Dissipativity-Based Stability Analysis of Networked Nonlinear Descriptor Systems and Its Application to Power Grids

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Abstract: This paper pursues to construct a theoretical framework which can efficiently capture the dynamics of large-scale heterogeneous power grids. We formulate a networked nonlinear descriptor system consisting of subsystems and network system as a mathematical abstraction of such grids. This descriptor representation of the system enables us to consider efficient analysis and control of the system while preserving its network topology. As a main result, we clarify the dissipativity of the systems and derive a sufficient condition for local asymptotic stability of partial states and synchronization based on the dissipativity. We apply these results to a power grid described by a structure-preserving model, showing their effectiveness in an engineering problem.

Key Words: hierarchical networked systems, descriptor systems, dissipativity, stability, power grids.

1. Introduction

A power grid is a typical large-scale complex system consisting of dynamic components. Such power grids can be regarded as large-scale heterogeneous systems where various kinds of power devices are connected, an example of which is illustrated by Fig. 1. From this viewpoint, it is desired to construct a theoretical framework for analysis and control of the grids mentioned above so that we can deal with a broader class of controller design (emergency control, load frequency control, economic load distribution, supply and demand operation, and etc.).

Fig. 1 Image of large-scale heterogeneous power grids.

The so-called hierarchical networked systems theory is a control-theoretic framework for analysis and control of large-scale complex systems. In the framework, depending on our purpose of analysis and control, we divide a target system into several layers with own scales in space and time, and we capture the system as a hierarchical interconnection of the subsystems. By doing so, it becomes possible to explore the analysis and control in a tractable and systematic way. In [6], the authors of this paper derived a hierarchical representation of the structure-preserving model of multi-machine power grids [7]. They pointed out that the heterogeneous power grid can be described as a system of both dynamical and static subsystems connected via a network given by static nonlinearities.

In modeling of a hierarchical networked system, an algebraic constraint frequently appears to represent the conservation law of physical quantities in the connection between subsystems [7]–[9]. To deal with such constraint appropriately, it becomes important to consider the analysis and control of the networked system based on so-called descriptor framework [10]. As this framework can preserve the network topology of the considered system because we do not need to rewrite the original system description to any particular representation, e.g., state-space representation and etc., we can exploit the network topology appropriately for the analysis and control.

In this paper, we aim at deriving a theoretical condition for asymptotic stability and synchronization for networked nonlinear descriptor systems. For this purpose, the contents of this paper are summarized below. Based on the above observation, we derive a fundamental result for analysis and control of the networked system based on so-called descriptor framework [10]. As this framework can preserve the network topology of the considered system because we do not need to rewrite the original system description to any particular representation, e.g., state-space representation and etc., we can exploit the network topology appropriately for the analysis and control.

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The notation $A^\top$ denotes the transpose of a matrix $A$. The symbol $\otimes$ is used to denote the Kronecker product. For a scalar function $H(x)$ with the indeterminate vector $x := [x_1 \ x_2 \ \cdots \ x_n]^\top$, its gradient $\nabla H(x)$ is defined by $\nabla H(x) := \left[\frac{\partial H(x)}{\partial x_1} \ \frac{\partial H(x)}{\partial x_2} \ \cdots \ \frac{\partial H(x)}{\partial x_n}\right]^\top$.

2. Dissipativity-Based Stability Condition

In this section, we first formulate a networked nonlinear descriptor system that can be regarded as a mathematical abstraction of power grids including the structure-preserving model [7]. Using this formulation, we derive a sufficient condition which guarantees local asymptotic stability of partial states and synchronization based on dissipativity [11]–[14].

2.1 Networked Nonlinear Descriptor System

Consider a system where nonlinear subsystems are interconnected via static nonlinearities as shown by Fig. 2. The nonlinearities are called network system in this paper. The network is connected and possesses an undirected graph structure $G = (V,E)$, where $V \subset \mathbb{Z}$ stands for the finite set of subsystems and $E \subset V \times V$ for the finite set of edges. Throughout this paper, the entire system can be viewed as a networked interconnection of subsystems via static nonlinearities to obtain a simple and relevant result.

![Fig. 2 Networked nonlinear descriptor system as a system of systems interconnected via nonlinear mapping, where $S_i$ ($i = 1, 2, \ldots, K$) and $N$ represent the $i$th subsystems and network system, respectively.](image)

2.1.1 Subsystem

We introduce the equations of the subsystem $S_i$ ($i = 1, 2, \ldots, K$) given as a nonlinear descriptor system [10]. The subsystem $S_i$ is described by the state-space equations

$$E_i \dot{x}_i = f_i(x_i, u_i), \quad y_i = h_i(x_i, u_i),$$

where $u_i \in \mathbb{R}^{p_i}, y_i \in \mathbb{R}^{p_i},$ and $x_i \in \mathbb{R}^{n_i}$ are the input, output, and state of $S_i$, and $E_i \in \mathbb{R}^{n_i \times n_i}$ is possibly singular, i.e., rank$E_i \leq n_i$. In this paper, the state of the network system is defined by the differences of the outputs of subsystems. Thus, we suppose that the dimensions of the inputs and outputs are same through all subsystems. In (1), $f_i : \mathbb{R}^{n_i \times p_i} \rightarrow \mathbb{R}^{n_i}$ and $h_i : \mathbb{R}^{n_i \times p_i} \rightarrow \mathbb{R}^{p_i}$ are locally Lipschitz functions satisfying $h_i(0,0) = 0$ and $f_i(x_i, u_i) = 0 \Rightarrow u_i = 0$.

