The Interplay of Charge and Spin in Quantum Dots: The Ising Case

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The physics of quantum dots is succinctly depicted by the Universal Hamiltonian, where only zero mode interactions are included. In the case where the latter involve charging and isotropic spin-exchange terms, this would lead to a non-Abelian action. Here we address an Ising spin-exchange interaction, which leads to an Abelian action. The analysis of this simplified yet non-trivial model shed some light on a more general case of charge and spin entanglement. We present a calculation of the tunneling density of states and of the dynamic magnetic susceptibility. Our results are amenable to experimental study and may allow for an experimental determination of the exchange interaction strength.

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I. INTRODUCTION

Significant progress in the study of the physics of quantum dots (QDs) has been achieved following the introduction of the Universal Hamiltonian (UH). The latter facilitated the simplification of intricate electron-electron interactions within a QD in a controlled way. Within that scheme interactions are represented as the sum of three spatially independent terms: charging, spin-exchange, and Cooper channel. Notably, even the inclusion of the first two terms turned out to be non-trivial: the resulting action is non-Abelian.

To understand the complexity of such a problem one can refer to the case of charging-only interaction. As was suggested by Kamenev and Gefen, one can take the following steps in solving that problem: start from a fermionic action which includes an interaction term quadratic in the (fermionic Grassman) variables, perform a Hubbard-Stratonovich transformation by introducing an auxiliary bosonic field, then perform a gauge transformation over the Grassman variables, and finally integrate them out. The resulting, purely bosonic, action is simple. In an imaginary time (Matsubara) picture the action is isotropic in QD have been reported earlier. Alhassid and Rupp, the latter model some quantities turn out to be particularly simple (e.g. the finite frequency spin susceptibility vanishes; evidently there is no difference between longitudinal and transverse spin susceptibility). This means that the analysis of a model with anisotropy in the spin interaction is called for. A perturbation expansion in spin anisotropy has been reported earlier, but it still remains desirable to consider an anisotropic model which can be analyzed exactly. By considering such a model one would be able to understand the entanglement between charge and spin degrees of freedom, and also see in detail how a non-vanishing, complex spin susceptibility arises. This is the focal point of the present analysis.

In bulk systems the exchange interaction competes with the kinetic energy leading to Stoner Instability (SI). In finite size systems mesoscopic Stoner unstable regime may be a precursor of bulk thermodynamic SI. We consider here an Ising spin interaction. Such a model is Abelian, and complications due to non-commutativity of different terms in the action do not arise here. Also such a model does not exhibit a mesoscopic Stoner unstable regime. This means that at zero temperature, as the dimensionless parameter \(J/\Delta\) (\(J\) being the exchange interaction strength and \(\Delta\) is the mean level spacing) the system abruptly switches from a paramagnetic to a (thermodynamic Stoner unstable) ferromagnetic phase. We stress that notwithstanding the simplicity of the model considered, spin-charge entanglement is present here, and non-trivial transverse a.c. susceptibility does arise. Some of our conclusions can in principle tested in QDs made of various transition metal alloys, Ni impurities in Pd host, Fe or Mn dissolved in rare earth materials, as well as new nearly ferromagnetic rare earth materials.

The outline of this paper is as follows. In Section II we introduce our model Hamiltonian and the subsequent imaginary time action. In Section III we employ the technique of zero-dimensional functional bosonization, which eventually allows us to express the single-particle Green’s Function as a product of the non-interacting Green’s Function and a term which depends on two bosonic fields. We then show how to reduce the problem to that of classical stochastic equations for the bosonic fields. In Section IV we express the grand-canonical partition function in terms of canonical ones, leading to both
a mathematical and physical simplification of the calculation. In Section V we calculate the tunneling density of states and in Section VI longitudinal and transverse spin susceptibilities. Section VII presents a summary of the main results with some perspectives. We include some more technical calculations in three Appendices.

II. HAMILTONIAN AND EFFECTIVE ACTION

We consider a normal-metal QD in the metallic regime, where the Thouless energy $E_{Th}$ and the mean level spacing $\Delta$ satisfy $g \equiv E_{Th}/\Delta \gg 1$ ($g$ is the dimensionless conductance) and a temperature $T \gg \Delta$. It is the regime where a description in terms of UH is viable.

We restrict ourselves to a simplified version of the UH where the interaction in the Cooper channel is set to zero and the spin-exchange term is chosen to be a fully anisotropic Ising-like term, $-J \hat{S}_z^2$, with a ferromagnetic exchange coupling, $J > 0$, $\hat{S}_Z$ is the total spin of the dot in the $\hat{z}$ direction. This form of interaction is sufficient to bring about the Stoner instability phenomenon and other spin-related effects, whilst avoiding calculational complexities inherent to a fully spin-symmetric model. Possible physical sources for such an anisotropy may include geometrical and/or molecular anisotropy, magnetic impurities in the system, or even the application of magnetic impurities in the system, or even the application of geometrical and/or molecular anisotropy, magnetic impurities in the system, or even the application of}

The complete form of the reduced UH is thus

$$
H = \sum_{\alpha, \sigma} \varepsilon_\alpha a^\dagger_{\alpha, \sigma} a_{\alpha, \sigma} + E_c \left[ \sum_{\alpha, \sigma} a^\dagger_{\alpha, \sigma} a_{\alpha, \sigma} - N_0 \right]^2 
- \frac{J}{4} \left[ \sum_{\alpha, \sigma} a^\dagger_{\alpha, \sigma} \sigma^z a_{\alpha, \sigma} \right]^2.
$$

Here $\{\varepsilon_\alpha\}$ is a set of electronic levels in the dot, and $N_0$ in the charging term represents a positive background charge controlled via an external gate. We assume that the QD is either isolated or weakly-coupled to the leads and in the Coulomb blockade regime. On the other hand, we will be considering the spin-disordered regime below the Stoner instability. So the parameters of the Hamiltonian (1) obey

$$
J < \Delta \ll T \ll E_c,
$$

where $T \equiv \beta^{-1}$ is the temperature.

