A Dichotomy for the Mackey Borel Structure

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Vienna, Preprint ESI 2171 (2009) August 24, 2009

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
A DICHOTOMY FOR THE MACKEY BOREL STRUCTURE

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Abstract. We prove that the equivalence of pure states of a separable C*-algebra is either smooth or it continuously reduces $[0, 1]^\mathbb{N}/\ell_2$ and it therefore cannot be classified by countable structures. The latter was independently proved by Kerr–Li–Pichot by using different methods. We also give some remarks on a 1967 problem of Dixmier.

If $E$ and $F$ are Borel equivalence relations on Polish spaces $X$ and $Y$, respectively, then we say that $E$ is Borel reducible to $F$ (in symbols, $E \leq_B F$) if there is a Borel-measurable map $f : X \to Y$ such that for all $x$ and $y$ in $X$ we have $xEy$ if and only if $f(x)Ff(y)$. A Borel equivalence relation $E$ is smooth if it is Borel-reducible to the equality relation on some Polish space. Recall that $E_0$ is the equivalence relation on $2^\mathbb{N}$ defined by $xE_0y$ if and only if $x(n) = y(n)$ for all but finitely many $n$. The Glimm–Effros dichotomy ([8]) states that a Borel equivalence relation $E$ is either smooth or $E_0 \leq_B E$.

One of the themes of the abstract classification theory is measuring relative complexity of classification problems from mathematics (see e.g., [12]). One can formalize the notion of ‘effectively classifiable by countable structures’ in terms of the relation $\leq_B$ and a natural Polish space of structures based on $\mathbb{N}$ in a natural way. In [10] Hjorth introduced the notion of turbulence for orbit equivalence relations and proved that an orbit equivalence relation given by a turbulent action cannot be effectively classified by countable structures.

The idea that there should be a small set $\mathcal{B}$ of Borel equivalence relations not classifiable by countable structures such that for every Borel equivalence relation $E$ not classifiable by countable structures there is $F \in \mathcal{B}$ such that $F \leq_B E$ was put forward in [11] and, in a revised form, in [4]. In this note we prove a dichotomy for a class of Borel equivalence relations corresponding to the spectra of C*-algebras by showing that one of the standard turbulent orbit equivalence relations, $[0, 1]^\mathbb{N}/\ell_2$, is Borel-reducible to every non-smooth spectrum.

Date: August 25, 2009.

1991 Mathematics Subject Classification. 03E15, 46L30, 22D25.

Key words and phrases. Borel equivalence relations, Mackey Borel structure.

Partially supported by NSERC. The work reported in this note was done in July 2008 while I was visiting IHES and it was presented at a mini-conference in set theory at the Institut Henri Poincaré in July 2008.

Filename: 2008g25-mackey.tex.
**States.** All undefined notions from the theory of C*-algebras and more details can be found in [2] or in [5]. Consider a separable C*-algebra \( A \). Recall that a functional \( \phi \) on \( A \) is *positive* if it sends every positive operator in \( A \) to a positive real number. A positive functional is a *state* if it is of norm \( \leq 1 \). The states form a compact convex set, and the extreme points of this set are the *pure states*. The space of pure states on \( A \), denoted by \( \mathcal{P}(A) \), equipped with the weak*-topology, is a Polish space ([14, 4.3.2]).

A C*-algebra \( A \) is *unital* if it has the multiplicative identity. Otherwise, we define the *unitization* of \( A \), \( \tilde{A} \), the canonical unital C*-algebra that has \( A \) as a maximal ideal and such that the quotient \( \tilde{A}/A \) is isomorphic to \( \mathbb{C} \) (see [5, Lemma 2.3]). If \( u \) is a unitary in \( A \) (or \( \tilde{A} \)) then  
\[(\text{Ad } u) a = u a u^* \]
defines an inner automorphism of \( A \).

Two pure states \( \phi \) and \( \psi \) are equivalent, \( \phi \sim_A \psi \), if there exists a unitary \( u \) in \( A \) (or \( \tilde{A} \)) such that \( \phi = \psi \circ \text{Ad } u \).

