Stability analysis based on monodromy matrix for switched dynamical systems

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Abstract: This study reports a stability analysis method based on the monodromy matrix for switched dynamical systems. First, we focus on nonlinear autonomous interrupted systems. In the electrical field, for example, most circuit equations are described in the form of a linear ordinary differential equation. However, with the growth of clean energy power generation devices, the nonlinear term appears in the circuit equation as represented by the photovoltaic module. From this point of view, we discuss a stability analysis method based on the monodromy matrix for nonlinear interrupted systems. Second, we consider impacting systems with a periodic threshold. The previous stability analysis method based on the monodromy matrix is only applicable to impacting systems with fixed threshold. We improved this method considering the periodic threshold. Many impacting systems in the engineering field have a periodic threshold; hence, we introduce this method herein with a mechanical application example. We aim to contribute to the development of the nonlinear theory using above-mentioned two topics.

Key Words: switched dynamical system, interrupted circuit, impacting system, stability analysis, monodromy matrix, saltation matrix

1. Introduction

Switched dynamical systems exhibit rich nonlinear phenomena because the continuous-time evolution is interrupted by discrete switching events. Analyzing the nonlinear phenomena observed in the system is important in understanding the qualitative characteristics of a system [1–4]. Moreover, the computation technique is applicable to system design and optimization [5, 6]. Therefore, developing the nonlinear theory and the computation technique is useful from the academic and engineering points of view.

Impacting systems fall into a class of switched dynamical systems [7–10]. In mechanical impacting systems, for example, an external force is often applied to a mass that impacts another mass or a wall. The mass velocities jump from one value to another through the impact, resulting in interrupted characteristics and causing nonlinear phenomena [11, 12].
Stability analysis, which is the process of deriving the characteristic multiplier of the Jacobian matrix, is an essential approach for understanding nonlinear phenomena. Typical examples of stability analysis methods that are applicable to switched dynamical systems are the Poincaré map method [13–17], trajectory sensitivity method [18, 19], TC-HAT computational toolbox [20], and monodromy matrix method [21, 22].

This study particularly focuses on the Poincaré map method and the monodromy matrix method. These two stability analysis methods have their advantages and disadvantages. For example, the Poincaré map method is applicable to almost all systems; however, the computation process may be complicated than that of the monodromy matrix method. Conversely, the stability analysis method based on the monodromy matrix is relatively simple; therefore, it is suitable to analyze the stability of the complicated or high-dimensional switched dynamical systems, which are a recent trend [23–25]. Moreover, a software for stability and bifurcation analysis of switched dynamical systems based on the monodromy matrix has been proposed [26]. This software is a MATLAB-based program with an appropriate GUI; therefore, it provides end users with a powerful computing tool to perform stability analysis. However, the monodromy matrix method requires a state-transition matrix calculated using the matrix exponential. Thus, the existing method based on the monodromy matrix is only applicable to a switched dynamical system or an impacting system with a linear ordinary differential equation.

An impacting system is categorized into two types, system with either a fixed or periodic threshold. For example, we consider the impact of two mass points. In the first type, we assume that the smaller mass vibrates because of the external force, whereas the larger mass does not move, even upon impact. This type of impacting system is called the system with a fixed threshold [27, 28]. In the second type, the larger mass is moved by the external force, and the smaller mass repeatedly impacts the larger mass. This system is called the system with a periodic threshold. A ball bouncing on a periodically vibrating table (i.e., bouncing ball problem) is a typical example of an impacting system with a periodic threshold [29, 30]. Many practical impacting systems have a periodic threshold, and we can calculate the stability of these systems using the Poincaré map method. However, the existing stability analysis method based on the monodromy matrix is only applicable to systems with a fixed threshold.

To overcome these shortfalls, we developed monodromy matrix methods for nonlinear switched dynamical systems presented in [31] and for impacting systems with a periodic threshold presented in [32]. Our developed method for nonlinear switched dynamical systems includes numerical integration algorithms, which are often used in the Poincaré map method. Therefore, these algorithms and the computation techniques of the Poincaré map method [15–17] are a key for developing the stability analysis method based on the monodromy matrix. Likewise, we should add the essence of the monodromy matrix method to Poincaré map method for simplifying the computation process and developing nonlinear theory. However, the existing detailed discussion is insufficient and the relationship between the two methods is unclear. Therefore, we focus on two types of switched dynamical systems, i.e., switching circuits and impacting systems, and clarify the relationship between the monodromy matrix method and Poincaré map method.

Section 2 introduces a stability analysis method for nonlinear autonomous switched dynamical systems, describes an $n$-dimensional system, and considers the evolution of the waveform perturbation and derivative with the initial value of the Poincaré map. Moreover, an example of the application is presented in an interrupted electric circuit with a nonlinear resistance. The method validity is then confirmed. Section 3 discusses the stability analysis method for the impacting system with a periodic threshold. We particularly focus on the perturbation evolution through the impact and define the saltation matrix, which is a key factor for calculating the stability of this type of impacting system. We show the application result in the impact oscillations of a rigid trolley-pantograph system. We clarify that the stability analysis method based on Poincaré map can be redefined in a manner similar as that of the monodromy matrix method.
2. Stability analysis for nonlinear autonomous switched dynamical systems

This section considers an \( n \)-dimensional autonomous switched dynamical system with a nonlinear ordinary differential equation. Assume that the system has two subsystems, and the motion equations are given by

\[
\frac{dx}{dt} = f(x, \lambda) = \begin{cases} f_1(x, \lambda_1), & \text{for subsystem-1} \\ f_2(x, \lambda_2), & \text{for subsystem-2} \end{cases},
\]

where \( x \in \mathbb{R}^n \) is a state variable and \( \lambda \) is a parameter. Let the initial value at \( t = kT \) be \( x_k \), where \( k = 0, 1, 2, \cdots \) and \( T \) denotes periodicity of an external force. The solution of Eq. (1) can be expressed in the form of

\[
x(t) = \varphi(t-kT, x_k, \lambda) = \begin{cases} \varphi_1(t-kT, x_k, \lambda_1), & \text{for subsystem-1} \\ \varphi_2(t-kT, x_k, \lambda_2), & \text{for subsystem-2} \end{cases}.
\]

Assume the following switching section:

\[
\Pi = \{ x \in \mathbb{R}^n \mid h(x(t), q(t)) = 0 \}.
\]

The switching event occurs when \( h(x(t), q(t)) = 0 \) is satisfied, where \( q(t) \) is a periodically moving threshold satisfying \( q(t) = q(t+T) \).

