Minimum Length Cutoff in Inflation and Uniqueness of the Action

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According to most inflationary models, fluctuations that are of cosmological size today started out much smaller than any plausible cutoff length such as the string or Planck lengths. It has been shown that this could open an experimental window for testing models of the short-scale structure of space-time. The observability of effects hinges crucially, however, on the initial conditions imposed on the new comoving modes which are continually being created at the cutoff length scale. Here, we address this question while modelling spacetime as obeying the string and quantum gravity inspired minimum length uncertainty principle. We find that the usual strategy for determining the initial conditions faces an unexpected difficulty because it involves reformulating the action and discarding a boundary term: we find that actions that normally differ merely by a boundary term can differ significantly when the minimum length is introduced. This is possible because the introduction of a minimum length comes with an ordering ambiguity much like the ordering ambiguity that arises with the introduction of $\hbar$ in the process of quantization.

1 Introduction

Some of the predictions of fundamental theories of physics can only be observed on energy scales as high as the Planck scale. The availability of such high energies in the early universe and the huge separation between conventional accelerator experiments and the Planck scale has led many to turn from accelerator-based experiments to cosmological observations in order to test such theories. Inflationary cosmology \cite{1} is one of the paradigms that may serve this purpose. There, it is assumed that quantum fluctuations of the inflaton are stretched by inflationary expansion to cosmological scales. About 60 e-folds of inflationary expansion are necessary to solve many of the puzzles of big bang cosmology but in most inflationary models the expansion is much larger. In models proposed by Linde \cite{2} the universe has expanded by a factor of $10^{10^{12}}$. An implication of these models is, therefore, that our observable universe was of sub-Planckian size at the beginning of the (last) inflationary period. This suggests that inflation could act as a magnifying glass for probing the short distance structure of space-time.

A similar question had arisen concerning a possible sensitivity of black hole radiation to transplanckian physics. There, it has been found that Hawking radiation is largely immune to transplanckian effects, see e.g. \cite{3}. In the case of inflation, however, it has been found that the inflationary predictions for the cosmic microwave background (CMB) do possess a small and possibly even observable sensitivity to modifications of quantum field theory in the ultraviolet. To this end, various examples of ultraviolet-modified dispersion
relations, some motivated by solid state analogs, have been tested for their effects on inflation, see [4, 5, 6, 7]. In particular, and this will be our interest here, the ultraviolet cutoff described by a lower bound in the formal uncertainty in position, $\Delta x_{\text{min}}$, has also been investigated for its implications in inflation, see [8, 9, 10, 11].

To model the small scale structure of space through a finite minimum position uncertainty $\Delta x_{\text{min}}$ is of interest because the corresponding modified uncertainty principle has been motivated to arise from quite general quantum gravity arguments as well as from string theory, see e.g. [12, 13, 14, 15]. In fact, any theory with this type of ultraviolet cutoff can be written, equivalently, as a continuum theory and as a lattice theory, see [16]. While in the continuum formulation the theory displays unbroken external symmetries, the theory’s ultraviolet regularity is displayed in its lattice formulation.

Indeed, it has been found that inflationary predictions for the CMB are sensitive to the natural ultraviolet cutoff if the cutoff is modelled through a finite minimum uncertainty in positions, $\Delta x_{\text{min}}$. The magnitude by which the cutoff affects the predicted scalar and tensor spectra in the CMB was found to depend crucially on the initial conditions when a mode’s evolution begins, which is when its proper wave length is the minimum length. These initial conditions determine how close the modes’ state is to the adiabatic vacuum during the period of adiabatic evolution before the mode crosses the Hubble horizon. If the modes are in the adiabatic vacuum during the phase of adiabatic evolution then the effects of Planck scale physics on inflationary predictions should be no bigger than of the order of $\sigma^2$, see [9], where:

$$\sigma = \frac{\Delta x_{\text{min}}}{L_{\text{Hubble}}}$$  \hspace{1cm} (1)

Here, $L_{\text{Hubble}}$ is the Hubble length during inflation. Thus, we have approximately $\sigma \approx 10^{-3}$ if the cutoff length, $\Delta x_{\text{min}}$, is at the string scale and $\sigma \approx 10^{-5}$ if the cutoff length is at the Planck scale. In principle, however, the cutoff can lead to arbitrarily large effects, namely if the modes’ state during the adiabatic phase differs strongly from the adiabatic vacuum (see also [17]). In this case, the modulus of the mode functions oscillates at horizon crossing and these oscillations translate into characteristic oscillations in the CMB spectra. This possibility is restricted, however, by the need to keep the back-reaction small [18]. Interestingly, Easther et.al. [10, 11] found that this constraint still allows nontrivial vacua with effects as large as of order $\sigma$. Effects of this magnitude might reach the threshold of observability.

So far, initial conditions have been proposed based on analyticity arguments [9] and based on similarity to the Bunch Davies vacuum [10, 11]. A further suggestion is to minimize the field uncertainties [19, 20]. Still, however, the crucial question how to determine initial conditions for the new comoving modes that are continually being created during an expansion has not been conclusively answered. The problem is of course equivalent to identifying the vacuum state.

Here, we address this problem by reconsidering how the vacuum state is usually identified within inflationary QFT without a minimum length. Namely, the usual strategy is to make use of the fact that the action can be rewritten so as to resemble the familiar action of a field on Minkowski space with time-dependent mass term. When quantizing, one then chooses the vacuum as one does for Minkowski space theories. We will find that this method is no longer reliable when there is a minimum length. The reason is that the reformulation of the action requires the neglect of a boundary term that ceases to be a boundary term once the minimum length is introduced. We find that the differences are small but noticeable both in the initial conditions and in the evolution equations. This shows that in any approach to introducing a minimum length
into inflation this will have to be taken into account: reformulations of an action that appear to be harmless due to neglect of a boundary term can lead to an unintended modification of the theory.

To see how this phenomenon can arise, let us recall that the particular model of a natural ultraviolet cutoff that we are considering is described by quantum mechanical uncertainty relations with correction terms in the ultraviolet, of the form

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \ldots \right) \]  

(2)

where \( \beta > 0 \) is a positive constant. In the simplest case, such an uncertainty relation arises from the modified commutation relation:

\[
[X, P] = i\hbar(1 + \beta P^2)
\]  

(3)

It is not difficult to show that the uncertainty relation then implies a finite lower bound to the position uncertainty \( \Delta x \):

\[ \Delta x_{\text{min}} = \hbar \sqrt{\beta} \]  

(4)

By choosing \( \beta \) appropriately we obtain a cutoff at the string or at the Planck scale. This type of ultraviolet cutoff was introduced into quantum field theory in \([21]\) and then into inflationary cosmology in \([8]\).

