Vanishing cycles and wild monodromy

Andrew Obus

Abstract

Let \( K \) be a complete discrete valuation field of mixed characteristic \((0, p)\) with algebraically closed residue field, and let \( f : Y \to \mathbb{P}^1 \) be a three-point \( G \)-cover defined over \( K \), where \( G \) has a cyclic \( p \)-Sylow subgroup \( P \). We examine the stable model of \( f \), in particular, the minimal extension \( K^{st}/K \) such that the stable model is defined over \( K^{st} \). Our main result is that, if \( |P| = p^n \) and the center of \( G \) has prime-to-\( p \) order, then the \( p \)-Sylow subgroup of \( \text{Gal}(K^{st}/K) \) has exponent dividing \( p^n - 1 \). This extends work of Raynaud in the case that \( |P| = p \).

1. Introduction

This paper investigates the stable reduction of three-point covers over complete discrete valuation fields of mixed characteristic. In particular, we examine the minimal extension of a field of definition of such a cover that is necessary to obtain the stable model.

Let \( G \) be a finite group. Let \( f : Y \to X \) be a branched \( G \)-Galois cover of curves defined over a complete discretely valued field \( K \) of characteristic 0 with valuation ring \( R \) whose residue field \( k \) is algebraically closed of characteristic \( p \). Suppose \( X \) has a smooth model over \( R \). Then, under mild hypotheses, there is a finite extension \( K^{st}/K \) with valuation ring \( R^{st} \) such that there is a minimal stable model \( f_{R^{st}} \) for \( f \times_K K^{st} \) whose special fiber \( \overline{f} : \overline{Y} \to \overline{X} \) (called the stable reduction) is reduced with only nodal singularities. The group \( G \) acts on \( \overline{Y} \). See \( \S 2.3 \) for more details.

The proofs of the existence of stable reduction are non-constructive, and the question of determining the minimal extension \( K^{st}/K \) over which the stable model can be defined is far from being answered in most cases. In the case that the branch points of \( f \) are defined over \( K \), the extension \( K^{st}/K \) is Galois, and we focus on the (unique) \( p \)-Sylow subgroup \( \Gamma_w \) of \( \text{Gal}(K^{st}/K) \), called the wild monodromy group of \( f \). This group acts faithfully on the stable reduction \( \overline{f} \).

The wild monodromy group has been studied by Lehr and Matignon in the case of a \( \mathbb{Z}/p \)-cover of \( \mathbb{P}^1 \) with any number of branch points (LM06), and by Raynaud in the case of a three-point \( G \)-cover (i.e., a cover of \( \mathbb{P}^1 \) branched exactly at 0, 1, and \( \infty \)) such that \( p \) exactly divides \( |G| \) (Ray99). Note that, in both cases, \( p \) exactly divides the order of the Galois group of \( f \). The methods of LM06 exploit the availability of explicit equations coming from having a Kummer cover, whereas the methods of Ray99 involve understanding combinatorial aspects of \( \overline{f} \). In particular, Raynaud proves a vanishing cycles formula (Ray99 3.4.2 (5)) which places restrictions on how complicated \( \overline{f} \) can be, which in turn places bounds on the wild monodromy group. Also, Ray99 relates the wild monodromy group to the question of good reduction of \( f \).

This paper builds on the ideas of Ray99. We place bounds on the wild monodromy group \( \Gamma_w \) of \( f \), when \( f \) is a three-point \( G \)-cover such that \( G \) has a cyclic \( p \)-Sylow subgroup of arbitrary order. Specifically, our main result is the following:

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Theorem 1.1. Suppose $G$ is a finite group with nontrivial cyclic $p$-Sylow subgroup $P$ of order $p^n$. Assume that $p$ does not divide the order of the center of $G$. If $f : Y \to X = \mathbb{P}^1$ is a three-point $G$-cover defined over $K$, then the wild monodromy group $\Gamma_w$ has exponent dividing $p^n - 1$.

Remark 1.2. (i) If $n = 1$, Theorem 1.1 says that $\Gamma_w$ is trivial, which is the result that appeared as [Ray99, Théorème 4.2.10 (1)] (for $Y$ having genus $\geq 2$).

(ii) If the branching indices of $f$ are prime to $p$ and there are no “new” étale tails (see §2.3), then $\Gamma_w$ is trivial, regardless of $n$ (Proposition 5.9).

(iii) We are not yet sure if, in fact, there exist examples where $n \geq 2$, where $p$ does not divide the order of the center of $G$, and where $\Gamma_w$ has exponent $p^n - 1$. Indeed, it is difficult to write down examples where there is any wild monodromy at all. We write down such an example in Appendix A where $p = 5$, $n = 3$, and $\Gamma_w \supseteq \mathbb{Z}/5$.

(iv) Using Raynaud’s relation between the wild monodromy group and good reduction, we exhibit a family of three-point $G$-covers with potentially good reduction to characteristic $p$, where $G$ has arbitrarily large cyclic $p$-Sylow subgroup. Specifically, $G \cong \text{PGL}_3(q)$, with $p^n | q^2 + q + 1$. See Example 5.12.

Several difficulties present themselves when we allow $G$ to have a $p$-Sylow subgroup of order greater than $p$. The first is that Raynaud’s vanishing cycles formula is proven only in the case where $p$ exactly divides $|G|$ (in fact, it is not immediately obvious what the generalization should be for arbitrary $G$). Our Theorem 3.14 extends this formula to the case where $G$ has a cyclic $p$-Sylow subgroup. The proof uses invariants of deformation data, which were used by Wewers in [Wew03b] to give an alternate proof of Raynaud’s vanishing cycles formula. Wewers associated deformation data to irreducible components of $Y$ on which $G$ acts with inertia group of order $p$. We extend this to the case where $G$ acts with cyclic inertia of any order, and introduce new effective invariants for these deformation data.

A more serious difficulty is that when $p$ divides the order of $|G|$ more than once, then $\overline{f} : \overline{Y} \to \overline{X}$ can have inseparable tails, i.e., irreducible components $\overline{W}$ of $\overline{X}$ that intersect the rest of $\overline{X}$ at one point such that $G$ acts with nontrivial inertia above $\overline{W}$. These tails are not present when $p$ exactly divides $|G|$, and they do not appear in Theorem 3.14. We prove a generalized vanishing cycles formula (Proposition 3.17) that takes these tails into account. This requires introducing what we call truncated effective invariants of deformation data. Using Proposition 3.17, we are able to prove Theorem 1.1.

There are proofs of Theorem 3.14 and Proposition 3.17 that do not use deformation data (see [Obu09, §3.1]), but the proof we give here is particularly nice, and the material on deformation data that we develop here will be used in the subsequent papers [Obu10a] and [Obu10b].

1.1 Section-by-section summary and walkthrough

In §2, we give preliminary results about group theory, stable reduction, and ramification. Many of these results are already known. In §3 we generalize the construction of deformation data given in [Hen99] and used in [Wew03b], and use these deformation data to prove the aforementioned vanishing cycles formulas. For a three-point $G$-cover $f : Y \to X$, results limiting the number and type of tails of the stable reduction $\overline{f} : \overline{Y} \to \overline{X}$ are given in §4. The main theorem is proved in §5, where we use arguments similar to, but more complicated than, those of [Ray99, §4.2] to obtain our restrictions on the wild monodromy of $f$. The connection to good reduction is discussed at the end of §5.
1.2 Notation and conventions

The following notations will be used throughout the paper: The letter \( p \) always represents a prime number. If \( G \) is a group, and \( H \) a subgroup, we write \( H \leq G \). We denote by \( N_G(H) \) the normalizer of \( H \) in \( G \) and by \( Z_G(H) \) the centralizer of \( H \) in \( G \). The order of \( G \) is written \( |G| \). If \( G \) has a cyclic \( p \)-Sylow subgroup \( P \), and \( p \) is understood, we write \( m_G = |N_G(P)/Z_G(P)| \).

If \( K \) is a field, \( \overline{K} \) is its algebraic closure. We write \( G_K \) for the absolute Galois group of \( K \). If \( H \leq G_K \), we write \( \overline{K}^H \) for the fixed field of \( H \) in \( \overline{K} \). Similarly, if \( \Gamma \) is a group of automorphisms of a ring \( A \), we write \( A^\Gamma \) for the fixed ring under \( \Gamma \). If \( K \) is discretely valued, then \( \overline{K}^{ur} \) is the completion of the maximal unramified algebraic extension of \( K \).

If \( x \) is a scheme-theoretic point of a scheme \( X \), then \( \mathcal{O}_{X,x} \) is the local ring of \( x \) on \( X \). If \( R \) is any local ring, then \( \widehat{R} \) is the completion of \( R \) with respect to its maximal ideal. If \( R \) is a discrete valuation ring with fraction field \( K \) of characteristic \( 0 \) and residue field \( k \) of characteristic \( p \), we normalize the valuation \( v \) on \( R \) so that \( v(p) = 1 \).

A branched cover \( f : Y \to X \) of smooth proper curves is a finite, surjective, generically étale morphism. All branched covers are assumed to be geometrically connected. If \( f \) is of degree \( d \) and \( G \) is a finite group of order \( d \) with \( G \cong \text{Aut}(Y/X) \), then \( f \) is called a Galois cover with (Galois) group \( G \). If we choose an isomorphism \( i : G \to \text{Aut}(Y/X) \), then the datum \((f, i)\) is called a \( G \)-Galois cover (or just a \( G \)-cover, for short). We will usually suppress the isomorphism \( i \), and speak of \( f \) as a \( G \)-cover.

The ramification index of a point \( y \in Y \) such that \( f(y) = x \) is the ramification index of the extension of complete local rings \( \mathcal{O}_{X, x} \to \mathcal{O}_{Y, y} \). If \( f \) is Galois, then the branching index of a closed point \( x \in X \) is the ramification index of any point \( y \) in the fiber of \( f \) over \( x \). If \( x \in X \) (resp. \( y \in Y \)) has branching index (resp. ramification index) greater than \( 1 \), then it is called a branch point (resp. ramification point).

If \( X \) is a smooth curve over a complete discrete valuation field \( K \) with valuation ring \( R \), then a semistable model for \( X \) is a relative curve \( X_R \to \text{Spec} R \) with \( X_R \times_R K \cong X \) and semistable special fiber (i.e., the special fiber is reduced with only ordinary double points for singularities).

For any real number \( r \), \( \lfloor r \rfloor \) is the greatest integer less than or equal to \( r \). Also, \( \langle r \rangle := r - \lfloor r \rfloor \).

2. Background Material

2.1 Finite groups with cyclic \( p \)-Sylow subgroups

In this section, we prove structure theorems about finite groups with cyclic \( p \)-Sylow subgroups. Throughout \( \section{2.1} \) \( G \) is a finite group with a cyclic \( p \)-Sylow subgroup \( P \) of order \( p^n \). Recall that \( m_G = |N_G(P)/Z_G(P)| \).

**Lemma 2.1.** Let \( Q \leq P \) have order \( p \). If \( g \in N_G(P) \) acts trivially on \( Q \) by conjugation, it acts trivially on \( P \). Thus \( N_G(P)/Z_G(P) \to \text{Aut}(Q) \), so \( m_G/(p-1) \).

**Proof.** (cf. \cite{Ray99} Remarque 3.1.8]) We know \( \text{Aut}(P) \cong (\mathbb{Z}/p^n)^\times \), which has order \( p^{n-1}(p-1) \), with a unique maximal prime-to-\( p \) subgroup \( C \) of order \( p-1 \). Let \( g \in N_G(P) \), and suppose that the image \( \overline{g} \) of \( g \) in \( N_G(P)/Z_G(P) \subseteq \text{Aut}(P) \) acts trivially on \( Q \). Since

\[ \mathbb{Z}/p^n \cong \text{Aut}(P) \to \text{Aut}(Q) \cong \mathbb{Z}/p \]

has \( p \)-group kernel we know that \( \overline{g} \) has \( p \)-power order. If \( \overline{g} \) is not trivial, then \( g \notin P \), and the subgroup \( \langle g, P \rangle \) of \( G \) has a non-cyclic \( p \)-Sylow subgroup. This is impossible, so \( \overline{g} \) is trivial, and \( g \) acts trivially on \( P \). \( \square \)

We state a theorem of Burnside:
2.2 Basic facts about (wild) ramification

for the upper numbering (resp. the upper numbering) $G_i = G^i = \{id\}$. Any $i$ such that $G^i \supseteq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an upper jump of the extension $L/K$. Likewise, if $G_i \supseteq G_{i+\epsilon}$, then $i$ is called a lower jump of $L/K$. If $i$ is a lower (resp. upper) jump and $i > 0$, then $G^i/G^{i+\epsilon}$ (resp. $G_i/G_{i+\epsilon}$) is an elementary abelian $p$-group. The lower jumps are all integers. The greatest upper jump (i.e., the greatest $i$ such that $G^i \neq \{id\}$) is called the conductor of higher ramification of $L/K$. The upper numbering is invariant under quotients (Ser79 IV, Proposition 14)). That is, if $H \leq G$ is normal, and $M = L^H$, then the $i$th higher ramification group for the upper numbering for $M/K$ is $G^i/(G^i \cap H)$.}

\[ 1 \rightarrow \Sigma \rightarrow \Gamma \rightarrow \Pi' \rightarrow 1, \]

where $\Pi \leq \Gamma$ maps isomorphically onto $\Pi'$. \hfill \Box

Corollary 2.4. (i) If $G$ has a normal subgroup of order $p$, then there exists a normal prime-to-$p$ subgroup $N < G$ such that $G/N \cong \mathbb{Z}/p^n \times \mathbb{Z}/m_G$.

(ii) If $G$ has a central subgroup of order $p$, then there exists a normal prime-to-$p$ subgroup $N < G$ such that $G/N \cong \mathbb{Z}/p^n$. In particular, $m_G = 1$.

Proof. In both cases, let $N$ be the maximal normal prime-to-$p$ subgroup of $G$. Then $G/N$ still has a normal subgroup of order $p$, and no nontrivial normal subgroups of prime-to-$p$ order. Part (i) then follows from Lemma 2.3. If $G$ has a central subgroup of order $p$, then by Lemma 2.1 we have $m_G = 1$. \hfill \Box

2.2 Basic facts about (wild) ramification

We state here some facts from Ser79 IV] and derive some consequences. Let $K$ be a complete discrete valuation field with algebraically closed residue field $k$ of characteristic $p > 0$. If $L/K$ is a finite Galois extension of fields with Galois group $G$, then $L$ is also a complete discrete valuation field with residue field $k$. Here $G$ is of the form $P \times \mathbb{Z}/m$, where $P$ is a $p$-group and $m$ is prime to $p$. The group $G$ has a filtration $G = G_0 \supseteq G_1 (i \in \mathbb{R}_{\geq 0})$ for the lower numbering, and $G \supseteq G^i$ for the upper numbering ($i \in \mathbb{R}_{> 0}$). If $i \leq j$, then $G_i \supseteq G_j$ and $G^i \supseteq G^j$ (see Ser79 IV, §1. §3). The subgroup $G_i$ (resp. $G^i$) is known as the $i$th higher ramification group for the lower numbering (resp. the upper numbering). One knows that $G_0 = G^0 = G$, and that for sufficiently small $\epsilon > 0$, $G_\epsilon = G^\epsilon = P$. For sufficiently large $i$, $G_i = G^i = \{id\}$. Any $i$ such that $G^i \supseteq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an upper jump of the extension $L/K$. Likewise, if $G_i \supseteq G_{i+\epsilon}$, then $i$ is called a lower jump of $L/K$. If $i$ is a lower (resp. upper) jump and $i > 0$, then $G^i/G^{i+\epsilon}$ (resp. $G_i/G_{i+\epsilon}$) is an elementary abelian $p$-group. The lower jumps are all integers. The greatest upper jump (i.e., the greatest $i$ such that $G^i \neq \{id\}$) is called the conductor of higher ramification of $L/K$. The upper numbering is invariant under quotients (Ser79 IV, Proposition 14)). That is, if $H \leq G$ is normal, and $M = L^H$, then the $i$th higher ramification group for the upper numbering for $M/K$ is $G^i/(G^i \cap H)$. \hfill \Box
Lemma 2.5. If \( P \) is abelian, then all upper jumps (in particular, the conductor of higher ramification) are in \( \frac{1}{m} \mathbb{Z} \).

