Yang-Mills connections on $G_2$-manifolds and Calabi-Yau 3-folds

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Abstract

We investigate Yang-Mills connections, $A$, with sufficient small $L^2$ curvature $F_A$ on a compact Riemannian n-manifold $M$, where $M$ is $G_2$-manifold or Calabi-Yau 3-fold. Suppose all flat connections on $M$ are nondegenerate, then we prove that when $M$ is a $G_2$-manifold, the Yang-Mills connection must be a $G_2$ instanton; when $M$ is a Calabi-Yau 3-fold, the vector bundle is holomorphic.

Keywords. Yang-Mills connection, $G_2$-instanton, holomorphic bundle

1 Introduction

Let $G$ be a compact Lie group and $E$ a principal $G$-bundle on a complete oriented Riemannian manifold $M$. Let $A$ denote a connection on $E$ and $\nabla_A$ the associated covariant derivative on the adjoint bundle $\text{ad}(E)$. The Yang-Mills energy of $A$ is

$$YM(A) := \|F_A\|_{L^2}^2$$

where $F_A$ denotes the curvature of $A$, $A$ connections is called a Yang-Mills connection if it is a critical point of the Yang-Mills functional.

In four dimensions, $F_A$ decomposes into its self-dual and anti-self-dual components,

$$F_A = F_A^+ + F_A^-$$

where $F_A^\pm$ denotes the projection onto the $\pm 1$ eigenspace of the Hodge star operator. A connection is called self-dual(respectively anti-self-dual) if $F_A = F_A^+$ (respectively $F_A = F_A^-$). A connection is called an instanton if is either self-dual or anti-self-dual. On compact oriented 4-manifolds, an instanton is always an absolute minimizer of the Yang-Mills energy. Not all Yang-Mills connections are instantons. See [13][14] for example of $SU(2)$ Yang-Mills connection on $S^4$ which are neither self-dual nor anti-self-dual.

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In higher dimensions, the instanton equation on $M$ can be introduced as follows. Assuming there is a closed $(n-4)$-form $\Omega$ on $M$. A connection, $A$, is called anti-self-dual instanton, when it satisfies the instanton equation

$$\ast F_A = - \ast (\Omega \wedge F_A).$$

Instantons on the higher dimension, proposed in [4] and studied in [7, 8, 11, 16, 21], are important both in mathematics [8, 16] and string theory [10]. It’s easy to see the instanton must be a Yang-Mills connection. But not all Yang-Mills connections could be instantons.

This leads to the question: Which time a Yang-Mills connection would be a instanton?

In the Stern [15], he considered the minimizing Yang-Mills connections on compact homogeneous 4-manifold, he proved that those connections were ether instantons or split into a sum of instantons on passage to the adjoint bundle. In this paper we consider the case of $G_2$-manifolds. On $G_2$-manifolds, the 2-forms decompose as

$$\Lambda^2(M) = \Lambda^2_7(M) + \Lambda^2_{14}(M).$$

where the fiber of $\Lambda^2_k$ is an irreducible $G_2$ representation of dimension $k$. Let $F_A = F^7_A + F^{14}_A$ be the corresponding decomposition of the curvature. Then we call a connection, $A$, is a $G_2$-instanton, if $F^7_A = 0$ (see [7, 11]). Our main result is the following theorem.

**Theorem 1.1.** Let $M$ be a compact $G_2$-manifold $M$, $A$ be a Yang-Mills connection on $M$. Suppose all flat connections on $M$ are nondegenerate, there exists a constant $\delta$ such that if $\|F_A\|_{L^2} \leq \delta$, then $F^7_A = 0$.

It means that the Yang-Mills connection must be a $G_2$-instanton.

On a Kähler n-manifold with Kähler form $\omega$ the curvature decomposes as

$$F_A = F^{2,0}_A + F^{1,1}_{A0} + \frac{1}{n}(\Lambda F_A)\omega + F^{0,2}_A,$$

where $\Lambda$ denotes the adjoint of exterior multiplication by $\omega$, and $F^{1,1}_{A0} = F^{1,1}_A - \frac{1}{n}(\Lambda F_A)\omega$.

**Theorem 1.2.** Let $A$ be a Yang-Mills connection on a vector bundle $E$ over a compact Calabi-Yau 3-fold. Suppose all flat connections on $M$ are nondegenerate, there exist a constant $\gamma$ such that if $\|F_A\|_{L^3} \leq \gamma$, then $F^{0,2}_A = 0$.

It means that the vector bundle is holomorphic.
2 Preliminaries and Basic estimates

2.1 Preliminaries

First, we recall some standard notations and definitions.

