Spin-textured Volkov–Pankratov states and their tunnel magnetoresistance response

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Abstract

Volkov–Pankratov (VP) states are a family of sub-gap states that appear at the smooth interface/domain wall between topologically distinct gaped states. We carry out quantum transport simulations on one- and two-dimensional lattice models to demonstrate the emergence of such states in the edge spectrum of a quantum spin Hall system subjected to a smoothly varying exchange field that switches its sign at a given spatial point. We show the VP states possess non-trivial spin textures that can be characterized by a winding number in real space. It is further demonstrated that the application of an electric field along the edge provides control of this spin texture without altering the winding number. Finally, we illuminate how these spin textures can be read off via the local tunnel magnetoresistance (TMR) response of spin-polarized tunnel probes attached to the edge and the TMR can be controlled by purely electrical means akin to a Datta–Das type spin transistor.

1. Introduction

Topological quantum matter exhibits a myriad of unconventional phenomena—perhaps the most celebrated one is the bulk-boundary correspondence [1, 2] when robust boundary modes emerge due to a topologically nontrivial bulk. Burgeoning experimental activities over the last decades have illuminated possibilities of leveraging this effect in device applications as well [3–7]. A gaped bulk can harbor a nontrivial topology when the underlying Hamiltonian admits certain spin–orbit interactions (leading to quantum spin Hall (QSH) insulators [4, 5, 7–10], topological insulators [11, 12]) or breaks time-reversal invariance (such as the case of a Chern insulator [13]). By tuning appropriate parameters of the Hamiltonian, one can invert the sign of the bulk gap \(\Delta\) about the Fermi energy, thereby creating a domain-wall type configuration in \(\Delta\) at a heterojunction setup comprising a junction between a topologically nontrivial bulk and a trivial one. Detecting these modes in experiments renders a direct avenue to probe the topology of the bulk.

For the specific case of a 1D system, an abrupt change (of the sign) in the profile of the gap \(\Delta(x)\) [e.g. when \(\Delta(x) \propto \text{sgn}(x)\)] would lead to the formation of bound states that are exponentially localized at the interface. A prominent example would be the Jackiw–Rebbi zero mode [1, 14–17] found either in the Su–Schrieffer–Heeger chain or at the magnetic domain walls in the helical edge states (HESs) of a QSH state (QSHS). In distinction, when \(\Delta(x)\) varies smoothly, sub-gap states can appear also at non-zero energies whose count depends on the width of the domain wall, identified as the Volkov–Pankratov (VP) states [18, 19]. Volkov and Pankratov noted these states while studying interfacial phenomena in ferroelectric compounds in the presence of a magnetic field. They further pointed out a supersymmetric structure [18] of the underlying effective Hamiltonian for a generic band inversion problem allowing for a prediction of such states without requiring any specific form of the potential at the transition region. Following that trail, numerous topological heterojunction setups have been theoretically studied which involve graphene...
nanoribbons, topological superconductors, Weyl semimetals, and recently a topological-trivial semiconductor heterojunction has been experimentally explored [20–25].

The central topic of this article is to investigate the emergence of VP states at the edges of a QSH insulator by employing quantum transport simulations on lattice models and to come up with a proposal for detecting these states within a quantum transport setting. A simple model to describe such a system is the Bernevig–Hughes–Zhang (BHZ) model [26] of HgTe quantum well [27] where adjusting the well width results in a band inversion, driving the system into a topological insulator state [24, 28, 29]. This is a time-reversal symmetric system with the edge modes having a conserved spin quantum number locked with their momentum, viz., if $+ \uparrow$ spins ($S_z = +1$) flow along $+k$, $- \downarrow$ spins ($S_z = -1$) would flow along $-k$. These are known as the HESs having linear dispersions around the $\Gamma$ point [30–35]. The HES of the QSHS are described by massless Dirac fermions where the Fermi sea carries a persistent spin current. Exposing them to a spatially varying transverse exchange field results in a spatially varying gap in the edge spectrum proportional to the strength of the exchange field and gives way to realizing edge-localized VP states.

The key results can be summarized as (i) establishing the emergence of stable VP states (using lattice transport simulations) at the edge of a QSH system due to the application of a spatially varying magnetic field with optimized strength across the bulk of the QSH even in presence of a finite but small amount of time-reversal symmetry breaking random disorder, (ii) showing the existence of nontrivial spin texture of the VP states in real space with helicoidal winding along the edge, characterized by winding numbers that are intimately connected to the energy quantum numbers, (iii) putting forward a proposal for possible experimental detection of our predicted nontrivial spin textures in tunnel magnetoresistance (TMR) response [36] of spin-polarized tunnel probes.

The rest of the paper is organized as follows. In section 2, we will present model calculations to demonstrate the appearance of the VP states in the edge spectrum of a QSHS setup in the presence of a linearly varying magnetic field $B(x)$ as well as a hyperbolic profile of $B(x)$. Alluding to the supersymmetric structure of the problem we will further discuss the effects of the inclusion of an electric field. We will show how it can be used to manipulate the spectrum of the bound states and their spin texture in real space. In section 3 we set up transport simulation for a one-dimensional spin–orbit coupled chain and the lattice model of a two-dimensional topological insulator and we benchmark it against the analytic findings of previous sections. Then we use these lattice simulations to carry out a feasibility study for our proposal for detecting these VP states using TMR. Finally, we conclude in section 4.

### 2. QSHS subjected to smooth potentials: the emergence of the VP states

The following model demonstrates the emergence of VP states for a smooth domain wall potential separating two phases related by a band inversion. As a specific setup, let us consider a QSHS where the HESs are subjected to a smoothly varying magnetic field $B(x)$ opening a spatially varying mass gap. The two phases here are characterized by the sign of this mass gap which can be inverted by changing the sign of the magnetic field. We construct a domain wall configuration of $B(x)$ around $x = 0$ such that $B(-x) = -B(x)$ separates the two phases at $x < 0$ and $x > 0$. The spins of the free HES are polarized along the $z$-direction which is perpendicular to the plane hosting the QSHS while the magnetic field is taken in-plane and oriented along the $x$-axis.

The Hamiltonian governing the dynamics of the free HES (extended from $x = -\infty$ to $x = +\infty$, $x$ representing an intrinsic one-dimensional coordinate along the edge) is a Dirac Hamiltonian

$$\mathcal{H}_{\text{QSH}} = \int dx \Psi \Gamma H \Psi ; H = -ihv_F \sigma_z \partial_x,$$

where $v_F$ is the Fermi velocity of the electrons on the edge and $\Psi \equiv (\psi_R \ \psi_L)^T$ denotes the annihilation operator for the right ($R$) and the left ($L$) moving electrons ($\sigma_i$ are the Pauli matrices). The in-plane magnetic field facilitating backscattering between the two helical edges, in this basis, takes the form

$$\mathcal{H}_B = \int dx \Psi \Gamma H_B (x) \Psi ; H_B (x) = g\mu_B B(x) \sigma_z,$$

where $g$ is the g-factor of the electron, and $\mu_B$ is the Bohr magneton. For the rest of the article, we will use the notion of a magnetic potential $M(x) = g\mu_B B(x)$ to simplify notations. Moreover, we will be interested in exploring how the finite-energy sub-gap states respond to an electric field that can either be applied externally or can arise naturally in the system from spontaneous defragmentation of the QSHS into regions of distinct spin–orbit fields respecting time-reversal symmetry but breaking $S_z$ conservation [37]. The result
of the electric field \( E(x) \) can be incorporated via including a space-dependent chemical potential \( \mu(x) \) (such that \( E = -\partial_x \mu \)) that leads to a total Hamiltonian of the form

\[
\mathcal{H} = \int dx \left[ \psi^\dagger \left( -i\hbar \gamma_\mu \partial_\mu + M(x) \gamma_\sigma + \mu(x) \right) \psi \right].
\]  

(3)

Note here both \( M(x) \) and \( \mu(x) \) are smooth functions of \( x \), and to obtain analytic solutions of the eigenvalue problem \( \mathcal{H} \psi = E \psi \), we will restrict to \( \mu(x) \propto M(x) \) when an electric field is applied.

