0 & 0 & -3\mu_c L_g^2 \end{pmatrix} \cdot v_1' + \begin{pmatrix} 0 & 2\mu_c & (3\lambda_c + 2\mu_c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v_1, \]

\[ \tilde{H}_1^2 = v_{2,t} \cdot \begin{pmatrix} - (\mu_c + \mu_c) & 0 & 0 \\ 0 & -2\mu_c L_g^2 & 0 \\ 0 & 0 & -2\mu_c L_g^2 \end{pmatrix} \cdot v_2' + \begin{pmatrix} 0 & 2\mu_c & -2\mu_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v_2, \]

\[ \tilde{H}_1^3 = v_{3,t} \cdot \begin{pmatrix} - (\mu_c + \mu_c) & 0 & 0 \\ 0 & -2\mu_c L_g^2 & 0 \\ 0 & 0 & -2\mu_c L_g^2 \end{pmatrix} \cdot v_3' + \begin{pmatrix} 0 & 2\mu_c & -2\mu_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot v_3, \]

6 Planar wave propagation in semi-infinite media

Before approaching the problem of reflection and transmission at specific Cauchy/relaxed and Cauchy/Mindlin interfaces, we present in the present section the problem of the study of wave propagation in semi-infinite Cauchy, relaxed micromorphic and Mindlin continua. In particular, we consider bulk propagation of plane waves and we show the dispersion relations for classical Cauchy, relaxed micromorphic and standard micromorphic continua. This will allow to establish again the result provided in [21, 22] according to which the relaxed micromorphic continuum is the only generalized model which allows the description of band-gaps in a continuum framework. Additionally to a summary of the results already presented in [21, 22], we propose a systematic study of the asymptotic properties of the dispersion curves for the relaxed micromorphic continuum. In fact, the characteristic behavior of the dispersion curves for \( k \to \infty \) is fundamental for the assessment of the band-gap existence in the framework of the relaxed micromorphic model. Analogous results are briefly presented also for the Cauchy and Mindlin continua, so showing that the relaxed micromorphic model is in fact the only one featuring band-gaps in the conventional, linear-elastic micromorphic framework.

To study bulk wave propagation, we assume in what follows that the involved unknown variables take the harmonic form

\[ u_1 = \alpha_1 e^{i(kx_1 - \omega t)}, \quad u_2 = \alpha_2 e^{i(kx_1 - \omega t)}, \quad u_3 = \alpha_3 e^{i(kx_1 - \omega t)}, \]

on the left side occupied by the Cauchy medium and

\[ v_1 = \beta_1 e^{i(kx_1 - \omega t)}, \quad v_2 = \beta_2 e^{i(kx_1 - \omega t)}, \quad v_3 = \beta_3 e^{i(kx_1 - \omega t)}, \]

\[ v_4 = \beta_4 e^{i(kx_1 - \omega t)}, \quad v_5 = \beta_5 e^{i(kx_1 - \omega t)}, \quad v_6 = \beta_6 e^{i(kx_1 - \omega t)}, \]

on the right side occupied by the relaxed micromorphic medium.

6.1 Classical Cauchy media

As far as bulk propagation in Cauchy media is concerned, we can find the standard dispersion relations for Cauchy media simply replacing the expressions (44) in the equations of motion (23) and simplifying, so obtaining

\[ \omega^2 = \frac{\lambda_{macro} + 2\mu_{macro}}{\rho} k^2, \quad \omega^2 = \frac{\mu_{macro}}{\rho} k^2, \quad \omega^2 = \frac{\mu_{macro}}{\rho} k^2, \]

or, equivalently

\[ \omega = \pm c_1 k, \quad \omega = \pm c_1 k, \quad \omega = \pm c_1 k, \]
where we denoted by

\[ c_l = \sqrt{\frac{\lambda_{\text{macro}} + 2\mu_{\text{macro}}}{\rho}}, \quad c_t = \sqrt{\frac{\mu_{\text{macro}}}{\rho}}, \]

the characteristic speeds in classical Cauchy media of longitudinal and transverse waves, respectively. Here, we have assumed

\[ \rho > 0, \quad \mu_{\text{macro}} \geq 0, \quad \lambda_{\text{macro}} + 2\mu_{\text{macro}} \geq 0. \] (48)

The dispersion relations (47) can be traced in the plane \((\omega, k)\), giving rise to the standard non-dispersive behavior for a classical Cauchy continuum (see Figure 3).

![Figure 3: Dispersion relations for a Cauchy continuum: (a) longitudinal waves and (b) transverse waves. The dispersion relations are straight lines (non-dispersive behavior) and \(c_l\) and \(c_t\) are their slopes for longitudinal and transverse waves respectively.](image)

It can be noticed in Figure 3 that we have one straight line for longitudinal waves and two for transverse waves (there are two superimposed lines in picture (b), according to Eqs. (47)).

On the other hand, the equations (47) can be inverted and we can state that the expressions of the wavenumbers as function of the frequency \(\omega\) can be found as

\[ k_1(\omega) = \pm \frac{1}{c_l} \omega, \quad k_2(\omega) = \pm \frac{1}{c_t} \omega, \quad k_3(\omega) = \pm \frac{1}{c_t} \omega. \] (49)

Based on equations (44), the solution for the displacement field can hence be rewritten as

\[ u_1 = \alpha_1 e^{i(k_1(\omega)x_1 - \omega t)}, \quad u_2 = \alpha_2 e^{i(k_2(\omega)x_1 - \omega t)}, \quad u_3 = \alpha_3 e^{i(k_3(\omega)x_1 - \omega t)}. \] (50)

Such relationships establish the solution for the displacement field for any real frequency \(\omega\) if the amplitudes \(\alpha_1, \alpha_2, \) and \(\alpha_3\) are known. Such amplitudes will be calculated in the following by imposing suitable boundary conditions.

We explicitly remark that, as far as second gradient continua are concerned, the dispersion relations are similar to those presented in figure 3, since the underlying kinematics is the same in both Cauchy and second gradient media (only the macroscopic displacement field). On the other hand, contrarily to Cauchy continua, second gradient media may exhibit dispersive behaviors in the sense that the dispersion relations analogous to those presented in figure 3 are not straight lines anymore (see [21]).

\[ \footnote{We explicitly remark that the positive or negative roots for the wavenumbers \(k_i\) must be chosen in Eq. (50) depending whether the wave travels in the \(x_1\) or \(-x_1\) direction.} \]
6.2 Relaxed micromorphic media

As far as the relaxed micromorphic model is concerned, we proceed in an analogous way and we replace the wave-forms \(\psi_{\alpha}^{(5)}\) for the unknown fields in the bulk equations \(\psi_{\alpha}^{(5)}\), so obtaining

\[
(k^2 A_{1}^{R} - \omega^2 I - i k B_{1}^{R} - C_{1}^{R}) \cdot \beta_{1} = 0 \quad \text{longitudinal waves (51)}
\]

\[
(k^2 A_{\alpha}^{R} - \omega^2 I - i k B_{\alpha}^{R} - C_{\alpha}^{R}) \cdot \beta_{\alpha} = 0, \quad \alpha = 2, 3, \quad \text{transverse waves (52)}
\]

\[
\omega^2 = A_{4}^{R} k^2 - C_{4}^{R}, \quad \omega^2 = A_{5}^{R} k^2 - C_{5}^{R}, \quad \omega^2 = A_{6}^{R} k^2 - C_{6}^{R}, \quad \text{uncoupled waves. (53)}
\]

6.2.1 Uncoupled waves.

The specific behavior of uncoupled waves can be easily studied, since, starting from Eq. (53), the frequency is explicitly obtained as a function of the wavenumber as:

\[
\omega(k) = \pm \sqrt{\omega^2_s + c_m^2 k^2}, \quad \omega(k) = \pm \sqrt{\omega^2_r + c_m^2 k^2}, \quad \omega(k) = \pm \sqrt{\omega^2_s + c_m^2 k^2},
\]

where we introduced the characteristic speed \(c_m\) and the characteristic frequencies \(\omega_s\) and \(\omega_r\) as

\[
c_m = \sqrt{A_{4}^{R}} = \sqrt{A_{5}^{R}} = \sqrt{A_{6}^{R}} = \sqrt{\frac{\mu e L^2}{\eta}}, \quad \omega_s = \sqrt{-C_{4}^{R}} = \sqrt{-C_{5}^{R}} = \sqrt{\frac{2(\mu e + \mu_{\text{micro}})}{\eta}}, \quad \omega_r = \sqrt{-C_{6}^{R}} = \sqrt{\frac{2\mu_c}{\eta}}.
\]

We have assumed

\[
\eta > 0, \quad \mu e \geq 0, \quad \mu_{\text{micro}} \geq 0, \quad \mu_c \geq 0.
\]

We start remarking that, since the equations for \(v_4\) and \(v_6\) are formally the same, we have two coinciding dispersion relations. Moreover, it is clear that, when \(k = 0\), one has that

\[
\omega(0) = \omega_s, \quad \omega(0) = \omega_r, \quad \omega(0) = \omega_s,
\]

which allows to determine the cut-off frequencies \(\omega_s\) and \(\omega_r\) for the uncoupled waves.