It is clear that similar to quantization, which changes the commutativity properties and therefore comes with a well-known ordering ambiguity, the introduction of a minimum length through an equation such as Eq. (3) changes the commutativity properties and therefore comes with an ordering ambiguity. In principle, of course, ordering ambiguities can be of arbitrary magnitude. For example, a classical system is unchanged by adding terms of the form \((xp - px)\) to its Hamiltonian \(H\). When promoting \(x\) and \(p\) to operators the resulting terms become proportional to \(\hbar\) and could be arbitrarily large and significant to the evolution. In the case we consider here, normally vanishing terms of the form \((xp - px - i\hbar)g(x, p)\) similarly become nonzero when \(\beta \neq 0\). Here, the new Hamiltonian is determined only up to terms that vanish when setting the minimum length to zero. Those terms can be arbitrarily large and, in principle, only experiments could decide which choice is correct. This is to be expected in any approach to introducing some form of a natural minimum length.

Of course, in the case of quantization it has proven to be a very reliable strategy to adopt the minimalist approach to resolving the ordering ambiguity: write the Hamiltonian in its most simple and symmetric form and leave it unchanged when introducing \(\hbar\), i.e. do not introduce terms by hand. The same minimalist approach has tacitly been applied in the literature when the minimum length uncertainty relation has been used in inflationary quantum field theory. We will now review this procedure, thereby uncovering potential pitfalls with implications for the determination of the vacuum.

2 Fluctuations in standard inflation

In inflation, see \([1]\), we consider the action of the scalar inflaton field, minimally coupled to gravity:

\[ S = \frac{1}{2} \int \left( \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right) \sqrt{-g} \, d^4x - \frac{1}{16\pi G} \int R \sqrt{-g} \, d^4x \]  

(5)

One assumes the background to be a homogenous isotropic Friedmann universe with zero spatial curvature. In comoving coordinates \(y\) and comoving time \(\tau\), the metric reads \(ds^2 = a^2(\tau) \left( d\tau^2 - \delta_{ij} dy^i dy^j \right)\). The
perturbations of the metric tensor can be decomposed into scalar, vector and tensor modes according to their transformation properties under spatial coordinate transformations on the constant-time hypersurfaces, namely $ds^2 = ds^2_S + ds^2_V + ds^2_T$, where:

$$
\begin{align*}
    ds^2_S &= a^2(\tau) \left((1 + 2\Phi) d\tau^2 - 2\partial_i B dy^i d\tau - [(1 - 2\Psi)\delta_{ij} + 2\partial_i \partial_j E] dy^i dy^j\right) \\
    ds^2_V &= a^2(\tau) \left(2r^2 + 2V_i dx^i d\tau - [\delta_{ij} + W_{i,j} + W_{j,i}] dx^i dx^j\right) \\
    ds^2_T &= a^2(\tau) \left(2r^2 - [\delta_{ij} + h_{ij}] dx^i dx^j\right)
\end{align*}
$$

This generalizes the decomposition of vector fields into a curl and a gradient field. Here, $\Phi, B, \Psi$ and $E$ are scalar fields, $V_i$ and $W_{ij}$ are 3-vector fields satisfying $V_{i,i} = W_{i,i} = 0$ and $h_{ij}$ is a symmetric three-tensor field satisfying $h_{i}^i = h_{ij,j} = 0$. The inflaton field $\phi(y, \tau)$ fluctuates about its spatially homogeneous background $\phi(y, \tau) = \phi_0(\tau) + \delta\phi(y, \tau)$, where $\phi_0(\tau)$ is the homogenous part of the scalar field that is driving the background expansion and the perturbation is assumed small: $|\delta\phi| \ll \phi_0$. In standard inflation, vector fluctuations are not amplified by the expansion but it should be interesting to reconsider if this still holds true in inflation with a minimum length. Here, we will focus on scalar and tensor fluctuations.

### 2.1 Scalar perturbations

It is the quantum fluctuations of the intrinsic curvature $\Re$ which are thought to have seeded what later became the dominant perturbations in the CMB. The intrinsic curvature $\Re$, which is gauge invariant, can be expressed as

$$
\Re = -\frac{a'}{a} \frac{\delta\phi}{\phi_0'} - \Psi.
$$

The prime denotes differentiation with respect to the conformal time $\tau$. Expanding the action to second order yields

$$
S^{(1)}_S = \frac{1}{2} \int d\tau d^3y \left[ (\partial_\tau \Re)^2 - \delta^{ij} \partial_i \Re \partial_j \Re \right]
$$

for the action of $\Re$, where

$$
z = \frac{a\phi_0'}{\alpha}, \quad \alpha = a'/a
$$

Clearly, the very simplest formulation that one can give for the action of the field $\Re$ is given in Eq. (10). The minimalist approach to dealing with ordering ambiguities therefore requires one to start from this formulation of the action when introducing $h$ and the minimum length $\Delta x_{min}$ while not introducing any ambiguous terms by hand. This was indeed tacitly the route taken in work that introduced the minimum length into inflationary QFT \[8, 9, 10, 11\].

In the vast literature on standard inflationary theory, however, a slight reformulation of the action is usually preferred as the starting point for quantization. Namely, one often introduces an auxiliary field variable, $u$, through

$$
u = -z\Re = a \left( \delta\phi + \frac{\phi_0' \Psi}{\alpha} \right)
$$

whose dynamics follows from the action:

$$
S^{(2)}_S = \frac{1}{2} \int d\tau d^3y \left[ (\partial_\tau u)^2 - \delta^{ij} \partial_i u \partial_j u + \frac{u''}{z} u^2 \right]
$$

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for the action of $\Re$, where

$$
z = \frac{a\phi_0'}{\alpha}, \quad \alpha = a'/a
$$
As long as we do not introduce a minimum length, the two actions \( S_S^{(1)} \) and \( S_S^{(2)} \) are equivalent. More precisely, they differ by a boundary term:

\[
S_S^{(1)} - S_S^{(2)} = \int d\tau \int d^3y \frac{d}{d\tau} \left( \frac{z''}{z} u^2 \right)
\] (14)

The reason why one often prefers to quantize starting from the action \( S_S^{(2)} \) rather than from the action \( S_S^{(1)} \) is that \( S_S^{(2)} \) possesses no overall time-dependent factor, and this gives it the appearance of an action of a free field theory on flat space. Its only nontrivial aspect is that the field \( u(y, \tau) \) has a time-varying “mass” \( z''/z \).

