Parker’s Stellar Wind Model for Polytropic Gas Flows

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Abstract

Parker’s [4] hydrodynamic stellar wind model is extended to polytropic gas flows. A compatible theoretical formulation is given and detailed systematic considerations are presented. The polytropic conditions are shown to lead to tenuous and faster wind flows and hence enable the stars to lose their angular momentum more quickly.
1. Introduction

Stellar wind is a continuous plasma outflow from a star (in the case of the Sun, this outflow typically emerges from coronal holes (Sakao et al. [2]) and carries a remnant of the stellar magnetic field that fills the space around the star (in the case of the Sun, this constitutes the heliosphere (Dialynas et al. [3])). Stellar winds carry off, especially when magnetized, a huge amount of angular momentum from the stars while causing a very negligible amount of mass loss from the stars. Weak to moderate stellar winds are generated by an expanding outer corona due to an extended active heating of the corona in conjunction with high thermal conduction. However, control of coronal base conditions and high thermal conduction are both inadequate to generate observed high wind speeds in the case of the Sun (Parker [8],[9]). This indicates the rationale for some additional acceleration mechanism to operate beyond the coronal base. Parker [8] gave an ingenious stationary model which provided for the smooth acceleration of the stellar wind through transonic speeds by continually converting the thermal energy into the kinetic energy of the wind. In the case of the Sun, the solar wind was confirmed and its properties were recorded by in situ observations (Neugebauer and Snyder [10], Hundhausen [11], Meyer-Vernet [12]).

One of the main assumptions in Parker’s [8] model is that the gas flow occurs under isothermal conditions (in standard rotation),

\[ p = a_0^2 \rho \]  

where \( a_0 \) is the constant speed of sound. However, the extended active heating of the corona may be represented in a first approximation by using the polytropic gas relation (Parker [13], Holzer [14], Keppens and Goedbloed [15]),

\[ p = C \rho^\gamma \]  

where \( C \) is an arbitrary constant and \( \gamma \) is the polytropic exponent, \( 1 < \gamma < 5/3 \). The modified Parker’s equation governing the acceleration of stellar wind of a polytropic gas given by the Holzer [14] seems to be erroneous. The purpose of this paper is to rectify this error and present a more compatible formulation and then make detailed systematic theoretical considerations to describe acceleration of stellar wind of a polytropic gas. The polytropic conditions appear to lead to tenuous and faster wind flows and hence enable the stars to lose their angular momentum more quickly.

2. Polytropic Gas Stellar Wind Model

In Parker’s hydrodynamic model [8] the stellar wind is represented by a steady and spherically symmetric flow so the flow variables depend only on \( r \), the distance from the star.

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1SOHO observations (Cho et al. [1]) revealed that the structure of the solar wind changes from the solar maximum to the solar minimum period.

2Parker [4]-[6] proposed that this is caused by the dissipation of plasma waves produced by microflares in the coronal holes. The details of the coronal heating mechanism are still controversial (Klimchuk [7]).

3In reality, observations of the solar wind (Wang and Sheeley [16]) suggested and (Kopp and Holzer [17]) proposed a rapidly-diverging superradial wind flow especially in some active regions like the coronal holes.
The flow velocity is further taken to be only in the radial direction - either inward (accretion model) or outward (wind model). We assume for analytical simplicity that the flow variables and their derivatives vary continuously so there are no shocks anywhere in the region under consideration.

The mass conservation equation is

\[
\frac{2}{r} + \frac{1}{\rho} \frac{d\rho}{dr} + \frac{1}{v_r} \frac{dv_r}{dr} = 0. \tag{3}
\]

Assuming the gravitational field to be produced by a central mass \(M_s\), Euler’s equation of momentum balance is

\[
\rho v_r \frac{dv_r}{dr} = -\frac{dp}{dr} - \frac{GM_s}{r^2} \rho \tag{4}
\]

\(G\) being the gravitational constant.

Using equations (2) and (3), equation (4) becomes

\[
(M^2 - 1) \frac{2v_r}{a^2} \frac{dv_r}{dr} = \frac{4}{r^2} (r - r_*) \tag{5}
\]

where,

\[
a^2 \equiv \frac{dp}{d\rho} = \frac{\gamma p}{\rho}, M \equiv \frac{v_r}{a}, r_* \equiv \frac{GM_s}{2a^2} \tag{6}
\]

Using (6), and noting the adiabatic relation,

\[
a^2 = \frac{1}{1 + \left(\frac{\gamma - 1}{2}\right) M^2} \tag{7}
\]

equation (5) becomes,

\[
\frac{M^2 - 1}{M^2 \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]} \frac{d}{dr} (M^2) = \frac{4}{r^2} (r - r_*) \tag{8}
\]

By introducing the total energy,

\[
E \equiv \frac{v_r^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM_s}{r} \tag{9}
\]