Let $T^*M$ be the cotangent bundle of $M$ and for $1 \leq p \leq n$, let $\Lambda^p(M)$ be the $p$-form bundles on $M$ with $T^*M = \Lambda^1 M$. One can form the associated bundle $E \otimes \Lambda^p$. Let $\Omega^p(E)$ be the set of sections of $E \otimes \Lambda^p$. Let $\mathfrak{g}$ be the Lie algebra of $G$, $Ad : G \to Aut(\mathfrak{g})$ be the adjoint representation and $adE$ be the associated adjoint vector bundle.

Denote $\Omega^p(ad(E)) = \Gamma(adE \otimes \Lambda^p(M))$. For a connection $A$ on $E$, we have exterior derivatives

$$d_A : \Omega^p(adE) \to \Omega^{p+1}(adE).$$

These are uniquely determined by the properties (see [6], p.35):

1. $d_A = \nabla_A$ on $\Omega^0(adE)$
2. $d_A(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^p \alpha \wedge d_A\beta$

for any $\alpha \in \Omega^p(adE)$, $\beta \in \Omega^q(adE)$.

The curvature $F_A \in \Omega^2(adE)$ of the connection $A$ is defined by

$$d_A d_A u = F_A u$$

for any section $u \in \Gamma(E)$. If $A$ is a connection on $E$, we can define covariant derivatives

$$\nabla_A : \Omega^p(E) \to \Gamma(\Lambda^pT^*M \otimes T^*M \otimes E)$$

For $\nabla_A$ and $d_A$, we have adjoint operators $\nabla_A^*$ and $d_A^*$. We also have Weitzenböck formula ([3], Theorem 3.2)

$$(d_A d_A^* + d_A^* d_A)\varphi = \nabla_A^* \nabla_A \varphi + \varphi \circ Ric + *\{*F_A, \varphi\}$$

(2.1)

where $\varphi \in \Omega^1(adE)$, $Ric$ is the Ricci tensor.

In a local orthonormal frame $(e_1, \ldots, e_n)$ of $TM$, the operator of $\varphi \circ Ric$ is defined by Bourguignon and Lawson [3] as follows.

$$\varphi \circ Ric(e_i) = \sum_{j=1}^n R_{ij}\varphi_j$$

We are interested in minima of the Yang-Mills energy

$$YM(A) = \|F_A\|^2_{L^2}.$$
where $F_A$ denotes the curvature of $A$. Critical points of this energy satisfy the Yang-Mills equation

$$d^*_A F_A = 0,$$

where $d^*_A$ denotes adjoint of $d_A$. In addition, all connections satisfy the Bianchi identity

$$d_A F_A = 0.$$

If $\psi \in \Omega^1_M(adE)$ then

$$F_A + \psi = F_A + d_A \psi + \psi \wedge \psi.$$

Here we note that our convention on exterior products of $adE$ valued form is normalized by

$$(dx^I \otimes v_I) \wedge (dx^J \otimes v_J) = \frac{1}{2} (dx^I \wedge dx^J) \otimes [v_I, v_J].$$

As a notional convenience, we will often use $L_\omega$ to denote exterior multiplication on the left by a form $\omega$. Its adjoint is denote $\Lambda_\omega$. Thus

$$L_\omega h := \omega \wedge h, \text{ and } \langle f, L_\omega h \rangle = \langle \Lambda_\omega f, h \rangle.$$

### 2.2 Estimates for Curvature of Yang-Mills connection

We have a priori estimate for the curvature of a Yang-Mills connection.

**Theorem 2.1.** ([17], Theorem 3.5) There exist constants $\varepsilon = \varepsilon(n)$ and $K = K(n)$ such that if $F_A$ is Yang-Mills field in $B_{2a_0}(x_0)$ and $\int_{B_{2a_0}(x_0)} |F_A|^2 < \varepsilon(n)$, then $|F_A(x)|$ is uniformly bounded in the interior of $B_{2a_0}(x_0)$ and

$$|F_A(x)|^2 \leq a^{-n} \int_{B_a(x)} |F_A|^2$$

for all $B_a(x) \subset B_{a_0}(x_0)$.

**Remark 2.2.** The Theorem 2.1 continues to hold for geodesic balls in a manifold $M$ endowed a non-flat Riemannian metric, $g$. The only difference in this more general situation is that the constants $K$ and $\varepsilon$ will depend on bounds on the Riemann curvature tensor over $B_{2a_0}(x_0)$ and the injectivity radius at $x_0 \in M$. Therefore, by employing a finite cover of $M$ by geodesic balls, $B_{a_0}(x_i)$, of radius $a_0 \subset (0, \rho/4]$, $\rho$ is the injectivity radius of the manifold $M$ and applying Theorem 2.1 to each ball $B_{2a}(x_i)$, we obtain a global version.