On the other hand, when \(k \to \infty\), one has

\[
\omega(k) \sim c_m k, \quad \omega(k) \sim c_m k, \quad \omega(k) \sim c_m k,
\]

which means that the dispersion curves for the uncoupled waves all have an asymptote on \(c_m\). The behavior of the dispersion curves for uncoupled waves is depicted in Figure 4.

![Figure 4: Dispersion curves for uncoupled waves. Such waves have cut-off frequencies \(\omega_s\) and \(\omega_r\) and they all have the same asymptote of slope \(c_m\). The curves (a) and (c) are superimposed.](image-url)
The solutions (54) can be inverted for $\omega > \max\{\omega_s, \omega_r\}$, so finding

$$
k_s(\omega) = \pm \frac{1}{c_m} \sqrt{\omega^2 - \omega_s^2}, \quad k_r(\omega) = \pm \frac{1}{c_m} \sqrt{\omega^2 - \omega_r^2}, \quad k_s(\omega) = \pm \frac{1}{c_m} \sqrt{\omega^2 - \omega_s^2},$$

(58)

where we remark that the wavenumbers $k_s$ and $k_r$ become purely imaginary for real frequencies $\omega < \omega_s$ and $\omega < \omega_r$, respectively.

Based on the wave-form assumption (45), the solution for the uncoupled waves can hence be written as

$$
v_4 = \beta_4 e^{i(k_s(\omega)x_1 - \omega t)}, \quad v_5 = \beta_5 e^{i(k_r(\omega)x_1 - \omega t)}, \quad v_6 = \beta_6 e^{i(k_s(\omega)x_1 - \omega t)}.
$$

(59)

The unknown amplitudes $\beta_4$, $\beta_5$ and $\beta_6$ can be calculated by imposing suitable boundary conditions as it will be shown later on.

6.2.2 Longitudinal waves.

We start by noticing that replacing the wave form (45) in (51) we get

$$
A_1 : \beta_1 = \left( k^2 A^R_1 - \omega^2 \mathbb{1} - i k B^R_1 - C^R_1 \right) \cdot \beta_1 = 0.
$$

(60)

In order to have non-trivial solutions of this algebraic system we must have

$$
\det A_1 = 0,
$$

that will furnish the solutions $\omega = \omega(k)$ which are usually known as dispersion relations. It can be checked that $\det A_1$ is a polynomial of the 6th order in $\omega$ and of the 4th order in $k$ and that only even powers of both $\omega$ and $k$ appear.

We assume

$$
\lambda_e + 2 \mu_e \geq 0, \quad \lambda_{\text{micro}} + 2 \mu_{\text{micro}} \geq 0.
$$

(61)

In order to get some preliminary information concerning the function $\det A_1$, we can start looking at its behaviour for $k \to 0$ and $k \to \infty$.

When $k = 0$ the system of algebraic equations (60) simplifies into

$$
(\omega^2 \mathbb{1} + C^R_1) \cdot \beta_1 = 0,
$$

which, recalling the definition (32) for $C^R_1$, equivalently reads

$$
\begin{pmatrix}
\omega^2 & 0 & 0 \\
0 & \omega^2 - \omega_s^2 & 0 \\
0 & 0 & \omega^2 - \omega_p^2
\end{pmatrix} \cdot \beta_1 = 0,
$$

where we introduced the new characteristic frequency

$$
\omega_p = \sqrt{(3\lambda_e + 2\mu_e) + (3\lambda_{\text{micro}} + 2\mu_{\text{micro}})}.
$$

This means that for very small values of $k$ the three dispersion equations become uncoupled and have cut-off frequencies which are respectively $\omega = 0$, $\omega = \omega_s$ and $\omega = \omega_r$.

On the other hand, letting $k \to \infty$, and considering the case for which the ratio $k/\omega$ remains finite, instead of studying the whole system of equations (60), we can consider the reduced system

$$
\left( k^2 A^R_1 - \omega^2 \mathbb{1} \right) \cdot \beta_1 = 0,
$$

(62)

\footnote{We explicitly remark that the positive or negative roots for the wavenumbers $k_s$ and $k_r$ must be chosen in Eq. (59) depending whether the wave travels in the $x_1$ or $-x_1$ direction. Moreover, when $\omega < \omega_s$ an exponential decaying with $x_1$ appears so that the solution is not periodic anymore and so-called standing waves appear for $v_4$ and $v_6$. The same happens for the variable $v_5$ when $\omega < \omega_r$.}
which, given the expression [32] for $A^R$, equivalently reads

$$
\begin{pmatrix}
    c_p^2 k^2 - \omega^2 & 0 & 0 \\
    0 & \frac{1}{3} c_m^2 k^2 - \omega^2 & -\frac{2}{3} c_m^2 k^2 \\
    0 & -\frac{1}{3} c_m^2 k^2 & \frac{2}{3} c_m^2 k^2 - \omega^2
\end{pmatrix} \cdot \beta_1 = 0,
$$

where we introduced the new characteristic speed $c_p$ as

$$c_p = \sqrt{\lambda e + \frac{2 \mu e}{\rho}}.
$$

It can be seen from this last system of algebraic equations that for high values of $k$, the first equation becomes uncoupled from the other two and implies that

$$\omega = c_p k.
$$

This means, in other words, that one of the longitudinal waves will have an asymptote in $c_p$. The last two equations remain coupled and, in order to let them have a non trivial solution, it must be set

$$\det \left( \begin{array}{ccc}
    \frac{1}{3} c_m^2 k^2 - \omega^2 & -\frac{2}{3} c_m^2 k^2 \\
    -\frac{1}{3} c_m^2 k^2 & \frac{2}{3} c_m^2 k^2 - \omega^2
\end{array} \right) = \omega^4 - c_m^2 k^2 \omega^2 = 0.
$$

This implies a solution $\omega = 0$ (which has to be excluded since it does not fall in the case $k/\omega$ finite for $k \to \infty$) and a solution $\omega = c_m k$. This means that for large values of $k$ one of the longitudinal dispersion curves will have an asymptote in $c_m$.

Once that the two asymptotes $c_m$ and $c_p$ are known, we can look for the horizontal asymptote $\omega_l$ by noticing that (for $k \to \infty$) the function $\det A_1$ can be factorized

$$\det A_1 = (\omega^2 - c_m^2 k^2)(\omega^2 - c_p^2 k^2)(\omega^2 - \omega_l^2) = 0.
$$

It can be checked that the horizontal asymptote is found to be

$$\omega_l = \sqrt{\frac{\lambda_{\text{micro}} + 2 \mu_{\text{micro}}}{\eta}}.
$$

The three dispersion relations for longitudinal waves are separately shown in Figure 5.
On the other hand, given the structure of the polynomial $\det A_1$, if it is solved in terms of $k = k(\omega)$, four solutions are found which can be formally written as:

$$\pm k_1^1(\omega), \quad \pm k_1^2(\omega).$$

We denote by $h_1^1$ and $h_1^2$ the eigenvectors associated to the eigenvalues $k_1^1(\omega)$ and $k_1^2(\omega)$ and verifying Eq. (60) with $\beta_1 = h_1^1$ and $\beta_1 = h_1^2$, respectively. According to Eq. (45), the solution for $v_1$ can finally be written as

$$v_1 = \beta_1^1 h_1^1 e^{i(\pm k_1^1(\omega)x_1 - \omega t)} + \beta_2^1 h_1^2 e^{i(\pm k_1^2(\omega)x_1 - \omega t)},$$

(63)

where the $+$ or $-$ sign must be chosen for the wavenumbers $k_1^1$ and $k_1^2$ depending whether the considered wave travels in the $x_1$ or $-x_1$ respectively. The two scalar unknowns $\beta_1^1$ and $\beta_2^1$ can be found by imposing suitable boundary conditions as it will be explained later on.

### 6.2.3 Transverse waves.

We start by noticing that replacing the wave form (45) in (52) we get

$$A_\alpha \cdot \beta_\alpha = (k^2 A^R_\alpha - \omega^2 \mathbf{1} - i k B^R_\alpha - C^R_\alpha) \cdot \beta_\alpha = 0.$$  

(64)

In order to have non-trivial solutions of this algebraic system we must have

$$\det A_\alpha = 0.$$

Following an equivalent reasoning with respect to what done for longitudinal waves, it can be checked that the solutions $\omega = \omega(k)$ of this equations are such that they have two asymptotes of slope $c_s$ and $c_m$ respectively and a horizontal asymptote at $\omega = \omega_t$, where we set

$$c_s = \sqrt{\frac{\mu + \mu_c}{\rho}}, \quad \omega_t = \sqrt{\frac{\mu_{\text{micro}}}{\eta}}.$$

The dispersion curves for transverse waves are depicted in Figure 6.