The similarity to a Minkowski space theory suggests that in this formulation the field can be quantized in the same way that one would quantize a field on flat space. This suggests that one can identify the vacuum state in the same way as one does in the case of Minkowski space theories. Concretely, the Euler Lagrange field equation reads:

\[
\hat{u}'' - \nabla^2 \hat{u} - \frac{z''}{z} \hat{u} = 0.
\] (15)

The momentum conjugate to \( u(y, \tau) \) is given by \( \pi(y, \tau) = \frac{\partial L}{\partial u'} = u'(y, \tau) \). To quantize, one promotes \( u \) and \( \pi \) to operators, \( \hat{u} \) and \( \hat{\pi} \), which satisfy canonical commutation relations on hypersurfaces of constant \( \tau \):

\[
[\hat{u}(\tau, y), \hat{u}(\tau, y')] = [\hat{\pi}(\tau, y), \hat{\pi}(\tau, y')] = 0
\] (16)

\[
[\hat{u}(\tau, y), \hat{\pi}(\tau, y')] = i\delta^3(y - y')
\] (17)

Employing the plane wave expansion

\[
\hat{u}(\tau, y) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ u_k(\tau) \hat{a}_ke^{ik\cdot y} + u_k^*(\tau) \hat{a}_k^\dagger e^{-ik\cdot y} \right]
\] (18)

the fields will obey the commutation relations Eqs. (16,17) if the operators \( \hat{a}_k \) obey the Fock commutation relations \( [\hat{a}_k, \hat{a}_k'] = [\hat{a}_k^\dagger, \hat{a}_k^\dagger] = 0, [\hat{a}_k, \hat{a}_k^\dagger] = i\delta^3(k - k') \) and if the mode functions \( u_k \) obey the Wronskian condition:

\[
u_k \frac{du_k}{d\tau} - u_k \frac{du_k^*}{d\tau} = -i.
\] (19)

Further, the field equation will be obeyed if the number-valued functions \( u_k(\tau) \) obey the mode equation:

\[
u_k'' + \left( k^2 - \frac{z''}{z} \right) u_k = 0.
\] (20)

At this point, initial conditions must be chosen for the solution of Eq. (20). This choice is crucial because it implies the identification of the vacuum state and this affects all predictions of the theory. Intuitively, one expects that if a mode can be followed back to when its proper wavelength was infinitesimally short then one sees the mode when it was virtually unaffected by curvature, i.e. here by the expansion. One should therefore be able to identify the correct solution of the mode equation at those early times, which then sets the initial conditions of the mode for all time. Indeed, in Eq. (20), one observes that \( z''/z \to 0 \) at early times, \( \tau \to -\infty \), i.e. when the mode’s proper wavelength was arbitrarily short. In this limit, Eq. (20) formally turns into \( u_k'' + k^2 u_k = 0 \) which is the zero mass wave equation for a Minkowski space theory. For such a theory the correct solution of the wave equation is known and one proceeds, therefore, to impose

\[
\hat{u}_k(\tau) \to \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad \text{for} \quad \tau \to -\infty
\] (21)
as the initial condition for Eq. (20). This identifies the initial vacuum of each mode as the incoming lowest energy vacuum. The mathematical problem for calculating $u$ is now well-posed. Finding $u$ then yields the mode function for the intrinsic curvature $\Re = -u/z$ and from it we finally obtain the observationally relevant power spectrum $P_{s}^{1/2}(k)$, of the intrinsic curvature's quantum fluctuations after horizon crossing:

$$P_{s}^{1/2}(k) = \sqrt{\frac{k^3}{2\pi^2}} |\Re_k| \quad \frac{k}{aH} \ll 1$$

(22)

To conclude: before introducing a minimum length the actions $S_{\Gamma}^{(1)}$ and $S_{\Gamma}^{(2)}$ differ merely by a boundary term. Thus, re-expressing the mode equation Eq. (20) that followed from $S_{\Gamma}^{(2)}$ in terms of the intrinsic curvature yields

$$\Re_k'' + \frac{2z'}{z} \Re_k' + k^2 \Re_k = 0$$

(23)

which is of course the same mode equation that one obtains as Euler Lagrange equation directly from the action $S_{\Gamma}^{(1)}$. The rationale for taking the detour via the action $S_{\Gamma}^{(2)}$ is that this route exhibits a similarity with QFT on Minkowski space, which suggests a particular choice of initial condition and thus of the vacuum.

2.2 Tensor perturbations

The situation for the tensor modes $h$ is similar. Their dynamics is determined by expanding the Einstein-Hilbert action to second order:

$$S_{\Gamma}^{(1)} = \frac{m_{Pl}^2}{64\pi} \int d\tau d^3y \ a^2(\tau) \ \partial_{\mu}h^{i}_{j} \ \partial^{\mu}h^{i}_{j}$$

(24)

The aim is to calculate the spectrum of the quantum fluctuations of $h$ after horizon crossing. This spectrum should become empirically testable through measurements of the CMB’s $B$-polarization spectrum, the first measurements of which may come from the upcoming PLANCK satellite telescope.

It is clear that the action $S_{\Gamma}^{(1)}$ is of precisely the same form as $S_{\Gamma}^{(2)}$, up to constants and the replacement of $z(\tau)$ by $a(\tau)$. This means that it is possible to reformulate also the tensor action to give it the appearance of a Minkowski space theory with variable mass term, thereby obtaining a criterion for picking the initial conditions. Therefore, instead of quantizing directly from $S_{\Gamma}^{(1)}$, one often prefers to introduce the re-scaled variable, $P^{i}_{j}$

$$P^{i}_{j}(y) = \sqrt{\frac{m_{Pl}^2}{32\pi}} a(\tau)h^{i}_{j}(y)$$

(25)

whose dynamics follows from the action:

$$S_{\Gamma}^{(2)} = \frac{1}{2} \int d\tau d^3y \left( \partial_{\tau}P^{i}_{j} \partial^{\tau}P^{i}_{j} - \delta^{rs}\partial_{\tau}P^{i}_{r} \partial_{s}P^{i}_{j} + \frac{a''}{a} P^{i}_{j} P^{i}_{j} \right)$$

(26)

Analogously to the case of scalar fluctuations, the two actions $S_{\Gamma}^{(1)}, S_{\Gamma}^{(2)}$ merely differ by a term which is a total time derivative

$$\Delta S_{\Gamma} = S_{\Gamma}^{(2)} - S_{\Gamma}^{(1)} = \frac{32\pi}{m_{Pl}^2} \int d\tau d^3y \left( \alpha P^{i}_{j} P^{i}_{j} \right)'$$

(27)

and therefore lead to the same equation of motion.

One proceeds by decomposing $P^{i}_{j}$ into its Fourier components

$$P^{i}_{j} = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^{3/2}} p_{k,\lambda}(\tau) \epsilon^{i}_{j}(\mathbf{k}; \lambda) e^{i\mathbf{k} \cdot \mathbf{y}}$$

(28)
where \( e^i_j(k; \lambda) \) is the polarization tensor, satisfying the conditions: \( \epsilon_{ij} = \epsilon_{ji}, \; \epsilon^i_i = 0, \; k^i \epsilon_{ij} = 0 \) and \( e^i_j(k; \lambda) e^{j*}_i(k; \lambda') = \delta_{\lambda \lambda'} \). There are two independent polarization states, usually denoted \( \lambda = +, \times \). It is convenient to choose \( \epsilon_{ij}(-k; \lambda) = \epsilon^{*}_j(k; \lambda) \) which implies that \( p_{k, \lambda} = p^{*}_{-k, \lambda} \). The action for tensor perturbations then takes the form:

\[
S^{(2)}_T = \sum_{\lambda = +, \times} \int d\tau d^3k \left( (\partial_\tau |p_{k, \lambda}|)^2 - \left( k^2 - \frac{a''}{a} \right) |p_{k, \lambda}|^2 \right)
\]

