Generalization Error of GAN from the Discriminator’s Perspective

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Abstract

The generative adversarial network (GAN) is a well-known model for learning high-dimensional distributions, but the mechanism for its generalization ability is not understood. In particular, GAN is vulnerable to the memorization phenomenon, the eventual convergence to the empirical distribution. We consider a simplified GAN model with the generator replaced by a density, and analyze how the discriminator contributes to generalization. We show that with early stopping, the generalization error measured by Wasserstein metric escapes from the curse of dimensionality, despite that in the long term, memorization is inevitable. In addition, we present a hardness of learning result for WGAN.

Keywords: probability distribution, generalization error, curse of dimensionality, early stopping, Wasserstein metric.

1 Introduction

The task of generative modeling in machine learning is as follows: Given \(n\) sample points of an unknown probability distribution \(P^*_\text{true}\), we would like to approximate \(P^*_\text{true}\) well enough to be able to generate new samples. The generative adversarial network (GAN) \cite{goodfellow2014generative} is among the most popular models for this task. It has found diverse applications such as image generation \cite{isola2017image}, photo editing \cite{wang2020image} and style transfer \cite{isola2016image}. More importantly, there are emerging scientific applications including inverse problems \cite{wang2020inverse}, drug discovery \cite{wang2021drug}, cosmological simulation \cite{wang2020cosmological}, material design \cite{wang2020material} and medical image coloration \cite{wang2020medical}, to name a few.

Despite these promising successes, we are far from a satisfactory theory. Arguably, one of the most important problems is the generalization ability of GANs, namely how they are able to estimate the underlying distributions well enough to be able to generate new samples that appear highly realistic. There are at least two difficulties with generalization:

1. Curve of dimensionality:

Let \(P_{\text{true}}^{(n)}\) be the empirical distribution associated with the \(n\) given sample points. Let \(W_2\) be the Wasserstein metric. It is known that for any absolutely continuous \(P^*_\text{true}\) \cite{villani2009optimal},

\[
W_2(P^*_\text{true}, P_{\text{true}}^{(n)}) \gtrsim n^{-\frac{1}{d^*}}
\]

for any \(\delta > 0\). The significance of \cite{yang2021generalization} is that it sets a lower bound for the generalization error of all possible models: Let \(\mathcal{A}\) be any algorithm (a mapping) that maps from an \(n\) sample set \(X_n = \{x_1, \ldots, x_n\}\) of \(P^*_\text{true}\) to an estimated distribution \(\mathcal{A}(X_n)\), then \cite{yang2021generalization}

\[
\inf_{\mathcal{A}} \sup_{P^*_\text{true}} \mathbb{E}_{X_n} [W_2^2(P^*_\text{true}, \mathcal{A}(X_n))] \gtrsim n^{-\frac{2}{d^*}}
\]

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where $P_*$ ranges among all distributions supported in $[0, 1]^d$. Thus, an exponential amount of samples $\epsilon^{-\Omega(d)}$ is needed to achieve an error of $\epsilon$, which is formidable for high-dimensional tasks (e.g. $d \geq 10^5$ for images). To overcome the curse of dimensionality, one needs to restrict to a smaller class of target distributions $P_*$ such that

$$
\inf_A \sup_{P_* \in P_*} \mathbb{E}_{X_n}[W_2^2(P_*, A(X_n))] \lesssim n^{-\alpha}
$$

for some constant $\alpha > 0$. Of course, to be relevant in practice, this restricted space of target distributions should not be too small.

2. Memorization during training:

There is another complication that we cannot avoid. It was argued in [22] that the true optimizer of the GAN model is simply $P_*^{(n)}$. This is disappointing since $P_*^{(n)}$ does not provide any new information beyond the data we already know. It suggests that if a training algorithm converges, most likely it converges to $P_*^{(n)}$, i.e. it simply memorizes the samples provided. This is known as the “memorization effect” and has been analyzed on a simpler model in [63]. Our best hope is there are intermediate times during the training process when the model provides a more useful approximation to the target distribution. However, this is a rather subtle issue. For instance, if we train a distribution $P_1$ by Wasserstein gradient flow over the loss

$$
L(P) = W_2^2(P, P_*^{(n)})
$$

then the training trajectory is exactly the Wasserstein geodesic from the initialization $P_0$ to $P_*^{(n)}$. Since the $W_2$ space has positive curvature (Theorem 7.3.2 [1]), the pairwise distances are comparable to those on a fat triangle (Figure 1 curve 1). This suggests that the generalization error $W_2(P_t, P_*)$ along the entire trajectory will be $n^{-O(1/d)}$ and suffers from the curse of dimensionality.

3. Mode collapse and mode dropping.

These are the additional difficulties that are often encountered in the training of GAN models.

Figure 1: Curve ① is the $W_2$ geodesic connecting $P_0$ and $P_*^{(n)}$. Curve ② is a training trajectory that generalizes.

Recently, an approach was proposed by [63] to establish the generalization ability of a density estimator (the bias-potential model). Its hypothesis space $P_*$ consists of Boltzmann distributions generated by kernel functions and satisfies universal approximation. Also, its training trajectory generalizes well: As illustrated by curve ② in Figure 1, the trajectory comes very close to $P_*$ before eventually turning toward memorizing $P_*^{(n)}$. Thus, the generalization error achieved by early-stopping escapes from the curse of dimensionality:

$$
\text{KL}(P_*||P) = O(n^{-1/4})
$$

This generalization ability is enabled by the function representation of the model. If the function class has small Rademacher complexity, then the model is insensitive to the sampling error $P_* - P_*^{(n)}$, and thus the generalization gap emerges very slowly.
The implication to GAN is that its generalization ability should be attributable to (the function representation of) both its generator and discriminator. This paper will focus on analyzing the discriminator. In place of the generator, we will deal directly with the probability distribution. Our result confirms the intuition of Figure 1 curve 2: Despite the eventual memorization, early-stopping achieves a generalization error of

\[ W_2(P_*, P) = O(n^{-\alpha}) \]  

with a constant exponent \( \alpha > 0 \).

This paper is structured as follows. Section 2 introduces our toy model of GAN. Sections 3 and 4 present the main results of this paper on generalization error (3) with a two-time-scale training model \((\alpha = 1/6)\) and a one-time-scale training model \((\alpha = 1/8)\) respectively. We also show that memorization always happens in the long time limit. Section 5 provides a supplementary result, that it is intractable to learn the Lipschitz discriminator of WGAN. Section 6 contains all the proofs. Section 7 concludes this paper with remarks on future directions.

1.1 Related work

- **GAN training:**
  Currently there is little understanding of the training dynamics and convergence property of GAN, or any distribution learning model with a generator such as the variational autoencoder [32] and normalizing flows [56]. The available results deal with either simplified models [40, 61, 21, 8] or convergence to local saddle points [43, 26, 37]. The situation is further complicated by training failures such as mode collapse [5, 11], mode dropping [64] and oscillation [48, 10]. Since our emphasis is on the discriminator, we omit the generator to make the analysis tractable.

- **GAN generalization:**
  Due to the difficulty of analyzing the convergence of GAN, generalization error estimates have been obtained only for simplified models whose generators are linear maps or their variants [21, 61, 35]. Another line of works [4, 65, 7] focuses on the “neural network distance”, which are the GAN losses defined by neural network discriminators, and shows that these distances have a sampling error of \( O(n^{-1/2}) \). Although [4] interprets this result as the inability of the discriminator to detect a lack of diversity, we show that this is in fact an advantage, such that the discriminator enables the training process to generalize well. Meanwhile, [25] proposes to use the neural network distance with held-out data to measure memorization, while [44] discussed the dependence of memorization on the number of latent samples drawn by GAN.

- **GAN design and regularization:**
  The improvement of GAN has mainly focused on three aspects: alternative loss functions (e.g. WGAN [4] and least-square GAN [38]), novel function representations (e.g. fully convolutional [48] and self-attention [30]), and new regularizations. There are roughly three kinds of regularizations: the regularizations on the function values (e.g. gradient penalty [24] and \( L^2 \) penalty [62]), the regularizations on the parameters (e.g. spectral normalization [41] and weight decay [34]) and the regularizations on the input values (e.g. batch normalization [28] and layer normalization [6]). See [51] for a comprehensive review. Our proofs indicate how function representation and regularization influence the generalization ability of GAN.

- **Function representation:**
  The function class is central to the theoretical analysis of machine learning models. A good function representation, such as the Barron space and flow-induced space [18] (which capture 2-layer networks and residual networks), is the key to the generalization ability of supervised learning models [15, 13, 16] and density estimator [63]. Broadly speaking, a supervised learning model can be studied as a continuous calculus of variations problem [17, 14] determined by four factors: its function representation, loss function, training rule, and their discretizations. Distribution learning has the additional factor of distribution representation [63], namely how the probability distribution is represented by functions.
2 Problem Setting

Consider the domain $\Omega = [0,1]^d$. Denote the space of probability measures by $\mathcal{P}(\Omega)$, and the space of finite signed Radon measures by $\mathcal{M}(\Omega)$. Denote the $p$-Wasserstein metric on $\mathcal{P}(\Omega)$ by $W_p$.

Denote the modeled distribution by $P$, the target distribution by $P_*$. Let $\{x_i\}_{i=1}^n$ be $n$ i.i.d. samples from $P$, and denote the empirical distribution by $P_n^* = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$.

If $P, P_*$ are absolutely continuous, denote their density functions by $p, p_*$. Conversely, given a density function $p$, denote the corresponding measure by $P$ (i.e. $p$ times the Lebesgue measure on $\Omega$). For convenience, we use density $p$ and measure $P$ interchangeably when there is no confusion.

We use $f \lesssim g$ or $f = O(g)$ to indicate that $\limsup_{x \to \infty} f(x)/g(x) < \infty$. We use $f = o(g)$ to indicate that $\lim_{x \to \infty} f(x)/g(x) = 0$. The notations $f \gtrsim g$ and $f = \Omega(g)$ and $\omega(g)$ are defined similarly. We use $f \asymp g$ to indicate that $f \lesssim g$ and $f \gtrsim g$.

2.1 Distribution Representation

As we have discussed in the introduction, the generalization ability of a model comes from the representation or parametrization of its component functions. Thus, for the GAN model, it is reasonable to conjecture that either the generator or the discriminator alone can enable GAN to generalize well.

Since this paper focuses on the discriminator, we prove that a discriminator with a good parametrization is sufficient for good generalization, without the help of the generator. To eliminate the confounding effect of the generator, one needs to construct a model whose generated distribution $P$ does not have any parametrization. The simplest way is to model $P$ by its density function $p$. We will consider $p$ as a function in $L^2(\Omega)$, and denote the space of probability densities by

$$\Delta = \{ p \in L^2(\Omega) \mid p \geq 0 \text{ a.e.}, \int p = 1 \}$$

Henceforth, this abstract model will be referred to as adversarial density estimation.

We will discuss how our results can be applied to the ordinary GAN in Section 7.1.

2.2 Function representation

To model the discriminator $D$, let us consider a simple function class that captures the two key properties of neural networks: universal approximation and dimension-independent complexity.

Specifically, we model $D$ by the random feature functions (or kernel functions) $[49, 18]$:

$$D(x) = E_{\rho_0(w, b)}[a(w, b)\sigma(w \cdot x + b)]$$

(4)

where $\sigma$ is the ReLU activation, $\rho_0$ is a fixed parameter distribution, and $a$ is the parameter function to be learned. Assume for convenience that $\rho_0$ has bounded support: $\text{sprt}\rho_0 \subseteq \{\|w\|_1 + |b| \leq 1\}$.

