Intuitionistic fuzzy dimension
of an intuitionistic fuzzy vector space

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Abstract: In the present paper the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space has been developed with the help of intuitionistic fuzzy basis.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy vector space, Intuitionistic fuzzy dimension.

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1 Introduction

The notion of intuitionistic fuzzy set (IFS) was introduced by Atanassov [1, 2, 3, 4] as a generalization of Zadeh’s fuzzy set [22]. There are situations where IFS theory is more appropriate to deal with [7]. IFS theory have successfully been applied in knowledge engineering, medical diagnosis, decision making, career determination, etc., [11, 21, 12]. Several researchers have extended various mathematical aspects such as groups, rings, topological spaces, metric spaces, topological groups, topological vector spaces etc. in IFS [6, 10, 13, 16, 17, 18, 19]. The notion of fuzzy vector subspaces has been introduced by Katsaras [14] and a notion of fuzzy bases and fuzzy dimension was studied by Shi \textit{et al.} [20]. We have introduced a notion of intuitionistic fuzzy vector space and intuitionistic fuzzy basis in [9]. As a continuation of our paper [9], here we introduced the notion of intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space with the help of intuitionistic fuzzy basis and studied some of its basic results.
2 Preliminaries

Definition 2.1. [1] Let X be a non-empty set. An intuitionistic fuzzy set (IFS for short) of X is defined as an object having the form $A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \}$, where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \}$.

In this paper, we use the symbols $a \land b = \min\{a, b\}$ and $a \lor b = \max\{a, b\}$.

Definition 2.2. [1] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a set X. Then

1. $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
3. $A^c = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \}$
4. $A \cap B = \{ (x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x)) \mid x \in X \}$.
5. $A \cup B = \{ (x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x)) \mid x \in X \}$.
6. $\square A = \{ (x, \mu_A(x), 1 - \mu_A(x)) \mid x \in X \}$, $\lozenge A = \{ (x, 1 - \nu_A(x), \nu_A(x)) \mid x \in X \}$.

Definition 2.3. [4] Let $A$ be an IFS in a set $X$. Then for $\lambda, \xi \in [0, 1]$ with $\lambda + \xi \leq 1$, the set $A^{[\lambda, \xi]} = \{ x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \xi \}$ is called $(\lambda, \xi)$-level subset of $A$.

Proposition 2.4. [4] Let $A$ be an IFS in a set $X$ and $(\lambda_1, \xi_1), (\lambda_2, \xi_2) \in \text{Im}(A)$. If $\lambda_1 \leq \lambda_2$ and $\xi_1 \geq \xi_2$, then $A^{[\lambda_1, \xi_1]} \supseteq A^{[\lambda_2, \xi_2]}$.

Definition 2.5. [15, 5] Let $X$ be a vector space over the field $\mathbb{K}$, the field of real and complex numbers, $\alpha \in \mathbb{K}$, $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of $X$. Then

1. the sum of $A$ and $B$ is defined to be the intuitionistic fuzzy set $A + B = (\mu_A + \mu_B, \nu_A + \nu_B)$ of $X$ given by

$$
\begin{align*}
\mu_{A+B}(x) &= \begin{cases} 
\sup_{x=a+b} \{ \mu_A(a) \land \mu_B(b) \} & \text{if } x = a + b \\
0 & \text{otherwise},
\end{cases} \\
\nu_{A+B}(x) &= \begin{cases} 
\inf_{x=a+b} \{ \nu_A(a) \lor \nu_B(b) \} & \text{if } x = a + b \\
1 & \text{otherwise}.
\end{cases}
\end{align*}
$$

2. $\alpha A$ is defined to be the IFS $\alpha A = (\mu_{\alpha A}, \nu_{\alpha A})$ of $X$, where

$$
\begin{align*}
\mu_{\alpha A}(x) &= \begin{cases} 
\mu_A(\alpha^{-1}x) & \text{if } \alpha \neq 0 \\
\sup_{y \in X} \mu_A(y) & \text{if } \alpha = 0, x = \theta \\
0 & \text{if } \alpha = 0, x \neq \theta,
\end{cases}
\end{align*}
$$
\[ v_{\alpha A}(x) = \begin{cases} v_A(\alpha^{-1}x) & \text{if } \alpha \neq 0 \\ \inf_{y \in X} v_A(y) & \text{if } \alpha = 0, x = \theta \\ 1 & \text{if } \alpha = 0, x \neq \theta. \end{cases} \]