The Euclidean action corresponding to the Hamiltonian (1) is given by

$$
S[\Psi, \bar{\Psi}] = \sum_\alpha \int_0^\beta d\tau \left\{ \bar{\Psi}_\alpha (\partial_\tau + \varepsilon_\alpha - \mu) \Psi_\alpha 
+ E_c \left[ \sum_\alpha \bar{\Psi}_\alpha \Psi_\alpha - N_0 \right]^2 
- \frac{J}{4} \left[ \sum_\alpha \bar{\Psi}_\alpha \sigma^z \Psi_\alpha \right]^2 \right\}.
$$

where we use spinor notations $\Psi_\alpha = (\bar{\Psi}_\uparrow_\alpha(\tau), \bar{\Psi}_\downarrow_\alpha(\tau))$. We introduce two auxiliary bosonic fields, $\varphi^c(\tau)$ and $\varphi^s(\tau)$, to decouple the Coulomb and exchange terms with the help of a standard Hubbard-Stratonovich (HS) transformation. This results in the following action:

$$
S = S^c + S^s + S^{mix},
$$

where

$$
S^c = \int_0^\beta d\tau \left[ \varphi^c(\tau)^2 + iN_0 \varphi^c(\tau) \right],
$$

$$
S^s = \int_0^\beta d\tau \frac{\varphi^s(\tau)^2}{J},
$$

$$
S^{mix} = \int_0^\beta d\tau \sum_\alpha \bar{\Psi}_\alpha \left[ \partial_\tau + \varepsilon_\alpha - \mu + i\varphi^c + \sigma^z \varphi^s \right] \Psi_\alpha.
$$

Here $\sigma^z$ is a Pauli matrix, the bosonic fields are periodic and the fermionic fields are antiperiodic in $\tau$ with period $\beta$. This action is the starting point for all the subsequent calculations. We will use the functional bosonization approach as developed in [3,11], first we gauge out the mixed fermionic-bosonic terms in the action (5) and then integrate over the fermionic field thus arriving at a purely bosonic action. After that, instead of dealing with this action directly we will use a stochastic bosonization as described in the following section.

III. FROM FUNCTIONAL TO STOCHASTIC BOSONIZATION

In order to gauge out the mixed fermionic-bosonic terms in the action (5), we introduce a generalized gauge transformation, $\bar{\Psi}_\alpha = \mathcal{T}^{-1} \Psi_\alpha$, $\bar{\Psi}_\alpha = \Psi_\alpha \mathcal{T}$ with

$$
\mathcal{T} = e^{i\theta(\tau)+\theta^*(\tau)} = \begin{pmatrix} e^{i\theta(\tau)+\theta^*(\tau)} & 0 \\
0 & e^{i\theta(\tau)-\theta^*(\tau)} \end{pmatrix}.
$$

“Gauging out” implies the following identity

$$
\bar{\Psi}_\alpha \left[ \partial_\tau + i\varphi^c(\tau) + \sigma^z \varphi^s(\tau) \right] \Psi_\alpha = \bar{\Psi}_\alpha \left[ \partial_\tau + A \right] \Psi_\alpha,
$$

where $A$ is some constant matrix. In order to fulfill (6) we require the gauge matrix $\mathcal{T}$ to obey

$$
\left[ \partial_\tau + i\varphi^c(\tau) + \sigma^z \varphi^s(\tau) \right] \mathcal{T} = \mathcal{T} A.
$$

Since the bosonic fields are real, this equation separates into real and imaginary parts, corresponding to the exchange and charge channels. Using the substitution $\mathcal{A} = A^s \sigma^z + iA^c$ for the constant matrix $A$ in the matrix gauge equation (7), we have

$$
\dot{\theta}^a(\tau) = A^a - \varphi^a(\tau)
$$

where $a$ stands either for charge, $c$, or for spin, $s$.

To determine the constants $A^s$ and $A^c$ we note that the antiperiodicity of the fermionic fields requires that $\mathcal{T}(\beta) = \mathcal{T}(0)$. This in turn implies $\theta^a(\beta) = \theta^a(0) + 2\pi n^a$
and \( \Theta'(\beta) = \Theta'(0) + 2\pi N \) with integer \( n^a \) and \( N \). Now we single out zero-Matsubara-frequency components of the bosonic fields \( \phi^a(\tau) \):

\[
\varphi^a(\tau) = \phi^a_0 + \tilde{\varphi}^a(\tau), \quad \beta \phi^a_0 = \int_0^\beta \! d\tau \varphi^a(\tau).
\] (9)

Integrating Eqs. (8) over \( \tau \) from 0 to \( \beta \) results in \( A^c = \phi^a_0 + (2\pi/\beta)N \) and \( A^a = \phi^a_0 + (2\pi i / \beta)n^a \) so that the gauge equations (8) reduce to the following form:

\[
\dot{\phi}^c(\tau) = \frac{2\pi i}{\beta} N - \tilde{\varphi}^c(\tau),
\] (10a)

\[
\dot{\phi}^a(\tau) = \frac{2\pi i}{\beta} n^a - \tilde{\varphi}^a(\tau).
\] (10b)

After the gauge transformation the mixed action in Eq. (5) is reduced to the following quadratic fermionic action in terms of the transformed fields:

\[
S_i = \int_0^\beta \! d\tau \sum_{\alpha} \bar{\Psi}_\alpha(\tau) \left[ \partial_\tau + \varepsilon_\alpha - \bar{\mu}_\alpha \right] \Psi_\alpha(\tau).
\] (11)

The zeroth components of the bosonic fields \([\ref{5}]\) enters Eq. (11) via the spin-dependent effective chemical potential \( \mu_\sigma \) given by

\[
\bar{\mu}_\sigma = \mu - i\phi^c_0 - \sigma\phi^a_0 - \frac{2\pi i}{\beta} (N + \sigma n^a),
\] (12)

where \( \sigma = \pm 1 \) for spin up/down respectively.

The gauge equations (10) become important for correlating functions which are not gauge invariant but depend on phase terms which are functions of the gauge parameters \( \Theta^c \) and \( \Theta^a \) (e.g. the Green’s function calculated in the following Section and Appendix A). These parameters are functionals of the bosonic fields \( \tilde{\varphi}^c(\tau) \) and \( \tilde{\varphi}^a(\tau) \) respectively. Thus, in order to calculate these phase terms, one should solve the gauge equations and then carry out the integration over the bosonic fields \([\ref{5}]\).

Here however, we consider an alternative method, which bypasses the need to carry out the functional integrals over \( \tilde{\varphi}^a(\tau) \). Even though in our case these integrations pose no great difficulty, the method we consider has general applicability and could be used in cases where such integrations are impossible to perform analytically.

Our approach is to view the gauge equations (10) as classical Langevin equations governing the stochastic dynamics of \( \Theta^c \) and \( \Theta^a \), with the bosonic fields playing the role of noise. The distribution of the noise is determined by the bosonic actions \( S^c \) and \( S^a \), Eq. (5).

The Langevin equations can be mapped, via the standard tools of classical stochastic analysis\([\ref{12}]\), to Fokker-Planck (FP) equations from which the time dependent distribution functions for \( \Theta^c \) and \( \Theta^a \) can be determined. As an example, the form of the FP equation derived from Eq. (10a) is

\[
\frac{\partial \Phi^c}{\partial \tau} = \left( \frac{2\pi i}{\beta} N - i\zeta \right) \frac{\partial \Phi^c}{\partial \theta} + E_c \frac{\partial^2 \Phi^c}{\partial \theta^2},
\] (13)

where \( \Phi^c \) is the distribution function for the gauge parameter \( \Theta^c \) and \( \zeta \) is a constant (details regarding the transition from Langevin to FP equations and their solution are given in Appendix C). Equation (13) is a standard diffusion equation with a drift term, the solution of which (with an appropriate initial condition) is simply a decaying Gaussian, explicitly given by Eq. (C5).

