GROUPS IN WHICH EVERY NON-ABELIAN SUBGROUP IS SELF-NORMALIZED

COSTANTINO DELIZIA, URBAN JEZERNIK, PRIMOŽ MORAVEC, AND CHIARA NICOTERA

Abstract. We study groups having the property that every non-abelian subgroup is equal to its normalizer. This class of groups is closely related to an open problem posed by Berkovich. We give a full classification of finite groups having the above property. We also describe all infinite soluble groups in this class.

1. Introduction

In his book [1], Berkovich posed the following problem:

Problem 1.1 ([1], Problem 9). Study the $p$-groups $G$ in which $C_G(A) = Z(A)$ for all non-abelian $A \leq G$.

Classification of such $p$-groups appears to be difficult, as there seem to be many classes of finite $p$-groups enjoying the above property. In a recent paper [5], the finite $p$-groups which have maximal class or exponent $p$ and satisfy Berkovich’s condition are characterized. In addition to that, the infinite supersoluble groups in which every non-abelian subgroup is self-centralized are completely classified. Some relaxations of Berkovich’s problem are considered in [3, 4] where locally finite or infinite supersoluble groups $G$ in which every non-cyclic subgroup $A$ satisfies $C_G(A) = Z(A)$ are described.

Recently, Pavel Zalesskii suggested to us another related problem:

Problem 1.2. Classify finite groups $G$ in which $N_G(A) = A$ for all non-abelian $A \leq G$.

Let us denote by $\Sigma$ the class of all groups $G$ satisfying the property stated in Problem 1.2 that is, that every non-abelian subgroup of $G$ is self-normalized. Motivation for considering this problem is twofold. Firstly, every group in $\Sigma$ clearly satisfies the property that every non-abelian subgroup is self-centralized. Thus the class $\Sigma$ fits into the framework set by the above mentioned Berkovich’s problem.

Key words and phrases. normalizer, non-abelian subgroup, self-normalized subgroup.
Secondly, the class of groups in which every non-abelian subgroup is self-normalized can be considered as a particular case of the following general situation. Fix a property $\mathcal{P}$ related to the subgroups of a given group and consider the class of all groups $G$ in which every subgroup $H$ either has the property $\mathcal{P}$ or is of bounded index in its normalizer $N_G(H)$. There is an abundance of literature studying restrictions these kind of conditions impose on groups. For example, taking $\mathcal{P}$ to be the property of being normal, the authors of [7] investigate finite $p$-groups in this class. If we take $\mathcal{P}$ to be commutativity and set the upper bound of $|N_G(H) : H|$ to 1, then we obtain the class $\Sigma$.

One of the purposes of this paper is to completely characterize the finite groups in $\Sigma$. This is done in Section 2. We show that these groups are either soluble or simple. Finite non-abelian simple groups in $\Sigma$ are precisely the groups $\text{Alt}(5)$ and $\text{PSL}(2, 2^n)$, where $2^n - 1$ is a prime (see Theorem 2.17). The structure of finite soluble non-nilpotent groups in $\Sigma$ is described in Theorem 2.13. In contrast with Berkovich’s problem, finite nilpotent non-abelian groups in $\Sigma$ are precisely the minimal non-abelian $p$-groups, whose structure is well-known.

In Section 3 we deal with infinite groups in $\Sigma$. We completely describe the structure of infinite soluble groups in $\Sigma$ in Theorems 3.2 and 3.3. On the other hand, the structure of infinite soluble groups with the property that every non-abelian subgroup is self-centralized is still not completely known, the paper [5] only deals with the supersoluble case. Finally, we prove in Theorem 3.4 that every infinite locally finite group in $\Sigma$ is metabelian.

2. Finite groups

The following three results hold in general for groups in $\Sigma$, not just for finite groups. They will be used throughout the paper without further reference.

**Proposition 2.1.** The following properties hold:

(i) The class $\Sigma$ is subgroup and quotient closed.

(ii) If $G \in \Sigma$, then every non-abelian subgroup of $G$ is self-centralized.

(iii) Non-abelian groups in $\Sigma$ are directly indecomposable.

**Proposition 2.2.** Let $G \in \Sigma$.

(i) Every proper subnormal subgroup of $G$ is abelian.

(ii) If $G$ is not perfect, then it is metabelian.

**Lemma 2.3.** Let $G \in \Sigma$ be a soluble group which is either infinite or non-nilpotent finite, and $F$ its Fitting subgroup. Then $F$ is abelian and of prime index in $G$. 

Proof. If $G$ is infinite then $F$ is abelian by [5, Theorem 3.1]. Otherwise, $F$ is abelian by (i) of Proposition 2.2. By (ii) of Proposition 2.2 we have $G' \leq F$, so $G/F$ is an abelian group having no non-trivial proper subgroups, and hence it has prime order.

The following easy observation immediately follows from the definition.

