GEODESIC CONNECTEDNESS OF SEMI-RIEMANNIAN MANIFOLDS.

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1. INTRODUCTION AND RIEMANNIAN MANIFOLDS

The problem of geodesic connectedness in semi-Riemannian manifolds (i.e. the question whether each two points of the manifold can be joined by a geodesic) has been widely studied from very different viewpoints. Our purpose is to review these semi-Riemannian techniques, and possible extensions. In the Riemannian case, it is natural to state this problem on (incomplete) manifolds with (possibly non-smooth) boundary, and we will discuss different conditions on this boundary in the remainder of this Section. In this case, geometrical as well as variational methods are applicable, and accurate results can be obtained by using the associated distance and related properties of positive-definiteness. For Lorentzian manifolds, the results cannot be so general, and very different techniques have been introduced which are satisfactory for some particular classes of Lorentzian manifolds. We start by considering several geometrical notions applicable to affine manifolds and, thus, to all semi-Riemannian manifolds, Section 2. Recall that variational methods are not applicable to affine manifolds, at least in a standard way: geodesics are the critical points of the action functional, but a metric tensor must be provided for the definition of this functional. In Section 3 some general facts on geodesic connectedness of Lorentzian manifolds are pointed out, and classical results about connectedness of spaceforms, which rely in the properties of actions of isometry groups, are summarized. In Section 4 we discuss variational methods applied to Lorentzian manifolds, which have been shown to be useful, mainly, to study stationary and splitting manifolds, with or without boundary. Finally, in Section 5 recent results, based on topological arguments and applicable to multiwarped spacetimes, are explained.

As pointed out by Gordon [33], the problem of the geodesic connectedness of a Riemannian manifold is important not only in its own right but also because of its relation, via the Jacobi metric, with the problem of connecting the points of the manifold by means of the trajectories determined by an autonomous potential. Moreover, it is also important because stronger results on geodesic connectedness among all semi-Riemannian manifolds are found for definite metrics. In fact, the following concept is especially useful: a Riemannian metric will be said to be convex if each two of its points can be joined by means of a distance-minimizing geodesic (not necessarily unique). By the Hopf-Rinow theorem all complete Riemannian manifolds are convex, and we discuss now when an incomplete one is either convex or geodesically connected.

We start with the simplest case. Let \((M, \langle \cdot, \cdot \rangle)\) be a \(n\)-dimensional Riemannian manifold (all the manifolds will be assumed to be connected and smooth, even though at most differentiability \(C^4\) will

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be needed), and \( \mathcal{D} \subset \mathcal{M} \) an open (connected) domain with differentiable (smooth) boundary \( \partial \mathcal{D} \); put \( \tilde{\mathcal{D}} = \mathcal{D} \cup \partial \mathcal{D} \). We will not assume a priori that \( \mathcal{M} \) is complete, because this assumption is not especially relevant for the indefinite case. The following definitions become natural:

(I) \( \partial \mathcal{D} \) is infinitesimally convex at \( p \in \partial \mathcal{D} \) (IC\(_p\)) if the second fundamental form \( \sigma_p \), with respect to the interior normal, is positive semidefinite. When \( \sigma_p \) is positive definite we say that \( \partial \mathcal{D} \) is strictly locally convex at \( p \) (SLC\(_p\)). Equivalently, take any differentiable function \( \phi : U \cap \mathcal{D} \rightarrow \mathbb{R} \), where \( U \subset \mathcal{M} \) is a neighborhood of \( p \) such that

\[
\phi^{-1}(0) = U \cap \partial \mathcal{D}, \quad \phi > 0 \text{ on } U \cap \mathcal{D} \quad \text{and} \quad d\phi_q \neq 0 \text{ for any } q \in U \cap \partial \mathcal{D}.
\]

Then, \( \partial \mathcal{D} \) is IC\(_p\) (resp. SIC\(_p\)) if and only if for one (and hence for all) function \( \phi \) satisfying (1.1):

\[
H_\phi(p)[v, v] \leq 0 \quad \text{resp. } < 0 \quad \forall v \in T_p \partial \mathcal{D}.
\]

(II) \( \partial \mathcal{D} \) is locally convex at \( p \in \partial \mathcal{D} \) (LC\(_p\)) if there exists a neighborhood \( U \subset \mathcal{M} \) of \( p \) such that

\[
\exp_p(T_p \partial \mathcal{D}) \cap (U \cap \mathcal{D}) = \emptyset.
\]

When \( \exp_p(T_p \partial \mathcal{D}) \cap (U \cap \tilde{\mathcal{D}}) = \{p\} \), then \( \partial \mathcal{D} \) is strictly locally convex at \( p \) (SLC\(_p\)).

It is easy to check

\[
\text{IC}_p \Leftrightarrow \text{LC}_p, \quad \text{SIC}_p \Rightarrow \text{SLC}_p.
\]

Clearly, the converse to the first implication is not true (\( \mathcal{M} = \mathbb{R}^2, \mathcal{D} = \{(x, y) : y > x^3\}, p = (0, 0) \)). Nevertheless, if there exists a neighborhood \( U \) of \( p \) such that \( \partial \mathcal{D} \) is IC\(_q\) for all \( q \in U \cap \partial \mathcal{D} \), then \( \partial \mathcal{D} \) is LC\(_q\) for all \( q \in U \cap \partial \mathcal{D} \). Do Carmo and Warner [23] realized that this problem is not as trivial as it sounds, and solved it (as a step for other computations) for the constant curvature case. Bishop [18] solved it in general, even though differentiability \( C^4 \) is explicitly used in his technique. Nor does the converse to the last implication (1.4) hold (\( \mathcal{M} = \mathbb{R}^2, \mathcal{D} = \{(x, y) : y > x^4\}, p = (0, 0) \)).

The previous definitions are applicable to each point \( p \) in the boundary \( \partial \mathcal{D} \). The following definitions are applicable to all \( \partial \mathcal{D} \):

1. Infinitesimally convex (IC): \( \partial \mathcal{D} \) is IC\(_p\) for all \( p \in \mathcal{D} \). This is equivalent to being:
   1A) locally convex (LC): LC\(_p\), \( \forall p \in \partial \mathcal{D} \) (because of Bishop’s result).
   1B) variationally convex (VC): for one (and hence for all) function \( \phi : \tilde{\mathcal{D}} \rightarrow \mathbb{R} \) satisfying (1.1) with \( U = \mathcal{M} \), inequality (1.2) holds, \( \forall \phi \in \partial \mathcal{D} \). For the equivalence between this definition and IC, it is enough to prove the following straightforward property [4, Cap. 3]: for any differentiable manifold with boundary \( \tilde{\mathcal{D}} \), a function \( \phi \) satisfying (1.1) with \( U = \mathcal{M} \) exists. (VC has been widely used by using variational methods, because function \( \phi \) allows direct penalization of the action functional (1.7).)