Figure 1 shows a conceptual diagram of the waveform behavior observed in the system. Suppose that the system was initially operated by subsystem-1, subsystem-1 switches to subsystem-2 if the waveform reaches switching section \( \Pi \). Subsequently, a periodic external force (e.g., a clock pulse) is added, and subsystem-2 switches back to subsystem-1.

Let the Poincaré section be \( \Sigma \) (Fig. 1). The waveform behavior during the periodic interval \( T \) is classified into two cases: cases A and B. In Case A, the system is operated by subsystem-1, whereas the switching event occurs in Case B.

Define the period-\( m \) waveform by

\[
x(kT) - x((k+l)T) \neq 0, \quad x(kT) - x((k+m)T) = 0,
\]

where \( l < m, l = 1, 2, 3, \cdots \), and \( m = 1, 2, 3, \cdots \). We use a superscript \( \ast \) for the periodic waveform (i.e., \( x^\ast(t) \) denotes the periodic waveform). Herein, we target the periodic waveform and analyze the stability. In the following, we rewrite \( x((k+m)T) \) and \( x^\ast((k+m)T) \) as \( x_{k+m} \) and \( x^\ast_{k+m} \).

2.1 Monodromy matrix

We consider herein the evolution of the periodic waveform perturbation. Let us denote an initial perturbation value as \( \Delta x_k \):

\[
\Delta x_k = x_k - x_k^\ast.
\]

Assume that the perturbation evolves during \( mT \) and the eventual perturbation at \( t = (k+m)T \) is given as

![Fig. 1. Waveform behavior of the nonlinear autonomous switched dynamical system.](image-url)
\[ \Delta x_{k+m} = x_{k+m} - x^*_k. \]  
\[ (6) \]

The \( \Delta x_k \) and \( \Delta x_{k+m} \) perturbations are related as
\[ \begin{align*}
\Delta x_{k+1} &= M_0 \Delta x_k, \\
\Delta x_{k+2} &= M_1 \Delta x_{k+1}, \\
&\quad \vdots \\
\Delta x_{k+m} &= M_{m-1} \Delta x_{k+m-1}.
\end{align*} \]
\[ (7) \]

Therefore, Eq. (7) can be rewritten as
\[ \Delta x_{k+m} = M \Delta x_k = M_{m-1} \cdots M_1 M_0 \Delta x_k. \]
\[ (8) \]

The matrices \( M_i \) for \( i = 0, 1, 2, \ldots, m-1 \) are called the monodromy matrices for each Poincaré section from \( t = (k+i)T \) to \( t = (k+i+1)T \). Moreover, \( M \) is the composite monodromy matrix for the complete Poincaré section from \( t = kT \) to \( t = (k+m)T \). The characteristic multipliers of \( M \) are obtained by solving the following equation:
\[ |M - \mu I| = 0. \]
\[ (9) \]

We can understand the waveform stability based on the characteristic multipliers.

We now explain the definition of the monodromy matrix for cases A and B. In Case A, based on Eq. (2), \( x_{k+1} \) is described as follows:
\[ x_{k+1} = \varphi_1 (T, x^*_k + \Delta x_k, \lambda_1). \]
\[ (10) \]

Equation (10) is rewritten as follows using Taylor expansion:
\[ x_{k+1} = \varphi_1 (T, x^*_k, \lambda_1) + \frac{\partial \varphi_1 (T, x^*_k, \lambda_1)}{\partial x^*_k} \Delta x_k. \]
\[ (11) \]

We obtain the monodromy matrix \( M \) for Case A as follows by substituting Eq. (11) into Eq. (6):
\[ M = \frac{\partial \varphi_1 (T, x^*_k, \lambda_1)}{\partial x^*_k}. \]
\[ (12) \]

Meanwhile, we have to define the monodromy matrix by considering the perturbation evolution through the switching event in Case B. Figure 2 shows a conceptual diagram of the monodromy matrix for Case B. The black waveform denotes the period-1 waveform, whereas the gray waveform is the perturbed waveform whose initial value, \( x_k \), exists in the vicinity of the fixed point \( x^*_k \). Assume that the switching event occurs at \( t = kT + \delta t_1 \) in the period-1 waveform but at \( t = kT + \bar{\delta} t_1 \) in the

**Fig. 2.** Conceptual diagram of the monodromy matrix. The black waveform denotes the period-1 waveform. The gray waveform is the perturbed waveform whose initial value, \( x_k \), exists in the vicinity of the fixed point \( x^*_k \).
perturbed waveform. Note that \(x^*(kT + \tilde{t}_1), x(kT + \tilde{t}_1), x^*(kT + \bar{t}_1), x(kT + \bar{t}_1), q(kT + \tilde{t}_1),\) and \(q(kT + \bar{t}_1)\) are rewritten as \(x^*_{i_1}, x_{i_1}, x^*_{i_1}, x_{i_1}, q_{i_1},\) and \(q_{i_1}\), respectively.