**Lemma 2.4.** Let $G$ be a finite group. Then $G$ belongs to $\Sigma$ if and only if the following holds: for every non-abelian subgroup $H$ of $G$, the number of conjugates of $H$ in $G$ is equal to $|G : H|$.

The next lemma gives a well-known characterization of minimal non-abelian finite $p$-groups, see [1, Exercise 8a, p. 29].

**Lemma 2.5.** Let $G$ be a finite $p$-group. The following are equivalent:

a) $G$ is minimal non-abelian.

b) $d(G) = 2$ and $|G'| = p$.

c) $d(G) = 2$ and $Z(G) = \Phi(G)$.

**Proposition 2.6.** Let $G$ be a nilpotent group. Then $G \in \Sigma$ if and only if $G$ is either abelian or a finite minimal non-abelian $p$-group for some prime $p$.

Proof. Let $G$ be a nilpotent group in $\Sigma$, and suppose that $G$ is not abelian. By [5, Theorem 3.1], $G$ has to be finite, and Proposition 2.1 implies that $G$ is a $p$-group. Let $H$ be a proper subgroup of $G$. Then $H < N_G(H)$ by [14, 5.2.4], hence $H$ is abelian. This proves that $G$ is minimal non-abelian. The converse is clear.

Finite soluble groups in which all Sylow subgroups are abelian are called $A$-groups, cf. [10, Seite 751]. Next we show that finite soluble non-nilpotent groups in $\Sigma$ are $A$-groups.

**Lemma 2.7.** Let $G \in \Sigma$ be a soluble non-nilpotent group and let $p = |G : F|$. Then the Sylow $p$-subgroups of $G$ are cyclic and, for primes $r \neq p$, the Sylow $r$-subgroups are abelian.

Proof. If $r \neq p$, then the Sylow $r$-subgroups of $G$ are contained in $F$ and so are abelian by [2.3]. Now let $P$ be a Sylow $p$-subgroup of $G$ and assume that $P$ is not cyclic. Since $C_G(F) = F$, there exists a prime $r$ and a Sylow $r$-subgroup $R$ of $G$ such that $P$ and $R$ do not commute. Since $R$ is normal in $G$, $X = PR$ is a subgroup of $G$. Since $P$ is not cyclic, there are maximal subgroups $P_1$ and $P_2$ of $P$ with $P_1 \neq P_2$. It follows that $X$ normalizes both $P_1R$ and $P_2R$. As $X \supset P_1R$ and $X > P_2R$, we conclude that $P_1R$ and $P_2R$ are abelian. But then $P = P_1P_2$ commutes with $R$, a contradiction. Hence $P$ is cyclic.

□
Suppose that an element $x$ acts on an abelian group $A$. Consider the induced homomorphism $\partial_x : A \to A$, $\partial_x(a) = a^{1-x}$. We will describe groups belonging to the class $\Sigma$ based on the following property of $\partial_x$:

\[
\text{if } B \leq A : (\partial_x(B) \leq B \implies \partial_x(B) = B).
\]

Note that $x$ acts fixed point freely on $A$ if and only if $\partial_x$ is injective. The property $\Pi$ implies injectivity of $\partial_x$, since taking $B = \ker \partial_x$ immediately gives $B = 1$. Furthermore, the property $\Pi$ implies that $\partial_x$ is an epimorphism by taking $B = A$. Therefore having property $\Pi$ implies that $\partial_x$ is an isomorphism.

The following proposition shows how property $\Pi$ is related to $\Sigma$.

**Proposition 2.8.** Let $G = \langle x \rangle \rtimes A$ with $x^p$ acting trivially on $A$ for some prime $p$ and $x$ acting fixed point freely on $A$. Then $G \in \Sigma$ if and only if $\partial_x$ has property $\Pi$.

**Proof.** Assume first that $G \in \Sigma$. If $\partial_x$ does not have property $\Pi$, then there exists a subgroup $B \leq A$ such that $\partial_x(B) \nsubseteq B$. Consider the subgroup $H = \langle x \rangle \rtimes \partial_x(B)$ of $G$. Take an element $b \in B \setminus \partial_x(B)$ and observe that $x^b = x\partial_x(b) \in H$. Therefore $b$ normalizes $H$ and does not belong to $H$. This implies that $H$ is abelian, and so $\partial_x(\partial_x(B)) = 1$. By injectivity of $\partial_x$, it follows that $B = 1$, a contradiction with $\partial_x(B) \nsubseteq B$. Therefore $\partial_x$ has property $\Pi$.