In what follows, we will discuss the effect of bounded and unbounded magnetic potential \( M(x) \), both in the absence and presence of an electric field. The strength of the electric field is such adjusted that \( \mu \) always stays within the gap set by the magnetic potential \( M(x) \). As we will see, the VP states display an interesting response to the electric field in distinction to the Jackiw–Rebbi mode, namely, the VP modes shift in the real space along the direction of the applied field in a highly nonlinear fashion.

2.1. Supersymmetric quantum mechanics

In the absence of an electric field, the Dirac problem at hand has a supersymmetric structure [18]. The Hamiltonian

\[
\mathcal{H} = \int dx \left[ \psi^\dagger \left( -i\hbar \gamma_\mu \partial_\mu + M(x) \gamma_\sigma \right) \psi \right]
\]

(4)

can be brought, via a unitary transformation \( U = e^{i\pi \sigma_\mu / 4} \), to a chiral form

\[
\mathcal{H} = \int dx \left[ \bar{\psi} \gamma_\sigma \left( A^\dagger \gamma_\mu \gamma_\sigma \right) \psi - \hbar \gamma_\mu \partial_\mu + M(x) \right].
\]

(5)

Denoting the chiral matrix in equation (5) as \( \tilde{H} \), the bound state solutions of \( \tilde{H} \psi = E \psi \), that are of the form \( \psi = [\theta \phi]^T \), are referred to as the VP states. We can decouple the equations for \( \theta \) and \( \phi \) by squaring \( \tilde{H} \) which produces two Schrödinger equations

\[
\begin{align*}
\left( \partial_\mu^2 - \left( \frac{M}{\hbar \gamma_\mu} \right)^2 + \partial_\mu \frac{M}{\hbar \gamma_\mu} - E^2 \right) \theta &= 0, \\
\left( \partial_\mu^2 - \left( \frac{M}{\hbar \gamma_\mu} \right)^2 - \partial_\mu \frac{M}{\hbar \gamma_\mu} - E^2 \right) \phi &= 0.
\end{align*}
\]

(6)

In the language of supersymmetric quantum mechanics, the function \( M/\hbar \gamma_\mu \) is known as the superpotential whereas the potentials \( (M/\hbar \gamma_\mu)^2 \pm \partial_\mu M/\hbar \gamma_\mu \) are the supersymmetric partner potentials [38]. The associated Schrödinger Hamiltonians admit only non-negative energy eigenvalues (which we index with an integer \( N \)) and at least one of them must have zero modes (signature of unbroken supersymmetry [39]).

We will now discuss two cases (i) an unbounded superpotential where \( M^2(x) \propto x^2 \)—a harmonic oscillator potential, and (ii) a bounded superpotential where \( M^2(x) \propto \tanh^2(x) \)—referred to as the Rosen–Morse potential [40, 41], and note distinct signatures in the resulting VP states. We will then apply an electric field in both setups to see how the energy spectrum and spatial texture of these states are influenced leaving identifiable signatures in transport measurements. Note these are some of the well-known examples of a reflection-less potential where quantum transport studies of the scattering states may also reveal interesting effects.

2.2. Unbounded \( M(x), \mu(x) = 0 \)

Let us first consider a simple unbounded superpotential \( M(x) = \alpha x \) to illustrate the physics associated with the emergence of VP states when no electric field is present i.e. \( \mu(x) = 0 \). Such a magnetic potential arises from an odd-parity magnetic field linearized around a given spatial point \( x = 0 \) and extended throughout the HES (from \( x = -\infty \) to \( x = +\infty \)), namely \( B(x) = B_0 x \). The operator \( A^\dagger \) in this case becomes proportional to the conventional bosonic ladder operator \( A^\dagger = \sqrt{2\alpha \hbar \gamma_\mu} a^\dagger \) where \([a, a^\dagger] = 1\). We, therefore, arrive at

\[
\begin{align*}
2\alpha \hbar \gamma_\mu a^\dagger a - E^2 &= 0, \\
2\alpha \hbar \gamma_\mu aa^\dagger - E^2 &= 0,
\end{align*}
\]

(7)

from which we can immediately obtain the spinor part of the solution \( \tilde{H} \psi = E \psi \) in terms of the eigenstates of the bosonic number operator \( \hat{N} = a^\dagger a \). The VP states for such a potential satisfy the energy quantization

\[
E_N = \text{sgn} (N) \sqrt{2\alpha \hbar \gamma_\mu |N|} ; \quad N \in \mathbb{Z}.
\]

(8)

We thus see that a linear potential, for massless Dirac particles, can lead to the genesis of discrete bound states at zero and non-zero energies. The zero-energy mode (for \( N = 0 \)), known as the Jackiw–Rebbi mode, is
extensively studied, known to carry fractional quantum numbers [42, 43], and whose existence signals a
topological phase transition for a sharp domain wall between two topologically distinct states.
The finite-energy VP states corresponding to \( N \neq 0 \), on the other hand, are a salient feature of a Dirac oscillator
[44–48] in a smoothly varying (here, linear) potential. We will later see the VP states foster nontrivial
windings in their spin texture (unlike the Jackiw–Rebbi mode), which can be manipulated by applying an
electric field.

The energy levels for the VP states in equation (8) are not equidistant reminding us of the Landau
quantization of Dirac fermions in graphene. The spatial profile of the VP states is obtained by solving
\[
\begin{align*}
\left[ \hbar^2 \psi'' + \left( E^2 - \alpha^2 x^2 + \alpha \hbar v_F \right) \right] \theta &= 0, \\
\left[ \hbar^2 \psi'' + \left( E^2 - \alpha^2 x^2 - \alpha \hbar v_F \right) \right] \phi &= 0.
\end{align*}
\]

The quantization in equation (8) implies the normalized solution for the \( N \)th bound state to be of the form
(see appendix A for details)
\[
\begin{align*}
\theta_{|N|}(x) &= A_{|N|} e^{-x^2/2|\xi|} H_{|N|}(x/\xi), \\
\phi_{|N|}(x) &= A_{|N|-1} e^{-x^2/2|\xi|} H_{|N|-1}(x/\xi),
\end{align*}
\]
where \( A_{|N|} = 2^{[N]} |N|! \pi^{-1/2} |\xi| \) and \( \xi = \sqrt{\hbar v_F/\alpha} \) (\( H_{|N|} \) are the Hermite polynomials). The parameter \( \xi \) is
analogous to the magnetic length in the Landau level problem (identifying \( \alpha/\hbar v_F \) with the magnetic field \( eB \))
over which the bound state wavefunctions localize along the edge.

From the explicit forms of the wavefunctions (equation (10)), we can straightforward calculate observables
like the probability and spin density associated with the VP spinor \( |\psi_{|N|}(x)\rangle \equiv [\theta_{|N|}(x), \phi_{|N|}(x)]^{T} / \sqrt{2} \)
keeping in mind that \( \phi_{|N|}(x) \) does not contribute to the zero mode. This has a natural interpretation in terms
of supersymmetry. The spectrum of \( \theta \) and \( \phi \) for finite energies are identical but the zero mode (for \( \theta \))—the
former has one while the latter does not as is the case of the supersymmetric quantum mechanics with
Witten index \( \nu = 1 \).

The probability density of a VP state with quantum number \( \pm N \) is \( f^{(|N|)}(x) = |\psi_{|N|}(x)|^2 \) which we plot in
figure 1(a) at \( \alpha = 0.5 \). For these wavefunctions involve a Gaussian component that decays over space
(equation (10)), the corresponding probability density \( f^{(|N|)}(x) \) dies off away from the gap closing point at
\( x = 0 \) with a characteristic length scale \( \xi \). However, we uncover an interesting signature when we measure the
spin density \( S_{|N|}(x) = \langle \psi_{|N|}(x)| \sigma_i |\psi_{|N|}(x)\rangle \) where \( \sigma_i \) are the Pauli matrices. For illustration, in figure 1(b),
we display the spatial profile of the normalized spin vector \( S^{(|N|)}(x) = (S_{z}^{(|N|)}, 0, S_{x}^{(|N|)}) / \sqrt{[S_{z}^{(|N|)}]^2 + [S_{x}^{(|N|)}]^2} \)
for \( N = 1, 2, 3 \) at \( \alpha = 0.5 \) (\( S_{z}^{(|N|)} = 0 \) as \( \theta_{|N|}, \phi_{|N|} \) in equation (10) are real). Notably, figure 1 reveals a neat
result: The quantum number \( N \) of the spinor \( |\psi_{|N|}(x)\rangle \) measures its winding along the extent of the helical edge
over which the associated wavefunctions localize. The winding number for the \( N \)th VP state is given by
\[
w^{(|N|)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \partial_x \arg \left[ S_{z}^{(|N|)} + iS_{x}^{(|N|)} \right].
\]
As one asymptotically approaches \( x \to \pm \infty \), the spin vectors associated with the VP spinors align or
anti-align along the \( z \)-axis (depending on the sign of \( \alpha \)). Note this is an invariant formulated in real space.
rather than the Fourier space since we do not have a translation-invariant HES [translation symmetry broken by $M(x)$] for which the momentum $k$ can serve as a good quantum number and casting the spinor on to a Bloch sphere can reveal the associated spin vector. Such kind of topological spin texture is known in the context of a quantum anomalous Hall (QAH) insulator set on a cylindrical geometry [49].

