A rich man’s derivation of scaling laws for the Kondo model

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We show how the one-loop “poor man’s scaling” equations for the Kondo model with arbitrary impurity spin can be obtained within the framework of the functional renormalization group approach for quantum spin systems recently developed by Krieg and Kopietz [arXiv:1807.02524]. We argue that our method supersedes the “poor man’s scaling” approach and can also be used to study the strong coupling regime of the Kondo model.

I. INTRODUCTION

In an influential paper entitled “A poor man’s derivation of scaling laws for the Kondo problem”, Anderson has derived the scaling laws for the Kondo problem using a cutoff renormalization technique. Although these scaling laws have been derived previously by means of a complicated space-time approach Anderson’s approach showed how these scaling laws emerge from the renormalization group and inspired many subsequent works. In his “poor man’s scaling” approach, Anderson calculates the scaling laws for the Kondo problem using a simple truncation of the average effective action. We use a specific implementation of the FRG formulation of the Wilsonian renormalization group, it is straightforward to obtain the scaling laws of the Kondo model.

II. EXACT RENORMALIZATION GROUP FOR THE KONDO MODEL

The Kondo model describes a single localized spin $S$, which is coupled to a bath of non-interacting conduction electrons in a metal The Kondo Hamiltonian is given by

$$\mathcal{H} = \sum_{ij\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + J S \cdot s_0,$$

$$= \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + J S \cdot s_0, \quad (1)$$

where the operators $c_{i\sigma}^\dagger$ and $c_{i\sigma}$ create and annihilate an electron with spin $\sigma = \uparrow, \downarrow$ at lattice site $i$, while the operator $S$ represents an additional spin with magnitude $S$ which couples via the exchange coupling $J$ to the electronic spin $s_0$ at the origin. In the second line of Eq. (1) we have transformed the electronic part of the Hamiltonian to momentum space,

$$c_{i\sigma} = \frac{1}{\sqrt{N}} \sum_k e^{ik \cdot r_i} c_{k\sigma}, \quad (2)$$

where $N$ is the number of lattice sites and the energy dispersion $\epsilon_k$ is related to the hopping energies $t_{ij}$ via

$$\epsilon_k = \frac{1}{N} \sum_{ij} e^{-i k \cdot (r_i - r_j)} t_{ij}. \quad (3)$$

The electronic spin at the site of the impurity spin is explicitly given by

$$s_0 = \sum_{\sigma\sigma'} c_{i=0,\sigma}^\dagger \sigma \sigma' c_{i=0,\sigma'} \quad = \frac{1}{N} \sum_{kk'} \sum_{\sigma\sigma'} c_{k\sigma}^\dagger \sigma \sigma' c_{k'\sigma'}, \quad (4)$$

where $\sigma$ is the vector of Pauli matrices. Note that with our normalization, the exchange coupling $J$ has units of energy and the density of states at the Fermi energy $\epsilon_F$,

$$\rho_0 = \frac{1}{N} \sum_k \delta(\epsilon_k - \epsilon_F), \quad (5)$$

has units of inverse energy. The components of the spin operator $S$ satisfy the $SU(2)$ commutation relations

$$[S^\alpha, S^\beta] = i \epsilon_{\alpha\beta\gamma} S^\gamma, \quad (6)$$

where the superscripts $\alpha, \beta, \gamma$ refer to the Cartesian components of the vector operator $S$ and $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric $\varepsilon$-tensor. Although in his “poor
man’s scaling” approach Anderson studied only the case $S = 1/2$, within our spin FRG it is straightforward to consider an arbitrary impurity spin $S$. Since the spin commutation relations are neither bosonic nor fermionic, the usual machinery of many-body perturbation theory is not directly applicable to the Kondo model. A popular solution to this problem is to represent the spin operators in terms of Abrikosov pseudofermions, but this requires an additional projection to eliminate the unphysical part of the enlarged Hilbert space. The crucial insight of our recent work is that the powerful FRG formalism can be directly applied to quantum spin systems without using any representation of the spin operators in terms of auxiliary variables. One simply has to define a proper cutoff scheme and write down the relevant generating functional in terms of a trace over the physical Hilbert space. For classical spin models and bosonic quantum lattice models, a similar strategy has been adopted earlier in Refs. [12–14].

