Averaging procedure in variable-$G$ cosmologies

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Abstract

Previous work in the literature had built a formalism for spatially averaged equations for the scale factor, giving rise to an averaged Raychaudhuri equation and averaged Hamiltonian constraint, which involve a backreaction source term. The present paper extends these equations to include models with variable Newton parameter and variable cosmological term, motivated by the nonperturbative renormalization program for quantum gravity based upon the Einstein-Hilbert action. We focus on the Brans-Dicke form of the renormalization-group improved action functional. The coupling between backreaction and spatially averaged three-dimensional scalar curvature is found to survive, and a variable-$G$ cosmic quintet is found to emerge. Interestingly, under suitable assumptions, an approximate solution can be found where the early universe tends to a FLRW model, while keeping track of the original inhomogeneities through three effective fluids. The resulting qualitative picture is that of a universe consisting of baryons only, while inhomogeneities average out to give rise to the full dark-side phenomenology.
I. INTRODUCTION

Over the last ten years, the use of the effective average action [1] and of the renormalization-group equations has made it possible to obtain encouraging evidence in favour of Einstein’s gravity being renormalizable at non-perturbative level [2, 3, 4, 5], with all running couplings having a finite limit at large momenta. The cornerstone of this program is the discovery of a new non-Gaussian ultraviolet fixed point, besides the trivial one at the origin [6, 7, 8, 9]. The resulting picture seems to be as follows: at sub-Planckian distances, spacetime is a fractal. It can be thought of as a self-similar hierarchy of superimposed Riemannian manifolds of any curvature. As one considers larger length scales where the renormalization-group running of the gravitational parameters comes to a halt, the spacetime ripples gradually disappear, and a classical four-dimensional spacetime manifold is recovered [6].

In the simplest possible terms, the renormalization group improvement consists in the modified Einstein equations [10]

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda(k)g_{\mu\nu} = 8\pi G(k)T_{\mu\nu}, \]  (1.1)

where the Newton parameter \( G \) and cosmological term \( \Lambda \) are now dependent on the scale \( k \), \( k \) being the running cut-off of the renormalization group equation [1]. In cosmology, the dynamical evolution is determined by a set of renormalization group equations by means of the cut-off identification \( k = k(t) \) which relates the energy scale of the running cutoff \( k \) of the renormalization group, with the cosmic time \( t \). In [11] it has been shown that, in a cosmological setting, the correct cutoff identification is \( k \propto t^{-1} \); it is thus possible to determine \( G(k(t)) \) and \( \Lambda(k(t)) \) in Eq. (1.1) once a renormalization-group trajectory is determined.

At a deeper level, one integrates out all fluctuations with momenta larger than a cutoff \( \bar{k} \), and one takes them into account by means of a modified dynamics for remaining fluctuation modes with momenta smaller than \( \bar{k} \). This modified dynamics is generated by a scale-dependent effective action \( \Gamma_k \), whose \( k \)-dependence is ruled by the exact renormalization-group equations. Flow equations can be used for a complete quantization of fundamental theories. On denoting by \( S \) their classical action, one imposes the initial condition \( \Gamma(k = \kappa) = S \) at the scale of the ultraviolet cutoff \( \kappa \), and one exploits the renormalization-group
equation to evaluate this averaged effective action $\Gamma_k$ for all $k < \kappa$, and one then takes the limits $k \to 0$ and $\kappa \to \infty$. In the case of fundamental theories, the continuum limit $\kappa \to \infty$ exists after having renormalized finitely many parameters in the action, and is evaluated at a non-Gaussian fixed point of the renormalization-group flow. Such a new fixed point replaces the Gaussian fixed point which, at least implicitly, underlies the construction of theories which are instead perturbatively renormalizable \cite{12}.

Notwithstanding its merits (see, however, the criticism expressed in \cite{13}), the renormalization group approach has only been applied, so far, to a strictly homogenous and isotropic universe. As is well known, however, the matter distribution in the observable universe may be considered homogenously distributed only on large scales so that it is interesting to investigate how this can affect the renormalization-group equations. Needless to say, lacking a detailed description of the matter distribution, we can only consider a kind of average universe obtained by averaging out the inhomogeneities. Fortunately, over the last decade, progress has been made on the longstanding problem of how to average a general inhomogeneous model. In particular, the work in \cite{14} considers an irrotational fluid motion with the associated Einstein equations (in $c = 1$ units)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G \rho u_\mu u_\nu. \quad (1.2)$$

A flow-orthogonal coordinate system $x^\mu = (t, X^k)$ is chosen, i.e. Gaussian normal coordinates comoving with the fluid. On writing $x^\mu = f^\mu(X^k, t)$, one has $u^\mu = \partial f^\mu / \partial t = (1, 0, 0, 0)$ and $u_\mu = \partial f^\mu / \partial t = (-1, 0, 0, 0)$, where $t$ is proper time. The spacelike hypersurfaces of constant $t$ are assumed to foliate the spacetime manifold, and their first fundamental form, i.e. the spatial 3-metric $g_{ij}$, is used to define

$$J(t, X^i) \equiv \sqrt{\det g_{ij}}. \quad (1.3)$$

The spatial averaging of a scalar field $\psi$ as a function of Lagrangian coordinates and time on an arbitrary compact portion $D$ of the fluid is defined by the volume integral (cf. \cite{15})

$$\langle \psi(t, X^i) \rangle_D \equiv \frac{1}{V_D} \int_D \psi(t, X^i) J d^3X, \quad (1.4)$$

where the volume $V_D$ is obtained as

$$V_D(t) \equiv \int_D J d^3X. \quad (1.5)$$
A key role in the formalism is played by a dimensionless effective scale factor

\[ a_D(t) \equiv \left( \frac{V_D(t)}{V_D(t_0)} \right)^{\frac{1}{3}}, \]  

in terms of which the averaged expansion rate takes the form (the dot being used for the total derivative with respect to \( t \))

\[ \langle \theta \rangle_D = \frac{\dot{V}_D}{V_D} = 3 \frac{\dot{a}_D}{a_D}. \]  

In [14], the scale factor \( a_D \) obeys the averaged Raychaudhuri equation

\[ 3 \frac{\ddot{a}_D}{a_D} + 4\pi G \langle \rho \rangle_D - \Lambda = Q_D = \frac{2}{3} \langle (\theta - \langle \theta \rangle_D)^2 \rangle_D - 2 \langle \sigma^2 \rangle_D, \]  

where \( Q_D \) is the backreaction source term and \( \sigma^2 \) is the rate of shear, and the Hamiltonian constraint

\[ 3 \left( \frac{\dot{a}_D}{a_D} \right)^2 - 8\pi G \langle \rho \rangle_D + \frac{1}{2} \langle (3) R \rangle_D - \Lambda = -\frac{1}{2} Q_D, \]  

where \( (3) R \) is the scalar curvature of the constant time spacelike hypersurfaces used in the spacetime foliation.