The “strict” concepts SIC, SLC and SVC can be defined analogously, and, clearly: SIC \( \Leftrightarrow \) SVC \( \Rightarrow \) SLC (but not the converse).

2. Geometrically convex (GC): for any \( p, q \in \mathcal{D} \), the range of any geodesic \( \gamma : [0, 1] \rightarrow \tilde{\mathcal{D}} \) such that \( \gamma(0) = p, \gamma(1) = q \) satisfies

\[
\gamma([0, 1]) \subset \mathcal{D}.
\]

If this also holds when \( p, q \in \partial \mathcal{D} \) the boundary is strictly geometrically convex (SGC). GC is a straightforward generalization to Riemannian manifolds of the usual notion of convexity given in Euclidean spaces. It is straightforward to check the implications LC \( \Rightarrow \) GC \( \Rightarrow \) IC; thus, by Bishop’s result, GC is equivalent to IC. Moreover, SGC \( \Leftrightarrow \) SLC. It is worth pointing out that, in the complete case, Germinario [23, Theorem 2.1], obtained by variational methods (with a weaker assumption on differentiability than Bishop), a direct proof of VC \( \Leftrightarrow \) GC.
then

If the answer is quite simple: generalize in a simple way. Now, it is natural to wonder if, in this case, the result is proven by using standard geometrical methods [5].

\[ D \subset M \] is connected, a domain \( D \) is convex if and only if \( \partial D \) is geodesically connected (as a consequence, the results by Gordon in [33] are reproven and generalized in a simple way). This definition is intrinsic to the (open domain of the) manifold. So, it is said that \( D \) (rather than \( \partial D \)) is PC (thus, it is natural to assume \( D = M \)).

Easily, \( PC \Rightarrow GC \). Nevertheless, PC is not implied by GC; in fact, a complete Riemannian manifold may be non-PC (for example: a complete surface with infinitely many holes). Summing up: at each point \( p \) implications (1.4) holds and, globally

\[ PC \Rightarrow GC \Leftrightarrow LC \Leftrightarrow VC \Leftrightarrow IC. \]

It is easy to check that, when \( \bar{D} \) is complete, there is no loss of generality assuming that so is \( M \): otherwise, a new Riemannian metric on all \( M \) can be defined such that it is complete and coincides with the original metric on \( \bar{D} \) [4, Cap. 3]. All the above equivalent conditions on convexity for \( \partial D \) provide different ways to prove that:

when \( M \) is complete, \( D \) is convex if and only if \( \partial D \) is convex.

Recall that the completeness assumption is essential. In fact, if \( M \) is incomplete and geodesically connected, a domain \( D \subset M \) with convex boundary may be non-geodesically connected (take as \( M \) a cylinder minus a small segment, and as \( D \) a small ball such that the segment lies inside it).

Now we are ready to examine the general case where \( \partial D \) is not differentiable or \( \bar{D} \) is not complete, which has been systematically studied in [5]. By the quoted result, it is clear that if there exists a sequence \( \{ \bar{D}_m \}, m \in \mathbb{N} \) of complete submanifolds with convex (differentiable) boundary such that

\[ \bar{D}_m \subset \bar{D}_{m+1} \quad \text{and} \quad D = \bigcup_{m \in \mathbb{N}} \bar{D}_m, \]

then \( D \) is geodesically connected (as a consequence, the results by Gordon in [33] are reproven and generalized in a simple way). Now, it is natural to wonder if, in this case, \( D \) must be convex. The answer is quite simple: if \( \bar{D} \) is complete then \( D \) is convex; otherwise, there are counterexamples. This result is proven by using standard geometrical methods [4].

Now, two questions arises naturally: (A) can the convexity of each \( \partial D_m \) be weakened? and (B) is there any intrinsic condition for the convexity of \( D \) (independent of its boundary in \( M \))? To answer these questions the (intrinsic) Cauchy boundary of \( D \) and a variational approach must be taken into account. Let \( \bar{D}^c \) be the canonical completion of \( D \) by using Cauchy sequences, and \( \partial_c D \) the corresponding boundary points, \( \bar{D}^c = D \cup \partial_c D \). Recall that \( \bar{D}^c \) is always complete as a metric space, but the boundary points in \( \partial_c D \) are not necessarily differentiable and, if they are, the metric may be non–extendible or degenerate there. Note that any point of \( \partial D \) determines naturally one or more points in \( \partial_c D \), and \( \bar{D} \) is complete if and only if all the points in \( \partial_c D \) are determined in this way by points of \( \partial D \). From the variational point of view, geodesics joining two fixed points \( p, q \in D \) are seen as critical points of the action functional

\[ f(x) = \frac{1}{2} \int_0^1 \langle \dot{x}(s), \dot{x}(s) \rangle ds \]

defined on the space of differentiable curves joining \( p \) and \( q \) or, technically better, on its \( H^1 \)– Sobolev completion, i.e., the Hilbert manifold \( \Omega^1(D, p, q) = \{ x \in H^{1,2}(0,1,D) \mid x(0) = p, x(1) = q \} \).

A first result [4] shows that a domain \( D \) as in \( \{4\} \) is convex, even if the boundary of each \( D_m \) is not convex (and, thus, each \( D_m \) may be non–geodesically connected), when a suitable local estimate of the loss of convexity of \( \partial D_m \) and boundness of the sequence is ensured.
Theorem 1.1. Let \((M, \langle \cdot, \cdot \rangle)\) be a complete Riemannian manifold and \(D\) an open domain of \(M\). Assume that there exists a positive differentiable function \(\phi\) on \(D\) such that

(i) \(\lim_{x \to \partial D} \phi(x) = 0\);
(ii) each \(y \in \partial D\) admits a neighbourhood \(U \subset M\) and constants \(a, b > 0\) such that
\[
a \leq \|\nabla \phi(x)\| \leq b \quad \forall x \in D \cap U;
\]
(iii) the first and second derivatives of the normalized flow of \(\nabla \phi\) are locally bounded close to \(\partial D\) (that is: each \(y \in \partial D\) admits a neighbourhood \(U \subset M\) such that the induced local flow of \(\nabla \phi/\|\nabla \phi\|^2\) on \(D \cap U\) have first and second derivatives with bounded norms), and
(iv) there exists a decreasing and infinitesimal sequence \(\{a_m\}\) such that each \(y \in \partial D\) admits a neighbourhood \(U \subset M\) and a constant \(M \in \mathbb{R}\) satisfying:
\[
H_\phi(x)[v, v] \leq M(v, v)\phi(x) \quad \forall x \in \phi^{-1}(a_m) \cap U, v \in T_x\phi^{-1}(a_m), m \in \mathbb{N}. \tag{1.8}
\]

Then \(D\) is convex.