2.1.2 Network system

We introduce the equation of network system $N$ described as static nonlinearities, which is the same formulation of the network system given by Arcak [15].

Let $M \in \mathbb{Z}$ be the total number of edges contained in $E$. We define the incidence matrix $D \in \mathbb{Z}^{K \times M}$ by

$$d_{ik} := \begin{cases} +1 & \text{(if } i\text{th node is the positive end of the } k\text{th edge)}, \\ -1 & \text{(if } i\text{th node is the negative end of the } k\text{th edge}), \\ 0 & \text{(otherwise)}, \end{cases}$$

where $d_{ik} \in \{-1, 0, +1\}$ ($i = 1, 2, \ldots, K; \ k = 1, 2, \ldots, M$) is the $(i,k)$th element of $D$. If the $i$th and $j$th nodes are connected by the $k$th edge, we define the difference variable $z_k \in \mathbb{R}^{p}$ by

$$z_k := \sum_{i=1}^{K} d_{ik} y_i = \begin{cases} y_i - y_j & \text{(if } i\text{th node is the positive end)}, \\ y_j - y_i & \text{(if } i\text{th node is the negative end)}. \end{cases}$$

We define the vectors $y \in \mathbb{R}^{Kp}$ and $z \in \mathbb{R}^{Mp}$ by $y := [y_1^\top \ y_2^\top \ \cdots \ y_K^\top]^\top$ and $z := [z_1^\top \ z_2^\top \ \cdots \ z_M^\top]^\top$, respectively. Then, these vectors satisfy the following algebraic equation:

$$z = (D^\top \otimes I_p) y.$$  \hfill (3)

We also define the output $u \in \mathbb{R}^{Kp}$ ($i = 1, 2, \ldots, L$) of $N$ by $u := [u_1^\top \ u_2^\top \ \cdots \ u_K^\top]^\top$, where $u_i \in \mathbb{R}^{p_i}$ is defined by $u_i = -\sum_{k=1}^{M} d_{ik} \phi_i(z_k)$. Moreover, $\phi_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}^{p_i}$ is a nonlinear function satisfying $\phi_i(z_k) = \nabla \Upsilon_i(z_k)$, \hfill (4)

where $\Upsilon_i : G_i \rightarrow \mathbb{R}$ is a nonnegative $C^2$ function and $G_i$ is an open set in which $z_k$ is allowed to evolve. The equation (4) implies that the gradient of the potential functions $\Upsilon_i(z_k)$ is defined by the nonlinear function $\phi_i(z_k)$. We define the vectors $\phi \in \mathbb{R}^{Mp}$ by $\phi := [\phi_1^\top \ \phi_2^\top \ \cdots \ \phi_K^\top]^\top$. Then, $u$ and $z$ satisfy the following algebraic equation to represent the static nonlinearities:

$$u = - (D \otimes I_p) \phi(z).$$  \hfill (5)

2.1.3 Problem formulation

In this paper, we derive a sufficient condition for the following two conditions.

(i) Each system $S_i$ ($i = 1, 2, \ldots, K$) is asymptotically stable. This implies that the partial states of subsystems converge to zero, i.e., $\lim_{t \rightarrow \infty} E_i x_i(t) = 0, \ \forall \ i = 1, 2, \ldots, K$.

(ii) The network system $N$ achieves a synchronization. This implies that the difference variables $z_k$ ($k = 1, 2, \ldots, M$) converge to a prescribed set $\mathcal{A}_k \subset \mathbb{R}^{p_i}$ i.e., $\lim_{t \rightarrow \infty} z_k(t) \in \mathcal{A}_k, \ \forall \ k = 1, 2, \ldots, M$.

Throughout this paper, we suppose the following assumption analogous to Arcak [15].

Assumption 1 The function $\Upsilon_k(z_k)$ satisfies the following properties for any $k = 1, 2, \ldots, M$:

$$\lim_{t \rightarrow \infty} \Upsilon_k(z_k) = \infty,$$  \hfill (6)

$$\Upsilon_k(z_k) = 0 \iff z_k \in \mathcal{A}_k,$$  \hfill (7)

$$\nabla \Upsilon_k(z_k) = 0 \iff z_k \in \mathcal{A}_k,$$  \hfill (8)

In Assumption 1, the divergence (6) means that $\Upsilon_k(z_k)$ is radially unbounded, which is a necessary technical assumption to prove asymptotic stability of nonlinear systems [16]. Moreover, the relationships (7) and (8) imply that the potential function
Remark 1

(i) The network system \(N\) formulated here is the same as Arak [15]. On the other side, the subsystems described by (1) are given as descriptor systems including algebraic constraints in their states. Since the subsystems in [15] do not contain such constraints, we can deal with a broader class of subsystems in this paper.

(ii) The structure preserving model of power grids [7] can preserve the network topology different from the network reduction model [9], which is frequently considered in transient stability of power grids. Thus, we can expect the analysis and control by utilizing the topology. In such a case, dynamics of the subsystems are described by differential-algebraic equations. The current formulation based on descriptor systems is suitable to the target analysis and control of power grids.

2.2 Dissipativity of Subsystems and Network System

In this subsection, we clarify dissipativity of the subsystems and the network system formulated in the last subsection. For this purpose, we define the following assumption of the subsystems based on dissipativity [11]–[14].

