THE ORTHOGONAL CHARACTER TABLE OF $\text{SL}_2(q)$

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Abstract. The rational invariants of the $\text{SL}_2(q)$-invariant quadratic forms on the real irreducible representations are determined. There is still one open question (see Remark 6.5) if $q$ is an even square.

1. Introduction

Throughout the paper let $G$ be a finite group. The isomorphism classes of $CG$-modules are parametrized by their characters. Our aim is to extend this connection in order to also determine the $G$-invariant quadratic forms from the character table of $G$. The ordinary character table displays the characters $\chi_V$ of the absolutely irreducible $CG$-modules $V$. For each $\chi_V$ let $K$ be the maximal real subfield of the character field of $V$ and $W$ the irreducible $KG$-module such that $V$ occurs in $W \otimes_K C$. Then the space

$$\mathcal{F}_G(W) := \left\{ F : W \times W \to K \mid F(v,w) = F(w,v) \text{ and } F(gw,gv) = F(w,v) \text{ for all } g \in G, v,w \in W \right\}$$

of $G$-invariant symmetric bilinear forms on $W$ is at least one-dimensional and every non-zero $F \in \mathcal{F}_G(W)$ is non-degenerate. The character $\chi_V$ also determines the $K$-isometry classes of the elements of $\mathcal{F}_G(W)$. The orthogonal character table additionally contains the invariants (see Section 2) that determine the $K$-isometry classes of $(W,F)$ for all non-zero $F \in \mathcal{F}_G(W)$.

For $G = \text{SL}_2(q)$ the ordinary character table was already known to Schur, [16]. This paper determines the orthogonal character tables of $\text{SL}_2(q)$ for all prime powers $q$. For $q = 2^n$ with $n$ even and the characters of degree $q + 1$ we could not specify which even primes ramify in the Clifford algebra (see Section 6).

This work grew out of the first author’s PhD thesis [2] written under the supervision of the second author. In this thesis, the first author also determines the ordinary orthogonal character tables for all (non-abelian) finite quasisimple groups of order up to $200,000$.

2. Invariants of quadratic spaces

Let $K$ be a field of characteristic 0, $V$ an $n$-dimensional vector space over $K$ and $F : V \times V \to K$ a non-degenerate symmetric bilinear form. The two most important invariants attached to such a space $(V,F)$ are the discriminant and the
Clifford invariant.

The discriminant of \((V, F)\) is

\[
d_{\pm}(V, F) := (-1)^{(n-1)/2} \det(V, F)
\]

where the determinant \(\det(V, F) \in K/(K^\times)^2\) is defined as the square class of the determinant of a Gram matrix of \(F\) with respect to any basis.

The Clifford algebra \(\mathcal{C}(V, F)\) is the quotient of the tensor algebra by the two-sided ideal \(\langle v \otimes v - \frac{1}{2} F(v, v) \cdot 1 \mid v \in V \rangle\). A \(K\)-basis of \(\mathcal{C}(V, F)\) is given by the ordered tensors \((b_{i_1} \otimes \ldots \otimes b_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n)\) of any basis \((b_1, \ldots, b_n)\) of \(V\), in particular \(\dim(\mathcal{C}(V, F)) = 2^n\). Put

\[
c(V, F) := \begin{cases} 
\mathcal{C}(V, F) & \text{if } n \text{ is even,} \\
\mathcal{C}_0(V, F) := \langle b_{i_1} \otimes \ldots \otimes b_{i_k} \mid k \text{ even} \rangle & \text{if } n \text{ is odd.}
\end{cases}
\]

Then \(c(V, F) \cong \mathcal{D}^\times\) is a central simple \(K\)-algebra with involution and therefore it has order 1 or 2 in the Brauer group. The Clifford invariant of \((V, F)\) is defined as the Brauer class of \(c(V, F)\):

\[
c(V, F) := [c(V, F)] = [\mathcal{D}] \in \text{Br}(K).
\]

A more detailed exposition of this material may be found e.g. in \([15]\).

Our interest in these two isometry invariants of quadratic spaces is mainly due to the following classical result by Helmut Hasse.

**Theorem 2.1** ([7]). Over a number field \(K\) the isometry class of a quadratic space is uniquely determined by its dimension, its determinant, its Clifford invariant and its signature at all real places of \(K\).

We are mainly interested in the case where \(K\) is a number field. Then \(\mathcal{D}\) is either \(K\) or a quaternion division algebra over \(K\). We use two notations for these \(\mathcal{D}\), either as a symbol algebra or by giving all the local invariants of \(\mathcal{D}\):

**Definition 2.2.** For \(a, b \in K\) let \((a, b) := \left[\left(\frac{a, b}{K}\right)\right] \in \text{Br}(K)\) where

\[
\left(\frac{a, b}{K}\right) := \langle 1, i, j, k \mid i^2 = a, j^2 = b, ij = -ji = k \rangle.
\]

By the Theorem of Hasse, Brauer, Noether, Albert (see [14, Theorem (32.11)]) any quaternion algebra \(\mathcal{D}\) over \(K\) is determined by the set of places \(\wp_1, \ldots, \wp_s\) (the ramified places) of \(K\), for which the completion of \(\mathcal{D}\) stays a division algebra. Therefore we also describe \(\mathcal{D} = \mathcal{Q}_{\wp_1, \ldots, \wp_s}\) by its ramified places, where we assume that the center \(K\) is clear from the context.