**Theorem 1.** Assume \( A \) is a separable C*-algebra. Then \( \sim_A \) is either smooth or there is a continuous map 
\[ \Phi: [0,1]^\mathbb{N} \to \tilde{A} \]
such that \( \alpha - \beta \in \ell_2 \) if and only if \( \Phi(A) \sim_A \Phi(B) \).

**Corollary 2.** Assume \( A \) is a separable C*-algebra. Then \( \sim_A \) is either smooth or it cannot be classified by countable structures.

**Proof.** By [10] it suffices to show that a turbulent orbit equivalence relation is Borel-reducible to \( \sim_A \) if \( \sim_A \) is not smooth. The equivalence relation \([0,1]^\mathbb{N}/\ell_2\) is well-known to be turbulent (e.g., [11]) and the conclusion follows by Theorem 1. \( \square \)

This result was independently proved in [13, Theorem 2.8] by directly showing the turbulence. As pointed out in [13, §3], it implies an analogous result of Hjorth ([9]) on irreducible representations of discrete groups, as well as its strengthening to locally compact groups.

1. **Proof of Theorem 1**

Recall that the CAR (Canonical Anticommutation Relations) algebra (also know as the Fermion algebra, or \( M_{2^\infty} \)) is defined as the infinite tensor product 
\[ M_{2^\infty} = \bigotimes_{n \in \mathbb{N}} M_2(\mathbb{C}) \]
where \( M_2(\mathbb{C}) \) is the algebra of \( 2 \times 2 \) matrices. Alternatively, one may think of \( M_{2^\infty} \) as the direct limit of \( 2^n \times 2^n \) matrix algebras \( M_{2^n}(\mathbb{C}) \) for \( n \in \mathbb{N} \).

The following analogue of the Glimm–Effros dichotomy is an immediate consequence of [6] (Notably, the key combinatorial device in the proof of [8] comes from Glimm).
Proposition 3. If $A$ is a separable C*-algebra then exactly one of the following applies.

1. $\sim_A$ is smooth.
2. $\sim_{M_2^{\infty}} \leq_B \sim_A$. □

We shall prove that $\sim_{M_2^{\infty}}$ is turbulent in the sense of Hjorth.

Lemma 4. If $\xi$ and $\eta$ are unit vectors in $H$ then

1. $\inf\{\|1-u\| : u \text{ unitary in } B(H) \text{ and } (u\xi|\eta) = 1\} = \sqrt{2(1-|\langle\xi|\eta\rangle|)}$.

Proof. Let $t = (\xi|\eta)$. Let $\xi' = \frac{1}{\|\text{proj}_\eta \xi\|}\text{proj}_\eta \xi$. Then the left-hand side of (1) is greater or equal than

$$\|\xi - \xi'\|^2 = \|\xi\|^2 + \|\xi'\|^2 - \frac{2}{(\xi|\eta)(\xi|\eta)\eta} = 2 - 2|t|.$$ 

For $\leq$ let $\zeta$ be the unit vector orthogonal to $\xi$ such that $\eta = t\xi + \sqrt{1-t^2}\zeta$ and let $u$ be the unitary given by $\begin{pmatrix} t & -\sqrt{1-t^2} \\ \sqrt{1-t^2} & t \end{pmatrix}$ on the span of $\xi$ and $\zeta$ and identity on its orthogonal complement. Then $u\xi = \eta$ and a straightforward computation gives $\|I-u\|^2 = 2-2t$ as required. □

If $\xi$ is a unit vector in a Hilbert space then by $\omega_\xi$ we denote the vector state $a \mapsto (a\xi|\xi)$. If $\xi_i$ is a unit vector in $H_i$ for $1 \leq i \leq m$ then $\xi = \bigotimes_{i=1}^m \xi_i$ is a unit vector in $H = \bigotimes_{i=1}^m H_i$ and $\omega_\xi$ is a vector state on $B(H)$.

