Elementary methods in the study of Deuring-Heilbronn Phenomenon

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Abstract. The aim of this work is to improve some elementary results regarding both the Deuring-Phenomenon and the Heilbronn-Phenomenon. We will give better estimates regarding both the influence of zeros of the Riemann zeta function on the exceptional zeros and that of the non-trivial zeros of arbitrary $L$-functions belonging to non-principal characters on the exceptional zeros.

1 Introduction

Let $L(s, \chi_D)$ be a Dirichlet $L$-function belonging to the real primitive character $\chi_D$ modulus $D$ satisfying $\chi_D(-1) = -1$. Let $h(-D)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$.

Two conjectures involving the class number $h(-D)$ of the imaginary quadratic field belonging to the fundamental discriminant $-D < 0$ were raised by Gauss, who published them in 1801 [6]. The first problem was about determining all the negative fundamental discriminants with class number one. The second problem was about proving that $h(-D) \to \infty$, as $D \to \infty$.

Regarding the second conjecture, in 1913, Gronwall [9] proved that if the function $L(s, \chi_D)$ belonging to the real primitive character $\chi_D(n) = (-D/n)$ has no zero in the interval $[1 - \frac{\alpha}{\log D}, 1]$, then $h(-D) > \frac{b(\alpha)\sqrt{\log D}}{\log D \sqrt{\log \log D}}$, where $\alpha$ is a constant and $b(\alpha)$ is a constant depending only on $\alpha$.

In 1918, Hecke [13] proved that, under the same hypotheses of Gronwall’s theorem, the inequality $h(-D) > \frac{b'(\alpha)\sqrt{\log D}}{\log D}$ holds, where $\alpha$ is a constant and $b'(\alpha)$ is a constant depending only on $\alpha$.

In 1933, Deuring [4] proved that under the assumption of the falsity of the classical Riemann Hypothesis the relation $h(-D) \geq 2$ holds for $D > D_0$, where $D_0$ is a constant. In 1934, Mordell [17] improved the result found by Deuring. Under the assumption of the falsity of the classical Riemann Hypothesis, Mordell proved that $h(-D) \to \infty$ as $D \to \infty$. These results showed an interesting connection between the possibly existing real zeros of special $L$-functions and the non-trivial zeros of the $\zeta$-function.

Better results regarding the influence of zeros of $\zeta(s)$ on the exceptional zeros, or equivalently, the Deuring phenomenon, were provided by the work of Pintz, who used a new approach involving some elementary methods. In 1976, Pintz [23] proved that, assuming a relatively strong upper bound for $h(-D)$, it is possible to determine, up to a factor $1 + o(1)$, the values of the corresponding $L$-function in a large domain of the critical strip.

Theorem. (Pintz) Given $0 < \varepsilon < 1/8$ and $D > D_1(\varepsilon)$, where $D_1(\varepsilon)$ is an effective constant depending on $\varepsilon$, we define the domain $H(\varepsilon, D)$, depending on $\varepsilon$ and on $D$, as the set

$$H(\varepsilon, D) = \{ s \in \mathbb{C}; |s| = 1 - \tau + it, |1 - s| \geq 1/\log^4 D, 0 \leq \tau \leq \frac{1}{2} - \varepsilon \}$$

where $\varrho = \max \{ \tau, D^{-\varepsilon/4} \}$. 

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If the inequality
\[ h(-D) \leq (\log D)^{3/4} \]
holds, then neither \( L(s, \chi_D) \) nor \( \zeta(s) \) has a zero in \( H(\varepsilon, D) \), and for \( s \in H(\varepsilon, D) \), we have
\[ L(s, \chi_D) = \frac{\zeta(2s)}{\zeta(s)} \prod_{p|D} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{8} (\log D)^{1/4} \right\} \right) \right] \]

An immediate consequence is that, except for the possible Siegel zero, neither \( L(s, \chi_D) \) nor \( \zeta(s) \) has a zero in this domain. Also, a weakened form of Mordell’s theorem follows, namely that if \( h(-D) \not\rightarrow \infty \) for \( D \rightarrow \infty \), then \( \zeta(s) \) has no zero in the half-plane \( \sigma > \frac{3}{4} \).

In 1984, Puglisi [24] made some improvements, extending further the domain of the critical strip in which it is possible to determine, up to a factor \( 1 + o(1) \), the values of the corresponding \( L \)-function.

**Theorem. (Puglisi)** Let \( \alpha, \lambda > 0 \) be real numbers with \( \alpha + \lambda < 1 \). Given
\[ \ell = (\log D)^{-\lambda}, \]
we define the following set
\[ H(\ell, D) = \left\{ s = \sigma + it : |1 - s| \geq (\log D)^{-\alpha}, 1/2 + \ell \leq \sigma \leq 1, |s| \leq D^{\ell/10} \right\} \]
If
\[ h(-D) \leq (\log D)^{\alpha} \]
then for each \( s \in H(\ell, D) \) the relation
\[ L(s, \chi_D) = \frac{\zeta(2s)}{\zeta(s)} \prod_{p|D} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3} (\log D)^{1-\alpha-\lambda} \right\} \right) \right] \]
holds.

An immediate consequence of Puglisi’s improvement is a reformulation of Mordell’s Theorem, that is, if \( \zeta(\beta + i\gamma) = 0 \) with \( \beta > \frac{3}{4} \), then, for every \( \varepsilon > 0 \), the relation \( h(-D) > (\log D)^{1-\varepsilon} \) holds, provided that \( D > D_0(\beta, \gamma, \varepsilon) \).

In 1934, Heilbronn [11] solved Gauss’ second conjecture. He proved that, under the assumption that the general Riemann Hypothesis is not true, \( h(-D) \rightarrow \infty \) if \( D \rightarrow \infty \). Heilbronn’s result is very important, as, combined with Hecke’s theorem, gives, without any assumption, that \( h(-D) \rightarrow \infty \) if \( D \rightarrow \infty \).

In 1935, Siegel [25] proved that \( h(-D) > D^{1/2-\varepsilon} \) for \( D > D_0(\varepsilon) \) for an arbitrary \( \varepsilon > 0 \), and with a constant \( D_0(\varepsilon) \) depending only on \( \varepsilon \), where the constant \( D_0(\varepsilon) \) is ineffective (for alternative proofs of Siegel’s Theorem see Estermann [5], Chowla [2], Goldfeld [7], Linnik [16], Pintz [19]).

Heilbronn played a fundamental role also in the attempt to prove Gauss’ first conjecture. In 1934, Heilbronn and Linfoot [12] showed that, except for the known values \(-D = -3, -4, -7, -8, -11, -19, -43, -67, -163\), there is at most a tenth negative fundamental discriminant with class number one.