**Example 2.3.** Let \((V, F)\) be a bilinear space and \(a \in K^\times\). Then the scaled space \((V, aF)\) has the following algebraic invariants (see [10, 5.(3.16)] for the Clifford invariant):

\[
d_{\pm}(V, aF) = \begin{cases} 
\quad d_{\pm}(V, F) & \text{if } \dim(V) \text{ is even,} \\
ad_{\pm}(V, F) & \text{if } \dim(V) \text{ is odd.}
\end{cases}
\]
and
\[ c(V, aF) = \begin{cases} c(V, F)(a, d_\pm(V, F)) & \text{if } \dim(V) \text{ is even}, \\ c(V, F) & \text{if } \dim(V) \text{ is odd}. \end{cases} \]

If
\[ (V, F) = (V_1, F_1) \perp (V_2, F_2) \]
is the orthogonal direct sum of two subspaces the determinant is just the product
\[ \det(V, F) = \det(V_1, F_1) \cdot \det(V_2, F_2). \]
The behavior of the Clifford invariant is more complicated, cf. [10]:
\[ c(V, F) = \begin{cases} c(V_1, F_1)c(V_2, F_2)(d_\pm(V_1, F_1), d_\pm(V_2, F_2)), & \dim(V_1) \equiv \dim(V_2) \pmod{2}, \\ c(V_1, F_1)c(V_2, F_2)(-d_\pm(V_1, F_1), d_\pm(V_2, F_2)), & \dim(V) \equiv \dim(V_1) \equiv 1 \pmod{2}. \end{cases} \]

**Example 2.4.** Let \( \mathbb{I}_n \) be the \( n \)-dimensional \( \mathbb{Q} \)-vector space that has an orthonormal basis \( (e_1, \ldots, e_n) \). Then \( d_\pm(\mathbb{I}_n) = (-1)^{(n-1)/2}(\mathbb{Q}^\times)^2 \) and
\[ c(\mathbb{I}_n) = \begin{cases} (1, 1) & n \equiv 0, 1, 2, 7 \pmod{8}, \\ (-1, -1) & n \equiv 3, 4, 5, 6 \pmod{8}. \end{cases} \]

The space \( \mathbb{A}_{n-1} := \langle \sum_{i=1}^n e_i \rangle^\perp \leq \mathbb{I}_n \) is the orthogonal complement of a space of discriminant \( n \) in \( \mathbb{I}_n \). This allows to compute the discriminant and Clifford invariant of \( \mathbb{A}_{n-1} \) using the formulas from the previous example: \( d_\pm(\mathbb{A}_{n-1}) = (-1)^{(n-1)(n-2)/2} n(\mathbb{Q}^\times)^2 \) and \( c(\mathbb{A}_{n-1}) \) depends on the value of \( n \) modulo 8:

| \( n \pmod{8} \) | 0, 1 | 2, 3 | 4, 5 | 6, 7 |
|------------------|-----|-----|-----|-----|
| \( c(\mathbb{A}_{n-1}) \) | 1 | \((-1, n)\) | \((-1, -1)\) | \((-1, -n)\) |

### 3. Methods

**3.1. Orthogonal character tables.** Let \( \chi \) be a complex irreducible character of the finite group \( G \) and let \( \hat{K} = \mathbb{Q}^{\chi^\times} \) be the maximal real subfield of the character field \( \mathbb{Q}(\chi) \). Let \( V \) be the irreducible \( CG \)-module affording the character \( \chi \) and let \( W \) be the irreducible \( KG \)-module such that \( V \) is a constituent of \( W_C := \mathbb{C} \otimes_K W \). Put
\[ \mathcal{F}_G(W) := \left\{ F : W \times W \to \hat{K} \mid F(v,w) = F(w,v) \text{ and } F(gw,gv) = F(w,v) \text{ for all } g \in \hat{G}, v,w \in W \right\} \]
the space of \( G \)-invariant symmetric bilinear forms on \( W \). As \( W \) is irreducible, all non-zero elements of \( \mathcal{F}_G(W) \) are non-degenerate and an easy averaging argument shows that \( \mathcal{F}_G(W) \) always contains a totally positive definite form \( F_0 \). We call \( W \) uniform if \( \mathcal{F}_G(W) = \langle F_0 \rangle_K \) is one-dimensional over \( K \).

**Remark 3.1.** There are three different situations to be considered:

(a) \( K = \mathbb{Q}(\chi) \) and \( V = W_C \): Then \( W \) is an absolutely irreducible \( KG \)-module and hence uniform.

(b) \( K = \mathbb{Q}(\chi) \) and \( W_C \cong V \oplus V \): Then the Schur index of \( \chi \) over \( K \) is \( 2 \), \( \chi(1) \) is even, and [18] tells us that \( d_\pm(F) \in (\hat{K}^\times)^2 \) for all non-zero \( F \in \mathcal{F}_G(W) \). If the real Schur index of \( \chi \) is one, then \( \dim(\mathcal{F}_G(W)) = 3 \).
If the real Schur index of $\chi$ is 2, then $W$ is uniform and \cite{18} Theorem B also gives the Clifford invariant of $(W,F)$:

$$c(W,F) = \begin{cases} 1 & \text{if } \dim_K(W) \equiv 0 \pmod{8} \\ [\text{End}_{FG}(W)] & \text{if } \dim_K(W) \equiv 4 \pmod{8}. \end{cases}$$

(c) $[Q(\chi) : K] = 2$. Then $\chi_W = m(\chi + \overline{\chi})$ for some $m \in \mathbb{N}$ and $W$ is uniform if and only if $m = 1$. Choose $\delta \in K$ such that $Q(\chi) = K(\sqrt{\delta})$, then $d_{\pm}(F) = \delta^{m\chi(1)}(K^*)^2$ for all $0 \neq F \in \mathcal{F}_G(W)$ (see \cite{15} Theorem 10.1.4, \cite{2} Theorem 4.3.9).

**Definition 3.2.** Let $\chi$, $K := Q(\chi)^+$, $W$ be as above. Put $n := \dim_K(W)$ and choose $0 \neq F \in \mathcal{F}_G(W)$. If $n$ is even then we define

$$d_{\pm}(\chi) := d_{\pm}(W,F).$$

If $n$ is odd, or $n$ is even, $W$ is uniform, and $d_{\pm}(\chi) = 1$, then we put

$$c(\chi) := c(W,F).$$

The orthogonal character table of $G$ is the complex character table of $G$ with this additional information added.

