Explanation of an unexpected occurrence of $\nu = \pm \frac{1}{2}$ fractional quantum Hall effect states in monolayer graphene

Janusz E Jacak

Department of Quantum Technologies, Wroclaw University of Science and Technology, Wyb. Wyspianskiego 27, 50-370 Wroclaw, Poland

E-mail: janusz.jacak@pwr.edu.pl

Received 29 May 2019, revised 26 July 2019
Accepted for publication 8 August 2019
Published 22 August 2019

Abstract

Recent experiment reveals the appearance of incompressible fractional quantum Hall effect states in monolayer graphene at $\nu = \pm \frac{1}{2}$ and $\pm \frac{1}{4}$ substituting the compressible Hall metal states at these fillings in the lowest Landau level in a narrow magnetic field window depending on the sample parameters. Simultaneously, none such behavior has been observed either for $\nu = \pm \frac{1}{2}$ or $\pm \frac{3}{4}, \pm \frac{5}{4}$. We propose an explanation of these observations in terms of homotopy of monolayer graphene in consistence with a general theory of correlated states in planar Hall systems.

Keywords: monolayer graphene, fractional quantum Hall effect (FQHE), even denominator filling rate, beyond composite fermions

1. Introduction

In the conventional two-dimensional (2D) Hall material, GaAs 2D electron system (2DES), at low temperatures (in mK scale) and at high magnetic field (of order of 10 T) the fractional quantum Hall effect (FQHE) [1, 2] is observed in great detail revealing the mysterious discrete hierarchy of Landau level fillings. This hierarchy was the subject of intensive theoretical studies and the consistence with experiments has been achieved within the homotopy cyclotron braid commensurability approach [3] which generalized the former composite fermion (CF) model [4]. The latter well predicted the main line of the FQHE hierarchy in the lowest Landau level (LLL) of GaAs except for the so-called enigmatic states [5]. Inclusion of nesting of multiloop cyclotron orbits with next-nearest neighbors in multiparticle correlations allowed, however, for explanation also of enigmatic states beyond the CF model restricted to only nearest neighbor correlations. Both, the CF model and the homotopy model, predicted the compressible multiparticle states of Hall metal type at $\nu = \frac{1}{2}, \frac{1}{4}$ in GaAs, which agrees with experimental observations in GaAs [2].

Nevertheless, in the recent paper [6] it has been reported a puzzling appearance of incompressible FQHE states at $\nu = \pm \frac{1}{2}$ and $\pm \frac{1}{4}$ in the first subband of the LLL in graphene monolayer at low temperatures ($\sim 300$ mK), whereas neither at $\nu = \pm \frac{3}{2}$ nor at $\pm \frac{5}{4}, \frac{7}{4}$. The even denominator FQHE states in the first subband of the LLL in monolayer graphene occurred only within some relatively narrow magnetic field windows (of a few T width around 28 T or 21 T, depending on a sample), i.e. they disappear beyond and beneath of this window borders—see. Figures 1 and 2. This behavior is
astonishing because in the LLL, both in spin-up and spin-down its subbands of GaAs 2DES, the Hall metal phase is observed at $\nu = \frac{1}{2}, \frac{1}{4}, \frac{3}{2}, \frac{3}{4}, \frac{5}{2}, \frac{5}{4}, \frac{7}{2}, \frac{7}{4}$, which agrees with the conventional predictions of hierarchy for FQHE in the LLL [3, 7].

