Adaptive Test of Conditional Moment Inequalities

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Abstract

In this paper, I construct a new test of conditional moment inequalities, which is based on studentized kernel estimates of moment functions with many different values of the bandwidth parameter. The test automatically adapts to the unknown smoothness of moment functions and has uniformly correct asymptotic size. The test has high power in a large class of models with conditional moment inequalities. Some existing tests have nontrivial power against $n^{-1/2}$-local alternatives in a certain class of these models whereas my method only allows for nontrivial testing against $(n/\log n)^{-1/2}$-local alternatives in this class. There exist, however, other classes of models with conditional moment inequalities where the mentioned tests have much lower power in comparison with the test developed in this paper.

Keywords: Conditional Moment Inequalities, Minimax Rate Optimality.

1 Introduction

Conditional moment inequalities (CMI) are often encountered both in economics and econometrics. In economics, they arise naturally in many models that include behavioral choice, see [Pakes (2010)] for a survey. In these models, an agent chooses...
the action that maximizes expected utility given her information set. Comparing the realized action with any other available action leads to CMI. In econometrics, they appear in the estimation problems with interval data and problems with censoring, e.g., see Manski and Tamer (2002). In addition, CMI offer a convenient way to study treatment effects in randomized experiments as described in Lee et al. (2011). In the next section, I provide three detailed examples of models with CMI.

Let $m : \mathbb{R}^d \times \mathbb{R}^k \times \Theta \to \mathbb{R}^p$ be a vector-valued known function. Let $(X, W)$ be a pair of $\mathbb{R}^d$ and $\mathbb{R}^k$-valued random vectors, and $\theta \in \Theta$ a parameter. The CMI can be written as

$$E[m(X, W, \theta)|X] \leq 0 \ a.s. \quad (1.1)$$

where inequalities are understood piecewise. I am interested in testing the null hypothesis, $H_0$, that $\theta = \theta_0$ against the alternative, $H_a$, that $\theta \neq \theta_0$ based on iid sample $(X_i, W_i)_{i=1}^n$ from the distribution of $(X, W)$. Note that I also allow for conditional moment equalities since they can be written as pairs of the CMI in model (1.1). Using CMI for inference is difficult because often these inequalities do not identify the parameter. Let

$$\Theta_I = \{ \theta \in \Theta : E[m(X, W, \theta)|X] \leq 0 \ a.s. \} \quad (1.2)$$

denote the identified set. The model is said to be identified if and only if $\Theta_I$ is a singleton. Otherwise, CMI do not identify the parameter $\theta$. For example, the latter may happen when the CMI arise from a game-theoretic model with multiple equilibria. Moreover, the parameter may be weakly identified. My approach leads to a test with the correct asymptotic size no matter whether the parameter is identified, weakly identified, or not identified.

Two approaches to robust CMI testing have been developed in the literature. One approach (Andrews and Shi (2010)), is based on converting CMI into an infinite number of unconditional moment inequalities using nonnegative weighting functions. The other approach (Chernozhukov et al. (2009)), is based on estimating moment functions nonparametrically. My method is inspired by the work of Andrews and Shi (2010). To motivate the test developed in this paper, consider two examples of CMI models. These models are highly stylized but convey main ideas. In the first model,
$m$ is multiplicatively separable in $\theta$, i.e. $m(X, W, \theta) = \theta \tilde{m}(X, W)$ for some $\tilde{m} : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ and $\theta \in \mathbb{R}$ with $E[\tilde{m}(X, W) | X] > 0$ almost surely. In the second model, $m$ is additively separable in $\theta$, i.e. $m(X, W, \theta) = \tilde{m}(X, W) + \theta$. The identified sets, $\Theta_I$, in these models are $\{ \theta \in \mathbb{R} : \theta \leq 0 \}$ and $\{ \theta \in \mathbb{R} : \theta \leq -\text{ess sup}_X E[\tilde{m}(X, W) | X] \}$ correspondingly. Andrews and Shi (2010) developed a test that has nontrivial power against alternatives of the form $\theta_0 = \theta_{0, n} = C / \sqrt{n}$ for any $C > 0$ in the first model, so their test has extremely high power in this model. It follows from Armstrong (2011a) that their test has low power in the second model, however (e.g., in comparison with the test of Chernozhukov et al. (2009)). In constrast, I construct a test that has high power in a large class of CMI models including models like that in the second example. At the same time, my test has virtually the same power in models like that described in the first example. The main difference between two approaches is that my test statistic is based on the studentized estimates of moments whereas theirs is not. More precisely, Andrews and Shi (2010) also consider studentization but they modify the variance term so that asymptotic power properties of their test are similar to those of the test with no studentization.

The test of Chernozhukov et al. (2009) also has high power in a large class of CMI models but it requires knowledge of certain smoothness properties of moment functions such as order of differentiability whereas the test developed in this paper does not. Moreover, my test automatically adapts to these smoothness properties selecting the most appropriate weighting function. This feature of the test is important because smoothness properties of moment functions are rarely known in practice. For this reason, I call the test adaptive.

The test statistic in this paper is based on kernel estimates of moment functions $E[m_j(X, W, \theta_0) | X]$ with many bandwidth values using positive kernels. Here $m_j(X, W, \theta)$ denotes $j$-th component of $m(X, W, \theta)$. I assume that the set of bandwidth values expands as the sample size $n$ increases so that the minimal bandwidth value converges to zero at an appropriate rate while the maximal one is fixed. Since the variance of the kernel estimators varies greatly with the bandwidth value, each

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1 Andrews and Shi (2010) developed tests based on both Cramer-von Mises and Kolmogorov-Smirnov test statistics. In this paper, I mainly refer to their test with Kolmogorov-Smirnov test statistic. Most statements are also applicable for Cramer-von Mises test statistic as well, however.

2 A kernel is said to be positive if the kernel function is positive on its support.
estimator is studentized, i.e. it is divided by its estimated standard deviation. The test statistic, \( \hat{T} \), is formed as the maximum of these studentized estimates, and large values of \( \hat{T} \) suggest that the null hypothesis is violated.

I develop a bootstrap method to simulate the critical value for the test. The method is based on the observation that the distribution of the test statistic, conditionally on the values \( \{X_i\}_{i=1}^n \), is asymptotically independent of the distribution of the noise \( \{m(X_i, W_i, \theta_0) - E[m(X_i, W_i, \theta_0)|X_i]\}_{i=1}^n \), apart from its second moment. For reasons similar to those discussed in Chernozhukov et al. (2007) and Andrews and Soares (2010), the distribution of the test statistic in large samples depends heavily on the extent to which CMI are binding. Moreover, the parameters that measure to what extent CMI are binding can not be estimated consistently. I develop a new approach to deal with this problem, which I refer to as the refined moment selection (RMS) procedure. The approach is based on the pretest that is used to decide what counterparts of the test statistic should be used in simulating the critical value for the test. In comparison with Andrews and Shi (2010), I use a model-specific critical value for the pretest, which is simulated as a high quantile of the appropriate distribution, whereas they use a deterministic threshold with no reference to the model. For comparison reasons, I also provide a plug-in critical value for the test. My proof of the bootstrap validity is interesting on its own right because it is not known whether the test statistic converges in distribution somewhere or not.

None of the tests in the literature including mine have power against alternatives in the set \( \Theta_I \). Therefore, I consider the alternatives of the form

\[
P\{E[m_j(X, W, \theta_0)|X] > 0\} > 0 \text{ for some } j = 1, ..., p
\]

To show that my test has good power properties in a large class of CMI models, I derive its power against alternatives of the form (1.3) assuming that \( E[m(X, W, \theta_0)|X] \) is some vector of unrestricted nonparametric functions. In other words, I consider nonparametric classes of alternatives. Once \( m(X, W, \theta) \) is specified, it is straightforward to translate my results into the parametric setting. The test developed in this paper is consistent against any fixed alternative outside of the set \( \Theta_I \). I also show that my method allows for nontrivial testing against \( (n/\log n)^{-1/2} \)-local one-directional
Finally, I prove that the test is minimax rate optimal against certain classes of smooth alternatives consisting of moment functions $E[m(X, W, \theta_0)|X]$ that are sufficiently flat at the points of maxima. Minimax rate optimality means that the test is uniformly consistent against alternatives in the mentioned class whose distance from the set of models satisfying (1.1) converges to zero at the fastest possible rate. The requirement that functions should be sufficiently flat can not be dropped because the test is based on the positive kernels.

The literature concerned with unconditional and conditional moment inequalities is expanding quickly. The list of published papers on unconditional moment inequalities includes Chernozhukov et al. (2007), Romano and Shaikh (2008), Rosen (2008), Andrews and Guggenberger (2009), Andrews and Han (2009), Andrews and Soares (2010), Bugni (2010), Canay (2010), Pakes (2010), and Romano and Shaikh (2010).

I note that there is also a large literature on partial identification which is close related to that on moment inequalities. Methods specific for conditional moment inequalities were developed in Khan and Tamer (2009), Kim (2008), Chernozhukov et al. (2009), Andrews and Shi (2010), Lee et al. (2011), Armstrong (2011a), and Armstrong (2011b). The case of CMI that point identify $\theta$ is treated in Khan and Tamer (2009). The test of Kim (2008) is closely related to that of Andrews and Shi (2010). Lee et al. (2011) developed a test based on the minimum distance statistic in the one-sided $L_p$-norm and kernel estimates of moment functions. The advantage of their approach comes from simplicity of their critical value for the test, which is an appropriate quantile of the standard Gaussian distribution. Their test is not adaptive, however, since only one bandwidth value is used. Armstrong (2011a) developed a new method for computing the critical value for the test statistic of Andrews and Shi (2010) which leads to a more powerful test than theirs but his method is not robust. In particular, his method can not be used in the CMI models like that described in the first example above. Armstrong (2011b) considered the test statistic similar to that used in this paper but he focused on estimation rather than inference.

Finally, an important related paper in the statistical literature is Dumbgen and Spokoiny (2001). They consider testing qualitative hypotheses in the ideal Gaussian white noise model where a researcher observes a stochastic process that can be represented as a

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3 In this paper, by one directional alternatives, I mean alternatives of the form $E[m(X, W, \theta_0)|X] = a_n f(X)$ for some sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ converging to zero where $f$ satisfies (1.3).
sum of the mean function and a Brownian motion. In particular, they developed a test for the null hypothesis that the mean function is (weakly) negative almost everywhere. Even though their test statistic is somewhat related to that used in this paper, the technical details of their analysis are quite different.

The rest of the paper is organized as follows. The next section elaborates on some examples of CMI models. Section 3 formally introduces the test. The main results of the paper are presented in section 4. A Monte Carlo simulation study is described in section 5. There I provide an example of an alternative with the well-behaved moment function such that the test developed in this paper rejects the null hypothesis with probability higher than 80% while the rejection probability of all competing tests does not exceed 20%. Brief conclusions are drawn in section 7. Finally, all proofs are contained in the Appendix.

2 Examples

In this section, I provide three examples where CMI arise naturally in economic and econometric models. The first two examples have function-valued parameters. In order to fit these examples into my framework, one can consider parametric approximations of corresponding functions.

Incomplete Models of English Auctions. My first example follows Haile and Tamer (2003) treatment of English auctions under weak conditions. The popular model of English auctions suggested by Milgrom and Weber (1982) assumes that each bidder is holding down the button while the price is going up continuously until she wants to drop out. The price at the moment of dropping out is her bid. In this model, it is well-known that the dominant strategy is to make a bid equal to her valuation of the object. In practice, participants usually call out bids, however. So, the price rises in jumps, and the bid may not be equal to person’s valuation of the object. In this situation, the relation between bids and valuations of the object depends crucially on the modeling assumptions. Haile and Tamer (2003) derived certain bounds on the distribution function of valuations based on minimal assumptions of rationality.

Suppose we have an auction with \( m \) bidders whose valuations of the object are drawn independently from the distribution \( F(\cdot, X) \) where \( X \) denotes observable
characteristics of the object. Let \( b_1, \ldots, b_m \) denote highest bids of each bidder. Let \( b_{1:m} \leq \ldots \leq b_{m:m} \) denote the ordered sequence of bids \( b_1, \ldots, b_m \). Assuming that bids do not exceed bidders’ valuations, Haile and Tamer (2003) derived the following upper bound on \( F(\cdot, X) \):

\[
E[I\{b_{i:m} \leq v\} - \phi^{-1}(F(v, X))] \geq 0 \text{ a.s.}
\]

for all \( v \in \mathbb{R} \) and \( i = 1, \ldots, m \) where \( \phi(\cdot) \) is a certain (known) function, see equation (3) in Haile and Tamer (2003). Similar lower bound follows from the assumption that bidders do not allow opponents to win at a price they would like to beat. Assuming we observe an iid sequence of auctions, these CMI can be used for inference on \( F(v, X) \).

**Interval Data.** In some cases, especially when data concerns personal information like individual income or wealth, one has to deal with interval data. Suppose we have a mean regression model

\[
Y = f(X, V) + \varepsilon
\]

where \( E[\varepsilon|X, V] = 0 \) a.s. and \( V \) is a scalar random variable. Suppose that we observe \( X \) and \( Y \) but we do not observe \( V \). Instead, we observe \( V_0 \) and \( V_1 \) called brackets such that \( V \in (V_0, V_1) \) a.s. In empirical analysis, brackets may arise because a respondent refuses to provide information on \( V \) but provides an interval to which \( V \) belongs. Following Manski and Tamer (2002) assume that \( f(X, V) \) is weakly increasing in \( V \) and \( E[Y|X, V] = E[Y|X, V, V_0, V_1] \). Then it is easy to see that

\[
E[I\{V_1 \leq v\}(Y - f(X, v))|X, V_0, V_1] \leq 0
\]

and

\[
E[I\{V_0 \geq v\}(Y - f(X, v))|X, V_0, V_1] \geq 0
\]

for all \( v \in \mathbb{R} \). If we observe an iid sample from the model, we can use these CMI for inference on \( f(X, V) \).

**Treatment Effects.** Suppose we have a randomized experiment where one group of people gets a new treatment while the control group gets a placebo. Let \( D = 1 \) if the person gets the treatment and 0 otherwise. Let \( p \) denote the probability that \( D = 1 \).
Let \( X \) denote person’s observable characteristics and \( Y \) denote a realized outcome. Finally, let \( Y_0 \) and \( Y_1 \) denote counterfactual outcomes had the person received a placebo or the new medicine respectively. Then \( Y = DY_1 + (1 - D)Y_0 \). The question of interest is whether the new medicine has a positive expected impact uniformly over all possible person’s characteristics \( X \). In other words, the null hypothesis, \( H_0 \), is that

\[
E[Y_1 - Y_0|X] \geq 0 \ a.s. \tag{2.5}
\]

Since in randomized experiments \( D \) is independent of \( X \), Lee et al. (2011) showed that

\[
E[Y_1 - Y_0|X] = E[DY/p - (1 - D)Y/(1 - p)|X] \tag{2.6}
\]

Combining (2.5) and (2.6) gives CMI.

### 3 The Test

In this section, I present the test statistic and give two bootstrap methods to simulate a critical value. Given nonparametric nature of the test, I use the corresponding terminology. For fixed \( \theta_0 \), let \( Y = m(X, W, \theta_0) \), \( f(X) = E[m(X, W, \theta_0)|X] \), and \( \varepsilon = Y - f(X) \) so that \( E[\varepsilon|X] = 0 \ a.s. \) Then under the null hypothesis,

\[
f(X) \leq 0 \ a.s. \tag{3.1}
\]

I refer to \( Y \) as a response variable, \( f \) as a vector-valued regression function, \( X \) as a design point, and \( \varepsilon \) as a disturbance. Components of \( f \) are denoted by \( f_1, ..., f_p \).

The analysis in this paper is conducted conditionally on the set of values \( \{X_i\}_{i=1}^n \) of the insrumental variable \( X \), so all probabilistic statements in this paper should be understood conditionally on \( \{X_i\}_{i=1}^n \) for almost all sequences \( \{X_i\}_{i=1}^n \). Lemma 4 in the Appendix provides certain conditions that insure that assumptions used in this paper hold for almost all sequences \( \{X_i\}_{i=1}^n \).

Section 3.1 defines the test statistic assuming that \( E[\varepsilon_i\varepsilon_i^T] = \Sigma_i \) is known for each \( i = 1, ..., n \). Section 3.2 gives two bootstrap methods to simulate a critical value. The first one is based on plug-in asymptotics, and the second one is based on the refined moment selection (RMS) procedure. Section 3.2 also provides some intuition.
of why these procedures lead to the correct asymptotic size of the test. When \( \Sigma_i \) is not known, it should be estimated from the data. Section 3.3 shows how to construct an appropriate estimator \( \hat{\Sigma}_i \) of \( \Sigma_i \). The feasible version of the test will be based on substituting \( \hat{\Sigma}_i \) for \( \Sigma_i \) both in the test statistic and in the critical value. 3.4 provides some notes on how to choose certain tuning parameters.

### 3.1 The Test Statistic

The test statistic in this paper is based on the kernel estimator of the vector-valued regression function \( f \). Let \( K : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) be some kernel. For bandwidth value \( h \in \mathbb{R}_+ \), denote \( K_h(x) = K(x/h)/h^d \). For each pair of observations \( i, j = 1, \ldots, n \), denote the weight function

\[
 w_h(X_i, X_j) = \frac{K_h(X_i - X_j)}{\sum_{k=1}^{n} K_h(X_i - X_k)}
\]

Then the kernel estimator of \( f_m(X_i) \) is

\[
 \hat{f}_{i,m,h} = \sum_{j=1}^{n} w_h(X_i, X_j)Y_{j,m}
\]

where \( Y_{j,m} \) denotes \( m \)-th component of response variable \( Y_j \). Conditionally on \( \{X_i\}_{i=1}^{n} \), the variance of the kernel estimator \( \hat{f}_{i,m,h} \) is

\[
 V_{i,m,h}^2 = \sum_{j=1}^{n} w_h^2(X_i, X_j)\Sigma_{j,mm}
\]

where \( \Sigma_{j,m_1m_2} \) denotes \( (m_1, m_2) \) component of \( \Sigma_j = E[\varepsilon_j \varepsilon_j^T] \).

Next, consider a finite set of bandwidth values \( H = \{h = h_{\max}a^k : h \geq h_{\min}, k = 0, 1, 2, \ldots\} \) for some \( h_{\max} > h_{\min} \) and \( a \in (0, 1) \). For simplicity, I assume that \( h_{\min} = h_{\max}a^k \) for some \( k \in \mathbb{N} \) so that \( h_{\min} \) is included in \( H \). I assume that as the sample size \( n \) increases, \( h_{\min} \) converges to zero while \( h_{\max} \) is fixed. For each bandwidth value \( h \in H \), choose a subset \( I_h \) of observations such that \( \|X_i - X_j\| > 2h \) for all \( i, j \in I_h \) with \( i \neq j \) and for each \( i = 1, \ldots, n \), there exist an element \( j(i) \in I_h \) such that \( \|X_i - X_{j(i)}\| \leq 2h \) where \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^d \). I refer to
$I_h$ as a set of test points. The choice of $I_h$ may be random, but it is important to select $I_h$ independently of response variables $\{Y_i\}_{i=1}^n$. So, conditionally on $\{X_i\}_{i=1}^n$, I assume that $I_h$ is nonstochastic. It will be assumed in the next section that $K(x) = 0$ for any $x \in \mathbb{R}^d$ such that $\|x\| > 1$. Thus, random variables $\{\hat{f}_{i,m,h}\}_{i \in I_h}$ are jointly independent for any fixed $m = 1, ..., p$ and $h \in H$ conditionally on $\{X_i\}_{i=1}^n$. This fact will play a key role in the derivation of the lower bound on the growth rate of the pdf of the test statistic, which is used in the analysis of size properties of the test.

Finally, denote $S = \{(i, m, h) : h \in H, i \in I_h, m = 1, ..., p\}$.