Conversely, assume now that $\partial_x$ has property $\Pi$. To prove that $G$ belongs to $\Sigma$, take a non-abelian subgroup $H$ of $G$. Note that $H$ must contain an element of the form $xa$ for some $a \in A$. Then since $\partial_x$ is surjective, we have $a = \partial_x(b)$ and so $xa = x^b$. After possibly replacing $H$ by $H^{b^{-1}}$, it suffices to consider the case when $b = 1$, and therefore $x \in H$. We can thus write $H = \langle x \rangle \rtimes B$ for some $B = H \cap A \leq A$. Let us show that $H$ is self-normalizing in $G$. To this end, take an element $x^j c \in N_G(H)$. Then $x^{x^j c} = x^{x c^{-x} c}$, and so we must have $\partial_x(c) \in B$. Conversely, any element $x^j c \in A$ with the property that $\partial_x(c) \in B$ normalizes $H$, since for any $b \in B$ we also have $b x^j c = (b x^j)c = b x^j c \in B$. Thus $N_G(H) = \langle x \rangle \rtimes \partial_x^{-1}(B)$. By property $\Pi$, we have $\partial_x^{-1}(B) = B$, which implies $N_G(H) = \langle x \rangle \rtimes B = H$, as required. \hfill $\square$

**Example 2.9.** Let $\zeta_p$ be a primitive complex $p$-th root of unity for some prime $p$. Then $\zeta_p$ acts by multiplication on the abelian group $\mathbb{C}$ and we can form $G = \langle \zeta_p \rangle \rtimes \mathbb{C}$. The group $G$ has a subgroup $H = \langle \zeta_p \rangle \rtimes \mathbb{Z}[\zeta_p]$. Now, note that $\zeta_p$ acts fixed point freely on $\mathbb{C}$, and multiplication by $1 - \zeta_p$ is invertible on $\mathbb{C}$. Therefore $\partial_{\zeta_p}$ is an isomorphism of $\mathbb{C}$. However, multiplication by $1 - \zeta_p$ maps $\mathbb{Z}[\zeta_p]$ into its augmentation $\{ \sum_i \lambda_i \zeta_p^i : \sum_i = 0 \}$. In particular, the restriction of $\partial_{\zeta_p}$ on $\mathbb{Z}[\zeta_p]$ is not surjective.
Therefore the isomorphism \( \partial_{\psi} \) does not have property \( \mathbb{I} \). In particular, neither \( G \) nor \( H \) belong to the class \( \Sigma \).

**Example 2.10.** Take \( C_p = \langle x \rangle \) for some prime \( p \) and consider a \( \mathbb{Z}[C_p] \)-module \( A \). Assume that \( \partial_x \) is an isomorphism of \( A \). To verify whether or not \( \partial_x \) has property \( \mathbb{I} \), it suffices to show that the restriction of \( \partial_x \) on every cyclic submodule of \( \mathbb{Z} \) is surjective. To do this, suppose that \( B \) is a cyclic \( \mathbb{Z}[C_p] \)-module with \( \partial_x \) having trivial kernel on \( B \). Therefore \( B \) is isomorphic to a quotient of the ring \( \mathbb{Z}[C_p] \) by some ideal \( J \). Denote \( D = x - 1 \) and \( N = x^{p-1} + x^{p-2} + \cdots + 1 \) as elements in \( \mathbb{Z}[C_p] \). Observe that injectivity of \( \partial_x \) is equivalent to saying that whenever \( D \cdot z \in J \) for some \( z \in B \), it follows that \( z \in J \). Now, we have that \( D \cdot N = 0 \), and so it follows that \( N \in J \). Therefore \( J \) is the preimage of an ideal in the ring \( \mathbb{Z}[C_p]/NZ[C_p] \cong \mathbb{Z}[\zeta_p] \), where \( \zeta_p \) is a complex primitive \( p \)-th root of unity. Note that multiplication by \( D \) is surjective on \( \mathbb{Z}[C_p]/J \) if and only if we have \( \text{im}D + J = \mathbb{Z}[C_p] \).

Consider two special cases. First let \( J = NZ[C_p] \). This corresponds to the module \( \mathbb{Z}[\zeta_p] \) from Example 2.9. Since \( \text{im}N = \ker D \) (see [15, Lemma 9.26]), we have that \( \text{im}D + J = \text{im}D + \text{im}N \). Dividing the polynomial \( N \) by \( D \) in \( \mathbb{Z}[\langle x \rangle] \), we get the remainder \( p \in \mathbb{Z} \). Whence \( \text{im}D + \text{im}N \) contains \( p\mathbb{Z}[C_p] \). On the other hand, \( \text{im}D + \text{im}N \) is not the whole of \( \mathbb{Z}[C_p] \), since \( \partial_{\psi} \) is not surjective on \( \mathbb{Z}[\zeta_p] \). Consider now the case when \( J \) is the ideal generated by \( N \) and a prime \( q \) distinct from \( p \). Thus \( \text{im}D + J \) contains \( p\mathbb{Z}[C_p] \) and \( q \). It follows that \( \text{im}D + J = \mathbb{Z}[C_p] \), and \( \partial_x \) is surjective in this case. Moreover, the map \( \partial_x \) will be surjective on all cyclic submodules of \( \mathbb{Z}[C_p]/J \), as the ideals corresponding to these submodules all contain \( J \). Therefore the group \( \langle x \rangle \ltimes \mathbb{Z}_q[\zeta_p] \) belongs to \( \Sigma \).

