Diffusion approximation for a simple kinetic model with asymmetric interface

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Abstract. We study a diffusion approximation for a model of stochastic motion of a particle in one spatial dimension. The velocity of the particle is constant but the direction of the motion undergoes random changes with a Poisson clock. Moreover, the particle interacts with an interface in such a way that it can randomly be reflected, transmitted, or killed, and the corresponding probabilities depend on whether the particle arrives at the interface from the left or right. We prove that the limit process is a minimal Brownian motion, if the probability of killing is positive. In the case of no killing, the limit is a skew Brownian motion. Moreover, we construct a cosine family related to the skew Brownian motion and provide a new derivation of transition probability densities for this process.

1. Introduction

The paper is devoted to the following model of stochastic motion of a particle on two copies of real line (see Fig. 1). When on the upper copy, denoted by \( \mathbb{R} \times \{1\} \), the particle moves deterministically to the right with a constant velocity, which we normalize to be equal to one; when on the lower copy \( \mathbb{R} \times \{-1\} \), it moves to the left with the same normalized velocity. At the points \((0, \pm 1)\), there is an interface, which randomly perturbs the deterministic motion and enables the particle to switch between the upper and lower copies of the real line. It is described by four nonnegative parameters \( p, p', q \) and \( q' \) such that both \( p + p' \leq 1 \) and \( q + q' \leq 1 \). A particle approaching the interface from the left, thus moving on the upper copy, filters through the interface with probability \( p \) and continues its motion to the right on \( \mathbb{R} \times \{1\} \). With probability \( p' \), the particle is reflected and starts moving to the left (from \((0, -1)\)) on the lower copy. Finally, with a possibly nonzero probability \( p_0 := 1 - p - p' \), the particle is killed and removed from the state space. Analogously, when approaching the interface from the right (on the lower copy), the particle filters through the interface with probability

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Figure 1. Random movements on two copies of real line with interface. Particles move to the left (on the lower line) and to the right (on the upper line), but may be reflected from or killed at the interface. Additionally, at random times, particles jump from the upper to the lower line and vice versa.

$q$, is reflected and continues its motion on the upper copy with probability $q'$, or is killed and removed from the state-space with probability $q_0 := 1 - q - q'$.

Additionally, we introduce the following scattering mechanism that allows the particle to randomly change its direction (thus switching between the copies of the lines) in the bulk (i.e., outside the interface): at the epochs of a Poisson process, independent of the random mechanism at the interface, the particle moving to the right changes its direction to the left, and vice versa, by jumping from one copy of the line to the other.

The entire random mechanism described above is somewhat related to the kinetic model of a motion of a phonon with an interface, studied in [4,26–28], and the telegraph process with elastic boundary at the origin [12,13].

After presenting, in Sect. 2, the semigroups that are involved in the model, we formulate two theorems on diffusion approximation. The first, Theorem 3.1, concerns the case where there is no killing at the interface, that is,

$$p + p' = q + q' = 1. \quad (1.1)$$

The result says that when diffusively scaled, the density of particles undergoing the motion described by the model is well-approximated by the density of population of particles on $\mathbb{R}$ performing independent Brownian motions with a trace of semi-permeable membrane at $x = 0$. The above means that in the space of absolutely integrable functions, the domain of the generator $\frac{1}{2} \Delta_{p,q}$ of such Brownian motions consists of functions $\phi$ satisfying the transmission conditions

$$p\phi(0-) = q\phi(0+) \quad \text{and} \quad \phi'(0+) = \phi'(0-), \quad (1.2)$$
and we have $\Delta_{p,q}\phi = \phi''$ for such $\phi$. For $p = q$, this is the generator of the standard
Brownian motion; for $p \neq q$, the related process is known as skew Brownian motion—
see [33, p. 45 Eq. (57)], [41, p. 107] and [35, pp. 115–117]. Analytic properties of
this process have been recently studied in [10], see also [11, Chapters 4 and 11]. A
transmission condition that is analogous to (1.2) appears, for a nonlinear parabolic
equation, in the hydrodynamic limit of a symmetric simple exclusion process, as
viewed from a tagged particle moving under the action of an external constant driving
force, see [31, Equation (1.4)]. The proof of Theorem 3.1 is given in Section 4.

Section 6, see Proposition 6.1 and Theorem 6.6, is devoted to the cosine families
generated by $\Delta_{p,q}$ and its dual. As an application of generation theorems presented
there, we provide a more detailed characterization of the process of the Brownian
motion with transmission condition (1.2). In particular, in Sect. 6.3, closed expres-
sions for transition probability densities of such process are derived. They reveal the
asymmetric nature of the apparently completely permeable membrane described in
the foregoing (see also Sect. 3).

In the second main theorem of our paper, Theorem 5.1, we study the kinetic model
in the case in which the killing is ‘effective,’ that is, condition $pq_0 + qp_0 + p_0q_0 \neq 0$
holds. This scenario includes the situation in which both probabilities $p_0$ and $q_0$
are strictly positive. In this case, the processes involved, if diffusively scaled, are well-
approximated by the minimal Brownian motion (i.e., the Brownian motion killed at
the interface), see also Remark 5.2 for more details.

We note that the case of no interface has been extensively studied and is well-
described in the literature, see for example [14, Chapter 12]. It is intriguingly related
to random evolutions of Griego and Hersh [19,20,37], the telegraph equation and the
corresponding stochastic process [17,23], see also [5]. The latter is referred throughout
the literature either as the telegraph or Poisson–Kac process [6,37], and turns out to be
a process with independent increments in the non-commutative Kisyński group, see
[6,25]. In general, the presence of an interface poses a technical challenge, due to the
fact that the domain of the generator of the semigroup describing the ‘free’ motion of
a particle has to be non-trivially modified to incorporate the transmission condition
that describes the interface. As exemplified in this paper, the presence of the interface
leads at the same time to new stochastic phenomena.

Throughout the paper, we adopt the following notation $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$,$\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (\infty, 0)$. Moreover, we use Iverson’s notation: the Iverson
bracket $[P]$ equals 1, or 0 iff $P$ is true, or false, respectively.

2. Semigroups featured in the model

We work in the space $L^1(S)$ of integrable functions on the Borel measurable space
$S := \mathbb{R} \times \{-1, 1\}$ in which each copy of the real line is equipped with the Lebesgue
measure $m$. 
2.1. The semigroup corresponding to the particle motion with no scattering in the bulk

2.1.1. Definition of the semigroup

For each $t \geq 0$, we consider an operator in $L^1(S)$ given by

$$T(t)\varphi(x, 1) = \varphi(x - t, 1)[x < 0] + \tilde{\varphi}(x - t, 1)[x > 0]$$

$$T(t)\varphi(x, -1) = \varphi(x + t, -1)[x > 0] + \tilde{\varphi}(x + t, -1)[x < 0], \ \text{for } \varphi \in L^1(S),$$

where for $x \in \mathbb{R}_+$,

$$\tilde{\varphi}(x, 1) := \varphi(x, 1)[x > 0] + (p\varphi(x, 1) + q^'\varphi(-x, -1))[x < 0],$$

$$\tilde{\varphi}(x, -1) := \varphi(x, -1)[x < 0] + (q\varphi(x, 1) + p^'\varphi(-x, 1))[x > 0].$$

The family $\{T(t), t \geq 0\}$ describes the evolution of the density of particles that move according to the rules presented in Introduction, provided that switching between the copies of the real line outside the interface is not yet allowed. The above means that a given particle moves either to the right or to the left in the bulk and its deterministic motion is perturbed only at the boundary. If $\varphi \in L^1(S)$ is the density of the population of such particles at time 0, then $T(t)\varphi$ is its density at time $t$. It is clear from this description, and a direct calculation confirms this, that $T(t)\varphi \geq 0$ and $\int_S T(t)\varphi \, dm \leq \int_S \varphi \, dm$, provided $\varphi \geq 0$, with equality holding when $p + p' = 1$ and $q + q' = 1$. This shows that each $T(t)$ is a sub-Markov, or a Markov operator, respectively, in the sense of [32], with the norm $\|T(t)\| \leq 1$ for all $t \geq 0$.

Moreover, a standard argument shows that $\lim_{t \to 0^+} \|T(t)\varphi - \varphi\|_{L^1(S)} = 0$, for all $\varphi$. A straightforward if somewhat tedious calculation shows furthermore that $T(t)T(s) = T(t + s)$. (This calculation is rather unrewarding; it is a wiser strategy to see the semigroup property as an immediate consequence of the obvious Markovian nature of the underlying process in the bulk, that is in $\mathbb{R}_+ \times \{\pm 1\}$.) The family $\{T(t), t \geq 0\}$ is therefore a strongly continuous semigroup of operators in $L^1(S)$.