The right hand side of equation (8) may be written alternatively as

\[
\frac{(M^2 - 1)}{M^2 \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]} \frac{d}{dr} (M^2) = \frac{4E + \left(\frac{4\gamma - 6}{\gamma - 1} - M^2\right) \frac{GM_s}{r}}{r \left( E + \frac{GM_s}{r} \right)} \tag{10}
\]

which shows that the corresponding result given by Holzer \[14\] is apparently erroneous. Introducing,

\[
r_{*0} \equiv \frac{GM_s}{2a_0^2} \tag{11}
\]
and using (7), equation (8) may be rewritten as

\[
\frac{(M^2 - 1)}{M^2 \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]} \frac{d}{dr} (M^2) = \frac{4}{r^2} \left[ r - \left\{ 1 + \left(\frac{\gamma - 1}{2}\right) M^2 \right\} r_{\ast o} \right].
\]  

(12)

We now consider some special asymptotic cases for which equation (12) facilitates simpler solutions.

**(i) Transonic Regime**

Near the sonic critical point \( r = r_{\ast} \), we may write,

\[
r = r_{\ast} + x, \quad M^2 = 1 + y.
\]  

(13)

Using (7) and (11), equation (12) then gives

\[
y \frac{dy}{dx} \approx \frac{8}{r_{\ast o}^2 (\gamma + 1)} x
\]  

(14)

from which,

\[
y \approx \frac{2/r_{\ast o}}{\sqrt{\gamma + 1}} x
\]  

(15a)

or

\[
(M^2 - 1) \approx \sqrt{2 (\gamma + 1)} \left( \frac{r}{r_{\ast}} - 1 \right).
\]  

(15b)

(15b) shows an enhanced acceleration of the polytropic wind (\( \gamma > 1 \)) past the sonic critical point.

**(ii) Near-star Regime**

For \( r/r_{\ast} \ll 1 \), equation (12) may be approximated by

\[
\frac{1}{M^2 \left[1 + \left(\frac{\gamma - 1}{2}\right) M^2\right]} \frac{d}{dr} (M^2) \approx \frac{4r_{\ast o}}{r^2}
\]  

(16)

from which,

\[
\frac{M^2}{1 + \left(\frac{\gamma - 1}{2}\right) M^2} \approx e^{-\left(\frac{4r_{\ast o}}{r}\right)}.
\]  

(17)

In the isothermal limit (\( \gamma \rightarrow 1 \)), (17) leads to

\[
M^2 \approx e^{-\left(\frac{4r_{\ast}}{r}\right)}.
\]  

(18)
Comparison of (17) with (18) shows again an enhanced acceleration of the polytropic wind \((\gamma > 1)\) near the star.

(iii) Far-star Regime

For \(r/r_* \gg 1\), equation (12) may be approximated by

\[
\frac{d}{dr} (M^2) \approx 2 \frac{(\gamma - 1)}{r} M^2
\]

from which,

\[
M^2 \approx r^{(2(\gamma - 1))}.
\]

(20) shows again an enhanced acceleration of the polytropic wind \((\gamma > 1)\) far from the star.

(iv) Effective de Laval Nozzle for the Polytropic Stellar Wind

The continuous acceleration of the polytropic stellar wind flow, seen above, from subsonic speeds at the coronal base to supersonic speeds away from the star implies a de Laval nozzle type situation operational near the star.

If \(A = A(r)\) is the cross section area of an effective de Laval nozzle associated with the polytropic stellar wind flow, we have from equations (5) and (8),

\[
\frac{(M^2 - 1)}{M^2} \left(\frac{2v_r}{a^2}\right) \frac{dv_r}{dr} = \frac{(M^2 - 1)}{M^2} \frac{1 + \left(\frac{\gamma - 1}{2}\right) M^2}{M^2} \frac{d}{dr} (M^2)
\]

\[
= 2 \frac{dA}{dr} = \frac{4}{r^2} (r - r_*)
\]

from which,

\[
A(r) = 4\pi r^2 e^{-2r_0} \int_{r_0}^{r} \frac{1}{r} \left[1 + \left(\frac{2 - 1}{2}\right) M^2\right] dr
\]

where \(r = r_0\) at the coronal base. (22) implies that the cross section area of the effective de Laval nozzle, for the polytropic wind \((\gamma > 1)\), varies faster, in consistency with the enhanced acceleration of the polytropic stellar wind.

5. Discussion

Thanks to the extended heating of the corona, a polytropic gas model is in order in dealing with stellar winds. In recognition of this, the present paper is aimed at putting forward a compatible theoretical formulation and providing a detailed systematic theoretical considerations to describe acceleration of stellar wind of a polytropic gas. The polytropic conditions have been shown to lead to tenuous and faster wind flows and hence enable the stars to lose this angular momentum more quickly.
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