Our aim here is to generalize the above averaging procedure to the variable-\( G \) cosmologies resulting from the renormalization-group approach, to gain a better understanding of the conditions under which a FLRW universe can be recovered from a renormalization-group approach (rather than imposing a FLRW symmetry as has been done so far). For this purpose, section 2 derives the Buchert average of the field equations obtained from a Brans-Dicke approach to the renormalization-group improved gravitational action. The integrability condition for the above equations is obtained in section 3, while a variable-\( G \) cosmic quintet is found to emerge in section 4. Section 5 studies accelerating patches and stationary models, while a solution formula for the spatially averaged scalar curvature is obtained in section 6. A particular solution of the averaged equations is then investigated in Section 7, where we also consider the case of a nearly homogenous universe through the useful introduction of a set of effective fluids clarifying the role of the different terms. Results and open problems are discussed in section 8. The basic identities used for the evaluation of Buchert averages are described in the appendix.
II. THE BUCHERT AVERAGING METHOD WITH VARIABLE $G$ AND $\Lambda$ IN A BRANS-DICKE APPROACH

In light of equations (1.3)–(1.9), it is clear that the Buchert averaging method aims at taking spatial averages of equations obtained from suitable contractions and traces of tensor field equations. In particular, within the framework of running-$G$ models, an interesting class is given by the Brans-Dicke approach developed in [16]. Here, the Einstein-Hilbert action is renormalization-group improved by replacing the Newton constant and the cosmological constant by scalar functions $G$ and $\Lambda$ in the corresponding Lagrangian density [16]. The position dependence of $G$ and $\Lambda$ is governed by a renormalization-group equation, and they have the status of externally prescribed background fields (whereas in [17] $G$ obeys an Euler-Lagrange equation), while the metric satisfies an effective Einstein equation similar to that of Brans-Dicke theory [16].

Following [16], we assume that the total action functional $S$ of our theory consists of the Einstein-Hilbert part $S_{EH}$ plus matter $S_M$ plus a term describing the four-momentum carried by the fields $G(x)$ and $\Lambda(x)$, i.e. (hereafter $\varphi$ denotes the collection of matter fields coupled to gravity)

$$S = S_{EH}(g, G, \Lambda) + S_M(g, \varphi) + S_\theta(g, G, \Lambda).$$

(2.1)

For the explicit form of the terms in (2.1) we refer the reader to Ref. [16], for length reasons. The resulting renormalization-group improved Einstein equation is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} + 8\pi GT_{\mu\nu} + \tau_{\mu\nu} + \vartheta_{\mu\nu},$$

(2.2)

where $\tau_{\mu\nu}$ results from the $x$ dependence of $G$ [16]:

$$\tau_{\mu\nu} = \frac{1}{G^2} \left\{ 2(\nabla_\mu G)(\nabla_\nu G) - G\nabla_\mu \nabla_\nu G - g_{\mu\nu} \left[ 2(\nabla_\rho G)(\nabla_\sigma G) - Gg^{\rho\sigma} \nabla_\mu \nabla_\nu G \right] \right\},$$

(2.3)

while $\vartheta_{\mu\nu}$ is obtained from the functional derivative of $S_\theta$ with respect to the metric, and can be taken to be of the form [16]

$$\vartheta_{\mu\nu} = -\frac{3}{2} \frac{1}{G^2} \left[ (\nabla_\mu G)(\nabla_\nu G) - \frac{1}{2} g_{\mu\nu} (\nabla_\rho G)(\nabla_\sigma G) \right],$$

(2.4)

which is a Brans-Dicke energy-momentum tensor for the field $1/G$. For our purposes it is convenient to add explicitly the tensors $\tau_{\mu\nu}$ and $\vartheta_{\mu\nu}$ to find

$$\Phi_{\mu\nu} \equiv \tau_{\mu\nu} + \vartheta_{\mu\nu}$$

$$= \frac{1}{G^2} \left\{ \frac{1}{2} (\nabla_\mu G)(\nabla_\nu G) - G \nabla_\mu \nabla_\nu G + g_{\mu\nu} \left[ -\frac{5}{4} (\nabla_\rho G)(\nabla_\sigma G) + G \Box G \right] \right\},$$

(2.5)
where \( g^{\alpha\beta}\nabla_\alpha \nabla_\beta \) is the standard notation for the wave operator.

The first equation we need on our way towards Buchert averages is the Hamiltonian constraint or \( G_{00} \) component, obtained from contraction of the renormalization-group improved Einstein equation (2.2) with \( u^\mu u^\nu \), i.e.

\[
u^\mu u^\nu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{1}{2} \left( R^{(3)} + K^2 - K_{ij} K^{ij} \right) = 8\pi G \rho + \Lambda + u^\mu u^\nu \Phi_{\mu\nu},
\]

where we denote by \( K_{ij} \) the extrinsic-curvature tensor of the spacelike hypersurfaces that foliate the spacetime manifold and have taken \( T_{\mu\nu} \) of the form \[14\]

\[
T_{\mu\nu} = \rho u_{\mu} u_{\nu},
\]

while

\[
u^\mu u^\nu \Phi_{\mu\nu} = \frac{1}{G^2} \left[ \frac{1}{2} G_{\alpha\beta}^2 - G G_{\alpha\beta} + \frac{5}{4} (\nabla_\rho G)(\nabla^\rho G) - G \Box G \right]
= -\frac{3}{4} \left( \frac{G_{,\rho}}{G} \right)^2 + \frac{\theta}{4} G_{,\rho} \frac{G_{,\rho}}{G} + \frac{5}{4} g^{ij} G_{,i} G_{,j} - \frac{\Delta G}{G},
\]

with \( \Delta \equiv g^{ij} \nabla_i \nabla_j \) the Laplacian operator (up to a sign). Hereafter, it is convenient to define

\[
\psi_G \equiv \log G,
\]

and bear in mind that the spatial part of the spacetime four-metric is, in our coordinates, the induced Riemannian three-metric \( h_{ij} dx^i \otimes dx^j \). The renormalization-group improved 00 component of the Einstein equation (2.2) reads therefore (we now write explicitly all traces, since the desired Buchert averages are for traces of the Einstein equations \[14\])

\[
\left( R^{(3)} + (\text{Tr} K)^2 - \text{Tr} K^2 \right)
= 16\pi G \rho + 2\Lambda - \frac{3}{2} \psi^2_G,0 + \theta \psi_G,0 + \frac{5}{4} h(\text{grad}\psi_G, \text{grad}\psi_G) - \frac{\Delta G}{G},
\]

where, by definition,

\[
(\text{grad}\psi_G)_i \equiv \frac{G_{,i}}{G}.
\]