One can define the kernel

$$k(x, x') = E_{\rho_0}[\sigma(w \cdot x + b)\sigma(w \cdot x' + b)]$$

(5)

The RKHS space $\mathcal{H}$ with kernel $k$ is generated by the norm

$$\|D\|_\mathcal{H} = \|a\|_{L^2(\rho_0)}$$

(6)

Regarding the universal approximation property, assume that $\text{sprt}\rho_0$ contains all directions: For any $(w, b) \neq 0$, there exists $\lambda > 0$ such that $\lambda(w, b) \in \text{sprt}\rho_0$ (e.g. let $\rho_0$ be the uniform distribution over the $L^1$ sphere). Then, [55] implies that the RKHS space $\mathcal{H}$ is dense in $C(\Omega)$ under the supremum norm. It follows that the kernel (5) is positive definite.

Regarding complexity, the Rademacher complexity of $\mathcal{H}$ escapes from the curse of dimensionality (Theorem 6 of [18]). Specifically, the following bound holds uniformly over any collection of $n$ points $\{x_i\}_{i=1}^n \subseteq [-1, 1]^d$,

$$\text{Rad}_n(\|D\|_\mathcal{H} \leq r) \leq 2r \frac{\sqrt{2\log 2d}}{\sqrt{n}}$$

This property eventually leads to our generalization error estimates.
2.3 Training loss

Denote by $L(P)$ a loss over the modeled distribution $P$. The GAN losses are constructed as dual norms over some family $D$ of discriminators.

The straightforward construction is the WGAN loss $[3]$,  
$$L(P) = \sup_{D \in D} \mathbb{E}_P [D(x)] - \mathbb{E}_P[D(x)] - R(D)$$  

(7)

where $R(D)$ is some regularization on $D$ such as gradient penalty $[24][33]$. The classical GAN loss $[22]$ is a weak version of the dual formulation of Jensen-Shannon divergence

$$L(P) = \sup_{D \in D} \mathbb{E}_P [\log \frac{e^{D(x)}}{1 + e^{D(x)}}] + \mathbb{E}_P [\log \frac{1}{1 + e^{D(x)}}]$$

There are many other constructions such as the $f$-divergence GAN loss $[45]$, energy-based GAN loss $[66]$, least-square GAN loss $[38]$ etc.

2.4 Training Rule

Rewrite the GAN loss (e.g. formula (7)) as a joint loss in $p$ and $D$

$$\min_p \max_D L(p, D)$$

To solve this min-max problem, we need to consider the relative time scales for the training of the variables $p$ and $D$:

1. One time scale training: The learning rates for the parameters of $D_t$ and for $p_t$ have the same magnitude.

2. Two time scale training: The learning rate for $D_t$ is much larger than that of $p_t$. Thus, on the time scale of $p_t$, the discriminator $D_t$ can be assumed optimal (at least when $L(p, D)$ is concave in the parameters of $D$). Effectively, $p_t$ is trained by gradient descent on sup$_p L(p, D)$. This dynamics has been shown to closely approximate two-time scale training $[9][26][37]$.

For the specific training rule, we will use continuous-time gradient flow. For the discriminator implemented by random feature functions $[4]$, we train its parameter function $a$ by gradient ascent

$$\frac{d}{dt} a_t(w, b) = \frac{\delta L(p, D_t)}{\delta a} = \int_\Omega \frac{\delta L(p, D_t)}{\delta D}(x) \sigma(w \cdot x + b)$$

where the variational gradients are taken in $L^2(\rho_0)$ and $L^2(\Omega)$. It follows that $D_t$ evolves by

$$\frac{d}{dt} D_t = \mathbb{E}_{\rho_0(w, b)} \left[ \frac{d}{dt} a_t(w, b) \sigma(w \cdot x + b) \right] = k * \frac{\delta L(p, D_t)}{\delta D}$$

(8)

where $k$ is the kernel defined in $[5]$ and $k*$ denotes the convolution over $L^2(\Omega)$

$$k * f(x) = \int_\Omega k(x, x') f(x')$$

(9)

Meanwhile, for the density function $p_t$, one option is the plain gradient descent

$$\frac{d}{dt} p_t = - \frac{\delta L(p_t, D_t)}{\delta p}$$

(10)

In this case, $p_t$ is not guaranteed to remain as a probability density, but becomes a signed measure in $\mathcal{M}(\Omega)$.

An alternative option is to perform projected gradient descent. Denote the tangent cone of the probability simplex $\Delta$ by

$$\forall p \in \Delta, \quad T_p \Delta = \{ q = q_+ - q_- \mid q_+ \geq 0, \quad q_- \in L^2(\Omega), \quad q_- \ll p \}$$

Let $\Pi_\Delta : L^2(\Omega) \to \Delta$ be the $L^2$ projection onto $\Delta$, and let $\Pi_{T_p \Delta}$ be the projection onto $T_p \Delta$. Then, the projected flow is given by

$$\frac{d}{dt} p_t = \Pi_{T_p \Delta} \left( - \frac{\delta L(p_t, D_t)}{\delta p} \right) = \lim_{\epsilon \to 0^+} \Pi_\Delta (p_t - \epsilon \delta_p L(p_t, D_t)) - p_t$$

(11)
## 2.5 Test Loss

The test loss is set to be the Wasserstein metric \( W_2 \) between the modeled distribution \( P \) and the target \( P_* \). If \( p \in L^2(\Omega) \), we consider its projection

\[
W_2(\Pi_\Delta(p), P_*)
\]

(12)

The \( W_2 \) metric is chosen for two reasons. First, a key advantage of \( W_2 \) is that it is sensitive to memorization, such that any solution that approximates \( P_*^{(n)} \) will exhibit the curse of dimensionality \(^1\). Thus, a natural criterion for generalization ability is that a good model should achieve a \( W_2 \) test error with a dimension-independent rate \(^3\), despite that it is trained using only \( P_*^{(n)} \).

Second, the \( W_2 \) metric can be seen as the \( L^2 \) regression loss for probability measures, and thus is a natural choice for the loss function. Specifically, it is the quotient metric on \( \mathcal{P}(\Omega) \) derived from the \( L^2 \) metric on the generators of GAN.

**Proposition** (Informal version of Proposition \(^2\)). *For any target distribution \( P_* \) and a distribution \( P \) generated by any generator \( G_* \),

\[
W_2(P, P_*) = \inf_{G_*} \| G - G_* \|_{L^2}
\]

where \( G_* \) is any generator that generates \( P_* \).

The details are given in Section \( 6.4 \).

### 3 Two Time Scale Training

First, we consider the setting with explicit regularization on the discriminator. Section \( 3.1 \) analyzes the generalization gap, and Section \( 3.2 \) analyzes the generalization error and early stopping.

The training loss is set to be the WGAN loss \(^7\). Instead of fixing the family \( D \), we simply penalize the RKHS norm \(^6\). Then, the loss associated with the discriminator in terms of the parameter function \( a \) becomes

\[
\max_a L_D(a) = \mathbb{E}_{P_*}[D] - \mathbb{E}_P[D] - \|D\|_{L^2}^2
\]

\[
= \int \mathbb{E}_{\rho_0(w, b)}[a(w, b)\sigma(w \cdot x + b)] \ d(P_* - P)(x) - \|a\|_{L^2(\rho_0)}^2
\]

(13)

This loss is strongly concave with a unique maximizer \( a_* \), obtainable by taking the variational derivative in \( a \)

\[
a_*(w, b) = \frac{1}{2} \int \sigma(w \cdot x' + b) \ d(P_* - P)(x')
\]

So the optimal discriminator \( D_* \) is given by

\[
D_*(x) = \mathbb{E}_{\rho_0(w, b)}[a_*(w, b)\sigma(w \cdot x + b)] = \frac{1}{2} \int k(x, x') \ d(P_* - P)(x')
\]

or \( D_* = \frac{1}{2} k*(P_* - P) \), where the kernel \( k \) is defined by \(^5\). With two-time-scale training and strong concavity, we can assume that the discriminator is always the maximizer \( D_* \), so the WGAN loss \(^7\) becomes

\[
L(P) = \mathbb{E}_{P_*}[D_*] - \mathbb{E}_P[D_*] = \frac{1}{2} \int \int k(d(P_* - P)^2
\]

(14)

which is the squared MMD metric \(^23\) with kernel \( k \). Similarly, we denote the empirical loss by

\[
L^{(n)}(P) = \frac{1}{2} \int \int k(d(P_*^{(n)} - P)^2
\]
3.1 Generalization Gap

The gradient descent training rule \( \frac{d}{dt} p_t = k \ast (P_\ast - P_t) \) and projected gradient descent \( \frac{d}{dt} p_t = \Pi T_p \Delta (k \ast (P_\ast - P_t)) \) can be written respectively as

\[
\frac{d}{dt} p_t = k \ast (P_\ast - P_t) \quad (15)
\]

\[
\frac{d}{dt} p_t = \Pi T_p \Delta (k \ast (P_\ast - P_t)) \quad (16)
\]

If the empirical loss \( L^{(n)} \) is used, we denote the training trajectory by \( p_{(n)}^t \).

**Proposition 1** (Generalization gap). With any target distribution \( P_\ast \in \mathcal{P}(\Omega) \) and with probability \( 1 - \delta \) over the sampling of \( P_{(n)}^* \),

1. If \( p_t, p_{(n)}^t \) are trained by gradient descent (15), then

\[
W_2(\Pi \Delta(p_t), \Pi \Delta(p_{(n)}^t)) \leq \sqrt{d} \frac{\sqrt{\log(2d) + 2\log(2/\delta)}}{\sqrt{n}} t
\]

2. If \( p_t, p_{(n)}^t \) are trained by projected gradient descent (16), then

\[
W_2(P_t, P_{(n)}^t) \leq \sqrt{d} \frac{\sqrt{\log(2d) + 2\log(2/\delta)}}{\sqrt{n}} t
\]

3.2 Generalization Error and Early-Stopping

Let \( p_{(n)}^t \) be trained by gradient descent (15) on the empirical loss \( L^{(n)} \).

**Theorem 2** (Generalization error). Given any target density function \( p_\ast \), with probability \( 1 - \delta \) over the sampling of \( P_{(n)}^* \), the generalization error of the trajectory \( p_{(n)}^t \) is bounded by

\[
W_2(P_\ast, \Pi \Delta(p_{(n)}^t)) \leq \sqrt{d} \frac{\|p_\ast - p_0\|_H}{\sqrt{t}} + \sqrt{d} \frac{4\sqrt{\log(2d) + 2\log(2/\delta)}}{\sqrt{n}} t
\]

This is a decomposition of the generalization error into training error plus generalization gap. It follows that with early stopping, we can escape from the curse of dimensionality.

**Corollary 3** (Early stopping). If we choose an early-stopping time \( T \) as follows

\[
T \asymp \|p_\ast - p_0\|_H^2 3 \left( \frac{n}{\log d} \right)^{1/3}
\]

then the generalization error obeys

\[
W_2(P_\ast, \Pi \Delta(p_T^{(n)})) \lesssim \sqrt{d}\|p_\ast - p_0\|_H^{2/3} \left( \frac{\log d}{n} \right)^{1/6}
\]

This result suggests that for the adversarial density estimation model, a polynomial amount of samples \( n = O(\epsilon^{-3}) \) is needed to achieve an error of \( \epsilon \), instead of an exponential amount \( \epsilon^{-\Omega(d)} \).

3.3 Memorization

Despite that early-stopping solutions perform well, this adversarial density estimation model eventually memorizes the samples.