By Example 2.3 and Remark 3.1 the values $d_{\pm}(\chi)$ and $c(\chi)$ are well defined, i.e. independent of the choice of the non-zero $F \in \mathcal{F}_G(W)$.

### 3.2. Clifford Orders

Let us now assume that $K$ is a local or global field of characteristic 0, i.e. $K$ is a finite extension of either $\mathbb{Q}_p$ or $\mathbb{Q}$, and let $R$ denote the ring of integers in $K$. Let $V$ be a finite dimensional vector space over $K$ and $F : V \times V \to K$ a symmetric bilinear form with associated quadratic form

$$Q_F : V \to K, v \mapsto Q_F(v) = \frac{1}{2} F(v,v).$$

**Definition 3.3.** A lattice $L$ in $V$ is a finitely generated $R$-submodule of $V$ that contains a $K$-basis of $V$. The lattice $L$ is called integral, if $F(L,L) \subseteq R$ and even, if $Q_F(L) \subseteq R$. The dual lattice of $L$ is $L^\# := \{v \in V \mid F(v,L) \subseteq R\}$ and $L$ is called unimodular, if $L = L^\#$.

Even unimodular lattices are called regular quadratic $R$-modules in \cite{9}. If $2 \not| R^\times$, then there are no regular $R$-modules $L$ of odd dimension. Kneser calls an even lattice $L$ of odd dimension such that $L^\#/L \cong R/2R$ a semi-regular quadratic $R$-module.

**Theorem 3.4** (\cite{9} Satz 15.8]). Assume that $R$ is a complete discrete valuation ring (with finite residue class field) and let $L$ be a regular or semi-regular quadratic $R$-module in $(V,Q_F)$. If $\dim(V) \geq 3$ then $L \cong \mathbb{H}(R) \perp M$ for some regular or semi-regular quadratic $R$-module $M$. Here $\mathbb{H}(R)$ is the hyperbolic plane, the regular free $R$-lattice with basis $(e,f)$ such that $Q_F(e) = Q_F(f) = 0$ and $F(e,f) = 1$.

As both invariants, the Clifford invariant and the discriminant of the hyperbolic plane $\mathbb{H}(K) = K\mathbb{H}(R)$ are trivial, we obtain the following corollary.
Corollary 3.5. Under the assumption of the theorem let \( \dim(V) \) be odd and \( L \) be a semi-regular lattice in \( V \). Then \( c(V, F) = 1 \).

Proof. We proceed by induction on the dimension of \( V \). If \( \dim(V) = 1 \) then \( c(V, F) = K \) and so \( c(V, F) = 1 \). So assume that \( \dim(V) \geq 3 \). Then \( L \cong H(R) \perp M \) and hence \( V \cong H(K) \perp KM \) for some semi-regular lattice \( M \) in \( KM \). By induction we have \( c(KM, F|KM) = 1 \). So

\[
c(V, F) = c(KM, F|KM)c(H(K))(-d_\pm(KM, F|KM), d_\pm(H(K))) = 1.
\]

Remark 3.6. Let \( L \) be an even lattice in \( V \). Then the Clifford order \( C(L, F) \) of \( L \) is defined to be the \( R \)-subalgebra of \( C(V, F) \) generated by \( L \). As \( Q_F(L) \subseteq R \), the Clifford order is an \( R \)-lattice in \( C(V, F) \), in particular finitely generated over \( R \). If \( L \) has an orthogonal basis \( (b_1, \ldots, b_n) \), then the ordered tensors \( (b_1 \otimes \cdots \otimes b_k | 1 \leq i_1 < \ldots < i_k \leq n) \) form an \( R \)-basis of \( C(L, F) \). In this case it is easy to compute the determinant of \( C(L, F) \) and of \( C_0(L, F) \) with respect to the reduced trace bilinear form (see [2, Theorem 7.2.2]): Up to some power of 2 they are both powers of \( Q_F(b_1) \cdots Q_F(b_n) \).

Corollary 3.7. Assume that \( K \) is a number field, \( 2 \neq p \in \mathbb{Z} \) is some odd prime and \( \wp \) is a prime ideal of \( K \) containing \( p \). Denote the completion of \( K \) at \( \wp \) by \( K_\wp \) and its valuation ring by \( R_\wp \). Assume that there is a lattice \( L \) in \( V \) such that \( L_\wp = R_\wp \otimes L \) is an even unimodular \( R_\wp \)-lattice. Then

\[
[c(V, F) \otimes K_\wp] = 1 \in \text{Br}(K_\wp).
\]

Proof. Since \( 2 \in R_\wp^\times \) the lattice \( L_\wp \) has an orthogonal basis and Remark 3.6 shows that the determinant of the Clifford order \( C(L_\wp, F) \) and also of \( C_0(L_\wp, F) \) is a unit in \( R_\wp \). In particular the determinant of a maximal order in \( c(V, F) \otimes K_\wp \) is a unit in \( R_\wp \), which shows that this central simple \( K_\wp \)-algebra is a matrix ring over \( K_\wp \) (see for instance [14, Theorem (20.3)]).

A bit more generally we may also compute the Clifford invariant of a bilinear space that contains a lattice of prime determinant:

Corollary 3.8. Keep the assumptions of Corollary 3.7 and let \( (W_\wp, E_\wp) \) be a 1-dimensional bilinear \( K_\wp \) vector space such that the \( \wp \)-adic valuation of the discriminant of \( E_\wp \) is odd. Then

\[
c((V \otimes K_\wp, F) \perp (W_\wp, E_\wp)) = 1 \in \text{Br}(K_\wp) \text{ if and only if } d_\pm(V \otimes K_\wp, F) \in (K_\wp^\times)^2.
\]

Proof. Clearly the Clifford invariant of the 1-dimensional space is trivial and also \( c(V \otimes K_\wp, F) \) is trivial by Corollary 3.7. So the formula in Example 2.3 gives us the Clifford invariant of the orthogonal sum as

\[
c((V \otimes K_\wp, F) \perp (W_\wp, E_\wp)) = (d_\pm(V \otimes K_\wp, F), u\pi)
\]

where \( u \) is a unit and \( \pi \) is a prime element in the valuation ring \( R_\wp \). As \( d := d_\pm(V \otimes K_\wp, F) \in R_\wp^\times \), this quaternion symbol is trivial if and only if \( d \) is a square.
3.3. A Clifford theory of orthogonal representations. Let $N \trianglelefteq G$ be a normal subgroup. Clifford theory explains the interplay between irreducible representations of $N$ and $G$ (see for instance [4, Section 11.1]). We want to describe the behavior of invariant forms under this correspondence.