This distribution, and a similar one for \( \Theta^a \), can now be used to calculate the averaging of any phase terms involving the gauge parameters in the calculation of non-gauge-invariant correlation functions. Thus we can, in effect, replace a functional integration with an integration over a finite number of parameters. This is an alternative method by which to integrate out the finite frequency components of the bosonic fields \( \varphi^c(\tau) \) and \( \varphi^a(\tau) \).

### IV. SINGLE PARTICLE GREEN’S FUNCTION: EFFECTIVE CHARGE QUANTIZATION

We begin with calculating the temperature Green’s function (GF) in the grand canonical ensemble and will show that in the Coulomb blockade regime it reduces essentially to one in the canonical ensemble. Our starting expression is:

\[
S^c(\tau, \mu) = \sum_{\alpha} S^c_{\alpha,\sigma}(\tau, \mu),
\] (14)

\[
S^c_{\alpha,\sigma} = \frac{1}{Z(\mu)} \int D[\Psi_\alpha] e^{-\int (N\Psi_\alpha^\dagger \Psi_\alpha + S[\Psi_\alpha])} \Psi_{\alpha,\sigma}(\tau) \Psi_{\alpha,\sigma}(\tau_f),
\]

where \( S_{\alpha,\sigma} \) is an auxiliary GF corresponding to a level \( \varepsilon_\alpha, S[\Psi_\alpha \Psi_\alpha^\dagger] \) is the \( \alpha \)-term in the Euclidean action \([\ref{4}]\) and \( \tau \equiv \tau_f - \tau_i \).

After the HS transformation and gauge transform \([\ref{6}]\), the Gaussian integration over the quadratic fermionic action \([\ref{11}]\) is straightforward. The resulting GF of non-interacting electrons corresponding to this action, \( S^c_{\alpha,\sigma}(\tau, \mu_\sigma) \), depends – via Eq. (12) – only on the zero-frequency component of the bosonic fields \( \phi^c_0 \). This allows us to subdivide the remaining functional integration with the bosonic part of the action \([\ref{4}]\) into that over the zero-frequency, \( \phi^c_0 \), and finite frequency, \( \tilde{\varphi}^c \), components, which results in the following expression:

\[
S^c_{\alpha,\sigma} = \Pi^c(\tau) \Pi^c(\tau_f) \mathbb{L}^{\alpha_0}(\mu_\sigma) \mathbb{L}^{\alpha_0}(\tau_f) \mathbb{L}^{\alpha_0}(\mu_\sigma) = \mathbb{L}^{\alpha_0}(\mu_\sigma) \mathbb{L}^{\alpha_0}(\tau_f) \mathbb{L}^{\alpha_0}(\mu_\sigma),
\] (15)

Here \( \Pi^c(\tau) \) are the phase correlation functions resulting from the functional averaging of the charge or spin phase factors over the finite-frequency components of the appropriate fields, \( \mathbb{L}^{\alpha_0}(\tau) \) stand for the functional integrals over the zeroth-component fields \( \phi^c_0 \) and \( \tilde{\varphi}^c \). All these functional integrals are defined in Eq. (A1)–(A3) in Appendix A. Then \( \mathbb{L}^{\alpha_0}(\mu_\sigma) = \mathbb{L}^{\alpha_0}(\mu_\sigma) \mathbb{L}^{\alpha_0}(\mu_\sigma) \) and \( \mathbb{L}^{\alpha_0}(\tau_f) \mathbb{L}^{\alpha_0}(\mu_\sigma) \) is the grand canonical partition function\([\ref{13}]\) of non-interacting electrons with the spin-dependent chemical potential \( \mu_\sigma \), defined by Eq. (12).
The charging effects can be fully accounted for by introducing winding numbers in the integration over $\varphi_0^c$:

$$\varphi_0^c = \omega_m + \frac{\varphi_0^c}{\beta}, \quad \omega_m = \frac{2\pi m}{\beta}$$

where $-\pi < \varphi_0^c < \pi$ and an integer $m$ is a winding number. In the original work of Gefen and Kamenev these were not considered, leading to an incorrect final result. They were first introduced in the context of the charging interaction on small metallic grains by Efetov and Tschersich within a Matsubara framework, and were finally correctly implemented by Sedlmayr, Yurkevich and Lerner within a Keldysh-technique framework.

The introduction of the winding numbers allows us to replace integration over $\varphi_0^c$ with summation over all integers $m$ and integration over $\varphi_0^c$. The sum over $m$ is performed using the Poisson formula, which results in a new summation of the form

$$\sum_N e^{-\beta E_c(N-N_0)^2} \times \mathcal{F}(N).$$

The Poisson resummation transforms summation over $m$ into summation over the conjugate variable, $N$. In our case $\varphi_0^c$ represents a phase, whose conjugate is evidently the particle number $N$. While the sum over the parameter $m$ had many contributions (since $(\beta E_c)^{-1} \ll 1$), the sum over $N$, under the conditions (2), only two terms $N = N_0 + \frac{1}{2}$, near the Coulomb peak ($N_0$ is half an integer) and one term in the Coulomb valleys (i.e., everywhere outside of the region of width $T$ near the peak): the contribution of all the other terms is exponentially suppressed. This is a manifestation of charge quantization in QDs.

In this way we perform the integration in Eq. (15) to find (see Appendix A):

$$\mathcal{G}_{\alpha,\sigma}(\tau, \mu) = \frac{\hat{\Pi}^c(\tau) \hat{\Pi}^s(\tau)}{Z(\mu)} \sum_N e^{-\beta E_c(N-N_0+\frac{\varphi}{2})} J_N, \quad (17)$$

$$J_N \equiv \int_{-\infty}^{\infty} d\varphi_0^c e^{i \varphi_0^c} \int_{-\pi}^{\pi} d\varphi_0^s \int_{-\infty}^{\infty} d\varphi_0^s \int_{-\infty}^{\infty} d\varphi_0^s \int_{-\infty}^{\infty} d\varphi_0^s |(N+\varphi)| \varphi_0^c Z^0(\frac{\varphi}{2}) \mathcal{G}_{\alpha,\sigma}(\tau, \mu_\sigma),$$

where the reduced phase correlation functions $\hat{\Pi}^a$ are defined in Eq. (A4). The effective charge quantization in Eq. (17) makes it natural to change over from grand canonical to canonical quantities for a given $N$, followed by a weighted summation over $N$, where required. Let us stress that the canonical quantities are auxiliary and we calculate in this way the grand canonical GF of Eq. (17).