To implement the bandwidth cutoff used in Anderson’s “poor man’s scaling” approach, we consider the following cutoff-dependent deformation of the Kondo Hamiltonian,

$$\mathcal{H}_\Lambda = \mathcal{H}_0 + V_\Lambda,$$

where $\mathcal{H}_0$ represents the local exchange interaction,

$$\mathcal{H}_0 = JS \mathbf{s} \cdot \mathbf{s}_0,$$

while the cutoff-dependent operator $V_\Lambda$ represents the regularized electronic kinetic energy,

$$V_\Lambda = \sum_{k\sigma} \left[ \xi_k + R_\Lambda(k) \right] c_{k\sigma}^\dagger c_{k\sigma},$$

where the dispersion $\xi_k = \epsilon_k - \epsilon_F$ is measured relative to the Fermi energy $\epsilon_F$. The regulator function $R_\Lambda(k)$ should be chosen such that it vanishes for $\Lambda \to 0$, so that in this limit we recover our original model. On the other hand, at some large initial cutoff scale $\Lambda_0$ the contribution from states with energies in some interval around the Fermi energy should be suppressed. A possible regulator with this property is

$$R_\Lambda(k) = \text{sign}(\xi_k)(\Lambda - |\xi_k|) \Theta(\Lambda - |\xi_k|).$$

The initial value $\Lambda_0$ of the cutoff should be identified with the total bandwidth of the dispersion, $\Lambda_0 = \max_k \{|\xi_k|\}$.

To derive exact FRG flow equations for the Kondo model, let us consider the generating functional of the cutoff-dependent connected correlation functions,

$$e^{\mathcal{G}_\Lambda[\eta, \eta, h]} = \text{Tr} \left[ e^{-\beta \mathcal{H}_0 T} e^{\xi \mathbf{c}^\dagger \mathbf{c}} + \int_0^\beta d\tau (h(\tau) \cdot S(\tau) - V_\Lambda(\tau)) \right], \quad (11)$$

where the trace is over the electronic Fock space as well as over the Hilbert space of the localized spin and we have used the following short notation for the source terms of the electron operators,

$$\langle \mathbf{c}, \mathbf{c}^\dagger \rangle = \int_0^\beta d\tau \sum_{k\sigma} \eta_{k\sigma}(\tau) c_{k\sigma}(\tau) + c_{k\sigma}^\dagger(\tau) \eta_{k\sigma}(\tau).$$

Here the sources $\eta_{k\sigma}(\tau)$ and $\eta_{k\sigma}(\tau)$ are Grassmann variables, $h(\tau)$ is a time-dependent external magnetic field, $\beta$ is the inverse temperature and $T$ is the Wick time-ordering operator in imaginary time, i.e., operators at larger values of $\tau$ in the expansion of the exponential should be moved to the left, with extra minus signs generated by the permutation of any pair of fermion operators. The time evolution of all operators under the time-ordering symbol in Eq. (11) is in the imaginary-time interaction picture with respect to $\mathcal{H}_0$, for example

$$S(\tau) = e^{R_0 \tau} S e^{-R_0 \tau}. \quad (13)$$

Taking variational derivatives of $\mathcal{G}_\Lambda[\eta, \eta, h]$ with respect to the sources and then setting the sources equal to zero, we can obtain all types of time-ordered connected correlation functions of the Kondo model. In particular, the magnetic moment of the localized spin is

$$m_\Lambda = \frac{\delta \mathcal{G}_\Lambda[0,0,h]}{\delta h(\tau)} \bigg|_{h=0} = \langle S \rangle, \quad (14)$$

and the time-ordered two-point connected correlation function of the localized spin is

$$F_{\Lambda}^{\alpha\beta}(\tau - \tau') = \frac{\delta^2 \mathcal{G}_\Lambda[0,0,h]}{\delta h_\alpha(\tau) \delta h_\beta(\tau')}, \quad (15)$$

where $\langle \ldots \rangle$ denotes the equilibrium expectation value with the deformed Hamiltonian $\mathcal{H}_\Lambda$ for vanishing sources, $\langle \ldots \rangle = \text{Tr} [e^{-\beta \mathcal{H}_\Lambda} \ldots]$.