Let $K$ be a totally real number field and $V$ an irreducible $KG$-module with a non-degenerate invariant form $F$. We will then call $(V, F)$ an orthogonal representation of $G$. Let $U$ be an irreducible $KN$-module occurring as a direct summand of $V|_N$ with multiplicity $e$. Let $I$ be the inertia group of $U$, of index $t := [G : I]$ in $G$, and let $G = \bigsqcup_{i=1}^t g_i I$ be a decomposition of $G$ into left $I$-cosets. We then have the following decomposition of $V|_N$ into pairwise non-isomorphic irreducible $KN$-modules $g_i U$ ($i = 1, \ldots, t$):

\begin{equation}
V|_N \cong \bigoplus_{i=1}^t (g_i U)^e,
\end{equation}

In this situation we obtain the following theorem

**Lemma 3.9.** The decomposition \([1]\) is orthogonal \[(V|_N, F) = (g_1 U^e, F_1) \perp (g_2 U^e, F_2) \perp \ldots \perp (g_t U^e, F_t)\]
and the forms $F_i$ are non-degenerate and pairwise $K$-isometric.

**Proof.** Clearly, the restriction of $F$ to the direct summand $g_i U^e$ is $N$-invariant. For $i \neq j$ we have

\[g_i U \cong g_i U^* \not\cong g_j U\]
so the summands $(g_i U)^e$ are orthogonal to each other and the $F_i$ are non-degenerate. The elements $g_j^{-1} g_i \in G \leq O(V, F)$ induce isometries between $F_i$ and $F_j$. \[\square\]

**Example 3.10.** Consider an odd prime $p$, a natural number $n$ and abbreviate $q := p^n$. Let $C_{(q-1)/2} \cong H \leq GL_n(F_p)$ be a subgroup acting with regular orbits on $F_p^n \setminus \{0\}$ in its natural action. Then the group $G := C_p^n \rtimes H$, which is isomorphic to the normalizer of a Sylow $p$-subgroup in $PSL_2(q)$ has $(q-1)/2$ linear characters and two non-linear characters $\psi_1, \psi_2$ of degree $(q-1)/2$ with Schur index $1$ and character field

\[Q(\psi_1) = Q(\psi_2) = \begin{cases} Q(\sqrt{q}) & \text{if } q \equiv 1 \pmod{4}, \\ Q(-\sqrt{q}) & \text{if } q \equiv 3 \pmod{4}. \end{cases}\]

Let $H_1$ be the unique subgroup of $H$ of order $p^{n-1}/2$ and put $N := C_p^n \rtimes H_1$. Then $N \trianglelefteq G$ and we will apply Theorem 3.9 to this normal subgroup in order to compute the discriminant $d_\pm(\psi_i)$ in the case $q \equiv 1 \pmod{4}$.

Let $\psi \in \{\psi_1, \psi_2\}$, $K = Q(\psi) = Q(\sqrt{q})$ and $(V, F)$ an orthogonal $KG$-module whose character is $\psi$. There is a character $1 \neq \chi \in \text{Irr}(C_p^n)$ such that $\psi = \text{ind}_{C_p^n}^G(\chi) = \text{ind}_N^G(\text{ind}_{C_p^n}^N(\chi))$. Ordinary Clifford theory shows that $\kappa := \text{ind}_{C_p^n}^N(\chi)$ is irreducible and an easy
application of Frobenius reciprocity reveals \((\psi|_N, \kappa)_N = 1\).

Thus we obtain an orthogonal decomposition

\[(V|_N, F) \cong (V_1, F_1) \perp \ldots \perp (V_t, F_t)\]

where \(F_1 \cong \ldots \cong F_t\) by Lemma \[3.3.2]\. We have \(t = \frac{q-1}{2} \frac{1}{p-1}\) if \(K = \mathbb{Q}\) and \(t = \frac{q-1}{p-1}\) if \(K = \mathbb{Q}(\sqrt{p})\).

Notice that \(\kappa\) is a faithful character of a group isomorphic to \(C_p \times C_{p-1}\). As the trace forms of cyclotomic fields are well understood (cf. [11] Section 3.3.2), we can find the determinants of the \((V_i, F_i)\) as

\[
\det(V_i, F_i) = \det(V_1, F_1) = \begin{cases} p(Q^x)^2 & \text{if } n \text{ is even} \\ u\sqrt{p}(Q(\sqrt{p})^x)^2 & \text{if } n \text{ is odd} \end{cases}
\]

for some unit \(u\) of the ring of integers of \(\mathbb{Q}(\sqrt{p})\). In conclusion, we obtain

\[
\det(\psi) = \begin{cases} 1(Q^x)^2 & \text{if } n \equiv 0 \pmod{4} \text{ or } p \equiv 3 \pmod{4}, \\ p(Q^x)^2 & \text{if } n \equiv 2 \pmod{4} \text{ and } p \equiv 1 \pmod{4}, \\ u\sqrt{p}(Q(\sqrt{p})^x)^2 & \text{if } n \equiv 1 \pmod{2}, \end{cases}
\]

In the case \(n \equiv 3 \pmod{4}\) the character \(\psi\) has non-real values and we find \(d_\pm(\psi) = -p(Q^x)^2\).