**Proposition 2.6.** [9] Let \( A, A_1, \ldots, A_n \) be intuitionistic fuzzy sets in a vector space \( X \) and \( \lambda_1, \ldots, \lambda_n \) be scalars. Then the following assertions are equivalent:

1. \( \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n \subseteq A. \)
2. For all \( x_1, x_2, \ldots, x_n \) in \( X \), we have
   \[
   \mu_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \geq \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\} \quad \text{and} \quad v_A(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \max\{v_{A_1}(x_1), v_{A_2}(x_2), \ldots, v_{A_n}(x_n)\}.
   \]

**Definition 2.7.** [9] An IFS \( V = (\mu_V, \nu_V) \) of a vector space \( X \) over the field \( K \) is said to be intuitionistic fuzzy vector space over \( X \) if

1. \( V + V \subseteq V \)
2. \( \alpha V \subseteq V \), for every scalar \( \alpha \).

We denote the set of all intuitionistic fuzzy vector spaces over a vector space \( X \) by \( \text{IFVS}(X) \).

**Remark 2.8.** [9] Let \( X \) be a vector space.

1. If \( \mu_V \) is a fuzzy subspace of \( X \), then \( V = (\mu_V, \mu_V^c) \in \text{IFVS}(X) \).
2. If \( V \in \text{IFVS}(X) \), then \( \mu_V \) and \( \nu_V^c \) are fuzzy vector subspace of \( X \).
3. If \( V \in \text{IFVS}(X) \), then \( \square V, \diamond V \in \text{IFVS}(X) \).

**Lemma 2.9.** [9] Let \( V \) be an intuitionistic fuzzy set in a vector space \( X \). Then, the following are equivalent:

1. \( V \) is an intuitionistic fuzzy vector space over \( X \).
2. For all scalars \( \alpha, \beta \), we have \( \alpha V + \beta V \subseteq V \).
3. For all scalars \( \alpha, \beta \) and for all \( x, y \in X \), we have
   \[
   \mu_V(\alpha x + \beta y) \geq \mu_V(x) \wedge \mu_V(y) \quad \text{and} \quad v_V(\alpha x + \beta y) \leq v_V(x) \vee v_V(y).
   \]

**Remark 2.10.** [9] Our definition of intuitionistic fuzzy vector space is equivalent to the definition of intuitionistic fuzzy subspace of [19] and [8].

**Proposition 2.11.** [8] If \( V, W \in \text{IFVS}(X) \), then \( V + W \in \text{IFVS}(X) \).

**Proposition 2.12.** [9] If \( V \in \text{IFVS}(X) \) \( \alpha \in K \), then \( \alpha V \in \text{IFVS}(X) \).

**Proposition 2.13.** [8] If \( \{V_i\}_{i \in I} \in \text{IFVS}(X) \), then \( \bigcap_{i \in I} V_i \in \text{IFVS}(X) \).
Proposition 2.14. [9] Let $V \in \text{IFVS}(X)$. Then $\mu_V(\theta) \geq \mu_V(x)$ and $\nu_V(\theta) \leq \nu_V(x)$, $\forall x \in X$.

Proposition 2.15. [9] Let $V \in \text{IFVS}(X)$. Then for each $(\lambda, \xi) \in [0, 1] \times [0, 1]$ with $\lambda + \xi \leq 1$, $\lambda \leq \mu_V(\theta)$ and $\xi \geq \nu_V(\theta)$, $V^{[\lambda, \xi]}$ is a subspace of the vector space $X$.

Definition 2.16. [9] For any $(a, b), (c, d) \in [0, 1] \times [0, 1]$ with $a + b \leq 1$, $c + d \leq 1$, we say that:

1. $(a, b) \geq (c, d)$ if $a \geq b$ and $c \leq d$.
2. $(a, b) \leq (c, d)$ if $a \leq b$ and $c \geq d$.
3. $(a, b) > (c, d)$ if $a > b$ and $c \leq d$ or if $a \geq b$ and $c < d$.
4. $(a, b) < (c, d)$ if $a < b$ and $c \geq d$ or if $a \leq b$ and $c > d$.
5. $(a, b) = (c, d)$ if $a = b$ and $c = d$.