Based on this notation, the test statistic is

$$T = \max_{s \in S} \frac{\hat{f}_s}{\hat{V}_s} \quad (3.5)$$

Let me now explain why the optimal bandwidth value depends on the smoothness properties of the components $f_1, ..., f_p$ of $f$. Without loss of generality, consider $j = 1$. Suppose that $f_1(X)$ is flat. Then $f_1(X)$ is positive on the large subset of its domain whenever its maximal value is positive. Hence, the maximum of $\hat{T}$ will correspond to a large bandwidth value because the variance of the kernel estimator, which enters the denominator of the test statistic, decreases with the bandwidth value. On the other hand, if $f_1(X)$ is allowed to have peaks, then there may not exist a large subset where it is positive. So, large bandwidth values may not yield large values of $\hat{T}$, and small bandwidth values should be used in such cases. I circumvent these problems by considering the set of bandwidth values jointly, and let the data determine the best bandwidth value. In this sense, my test adapts to the smoothness properties of $f(X)$.

This allows me to construct a test with good uniform power properties over possible smoothness of $f(X)$.

When $\Sigma_i$ is not observed, which is usually the case in practice, one can define $\hat{V}_{i,m,h}^2 = \sum_{j=1}^n w_h^2(X_i, X_j) \hat{\Sigma}_{j,m,m}$ and use

$$\hat{T} = \max_{s \in S} \frac{\hat{f}_s}{\hat{V}_s} \quad (3.6)$$

Although my argument in the derivation of the lower bound is based on the fact that $\{\hat{f}_{i,m,h}\}_{i \in I_h}$ are jointly independent, I believe that the same lower bound can be obtained even for the case $I_h = \{1, ..., n\}$. If this statement is true, one can use $I_h = \{1, ..., n\}$ in the definition of the test statistic.
instead of $T$ where $\hat{\Sigma}_j$ is some estimator of $\Sigma_j$. Some possible estimators are discussed in section 3.3.

### 3.2 Critical Values

Suppose we want to construct a test of size $\alpha$. This subsection explains how to simulate a critical value $t_{1-\alpha}$ for the statistic $\hat{T}$ based on two bootstrap methods. One method is based on the plug-in asymptotics, and the other one is based on the refined moment selection (RMS) procedure. Both methods have deterministic and randomized versions. For the randomized versions, one first determines some small interval, say $[c, c+\beta]$ with $\beta > 0$, where the critical value belongs. Then one draws the critical value from a certain distribution with the support $[c, c+\beta]$. This randomization comes from my proof technique, which is based on the Linderberg method. Under somewhat stronger conditions, I also prove the validity of both methods with $\beta = 0$, which corresponds to their deterministic versions. The test will be of the following form: reject the null hypothesis if and only if $\hat{T} > t_{1-\alpha}$.

Let $\beta$ be either zero or some small positive number. Let $g_0$ be a thrice differentiable function from $\mathbb{R}$ into $[0, 1]$ such that $g_0(x) = 1$ for all $x \leq 0$ and $g_0(x) = 0$ for all $x \geq 1$. Denote $g(x) = g_0((x-c)/\beta)$ for some $c \in \mathbb{R}$. Since $g(x) \in [0, 1]$ for all $x \in \mathbb{R}$, $g(\cdot)$ gives a randomized test: upon observing the test statistic $\hat{T} = x$, one accepts the null hypothesis with probability $g(x)$. I will choose $c$ so that, under the null hypothesis, $E[g(\hat{T})] \geq 1 - \alpha + o(1)$ as $n \to \infty$, which leads to the correct asymptotic size of this randomized test. An equivalent way to describe this test is as follows. Let $U$ be a random variable independent of the data with uniform distribution on $[0, 1]$. Define the critical value $t_{1-\alpha}$ for the test from the equation $g(t_{1-\alpha}) = U$. Since $g(x)$ is decreasing in $x$, this equation has the unique solution so that $t_{1-\alpha}$ is well-defined. Lemma 1 in the Appendix shows that $E[g(\hat{T})] = P\{\hat{T} \leq t_{1-\alpha}\}$, which means that the randomized test is equivalent to the test based on the critical value $t_{1-\alpha}$. Note that the latter formulation is more convenient for the confidence set construction: one can use the same $U$ for all possible values of $\theta_0$. For the purposes of presentation, the former formulation is suitable, however. I refer to $g(\cdot)$ as a test function.

Let me now describe two possible bootstrap methods to simulate $c$. The first method is based on plug-in asymptotics. It relies on two observations. First, it is
easy to see that, for a fixed distribution of disturbances \( \{\varepsilon_i\}_{i=1}^n \), the maximum of \( 1 - \alpha \) quantile of the test statistic \( \hat{T} \) over all possible functions \( f \) satisfying \( f \leq 0 \) almost surely corresponds to \( f = 0_p \). Second, lemmas 9 and 11 in the Appendix show that the distribution of the statistic \( \hat{T} \) is asymptotically independent of the distribution of disturbances \( \{\varepsilon_i : i = 1, \ldots, n\} \) apart from their second moments \( \{\Sigma_i : i = 1, \ldots, n\} \).

These observations suggest that one can simulate \( c \) by the following procedure:

1. For each \( i = 1, \ldots, n \), simulate \( \tilde{Y}_i \sim N(0_p, \hat{\Sigma}_i) \) independently across \( i \).
2. Calculate \( T^{PIA} = \max_{(i,m,h) \in S} \sum_{j=1}^n w_h(X_i, X_j) \tilde{Y}_{j,m}/\hat{V}_{i,m,h} \).
3. Repeat steps 1 and 2 independently \( B \) times for some large \( B \) to obtain \( \{T_b^{PIA} : b = 1, \ldots, B\} \).
4. Find \( c_1^{PIA} \) such that \( \sum_{b=1}^B g_0((T_b^{PIA} - c_1^{PIA})/\beta)/B = 1 - \alpha \).

Then plug-in test function \( g_1^{PIA} : \mathbb{R} \rightarrow [0, 1] \) is given by \( g_1^{PIA}(x) = g_0((x - c_1^{PIA})/\beta) \) for all \( x \in \mathbb{R} \).

The second method is based on the refined moment selection (RMS) procedure. It gives a less conservative critical value while maintaining the required size of the test. The method is based on the observation that \( |\hat{T}| = O_p(\sqrt{\log n}) \) if \( f = 0_p \) (see lemmas 8, 9, 11 in the Appendix) while \( \hat{f}_{i,m,h}/\hat{V}_{i,m,h} \rightarrow -\infty \) with a polynomial rate if \( f_m(X_i) < 0 \) and \( h \rightarrow 0 \). Such terms will have asymptotically negligible effect on the distribution of \( \hat{T} \), so we can ignore corresponding terms in the simulated statistic. Specifically, let \( \gamma < \alpha/2 \) be some small positive number. First, use the plug-in bootstrap to find \( c_{1-\gamma}^{PIA} \). Denote

\[
S^{RMS} = \{s \in S : \hat{f}_s/\hat{V}_s > -2(c_{1-\gamma}^{PIA} + \beta)\} \quad (3.7)
\]

Second, run the following procedure:

1. For each \( i = 1, \ldots, n \), simulate \( \tilde{Y}_i \sim N(0_p, \hat{\Sigma}_i) \) independently across \( i \).
2. Calculate \( T^{RMS} = \max_{(i,m,h) \in S^{RMS}} \sum_{j=1}^n w_h(X_i, X_j) \tilde{Y}_{j,m}/\hat{V}_{i,m,h} \).
3. Repeat steps 1 and 2 independently \( B \) times for some large \( B \) to obtain \( \{T_b^{RMS} : b = 1, \ldots, B\} \).
4. Find $c_{1-\alpha+2\gamma}^{RMS}$ such that $\sum_{b=1}^{B} g_0((T_b^{RMS} - c_{1-\alpha+2\gamma}^{RMS})/\beta)/B = 1 - \alpha + 2\gamma$.

Then RMS test function $g_{1-\alpha}^{RMS} : \mathbb{R} \to \mathbb{R}$ is given by $g_{1-\alpha}^{RMS}(x) = g_0((x - c_{1-\alpha+2\gamma}^{RMS})/\beta)$ for all $x \in \mathbb{R}$. The additional term $2\gamma$ can be interpreted as a correction for the truncation procedure introduced in $S^{RMS}$.

### 3.3 Estimating $\Sigma_i$

Let me now explain how one can estimate $\Sigma_i$. The literature on estimating $\Sigma_i$ is huge. Among other papers, it includes Rice (1984), Muller and Stadtmuller (1987), Hardle and Tsybakov (1997), and Fan and Yao (1998). For scalar-valued response variables, a variety of such estimators is described in Horowitz and Spokoiny (2001). All those estimators can be immediately generalized to vector-valued response variables. For completeness, I describe one estimator here. For $i = 1, \ldots, n$ define $j(i)$ by the following recursion:

$$j(1) = \arg \min_{j=2,\ldots,n} \|X_j - X_1\|$$  \hspace{1cm} (3.8)

and

$$j(i) = \arg \min_{j \neq i, j(1), \ldots, j(i-1)} \|X_j - X_i\|$$  \hspace{1cm} (3.9)

Then variance $\Sigma_i$ can be estimated by

$$\hat{\Sigma}_i = \frac{\sum_{k=1}^{n}(Y_k - Y_{j(k)})(Y_k - Y_{j(k)})^T I(\|X_k - X_i\| \leq b_n)}{2 \sum_{k=1}^{n} I(\|X_k - X_i\| \leq b_n)}$$  \hspace{1cm} (3.10)

where $b_n$ denotes some bandwidth value. This estimator will be uniformly consistent for $\Sigma_i$ over $i = 1, \ldots, n$ with rate $(\log n/n)^{(1/(2+d))}$, i.e.

$$\max_{i=1,\ldots,n} \|\hat{\Sigma}_i - \Sigma_i\|_o = O_p \left( \frac{\log n}{n} \right)^{1/(2+d)}$$  \hspace{1cm} (3.11)

if (i) $b_n \approx (\log n/n)^{(1/(2+d))}$ and (ii) assumptions from section 4.1 hold where $\| \cdot \|_o$ denotes the spectral norm on the space of $p \times p$-dimensional symmetric matrices corresponding to Euclidean norm on $\mathbb{R}^p$. To choose bandwidth value $b_n$ in practice, one can use any type of the cross validation. An advantage of this estimator is that
it is fully adaptive with respect to smoothness properties of regression function \( f \).

The intuition behind this estimator is based on the following argument. Note that \( j(k) \) is chosen so that \( X_{j(k)} \) is close to \( X_k \). If regression function \( f \) is continuous,

\[
Y_k - Y_{j(k)} = f(X_k) - f(X_{j(k)}) + \varepsilon_k - \varepsilon_{j(k)} \approx \varepsilon_k - \varepsilon_{j(k)} \tag{3.12}
\]

so that

\[
E[(Y_k - Y_{j(k)})(Y_k - Y_{j(k)})^T] \approx \Sigma_k + \Sigma_{j(k)} \tag{3.13}
\]

since \( \varepsilon_k \) is independent of \( \varepsilon_{j(k)} \). If \( b_n \) is small enough and \( \Sigma(X) \) is continuous, \( \Sigma_k + \Sigma_{j(k)} \approx 2\Sigma_i \) since only \( X_k \) satisfying \( \|X_k - X_i\| \leq b_n \) are used in estimating \( \Sigma_i \).

### 3.4 Remarks on the Choice of Testing Parameters

Implementing the deterministic version of the test requires choosing minimal and maximal bandwidth values \( h_{\text{min}} \) and \( h_{\text{max}} \) and the parameter \( \gamma \). The randomized version of the test also use the parameter \( \beta \) and the function \( g_0 : \mathbb{R} \to [0, 1] \). In this section, I provide some notes on how to choose these objects for the randomized test to make sure that the test maintains the required size.

First, I recommend to set \( h_{\text{max}} = \max_{i,j=1,...,n} \|X_i - X_j\|/2 \) as a normalization. Second, it follows from theorem 11 that the test with RMS test function is not conservative asymptotically only if \( \gamma = \gamma_n \to 0 \) as \( n \to 0 \). So, I recommend to set \( \gamma \) as a small fraction of \( \alpha \), for example \( \gamma = 0.01 \) for \( \alpha = 0.05 \). Alternatively, one can set \( \gamma = 0.1 / \log(n) \) similarly the corresponding choice in Chernozhukov et al. (2009).

Next, consider how to choose \( g_0 \), \( h_{\text{min}} \), and \( \beta \). It follows from theorems 1 and 6 and lemma 11 that the test maintains the required size if

\[
\Delta = \frac{3}{6^{1/3} \beta^{2/3} pbn^{1/3}} \left( \frac{\|g''_0\|_{\infty}}{\beta^3} + \frac{3\|g'_0\|_{\infty}}{\beta^2} + \frac{\|g'_0\|_{\infty}}{\beta} \right)^{1/3} \left( \|g'_0\|_{\infty} \log |S| \right)^{2/3} F \tag{3.14}
\]

is small in comparison with \( \alpha \) (required size) where

\[
F = \left( \max E[|\varepsilon_{i,m}^3|] + \max \sqrt{8/\pi \Sigma_{i,m}^{3/2}} \right)^{1/3} \tag{3.15}
\]
with both maxima taken over \(i = 1, \ldots, n\) and \(m = 1, \ldots, p\) and

\[
b = \max_{(i,m,h) \in S; j = 1, \ldots, n} \frac{w_h(X_i, X_j)}{V_{i,m,h}}
\]  

(3.16)

If \(\beta \ll 1\), the good choice of \(g_0\) is given by

\[
g_0(x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
1 - (16/3)x^3 & \text{if } x \in (0, 1/4] \\
7/6 - x - 4(x - 1/4)^2 + (16/3)(x - 1/4)^3 & \text{if } x \in (1/4, 3/4] \\
(16/3)(1 - x)^3 & \text{if } x \in (3/4, 1] \\
0 & \text{if } x > 1 
\end{cases}
\]  

(3.17)

This function is chosen so that \(g_0'''(x) = -32\) for \(x \in (0, 1/4]\), \(+32\) for \(x \in (1/4, 3/4]\), and \(-32\) for \(x \in (3/4, 1]\). Given this function, if \(\beta \leq 1\), it is enough to set parameters so that

\[
1.8pbn^{1/3}(\log |S|)^{2/3}F/\beta^{5/3} \ll \alpha
\]  

(3.18)

Given \(h_{\text{min}}\), \(b\) and \(F\) can be estimated from the data. Then one can choose \(\beta\) so that the inequality above is satisfied. Note that there is a trade-off between choosing small \(\beta\) and small \(h_{\text{min}}\) since \(b\) is a decreasing function of \(h_{\text{min}}\).

I note that the inequality (3.18) guarantees good size properties of the test uniformly over a large set of the true distributions of disturbances \(\{\varepsilon_j\}_{j=1}^n\). In particular, this set includes discrete distributions, which lead to the distributions of the test statistic that are difficult to approximate using Gaussian disturbances. Therefore, this inequality is difficult to satisfy in sample sizes typical for economic data. Nevertheless, this inequality is still useful because it gives a starting point in choosing testing parameters.

\footnote{Similar phenomenon is also known in the classical theory of Central Limit Theorems, see Ibragimov and Linnik (1971)}
4 The Main Results

This section presents my main results. Section 4.1 gives regularity conditions. Section 4.2 describes size properties of the test. Section 4.3 explains the behavior of the test under a fixed alternative. Section 4.4 derives the rate of consistency of the test against one-directional alternatives mentioned in the introduction. Section 4.5 shows the rate of uniform consistency against certain classes of smooth alternatives. Section 4.6 presents the minimax rate-optimality result.

4.1 Assumptions

Let $M_h(X_i)$ be the number of elements in the set $\{X_j : \|X_j - X_i\| \leq h, j = 1, \ldots, n\}$. In what follows, I will write $C$ and its variants for a generic constant whose value may vary depending on the context. Results in this paper will be proven under the following regularity assumptions.

Assumption 1. (i) Design points $\{X_i\}_{i=1}^n$ are nonstochastic. (ii) For some constant $0 < \bar{C} < \infty$ and all $i = 1, \ldots, n$, $\|X_i\| < \bar{C}$. (iii) For some constants $0 < C_1 < C_2 < \infty$, $C_1 nh^d \leq M_h(X_i) \leq C_2 nh^d$ for all $i \in \mathbb{N}$ and $h \in H = H_n$.

The design points are nonstochastic because the analysis is conducted conditionally on $\{X_i\}_{i=1}^n$. Assumption 1 also states that the design points have bounded support, which is a mild assumption. In addition, it states that the number of design points in certain neighborhoods of each design point is proportional to the volume of the neighborhood with the coefficient of proportionality bounded from above and away from zero. It is stated in Horowitz and Spokoiny (2001) that assumption 1 holds in an iid setting with probability approaching one as the sample size increases if the distribution of $X_i$ is absolutely continuous with respect to Lebesgue measure, has bounded support, and has the density bounded away from zero on the support. This statement is actually wrong unless one makes some extra assumptions. Lemma 3 in the Appendix gives a counter-example. Instead, lemma 4 shows that assumption 1 holds for large $n$ almost surely if, in addition, I assume that the density of $X_i$ is bounded from above, and that the support of $X_i$ is a convex set. Necessity of the density boundedness is obvious. Convexity of the support is not necessary for assumption 1 but it gives a good trade-off between generality and simplicity. In
general, one should deal with some smoothness properties of the boundary of the support. Note that the statement “for large $n$ almost surely” is stronger than “with probability approaching one”. Note also that assumption 1(iii) requires inequalities to hold for all $i \in \mathbb{N}$, not just for $i = 1, \ldots, n$.

**Assumption 2.** (i) Disturbances $\{\varepsilon_i : i = 1, \ldots, n\}$ are independent $\mathbb{R}^p$-valued random variables with $\mathbb{E}[\varepsilon_{i,m_1}] = 0$, $\mathbb{E}[\varepsilon_{i,m_1}\varepsilon_{i,m_2}] = \Sigma_{i,m_1,m_2} < \infty$, and $\mathbb{E}[\varepsilon_{i,m_1}\varepsilon_{i,m_2}\varepsilon_{i,m_3}\varepsilon_{i,m_4}] = s_{i,m_1,m_2,m_3,m_4}^4 < \infty$ for all $i = 1, \ldots, n$ and $m_1, m_2, m_3, m_4 = 1, \ldots, p$. (ii) For some constants $0 < C < \infty$ and $\delta > 0$, $\mathbb{E}[|\varepsilon_{i,m}|^{4+\delta}] \leq C$ for all $i = 1, \ldots, n$ and $m = 1, \ldots, p$. (iii) For some constant $0 < C < \infty$, $|\Sigma_{i,m_1,m_2} - \Sigma_{j,m_1,m_2}| \leq C\|X_i - X_j\|$ and $|s_{i,m_1,m_2,m_3,m_4}^4 - s_{j,m_1,m_2,m_3,m_4}^4| \leq C\|X_i - X_j\|$ for all $i, j = 1, \ldots, n$ and $m_1, m_2, m_3, m_4 = 1, \ldots, p$. (iv) For some constant $0 < C < \infty$, $\Sigma_{i,m_1} \geq C$ for all $i = 1, \ldots, n$ and $m = 1, \ldots, p$.

The reason for imposing assumption 2 is threefold. First, finite third moment of disturbances is used in the derivation of a certain invariance principle with the rate of convergence. As in the classical central limit theorem, finite two moments are sufficient to prove weak convergence but more finite moments are necessary if we are interested in the rate of convergence. Second, finite $4+\delta$ moment of disturbances and Lipshitz continuity properties are used to make sure that $\hat{\Sigma}_i$ converges in probability to $\Sigma_i$ uniformly over $i = 1, \ldots, n$ for a particular estimator $\hat{\Sigma}_i$ of $\Sigma_i$ described in section 3.3 at an appropriate rate. Finally, I assume that the variance of each component of disturbances is bounded away from zero for simplicity of the presentation. Since I use a studentization of kernel estimators, without this assumption, it would be necessary to truncate the variance of the kernel estimators from below with truncation level slowly converging to zero. That would complicate the derivation of the main results without changing main ideas.