Lemma 5. If $H_i$ is a Hilbert space and $\xi_i, \eta_i$ are unit vectors in $H_i$ for $1 \leq i \leq m$ then

$$\inf\{\|I-u\| : u \text{ unitary and } \omega_{\bigotimes_{i=1}^m \xi_i} = \omega_{\bigotimes_{i=1}^m \eta_i} \circ \Ad u\} = 2\sqrt{2(1-\prod_{i=1}^m |\langle\xi_i|\eta_i\rangle|)}.$$

Proof. The case when $m = 1$ follows from Lemma 4 and the fact that $\omega_\xi = \omega_{\alpha\xi}$ when $|\alpha| = 1$. Since $(\bigotimes_{i=1}^m \xi_i|\bigotimes_{i=1}^m \eta_i) = \prod_{i=1}^m (\xi_i|\eta_i)$, the general case is an immediate consequence of Lemma 4. □

Theorem 6. There is a continuous map $\Phi : (-\frac{\pi}{2}, \frac{\pi}{2})^N \to \mathbb{P}(M_2^{\infty})$ such that for all $\alpha$ and $\beta$ in the domain we have

$$\sum_n (\alpha_n - \beta_n)^2 < \infty \iff \Phi(\alpha) \sim \Phi(\beta).$$

Proof. Consider the standard representation of $M_2(\mathbb{C})$ on $\mathbb{C}^2$. Then the pure states of $M_2(\mathbb{C})$ are of the form $\omega_{(\cos\alpha, \sin\alpha)}$ for $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Let $\Phi(\alpha) = \bigotimes_{n=1}^\infty \omega_{(\cos\alpha_n, \sin\alpha_n)}$. This map is continuous: If $a \in M_2^{\infty}$ and $\varepsilon > 0$, fix $m$ and $a' \in M_{2m}$ such that $\|a - a'\| < \varepsilon/2$. Then $\Phi(\alpha)(a')$ depends only on $\alpha_j$ for $j \leq m$, and in a continuous fashion.
Recall that for \( 0 < t_j < 1 \) we have \( \prod_{j=1}^{n} t_j > 0 \) if and only if \( \sum_{j=1}^{n} (1-t_j) < \infty \). Therefore

\[
\sum_{n=1}^{\infty} (\alpha_n - \beta_n)^2 < \infty \Leftrightarrow \sum_{n=1}^{\infty} \sin^2 \left( \frac{\alpha_n - \beta_n}{2} \right) < \infty \Leftrightarrow \prod_{n=1}^{\infty} \cos(\alpha_n - \beta_n) > 1.
\]

Assume \( \prod_{n=1}^{\infty} \cos(\alpha_n - \beta_n) > 0 \). In the \( n \)-th copy of \( M_2 \) in \( M_{2^\infty} = \bigotimes_{n=1}^{\infty} M_2 \) pick a unitary \( u_n \) such that

\[
\|1 - u_n\| < \sqrt{2(1 - \prod_{j=m}^{\infty} \cos(\alpha_j - \beta_j))}
\]

and \( u_n(\cos \alpha_n, \sin \alpha_n) = (\cos \beta_n, \sin \beta_n) \). Note that

\[
(\cos \alpha_n, \sin \alpha_n)(\cos \beta_n, \sin \beta_n) = \cos(\alpha_n - \beta_n).
\]

Let \( v_n = \bigotimes_{j=1}^{n} u_j \). Then \( v_n \) for \( n \in \mathbb{N} \) form a Cauchy sequence, because

\[
v_m - v_{m+n} = v_m(1 - \bigotimes_{j=m+1}^{n} u_j) \text{ and therefore}
\]

\[
\|v_m - v_n\| < \sqrt{2(1 - \prod_{j=m}^{\infty} \cos(\alpha_j - \beta_j))}.
\]

Let \( v \in M_{2^\infty} \) be the limit of this Cauchy sequence. Since for each \( m \) and \( a \in M_{2^m} \) we have \( \Phi(\vec{\alpha})(a) = \Phi(\vec{\beta})(v_na^*_n) \) for any \( n \geq m \), we have \( \Phi(\vec{\alpha}) = \Phi(\vec{\beta}) \circ \text{Ad} \ v \).