It is straightforward to check that, if \(\partial D\) is differentiable and convex, all the conditions (i)–(iv) are automatically satisfied.

To prove this result, the action functional \(f\) is penalized with a term depending on a positive parameter \(\epsilon\): \(f_\epsilon(x) = f(x) + \epsilon \int_0^1 \phi^{-2}(\dot{x}(s))ds\). Each penalized functional is bounded from below and satisfies the condition of Palais-Smale; so, from standard variational arguments (see, for example, [35]), \(f_\epsilon\) admits a critical (minimum) point. The crucial point is to prove that a critical point of \(f_\epsilon\) in a sublevel of \(f_\epsilon\) is uniformly far (with respect to \(\epsilon\)) from \(\partial D\). In the proof the critical points of the penalized functionals are projected (using the normalized flow of \(\nabla \phi\)) on the hypersurface \(\phi^{-1}(a_m)\) for \(m\) large enough. This makes possible to get critical points of \(f\) (i.e. geodesics) not touching \(\partial D\) by means of a limit process.

Technical condition (iii) and even the completeness of the ambient manifold \(M\) can be weakened if (iv) is imposed on all points and directions enough close to the boundary. So, a straightforward consequence of the technique in for Theorem 1.1 is the following result [3] (compare with [13]):

Theorem 1.2. Let \((M, \langle \cdot, \cdot \rangle)\) be a Riemannian manifold, \(D \subset M\) an open domain, and \(\bar{D}^c = D \cup \partial D\) its canonical Cauchy completion. Assume that there exists a positive differentiable function \(\phi\) on \(D\) such that

(i) \(\lim_{x \to \partial D} \phi(x) = 0\);
(ii) each \(y \in \partial D\) admits a neighbourhood \(U \subset \bar{D}^c\) and constants \(a, b > 0\) such that
\[
a \leq \|\nabla \phi(x)\| \leq b \quad \forall x \in D \cap U;
\]
(iii) each \(y \in \partial D\) admits a neighbourhood \(U \subset \bar{D}^c\) and a constant \(M \in \mathbb{R}\) such that inequality (1.8) holds for all \(x \in D \cap U\) and for all \(v \in T_xM\).

Then \(D\) is convex.

It is worth pointing out that in both previous theorems a multiplicity result can be also obtained, when the topology of the fiber is not homotopically trivial: if \(D\) is not contractible (in itself), then for any \(p, q \in D\) there exists a sequence \(\{x_m\}\) of geodesics in \(D\) joining them such that \(\lim_{m \to \infty} f(x_m) = \infty\). For the proof one uses that the Lusternik-Schnirelman category of \(D\) is infinite, which implies the existence of infinitely many connecting critical points of \(f_\epsilon\) with diverging lengths. We refer to [3] for a deeper discussion of these results, and for examples, (see also [2]).

2. AFFINE CONNECTIONS

Now, consider a manifold \(M\) endowed with an affine connection \(\nabla\); thus, all the properties of its geodesics will hold for all semi-Riemannian manifolds. As we are interested in geodesics, there is no
loss of generality assuming that $\nabla$ is symmetric; otherwise, it is well-known that there exists another affine connection with the same geodesics and torsion-free.

All previous notions about the convexity of the (differentiable) boundary of a domain $\mathcal{D}$ are directly extendible to the affine case except $\text{IC}_p$, $\text{IC}$, because the second fundamental form $\sigma_p$ is not canonically defined. Nevertheless, the characterization of $\text{IC}_p$ in terms of the function $\phi$ satisfying (1.1) does make sense. In fact, it is not difficult to check that (1.2) still holds for a function $\phi$ satisfying (1.1) if and only if it holds for each such $\phi$. Thus, this will be the natural definition of $\text{IC}_p$ for affine manifolds. In principle, Bishop’s result holds just in the Riemannian case and, so, the implications which hold are (1.4) and: $\text{PC}$ or $\text{LC} \Rightarrow \text{GC} \Rightarrow \text{VC} \Leftrightarrow \text{IC}$, and $\text{SLC} \Rightarrow \text{SGC} \Leftrightarrow \text{SVC} \Leftrightarrow \text{SIC}$.

However, these concepts (except for $\text{PC}$) are not so useful in the affine case, because even when $\mathcal{D}$ is complete and $\partial \mathcal{D}$ is convex in the strongest sense, $\mathcal{D}$ may be non-geodesically connected. In fact, the first problem to be solved is what conditions on a manifold without boundary must be imposed to obtain geodesic connectedness. The following example by Bates [8] shows that a compact and complete affine manifold may be non-geodesically connected.

**Example 2.1.** Consider the moving frame $(X_1 = \cos x \partial_x + \sin x \partial_y, X_2 = -\sin x \partial_x + \cos x \partial_y)$ on $\mathbb{R}^2$, and the affine connection $\nabla$ such that $X_1, X_2$ are parallel. The geodesics of $\nabla$ are the integral curves of the (complete) vector fields $X = aX_1 + bX_2, a, b \in \mathbb{R}$. So, any geodesic $\gamma(s) = (x(s), y(s))$ is complete and $x(s)$ lies in an interval of length $\leq 2\pi$. Thus, a required example is the quotient torus $T^2 = \mathbb{R}^2/4\pi\mathbb{Z}^2$, with the induced connection.

Given $p \in \mathcal{M}$ let $\text{Con}(p) \subseteq \mathcal{M}$ be the subset containing the points of $\mathcal{M}$ which can be connected with $p$ by means of a geodesic. The previous example also shows that, even under strong hypotheses, $\text{Con}(p)$ may be non-closed. To study this more in depth, consider the following concepts [34], [13].