Before stating assumption 3, let me give formal definitions of Holder smoothness class $F(\tau, L)$ and its subsets $F_{\varsigma}(\tau, L)$. For $d$-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \ldots + \alpha_d$, function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, denote

$$D^\alpha g(x) = \frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}(x)$$

whenever it exists. For $\tau > 0$, it is said that the function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ belongs to
the class $\mathcal{F}(\tau, L)$ if it has continuous partial derivatives up to order $[\tau]$ and for any $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $|\alpha| = [\tau]$ and $x, y \in \mathbb{R}^d$,

$$|D^\alpha g(x) - D^\alpha g(y)| \leq \|x - y\|^{\tau - [\tau]} \tag{4.2}$$

Here $[\tau]$ denotes the largest integer strictly smaller than $\tau$. For any $g \in \mathcal{F}(\tau, L)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, and $l = (l_1, \ldots, l_d) \in \mathbb{R}^d$ satisfying $\sum_{m=1}^d l_m^2 = 1$, let $g^{(k,l)}(x)$ denote $k$-th derivative of function $f$ in direction $l$ at point $x$ whenever it exists. For $\zeta = 1, \ldots, [\tau]$, let $\mathcal{F}_\zeta(\tau, L)$ denote the class of all elements of $\mathcal{F}(\tau, L)$ such that for any $g \in \mathcal{F}_\zeta(\tau, L)$ and $l = (l_1, \ldots, l_d) \in \mathbb{R}^d$ satisfying $\sum_{m=1}^d l_m^2 = 1$, $f^{(k,l)}(x) = 0$ for all $k = 1, \ldots, \zeta$ whenever $f^{(1,l)}(x) = 0$, and there exist $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $l = (l_1, \ldots, l_d) \in \mathbb{R}^d$ satisfying $\sum_{m=1}^d l_m^2 = 1$ such that $f^{(\zeta+1,l)}(x) \neq 0$ and $f^{(1,l)}(x) = 0$. If $\tau \leq 1$, I set $\zeta = 0$ and $\mathcal{F}_\zeta(\tau, L) = \mathcal{F}(\tau, L)$.

**Assumption 3.** (i) For some $\tau \geq 1/4$, $L > 0$, and $\zeta = 1, \ldots, [\tau]$, regression functions $f_m(\cdot) = f_{m,n}(\cdot)$ belong to the class $\mathcal{F}_\zeta(\tau, L)$ for all $m = 1, \ldots, p$. (ii) If $\zeta < [\tau]$, then for any $x \in \mathbb{R}^d$ and all $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $|\alpha| = \zeta + 1$, $|D^\alpha f_m(x)| \leq C$ for some constant $C > 0$ and all $m = 1, \ldots, p$.

For simplicity of notation, I assume that all components of $f$ have the same smoothness properties. This assumption is used in the derivation of the power properties of the test. The restriction $\tau \geq 1/4$ is also needed to make sure that $\hat{\Sigma}_i$ converges in probability to $\Sigma_i$ uniformly over $i = 1, \ldots, n$ at an appropriate rate. I allow regression functions to depend on $n$ to perform a local power analysis.

**Assumption 4.** Set of bandwidth values has the following form: $H = H_n = \{h = h_{\text{max}}a^k : h \geq h_{\text{min}}, k = 0, 1, 2, \ldots\}$ where $a \in (0, 1)$, $h_{\text{max}} = \bar{C}$ and $h_{\text{min}} = h_{\text{min,n}} \to 0$ as $n \to \infty$ such that $|H_n| \leq C \log n$ for some constant $C > 0$.

According to this assumption, maximal bandwidth value, $h_{\text{max}}$, is independent of $n$. Its value is chosen to match the radius $\bar{C}$ of the support of design points. It is intended to detect deviations from the null hypothesis in the form of flat alternatives. Minimal bandwidth value, $h_{\text{min}}$, converges to zero as the sample size increases in such a way that the number of bandwidth values in the set $H_n$ is growing at a logarithmic rate or slower. This assumption will be satisfied if $h_{\text{min}}$ converges to zero.
at a polynomial rate. Minimal bandwidth value is intended to detect deviations from
the null hypothesis in the form of alternatives with peaks.

**Assumption 5.** Estimators $\hat{\Sigma}_i$ of $\Sigma_i$ satisfy $\max_{i=1, \ldots, n} \|\hat{\Sigma}_i - \Sigma_i\|_\omega = o_p(n^{-\kappa})$ with $\kappa = 1/(2 + d) - \phi$ for arbitrarily small $\phi > 0$ where $\| \cdot \|_\omega$ denotes the spectral norm on the space of $p \times p$-dimensional symmetric matrices corresponding to the Euclidean norm on $\mathbb{R}^p$.

As follows from Muller and Stadtmuller (1987), under assumptions 2 and 3, assumption 5 is satisfied for the estimators $\hat{\Sigma}_i$ of $\Sigma_i$ described in section 3.3. In practice, due to the course of dimensionality, it might be useful to use some parametric or semi-parametric estimators of $\Sigma_i$ instead of the estimator described in section 3.3. For example, if we assume that $\Sigma_i = \Sigma_j$ for all $i, j = 1, \ldots, n$, then the estimator of Rice (1984) (or its multivariate generalization) is $1/\sqrt{n}$-consistent. In this case, assumption 5 will be satisfied with $\kappa = 1/2 - \phi$ for arbitrarily small $\phi > 0$.

**Assumption 6.** (i) The kernel $K$ is positive and supported on $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$. (ii) For some constant $0 < C < 1$, $K(x) \leq 1$ for all $x \in \mathbb{R}^d$ and $K(x) \geq C$ for all $\|x\| \leq 1/2$.

I assume that the kernel function is positive on its support. Many kernels satisfy this assumption. For example, one can use rectangular, triangular, parabolic, or biweight kernels. See Tsybakov (2009) for the definitions. On the other hand, the requirement that the kernel is positive on its support excludes higher-order kernels, which are necessary to achieve minimax optimal testing rate over large classes of smooth alternatives. I require positive kernels because of their negativity-invariance property, which means that any kernel smoother with a positive kernel maps the space of negative functions into itself. This property is essential for obtaining a test with the correct asymptotic size when smoothness properties of moment functions are unknown. With higher-order kernels, one has to assume undersmoothing so that the bias of the estimator is asymptotically negligible in comparison with its standard deviation. Otherwise, large values of $\hat{T}$ might be caused by large values of the bias term relative to the standard deviation of the estimator even though all components of $f(X)$ are negative. However, for undersmoothing, one has to know the smoothness properties of $f(X)$. In contrast, with positive kernels, the set of bandwidth values can
be chosen without reference to these smoothness properties. In particular, the largest bandwidth value can be chosen to be independent of the sample size \( n \). Nevertheless, the test developed in this paper will be rate optimal in the minimax sense against class \( \mathcal{F}_{[\tau]}(\tau, L) \) when \( \tau > d \).

**Assumption 7.** (i) For some constant \( C > 0 \), \( \beta = \beta_n \leq C \). (ii) \( (\log n)^4/ (\beta_n^{10} h_{\min}^{3d} n) \to 0 \) as \( n \to \infty \).

Assumption 7 establishes the trade-off between choosing small value of \( \beta \) and small value of \( h_{\min} \). It is a key condition used to establish an invariance principle that shows that asymptotic distribution of \( \hat{T} \) depends on the distribution of disturbances \( \{\varepsilon_i : i = 1, \ldots, n\} \) only through their covariances \( \{\Sigma_i : i = 1, \ldots, n\} \). Under somewhat stronger conditions, corollary 1 shows that I can set \( \beta = 0 \), which corresponds to the deterministic version of the test. Note that from assumption 7(ii), it follows that \( h_{\min} \) converges to zero at a polynomial rate which is consistent with assumption 4.

**Assumption 8.** (i) For every \( h \in H_n \), set of test points \( I_h = I_{h,n} \) is such that \( \|X_i - X_j\| > 2h \) for all \( i, j \in I_{h,n} \) with \( i \neq j \) and for each \( i = 1, \ldots, n \), there exists an element \( j(i) \in I_{h,n} \) such that \( \|X_i - X_{j(i)}\| \leq 2h \). (ii) \( S = S_n = \{(i, m, h) : h \in H_n, i \in I_{h,n}, m = 1, \ldots, p\} \).

Denote the class of models satisfying assumptions 2 and 3 for some fixed values of all constants by \( \mathcal{G} \). Each element \( w \in \mathcal{G} \) consists of a pair \((f^w, \varepsilon^w)\), where \( f^w \) denotes the regression function and \( \varepsilon^w \) denotes all the information about the distribution of disturbances in model \( w \). Denote the subset of models satisfying \( f \leq 0 \) almost surely by \( \mathcal{G}_0 \).

### 4.2 Size Properties of the Test

Analysis of size properties of the test is complicated because the asymptotic distribution of the test statistic is unknown. Instead, I use a finite sample approach based on the Lindeberg method. For each sample size \( n \), this method gives an upper error bound on approximating the expectation of smooth functionals of the test statistic by its expectation calculated assuming Gaussian noise \( \{\varepsilon_i\}_{i=1}^n \). I also derive a simple lower bound on the growth rate of the pdf of the test statistic to show that the expectation of smooth functionals can be used to approximate the expectation of indicator
functions. Combining these results leads to the approximation of the cdf of the test statistic by its cdf calculated assuming Gaussian disturbances with an explicit error bound. This allows me to derive certain conditions which insure that the error converges to zero as the sample size $n$ increases, which is a key step in establishing the bootstrap validity.

The first theorem states that the test has correct asymptotic size uniformly over the class of models $\mathcal{G}_0$ both for plug-in and RMS test functions. In addition, the test with the plug-in test function is nonconservative as the size of the test converges to the required level $\alpha$ uniformly over the class of models $\mathcal{G}_0$ with $f^w \equiv 0_p$. When I set $\gamma = \gamma_n \to 0$, the same holds for the test with the RMS test function.

**Theorem 1.** Let assumptions 1-8 hold. Then for $P = \text{PIA}$ or $\text{RMS}$,

$$\inf_{w \in \mathcal{G}_0} E_w[g^P_{1-\alpha}(\hat{T})] \geq 1 - \alpha + o(1) \quad (4.3)$$

In addition,

$$\sup_{w \in \mathcal{G}_0, f^w \equiv 0_p} E_w[g^{\text{PIA}}_{1-\alpha}(\hat{T})] = 1 - \alpha + o(1) \quad (4.4)$$

and if $\gamma_n \to 0$, then

$$\sup_{w \in \mathcal{G}_0, f^w = 0_p} E[g^{\text{RMS}}_{1-\alpha}(\hat{T})] = 1 - \alpha + o(1) \quad (4.5)$$

as well.

Proofs of all results are presented in the Appendix. From the proof of theorem 1 I also have

**Corollary 1.** If instead of (ii) we assume $(\log n)^{19}/(h_{\min,n}^{3d}) \to 0$, then theorem 1 holds with $\beta = \beta_n = 0$.

The case $\beta = 0$ corresponds to the deterministic version of the test, which rejects the null if and only if $\hat{T} > c^P_{1-\alpha}$ for $P = \text{PIA}$ or $\text{RMS}$. However, I can guarantee that this test maintains the required size only if $h_{\min}$ converges to zero very slowly since $(\log n)^{19}$ is a very large number for reasonable sample sizes.
4.3 Consistency Against a Fixed Alternative

Let me introduce a distance between model \( w \in G \) and the null hypothesis:

\[
\rho(w, H_0) = \sup_{i=1, \ldots, \infty; m=1, \ldots, p} \left[ f^w_m(X_i) \right]_+ \tag{4.6}
\]

For any alternative outside of the set \( \Theta_I \), \( \rho(w, H_0) > 0 \). In this section, I state the result that the test is consistent against any fixed alternative \( w \) with \( \rho(w, H_0) > 0 \) satisfying assumptions 1-8. Moreover, I show that the test is consistent uniformly against alternatives whose distance from the null hypothesis is bounded away from zero. For \( \rho > 0 \), let \( G_\rho \) denote the subset of all elements of \( G \) such that \( \rho(w, H_0) \geq \rho \) for all \( w \in G_\rho \). Then

**Theorem 2.** Let assumptions 1-8 hold. Then for \( P = PIA \) or RMS,

\[
\sup_{w \in G_\rho} E_w \left[ g^P_{1-\alpha}(\hat{T}) \right] \to 0 \tag{4.7}
\]

as \( n \to \infty \).

4.4 Consistency Against One-Directional Alternatives

Let \( w(0) \in G \) be such that \( \rho(w(0), H_0) > 0 \). For some sequence \( \{a_n\}_{n=1}^\infty \) of positive numbers converging to zero, let \( f^n = a_n f^w(0) \) be a sequence of local alternatives. I refer to such sequences as local one-directional alternatives. This section establishes the consistency of the test against such alternatives whenever \( \sqrt{n/\log n} a_n \to \infty \).

**Theorem 3.** Let assumptions 1-8 hold. Then for \( P = PIA \) or RMS,

\[
\sup_{w \in G, f^w = f^n} E_w \left[ g^P_{1-\alpha}(\hat{T}) \right] \to 0 \tag{4.8}
\]

as \( n \to \infty \) if \( \sqrt{n/\log n} a_n \to \infty \).

**Remark.** Recall the CMI model from the first example mentioned in the introduction where \( m(X, W, \theta) = \theta \tilde{m}(X, W) \) and \( E[\tilde{m}(X, W)|X] > 0 \) almost surely. The theorem above shows that the test developed in this paper is consistent against sequences of alternatives \( \theta_0 = \theta_{0,n} \) whenever \( \sqrt{n/\log n} \theta_{0,n} \to \infty \) in this model. So, my test is
consistent against virtually the same set of alternatives in this model as the test of Andrews and Shi (2010).

4.5 Uniform Consistency Against Holder Smoothness Classes

In this section, I present the rate of uniform consistency of the test against the class $\mathcal{F}_\zeta(\tau, L)$ under certain additional constraints. These additional constraints are needed to deal with some boundary effects. Let $S = \text{cl}\{X_i : i \in \mathbb{N}\}$ denote the closure of the infinite set of design points. For any $\vartheta > 0$, let $S_{\vartheta}$ be the subset of $S$ such that for any $x \in S_{\vartheta}$, the ball with center at $x$ and radius $\vartheta$, $B_\vartheta(x)$, is contained in $S$, i.e. $B_\vartheta(x) \subset S$. Denote $\zeta = \min(\zeta + 1, \tau)$. When $\zeta \leq d$, set $\vartheta = \vartheta_n = 4\sqrt{d}h_{\text{min}}$. When $\zeta > d$, set $\vartheta = \vartheta_n = 4\sqrt{d}\left(\log n/n\right)^{1/(2\zeta + d)}$. Let $N_{\vartheta_n} = \{i \in \mathbb{N} : X_i \in S_{\vartheta_n}\}$. For any $w \in \mathcal{G}$, let

$$\rho_{\vartheta_n}(w, H_0) = \sup_{i \in N_{\vartheta_n}, m=1,\ldots,p} [f_{m}^w(X_i)]_+$$

(4.9)

denote the distance between $w$ and $H_0$ over set $S_{\vartheta_n}$. For the next theorem, I will use $\rho_{\vartheta_n}$-metric (instead of $\rho$-metric) to measure the distance between alternatives and the null hypothesis. Such restrictions are quite common in the literature. See, for example, Dumbgen and Spokoiny (2001) and Lee et al. (2011). Let $\mathcal{G}_{\vartheta}$ be the subset of all elements of $\mathcal{G}$ such that $\inf_{w \in \mathcal{G}_{\vartheta}} \rho_{\vartheta_n}(w, H_0) \geq Ch_{\text{min}}^\zeta$ for some large constant $C$ if $\zeta \leq d$ and $\inf_{w \in \mathcal{G}_{\vartheta}} \rho_{\vartheta_n}(w, H_0) (n/\log n)^{\zeta/(2\zeta + d)} \to \infty$ if $\zeta > d$. Then

**Theorem 4.** Let assumptions 1-8 hold. For $P = PIA$ or RMS, if (i) $\zeta \leq d$ or (ii) $\zeta > d$ and $h_{\text{min}} < (\log n/n)^{1/(2\zeta + d)}$ for large enough $n$, then

$$\sup_{w \in \mathcal{G}_{\vartheta}} E_w[g_{1-\alpha}^P(\hat{T})] \to 0$$

(4.10)

as $n \to \infty$.

**Remark.** Recall the CMI model from the second example mentioned in the introduction where $m(X, W, \theta) = \hat{m}(X, W) + \theta$. Assume that $X \in \mathbb{R}$ and $E[\hat{m}(X, W)|X] = -|X|\nu$ with $\nu > 1$. In this model, the identified set is $\Theta_I = \{\theta \in \mathbb{R} : \theta \leq 0\}$. The theorem above shows that the test developed in this paper is consistent against sequences of alternatives $\theta_0 = \theta_0, n$ whenever $(n/\log n)^{\nu/(2\nu + 1)} \theta_0 n \to \infty$. At the same time, it follows from Armstrong (2011a), the test of Andrews and Shi (2010) is con-
sistent only if \( n^{\nu/(2(\nu+1))} \theta_{n,0} \to \infty \), so their test has a slower rate of consistency than that developed in this paper.

### 4.6 Lower Bound on the Minimax Rate of Testing

In this section, I give a lower bound on the minimax rate of testing. For \( S_{\theta} \) defined in the previous section, let \( N(h, S_{\theta}) \) be the largest \( m \) such that there exists \( \{x_1, ..., x_m\} \subset S_{\theta} \) with \( \|x_i - x_j\| \geq h \) for all \( i, j = 1, ..., m \) if \( i \neq j \). I will assume that \( N(h, S_{\theta}) \geq Ch^{-d} \) for all \( h \in (0, 1) \) and large enough \( n \) for some constant \( C > 0 \). This condition holds almost surely under the conditions of lemma 4. Let \( \phi_n(Y_1, ..., Y_n) \) denote a sequence of tests, i.e. \( \phi_n(Y_1, ..., Y_n) \) equals the probability of rejecting the null hypothesis upon observing sample \( Y = (Y_1, ..., Y_n) \).

**Theorem 5.** Let assumptions 1-8 hold. Assume that (i) \( N(h, S_{\theta}) \geq Ch^{-d} \) for all \( h \in (0, 1) \) and large enough \( n \) for some constant \( C > 0 \), (ii) \( \zeta = \tau \), and (iii) \( r_n(n/\log n)^{\tau/(2\tau+d)} \to 0 \) as \( n \to \infty \) for some sequence of positive numbers \( r_n \). Then for any sequence of tests \( \phi_n(Y_1, ..., Y_n) \) with \( \sup_{w \in G_0} E_w[\phi_n(Y_1, ..., Y_n)] \leq \alpha \),

\[
\limsup_{n \to \infty} \inf_{w \in G_0, \rho \theta(H_0) \geq Cr_n} E_w[\phi_n(Y_1, ..., Y_n)] \leq \alpha \quad (4.11)
\]

Since \( F_{[\tau]}(\tau, L) \subset F(\tau, L) \), the same lower bound applies for the class \( F(\tau, L) \) as well. Comparing this result with theorem 4 shows that the test presented in this paper is minimax rate optimal if \( \zeta = \tau > d \) and \( h_{\min} \) is chosen to converge to zero fast enough. When \( \zeta = \tau = d \) and \( \beta_n \) is set to be constant, the test is rate optimal up to some logarithmic factors if \( h_{\min} \) is chosen to converge to zero as fast as possible satisfying assumption 7. When \( \tau < d \), the test is not rate optimal since the rate of consistency does not match the lower bound.