Since the left-hand side of (2.10) has the same functional form as in general relativity, the Buchert average (1.4) of (2.10) leads to (see appendix)

\[
3 \left( \frac{\dot{a}_D}{a_D} \right)^2 - 8\pi \langle G \rho \rangle_D + \frac{1}{2} (\langle R \rangle_D - \langle \Lambda \rangle_D
= -\frac{1}{2} \dot{Q}_D + \langle \theta \psi_G,0 \rangle_D - \frac{3}{4} \psi^2_G,0_D + \frac{5}{4} \langle h(\text{grad}\psi_G, \text{grad}\psi_G) \rangle_D - \frac{\langle \Delta G \rangle_D}{G}. \tag{2.12}
\]
The second equation we need can be obtained by contracting (2.2) with the contravariant spacetime metric $g^{\mu \nu}$. Of course, this contains also $R^0_0$ already encoded, up to a sign, in (2.10), but the difference between the full scalar curvature and $R^0_0$ is the new equation we need. We point out that, from (2.2) and (2.5), and exploiting the Arnowitt-Deser-Misner identity with unit lapse function and vanishing shift vector

$$\mathcal{R} = 8\pi G \rho + 4\Lambda - \frac{9}{2} \psi^{2}_{G,0} + 3\theta_{G,0} + \frac{3}{G} \frac{G_{,00}}{G},$$  

(2.13)

one finds, by adding and subtracting $\text{Tr} K^2$ in (2.13),

$$\mathcal{R} = \left[ (3) R + (\text{Tr} K)^2 - \text{Tr} K^2 \right] + 2\text{Tr} K^2 - 2 \frac{\partial}{\partial t} \text{Tr} K$$

$$= 8\pi G \rho + 4\Lambda - \frac{9}{2} \psi^{2}_{G,0} + 3\theta_{G,0} + \frac{3}{G} \frac{G_{,00}}{G} + \frac{9}{2} \{h(\text{grad} \psi_{G}, \text{grad} \psi_{G}) - 3 \frac{\Delta G}{G} \},$$  

(2.14)

where we can exploit (2.10) and then take the Buchert average to obtain eventually (see appendix)

$$\frac{2}{3} \left[ \left< \text{Tr} K^2 - \frac{\partial}{\partial t} \text{Tr} K \right>_{D} \right]$$

$$= 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{2}{3} \frac{\partial}{\partial t} (\theta)_{D} - \frac{2}{3} Q_{D}$$

$$= -\frac{8\pi}{3} \langle G\rho \rangle_{D} + \frac{2}{3} \langle \Lambda \rangle_{D} - \langle \psi^{2}_{G,0} \rangle_{D} + \frac{1}{3} \langle \theta_{G,0} \rangle_{D} + \frac{\langle G_{,00} \rangle}{G} \right)_{D}$$

$$+ \frac{2}{3} \langle h(\text{grad} \psi_{G}, \text{grad} \psi_{G}) \rangle_{D} - \frac{1}{3} \left< \frac{\Delta G}{G} \right>_{D}.$$  

(2.15)

If the FLRW symmetry is imposed, our equations (2.10) and (2.14) are in full agreement with (3.18a) and (3.18b) of [16], respectively, i.e. ($H$ being the Hubble parameter $\frac{\dot{a}}{a}$, and $\chi = 1, 0, -1$ the curvature parameter, while bearing in mind that our pressure parameter vanishes)

$$H^2 + \frac{\chi}{a^2} = \frac{1}{3} \Lambda + \frac{8\pi}{3} G \rho + \psi_{G,0} H - \frac{1}{4} \psi^{2}_{G,0},$$

$$H^2 + 2 \frac{\dot{a}}{a} + \frac{\chi}{a^2} = \Lambda - \frac{5}{4} \psi^{2}_{G,0} + \frac{G_{,00}}{G} + 2 \psi_{G,0} H.$$  

Otherwise they express corrections involving the spatial gradient of $G$. Note that $8\pi \langle G\rho \rangle_{D}$ can be eliminated in (2.15) with the help of (2.12), but this step is inessential.

Last, but not least, we have to take the Buchert average of the on-shell consistency equation derived in [16], i.e.

$$4\pi \langle GT_{\nu}^{\nu} \nabla_{\mu} G \rangle_{D} = \langle \nabla^{\mu} (G \Lambda) \rangle_{D},$$  

(2.16)
where \( u^\mu \) is the same vector used in (2.6).

III. INTEGRABILITY CONDITION

The integrability condition for our coupled system (2.12), (2.15) and (2.16) is obtained after re-expressing (2.12) and (2.15) in the form

\[
3 \left( \frac{\dot{a}_D}{a_D} \right)^2 - 8\pi \langle G\rho \rangle_D + \frac{1}{2} \langle (3) R \rangle_D - \langle \Lambda \rangle_D = -\frac{1}{2} Q_D + \langle F_1 \rangle_D, \tag{3.1}
\]

\[
3 \frac{\ddot{a}_D}{a_D} + 4\pi \langle G\rho \rangle_D - \langle \Lambda \rangle_D = Q_D + \frac{3}{2} \langle F_2 \rangle_D, \tag{3.2}
\]

where we have defined

\[
F_1 \equiv \theta \psi_{G,0} - \frac{3}{4} \psi_{G,0}^2 + \frac{5}{4} h(\text{grad} \psi_G, \text{grad} \psi_G) - \frac{\Delta G}{G}, \tag{3.3}
\]

\[
F_2 \equiv \frac{1}{3} \theta \psi_{G,0} - \psi_{G,0}^2 + \frac{G_{00}}{G} + \frac{2}{3} h(\text{grad} \psi_G, \text{grad} \psi_G) - \frac{1}{3} \frac{\Delta G}{G}. \tag{3.4}
\]

Following [18], we now take the time derivative of (3.1) and then use again (3.1) and (3.2), i.e. (hereafter, all partial time derivatives acting on Buchert averages coincide with total time derivatives of such spatial averages)

\[
\frac{\partial}{\partial t} 3 \left( \frac{\dot{a}_D}{a_D} \right)^2 = 2 \frac{\dot{a}_D}{a_D} \left( 3 \frac{\ddot{a}_D}{a_D} - 3 \left( \frac{\dot{a}_D}{a_D} \right)^2 \right)
\]

\[
= 2 \frac{\dot{a}_D}{a_D} \left[ -4\pi \langle G\rho \rangle_D + \langle \Lambda \rangle_D + Q_D + \frac{3}{2} \langle F_2 \rangle_D 
- 8\pi \langle G\rho \rangle_D + \frac{1}{2} \langle (3) R \rangle_D - \langle \Lambda \rangle_D + \frac{1}{2} Q_D - \langle F_1 \rangle_D \right]
\]

\[
= 2 \frac{\dot{a}_D}{a_D} \left[ -12\pi \langle G\rho \rangle_D + \frac{3}{2} Q_D + \frac{1}{2} \langle (3) R \rangle_D + \left( \frac{3}{2} F_2 - F_1 \right) \right] \tag{3.5}
\]

