Approximation of Feasible Power Injection Regions in Distribution Networks via Dual SOCP

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Abstract—We develop an optimization method to approximate the region of feasible power injections in distribution networks. Based on the nonlinear Dist-Flow model of an alternating-current (AC) network with a radial structure, we first formulate a power-injection feasibility problem considering voltage and current limits. The feasibility problem is then relaxed to a convex second-order cone program (SOCP). We utilize the strong dual problem of the SOCP to construct a convex polytopic approximation of the SOCP-relaxed feasible power injection region. We further develop a heuristic method to approximately remove the power injections that make the SOCP relaxation inexact, thus establishing an approximate polytope of solvable and safe power injections. Numerical results demonstrate a satisfactory balance reached by the proposed method between the accuracy of approximation and the simplicity of computation.

Index Terms—AC power flow, distribution networks, feasibility, optimization, second-order cone program

I. INTRODUCTION

The growing volatility of end-use energy demands and distributed renewable energy sources motivates the investigation of generation and load capacities that can be safely hosted by power systems, especially distribution networks. The key to this problem lies in finding the maximum feasible region of net power injections, i.e., generation minus load power, across network nodes. For any power injection vector within this region, the power flow equation is solvable and its solution satisfies safety limits for voltages and currents. The feasible power injection region allows us to know the generation and load capacities that can be handled by the power system prior to its actual operation and provides some guidance on the future planning of generations and lines.

Existing methods to determine such a feasible region can be divided into two categories according to their underlying power flow models. The first category rests on the lossless direct current (DC) power flow model [1]–[3]. The second category retains more accurate nonlinear AC models [4]–[8]. For distribution networks, the lossless DC model is not accurate enough since the distribution lines have higher resistance to reactance ratios. Therefore, this paper focuses on the second category of work by adopting the nonlinear Dist-Flow model [9], [10] for single-phase equivalent distribution networks represented by radial (i.e., tree) graphs.

In our context, feasibility of the net power injections requires solvability of the power flow equations and satisfaction of safety limits for voltages and currents. For the former requirement, some literature proved sufficient conditions under which the AC power flow equations are solvable, by utilizing Banach fixed-point theorem for contraction mappings [11], [12] or Brouwer fixed-point theorem for continuous mappings over compact convex sets [13], [14]. Those methods cannot be readily applied to the analysis of feasible region, whose model consists not only of equations but inequality safety constraints.

With further consideration of voltage and current limits, the feasible regions can be more restricted than the solvable regions, and often rely on optimization methods to compute. For instance, reference [5] solved nonlinear programs to get a set of boundary points that each make a different safety limit binding, and then built a feasible region heuristically as the convex hull of those boundary points. Certified inner approximations of feasible regions were solved from convex programs based on a tightened-relaxed second-order cone approximation [6] or refined linear approximations [7], [8] to AC power flow. In particular, the refined linearizations [7], [8] may be more accurate than the DC models [1]–[3], by replacing the nonlinear terms with their improved constant estimates rather than zero. However, such estimation typically only works for a specific objective function that merely explores the feasible region towards a single direction or with a specific shape of the power injection vector. This may limit its application scenarios and the information that can be provided.

In this paper, we propose an alternative optimization method to complement the literature above. Specifically:

• A nonlinear Dist-Flow model based optimization problem is developed to generate the feasible region of power injection vectors without specifying their directions or shapes. Compared with previous work [4]–[8], our formulation is applicable to a wider range of scenarios.
• To cope with the difficulty of solving the above optimization problem due to its nonconvexity, we first relax the problem to a convex second-order cone program (SOCP). Then its strong dual problem (also a SOCP) is derived based on which we can further estimate the feasible region. Approximating the dual SOCP instead of linearizing the primal SOCP directly can retain the nonlinearity feature of power flow as far as possible and is more accurate.

• We construct a polyhedral approximation of the feasible region by extending the algorithm in [2], [3] from linear programs to the dual SOCP. We proved that if the algorithm terminates in a finite number of iterations, the approximation is exact. Furthermore, we propose a heuristic method to remove the power injections that make the primal SOCP relaxation inexact so that we can build a tighter approximation of the feasible power injection region.

