A Program for Geometric Arithmetic

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In this article, we originate a program for what I call Geometric Arithmetic. Such a program would consist of four parts, if I were able to properly understand the essentials now. Namely, (1) Non-Abelian Class Field Theory; (2) Geo-Ari Cohomology Theory; (3) New Non-Abelian Zeta Functions; and (4) Riemann Hypothesis. However, here I could only provide the reader with \( \frac{1+1(\frac{1}{4}+\frac{1}{2})+1}{4} \) of them. To be more precise, discussed in this article are the following particulars;

(A) Representation of Galois Group, Stability and Tannakian Category;
(B) Moduli Spaces, Riemann-Roch, and New Non-Abelian Zeta Function; and
(C) Explicit Formula, Functional Equation and Geo-Ari Intersection.

So what are these ABC of the Geometric Arithmetic?!

As stated above, (A) is aimed at establishing a Non-Abelian Class Field Theory. The starting point here is the following classical result: Over a compact Riemann surface, a line bundle is of degree zero if and only if it is flat, i.e., induced from a representation of fundamental group of the Riemann surface. Clearly, being a bridge connecting divisor classes and fundamental groups, this result may be viewed as and is indeed a central piece of the classical (abelian) class field theory. (See e.g., [Hilbert] and [Weil].) Thus it is then only natural to give a non-abelian generalization of it in order to offer a non-abelian class field theory. This was first done by Weil. In his fundamental paper on generalization of abelian functions [Weil1], Weil showed that over a compact Riemann surface, a vector bundle is of degree zero if and only if it is induced from a representation of fundamental group of the surface.

Thus far, two new aspects naturally emerge. That is, unitary representations and non-compact Riemann surfaces, reflecting finite quotients of Galois groups and ramifications in Class Fields Theory, CFT for short, respectively: In a (complex) representation class of a finite group, there always exists a unitary one, while a discussion for compact Riemann surfaces results only unramified CFT. Thus mathematics demands new results to couple with them. As it is well-known that to this end we then have (i) Mumford’s stability of vector bundles in terms of intersection; (ii) Narasimhan-Seshadri’s correspondence; and (iii) Seshadri’s parabolic analog of (i) and (ii). That is to say, now the above result of Weil is further refined to the follows: Over (punctured) Riemann surfaces, (Seshadri) equivalence classes of semi-stable parabolic bundles of parabolic degree zero correspond naturally in one-to-one to isomorphism classes of unitary representations of fundamental groups.

On the other hand, the above results, while central, do only parts of the CFT – at its best, the Weil-Narasimhan-Seshadri correspondence reflects a micro reciprocity law. What CFT really stands should not be a relation between a single representation and an isolated bundle, instead, CFT should expose Galois groups intrinsically in terms of representation classes of bundles globally. Thus an integration process aiming at constructing a global theory becomes a great necessity.

It is at this point where the theory of Tannakian category enters into the picture. Recall that the existing theory of Tannakian category takes the following forms: (i) groups may be reconstructed from their associated categories of representations; (ii) Fiber functors equipped Tannakian categories are clone categories of (i), i.e., are equivalent to the categories of representations; and (iii) original groups may be recovered from the automorphism groups of fiber functors.

At it turns out, with this strongest form of the standard theory of Tannakian category, we have little hope to match it perfectly with the CFT we are looking for. Fortunately, there are still room to manoeuvre, since in CFT we only care about finite quotients of the associated groups, and in terms of representations
finite quotients correspond to what we call finitely completed Tannakian subcategories according to Tannaka duality and van Kampen completeness theorem. In this way, we finally establish a non-abelian CFT for Riemann surfaces, or better, for function fields over complex numbers successfully. Main results include the Existence Theorem, the Conductor Theorem and the Reciprocity Law. See e.g. Theorem A.2.4.2.

By establishing a CFT for Riemann surfaces as above, possibly, we may give the reader an impression that everything works smoothly. No, practically, it is not the case. For example, we do not need all unitary representations. Or put this in another way, all semi-stable parabolic bundles of parabolic degree zero lead us to nowhere. Consequently, we must carefully select among these semi-stable objects a handful portion so that (i) the standard theory of Tannakian category could be applied; and (ii) there are still rooms for us to manoeuvre, along the line of Tannaka duality and van Kampen completeness theorem. This then leads to what we call geo-ari representations and geo-ari bundles.

Remark. By definition, as a direct consequence of the Narasimhan-Seshadri correspondence, the correspondence between geo-ari representation and geo-ari bundles for function fields over complex numbers holds more or less trivially. However, the situation changes dramatically for global fields. For example, for curves over finite fields, we need to introduce a new principal, called the Harder-Narasimhan correspondence, to tackle this.

The experienced reader here naturally would ask how we overcome the difficulty about tensor products of geo-ari bundles, since, generally speaking, to show the tensor operation is closed is the key to apply the theory of Tannakian category. Here for Riemann surfaces, two approaches are available. For one, we use the Narasimhan-Seshadri correspondence, as easily one sees that tensors of unitary representations are again unitary. But this analytic approach is not a genuine one, since a micro reciprocity law, i.e., the Weil-Narasimhan-Seshadri correspondence is used. Thus a purely algebraic proof should be pursued. This then leads to the works of Kempf and Ramanan-Ramanathan on instability flags, which I call the $KR^2$-trick. Moreover, as the original $KR^2$-trick only works for bundles without parabolic structures, so to stylize the non-abelian CFT (for Riemann surfaces), we ask for a parabolic version of the $RK^2$-trick. To achieve this, we follow a supplementary work of Faltings and Tataro: First, as in [Fa], rewrite any geo-ari subbundle in terms of filtrations over certain points on the surface, disjoint from parabolic points; then use the GIT stability to check whether the associated point for a subbundle of the tensor is semi-stable. If so, we are done by definition. If not, by the instability flag of Mumford-Kempf, we obtain a modified GIT stable point according to Ramanan-Ramanathan, from which, the intersection stability for the tensor may finally be proved by using the intersection stability of all the components in the tensor product as in [To].

Motivated by such a success in non-abelian CFT for function fields over complex numbers, we anticipate that in principal, the non-abelian CFT for local and global fields works similarly. So the building blocks of our program for a non-abelian CFT then are the follows:

1. there should be a suitable type of representations of Galois groups, which we call geometric representations and a suitable type of intersection stability for bundles which we call geometric parabolic bundles such that an analog of the Weil-Narasimhan-Seshadri Correspondence holds;
2. there should exist subclasses of geometric representations and geometric parabolic bundles, which we call geo-ari representations and geo-ari bundles, respectively, such that (i) these classes form naturally two abelian categories, (ii) an analog of Harder-Narasimhan Correspondence holds; and (iii) an analog of $KR^2$-trick works. Thus in particular, by (1) and (2), we obtain two equivalent (generalized) Tannakian categories together with natural fiber functors;
3. The (generalized) Tannakian categories contain systems of the so-called finitely completed Tannakian subcategories, so that via an analog of Tannaka Duality and van Kampen completeness theorem, we obtain the so-called fundamental theorem of non-abelian CFT such as the existence theorem, the conductor theorem and the reciprocity law.

To end this brief discussion on Part (A), we would like to point out that to realize the above mentioned 123 for our non-abelian CFT, standard theories on GIT, Tannakian category and representations of Galois groups are far from being enough. For examples, to achieve (1), we require (i) a Geometric Invariant Theory over integral bases in the spirit of Arakelov; (ii) a deformation theory for geometric representations of Galois
groups; and (iii) a suitable completeness for representation and stability along the line of Fountaine and Langton, respectively; and to achieve (3), we require a theory of Tannakian category over integral bases.

Our next main scheme is devoted to non-abelian zeta functions. As stated at the very beginning, Part (B) is a combination of our partial understanding of our new non-abelian zeta functions and what we call geo-ari cohomology. This part to a large extent is practical rather than theoretical, due to the fact that not only all studies here are based on practical constructions, but we have not yet understood the mathematics involved theoretically.

Unlike for the classical Weil zeta functions, instead of working on general algebraic varieties and counting their rational points (over finite fields) in a very primitive way, for our non-abelian zeta functions, we concentrate our attentions to moduli spaces of semi-stable bundles and count their rational points from moduli point of view, in a similar way as what Shimura does for Shimura varieties.

To be more precise, consider function fields over finite fields first. Then, for each fixed natural number \( r \), we, by using a work of Mumford-Seshadri, obtain the associated moduli spaces of rank \( r \) semi-stable bundles. In particular, with the so-called Harder-Narasimhan correspondence, which claims that the rationalities of bundles and moduli points coincide, we could then introduce a new type of zeta functions by considering rational points of the moduli space as moduli points associated with rational semi-stable bundles.

This approach, while different from that of Weil, is indeed a natural generalization of that of Artin: When \( r = 1 \), our construction recovers the classical Artin zeta functions. Moreover, just like classical abelian zeta functions, our non-abelian version satisfies rationality and a standard type of function equation as well. Since we even can give uniform bounds for the coefficients of these (local) zeta functions, so via an Euler product, we further introduce a more global non-abelian zeta functions for curves defined over number fields. Needless to say, when \( r = 1 \), these global zeta functions are nothing but the classical Hasse-Weil zeta functions for curves. So non-abelian arithmetic aspect of curves is supposed to be reflected by these new zeta functions.

Well, while this latest general statement should finally lead us to a mathematics wonderland, we have no yet found our theoretical feet. For this purpose, I then turn my attention to some concrete examples. This directs us to the study of what I call the refined Brill-Noether locus and their intersections: Beyond the classical consideration, the refined Brill-Noether locus measures how automorphisms of the associated bundles change too. As a direct consequence, we obtain a concrete reciprocity law for elliptic curves in ranks 2 and 3.

It now becomes quite apparent that key points for our construction of non-abelian zeta functions are the follows: (i) moduli spaces of semi-stable bundles admit naturally algebraic variety structures; (ii) there exists a well-established cohomology theory, in which the so-called Serre Duality and Riemann-Roch hold. So to construct non-abelian zeta functions of number fields, we should carry out some basic researches since both (i) and (ii) above seem to be virgin lands in number theory.

However, we cannot offer the reader a very satisfied GIT and a completed cohomology theory over number fields now. Fortunately, in this article, we manage successfully to obtain some practical items which are sufficient to the construction of our new non-abelian zeta functions for number fields. To say the truth, the outcome turns out to be equally nice: Not only the non-abelian zeta functions for number fields could be defined as a natural generalization of the classical Dedekind zeta functions, these new zeta functions are as canonical as they should be – they satisfy the functional equation, and the residues of them at simple poles are nothing but the volumes of what I call the Tamagawa measures of the associated moduli spaces of semi-stable bundles (over number fields). In particular, when rank is one, our work essentially recovers Iwasawa’s ICM talk at MIT about Dedekind zeta functions.

By saying this, I have no intention to claim that we are satisfied with what we have achieved. Far from being it, we have little understanding of these new zeta functions. For examples,

1. we have no idea now on how the non-abelian reciprocity law, which, by (A), are supposed to hold naturally, could be read from our non-abelian zeta functions;

2. generally speaking, we are less sure about the meaning of special values of our non-abelian zeta functions – We meet essential difficulties when trying to explain our non-abelian zeta functions in terms of

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3
the existing motivic language.

(Recall that all classical abelian zeta functions are supposed to be motivic in the sense of Grothendieck, Shimura, Deligne and Langlands, and hence that the associated special values of classical zeta functions are supposed to have motivic interpretations as conjectured by Beilinson, and Bloch-Kato, based on the fundamental works of Euler, Riemann, Borel, Quillen and Tate, among others.)

As to (i) and (ii) above for number fields, what is accomplished here is the introduction of intersection stability, a construction of the corresponding moduli spaces, and a practical formation of one dimensional geo-ari cohomology for which the duality and Riemann-Roch hold. In fact, the intersection stability may be dated back earlier from the works of Stuhler and Grayson, despite the fact that we work out this independently.

Note that also for the construction of non-abelian zeta functions for number fields, main properties for moduli spaces we need are the compactness and the existence of natural measures. So, based on Arakelov intersection theory and Chevalley-Weil’s adeles, we may easily generalize (i) to number fields. In comparison, (ii), the key to the convergence, the functional equation, remains very challenging. However, in this article, based on the earlier works in particular, that of Tate [Tate] and Schoof-van der Geer, (see also Lang [Lang], Arakelov, Szpiro, Neukirch), we are able to offer a practical definition of one dimensional geo-ari cohomology in terms of Chevalley idelic language via Fourier analysis. In our definition of geo-ari $h^i$’s, we make a clear distinction between algebraic and arithmetic aspects – algebraically, cohomology groups are finite generated abelian groups, while arithmetically, geo-ari cohomology is a finite definite quantities measuring geo-arithmetical complexities by counting all and hence infinitely many elements in the about algebraic cohomology groups (with the help of Fourier analysis). Thus, it would be extremely interesting to compare our sheaf theoretic approach with Deninger’s Betti cohomology approach in which infinite dimensional spaces are used with the help of the regularized determinant formalism.

Our practical cohomology works only in dimension one. However, based on linear compacity of Chevalley, as given in Iwasawa’s Princeton lectures notes, and Parshin’s approach to duality and residue in dimension two, we at the end of Part (C) provide with the reader an program for what I call a half-theoretical geo-ari cohomology in lower dimensions, which should play a key role in establishing the Hodge Index Theorem for our geo-ari intersection introduced in (C).

Part (C) of the program is designed to give a geometric justification of the formal summation \( \sum_{s \xi(s) = 0} \rho^s \) for \( \rho \in \mathbb{R} \). For this purpose, we propose a new two dimensional geo-ari intersection theory. This geo-ari intersection turns out to be very interesting, since the Riemann Hypothesis may be naturally studied within the framework of this model along with the line of Weil’s original proof of the so-called Hasse-Weil Theorem, or better, the Riemann Hypothesis for Artin zeta functions. Due to the facts that the Cramer formula is behind the above summation for zeros of Riemann zeta and that the Explicit Formula of Weil is behind the above proposed geo-ari intersection, our approach to the Riemann Hypothesis is in appearance different from but in essence related to that of Deninger and Quillen.

More precisely, to introduce our model on a two dimensional geo-ari intersection, first we assume that there exist two dimensional mathematics sites, which I call geo-ari surfaces; Then, as in geometry, we assume that on these geo-ari surfaces, there are naturally divisors, which I call micro divisors; (Unlike in the geometric case, these micro divisors are assumed to be parametrized by \( \mathbb{R} \).) With this, motivated by the standard properties of intersections, the Riemann-Roch in dimension one, the adjunction formula, the global functional equation, and the Weil explicit formula, we introduce six simple axioms for the intersections, consisting of one for (permutation) symmetry, one for mirror symmetry, two for fixed points, one for micro explicit formula, and one for normalization.

The advantage of having this mathematics model on geo-ari intersection is that, as in geometry, then the Riemann Hypothesis may be deduced from an analog of the Hodge index theorem. Thus, motivated by what happens in geometry, we should also search for a good geo-ari cohomology in dimension two. As stated above, such a program is proposed at the end of (C).
Contents

A. Representation of Galois Group, Stability and Tannakian Category

A.1. Summary

A.2. Non-Abelian CFT for Function Fields over C

A.2.1. Weil-Narasimhan-Seshadri Correspondence
  A.2.1.1. Unitary Representations of Fundamental Groups
  A.2.1.2. Semi-Stable Parabolic Bundles
  A.2.1.3. Weil-Narasimhan-Seshadri Correspondence: A Micro Reciprocity Law

A.2.2. Rationality: Geo-Ari Representations and Geo-Ari Bundles
  A.2.2.1. Branched Coverings of Riemann Surfaces
  A.2.2.2. Geo-Ari Representations and Geo-Ari Bundles

A.2.3. $KR^2$-Trick and Completed Tannakian Categories
  A.2.3.1. Completed Tannakian Category and van Kampen Completeness Theorem
  A.2.3.2. $KR^2$-Trick

A.2.4. Non-Abelian CFT for Function Fields over Complex Numbers
  A.2.4.1. Micro Reciprocity Law, Tannakian Duality and the Reciprocity Map
  A.2.4.2. Non-Abelian CFT

A.2.5. Classical (abelian) CFT: An Example of Kwada-Tata and Kawada
  A.2.5.1. Class Formation
  A.2.5.2. The Work of Kawada and Tate
  A.2.5.3. Abelian CFT for Riemann Surfaces In Terms of Geo-Ari Bundles

A.3. Towards Non-Abelian CFT for Global Fields

A.3.1. Weil-Narasimhan-Seshadri Type Correspondence
  A.3.1.1. Geometric Representations
  A.3.1.2. Semi-Stability in terms of Intersection
  A.3.1.3. Weil-Narasimhan-Seshadri Type Correspondence

A.3.2. Harder-Narasimhan Correspondence

A.3.3. $KR^2$-Trick

A.3.4. Tannakian Category Theory over Arbitrary Bases

A.3.5. Non-Abelian CFT for Global Fields
B. Moduli Spaces, Riemann-Roch, and New Non-Abelian Zeta functions

B.1. New Local and Global Non-Abelian Zeta Functions for Curves

B.1.1. Local Non-Abelian Zeta Functions

B.1.1.1. Artin Zeta Functions for Curves

B.1.1.2. Too Different Generalizations: Weil’s Zeta Functions and A New Approach

B.1.1.3. Moduli Spaces of Semi-Stable Bundles

B.1.1.4. New Local Non-Abelian Zeta Functions

B.1.1.5. Basic Properties for Non-Abelian Zeta Functions

B.1.2. Global Non-Abelian Zeta Functions for Curves

B.1.2.1. Preparations

B.1.2.2. Global Non-Abelian Zeta Functions for Curves

B.1.2.3. Working Hypothesis

B.1.3. Refined Brill-Noether Locus for Elliptic Curves: Towards A Reciprocity Law

B.1.3.1 Results of Atiyah

B.1.3.2. Refined Brill-Noether Locus

B.1.3.3. Towards A Reciprocity Law: Measuring Refined Brill-Noether Locus Arithmetically

B.1.3.4. Examples In Ranks Two and Three: A Precise Reciprocity Law

B.1.3.5. Why Use only Semi-Stable Bundles

Appendix to B.1: Weierstrass Groups

1. Weierstrass Divisors

2. K-Groups

3. Generalized Jacobians

4. Galois Cohomology Groups

5. Deligne-Beilinson Cohomology

B.2. New Non-Abelian Zeta Functions for Number Fields

B.2.1. Iwasawa’s ICM Talk on Dedekind Zeta Functions

B.2.2. Intersection Stability

B.2.2.1. Classification of Unimodular Lattices: A Global Approach

B.2.2.2. Semi-Stable Bundles over Number Fields

B.2.2.3. Adelic Moduli and Its Associated Tamagawa Measure
B.2.3. Geo-Ari Duality and Riemann-Roch: A Practical Geo-Ari Cohomology following Tate

B.2.3.1. An Example
B.2.3.2. Canonical Divisors and Space of Different Forms
B.2.3.3. Algebraic Cohomology for Matrix Divisors
B.2.3.4. Geo-Arit Cohomology and Its Associated Riemann-Roch

B.2.4. Non-Abelian Zeta Function For Number Fields

B.2.4.1. The Construction
B.2.4.2. Basic Properties

C. Explicit Formula, Functional Equation and Geo-Ari Intersection

C.1. The Riemann Hypothesis for Curves

C.1.1. Weil’s Explicit Formula: the Reciprocity Law
C.1.2. Geometric Version of Explicit Formula
C.1.3. Riemann Hypothesis for Function Fields

C.2. Geo-Ari Intersection in Dimension Two: A Mathematics Model

C.2.1. Motivation from Cramér’s Formula
C.2.2. Micro Divisors
C.2.3. Global Divisors and Their Intersections: Geometric Reciprocity Law
C.2.4. The Riemann Hypothesis
C.2.5. Not so serious Convergence Problem
C.2.6. Weil’s Explicit Formula and Two Dimensional Geometric Arithmetic Intersections

C.3. Towards A Geo-Ari Cohomology in Lower Dimensions

C.3.1. Classical Approach in Dimension One
C.3.2. Chevalley’s Linear Compacity
C.3.3. Adelic Approach in Geometric Dimension Two
A. Representation of Galois Group, Stability and Tannakian Category

A.1. Summary

Our Program for a non-abelian class CFT may be roughly summarized in the following table.

| Galois Aspect       | Non-Abelian Class Field Theory: A Program | Bundle Aspect |
|---------------------|------------------------------------------|--------------|
| Geometric Reps     | Narasimhan-Seshadri Correspondence       | S. Stable Parabolic Bundles |
| ↓ Rationality      | Vanishing of Brauer Groups               | Rationality↓ |
| Geo-Ari Reps       | Harder-Narasimhan Correspondence         | Geo-Ari Bundles |
| ↓ ⊗                | K R² Trick                               | Clone Tannakian Category |
| Tannakian Category | Tannaka Duality, van Kampen Cpt Th       | Aut⊗         |
| Galois Group       | Reciprocity Map                          |              |
| Finite Quotient    | Existence Theorem, Reciprocity Law       | Finitely Completed Module |

A.2. Non-Abelian CFT for Function Fields over C

A.2.1. Unitary Representations of Fundamental Groups

Let $M^0$ be a punctured Riemann surface of signature $(g, N)$ with $M$ the smooth compactification. Then, $M^0 = M \setminus \{P_1, \ldots, P_N\}$, and $M$ is of genus $g$ with $P_1, \ldots, P_N$ pairwise distinct points on $M$. Suppose that $2g - 2 + N > 0$. From the uniformization theorem, $M^0$ can be represented as a quotient $\Gamma \backslash \mathcal{H}$ of the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \Im z > 0\}$ modulo an action of a torsion-free Fuchsian group $\Gamma \in PSL_2(\mathbb{R})$, generated by $2g$ hyperbolic transformations $A_1, B_1, \ldots, A_g, B_g$ and $N$ parabolic transformations $S_1, \ldots, S_N$ satisfying a single relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} S_1 \cdots S_N = 1.$$ 

Denote the fixed points, the so-called cusps, of the parabolic elements $S_1, \ldots, S_N$ by $z_1, \ldots, z_N$ respectively. Then, images of the cusps $z_1, \ldots, z_N \in \mathbb{R} \cup \{\infty\}$ under the projection $p : \mathcal{H}^* := \mathcal{H} \cup \mathbb{R} \cup \{\infty\} \to \Gamma \backslash \mathcal{H}^* = M$ result the punctures $P_1, \ldots, P_N \in M$. For each $i = 1, \ldots, N$, denote by $\Gamma_i = \Gamma_{z_i}$ the stabilizer of $z_i$ in $\Gamma$. Then $\Gamma_i$ is a cyclic subgroup in $\Gamma$ generated by $S_i$. Moreover, for an element $\sigma_i \in PSL_2(\mathbb{R})$ such that $\sigma_i \infty = z_i$, we have $\sigma_i^{-1} S_i \sigma_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, and hence $\langle \sigma_i^{-1} S_i \sigma_i \rangle = \Gamma_{z_i}$. (For simplicity, from now on, when only a local discussion is involved, we always assume that $z_i = \infty$.)

For a representation $\rho : \pi_1(M^0) \simeq \Gamma \to GL(n, \mathbb{C})$ of $\Gamma$ into a complex vector space $V$, the vector bundle $V := \mathcal{H} \times V$ on $\mathcal{H}$ admits a natural $\Gamma$-vector bundle structure via $\gamma(z, v) = (\gamma(z), \rho(\gamma)v)$ for $\gamma \in \Gamma, z \in \mathcal{H}$ and $v \in V$. The quotient of $\mathcal{H} \times V$ modulo the action of $\Gamma$ is then a vector bundle of rank $n$ over $M^0$. Moreover, since the same representation $\rho$ defines also a $\Gamma$-vector bundle structure on $\mathcal{H}^* \times V$, we obtain a vector bundle $V_\rho$ on $M = \Gamma \backslash \mathcal{H}^*$ as well.

Next assume that $\rho$ is unitary. Then with respect to a suitable basis of $V$,

$$\rho(S_i) = \text{diag}(\exp(2\pi i \alpha_{i1}), \ldots, \exp(2\pi i \alpha_{in}))$$

where $\alpha_{ij} \in [0, 1)$ for all $i = 1, \ldots, N, j = 1, \ldots, n$. Hence, $V_\rho$, or better, the associated sheaf $p_\rho^*(V)$ of sections may be interpreted as follows: On $M^0$, it corresponds to the $\Gamma$-invariant sections of $V$, while near parabolic punctures $P_j = P_{z_j} \in M$, over a neighbourhood $U$ of $P_j$ of the form $\mathcal{H}_\delta \backslash \Gamma_{\infty}$ where $\mathcal{H}_\delta := \{z = x + iy : y > \delta > 0\}$, the sections are all bounded $\Gamma_{\infty}$-invariant sections of $V$ on $\mathcal{H}_\delta$. Thus, as an $\mathcal{O}_{M,P_j}$-module, a basis of $p_\rho^*(V)$ at $P_j$ is given by the $\Gamma_{\infty}$ sections $\theta_j : z \mapsto \exp(2\pi i \alpha_{j1} z) e_j$ where $\{e_1, \ldots, e_n\}$ is a basis of $V$ such that $S_i(e_j) = \exp(2\pi i \alpha_{ij}) e_j$. 

8
As a direct consequence, in addition to the associated bundles $V_\rho$ on $M$, there exist as well the following structures on the fibers of $V_\rho$ at punctures $P_1, \ldots, P_N$: Over $P = P_i$, we obtain real numbers
\[
\alpha_{i1} = \alpha_{i2} = \ldots = \alpha_{ik_i} < \alpha_{i,k_i+1} = \alpha_{i,k_i+2} = \ldots = \alpha_{i,k_r} < \ldots = \alpha_{i,k_r},
\]
and a decreasing flag of $V_\rho|_P$ defined by
(i) $F_1(V_\rho|_P) := V_\rho|_P$;
(ii) $F_2(V_\rho|_P)$, the subspace spanned by $\theta_{k_1+1}, \ldots, \theta_n$;
(iii) $F_3(V_\rho|_P)$ the subspace spanned by $\theta_{k_2+1}, \ldots, \theta_n$, etc.
Clearly $k_1 = \dim F_1(V_\rho|_P) - \dim F_2(V_\rho|_P), \ldots, k_r = \dim F_r(V_\rho|_P)$, and these additional structures are indeed determined by $\alpha'_{ij} := \alpha_{ik_j}, j = 1, \ldots, r_i$, and $k_j$’s.

**Proposition.** (Seshadri) *With the same notation as above, we have $\deg(V_\rho) = -\sum_{i,j=1}^{N,r_i} k_{ij}\alpha'_{ij}$.*

### A.2.1.2. Semi-Stable Parabolic Bundles

Following Seshadri, by definition, a parabolic structure on a vector bundle $E$ over a compact Riemann surface is given by the following data:

1. A finite collection of points $P_1, \ldots, P_N \in M$; and for each $P = P_i$,
2. A flag $E_P = F_1E_P \supset F_2E_P \supset \ldots \supset F_rE_P$; and
3. A collection of parabolic weights $\alpha_1, \ldots, \alpha_r$ attached to $F_1E_P, \ldots, F_rE_P$ such that $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_r < 1$.

Often $k_1 = \dim F_1E_P - \dim F_2E_P, \ldots, k_r = \dim F_rE_P$ are called the multiplicities of $\alpha_1, \ldots, \alpha_r$; and a bundle $E$ together with a parabolic structure
\[
(P = P_i; E_P = F_1E_P \supset F_2E_P \supset \ldots \supset F_rE_P; \alpha_1 = \alpha_{i1}, \ldots, \alpha_r = \alpha_{ir_i})_{i=1}^{N}
\]
is called a parabolic bundle and is written as
\[
\Sigma(E) := \Sigma := \left(E; (P = P_i; E_P = F_1E_P \supset F_2E_P \supset \ldots \supset F_rE_P; \alpha_1 = \alpha_{i1}, \ldots, \alpha_r = \alpha_{ir_i})_{i=1}^{N}\right).
\]

Trivially, if $W$ is a subbundle of $E$, then $\Sigma$ induces a natural parabolic structure $\Sigma(W)$ on $W$.

For parabolic bundles, its associated parabolic degree is defined to be
\[
\text{para deg}(\Sigma) := \deg E + \sum_{i=1}^{N} \left(\sum_{j=1}^{r_i} k_{ij}\alpha'_{ij}\right).
\]

So, in particular, if $\Sigma$ is induced from a unitary representation of fundamental group of a Riemann surface, its associated para degree is zero. By definition, a parabolic bundle $\Sigma$ is called (Mumford-Seshadri) semi-stable (resp. stable) if for any subbundle $W$ of $E$,
\[
\frac{\text{para deg}(\Sigma(W))}{\text{rank}(W)} \leq (\text{resp. <}) \frac{\text{para deg}(\Sigma(E))}{\text{rank}(E)}.
\]

**Proposition.** (Seshadri) *With the same notation as above, if $\Sigma$ is induced from a unitary representation of the corresponding fundamental group, then $\Sigma$ is semi-stable. Moreover, if the representation is irreducible, then $\Sigma$ is in fact stable.*

### A.2.1.3. Weil-Narasimhan-Seshadri Correspondence: A Micro Reciprocity Law

The real surprising result is the inverse of Proposition 2.1.2. The starting point for all this is the following classical result on line bundles: Over a compact Riemann surface, a line bundle is of degree zero if and only if it is flat, i.e., it is induced from a representation of fundamental group. It is Weil who first generalized this to vector bundles in his fundamental paper on Generalisation des fonctions abéliennes dated in 1938: Over a
compact Riemann surface, a vector bundle is of degree zero if and only if it is induced from a representation of fundamental group. (The reader may find a modern proof in Gunning’s Princeton lecture notes.)

It is said that Weil’s primitive motivation is to develop a non-abelian CFT for Riemann surfaces. Granting this, clearly, the next step is to study what happens for general Riemann surfaces, which need not to be compact. This then leads to Weil and Toyama’s theory on matrix divisors.

While all this seems to be essentially in the right direction, still many crucial points are missing in these earlier studies.

Recall that the reciprocity law in CFT is essentially the one for finite field extensions and that for a finite group, in any equivalence class of (finite dimensional complex) representations there always exists a unitary one. Thus naturally we should strengthen Weil’s theorem from any representation to that of unitary representation. For doing so, the first difficulty appears in algebraic side. That is, how to give a corresponding algebraic condition? This is solved by Mumford with his famous intersection stability. In fact, not only Mumford introduces the intersection stability, he also studies the associated deformation theory via his fundamental work on GIT stability.

On the other hand, for Riemann surfaces, fundamental groups may be described very precisely, and hence deformations of the associated unitary representations may be quantitatively studied. All this, together with certain completeness for both unitary representations and Mumford’s intersection stability, the so-called Langton’s Principal, then leads Narasimhan and Seshadri to prove that over a compact Riemann surface, Mumford semi-stable vector bundles of degree zero are naturally associated with unitary representations of the fundamental group of the surface. Later on, Seshadri first generalizes this result to $\pi$-bundles, and then to parabolic bundles.

Theorem. (Seshadri) There is a natural one-to-one correspondence between isomorphism classes of unitray representations of fundamental groups of punctured Riemann surfaces and equivalence classes of semi-stable parabolic bundles of parabolic degree zero.

We would like call this result the Weil-Narasimhan-Seshadri correspondence. Due to the fact that it reveals an intrinsic relation between fundamental groups and vector bundles, often we call it a micro reciprocity law as well. (Over higher dimensional compact Kähler manifold, similar correspondence is named the Kobayashi-Hitchin correspondence, which is established by Ulenberk-Yau. See also Donaldson for projective manifolds.)

Note also that, as stated above, a proof is based on
(a) Geometric Invariant Theory;
(b) Deformation of Representations; and
(c) completeness of semi-stable parabolic bundles –the Langton principal.

Consequently, all this is supposed to play a crucial role in our program for non-abelian CFT, the details of which will be discussed later.

While the Narasimhan-Seshadri correspondence is a kind of Reciprocity Law, it is only a micro one. Thus to find the genuine one, we need to study it globally. Thus, the following result for finite groups naturally enters into the picture: Any finite group is determined by its characters and vice versa. Hence, if we have a similar result for more general groups, we could establish our non-abelian CFT for Riemann surfaces. So naturally, we are led to the so-called Tannakian category. But before that we still need to make sure that coverings and parabolic bundles are under control. It is for this purpose, we introduce the Rationality discussion in our program.