We consider a family of connections near a flat connection $\Gamma$,

$$A_N(M) = \{ A \in \Omega^1_M(adE) : \| A - \Gamma \|_{L^n} \leq N \| F_A \|_{L^2}, \text{ } N \text{ is a bounded constant} \}.$$
Theorem 2.3. Let \( A \in A_N(M) \) be a connection on the bundle \( E \) over \( M = M^n \) \((n \geq 2)\) be a compact Ricci-flat manifold. Suppose all flat connections over \( M \) are nondegenerate. There are constants \( \eta \) and \( \lambda \) such that if \( \| F_A \|_{L^2} \leq \eta \), then
\[
\| \nabla_A \varphi \|_{L^2} \geq \lambda \| \varphi \|_{L^1}
\]
where \( \varphi \in \Omega^1_M(adE) \).

Proof. Since \( \Gamma \) is a flat connection, then the cohomology group \( H^1(M, \Gamma) \) is zero. The basic elliptic estimate for the operator \( d_\Gamma + d_\Gamma^* \) on 1-forms gives a bound of the form
\[
\| \varphi \|_{L^2}^2 \leq c_1 (\| d_\Gamma \varphi \|_{L^2}^2 + \| d_\Gamma^* \varphi \|_{L^2}^2).
\]
where \( c_1 \) is a constant.

Now \( d_A \varphi = d_\Gamma \varphi + [A - \Gamma, \varphi] \), and \( d_A^* \varphi = d_\Gamma^* \varphi - * [A - \Gamma, * \varphi] \). Using the Sobolev embedding theorem
\[
\| \varphi \|_{L^{2n}} \leq \text{const.} \| \varphi \|_{L^1}
\]
We get
\[
\| d_A \varphi \|_{L^2} \geq \| d_\Gamma \varphi \|_{L^2} - 2\| A - \Gamma \|_{L^n} \| \varphi \|_{L^{2n}}^2 + \| d_\Gamma^* \varphi \|_{L^2} - \| A - \Gamma \|_{L^n} \| \varphi \|_{L^2}^2,
\]
and
\[
\| d_A^* \varphi \|_{L^2} \geq \| d_\Gamma^* \varphi \|_{L^2} - c_2 \| A - \Gamma \|_{L^n} \| \varphi \|_{L^2}^2.
\]
By Weitzenböck formula (2.1) and \( M \) is a Ricci-flat manifold, we have
\[
\| \nabla_A \varphi \|_{L^2} \geq (\| d_A \varphi \|_{L^2} + \| d_A^* \varphi \|_{L^2}) - 2\langle F_A, \varphi \wedge \varphi \rangle + (\| d_\Gamma \varphi \|_{L^2} + \| d_\Gamma^* \varphi \|_{L^2}) - (c_2 + c_3) \| A - \Gamma \|_{L^n} \| \varphi \|_{L^2}^2 - c_4 \| F_A \|_{L^2} \| \varphi \|_{L^2}^2 + c_4 \| F_A \|_{L^2} \| \varphi \|_{L^2}^2
\]
Here we used the fact
\[
| \langle F_A, \varphi \wedge \varphi \rangle | \leq \| F_A \|_{L^2} \| \varphi \|_{L^{2n}}^2 \leq c_4 \| F_A \|_{L^2} \| \varphi \|_{L^2}^2
\]
If \( \| F_A \|_{L^2} \leq \varepsilon \) such that \((c_2 + c_3) N^2 \| F_A \|_{L^2}^2 - c_4 \| F_A \|_{L^2} \leq \frac{1}{2} c_1 \), we can re-arrange this as
\[
\| \nabla_A \varphi \|_{L^2} \geq \frac{1}{2} c_1 \| \varphi \|_{L^2}^2.
\]
So the result holds with \( \eta = \varepsilon \) and \( \lambda = \sqrt{\frac{1}{2} c_1} \). \( \square \)
A connection $A$ belongs to $A_N(M)$ not always exist in a compact Riemannian $n$-manifold $M$. But thanks for the Uhlenbeck’ work:

**Theorem 2.4.** ([18] Corollary 4.3) If $2p > n$ and $M = M^n$ be a compact manifold, then there exists an $\varepsilon(p, M, G) > 0$ such that if $A$ is a connection with $\int_M |F_A|^p \leq \varepsilon$, then there exists a flat connection $\Gamma$ on $M$ and a gauge transformation $u$ such that

$$\|u^*(A) - \Gamma\|_{L^p(M)}^p \leq K \int_M |F_A|^p.$$

So if we can prove $L^p$-norm of the curvature of Yang-Mills connection can be estimate by $L^\frac{2}{p}$-norm when the $L^\frac{2}{p}$-norm is sufficiently small. The Theorem 2.4 is hold for the case of $p = \frac{n}{2}$.