Proposition 2.17. [9] Let $V \in \text{IFVS}(X)$ with $\dim X = m$. Then $\text{Im}(V)$ contains at most $m + 1$ points of $[0, 1] \times [0, 1]$.

Definition 2.18. [9] Let $V = (\mu_V, \nu_V) \in \text{IFVS}(X)$. Then for any $\lambda \in \mu_V(X), \xi \in \nu_V(X)$ we define $\mu_V^{[\lambda]} = \{x \in X : \mu_V(x) \geq \lambda\}$ and $\nu_V^{[\xi]} = \{x \in X : \nu_V(x) \leq \xi\}$, $[\lambda, \mu_V^{[\lambda]}](x) = \begin{cases} \lambda, & \text{if } x \in \mu_V^{[\lambda]} \\ 0, & \text{otherwise} \end{cases}$, $[\xi, \nu_V^{[\xi]}](x) = \begin{cases} \xi, & \text{if } x \in \nu_V^{[\xi]} \\ 1, & \text{otherwise} \end{cases}$.

Theorem 2.19. [9] (Representation Theorem) Let $V \in \text{IFVS}(X)$ with $\dim X = m$ and $\text{Im}(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \ldots, (\lambda_k, \xi_k)\}, k \leq m$ such that $(1, 0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \ldots > (\lambda_k, \xi_k)$ $\geq (0, 1)$. Then there exists nested collection of subspaces of $X$ as $\{\theta\} \subseteq V^{[\lambda_0, \xi_0]} \subseteq V^{[\lambda_1, \xi_1]} \subseteq \ldots \subseteq V^{[\lambda_k, \xi_k]}$ $= X$ such that $\mu_V = \lambda_0^{1/\mu_V^{[\lambda_0]}}, \lambda_1^{1/\mu_V^{[\lambda_1]}}, \ldots, \lambda_k^{1/\mu_V^{[\lambda_k]}}$ and $\nu_V = \xi_0^{1/\nu_V^{[\xi_0]}}, \xi_1^{1/\nu_V^{[\xi_1]}}, \ldots, \xi_k^{1/\nu_V^{[\xi_k]}}$. Also,

1. If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i] \times [\xi_{i+1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta, \rho]} = V^{[\eta, \sigma]} = V^{[\lambda_i, \xi_i]}$.
2. If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i] \times [\xi_{i+1}, \xi_i)$, $(\eta, \sigma) \in (\lambda_i, \lambda_{i-1}] \times [\xi_{i-1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $V^{[\zeta, \rho]} \supseteq V^{[\eta, \sigma]}$.

Definition 2.20. [9] Let $V \in \text{IFVS}(X)$ with $\dim X = m$. Consider Theorem 2.19. Let $B_{V_i}$ be the basis of $V^{[\lambda_i, \xi_i]}, i = 0, 1, \ldots, k$ such that

\[ B_{V_0} \supseteq B_{V_1} \supseteq \ldots \supseteq B_{V_k}. \]  

\[(*)\]

If $V^{(\lambda_0, \xi_0)} = \{\theta\}$, we start with $V^{(\lambda_1, \xi_1)}$.

Define a map $\mathbb{B}$ from $X$ to $[0, 1] \times [0, 1]$ by

$\mu_\mathbb{B}(x) = \begin{cases} \vee \{\lambda_i : x \in B_{V_i}\} & \text{and } \nu_\mathbb{B}(x) = \begin{cases} \wedge \{\xi_i : x \in B_{V_i}\} & 1, \text{otherwise} \end{cases} \end{cases}$.
Let $\mu_B(x) = \lambda_j$. Then $x \in B_{V_i}$ and $x \notin B_{V_{i,j}}$, i.e. $x \in V^{[\lambda_j, \xi_i]}$ and $x \notin V^{[\lambda_{i-1}, \xi_{i-1}]}$. Thus $\mu_V(x) \geq \lambda_j$ and $\nu_V(x) \leq \xi_j$. If $\mu_V(x) > \lambda_j$, then $\mu_V(x) = \lambda_l$ for some $l < j$. Then $x \in V^{[\lambda_j, \xi_j]}$ and $\mu_B(x) = \lambda_l$, which is a contradiction. Therefore $\mu_V(x) = \lambda_j$. Then $\nu_V(x) = \xi_j$ i.e. $v_B(x) = \xi_j$. Therefore $B$ is an intuitionistic fuzzy set and it is called intuitionistic fuzzy basis of $V$ corresponding to $(\ast)$. 