Similarly, by taking derivatives with respect to the Grassmann sources $\eta$ and $\bar{\eta}$, we obtain the time-ordered connected correlation functions of the conduction electrons. For example, the electronic single-particle Green function is given by

$$G_\Lambda^{\sigma}(k, k', \tau - \tau') = \frac{\delta^2 \mathcal{G}_\Lambda[\bar{\eta}, \eta, 0]}{\delta \eta_{k\sigma}(\tau) \delta \eta_{k'\sigma}(\tau')} \bigg|_{\eta=\bar{\eta}=0} = -\langle (c_{k\sigma}(\tau) c_{k'\sigma}(\tau')) \rangle. \quad (17)$$

Following Ref. [13], we can derive an exact flow equation for the generating functional $\mathcal{G}_\Lambda[\bar{\eta}, \eta, h]$ by simply taking the derivative of Eq. (11) with respect to the cutoff parameter $\Lambda$. We then obtain the exact FRG flow equation

$$\partial_\Lambda \mathcal{G}_\Lambda[\bar{\eta}, \eta, h] = \int_0^\beta d\tau \sum_{k\sigma} \left[ \delta^2 \mathcal{G}_\Lambda[\bar{\eta}, \eta, h] \right] \frac{\delta^2 \mathcal{G}_\Lambda[\bar{\eta}, \eta, h]}{\delta \eta_{k\sigma}(\tau) \delta \eta_{k\sigma}(\tau)} + \frac{\delta^2 \mathcal{G}_\Lambda[\bar{\eta}, \eta, h]}{\delta \eta_{k\sigma}(\tau) \delta \eta_{k\sigma}(\tau)} \bigg|_{\eta=\bar{\eta}=0} = \partial_\Lambda \mathcal{G}_\Lambda[\bar{\eta}, \eta, h]. \quad (18)$$

Finally, we introduce the generating functional of the cutoff-dependent irreducible vertices,

$$\Gamma_\Lambda[\bar{\psi}, \psi, \mathbf{m}] = \langle \bar{\psi}(\tau) \psi(\tau) + \bar{\psi}(\tau) \psi(\tau) + \int_0^\beta d\tau h(\tau) \cdot \mathbf{m}(\tau) \rangle$$

$$-\mathcal{G}_\Lambda[\bar{\eta}, \eta, h] - \int_0^\beta d\tau \sum_{k\sigma} R_\Lambda(k) \bar{\psi}_{k\sigma}(\tau) \psi_{k\sigma}(\tau).$$