**Lemma 2.11.** Let \( x \) be an automorphism of order \( p \) of an abelian group \( A \). If \( \partial_x \) is surjective, then \( A = pA \), i.e., \( A \) is \( p \)-divisible.

**Proof.** Consider \( A \) as a \( \mathbb{Z}[\langle x \rangle] \)-module. In this sense, the operator \( \partial_x \) corresponds to the element \( 1 - x \in \mathbb{Z}[\langle x \rangle] \). We have \( (1 - x)^p \equiv 0 \) modulo \( p\mathbb{Z}[\langle x \rangle] \), and so the image of \( (\partial_x)^p : A \to A \) is a subgroup of \( p\mathbb{Z}[\langle x \rangle]A = pA \). As \( \partial_x \) is assumed to be surjective, it follows that \( A = pA \). \( \square \)

**Corollary 2.12.** Let \( G = \langle x \rangle \ltimes A \) with \( x^p \) acting trivially on \( A \) for some prime \( p \). Assume that \( A \) is free abelian of finite rank. Then \( G \) does not belong to \( \Sigma \).

**Proof.** By Lemma 2.11, the map \( \partial_x \) is not surjective, and so \( \partial_x \) does not have property \( \mathbb{I} \). It follows from Proposition 2.8 that \( G \) does not belong to \( \Sigma \). \( \square \)
Theorem 2.13. Let $G$ be a finite soluble non-nilpotent group. Then $G \in \Sigma$ if and only if $G$ splits as $G = \langle x \rangle \rtimes A$, where $\langle x \rangle$ is a $p$-group for some prime $p$, $A$ is an abelian $p'$-group, $x^p$ is central and $x$ acts fixed point freely on $A$.

Proof. Assume first that $G \in \Sigma$. By Lemma 2.11, all Sylow subgroups of $G$ are abelian. It follows that $G' \cap Z(G) = 1$ by [11, 10.1.7], and $G$ splits as $G = \langle x \rangle \rtimes G'$ with $x^p$ in the Fitting subgroup of $G$ for some prime $p$. Whence $\langle x^p \rangle \leq Z(G)$. If an element $xa \in G$ is central, then $a^x = a$ and so $a$ must be central. As $x$ is not central, we must have $Z(G) = \langle x^p \rangle$ and $C_G(x) = \langle x \rangle$. Observe that as $G$ belongs to $\Sigma$, the map $\partial_x$ is surjective on $G'$, and so by Lemma 2.11 the group $G'$ must be of $p'$-order. Now, if $\langle x \rangle$ is not of prime power order, then it splits as a product $\langle x \rangle = A_p \rtimes A_p'$ with $A_p$ a $p$-group and $A_p'$ a $p'$-group. Then $A_p \rtimes G'$ is a non-abelian proper normal subgroup of $G$, a contradiction. Whence $\langle x \rangle$ is of $p$-power order. Note that $x$ acts fixed point freely on $G'$ since $C_G(x) \cap G' = 1$. Thus $G/Z(G)$ is a Frobenius group with complement of order $p$.

Conversely, take $G = \langle x \rangle \rtimes A$ with the stated properties. Therefore $x^p$ acts trivially on $A$ and $\partial_x$ is an injective endomorphism of $A$. As $x$ is finite, it immediately follows that $\partial_x$ is surjective and that it satisfies property II. It now follows from Lemma 2.8 that $G$ belongs to $\Sigma$. \hfill \Box

Notice that in Theorem 2.13 we have that $\langle x \rangle$ is the Sylow $p$-subgroup of $G$ and $F = \langle x^p \rangle \rtimes A$, so $A = F/\langle x^p \rangle$.

Theorem 2.14. Let $G$ be a finite group in $\Sigma$. Then $G$ is either soluble or simple.

Proof. By induction on the order of $G$. Suppose that $G$ is not simple, and let $A$ be a maximal normal subgroup of $G$. Then $A$ is abelian since $G$ is in $\Sigma$, hence $C_G(A)$ contains $A$ and is normal in $G$. It follows that either $C_G(A)$ is abelian or $C_G(A) = G$.

Assume first that $C_G(A)$ is abelian. Thus $A = C_G(A)$ by the maximality. Let $P/A$ be a Sylow subgroup of $G/A$, and choose $x \in P \setminus A$. Then $A\langle x \rangle$ is not abelian and subnormal in $P$, so $A\langle x \rangle = P$ by (i) of Proposition 2.2. This shows that every Sylow subgroup of $G/A$ is cyclic, hence $G/A$ is soluble. Therefore $G$ is soluble.