A.2.2. Rationality: Geo-Ari Representations and Geo-Ari Bundles

A.2.2.1. Branched Coverings of Riemann Surfaces

The advantage in developing a non-abelian CFT for Riemann surfaces is that all the time we have concrete geometric models ready to use. As there is no additional cost and also for later discussion on non-abelian CFT for higher dimensional function fields over complex numbers, we next recall some basics for branched coverings of complex manifolds.
Let $M$ be an $n$-dimensional connected complex manifold. A branched covering of $M$ is by definition an $n$-dimensional irreducible normal complex space $X$ together with a surjective holomorphic mapping $\pi : X \to M$ such that

1. every fiber of $\pi$ is discrete in $X$;
2. $R_\pi := \{ q \in X : \pi^* : O_{\pi(q), M} \to O_q, X \text{ is not isomorphic} \}$ and $B_\pi := \pi(R_\pi)$ are hypersurfaces of $X$ and $M$ respectively. As usual, we call $R_\pi$ and $B_\pi$ ramification locus and branch locus respectively;
3. $\pi : X \setminus \pi^{-1}(B_\pi) \to M \setminus B_\pi$ is an unramified covering; and
4. for any $p \in M$, there is a connected open neighbourhood $W$ of $p$ in $M$ such that for every connected component $U$ of $\pi^{-1}(W)$, (i) $\pi^{-1}(p) \cap U = \{ q \}$ is one point and (ii) $\pi_U := \pi|_U : U \to W$ is surjective and proper. Thus in particular, the induced map $\pi : X \setminus \pi^{-1}(B_\pi) \to M \setminus B_\pi$ is a topological covering and $\pi_U$ is finite.

For example, if $\pi : X \to M$ is a surjective proper finite holomorphic map, $\pi$ is a finite branched covering of $M$.

Two branch coverings $\pi : X \to M$ and $\pi' : X' \to M$ are said to be equivalent if there is a biholomorphic map $\phi : X \to X'$ such that $\pi = \pi' \circ \phi$. In this case we write $\pi \geq \pi'$ or $\pi' \leq \pi$. The set of all automorphisms of $\pi$ forms a group $G_\pi$ naturally. One checks easily that if we denote by $\pi_1$ the restriction of $\pi$ to $X \setminus \pi^{-1}(B_\pi)$, then $G_\pi$ is canonically isomorphic to $G_{\pi_1}$. By definition, $\pi$ is called a Galois covering if $G_\pi$ acts transitively on every fibre of $\pi$; and $\pi$ is called abelian if $\pi$ is Galois and $G_\pi$ is abelian.

**Theorem.** If $\pi : X \to M$ is a Galois covering, then

1. for every subgroup $H$ of $G_\pi$, there is a branched covering $\pi_H : X/H \to M$ such that $\pi_H \leq \pi$;
2. the correspondence $H \to \pi' = \pi_H$ gives a bijection between subgroups $H$ and equivalence classes of branched coverings $\pi'$ of $M$ such that $\pi' \leq \pi$; and
3. $H$ is normal if and only if $\pi_H$ is a Galois covering, for which $G_{\pi_H}$ is isomorphic to $G_\pi/H$.

Note that for any branched covering $\pi : X \to M$, if $p, q$ are points of $B_\pi$ and $\pi^{-1}(B_\pi)$ respectively such that

1. $B_\pi$ is normally crossing at $p, q \in \pi^{-1}(p)$;
2. $X$ is smooth at $q$; and
3. $\pi^{-1}(B_\pi)$ is normally crossing at $q$,

then, for a sufficiently small connected open neighbourhood of $p$ with a coordinate system $(w_1, \ldots, w_n)$ such that $p = 0$ and $B_\pi \cap W = \{(w_1, \ldots, w_n) : w_k \ldots w_n = 0\}$ for some $k$, there is a coordinate system $(z_1, \ldots, z_n)$ in the connected component $U$ of $\pi^{-1}(W)$ with $q \in U$ such that $q = 0, \pi^{-1}(B_\pi) \cap U = \{(z_1, \ldots, z_n) \in U : z_k \ldots z_n = 0\}$ and $\pi_U(z_1, \ldots, z_n) = (z_1, z_{k-1}, z_{k}^2, \ldots, z_{n})$. Often we call $e_j, j = k, \ldots, n$ the ramification index of the irreducible $C_j$ of $\pi^{-1}(B_\pi)$ such that $C_j \cup U = \{(z_1, \ldots, z_n) : z_j = 0\}$.

Suppose now that $B_\pi = D_j \cup \ldots \cup D_N$ is the irreducible decomposition of $B_\pi$. Let $D = \sum_{i=1}^N e_i D_i$ be an effective divisor on $M$. By definition, the branched covering $\pi : X \to M$ is called branched at $D$ (resp. at most at $D$) if for every irreducible component $C$ of $\pi^{-1}(B)$ with $\pi(C) = D_j$, the ramification index of $\pi$ at $C$ is $e_j$ (resp. divides $e_j$). In particular, a branched at $D$ Galois covering $\pi : X \to M$ is called maximal if for any branched covering $\pi' : X' \to M$ which branches at most at $D$, $\pi' \leq \pi$.

While, in general, it is complicated to describe maximal branched covering for higher dimensional complex manifolds (see however [Kato] and [Namba]), for Riemann surfaces, it may be simply stated as follows.

A result of Bundgaard-Nielsen-Fox says that there is no finite Galois covering $\pi : X \to M$ branched at $D = \sum_{j=1}^N e_j P_j$ if and only if either (i) $g = 0$ and $N = 1$ or (ii) $g = 0, N = 2$ and $e_1 \neq e_2$. Here we write $D_j$ as $P_j, j = 1, \ldots, N$. Thus from now on, we always assume that we are not in these exceptions. Also we assume that $e_j \geq 2$. (Otherwise, we may omit it from the beginning as there is no ramification for the corresponding points $P_j$ then.)

Let $J(D)$ be the smallest normal subgroup of $\pi_1(M^0)$ which contains $S_{1}^{e_1}, \ldots, S_{N}^{e_N}$. Then (as used in the proof of the above result of Bundgaard-Nielsen-Fox,) the normal subgroup $J(D)$ satisfies the following condition:

**Condition (*)**: If $S_{j}^{d} \in J(D)$, then $d|e_j$ for $j = 1, \ldots, N$. 11
As a direct consequence, we have the following

**Theorem.** (Bundgaard-Nielsen-Fox) With the same notation as above,
(1) there is a maximal covering $\tilde{\pi} : \tilde{M}(D) \to M$ which branches at $D$. In particular, $\tilde{M}(D)$ is simply connected;
(2) there is a canonical one-to-one correspondence between subgroups (resp. normal subgroups) $H$ of the quotient group $\pi_1(M^0)/J(D)$ and equivalence classes of branched coverings (resp. Galois coverings) $\pi : X \to M$ which branch at most at $D$;
(3) $\pi : X \to M$ branches at $D$ if and only if the following condition for $K$ is satisfied (here $H=K/J(D)$): if $S_j^0 \in K$, then $d_j e_j, i = 1, \ldots, N$.

### A.2.2.2. Geo-Ari Representations and Geo-Ari Bundles

Motivated by the above discussion, we now introduce what we call geo-ari representations of fundamental groups.

As before, let $M$ be a compact Riemann surface of genus $g$ with marked points $P_1, \ldots, P_N$. Set $M^0 = M \setminus \{P_1, \ldots, P_N\}$. Then, $\pi_1(M^0)$ is generated by hyperbolic elements $A_1, B_1, \ldots, A_g, B_g$ and parabolic elements $S_1, \ldots, S_N$ such that $[A_1, B_1][A_2, B_2][S_1] \cdots [S_N] = 1$. Fix an effective divisor $D = \sum_{i=1}^N e_i P_i$ on $M$. By definition, a geometric arithmetic representation, a geo-ari representation for short, of the fundamental group $\pi_1(M^0)$ along with $D$ is a unitary representation $\rho : \pi_1(M^0) \to U(l)$ such that

(i) $\rho(S_i) = \text{diag}\left(\exp(2\pi i \beta_{i1}), \ldots, \exp(2\pi i \beta_{il})\right)$ for all $i = 1, \ldots, N$; and
(ii) there exist integers $\gamma_{ij} > 0$ and $\delta_{ij} \geq 0$ such that
(a) $\gamma_{ij} e_j$;
(b) $(\gamma_{ij}, \delta_{ij}) = 1$; and
(c) $\delta_{ij} = \frac{\delta_{ij}}{\gamma_{ij}}$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, l$.

Parallelly, we define a geo-ari bundle on $M$ along $D$ to be a parabolic degree zero semi-stable parabolic bundles

$$\Sigma = \left( E; (P = P_i; E_P = F_1 E_P \supset F_2 E_P \cdots \supset F_r E_P; \alpha_1 = \alpha_{i1}, \ldots, \alpha_r = \alpha_{ir}) \right)$$

such that the following conditions are satisfied: there exist integers $\gamma_{ij} > 0$ and $\delta_{ij} \geq 0$ such that

(a) $\gamma_{ij} e_j$;
(b) $(\gamma_{ij}, \delta_{ij}) = 1$; and
(c) $\alpha_{ij} = \frac{\delta_{ij}}{\gamma_{ij}}$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, r_i$.

Obviously, by applying Theorem 2.1.3, we obtain the following

**Theorem’.** With the same notation as above, there exists a natural one to one correspondence between isomorphic classes of geo-ari representations of $\pi_1(M^0)$ along with $D$ and (Seshadri) equivalence classes of geo-ari bundles along $D$ over $M$.

**Remark.** We call this result the Harder-Narasimhan Correspondence, despite the fact that in the situation now, i.e., over complex numbers, the Harder-Narasimhan correspondence is simply the direct consequence of Narasimhan-Seshadri correspondence. Later we will see that when the constant field is finite, such a correspondence, first established by Harder-Narasimhan, is based on the vanishing of the related Brauer groups.

Clearly, if $D' = \sum_{j=1}^r e'_j P_j$ is an effective divisor such that $e_j e'_j$ for all $j = 1, \ldots, N$, then geo-ari representations of $\pi_1(M^0)$ (resp. geo-ari bundles) along with $D$ are also geo-ari representations of $\pi_1(M^0)$ (resp. geo-ari bundles) along with $D'$. (Usually, we write $D|D'$.) Thus if we denote $U(M; D)$ the category of equivalences classes of geo-ari representations of $\pi_1(M^0)$ along with $D$, (see, e.g., 2.3.1 below for a brief discussion on categories,) and $M(M; D)$ the category of geo-ari bundles along $D$ over $M$, then by using a result of Mehta-Seshadri, see e.g., Prop. 1.15 of [MS], we have the following

**Proposition.** With the same notation as above, $U(M; D)$ and $M(M; D)$ are equivalent abelian categories. Moreover if $D|D'$, then $U(M; D)$ and $M(M; D)$ are abelian subcategories of $U(M; D')$ and $M(M; D')$. 

12
such that \( \text{dom}(g) \text{x} \) is a morphism 1

(3) There is a unique identity object 1 such that

S

following conditions are satisfied:

(1) for any object

abelian category if called completed if

\( \text{dom}(g \circ f) = \text{dom}(f) \) and \( \text{cod}(g \circ f) = \text{cod}(g) \), \( f \circ (g \circ h) = (f \circ g) \circ h \), and for every object \( x \) there is a morphism \( 1_x : x \to y \) such that \( f \circ 1_x = 1_x \circ f = f \). Thus we may form the set \( \text{Hom}(x, y) \) by taking the collection of all \( f : x \to y \). By definition, if moreover the hom sets are all abelian groups such that compositions are bilinear, we call it a preadditive category.

Among two categories, a functor \( T : A \to B \) is defined to be a pair of maps \( \text{Obj}(A) \to \text{Obj}(B) \) and \( \text{Arr}(A) \to \text{Arr}(B) \) such that if \( f : c \to c' \) is an morphism in \( A \), then \( T(f) \) is a morphism \( T(f) : T(c) \to T(c') \) in \( B \), and that \( T(1_c) = 1_{T(c)} \), \( T(g \circ f) = T(g) \circ T(f) \); and a natural transformation \( \tau \) between two functors \( S, T : A \to B \) is defined to be a collection of morphisms \( \tau_c : T(c) \to S(c) \) in \( B \) such that for any \( f : c \to c' \) in \( A \), \( \tau_c(T(f)) = S(f) \tau_{c'} \); if moreover all \( \tau_c \) have inverses, then the natural transformation is called a functorial isomorphism.

Among categories, abelian categories are of special importance. By definition, an abelian category is a preadditive category such that

(1) there is a unique object called zero object such that it is the initial as well as the final object of the category;
(2) direct product construction exists; and
(3) associated to any morphism are kernel and cokernel which are objects of the category as well; moreover, every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and
(4) every morphism can be factored into an epimorphism followed by a monomorphism.

To facilitate ensuing discussion, we introduce some new concepts in category theory. By definition, an object \( x \) in an abelian category is called decomposable if there exist objects, \( y, z \) different from zero, such that \( x = y \oplus z \); and \( x \) is called irreducible if it is not decomposable. Moreover, an abelian subcategory of an abelian category if called completed if

(1) for any object \( x \), there is a unique finite decomposition \( x = \oplus x_i \) with \( x_i \) irreducible, the irreducible components of \( x \);
(2) the subcategory contains all of its irreducible components of its objects.

**Proposition.** With the same notation as above, the categories \( U(M; D) \) and \( M(M; D) \) are completed abelian categories.

The reader may prove this proposition from the following facts:

(1) Among two stable parabolic bundles of the same parabolic degree, homomorphisms are either zero or an isomorphism;
(2) There exist Jordan-Hölder filtrations for parabolic semi-stable bundles;
(3) The associated Jordan-Hölder graded parabolic bundles in (2) is unique.

**Remark.** Motivated by this proposition, the reader may give a more abstract criterion to check when an abelian subcategory is completed.

Next, let us recall what a tensor category should be. By definition, a category is called a tensor category if there is an operation, called tensor product \( A \otimes B \) for any two objects \( A, B \) of the category, such that the following conditions are satisfied:

(1) There are natural isomorphisms \( S : A \otimes B \to B \otimes A \) and \( T : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \);
(2) \( S \) and \( T \) satisfy the so-called pentagon and hexagon axioms;
(3) There is a unique identity object 1 such that \( A \simeq A \otimes 1 \simeq 1 \otimes A \) for all object \( A \).

Clearly, for tensor categories, we may introduce the so-called tensor operation for any finite number of objects. Usually, there are many ways to do so, but the above conditions for tensor category implies that
these different ways are all the same. Moreover, if for any objects \( x, y \) of \( A \), \( \text{Hom}(x, y) \) is again an object in \( A \) satisfies the following conditions, we call \( A \) a rigid tensor category:

(4) there exists a morphism, the evaluation map, \( ev_{x,y} : \text{Hom}(x, y) \otimes x \to y \) such that for any object \( t \) and any morphism \( g : t \otimes x \to y \), there exists a unique morphism \( f : t \to \text{Hom}(x, y) \) such that the following commutative diagram commutes

\[
\begin{array}{ccc}
t \otimes x & \xrightarrow{f \otimes 1_x} & \text{Hom}(x, y) \otimes x \\
\downarrow g & & \downarrow ev_{x,y} \\
y & & y
\end{array}
\]

(5) There exists natural isomorphism

\[
\text{Hom}(x_1, y_1) \otimes \text{Hom}(x_2, y_2) \simeq \text{Hom}(x_1 \otimes x_2, y_1 \otimes y_2)
\]

which is compatible with (4);

(6) For any object \( x \), by (5), if we set \( x^\vee := \text{Hom}(x, 1) \), then there exists a natural isomorphism \( x \simeq (x^\vee)^\vee \).

Now we are ready to introduce Tannakian categories. By definition, a Tannakian category is a category which is both an abelian category and a rigid tensor category such that the tensor operation is bilinear. Clearly then in such categories, \( \text{Hom}(x, y) \simeq x \otimes y^\vee \). For example, the category \( \text{Vec}_k \) of finite dimensional vector spaces over a field \( k \) is a Tannakian category. By definition, a functor \( \omega : A \to \text{Vec}_k \) from a Tannakian category \( A \) to the category \( \text{Vec}_k \) is called a fiber functor, if \( \omega \) is an exact faithful tensor functor. (Recall that faithful means there is a natural injection \( \omega(\text{Hom}(A(x,y))) \hookrightarrow \text{Hom}_{\text{Vec}_k}(\omega(x),\omega(y)) \) which is induced from \( \omega \)).

Associated to a fiber functor \( \omega \) is naturally its automorphic group \( \text{Aut}^\otimes \omega \). In particular, then we have the following fundamental theorem of Tannakian category:

**Theorem.** (Tannaka, Grothendieck, Saavedra Rivano) *With the same notation as above, assume that \( (A, \omega : A \to \text{Vec}_k) \) consists of a Tannakian category and a fiber functor such that the field \( k \) is canonically isomorphic to \( 1^\vee_A \), then \( A \) is equivalent to the category of representations of the group \( \text{Aut}^\otimes \omega \).

A Proof of this theorem may be deduced from the following facts:

(1) Category of all representations forms naturally a Tannakian category together with a fiber functor;

(2) A knowledge of the representations of a group is equivalent to a knowledge of the group; and

(3) Any Tannakian category is in fact a clone of (1), via the so-called Tannaka Duality Principal.

Now we are ready to introduce our own Tannakian categories which are completed and equipped with natural fiber functors. There are two of them, i.e., the one for geo-ari representations and the one for geo-ari bundles. Unlike what we did before, for convinence, here we make no clear distinctions between them.

By Proposition 2.2.2, it is enough to introduce the fiber functors and show that geo-ari representations and geo-ari bundles are closed under the tensor operation. But all this is quite straightforward: By definition, tensors of unitary representations are again unitary, while for geo-ari bundles, the fiber functor may be defined by taking special fibers for the bundles at any point which is not a marked one. Note that morphisms between geo-ari bundles are either zero or isomorphisms, so the latest defined functor is faithful. Therefore, we obtain the following

**Key Proposition.** *With the same notation as above, \( U(M; D) \) and hence \( M(M; D) \) are completed Tannakian categories equipped with natural fiber functors to \( \text{Vec}_C \).

**Remark.** While it is enough for the purpose to develop a non-abelian CFT for Riemann surfaces to use the above standard theory of Tannakian categories, in general, such a theory for Tannakian category (over base fields) is not adequate. See ??? for details.

We end this brief discussion on Tannakian category by the so-called van Kampen completeness theorem, which will be used in the proof of the fundamental theorem of CFT for Riemann surfaces later.

For any group \( G \), denote its associated Tannakian category of equivalence classes \( [\rho] \) of unitary representations \( \rho : G \to \text{Aut}(V_\rho) \) by \( U(G) \). Fix for all classes \( [\rho] \) a representative \( \rho \) once and for all. A subset \( Z \)
of $U(G)$ is said to contain sufficiently many representations if for any two distinct elements $g_1, g_2$ of $G$, there exists $[\rho]$ in $Z$ such that $\rho(g_1) \neq \rho(g_2)$.

**van Kampen Completeness Theorem.** With the same notation as above, if $G$ is compact, then, as a completed Tannakian category, $U(G)$ may be generated by any collection of objects which contains sufficiently many representations.

### A.2.3.2. $KR^2$-Trick

While it is quite nice to have a proof of Key Proposition 2.3.1, we are by no means satisfied with it. The proposition consists of two aspects: the algebraic one and the analytic one. So, what we should do is the follows:

(1) From analytic point of view, to prove that
   (a) the tensor operation is closed; and
   (b) the so-called forgetful functor is faithful; and

(2) From algebraic point of view, to show that
   (a) the tensor operation is closed; and
   (b) the functor introduced in 2.3.1 is faithful.

However in our proof outlined above, only (1.a) and (2.b) are shown. That is to say, with the help of the so-called Narasimhan-Seshadri correspondence, a micro reciprocity law, we make no distinction between algebraic and analytic aspects. Thus, a purely analytic proof for (1.a) and a purely algebraic proof for (2.a) should be pursued.

The proof of (1.a) is a simple one since unitary representations for our fundamental groups, or better the unitary representations of quotients of fundamental groups by (normal) subgroups generated by weighted parabolic generators have a semi-simplification. We leave the details for the reader. Thus the real challenge here is an algebraic proof of (2.a), i.e., an proof for that tensor products of geo-ari bundles are again geo-ari bundles. It is here we should use another central concept in GIT, the so-called instability flag of Mumford-Kempf (with the refined version given by Ramanan and Ramanathan). This goes as follows.

In his study of GIT stabilities, Mumford conjectures that if a point is not semi-stable, then there should exist a parabolic subgroup which takes responsibility, in the view of the so-called Hilbert-Mumford criterion. This is confirmed by Kempf. (In Kempf’s study, as suggested by Mumford, the rationality problem is also treated successfully, at least when constant fields are perfect.) Kempf’s result then motivates Ramanan and Ramanathan to show that even though, for the original action, the corresponding point is not semi-stable, but if a certain type well-controlled modification is allowed (with the aim to cancel the instability contribution in the original action), a new (yet well-associated) point could be constructed such that with respect to the natural induced action this new point becomes semi-stable. As a direct consequence, in the case for semi-stable vector bundles without parabolic structures, since the new well-associated action may be associated to the intersection stability condition for bundles naturally, we may then obtain an algebraic proof of (2.a) for bundles.

**Remark.** Over complex numbers, the rationality is not a serious problem. However, over finite fields, this turns to be a difficult one. As a matter of fact, we then need to use the Frobenius to tackle the rationality problem. For details, see ???.

Therefore, to develop a non-abelian CFT for function fields over complex numbers, we need to find a more general version, what we call the $KR^2$-trick, of the above result of Kempf, and Ramanan and Ramanathan, which works for parabolic bundles. For this purpose, we may follow a more down-to-earth approach of Faltings and Tataro. That is to say, following Faltings, we first write any parabolic sub-bundle of the tensor product in terms of filtrations over some points, disjoint from parabolic ones; then introduce a certain GIT stability for general filtrations to check whether the induced filtrations from subbundles and the original parabolic filtration are GIT stable. If so, we by definition are done. Otherwise, following Totaro, from the associated instability flag of Mumford-Kempf, we can complete the proof by using the intersection stability of each of the components of the tensor product.

### A.2.4. Non-Abelian CFT for Function Fields over Complex Numbers
A.2.4.1. Micro Reciprocity Law, Tannakian Duality and the Reciprocity Map

Let $M$ be a compact Riemann surface of genus $g$ with marks $P_1, \ldots, P_N$. Set $M^0 := M \setminus \{P_1, \ldots, P_N\}$. Then, the fundamental group $\pi_1(M^0)$ is generated by $2g$ hyperbolic generators $A_1, B_1, \ldots, A_g, B_g$ and $N$ parabolic generators $S_1, \ldots, S_N$ which satisfy one single relation $\prod_{i=1}^g [A_i, B_i] \cdot \prod_{j=1}^N S_j = 1$. Moreover, with respect to an effective divisor $D = \sum_{i=1}^N e_i P_i$, we have a Tannakian category $\mathcal{U}(M; D)$ consisting of equivalent classes of unitary representations $[\rho : \pi_1(M^0) \to \text{Aut}(V_0)]$ of $\pi_1(M^0)$ such that

(i) $\rho(S_i) = \text{diag}(\text{exp}(2\pi i \beta_{i1}), \ldots, \text{exp}(2\pi i \beta_{il}))$ for all $i = 1, \ldots, N$; and
(ii) there exist integers $\gamma_{ij} > 0$ and $\delta_{ij} \geq 0$ such that
   (a) $\gamma_{ij} | e_j$;
   (b) $(\gamma_{ij}, \delta_{ij}) = 1$; and
   (c) $\beta_{ij} = \frac{\gamma_{ij}}{\delta_{ij}}$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, l$.

Now in each equivalence class $[\rho]$, fix a unitary representation, denoted also by $\rho$, once and hence for all. Clearly, $\rho$ induces a unitary representation of the group $\pi_1(M^0)/J(D)$, where $J(D)$ denotes the normal subgroup of $\pi_1(M^0)$ generated by $S_1^{\gamma_{11}}, \ldots, S_N^{\gamma_{Nl}}$. Call this latest representation $\rho$ as well. Then, for any element $g \in \pi_1(M^0)/J(D)$, $\rho(g)$ induces for each object $[\rho : \pi_1(M^0) \to \text{Aut}(V_0)]$ in $\mathcal{U}(M; D)$ an automorphism of $V_0$. As a direct consequence, we obtain a natural group morphism from $\pi_1(M^0)/J(D)$ to the automorphism group of the corresponding fiber functor $\mathcal{U}(M; D) \to V_{eC}$.

On the other hand, for the Tannakian category $\mathcal{M}(M; D)$ of geo-ari bundles on $M$ along $D$ together with the fiber functor $\omega(M; D) : \mathcal{M}(M; D) \to V_{eC}$, by Tannakian Duality, we conclude that $\omega(M; D) : \mathcal{M}(M; D) \to V_{eC}$ is equivalent to the Tannakian category of the representations of $\text{Aut}^\otimes \omega$. Therefore, by the Narasimhan-Seshadri correspondence (and the Harder-Narasimhan correspondence), the so-called micro reciprocity law, we obtain a canonical group morphism

$$\Omega(D) : \pi_1(M^0)/J(D) \to \text{Aut}^\otimes(\omega(M; D)).$$

We will call $\Omega(D)$ the reciprocity map associated with $(M, D)$.

A.2.4.2. Non-Abelian CFT

By definition, a subcategory $S$ of a Tannakian category with respect to a fiber functor $\omega$ is called a finitely completed Tannakian subcategory, if

(1) it is a completed Tannakian subcategory;
(2) there exist finitely many objects which generated $S$ as an abelian tensor subcategory;
(3) $\text{Aut}^\otimes(\omega|_S)$ is a finite group.

With this, by using the van Kampen completeness theorem and the above reciprocity map, we then can manage to obtain the following fundamental theorem on non-abelian class field theory for function fields over complex numbers.

**Fundamental Theorem in Non-Abelian Class Field Theory for Riemann Surfaces.**

(1) (Existence and Conductor Theorem) There is a natural one-to-one correspondence $\omega_{M,D}$ between

$$\{S : \text{finitely completed Tannakian subcategory of } \mathcal{M}(M; D)\}$$

and

$$\{\pi : M' \to M : \text{finite Galois covering branched at most at } D\};$$

(2) (Reciprocity Law) There is a natural group isomorphism

$$\text{Aut}^\otimes(\omega(M; D)|_S) \simeq \text{Gal}(\omega_{M,D}(S)).$$

We end this discussion by pointing out that, as an application, one may use this fundamental theorem to solve the geometric inverse Galois problem.
A.2.5. Classical (abelian) CFT: An Example of Kwada-Tata and Kawada

A.2.5.1. Class Formation

Classical abelian class field theory was first formulated axiomatically in Artin-Tate seminar in terms of cohomology of groups. As an example of this formulation, later Kawada-Tate and Kawada studied function fields over complex numbers. To make a comparison between this classical approach of CFT and what we outlined above, next we recall the works in Kawada-Tate and Kawada.

Let $k_0$ be a given ground field and $\Omega$ a fixed infinite separable normal algebraic extension of $k_0$. Let $R$ be the set of all finite extensions of $k_0$ in $\Omega$. By definition we call a collection $\{E(K) : K \in R\}$ of abelian groups $E(K)$ a formulation if the following conditions are satisfied:

F1. If $k \subset K$ then there is an injective morphism $\phi_{k/K} : E(k) \hookrightarrow E(K)$;

F2. If $k \subset l \subset K$, then $\phi_{l/K} \circ \phi_{k/l} = \phi_{k/K}$;

F3. If $K/k$ is normal and $G = Gal(K/k)$ is its Galois group, then $G$ acts on $E(K)$ and $\phi_{k/K}(E(k)) = E(K)^G$;

F4. If $k \subset L \subset K$ and $L/k, K/k$ are both normal, then the Galois group $F = Gal(L/k)$ is a quotient group of $G = Gal(K/k)$. Denote by $\lambda_{G/F} : \Rightarrow F$ the canonical quotient map. Then for any $\sigma \in G$ and $f \in E(L)$, $\sigma \circ \phi_{l/K}(f) = \phi_{L/K} \circ (\lambda_{G/F})(f)$.

Moreover if a formulation is called a class formation if it satisfies the following additional conditions on group cohomology:

C1: $H^1(G,E(K)) = 0$; and

C2: $H^2(G,E(K)) \cong [K:k]Z$.

By a theorem of Tate, C1 and C2 imply and hence are equivalent to the following stronger condition C. In a class formation, for all $r \in Z$, $H^r(G,E(K)) \cong H^{r-2}(G,Z)$ for all $r$. In particular, if a 2-cocycle $\alpha(K)$ generates the cyclic group $H^2(G,E(K))$, then the cup-product $g \mapsto \alpha(K) \cup g$ induces the isomorphism.

Note that by definition, $H^{-2}(G,Z) = G^{ab} := G/[G,G]$, and the 0-th cohomology group $H^0(G,E(K))$ is nothing but $E(K)^G/T_G(E(K))$, (where $T_G = \sum_{\sigma \in G} \sigma(a)$). So for a class formation, we obtain then the reciprocity law

$$E(k) \phi^{-1}(T_G E(K)) \cong G^{ab}.$$

Furthermore, for a class formation, with respect to the so-called res (restriction), infl (inflation) and ver (Verlagerung) operations of group cohomology, we have the follows;

Case 1. For $k \subset l \subset K$ with $K/k$ normal, $G := Gal(K/k)$ and $H := Gal(k/l)$, in $H^2(G,E(K))$, $\text{ver}_{H/G} \circ \text{res}_{G/H} = [G : H] \cdot 1$. So, $\text{res}_{G/H}$ is surjective and $\text{ver}_{H/G}$ injective;

Case 2. For $k \subset L \subset K$ with $L/k, K/k$ normal, and $G := Gal(K/k)$, $H := Gal(k/L)$, then we have the exact sequences

$$0 \rightarrow H^2(F,E(L)) \xrightarrow{i^{inf}} H^2(G,E(K)) \xrightarrow{\text{res}_{G/H}} H^2(H,E(K)).$$

Therefore, there exist 2-cocycles $\{\alpha(K)\}$ of $G(K/k)$ over $E(k)$ such that

D1. $\alpha(K/k)$ are generators of the cyclic groups $H^2(G,E(K))$;

D2. In Case 1,

$$\text{res}_{G/H}(\alpha(K/k)) \sim f(K/l); \quad \text{ver}_{H/G}(\alpha(K/k)) \sim [G : H] \cdot \alpha(K/k);$$

D3. In Case 2, $i^{inf}_{F/G}(\alpha(L/k)) \sim [K : L] \cdot \alpha(K/k)$. Here $\sim$ means cohomologous.

Often, we call such a system $\{\alpha(K/k)\}$ the canonical 2-cocycle of $G(K/k)$ over $E(K)$ which may be used to write down the reciprocity law as follows: Introduce $\left( \frac{K/k}{\sigma} \right) \in E(k)$, $\sigma \in G := Gal(K/k)$ for normal extension $K/k$ by

$$\left( \frac{K/k}{\sigma} \right) = \phi_{k/k}^{-1} \left( \prod_{\rho \in \hat{G}} \alpha_{K/k}(\rho,\sigma) \right) \mod \phi_{K/k}^{-1}(T_G E(K/k)),$$

Then by a result of Nakayama, the symbol $(a,K/k) \in G^{ab}, a \in E(k)$ defined by

$$(a,K/k) := \sigma \mod [G,G] \quad \text{when} \quad \left( \frac{K/k}{\sigma} \right) = a \mod \phi_{K/k}^{-1}(T_G E(K)),$$
satisfies all the properties of the norm residue symbol in number theory.

Let \( k \in \mathcal{R} \) be fixed and \( \Omega^o(k) \) be the maximal abelian extension of \( k \) in \( \Omega \). Set \( \mathcal{R}^a(k) \) be the collection of finite abelian extensions of \( k \) (in \( \Omega^o \)). Then for \( K \in \mathcal{R}^a(k) \) define a subgroup \( A(K/k) \) of \( E(k) \) to be \( \phi_{k/K}(T_{\text{Gal}}(K/k)E(K)) \). By definition, a subgroup \( F \) of \( E(k) \) is called admissible if \( F = A(K/k) \) for some \( K \in \mathcal{R}^a(k) \). Denote the set of all admissible subgroups of \( E(k) \) by \( \mathcal{U}(E(k)) \). Then one checks, by the properties of norm residue symbol, that

1. (Combination Theorem) \( A(K_1 \cdot K_2/k) = A(K_1/k) \cap A(K_2/k) \), \( A(K_1 \cap K_2/k) = A(K_1/k) \cdot A(K_2/k) \);
2. (Ordering Theorem) \( A(K_1/k) \supset A(K_2/k) \) if and only if \( K_1 \subset K_2 \);
3. (Uniqueness Theorem) \( A(K_1/k) = A(K_2/k) \) if and only if \( K_1 = K_2 \) for \( K_1, K_2 \in \mathcal{R}^a(k) \).