**Lemma 2.5.** Let $M = M^n$ be a compact Riemannian manifold, $n \geq 2$, $A$ be a Yang-Mills connections with curvature $F_A$, for $2p \geq n$, there exist constant $\varepsilon$ and $C$ such that $\|F_A\|_{L^\frac{2}{p}} \leq \varepsilon$, then

$$\|F_A\|_{L^p} \leq C\|F_A\|_{L^\frac{2}{p}}.$$

**Proof.** Form Theorem 2.1 we have

$$\|F_A\|_{L^\infty} \leq C\|F_A\|_{L^2}.$$

For $n \geq 4$, by $L^p$ interpolation, we have

$$\|F_A\|_{L^2} \leq (Vol(M))^{1-\frac{2}{p}}\|F_A\|_{L^\frac{2}{p}}.$$

Then

$$\|F_A\|_{L^p}^p \leq \|F_A\|_{L^\infty}^{\frac{p-2}{2}}\|F_A\|_{L^\frac{2}{p}}^\frac{2}{2} \leq C^{p-\frac{2}{p}}\|F_A\|_{L^2}^{\frac{p-2}{2}}\|F_A\|_{L^\frac{2}{p}}^\frac{2}{2} \leq C^{p-\frac{2}{p}}(Vol(M))^{(1-\frac{2}{p})(p-\frac{2}{p})}\|F_A\|_{L^\frac{2}{p}}^p.$$

Thus

$$\|F_A\|_{L^p} \leq K\|F_A\|_{L^\frac{2}{p}},$$

where $K^p = C^{p-\frac{2}{p}}(Vol(M))^{(1-\frac{2}{p})(p-\frac{2}{p})}$.

For $n = 2, 3$, then $L^p$ interpolation implies that

$$\|F_A\|_{L^2}^2 \leq \|F_A\|_{L^\infty}^{2-\frac{2}{p}}\|F_A\|_{L^\frac{2}{p}}^{\frac{2}{p}} \leq (C\|F_A\|_{L^2})^{2-\frac{2}{p}}\|F_A\|_{L^\frac{2}{p}}^{\frac{2}{p}} \leq (2.3)$$

Thus

$$\|F_A\|_{L^2} \leq C^{\frac{4-n}{n}}\|F_A\|_{L^\frac{2}{p}}.$$
Yang-Mills connections on $G_2$-manifolds and Calabi-Yau 3-folds

And we have
\[
\|F_A\|_{L^p} \leq \|F_A\|_{L^\infty(M)} (Vol(M))^{\frac{1}{p}} \\
\leq C(Vol(M))^{\frac{1}{p}} \|F_A\|_{L^2}.
\]

Then we obtain
\[
\|F_A\|_{L^p} \leq \left(\frac{\|F_A\|_{L^\infty}}{Vol(M)}\right)^{\frac{1}{p}} C^\frac{4}{n} \|F_A\|_{L^2}.
\]

From the Lemma 2.5, the Theorem 2.4 is hold for $p = \frac{n}{2}$. By the Sobolev embedding theorem, $\|A - \Gamma\|_{L^n} \leq const. \|A - \Gamma\|_{L^n}$, then there exist a gauge transformation $u$ such that $\|u^*(A) - \Gamma\|_{L^n} \leq K \|F_A\|_{L^\frac{n}{2}}$. Then from Theorem 2.3, we have

**Corollary 2.6.** Let $A$ be a Yang-Mills connection on the bundle $E$ over $M$, where $M = M^n$ ($n \geq 2$) be a compact Ricci-flat manifold. Suppose all flat connections over $M$ are nondegenerate. There are constants $\eta$ and $\lambda$ such that if $\|F_A\|_{L^\frac{n}{2}} \leq \eta$, then there exists a gauge transformation $u$ such that
\[
\|\nabla u^*(A)\varphi\|_{L^2} \geq \lambda \|\varphi\|_{L^1}
\]
where $\varphi \in \Omega^1_M(adE)$.

## 3 Yang-Mills connection and $G_2$-instanton

### 3.1 $G_2$-manifolds

In this section, we collect some basic fact about of $G_2$-manifold $M$. For detail, see [2, 20].

**Definition 3.1.** [20] Let $M$ be a 7-dimensional smooth manifold, and $\phi \in \Lambda^3(M)$ a 3-form. $(M, \phi)$ is called a $G_2$-manifold if $\phi$ is non-degenerate and positive everywhere on $M$. We consider $M$ as a Riemannian manifold, with the Riemannian structure determined by $\phi$ as above. The manifold $(M, g, \phi)$ is called a holonomy $G_2$-manifold if $\phi$ is parallel with respect to the Levi-Civita connection associated with $g$. Further on, we shall consider only holonomy $G_2$-manifolds, and (abusing the language) omit the word holonomy.