4. THE ORTHOGONAL CHARACTER TABLE OF \(SL_2(q)\) FOR ODD \(q\)

Let \(p\) be an odd prime, \(n\) a natural number, put \(q := p^n\) and let \(G := SL_2(q)\) be the group of all \(2 \times 2\) matrices of determinant 1 over the field with \(q\) elements. A reference for the ordinary (and modular) representation theory of this group is, for example [11]. We use the ordinary character table and the notation of the absolutely irreducible characters from [5]:

**Theorem 4.1** ([5] Theorem 38.1). Let \(\langle \nu \rangle = \mathbb{F}_q^\times\). Consider

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}
\]

and let \(b \in SL_2(q)\) be an element of order \(q+1\).

For \(x \in SL_2(q)\), let \(\langle x \rangle\) denote the conjugacy class containing \(x\). \(SL_2(q)\) has the following \(q+4\) conjugacy classes of elements, listed together with the size of the classes.

| \(x\) | 1 | \(z\) | \(c\) | \(d\) | \(z_c\) | \(z_d\) | \(a\) | \(b^m\) |
|-----|---|---|---|---|---|---|---|---|
| \(|(x)\)| 1 | 1 | \(\frac{1}{2}(q^2-1)\) | \(\frac{1}{2}(q^2-1)\) | \(\frac{1}{2}(q^2-1)\) | \(\frac{1}{2}(q^2-1)\) | \(q(q+1)\) | \(q(q-1)\) |

where \(1 \leq \ell \leq \frac{q-3}{2}, 1 \leq m \leq \frac{q-1}{2}\).

Put

\[
\varepsilon := (-1)^{(q-1)/2}, \quad \zeta_r := \exp(2\pi i/r) \quad \text{and} \quad \psi_r(s) := \zeta_r^s + \zeta_r^{-s} \text{ for } r, s \in \mathbb{N}.
\]
Then the character table of $\text{SL}_2(q)$ reads as

| $\chi$ | $K$ | $\dim_K(W)$ | $c(\chi)$ | $d_{\pm}(\chi)$ | $q$ |
|-------|-----|-------------|-----------|----------------|----|
| $1$   | $Q$ | 1           | 1         | -              | all|
| $\psi$ | $Q$ | $q$         | $c(A_q)$  | -              | all|
| $\chi_{i\text{ even}}$ | $Q(\sqrt{q})$ | $q + 1$ | - | $\varepsilon(\varepsilon - 2)q$ | all|
| $\chi_{i\text{ odd}}$ | $Q(\sqrt{q})$ | $2(q + 1)$ | $[\text{End}_{K_G}(W)]$ | 1 | 1 (mod 4) |
| $\theta_{j\text{ even}}$ | $Q(\sqrt{q})$ | $q - 1$ | 1 if $q = 2$ | $\varepsilon q$ | all|
| $\theta_{j\text{ odd}}$ | $Q(\sqrt{q})$ | $2(q - 1)$ | 1 | $[\text{End}_{K_G}(W)]$ | 1 | 1 (mod 4) |
| $\xi_1, \xi_2$ | $Q(\sqrt{q})$ | $\frac{q+1}{2}$ | 1 | - | $q \equiv 1,-3 (\text{mod 16})$ |
| $\xi_1 = \xi_2$ | $Q$ | $q + 1$ | 1 | 1 | 3 (mod 8) |
| $\eta_1, \eta_2$ | $Q(\sqrt{q})$ | $q - 1$ | $[\text{Q}_{p,\infty} \otimes K]$ | 1 | 1 (mod 4) |
| $\eta_1 = \eta_2$ | $Q$ | $q - 1$ | - | $-q$ | 3 (mod 4) |

where $1 \leq i \leq \frac{q-3}{2}$, $1 \leq j \leq \frac{q-1}{2}$, $1 \leq \ell \leq \frac{q-3}{2}$, $1 \leq m \leq \frac{q-1}{2}$.

The columns for the classes $(zc)$ and $(zd)$ are omitted because for any irreducible character $\chi$ the relation $\chi(zc) = \chi(zd)\chi(1)$ holds.

**Theorem 4.2.** The following table gives the orthogonal character table of $\text{SL}_2(q)$.

We use the abbreviations introduced in Theorem 4.1. As before $K$ is the maximal real subfield of the character field and $W$ the irreducible $K_G$-module, whose character contains $\chi$. 
5. The proof of Theorem 4.2

5.1. The faithful characters of $G$. The faithful irreducible characters of $\text{SL}_2(q)$ either have real Schur index 2 or they take values in an imaginary quadratic number field. Janusz [8, Theorem] contains an explicit description of the endomorphism rings $\text{End}_{KG}(W)$. In particular their discriminants and Clifford invariants can be read off from Remark 3.1 (b) and (c).

5.2. The non-faithful characters $\eta_i$. If $q \equiv 3 \pmod{4}$ then the characters $\eta_1$ and $\eta_2$ of degree $(q-1)/2$ have character field $\mathbb{Q}(\sqrt{q}) = \mathbb{Q}(\sqrt{-p})$ and Schur index 1. So Remark 3.1 (c) yields their discriminant.

5.3. The Steinberg character. The character $\psi$ is a non-faithful character of degree $q$. As $1 + \psi$ is the character of a 2-transitive permutation representation of $G$, the invariants of $\psi$ are those of $A_q$ as given in Example 2.4.