Assumption 2 Define the subsystem \(S_i\) \((i = 1, 2, \ldots, K)\) by (1). Then, \(S_i\) is dissipative with respect to the supply rate \(\Phi_i: \mathbb{R}^{2p} \rightarrow \mathbb{R}\) defined by the input \(u_i\) and output \(y_i\). This implies that there exists a \(C^1\), nonnegative definite, radially unbounded function \(\Psi_i: \mathbb{R}^n_i \rightarrow \mathbb{R}\) and a positive definite function \(\Lambda_i: \mathbb{R}^n_i \rightarrow \mathbb{R}\) satisfying the dissipation inequality

\[
\frac{d}{dt} \Psi_i(x_i) \leq \Phi_i(u_i, y_i) - \Delta_i(x_i)
\]

along the trajectory of \(S_i\), for any \(i = 1, 2, \ldots, K\) [17]–[20]. The functions \(\Psi_i(x_i)\) and \(\Delta_i(x_i)\) are called storage function and dissipation rate with respect to the supply rate \(\Phi_i(u_i, y_i)\), respectively. Moreover, \(\Psi_i(x_i)\) is assumed to satisfy the following two properties (i) and (ii).

(i) There exists a \(C^2\) function \(\Psi_i: \mathbb{R}^n_i \rightarrow \mathbb{R}\) satisfying the equality \(\frac{\partial \Psi_i}{\partial x_i}(x_i) = \Psi_i(x_i)^T E_i\).

(ii) The Jacobian \(\frac{\partial \Psi_i}{\partial x_i}(x_i) \in \mathbb{R}^{n_i \times n_i}\) of \(\Psi_i(x_i)\) satisfies the inequality \(\left(\frac{\partial \Psi_i}{\partial x_i}(x_i)\right)^T E_i = E_i^T \frac{\partial \Psi_i}{\partial x_i}(x_i) \geq 0\).

In the remainder of this paper, we set the following assumption on the supply rates of the subsystems and the network system.

Assumption 3 Define the subsystem \(S_i\) \((i = 1, 2, \ldots, K)\) and the network system \(N\) by (1) and (3), (5), respectively. Then, the following inequality holds along the trajectories of \(S_1, S_2, \ldots, S_K\) and \(N\):

\[
\sum_{i=1}^{K} \Phi_i(u_i, y_i) - u_i^T y_i \leq 0.
\]  

Remark 2 By setting \(\Phi_i(u_i, y_i) := u_i^T y_i\). Assumption 2 includes the case, considered in [15], where \(S_i\) is passive as a special case. If all the subsystems \(S_i\) are passive, then the left-hand side of (10) is identically equal to zero. This implies that Assumption 3 vanishes in the passivity case.

2.3 Dissipativity-Based Stability Condition

We define the set of equilibria by

\[
S := \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \in \mathbb{R}^{\Sigma^1_{i=1} n_i + Mp} \left| \begin{array}{l}
E_i x_i = 0 \forall i = 1, 2, \ldots, K, \\
z \in \mathcal{R}(D^T \otimes I_p)
\end{array} \right. \right\}
\]

where \(\mathcal{R}(D^T \otimes I_p)\) denotes the range space of \(D^T \otimes I_p\). We set the following assumption which guarantees that all equilibrium points exist inside of a prescribed set \(A_k \subset \mathbb{R}^p\) \((k = 1, 2, \ldots, M)\).

Assumption 4 The equality \((D \otimes I_p) \phi(z) = 0\) and the constraint \(z \in \mathcal{R}(D^T \otimes I_p)\) imply \(z \in A_1 \times \cdots \times A_M\).

If the columns of \(D\) are linearly independent, Assumption 4 holds because the null space of \(D \otimes I_p\) is trivial. Thus, we have \(\phi(z) = 0\) from \((D \otimes I_p) \phi(z) = 0\). Then, \(z \in A_k\) follows from the properties (4) and (8) in Assumption 1 for any \(k = 1, 2, \ldots, M\). On the other hand, when the columns are linearly dependent, whether Assumption 4 holds or not depends on the sets \(A_k\).

To give our main result, we define the set \(\mathcal{A} \subset \mathbb{R}^{\Sigma^1_{i=1} n_i + Mp}\) by

\[
\mathcal{A} = \left\{ \begin{bmatrix} x, z \end{bmatrix} \in \mathbb{R}^{\Sigma^1_{i=1} n_i + n_M} \left| \begin{array}{l}
E x = 0, \\
z \in A_1 \times \cdots \times A_M \cap \mathcal{R}(D^T \otimes I_p)
\end{array} \right. \right\}
\]

where \(E := \text{diag}(E_1, E_2, \ldots, E_K) \in \mathbb{R}^{\Sigma^1_{i=1} n_i \times \Sigma^1_{i=1} n_i}\) is a block diagonal matrix. Then, we have a sufficient condition which guarantees local asymptotic stability of partial states and synchronization as a main result. The proof is based on the energy function \(\Psi: \mathbb{R}^{\Sigma^1_{i=1} n_i + Mp} \rightarrow \mathbb{R}\) of the entire system defined by

\[
\Psi(x, z) := \sum_{i=1}^{K} \Psi_i(x_i) + \sum_{k=1}^{M} \Psi_k(z_k),
\]

where \(x := [x_1^T \ x_2^T \ \cdots \ x_K^T] \in \mathbb{R}^{\Sigma^1_{i=1} n_i}\) is the state of the entire system.