(29)

To quantize, one promotes \( p_{k, \lambda} \) to an operator \( \hat{p}_{k, \lambda} \) and expands it in terms of creation and annihilation operators, \( \hat{p}_{k, \lambda} = p_k(\tau) \hat{a}_{k, \lambda} + p_k^*(\tau) \hat{a}^\dagger_{k, \lambda} \) to obtain the wave equation

\[
p_k'' + \left( k^2 - \frac{a''}{a} \right) p_k = 0,
\]

(30)

for the mode functions \( p_k(\tau) \) (omitting the index \( \lambda \)). Analogously to scalar fluctuations, also the \( p_k(\tau) \) must obey the Wronskian condition [11]. As for scalar modes, the similarity with the zero mass Minkowski space wave equation at early times suggests to impose the initial condition that the field takes the form given in [21] for \( k/aH \to \infty \). The mathematical problem is then well defined and \( p \) can be calculated. From \( p \) one obtains the tensor mode \( h^j_i = \sqrt{\frac{32\pi}{m_p^2 a^2}} \hat{p}^j_i \) and finally the spectrum of tensor quantum fluctuations \( h_k \) after horizon crossing:

\[
P^{1/2}_T = \sqrt{\frac{k^3}{2\pi^2}} |h_k| \bigg|_{\frac{k}{a} \ll 1}
\]

(31)

To summarize, as in the case of scalar fluctuations, one starts quantization from the action \( S^{(2)}_T \) so as to exploit the similarity with Minkowski space QFT for identifying the initial conditions and thus the vacuum state for tensor modes.

### 3 Inflation with the minimum length uncertainty relation

The minimum length uncertainty principle was first introduced into inflation in [8]. One starts by implementing the uncertainty relations in first quantization through modifications of the canonical \( x, p \) commutation relations, as in Eq. [8]. The first quantization commutation relations then carry over to quantum field theory. Note that since momentum space is unaffected by the minimum length uncertainty relations the field commutators in momentum space remain unchanged by the procedure.

This program was carried out explicitly [8] for an action of the form of the tensor action \( S^{(1)}_T \). This showed how \( \beta > 0 \) generalizes the action \( S^{(1)}_T \) to a new action \( S^{(1)}_{T, \beta} \) and how the tensor fluctuations’ equation of motion changes correspondingly. Those results immediately also translate to the case of the scalar action \( S^{(1)}_S \) since this action differs merely by overall constants and by suitably replacing \( a(\tau) \) with \( z(\tau) \). These equations of motion have been further investigated in [10][11]. We will explicitly list those equations of motion below.

Our aim now is to carry out the same program for introducing the minimum length, starting, however, from the often-used actions \( S^{(2)}_S \) and \( S^{(2)}_T \) to derive the generalized actions \( S^{(2)}_{S, \beta} \) and \( S^{(2)}_{T, \beta} \) and the correspondingly generalized equations of motion for scalar and tensor fluctuations. We will find that the generalized actions \( S^{(1)}_{S, \beta} \) and \( S^{(1)}_{T, \beta} \) as well as the generalized actions \( S^{(2)}_{T, \beta} \) and \( S^{(2)}_{T, \beta} \) no longer differ merely by boundary terms, are therefore not equivalent and lead to slightly different equations of motion.
3.1 Scalar fluctuations with minimum length

The minimum length is to be introduced as a minimum proper length in the CMB rest frame. To this end, we transform the action \( S_S^{(2)} \) as given in Eq. (13) from comoving coordinates \( y^i \) and time \( \tau \) to proper coordinates \( x^i \) and time \( \tau \), where \( x^i = a(\tau)y^i \). Since the transformation is time-dependent, the chain rule leads to a nontrivial transformation of the derivative \( \partial_\tau \) on fields and we obtain:

\[
S_S^{(2)} = \int d\tau \frac{d^3x}{2a^3} \left\{ \left( \partial_\tau + \frac{a'}{a} \sum_{i=1}^{3} \partial_{x^i} x^i - 3 \frac{a'}{a} \right) u \right\}^2 - a^2 \sum_{i=1}^{3} (\partial_{x^i} u)^2 + \frac{z''}{z} u^2 \right\} \tag{32}
\]

We identify \(-i\partial_{x^i}\) as the momentum operator, \( P^i \), and \( x^i \) as the position operator \( X^i \). These operators are defined on a Hilbert space of fields (not states) with:

\[
(u_1, u_2) = \int d^3x \ u_1^*(x) u_2(x) \\
X^i u(x) = x^i u(x) \\
P^i u(x) = -i\partial_{x^i} u(x). \tag{33}
\]

The fields thus form a Hilbert space representation of the commutation relations:

\[
[X^i, P^j] = i\delta^{ij}, \quad [X^i, X^j] = 0, \quad [P^i, P^j] = 0 \tag{34}
\]

This merely expresses the fact that the canonical commutation relations of first quantization are present also in second quantization. For example, in quantum field theory the \( \hbar \) of the Fourier factor \( e^{ixp/\hbar} \) directly derives from the \( \hbar \) in the commutation relations of first quantization. Of course, the operators \( X^i \) and \( P^j \) no longer possess a simple interpretation as observables. We see from Eqs. (13,32) that under the time-dependent mapping from comoving to proper positions, the chain rule makes the action of \( \partial_\tau \) on fields in comoving coordinates transform into a new action on fields in proper coordinates, namely \( \partial_\tau \to A(\tau) \), where:

\[
A(\tau) = \left( \partial_\tau + \frac{a'}{a} \sum_{i=1}^{3} P^i X^i - 3 \frac{a'}{a} \right) \tag{35}
\]

Using the operators \( X \) and \( P \) we can write the action (32) in representation-independent form

\[
S_S^{(2)} = \int d\tau \frac{d^3x}{2a^3} \left( u, A^\dagger(\tau) A(\tau) u \right) - a^2 \left( u, P^2 u \right) + \frac{z''}{z} \left( u, u \right), \tag{36}
\]

meaning that \( 36 \) is true without referring to the position representation, momentum representation or any other representation of the fields. Following \( 38 \), our strategy now is to maintain this representation-independent form of the action while introducing a cutoff by modifying the underlying position-momentum commutation relations \( 37 \). The modified commutation relations should break neither translation nor rotation invariance and should introduce a finite minimum position uncertainty \( \Delta x_{\text{min}} \) in all three position variables. It has been shown \( 22 \) that all such commutation relations must have the following form

\[
[X^i, P^j] = i \left( \frac{2\beta p^2}{\sqrt{1 + 4\beta p^2}} \right) \delta^{ij} + 2\beta P^i P^j, \quad [X^i, X^j] = 0, \quad [P^i, P^j] = 0 \tag{37}
\]
to first order in the parameter $\beta$, which is chosen positive, see \[22\]. The minimum position uncertainty $\Delta x_{\min}$ in every coordinate is given by

$$\Delta x_{\min} = \frac{\sqrt{\beta}}{2} (1 + d/2)^{1/4} \left( \sqrt{1 + d/2} + 1 \right)$$