### 5 Models with Infinitely Many CMI

In this section, I briefly outline an extension of the test to the case of infinitely many CMI. Suppose that the parameter \( \theta \) is restricted by a countably infinite number of CMI, i.e. \( p = \infty \). As before, I am interested in testing the null hypothesis, \( H_0 \), that \( \theta = \theta_0 \) against the alternative, \( H_a \), that \( \theta \neq \theta_0 \). One possible approach to testing...
in this model is to construct a test as described in section 3 based on some finite subset of CMI assuming that as the sample size $n$ increases, this subset expands covering all CMI in the asymptotics. The advantage of the finite sample approach used in this paper is that it immediately gives certain conditions that insure that such a test maintain the required size asymptotically. Assume that the test is based on $K = K_n \to \infty$ inequalities. Then

**Corollary 2.** Let assumptions 1-4, 6 and 8 hold. In addition, assume that (i) \( \max_{i=1,\ldots,n} \| \hat{\Sigma}_i - \Sigma_i \|_o = o_p(n^{-\kappa}) \) for some $\kappa > 0$, (ii) $K_n \log n/n^{\kappa/4} \to 0$, (iii) $\beta = \beta_n \leq C$, and (iv) $K_n^6(\log n)^4/\left(\beta_n^{10} h_{\min}^3 n\right) \to 0$ as $n \to \infty$. Then for $P = PIA$ or RMS,

$$\inf_{w \in \mathcal{G}_0} E_w[g_P^{1-\alpha}(\hat{T})] \geq 1 - \alpha + o(1)$$

as $n \to \infty$. In addition,

$$E_w[g_P^{1-\alpha}(\hat{T})] \to 0$$

for any $w \in \mathcal{G}_\rho$ with $\rho > 0$.

This corollary shows that the randomized test has correct asymptotic size both with plug-in and RMS critical values and is consistent against fixed alternatives outside of the set $\Theta_I$. Note that $\kappa$ appearing in condition (i) in this corollary will generally be different from $\kappa$ used in assumption 5 because of increasing number of moment functions. Results concerning the test with deterministic critical values and local power of the test, with suitable modifications, can also be easily obtained using arguments similar to those used in the proofs of corollary 1 and theorems 3 and 4. For brevity, I do not discuss these results.

## 6 Monte Carlo Results

In this section, I present results of Monte Carlo simulations. The aim of these simulations is twofold. First, I demonstrate that my test accurately maintain size in finite samples reasonably well. Second, I compare relative advantages and disadvantages of my test and the tests of [Andrews and Shi (2010)](https://doi.org/10.1093/ecta/77.5.883), [Chernozhukov et al. (2009)](https://doi.org/10.1093/jjse/bsp022), and [Lee et al. (2011)](https://doi.org/10.1006/jmva.2008.0790). The methods of [Andrews and Shi (2010)](https://doi.org/10.1093/ecta/77.5.883) and [Lee et al. (2011)](https://doi.org/10.1006/jmva.2008.0790) are most appropriate for detecting flat alternatives, which represent one-directional local
alternatives. These methods have low power against alternatives with peaks, however. The test of Chernozhukov et al. (2009) has higher power against such alternatives, but it requires knowing smoothness properties of the moment functions. The authors suggest certain rule-of-thumb techniques to choose a bandwidth value. Finally, the main advantage of my test is its adaptiveness. In comparison with Andrews and Shi (2010) and Lee et al. (2011), my test has higher power against alternatives with peaks. In comparison with Chernozhukov et al. (2009), my test has higher power when their rule-of-thumb techniques lead to an inappropriate bandwidth value. For example, this happens when the underlying regression function is mostly flat but varies significantly in the region where the null hypothesis is violated (the case of spatially inhomogeneous alternatives, see Lepski and Spokoiny (1999)).

The data generating process in the experiments is

\[ Y = L(M - |X|)_+ - m + \varepsilon \]  

(6.1)

where \( X, Y, \) and \( \varepsilon \) are scalar random variables and \( L, M, \) and \( m \) are some constants. \( X \) is distributed uniformly on \((-2, 2)\). Depending on the experiment, \( \varepsilon \) is distributed according to \( 0.1 \cdot N(0, 1) \) or \((\xi \cdot 0.07 + (1 - \xi) \cdot 0.18) \cdot N(0, 1)\) where \( \xi \) is a Bernoulli random variable with \( p(\xi = 1) = 0.8 \) and \( p(\xi = 0) = 0.2 \) independent of \( N(0, 1) \). In both cases, \( \varepsilon \) is independent of \( X \). I consider the following specifications for parameters. Case 1: \( L = M = m = 0 \). Case 2: \( L = 0.1, M = 0.2, m = 0.02 \). Case 3: \( L = M = 0, m = -0.02 \). Case 4: \( L = 2, M = 0.2, m = 0.2 \). Note that \( E[Y|X] \leq 0 \) almost surely in cases 1 and 2 while \( P\{E[Y|X] > 0\} > 0 \) in cases 3 and 4. In case 3, the alternative is flat. In case 4, the alternative has a peak in the region where the null hypothesis is violated. I have chosen parameters so that rejection probabilities are strictly greater than 0 and strictly smaller than 1 in most cases so that meaningful comparisons are possible. I generate samples \((X_i, Y_i)_{i=1}^n\) of size \( n = 250 \) and 500 from the distribution of \((X, Y)\). In all cases, I consider tests with the nominal size 10%. The results are based on 1000 simulations for each specification.

For the test of Andrews and Shi (2010), I consider their Kolmogorov-Smirnov test statistic with boxes and truncation parameter 0.05. I simulate both plugin (AS, plugin) and GMS (AS, GMS) critical values based on the bootstrap suggested in their paper. I use the support of the empirical distribution of \( X \) to choose a set of
weighting functions. All other tuning parameters are set as prescribed in their paper. Implementing all other tests requires selecting a kernel function. In all cases, I use the following kernel function

\[ K(x) = 1.5(1 - 4x^2)_+ \]  

(6.2)

For the test of Chernozhukov et al. (2009), I use their kernel type test statistic with critical values based on the multiplier bootstrap both with (CLR, \( \hat{V} \)) and without (CLR, \( V \)) the set estimation. Both Chernozhukov et al. (2009) and Lee et al. (2011) (LSW) circumvent edge effects of kernel estimators by restricting their test statistics to the proper subsets of the support of \( X \). So, I select 10 and 90% quantiles of the empirical distribution of \( X \) as bounds for the set over which the test statistics are calculated. Both tests are nonadaptive. In particular, there is no formal theory on how to choose bandwidth values in their tests. I use their suggestions to choose bandwidth values. For the test of Lee et al. (2011), I use their test statistic based on one-sided \( L_1 \)-norm.

Let me now describe the choice of parameters for the test developed in this paper. The largest bandwidth value, \( h_{\text{max}} \), is set to be one half of the length of the support of the empirical distribution. I choose the smallest bandwidth value, \( h_{\text{min}} \), so that the kernel estimator uses on average 15 data points when \( n = 250 \) and 20 data points when \( n = 500 \). The scaling parameter, \( a \), equals 0.8 so that the set of bandwidth values is

\[ H_n = \{ h = h_{\text{max}} 0.8^k : h \geq h_{\text{min}}, k = 0, 1, 2, \ldots \} \]  

(6.3)

My test requires choosing the set \( S_n \). For each bandwidth value, \( h \), I select the largest subset, \( S_{n,h} \), of \( X_i \)'s such that \( X_i - X_j \geq h \) for any nonequal elements in \( S_{n,h} \), and the smallest \( X_i \) is always in \( S_{n,h} \). Then \( S_n = \{ (i, h) : h \in H_n, X_i \in S_{n,h} \} \). In all cases, I set \( \beta = 0 \) so that the deterministic version of the critical values is used. Finally, for the RMS critical value, I set \( \gamma = 0.1 / \log(n) \) to make meaningful comparisons with the test of Chernozhukov et al. (2009). In all bootstrap procedures, for all tests, I use 1000 repetitions when \( n = 250 \) and 500 repetitions when \( n = 500 \).

The results of the experiments are presented in table 1 for \( n = 250 \) and in table 2 for \( n = 500 \). In both tables, my test is denoted as Adaptive test with plug-in and
Table 1: Results of Monte Carlo Experiments, $n = 250$

| Distribution $\varepsilon$ | Case | Probability of Rejecting Null Hypothesis |
|-----------------------------|------|------------------------------------------|
|                             | AS, plugin | AS, GMS | LSW | CLR, $\hat{V}$ | CLR, $\hat{V}$ | Adaptive test, plugin | Adaptive test, RMS |
| Normal                      | 1     | 0.099  | 0.102 | 0.124 | 0.151 | 0.151 | 0.101 | 0.101 |
|                             | 2     | 0.002  | 0.007 | 0.000 | 0.008 | 0.008 | 0.009 | 0.009 |
|                             | 3     | 0.910  | 0.910 | 0.941 | 0.808 | 0.808 | 0.723 | 0.723 |
|                             | 4     | 0.000  | 0.143 | 0.000 | 0.122 | 0.191 | 0.589 | 0.821 |
| Mixture                     | 1     | 0.078  | 0.086 | 0.107 | 0.134 | 0.134 | 0.124 | 0.124 |
|                             | 2     | 0.002  | 0.002 | 0.000 | 0.010 | 0.010 | 0.016 | 0.016 |
|                             | 3     | 0.904  | 0.905 | 0.925 | 0.833 | 0.833 | 0.692 | 0.692 |
|                             | 4     | 0.000  | 0.121 | 0.000 | 0.111 | 0.197 | 0.555 | 0.808 |

RMS critical values. Consider first results for $n = 250$. In case 1, where the null hypothesis holds, all tests have rejecting probabilities close to the nominal size 10% both for normal and mixture of normals disturbances. In particular, RMS procedure for my test, GMS procedure for the test of Andrews and Shi (2010) and the test of Chernozhukov et al. (2009) with the set estimation do not overreject, which might be concerned based on the construction of these tests. In case 2, where the null hypothesis holds but the underlying regression function is mainly strictly below the borderline, all tests are conservative. When the null hypothesis is violated with a flat alternative (case 3), the tests of Andrews and Shi (2010) and Lee et al. (2011) have highest rejection probabilities as expected from the theory. In this case, my test is less powerful in comparison with these tests and somewhat similar to the method of Chernozhukov et al. (2009). This is compensated in case 4 where the null hypothesis is violated with the peak-shaped alternative. In this case, the power of my test is much higher than that of competing tests. This is especially true for my test with RMS critical values whose rejection probability exceeds 80% while rejection probabilities of competing tests do not exceed 20%. Note that all results are stable across distributions of disturbances. Also note that my test with RMS critical values has much higher power than the test with plugin critical values in case 4. So, among these two tests, I recommend the test with RMS critical values. Results for $n = 500$ indicate a similar pattern. Concluding this section, I note that all simulation results are consistent with the presented theory.
Table 2: Results of Monte Carlo Experiments, $n = 500$

| Distribution $\varepsilon$ | Case | Probability of Rejecting Null Hypothesis | Probability of Rejecting Null Hypothesis |
|-----------------------------|------|-----------------------------------------|-----------------------------------------|
|                             |      | AS, plugin | AS, GMS | LSW | CLR, $V$ | CLR, $\hat{V}$ | Adaptive test, plugin | Adaptive test, RMS |
| Normal                      | 1    | 0.095      | 0.104   | 0.119 | 0.126 | 0.126 | 0.103 | 0.103 |
|                             | 2    | 0.000      | 0.001   | 0.000 | 0.002 | 0.002 | 0.008 | 0.008 |
|                             | 3    | 0.997      | 0.997   | 0.996 | 0.954 | 0.954 | 0.903 | 0.903 |
|                             | 4    | 0.008      | 0.587   | 0.000 | 0.497 | 0.694 | 0.976 | 0.999 |
| Mixture                     | 1    | 0.120      | 0.123   | 0.130 | 0.117 | 0.117 | 0.119 | 0.119 |
|                             | 2    | 0.000      | 0.001   | 0.000 | 0.000 | 0.000 | 0.010 | 0.010 |
|                             | 3    | 0.993      | 0.993   | 0.996 | 0.949 | 0.949 | 0.903 | 0.903 |
|                             | 4    | 0.005      | 0.549   | 0.000 | 0.456 | 0.625 | 0.978 | 0.997 |

7 Conclusions

In this paper, I developed a new test of conditional moment inequalities. In contrast to some other tests in the literature, my test is directed against general nonparametric alternatives, which gives high power in a large class of CMI models. Considering kernel estimates of moment functions with many different values of the bandwidth parameter allows me to construct a test that automatically adapts to the unknown smoothness of moment functions and selects the most appropriate testing bandwidth value. The test developed in this paper has uniformly correct asymptotic size, no matter whether the model is identified, weakly identified, or not identified, and is uniformly consistent against certain, but not all, large classes of smooth alternatives whose distance from the null hypothesis converges to zero at a fastest possible rate. The tests of Andrews and Shi (2010) and Lee et al. (2011) have nontrivial power against $n^{-1/2}$-local one-directional alternatives whereas my method only allows for nontrivial testing against $(n/\log n)^{-1/2}$-local alternatives of this type. Additional $(\log n)^{1/2}$ factor should be regarded as a price for having fast rate of uniform consistency. There exist sequences of local alternatives against which their tests are not consistent whereas mine is. Monte Carlo experiments give an example of a CMI model where finite sample power of my test greatly exceeds that of competing tests.
A Appendix

This Appendix contains proofs of all results stated in the main part of the paper. Section A.1 explains the equivalent representations for the randomized test. Section A.2 derives a bound on the modulus of continuity in the operator norm of the square root operator on the space of symmetric positive semidefinite matrices. Section A.3 gives a straightforward generalization of results in Chatterjee (2005) to the case of multidimensional random variables. They are concerned with conditions when the distribution of some function of several independent random variables with unknown distributions can be approximated by substituting Gaussian distributions with the same first two moments. They are based on the Linderberg’s argument. The result is specialized to the situation when the function of interest can be written in the form of the maximum of linear functions of the data. These results have their own value as they can be used as an alternative to results on stochastic approximation from empirical process theory. They are also useful because they give an explicit bound on the approximation error. Section A.4 gives sufficient conditions for assumption 1 in the main part of the paper. Section A.5 presents an anticoncentration inequality for the maximum of Gaussian random variables with unit variance. Section A.6 describes a result on Gaussian random variables which is used in the proof of lower bound on the minimax rate. Section A.7 develops some preliminary technical results necessary for the proofs of the main theorems. Finally, section A.8 presents the proofs of the theorems stated in the main part of the paper.

Note that all convergence results proven in this Appendix hold uniformly over the class of models $\mathcal{G}$. This fact will not be stated seperately in each special case, but it is assumed everywhere in this Appendix.

A.1 Lemma on the equivalent representation of the test

The lemma below was used in section 3.2 to show that the randomized test is equivalent to the test with the random critical value.

Lemma 1. $E[g(\hat{T})] = P\{\hat{T} \leq t_{1-\alpha}\}$. 
Proof. Since \( g(\hat{T}) \in [0, 1] \) almost surely,
\[
E[g(\hat{T})] = \int_0^1 P\{g(\hat{T}) \geq x\} \, dx
\]  
(A.1)

Given that \( U \) is independent of the data and, hence, of \( \hat{T} \),
\[
\int_0^1 P\{g(\hat{T}) \geq x\} \, dx = P\{g(\hat{T}) \geq U\}
\]  
(A.2)

Finally, note that \( \{g(\hat{T}) \geq U\} \) is equivalent to \( \{\hat{T} \leq t_{1-\alpha}\} \) so that
\[
P\{g(\hat{T}) \geq U\} = P\{\hat{T} \leq t_{1-\alpha}\}
\]  
(A.3)

Combining (A.1), (A.2), and (A.3) gives the result. \( \Box \)

A.2 Continuity of the square root operator on the set of positive semidefinite matrices

Lemma 2. Let \( A \) and \( B \) be \( p \times p \)-dimensional symmetric positive semidefinite matrices. Then \( \|A^{1/2} - B^{1/2}\|_o \leq p^{1/2}\|A - B\|_o^{1/2} \) where \( \| \cdot \|_o \) means the spectral norm corresponding to the Euclidean norm on \( \mathbb{R}^p \).

Proof. Let \( a_1, \ldots, a_p \) and \( b_1, \ldots, b_n \) be orthogonal eigenvectors of matrices \( A \) and \( B \) correspondingly. Without loss of generality, I can and will assume that \( \|a_i\| = \|b_i\| = 1 \) for all \( i = 1, \ldots, p \) where \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^p \). Let \( \lambda_1(A), \ldots, \lambda_p(A) \) and \( \lambda_1(B), \ldots, \lambda_p(B) \) be corresponding eigenvalues. Let \( f_{i1}, \ldots, f_{ip} \) be coordinates of \( a_i \) in the basis \( (b_1, \ldots, b_p) \) for all \( i = 1, \ldots, p \). Then \( \sum_{j=1}^{p} f_{ij}^2 = 1 \) for all \( i = 1, \ldots, p \).

For any \( i = 1, \ldots, p \),
\[
\sum_{j=1}^{p} (\lambda_i(A) - \lambda_j(B))^2 f_{ij}^2 = \| \sum_{j=1}^{p} (\lambda_i(A) - \lambda_j(B)) f_{ij} b_j \|^2
\]
\[
= \| \lambda_i(A) a_i - \sum_{j=1}^{p} \lambda_j(B) f_{ij} b_j \|^2
\]
\[
= \| (A - B)a_i \|^2
\]
\[
\leq \| A - B \|_o^2
\]
since \( \|(A - B)a_i\| \leq \|A - B\|_o \|a_i\| = \|A - B\|_o. \)

For \( P = A, B \), \( P^{1/2} \) has the same eigenvectors as \( P \) with corresponding eigenvalues equal to \( \lambda_1^{1/2}(P), \ldots, \lambda_n^{1/2}(P) \). Therefore, for any \( i = 1, \ldots, p \),

\[
\|(A^{1/2} - B^{1/2})a_i\|^2 = \sum_{j=1}^p (\lambda_i^{1/2}(A) - \lambda_j^{1/2}(B))^2 f_{ij}^2 \\
\leq \sum_{j=1}^p |\lambda_i(A) - \lambda_j(B)| f_{ij}^2 \\
\leq \left( \sum_{j=1}^p (\lambda_i(A) - \lambda_j(B))^2 f_{ij}^2 \right)^{1/2} \\
\leq \|A - B\|_o
\]

where the last line used the inequality derived above. For any \( c \in \mathbb{R}^p \) with \( \|c\| = 1 \), let \( d_1, \ldots, d_p \) be coordinates of \( c \) in the basis \((a_1, \ldots, a_p)\). Then

\[
\|(A^{1/2} - B^{1/2})c\| = \|(A^{1/2} - B^{1/2}) \sum_{i=1}^p d_i a_i\| \\
\leq \sum_{i=1}^p |d_i| \|(A^{1/2} - B^{1/2})a_i\| \\
\leq \sum_{i=1}^p |d_i| \|A - B\|_o^{1/2} \\
\leq p^{1/2} \|A - B\|_o^{1/2}
\]

since \( \sum_{i=1}^p d_i^2 = 1 \). Thus, \( \|A^{1/2} - B^{1/2}\|_o \leq p^{1/2} \|A - B\|_o^{1/2}. \)

\[\square\]

### A.3 Invariance principle

In this section, I generalize results of Chatterjee (2005) to the case of random vectors \((p > 1)\). I also specialize results for the case of linear functions because it allows to greatly improve some constants in Chatterjee's derivation. Let \( Z_1, \ldots, Z_n \) be a sequence of independent \( p \)-dimensional random vectors with \( E[Z_j] = 0 \) for all \( j = 1, \ldots, n \). Denote \( Z = (Z_1, \ldots, Z_n) \). For each \( k = 1, \ldots, K \) and \( m = 1, \ldots, p \), let \( f_{km}(Z) = \sum_{j=1}^n a_{kjm} Z_{j,m} \) be some linear function of \( Z \) where \( a_{kjm} \geq 0 \) for each \( k = 1, \ldots, K \),
Let $j = 1, \ldots, n$, and $m = 1, \ldots, p$, and $Z_{j,m}$ denotes $m$-th component of vector $Z_j$. Let $U_1, \ldots, U_n$ be a sequence of independent normal $p$-dimensional random vectors such that $E[U_j] = 0$ and $E[Z_j Z_j^T] = E[U_j U_j^T]$ for each $j = 1, \ldots, n$. Denote $U = (U_1, \ldots, U_n)$ and

$$C(g) = \|g''\|_\infty + 3\|g''\|_\infty + \|g'\|_\infty$$

(A.4)

Denote $a = \max_{k,j,m} a_{kjm}$. Then

**Theorem 6.** For any thrice differentiable function $g$ on $\mathbb{R}$,

$$E[g(\max_{k,m} f_{km}(Z))] - E[g(\max_{k,m} f_{km}(U))] \leq \left(\frac{3}{6^{1/3}}\right) a \left(\frac{C(g)n}{2/3} \log(Kp)\right)^{2/3} \left(\max_{j,m} E[Z_{j,m}^3] + \max_{j,m} E[U_{j,m}^3]\right)^{1/3}$$

**Remark.** The constant in the inequality above can be improved somewhat by using expressions for $A_1$, $A_2$, and $A_3$ in the proof given below. I do not follow this step because that would mess up the statement of the theorem significantly.