Theorem 1 Define the subsystem \(S_i\) \((i = 1, 2, \ldots, K)\) and the network system \(N\) by (1) and (3), (5), respectively. Suppose that Assumptions 1-3 hold. Then, the following statements (i) and (ii) hold.

(i) If Assumption 4 holds, the set \(\mathcal{A}\) is uniformly asymptotically stable with the region of attraction defined by

\[
\mathcal{G} = \left\{ \begin{bmatrix} x \\ z \end{bmatrix} \left| \begin{array}{l}
x \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_K}, \\
z \in \{G_1 \times \cdots \times G_M\} \cap \mathcal{R}(D^T \otimes I_p)
\end{array} \right. \right\}
\]

(ii) If Assumption 4 does not hold, all trajectories \((x(t), z(t))\) starting in \(\mathcal{G}\) converge to the set of equilibria \(S\). This implies that the convergence lim_{t \to \infty} z_k(t) = z_k^*\) holds for any edge \(k \in E\) \((k = 1, 2, \ldots, K)\), where \(z_k^*\) denotes the \(k\)th element of \(z^* \in S\).
3. Application to Power Grid Described by Structure-Preserving Model

In this section, we apply the dissipativity-based stability condition in the last section to a power grid described by the structure-preserving model [7].
Also, let \( \theta_i \in \mathbb{R} \) denote the voltage phase of the bus. Moreover, \( \omega_i \in \mathbb{R} \) is the reference frequency. Thus, the electromechanical dynamics of \( i \) are described by the following nonlinear differential equations, called \textit{swing equations} of \( i \) [23]:
\[
\dot{\delta}_i = \omega_i - \omega_s, \\
M_i \ddot{\omega}_i = -D_i (\omega_i - \omega_s) + P_{m,i} - P_e(i, \theta_{Kc,i}).
\]
(13)
In (13), \( P_{m.i} \in \mathbb{R} \) is the constant mechanical input power which drives \( l_i \), and \( P_e(i, \theta_{Kc,i}) \in \mathbb{R} \) is the electrical output power which is delivered to \( E_i \) and is defined by \( P_e(i, \theta_{Kc,i}) := F_k V_{Kc} \sin (\delta_i - \theta_{Kc,i}) \). Finally, \( M_i > 0 \) is the inertia constant of \( l_i \), \( D_i > 0 \) is the damping constant of \( l_i \), \( F_i \in \mathbb{R} \) is the constant voltage behind transient reactance, \( V_{Kc,i} \in \mathbb{R} \) is the constant bus voltage, and \( L_i > 0 \) is the sum of reacance elements including transmcers.

### 3.1.3 External bus of generator

Let us denote \( N_i \) the set of all buses directly joining to the external bus \( E_i \) (\( i \in V_E \)) of \( G_{-Kc} \), or the load \( L_i \) (\( i \in V_L \)) via a single power line. For \( E_i \), we have:
\[
P_{e,i}(\delta_{i-Kc,i}, \theta) - \sum_{j \in N_i} V_{i-Kc,j} B_{ij} \sin (\theta_i - \theta_j) = P_{L,i},
\]
(14)
where \( B_{ij} \in \mathbb{R} \) is the transfer susceptance of the \( (i,j) \)th element in the reduced admittance matrix of the grid, and \( V_i \in \mathbb{R} \) is the constant bus voltage.

### 3.1.4 Load bus

For the load bus \( L_i \) (\( i \in V_L \)), we have:
\[
- \sum_{j \in N_i} V_j B_{ij} \sin (\theta_i - \theta_j) = P_{L,i}.
\]
(15)
The parameter \( P_{L,i} \in \mathbb{R} \) stands for the active power consumption at the load.

### 3.2 Descriptor Representation and Dissipativity

In this subsection, we describe the structure-preserving model by the state-space equations of the networked nonlinear descriptor systems introduced in Section 2. We also derive their storage and potential functions to prove their dissipativity. Before moving to the aforementioned formulation, we define \( (\omega^*, \theta^*, z^*) \in \mathbb{R}^{k+m} \) as the stable equilibrium point (SEP) of the entire grid, which is a solution to the algebraic equations
\[
P_{m,j} - P_{e,j}(\delta_{j-Kc,j}, \theta) = 0 (i \in V_L), \\
P_e(i, \theta_{Kc,i}) - \sum_{j \in N_i} V_{i-Kc,j} B_{ij} \sin (\theta_i - \theta_j) = 0 (i \in V_E), \\
- \sum_{j \in N_i} V_j B_{ij} \sin (\theta_i - \theta_j) = 0 (i \in V_L).
\]
(16)

#### 3.2.1 Subsystems

1. **Internal bus of generator**

We define the electrical inputs \( u_i \in \mathbb{R} \) (\( i \in V_L \)) of \( l_i \) by \( u_i := P_{m,i} - P_{e,i}(\delta_{i-Kc,i}, \theta_{Kc,i}) \). From the equations (13) of \( l_i \) (\( i \in V_L \)), \( l_i \) is described by the state-space equation as follows:
\[
M_i \ddot{\omega}_i = -D_i (\omega_i - \omega_s) + u_i, \quad y_i = \delta_i.
\]
(16)
Define the storage function \( \Psi_i : \mathbb{R} \to \mathbb{R} \) of \( l_i \) by \( \Psi_i(\omega_i) = \frac{1}{2} M_i (\omega_i - \omega_s)^2 \). This function is a positive definite function. The derivative of the function along the trajectory of \( l_i \) satisfies the inequality \( \frac{d}{dt} \Psi_i(\omega_i) \leq u_i \delta_i \) along the trajectory of \( l_i \). Hence, \( l_i \) is passive with respect to the input \( u_i \) and the output \( \dot{y}_i = \dot{\delta}_i \).