These unusual states at fractions with denominators 2 and 4 in the first subband of the LLL in monolayer graphene do not find any explanation in the framework of a conventional theory of FQHE in the LLL of monolayer graphene [6, 8]. The authors of the experiment [6] suggest that the occurrence of even denominator FQH phases closer to $\nu = 0$ (in $(0, 1)$ range) but not in the more distant region $(1, 2)$, is connected with magnetic field induced competition or influence of various other correlated states which has been theoretically identified recently [9] at $\nu = 0$ in monolayer graphene exposed to the magnetic field—as presented in figure 3. This competition of magnetic-spin correlated phases mixed with valley pseudospin structure in monolayer graphene leads to phase diagram for $\nu = 0$ as shown in figure 3(d). By comparison of the positions of magnetic field windows at which the state $\nu = \pm \frac{1}{2}$ occurred (in various samples) versus sublattice splitting energy $\Delta_{AB}$—see. Figure 3, one can notice that the so-called valley-coherent partially sublattice polarized phase (PSP) competing with the canted antiferromagnet (CAF) might have something in common with the appearance of even denominator FQH phases closer to $\nu = 0$ (in $(0, 1)$ range) but not in the more distant region $(1, 2)$. None arguments have been, however, drawn out to support the existence of even denominator FQH states in the first subband of the LLL.
FQHE at $\nu = \frac{1}{2}$ or $\frac{3}{2}$ [7], whereas the homotopy theory of FQHE [3] offers the energy competition of various commensurability patterns at the same $\nu$ including nesting of multiloop cyclotron braids with nearest and next-nearest neighbors in Wigner-type web of electrons induced by Coulomb repulsion of electrons. Some homotopy patterns for multiparticle correlations in planar system in the presence of magnetic field result also in correlated incompressible FQHE states at $\nu = \frac{1}{2}$, $\frac{3}{2}$ besides Hall metal states at these fillings, and the realization of the compressible Hall metal phase is the matter of the energy competition between various possible multiparticle phase organizations. This energy competition is conditioned by the particular shape of multi-particle wave functions corresponding to different homotopy patterns of correlations. The multiparticle wave functions for the homotopy patterns in GaAs and in graphene monolayer are different and this is probably the source of reported in [6] oddness with states at $\nu = \pm \frac{1}{2}$ and $\pm \frac{3}{2}$ in graphene monolayer in comparison to GaAs.

The paper is organized as follows, in the following paragraph, the main points of the homotopy cyclotron braid group approach to FQHE hierarchy are outlined in comparison to the conventional CF model. Next, the method of construction of trial multiparticle wave functions for various correlation homotopy patterns is described for both materials, GaAs and monolayer graphene. Next, the discussion of the states at $\nu = \frac{1}{2}$, $\frac{3}{2}$ in graphene monolayer is addressed to elucidate observations reported in the paper [6]. Some particular explanation is shifted to appendix.

2. Main points of the homotopy approach to FQHE

FQHE is observed at large magnetic field, $B$, for which the cyclotron orbit size, $\frac{e}{h} S \left( \frac{2}{3} \right)$ is the magnetic field quantum is smaller than the particle separation, $\frac{\hbar}{e B}$ (here, $S$ is the planar sample surface, $N$ is the number of electrons) i.e. when,

$$\frac{h}{eB} < \frac{S}{N},$$

(1)

Note that for $B_0$ at which,

$$\frac{h}{eB_0} = \frac{S}{N},$$

(2)

we deal with the integer quantum Hall effect (IQHE), because the degeneracy of LLs at the field $B$ equals to,

$$N_0 = \frac{BSe}{h},$$

(3)

and for $B = B_0$ it is the same equation as equation (2) taken at $N = N_0$, thus $\nu = \frac{N}{N_0} = 1$.

However, for $B_{1/3} = 3B_0$ (at $S$ and $N$ kept constant) equation (2) does not hold and the inequality (1) might be lifted to the equality by 3-fold enhancement the magnetic field flux quantum, $\frac{2}{3} \rightarrow \frac{2}{3} \frac{B}{B_{1/3}}$. An increase of the magnetic field flux quantum is proven for multiloop trajectories (the formal proof goes via the Bohr–Sommerfeld rule [3]), resulting in the magnetic flux quantum, $\Phi_k = \frac{(2k+1)h}{e}$, for planar braids with additional $k$ loops—see. Appendix. Braids define exchanges of identical indistinguishable particles in the classical multiparticle configuration space and must nest to at least neighboring particles uniformly distributed on the plane due to the Coulomb repulsion. Shorter braids cannot be defined. For planar electron systems exposed to the perpendicular magnetic field, braids are of finite range because of cyclotron effect, i.e. are of size $\frac{\Phi_k}{B}$, Ordinary braids (without any loops, i.e. with $k = 0$ for corresponding magnetic field flux quantum, $\Phi_0 = \frac{h}{e}$) are too short to reach neighbors at $B_{1/3}$, but in this case the larger braids with an additional loop, i.e. for $k = 1$, of size $\frac{\Phi_1}{B}$, the braids with one additional loop perfectly fit to particle separation,

$$\frac{S}{N} = \frac{3h}{e3B_0},$$

(4)