Expressing $J_N$ via canonical quantities leads to an extra summation since $Z = \sum_n e^{\beta n} Z_n$, etc. This calculation is detailed in Appendix B. The resulting full single particle GF in imaginary time (following summation over all single particle energy states) is given by

$$\mathcal{G}(\tau, \mu) = \frac{\pi T}{\Delta} e^{-\beta E_c(\Delta-J)/2 |\tau|} \frac{F(\tau)}{\sin(\pi |\tau| T)} \frac{F(0)}{F(0)}, \quad (19)$$

where

$$F(\tau) = \sum_{N} e^{-\beta E_c(\delta N)^2} \sum_{M=-N}^{N} e^{-\frac{1}{4}(\Delta-J)M^2 - \tau E_{N,M}}, \quad (20)$$

$$\delta N \equiv N - N_0 - \frac{\mu}{2E_c}, \quad E_{N,M} \equiv 2E_c \delta N - \frac{J M}{2T}. \quad (21)$$

The double summation above arises from replacing the grand canonical partition function in terms of the sum over canonical ones, $Z(\mu) = \sum_n e^{\beta n} Z_n$. The summation parameters are the electron number, $N$, and the total spin of the dot (in the units of $\hbar/2$), $M$. Naturally, the GF is spin independent: we are considering the regime of parameters, Eq. (2), below the Stoner instability where there is no symmetry breaking to distinguish opposite spin polarizations. Note that this result is valid in the regime of conditions (2), provided that

$$N\Delta \gg T, \quad (N - |M|)\Delta \gg T, \quad (21)$$

i.e., when the QD contains many electrons and is not very close to the Stoner instability. Moreover, under these conditions the sum over $M$ in Eqs. (19) and (20) can be replaced by an integral from $-\infty$ to $+\infty$ and the exponent of $\frac{J^2 \tau^2}{4(\Delta-J)^2}$ resulting from this integration can be totally neglected. With the same accuracy, we should neglect the exchange energy $J$ in the exponent in Eq. (19). Thus we find

$$\mathcal{G}(\tau, \mu) = \frac{\pi e^{-\beta \varepsilon_c |\tau|}}{\beta \Delta \sin(\frac{\pi |\tau|}{\beta})} \frac{1}{Z} \sum_N e^{-\beta \varepsilon_c |N|} \int_{-\infty}^{\infty} d\varphi_0^c e^{i \varphi_0^c} \int_{-\infty}^{\infty} d\varphi_0^s \int_{-\infty}^{\infty} d\varphi_0^s \int_{-\infty}^{\infty} d\varphi_0^s |(N+\varphi)| \varphi_0^c Z^0(\frac{\varphi}{2}) \mathcal{G}_{\alpha,\sigma}(\tau, \mu_\sigma),$$

so that under conditions (2) and (21) — not surprisingly — the one-particle GF is independent of the exchange part of the universal Hamiltonian (1). Such a dependence would emerge only very close to the Stoner instability, when $|\Delta - J|/J \ll 1$ but this parametric region is beyond the scope of the presented technique.

V. TUNNELING DENSITY OF STATES

The tunneling density of states (TDoS), $\nu(\varepsilon)$, can be directly related to the conductance of the QD in the limit of weak coupling to the leads and is thus a quantity of great importance. The TDoS is given by $\nu(\varepsilon) = -\frac{1}{\pi} \text{Im} \mathcal{G}^R(\varepsilon)$, where the retarded GF, $G^R(\varepsilon)$, is a Fourier transform of the GF in real time, $G(t, \mu)$, obtained from Eq. (22) by the straightforward analytical continuation from the upper half-plane. Since $\mathcal{G}(\tau, \mu)$ is independent of the exchange energy under the conditions (2) and Eq. (21), so is the TDoS $\nu(\varepsilon)$.

For tutorial purposes, we use the results of Appendices A and B to derive a more general expression for $\nu(\varepsilon)$, valid for any relation between the parameters in Eqs. (2) and (21) and show how it goes over to the known expression under conditions (2) and (21).
Using the GF in the \( \epsilon \)-representation, Eq. (27), and performing the summation over all the levels as described at the end of Appendix 13 we find

\[
\frac{\nu(\epsilon)}{\nu_0} = \frac{1}{Z} \sum_{N} \sum_{M=-N}^{N} e^{-\beta E_c(N-\tilde{N}_0)^2 - \frac{1}{2} \beta (\Delta - J) M^2} \times \\
\left[ 1 - n(\epsilon - \tilde{\mu} - \xi_{N,M}) + n(\epsilon - \tilde{\mu} - \xi_{N-1,M-1}) \right],
\]

(23)

where we have defined

\[
\xi_{N,M} \equiv 2E_c(N - \tilde{N}_0 + \frac{1}{2}) - \frac{1}{2} J(M + \frac{1}{2}),
\]

(24)

and

\[
\tilde{\mu} \equiv \frac{1}{2} \Delta(N + M), \quad \tilde{N}_0 \equiv N_0 + \frac{\mu}{2E_c},
\]

(25)

while \( \nu_0 = 2/\Delta \) is the TDoS in the absence of interactions, \( n(\epsilon) \equiv [1 + e^{\beta \epsilon}]^{-1} \).

Equation (23) is the general expression for the TDoS for any combination of parameters for a many-electron dot. When the inequalities (21) and (24) are satisfied, we can easily sum over \( M \) as described at the end of the previous section and then limit the summation over \( N \) to the two terms for which the value of \( |N - N_0| \) is minimal (although deep in the Coulomb valley only one term is actually contributing). The resulting TDoS is independent of \( J \) (or, more precisely, tiny \( J \)-dependent corrections are beyond the accuracy of current calculations and thus omitted) and coincides with that obtained in Ref. 13.

\[
\frac{\nu(\epsilon)}{\nu_0} = U(\epsilon - \xi_N) + e^{-\beta (\xi_N - \tilde{\mu})} U(\epsilon - \xi_{N+1}) \frac{1 + e^{-\beta (\xi_N - \tilde{\mu})}}{1 + e^{-\beta (\xi_N - \tilde{\mu})}},
\]

(26)

where \( U(\epsilon - \xi_N) \equiv n(\epsilon - \xi_{N-1} - \tilde{\mu}) + 1 - n(\epsilon - \xi_N - \tilde{\mu}) \), and \( \xi_N \) is obtained from \( \xi_{N,M} \) by putting \( J = 0 \) in Eq. (25).

We illustrate the dependence of \( \nu \) on energy for integer, half-integer and intermediate values of \( \tilde{N}_0 \) in Fig. 1 for a specific choice of parameter values \( T \) and \( \Delta \). Its dependence on temperature at the bottom of a Coulomb blockade valley is depicted in Fig. 2. It is important to note that the TDoS obtained in the Coulomb valleys is not physical since we neglect co-tunneling contributions; however, the \( T \)-dependence near the peak will be obtained as a linear combinations of those shown in Fig. 2.

Note that for any given set of parameters the center of the TDoS curve is at \( \epsilon_0 = \frac{\Delta}{2} \tilde{N}_0 - 2E_c \left( N - \tilde{N}_0 \right) \) and thus a function of \( \tilde{N}_0 \), Eq. (23). This moving from one Coulomb valley to the next, the TDoS curve is shifted by \( \Delta/2 \) due to adding an extra electron to the dot, which raises the effective chemical potential and thus shift the TDoS curve. That is the reason for the ‘half-gap’ in TDoS at the degeneracy point.

**VI. MAGNETIC SUSCEPTIBILITY**

We now turn to calculating the longitudinal and transverse magnetic susceptibilities of the system.