**Proposition 2.21.** [9] Let $B$ be an intuitionistic fuzzy basis of $V$ corresponding to $(\ast)$ of Definition 2.20. Then

(1) If $(\zeta, \rho), (\eta, \sigma) \in (\lambda_{i+1}, \lambda_i) \times [\xi_i, \xi_{i+1})$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $B[\zeta, \rho] = B[\eta, \sigma] = B_{V_i}$.

(2) If $(\zeta, \rho) \in (\lambda_{i+1}, \lambda_i) \times [\xi_i, \xi_{i+1})$, $(\eta, \sigma) \in (\lambda_i, \lambda_{i-1}) \times [\xi_{i-1}, \xi_i)$ with $\zeta + \rho \leq 1, \eta + \sigma \leq 1$, then $B[\zeta, \rho] = B[\eta, \sigma]$. 

(3) $B^{[\lambda, \xi]}$ is a basis of $V^{[\lambda, \xi]}$ for $\lambda \in (0,1], \xi \in [0,1)$ with $\lambda + \xi \leq 1$.

**Proposition 2.22.** Let $B$ be an intuitionistic fuzzy basis of $V$ corresponding to $(\ast)$ of Definition 2.20. Then $\mu_B^{[\lambda]} = B_{V_{i}} = v_{B_{i}}^{[\xi]}$, for $i = 0,1,2,...,k$.

**Proof**. Let $x \in \mu_B^{[\lambda]} \Rightarrow \mu_B(x) \geq \lambda_i$. Let $\mu_B(x) = \lambda_j \Rightarrow x \in B_{V_i} \subset B_{V_i}$. Thus $\mu_B^{[\lambda]} \subseteq B_{V_i}$. Conversely, let $x \in B_{V_i} \Rightarrow \mu_V(x) \geq \lambda_i$. Let $\mu_V(x) = \lambda_j$. If $\lambda_j > \lambda_i$, then $\mu_B(x) = \lambda_j$.

If $\lambda_j = \lambda_i$, then $\mu_B(x) \geq \lambda_i$. Therefore, in any case $x \in \mu_B^{[\lambda]}$.

Thus $B_{V_i} \subseteq \mu_B^{[\lambda]}$. Hence $\mu_B^{[\lambda]} = B_{V_i}$.

Similarly, it can be proved that $B_{V_i} = v_{B_{i}}^{[\xi]}$. \hfill \Box

**Proposition 2.23.** Let $V \in IFVS(X)$ with $dimX = m$ and $Im(V) = \{(\lambda_0, \xi_0), (\lambda_1, \xi_1), \ldots (\lambda_k, \xi_k)\}, k \leq m$ such that $(1,0) \geq (\lambda_0, \xi_0) > (\lambda_1, \xi_1) > \ldots > (\lambda_k, \xi_k) \geq (0,1)$. Then for $i = 0, 1,...,k$, $V^{[\lambda_i, \xi_i]} = \mu_V^{[\lambda_i]} = v_{V_i}^{[\xi_i]}$.

**Proof**. Obviously, $V^{[\lambda_i, \xi_i]} \subseteq \mu_V^{[\lambda_i]}$.

Let $x \in \mu_V^{[\lambda_i]}$.

$\Rightarrow \mu_V(x) \geq \lambda_i$.

Let $\mu_V(x) = \lambda_j$. Then $\nu_V(x) = \xi_j$.

$\Rightarrow x \in V^{[\lambda_j, \xi_j]}$ [as either $(\lambda_j, \xi_j) = (\lambda_i, \xi_i)$ or $(\lambda_j, \xi_j) > (\lambda_i, \xi_i)$].

Thus $\mu_B^{[\lambda_i]} \subseteq V^{[\lambda_i, \xi_i]}$. Therefore $V^{[\lambda_i, \xi_i]} = \mu_V^{[\lambda_i]}$.

Similarly we have $V^{[\lambda_i, \xi_i]} = v_{V_i}^{[\xi_i]}$. \hfill \Box

**Proposition 2.24.** Let $B$ be an intuitionistic fuzzy basis of $V$ corresponding to $(\ast)$ of Definition 2.20. Then $|\mu_B^{[\lambda]}| = dim(\mu_V^{[\lambda]})$ and $|v_B^{[\xi]}| = dim(v_{V_{i}}^{[\xi]})$, for $i = 0,1,2,...,k$.