The Jackiw–Rebbi mode does not wind since the corresponding spin vector $S^{(0)}(x)$ only has a $z$ component. In distinction, localized periodic windings of the associated spin vector $S^{(N)}(x)$ are observed for any VP state with index $N$ (with $N \neq 0$) as one traces it along the HES. Reminiscent of helicoids in the real space, this is a noteworthy signature of the VP states while the centers of the windings for the $N$th VP state are given by the locations of the zeros of the Hermite polynomial $H_{N-1}(x/\xi)$ and coincide with the local minima of the associated probability density. This is remarkable since by tuning the strength of the magnetic field, thereby changing $\xi$, one can alter the local spin polarization $S^{(N)}(x)$ and vary the distance between the winding centers of $S^{(N)}$ for $N \geq 2$. Mapping from the extended real line $\mathbb{R} : [-\infty, +\infty]$ to the spin space $S_2$ (surface of a unit sphere), we observe the $N$th VP state encircles an equatorial plane (in $S_2$) $N$ times accumulating a spin-Berry phase of $\pi N$.

Let us now investigate the emergence of the VP states in a smoothly varying but bounded potential that is more realistic to appear in an actual physical junction. This alters the spatial profile of the VP states and the associated energy quantization, and further and more important, limits the maximum number of such states within the gap due to the magnetic potential as will be demonstrated below.

2.3. Bounded $M(x), \mu(x) = 0$

Here we will assume no electric field is present by setting $\mu(x) = 0$ and model the smooth profile of the bounded magnetic potential by a tan-hyperbolic function, namely, $M(x) = M_0 \tanh(x/L)$ [50], which is also referred to as the Rosen–Morse potential in the literature, where $M_0$ denotes the strength of the potential and the characteristic length $L$ the saturation length beyond which $M(x)$ attains the saturation values $\pm M_0$ on either side of $x = 0$. Such type of magnetic potential is of relevance to modeling the physical interfaces in heterojunction-setups between topologically distinct phases and has been studied in graphene-based systems [51].

In the presence of such a bounded potential, the spatial profiles of the spinor components $\theta$ and $\phi$ are modified as follows. Substituting $M(x) = M_0 \tanh(x/L)$ in equation (6) and introducing the variable $z = \tanh(x/L)$, we obtain

$$\left[ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + l(l + 1) - \frac{\lambda^2}{1 - z^2} \right] \theta = 0,$$

$$\left[ (1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + l(l - 1) - \frac{\lambda^2}{1 - z^2} \right] \phi = 0,$$

where $l = M_0 L / \hbar v_F = L / \zeta$, $\zeta$ being a characteristic length scale inversely proportional to the saturation value of the magnetic potential $M_0$ and $\lambda = L / \sqrt{M_0^2 - \hbar^2 v_F^2}$. For the bound states, in order to express $\theta, \phi$ in terms of polynomials, $l, \lambda$ must be integers with $0 \leq \lambda \leq l$. These conditions together imply a finite number of VP states to exist within the gap $[-M_0, M_0]$ which is determined by the integer $l$ while their energy quantization is dictated by $\lambda$. Defining another integer-valued index $N \equiv l - \lambda$, the energy quantization for the VP states, which are the solution of equation (12), for the hyperbolic potential reads (see appendix A for details)

$$E_N = \text{sgn} \, (N) M_0 \left[ \frac{2|N|}{l} \frac{N^2}{l^2} \right]^{1/2},$$

while the spinor solution for the $N$th bound state is given by the associated Legendre polynomials

$$\theta \sim P_l^N (z) ; \phi \sim P_{l-1}^N (z).$$

The zero mode corresponds to $N = 0$ and is of the form $\theta \sim \text{sech}^l (x/L)$ while the VP states at the edge of the spectrum, i.e. $N = l$ are non-normalizable and referred to as the half-bound states [52].

The localization of the VP states is governed by the length scale $L$. As far as observables like the probability density and spin density are concerned, their behavior qualitatively remains the same as the linear case $M(x) \propto x$ but, to note, unlike the linear case, the winding number for the $N$th VP state is close to but not exactly $N$. This is because of the potential being bounded, the spinor does not perfectly line up along the $z$-axis as $x \to \pm \infty$ (for the linear gradient this was not the case since the potential was unbounded and $S^{(N)}_S \to 0$ as $x \to \pm \infty$). For the hyperbolic gradient case, $S^{(N)}_S$ for the $N$th VP state attains a non-zero value as $x \to \pm \infty$ multiplied by $\text{sgn}(x)$. This finite value decreases with $N$ and so does the deviation from $N$. We will comment on this issue in detail later at the end of this section.
2.4. Unbounded $M(x)$, $\mu(x) \neq 0$

We now embark on studying the response of the VP states to an external electric field. Let us first discuss the fate of the VP states that arise in a linear magnetic potential, additionally subject to a uniform electric field $E$ that leads to a chemical potential $\mu(x) = -EX$. The spatial profile of the spinor components $\theta$ and $\phi$, in this case, are obtained from the decoupled differential equations

\[
\begin{align*}
\left[ \partial_x^2 + \frac{E^2}{\hbar^2 v_F^2} + A(E) + \frac{2E^2}{\hbar^2 v_F^2} x - A^2(E) x^2 \right] \theta &= 0, \\
\left[ \partial_x^2 + \frac{E^2}{\hbar^2 v_F^2} - A(E) + \frac{2E^2}{\hbar^2 v_F^2} x - A^2(E) x^2 \right] \phi &= 0,
\end{align*}
\]

where $A(E) = \sqrt{\alpha^2 - E^2}/\hbar v_F$. Note the condition for the chemical potential to remain within the magnetic gap implies $E \leq \alpha$.

For bound states, we seek the solutions to be expressed in terms of polynomials which enforces a quantization of their energy (see appendix A for details)

\[
E_N(E) = \text{sgn}(N) \sqrt{2\alpha \hbar v_F |N|} \left( 1 - \frac{E^2}{\alpha^2} \right)^{3/4},
\]

depending on the electric field strength $E$. The spatial forms of $\theta$ and $\phi$ of the $N$th bound state are given by

\[
\begin{align*}
\theta_{[N]}(x) &= B_{[N]} e^{-\left(x-x_0\right)^2/2\xi_0^2} H_{[N]} \left[ \left( x - x_0 \right) / \xi_0 \right], \\
\phi_{[N]}(x) &= B_{[N]}^{-1} e^{-\left(x-x_0\right)^2/2\xi_0^2} H_{[N]-1} \left[ \left( x - x_0 \right) / \xi_0 \right],
\end{align*}
\]

where $B_{[N]}^{-2} = 2^{[N]} |N|! \pi^{1/2} \xi_0$, $\xi_0 = \sqrt{1/A(E)}$, and $x_0 = E_N \alpha / (\alpha^2 - E^2)$.

The application of an electric field unfolds an intriguing interplay signaling a transition at a critical field $E_0 = \alpha$ that has striking signatures in the spectrum as well as the spatial profiles of the probability density $J(x)$ and the (normalized) spin density $S(x)$.

Firstly, the energy spacing between the VP states decreases with the electric field $E$ (as per the scaling noted in equation (16)) as $E \to \alpha$ and the entire tower collapses at the transition point $E = \alpha$. This is a remarkable phenomenon from the viewpoint of supersymmetric quantum mechanics—the associated Witten index ceases to qualify for a topological invariant of the full supersymmetric theory as the number of zero modes alters drastically at this critical point.