2. **External bus of generator**

We define the electrical input \( u_i \in \mathbb{R} \) (\( i \in V_E \)) at each bus by
\[
u_i := P_{e,i}(\delta_{i-Kc,i}, \theta) - \sum_{j \in N_i} V_j B_{ij} \sin (\theta_i - \theta_j) - P_{L,i}.
\]
Then, the state-space equations of the external bus \( E_i \) of \( G_{-Kc} \) (\( i \in V_E \)) are described by
\[
0 = u_i, \quad y_i = \theta_i.
\]
(17)
From (17), we have \( u_i \dot{y}_i = 0 \), which implies that \( E_i \) is lossless with respect to the input \( u_i \) and the output \( y_i = \theta_i \).

3. **Load bus**

We define the electrical inputs \( u_i \in \mathbb{R} \) (\( i \in V_L \)) at each bus by
\[
u_i := - \sum_{j \in N_i} V_j B_{ij} \sin (\theta_i - \theta_j) - P_{L,i} (i \in V_L).
\]
The load bus \( L_i \) (\( i \in V_L \)) can be described by
\[
0 = u_i, \quad y_i = \theta_i.
\]
(18)
We can prove that \( L_i \) is lossless from \( u_i \) to \( \dot{y}_i = \dot{\theta}_i \) by an analogous discussion the case of external bus of generators.

#### 3.2.2 Network system

The transmission lines can be decomposed to \( k \in (V_L, V_E) \) and \( k \in (V_E \cup V_L, V_E \cup V_L) \) from the assumption on its topology. In the following, we derive their descriptor representations and corresponding potential functions. They yield an entire description of the network system \( N \).

1. **Transmission lines between internal and external buses**

The nonlinear function corresponding to the lossless transmission line \( k \in (V_L, V_E) \) is given by
\[
\phi_k(z_k) := \frac{F_k V_{Kc,k}}{\omega_s L_k} \left( \sin z_k - \sin z_k^* \right),
\]
\[
z_k := \delta_k - \theta_{Kc,k} (k \in (V_L, V_E)).
\]
Then, the potential function of this line is given by
\[
\tau_k(z_k) = - \frac{F_k V_{Kc,k}}{\omega_s L_k} \left( \cos z_k - \cos z_k^* \right) + \left( \sin z_k^* \right) (z_k - z_k^*).
\]

2. **Transmission lines between external and load buses**

The nonlinear function corresponding to the lossless transmission line \( k \in (V_E \cup V_L, V_E \cup V_L) \) is given by
\[
\phi_k(z_k) := V_j B_{ij} \left( \sin z_k - \sin z_k^* \right),
\]
\[
z_k := \theta_i - \theta_j.
\]
Then, the potential function of this transmission line is described by
\[
\tau_k(z_k) := - V_j B_{ij} \left( \cos z_k - \cos z_k^* \right) + \left( \sin z_k^* \right) (z_k - z_k^*).
Remark 5

$\dot{\gamma}$

3.3 Local Asymptotic Stability and Synchronization of Power Grid

When the passivity of the subsystems $I_i$, $E_i$, and $L_i$ holds, Assumption 3 is satisfied because the left-hand side of (10) is identically equal to zero. Thus, we have the following corollary by applying these facts to Theorem 1 and using the following $\Psi : \mathbb{R}^{K+M} \to \mathbb{R}$ as an energy function of the entire power grid [8], [9]:

$$\Psi(x, z) = \sum_{i \in V} \Psi_i(\omega_i) + \Upsilon(z).$$

Corollary 1 Consider a power grid (13)-(15) described by the structure-preserving model. Define the subsystem $S_i$ $(i = 1, 2, \ldots, K)$ and the network system $N$ by (16)-(18) and (19), respectively. Then, the following statements (i)-(iii) hold.

(i) $\lim_{t \to \infty} \omega_i(t) = \omega_i$ holds for any generator $G_i (i \in V_G)$.

(ii) $\lim_{t \to \infty} z_k(t) = z_k^*$ holds for any edge $k \in E$.

(iii) $\lim_{t \to \infty} \delta_i(t) = \omega_i$ holds for any load $L_i (i \in V_L)$.

Remark 5

(i) Theorem 1 does not directly imply the statement (iii) of Corollary 1. The statement follows by combining the statements (i) and (ii) in Corollary 1, which guarantees the synchronization of the voltage phases.

(ii) The statement (i) of Corollary 1 shows that the synchronization of the angular frequencies is achieved. However the consensus of the derivatives of the voltage phases is not guaranteed because the subsystems are described as descriptor systems. On the other hand, the statements (ii) and (iii) imply that all the differences of the angular positions and voltage phases converge to the steady state value determined by the SEP $(\omega^*, \theta^*, z^*)$. This corresponds to the convergence of active power transferred between buses to the steady state value, which can be utilized the design of power flows. It remains a future work to consider this point in detail.