Also, if let \( \Gamma(k) \) be the compact Galois group \( \Omega^o(k)/k \), then \( \Gamma(k) \) is the inverse limit group of \( \{ G(K/k) : K \in \mathcal{R}^a(k) \} \). Moreover, for \( a \in E(k) \), the limit, called the generalized norm residue symbol, \( (a, k) := \lim_{K \in \mathcal{R}^a(k)} (a, K/k) \in \Gamma(k) \) exists. Set \( \mathcal{T}(k) := (E(k), k) \subset \Gamma(k) \) and \( \mathcal{R}(k) := \{(a \in E(k) : (a, k) = 1) \subset E(k) \} \). Then the mapping \( a \mapsto (a, k) \) induces an isomorphism

\[
E(k)/\mathcal{R}(k) \simeq \mathcal{T}(k) \subset \Gamma(k),
\]

where \( \mathcal{T}(k) \) is dense in \( \Gamma(k) \).

For examples, classical (abelian) CFT for global fields are all class formations. More precisely,
(A) For a number field \( k \), we may take \( E(k) \) to be the idele class group \( C_k \) and \( \phi_{k/K} \) the natural inclusion. In particular, \( \mathcal{R}(k) \) is then the connected component of the unity of \( C_k \), \( \Gamma(k) = \mathcal{T}(k) \), and \( \mathcal{U}(E(k)) \) is the set of all open subgroup of \( E(k) \) of finite index;
(B) For a function field \( k \) of one variable over a finite field, we may take \( E(k) \) to be the idele class group \( C_k \) and \( \phi_{k/K} \) the natural inclusion. In particular, \( \mathcal{R}(k) = 1, \Gamma(k)/\mathcal{T}(k) \) is a uniquely divisible group isomorphic to \( \mathbb{Z}/\mathbb{Z} \), and \( \mathcal{U}(E(k)) \) is the set of all open subgroup of \( E(k) \) of finite index.

### A.2.5.2. The Work of Kawada and Tate

Choose \( k_0 \) to be an algebraic function field of one variable over complex numbers with \( \Omega \) the algebraic closure of \( k_0 \). Let \( D(k) \) be the group of all fractional divisors \( \prod_v P_v^{r_v} \), where \( r_v \in \mathbb{Q} \), and \( r_v = 0 \) for almost all \( v \), and \( P(k) \) the group of all principal divisors. Let \( D(k) = D(k)/P(k) \).

Define \( E(k) := D(k)^{\vee} = \text{Hom}(D(k), \mathbb{R}/\mathbb{Z}) \), the group of characters of \( D(k) \), then \( \{ E(k) \} \) with the conorm map \( \phi_{k/K} \) is a class formation such that the norm residue map \( \Phi_k : E(k) \rightarrow A(k) \) is surjective but \( F(k) := \ker \Phi_k \neq 0 \). As a direct consequence, if \( T \mathcal{D}(k) \) denotes the torsion subgroup of \( \mathcal{D}(k) \), \( \{ E(k)^* = E(k)/F(k) = (T \mathcal{D}(k))^{\vee} \} \) gives also a class formation.

On the other hand, easily,
(i) the character of \( k_0 \) is zero;
(ii) \( k_0 \) contains all the roots of unity; and
(iii) for any finite normal extension \( K/k \) with \( k \supset k_0 \) and \( K/k \) finite, \( N_{K/k}K = k \).

So, by applying the so-called Kummer theory, \( E(k) := (k^* \otimes \mathbb{Q}/\mathbb{Z})^{\vee} \), and the conorm \( \phi_{k/K} : E(k) \rightarrow E(K) \), i.e., \( \phi_{k/K}(\chi)(A) = \chi(N_{K/k}A) \) for \( \chi \in E(k), A \in K^* \otimes \mathbb{Q}/\mathbb{Z} \), give a class formation with \( E(k) = A(k) \).

Thus, in particular, by comparing these class formations, we obtain a canonical isomorphism \( k^* \otimes \mathbb{Q}/\mathbb{Z} \simeq T \mathcal{D}(k) \). Moreover, if \( \Omega \) is a maximal unramified extension of \( k_0 \), \( t := t(\Omega/k) \) and \( E(k)_t := (\mathcal{D}_S(k)^{\vee} \) with \( \mathcal{D}_S(k) \) the divisor class group of the usual sense, i.e., with integral coefficients, then \( \{ E(k)_t \} \) gives a class formation for \( t \). On the other hand, for a fixed finite set \( S \neq 0 \) of prime divisors of \( k_0 \), let \( \Omega_S \) be the maximal \( S \)-ramified extension of \( k_0 \). Put \( t_S := t(\Omega_S/k_0) \) and \( E(k)_S := (\mathcal{D}_S(k)^{\vee} \) with \( \mathcal{D}_S(k) \) the \( S \)-fractional divisor class group of \( k \). Then \( \{ E(S)_t \} \) forms also a class formation for \( t_S \) such that \( E(k)_S^{\vee} \simeq A(k) \).

### A.2.5.3. Abelian CFT for Riemann Surfaces In Terms of Geo-Ari Bundles
However, it is via the third approach of a class formation for Riemann surfaces due to Kawada, these classical results are related with our approach to the CFT. It goes as follows. For \( k_0 < k \), let \( S(k) \) denote the set of prime divisors of \( k \) which are extensions of a prime divisor of \( k_0 \) contained in \( S \). Let \( R(k) \) be the Riemann surface of \( k \) and \( R_S(k) = R(k)|S(k) \). Let \( E(k) \) be the one dimensional integral homology group \( H_1(R_S(k), \mathbb{Z}) \) of \( R_S(k) \), and define \( \phi_{k/K} : E(k) \to E(K) \) by \( \gamma \mapsto V\gamma \) where \( V\gamma \) is the covering path of \( \gamma \in R_S(k) \) on the unramified covering surface \( R_S(K) \) of \( R_S(k) \). Then \( \{E(k)\} \) forms again a class formation for \( t_S \) with \( Im \Phi_k \) dense in \( A(k) \) and \( Ker \Phi_k = 0 \). Indeed, one may obtain this by looking at the canonical pairing, the micro reciprocity law in this context, \( H_1(R_S(k), \mathbb{Z}) \times TD_S(k) \to S^1 \subset \mathbb{C} \) defined by \( (\eta, A)_S \mapsto \exp(\int_{\gamma} d\log A) \) for 1-cycles \( \eta \) on \( R_S(k) \) and \( A \in TD_S(k) \). Here \( d\log A \) denotes the abelian differential of the third kind on \( R_S(k) \) corresponding to a divisor \( A \).

On the other hand, over a compact Riemann surface \( M \) with punctures \( P_1, \ldots, P_N \) with respect to an effective divisor \( D = \sum e_j P_j \), we may introduce the group \( \text{Div}_{\mathbb{Q}}(M,D) \) of degree zero \( \mathbb{Q} \)-divisors along \( D \) on \( M \) by collecting all degree zero \( \mathbb{Q} \)-divisors of the form \( \sum a_j P_j + E \) with \( a_j \in \mathbb{Z} \) and \( E \) a ordinary integral divisor on \( M \). Denote the induced (rational equivalence) divisor class group \( \text{Cl}_{\mathbb{Q}}^Q(M,D) \). Then, \( \text{Cl}_{\mathbb{Q}}^Q(M,D) \) is simply the collection of all geometric bundles of rank 1 introduced in 2.2.2. Hence Theorem 2.4.2 then implies the following

**Theorem.** With the same notation as above, there is a one-to-one correspondence between the set of all isomorphism classes of finite abelian coverings \( \pi : X \to M \) branched at most at \( D \) and the set of all finite subgroups \( S \) of \( \text{Cl}_{\mathbb{Q}}^Q(M,D) \). Moreover, the correspondence \( \pi \mapsto S(\pi) \) satisfies that
(i) (Reciprocity Law) \( S(\pi) \simeq G_\pi \); and
(ii) (Ordering Theorem) \( \pi \leq \pi' \) if and only if \( S(\pi) \subset S(\pi') \).

Therefore, it seems to be very crucial to understand the precise relation between Kawada-Tate’s results and this latest theorem, in order to develop a (non-abelian) CFT for global fields.

### A.3. Towards Non-Abelian CFT for Global Fields

#### A.3.1. Weil-Narasimhan-Seshadri Type Correspondence

##### A.3.1.1. Grometric Representations

From what discussed above, in order to develop a non-abelian CFT for local and global fields, the first step should be the one to establish a micro reciprocity law, i.e., a Weil-Narasimhan-Seshadri type correspondence. Therefore, we are supposed to

1. introduce suitable classes of representations of Galois groups;
2. find corresponding classes for bundles in terms of intersection;
3. establish natural correspondences between classes in (1) and (2).

Hence, it is then more practicable to divide the problem into two. Namely, a general one in the sense of Weil Correspondence, and a refined one in the sense of Narasimhan-Seshadri correspondence.

We start with the Weil Correspondence. Here, we are then supposed to first introduce a general notion for representation of Galois groups such that, naturally, associated to such representations are special vector bundles together with additional structure.

As an example, let me explain what I have in mind in the case for number fields. So, let \( F \) be a number field with a finite subset \( S \) of places of \( F \). Denote the corresponding Galois group by \( G_{F,S} \). Naturally, by a representation, it should be first a continuous group homomorphism \( \rho : G_{F,S} \to \text{GL}_n(A_{F,S}) \), where \( A_{F,S} \) denotes the ring of \( S \)-adeles. Moreover, among others, we should assume that

1. for all places \( v \) of \( F \), the induced representations \( \rho_v : G_{F,S} \to \text{GL}_n(A_{F,v}) \) are unramified almost everywhere;
2. for a fixed places \( p \) of \( \mathbb{Q} \), there should be a compactibility condition for all places \( v \) above \( p \).

In addition, it also seems to be very natural to assume that

- for all induced \( \rho_v : G_v \to \text{GL}_n(F_v) \), there are invariant lattices \( M_v \) of \( F_v \) which are induced from a global lattice \( M \) over \( F \); and
- at places \( v \in S \), there should be naturally a certain weighted filtration induced by the action of Frobenius.

Thus in particular, associated to such a representation, is naturally a well-defined vector bundle equipped with a parabolic structure. We are expecting that geometric bundles are of Arakelov degree zero.
Based on the Weil correspondence, we should be able to enter the level of Narasimhan-Seshadri type correspondence. Here essential difficulties should appear. Chiefly, what should be a natural analog of being unitary? A suitable candidate seems to be that of Fontaine’s semi-stability at finite places and (unitary) at infinite places. However, we are not very sure about this, as somehow we believe that the condition of semi-stable representation is too restricted. For this reason, we propose the follows:

(e) for all places \( v \), the images of the induced representation \( \rho_v : G_v \to \text{Gl}_n(F_v) \) are contained in maximal compact subgroups; moreover, certain compactibility conditions are satisfied by \( \rho_v \) for all \( v \) over a fixed place of \( \mathbb{Q} \);

(f) there should be a natural deformation theory for these representations such that

(i) the size of all equivalence classes of these representations can be controlled;

(ii) a certain completeness holds.

Clearly, here the reader would find the work of Rapoport and Zink [RZ] and the lecture notes of Tilouine [Ti] are of great use. The cases for local fields and function fields may be similarly discussed. If success, we call such a representation a geometric representation.

A.3.1.2. Semi-Stability in terms of Intersection

This is the algebraic side of the micro reciprocity law. We here only study what happens for function fields over finite fields, while leave a detailed definition for number fields in Part (B).

With this restriction of fields, the situation becomes much simpler: We may use the existing stability condition of Seshadri for parabolic bundles.

That is to say, what we care here are the so-called parabolic semi-stable bundles of parabolic degree zero defined on algebraic curves over finite fields, introduced by Seshadri. However, in doing so, we are afraid that our program leads only a non-abelian CFT with tame ramifications. So it seems that, to deal with wild ramifications, additional works are needed. Therefore, in general, for this algebraic aspect, we propose the follows:

(0) Once we have an intersection semi-stabilities, the analog of the existence and uniqueness of Harder-Narasimhan filtration, the existence of Jordan-Hölder filtration and the uniqueness of the graded Jordan-Hölder objects should hold; moreover, morphisms of stable objects should be either zero or isomorphisms;

(1) The so-called Langton’s Completeness Principal should hold for such intersection semi-stabilities;

(2) The intersection semi-stability should be naturally related with a GIT stability. As a direct consequence, then moduli spaces can be formed naturally, and by (1), are indeed compact;

(3) With the help of the Frobenius, we should be able to define a special subset (of points) of moduli spaces, such that tensor products of the resulting bundles are represented again by points in this special subset.

If success, we will call these objects geometric bundles.

A.3.1.3. Weil-Narasimhan-Seshadri Type Correspondence

For a Weil type correspondence, from the classical proof, say, the one given in Gunning’s Princeton lecture notes, we should develop an analog of the de Rham, the Dolbeault cohomologies as well as a Hodge type theory. While, in general, it is out of reach, note that our base is of dimension one, it is still hopeful. For example, for curves over finite fields, we understand that all this is known to experts.

Next, let us consider a Narasimhan-Seshadri type correspondence. Here, we should establish a natural correspondence between the equivalence classes of geometric representations and the (Seshadri) equivalence classes of geometric bundles. Hence, the key points are the follows:

(1) By definition, a geometric representation should naturally give a geometric bundle;

(2) A Weil type correspondence holds. In particular, this implies that, by definition, geometric bundles come naturally from representations of Galois groups;

(3) Representations resulting from geometric bundles in (2) should be geometric. To establish this, we should use the deformation theories of geometric representations and geometric bundles as proposed in 3.1.1 and 3.1.2 above. For example, the final justification should be based on the compactness of both geometric representations and geometric bundles, via a direct counting. (In geometry, the counting is possible since
we know the structure of the fundamental group. But in arithmetic, this is quite difficult. For this, we find that the work of Fried and Völklein [FV] seems to be useful.)

A.3.2. Harder-Narasimhan Correspondence

This is specially designed to select special subclasses of geometric representations and geometric bundles so as to get what we call geo-ari representations and geo-ari bundles. There are two main reasons for doing so. The first is based on the facts that for function fields, there are examples of stable bundles whose tensor products are no longer semi-stable; and more importantly that if the bundles and their associated Frobenius twists are all semi-stable, then, so are their tensor products. The second comes from our construction of non-abelian zeta functions in Part (B). To define such non-abelian zeta functions, we use only rational points (over constant fields) of the moduli spaces, based on a result of Harder and Narasimhan, which guarantees the coincidence of the rationality of moduli points and the rationality of the corresponding bundles.

Thus, for example, for function fields over finite fields, a geo-ari bundle is defined to a geometric bundle which satisfies not only all the conditions for geo-ari bundles over Riemann surfaces, but an additional one, which says that all its Frobenius twists are semi-stable as well.

So for our more general purpose, key points here are the follows:
(1) to give a proper definition of geo-ari representations so that they form a natural abelian category;
(2) to give a suitable definition of geo-ari bundles so that they are closed under tensor product; and
(3) to establish a natural correspondence between (1) and (2). If success, we call such a result a Harder-Narasimhan type correspondence.

A.3.3. $KR^2$-Trick

This is specially designed to give an algebraic proof of the following key statement appeared as 3.2.(2):
The tensor products of geo-ari bundles are again geo-ari bundles.

We start with geo-ari bundles over function fields. When no parabolic structures is involved, this is solved by Kempf and Ramanan-Ramanthan. In their proof, key points are the follows.
(1) Rationality and uniqueness of instability flags, where the Frobenius twists are used;
(2) Existence of a GIT stable modification associated to any non semi-stable one; and
(3) Existence of GIT points for bundles such that GIT stability implies intersection stability by definition;
(4) Relation between the GIT modification in (2) and the intersection stability, where the intersection stability for each component of the tensor product plays a key role.

More precisely, if the tensor is not intersection stable, by (3), the associated GIT point would not be GIT stable. Thus from (2) there exists a GIT stable modification which is well understood by (1). Therefore, finally, by (4), i.e., the intersection stability of each components, we may finally conclude the intersection stability for the tensor product with the help of (2).

Thus, the problem left here is to see how the work of Ramanan-Ramanathan could be developed to deal with parabolic structure. In theory, this may be done by working on product of varieties instead of just a single one. As this process is more or less similar to what happens in constructing moduli space of parabolic semi-stable bundles, an expert should be able to carry the details out.

However, there is a more elementary approach as well, essentially given in the supplementary works of Faltings and Totaro. It goes as follows. First, as what Faltings does, write parabolic subbundles for the tensor in terms of weighted filtrations on the fibers (supported over points which are disjoint from punctures); then introduce a GIT stability with respect to weighted filtrations. So there are two possibilities: if the weighted filtration resulted from the parabolic subbundle is GIT stable, then by (3) above, we get the intersection stability of the tensor product. Otherwise, by (1) and (2) above, we may obtain a well-associated modification which is then GIT stable. This then by applying (4) completes the proof.

Before going to number fields, I would like to draw the attention of the reader to a relation between the stability of the generic point and the stability of special points, due to Mumford: GIT stability for objects over the generic point implies that over almost all but finitely many special points, the associated points are GIT semi-stable.

Now we come to number fields. For this, first, we should develop an Arakelov style GIT. This sounds
difficult, but due to the following fact we expect that it is still workable: Kempf’s functional related to the Hilbert-Mumford criterion for GIT stability is essentially compatible with Arakelov theory. (See e.g., [Bu] and [Zh].)

Based on such a new GIT, then we should find a relation between the intersection stability in the definition of geo-ari bundles and this new type of GIT stability. In particular, we expect that an analog of $KR^2$-trick works. Thus the final key point should be

(3) An Arakelov style GIT exists such that (1), (2), (3) and (4) above work equally well under the framework of this new GIT over number fields.

Note also that not only for the purpose of $KR^2$-trick, a new GIT is needed, for the construction of moduli spaces of geometric bundles, such a new GIT is supposed to be crucial. For the related point, see also ?? in Part (B).

A.3.4. Tannakian Category Theory over Arbitrary Bases

While for a non-abelian class field theory of function fields over complex numbers, the standard theory of Tannakian category may be applied directly, I do not expect this in general. For example, for number fields, we need to develop a more general theory, a theory of Tannakian categories over the ring of integers. The key points for this new type of Tannakian categories are supposed to be the follows.

(1) Fiber functors should give us a group. Thus whether fiber functors are to categories of vector spaces over a field is not really important. In fact, for number fields, fiber functors should be faithful exact tensor functor to the categories of finitely generated projective modules of the corresponding ring of integers, among others;

(2) An analog of Tannakian duality holds; and

(3) Tannakian category should contain the so-called finitely completed Tannakian subcategories with respect to which an analog of van Kampen Completeness Theorem holds.

If success, as a direct consequence, by (1) and (2), using Narasimhan-Seshadri and Harder-Narasimhan correspondences, we get naturally a reciprocity map. This then by (3), leads to a completed non-abelian CFT.

A.3.5. Non-Abelian CFT for Local and Global Fields

All in all, from the above discussion, what we then expect is the following conjecture, or better,

**Working Hypothesis.** For local and global fields,

(1) there are well-defined geometric representations and geometric bundles such that a Weil-Narasimhan-Seshadri type correspondence holds;

(2) there are refined well-defined geo-ari representations and geo-ari bundles such that a Harder-Narasimhan type correspondence holds;

(3) there are well-defined GIT type stability such that the intersection stability as appeared in the definition of geo-ari bundles could be understood in terms of this new GIT stability. Moreover, an analog of $KR^2$-trick works;

(4) there is a well-established Tannakian type category theory, for which a Tannaka type duality and van Kampen type completeness theorem hold; moreover the category of geo-ari objects forms naturally such Tannakian type categories.

In particular, the fundamental results in non-abelian class field theory, such as existence theorem, conductor theorem and reciprocity law, hold.
B. Moduli Spaces, Riemann-Roch, and New Non-Abelian Zeta functions

B.1. New Local and Global Non-Abelian Zeta Functions for Curves

B.1.1. Local Non-Abelian Zeta Functions

B.1.1.1. Artin Zeta Functions for Curves

We start this program by recalling the construction and basic properties of the classical Artin zeta functions of curves defined over finite fields.

Let \( C \) be a projective irreducible reduced regular curve of genus \( g \) defined over a finite field \( k := \mathbb{F}_q \) with \( q \) elements. Then the arithmetic degree \( d(P) \) of a closed point \( P \) on \( C \) is defined to be \( d(P) := [k(P) : k] \), where \( k(P) \) denotes the residue class field of \( C \) at \( P \). So \( q^{d(P)} \) is nothing but the number \( N(P) \) of elements in \( k(P) \). Extending this to all divisors, we then may define the Artin zeta function \( \zeta_C(s) \) for curve \( C \) over \( k \) by setting

\[
\zeta_C(s) := \prod_{P} (1 - N(P)^{-s})^{-1} = \sum_{D \geq 0} N(D)^{-s} = \sum_{D \geq 0} (q^{-s})^{d(D)}, \quad \text{Re}(s) > 1,
\]

where the sum is taken over all the effective divisors \( D \) on \( C \). As usual, set \( t = q^{-s} \), and \( Z_C(t) = \sum_{D \geq 0} t^{d(D)} \).

Note that the number of positive divisors which are rational equivalent to a fixed divisor \( D \) is \( (q^{h_0(C,D)} - 1)/(q - 1) \). So,

\[
Z_C(t) = \sum_{D \geq 0} t^{d(D)} = \sum_{d \geq 0} \sum_{D \in \mathcal{D}} \sum_{d \in D} t^d = h(C) \sum_{d \geq 0} \left( \frac{q^{h_0(C,D)} - 1}{q - 1} \right) \cdot t^d,
\]

where the sum \( \sum_{D} \) is taken over the rational divisor classes of degree \( d \), and \( h(C) \) denotes the cardinality of \( \text{Div}_0(C)/\text{Div}_0(C) \), the group of degree zero divisors modulo the subgroup of principal divisors. Thus by Riemann-Roch theorem, for a positive divisor \( D \), \( h_0(C,D) \leq d(D) + 1 \). So up to finitely many convergent terms, the convergence of \( Z_C(t) \) is the same as that for \( \sum_{d \geq 0} (d + 1)(qt)^d \). Hence \( \zeta_C(s) \) is well-defined.

Being well-defined,

\[
(q - 1)^{-1} \zeta_C(s) = \sum_{d \geq 0} \sum_{D} (q^{h_0(D)} - 1)q^{-ds} = \sum_{d \geq 0} \sum_{D} q^{h_0(D)} - ds = \frac{h(C)}{1 - q^{-s}}.
\]

Set \( F(q^{-s}) := \sum_{d \geq 0} \sum_{D} q^{h_0(D)} - ds \cdot \frac{h(C)}{1 - q} \) and \( G(q^{-s}) := \sum_{d \geq 0} \sum_{D} q^{h_0(D)} - ds \), where \( \sum' \) and \( \sum'' \) are taken over \( d \geq 2g - 2 + 1 \) and \( 0 \leq d \leq 2g - 2 \) respectively. Then,

(i) \( (q - 1)Z_C(t) = F(t) + G(t) \);
(ii) Being a sum of finite many terms, \( G(t) \) is rational; and moreover, by Riemann-Roch,
(iii) \( F(t) = h(C) \sum' q^{d-g+1}t^d - h(C)q^{1 - g}(qt)^{2g-2} \cdot \frac{1}{1 - qt} = h(C)q^{1 - g}(qt)^{2g-2} - h(C) \cdot \frac{1}{1 - qt} \).

So, \( Z_C(t) \) is indeed a rational function of \( t \). Further, from (iii), \( F(t) = q^{g-1}t^{2g-2}F \left( \frac{1}{qt} \right) \). Note also that in \( G(t) \), the sum is taken the sum over all divisors whose degrees are between 0 and \( 2g - 2 \). Thus by the duality and Riemann-Roch, for any canonical divisor \( K_C \) of \( C \),

\[
G(t) = \sum_{D} q^{h_0(D)}t^d = \sum_{D} q^{d(D)-g+1+h(K_C-D)}t^d = \sum_{D} q^{(q-1)h_0(K_C-D)}q^{-g+1}q^{h_0(K_C-D)} = q^{-1}q^{2g-2}G \left( \frac{1}{qt} \right).
\]

That is to say, \( \zeta_C(s) \) satisfies the functional equation. So we have the following

**Theorem.** With the same notation as above, \( \zeta_C(s) \) is well-defined, admits a meromorphic to the whole complex s-plane. Moreover,

\[
\zeta_C(s) = N(K_C)^{1/2-s} \zeta_C(1-s)
\]

and there exists a polynomial \( P_C(t) \) of degree \( 2g \) such that

\[
Z_C(t) = \frac{P_C(t)}{(1-t)(1/qt)}.
\]
B.1.1.2. Too Different Generalizations: Weil Zeta Functions and A New Approach

It is well-known that based on the so-called reciprocity law the above Artin zeta function may be written as

\[ Z_C(t) = \exp \left( \sum_{m \geq 1} \frac{N_m}{m} t^m \right) \]

where \( N_m := \#C(F_{q^m}) \) denotes the number of \( F_{q^m} \)-rational points of \( C \). This then leads to a far reaching generalization to the so-called Weil zeta functions of higher dimensional varieties defined over finite fields, the study of which dominantes what we call Arithmetic Geometry in the second half of 20th century.

On the other hand, for the purpose of developing a non-abelian zeta function theory, we here do it differently. In terms of Artin zeta functions, the key then is the following observations;

1. \( \sum \) in 1.1.1 may be viewed as taking summation over the degree \( d \) Picard group of the curve;
2. Picard groups for curves may be viewed as moduli spaces of (semi-stable) line bundles;
3. moduli spaces of semi-stable bundles exist and all the terms appeared in the summation for Artin zeta functions, such as \( h \) and degree, make sense for vector bundles as well.

B.1.1.3. Moduli Spaces of Semi-Stable Bundles

Let \( C \) be a regular, reduced and irreducible projective curve defined over an algebraically closed field \( \bar{k} \). Then according to Mumford [M], a vector bundle \( V \) on \( C \) is called semi-stable (resp. stable) if for any proper subbundle \( V' \) of \( V \),

\[ \mu(V') := \frac{d(V')}{r(V')} \leq (\text{resp. } <) \frac{d(V)}{r(V)} =: \mu(V). \]

Here \( d \) denotes the degree and \( r \) denotes the rank.

**Proposition 1.** Let \( V \) be a vector bundle over \( C \). Then

(a) ([HN]) there exists a unique filtration of subbundles of \( V \), the so-called Harder-Narasimhan filtration of \( V \),

\[ \{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{s-1} \subset V_s = V \]

such that for \( 1 \leq i \leq s-1 \), \( V_i/V_{i-1} \) is semi-stable and \( \mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_i) \);

(b) (see e.g. [Se]) if moreover \( V \) is semi-stable, there exists a filtration of subbundles of \( V \), a Jordan-Hölder filtration of \( V \),

\[ \{0\} = V^{t+1} \subset V^t \subset \ldots \subset V^1 \subset V^0 = V \]

such that for all \( 0 \leq i \leq t \), \( V^i/V^{i+1} \) is stable and \( \mu(V^i/V^{i+1}) = \mu(V) \). Moreover, \( \text{Gr}(V) := \oplus_{i=0}^{t} V^i/V^{i+1} \), the associated (Jordan-Hölder) graded bundle of \( V \), is determined uniquely by \( V \).

Following Seshadri, two semi-stable vector bundles \( V \) and \( W \) are called \( S \)-equivalent, if their associated Jordan-Hölder graded bundles are isomorphic, i.e., \( \text{Gr}(V) \simeq \text{Gr}(W) \). Applying Mumford’s general result on geometric invariant theory ([M]), Seshadri proves the following

**Theorem 2.** ([Se]) Let \( C \) be a regular, reduced, irreducible projective curve defined over an algebraically closed field. Then over the set \( \mathcal{M}_{C,r}(d) \) of \( S \)-equivalence classes of rank \( r \) and degree \( d \) semi-stable vector bundles over \( C \), there is a natural normal, projective algebraic variety structure.

Now assume that \( C \) is defined over a finite field \( k \). Naturally we may talk about \( k \)-rational bundles over \( C \), i.e., bundles which are defined over \( k \). Moreover, from geometric invariant theory, projective varieties \( \mathcal{M}_{C,r}(d) \) are defined over a certain finite extension of \( k \). Thus it makes sense to talk about \( k \)-rational points of these moduli spaces too. The relation between these two types of rationality is given by Harder and Narasimhan based on a discussion about Brauer groups:

**Proposition 3.** ([HN]) There exists a finite field \( F_q \) such that for any \( d \), the subset of \( F_q \)-rational points of \( \mathcal{M}_{C,r}(d) \) consists exactly of all \( S \)-equivalence classes of \( F_q \)-rational bundles in \( \mathcal{M}_{C,r}(d) \).
From now on, without loss of generality, we always assume that finite fields $\mathbb{F}_q$ (with $q$ elements) satisfy the property in this Proposition. Also for simplicity, we write $\mathcal{M}_{C,r}(d)$ for $\mathcal{M}_{C,r}(d)(\mathbb{F}_q)$, the subset of $\mathbb{F}_q$-rational points, and call them moduli spaces by an abuse of notations.

### B.1.1.4. New Local Non-Abelian Zeta Functions

Let $C$ be a regular, reduced, irreducible projective curve defined over the finite field $\mathbb{F}_q$ with $q$ elements. Define the rank $r$ non-abelian zeta function $\zeta_{C,r,\mathbb{F}_q}(s)$ by setting

$$\zeta_{C,r,\mathbb{F}_q}(s) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(d), d \geq 0} q^{h^0(C,V)} - \frac{1}{\#\text{Aut}(V)} (q^{-s})^{d(V)}, \quad \text{Re}(s) > 1.$$

Clearly, we have the following

**Fact.** With the same notation as above, $\zeta_{C,1,\mathbb{F}_q}(s)$ is nothing but the classical Artin zeta function for curve $C$.

### B.1.1.5. Basic Properties for Non-Abelian Zeta Functions

Many basic properties for classical Artin zeta functions are satisfied by our non-abelian zeta functions as well. More precisely, we have the following

**Theorem.** With the same notation as above,

1. The non-abelian zeta function $\zeta_{C,r,\mathbb{F}_q}(s)$ is well-defined for $\text{Re}(s) > 1$, and admits a meromorphic extension to the whole complex $s$-plane;
2. (Rationality) If we set $t := q^{-s}$ and introduce the non-abelian $Z$-function of $C$ by setting

$$\zeta_{C,r,\mathbb{F}_q}(s) =: Z_{C,r,\mathbb{F}_q}(t) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(d), d \geq 0} q^{h^0(C,V)} - \frac{1}{\#\text{Aut}(V)} t^{d(V)}, \quad |t| < 1,$$

then there exists a polynomial $P_{C,r,\mathbb{F}_q}(s) \in \mathbb{Q}[t]$ such that

$$Z_{C,r,\mathbb{F}_q}(t) = \frac{P_{C,r,\mathbb{F}_q}(t)}{(1-t^r)(1-q^rt^r)};$$

3. (Functional Equation) If we set the rank $r$ non-abelian $\xi$-function $\xi_{C,r,\mathbb{F}_q}(s)$ by

$$\xi_{C,r,\mathbb{F}_q}(s) := \zeta_{C,r,\mathbb{F}_q}(s) \cdot (q^s)^r(q-1),$$

then

$$\xi_{C,r,\mathbb{F}_q}(s) = \xi_{C,r,\mathbb{F}_q}(1-s).$$

One may prove this theorem by using the vanishing theorem, duality, and the Riemann-Roch theorem. See e.g., ?? for details.