which gives,

$$\nu = \frac{N}{N_0} = \frac{Nh}{3B_0Se} = \frac{1}{3}$$

(5)
The hierarchy (8) reproduces all experimentally observed FQHE filling ratios in GaAs in the LLL, including enigmatic states, \( \nu = \frac{q}{N} \), \( q = 2k + 1 \) odd integer and \( y \geq x \geq 1 \) positive integers \( 1, 2, \ldots \). The hierarchy (8) reproduces the CF function of \( q = 2k + 1 \) with \( k \) denoting number of flux quanta pinned to CFs, \( y = 1, 2, 3, \ldots \) — positive integer, and \( \pm \) indicates the orientation of the resultant magnetic field (the external magnetic field \( B \) reduced by averaged field of CF fluxes) conformal (+) or opposite (−) with respect to the external \( B \) field orientation. One can notice that the formula (8) reproduces the CF hierarchy \( x = 1 \). Thus, the CF model is an effective illustration of multiloop cyclotron orbits and each flux quantum of the auxiliary field pinned to the CF indicates a sole loop in multiloop structure described above. The CF hierarchy agrees with the hierarchy (8) at \( x = 1 \), which means that the model construction of CFs cannot account for all types of homotopy patterns when the nesting with next-nearest neighbors with \( x > 1 \) of \( q - 1 \) first loops contribute to the commensurability condition (7). CFs neglect thus the possibility of a more complicated homotopy braid commensurability patterns involving next-nearest neighbors.

The above presented cyclotron braid commensurability scheme can be generalized by inclusion of the nesting of braids also with next-nearest neighbors and for each loop of the multiloop braid separately, which leads to the commensurability condition [3, 11],

\[
\frac{S}{N} = \frac{h}{eBx_1} + \frac{h}{eBx_2} + \cdots + \frac{h}{eBx_q},
\]

where \( q = 2k + 1 \) is the number of cyclotron loops (\( k \) is the number of loops in the braid), \( x_i \geq 1 \) denotes the order of next-nearest neighbors commensurate with \( i \)th loop of the multiloop orbit (\( x_1 \) denotes the nearest neighbor commensurability of \( i \)th loop), \( \pm \) indicates possible congruent, +, or inverted, −, orientation of the subsequent loop with respect to the preceding one, (i.e. of the eight-digit shape orbit in the case of −, or, in other words, inverse to the external field flux). Due to energy preference discussed in [3, 11] one can simplify (for GaAs 2DES case) the above condition assuming \( x_1 = x_2 = \ldots = x_{q-1} = x \) and \( x_q = y \), and express it in the following form,

\[
\frac{S}{N} = \frac{(q - 1)h}{eBx} \pm \frac{h}{eB_y},
\]

which defines the hierarchy of FQHE in the LLL (when \( N_0 \) is given by equation (3)).
singular) factor of N-fold product of single-particle exponents \( e^{-|z|^2/4\ell_b^2} \) (where \( z \) is the complex representation of \( i \)th particle position on the plane and \( \ell_b = \sqrt{\frac{eB}{2m}} \) is the magnetic length). The form of exponential factor is the same as in the gas system and is maintained for an arbitrary state when interparticle interaction is switched-on (like for GaAs). In graphene the single-particle Landau states are, however, modified by graphene crystal field, which affect the envelope multiparticle factor besides the polynomial part, the same as in GaAs and conditioned by the selected cyclotron braid group acc. to the commensurability pattern. Scalar unitary representations of the cyclotron braid groups defining multiparticle correlations in corresponding homotopy phases in the LLL determine the shape of the related multiparticle holomorphic wave function (the polynomial part of the multiparticle wave function) in an unambiguous manner, as illustrated in [3]. This procedure is exact in contrary to the so-called projection on the LLL of higher LL wave functions in order to remove singularities in the CF model (being not unambiguously defined [7]). The envelope of the multiparticle wave function, invariant with respect to any braid exchanges of particles has the form \( e^{-\sum_{x\in\mathbb{Z}} |x|^2/4\ell_b^2} \), but only in GaAs case (when the Hilbert space of the \( N \)-particle system is spanned by antisymmetrized products of single-particle gaseous Landau functions (at symmetric gauge)).

The hierarchy of Hall-metal states can be determined by the limit \( y \to \infty \) of the FQHE general hierarchy (8),

\[
\nu_{H\text{m}} = \lim_{y \to \infty} \frac{A_y}{(q-1)y+x} = \frac{x}{q-1},
\]

the limit \( y \to \infty \) reflects the range of the last loop of the multi-loop orbit not limited as in the ordinary compressible Fermi liquid—thus defines the Hall metal state. From (9) one can get at \( x = 1 \), \( \nu_{H\text{m}} = \frac{1}{2}, \frac{2}{2}, \cdots \), (\( \nu_{H\text{m}} = \frac{1}{3} \) is the filling ratio for a dual Hall metal state of LL-band holes in the LLL).