• Numerical results show that the proposed method can approximate the complicated feasible region with a simple polytope after moderate computation, while preserving relatively good accuracy.

Section II below introduces the power network model we use. Section III formulates the power-injection feasibility problem and its SOCP relaxation. Section IV elaborates our method to approximate the feasible region. Section V reports numerical experiments, and Section VI concludes the paper.

II. POWER NETWORK MODEL

Consider the single-phase equivalent model of a distribution network, which is a radial graph with a set \( N \) of nodes and a set \( L \) of lines. Index the nodes as \( N = \{0, 1, \ldots, N\} \), where 0 represents the root node (slack bus). For convenience, we treat the lines as directed; for example, if a line connects nodes \( i, j \), where node \( i \) is closer to the root than node \( j \), then the line directs from \( i \) to \( j \) and is denoted by \( i \rightarrow j \).

The power flow in the network at a particular time instant can be modeled by the classic Dist-Flow equations purely in real numbers [9], [10]. Specifically, let \( v_i, p_i, q_i \) denote the squared voltage magnitude, the net active power injection, and the net reactive power injection at node \( i \in N \), respectively, where net injection means supply minus demand. Let \( \ell_{ij} \) denote the squared current magnitude through line \( i \rightarrow j \). Let \( P_{ij} \) and \( Q_{ij} \) denote the net active and reactive power, respectively, that are sent by node \( i \) onto line \( i \rightarrow j \); note that they are different from the net power arriving at node \( j \) due to power loss, and can be negative to indicate node \( i \) receiving power from line \( i \rightarrow j \). Let \( r_{ij}, x_{ij} \) denote the constant resistance and reactance of line \( i \rightarrow j \), respectively. Consider the following quantities:

- Power injections \((p, q)\) stacked as a column vector, where \( p = (p_0, \ldots, p_N) \) and \( q = (q_0, \ldots, q_N) \);
- State variables \( x := (v, \ell, P, Q) \), where each of \( v, \ell, P, Q \) is a column vector indexed by \( \{1, \ldots, N\} \).

**Remark:** Without loss of generality, we assume there is only one node, indexed as node 1, connected to the root node 0.

\[\begin{align*}
\forall i \rightarrow j: & \quad P_{ij} - r_{ij}\ell_{ij} - \sum_{k: j \rightarrow k} P_{jk} + p_j = 0 \quad (1a) \\
& \quad Q_{ij} - x_{ij}\ell_{ij} - \sum_{k: j \rightarrow k} Q_{jk} + q_j = 0 \quad (1b) \\
& \quad v_i - v_j - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij} = 0 \quad (1c) \\
& \quad P_{ij}^2 + Q_{ij}^2 - v_i\ell_{ij} = 0. \quad (1d)
\end{align*}\]

Given \((p, q) \in \mathbb{R}^{2N}\), equation (1) is a set of \((4N)\) equations with \((4N)\) real variables \( x = (v, \ell, P, Q) \).

In addition, power system operations require the following safety limits to be satisfied:

\[\begin{align*}
\underline{v}_i \leq v_i \leq \bar{v}_i, & \quad \forall i = 1, \ldots, N \quad (2a) \\
0 \leq \ell_{ij} \leq \bar{\ell}_{ij}, & \quad \forall i \rightarrow j \quad (2b)
\end{align*}\]

where the voltage limits \(\underline{v}_i, \bar{v}_i\) for all nodes \(i\) and the current limits \(\bar{\ell}_{ij}\) for all lines \(i \rightarrow j\) are given as positive constants.

With the model above, we can formalize the power-injection feasibility problem in the next section.

III. FEASIBILITY PROBLEM AND RELAXATION

Consider the net active and/or reactive power injections at some nodes to be known constant numbers such as zero. Let subvector \(d \in \mathbb{R}^U\) collect all such known constant elements in \((p, q) \in \mathbb{R}^{2N}\), and let \(u \in \mathbb{R}^U\) collect all the other elements, i.e., the unknown variable power injections, with \(D + U = 2N\). In practice, \(U\) can be much smaller than \((2N)\) [2]-[4].