\[
= 8\pi \left[ \frac{\partial}{\partial t} \langle G\rho \rangle_D + \langle G\rho \theta \rangle_D - 3 \frac{\dot{a}_D}{a_D} \langle G\rho \rangle_D \right]
+ \frac{\partial}{\partial t} \langle \Lambda \rangle_D - \frac{1}{2} \frac{\partial}{\partial t} \langle (3) R \rangle_D - \frac{1}{2} \frac{\partial Q_D}{\partial t} + \frac{\partial}{\partial t} \langle F_1 \rangle_D,
\]

where use has been made of the Buchert identity [14]

\[
\frac{\partial}{\partial t} \langle \psi \rangle_D - \left\langle \frac{\partial \psi}{\partial t} \right\rangle_D = \langle \psi \theta \rangle_D - \langle \psi \rangle_D \langle \theta \rangle_D \tag{3.6}
\]

with \( \psi = G\rho \). Now both sides of (3.5) contain the term \(-24\pi (\dot{a}_D/a_D) \langle G\rho \rangle_D\), leading eventually to the desired integrability condition

\[
\frac{\partial Q_D}{\partial t} + \frac{6}{a_D} \dot{a}_D Q_D + \frac{\partial}{\partial t} \langle (3) R \rangle_D + 2 \frac{\dot{a}_D}{a_D} \langle (3) R \rangle_D
\]
\[ a^6 D \left\{ \frac{\partial}{\partial t} (a^6 D^6 D^6 Q^D D^6) + a^4 D \frac{\partial}{\partial t} \left( a^2 D^6 (3) R^D \right) \right\} = 16 \pi \left[ \left( \frac{\partial}{\partial t} (G \rho) \right)_D + \left( G \rho \theta \right)_D \right] + 2 \frac{\partial}{\partial t} \langle \Lambda \rangle_D + 2 \left[ \frac{\partial}{\partial t} \langle F_1 \rangle_D + \frac{\dot{a}_D}{a_D} \langle (2F_1 - 3F_2) \rangle_D \right], \]  

where, from the definitions (3.3) and (3.4),

\[ 2F_1 - 3F_2 = \theta \psi_{G,0} + \frac{3}{2} \psi_{G,0}^2 - \frac{G_{00}}{G} + \frac{1}{2} h(\text{grad}\psi_G; \text{grad}\psi_G) - \frac{\Delta G}{G}. \]  

**IV. THE VARIABLE-G COSMIC QUINTET AND COSMIC EQUATIONS**

Inspired by the work in [18], it is convenient to introduce the following dimensionless quantities:

\[ \Omega^D_m = \frac{8\pi \langle G \rho \rangle_D}{3H^2_D}, \quad \Omega^D_\Lambda = \frac{\langle \Lambda \rangle_D}{3H^2_D}, \quad \Omega^D_R = -\frac{\langle (3) R \rangle_D}{6H^2_D}, \quad \Omega^D_Q = -\frac{Q_D}{6H^2_D}, \quad \Omega^D_G = \frac{\langle F_1 \rangle_D}{3H^2_D}, \]  

where the definition of \( \Omega^D_G \) has been suggested by the averaged Hamiltonian constraint (3.1), and hereafter \( H_D = \dot{a}_D/a_D \) is the effective Hubble parameter. In a homogeneous and isotropic universe with constant \( G \) and \( \Lambda \), \( \Omega^D_m \) and \( \Omega^D_\Lambda \) reduce to the usual matter and cosmological constant density parameters, \( \Omega^D_R \) reduces to \( \Omega_k \), while \( \Omega^D_Q \) and \( \Omega^D_G \) vanish, so that the standard FLRW cosmology is then recovered. Equations (4.1) represent therefore the definition of the **cosmic quintet** of density parameters for variable-\( G \) inhomogeneous models of the Brans-Dicke type. Such an interpretation is also confirmed on noting that, by virtue of (3.1), one finds

\[ \Omega^D_m + \Omega^D_\Lambda + \Omega^D_R + \Omega^D_Q + \Omega^D_G = 1. \]  

Pursuing the analogy with FLRW cosmology, it is instructive to recast (3.1), (3.2) and (3.7) into an almost Friedmann form. To this aim, we look for an effective density \( \rho^D_{\text{eff}} \) and effective pressure \( p^D_{\text{eff}} \) such that (3.1) and (3.2) read as

\[ 3H^2_D = \langle \Lambda \rangle_D + 8\pi \langle G \rangle_D \rho^D_{\text{eff}}, \]  

\[ 3\frac{\ddot{a}_D}{a_D} = \langle \Lambda \rangle_D - 4\pi \langle G \rangle_D \left( \rho^D_{\text{eff}} + 3p^D_{\text{eff}} \right). \]
By comparison with (3.1) and (3.2), we solve for $\rho_{\text{eff}}^D$ and $p_{\text{eff}}^D$ to find

$$
\langle G \rangle_D \rho_{\text{eff}}^D = \langle G \rho \rangle_D - \frac{1}{16\pi} \langle^{(3)}R \rangle_D - \frac{1}{16\pi} Q_D + \frac{\langle F_1 \rangle_D}{8\pi},
$$

(4.5)

$$
\langle G \rangle_D p_{\text{eff}}^D = -\frac{1}{16\pi} Q_D + \frac{1}{48\pi} \langle^{(3)}R \rangle_D - \frac{1}{24\pi} \langle (F_1 + 3F_2) \rangle_D.
$$

(4.6)

V. ACCELERATING PATCHES AND STATIONARY MODELS

From equation (3.2), inspired by the work in [18], we see that the condition for an accelerating patch $D$ of the universe reads as

$$
Q_D > 4\pi \langle G \rho \rangle_D - \langle \Lambda \rangle_D - \frac{3}{2} \langle F_2 \rangle_D.
$$

(5.1)

The definition of the density parameters in (4.1) can be used to re-express (5.1) in the form

$$
-\Omega_Q^D > \frac{1}{4} \Omega_m^D - \frac{1}{2} \Omega^D_{\Lambda} - \frac{3}{4} \frac{\langle F_2 \rangle_D}{\langle F_1 \rangle_D} \Omega_{G}^D.
$$

(5.2)

If the domain $D$ is taken to be as large as our observable universe [18], the Hamiltonian constraint (4.2), jointly with (5.2), yields

$$
\frac{3}{2} \Omega_{\Lambda}^D + \Omega_{R}^D > 1 - \frac{3}{4} \Omega_m^D - \left( \frac{3}{4} \frac{\langle F_2 \rangle_D}{\langle F_1 \rangle_D} + 1 \right) \Omega_{G}^D.
$$

(5.3)

A. Stationary models

In our approach, in order to evaluate the quantities occurring in the equations, stationarity of the whole universe model is assumed. Within this framework, the manifold $\Sigma$ we refer to hereafter must be a finite-volume compact manifold, so that the global average makes sense. We find therefore the global stationarity conditions (cf [18])

$$
Q_\Sigma = 4\pi \langle G \rho \rangle_\Sigma - \langle \Lambda \rangle_\Sigma - \frac{3}{2} \langle F_2 \rangle_\Sigma,
$$