**Corollary.** With the same notation as above,

1. $P_{C,r,\mathbb{F}_q}(t) \in \mathbb{Q}[t]$ is a degree $2rg$ polynomial;
2. Denote all reciprocal roots of $P_{C,r,\mathbb{F}_q}(t)$ by $\omega_{C,r,\mathbb{F}_q}(i), i = 1, \ldots, 2rg$. Then after a suitable rearrangement,

$$\omega_{C,r,\mathbb{F}_q}(i) \cdot \omega_{C,r,\mathbb{F}_q}(2rg - i) = q, \quad i = 1, \ldots, rg;$$

3. For each $m \in \mathbb{Z}_{\geq 1}$, there exists a rational number $N_{C,r,\mathbb{F}_q}(m)$ such that

$$Z_{r,\mathbb{F}_q}(t) = P_{C,r,\mathbb{F}_q}(0) \cdot \exp \left( \sum_{m=1}^{\infty} N_{C,r,\mathbb{F}_q}(m) \frac{t^m}{m} \right).$$
Moreover,
\[ N_{C,r,F_q}(m) = \begin{cases} 
  r(1 + q^m) - \sum_{i=1}^{2r} \omega_{C,r,F_q}(i)^m, & \text{if } r \mid m; \\
  -\sum_{i=1}^{2r} \omega_{C,r,F_q}(i)^m, & \text{if } r \not\mid m;
\end{cases} \]

(4) For any \( a \in \mathbb{Z}_{>0} \), denote by \( \zeta_a \) a primitive \( a \)-th root of unity and set \( T = t^a \). Then
\[ \prod_{i=1}^{a} Z_{C,r}(\zeta_i t) = (P_{C,r,F_q}(0))^a \cdot \exp \left( \sum_{m=1}^{\infty} N_{r,C,F_q}(ma) \frac{T^m}{m} \right). \]

B.1.2. Global Non-Abelian Zeta Functions for Curves

B.1.2.1. Preparations

Let \( C \) be a regular, reduced, irreducible projective curve of genus \( g \) defined over the finite field \( F_q \) with \( q \) elements. Then the rationality of \( \zeta_{C,r,F_q}(s) \) says that there exists a degree \( 2rg \) polynomial \( P_{C,r,F_q}(t) \in \mathbb{Q}[t] \) such that
\[ Z_{C,r,F_q}(t) = \frac{P_{C,r,F_q}(t)}{(1-t^r)(1-q^{-r}t^r)}. \]
Set
\[ P_{C,r,F_q}(t) = \sum_{i=0}^{2rg} a_{C,r,F_q}(i)t^i. \]

By the functional equation for \( \zeta_{C,r,F_q}(t)(s) \), we have
\[ P_{C,r,F_q}(t) = P_{C,r,F_q}(\frac{1}{q^r}) \cdot q^r \cdot t^{2rg}. \]

So, for \( i = 0, 1, \ldots, rg - 1 \), \( a_{C,r,F_q}(2rg - i) = a_{C,r,F_q}(i) \cdot q^{r-g-i} \).

To further determine these coefficients, following Harder and Narasimhan (see e.g. [HN] and [DR]), who first consider the \( \beta \)-series invariants below, we introduce the following invariants:
\[ \alpha_{C,r,F_q}(d) := \sum_{V \in [V] \in \mathcal{M}_{r,C}(d)(F_q)} \frac{q^{h^0(C,V)}}{\#\text{Aut}(V)}, \quad \beta_{C,r,F_q}(d) := \sum_{V \in [V] \in \mathcal{M}_{r,C}(d)(F_q)} \frac{1}{\#\text{Aut}(V)}. \]

and \( \gamma_{C,r,F_q}(d) := \alpha_{C,r,F_q}(d) - \beta_{C,r,F_q}(d) \). One checks that all \( \alpha_{C,r,F_q}(d), \beta_{C,r,F_q}(d) \) and \( \gamma_{C,r,F_q}(d) \)'s may be calculated from \( \alpha_{C,r,F_q}(i), \beta_{C,r,F_q}(j) \) with \( i = 0, \ldots, r(g-1) \) and \( j = 0, \ldots, r-1 \).

An Ugly Formula With the same notation as above,
\[ a_{C,r,F_q}(i) = \begin{cases} 
  \alpha_{C,r,F_q}(d) - \beta_{C,r,F_q}(d), & \text{if } 0 \leq i \leq r-1; \\
  \alpha_{C,r,F_q}(d) - (q^r + 1)\alpha_{C,r,F_q}(d-r) + q^r \beta_{C,r,F_q}(d-r), & \text{if } r \leq i \leq 2r-1; \\
  -\alpha_{C,r,F_q}(d) + (q^r + 1)\alpha_{C,r,F_q}(d-r) + q^r \beta_{C,r,F_q}(d-2r), & \text{if } 2r \leq i \leq r(g-1)-1; \\
  -(q^r + 1)\alpha_{C,r,F_q}(r(g-2)) + q^r \alpha_{C,r,F_q}(r(g-3)) + \alpha_{C,r,F_q}(r(g-1)), & \text{if } i = r(g-1); \\
  \alpha_{C,r,F_q}(d) - (q^r + 1)\alpha_{C,r,F_q}(d-r) + q^r \beta_{C,r,F_q}(d-2r)q^r, & \text{if } r(g-1) + 1 \leq i \leq rg-1; \\
  2q^r \alpha_{C,r,F_q}(r(g-2)) - (q^r + 1)\alpha_{C,r,F_q}(r(g-1)), & \text{if } i = rg.
\end{cases} \]

B.1.2.2. Global Non-Abelian Zeta Functions for Curves

Let \( C \) be a regular, reduced, irreducible projective curve of genus \( g \) defined over a number field \( F \). Let \( S_{\text{bad}} \) be the collection of all infinite places and these finite places of \( F \) at which \( C \) does not have good reductions. As usual, a place \( v \) of \( F \) is called good if \( v \not\in S_{\text{bad}} \).
Thus, in particular, for any good place \( v \) of \( F \), the \( v \)-reduction of \( C \), denoted as \( C_v \), gives a regular, reduced, irreducible projective curve defined over the residue field \( F(v) \) of \( F \) at \( v \). Denote the cardinal number of \( F(v) \) by \( q_v \). Then, by 1.1, we obtain the associated rank \( r \) non-abelian zeta function \( \zeta_{C_v,r,F_{q_v}}(s) \). Moreover, from the rationality of \( \zeta_{C_v,r,F_{q_v}}(s) \), there exists a degree \( 2rg \) polynomial \( P_{C_v,r,F_{q_v}}(t) \in \mathbb{Q}[t] \) such that

\[
Z_{C_v,r,F_{q_v}}(t) = \frac{P_{C_v,r,F_{q_v}}(t)}{(1-t^g)(1-q_v t^g)}.
\]

Clearly, \( P_{C_v,r,F_{q_v}}(0) = \gamma_{C_v,r,F_{q_v}}(0) \neq 0 \). Thus it makes sense to introduce the polynomial \( \tilde{P}_{C_v,r,F_{q_v}}(t) \) with constant term 1 by setting

\[
\tilde{P}_{C_v,r,F(v)}(t) := \frac{P_{C_v,r,F(v)}(t)}{P_{C_v,r,F(v)}(0)}
\]

Now by definition, the rank \( r \) non-abelian zeta function \( \zeta_{C,r,F(s)} \) of \( C \) over \( F \) is the following Euler product

\[
\zeta_{C,r,F}(s) := \prod_{v: \text{good}} \frac{1}{\tilde{P}_{C_v,r,F_{q_v}}(q_v^{-s})}, \quad \text{Re}(s) >> 0.
\]

Clearly, when \( r = 1 \), \( \zeta_{C,r,F}(s) \) coincides with the classical Hasse-Weil zeta function for \( C \) over \( F \).

**Conjecture.** For a regular, reduced, irreducible projective curve \( C \) of genus \( g \) defined over a number field \( F \), its associated rank \( r \) global non-abelian zeta function \( \zeta_{C,r,F}(s) \) admits a meromorphic continuation to the whole complex \( s \)-plane.

Recall that even when \( r = 1 \), i.e., for the classical Hasse-Weil zeta functions, this conjecture is still open. However, in general, we have the following

**Theorem 1.** Let \( C \) be a regular, reduced, irreducible projective curve defined over a number field \( F \). When \( \text{Re}(s) > 1 + g + (r^2 - r)(g - 1) \), the associated rank \( r \) global non-abelian zeta function \( \zeta_{C,r,F}(s) \) converges.

This theorem may be deduced from a result of (Harder-Narasimhan) Siegel on Tamagawa numbers of \( SL_r \), the ugly yet very precise formula for local zeta function in 1.2.1, Clifford Lemma for semi-stable bundles, and Weil’s theorem on the Riemann hypothesis for Artin zeta functions. In fact we have the following

**Proposition 2.** With the same notation as above, when \( q \to \infty \),

(a) For \( 0 \leq d \leq r(g - 1) \),

\[
\frac{\alpha_{C,r,F}(d)}{q^{d/2 + r + r^2(g - 1)}} = O(1);
\]

(b) For all \( d \),

\[
\beta_{C,r,F}(d) = O\left(q^{r^2(g - 1)}\right);
\]

(c)

\[
\frac{q^{r - 1}(g - 1)}{\gamma_{C,r,F}(0)} = O(1).
\]

**B.1.2.3. Working Hypothesis**

Like in the theory for abelian zeta functions, we want to use our non-abelian zeta functions to study non-abelian aspect of arithmetic of curves. Motivated by the classical analytic class number formula for Dedekind zeta functions and its counterpart BSD conjecture for Hasse-Weil zeta functions of elliptic curves, we expect that our non-abelian zeta function could be used to understand the Weil-Petersson volumes of moduli spaces of stable bundles.

For doing so, we then also need to introduce local factors for ‘bad’ places. This may be done as follows. For \( \Gamma \)-factors, we take these coming from the functional equation for \( \zeta_F(r s) \cdot \zeta_F(r(s - 1)) \), where \( \zeta_F(s) \) denotes the standard Dedekind zeta function for \( F \); while for finite bad places, first, use the semi-stable reduction for curves to find a semi-stable model for \( C \), then use Seshadri’s moduli spaces of parabolic bundles to construct polynomials for singular fibers, which usually have degree lower than \( 2rg \). With all this being done, we then
can introduce the so-called completed rank \( r \) non-abelian zeta function for \( C \) over \( F \), or better, the completed rank \( r \) non-abelian zeta function \( \xi_{X,r,\mathcal{O}_F}(s) \) for a semi-stable model \( X \to \text{Spec}(\mathcal{O}_F) \) of \( C \). Here \( \mathcal{O}_F \) denotes the ring of integers of \( F \). (If necessary, we take a finite extension of \( F \).)

**Conjecture.** \( \xi_{X,r,\mathcal{O}_F}(s) \) is holomorphic and satisfies the functional equation

\[
\xi_{X,r,\mathcal{O}_F}(s) = \pm \xi_{X,r,\mathcal{O}_F}(1 + \frac{1}{r} - s).
\]

Moreover, we expect that for certain classes of curves, the inverse Mellin transform of our non-abelian zeta functions are naturally associated to certain modular forms of weight \( 1 + \frac{1}{r} \).

**Example.** For elliptic curves \( E \) defined over \( \mathbb{Q} \), we obtain the following ‘absolute Euler product’ for rank 2 zeta functions of elliptic curves

\[
\zeta_{E,2,\mathbb{Q}}(s) = \zeta_2(s) = \prod_{p \text{ prime}} \frac{1}{1 + (p-1)p^{-s} + (2p-4)p^{-2s} + (p^2-p)p^{-3s} + p^{2s}p^{-4s}}.
\]

At this point, it may be better to recall the following result of Andrianov ([An]). (We thank Kohnen for drawing our attention to this point.) The so-called genus two spinor \( L \)-function stands in the form

\[
\prod_p \frac{1}{1 - \lambda(p)p^{-s} + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})p^{-2s} - \lambda(p)p^{2k-3}p^{-3s} + p^{4k-6}p^{-4s}}.
\]

Clearly, if we set \( k = 2 \), \( \lambda(p) = 1 - p \) and \( \lambda(p^2) = p^2 - 4p + 4 \), we see that formally the above two zeta functions coincide. This suggests that there might be a close relation between them. The following is a speculation I made after the discussion with Deninger and Kohnen.

The relation between the above two zeta functions should be in the same style as the Shimura correspondence for half weight and integral weight modular forms.

To convince the reader, let me point out the following facts:

1. Andrianov’s zetas have a Hecke theory, are coming from certain weight 2 modular forms, and have the local factors

\[
1 - \lambda(p)t + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})t^2 - \lambda(p)p^{2k-3}t^3 + p^{4k-6}t^4
\]

\[
1 - p^{2k-4}t^2
\]

2. Our working hypothesis concerning weight 3/2 modular forms are made mainly from the fact that our local factor takes the form

\[
1 + (p-1)t + (2p-4)t^2 + (p^2-p)t^3 + p^2t^4
\]

\[
(1-t^2)(1-p^2t^2)
\]

in which an additional factor \( 1 - p^2t^2 \) appears in the denominator.

**B.1.3. Refined Brill-Noether Locus for Elliptic Curves: Towards A Reciprocity Law**

1.3.1 Results of Atiyah

Let \( E \) be an elliptic curve defined over \( \overline{\mathbb{F}}_q \), an algebraic closure of the finite field \( \mathbb{F}_q \) with \( q \)-elements.

Recall that a vector bundle \( V \) on \( E \) is called indecomposable if \( V \) is not the direct sum of two proper subbundles, and that every vector bundle on \( E \) may be written as a direct sum of indecomposable bundles, where the summands and their multiplicities are uniquely determined up to isomorphism. Thus to understand vector bundles, it suffices to study the indecomposable ones. To this end, we have the following result of Atiyah [At]. In the sequel, for simplicity, we always assume that the characteristic of \( \mathbb{F}_q \) is strictly bigger than the rank of \( V \).

**Theorem 1.** (Atiyah) (a) For any \( r \geq 1 \), there is a unique indecomposable vector bundle \( I_r \) of rank \( r \) over \( E \), all of whose Jordan-Hölder constituents are isomorphic to \( \mathcal{O}_E \). Moreover, the bundle \( I_r \) has a canonical filtration

\[
\{0\} \subset F^1 \subset \ldots \subset F^r = I_r
\]

28
with $F^i = I_i$ and $F^{i+1}/F^i = \mathcal{O}_E$;
(b) For any $r \geq 1$ and any integer $a$, relative prime to $r$ and each line bundle $\lambda$ over $E$ of degree $a$, there exists up to isomorphism a unique indecomposable bundle $W_r(a; \lambda)$ over $E$ of rank $r$ with $\lambda$ the determinant;
(c) The bundle $I_r(W_r(a; \lambda)) = I_r \otimes W_r(a; \lambda)$ is indecomposable and every indecomposable bundle is isomorphic to $I_r(W_r(a; \lambda))$ for a suitable choice of $r, r', \lambda$. Every bundle $V$ over $E$ is a direct sum of vector bundles of the form $I_r(W_r(a; \lambda))$, for suitable choices of $r_i, r_i'$ and $\lambda_i$. Moreover, the triples $(r_i, r_i', \lambda_i)$ are uniquely specified up to permutation by the isomorphism type of $V$.

Here note in particular that $W_r(0, \lambda) \simeq \lambda$, and that indeed $I_r(W_r(a; \lambda))$ is the unique indecomposable bundle of rank $rr'$ such that all of whose successive quotients in the Jordan-Hölder filtration are isomorphic to $W_r(a; \lambda)$. Now for a vector bundle $V$ over $E$, define its slope $\mu(V)$ by $\mu(V) := \deg(V)/\rank(V)$. Then, $I_r(W_r(a; \lambda))$ is semi-stable with $\mu(I_r(W_r(a; \lambda))) = a/r'$.

**Theorem 2.** (a) (Atiyah) Every bundle $V$ over $E$ is isomorphic to a direct sum $\bigoplus_i V_i$ of semi-stable bundles, where $\mu(V_i) > \mu(V_{i+1})$;
(b) (Atiyah) Let $V$ be a semi-stable bundle over $E$ with slope $\mu(V) = a/r'$ where $r'$ is a positive integer and $a$ is an integer relatively prime to $r'$. Then $V$ is a direct sum of bundles of the form $I_r(W_r(a; \lambda))$, where $\lambda$ is a line bundle of degree $a$;
(c) (Atiyah, Mumford-Seshadri) There exists a natural projective algebraic variety structure on

$$\mathcal{M}_{E,r}(\lambda) := \{ V : \text{semi-stable, } \det V = \lambda, \rank(V) = r \}/\sim_S.$$ 

Moreover, if $\lambda \in \text{Pic}^0(E)$, then $\mathcal{M}_{E,r}(\lambda)$ is simply the projective space $\mathbf{P}^{r-1}_{\mathbf{F}_q}$.

### 1.3.2. Refined Brill-Noether Locus

Now let $E$ be an elliptic curve defined over a finite field $\mathbf{F}_q$. Then over $\overline{E} = E \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$, from 1.3.1, we have the moduli spaces $\mathcal{M}_{E, r}(\lambda)$ (resp. $\mathcal{M}_{E, r}(d)$) of semi-stable bundles of rank $r$ with determinant $\lambda$ (resp. degree $d$) over $\overline{E}$. As algebraic varieties, we may consider $\mathbf{F}_q$-rational points of these moduli spaces. Clearly, by definition, these rational points of moduli spaces correspond exactly to these classes of semi-stable bundles which themselves are defined over $\mathbf{F}_q$. (In the case for $\mathcal{M}_{E, r}(\lambda)$, $\lambda$ is assumed to be rational over $\mathbf{F}_q$.) Thus for simplicity, we simply write $\mathcal{M}_{E, r}(\lambda)$ or $\mathcal{M}_{E, r}(d)$ for the corresponding subsets of $\mathbf{F}_q$-rational points. For example, we then simply write $\text{Pic}^0(E)$ for $\text{Pic}^0(E)(\mathbf{F}_q)$.

Note that if $V$ is semi-stable with strictly positive degree $d$, then $h^0(E, V) = d$. Hence the standard Brill-Noether locus is either the whole space or empty. In this way, we are lead to study the case when $d = 0$.

For this, recall that for $\lambda \in \text{Pic}^0(E)$,

$$\mathcal{M}_{E, r}(\lambda) = \{ V : \text{semi-stable, } \rank(V) = r, \det(V) = \lambda \}/\sim_S$$

is identified with

$$\{ V = \bigoplus_{i=1}^{r} L_i : \otimes_i L_i = \lambda, L_i \in \text{Pic}^0(E), i = 1, \ldots, r \}/\sim_{\text{iso}} \simeq \mathbf{P}^{r-1}$$

where $/\sim_{\text{iso}}$ means modulo isomorphisms.

Now introduce the standard Brill-Noether locus

$$W_{E, r}^a(\lambda) := \{ [V] \in \mathcal{M}_{E, r}(\lambda) : h^0(E, \text{gr}(V)) \geq a \}$$

and its ‘stratification’ by

$$W_{E, r}^a(\lambda)^0 := \{ [V] \in W_{E, r}(\lambda) : h^0(E, \text{gr}(V)) = a \} = W_{E, r}^a(\lambda) \setminus \bigcup_{b \geq a+1} W_{E, r}^b(\lambda).$$

One checks easily that $W_{E, r}^a(\lambda) \simeq \mathbf{P}^{(r-a)-1}$, $W_{E, r+1}(\lambda) \simeq W_{E, r}^a(\lambda)$, and $W_{E, r+1}(\lambda)^0 \simeq W_{E, r}^a(\lambda)^0$. 29
The Brill-Noether theory is based on the consideration of $h^0$. But in the case for elliptic curves, for arithmetic consideration, such a theory is not fine enough: not only $h^0$ plays a crucial role, the automorphism groups are important as well. Based on this, we introduce, for a fixed $(k+1)$-tuple non-negative integers $(a_0; a_1, \ldots, a_k)$, the subvariety of $W_{E,r}^{a_0}$ by setting

$$W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda) := \{[V] \in W_{E,r}^{a_0}(\lambda) : \text{gr}(V) = \mathcal{O}_E^{(a_0)} \oplus \bigoplus_{i=1}^k \mathcal{O}_E^{(a_i)} \otimes L_i^{\otimes a_i} = \lambda, \ L_i \in \text{Pic}^0(E), \ i = 1, \ldots, k \}.$$  

Moreover, we define the associated ‘stratification’ by setting

$$W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda)^0 := \{[V] \in W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda) : \#(\mathcal{O}_E, L_1, \ldots, L_k) = k + 1 \}.$$  

Easily we see that $W_{E,r}^{a_0+1; a_1, \ldots, a_k}(\lambda) \simeq W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda),$ $W_{E,r+1}^{a_0; a_1, \ldots, a_k}(\lambda)^0 \simeq W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda)^0$, and $M_{E,r}(\lambda) = \bigcup_{a_0; a_1, \ldots, a_k} W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda)^0$, where the union is a disjoint one.

Now recall that for elliptic curves $E$,

1. The quotient space $E(n)/S_n$ is isomorphic to the $\mathbb{P}^{n-1}$-bundle over $E$; and
2. The quotient of $E(n-1)/S_n$ is isomorphic to $\mathbb{P}^{n-1}$. Here we embed $E(n-1)$ as a subspace of $E(n)$ under the map:

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n)$$

with $x_n = \lambda - (x_1 + x_2 + \ldots + x_{n-1})$. Thus, $W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda)$ may be described explicitly as follows.

**Proposition.** With the same notation as above, regroup $(a_0; a_1, \ldots, a_k)$ as $(a_0; b_1^{(s_1)}, \ldots, b_l^{(s_l)})$ with the condition that $b_1 > b_2 > \ldots > b_l$ and $s_1, s_2, \ldots, s_l \in \mathbb{Z}_{>0}$, then

1. if $b_l = 1$,

$$W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda) \simeq \prod_{i=1}^{l-1} \mathbb{P}^{s_i-1}_E \times \mathbb{P}^{s_l};$$

2. if $b_l > 1$,

$$W_{E,r}^{a_0; a_1, \ldots, a_k}(\lambda) \simeq \prod_{i=1}^l \mathbb{P}^{s_i}_E.$$  

**Remark.** When $\lambda = \mathcal{O}_E$, the refined Brill-Noether loci $W_{E,r}^{a_0; a_1, \ldots, a_k}(\mathcal{O}_E)$ are isomorphic to products of (copies of) projective bundles over $E$ and (copies of) projective spaces, which are special subvarieties in $M_{E,r}(\mathcal{O}_E) = \mathbb{P}^{r-1}$. It appears that the intersections among these refined Brill-Noether loci are quite interesting. So define the Brill-Noether tautological ring $BN_{E,r}(\mathcal{O}_E)$ to be the subring generated by all the associated refined Brill-Noether loci (in the corresponding Chow ring). What can we say about it? As an example, we consider cases when $r = 2, 3$.

1. If $r = 2$, then this ring consists of only two elements: 1-dimensional one $W_{E,2}^{2,0}(\mathcal{O}_E) = \{[\mathcal{O}_E \oplus \mathcal{O}_E]\}$ and the whole $\mathbb{P}^1$. So everything is simple;
2. If $r = 3$, then (generators of) this ring contains five elements: 2 of 0-dimensional objects: $W_{E,3}^{3,0}(\mathcal{O}_E) = \{[\mathcal{O}_E]\}$ and $W_{E,3}^{1,2}(\mathcal{O}_E) = \{[\mathcal{O}_E \oplus T_2^{(2)}] : T_2 \in E_2\}$ containing 4 elements; 2 of 1-dimensional objects:

$$W_{E,3}^{1,1,1} = \{[\mathcal{O}_E \oplus L \otimes L^{-1}] : L \in \text{Pic}^0(E)\} \simeq \mathbb{P}^1,$$

a degree 2 projective line contained in $\mathbb{P}^2 = M_{E,3}(\mathcal{O}_E)$; and

$$W_{E,3}^{0,2,1} = \{[T_2 \otimes L^{-2}] : L \in \text{Pic}^0(E)\} \text{ a degree 3 curve which is isomorphic to } E; \text{ and finally the whole space. Moreover, the intersection of } W_{E,3}^{1,1,1} = \mathbb{P}^1 \text{ and } W_{E,3}^{0,2,1} = E \text{ are supported on 0-dimensional locus } W_{E,3}^{1,1,1}, \text{ with the multiplicity 3 on the single point locus } W_{E,3}^{3,0}(\mathcal{O}_E) \text{ and 1 on the completment of the points in } W_{E,3}^{1,1,1}.\$$

1.3.3. Towards A Reciprocity Law: Measuring Refined Brill-Noether Locus Arithmetically
To measure the Brill-Noether locus, we introduce the following arithmetic invariant $\alpha_{E,r}(\lambda)$ by setting

$$\alpha_{E,r}(\lambda) := \sum_{V \in [V] \in M_{E,r}(\lambda)} \frac{q^{h^{0}(E,V)}}{\#\text{Aut}(V)}.$$  

Also set

$$\alpha_{\lambda}^{a_0+1,a_1,\ldots,a_k}(\lambda) := \sum_{V \in [V] \in W_{E,r}^{a_0+1,a_1,\ldots,a_k}(\lambda)} \frac{q^{h^{0}(E,V)}}{\#\text{Aut}(V)}.$$  

Before going further, we remark that above, we write $V \in [V]$ in the summation. This is because in each $S$-equivalence class $[V]$, there are usually more than one vector bundles $V$. For example, $[\mathcal{O}_{E}^{(1)}]$, $\mathcal{O}_{E}^{(2)} \oplus I_2$, $I_2 \oplus I_2$, $\mathcal{O}_{E} \oplus I_3$, and $I_4$

Due to the importance of automorphism groups, following Harder-Narasimhan, and Desale-Ramanan, we introduce the following $\beta$-series invariants $\beta_{E,r}(d)$, $\beta_{E,r}(\lambda)$ and $\beta_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda)$ by setting

$$\beta_{E,r}(d) := \sum_{V \in [V] \in M_{E,r}(d)} \frac{1}{\#\text{Aut}(V)}, \quad \beta_{E,r}(\lambda) := \sum_{V \in [V] \in M_{E,r}(\lambda)} \frac{1}{\#\text{Aut}(V)},$$

and

$$\beta_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda) := \sum_{V \in [V] \in W_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda)} \frac{1}{\#\text{Aut}(V)}.$$  

In particular, we have the following deep

**Theorem.** ([HN] & [DR]) For all $\lambda, \lambda' \in \text{Pic}^{d}(E)$,

$$\beta_{E,r}(\lambda) = \beta_{E,r}(\lambda').$$

Moreover,

$$N_{1} \cdot \beta_{E,r}(\lambda) = \frac{N_{1}}{q-1} \cdot \prod_{i=2}^{r} \zeta_{E}(i) - \sum_{\Sigma_{i,r} = \lambda, i = d_{i} \geq 1} \prod_{i < j}^{r} \beta_{E,r}(d_{i}) \frac{1}{q^{2} - q^{2}}.$$  

Here $N_{1}$ denotes $[E] := [E(\mathbb{F}_{q})]$ and $\zeta_{E}(s)$ denotes the Artin zeta function for elliptic curve $E/\mathbb{F}_{q}$.

**Remark.** I would like to thank Ueno here, who many years ago draw my attentions to Atiyah and Bott’s comments about their Morse theoretical approach and Harder-Narasimhan’s adelic approach towards Poincaré polynomials of the associated moduli spaces.

Thus, we are lead to introduce the $\gamma$-series invariants $\gamma_{E,r}(\lambda)$ and $\gamma_{E,r}^{a_0+1,a_1,\ldots,a_k}(\lambda)$ by setting

$$\gamma := \alpha - \beta.$$  

That is to say,

$$\gamma_{E,r}(\lambda) := \sum_{V \in [V] \in M_{E,r}(\lambda)} \frac{q^{h^{0}(E,V)} - 1}{\#\text{Aut}(V)}, \quad \gamma_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda) := \sum_{V \in [V] \in W_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda)} \frac{q^{h^{0}(E,V)} - 1}{\#\text{Aut}(V)}.$$  

Clearly, for $\lambda \in \text{Pic}^{0}(E),$

$$\alpha_{E,r}(\lambda) = \sum_{a_0,a_1,\ldots,a_k} \alpha_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda), \quad \beta_{E,r}(\lambda) = \sum_{a_0,a_1,\ldots,a_k} \beta_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda)$$
and hence
\[ \gamma_{E,r}(\lambda) = \sum_{a_0:a_1,\ldots,a_k} \gamma_{E,r}^{a_0,a_1,\ldots,a_k}(\lambda). \]

Motivated by the above theorem, we make the following

**Conjecture.** For all \( \lambda \in \text{Pic}^0(E) \), \( \alpha_{E,r}(\lambda) = \alpha_{E,r}(\mathcal{O}_E) \). Hence also \( \gamma_{E,r}(\lambda) = \gamma_{E,r}(\mathcal{O}_E) \).

### 1.3.4. Examples In Ranks Two and Three: A Precise Reciprocity Law

Let \( E \) be an elliptic curve defined over the finite field \( \mathbb{F}_q \).

(i) If rank \( r \) is two, we need only to calculate \( \beta_{E,2}(0), \beta_{E,2}(1) \) and \( \gamma_{E,r}(0) \).

First consider \( \beta_{E,2}(0) \). By our discussion on Brill-Noether locus, it suffices to calculate \( \beta_{E,2}(\mathcal{O}_E) \). Now
\[ \mathcal{M}_{E,2}(\mathcal{O}_E) = W_{E,2}^{2,0}(\mathcal{O}_E)^0 \cup W_{E,2}^{0,2}(\mathcal{O}_E)^0 \cup W_{E,2}^{0,1,1}(\mathcal{O}_E)^0. \]
Clearly,
\[ W_{E,2}^{2,0}(\mathcal{O}_E)^0 = \{ [V] : \text{gr}(V) = \mathcal{O}_E^{(2)} \} \]
consisting of just 1 element;
\[ W_{E,2}^{0,2}(\mathcal{O}_E)^0 = \{ [V] : \text{gr}(V) = T_{2}^{(2)}, T_{2} \in E_{2}, T_{2} \neq \mathcal{O}_E \} \]
consisting of 3 elements coming from non-trivial \( T_{2} \in E_{2}, 2\)-torsion subgroup of \( E \); while
\[ W_{E,2}^{0,1,1}(\mathcal{O}_E)^0 = \{ [V] : \text{gr}(V) = L \oplus L^{-1}, L \in \text{Pic}^0(E), L \neq L^{-1} \} \]
is simply the complement of the above 4 points in \( P^1 \). With this, one checks that
\[ \beta_{E,2}(0) = \left( \frac{1}{(q^2-1)(q^2-q)} + \frac{1}{(q-1)q} \right) + 3 \left( \frac{1}{(q^2-1)(q^2-q)} + \frac{1}{(q-1)q} \right) + (q+1-(3+1)) \frac{1}{(q-1)^2} = \frac{q+3}{q^2-1}. \]

And hence
\[ \beta_{E,2}(0) = N_1 \cdot \frac{q+3}{q^2-1}. \]

As for \( \beta_{E,2}(1) \), it is very simple, since any degree one rank two semi-stable bundle is stable. Moreover, by the result of Atiyah cited in 1.3.1, there is exactly one stable rank two bundle whose determinant is the fixed line bundle. Thus
\[ \beta_{E,2}(1) = N_1 \cdot \frac{1}{q-1}. \]

Finally, we study \( \gamma_{E,2}(0) \). Clearly if \( \lambda \neq \mathcal{O}_E \), then \( \gamma_{E,2}(\lambda) \) is supported on
\[ W_{E,2}^{1,1}(\lambda) = \{ [V] : \text{gr}(V) = \mathcal{O}_E \oplus \lambda \} \]
consisting only one element with \( V = \text{gr}(V) = \mathcal{O}_E \oplus \lambda \). So
\[ \gamma_{E,2}(\lambda) = \frac{q-1}{(q-1)^2} = \frac{1}{q-1}. \]

On the other hand, \( \gamma_{E,2}(\mathcal{O}_E) \) is supported on
\[ W_{E,2}^{2,0}(\mathcal{O}_E) = \{ [V] : \text{gr}(V) = \mathcal{O}_E^{(2)} \} \]
consisting only one element too. Since in the class $[V]$ with $\text{gr}(V) = \mathcal{O}_E^{(2)}$, there are two elements, i.e., $\mathcal{O}_E^{(2)}$ and $I_2$,

$$\gamma_{E,2}(\mathcal{O}_E) = \frac{q^2 - 1}{(q^2 - 1)(q^2 - q)} + \frac{q - 1}{(q - 1)q} = \frac{1}{q - 1} = \beta_{E,1}(\mathcal{O}_E).$$

Thus we have the following

**Proposition.** With the same notation as above,

$$Z_{E,2,\mathcal{F}_s}(t) = \frac{N_1}{q - 1} \cdot \frac{1 + (q - 1)t + (2q - 4)t^2 + (q^2 - q)t^3 + q^2t^4}{(1 - t^2)(1 - q^2t^2)}.$$

This then gives the beautiful absolute Euler product mentioned in 1.2.3.