Definition 1. Given constant power-injection vector \(d\), the variable power-injection vector \(u\) is feasible if there exists \(x = (v, \ell, P, Q)\) such that \((x; d, u)\) satisfies power flow equations (1) and safety limits (2). The feasible power injection region is defined as:

\[ U := \{u \in \mathbb{R}^U \mid u \text{ is feasible} \} \]

**Remark:** At every node \(i\) with unknown variable power supply \(u^i := (p_i^u, q_i^u)\) and/or demand \(u_i^d := (p_i^d, q_i^d)\), its net power injection is \(u_i = u_i^u - u_i^d\). From the feasible region \(U\) of \(u\), one can derive the feasible region \(U^d\) for supply \(u^d\) if the operating region \(U^d\) for demand \(u^d\) is known, and vice versa. Actually \(U^d = U + U^d\), the Minkowski sum. Oftentimes \(U^d\) is specified as \(u^d \leq u^d \leq \pi^d\) (element-wise); if \(U\) is approximated by a finite union of convex polytopes, as discussed in Section V, then the said Minkowski sum can be efficiently and accurately computed [15].

For conciseness, we rewrite the linear program (1a)-(1c) of Dist-Flow equations as \(A_f x + B_f u + \gamma_f = 0\) and safety limits (2) as \(A_s x + \gamma_s \leq 0\), where both equality and inequality are element-wise, and constant matrices and vectors \(A_f, B_f, \gamma_f, A_s, \gamma_s\).1

1We also denote reactance by \(x\), expecting that would not cause confusion.
are provided in Appendix-A. Given any \( u \), we introduce its 
feasibility problem as the following optimization program:

\[
\text{FP}(u) : \min \ 1^T \tilde{z} \quad (3a) \\
\text{over } x = (v, \ell, P, Q), \quad \tilde{z} = (z_s, z_q, \tilde{z}_q) \geq 0 \\
\text{s. t. } A_f x + B_f u + \gamma_f = 0 \quad (3b) \\
A_x x + \gamma_s \leq z_s \\
P_{ij}^2 + Q_{ij}^2 - v_i \ell_i j \leq z_{q,ij}, \ \forall i \rightarrow j \\
v_i \ell_i j - (P_{ij}^2 + Q_{ij}^2) \leq \tilde{z}_{q,ij}, \ \forall i \rightarrow j \quad (3e)
\]

where \( 1^T \) in objective (3a) is a row vector of all ones. Any element of the slack variable \( \tilde{z} \) can increase as needed to satisfy the corresponding inequality constraint, but only \( \tilde{z} = 0 \) can guarantee feasibility in terms of (3b). Therefore, denoting the minimum objective value of \( \text{FP}(u) \) as \( \text{fp}(u) \), the feasible power injection region is equivalently:

\[
\mathcal{U} = \{ u \in \mathbb{R}^U \mid \text{fp}(u) = 0 \}.
\]

Due to quadratic constraint (3c), problem \( \text{FP}(u) \) is nonconvex and thus hard to analyze. By removing (3c) and rewriting (3d), we relax \( \text{FP}(u) \) to a convex SOCP:

\[
\text{FP}'(u) : \min \ 1^T z \\
\text{over } x, \ y, \ z = (z_s, z_q) \geq 0 \\
\text{s. t. } (3b) - (3c) \\
\|y_{ij}\|_2 \leq c_q x + \gamma_{q,ij} + z_{q,ij}, \ \forall i \rightarrow j
\]

where \( y \in \mathbb{R}^{3N}, A_y \in \mathbb{R}^{(3N) \times 4(N)}, \) and \( b_y \in \mathbb{R}^{3N} \) vertically stack \( y_{ij} \in \mathbb{R}^3, A_y^{ij} \in \mathbb{R}^{(3N) \times 4(N)}, \) and \( b_y^{ij} \in \mathbb{R}^3 \) respectively for all lines \( i \rightarrow j \). Row vector \( c_q^{ij} \in \mathbb{R}^{1 \times 4(N)} \) and number \( \gamma_{q,ij} \in \mathbb{R} \) are also stacked vertically for all \( i \rightarrow j \) as \( c_q \in \mathbb{R}^{N \times 4(N)} \) and \( \gamma_q \in \mathbb{R}^N \). The constant matrices and vectors \( A_y, b_y, c_q, \gamma_q \) are provided in Appendix-B, which make:

\[
A_y^{ij} x + b_y^{ij} = [2P_{ij}, 2Q_{ij}, v_i - \ell_i j]^T, \ \forall i \rightarrow j \\
c_q^{ij} x + \gamma_{q,ij} = v_i + \ell_i j, \ \forall i \rightarrow j
\]

and thus make (4b)–(4c) equivalent to (3d, e)²

Problem \( \text{FP}'(u) \) facilitates the definition of an \( \text{SOCP}\)-relaxed feasible power injection region:

\[
\mathcal{U}' := \{ u \in \mathbb{R}^U \mid \text{fp}'(u) = 0 \}
\]

where \( \text{fp}'(u) \) is the minimum objective value of \( \text{FP}'(u) \). It is obvious that \( \mathcal{U} \subseteq \mathcal{U}' \), i.e., \( \mathcal{U}' \) is a relaxation of \( \mathcal{U} \).

A common practice to further simplify the feasible-region characterization is to outer approximate the second-order cone (4c) with a polytopic cone, which can achieve arbitrary precision by constructing sufficiently many planes tangent to the surface of the second-order cone [16, 17]. Consequently, \( \text{FP}'(u) \) is relaxed to a linear program, and then the algorithm in [2, 3] can be employed to get a convex polytopic outer approximation of \( \mathcal{U}' \). In this work, we propose an alternative method that does not rely on such linearization. Instead, we work directly on the SOCP \( \text{FP}'(u) \) and its dual problem to preserve the intrinsic nonlinearity of the AC power flow model.

IV. APPROXIMATING FEASIBLE REGION

To offer a closed-form approximation of feasible region \( \mathcal{U} \), we first develop a convex polytopic approximation of its relaxation \( \mathcal{U}' \) via the dual problem of SOCP \( \text{FP}'(u) \). We then develop a heuristic method to approximately remove the power injections that make the SOCP relaxation inexact, resulting in a tighter approximation of \( \mathcal{U} \).

A. Dual SOCP

Let \( \mu := (\mu_f, \mu_y) \) denote the dual variables for the equality constraints in problem \( \text{FP}'(u) \), with \( \mu_f \in \mathbb{R}^{3N} \) for (3b) \( 2 \) and \( \mu_y \in \mathbb{R}^{3N} \) for (3c) vertically stacking \( \mu_{y,ij} \in \mathbb{R}^3, \ \forall i \rightarrow j \). Let \( \lambda := (\lambda_s, \lambda_q) \) denote the dual variables for the inequality constraints, with \( \lambda_s \in \mathbb{R}^{4N} \) for (3e) and \( \lambda_q \in \mathbb{R}^N \) for (3f). Then the Lagrangian of \( \text{FP}'(u) \) is:

\[
L_u = 1^T z + \mu_f^T (A_f x + B_f u + \gamma_f) + \lambda_s^T (A_s x + \gamma_s - z_s) + \mu_y^T (y - A_y x - b_y) + \sum_{i \rightarrow j} \lambda_{q,ij} (\|y_{ij}\|_2 - c_q^{ij} x - \gamma_{q,ij} - z_{q,ij}) \\
= z^T (1 - \lambda) + \sum_{i \rightarrow j} (y_{ij}^{T} \mu_{y,ij} + \|y_{ij}\|_2 \lambda_{q,ij}) + x^T \left( A_f^T \mu_f + A_s^T \lambda_s - A_y^T \mu_y - c_q^T \lambda_q \right) + \mu_y^T (B_f u + \gamma_f) + \lambda_s^T \gamma_s - \mu_f^T b_f - \lambda_q^T \gamma_q. \quad (5)
\]

Through \( \text{min}_{u \geq 0, x, y, \lambda} L_u(x, y, z; u, \lambda) \) we can get the dual objective function. By (5), \( L_u \) can only attain a finite minimum over \( (z \geq 0, x, y) \) when the dual variables satisfy:

\[
0 \leq \lambda \leq 1 \\
A_f^T \mu_f + A_s^T \lambda_s = A_y^T \mu_y + c_q^T \lambda_q \\
\|\mu_y\|_2 \leq \lambda_{q,ij}, \ \forall i \rightarrow j
\]