![Diagram](attachment:image.png)

**FIG. 1.** TDoS (in units of \( \nu_0 \)) as a function of \( \epsilon \equiv \epsilon/E_c \) for \( T = 0.2E_c \) and \( \Delta/T = 0.1 \) in (a) a CB valley (\( \tilde{N}_0 = 100 \)), (b) an intermediate region (\( \tilde{N}_0 = 100.35 \)), (c) a CB peak (\( \tilde{N}_0 = 100.5 \)).

It is clear that only the static component of the longitudinal susceptibility is non-zero due to the lack of spin flip processes in the Ising model. A direct calculation of the correlation function \( \langle S_z(\tau)S_z(0) \rangle \) shows this to be \( \tau \)-independent, as expected. The static susceptibility is given by

\[
\chi_{zz} = \frac{1}{\beta} \lim_{h \to 0} \frac{d^2}{dh^2} \ln \mathcal{Z}(h),
\]

(27)

where \( \mathcal{Z}(h) \) is the partition function of the system calculated in the presence of the following source term in the
The final outcome of this calculation is follows to that of the calculation of the GF described in Section IV. The calculation is straightforward, leading to the result

$$\mathcal{Z}(h) = \kappa \exp \left\{ \frac{\beta^2 h^2}{4\beta(\Delta - J)} \right\},$$

with $\kappa$ being some irrelevant constant. Plugging this into the definition (27) yields the well known expression

$$\chi_{zz}(\omega = 0) = \frac{1}{2} \frac{1}{\Delta - J}.$$

As expected, the static susceptibility is independent of the number of particles on the dot, external gate voltage, charging effects, etc.

We now turn to a calculation of the transverse magnetic susceptibility. This quantity is inherently different from the longitudinal one since it is dynamic: the model allows for transitions between different transverse spin polarization states.

We define the dynamic transverse susceptibility in imaginary time as

$$\frac{1}{\beta} \chi^+-(\tau) = \langle \sigma^+(0)\sigma^- (\tau) \rangle,$$

where $\sigma^+ = \sum_{\alpha} \overline{\Psi}_{\alpha\uparrow} \Psi_{\alpha\downarrow}$ and $\sigma^- = \sum_{\alpha} \overline{\Psi}_{\alpha\downarrow} \Psi_{\alpha\uparrow}$. Thus we need to calculate the functional average of

$$\sum_{\alpha,\beta} \overline{\Psi}_{\alpha\uparrow}(0) \Psi_{\alpha\downarrow}(0) \overline{\Psi}_{\beta\downarrow}(\tau) \Psi_{\beta\uparrow}(\tau),$$

with the action given by Eq. (3). The procedure closely follows that of the calculation of the GF described in Section IV. The final outcome of this calculation is

$$\chi^+-(\tau) = \frac{\beta e^{i\tau}}{Z(\mu)} \sum_N e^{-\beta E_c(N-\tilde{N}_0)^2} \sum_{M=-N}^{N} \left\{ e^{-\frac{1}{2}\beta(\Delta-J)M^2} \times e^{i\tau M} \sum_{\alpha} [1 - n_\alpha(\tilde{\mu}_\uparrow)] n_\alpha(\tilde{\mu}_\downarrow) \right\},$$

where $\tilde{\mu}_\sigma \equiv N_\sigma \Delta$ and $N_\sigma$ is the total number of electrons with the spin projection $\sigma = \uparrow, \downarrow$.

Fourier-transforming the result of Eq. (31) to Matsubara frequencies and then performing a simple analytic continuation, we find the imaginary part of the physical response function $\chi^+-(\omega)$:

$$\text{Im}\chi^+-(\omega) = \frac{\sqrt{\pi} \beta(\Delta-J)}{2J} e^{\frac{\pi}{2\beta} \frac{(\Delta+J)-(\Delta-J)^2}{\Delta}} \times \left(1 + \frac{\omega}{J} \right) \frac{\sinh \left[ \frac{\beta \omega}{2J} \right]}{\sinh \left[ \frac{\beta \Delta}{2J} (1 + \frac{\omega}{J}) \right]}$$

This function is depicted in Fig. 3. The most salient features are a linear dependence at the origin and the existence of a peak at a certain $\omega_0$. Both the slope at the origin and the value of $\omega_0$ can be used to characterize an experimentally obtained curve of the transverse magnetic susceptibility as a function of frequency. We
The frequency is given by inequality (2). Under the same condition, the peak frequency is

\[
\omega_0 \approx \sqrt{\frac{2J^2}{\beta(\Delta - J)}}.
\]

Yet another parameter of interest is the full-width-at-half-maximum (FWHM). Numerical analysis shows that it is proportional to the resonance frequency: \(FWHM \approx 1.59\omega_0\). This result was derived by numerically obtaining the FWHM for various values of \(\omega_0\) and fitting the results to a linear curve, as shown in Fig. 4.

The imaginary part of the susceptibility represents the systems capacity to absorb and dissipate magnetic energy at a nonzero frequency. For the static susceptibility only a real part is finite. A simple calculation leads to

\[
\frac{1}{\beta} \text{Im} \chi^{+-}(\omega \to 0) \approx \frac{\omega}{2J} \sqrt{\frac{\pi}{\beta\Delta}},
\]

where the approximation was made consistent with the inequality (2). Under the same condition, the peak frequency is given by

\[
\omega_0 \approx \sqrt{\frac{2J^2}{\beta(\Delta - J)}}.
\]

Yet another parameter of interest is the full-width-at-half-maximum (FWHM). Numerical analysis shows that it is proportional to the resonance frequency: \(FWHM \approx 1.59\omega_0\). This result was derived by numerically obtaining the FWHM for various values of \(\omega_0\) and fitting the results to a linear curve, as shown in Fig. 4.

The imaginary part of the susceptibility represents the systems capacity to absorb and dissipate magnetic energy at a nonzero frequency. For the static susceptibility only a real part is finite. A simple calculation leads to

\[
\text{Re} \chi^{+-}(\omega = 0) = \frac{1}{\Delta} e^{\Delta(J+J)} \approx \frac{1}{\Delta}.
\]

Note that in the limit \(J = 0\) we recover the well known identity \(\chi^{+-} = 2\chi_{zz}\) for the static susceptibilities. The real part of \(\chi^{+-}\) at finite frequencies can be found either directly or via the Kramers-Kronig relations but we do not present the result here as it has little physical relevance.

As in the case of the longitudinal magnetic susceptibility, it is clear that the transverse susceptibility is not affected by the charging interaction in the dot. Once again we see that under conditions (2) and (21) the charge and spin degrees of freedom are effectively decoupled.