Now assume \( \Phi(\vec{\alpha}) \sim \Phi(\vec{\beta}) \) and, for the sake of obtaining a contradiction, that \( \prod_{n=1}^{\infty} \cos(\alpha_n - \beta_n) = 0 \). There is \( m \) and a unitary \( u \in M_{2^m} \) such that

\[
\|\Phi(\vec{\alpha}) - \Phi(\vec{\beta}) \circ \text{Ad} \ u\| < \frac{1}{2},
\]

(by e.g., [6]). However, we can find \( n > m \) large enough so that with

\[
\xi_n = \bigotimes_{j=m}^{n} (\cos \alpha_j, \sin \alpha_j) \text{ and } \eta_n = \bigotimes_{j=m}^{n} (\cos \beta_j, \sin \beta_j)
\]

the quantity

\[
(\xi_n | \eta_n) = \prod_{j=m}^{n} \cos(\alpha_j - \beta_j)
\]

is as close to zero as desired. Then \( \|\omega_{\xi_n} - \omega_{\eta_n}\| \) is as close to 2 as desired, since \( a_n = \text{proj}_{\xi_n} \omega_{\xi_n} - \text{proj}_{\eta_n} \omega_{\eta_n} \) has norm close to 1 and \( \omega_{\xi_n}(a_n) \) is close to 1 while \( \omega_{\eta_n}(a_n) \) is close to \(-1\).

**Proof of Theorem 1.** Assume \( \sim_A \) is not smooth. The conclusion follows by Glimm’s Proposition 3 and Theorem 6. \( \square \)

2. Concluding Remarks

We note that the class of equivalence relations corresponding to spectra of \( C^* \)-algebras is restrictive in another sense. The following proposition was probably well-known (cf. [9, Corollary 1.3]).

**Proposition 7.** If \( A \) is a separable \( C^* \)-algebra then the relation \( \phi \sim_A \psi \) on \( \mathcal{P}(A) \) is \( F_\sigma \).
Proof. By replacing $A$ with its unitization if necessary we may assume $A$ is unital. Fix a countable dense set $U$ in the unitary group of $A$ and a countable dense set $D$ in $A_{\leq 1}$. We claim that

$$\phi \sim \psi \iff (\exists u \in U)(\forall a \in D)|\phi(a) - \psi(ua^*)| < 1.$$ 

Assume $\phi \sim \psi$ and fix $v$ such that $\phi = \psi \circ \text{Ad } v$. If $u \in U$ is such that $\|v - u\| < 1/2$ then

$$|\psi(ua^* - va^*)| = |\psi((u - v)au^* - va(u^* - v^*))| < 1$$

for all $a \in A_{\leq 1}$.

Now assume $u \in U$ is such that $|\phi(a) - \psi(ua^*)| < 1$ for all $a \in D$. Then $\|\phi - \psi \circ \text{Ad } u\| < 2$ and by [7] we have $\phi \sim \psi$. □

For a Hilbert space $H$ by $\mathcal{B}(H)$ we denote the algebra of its bounded linear operators. Let $\pi_1: A \to \mathcal{B}(H_1)$ and $\pi_2: A \to \mathcal{B}(H_2)$ be representations of $A$. We say $\pi_1$ and $\pi_2$ are (unitarily) equivalent and write $\pi_1 \sim \pi_2$ if there is a Hilbert space isomorphism $u: H_1 \to H_2$ such that the diagram

$$\begin{array}{ccc}
\mathcal{B}(H_1) & \xrightarrow{\pi_1} & A \\
\downarrow & & \downarrow \text{Ad } u \\
\mathcal{B}(H_2) & \xrightarrow{\pi_2} & \text{Ad } u(a) = uau^*
\end{array}$$

commutes.

A representation of $A$ on some Hilbert space $H$ is irreducible if there are no nontrivial closed subspaces of $H$ invariant under the image of $A$. The spectrum of $A$, denoted by $\hat{A}$, is the space of all equivalence classes of irreducible representations of $A$. The GNS construction associates a representation $\pi_\phi$ of $A$ to each state $\phi$ of $A$ (see e.g., [5, Theorem 3.9]). Moreover, $\phi$ is pure if and only if $\pi_\phi$ is irreducible ([5, Theorem 3.12]) and for pure states $\phi_1$ and $\phi_2$ we have that $\phi_1$ and $\phi_2$ are equivalent if and only if $\pi_{\phi_1}$ and $\pi_{\phi_2}$ are equivalent ([5, Proposition 3.20]).