Note that \( \lambda \geq 0 \) in (6a) is a general requirement for all the dual variables associated with inequality constraints, and (6c) must hold by noticing

\[
y_{ij}^T \mu_{y,ij} + \|y_{ij}\|_2 \lambda_{q,ij} \geq (\lambda_{q,ij} - \|\mu_y\|_2) \|y_{ij}\|_2.
\]

When (6) is satisfied, all the terms containing \( x, y, z \) in (5) attain their minimum value zero, and hence we obtain the dual problem for \( \text{FP}'(u) \), which is also an SOCP:

\[
\text{DP}'(u) : \max_{\mu, \lambda} \mu_y^T (B_f u + \gamma_f) + \lambda_s^T \gamma_s - \mu_f^T b_f - \lambda_q^T \gamma_q \\
\text{s. t. } (6)
\]

Let \( D_u(\mu, \lambda) \) denote the objective function and \( \text{dp}'(u) \) denote the maximum objective value of \( \text{DP}'(u) \). The following result lays the foundation for approximating the SOCP-relaxed feasible region \( \mathcal{U}' \) via the dual SOCP \( \text{DP}'(u) \).

**Proposition 1.** For all \( u \in \mathbb{R}^U \), strong duality holds between \( \text{FP}'(u) \) and \( \text{DP}'(u) \), i.e., their optimal values \( \text{fp}'(u) = \text{dp}'(u) \).

**Proof:** Consider an arbitrary \( u \in \mathbb{R}^U \). Since problem \( \text{FP}'(u) \) is convex, it is sufficient to prove Slater’s condition

²Given \( x \), the values of \( z_q \) in (3c) and (4c) are generally not equal, but we do not differentiate notation due to their identical role as slack variables.
Algorithm 1: Approximate relaxed feasible region $\mathcal{U}'$

1. Initialization: $\mathcal{U}'_{\text{poly}} = \{ u \in \mathbb{R}^U \mid u \leq u \leq \pi \}$ for sufficiently low $u$ and high $\pi$; $\mathcal{V}_{\text{safe}} = \emptyset$; $c = 0$;
2. update vertices $\mathcal{V}(\mathcal{U}'_{\text{poly}})$. Let $d_{\text{max}} = 0$;
3. for $u \in \mathcal{V}(\mathcal{U}'_{\text{poly}})$ and $u \notin \mathcal{V}_{\text{safe}}$ do
   4. solve $\text{DP}(u)$ to obtain an optimal solution $(\mu^*, \lambda^*)$ and maximum objective value $d_p(u)$;
   5. if $d_p(u) > d_{\text{max}}$ then
      6. $d_{\text{max}} \leftarrow d_p(u)$;
      7. $(\mu_{\text{max}}, \lambda_{\text{max}}) \leftarrow (\mu^*, \lambda^*)$;
   8. else if $d_p(u) \leq 0$ then $\mathcal{V}_{\text{safe}} = \mathcal{V}_{\text{safe}} \cup \{u\}$;
9. end
10. if $d_{\text{max}} = 0$ or $c = C_{\text{max}}$ then
   11. return $\mathcal{U}'_{\text{poly}}$.
   12. else
   13. add to $\mathcal{U}'_{\text{poly}}$ a cutting plane:
      \[ (\mu^*)_t \max (By + \gamma) + \lambda^* \max \gamma = 0 \]
   14. $c \leftarrow c + 1$;
   15. go back to Line 2;
16. end

Proposition 3. The output $\mathcal{U}'_{\text{poly}}$ is an outer approximation of $\mathcal{U}'$.

Proof: Note the initial $\mathcal{U}'_{\text{poly}}$ contains $\mathcal{U}'$. We next prove that any cutting plane added in Line 13 would not remove any point in $\mathcal{U}'$. To show that, consider an arbitrary $u$ removed by a cutting plan whose coefficients are $(\mu_{\text{max}}, \lambda_{\text{max}})$. Then there must be $D_u(\mu_{\text{max}}, \lambda_{\text{max}}) > 0$. Since $(\mu_{\text{max}}, \lambda_{\text{max}})$ is dual feasible satisfying (4), we have $u \notin \mathcal{U}'$ by (7).