2.1.2. The generator of $\{T(t), t \geq 0\}$

Turning to the task of describing the generator of this semigroup, we let the operator $A$ in $L^1(S)$ be defined as follows. Its domain $\mathcal{D}(A)$ is made of $\varphi \in L^1(S)$ of the form

$$\varphi(x, 1) = (C_1 + \int_x^0 \psi(y, 1) \, dy)[x < 0] + (C_2 - \int_0^x \psi(y, 1) \, dy)[x > 0],$$

$$\varphi(x, -1) = (C_3 - \int_x^0 \psi(y, -1) \, dy)[x < 0] + (C_4 + \int_0^x \psi(y, -1) \, dy)[x > 0].$$

(2.1)

where $\psi \in L^1(S)$, and the constants $C_1, C_2, C_3,$ and $C_4$ satisfy

$$C_2 = pC_1 + q' C_4 \quad \text{and} \quad C_3 = p' C_1 + q C_4.$$  

(2.2)
In other words, a \( \varphi \in \mathcal{D}(A) \) is absolutely continuous in each of the sets \( \mathbb{R}_+ \times \{i\} \) and \( \mathbb{R}_- \times \{i\}, i \in \{-1, 1\} \) separately, with absolutely integrable derivatives there, and possesses finite both right and left limits at the points \((0, \pm 1)\) that satisfy

\[
\varphi(0+, 1) = p\varphi(0-, 1) + q'\varphi(0+, -1),
\]

\[
\varphi(0-, -1) = p'\varphi(0-, 1) + q\varphi(0+, -1).
\]

(2.3)

For such \( \varphi \), we define

\[
A\varphi(x, i) := -i \varphi'(x, i), \quad (x, i) \in \mathbb{R}_+ \times \{-1, 1\}.
\]

**Proposition 2.1.** The operator \( A \) defined above is the generator of the semigroup \( \{T(t), t \geq 0\} \).

**Proof.** We check first that for any \( \varphi \in \mathcal{D}(A) \), the limit \( \lim_{t \to 0^+} t^{-1}(T(t)\varphi - \varphi) \) exists and equals \( A\varphi \). To this end, we consider a \( \varphi \) of the form (2.1) and note that

\[
T(t)\varphi(x, 1) - \varphi(x, 1) = \int_{x-t}^t \psi(y, 1) \, dy,
\]

provided that either \( x < 0 \), or \( x > t \). By a direct calculation, similar to that presented in [36, p. 10] or [9, pp. 56–58], one can show that

\[
\int_{\mathbb{R} \setminus (0, t)} \left| \frac{1}{t} \left[ T(t)\varphi(x, 1) - \varphi(x, 1) \right] - \psi(x, 1) \right| \, dx \longrightarrow 0. \quad (2.4)
\]

Next, since for \( x \in (0, t) \), we have

\[
T(t)\varphi(x, 1) - \varphi(x, 1) = p \left[ C_1 + \int_{x-t}^0 \psi(y, 1) \, dy \right] + q' \left[ C_4 + \int_0^{t-x} \psi(y, 1) \, dy \right] - C_2 + \int_x^t \psi(y, 1) \, dy
\]

and the constants cancel out by the first condition in (2.2), we show that the following three integrals

\[
I_1(t) := \frac{1}{t} \int_0^t \left| \int_x^0 \psi(y, 1) \, dy \right| \, dx,
\]

\[
I_2(t) := \frac{1}{t} \int_0^t \left| \int_0^{t-x} \psi(y, 1) \, dy \right| \, dx,
\]

\[
I_3(t) := \frac{1}{t} \int_0^t \left| \int_x^t \psi(y, 1) \, dy \right| \, dx,
\]

converge to zero, as \( t \to 0^+ \). Fortunately, all of them are of similar type, and the argument is pretty much the same in each case. For example, using Fubini’s theorem, we can write
\[ I_1(t) \leq \frac{1}{t} \int_{-t}^{0} \int_{0}^{y+t} \mathsf{d}x|\psi(y, 1)| \mathsf{d}y \]
\[ \leq \frac{1}{t} \int_{-t}^{0} |y + t||\psi(x, 1)| \mathsf{d}y \leq \int_{-t}^{0} |\psi(x, 1)| \mathsf{d}y \quad \text{as } t \to 0+. \]

Convergence of each \( I_j(t) \) to 0 allows to conclude that condition (2.4) holds also when \( \mathbb{R} \setminus (0, t) \) is replaced by the entire real line. Since the proof of convergence of \( \frac{1}{t}[T(t)\varphi(\cdot, -1) - \varphi(\cdot, -1)] \) to \( \varphi(\cdot, -1) \) on the other copy of the real line is similar, we omit it.

This establishes the fact that the generator of \( \{T(t), t \geq 0\} \) is an extension of \( A \). We conclude the proof by showing that, in fact, it coincides with \( A \). Since for \( \lambda > 0 \), the resolvent equation for the generator has precisely one solution; it suffices to demonstrate that for any \( \psi \in L^1(S) \)

\[ \text{there exists a } \varphi \in \mathcal{D}(A) \text{ such that } \lambda \varphi - A \varphi = \psi. \quad (2.5) \]

To this end, given \( \psi \in L^1(S) \), we define

\[ \varphi(x, -1) := e^{\lambda x} \int_{x}^{\infty} e^{-\lambda y} \psi(y, -1) \mathsf{d}y \quad \text{for } x > 0. \quad (2.6) \]

It is straightforward to verify that

\[ \lambda \int_{0}^{\infty} |\varphi(x, -1)| \mathsf{d}x \leq \int_{0}^{\infty} |\psi(x, -1)| \mathsf{d}x. \]

Furthermore, computing \( \lambda \int_{0}^{x} \varphi(y, -1) \mathsf{d}y \) from (2.6), we obtain that:

\[ \varphi(x, -1) = C_4 + \int_{0}^{x} [\lambda \varphi(y, -1) - \psi(y, -1)] \mathsf{d}y, \quad x > 0, \quad (2.7) \]

with \( C_4 := \int_{0}^{\infty} e^{-\lambda y} \psi(y, -1) \mathsf{d}y \). Analogously, the function \( \mathbb{R}_+ \ni x \mapsto \varphi(x, 1) \) defined by

\[ \varphi(x, 1) := e^{-\lambda x} \int_{-\infty}^{x} e^{\lambda y} \psi(y, 1) \mathsf{d}y \]

is absolutely integrable, and satisfies

\[ \varphi(x, 1) = C_1 + \int_{x}^{0} [\lambda \varphi(y, 1) - \psi(y, 1)] \mathsf{d}y, \quad x < 0, \quad (2.8) \]

with \( C_1 := \int_{-\infty}^{0} e^{\lambda y} \psi(y, 1) \mathsf{d}y \). Finally, we define

\[ \varphi(x, 1) := (pC_1 + qC_4)e^{-\lambda x} + \int_{0}^{x} e^{-\lambda (x-y)} \psi(y, 1) \mathsf{d}y, \quad x > 0, \]

\[ \varphi(x, -1) := (p'C_1 + qC_4)e^{\lambda x} + \int_{x}^{0} e^{\lambda (x-y)} \psi(y, -1) \mathsf{d}y, \quad x < 0, \]
and verify that these functions are absolutely integrable on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Calculating as above we establish that

$$\varphi(x, 1) = (pC_1 + q' C_4) - \int_0^x [\lambda \varphi(y) - \psi(y)] \, dy, \quad x > 0,$$

$$\varphi(x, -1) = (p' C_1 + q C_4) - \int_x^0 [\lambda \varphi(y) - \psi(y)] \, dy, \quad x < 0.$$ 

These relations, together with (2.7) and (2.8), prove that $\varphi$ satisfies condition (2.5), which concludes the proof of the proposition. □

2.2. The semigroup with scattering

It is our next goal to study the evolution of the density of particles when the possibility of random scattering (jumps between the copies of the real lines) in the bulk is allowed. This evolution shall be described by a semigroup corresponding to a generator obtained by a bounded perturbation of the generator $A$. For that purpose, we define a bounded linear operator $B = I$, where $I$ is the identity operator in $L^1(S)$, whereas

$$B \varphi(x, i) = \varphi(x, -i), \quad (x, i) \in \mathbb{R} \times \{-1, 1\}, \varphi \in L^1(S).$$

More precisely, if particles perform only jumps between the copies of the real lines at the epochs of a Poisson process with intensity $\lambda$, then the dynamics of its law is governed by the semigroup of Markov operators generated by $\lambda(B - I)$. Since $B$ is bounded, the Phillips Perturbation Theorem asserts that the operator

$$G := A + \lambda(B - I), \quad (2.9)$$

is a generator of a strongly continuous semigroup $\{e^{tG}, t \geq 0\}$ on $L^1(S)$. Since both $T(t)$ and $B$ are sub-Markovian so is also each $e^{tG}$ (see e.g., [32, pp. 236–239]). This semigroup describes the evolution of the density of particles undergoing the random scattering (switching between $\mathbb{R} \times \{\pm 1\}$), both at the interface and in the bulk, as described in the foregoing.

2.3. Diffusive scaling of the model

Finally, to complete the setup, we consider the asymptotics of the population density under the macroscopic scaling of both temporal and spatial variables. To this end, we set $\lambda = 1$ in (2.9) and introduce the diffusive space-time scaling $t' = \varepsilon^2 t$ and $x' = \varepsilon x$, where $\varepsilon > 0$ is a small parameter that will eventually tend to 0. Here, $(t', x')$ and $(t, x)$ represent the macro- and microscopic variables. The deterministic velocity is not scaled and remains equal to 1. The evolution of the particle density profile in the macroscopic variables is then governed by the generator

$$G_{\varepsilon} := \frac{1}{\varepsilon} A + \frac{1}{\varepsilon^2} (B - I), \quad (2.10)$$

defined on $D(G) := D(A)$, with the respective semigroup $\{e^{tG_{\varepsilon}}, t \geq 0\}$. In what follows, we shall investigate the asymptotics of $\{e^{tG_{\varepsilon}}, t \geq 0\}$, as $\varepsilon \to 0$. 