### 3. Explanation of even denominators in FQHE hierarchy in the LLL of monolayer graphene by homotopy-patterns

In graphene the crystal structure with two carbon atoms per elementary cell leads to spin-valley four-fold structure of LLs in magnetic field [8]. The crystal field in graphene monolayer influences ordinary quantum Landau quantization and instead of the single-particle Landau dispersion, \( E_n = \hbar\omega_B (n + \frac{1}{2}) \) with \( \omega_B = \frac{eB}{2m} \), as in the electron gas, the LL dispersion in monolayer graphene is, \( E_n = \hbar\omega'\sqrt{n} \) with \( \omega' = \frac{2\nu_F}{\ell_b} \). (\( \nu_F \) is the Fermi velocity) [8]. This difference in the LL energy is caused by the relativistic-type electron spectrum near Dirac cones in corners of the hexagonal Brillouin zone of the graphene monolayer. The crystal field does not change, however, the homotopy of braid trajectories. The latter is governed in the presence of the perpendicular magnetic field by the commensurability of cyclotron orbits of bare electrons on the plane with Wigner-type electron crystal lattice. The bare kinetical energy core of electrons at magnetic field presence in graphene monolayer is the same as in the gas, \( \hbar\omega_B (n + \frac{1}{2}) \), and only due to the crystal field (electric-type interaction of electrons with the graphene crystal lattice) the dressed energy attains the form \( \hbar\omega'\sqrt{n} \). The homotopy classes are robust against the crystal field and the latter does not modify the FQHE hierarchy resulted from the cyclotron braid commensurability restrictions, the same as in 2DES. In the case of 2DES, the Wigner crystal is the triangle planar lattice—the classical lowest energy distribution of electrons on jellium at \( T = 0 \) K (when classical kinetetical energy vanishes). The self-energies of particular homotopy classes depend, however, of the crystal field dressing electrons in graphene in different manner than in GaAs. Thus despite the same FQHE hierarchy the experimental its manifestation may differ in various materials due to energy competition between various homotopy phases admissible at the same filling rate. The self-energy depends on the shape of the multiparticle wave function defined in the LLL by the same symmetry (scalar unitary representations of particular cyclotron braid subgroups), however, in different subspace of multiparticle Hilbert space because of different envelope part of wave functions. The subspace spanned by combinations of single-particle Landau wave functions are different in graphene than in GaAs.

To account this effect qualitatively let us invoke the study of electron distribution in graphene monolayer at \( \nu = 0 \) with respect to the magnetic field [9], which reveals a competition of various phases including spin and valley-pseudospin. Of particular interest is the charge density wave type distribution just in the similar window for the magnetic field as that one for which the astonishing even denominator FQHE states occur [6]—as imagined in figure 3. When the electron density wave with periodicity scale of order of elementary cell of graphene interferes with the ideal Wigner crystal distribution with larger spatial scale governed by the magnetic length, the correlations of next nearest neighbors may be energetically favored, at least close to \( \nu = 0 \). Preferring by the wave function of some kind of correlation by commensurate density concentration resolves itself to the increase of number of loops linking favored next-nearest neighbors, because this enhances multiplicity of zeros (as in Laughlin function) for

| Table 1. Examples of homotopy patterns according to equation (10) and corresponding filling fractions \( \nu \) |
|----------------|----------------|----------------|
| \( x_1, x_2, x_3 \) | \( \nu = N/N_0 \) | Type |
| 1, 1, 1 | 1/3 | CF |
| 1, 1, 2 | 2/5 | CF |
| 1, 1, 2 | 1/2 | Not CF |
| \( x_1, x_2, x_3, x_4, x_5 \) | \( \nu = N/N_0 \) | Type |
| 1, 2, 2, 2, 2 | 1/3 | Not CF |
| 1, 2, 2, 2, -2 | 1/2 | Not CF |
| 1, 1, 1, 2, 2 | 1/4 | Not CF |
| 1, 1, 1, 1, -1 | 1/3 | CF |
| 1, 1, 1, 2, -2 | 1/3 | Not CF |
| 1, 2, 2, 2, -2 | 1/2 | Not CF |
| 1, 1, 2, 2, -2 | 2/5 | Not CF |
| \( x_1, x_2, x_3, x_4, x_5, x_6, x_7 \) | \( \nu = N/N_0 \) | Type |
| 1, 2, 2, 2, 2, 2 | 1/4 | Not CF |
selected group of electrons, which reduces repulsion energy of these electrons. In such a case, various commensurability parameter sets, \( x_1, \ldots, x_q \) in equation (6), giving the same \( \nu \) will not result finally in the lower energy the larger amount of \( x_1 = 1 \) occurs, as it was the case in GaAs, but inversely, for as most as possibly \( x_i > 1 \) occur corresponding to the next-nearest neighbors. If the influence of the charge density wave stable at \( \nu = 0 \) weakens for more distant \( \nu \), the privilege of next nearest neighbors also diminishes. It will happen at lower magnetic field (as in the next subband of the LLL in graphene monolayer) where the Wigner crystal concerns only \( \sqrt{2}/2 \) of electrons and the magnetic length (\( \sim \frac{1}{\sqrt{B}} \)) grows resulting in the escape from the interference with of much smaller spatial scale charge density wave stable at \( \nu = 0 \).