(II) When the rank is three, first, by the fact that $\text{Aut}(\mathcal{O}_E \oplus I_2) = (q - 1)^2q^3$, we have

$$\sum_{V, \text{gr}(V) = \mathcal{O}_E^{(2)}} \frac{1}{\# \text{Aut}(V)} = \frac{1}{(q^2 - 1)(q^2 - q)} + \frac{1}{(q - 1)q}$$

and

$$\sum_{W, \text{gr}(W) = \mathcal{O}_E^{(3)}} \frac{q^{6h}(E, V) - 1}{\# \text{Aut}(V)} = \frac{q^3 - 1}{(q^3 - 1)(q^3 - q)(q^3 - q^2)} + \frac{q^2 - 1}{(q - 1)^2q^3} + \frac{q - 1}{(q - 1)q^2}.$$

Consequently,

$$\gamma_{E,3}(0) = N_1 \cdot \gamma_{E,3}(\mathcal{O}_E) = N_1 \cdot \beta_{E,3}(\mathcal{O}_E) = N_1 \cdot \frac{q + 3}{q^3 - 1}.$$

So we are left to study $\beta_{E,3}(d)$, $d = 0, 1, 2$. Easily,

$$\beta_{E,3}(1) = \beta_{E,3}(2) = N_1 \cdot \frac{1}{q - 1}$$

since here all semi-stable bundles become stable. Thus we are led to consider only $\beta_{E,3}(0)$, and hence $\beta_{E,3}(\lambda)$ for any $\lambda \neq \mathcal{O}_E$. (Despite the fact that $\beta_{E,r}(\lambda) = \beta_{E,r}(\mathcal{O}_E)$ for any $\lambda \in \text{Pic}^0(E)$, in practice, the calculation of $\beta_{E,r}(\lambda)$ with $\lambda \neq \mathcal{O}_E$ is easier than that for $\beta_{E,r}(\mathcal{O}_E)$.)

Now

$$\mathcal{M}_{E,3}(\lambda) = \left(\mathcal{W}_{E,3}^{2;1}(\lambda)^0 \cup \mathcal{W}_{E,3}^{1;2}(\lambda)^0 \cup \mathcal{W}_{E,3}^{1;1,1}(\lambda)^0\right) \cup \mathcal{W}_{E,3}^{0;3}(\lambda)^0 \cup \mathcal{W}_{E,3}^{0;2,1}(\lambda)^0 \cup \mathcal{W}_{E,3}^{0;1,1,1}(\lambda)^0.$$

Moreover, we have

1. $\mathcal{W}_{E,3}^{2;1}(\lambda)^0$ consists of a single class $[V]$, i.e., the one with $\text{gr}(V) = \mathcal{O}_E^{(2)} \oplus \lambda$, which contains two vector bundles, i.e., $\mathcal{O}_E^{(2)} \oplus \lambda$ and $I_2 \oplus \lambda$;

2. $\mathcal{W}_{E,3}^{2;1}(\lambda)^0 \cup \mathcal{W}_{E,3}^{1;2}(\lambda)^0 \cup \mathcal{W}_{E,3}^{1;1,1}(\lambda)^0 \simeq \mathbb{P}^1$ with $\mathcal{W}_{E,3}^{1;2}(\lambda)^0$ consists of 4 classes $[V]$, i.e., these such that $\text{gr}(V) = \mathcal{O}_E^{(2)} \oplus \left(\lambda^\dagger\right)^{(2)}$, where $\lambda^\dagger$ denotes any of the four square roots of $\lambda$. Clearly then in each class $[V]$, there are also two vector bundles $\mathcal{O}_E^{(2)} \oplus \lambda \oplus \lambda^\dagger$.

3. $\mathcal{W}_{E,3}^{0;3}(\lambda)^0 \cup \mathcal{W}_{E,3}^{0;2,1}(\lambda)^0 \cup \mathcal{W}_{E,3}^{0;1,1,1}(\lambda)^0 = \mathbb{P}^2 \setminus \mathbb{P}^1$.

3.a) $\mathcal{W}_{E,3}^{0;3}(\lambda)^0$ consists of 9 classes $[V]$, i.e., these $[V]$ with $\text{gr}(V) = \left(\lambda^\dagger\right)^{(3)}$ where $\lambda^\dagger$ denotes any of the 9 triple roots of $\lambda$. Moreover, in each $[V]$, there are three bundles, i.e., $\left(\lambda^\dagger\right)^{(3)}$, $\lambda^\dagger \oplus I_2 \oplus \lambda^\dagger$ and $I_3 \oplus \lambda^\dagger$.

3.b) $\left(\mathcal{W}_{E,3}^{2;1}(\lambda)^0 \cup \mathcal{W}_{E,3}^{1;2}(\lambda)^0\right) \cup \left(\mathcal{W}_{E,3}^{0;3}(\lambda)^0 \cup \mathcal{W}_{E,3}^{0;2,1}(\lambda)^0\right)$ is isomorphic to $E$. Moreover, each class $[V]$ in $\mathcal{W}_{E,3}^{0;2,1}(\lambda)^0$ consists of two bundles, i.e., $L^{(2)} \oplus \lambda \oplus L^{-2}$ and $I_2 \oplus L \oplus \lambda \oplus L^{-2}$ when $\text{gr}(V) = L^{(2)} \oplus \lambda \oplus L^{-2}$.

(One checks that in fact the refined Brill-Noether loci $\mathbb{P}^2$ and $E$ appeared above are embedded in $\mathbb{P}^2$ as degree 2 and 3 regular curves. And hence the intersection should be 6: The intersection points are at $[V]$.)
with \( \text{gr}(V) = \mathcal{O}_E^{(2)} \oplus \lambda \) with multiplicity 2, and \( \mathcal{O}_E \oplus (\lambda^2)^{(2)} \) corresponding to four square roots of \( \lambda \) with multiplicity one. That is to say, the intersection actually are supported on \( W_{E,3}^{2,1}(\lambda)^0 \cup W_{E,3}^{1,2}(\lambda)^0 \). So it would be very interesting in general to study the intersections of the refined Brill-Noether loci as well, as stated in 1.3.2.)

From this analysis, we conclude that

\[
\beta_{E,3}(\lambda) = \left( \frac{1}{(q^2 - 1)(q^2 - q)(q - 1)} + \frac{1}{(q - 1)q(q - 1)} \right) + 4 \left( \frac{1}{(q^2 - 1)(q^2 - q)(q - 1)} + \frac{1}{(q - 1)q(q - 1)} \right) + (q - 4) \cdot \left( \frac{1}{(q - 1)^3} \right) 
\]

\[
+ 9 \left( \frac{1}{(q^3 - 1)(q^3 - q)(q^2 - q)} + \frac{1}{(q - 1)^2q^3} + \frac{1}{(q - 1)q^2} \right) + (N_1 - (9 + 4 + 1)) \cdot \left( \frac{1}{(q - 1)(q^2 - 1)(q^2 - q)} + \frac{1}{(q - 1)(q - 1)q} \right) 
\]

\[
+ (q^2 - (N_1 - 4 - 1)) \cdot \frac{1}{(q - 1)^3}.
\]

Therefore, to finally write down the associated non-abelian zeta function, it suffices to use the ugly formula. We leave this to the reader.

### 1.3.5. Why Use only Semi-Stable Bundles

At the first glance, it seems that in the definition of non-abelian zeta functions we should consider all vector bundles, just as what happens in the theory of automorphic \( L \)-functions. However, we here use an example with \( r = 2 \) to indicate the opposite.

Thus we first introduce a new zeta function \( \zeta_{E,r}^{\text{all}}(s) \) by

\[
\zeta_{E,r}^{\text{all}}(s) := \sum_{V: \text{rank}(V) = 2} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} \cdot q^{-sd(V)}.
\]

Then by our discussion on the non-abelian zeta functions associated to semi-stable bundle, we only need to consider the contribution of rank 2 bundles which are not semi-stable.

Assume that \( V \) is not semi-stable of rank 2. Let \( L_2 \) be the line subbundle of \( V \) with maximal degree, then \( V \) is obtained from the extension of \( L_2 := V/L_1 \) by \( L_1 \)

\[
0 \rightarrow L_1 \rightarrow V \rightarrow L_2 \rightarrow 0.
\]

But \( V \) is not semi-stable implies that all such extensions are trivial. Thus \( V = L_1 \oplus L_2 \). For later use, set \( d_i \) to be the degree of \( L_i, i = 1, 2 \). Then \( d_1 + d_2 = d \) the degree of \( V \), and

\[
\# \text{Aut}(V) = (q - 1)^2 \cdot q^{h^0(E,L_1 \otimes L_2^{-1})} = (q - 1)^2 \cdot q^{d_1 - d_2}.
\]

Next we study the the contribution of degree 0 vector bundles of rank 2 which are not semi-stable. Note that the support of the summation should have non-vanishing \( h^0 \). Thus \( V = L_1 \oplus L_2 \) where \( L_1 \in \text{Pic}^{d_1}(E) \) with \( d_1 > 0 \). So the contributions of these bundles are given by

\[
\zeta_{E,2}^{\text{all}}(s) = Z_{E,2}(t)
\]

\[
= \sum_{d=1}^{\infty} \sum_{L_1 \in \text{Pic}^{d_1}(E), L_2 \in \text{Pic}^{-d}(E)} \frac{q^{h^0(L_1)} - 1}{(q - 1)^2 q^{h^0(L_1 \otimes L_2^{-1})}} = \frac{N_1^2}{(q - 1)^2} \cdot \sum_{d=1}^{\infty} \frac{q^d - 1}{q^{2dt}}
\]

\[
= \frac{qN_1^2}{(q^2 - 1)(q - 1)^2}.
\]

34
Now we consider all degree strictly positive rank 2 vector bundles which are not semi-stable. From above we see that \( V = L_1 \oplus L_2 \) with \( d_1 > d_2 \). Thus for \( h^0(E, V) \), there are three cases:

(i) \( d_2 > 0 \). Then \( h^0(E, V) = d_1 \);
(ii) \( d_2 = 0 \). Here there are two subcases, namely, (a) if \( L_2 = \mathcal{O}_E \), then \( h^0(E, V) = d_1 + 1 \); (b) if \( L_2 \neq \mathcal{O}_E \), then \( h^0(E, V) = d_1 \);
(iii) \( d_2 < 0 \). Then \( h^0(E, V) = d_1 \).

Therefore, all in all the contribution of strictly positive degree rank 2 bundles which are not semi-stable to the zeta function \( \zeta_{E,2}^\ell(s) \) is given by

\[
\zeta_{E,2}^\ell(s) = Z_{E,2}^\ell(t) = \left( \sum_{(i)} + \sum_{(ii,a)} + \sum_{(ii,b)} + \sum_{(iii)} \right) \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} t^d
\]

where \( \sum_{(*)} \) means the summation is taken for all vector bundles in case (*). Hence, we have

\[
\sum_{(i)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_1^2 \cdot \sum_{d = 1}^{\infty} \sum_{d_1 + d_2 = d, d > 0} \frac{q^d - 1}{(q - 1)^2q^{d_1 - d_2}} t^d,
\]

\[
\sum_{(ii,a)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_1 \cdot \sum_{d = 1}^{\infty} \frac{q^{d+1} - 1}{(q - 1)^2q^d} t^d,
\]

\[
\sum_{(ii,a)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_1(N_1 - 1) \cdot \sum_{d = 1}^{\infty} \frac{q^d - 1}{(q - 1)^2q^d} t^d,
\]

\[
\sum_{(iii)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_2^2 \cdot \sum_{d = 1}^{\infty} \sum_{d_1 + d_2 = d, d > 0} \frac{q^{d_1} - 1}{(q - 1)^2q^{d_1 - d_2}} t^d.
\]

By a direct calculation, we find that

\[
\sum_{(i)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_1^2 t^3 \cdot \frac{q^2 + q + 1 + q^2 t}{(1 - t^2)(1 - q^2t^2)(q - t)},
\]

\[
\sum_{(ii,a)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_1 t \cdot \frac{q^1 + 1 - t}{(q - t)(1 - t)},
\]

\[
\sum_{(ii,a)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_1(N_1 - 1) \cdot \frac{t}{(q - t)(1 - t)},
\]

\[
\sum_{(iii)} \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} = N_2^2 t \cdot \frac{q^2 + q + 1 - qt}{(q - 1)^2(q^2 - 1)(1 - t)(q - t)}.
\]

Finally we consider the contribution of bundles with strictly negative degree. First we have the following classification according to \( h^0(E, V) \).

(i) \( d_1 > 0 > d_2 \). Then \( h^0(E, V) = d_1 \);
(ii) \( d_1 = 0 > d_2 \). Here two subcases. (a) \( L_1 = \mathcal{O}_E \), then \( h^0(E, V) = 1 \); (b) \( L_1 \neq \mathcal{O}_E \), then \( h^0(V) = 0 \);
(iii) \( 0 > d_1 > d_2 \). Here \( h^0(V) = 0 \).

Thus note that the support of \( h^0(E, V) \) is only on the cases (i) and (ii,a), we see that similarly as before, the contribution of strictly negative degree rank 2 bundles which are not semi-stable to the zeta function is given by

\[
\zeta_{E,2}^{<0}(s) = Z_{E,2}^{<0}(t) = \left( \sum_{(i)} + \sum_{(ii,a)} \right) \frac{q^{h^0(V)} - 1}{\# \text{Aut}(V)} t^d,
\]
which may be checked to be
\[
\zeta_{E,2}^{0}(s) = Z_{E,2}^{0}(t) = N_2^{-1} \sum_{d=1}^{-\infty} \sum_{d_1 > d_2, d_1 + d_2 = d} \frac{q^{d_1} - 1}{(q-1)^2 q^{d_1-d_2} t^d} + N_1 \sum_{d=-1}^{-\infty} \frac{q-1}{(q-1)^2 q^{-d} t^d}
\]

\[
= \frac{N_2^2}{(q-1)^2} \cdot \frac{q}{(qt-1)(q^2-1)} + \frac{N_1}{q-1} \cdot \frac{1}{qt-1}.
\]

I hope now the reader is fully convinced that our definition of non-abelian zeta function by using moduli space of semi-stable bundles is much better: Not only our semi-stable zeta functions have much neat structure, we also have well-behavior geometric and hence arithmetic spaces ready to use. In a certain sense, we think the picture of our non-abelian zeta function is quite similar to that the so-called new forms: Only after removing these not-semi-stable contributions, we can see the intrinsic beautiful structures.

Remark. Another way to introduce non-abelian zeta functions using semi-stable bundles is that when taking the summation, do not take all elements in a single Seshadri equivalence class; instead, choose only one single representative, say the one with maximal automorphism group. We leave the details to the reader.

Appendix to B.1: Weierstrass Groups

Motivated by Kato’s construction of Euler systems for elliptic curves in terms of elements in $K_2$ using torsion points, we here introduce what I call Weierstrass groups using Weierstrass divisors for curves.

1. Weierstrass Divisors

(1.1) Let $M$ be a compact Riemann surface of genus $g \geq 2$. Denote its degree $d$ Picard variety by $\text{Pic}^d(M)$. Fix a Poincaré line bundle $\mathcal{P}_d$ on $M \times \text{Pic}^d(M)$. (One checks easily that our constructions do not depend on a particular choice of the Poincaré line bundle.) Let $\Theta$ be the theta divisor of $\text{Pic}^{g-1}(M)$, i.e., the image of the natural map $M^{g-1} \to \text{Pic}^{g-1}(M)$ defined by $(P_1, \ldots, P_{g-1}) \mapsto [\mathcal{O}_M(P_1 + \ldots + P_{g-1})]$. Here $[\cdot]$ denotes the class defined by $\cdot$. We will view the theta divisor as a pair $(\mathcal{O}_{\text{Pic}^{g-1}(M)}(\Theta), 1_\Theta)$ with $1_\Theta$ the defining section of $\Theta$ via the structure exact sequence $0 \to \mathcal{O}_{\text{Pic}^{g-1}(M)} \to \mathcal{O}_{\text{Pic}^{g-1}(M)}(\Theta)$.

Denote by $p_i$ the $i$-th projection of $M \times M$ to $M$, $i = 1, 2$. Then for any degree $d = g-1+n$ line bundle on $M$, we get a line bundle $p_i^*L(-n\Delta)$ on $M \times M$ which has relative $p_2$-degree $q-1$. Here, $\Delta$ denotes the diagonal divisor on $M \times M$. Hence, we get a classifying map $\phi_L : M \to \text{Pic}^{g-1}(M)$ which makes the following diagram commute:

\[
\begin{array}{ccc}
M \times M & \to & M \times \text{Pic}^{g-1}(M) \\
p_2 & \downarrow & \downarrow \pi \\
M & \phi_L & \text{Pic}^{g-1}(M).
\end{array}
\]

One checks that there are canonical isomorphisms

\[
\lambda_\pi(\mathcal{P}^{g-1}) \simeq \mathcal{O}_{\text{Pic}^{g-1}(M)}(-\Theta)
\]

and

\[
\lambda_p(\phi_L^*L(-n\Delta)) \simeq \phi_L^* \mathcal{O}_{\text{Pic}^{g-1}(M)}(-\Theta).
\]

Here, $\lambda_\pi$ (resp. $\lambda_p$) denotes the Grothendieck-Mumford cohomology determinant with respect to $\pi$ (resp. $p_2$). (See e.g., [L].)

Thus, $\phi_L^*1_\Theta$ gives a canonical holomorphic section of the dual of the line bundle $\lambda_p(\phi_L^*L(-n\Delta))$, which in turn gives an effective divisor $W_L(M)$ on $M$, the so-called Weierstrass divisor associated to $L$.

Example. With the same notation as above, take $L = K_2^{\otimes m}$ with $K_M$ the canonical line bundle of $M$ and $m \in \mathbb{Z}$. Then we get an effective divisor $W_{K_2^{\otimes m}}(M)$ on $M$, which will be called the $m$-th Weierstrass divisor associated to $M$. For simplicity, denote $W_{K_2^{\otimes m}}(M)$ (resp. $\phi_{K_2^{\otimes m}}$) by $W_m(M)$ (resp. $\phi_m$).
One checks easily that the degree of $W_m(M)$ is $g(g - 1)^2(2m - 1)^2$ and we have an isomorphism $O_M(W_m(M)) \simeq K^{\otimes g(g - 1)(2m - 1)^2/2}$. Thus, in particular,

$$f_{m,n} := \frac{(\phi_m^* 1_{\Theta})^{\otimes (2m - 1)^2}}{(\phi_n^* 1_{\Theta})^{\otimes (2m - 1)^2}}$$

gives a canonical meromorphic function on $M$ for all $m, n \in \mathbb{Z}$.

**Remark.** We may also assume that $m \in \frac{1}{2} \mathbb{Z}$. Furthermore, this construction has a relative version as well, for which we assume that $f : X \to B$ is a semi-stable family of curves of genus $g \geq 2$. In that case, we get an effective divisor $(O_X(W_m(f)), 1_{W_m(f)})$ and canonical isomorphism

$$(O_X(W_m(f)), 1_{W_m(f)}) \simeq (O_X(W_1(f)), 1_{W_1(f)})^{\otimes (2m - 1)^2} \otimes (O_X(W_2(f)), 1_{W_2(f)})^{\otimes 4m(1-m)}.
$$

The proof may be given by using Deligne-Riemann-Roch theorem, which in general, implies that we have the following canonical isomorphism:

$$(O_X(W_L(f)), 1_{W_L(f)}) \otimes f^* \lambda_f(L) \simeq L^{\otimes n} \otimes K_f^{\otimes n(n-1)/2}.
$$

(See e.g. [Bu].) To allow $m$ to be a half integer, we then should assume that $f$ has a spin structure. Certainly, without using spin structure, a modified canonical isomorphism, valid for integers, can be given.

### 2. K-Groups

(2.1) Let $M$ be a compact Riemann surface of genus $g \geq 2$. Then by the localization theorem, we get the following exact sequence for $K$-groups

$$K_2(M) \xrightarrow{\lambda} K_2(C(M)) \xrightarrow{\bigoplus_{p \in P} \partial_p} \prod_{p \in M} C^*_p.
$$

Note that the middle term may also be written as $K_2(C(M\setminus S))$ for any finite subset $S$ of $M$, we see that naturally by a theorem of Matsumoto, the Steinberg symbol $\{f_{m,n}, f_{m',n'}\}$ gives a well-defined element in $K_2(C(M))$. Denote the subgroup generated by all $\{f_{m,n}, f_{m',n'}\}$ with $m, n, m', n' \in \mathbb{Z}_{>0}$ in $K_2(C(M))$ as $\Sigma(M)$.

**Definition.** With the same notation as above, the *first Weierstrass group* $W_I(M)$ of $M$ is defined to be the $\lambda$-pull-back of $\Sigma(M)$, i.e., the subgroup $\lambda^{-1}(\Sigma(M))$ of $K_2(M)$.

(2.2) For simplicity, now let $C$ be a regular projective irreducible curve of genus $g \geq 2$ defined over $\mathbb{Q}$. Assume that $C$ has a semi-stable regular module $X$ over $\mathbb{Z}$ as well. Then we have a natural morphism $K_2(X) \xrightarrow{\phi} K_2(M)$. Here $M := C(\mathbb{C})$.

**Conjecture I.** With the same notation as above, $\phi(K_2(X))_\mathbb{Q} = W_I(M)_\mathbb{Q}$.

### 3. Generalized Jacobians

(3.1) Let $C$ be a projective, regular, irreducible curve. Then for any effective divisor $D$, one may canonically construct the so-called generalized Jacobian $J_D(C)$ together with a rational map $f_D : C \to J_D(C)$.

More precisely, let $C_D$ be the group of classes of divisors prime to $D$ modulo these which can be written as $\text{div}(f)$. Let $C^n_D$ be the subgroup of $C_D$ which consists of all elements of degree zero. For each $p_i$ in the support of $D$, the invertible elements modulo those congruent to 1 (mod $D$) form an algebraic group $R_{D,p_i}$ of dimension $n_i$, where $n_i$ is the multiplicity of $p_i$ in $D$. Let $R_D$ be the product of these $R_{D,p_i}$. One checks
easily that \( \mathbf{G}_m \), the multiplicative group of constants naturally embeds into \( R_D \). It is a classical result that we then have the short exact sequence

\[
0 \to R_D/\mathbf{G}_m \to C_D^0 \to J \to 0
\]

where \( J \) denotes the standard Jacobian of \( C \). (See e.g., [S].) Denote \( R_D/\mathbf{G}_m \) simply by \( R_D \).

Now the map \( f_D \) extends naturally to a bijection from \( C_D^0 \) to \( J_D \). In this way the commutative algebraic group \( J_D \) becomes an extension as algebraic groups of the standard Jacobian by the group \( R_D \).

**Example.** Take the field of constants as \( C \) and \( D = W_m(M) \), the \( m \)-th Weierstrass divisor of a compact Riemann surface \( M \) of genus \( g \geq 2 \). By (1.1), \( W_m(M) \) is effective. So we get the associated generalized Jacobian \( J_{W_m(M)} \). Denote it by \( WJ_m(M) \) and call it the \( m \)-th Weierstrass-Jacobian of \( M \). For example, if \( m = 0 \), then \( WJ_0(M) = J(M) \) is the standard Jacobian of \( M \). Moreover, one knows that the dimension of \( R_{W_m(M),p} \) is at most \( g(g + 1)/2 \). For later use denote \( R_{W_m(M)} \) simply by \( R_m \).

(3.2) The above construction works on any base field as well. We leave the detail to the reader while point out that if the curve is defined over a field \( F \), then its associated \( m \)-th Weierstrass divisor is rational over the same field as well. (Obviously, this is not true for the so-called Weierstrass points, which behavior in a rather random way.) As a consequence, by the construction of the generalized Jacobian, we see that the \( m \)-th Weierstrass-Jacobians are also defined over \( F \). (See e.g., [S].)

### 4. Galois Cohomology Groups

(4.1) Let \( K \) be a perfect field, \( \overline{K} \) be an algebraic closure of \( K \) and \( G_{\overline{K}/K} \) be the Galois group of \( \overline{K} \) over \( K \). Then for any \( G_{\overline{K}/K} \)-module \( M \), we have the Galois cohomology groups \( H^0(G_{\overline{K}/K}, M) \) and \( H^1(G_{\overline{K}/K}, M) \) such that if

\[
0 \to M_1 \to M_2 \to M_3 \to 0
\]

is an exact sequence of \( G_{\overline{K}/K} \)-modules, then we get a natural long exact sequence

\[
0 \to H^0(G_{\overline{K}/K}, M_1) \to H^0(G_{\overline{K}/K}, M_2) \to H^0(G_{\overline{K}/K}, M_3) \\
\to H^1(G_{\overline{K}/K}, M_1) \to H^1(G_{\overline{K}/K}, M_2) \to H^1(G_{\overline{K}/K}, M_3).
\]

Moreover, if \( G \) is a subgroup of \( G_{\overline{K}/K} \) of finite index or a finite subgroup, then \( M \) is naturally a \( G \)-module. This leads a restriction map on cohomology \( \text{res} : H^1(G_{\overline{K}/K}, M) \to H^1(G, M) \).

(4.2) Now let \( C \) be a projective, regular irreducible curve defined over a number field \( K \). Then for each place \( p \) of \( K \), fix an extension of \( p \) to \( \overline{K} \), which then gives an embedding \( \overline{K} \subset K_p \) for the \( p \)-adic completion \( K_p \) of \( K \) and a decomposition group \( G_p \subset G_{\overline{K}/K} \).

Now apply the construction in (3.1) to the short exact sequence

\[
0 \to R_m \to WJ_m(C) \to J(C) \to 0
\]

over \( K \). Then we have the following long exact sequence

\[
0 \to R_m(K) \to WJ_m(K) \to J(K) \\
\to H^1(G_{\overline{K}/K}, R_m(K)) \to H^1(G_{\overline{K}/K}, WJ_m(K)) \stackrel{\psi}{\to} H^1(G_{\overline{K}/K}, J(K)).
\]

Similarly, for each place \( p \) of \( K \), we have the following exact sequence

\[
0 \to R_m(K_p) \to WJ_m(K_p) \to J(K_p) \\
\to H^1(G_p, R_m(K_p)) \to H^1(G_p, WJ_m(K_p)) \stackrel{\psi}{\to} H^1(G_p, J(K_p)).
\]

38
Now the natural inclusion $G_p \subset G_{K/K}$ and $\overline{K} \subset \overline{K}_p$ give restriction maps on cohomology, so we arrive at a natural morphism

$$\Phi_m : \psi \left( H^1(G_{K/K}, WJ_m(K)) \right) \rightarrow \prod_{p \in M_K} \psi_p \left( H^1(G_p, WJ_m(K_p)) \right).$$

Here $M_K$ denotes the set of all places over $K$.

**Definition.** With the same notation as above, the second Weierstrass group $W_{II}(C)$ of $C$ is defined to be the subgroup of $H^1(G_{K/K}, J(C)(K))$ generated by all Ker $\Phi_m$, the kernel of $\Phi_m$, i.e., $W_{II}(C) := \langle \text{Ker} \Phi_m : m \in \mathbb{Z}_{>0} \rangle$.

**Conjecture II.** With the same notation as above, the second Weierstrass group $W_{II}(C)$ is finite.

5. Deligne-Beilinson Cohomology

(5.1) Let $C$ be a projective regular curve of genus $g$. Let $P$ be a finite set of $C$. For simplicity, assume that all of them are defined over $\mathbb{R}$. Then we have the associated Deligne-Belinson cohomology group $H^1_D(C \setminus P, \mathbb{R}(1))$ which leads to the following short exact sequence:

$$0 \rightarrow \mathbb{R} \rightarrow H^1_D(C \setminus P, \mathbb{R}(1)) \xrightarrow{\text{div}} \mathbb{R}[P]^0 \rightarrow 0$$

where $\mathbb{R}[P]^0$ denotes (the group of degree zero divisors with support on $P$)$\mathbb{R}$.

The standard cup product on Deligne-Beilinson cohomology leads to a well-defined map:

$$\cup : H^1_D(C \setminus P, \mathbb{R}(1)) \times H^1_D(C \setminus P, \mathbb{R}(1)) \rightarrow H^2_D(C \setminus P, \mathbb{R}(2)).$$

Furthermore, by Hodge theory, there is a canonical short exact sequence

$$0 \rightarrow H^1(C \setminus P, \mathbb{R}(1)) \cap F^1(C \setminus P) \rightarrow H^2_D(C \setminus P, \mathbb{R}(2)) \xrightarrow{p_D} H^2_D(C, \mathbb{R}(2)) \rightarrow 0$$

where $F^1$ denotes the $F^1$-term of the Hodge filtration on $H^1(C \setminus P, \mathbb{C})$.

All this then leads to a well-defined morphism

$$[\cdot, \cdot]_D : \wedge^2 \mathbb{R}[P]^0 \rightarrow H^2_D(C, \mathbb{R}(2)) = H^1(C, \mathbb{R}(1))$$

which make the associated diagram coming from the above two short exact sequences commute. (See e.g. [Bei].)

(5.2) Now applying the above construction with $P$ being the union of the supports of $W_1$, $W_m$ and $W_n$ for $m, n > 0$. Thus for fixed $m, n$, in $\mathbb{R}[P]^0$, we get two elements $\text{div}(f_{1,m})$ and $\text{div}(f_{1,n})$. This then gives $[\text{div}(f_{1,m}), \text{div}(f_{1,n})]_D \in H^1(X, \mathbb{R}(1))$. Thus, by a simple argument using the Stokes formula, we obtain the following

**Lemma.** For any holomorphic differential 1-form $\omega$ on $C$, we have

$$\langle [\text{div}(f_{1,m}), \text{div}(f_{1,n})]_D, \omega \rangle := -\frac{1}{2\pi i} \int \text{div}(f_{1,m}), \text{div}(f_{1,n})]_D \wedge \bar{\omega}$$

$$= -\frac{1}{2\pi i} \int g(D, z) dx(D, z) \wedge (f_{1,n}) \wedge \bar{\omega},$$

Here $g(D, z)$ denotes the Green’s function of $D$ with respect to any fixed normalized (possibly singular) volume form of quasi-hyperbolic type. (See e.g. [We]).

(5.3) With exactly the same notation as in (4.2), then in $H^1(X, \mathbb{R}(1))$ we get a collection of elements $\text{div}(f_{1,m}), \text{div}(f_{1,n})]_D$ for $m, n \in \mathbb{Z}_{>0}$.

**Definition.** With the same notation as above, assume that $C$ is defined over $\mathbb{Z}$. Define the first quasi-Weierstrass group $W'_{-1}(C)$ of $C$ to be the subgroup of $H^1(X, \mathbb{R}(1))$ generated by $[\text{div}(f_{1,m}), \text{div}(f_{1,n})]_D$ for
all \( m, n \in \mathbb{Z}_{\geq 0} \) and call \( W_{-1}(C)_Q \) the \( \text{-first Weierstrass group} W_{-1}(C) \) of \( C \). That is to say, \( W_{-1}(C) := \langle \text{div}(f_{1,m}), \text{div}(f_{1,n}) \rangle : m, n \in \mathbb{Z}_{\geq 0} \rangle_Q \).

**Conjecture III.** With the same notation as above, \( W_{-1}(C)_R \) is the full space, i.e. equals to \( H^1(X, \mathbb{R}(1)) \).

That is to say, Weierstrass divisors should give a new rational structure for \( H^1(X, \mathbb{R}(1)) \), and hence the corresponding regulator should give the leading coefficient of the \( L \)-function of \( C \) at \( s = 0 \), up to rationals.

### B.2. New Non-Abelian Zeta Functions for Number Fields

#### B.2.1. Iwasawa’s ICM Talk on Dedekind Zeta Functions

As for function fields, here we start with a discussion on abelian zeta functions for number fields, i.e., Dedekind zeta functions. However, we will not adapt the classical approach, rather we would like to recall Iwasawa’s interpretation. (Based on the fact that Iwasawa’s original choice of certain auxiliary functions do not naturally lead to any meaningful cohomology, some subtle changes are made.)

Let \( F \) be a number field. Denote by \( S \) the collection of all (unequivalent) normalized places of \( F \). Set \( S_\infty \) to be the collection of all Archimedean places of \( F \) and \( S_{\text{fin}} := S \setminus S_\infty \).