Under the action of $G_2$, the space $\Lambda^*(M)$ splits into irreducible representations, as follows.
\[
\Lambda^2(M) = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M) \\
\Lambda^3(M) = \Lambda^3_1(M) \oplus \Lambda^3_2(M) \oplus \Lambda^3_{27}(M)
\]

where $\Lambda^p_d$ denotes an irreducible $G_2$ representation of dimension $d$. Clearly $\Lambda^*(M) = \Lambda^{7-*}(M)$ as a $G_2$-representation, and the space $\Lambda^4(M)$ and $\Lambda^5(M)$ split in a similar fashion. The space $\Lambda^0$ and $\Lambda^1$ are irreducible.
These summands for $\Lambda^2(M)$ can be characterized as follows:

$$\Lambda^2_7(M) = \{ \alpha \in \Lambda^2(M) \mid \alpha \wedge \phi = 2 * \phi \}$$

$$\Lambda^2_{14}(M) = \{ \alpha \in \Lambda^2(M) \mid \alpha \wedge \phi = -* \phi \}$$

We define a projective map $\Pi^2_7 : \Lambda^2(M) \to \Lambda^2_7(M)$.

**Proposition 3.2.**

$$\Pi^2_7 (\cdot) = \frac{1}{3} * (\cdot \wedge \phi) \wedge \phi$$

**Proof.** First we write $\forall f \in \Lambda^2(M)$ to

$$f = f^7 + f^{14}.$$  

where $f^7 \in \Lambda^2_7(M)$, $f^{14} \in \Lambda^2_{14}(M)$.

The operators in both sides are linear, so we only to prove

$$f^7 = \frac{1}{3} * (\cdot (f^7 \wedge \phi) \wedge \phi)$$

and

$$0 = \frac{1}{3} * (\cdot (f^{14} \wedge \phi) \wedge \phi).$$

There exists $\alpha \in \Lambda^1(M)$ such that

$$f^7 = * (\alpha \wedge * \phi).$$

then

$$\frac{1}{3} * (\cdot (f^7 \wedge \phi) \wedge \phi) = \frac{1}{3} * (\cdot (\alpha \wedge * \phi) \wedge \phi) = \frac{1}{3} * (3 \alpha \wedge * \phi) = * (\alpha \wedge * \phi) = f^7.$$

Here we use a identity holds for all $\alpha \in \Lambda^1(M)$ (see [2])

$$* (\cdot (\alpha \wedge * \phi) \wedge \phi) = 3 \alpha.$$

In local orthonormal coframe $\{e^j\}_{j=1}^7$ in which

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

here we write $e^{ijk}$ for the wedge product $e^i \wedge e^j \wedge e^k$. Every element in $\Lambda^2_{14}(M)$ is conjugate to an element of the form (see [2])

$$\beta = \lambda_1 e^{23} + \lambda_2 e^{45} - (\lambda_1 + \lambda_2) e^{67}.$$

Then compute in direct way, we get

$$* (\cdot (\beta \wedge \phi) \wedge \phi) = 0.$$
And these summands for $\Lambda^3(V^*)$ can be characterized as follows:

$$\Lambda^3_1(M) = \{r\phi \mid r \in \mathbb{R}\}$$

$$\Lambda^3_2(M) = \{*(\alpha \wedge \phi) \mid \alpha \in \Lambda^1(V^*)\}$$

$$\Lambda^3_2(M) = \{\alpha \in \Lambda^3(M) \mid \alpha \wedge \phi = \alpha \wedge \phi = 0\}$$

As above, we define a projective map $\Pi^3_1 : \Lambda^3(M) \to \Lambda^3_1(M)$.

**Proposition 3.3.**

$$\Pi^3_1(\cdot) = \frac{1}{7} * (\cdot \wedge \phi) \phi$$

(3.2)

### 3.2 $G_2$ instantons

We return to consider the Yang-Mills connection over $G_2$ manifolds. Let $A$ be a Yang-Mills connection, then $F_A$ is a harmonic $adE$ value 2-form on $M$. Write $F_A = F^7_A + F^{14}_A$, where $F^7_A \in \Lambda^2 \otimes g$ and $F^{14}_A \in \Lambda^4 \otimes g$. Then we have

$$F^7_A = \frac{1}{3} (F_A + *(F_A \wedge \phi)).$$

Hence

$$d_A^* F^7_A = \frac{1}{3} * d_A(F_A \wedge \phi) = 0.$$

Then it’s easy to see

$$d_A^* F^{14}_A = 0.$$

Following Verbitsky [20], on $G_2$ manifold we can define the structure operator, $C : \Lambda^*(M) \to \Lambda^{*+1}(M)$, which satisfies:

1. $C \mid_{\Lambda^0} = 0$
2. $C \mid_{\Lambda^1}(\cdot) = *(\phi \wedge \cdot)$
3. $C(\alpha \wedge \beta) = C(\alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge \beta$.