Unlike (3), (4) based on linear programs, the SOCP-relaxed feasible region $\mathcal{U}'$ may not be the intersection of a finite number of cutting planes (i.e., a convex polytope). Therefore, Algorithm 1 may not guarantee $d_p(u) = 0$ for all vertices $u \in \mathcal{V}(\mathcal{U}'_{\text{poly}})$ in a finite number of iterations. However, if it does so, as what happens in our numerical experiments, it will produce a nice outcome as follows.

Proposition 4. If Algorithm 1 terminates with $d_{\text{max}} = 0$ in a finite number of iterations, then it returns a convex polytope $\mathcal{U}'_{\text{poly}} = \mathcal{U}'$.

Proof: Proposition 3 has shown $\mathcal{U}' \subseteq \mathcal{U}'_{\text{poly}}$. If Algorithm 1 terminates with $d_{\text{max}} = 0$ after adding a finite number of cutting planes in Line 13, then it returns a convex polytope $\mathcal{U}'_{\text{poly}}$. Moreover, all the vertices $u \in \mathcal{V}(\mathcal{U}'_{\text{poly}})$ satisfy $d_p(u) = 0$, i.e., $u \in \mathcal{U}'$ by (7). This fact, together with the convexity of $\mathcal{U}'$ shown in Proposition 2 implies $\mathcal{U}'_{\text{poly}} \subseteq \mathcal{U}'$. Thus we have proved $\mathcal{U}'_{\text{poly}} = \mathcal{U}'$.

An immediate corollary of Proposition 4 is that if $\mathcal{U}'$ is not a polytope, then Algorithm 1 cannot terminate in a finite number of iterations with $d_{\text{max}} = 0$. If that happens, one can terminate Algorithm 1 when reaching the maximum number of iterations $C_{\text{max}}$, to obtain a convex polytope outer approximation of $\mathcal{U}'$. In this sense, the outcome of Algorithm 1 serves as a posterior indicator of the structure of $\mathcal{U}'$.

C. Removing SOCP-inexact power injections

Remember our goal is to approximate the feasible power injection region $\mathcal{U}$, whereas $\mathcal{U}'$ studied so far is a relaxation of $\mathcal{U}$ based on SOCP. The following result suggests limited cases where the relaxation $\mathcal{U}'$ is useful.

Proposition 5. For every $u' \in \mathcal{U}'$, there must be $u \leq u'$ (element-wise), such that $u \in \mathcal{U}$.

Proof: We just consider the nontrivial case $u' \in \mathcal{U}' \setminus \mathcal{U}$. By definitions of $\mathcal{U}$ and $\mathcal{U}'$, there exists $x' = (v', \ell', P', Q')$ that satisfies (1a)–(1c), (2), (3), (4), and
\[ v'^i_j \geq P^2_{ij} + Q^2_{ij}, \quad \forall i \to j \]
with strict inequality for at least one line. Then one can construct a new power injection \( u \) from \( u' \), in the same way as the proof of [10] Theorem 1] (exactness of SOCP relaxation when load over-satisfaction is allowed). Specifically, pick a particular line \( m \rightarrow n \) where (8) strictly holds, make:

\[
p_m = p_m' - \frac{\varepsilon r_{mn}}{2}, \quad q_m = q_m' - \frac{\varepsilon x_{mn}}{2},
\]

\[
p_n = p_n' - \frac{\varepsilon r_{mn}}{2}, \quad q_n = q_n' - \frac{\varepsilon x_{mn}}{2},
\]

for some \( \varepsilon > 0 \), and keep other elements of \( u \) equal to \( u' \).