In 1935 Landau [14] proved that if \( h(-D) = h \), then the inequality \( D \leq D(h) = Ch^6 \log^6(3h) \) holds, where \( C \) is an absolute effective constant, with the possible exception of at most one negative fundamental discriminant.

In 1950 Tatzuwa [27] proved Landau’s theorem mentioned above with \( D(h) = Ch^2 \log^2(13h) \). Furthermore, Tatzuwa made some improvements regarding the effectivization of Siegel’s Theorem, showing that if \( h(-D) \leq D^{1/2-\varepsilon} \), then the inequality \( D \leq D_0(\varepsilon) = \max \{ e^{12}, e^{1/\varepsilon} \} \) holds, with the possible exception of at most one negative fundamental discriminant.

Finally, in 1966-1967, Baker [1] and Stark [20] proved independently that there is no tenth imaginary quadratic field with class number one.
The results found by Deuring [4] and Heilbronn [11] regarding the influence of the non-trivial zeros of both \( \zeta(s) \) and \( L(s, \chi) \) (where \( \chi \) is an arbitrary real or complex character) on the real zeros of other real \( L \)-functions caught the interest of Linnik, who deeply analyzed this phenomenon, known as the Deuring-Heilbronn phenomenon, in his work concerning the least prime in an arithmetic progression, finding important new results [14].

**Theorem. (Linnik)** If an \( L \)-function belonging to a real non-principal character modulus \( D \) has a real zero \( 1 - \delta \) with

\[
\delta \leq \frac{A_1}{\log D},
\]

then all the \( L \)-functions belonging to characters modulus \( D \) have no zero in the domain

\[
\sigma \geq 1 - \frac{A_2}{\log D(|t| + 1)} \log \left( \frac{e A_1}{\delta \log D(|t| + 1)} \right), \quad \delta \log D(|t| + 1) \leq A_1,
\]

where \( A_1 \) and \( A_2 \) are absolute constants.

Some improvements related to the Heilbronn phenomenon were found by Pintz in 1975 [22]. In particular, using elementary methods, he proved the following result.

**Theorem (Pintz)** Let \( L(s, \chi_k) \) be a Dirichlet’s \( L \)-function belonging to the non principal character (real or complex) \( \chi_k \) modulus \( k \). Suppose that \( L(s, \chi_k) \) has a zero \( s_0 = 1 - \gamma + it \) with \( \gamma < 0.05 \). Then, for an arbitrary real non-principal character \( \chi_D \) mod \( D \) (for which \( \chi_k \chi_D \) is also non-principal) the inequality

\[
L(1, \chi_D) > \frac{1}{140U^{6\gamma} \log^3 U}
\]

holds, where \( U = k|s_0|D \).

The aim of this work is to further investigate both the Deuring phenomenon and the Heilbronn phenomenon. We will find better estimates regarding the influence of zeros of \( \zeta(s) \) on the exceptional zeros and that of the non-trivial zeros of arbitrary \( L \)-functions belonging to non-principal characters on the exceptional zeros, respectively.

Regarding the Deuring phenomenon, combining elementary methods with some tools of complex analysis based on Pintz’s [23] and Puglisi’s [24] approach, we will go further into the critical strip. More precisely, we will prove the following theorem, provided that \( L(s, \chi) \) is a Dirichlet \( L \)-function belonging to the real primitive character \( \chi \) modulus \( q \) satisfying \( \chi(-1) = -1 \) and \( h(-q) \) is the number of classes of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-q}) \).

**Theorem 1.** Let \( \eta, \mu > 0 \) be real numbers with \( \eta > \max(\mu, 1) \). Given

\[
\ell = (\log \log q)^{-\mu},
\]

we define the following set

\[
H(\ell, q) = \left\{ s = \sigma + it : |1 - s| \geq (\log q)^{-4}, 1/2 + \ell \leq \sigma \leq 1, |s| \leq q^{\ell/10} \right\}
\]

If

\[
h(-q) \leq \frac{\log q}{(\log \log q)^{1/2}},
\]

then for each \( s \in H(\ell, q) \) the relation

\[
L(s, \chi) = \frac{\zeta(2s)}{\zeta(s)} \prod_{p|q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\eta-\mu} \right\} \right) \right]
\]

holds.
As an immediate consequence, a new reformulation of Mordell’s Theorem follows from Theorem 1.

**Corollary 1.** If $\zeta(\beta + i\gamma) = 0$ with $\beta > 1/2$, then for every $\eta > 1$ the relation

$$ h(-q) > \frac{\log q}{(\log \log q)^\eta} $$

holds, provided that $q > q_0(\beta, \gamma, \eta)$.

The improvements regarding the Deuring phenomenon stated above make sense, as the inequality $h(-q) > c \log q/(\log \log q)^{\eta}$ had never been generalized to an arbitrary modulus $q$, but it was valid only for $q$ prime ($[8]$, $[10]$).

Regarding the Heilbronn phenomenon, we will improve Pintz’s theorem stated above, showing that it is possible to extend the range of values for $\gamma$ to $0 < \gamma < \frac{1}{8}$ if $\chi_k$ is real and to $0 < \gamma \leq \frac{1}{8}$ if $\chi_k$ is complex. More precisely, we will use elementary methods based on Pintz’s approach $[22]$ to prove the following theorem.

**Theorem 2.** Let $L(s, \chi_k)$ be a Dirichlet $L$-function belonging to the real non-principal character $\chi_k$ modulus $k$. Suppose that $L(s, \chi_k)$ has a zero $s_0 = 1 - \gamma + it$ with $0 < \gamma < \frac{1}{4}$ if $\chi_k$ is real or $\gamma \leq \frac{1}{8}$ if $\chi_k$ is complex, and another $L$-function belonging to the real non-principal character $\chi_D$ mod $D$ (for which $\chi_k \chi_D$ is also non-principal) the inequality

$$ L(1, \chi_D) \geq \frac{c_1}{U^{b\gamma} \log^5 U}, \quad \text{for} \quad \frac{1}{2(1-3\gamma)} < b < \frac{1}{2\gamma} $$

holds, where $U = k |s_0| D$ and $c_1$ is an effective constant.

The same result can be obtained if $\chi_k$ is a complex non-principal character, provided that $0 < \gamma \leq \frac{1}{8}$.

Theorem 2 has some important consequences. First of all, we can deduce that a zero in the half-plane $\sigma > \frac{3}{4}$ for real characters or a zero in the half-plane $\sigma \geq \frac{7}{8}$ for complex characters implies that $h(-D) \to \infty$.

Furthermore, a weakened form of Linnik Theorem $[15]$ can be deduced (the following theorem is an improvement of Theorem 2 of $[22]$).