![Figure 6: Dispersion relations for transverse waves. Two asymptotes of slope $c_s$ and $c_m$ are identified, together with a horizontal asymptote at $\omega = \omega_t$. Two cut-off frequencies $\omega_r$ and $\omega_s$ are identified for the optic waves.](image)

As done for the longitudinal waves, given the structure of the polynomial $\det A_\alpha$, if it is solved in terms of $k = k(\omega)$, four solutions are found which can be formally written as:

$$\pm k_\alpha^1(\omega), \quad \pm k_\alpha^2(\omega), \quad \alpha = 2, 3.$$

$^6$To the sake of simplicity, we do not show the explicit form of the functions $k_1^1(\omega)$ and $k_1^2(\omega)$.

$^7$To the sake of simplicity, we do not show the explicit form of the functions $k_\alpha^1(\omega)$ and $k_\alpha^2(\omega)$. 
We denote by $h_1^\alpha$ and $h_2^\alpha$ the eigenvectors associated to the eigenvalues $k_1^\alpha(\omega)$ and $k_2^\alpha(\omega)$ and verifying Eq. (64) with $\beta_\alpha = h_1^\alpha$ and $\beta_\alpha = h_2^\alpha$, respectively. According to Eq. (45), the final solution for $v_\alpha$ can be finally written as

$$v_\alpha = \beta_1^\alpha h_1^\alpha e^{i(\pm k_1^\alpha(\omega)x_1 - \omega t)} + \beta_2^\alpha h_2^\alpha e^{i(\pm k_2^\alpha(\omega)x_1 - \omega t)}, \quad \alpha = 2, 3. \quad (65)$$

where the + or − sign must be chosen for the wavenumbers $k_1^\alpha$ and $k_2^\alpha$ depending whether the considered wave travels in the $x_1$ or $-x_1$ respectively. The scalar unknowns $\beta_1^\alpha$ and $\beta_2^\alpha$ can be found by imposing suitable boundary conditions as it will be explained in the next section.

6.2.4 A summary concerning dispersion curves in relaxed micromorphic media

We now want to summarize our findings concerning the characteristic behavior of the dispersion curves in relaxed micromorphic media. To do so, we start presenting in figure 7 all the obtained dispersion curve diagrams for uncoupled, longitudinal and transverse waves, so highlighting the presence of a complete band gap.

Figure 7: Dispersion relations for all the considered uncoupled, longitudinal and transverse waves. Identification of the complete band-gap.

Moreover, we summarize in the following formulas all the characteristic quantities appearing in Figure 7 whose expressions in terms of the constitutive parameters have been identified in the preceding subsections:

$$\omega_s = \sqrt{\frac{2(\mu_e + \mu_{\text{micro}})}{\eta}}, \quad \omega_r = \sqrt{\frac{2\mu_e}{\eta}}, \quad \omega_p = \sqrt{\frac{(3\lambda_e + 2\mu_e) + (3\lambda_{\text{micro}} + 2\mu_{\text{micro}})}{\eta}},$$

$$\omega_l = \frac{\lambda_{\text{micro}} + 2\mu_{\text{micro}}}{\eta}, \quad \omega_{\text{t}} = \sqrt{\frac{2\mu_{\text{micro}}}{\eta}},$$

$$c_m = \sqrt{\frac{\mu_e L_e^2}{\eta}}, \quad c_p = \sqrt{\frac{\lambda_e + 2\mu_e}{\rho}}, \quad c_s = \sqrt{\frac{\mu_e + \mu_e}{\rho}}, \quad (66)$$

We explicitly remark that the relaxed micromorphic model is intrinsically a macroscopic model which is able to account for the presence of the microstructure in an effective sense at the continuum level. This means that it can be usefully employed as far as the wavelength of the considered waves is bigger than the characteristic size of the unit cell of the underlying microstructure. For this reason, the dispersion relations shown in Fig. 7 can be used for the description of band-gap metamaterials as far as the wavenumber $k$ is smaller than a threshold value corresponding to the situation in which the wavelength of the traveling wave is equal to the characteristic size of the unit cell of the considered metamaterial.
The repercussions on the dispersion curves shown in Figure 7 are that the oblique asymptote becomes horizontal, with the effect that the associated dispersion curves flatten in order to become horizontal for $k \to \infty$. A non-negligible class of band-gap metamaterials can, at least in a first approximation, be modeled by means of such an internal variable model. Suggestive results in this sense have been provided in the recent paper [34] in which numerical homogenization techniques are applied in order to obtain an internal variable continuum model instead of a relaxed micromorphic one. The relaxed micromorphic model is obtained instead of a relaxed micromorphic one. The relaxed micromorphic model is obtained instead of a relaxed micromorphic one. The relaxed micromorphic model is obtained instead of a relaxed micromorphic one. The relaxed micromorphic model must be the target for describing band-gap metamaterials with non-local effect.

It can be checked that, as soon as $L_c = 0$ (or equivalently $c_m = 0$):

- The cut-off frequencies $\omega_p$, $\omega_s$, and $\omega_r$ remain the same as in the general case $L_c \neq 0$.
- The uncoupled waves all become horizontal straight lines (see eq. 54) and, in particular
  \[ \omega(k) = \omega_s, \quad \omega(k) = \omega_r, \quad \omega(k) = \omega_s. \]  
  \[ (68) \]

- The characteristic equation $\det A_1 = 0$ for longitudinal waves is modified in the sense that it becomes of the second order in $k$ (instead that of the 4th order for the case $L_c \neq 0$). This means that the oblique asymptote $c_p$ of the previous case is preserved, while two horizontal asymptotes $\omega^1_L$ and $\omega^2_L$ can be found that are defined as
  \[ \omega_L^{1,2} = \frac{a \pm \sqrt{a^2 - b}}{2 \eta (\lambda_e + 2 \mu_e)}, \]
  with
  \[ a = 6 \lambda_e \mu_e + 4 \mu_e (\mu_e + 2 \mu_{micro}) + \lambda_e (3 \lambda_{micro} + 6 \mu_e + 4 \mu_{micro}), \]
  \[ b = 8 (\lambda_e + 2 \mu_e) \left[ \lambda_e (3 \lambda_{micro} (\mu_e + \mu_{micro}) + 2 \mu_{micro} (3 \mu_e + \mu_{micro})) + 2 \mu_e (2 \mu_{micro} (\mu_e + \mu_{micro}) + \lambda_{micro} (3 \mu_e + 3 \mu_{micro})) \right] \]

Although not immediately evident, it can be checked that under the considered hypotheses on the constitutive parameters (definite positiveness of the energy and positive macro and micro mass densities), the obtained values of the longitudinal asymptotes are real and positive.

- The characteristic equation $\det A_\alpha = 0$ for transverse waves is modified in the sense that it also becomes of the second order in $k$ (instead that of the 4th order for the case $L_c \neq 0$). This means that the oblique asymptotes $c_s$ of the previous case is preserved, while two horizontal asymptotes $\omega^1_T$ and $\omega^2_T$ can be found that are defined as
  \[ \omega_T^{1,2} = \sqrt{\frac{2 \mu_e \mu_e + (\mu_e + \mu_e) \mu_{micro} \pm \sqrt{(2 \mu_e \mu_e + (\mu_e + \mu_e) \mu_{micro})^2 - 4 \mu_e \mu_e \mu_{micro} (\mu_e + \mu_e)}}}{\eta (\mu_e + \mu_e)}, \]
  with
  \[ a = 6 \lambda_e \mu_e + 4 \mu_e (\mu_e + 2 \mu_{micro}) + \lambda_e (3 \lambda_{micro} + 6 \mu_e + 4 \mu_{micro}), \]
  \[ b = 8 (\lambda_e + 2 \mu_e) \left[ \lambda_e (3 \lambda_{micro} (\mu_e + \mu_{micro}) + 2 \mu_{micro} (3 \mu_e + \mu_{micro})) + 2 \mu_e (2 \mu_{micro} (\mu_e + \mu_{micro}) + \lambda_{micro} (3 \mu_e + 3 \mu_{micro})) \right] \]

Although not immediately evident, it can be checked that under the considered hypotheses on the constitutive parameters (definite positiveness of the energy and positive macro and micro mass densities), the obtained values of the transverse asymptotes are real and positive.

The considered conditions on the parameters are dictated by the positive definiteness of the strain energy density as well as by the fact of considering positive macro and micro mass densities.

\[ \rho > 0, \quad \eta > 0, \quad \mu_e > 0, \quad \mu_{micro} > 0, \quad \mu_c \geq 0, \quad \lambda_e + 2 \mu_e > 0, \quad \lambda_{micro} + 2 \mu_{micro} > 0. \]  

(67)
This internal variable model, seen as a degenerate singular limit case of the relaxed micromorphic model, will be discussed in more detail in future contributions in which the interest of using a relaxed micromorphic model at the continuum level will be discussed. It is clear that, whenever possible, the use of a relaxed micromorphic model for the description of band-gap metamaterials is preferable to the use of an internal variable model, at least for two reasons:

- it allows to account for non-local effects, i.e. for the description of metamaterials in which the unit cells can interact
- the relationships between the characteristic frequencies and velocities which appear as basic quantities in the dispersion curves are more easily related to the constitutive parameters than in the case of the internal variable model.