Proof. Let \( L_0 \subset L \) be the fixed field of \( L \) under \( P \). By the Hasse-Arf theorem ([Ser79, V, Theorem 1]), the upper jumps for the \( P \)-extension \( L/L_0 \) are integers. By Herbrand’s formula ([Ser79, IV, §3]), the upper jumps for \( L/K \) are \( \frac{1}{m} \) times those for \( L/L_0 \). The lemma follows. \( \square \)

The following lemma will be useful in §5. In the case \( G \cong \mathbb{Z}/p \times \mathbb{Z}/m \), it is essentially [Ray99, Propositions 1.1.4, 1.1.5].

Lemma 2.6. Fix an algebraic closure \( \overline{K} \) of \( K \). Suppose \( L/K \) is a finite \( G \)-Galois extension with conductor \( \sigma \), and \( K'/K \) is a \( \mathbb{Z}/p \)-Galois extension with conductor \( \tau \leq \sigma \). Assume we can embed \( L/K \) and \( K'/K \) into \( \overline{K}/K \) so that they are linearly disjoint, and write \( L' \) for the compositum \( LK' \). Write \( \rho \) for the conductor of \( L'/K' \). Then \( \rho - \tau = p(\sigma - \tau) \).

Proof. Since \( L \) and \( K' \) are linearly disjoint, \( G' := \text{Gal}(L'/K) \cong G \times \mathbb{Z}/p \). Also, write \( G \cong G' := \text{Gal}(L'/K') \). Since the upper numbering is invariant under quotients, the conductor of \( L'/K \) is equal to \( \text{max}(\sigma, \tau) = \sigma \). Let \( H := \text{Gal}(L'/L) \subset \text{Gal}(L'/K) \), and let \( i \) be the greatest integer such that \( G'' \ni H \) but \( G''_{i+1} \ni H \). Clearly \( H \cong \mathbb{Z}/p \). Since the lower numbering is preserved under subgroups, we have that \( G'_j = G''_j \) for \( j > i \). Furthermore, \( G''_j = G''_j \times H \) for \( j \leq i \). If \( \phi' \) and \( \phi'' \) are the respective Herbrand functions of \( L'/K' \), \( L'/K \) ([Ser79, IV, §3]), then \( \phi'(i) = \phi''(i) = \tau \). By the definition of the upper numbering, for \( j > i \), we have \( \phi'(j) - \phi'(i) = p(\phi''(j) - \phi''(i)) \) for \( j > i \). Taking \( j \) to be maximal such that \( G'_j \) is nontrivial, we obtain \( \rho - \tau = p(\sigma - \tau) \). \( \square \)

If \( A, B \) are the valuation rings of \( K, L \), respectively, sometimes we will refer to the conductor or higher ramification groups of the extension \( B/A \).

2.2.1 Smooth Curves. Let \( f : Y \to X \) be a branched cover of smooth, proper, integral curves over \( k \). The Hurwitz formula ([Har77, IV §2]) states that

\[ 2g_Y - 2 = (\deg f)(2g_X - 2) + |\Delta|, \]

where \( \Delta \) is the ramification divisor and \( |\Delta| \) is its degree (recall that \( \Delta = \sum_{y \in Y} d_y y \), where \( d_y \) is the length of \( \Omega_{Y/X} \) at \( y \)). For each point \( y \in Y \) with image \( x \in X \), the degree of \( \Delta \) at \( y \) can be related to the higher ramification filtrations of \( \text{Frac}(\hat{O}_{Y,y})/\text{Frac}(\hat{O}_{X,x}) \) ([Ser79, IV, Proposition 4]). In particular, if the ramification index \( e_y \) of \( y \) is prime to \( p \), then the degree of \( \Delta \) is \( e_y - 1 \).

In particular, suppose the Galois group \( G \) of \( \hat{O}_{Y,y}/\hat{O}_{X,x} \) is isomorphic to \( P \times \mathbb{Z}/m \) with \( P \) cyclic of order \( p^n \). Then all subgroups of \( P \) must occur as higher ramification groups (the subquotients of the higher ramification filtration having exponent \( p \)). For \( 1 \leq i \leq n \), write \( u_i \) (resp. \( j_i \)) for the upper (resp. lower) jump such that \( G^{u_i} \) (resp. \( G^{j_i} \)) is isomorphic to \( \mathbb{Z}/p^{n-i+1} \). Write \( u_0 = j_0 = 0 \). Then \( 0 = u_0 < u_1 < \cdots < u_n \) and \( 0 = j_0 < j_1 < \cdots < j_n \). We will sometimes call \( j_i \) (resp. \( u_i \)) the \( i \)th lower jump (resp. upper jump) of the extension \( \hat{O}_{Y,y}/\hat{O}_{X,x} \). Let \( |\Delta_y| \) be the degree of \( \Delta \) at \( y \).

Lemma 2.7.

(i) In terms of the lower jumps, we have

\[ |\Delta_y| = p^n m - 1 + \sum_{i=1}^{n} j_ip^{n-i}(p-1) = p^n m - 1 + \sum_{i=1}^{n} (p^{n-i+1} - 1)(j_i - j_{i-1}). \]

(ii) In terms of the upper jumps, we have

\[ |\Delta_y| = p^n m - 1 + \sum_{i=1}^{n} mp^{i-1}(p^{n-i+1} - 1)(u_i - u_{i-1}). \]
2.9 Proposition
Proof. Choose \( \Gamma \) acts trivially on \( K \) defined over \( \sigma \). By [Har77, IV §2], \( |\Delta| \) is equal to the valuation of the different of the extension \( \mathcal{O}_{Y,\gamma}/\mathcal{O}_{X,\gamma} \), where a uniformizer of \( \mathcal{O}_{Y,\gamma} \) is given valuation 1. By [Ser79, IV, Proposition 4], this different is equal to \( \sum_{r=0}^{\infty} (|G_r| - 1) \). Now it is a straightforward exercise to show that (i) holds. Part (ii) follows from part (i) by Herbrand’s formula (essentially, the definition of the upper numbering). \( \square \)

Remark 2.8. In the above context, it follows from Herbrand’s formula that the conductor \( u_n \) is equal to \( \frac{1}{m} \sum_{i=1}^{n} \frac{j_i - 1}{p^i - 1} \), which can also be written as \( \left( \sum_{i=1}^{n-1} \frac{1}{p^i} \right) + \frac{1}{p^n} j_n \).

2.3 Stable reduction
We now introduce some notation that will be used for the remainder of this section. Let \( X/K \) be a smooth, proper, geometrically integral curve of genus \( g_X \), with \( K \) is a characteristic zero complete discrete valued field, with algebraically closed residue field \( k \) of characteristic \( p > 0 \) (e.g., \( K = \mathbb{Q}_p \)). Let \( R \) be the valuation ring of \( K \). Write \( v \) for the valuation on \( R \). We normalize by setting \( v(p) = 1 \).

For the rest of this section, assume that \( X \) has a smooth model \( X_R \) over \( R \). Let \( f : Y \to X \) be a \( G \)-Galois cover defined over \( K \), with \( G \) any finite group, such that the branch points of \( f \) are defined over \( K \) and their specializations do not collide on the special fiber of \( X_R \). Assume that \( 2g_X - 2 + r > 1 \), where \( r \) is the number of branch points of \( f \). By a theorem of Deligne and Mumford ([DM69, Corollary 2.7]), combined with work of Raynaud ([Ray90], [Ray99]) and Liu ([Liu06]), there is a minimal finite extension \( K^s/K \) with ring of integers \( R^s \), and a unique model \( Y^s \) of \( Y_K \) (called the stable model) such that

- The special fiber \( \overline{Y} \) of \( Y^s \) is semistable (i.e., it is reduced, and has only nodes for singularities).
- The ramification points of \( f_{K^s} = f \times_K K^s \) specialize to distinct smooth points of \( \overline{Y} \).
- Any genus zero irreducible component of \( \overline{Y} \) contains at least three marked points (i.e., ramification points or points of intersection with the rest of \( \overline{Y} \)).

Since the stable model is unique, it is acted upon by \( G \), and we set \( X^s = Y^s/G \). Then \( X^s \) can be naturally identified with a blowup of \( X \times_R R^s \) centered at closed points. Furthermore, the nodes of \( \overline{Y} \) lie above nodes of the special fiber \( \overline{X} \) of \( X^s \).

The map \( f^s : Y^s \to X^s \) is called the stable model of \( f \) and the field \( K^s \) is called the minimal field of definition of the stable model of \( f \). Note that our definition of the stable model is the definition used in [Wew03b]. This differs from the definition in [Ray99] in that [Ray99] allows the ramification points to coalesce on the special fiber. If we are working over a finite extension \( K'/K^s \) with ring of integers \( R' \), we will sometimes abuse language and call \( f^s \times_R R' \) the stable model of \( f \).

For each \( \sigma \in G_K \), \( \sigma \) acts on \( \overline{Y} \) and this action commutes with \( G \). Let \( \Gamma^s \subseteq G_K \) consist of those \( \sigma \in G_K \) such that \( \sigma \) acts trivially on \( \overline{Y} \).

Proposition 2.9. The extension \( K^s/K \) is the extension cut out by \( \Gamma^s \subseteq G_K \). In other words, \( \Gamma^s = G_{K^s} \).

Proof. Choose \( \gamma \in G_{K^s} \). By Hensel’s lemma, each smooth point \( \overline{y} \) of \( \overline{Y} \) is the specialization of a \( K^s \)-rational point \( y \) of \( Y^s \). Since \( \gamma \) fixes \( y \), it fixes \( \overline{y} \). Since the smooth points of \( \overline{Y} \) are dense, \( \gamma \) acts trivially on \( \overline{Y} \), so \( \gamma \in \Gamma^s \).

Now choose \( \gamma \in \Gamma^s \). By [Liu06, Remark 2.21], the extension \( K^s/K \) is the compositum of two extensions: the minimal extension \( K'/K \) leading to the stable reduction \( \overline{f} : \overline{Y} \to \overline{X}' \) of \( f \) under the definition of [Ray99] (where we allow the ramification points to coalesce), as well as the minimal extension \( K''/K \) over which all of the ramification points of \( f \) are defined. Since the ramification points of \( f \) specialize to distinct points on \( \overline{Y} \), it follows that \( \gamma \) does not permute these points nontrivially. But any nontrivial element of \( G(K''/K) \) does permute the ramification points.
nontrivially. Thus \( \gamma \in G_{K'} \). On the other hand, since \( \gamma \) acts trivially on \( Y \) (which dominates \( \overline{Y} \)), it acts trivially on \( \overline{Y} \). By [Ray99, Proposition 2.2.2], \( \gamma \in G_{K'} \). Since \( G_{K'} \cap G_{K''} = G_{K^{st}} \), we have \( \gamma \in G_{K^{st}} \). \( \square \)

Since \( \Gamma^{st} \) is the kernel of the homomorphism \( G_k \to \text{Aut}(\overline{Y}) \), it follows from Proposition 2.9 that \( K^{st} \) is Galois over \( K \).

If \( \overline{Y} \) is smooth, the cover \( f : Y \to X \) is said to have potentially good reduction. If \( \overline{Y} \) can be contracted to a smooth curve by blowing down components of genus zero, then the curve \( Y \) is said to have potentially good reduction. If \( f \) or \( Y \) does not have potentially good reduction, it is said to have bad reduction. In any case, the special fiber \( \overline{f} : \overline{Y} \to \overline{X} \) of the stable model is called the stable reduction of \( f \). The action of \( G \) on \( Y \) extends to the stable reduction \( \overline{Y} \) and \( \overline{Y}/G \approx \overline{X} \). The strict transform of the special fiber of \( X_{K^{st}} \) in \( \overline{X} \) is called the original component, and will be denoted \( \overline{X}_0 \).

2.3.1 The graph of the stable reduction As in [Wew93], we construct the (unordered) dual graph \( \mathcal{G} \) of the stable reduction of \( \overline{X} \). An unordered graph \( \mathcal{G} \) consists of a set of vertices \( V(\mathcal{G}) \) and a set of edges \( E(\mathcal{G}) \). Each edge has a source vertex \( s(e) \) and a target vertex \( t(e) \). Each edge has an opposite edge \( \overline{e} \), such that \( s(e) = t(\overline{e}) \) and \( t(e) = s(\overline{e}) \). Also, \( \overline{\overline{e}} = e \).

Given \( f, \overline{f}, \overline{Y}, \) and \( \overline{X} \) as in this section, we construct two unordered graphs \( \mathcal{G} \) and \( \mathcal{G}' \). In our construction, \( \mathcal{G} \) has a vertex \( v \) for each irreducible component of \( \overline{X} \) and an edge \( e \) for each ordered triple \( (\overline{e}, \overline{W}', \overline{W}'') \), where \( \overline{W}' \) and \( \overline{W}'' \) are irreducible components of \( \overline{X} \) whose intersection is \( \overline{e} \). If \( e \) corresponds to \( (\overline{e}, \overline{W}', \overline{W}'') \), then \( s(e) \) is the vertex corresponding to \( \overline{W}' \) and \( t(e) \) is the vertex corresponding to \( \overline{W}'' \). The opposite edge of \( e \) corresponds to \( (\overline{e}, \overline{W}'', \overline{W}') \). We denote by \( \mathcal{G}' \) the augmented graph of \( \mathcal{G} \) constructed as follows: consider the set \( B_{\text{wild}} \) of branch points of \( f \) with branching index divisible by \( p \).

For each \( x \in B_{\text{wild}} \), we know that \( x \) specializes to a unique irreducible component \( \overline{W}_x \) of \( \overline{X} \), corresponding to a vertex \( A_x \) of \( \mathcal{G} \). Then \( V(\mathcal{G}') \) consists of the elements of \( V(\mathcal{G}) \) with an additional vertex \( V_x \) for each \( x \in B_{\text{wild}} \). Also, \( E(\mathcal{G}') \) consists of the elements of \( E(\mathcal{G}) \) with two additional opposite edges for each \( x \in B_{\text{wild}} \), one with source \( V_x \) and target \( A_x \), and one with source \( A_x \) and target \( V_x \). We write \( v_0 \) for the vertex corresponding to the original component \( \overline{X}_0 \).

An irreducible component of \( \overline{X} \) corresponding to a leaf of \( \mathcal{G} \) that is not \( \overline{X}_0 \) is called a tail of \( \overline{X} \). All other components are called interior components. We partially order the vertices of \( \mathcal{G}' \) such that \( v_1 \preceq v_2 \) iff \( v_1 = v_2, v_1 = v_0, \) or \( v_0 \) and \( v_2 \) are in different connected components of \( \mathcal{G}' \setminus v_1 \) (we order “outward” from the original component). Similarly, we can compare edges with each other, and edges with vertices. For this we overload the symbol \( \preceq \). The set of irreducible components and singular points of \( \overline{X} \) inherits the partial order \( \preceq \).

2.3.2 Inertia Groups of the Stable Reduction.

Proposition 2.10 ([Ray99, Proposition 2.4.11]). The inertia groups of \( \overline{f} : \overline{Y} \to \overline{X} \) at points of \( \overline{Y} \) are as follows (note that points in the same \( G \)-orbit have conjugate inertia groups):

(i) At the generic points of irreducible components, the inertia groups are \( p \)-groups.

(ii) At each node, the inertia group is an extension of a cyclic, prime-to-\( p \) order group, by a \( p \)-group generated by the inertia groups of the generic points of the crossing components.

(iii) If a point \( y \in Y \) above a branch point \( x \in X \) specializes to a smooth point \( \overline{y} \) on a component \( \overline{V} \) of \( \overline{Y} \), then the inertia group at \( \overline{y} \) is an extension of the prime-to-\( p \) part of the inertia group at \( y \) by the inertia group of the generic point of \( \overline{V} \).
(iv) At all other points \( q \) (automatically smooth, closed), the inertia group is equal to the inertia group of the generic point of the irreducible component of \( \overline{Y} \) containing \( q \).

If \( \overline{V} \) is an irreducible component of \( \overline{Y} \), we will always write \( I_{\overline{V}} \leq G \) for the inertia group of the generic point of \( \overline{V} \), and \( D_{\overline{V}} \) for the decomposition group.