**Theorem 3.** If an $L$-function belonging to a non-principal character $\chi_k$ modulus $k$ has a zero $s_0 = 1 - \gamma + it$ with $\gamma < \frac{1}{4}$ if $\chi_k$ is real or $\gamma \leq \frac{1}{8}$ if $\chi_k$ is complex, and another $L$-function belonging to the real non-principal character $\chi_D$ (for which $\chi_k \chi_D$ is also non-principal) modulus $D$ has a real exceptional zero $1 - \delta$, then the inequality

$$ \delta > \frac{c_1}{U^{b\gamma} \log^5 U} \quad \text{for} \quad \frac{1}{2(1-3\gamma)} < b < \frac{1}{2\gamma} $$

holds, where $U = k |s_0| D$ and $c_1$ is the constant of Theorem 2.

An immediate consequence is Linnik’s Theorem, stated above, in the following form.

**Corollary 2.** If an $L$-function belonging to a real non-principal character modulus $D$ has a real zero $1 - \delta$ with

$$ \delta = O_\varepsilon \left( \frac{1}{\log^{3+\varepsilon} D} \right) \quad (\varepsilon > 0) $$

then all the $L$-functions belonging to characters modulus $D$ have no zero in the domain

$$ \sigma \geq 1 - \frac{1}{b \log U} \log \left( \frac{c_1}{\delta \log^5 U} \right) $$

where $c_1$ and $b$ have been defined in Theorem 2.
where both $c_1$ and $b$ have been defined in Theorem 2.

2 Proof of Theorem 1

In order to prove Theorem 1 following Pintz’s [23] and Puglisi’s [24] approach to the Deuring-phenomenon, we need some lemmas.

First of all, we define the function $g(n) = \sum_{d|n} \chi(d)$.

**Lemma 1.** Given $\frac{1}{2} + \ell \leq \sigma \leq \frac{x}{2^\ell}$ and $x \gg q$, the relation

$$\sum_{n \leq x} \frac{g(n)}{n^s} \left(1 - \frac{n}{x}\right)^2 = L(s, \chi) \zeta(s) + \frac{2x^{1-s}L(1, \chi)}{(1-s)(2-s)(3-s)} + O\left(|s| \log^2(2 + |s|) \exp\left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^\mu} \right\} \right)$$

holds.

**Proof.** Following exactly the proof of Lemma 2 of [24], we obtain again that

$$\frac{1}{2} \sum_{n \leq x} \frac{g(n)}{n^s} \left(1 - \frac{n}{x}\right)^2 = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} L(s + w, \chi) \zeta(s + w) x^w dw + \frac{L(s, \chi) \zeta(s)}{2} + \frac{x^{1-s}L(1, \chi)}{(1-s)(2-s)(3-s)}$$

Now, using both the hypothesis of Lemma 4 and the classical estimates that were already used in the proof of Lemma 2 of [24], namely

$$\zeta(it) \ll \sqrt{|t| + 1 \log(|t| + 2)}$$

$$L(it, \chi) \ll q(|t| + 1) \log(q(|t| + 1)),$$

we get

$$\left| \int_{-\infty}^{\infty} \frac{x^{\sigma + iu} \sqrt{q} |t + u| \log q \log^2(|t + u| + 2)}{(-\sigma + iu)(-\sigma + 1 + iu)(-\sigma + 2 + iu)} du \right| \ll$$

$$\ll |s|q^{-\ell} \log q \log^2(2 + |s|) \ll$$

$$\ll |s| \log^2(2 + |s|) \exp\left\{ -\frac{\log q}{(\log \log q)^\mu} + \log \log q \right\} \ll$$

$$\ll |s| \log^2(2 + |s|) \exp\left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^\mu} \right\}$$

Finally, from the above estimate we can conclude that the relation

$$\sum_{n \leq x} \frac{g(n)}{n^s} \left(1 - \frac{n}{x}\right)^2 = L(s, \chi) \zeta(s) + \frac{2x^{1-s}L(1, \chi)}{(1-s)(2-s)(3-s)} + O\left(|s| \log^2(2 + |s|) \exp\left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^\mu} \right\} \right)$$

holds for $\frac{1}{2} + \ell \leq \sigma \leq \frac{x}{2^\ell}$ and $x \gg q$. $\square$
Lemma 2. If \( s \in H(\ell, q) \) and the following inequality

\[
h(-q) \leq \frac{\log q}{(\log \log q)^{\eta}}
\]

holds, then the relation

\[
\sum_{n \leq q} \frac{g(n)}{n^s} = L(s, \chi)\zeta(s) + O \left( \exp \left\{ -\frac{1}{3} \frac{\log q}{(\log \log q)^{\mu}} \right\} \right)
\]

holds.

Proof. First of all, we suppose that \( \frac{1}{2} + \ell \leq \sigma \leq \frac{7}{8} \).
We know that

\[
\sum_{n \leq q} \frac{g(n)}{n^s} \left( 1 - \frac{n}{q} \right)^2 = \sum_{n \leq q} g(n) \frac{1}{n^s} - \frac{2}{q} \sum_{n \leq q} g(n) \frac{1}{n^s} + \frac{1}{q^2} \sum_{n \leq q} g(n) n^2
\]

Using Lemma 1 we have

\[
\sum_{n \leq q} \frac{g(n)}{n^s} = L(s, \chi)\zeta(s) + \frac{2q^{1-s}L(1, \chi)}{(1 - s)(2 - s)(3 - s)} +
\]

\[
+ O \left( |s| \log^2(1 + |s|) \exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^{\mu}} \right\} \right) +
\]

\[
+ \frac{2}{q} \sum_{n \leq q} g(n) \frac{1}{n^{s-1}} - \frac{1}{q^2} \sum_{n \leq q} g(n) \frac{1}{n^{s-2}}
\]

Now, if we use Dirichlet’s Class Number Formula (see Davenport [3], chapter 6) and
Lemma 1 of [24], it follows that

\[
\frac{1}{q} \sum_{n \leq q} \frac{g(n)}{n^{s-1}} \ll q^{-\frac{1}{2} - \ell} \sum_{n \leq q} g(n) \ll q^{-\frac{1}{2} - \ell} L(1, \chi) \ll q^{-\ell} h(-q) \ll q^{-\ell} \frac{\log q}{(\log \log q)^{\eta}} =
\]

\[
= \exp \left\{ -\ell \log q + \log q \log \log q - \eta \log \log \log q \right\} \ll
\]

\[
\ll \exp \left\{ -\frac{\log q}{(\log \log q)^{\mu}} + \log q \right\} \ll \exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^{\mu}} \right\}
\]

In the same way, since

\[
\frac{1}{q^2} \sum_{n \leq q} \frac{g(n)}{n^{s-2}} \ll q^{\frac{1}{2} - \ell} L(1, \chi)
\]

\[
\frac{2q^{1-s}L(1, \chi)}{(1 - s)(2 - s)(3 - s)} \ll q^{\frac{1}{2} - \ell} L(1, \chi),
\]

the same estimate as before holds.
Furthermore, we observe that

\[
|s| \log^2(1 + |s|) \exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^{\mu}} \right\} \leq q^{\frac{1}{10}} \left( \frac{\log q}{(\log \log q)^{\mu}} \right)^2 \exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^{\mu}} \right\}
\]