Denote by \( I \) the idele group of \( F \), \( N : I \to \mathbb{R}_{\geq 0} \) and \( \text{deg} : I \to \mathbb{R} \) the norm map and the degree map on ideles respectively. Also introduce the following subgroups of \( I \):

\[
I^0 := \{ (a_v) \in I : \text{deg}(a_v) = 0 \},
\]

\[
F^* := \{ (a_v) \in I^0 : a_v = \alpha \in F \setminus \{0\}, \forall v \in S \},
\]

\[
U := \{ (a_v) \in I^0 : |a_v|_v = 1 \forall v \in S \},
\]

\[
I_{\text{fin}} := \{ (a_v) \in I : a_v = 1 \forall v \in S_{\text{fin}} \},
\]

\[
I_\infty := \{ (a_v) \in I : a_v = 1 \forall v \in S_\infty \}.
\]

Set \( U_{\text{fin}} := U \cap I_{\text{fin}} \). Then, with respect to the natural topology on \( I \), we have

1. \( F \to I \) is discrete and \( I^0/F^* \) is compact. Write \( \text{Pic}(F) = I/F^* \);
2. \( U \to I \) is compact;
3. \( U_{\text{fin}} \to I_{\text{fin}} \) is both open and compact. Moreover, the morphism \( I : [a = (a_v)] \to I(a) := \prod_{v \in S_{\text{fin}}} F_{\mathbb{Q}_v}^{\text{ord}_v(a_v)} \) induces an isomorphism between \( I_{\text{fin}}/U_{\text{fin}} \) and the ideal group of \( F \), where \( P_v \) denotes the maximal ideal of the ring of integers \( \mathcal{O}_F \) corresponding to the place \( v \); and, \( N(a) = \prod_{v \in S} |a_v|_v^{N_v := [\mathbb{F}_v: \mathbb{Q}_v]} = N(I(a))^{-1} \) with \( N(I(a)) \) the norm of the ideal \( I(a) \);
4. \( I = I_{\text{fin}} \times I_\infty \). Hence we may write an idele \( a \) as \( a = a_{\text{fin}} \cdot a_\infty \) with \( a_{\text{fin}} \in I_{\text{fin}} \) and \( a_\infty \in I_\infty \) respectively. In particular, if \( d\mu(a) \) denotes the normalized Haar measure on \( I \) as say in Weil’s Basic Number Theory, we have

\[
d\mu(a) = d\mu(a_{\text{fin}}) \cdot d\mu(a_\infty)
\]

corresponding to the decomposition \( I = I_{\text{fin}} \times I_\infty \).

Set

\[
e(a_{\text{fin}}) := \begin{cases} 1, & \text{if } I(a_{\text{fin}}) \subset \mathcal{O}_F, \\ 0, & \text{otherwise,} \end{cases}
\]

\[
e(a_\infty) = \exp(-\pi \sum_{v \in \mathbb{R}} a_v^2 - 2\pi \sum_{v \in \mathbb{C}} |a_v|^2)
\]

and

\[
e(a_{\text{fin}} \cdot a_\infty) = e(a_{\text{fin}}) \cdot e(a_\infty),
\]

for \( a_{\text{fin}} \in I_{\text{fin}} \) and \( a_\infty \in I_\infty \). Denote by \( \Delta_F \) (the absolute values of) the discriminant of \( F \), \( r_1 \) and \( r_2 \) the number of real and complex places in \( S_\infty \) as usual.

Now we are ready to write down Iwasawa’s interpretation of the Dedekind zeta function for \( F \) in the form suitable for our later study. This goes as follows.

For \( s \in \mathbb{C}, \text{Re}(s) > 1 \),

40
\[
\xi_F(s) := \Delta_F^s (2\pi^{-s} \Gamma(\frac{s}{2}))^{r_1} ((2\pi)^{-s} \Gamma(s))^{r_2} \sum_{0 \neq I \subset \mathcal{O}_F} N(I)^{-s}
\]

\[
= \Delta_F^s \cdot \sum_{0 \neq I \subset \mathcal{O}_F} N(I)^{-s} \int_{v_+ \in F_v, v_+ \in S_{\infty}} \prod_v |t_v|_v^s \exp(-\pi \sum_{v \in \mathbb{R}} t_v^2 - 2\pi \sum_{v \in \mathbb{C}} |t_v|^2) \prod_v d^* t_v
\]

\[
= \Delta_F^s \cdot \sum_{0 \neq I \subset \mathcal{O}_F} N(I)^{-s} \int_{I_\infty} N(a_\infty)^s e(a_\infty) d\mu(a_\infty)
\]

\[
= \Delta_F^s \cdot \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \left( \int_{I_{\text{fin}}} N(a_{\text{fin}})^s e(a_{\text{fin}}) d\mu(a_{\text{fin}}) \cdot \int_{I_\infty} N(a_\infty)^s e(a_\infty) d\mu(a_\infty) \right)
\]

\[
= \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \Delta_F^s \cdot \int_{I_{\text{fin}} \times I_\infty} \left( N(a_{\text{fin}}) N(a_\infty) \right)^s \left( e(a_{\text{fin}}) e(a_\infty) \right) \left( d\mu(a_{\text{fin}}) d\mu(a_\infty) \right)
\]

Now denote by \(d\mu([a])\) the induced Haar measure on the Picard group \(\text{Pic}(F) := I/F^*\). Note that

\[
e(a_{\text{fin}} a_\infty) := \begin{cases} 
e(a_\infty), & \text{if } I(a_{\text{fin}}) \subset \mathcal{O}_F, \\ 0, & \text{otherwise}, \end{cases}
\]

and that by (1) above, \(F^* \hookrightarrow I\) is discrete, (hence taking integration over \(F^*\) means taking summation), we get

\[
\xi_F(s) = \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \Delta_F^s \cdot \int_{I/F^*} \left( \int_{F^*} N(\alpha)^s e(\alpha a) d\mu([a]) \right)
\]

(by the product formula)

\[
= \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \Delta_F^s \cdot \int_{I/F^*} N([a])^s d\mu([a]) \cdot \sum_{\alpha \in F^*} e(\alpha a_{\text{fin}}) e(\alpha a_\infty)
\]

\[
= \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \Delta_F^s \cdot \int_{I/F^*} N([a])^s d\mu([a]) \cdot \sum_{\alpha \in F^*, I(a_{\text{fin}}) \subset \mathcal{O}_F} e(\alpha a_\infty)
\]

Now for an idele \(L = (a_v)\), define the 0-th algebraic cohomology group of the idele \(L^{-1}\) by

\[
H^0(F, L^{-1}) := \{ \alpha \in F^*, \alpha a_v \subset \mathcal{O}_v, \forall v \in S_{\text{fin}} \} = I(L^{-1})
\]

moreover for the associated idele class, introduce its associated geometric 0-th geo-arithmetic cohomology via

\[
h^0(F, L^{-1}) := \log \left( \sum_{\alpha \in H^0(F, L^{-1}) \setminus \{0\}} \exp \left( -\pi \sum_{v \in \mathbb{R}} |g_v \alpha|^2 - 2\pi \sum_{v \in \mathbb{C}} |g_v \alpha|^2 \right) N(L)^s d\mu(L) \right)
\]

With this, then easily, we have

\[
\xi_F(s) = \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \Delta_F^s \cdot \int_{\text{Pic}(F)} \frac{1}{w_F} \sum_{\alpha \in H^0(F, L^{-1}) \setminus \{0\}} \exp \left( -\pi \sum_{v \in \mathbb{R}} |g_v \alpha|^2 - 2\pi \sum_{v \in \mathbb{C}} |g_v \alpha|^2 \right) N(L)^s d\mu(L)
\]

\[
= \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \frac{1}{w_F} \cdot \Delta_F^s \cdot \int_{\text{Pic}(F)} \left( e^{\log_2(1)} N(L)^s d\mu(L) \right)
\]

\[
= \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \frac{1}{w_F} \cdot \Delta_F^s \cdot \int_{\text{Pic}(F)} \left( e^{\log_2(1)} - 1 \right) N(L)^{-s} d\mu(L)
\]

\[
= \frac{1}{\text{vol}(U_{\text{fin}})} \cdot \Delta_F^s \cdot \int_{\text{Pic}(F)} \left( e^{\deg(L)} - 1 \right) \left( e^{\deg(L)} \right)^{-s} d\mu(L).
\]
Here \( w_F \) denotes the number of units of \( F \), and \( \text{Aut}(L)(= w_F) \) denotes the number of automorphisms of \( L \).

It is very clear that, formally, this version of (the completed) Dedekind zeta functions for number fields stands exactly the same as our interpretation of Artin zeta functions for function fields. So to introduce an non-abelian zeta function for number fields, key points are the follows:

1. A suitable stability in terms of intersection for bundles over number fields should be introduced;
2. Stable bundles over number fields should form moduli spaces, over which there exist natural measures; and
3. There should be a geo-ari cohomology such that duality and Riemann-Roch type results hold.

B.2.2. Intersection Stability

B.2.2.1. Classification of Unimodular Lattices: A Global Approach

Even though it has not yet been very popular, intersection stability in arithmetic is indeed quite fundamental, as what we are going to see.

Recall that a full rank lattice \( \Lambda \subset \mathbb{R}^r \) is said to be integral if for any \( x \in \Lambda \), \((x, x)\) is an integer, and that an integral lattice \( \Lambda \) is called unimodular, if the volume of its fundamental domain is one. It is a classical yet still very challenging problem to classify all unimodular lattices.

Roughly speaking, classifications of unimodular lattices consist of two different aspects, i.e., the local and the global one. For the local study, we are mainly interested in enumerating all unimodular lattices, which in recent years proves to be very fruitful. However for the global study, besides the pioneer works done by Minkowski and Siegel, such as the mass formula and asymptotic upper bounds for the numbers of unimodular lattices in terms of volumes of Siegel domains, less progress has been recorded.

One of the main difficulties in the global study is that Siegel domains are hard to be understood. Thus, it seems to be very essential to find a natural method to divide these domains into certain well-behavior blocks. Motivated by what happens for bundles over function fields, in particular, the so-called Mumford stability and the associated Harder-Narasimhan filtration, we introduce the following

**Definition.** A lattice \( \Lambda \) is called stable (resp. semi-stable) if for any proper sublattice \( \Lambda' \),

\[
\text{Vol}(\Lambda')^{\text{rank}(\Lambda)} > \quad (\text{resp.} \geq) \quad \text{Vol}(\Lambda)^{\text{rank}(\Lambda')}.
\]

Standard properties about Harder-Narasimhan filtrations and Jordan-Hölder filtrations hold here as well. That is to say, we have the following:

**Proposition.** Let \( \Lambda \) be a lattice. Then

1. There exists a unique filtration of proper sublattices,

\[
0 = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_s = \Lambda
\]

such that \( \Lambda_s/\Lambda_{s-1} \) is semi-stable and

\[
\text{Vol}(\Lambda_{i+1}/\Lambda_i)^{\text{rank}(\Lambda_s/\Lambda_{i-1})} > \text{Vol}(\Lambda_i/\Lambda_{i-1})^{\text{rank}(\Lambda_{i+1}/\Lambda_i)}.
\]

2. If moreover \( \Lambda \) is semi-stable, then there exists a filtration of proper sublattices,

\[
0 = \Lambda^{i+1} \subset \Lambda^i \subset \ldots \subset \Lambda^0 = \Lambda
\]

such that \( \Lambda^j/\Lambda^{j+1} \) is stable and

\[
\text{Vol}(\Lambda^j/\Lambda^{j+1})^{\text{rank}(\Lambda^{i-1}/\Lambda^i)} = \text{Vol}(\Lambda^{j-1}/\Lambda^j)^{\text{rank}(\Lambda^{i+1}/\Lambda^{j+1})}.
\]

Furthermore, the graded lattice \( \text{Gr}(\Lambda) := \oplus \Lambda^{i-1}/\Lambda^i \), the so-called Jordan-Hölder graded lattice of \( \Lambda \), is uniquely determined by \( \Lambda \).
Thus, in particular, for unimodular lattices, we have the following

**Corollary.** Unimodular lattices are semi-stable. Moreover, a unimodular lattice is stable if and only if it contains no proper unimodular sublattice.

In this sense, to classify all unimodular lattices, it suffices to classify all stable unimodular lattices. This then leads to the following consideration.

Denote by \( \mathcal{M}_{\mathbb{Q}, r}(1) \) the collection, or better, the moduli space, of all rank \( r \) semi-stable lattices of volume one. Then one checks that \( \mathcal{M}_{\mathbb{Q}, r}(1) \) admits a natural metric and is indeed compact.

**Example.** With the help of the reduction theory from geometry of numbers,

\[
\mathcal{M}_{\mathbb{Q}, 2}(1) \simeq \left\{ \left( \begin{array}{cc} a & 0 \\ b & \frac{1}{a} \end{array} \right) : 1 \leq a \leq \sqrt{\frac{2}{\sqrt{3}}} \sqrt{a^2 - a^{-2}} \leq b \leq a - \sqrt{a^2 - a^{-2}} \right\} : \text{SO(2)}.
\]

Moreover,

\[
\left\{ \left( \begin{array}{cc} a & 0 \\ b & \frac{1}{a} \end{array} \right) : 1 \leq a \leq \sqrt{\frac{2}{\sqrt{3}}} \sqrt{a^2 - a^{-2}} \leq b \leq a - \sqrt{a^2 - a^{-2}} \right\}
\]

may be viewed as a closed bounded domain in the upper half plane. As a direct consequence, \( \mathcal{M}_{\mathbb{Q}, 2}(1) \) admits a natural metric as well, induced from the Poincaré metric.

To go back to unimodular lattices, we may now view them naturally as certain special points in our geometric moduli spaces. (Recall that unimodular lattices are integral.) In this way, the problem of classifying unimodular lattices looks very much similar to that of finding rational points in algebraic varieties. Thus, along with the line of Minkowski’s geometry of numbers, the first thing we have to do is to evaluate volumes of these moduli spaces with respect to the associated natural metrics. It is for this purpose that we introduce our non-abelian zeta functions for number fields: Theory of Dedekind zeta functions tells us that, regulators of number fields, or better, volumes of the lattices generated by fundamental units, may be read from the residues of Dedekind zeta functions at the simple poles \( s = 1 \).

**B.2.2.2. Semi-Stable Bundles over Number Fields**

Over general number fields, we may also introduce the intersection-stability for (parabolic \( G \)) bundles according to the following observations:

(1) there exists a well-developed Arakelov theory, from which in particular we have the concept like hermitian vector sheaves, rank and degree;

(2) over each local fields, the relation between 1-PS and weighted filtration is well-understood for reductive groups.

For example, in terms of Arakelov theory, let \((\mathcal{E}, \rho)\) be a hermitian vector sheaf over a number field \( F \), or, better over, the spectrum of the ring of integers. Then the rank and the Arakelov degree makes sense. Introduce the (Arakelov) \( \mu \)-invariant by \( \mu(\mathcal{E}, \rho) := \frac{\deg(\mathcal{E}, \rho)}{\text{rank}(\mathcal{E}, \rho)} \). Then by definition, \((\mathcal{E}, \rho)\) is called semi-stable (resp. stable) if for any metrized vector subsheaf \((\mathcal{E}_1, \rho_1 = \rho|_{\mathcal{E}_1, \infty})\),

\[
\mu(\mathcal{E}_1, \rho_1) \leq \text{ (resp. <) } \mu(\mathcal{F}, \rho).
\]

Moreover, just as in 2.2.1 over \( \mathbb{Q} \), in general, standard properties about Harder-Narasimhan filtration and Jordan-Hölder filtrations holds here as well, based on the fact that, as a subset of \( \mathbb{R} \),

\[
\{ \mu(\mathcal{E}_1, \rho_1) : \mathcal{E}_1 \text{ is a vector subsheaf of } \mathcal{E}, \text{ and } \rho_1 = \rho|_{\mathcal{E}_1, \infty} \}
\]

is discrete and bounded from above.

We will leave the corresponding generalization to parabolic \( G \)-bundles to the reader. Instead, we want to introduce an adelic version of the stability with the aim to construct the corresponding moduli spaces and the associated Tamagawa measures.
B.2.2.3. Adelic Moduli and Its Associated Tamagawa Measure

As above, let $F$ be a number field, i.e., a finite extension of $\mathbb{Q}$. Denote by $S = S_F$ the collection of all (unequivalent) normalized places of $F$. Set $S_\infty$ be the collection of all Archimedean places in $S$, and $S_{\text{fin}} := S \setminus S_\infty$.

For each $v \in S$, denote by $F_v$ the $v$-completion of $F$. If $v \in S_\infty$, $F_v$ is $\mathbb{R}$ or $\mathbb{C}$. We will then call $v$ (resp. denote $v$) a real or a complex place (resp. $\mathbb{R}$ or $\mathbb{C}$) accordingly. If $v \in S_{\text{fin}}$, denote by $\mathcal{O}_v$ the ring of $v$-adic integers of $F_v$, and $\mathcal{M}_v$ its maximal ideal. Fix also a generator $\pi_v$ of $\mathcal{M}_v$.

Fix a positive integer $r$. For all $v \in S$, let $G_v := \text{GL}_r(F_v)$. If $g_v = (g_{v,ij}) \in G_v$, denote its inverse by $g_v^{-1} = (g_{v}^{ij})$. With this, for $v \in S_{\text{fin}}$, introduce a subgroup $U_v$ of $G_v$ by setting

$$U_v := \{g_v \in G_v : g_v = (g_{v,ij}) \text{ with } g_{v,ij} \in \mathcal{O}_v \text{ and } g_{v}^{ij} \in \mathcal{O}_v\}.$$  

Now, following Weil, define the associated adelic group $G(A_F)$ via

$$G(A_F) := \text{GL}_r(A_F) := \{g = (g_v)_{v \in S} : g_v \in G_v \text{ s.t. for almost all but finitely many } v \in S_{\text{fin}}, g_v \in U_v\}.$$  

Note that $G(F) := \text{GL}_r(F)$ may be naturally embedded into $\text{GL}_r(A_F)$ via the diagonal map $\alpha \mapsto (\alpha, ..., \alpha, ...)$.

One checks that with respect to the natural topology on $G(A_F)$, $G_r(F)$ is a discrete subgroup of $G_r(A_F)$. So we may form the quotient group $G_r(F) \backslash G_r(A_F)$. By definition, a rank $r$ pre vector bundle on a number field $F$ is an element $[g] \in \text{GL}_r(F) \backslash \text{GL}_r(A_F)$. Also for our own convenience, we call an element $g \in \text{GL}_r(A_F)$ a rank $r$ matrix divisor on $F$. Two rank $r$ matrix divisors $g = (g_v)$ and $g' = (g'_v)$ are said to be (rationally) equivalent if there is an element $\alpha \in G(F)$ such that $g_v = \alpha \cdot g'_v$ for all $v \in S$.

For example, if $r = 1$, then $G(A_F)$ is simply the collection of invertible elements in $A_F$, the ring of adeles of $F$, i.e., $\text{GL}_1(A_F) = I_F$, the group of ideles of $F$. Hence a pre line bundle on $F$ is indeed an element in $F^* \backslash I_F$. Moreover, we may view elements in $I_F$, the rank 1 matrix divisors on $F$ in our language, as divisors on $F$.

Associated to a rank $r$ matrix divisor $g = (g_v)$ is naturally a hermitian vector sheaf $(\mathcal{E}(g), \rho(g))$ over $F$. Indeed, we may set $\mathcal{E}(g)$ to be $\{\alpha \in F^r : g^{-1}_v \cdot \alpha \in \mathcal{O}_v, \forall v \in S_{\text{fin}}\}$ which is a rank $r$ vector sheaf on $\text{Spec}(O_F)$ and may be naturally embedded into $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^r$, where as usual, $r_1$ and $r_2$ denote the real and complex embeddings of $F$ respectively. View $(g_{\sigma})_{\sigma \in S_\infty}$ as an isomorphism of $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^r$, and then define $\rho(g)$ as the natural metric on $\mathcal{E}(g)_{\infty}$ induced from the Euclidean metric on this latest $(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})^r$. Clearly if $g$ and $g'$ are rational equivalent, then the associated hermitian vector sheaves $(\mathcal{E}(g), \rho(g))$ and $(\mathcal{E}(g'), \rho(g'))$ are isometric to each other. Hence it makes sense to talk the associated hermitian vector sheaf for a pre vector bundle. If the Arakelov degree of $(\mathcal{E}(g), \rho(g))$ is $d$, the $g$ and $[g]$ are said to be of degree $d$.

By definition, a pre vector bundle $[g]$ of a number field $F$ is semi-stable (resp. stable), if its associated hermitian vector sheaf $(\mathcal{E}(g), \rho(g))$ is semi-stable (resp. stable); and the adelic moduli space $\mathcal{M}_{A_F, r}(d)$ of semi-stable pre vector bundles of rank $r$ and degree $d$ is a collection of all semi-stable pre vector bundles of rank $r$ and degree $d$.

Clearly, semi-stability is a closed condition. Moreover, for a fixed degree, semi-stability is also a bounded condition. Thus, we conclude that the moduli space $\mathcal{M}_{A_F, r}(d)$ is indeed compact.

The advantage of using the adelic moduli space $\mathcal{M}_{A_F, r}(d)$ is that then we may obtain a natural measure, the Tamagawa one. In fact, as a compact subset in $\text{GL}_r(F) \backslash \text{GL}_r(A_F)$, $\mathcal{M}_{A_F, r}(d)$ inherits a natural measure from the Tamagawa measure on the total space. For simplicity, we simply call this induced measure on $\mathcal{M}_{A_F, r}(d)$ as the (associated) Tamagawa measure.

On the other hand, using Seshadri type equivalence, we may also introduce the so-called moduli space $\mathcal{M}_{F, r}(d)$ of semi-stable vector sheaves of rank $r$ and degree $d$ on $F$. By the uniqueness of the Jordan-Hölder graded hermitian vector sheaves, we obtain a well-defined continuous map

$$\pi_F : \mathcal{M}_{A_F, r}(d) \to \mathcal{M}_{F, r}(d).$$  

Now by the so-called finiteness result of Borel on adelic groups over number fields, we see that the fiber of $\Pi_F$ is indeed compact. Thus, naturally, we get a natural finite measure on $\mathcal{M}_{F, r}(d)$ as well. (In fact,
for vector bundles, only a weak version of Borel’s result is needed here. But if we want to study parabolic \( G \)-bundles, then a full version of Borel’s result has to be used.)

**Remark.** As suggested in Part (A), we should develop a GIT in terms of Arakelov theory. If so, then we may have a new construction of the moduli space of semi-stable vector bundles, for which, the above map \( \Pi_F \) may be viewed as an analog of the moment map.

Clearly the relation between the Tamagawa volume of \( M_{A_F,r}(d) \) and the volume of \( M_{F,r}(d) \) (with respect to the measure induced from that of Tamagawa measure via \( \pi_F \)) deserves a thoughtful study. For example, when \( r=1 \), this is carried by Tate in his thesis. In this sense, what we just ask is a non-abelian generalization of what Tate does. So it would be quite interesting to see how our problem is related to Bloch’s work on the so-called Tamagawa numbers as well.

### B.2.3. Geo-Ari Duality and Riemann-Roch: A Practical Geo-Ari Cohomology following Tate

#### B.2.3.1. An Example

To introduce a more general zeta function, from Artin’s definition of (abelian) zeta function for curves defined over finite fields ([A]), and Iwasawa’s interpretation of Dedekind zeta function ([Iw]), we see that it is better to have a cohomology theory in arithmetic such that the duality and the Riemann-Roch are satisfied. We claim that this can be rigorously developed following Tate’s Thesis at least in geo-ari dimension one. To explain the basic idea, we offer the following simplest example.

Consider only rank 1 lattices over \( \mathbb{Q} \): They are parametrized by \( \mathbb{R}_{>0} \), say the lattices with the forms \( \Lambda_t := \mathbb{Z} \cdot \sqrt{t}, t \in \mathbb{R}_{>0} \). For \( \Lambda_t \), the Poisson summation formula says that

\[
\sum_{n \in \mathbb{Z}} e^{-\pi tn^2} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t}.
\]

Namely,

\[
\sum_{\alpha \in \Lambda_t} e^{-\pi |\alpha|^2} = \frac{1}{\text{Vol}(\Lambda_t)} \sum_{\beta \in \Lambda_t^\vee} e^{-\pi |\beta|^2}.
\]

Here \( \Lambda_t^\vee \) denotes the dual lattice of \( \Lambda_t \). Thus, if we set \( h^0(Q, \Lambda_t) := \log \left( \sum_{\alpha \in \Lambda_t} e^{-\pi |\alpha|^2} \right) \), then

\[
h^0(Q, \Lambda_t) = h^0(Q, \Lambda_t^\vee) = \deg(\Lambda_t).
\]

This is simply the analogue of the Riemann-Roch in geometry. Indeed, as for \( Q \), the metrized dualizing sheaf is simply the standard lattice \( Z \subset R = Q_\infty \). (See e.g. [La2].) So (*) becomes

\[
h^0(Q, \Lambda_t) - h^0(Q, K_Q \otimes \Lambda_t^\vee) = \deg(\Lambda_t) - \frac{1}{2} \deg K_Q,
\]

where \( \deg K_Q = \log |D_Q| = \log 1 = 0 \) with \( \Delta_Q \) the discriminant of \( Q \).

#### B.2.3.2. Canonical Divisors and Space of Different Forms

Let \( F \) be a number field, i.e., a finite extension of \( Q \). Denote by \( S = S_F \) the collection of all (unequivalent) normalized places of \( F \). Set \( S_\infty \) be the collection of all Archimedean places in \( S \), and \( S_\text{fin} := S \setminus S_\infty \).

For any \( v \in S_\text{fin} \), denote by \( \lambda_v \) the composition of natural morphisms

\[
Q_p \to Q_p/\mathbb{Z}_p \hookrightarrow Q/\mathbb{Z} \hookrightarrow R/\mathbb{Z}.
\]

Then we get a natural map \( \lambda_v : F_v \to R/\mathbb{Z} \) defined by \( \lambda_v := \lambda_0 \circ \text{Tr}_{F_v/Q_v} \). Here \( p \) is the place of \( Q \) under \( v \), and \( \text{Tr}_{F_v/Q_v} : F_v \to Q_p \) denotes the local trace. With this, the local different \( \partial_v \) of \( F \) at \( v \) is characterized by

\[
\partial_v^{-1} := \{ \alpha_v \in F_v : \lambda_v(\alpha \cdot \mathcal{O}_v) = 0 \}.
\]
Moreover, being an ideal of the discrete valuation ring \( O_v \) with a parameter \( \pi_v, \partial_v = \pi_v^{\text{ord}_v(\partial_v)} \cdot O_v \).

By definition, two idèles \( a = (a_v) \) and \( b = (b_v) \) are called strictly equivalent, written as \( a \sim_{\text{st}} b \), if, for all \( v \in S_{\infty}, a_v = b_v \); while for all \( v \in S_{\text{fin}} \), there exists \( v \)-adic units \( u_v \) such that \( a_v = u_v b_v \). By an abuse of notation, denote the associated equivalence class of \( a \) by \([a]_{\text{st}}, [a]\), or even \( a \), and denote \( I_F/\sim_{\text{st}} \) also by \( I_F \).

Now define an idelic canonical element \( \omega_F \) of the number field \( F \) as the (strictly) equivalence class associated to the idele \((\omega_v)\) of \( F \). Here for each \( v \in S_{\infty}, \omega_v := 1 \); while for each \( v \in S_{\text{fin}}, \omega_v := \pi_v^{-\text{ord}_v(\partial_v)} \).

One checks easily that \([\omega_v]_{\text{st}}\) is well-defined. We often call \([\omega_v]_{\text{st}}\) a canonical divisor of \( F \) as well. For our own convenience, set \( \pi_v^{-\text{ord}_v(\partial_v)} := 1 \) for all \( v \in S_{\infty} \), despite that we do not have \( \pi_v \) when \( v \in S_{\infty} \).

Motivated by the study for function fields, we then define the space of rational differentials of \( F \) by

\[
\Omega^1_F := \left\{ \left[ (\alpha \cdot \pi_v^{\text{ord}_v(\partial_v)})^t \right]_{\text{st}} : \alpha \in F \right\}.
\]

Here \( \cdot^t \) denotes the transpose of \( \cdot \).

**B.2.3.3. Algebraic Cohomology for Matrix Divisors**

Let \( F \) be a number field. For every rank \( r \) matrix divisor \( g = (g_v) \) of \( F \), define its 0-th (cohomology) group

\[
H^0(\text{Spec}(O_F), g) := \left\{ \alpha = (\alpha_1, \ldots, \alpha_r)^t \in F^r : g_v^{-1} \cdot \alpha \in (O_v)^r, \forall v \in S_{\text{fin}} \right\}.
\]

(In this part, for any set \( A \), let \( A_r \) to be the collection of vectors \((a_1, \ldots, a_r)\) with \( a_i \in A \) and \( A^r \) to be the collection of vectors \((a_1, \ldots, a_r)^t\) with \( a_i \in A \).)

In particular, one sees that, if \( r = 1 \),

\[
H^0(\text{Spec}(O_F), g) := \left\{ \alpha \in F : g_v^{-1} \cdot \alpha \in O_v, \forall v \in S_{\text{fin}} \right\}
\]

has the following interpretation.

Let \( \text{Cl}(O_F) \) denote the ideal class group of \( F \). Then there exists a natural morphism

\[
\psi : I_F / \sim_{\text{st}} \to \text{Cl}(O_F) \quad g = (g_v) \mapsto \prod_{v \in S_{\text{fin}}} P_v^{\text{ord}_v(g_v)}.
\]

Here \( P_v \) denotes the prime ideal of \( O_F \) corresponding to the place \( v \). One checks easily that

\[
H^0(\text{Spec}(O_F), g) = \psi(g)
\]

which is nothing but the global section of the line bundle \( O\left( \sum_v -\text{ord}_v(g_v)[v] \right) \) on \( \text{Spec}(O_F) \).

Next let us define the 1-st (cohomology) group of a rank \( r \) matrix divisor \( g \) on \( F \). To make the picture more clear, we start with \( r = 1 \). In this case, \( H^1(\text{Spec}(O_F), g) \) should be a collection of rational differentials on \( F \). Thus, as over function fields, for a pre-line bundle \( g \) over \( F \), naturally we define its first cohomology group by setting

\[
H^1(\text{Spec}(O_F), g) := \left\{ \beta \in \Omega^1_F : \beta_v \cdot g_v \in O_v, \forall v \in S_{\text{fin}} \right\}.
\]

From this definition, we note that there is a natural isomorphism between \( H^1(\text{Spec}(O_F), g) \) and

\[
\left\{ \alpha \in F : g_v \cdot \alpha \cdot \pi_v^{\text{ord}_v(\partial_v)} \in O_v, \forall v \in S_{\text{fin}} \right\}
\]

which is simply

\[
\left\{ \alpha \in F : \left( \pi_v^{-\text{ord}_v(\partial_v)} \cdot g_v^{-1} \right)^{-1} \alpha \in O_v, \forall v \in S_{\text{fin}} \right\},
\]

i.e., \( H^0(\text{Spec}(O_F), \omega_F \otimes g^{-1}) \). (Here and later, the tensor product is defined as usual for matrices.)
We now study how \( H^i \)'s depend on rational equivalence classes. Assume \( g = (g_v) \sim_{ra} g' = (g'_v) \). So there exists an \( \alpha \in F^* \) such that \( g_v = \alpha \cdot g'_v \) for all \( v \in S \). Hence

\[
H^0(\text{Spec}(\mathcal{O}_F), g) = \left\{ x \in F : g_v^{-1} \cdot x \in \mathcal{O}_v, \ \forall v \in S_{\text{fin}} \right\} = \left\{ x \in F : (g'_v)^{-1} \cdot (\alpha^{-1} \cdot x) \in \mathcal{O}_v, \ \forall v \in S_{\text{fin}} \right\}.
\]

That is to say, for every element \( x \in H^0(\text{Spec}(\mathcal{O}_F), g) \), \( \alpha^{-1} \cdot x \) is an element in \( H^0(\text{Spec}(\mathcal{O}_F), g') \). This then gives an effective isomorphism between these two 0-th cohomology groups

\[
H^0(\text{Spec}(\mathcal{O}_F), g) \xrightarrow{\alpha^{-1}} H^0(\text{Spec}(\mathcal{O}_F), g').
\]

Similarly, we have the canonical isomorphism

\[
H^1(\text{Spec}(\mathcal{O}_F), g) \xrightarrow{\alpha} H^1(\text{Spec}(\mathcal{O}_F), g').
\]

With this, we are ready to come back to the general situation, i.e., that for matrix divisors. By definition, for a rank \( r \) matrix divisor \( g \) over \( F \), define its first cohomology groups by setting

\[
H^1(\text{Spec}(\mathcal{O}_F), g) := \left\{ \beta = (\beta_1, \ldots, \beta_r) \in (\Omega^1_F)_r : \beta \cdot g_v^t \in (\mathcal{O}_v)_r, \ \forall v \in S_{\text{fin}} \right\}.
\]

One checks easily that the above discussion for rank 1 vector divisors also holds for matrix divisors.