For the rest of this subsection, assume \( G \) has a cyclic \( p \)-Sylow subgroup. When \( G \) has a cyclic \( p \)-Sylow subgroup, the inertia groups above a generic point of an irreducible component \( \overline{W} \subset \overline{X} \) are conjugate cyclic groups of \( p \)-power order. If they are of order \( p^i \), we call \( \overline{W} \) a \( p^i \)-component. If \( i = 0 \), we call \( \overline{W} \) an \( \acute{e}tale \) component, and if \( i > 0 \), we call \( \overline{W} \) an inseparable component. For an inseparable component \( \overline{W} \), the morphism \( Y \times_X \overline{W} \to \overline{W} \) corresponds to an inseparable extension of the function field \( k(\overline{W}) \). This is because, since \( \overline{Y} \) is reduced, the inertia of \( f \) at the local ring of the generic point of an irreducible component of \( \overline{Y} \) above \( \overline{W} \) must come from an inseparable extension of residue fields.

An \( \acute{e}tale \) tail of \( \overline{X} \) is called primitive if it contains a branch point other than the point at which it intersects the rest of \( \overline{X} \). Otherwise it is called new. This follows \cite{Wew03b}. An inseparable tail that does not contain the specialization of any branch point will be called a new inseparable tail.

A inseparable tail that is a \( p^i \)-component will also be called a \( p^i \)-tail (a new \( p^i \)-tail if it is new).

**Corollary 2.11.** If \( \overline{V} \) and \( \overline{V}' \) are two adjacent irreducible components of \( \overline{Y} \), then either \( I_{\overline{V}} \subset I_{\overline{V}'} \) or vice versa.

**Proof.** Let \( q \) be a point of intersection of \( \overline{V} \) and \( \overline{V}' \) and let \( I_q \) be its inertia group. Then the \( p \)-part of \( I_q \) is a cyclic \( p \)-group, generated by the two cyclic \( p \)-groups \( I_{\overline{V}} \) and \( I_{\overline{V}'} \). Since the subgroups of a cyclic \( p \)-group are totally ordered, the corollary follows.

**Corollary 2.12.** Let \( \overline{S} \subset \overline{Y} \) be a union of irreducible components of \( \overline{Y} \), all of which lie above inseparable components of \( \overline{X} \). Suppose that \( \overline{S} \) is connected. Let \( D_{\overline{S}} \subset G \) be the decomposition group of \( \overline{S} \) (i.e., the maximal subgroup of \( G \) such that \( D_{\overline{S}}(\overline{S}) = \overline{S} \)). Then \( D_{\overline{S}} \) has a normal subgroup of order \( p \).

**Proof.** Pick any irreducible component \( \overline{V} \) of \( \overline{S} \), and let \( Q \) be the unique subgroup of order \( p \) of \( I_{\overline{V}} \). Let \( g \in D_{\overline{S}} \), and write \( \overline{V}' = g\overline{V} \). Then \( gQg^{-1} \) is the unique subgroup of order \( p \) of the inertia group of the generic point of \( \overline{V}' \). Since \( \overline{S} \) is connected, there exists a sequence \( \Sigma \) of components of \( \overline{S} \), starting with \( \overline{V} \) and ending with \( \overline{V}' \), such that each component in \( \Sigma \) intersects the preceeding and the following component. The components in \( \Sigma \) all lie above inseparable components of \( \overline{X} \). Then the inertia group of the generic point of each component in \( \Sigma \) has a unique subgroup of order \( p \). We know from Corollary 2.11 that for any two such adjacent components, the inertia group of one contains the inertia group of the other. Thus, both inertia groups contain the same subgroup of order \( p \). So \( Q = gQg^{-1} \), and we are done.

**Proposition 2.13.** If \( x \in X \) is branched of index \( p^a s \), where \( p \nmid s \), then \( x \) specializes to a \( p^a \)-component.

**Proof.** By Proposition 2.10 (iii) and the definition of the stable model, \( x \) specializes to a smooth point of a component whose generic inertia has order at least \( p^a \). Because our definition of the stable model requires the specializations of the \( |G|/p^a s \) ramification points above \( a \) to be disjoint, the specialization of \( x \) must have a fiber with cardinality a multiple of \( |G|/p^a s \). This shows that \( x \) must specialize to a component with inertia at most \( p^a \).

**Remark 2.14.** It follows from Proposition 2.10 (iii) and the proof of Proposition 2.13 that if \( y \) is a ramification point above \( x \), then the specialization \( \overline{y} \) of \( y \) also has inertia group in \( G \) cyclic of order \( p^a s \).
Lemma 2.15 ([Ray99], Proposition 2.4.8). If \( W \) is an étale component of \( \overline{X} \), then \( W \) is a tail.

Lemma 2.16. If \( W \) is a \( p^a \)-tail of \( \overline{X} \), then the component \( W' \) that intersects \( W \) is a \( p^b \)-component with \( b > a \).

Proof. The proof is essentially the same as the proof of [Ray99, Lemme 3.1.2]. Assume that the proposition is false. Let \( V \) be an irreducible component of \( W \) lying above the genus zero component \( \overline{W} \). By Proposition 2.10 and our assumption, the map \( g : V \to W \) is the composition \( h \circ q \) of a tamely ramified, generically étale morphism \( h \) with a radicial morphism \( q \) of degree \( p^a \). Now, \( h \) can only be branched at the intersection \( \overline{w} \) of \( W \) and \( W' \), and the specialization of, at most, one point \( a_i \) to \( \overline{W} \). Since there are at most two branch points, and they are tame, then \( h \) is totally ramified at these points. So \( q(\overline{V}) \) has genus zero and has only one point above \( \overline{w} \). Since \( q \) is radicial, the same is true for \( \overline{V} \). This contradicts the definition of the stable model, as \( \overline{V} \) has genus zero and insufficiently many marked points.

Note that Lemma 2.16 shows that if \( p \) exactly divides \( |G| \), then there are no inseparable tails. But there can be inseparable tails if a higher power of \( p \) divides \( |G| \).

Notation 2.17. Let \( x \) be the intersection point of two components \( \overline{W} \) and \( \overline{W}' \) of \( \overline{X} \), and let \( y \) lie above \( x \), on the intersection of two components \( V \) and \( V' \) of \( \overline{Y} \). Assume that \( W \) is a \( p^r \)-component and \( \overline{W}' \) is a \( p^{r'} \)-component, \( r \geq r' \). The inclusion \( \hat{O}_{\overline{W}',x} \hookrightarrow \hat{O}_{\overline{V}',y} \) induced from the cover is a composition

\[
\hat{O}_{\overline{W}',x} \hookrightarrow S \hookrightarrow \hat{O}_{\overline{V}',y}
\]

where \( \hat{O}_{\overline{W}',x} \hookrightarrow S \) is a totally ramified Galois extension with group \( J \cong \mathbb{Z}/p^{-r'} \times \mathbb{Z}/m_x \) and \( S \hookrightarrow \hat{O}_{\overline{V}',y} \) is a purely inseparable extension of degree \( p^{r'} \). The extension \( \hat{O}_{\overline{W}',x} \hookrightarrow S \) and the group \( J \) depend only on \( x \), up to isomorphism, so we denote them by \( \hat{O}_{\overline{W}',x} \hookrightarrow S_x \), and \( J_x \), respectively.

Definition 2.18. For \( r > r' \), let \( B_{r',r} \) index the set of intersection points of a \( p^r \)-component \( \overline{W} \) with a \( p^{r'} \)-component \( \overline{W}' \) of \( \overline{X} \). For \( b \in B_{r',r} \), let \( x_b \) be the corresponding point of intersection. For any \( r' \leq \alpha < r \), let \( J_{b,\alpha} \) be the unique subgroup of \( J_x \) of order \( p^{\alpha-r'} \). Finally, for \( r' \leq \alpha < r \), let \( \sigma_{b,\alpha} \) be the conductor of the extension \( \hat{O}_{\overline{W}',x_b} \hookrightarrow S_{x_b}^{I_{\alpha}} \) (2.2). If \( \alpha = r' \), we will often just write \( \sigma_{x_b}^{r'} \) for \( \sigma_{x_b}^{\alpha} \). Furthermore, if \( x_b \) lies on a tail \( \overline{X}_b \), we will simply write \( \sigma_b \) (resp. \( \sigma_b^{r'} \)) for \( \sigma_{x_b} \) (resp. \( \sigma_{x_b}^{r'} \)).

We call the \( \sigma_{x_b}^{\alpha} \) the truncated effective ramification invariants (of \( T \)) at \( x_b \). We call \( \sigma_{x_b} \) simply the effective ramification invariant (of \( T \)) at \( x_b \). If \( \overline{X}_b \) is a tail of \( \overline{X} \), then \( \sigma_{x_b}^{\alpha} \) (resp. \( \sigma_{x_b} \)) is called the truncated effective ramification invariant (resp. effective ramification invariant) of the tail \( \overline{X}_b \).

Remark 2.19. In the case \( r = 1, r' = 0 \), the \( \sigma_b \) for tails \( \overline{X}_b \) are the same as those defined in [Ray99] and [Wew03b].

Lemma 2.20. The effective ramification invariants \( \sigma_{x_b} \) lie in \( \frac{1}{m_{J_{x_b}} \mathbb{Z}} \). In particular, they lie in \( \frac{1}{m_G} \mathbb{Z} \).

Proof. The extension \( S^{I_{\alpha}}_{x_b}/\hat{O}_{\overline{X}_b,x_b} \) has Galois group \( J_{x_b}/I_\alpha \), which is isomorphic to \( \mathbb{Z}/p^d \times \mathbb{Z}/\ell \) for some \( d, \ell \). The quotient of \( J_{x_b}/I_\alpha \) by its maximal prime-to-\( p \) central subgroup \( H \) is \( \mathbb{Z}/p^d \times \mathbb{Z}/m_{J_{x_b}} \). The effective ramification invariants over \( x_b \) are not affected by quotienting out by \( H \), as the upper numbering is invariant under taking quotients. So \( \sigma_{x_b} \in \frac{1}{m_{J_{x_b}}} \mathbb{Z} \). Since \( J_{x_b} \) is a subquotient of \( G \), it is easy to see that \( m_{J_{x_b}} m_G \), showing the second statement of the lemma.

We give one more definition:
**Remark** 3.2. (i) In [Hen99], Proposition 3.1 is stated for \(X \to \text{Spec } R\) with dimension 1 fibers, but the proof carries over without change to the case of dimension 0 fibers as well (i.e., the case where \(A\) is a discrete valuation ring containing \(R\)). It is this case that will be used in §3.2 to define deformation data.
In the cases of multiplicative and additive reduction, the map \( Y_k \to X_k \) is seen to be inseparable.

### 3.2 Deformation Data

Deformation data arise naturally from the stable reduction of covers. Say \( f : Y \to X \) is a branched \( G \)-cover as in \( \text{[2.3]} \) with stable model \( f^{st} : Y^{st} \to X^{st} \) and stable reduction \( f : \widetilde{Y} \to \widetilde{X} \). Much information is lost when we pass from the stable model to the stable reduction, and deformation data provide a way to retain some of this information. This process is described in detail in \( \text{[Hen99]} \) 5, §1 in the case where the inertia group of a component has order \( p \). In Construction 3.4, we generalize it to the case where the inertia group is cyclic of order \( p' \).

#### 3.2.1 Generalities

Let \( \bar{W} \) be any connected smooth proper curve over \( k \). Let \( H \) be a finite group and \( \chi \) a 1-dimensional character \( H \to \mathbb{F}_p^\times \). A deformation datum over \( \bar{W} \) of type \((H, \chi)\) is an ordered pair \((\bar{V}, \omega)\) such that: \( \bar{V} \to \bar{W} \) is an \( H \)-Galois branched cover; \( \omega \) is a meromorphic differential form on \( \bar{V} \) that is either logarithmic or exact (i.e., \( \omega = du/u \) or \( du \) for \( u \in k(\bar{V}) \)); and \( \eta^* \omega = \chi(\eta) \omega \) for all \( \eta \in H \). If \( \omega \) is logarithmic (resp. exact), the deformation datum is called multiplicative (resp. additive). When \( \bar{V} \) is understood, we will sometimes speak of the deformation datum \( \omega \).

If \((\bar{V}, \omega)\) is a deformation datum, and \( w \in \bar{W} \) is a closed point, we define \( m_w \) to be the order of the prime-to-\( p \) part of the ramification index of \( \bar{V} \to \bar{W} \) at \( w \). Define \( h_w \) to be \( \text{ord}_v(\omega) + 1 \), where \( v \in \bar{V} \) is any point which maps to \( w \in \bar{W} \). This is well-defined because \( \omega \) transforms nicely via \( H \). Lastly, define \( \sigma_w = h_w/m_w \). We call \( w \) a critical point of the deformation datum \((\bar{V}, \omega)\) if \((h_w, m_w) \neq (1, 1) \). Note that every deformation datum contains only a finite number of critical points. The ordered pair \((h_w, m_w)\) is called the signature of \((\bar{V}, \omega)\) (or of \( \omega \), if \( \bar{V} \) is understood) at \( w \), and \( \sigma_w \) is called the invariant of the deformation datum at \( w \).

**Proposition 3.3.** Let \((\bar{V}, \omega)\) be a deformation datum of type \((H, \chi)\). Let \( v \in \bar{V} \) be a tamely ramified point lying over \( w \in \bar{W} \), and write \( I_v \) for the inertia group of \( \phi : \bar{V} \to \bar{W} \) at \( v \). If \(|I_v/(I_v \cap \ker(\chi))| = \mu \), then \( \sigma_w \in \frac{1}{\mu} \mathbb{Z} \).

**Proof.** In a formal neighborhood of \( v \), we can use Kummer theory to see that \( \phi \) is given by the equation \( k[[t]] \to k[[t]]/\mathbb{Z} m_w t \), where \( t \) is a local parameter at \( w \) and \( \tau \) is a local parameter at \( v \). Expanding \( \omega \) out as a Laurent series in \( \tau \), we can write

\[
\omega = \left( c \tau^{h_w-1} + \sum_{i=1}^{\infty} c_i \tau^{h_w-1+i} \right) d\tau.
\]

Let \( g \) be a generator of \( I_v \) such that \( g^*(\tau) = \zeta_{m_w} \tau \). Since \( g^* \in \ker(\chi) \), we have that \( (g^*)^* \omega = \omega \). Thus \((g^*)^*(\tau^{h_w-1} d\tau) = \tau^{h_w-1} d\tau \). So \( \mu h_w \) is a multiple of \( m_w \). Therefore, \( \sigma_w = \frac{h_w}{m_w} \in \frac{1}{\mu} \mathbb{Z} \). \( \square \)

#### 3.2.2 Deformation data arising from stable reduction

Maintain the notations of \( \text{[2.3]} \). For each irreducible component of \( \bar{Y} \) lying above a \( p' \)-component of \( \bar{X} \) with \( r > 0 \), we will construct \( r \) different deformation data. For this construction, we can replace \( K^{st} \) with a finite extension \( K' \) that is as large as we wish. In particular, we work over \( K' \) containing a \( p' \)th root of unity and having ring of integers \( R' \). By abuse of notation, we write \( Y^{st} \) for \( Y^{st} \times_{K^{st}} K' \).

**Construction 3.4.** Let \( \bar{V} \) be an irreducible component of \( \bar{Y} \) with generic point \( \eta \) and nontrivial generic inertia group \( I \cong \mathbb{Z}/p'^k \subset G \). Write \( B = \mathcal{O}_{Y^{st}, \eta} \), and \( C = B[I] \), the invariants of \( B \) under the action of \( I \). Then \( B \) (resp. \( C \)) is a complete, mixed characteristic, discrete valuation ring with residue field \( k(\bar{V}) \) (resp. \( k(\bar{V})^{p'} \)). The group \( I \cong \mathbb{Z}/p'^k \) acts on \( B \); for \( 0 \leq i \leq r \), we write \( I_i \) for the subgroup of order \( p'^i \) in \( I \), and we write \( C_i \) for the fixed ring \( B^{I_{r-i+1}} \). Thus \( C_1 = C \). Then for \( 1 \leq i \leq r \), the extension \( C_i \to C_{i+1} \) is an extension of complete discrete valuation rings satisfying
the conditions of Proposition 3.1, but with relative dimension 0 instead of 1 over $R'$ (see Remark 3.2 (i)). On the generic fiber, the extension is given by an equation $y^p = z$, where $z$ is well-defined up to raising to a prime-to-$p$ power in $C_i^\times/(C_i^\times)^p$. We make $z$ completely well-defined in $C_i^\times/(C_i^\times)^p$ by fixing a $p$th root of unity $\mu$ and a generator $\alpha$ of $\text{Aut}(C_{i+1}/C_i)$ and forcing $\alpha(y) = \mu y$. In both the case of multiplicative and additive reduction, Proposition 3.1 yields an element

$$\pi \in C_i \otimes_{R'} k = k(\nabla)^{p^{\prime}-i+1} \cong k(\nabla)^{p^r},$$

the last isomorphism coming from raising to the $p^{\prime}$th power. In the case of multiplicative reduction, set $\omega_i = d\pi/\pi$, and in the case of additive reduction, set $\omega_i = d\pi$. In both cases, $\omega_i$ can be viewed as a differential form on $k(\nabla)^{p^r}$. Write $\nabla'$ for the curve whose function field is $C \otimes_{R'} k = k(\nabla)^{p^r} \subset k(\nabla)$. Then each $\omega_i$ is a meromorphic differential form on $\nabla'$.