We next prove that for \( u \leq u' \) above, there exists \( x = (v, \ell, F, Q) \) that satisfies (1a)-(1c), (2), (8), and moreover:

\[
v_m \ell_{mn} = P^2_{mn} + Q^2_{mn} \quad (9)
\]

for the said particular line \( m \rightarrow n \). How to find such an \( x \) and why it satisfies (1a)-(1c), (2a), (8) are straightforward from the proof of [10] Theorem 1]. Moreover, that proof shows:

\[
v_m \ell_{mn} - P^2_{mn} - Q^2_{mn} = -\frac{r_{mn}^2 + x_{mn}^2}{4} - (v_m' - r_{mn}P_m' - x_{mn}Q_m') \varepsilon + \left(v_m' \ell_{mn}' - P^2_{mn} - Q^2_{mn}\right)
\]

which is a parabola of \( \varepsilon \) that opens down and passes through:

\[
\left(0, \ell_{mn}'\right), \quad \left(P^2_{mn} - \frac{x_{mn}^2}{2}, Q^2_{mn} > 0\right)
\]

and

\[
\left(P^2_{mn} - \frac{x_{mn}^2}{4}, Q^2_{mn} \right) \leq -\frac{2}{2} \left(\frac{2}{2}\right)
\]

Therefore, one can always find \( 0 < \varepsilon \leq \ell_{mn}' \), at which the parabola attains zero, i.e., (9) is satisfied, while at the same time \( \ell_{mn}' = \ell_{mn}' - \varepsilon \) satisfies current limit (2b).

Repeat the process above for every line that strictly satisfies (8), to make equality (9) hold for that line, until we reach a power injection \( u \leq u' \) and a corresponding \( x \) that satisfy (1a)-(1c), i.e., \( u \in U \). This completes the proof.

Proposition 5 implies that \( U' \) expands \( U \) towards the directions where the net power injections are non-decreasing at all the nodes. Therefore, if we explore the maximum demand and minimum generation across all the nodes simultaneously in the relaxed region \( U' \), the result will surely fall in the exact feasible region \( U \). However, in future power systems with highly heterogeneous sources and loads, it would be a common task to simultaneously evaluate the maximum demand at some nodes and the maximum generation at others, in which case searching in \( U' \) may deliver an infeasible result.

To overcome this drawback, we design a heuristic to approximate remove \( U := U' \backslash U \) from \( U' \). The power injections \( u \in U \) are feasible in terms of the SOCP relaxation \( FP' \) but infeasible in terms of \( FP \), as formally defined below.

**Definition 2.** A power injection \( u \in U' \) is **SOCP-inexact**, if every optimal solution of \( FP'(u) \) satisfies:

\[
\|y_{ij}\|_2 < c_{q,ij}x + \gamma_{q,ij} \quad \text{for some } i \rightarrow j.
\]

The **SOCP-inexact power injection region** is:

\[
\tilde{U} = \{u \in U' | u \text{ is SOCP-inexact}\}.
\]

Our next focus is to build an approximation of \( \tilde{U} \). For that, we consider the following set defined on the dual SOCP:

\[
\tilde{U}_d := \{u \in U | \text{Every optimal solution of } DP'(u) \text{ satisfies } \lambda_{q,ij} = 0 \text{ for some } i \rightarrow j\}.
\]

The SOCP-inexact power injection region is:

\[
\tilde{U} = \{u \in U' | u \text{ is SOCP-inexact}\}.
\]

**Algorithm 2:** Approximate \( \tilde{U} \) (or SOCP-inexact \( \tilde{U} \))

1. **Initialization:** \( U_{poly} = U_{poly} \) returned by Algorithm 1
2. **Update vertices \( V(\tilde{U}_{poly}) \):** Let \( dp''_{max} = -\eta \).
3. **for** \( u \in V(\tilde{U}_{poly}) \) **and** \( u \notin V_{safe} \) **do**
   4. **solve** \( DP''(u, \delta) \) **to obtain an optimal solution** \( (\mu^*, \lambda^*) \) and maximum objective value \( dp''(u, \delta) \).
   5. **if** \( dp''(u, \delta) > dp''_{max} \) **then**
      6. **dp''_{max} \leftarrow dp''(u, \delta) ;**
      7. \( (\mu_{max}, \lambda_{max}) \leftarrow (\mu^*, \lambda^*) ;
   8. **else if** \( dp''(u, \delta) \leq -\eta \) **then** \( V_{safe} = V_{safe} \cup \{u\} \);
9. **end**
10. **if** \( dp''_{max} = -\eta \) **or** \( c = C_{max} \) **then**
11. **return** \( U_{poly} \).
12. **else**
13. **add to** \( \tilde{U}_{poly} \) **a** cutting plane:
      14. \( \mu^*_{q,ij} B_{fj} + \lambda^*_{q,ij} \gamma_s \leq -\eta ;
      15. \( c \leftarrow c + 1 ;
      16. **go back to Line 2 ;**
17. **end**