VII. SUMMARY

The main results of this work fall into three basic categories. These are the single particle GF, the TDoS, and the magnetic susceptibilities. The results for all three classes of correlation functions were obtained by means of the functional bosonization approach combined with the solution of classical stochastic equations for the bosonic fields. We considered the Ising version of the Universal Hamiltonian for description of the interplay between the spin and charge degrees of freedom in zero-dimensional systems. Such model is Abelian and therefore does not include the physics of non-commutative variables. It also does not exhibit the mesoscopic Stoner instability regime. Nevertheless, the spin-charge entanglement is present being manifested in e.g. non-trivial AC spin susceptibility. The model, being a simplified version of the quantum Universal Hamiltonian model gives qualitatively correct description of the thermodynamics and transport through nanostructures in the vicinity of thermodynamic Stoner Instability point. The Stochastic Bosonization appears to be very powerful tool for a treatment of Abelian gauge theories and a promising method for solving non-Abelian models corresponding to isotropic/anysotropic quantum limits of the Universal Hamiltonian. The theory of thermodynamic Stoner Instability and its influence on the transport through single electron transistor can be tested experimentally in quantum dot devices and granular systems.

We summarize below the central results and key observations reported in the paper.

- **Canonical variables and charge quantization.** In our calculation of the GF, the tools we used and the choices made not only allowed us to carry out a non-perturbative calculation, but also had physical significance. The use of functional bosonization and generalized gauge transformations and the implementation of winding numbers, as well as the transformation to conjugated variables via the Poisson re-summation, led us to employ canonical quantities. The latter is a consequence of strong charging interaction.

- **Regimes of validity.** The transition to canonical quantities, namely the introduction of the canonical partition function, also led to further insight with regard to the various physical regimes the system may be found in. Our calculation of the canonical partition function itself (and the associated quantity \(Z_X(\bar{f}_a)\)) imposed limitations on the physical parameters involved. We found that the system must be large enough (meaning a large number of electrons), and far below the Stoner instability point. We had to self consistently assume that the fluctuations in the systems magnetization were much smaller than the system size. This corresponds to a requirement that the system be far...
from a phase transition point, which in our case is the SI point.

- **Spin-charge entanglement.** Introduction of the canonical partition functions led directly to a summation over all possible values of the magnetization. These are of course limited to |M| < N. Since the number of particles itself is controlled by the charging interaction when in the CB regime, and the fluctuations of the magnetization are influenced by the exchange interaction, this can be seen as a form of coupling between the charge and spin degrees of freedom. The coupling between the two interaction channels becomes important as the magnitude of magnetization fluctuations increases, i.e. as one approaches the SI point. Only then do values of M which approach the system size become accessible and, consequently, of physical importance. Far below the SI point, the spin-charge coupling is very weak, and effects of interplay are minimal. Our calculation of the TDoS showed the exchange interaction to have an extremely negligible effect. The magnetic susceptibilities in turn showed no dependence on the charging interaction.

- **Determining J and ∆.** The calculation of the transverse magnetic susceptibility is, to our knowledge, a new result, and perhaps the most important in this work. As we have discussed previously, the importance of this result is that it provides an experimental method to determine the values of the parameters J and ∆. Our result is a direct prediction of the absorption spectrum of the system, and as such should be amenable to experimental measurement. The various curve characteristics which we derived, including the slope at ω → 0, the location of the resonance frequency and the FWHM, should in principal, through their dependence on J and ∆, allow these values to be ascertained from such a measurement.

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**Appendix A: Grand Canonical Single Particle Green’s Function**

In this appendix we present a detailed non-perturbative calculation of the single particle Green’s Function (GF) for our model system (1). The GF itself was used in order to derive the tunneling density of states (TDoS), but its calculation also serves to show the methodology used in calculating the various other quantities considered in this work.

As discussed in section II, a HS transformation is applied, reducing the action to the form presented in Eqs. (1) and (3). Carrying out the Gaussian integration over the fermionic fields after the gauge transformation (10), we obtain the GF as follows:

\[
\mathcal{G}_{\alpha,\sigma}(\tau,\mu) = \Pi^c(\tau)\Pi^a(\tau) \times \\
\int_{-\infty}^{\infty} d\varphi_c^c e^{-S_0([\varphi_c^c])} \int_{-\infty}^{\infty} d\varphi_s^a e^{-S_0([\varphi_s^a])} \int_{-\infty}^{\infty} d\varphi_c^s e^{-S_0([\varphi_c^s])} \mathcal{Z}^0(\mu) \\
\int_{-\infty}^{\infty} d\varphi_s^s e^{-S_0([\varphi_s^s])} \mathcal{Z}^0(\bar{\mu})
\]

(A1)

Here \( \mathcal{Z}^0(\mu) = \mathcal{Z}^0(\bar{\mu}_1)\mathcal{Z}^0(\mu_1) \) and \( \mathcal{S}^0_{\alpha,\sigma}(\tau,\bar{\mu}) \) are the grand canonical partition function and GF of non-interacting electrons with the spin-dependent chemical potential \( \bar{\mu}_\sigma \), defined by Eq. (12). Both \( \mathcal{Z}^0 \) and \( \mathcal{S}^0 \) are functions of the zero-Matsubara components \( \varphi_0^c \) and \( \varphi_0^s \) of the bosonic fields, over which the integration in Eq. (A1) is carried out with

\[
S_0^c = \frac{\beta|\varphi_0^c|^2}{4E_c} - i\beta N_0 \varphi_0^c \quad S_0^s = \frac{\beta|\varphi_0^s|^2}{J}.
\]

(A2)

The functional integration over the remaining components of the bosonic fields results in the appearance of the phase correlation functions:

\[
\Pi^c(\tau) = \left\langle e^{i\varphi^c(\tau) - \varphi^c(\tau)} \right\rangle_{\varphi^c} \\
\Pi^a(\tau) = \left\langle e^{i\varphi^a(\tau) - \varphi^a(\tau)} \right\rangle_{\varphi^a}
\]

(A3)

The functional averaging above is carried out with the weights exp[−\( \mathcal{S}^{c,s} \)], where \( \mathcal{S}^{c,s} \) are obtained from the appropriate bosonic action in Eq. (3) by subtracting the zeroth Matsubara components of Eq. (A2).

The calculation of the the correlation functions of Eq. (A3) is carried out in Appendix C using the tools of stochastic analysis. The results are:

\[
\Pi^c(\tau) = e^{-E_c} \left( |\tau| - \frac{\pi^2}{\beta} \right)^{2\pi i N} \mathcal{Z}^0 \equiv \Pi^c(\tau) e^{2\pi i N} \\
\Pi^a(\tau) = e^{-\frac{1}{4} \left( |\tau| - \frac{\pi^2}{\beta} \right)^{2\pi i N} \mathcal{Z}^{s}} \equiv \Pi^a(\tau) e^{2\pi i N}
\]

(A4)

At this point we introduce the winding numbers, as discussed in section (A1) of the main text. Following the transition \( \varphi^c = \omega_m + \frac{\pi^2}{\beta} \), and utilizing the identities \( \mathcal{Z}^0(\mu -
\[ i\omega_m = 2^0(\mu) \text{ and } G^0(\tau, \mu - i\omega_m) = e^{-i\omega_m\tau}g^0(\tau, \mu), \] we end up with
\[ G_{\alpha,\sigma}(\tau, \mu) = \frac{\tilde{\pi}(\tau)\tilde{\pi}(\tau)}{Z(\mu)} \sum_m e^{2\pi i(N_0 - \frac{m}{\beta})} e^{-\frac{x^2\mu^2}{2\beta}}. \]
\[ \mathcal{J}_m \equiv \int d\tilde{\varphi}_0 e^{-\frac{(\tilde{\varphi}_0)^2}{2\beta} + \tilde{\varphi}_0(N_0 - \frac{m}{\beta})} 2^0(\tilde{\mu})G_{\alpha,\sigma}. \]

The grand partition function \( \tilde{Z}(\mu) \) above is represented by the same double-integral and sum with \( g^0 \) replaced by 1. The exponential factors involving \( N \) and \( n^s \) arising from the phase correlation functions and the non-interacting GF cancel each other out exactly. This is hardly surprising as they are completely arbitrary.