(38)

where $d$ is the number of space dimensions, see \[23\]. Here $d = 3$, so that $\Delta x_{\min} \approx 1.62 \sqrt{\beta}$. Correspondingly, $\sigma \approx 1.62 \sqrt{\beta}H$, where $H$ is the Hubble parameter. A convenient Hilbert space representation of the modified commutation relations \[34\], is given by

$$X^i \psi(\rho) = i \partial_{\rho^i} \psi(\rho)$$

(39)

$$P^i \psi(\rho) = \frac{\rho^i}{1 - \beta \rho^2} \psi(\rho)$$

(40)

with the scalar product:

$$(\psi_1, \psi_2) = \int_{\rho^2 < \beta^{-1}} d^3 \rho \, \psi_1^\dagger(\rho) \psi_2(\rho)$$

(41)

Thus, the operator $A(\tau)$ changes as the minimum length is introduced, i.e. when $\beta > 0$. The action for scalars, Eq. \[36\], then also changes to become:

$$S_{S,\beta}^{(2)} = \int d\tau \int_{\rho^2 < \beta^{-1}} d^3 \rho \, \frac{1}{2 \alpha^3} \left\{ \left( \partial_{\tau} - \frac{a'}{a} \frac{\rho^i}{1 - \beta \rho^2} \partial_{\rho^i} - \frac{3 a''}{a} \right) u \right\}^2 - \frac{a^2 \rho^2 |u|^2}{(1 - \beta \rho^2)} + \frac{z''}{z} |u|^2 \right\}$$

(42)

The presence of $\rho$ derivatives means that the $\rho$ modes are coupled. Conveniently, in the new variables $(\tilde{\tau}, \tilde{k})$

$$\tilde{\tau} = \tau,$$

$$\tilde{k}^i = a \rho^i e^{-\beta \rho^2 / 2}$$

(43)

the $\tilde{k}$ modes decouple. To see this, note that:

$$\partial_{\tilde{\tau}} - \frac{a'}{a} \frac{\rho^i}{1 - \beta \rho^2} \partial_{\rho^i} = \partial_{\tau}.$$

(44)

We will use the common index notation $\tilde{u}_k$ for the decoupling modes. The $\tilde{k}$ modes only coincide with the usual comoving modes on large scales, i.e., only for small $\rho^2$. This means that the precisely comoving $k$ modes, obtained by scaling $k^i = a \rho^i$, decouple only at large distances; at distances close to the cutoff scale they couple. The action now takes the decoupled form (i.e. there are no $\tilde{k}$ derivatives):

$$S_{S,\beta}^{(2)} = \int d\tilde{\tau} \int_{\tilde{k}^2 < a^2 / e^\beta} d^3 \tilde{k} \, a^{-a} \frac{\kappa}{2} \left\{ \left( \partial_{\tilde{\tau}} - \frac{3 a''}{a} \right) \tilde{u}_k(\tilde{\tau}) \right\}^2 - \mu |\tilde{u}_k|^2 + \frac{z''}{z} |\tilde{u}_k|^2 \right\}$$

(45)

with the functions $\mu$ and $\kappa$ are defined through

$$\mu(\tau, \tilde{k}) = -\frac{a^2}{\beta} \frac{W(-\beta \tilde{k}^2 / a^2)}{1 + W(-\beta \tilde{k}^2 / a^2)^2}$$

(46)

$$\kappa(\tau, \tilde{k}) = \frac{e^{-\frac{3}{2} W(-\beta \tilde{k}^2 / a^2)}}{1 + W(-\beta \tilde{k}^2 / a^2)}$$

(47)

where $W$ is the Lambert $W$ function (see e.g. \[24\]), which is defined so that $W(x) e^{W(x)} = x$. As expected, each comoving mode $\tilde{k}$ has a starting time, $\tau_c$, namely the time at which $a(\tau_c) = e^{\beta \tilde{k}^2}$, which is when
\( \rho^2 = 1/\beta \), which is when the mode’s proper wave length is the cutoff length. The equation of motion that follows from the action \( S_{S,\beta}^{(2)} \) is:

\[
\dddot{\bar{u}}_k + \left( \frac{\kappa'}{\kappa} - 6 \frac{a'}{a} \right) \dot{\bar{u}}_k + \left( \mu - 3 \frac{\kappa' a'}{\kappa a} - 3 \left( \frac{a'}{a} \right)' - 9 \frac{\kappa''}{\kappa} + \frac{z''}{z} \right) \bar{u}_k = 0
\]  

(48)

The equation of motion contains a number of terms that involve the scale factor \( a \) and appears rather complicated. This is not a consequence of the introduction of the minimum length. Instead, it is merely due to our choice of variables. To see this, note first that the functions \( \mu \) and \( \kappa \) are simpler in the variables \( \tau \) and \( \rho \):

\[
\mu(\tau, \rho) = \frac{a^2 \rho^2}{(1 - \beta \rho^2)^2}
\]

(49)

\[
\kappa(\rho) = \frac{e^{3\beta \rho^2}}{1 - \beta \rho^2}
\]

(50)

Thus, as the cutoff is removed, \( \beta \rightarrow 0 \), we have that \( \mu \rightarrow k^2 \) and \( \kappa \rightarrow 1 \). The action (45) thus turns into a conventional-looking action, except for an overall factor of \( a^{-6} \). The many terms of \( a \) and \( a' \) in the equation of motion (48) trace back to this pre-factor \( a^{-6} \) in the action (45). The occurrence of the factor \( a^6 \) might be surprising since we had started with the action \( S_{S}^{(2)} \) as given in (13), which of course does not possess a time-dependent pre-factor. The reason for the occurrence of this pre-factor is that the operations of Fourier transforming and of scaling do not commute: We did not directly Fourier transform the original action (13) from comoving positions to comoving momenta, as is usually done. Instead, we first scaled the comoving position coordinates to proper coordinates (where we introduced the minimum length), then Fourier transformed to proper momenta, and finally scaled to comoving momenta. The field variable \( \bar{u}_k \) therefore differs from the usual field variable \( \tilde{u}_k \) by a factor of \( a^{-3} \):

\[
\tilde{u}_k = a^{-3} \bar{u}_k
\]

(51)

In this commonly used field variable, the action \( S_{S,\beta}^{(2)} \) for scalar fluctuations, Eq. (45), then takes the more familiar-looking form

\[
S_{S,\beta}^{(2)} = \int d\tau \int_{k^2 < a^2/\epsilon \beta} d^3 \tilde{k} \frac{1}{2} \kappa \left( u''_k u'_k - \left( \mu - \frac{z''}{z} \right) u^2_k \right)
\]

(52)

and also yields the equation of motion in a simpler form:

\[
\dddot{u}_k + \frac{\kappa'}{\kappa} \dot{u}_k + \left( \mu - \frac{z''}{z} \right) u_k = 0, \quad \text{(derived from } S_{S,\beta}^{(2)}\text{)}
\]