Let $G(\mathcal{M})$ be the space of the geodesics of $(\mathcal{M}, \nabla)$, that is, the projective bundle $PM \otimes \text{T}M$ (obtained by identifying two vectors of the reduced bundle $T'\mathcal{M} \otimes \text{T}M \setminus \text{zero section}$) if they are proportional) where two classes of non-zero vectors $[v], [w] \in PM$ are identified if there exists a geodesic $\gamma : [a, b] \rightarrow \mathcal{M}$ such that $\gamma'(t_0) \in [v], \gamma'(t_1) \in [w]$ for some $t_0, t_1 \in [a, b]$. Thus, each geodesic $\gamma$ determines a unique class $[\gamma] \in G(\mathcal{M})$. In the remainder of this Section, all the geodesics will be assumed inextendible.

The natural quotient topology of $G(\mathcal{M})$ can be characterized as follows. Consider first a sequence of geodesics $\gamma_n : [a_n, b_n] \rightarrow \mathcal{M}$ and a geodesic $\gamma : [a, b] \rightarrow \mathcal{M}$. If there exists $t_0 \in [a, b]$, contained in all but a finite number of $[a_n, b_n]$, and $\{\gamma'_n(t_0)\} \rightarrow \gamma'(t_0)$, then $\limsup \{a_n\} \leq a < b \leq \liminf \{b_n\}$ and $\{\gamma'_n\}$ converges uniformly on compact subsets of $[a, b]$ to $\gamma'$ (for any distance on $\text{T}M$ compatible with its topology) [40], Prop. 2.1. Now, a sequence of geodesics $\{\beta_n\}$ is said to converge tangentially to a geodesic $\beta$ if there exists $\gamma_n \in [\beta_n], \gamma \in [\beta]$ and a $t_0$ such that $\{\gamma'_n(t_0)\} \rightarrow \gamma'(t_0)$; this convergence holds if and only if $\{[\beta_n]\} \rightarrow [\beta]$ in $G(\mathcal{M})$, [13].

It is worth pointing out:

(I) Tangential convergence is completely independent of completeness, even if $\mathcal{M}$ is compact. In fact, even if $\gamma_n$ converges tangentially to a unique $\gamma$, all $\gamma_n$ may be incomplete and $\gamma$ complete, or vice versa; counterexamples can be found in Lorentzian tori [40].

(II) The sequence $\gamma_n$ may converge tangentially to more than one limit and, in this case, $G(\mathcal{M})$ is not Hausdorff. Remarkably, this happens when a point $p$ of any affine manifold is removed, if not all the geodesics starting at $p$ are closed. But, unfortunately, this is not by any means the only case. In fact, consider the standard flat torus $\mathbb{R}^2/\mathbb{Z}^2$; the sequence of geodesics constantly equal to $\alpha$, where $\alpha$ is induced on the torus by a geodesic in $\mathbb{R}^2$ with irrational slope, has infinitely many tangential limits.

Recall that $(\mathcal{M}, \nabla)$ is called disprisoning if given any geodesic $\gamma : [a, b] \rightarrow \mathcal{M}$ and any compact subset...
Of \( \mathcal{M} \) there are sequences \( \{ t_n \} \rightarrow a^+, \{ s_n \} \rightarrow b^- \) such that \( \gamma(t_n) \) and \( \gamma(s_n) \) do not lie in \( K \). Disprisonment, pseudoconvexity and the topology of \( G(\mathcal{M}) \) have proven to be fruitful in order to study some geometrical properties, including a Cartan-Hadamard type theorem and applications to Relativity, see \([8],[10],[11]\). For geodesic connectedness, the following result by Beem and Parker holds \([12],[13]\).

Theorem 2.2. Let \( (\mathcal{M}, \nabla) \) be an affine manifold; the following implications are fulfilled:

\begin{equation*}
\text{Disprisonment and pseudoconvexity } \Rightarrow \ G(\mathcal{M}) \text{ is Hausdorff } \Rightarrow \ \text{Con}(p) \text{ is closed, } \forall p \in \mathcal{M}.
\end{equation*}

Moreover, if this last property holds and there are no conjugate points, then \( (\mathcal{M}, \nabla) \) is geodesically connected.

The last assertion is obvious because the absence of conjugate points implies that Con(\( p \)) is open for all \( p \in \mathcal{M} \); thus, the two implications in Theorem 2.2 yield sufficient conditions for geodesic connectedness.

### 3. THE INDEFINITE SEMI-RIEMANNIAN CASE. SPACEFORMS

We refer to the standard books \([10],[38]\) for definitions and general background about semi-Riemannian, and especially Lorentzian, manifolds. For indefinite metrics, the absence of an associated canonical distance and, so, of any analog to the Hopf-Rinow theorem, makes the problem of geodesical connectedness very subtle. Perhaps the only non-trivial result with a clear resemblance to the Riemannian case is that of Avez \([1]\) and Seifert \([10]\): in a globally hyperbolic Lorentzian manifold, any pair of causally related points (i.e. which can be joined with a causal curve) can be joined by means of a causal geodesic. For this result is essential that, in the Lorentzian case, causal geodesics maximize locally the “time-separation” (or “Lorentzian distance”) between causally related points. Global hyperbolicity introduces then a sort of compactness in the space of causal curves \( C^\text{cau}_{p,q} \) joining any two fixed points \( p, q \), in such a way that the lengths of curves in \( C^\text{cau}_{p,q} \) are bounded, and its supremum at every connected part of \( C^\text{cau}_{p,q} \) is reached by a curve (necessarily pregeodesic) therein.

Nevertheless, neither compactness nor completeness implies geodesic connectedness. It is interesting to study the geodesic connectedness of Lorentzian surfaces (Smith, \([51]\)), in comparison with the affine case. Recall first that a plane \( S \) endowed with a Lorentzian metric (or conformal class of metrics) is called normal if there exists a diffeomorphism of \( S \) onto \( \mathbb{R}^2 \) which takes every null-geodesic into an axis-parallel line; this can be characterized in terms of the absence of barriers \([51]\) (see also \([52]\)). Moreover, a null-complete plane is normal if its Gaussian curvature does not change sign off a compact subset, and the integral of its curvature is finite.

Theorem 3.1. (1) A normal Lorentzian plane is geodesically connected \([51]\). (2) The universal covering of a complete Lorentzian torus with a Killing vector field \( K(\neq 0) \) is normal \([17]\).