In summary, we can say that the case \( L_c = 0 \) is a degenerate case in the sense that the response of the material with respect to wave propagation becomes completely different from the response that can be observed in the relaxed micromorphic model with infinitely small, but non-vanishing, \( L_c \). We leave to a subsequent work the task of showing in more detail how the degenerate case \( L_c = 0 \) differs from the limit for very small \( L_c \) of our relaxed micromorphic model. We limit here ourselves to remark that non-local effects might be taken into account when dealing with metamaterials with heterogeneous microstructures and strong contrasts in the micro-mechanical properties, even if in some particular cases they may be very small.

6.2.6 The degenerate case \( \mu_c = 0 \) and \( L_c = 0 \)

In our previous work [21] we have shown that as soon as non-local effects are present (\( L_c \neq 0 \)), then the Cosserat couple modulus \( \mu_c \) must be non-vanishing (and larger than a given threshold) in order to have a complete band-gap appearing. The case with \( L_c > 0 \) and \( \mu_c = 0 \) still remains well-posed but no band-gaps are allowed in this case. We explicitly remark here that if we consider the degenerate case \( \mu_c = 0 \) and simultaneously \( L_c = 0 \), then the skew-symmetric part of the micro-distortion tensor \( P \) cannot be controlled anymore since the associated governing equations reduce to skew \( P_{\mu} = 0 \). *Mutatis mutandis*, we claim that such degenerate model is not able to describe the rotational vibrations in band-gaps metamaterials, even if the model could be able to provide band-gap behaviors. The phenomenological consistency of such degenerate model is thus questionable, since one should individuate a material in which microscopic rotations are forbidden inside the unit cell in order to let it be applicable. On the other hand, the case \( \mu_c \neq 0 \) and \( L_c = 0 \) presented in the previous Subsection can describe a certain class of band-gap metamaterials in the very special case in which non-local which allow for rotational vibrations at relatively high frequencies, but not for non-local effects.

6.3 The standard micromorphic continuum

For the standard micromorphic model, we proceed in an analogous way and we replace the wave-forms (45) for the unknown fields in the bulk equations (39), so obtaining

\[
(k^2 A_{\alpha}^C - \omega^2 I - i k B_{\alpha}^C - C_{\alpha}^C) \cdot \beta_{\alpha} = 0, \quad \alpha = 2, 3, \text{ transverse waves} \tag{70}
\]

\[
(\omega^2 = A_4^C k^2 - C_4^C, \quad \omega^2 = A_5^C k^2 - C_5^C, \quad \omega^2 = C_6^R k^2 - C_6^C, \text{ uncoupled waves.} \tag{71}
\]

The assumptions on the considered constitutive coefficients are the same given in eq. (67) for the relaxed micromorphic case.

6.3.1 Uncoupled waves.

The dispersion relations for uncoupled waves in standard micromorphic media are the same as those obtained for the relaxed micromorphic continuum and read:

\[
\omega(k) = \sqrt{\omega_0^2 + c_3^2 k^2}, \quad \omega(k) = \sqrt{\omega_0^2 + c_3^2 k^2}, \quad \omega(k) = \sqrt{\omega_0^2 + c_3^2 k^2}, \tag{72}
\]
where we set
\[ c_g = \sqrt{\frac{\mu c L_2^2}{\eta}}. \]

On the basis of what has been explained before, such curves have cut-off frequencies \( \omega_s \) and \( \omega_r \) and they all share the same asymptote of slope \( c_g \).

### 6.3.2 Longitudinal waves.

We start by noticing that replacing the wave form (45) in (51) we get
\[ \tilde{A}_1 = (k^2 A^C_1 - \omega^2 \mathbb{I} - i k B^C_1 - C^C_1) \cdot \beta_1 = 0. \]

Following the same steps presented in Subsection 6.2.2, we can establish that for \( k \to 0 \) the longitudinal waves have the same cut-off frequencies than those evaluated for the relaxed micromorphic case, in particular: \( \omega = 0 \), \( \omega = \omega_s \) and \( \omega = \omega_r \).

In the case \( k \to \infty \) with \( k/\omega \) finite, on the other hand, one finds that in order to have non-trivial solutions it must be
\[ \text{det} \left( k^2 A^C_1 - \omega^2 \mathbb{I} \right) = \text{det} \begin{pmatrix} c_p^2 k^2 - \omega^2 & 0 & 0 \\ 0 & c_g^2 k^2 - \omega^2 & 0 \\ 0 & 0 & c_g^2 k^2 - \omega^2 \end{pmatrix} = 0. \]

This means that the three dispersion curves become uncoupled for big wavenumbers and they have asymptotes \( c_p \), \( c_g \) and \( c_g \) respectively, where we define
\[ c_g = \sqrt{\frac{\mu c L_2^2}{\eta}}. \]

No horizontal asymptote is found for longitudinal waves in standard micromorphic media.

On the other hand, given the structure of the polynomial \( \text{det} A_1 \), if it is solved in terms of \( k = k(\omega) \), six solutions are found which can be formally written as\[^{10}\]
\[ \pm \tilde{k}_1^1(\omega), \quad \pm \tilde{k}_1^2(\omega), \quad \pm \tilde{k}_1^3(\omega). \]

We denote by \( \tilde{h}_1^i \) and \( \tilde{h}_2^i \) the eigenvectors associated to the eigenvalues \( \tilde{k}_1^1(\omega), \tilde{k}_1^2(\omega) \) and \( \tilde{k}_1^3(\omega) \) and verifying (60) with \( \beta_1 = \tilde{h}_1^1 \) and \( \beta_1 = \tilde{h}_2^1 \), respectively. According to (45), the final solution for \( \tilde{h}_1 \) can be finally written as
\[ \tilde{v}_1 = \beta_1^1 \tilde{h}_1^1 e^{i(\pm \tilde{k}_1^1(\omega)x_1 - \omega t)} + \beta_1^2 \tilde{h}_1^2 e^{i(\pm \tilde{k}_1^2(\omega)x_1 - \omega t)} + \beta_1^3 \tilde{h}_1^3 e^{i(\pm \tilde{k}_1^3(\omega)x_1 - \omega t)}, \]

where the + or − sign must be chosen for the wavenumber depending whether the considered wave travels in the \( x_1 \) or \( -x_1 \) direction, respectively. The three scalar unknowns \( \beta_1^1, \beta_1^2 \) and \( \beta_1^3 \) can be found by imposing suitable boundary conditions as it will be explained later on.

### 6.3.3 Transverse waves.

We start by noticing that replacing the wave form (45) in (52) we get
\[ \tilde{A}_\alpha = (k^2 A^C_\alpha - \omega^2 \mathbb{I} - i k B^C_\alpha - C^C_\alpha) \cdot \beta_\alpha = 0. \]

Following the same steps as before, it is found that the transverse waves have asymptotes \( c_s \), \( c_g \) and \( c_g \) respectively and no horizontal asymptote is identified.

We conclude that, according to what has been presented in [21], no complete band gaps are possible in standard micromorphic continua due to the loss of the horizontal asymptotes for longitudinal and transverse waves.

[^10]: To the sake of simplicity, we do not show the explicit form of the functions \( \tilde{k}_1^1(\omega), \tilde{k}_1^2(\omega) \) and \( \tilde{k}_1^3(\omega) \).
As done for the longitudinal waves, given the structure of the polynomial \( \det \tilde{A}_\alpha \), if it is solved in terms of \( k = k(\omega) \), six solutions are found which can be formally written as
\[
\pm \tilde{k}_1^1(\omega), \quad \pm \tilde{k}_2^2(\omega), \quad \pm \tilde{k}_3^3(\omega), \quad \alpha = 2, 3.
\]

We denote \( \tilde{h}_1^1(\omega), \tilde{h}_2^2(\omega) \) and \( \tilde{h}_3^3(\omega) \) the eigenvectors associated to the eigenvalues \( \tilde{k}_1^1(\omega), \tilde{k}_2^2(\omega) \) and \( \tilde{k}_3^3(\omega) \) and verifying (75) with \( \beta_\alpha = \tilde{h}_1^1, \beta_\alpha = \tilde{h}_2^2, \beta_\alpha = \tilde{h}_3^3 \), respectively. According to (45), the final solution for \( \tilde{v}_\alpha \) can be finally written as
\[
\tilde{v}_\alpha = \beta_1^1 \tilde{h}_1^1 e^{i(\pm \tilde{k}_1^1(\omega)x_1-\omega t)} + \beta_2^2 \tilde{h}_2^2 e^{i(\pm \tilde{k}_2^2(\omega)x_1-\omega t)} + \beta_3^3 \tilde{h}_3^3 e^{i(\pm \tilde{k}_3^3(\omega)x_1-\omega t)}, \quad \alpha = 2, 3. \tag{76}
\]
where the + or − sign must be chosen for the wavenumber depending whether the considered wave travels in the \( x_1 \) or \( -x_1 \) respectively. The scalar unknowns \( \beta_1^1, \beta_2^2, \beta_3^3 \) can be found by imposing suitable boundary conditions as it will be explained in the next section.

Figure 8: Dispersion relations for all the considered uncoupled, longitudinal and transverse waves in the case of standard Mindlin micromorphic continua. No complete band-gap can be identified.