As a consequence, combining all the previous estimates, we can conclude that

\[
\sum_{n \leq q} \frac{g(n)}{n^s} = L(s, \chi)\zeta(s) + O \left( \exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^{\mu}} \right\} \left( 1 + q^{\frac{1}{10}} \left( \frac{\log q}{(\log \log q)^{\mu}} \right)^2 \right) \right)
\]

On the other hand, we have

\[
q^{\frac{7}{10}} \left( \frac{\log q}{(\log \log q)^{\mu}} \right)^2 = \exp \left\{ -\frac{1}{10} \log q + 2 \log \log q - 2 \mu \log \log q \right\} \ll \exp \left\{ \frac{\log q}{10(\log \log q)^{\mu}} \right\}
\]
As a result, the following estimate
\[
\exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^\mu} \right\} \left( q^{1/10} \left( \frac{\log q}{(\log \log q)^\mu} \right)^2 \right) \ll \exp \left\{ -\frac{1}{2} \frac{\log q}{(\log \log q)^\mu} \right\} \ll \exp \left\{ -\frac{1}{3} \frac{\log q}{(\log \log q)^\mu} \right\}
\]
holds.
So, we proved the claim for \( \frac{1}{2} + \ell \leq \sigma \leq \frac{7}{8} \).
If \( \frac{7}{8} < \sigma < 1 \), we can conclude as in Lemma 3 of [24].

Now, we define the same sets used by Pintz [23] and Puglisi [24]:
\[
A_j = \{ n \in \mathbb{N} : p | n \Rightarrow \chi(p) = j \} \quad (j = -1, 0, 1)
\]
\[
R = \{ r = bm : b \in A_0, m \in A_{-1} \}
\]

**Lemma 3.** If
\[
\sum_{a \in A_1, 1 < a \leq \sqrt{q}/2} 1 \leq h(-q)
\]
then
\[
\chi(p) = 1 \Rightarrow p \geq \frac{1}{2} \exp \left\{ \frac{\log q}{2(h(-q) + 1)} \right\}
\]

**Proof.** By contradiction, we suppose that
\[
\chi(p) = 1 \Rightarrow p \leq \frac{1}{2} \exp \left\{ \frac{\log q}{2(h(-q) + 1)} \right\}
\]
Since \( h(-q) \geq 1 \), we have
\[
p^{h(-q)+1} \leq \frac{1}{2^{h(-q)+1}} \exp \left\{ \frac{\log q}{2} \right\} \leq \frac{1}{4} \sqrt{q} \leq \frac{1}{2} \sqrt{q}
\]
Then, we consider \( p, p^2, \ldots, p^{h(-q)+1} \). Under these conditions, the sum
\[
\sum_{a \in A_1, 1 < a \leq \sqrt{q}/2} 1
\]
has at least \( h(-q) + 1 \) terms. Indeed, taken \( a = p^j \) with \( j = 1, \ldots, h(-q) + 1 \), we have \( p|p^j \) and \( \chi(p) = 1 \) by hypothesis.
However, we have a contradiction because we got that \( h(-q) + 1 \leq h(-q) \).

**Lemma 4.** If \( \sigma \geq \frac{1}{2} + \ell \) and the inequality
\[
h(-q) \leq \frac{\log q}{(\log \log q)^\eta}
\]
holds, then the relation
\[
\sum_{a \in A_1, 1 < a \leq q} g(a)a^{-\sigma} \ll \exp \left\{ -\frac{1}{10}(\log \log q)^\eta \right\}
\]
holds.
Proof. We know that

\[ \sum_{a \in A_1, 1 < a \leq q} g(a)a^{-\sigma} \leq \exp \left\{ C \sum_{p \leq \sqrt{q}} p^{-\sigma} \right\} - 1 \quad (C > 0) \]

Furthermore, from Lemma 3 if

\[ \sum_{a \in A_1, 1 < a \leq \sqrt{q}/2} 1 \leq h(-q) \]

then

\[ \chi(p) = 1 \Rightarrow p > \frac{1}{2} \exp \left\{ \frac{\log q}{2(h(-q) + 1)} \right\} = R_0 \]

As a result, since \( \frac{1}{2} \leq \sigma < 1, \eta > \max(\mu, 1) \) and the inequalities

\[ 1 \leq h(-q) \leq \frac{\log q}{(\log \log q)^\eta} \]

hold, we can conclude that

\[ \sum_{p \leq \sqrt{q}/2} \frac{\log q}{(\log \log q)^\eta} \exp \left\{ - \frac{\sigma \log q}{2(h(-q) + 1)} \right\} \leq 2 \frac{\log q}{(\log \log q)^\eta} \exp \left\{ - \frac{(\frac{1}{2} + \ell) \log q}{2 + 2\log q/(\log \log q)^\eta} \right\} \leq \]

\[ \leq 2 \frac{\log q}{(\log \log q)^\eta} \exp \left\{ - \frac{1}{8} (\log \log q) \eta - \frac{1}{2} (\log \log q)^{\eta-\mu} \right\} \leq \]

\[ \leq 2 \frac{\log q}{(\log \log q)^\eta} \exp \left\{ - \frac{1}{8} (\log \log q)^{\eta} \right\} \ll \exp \left\{ - \frac{1}{10} (\log \log q)^{\eta} \right\} \]

Furthermore, for \( \sigma \geq \frac{1}{2} + \ell \) we have

\[ \sum_{\sqrt{q}/2 < p \leq q, \chi(p) = 1} p^{-\sigma} \leq \sum_{\sqrt{q}/2 < n \leq q} g(n)n^{-\sigma} \ll q^{-\frac{\ell}{2}} \sum_{\sqrt{q}/2 < n \leq q} g(n)n^{-\frac{1}{2}} \]

Even more, using Lemma A of [24] (for the proof see Goldfeld [8], p. 637) with \( \varepsilon = \frac{1}{11} \), we have, for \( 0 < 10\eta < x \),

\[ \sum_{y < n \leq x} \frac{g(n)}{\sqrt{n}} = \sum_{d \leq \sqrt{x}} \frac{1}{d^2} \sum_{g/d^2 < k \leq x/d^2} \nu(k)k^{-1/2} \ll L(1, \chi) \left\{ \frac{\sqrt{q}}{\sqrt{y}} + \sqrt{x} + x^{\frac{1}{2} - \frac{\varepsilon}{2}} q^{\frac{1}{11}} \right\} \]