Hence we can assume that $C_G(A) = G$, and so $A \leq Z(G)$. Moreover, $A$ is contained in every maximal subgroup $M$ of $G$. Namely, if this is not the case, then $G = MA$ implies that $G' = M' < G$, so $G'$ is abelian, hence $G$ is soluble and we are done. Therefore every maximal subgroup of $G$ is non-simple, so it is soluble by the induction hypothesis. Suppose that $G$ has a maximal subgroup which is nilpotent. Then
Proposition 2.6 implies that its Sylow 2-subgroup has nilpotency class $\leq 2$, therefore it follows from \[11\] that $G$ is soluble. Hence we can assume that every maximal subgroup of $G$ is not nilpotent. In this case, every Sylow subgroup of $G$ is abelian by Lemma 2.7. Then we have $G' \cap Z(G) = 1$, hence $G' < G$. Therefore $G'$ is abelian, $G$ is soluble and the proof is complete. $\square$

Theorem 2.14, Proposition 2.6 and Theorem 2.13 show that, in order to obtain a full classification of all finite groups in the class $\Sigma$, it only remains to describe the finite simple groups in $\Sigma$. At first we need a couple of auxiliary results.

Lemma 2.15. Let $n > 2$. The dihedral group $\text{Dih}(n)$ of order $2n$ belongs to $\Sigma$ if and only either $n = 4$ or $n$ is odd.

Proof. Denote $G = \text{Dih}(n) = \langle x, y \mid x^n = y^2 = 1, xy = x^{-1} \rangle$, where $n \neq 4$. Suppose that $G \in \Sigma$. Then $n$ is not a power of 2 by Proposition 2.6. Let $p$ be an odd prime dividing $n$, and assume that $n$ is even. Denote $H = \langle y, x^{n/p} \rangle$. Then $x^{n/2} \in Z(G) \setminus H$, hence $H$ is not self-normalized. Therefore $n$ is odd.

Conversely, clearly $\text{Dih}(4)$ belongs to $\Sigma$ since it is minimal non-abelian. Suppose now $n$ is odd. Let $H$ be a non-abelian subgroup of $\text{Dih}(n)$ of index $m$. Then $H$ is conjugate to $K = \langle x^m, y \rangle$. Take $z = x^iy^j \in N_G(K)$, $0 \leq i < n$, $0 \leq j \leq 1$. Then $x^{i}y^{j} = x^{2^{i}(j+1)}y \in K$ if and only $m$ divides $i$, that is, $z \in K$. This shows that $\text{Dih}(n) \in \Sigma$. $\square$

Lemma 2.16. If $q \neq 3, 5$ is an odd prime power, then $\text{PSL}(2, q)$ does not belong to $\Sigma$.

Proof. Let $G = \text{PSL}(2, q)$. Since $q$ is odd, it follows from [6] that $G$ contains a subgroup $H$ isomorphic to $\text{Dih}((q - 1)/2)$, and a subgroup $K$ isomorphic to $\text{Dih}((q + 1)/2)$. If $q \equiv 1 \mod 4$, then $H$ is not in $\Sigma$, unless $q = 5$, whereas if $q \equiv 3 \mod 4$, then $K \notin \Sigma$, unless $q = 3$ or $q = 7$. Notice that $\text{PSL}(2, 7) \notin \Sigma$ since it has a subgroup isomorphic to $\text{Sym}(4)$ (see [10, Theorem 8.27]), and therefore a non-abelian subgroup which is isomorphic to $\text{Alt}(4)$ and is not self-normalizing. $\square$

Theorem 2.17. A finite non-abelian simple group $G$ belongs to $\Sigma$ if and only if it is isomorphic to $\text{Alt}(5)$ or $\text{PSL}(2, 2^n)$, where $2^n - 1$ is a prime.

Proof. Let $G \in \Sigma$ be finite non-abelian simple. Let $P_p$ be a Sylow $p$-subgroup of $G$. It follows from [9, Theorem 1.1] that if $p > 3$, then $P_p$ is abelian. For $p = 3$, the same result implies that either $P_3$ is abelian or $G \cong \text{PSL}(2, 3^{3a})$, where $a \geq 1$. The latter cannot happen
by Lemma 2.16. Hence we conclude that $P_3$ needs to be abelian. If $P_2$ is also abelian, then all Sylow subgroups of $G$ are abelian, and it follows from [2] that $G$ belongs to one of the following groups: $J_1$, or $\text{PSL}(2,q)$, where $q > 3$ and $q \equiv 0, 3, 5 \mod 8$. Note that the latter condition can be reduced to $q \equiv 0 \mod 8$ or $q = 5$ by Lemma 2.16. If $P_2$ is non-abelian, then it is minimal non-abelian by Proposition 2.6, and hence $P_2$ is nilpotent of class two. By [8], $G$ is isomorphic to one of the following groups: $\text{PSL}(2,q)$, where $q \equiv 7, 9 \mod 16$, $\text{Alt}(7)$, $\text{Sz}(2^n)$, $\text{PSU}(3,2^n)$, $\text{PSL}(3,2^n)$ or $\text{PSp}(4,2^n)$, where $n \geq 2$. The first family can be ruled out by Lemma 2.16. It suffices to see which of the above listed groups belong to $\Sigma$. It follows from ATLAS that the Janko group $J_1$ has a subgroup isomorphic to $\text{Dih}(3) \times \text{Dih}(5)$, hence it is not in $\Sigma$ by Proposition 2.1. Also, note that $\text{Alt}(7)$ has a subgroup isomorphic to $\text{Sym}(4)$, therefore $\text{Alt}(7) \not\in \Sigma$. If $G = \text{Sz}(2^n)$ and $P_2$ is its Sylow 2-subgroup, then $|P_2'| = 2^n$ by [16], hence the Suzuki groups do not belong to $\Sigma$. Similarly, if $G$ is $\text{PSU}(3,2^n)$ or $\text{PSL}(3,2^n)$, then the derived subgroup of a Sylow 2-subgroup of $G$ has order $2^n$, whereas if $G = \text{PSp}(4,2^n)$, then $|P_2'| = 2^{2n}$. This shows that neither of these groups belongs to $\Sigma$.