Let the perturbations around the switching events be

\[
\Delta x_+ = x_{i_1} - x^*_{i_1},
\]

and

\[
\Delta x_- = x^*_{i_1} - x_{i_1},
\]

where \(\bar{t}_1 - \tilde{t}_1 = \Delta t\). (15)

Based on Eq. (2), \(x_{i_1}\) is given as

\[
x_{i_1} = \varphi_1 \left( \tilde{t}_1, x^*_{i_1} + \Delta x_k, \lambda_1 \right).
\]

Using the Taylor expansion, Eq. (16) can be rewritten as

\[
x_{i_1} = \varphi_1 \left( \tilde{t}_1, x^*_{i_1}, \lambda_1 \right) + \frac{\partial \varphi_1}{\partial x^*_k} \left( \tilde{t}_1, x^*_{i_1}, \lambda_1 \right) \Delta x_k
\]

Substituting Eq. (17) into Eq. (13), we obtain

\[
M = \frac{\partial \varphi_1}{\partial x^*_k} \left( \tilde{t}_1, x^*_{i_1}, \lambda_1 \right)
\]

Likewise, \(x_{k+1}\) is

\[
x_{k+1} = \varphi_2 \left( T - \bar{t}_1, x^*_{i_1} + \Delta x_+, \lambda_2 \right).
\]

Using the Taylor expansion, Eq. (19) becomes

\[
x_{k+1} = \varphi_2 \left( T - \bar{t}_1, x^*_{i_1}, \lambda_2 \right) + \frac{\partial \varphi_2}{\partial x^*_{i_1}} \left( T - \bar{t}_1, x^*_{i_1}, \lambda_2 \right) \Delta x_+
\]

Because \(\Delta x_{k+1} = x_{k+1} - x^*_{k+1}\), Eq. (20) takes the form of

\[
M = \frac{\partial \varphi_2}{\partial x^*_{i_1}} \left( T - \bar{t}_1, x^*_{i_1}, \lambda_2 \right).
\]

Thus, the monodromy matrix, \(M\), for case-B is defined as

\[
M = M_+ S M_- = \frac{\partial \varphi_2}{\partial x^*_{i_1}} \left( T - \bar{t}_1, x^*_{i_1}, \lambda_2 \right) S \frac{\partial \varphi_1}{\partial x^*_k} \left( \tilde{t}_1, x^*_{i_1}, \lambda_1 \right),
\]

where \(S\) is called the saltation matrix [22].

The perturbed waveform \(x(t)\) exists near the period-\(m\) waveform \(x^*(t)\); hence, we assume that \(f(x^*, \lambda) \approx f(x, \lambda)\) just before and after the switching event. In the following section, \(f_1\) denotes the equation of motion just before the switching event, whereas \(f_2\) indicates that of just after the event. Suppose that the period-1 waveform behaves as subsystem-2 during \(\Delta t\), whereas the perturbed waveform behaves as subsystem-1 (Fig. 2). We obtain

\[
x^*_{i_1} = x^*_{i_1} + f_2 \Delta t
\]

and

\[
x_{i_1} = x_{i_1} + f_1 \Delta t = x^*_{i_1} + \Delta x_- + f_1 \Delta t.
\]

Therefore, we get

\[
\Delta x_+ = \Delta x_- + (f_1 - f_2) \Delta t.
\]

241
The switching conditions are described as
\[ h(x^*_i, q_i) = 0 \] (26)
and
\[ h(x_i, q_i) = 0. \] (27)
Thus, we obtain
\[ 0 = h(x_i, q_i) + \left. \frac{\partial h}{\partial t} \right|_{t=i} \Delta t \]
\[ = n^\top (x_i - q_i) + \left. \frac{\partial h}{\partial t} \right|_{t=i} \Delta t \]
\[ = n^\top (x^*_i + \Delta x_+ + f_1 \Delta t - q_i) + \left. \frac{\partial h}{\partial t} \right|_{t=i} \Delta t \]
\[ = n^\top (\Delta x_+ + f_1 \Delta t) + \left. \frac{\partial h}{\partial t} \right|_{t=i} \Delta t, \] (28)
where
\[ n = \nabla h(x(t), q(t)), \quad q_i \approx q_i + \left. \frac{\partial h}{\partial t} \right|_{t=i} \Delta t, \text{ and } x^*_i - q_i = h(x^*_i, q_i) = 0. \] Thus, \( \Delta t \) is
\[ \Delta t = -\frac{n^\top \Delta x_-}{n^\top f_1 + \left. \frac{\partial h}{\partial t} \right|_{t=i}}. \] (29)
Substituting Eq. (29) into Eq. (25), we obtain
\[ \Delta x_+ = \Delta x_- - \frac{(f_1 - f_2) n^\top \Delta x_-}{n^\top f_1 + \left. \frac{\partial h}{\partial t} \right|_{t=i}}. \] (30)
The saltation matrix \( S \) satisfies
\[ \Delta x_+ = S \Delta x_- \] (31)
yielding
\[ S = I + \frac{(f_2 - f_1) n^\top}{n^\top f_1 + \left. \frac{\partial h}{\partial t} \right|_{t=i}}, \] (32)
where the saltation matrix can be defined if the motion equations are explicitly described. We can calculate the stability of the period-\( m \) waveform based on Eqs. (8), (9), (12), (22), and (32).

2.2 Poincaré map approach
We consider the following local mappings.
\[ P^0_s : \Sigma \to \Sigma \]
\[ x_k \mapsto x_{k+1} = \varphi_1 (T, x_k, \lambda_1) \] (33)
\[ P^1_s : \Sigma \to \Pi \]
\[ x_k \mapsto x_t = \varphi_1 (t_s, x_k, \lambda_1) \] (34)
\[ P^2_s : \Pi \to \Sigma \]
\[ x_t \mapsto x_{k+1} = \varphi_2 (T - t_s, x_t, \lambda_2) \] (35)
Note that the switching event occurs at \( t = kT + t_s \). Let the Poincaré map of each clock interval be
\[ P^s = P^0_s \] (36)
and
\[ P^s = P_2^s \circ P_1^s, \]

where \( t_s > T \) for Eq. (36) and \( t_s \leq T \) for Eq. (37).