In Figure 8 we show the dispersion relations for classical Mindlin media grouped as uncoupled, longitudinal and transverse waves.

7 Reflection and transmission at Cauchy/relaxed-micromorphic and Cauchy/Mindlin interfaces

In order to perform suitable numerical simulations based on the theoretical framework developed in the previous sections, we study reflection and transmission of plane waves at a surface of discontinuity between a classical Cauchy continuum and a micromorphic (relaxed or standard) one which is located at \( x_1 = 0 \). To this purpose we introduce the quantities
\[
J_i = \frac{1}{\tau} \int_0^\tau H_i(0, t) \, dt, \quad J_r = \frac{1}{\tau} \int_0^\tau H_r(0, t) \, dt, \quad J_t = \frac{1}{\tau} \int_0^\tau H_t(0, t) \, dt,
\]
where \( \tau \) is the period of the traveling plane wave and \( H_i, H_r \) and \( H_t \) are the energy fluxes of the incident, reflected and transmitted energies, respectively. The reflection (\( R \)) and transmission (\( T \)) coefficients can hence be defined as
\[
R = \frac{J_r}{J_i}, \quad T = \frac{J_t}{J_i}. \tag{77}
\]

\(^{11}\)To the sake of simplicity, we do not show the explicit form of the functions \( \tilde{k}_1^1(\omega), \tilde{k}_2^2(\omega) \) and \( \tilde{k}_3^3(\omega) \).
Since the considered system is conservative, one must have \(R + T = 1\).

For all the numerical simulations presented in this section we chose the set of parameters shown in Table 1 if not differently specified.

| \( \rho \) | \( \eta \) | \( \mu_\text{e} \) | \( \mu_\text{micro} \) | \( \lambda_\text{micro} \) | \( \lambda_\text{macro} \) | \( \mu_\text{macro} \) | \( L_\text{c} \) | \( L_\text{g} \) |
|---|---|---|---|---|---|---|---|---|
| \([Kg/m^3]\) | \([Kg/m]\) | \([Pa]\) | \([Pa]\) | \([Pa]\) | \([Pa]\) | \([Pa]\) | \([m]\) | \([m]\) |
| 2000 | \(10^{-2}\) | \(2 \times 10^9\) | \(2 \times 10^8\) | \(10^8\) | \(4 \times 10^8\) | \(4 \times 10^8\) | \(2 \times 10^8\) | \(10^{-2}\) |

Table 1: Numerical values of the constitutive parameters used for the numerical simulations.

### 7.1 Interface between a Cauchy and a relaxed micromorphic medium

When considering a classical Cauchy continuum on the \(-\) side and a relaxed micromorphic one on the \(+\) side, we have that, recalling Eqs. (50), (63), (65), (69) the solution can be written in terms of the calculated eigenvalues and eigenvectors as:

\[
\begin{align*}
    u^-_1 &= u_i^1 + u_1^r, & u^-_2 &= u_2^1 + u_2^r, & u^-_3 &= u_3^1 + u_3^r, \\

\end{align*}
\]

where we set for compactness of notation

\[
\begin{align*}
    u_i^1 &= \bar{\alpha}_1 e^{i(\omega/cm \cdot x_1 - \omega t)}, & u_1^r &= \alpha_1 e^{i(\omega/cm \cdot x_1 + \omega t)}, \\
    u_2^1 &= \bar{\alpha}_2 e^{i(\omega/cm \cdot x_1 - \omega t)}, & u_2^r &= \alpha_2 e^{i(\omega/cm \cdot x_1 + \omega t)}, \\
    u_3^1 &= \bar{\alpha}_3 e^{i(\omega/cm \cdot x_1 - \omega t)}, & u_3^r &= \alpha_3 e^{i(\omega/cm \cdot x_1 + \omega t)},
\end{align*}
\]

and moreover

\[
\begin{align*}
    v_1^+ &= \beta_1^1 h_1^1 e^{i(k_1^1(\omega)x_1 - \omega t)} + \beta_2^{1} h_1^{1} e^{i(k_1^{2}(\omega)x_1 - \omega t)}, & v_\alpha^+ &= \beta_1^\alpha h_\alpha^1 e^{i(k_\alpha^1(\omega)x_1 - \omega t)} + \beta_2^\alpha h_\alpha^{2} e^{i(k_\alpha^2(\omega)x_1 - \omega t)}, & \alpha = 2, 3, \\
    v_4^+ &= \beta_4 e^{i(1/cm \cdot \sqrt{\omega^2 - \omega_3^2} x_1 - \omega t)}, & v_5^+ &= \beta_5 e^{i(1/cm \cdot \sqrt{\omega^2 - \omega_4^2} x_1 - \omega t)}, & v_6^+ &= \beta_6 e^{i(1/cm \cdot \sqrt{\omega^2 - \omega_5^2} x_1 - \omega t)},
\end{align*}
\]

Given the frequency \(\omega\) of the traveling wave, the solution is hence known except for the amplitudes \(\alpha_i\) and \(\beta_i\). If we count, we have 12 unknown amplitudes to be determined. We focus here on the study of wave reflection and transmission concerning two of out of the six possible types of connection between such media (see Section 3.4), because the remaining four constraints always give rise to complete reflection. In particular, we will study wave transmission and reflection at surfaces of discontinuity between a Cauchy medium and a relaxed micromorphic one connected with the following constraints

- macro internal clamp with fixed microstructure,
- macro internal clamp with free microstructure.

With the introduced notations, the flux associated to the incident, reflected and transmitted waves can be computed recalling Eqs. (21), (78), (37) as

\[
\begin{align*}
    H_i &= -u_{1,t}^i \left[ (\lambda_\text{macro} + 2\mu_\text{macro}) u_{1,1}^i \right] - u_{2,t}^i \left[ \mu_\text{macro} u_{2,1}^i \right] - u_{3,t}^i \left[ \mu_\text{macro} u_{3,1}^i \right], \\
    H_r &= -u_{1,t}^r \left[ (\lambda_\text{macro} + 2\mu_\text{macro}) u_{1,1}^r \right] - u_{2,t}^r \left[ \mu_\text{macro} u_{2,1}^r \right] - u_{3,t}^r \left[ \mu_\text{macro} u_{3,1}^r \right], \\
    H_t &= H_1^1 + H_2^2 + H_3^3 + H_1^4 + H_3^5 + H_6^6,
\end{align*}
\]

where we explicitly notice that Eqs. (38) and (79) are used for the computation of the transmitted flux.

### 7.1.1 Macro internal clamp with fixed microstructure

We recall from Subsection (3.4.1) that the 12 jump conditions to be imposed at \(x_1 = 0\) for the considered connection are

\[
\begin{align*}
    [u_i] &= 0, & t_i - f_i &= 0, & P_{ij}^r &= 0, & i = 1, 2, 3, & j = 2, 3,
\end{align*}
\]

\[\text{We suppose that the amplitudes } \bar{\alpha}_1, \bar{\alpha}_2 \text{ and } \bar{\alpha}_3 \text{ of the incident waves traveling in the Cauchy continuum are known.}\]
which in terms of the new variables (see Eqs. (9), (35), (28), (29) and (30)), recalling that \( n = (1, 0, 0) \) and introducing the tangent vectors \( \tau_1 = (0, 1, 0) \) and \( \tau_2 = (0, 0, 1) \), read

\[
v_1^+ \cdot n - u_1^- = 0, \quad v_2^+ \cdot n - u_2^- = 0, \quad v_3^+ \cdot n - u_3^- = 0,
\]

\[
\begin{pmatrix}
\lambda_e + 2\mu_e \\
0 \\
0
\end{pmatrix}
\cdot (v_1^+)' +
\begin{pmatrix}
0 \\
-2\mu_e \\
-(3\lambda_e + 2\mu_e)
\end{pmatrix}
\cdot v_1^+ = (\lambda_{\text{macro}} + 2\mu_{\text{macro}}) (u_1^-)',
\]

\[
\begin{pmatrix}
\mu_e + \mu_c \\
0 \\
0
\end{pmatrix}
\cdot (v_2^+)' +
\begin{pmatrix}
0 \\
-2\mu_e \\
2\mu_e
\end{pmatrix}
\cdot v_2^+ = \mu_{\text{macro}} (u_2^-)',
\]

\[
\begin{pmatrix}
\mu_e + \mu_c \\
0 \\
0
\end{pmatrix}
\cdot (v_3^+)' +
\begin{pmatrix}
0 \\
-2\mu_e \\
2\mu_e
\end{pmatrix}
\cdot v_3^+ = \mu_{\text{macro}} (u_3^-)',
\]

(83)

\[
P_{22} = \frac{1}{2} (v_6^+ + 2v_1^+ \cdot \tau_2 - v_1^+ \cdot \tau_1) = 0, \quad P_{23} = v_4^+ + v_5^+ = 0, \quad P_{32} = v_4^+ - v_5^+ = 0,
\]

(84)

\[
P_{33} = \frac{1}{2} (-v_6^+ + 2v_1^+ \cdot \tau_2 - v_1^+ \cdot \tau_1) = 0, \quad P_{32} = v_2^+ \cdot \tau_1 + v_3^+ \cdot \tau_2 = 0, \quad P_{34} = v_4^+ \cdot \tau_1 + v_5^+ \cdot \tau_2 = 0.
\]

Inserting the wave-form solutions (78) and (79) with \( x_1 = 0 \) in such expressions for the jump conditions the unknown amplitudes can be determined. We do not report here their explicit form since it is rather complex and it does not add essential information to the present treatise. Nevertheless, we can easily notice that from the boundary conditions \( P_{23} = P_{32} = 0 \), it follows that \( \beta_4 = \beta_5 = 0 \) and hence \( v_4 = v_5 = 0 \ \forall x_1 \). This means that the modes \( v_4 \) and \( v_5 \) cannot be activated at the considered interface between a relaxed micromorphic medium and a Cauchy continuum.