Secondly, when the chemical potential due to the applied electric field is within the gap $[-M_0, M_0]$ set by the magnetic potential $M(x)$, i.e. $E < \alpha$, the effect of the electric field manifests as a shift of the spatial profile of the observables such as the probability density and the spin texture of the VP states for $N \neq 0$ in as we see in the plot of $J(x)$ in figure 2(a). Here we have plotted $J(x)$ for the $N = 2$ VP state at different values of $E$. The shift $x_0$ evidently increases with $E$ but so does the localization length $\xi_0$ and as a result, the magnitude of $J(x)$
is diminished. The Jackiw–Rebbi mode remains unaffected by the electric field since the shift parameter \( x_0 \) vanishes for the zero mode.

The electric field also influences the winding pattern of the spin vector associated with each VP state. The locations of the winding centers are shifted along the direction of the electric field while the separation between two successive winding centers decreases with the electric field. This is shown in figure 2(b) for the VP states with \( N = 2, 3 \) at different values of \( E \). Because of such a shift of the winding structure associated with the eigenstates \( \psi \), when we compute the overlap of a given eigenstate, indexed by \( N \), at a finite value of \( E \) with that for \( E = 0 \), we observe a periodic modulation in the profile of \( O(\mathcal{E}) \equiv |\langle \psi(\mathcal{E})|\psi(0)\rangle|^2 \) as a function of \( E/\alpha \). Notably, each VP state at the critical value of the electric field \( E = \alpha \), when they are part of a degenerate manifold of zero modes, becomes orthogonal to itself at \( E = 0 \). This is visible in figure 2(c). The zeros of the overlap function \( O(\mathcal{E}) \) are given by the values of \( E \) at which the shifted spin texture is orthogonal to that at \( E = 0 \). The number of such zeros for the \( N \)th VP state is, therefore, \( N \) for it has \( N \) winding centers.

We will now investigate the effect of a spatially textured electric field \( \mathcal{E}(x) \) such that the resultant chemical potential \( \mu(x) \propto M(x) = \kappa M_0 \tanh(x/L) \) with \( |\kappa| \leq 1 \). The decoupled equations for the spinor components \( \theta \) and \( \phi \) then read:

\[
\begin{align*}
-\partial_x^2 + W(x) - \partial_y W(x) \theta &= \varepsilon \theta, \\
-\partial_x^2 + W(x) + \partial_y W(x) \phi &= \varepsilon \phi, \\
\end{align*}
\]  

where

\[
W(x) = \frac{\sqrt{1 - \kappa^2}}{\hbar v} \left[ M(x) + \frac{\kappa}{1 - \kappa^2} E \right],
\]  

and

\[
\varepsilon = E^2 / \left[ \hbar^2 v^2 \left( 1 - \kappa^2 \right) \right].
\]

Equation (18) is reminiscent of the supersymmetry problem posed in the beginning in equation (6). Here we will elicit the concept of shape invariance [40] of the supersymmetric partner potentials \( W^2 \pm \partial_y W \) to analytically compute the bound state spectrum since the superpotential \( W(x) \) can be cast to the Rosen–Morse form \( W(x) = A \tanh(x/L) + B/A \) [40, 41]. The spectrum of the VP states, in this case, turns out to be (see appendix B for details):

\[
E_N = \text{sgn}(N) \sqrt{\frac{M_0^2(1 - \kappa^2) - (M_0 \sqrt{1 - \kappa^2} - \hbar v |N|/L)^2}{1 + M_0^2 \kappa^2 / (M_0 \sqrt{1 - \kappa^2} - \hbar v |N|/L)^4}}.
\]

We can immediately retrieve the spectrum for the zero electric field case (equation (13)) upon setting \( \kappa = 0 \) in the above equation.

To obtain the spatial profile of the VP states, we cast equation (18) as:

\[
\begin{align*}
(1 - z^2) \frac{d^2 \theta}{dz^2} + 2z \frac{d \theta}{dz} + \left[ l' (l' + 1) - \frac{\lambda^2}{1 - z^2} \right] \theta &= 0, \\
(1 - z^2) \frac{d^2 \phi}{dz^2} + 2z \frac{d \phi}{dz} + \left[ l' (l' - 1) - \frac{\lambda^2}{1 - z^2} \right] \phi &= 0,
\end{align*}
\]

where the variable \( z = \tanh(x/L) \), \( l' = M_0 L \sqrt{1 - \kappa^2}/\hbar v \), and \( \lambda' = L \sqrt{(1 - \kappa^2) M_0^2 - E^2 + 2 E \hbar M_0 x^2}/\hbar v \). We identify the parameters \( l', \lambda' \) with \( l, \lambda \) for the zero electric field case when \( \kappa = 0 \). The solutions of equation (22) are given in terms of Jacobi polynomials when \( l' \) is an integer that physically represents the number of VP states present in the system. Thus we obtain the spatial profile of the spinor components \( \theta \) and \( \phi \) of the \( N \)th bound state as:

\[
\begin{align*}
\theta_{[N]}(z) &\sim (1 - z)^{a_{[N]}/2} (1 + z)^{b_{[N]}/2} P_{[N]}^{a_{[N]}/2, b_{[N]}/2}(z), \\
\phi_{[N]}(z) &\sim (1 - z)^{a_{[N]}/2} (1 + z)^{b_{[N]}/2} P_{[N]}^{a_{[N]}/2, b_{[N]}/2}(z),
\end{align*}
\]
where $P^{(a,b)}_N$ are the Jacobi polynomials, $a_N = l' - N + l' E \hbar / (\hbar v_F (l' - N)^2 - 1)$, $b_N = 2(l' - N) - a_N$, $a'_N = a_N (l' \to l' - 1)$, $b'_N = 2(l' - N - 1) - a'_N$. Here the index $N$ is bounded by $|N| \leq l'$ and the zero mode ($N = 0$) exists as a solution only for $\theta$. Therefore, the total number of non-zero energy bound states in the system amounts to $2l'$.

With this, we can compute the probability density $f^{(N)}(x)$ which we plot in figure 3. For the plot, we consider $M_0 / \hbar v_F = 10$ and $L = 1$ setting the maximum number of VP states allowed in the system to $2l' = 20$ when no electric field is applied i.e. $\kappa = 0$. We then apply an electric field specified with the dimensionless parameter $\kappa \approx 0.44$ that reduces $l'$ to 9, or equivalently, $2l' = 18$. We plot $f^{(N)}(x)$ for $N = 1$ (figure 3(a)) and $N = 2$ (figure 3(b)) both evincing the spatial shift due to a finite electric field and bounded chemical potential.

Finally, we would like to highlight how the VP wavefunctions for the bounded case i.e. with a finite number of bound states ($l'$) differ from the unbounded ones by computing their overlap for a given index $N$, both in presence and absence of the electric field. This is shown in figure 4 for $N = 1, 2$. That the wavefunctions for the bounded case lack an integer-valued winding number to characterize their spin texture is evident from their overlap with the unbounded case deviating from 1. The overlap function $O(l') \equiv |\langle \psi(\infty) | \psi(l') \rangle|^2$ asymptotically approaches 1 as $l' \to \infty$, however, the growth of $O(l')$ toward 1 gets progressively slower with the electric field increase noticeable in figure 4. Since $l'$ is directly proportional to the mass gap $M_0$, this supports our claim that when the mass gap becomes quite large, the spin texture of the VP states for the bounded case almost attains complete windings determined by their quantization index $N$.

3. Detecting the VP states in a lattice model simulation

To demonstrate the emergence of the VP states in a more realistic setup, we consider transport simulations performed on two lattice models, one of a one-dimensional spin–orbit coupled chain and the other of a two-dimensional topological insulator hosting one-dimensional HESs at its boundary, using the package KWANT [53]. The energy-resolved transmission probability renders evidence for the existence of the bound states (the VP states) and demonstrates the parametric dependence of their spectrum on the magnetic field and the electric field, which is in tune with the results obtained from the analytical results of one-dimensional helical edge model discussed in section 2.