Fix a separable C*-algebra $A$. Let $\text{Irr}(A, H_n)$ denote the space of irreducible representations of $A$ on a Hilbert space $H_n$ of dimension $n$ for $n \in \mathbb{N} \cup \{\aleph_0\}$. Each $\text{Irr}(A, H_n)$ is a Polish space with respect to the weakest topology making all functions $\text{Irr}(A, H_n) \ni \pi \mapsto (\pi(a)\xi|\eta) \in \mathbb{C}$, for $a \in A$ and $\xi, \eta \in H_n$, continuous. In other words, a net $\pi_\lambda$ converges to $\pi$ if and only if $\pi_\lambda(a)$ converges to $\pi(a)$ for all $a \in A$. Since $A$ is separable, each irreducible representation of $A$ has range in a separable Hilbert space, and therefore $\hat{A}$ can be considered as a quotient space of the direct sum of $\text{Irr}(A, H_n)$ for $n \in \mathbb{N} \cup \{\aleph_0\}$. Therefore $\hat{A}$ carries a Borel structure (known as the Mackey Borel structure) inherited from a Polish space. For type $I$
C*-algebras (also called GCR or postliminal) this space is a standard Borel space. (All of these notions are explained in [1, §4].)

Since pure states correspond to irreducible representations, we can identify the Mackey Borel structure of $A$ with a $\sigma$-algebra of sets in $\hat{A}$. It is easy to check that this $\sigma$-algebra consists exactly of those sets whose preimages in $\mathbb{P}(A)$ are Borel subsets in $\mathbb{P}(A)$.

Glimm proved ([6], [14, §6.8]) that the Mackey Borel structure of a C*-algebra $A$ is smooth (i.e., isomorphic to a standard Borel space) if and only if $A$ is a type I C*-algebra. Proposition 3 is a consequence of this result.

**Problem 8** (Dixmier, 1967). Is the Mackey Borel structure on the spectrum of a simple separable C*-algebra always the same when it is not standard?

G. Elliott generalized Glimm’s result and proved that the Mackey Borel structures of simple AF algebras are isomorphic ([3]). (A C*-algebra is an AF (approximately finite) algebra if it is a direct limit of finite-dimensional algebras.) One reformulation of Elliott’s result is that for any two simple separable AF algebras $A$ and $B$ there is a Borel isomorphism $F: \mathbb{P}(A) \to \mathbb{P}(B)$ such that $\phi \sim_A \psi$ if and only if $F(\phi) \sim_B F(\psi)$ (see [3, §6]. Also, [3, Theorem 2] implies that if $A$ is a simple separable AF algebra and $B$ is a non-Type I simple separable algebra we have $\sim_A \leq_B \sim_B$.

With this definition the quotient structure $\text{Borel}(\mathbb{P}(A))/\sim_A$ is isomorphic to the Mackey Borel structure of $A$. Note that $\sim_A$ is smooth exactly when the Mackey Borel structure of $A$ is smooth.

Note that Mackey Borel structures of $A$ and $B$ of separable C*-algebras are isomorphic if and only if there is a Borel isomorphism $f: \hat{X} \to \hat{X}$ such that $\pi_1 \sim_A \pi_2$ if and only if $f(\pi_1) \sim_B f(\pi_2)$. Hence Problem 8 is rather close in spirit to the theory of Borel equivalence relations.

N. Christopher Phillips suggested more general problems about the Mackey Borel structure of simple separable C*-algebras, motivated by his discussions with Masamichi Takesaki. There are two (related) kinds of questions: Can one do anything sensible, and, from the point of view of logic, how bad is the problem?

**Problem 9.** Does the complexity of the Mackey Borel structure of a simple separable C*-algebra increase as one goes from nuclear C*-algebras to exact ones to ones that are not even exact?

For definitions of nuclear and exact C*-algebras see e.g., [2].

**Problem 10.** Assume $A$ and $B$ are C*-algebras and $\sim_A$ is Borel-reducible to $\sim_B$. What does this fact imply about the relation between $A$ and $B$?

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