Once the amplitudes (and hence the solution) have been determined, the incident, reflected and transmitted flux can be computed according to equations (80)-(82), in which the amplitudes calculated for the constraint considered in this Subsection are used. The reflection and transmission coefficients \( R \) and \( T \) can then be computed according to Eqs. (77).

Figure 9: Cauchy-relaxed-micromorphic interface: macro clamp with fixed microstructure. Reflection coefficient as function of frequency for incident P waves (\( \bar{\alpha}_1 = 1, \bar{\alpha}_2 = \bar{\alpha}_3 = 0 \)). Complete reflection is triggered in the frequency range for which band-gaps are known to occur in bulk wave propagation.
Figure 10: Cauchy/relaxed-micromorphic interface: macro clamp with fixed microstructure. Reflection coefficient as function of frequency for incident S waves ($\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 0$, or equivalently $\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = 1$). Complete reflection is triggered in the frequency range for which band-gaps are known to occur in bulk wave propagation.

Figure 11: Cauchy/relaxed-micromorphic interface: macro clamp with fixed microstructure. Reflection coefficient as function of frequency for incident P+S wave ($\bar{\alpha}_1 = 1$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 1$). Complete reflection is triggered in the frequency range for which band-gaps are known to occur in bulk wave propagation.

Figures 9, 10 and 11 show the reflection coefficient $R$ plotted as a function of the frequency $\omega$ for the case of incident longitudinal, transverse or generic (longitudinal + transverse) waves and for the considered constraint. We explicitly comment only the plot referring to the generic incident wave (Figure 11), being the remarks concerning pure longitudinal and pure transverse incident waves completely analogous. Hence, with reference to Figure 11 it can be noticed that for the frequency range $[\omega_l, \omega_s]$ a complete reflection takes place. This is quite sensible considered that the frequency interval $[\omega_l, \omega_s]$ is the one for which a complete band gap occurs when considering bulk wave propagation (see Figure 7); since no wave can propagate in the relaxed medium in such frequency range, all the energy carried by the incident wave is completely reflected and travels back in the Cauchy medium. On the other hand, it can be inferred that, outside the band-gap frequency interval, the amount of reflected energy drops drastically, which means that a non-negligible amount of the energy initially transported by the incident wave is transmitted in the relaxed medium. In other words we can say that, outside the range of frequencies for which band-gaps occur, the continuity of the macroscopic displacement allows the connection between the Cauchy medium and the relaxed micromorphic medium in such a way that a considerable amount of energy is transferred to the relaxed continuum.
7.1.2 Macro internal clamp with free microstructure

We recall from Subsection (3.4.1) that the 12 jump conditions to be imposed at $x_1 = 0$ for the considered connection are

\[
\| u_i \| = 0, \quad t_i - f_i = 0, \quad \tau_{ij}^+ = 0, \quad i = 1, 2, 3, \quad j = 2, 3,
\]

which in terms of the new variables (see Eqs. (9), (35), (28), (29) and (30)) and introducing the tangent vectors $\tau_1 = (0, 1, 0)$ and $\tau_2 = (0, 0, 1)$, read

\[
\| u_1 \| = 0, \quad \| u_2 \| = 0, \quad \| u_3 \| = 0,
\]

\[
\begin{pmatrix}
\lambda_e + 2\mu_e & 0 & 0 \\
0 & -2\mu_e & -(3\lambda_e + 2\mu_e)
\end{pmatrix}
\cdot
(v_1^+)' + (v_1^+)' = \left(\lambda_{macro} + 2\mu_{macro}\right)(u_1^-)',
\]

\[
\begin{pmatrix}
\mu_e + \mu_c & 0 & 0 \\
0 & -2\mu_c & 2\mu_c
\end{pmatrix}
\cdot
(v_\alpha^+)' + (v_\alpha^+)' = \mu_{macro}(u_\alpha^-)', \quad \alpha = 2, 3,
\]

\[
\tau_{22} = \left(-\frac{\mu_e L_e^2}{2}\right) \cdot (v_1^+)' + \frac{\mu_e L_e^2}{2} (v_6^+)' = 0, \quad \tau_{23} = \mu_e L_e^2 \left((v_4^+)' + (v_5^+)'\right), \quad \tau_{12} = \left(\frac{\mu_e L_e^2}{\mu_e L_e^2}\right) \cdot (v_2^+)' = 0,
\]

\[
\tau_{33} = \left(-\frac{\mu_e L_e^2}{2}\right) \cdot (v_1^+)' - \frac{\mu_e L_e^2}{2} (v_6^+)' = 0, \quad \tau_{32} = \mu_e L_e^2 \left((v_4^+)' - (v_5^+)'\right) = 0, \quad \tau_{13} = \left(\frac{\mu_e L_e^2}{\mu_e L_e^2}\right) \cdot (v_3^+)' = 0.
\]

Inserting the wave-form solutions (78) and (79) with $x_1 = 0$ in these expressions for the jump conditions the unknown amplitudes can be determined. As for the previous case, we do not report here their explicit form for the sake of conciseness. Nevertheless, we can notice that from the boundary conditions $\tau_{23} = \tau_{32} = 0$, it follows that $\beta_4 = \beta_5 = 0$ and hence $v_4 = v_5 = 0$ for all $x_1$. This means that the modes $v_4$ and $v_5$ cannot be activated at the considered interface between a relaxed micromorphic medium and a Cauchy continuum.

Once that the amplitudes (and hence the solution) have been determined, the incident, reflected and transmitted flux can be computed according to equations (80)-(82), in which the amplitudes calculated for the constraint considered in this Subsection are used.

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Figure 12: Cauchy/relaxed-micromorphic interface: macro clamp with free microstructure. Reflection coefficient as function of frequency for incident P waves ($\bar{\alpha}_1 = 1$, $\bar{\alpha}_2 = \bar{\alpha}_3 = 0$). Complete reflection is triggered in the frequency range for which band-gaps are known to occur in bulk wave propagation. A local resonance frequency can be recognized at $\omega_p$ due to the fact that the microstructure is free to vibrate.

Figure 13: Cauchy/relaxed-micromorphic interface: macro clamp with free microstructure. Reflection coefficient as function of frequency for incident S waves ($\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 0$, or equivalently $\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = 1$). Complete reflection is triggered in the frequency range for which band-gaps are known to occur in bulk wave propagation. A local resonance frequency can be recognized at $\omega_r$ due to the fact that the microstructure is free to vibrate.
Figure 14: Cauchy/relaxed-micromorphic interface: macro clamp with free microstructure. Reflection coefficient as function of frequency for incident P+S wave ($\bar{\alpha}_1 = 1$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 1$). Complete reflection is triggered in the frequency range for which band-gaps are known to occur in bulk wave propagation. Two local resonance frequencies can be recognized at $\omega_p$ and $\omega_r$ due to the fact that the microstructure is free to vibrate.

In Figures 12, 13 and 14 we show the behavior of the reflection coefficient $R$ as a function of frequency for longitudinal, transverse and generic (longitudinal + transverse) incident waves. Once again, being the case of pure longitudinal and pure transverse incident wave completely analogous, we only comment here the behavior of the reflection coefficient for generic (longitudinal + transverse) waves, with reference to Figure 14. It can be seen that a complete reflection can be observed in the band-gap frequency interval. On the other hand, some phenomena of localized resonances occur at $\omega_p$ and $\omega_r$ for longitudinal and transverse waves, respectively. This means that the cut-off frequencies $\omega_p$ and $\omega_r$ are indeed resonance frequencies for the considered free microstructure. Such peaks of reflected energy can hence be completely associated to the characteristics of the considered microstructures and to their characteristic resonant behaviors.

We explicitly remark that, suitably tuning the values of the parameters shown in Table 1, the local resonant behaviors presented in Figures 12, 13 and 14 can be modified and may eventually lead to the formation of a second band gap entirely due to the presence of the interface. The fact of creating a second band gap can be directly related to both i) the values of the constitutive parameters presented in Table 1 and ii) the type of the chosen connection, namely the internal clamp with free microstructure. Suitably tuning the parameters of the relaxed micromorphic model can thus lead to the description of such double band gap profiles which, as shown in [23], may be of help for the modeling of real phononic crystals of the type studied in [17].