The derivative with the initial value of the Poincaré map for \( t_s \leq T \) is given by

\[
\frac{\partial x_{k+1}}{\partial x_k} = \frac{\partial \varphi_2}{\partial x_t} \frac{\partial x_t}{\partial x_k} - \frac{\partial \varphi_2}{\partial t} \frac{\partial t_s}{\partial x_k} - f_2 \frac{\partial t_s}{\partial x_k},
\]

where

\[
\frac{\partial x_t}{\partial x_k} = \frac{\partial \varphi_1}{\partial x_k} + \frac{\partial \varphi_1}{\partial t} \frac{\partial t_s}{\partial x_k} = \frac{\partial \varphi_1}{\partial x_k} + f_1 \frac{\partial t_s}{\partial x_k}
\]

and

\[
\frac{\partial t_s}{\partial x_k} = -\frac{n^\top \partial \varphi_1}{n^\top f_1 + \partial h / \partial t|_{t=t_s}}
\]

Therefore, Eq. (38) is rewritten as follows:

\[
\frac{\partial x_{k+1}}{\partial x_k} = \frac{\partial \varphi_2}{\partial x_t} \left( \frac{\partial \varphi_1}{\partial x_k} + f_1 \frac{\partial t_s}{\partial x_k} \right) - f_2 \frac{\partial t_s}{\partial x_k} = \left( \frac{\partial \varphi_2}{\partial x_t} - \left( f_2 - f_1 \right) \frac{n^\top \partial \varphi_1}{n^\top f_1 + \partial h / \partial t|_{t=t_s}} \right) \frac{\partial \varphi_1}{\partial x_k}.
\]

Equation (41) is the well known form of the stability analysis method based on the Poincaré map [15–17].

Further, let \( f_2 = f_2|_{t=T} \) in Eq. (41) be

\[
f_2 = \frac{\partial \varphi_2}{\partial x_t} f_2',
\]

where \( f_2' = f_2|_{t=t_s} \). Substituting Eq. (42) into Eq. (41), we get

\[
\frac{\partial x_{k+1}}{\partial x_k} = \frac{\partial \varphi_2}{\partial x_t} \left( \frac{\partial \varphi_1}{\partial x_k} - (f_2 - f_1) \frac{n^\top \partial \varphi_1}{n^\top f_1 + \partial h / \partial t|_{t=t_s}} \right) \frac{\partial \varphi_1}{\partial x_k},
\]

where we rewrite \( f_2' \) as \( f_2 \).

The derivative with the initial value of the Poincaré map for \( t_s > T \) is shown in Eq. (12). Therefore, we can obtain the same form of the equation as shown in Eqs. (12), (22), (32), and (43) through different approaches, which include the stability analysis method based on the monodromy matrix and that based on the Poincaré map.

### 2.3 Example of application

We applied the proposed method to an interrupted circuit with a nonlinear characteristic. Figure 3 shows the schematic of the circuit model simulated using LTspice. The circuit equation is given by

\[
\begin{align*}
L \frac{di}{dt} &= -ri - v, \\
C \frac{dv}{dt} &= i - G(v) + \frac{E_1 - v}{R_0 + R_1}, \quad \text{for switch-1,}
\end{align*}
\]

243
Fig. 3. Interrupted circuit with a nonlinear characteristic.

\[
\begin{align*}
L \frac{di}{dt} & = -ri - v \\
C \frac{dv}{dt} & = i - G(v) + \frac{E_2 - v}{R_0 + R_2}
\end{align*}
\]  

(45)

where \( G(v) \) denotes the nonlinear resistor described as

\[
G(v) = -a \tanh(bv).
\]  

(46)

The circuit parameters are as follows:

\[
\begin{align*}
L & = 50\,[\text{mH}] , & C & = 0.1\,[\text{\mu F}] , & E_1 & = 3.46\,[\text{V}] , & E_2 & = 1.82\,[\text{V}] \\
r & = 70.7\,[\Omega] , & R_0 & = 0\,[\Omega] , & R_1 & = 987\,[\Omega] , & R_2 & = 281\,[\Omega] \\
v_{\text{ref}} & = -2.32\,[\text{V}] , & a & = 6.16956 \times 10^{-3} , & b & = 0.36598
\end{align*}
\]  

(47)

Figure 4 shows the waveform behavior, where \( i, v, v_{\text{ref}}, v_{\text{sw}}, v_{\text{clk}}, \) and \( v_{\text{com}} \) denote the inductor current, capacitor voltage, reference voltage (threshold), switching signal, clock pulse, and output of the comparator, respectively. Suppose that the switch is initially connected to position-1. When the capacitor voltage reaches the reference voltage, the switch changes to position-2. Subsequently, the switch turns to position-1 when a clock pulse appears.

Fig. 4. Waveform behavior. \( i, v, v_{\text{ref}}, v_{\text{sw}}, v_{\text{clk}}, \) and \( v_{\text{com}} \) denote the inductor current, capacitor voltage, reference voltage (threshold), switching signal, clock pulse, and output of the comparator, respectively.
Let the solution of Eq. (44) be

\[ x(t) = \varphi_1(t - kT, x_k, \lambda) = \begin{cases} \ i(t) = \varphi_{11}(t - kT, x_k, \lambda_1) \\ \ v(t) = \varphi_{12}(t - kT, x_k, \lambda_2) \end{cases}, \]  

(48)

and that of Eq. (45) be

\[ x(t) = \varphi_2(t - kT, x_k, \lambda) = \begin{cases} \ i(t) = \varphi_{21}(t - kT, x_k, \lambda_1) \\ \ v(t) = \varphi_{22}(t - kT, x_k, \lambda_2) \end{cases}. \]  