Furthermore, let $D := D_{\nabla'}$, and write $H = D/I$. Then if $\nabla$ is the component of $X$ lying below $\nabla'$, we have maps $\nabla \rightarrow \nabla' \rightarrow \nabla$, with $\nabla = \nabla'/H$. The curves $\nabla$ and $\nabla'$ are abstractly isomorphic. Any $g \in H$ has a canonical conjugation action on $I$, and also on the subquotient of $I$ given by $\text{Aut}(C_{i+1}/C_i)$. This action gives a homomorphism $\chi : H \rightarrow \mathbb{F}_p^\times$. We claim to have constructed, for each $i$, a deformation datum $(\nabla', \omega_i)$ of type $(H, \chi)$ over $\nabla$.

Everything is clear except for the transformation property, so let $g \in H$. Then for $z$ as in the construction, taking a $p$th root of $z$ and of $g^*z$ must yield the same extension, so $g^*z = c^pz^q$ with $c \in C_i$ and $q \in \{1, \ldots, p-1\}$. It follows that $g^*y = \zeta cy^q$ for $\zeta$ some $p$th root of unity. It also follows that $g^*(\omega_i) = q\omega_i$. If $\alpha$ is a generator of $\text{Aut}(C_{i+1}/C_i)$ as before, then we must show that $g\alpha g^{-1} = \alpha q^g$.

Write $\alpha^*y = \mu y$ for some, possibly different, $p^\prime$th root of unity $\mu$. Then

$$(g\alpha g^{-1})^*(\alpha^*y) = (g^{-1})^*\alpha^*g^*y = (g^{-1})^*\alpha^*\zeta cy^q = (g^{-1})^*\mu^q\zeta cy^q = \mu^qy.$$

Thus $g\alpha g^{-1} = \alpha q^g$. This completes Construction 3.4.

For the rest of this section, we will only concern ourselves with deformation data that arise from the stable reduction of branched $G$-covers $Y \rightarrow X = \mathbb{P}^1$ where $G$ has a cyclic $p$-Sylow subgroup, via Construction 3.4. We will use the notations of (2.9) and Construction 3.4 throughout this section.

From [Wew03], Proposition 1.7], we have the following result in the case of inertia groups of order $p$. The proof is the same in our case, and we omit it.

**Lemma 3.5.** Say $(\nabla', \omega)$ is a deformation datum arising from the stable reduction of a cover as in Construction 3.4, and let $\nabla$ be the component of $X$ lying under $\nabla'$. Then a critical point $x$ of the deformation datum on $\nabla$ is either a singular point of $X$ or the specialization of a branch point of $Y \rightarrow X$ with ramification index divisible by $p$. In the first case, $\sigma_x \neq 0$, and in the second case, $\sigma_x = 0$ and $\omega$ is logarithmic.

The next result, Proposition 3.6, generalizes one part of the theorem [Hen99, 5, Theorem 1.10]. It provides the inner workings behind the cleaner interface given by Lemma 3.11. We assume the situation of Notation 2.17 (in particular, the notations $x$, $y$, $\nabla$, $\nabla'$, $\nabla$, $r$, $r'$, $S_x$, and $J_x$). By Lemma 2.15, $r \geq 1$. For each $i$, $1 \leq i \leq r$, there is a deformation datum with differential form $\omega_i$ associated to $\nabla$. For each $i'$, $1 \leq i' \leq r'$, there is a deformation datum $\omega_i'$ associated to $\nabla'$. Let $m_x$ be the prime-to-$p$ part of the ramification index at $x$. Let $I$ be the inertia group of $y$ in $G$, and let $I_i$ (resp. $J_i$) be the unique subgroup of order $p^i$ in $I$ (resp. $J_x$). The following proposition gives a compatibility between deformation data, and also relates deformation data to the geometry of $\nabla$.
Proposition 3.6. With $x$ as above, let $(h_{i,x}, m_x)$ (resp. $(h'_{i,x}, m_x)$) be the signature of $\omega_i$ (resp. $\omega'_i$) at $x$. Write $\sigma_{i,x} = h_{i,x}/m_x$ and $\sigma'_{i,x} = h'_{i,x}/m_x$. Then the following hold:

(i) If $i = i' + r - r'$, then $h_{i,x} = -h'_{i',x}$ and $\sigma_{i,x} = -\sigma'_{i',x}$.
(ii) If $i < r - r'$, then $h_{i,x} = h$, where $h$ is the upper (equivalently lower) jump in the extension $S_{l-r'-i}^{r-r'-i} \hookrightarrow S_{l-r'-i}^{r-r'-i+1}$. Also, $\sigma_{i,x} = \sigma$, where $\sigma$ is the upper jump in the extension $S_{l-r'-i}^{r-r'-i} \hookrightarrow S_{l-r'-i}^{r-r'-i+1}$.

Proof. (cf. [Wew03b, Proposition 1.8]) The group $I$ acts on the annulus $A = \text{Spec } \mathcal{O}_{Y^*}$. The statements about $h_{i,x}$ follow from [Hen99, 5, Proposition 1.10] applied to the form al annulus and an automorphism given by a generator of $I_{r-i}/I_{r-i}$ considered as a subquotient of $J_x$. The statements about $\sigma_{i,x}$ follow from dividing the statements about $h_{i,x}$ by $m_x$. Note that what we call $h_{i,x}$, [Hen99] calls $-m$.

Remark 3.7. For $r' \leq \alpha < r$, consider the $\mathbb{Z}/p^{r'-\alpha} \times \mathbb{Z}/m_x$-extension $\mathcal{O}_{\overline{W}_x} \hookrightarrow S_{l}^{l}$. If the $j_i$ are its lower jumps (see (2.2), then Proposition 3.6, combined with [OP08, Lemma 3.1], shows that $j_i = h_{i,x}$. By Remark 2.8, the conductor of this extension is equal to

$$\left(\sum_{i=1}^{r-\alpha-1} \frac{p-1}{p} \sigma_{i,x}\right) + \frac{1}{p^{r-\alpha-1}} \sigma_{r-\alpha,x}.$$ 

We set up the local vanishing cycles formula. Our first version, Equation (3.1), will be unwieldy, but it will be used to prove our second version, the much cleaner Equation (3.2). Let $(\nabla, \omega)$ be a deformation datum of type $(H, \chi)$ (not necessarily coming from the stable reduction of a cover). Let $B$ (resp. $B'$) be the set of critical points of $(\nabla, \omega)$ where $\nabla \to \overline{W}$ is tamely (resp. wildly) ramified. Let $g_{\nabla}$ be the genus of $\overline{W}$.

For each $w \in B'$, suppose that the inertia group of a point $v$ above $w$ is $\mathbb{Z}/p^v \times \mathbb{Z}/m_w$ with $p \nmid m_w$. For $1 \leq i \leq n_w$, let $h_{i,w}$ be the $i$th lower jump of the extension $\mathcal{O}_{\overline{W}_v} \hookrightarrow \mathcal{O}_{\overline{W}_w}$, and let $\sigma_{i,w} = h_{i,w}/m_w$. We maintain the notation $(h_{i,w}, m_w)$ for the signature of $\omega$ at $w$, and $\sigma_{i,w}$ for the invariant at $w$. Note that there is not necessarily any relation between the $\sigma_{i,w}$ and $\sigma_{w}$. Then we have

Lemma 3.8 (Local vanishing cycles formula).

$$\sum_{w \in B'} \left(\frac{\sigma_w}{p^n_w} - 1 - \sum_{i=1}^{n_w} \frac{p-1}{p} \sigma_{i,w}\right) + \sum_{b \in B} (\sigma_b - 1) = 2g_{\nabla} - 2. \quad (3.1)$$

Proof. Let $g_{\nabla}$ be the genus of $\nabla$, and $d$ the degree of the map $\nabla \to \overline{W}$. The Hurwitz formula, along with Lemma 2.7 (i), yields that

$$2g_{\nabla} - 2 = d(2g_{\nabla} - 2) + d \sum_{b \in B} (1 - \frac{1}{m_b}) + \sum_{w \in B'} \frac{d}{p^n_w m_w} (p^n_w m_w - 1 + \sum_{i=1}^{n_w} h_{i,w} p^{n_w-i} (p-1)).$$

Furthermore, the degree of a differential form on $\nabla$ is

$$\sum_{b \in B} \frac{d}{m_b} (h_b - 1) + \sum_{w \in B'} \frac{d}{p^n_w m_w} (h_w - 1).$$

Substituting this for $2g_{\nabla} - 2$ and rearranging yields the formula.

Remark 3.9. In the case $B' = \emptyset$, the local vanishing cycles formula (3.1) reduces to that found in [Wew03b, p. 998].
Let us resume the assumption that all of our deformation data come from Construction 3.4. Recall that \( \mathcal{G}' \) is the augmented dual graph of \( \overline{X} \). To each edge \( e \) of \( \mathcal{G}' \) we will associate an invariant \( \sigma^\text{eff}_e \), called the effective invariant.

For each \( 0 \leq j < n \), write \( \mathcal{G}'_j \) for the subgraph of \( \mathcal{G}' \) consisting of: those vertices corresponding to \( p^s \)-components for \( s > j \); those corresponding to specializations of branch points where \( p^{j+1} \) divides the branching index; and the edges incident to at least one of these vertices. Write \( \mathcal{G}_j = \mathcal{G}'_j \cap \mathcal{G} \).

Note that an edge in \( E(\mathcal{G}'_j) \) might have a source or a target not in \( V(\mathcal{G}'_j) \); these edges correspond to points of \( B_{r,r'} \) with \( r > j \leq r' \) (see Definition 2.13). Note further that \( E(\mathcal{G}'_0) = E(\mathcal{G}') \). For each edge in \( \mathcal{G}'_j \), we associate a set of invariants \( \sigma^\text{eff,} \alpha \), \( 0 \leq \alpha < j \), called the truncated effective invariants. The effective invariant \( \sigma^\text{eff}_e \) will be equal to the truncated effective invariant \( \sigma^\text{eff,} 0 \).

**Definition 3.10.** If \( s(e) \) corresponds to a \( p^r \)-component \( W \), and \( t(e) \) corresponds to a \( p^{r'} \)-component \( W' \) with \( r \geq r' \), then \( r \geq 1 \) by Lemma 2.15. Let \( \omega_i, 1 \leq i \leq r \), be the deformation data above \( W \). If \( \{ w \} = W \cap W' \), then

\[
\sigma^\text{eff}_e := \left( \sum_{i=1}^{r-1} \frac{p^i}{p^{i+1}} \sigma_{i,w} \right) + \frac{1}{p^{r-1}} \sigma_{r,w}.
\]

Note that this is a weighted average of the \( \sigma_{i,w} \)'s. Furthermore, we write

\[
\sigma^\text{eff,} \alpha_e := \left( \sum_{i=1}^{r-\alpha-1} \frac{p^{-\alpha}}{p^{i}} \sigma_{i,w} \right) + \frac{1}{p^{r-\alpha-1}} \sigma_{r-\alpha,w}
\]

for all \( 0 \leq \alpha < r \).

- If \( s(e) \) corresponds to a \( p^r \)-component and \( t(e) \) corresponds to a \( p^{r'} \)-component with \( r < r' \), then \( \sigma^\text{eff}_e := -\sigma^\text{eff}_e \). Also, \( \sigma^\text{eff,} \alpha_e := -\sigma^\text{eff,} \alpha_e \) for all \( \alpha < r' \).
- If either \( s(e) \) or \( t(e) \) is a vertex of \( \mathcal{G}' \) but not \( \mathcal{G} \), then \( \sigma^\text{eff}_e := 0 \). If, additionally, \( e \in E(\mathcal{G}'_j) \), then \( \sigma^\text{eff,} \alpha_e := 0 \) for all \( \alpha < j \).

Essentially, the truncated effective invariants are the same as the regular effective invariants, but we ignore the “top” \( \alpha \) differential forms (and thus the truncated invariants are not defined unless Construction 3.4 associates more than \( \alpha \) differential forms to the component in question).

**Lemma 3.11.** (i) For any \( e \in E(\mathcal{G}') \), we have \( \sigma^\text{eff,} \alpha_e = -\sigma^\text{eff,} \alpha_e \).

(ii) Suppose \( e \) corresponds to a point \( x \), \( s(e) \) corresponds to a \( p^r \)-component, \( t(e) \) corresponds to a \( p^{r'} \)-component, and \( r > r' \). Then \( \sigma^\text{eff,} \alpha_e = \sigma_x \) for any \( r' < \alpha < r \) (Definition 2.15).

(iii) In particular, if \( t(e) \) corresponds to an étale tail \( \overline{X}_b \), then \( \sigma^\text{eff}_e = \sigma_b \).

**Proof.** To (i): This needs proof only when \( e \) corresponds to the intersection of two \( p^r \)-components. But then the result follows immediately from Proposition 3.6 (setting \( r = r' \)).

To (ii): In this case, \( \sigma^\text{eff,} \alpha_e = \left( \sum_{i=1}^{r-\alpha-1} \frac{p^{-\alpha}}{p^{i}} \sigma_{i,x} \right) + \frac{1}{p^{r-\alpha-1}} \sigma_{r-\alpha,x} \). Remark 3.7 shows that this is equal to \( \sigma_x \).

To (iii): By Lemma 2.16 this is (ii) in the case \( \alpha = r' = 0 \).

**Lemma 3.12** (Effective local vanishing cycles formula). Let \( v \in V(\mathcal{G}') \) correspond to a \( p^r \)-component \( \overline{W} \) of \( \overline{X} \) with genus \( g_v \). Then for all \( \alpha < r \),

\[
\sum_{s(e) = v} (\sigma^\text{eff,} \alpha_e - 1) = 2g_v - 2.
\]
Vanishing cycles and wild monodromy

Proof. Each \( e \in E(G') \) with \( s(e) = v \) corresponds to a point \( w_e \) on \( \overline{W} \). Write

\[ B = \{ e \in E(G') \mid s(e) = v \text{ and } t(e) \text{ does not correspond to a } p^a\text{-component with } a > r \}. \]

Write \( B' = \{ e \in E(G') \mid s(e) = v \} \setminus B \). For \( e \in B \), let \( \sigma_{i,w_e} \) be the invariant of the \( i \)th deformation datum above \( \overline{W} \) at \( w_e \). For \( e \in B' \), let \( \sigma_{w_e} \) be the invariant of the \( i \)th deformation above \( \overline{W} \) at \( w_e \), where \( \overline{W} \) is the component that intersects \( \overline{W} \) at \( w_v \), and write \( n_{w_e} \) as in Lemma 3.8. Then for \( 1 \leq i \leq r \), Equation (3.1), along with Proposition 3.6, shows that

\[
\sum_{e \in B'} \left( -\frac{\sigma_{i+n_{w_e},w_e}}{p^{n_{w_e}}} - 1 - \sum_{j=1}^{n_{w_e}} \frac{p-1}{p^j} \sigma_{j,w_e} \right) + \sum_{e \in B} (\sigma_{i,w_e} - 1) = 2g_v - 2. \tag{3.3}
\]

For \( 1 \leq i \leq r - \alpha - 1 \), we multiply the \( i \)th equation (3.3) by \( \frac{p-1}{p^i} \) to obtain an equation \( E_i \). For \( i = r - \alpha \), we multiply it by \( \frac{1}{p-1} \) to obtain \( E_{r-\alpha} \). Note that these coefficients add up to 1. Adding up the equations \( E_i \) yields

\[
\sum_{e \in B'} (-\sigma_{v,\alpha}^{\text{eff}} - 1) + \sum_{e \in B} (\sigma_{e,\alpha}^{\text{eff}} - 1) = 2g_v - 2.
\]

Combining and using Lemma 3.11 (i) proves the lemma. \( \square \)

Remark 3.13. Compare (3.2) to the vanishing cycles formula in [Wew03b, p. 998].