Thus, a complete Lorentzian torus with such a \( K \) is geodesically connected (this can be also checked more directly, \([17]\)); the completeness in assertion (2) can be replaced by any of the following conditions (a posteriori equivalent): (i) \( K \) has a definite causal sense on all the torus (timelike or null or spacelike), or (ii) the metric is (globally) conformally flat. Now, consider the following Lorentzian torus \([51, \text{Sect. 5}]\) (see also \([3, \text{Sect. 3}],[17, \text{Sect. 5}]\)).

Example 3.2. Take on \( \mathbb{R}^2 \) the moving frame \( X_1, X_2 \) as in Example 2.1, and consider the Lorentzian metric \( g \) such that \( X_1 \) and \( X_2 \) are null and \( g(X_1, X_2) = -1 \). The velocities of any timelike or spacelike curve \( (x(s), y(s)) \) must remain in any of the four cones continuously determined by \( X_1, X_2 \); so, the length of the projection of \( x(s) \) is again bounded. Thus, a geodesically disconnected Lorentzian torus is induced.

It is worth pointing out that in this example \( K = \partial_y \) is a Killing vector field (in Example 2.1 \( K \) is an affine vector field) inducible on the torus. So, this Lorentzian torus is geodesically incomplete.
(the causal character of $K$ changes); we do not know any example of complete and geodesically disconnected torus.

The fact that a complete semi-Riemannian manifold may be geodesically disconnected can be stressed studying spaceforms. We say that a semi-Riemannian $n$-dimensional manifold $\mathcal{M}$ of index $\nu$ is a spaceform if it is complete with constant curvature $C$. In this case, $\mathcal{M}$ is covered by the corresponding model (1-connected) space $M(n, \nu, C)$, that is, $\mathcal{M} = M(n, \nu, C)/\Gamma$, where $\Gamma$ is the fundamental group of $\mathcal{M}$. The case $C = 0$ is trivial (the model space $M(n, \nu, 0) \equiv \mathbb{R}_\nu^n$ is geodesically connected) and, up to a homothety, we can assume $C = 1$ (the homothetic factor may be positive as well as negative; recall that the Levi-Civita connection remains unchanged). If $n \geq 3$, the model space is then the pseudosphere $S^\nu_0$ (spacelike vectors of norm 1 in $\mathbb{R}^\nu_0$); the Lorentzian pseudosphere $S^\nu_1$ is also called de Sitter spacetime, and it is globally hyperbolic. Recall that an affine manifold is called starshaped from a point $p$ if $\exp_p$ is onto. It is not difficult to check [38, Propos. 5.38] that no indefinite pseudosphere $S^\nu_\nu$, $0 < \nu < n$ is geodesically connected. The following result by Calabi and Markus [19], in particular, solves completely the geodesic connectedness of Lorentzian spaceforms of positive curvature with $n \geq 3$.

**Theorem 3.3.** For $n \geq 2$:

1. Two points $p, q \in S^\nu_1$ are connectable by a geodesic if and only if $\langle p, q \rangle_1 > -1$, where $\langle \cdot, \cdot \rangle_1$ is the usual Lorentzian inner product of $\mathbb{L}^{n+1}_1 \equiv \mathbb{R}^{n+1}_1$.

2. Every spaceform $\mathcal{M} = S^\nu_1/\Gamma, \mathcal{M} \neq S^\nu_1$ is starshaped from some point $p \in \mathcal{M}$.

3. A spaceform $\mathcal{M} = S^\nu_1/\Gamma$ is geodesically connected if and only if it is not time-orientable.

The proof of (1) follows by a direct computation of the geodesics. For the remainder, it is essential that, whenever $2\nu \leq n$, the group of isometries $\Gamma$ is finite. Then, up to conjugacy, $\Gamma \subset O(1) \times O(n) \subset O_1(n+1)$, and the proof follows by studying the barycenter of the orbits, which must lie in the timelike axis of $\mathbb{R}^{n+1}_1$.

For arbitrary index $\nu$ (including the case $\nu = n - 1$, which is equivalent to the Lorentzian case of constant negative curvature) the only general results we know are extensions of Theorem 3.3, with $n \geq 3$, and assuming as an additional hypothesis (when $2\nu > n$) that the fundamental group $\Gamma$ is finite. For these extensions, the non time-orientability must be replaced by the inexistence of a proper time-axis [33, Theor. 11.2.3]. We recall that a time-axis $T$ is a one-dimensional $\Gamma$-invariant negative-definite linear subspace of $\mathbb{R}^{n+1}_\nu$. $T$ is proper if $\Gamma$ acts trivially on $T$; that is, if $T$ is not proper then $\Gamma$ also acts as a multiplication by -1.

**4. VARIATIONAL METHODS.**

The systematic application of variational methods on infinite-dimensional manifolds to Lorentzian Geometry started with a seminal paper by Benci and Fortunato [14] (see also [15]), who studied geodesic connectedness of standard stationary spacetimes. Since then, the geodesic connectedness of stationary as well as splitting spacetimes, with or without boundary, has been widely studied by variational methods. We will review briefly these results, explaining mainly the stationary case, and giving some references and comments for the splitting case (see also the next Section). We refer to [15] for general background on variational methods and applications to this and other problems in Lorentzian Geometry, and to [11] for properties of Killing fields which will be borne in mind.

For stationary manifolds we will follow the approach in [31], because it makes more intrinsic assumptions and recovers the previous results. Recall that a Lorentzian manifold $\langle \mathcal{M}, \langle \cdot, \cdot \rangle \rangle$ is called stationary if it admits a globally defined timelike Killing vector field $K$. Fixing two points $p, q \in \mathcal{M}$, geodesics joining them are, still in the semi-Riemannian case, the critical points of the action functional $f$ on $\Omega^1(\mathcal{M}, p, q)$ given in (1.7). But now this functional is bounded neither from above nor
from below, and it may not satisfy Palais-Smale condition. Nevertheless, this problem can be solved by taking into account that any critical point of $f$ (or geodesic) $z$ satisfies $\langle K, z' \rangle \equiv C_z$, where $C_z$ is a constant. This suggests that variations in the $K$–direction are irrelevant, and that, under the orthogonal splitting of the tangent bundle $TM = \text{Span}(K) \oplus \text{Span}(K)^\perp$, only projections on the spatial part $\text{Span}(K)^\perp$ are important (similar ideas work for geodesic completeness [22, 11]).