3. A limit theorem: no loss of probability mass at the interface

Our main theorem of this section says that in the case where condition (1.1) is satisfied, the scaling of (2.10) leads in the limit, as $\varepsilon \to 0^+$, to a diffusion on a real line with a trace of semi-permeable membrane at $x = 0$. (i.e., to the skew Brownian motion).

Suppose that at least one of the two numbers $p$ and $q$ is nonzero. We define the domain of an operator $\Delta_{p,q}$ in $L^1(\mathbb{R})$ to consist of $\phi$ of the form

$$\phi(x) = \begin{cases} qC + Dx - \int_x^0 (x - y)\psi(y)\,dy & [x < 0] \\ pC + Dx + \int_0^x (x - y)\psi(y)\,dy & [x > 0] \end{cases}, \quad x \in \mathbb{R}^*, \quad (3.1)$$

where $C$ and $D$ are constants and $\psi$ belongs to $L^1(\mathbb{R})$. For any such $\phi$ we let

$$\Delta_{p,q}\phi = \psi := \phi''.$$

In other words, $\phi \in \mathcal{D}(\Delta_{p,q})$ is continuously differentiable on each of the two half-lines $\mathbb{R}_\pm$ (separately) and $\phi'$ is absolutely continuous with absolutely integrable $\phi''$. Moreover, the right and left limits of both $\phi$ and $\phi'$ at $x = 0$ exist, are finite and related by condition (1.2).

As proved in [10], see also [11, Chapters 4 and 11], $\Delta_{p,q}$ is the generator of a strongly continuous semigroup of Markov operators in $L^1(\mathbb{R})$. This operator plays a crucial role in what follows. More precisely, its isomorphic image will be shown to govern the diffusion approximation to our model.

To bring the operator $\Delta_{p,q}$ closer to a reader, we note first that in the symmetric case when $p = q$, it reduces to the generator of the standard heat semigroup in $L^1(\mathbb{R})$, see e.g., [32, pp. 232-234]. This agrees with the intuition that in the limit, the interface is completely invisible, being totally permeable. However, the asymmetric case is remarkably different. In particular, in contrast to the standard Brownian motion, transition probabilities for the Markov process governed by $\frac{1}{2}\Delta_{p,q}$ are skewed either to the left or to the right, depending on whether $p$ is smaller or larger than $q$. The related stochastic process has been known in the literature since 1970s (see [22,40]) as a skew Brownian motion; it behaves like a Brownian motion except that the sign of each excursion is chosen using an independent Bernoulli random variable—see the already cited survey article of A. Lejay [33] for more information.

Results presented in [10] and [11, Chapters 4 and 11], allow interpreting the skew Brownian motion differently, as a Brownian motion with a trace of semi-permeable membrane at $x = 0$. Namely, the process can be obtained as a limit of snapping out Brownian motions in which $x = 0$ is a semi-permeable membrane characterized by two permeability coefficients: one for filtering from the left to the right and another one for filtering from the right to the left, equal to, say, $\lambda_p$ and $\lambda_q$, respectively (see [11, Chapter 11] and [34] for more on such Brownian motions). The process related
to $\frac{1}{2} \Delta_{p,q}$ is obtained in the limit, as both $\lambda_p$ and $\lambda_q$ tend to infinity in such a way that $\lambda_p/\lambda_q = p/q$. Therefore, although $x = 0$ seems to be totally permeable, there is still asymmetry between the way particles filter from the right to the left and in the opposite direction. We refer to the works cited above for more details; more information is provided in Sect. 6.

Before stating the main result of this section, we make two additional observations. First, obviously, the space $L^1(\mathbb{R})$ is isometrically isomorphic to the subspace $L^0$ of $L^1(S)$ made of functions $\varphi$ such that $\varphi(x, 1) = \varphi(x, -1), x \in \mathbb{R}$. The isomorphism we have in mind is $J : L^1(\mathbb{R}) \rightarrow L^0$ given by

$$J \varphi(x, i) = \frac{1}{2} \varphi(x), \quad (x, i) \in \mathbb{R} \times \{-1, 1\}, \varphi \in L^1(\mathbb{R}),$$

with $J^{-1} \varphi(x) = 2 \varphi(x, 1), x \in \mathbb{R}, \varphi \in L_0$. It follows that the operators

$$S(t) := Je^{\frac{1}{2} \Delta_{p,q}} J^{-1}, \quad t \geq 0$$

form a strongly continuous semigroup of operators in $L_0$. Its generator is

$$\frac{1}{2} \widetilde{\Delta}_{p,q} := \frac{1}{2} J \Delta_{p,q} J^{-1},$$

with the domain equal to $\mathcal{D}(\widetilde{\Delta}_{p,q}) = J(\mathcal{D}(\Delta_{p,q}))$. That is, a $\varphi \in L_0$ belongs to $\mathcal{D}(\widetilde{\Delta}_{p,q})$ iff $J^{-1} \varphi$ is in $\mathcal{D}(\Delta_{p,q})$ and then $\Delta_{p,q} \varphi = J \Delta_{p,q} J^{-1} \varphi$, see e.g., [6, Section 7.4.22].

Secondly, observe that $B^2 = I$. Therefore,

$$e^{t(B - I)} = (e^{-t} \sinh t) B + (e^{-t} \cosh t) I$$

implying that the strong limit

$$\lim_{t \to \infty} e^{t(B - I)} =: P$$

exists and equals $\frac{1}{2} (B + I)$. We note also that

$$P \varphi(x, i) = \frac{1}{2} (\varphi(x, 1) + \varphi(x, -1)), \quad (x, i) \in \mathbb{R} \times \{-1, 1\}.$$  

Hence, $P : L^1(S) \rightarrow L^1(S)$ is a projection of $L^1$ onto $L_0$, which preserves the norm of nonnegative elements of $L^1(S)$.

**Theorem 3.1.** Assume that condition (1.1) is satisfied and $p + q > 0$. Let $\widetilde{\Delta}_{p,q}$ be the isomorphic image of $\Delta_{p,q}$ in $L_0$, defined in (3.3). Then,

$$\lim_{\varepsilon \to 0^+} e^{tG_\varepsilon} \varphi = e^{\frac{1}{2} \widetilde{\Delta}_{p,q}} P \varphi, \quad t > 0, \varphi \in L^1(S),$$

strongly in the norm of $L^1(S)$, and the limit is uniform in $t$ on compact subsets of $(0, \infty)$. For $\varphi \in L_0$, the limit holds also for $t = 0$ and is uniform in $t$ on compact subsets of $[0, \infty)$. 


This theorem is proved in Sect. 4.

Remark 3.2. Suppose that $\varphi \in L^1(S)$ is a probability density. Then, according to Theorem 3.1, both $2e^{tG_\varphi(x, i)}$, $i \in \{-1, 1\}$ become asymptotically equal to the density of a Brownian motion with a trace of semi-permeable membrane at $x = 0$.

4. Proof of Theorem 3.1

Let $\mathcal{A}$ be the extension of the operator $A$ to the domain made of $\varphi$ of the form (2.1), which need not satisfy (2.3), and given by

$$\mathcal{A}\varphi(x, i) = -i\varphi'(x, i), \quad (x, i) \in \mathbb{R}_* \times \{-1, 1\}. \quad (4.1)$$

Also, we let

$$G_\varepsilon := \frac{1}{\varepsilon} A + \frac{1}{\varepsilon^2} (B - I) \quad (4.2)$$

be the extension of $G_\varepsilon$ defined in (2.10) to $\mathcal{D}(\mathcal{A})$.