3.3 Vanishing cycles formulas

Recall that \( m_G = |N_G(P)/Z_G(P)| \), and that we will write \( m \) instead of \( m_G \) when \( G \) is understood. The vanishing cycles formula ([Ray99, 3.4.2 (5)], [Wew03b, Corollary 1.11]) is a key formula that helps us understand the structure of the stable reduction of a branched \( G \)-cover of curves in the case where \( p \) exactly divides the order of \( G \). Here, we generalize the formula to the case where \( G \) has a cyclic \( p \)-Sylow group of arbitrary order. For any étale tail \( X_b \), recall that \( \sigma_b \) is the effective ramification invariant at the point of intersection \( x_b \) of \( X_b \) with the rest of \( X \) (Definition 2.18).

Theorem 3.14 (Vanishing cycles formula). Let \( f : Y \to X \), \( X \) not necessarily \( \mathbb{P}^1 \), be a \( G \)-Galois cover as in (2.3) where \( G \) has a cyclic \( p \)-Sylow subgroup. As in (2.3) there is a smooth model \( X_R \) of \( X \) where the specializations of the branch points do not collide, \( f \) has bad reduction, and \( f : \overline{Y} \to \overline{X} \) is the stable reduction of \( f \). Let \( \Pi \) be the set of branch points of \( f \) that have branching index divisible by \( p \). Let \( B_{\text{new}} \) be an indexing set for the new étale tails and let \( B_{\text{prim}} \) be an indexing set for the primitive tails. Let \( B_{\text{et}} = B_{\text{new}} \cup B_{\text{prim}} \). Let \( g_X \) be the genus of \( X \). Then we have the formula

\[
2g_X - 2 + |\Pi| = \sum_{b \in B_{\text{et}}} (\sigma_b - 1). \tag{3.4}
\]

Theorem 3.14 has the immediate corollary:

Corollary 3.15. Assume further that \( f \) is a three-point cover of \( \mathbb{P}^1 \). Then

\[
1 = \sum_{b \in B_{\text{new}}} (\sigma_b - 1) + \sum_{b \in B_{\text{prim}}} \sigma_b. \tag{3.5}
\]

Proof (of the theorem, cf. [Wew03b, Corollary 1.11]). Let \( G \) (resp. \( G' \)) be the dual graph (resp. augmented dual graph) of the stable reduction of \( f \). For any collection of vertices \( H \subseteq V(G) \) containing \( v_0 \), let \( B(H) \) be the collection of edges \( e \in E(G') \) such that \( s(e) \in H \), but \( t(e) \not\in H \). We define \( F(H) = \sum_{e \in B(H)} (\sigma_e^{\text{eff}} - 1) \) (see Definition 3.10). Then, if \( H = \{ v_0 \} \), \( F(H) = 2g_X - 2 \) by
By outward induction, using Lemmas 3.12 and 3.11 (i), we can add one adjacent vertex at a time to $H$ without changing the value of $F(H)$, so long as the vertex corresponds to an inseparable component. Thus, if $H = V(G_0)$ (p. 14), then $F(H) = 2g_X - 2$. By Lemma 3.11 (iii), we obtain

$$\sum_{b \in B_{\alpha}} (\sigma_b - 1) - |\Pi| = 2g_X - 2.$$  

Remark 3.16. One can also construct a proof analogous to that of [Ray99, 3.4.2 (5)], using the auxiliary cover. This is done in the author’s thesis [Obu09, §3.1].

The above formula can be generalized. For every $i, 1 \leq i \leq n$, write $\Pi_i$ for the set of branch points of $f$ which have branching index divisible by $p^i$. Let $B_{r,r'}$, $r > r'$, be as in Definition 2.18 each $b \in B_{r,r'}$ corresponding to a point $x_b$. Then we have the following formula:

**Proposition 3.17.** Fix $\alpha$ such that $0 \leq \alpha \leq n - 1$ and there exists some nonempty $\mathcal{B}_{r,r'}$ with $r' \leq \alpha < r$. Let $\mathcal{B}^0_{r,r'} \subset \mathcal{B}_{r,r'}$ be the subset consisting of those $b$ such that the vertex corresponding to the $p^r$-component containing $x_b$ is a maximal vertex for $G_\alpha$ (with ordering induced from $G'$, see p. 14). Then

$$2g_X - 2 + |\Pi_{\alpha+1}| \geq \sum_{b \in \mathcal{B}^0_{r,r'}} \sum_{\alpha < r'} (\sigma_b^\alpha - 1).$$  

(3.6)

If $\overline{f}$ is monotonic, we have equality in (3.6).

Proof. Let $U_i$, $i \in I$, be the set of connected components of $G_\alpha$. For each $i \in I$, let $B(U_i) \subset E(G')$ be the set of those edges $e$ such that $e(\alpha) \in V(U_i)$ but $e(\beta) \notin V(U_i)$. Define $F(U_i) = \sum_{e \in B(U_i)} (\sigma^\alpha_{e,\alpha} - 1)$. Then, as in the proof of Theorem 3.14, we have $F(U_i) = 2g_X - 2$ if $v_0 \in U_i$, and $F(U_i) = -2$ otherwise.

Set $\delta = 1$ if $v_0 \in V(G_\alpha)$, and $\delta = 0$ otherwise. Then we know

$$\sum_{i \in I} \sum_{e \in B(U_i)} (\sigma^\alpha_{e,\alpha} - 1) = \sum_{i \in I} F(U_i) = -2|I| + 2\delta g_X.$$  

(3.7)

Lemma 3.11 (ii) shows that, for $e \in B(U_i)$ corresponding to $b$ in some $\mathcal{B}^0_{r,r'}$, we have $\sigma^\alpha_{e,\alpha} = \sigma_b^\alpha$. In any case, for $e \in B(U_i)$, Lemma 3.11 (ii)–(iii) shows that $\sigma^\alpha_{e,\alpha} \geq 0$, with equality holding iff $t(e) \in G' \setminus G$. Also, an easy combinatorial argument shows that

$$|\bigcup_{i \in I} B(U_i)| - |\bigcup_{r' \leq \alpha < r} \mathcal{B}^\alpha_{r,r'}| = |\Pi_{\alpha+1}| + 2|I| - 2.$$  

We expand out $F(U_i)$ in (3.7), using the inequality $\sigma^\alpha_{e,\alpha} - 1 \geq -1$ for those $e \in B(U_i)$ not corresponding to elements of $\mathcal{B}^\alpha_{r,r'}$. This yields

$$\left( \sum_{r,r'} \sum_{b \in \mathcal{B}^\alpha_{r,r'}} (\sigma_b^\alpha - 1) \right) - (|\Pi_{\alpha+1}| + 2|I| - 2) \leq -2|I| + 2g_X,$$

with equality iff $\delta = |I| = 1$, i.e., $G'_\alpha$ is connected. A simple rearrangement yields Equation (3.6).

Lastly, if $\overline{f}$ is monotonic, then clearly $\delta = |I| = 1$, so we have equality.

Remark 3.18. The case $\alpha = 0$ of the generalized vanishing cycles formula (3.6) is the vanishing cycles formula (3.4).

4. Properties of tails of the stable reduction

We maintain the assumptions of [2.3] along with the assumption that a $p$-Sylow subgroup of $G$ is cyclic of order $p^n$. Throughout, we will use the abbreviation $m = m_G$, as well as the notations $B_{r,r'}$.
and the variants on $\sigma_{b}$ from Definition 2.18

Lemma 4.1. If $b \in B_{r, r'}$, indexes an inseparable tail $\overline{X}_{b}$ (so $r' > 0$), then all $\sigma_{b}$’s ($r' \leq \alpha < r$) are integers. In particular, $\sigma_{b} \in \mathbb{Z}$.

Proof. Consider an irreducible component $Y$ of $\overline{Y}$ lying above $\overline{X}_{b}$. By Corollary 2.12, its decomposition group $D_{Y}$ has a normal subgroup of order $p$. Thus, by Corollary 2.1(i), there exists $N < D_{Y}$ with $p \nmid |N|$ such that $D_{Y}/N$ is of the form $\mathbb{Z}/p^{a} \rtimes \mathbb{Z}/\ell$ (for some $a > r'$ and $\ell$ with $p \nmid \ell$). Now, by Remark 2.14 if $y$ is a ramification point of $f$ specializing to $\overline{y}$ on $Y$, then the inertia group of $\overline{y}$ in $Y_{b} \to \overline{X}_{b}$ is cyclic of order $p^{r'}$, with $p \nmid s$. Then the inertia group of the image of $\overline{y}$ in $Y_{b}/N \to \overline{X}_{b}$ is also cyclic. Since $p \nmid |N|$, the order of this inertia group must be $p^{r'}$.

Let $H$ be the unique subgroup of $\text{Aut}((\overline{Y}_{b}/N)/\overline{X}_{b})$ of order $p^{a}$. Then, by Proposition 2.10 the map $(\overline{Y}_{b}/N)/H \to \overline{X}_{b}$ can only be ramified at the intersection point $x_{b}$ of $\overline{X}_{b}$ with the rest of $\overline{X}$. Thus it is a trivial cover, and we see that $\ell = 1$. So $Y_{b}/N \to \overline{X}_{b}$ is a cyclic cover of order $p^{\alpha}$, branched at $y_{b}$. By the Hasse-Arf Theorem ([Ser79, V, Theorem 1]), the upper jumps of higher ramification at $y_{b}$ are all integral. Since the upper numbering is invariant under quotients, we conclude that $\sigma_{b}$ is integral for all $r' \leq \alpha < r$. □

Lemma 4.2. (i) A new tail $\overline{X}_{b}$ (étale or inseparable) has $\sigma_{b} \geq 1 + 1/m$.

(ii) If $r' > 0$, and $\overline{X}_{c}$ is any $p'$-tail that borders a $p'$-component, then $\sigma_{b} \geq p'^{r'-1}$.

(iii) If $\overline{X}_{b}$ is a primitive étale tail that borders a $p'$-component, then $\sigma_{b} \geq p^{r'-1}/m$.

Proof. For part (i), assume that $\overline{X}_{b}$ is a $p'$-component. If we assume that (i) is false, then either $\overline{X}_{b}$ borders a $p'$-component with $r - r' \geq 2$ or [Pri06, Lemma 1.1.6] shows that each irreducible component above $\overline{X}_{b}$ is a genus zero Artin-Schreier cover of $\overline{X}_{b}$. In the first case, we cite [Pri06, Lemma 19], which shows that $\sigma_{b} \geq p\alpha^{a+1}$ for all $\alpha$ where the statement makes sense. Then $\sigma_{b} := \sigma_{b}^{'r'} \geq p^{r'-1}/m$, so $p^{r'-1}/m \geq 1 + 1/m$, as $m|(p - 1)$. In the second case, since no ramification points specialize to the components over $\overline{X}_{b}$, this contradicts the three-point condition of the stable model.

For parts (ii) and (iii), we have $\sigma_{b} = \sigma_{b}^{'r'}$. If $\overline{X}_{b}$ is inseparable, then $1 \leq \sigma_{b}^{r'-1} \in \mathbb{Z}$ (by Lemma 4.1). So $\sigma_{b}^{r'-1} \geq p^{r'-1}$ (again, using [Pri06, Lemma 19]), proving (ii). Also, $\sigma_{b}^{r'-1} \geq 1/m$ for $\overline{X}_{b}$ étale and primitive, by Lemma 2.20. Then (iii) follows using [Pri06, Lemma 19]. □

Remark 4.3. Lemma 4.2 (i) and (iii) show that all terms on the right-hand side of the vanishing cycles formula (3.5) are positive.

We now give some sufficient criteria for the stable reduction of $f$ to be monotonic.

Proposition 4.4. For any $G$, if $\overline{T}$ is a component of $\overline{X}$ such that there are no étale tails $\overline{X}_{b} \succ \overline{T}$, then $\overline{f}$ is monotonic from $\overline{T}$.

Proof. Suppose $\overline{T}$ is a $p^{i}$-component, and there are no étale tails lying outward from $\overline{T}$. For a contradiction, suppose $t \in \overline{T}$ is a point such that there exists a $p^{i}$-component $\overline{W}$, with $j > i$, lying outward from $t$. Consider the morphism $X^{st} \to X'$ that is “the identity” on $X_{K^{st}}$ and contracts $\overline{U}$, the union of all components of $\overline{X}$ outward from $t$. If $Y'$ is the normalization of $X'$ in $K^{st}(Y)$, then $Y'$ is obtained from $X^{st}$ by contracting all of the components of the special fiber above those components contracted by $X^{st} \to X'$. Let $y$ be a point of $Y'$ lying over the image of $t$ in $X'$ (which we call $t$, by abuse of notation), and consider the map of complete germs $Y_{y} \to \overline{X}_{t}$. This map is Galois. Its Galois group $G'$ contains the decomposition group of the connected component of $\overline{f}^{-1}(\overline{U})$ containing a preimage of $y$ in $\overline{Y}$. By Corollaries 2.1(i) and 2.12, there exists $N \leq G'$ such that $G'/N \cong \mathbb{Z}/p^{a} \rtimes \mathbb{Z}/\ell$, with $p \nmid \ell$ (this is the only place we use the assumption that there are no étale tails lying outward from $\overline{T}$). Also, $a \geq j > i$ by our assumption on $t$. 17
Let \( \tilde{V}_v \) be the quotient of \( \tilde{Y}_y \) by the subgroup of \( G' \) that contains \( N \) and whose image in \( G/N \) has order \( p^i \). Then \( \phi : \tilde{V}_v \to \tilde{X}'_y \) is Galois with Galois group \( \mathbb{Z}/p^{a-i} \rtimes \mathbb{Z}/\ell \). Note that \( t \) is a smooth point of \( X' \), as we have contracted only a tree of projective lines (of course, \( y \) may be quite singular, but it is still a normal point of \( Y' \)).

Now, \( \phi \) is totally ramified above the point \( t \), but it is unramified above the height 1 prime \( (\pi) \), where \( \pi \) is a uniformizer of \( R \), because we have quotiented out the generic inertia of \( \mathbb{T} \). Using purity of the branch locus ([Szabó 09, Theorem 5.2.13]), we see that \( \phi \) must be ramified over some height 1 prime \( (u) \) such that the scheme cut out by \( u \) intersects the generic fiber. Since we have been assuming from the beginning that the branch points of \( Y_K \to X_K \) do not collide on the special fiber \( \overline{X}_0 \), and we have not contracted \( \overline{X}_0 \), there is at most one branch point on the generic fiber that can specialize to \( t \). Thus \( (u) \) cuts the generic fiber in exactly one point, and it is the only height 1 prime above which \( \phi \) is ramified. So \( \phi \) is étale outside of the scheme cut out by \( (u) \). We are now in the situation of [Raynaud 94, Lemme 6.3.2], and we conclude that the ramification index at \( (u) \) is prime to \( p \). But this contradicts the fact that the ramification index above \( t \) is divisible by \( p^{a-i} \). \( \square \)

**Remark 4.5.** The proof above shows that, if \( G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m \), then \( \overline{f} \) is monotonic from \( \mathbb{T} \), even if there are étale tails lying outward from \( \mathbb{T} \).

For the rest of this section, assume that \( f : Y \to X \) is a three-point cover of \( \mathbb{P}^1 \) with bad reduction.

**Proposition 4.6.** The stable reduction \( \overline{X} \) has fewer than \( p \) étale tails. For \( d \geq 1 \), the number of \( p^d \)-tails of \( \overline{X} \) is less than \( p^d \).

**Proof.** We proceed by strong induction on \( d \), proving the base cases \( d = 0 \) (i.e., the étale tails) and \( d = 1 \) separately.

For \( d = 0 \), Equation (3.5) gives

\[
1 = \sum_{b \in \mathcal{B}_{\text{new}}} (\sigma_b - 1) + \sum_{b \in \mathcal{B}_{\text{prim}}} \sigma_b.
\]

Each term on the right-hand side above is at least \( 1/m \), by Lemma 4.2. So there are at most \( m \) étale tails. Since \( m \mid (p - 1) \), the case \( d = 0 \) is proved.