We are left with the groups $G = \text{PSL}(2,q)$, where $q = 5$ or $q \equiv 0 \mod 8$. The subgroup structure of $G$ is described in [3]. It is straightforward to verify that $\text{PSL}(2,5) = \text{Alt}(5) \in \Sigma$. Consider now $q = 2^n$ with $n \geq 3$. Suppose first $q - 1$ is not a prime. Let $d$ be a proper divisor of $q - 1$. Then it follows from [6] section 250 that $G$ has a single conjugacy class of size $q + 1$ of a subgroup $H$ isomorphic to $C_2^n \rtimes C_d$. Therefore $|G : N_G(H)| = q + 1$, and $|G : H| = (q^2 - 1)/d$. As $d < q - 1$, it follows that $H \neq N_G(H)$, therefore $G \not\in \Sigma$. On the other hand, let $q - 1$ now be a prime. Going through the list of subgroups of $G$ given in [6], along with the given data on the number of conjugates of these subgroups, we see that apart from the subgroups in section 250, one has that for every non-abelian subgroup $H$ of $G$, the number of conjugates of $H$ is equal to $|G : H|$. As for the remaining subgroups, note that they must be of order $2^md$ for some integer $m$ and some divisor $d$ of $q - 1$. There are only two such options, one corresponding to abelian groups of order $2^m$ and the other corresponding to subgroups of order $q(q - 1)$. Each of the latter belongs to a system of $(q^2 - 1)2^{n-m}/(2^k - 1)$ conjugate groups for some $k$ dividing $n$. Note that since $q - 1$ is a prime, $n$ must also be a prime. When $k = 1$, the group under consideration is abelian; therefore $k = n$. We thus have that the number of conjugates of each of these subgroups is equal to their index in $G$. By Lemma 2.4 this shows that if $2^n - 1$ is a prime, then $\text{PSL}(2,2^n)$ belongs to $\Sigma$. □
3. Infinite groups

Let $G \in \Sigma$ be an infinite finitely generated soluble group, and let $F$ denote the Fitting subgroup of $G$. Then $F$ is polycyclic and $G = \langle x \rangle F$ for every element $x \in G \setminus F$, by Lemma 2.3. We will denote by $h(F)$ the Hirsch length of $F$. In what follows, the set of all periodic elements of a group $G$ will be denoted by $T(G)$. For a prime $p$, let $T_p(G)$ be the set of elements in $G$ of $p$-power order, and let $T_p'(G)$ be the set of elements in $G$ of order coprime to $p$.

Lemma 3.1. Let $G \in \Sigma$ be an infinite finitely generated soluble group, and suppose $h(F) = 1$. Then $G$ is abelian.

Proof. Assume not. Since $h(F) = 1$ the group $G$ is infinite cyclic-by-finite. It easily follows that there exists a finite normal (and hence abelian) subgroup $N$ such that $G/N$ is either infinite cyclic or infinite dihedral. As the latter group is not in $\Sigma$, we can write $G = \langle x \rangle \rtimes N$ where $x$ aperiodic. Since $N \leq F$ and $G$ is not abelian, we conclude that $x \notin F$. By Lemma 2.3 there exists a prime number $p$ such that $x^p \in F$. Then $x^p \in Z(G)$. Then $G/\langle x^p \rangle$ is finite. Moreover, it is not nilpotent by [5, Theorem 3.1]. The Fitting subgroup of $G/\langle x^p \rangle$ is $F/\langle x^p \rangle = \langle x^p \rangle \times T(F)$, so it equals $T(F)$ since $h(F) = 1$. Now by Theorem 2.13 it follows that $G/\langle x^p \rangle$ is a normal proper subgroup of the factor group $G/\langle x^{pq} \rangle$. Note that $\langle x^p \rangle \times T(F)$ is not abelian since $x \notin Z(G)$. Therefore $G/\langle x^{pq} \rangle \notin \Sigma$, a contradiction.

Theorem 3.2. Let $G \in \Sigma$ be an infinite finitely generated soluble group. Then $G$ is abelian.