**Proposition 4** (Memorization). Given the condition of Theorem 2, \( P_{(n)}^t \) converges weakly to \( P_{(n)}^* \).

We show a stronger result in Lemma 13 that this model can be trained to converge to any distribution.
3.4 Remarks

Remark 1 (Outside the Hypothesis Space). Theorem 2 requires that the target density belongs to the hypothesis space
\[ \mathcal{P}_* = \{ p_* \in \Delta \mid \| p_* - p_0 \|_H < \infty \} \]
It is a weak condition, because given any \( p_* \in \Delta \), the set \( p_* + \mathcal{H} \) is dense in \( L^2(\Omega) \), so there are plenty of initializations \( p_0 \) that can work. Even if \( p_* - p_0 \notin \mathcal{H} \), it is straightforward to show that,
\[ \| p_* - p_t \|_{L^2} \leq \inf_{q} \frac{\| q - p_0 \|_H}{\sqrt{t}} + \| p_* - q \|_{L^2} \]
So if the target \( p_* \) satisfies
\[ \inf_{\| p_0 - q \|_H \leq R} \| p_* - q \|_{L^2} \lesssim R^{-\beta} \]
for some \( \beta > 0 \), then the training error can be bounded by
\[ \| p_* - p_t \|_{L^2} \lesssim t^{-\frac{\beta}{2(1+\beta)}} \]
and the early-stopping generalization error becomes \( O(n^{-\frac{\beta}{6(\beta+4)}}) \).

Remark 2 (Finite neurons and approximation error). The discriminator (4) is defined as an average of a possibly infinite collection of feature functions, but in practice, there is only a finite number \( m \) of neurons:
\[ D^{(m)}(x) = \frac{1}{m} \sum_{j=1}^{m} a_j \sigma(w_j \cdot x + b_j) \]
where each \((w_j, b_j)\) is sampled from the parameter distribution \( \rho_0 \). The model and its training dynamics (13, 14, 15) can be adapted to this finite-neuron setting, and we denote the empirical training trajectory by \( p^{(n,m)}_t \). The generalization bound of Theorem 2 continues to hold with an additional term of the approximation error, which scales as \( O(\sqrt{t/m}) \). Specifically, with probability \( 1 - 2\delta \) over the sampling of \( P^{(n)}_* \) and \( \rho^{(m)}_0 \),
\[ W^2_2(\mathcal{P}_*, \Pi_\Delta(p^{(n,m)}_t)) \leq \frac{\| p_* - p_0 \|_H}{\sqrt{t}} + \frac{\| p_* - p_0 \|_H [4 + \sqrt{2\log(4/\delta)}]}{\sqrt{m}} \sqrt{t} + \frac{4\sqrt{2\log 2d} + \sqrt{2\log(2/\delta)}}{\sqrt{n}} t \]
The proof is given in Section 6.5. In particular, this finite-neuron model is still able to avoid the curse of dimensionality.

4 One Time Scale Training

The previous section demonstrates that, with an explicit regularization \( \| D \|_H \), we can bound the Rademacher complexity of the discriminators and enable the adversarial density estimator to generalize well. This section shows that even if we do not explicitly bound the complexity of \( D \), the early-stopping solutions still enjoy good generalization accuracy.

4.1 Ill-posedness

Recall that we are using the WGAN loss (7), which we write as a min-max problem:
\[ \min_p \max_D L(p, D) = \min_p \max_D \mathbb{E}_{p_*}[D] - \mathbb{E}_p[D] - R(D) \]
Instead of penalizing the parameters of the discriminator \( R(D) = \| D \|_H^2 = \| a \|_{L^2(\Omega)}^2 \), we consider weaker regularizations on its function value. For instance,
- The \( L^2 \) penalty proposed by (12)
\[ R(D) = \| D \|_{L^2(\Omega)}^2 \]
• Gradient penalty
\[ R(D) = \| \nabla D \|^2_{L^2(\Omega)} \] (18)

• The Lipschitz penalty proposed by [24]. We present a simplified form for better illustration
\[ R(D) = \| 1 - \| \nabla D \| \|^2_{L^2(\Omega)} \] (19)

• Other Lipschitz penalties [33, 46] in simplified forms
\[ R(D) = \| \max(0, \| \nabla D \| - 1) \|^2_{L^2(\Omega)} \] or \[ \| \max(0, \| \nabla D \|^2 - 1) \|^2_{L^1(\Omega)} \] (20)

None of the above regularizations lead to a well-defined GAN loss.

**Proposition 5** (Ill-posedness). Consider the empirical loss function
\[ L^{(n)}(P) = \sup_{D \in C_1(\Omega)} E_{P^{(n)}}[D] - E_P[D] - cR(D) \]

with any \( c \geq 0 \), where the regularization \( R \) is any of \([17, 18, 19, 20]\). Suppose the dimension \( d \geq 3 \). Then, for any distribution \( P \neq P^{(n)} \)
\[ L^{(n)}(P) = \infty \]

Heuristically, these regularizations are too weak to control the complexity of \( D \), so that \( D \) can diverge around the sample points of \( P^{(n)} \), and thus the loss blows up. By the universal approximation property [27], this result holds if we implement \( D \) by neural networks or random feature functions.

It follows that the two-time-scale training, with \( D \) trained to optimality, is not applicable.

### 4.2 Generalization error

Nevertheless, we show that one-time-scale training still performs well and achieves a small generalization error.

For simplicity, we focus on the \( L^2 \) regularization [17],
\[ \min_p \max_D L(p, D) = \min_p \max_D E_{P^{(n)}}[D] - E_P[D] - \frac{c}{2} \| D \|^2_{L^2(\Omega)} \] (21)

with some \( c > 0 \). As usual we model \( D \) by the random feature function [4].

The one-time-scale gradient descent-ascent [10, 3] can be written as
\[ \frac{d}{dt} p_t = D_t, \quad \frac{d}{dt} D_t = k * (P_\ast - P_t) - ck * D_t \] (22)

Again, denote by \( p_t^{(n)}, D_t^{(n)} \) the training trajectory on the empirical loss \( L^{(n)} \).

**Theorem 6** (Generalization error). Suppose \( 0 < c \leq \sqrt{2} \). Initialize \( p_t^{(n)} \) by \( p_0 \) and the parameter function of the discriminator \( D_t^{(n)} \) by \( a_0^{(n)} \equiv 0 \). With probability \( 1 - \delta \) over the sampling of \( P_\ast^{(n)} \), we have
\[ W_2(\Pi_\Delta(p_t^{(n)}), P_\ast) \leq \sqrt{\frac{d}{c}} \| p_\ast - p_0 \|_{\mathcal{H}} \sqrt{k} + \sqrt{\frac{d}{c}} \frac{4\sqrt{2\log 2d} + \sqrt{2\log(2/\delta)}}{\sqrt{n}} t^{1/2} \]

The condition \( c \leq \sqrt{2} \) is imposed for convenience. Otherwise, the “friction” is too large, so the convergence becomes slower and the formula becomes more complicated.

Similar to Corollary 3, we show that early stopping can escape from the curse of dimensionality.

**Corollary 7** (Early stopping). Given the condition of Theorem 4 if we choose an early-stopping time \( T \) as follows
\[ T \gg \| p_\ast - p_0 \|_{\mathcal{H}} \frac{1}{\sqrt{d}} \left( \frac{n}{\log d} \right)^{1/4} \]
then the generalization error obeys
\[ W_2(P_\ast, \Pi_\Delta(p_T^{(n)})) \lesssim \sqrt{d} \| p_\ast - p_0 \|_{\mathcal{H}}^{3/4} \left( \log d \right)^{3/8} \]

Finally, we can also establish memorization in the long time limit.

**Proposition 8** (Memorization). Given the condition of Theorem 4, \( P_t^{(n)} \) converges weakly to \( P_\ast^{(n)} \).
5 Slow Deterioration

In the previous sections, we demonstrated that the generalization ability of the adversarial density estimation models, $\{[13]\}$ and $\{[21]\}$, can be attributed to the dimension-independent complexity of the discriminators during training (e.g. Rademacher complexity). This section provides a supplementary view, that because the complexity of $D_t$ grows slowly, it would take a very long time for $D_t$ to deteriorate to the optimal Lipschitz discriminator of WGAN $\{[3]\}$. Heuristically, this slow deterioration is beneficial, because as illustrated in Figure 1 curve $\{[7]\}$, training on the Wasserstein landscape suffers from the curse of dimensionality.

Consider the empirical WGAN loss with Lipschitz discriminator

$$\max_{\|D\|_{\text{Lip}} \leq 1} \mathbb{E}_{P_\omega(n)}[D] - \mathbb{E}_P[D]$$

The Kantorovich-Rubinstein theorem $\{[57]\}$ tells us that this is the $W_1$ metric between $P$ and $P_\omega(n)$, while the Arzelà–Ascoli theorem implies that the maximizers $D_\star$ exist (over the compact domain $\Omega$). It has been the focus of several GAN models (e.g. $\{[3, 24, 33, 46]\}$) to try to learn these $D_\star$ by neural networks.

For convenience, suppose our modeled distribution $P$ is exactly $P_\star$ and $P_\star$ is the uniform distribution over $\Omega = [0, 1]^d$. By Theorem 5.1 of $\{[12]\}$, we have

$$\max_{\|D\|_{\text{Lip}} \leq 1} \mathbb{E}_{P_\omega(n)}[D] - \mathbb{E}_P[D] = W_1(P_\star, P_\omega(n)) \simeq n^{-1/d} \quad (23)$$

The large gap $n^{-1/d}$ indicates that $D_\star$ well separates $P_\star$ and $P_\omega(n)$, suggesting that during GAN training, $D_\star$ can quickly drive $P$ away from $P_\star$ and towards memorizing $P_\omega(n)$.

Let us consider the loss associated with the discriminator

$$\max_a L^{(n)}(a) = \mathbb{E}_{P_\omega(n)}[D] - \mathbb{E}_P[D] - R(a)$$

$$D(x) = \mathbb{E}_{P_\omega}(a(w, b) \sigma(w \cdot x + b))$$

where $D$ is again modeled as a random feature function. Assume two-time-scale training, so that $P = P_\star$ is fixed as we train $D_t$. The regularization $R$ can be, for instance, the Lipschitz constraint

$$R(a) = \begin{cases} 0 & \text{if } \|D\|_{\text{Lip}} \leq 1 \\ \infty & \text{else} \end{cases} \quad (24)$$

or a Lipschitz penalty, as in WGAN-GP $\{[24, 33, 46]\}$

$$R(a) = c \max(0, \|D\|_{\text{Lip}} - 1) \quad (25)$$

with $c \gg 1$.

The following result indicates that $D_t$ cannot approximate $D_\star$ efficiently.

**Proposition 9** (Slow deterioration). Suppose $R$ is any regularization term such that $L^{(n)}$ is bounded above and that we can train $a$ by continuous-time (sub)gradient flow. Let $a_t$ be the gradient ascent trajectory with any initialization $a_0 \in L^2(P_\omega)$, let $D_t$ be the discriminator, and let $D_\star$ be any maximizer of $\{24\}$. With probability $1 - \delta$ over the sampling of $P_\omega(n)$, we have

$$\|D_t - D_\star\|_{L^\infty(\Omega)} \geq \frac{3}{40}n^{-1/d} - \frac{2\sqrt{2}\log 2d + \sqrt{\log(2/\delta)/2}}{\sqrt{n}} o(\sqrt{t})$$

Both $\{24\}$ and $\{25\}$ satisfy the condition of Proposition 9 since $-L^{(n)}$ becomes a proper closed convex function.