More precisely, let $C_{p,q} = \{ z \in C^1([0,1], \mathcal{M}) \mid z(0) = p, z(1) = q, \langle K, z' \rangle \equiv C_z \}$ and $\mathcal{N}_{p,q}$ its Sobolev $H^{1,2}$-completion; for technical computations, the Riemannian metric $g_R$ obtained by reversing $\langle \cdot, \cdot \rangle$ on $\text{Span}(K)$ can be used. The action functional $f$ can be written as the sum of two functionals $f_1, f_2$,

$$f_1(z) = \frac{1}{2} \int_0^1 \left( \langle \dot{z}(s), \dot{z}(s) \rangle - \frac{\langle \dot{z}(s), K \rangle^2}{\langle K, K \rangle} \right) ds, \quad f_2(z) = \frac{1}{2} \int_0^1 \frac{\langle \dot{z}(s), K \rangle^2}{\langle K, K \rangle} ds.$$

It is not difficult to check that $f'_1$ vanishes on all tangent vectors $\zeta$ to $\Omega^1(\mathcal{M}, p, q)$ which are pointwise parallel to $K$, that is, $f_1$ is the “spatial” part with respect to $K$ of the functional $f$. Moreover, $\mathcal{N}_{p,q}$ can be characterized as the set of $z \in \Omega^1(\mathcal{M}, p, q)$ such that $f'(z)[\zeta] = f'_2(z)[\zeta] = 0$, for any such $\zeta$ parallel to $K$; by the implicit function theorem, $\mathcal{N}_{p,q}$ is a submanifold of $\Omega^1(\mathcal{M}, p, q)$. Finally, one can check that critical points of $f$ on $\Omega^1(\mathcal{M}, p, q)$ coincide with the critical points of the restriction $J$ of $f$ to $\mathcal{N}_{p,q}$ (and thus to $C_{p,q}$).

Summing up, if some conditions are imposed on $C_{p,q}$ so that the restriction $J$ of $f$ to $\mathcal{N}_{p,q}$ must reach a critical point, then a geodesic connecting $p$ and $q$ is obtained. And the most natural variational such conditions are:

$$C_{p,q} \text{ is (i) non-empty and (ii) } c\text{–precompact, for some } c > \text{Inf}_{C_{p,q}} f(z)$$

($C_{p,q}$ is said $c$–precompact if every sequence $\{ z_n \} \subset C_{p,q}$ with $f(z_n) \leq c$ has a uniformly convergent subsequence in $\mathcal{M}$). Thus, if (i) and (ii) hold for all $p, q \in \mathcal{M}$, then $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ is geodesically connected (one can check that $J$ is bounded from below, its sublevels $J^{c'}$ are complete metric subspaces of $\mathcal{N}_{p,q}$, for all $c' \leq c$, and Palais-Smale condition is fulfilled).

Now, we can wonder when (i) and (ii) hold. If the restriction of $f$ to $C_{p,q}$ is pseudocoercive (that is, $c$–precompact for all $c \geq \text{Inf}_{C_{p,q}} f(z)$) for any $p, q \in \mathcal{M}$ then $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ is globally hyperbolic, but the converse is not true. Condition (i) holds if either $p$ and $q$ are causally related or $K$ is complete (this happens, for example, when the auxiliary Riemannian metric $g_R$ is complete, and $g_R(K, K)(= -(K, K))$ is bounded); clearly, the converse does not hold. Assume that $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ is a standard stationary spacetime, that is, $\mathcal{M}$ is a product manifold $\mathcal{M} = \mathbb{R} \times \mathcal{M}_0$ ($\mathcal{M}_0$ any manifold) and $\langle \cdot, \cdot \rangle$ can be written, with natural identifications, as:

$$\langle \cdot, \cdot \rangle = -\beta dt^2 + 2\omega \otimes dt + g_0, \quad (4.1)$$

where $dt^2$ is the usual metric on $\mathbb{R}$, and $g_0$, $\beta$, $\omega$ are, resp., a Riemannian metric, a positive function and a 1–form, all on $\mathcal{M}_0$ (locally, stationary spacetimes look like standard stationary ones).

**Theorem 4.1.** A standard stationary spacetime is geodesically connected, if: (a) $g_0$ is complete, (b) $0 < \text{Inf}(\beta) \leq \text{Sup}(\beta) < \infty$, and (c) the $g_0$-norm of $\omega(x)$ has a sublinear growth in $\mathcal{M}_0$.

(For (c), we mean that the norm of $\omega(x)$ has an upper bound $A \cdot d_0(x, p_0)^\alpha + B$, for some $A, B \in \mathbb{R}, \alpha \in (0, 1], p_0 \in \mathcal{M}_0$, where $d_0$ is the $g_0$–distance on $\mathcal{M}_0$). In fact, under these three conditions, assumptions (i) and (ii) are always satisfied; we refer to [31, Prop. A.3] for a more intrinsic way to express these conditions, in terms of stationary manifolds admitting a differentiable time function (which are standard stationary a posteriori). In the standard static case ($\omega \equiv 0$) condition $0 < \text{Inf}(\beta)$
can be dropped (see [2], [3]); however, we remark that the imposed inequalities always imply global hyperbolicity [19, Cor. 3.4, 3.5].

It is also worth pointing out:

(I) This technique provides also consequences for the existence of infinitely many connecting geodesics or timelike geodesics when \( \mathcal{M} \) is not contractible. In fact,

*two points* \( p, q \) *of a stationary manifold can be joined by a sequence of spacelike geodesics with divergent lengths if \( K \) is complete, \( \mathcal{C}_{p,q} \) is pseudocoercive and \( \mathcal{M} \) is non-contractible;*

(the essential step for the proof is that \( \Omega_{p,q}(M) \) is homotopically equivalent to \( \mathcal{N}_{p,q} \) and, thus, the Ljusternik-Schnirelman category of \( \mathcal{N}_{p,q} \) is infinite).