(53)

Note that the introduction of the minimum length did leave us with a time-dependent pre-factor \( \kappa(\tau, \tilde{k}) \) in the action Eq. (52), a fact that we will revisit. The mode equation (53) generalizes Eq. (20) in the presence of the minimal length cutoff when starting from the action \( S_{S}^{(2)} \). We need to add that the canonical commutation relations between \( u_k \) and its conjugate momentum, \( \pi_k = \kappa u'_k \), namely

\[
[u_k, \pi_{k'}] = i\delta^3(\tilde{k} - \tilde{k'})
\]

(54)

require that the solutions to equation (53) also obey the slightly generalized Wronskian condition

\[
u_k(\tau)u^\tau_k(\tau) - u_k(\tau)u^\tau_k(\tau) = i\kappa^{-1}.
\]

(55)
Expressing the equation of motion in terms of the intrinsic curvature, \( \mathcal{R} = -\frac{u}{z} \), we obtain:

\[
\mathcal{R}_{\tilde{k}}'' + \left( \frac{\kappa'}{\kappa} + \frac{2s'}{z} \right) \mathcal{R}_{\tilde{k}}' + \left( \mu + \frac{z'\kappa'}{z\kappa} \right) \mathcal{R}_{\tilde{k}} = 0 \quad \text{(derived from } S^{(2)}_{S,\beta} \text{)} \quad (56)
\]

It is straightforward to show that the wave equation and Wronskian equation reduce to the usual wave equation \( (20) \) and Wronskian condition \( (19) \) in the limit \( \beta \to 0 \), i.e. when the minimum length cutoff is removed.

To summarize, we calculated the generalization of the action \( S^{(2)}_S \) to the action \( S^{(2)}_{S,\beta} \) and found the corresponding equation of motion \( (53) \).

Let us now compare with the result of introducing the minimum length uncertainty relation into the action \( S^{(1)}_S \) to obtain \( S^{(1)}_{S,\beta} \). We read off from \( [8] \) that the action \( S^{(1)}_{S,\beta} \) yields the wave equation:

\[
\mathcal{R}_{\tilde{k}}'' + \left( \frac{\kappa'}{\kappa} + \frac{2s'}{z} \right) \mathcal{R}_{\tilde{k}}' + \mu \mathcal{R}_{\tilde{k}} = 0 \quad \text{(derived from } S^{(1)}_{S,\beta} \text{)} \quad (57)
\]

The results of \( [8] \) also show that this field \( u_{\tilde{k}} \) satisfies the same Wronskian condition \( (55) \). Clearly, the equations of motion \( (53) \) and \( (58) \) differ and we will need to investigate the origin and extent of the difference.

### 3.2 Tensor fluctuations with minimum length

From the case of scalar fields we find the corresponding two actions \( S^{(1)}_{T,\beta} \) and \( S^{(2)}_{T,\beta} \) for tensor perturbations, namely by inserting suitable constants and by replacing occurrences of \( z \) by \( a \). For \( \beta = 0 \) the two actions are of course equivalent, differing merely by a boundary term. For \( \beta > 0 \), however, we find that they yield slightly different equations of motion:

\[
\begin{align*}
\mathcal{R}_{\tilde{k}}'' + \left( \frac{\kappa'}{\kappa} + \frac{2a'}{a} \right) \mathcal{R}_{\tilde{k}}' + \mu \mathcal{R}_{\tilde{k}} &= 0 \quad \text{(derived from } S^{(1)}_{T,\beta} \text{)} \quad (59) \\
p_{\tilde{k}}'' + \left( \frac{\kappa'}{\kappa} + \frac{2a'}{a} \right) p_{\tilde{k}}' + \left( \mu - \frac{a''}{a} - \frac{a'\kappa'}{a\kappa} \right) p_{\tilde{k}} &= 0 \quad \text{(derived from } S^{(1)}_{T,\beta} \text{)} \quad (60) \\
h_{\tilde{k}}'' + \left( \frac{\kappa'}{\kappa} + \frac{2a'}{a} \right) h_{\tilde{k}}' + \left( \mu + \frac{a'\kappa'}{a\kappa} \right) h_{\tilde{k}} &= 0 \quad \text{(derived from } S^{(2)}_{T,\beta} \text{)} \quad (61) \\
p_{\tilde{k}}'' + \left( \frac{\kappa'}{\kappa} + \frac{2a'}{a} \right) p_{\tilde{k}}' + \left( \mu - \frac{a''}{a} \right) p_{\tilde{k}} &= 0 \quad \text{(derived from } S^{(2)}_{T,\beta} \text{)} \quad (62)
\end{align*}
\]

### 3.3 Origin of the differences in the mode equations

In order to trace the inequivalence of the obtained equations of motion, we begin by noting that we encountered an ordering ambiguity in Eqs. \( (59, 60) \) when modifying the commutation relations: consider the formal position and momentum operators in the operator \( A(\tau) \). We could have used the first quantization’s canonical commutation relations to arbitrarily re-order the positions and momenta. Clearly, it does matter, however, whether we do this before or after we change the first quantization’s commutation relations. In

\[11\]
this way, for example by adding terms of the form $(xp - px - i\hbar)f(x, p)$ before changing the commutation relations, we could have introduced into the action arbitrary terms that vanish as the minimum length is set to zero, i.e. as $\beta \to 0$.

Of course, in any theory that generalizes quantum field theory by introducing a minimum length parameter one can guess Hamiltonians and actions etc. only up to terms that vanish as the minimum length parameter vanishes - much like quantum Hamiltonians can be guessed from classical Hamiltonians only up to terms that vanish as $\hbar \to 0$. We encountered essentially an instance of Dirac’s observation that quantization removes degeneracy. As in the case of quantization, the minimalist approach to dealing with the ambiguity is to bring the action into a simple form and to not use the ambiguity to introduce by hand any such terms that would vanish as the minimum length is set to zero. This was the approach tacitly adopted in $^8$ and we here also adopted the same minimalist approach when we introduced the minimum length into $S^{(1)}_S, S^{(2)}_S, S^{(1)}_T$ and $S^{(2)}_T$. We then found that actions $S^{(1)}_S$ and $S^{(2)}_S$ (and similarly $S^{(1)}_T, S^{(2)}_T$) yield differing equations of motion. How could this happen, given that the two actions $S^{(1)}_S$ and $S^{(2)}_S$ (and similarly $S^{(1)}_T, S^{(2)}_T$) are equivalent?