Hence, it takes at least $\omega(n^{1-\frac{1}{2d}})$ time to learn the optimal Lipschitz discriminator. The generality of Proposition 9 indicates that it could be futile to search for a Lipschitz regularization for WGAN.

---

$^1$ Technically, Figure 1 curve $\{[7]\}$ concerns the $W_2$ loss, but it is reasonable to believe that training on $W_1$ or any $W_p$ loss cannot escape from the curse of dimensionality either.
6 Proofs

The results from Sections 3, 4 and 5 are proved in the following three subsections respectively.

6.1 Two time scale training

Definition 1. Let $k$ be the kernel defined in (3). Consider the convolution (9) as a symmetric compact operator over $L^2(\Omega)$. By universal approximation (Section 2.2), $k$ is positive definite. It follows that we can construct an orthonormal basis of eigenvectors $\{e_i\}_{i=1}^\infty$ with eigenvalues $\lambda_i > 0$. Denote

$$\tilde{e}_i = \frac{e_i}{\sqrt{\lambda_i}}$$

Then, $\{\tilde{e}_i\}_{i=1}^\infty$ is an orthonormal basis of $\mathcal{H}$.

Regarding the Wasserstein test loss (12), we have the following convenient bound.

Lemma 10. For any $p, q \in L^2(\Omega)$,

$$W_2(\Pi_\Delta(p), \Pi_\Delta(q)) \leq \sqrt{d} \|p - q\|_{L^2(\Omega)}$$

Proof. The $L^1$ difference between two probability densities is equivalent to an optimal transport distance with the loss

$$c(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$$

Since $\|x - y\| \leq \text{diam}(\Omega)c(x, y)$ for all $x, y \in \Omega$, the $W_2$ metric is dominated by the $L^1$ distance

$$W_2(\Pi_\Delta(p), \Pi_\Delta(q)) \leq \sqrt{d}\|\Pi_\Delta(p) - \Pi_\Delta(q)\|_{L^1(\Omega)}$$

Meanwhile, since $\Omega = [0, 1]^d$ has unit volume

$$\|\Pi_\Delta(p) - \Pi_\Delta(q)\|_{L^1(\Omega)} \leq \|\Pi_\Delta(p) - \Pi_\Delta(q)\|_{L^2(\Omega)} \leq \|p - q\|_{L^2(\Omega)}$$

\hfill \Box

We also need the following lemma from [63].

Lemma 11. For any distribution $P_\ast \in \mathcal{P}(\Omega)$ and any $\delta \in (0, 1)$, with probability $1 - \delta$ over the i.i.d. sampling of $P_{\ast}^{(n)}$,

$$\sup_{\|w\|_1 + \|b\| \leq 1} \left| \int \sigma(w \cdot x + b) d(P_\ast - P_{\ast}^{(n)})(x) \right| \leq \frac{4\sqrt{2\log 2d + \sqrt{2\log(2/\delta)}}}{\sqrt{n}}$$

6.1.1 Proof of generalization gap

Proof of Proposition 7. First, for the plain gradient flow (15), we have

$$\frac{d}{dt}\|p_t^{(n)} - p_t\|_{L^2(\Omega)} = \left\langle \frac{p_t^{(n)} - p_t}{\|p_t^{(n)} - p_t\|}, k \ast (p_\ast^{(n)} - p_t^{(n)}) - k \ast (p_\ast - p_t) \right\rangle_{L^2(\Omega)}$$

$$= \left\langle \frac{p_t^{(n)} - p_t}{\|p_t^{(n)} - p_t\|}, k \ast (p_\ast^{(n)} - p_\ast) - k \ast (p_t^{(n)} - p_t) \right\rangle_{L^2(\Omega)}$$

$$\leq \left\langle \frac{p_t^{(n)} - p_t}{\|p_t^{(n)} - p_t\|}, k \ast (p_\ast^{(n)} - p_\ast) \right\rangle_{L^2(\Omega)}$$

$$\leq \|k \ast (p_\ast^{(n)} - p_\ast)\|_{L^2(\Omega)}$$

$$\leq \sup_{x \in \Omega} \left| E_{P_\ast}(w, b) \left[ \sigma(w \cdot x + b) \int \sigma(w' \cdot x' + b) d(P_{\ast}^{(n)} - P_\ast)(x') \right] \right|$$
Since $\Pi$ is a projection onto a convex set, we can choose $\epsilon_k$ such that

$$
\Pi \Delta p_t \leq \Pi \Delta p_t
$$

Then, Lemma 11 implies that with probability $1 - \delta$,

$$
\|p_t - p_t\|_{L^2(\Omega)} \leq \int_0^t 4\sqrt{2} \log 2d + \sqrt{2 \log (2/\delta)} \sqrt{n}
$$

One can conclude by Lemma 10 that

$$
W_2(\Pi(\Delta p_t^{(n)}), \Pi(\Delta p_t)) \leq 4\sqrt{2} \log 2d + \sqrt{2 \log (2/\delta)} t
$$

Next, for the projected gradient flow [16], we have

$$
\frac{d}{dt} \|p_t^{(n)} - p_t\| = \left\langle \frac{p_t^{(n)} - p_t}{\|p_t^{(n)} - p_t\|}, \Pi T \Delta (k \ast (p_t^{(n)} - p_t)) - \Pi T \Delta (k \ast (p_t - p_t)) \right\rangle
$$

where

$$
p_{t,\epsilon} := p_t + \epsilon k \ast (p_t - p_t), \quad p_t^{(n)} := p_t^{(n)} + \epsilon k \ast (p_t^{(n)} - p_t^{(n)})
$$

Since $\Pi$ is a projection onto a convex set,

$$
\left\langle p_t^{(n)} - p_t, p_{t,\epsilon} - \Pi \Delta (p_{t,\epsilon}) \right\rangle \leq \left\langle \Pi \Delta (p_{t,\epsilon}) - p_t, p_{t,\epsilon} - \Pi \Delta (p_{t,\epsilon}) \right\rangle = O(\epsilon^2)
$$

$$
\left\langle p_t - p_t^{(n)}, p_{t,\epsilon}^{(n)} - \Pi \Delta (p_{t,\epsilon}^{(n)}) \right\rangle = O(\epsilon^2)
$$

It follows that

$$
\frac{d}{dt} \|p_t^{(n)} - p_t\| \leq \lim_{\epsilon \to 0^+} \left\langle \frac{p_t^{(n)} - p_t}{\|p_t^{(n)} - p_t\|}, \frac{p_{t,\epsilon}^{(n)} - p_t^{(n)}}{\epsilon} - \frac{p_{t,\epsilon} - p_t}{\epsilon} \right\rangle + O(\epsilon)
$$

$$
= \left\langle \frac{p_t^{(n)} - p_t}{\|p_t^{(n)} - p_t\|}, k \ast (p_t - p_t) - k \ast (p_t^{(n)} - p_t^{(n)}) \right\rangle
$$

The proof is completed using the same argument for plain gradient flow.

\[\square\]

\textbf{6.1.2 Proof of generalization error}

**Lemma 12** (Training error, two-time-scale). If $p_t$ is trained by the gradient flow [15] with any target distribution $p_t \in L^2(\Omega)$, we have

$$
\|p_t - p_t\|_{L^2(\Omega)}^2 \leq \frac{\|p_t - p_t\|_{H^2}^2}{t}
$$

**Proof.** We show that the training rule [15] coincides with the training trajectory of RKHS regression: For any $a \in L^2(\rho_0)$, define the function

$$
f_a(x) = \mathbb{E}_{\rho_0(w,b)}[a(w,b)\sigma(w \cdot x + b)]
$$

If $\|p_0 - p_t\|_{H^2} < \infty$, we can choose $a_0$ such that $f(a_0) = p_0 - p_t$. Then, if we train $a_t$ by gradient flow with initialization $a_0$ on the regression loss

$$
\min_a \Gamma(a) = \frac{1}{2} \|f_a\|_{L^2(\Omega)}^2,
$$

\[12\]
the function \( f_{a_t} \) evolves by
\[
\frac{d}{dt} f_{a_t}(x) = E_{p_0} \left[ \frac{d}{dt} a_t \sigma(w \cdot x + b) \right] \\
= E_{p_0} \left[ \int_\Omega f_{a_t}(x') \sigma(w \cdot x' + b) dx' \right] \\
= - \int_\Omega f_{a_t}(x') k(x, x') dx'
\]
or equivalently
\[
\frac{d}{dt} f_{a_t} = - k * f_{a_t}
\]
So the training dynamics of \( f_{a_t} \) is the same as the training rule (15) for the function \( p_t - p_* \). Since \( f_{a_0} = p_0 - p_* \), we have \( f_{a_t} = p_t - p_* \) for all \( t \geq 0 \).

It follows from the convexity of \( \Gamma \) that
\[
\| p_t - p_* \|_{L^2(\Omega)}^2 = \| f(a_t) \|_{L^2(\Omega)}^2 \leq \frac{\| a_0 \|_{\mathcal{H}}^2}{t} = \frac{\| p_0 - p_* \|_{\mathcal{H}}^2}{t}
\]

**Proof of Theorem**
Decompose the generalization error into training error + generalization gap:
\[
\| p_* - p_t^{(n)} \|_{L^2(\Omega)} \leq \| p_* - p_t \|_{L^2} + \| p_t - p_t^{(n)} \|_{L^2}
\]
The first term is bounded by Lemma 12 and the second term is bounded by (26). Therefore,
\[
\| p_* - p_t^{(n)} \|_{L^2(\Omega)} \leq \frac{\| p_0 - p_* \|_{\mathcal{H}}}{\sqrt{t}} + \frac{4\sqrt{2} \log 2d}{\sqrt{n}}\frac{\sqrt{2} \log(2d)}{\sqrt{n}}\frac{1}{t}
\]
Then, we conclude by Lemma 10.

**6.1.3 Proof of memorization**

Proposition 4 is a corollary of the following lemma.

**Lemma 13** (Universal convergence, two-time-scale). Given any signed measure \( \tilde{P} \in \mathcal{M}(\Omega) \) and any initialization \( p_0 \in L^2(\Omega) \), if we define \( P_t \) by
\[
\frac{d}{dt} p_t = k * (\tilde{P} - P_t)
\]
with any initialization \( p_0 \in L^2(\Omega) \), then \( P_t \) converges weakly to \( \tilde{P} \).

**Proof.** Let \( \lambda_i \) and \( \tilde{e}_i \) be the eigendecomposition in Definition 1. Denote \( f_t = k * P_t \) and \( \tilde{f} = k * \tilde{P} \). Decompose \( f_t - \tilde{f} \) into
\[
f_t - \tilde{f} = \sum_{i=1}^{\infty} y_i^* \tilde{e}_i
\]
Then,
\[
\frac{d}{dt} (f_t - \tilde{f}) = k * [k * (\tilde{P} - P_t)] = -k * (f_t - \tilde{f}) = -\sum_{i=1}^{\infty} y_i^* \lambda_i \tilde{e}_i
\]
It follows that \( y_i^* = y_0^* e^{-\lambda_i t} \). Given that
\[
\sum_{i=1}^{\infty} (y_0^*)^2 = \| f_0 - \tilde{f} \|_{\mathcal{H}}^2 = E_{(p_0 - \tilde{P})^2}[k] \leq (\| p_0 \|_{TV} + \| \tilde{P} \|_{TV})^2 \| k \|_{C(\Omega \times \Omega)} < \infty
\]
we can apply dominated convergence theorem to obtain
\[
\lim_{t \to \infty} \| f_t - \tilde{f} \|_{\mathcal{H}}^2 = \lim_{t \to \infty} \sum_{i=1}^{\infty} (y_0^*)^2 e^{-2\lambda_i t} = 0
\]
Thus, \( f_t \to \tilde{f} \) in \( \mathcal{H} \), which implies
\[
\forall f \in \mathcal{H}, \quad \lim_{t \to \infty} \int f \, d(P_t - \tilde{P}) = \lim_{t \to \infty} \langle f_t - \tilde{f}, f \rangle_{\mathcal{H}} = 0
\]
As discussed in Section 2.2, the RKHS space \( \mathcal{H} \) is dense in \( C(\Omega) \) under the supremum norm. It follows that
\[
\forall f \in C(\Omega), \quad \lim_{t \to \infty} \int f \, dP_t = \int f \, d\tilde{P}
\]
Hence, \( P_t \) converges weakly to \( \tilde{P} \).