5.4. The characters $\theta_j$, $j$ even. For even $j$, the character $\theta_j$ is a non-faithful character of even degree $q-1$ with totally real character field $K$ and Schur index 1. Let $(W, F)$ be the orthogonal $KG$-module affording the character $\theta_j$. Then the restriction of $W$ to the Borel subgroup $B \cong (C_{(q)}^n \rtimes C_{(q-1)/2})$ of $\text{PSL}_2(q)$ has character $\psi_1 + \psi_2$ from Example 3.10. As $d_\pm(\psi_1)$ and $d_\pm(\psi_2)$ are Galois conjugate, the formula for $d_\pm(\psi_1)$ in Example 3.10 yields

$$d_\pm(\theta_j) = \begin{cases} 1(K^\times)^2 & n \text{ even} \\ \varepsilon p(K^\times)^2 & n \text{ odd}. \end{cases}$$

If $n$ is even then we can also deduce the Clifford invariant of $(W, F)$: In this case $q \equiv 1 \pmod{4}$ so $-\zeta_{q+1}^2$ is a primitive $q+1$st root of unity and hence all characters of degree $q-1$ of the group $\text{PSL}_2(q)$ extend to characters of $\text{PGL}_2(q)$ with the same character field (see [17, Table III] for a character table) and of Schur index 1 (see [4]). So $(W, F)$ is also an orthogonal representation of $\text{PGL}_2(q)$ and restricting $(W, F)$ to $B$, we obtain the orthogonal sum of two isometric spaces $(W, F) \cong (V_1, F_1) \perp (V_2, F_2)$ because the normalizer of $B$ in $\text{PGL}_2(q)$ interchanges $V_1$ and $V_2$. By Example 3.10 we have $d_\pm(V_i, F_i) = p$ if $n \equiv 2 \pmod{4}$ and $p \equiv 1 \pmod{4}$ and $d_\pm(V_i, F_i) = 1$ otherwise ($i = 1, 2$). In both cases $(d_\pm(V_1, F_1), d_\pm(V_2, F_2)) = 1 \in \text{Br}(\mathbb{Q})$ and so by Example 2.3 $\mathfrak{c}(W, F) = \mathfrak{c}(V_1, F_1)\mathfrak{c}(V_2, F_2) = \mathfrak{c}(V_1, F_1)^2 = 1$.

5.5. The characters $\chi_i$, $i$ even. For even $i$, the character $\chi_i$ is a non-faithful character of even degree $q+1$ with totally real character field $K$ and Schur index 1. As before we restrict $\chi_i$ to the Borel subgroup and obtain

$$\chi_i|_B = \psi_1 + \psi_2 + \alpha + \overline{\alpha}$$

where $\psi_1, \psi_2$ are as in 5.3 and $\alpha$ is a complex linear character of $B$. Comparing character values we obtain that $\alpha(y) = \zeta_{q-1}^i$ for a suitably chosen generator $y$ of $C_{(q-1)/2} \leq B$. In particular $\mathbb{Q}(\alpha) = \mathbb{Q}(\zeta_{q-1}^i) = \mathbb{K}(\sqrt{\theta_{q-1}^{(2i)}} - 2)$ and hence Remark 3.1 (c) tells us that $d_\pm(\alpha) = \sqrt{\theta_{q-1}^{(2i)}} - 2$. The discriminant of $\psi_1$ and $\psi_2$ behave as in 5.4 and hence we compute the discriminant $d_\pm(\chi_i) = \varepsilon(\theta_{q-1}^{(2i)} - 2)q$. 


5.6. The characters $\xi_1$, $\xi_2$ for $q \equiv 1 \pmod{4}$. Assume that $q = p^n \equiv 1 \pmod{4}$. Then the two characters $\xi_1$ and $\xi_2$ of odd degree $\frac{q+1}{2}$ factor through $\text{PSL}_2(q)$ and have a totally real character field $K = \mathbb{Q}(\chi_1) = \mathbb{Q}(\chi_2) = \mathbb{Q}(\sqrt{q})$.

**Proposition 5.1.** There are the following two possibilities for $c(\xi_1) = c(\xi_2)$:

| $q \pmod{16}$ | $n$ even | $n$ odd |
|---------------|---------|---------|
| 1             | $\left[ \mathbb{Q}_{p,\infty} \right]$ | 1       |
| 9             | $\left[ \mathbb{Q}_{2,p} \right]$ | $\left[ \mathbb{Q}_{2,\infty} \right]$ |
| 1             | 1       | 1       |
| −3            | $\left[ \mathbb{Q}_{2,\infty} \right]$ | $\left[ \mathbb{Q}_{2,\infty,\varphi_1,\varphi_2} \right]$ |
| 9             | $\left[ \mathbb{Q}_{\infty,\varphi_1,\varphi_2} \right]$ | 5       |
| 5             | $\left[ \mathbb{Q}_{\infty,\varphi_1,\varphi_2} \right]$ | $\left[ \mathbb{Q}_{\infty,\varphi_1,\varphi_2} \right]$ |

Here, for $p \equiv 1 \pmod{8}$ and $n$ odd, $\varphi_1$ and $\varphi_2$ denote the two places of $K = \mathbb{Q}(\sqrt{p})$ that divide 2.

**Proof.** Let $\xi$ be one of $\xi_1$ or $\xi_2$, $K = \mathbb{Q}(\sqrt{q})$ and $W$ the $K\mathbb{G}$-module affording the character $\xi$. Since $\mathcal{F}_G(W)$ always contains a totally positive definite form, we know that $c(\xi) \otimes \mathbb{R} = 1$ if $q \equiv 1, -3 \pmod{16}$ and $c(\xi) \otimes \mathbb{R} \neq 1$ otherwise, for all real places of $K$. If $K \neq \mathbb{Q}$ then $\xi_1$ and $\xi_2$ are Galois conjugate and so are $c(\xi_1)$ and $c(\xi_2)$. The outer automorphism of $G$ interchanges $\xi_1$ and $\xi_2$ which also shows that $c(\xi_1) = c(\xi_2)$, so this algebra is stable under the Galois group of $K$. Moreover the only possible finite primes of $K$ that ramify in $c(\xi)$ are those dividing $p$ or 2. This is seen as follows: The representation $\xi$ is irreducible modulo all other primes $\ell$ (see [11, Section 9.3]) so in particular there is a $G$-invariant lattice $L$ in $W$ whose determinant is not divisible by $\ell$ and hence $\ell$ does not ramify in $c(W,F)$ by Remark 5.6. Noting that 2 is decomposed in $K$ if and only if $n$ is odd and $p \equiv 1 \pmod{8}$, we are left with the possibilities for $c(\xi)$ as stated. □