The one-dimensional tight-binding model is subjected to a uniform spin–orbit coupling that breaks the spin degeneracy of the dispersions and splits them laterally forming a Dirac cone about $E = 0, k = 0$ in the
with $M = a$ being the lattice constant. Each of the terms, $\sigma$, chemical potential $\mu$ in terms of the $-\sigma$, $-\sigma = M \mod a$ modifies the diagonal term to $\sigma$ denotes the set of creation operators for the electrons with $H_M + \hbar \sigma \tanh 1 \text{nm}$, $= -\sigma$ spins at site $i$, where the $V \alpha$ being the Fermi $0$ arising from the $\sigma$ orbit coupled chain (blue sites are the lattice sites of concern and the red ones are of the leads) exposed to a spatially varying magnetic field potential $M(x) = M_0 \tanh[(x - L)/L]$ where the total length of the chain is $2L$. Shown is the spectrum in the absence of the magnetic field which features a Dirac cone at $E = 0, k = 0$ (the shaded area is bounded between $[-M_0 \tanh(1), +M_0 \tanh(1)]$, well within the linear region of the dispersions around $E = 0$). The energy-resolved transmission probability $T(E)$ reveals the existence of the VP states within this energy window.

Figure 5. Schematic of the setup used in the simulation of a 1D spin–orbit coupled chain (blue sites are the lattice sites of concern and the red ones are of the leads) exposed to a spatially varying magnetic field potential $M(x) = M_0 \tanh[(x - L)/L]$ where the total length of the chain is $2L$. Shown is the spectrum in the absence of the magnetic field which features a Dirac cone at $E = 0, k = 0$ (the shaded area is bounded between $[-M_0 \tanh(1), +M_0 \tanh(1)]$, well within the linear region of the dispersions around $E = 0$). The energy-resolved transmission probability $T(E)$ reveals the existence of the VP states within this energy window.

Brillouin zone. A constant Zeeman exchange field would trivially gap out this Dirac cone (leaving dispersing modes within the gap with a large momentum separation between them $\delta k = 2k_F$, $k_F$ being the Fermi momentum); we instead apply a spatially varying but bounded magnetic field potential $M(x) = M_0 \tanh[(x - L)/L]$ coupling the two spin species at the Dirac cone, whose strength is lying within the linear energy window about the Dirac cone. Even in this finite bandwidth case, we observe the genesis of massive mid-gap bound states identified as the VP states.

In the two-dimensional lattice model, we will be considering (i) the case of a linearly varying magnetic field potential $M(x, y)$ and chemical potential $\mu(x, y)$ resulting from a uniform electric field $E$, and (ii) the case of the hyperbolic magnetic field potential and chemical potential arising from a non-uniform electric field. Lately, it is important to note that we apply magnetic and electric fields throughout the two-dimensional lattice and not just on the edge. We tune the strength of these fields such that they do not tamper with the topological properties of the bulk (i.e. keep the bulk gap intact) while influencing the edge modes in a desired fashion.

3.1. A one-dimensional lattice model

The one-dimensional spin–orbit coupled chain is specified by the Hamiltonian

$$H_{1D} = \sum_{j} \left( c_{j}^\dagger H_{j,j+a} c_{j+a} + \text{h.c.} + c_{j}^\dagger H_{j,j} c_{j} \right),$$

where $c_{j}^\dagger = (c_{j}^\dagger, c_{j}^\dagger)$ denotes the set of creation operators for the electrons with $\uparrow$ and $\downarrow$ spins at site enumerated by $j$ with $a$ being the lattice constant. Each of the terms, $H_{j,j}$ and $H_{j,j+a}$, is a $2 \times 2$ matrix defined by

$$H_{j,j} = \frac{2t}{a^2} \sigma_0$$

$$H_{j,j+a} = -\frac{t}{a^2} \sigma_0 + \frac{\alpha}{2a} \sigma_y,$$

$t$ denoting the spin-independent hopping strength and $\alpha$, the spin–orbit coupling. Adding an in-plane magnetic field potential of the form $M(x) = M_0 \tanh[(x - L)/L]$ modifies the diagonal term to

$$H_{j,j} = \frac{2t}{a^2} \sigma_0 + M_j \sigma_x,$$

where $M_j = M_0 \tanh[(x_j - L)/L]$, $x_j$ being the coordinate of the $j$th lattice site. Upon simulating this model using the KWANT package, we obtain the spectrum of the bound states as shown in figure 5 in terms of the energy-resolved transmission probability. This simple lattice model neatly elucidates the emergence of VP states in a smoothly varying magnetic field acting upon Dirac electrons with a bounded spectrum (the asymmetry in the plot of $T(E)$ about $E = 0$ is due to the spectral asymmetry about $E = 0$ arising from the finite bandwidth). The plots are obtained with the parameter values $2L = 200a$, $a = 1 \text{ nm}$, $M_0 = 0.05 \text{ meV}$, $t = 2 \text{ meV} \cdot \text{nm}^2$, $\alpha = 1 \text{ meV} \cdot \text{nm}$. 
Figure 6. (a) The schematic of the lattice model used for KWANT simulation. The QSH region is subjected to a magnetic potential $M(x)$ that varies linearly from $-M_0$ to $M_0$ over an extent of $2L$. (b) The energy spacing of the block matrix defined by $(\sin \theta \sigma_z) = a \cos \theta$, coincides with the bound state energies due to the overlap of bound state wavefunction with the states at the same energies in the source and drain electrodes. Both top and bottom edges host VP states which are time reversal partners, resonant transmission for each of them corresponds to independent $e^2/h$ conductance, together leading to $G_0 = 2e^2/h$ conductance peak value. Shown in (b) are the resonances for various values of the ramp parameter $\tilde{a} \equiv \alpha h \tilde{E}$ when no electric field is present. The energy spacing of the neighboring VP states $\Delta E$ increases with $\tilde{a}$ following $\Delta E \sim \tilde{a}^2$. (c) The same in the presence of a uniform electric field of strength $\tilde{E} \ll \alpha \tilde{E}$ (the effective electric field is denoted as $\tilde{E} = \tilde{E} h \tilde{E}$) which qualitatively shows the shift of the transmission peaks toward zero energy. Here $\Delta E$ decreases with $\tilde{E}$ with the peaks of the resonances gradually diminishing.

3.2. The two-dimensional lattice model

For the two-dimensional lattice model, we consider a square lattice of rectangular geometry with a length of $2L$ and width $W = 200a$ (figure 6(a)), where $a$ is the lattice spacing (explicit values are mentioned later). A Zeeman field is applied along the length of the rectangular sample which is along the $x$-axis, such that the strength of the field is given by $M(x, y) = \alpha(x - x_0)$, $x_0$ being the center of the rectangle situated at $x = L$. The parameter $\alpha$ is fixed by a cutoff $M_0$, which lies well within the bulk gap, such that $M_0 = \alpha L$.

Let us now specify the model which is simulated using the KWANT package. The bulk, in the pristine form, is described by the BHZ Hamiltonian for the HgTe/CdTe quantum well [26]

$$H_{\text{BHZ}} = -Dk_x^2 + Ak_x \sigma_z \tilde{x} + Ak_y \tilde{y} + (M - Bk_y^2) \tilde{z},$$

where $\sigma$ and $\tilde{x}$ denote the Pauli matrices to describe the spins (up or down along $S_z$) and the orbitals (electron type or hole type) respectively and $A, B, D$ and $M$ are material dependent parameters. We discretize the Hamiltonian in equation (28) to obtain a tight-binding version on our square lattice but with a basis of two sites (to represent the two-level system of orbitals). Using $k_x = a^{-1} \sin(k_xa)$ and $k_y = a^{-1} \sin(k_ya)$ the tight-binding Hamiltonian reads

$$H_{\text{TB}} = \sum_j (c_j^\dagger H_{i,j+\alpha_x,\alpha_y} c_j + c_j H_{i,j+\alpha_x,\alpha_y} + \text{h.c.} + c_j^\dagger H_{i,j} c_j),$$

where $c_j^\dagger \equiv (c_j^\dagger, c_j^\dagger, c_j^\dagger, c_j^\dagger)\nonumber$ denotes the set of creation operators for the electrons in $s$ and $p$ orbital with $\uparrow$ and $\downarrow$ spins at site $j$ with coordinates $j = (j_x, j_y)$; $a_x = a(1,0)$ and $a_y = a(0,1)$ are the lattice vectors with $a$ being the lattice constant. Each of the terms, $H_{i,j}$ and $H_{i,j+\alpha_x,\alpha_y}$, is a $4 \times 4$ block matrix defined by...
\[ H_{jj} = -\frac{4D}{a^2} - \frac{4B}{a^2} \sigma_z + M \sigma_z, \]
\[ H_{ij+x} = \frac{D + B \sigma_x}{a^2} + \frac{A \sigma_x \sigma_z}{2ia}, \]
\[ H_{ij+y} = \frac{D + B \sigma_x}{a^2} + \frac{iA \sigma_y}{2a}. \]