6.2 One time scale training

6.2.1 Proof of ill-posedness

Proof of Proposition 5. Let \( \{x_i\}_{i=1}^n \) be the sample points of \( P^*_n \). Define the function
\[
D(x) = \sum_{i=1}^n \|x - x_i\|^{-d/2+1.1}
\]
Let \( \eta \) be a mollifier (\( \eta \) is smooth, non-negative, supported in the unit ball and \( \int \eta = 1 \)). Define
\[
D_\epsilon = \int \eta(y) D(x - \epsilon y) dy
\]
Then, \( \sup_{\epsilon > 0} R(D_\epsilon) < \infty \), while for any \( P \neq P^*_n \),
\[
\lim_{\epsilon \to 0^+} E_{P^*_n}[D_\epsilon] - E_P[D_\epsilon] = \infty
\]
It follows that
\[
\sup_{D \in C^1} E_{P^*_n}[D] - E_P[D] - R(D) = \infty
\]

6.2.2 Proof of generalization error

As usual, we try to bound the generalization error by the training error plus the generalization gap, and estimate them separately.

Lemma 14 (Duhamel’s integral). Consider the one-dimensional second-order ODE
\[
\ddot{x} + bx + ax = q \\
x(0) = x_0, \quad \dot{x}(0) = 0
\]
where \( a, b \) are constants and \( q \) is a locally integrable function in \( t \). If \( 4a > b^2 \), then the solution is given by
\[
x(t) = x_0 e^{-\frac{b}{4} t} \left[ \cos \left( t \sqrt{a - \frac{b^2}{4}} \right) + \frac{b}{2} \sin \left( t \sqrt{a - \frac{b^2}{4}} \right) \right] + \int_0^t q(s) e^{-\frac{b}{4} (t-s)} \frac{\sin \left( (t-s) \sqrt{a - \frac{b^2}{4}} \right)}{\sqrt{a - \frac{b^2}{4}}} ds
\]
Proof. Direct verification.

Lemma 15. Let \( k \) be the kernel defined in (9). Assume that the support of the parameter distribution \( \rho_0 \) is contained in \( \{ \|w\|_1 + |b| \leq 1 \} \). Then, the operator norm of the convolution \( \langle \rangle \) over \( L^2(\Omega) \) is bounded by 1.
Proof.

\[ \|k\|_\text{op} = \sup_{\|f\|_{L^2(\Omega)} \leq 1} \langle k * f, f \rangle_{L^2(\Omega)} \]
\[ = \sup_{\|f\|_{L^2(\Omega)} \leq 1} \mathbb{E}_{p_0(w,b)} \left[ \left( \int_\Omega f(x) \sigma(w \cdot x + b) \right)^2 \right] \]
\[ \leq \mathbb{E}_{p_0(w,b)} \left[ \|\sigma(w \cdot x + b)\|^2_{L^2} \right] \]
\[ \leq \sup_{\|w\|_1 + |b| \leq 1, x \in \Omega} \|\sigma(w \cdot x + b)\|^2 \]
\[ \leq 1 \]

Lemma 16 (Training error, one-time-scale). Given any \(0 < c \leq \sqrt{2}\), any target density \(p_\ast\) and any initialization \(p_0\) such that \(\|p_\ast - p_0\|_\mathcal{H} < \infty\), let \(p_t, D_t\) be the training trajectory (22) with \(D_0 \equiv 0\), then we have

\[ \|p_\ast - p_t\|_{L^2} \leq \frac{\|p_\ast - p_0\|_\mathcal{H}}{\sqrt{ct}} \]

Proof. Let \(\{\lambda_i, \tilde{e}_i\}_{i=1}^\infty\) be the eigendecomposition given by Definition 1. Lemma 15 implies that all \(\lambda_i \in (0, 1]\). Then, the condition \(0 < c \leq \sqrt{2}\) implies that \(2\lambda_i \geq c^2 \lambda_i^2\).

For any \(t\), define the orthonormal decomposition

\[ p_t - p_\ast = \sum_{i=1}^\infty x^i(t)\tilde{e}_i, \quad \sum_{i=1}^\infty x^i(0)^2 = \|p_0 - p_\ast\|_\mathcal{H}^2 < \infty \]

Denote \(u(t) = p_t - p_\ast\). The training rule (22) can be rewritten as

\[ \ddot{u} + c\lambda \ast \dot{u} + k \ast u = 0 \]

Taking RKHS inner product with \(\tilde{e}_i\) for each \(i\), we obtain

\[ \forall i, \quad \ddot{x}^i + c\lambda_i \dot{x}^i + \lambda_i x^i = 0 \]
\[ x^i(0) = x_0^i, \quad \dot{x}^i(0) = 0 \]

Since \(4\lambda_i > c^2 \lambda_i^2\), Lemma 14 implies that

\[ \forall i, \quad x^i(t) = x_0^i e^{-\frac{\lambda_i}{2} t} \left[ \cos \left( t \sqrt{\lambda_i - \frac{c^2 \lambda_i^2}{4}} \right) + \frac{\lambda_i}{\sqrt{\lambda_i - \frac{c^2 \lambda_i^2}{4}}} \sin \left( t \sqrt{\lambda_i - \frac{c^2 \lambda_i^2}{4}} \right) \right] \]

Since \(\frac{1}{\sqrt{\lambda_i}} - 1 \geq 1\),

\[ |x^i(t)| \leq |x_0^i| e^{-\frac{\lambda_i}{2} t} \left| \cos \left( t \sqrt{\lambda_i - \frac{c^2 \lambda_i^2}{4}} \right) + \sin \left( t \sqrt{\lambda_i - \frac{c^2 \lambda_i^2}{4}} \right) \right| \leq \sqrt{2}|x_0^i| e^{-\frac{\lambda_i}{2} t} \]

It follows that

\[ \|p_t - p_\ast\|_{L^2}^2 = \sum_{i=1}^\infty \lambda_i |x^i(t)|^2 \leq \sum_{i=1}^\infty 2\lambda_i (x_0^i)^2 e^{-\lambda_i t} \]
\[ \leq \sum_{i=1}^\infty \sup_{\lambda_i > 0} 2\lambda_i (x_0^i)^2 e^{-\lambda_i t} \leq \sum_{i=1}^\infty 2 \frac{1}{e \sqrt{ct}} (x_0^i)^2 \]
\[ \leq \frac{\|p_\ast - p_0\|_\mathcal{H}^2}{ct} \]

\[ \square \]
Recall that the one-time-scale training over the empirical loss \( L^{(n)} \) is given by the following dynamics

\[
\frac{d}{dt} p_t^{(n)} = D_t^{(n)}, \quad \frac{d}{dt} D_t^{(n)} = k \ast (P_s^{(n)} - P_t^{(n)}) - ck \ast D_t^{(n)}
\]  

(27)

Lemma 17 (Generalization gap, one-time-scale). Given any \( 0 < c < 2 \) and any target distribution \( P_* \), let \( p_t \) and \( p_t^{(n)} \) be the trajectory of the dynamics [23] and [27] with the same initialization \( p_0 = p_0^{(n)} \) and \( D_0 = D_0^{(n)} = 0 \). Then, with probability \( 1 - \delta \) over the sampling of \( P_s^{(n)} \), we have

\[
\|p_t - p_t^{(n)}\|_{L^2(\Omega)} \leq \frac{4\sqrt{2} \log 2d + \sqrt{2 \log(2/\delta)}}{\sqrt{n}} t^{3/2}
\]

(27)

Proof. Let \( \{\lambda_i, \hat{e}_i\}_{i=1}^{\infty} \) be the eigendecomposition given by Definition 1. Lemma 15 implies that \( 4\lambda_i > c^2 \lambda_i^2 \).

For any \( t \), define the orthonormal decompositions

\[
p_t - p_t^{(n)} = \sum_{i=1}^{\infty} y^i(t) \hat{e}_i
\]

\[
k \ast (P_* - P_s^{(n)}) = \sum_{i=1}^{\infty} q^i \hat{e}_i
\]

Denote \( u(t) = p_t - p_t^{(n)} \). Then, \( u(0) = \dot{u}(0) \equiv 0 \). The training rules (22, 27) imply that

\[
\ddot{u} + ck \ast \dot{u} + k \ast u = k \ast (P_* - P_s^{(n)})
\]

Taking RKHS inner product with \( \hat{e}_i \) for each \( i \), we obtain

\[
\forall i, \quad \ddot{y}^i + c\lambda_i \dot{y}^i + \lambda_i y^i = q^i
\]

\[
y^i(0) = \dot{y}^i(0) = 0
\]

Since \( 4\lambda_i > c^2 \lambda_i^2 \), Lemma 14 implies that

\[
\forall i, \quad y^i(t) = q^i \int_0^t \sin \left( (t-s) \sqrt{\frac{\lambda_i - c^2 \lambda_i^2}{4}} \right) ds
\]

Then

\[
\sqrt{\lambda_i} |y^i(t)| \leq |q^i| \sqrt{\lambda_i} \int_0^t e^{-\frac{c\lambda_i}{2} (t-s)} (t-s) ds
\]

\[
\leq |q^i| \sqrt{\lambda_i} \left( \frac{c\lambda_i}{2} \right)^{-2} \left( 1 - e^{-\frac{c\lambda_i}{2} t} - e^{-\frac{c\lambda_i}{2} t} (\frac{c\lambda_i}{2} t) \right)
\]

\[
\leq |q^i| \sqrt{\frac{2}{c}} e^{3/2} z_i^{-3/2} [1 - e^{-z_i} (1 + z_i)], \quad z_i := \frac{c\lambda_i}{2} t
\]

\[
\leq |q^i| \sqrt{\frac{2}{c}} e^{3/2} \sup_{z > 0} z^{-3/2} [1 - e^{-z} (1 + z)]
\]

\[
\leq |q^i| \frac{t^{3/2}}{\sqrt{c}}
\]

It follows that

\[
\|p_t - p_t^{(n)}\|_{L^2(\Omega)}^2 \leq \sum_{i=1}^{\infty} \lambda_i |y^i(t)|^2 \leq \sum_{i=1}^{\infty} (q^i)^2 \frac{t^{3/2}}{c} = \frac{t^{3}}{c} \|k \ast (P_* - P_s^{(n)})\|_H^2
\]

Lemma 11 implies that with probability \( 1 - \delta \),

\[
\|k \ast (P_* - P_s^{(n)})\|_H^2 = \int \int k d(P_* - P_s^{(n)})^2
\]
Lemma 16 and the second term is bounded by Lemma 17. Therefore, we can apply dominated convergence theorem to obtain

\[ \|p_t - p_t^{(n)}\|_{L^2(\Omega)}^2 \leq \left( \frac{4\sqrt{2} \log 2d + \sqrt{2} \log(2/\delta)}{\sqrt{n}} \right)^2 \]

Hence,

\[ \|p_t - p_t^{(n)}\|_{L^2(\Omega)}^2 \leq \left( \frac{4\sqrt{2} \log 2d + \sqrt{2} \log(2/\delta)}{\sqrt{n}} \right)^2 t^3 \frac{c}{\sqrt{2}} \]

Proof of Theorem 6. As usual, we bound the generalization error by training error plus generalization gap:

\[ \|p_* - p_t^{(n)}\|_{L^2(\Omega)} \leq \|p_* - p_t\|_{L^2} + \|p_t - p_t^{(n)}\|_{L^2} \]

where we assume the same initialization \( p_0 = p_0^{(n)} \) and \( a_0 = a_0^{(n)} \equiv 0 \). The first term is bounded by Lemma 16 and the second term is bounded by Lemma 17. Therefore,

\[ \|p_* - p_t^{(n)}\|_{L^2(\Omega)} \leq \frac{\|p_* - p_0\|_H}{\sqrt{c} t} + 4 \frac{2 \log 2d + \sqrt{2} \log(2/\delta)}{\sqrt{n}} t^{3/2} \]

The proof is completed using Lemma 10.