**Proof.** As in Chatterjee (2005), for $\alpha \geq 1$, let $F_\alpha : \mathbb{R}^{p \times n}$ be such that

$$F_\alpha(x) = \alpha^{-1} \log\left(\sum_{k,m} \exp(\alpha f_{km}(x))\right)$$

(A.5)

for all $x \in \mathbb{R}^{p \times n}$. Then

$$\max_{k,m} f_{km}(x) = \alpha^{-1} \log(\exp(\alpha \max_{k,m} f_{km}(x)))$$

$$\leq \alpha^{-1} \log(\sum_{k,m} \exp(\alpha f_{km}(x)))$$

$$\leq \alpha^{-1} \log(Kp \exp(\alpha \max_{k,m} f_{km}(x)))$$

$$\leq \alpha^{-1} \log(Kp) + \max_{k,m} f_{km}(x)$$

So,

$$|\max_{k,m} f_{km}(x) - F_\alpha(x)| \leq \alpha^{-1} \log(Kp)$$

(A.6)
Thus,

\[ |E[g(\max_{k,m} f_{km}(Z))] - E[g(\max_{k,m} f_{km}(U))]| \leq 2\|g'\|_{\infty}^{-1} \log(Kp) + |E[g(F_{\alpha}(Z))] - E[g(F_{\alpha}(U))]| \]

For any \( j = 0, ..., n \), denote \( Z^j = (Z_1, ..., Z_j, U_{j+1}, ..., U_n) \). Then

\[ |E[g(F_{\alpha}(Z))] - E[g(F_{\alpha}(U))]| \leq \sum_{j=1}^{n} |E[g(F_{\alpha}(Z^j))] - E[g(F(Z^{j-1}))]| \quad \text{(A.7)} \]

For \( Z_1, ..., Z_{j-1}, U_{j+1}, ..., U_n \) fixed, denote \( l(Z_j) = g(F_{\alpha}(Z^j)) \). By Taylor formula,

\[
g(F_{\alpha}(Z^j) - g(F_{\alpha}(Z^{j-1})) = l(Z_j) - l(U_j) = \sum_{m_1} \frac{\partial l(0)}{\partial Z_{jm_1}} (Z_{jm_1} - U_{jm_1})
\]
\[ + \frac{1}{2} \sum_{m_1, m_2} \frac{\partial^2 l(0)}{\partial Z_{jm_1} \partial Z_{jm_2}} (Z_{jm_1} Z_{jm_2} - U_{jm_1} U_{jm_2})
\]
\[ + \frac{1}{6} \sum_{m_1, m_2, m_3} \frac{\partial^3 l(\tilde{Z})}{\partial Z_{jm_1} \partial Z_{jm_2} \partial Z_{jm_3}} Z_{jm_1} Z_{jm_2} Z_{jm_3}
\]
\[ - \frac{1}{6} \sum_{m_1, m_2, m_3} \frac{\partial^3 l(\tilde{U})}{\partial Z_{jm_1} \partial Z_{jm_2} \partial Z_{jm_3}} U_{jm_1} U_{jm_2} U_{jm_3} \]

where \( \tilde{Z} \) and \( \tilde{U} \) are on the lines connecting 0 and \( Z_j \) and 0 and \( U_j \) correspondingly. By independence,

\[
|E[g(F_{\alpha}(Z^j))] - E[g(F(Z^{j-1}))]| \leq (1/6) \sum_{m_1, m_2, m_3} \sup_{X \in \mathbb{R}^{p \times n}} \left| \frac{\partial^3 g(F_{\alpha}(X))}{\partial X_{jm_1} \partial X_{jm_2} \partial X_{jm_3}} \right| (E[|Z_{jm_1} Z_{jm_2} Z_{jm_3}|] + E[|U_{jm_1} U_{jm_2} U_{jm_3}|])
\]

By Holder inequality,

\[
E[|Z_{jm_1} Z_{jm_2} Z_{jm_3}|] \leq \max_m E[|Z_{jm}|^3] \quad \text{(A.8)}
\]
and
\[
E[|U_{jm1} U_{jm2} U_{jm3}|] \leq \max_m E[|U_{jm}|^3] \tag{A.9}
\]

Denote
\[
A_1 = \sup_{X \in \mathbb{R}^{p \times n}} \left| \frac{\partial F_\alpha(X) \partial F_\alpha(X) \partial F_\alpha(X)}{\partial X_{jm1} \partial X_{jm2} \partial X_{jm3}} \right| \tag{A.10}
\]

\[
A_2 = \sup_{X \in \mathbb{R}^{p \times n}} \left| \frac{\partial F_\alpha(X) \partial^2 F_\alpha(X)}{\partial X_{jm1} \partial X_{jm2} \partial X_{jm3}} \right| + \sup_{X \in \mathbb{R}^{p \times n}} \left| \frac{\partial F_\alpha(X) \partial^2 F_\alpha(X)}{\partial X_{jm1} \partial X_{jm2} \partial X_{jm3}} \right| + \sup_{X \in \mathbb{R}^{p \times n}} \left| \frac{\partial F_\alpha(X) \partial^2 F_\alpha(X)}{\partial X_{jm1} \partial X_{jm2} \partial X_{jm3}} \right| \tag{A.11}
\]

Then
\[
\sup_{X \in \mathbb{R}^{p \times n}} \left| \frac{\partial^3 g(F_\alpha(X))}{\partial X_{jm1} \partial X_{jm2} \partial X_{jm3}} \right| \leq \|g''\|_\infty A_1 + \|g''\|_\infty A_2 + \|g'\|_\infty A_3 \tag{A.12}
\]

So, it only remains to bound partial derivatives of \(F_\alpha\).

To simplify notation, denote \(B_{km} = \exp(\alpha f_{km}(X))\) for \(k = 1, ..., K\) and \(m = 1, ..., p\). Then
\[
\frac{\partial F_\alpha(X)}{\partial X_{jm1}} = \frac{\sum_k B_{km} a_{kjm1}}{\sum_{k,m} B_{km}} \tag{A.13}
\]

The expression on the right hand side of the formula above is the expectation of a random variable which takes value \(a_{kjm1}\) with probability \(B_{km} / \sum_{km} B_{km}\) for \(k = 1, ..., K\) and 0 with probability \(1 - \sum_k B_{km} / \sum_{km} B_{km}\). If \(m_1, m_2,\) and \(m_3\) are all different, then
\[
\frac{\partial F_\alpha(X)}{\partial X_{jm1}} \frac{\partial F_\alpha(X)}{\partial X_{jm2}} \frac{\partial F_\alpha(X)}{\partial X_{jm3}} \tag{A.14}
\]

will be the product of expectations of 3 random variables with nonintersecting supports. It is easy to see that this product will be not greater than \(a^3/27\). All other cases can
be treated by the same argument. We have

\[
A_1 \leq \begin{cases} 
  a^3/27 & \text{if } m_1, m_2, \text{ and } m_3 \text{ are all different} \\
  4a^3/27 & \text{if } m_1 = m_2 \neq m_3 \\
  a^3 & \text{if } m_1 = m_2 = m_3 
\end{cases}
\] (A.15)

If \( m_1, m_2, \text{ and } m_3 \) are all different, then

\[
\frac{\partial^2 F_\alpha(X)}{\partial X_{jm_1} \partial X_{jm_2}} = -\alpha \frac{\sum_k B_{km_1} a_{kj_1jm_2} \sum_k B_{km_2} a_{kj_2jm_2}}{\left(\sum_k B_{km}\right)^2}
\] (A.16)

and

\[
\frac{\partial^3 F_\alpha(X)}{\partial X_{jm_1} \partial X_{jm_2} \partial X_{jm_3}} = 2\alpha^2 \frac{\sum_k B_{km_1} a_{kj_1jm_3} \sum_k B_{km_2} a_{kj_2jm_2} \sum_k B_{km_3} a_{kj_3jm_3}}{\left(\sum_k B_{km}\right)^3}
\] (A.17)

If \( m_1 = m_2 \neq m_3 \), then

\[
\frac{\partial^2 F_\alpha(X)}{\partial X_{jm_1} \partial X_{jm_2}} = -\alpha \frac{(\sum_k B_{km_1} a_{kj_1jm_3})^2}{\left(\sum_k B_{km}\right)^2} + \alpha \frac{\sum_k B_{km_1} a_{kj_1jm_1}}{\sum_k B_{km}}
\] (A.18)

and

\[
\frac{\partial^3 F_\alpha(X)}{\partial X_{jm_1} \partial X_{jm_2} \partial X_{jm_3}} = 2\alpha^2 \frac{(\sum_k B_{km_1} a_{kj_1jm_3})^2 \sum_k B_{km_3} a_{kj_3jm_3}}{\left(\sum_k B_{km}\right)^3} - \alpha^2 \frac{\sum_k B_{km_1} a_{kj_1jm_1} \sum_k B_{km_3} a_{kj_3jm_3}}{\left(\sum_k B_{km}\right)^2}
\]

If \( m_1 = m_2 = m_3 \), then

\[
\frac{\partial^3 F_\alpha(X)}{\partial X_{jm_1} \partial X_{jm_2} \partial X_{jm_3}} = \alpha^2 \frac{\sum_k B_{km_1} a_{kj_1jm_1}^3}{\left(\sum_k B_{km}\right)} - 3\alpha^2 \frac{\sum_k B_{km_1} a_{kj_1jm_1} \sum_k B_{km_1} a_{kj_1jm_1}}{\left(\sum_k B_{km}\right)^2} + 2\alpha^2 \frac{(\sum_k B_{km_1} a_{kj_1jm_1})^3}{\left(\sum_k B_{km}\right)^3}
\]
So,

\[
A_2 \leq \begin{cases} 
3\alpha a^3/27 & \text{if } m_1, m_2, \text{ and } m_3 \text{ are all different} \\
59\alpha a^3/108 & \text{if } m_1 = m_2 \neq m_3 \\
3\alpha a^3 & \text{if } m_1 = m_2 = m_3
\end{cases} \tag{A.19}
\]

and

\[
A_3 \leq \begin{cases} 
2\alpha^2 a^3/27 & \text{if } m_1, m_2, \text{ and } m_3 \text{ are all different} \\
8\alpha^2 a^3/27 & \text{if } m_1 = m_2 \neq m_3 \\
\alpha^2 a^3 & \text{if } m_1 = m_2 = m_3
\end{cases} \tag{A.20}
\]

Therefore,

\[
|E[g(\max_{k,m} f_{km}(Z))] - E[g(\max_{k,m} f_{km}(U))]| \\
\leq 2\|g'\|_\infty \alpha^{-1} \log(Kp) + \frac{np^3\alpha^2a^3}{6} C(g) \left[ \max_{j,m} E[|Z_{jm}|^3] + \max_{j,m} E[|U_{jm}|^3] \right]
\]

Optimizing with respect to $\alpha$ yields the result. \hfill \square

### A.4 Primitive Conditions for Assumption 1

In this section, I give a counter-example for the statement that for assumption [I] to hold, it suffices to assume that $\{X_i : i = 1, \ldots, n\}$ are sampled from a distribution that is absolutely continuous with respect to Lebesgue measure, has bounded support, and whose density is bounded from above and away from zero on the support. I also prove that assumption [I] holds if, in addition to above conditions, one assumes that the support is a convex set.

**Lemma 3.** There exist a probability distribution on $[-1, 1]^2$ which is uniform on its support such that if $\{X_i : i = 1, \ldots, n\}$ are sampled from this distribution, then assumption [I] fails.

**Proof.** As an example of such a probability distribution, consider the uniform distribution on

\[
S = \{(x_1, x_2) \in [-1, 1]^2 : x_1 \geq 0; -(1 + \alpha)x_1^\alpha/2 \leq x_2 \leq (1 + \alpha)x_1^\alpha/2\} \tag{A.21}
\]
for some $\alpha > 0$. For fixed $i$, the probability that $X_{i,1} \leq \underline{h}$ is $p = \frac{\underline{h}}{1 + \alpha}$, and the probability that $X_{i,1} > \overline{h}$ is $\overline{p} = 1 - \frac{\overline{h}}{1 + \alpha}$. Let $A_n$ be an event that $X_{i,1} \leq \underline{h}$ for exactly one $i = 1, \ldots, n$ whereas $X_{i,1} > \overline{h}$ for all other $i = 1, \ldots, n$ with $\underline{h} < \overline{h}$. The probability of this event is

$$P(A_n) = np\overline{p}^{n-1} = n\frac{\underline{h}}{1 + \alpha} \left(1 - \frac{\overline{h}}{1 + \alpha}\right)^{n-1} \tag{A.22}$$

Set $\underline{h} = \left(\frac{C_1}{n}\right)^{1/(1+\alpha)}$ and $\overline{h} = \left(\frac{C_2}{n}\right)^{1/(1+\alpha)}$ with $0 < C_1 < C_2 < 1$. Then we can find the limit of $P(A_n)$ as $n \to \infty$:

$$\lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} C_1(1 - C_2/n)^{n-1} = C_1 e^{-C_2} > 0 \tag{A.23}$$

Note that on $A_n$, there is an observation $X_i$ such that there is no other observations in the ball with center at $X_i$ and radius $\left(\frac{C_1}{n}\right)^{1/(1+\alpha)}$. The result now follows by choosing $\alpha$ sufficiently large such that $n^{-1/(1+\alpha)}$ converges to zero slower then $h_{\min}$.

Now I give a sufficient primitive condition for assumption 1.

**Lemma 4.** If $\{X_i : i = 1, \ldots, n\}$ are sampled from a distribution which is absolutely continuous with respect to Lebesgue measure, has bounded and convex support $S \subset \mathbb{R}^d$, and whose density is bounded from above and away from zero on the support, then assumption 1 holds for large $n$ almost surely.

**Proof.** Consider sets of the following form: $I(a_1, \ldots, a_d, c) = S \cap \{x : a_1 x_1 + \ldots + a_d x_d = c\}$ with $a_1^2 + \ldots + a_d^2 = 1$. These are convex sets. It follows from the fact that the density is bounded from above that $\inf_{a_1, \ldots, a_d} \sup_c D(I(a_1, \ldots, a_d, c)) > 0$ where $D(\cdot)$ denotes the diameter of the set. So, there exists some constant $0 < C \leq 1$ such that for all $r < 1$ and all $x \in S$, each ball with center at $x$ and radius $r$ has at least fraction $C$ of its Lebesgue measure inside of the support $S$: $\lambda(B(x, r) \cap S)/\lambda(B(x, r)) > C$.

Note that $\delta$-covering numbers of the set $S$ satisfy $N(\delta) \lesssim \delta^d$ as $\delta \to 0$, i.e. there exists some constant $C > 0$ such that $N(\delta, S) < C/\delta^d$. Consider the lower bound. For each $h \in H_n$, consider the set of covering balls with centers $G_{h,1}, \ldots, G_{h,N(h)}$ and radii $\delta_h = h/2$. Then for each $X_i$ and $h \in H_n$, there exists some $j \in \{1, \ldots, N(h)\}$ such that $B(X_i, h) \supset B(G_{h,j}, \delta_h)$. Thus, it is enough to prove the lower bound for the
number of observations dropping into these covering balls. Since the density is bounded away from zero, there exists some constant $C > 0$ such that for each $h \in H_n$ and $j = 1, ..., N(h)$, $P(X_i \in B(G_{h,j}, \delta_h)) > C h^d$. Denote $I_{h,j}(X_i) = I\{X_i \in B(G_{h,j}, \delta_h)\}$. A Hoeffding inequality (see proposition 1.3.5 in Dudley (1999)) gives

$$P\left\{ \frac{1}{n} \sum_{i=1}^{n} I_{h,j}(X_i) < \frac{Ch^d}{2} \right\} \leq P\left\{ \frac{1}{n} \sum_{i=1}^{n} I_{h,j}(X_i) < E[I_{h,j}(X_i)] < -\frac{Ch^d}{2} \right\} \leq C \exp(-C n h^d)$$

(A.24)

Then by union bound,

$$P(\bigcup_{h \in H_n, j=1, ..., N(h)} \frac{1}{n} \sum_{i=1}^{n} I_{h,j}(X_i) < \frac{Ch^d}{2}) \leq C h_{\min}^{-d} \log n \exp(-C n h_{\min}^d) \rightarrow 0$$

(A.25)

as $n \rightarrow \infty$. Summing the probabilities above over $n$, we conclude, by the Borel-Cantelli lemma, that the lower bound in assumption 1(iii) holds for large $n$ almost surely. A similar argument gives the upper bound.

\[ \square \]

### A.5 Anticoncentration Inequality for the Maximum of Gaussian Random Variables

In this section, I derive an upper bound for the pdf of the maximum of correlated Gaussian random variables satisfying certain assumptions. Let $\{Z_i : i = 1, ..., S\}$ be a set of standard Gaussian random variables. Assume that this set contains at least $M$ independent random variables. Define $W = \max_{i=1,...,S} Z_i$. Let $m$ denote the median of $W$ and $f_W(\cdot)$ denote its pdf. Then

**Lemma 5.** $\sup_{w > m} f_W(w) \leq C \sqrt{\log(M + 1) S / M}$ for some universal constant $C$.

**Proof.** The case $M = 1$ is trivial. So, assume that $M > 1$. Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cdf and the pdf of the standard Gaussian distribution. Since there is at least $M$ independent standard Gaussian random variables, $\Phi^M(m) \geq 1/2$ and $m > 0$. So, there exists some constant $C > 0$ such that $\Phi(x) \leq 1 - \phi(x)/(Cx)$ for any $x \geq m$ (see proposition 2.2.1 in Dudley (1999)) and

$$\left(1 - \frac{\phi(m)}{Cm}\right)^M \geq \frac{1}{2}$$

(A.26)
Let $y$ denote the unique positive real number such that

$$\left(1 - \frac{\phi(y)}{Cy}\right)^M = \frac{1}{2}$$  \hspace{1cm} (A.27)

Note that $y \leq m$. In addition, $y$ is increasing in $M$, so there exists some constant $C_1 > 0$ such that $y > C_1$ for any $M \geq 2$. Taking logs of both sides of equation (A.27) and noting that $\log(1 + x) \leq x$ for any $x \in \mathbb{R}$, we obtain $\phi(y) \leq y\log C/M$ for some constant $C > 1$. On the other hand, $\phi(y)/(Cy) < 1/2$. So inequality $\log(1 + x) \geq 2x$ for any $x \in (-1/2, 0]$ gives

$$\frac{2M\phi(y)}{Cy} \geq \log 2$$ \hspace{1cm} (A.28)

Combining this inequality with $y > C_1$ yields $y \leq C\sqrt{\log(M + 1)}$ for any $M$ if $C$ is sufficiently large. Therefore, $\phi(y) \leq C\sqrt{\log(M + 1)}/M$ and for any $w > m$,

$$f_W(w) \leq S\phi(w) \leq S\phi(m) \leq S\phi(y) \leq C\sqrt{\log(M + 1)}S/M$$ \hspace{1cm} (A.29)

\[\square\]

### A.6 Result on Gaussian Random Variables

In this section, I state a result on Gaussian random variables which will be used in the derivation of the lower bound on the rate of uniform consistency.