In the region where the spatially varying in-plane magnetic field and a uniform electric field are applied along the \( x \)-direction, the diagonal term in equation (30) is modified to

\[ H_{jj} = -\frac{4D}{a^2} - \frac{4B}{a^2} \sigma_z + M \sigma_z + M_j \sigma_x + \mathcal{E} (x_j - x_c), \]  

where \( M_j = \alpha(x_j - x_c) \), \( x_c \) being the \( x \)-coordinate of the \( j \)th lattice site and \( \mathcal{E} \) the strength of the electric field. The last term in equation (31) is the discretized form of the chemical potential \( \mu(x, y) = -\mathcal{E}(x - x_c) \) which is present all over the lattice owing to the electric field.

The standard parameters for the HgTe/CdTe quantum wells that are used in equation (30) are \([27]\)

\( A = \hbar \nu_F = 364.5 \text{ nm meV}, B = -686 \text{ nm}^2 \text{ meV}, \) and \( M = -15 \text{ meV} \) while \( D \) is set to zero to place the Dirac cone at zero energy, and the lattice constant \( a = 3 \text{ nm} \). The length \( L \) over which the magnetic and the electric field are operating is decided by setting the parameter \( \alpha = M_0/L \) to different values (here we consider \( \alpha \hbar \nu_F = 0.5 \) and \( \alpha \hbar \nu_F = 0.8 \) which yield \( L = 486a \) and \( L = 303a \) respectively) with \( M_0 = 2 \text{ meV} \), much less than the bulk gap determined by \( M \). The magnetic field is then ramped from \(-M_0 \) to \( M_0 \) over a length of \( 2L \) (figure 6(a)). Hence the application of the spatially varying magnetic field hardly has any effect on the bulk, while it creates a smoothly varying mass domain wall appearing only at the edge spectrum of the QSHS. As a result, no tunneling between the upper and lower edge occurs. When present, the values of the electric field are chosen such that \( \mathcal{E} < \alpha \).

3.3. Spectrum of the VP states

With the essentials of our setup provided, we turn to observables such as the transmission probability \( T \) (or equivalently the differential conductance in units of \( G \)) as a function of the incidence energy \( E \). Such a plot will feature resonances from which the existence of the VP states can be readily verified. We do so at distinct values of the magnetic ramp parameter \( \alpha \) and the electric field \( \mathcal{E} \) to elucidate the parametric dependence of the energy spectrum of these bound states.

The plot in figure 6(b) displays the behavior of \( T(E) \) at different values of \( \alpha \) (with \( \mathcal{E} = 0 \)) from which we observe the VP states appearing at quantized values of \( E \) with sharp resonance peaks when a linearly varying magnetic field is forming a smooth interface between two topologically distinct regions identified with \( \text{sgn}[M(x)] \). The Jackiw–Rebbi mode is visible at \( E = 0 \), the energy spacing between neighboring bound states increases with \( \alpha \) following equation (8). With \( \alpha \) taking higher values, the interface becomes sharper, and as a result, the VP states gradually disappear into the bulk with the Jackiw–Rebbi mode remaining pinned to \( E = 0 \).

In addition, when a uniform electric field of strength \( \mathcal{E} \) is applied, the expression for the bound state energy spectrum is modified according to equation (16). A reverse phenomenon is now observed. The energy spacing between the neighboring bound states starts shrinking with \( \mathcal{E} \) and the height of the resonance peaks diminished as visible in figure 6(c). Here we have set \( \alpha \hbar \nu_F = 0.5 \) and considered various values of \( \mathcal{E} \) such that \( \mathcal{E} < \alpha \) or equivalently, \( \mathcal{E} \hbar \nu_F < 0.5 \). The bound states start collapsing onto the Jackiw–Rebbi mode as \( \mathcal{E} \to \alpha \) and exactly at \( \mathcal{E} = \alpha \) they form a highly degenerate manifold of zero modes. Together, these results illuminate how one can engineer such finite-energy sub-gap states at the interface of distinct topological states via the elegant interplay of an in-plane electric field and the spatially-textured magnetic field of the interface.

3.4. Spin texture detection of the VP states via transport simulation

As elucidated previously, the VP states come with spatially varying charge density and spin textures, the latter characterized by nontrivial winding. This can be readily observed in our simulation when augmented by a spin-polarized tunnel probe. As a probe, we use the spin-polarized edge state of a QAH system which can be obtained from the QSHS by inducing a topological phase transition driven by an exchange field of strength \( g_0 \) in the BHZ Hamiltonian with \( g_0 > |M| \) \([54, 55]\). The spin polarization of the surviving chiral edge is determined in the following way: for \( g_0 < M \) (note \( M < 0 \), it is spin-\( \uparrow \)-polarized whereas for \( g_0 > -M \), it is spin-\( \downarrow \)-polarized. The model Hamiltonian describing the corresponding lattice model is
In our setup, this QAH system (of dimensions $55a \times 200a$) is attached to the left of the QSH system (of dimensions $967a \times 200a$) via an insulating tunnel probe (of dimensions $11a \times 12a$) to facilitate spin-$\uparrow$ injection, achieved by setting $g_0 = -21$ meV (figure 7). The insulating region is a trivial insulator characterized by the Hamiltonian in equation (30) with the sign of $M$ flipped.

The bulk Hamiltonian of the QSHS is also modified such that upon applying a suitable magnetic field in the $x$-direction, the Jackiw–Rebbi mode is formed polarized along the $z$-axis. For this, the pristine BHZ model must host the helical edge modes polarized along the $\pm y$ direction (per our original theoretical model). The magnetic potential $M(x)$ varies linearly from $-M_0$ meV to $+M_0$ for $M_0 = 2$ meV along $x$-direction and is distributed all over the bulk, such that the upper and bottom edges host independent time reversal partner VP states with opposite spin textures. The strength of the magnetic potential is tuned such that the bulk topological gap stays protected hence ensuring the presence of a robust edge state while the edge spectrum hosts the VP states. Therefore, our modified BHZ Hamiltonian to describe the QSHS reads

$$H_{j,j} = -\frac{4D}{a^2} - \frac{4B}{a^2} \sigma_z + \mathcal{M} \sigma_z + g_0 \sigma_y \sigma_z,$$

$$H_{j,j+a_x} = \frac{D + B \sigma_z}{a^2} + \frac{A \sigma_y \sigma_z}{2ia},$$

$$H_{j,j+a_y} = \frac{D + B \sigma_z}{a^2} + \frac{i A \sigma_y}{2a}. \quad (32)$$

On the bottom of the QSHS (see figure 7), another QAH system of dimensions $200a \times 200a$ is attached that hosts a spin-$\downarrow$ channel (at $g_0 = 21$ meV). A small insulating region (of dimensions $12a \times 11a$) between this QAH probe and the main QSH region is kept to ensure that we are at a weak tunneling limit so that the tunneling current from QAH edge state via its magnetoresistance (TMR) response reads off the spin texture of the VP states. We further attach semi-infinite normal QSH leads at the open ends of both the QAH regions as shown in figure 7.