6.2.3 Proof of memorization

Proposition 8 is a corollary of the following lemma.

**Lemma 18** (Universal convergence, one-time-scale). Given any signed measure \( \tilde{P} \in \mathcal{M}(\Omega) \), if we define \( p_t \) by the dynamics

\[ \tilde{p}_t = -k * (p_t - \tilde{P}) - ck \tilde{p}_t \]

with any initialization \( p_0 \in L^2(\Omega) \) and \( \tilde{p}_0 \equiv 0 \), then \( p_t \) converges weakly to \( \tilde{P} \).

Proof. Let \( \lambda_i \) and \( \tilde{e}_i \) be the eigendecomposition from Definition 1. Define \( u(t) = k * (P_t - \tilde{P}) \) and decompose \( u(t) \) into

\[ u(t) = \sum_{i=1}^{\infty} y^i(t) \tilde{e}_i \]

Then,

\[ \dot{u} = k * [-k * (p_t - \tilde{P}) - ck \tilde{p}_t] = -k * u - ck \dot{u} = \sum_{i=1}^{\infty} -\lambda_i (y^i + cy^i) \tilde{e}_i \]

It follows that

\[ \forall i, \quad \dot{y}^i + c\lambda_i y^i + \lambda_i y^i = 0 \]

Using the argument from the proof of Lemma 16, we obtain

\[ |y^i(t)| \leq \sqrt{2} |y^i(0)| e^{-\frac{c\lambda_i t}{2}} \]

Since

\[ \sum_{i=1}^{\infty} y_i(0)^2 = \|u_0\|_{\mathcal{H}}^2 = \mathbb{E}(p_0 - \tilde{P})_2[k] \leq \|P_0 - \tilde{P}\|_{TV}^2 \|k\|_{C(\Omega \times \Omega)} < \infty \]

we can apply dominated convergence theorem to obtain

\[ \lim_{t \to \infty} \|u_t\|_{\mathcal{H}}^2 = \lim_{t \to \infty} 2 \sum_{i=1}^{\infty} y_i(0)^2 e^{-c\lambda_i t} = 0 \]

Thus, \( u_t \to 0 \) in \( \mathcal{H} \), which implies

\[ \forall f \in \mathcal{H}, \quad \lim_{t \to \infty} \int f d(P_t - \tilde{P}) = \lim_{t \to \infty} \langle u_t, f \rangle_{\mathcal{H}} = 0 \]

Since \( \mathcal{H} \) is dense in \( C(\Omega) \) in the supremum norm, \( P_t \) converges weakly to \( \tilde{P} \).
6.3 Slow deterioration

**Lemma 19.** With any set of \( n \) points \( x_i \in \Omega \),

\[
\sup_{\|D\|_{L^p} \leq 1} \left| \mathbb{E}_{P_n} [D] - \frac{1}{n} \sum_{i=1}^{n} D(x_i) \right| \geq \frac{3}{20} n^{-1/d}
\]

*Proof.* By Lemma 3.1 of [19], we have

\[
\sup_{\|D\|_{L^p} \leq 1} \left| \mathbb{E}_{P_n} [D] - \frac{1}{n} \sum_{i=1}^{n} D(x_i) \right| \geq \frac{d}{d+1} \left( \frac{1}{(d+1)\omega_d} \right)^{1/d} \frac{1}{\text{diam}(\Omega)} n^{-1/d}
\]  

(28)

where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \),

\[
\omega_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \approx \frac{1}{\sqrt{\pi d}} \left( \frac{2\pi e}{d} \right)^{d/2}
\]

Denote the right hand side of (28) by \( C_d n^{-1/d} \). By direct computation, \( C_d \geq 3/20 \).

**Lemma 20.** Given the condition of Proposition 9 we have

\[
\lim_{t \to \infty} \frac{\|a_t\|_{L^2(\rho_0)}}{\sqrt{t}} = 0
\]

*Proof.* The arguments are adapted from Lemma 3.3 of [59]. Define the value

\[
L^{(n)}_{\infty} = \lim_{t \to \infty} L^{(n)}(a_t) \leq \sup_{a} L^{(n)} < \infty
\]

which is assumed bounded by Proposition 9. Meanwhile, we also have

\[
\frac{d}{dt} \|a_t\|_{L^2(\rho_0)} = \langle \frac{a_t}{\|a_t\|}, \delta_a L^{(n)}(a_t) \rangle \leq \|\delta_a L^{(n)}(a_t)\|_{L^2} = \sqrt{\frac{d}{dt} L^{(n)}(a_t)}
\]

Thus, for any \( t > t_0 \geq 0 \), we have

\[
\|a_t\| - \|a_{t_0}\| \leq \int_{t_0}^{t} \sqrt{\frac{d}{dt} L^{(n)}(a_s)} ds \leq \sqrt{t - t_0} \sqrt{L^{(n)}(a_t) - L^{(n)}(a_{t_0})}
\]

\[
\leq \sqrt{t - t_0} \sqrt{L^{(n)}_{\infty} - L^{(n)}(a_{t_0})}
\]

By choosing \( t_0 \) sufficiently large, the term \( \sqrt{L^{(n)}_{\infty} - L^{(n)}(a_{t_0})} \) can be made arbitrarily small.

**Proof of Proposition 9** Lemma 11 implies that with probability \( 1 - \delta \),

\[
\forall \gamma > 0, \quad \sup_{\|D\|_{L^\infty} \leq \gamma} \left| \mathbb{E}_{P_n} [D] - \mathbb{E}_{P_{n}(\gamma)} [D] \right|
\]

\[
= \sup_{\|a\|_{L^2(\rho_0)} \leq \gamma} \left| \int \left[ \mathbb{E}_{\rho_0} [a(w, b)\sigma(w \cdot x + b)] d(P_* - P_{n}(\gamma)) (x) \right]
\]

\[
\leq \sup_{\|a\|_{L^2(\rho_0)} \leq \gamma} \|a\|_{L^1(\rho_0)} \left| \int \sigma(w \cdot x + b) d(P_* - P_{n}(\gamma)) (x) \right|_{L^\infty(\rho_0)}
\]

\[
\leq \gamma \left( \sqrt{\frac{2\log 2d}{n}} + \sqrt{\frac{2\log(2/\delta)}{n}} \right)
\]

Meanwhile, Lemma 19 implies that

\[
\|D_t - D_*\|_{L^\infty(\Omega)} \geq \frac{1}{2} \left| \int D_t - D_* \, d(P_* - P_{n}(\gamma)) \right|
\]

18
Recall that GAN, as well as other generative models, adopts the generator representation for its
metric and

\[ W_6.4 \text{ Relation between } W_2 \text{ metric and } L^2 \text{ loss} \]

Recall that GAN, as well as other generative models, adopts the generator representation for its modeled distribution:

\[ P = G \# \mathcal{N} := \text{law}(X), \quad X = G(Z), \quad Z \sim \mathcal{N} \]

where \( \mathcal{N} \) is some fixed input distribution and \( G \) is the generator.

Denote by \( L^2(\mathcal{N}; \mathbb{R}^d) \) the space of \( L^2(\mathcal{N}) \) functions from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). Denote by \( \mathcal{P}_{ac}(\mathbb{R}^d) \) the space of absolutely continuous probability measures, and by \( \mathcal{P}_2(\mathbb{R}^d) \) probability measures with finite second moments.

The \( L^2 \) regression loss on \( G \) induces the \( L^2 \) metric on \( P \).

**Proposition 21.** Given any target distribution \( P_\ast \in \mathcal{P}_2(\mathbb{R}^d) \), any input distribution \( \mathcal{N} \in \mathcal{P}_{ac}(\mathbb{R}^d) \), and any \( L^2(\mathcal{N}; \mathbb{R}^d) \) function \( G \), denote \( P = G \# \mathcal{N} \), then

\[ W_2(P, P_\ast) = \inf_{G_\ast} \| G - G_\ast \|_{L^2(\mathcal{N})} \] (30)

where \( G_\ast \) ranges among all \( L^2(\mathcal{N}; \mathbb{R}^d) \) functions such that \( P_\ast = G_\ast \# \mathcal{N} \).

**Proof.** We have \( P \in \mathcal{P}_2(\mathbb{R}^d) \) because

\[ E_P[\| x \|^2] = E_{\mathcal{N}}[\| G(x) \|^2] = \| G \|^2_{L^2(\mathcal{N})} < \infty \]

The set \( \{ G_\ast \in L^2(\mathcal{N}; \mathbb{R}^d) \mid G_\ast \# \mathcal{N} = P_\ast \} \) is nonempty by Theorem 2.12 of [57], so the right hand side of (30) is well-defined. Furthermore, this term is continuous over \( G \in L^2(\mathcal{N}; \mathbb{R}^d) \) by triangle inequality.

First, consider the simple case when \( P \in \mathcal{P}_{2,ac}(\mathbb{R}^d) \). By Theorem 2.12 of [57], there exists an optimal transport map \( h \) from \( P \) to \( P_\ast \). Then,

\[ W_2(P, P_\ast) = \| I - h \|_{L^2(\mathcal{N})} = \| G - h \circ G \|_{L^2(\mathcal{N})} \]

It follows that

\[ W_2(P, P_\ast) = \inf_{G, \# \mathcal{N} = P_\ast} \| G - G_\ast \|_{L^2(\mathcal{N})} \]

Meanwhile, given any \( G_\ast \in L^2(\mathcal{N}; \mathbb{R}^d) \) \( (G_\ast \# \mathcal{N} = P_\ast) \), define a joint distribution by \( \pi = (G, G_\ast) \# \mathcal{N} \), which is a coupling between \( P \) and \( P_\ast \). Then,

\[ W_2(P, P_\ast) \leq E_{\pi}(\| x - x' \|^2) = \| G - G_\ast \|_{L^2(\mathcal{N})} \]

Taking infimum over \( G_\ast \), we obtain (30).