Lemma 4.1. For any $\lambda > 0$ and $\varepsilon > 0$, the kernel of $\lambda - G_\varepsilon$ is a two-dimensional linear space spanned by $\varphi_- = \varphi_-(\varepsilon, \lambda)$ and $\varphi_+ = \varphi_+(\varepsilon, \lambda)$ defined by:

$$\varphi_-(x, i) := e^{\mu x} \left[1[i = 1] + w_+[i = -1]\right] \quad [x < 0],$$

$$\varphi_+(x, i) := e^{-\mu x} \left[1[i = 1] + w_-[i = -1]\right] \quad [x > 0], \quad (4.3)$$

where

$$\mu := \mu(\varepsilon, \lambda) = \sqrt{\frac{\lambda(\lambda \varepsilon^2 + 2)}{\lambda}} \quad (4.4)$$

and

$$w_\pm = w_\pm(\varepsilon, \lambda) := \lambda \varepsilon^2 \pm \mu \varepsilon + 1. \quad (4.5)$$

Remark 4.2. A simple direct calculation shows that

$$w_+ w_- = 1. \quad (4.6)$$

Moreover,

$$\lim_{\varepsilon \to 0^+} w_\pm(\varepsilon, \lambda) = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \mu(\varepsilon, \lambda) = \sqrt{2\lambda}. \quad (4.7)$$

Proof of Lemma 4.1. For a $\varphi \in \mathcal{D}(\mathcal{A})$ to belong to the kernel of $\lambda - G_\varepsilon$, we need to have

$$(\lambda \varepsilon^2 + 1)\varphi - \varepsilon A\varphi = B\varphi.$$
In other words, because of (4.1),

\[(\lambda \varepsilon^2 + 1)\varphi(x, 1) + \varepsilon \varphi'(x, 1) = \varphi(x, -1),\]

\[(\lambda \varepsilon^2 + 1)\varphi(x, -1) - \varepsilon \varphi'(x, -1) = \varphi(x, 1), \quad x \in \mathbb{R}_.\]  \hspace{1cm} (4.8)

These relations and the definition of $\mathcal{D}(\mathcal{A})$ (see (2.1)) imply that the functions $R^*_\gamma \ni x \to \varphi'(x, i), i \in \{-1, 1\}$ are absolutely continuous in each half-line $\mathbb{R}_\pm$ separately, with $\varphi'' \in L^1(\mathbb{R})$. Thus, substituting $\varphi(x, -1)$ from the first equation of (4.8) into the second one and recalling the definition of $\mu$ (cf (4.4)), we obtain

$$\mu \varphi(x, 1) = \varphi''(x, 1), \quad x \in \mathbb{R}_\gamma.$$ 

Solving this ordinary differential equation, under the constrain that $\varphi \in L^1(S)$, we conclude that

$$\varphi(x, 1) = C_1 e^{\mu x}[x < 0] + C_2 e^{-\mu x}[x > 0]$$

for some constants $C_1$ and $C_2$. Because of the first equation in (4.8), it is immediate that $\varphi$ must be a linear combination of $\varphi_-$ and $\varphi_+$. Furthermore, by a direct calculation, using (4.6), it can be verified that both $\varphi_-$ and $\varphi_+$ belong to the kernel of $\lambda - \mathcal{G}_\epsilon$. □

**Proof of Theorem 3.1.** Relation (3.4) allows us to work in the framework of the singular perturbation theorem of T. G. Kurtz ([14,29,30] or [11], Theorem 42.1). To prove Theorem 3.1, we need to show that

(i) for any $\varphi_0 \in \mathcal{D}(\widetilde{\Delta}_{p,q})$, there are $\varphi_\epsilon \in \mathcal{D}(\mathcal{G}_\epsilon) = \mathcal{D}(\mathcal{A})$ such that

$$\lim_{\epsilon \to 0+} \varphi_\epsilon = \varphi_0 \quad \text{and} \quad \lim_{\epsilon \to 0+} \mathcal{G}_\epsilon \varphi_\epsilon = \frac{1}{2} \Delta_{p,q} \varphi_0.$$  \hspace{1cm} (4.11)

(ii) for any $\phi \in \mathcal{D}(\mathcal{A})$, we have

$$\lim_{\epsilon \to 0+} \varepsilon^2 \mathcal{G}_\epsilon \phi = (B - I)\phi.$$ 

Since (ii) follows immediately from (2.10), we are left with showing (i).

Let $\varphi_0$ belong to $\mathcal{D}(\widetilde{\Delta}_{p,q})$ and let

$$\phi := J^{-1} \varphi_0,$$  \hspace{1cm} (4.9)

so that (3.1) holds for some $C, D \in \mathbb{R}$ and $\psi \in L^1(\mathbb{R})$. Since both $\phi$ and $\phi''$ are in $L^1(\mathbb{R})$, so is $\phi'$ (see e.g., [24] p. 192). Following [14] p. 471, we define

$$\tilde{\varphi}_\epsilon(x, i) := \frac{1}{2} \phi(x) - \frac{\varepsilon i}{4} \phi'(x)$$

$$= \varphi_0(x, i) - \frac{\varepsilon i}{2} \varphi'_0(x, i), \quad (x, i) \in \mathbb{R}_\gamma \times \{-1, 1\}.$$  \hspace{1cm} (4.10)

Then, all $\tilde{\varphi}_\epsilon, \varepsilon > 0$ belong to $\mathcal{D}(\mathcal{A}) \subset L^1(S)$. A straightforward computation shows that

$$\mathcal{G}_\epsilon \tilde{\varphi}_\epsilon = \frac{1}{2} \Delta_{p,q} \varphi_0$$  \hspace{1cm} (4.11)
(see (4.2) for the definition of $G_\varepsilon$), whereas obviously from the definition,
\[
\lim_{\varepsilon \to 0^+} \tilde{\varphi}_\varepsilon = \varphi_0. \tag{4.12}
\]
Thus, we would be done, were it not for the fact that in general $\tilde{\varphi}_\varepsilon \not\in\mathcal{D}(A)$. Therefore, we need to modify the definition of $\tilde{\varphi}_\varepsilon$ in order to obtain $\varphi_\varepsilon \in \mathcal{D}(A)$ such that (4.12) is satisfied and (4.11) holds at least asymptotically.

The following construction uses the ideas of [18] (see also Theorem 3.1 in the more recent [21]), [2, pp. 230–232] and [3, Lemma 2.3]. Let $F : \mathcal{D}(A) \to \mathbb{R}^2$ be a linear mapping given by (c.f. (2.3))
\[
F\varphi = \begin{pmatrix}
\varphi(0^+, 1) - p\varphi(0^-, 1) - q'\varphi(0^+, -1) \\
\varphi(0^-, -1) - p'\varphi(0^-, 1) - q\varphi(0^+, -1)
\end{pmatrix}, \quad \varphi \in \mathcal{D}(A). \tag{4.13}
\]
Note that
\[
\varphi \in \mathcal{D}(A) \text{ belongs to } \mathcal{D}(A) \text{ iff } F\varphi = 0. \tag{4.14}
\]

Let $\lambda > 0$ be fixed. By Lemma 4.1, the kernel of $\lambda - G_\varepsilon$ is a two-dimensional subspace of $L^1(S)$, and is therefore isomorphic to $\mathbb{R}^2$. In other words, any member of this kernel that is of the form
\[
\varphi = C_1\varphi_- + C_2\varphi_+
\]
(4.15)
can be identified with the pair $(C_1, C_2) \in \mathbb{R}^2$. The lemma implies further that for $\varphi$ of this form, we have
\[
F\varphi = \begin{pmatrix}
-pC_1 + (1 - q'w_-)C_2 \\
(w_+ - p')C_1 - qw_-C_2
\end{pmatrix}.
\]
By (1.1) and (4.6), it follows that the determinant of the matrix $M_\varepsilon$ representing the linear mapping $\mathbb{R}^2 \ni (C_1, C_2) \mapsto \varphi \mapsto F\varphi \in \mathbb{R}^2$ equals
\[
\det(M_\varepsilon) = pqw_- + (1 - q'w_-)(p' - w_+) = (1 - p - q)(1 - w_-) + 1 - w_+.
\]
Using (4.4) and (4.5), we conclude that
\[
\det(M_\varepsilon) = -2\lambda\varepsilon^2 - \varepsilon(p + q)(\mu - \lambda\varepsilon). \tag{4.16}
\]
The determinant is strictly negative because $\mu > \varepsilon\lambda$, see (4.4). Thus, we infer that for any $u = (u_1, u_2) \in \mathbb{R}^2$, there exists a unique $\varphi =: K_{\lambda, \varepsilon}u$ in the kernel of $\lambda - G_\varepsilon$ such that
\[
FK_{\lambda, \varepsilon}u = u, \quad u \in \mathbb{R}^2 \tag{4.17}
\]
and it is given by (4.15) with 

\[ C_1 = C_1(\varepsilon) := \frac{\det(M_{\varepsilon,1})}{\det(M_{\varepsilon})} := \frac{-qw_-u_1 - (1 - q'w_-)u_2}{\det(M_{\varepsilon})}, \]

\[ C_2 = C_2(\varepsilon) := \frac{\det(M_{\varepsilon,2})}{\det(M_{\varepsilon})} := \frac{-(w_+ - p')u_1 - pu_2}{\det(M_{\varepsilon})}. \]

Here, \( M_{\varepsilon,j}, j = 1, 2 \) is the matrix formed by replacing the \( j \)-th column of \( M_{\varepsilon} \) by the column vector formed by the coordinates of \( u \). This allows us to define

\[ \varphi_{\varepsilon} := \tilde{\varphi}_{\varepsilon} - K_{\lambda, \varepsilon}F \tilde{\varphi}_{\varepsilon}, \quad \varepsilon > 0. \] (4.18)

Then, \( \varphi_{\varepsilon} \) belongs to \( \mathcal{D}(A) \) as the difference of two elements of this subspace. Moreover, \( F \varphi_{\varepsilon} = 0 \), by (4.17), that is, \( \varphi_{\varepsilon} \in \mathcal{D}(A) \), see (4.14). Also, since \( \phi \) of (4.9) is of the form (3.1), by (4.10), we have:

\[ \tilde{\varphi}_{\varepsilon}(0+, i) = \frac{1}{2}(pC - \varepsilon i D) \quad \text{and} \quad \tilde{\varphi}_{\varepsilon}(0-, i) = \frac{1}{2}(qC - \varepsilon i D). \]