We already indicated that the answer traces back to the fact that the actions in the two formulations of types $S^{(1)}$ and $S^{(2)}$ are equivalent only up to a boundary term. After the minimum length is introduced these terms are no longer boundary terms. To see that this is the case, consider the scalar actions $S^{(2)}_S, \beta$ in (42) as expressed in terms of the field $u_{\tilde{k}}$. Note that it possesses in its integration measure a time-dependent factor

$$\int d\tau d^3\tilde{k} \kappa(\tau, \tilde{k})$$

If we remove the minimum length, $\beta \to 0$, we obtain of course $\kappa \to 1$. In the case $\beta > 0$, however, if

$$\int d\tau d^3\tilde{k} \Delta\mathcal{L} = \int d\tau d^3\tilde{k} \frac{d}{d\tau} f(\tau)$$

is a negligible boundary term arising from a total time derivative, then in the presence of the minimum length uncertainty relation

$$\int d\tau d^3\tilde{k} \kappa(\tau, \tilde{k}) \Delta\mathcal{L} = \int d\tau d^3\tilde{k} \kappa(\tau, \tilde{k}) \frac{d}{d\tau} f(\tau)$$

is not a boundary term. The same phenomenon occurs for tensor fluctuations: the two actions which are normally equivalent because differing merely by the total time derivative $\Delta S_T$ given in Eq. 27 now yield different equations of motion. Indeed, as expected, when the minimal length is introduced the two actions differ by:

$$S^{(2)}_{T,\beta} - S^{(1)}_{T,\beta} = \int d\tau d^3\tilde{k} \left( a' a \hbar^2 \right)' \kappa(\tau, \tilde{k})$$

The integrant is generally not a total time derivative due to the presence of the function $\kappa(\tau, \tilde{k})$.

### 3.4 Comparison of the equations of motion

The equations of motion that arise from the actions of type $S^{(1)}$ and type $S^{(2)}$ differ merely in their “mass terms”, i.e. in the terms that multiply the undifferentiated field. The mass terms differ by the terms

$$e(\tau, \tilde{k}) = \frac{z' \kappa'}{z \kappa}, \quad \text{and} \quad d(\tau, \tilde{k}) = \frac{a' \kappa'}{a \kappa}$$
Figure 1: Comparison of the three “mass terms” \( \ln(\mu) \) (dashed), \( \ln(|d|) \) (dotted) and \( \ln(a''/a) \) (solid) versus conformal time. The term \( d \) of the ambiguity is dominated by \( \mu \) and \( a''/a \) throughout the evolution.

in the scalar and tensor cases respectively. Let us now compare, for example, the equations of motion for tensors - the situation is completely analogous for scalars. How significant is the term \( d(\tau, \tilde{k}) = a'' \kappa a'' \) by which the two equations (60) and (62) differ? The term \( d \) competes with the two other “mass” terms, \( a''/a \) and \( \mu \). In Fig.1, the magnitudes of the terms \( d, \frac{a''}{a} \) and \( \mu \) are compared by plotting the logarithm of their absolute values against conformal time. We chose the de Sitter background with a realistic Planck length to Hubble length ratio of \( \sigma = 10^{-5} \). In de Sitter space all \( \tilde{k} \) modes evolve in the same way and we arbitrarily chose \( \tilde{k} = 1 \). The curves start at the creation time \( \tau_c \) of the mode and end at future infinity \( \tau = 0 \). There are three distinct phases in a mode’s evolution:

A) In the initial phase close to the creation time, the behavior of the differential equation is dominated by the terms \( \mu \) and \( d \) which both appear to diverge. (The function \( a'/a \) is regular at \( \tau_c \) since the creation time of a particular mode is not a special time for the scale factor.) The behavior at creation time is crucial, however, because this is where the initial conditions for the mode are to be set by some suitable criterion for choosing the vacuum state of the system. We need to determine, therefore, the relative magnitudes of
\(d\) and \(\mu\) as \(\tau \to \tau_c^+\). To this end, let us consider \(\mu\) as a function of the inverse proper wavelength \(\rho\), as in Eq. (50). This shows that \(\mu\) is indeed divergent when the proper wavelength \(1/\rho\) approaches the minimum wavelength \(\sqrt{3}\), i.e. at the creation time \(\tau_c\) (Eq. (520) also shows that \(\mu \to k^2\) at late times, as expected).

Now a straightforward calculation yields for the ratio of the functions \(d\) and \(\mu\):

\[
\frac{d(\tau, \tilde{k})}{\mu(\tau, k)} = -\sigma^2 \left(5 + 3 W(-k^2 \sigma^2)\right)
\]  

(68)

The range of the Lambert W function is the finite interval \([-1, 0]\). Thus, since \(\mu\) is divergent at \(\tau_c\), also \(d\) is divergent at \(\tau_c\). However, Eq. (58) also implies that at all times the term \(d\) is much smaller than the term \(\mu\), namely by a factor \(\sigma^2\), up to a pre-factor of order one. Since \(\mu\) dominates \(d\) by a factor of order \(\sigma^2\), criteria for determining the mode’s initial condition at \(\tau_c\) are generally only correspondingly weakly affected by the presence or absence of the term \(d\).

B) The initial period is followed by an adiabatic period in which all three terms \(\mu, d\) and \(a''/a\) are slowly varying. This phase lasts until horizon crossing. In order to estimate the effect of the term \(d\) on the time evolution in this phase we compare the oscillation frequencies \(\Omega_d = \sqrt{\mu - a''/a - d}\) and \(\Omega_0 = \sqrt{\mu - a''/a}\) in the adiabatic phase with and without the term \(d\) respectively. During the adiabatic phase we can neglect the term \(a''/a\) and we can set \(\mu \approx \tilde{k}^2\) to obtain: \(\Omega_d/\Omega_0 = (1 - d/(2\mu)) + \mathcal{O}(d^2/\mu^2)\), i.e. \(\Omega_d \approx \Omega_0(1 + b\sigma^2)\) and thus \(\Delta \Omega \approx b\tilde{k}\sigma^2\) where \(b\) is of order one. In principle, this frequency shift alters the number \(N\) of oscillations during the duration of the second phase. In order to estimate \(N\), we note that the \(\tilde{k}\) mode is created at the time \(\tau_c = -1/(H\sqrt{\sigma^3})\); the duration, \(T\), of the second phase is comparable to this and so is of the order of \(T \approx 1/(H\sqrt{\sigma^3})\). Recall that \(\sigma \approx 1.62 \sqrt{3}H\), which implies \(T \approx 1.62/(\tilde{k}\sigma\sqrt{\sigma})\). Thus, in the course of the adiabatic second phase, the presence or absence of the term \(d\) implies approximately \(N\) more or fewer oscillations, where

\[
N \approx \Delta \Omega T \approx \frac{1.62 b \tilde{k} \sigma^2}{\tilde{k}\sigma\sqrt{\sigma}} \approx \sigma
\]

(69)

which is far less than a single oscillation.

C) The last period, from horizon crossing to the infinite future \(\tau \to 0\) is not clearly resolved in our plot. It is the period when the term \(a''/a\) diverges and entirely dominates the \(\mu\) and \(d\) terms since they stay finite.

4 Conclusions

While the framework of quantum field theory is well-tested down to distances of about \(10^{-18} m\), it is generally expected that there are corrections due to quantum gravity when approaching the Planck length of about \(10^{-35} m\) which may well constitute a fundamental smallest length in nature.