The summation over the winding numbers above can be performed using the Poisson formula
\[ \sum_{k=-\infty}^\infty f(2\pi k) = \frac{1}{2\pi} \sum_{m=-\infty}^\infty \int_{-\infty}^\infty e^{inx} f(x) dx. \]

This results in the expression for GF given by Eqs. 17 and 18 in the main text.

Appendix B: Calculations in AuxiliaryCanonical Ensemble

We express \( Z^0(\mu) \) in Eqs. 17 and 18 via the sum of the canonical partition functions for a system of \( n \) non-interacting electrons, \( Z_n^0 \), using the standard relation
\[ Z^0(\mu) = \prod_\alpha \left[ 1 + e^{-\beta(\varepsilon_\alpha - \mu)} \right] = \sum_n e^{\beta\mu n}Z_n^0, \]
and the canonical partition functions with one level, \( \varepsilon_\alpha \), excluded:
\[ Z^0(\tilde{f}_\alpha, \mu) = \prod_{\alpha \neq \alpha} \left[ 1 + e^{-\beta(\varepsilon_\alpha - \mu)} \right] = \sum_n e^{\beta\mu n}Z_n^0(\tilde{f}_\alpha). \]

Then we substitute into Eq. 18 the finite temperature GF of non-interacting fermions
\[ G^0_{\alpha,\sigma}(\tau > 0, \mu) = e^{-\beta(\varepsilon_\alpha - \mu)^2} \left( 1 - n_{\alpha, \sigma}(\mu) \right), \]
where \( n_{\alpha, \sigma}(\mu) \) is the Fermi-Dirac occupation factor. We limit the calculation to \( G(\tau > 0) \), since \( G(\tau) = -G(\tau + \beta) \). Recalling that \( Z^0(\tilde{\mu}) = 2^0_1(\tilde{\mu})\tilde{Z}^0_0(\tilde{\mu}) \) we cast Eq. 18 into the form
\[ J_N = \int_{-\infty}^\infty d\tilde{\varphi}_0 e^{-\frac{(\tilde{\varphi}_0)^2}{2\beta}} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{iN\tilde{\varphi}_0} e^{-\frac{(\varepsilon_\alpha - \mu + \sigma\varphi)^2}{2\beta}} \]
\[ \times \sum_{m,n} e^{\beta\mu(m+n) - \varphi_0(m+n) - \sigma(m-n)\varphi_0} \sum_{\sigma,m}(\tilde{f}_\alpha)Z_{\sigma,m}. \]

Carrying out the integration over \( \tilde{\varphi}_0 \) yields a Kronecker delta \( \delta_{N,n+m} \). Performing the Gaussian integration over \( \tilde{\varphi}_0 \) and defining \( M = m - n \) we find
\[ J_N = \sum_{M=-N}^N e^{\beta\mu N + \frac{1}{2}\beta J_0(M^2/\beta) - (\varepsilon_\alpha - \mu)^2} Z_{N+M}(\tilde{f}_\alpha)Z_{N-M}. \]

Substituting this into Eq. 17 yields after straightforward algebraic manipulations
\[ G_{\alpha,\sigma}(\tau > 0, \mu) = e^{-\beta\Delta T^2} \]
\[ Z_N(\tilde{f}_\alpha) = [1 - n_\alpha(\tilde{\mu})] Z_N, \]
where the Fermi factor for the \( \alpha^{th} \) level, \( n_\alpha(\tilde{\mu}) \equiv \frac{1}{1 + e^{\beta(\varepsilon_\alpha - \mu)}} \), is taken with the auxiliary chemical potential \( \tilde{\mu} \equiv N\Delta \).

Substituting Eq. 25 into Eq. 24, we find:
\[ G_\alpha(\tau > 0, \mu) = \frac{1}{Z} \sum_{N} e^{-\beta E_N(N - \tilde{\mu})^2} \] 
\[ \sum_{M=-N}^N e^{-\frac{1}{2}\beta(\Delta - J)M^2} \left[ 1 - n_\alpha(\tilde{\mu}) \right] e^{-\xi_\alpha(N,M)^2} \]
where \( \mu \) and \( \tilde{N}_0 \) are defined in Eq. 25 and the auxiliary partition function \( 	ilde{Z} = F(0) \) is given by Eq. 26 in the main text. The result is naturally spin-independent. Technically, the formal spin dependence vanished when calculating the integral \( J_N \), Eq. 18. The GF for negative \( \tau \) can be obtained from Eq. 18 using \( G(\tau) = \) \( G(-\tau) \).

Now we find the full GF by summing over all single particle states \( \varepsilon_\alpha \). This summation is carried out in the usual way by making the substitution \( \sum_\alpha G_\alpha \rightarrow \Delta^{-1} \int_0^\infty G(\varepsilon_\alpha) d\varepsilon_\alpha \), i.e. effectively by averaging over disorder by introducing the mean level spacing \( \Delta \). This leads to Eq. 19 in the main text.

Finally, we write the GF in the energy representation. Making the standard analytical continuation to the real time, \( \tau \rightarrow \tau \), and Fourier transforming the GF to the energy domain we obtain the retarded GF used in the calculation of the TDoS as follows:
\[ G_\alpha^R(\varepsilon) = \frac{1}{Z} \sum_{N} \sum_{M=-N}^N e^{-\beta E_N(N - \tilde{\mu})^2 - \frac{1}{2}\beta(\Delta - J)M^2} \]
\[ \times \left[ \frac{1 - n_\alpha(\tilde{\mu})}{\varepsilon - \varepsilon_\alpha - \xi_{N,M} + \tilde{i}0} + \frac{n_\alpha(\tilde{\mu})}{\varepsilon - \varepsilon_\alpha - \xi_{N-1,M-1} + \tilde{i}0} \right]. \]
Appendix C: Phase Correlation Functions and Stochastic Analysis

Here we use stochastic analysis to calculate the phase correlation function $\Pi^\ast$ defined in Eq. \((A3)\). $\Pi^\ast$ has been calculated in exactly the same manner.