Hence, it follows that 

\[ u_{\varepsilon} := F \tilde{\varphi}_{\varepsilon} \text{ (see (4.13)) is given by} \]

\[ u_{\varepsilon} = \frac{C}{2} \left( \frac{p(1 - q - q')}{q(1 - p' - p)} + \frac{\varepsilon D}{4} \left( \frac{-(1 - p + q')}{(1 - q + p')} - \frac{1}{(1 - q + p')} \right) \right) \]

because of assumption (1.1). The respective \( \det(M_{\varepsilon,j}), j = 1, 2 \) satisfy therefore, by (4.7),

\[ \lim_{\varepsilon \to 0^+} \frac{\det(M_{\varepsilon,1})}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{D}{4} (qw_-(1 - p + q') - (1 - q'w_-(1 - q + p')) = \frac{\varepsilon D}{4} ((1 - p + q') - (1 - q + p')) = 0, \]

and similarly

\[ \lim_{\varepsilon \to 0^+} \frac{\det(M_{\varepsilon,2})}{\varepsilon} = \frac{pD}{4} ((1 - p + q') - (1 - q + p')) = 0. \]

Note also that, see (4.16) and (4.4),

\[ \lim_{\varepsilon \to 0^+} \frac{\det(M_{\varepsilon})}{\varepsilon} = -\sqrt{2\lambda(p + q)} \neq 0. \]

Thus, \( \lim_{\varepsilon \to 0^+} \frac{C_j(\varepsilon)}{\varepsilon} = 0, \quad j = 1, 2. \)

Recall that \( \varphi_{\varepsilon}(\varepsilon, \lambda) \) are the functions defined in (4.3). By (4.7), their limits \( \lim_{\varepsilon \to 0^+} \varphi_{\varepsilon}(\varepsilon, \lambda) \) exist. We conclude therefore that

\[ \lim_{\varepsilon \to 0^+} K_{\lambda, \varepsilon}F \tilde{\varphi}_{\varepsilon} = 0, \] (4.19)
$$K_{\lambda, \varepsilon} F \tilde{\varphi}_{\varepsilon} = C_1(\varepsilon) \varphi_-(\varepsilon, \lambda) + C_2(\varepsilon) \varphi_+(\varepsilon, \lambda).$$

In consequence, see (4.12) and (4.18),

$$\lim_{\varepsilon \to 0^+} \varphi_{\varepsilon} = \lim_{\varepsilon \to 0^+} \tilde{\varphi}_{\varepsilon} = \varphi_0.$$

Finally, since $$K_{\lambda, \varepsilon} F \tilde{\varphi}_{\varepsilon} \in \ker(\lambda - G_{\varepsilon})$$, we can write, thanks to (4.11) and (4.19), that

$$\lim_{\varepsilon \to 0^+} G_{\varepsilon} \varphi_{\varepsilon} = \lim_{\varepsilon \to 0^+} G_{\varepsilon} (\tilde{\varphi}_{\varepsilon} - K_{\lambda, \varepsilon} F \tilde{\varphi}_{\varepsilon}) = \lim_{\varepsilon \to 0^+} (G_{\varepsilon} \tilde{\varphi}_{\varepsilon} - \lambda K_{\lambda, \varepsilon} F \tilde{\varphi}_{\varepsilon}) = \lim_{\varepsilon \to 0^+} (G_{\varepsilon} \tilde{\varphi}_{\varepsilon} - \lambda K_{\lambda, \varepsilon} F \tilde{\varphi}_{\varepsilon}) = \lim_{\varepsilon \to 0^+} G_{\varepsilon} \tilde{\varphi}_{\varepsilon} = \tilde{\varphi}_{0, p, q} \varphi_0,$$

as required in (i).

\[\square\]

5. A limit theorem: loss of probability mass at the interface

In this section, we study our model in the case where condition (1.1) is violated.

Let $$C_b[0, +\infty)$$ be the space of continuous functions $$f$$ on $$[0, +\infty)$$ such that $$f(0) = 0$$ and the limit $$\lim_{x \to +\infty} f(x)$$ exists and is finite. The formula

$$S(t) f(x) = \int_0^\infty q_t(x, y) f(y) \, dy, \quad t > 0, \ f \in C_b[0, +\infty],$$

where

$$q_t(x, y) := \frac{1}{\sqrt{2\pi t}} \left( e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right), \quad t > 0, \ x, y \in \mathbb{R}^+,$$

defines a strongly continuous semigroup of operators in $$C_b[0, +\infty]$$ (see e.g., [6] Section 8.1.22, or [15] pp. 341 and 477). This semigroup describes the so-called minimal (or killed) Brownian motion on $$\mathbb{R}^+$$ in which a particle starting at $$x > 0$$ initially behaves according to the rules of the standard Brownian motion, but is killed and removed from the state space upon touching $$x = 0$$ for the first time.

The generator $$H$$ of $$\{S(t), t \geq 0\}$$ is defined as follows: its domain $$\mathcal{D}(H)$$ consists of twice continuously differentiable $$f \in C_b[0, +\infty]$$ such that $$f'' \in C_b[0, +\infty]$$, and $$Hf = \frac{1}{2} f''$$. An explicit formula for $$R_\lambda := (\lambda - H)^{-1}$$ reads (see again [15] p. 477):

$$R_\lambda f(x) = \frac{1}{\sqrt{2\lambda}} \int_0^\infty \left( e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}(x+y)} \right) f(y) \, dy, \quad (5.1)$$

for $$\lambda > 0$$ and $$f \in C_b[0, +\infty]$$.

It can be checked that $$L^1(\mathbb{R}^+)$$, treated as a subspace of the dual space $$(C_b[0, +\infty])^*$$, is invariant under the adjoint operator $$R_\lambda^*$$, and, for any $$\phi \in L^1(\mathbb{R}^+)$$, the formula for $$R_\lambda^* \phi$$ is given by the same expression as $$R_\lambda f$$, with $$f$$ replaced by $$\phi$$. This fact can be used to verify that $$R_\lambda^*$$ (as restricted to $$L^1(\mathbb{R}^+)$$) is the resolvent of the operator $$H^*$$.
defined on the domain $D(H^*)$ consisting of functions $\phi \in L^1(\mathbb{R}_+)$ that are of the form
\[
\phi(x) = Cx + \int_0^x (x - y)\psi(y)\,dy, \quad x > 0,
\]
where $C \in \mathbb{R}$ and $\psi \in L^1(\mathbb{R}_+)$. It coincides with the space of functions $\phi \in L^1(\mathbb{R}_+)$ that have two generalized derivatives in $L^1(\mathbb{R}_+)$ and satisfy $\phi(0+) = 0$. We have $H^* \phi = \frac{1}{2}\phi'' = \frac{1}{2}\psi$.

Since $H^*$ is densely defined, and $\lambda R^\lambda_\lambda$, $\lambda > 0$ are sub-Markov operators, $H^*$ is the generator of a strongly continuous semigroup of sub-Markov operators in $L^1(\mathbb{R}_+)$, which we denote $\{e^{tH^*}, t \geq 0\}$ (see e.g., [32, Corollary 7.8.1]). As before, we verify that $e^{tH^*} \phi$ is formally given by the same formula as $S(t)f$ (with $f$ replaced by $\phi$). Operators $e^{tH^*}, t \geq 0$ should be interpreted as follows: if $\phi$ is the initial distribution of the minimal Brownian motion, then $e^{tH^*} \phi$ is its distribution at time $t \geq 0$.

Let the domain $D(\Delta_0)$ of the operator $\Delta_0$ in $L^1(\mathbb{R})$ be composed of functions $\phi$ of the form
\[
\phi(x) = (Cx - \int_x^0 (x - y)\psi(y)\,dy)[x < 0] + (Dx + \int_0^x (x - y)\psi(y)\,dy)[x > 0]. \tag{5.2}
\]
Here, $C$ and $D$ are real constants and $\psi$ is in $L^1(\mathbb{R})$. We let $\Delta_0 \phi := \psi(= \phi'')$. Then, the analysis presented above shows that $\Delta_0$ is the generator of a strongly continuous semigroup in $L^1(\mathbb{R})$. As before, the semigroup generated by $\frac{1}{2}\Delta_0$ describes the minimal Brownian motion, but this time the state-space is composed of two disjoint half-axes: $\mathbb{R}_-$ and $\mathbb{R}_+$. In this process, a particle starting at an $x \neq 0$ performs a standard Brownian motion until the first time when it touches 0; at this moment the particle is killed and removed from the state space.

As in Sect. 3, in the space $L_0 \subset L^1(S)$, there is an isomorphic image of the semigroup generated by $\Delta_0$. Denoting the generator of this image by $\tilde{\Delta}_0$, we obtain the following counterpart of Theorem 3.1. Its formulation involves the probabilities
\[
p_0 := 1 - p - p' \quad \text{and} \quad q_0 := 1 - q - q' \tag{5.3}
\]
that a particle passing through the interface is killed and removed from the state-space.