(49)

The monodromy matrix without a switching event during the clock interval, which corresponds to Case A, is calculated by numerically integrating the variational equation for the interval \( t = kT \) to \( t = (k+1)T \), yielding

\[ M = \frac{\partial \varphi_1}{\partial x_k}(T, x_k^*, \lambda) S \frac{\partial \varphi_1}{\partial x_k}(\bar{t}_1, x_k^*, \lambda). \]  

(50)

Assume that the switching event for the perturbed waveform occurs at \( t = kT + \bar{t}_1 \), whereas that for the periodic waveform occurs at \( t = kT + \bar{t}_1 \). In the following, we rewrite \( x^*(kT + \bar{t}_1) \) as \( x_k^* \). Therefore, the monodromy matrix corresponding to Case B is given by

\[ M = \frac{\partial \varphi_2}{\partial x_k}(T - \bar{t}_1, x_k^*, \lambda) S \frac{\partial \varphi_1}{\partial x_k}(\bar{t}_1, x_k^*, \lambda), \]  

(51)

where

\[ \frac{\partial \varphi_1}{\partial x_k}(\bar{t}_1, x_k^*, \lambda) = \begin{bmatrix} \frac{\partial \varphi_{11}}{\partial \bar{t}_1}(\bar{t}_1, x_k^*, \lambda_1) \\ \frac{\partial \varphi_{12}}{\partial \bar{t}_1}(\bar{t}_1, x_k^*, \lambda_2) \end{bmatrix}, \]  

(52)

and

\[ \frac{\partial \varphi_2}{\partial x_k}(T - \bar{t}_1, x_k^*, \lambda) = \begin{bmatrix} \frac{\partial \varphi_{21}}{\partial \bar{t}_1}(T - \bar{t}_1, x_k^*, \lambda_1) \\ \frac{\partial \varphi_{22}}{\partial \bar{t}_1}(T - \bar{t}_1, x_k^*, \lambda_2) \end{bmatrix}. \]  

(53)

Similarly, we can calculate the stability of the period-\( m \) waveform considering the waveform periodicity.

We obtained Eqs. (50), (52), and (53) using the numerical integration method, wherein the variational equations used for the calculation are shown in Tables I and II.

The saltation matrix is defined by Eq. (32), where

\[ f_1 = \begin{bmatrix} -ri - \frac{L}{L} + \frac{E_1 - v}{R_0 + R_1} \end{bmatrix} \]  

(54)

and

\[ f_2 = \begin{bmatrix} -ri - \frac{L}{L} + \frac{E_2 - v}{R_0 + R_2} \end{bmatrix} \]  

(55)

The switching condition is given by
The bifurcation phenomena clearly occur in the circuit. We particularly focus on the local bifurcation \( f \) around the bifurcation point. We observed the period-2 waveform at \( \lambda = 3 \). Figure 5 shows the 1-parameter bifurcation diagram of \( v \) vs \( f \) plane in the numerical simulation. The bifurcation phenomena clearly occur in the circuit. We particularly focus on the local bifurcation phenomena observed at \( f = 3.25 \) kHz. Figure 6 shows examples of the waveform calculated using LTspice around the bifurcation point. We observed the period-2 waveform at \( f = 3.4 \) kHz and the
period-4 waveform at $f = 3.1$ kHz. Table III shows the stability of the period-2 waveform. The period-doubling bifurcation occurred at $f = 3.2729$ kHz. The bifurcation phenomenon was observed at this parameter, confirming the validity of the method.

3. Stability analysis for linear autonomous impacting systems

Assume an $n$-dimensional autonomous impacting system with a linear ordinary differential equation. Figure 7 shows the conceptual diagram of the waveform behavior. The equation of motion takes the following form:

$$\frac{dx}{dt} = f(x, \lambda) = Ax + B,$$

(60)

where $x \in \mathbb{R}^n$, $A$, and $B$ are the state matrices and $\lambda$ is a parameter. Let the solution of Eq. (60) be

$$x(t) = \phi(t - kT, x_k, \lambda),$$

(61)

where $x_k$ is an initial value. Assume that the impacting phenomenon occurs at a surface $\Pi$, which is defined through Eq. (3).
Fig. 7. Waveform of the linear autonomous impacting system with a periodic threshold. The horizontal axis denotes a time. The vertical axis denotes the waveform. $\Pi$: switching section. $J$: state-jump caused by the impact. $\Sigma$: Poincaré section. $x_-$ and $x_+$ denote the states immediately before and after the jumping phenomenon.

The state-jump caused by the impact is described as

$$J : \Pi \mapsto \Pi,$$
$$x_- \mapsto x_+ = r(t_i, x_-, \lambda) = Cx_- + D_{t=t_i},$$

where $x_-$ and $x_+$ denote the states immediately before and after the jumping phenomenon, respectively. We assume that the jumping phenomenon occurs at $t = kT + t_i$. Matrices $C$ and $D_{t=t_i}$ correspond to the jumping phenomenon. Note that $h(x_-, q(t_i)) = 0$ and $h(x_+, q(t_i)) = 0$ are held, because the position of an object, which impacts the periodic threshold, does not change at the impacting point in most mechanical impacting systems.

We show an example here to understand in detail the state-jump caused by the impact. We considered a two-dimensional impacting system with a restitution coefficient $\alpha$. Assume that the state vector $x$ comprises the position $y$ and the velocity $v$. We then obtain

$$y_+ = y_-, \quad v_+ - \frac{dq}{dt}_{t=t_i} = -\alpha \left( v_- - \frac{dq}{dt}_{t=t_i} \right),$$

which yields

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix} \quad \text{and} \quad D_{t=t_i} = \begin{bmatrix} 0 \\ (1 + \alpha) \frac{dq}{dt}_{t=t_i} \end{bmatrix}.$$

### 3.1 Monodromy matrix

The monodromy matrix has been defined in Section 2; therefore, we will only focus on the perturbation evolution around the impact phenomenon here.