The spin texture of the VP states can either be read off directly from the spin density plot of $S_{x}^{(N)}$ in figure 8 or the energy-resolved transmission in the TMR response as in figure 9. Figure 8 shows the spin density plots corresponding to the situation where the source lead is connected to the QSH region via the QAH region (zoomed in figure 8(a)), while the drain lead is kept disconnected from the QSH region by increasing the potential barrier between the QAH region next to the drain and the QSH region. This corresponds to a situation where a spin-polarized electron from the QSH region tunnels through the narrow junction (current injection point in figure 8(a)) into the tails of the wavefunctions of the zero modes of the upper and the lower edge, hence resulting in a finite spin density which is apparent from the plot given in figure 8. It is important to note that the zero modes, which are primarily localized in the upper and the lower edge, are spin-$\uparrow$ and spin-$\downarrow$ polarized respectively, while the vertical edge (along the $y$-direction) of the QSH.
Figure 8. Spatial distribution of the spin density $S_z(x,y)$ for the Jackiw–Rebbi mode $N = 0$ in (a) and the second VP state $N = 2$ in (b) showing prominent variations along the edge reveals that the spin texture for the $N$th VP state winds exactly $N$ times ($x$ and $y$-axis in nm). All the plots are obtained considering linearly varying magnetic field as also depicted in figure 7. The energy of the incident electrons is tuned at the zero mode energy $E = 0$ meV in (a) and at the second VP state energy $E = 1.424$ meV in (b).

Figure 9. The spatial profile of the transmission probability $T$ detected by the spin-$\downarrow$ edge of the QAH of the drain electrode (see figure 7) is represented by black solid line for $N = 0$ in (a) and by red solid line for $N = 2$ in (b). The black dashed line in (b) indicates the behavior of the TMR response obtained from analytical calculation with a scaling constant $k_1 = 40$. In (c), the same profile of $N = 2$ is plotted in the presence of a random disorder in $M(x)$ with values ranging from $[-0.125M_0, 0.125M_0]$ in addition to the linearly varying magnetic potential. The error bars are larger at higher values of the transmission probability. In the plots, $x$ represents the direction along the HES in the nm unit. In (d), shown is the spatial shift of the transmission probability in the presence of a uniform electric field. The dashed line at the back represents the no electric field case with respect to which the shift is measured. The energy of the incident electrons is set at the zero mode energy $E = 0$ meV in (a), at the second VP state energy $E = 1.424$ meV in (b) and (c), and the second VP state energy in presence of electric field which is $E = 1.394$ meV in (d).

region, which is next to the injection point, sees a constant magnetic potential ($M(x)$ left boundary), as the profile of the magnetic field has a gradient only along the $x$-direction) and has a uniform gap. The $z$-polarized zero modes on the upper and the lower edge penetrate into the gaped vertical edge mode of the QSH system and weakly hybridize with each other. This weak hybridization of the zero modes allows the tunneled electrons to induce a finite spin-density both in the upper and lower edge as shown in figure 8. For the second VP state, the $S_z$ spin density flips the sign four times corresponding to a winding number two. For the lower edge, the sign of the magnetic field gradient is opposite which results in an inverted spin polarization of the VP states retaining the winding number. The anti-correlation between the spin polarization of the lobes on the upper and the lower edge indicates they are time-reversed partners of each other. These results altogether numerically reconfirm our claim in section 2.2 that the spin texture of the $N$th VP states winds exactly $N$ times along an edge.

From the perspective of experimental detection of the spin textures of the VP states, a study pertaining to its detection via a local spin-polarized tunnel probe is desirable. Motivated by this fact, we study the TMR response of a local spin-polarized probe as a possible readout for the spin textures. We propose to numerically implement the spin-polarized probe in our transport simulations via the spin-polarized QAH edge state as shown as the drain electrode in figure 7 where the position of the tunnel junction (represented by the orange color rectangular region in figure 7) may be shifted along the edge, thereby enabling a local scan of the TMR. The results of our simulations are presented in figure 9. To develop a physical intuition of
Conveniently, the magnetic potential increases gradually from $k$ to $\tanh(k)$ potential, which assumes random $0\%$ domain wall in the absence ($\mu_0 = 0$) and the presence ($\mu_0 = 0.3\text{ meV}$) of an electric field (strength denoted by $\mu_0$).

Before concluding, we briefly discuss the situation when the magnetic domain wall has a tanh-hyperbolic profile. In this case, we apply a magnetic potential throughout the sample such that $M(x, y) = M_0 \tanh((x - L)/L)$ with $M_0 = 2\text{ meV}$, where $2L$ denotes the spatial extent of the QSH region along the direction of the edge. Set $2L = 1214a$ conveniently, the magnetic potential increases gradually from $-M_0 \tanh(1)$ to $M_0 \tanh(1)$.

We also include a chemical potential that varies along the length of the sample as $\mu(x, y) = -\mu_0 \tanh((x - L)/L)$ with $\mu_0 = \kappa M_0$. In figure 10, the peaks of the transmission probability $T(E)$ signal the presence of the VP states, which obey the energy quantization described in equation (13). In the presence of the chemical potential ($\mu_0 \neq 0$), the Jackiw–Rebbi mode remains pinned at zero energy while the energies of the other VP states get pushed toward zero as expected from equation (21). When $\kappa = 1$, all the VP states collapse onto the Jackiw–Rebbi mode, similar to what happens for the linear potential case.

Collectively, these results illuminate the possibility of engineering finite-energy sub-gap states arising at the interface of distinct topological states via an elegant interplay of in-plane electric and magnetic fields that may find potential application in the field of spintronics.
4. Conclusion

In this work, we discuss the physics of a smooth interface at a topological-trivial insulator heterojunction across which a band inversion occurs. Such interfaces between distinct topological states, that are representative of a new class of quantum matter, known as the designer quantum materials, can foster intriguing electronic and transport phenomena with implications for developing novel electronic devices, such as topological transistors. Our study provides numerical evidence via lattice simulation on topological insulators that their edge states can host massive bound states, called the VP states, which are stable against finite but weak bulk disorder. We illuminate that these VP states have interesting spin textures with real space winding numbers directly connected to their energy eigenvalues. We then put forward a proposal for the direct detection of the VP states via the spin textures by exploiting their local TMR response and provide strong support to its feasibility by carrying out lattice simulations.

We also show the TMR response could be controlled using an in-plane electric field along the edge and hence the proposed setup may find application as a spin transistor. As far as the experimental implication of the spatially varying magnetic field is concerned, in [57], the authors engineered a periodic array of nanomagnets on the edge of a $\nu = 2$ quantum Hall insulator such that a resonant conversion of spin-up to spin-down electrons is facilitated. Given such fine engineering of magnetic field profiles at the edge of a quantum Hall system has already been realized experimentally, it is quite conceivable that our proposal can also be implemented by exploiting a similar route. This is in distinction to some of the previous works along this direction, such as [21] where the (particle–hole) pseudospin texture of the VP states can pose onerous challenges to experimental detection.

Data availability statement

The data generated and/or analyzed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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Author contribution

The first two authors, V A and S C have contributed equally to this work.

Appendix A. Bound state spectrum of the VP states

In this appendix, we will show how to derive the bound state spectrum of the VP states from the associated wavefunctions. The four distinct cases are as follows.

A.1. Unbounded $M(x), \mu(x) = 0$

The decoupled equations for $\theta$ and $\phi$, as given in equation (9), read

$$\frac{d^2 y}{dx^2} - k_1 y + k_2 y = 0,$$

where, $k_1 = \alpha^2 / (\hbar v_F^2)$ and $k_2 = (E^2 + \sigma \alpha \hbar v_F^2) / (\hbar^2 v_F^2)$ with $\sigma = +1, -1$ for $y = \theta$ and $y = \phi$ respectively. A general form of the solution to the above equation is

$$y(x) \sim e^{-\sqrt{\pi/2} x} H_{\frac{1}{2}} (k_1/\sqrt{\pi}) \left( k^{1/4} x \right).$$

A polynomial form of the solution (representing a bound state) then demands the index of the Hermite polynomial to be an integer implying
for a bound state solution, we find

\[
\frac{1}{2} \left( \frac{k_1}{\sqrt{v}} - 1 \right) = |N| \quad \Rightarrow \quad E = \text{sgn} \,(N) \sqrt{2\alpha \hbar \nu_E |N|},
\]  

(A3)

yielding the spectrum of the bound states as noted in equation (8).