(5.4)

$$
\langle^{(3)}R \rangle_\Sigma = 12\pi \langle G \rho \rangle_\Sigma - 6H_\Sigma^2 + 3\langle \Lambda \rangle_\Sigma + \left\langle \left( \frac{3}{2} F_2 + 2 F_1 \right) \right\rangle_\Sigma,
$$

(5.5)

$$
H_\Sigma = \frac{\dot{a}_\Sigma}{a_\Sigma} = \frac{C}{a_\Sigma},
$$

(5.6)

where $Q_\Sigma$ and $\langle^{(3)}R \rangle_\Sigma$ are now the global kinematical backreaction and the globally averaged three-dimensional spatial curvature, respectively, while the constant $C$ can be obtained from
the Hamiltonian constraint evaluated at the initial time. Interestingly, if we now insert the
global stationarity conditions (5.4)–(5.6) into the formulae (4.5) and (4.6) for effective density
and effective pressure, we find some remarkable cancellations (i.e. vanishing coefficients of
\langle G\rho \rangle^D, \langle F_1 \rangle^D, \langle F_2 \rangle^D), leading to

\begin{align}
\langle G \rangle^D p^D_{\text{eff}} &= \frac{1}{8\pi} \left( \langle \Lambda \rangle^D - H^2 \right), \quad (5.7) \\
\langle G \rangle^D \rho^D_{\text{eff}} &= \frac{1}{8\pi} \left( 3H^2 - \langle \Lambda \rangle^D \right). \quad (5.8)
\end{align}

Hence we find the cosmic equation of state

\begin{align}
\langle G \rangle^D \left( p^D_{\text{eff}} + \frac{1}{3} \rho^D_{\text{eff}} \right) &= \frac{1}{12\pi} \langle \Lambda \rangle^D. \quad (5.9)
\end{align}

On taking the time derivative of \(Q^D\) and \(\langle (3)R \rangle^D\) in (5.4) and (5.5), and then using the
identity (3.6) with \(\psi = G\rho, D = \Sigma,\) and again (5.4)–(5.6), we find

\begin{align}
\left( \frac{\partial}{\partial t} + 3\frac{C}{a^D} \right) \left( Q^D + \langle \Lambda \rangle^D + \frac{3}{2} \langle F_2 \rangle^D \right) &= 4\pi \left[ \left\langle \frac{\partial}{\partial t} (G\rho) \right\rangle^D + \langle G\rho\theta \rangle^D \right], \quad (5.10)
\end{align}

\begin{align}
\frac{\partial}{\partial t} \langle (3)R \rangle^D + 9\frac{C}{a^D} Q^D + 6 \frac{\partial}{\partial t} H^2 &= 12\pi \left[ \left\langle \frac{\partial}{\partial t} (G\rho) \right\rangle^D + \langle G\rho\theta \rangle^D \right] \\
&- \frac{9C}{a^D} \left( \langle \Lambda \rangle^D + \frac{3}{2} \langle F_2 \rangle^D \right) + 3 \frac{\partial}{\partial t} \langle \Lambda \rangle^D + \frac{\partial}{\partial t} \left\langle \left( \frac{3}{2} F_2 + 2F_1 \right) \right\rangle^D. \quad (5.11)
\end{align}

The complete integral of (5.10) is given by the complete integral of the homogeneous
equation [18]

\begin{align}
\left( \frac{\partial}{\partial t} + 3\frac{C}{a^D(t)} \right) Q^D = 0
\end{align}

plus a particular integral of the full equation, so that we can write (the initial value of the
left-hand side below being denoted by \(J^D(t_i)\))

\begin{align}
\left( Q^D + \langle \Lambda \rangle^D + \frac{3}{2} \langle F_2 \rangle^D \right) (t) &= \frac{J^D(t_i)}{a^D(t)} \\
+ 4\pi \int_{t_i}^{t} \gamma(t,t') \left[ \left\langle \frac{\partial}{\partial t'} (G\rho) \right\rangle^D (t') + \langle G\rho\theta \rangle^D(t') \right] dt', \quad (5.12)
\end{align}

where \(\gamma(t,t')\) is the Green function of the operator \(\frac{\partial}{\partial t} + 3\frac{C}{a^D(t)}\). Since we are here assuming
\(\dot{a}^D = C\), we are actually dealing with the first-order operator \(\frac{\partial}{\partial t} + \frac{3C}{(Ct+a_i)}\), with \(a_i \equiv a^D(t_i)\).
The desired Green function solves the equation

\begin{align}
\left( \frac{\partial}{\partial t} + \frac{3C}{(Ct+a_i)} \right) \gamma(t,t') = 0 \forall t \neq t', \quad (5.13)
\end{align}
and suffers a jump at \( t = t' \) given by

\[
\lim_{t \to t^+} \gamma(t, t') - \lim_{t \to t^-} \gamma(t, t') = 1.
\] (5.14)

It therefore reads as (hereafter \( \Theta \) is the step function)

\[
\gamma(t, t') = (C t + a_i)^{-3C} \left[ \Theta(t - t') h_1(t') + \Theta(t' - t) h_2(t') \right].
\] (5.15)

where

\[
h_1(t') = (C t' + a_i)^{3C}, \quad h_2(t') = 0,
\] (5.16)
i.e. one finds, \( \forall t \neq t' \),

\[
\gamma(t, t') = \Theta(t - t') \left( \frac{t' + a_i}{C} \right)^{3C} \left( t + a_i/C \right).
\] (5.17)

VI. A SOLUTION FORMULA FOR THE SPATIALLY AVERAGED SCALAR CURVATURE

Note now that, by virtue of (5.10), the equation (5.11) can be recast in the form

\[
\frac{\partial}{\partial t} \left( \langle (3)R \rangle_{\Sigma} - 3Q_{\Sigma} - 6\langle \Lambda \rangle_{\Sigma} - \langle (6F_2 + 2F_1) \rangle_{\Sigma} \right) = -6 \frac{\partial}{\partial t} H^2_{\Sigma} = 12 \frac{C^3}{a^3},
\] (6.1)

where a remarkable cancellation of coefficients for \( \frac{C}{a^2}Q_{\Sigma} \) has occurred. On denoting by \( \chi \) an integration constant, and bearing in mind that \( a_{\Sigma}(t) = Ct + a_i \), equation (6.1) is solved by

\[
\langle (3)R \rangle_{\Sigma} = 3Q_{\Sigma} + 6\langle \Lambda \rangle_{\Sigma} + \langle (6F_2 + 2F_1) \rangle_{\Sigma} + \chi + 12\int_{t_i}^{t} \frac{\Theta(t - t')}{(C t' + a_i)^3} dt',
\] (6.2)

where \( Q_{\Sigma} \) is given by the solution formula (5.12). The coupling between spatially averaged scalar curvature and backreaction is therefore found to survive.