**Proposition 3.4.** [20] Let $(M, \phi)$ be a parallel $G_2$ manifold, and $C$ its structure operator. Then $C$ induces isomorphisms

$$\Lambda^i_7 \to \Lambda^{i+1}_7, \ (i = 2, 3, 4, 5).$$

For above proposition, there exists $\psi_A \in \Lambda^1(M) \otimes g$ such that

$$C(\psi_A) = F^7_A.$$

This means that

$$*(\phi \wedge \psi_A) = F^7_A.$$
Applying $d^*_A$ to each side gives

$$\ast (d_A \psi_A \wedge \ast \phi) = 0$$  \hspace{1cm} (3.3)

Then form Prop.3.2 and (3.3), we have

$$\Pi^2_1(d_A \psi_A) = 0$$  \hspace{1cm} (3.4)

There exists an identity always hold for $\forall \alpha \in \Lambda^1(M)$ (see [2])

$$\ast (\ast (\alpha \wedge \ast \phi) \wedge \ast \phi) = 3\alpha.$$  \hspace{1cm} (3.5)

By the definition of $\psi_A$, then

$$\psi_A = \frac{1}{3} \ast (\ast (F^7_A \wedge \ast \phi)).$$  \hspace{1cm} (3.5)

From (3.5), applying $d^*_A$ to $\psi_A$ gives

$$d^*_A \psi_A = \frac{1}{3} \ast d_A (F^7_A \wedge \phi) = \frac{1}{3} \ast (d_A F^7_A \wedge \phi)$$  \hspace{1cm} (3.6)

Next we want to prove $d^*_A \psi_A = 0$. First we denote the spaces of differential forms $\Lambda^0(M) = \Omega_1$, $\Lambda^1(M) = \Omega_7$, $\Lambda^2_{14}(M) = \Omega_{14}$ and $\Lambda^3_{27} = \Omega_{27}$. Then for all $p, q \in \{1, 7, 14, 27\}$, there exists a first order differential operator $d^p_q : \Omega_p \rightarrow \Omega_q$. In this article, we only use the identity

$$d\beta = \frac{1}{4} \ast (d_{14}^1 \beta \wedge \phi) + d_{27}^1 \beta.$$  \hspace{1cm} (3.7)

where $\beta \in \Omega_{14}$. For detail, see ([2] Proposition 3).

**Lemma 3.5.** Let $A$ be a Yang-Mills connection on a $G_2$-manifold $M$, then

$$\Pi^3_1(d_A F^7_A) = 0.$$  

**Proof.** First from the Bianchi identity $d_A F_A = 0$, we have

$$\Pi^3_1(d_A F_A) = \Pi^3_1(d_A F^7_A) + \Pi^3_1(d_A F_{14}^1) = 0$$

So we only need to proof $d_A F_{14}^1 = 0$. In the other way,

$$\Pi^3_1(d_A F_{14}^1) = \Pi^3_1(dF_{14}^1) + \Pi^3_1([A, F_{14}^1])$$

$$= \frac{1}{7} \ast ([A, F_{14}^1] \wedge \phi) \cdot \phi$$

$$= \frac{1}{7} \ast ([A \wedge \phi, F_{14}^1]) \cdot \phi = 0$$

We use the fact $\Pi^3_1(dF_{14}^1) = 0$, this can be obtain easily form (3.7).

And $[A \wedge \phi, F_{14}^1] = 0$, since $\ast (A \wedge \phi) \in \Lambda^2_{14}(M) \otimes g$. \hfill \Box
From the lemma 3.5 and (3.6), we can obtain
\[ d^*_A \psi_A = 0. \] (3.8)

On a $G_2$-manifold, we can express the Yang-Mills energy as
\[ \| F_A \|^2_{L^2} = \int_M tr(F_A \wedge \ast F_A) = \int_M tr(F^7_A \wedge \ast F^7_A + F^{14}_A \wedge \ast F^{14}_A) \]
\[ = \int_M tr(\frac{1}{2} F^7_A \wedge F^7_A \wedge \phi - F^{14}_A \wedge F^{14}_A \wedge \phi) \]
\[ = 3\| F^7_A \|^2_{L^2} - \int_M tr(F^2_A) \wedge \phi \]

The last integral is independent of the connection. We consider the variation $A + t\psi_A$. We have
\[ \| F_{A+t\psi_A} \|^2_{L^2} = 3\| F^7_{A+t\psi_A} \|^2_{L^2} + \text{topological constant}. \] (3.9)

From (3.4), we have
\[ F^7_{A+t\psi_A} = F^7_A + t\Pi^7_A(\ast d_A \psi_A) + t^2 \Pi^7_A(\ast \psi_A \wedge \psi_A) \]
\[ = F^7_A + t^2 \Pi^7_A(\ast \psi_A \wedge \psi_A) \]

We compare the terms of $t^2$ in (3.9), hence
\[ \| d_A \psi_A \|^2_{L^2} + 2\langle F_A, \psi_A \wedge \psi_A \rangle = 6\langle F^7_A, \psi_A \wedge \psi_A \rangle \] (3.10)

We using Weitzenböck formula (2.1) and the vanishing of the Ricci curvature on $G_2$-manifold, then
\[ \| d_A \psi_A \|^2_{L^2} = \| \nabla_A \psi_A \|^2_{L^2} + 2\langle F_A, \psi_A \wedge \psi_A \rangle \] (3.11)