**Proof**. $|\mu_B^{[\lambda]}| = |B_{V_i}| = dim(V^{[\lambda_i, \xi_i]}) = dim(\mu_V^{[\lambda_i]})$ [By Proposition 2.22 and 2.23]. The rest part is similar. \hfill \Box

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3 Intuitionistic fuzzy dimension

Definition 3.1. Let $A$ be an intuitionistic fuzzy set over $X$. Define a map $|A| : \mathbb{N} \to [0,1] \times [0,1]$ such that $\forall n \in \mathbb{N}, \mu_A(n) = \bigvee \{ a : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\} \text{ with } a+b \leq 1 \text{ and } |A^{(a,b)}| \geq n \}$ and $v_A(n) = \bigwedge \{ b : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\} \text{ with } a+b \leq 1 \text{ and } |A^{(a,b)}| \geq n \}$. Then $|A|$ is an intuitionistic fuzzy set over $\mathbb{N}$, which is called the cardinality of $A$.

Definition 3.2. For two IFS $A, B$ over $X$, the addition $|A| + |B|$ of $|A|$ and $|B|$ is defined as follows: for any $n \in \mathbb{N}, \mu_{(A|+|B)}(n) = \bigvee_{k+l=n} (\mu_A(k) \land \mu_B(l))$ and $v_{(A|+|B)}(n) = \bigwedge_{k+l=n} (v_A(k) \lor v_B(l))$.

Proposition 3.3. For two IFS $|A|, |B|$ over $\mathbb{N}$ and for any $(a,b) \in [0,1] \times [0,1]$ with $a+b \leq 1$, $\mu_{(A|+|B)}(n) = \mu_A(n) + \mu_B(n)$ and $v_{(A|+|B)}(n) = v_A(n) + v_B(n)$.

Proof. First we prove that $\mu_{(A|+|B)}(n) \leq \mu_A(n) + \mu_B(n)$.

Let $n \in \mu_{(A|+|B)}(n)$. Then $\mu_{(A|+|B)}(n) = \bigvee_{k+l=n} (\mu_A(k) \land \mu_B(l)) \geq a$.

Hence there exist $k, l$ such that $n = k+l$ and $\mu_A(k) \land \mu_B(l) \geq a$. Then $k \in \mu_A(n)$ and $l \in \mu_B(n)$, i.e., $n = k+l \in \mu_A(n) + \mu_B(n)$. Similarly, it can be proved that $v_{(A|+|B)}(n) \leq v_A(n) + v_B(n)$.

Conversely suppose that $n \in \mu_A(n) + \mu_B(n)$.

Then there exist $k, l$ such that $n = k+l$ with $k \in \mu_A(n), l \in \mu_B(n)$. Then $(\mu_A(n))(k) \geq a, (\mu_B(n))(l) \geq a$.

Therefore $\mu_{(A|+|B)}(n) = \bigvee_{k+l=n} (\mu_A(k) \land \mu_B(l)) \geq a$. Thus $n \in \mu_{(A|+|B)}(n)$.

Hence $\mu_A(n) + \mu_B(n) \leq \mu_{(A|+|B)}(n)$.

Similarly, we have $v_A(n) + v_B(n) \leq v_{(A|+|B)}(n)$. Hence proved. $\square$

Definition 3.4. Let $V \in IFVS(X)$ with an intuitionistic fuzzy basis $B$. Define $\text{dim}(V) = |B|$. Then $\text{dim}(V)$ is called intuitionistic fuzzy dimension of $V$.

Proposition 3.5. Let $B$ and $B'$ be two intuitionistic fuzzy bases of an intuitionistic fuzzy vector space $V \in IFVS(X)$. Then $|B| \geq |B'|$.

Proof. By Proposition 2.21, $\mathbb{B}^{(a,b)}$ and $\mathbb{B}'^{(a,b)}$ are bases of $V^{(a,b)}$ for $a \in (0,1), b \in [0,1]$ with $a+b \leq 1$. Then $|\mathbb{B}^{(a,b)}| \geq |\mathbb{B}'^{(a,b)}|$.