4. Numerical Example

We consider a symmetric and simple power grid benchmark consisting of four generators and twelve buses depicted by Fig. 6. The grid has an analogous network topology to IEEE 9-bus test system [9] and can be regarded as a simplification of the test system. This is expressed as the structure-preserving model which is described by a networked nonlinear descriptor system. In this case, the set of buses are given by $V_i := \{1, 2, 3\}$, $V_E := \{4, 5, 6\}$, and $V_L := \{7, 8, 9\}$. We also define the indices of the edges corresponding to the nodes by Table 1.

![Fig. 6 Power grid consisting of four generators and twelve buses.](Image 347x407 to 506x554)

| edge | nodes | nodes | nodes |
|------|-------|-------|-------|
| 1    | [1, 5] | 5     | 9     |
| 2    | [2, 6] | 6     | 10    |
| 3    | [3, 7] | 7     | 11    |
| 4    | [4, 8] | 8     | 12    |

By using an existing network reduction method, e.g., Kron reduction [24], this grid can be described by ordinary state-space systems given by differential equations. However, the grid is also reduced to a network whose topology is given by a complete graph as depicted by Fig. 7 in general. Since the original topology is not preserved after the reduction, it is impossible for us to utilize the network topology appropriately for the analysis and control.

For simplicity of the demonstration, we set the parameters of the internal and external buses and the transmission lines in the above grid, and normalize some parameters as follows:

- $M_i = 1$, $D_i = 0.06$, $F_i = 1 (i \in V_i)$,
- $E_i = 1 (i \in V_E \cup V_L)$,
- $\omega_k = 50$, $L_i = 0.002$, $b_i = 10 (i \in V_i)$,
- $B_{ij} = B_{ji} = b_i = 3 (i, j \in V_E \times V_L)$,
- $P_{m,i} = 0.95 (i \in V_i)$, $P_{l,i} = \begin{cases} 0 & (i \in V_E) \\ -p & (i \in V_L) \end{cases}$,
In particular, $b_{i} > 0$ is the critical transmission powers between the internal and external buses, and $b_{i} > 0$ is the critical transmission powers between the external and load buses. These are supposed to satisfy the inequalities $b_{i} > p$ and $b_{i} > p$.

Under the above setting, the state-space equations of the internal buses $l_i$ ($i \in \mathcal{V}_I$) are described by

$$
\dot{\omega}_i = -D_i (\omega_i - \omega_i) + u_i,
$$

$$
u_i = p - b_i \sin (\delta_i - \delta_{i+4}), \quad \gamma_i = \omega_i - \omega_i.
$$

Moreover, the external buses $E_i$ ($i \in \mathcal{V}_E$) are described by the state-space equations as follows:

$$
0 = u_i,
$$

$$
u_i = -b_i \sin (\delta_i - \delta_{1}) - b_i \sin (\delta_i - \delta_{12}) - b_i \sin (\delta_i - \delta_{5}) \quad (i = 5),
$$

$$
u_i = -b_i \sin (\delta_i - \delta_{5}) - b_i \sin (\delta_i - \delta_{12}) - b_i \sin (\delta_i - \delta_{16}) \quad (i \in \mathcal{V}_E \setminus \{5\}),
$$

$$
\gamma_i = \theta_i.
$$

Finally, the load buses $L_i$ ($i \in \mathcal{V}_L$) are described by the state-space equations as follows:

$$
0 = u_i,
$$

$$
u_i = -p - b_i \sin (\delta_i - \delta_{12}) - b_i \sin (\delta_i - \delta_{16}) \quad (i \in \mathcal{V}_E \setminus \{12\}),
$$

$$
u_{12} = -p - b_i \sin (\delta_{12} - \delta_{5}) - b_i \sin (\delta_{12} - \delta_{5}) \quad (i = 12),
$$

$$
\gamma_i = \theta_i.
$$

In these equations, the state $x \in \mathbb{R}^9$ and the input $u \in \mathbb{R}^9$, the output $y \in \mathbb{R}^9$ of the buses are defined by

$$
x := \begin{bmatrix} \omega_1 - \omega_i \cdots \omega_i \end{bmatrix}, \quad u := \begin{bmatrix} u_1 \\ \vdots \\ u_{12} \end{bmatrix}, \quad y := \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_8 \end{bmatrix},
$$

respectively.

The difference variable $z \in \mathbb{R}^9$ is described as follows:

$$
z = D^z y, \quad D := \begin{bmatrix} I_4 & 0_{6 \times 4} & 0_{6 \times 4} \\ -I_4 & I_4 & C_4 \\ 0_{6 \times 4} & -I_4 & I_4 \end{bmatrix},
$$

$$
C_4 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}.
$$

Then, the network system $N$ is described by the static state-space equations $u = -D \phi(z)$, where $\phi_k(z_k) \ (k = 1, 2, \ldots, 9)$ are defined by

$$
\phi_k(z_k) := \begin{cases} b_i \sin z_k - p & (k = 1, 2, 3, 4), \\
\frac{b_i}{2} \sin z_k - \frac{p}{2} & (k = 5, 6, 7, 8), \\
\frac{b_i}{2} \sin z_k + \frac{p}{2} & (k = 9, 10, 11, 12). 
\end{cases}
$$

Under the above setting, the SEP $(x^*, z^*)$ is given by

$$
x^* := \omega_1 \mathbf{1}_{12} \quad \text{and} \quad z^* := \begin{bmatrix} \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \\ \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \\ - \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \end{bmatrix}.
$$