From (3.10) and (3.11), we get
\[ \| \nabla_A \psi_A \|^2_{L^2} = 2\langle F^7_A, \psi_A \wedge \psi_A \rangle - 4\langle F^{14}_A, \psi_A \wedge \psi_A \rangle \] (3.12)

**Theorem 3.6.** Let $M$ be a compact $G_2$-manifold $M$ with $H^1(M) = 0$. Let $A$ be a Yang-Mills connection on $M$. Suppose all flat connections on $M$ are nondegenerate, there exists a constant $\delta$ such that if $\| F_A \|_{L^\infty} \leq \delta$, then the Yang-Mills connection must be a instanton.

**Proof.** If $\| F_A \|_{L^\infty} \leq \delta$, $\delta$ sufficiently small, then from the Corollary 2.6 there exists a flat connection $\Gamma$ and a gauge transformation $u$ (we also denote $u^*(A)$ to $A$) such that
\[ \| \nabla_A \psi_A \|^2_{L^2} \geq \lambda^2 \| \psi_A \|^2_{L^2}. \]

The identity (3.12) is invariant under gauge transformation, hence
\[ \| \nabla_A \psi_A \|^2_{L^2} = 2\langle F^7_A, \psi_A \wedge \psi_A \rangle - 4\langle F^{14}_A, \psi_A \wedge \psi_A \rangle \]
\[ \leq 4\| F_A \|_{L^\infty} \| \psi_A \|^2_{L^\infty} \]
\[ \leq c_5 \| F_A \|_{L^\infty} \| \psi_A \|^2_{L^2}. \]
here we use the Sobelov imbedding theorem $\|\psi_A\|_{L^\infty} \leq \text{const.} \|\psi_A\|_{L^1}^2$.

If $\|F_A\|_{L^2} \leq \min\{\frac{\lambda^2}{2c_5}, \delta\}$, then

$$\|\psi_A\|^2_{L^2} \leq \frac{1}{2} \|\psi_A\|_{L^2}^2.$$  

Then in $M$ $\psi_A$ is vanish, it implies that $F^7_A = 0$. 

\section{Yang-Mills connection and holomorphic bundle}

Let $M$ be a compact Calabi-Yau 3-fold, with Kähler form $\omega$ and nonzero covariant constant (3,0) form $\Omega$ \[9\]. Let $A$ be a connection on a $G$-bundle $E$ over $M$.

Decompose the curvature, $F_A$ as

$$F_A = F^{2,0}_A + F^{1,1}_A + \phi_A \omega + F^{0,2}_A$$

where $\phi_A = \frac{1}{3} (\Lambda F_A)$.

The Kähler identity

$$\omega \wedge F_A = *(F^{2,0}_A + 2\phi_A \omega - F^{1,1}_A + F^{2,0}_A)$$

implies, after wedging with $F_A$, taking the trace, and integrating, that

$$4\|F^{0,2}_A\|^2 + 9\|\phi_A\|^2 - \|F_A\|^2 = - \int_M \text{tr}(F^2_A) \wedge \omega$$

and is therefore independent of the connection. Then we have the identity

$$YM(A) = 4\|F^{0,2}_A\|^2 + \|\Lambda F_A\|^2 + \text{topological constant.} \quad (4.1)$$

The energy functional $\|\Lambda F_A\|^2$ plays an important role in the study of Hermitian-Einstein connections \[5\], \[19\].

\begin{lemma}
Let $M$ be a Kähler m-fold, $A$ be a Yang-Mills connection, then

$$\bar{\partial}^* F^{0,2}_A = 0. \quad (4.2)$$

\end{lemma}

\textbf{Proof.} Using Kähler identity again, we can obtain a identity the same to (4.1) for any Kähler m-fold.

$$YM(A) = 4\|F^{0,2}_A\|^2 + \|\Lambda F_A\|^2 + \text{topological constant}$$

When $A$ is a Yang-Mills connection, we have

$$\frac{1}{2} \frac{d}{dt} YM(A(t)) |_{t=0} = 0$$
where \( A(0) = A \). We can choose that \( A(t) = A + t(\psi + \bar{\psi}) \), \( \psi \in \Omega^{0,1}_M(ad(E)) \) and \( d_A^* \psi = d_A^* \bar{\psi} = 0 \), then \( \Lambda d_A(\psi + \bar{\psi}) = 0 \), so that \( \Lambda F_{A(t)} = \Lambda F_A + O(t^2) \). Then

\[
\frac{1}{2} \frac{d}{dt} YM(A(t))|_{t=0} = 4 \int_M \langle F_{A}^{0,2}, \bar{\partial}_A \psi \rangle = 4 \int_M \langle \bar{\partial}_A F_A^{0,2}, \psi \rangle = 0
\]

We have \( \bar{\partial}_A \bar{\partial}_A F_A^{0,2} = 0 \). It means that \( \bar{\partial}_A F_A^{0,2} \in ker \bar{\partial}_A \). Then we obtain that

\[
\bar{\partial}_A F_A^{0,2} = 0.
\]

Define an \( ad(E) \) valued (0,1) form \( \psi_A \), so that

\[
\Lambda_{\bar{\Omega}}(\psi_A) = F_A^{0,2}
\]

where \( \Lambda_{\bar{\Omega}} \) is the dual of \( L_{\bar{\Omega}} : \eta \rightarrow \bar{\Omega} \wedge \eta \).