### 7.2 Interface between a Cauchy and a standard micromorphic medium

When considering a classical Cauchy continuum on the $-$ side and a standard micromorphic one on the $+$ side, we have that, recalling Eqs. (50), (63), (65), (59) that the solution can be written in terms of the calculated eigenvalues and eigenvectors as

\[ u_1^- = u_1^i + u_1^r, \quad u_2^- = u_2^i + u_2^r, \quad u_3^- = u_3^i + u_3^r, \]

where we set for compactness of notation

\[ u_1^i = \bar{\alpha}_1 e^{i(\omega/c_1 x_1 - \omega t)}, \quad u_1^r = \alpha_1 e^{i(-\omega/c_1 x_1 - \omega t)}, \]

\[ u_2^i = \bar{\alpha}_2 e^{i(\omega/c_1 x_1 - \omega t)}, \quad u_2^r = \alpha_2 e^{i(-\omega/c_1 x_1 - \omega t)}, \]

\[ u_3^i = \bar{\alpha}_3 e^{i(\omega/c_1 x_1 - \omega t)}, \quad u_3^r = \alpha_3 e^{i(-\omega/c_1 x_1 - \omega t)}, \]

\[ 13\text{ We suppose that the amplitudes } \bar{\alpha}_1, \bar{\alpha}_2 \text{ and } \bar{\alpha}_3 \text{ of the incident waves traveling in the Cauchy continuum are known.} \]
and moreover
\[
\tilde{v}_1^+ = \beta_1 \tilde{v}_1 e^{i(k_1(\omega)x_1 - \omega t)} + \beta_2 \tilde{v}_1 e^{i(k_2(\omega)x_1 - \omega t)} + \beta_3 \tilde{v}_1 e^{i(k_3(\omega)x_1 - \omega t)},
\]
\[
\tilde{v}_\alpha^+ = \beta_4 e^{i(k_\alpha(\omega)x_1 - \omega t)} + \beta_5 e^{i(k_\alpha(\omega)x_1 - \omega t)} + \beta_6 e^{i(k_\alpha(\omega)x_1 - \omega t)}, \quad \alpha = 2, 3,
\]
(88)
\[
\tilde{v}_4^+ = \beta_4 e^{i(1/c_m \sqrt{\omega^2 - \omega_2^2} x_1 - \omega t)}, \quad \tilde{v}_5^+ = \beta_5 e^{i(1/c_m \sqrt{\omega^2 - \omega_2^2} x_1 - \omega t)}, \quad \tilde{v}_6^+ = \beta_6 e^{i(1/c_m \sqrt{\omega^2 - \omega_2^2} x_1 - \omega t)}.
\]

Given the frequency $\omega$ of the traveling wave, the solution is hence known except for the amplitudes $\alpha$ and $\beta$. If we count, we have 15 unknown amplitudes to be determined. We focus here on the study of wave reflection and transmission concerning two out of the six possible types of connection between such media (see Section (3.4)), because the remaining four constraints always give rise to complete reflection. In particular, we will study wave transmission and reflection at surfaces of discontinuity between a Cauchy medium and a relaxed micromorphic one connected with the following constraints

- macro internal clamp with fixed microstructure,
- macro internal clamp with free microstructure.

With the introduced notations, the flux associated to the incident, reflected and transmitted waves can be computed recalling Eqs. (24), (87), (37), as
\[
H_i = -u_{1,t} [\lambda_{\text{macro}} + 2\mu_{\text{macro}}} u_{1,t} - u_{2,t} [\mu_{\text{macro}} u_{2,t} - u_{3,t} [\mu_{\text{macro}} u_{3,t}]] \tag{89}
\]
\[
H_r = -u_{1,t} [\lambda_{\text{macro}} + 2\mu_{\text{macro}}} u_{1,t} - u_{2,t} [\mu_{\text{macro}} u_{2,t} - u_{3,t} [\mu_{\text{macro}} u_{3,t}]] \tag{90}
\]
\[
H_t = H_1^1 + H_2^2 + H_3^3 + H_4^4 + H_5^5 + H_6^6 \tag{91}
\]

where we explicitly notice that Eqs. (38) and (88) are used for the computation of the transmitted flux.

7.2.1 Macro internal clamp with fixed microstructure

We recall from Subsection (3.4.1) that the 12 jump conditions to be imposed at $x_1 = 0$ for the considered connection are
\[
[u] = 0, \quad t_i - f_i = 0, \quad P_{ij}^+ = 0, \quad i = 1, 2, 3, \quad j = 2, 3,
\]
which in terms of the new variables (see Eqs. (9), (35), (28), (29) and (30), recalling that $n = (1, 0, 0)$ and introducing the tangent vectors $\tau_1 = (0, 1, 0)$ and $\tau_2 = (0, 0, 1)$, read
\[
v_1^+ \cdot n - u_1^- = 0, \quad v_2^+ \cdot n - u_2^- = 0, \quad v_3^+ \cdot n - u_3^- = 0,
\]
\[
\begin{pmatrix}
\lambda_e + 2\mu_e \\
0 \\
0
\end{pmatrix}
\cdot (v_1^+)^\prime + \begin{pmatrix}
0 \\
-2\mu_e \\
-(3\lambda_e + 2\mu_e)
\end{pmatrix}
\cdot v_1^+ = (\lambda_{\text{macro}} + 2\mu_{\text{macro}}) (u_1^-)^\prime,
\]
(92)
\[
\begin{pmatrix}
\mu_e + \mu_e \\
0 \\
0
\end{pmatrix}
\cdot (v_2^+)^\prime + \begin{pmatrix}
0 \\
-2\mu_e \\
2\mu_e
\end{pmatrix}
\cdot v_2^+ = \mu_{\text{macro}} (u_2^-)^\prime,
\]
\[
\begin{pmatrix}
\mu_e + \mu_e \\
0 \\
0
\end{pmatrix}
\cdot (v_3^+)^\prime + \begin{pmatrix}
0 \\
-2\mu_e \\
2\mu_e
\end{pmatrix}
\cdot v_3^+ = \mu_{\text{macro}} (u_3^-)^\prime,
\]

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\[ P_{22} = \frac{1}{2} \left( v_0^+ + 2 v_1^+ \cdot \tau_2 - v_1^+ \cdot \tau_1 \right) = 0, \quad P_{23} = v_4^+ + v_5^+ = 0 \quad P_{32} = v_4^+ - v_5^+ = 0, \]

\[ P_{33} = \frac{1}{2} \left( -v_0^+ + 2 v_1^+ \cdot \tau_2 - v_1^+ \cdot \tau_1 \right) = 0, \quad P_{12} = v_2^+ \cdot \tau_1 + v_2^+ \cdot \tau_2 = 0, \quad P_{13} = v_3^+ \cdot \tau_1 + v_3^+ \cdot \tau_2 = 0. \]

Inserting the wave-form solutions (87) and (88) with \( x_1 = 0 \) in such expressions for the jump conditions the unknown amplitudes can be determined. We do not report here their explicit form since it is rather complex and it does not add essential information to the present treatise. Nevertheless, we can easily notice that from the boundary conditions \( P_{23} = P_{32} = 0 \), it follows that \( \beta_4 = \beta_5 = 0 \) and hence \( v_4 = v_5 = 0 \) for all \( x_1 \). This means that the modes \( v_4 \) and \( v_5 \) cannot be activated at the considered interface between a relaxed micromorphic medium and a Cauchy continuum.

Once that the amplitudes (and hence the solution) have been determined, the incident, reflected and transmitted flux can be computed according to equations (89)-(91), in which the amplitudes calculated for the constraint considered in this Subsection are used.

![Figure 15: Cauchy/Mindlin interface: macro clamp with fixed microstructure. Reflection coefficient as function of frequency for incident P waves (\( \tilde{\alpha}_1 = 1, \tilde{\alpha}_2 = \tilde{\alpha}_3 = 0 \)). Due to the absence of band-gaps in Mindlin continua, a very low amount of energy is reflected for any value of the frequency in the observed interval.](image)

![Figure 16: Cauchy/Mindlin interface: macro clamp with fixed microstructure. Reflection coefficient as function of frequency for incident S waves (\( \tilde{\alpha}_1 = 0, \tilde{\alpha}_2 = 1, \tilde{\alpha}_3 = 0 \), or equivalently \( \tilde{\alpha}_1 = 0, \tilde{\alpha}_2 = 0, \tilde{\alpha}_3 = 1 \)). Due to the absence of band-gaps in Mindlin continua, a very low amount of energy is reflected for any value of the frequency in the observed interval.](image)
Figure 17: Cauchy/Mindlin interface: macro clamp with fixed microstructure. Reflection coefficient as function of frequency for incident P+S wave ($\bar{\alpha}_1 = 1$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 1$). Due to the absence of band-gaps in Mindlin continua, a very low amount of energy is reflected for any value of the frequency in the observed interval.