Hence for any $n \in \mathbb{N}$,

$\mu_{(B)}(n) = \bigvee \{ a : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\} \text{ with } a+b \leq 1 \text{ and } |\mathbb{B}^{(a,b)}| \geq n \}$

$= \bigvee \{ a : (a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\} \text{ with } a+b \leq 1 \text{ and } |\mathbb{B}'^{(a,b)}| \geq n \}$

$= \mu_{(B')} (n)$. Similarly, for any $n \in \mathbb{N}, v_{(B)}(n) = v_{(B')} (n)$. Hence proved. $\square$

Remark 3.6. Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space is independent of intuitionistic fuzzy basis.

Proposition 3.7. Let $X$ be a vector space with $\text{dim}X = m$ and $V \in IFVS(X)$. Then for any $(a,b) \in [0,1] \times [0,1] \setminus \{(0,1)\}$ with $a+b \leq 1$ and $n \in \mathbb{N}$, $n \in \mu_{\text{dim}(V)} \iff n \leq \text{dim}(\mu_{\text{dim}(V)})$ and $n \in v_{\text{dim}(V)} \iff n \leq \text{dim}(v_{\text{dim}(V)})$.
**Proposition 3.9.** Let \( \lambda \subseteq V^{[\lambda, \xi]} \) for each \( \lambda \). Then there exists a nested collection of subspaces of \( X \) as \( \{ \emptyset \} \subseteq V^{[\emptyset, \xi]} \subseteq V^{[\lambda_1, \xi_1]) \subseteq \ldots \subseteq V^{[\lambda_k, \xi_k]} \subseteq X \).

Let \( B_i \) be the basis of \( V^{[\lambda_i, \xi_i]} \), \( i = 0, 1, \ldots, k \) such that \( B_0 \subseteq B_1 \subseteq \ldots \subseteq B_k \).

Let \( B \) be an intuitionistic fuzzy basis corresponding to \( (\ast) \) defined as in Definition 2.20. Let \( n \in \mu_{\text{dim}(V)}^a(n) \Rightarrow \mu_{\text{dim}(V)}^a(n) \geq a \Rightarrow \forall \{ c : (c, d) \in [0, 1] \} \text{ with } c + d \leq 1 \text{ and } | \mathbb{B}_{c, d} | \geq n \geq a.

Then there exists \( (c, d) \in [0, 1] \times [0, 1] \) \( \{ (0, 0) \} \) with \( c + d \leq 1 \) such that \( c \geq a \) and \( | \mathbb{B}_{c, d} | \geq n \).

Now \( \dim(\mu_{\text{dim}(V)}^a) = | \mu_{\text{dim}(V)}^a | = | \mathbb{B}_{c, d} | = | \mathbb{B}_{c, d} | = n \).

Conversely suppose that \( n \leq \dim(\mu_{\text{dim}(V)}^a) = | \mu_{\text{dim}(V)}^a | \). Now \( a \in (\lambda_{i+1}, \lambda_i) \), for some \( i \). Hence \( | \mu_{\text{dim}(V)}^a | = | \mu_{\text{dim}(V)}^a | \). Then \( \mu_{\text{dim}(V)}^a = \forall \{ c : (c, d) \in [0, 1] \times [0, 1] \} \text{ with } c + d \leq 1 \text{ and } | \mathbb{B}_{c, d} | \geq n \geq a \).

Hence \( n \in \mu_{\text{dim}(V)}^a \Leftrightarrow n \leq \dim(\mu_{\text{dim}(V)}^a) \).

Similarly it can be proved that \( n \in \nu_{\text{dim}(V)}^b \Leftrightarrow n \leq \dim(\nu_{\text{dim}(V)}^b) \). \( \square \)

**Proposition 3.8.** Let \( X \) be a vector space with \( \dim X = m \) and \( V_1, V_2 \in \text{IFVS}(X) \). Then we have the following results:

1. For all \( (a, b) \in [0, 1] \times [0, 1] \) with \( a + b \leq 1 \), \( \mu_{\text{dim}(V_1)}^a \cap \mu_{\text{dim}(V_2)}^b = \nu_{\text{dim}(V_1)}^b \cap \nu_{\text{dim}(V_2)}^b \).