The scenario described above agrees with the experimental observations [6]. The Hall metal stable in the LLL of GaAs at \( \nu = \frac{1}{3}, \frac{1}{2} \) (and \( \frac{3}{2} \) for LL band holes) and corresponding to the limit \( y \to \infty \) of the condition (8) but with \( x_1 = \ldots = x_{q-1} = x = 1 \) (nearest neighbors) is substituted in graphene by another homotopy patterns corresponding to \( \nu = \frac{1}{3}, \frac{1}{2} \) with as much as possible loops commensurate with next-nearest neighbors. Especially frequently \( \frac{1}{3} \) and \( \frac{1}{2} \) occur as filling ratios for homotopy patterns with all finite \( x_i \) (including \( y = x_q \)) for hierarchy family \( (q = 3, 5, 7) \),

\[
\nu = \left(1/x_1 + 1/x_2 + \cdots + 1/x_q\right)^{-1},
\]

(10)

with \( x_i \geq 2 \) for \( i > 1 \) (equation (10) is equation (6) with equation (3) with \( \pm \) incorporated in \( x_q \)). This situation is exemplified in table 1 and illustrated in figure 5. Note that homotopy patterns with several \( x_i > 1 \) do not allow for the CF picture which is restricted only to \( x = 1 \) in hierarchy (8). These states are, however, incompressible (with stiffer fixed surface \( S \) required for commensurability condition) FQHE states as observed in the experiment [6].

This behavior is linked with the property that for the same filling ratio \( \nu \) various homotopy patterns (denoted by \( (x_i, i = 1, \ldots, q) \)) are possible in general, and the eigen-energy decides which pattern is stable. The lower energy assigns the ground state at particular \( \nu \). The Hall metal state at \( \nu = \pm \frac{1}{2} \) or at \( \nu = \pm \frac{1}{2}, \pm \frac{3}{2} \) (\( \pm \) reflects here the mirrored particle-hole states in band structure of the monolayer graphene, symmetric with respect to the zeroth energy at the center of the LLL four-fold degenerated in spin-valley degrees of freedom [8]) have the lower energy than other available (according to equation (10)) homotopy patterns, unless these Hall states are overcome in energy gain by homotopy patterns privileging next-nearest neighbors. In the window for magnetic field as indicated in figures 1 and 2 such states are incompressible FQHE state with several \( x_i > 1 \), as shown in table 1. Simultaneously, other fractional states are slightly changed in stability because the homotopy patterns with several \( x_i > 1 \) substitute patterns with \( x = 1 \) at the same \( \nu \) (as in table 1 shown for \( \nu = \frac{1}{3}, \frac{2}{3} \), see figure 6). This small decrease of the stability of these states is also noticeable in the experiment [6]. All states corresponding to homotopy patterns with several \( x_i > 1 \) are not of CF type.

---

**Figure 5.** (A) Cartoon visualization of three first homotopy patterns for three-loop cyclotron trajectories as listed in table 1: the first pattern with the signature, \( x_1 = 1, x_2 = 1, x_3 = 1 \), corresponds to CF-type FQHE state at filling rate \( \nu = \frac{1}{3} \)—commensurabilities of all three loops concern nearest neighboring electrons; the second pattern with the signature, \( x_1 = 1, x_2 = 1, x_3 = 2 \), corresponds also to CF-type FQHE state at filling rate \( \nu = \frac{2}{5} \)—commensurabilities of two loops concern nearest neighbors, whereas of the third one concerns next-nearest neighbors of order 2; the third pattern with the signature, \( x_1 = 1, x_2 = 2, x_3 = 2 \), does not correspond to CF-type FQHE state at filling rate \( \nu = \frac{1}{2} \) that which is observed in experiment [6]—commensurability of only one loop concerns nearest neighbors, whereas of the remaining two loops concern next-nearest neighbors of order 2. (B) Schematic visualisation of the commensurability of Wigner-type electron distribution in graphene monolayer (the triangle Wigner 2D lattice of electrons indicated as blue balls)—three situations are shown, the first one is the sigleloop cyclotron orbit commensurability with nearest neighbors, which corresponds to IQHE at field \( B_0 \), the second one concerns three-times larger field \( 3B_0 \) at which sigloople orbit is too short to match nearest neighbors in the Wigner lattice—but the three-loop orbit perfectly fits to nearest neighbors, which corresponds to FQHE at \( \nu = \frac{1}{3} \) and the third case, which concerns the field \( 2B_0 \) at which singleloop orbit is also too short in comparison to electron separation, but the homotopy pattern with \( x_1 = 1, x_2 = 2, x_3 = 2 \) (see table 1) allows the braod commensurability with electron distribution including next-nearest neighbors for two loops—this state is of incompressible FQHE type at \( \nu = \frac{1}{3} \), beyond the CF model.