**Lemma 5.2.** $c(\xi_1) = c(\xi_2)$ is given in line * of Proposition 5.1.

**Proof.** Let $\xi$ be one of $\xi_1$ or $\xi_2$. By Proposition 5.1 it suffices to show that the primes of $K$ that divide 2 do not ramify in $c(\xi)$. So let $\varphi$ be a prime ideal of $K$ that contains 2 and let $K_\varphi$ be the valuation ring in the completion $K_\varphi$ (so $K_\varphi \cong \mathbb{Z}_2$ if $q \equiv 1 \pmod{8}$ and $K_\varphi \cong \mathbb{Z}_2[\zeta_q]$ if $q \equiv 5 \pmod{8}$). By [13, Theorem VII.12 and Theorem VII.4] the image of $R_\varphi G$ in $\text{End}(K_\varphi \otimes W)$ is isomorphic to

$$\Delta(\xi(R_\varphi G)) = \begin{pmatrix} R_\varphi & 2R_\varphi^{1\times(q-1)/2} \\ R_\varphi^{(q-1)/2\times1} & R_\varphi^{(q-1)/2\times(q-1)/2} \end{pmatrix}.$$

In particular the $R_\varphi G$-lattices in $K_\varphi \otimes W$ form a chain

$$\ldots \supset L' \supset L \supset 2L' \supset 2L \ldots$$

with $L'/L \cong R_\varphi/2R_\varphi$. If $F \in \mathcal{F}_G(W)$ is non-degenerate and $L$ is $G$-invariant, then also its dual lattice is $G$-invariant. This shows that there is some $F \in \mathcal{F}_G(W)$ such that $L'$ is the dual lattice of $L$. But then $Q_F(L) \subseteq R_\varphi$ because otherwise the even sublattice of $L$ would be a $G$-invariant sublattice of index 2 in $L$. So $L$ is a semi-regular quadratic $R_\varphi$-module in $(K_\varphi \otimes W,F)$ and by Corollary 5.3 this implies that $c(K_\varphi \otimes W,F) = 1$. □

Note that for $n = 1$ and $n = 2$ it is also possible to deduce this lemma using the character theoretic method from [12] (see [2, Section 6.4]).
6. The orthogonal character table of \( \text{SL}_2(2^n) \)

We now assume that \( q = 2^n \) with \( n \geq 2 \) and put \( G := \text{SL}_2(q) \). Then the ordinary character table of \( G \) is given in \([5, \text{Theorem } 38.2]\):

**Theorem 6.1** (\([5, \text{Theorem } 38.2]\)). Let \( \nu \) be a generator of \( F_2^\times \) and consider the elements

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a := \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}
\]

of \( G \). The group also contains an element \( b \) of order \( q+1 \). The character table of \( G \) is

| Character | Invariant |
|-----------|-----------|
| \( \psi \) | \( d_\pm(\psi) = q + 1 \) |
| \( \chi_i, 1 \leq i \leq \frac{q-2}{2} \) | \( c(\chi_i) = \begin{cases} 1 \in \text{Br}(\mathbb{Q}(\chi_i)) & \text{if } n \text{ is odd, see Theorem } 6.4 \\ \text{see Theorem } 6.3 & \text{if } n \text{ is even} \end{cases} \) |
| \( \theta_j, 1 \leq j \leq \frac{q}{2} \) | \( c(\theta_j) = \begin{cases} (-1, -1) \in \text{Br}(\mathbb{Q}(\sqrt{5})) & \text{if } q = 4, \\ 1 \in \text{Br}(\mathbb{Q}(\theta_j)) & \text{if } q \geq 8. \end{cases} \) |

**Proof.** For the Steinberg character \( \psi \) we again have that \( \psi + 1 \) is the character of a 2-transitive permutation representation. In particular \( d_\pm(\psi) = d_\pm(A_q) = q + 1 \). For the characters \( \theta_j \) of degree \( q-1 \) we note that the restriction of these characters to the normalizer \( \tilde{B} \cong C_2^n \rtimes C_{q-1} \) of the Sylow-2-subgroup of \( G \) is the character of an irreducible rational monomial representation \( V \). So \( V \) has an orthonormal basis and hence \( c(\theta_j) = c(I_q \otimes K) \) is given in Example 2.4.

To describe the Clifford invariant of the characters \( \chi_i \) of degree \( q+1 \) note that for the infinite places of \( K \) the invariant \( [c(\chi_i) \otimes_K \mathbb{R}] \in \text{Br}(\mathbb{R}) \) is non-trivial if and only if \( q = 4 \), because in all other cases, the character degree is 1 (mod 8).

For the odd finite primes of \( K \), the Clifford invariant of \( \chi_i \) is given in the next theorem:
Theorem 6.3. Let \( 1 \leq i \leq (q - 2)/2 \), \( K = \mathbb{Q}(\chi_i) = \mathbb{Q}[\varphi(q)] \), and let \( \varphi \) be some maximal ideal of \( \mathbb{Z}_K \) such that \( \varphi \cap \mathbb{Z} = p\mathbb{Z} \) for some odd prime \( p \). Then \( [c(\chi_i) \otimes K_p] \in \text{Br}(K_p) \) is not trivial if and only if

(i) \( p \equiv \pm 3 \pmod{8} \), and (ii) \( (q - 1)/(\gcd(q - 1, i)) \) is a power of \( p \).