Proof. We have $G = \langle x \rangle F$ with $x^p \in F \cap Z(G)$. Consider first the case when $x^p = 1$. Thus $G = \langle x \rangle \rtimes F$.

Let us show that $x$ acts fixed point freely on $F$. To this end, let $f \in F$ be an element with $f^x = f$. For any positive integer $k$, the quotient groups $G/F^k$ are finite and belong to $\Sigma$. This shows that $f^xF^k = fF^k$, and thus Theorem 2.13 gives that $fF^k$ is trivial in $G/F^k$. This means that $f \in \bigcap_k F^k = 1$. Therefore $x$ acts fixed point freely on $F$ and $\partial_x$ is injective.

Since $G \in \Sigma$, the group $F$ can not be free abelian by Corollary 2.12. Thus the torsion subgroup $T(F)$ is not trivial. The factor group $G/T(F)$ belongs to $\Sigma$, and so the action of $x$ on this group is trivial by Corollary 2.12. Therefore $\partial_x(F)$ is trivial in $G/T(F)$. So the image of
Theorem 3.3. Let \( \Sigma \) result completes the description of all infinite soluble groups in non-periodic soluble group in \( \Sigma \) Then \( G \) that every aperiodic element of the map \( \partial \) since the group \( G/\langle z \rangle \) in \( p \) group for some prime \( p \) proving the converse. \( \Sigma \) decomposition belongs to It follows from Proposition 2.8 that a group \( h \) with an element of order \( p \) \( \langle x \rangle \) that is normalized by the element \( z \) trivially on \( A \) trivial, as claimed. Hence the group \( \langle x \rangle \) \( \langle x \rangle \) is contained in the finite group \( G/\langle x \rangle \) is an infinite finitely generated soluble group with an element of order \( p \) outside its Fitting subgroup. Therefore \( G/\langle x \rangle \) must be abelian by the above argument. As \( \langle x \rangle \) is contained in \( Z(G) \), it follows that \( G \) is nilpotent, and so \( G \) must be abelian. The proof is now complete. \( \Box \)

Let \( G \in \Sigma \) be a soluble group. It follows easily from Theorem 3.2 that every aperiodic element of \( G \) is central. As a consequence, every non-periodic soluble group in \( \Sigma \) is abelian. Therefore the following result completes the description of all infinite soluble groups in \( \Sigma \).

**Theorem 3.3.** Let \( G \) be an infinite non-abelian soluble periodic group. Then \( G \) belongs to \( \Sigma \) if and only if \( G \) splits as \( \langle x \rangle \ltimes A \) with \( \langle x \rangle \) a \( p \)-group for some prime \( p \), \( A \) is a \( p' \)-group, \( x^p \) acts trivially on \( A \) and \( \partial_x \) has property \( \mathbb{I} \).

**Proof.** It follows from Proposition 2.8 that a group \( G \) with the above decomposition belongs to \( \Sigma \). Therefore we are only concerned with proving the converse.

Assume that \( G \in \Sigma \) is an infinite non-abelian soluble periodic group. We have that \( G = \langle x \rangle F \) with \( x^p \in F \cap Z(G) \) for some prime \( p \). Note that since \( x \) is of finite order, say, \( p^k \beta \) for some \( \beta \) coprime to \( p \), we may replace \( x \) by \( x^\beta \) and assume from now on that \( \langle x \rangle \subseteq T_p(F) \).

Let us first prove that \( T_p(F) = \langle x^p \rangle \). To this end, it suffices to consider the factor group \( G/\langle x^p \rangle \), and therefore we can assume that \( x^p = 1 \). Thus \( G = \langle x \rangle \ltimes F \), and so \( G = (\langle x \rangle \ltimes T_p(F)) \ltimes T_p'(F) \). If the group \( \langle x \rangle \ltimes T_p(F) \) is not cyclic, then there is an element \( z \in T_p(F) \) that commutes with \( x \). It follows that the group \( \langle x \rangle \ltimes T_p(F) \) contains the subgroup \( \langle x \rangle \ltimes \langle z \rangle \cong C_p \times C_p \). Now \( G \) contains the subgroup \( \langle x \rangle \ltimes T_p(F) \) that is normalized by the element \( z \). This is a contradiction with \( G \in \Sigma \). Hence the group \( \langle x \rangle \ltimes T_p(F) \) is cyclic. This is possible only if \( T_p(F) \) is trivial, as claimed.