We obtain the storage function $I_i$ ($i \in \mathcal{V}_I$) and the potential functions of the transmission line $k (k \in (\mathcal{V}', \mathcal{V}'))$ by

$$
\Psi_i(\omega_i) = \frac{1}{2} (\omega_i - \omega_i)^2,
$$

$$
\Upsilon_i(z_k) = \begin{cases} -b_i \cos z_k - \cos z_k - p (z_k - z_k) & (k = 1, 2, 3, 4), \\
-b_i \cos z_k - \cos z_k + \frac{1}{2} p (z_k - z_k) & (k = 5, 6, 7, 8), \\
-b_i \cos z_k - \cos z_k + \frac{1}{2} p (z_k - z_k) & (k = 9, 10, 11, 12), 
\end{cases}
$$

respectively, where $z_k \in \mathbb{R}$ is the $k^{th}$ component of $z_k$. From the definitions of $\Upsilon_i(z_k)$, the potential energy of $N$ is described by

$$
\Upsilon(z) = \sum_{k=1}^{9} \Upsilon_i(z_k).
$$

The Hessian of $\Upsilon(z)$ at the SEP $(x^*, z^*)$ is computed by

$$
\text{diag} \left( \begin{bmatrix} b_i \cos \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \\ b_i \cos \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \end{bmatrix} \right),
$$

which is positive definite from $\cos \sin^{-1} \left( \frac{p}{b_i} \right) > 0$ and $\cos \sin^{-1} \left( \frac{p}{b_i} \right) > 0$. Since the storage functions $\Psi_i(\omega_i)$ are also positive definite along the trajectory of $l_i$, the energy function

$$
\Psi(x, z) = \sum_{i \in \mathcal{V}_I} \Psi_i(\omega_i) + \Upsilon(z)
$$

of the entire power grid is positive definite. We can also prove the negative definiteness of its derivative from the passivity of all buses. Hence, the statements (i)-(iii) of Corollary 1 are satisfied for this power grid. Thus, we have the local asymptotic stability of the state and the synchronization in Corollary 1 as

$$
\lim_{t \to \infty} \omega_i(t) = \lim_{t \to \infty} \dot{\theta}_i(t) = \omega_i \quad \forall i \in \mathcal{V}_I \cup \mathcal{V}_E \cup \mathcal{V}_L,
$$

$$
\lim_{t \to \infty} z_k(t) = z_k^* \quad \forall k \in (\mathcal{V}', \mathcal{V}').
$$

In the following, we see that the statements of Corollary 1 are achieved via numerical simulation. We injected a disturbance $(\delta_3, \theta_3)$ to the angular positions and voltage phases corresponding to the SEP $(x^*, z^*)$ at $t = 0$ as follows:

$$
\delta(0) = \delta^\ast + \theta^\ast \quad \theta^\ast := \begin{bmatrix} \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \\ \sin^{-1} \left( \frac{p}{b_i} \right) \mathbf{1}_4 \end{bmatrix},
$$

$$
\delta_d := \begin{bmatrix} -0.0211 \\ -0.1026 \\ -0.0851 \\ -0.3098 \end{bmatrix}, \quad \theta_d := \begin{bmatrix} 0.0136 \\ 0.0172 \\ 0.5082 \\ 0.4717 \\ 0.2309 \\ -0.1942 \\ 0.4985 \\ 0.0525 \end{bmatrix}.
$$

Fig. 7 Power grid expressed as a network with a complete graph consisting of four internal buses through a network reduction.
5. Conclusions

In this paper, we have formulated a networked nonlinear descriptor system as a mathematical abstraction of power grids consisting of generators and loads. We have clarified the passivity of the subsystems and network system, and have shown local asymptotic stability of the partial states and synchronization based on the dissipativity. This result is a generalization of the stability condition in [15] in the sense that the subsystems...
are represented by descriptor systems. We have applied the stability result to a power grid described by structure preserving model.

As future work, it is desired to apply the derived stability and synchronization condition for analysis and control of power grids including photovoltaic generation units and derive its theoretical and unified framework. The reason is that algebraic constraints due to DC circuits in the units can be explicitly described by a descriptor system. Toward this direction, it can be considered to generalize the condition to the framework of robust control synthesis by estimating parametric uncertainties of power grids including photovoltaic generation units. In addition, it is also desired to derive another quantitative condition of power grids including the units in the sense of time evolutions of functions related to the grids. Regarding this point, the authors of this paper have derived a quantitative and hierarchical evaluation for swing instability of power grids [25] measuring time evolutions of so-called energy functions constructed by a hierarchical manner. A combination of the qualitative conditions of [25] on qualitative and theoretical conditions of this paper is expected to be effective to evaluate a stability margin of the aforementioned power grids in the sense of time evolutions. This condition is related to clarifying how dissipativity of subsystems including photovoltaic generation units is likely to be guaranteed. It can be another direction as an application of the derived stability and synchronization conditions of this paper to control system design.

Acknowledgments

This work was supported by JST CREST Grant Number JP-MJCR15K1, Japan, and JSPS Grant-in-Aids for Grant-in-Aid for Scientific Research (C) 17K06487.