Next, for the general case with \( P \in \mathcal{P}_2(\mathbb{R}^d) \), define the random variable \( Z \sim P \) and an independent random variable \( W \) with unit Gaussian distribution. Since \( \mathcal{N} \) is absolutely continuous, Theorem 16 from Chapter 15 of [50] implies that the measure space \( (\mathcal{N}, \mathbb{R}^d) \) is isomorphic to \( [0, 1] \) with Lebesgue measure, which is isomorphic to \( \mathbb{R}^d \) with the unit Gaussian distribution. Thus, we can consider \( W \) as a random variable defined on the measure space \((\mathbb{R}^d, \mathcal{N})\). For any \( \epsilon > 0 \), define

\[ G_\epsilon = G + \epsilon W, \quad P_\epsilon = G_\epsilon \# \mathcal{N} \]

As \( \epsilon \to 0^+ \), the map \( G_\epsilon \) converges to \( G \) in \( L^2(\mathcal{N}; \mathbb{R}^d) \) and \( P_\epsilon \) converges to \( P \) in \( W_2 \). Since \( P_\epsilon \in \mathcal{P}_{2,ac}(\mathbb{R}^d) \), our preceding result implies that

\[ W_2(P, P_\ast) = \inf_{G, \# \mathcal{N} = P_\ast} \| G_\epsilon - G_\ast \|_{L^2(\mathcal{N})} \]

Taking the limit \( \epsilon \to 0^+ \), we obtain (30) by continuity.

\[ \square \]
6.5 Finite-neuron discriminator

Recall that in Remark 2, the finite-neuron discriminator is defined by

\[ D^{(m)}(x) = \mathbb{E}_{\rho_0^{(m)}(w,b)} [a(w, b) \sigma(w \cdot x + b)], \quad \rho_0^{(m)} = \frac{1}{m} \sum_{j=1}^{m} \delta(w_j, b_j), \quad a(w_j, b_j) = a_j \]

The WGAN loss (13) is modified into

\[ L_D^{(m)}(a) = \mathbb{E}_{P_*}[D^{(m)}] - \mathbb{E}_{P}[D^{(m)}] - \|a\|_{L^2(\rho_0^{(m)})}^2 \]

\[ = \frac{1}{m} \sum_{j=1}^{m} \int a_j \sigma(w_j \cdot x + b_j) d(P_* - P)(x) - \frac{1}{m} \sum_{j=1}^{m} a_j^2 \]

Then, it is straightforward to check that the loss (14) becomes

\[ L^{(m)}(P) = \frac{1}{2} \int \int k^{(m)}(P_* - P)^2, \quad L^{(m,n)}(P) = \frac{1}{2} \int \int k^{(m)}(P_*^{(n)} - P)^2 \]

where the kernel is given by

\[ k^{(m)}(x, x') = \mathbb{E}_{\rho_0^{(m)}(w,b)} [\sigma(w \cdot x + b) \sigma(w \cdot x' + b)] = \frac{1}{m} \sum_{j=1}^{m} \sigma(w_j \cdot x + b_j) \sigma(w_j \cdot x' + b_j) \]

Similar to (15), the training dynamics of the population-loss trajectory \( p_t^{(m)} \) and the empirical-loss trajectory \( p_t^{(m,n)} \) are given by

\[ \frac{d}{dt} p_t^{(m)} = k^{(m)} * (P_* - p_t^{(m)}) \]

\[ \frac{d}{dt} p_t^{(m,n)} = k^{(m)} * (P_*^{(n)} - p_t^{(m,n)}) \]

**Lemma 22.** With probability 1 \( - \delta \) over the sampling of \( \rho_0^{(m)} \), the operator norm of \( k - k^{(m)} \) over \( L^2(\Omega) \) is bounded by

\[ \|k - k^{(m)}\|_{op} \leq \frac{2 + \sqrt{\log(4/\delta)}/2}{\sqrt{m}} \]

**Proof.** For convenience, denote \( \bar{w} = (w, b) \) and \( \bar{x} = (x, 1) \). Since \( (k - k^{(m)}) * \) is a symmetric operator,

\[ \|k - k^{(m)}\|_{op} = \sup_{\|h\|_{L^2(\Omega)} \leq 1} \left| \int (k - k^{(m)}) * h \right| \]

\[ = \sup_{\|h\|_{L^2(\Omega)} \leq 1} \left| \int \left( \int \Omega h(x) \sigma(\bar{w} \cdot \bar{x}) d\bar{x} \right)^2 d(\rho_0 - \rho_0^{(m)})(\bar{w}) \right| \]

\[ = \sup_{f_0 \in \mathcal{F}_0} \left| \int f_0(\bar{w}) d(\rho_0 - \rho_0^{(m)})(\bar{w}) \right| \]

where we define the function families

\[ \mathcal{F}_0 = \left\{ \bar{w} \mapsto \left( \int \Omega h(x) \sigma(\bar{w} \cdot \bar{x}) d\bar{x} \right)^2 \mid \|h\|_{L^2(\Omega)} \leq 1 \right\} \]

\[ \mathcal{F}_1 = \left\{ \bar{w} \mapsto \int \Omega h(x) \sigma(\bar{w} \cdot \bar{x}) d\bar{x} \mid \|h\|_{L^2(\Omega)} \leq 1 \right\} \]

Given any set \( W = \{\bar{w}_j\}_{j=1}^{m} \subseteq \text{sprt} \rho_0 \), the Rademacher complexity is defined as

\[ \text{Rad}_m(\mathcal{F} \circ W) = \frac{1}{m} \mathbb{E}_{\xi \in \{-1, 1\}^m} \left[ \sup_{f \in \mathcal{F}} \sum_{j=1}^{m} \xi_j f(\bar{w}_j) \right] \]
where $\xi_j$ are i.i.d. random variables with a uniform distribution over $\{\pm 1\}$.
Recall that $\sigma$ is ReLU, $\Omega = [0, 1]^d$ and $\text{sprt}\rho_0 \subseteq \{\|\tilde{w}\|_1 \leq 1\} \subseteq \mathbb{R}^{d+1}$. It follows that
\[
\sup_{\tilde{w} \in W} \sup_{f_j \in F} |f_1(\tilde{w})| \leq \sup_{\tilde{w} \in \text{sprt}\rho_0} \sup_{x \in \Omega} |\sigma(\tilde{w} \cdot \tilde{x})| \leq 1
\]
Thus the contraction property of $\text{Rad}_m$ (Lemma 26.9 [53]) implies that
\[
\text{Rad}_m(F_0 \circ W) \leq \text{Rad}_m(F_1 \circ W)
\]
Meanwhile, by Jensen’s inequality
\[
\text{Rad}_m(F_1 \circ W) = \frac{1}{m} \mathbb{E}_{\xi \in \{\pm 1\}^m} \left[ \sup_{\|h\|_{L^2(\Omega)}} \sum_{j=1}^m |\xi_j| \int_{\Omega} h(x) \sigma(\tilde{w}_j \cdot \tilde{x}) \right]
\]
\[
= \frac{1}{m} \mathbb{E}_{\xi \in \{\pm 1\}^m} \left[ \sup_{\|h\|_{L^2(\Omega)}} \int_{\Omega} h(x) \sum_{j=1}^m |\xi_j| \sigma(\tilde{w}_j \cdot \tilde{x}) \right]
\]
\[
\leq \frac{1}{m} \mathbb{E}_{\xi \in \{\pm 1\}^m} \left[ \|\sum_{j=1}^m |\xi_j| \sigma(\tilde{w}_j \cdot \tilde{x})\|_{L^2(\Omega)} \right]
\]
\[
\leq \frac{1}{m} \left( \mathbb{E}_{\xi \in \{\pm 1\}^m} \left[ \left( \int_{\Omega} \sum_{j=1}^m |\xi_j| \sigma(\tilde{w}_j \cdot \tilde{x})\right)^2 \right] \right)^{1/2}
\]
\[
= \frac{1}{m} \left( \int_{\Omega} \sum_{j=1}^m |\sigma(\tilde{w}_j \cdot \tilde{x})|^2 \right)^{1/2}
\]
\[
\leq \frac{1}{m} \left( \int_{\Omega} \sum_{j=1}^m \frac{1}{2} \right)^{1/2}
\]
\[
\leq \frac{1}{\sqrt{m}}
\]
Since the calculations hold for any subset $W$, we have
\[
\sup \{\text{Rad}_m(F_0 \circ W) \mid W = \{\tilde{w}_j \}_{j=1}^m \subseteq \text{sprt}\rho_0\} \leq \frac{1}{\sqrt{m}}
\]
Note that for all $f_0 \in F_0$, the function $f_0$ ranges in $[0, 1]$. Then, Theorem 26.5 of [53] implies that with probability $1 - \delta$ over the sampling of $\rho_0^{(m)}$,
\[
\sup_{f_0 \in F_0} \left| \int_{\Omega} f_0(\tilde{w}) d(\rho_0 - \rho_0^{(m)})(\tilde{w}) \right|
\]
\[
\leq 2 \mathbb{E} [\text{Rad}_m(F_0 \circ W) \mid (\tilde{w}_1, \ldots, \tilde{w}_m) \sim \rho_0^m] + \sqrt{\frac{\log(2/\delta)}{2m}}
\]
\[
\leq 2 \sup \{\text{Rad}_m(F_0 \circ W) \mid W = (\tilde{w}_1, \ldots, \tilde{w}_m) \subseteq \text{sprt}\rho_0\} + \sqrt{\frac{\log(2/\delta)}{2m}}
\]
\[
\leq 2 + \sqrt{\log(2/\delta)/2m}
\]
Applying the same argument to the function family $\{-f_0 \mid f_0 \in F_0\}$, we obtain that with probability $1 - 2\delta$,
\[
\sup_{f_0 \in F_0} \left| \int_{\Omega} f_0(\tilde{w}) d(\rho_0 - \rho_0^{(m)}) \right| \leq 2 + \sqrt{\log(2/\delta)/2m}
\]
which concludes the proof.
It remains to compare the training dynamics of \( p_t \) and \( p_t^{(m)} \):

\[
\frac{d}{dt} \| p_t - p_t^{(m)} \|_{L^2(\Omega)} = \langle \frac{p_t - p_t^{(m)}}{\| p_t - p_t^{(m)} \|}, k \ast (p_\ast - p_t) - k^{(m)} \ast (p_\ast - p_t^{(m)}) \rangle \\
= \langle \frac{p_t - p_t^{(m)}}{\| p_t - p_t^{(m)} \|}, (k - k^{(m)}) \ast (p_\ast - p_t) \rangle \\
\leq \langle \frac{p_t - p_t^{(m)}}{\| p_t - p_t^{(m)} \|}, (k - k^{(m)}) \ast (p_\ast - p_t) \rangle \\
\leq \| k - k^{(m)} \|_{op} \| p_\ast - p_t \|_{L^2(\Omega)} \\
\leq \| k - k^{(m)} \|_{op} \| p_\ast - p_0 \|_{H} \frac{t}{\sqrt{t}}
\]

with the last step given by Lemma \[12\]. Then, Lemma \[22\] implies that with probability \( 1 - \delta \),

\[
\| p_t - p_t^{(m)} \|_{L^2(\Omega)} \leq 2\sqrt{t} \| p_\ast - p_0 \|_{H} \| k - k^{(m)} \|_{op} \\
\leq \| p_\ast - p_0 \|_{H} \frac{4 + \sqrt{2 \log(4/\delta)}}{\sqrt{n}} \sqrt{t}
\]

Moreover, it is straightforward to check that the proof of inequality \[26\] continues to hold if the kernel \( k \) is replaced by \( k^{(m)} \), so that \[26\] can be directly modified into

\[
\| p_t^{(m)} - p_t^{(m,n)} \|_{L^2} \leq \frac{4\sqrt{2 \log 2d + \sqrt{2 \log(2/\delta)}}}{\sqrt{n}} t
\]

Hence, we can conclude with Lemma \[12\] that, with probability \( 1 - 2\delta \),

\[
\| p_\ast - p_t^{(m,n)} \|_{L^2(\Omega)} \leq \| p_\ast - p_t \|_{L^2} + \| p_t - p_t^{(m)} \|_{L^2} + \| p_t^{(m)} - p_t^{(m,n)} \|_{L^2} \\
\leq \| p_\ast - p_0 \|_{H} \frac{4 + \sqrt{2 \log(4/\delta)}}{\sqrt{n}} \sqrt{t} + \frac{4\sqrt{2 \log 2d + \sqrt{2 \log(2/\delta)}}}{\sqrt{n}} t
\]

Then, Remark\[2\] follows from Lemma \[10\].