If, therefore, there exists a finite minimum wavelength then, during inflation, comoving modes are continually being created. Initially, a new comoving mode will evolve under the influence of Planck scale effects but it is clear that at late times a comoving mode’s equation of motion will reduce to the usual low-energy mode equation, namely when the mode’s proper wavelength becomes much larger than the minimum length. Thus, as was pointed out in [25], effects of the Planck scale can propagate into the observable low energy realm essentially only by selecting a solution of the mode equation which at late times differs from the usually assumed solution for the usual mode equation.
This suggests a simple technique for exploring possible effects that Planck scale physics could have on inflationary predictions for the CMB. Assume that standard quantum field theory holds unchanged down to the minimum wavelength where modes are being created. Then, consider a variety of possible initial conditions for the newly created modes by applying candidate criteria for identifying the vacuum state. It is clear that in the time translation invariant de Sitter case all effects reduce to merely an overall re-normalization of the flat spectrum (if each mode’s initial condition is chosen by applying the same criterion).

When the Hubble parameter varies, however, then modes oscillate a variable number of times before crossing the horizon. Thus, generically, a mode’s amplitude will be alternatingly large and small when crossing the horizon. This can lead to potentially observable characteristic oscillations in the spectrum \[20\]. In this approach, quantum field theory is implicitly assumed to hold unchanged down to the Planck scale and therefore Planck scale physics is modelled so as to affect the predictions of inflation, for any given evolution of the scale factor \(a(\tau)\), merely through the initial conditions. In any realistic model, of course, the quantum field theoretic mode equations will be modified when approaching the Planck scale. This too will have an effect on the number of oscillations that a mode undergoes before horizon crossing and it will therefore contribute to the predicted oscillations of the CMB spectra.

Here, we considered a concrete model for how quantum field theory is modified when approaching the Planck length, namely by introducing the minimum length uncertainty principle. The equations of motion then indeed became modified at scales close to the Planck scale. As was shown in \[9, 10, 11\], the inflationary predictions for the CMB are to some extent affected, possibly leading to observable oscillations in the fluctuation spectra. However, while the equations of motion are known, the details of the predictions still significantly depend on precisely which initial condition is chosen, i.e. on the identification of the vacuum.

As yet, it is not fully understood in any model how Planck scale physics determines the initial conditions of modes as they are being created, i.e. when their proper wave length is the minimum length. Within our model of spacetime, in which there is a minimum length uncertainty relation, the problem of determining the initial conditions for new comoving modes is further complicated by the fact that the mode equation possesses an irregular singular point at the initial time, see \[8, 9, 10, 11\]. So far, in the literature, a mathematical argument based on analyticity \[9\] and a physical argument based on similarity to the Bunch Davies vacuum \[10, 11\] have been discussed and the implications for the CMB have been investigated. Nevertheless, the crucial problem of determining the initial state of modes when they emerge from the Planck scale in an expansion is still essentially unsolved.

Therefore, we here reconsidered the conventional approach to fixing the vacuum: introduce new variables in terms of which the action resembles that of a Minkowski space theory with variable mass - a theory for which the correct vacuum is known. Interestingly, we found that introducing the minimum length into this reformulated action does not yield the same theory - the action and the equations of motion differ slightly. This raised the question as to the extent to which any predictions are affected by this ambiguity.

Let us, therefore, recall the mechanism that leads to the prediction of characteristic oscillations in the CMB spectra due to the minimum length uncertainty relation. The modulus of the mode solution oscillates before horizon crossing if it deviates from the adiabatic solution. In the case of non-de Sitter inflation, modes then cross the Hubble horizon alternatingly with small and large amplitudes and this translates into the prediction of oscillations in the CMB spectra. The oscillations are the more pronounced the more the
solutions deviate from the usual adiabatic vacuum in the phase preceding horizon crossing. For the minimum length uncertainty relation an effect of order $\sigma$ appears possible \cite{10, 11}. This depends, of course, on the choice of initial condition.

On the basis of the current experimental approaches, it is clear that, at best, only effects on the CMB spectrum that are of order $\sigma$ might become observable over the foreground. Effects of order $\sigma^2$ must be expected to compete with numerous other effects such as those due to backreaction and the nonlinearity of gravity. We found that in the mode equation the term $\mu$ dominates the ambiguous term $d$ by a factor $\sigma^2$.

On the one hand, this means that the ambiguous term's effect on the amplitude of the minimum-length-induced characteristic oscillations in the CMB spectra is very small. This is because any generic criterion for choosing the modes' initial condition should be affected only to the order of $\sigma^2$. We note, however, that one might of course conceive of special initial conditions that are arbitrarily different for the two cases. For any generic criterion for initial conditions, the modes' deviation from the adiabatic solution is, therefore correspondingly little affected by whether one starts from the actions of type $S^{(1)}$ or those of the type $S^{(2)}$. On the other hand, this leaves open the possibility that the phases of the predicted characteristic oscillations in the CMB spectra are affected by the ambiguous term $d$. Indeed, the term $d$ implies a small frequency shift for the mode in the period before horizon crossing. We found, however, that the shift in the oscillation frequency accumulates only to a small fraction ($\sigma \approx 10^{-5}$) of a single oscillation (whose amplitude is itself at most of order $\sigma$) over the entire adiabatic phase.

To summarize, choosing either the actions of type $S^{(1)}$ or the actions of type $S^{(2)}$ generically leads to practically indistinguishable predictions for the effect of the minimum length uncertainty relation on the CMB. Nevertheless, the two actions $S^{(1)}$ or $S^{(2)}$ are actually different and the much-sought-after precise physical criterion for identifying the newly created modes' initial condition may be sensitive to the correct choice of equation of motion. We draw the lesson, therefore, that great care must be taken when introducing a further fundamental constant such as a minimum length into quantum field theory: even if, as usual, ordering ambiguities are resolved by adopting the minimalist approach, previously harmless reformulations of the theory up to a boundary term can lead to an actual change in the theory.

In the particular case at hand one may try to circumvent the problem by starting from the most plausible actions, i.e. those of type $S^{(1)}$, introducing the minimum length and only then reformulating the theory such as to resemble a Minkowski space theory with variable mass. Even this approach faces difficulties due to the introduction of a minimum length scale, however. To see this, note that, usually, modes can be followed arbitrarily far back in the past to when their proper wavelength was arbitrarily short and therefore essentially unaffected by curvature, yielding an arbitrarily close match with Minkowski space. With a minimum length, however, the initial conditions must be chosen when the proper wavelength of the mode is the cutoff length. At that time, unavoidably, the match with the Minkowski case is imprecise. In addition, a further redefinition of the field variable followed by a suitable redefinition of the time variable could preserve the appearance of a free field theory on flat space with a time-dependent mass. This would modify, however, the generator of time evolution, i.e. the Hamiltonian. This can affect Hamiltonian-based criteria for picking the vacuum state; a similar point was made in \cite{20}. An example of a criterion for choosing the vacuum that does not rely on the Hamiltonian, but on field uncertainties instead, was given in \cite{19, 20}. It should be interesting to apply this criterion in the case of the minimum length uncertainty relation.

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