We note that the gauge equation \((10a)\) can be viewed as a Langevin equation wherein the field $\varphi^\ast(\tau)$ plays the role of the stochastic force (noise), the distribution of which is governed by the action $S_{\varphi^\ast}$ obtained from the appropriate bosonic action in Eq. \((5)\) by subtracting the zeroth Matsubara components of Eq. \((A2)\). The noise correlation function is given by

$$\langle \varphi^\ast(\tau) \varphi^\ast(\tau') \rangle = 2E_c \left[ \delta (\tau - \tau') - \frac{1}{\beta} \right], \quad \text{(C1)}$$

which follows from the expansion of $\varphi^\ast(\tau)$ in terms of Matsubara components: $\varphi^\ast(\tau) = \sum_{m \neq 0} \varphi^\ast_m e^{-i\omega_m \tau}$. Indeed, the functional distribution of $\varphi^\ast_m$ is

$$\int \mathcal{D}[\varphi^\ast(\tau)] e^{-\int_0^\beta d\tau \{ \varphi^\ast(\tau) \mid E(\tau) \}^{-1} \varphi^\ast(\tau)} = \prod_{m \neq 0} \varphi^\ast_m \sum_{m, n \neq 0} \varphi^\ast_m [\delta_{m, n} - \omega_m \omega_n] \varphi^\ast_n, \quad \text{(C2)}$$

which corresponds to $\langle \varphi^\ast_m \varphi^\ast_n \rangle = 2E_c \delta_{m, n} - 1$, immediately leading to Eq. \((C1)\).

It is convenient to represent the noise field as $\varphi^\ast(\tau) = \eta(\tau) + i\zeta$ with $\eta(\tau)$ a random function and $\zeta$ a Gaussian random variable satisfying $\langle \eta(\tau) \rangle = 0$, $\langle \hat{\eta}(\tau) \hat{\eta}(\tau') \rangle = 2E_c \delta (\tau - \tau')$ and $\langle \zeta^2 \rangle = 2E_c / \beta$. As $\eta(\tau)$ is standard white noise, we follow the standard procedure to map the Langevin equation \((10a)\) to a Fokker-Planck (FP) equation:

$$\frac{\partial^2 \varphi}{\partial \theta^2} = \left( \frac{2\pi}{\beta} N - i\zeta \right) \frac{\partial \varphi}{\partial \theta} + E_c \frac{\partial^2 \varphi}{\partial \zeta^2}. \quad \text{(C3)}$$

Here $\varphi^\ast(\theta, \tau; \theta', \tau')$ is the conditional transition probability function for a given $\zeta$, formally defined by

$$\varphi^\ast(\theta, \tau; \theta', \tau') = \left\langle \delta [\theta(\tau) - \theta] \delta [\theta'(\tau') - \theta'] \right\rangle_{\eta, \zeta}, \quad \text{(C4)}$$

i.e. $\varphi^\ast = \varphi^\ast(\zeta)$. Equation \((C3)\) is a standard diffusion equation with a drift term. Its solution, with the natural boundary condition $\varphi^\ast(\theta, \tau; \theta', \tau')|_{\zeta = 0} = \delta(\theta - \theta')$, is a decaying Gaussian:

$$\varphi^\ast(\theta, \tau; \theta', \tau')|_{\zeta = 0} = \exp \left\{ - \frac{[(\theta - \theta')^2 + (N - i\zeta)(\tau - \tau')]^2}{4E_c (\tau - \tau')} \right\}. \quad \text{(C5)}$$

Now we write $\Pi^\ast(\tau)$, defined in Eq. \((A4)\), in terms of the transition probability function \((A4)\):

$$\left\langle e^{i[\theta(\tau) - \theta'(\tau)]} \right\rangle_{\zeta} = \int_{-\infty}^{\infty} d\theta d\theta' \Pi^\ast(\theta, \tau; \theta', \tau') e^{-i(\theta - \theta')}.$$

Substituting here the solution \((C5)\), we find the conditional (for a given $\zeta$) phase correlation function as

$$\Pi^\ast_\zeta(\tau) = e^{-E_c \tau} e^{-\zeta \tau} e^{-i2\pi N^2 \tau}, \quad \text{(C6)}$$

where we defined $\tau = |\tau_f - \tau_i|$. Finally, the averaging over the quenched random variable $\zeta$ results in the first of Eqs. \((A4)\). The second one, for $\Pi^\ast(\tau)$, has obtained by applying, step by step, exactly the same procedure.

Appendix D: Calculation of the Canonical Partition Function

In this appendix we evaluate the canonical partition functions $Z_N$ and $Z_N(\epsilon_f)$ defined in Eqs. \((B1)\) and \((B2)\). It follows from Eq. \((B2)\) that

$$Z_N(\epsilon_f) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{N \phi} Z_0^{\phi}(\mu = -i\epsilon_f / \beta) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{N \phi} \prod_{\alpha' \neq \alpha} \left( 1 + e^{-\beta \epsilon_{\alpha'} - i\epsilon_f} \right). \quad \text{(D1)}$$

We calculate $Z_N(\epsilon_f)$ (and thus $Z_N$) in the saddle-point approximation:

$$Z_N(\epsilon_f) \approx e^{-S_\alpha(\epsilon_f)} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{-\frac{1}{4}[S_\alpha'(\epsilon_f)]^2} \text{e}^{-\frac{1}{2}[\epsilon_f - \epsilon_f(0)]^2}, \quad \text{(D2)}$$

where

$$S_\alpha(\epsilon_f) = -iN\epsilon_f - \ln \prod_{\alpha' \neq \alpha} \left( 1 + e^{-\beta \epsilon_{\alpha'} - i\epsilon_f} \right). \quad \text{(D3)}$$

The saddle-point equation, $S'_\alpha = 0$, is convenient to write by replacing $\sum_{\alpha'} f(\epsilon_{\alpha'})$ with $\Delta^{-1} \int_0^\infty d\epsilon f(\epsilon)$ as in Appendix \((A2)\). This gives, after calculating the integral, the following equation for finding $\epsilon_f^0$:

$$N + \frac{1}{1 + e^{\beta \epsilon_f^0 + i\epsilon_f^0}} = \frac{1}{\beta \Delta} \ln \left( 1 + e^{-i\epsilon_f^0} \right). \quad \text{(D4)}$$

The Fermi-factor there, being of order 1, can be neglected, which means that the same saddle-point we would find in a calculation of $Z_N$ for large enough $N$ the saddle-point is unaltered by the exclusion of a single state. Assuming also that $N$ is so large that $\beta N \Delta \gg 1$, we find from Eq. \((D4)\):

$$-i\epsilon_f^0 = \beta N \Delta \equiv \beta \bar{\mu}. \quad \text{(D5)}$$
In the same approximation \( S_\alpha(\varphi^c_\alpha) = 1/(\beta \Delta) \), so that calculating the Gaussian integral in Eq. \( \text{(D2)} \) gives

\[
Z_N = e^{-\frac{1}{2} \beta \mu N}, \quad (\text{D6})
\]

while \( Z_N(\not{\epsilon}\alpha) \) differs only by the exclusion of the level \( \alpha \):

\[
Z_N(\not{\epsilon}\alpha) = [1 - n_\alpha(\bar{\mu})] Z_N. \quad (\text{D7})
\]

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