Following the argument used by Puglisi in [24], if we take \( H = \frac{\log 4q}{\log 121} \), we obtain that

\[ \sum_{\sqrt{q}/2 < p \leq q, \chi(p) = 1} p^{-\sigma} \ll q^{-\frac{\ell}{2}} \sum_{h \leq H} \sum_{\sqrt{q}/2 < n \leq \sqrt{q}/2} g(n)n^{-\frac{1}{2}} \ll \]

\[ \ll q^{-\frac{\ell}{2}} L(1, \chi) \sum_{h \leq H} \left\{ \sqrt{q} + \sqrt{q(11)^h} + \left( \frac{\sqrt{q(11)^h}}{2} \right)^{\frac{1}{2} - \frac{\varepsilon}{2}} q^{\frac{1}{11}} \right\} \ll \]

\[ \ll Hq^{-\frac{\ell}{2}} L(1, \chi) = \frac{\log 4q}{\log 121} q^{\frac{1}{2}} q^{-2(\log \log q)^{\mu}} L(1, \chi) \ll \exp \left\{ - \frac{1}{8} \frac{\log q}{(\log \log q)^{\mu}} \right\} \]

where we used the estimate

\[ q^{-\ell} h(-q) \ll \exp \left\{ - \frac{1}{2} \frac{\log q}{(\log \log q)^{\mu}} \right\} \]

Adding both the terms, the conclusion follows. \( \square \)
Lemma 5. If $\sigma \geq \frac{1}{2} + \ell$ and the inequality

$$h(-q) \leq \frac{\log q}{(\log \log q)^\eta}$$

holds, then the relation

$$\sum_{n \leq q} g(n)n^{-s} = \sum_{r \in R, r \leq q} g(r)r^{-s} + O \left( \exp \left\{ -\frac{1}{16} (\log \log q)^\eta \right\} \right)$$

holds.

Proof. First of all, we observe that

$$\sum_{n \leq q} g(n)n^{-s} = \sum_{r \in R, r \leq q} g(r)r^{-s} + O \left( \sum_{r \in R, r \leq q} g(r)r^{-\sigma} \sum_{a \in A_1, 1 < a \leq q} g(a)a^{-\sigma} \right)$$

and

$$\sum_{r \in R, r \leq q} g(r)r^{-\sigma} \leq \sum_{b \in A_0} \frac{\mu^2(b)}{b^{2\ell+1}} \sum_{k \geq 1} k^{-1-2\ell} \ll \frac{1}{\ell} \exp \left\{ \sum_{p | q} \frac{1}{\sqrt{p}} \right\}$$

where $\mu$ is M"obius’ Function.

Since

$$h(-q) \leq \frac{\log q}{(\log \log q)^\eta}$$

and

$$\sum_{p | q} 1 \leq 1 + \frac{\log(h(-q))}{\log 2},$$

then

$$\exp \left\{ \sum_{p | q} \frac{1}{\sqrt{p}} \right\} \leq \exp \left\{ 1 + \frac{\log(h(-q))}{\log 2} \right\} \leq 3 \exp \left\{ \frac{\log(h(-q))}{\log 2} \right\} \ll \left( \frac{\log q}{(\log \log q)^\eta} \right)^{\frac{1}{\log 2}}$$

It follows that

$$\sum_{r \in R, r \leq q} g(r)r^{-\sigma} \ll \frac{1}{\ell} \left( \frac{\log q}{(\log \log q)^\eta} \right)^{\frac{1}{2\eta}}$$

As a result, we have

$$\sum_{r \in R, r \leq q} g(r)r^{-\sigma} \sum_{a \in A_1, 1 < a \leq q} g(a)a^{-\sigma} \ll (\log \log q)^{\mu - \frac{\eta}{2}} (\log q)^{\frac{1}{\mu}} \left( \exp \left\{ -\frac{1}{10} (\log \log q)^\eta \right\} \right) \ll \exp \left\{ -\frac{1}{16} (\log \log q)^\eta \right\}$$

Lemma 6. If $\sigma \geq \frac{1}{2} + \ell$ and the inequality

$$h(-q) \leq \frac{\log q}{(\log \log q)^\eta}$$

holds, then the relation

$$\sum_{r \leq q} \frac{g(r)}{r^s} = \zeta(2s) \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3} (\log \log q)^{\eta - \mu} \right\} \right) \right] +$$

$$+ O \left( \exp \left\{ -\frac{1}{2} (\log \log q)^{\mu} \right\} \right)$$

holds.
Proof. We have already seen that
\[ \frac{1}{\ell} \sum_{h \mid q} \frac{\mu^2(h)}{\sqrt{h}} = \frac{1}{\ell} \prod_{p \mid q} \left( 1 + \frac{1}{\sqrt{p}} \right) \ll (\log \log q)^{\mu - \frac{\eta}{\log 2}} (\log q)^{\frac{1}{\log 2}} \]

Furthermore, if \( n \not\in R \), then \( n > R_0 \).
It follows that
\[ \sum_{r \in R, r \leq q} g(r) r^s = \sum_{h \mid q} \frac{\mu^2(h)}{h^s} \sum_{r \in R, r \leq \sqrt{q/h}} r^{-2s} = \]
\[ = \sum_{h \mid q} \frac{\mu^2(h)}{h^s} \left[ \zeta(2s) + O \left( \sum_{r > R_0} r^{-1-2t} \right) \right] + O \left( \sum_{h \mid q} \frac{\mu^2(h)}{h^s + \ell} \sum_{r > \sqrt{q/h}} r^{-1-2t} \right) = \]
\[ = \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ \zeta(2s) + O \left( \frac{1}{\ell} \exp \left\{ -\frac{\log q}{2(\log \log q)^{\mu}(\log q)^{\frac{1}{\log 2}}} \right\} \right) \right] + \]
\[ + O \left( \frac{q^{-\ell}}{\ell} \sum_{h \mid q} \frac{\mu^2(h)}{\sqrt{h}} \right) = \]
\[ = \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ \zeta(2s) + O \left( \frac{1}{\ell} \exp \left\{ -\frac{1}{2}(\log \log q)^{\eta-\mu} \right\} \right) \right] + \]
\[ + O \left( \frac{q^{-\ell}(\log \log q)^{\mu-\eta}}{\log 2} \right) = \]
\[ = \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ \zeta(2s) + O \left( (\log \log q)^{\mu} \exp \left\{ -\frac{1}{2}(\log \log q)^{\eta-\mu} \right\} \right) \right] + \]
\[ + O \left( \exp \left\{ -\frac{\log q}{(\log \log q)^{\mu}} + \frac{1}{\log 2} \log \log q + \left( \mu - \frac{\eta}{\log 2} \right) \log \log \log q \right\} \right) = \]
\[ = \zeta(2s) \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\eta-\mu} \right\} \right) \right] + \]
\[ + O \left( \exp \left\{ -\frac{1}{2}(\log \log q)^{\mu} \right\} \right) \]