We now have a splitting \( G = \langle x \rangle \ltimes A \) with \( A = T_p'(F) \), \( x \) acts non-trivially on \( A \) and \( x^p \) is central. Let us now show that \( x \) acts fixed point freely on \( A \). It will then follow from Proposition 2.8 that \( \partial_x \) has property \( \mathbb{I} \). To see this, assume that \( z \in A \) is a fixed point of \( x \). Thus \( z \in Z(G) \). In particular, as \( G \in \Sigma \), we have that \( z \) must be contained in every non-abelian subgroup of \( G \). Now, as \( x \) acts non-trivially on \( A \), there is an element \( b \in A \) with \( b^x \neq b \). Set \( B = \langle b, b^x, \ldots, b^{x^{p-1}} \rangle \),
this is an $x$-invariant finite subgroup of $G$. Therefore $G$ possesses the finite non-abelian subgroup $\langle x \rangle \rtimes B$. By Theorem 2.13 we have that $x$ acts fixed point freely on $B$. On the other hand, we must have that $z \in \langle x \rangle \rtimes B$, and so $z \in B$. This implies that $z$ is trivial, as required. The proof is complete. □

Let $G$ be an infinite group in $\Sigma$, and suppose that $G$ is not soluble. Then $G$ is perfect by (ii) of Proposition 2.2. Our last result gives information on the structure of such a group $G$ provided that it is locally finite.

**Theorem 3.4.** Let $G$ be an infinite locally finite group in $\Sigma$. Then $G$ is metabelian.

**Proof.** Let $G \in \Sigma$ be locally finite, and suppose that $G$ is not metabelian. Then $G$ contains a finite insoluble subgroup, say $H_0$. It follows from Theorem 2.14 and Theorem 2.17 that $H_0$ is isomorphic to either $\text{Alt}(5)$ or some $\text{PSL}(2, 2^n)$ with $n$ a prime. Pick an element $x_1 \in G$ that does not belong to $H_0$, and set $H_1 = \langle x_1, H_0 \rangle$. We now have that the group $H_1$ is isomorphic to some $\text{PSL}(2, 2^m)$ with $m$ a prime. Finally let $x_2 \in G$ be an element not in $H_1$, and set $H_2 = \langle x_2, H_1 \rangle$. The group $H_2$ is isomorphic to some $\text{PSL}(2, 2^k)$ with $k$ a prime. Now, as $H_2 = \text{PSL}(2, 2^k)$ properly contains $H_1 = \text{PSL}(2, 2^m)$, we must have $m \mid k$, which is impossible since $m$ and $k$ are distinct primes. □

**References**

[1] Y. Berkovich, Groups of prime power order, Vol. 1, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[2] A. M. Broshi, *Finite groups whose Sylow subgroups are abelian*, J. Algebra **17** (1971), 74–82.

[3] C. Delizia, U. Jezernik, P. Moravec and C. Nicotera, *Groups in which every non-cyclic subgroup contains its centralizer*, J. Algebra Appl. **13** (2014), no. 5, 1350154 (11 pages).

[4] C. Delizia, U. Jezernik, P. Moravec, C. Nicotera and C. Parker, *Locally finite groups in which every non-cyclic subgroup is self-centralizing*, J. Pure Appl. Algebra, DOI 10.1016/j.jpaa.2016.06.015.

[5] C. Delizia, H. Dietrich, P. Moravec and C. Nicotera, *Groups in which every non-abelian subgroup is self-centralizing*, J. Algebra **462** (2016), 23–36.

[6] L. E. Dickson, *Linear groups, with an exposition of the Galois field theory*, Teubner, Leipzig, 1901.

[7] G. Fernández-Alcober, L. Legarreta, A. Tortora and M. Tota, *Some restrictions on normalizers or centralizers in finite $p$-groups*, Israel J. Math. **208** (2015), 193–217.

[8] R. Gilman and D. Gorenstein, *Finite groups with Sylow 2-subgroups of class two. I*, Trans. Amer. Math. Soc. **207** (1975), 1–101.
[9] R. M. Guralnick, G. Malle and G. Navarro, *Self-normalizing Sylow subgroups*, Proc. Amer. Math. Soc. **132** (2003), no. 4, 973–979.
[10] B. Huppert, *Endliche Gruppen I*, Springer Verlag, 1967.
[11] Z. Janko, *Verallgemeinerung eines Satzes von B. Huppert und J. G. Thompson*, Arch. Math. **12** (1961), 280–281.
[12] O.H. Kegel and B.A.F. Wehrfritz, *Locally Finite Groups*, North-Holland Publishing Company, 1973.
[13] C. R. Leedham-Green and S. McKay, *The structure of groups of prime power order*, Oxford University Press, 2002.
[14] D. J. S. Robinson, *A course in the theory of groups*, 2nd Edition, Springer-Verlag, 1996.
[15] J. J. Rotman, *An introduction to homological algebra*, Universitext, 2nd Edition, Springer, New York, 2009.
[16] M. Suzuki, *On a class of doubly transitive groups*, Ann. Math. **75** (1962), 105–145.

University of Salerno, Italy
*E-mail address:* cdelizia@unisa.it

University of Ljubljana, Slovenia
*E-mail address:* urban.jezernik@fmf.uni-lj.si

University of Ljubljana, Slovenia
*E-mail address:* primoz.moravec@fmf.uni-lj.si

University of Salerno, Italy
*E-mail address:* cnicoter@unisa.it