The state transition matrix across each jump, which is called the saltation matrix, must be calculated to derive the monodromy matrix because the impact phenomenon occurs. The stability of the impacting systems strongly depends on the saltation matrix. We introduce the saltation matrix for the impacting systems with a periodically moving threshold in the following discussion.

Assume that a small perturbation is added to the state immediately before the jumping phenomenon. We study how the perturbation evolves as the waveform travels across the impact. Figure 8 shows a conceptual diagram of the monodromy and saltation matrices, where the black waveform denotes the period-1 waveform, and the gray waveform denotes the perturbed waveform. We assume that the period-1 waveform reaches the threshold at $t = t_i$, whereas the perturbed waveform reaches the threshold at $t = \tilde{t}_i$. Note that we rewrote $x(kT), x((k+m)T), x^*(kT), x^*((k+m)T), x(kT+t_i), x^*(kT+t_i), q(kT+t_i)$, and $q(kT+t_i)$ as $x_k, x_{k+m}, x^*_k, x^*_{k+m}, x_{t_1}, x^*_{t_1}, x_{t_1},$ and $q_{t_1}$, respectively for simplicity.

Based on the impact-related matrices $C$, $D_{t=t_i}$, and $D_{t=\tilde{t}_i}$, the state immediately after the impact is expressed as $x^*_{t_1+}$ and $x^*_{t_1+}$, which are given by
Fig. 8. Conceptual diagram of the monodromy matrix where the black waveform denotes the period-\(m\) waveform \(x^*(t)\) and the gray waveform denotes the perturbed waveform \(x(t)\).

\[ x_{t_i+}^* = Cx_{t_i-}^* + D|_{t=t_i}, \]

\[ x_{t_i+}^* = Cx_{t_i-} + D|_{t=t_i}, \]

where \(x_{t_i-}^*\) and \(x_{t_i-}\) denote the state immediately before the jumping phenomenon occurs. \(\Delta x_+\) is expressed as

\[ \Delta x_+ = x_{t_i+}^* - x_{t_i-}^*, \]

where

\[ x_{t_i-}^* = x_{t_i+}^* + f(x_{t_i+}, \lambda)\Delta t, \]

\[ = Cx_{t_i-}^* + f(x_{t_i+}, \lambda)\Delta t, \]

and

\[ x_{t_i+} = Cx_{t_i-} + D|_{t=t_i}, \]

\[ = C(x_{t_i} + f(x_{t_i}, \lambda)\Delta t) + D|_{t=t_i} \]

\[ = C(x_{t_i-} + \Delta x_- + f(x_{t_i}, \lambda)\Delta t) + D|_{t=t_i}, \]

because

\[ x_{t_i} = x_{t_i-} + \Delta x_- . \]

In Eqs. (66) and (67), \(\Delta t = \bar{t}_i - t_i\). Assume that \(f(x_{t_i+}, \lambda) \approx f(x_{t_i+}, \lambda)\) because the perturbed waveform \(x(t)\) exists near the period-\(m\) waveform \(x^*(t)\). Note that we rewrote \(f(x_{t_i-}, \lambda)\) and \(f(x_{t_i+}, \lambda)\) as \(f_{\text{i-}}\) and \(f_{\text{i+}}\), respectively. Thus, \(\Delta x_+\) is provided as follows:

\[ \Delta x_+ = x_{t_i+} - x_{t_i}, \]

\[ = C\Delta x_- + (Cf_{\text{i-}} - f_{\text{i+}})\Delta t + D|_{t=t_i} - D|_{t=t_i} \]

\[ = C\Delta x_- + \left(Cf_{\text{i-}} - f_{\text{i+}} + \frac{\partial D}{\partial t}|_{t=t_i}\right)\Delta t, \]

because

\[ D|_{t=t_i} - D|_{t=t_i} = \frac{\partial D}{\partial t}|_{t=t_i}\Delta t. \]

The switching conditions for the period-\(m\) and perturbed waveforms immediately before and after the impact are

\[ h(x_{t_i-}, q_{t_i}) = 0, \quad h(x_{t_i+}, q_{t_i}) = 0, \]

\[ h(x_{t_i-}, q_{t_i}) = 0, \quad h(x_{t_i+}, q_{t_i}) = 0. \]

Using the Taylor expansion, we obtain the following equation:
\[
0 = h(x_{t_i+}, q_{t_i}) + \frac{\partial h}{\partial t} \bigg|_{t=t_i} \Delta t
= n^\top (x_{t_i+} - q_{t_i}) + \frac{\partial h}{\partial t} \bigg|_{t=t_i} \Delta t
= n^\top \left( C (x^*_i - \Delta x_i + f_i \Delta t) + D|_{t=t_i} - q_{t_i} \right) + \frac{\partial h}{\partial t} \bigg|_{t=t_i} \Delta t,
\]
(72)

where \( C x^*(t_i) - D|_{t=t_i} - q(t_i) = h(x^*(t_i) + q(t_i)) = 0 \), and \( n = \nabla h(x(t_i), q(t_i)) \). Thus, \( \Delta t \) is
\[
\Delta t = - \frac{n^\top C \Delta x_i}{n^\top \left( C f_i - f_i \right) + \frac{\partial D}{\partial t} \bigg|_{t=t_i} + \frac{\partial h}{\partial t} \bigg|_{t=t_i}}.
\]
(73)

Consequently, Eq. (69) takes the following form:
\[
\Delta x_+ = C \Delta x_i - \left( C f_i - f_i \right) \frac{n^\top C \Delta x_i}{n^\top \left( C f_i - f_i \right) + \frac{\partial D}{\partial t} \bigg|_{t=t_i} + \frac{\partial h}{\partial t} \bigg|_{t=t_i}}.
\]
(74)

Here, the saltation matrix \( S \) satisfies Eq. (31). Therefore, based on Eq. (74), we obtain
\[
S = C - \left( C f_i - f_i \right) \frac{n^\top C}{n^\top \left( C f_i - f_i \right) + \frac{\partial D}{\partial t} \bigg|_{t=t_i} + \frac{\partial h}{\partial t} \bigg|_{t=t_i}}.
\]
(75)

If we assume a zero amplitude of the threshold oscillation (i.e., a fixed threshold), Eq. (75) reduces to
\[
S = I - \left( f_i - f_i \right) \frac{n^\top}{n^\top f_i}.
\]
(76)

Equation (76) is the saltation matrix for impacting systems with a fixed threshold [9].