For timelike geodesics, recall that, under our type of assumptions, Avez-Seifert’s technique is appliable and chronologically related points can be joined by timelike geodesics. Let \( p, q \in \mathcal{M} \), and \( \gamma_q(t) \) be an integral curve of \( K \) starting at \( q \); when \( K \) is complete then \( p \) belongs to the chronological past of \( \gamma_q(t) \) for \( t \) big enough (a direct proof is not difficult, see also [10, Sect. 4]). Then, under precompactness, \( \mathcal{N}_{p,\gamma_q(t)} \) contains at least a timelike geodesic for \( t \) big enough and, when the topology is not homotopically trivial, the following result on multiplicity holds (see [31, Theorem 1.4]):

*If \( \mathcal{M} \) is non-contractible, \( K \) is complete and there exist \( c_0 < 0, t_0 > 0 \) such that \( \mathcal{N}_{p,\gamma_q(t)} \) is \( c_0 \) precompact for all \( t > t_0 \), then the number of timelike geodesics joining \( p \) and \( \gamma_q(t) \) goes to \( \infty \) when \( t \to \infty \).*

(II) Analogous techniques should work if a semi-Riemannian manifold \( (\mathcal{M}, g) \) of index \( s \) admits \( s \) Killing vector fields \( K_1, \ldots, K_s \) independent at each \( p \in \mathcal{M} \) such that \( g \) restricted to \( K = \text{Span}\{K_1, \ldots, K_s\} \) is negative definite. We have now the natural splitting \( TM = K \oplus K^\perp \), and, for each geodesic \( z \), the projection of \( \dot{z} \) on \( K \) can be recovered from the constants \( C_{z,i} \equiv g(\dot{z}, K_i) \) (the analogous problem for geodesic completeness was solved in [11]).

Moreover, even in the Lorentzian case, one can consider the case when there exist two pointwise independent Killing vector fields \( K_1, K_2 \) such that \( \{K_1(p), K_2(p)\} \) spans a Lorentzian plane at each \( p \in \mathcal{M} \) but neither \( K_1 \) nor \( K_2 \) are timelike on (all) \( \mathcal{M} \). Remarkably, this happens in Gödel type spacetimes; for the modifications of the technique in this case, see [21], [22].

(III) Let us discuss the case with boundary briefly (see also [4], [5], [27]). Consider first a standard stationary spacetime \( \mathcal{M} = \mathbb{R} \times \mathcal{M}_0 \) satisfying the assumptions of Theorem 4.1 (thus, geodesically connected), and let \( \mathcal{D}_0 \subset \mathcal{M}_0 \) be an open domain with differentiable boundary \( \partial \mathcal{D}_0 \). We have seen that, in general, a domain \( \mathcal{D} \) of a complete or geodesically connected semi-Riemannian manifold does not inherit good properties for geodesic connectedness. Nevertheless, if \( \mathcal{D} = \mathbb{R} \times \mathcal{D}_0 \subset \mathcal{M} \) is variationally convex (VC), then the corresponding function \( \Phi \) can be chosen independent of \( t \), and the functionals \( f, J \), can be penalized in a similar way to the Riemannian case. So, penalized functionals \( J_\nu \) do satisfy Palais-Smale condition, and one obtains geodesic connectedness. Moreover, one obtains again that \( \partial \mathcal{D} \) is VC if and only if it is GC [5]. Of course, if we are interested just in \( \mathcal{D} \), it is not exactly relevant for \( \mathcal{M} \) to fulfill assumptions of Theorem 4.1: one needs just the convexity of \( \partial \mathcal{D} \) and the possibility to extend the metric on \( \mathcal{D} \) to all \( \mathcal{M} \), in such a way that the assumptions of Theorem 4.1 are fulfilled; this possibility can be expressed as more intrinsic conditions on \( \mathcal{D} \). Furthermore, the assumption that the stationary manifold is standard can also be dropped. In fact, in order to penalize the functionals \( f, f_1, f_2 \) as in the Riemannian case, one needs only that the boundary \( \partial \mathcal{D} \) can be determined by a function \( \phi \) as in the definition of VC, which is also invariant by the flow of \( K \).

When the boundary is not smooth, the problem has been studied in the standard *static* case \( (\omega \equiv 0) \). A result in the spirit of Theorem 1.2 can be obtained, proving the geodesic connectedness of spacetimes such as outer Reissner-Nordström’s and Schwarzschild’s [16] (see also [2, Cap. 6]). On the other hand, for the standard static case, a different variational principle in [14] solves completely
the problem of connecting a point and a integral curve of \( \partial_t \), for arbitrary \( \mathcal{M}_0 \) (or \( \mathcal{D}_0 \)).

We will mean by a splitting spacetime a product manifold \( \mathcal{M} = \mathbb{R} \times \mathcal{M}_0 \) endowed with a Lorentzian metric as \( (4.1) \) but allowing \( g_0, \beta, \) and \( \omega \) to depend differentiably on the time variable \( t \). In this case, bounds on the derivatives with respect to \( t \) of these elements must be also imposed in order to obtain geodesic connectedness. As a typical result we have \( [30] \):

**Theorem 4.2.** A splitting spacetime \( (\mathcal{M}, \langle \cdot, \cdot \rangle = -\beta(t, x)dt^2 + 2\omega(t, x) \otimes dt + g_t(x)) \) is geodesically connected, if:

1. \( g_0 \) is complete and there exists \( \lambda > 0 \) such that \( g_t > \lambda g_0 \) for all \( t \).
2. \( 0 < \inf(\beta) \) and \( \beta(x, 0), \omega(x, 0) \) are bounded.
3. \( g_t/\beta(t, x) \) and \( \omega/\beta(t, x) \) are bounded by a function on \( \mathcal{M} \) type: \( b_0(x) + b_1(x)|t|^\mu, \) \( (\mu \in [0, 1], \) and \( \mu \in [0, 2], \) resp.)
4. Consider the natural derivatives \( \partial_\alpha, \partial_\beta, \partial_\delta \) of \( \alpha, \beta, \delta \) with respect to \( t \), resp. Then the \( g_t \)-norms of \( \partial_\alpha/\alpha, \partial_\beta/\beta, \partial_\delta/\delta \) are bounded at each hypersurface with constant \( t \), and its supremum when \( t \to \pm \infty \) goes to 0.

Moreover, domains of type \( \mathcal{D} = [a, b] \times \mathcal{M}_0 \) (strips) are shown to inherit geodesic connectedness, provided that \( \partial \mathcal{D} \) is VC (see \( [17] \) for orthogonal splittings \( \omega \equiv 0 \), \( [30] \) for non-orthogonal splittings, and \( [30] \) for strips; see also \( [15] \)).