(19)
which differs from the regular Legendre transform by the subtraction of the regulator term. On the right-hand side of Eq. (19) it is understood that the sources \( \eta_{k}\sigma(\tau) = \eta_{k}\sigma(\tau; \psi, \bar{\psi}, \psi, \bar{\psi}) \) and \( h(\tau) = h(\tau; \psi, \bar{\psi}, \psi, \bar{\psi}) \) should be considered as functionals of the expectation values \( \psi_{k}\sigma(\tau) = \langle Tc_{k}\sigma(\tau) \rangle \) and \( m(\tau) = \langle TS(\tau) \rangle \) by inverting the relations
\[
\psi_{k}\sigma(\tau; \bar{\eta}, \eta, h) = \langle Tc_{k}\sigma(\tau) \rangle = \frac{\delta G_{\Lambda}[\bar{\eta}, \eta, h]}{\delta \eta_{k}\sigma(\tau)}, \quad (20)
\]
\[
\bar{\psi}_{k}\sigma(\tau; \bar{\eta}, \eta, h) = \langle Tc_{k}\sigma(\tau) \rangle = -\frac{\delta G_{\Lambda}[\bar{\eta}, \eta, h]}{\delta \eta_{k}\sigma(\tau)}, \quad (21)
\]
\[
m^{\alpha}(\tau; \bar{\eta}, \eta, h) = \langle TS^{\alpha}(\tau) \rangle = \frac{\delta G_{\Lambda}[\bar{\eta}, \eta, h]}{\delta h^{\alpha}(\tau)}. \quad (22)
\]
Note that by construction, the sources as functionals of the expectation values can be obtained from the derivatives of the functional \( \Gamma_{\Lambda}[\bar{\psi}, \psi, m] \) as follows,
\[
\eta_{k}\sigma(\tau; \bar{\psi}, \psi, m) - R_{A}(k)\psi_{k}\sigma(\tau) = -\frac{\delta \Gamma_{\Lambda}[\bar{\psi}, \psi, m]}{\delta \psi_{k}\sigma(\tau)}, \quad (23)
\]
\[
\bar{\eta}_{k}\sigma(\tau; \bar{\psi}, \psi, m) - R_{A}(k)\bar{\psi}_{k}\sigma(\tau) = -\frac{\delta \Gamma_{\Lambda}[\bar{\psi}, \psi, m]}{\delta \psi_{k}\sigma(\tau)}, \quad (24)
\]
\[
h^{\alpha}(\tau; \bar{\psi}, \psi, m) = \frac{\delta \Gamma_{\Lambda}[\bar{\psi}, \psi, m]}{\delta h^{\alpha}(\tau)}. \quad (25)
\]
Taking the derivative of Eq. (19) with respect to \( \Lambda \) and using the flow equation \( \{18\} \), we find that \( \Gamma_{\Lambda}[\bar{\psi}, \psi, m] \) satisfies
\[
\partial_{\Lambda} \Gamma_{\Lambda}[\bar{\psi}, \psi, m] = -\int_{0}^{\beta} d\tau \sum_{k \sigma} (\partial_{\Lambda} R_{A}(k)) \times \frac{\delta^{2} G_{\Lambda}[\bar{\eta}, \eta, h]}{\delta \eta_{k}\sigma(\tau) \delta \eta_{k}\sigma(\tau)}, \quad (26)
\]
At this point it is convenient to collect all fields into a seven-component superfield \( \Phi_{\Lambda} = (\psi_{1}, \bar{\psi}_{1}, \psi_{2}, \bar{\psi}_{2}, m_{1}, m_{2}, \bar{m}_{2}) \) where the label \( \alpha \) denotes all parameters which are necessary to specify the field configuration, including the field type. The functional derivative in the last line of Eq. (26) can then be transformed as follows,
\[
\frac{\delta^{2} G_{\Lambda}[\bar{\eta}, \eta, h]}{\delta \eta_{k}\sigma(\tau) \delta \eta_{k}\sigma(\tau)} = \left[ \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right] \partial_{\Lambda} \Gamma_{\Lambda}[\Phi] + \left[ \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right] R_{A}, \quad \text{where the matrix elements of the derivative operator} \frac{\delta}{\delta \Phi} \text{are defined by}
\]
\[
\left[ \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right]_{\alpha\beta} = \frac{\delta}{\delta \Phi_{\alpha}} \frac{\delta}{\delta \Phi_{\beta}}, \quad \left[ \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right]_{\alpha\beta}^{T} = \frac{\delta}{\delta \Phi^{\beta}} \frac{\delta}{\delta \Phi_{\alpha}}. \quad (28)
\]
The statistics matrix \( Z \) is diagonal, \( Z_{\alpha\alpha'} = \delta_{\alpha\alpha'} \zeta_{\alpha} \), where \( \zeta_{\alpha} = 1 \) if \( \alpha \) refers to one of the components of the magnetization field \( m \) and \( \zeta_{\alpha} = -1 \) if \( \alpha \) labels a fermionic field. The regulator matrix \( R_{A} \) is defined by writing the regulator term in superfield notation as
\[
\int_{0}^{\beta} d\tau \sum_{k\sigma} R_{A}(k)\bar{\psi}_{k}\sigma(\tau)\psi_{k}\sigma(\tau) = \frac{1}{2} \int_{0}^{\beta} \Phi_{\alpha} \left[ R_{A} \right]_{\alpha\beta}^{\Phi_{\alpha}} \Phi_{\beta}. \quad (29)
\]
and the matrix \( \Gamma_{\Lambda}[\Phi] \) is defined by
\[
\Gamma_{\Lambda}[\Phi] = Z \left[ \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right] \Gamma_{\Lambda}[\Phi] = \left( \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right) G_{\Lambda}[\Phi]. \quad (30)
\]
The flow equation \( \{26\} \) can therefore be written in the compact matrix form
\[
\partial_{\Lambda} \Gamma_{\Lambda}[\Phi] = \frac{1}{2} \text{Str} \left\{ (\partial_{\Lambda} R_{A}) \left[ \Gamma_{\Lambda}[\Phi] + R_{A} \right]^{-1} \right\} \]
\[
= \frac{1}{2} \text{Str} \left\{ (\partial_{\Lambda} R_{A}) \left[ \left( \frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \right) G_{\Lambda}[\Phi] + R_{A} \right]^{-1} \right\}, \quad (31)
\]
where the supertrace is defined by \( \text{Str}\{\cdots\} = \text{Tr}\{Z\cdots\} \). Eq. (31) is a generalized Wetterich equation\(\{2\} \) for the Kondo model. As already emphasized in Ref. [10], the spin degrees of freedom appear in Eq. (31) in the same way as a bosonic field, which is a consequence of the fact that time-ordered spin correlation functions satisfy bosonic Kubo-Martin-Schwinger boundary conditions. We emphasize that the FRG flow equation (31) is formally exact and encodes the renormalization group flow of all correlation functions of the Kondo model.