**Lemma 6.** Let $\xi_n$, $n = 1, ..., \infty$, be a sequence of independent standard Gaussian random variables and $w_{i,n}$, $i = 1, ..., n$, $n = 1, ..., \infty$, be a triangular array of positive numbers. If $w_{i,n} < C\sqrt{\log n}$ with $C \in (0, 1)$ for all $i = 1, ..., n$, $n = 1, ..., \infty$, then

$$\lim_{n \to \infty} E\left[|n^{-1} \sum_{i=1}^{n} \exp(w_{i,n}\xi_i - w_{i,n}^2/2) - 1|\right] = 0$$ \hspace{1cm} (A.30)

**Proof.** The proof is based on the generalization of lemma 6.2 in Dumbgen and Spokoiny (2001). Denote $Z_{i,n} = \exp(w_{i,n}\xi_i - w_{i,n}^2/2)$ and $t_n = (E[\sum_{i=1}^{n} Z_{i,n}/n - 1])^{1/2}$. Note
that $EZ_{i,n} = 1$ and $EZ_{i,n}^2 = \exp(w_{i,n}^2)$. Thus,

$$t_n^2 = \left( \sum_{i=1}^{n} (EZ_{i,n}^2 - (EZ_{i,n})^2) \right)/n^2 \leq \sum_{i=1}^{n} \exp(w_{i,n}^2)/n^2 \to 0 \quad (A.31)$$

if $\max_{i=1,...,n} \exp(w_{i,n}^2)/n \to 0$. The last condition holds by assumption. So,

$$E|n^{-1} \sum_{i=1}^{n} \exp(w_{i,n}\xi_i - w_{i,n}^2/2) - 1| = \int_{t_n}^{\infty} P(|n^{-1} \sum_{i=1}^{n} Z_{i,n} - 1| > t)dt \leq t_n + \int_{t_n}^{\infty} t^2/n^2 dt \leq 2t_n \to 0$$

\[\square\]

**A.7 Preliminary Technical Results**

In this section, I derive some necessary preliminary results that are used in the proofs of the theorems stated in the main part of the paper. It is assumed throughout that assumptions 1-8 hold. I will use the following additional notation. Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\psi_n \geq C\psi p\log n/n\kappa/4$ for some large constant $C\psi > 0$ and $\psi_n \to 0$ as $n \to \infty$. For any $\lambda \in (0,1)$, define $c_{1-\lambda}^{PIA,0} \in \mathbb{R}$ and $g_{1-\lambda}^{PIA,0} : \mathbb{R} \to [0,1]$ by analogy with $c_{1-\lambda}^{PIA}$ and $g_{1-\lambda}^{PIA}$ with $\Sigma_i$ used instead of $\hat{\Sigma}_i$ for all $i = 1, ..., n$. Denote $S_n^{D} = \{s \in S_n : f_s/V_s > -(c_{1-\gamma_n-\psi_n}^{PIA,0} + \beta_n)\}$. For any $\lambda \in (0,1)$, define $c_{1-\lambda}^{D} \in \mathbb{R}$ and $g_{1-\lambda}^{D} : \mathbb{R} \to [0,1]$ by analogy with $c_{1-\lambda}^{RMS}$ and $g_{1-\lambda}^{RMS}$ with $S_n^{D}$ used instead of $S_n^{RMS}$. Let $\{\epsilon_i : i = 1, ..., n\}$ be an iid sequence of $p$-dimensional standard Gaussian random vectors that are independent of the data. Denote $\hat{e}_j = \hat{\Sigma}^{1/2}\epsilon_j$ and $e_j = \Sigma^{1/2}\epsilon_j$. Note that $\hat{e}_j$ is equal in distribution to $\tilde{Y}_j$. Finally, denote

$$\varepsilon_{i,m,h} = \sum_{j=1}^{n} w_h(X_i, X_j)\varepsilon_{j,m} \quad (A.32)$$

$$f_{i,m,h} = \sum_{j=1}^{n} w_h(X_i, X_j)f_m(X_j) \quad (A.33)$$
\[ e_{i,m,h} = \sum_{j=1}^{n} w_h(X_i, X_j) e_j \]  
(A.34)

\[ \hat{e}_{i,m,h} = \sum_{j=1}^{n} w_h(X_i, X_j) \hat{e}_j \]  
(A.35)

\[ T^{PIA} = \max_{s \in S_n} (\hat{e}_s / \hat{V}_s) \]  
(A.36)

\[ T^{PIA,0} = \max_{s \in S_n} (e_s / V_s) \]  
(A.37)

Note that \( T^{PIA} \) is equal in distribution to the simulated statistic.

I start with a result on bounds for weights and variances of the kernel estimator. The same result can be found in [Horowitz and Spokoiny (2001)].

**Lemma 7.** There exist constants \( C > 0 \) and \( 0 < C_1 < C_2 < \infty \) such that, for any \( i, j = 1, \ldots, n \), \( m = 1, \ldots, p \), and \( h \in H_n \),

\[ w_h(X_i, X_j) \leq C/(nh^d) \]  
(A.38)

and

\[ C_1/\sqrt{nh^d} \leq V_{i,m,h} \leq C_2/\sqrt{nh^d} \]  
(A.39)

**Proof.** By assumptions 1 and 6, for any \( i = 1, \ldots, n \) and \( h \in H_n \),

\[ C_1 nh^d \leq CM_h/2(X_i) \leq \sum_{k=1}^{n} K(X_i - X_k) \leq M_h(X_i) \leq C_2 nh^d \]  
(A.40)

and

\[ C_1 nh^d \leq \sum_{k=1}^{n} K^2(X_i - X_k) \leq C_2 nh^d \]  
(A.41)

for some constants \( C > 0 \) and \( 0 < C_1 < C_2 < \infty \). In addition, \( K(X_i - X_j) \leq 1 \) for any \( j = 1, \ldots, n \). So,

\[ w_h(X_i - X_j) = K(X_i - X_j) / \sum_{k=1}^{n} K(X_i - X_k) \leq C/(nh^d) \]  
(A.42)
By assumption \[2\] since \(\sum_{j=1}^{n} w_h(X_i, X_j) = 1\),

\[
V_{i,m,h} = \left( \sum_{j=1}^{n} w_h^2(X_i, X_j) \Sigma_{j,mm} \right)^{1/2}
\leq C \left( \sum_{j=1}^{n} w_h^2(X_i, X_j) \right)^{1/2}
\leq C \max_{j=1, \ldots, n} w_h^{1/2}(X_i, X_j)
\leq C/\sqrt{nh^d}
\]

and

\[
V_{i,m,h} \geq C \left( \sum_{j=1}^{n} w_h^2(X_i, X_j) \right)^{1/2} \geq (C/\sqrt{nh^d}) \left( \sum_{j=1}^{n} K^2(X_i - X_j) \right)^{1/2} \geq C/\sqrt{nh^d}
\quad \text{(A.43)}
\]

Lemma 8. \(E[\max_{s \in S_n} |e_s/V_s|] \leq C(\log n)^{1/2}\).

**Proof.** For any \(s \in S_n\), \(e_s/V_s\) is a standard Gaussian random variable. Denote \(\psi = \exp(x^2) - 1\). Let \(\| \cdot \|_\psi\) denote \(\psi\)-Orlicz norm. It is easy to check that \(\|e_s/V_s\|_\psi < C < \infty\). So, by lemma 2.2.2 in Van der Vaart and Wellner (1996),

\[
E[\max_{s \in S_n} |e_s/V_s|] \leq C \| \max_{s \in S_n} |e_s/V_s|\|_\psi \leq C(\log n)^{1/2}
\quad \text{(A.44)}
\]

since \(|S_n| \leq Cn^\phi\) for some \(\phi > 0\). \(\square\)

Lemma 9. \(\max_{s \in S_n} |\hat{V}_s/V_s - 1| = o_p(n^{-\kappa})\) and \(\max_{s \in S_n} |V_s/\hat{V}_s - 1| = o_p(n^{-\kappa})\).

**Proof.** By assumption \[2\] for any \((i, m, h) \in S_n\),

\[
V_{i,m,h}^2 = \sum_{j=1}^{n} w_h^2(X_i, X_j) \Sigma_{j,mm} \geq C \sum_{j=1}^{n} w_h^2(X_i, X_j)
\quad \text{(A.45)}
\]

In addition,

\[
|\hat{V}_{i,m,h}^2 - V_{i,m,h}^2| \leq \sum_{j=1}^{n} w_h^2(X_i, X_j) |\hat{\Sigma}_{j,mm} - \Sigma_{j,mm}|
\quad \text{(A.46)}
\]
So,

\[
\max_{s \in S_n} \left| \frac{\hat{V}_s^2}{V_s^2} - 1 \right| \leq C \max_{m=1, \ldots, p} \max_{j=1, \ldots, n} \left| \hat{\Sigma}_{j,mm} - \Sigma_{j,mm} \right|
\]

\[
\leq C \max_{j=1, \ldots, n} \| \hat{\Sigma}_j - \Sigma_j \|_o
\]

Assumption 5 gives \( \max_{j=1, \ldots, n} \| \hat{\Sigma}_j - \Sigma_j \|_o = o_p(n^{-\kappa}) \). So, \( \max_{s \in S_n} \left| \frac{\hat{V}_s^2}{V_s^2} - 1 \right| = o_p(n^{-\kappa}) \). Combining this result with inequality \(|x-1| \leq |x^2-1|\), which holds for any \( x > 0 \), yields the first result of the lemma. The second result follows from the first one and the inequality \(|1/x-1| < 2|x-1|\), which holds for any \(|x-1| < 1/2\).

**Lemma 10.** \( P\{c_{1-\nu_n-\psi_n}^{PIA,0} > c_1^{PIA} \} = o(1) \) and \( P\{c_{1-\nu_n+\psi_n}^{PIA,0} < c_1^{PIA} \} = o(1) \) for any sequences \( \{\nu_n\}_{n=1}^{\infty} \) and \( \{\psi_n\}_{n=1}^{\infty} \) of positive numbers satisfying \( \nu_n + \psi_n \leq 1/2 \) and \( \psi_n \geq C_{\psi} p \log n / n^{\kappa/4} \) with large enough \( C_{\psi} > 0 \).

**Proof.** Denote

\[
p_1 = \max_{s \in S_n} \left| \frac{\hat{V}_s}{V_s} \right| \max_{s \in S_o} \left| \frac{V_s}{\hat{V}_s} - 1 \right|
\]

and

\[
p_2 = \max_{(i,h,m) \in S_n} \left| \sum_{j=1}^{n} w_h(X_i, X_j) \left( \left( \hat{\Sigma}_j^{1/2} - \Sigma_j^{1/2} \right) \epsilon_j \right)_m / V_{i,m,h} \right|
\]

Then

\[
|T^{PIA} - T^{PIA,0}| \leq p_1 + p_2
\]

Let \( A \) denote the event \( \{ \max_{j=1, \ldots, n} \| \hat{\Sigma}_j - \Sigma_j \|_o < n^{-\kappa} \} \). By assumption 5, \( P(A) \rightarrow 1 \) as \( n \rightarrow \infty \). Thus, it is enough to show that \( c_{1-\nu_n-\psi_n}^{PIA,0} \leq c_1^{PIA} \) and \( c_{1-\nu_n+\psi_n}^{PIA,0} \geq c_1^{PIA} \) on \( A \).

As in the proof of lemma 5, \( \max_{s \in S_n} |V_s/\hat{V}_s - 1| \leq C n^{-\kappa} \) on \( A \). By lemma 5, \( E[\max_{s \in S_n} e_s/V_s] \leq C \sqrt{\log n} \). So, Markov inequality gives for any \( B > 0 \), on \( A \),

\[
P(p_1 > C \sqrt{\log nn^{-\kappa}} B | Y_1^n) \leq 1/B
\]
where $Y_1^n$ is a shorthand for $\{Y_i\}_{i=1}^n$. Consider $p_2$. For any $j = 1, ..., n$ and $m = 1, ..., p$

$$E[\|(\hat{\Sigma}_j^{1/2} - \Sigma_j^{1/2})\epsilon_j\|_m^2|Y_1^n] \leq E[\|(\hat{\Sigma}_j^{1/2} - \Sigma_j^{1/2})\epsilon_j\|_0^2|Y_1^n] \leq E[\hat{\Sigma}_j^{1/2} - \Sigma_j^{1/2}]^2_0 \|\epsilon_j\|^2|Y_1^n] \leq p^2(\hat{\Sigma}_j - \Sigma_j)_o$$

where the last line follows from lemma\textsuperscript{2}. So, conditionally on $Y_1^n$, on $A$, $\sum_{j=1}^n w_h(X_i, X_j)((\hat{\Sigma}_j^{1/2} - \Sigma_j^{1/2})\epsilon_j)_m/V_{i,m,h}$ is mean-zero Gaussian random variable with variance bounded by $p^2n^{-\kappa}$ for any $(i, m, h) \in S_n$. In addition, on $A$, $\max_{s \in S_n} V_s/\hat{V}_s \leq 2$ for large $n$. Thus, Markov inequality and the argument like that used in lemma\textsuperscript{8} yield

$$P(p_2 > C\sqrt{\log n} p n^{-\kappa/2} B | Y_1^n) \leq 1/B \quad \text{(A.51)}$$

on $A$. Take $B = n^{\kappa/4}/(p \log n)$. Recall that $\psi_n \geq C_\psi p \log n/n^{\kappa/4}$. So, $\psi_n > \max(4/B, C_1p^2(\log n)^2n^{-\kappa/2}B)$ for some large $C_1 > 0$ whenever $C_1 < C_\psi$.

Note that $T^{PIA,0}$ is the maximum over $|S_n|$ standard Gaussian random variables. In addition, for fixed $m = 1, ..., p$ and $h \in H_n$, random variables $\{e_{i,m,h}/V_{i,m,h} : (i, m, h) \in S_n\}$ are mutually independent, $|H_n| \leq C \log n$. So, lemma\textsuperscript{5} gives $c_{1-\nu_n-\psi_n/2}^{PIA,0} - c_{1-\nu_n-\psi_n}^{PIA,0} \geq C\psi_n/(p(\log n)^{3/2})$. I will assume that $C$ in the last inequality is smaller than $C_1$.

Now the first part of the lemma follows from

$$E[g_{1-\nu_n-\psi_n}^{PIA,0}(T^{PIA})|Y_1^n] \leq E[g_{1-\nu_n-\psi_n}^{PIA,0}(T^{PIA} - p_1 - p_2)|Y_1^n] \leq E[g_{1-\nu_n-\psi_n}^{PIA,0}(T^{PIA} - C\sqrt{\log n} n^{-\kappa/2} B)|Y_1^n] + 2/B \leq E[g_{1-\nu_n-\psi_n/2}^{PIA,0}(T^{PIA,0})|Y_1^n] + 2/B = 1 - \nu_n - \psi_n/2 + 2/B \leq 1 - \nu_n$$

on $A$. The second part of the lemma follows from a similar argument. \hfill \Box

**Lemma 11.** $E[g_{1-\nu_n}^{PIA,0}(\max_{s \in S_n}(\varepsilon_s/V_s))] = 1 - \nu_n + o(1)$ and $E[g_{1-\nu_n}^{PIA,0}(-\max_{s \in S_n}(\varepsilon_s/V_s))] = 1 - \nu_n + o(1)$ for any sequence $\{\nu_n\}_{n=1}^\infty$ such that $\nu_n \in (0, 1)$. 45
Proof. By lemma 7 for any \((i, m, h) \in S_n\) and any \(j = 1, \ldots, n\),

\[
  w_h(X_i, X_j)/V_{i,m,h} \leq C/\sqrt{nh^d} \leq C/\sqrt{nh^d_{\min}} \quad (A.52)
\]

Recall the definition of \(C(\cdot)\) given before theorem 6. By assumption 7, \(\beta = \beta_n \leq C\) for some constant \(C > 0\). So, \(C(g_{1-\alpha}^{\text{PLA}_0}) \leq C/\beta^3\). In addition, \(\|g_{1-\alpha}^{\text{PLA}_0}\|_{\infty} \leq C/\beta\).

Given assumption 7, the result follows by applying theorem 6 with \(\text{Lemma 13}\).

**Lemma 12.** \(\max_{s \in S_n} |\varepsilon_s/V_s| = O_p(\sqrt{\log n})\) and \(\max_{s \in S_n} |\varepsilon_s/\hat{V}_s| = O_p(\sqrt{\log n})\).

**Proof.** Combining the definition of \(g_0\), lemma 11 and \(\beta_n \leq C\) for some constant \(C > 0\) gives

\[
P\{\max_{s \in S_n} (\varepsilon_s/V_s) > C\sqrt{\log n}\} \leq 1 - E[g_0((\max_{s \in S_n} (\varepsilon_s/V_s) + \beta_n - C\sqrt{\log n})/\beta_n)]
\]

\[
= 1 - E[g_0((\max_{s \in S_n} (\varepsilon_s/V_s) + \beta_n - C\sqrt{\log n})/\beta_n)] + o(1)
\]

\[
\leq P\{\max_{s \in S_n} (\varepsilon_s/V_s) > C\sqrt{\log n} - \beta_n\} + o(1)
\]

\[
\leq P\{\max_{s \in S_n} (\varepsilon_s/V_s) > (C/2)\sqrt{\log n}\} + o(1)
\]

By lemma 8 \(\max_{s \in S_n} (\varepsilon_s/V_s) = O_p(\sqrt{\log n})\). So, by choosing \(n\) large enough and then \(C\) large enough, we can make \(P\{\max_{s \in S_n} (\varepsilon_s/V_s) > C\sqrt{\log n}\}\) arbitrarily small uniformly in \(n\). The same reasoning gives the lower as well. We conclude that \(\max_{s \in S_n} (\varepsilon_s/V_s) = O_p(\sqrt{\log n})\). The second result follows from

\[
\max_{s \in S_n} |\varepsilon_s/\hat{V}_s| \leq \max_{s \in S_n} |\varepsilon_s/V_s| \max_{s \in S_n} (V_s/\hat{V}_s) = O_p(\sqrt{\log n}) \quad (A.53)
\]

since \(\max_{s \in S_n} (V_s/\hat{V}_s) = O_p(1)\) by lemma 9.