### 3.2 Poincaré map approach

We consider the following local mappings:
\[
P_0^i : \Sigma \to \Sigma \quad \quad x_k \mapsto x_{k+1} = \phi(T, x_k, \lambda)
\]
(77)
\[
P_1^i : \Sigma \to \Pi \quad \quad x_k \mapsto x_{t_i} = \phi(t_i, x_k, \lambda)
\]
(78)
\[
P_2^i : \Pi \to \Sigma \quad \quad x_{t_i} \mapsto x_{k+1} = \phi(T - t_i, x_{t_i}, \lambda)
\]
(79)

Let the Poincaré map of each clock interval be
\[
P^i = P_0^i
\]
(80)

and
\[
P^i = P_2^i \circ J \circ P_1^i,
\]
(81)
where \( t_i > T \) for Eq. (80) and \( t_i \leq T \) for Eq. (81). \( J \) is the state-jump as shown in in Eq. (62).

The derivative with the initial value of the Poincaré map for \( t_i \leq T \) is given by

\[
\frac{\partial x_{k+1}}{\partial x_k} = \frac{\partial \phi}{\partial x_{t_{i+1}}} \left|_{t=t_i} \right. \frac{\partial r}{\partial x_k} - \frac{\partial \phi}{\partial t} \left|_{t=t_i} \right. \frac{\partial t_i}{\partial x_k} = \frac{\partial \phi}{\partial x_{t_{i+1}}} \left( \frac{\partial r}{\partial x_{t_{i+1}}} \left|_{t=t_i} \right. + \frac{\partial r}{\partial t} \left|_{t=t_i} \right. \frac{\partial t_i}{\partial x_k} - \frac{\partial \phi}{\partial t} \left|_{t=t_i} \right. \frac{\partial t_i}{\partial x_k} \right) - \frac{\partial \phi}{\partial x_{t_{i+1}}} \left( C \left( \frac{\partial \phi}{\partial x_k} + f_i \left|_{t=t_i} \right. \right) + \frac{\partial D}{\partial t} \left|_{t=t_i} \right. \frac{\partial t_i}{\partial x_k} \right) - \frac{\partial \phi}{\partial x_{t_{i+1}}} \left( C f_i - f_i + \frac{\partial D}{\partial t} \left|_{t=t_i} \right. \right) \frac{\partial t_i}{\partial x_k},
\]

(82)

where we assume that the following equation holds:

\[
\frac{\partial \phi}{\partial t} \left|_{t=t_i} \right. = \frac{\partial \phi}{\partial x_{t_{i+1}}} f_i +.
\]

(83)

Based on the switching condition of \( h(x_{t_{i+1}}, q_i) = 0 \), we get

\[
\frac{\partial h}{\partial x_k} = \frac{\partial h}{\partial t} \left|_{t=t_i} \right. \frac{\partial t_i}{\partial x_k} + n^\top \frac{\partial x_{t_{i+1}}}{\partial x_k} = 0,
\]

(84)

where

\[
\frac{\partial x_{t_{i+1}}}{\partial x_k} = \frac{\partial r}{\partial x_{t_{i+1}}} \left|_{t=t_i} \right. + \frac{\partial x_{t_{i+1}}}{\partial t} \left|_{t=t_i} \right. \frac{\partial t_i}{\partial x_k} = C \frac{\partial \phi}{\partial x_k} + \left( C f_i - f_i + \frac{\partial D}{\partial t} \left|_{t=t_i} \right. \right) \frac{\partial t_i}{\partial x_k}.
\]

(85)

Therefore, the following equation holds from Eqs. (84) and (85).

\[
\frac{\partial t_i}{\partial x_k} = - \frac{n^\top C \frac{\partial \phi}{\partial x_k}}{n^\top \left( C f_i - f_i + \frac{\partial D}{\partial t} \left|_{t=t_i} \right. \right) + \frac{\partial h}{\partial t} \left|_{t=t_i} \right.}
\]

(86)

Substituting Eq. (86) into Eq. (82), the following equation is defined.

\[
\frac{\partial x_{k+1}}{\partial x_k} = \frac{\partial \phi}{\partial x_{t_{i+1}}} \left( C - \frac{n^\top C \frac{\partial \phi}{\partial x_k}}{n^\top \left( C f_i - f_i + \frac{\partial D}{\partial t} \left|_{t=t_i} \right. \right) + \frac{\partial h}{\partial t} \left|_{t=t_i} \right.} \right) \frac{\partial \phi}{\partial x_k}
\]

(87)

It is clear that Eq. (87) has the same form as the monodromy matrix.

### 3.3 Example of application

In this section, we confirm the validity of the method by applying it to an impacting system with a periodic thresholds.

Figure 9 provides a schematic of an impacting system with a periodic threshold [30]. The equation of motion is given by

\[
\begin{align*}
\frac{dy}{dt} &= v \\
\frac{dv}{dt} &= -\frac{1}{m} (ky + cv),
\end{align*}
\]

(88)
where $k$ denotes the spring constant, $c$ denotes the damping coefficient, and $y$ denotes the mass displacement from the unstretched position of the spring. Moreover, a periodic function (i.e., periodic threshold) is defined as

$$p(t) = e \sin(\omega t) + d,$$  \hspace{1cm} (89)

where $e$ is the amplitude, $\omega$ is the angular frequency, and $d$ is the baseline. If the mass impacts the periodic threshold, the velocity instantly changes from $x_-$ to $x_+$ with reflection coefficient $\alpha$.