Now, we are ready to prove Theorem II
Using all the results we found previously, we can conclude that
\[ \zeta(s) = \sum_{n \leq q} g(n) n^s + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\mu} \right\} \right) = \]
\[ = \sum_{r \in R, r \leq q} g(r) r^{-s} + O \left( \exp \left\{ -\frac{1}{16}(\log \log q)^{\eta} \right\} \right) = \]
\[ = \zeta(2s) \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\eta-\mu} \right\} \right) \right] + \]
\[ + O \left( \exp \left\{ -\frac{1}{16}(\log \log q)^{\eta} \right\} \right) = \]
\[ = \zeta(2s) \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\eta-\mu} \right\} \right) \right] \]

\[ \square \]

Now, we are ready to prove Theorem II
Using all the results we found previously, we can conclude that
\[ L(s, \chi) = \sum_{n \leq q} g(n) n^s + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\mu} \right\} \right) = \]
\[ = \sum_{r \in R, r \leq q} g(r) r^{-s} + O \left( \exp \left\{ -\frac{1}{16}(\log \log q)^{\eta} \right\} \right) = \]
\[ = \zeta(2s) \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\eta-\mu} \right\} \right) \right] + \]
\[ + O \left( \exp \left\{ -\frac{1}{16}(\log \log q)^{\eta} \right\} \right) = \]
\[ = \zeta(2s) \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \left[ 1 + O \left( \exp \left\{ -\frac{1}{3}(\log \log q)^{\eta-\mu} \right\} \right) \right] \]

\[ \square \]
3 Proof of Theorem 2

Following exactly Pintz’s proof of Theorem 1 of [22], we define the following sets

\[ A_\nu = \{ n \in \mathbb{N}; \, p|n, p \text{ prime} \rightarrow \chi_D(p) = \nu \} \quad (\nu = -1, 0, 1) \]

\[ C = \{ c; c = uv, u \in A_1, v \in A_0 \} \]

and the following two multiplicative functions

\[ g_\lambda(n) = \sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if} \quad n = l^2 \\ 0, & \text{if} \quad n \neq l^2 \end{cases} \]

(where \( \lambda(n) \) denotes Liouville’s \( \lambda \)-function) and

\[ g_D(n) = \sum_{d|n} \chi_D(d) = \prod_{p^n|n} (1 + \chi_D(p) + \cdots + \chi_D^n(p)) \geq 0 \quad (1) \]

Again, from Pintz’s proof of Theorem 1 in [22], for \( n = uvm = cm, \quad u \in A_1, v \in A_0, m \in A_{-1} \), we get

\[ g_\lambda(n) = g_\lambda(u)g_\lambda(v)g_\lambda(m) = \sum_{c_l|c, \, c_l = u_1r_1} 2^\omega(u_l) \lambda(c_l) g_D \left( \frac{n}{c_l} \right) \quad (2) \]

(where \( \omega(n) \) denotes the number of distinct prime divisors of \( n \)) and, for \( c \in C, c = uv, \, u \in A_1, \, v \in A_0 \), we have

\[ 2^\omega(u) \leq g_D(c) \leq d(c) \quad (3) \]

Now, let \( b, h \) two positive real numbers, with \( 1 < h < 2b \). Thus, considering (1), (2) and (3) we have

\[
\left| \sum_{n \leq U^b} \frac{\chi_k(n)}{n^{\gamma_0}} \right| = \left| \sum_{n \leq U^b} \frac{\chi_k(n)}{n^{\gamma_0}} g_\lambda(n) \right| = \left| \sum_{n \leq U^b} \frac{\chi_k(n)}{n^{\gamma_0}} \sum_{c \in C, \, c|m} 2^\omega(u) \lambda(c) g_D \left( \frac{n}{c} \right) \right| = \left| \sum_{c \in C, \, c|m} 2^\omega(u) \lambda(c) g_D \left( \frac{n}{c} \right) \right| + \left| \sum_{r \leq U^b/c} \frac{\chi_k(r)}{r^{\gamma_0}} g_D(r) \right| \leq \left| \sum_{n \leq U^{b/h}} \frac{d(n)}{n^{1-\gamma}} \sum_{r \leq U^{b/n}} \frac{\chi_k(r)}{r^{\gamma_0}} g_D(r) \right| + \sum_{r \leq U^{b/n}} \frac{g_D(n)}{n^{1-\gamma}} \sum_{r \leq U^{b/n}} \frac{d(r)}{r^{1-\gamma}} = \sum_{1} + \sum_{2}
\]

Before trying to estimate both the two sums in (4), we find a lower bound for

\[
\left| \sum_{n \leq U^b} \frac{\chi_k(n)}{n^{\gamma_0}} \right|
\]
In order to do this, we consider the two cases ($\chi_k$ real or complex) separately. We start with $\chi_k$ real and non principal. We observe that

$$\sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^{s_0}} = \sum_{l=1}^{\infty} \frac{1}{l^{2s_0}} =$$

$$= \sum_{l=1}^{\infty} \frac{\chi_{0,k}(l)}{l^{2s_0}} = L(2s_0, \chi_{0,k}) =$$

$$= \prod_{p \mid k} \left(1 - \frac{1}{p^{2s_0}}\right)^{-1}$$

where in the last inequality we used Euler’s identity.

Hence,

$$\left| \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^{s_0}} \right| = \left| \prod_{p \mid k} \left(1 - \frac{1}{p^{2s_0}}\right)^{-1} \right| \geq$$

$$\geq \prod_{p \mid k} \frac{1}{1 + \frac{1}{p^{2(1-\gamma)}}} \geq$$

$$\geq \prod_{p} \frac{1}{1 + \frac{1}{p^{2(1-\gamma)}}} =$$

$$= \frac{\zeta(4(1-\gamma))}{\zeta(2(1-\gamma))} >$$

$$> \frac{\zeta(4)}{\zeta(2)} > \frac{\pi^4}{270} > 0.36$$

since $0 < \gamma < \frac{1}{4}$.

Now, we turn to the case where $\chi_k$ is complex and non principal. Since $0 < \gamma \leq \frac{1}{8}$, we observe that

$$\left| \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^{s_0}} \right| \geq 1 - \sum_{l=2}^{\infty} \frac{1}{l^{2(1-\gamma)}} \geq$$

$$\geq 1 - \sum_{l=2}^{10} \frac{1}{l^{7/4}} - \int_{10}^{\infty} \frac{dl}{l^{7/4}} \geq 0.029$$

Now, we separately estimate the two sums of (4).