**Lemma 13.** \(P\{\max_{s \in S_n \backslash S_0} \hat{f}_s/\hat{V}_s > 0\} \leq \gamma_n + o(1)\).

**Proof.** By lemma 11

\[
P\{\max_{s \in S_n} (\varepsilon_s/V_s) \leq c_{1-\gamma_n-\psi_n}^{\text{PLA}_0} + \beta_n\} \geq E[g_{1-\gamma_n-\psi_n}^{\text{PLA}_0} (\max_{s \in S_n} (\varepsilon_s/V_s))] = 1 - \gamma_n - \psi_n + o(1) \quad (A.54)
\]
Since for any $s \in S_n \setminus S_n^D$, $f_s/V_s \leq -(c^{PIA,0}_{1-\gamma_n-\psi_n} + \beta_n)$,

$$P\{ \max_{s \in S_n \setminus S_n^D} (\hat{f}_s/\hat{V}_s) > 0 \} = P\{ \max_{s \in S_n \setminus S_n^D} (\hat{f}_s/V_s) > 0 \}$$
$$= P\{ \max_{s \in S_n \setminus S_n^D} (f_s/V_s + \varepsilon_s/V_s) > 0 \}$$
$$\leq P\{ \max_{s \in S_n \setminus S_n^D} (-c^{PIA,0}_{1-\gamma_n-\psi_n} - \beta_n - \varepsilon_s/V_s) > 0 \}$$
$$\leq P\{ \max_{s \in S_n} (\varepsilon_s/V_s) > c^{PIA,0}_{1-\gamma_n-\psi_n} + \beta_n \}$$
$$\leq 1 - (1 - \gamma_n - \psi_n) + o(1)$$
$$= \gamma_n + \psi_n + o(1)$$

Noting that $\psi_n = o(1)$ yields the result. \hfill \qed

**Lemma 14.** $P\{S_n^D \subset S_n^{RMS} \} \geq 1 - \gamma_n + o(1)$.

**Proof.** By lemma 10, $P\{c^{PIA,0}_{1-\gamma_n-\psi_n} > c^{PIA}_{1-\gamma_n} \} = o(1)$. In addition, for any $x \in (-1, 1)$,

$$2/(1 + x) - 1 \geq 2(1 - x) - 1 \geq 1 - 2x \geq 1 - 2|x| \quad (A.55)$$

So,

$$P\{S_n^D \subset S_n^{RMS} \} = P\{ \min_{s \in S_n^D} (\hat{f}_s/\hat{V}_s) > -2(c^{PIA}_{1-\gamma_n} + \beta_n) \}$$
$$\geq P\{ \min_{s \in S_n^D} (\hat{f}_s/V_s) \max_{s \in S_n^D} (V_s/\hat{V}_s) > -2(c^{PIA}_{1-\gamma_n} + \beta_n) \}$$
$$\geq P\{ \min_{s \in S_n^D} (-c^{PIA,0}_{1-\gamma_n-\psi_n} - \beta_n + \varepsilon_s/V_s) \max_{s \in S_n^D} (V_s/\hat{V}_s) > -2(c^{PIA}_{1-\gamma_n} + \beta_n) \}$$
$$= P\{ \min_{s \in S_n^D} (\varepsilon_s/V_s) > c^{PIA,0}_{1-\gamma_n-\psi_n} + \beta_n - 2(c^{PIA}_{1-\gamma_n} + \beta_n)/ \max_{s \in S_n^D} (V_s/\hat{V}_s) \}$$
$$\geq P\{ \max_{s \in S_n} (-\varepsilon_s/V_s) < -c^{PIA,0}_{1-\gamma_n-\psi_n} - \beta_n + 2(c^{PIA}_{1-\gamma_n} + \beta_n)/ \max_{s \in S_n^D} (V_s/\hat{V}_s) \} + o(1)$$
$$\geq P\{ \max_{s \in S_n} (-\varepsilon_s/V_s) < (c^{PIA,0}_{1-\gamma_n-\psi_n} + \beta_n)(1 - 2|\max_{s \in S_n^D} (V_s/\hat{V}_s) - 1|) \} + o(1)$$

Combining lemma 8 and Markov inequality yields

$$\gamma_n + \psi_n = 1 - E[g^{PIA,0}_{1-\gamma_n-\psi_n}(\max_{s \in S_n} (e_s/V_s))]$$
$$\leq P\{ \max_{s \in S_n} (e_s/V_s) > c^{PIA,0}_{1-\gamma_n-\psi_n} \}$$
$$\leq C(\log n)^{1/2}/c^{PIA,0}_{1-\gamma_n-\psi_n}$$
So, \( c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} \leq C(\log n)^{1/2}/(\gamma_n + \psi_n) \). By lemma 9, \(|\max_{s \in S_n^D}(V_s/\hat{V}_s) - 1| < Cn^{-\kappa}\) wpal. So, wpal,

\[
(c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)(1 - 2|\max_{s \in S_n^D}(V_s/\hat{V}_s) - 1|) \geq c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n - C(\log n)^{1/2}n^{-\kappa}/(\gamma_n + \psi_n)
\]

(A.56)

Take \( \chi_n = Cp(\log n)^2n^{-\kappa}/(\gamma_n + \psi_n) \). Then \( \chi_n = o(1) \) by the choice of \( \psi_n \). By lemma 5

\[
c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n - C(\log n)^{1/2}n^{-\kappa}/(\gamma_n + \psi_n) \geq c_{1-\gamma_n-\psi_n-\chi_n + \beta_n}^{\text{PIA,0}}
\]

Therefore,

\[
P\{S_n^D \subset S_n^{RMS}\} \geq P\{\max_{s \in S_n}(-\varepsilon_s/V_s) < c_{1-\gamma_n-\psi_n-\chi_n + \beta_n}^{\text{PIA,0}} + o(1)
\]

\[
\geq 1 - \gamma_n - \psi_n - \chi_n + o(1)
\]

\[
= 1 - \gamma_n + o(1)
\]

since \( \psi_n + \chi_n = o(1) \).

\[\square\]

**Lemma 15.** If \( f = 0_p \), then \( P\{S_n^{RMS} = S_n\} \geq 1 - \gamma_n + o(1) \).

**Proof.** By lemma 10, \( P\{c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} > c_{1-\gamma_n}^{\text{PIA}}\} = o(1) \). By lemma 9, \( \max_{s \in S_n}(V_s/\hat{V}_s) \leq 1 + n^{-\kappa} \) wpal as \( n \to \infty \). If \( f = 0_p \), then for any \( s \in S_n, \hat{f}_s = \varepsilon_s \). So,

\[
P\{S_n^{RMS} = S_n\} = P\{\min_{s \in S_n}(\varepsilon_s/\hat{V}_s) > -2(c_{1-\gamma_n}^{\text{PIA}} + \beta_n)\}
\]

\[
\geq P\{\min_{s \in S_n}(\varepsilon_s/\hat{V}_s) > -2(c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)\} + o(1)
\]

\[
\geq P\{\min_{s \in S_n}(\varepsilon_s/\hat{V}_s) \max_{s \in S_n}(V_s/\hat{V}_s) > -2(c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)\} + o(1)
\]

\[
\geq P\{\min_{s \in S_n}(\varepsilon_s/\hat{V}_s)(1 + n^{-\kappa}) > -2(c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)\} + o(1)
\]

\[
\geq P\{\min_{s \in S_n}(\varepsilon_s/V_s) > -2(c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)(1 - n^{-\kappa})\} + o(1)
\]

\[
\geq P\{\min_{s \in S_n}(\varepsilon_s/V_s) > -(c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)\} + o(1)
\]

\[
\geq P\{\max_{s \in S_n}(-\varepsilon_s/V_s) < (c_{1-\gamma_n-\psi_n}^{\text{PIA,0}} + \beta_n)\} + o(1)
\]

\[
\geq E[g_{1-\gamma_n-\psi_n}^{\text{PIA,0}}(\max_{s \in S_n}(-\varepsilon_s/V_s))] + o(1)
\]

48
Combining these results with lemma 11 yields

\[ P\{S_n^{\text{RMS}} = S_n\} \geq 1 - \gamma_n - \psi_n + o(1) \quad \text{(A.58)} \]

The result follows by noting that \( \psi_n = o(1) \).

**Lemma 16.** \( c_{1-\alpha}^{\text{RMS}} + \beta_n \leq c_{1-\alpha}^{\text{PIA}} + \beta_n = O_\nu(\sqrt{\log n}). \)

**Proof.** Since \( S_n^{\text{RMS}} \subseteq S_n \), \( c_{1-\alpha}^{\text{RMS}} + \beta_n \leq c_{1-\alpha}^{\text{PIA}} + \beta_n \). By lemma 10, \( P\{c_{1-\alpha/2}^{\text{PIA}} < c_{1-\alpha}^{\text{PIA}}\} = o(1) \). By assumption 7, \( \beta_n \leq C \) for some \( C > 0 \). Markov inequality and lemma 8 give \( c_{1-\alpha/2}^{\text{PIA}} \leq C\sqrt{\log n} \) for \( C \) large enough. Combining these results yields the statement of the lemma.

**Lemma 17.** Let \( \tau > 1 \), \( L > 0 \), \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( h = (h_1, \ldots, h_d) \in \mathbb{R}^d \), and \( f \in \mathcal{F}(\tau, L) \) for some \( \zeta = 1, \ldots, [\tau] \). If \( \zeta < [\tau] \), assume that for any \( x \in \mathbb{R}^d \) and all \( d \)-tuples of nonnegative integers \( \alpha = (\alpha_1, \ldots, \alpha_d) \) satisfying \( |\alpha| = \zeta + 1 \), \( |D^\alpha f(x)| \leq C \) for some constant \( C > 0 \). Then \( \partial f(x_1, \ldots, x_d)/\partial x_m \geq 0 \) for all \( m = 1, \ldots, d \) implies that for any \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \) satisfying \( 0 \leq y \leq h \),

\[ f(x + y) - f(x) \geq -\frac{\max(L^{\zeta-[\tau]}, C)}{\zeta} \|h\|^{\zeta} \quad \text{(A.59)} \]

for \( \zeta = \min(\zeta + 1, [\tau]) \).

**Proof.** For any \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \) satisfying \( 0 \leq y \leq h \), choose a direction \( l = (l_1, \ldots, l_d) \in \mathbb{R}^d \) by setting \( l_m = y_m/\sqrt{\sum_{j=1}^d y_j^2} \) for all \( m = 1, \ldots, d \). Let \( f^{(k,l)}(x) \) denotes \( k \)-th derivative of \( f \) in direction \( l \) evaluated at point \( x \). Then \( f^{(1,l)}(x) \geq 0 \). If \( f^{(1,l)}(x + ty) \geq 0 \) for all \( t \in (0, 1) \), then the result is obvious. If \( f^{(1,l)}(x + t_0 y) = 0 \) for some \( t_0 \in (0, 1) \), then \( f^{(k,l)}(x + t_0 y) = 0 \) for all \( k = 1, \ldots, \zeta \). If \( \zeta = [\tau] \), then by Holder smoothness, \( f^{(\zeta,l)}(x + ty) \geq -L(t - t_0)\|y\|^\tau \zeta \). Integrating it \( \zeta \) times gives

\[ f(x + y) - f(x) \geq -\frac{L^{\zeta-[\tau]}}{(\gamma - \zeta + 1)\cdots(\gamma - \zeta + K)} \|y\|^{\zeta} \quad \text{(A.60)} \]

since \( \zeta = \tau \) in this case. If \( \zeta < [\tau] \), then \( f^{(\zeta,l)}(x + ty) \geq -C(t - t_0)\|y\| \). Integrating it \( \zeta \) times gives the inequality similar to (A.60) with \( \zeta + 1 \) instead of \( \zeta \) and \( C \) instead of \( L^{\tau-\zeta} \). The result follows by noting that \( \|y\| \leq \|h\| \).
A.8 Proofs of Theorems

Proof of Theorem 1. Under the null hypothesis, for any \( s \in S_n \), \( f_s \leq 0 \) since the kernel \( K \) is positive by assumption. By lemma 10, \( P(c_{1-\alpha-\psi_n}^{\text{PIA}} > c_{1-\alpha-\psi_n}^{\text{PIA}}) = o(1) \). By lemma 9, \( \max_{s \in S_n} (V_s / \hat{V}_s) \leq 1 + n^{-\kappa} \) wp1 as \( n \to \infty \). So,

\[
E[g_{1-\alpha}^{\text{PIA}}(\hat{T})] = E[g_{1-\alpha}^{\text{PIA}}(\max_{s \in S_n} (\hat{f}_s / \hat{V}_s))] \geq E[g_{1-\alpha}^{\text{PIA}}(\max_{s \in S_n} (\varepsilon_s / \hat{V}_s))] \geq E[g_{1-\alpha}^{\text{PIA}}(\max_{s \in S_n} (\varepsilon_s / V_s) \max_{s \in S_n} (V_s / \hat{V}_s))] + o(1) \geq E[g_{1-\alpha-\psi_n}^{\text{PIA}}(\max_{s \in S_n} (\varepsilon_s / V_s)(1 + n^{-\kappa}))] + o(1) \geq E[g_0(\max_{s \in S_n} (\varepsilon_s / V_s)(1 + n^{-\kappa}) - c_{1-\alpha-\psi_n}^{\text{PIA}}) / \beta_n)] + o(1)
\]

Denote \( \delta_n = (\log n / n^\kappa)^{1/2} \). Two different cases will be considered depending on whether \( \beta_n > \delta_n \) or \( \beta_n \leq \delta_n \). Divide the sequence \( \{n\}_{n=1}^\infty \) into two subsequences, \( \{n_1^1\}_{k=1}^\infty \) and \( \{n_2^1\}_{k=1}^\infty \), so that \( \beta_{n_1^k} > \delta_{n_1^k} \) and \( \beta_{n_2^k} \leq \delta_{n_2^k} \) for all \( k \in \mathbb{N} \). First, consider the subsequence \( \{n_1^1\}_{k=1}^\infty \). For simplicity of notation, I will drop indices writing \( n \) instead of \( n_k^1 \). By lemma 12, \( \max_{s \in S_n} |\varepsilon_s / V_s| = O_p(\sqrt{\log n}) \). So, \( \max_{s \in S_n} |\varepsilon_s / V_s| / (n^\kappa \beta_n) < n^{-\kappa / 4} \) wp1 as \( n \to \infty \). Since \( g_0 \) has bounded first derivative,

\[
E[g_0(\max_{s \in S_n} (\varepsilon_s / V_s)(1 + n^{-\kappa}) - c_{1-\alpha-\psi_n}^{\text{PIA}}) / \beta_n)] = E[g_0(\max_{s \in S_n} (\varepsilon_s / V_s) - c_{1-\alpha-\psi_n}^{\text{PIA}}) / \beta_n)] + o(1)
\]

The last expression equals \( E[g_{1-\alpha-\psi_n}^{\text{PIA}}(\max_{s \in S_n} (\varepsilon_s / V_s))] + o(1) \). Combining these results and lemma 11 yields

\[
E[g_{1-\alpha}^{\text{PIA}}(\hat{T})] \geq 1 - \alpha - \psi_n + o(1) = 1 - \alpha + o(1) \quad \text{(A.61)}
\]

Next, consider the subsequence \( \{n_2^1\}_{k=1}^\infty \). Again, I will write \( n \) instead of \( n_2^2 \). Take \( \chi_n = Cp(\log n)^2 n^{-\kappa / 2} \) with large enough \( C \). Note that \( \chi_n = o(1) \). As in lemma 14,

\[
c_{1-\alpha-\psi_n}^{\text{PIA}}(1 - n^{-\kappa}) - \beta_n \geq c_{1-\alpha-\psi_n}^{\text{PIA}} - \chi_n \quad \text{(A.62)}
\]
Continuing the chain of inequalities from above gives

\[
E[g_{1-\alpha}^{PIA}(\hat{T})] \geq P\{\max_{s \in S_n}(\epsilon_s/V_s)(1 + n^{-\kappa}) \leq c_{1-\alpha-\psi_n}^{PIA,0} \} + o(1)
\]

\[
\geq P\{\max_{s \in S_n}(\epsilon_s/V_s) \leq c_{1-\alpha-\psi_n}^{PIA,0} (1 - n^{-\kappa}) \} + o(1)
\]

\[
\geq P\{\max_{s \in S_n}(\epsilon_s/V_s) - \beta_n \leq c_{1-\alpha-\psi_n-\chi_n}^{PIA,0} \} + o(1)
\]

\[
\geq E[g_{1-\alpha-\psi_n-\chi_n}^{PIA,0}(\max_{s \in S_n}(\epsilon_s/V_s))]
\]

An application of lemma 11 yields

\[
E[g_{1-\alpha}^{PIA}(\hat{T})] \geq 1 - \alpha - \psi_n - \chi_n + o(1) = 1 - \alpha + o(1) \quad (A.63)
\]

Now consider the RMS test function. By lemma 14, \( P\{c_{1-\alpha+2\gamma_n}^{D} > c_{1-\alpha+2\gamma_n}^{RMS} \} \leq \gamma_n + o(1) \). By lemma 13, \( P\{\max_{s \in S_n \setminus S_n^D} \hat{f}_s/V_s > 0 \} \leq \gamma_n + o(1) \). So,

\[
E[g_{1-\alpha+2\gamma_n}^{RMS}(\hat{T})] = E[g_{1-\alpha+2\gamma_n}^{RMS}(\max_{s \in S_n}(\hat{f}_s/V_s))]
\]

\[
\geq E[g_{1-\alpha+2\gamma_n}^{D}(\max_{s \in S_n}(\hat{f}_s/V_s))] - \gamma_n + o(1)
\]

\[
\geq E[g_{1-\alpha+2\gamma_n}^{D}(\max_{s \in S_n^D}(\hat{f}_s/V_s))] - 2\gamma_n + o(1)
\]

Since \( S_n^D \) is nonstochastic, from this point, the argument similar to that used in the proof for the plug-in test function with \( S_n^D \) instead of \( S_n \) yields the result for the RMS critical values.

Next assume that \( f = 0_p \). By lemma 10, \( P\{c_{1-\alpha+\psi_n}^{PIA,0} < c_{1-\alpha}^{PIA} \} = o(1) \). By lemma 9, \( \min_{s \in S_n}(V_s/\hat{V}_s) \geq 1 - n^{-\kappa} \) wp1 as \( n \to \infty \). So,

\[
E[g_{1-\alpha}^{PIA}(\hat{T})] = E[g_{1-\alpha}^{PIA}(\max_{s \in S_n}(\hat{f}_s/V_s))]
\]

\[
= E[g_{1-\alpha}^{PIA}(\max_{s \in S_n}(\epsilon_s/\hat{V}_s))]
\]

\[
\leq E[g_{1-\alpha+\psi_n}^{PIA,0}(\max_{s \in S_n}(\epsilon_s/\hat{V}_s))] + o(1)
\]

\[
\leq E[g_{1-\alpha+\psi_n}^{PIA,0}(\max_{s \in S_n}(\epsilon_s/V_s) \min_{s \in S_n}(V_s/\hat{V}_s))] + o(1)
\]

\[
\leq E[g_{1-\alpha+\psi_n}^{PIA,0}(\max_{s \in S_n}(\epsilon_s/V_s)(1 - n^{-\kappa}))] + o(1)
\]

\[
= E[g_{0}(\max_{s \in S_n}(\epsilon_s/V_s)(1 - n^{-\kappa}) - c_{1-\alpha+\psi_n}^{PIA,0}/\beta_n)] + o(1)
\]
For the subsequence \( \{n_k^1\}_{k=1}^{\infty} \), writing \( n \) instead of \( n_k^1 \),

\[
E[g_0(\max_{s \in S_n}(\varepsilon_s/V_s)(1 - n^{-\kappa} - c_{\text{PIA},0}^{1-\alpha+\psi_n}))/\beta_n]
= E[g_0((\max_{s \in S_n}(\varepsilon_s/V_s) - c_{\text{PIA},0}^{1-\alpha+\psi_n})/\beta_n)] + o(1)
\]

So, the result that \( E[g_{1-\alpha}^{\text{PIA}}(\hat{T})] \leq 1 - \alpha + o(1) \) follows by applying \( \Box \). For the subsequence \( \{n_k^2\}_{k=1}^{\infty} \), with the same choice of \( \chi_n \),

\[
(c_{1-\alpha+\psi_n}^{\text{PIA},0} + \beta_n)(1 + 2n^{-\kappa}) \leq c_{1-\alpha+\psi_n+\chi_n}^{\text{PIA},0}
\]

where I again write \( n \) instead of \( n_k^2 \). In addition, for any \( x \in (0, 1/2) \),

\[
1/(1 - x) < 1 + 2x
\]

So,

\[
E[g_{1-\alpha}^{\text{PIA}}(\hat{T})] \leq P\{\max_{s \in S_n}(\varepsilon_s/V_s)(1 - n^{-\kappa}) \leq c_{1-\alpha+\psi_n}^{\text{PIA},0} + \beta_n\} + o(1)
\]

\[
\leq P\{\max_{s \in S_n}(\varepsilon_s/V_s) \leq (c_{1-\alpha+\psi_n}^{\text{PIA},0} + \beta_n)(1 + 2n^{-\kappa})\} + o(1)
\]

\[
\leq P\{\max_{s \in S_n}(\varepsilon_s/V_s) \leq c_{1-\alpha+\psi_n+\chi_n}^{\text{PIA},0}\} + o(1)
\]

\[
\leq E[g_{1-\alpha+\psi_n+\chi_n}^{\text{PIA},0}(\max_{s \in S_n}(\varepsilon_s/V_s))]
\]

Again, the result that \( E[g_{1-\alpha}^{\text{PIA}}(\hat{T})] \leq 1 - \alpha + o(1) \) follows by applying lemma \( \Box \).