For \( d = 1 \), consider (3.6) for \( \alpha = 1 \). In the notation of (3.6), let \( \mathcal{B}' \subset \bigcup_{r>1} B_{r,1} \) be the set of tails which are \( p \)-components and which contain the specialization of a branch point of \( f \). Then we obtain

\[
1 \geq \sum_{b \in \bigcup_{r>1} B_{r,1}^{B} \setminus B'} (\sigma_b - 1) + \sum_{b \in B'} \sigma_b.
\]

We note that the number of points indexed by \( \bigcup_{r>1} B_{r,1}^{B} \) that do not lie on a \( p \)-tail is bounded by the number of étale components, i.e., there can be no more than \( p - 1 \) of them (this is because each such point lies on a \( p \)-component that has an étale tail lying outward from it, and two such \( p \)-components do not share the same étale tail). On the right-hand side of (4.1), each term corresponding to such a \( p \)-component contributes at least \( \frac{1}{m} \), thus at least \( \frac{1}{p-1} \), at least \( \frac{2}{p-1} \), At least \( \frac{2}{p-1} \). By Lemma 4.2 (ii), each other term on the right-hand side of (4.1) is nonnegative. Also, each tail which is a \( p \)-component corresponds to a term on the right-hand side of (4.1), and each such term is at least \( 1 \), by Lemma 4.2. Thus the right-hand side is at least \( \frac{2}{p-1} (p - 1) + \nu \), where \( \nu \) is the number of tails which are \( p \)-components. We conclude from (4.1) that \( \nu \leq p - 1 \), proving the case \( d = 1 \).

Now, assume the lemma holds up through \( d = \delta \). The number of \( p^{\delta+1} \)-components \( \bigcup_{r>\delta} B_{r,\delta}^{B} \) which are not tails is bounded by the number of tails which are \( p^a \)-components for some \( a \leq \delta \). By
the inductive hypothesis, this is bounded by
\[ M = (p - 1) + (p - 1) + (p^2 - 1) + \cdots + (p^\delta - 1) = \frac{p^{\delta+1} - 1}{p - 1} + p - \delta - 2. \]

Some calculation shows that this is less than \((p^{\delta+1} - 1)p^{-1}/p - 2\) for \(p > 2\).

Analogously to the case of \(d = 1\), Equation (3.6) for \(\alpha = \delta + 1\) yields the inequality
\[ 1 \geq \frac{2 - p}{p - 1} M + \nu > 1 - p^{\delta+1} + \nu, \]
where \(\nu\) is the number of tails which are \(p^{\delta+1}\)-components (if \(p = 2\), the second inequality holds without any condition on \(M\)). We conclude that \(\nu < p^{\delta+1}\).

**Corollary 4.7.** For any \(\mu \in \mathbb{R}, d > 0\), let \(S_{d,\mu}\) be the set of \(p^d\)-tails \(\overline{X}_b\) that satisfy \(\sigma_b - 1 \geq p^\mu\). Then the cardinality of \(S_{d,\mu}\) is less than \(p^{d-\mu}\).

**Proof.** We carry the proof of Proposition 4.6 through. In the \(d = 1\) step, if we let \(\nu = |S_{1,\mu}|\) (as opposed to the total number of \(p\)-tails), then we obtain from (4.1) that \(p^\mu \nu < p - 1\). This gives the corollary for \(d = 1\). In the inductive step, we set \(\nu = |S_{d+1,\mu}|\) as opposed to the total number of \(p^{d+1}\)-tails. Again, we conclude that \(p^\mu \nu < p^{d+1}\), which gives the corollary. \(\square\)

The following proposition will be useful for Example 5.12.

**Proposition 4.8.** Suppose \(\overline{X}\) has no new étale tails. Then it has no new inseparable tails.

**Proof.** Assume, for a contradiction, that \(i \geq 1\) is minimal such that there is a new \(p^i\)-tail \(\overline{X}_b\).

Applying (3.6) for \(\alpha = i\) yields (in the notation of (3.6))
\[-2 + |\Pi_{i+1}| \geq \sum_{r,r' \leq i} \sum_{b \in B_{r,r'}} (\sigma_b^i - 1).\]

On the right-hand side, using Proposition 4.4, all terms \(b\) that do not correspond to \(p^i\)-tails correspond to points with primitive étale tails lying outward from them. There are at most \(3 - |\Pi_{i+1}|\) such tails (and at least 1, by the vanishing cycles formula (3.5)), so there are at most \(3 - |\Pi_{i+1}|\) such \(b\), for which \(\sigma_b^i - 1 > -1\) in each case. By Lemmas 4.1 and 4.2, the \(b\) corresponding to new \(p^i\)-tails (of which there is at least 1) satisfy \(\sigma_b^i - 1 \geq 1\) and those corresponding to other \(p^i\)-tails satisfy \(\sigma_b^i - 1 \geq 0\). Putting this all together, we see that the right-hand side is strictly greater than \(-(3 - |\Pi_{i+1}|) + 1 = -2 + |\Pi_{i+1}|\). This is a contradiction. \(\square\)

5. Wild Monodromy and Stable Reduction

5.1 The main theorem

We maintain the assumptions and notations of 2.3. In particular, \(G\) is a finite group with cyclic \(p\)-Sylow subgroup \(P\) of order \(p^n\), and \(m_G = |N_G(P)/Z_G(P)|\). Assume \(n \neq 0\) (so that \(p\) divides the order of \(G\)). We make the additional (important!) assumption that \(p\) does not divide the order of the center of \(G\). Let \(K_0 = \text{Frac}(W(k))\), where \(k\) is an algebraically closed field of characteristic \(p\). Let \(f : Y \to X = \mathbb{P}^1\) be a three-point \(G\)-cover defined over a finite extension \(K/K_0\). Write \(K^{st}/K\) for the smallest extension of \(K\) over which the stable model of \(f\) can be defined. Then \(\Gamma := \text{Gal}(K^{st}/K)\) is called the monodromy group, and its (unique) \(p\)-Sylow subgroup \(\Gamma_w\) is called the wild monodromy group. Recall from 2.3 that \(\Gamma\) is the largest quotient of \(G_K\) that acts faithfully on the stable reduction \(\overline{f} : \overline{Y} \to \overline{X}\) of \(f\). So \(\Gamma\) acts on \(\overline{Y}\) and the action descends to an action on \(\overline{X}\). Furthermore, the action of \(\Gamma\) commutes with the action of \(G\). Theorem 1.1 states that \(\Gamma_w\) has exponent dividing \(p^{n-1}\). In other words, for any \(g \in \Gamma_w\), \(g^{p^{n-1}} = 1\).
The proof of Theorem 1.1 which is spread out over \[5.2\] relies on methods similar to those used in \[Ray99\]. The main idea is to examine possible $p$-power order actions on the stable reduction of $f$ in detail, and to show that actions of order $p^n$ cannot be induced by elements of $\Gamma_w$. Our main tools are the generalized vanishing cycles formula \[3.6\] and Proposition 4.6.

5.2 Proof of Theorem 1.1

Assume first that $f$ has potentially good reduction. Then, by \[Ray99\], Proposition 4.2.2], the wild monodromy group $\Gamma_w$ is isomorphic to a subgroup of the $p$-Sylow subgroup $Q$ of the center of $G$, which is trivial by assumption.

Now assume that $f$ does not have potentially good reduction. We will use the notations $B_{r,r'}$, $J_x$ as well as $\sigma^w_{x_b}$ and its variants from Notation 2.17 and Definition 2.18.

We study how $\Gamma_w$ acts on different parts of $\overline{Y}$ and $\overline{X}$. We start with an easy lemma:

**Lemma 5.1.** If $\gamma \in \Gamma_w$ acts on a component $\overline{W}$ of $\overline{X}$, then it fixes pointwise any component $\overline{W}' \prec \overline{W}$ of $\overline{X}$.

**Proof.** Since $k$ is algebraically closed, all elements of $G_K$ commute with the reduction from $R$ to $k$. Thus $\Gamma_w$, which is a subquotient of $G_K$, acts trivially on $\overline{X}_0$, which is the reduction of the standard model of $\mathbb{P}^1_R$ to $k$. So we may assume $\overline{W} \neq \overline{X}_0$. By continuity, $\gamma$ fixes the singular points of $\overline{X}$ lying on $\overline{W}$ in the directions of $\overline{X}_0$ and $\overline{W}$. Then $\gamma$ acts on $\overline{W} \cong \mathbb{P}^1$ with at least two fixed points and $p$-power order. So $\gamma$ acts trivially. 

We will look separately at the action of $\gamma$ on the étale tails, and then on the inseparable tails.

5.2.1 The étale tails. We first examine how the action of $\Gamma_w$ interacts with the étale tails.

**Lemma 5.2.** The action of $\Gamma_w$ permutes the étale tails of $\overline{X}$ trivially.

**Proof.** (cf. \[Ray99\] p. 112) By Proposition 4.6 there are at most $p-1$ étale tails. But $\Gamma_w$ is a $p$-group, and thus must permute them trivially. \[\square\]

**Corollary 5.3.** The group $\Gamma_w$ acts trivially on the components $\overline{W}$ of $\overline{X}$ such that there exists an étale tail $\overline{X}_b \succ \overline{W}$.

**Proof.** This follows from Lemmas 5.1 and 5.2. \[\square\]

**Lemma 5.4.** Let $\overline{Y}_b$ be a component of $\overline{Y}$ lying above an étale tail $\overline{X}_b$ of $\overline{X}$. If $\gamma \in \Gamma_w$ acts trivially on $\overline{X}_b$ and acts on $\overline{Y}_b$, then it acts trivially on $\overline{Y}_b$.

**Proof.** Write $D := D_{\overline{Y}_b}$, the decomposition group of $\overline{Y}_b \to \overline{X}_b$. Suppose that $\gamma$ acts nontrivially on $\overline{Y}_b$. Then we can view $\gamma$ as an element of the center of $D$, and replacing $\gamma$ with some power, we may assume that $\gamma$ has order $p$ in $D$. Let $Q \subseteq D$ be the central subgroup of order $p$ generated by $\gamma$. We will show that $Q$ is central in $G$, contradicting the running assumption of this section.

Corollary 2.4 (ii) shows that $m_D = 1$. By Lemma 2.20 we have $\sigma_b \in \mathbb{Z}$ for $b$ corresponding to $\overline{X}_b$. By the vanishing cycles formula, we have $\sigma_b = 1$ if $\overline{X}_b$ is primitive and $\sigma_b = 2$ if $\overline{X}_b$ is new. By Lemma 4.2 $\overline{X}_b$ borders a $p$-component, unless $p = 2$, in which case $\overline{X}_b$ can border a $p^2$-component if it is new. In all cases, \[5.5\] shows that $\overline{X}_b$ is the only étale tail.

We claim that $\overline{Y}_b \to \overline{X}_b$ is totally ramified above the singular point $\overline{x}_b$ of $\overline{X}$ on $\overline{x}_b$. To prove the claim, assume first that $\overline{X}_b$ borders a $p$-component. Since $Q$ is central in $D$, it is contained in every $p$-subgroup of $D$, in particular, every wild inertia group of a wildly ramified point of $\overline{Y}_b$. By Proposition 2.10 (ii), $Q$ is the $p$-Sylow subgroup of all these inertia groups, and thus $h : \overline{Y}_b/Q \to \overline{X}_b$ is tamely ramified, branched at at most two points. So $h$ is a cyclic, totally ramified cover of degree
\(|D/Q|\), and \(\overrightarrow{Y_b}/Q\) has genus zero. Therefore, \(\overrightarrow{Y_b} \to \overrightarrow{Y_b}/Q\) is an Artin-Schreier cover, totally ramified at its unique ramification point, proving the claim in this case.

Now, assume \(\overrightarrow{X_b}\) borders a \(p^2\)-component (so \(p = 2\) and \(\overrightarrow{X_b}\) is new). By [PT06] Lemma 19, \(\sigma_b^1 = 1\). Again, \(Q\) is contained in every inertia group, so \(h : \overrightarrow{Y_b}/Q \to \overrightarrow{X_b}\) is a \(D/Q\)-cover with inertia groups with 2-Sylow subgroup of order 2. But \(m_{D/Q} = 1\), so \(D/Q\) has a subgroup \(N\) of odd order such that \(|(D/Q)/N|\) is cyclic of 2-power order (Lemma 2.2). Since the inertia groups of \(j : (\overrightarrow{Y_b}/Q)/N \to \overrightarrow{X_b}\) have order 2, we must have that \(|(D/Q)/N| = 2\), and then \(j\) is totally ramified above \(\overrightarrow{Y_b}\). Since \(\sigma_1 = 1\), a Hurwitz formula calculation shows that the genus of \((\overrightarrow{Y_b}/Q)/N\), being branched at only one point with conductor 1, is zero. This implies that \(N\) is trivial, as \(\overrightarrow{Y_b}/Q \to (\overrightarrow{Y_b}/Q)/N\) is a prime-to-2 cover of \(\mathbb{P}^1\) branched at one point. So \(|D| = 4\), and we have total ramification above the singular point of \(\overrightarrow{X}\). The claim is proved.

Our next claim is that \(Q\) is normal in \(G\). To show this, we note that because \(\overrightarrow{Y_b} \to \overrightarrow{X_b}\) is totally ramified above \(\overrightarrow{Y_b}\), then \(\overrightarrow{Y_b}/\overrightarrow{Y_b} = \overrightarrow{Y_b}\) is still connected. Clearly, we may remove all of the other components above \(\overrightarrow{X_b}\) while preserving this connectivity. Since \(G\) acts on what remains of \(\overrightarrow{Y}\), and all of the remaining components have nontrivial inertia, then Corollary 2.12 shows that \(G\) has a normal subgroup of order \(p\). But since \(G\) has cyclic \(p\)-Sylow subgroup, it can only have one subgroup of order \(p\), which must be \(Q\). The claim is proved.

Lastly, we show that \(Q\) is central in \(G\). Now, since \(\overrightarrow{X_b}\) is the only étale tail, there is at most one branch point of \(f : Y \to X\) with prime-to-\(p\) branching index (it exists iff \(\overrightarrow{X_b}\) is primitive). By Corollary 2.3 (i), there exists a subgroup \(N \leq G\) such that \(p \nmid |N|\) and \(G/N \cong \mathbb{Z}/p^n \times \mathbb{Z}/m_G\), with faithful action. So \(Y/N \to X\) also has at most one branch point with prime-to-\(p\) branching index. Let \(P \leq G/N\) be the unique \(p\)-Sylow subgroup. Then \((Y/N)/P \to X\) is a \(\mathbb{Z}/m_G\)-Galois cover, namely ramified at at most one point. Thus it is trivial, \(m_G = 1\), and \(Q\) maps isomorphically onto its image \(Q/N\) in \(G/N \cong \mathbb{Z}/p^n\). The conjugation action of \(G\) on \(Q/N\) is well defined, and clearly trivial. Hence, the action of \(G\) on \(Q/N\) is trivial, so the action of \(G\) on \(Q\) is trivial. This completes the proof of the lemma.

**Lemma 5.5.** Let \(\overrightarrow{X_b}\) be an étale tail of \(\overrightarrow{X}\) intersecting the rest of \(\overrightarrow{X}\) at \(x_b\), and let \(\overrightarrow{Y_b}\) be a component of \(\overrightarrow{Y}\) lying above \(\overrightarrow{X_b}\). Write \(\Sigma\) for the set of singular points of \(\overrightarrow{Y}\) lying on \(\overrightarrow{Y_b}\). Suppose that \(\gamma \in \Gamma_w\) acts nontrivially on \(\overrightarrow{Y_b}\). Then \(\gamma\) acts with order \(p\) on \(\overrightarrow{Y_b}\), and fixes no point of \(\Sigma\).