---

**Table 1.** Commensurability of loops of FQHE state at filling rate \( \nu = \frac{1}{3} \), beyond the CF model.
For lower magnetic field, when the magnetic length $l_B = \sqrt{\frac{\hbar}{eB}}$ grows, the modification of the Wigner crystal accommodated in the second subband of the LLL to the half of the electron amount, $N/2$, is of lower significance as the interference with charge density wave diminishes. In this range of $\nu$ the nearest neighbors ($x = 1$) are again favored and the CF-type homotopy patterns energetically prevail in energy over next-nearest neighbor nesting. Moreover, it is reasonably to expect that the charge density wave trace in the multiparticle wave functions in the range $\nu \in (1, 2)$ is weaker than in the region $\nu \in (0, 1)$ due to larger departure from $\nu = 0$ at which the charge density wave is stable. The window with partially sublattice polarized (PSP) corresponding to the stable charge density wave does not overlap with smaller magnetic field corresponding to the second subband of the LLL in graphene monolayer [9, 16] and CF-type FQHE hierarchy with $x = 1$ is restored in $\nu \in (1, 2)$ sector, which agrees with the observations [6].

4. Conclusions

It has been demonstrated that the FQHE incompressible states in the LLL can occur also at filling fractions with even denominators, including 2 and 4, as the commensurability condition admits such filling fraction though not of CF type. Because of the energy competition between various homotopy phases at the same filling fraction the occurrence in the experiment of particular hierarchy depends on the shape of multiparticle wave functions corresponding to symmetry imposed by the commensurability patterns including various divisions of correlation among nearest and next-nearest neighbor correlations, then not CF states at $\nu = \frac{1}{4}$, $\frac{5}{12}$ are more stable than competitors of CF-type at these filling rates.

Appendix. Multi-loop cyclotron braids have larger size than single-loop ones—quantum of the magnetic field flux is not universally defined

Here we prove that the magnetic field flux quantum changes its value in various homotopy phases. In particular, for multi-loop cyclotron braid orbits the larger field flux quantum defines larger dimension of the orbit, and it can reach particles too distant for single-loop orbits. This fact is the origin of the FQHE.

Let us consider the Bohr–Sommerfeld rule, which links the area of the 1D phase space ranged by the classical phase trajectory loop with the corresponding number of quantum states. The quasiclassical wave function in a 1D well, $U(x)$, with turning points $a$ and $b$ has the form,

Figure 6. (A) Schematic visualisation of the commensurability of Wigner-type electron distribution in graphene monolayer (the triangle Wigner 2D lattice of electrons indicated as blue balls)—two different homotopy patterns for $\nu = \frac{1}{3}$ are shown, the first one is of CF-type (i.e. with $x_1 = x_2 = x_3 = 1$, see table 1), whereas the second pattern is not of CF-type ($x_1 = 1, x_2 = x_3 = x_4 = x_5 = 2$, see table 1). The latter prefers next-nearest neighbors as in [6]. (B) Schematic visualisation of the commensurability of Wigner-type electron distribution in graphene monolayer—two different homotopy patterns for $\nu = \frac{5}{12}$ are shown, the first one is of CF-type (i.e. with $x_1 = x_2 = 1, x_3 = 2$, whereas the second pattern is not of CF-type ($x_1 = x_2 = 1, x_3 = x_4 = 2, x_5 = -2$, see table 1, minus indicates the opposite orientation of the loop with respect to preceding one, marked by opposite arrow in the figure). The latter prefers next-nearest neighbors as in [6]. Various homotopy patterns for the same $\nu$ correspond to states with different activation energy. If envelope function of multiparticle states favors next-nearest neighbor correlations, then not CF states at $\nu = \frac{1}{4}$, $\frac{5}{12}$ are more stable than competitors of CF-type at these filling rates.