VII. A PARTICULAR ASSUMPTION ON THE AVERAGED EQUATIONS

In order to work out a general solution of the cosmic equations, we should know how matter is spatially distributed and assume a metric for the inhomogenous universe to compute the averaged quantities. Moreover, an ansatz should be made for the spatial variation of \( G \), possibly consistent with the renormalization-group equations, so that \( F_1 \) and \( F_2 \) can be evaluated. Since such an information is lacking by virtue of severe technical difficulties in deriving \( G(x, t) \), we can only look for a particular solution under some reasonable assumptions.
A first step along this road can be done by differentiating both sides of Eq. (4.3) with respect to the cosmic time $t$, thus yielding

$$3 \ddot{\alpha}_D = \langle \Lambda \rangle_D + 4\pi \langle G \rangle_D \left( \frac{1}{H_D} \frac{\partial \rho_{\text{eff}}^D}{\partial t} + 3\rho_{\text{eff}}^D \right) + \frac{1}{2H_D} \left( \frac{\partial \langle \Lambda \rangle_D}{\partial t} + 8\pi \rho_{\text{eff}}^D \frac{\partial \langle G \rangle_D}{\partial t} \right).$$

In a FLRW universe, the effective fluid should be replaced by the standard source (matter and radiation) term. In this case, the standard continuity equation holds and the above relation identically reduces to the Raychaudhuri equation. Let us assume that our effective fluid can still satisfy, to a first approximation, the standard continuity equation, i.e.

$$\frac{\partial \rho_{\text{eff}}^D}{\partial t} + 3H_D(\rho_{\text{eff}}^D + p_{\text{eff}}^D) = 0.$$ (7.1)

In classical general relativity, this equation holds by construction of the effective equations. In our approach, consistency of the theory seems to demand that Eq. (7.1) should hold, but for example we cannot yet say what would one expect if (7.1) had source terms. To study (tiny) departures from (7.1), one might try to assume analyticity of $G(x,t), \Lambda(x,t)$ and hence set up a perturbative scheme for the evaluation of all averaged equations. This is an important topic that deserves attention in a separate paper.

On solving with respect to $(1/H_D)\partial \rho_{\text{eff}}^D/\partial t$, the above relation becomes

$$3 \ddot{\alpha}_D = \langle \Lambda \rangle_D - 4\pi \langle G \rangle_D(\rho_{\text{eff}}^D + 3p_{\text{eff}}^D) + \frac{1}{2H_D} \left( \frac{\partial \langle \Lambda \rangle_D}{\partial t} + 8\pi \rho_{\text{eff}}^D \frac{\partial \langle G \rangle_D}{\partial t} \right).$$

By equating to Eq. (4.4), we get the remarkable relation

$$\frac{\partial \langle \Lambda \rangle_D}{\partial t} + 8\pi \rho_{\text{eff}}^D \frac{\partial \langle G \rangle_D}{\partial t} = 0,$$ (7.2)

which, for a homogenous and isotropic universe, reduces to

$$\dot{\Lambda} + 8\pi \rho \dot{G} = 0,$$

which has already been obtained in the literature [1].

It is worth wondering whether Eq. (7.1) can lead to a constraint also on the functions $F_1$ and $F_2$. For this purpose, let us first note that, by virtue of the definitions of the effective fluid energy density and pressure, one has

$$\langle G \rangle_D \frac{\partial \rho_{\text{eff}}^D}{\partial t} = \frac{\partial \langle G \rho \rangle_D}{\partial t} - \frac{1}{16\pi} \frac{\partial Q_D}{\partial t} - \frac{1}{16\pi} \frac{\partial \langle (3) R \rangle_D}{\partial t} + \frac{1}{8\pi} \frac{\partial \langle F_1 \rangle_D}{\partial t} - \rho_{\text{eff}}^D \frac{\partial \langle G \rangle_D}{\partial t},$$

$$\langle G \rangle_D (\rho_{\text{eff}}^D + p_{\text{eff}}^D) = \langle G \rho \rangle_D - \frac{1}{8\pi} Q_D - \frac{1}{24\pi} \langle (3) R \rangle_D + \frac{1}{8\pi} \langle F_1 \rangle_D - \frac{1}{24\pi} \langle (F_1 + 3F_2) \rangle_D.$$
Multiplying by $\langle G \rangle_D$ and inserting these relations, the continuity equation (7.1) for the effective fluid becomes

\[
\left( \frac{\partial Q_D}{\partial t} + 6H_D Q_D \right) + \left( \frac{\partial \langle (3) R \rangle_D}{\partial t} + 2H_D \langle (3) R \rangle_D \right) = 16\pi \left( \frac{\partial \langle G \rho \rangle_D}{\partial t} + 3H_D \langle G \rho \rangle_D \right)
\]
\[
+ 2 \left( \frac{\partial \langle F_1 \rangle_D}{\partial t} + 3H_D \langle F_1 \rangle_D \right)
\]
\[
- 2H_D \langle (F_1 + 3F_2) \rangle_D = 16\pi \rho_{\text{eff}} \frac{\partial \langle G \rangle_D}{\partial t}.
\]

Comparing this relation with the integrability condition (3.7) and making use of the Buchert identity (3.6) with $\psi = G\rho$ and $\langle \theta \rangle_D = 3H_D$, we finally get

\[
\frac{\partial \langle \Lambda \rangle_D}{\partial t} + \frac{8\pi}{16\pi} \rho_{\text{eff}} \frac{\partial \langle G \rangle_D}{\partial t} = H_D \left( 3\langle F_1 \rangle_D - 2\langle (F_1 + 3F_2) \rangle_D - \langle (2F_1 - 3F_2) \rangle_D \right),
\]

so that, since the left-hand side vanishes because of Eq. (7.2), we obtain

\[
3\langle F_1 \rangle_D - 2\langle (F_1 + 3F_2) \rangle_D - \langle (2F_1 - 3F_2) \rangle_D = -\langle (F_1 + 3F_2) \rangle_D = 0,
\]

having made use of the linearity of the mean operator. Summarizing, we can therefore conclude that the conservation of the effective fluid implies two remarkable relations. The first one, Eq. (7.2) is the counterpart for an inhomogenous model of the renormalization-group condition on the $G$ and $\Lambda$ time-derivatives. On the contrary, Eq. (7.3) is an interesting new constraint closely relating the spatial variation of the two auxiliary functions $F_1$ and $F_2$. Indeed, whatever is the exact spatial variation of $G$, the condition $\langle F_1 \rangle_D = -3\langle F_2 \rangle_D$ must hold. It is worth stressing, however, that this constraint only refers to the averaged quantities, so that it is impossible to infer any information on the spatial variation of $G$.