We begin with the first one:

$$\sum_{l=1}^{U^h/n} \frac{d(n)}{n^{1-\gamma}} \sum_{r \leq U^h/n} \frac{\chi_k(r)}{r^{s_0}} g_D(r)$$

We start by considering the inner sum, that is

$$\left| \sum_{r \leq U^h/n} \frac{\chi_k(r)}{r^{s_0}} g_D(r) \right|$$

Let $y \geq U^h/n$ be a fixed number and let $z$ be a parameter we will choose later. Since $U = kD|s_0|$, we have
At this point, we observe that

\[
\left| \sum_{r \leq y} \frac{\chi_k(r)}{r^{s_0}} g_D(r) \right| \leq
\]

\[
\leq \sum_{d \leq z} \frac{1}{d^{1-\gamma}} \cdot \left( \frac{y}{d} \right)^{1-\gamma} + \sum_{l \leq y/z} \frac{1}{l^{1-\gamma}} \cdot 2|s_0| \sqrt{k \log(kD)} \leq (7)
\]

\[
\leq \frac{z \cdot 2|s_0| \sqrt{k \log k}}{y^{1-\gamma}} + 2|s_0| \sqrt{k \log(kD)} \cdot \sum_{l \leq y/z} \frac{1}{l^{1-\gamma} l^\gamma} \leq \frac{z \cdot 2|s_0| \sqrt{k \log k}}{y^{1-\gamma}} + 2|s_0| \sqrt{k \log(kD)} \cdot \frac{y^\gamma \log \left( \frac{y}{z} \right)}{z}
\]

where in the second step we used the Polya-Vinogradov inequality, while in the last inequality we used the fact that

\[
\sum_{l \leq y/z} \frac{1}{l^{1-\gamma} l^\gamma} = \frac{y^\gamma}{z} \sum_{l \leq y/z} \frac{1}{l^{1-\gamma} \left( \frac{y}{z} \right)^\gamma} < \frac{y^\gamma}{z} \sum_{l \leq y/z} \frac{1}{l^{1-\gamma} l^\gamma} \leq \frac{y^\gamma \log \left( \frac{y}{z} \right)}{z}
\]

Now, we choose \( z \) such that

\[
z = \frac{\sqrt{D} y^\gamma}{y^{1-\gamma}}
\]

or equivalently,

\[
z = y^{\frac{1}{2}} D^{\frac{1}{4}}
\]

Using this value for \( z \), the relation (7) becomes

\[
\left| \sum_{r \leq y} \frac{\chi_k(r)}{r^{s_0}} g_D(r) \right| \leq 2 \cdot y^{\gamma - \frac{1}{2}} D^{\frac{3}{4}} |s_0| \sqrt{k \log k} + 2|s_0| \sqrt{k} \cdot D^{\frac{1}{4}} \cdot y^{\gamma - \frac{1}{2}} \cdot \log(kD) \log \left( \frac{\sqrt{y}}{D^{1/4}} \right)
\]

Now, we consider \( y = U^b/n \). It follows that

\[
\sum_{1} = \sum_{n \leq U^{b/h}} \frac{d(n)}{n^{1-\gamma}} \sum_{r \leq U^b/n} \frac{\chi_k(r)}{r^{s_0}} g_D(r) \ll \\
\ll \sum_{n \leq U^{b/h}} \frac{d(n)}{n^{1-\gamma}} \cdot \left( \frac{U^b}{n} \right)^{\gamma - \frac{3}{2}} D^{\frac{1}{4}} \left( 2|s_0| \sqrt{k \log k} + 2|s_0| \sqrt{k} \cdot D^{\frac{1}{4}} \cdot y^{\gamma - \frac{1}{2}} \cdot \log(kD) \log \left( \frac{\sqrt{y}}{D^{1/4}} \right) \right) \ll \\
\ll 2|s_0| \sqrt{k} \cdot U^{b(\gamma - \frac{3}{2}) + \frac{3}{4}} \cdot \log^2 U \cdot \sum_{n \leq U^{b/h}} \frac{d(n)}{\sqrt{n}} \ll \\
\ll 2|s_0| \sqrt{k} \cdot U^{b(\gamma - \frac{1}{2}) + \frac{1}{4}} \cdot \log^3 U \cdot \left( U^{\frac{b}{2}} \right)^{\frac{3}{4}} = \\
= 2|s_0| \sqrt{k} \cdot U^{b(\gamma - \frac{1}{2} + \frac{1}{4}) + \frac{1}{4}} \cdot \log^3 U
\]

At this point, we observe that

\[
b \left( \gamma - \frac{1}{2} + \frac{1}{2h} \right) + \frac{1}{4} < 0
\]

if and only if

\[
b > \frac{1}{4} \cdot \left( \frac{1}{2} - \gamma - \frac{1}{2h} \right) \quad \text{and} \quad \frac{1}{2} - \gamma - \frac{1}{2h} > 0 \quad (8)
\]
Under these conditions we can conclude that the estimate
\[
\sum_1 \ll 2|s_0| \sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{10}) + \frac{1}{4} \cdot \log^3 U
\]  
(9)
is non trivial, if \( U \geq U_0(\gamma) \), where \( U_0(\gamma) \) is a constant depending on \( \gamma \).

Now, we turn our attention to the second sum of (4), that is
\[
\sum_2 = \sum \frac{g_D(n)}{n^{1-\gamma}} \sum \frac{d(r)}{r^{1-\gamma}}
\]
Since
\[
\sum_{r \leq U^{b/h}} \frac{d(r)}{r^{1-\gamma}} \ll \left( \frac{U^{b/h}}{n} \right)^\gamma \sum_{r \leq U^{b/h}} \frac{d(r)}{r} \ll \left( \frac{U^{b/h}}{n} \right) \cdot \left( \frac{1}{2} + o(1) \right) \log^2 U
\]
we have
\[
\sum_2 \ll U^{b\gamma} \cdot \log^2 U \cdot \left( \frac{1}{2} + o(1) \right) \sum \frac{g_D(n)}{n}
\]
However, from Lemma 1 of [21], we know that
\[
\sum_{U^{b/h} < n \leq U^b} \frac{g_D(n)}{n} = b \left( 1 - \frac{1}{h} \right) \log U \cdot L(1, \chi_D) + O \left( \sqrt{D} \log D \log \frac{U}{h^2} \right) =
\]
\[
= b \left( 1 - \frac{1}{h} \right) \log U \cdot L(1, \chi_D) + O \left( U^{-\frac{b}{2h} - \frac{1}{2}} \right) \log U =
\]
\[
= \log U \cdot \left( b \left( 1 - \frac{1}{h} \right) L(1, \chi_D) + O \left( U^{-\frac{b}{2h} - \frac{1}{2}} \right) \right)
\]
which is well defined, as we supposed that \( 1 < h < 2b \).