Acknowledgments

Author thanks Zlatko Papic for suggestion of the problem. Research supported by NCN project P.2016/21/D/ST3/00958.
\[ \Psi(x) = \begin{cases} \frac{e}{\sqrt{2m}} \sin \frac{1}{\beta} \int_a^b \text{d}x, & \text{for } \Psi(a) = 0, \\ \frac{e}{\sqrt{2m}} \sin \frac{1}{\beta} \int_a^b \text{d}x, & \text{for } \Psi(b) = 0, \end{cases} \]  

where \( p(x) = \sqrt{2m(E-U(x))} \) (for simplicity, assuming vertical infinite borders of the well). Uniqueness of the wave function requires,

\[
2 \int_a^b \text{d}x = \oint \text{d}x = S_{yp} = n(2k+1)h = n(2k+1)h,
\]

which is the Bohr–Sommerfeld quantization rule (\( h \) is Planck constant, \( n \) is an integer; for non-vertical infinite borders, \( S_{yp} = (n + \frac{1}{2})h \) [17]). The above has been derived upon the condition that the trajectory is single-loop. For a different homotopy class and for a multi-loop trajectory one obtains, however,

\[
2 \int_a^b \text{d}x = \oint \text{d}x = S_{pm} = (2k + 1)n2\pi h = n(2k + 1)h,
\]

for a trajectory \((a, b)\) with additional \( k \) loops. Each loop of all 2k loops symmetrically pinned (by \( k \) loops) to both branches, ‘upper’ (\( +p \)) and ‘lower’ (\( -p \)), of the closed trajectory between \( a \) and \( b \) in the integral \( \oint \text{d}x \) adds \( 2\pi \).

This is of particular importance when the Bohr–Sommerfeld rule is applied to an effective 1D phase-space \((Y, P_y)\) of \( x, y \) components of the 2D kinematic momentum in the presence of a perpendicular magnetic field. The kinematic momentum components (at the Landau gauge, \( A = (0, Bx, 0) \)),

\[
P_x = -i\hbar \frac{\partial}{\partial x}, \quad P_y = -i\hbar \frac{\partial}{\partial y} - eBx,
\]

do not commute,

\[
[P_x, P_y] = i\hbar eB.
\]

The pair of operators, \( Y = \frac{1}{i\hbar}P_x \) and \( P_y \), can be treated as operators of canonically conjugated generalized position \( Y \) and momentum \( P_y \), because \( [Y, P_y] = i\hbar \). Thus, the 1D effective phase space, \((Y, P_y)\), is actually the 2D space, \((P_x, P_y)\). The latter 2D kinematic momentum space is, on the other hand, the ordinary 2D space \((x, y)\) renormalized by the factor \( \frac{1}{\sqrt{2\pi \hbar}} \) and turned in plane by \( \pi/2 \), which is noticeable due to the Lorentz force, \( \mathbf{F} = \frac{\partial \Psi}{\partial \mathbf{r}} = e\frac{\partial \Psi}{\partial y} \times \mathbf{B} \), which gives \( \text{d}P_x = eB\text{d}y \) and \( \text{d}P_y = -eB\text{d}x \).

In 2D position space, trajectories \((x, y)\) may belong to different homotopy classes and may be attributed to non-contractible additional loops (as in charged multiparticle planar systems at sufficiently strong magnetic field). Hence, in this homotopy-rich 2D case, from the generalized Bohr–Sommerfeld rule one obtains,

\[
S_{yp} = n(2k + 1)h,
\]

or rewritten to \((x, y)\) space,

\[
S_{xy} = \frac{(2k + 1)nh}{eB},
\]

which defines the generalized quantum of magnetic field flux,

\[
\Phi_k = \Delta S_{xy}B = \frac{(2k + 1)h}{e},
\]

\( \Delta S_{xy} \) is the change of \( S_{xy} \) in equation (A.7) when \( n \) is changed by 1. Only for \( k = 0 \), i.e. for the homotopy class without additional loops, the flux quantum equals to \( \Phi_0 = \frac{B}{e} \).