2. For all \( (a, b) \in [0, 1] \times [0, 1] \) with \( a + b \leq 1 \), \( \mu_{\text{dim}(V_1 + V_2)}^a = \mu_{\text{dim}(V_1)}^a + \mu_{\text{dim}(V_2)}^b \) and \( \nu_{\text{dim}(V_1 + V_2)}^b = \nu_{\text{dim}(V_1)}^b + \nu_{\text{dim}(V_2)}^b \).

**Proof.** We only give the proof of (2). For any \( (a, b) \in [0, 1] \times [0, 1] \) with \( a + b \leq 1 \), we have \( a \in \mu_{\text{dim}(V_1 + V_2)}^a \Leftrightarrow \sup \{ \mu_{\text{dim}(V_1)}^a \cap \mu_{\text{dim}(V_2)}^b \} \geq a \).

\( \Leftrightarrow \) there exist \( x_1, x_2 \) such that \( x_1 + x_2 = x \) and \( \mu_{\text{dim}(V_1)}^a \cap \mu_{\text{dim}(V_2)}^b \geq a \).

\( \Leftrightarrow \) there exist \( x_1, x_2 \) such that \( x_1 + x_2 = x \) and \( x_1 \in \mu_{\text{dim}(V_1)}^a \) and \( x_2 \in \mu_{\text{dim}(V_2)}^b \).

Similarly it can be proved that \( \nu_{\text{dim}(V_1 + V_2)}^b = \nu_{\text{dim}(V_1)}^b + \nu_{\text{dim}(V_2)}^b \). \square

**Proposition 3.9.** Let \( X \) be a vector space with \( \dim X = m \) and \( V_1, V_2 \in \text{IFVS}(X) \). Then \( \dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) \).

**Proof.** For any \( (a, b) \in [0, 1] \times [0, 1] \) with \( a + b \leq 1 \), let \( n \in \mu_{\text{dim}(V_1 + V_2)}^a \Leftrightarrow \sup \{ \mu_{\text{dim}(V_1)}^a \cap \mu_{\text{dim}(V_2)}^b \} \geq a \). \( \Leftrightarrow \) there exist \( k, l \) such that \( n = k + l \) and \( k \in \mu_{\text{dim}(V_1 + V_2)}^a \) and \( l \in \mu_{\text{dim}(V_1 \cap V_2)}^a \). Then by Proposition 3.7, \( k \leq \dim(\mu_{\text{dim}(V_1 + V_2)}^a) = \dim(\mu_{\text{dim}(V_1)}^a + \mu_{\text{dim}(V_2)}^a) \) and \( l \leq \dim(\mu_{\text{dim}(V_1 \cap V_2)}^a) = \dim(\mu_{\text{dim}(V_1)}^a \cap \mu_{\text{dim}(V_2)}^a) \). Thus \( n \leq \dim(\mu_{\text{dim}(V_1)}^a + \mu_{\text{dim}(V_2)}^a) + \dim(\mu_{\text{dim}(V_1)}^a \cap \mu_{\text{dim}(V_2)}^a) \).

Then there exist \( k' \) and \( l' \) such that \( n = k' + l' \) and \( k' \leq \dim(\mu_{\text{dim}(V_1)}^a) \) and \( l' \leq \dim(\mu_{\text{dim}(V_2)}^a) \). Now by Proposition 3.7, \( k' \in \mu_{\text{dim}(V_1)}^a \) and \( l' \in \mu_{\text{dim}(V_2)}^a \). Therefore \( n = k' + l' \in \mu_{\text{dim}(V_1)}^a + \mu_{\text{dim}(V_2)}^a \).

Similarly, \( \nu_{\text{dim}(V_1 + V_2)}^b \leq \nu_{\text{dim}(V_1)}^b + \nu_{\text{dim}(V_2)}^b \). \( \square \)

Also, it can be proved that for any \( (a, b) \in [0, 1] \times [0, 1] \) with \( a + b \leq 1 \), \( \mu_{\text{dim}(V_1)}^a + \mu_{\text{dim}(V_2)}^b \leq \mu_{\text{dim}(V_1 + V_2)}^a + \mu_{\text{dim}(V_1 \cap V_2)}^a \) and \( \nu_{\text{dim}(V_1)}^b + \nu_{\text{dim}(V_2)}^b \leq \nu_{\text{dim}(V_1 + V_2)}^b + \nu_{\text{dim}(V_1 \cap V_2)}^b \). Thus for any \( (a, b) \in [0, 1] \times [0, 1] \) with \( a + b \leq 1 \), \( \mu_{\text{dim}(V_1)}^a + \mu_{\text{dim}(V_2)}^b = \mu_{\text{dim}(V_1 + V_2)}^a + \mu_{\text{dim}(V_1 \cap V_2)}^a \) and \( \nu_{\text{dim}(V_1)}^b + \nu_{\text{dim}(V_2)}^b = \nu_{\text{dim}(V_1 + V_2)}^b + \nu_{\text{dim}(V_1 \cap V_2)}^b \). Hence \( \dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) \). \( \square \)
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References

[1] Atanassov, K. T. (1986) Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 87–96.