More explicitly, in a local special unitary frame

\[
\psi_A = F_{23}^{2,0} d\bar{z}^1 + F_{31}^{2,0} d\bar{z}^2 + F_{12}^{2,0} d\bar{z}^3
\]

Applying \( \bar{\partial}_A \) to each side of (4.3) gives

\[
\Lambda_{\bar{\Omega}}(\bar{\partial}_A \psi_A) = 0
\]

and therefore

\[
\bar{\partial}_A \psi_A = 0 \quad (4.4)
\]

The Bianchi identity implies \( \bar{\partial}_A F_A^{0,2} = 0 \), which is equivalent to

\[
\bar{\partial}_A \psi_A = 0 \quad (4.5)
\]

We consider the connection \( A_t = A + t(\psi_A + \bar{\psi}_A) \). We denote \( \eta_A = \psi_A + \bar{\psi}_A \). From (4.1), we have

\[
YM(A_t) = 4\| F_{A_t}^{0,2} \|^2 + \| \Lambda F_{A_t} \|^2 + \text{topological constant}.
\]

Hence both sides are quadratic polynomials on \( t \). Compare the terms of \( t^2 \), we have

\[
\| d_A \eta_A \|^2_{L^2} + 2Re\langle F_A, \eta_A \wedge \eta_A \rangle = 8Re\langle F_A^{0,2}, \eta_A \wedge \eta_A \rangle + 6Re\langle \phi_A \omega, \eta_A \wedge \eta_A \rangle. \quad (4.6)
\]

From (4.5), we get \( d_A^* \eta_A = 0 \). We using Weitzenböck formula (2.1) and the vanishing of the Ricci curvature on Calabi-Yau manifold, then

\[
\| d_A \eta_A \|^2_{L^2} = \| \nabla_A \eta_A \|^2_{L^2} + 2Re\langle F_A, \eta_A \wedge \eta_A \rangle
\]

So (4.6) become to

\[
\| \nabla_A \eta_A \|^2_{L^2} = 4Re\langle F_A^{2,0} + F_A^{1,1}, \eta_A \wedge \eta_A \rangle - 4Re\langle F_A^{0,2}, \eta_A \wedge \eta_A \rangle - 6Re\langle \phi_A \omega, \eta_A \wedge \eta_A \rangle. \quad (4.7)
\]
**Theorem 4.2.** Let $A$ be a Yang-Mills connection on a vector bundle $E$ over a compact Calabi-Yau 3-fold. Suppose all flat connections on $M$ are nondegenerate, there exist a constant $\gamma$ such that if $\|F_A\|_{L^3} \leq \gamma$, then the bundle is holomorphic.

**Proof.** If $\|F_A\|_{L^3} \leq \delta$, $\delta$ sufficiently small, then from the Corollary 2.6 there exists a flat connection $\Gamma$ and a gauge transformation $u$ (we also denote $u^*(A)$ to $A$) such that

$$\|\nabla_A \psi_A\|_{L^2}^2 \geq \xi^2 \|\psi_A\|_{L^1}^2.$$

The identity (4.7) is invariant under gauge transformation, hence

$$\|\nabla_A \eta_A\|^2_{L^2} = 4Re\langle F_A^{2.0} + F_A^{1.1}, \eta_A \wedge \eta_A \rangle - 4Re\langle F_A^{0.2}, \eta_A \wedge \eta_A \rangle - 6Re\langle \phi_A \omega, \eta_A \wedge \eta_A \rangle$$

$$\leq 4\|F_A\|_{L^3}\|\eta_A\|^2_{L^3}$$

$$\leq c_6\|F_A\|_{L^3}\|\eta_A\|^2_{L^2}.$$

here we use the Sobolev imbedding theorem $\|\eta_A\|^2_{L^4} \leq const.\|\eta_A\|^2_{L^2}$.

If $\|F_A\|_{L^3} \leq \min\{\frac{\lambda^2}{2c_6}, \lambda\}$, then

$$\|\eta_A\|^2_{L^2} \leq \frac{1}{2}\|\eta_A\|^2_{L^1}.$$

it’s implies that $\eta_A = 0$, then $F_A^{0.2} = 0$. 

\[\blacksquare\]

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