**Proof.** By Lemma 5.3, we see that the action of \(\gamma\) on \(\overrightarrow{Y_b}\) descends faithfully to \(\overrightarrow{X_b}\). If \(\overrightarrow{X_b}\) is a primitive tail, then \(\gamma\) fixes two points on the tail and thus acts trivially, so we may assume \(\overrightarrow{X_b}\) is a new tail. Since \(\overrightarrow{X_b} \cong \mathbb{P}^1\), the action of \(\gamma\) on \(\overrightarrow{X_b}\) has order \(p\). Now, suppose \(\gamma\) fixes a point \(y_b \in \overrightarrow{Y_b}\) that is a singular point of \(\overrightarrow{Y}\), lying above \(x_b \in \overrightarrow{X}\). Consider the cover \(\overrightarrow{Y_b} \to \overrightarrow{X_b}\) given by quotienting \(\overrightarrow{Y_b} \to \overrightarrow{X_b}\) by the group generated by \(\gamma\). Let \(\sigma'_b\) be the corresponding effective ramification invariant for \(\overrightarrow{Y_b} \to \overrightarrow{X_b}\). Since \(\overrightarrow{X_b} \to \overrightarrow{X}\) is an Artin-Schreier extension with conductor 1, Lemma 2.6 (with \(r = 1\)) shows that \(\sigma_b - 1 = p(\sigma'_b - 1)\). But \(\sigma'_b \neq 1\), because if \(\sigma'_b = 1\) then \(\sigma_b = 1\), and we know \(\sigma_b > 1\) (Lemma 4.2). So \(\sigma'_b > 1\) ([Ray99] Lemme 1.1.6), and in particular, \(\sigma'_b \geq 1 + \frac{1}{m} \geq 1 + \frac{1}{p-1}\) (Lemmas 2.5 and 2.1). So \(\sigma_b - 1 = p(\sigma'_b - 1) \geq \frac{p}{p-1} > 1\), which contradicts the vanishing cycles formula (4.5). \(\square\)

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5.2.2 The inseparable tails. Suppose \(\overrightarrow{X_b}\) is an inseparable tail whose intersection \(x_b\) with the rest of \(\overrightarrow{X}\) is in \(B_{r,r'}\) (by Lemma 2.16 \(r > r'\)). Pick any component \(\overrightarrow{Y_b}\) lying above \(\overrightarrow{X_b}\). Let \(y_b\) be a point of intersection of \(\overrightarrow{Y_b}\) with the rest of \(\overrightarrow{Y}\). Write \(Q\) for a \(p\)-Sylow subgroup of \(\Gamma_w\) which fixes \(y_b\) and acts faithfully on \(\overrightarrow{Y_b}\). If \(D\) contains an element \(\gamma\) such that \(\gamma^p\) generates \(Q\), then we set \(e_b = 1\) (this implies that
The second lower jump of wild ramification of \( \hat{v}_b \to \hat{x}'_b \) is equal to the lower jump of wild ramification of \( \hat{v}_b \to \hat{x}_b \), because the lower numbering respects subgroups. Since the wild ramification of \( \hat{v}_b \to \hat{x}_b \) is of order \( p \), this is the same as the upper jump, which is what we wish to calculate. Now, the first upper jump of \( \hat{v}_b \to \hat{x}'_b \) is 1, because \( \hat{X}_b \to \hat{X}'_b \) is an Artin-Schreier cover of conductor 1. By [Pri06 Lemma 4.2], the second upper jump of wild ramification of \( \hat{v}_b \to \hat{x}'_b \) is at least \( p \), and thus, from the definition of the upper numbering, the second lower jump is at least \( p^2 - p + 1 \). So \( \sigma_b^{-1} \geq p^2 - p + 1 \geq p + 1 \), which proves the lemma.

5.2.3 The Wild Monodromy Group. Finally, we prove Theorem 1.1 i.e., we show that for every \( \gamma \in \Gamma_w \), \( \gamma^{p^{n-1}} = 1 \). Since \( \Gamma_w \) acts faithfully on \( \overline{Y} \), it suffices to show that, for an arbitrary point \( y \in \overline{Y} \) and an arbitrary \( \gamma \in \Gamma_w \), we have \( \gamma^{p^{n-1}}(y) = y \).

Let us first assume that \( y \) is a point of \( \overline{Y} \) which lies on an irreducible component \( \overline{Y}_b \) above an étale tail \( \overline{X}_b \). By Lemma 5.2, the tail \( \overline{X}_b \) is acted on by \( \gamma \). By Lemma 2.16, \( \overline{X}_b \) intersects an inseparable component, and thus the inertia groups of \( \overline{f} \) above the point of intersection are of order divisible by \( p \). So the number of irreducible components \( N \) lying above \( \overline{X}_b \) is divisible at most by \( p^{n-1} \), not by \( p^n \). Thus \( \gamma^{p^{n-1}}(y) \in \overline{Y}_b \). Furthermore, if any \( \gamma \in \Gamma_w \) acts nontrivially on \( \overline{Y}_b \), then Lemma 5.3 shows that \( \gamma \) acts with order \( p \) on \( \overline{Y}_b \), and \( p \) divides the number of intersection points of \( \overline{Y}_b \) with the rest of \( \overline{Y} \). In this case, \( v_p(N) \leq n - 2 \), and \( \gamma^{p^{n-2}}(y) \in \overline{Y}_b \). In any case, \( \gamma^{p^{n-1}}(y) = y \).

Now, assume \( y \) is a point of \( \overline{Y}_b \), where \( \overline{Y}_b \) is an irreducible component of \( \overline{Y} \) lying above an inseparable \( p^{r'} \)-tail \( \overline{X}_b \) intersecting a \( p^{r'} \)-component. Let \( \epsilon_b = 0 \) or 1 as in Lemma 5.6. If \( \overline{X}_b \) is new, then by Lemmas 4.1 4.2(i), and 5.6 we have \( \sigma_b - 1 \geq p^h \). Combining this with Corollary 4.7, we see that there are fewer than \( p^{r - \epsilon_b} - 1 \) tails of type \( (r, r') \) with the same \( \sigma_b \), and thus \( \gamma^{p^{r - \epsilon_b - 1}}(\overline{X}_b) = \overline{X}_b \). If \( \overline{X}_b \) is not new, then \( \gamma \) fixes \( \overline{X}_b \) and \( \epsilon_b = 0 \), so we also have \( \gamma^{p^{r - \epsilon_b - 1}}(\overline{X}_b) = \overline{X}_b \).

Suppose a \( p \)-Sylow subgroup \( Q \) of \( D_{\overline{X}_b} / \overline{Y}_b \) has order \( p^t \). Then there are at most \( p^{n-a-r} \) irreducible components lying above \( \overline{X}_b \). So \( \gamma' = \gamma^{p^{n-a-\epsilon_b-1}} \) satisfies \( \gamma'(\overline{Y}_b) = \overline{Y}_b \). But, as was remarked
before Lemma 5.6, we must have \((\gamma')^p a+b(y) = y\). So \(\gamma^{p-1}(y) = y\), as we wished to prove.

Lastly, assume the remaining case, i.e., that \(y\) lies over some point \(x\) on an interior component \(\overline{W}\) of \(\overline{X}\). Suppose first that there exists an étale tail \(\overline{X}_b \supset \overline{W}\). Then, for any \(\gamma \in \Gamma_w\), Corollary 5.3 shows that \(\gamma\) fixes \(\overline{W}\) pointwise. Then \(\gamma\) acts on the fiber above \(x\). By Lemma 2.14, the component \(\overline{W}\) must be inseparable. Thus \(p^a\) does not divide the cardinality of the fiber above \(x\), so \(\gamma^{p^a-1}(y) = y\). Now suppose that there does not exist any étale tail lying outward from \(\overline{W}\). In this case, let \(\overline{X}_b \supset \overline{W}\) be a \(p^a\)-tail. Then \(\gamma^{p^a-1}(\overline{X}_b) = \overline{X}_b\), and \(\gamma^{p^a-1}\) acts on the fiber above \(x\). By Lemma 2.16 and Proposition 4.4, the generic inertia above \(\overline{W}\) has order divisible by \(p^a+1\), so \(p^a\) does not divide the cardinality of this fiber. Then \(\gamma^{p^a-1+n-1}(y) = \gamma^{p^a-2}(y) = y\), finishing the proof of Theorem 1.1.

5.3 Further Restrictions on the Wild Monodromy

We state here some stronger results than Theorem 1.1 which will be useful for Example 5.12. We maintain the notation of §5.

**Lemma 5.7.** Let \(\overline{W}\) be an inseparable component of \(\overline{X}\). Suppose there exists \(\gamma \in \Gamma_w\) that acts trivially on \(\overline{W}\), but non-trivially above \(\overline{W}\). Then, for any irreducible component \(\overline{V}\) of \(\overline{Y}\) above \(\overline{W}\) with decomposition group \(D_{\overline{V}} \leq G\), we have \(m_{D_{\overline{V}}} = 1\).

**Proof.** We may assume that the action of \(\gamma\) above \(\overline{W}\) is of order \(p\). Since the \(D_{\overline{V}}\) are conjugate for each \(\overline{V}\) above \(\overline{W}\), it suffices to prove the lemma for any one \(\overline{V}\). Let \(I_{\overline{V}} \leq D_{\overline{V}}\) be the inertia group of \(\overline{V}\). We have three cases to consider:

**Case (1): There exists \(\overline{V}\) above \(\overline{W}\) on which \(\gamma\) acts.**

We can think of \(\gamma\) as a (central) element of \(D := D_{\overline{V}} / I_{\overline{V}}\). Since \(\gamma\) is central in \(D\), Corollary 2.4(ii) shows that \(m_D = 1\). Since \(D\) is a quotient of \(D_{\overline{V}}\) by a \(p\)-group, it is easy to see that \(m_{D_{\overline{V}}} = m_D = 1\).

**Case (2a): There is no \(\overline{V}\) above \(\overline{W}\) on which \(\gamma\) acts, and \(p \nmid |D_{\overline{V}} / I_{\overline{V}}|\).**

Pick some \(\overline{V}\) above \(\overline{W}\), and some point \(y \in \overline{V}\) that is a smooth point of \(\overline{Y}\). By Proposition 2.10, the inertia group of \(y\) is \(I_{\overline{V}}\). Since \(p \nmid |D_{\overline{V}} / I_{\overline{V}}|\), then \(D_{\overline{V}} = I_{\overline{V}} \times H\), where \(H\) has prime-to-p order.

Now, \(\gamma\) fixes \(\overline{W}\), so there is some element \(g \in G\) such that \(g(y) = \gamma(y)\). Since \(\Gamma_w\) commutes with \(G\), this also implies \(g^a(y) = \gamma^a(y)\), for all \(a \in \mathbb{Z}\). We claim that \(g\) normalizes \(D_{\overline{V}}\). In fact, even more is true: \(ghg^{-1}(v) = h(v)\) for all \(h \in D_{\overline{V}}\) and \(v \in \overline{V}\). This is because \(ghg^{-1}(y) = gh\gamma^{-1}(y) = \gamma^{-1}gh(y)\), and \(\gamma^{-1}g\), being a Galois automorphism of \(\overline{V} \to \overline{W}\) with the fixed point \(y\), fixes \(\overline{V}\) pointwise. So \(ghg^{-1}(y) = h(y)\), and \(ghg^{-1}\) and \(h\), both being elements of \(D_{\overline{V}}\) which act the same way on \(y\), must act the same way on all \(v \in \overline{V}\). Thus, conjugation by \(g\) induces the identity on \(D_{\overline{V}} / I_{\overline{V}}\), and \(gHg^{-1}\) is a lift of \(H\) in \(D_{\overline{V}}\). Since \(H^1(H, I_{\overline{V}}) = 0\) by the Schur-Zassenhaus theorem, all such lifts differ only by conjugation by an element of \(I_{\overline{V}}\), so we have that there is some \(i \in I_{\overline{V}}\) such that conjugation by \(g\) and conjugation by \(i\) act identically on \(H\). In particular, \(gi^{-1}\) centralizes \(H\) and normalizes \(D_{\overline{V}}\) and \(I_{\overline{V}}\). By replacing our choice of \(g\) with \(gi^{-1}\), we may even assume that \(g\) centralizes \(H\).

In any case, we know that \(\gamma^p(y) = y\), so \(g^p(y) = y\). This implies \(g^p \in I_{\overline{V}}\). But \(g \notin I_{\overline{V}}\), so we must have that \(g^p\) generates \(I_{\overline{V}}\) (if not, then \(g\) and \(I_{\overline{V}}\) generate a non-cyclic \(p\)-group). Since \(g\) centralizes \(H\), so does \(g^p\). Then \(I_{\overline{V}}\) commutes with \(H\), and \(m_{D_{\overline{V}}} = 1\).

**Case (2b): There is no \(\overline{V}\) above \(\overline{W}\) on which \(\gamma\) acts, and \(p \mid |D_{\overline{V}} / I_{\overline{V}}|\).**

We show that this case does not arise. Take \(\overline{V}\), \(g\), and \(y\) as in Case (2a). As in Case (2a), we have that \(g\) normalizes both \(D_{\overline{V}}\) and \(I_{\overline{V}}\), that \(g\) centralizes \(D_{\overline{V}} / I_{\overline{V}}\), and that \(g^p \in I_{\overline{V}}\). Now, consider the group \(M \leq G\) generated by \(D_{\overline{V}}\) and \(g\). The subgroup \(I_{\overline{V}}\) is normal in \(M\), so let \(M' \cong M / I_{\overline{V}}\).
Then the image of $g$ has order $p$ in $M'$ and centralizes the image of $D_{\Gamma}$ in $M'$. But the image of $D_{\Gamma}$ in $M'$ has a nontrivial element $d$ of order $p$, and so the $p$-subgroup of $M'$ generated by $d$ and the image of $g$ is elementary abelian. This is a contradiction, as $G$ has cyclic $p$-Sylow group. □

**Lemma 5.8.** As in Lemma 5.7, let $W$ be an inseparable component of $X$. If there exists $\gamma \in \Gamma_w$ that acts trivially on $W$, but non-trivially above $W$, then for each singular point $w$ of $X$ on $W$, either no étale tail lies outward from $w$ or every étale tail lies outward from $w$.

**Proof.** We first claim that

$$\sigma_e^{\text{eff}} \in Z[\frac{1}{p}]$$

for $e \in E(G)$ such that $e$ corresponds to $w$ and $s(e)$ corresponds to $W$. Suppose that $e$ is such an edge. Write $W'$ for the component corresponding to $t(e)$, write $W$ for a component of $Y$ above $W'$ intersecting $V$, and pick $v \in \overline{V} \cap \overline{V'}$. Write $I_v$ for the inertia group in $G$ at $v$. By Lemma 5.7, we have $m_{D_{\Gamma}} = 1$. Since $I_v \subseteq D_{\Gamma}$, it follows that $m_{I_v} = 1$.

Assume that $I_{\Gamma}$ contains the generic inertia group of $I_{\Gamma'}$ in $G$. Then $\sigma_e^{\text{eff}}$ is defined using deformation data on $Z$, where $V \xrightarrow{\phi} Z \xrightarrow{\psi} X$ is such that $\phi$ is radical and $\psi$ is tamely ramified at $\phi(v)$. By Proposition 5.3, the invariants of all of these deformation data above $w$ are integers. Then the definition of the effective invariant $\sigma_e^{\text{eff}}$ at $w$ shows that it is in $Z[\frac{1}{p}]$. If, instead, $I_{\Gamma'}$ contains $I_{\Gamma}$, the same argument shows that $\sigma_e^{\text{eff}} \in Z[\frac{1}{p}]$, and so $\sigma_e^{\text{eff}} = -\sigma_e^{\text{eff}} \in Z[\frac{1}{p}]$. The claim is proved.

On the other hand, it is not hard to see from the effective local vanishing cycles formula (3.2) and Lemma 3.11 that

$$\langle \sigma_e^{\text{eff}} \rangle = \langle \sum_{b \in B_{\gamma}} \sigma_b \rangle.$$  

(5.1)

The right-hand side of (5.1) is in $\frac{1}{m_{\Gamma}} Z \subset \frac{1}{p-1} Z$. The vanishing cycles formula (3.5) shows that it is in $Z$ iff the right-hand side counts either no étale tails or every étale tail. Since $Z[\frac{1}{p}] \cap \frac{1}{p-1} Z = Z$, this must be the case.

The following proposition is the main result of this section:

**Proposition 5.9.** If the branching indices of $f$ are prime to $p$ and $X$ has no new étale tails, then the wild monodromy $\Gamma_w$ is trivial.