### III. SCALING EQUATIONS FOR THE ANISOTROPIC SPIN-S KONDO MODEL

In this section we show how to obtain from Eq. (31) the leading-order scaling equations for the Kondo model for arbitrary impurity spin \( S \). It is instructive to consider here the more general anisotropic Kondo Hamiltonian,
\[
\mathcal{H} = \sum_{k\sigma} c_{k\sigma}^{\dagger} c_{k\sigma} + J^{\pm} S^{\sigma} s_{0}{_{\sigma}}^{\mp} + J^{x} (s_{0}{_{x}}^{y} s_{0}{_{y}}^{y}). \quad (32)
\]
By expanding the generating functional \( \Gamma_{\Lambda}[\Phi] = \Gamma_{\Lambda}[\bar{\psi}, \psi, m] \) in powers of the fields, we can reduce the
flow equation (31) to an infinite hierarchy of coupled flow equations for the irreducible vertices. For the anisotropic Kondo model, the expansion of $\Gamma_\Lambda[\bar{\psi}, \psi, \mathbf{m}]$ up to third order in the fields is of the form

$$
\Gamma_\Lambda[\bar{\psi}, \psi, \mathbf{m}] = \Gamma_\Lambda^{(0)} + \frac{1}{N\beta} \sum_{k k' \sigma} \sum_{\omega} \left[ N\delta_{k, k'} (\xi_k - i\omega) + \Sigma_\Lambda(k, k', \omega) \right] \bar{\psi}_{\mathbf{k}\omega\sigma} \psi_{k'\omega\sigma}
$$

$$
+ \frac{1}{\beta} \sum_{\nu} \left[ \frac{1}{2} \left[ F^\parallel_\Lambda(\nu) \right]^{-1} m^-_\nu m^+_\nu + \left[ F^\perp_\Lambda(\nu) \right]^{-1} m^-_\nu m^+_\nu \right]
$$

$$
+ \frac{1}{N\beta^2} \sum_{k k' \omega' \omega} J^\parallel_\Lambda(k, \omega, k', \omega') \left[ \bar{\psi}_{k\omega\uparrow} \psi_{k'\omega'\uparrow} - \bar{\psi}_{k\omega\downarrow} \psi_{k'\omega'\downarrow} \right] m^-_\omega m^+_{\omega'}
$$

$$
+ \frac{1}{N\beta^2} \sum_{k k' \omega' \omega} J^\perp_\Lambda(k, \omega, k', \omega') \left[ \bar{\psi}_{k\omega\uparrow} \psi_{k'\omega'\uparrow} m^-_{\omega} m^+_{\omega'} + \bar{\psi}_{k\omega\downarrow} \psi_{k'\omega'\downarrow} m^-_{\omega} m^+_{\omega'} \right]
$$

$$
+ \frac{1}{\beta^2} \sum_{\nu_1 \nu_2 \nu_3} \delta_{\nu_1 + \nu_2 + \nu_3, 0} \Gamma^{\Lambda+\Sigma}_\Lambda(\nu_1, \nu_2, \nu_3) m^-_{\nu_1} m^+_{\nu_2} m^-_{\nu_3} + \ldots,
$$

(33)

where the ellipsis represents terms involving more than three powers of the fields and the transverse magnetization is expressed in terms of its spherical components $m^\pm = m^x \pm i m^y$. The Fourier transform to frequency space is defined as follows,

$$
\psi_{k\sigma}(\tau) = \frac{1}{\beta} \sum_{\omega} e^{-i\omega\tau} \psi_{k\omega\sigma},
$$

(34)

$$
\mathbf{m}(\tau) = \frac{1}{\beta} \sum_{\nu} e^{-i\nu\tau} \mathbf{m}_\nu,
$$