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Appendix

Proof of Theorem 1

We first prove dissipativity of a set of the subsystems $\{S_1, S_2, \ldots, S_K\}$ with respect to the supply rate $\sum_{i=1}^{K} \Phi_i(u_i, y_i)$. From Assumption 2, there exists a $C^1$, nonnegative definite, radially unbounded function $\Psi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ and a positive definite function $\Delta_i : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the dissipation inequality (9) along the trajectory of $S_i$, for any $i = 1, 2, \ldots, K$. Moreover, $\Psi_i(x_i)$ satisfies Assumption 2 (i), (ii). From (9), the derivative of the summation of all $\Psi_i(x_i)$ satisfies the inequality along the trajectory of $\{S_1, S_2, \ldots, S_K\}$. Since $\sum_{i=1}^{K} \Phi_i(u_i, y_i)$ is a positive definite function, $\{S_1, S_2, \ldots, S_K\}$ is dissipative with respect to the supply rate $\sum_{i=1}^{K} \Phi_i(u_i, y_i)$.
Define the energy function \( \Psi : \mathbb{R}^{\sum_{i=1}^{n,+M_p}} \rightarrow \mathbb{R} \) of the entire system by (12). Along the trajectory of the system, the derivative of \( \Psi(x, z) \) satisfies the equality

\[
\frac{d}{dt} \Psi(x, z) = \sum_{i=1}^{K} \frac{d}{dt} \Psi_i(x_i) + \sum_{k=1}^{M} \Delta \chi_k.
\]  
(A.1)

From Theorem 1 in [15], \( N \) is passive with respect to the input \( -\dot{y} \) and the output \( u \). From (4), we have the equality \( \sum_{i=1}^{K} \frac{d}{dt} \Psi_i(z_i) = -\dot{y}^T u \). By applying the right-hand side of the above equality to (A.1), we have

\[
\frac{d}{dt} \Psi(x, z) \leq \sum_{i=1}^{K} \Phi_0(u_i, y_i) - \dot{y}^T u - \sum_{i=1}^{K} \Delta \chi_i.
\]

Applying Assumption 3 to the first and second terms of the right-hand side, we obtain the inequality \( \frac{d}{dt} \Psi(x, z) \leq -\sum_{i=1}^{K} \Delta \chi_i \). This inequality implies the negative definiteness of \( \frac{d}{dt} \Psi(x, z) \). Moreover, we can prove the positive definiteness of \( \Psi(x, z) \) along the trajectory of the entire system from Assumption 2 and (3). It follows from these results that the convergence \( \lim_{t \to \infty} \Psi(x, z) = 0 \) holds. From the definition of \( \Psi(x, z) \) and the positive definiteness of \( \Psi_i(x_i) \), \( \Delta \chi_k \) for all \( i = 1, 2, \ldots, K; k = 1, 2, \ldots, M \), the convergences

\[
\lim_{t \to \infty} \Psi_i(x_i) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Delta \chi_k = 0 \quad \text{(A.2)}
\]

hold for all \( i = 1, 2, \ldots, K \) and \( k = 1, 2, \ldots, M \).

In the following, we suppose that the set \( \mathcal{A} \) is not stable, and deduce a contradiction. Firstly, we assume that there exists an index \( i \in \mathbb{N} \) such that \( E_i x_i(t) \) grows unbounded as \( (x(t), z(t)) \) approaches \( \mathcal{G}^\infty \) contrary to the relationship (7) in Assumption 1. From the property (i) in Assumption 2, there exists a \( C^2 \) and positive definite matrix function \( \Psi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) satisfying \( \Psi_j(x_j) = x_j^T E_j^T \Psi_j(x_j) E_j \). Then, the first convergence in (A.2) implies that the trajectories of \( E_i x_i(t) \) are bounded for any \( t \in [0, T] \) and any \( T \) within the maximal interval \( [0, t_i] \) where the differential-algebraic equations (1) and (3), (5) are defined. Since this bound does not depend on \( T \), we can find a bound on \( E_i x_i(t) \) increasing linearly in \( T \). If \( t_i \) is finite, we can prove the existence of \( E_i x_i(t) \) by letting \( T \to t_i \). This implies a contradiction. Next, we assume that there exists an index \( k \in \mathbb{N} \) such that \( z_k(t) \) grows unbounded as \( (x(t), z(t)) \) approaches \( \mathcal{G}^\infty \) contrary to the relationship (7) in Assumption 1. Then, we can also deduce a contradiction based on an analogous discussion to \( E_i x_i(t) \). Thus, we can conclude that there is no finite escape time as \( (x(t), z(t)) \) approaches \( \mathcal{G}^\infty \). This shows the existence of solutions of the differential-algebraic equations (1) and (3), (5) for all \( t \geq 0 \). Thus, we conclude the stability of the set \( \mathcal{A} \).

Finally, we prove the attractivity of \( \mathcal{A} \). In the following, we investigate the invariant set where \( \frac{d}{dt} \Psi(x, z) = 0 \) holds. Note that, if \( x_i = 0 \) holds identically, then we have \( u_i = 0 \) from (2). We use the invariance principle (Lemma 4.1 of [16]) because the right-hand side of (A.1) vanishes on a superset of \( \mathcal{A} \). Then, the trajectories \( (x(t), z(t)) \) converges to the set \( S \) in (11) as \( t \) goes to infinity. When Assumption 4 holds, since \( S \) coincides with \( \mathcal{A} \), the asymptotic stability of \( \mathcal{A} \) with region of attraction \( \mathcal{G} \) holds. This concludes the proof of the statement (i). On the other hand, if Assumption 4 fails, the uniformity of asymptotic stability follows from the time invariance of the dynamics of \( (x(t), z(t)) \). This concludes the proof of the statement (ii).