Proof. We first note that condition (ii) implies that \( p \) divides \( q - 1 \). If condition (ii) is not fulfilled, then the reduction of \( \chi_i \) modulo \( \varphi \) is an irreducible Brauer character (see for instance [3]). In particular the orthogonal \( K_p G \)-module \( V \) affording the character \( \chi_i \) contains an (even) unimodular \( R_p \)-lattice. So Corollary 3.3 tells us that \( [c(\chi_i) \otimes K_p] = 1 \in \text{Br}(K_p) \). If the condition (ii) is satisfied, then \( \varphi \) is the unique prime ideal of \( K \) that contains \( p \), the extension \( K_p/\mathbb{Q}_p \) is totally ramified, and (again by [3]) the \( \varphi \)-modular Brauer tree of the block containing \( \chi_i \) is given as

```
1 -- q --- ψ
   \χ
```

where the multiplicity of the exceptional vertex \( \chi \) is \( \frac{p - 1}{2} \) with \( a = \nu_p(q - 1) \). In particular [13, Theorem (VIII.3)] yields that the \( R_p \)-order \( R_p G \) acts on \( V \) as

\[
\Delta_{\chi_i}(R_p G) = \begin{pmatrix} R_p & \varphi R_{q}^{1 \times q} \\ R_p^{1 \times q} & R_p^{q \times q} \end{pmatrix}.
\]

As in the proof of Lemma 5.2 the \( R_p G \)-invariant lattices in \( V \) form a chain:

\[
\ldots \supset L' \supset L \supset \varphi L' \supset \varphi L \ldots
\]

with \( L'/L \cong R_p/\varphi R_p \). So there is a \( G \)-invariant form \( F \) on \( V \) such that \( L' = L^# \), in particular the \( \varphi \)-adic valuation of the determinant of \( L \) is 1. Choose \( (b_1, \ldots, b_q) \in L^q \) such that the images form a basis \( \overline{B} \) of \( L/\varphi L' \) and put \( W := \langle b_1, \ldots, b_q \rangle_{K_p} \leq V \). The modular representation \( L/\varphi L' \) is isomorphic to the \( \varphi \)-modular reduction of the Steinberg module \( \psi \). In particular the determinant of the Gram matrix of \( \overline{B} \) is \( q + 1 \in \mathbb{Z}/p\mathbb{Z} \cong R_p/\varphi R_p \). As \( \varphi \) is odd and \( q + 1 \in R_p^{\times} \), this gives the discriminant of the bilinear \( K_p \)-module

\[
d_{\pm}(W, F|_W) = (q + 1)(K_p^{\times})^2 = 2(K_p^{\times})^2
\]

because \( q + 1 \equiv 2 \pmod{p} \) since \( p \) divides \( q - 1 \). We can now apply Corollary 3.3 to conclude that the Clifford invariant of \( (V, F) \) is non-trivial, if and only if 2 is not a square in \( K_p \), if and only if 2 is not a square in \( F_p = R_p/\varphi \) which is equivalent to condition (i) by quadratic reciprocity.

□

Theorem 6.4. If \( q = 2^n \) and \( n \) is odd then \( c(\chi_i) = 1 \in \text{Br}(\mathbb{Q}(\chi_i)) \) for all \( 1 \leq i \leq \frac{q - 2}{2} \).

Proof. Let \( M := \mathbb{Q}_2[\zeta_{2^n - 1}] \) be the unramified extension of \( \mathbb{Q}_2 \) of degree \( n \). Then \( M \) is a splitting field for \( G \). Moreover the \( M \)-representation \( V_M \) affording the character \( \chi_i \) is induced up from a linear \( M \)-representation of the normalizer \( B = C_2^n \rtimes C_{2^n - 1} \) of the Sylow-2-subgroup of \( G \). In particular \( V_M \) is an irreducible monomial representation and hence the standard form \( F_M \) is \( G \)-invariant,
so \((V_M, F_M) \cong I_{2^{n+1}} \otimes M\). For \(n \geq 3\) the dimension of \(V_M\) is \(\equiv 1 \pmod{8}\) and so by Example 2.4 the Clifford invariant of \((V_M, F_M)\) is trivial in \(\text{Br}(M)\). Now let \(K = \mathbb{Q}(\chi_i)\) an orthogonal KG-module affording the character \(\chi_i\), and let \(\varphi\) be some prime ideal of \(K\) dividing 2. As \(K \subseteq \mathbb{Q}(\zeta_{2^n})\) the completion of \(K\) at \(\varphi\) is contained in \(M\) and, by the same argument as before, \((V \otimes M, F) \cong (V, aF(M))\) for some non-zero \(a \in M\). In particular \(c(V \otimes M, F) = 1\) in \(\text{Br}(M)\). As \([M : \mathbb{Q}_2] = n\) is assumed to be odd, also \([M : K_\varphi]\) is odd and hence \(c(V \otimes K_\varphi, F) = 1\) in \(\text{Br}(K_\varphi)\). This argument shows that no even prime \(\varphi\) of \(K\) ramifies in \(c(V, F)\). Also the real primes do not ramify because \(\dim(V) \equiv 1 \pmod{8}\). So by Theorem 6.3 there is at most one prime ideal of \(K\) that ramifies in \(c(V, F)\). But the number of ramified primes is even, which shows that \(c(\chi_i) = 1\) in the Brauer group of \(K\).

Note that Theorem 6.4 together with Theorem 6.3 implies the well known fact that if \(n\) is odd then all primes \(p\) dividing \(2^n - 1\) satisfy \(p \equiv \pm 1 \pmod{8}\) (because then \(2^{(n+1)/2}\) is a square root of 2 modulo \(p\)).

**Remark 6.5.** In the situation of Theorem 6.3 if \([c(\chi_i) \otimes K_\varphi] \in \text{Br}(K_\varphi)\) is non-trivial and \(q \neq 4\), then an odd number of even primes of \(K\) also ramify in \(c(\chi_i)\). However, we did not determine in general which even primes of \(K\) ramify in \(c(\chi_i)\) for the case that \(n\) is even. Of course the same argument as in the proof of Theorem 6.4 works if the primes above 2 are decomposed in \(\mathbb{Q}(\zeta_{q-1})/\mathbb{Q}(\zeta_{q-1})\).

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