Hence, we can conclude that
\[
\sum_2 \ll U^{b\gamma} \cdot \log^2 U \cdot \left( \frac{1}{2} + o(1) \right) \sum \frac{g_D(n)}{n} \ll
\]
\[
\ll U^{b\gamma} \cdot \log^2 U \cdot \left( \frac{1}{2} + o(1) \right) \cdot \log U \cdot \left( b \left( 1 - \frac{1}{h} \right) L(1, \chi_D) + O \left( U^{-\frac{b}{2h} - \frac{1}{2}} \right) \right) \leq
\]
\[
\leq c_0 U^{b\gamma} \log^3 U \cdot L(1, \chi_D)
\]  
(10)
if \( U \geq U_0'(\gamma) \), where \( U_0'(\gamma) \) is a constant depending on \( \gamma \) and \( c_0 \) is an effective constant.

At this point, if \( \chi_k \) is a real character, combining (5), (9), (10), under the conditions seen above, we get
\[
0.36 \leq 2|s_0| \sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{10}) + \frac{1}{4} \cdot \log^3 U + c_0 U^{b\gamma} \log^3 U \cdot L(1, \chi_D)
\]
or equivalently,
\[
0.36 - 2|s_0| \sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{10}) + \frac{1}{4} \cdot \log^3 U \leq c_0 U^{b\gamma} \log^3 U \cdot L(1, \chi_D)
\]
Furthermore, for \( U \geq U_0(\gamma) \) sufficiently large, and so \( D \geq D_0(\gamma) \) sufficiently large, we have
\[
2|s_0| \sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{10}) + \frac{1}{4} \cdot \log^3 U \leq \frac{3}{10
\]
Hence, if \( \chi_k \) is a real non principal character, we can conclude that
\[
L(1, \chi_D) \geq \frac{1}{c_0 U^{b\gamma} \log^3 U} \geq \frac{c_1}{U^{b\gamma} \log^3 U}
\]

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where \( c_1 \) is an effective constant.  

On the other hand, if \( \chi_k \) is a complex character, combining (3), (9), (10), under the conditions (8) seen above, we get

\[
0.029 \leq 2|s_0|\sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{2\pi}) + \frac{1}{4} \cdot \log^3 U + c_0 U^{b\gamma} \log^3 U \cdot L(1, \chi_D)
\]

or equivalently,

\[
0.029 - 2|s_0|\sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{2\pi}) + \frac{1}{4} \cdot \log^3 U \leq c_0 U^{b\gamma} \log^3 U \cdot L(1, \chi_D)
\]

Furthermore, for \( U \geq U_0(\gamma) \) sufficiently large, and so \( D \geq D_0(\gamma) \) sufficiently large, we have

\[
2|s_0|\sqrt{k} \cdot U^b(\gamma - \frac{1}{2} + \frac{1}{2\pi}) + \frac{1}{4} \cdot \log^3 U \leq \frac{1}{50}
\]

Hence, if \( \chi_k \) is a complex non principal character, we can conclude that

\[
L(1, \chi_D) \geq \frac{1}{c_0 U^{b\gamma} \log^3 U} \geq \frac{c_1}{U^{b\gamma} \log^3 U}
\]

where \( c_1 \) is an effective constant.

Finally, we observe that, in order to have a non trivial estimate, \( b \) shall satisfy \( b < \frac{1}{2\gamma} \).

However, due to conditions (3), we already know that

\[
b > \frac{1}{4} \cdot \frac{1}{(\frac{1}{2} - \gamma - \frac{1}{2\pi})}
\]

and

\[
\frac{1}{2} - \gamma - \frac{1}{2h} > 0
\]

or equivalently,

\[
\gamma < \frac{1}{2} - \frac{1}{2h}
\]

where \( 1 < h < 2b \).

Hence, we shall have

\[
\frac{1}{4} \cdot \frac{1}{(\frac{1}{2} - \gamma - \frac{1}{2\pi})} < \frac{1}{2\gamma}
\]

or equivalently,

\[
\gamma < \frac{1}{3} - \frac{1}{3h}
\]

Now, we observe that, for \( h > 1 \), the inequality

\[
\frac{1}{3} - \frac{1}{3h} < \frac{1}{2} - \frac{1}{2h}
\]

is always satisfied. As a result, provided that \( h > 1 \) as we supposed before, \( b, \gamma \) and \( h \) shall satisfy simultaneously only the following three relations:

\[
b < \frac{1}{2\gamma} \quad (11)
\]

\[
1 < h < 2b \quad (12)
\]

\[
\gamma < \frac{1}{3} - \frac{1}{3h} \quad (13)
\]

Now, we observe that, from (11) and (12), the inequality

\[
h < \frac{1}{\gamma}
\]
holds.
On the other hand, from (13) we have
\[ h > \frac{1}{1 - 3\gamma} \]
As a result, we get
\[ \frac{1}{\gamma} > h > \frac{1}{1 - 3\gamma} \] (14)
or even better,
\[ \frac{1}{\gamma} > 2b > h > \frac{1}{1 - 3\gamma} \] (15)
From (14) it follows that
\[ \gamma < \frac{1}{4}, \]
which makes sense, since it is stated in the hypotheses for the real case, while \( \gamma \leq \frac{1}{8} < \frac{1}{4} \) for the complex case.
On the other hand, (15) implies that
\[ \frac{1}{2(1 - 3\gamma)} < b < \frac{1}{2\gamma} \]
Hence, having fixed \( b \) such that
\[ \frac{1}{2(1 - 3\gamma)} < b < \frac{1}{2\gamma}, \]
if we choose \( h \) such that
\[ \frac{1}{1 - 3\gamma} < h < 2b, \]
we have the inequality
\[ L(1, \chi_D) \geq \frac{c_1}{U^{b\gamma} \log^3 U} \]
if \( \chi_k \) is real, or
\[ L(1, \chi_D) \geq \frac{c_1'}{U^{b\gamma} \log^4 U} \]
if \( \chi_k \) is complex, where \( U = k |s_0| D \) and \( c_1, c_1' \) are effective constants.
The proof of Theorem 2 is complete.

4 Proof of Theorem 3

As in the proof of Theorem 2 of [22], by a result of Page 18, given \( \chi_D \) a real non-principal character mod \( D \), we know that the greatest real zero \( 1 - \delta \) of \( L(s, \chi_D) \) satisfies
\[ \frac{L(1, \chi_D)}{\delta} \leq \log^2 D \]
Furthermore, since \( U = k |s_0| D \) by hypothesis, then \( \log^2 D \leq \log^2 U \). Hence,
\[ \frac{L(1, \chi_D)}{\delta} \leq \log^2 U \]
Now, using Theorem 2 it follows that
\[ \delta > \frac{c_1}{U^{b\gamma} \log^5 U} \quad \text{for} \quad \frac{1}{2(1 - 3\gamma)} < b < \frac{1}{2\gamma}. \]
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