(35)

where $\omega = 2\pi(n + 1/2)T$ is a fermionic Matsubara frequency and $\nu = 2\pi n T$ is a bosonic one. In the first line of Eq. (33), the function $\Sigma_\Lambda(k, k', \omega)$ is the electronic self-energy generated by the coupling to the impurity spin. In the second line, the functions $F^\parallel_\Lambda(\nu)$ and $F^\perp_\Lambda(\nu)$ can be identified with the longitudinal and the transverse dynamic susceptibility of the impurity spin. The functions $J^\parallel_\Lambda(k, \omega, k', \omega')$ and $J^\perp_\Lambda(k, \omega, k', \omega')$ can be identified with renormalized longitudinal and transverse exchange interactions, which in general depend on the momenta and frequencies of the incoming and outgoing fermions. Finally, the vertex $\Gamma^{\Lambda+\Sigma}_\Lambda(\nu_1, \nu_2, \nu_3)$ describes dynamic correlations between the components of the impurity spin. We represent the different types of three-legged vertices in the expansion (33) by the graphical symbols shown in Fig. 1.

It is now straightforward to write down exact flow equations for the vertices appearing in the expansion (33). These flow equations depend on various higher-order vertices. Fortunately, in order to derive Anderson’s “poor man’s scaling” results, it is sufficient to consider only the flow of the mixed three-legged vertices, $J^\parallel_\Lambda(k, \omega, k', \omega')$ and $J^\perp_\Lambda(k, \omega, k', \omega')$, which can be identified with the renormalized exchange couplings provided their momentum and frequency dependence can be ignored. Moreover, in the weak coupling limit we may truncate the flow equations by retaining only those diagrams which depend quadratically on the exchange couplings. The corresponding diagrams are shown graphically in Fig. 2. Explicitly, the corresponding flow equations for the generalized exchange couplings are
The three-legged vertex $\Gamma_{\Lambda}^{-z}(\nu,-\nu,0)$ appearing in the above expressions is related to the corresponding connected spin correlation function $G_{\Lambda}^{z-z}(\nu,-\nu,0)$ via the usual tree expansion\cite{3} implying

$$[F_{\Lambda}^{z}(\nu)]^{2}\Gamma_{\Lambda}^{z-z}(\nu,-\nu,0) = -\frac{G_{\Lambda}^{z-z}(\nu,-\nu,0)}{F_{\Lambda}^{z}(0)},$$

$$F_{\Lambda}^{z}(\nu)\Gamma_{\Lambda}^{z-z}(0,-\nu,0) = -\frac{G_{\Lambda}^{z-z}(0,\nu,-\nu)}{F_{\Lambda}^{z}(0)}.$$  

In the weak coupling limit, it is sufficient to approximate the inverse spin propagators and the three-point spin correlation functions on the right-hand side of Eqs. $[41]$ and $[42]$ by the corresponding expressions describing a free spin with magnitude $S$. The static spin propagators are then approximated by