**Proposition.** With the same notation as above,

1. If \( g = \alpha \cdot g' \) for \( \alpha \in G(F) \),

\[
H^0(\text{Spec}(\mathcal{O}_F), g) \xrightarrow{\alpha^{-1}} H^0(\text{Spec}(\mathcal{O}_F), g').
\]

2. If \( g = (g_v) \) denote \( g^{-1} := (g_v^{-1}) \) the inverse of \( g \), then

\[
H^0(\text{Spec}(\mathcal{O}_F), g) \simeq \left( H^1(\text{Spec}(\mathcal{O}_F), g^{-1} \otimes \omega_F) \right)^t.
\]

In particular, \( H^1(\text{Spec}(\mathcal{O}_F), g) \) is canonically isomorphic to

\[
\left\{ \alpha \in F^r : (g_v(\alpha)) \in (\partial_v^{-1})^r, \ \forall v \in S_{\text{fin}} \right\}.
\]

### B.2.3.4. Geo-Arit Cohomology and Its Associated Riemann-Roch

It is well-known that in geometry, once we have cohomology groups, naturally, we use (their ranks or) their dimensions over the base field to define the corresponding \( h^0 \) and \( h^1 \). Yet, for arithmetic setting, we must do it very differently. (By saying this, we do not mean that the original geometric counting has no implication in arithmetic setting: recently Deninger ([D]) proposes a formalism of Betti type (co)homology theory, where he essentially uses the original geometric way to count (infinite dimensional spaces). It would be quite interesting to understand the relation between Deninger’s geometric way of counting and the one used here, which we call an arithmetic counting over finitely generated cohomology groups \( H^0 \) and \( H^1 \).)

Let \( F \) be a number field and \( g \) be a rank \( r \) matrix divisor on \( F \), i.e., \( g \in \text{GL}_r(A_F) \). Then the 0-th cohomology group and the 1-st cohomology group of \( g \) are well-defined. Note that in particular, if \( \beta = (\beta_v) \in H^1(\text{Spec}(\mathcal{O}_F), g) \), then for all \( v \in S_{\infty} \), \( \beta_v \in (F_v)_r \) are simply real or complex \( r \)-vectors. With this in mind, we define the geometric arithmetic cohomology of \( g \) as follows.

First, define the 0-th geometric arithmetic cohomology \( h^0(F, g) \) via

\[
h^0(F, g) := \log \left( \sum_{\alpha \in H^0(\text{Spec}(\mathcal{O}_F), g)} \exp \left( -\pi \left( \sum_{v: \text{real}} |g_v^{-1} \cdot \alpha_v|_v^2 + 2 \sum_{v: \text{complex}} |g_v^{-1} \cdot \alpha_v|_v^2 \right) \right) \right),
\]

47
and define the 1-st geometric arithmetic cohomology $h^1(F,g)$ via
\[ h^1(F,g) := \log \left( \sum_{\beta \in H^1(\text{Spec}(O_F),g)} \exp \left( -\pi \left( \sum_{v \text{ real}} |\beta_v \cdot g_v|^2 + 2 \sum_{v \text{ complex}} |\beta_v \cdot g_v|^2 \right) \right) \right). \]

With this, we obtain the following

**Proposition.** With the same notation as above,
(1) $h^0(F,g)$ and $h^1(F,g)$ are well-defined, i.e., the summations on the right hand sides are convergent;
(2) If $g \sim_{\text{r.e.}} g'$, then $h^i(F,g') = h^i(F,g)$ for $i = 0, 1$;
(3) $h^0(F,g) = h^1(F,g^{-1} \otimes \omega_F)$.

Hence, for a rank $r$ vector bundle $[g]$ over $F$, we define $h^0(F,[g])$ and $h^1(F,[g])$, the 0-th and the 1-st arithmetic cohomology of $[g]$, to be $h^0(F,g)$ and $h^1(F,g)$ respectively, for any representative $g \in \text{GL}(A_F)$ (of $[g]$). By Proposition (2) above, they are well-defined.

With all this, surely, to state the Riemann-Roch theorem, we still need to define the degree for a vector bundle. This may be done as over function fields. That is to say, if $g = (g_v) \in \text{GL}_r(A_F)$ is a matrix divisor, denote its determinant by $\det(g) = (\det(g_v))$, which is simply an idele of $F$. Moreover, choose a Haar measure $da$ on $A_F$. Then for any idele $b$ of $F$, set $N(b)$ or $\|b\|$ to be the unique positive number such that $d(b \cdot a) =: N(b) \cdot da$. (This is a global way to understand $N(b)$. Locally, $N(b) = \|b\|$ may be defined as follows: If $b = (b_v)$, then
\[ \|b\| := \prod_{v \in S} \|b_v\|_v =: \prod_{v \in S} |b_v|_v^{N_v}. \]
Here $N_v := [F_v : Q_p]$ denotes the local degree of the place of $v$ and $p$ is the place of $Q$ under $v$.) Finally, define the degree of $g$, denoted by $\deg(g)$, by
\[ \deg(g) := \log \left( N(\det(g)) \right). \]

By using the product formula, one checks that for any rank $r$ vector bundle $[g]$ of $F$
\[ \deg([g]) := \deg(g) \]
is well-defined. We will call this real number the degree of the rank $r$ vector bundle $[g]$. For example, one checks easily that the degree of the canonical divisor $\omega_F$ is simply $\log |\Delta_F|$, where $\Delta_F$ denotes the discriminant of $F$.

**Geo-Ari Riemann-Roch Theorem over Number Fields.** Let $F$ be a number field. Then for any vector bundle $E$ over $F$, we have
\[ \chi_{\text{geo}}(F,E) := h^0(F,E) - h^1(F,E) = \deg(E) - \text{rank}(E) \cdot \frac{1}{2} \deg(\omega_F). \]

One may prove this result as follows following Tate. First, recall the standard Poisson summation formula to the pair $(F^n, A^n)$.

**Poisson summation formula.** Let $f$ be continuous and in $L^1(A^n)$. Assume that $\sum_{a \in F^n} |f(x + a)|$ is uniformly convergent for $x$ in a compact subset of $A^n$, and that $\sum_{a \in F^n} \hat{f}(a)$ is convergent, where $\hat{f}$ denotes the Fourier transform of $f$. Then
\[ \sum_{a \in F^n} \hat{f}(a) = \sum_{a \in F^n} f(a). \]

But, for any element $g \in \text{GL}_r(A_F)$, set $h(x) := f(gx)$, then one checks that $\hat{h}(x) = \frac{1}{\|\det(g)\|} \hat{f}(g^{-1}x)$. Thus, from the Poisson summation formula above, we get the following more suitable version for our application:
\[ \frac{1}{\|\det(g)\|} \sum_{a \in F^r} \hat{f}(g^{-1} \cdot a) = \sum_{a \in F^r} f(g \cdot a). \]
Now set \( f(x) := \prod_{v \in S} f_v(x_v) \) with \( f_v \) as follows:

(i) If \( v \) is real, then \( f_v(x_v) := e^{-\pi|x_v|^2} \), where for \( x_v = (x_v^{(1)}, \ldots, x_v^{(r)}) \), \( |x_v|^2 = \sum_i (x_v^{(i)})^2 \). Obviously, we have \( f_v = \hat{f}_v \).

(ii) If \( v \) is complex, then \( f_v(x_v) := e^{-2\pi|x_v|^2} \), where for \( x_v = (x_v^{(1)}, \ldots, x_v^{(r)}) \), \( |x_v|^2 = \sum_i x_v^{(i)} \cdot \overline{x_v^{(i)}} \). Similarly, \( f_v = \hat{f}_v \).

(iii) If \( v \) is finite, then \( f_v \) is defined to be the characteristic function of \( (\partial^{-1}_v)^r \). One checks that \( \hat{f}_v \) equals to \( (N(\partial_v))^{r/2} \) times the characteristic function of \( \mathcal{O}_v^r \).

With this, what we have is the following formula:

\[
\frac{1}{|\text{det}(g)|} \cdot (N(\omega_F))^{r/2} \cdot \sum_{g^{-1} \in \mathcal{O}_v^r, \forall v \in S} \left( \prod_{v: \text{real}} e^{-\pi|g^{-1}(\alpha)|^2} \prod_{v: \text{complex}} e^{-2\pi|g^{-1}(\alpha)|^2} \right)
= \sum_{g^{-1} \in (\partial^{-1}_v)^r, \forall v \in S} \left( \prod_{v: \text{real}} e^{-\pi|g(\alpha)|^2} \prod_{v: \text{complex}} e^{-2\pi|g(\alpha)|^2} \right).
\]

This then gives a proof of the theorem by definition and the Proposition 2.3.3.

Remarks. (1) The practical geo-ari cohomology and Riemann-Roch theorem here have indeed a theoretical treatment along the line in C.3. We are going to include this in the yet to be released second version of [We2].

(2) We may understand van der Geer and Schoof’s work as follows:

(i) For any idele \( a \), define the associated Arakelov divisor \( \text{div}_{\mathcal{A}}(a) \) as follows:

\[
\text{div}_{\mathcal{A}}(a) := - \sum_{v \in S_{\infty}} \text{ord}_v(a_v) \cdot [v] + \sum_{v \in S_{\infty}} N_v \log |a_v| \cdot [v].
\]

Obviously, all Arakelov divisor may be constructed in this way.

(ii) Define \( H^0(\text{Spec}\mathcal{O}_F, \text{div}_{\mathcal{A}}(a)) := H^0(\text{Spec}\mathcal{O}_F, a) \) and setting \( h^0(\text{Spec}\mathcal{O}_F, \text{div}_{\mathcal{A}}(a)) := h^0(\text{Spec}\mathcal{O}_F, a) \).

One checks then that this definition coincides with that of van der Geer and Schoof. Hence, we have also; the van der Geer-Schoof’s Riemann-Roch theorem: For any Arakelov divisor \( D \) over \( \text{Spec}(\mathcal{O}_F) \),

\[
h^0(\text{Spec}(\mathcal{O}_F), D) - h^0(\text{Spec}(\mathcal{O}_F), K_{\text{Spec}(\mathcal{O}_F)} - D) = \deg(D) - \frac{1}{2}\deg(\omega_F).
\]

B.2.4. Non-Abelian Zeta Function For Number Fields

B.2.4.1. The Construction

Let \( F \) be a number fields with discriminant \( \Delta_F \). Denote by \( \mathcal{M}_{A,F,r}(d) \) the (adelic) moduli space of rank \( r \) and degree \( d \) semi-stable pre vector bundles over \( F \), and \( d\mu \) its associated Tamagawa measure. Then we define the rank \( r \) non-abelian (completed) zeta function \( \xi_{F,r}(s) \) of \( F \) by

\[
\xi_{F,r}(s) := \left(|\Delta_F|^{\frac{r}{2}}\right)^s \int_{g \in \mathcal{M}_{A,F,r}(d), t \in \mathbb{R}_{>0}} \left(e^{h^0(A_F,g)} - 1\right)(e^{-s} d\mu(g)) \cdot \omega(g), \quad \text{Re}(s) > 1.
\]

Following 2.1, i.e., Iwasawa’s interpretation of Dedekind zeta functions, we have \( \xi_{F,1}(s) = w_F \cdot \xi_F(s) \) where \( w_F \) denotes the number of roots of unity in \( F \) and \( \xi_F(s) \) denotes the completed Dedekind zeta function for \( F \). Note also that the terms appeared in our construction above, such as the degree, the geo-ari cohomology \( h^0 \), the moduli space, and the Tamagawa measure are all canonically and naturally associated with number fields. Hence, our non-abelian zeta should be genuinely related with non-abelian arithmetic properties of number fields.

Remark. Recall that there is an algebraic moment map \( \pi_F : \mathcal{M}_{A,F,r}(d) \to \mathcal{M}_{F,r}(d) \). Thus our above construction of the zetas may be understood as a kind of algebraic version of Feynman type integral. On
the other hand, in Part (A), we propose a Weil-Narasimhan-Seshadri type correspondence, namely, a micro reciprocity law. So it is not unreasonable to expect that our zetas may be written in terms of analytic Feynman type integrals, and that the global (non-abelian) reciprocity law may be obtained from our zeta functions. I would like to thank Nitta and Okada for their discussion here.

B.2.4.2. Basic Properties

Just as for Dedekind zeta functions, our non-abelian zeta functions are well-defined, satisfy functional equation as well. Moreover, the residues of these zeta functions may be calculated in terms of the volumes of the moduli space. More precisely, we have the following

Theorem. With the same notation as above, we have

1. \( \xi_{F,r}(s) = \left( |\Delta_F|^r \right)^s \int_{g \in M_{A_{F,r}}(t), t \in \mathbb{R}_{>0}} \left( e^{h^0(A_{F,r}, g)} - 1 \right) (e^{-s})^{\deg(g)} \cdot d\mu(g) \) converges absolutely and uniformly when \( \Re(s) \geq 1 + \delta \) for any \( \delta > 0 \);
2. \( \xi_{F,r}(s) \) admits a unique meromorphic continuation to the whole complex \( s \)-plane with only two simple poles at \( s = 0, 1 \) whose residues are \( \text{Vol}(M_{A_{F,r}}(t)) \) for one and hence for all \( t \);
3. (Functional Equation) \( \xi_{F,r}(s) = \xi_{F,r}(1-s) \).

Remark. Most suitable definition for non-abelian zeta functions of number fields should be

\[
\xi_{F,r}(s) := \left( |\Delta_F|^r \right)^s \int_{\Lambda \in M_{A_{F,r}}(t), t \in \mathbb{R}_+} \frac{e^{h^0(A_{F,r}, \Lambda)} - 1}{\#\text{Aut}(\Lambda)} (e^{-s})^{\deg(\Lambda)} \cdot d\mu(\Lambda), \quad \Re(s) > 1
\]

where Aut denotes the automorphism group. Moreover, instead of using the adelic moduli spaces, we may introduce the ‘standard’ version of non-abelian zeta functions using integrations over moduli spaces of semi-stable lattices. In this way, we may then also see what are the ‘Gamma’-factors and a non-abelian version of Tate’s calculation on the so-called analytic class number formula.
C. Explicit Formula, Functional Equation and Geo-Ari Intersection

C.1. The Riemann Hypothesis for Curves

C.1.1. Weil’s Explicit Formula: the Reciprocity Law

Let $C$ be a projective irreducible reduced regular curve of genus $g$ defined over a finite field $k := \mathbb{F}_q$. Denote by $\zeta_C(s)$ the associated Artin zeta function. Set $t = q^{-s}$ and $Z_C(t) := \zeta_C(s)$. Then by the rationality, there exists a polynomial $P_C(t)$ of degree $2g$ such that

$$Z_C(t) = \frac{P_C(t)}{(1-t)(1-qt)}.$$

Now let $C_n$ be the curve obtained from $C$ by extending the field of constants from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$. Then by a discussion on covering of curves, we obtain the following

**Reciprocity Law.** With the same notation as above, $Z_{C_n}(t^n) = \prod_\zeta Z_C(\zeta t)$ where the product is taken over all the $n$-th roots of 1.

Moreover, by the Euler product,

$$t \frac{Z_C'}{Z_C}(t) = \sum_{n=1}^{\infty} \sum_P d(P) t^{nd(P)} = \sum_{m=1}^{\infty} \left( \sum_P d(P) \right) t^m$$

where $\sum_P$ is taken over those closed points rational over $\mathbb{F}_q$ whose degree divides $m$. Hence,

$$Z_C(t) = \exp \left\{ \sum_{m=1}^{\infty} N_m \frac{t^m}{m} \right\}$$

where $N_m = \sum_P d(P) |_{m} d(P)$. Clearly when $m = 1$, the sum $\sum P, d(P)|1$ simply counts the number of closed points of $C$ rational over $\mathbb{F}_q$. Thus, by the reciprocity law above,

$$Z_{C_n}(t^n) = \exp \left\{ \sum_{m=1}^{\infty} M_m \frac{t^{nm}}{m} \right\} = \exp \left\{ \sum_{m=1}^{\infty} N_m \frac{t^m}{m} \left( \sum_\zeta \zeta^m \right) \right\}.$$

This implies that $N_n = N_1(C_n)$, i.e., $N_n$ is the number of closed points on $C$ which are rational over $\mathbb{F}_{q^n}$. Therefore, we have the following

**Weil’s Explicit Formula.** With the same notation as above,

$$N_n = q^n + 1 - \sum_{\zeta_C(p)=0} \rho^n,$$

where $\rho_1, \ldots, \rho_{2g}$ denotes the reciprocals of the roots of $P_C(t)$.

C.1.2. Geometric Version of Explicit Formula

As above, let $C$ be an algebraic curve defined over $\mathbb{F}_p$, the finite field with $p$ elements. Over $\mathbb{C} \times C$, for $n \in \mathbb{Z}$, introduce (micro) divisors $A_n$ (via algebraic correspondence) as follows:

$$A_n := \begin{cases} \{(x, x^p^n) : x \in C\}, & \text{if } n \geq 0; \\ \{x^{p^{-n}}, x) : x \in C\}, & \text{if } n \leq 0. \end{cases}$$
Clearly, we have the following relations for the intersections among $A_n$'s.

(i) If $n \geq m \geq 0$,
\[
\langle A_n, A_m \rangle = p^m \langle A_{n-m}, A_0 \rangle;
\]

(ii) If $m \geq n \geq 0$,
\[
\langle A_n, A_m \rangle = p^n \langle A_{m-n}, A_0 \rangle;
\]

(iii) If $m \geq 0 \geq n$,
\[
\langle A_n, A_m \rangle = \langle A_{n-m}, A_0 \rangle;
\]

(iv) If $n \leq m \leq 0$,
\[
p^m \langle A_n, A_m \rangle = \langle A_{m-n}, A_0 \rangle;
\]

(v) If $m \leq n \leq 0$,
\[
p^n \langle A_n, A_m \rangle = \langle A_{n-m}, A_0 \rangle;
\]

(vi) If $n \geq 0 \geq m$,
\[
\langle A_n, A_m \rangle = \langle A_{n-m}, A_0 \rangle.
\]

Hence, we have the following

Lemma 1. With the same notation as above, we have

1. For all $m, n \in \mathbb{Z}$,
\[
\langle A_n, A_m \rangle = \langle A_m, A_n \rangle;
\]

2. For all $m, n \in \mathbb{Z}$,
\[
\langle A_{-n}, A_{-m} \rangle = \langle A_m, A_n \rangle;
\]

3. For all $n \geq m \geq 0$ in $\mathbb{Z}$,
\[
\langle A_n, A_m \rangle = p^m \langle A_{n-m}, A_0 \rangle;
\]

4. For all $n \geq 0 \geq m$ in $\mathbb{Z}$,
\[
\langle A_n, A_m \rangle = \langle A_{n-m}, A_0 \rangle.
\]

Obviously, (1) \sim (4) are equivalent to (i) \sim (vi) above. Therefore, in order to understand $\langle A_n, A_m \rangle$ for all $n, m \in \mathbb{Z}$, we only need to know $\langle A_n, A_0 \rangle$ for $n \geq 0$.

For this latest purpose, first, note that $A_0$ is simply the diagonal. Hence, by definition, for $n > 0$, $\langle A_n, A_0 \rangle$ is $N_n(C)$, the number of closed points of $C$ which are rational over $F_{p^n}$.

Secondly, by the functional equation of the zeta function $\zeta_C(s)$ of $C$, which itself is a direct consequence of the Riemann-Roch theorem for the curve $C$, (and the rationality of $\zeta_C(s)$), we have the following

Lemma 2. (Explicit Formula of Weil) With the same notation as above, if $n \in \mathbb{Z}_{\geq 0}$,
\[
\langle A_n, A_0 \rangle = \langle A_n, F_1 \rangle + \langle A_n, F_2 \rangle - \sum_{\zeta_C(s) = 0} s^n.
\]

Here $F_1$ and $F_2$ denotes the fibers in two directions of $C \times C$ respectively, and the sum is taken over all zeros of $\zeta_C(s)$.

Remark. From now on, we may from place to place have some sign problems, say $s^n$ may well mean $s^{-n}$.

C.1.3. Riemann Hypothesis for Function Fields

Now, following Weil again, we prove the (Artin-)Riemann hypothesis. (See e.g., [Ha].)

Let $f : p^\mathbb{Z} \to \mathbb{Z}$ be a function with finite supports. Define its Mellin transform via $\hat{f}(s) := \sum_n f(n)p^{-ns}$. Clearly, if $f^*(p^n) := f(p^{-n})p^{-n}$, $\hat{f}(s) = \hat{f}(1-s)$; moreover, if $(f \ast g)(p^n) := \sum_m f(p^n)g(p^{n-m})$, $\hat{f} \ast \hat{g}(s) = \hat{f}(s) \cdot \hat{g}(s)$ in particular, $\hat{f} \ast \hat{g}^*(s) = \hat{f}(s) \cdot \hat{g}(1-s)$.
With $f$, we may use the divisors $A_n$'s above to define a (global) $\mathbb{Q}$-divisor $D_f$ on $C \times C$ as follows:

$$D_f := \sum_{n>0} f(p^n)A_n + \sum_{n \geq 0} f(p^{-n})p^{-n}A_n.$$  

**Theorem.** (Weil) *With the same notation as above,*

(1) *Relative Degrees*

$$\langle D_f, F_1 \rangle = \hat{f}(1); \quad \langle D_f, F_2 \rangle = \hat{f}(0);$$

(2) *Fixed Points Formula*

$$\langle D_f, D_g \rangle = \langle D_{\hat{f} \hat{g}}, \text{Diag} \rangle,$$

where Diag denotes the diagonal of $C \times C$;

(3) *Explicit Formula*

$$\hat{f}(0) + \hat{f}(1) - \sum_{\zeta(s)=0} \hat{f}(s) = \langle D_f, \text{Diag} \rangle.$$  

Indeed, by definition,

$$\langle D_f, F_1 \rangle = \sum_{n>0} f(p^n) \cdot p^n + \sum_{n \geq 0} f(p^{-n})p^{-n} \cdot 1 = \hat{f}(1);$$

$$\langle D_f, F_2 \rangle = \sum_{n>0} f(p^n) + \sum_{n \geq 0} f(p^{-n})p^{-n} \cdot p^n = \hat{f}(0).$$

This gives (1). (2) is a direct consequence of the relations (i) $\sim$ (vi) for the intersections of $A_n$'s, and hence comes from Lemma 12.1. Finally, (3) is simply Lemma 1.2.2 by definition.

Next we apply the two dimensional intersection theory, in particular, the Hodge Index Theorem. Since

$$\langle F_1 + F_2, D_f - \hat{f}(1)F_2 - \hat{f}(0)F_1 \rangle = 0,$$

$$\langle D_f - \hat{f}(1)F_2 - \hat{f}(0)F_1, D_f - \hat{f}(1)F_2 - \hat{f}(0)F_1 \rangle \leq 0.$$  

That is,

$$\hat{f}(0) \cdot \hat{f}(1) \geq \frac{1}{2} \langle D_f, D_f \rangle.$$  

Thus by Theorem above, this last equality is equivalent to

$$\sum_{\zeta(s)=0} \hat{f}(s) \cdot \hat{f}(1-s) \geq 0.$$  

From this, easily, we get the following Riemann Hypothesis for Artin zeta functions of curves over finite fields.

**Theorem.** (Hasse-Weil) *Let $\zeta(s)$ be the zeta function for a curve defined over a finite field. If $\zeta(s) = 0,$ then $\text{Re}(s) = \frac{1}{2}.$*  

**C.2. Geo-Ari Intersection in Dimension Two: A Mathematics Model**

**C.2.1. Motivation from Cramér’s Formula**

In the above discussion, the summation $\sum_{\zeta(s)=0} s^n$ plays a key role in understanding Artin-Riemann Hypothesis, via the so-called micro explicit formula of Weil. So naturally, we want to know whether this approach works for number fields, and are led to study the formal summation

$$\sum_{\zeta(s)=0} x^n, \quad \text{for } x \in [1, \infty).$$  

(*)
Here $\xi_Q(s)$ denotes the completed Riemann zeta function, i.e., $\xi_Q(s) := \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta_Q(s)$ with $\Gamma(s)$ the standard Gamma function and $\zeta_Q(s)$ the Riemann zeta function.

The reader at this point certainly would reject $(\ast)$, since the summation does not make any sense. How could I write such a monster down?! Well, do not be so panic!!! After all, in the study of the prime distributions, the conditional convergent summation $\sum_{\xi_Q(\rho)=0} x^{\rho}$ does appear. Recall that we have the following Riemann-von Mangoldt formula

$$\sum_{p \leq x} \log p = x - \sum_{\xi_Q(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^2),$$

which itself motivates and hence stands as a special form of the more general form of explicit formulas. See e.g., Jorgenson and Lang’s lecture notes on Explicit Formulas. More generally, in various discussions about prime distributions, we do use the summations such as $\sum_{\xi_Q(\rho)=0} \frac{x^\rho}{\rho}$. In this sense, the problem is not whether we should introduce $(\ast)$, rather, it should be how to justify it.

Anyway, let me make a change of variables $x := e^t$ with $t \geq 0$. Then $(\ast)$ becomes

$$V(t) := \sum_{\xi_Q(\rho)=0} e^{t\rho}.$$  (**)

So, following Riemann, we may further view $V(t)$ as a function of complex variable $z$, i.e.,

$$V(z) := \sum_{\xi_Q(\rho)=0} e^{z\rho}.$$  (**')

Now, we claim that there is a nice way to regularize $(\ast)$.

To explain this in a simpler form, which in fact would not really make our life any easier, we instead consider the partial sum $V_+(z)$ defined by

$$V_+(z) := \sum_{\xi_Q(\rho)=0, \text{Re}(\rho) > 0} e^{z\rho}.$$  (++)

Then we have the following result, dated in 1919.

**Theorem.** (Cramér) The function $V_+(z)$ converges absolutely for $\text{Im}(z) > 0$. Moreover,

$$2\pi i V_+(z) - \log \frac{z}{1 - e^z}$$

has a meromorphic continuation to $\mathbb{C}$, with simple poles at the points $\pm \pi \pi n$ for all integers $n$, and at the points $\pm \log p^m$ for all powers of primes.

We claim that this theorem actually offers us a natural analytic way to normalize the formal summation $\sum_{\xi_Q(\rho)=0} x^\rho$ for $x \in [1, \infty)$, by using [C], [JL1,2,3], and [DS]. In a sense, this is in a similar way as what we do when normalizing $\infty!$. By the Stirling formula

$$n! = \sqrt{2\pi} \cdot \sqrt{n} \cdot \left(\frac{n}{e}\right)^n \cdot \exp\left(\frac{\theta_n}{12}\right)$$

for $n$ sufficient large with $|\theta_n| < 1$, we set $\infty! = \sqrt{2\pi}$.

However, in this article, we do it very differently – We are going to construct a mathematics model to normalize this formal summation geometrically.

**C.2.2. Micro Divisors.**

We will not use Grothendieck’s scheme language. Instead, formally, we call (the set theoretical product) $S := \text{Spec}(\mathbb{Z}) \times \text{Spec}(\mathbb{Z})$ a geometric arithmetic base (surface).

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54
Similarly as in 1.2, associated to all \( x \in [0, \infty] \) are symbols \( D_x \) which will be called micro divisors. Associated to any two micro divisors \( D_x, D_y \) is the intersection number \( \langle D_x, D_y \rangle \in \mathbb{R} \). Assume that the following fundamental relations are satisfied by our micro intersections:

1. (Symmetry) For any \( x, y \in [0, \infty] \),
   \[
   \langle D_x, D_y \rangle = \langle D_y, D_x \rangle;
   \]

2. (Mirror Image) For any \( x, y \in [0, \infty] \),
   \[
   \langle D_x, D_y \rangle = \langle D_{\frac{x}{y}}, D_{\frac{y}{x}} \rangle;
   \]

3. (Fixed Points 1) If \( 0 \leq x \leq y \leq 1 \), then
   \[
   \langle D_x, D_y \rangle = y \langle D_{\frac{x}{y}}, D_1 \rangle;
   \]

4. (Fixed Points 2) If \( 0 \leq x \leq 1 \leq y \leq \infty \), then
   \[
   \langle D_x, D_y \rangle = \langle D_{\frac{x}{y}}, D_1 \rangle.
   \]

Note that \([0, \infty] \times [0, \infty]\) is simply the union of \( \{0 \leq x \leq y \leq 1\}, \{0 \leq y \leq x \leq 1\}, \{1 \leq x \leq y \leq \infty\}, \{1 \leq y \leq x \leq \infty\}, \{0 \leq x \leq 1, 1 \leq y \leq \infty\} \) and \( \{0 \leq y \leq 1, 1 \leq x \leq \infty\} \). Thus by the above relations, if we define the precise intersection \( \langle D_x, D_1 \rangle \) for all \( x \in [0, 1] \), then we have all the intersections \( \langle D_x, D_y \rangle \) for all \( x, y \in [0, \infty] \).

5. (Explicit Formula 1) Denote the completed Riemann zeta function by \( \xi_Q(s) \). Then, for all \( x \in [0, 1] \),
   \[
   \langle D_x, D_1 \rangle = \langle D_0, D_x \rangle + \langle D_\infty, D_x \rangle - \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^s.
   \]

Here, as said in 2.1, we certainly encounter with the convergence problem of \( \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^s \). Instead of solving it, in our model, we simply assume that our micro intersection offers a natural normalization of \( \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^s \) via the (5). It is in this sense we say that our model gives a geometric way to normalize \( \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^s \).

**Remark.** The compactibility among \((i)\)'s, \( i = 1, 2, 3, 4, 5 \), is guaranteed by the functional equation for Riemann zeta function. Moreover, if \( x \geq 1 \), then by the Mirror principal, we see that \( \langle D_x, D_1 \rangle = \langle D_{\frac{x}{y}}, D_1 \rangle \).

So, by using the Explicit Formula 1, i.e., (5) above, together with the Relations II below, we have
\[
\langle D_{\frac{x}{y}}, D_1 \rangle = 1 + \frac{1}{x} - \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^{-s}.
\]

Multiplying both sides by \( x \), we get
\[
x \langle D_{\frac{x}{y}}, D_1 \rangle = 1 + x - \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^{1-s} = 1 + x - \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^s,
\]
by the functional equation. On the other hand, by the Fixed Point 1, i.e., (3) above, we get
\[
\langle D_x, D_1 \rangle = x \langle D_{\frac{x}{y}}, D_1 \rangle.
\]
That is to say, formally, for all \( x \),
\[
\langle D_x, D_1 \rangle = 1 + x - \sum_{\xi_Q(s)=0}^{\xi_Q(s)=x} x^s.
\]

From (1), (2), (3) and (4), we may formally get the following relations.
Relations I. (i) \( \langle D_0, D_0 \rangle = \langle D_\infty, D_\infty \rangle \);
(ii) \( \langle D_0, D_1 \rangle = \langle D_\infty, D_1 \rangle = \langle D_0, D_\infty \rangle \).

Note that \( D_0 \) and \( D_\infty \) are supposed to be the fibers in two directions of \( \text{Spec}(\mathbb{Z}) \times \text{Spec}(\mathbb{Z}) \) over \( \infty \) in \( \text{Spec}(\mathbb{Z}) \), so we normalize our intersection further by the following

(6) (Normalization 1) \( \langle D_0, D_0 \rangle = 0, \quad \langle D_0, D_1 \rangle = 1 \).

With this, formally we may further get the following

Relations II. (i) For \( \langle D_0, D_x \rangle \),

\[
\langle D_0, D_x \rangle = \begin{cases} 
  x, & \text{if } x \in [0, 1]; \\
  1, & \text{if } x \in [1, \infty];
\end{cases}
\]

(ii) For \( \langle D_\infty, D_x \rangle \),

\[
\langle D_\infty, D_x \rangle = \begin{cases} 
  1, & \text{if } x \in [0, 1]; \\
  \frac{1}{x}, & \text{if } x \in [1, \infty];
\end{cases}
\]

In particular, for \( x \in [0, 1] \),

\[
\langle D_x, D_1 \rangle = 1 + x - \sum_{\xi \in \mathbb{Q}(s) = 0} x^s.
\]

Remark 2. In (iii) above, taking \( x = 1 \), we get

\[
\langle D_1, D_1 \rangle = 2 + \sum_{\xi \in \mathbb{Q}(s) = 0} 1.
\]

Thus, via the so-called Adjunction Formula, which should be one of the fundamental results for our intersection, \( \sum_{\xi \in \mathbb{Q}(s) = 0} 1 \) is supposed to be related to the canonical divisor for our arithmetic dimension one base.