In order to gain further insight, it is interesting to consider the case of a nearly homogenous and isotropic universe. As a first step, we introduce three auxiliary effective fluids with energy densities

\[
\begin{aligned}
\rho^D_M &\equiv \frac{\langle G \rho \rangle_D}{\langle G \rangle_D} - \frac{3}{8\pi} \frac{\langle F_2 \rangle_D}{\langle G \rangle_D}, \\
\rho^D_Q &\equiv -\frac{1}{16\pi} \frac{Q_D}{\langle G \rangle_D}, \\
\rho^D_R &\equiv -\frac{1}{16\pi} \frac{\langle (3) R \rangle_D}{\langle G \rangle_D} + \frac{1}{8\pi} \frac{\langle (F_1 + 3F_2) \rangle_D}{\langle G \rangle_D}.
\end{aligned}
\]
and pressure given as

\[
\begin{align*}
p_{M}^{D} & \equiv 0, \\
p_{Q}^{D} & \equiv -\frac{1}{16\pi} \frac{Q_{D}}{\langle G \rangle_{D}}, \\
p_{R}^{D} & \equiv \frac{1}{48\pi} \frac{\langle (3)R \rangle_{D}^{D}}{\langle G \rangle_{D}} - \frac{1}{24\pi} \frac{\langle (F_{1} + 3F_{2}) \rangle_{D}}{\langle G \rangle_{D}},
\end{align*}
\](7.5)

so that the equations of state are \((w_{M}^{D}, w_{Q}^{D}, w_{R}^{D}) = (0, 1, -1/3)\). As can be straightforwardly checked, one has

\[
\begin{align*}
\rho_{\text{eff}}^{D} & = \rho_{M}^{D} + \rho_{Q}^{D} + \rho_{R}^{D}, \\
p_{\text{eff}}^{D} & = p_{M}^{D} + p_{Q}^{D} + p_{R}^{D},
\end{align*}
\](7.6)

which shows that the effective fluid may be viewed as consisting of three distinct components. The first one, \(\rho_{M}^{D}\), has zero pressure and reduces to the standard matter term when we turn off the spatial variation of \(G\) (so that \(F_{1} = F_{2} = 0\)) and replace averaged quantities with standard ones in a FLRW universe. We can therefore think of it as the matter term of our model although, since we do not know anything about \(F_{1}\) and \(F_{2}\), we cannot rule out a priori that \((-3/8\pi)(\langle F_{2} \rangle_{D}/\langle G \rangle_{D})\) overcomes \(\langle G \rho \rangle_{D}/\langle G \rangle_{D}\), thus giving rise to a negative energy density. The second component, with energy density \(\rho_{Q}^{D}\), has the same equation of state as the stiff matter. However, should the backreaction term \(Q_{D}\) be positive, its energy density is negative thus leading to a negative pressure acting as a variable cosmological constant (eventually driving cosmic speed up). Finally, the term with energy density \(\rho_{R}^{D}\) has a negative equation of state, but we cannot say whether it acts as a speeding up factor. Indeed, we do not know whether the curvature term overcomes the other term giving a positive or negative energy density. Note that, should we assume the conservation of the effective fluid, then Eq. (7.3) follows and \(\rho_{R}^{D}\) is positive-definite. However, to obtain results of general nature, we prefer not to assume here a priori the validity of Eq. (7.1). In other words, at this stage, we are considering effective fluids which make it unnecessary to consider Eq. (7.1).

On denoting with \(w_{\text{eff}}^{D} \equiv p_{\text{eff}}^{D}/\rho_{\text{eff}}^{D}\) the effective fluid equation of state, and inverting Eqs.
(7.6), we get
\[
\begin{align*}
\rho_{Q}^{D} &= \frac{\rho_{D}^{M}}{4} \left[ -1 + (1 + 3 w_{\text{eff}}^{D}) \frac{\rho_{\text{eff}}^{D}}{\rho_{M}} \right], \\
\rho_{R}^{D} &= \frac{3 \rho_{D}^{M}}{4} \left[ -1 + (1 - w_{\text{eff}}^{D}) \frac{\rho_{\text{eff}}^{D}}{\rho_{M}} \right],
\end{align*}
\]
(7.7)
which are fully general. Let us now assume that the universe is nearly homogenous and isotropic as it is expected to be, for instance, in its infancy when perturbations had still to grow. In this case, we can write
\[
G \rho - (3/8 \pi) F_2 = G_N \rho_M^{FLRW} [1 + \Delta_M(t, X^i)],
\]
(7.8)
\[
G = G_N [1 + \Delta_G(t, X^i)],
\]
(7.9)
where $G_N$ is the Newtonian gravitational constant, $\rho_M^{FLRW}$ the dust matter energy density in a FLRW universe, and $\Delta_M(t, X^i)$ and $\Delta_G(t, X^i)$ two unknown functions accounting for the small perturbations induced by the inhomogeneities. It is worth noting that Eqs. (7.8) and (7.9) may still be formally written even if deviations from the FLRW universe are severe, the only difference being that $\Delta_M(t, X^i)$ and $\Delta_G(t, X^i)$ are no longer much smaller than 1.

On averaging Eqs. (7.8) and (7.9), the energy density of the matter-like term reads as
\[
\rho_{D}^{M} = \frac{\langle G \rho - (3/8 \pi) F_2 \rangle_D}{\langle G \rangle_D} = \rho_M^{FLRW} \frac{1 + \langle \Delta_M \rangle_D}{1 + \langle \Delta_G \rangle_D},
\]
(7.10)
while Eqs. (4.5) and the average of (7.9) give
\[
\rho_{\text{eff}}^{D} = \frac{3 H_D^2 - \langle \Lambda \rangle_D}{8 \pi G_N (1 + \langle \Delta_G \rangle_D)}.
\]
(7.11)
On inserting Eqs. (7.10) and (7.11) into Eqs. (7.7), we get
\[
\begin{align*}
\rho_{Q}^{D} &= \frac{1}{4} \left( \frac{1 + \langle \Delta_M \rangle_D}{1 + \langle \Delta_G \rangle_D} \right) \left[ -1 + \frac{1 + \langle \Delta_G \rangle_D (3 H_D^2 - \langle \Lambda \rangle_D)}{1 + \langle \Delta_M \rangle_D (8 \pi G_N \rho_M^{FLRW})} (1 + 3 w_{\text{eff}}^{D}) \right] \rho_M^{FLRW}, \\
\rho_{R}^{D} &= \frac{3}{4} \left( \frac{1 + \langle \Delta_M \rangle_D}{1 + \langle \Delta_G \rangle_D} \right) \left[ -1 + \frac{1 + \langle \Delta_G \rangle_D (3 H_D^2 - \langle \Lambda \rangle_D)}{1 + \langle \Delta_M \rangle_D (8 \pi G_N \rho_M^{FLRW})} (1 - w_{\text{eff}}^{D}) \right] \rho_M^{FLRW}.
\end{align*}
\]
(7.12)