A.2. Bounded \( M(x), \mu(x) = 0 \)

The solutions of the differential equations in equation (12) are given by the associated Legendre polynomials:

\[
\theta \sim P_l^\mu(z) \quad \text{and} \quad \phi \sim P_{l-1}^\mu(z),
\]

where \( z = \text{tanh} \,(x/L), l = M_0 \hbar /\hbar \nu_E, \) and \( \lambda = L \sqrt{M_0^2 - \lambda^2}/\hbar \nu_E. \) To have bound state solutions, the condition to be satisfied is that \( l \) and \( \lambda \) take integer values while \( \lambda \leq l \) for \( \theta \) and \( \lambda \leq l - 1 \) for \( \phi. \) Therefore, we can write

\[
\lambda = l - |N| \quad \Rightarrow \quad E = \text{sgn} \,(N)M_0 \left[ \frac{2|N|}{l} - \frac{N^2}{l^2} \right]^{1/2}.
\]  

(A4)

For \( N = 0 \), we obtain \( \lambda = l \) and \( E = 0 \) which corresponds to the zero mode whose spatial profile is dictated by \( P_l(z) \sim (1 - z^2)^{l/2} \) \( = \text{sech}^l(x/L). \) On the other hand, for \( \lambda = 0 \), we obtain the half-bound state solutions at the edge of the spectrum \( E = \pm M_0 \) [52].

A.3. Unbounded \( M(x), \mu(x) \neq 0 \)

Equation (15) can be expressed as

\[
\frac{d^2y}{dx^2} + (-ax^2 + bx + c) y = 0,
\]  

(A5)

where

\[
a = \left( \frac{\alpha^2}{\hbar^2 \nu_E^2} - \frac{\mathcal{E}^2}{\hbar^2 \nu_E^2} \right),
\]

\[
b = \frac{2E\mathcal{E}}{\hbar^2 \nu_E},
\]

\[
c = \left( \frac{E^2}{\hbar^2 \nu_E^2} + \sigma \sqrt{\frac{\alpha^2}{\hbar^2 \nu_E^2} - \frac{\mathcal{E}^2}{\hbar^2 \nu_E^2}} \right),
\]

and \( \sigma = +1, -1 \) for \( \theta = y \) and \( \phi = \varphi \) respectively. The solution of the above equation is given by

\[
y(x) \sim e^{-\left( \frac{(ax + \mathcal{E})^2}{\sqrt{\alpha^2 \nu_E^2 + \mathcal{E}^2 \nu_E^2}} \right)^{1/2}} \left( \frac{2ax - b}{2a^{3/4}} \right).
\]  

(A7)

Therefore, applying the condition that the index of the Hermite polynomial will be a non-negative integer for a bound state solution, we find

\[
\frac{b^2 - 4\alpha^{3/2} + 4ac}{8a^{3/2}} = |N|
\]

\[
\Rightarrow E = \text{sgn} \,(N) \sqrt{2\alpha \hbar \nu_E |N|} \left( 1 - \frac{\mathcal{E}^2}{\alpha^2} \right)^{3/4},
\]  

(A8)

as noted in equation (16).

Appendix B. Bound state spectrum from supersymmetric shape invariance

For details on the shape invariance of solvable supersymmetric potentials, the reader is referred to [40]. Here we provide a sketch of the underlying concept and its illustration with examples relevant to the QSHS problem at hand.

In quantum mechanics, a pair of systems described by the Hamiltonians

\[
\mathcal{H}_\pm = -\frac{d^2}{dx^2} + V_\pm(x, a),
\]  

(B1)
with the parameter \(a\) specifying the potentials, are called supersymmetric partners of each other with an identical spectrum (up to zero modes). The two potentials, called the partner potentials, \(V_{\pm}(x,a)\) derive from a superpotential \(W(x,a)\) as

\[
V_{\pm}(x,a) = W^2(x,a) \pm \frac{dW}{dx}.
\]

(B2)

The supersymmetric shape invariance condition applies to a specific class of superpotentials \(W(x,a)\) such that the two (shape invariant) partner potentials are related by

\[
V_+(x,a_0) - V_-(x,a_1) = g(a_1) - g(a_0),
\]

(B3)

where \(g(a_1) - g(a_0)\) is an additive constant and \(a_1 = f(a_0)\), some function of \(a_0\), generalizing to \(a_{i+1} = f(a_i)\). Note the coordinate dependence in \(V_+(x,a_0)\) and \(V_-(x,a_1)\) is the same but the parameter values are different.

At the level of the spectra for the two systems, characterized by the energy eigenvalues \(E_n^+\) and \(E_n^-\) for \(\mathcal{H}_+\) and \(\mathcal{H}_-\) respectively, equation (B3) implies

\[
E_n^+(a_0) - E_n^-(a_1) = g(a_1) - g(a_0), \quad \forall n
\]

(B4)

with \(E_0^-(a_i) = 0\), \(i = 0, 1, \ldots, n\) for an unbroken supersymmetry. With the method of induction, it is straightforward to show that

\[
E_n^-(a_0) = g(a_n) - g(a_0),
\]

(B5)

and the supersymmetry demands \(E_n^+(a_i) = E_{n+1}^-(a_i)\). Therefore, knowing \(W(x,a)\) and identifying \(a_0, a_1, \) and \(g(a_0)\) suffice to compute the entire spectrum of \(\mathcal{H}_\pm\).

For example, if we try to compute the spectrum of \(\mathcal{H}_-(x,a_0)\), the starting point is to note \(E_0^-(a_0) = 0\). Then the next eigenvalue will be given by

\[
E_1^-(a_0) = E_0^+(a_0) = E_0^-(a_1) + g(a_1) - g(a_0)
\]

\[
= g(a_1) - g(a_0),
\]

(B6)

(using equation (B4)). Similarly,

\[
E_n^-(a_0) = E_1^+(a_0) = E_1^+(a_1) + g(a_1) - g(a_0)
\]

\[
= E_0^+(a_1) + g(a_1) - g(a_0)
\]

\[
= E_0^-(a_2) + g(a_1) - g(a_0) + g(a_1) - g(a_0)
\]

\[
= g(a_2) - g(a_0).
\]

(B7)

Continuing this way, we end up at equation (B5).

Let us now illustrate how we can use the concept of shape invariance to compute the spectrum of the VP states in the presence of an electric field. The following two cases are of concern.

**B.1. Unbounded \(M(x), \mu(x) \neq 0\)**

Here we consider \(M(x) = \alpha x\) and \(\mu(x) = -E x\). Equation (15) can be expressed as

\[
\left[-\partial_x^2 + W^2(x) - \partial_x W(x)\right] \theta = \varepsilon \theta,
\]

\[
\left[-\partial_x^2 + W^2(x) + \partial_x W(x)\right] \phi = \varepsilon \phi,
\]

(B8)

where \(W(x) = Ax + B\) with

\[
A = \sqrt{\alpha^2 - \varepsilon^2}/h\nu, \quad B = E \frac{\varepsilon}{(\hbar^2 v_F^2 A)},
\]

(B9)

and

\[
\varepsilon = E \frac{\alpha^2}{[\hbar^2 v_F^2 (\alpha^2 - \varepsilon^2)]}.
\]

(B10)
It is straightforward to identify \( g(a_i) = 2Aa_i = a_0 + i, \) \( i = 0, 1, \ldots, |N|, \) and so,

\[
\varepsilon = 2A|N|, \quad (B11)
\]

from which equation (16) follows. The case for \( \mu(x) = 0 \) is obtained simply by setting \( \varepsilon = 0 \) is equation (B10).

B.2. Bounded \( M(x), \mu(x) \neq 0 \)

In this case, the superpotential is of the form

\[
W(x) = A \tanh \left( \frac{x}{L} \right) + B \quad (B12)
\]

with

\[
A = \frac{\sqrt{1 - \kappa^2 M_0}}{\hbar v_F} \quad ; \quad B = \frac{\kappa E}{\hbar v_F \sqrt{1 - \kappa^2}} \quad (B13)
\]

Here, we find

\[
g(a_i) = -a_i^2 - A^2 B^2 / a_i^2, \quad (B14)
\]

with \( a_0 = A \) and \( a_i = a_0 + i, i = 0, 1, \ldots, |N|, \) which yields the energies of the partner Hamiltonians [41]

\[
\varepsilon = A^2 + B^2 - (A - |N|/L)^2 - \frac{A^2 B^2}{(A - |N|/L)^2}. \quad (B15)
\]

Noting, in this case

\[
\varepsilon = E^2 / \left[ \hbar^2 v_F^2 \left( 1 - \kappa^2 \right) \right], \quad (B16)
\]

we arrive at equation (21).

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