Nevertheless, now the t-dependence does not allow a reduction to an equivalent Riemannian problem, as in the stationary case. Instead, Rabinowitz’s saddle point theorem \( [33] \) is used, but two technical complications must be circumvented: (A) \( f \) does not satisfy a Palais-Smale condition, which is solved by approximating by a family of functionals \( f_\eta, \eta \geq 0, f_0 = f \), and making some a priori estimates of the critical points of \( f_\eta, \eta > 0 \) in order to ensure a good behavior under the limit \( \eta \to 0 \), and (B) in Rabinowitz’s theorem, the independent directions where the functional goes to \(-\infty \) are finite; so, a Galerkin finite-dimensional approximation is carried out. For the existence of infinitely many connecting geodesics when \( \mathcal{M} \) is not contractible, the relative category, a topological invariant somewhat subtler than the Ljusternik-Schnirelman category, is used.

Finally, it is worth pointing out that the problem of geodesic connectedness is naturally generalized to others like: (1) the connectedness of two submanifolds by normal geodesics, studied for splitting manifolds in \( [20] \), or (2) the connection by trajectories of some more general Lagrangian systems, studied for stationary manifolds and potential vector fields independent of time in \( [8] \).

5. **Multiwarped Spacetimes. A Topological Method**

A multiwarped spacetime is a product manifold \( I \times F_1 \times \cdots \times F_m, I = [a, b] \subseteq \mathbb{R} \) endowed with a metric \( g = -dt^2 + \sum_{i=1}^m f_i^2(t)g_i, t \in I \), where \( f_1, \ldots, f_m \) are positive functions on \( I \), and each \( g_i \) is a Riemannian metric on the manifold \( F_i \). These spacetimes include classical examples of spacetimes: when \( m = 1 \) they are the Generalized Robertson-Walker (GRW) spacetimes, standard models of inflationary spacetimes \( [15] \); when \( m = 2 \), the intermediate zone of Reissner-Nördström spacetime and the interior of Schwarzschild spacetime appear as particular cases \( [16] \); moreover, multiwarped spacetimes may also represent relativistic spacetimes together with internal spaces attached at each point (see \( [37] \)).

The geodesic connectedness of this type of spacetimes with \( m = 2 \) have been studied by using variational methods in manifolds without and with boundary \( [28], [29] \). Nevertheless, more accurate results are obtained by using a topological method introduced in \( [24] \); in fact, the following result is proven:

**Theorem 5.1.** (1) In a multiwarped spacetime with convex fibers \( (F_1, g_1), \ldots, (F_m, g_m) \) each two causally related points can be joined by a causal geodesic.
(2) The multiwarped spacetime is geodesically connected if the fibers are and:

\[ f_i^b f_i^{-2} (f_1^{-2} + \cdots + f_m^{-2})^{-1/2} = \infty \quad \text{for all } i \text{ and for some } c \in (a, b). \]

Moreover, if one of the fibers \( F_j \) is not contractible then:

(a) each two points can be joined by infinitely many geodesics, and

(b) for any \( z \in I \times F_1 \times \cdots \times F_m, \) and \( x \in F_1 \times \cdots \times F_m, \) the number of timelike geodesics joining \( z \) and \( (t, x) \) goes to \( \infty \) when \( t \) goes to an endpoint of the interval \( I. \)

It is worth pointing out:

(I) Equality (5.1) is equivalent to the following condition: any point \( z_0 \) of the spacetime can be joined with any line \( L[x] \) by means of both, a future directed and a past directed causal curve. Nevertheless, Theorem 5.1 does not cover all the possibilities of the technique, and more general versions of this theorem can be given. These general versions give very accurate results; in fact, a necessary and sufficient condition for geodesic connectedness when \( m = 1 \) can be given with a reasonably long distinction of cases [23] (multiplicity, existence of timelike geodesics, conjugate points, etc. are also completely characterized in this reference; see also [45]). In particular, geodesic connectedness of Reissner-Nordström Intermediate spacetime is reproven (previous proofs and results in this direction were obtained in [29], [46]). The accuracy of the technique is shown by proving the geodesic connectedness of Schwarzschild inner spacetime; in fact, a good behaviour of the warping functions yields geodesic connectedness, but the warping functions of Schwarzschild inner spacetime do not have such good behaviour. Nevertheless, in this case, this problem can be skipped because one of the fibers of Schwarzschild’s is a sphere (and so, each pair of its points can be joined by geodesics of arbitrarily large length); if this fiber is replaced by a plane, the resulting spacetime is not geodesically connected.

(II) If the fibers are assumed to have boundary, the problem is reduced to considering when this boundary implies geodesic connectedness or convexity (Section 1). The general technique also works when a strip \( [a', b'] \subset I \) is considered; so, the problem with boundary is also solved, for boundaries which preserve the multiwarped structure.

For the proof of Theorem 5.1, the Avez-Seifert type result (1) relies on a partial integration of the geodesics. For (2), the idea is the following, under an assumption somewhat stronger than (5.1) (under (5.1) some technicalities must be also taken into account). As there are points in each \( L[x] \) both, future and past related with \( z, \) these points can be joined with causal geodesics. Thus, we have just to connect \( z = (z_0, z_1, \ldots z_m) \) and \( (t, x), x = (x_1, \ldots x_m) \) for \( t \) in a compact interval. For this: (A) fixing the geodesics in the fibers joining each \( z_i \) and \( x_i, \) a continuous map \( \bar{\mu}(c, K), \) \( \bar{\mu} : [0, 1]^{m-1} \times [K^-, K^+] \rightarrow \mathbb{R}^{m-1}, \) is constructed in such a way that each zero of \( \bar{\mu} \) represents the initial condition of a connecting geodesic, (B) for \( K = K^- \) and \( K = K^+, \) the result above on causal geodesics ensures the existence of at least one zero of (a continuous extension of) \( \bar{\mu} \) for some \( c \in [0, 1]^{m-1}; \) then, the problem is solved if a continuous set of zeros \( Z \) containing these two zeros is found, (C) for each \( K \in ]K^-, K^+[ \) the behavior of \( \bar{\mu}(c, K) \) at the boundary of \( [0, 1]^{m-1}, \) allows to find \( Z, \) by means of arguments on continuity of solutions of equations depending on a parameter, based in Brower’s topological degree. When \( F_j \) is not contractible, the result follows by applying the technique for each geodesic joining \( z_j \) and \( x_j. \)
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