The generalized magnetic field flux quanta \( \Phi_k \) define different sizes of multi-loop cyclotron orbits, \( \Phi_k/B \). The IQHE corresponds to \( k = 0 \) (the homotopy class of single-loop cyclotron orbits) and the cyclotron orbit size for \( k = 0 \) equals to \( \Delta S_{xy} = \frac{2\pi h}{B} = \frac{2\pi}{\nu} \), which gives \( \nu = \frac{N}{N_0} = 1 \). (\( N_0 = \frac{B\pi}{h} \) is the LL degeneracy taken here for \( B_0 \), \( S \) is the sample surface size, \( N \) is the number of electrons, \( B_0 \) is the magnetic field for \( \nu = 1 \).)

The FQHE-main line corresponds to \( k = 1, 2, \ldots \) (the homotopy classes with \( q = (2k + 1) \)-loop cyclotron orbits or braids with \( k \) additional loops); e.g. for \( k = 1 \) (the simplest Laughlin state), the triple-loop cyclotron orbit has the size \( \Delta S_{xy} = \frac{2\pi}{3B} \). This orbit for \( B = 3B_0 \) fits to interparticle separation \( \frac{2\pi}{3B_0} \) —hence, from the commensurability condition \( \frac{2\pi}{3B_0} = \frac{2\pi}{N} \), one obtains, \( \nu = \frac{N}{N_0} = \frac{N}{B_0S/4\pi} = \frac{1}{3} \).

Bohr–Sommerfeld quantization applied above to many particle systems is interaction independent, i.e. it holds for arbitrarily strongly interacting multiparticle systems. The sizes of the generalized magnetic flux quanta are thus also interaction independent for different homotopy classes, although the existence of nonhomotopic trajectories in \((x, y)\) space is conditioned by the Coulomb interaction of 2D charged particles (via the cyclotron commensurability condition in Wigner-type crystal of electrons). In a gas system of noninteracting particles their mutual positions are arbitrary, which dismisses any correlations and related homotopies.

**ORCID IDs**

Janusz E Jacak
https://orcid.org/0000-0002-6946-2495

**References**

[1] Tsui D C, Stöhrm H L and Gosard A C 1982 Two-dimensional magnetotransport in the extreme quantum limit *Phys. Rev. Lett.*, **48** 1559

[2] Pan W, Stöhrm H L, Tsui D C, Pfeiffer L N, Baldwin K W and West K W 2003 Fractional quantum Hall effect of composite fermions *Phys. Rev. Lett.*, **90** 016801

[3] Jacak J 2018 Application of the path integral quantization to indistinguishable particle systems topologically confined by a magnetic field *Phys. Rev. A*, **97** 012108

[4] Jain J K 1989 Composite-fermion approach for the fractional quantum Hall effect *Phys. Rev. Lett.*, **63** 199

[5] Mukherjee S, Mandal S S, Wu Y H, Wójs A and Jain J K 2014 Enigmatic 4/11 state: a prototype for unconventional fractional quantum Hall effect *Phys. Rev. Lett.*, **112** 016801

[6] Zibrov A A, Spanton E M, Zhou H, Kometter C, Taniguchi T, Watanabe K and Young A F 2018 Even denominator
fractional quantum Hall states at an isospin transition in monolayer graphene Nat. Phys. 14 930

[7] Jain J K 2007 Composite Fermions (Cambridge: Cambridge University Press)

[8] Goerbig M O 2011 Electronic properties of graphene in a strong magnetic field Rev. Mod. Phys. 83 1193

[9] Wu F, Sodemann I, Araki Y, Mac-Donald A and Jolicoeur T 2014 SO(5) symmetry in the quantum Hall effect in graphene Phys. Rev. B 90 235432

[10] Laughlin R B 1983 Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations Phys. Rev. Lett. 50 1395

[11] Jacak J 2017 Unconventional fractional quantum Hall effect in bilayer graphene Sci. Rep. 7 8720

[12] Łydźba P, Jacak L and Jacak J 2015 Hierarchy of fillings for the FQHE in monolayer graphene Sci. Rep. 5 14287

[13] Birman J S 1974 Braids, Links and Mapping Class Groups (Princeton, NJ: Princeton University Press)

[14] Wu Y S 1984 General theory for quantum statistics in two dimensions Phys. Rev. Lett. 52 2103

[15] Jacak J, Gonczarek R, Jacak L and Jóźwiak I 2012 Application of Braid Groups in 2D Hall System Physics: Composite Fermion Structure (Singapore: World Scientific)

[16] Nomura K, Ryu S and Lee D 2009 Field-induced Kosterlitz–Thouless transition in the $N = 0$ Landau level of graphene Phys. Rev. Lett. 103 216801

[17] Landau L D and Lifshitz E M 1965 Quantum Mechanics, No-Relativistic Theory (Oxford: Pergamon)