**Theorem 5.1.** Suppose that
\[
\gamma := pq_0 + qp_0 + p_0q_0 \neq 0. \tag{5.4}
\]
Then,
\[
\lim_{\varepsilon \to 0^+} e^{tG_\varepsilon} \phi = e^{t\frac{1}{2}\tilde{\Delta}_0} P \phi, \quad t > 0, \quad \phi \in L^1(S),
\]
(in the norm of $L^1(S)$) and the limit is uniform for $t$ in compact subsets of $(0, \infty)$. For $\phi \in L_0$, the limit extends to $t = 0$ and is uniform for $t$ in compact subsets of $[0, \infty)$. 
Proof. We follow the argument presented in the proof of Theorem 3.1 and define functions $\tilde{\psi}_\varepsilon$ and mapping $F$ by formulas (4.10) and (4.13), respectively. Relation (4.16) appearing there has been derived under assumption (1.1), and therefore requires modification based on relations (5.3). In general, when (1.1) need not be true, 
\[
\det(M_\varepsilon) = -2\lambda\varepsilon^2 - (p + q)\varepsilon(\mu - \lambda\varepsilon) + [q_0(1 - p) + p_0(1 - q) - p_0q_0]w_– - p_0 - q_0.
\]
Hence, in contrast to the case considered before, $\det(M_\varepsilon)$ is not of the order of $\varepsilon$. Rather, $\lim_{\varepsilon \to 0^+} \det(M_\varepsilon) = -\gamma$, which is nonzero by assumption (5.4). In particular, $\det(M_\varepsilon)$ is nonzero for $\varepsilon$ small enough, and thus the operator $K_{\lambda, \varepsilon}$ is well-defined.

Since $\phi$ is of the form (5.2), we have
\[
\tilde{\psi}_\varepsilon(0+, i) = -\frac{\varepsilon i}{4} D \quad \text{and} \quad \tilde{\psi}_\varepsilon(0-, i) = -\frac{\varepsilon i}{4} C,
\]
implying (see (4.13))
\[
F\tilde{\psi}_\varepsilon = \frac{\varepsilon}{4} \left( -D + pC - q'D \right)
+ p'C - qD \right),
\]
and $\lim_{\varepsilon \to 0^+} F\tilde{\psi}_\varepsilon = 0$. This in turn proves that $\lim_{\varepsilon \to 0^+} \det(M_{\varepsilon, i}) = 0$, for $i = 1, 2$ and then, as in the proof of Theorem 3.1 that $\lim_{\varepsilon \to 0^+} K_{\lambda, \varepsilon} F\tilde{\psi}_\varepsilon = 0$. The rest of the proof runs as in the aforementioned theorem. □

Remark 5.2. The assumption that $\gamma \neq 0$ is natural. It is automatically satisfied if both $p_0$ and $q_0$ are positive; this agrees with our intuitions well because this is the case in which killing is possible for particles approaching the interface from both sides. Condition (5.4) is also fulfilled in the following two cases: (a) $p_0 = 0$ but $q_0 \neq 0$ and $p > 0$, and (b) $q_0 = 0$ but $p_0 \neq 0$ and $q > 0$. To explain the case (a): condition $p_0 = 0$ combined with $p = 0$ describes the scenario in which all particles approaching the interface from the left are reflected. Since such particles can never filter to the right half-axis to be possibly killed there, it is clear that the minimal Brownian motion is not a good approximation for a process with $p_0 = p = 0$ even if $q_0 \neq 0$. Interpretation of (b) is similar.

6. Transition probabilities for the process governed by $\frac{1}{2} \Delta_{p,q}$

The operator $\Delta_{p,q}$ of Sect. 3 turns out to be not only the generator of a semigroup, but also the generator of a bounded cosine family: there is a strongly continuous family $\{C_{p,q}(t), t \in \mathbb{R}\}$ of equibounded operators such that $C_{p,q}(0) = I$, $C_{p,q}(t) = C_{p,q}(-t)$ for $t > 0$, and
\[
\lambda(\lambda^2 - \Delta_{p,q})^{-1} = \int_0^\infty e^{-\lambda t} C_{p,q}(t) \, dt, \quad \lambda > 0.
\]
We recall that by [1, Proposition 3.14.4], this relation implies the cosine family functional equation:

\[ 2C_{p,q}(t)C_{p,q}(s) = C_{p,q}(s + t) + C_{p,q}(t - s), \quad s, t \in \mathbb{R}. \]

We will find an explicit formula for \( \{C_{p,q}(t), t \in \mathbb{R}\} \) using Lord Kelvin’s method of images and, as an application, will obtain closed expressions for transition probability densities for the Markov process governed by \( \frac{1}{2} \Delta_{p,q}. \)

### 6.1. The cosine family generated by \( \Delta_{p,q} \)

The basic idea of the method (see [7,8]) is to represent a cosine family generated by the Laplace operator, with the domain described by a boundary condition, by means of the basic cosine family

\[ C(t)\phi(x) = \frac{1}{2}(\phi(x + t) + \phi(x - t)), \quad x, t \in \mathbb{R}, \phi \in L^1(\mathbb{R}) \quad (6.1) \]

and of a unique extension operator that is associated with the boundary condition. As explained in [10], in the case of transmission conditions the method should be appropriately modified, and in particular requires constructing two, and not just one, extension operators. Here are the details (see Fig. 2) pertaining to the transmission conditions (1.2).

Let \( \{C_{p,q}(t), t \in \mathbb{R}\} \) denote the searched for cosine family generated by \( \Delta_{p,q}. \) To find \( C_{p,q}(t)\phi \) for a given \( \phi \in L^1(\mathbb{R}) \) and \( t \in \mathbb{R}, \) we first discard the part of \( \phi \) on the negative half-axis, and then find a way to extend \( \phi \) to the entire \( \mathbb{R} \) so that

\[ C_{p,q}(t)\phi(x) = C(t)\tilde{\phi}_{\text{right}}(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}_+. \quad (6.2) \]

Here, \( C(t) \) is given by (6.1) and \( \tilde{\phi}_{\text{right}} : \mathbb{R} \to \mathbb{R} \) is an (yet unknown) extension of the right part of the graph of \( \phi. \) We stress that this formula is supposed to be valid only for \( x > 0. \) Its counterpart for \( x < 0 \) is obtained similarly. We cut off the part of \( \phi \) on
the positive half-axis, and then extend the remaining graph to that of a function on the entire real line in such a way that

\[ C_{p,q}(t)\phi(x) = C(t)\tilde{\phi}_{\text{left}}(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}_-. \]  

(6.3)

Here, \( \tilde{\phi}_{\text{left}} \) is an (also unknown) extension of the left part of the graph of \( \phi \). Formulas (6.2) and (6.3) define then the cosine family on the entire real line in such a way that the positive half-axis, and then extend the remaining graph to that of a function on the entire real line in such a way that

Thus, \( \tilde{\phi}_{\text{left}} \) is an (also unknown) extension of the left part of the graph of \( \phi \). Formulas (6.2) and (6.3) define then the cosine family on the entire real line in such a way that the positive half-axis, and then extend the remaining graph to that of a function on the entire real line in such a way that

Similarly, \( \psi_{\text{left}}(t) = \frac{2q}{p + q} \phi(t) + \frac{q - p}{p + q} \phi(-t), \quad t > 0. \)  

(6.5)

At this point, we forget about the fact that these formulae have been derived under the assumption that \( \phi \in \mathcal{D}(\Delta_{p,q}) \) and make an ansatz that for \( t \geq 0 \) the searched for cosine family, generated by \( \Delta_{p,q} \), is given by

\[ 2C_{p,q}(t)\phi(x) := \begin{cases} 
\phi(x - t) + \phi(x + t), & x \in (-\infty, -t) \cup (t, \infty), \\
\phi(x - t) + \psi_{\text{left}}(x + t), & x \in (-t, 0), \\
\psi_{\text{right}}(t - x) + \phi(x + t), & x \in (0, t). 
\end{cases} \]
By (6.1), (6.4) and (6.5), this formula has the following equivalent form:

\[ C_{p,q}(t)\phi(x) = C(t)\phi(x) \quad \text{as long as } |x| > t, \]

and for the remaining \(x\)s, we have

\[
C_{p,q}(t)\phi(x) = \begin{cases} 
C(t)\phi(x) + \frac{q-p}{2(p+q)}[\phi(-x-t) + \phi(x+t)], & x \in (-t,0), \\
C(t)\phi(x) + \frac{p-q}{2(p+q)}[\phi(t-x) + \phi(x-t)], & x \in (0,t).
\end{cases}
\]

(6.7)

Obviously, for \(p = q\), we have \(C_{p,q}(t)\phi = C(t)\phi\). Formula (6.7) shows that the same holds also when \(\phi\) is an odd function. On the other hand, if \(\phi\) is even, we can rewrite (6.7) in the form

\[
C_{p,q}(t)\phi(x) = C(t)\phi(x) + \frac{q-p}{p+q}[\phi(x+t)1_{(-t,0)}(x) - \phi(x-t)1_{(0,t)}(x)], \quad t > 0.
\]

(6.8)

**Proposition 6.1.** Suppose that \(p + q > 0\). Then, \(\{C_{p,q}(t), t \in \mathbb{R}\}\) is a strongly continuous cosine family and its generator is \(\Delta_{p,q}\).