### 6.6 Gradient analysis for the generator

This subsection provides the details of the calculation in Section \[7.1\].

First, to derive the formula

\[
\frac{\delta L}{\delta G} = \nabla_x \frac{\delta L}{\delta P} \circ G
\]

one simply note that for any perturbation \( h \in L^2(\mathcal{N}; \mathbb{R}^d) \)

\[
\left\langle h, \frac{\delta L}{\delta G} \right\rangle_{L^2(\mathcal{N})} = \lim_{\epsilon \to 0} \frac{L((G + \epsilon h)\#\mathcal{N}) - L(G\#\mathcal{N})}{\epsilon} \\
= \lim_{\epsilon \to 0} \int \frac{\delta L}{\delta P} \bigg|_{P = G \# \mathcal{N}} \frac{d((G + \epsilon h)\#\mathcal{N} - G\#\mathcal{N})}{\epsilon} \\
= \lim_{\epsilon \to 0} \int \frac{1}{\epsilon} \left[ \frac{\delta L}{\delta P}(G + h) - \frac{\delta L}{\delta P}(G) \right] d\mathcal{N} \\
= \int \nabla_x \frac{\delta L}{\delta P}(G) \cdot h \ d\mathcal{N} \\
= \left\langle h, \nabla_x \frac{\delta L}{\delta P} \circ G \right\rangle_{L^2(\mathcal{N})}
\]

Next, we try to bound the norm of the following term

\[
\frac{\delta L}{\delta G} - \frac{\delta L^{(n)}}{\delta G} = \nabla_x \left( \frac{\delta L}{\delta P} - \frac{\delta L^{(n)}}{\delta P} \right) \circ G = \nabla_1 k \ast (P_\ast - P_\ast^{(n)}) \circ G
\]

\[
22
\]
where $\nabla_1$ means taking the gradient in the first entry of $k$. Note that

$$
\|\nabla_1 k \ast (P_\ast - P^{(n)}_\ast) \circ G\|_{L^2(N)} = \|\nabla_1 k \ast (P_\ast - P^{(n)}_\ast)\|_{L^2(P)}
$$

$$
= \left\| \mathbb{E}_{\rho_0(w,b)} \left[ \nabla_\mathbf{x} \sigma(\mathbf{w} \cdot \mathbf{x} + b) \int \sigma(\mathbf{w} \cdot \mathbf{x} + b) d(P_\ast - P^{(n)}_\ast) \right] \right\|_{L^2(P)}
$$

$$
\leq \sup_{\mathbf{w},b \in \operatorname{spec} \rho_0} \left\| \|\mathbf{w}\|_2 \|\sigma\|_{L^p} \int \sigma(\mathbf{w} \cdot \mathbf{x} + b) d(P_\ast - P^{(n)}_\ast) \right\|
$$

$$
\leq \sup_{\|\mathbf{w}\|_1 + |b| \leq 1} \left| \int \sigma(\mathbf{w} \cdot \mathbf{x} + b) d(P_\ast - P^{(n)}_\ast) \right|
$$

Then, Lemma 11 implies that with probability $1 - \delta$,

$$
\left\| \frac{\delta L}{\delta G} - \frac{\delta L^{(n)}}{\delta G} \right\|_{L^2(N)} \leq \frac{4\sqrt{2 \log 2d + \sqrt{2 \log (2/\delta)}}}{\sqrt{n}}
$$

7 Discussion

Let us conclude with some of the insights obtained in this paper:

- Good generalization is achievable in high dimensions by early stopping and the error estimate escapes from the curse of dimensionality, whereas in the long term ($t \to \infty$), the trained distribution slowly deteriorates to the empirical distribution and exhibits memorization. This is an implicit regularization result, such that the progress toward $P_\ast$ and the deterioration due to $P_\ast - P^{(n)}_\ast$ occur on two time scales.

- The mechanism for generalization is the dimension-independent complexity of the function representation of the discriminator, which in our setting is the Rademacher complexity of random feature functions. It renders the loss landscape insensitive to the sampling error $P_\ast - P^{(n)}_\ast$, and thereby delays the onset of memorization. Our hardness of learning result for WGAN also follows from this small complexity.

- The regularization $R(D)$ is crucial for establishing the convergence of training. As demonstrated by the proof of Theorem 6, the role of $R(D)$ is to introduce “friction” into the min-max training of $P_t$ and $D_t$ and dampens their oscillatory dynamics. Furthermore, Theorem 2 demonstrates that regularizations on parameters may perform better than regularizations on function value. The former imposes a tighter control on the complexity of the discriminator, and thus the growth of the generalization gap is slower. Beyond the RKHS norm [4], one can also consider the spectral norm [11] or the Barron norm and flow-induced norm [13].

Finally, we discuss several possible directions for future research.

7.1 The generator

So far this paper has focused on analyzing the discriminator with the generator omitted. Here we show that part of our proof of generalization can be extended to the ordinary GAN with a generator.

One key step in our argument is the comparison of the landscapes of the population loss $L$ and the empirical loss $L^{(n)}$: For any finite measure $P$, with high probability, we can compare the variational derivatives

$$
\left\| \frac{\delta L}{\delta P} - \frac{\delta L^{(n)}}{\delta P} \right\|_{L^2(\Omega)} = \|k \ast (P_\ast - P^{(n)}_\ast)\|_{L^2(\Omega)} = O(n^{-1/2})
$$

(See the proof of Proposition 1 for more details.) It is this closeness between the two derivatives that eventually leads to the estimate of the generalization gap.

For the ordinary GAN, the loss becomes $L(P), P = G \# N$, where $N$ is some input distribution such as the unit Gaussian. The variational gradient of the generator $G$ over $L^2(N; \mathbb{R}^d)$ is simply

$$
\frac{\delta L}{\delta G} = \nabla_x \frac{\delta L}{\delta P} \circ G
$$
It follows that the population loss and empirical loss landscapes differ by
\[
\begin{align*}
\left\| \frac{\delta L}{\delta G} - \frac{\delta L^{(n)}}{\delta G} \right\|_{L^2(\mathcal{N})} &= \left\| \nabla_x \left( \frac{\delta L}{\delta P} - \frac{\delta L^{(n)}}{\delta P} \right) \circ G \right\|_{L^2(\mathcal{N})} \\
&= \left\| \nabla_1 k^* (P_* - P^{(n)}_*) \right\|_{L^2(P)} \\
&= O(n^{-1/2})
\end{align*}
\]
(See Section 6.5 for more details.) Then, one can conclude heuristically that the generalization gap for the ordinary GAN should also scale as \(O(t/\sqrt{n})\). Nevertheless, the gap between this heuristic argument and a rigorous proof is that the loss landscape for \(G\) is generally nonconvex, so the proof routines in Sections 6.1.1 and 6.1.2 need to be modified.

### 7.2 Wasserstein gradient flow

As discussed in Section 2.1, the distribution \(P\) is modeled as a density function in order to remove any parametrization in the generator. Its training depends on the linear topology of \(P(\Omega)\), such that the density function \(p_t\) is updated vertically \([15, 16]\). However, to better capture the training of GAN (and generative models in general), one can try to use horizontal updates: Consider the distribution \(P_t = G_t \# \mathcal{N}\), where \(G_t\) is the generator during training, then the trajectory \(P_{[0,T]}\) can be seen as a random smooth path
\[
P_{[0,T]} = \text{law}(x_{[0,T]}), \quad x_{[0,T]} = G_{[0,T]}(z) \in C^1([0, T] \to \mathbb{R}^d), \quad z \sim \mathcal{N}
\]
In particular, \(P_t\) satisfies the conservation of local mass, unlike the vertical updates that teleport mass.

One natural way to perform horizontal updates without any parametrization in \(G\) is to use the Wasserstein gradient flow
\[
\partial_t P_t = \nabla \cdot \left( P_t \nabla \frac{\delta L(P_t)}{\delta P} \right) = \nabla \cdot \left( P_t \nabla k^* (P_t - P_*) \right)
\]
One can try to bound its generalization gap as in Proposition 1.

**Conjecture 1.** Let \(P_t, P_t^{(n)}\) be the population-loss trajectory and empirical-loss trajectory. Then, for any \(\delta \in (0, 1)\), with probability \(1 - \delta\) over the sampling of \(P_*^{(n)}\),
\[
W_2(P_t, P_t^{(n)}) \lesssim \sqrt{\log d + \log \frac{1}{\delta} \sqrt{n} t}
\]

The difficulty for horizontal updates is that one typically needs geodesic convexity in \(W_2\) space, which is not known to hold for any of the GAN losses. For instance, the geodesic nonconvexity of the MMD metric \([14]\) has been analyzed in \([2]\).

### 7.3 Sophisticated discriminators

Instead of the WGAN loss with random feature functions, one can study more general set-ups with diverse losses and discriminators.

For the function representation of \(D\), ideally one would like to consider any family \(\mathcal{D}\) whose Rademacher complexity scales as \(O(n^{-1/2})\). Examples include 2-layer networks with bounded Barron norm \([18]\), deep residual networks with bounded flow-induced norm \([18]\) and multilayer networks with bounded path norm \([20]\).

For the training loss, one can consider the classical GAN loss \([22]\), the \(f\)-GAN loss \([45]\), the energy-based GAN loss \([66]\), the least-square GAN loss \([38]\), and any other losses defined as the dual over a discriminator family \(\mathcal{D}\).

Then, one can try to extend the generalization gap from Proposition 1 to these settings. The added difficulty, however, is that one no longer has a close-form formula for the dynamics of \(p_t\) as in \([15]\), making the comparison of the trajectories over the population loss and empirical loss less explicit.
7.4 Slower deterioration

A shortcoming of Proposition 9 is that its proof (in particular, inequality 29) does not utilize the fact that the discriminator $D_t$ during training has bounded Lipschitz norm

$$\forall t > 0, \quad \|D_t\|_{\text{Lip}} \lesssim 1$$

It seems reasonable that a more refined analysis would lead to a much stronger lower bound. For instance, one might conjecture that with probability $1 - \delta$,

$$\inf_D \{\|\nabla D - \nabla D_*\|_{L^2(\Omega)} \mid \|D\|_H \leq R, \|D\|_{\text{Lip}} \leq 1\} \gtrsim \delta \frac{1}{R^{d/2} n^{-d/2} - \frac{\sqrt{\log 2d + \sqrt{\log 2}/\delta}}{\sqrt{n}}}$$

The exponent $\frac{2}{d-2}$ comes from Corollary 3.4 of \[19\], and we estimate $\|\nabla D - \nabla D_*\|$ because it is the gradient field $\nabla D_t$ that drives the distribution during GAN training.

Then, Lemma 20 implies that it would take at least $n^{\Omega(d^2)}$ amount of time for $D_t$ to learn $D_*$. 

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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