Figures 15, 16 and 17 show the behavior of the reflection coefficient as a function of frequency for longitudinal, transverse and generic (longitudinal + transverse) incident waves, respectively. It can be noticed that no frequency interval can be identified for which complete reflection occur, due to the fact that no band-gaps are allowed in standard micromorphic media (see Figure 8). More than this, it can be noticed that, very few energy is reflected back in the Cauchy medium for all values of frequencies. This means that the fact of fixing the microstructure at the interface, does not allow for micro resonant modes that trigger energy reflection. As a result, the standard micromorphic medium almost behaves as a Cauchy continuum with respect to wave reflection and transmission since no microstructure-related local resonances are activated.

7.2.2 Macro clamp with free microstructure

We recall from Subsection (3.4.1) that the 12 jump conditions to be imposed at $x_1 = 0$ for the considered connection are

$$[u_i] = 0, \quad t_i - f_i = 0, \quad \tau_{ij}^+ = 0, \quad i = 1, 2, 3, \quad j = 2, 3,$$

which in terms of the new variables (see Eqs. (9), (35), (28), (29) and (30)) and introducing the tangent vectors $\tau_1 = (0, 1, 0)$ and $\tau_2 = (0, 0, 1)$, read

$$\nu_1 = 0, \quad [u_2] = 0, \quad [u_3] = 0,$$

$$\begin{pmatrix} \lambda_e + 2\mu_e \\ 0 \\ 0 \end{pmatrix} \cdot (v_i^+) + \begin{pmatrix} 0 \\ -2\mu_e \\ -(3\lambda_e + 2\mu_e) \end{pmatrix} \cdot v_1^+ = (\lambda_{macro} + 2\mu_{macro}) (u_1^-)'$$

$$\begin{pmatrix} \mu_e + \mu_c \\ 0 \\ 0 \end{pmatrix} \cdot (v_i^+) + \begin{pmatrix} 0 \\ -2\mu_e \\ 2\mu_e \end{pmatrix} \cdot v_\alpha^+ = \mu_{macro} (u_\alpha^-)', \quad \alpha = 2, 3,$$

(94)
\[ \begin{align*}
\tilde{\tau}_{11} &= \left( \begin{array}{c} 0 \\
\frac{\mu_e L_g^2}{\mu_e L_g^2} \end{array} \right) \cdot (v_1^+) = 0, \\
\tilde{\tau}_{12} &= \left( \begin{array}{c} 0 \\
\frac{\mu_e L_g^2}{\mu_e L_g^2} \end{array} \right) \cdot (v_2^+) = 0, \\
\tilde{\tau}_{13} &= \left( \begin{array}{c} 0 \\
\frac{\mu_e L_g^2}{\mu_e L_g^2} \end{array} \right) \cdot (v_3^+) = 0, \\
\tilde{\tau}_{22} &= \left( \begin{array}{c} 0 \\
\frac{-\mu_e L_g^2}{\mu_e L_g^2} \end{array} \right) \cdot (v_1^+) + \frac{\mu_e L_g^2}{\mu_e L_g^2} (v_6^+) = 0, \\
\tilde{\tau}_{21} &= \left( \begin{array}{c} 0 \\
\frac{\mu_e L_g^2}{-\mu_e L_g^2} \end{array} \right) \cdot (v_2^+) = 0, \\
\tilde{\tau}_{23} &= \mu_e L_g^2 \left( (v_4^+) + (v_5^+) \right) = 0, \\
\tilde{\tau}_{33} &= \left( \begin{array}{c} 0 \\
\frac{-\mu_e L_g^2}{\mu_e L_g^2} \end{array} \right) \cdot (v_1^+) - \frac{\mu_e L_g^2}{\mu_e L_g^2} (v_6^+) = 0, \\
\tilde{\tau}_{32} &= \mu_e L_g^2 \left( (v_4^+) - (v_5^+) \right) = 0, \\
\tilde{\tau}_{31} &= \left( \begin{array}{c} 0 \\
\frac{\mu_e L_g^2}{-\mu_e L_g^2} \end{array} \right) \cdot (v_3^+) = 0.
\end{align*} \]

\[(95)\]

Inserting the wave-form solutions (87) and (88) with \( x_1 = 0 \) in such expressions for the jump conditions the unknown amplitudes can be determined. As for the previous case, we do not report here their explicit form for the sake of conciseness. Nevertheless, we notice that from the boundary conditions \( \tau_{23} = \tau_{32} = 0 \), it follows that \( \beta_4 = \beta_5 = 0 \) and hence \( v_4 = v_5 = 0 \) for all \( x_1 \). This means that, as for the previous constraint, the modes \( v_4 \) and \( v_5 \) cannot be activated at the considered interface between a relaxed micromorphic medium and a Cauchy continuum. Once that the amplitudes (and hence the solution) have been determined, the incident, reflected and transmitted flux can be computed according to Eqs. (89)-(91), in which the amplitudes calculated for the constraint considered in this Subsection are used.

Figure 18: Cauchy/Mindlin interface: macro clamp with free microstructure. Reflection coefficient as function of frequency for incident P waves (\( \bar{\alpha}_1 = 1, \bar{\alpha}_2 = \bar{\alpha}_3 = 0 \)). A local resonance can be observed corresponding to the frequency \( \omega_p \) due to the fact that the microstructure is free to vibrate.
Figure 19: Cauchy/Mindlin interface: macro clamp with free microstructure. Reflection coefficient as function of frequency for incident S waves ($\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 0$, or equivalently $\bar{\alpha}_1 = 0$, $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = 1$). A local resonance can be observed corresponding to the frequency $\omega_r$ due to the fact that the microstructure is free to vibrate.

Figure 20: Cauchy/Mindlin interface: macro clamp with free microstructure. Reflection coefficient as function of frequency for incident P+S wave ($\bar{\alpha}_1 = 1$, $\bar{\alpha}_2 = 1$, $\bar{\alpha}_3 = 1$). Two local resonances can be observed corresponding to the frequencies $\omega_p$ and $\omega_r$ due to the fact that the microstructure is free to vibrate.

Figures 18, 19 and 20 show the behavior of the reflection coefficient as function of the frequency for longitudinal, transverse and generic (longitudinal + transverse) waves, respectively. As for the previous case, no complete reflection frequency intervals can be identified, due to the impossibility of standard micromorphic models to predict frequency band-gaps. On the other hand, due to the fact that the microstructure is left free to vibrate at the considered interface some local resonances can be identified at $\omega_p$ and $\omega_r$ for longitudinal and transverse waves respectively. This is coherent with the results obtained in Subsection 7.1.2 for the analogous constraint imposed at a Cauchy/relaxed-micromorphic interface. Moreover, some local resonances also occur in the vicinity of the cut-off frequency $\omega_s$ which, in the case of relaxed micromorphic continuum, was the upper bound of the band-gap. This means that, in a standard Mindlin continuum, the microstructural elements do not possess enough freedom to vibrate independently of the matrix so that no complete frequency band-gap can be triggered.

8 Conclusions

In this paper we present a comprehensive treatise on the setting-up of jump conditions to be imposed at surfaces of discontinuity in relaxed micromorphic and in standard Mindlin micromorphic continua, also recalling the
analogous conditions for classical Cauchy media. This general theoretical framework allows the correct setting up of:

- the jump conditions to be imposed at internal surfaces embedded in relaxed micromorphic, Mindlin or Cauchy continua,
- as a particular case of the previous point, the possible connections between any combination of Mindlin, relaxed micromorphic or Cauchy media.

We hence focus our attention on the particular case of interfaces between a classical Cauchy continuum, on one side, and a relaxed micromorphic one (or, alternatively, a standard Mindlin one) on the other side. We study all the different types of possible connections at such interfaces, with particular attention to what we call “macro internal clamp with fixed microstructure” and “macro internal clamp with free microstructure”. Both these constraints guarantee continuity of the macroscopic displacement at the considered interface and differ for the type of boundary conditions which are imposed at the level of the microstructure. In particular, the microstructure is kept fixed in the first constraint, while it is free to vibrate in the second one.

We show that, independently of the chosen constraint, the relaxed micromorphic medium always presents frequency intervals for which complete reflection occurs, while the standard Mindlin’s model does not allow for this possibility. This is directly related to the fact that frequency band-gaps are allowed in the relaxed micromorphic model, while they are not possible in standard Mindlin continua.

On the other hand, the different boundary conditions that are imposed at the level of the microstructure, may allow for extra resonant behaviors producing reflection peaks at higher frequencies. In particular, such additional microstructure-related resonance frequencies can be identified when the microstructure is left free to vibrate at the considered interface.

Even if this is not the main aim of the present paper, we explicitly remark that the suitable tuning of the constitutive parameters of the relaxed micromorphic model may modify the local resonant behaviors presented in Figures [12][14] eventually giving rise to a second band gap for the considered constraint of macro internal clamp with free microstructure. The possibility of modeling an additional band-gap by suitably tuning the parameters of the relaxed model is worth of note, since inverse measurements may be envisaged for real band-gap metamaterials [23].

Acknowledgements

Angela Madeo thanks INSA-Lyon for the funding of the BQR 2016 "Caractérisation mécanique inverse des mé-tamatériaux: modélisation, identification expérimentale des paramètres et évolutions possibles". The work of I.D. Ghiba was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-II-RU-TE-2014-4-1109.

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