C.2.3. Global Divisors and Their Intersections: Geometric Reciprocity Law

Motivated by C.1, i.e., the discussion about Artin-Riemann Hypothesis, we start with a standard construction in function theory. Let \( f : \mathbb{R}^+ \to \mathbb{R} \) be a smooth, compactly supported function. Define its Mellin transform via

\[
\hat{f}(s) := \int_0^\infty f(x) x^s \frac{dx}{x}.
\]

Then, if \( f^*(x) := f(\frac{1}{x}) \cdot \frac{1}{x} \),

\[
\widehat{f^*}(s) = \hat{f}(1 - s);
\]

and if \( (f * g)(x) := \int_0^\infty f(y) g(\frac{x}{y}) \frac{dy}{y} \) denotes the standard multiplicative convolution,

\[
\hat{f} * g(s) = \hat{f}(s) \cdot \hat{g}(s).
\]

In particular,

\[
\hat{f} * g^*(s) = \hat{f}(s) \cdot \hat{g}(1 - s).
\]

Next, we give a parallel construction for our divisors. Standard wisdom says that we should use linear combinations of generalized divisors \( D_x, x \in [0, \infty] \) to form new type of divisors. But as we clearly see that such a conventional way does not result sufficiently many divisors, we do it very differently.

Starting from divisors \( D_x \), for any might-be-interesting function \( f \), formally define the associated global divisor \( D_f \) by setting

\[
D_f := \int_0^1 f(x) \cdot D_x \cdot \frac{dx}{x} + \int_1^\infty f(x) \cdot x D_x \cdot \frac{dx}{x}.
\]
Remark. In all formal discussions here, we pay no attention to the convergence problem. See however 2.5 below. In other words, we assume also in our model that global divisors do exist and their relations with micro divisors are given as above.

With this definition, we may extend the intersection in 2.2 to $D_f$’s by linearity. For example, the intersection $(D_f, D_1)$ is by definition given by

$$
(D_f, D_1) = \int_0^1 f(x) \cdot (D_x, D_1) \cdot \frac{dx}{x} + \int_1^{\infty} f(x) \cdot x(D_x, D_1) \cdot \frac{dx}{x}.
$$

Then formally, we have the following

**Key Relations. With the same notation as above,**

(i) *(Relative Degrees in Two Fiber Directions)*

$$
\deg_1 D_f := \langle D_0, D_f \rangle = 1 \hat{f}(1)
$$

and

$$
\deg_2 D_f := \langle D_\infty, D_f \rangle = 0 \hat{f}(0).
$$

(ii) *(Fixed Point Formula)*

$$
\langle D_f, D_g \rangle = \langle D_f \ast g, D_1 \rangle;
$$

(iii) *(Explicit Formula 2)*

$$
\langle D_f, D_1 \rangle = 0 \hat{f}(0) + 1 \hat{f}(1) - \sum_{\xi(s)=0} \hat{f}(s).
$$

In fact, this may be formally checked as follows using axioms. First consider the Fixed Point Formula. Set $D_x' := D_x^\perp$. Then, by definition,

$$
\langle D_f \ast g, D_1 \rangle = \langle \int_0^1 f \ast g(x)D_x \frac{dx}{x} + \int_0^1 f \ast g\left(\frac{1}{x}\right)1xD_x \frac{dx}{x}, D_1 \rangle
$$

By changing variables $x' := \frac{x}{y}$, the latest quantity is simply

$$
= \langle \int_0^\infty f(y)g\left(\frac{x}{y}\right)\frac{dy}{y}D_x \frac{dx}{x} + \int_0^1 f(y)g\left(\frac{1}{xy}\right)xy\frac{dy}{y}xD_x' \frac{dx}{x}, D_1 \rangle
$$

But for $x \geq y$, we may split the region into three parts, i.e.,

(1.1) $1 \geq x \geq 1, 1 \geq y \geq 0$ and $x \geq y$;

(2.1) $1 \geq x \geq 0, 1 \geq y \geq 0$ and $x < y$;

(3.1) $\infty \geq x \geq 1, \infty \geq y \geq 1$ and $x \geq y$;

(3.2) $\infty \geq x \geq 1, 1 \geq y \geq 0$ and $x < y$;

(3.3) $\infty \geq y \geq 1$ and $1 \geq x \geq 0$.
Hence the latest quantity is simply, by writing according to (2.1), (2.2) and (2.3) (resp. (1.1), (1.2) and (1.3)) for the first term (resp. second term),

\[
\langle \left( \int_0^1 f(y) \frac{dy}{y} \int_0^y g(x)x D_x \frac{dx}{x} \right) + \left( \int_0^1 f(y) \frac{dy}{y} \int_1^y g(x)x D_x \frac{dx}{x} \right), D_1 \rangle
\]

\[
\langle \left( \int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x)x D_x \frac{dx}{x} \right) + \left( \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x)x D_x \frac{dx}{x} \right), D_1 \rangle
\]

\[
\langle \left( \int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x)x D_x \frac{dx}{x} \right) + \left( \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x)x D_x \frac{dx}{x} \right), D_1 \rangle
\]

Now accordingly call each of the terms from the beginning as (3.1), (3.2), (4.1), (4.2), (5.1) and (5.2), we then see that for (3.1), (4.1), (5.1) and (5.2) we have \(0 \leq x \leq y \leq 1\), (resp. \(0 \leq y \leq x \leq 1\), \(1 \leq x \leq y \leq \infty\) and \(\infty \geq x \geq y \geq 1\)), hence by our axioms, for \(0 \leq x \leq y \leq 1\)

\[
\langle y(xD_x, D_1) = \langle D_y, D_x \rangle \rangle
\]

\[
\langle x(yD_x, D_1) = \langle D_x, D_y \rangle \rangle
\]

\[
\langle y(xD_x, D_1) = \langle D_x, D_y \rangle \rangle
\]

\[
\langle x(yD_x, D_1) = \langle D_x, D_y \rangle \rangle
\]

We see that the latest combination is simply

\[
\left( \int_0^1 f(y) \frac{dy}{y} \int_0^y g(x) D_y x \frac{dx}{x} \right) + \left( \int_0^1 f(y) \frac{dy}{y} \int_1^y g(x) D_y x \frac{dx}{x} \right)
\]

\[
\left( \int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \right) + \left( \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x) D_y x \frac{dx}{x} \right)
\]

\[
\left( \int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \right) + \left( \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x) D_y x \frac{dx}{x} \right)
\]

which certainly is nothing but

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot D_1 + \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x) D_y x \frac{dx}{x} \cdot x \cdot D_1
\]

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot y \cdot D_1 + \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x) D_y x \frac{dx}{x} \cdot y \cdot D_1
\]

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot x \cdot y \cdot D_1
\]

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot y \cdot D_1 + \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x) D_y x \frac{dx}{x} \cdot y \cdot D_1
\]

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot x \cdot y \cdot D_1
\]

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot y \cdot D_1 + \int_0^1 f(y) \frac{dy}{y} \int_1^1 g(x) D_y x \frac{dx}{x} \cdot y \cdot D_1
\]

\[
\int_0^1 f(y) \frac{dy}{y} \int_0^1 g(x) D_y x \frac{dx}{x} \cdot x \cdot y \cdot D_1
\]
Now by definition, this latest combination is simply

$$\langle D_f, D_g \rangle.$$ 

This then completes the proof of the Fixed Point Formula.

To see the relative degree relation, we have the following formal arguments.

$$\langle D^\hat{f}, D^0 \rangle = \int_0^1 f(x)\langle D_x, D^0 \rangle \frac{dx}{x} + \int_1^\infty f(x)\langle D_x, D^0 \rangle \frac{dx}{x}$$

$$= \int_0^1 f(x)\frac{dx}{x} + \int_1^\infty f(x)\frac{dx}{x} = \int_0^\infty f(x)\frac{dx}{x} = \hat{f}(1)$$

and

$$\langle D^\hat{f}, D^\infty \rangle = \int_0^1 f(x)\langle D_x, D^\infty \rangle \frac{dx}{x} + \int_1^\infty f(x)\langle D_x, D^\infty \rangle \frac{dx}{x}$$

$$= \int_0^\infty f(x)\frac{dx}{x} = \hat{f}(0).$$

So here a standard regularization is needed. For details, see 2.5 below on Not So Serious Convergence Problems.

Finally, let see how the explicit formula is established. Here the functional equation plays a key role as in function fields case.

Indeed, by definition,

$$\langle D^\hat{f}, D^1 \rangle = \int_0^1 f(x)\langle D_x, D^1 \rangle \frac{dx}{x} + \int_1^\infty f(x)\langle D_x, D^1 \rangle \frac{dx}{x}.$$ 

Now by the micro explicit formula, we have

$$\langle D^\hat{f}, D^1 \rangle$$

$$= \int_0^1 f(x)\langle D_x, D^0 \rangle + \langle D_x, D^\infty \rangle - \sum_{\xi Q(s)=0} x^s \frac{dx}{x} + \int_1^\infty f(x)\langle D_x, D^0 \rangle + \langle D_x, D^\infty \rangle - \sum_{\xi Q(s)=0} x^{s-1} \frac{dx}{x}.$$ 

This is because by the local explicit formula, we have for $x \in [0,1],$

$$\langle D_x, D^1 \rangle = \langle D_x, D^0 \rangle + \langle D_x, D^\infty \rangle - \sum_{\xi Q(s)=0} x^s.$$ 

Hence, if $x \geq 1,$ we have

$$x\langle D_x, D^1 \rangle = x\langle D_x^1, D^1 \rangle = x\langle D_x^1, D^0 \rangle + x\langle D_x^1, D^\infty \rangle - \sum_{\xi Q(s)=0} x^{1-s} = x\langle D_x^1, D^0 \rangle + x\langle D_x^1, D^\infty \rangle - \sum_{\xi Q(s)=0} x^s$$

where in the last step, we use the functional equation for the Riemann zeta function. Thus in particular, we see that for $x \geq 1,$

$$\langle D_x, D^1 \rangle = \langle D_x, D^0 \rangle + \langle D_x, D^\infty \rangle - \sum_{\xi Q(s)=0} x^s.$$
holds as well. Certainly, then
\[ \langle D_f, D_1 \rangle = \langle D_f, D_0 \rangle + \langle D_f, D_\infty \rangle + \int_0^\infty f(x) \sum_{\xi \mathbb{Q}(s) = 0} x \frac{dx}{x} = \hat{f}(0) + \hat{f}(1) - \sum_{\xi \mathbb{Q}(s) = 0} \hat{f}(s). \]

C.2.4. The Riemann Hypothesis

To finally relate our intersection with the Riemann Hypothesis, we should have a certain positivity. That is to say, we need an analog of the so-called Hodge Index Theorem.

**A Weak Version of Hodge Index Theorem.** With the same notation as above, the self-intersection of the global divisor \( L_f := D_f - \hat{f}(1) D_\infty - \hat{f}(0) D_0 \) is non-positive, i.e.,
\[ \langle L_f, L_f \rangle \leq 0. \] (*)

**Remark.** We call (*) a weak version of Hodge Index Theorem, since
\[ \langle L_f, D_0 + D_\infty \rangle = 0. \]

From now on, let us assume that (*) holds. Then by the Key Relations in the previous section, with a direct calculation, we certainly will arrive at
\[ \sum_{\xi \mathbb{Q}(s) = 0} \hat{f}(s) \cdot \hat{f}(1 - s) \geq 0. \]

This is very nice, since then, following Weil, we may get the Riemann Hypothesis from this latest inequality. (See e.g., page 342 of the second edition of Lang’s Algebraic Number Theorm for more details.)

C.2.5. Not so serious Convergence Problem

We add some remarks on the formal calculation appeared in 2.3. Roughly speaking, to justify them, what we meet is a certain regularized process. This may be done as what Jorgenson and Lang do in their lecture notes on Basic Analysis of Regularized Series and Product. More precisely, motivated by the definitions of hyperbolic Green’s functions (of Selberg, Hejhal, Groos and Zagier,) Ray-Singer’s analytic torsions, we may first introduce imaginary divisors \( D_{f,s} \) by setting
\[ D_{f,s} = \frac{1}{\Gamma(s)} \left( \int_0^1 f(x) D_x x^s \frac{dx}{x} + \int_1^\infty f(x) x D_x x^s \frac{dx}{x} \right) \]
for certain type of suitable functions \( f \) for \( s \) whose real parts are sufficiently large; then assume that in our model \( D_{f,s} \) has a meromorphic continuation to the half space \( \text{Re}(s) \geq -\varepsilon \) with \( \varepsilon > 0 \), from which we could finally get a well-defined \( D_f \) after removing the singularity at \( s = 0 \).

C.2.6. Weil’s Explicit Formula and Two Dimensional Geometric Arithmetic Intersections

I still have not explain why we say the above intersection is indeed an intersection over the geometric arithmetic surface \( \text{Spec}(\mathbb{Z}) \times \text{Spec}(\mathbb{Z}) \). To understand this, we have to use yet another fundamental result of Weil, the Weil Explicit Formula.

Note that by the Key Relations, we obtain the following crucial formula:
\[ \langle D_f, D_1 \rangle = \hat{f}(0) + \hat{f}(1) - \sum_{\xi \mathbb{Q}(s) = 0} \hat{f}(s). \] (*)
On the other hand, as an intersection over $\text{Spec}(\mathbb{Z}) \times \text{Spec}(\mathbb{Z})$, $\langle D_f, D_1 \rangle$ should be counted locally over each point $(p, q) \in \text{Spec}(\mathbb{Z}) \times \text{Spec}(\mathbb{Z})$, i.e., we should have the decomposition

$$\langle D_f, D_1 \rangle = \sum_{p, q \leq \infty} \langle D_f, D_1 \rangle_{(p, q)}.$$ 

Now note that $D_1$ is the diagonal, so besides the points $(p, p)$, $p \leq \infty$ on the diagonal, $\langle D_f, D_1 \rangle_{(p, q)}$ is naturally zero. With this, then we would have

$$\langle D_f, D_1 \rangle = \sum_{p \leq \infty} \langle D_f, D_1 \rangle_{(p, p)}.$$ 

Clearly, this latest expression suggests that via $(\ast)$ above

$$\hat{f}(0) + \hat{f}(1) - \sum_{\xi Q(s) = 0} \hat{f}(s) = \sum_{p \leq \infty} W_p(f),$$ 

where for each places $p$ of $\mathbb{Q}$, i.e., the primes $p$ and the Archimedean place $\infty$. Without any mistake, it is then nothing but Weil’s explicit formula.

On the other hand, this interpretation then naturally leads to a question about the explicit formula for the micro intersection. Recall that one of the key assumption for our micro intersection is that, for $x \in [0, 1]$,

$$\langle D_x, D_1 \rangle = \langle D_x, D_0 \rangle + \langle D_x, D_\infty \rangle - \sum_{\xi Q(s) = 0} x^s.$$ 

Therefore, if we believe that $\langle \cdot, \cdot \rangle$ is indeed a two dimensional intersection on $\text{Spec}(\mathbb{Z}) \times \text{Spec}(\mathbb{Z})$, then we similarly should have

$$1 + x - \sum_{\xi Q(s) = 0} x^s = \sum_{p \leq \infty} W_p(x), \quad \text{for } x \in [0, 1].$$ 

We will call $(\ast \ast \ast)$ the micro explicit formula of Cremér, motivated by Jongenson and Lang’s ‘ladder principle’.

Remark. There are also fundamental works of Deninger and Quillen on the Riemann Hypothesis, based on certain cohomological consideration. While these approaches appear quite different, they share one common part, the Weil Explicit Formula.

C.3. Towards A Geo-Ari Cohomology in Lower Dimensions

C.3.1. Classical Approach in Dimension One

It is clear that we should go beyond a geo-ari intersection: To complete the picture, for example, we need to establish an analog of Hodge index theorem in Geometric Arithmetic, since it is where the positivity comes. Thus, from our experiences with Algebraic Geometry and Arakelov Theory, we are led to develop a corresponding geo-ari cohomology theory.

It is our belief that a general yet well-behavior cohomology theory is at the present time beyond our reach. However this does not mean that we cannot do anything about it. After all, what we need is a practical yet uniform cohomology (and intersection) in dimensions one and two, such that duality, adjunction formula and Riemann-Roch are satisfied.

With this in mind, we recall what happens in geometry for cohomology in dimension one. Classical approaches (to cohomology), such as the one cited in Serre’s GTM on Algebraic Groups and Class Fields, consist of two aspects, namely, the algebraic one and the analytic one. Moreover, with an algebraic approach, we may develop a general sheaf cohomology theory, thanks to the work of Grothendieck. Relatively speaking, for analytic aspect, we have achieved very little. Thus, we want to explore it, since we understand that a geo-ari cohomology should be based on the analytic discussion.
Let $C$ be a regular irreducible reduced projective curve of genus $g$ defined over a field $k$ with $D$ a divisor on $C$. Denote by $F$ its associated function fields. Then the keys to an algebraic cohomology theory may be summarized as follows.

(1) by definition, the 0-th cohomology group of (the divisor class associated to) $D$ is given by $H^0(C, D) := \{f \in F : \text{div}(f) + D \geq 0\};$
(2) From the short exact sequence of sheaves
\[
0 \rightarrow \mathcal{O}_C(D) \rightarrow F \rightarrow F/\mathcal{O}_C(D) \rightarrow 0,
\]
where $F$ denotes the constant sheaf on $C$ associated to the function field $F$, we get a long exact sequence
\[
0 \rightarrow H^0(C, \mathcal{O}_C(D)) \rightarrow F \rightarrow H^0(C, F/\mathcal{O}_C(D)) \rightarrow H^1(C, \mathcal{O}_C(D)) \rightarrow 0.
\]
Thus, by definition, there should be canonical isomorphisms
\[
H^1(X, \mathcal{O}_C(D)) \simeq \mathbb{A}/(\mathbb{A}(D) + F).
\]
Here $\mathbb{A}$ denotes the associated adelic ring, and
\[
\mathbb{A}(D) := \{(r_p) \in \mathbb{A} : \text{ord}_p(r_p) + \text{ord}_D(p) \geq 0\};
\]
(3) For any point $p$, from the structural exact sequence of sheaves $0 \rightarrow \mathcal{O}_C(D) \rightarrow \mathcal{O}_C(D+p) \rightarrow \mathcal{O}_C(D)|_p \rightarrow 0$, we get a long exact sequence of cohomology
\[
0 \rightarrow H^0(C, \mathcal{O}_C(D)) \rightarrow H^0(C, \mathcal{O}_C(D+p)) \rightarrow H^0(C, \mathcal{O}_C(D)|_p) \rightarrow H^1(C, \mathcal{O}_C(D)) \rightarrow H^1(C, \mathcal{O}_C(D+p)) \rightarrow 0;
\]
(4) By studying the residue pairing, we get a canonical isomorphism
\[
\mathbb{A}/(\mathbb{A}(D) + F) \simeq (H^0(C, \mathcal{O}_C(K_C - D)))^\vee
\]
which in particular implies that there exists a natural duality between $H^0(C, D)$ and $H^1(C, K_C - D)$;
(5) Cohomology groups $H^0$ and $H^1$ are all finite dimensional vector spaces. Thus in particular,
\[
h^0(C, D) - h^1(C, D) = h^0(C, D + p) - h^1(C, D + p) - 1.
\]
This then implies the duality and the Riemann-Roch
\[
h^0(C, D) - h^1(C, D) = d(D) - (g - 1).
\]

Next, we describe the analytic aspect of the above cohomology, which is based on a study about certain quotient and sub spaces associated to the adelic ring $\mathbb{A}$.

(1') For any divisor $D$ on $C$, define its associated cohomology groups by $H^0(X, \mathcal{O}_C(D)) = \mathbb{A}(D) \cap F$ and $H^1(C, D) := \mathbb{A}/\mathbb{A}(D) + F$;
(2') There is the following commutative 9-diagram $\Sigma(D)$, whose rows and columns are all exact:
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{A}(D) \cap F & \rightarrow & \mathbb{A}(D) & \rightarrow & \mathbb{A}(D)/\mathbb{A}(D) \cap F \simeq \mathbb{A}(D) + F/F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & F & \rightarrow & \mathbb{A}/F & \rightarrow & \mathbb{A}/F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & F/\mathbb{A}(D) \cap F & \rightarrow & \mathbb{A}(D)/\mathbb{A}(D) & \rightarrow & \mathbb{A}/\mathbb{A}(D) + F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0.
\end{array}
\]
So, we get the exact sequences

\[ 0 \to H^0(C, \mathcal{O}_C(D)) \to F \to F/\mathbf{A}(D) \cap F \to 0, \quad 0 \to F/\mathbf{A}(D) \cap F \to \mathbf{A}/\mathbf{A}(D) \to H^1(C, \mathcal{O}_C(D)) \to 0 \]

which clearly are equivalent to the exact sequence (2) above;

(3') For any \( p \in C \), there exists a natural morphism from \( \Sigma(D) \to \Sigma(D+p) \). Thus, by the five lemma, based on the fact that \( F \) and \( A \) are the same in these two 9-diagrams, we obtain the exact sequences

\[ 0 \to A(D) \cap F \to A(D+p) \cap F \to A(D+p) \cap F/A(D) \cap F \to 0 \]
\[ 0 \to A(D+p) \cap F/A(D) \cap F \to A(D+p)/A(D) \to A(D+p) + F/A(D) + F \to 0 \]
\[ 0 \to A(D+p) + F/A(D) + F \to A/A(D) + F \to A/A(D+p) + F \to 0. \]

Clearly, these are equivalent to the exact sequence (3) above;

(4') Residue pairing works at the level of adelic language as well by the self-dual property of \( A \).

Therefore, provided that we know how to count the terms involved, namely, that we have an analog of (5) above, we can develop a cohomology theory using only adelic language (which satisfies duality and the Riemann-Roch theorem).

However, generally speaking, the counting is a very difficult one. In algebraic approach, this is based on the fact that all coherent sheaves are locally finitely generated. In analytic approach, the counting will be based on the spacial analytic properties of \( A \). To explain it, as an example, we now consider the simplest case, namely, when the constant field \( k \) is \( \mathbb{F}_q \), the finite field with \( q \) elements. (Over number fields, we count the geo-ari cohomology by Tate’s Fourier analysis over \( A \).)

Recall that with respect to the natural topology on \( A \), \( F \) is discrete and \( A(D) \) is compact. Thus in particular, \( H^0(C, \mathcal{O}_C(D)) = A(D) \cap F \) is finite. Similarly, since \( A(D) \) is compact and \( A/F \) is compact so \( A(D) + F/F \) is again compact. But \( k \) is a finite fields, so compactness implies finiteness, and the number of elements in a finite dimensional space is simply \( q \) to the power of the corresponding dimension. Thus, with respect to the natural Haar measures on the associated groups induced from that on \( A \),

\[ \text{Vol}(A(D)) = q^{h^0(C, \mathcal{O}_C(D))} \cdot \text{Vol}(A(D) + F/F) \]

and

\[ \text{Vol}(A/F) = q^{h^1(C, \mathcal{O}_C(D))} \cdot \text{Vol}(A(D) + F/F). \]

Therefore,

\[ q^{h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D))} = \frac{\text{Vol}(A(D))}{\text{Vol}(A/F)}. \]

Easily by definition,

\[ \text{Vol}(A(D)) = q^{d(D)}, \quad \text{Vol}(A/F) = q^{g-1}, \]

so we obtain an analytic proof of the Riemann-Roch theorem

\[ h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = d(D) - (g - 1). \]

**C.3.2. Chevalley’s Linear Compacity**

The above discussion for curves over finite fields is not valid for general curves, since even the space is finite-dimensional, it is not finite itself. To overcome this difficult, Chevalley introduced his linear compacity. Next, we indicate how Chevalley’s method works. Instead of recalling all the details, I indicate what are the essential points involved. (For details, we recommend the reader to consult Iwasawa’s Princeton lecture notes.)

(0) There exists the 9-diagram as above:
(1) (Additive Structure) Among subquotient groups of the adelic ring are topological spaces called discrete objects and linearly compact objects. Moreover, all groups used in our 9-diagram are supposed to be locally linearly compact, i.e., they are either extensions of discrete objects by linearly compact objects, or extensions of linearly compact objects by discrete objects, or simply generated by finitely many discrete objects and linearly compact objects. In particular, \( A \) is selfdual and locally linearly compact.

(2) Discrete objects and linearly compact objects are dual to each other. Thus, if an object is both locally linearly compact and discrete, it is then isomorphic to a finite dimensional \( k \) vector space, and hence the dimension may be counted;

(3) (Multiplicative Structure) Under the multiplication, \( A \) is self-dual. As a direct consequence, we get the duality.

As such, the counting may be proceed as follows to offer the Riemann-Roch:

(i) By definition, \( \dim(A(0) \cap F) = 1 \) and \( \dim(A/A(0) + F) = g \);

(ii) By counting local contribution, \( [A(D + p) : A(D)] = d(D + p) - d(D) = 1 \), despite the fact that \( A(D) \) and \( A(D + p) \) cannot be counted;

(iii) By definition, \( [A(D + p) : A(D)] = \dim(A(D + p)/A(D)) \);

(iv) The fundamental theorem of isomorphisms for groups implies that

\[
\dim(A(D + p)/A(D)) = \dim(A(D + p) + F/A(D) + F) + \dim(A(D + p) \cap F/A(D) \cap F)
= \dim(A/A(D) + F - \dim(A/A(D) + p) + F + \dim(A(D + p) + F - \dim(A(D) + F)
= (h^0(C, D + p) - h^1(C, D + p)) - (h^0(C, D) - h^1(C, D)).
\]

At this point, I would like to point out that it is quite essential to combine the approach here with our approach to geo-arith cohomology for number fields via Fourier analysis.

C.3.3. Adelic Approach in Geometric Dimension Two

Now we consider two dimensional case. Besides the definition of cohomology groups and the counting of these cohomology groups, from algebraic geometry and Arakelov theory, we know that the (weak) Riemann-Roch theorem and duality may be obtained via the adjunction formula, a one dimensional Riemann-Roch and a long exact sequence of cohomology groups resulting from a short exact sequence of sheaves

\[
0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D + C) \rightarrow \mathcal{O}(D)|_{C} \rightarrow 0
\]

for a divisor \( D \) and a regular curve \( C \) on the surface, and a residue discussion.

On the other hand, just as for curves, we want to develop a two dimensional cohomology theory using only adelic language. So two parts are involved:

(I) (Algebraic Structure) Definition of cohomology groups and some associated structural exact sequences;

(II) (Analytic Structure) Counting of the cohomology groups in (1).

We start with the algebraic structure. In algebraic geometry, this is essentially given by Parshin as a by-product of his discussion on residues. As above, we here only indicate the main points. (Interesting reader may consult Prshin’s original paper for the details.)

So for a surface \( S \) with function field \( F \), we introduce its associated ring of adeles \( A \). Inside \( A \) are two subrings, which we denote by \( A_0 \) and \( A_1 \), respectively. Similarly, for a divisor \( D \) on \( S \), introduce its associated subgroup \( A(D) \) as in curve case. In particular, then we have the following three 9-diagrams:
for which isomorphisms

\[ A(D) / A(D) \cap (A_0 \cap A_1) \simeq A(D) + A_0 / A_0 \cap A_1 \]

and

\[ A_0 \cap A_1 / A(D) \cap A_0 \cap A_1 \simeq A(D) + (A_0 \cap A_1) / A(D) \]

are used; and

\[
\begin{array}{cccccc}
0 & \rightarrow & A(D) \cap (A_0 \cap A_1) & \rightarrow & A(D) \cap A_1 & \rightarrow & A(D) \cap (A_0 \cap A_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A(D) \cap A_0 & \rightarrow & A(D) \cap (A_0 + A_1) & \rightarrow & A(D) \cap (A_0 + A_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{A(D) \cap A_0}{A(D) \cap (A_0 \cap A_1)} & \rightarrow & \frac{A(D) \cap (A_0 + A_1)}{A(D) \cap (A_0 \cap A_1)} & \rightarrow & \frac{A(D) \cap (A_0 + A_1)}{A(D) \cap (A_0 + A_1)} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

for which the isomorphisms

\[ A(D) \cap A_1 / A(D) \cap (A_0 \cap A_1) \simeq A(D) \cap A_0 + A(D) \cap A_1 / A(D) \cap A_0 \]

and

\[ A(D) \cap A_0 / A(D) \cap (A_0 \cap A_1) \simeq A(D) \cap A_0 + A(D) \cap A_1 / A(D) \cap A_1 \]

are used; and

\[
\begin{array}{cccccc}
0 & \rightarrow & A(D) \cap (A_0 + A_1) & \rightarrow & A(D) & \rightarrow & A(D) / A(D) \cap (A_0 + A_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & A_0 + A_1 & \rightarrow & A & \rightarrow & A / (A_0 + A_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (A_0 + A_1) / A(D) \cap (A_0 + A_1) & \rightarrow & A / A(D) & \rightarrow & A / A(D) + (A_0 + A_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]

for which the isomorphisms

\[ A(D) / A(D) \cap (A_0 + A_1) \simeq A(D) + (A_0 + A_1) / (A_0 + A_1) \]

and

\[ (A_0 + A_1) / A(D) \cap (A_0 + A_1) \simeq A(D) + (A_0 + A_1) / A(D) \]

are used.

Now, define the cohomology groups \( H^i, i = 0, 1, 2 \) by setting

\[ H^0(S, D) := A(D) \cap (A_0 \cap A_1); \]

\[ H^1(S, D) := A(D) \cap (A_0 + A_1) / A(D) \cap A_0 + A(D) \cap A_1; \]

and

\[ H^2(S, D) := A / A(D) + (A_0 + A_1). \]
Moreover, by working with $D + C$ for regular curve $C$ on $S$, we could get another set of three 9-diagrams, to which there are natural morphisms from the three 9-diagrams for $D$ above. As a direct consequence, by a snake chasing, we arrive at the following long exact sequence of cohomologies, which as stated above, plays a key role in the induction process:

$$0 \rightarrow H^0(S, D) \rightarrow H^0(S, D + C) \rightarrow H^0(C, D + C|_C)$$
$$\rightarrow H^1(S, D) \rightarrow H^1(S, D + C) \rightarrow H^1(C, D + C|_C) \rightarrow H^2(S, D) \rightarrow H^2(S, D + C) \rightarrow 0.$$  

In this way, provided that a good counting is available, we then are able to give an adelic approach to the weak Riemann-Roch in dimension two by using the Riemann-Roch for curves with the help of the adjunction formula.

To understand the analytic structure, motivated by Chevalley’s theory on linear compactness for adelic rings of algebraic curves, we need to study the terms used in the above algebraic discussion. The key is the self-dual property of $A$. In fact, we have the following canonical isomorphisms:

(a) $A^\perp_0 \cong A_1, A^\perp_1 \cong A_0$, and $A(D)^\perp = A(K_S - D)$, where $K_S$ denotes a canonical divisor of $S$. In particular,

$$(A_0 + A_1)^\perp \cong A_1 \cap A_2, \quad (A_1 \cap A_2)^\perp \cong A_0 + A_1.$$

(b) Duality between $H^0(S, D)$ and $H^2(S, K_S - D)$

$$(A(D) \cap A_0 \cap A_1)^\perp \cong A/(A(D) \cap A_0 \cap A_1)^\perp$$
$$\cong A/A(D)^\perp + A^\perp_0 + A^\perp_1 \cong A/A(K_S - D) + (A_0 + A_1),$$

(c) Note that $A_0 \cap A_1 = F$ is nothing but the function field of $S$. Moreover, one checks that algebraically

$$A(D) \cap (A_0 + A_1)/A(D) \cap A_0 + A(D) \cap A_1$$
$$\cong A_0 \cap (A(D) + A_1)/(A(D) \cap A_0 + A_0 \cap A_1)$$
$$\cong A_1 \cap (A(D) + A_0)/(A(D) \cap A_1 + A_0 \cap A_1);$$

(d) Duality between $H^1(S, D)$ and $H^1(S, K_S - D)$:

$$\left((A_0 \cap (A(D) + A_1)/(A(D) \cap A_0 + A_0 \cap A_1))\right)^\perp$$
$$\cong (A(D) \cap A_0 + A_0 \cap A_1)^\perp/(A_0 \cap (A(D) + A_1))^\perp$$
$$\cong (A(K_S - D) + A_1) \cap (A_0 + A_1)/(A_1 + A(K_S - D) \cap A_0)$$
$$\cong (A(K_S - D) \cap A_0 + A_1)/A(K_S - D) \cap A_0 + A(D) \cap A_1.$$

Thus, to attack (II), the analytic structure aiming at a reasonable counting, we should introduce a geo-ari theory for surfaces which is compatible with (a), (b), (c) and (d) above, similar to that of linear compactness of Chevalley used in proving the Riemann-Roch in dimension one. For example, we should have a notion of geo-ari compactness such that if a space is both discrete and geo-ari compact, it should be of finite dimension; moreover, the duality should transform discrete spaces to geo-ari compact spaces and vice versa.

Note that for curves, if the base field is finite, then we can equally use Fourier analysis to do the counting. Thus, for two dimensional surfaces, a similar discussion should work as well. All this then inevitably leads to a geo-ari cohomology in dimensions one and two over number fields, our primary goal.
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