**Remark 6.2.** The proof follows closely the proof of Proposition 2.1, and hence we omit it. The cosine family functional equation for \(\{C_{p,q}(t), t \in \mathbb{R}\}\) is a direct consequence of Proposition 6.6 presented in Sect. 6.2.

6.2. Dual cosine family in \(C[-\infty, +\infty]\)

Let \(C[-\infty, +\infty]\) be the space of continuous functions on \(\mathbb{R}\) that have finite limits at both \(+\infty\) and \(-\infty\). A direct computation, using (6.6) and (6.7), shows that the dual operators to \(C_{p,q}(t)\) in \(L^\infty(\mathbb{R})\) are given by

\[
C_{p,q}^*(t) f(x) = \begin{cases} 
C^*(t) f(x), & |x| \geq t, \\
C^*(t) f(x) + \frac{q-p}{2(p+q)}[f(-x-t) - f(x+t)], & -t < x \leq 0, \\
C^*(t) f(x) + \frac{p-q}{2(p+q)}[f(t-x) - f(x-t)], & 0 < x < t.
\end{cases}
\]

(6.9)

**Remark 6.3.** Here, \(C^*(t)\) is the dual to \(C(t)\), given also by formula (6.1). Interestingly, the family \(\{C_{p,q}^*(t), t \in \mathbb{R}\}\) leaves the subspace \(C[-\infty, +\infty]\) of \(L^\infty(\mathbb{R})\) invariant. Formula (6.7) could be used to define a cosine family on \(L^\infty(\mathbb{R})\), however, then the subspace \(C[-\infty, +\infty]\) would not be invariant under \(\{C_{p,q}(t), t \in \mathbb{R}\}\). Moreover, such a family would not be continuous in the strong topology of \(L^\infty(\mathbb{R})\).
Remark 6.4. Interestingly, if \( q > p \), the expression in the second line of (6.9) can be interpreted in probabilistic terms, but that in the third line cannot. Vice versa, if \( q < p \), the third line can be interpreted probabilistically, but the second cannot.

To wit, the second line can be equivalently written as

\[
C(t)f(x) = \frac{1}{2}f(x-t) + \frac{1}{2} \left( \frac{2p}{p+q}f(x+t) + \frac{q-p}{p+q}f(-x-t) \right), \quad -t < x \leq 0. 
\]

The arguments \( x-t, x+t \) and \( -x-t \) used here are possible positions at time \( t \) of a particle which at time 0 starts at a point \( x \in (-t, 0) \) and moves according to the following rules. (a) With probability \( \frac{1}{2} \), it moves to the left with constant speed equal to 1. (b) With the same probability, it moves to the right with the same speed, but when it hits an interface at 0, it either filters through the interface (with conditional probability \( \frac{2p}{p+q} \)), or is reflected (with conditional probability \( \frac{q-p}{p+q} \)), and starts moving to the left with the same velocity as before.

A similar interpretation of the third line is impossible, since the factor \( \frac{p-q}{2(p+q)} \) appearing there is in the case under consideration negative, and thus cannot be thought of as a probability. As shown in Sect. 6.3, the entire formula for the semigroup generated by \( \frac{1}{2} \Delta^*_{p,q} \) has a natural probabilistic interpretation. \( \square \)

Before we prove that \( \{C^*_p(q)(t), t \in \mathbb{R}\} \) is a cosine family we turn to a description of an operator that later on will be shown to be the generator of \( \{C^*_p(q)(t), t \in \mathbb{R}\} \). Namely, we say that an \( f \in C[\infty, +\infty] \) belongs to \( \mathcal{D}(\Delta^*_{p,q}) \) if the following three conditions are satisfied:

(a) \( f \) is twice continuously differentiable in both \( (-\infty, 0) \) and \( [0, \infty) \), separately, with left-hand and right-hand derivatives at \( x = 0 \), respectively,

(b) both the limits \( \lim_{x \to -\infty} f''(x) \) and \( \lim_{x \to +\infty} f''(x) \) exist and are finite (it follows that, in fact, they have to be equal to 0), and

(c) \( pf'(0+) = qf'(0-) \) and \( f''(0+) = f''(0-) \). Note that this condition implies that although \( f'(0) \) need not exist, it is meaningful to speak of \( f''(0) \).

Furthermore, we define \( \Delta^*_{p,q} f := f'' \).

Lemma 6.5. For any \( \lambda > 0 \) and \( g \in C[\infty, +\infty] \), the resolvent equation

\[
\lambda^2 f - \Delta^*_{p,q} f = g
\]

has a unique solution \( f \in \mathcal{D}(\Delta^*_{p,q}) \) given by

\[
f(x) = \begin{cases} 
C_+ e^{\lambda x} + D_+ e^{-\lambda x} - \lambda^{-1} \int_0^x \sinh[\lambda(x-y)] g(y) \, dy, & x \geq 0, \\
C_- e^{-\lambda x} + D_- e^{\lambda x} + \lambda^{-1} \int_x^0 \sinh[\lambda(x-y)] g(y) \, dy, & x \leq 0,
\end{cases}
\tag{6.10}
\]

where

\[
C_- := \frac{1}{2\lambda} \int_{-\infty}^0 e^{\lambda y} g(y) \, dy \quad \text{and} \quad C_+ := \frac{1}{2\lambda} \int_0^\infty e^{-\lambda y} g(y) \, dy \tag{6.11}
\]
whereas
\[ D_- := \frac{2p}{p+q} C_+ + \frac{q-p}{p+q} C_- \quad \text{and} \quad D_+ := \frac{p-q}{p+q} C_+ + \frac{2q}{p+q} C_- . \] (6.12)

**Proof.** On the right half-axis, a solution \( f \) to the resolvent equation solves the differential equation
\[ \lambda^2 f(x) - f''(x) = g(x) \] (6.13)
and is thus of the form given in the upper part of (6.10). Since we stipulate that the limit \( \lim_{x \to \infty} f(x) \) exists and is finite, we must take \( C_+ \) as in (6.11). Analogous argument establishes the lower part of (6.10) with \( C_- \) of Eq. (6.11). Since \( f \) is to belong to \( C[\infty, +\infty] \), we should have \( f(0-) = f(0+) \) and by condition (c) of the definition of \( D(\Delta^*_p, q) \), we should have \( qf'(0-) = pf'(0+) \). These two conditions hold iff
\[ C_- + D_- = C_+ + D_+ \quad \text{and} \quad p(C_+ - D_+) = q(D_- - C_-) \]
and the unique solution to this system for unknown \( D_- \) and \( D_+ \) is given by (6.12).

We note that due to the fact that \( f \) satisfies the differential Eq. (6.13) on each half axis \( \mathbb{R}_\pm \), the condition \( f(0-) = f(0+) \) implies that \( f''(0-) = f''(0+) \). This proves that the function \( f \) defined above is indeed a member of \( D(\Delta^*_p, q) \). \( \square \)

For a future reference, we note that our lemma immediately implies the following alternative formula for \( f = (\lambda^2 - \Delta^*_p, q)^{-1} g \):
\[ f(x) = \begin{cases} 
D_+ e^{-\lambda x} + \frac{1}{2\lambda} \int_0^\infty e^{-\lambda|x-y|} g(y) \, dy, & x \geq 0, \\
D_- e^{\lambda x} + \frac{1}{2\lambda} \int_{-\infty}^0 e^{-\lambda|x-y|} g(y) \, dy, & x \leq 0.
\end{cases} \] (6.14)

**Theorem 6.6.** Suppose that \( p + q > 0 \). Then, \( \{C^*_p, q(t), t \in \mathbb{R}\} \) is a strongly continuous cosine family with generator \( \Delta^*_p, q \). In addition
\[ \|C^*_p, q(t)\| \leq M, \quad t \in \mathbb{R}, \] (6.15)
where
\[ M = M(p, q) := \frac{2\max(p, q)}{p+q} . \]

**Proof.** Formula (6.9) implies that for each \( g \in C[-\infty, +\infty] \) and \( x \in \mathbb{R} \), the function \( [0, \infty) \ni t \mapsto C^*_p, q(t) g(x) \) is continuous and bounded by \( M\|g\| \). A direct calculation, using (6.9), leads to the following formula for the Laplace transform:
\[ \int_0^\infty e^{-\lambda t} C^*_p, q(t) g(x) \, dt = \frac{1}{2} \int_{-\infty}^\infty e^{-\lambda|x-s|} \left[ \text{g}_{\text{right}}(s) \right] \, ds, \]
where $\tilde{g}_{\text{right}}(s) := g(s)$ for $s \geq 0$ and

$$
\tilde{g}_{\text{right}}(s) := \frac{2q}{p+q} g(s) + \frac{p-q}{p+q} g(-s) \quad \text{for } s < 0.
$$

We note that $\tilde{g}_{\text{right}}$ is a continuous function on $\mathbb{R}$ that extends the right part of $g$.