$$F_{0}^{z}(0) \approx \beta b'_{0},$$

$$F_{0}^{z}(0) \approx 2\beta b'_{0},$$

where

$$b'_{0} = \frac{S(S+1)}{3}.$$  

The three-legged vertex $\Gamma_{\Lambda}^{z-z}(0,-\nu,0)$ of a single isolated spin in an external magnetic field has
been derived by Vaks, Larkin, and Pikin. Here we only need the limit of vanishing external magnetic field,
\[ G_0^{\pm 
abla} (\nu_1, \nu_2, \nu_3) = -2 \beta \rho_0 (1 - \delta_{\nu_1,0} \delta_{\nu_2,0} \delta_{\nu_3,0}) \times \left[ \delta_{\nu_1,0} \nu_2 + \delta_{\nu_2,0} \nu_3 + \delta_{\nu_3,0} \nu_1 \right]. \] (46)
We conclude that
\[ [F_0^{\pm}(\nu)]^2 \Gamma_0^{\pm 
abla} (-\nu, \nu, 0) = 2 F_0^{\pm}(\nu) F_0^{\parallel}(\nu) \Gamma_0^{- \nabla} (0, -\nu, \nu) \]
\[ = \begin{cases} 0 & \text{for } \nu = 0, \\ 2/(i\nu) & \text{for } \nu \neq 0, \end{cases} \] (47)
which is independent of the spin $S$. Substituting the free-spin result \cite{47} for the flowing spin vertices in Eqs. \eqref{eq:39} and \eqref{eq:40} we obtain the flow equations
\[ \partial_{\Lambda} J_{\Lambda}^{\parallel} = -A_{\Lambda} (J_{\Lambda}^{\perp})^2, \] (48)
\[ \partial_{\Lambda} J_{\Lambda}^{\perp} = -A_{\Lambda} J_{\Lambda}^{\perp} J_{\Lambda}^{\parallel}, \] (49)
where
\[ A_{\Lambda} = \frac{2}{\beta N} \sum_q \sum_{\nu \neq 0} \frac{\hat{G}_{\Lambda}(q, \nu)}{i\nu}. \] (50)
To proceed, we have to specify our regulator. The bandwidth cutoff scheme adopted by Anderson in his “poor man’s scaling” approach can be implemented in the FRG via the Litim regulator given in Eq. \eqref{eq:10}, which suppresses the propagation of electrons in an energy shell around the Fermi surface. The single-scale propagator as defined in Eq. \eqref{eq:38} is then of the form
\[ \hat{G}_{\Lambda}(k, \omega) = \frac{\Theta(\Lambda - |\xi_k|)}{[i\omega - \xi_k - R_{\Lambda}(k)]^2}. \] (51)
Neglecting the energy dependence of the density of states, we can easily carry out the momentum integration in Eq. \eqref{eq:50},
\[ \frac{1}{N} \sum_q \hat{G}_{\Lambda}(q, \nu) = \rho_0 \frac{4i\nu \Lambda^2}{(\nu^2 + \Lambda^2)^2}, \] (52)
where $\rho_0$ is the density of states at the Fermi energy per spin projection, see Eq. \eqref{eq:5}. In the zero-temperature limit, the Matsubara sum in Eq. \eqref{eq:50} then becomes elementary and we obtain
\[ A_{\Lambda} = 2 \rho_0 / \Lambda. \] (53)
Introducing the logarithmic flow parameter $l = \ln(\Lambda_0 / \Lambda)$ and using $\partial_l = -\Lambda \partial_{\Lambda}$ we finally arrive at the well-known scaling equations for the anisotropic Kondo model \cite{24,25,26}
\[ \partial_l J_{\Lambda}^{\parallel} = 2 \rho_0 (J_{\Lambda}^{\perp})^2, \] (54)
\[ \partial_l J_{\Lambda}^{\perp} = 2 \rho_0 J_{\Lambda}^{\perp} J_{\Lambda}^{\parallel}. \] (55)
Note that our derivation is valid for arbitrary impurity spin $S$. The fact that the spin cancels in the weak coupling scaling equations indicates that the Kondo energy where the system crosses over to the strong coupling regime is independent of $S$. This is consistent with the exact Bethe-ansatz solution of the spin-$S$ Kondo model. \cite{20}

IV. CONCLUSIONS

In this work we have shown how to derive the one-loop scaling equations for the anisotropic Kondo model within the functional renormalization group approach for quantum spin systems recently proposed in Ref. \cite{10}. Of course, the scaling laws for the Kondo model are well known and have been derived a long time ago using less sophisticated methods. Nevertheless, it is conceptually important to show how these scaling laws can be derived within the framework of the FRG because the FRG claims to unify different implementations of the renormalization group idea. \cite{51-55} In the same way that the “poor man’s scaling” approach has superseded the more complicated space-time approach adopted earlier by Anderson, Yuval, and Hamann \cite{52} the FRG approach supersedes the “poor man’s scaling” approach, since it avoids the unconventional $T$-matrix renormalization and embeds the renormalization group theory for the Kondo model into the established framework of the FRG. Although it requires some effort to become acquainted with the FRG formalism, once this is achieved one can transcend the “poor man’s scaling” approach in several directions. For example, by adopting a cutoff scheme where at the initial scale the electronic bandwidth vanishes, the spin FRG can also be used to study the strong coupling regime of the Kondo model, where for $S = 1/2$ the impurity spin is completely screened and the conduction electrons interact via an induced two-body interaction. \cite{22} Moreover, within the FRG it is straightforward to keep track of the renormalization group flow of the electronic self-energy $\Sigma_k(k, k', \omega)$, which contains information about the spatial distribution of the charge density in the vicinity of the impurity spin. \cite{22} In principle, our approach can also be used to study multi-channel Kondo models or other impurity models where the strong coupling phase is not a Fermi liquid. Finally, let us emphasize again that our spin FRG approach works directly with the physical spin operators, thus avoiding the complications which arise if the impurity spin is represented in terms of auxiliary fermions, such as Abrikosov pseudofermions \cite{51,52,55} or Majorana fermions. \cite{24,25}

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