The physical parameters are as follows:

$$m = 285[g], \quad k = 200[N/m], \quad c = 0.7399[Ns/m], \quad e = 0.989[mm], \quad d = 50[mm], \quad \alpha = 0.155.$$  \hspace{1cm} (90)

Assume $x = (y, v)$; therefore, Eq. (88) is expressed as $\dot{x} = Ax + B$, where

$$A = \begin{bmatrix} 0 & 1 \\ \frac{k}{m} & -\frac{c}{m} \end{bmatrix}.$$  \hspace{1cm} (91)

and

$$B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  \hspace{1cm} (92)

$x_- = (y_-, v_-)$ and $x_+ = (y_+, v_+)$ are the states immediately before and after the jumping phenomenon, where $y_- = y_+$. Assume that the jumping phenomenon occurs at $t = t_i$. Matrices $C$ and $D|_{t=t_i}$ corresponding to the jumping phenomenon are defined as

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}.$$  \hspace{1cm} (93)

and
Fig. 11. Examples of the typical waveform observed in a 1-parameter bifurcation diagram.

**Table IV.** Stability of the period-2 waveform.

| \( f \) [Hz] | \( \mu_1 \)  | \( \mu_2 \)  | Remarks                        |
|-----------|-------|-------|-------------------|
| 20.1918   | −0.9889 | −0.0005 | Stable period-2 waveform |
| 20.1944   | −0.9950 | −0.0004 | Stable period-2 waveform |
| ...       | ...    | ...    | ...               |
| 20.1965   | −1.0000 | −0.0004 | Period-doubling bifurcation point |
| ...       | ...    | ...    | ...               |
| 20.2007   | 1.0100  | −0.0004 | Unstable period-2 waveform |
| 20.2176   | −1.0501 | −0.0004 | Unstable period-2 waveform |

\[
D = \begin{bmatrix}
0 \\
(1 + \alpha) (\omega \cos (\omega t))
\end{bmatrix}.
\] (94)

Figure 10 shows an example of 1-parameter bifurcation diagrams upon varying the bifurcation parameter \( f \), which is the frequency of the periodic threshold, from \( f = 19 \text{ Hz} \) to \( f = 21 \text{ Hz} \). Figure 11 depicts the typical waveform observed in the 1-parameter bifurcation diagrams. The period-2 waveform bifurcates to the period-4 waveform near \( f = 20.0 \text{ Hz} \). We focused on this bifurcation phenomenon and calculated the stability of the period-2 waveform as an example.

The motion equation is described in the form of a linear ordinary differential equation; thus, the perturbation evolution can be easily calculated using the matrix exponential. Conversely, the saltation matrix shown in Eq. (75) is given by
\[ S = \begin{pmatrix}
1 - \frac{v_- - v_+}{v_- - e\omega \cos(\omega t_i)} & 0 \\
\frac{\alpha}{m}(ky + cv_) + \frac{1}{m}(ky + cv_+) - (1 + \alpha)e\omega^2 \sin(\omega t_i) & \frac{v_- - e\omega \cos(\omega t_i)}{-\alpha}
\end{pmatrix}. \quad (95)

Similarly, we can calculate the characteristic multipliers \( \mu \) considering the impact times.

Table IV lists the stability characteristics of the period-2 waveform, indicating that period-doubling bifurcations appeared at \( f = 20.1965 \) Hz. The 1-parameter bifurcation diagrams in Fig. 10 clearly show that period-doubling bifurcation occurs at this parameter value.

4. Conclusions

This paper reported a stability analysis method based on the monodromy matrix for switched dynamical systems. We particularly focused on two types of systems: nonlinear autonomous system and impacting system.

The existing stability analysis method based on the monodromy matrix requires the matrix exponential; thus, the method is only applicable to the switched dynamical system with a linear ordinary differential equation. However, many switched dynamical systems in the engineering field have a nonlinear term in the motion equation. Thus, in the first topic, we explained the stability analysis method based on the monodromy matrix without a matrix exponential. The method is a fusion of the stability analysis methods between the Poincaré map method and the monodromy matrix method. The evolution of the perturbation during without the switching event was calculated using the numerical integration with variational equations, as it is done in the Poincaré map method. Conversely, that during the switching event was calculated using the saltation matrix, which is a key for calculating the waveform stability in the monodromy matrix method. We explained the composite monodromy matrix based on the above-mentioned idea. We also showed an example of the application in an interrupted electric circuit with a nonlinear ordinary differential equation.

The stability analysis method based on the monodromy matrix is applicable to impacting systems. Conversely, the existing method is only applicable to impacting systems with a fixed threshold. Thus, in the second topic, we introduced the extension theory applicable to the impacting system with a periodic threshold. We particularly focused on the saltation matrix considering the periodic threshold and explained the derivation process.

The first topic is mainly applicable to switching circuits. The recent development in clean energy devices will expand the demand for this method because some have nonlinear characteristics. The second topic will contribute to the stability analysis in impacting systems in the mechanical, biological, and ecological fields, among others. We particularly confirmed that the stability analysis method based on Poincaré maps can be redefined in the same form as that of the monodromy matrix method. We consider that this new form in the Poincaré map method is suitable for computation compared with the old form.

Controlling unstable periodic waveforms is important from the engineering point of view, but the method requires derivatives of the Poincaré map with initial and parameter values. Therefore, in future studies, we will derive perturbation evolution of the parameter value in the switched dynamical system based on the idea of the derivation process of monodromy and saltation matrices.

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