For the RMS test function, note that by lemma \( \Box \), \( P\{S_n^{\text{RMS}} = S_n\} \geq 1 - \gamma_n + o(1) \) whenever \( f = 0_p \). If \( \gamma_n = o(1) \), then

\[
E[g_{1-\alpha}^{\text{RMS}}(\hat{T})] = E[g_{1-\alpha+2\gamma_n}^{\text{PIA}}(\hat{T})] + o(1) \leq 1 - \alpha + o(1)
\]  

\[
(A.66)
\]

Proof of Corollary 1:

Proof of Corollary 1. If \((\log n)^{19}/(h_{\text{min}}^d n) \to 0\), then one can set \( \varrho_n \) so that \( \varrho_n(\log n)^{3/2} \to 0 \) and \((\log n)^{4}/(\varrho_n^{10}h_{\text{min}}^d n) \to 0 \). Then \( \varrho_n \) satisfies assumption \( \Box \). So, the result of theorem \( \Box \) holds for \( \varrho_n \) instead of \( \beta_n \). Let \( c_x^{\text{PIA},0,e} \) denote the value of \( c_x^{\text{PIA},0} \) evaluated...
with \( \varrho_n \) instead of \( \beta_n \) for all \( x \in (0, 1) \). By lemma 10, \( P\{c_{1-\alpha-\psi_n}^{PIA} > c_{1-\alpha}^{PIA}\} = o(1) \). By lemma 5, \( c_{1-\alpha-\psi_n}^{PIA,0} \leq c_{1-\alpha-\psi_n}^{PIA} \) for \( C \) large enough. So,

\[
P\{\hat{T} \leq c_{1-\alpha}^{PIA}\} \geq E[g_0((\hat{T} + \varrho_n) - c_{1-\alpha}^{PIA})/\varrho_n)]
\[
\geq E[g_0((\hat{T} + \varrho_n - c_{1-\alpha-\psi_n}^{PIA,0})/\varrho_n)] + o(1)
\[
\geq E[g_0((\hat{T} - c_{1-\alpha-\psi_n}^{PIA,0})/c_{\varrho_n}(\log n)^{3/2})/\varrho_n)] + o(1)
\]

From this point, the argument like that used in the proof of theorem 1 with \( \varrho_n \) instead of \( \beta_n \) leads to \( P\{\hat{T} \leq c_{1-\alpha}^{PIA}\} \geq 1 - \alpha + o(1) \). All other statements of theorem 1 follow from similar arguments. \( \square \)

Proof of Theorem 2:

Proof of Theorem 2. For any \( w \in G_p \), there exist \( i(w) \in \mathbb{N} \) and \( m(w) = 1, \ldots, p \) such that \( f^{w}_{m(w)}(X_{i(w)}) \geq \rho \). For simplicity of notation, I will drop index \( w \). By assumption 3 there exists a ball \( B_\delta(X_i) \) with center at \( X_i \) and radius \( \delta \) such that \( f_{m}(X_j) \geq \rho/2 \) for all \( X_j \in B_\delta(X_i) \). Note that \( \delta \) can be chosen independently of \( w \). So, for some \( \alpha = \alpha(N) \), \( n \geq N \), there exists a triple \( s_n = (i_n, m, h_n) \in S_n \) with \( h_n \) bounded away from zero such that \( f_{m}(X_j) \geq \rho/2 \) for all \( X_j \in B_{h_n}(X_{i_n}) \). Hence, \( f_{s_n} \geq \rho/2 \). Lemma 7 gives \( V_{s_n} \leq n^{-\phi} \) for some \( \phi > 0 \), so \( f_{s_n}/V_{s_n} > Cn^\phi \). By lemma 9, \( \hat{V}_{s_n}/V_{s_n} - 1 = o_p(1) \). So, for any \( \tilde{C} < C \), \( P\{f_{s_n}/\hat{V}_{s_n} > \tilde{C}n^\phi\} \to 1 \). Thus,

\[
E[g_{1-\alpha}^p(\hat{T})] \leq P\{\hat{T} \leq c_{1-\alpha}^p + \beta_n\}
\leq P\{f_{s_n}/\hat{V}_{s_n} \leq c_{1-\alpha}^p + \beta_n + \max_{s \in S_n} |\varepsilon_s/\hat{V}_s|\}
\leq P\{c_{1-\alpha}^p + \beta_n + \max_{s \in S_n} |\varepsilon_s/\hat{V}_s| > Cn^\phi\} + o(1)
\]

The result follows by noting that from lemmas 12 and 16, \( c_{1-\alpha}^p + \beta_n + \max_{s \in S_n} |\varepsilon_s/\hat{V}_s| = O_p(\sqrt{\log n}) \). \( \square \)

Proof of Theorem 3:

Proof of Theorem 3. As in the proof of theorem 2, since \( \rho(w,H_0) > 0 \), there exists \( i \in \mathbb{N} \) such that \( f^{w}_{m}(X_i) \geq \rho \) for some \( m = 1, \ldots, p \) and \( \rho > 0 \). In addition, by assumption 3 there exists a ball \( B_\delta(X_i) \) such that \( f^{w}_{m}(X_j) \geq \rho/2 \) for all \( X_j \in B_\delta(X_i) \).
So, for some $N \in \mathbb{N}$ and any $n \geq N$, there exists a triple $s_n = (i_n, m, h) \in S_n$ such that $f_{m}^{w}(X_j) \geq \rho/2$ for all $X_j \in B_h(X_{i_n})$. Hence, $f_{s_n}^{n} \geq a_n \rho/2$. Note that in contrast with theorem 2, now we choose fixed bandwidth value $h$. By lemma $[\text{lemma number}]$, $V_{s_n} \leq C/\sqrt{n}$. Then lemma $[\text{lemma number}]$ gives $P\{f_{s_n}^{n}/\hat{V}_{s} > \tilde{C} a_n / \sqrt{n}\} \to 1$ for some $\tilde{C} > 0$. The same argument as in the proof of theorem 2 yields

$$E[g_{1-\alpha}(\hat{T})] \leq P\{c_{1-\alpha}^P + \beta_n + \max_{s \in S_n} |\varepsilon_s / \hat{V}_s| > \tilde{C} a_n \sqrt{n}\} + o(1) \quad (A.67)$$

Combining $c_{1-\alpha}^P + \beta_n + \max_{s \in S_n} |\varepsilon_s / \hat{V}_s| = O_p(\sqrt{\log n})$ and $a_n \sqrt{n/ \log n} \to \infty$ gives the result.

Proof of Theorem 4:

Proof of Theorem 4. First, consider $\tau \leq 1$ case. In this case, $\zeta = \tau$. Since $d \geq 1$, we are in the situation $\zeta \leq d$. For any $w \in G_\theta$, there exist $i(w) \in \mathbb{N}_\theta$ and $m(w) = 1, \ldots, p$ such that $f_{m(w)}^{w}(X_{i(w)}) \geq (C/2)h^e_{\min}$. By assumptions $[\text{lemma number}]$ and $[\text{lemma number}]$ there exists $j(w) = 1, \ldots, n$ such that $\|X_{i(w)} - X_{j(w)}\| \leq \delta_{\min}$ and $s_n(w) = (j(w), m(w), h_{\min}) \in S_n$. By assumption $[\text{lemma number}]$, $f_{m(w)}^{w}(X_{l}) \geq \tilde{C} h_{\min}^\zeta$ for all $l = 1, \ldots, n$ such that $X_l \in B_{h_{\min}}(X_{j(w)})$ for some constant $\tilde{C}$. So, $f_{s_n(w)}^{w} \geq \tilde{C} h_{\min}^\zeta$. By assumption $[\text{lemma number}]$, $n h_{\min}^{3d} / \log n \to \infty$ as $n \to \infty$. By lemma $[\text{lemma number}]$, $V_{s_n(w)} \leq C/\sqrt{nh_{\min}^d}$. So,

$$f_{s_n(w)}^{w} / (V_{s_n(w)} \sqrt{\log n}) \geq (\tilde{C}/C) \sqrt{nh_{\min}^{2\zeta + d} / \log n} \geq (\tilde{C}/C) \sqrt{nh_{\min}^{3d} / \log n} \to \infty \quad (A.68)$$

uniformly in $w \in G_\theta$. The result follows from the same argument as in the proof of theorem 2.

Consider $\tau > 1$ case. Suppose $\zeta \leq d$. For any $w \in G_\theta$, there exist $i(w) \in \mathbb{N}_\theta$ and $m(w) = 1, \ldots, p$ such that $f_{m(w)}^{w}(X_{i(w)}) \geq (C/2)h^e_{\min}$. For $m = 1, \ldots, d$, set $e_m = 4h_{\min}$ if $\partial f_{m(w)}^{w}(X_{i(w)}) / \partial x_m \geq 0$ and $-4h_{\min}$ otherwise. Consider the cube $C$ whose edges are parallel to axes and that contains vertices $(X_{i(w),1}, \ldots, X_{i(w),d})$ and $(X_{i(w),1} + 2e_1, \ldots, X_{i(w),d} + 2e_d)$. By lemma $[\text{lemma number}]$, for all $x \in C$, $f_{m(w)}^{w}(x) \geq \tilde{C} h_{\min}^\zeta$ for some constant $\tilde{C}$. By the definition of $\mathbb{N}_\theta$ and assumption $[\text{lemma number}]$ there exists $l(w) = 1, \ldots, n$ such that $X_l(w) \in B_{h_{\min}}(X_{i(w),1} + e_1, \ldots, X_{i(w),d} + e_d)$. By assumption $[\text{lemma number}]$ there exists $j(w) = 1, \ldots, n$ such that $X_{j(w)} \in B_{3h_{\min}}(X_{i(w),1} + e_1, \ldots, X_{i(w),d} + e_d)$ and $s_n(w) = (j(w), m(w), h_{\min}) \in S_n$. So, $f_{m(w)}^{w}(X_l) \geq \tilde{C} h_{\min}^\zeta$ for all $l = 1, \ldots, n$ such that $X_l \in
Suppose $\zeta > d$. The only difference between this case and the previous one is that now optimal testing bandwidth value is greater than $h_{\min}$. Let $h_0$ be the largest bandwidth value in the set $S_n$ which is smaller than $(\log n/n)^{1/(2\zeta + d)}$. For any $w \in G$, the same construction as above gives $s_n(w) = (j(w), m(w), h_0) \in S_n$ such that $f^{w}_{m(w)}(X_l) \geq \rho_0(w, H_0) - \tilde{C}h_0^\zeta$ for all $l = 1, \ldots, n$ such that $X_l \in B_{h_0}(X_{j(w)})$. Since $\rho_0(w, H_0) \geq b_n(\log n/n)^{\zeta/(2\zeta + d)}$ for some sequence of real numbers $\{b_n\}_{n=1}^{\infty}$ such that $b_n \to \infty$ as $n \to \infty$, $f^{w}_{s_n(w)} \geq (b_n - \tilde{C})(\log n/n)^{\zeta/(2\zeta + d)}$. By lemma 7, $V_{s_n(w)} \leq C/\sqrt{nh_0^d}$. Then

$$f^{w}_{s_n(w)}/(V_{s_n(w)}\sqrt{\log n}) \geq (b_n - \tilde{C})/(2C) \to \infty \quad (A.69)$$

The result follows as above. \hfill \Box

**Proof of Theorem 5:**

Define $v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ as follows. Set $v(x, h) = 0$ if $x < 0$ or $x > 2$ for all $h \in \mathbb{R}_+$.

First, define functions $b_1, \ldots, b_K$ on $(0, 1]$ for some $K$ to be chosen below by the following induction. Set $b_1(x) = +1$ for $x \in (0, 1/2]$ and $-1$ for $x \in (1/2, 1]$. Given $b_1, \ldots, b_{k-1}$, for $i = 1, 3, \ldots, 2^k-1$ and $x \in ((i-1)2^{-k}, i2^{-k}]$, set $b_k(x) = +1$ if $b_{k-1}(y) = +1$ for $y \in ((i-1)2^{-k}, (i+1)2^{-k}]$ and $-1$ otherwise. For $i = 2, 4, \ldots, 2^k$ and $x \in ((i-1)2^{-k}, i2^{-k}]$, set $b_k(x) = -1$ if $b_{k-1}(y) = +1$ for $y \in ((i-2)2^{-k}, i2^{-k}]$ and $+1$ otherwise. By induction, define $b_1, \ldots, b_K$ where $K$ is the largest integer strictly smaller than $\tau$, i.e. $K = [\tau]$.

Now let us define $\nu: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$. Set $v(x, h) = 0$ if $x < 0$ or $x > 2$ for all $h \in \mathbb{R}_+$. For $x \in [0, 2]$, $\nu$ will be defined through its derivatives. Set $\partial^k v(0, h)/\partial x^k = 0$ for all $k = 0, \ldots, K$. For $i = 1, \ldots, 2^K$, once function $\partial^K v(x, h)/\partial x^K$ is defined for $x \in [0, (i-1)2^{-K}]$, set

$$\partial^K v(x, h)/\partial x^K = \partial^K v((i-1)2^{-K}, h)/\partial x^K + b_K(x)h^K L(x - (i-1)2^{-K})^{\tau - K} \quad (A.70)$$

for $x \in ((i-1)2^{-K}, i2^{-K}]$. These conditions define function $v(x, h)$ for $x \in [0, 1]$ and $h \in \mathbb{R}_+$. For $x \in (1, 2]$ and $h \in \mathbb{R}_+$, set $v(x, h) = v(2-x, h)$ so that $v$ is symmetric in $x$ around $x = 1$. It is easy to see that for fixed $h \in \mathbb{R}_+$, $v(\cdot / h, h) \in F_\tau(\tau, L)$ and
sup_{x \in \mathbb{R}} v(x/h, h) \in (C_1 h^r, C_2 h^r) for some positive constants C_1 and C_2 independent of h.

Let q : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+ be given by q(x, h) = v(||x||/h + 1, h) for all (x, h) \in \mathbb{R}^d \times \mathbb{R}_+. Note that for fixed h \in \mathbb{R}_+, q(\cdot, h) \in \mathcal{F}_{[r]}(\tau, L), q(x, h) = 0 if ||x|| > h, and q(0, h) = \sup_{x \in \mathbb{R}^d} q(x, h) \in (C_1 h^r, C_2 h^r).

Since r_n(n/\log n)^{r/(2r+d)} \to 0, there exists a sequence of positive numbers \{\psi_n\}_{n=1}^\infty such that r_n = \psi_n(n/\log n)^{r/(2r+d)} and \psi_n \to 0. Set h_n = \psi_n(n/\log n)^{1/(2r+d)}. By the assumption on packing numbers N(h, S_\theta), there exists a set \{j(l) \in \mathbb{N}_\theta : l = 1, ..., N_n\} such that \|X_{j(l_1)} - X_{j(l_2)}\| > 2h_n for l_1, l_2 = 1, ..., N_n if l_1 \neq l_2 and N_n > C h_n^d for some constant C. For l = 1, ..., N_n, define function f^l : \mathbb{R}^d \to \mathbb{R}^p given by f^l(x) = q(x - X_{j(l)}, h_n) and f^l_m(x) = 0 for all m = 2, ..., p for all x \in \mathbb{R}^d. Note that functions \{f^l\}_{l=1}^{N_n} have disjoint supports. Moreover, for every l = 1, ..., N_n and m = 1, ..., p, f^l_m \in \mathcal{F}_{[r]}(\tau, L). Let \{\xi_i\}_{i=1}^n be a sequence of independent standard Gaussian random vectors N(0, I_p). For l = 1, ..., N_n, define an alternative, w_l, with the regression function f^l and disturbances \{\xi_i\}_{i=1}^n. Note that \rho_\theta(w_l, H_0) \geq C r_n for all l = 1, ..., N_n for some constant C. In addition, let w_0 denote the alternative with zero regression function and disturbances \{\xi_i\}_{i=1}^n.

As in the proof of lemma 6.2 in [Dumbgen and Spokoiny (2001)], for any sequence \phi_n = \phi_n(Y_1, ..., Y_n) of tests with \sup_{w \in \mathcal{G}_0} E_w[\phi_n] \leq \alpha,

\begin{align*}
\inf_{w \in \mathcal{G}_0, \rho_\theta(w, H_0) \geq C r_n} E_w[\phi_n] - \alpha & \leq \min_{l=1, ..., N_n} E_{w_l}[\phi_n] - E_{w_0}[\phi_n] \\
& \leq \sum_{i=1}^{N_n} E_{w_i}[\phi_n]/N_n - E_{w_0}[\phi_n] \\
& \leq E_{w_0}[(\sum_{i=1}^{N_n} (dP_{w_i}/dP_{w_0})/N_n - 1)\phi_n] \\
& \leq E_{w_0}[(\sum_{i=1}^{N_n} dP_{w_i}/dP_{w_0}/N_n - 1)]
\end{align*}

where dP_{w_l}/dP_{w_0} denotes a Radon-Nykodim derivative. For l = 1, ..., N_n, denote \omega_l = (\sum_{i=1}^n (f^l_1(X_i))^2)^{1/2} and \xi_l = \sum_{i=1}^n f^l_1(X_i)Y_{i,1}/\omega_l. Then

\begin{align*}
dP_{w_l}/dP_{w_0} = \exp(w_l\xi_l - \omega_l^2/2) \\
(A.71)
\end{align*}

56
Note that $\omega_t \leq C n^{1/2} h^{\tau + d/2}$. In addition, under the model $w_0$, $\xi_t$ are independent standard Gaussian random variables. So, an application of lemma 6 gives

$$E_{w_0}[\sum_{i=1}^{N_n} dP_{w_i}/dP_{w_0}/N_n - 1] \to 0$$

(A.72)

if $C n^{1/2} h^{\tau + d/2} < \tilde{C}(\log N_n)^{1/2}$ for some constant $\tilde{C} \in (0, 1)$ for all large enough $n$. The result follows by noting that $n^{1/2} h^{\tau + d/2} = o(\sqrt{\log n})$ and $\log N_n \geq C \log n$ for some constant $C$.

Proof of Corollary 2:

Proof of Corollary 2. Replace $p$ by $K_n$ both in $\psi_n$ and $\chi_n$ in all preliminary results and theorem 1. Then all preliminary results except lemma 11 hold for the test with $K_n \to \infty$. Lemma 11 holds with conditions (iii) and (iv) in the corollary replacing assumption 7. So, the first result follows from the same argument as in theorem 1. For any $w \in G_\rho$, there exists some $m(w) \in \mathbb{N}$ such that $\sup_{i \in \mathbb{N}} [f^w_{m(w)}(X_i)]_+ > 0$. Once $m(w)$ is included in the test statistic, the second result follows as in the proof of theorem 2.

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