**Proof.** By Proposition 4.8, there are now new inseparable tails. By Proposition 2.13, there are no inseparable tails at all. So all tails are primitive étale. Pick $\gamma \in \Gamma_w$. Then $\gamma$ fixes $X$ pointwise (it fixes the interior components by Corollary 5.3, and it fixes the primitive tails because it fixes two points on them). Now, since every tail is primitive, we see that each singular point of $X$ has exactly one of the three étale tails lying outward from it. So $\gamma$ acts trivially above the inseparable components of $X$ (Lemma 5.8), and thus does not permute the components of $Y$ above the étale tails. Lastly, by Lemma 5.4, the action of $\gamma$ above the étale tails is trivial. So $\gamma$ is the identity. □

### 5.4 Good reduction

Recall that $f : Y \to X \cong \mathbb{P}^1$ is a three-point cover defined over $K$. The proposition below motivates Proposition 5.9 above:

**Proposition 5.10.** If the absolute ramification index $e$ of $K$ is less than $(p - 1)/m_G$, and if $f$ has bad reduction, then $\Gamma_w$ is non-trivial.

**Proof.** This is essentially the argument of [Ray99] §5.1.

This has the following consequence:
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Corollary 5.11. Let $X = \mathbb{P}^1_K$, where $K$ is a complete discretely valued field of mixed characteristic $(0, p)$ and absolute ramification index $e$. Let $f : Y \to X$ be a smooth, geometrically connected, Galois cover branched only at $\{0, 1, \infty\}$ with Galois group $G$, defined over $K$. Assume that $G$ has a cyclic $p$-Sylow subgroup, and that $e < (p - 1)/m_G$. Assume further that there are no new étale tails in the stable reduction of $f$. Then $f$ has potentially good reduction.

Proof. By [Ray99, Corollaire 4.2.13], the ramification indices of $f$ are of prime-to-$p$ order. The Corollary follows from Propositions 5.9 and 5.10.

Example 5.12. We exhibit a family of covers with arbitrarily large cyclic $p$-Sylow subgroup that have good reduction by Corollary 5.11. Fix a prime $p \equiv 1 \pmod{3}$, and $n \geq 1$. Let $q$ satisfy

$$q^2 + q + 1 \equiv 0 \pmod{p^n}. \quad (5.2)$$

We note that, for $n = 1$, this is satisfied whenever $q^3 \equiv 1 \pmod{p}$ and $q \not\equiv 1 \pmod{p}$. Since $p \equiv 1 \pmod{3}$, there are solutions of (5.2) for $n = 1$. By Dirichlet’s theorem there are infinitely many prime solutions $q$ (and perhaps some higher prime power solutions, too). Once there is a solution of (5.2) for $n = 1$, Hensel’s lemma gives solutions for all $n$. Since these solutions are given by congruence conditions (mod $p^n$), there are also infinitely many prime solutions $q$ for any fixed $n$ and $p \equiv 1 \pmod{3}$ (and perhaps some higher prime powers). The smallest solution for $n > 1$ and $q$ a prime power is $p = 7$, $n = 2$, and $q = 67$.

Assume $q = \ell^f$ is a prime power satisfying (5.2). Let $G \cong PGL_3(q)$. We know by [Hup67, Satz 7.3] that $G$ has a cyclic $p$-Sylow subgroup of order $p^{v_p(q^2+q+1)} > p^n$, with $m_G = 3$. We will construct a three-point $G$-cover defined over $K_0$ with potentially good reduction to characteristic $p$.

Consider $H := GL_3(q) \rtimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$-action is inverse-transpose. In [MM99, II, Proposition 6.4 and Theorem 6.5], a rigid class vector $(\tilde{C}_0, \tilde{C}_1, \tilde{C}_\infty)$ is exhibited for $H/\{\pm 1\}$, where $\tilde{C}_0$ has order 2, $\tilde{C}_1$ has order 4, and $\tilde{C}_\infty$ has order $(q - 1)\ell^a$ for some $a$ (this is because the characteristic polynomial for the elements of $\tilde{C}_\infty$ has eigenvalues of order $q - 1$). Since $p$ does not divide the order of any of the ramification indices, this triple is rational over $K_0$, so the corresponding $H/\{\pm 1\}$-cover is defined over $K_0$. Thus there is a quotient $G \rtimes \mathbb{Z}/2$-cover $h : Y \to \mathbb{P}^1$ defined over $K_0$.

Let $X \to \mathbb{P}^1$ be the quotient cover of $h : Y \to \mathbb{P}^1$ corresponding to the group $G$. Then $X \to \mathbb{P}^1$ is a cyclic cover of degree 2, branched at 0 and 1 (this comes from the proof of [MM99, II, Proposition 6.4]). This means that $X \cong \mathbb{P}^1$, and $Y \to X$ is branched at three points (the two points above $\infty$, and the unique point above 1). So we have constructed a three-point $G$-cover $f : Y \to X \cong \mathbb{P}^1$ defined over $K_0$, and all three branch points have prime-to-$p$ branching index. Thus, there are three primitive tails (Proposition 2.13 and Lemma 2.15).

Since $m_G = 3$, each primitive tail $\nabla_0$ satisfies $\sigma_0 \geq \frac{1}{3}$. By the vanishing cycles formula (3.5), all $\sigma_0$ are equal to $\frac{1}{3}$ and there are no new (étale) tails. We note that the absolute ramification index $e$ of $K_0$ is 1, and, since $p > 4$, we have $e < \frac{p - 1}{m_G}$. We conclude using Corollary 5.11.

Remark 5.13. There are no examples satisfying the hypotheses of Corollary 5.11 where $G$ is p-solvable. This is because there would be a quotient cover with Galois group $\mathbb{Z}/p \rtimes \mathbb{Z}/m_G$ and prime-to-$p$ ramification indices ([Obu10a, Proposition 2.1]). This cannot be defined as a Galois cover over a field of ramification index $e < (p - 1)/m_G$ (see [Wew05a, Cor. 1.5]; in fact, one exactly needs $e = (p - 1)/m_G$).

Appendix A. An example of wild monodromy

Throughout this appendix, let $G = SL_2(251)$, and let $k$ be an algebraically closed field of characteristic $p = 5$. Let $R_0 = W(k)$ and $K_0 = \text{Frac}(R_0)$. Lastly, let $K = K_0(\mu_{5\infty})$ (that is, we adjoin all 5th-power roots of unity to $K$), and let $R$ be the valuation ring of $K$. Note that $G$ has a cyclic
5-Sylow subgroup of order $5^3 = 125$ and $m_G = 2$. Our example of a three-point $G$-cover with nontrivial wild monodromy (and such that 5 does not divide the order of the center of $G$) depends on intricate calculations from [Obu10b]. We normalize all valuations on $R_0$, $K_0$, or any extensions thereof so that $v(5) = 1$.

**Proposition A.1.** There exists a three-point cover $f : Y \to X = \mathbb{P}^1_K$, defined over $K$, such that the branching indices of the three branch points are $e_1$, $e_2$, and $e_3$, with $v_5(e_1) = 0$, $v_5(e_2) = 2$, and $v_5(e_3) = 3$.

**Proof.** We show that such a cover can be defined over $Q^{ab}$. Since $Q^{ab} \hookrightarrow K$, this will prove the proposition.

Let $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$. This has order 251. We claim there exists $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ satisfying the following properties:

- The order of $\beta$ is 250.
- The order of $\alpha \beta$ is 50.
- The matrices $\alpha$ and $\beta$ generate $SL_2(251)$.

To prove the claim, first note that any $GL_2(251)$-conjugacy class in $G$ is determined by the trace of the matrices it contains, unless the trace is $\pm 2$. In particular, the trace of a matrix determines its order if it is not $\pm 2$. Let $\tau$ be the trace of the matrices in some conjugacy class of order 250, and let $\rho$ be the trace of the matrices in some conjugacy class of order 50. Then $\tau$, $\rho$, 2, and $-2$ are pairwise distinct. Choose $a$, $b$, $c$, and $d$ in $F_{251}$ solving the (clearly solvable) system of equations:

\[
\begin{align*}
    a + d &= \tau \\
    a + c + d &= \rho \\
    ad - bc &= 1
\end{align*}
\]

Since the trace of $\alpha \beta$ is $a + c + d$, these equations ensure that $\beta$ and $\alpha \beta$ have the desired orders. Let $\overline{\alpha}$ and $\overline{\beta}$ be the images of $\alpha$ and $\beta$ in $H := PSL_2(251)$. Since $c \neq 0$, one checks that $\overline{\beta}$ does not normalize the subgroup generated by $\overline{\alpha}$. Then, by [Hup67, II, Hauptsatz 8.27], we have that $\overline{\alpha}$ and $\overline{\beta}$ generate $H$. Furthermore, since $\beta$ is diagonalizable over $GL_2(251)$ and has eigenvalues of order 250, then $\beta^{125} = -I_2$. Since $\overline{\alpha}$ and $\overline{\beta}$ generate $H$, and $\beta$ generates $\ker(G \to H)$, then $\alpha$ and $\beta$ generate $G$.

Consider the triple $([\alpha], [\beta], [\alpha \beta]^{-1})$ of conjugacy classes of $G$. By [MM99, I, Theorem 5.10 and Remark afterward], this triple is rigid. By [MM99, I, Theorem 4.8], there exists a three-point $G$-cover of $\mathbb{P}^1$, defined over $Q^{ab}$, with branching indices $e_1 = \ord(\alpha) = 251$, $e_3 = \ord(\beta) = 250$, and $e_2 = \ord((\alpha \beta)^{-1}) = 50$. This completes the proof of the proposition. \hfill $\Box$

**Proposition A.2.** If $f : Y \to X = \mathbb{P}^1_K$ is a cover satisfying the properties of Proposition A.1, then $f$ has nontrivial wild monodromy $\Gamma_w$.

**Proof.** To fix notation, we assume $f$ is branched at $x = 0$, $x = 1$, and $x = \infty$ of index $e_1$, $e_2$, and $e_3$, respectively, with $v_5(e_1) = 0$, $v_5(e_2) = 2$, and $v_5(e_3) = 3$. By [Obu10c, Lemma 3.2], the stable reduction of $f$ has both a primitive étale tail and a new étale tail. Construct the strong auxiliary cover $f^{str} : Y^{str} \to X$ of $f$ ([Obu10b §2.5]). This is a four-point $G^{str}$-cover, with $G^{str} \cong \mathbb{Z}/125 \rtimes \mathbb{Z}/2$ such that the action of $\mathbb{Z}/2$ is faithful. By [Obu10b, p. 22, (3.1), (3.2)], this cover given by equations

\[
\begin{align}
    z^2 &= \frac{x - a}{x} \\
    y^{125} &= g(z) := \left( \frac{z + 1}{z-1} \right)^r \left( \frac{z + \sqrt{1-a}}{z - \sqrt{1-a}} \right)^s
\end{align}
\]  

(A.1) (A.2)
where \( r \) and \( s \) are integers satisfying \( v_5(r) = 0 \) and \( v_5(s) = 1 \). Replacing \( y \) with a prime-to-\( p \) power, we can assume \( s = 5 \). By [Obu10b, Lemma 3.7], we have \( v(1 - a) > 0 \) in \( K(a)/K \), and then [Obu10b] Lemmas 3.22 and 3.26(i) show that we can take \( a = 1 - \frac{25}{r} \). In particular, \( f^{str} \) is defined over \( K \). By [Obu10b, Proposition 3.31], the stable model \((f^{str})^{st} : (Y^{str})^{st} \to X^{st} \) of \( f^{str} \) has a new inseparable \( p \)-tail \( \overline{X}_c \). We claim that there is an extension \( L/K \) such that \( \text{Gal}(L/K) \) acts nontrivially of order 5 on the stable reduction \( f^{str} : Y^{str} \to X^{str} \) of \( f^{str} \) above an étale neighborhood of \( \overline{X}_c \). Where \( \alpha \) is a \( 5 \)-th power, in \( K \) if \( \alpha = \sqrt{5} \). This follows from Lemma A.3 below.

Let \( \alpha (\sqrt{5})/K(\sqrt{5}) \) be the rational function in (A.2), with \( d \) the \( 5 \)-th power, in \( K \) if \( \alpha = \sqrt{5} \). This is clearly not a 5-th power in \( K \). We claim that there is an extension \( L/K(\sqrt{5}) \) such that \( \text{Gal}(L/K(\sqrt{5})) \) will permute these 25 points in orbits of order 5, and we will be done. This follows from Lemma [A.3] below. \( \square \)

**Lemma A.3.** Let \( d = \frac{25^{7/5}}{r} \), where \( r \) is a prime-to-\( p \) integer and we choose any \( p \)-th root of 5. Let \( g \) be the rational function in (A.2), with \( s = 5 \) and \( a = 1 - \frac{25}{2} \). Then \( g(d) \) is a 5-th power, but not a 25-th power, in \( K(\sqrt{5}) \).

**Proof.** Fix a 5-th root of 5 in \( \overline{K} \), which we will denote by either \( \sqrt[5]{5} \) or \( 5^{1/5} \). We first note that \( g(d) \in K_0(\sqrt[5]{5}) \). By [Obu10b] p. 38, equation after (3.18), we have

\[
g(d) = \pm \left( 1 - \frac{8r^3}{75} d^3 - \frac{32r^5}{5^5} d^5 \right) + o(5^{9/4}).
\]

Upon plugging in \( d \) and simplifying, this gives

\[
g(d) = \pm (1 - 3 \cdot 5^{11/5} - 4 \cdot 5^2) + o(5^{9/4}).
\]

Using the binomial theorem, we see that \( g(d) \) has a 5-th root \( \delta \) in \( K_0(\sqrt[5]{5}) \), and

\[
\delta = \pm (1 - 3 \cdot 5^{6/5} - 20) + o(5^{9/4}).
\]

We wish to show that \( \delta \) is not a 5-th power in \( K(\sqrt[5]{5}) \).

Now, since \( K(\sqrt[5]{5})/K_0(\sqrt[5]{5}) \) is abelian, any subextension is Galois. So if \( \delta \) is a 5-th power in \( K(\sqrt[5]{5}) \), then taking a 5-th root of \( \delta \) must generate a Galois extension of \( K_0(\sqrt[5]{5}) \). This is clearly not the case unless \( \delta \) is already a 5-th power in \( K_0(\sqrt[5]{5}) \), so it suffices to show that \( \delta \) is not a 5-th power in \( K_0(\sqrt[5]{5}) \). Since \(-1\) is a 5-th power in \( K_0 \), we may assume that \( \delta = 19 + 3 \cdot 5^{6/5} + o(5^{5/4}) \).

Suppose that \( \epsilon \in K_0(\sqrt[5]{5}) \) such that \( \epsilon^5 = \delta \), and write

\[
\epsilon = \alpha + \beta \cdot 5^{1/5} + \gamma \cdot 5^{2/5} + \eta \cdot 5^{3/5} + \theta \cdot 5^{4/5},
\]

where \( \alpha, \beta, \gamma, \eta, \) and \( \theta \) are in \( K_0 \). By Gauss’ Lemma, \( \epsilon \in R_0[\sqrt[5]{5}] \). Equating coefficients of 1 and \( 5^{6/5} \) gives the equations

\[
\alpha^4 \beta \equiv 3 \pmod{5}
\]

\[
\alpha^5 + 5 \beta^5 \equiv 19 \pmod{25}
\]

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The second equation yields $\alpha \equiv 4 \pmod{5}$, and then the first equation yields $\beta \equiv 3 \pmod{5}$. But then $\alpha^5 + 5\beta^5 \equiv 24 + 5 \cdot 18 \equiv 14 \not\equiv 19 \pmod{25}$. So $\epsilon$ cannot exist, and we are done.

**Remark A.4.** The example above is quite complicated, and is not generalizable in any meaningful way (for instance, it depends critically on having $p = 5$). One hopes for easier examples, but they are difficult to come by. For instance, results of [Obu10a, §4.1] show that no examples of three-point $G$-covers with nontrivial wild monodromy can exist when $G$ is $p$-solvable and $m_G > 1$. So if one wants to find an easier example where $p$ does not divide the order of the center of $G$, one needs to look either at a group that is not $p$-solvable, or at a group of the form $G \cong H \rtimes \mathbb{Z}/p^n$, where the action of $\mathbb{Z}/p^n$ on $H$ is faithful.

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Andrew Obus  obus@math.columbia.edu
Department of Mathematics, Columbia University, MC 4403, 2990 Broadway, New York, NY 10027