A glance at (6.11) and (6.12) shows that the upper part of (6.14) can be rewritten in the form

$$
f(x) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} e^{-\lambda|x-y|}\tilde{g}_{\text{right}}(y) \, dy, \quad x \geq 0.
$$

Therefore, for $x \geq 0$,

$$
\int_{0}^{\infty} e^{-\lambda t} C_{p,q}^*(t) g(x) \, dt = \lambda(\lambda^2 - \Delta_{p,q}^*)^{-1} g(x), \quad (6.16)
$$

and a similar analysis on the left half-axis, involving the extension $\tilde{g}_{\text{left}}$ of the left part of $g$ which is given by

$$
\tilde{g}_{\text{left}}(s) = \frac{q-p}{p+q} g(-s) + \frac{2p}{p+q} g(s) \quad \text{for } s \geq 0,
$$

shows that (6.16) can be extended to the entire $\mathbb{R}$.

Equation (6.16) is a key to the proof of the theorem. Since the mapping $\mathbb{R}_+ \ni \lambda \mapsto (\lambda - \Delta_{p,q}^*)^{-1}$ is infinitely differentiable, even in the operator norm, so is $\lambda \mapsto \lambda(\lambda^2 - \Delta_{p,q}^*)^{-1}$. Because the convergence in the norm of $C[-\infty, +\infty]$ implies the pointwise convergence, we conclude that

$$
\left[ \frac{d^n}{d\lambda^n} \lambda(\lambda^2 - \Delta_{p,q}^*)^{-1} g \right](x) = \int_{0}^{\infty} e^{-\lambda t} (-t)^n C_{p,q}^*(t) g(x) \, dt,
$$

for $x \in \mathbb{R}$, $\lambda > 0$ and $n \in \mathbb{N}$. This in turn yields

$$
\left\| \frac{d^n}{d\lambda^n} \lambda(\lambda^2 - \Delta_{p,q}^*)^{-1} g \right\| \leq M\|g\| \int_{0}^{\infty} e^{-\lambda t} t^n \, dt = \frac{M\|g\|n!}{\lambda^{n+1}},
$$

showing that the estimates of the Sova–Da Prato–Giusti generation theorem are satisfied, see e.g., [16, p. 119], or the original papers [38] and [39]. Since $\Delta_{p,q}^*$ is densely defined, we conclude that this operator is a generator of a cosine family, say $\{C(t), t \in \mathbb{R}\}$ such that $\|C(t)\| \leq M$, $t \in \mathbb{R}$. Now, the relation

$$
\lambda(\lambda^2 - \Delta_{p,q}^*)^{-1} = \int_{0}^{\infty} e^{-\lambda t} C(t) \, dt,
$$

see the already cited [1, Proposition 3.14.4], combined with the fact that the convergence in $C[-\infty, +\infty]$ implies the pointwise convergence allows us to infer that

$$
\left[ \lambda(\lambda^2 - \Delta_{p,q}^*)^{-1} g \right](x) = \int_{0}^{\infty} e^{-\lambda t} C(t) g(x) \, dt
$$
for all $x \in \mathbb{R}$, $\lambda > 0$ and $g \in C[-\infty, +\infty]$. But, in view of (6.16), this means that two continuous functions: $t \mapsto \mathcal{C}(t)g(x)$ and $t \mapsto C_{p,q}^{*}(t)g(x)$ (we think of $g$ and $x$ as temporarily fixed) have identical Laplace transforms, and consequently these functions coincide. This proves that $\mathcal{C}(t)g = C_{p,q}^{*}(t)g$ for all $t \geq 0$ and $g \in C[-\infty, +\infty]$, showing in turn that $\{C_{p,q}^{*}(t), t \in \mathbb{R}\}$ is identical to $\{\mathcal{C}(t), t \in \mathbb{R}\}$. In particular, the former is a cosine family generated by $\Delta_{p,q}^{*}$ and estimate (6.15) follows, as claimed. \hfill \Box

6.3. Transition probability densities for the process governed by $\frac{1}{2}\Delta_{p,q}^{*}$

As the generator of a cosine family, $\Delta_{p,q}^{*}$ is also the generator of a semigroup. The Weierstrass formula (see e.g., [1, p. 219] or [16, p. 120]) tells us that

$$e^{t \frac{1}{2} \Delta_{p,q}^{*}} f(x) = \sqrt{\frac{2}{\pi t}} \int_{0}^{\infty} e^{-\frac{s^2}{2t}} C_{p,q}^{*}(s) f(x) \, ds$$

for $t > 0$, $x \in \mathbb{R}$ and $f \in C[-\infty, +\infty]$. Here, again, we use the fact that norm convergence in $C[-\infty, +\infty]$ implies pointwise convergence. The semigroup $\{e^{t \frac{1}{2} \Delta_{p,q}^{*}}, t \geq 0\}$ describes the same stochastic process as $\{e^{t \frac{1}{2} \Delta_{p,q}}, t \geq 0\}$ but from a different perspective: whereas the later determines evolution of the densities, the former determines the evolution in time of the expected values of a family of observables of the value of the process. More specifically,

$$e^{t \frac{1}{2} \Delta_{p,q}^{*}} f(x) = E_{x} f(w_{p,q}(t)) \quad (6.17)$$

where $w_{p,q}(t), t \geq 0$ is the underlying process, $E_{x}$ is the expected value conditional on the process starting at $x$, and $f \in C[-\infty, +\infty]$.

Our last goal is to combine these two relations with the explicit form of $C_{p,q}^{*}(t)$ given in (6.9), to find closed form formulae for transition probability densities of the process $w_{p,q}(t), t \geq 0$, that corresponds to the generator $\Delta_{p,q}$. To start with, for $x > 0$, by (6.9), we can write

$$\sqrt{2\pi t} e^{\frac{1}{2} \Delta_{p,q}^{*}} f(x) = \int_{0}^{\infty} e^{-\frac{s^2}{2t}} \left[ f(x+s) + f(x-s) \right] ds$$

$$+ \frac{p-q}{p+q} \int_{-\infty}^{0} e^{-\frac{s^2}{2t}} \left[ f(s) - f(x-s) \right] ds$$

$$= \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{2t}} f(y) \, dy$$

$$+ \frac{p-q}{p+q} \left[ \int_{0}^{\infty} e^{-\frac{(y-x)^2}{2t}} f(y) \, dy - \int_{-\infty}^{0} e^{-\frac{(y-x)^2}{2t}} f(y) \, dy \right].$$

Hence, by (6.17), for $x > 0$ and $t \geq 0$,

$$E_{x} f(w_{p,q}(t)) = \int_{-\infty}^{\infty} \gamma_{p,q}^{+}(t, x, y) f(y) \, dy,$$
where

\[
\gamma^+_{p,q}(t, x, y) := \begin{cases} 
\frac{1}{\sqrt{2\pi t}} \left( e^{-\frac{(y-x)^2}{2t}} + \frac{p-q}{p+q} e^{-\frac{(y+x)^2}{2t}} \right), & y > 0, \\
\frac{1}{\sqrt{2\pi t}} \frac{2q}{p+q} e^{-\frac{(y-x)^2}{2t}}, & y < 0,
\end{cases}
\]

is the probability density for the position of \( w_{p,q}(t) \), given that the process starts at \( x > 0 \). Similarly, for \( x < 0 \),

\[
Ex f(w_{p,q}(t)) = \int_{-\infty}^{\infty} \gamma^-_{p,q}(t, x, y) f(y) \, dy,
\]

where

\[
\gamma^-_{p,q}(t, x, y) := \begin{cases} 
\frac{1}{\sqrt{2\pi t}} \left( e^{-\frac{(y-x)^2}{2t}} + \frac{q-p}{p+q} e^{-\frac{(y+x)^2}{2t}} \right), & y < 0, \\
\frac{1}{\sqrt{2\pi t}} \frac{2p}{p+q} e^{-\frac{(y-x)^2}{2t}}, & y > 0,
\end{cases}
\]

These formulae confirm the already announced fact that even though the membrane situated at \( x = 0 \) is apparently completely permeable for the process governed by \( \frac{1}{2} \Delta^*_{p,q} \), there is some residual asymmetry between the way particles filter from the right to the left and in the opposite direction.

To see this, consider, for example, the case of \( p > q \) in which filtering to the right is ‘easier’ than filtering to the left. Then, the probability that a particle starting at an \( x \neq 0 \) will be in a subset of \((0, \infty)\) at time \( t \) is larger than the same probability in the standard, that is, symmetric Brownian motion. This is due to the fact that

\[
\gamma^-_{p,q}(t, x, y) > \gamma^-_{p,p}(t, x, y) \quad \text{and} \quad \gamma^+_{p,q}(t, x, y) > \gamma^+_{p,p}(t, x, y), \quad y > 0;
\]

of course, this comes at the cost of reversing these inequalities in the left half-axis.

It is also instructive to look at the particular subcase of \( q = 0 \) (total impermeability from the right); then, \( \gamma^+_{p,0} \) coincides with the transition probability density of the reflected Brownian motion on the right half-axis, and is zero on the left half-axis. At the same time, \( \gamma^-_{p,0} \) coincides with the transition probability density of a minimal Brownian motion on the left half-axis, and all the probability mass that is lost at \( x = 0 \) is transferred to the right half-axis.

Remark 6.7. We note that the formulae for the transition probability densities of a skew Brownian motion have been derived in a different way in [33, eq. (17) p. 420].
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