[2] Atanassov, K. T. (1994) New operations defined over intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61(2), 137–142.

[3] Atanassov, K. T. (1999) *Intuitionistic Fuzzy Sets: Theory and Applications*, Studies in Fuzziness and Soft Computing, Vol. 35, Springer Physica-Verlag, Heidelberg.

[4] Atanassov, K. T., (2012) *On Intuitionistic Fuzzy Sets Theory*, Studies in Fuzziness and Soft Computing, Vol. 283, Springer, Berlin.

[5] Atanassova, L. (2007) On intuitionistic fuzzy versions of L. Zadeh’s extension principle, *Notes on Intuitionistic Fuzzy Sets*, 13(3), 33–36.

[6] Biswas, R. (1989) Intuitionistic fuzzy subgroups, *Math. Forum*, 10, 37–46.

[7] Biswas, R. (1997) On fuzzy sets and intuitionistic fuzzy sets, *Notes on Intuitionistic Fuzzy Sets*, 3(1), 3–11.

[8] Chen, W., & Zhang, S. (2009) Intuitionistic fuzzy Lie sub-superalgebras and intuitionistic fuzzy ideals, *Computers and Mathematics with Applications*, 58, 1645–1661.

[9] Chiney, M., & Samanta, S. K., Intuitionistic fuzzy basis of an intuitionistic fuzzy vector space, *Notes on Intuitionistic Fuzzy Sets* (Accepted).

[10] Coker, D. (1997) An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems*, 88, 81–89.

[11] De, S. K., Biswas, R., & Roy, A. R. (2001) An application of intuitionistic fuzzy sets in medical diagnostic, *Fuzzy Sets and Systems*, 117(2), 209–213.

[12] Ejegwa, P. A., Akubo, A. J., & Joshua, O. M. (2014) Intuitionistic fuzzy set and its application in career determination via normalized euclidean distance method, *European Scientific Journal*, 10(15), 529–536.
[13] Hur, K., Kang, H.W., & Song, H. K. (2003) Intuitionistic fuzzy subgroups and subrings, *Honam Math. J.*, 25, 19–41.

[14] Katsaras, A. K., & Liu, D. B. (1977) Fuzzy vector spaces and fuzzy topological vector spaces, *J. Math. Anal. Appl.*, 58, 135–146.

[15] Mohammed, M. J., & Ataa, G. A. (2014) On Intuitionistic fuzzy topological vector space, *Journal of College of Education for Pure Sciences*, 4, 32–51.

[16] Mondal, K. K., & Samanta, S. K. (2013) A study on intuitionistic fuzzy topological spaces, *Notes on Intuitionistic Fuzzy Sets*, 9(1), 1–32.

[17] Park, J. H. (2004) Intuitionistic fuzzy metric spaces, *Chaos Solitons Fractals*, 22, 1039–1046.

[18] Padmapriya, S., Uma, M. K., & Roja, E. (2014) A study on intuitionistic fuzzy topological groups, *Annals of Fuzzy Mathematics and Informatics*, 7(6), 991–1004.

[19] Pradhan, R., & Pal, M. (2012) Intuitionistic fuzzy linear transformations, *Annals of Pure and Applied Mathematics*, 5(1), 57–68.

[20] Shi, F. G., & Huang, C. E. (2010) Fuzzy bases and the fuzzy dimension of fuzzy vector spaces, *Math. Commun.*, 15(2), 303–310.

[21] Szmidt, E., & Kacprzyk, J. (1996) Intuitionistic fuzzy sets in group decision making, *Notes on Intuitionistic Fuzzy Sets*, 2(1), 11–14.

[22] Zadeh, L. A. (1965) Fuzzy sets, *Information and Control*, 8, 338–353.