We can now work out an interesting property of these two effective fluids by assuming a phenomenological ansatz for the term $3 H_D^2 - \langle \Lambda \rangle_D$ and for the effective fluid equation of state $w_{\text{eff}}^{D}$. It is reasonable to assume that, for very large $z$,
\[
\begin{align*}
w_{\text{eff}}^{D}(z) &\simeq 0, \\
3 H_D^2(z) - \langle \Lambda \rangle_D(z) &\simeq 8 \pi G_N \rho_M^{FLRW}(z).
\end{align*}
\]
Equation (7.12) shows that, in this limit, both $\rho_Q^D$ and $\rho_R^D$ vanish so that the dust matter FLRW case is recovered in the early universe whatever is the exact shape of the correction functions $\Delta_M$ and $\Delta_G$. In other words, we find that the early universe tends to a FLRW model, but keeps track of the original inhomogeneities through the two effective fluids with density $\rho_Q^D$ and $\rho_R^D$. Actually, depending on the functional expression adopted for $H_D$, it is also possible that $\langle \Delta_M \rangle$ vanishes identically at all redshifts. We stress that the constraint $\langle \Delta_M \rangle_D(z) = 0$ does not imply that the universe is homogenous, but only that the matter inhomogeneities average out to zero. As is clear from Eqs. (7.12), in such a case, the two fictitious fluids $\rho_Q^D$ and $\rho_R^D$ still contribute to the cosmic dynamics. In particular, should $H_D$ lead to cosmic acceleration, we can therefore argue that these two fluids arising from inhomogeneities drive the cosmic speed up in a matter dominated universe.

VIII. CONCLUDING REMARKS AND OPEN PROBLEMS

Our original equations (2.12), (2.15), (2.16), (3.7), (4.5), (4.6), (5.2), (5.4), (5.5), (5.10), (5.12), (6.2) provide a simple but nontrivial application of the Buchert method for spatial averages [14, 18] to renormalization-group improved action functionals with variable $G$ and $\Lambda$ of the Brans–Dicke type [16]. It now remains to be seen whether our averaged equations agree with the qualitative picture in general relativity [18, 20], according to which backreaction effects point to a global instability of the standard cosmological model, with exact solutions and perturbative results modeling this instability lying in the right sector to account for dark energy from inhomogeneities [21, 22] (for a critical view, see however the work in [23]).

The definition of the effective fluids in section 7 makes it clear what is the role played by the different inhomogeneity terms introduced by the averaging procedure. In the early universe, only the $\rho_M^D$ term survives and reduces to the standard dust matter, so that the usual FLRW case is recovered. It is however intriguing to note that, in the late epochs, the effective matter fluid energy density $\rho_M^D$ is increased with respect to the standard one because of the $(-3/8\pi)\langle F_2 \rangle_D/\langle G \rangle_D$ term. As such, one could speculate that this latter component mimics an effective dark matter component, while $\langle G\rho \rangle_D/\langle G \rangle_D$ accounts for the baryons. On the other hand, the two additional fluids $\rho_Q^D$ and $\rho_R^D$ can both provide a negative pressure so that they can drive accelerated expansion. The resulting qualitative picture is that of a universe consisting of baryons only, while inhomogeneities average out to give rise to the full
dark-side phenomenology. Needless to say, such a qualitative picture must be substantiated by a detailed comparison with the data on both the background expansion and the growth of structure, and hence we cannot draw any definite conclusion at the moment. An alternative possibility, from this point of view, is to compare our effective fluids’ picture with the recent literature on the morphon field [24, 25] in order to work out fruitful analogies.

We may be criticized for having used the renormalization-group approach only as a motivation, without ever exploiting it for the explicit evaluation of the Buchert averages involving \( G \) and \( \Lambda \). On the other hand, as far as we know, no-one has succeeded so far in obtaining \( G(x, t) \) and \( \Lambda(x, t) \) from the exact integration of the renormalization-group equation, and previous attention had always focused on FLRW models where, by symmetry, \( G \) and \( \Lambda \) can only depend on the time variable. Without an explicit knowledge of the desired \( G(x, t) \) and \( \Lambda(x, t) \), we cannot yet provide examples of our averaging procedure. However, as we have stressed after Eq. (7.1), a promising way out might be obtained by studying a power-series expansion of \( G \) and \( \Lambda \), which is independent of any fixed-point assumption. We hope to be able to return to this point in a separate paper.

Another interesting issue is whether our averaged equations with variable Newton parameter can be relevant for the theoretical scheme proposed in [26, 27, 28] as yet another alternative to dark energy. It would be also quite important to repeat our analysis with the help of the covariant technique developed in [29].

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**APPENDIX A: BASIC IDENTITIES FOR THE BUCHERT AVERAGES**

Relying upon [14], we express the extrinsic-curvature tensor \( K_{ij} \) of the spacelike hypersurfaces foliating the spacetime manifold in the form

\[
K_{ij} = -\sigma_{ij} - \frac{\theta}{3}g_{ij},
\]  

\[(A1)\]
where the shear tensor is symmetric and traceless. Hence one finds the simple but fundamental identities (hereafter $\sigma^2 \equiv \frac{1}{2} \sigma_{ij} \sigma^{ij}$)

$$\text{Tr} K = -\theta,$$

$$\text{Tr} K^2 = 2\sigma^2 + \frac{\theta^2}{3},$$

$$\frac{1}{2} \left( (\text{Tr} K)^2 - \text{Tr} K^2 \right) = \frac{\theta^2}{3} - \sigma^2.$$  \hfill (A2)

(A3)

When we evaluate the Buchert average of the Hamiltonian constraint (2.10), we exploit the identity (1.7) and the definition (1.8) to write

$$\frac{1}{3} \langle \theta^2 \rangle_D - \langle \sigma^2 \rangle_D = \frac{Q_D}{2} + \frac{1}{3} \langle \theta \rangle_D^2 = \frac{Q_D}{2} + 3 \left( \frac{\dot{a}_D}{a_D} \right)^2.$$  \hfill (A5)

Along the same lines, the first equality in (2.15) is obtained by virtue of

$$\left\langle \text{Tr} K^2 - \frac{\partial}{\partial t} \text{Tr} K \right\rangle_D = 2 \langle \sigma^2 \rangle_D + \frac{1}{3} \left[ \langle \theta^2 \rangle_D - \langle \theta \rangle_D^2 \right] + \frac{1}{3} \langle \theta \rangle_D^2 + \left\langle \frac{\partial \theta}{\partial t} \right\rangle_D,$$  \hfill (A6)

where

$$\left\langle \frac{\partial \theta}{\partial t} \right\rangle_D = \frac{\partial}{\partial t} \langle \theta \rangle_D - \langle \theta^2 \rangle_D + \langle \theta \rangle_D^2.$$  \hfill (A7)

The identities (1.7), (A6) and (A7) lead eventually to the first equality in (2.15).

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