By complementary slackness [18], Section 5.5.2], for every primal-dual optimal of \( FP'(u) \) and \( DP'(u) \), there is:

\[
\lambda_{q,ij} = \frac{\|y_{ij}\|_2 - c_{q,ij}x - \gamma_{q,ij}}{\lambda_{q,ij}} = 0, \quad \forall i \rightarrow j.
\]

This implies \( \tilde{U} \subseteq U_d \). Although \( \tilde{U} = U_d \) may not hold, their difference can only occur under rare circumstances where \( \lambda_{q,ij} = \frac{\|y_{ij}\|_2 - c_{q,ij}x - \gamma_{q,ij}}{\lambda_{q,ij}} = 0 \) at a primal-dual optimal. Hence we focus on \( U_d \) as an approximation of \( \tilde{U} \).

Given an arbitrary \( u \in U_d \subseteq U' \), the maximum objective value of \( DP'(u) \) is \( dp'(u) \). Now we add the following constraint to tighten the dual feasible set (9):

\[
\lambda_{q} \geq \delta
\]

where the inequality is element-wise and \( \delta \in \mathbb{R}^n \) is a vector of strictly positive parameters, whose design will be elaborated later. Consider the tightened dual SOCP:

\[
DP''(u, \delta) : \max_{\mu, \lambda} \mu^*_{ij} (B_{fj} u + \gamma_{f}) + \lambda^*_{ij} \gamma_s - \mu^*_{q,y} b_y - \lambda^* q \gamma_q \quad \text{s. t.} \quad \delta, (10)
\]

and let \( dp''(u, \delta) \) denote its maximum objective value. For \( u \in U_d \), there must be \( dp''(u, \delta) < 0 \), because otherwise \( DP'(u) \) would have an optimal solution that satisfies (10), contradicting the definition of \( U_d \). Actually \( dp''(u, \delta) \leq -\eta \) for some \( \eta > 0 \) that depends on \( u \) and \( \delta \).
The idea above inspires our design of Algorithm 2, which follows a similar procedure to Algorithm 1 to approximate $\mathcal{U}_d$ (or $\mathcal{U}$). If Algorithm 2 terminates with $d_{\mu_{\max}} = -\eta$ in a finite number of iterations, then it returns a convex polytope $\mathcal{U}_{\text{poly}} \subseteq \mathcal{U}_d$ that guarantees $d_{\mu^*}(u, \delta) \leq -\eta < 0$ for all $u \in \mathcal{U}_{\text{poly}}$. To make Algorithm 2 more robust, we may choose $\eta' > \eta$ for the added cutting plane in Line 13.

We observe in numerical experiments that Algorithm 2 is quite sensitive to parameters $\delta$ and $\eta$. A general guideline is that (i) given $\delta$, choosing a smaller $\eta$ and (ii) given $\eta$, choosing a bigger $\delta$ will both make $\mathcal{U}_{\text{poly}}$ bigger and lead to a smaller (more conservative) approximation of $\mathcal{U} = \mathcal{U}_d \setminus \mathcal{U}$.

It is often difficult for Algorithm 2 to use a single convex polytope $\mathcal{U}_{\text{poly}}$ to accurately approximate the most likely nonconvex $\mathcal{U}$. To deal with this difficulty, we propose to run Algorithm 2 multiple times with different $\delta \in \mathbb{R}_+^n$. As a result, we obtain multiple convex polytopes whose union serves as a better approximation of $\mathcal{U}$. Those vectors $\delta$ are selected in the following way. We traverse the vertices of $\mathcal{U}_d$ and select one vertex $u$ each time to run Algorithm 2. Solve the dual SOCP $\text{DP}'(u)$ to get an optimal solution $(\mu^*, \lambda^*)$. In $\lambda^*_u$, keep all the strictly positive elements as they are, and add a small positive number to all the zero elements. The result is $\delta$ for that run of Algorithm 2.

**Concluding remark for Section IV** As discussed, the feasible power injection region $\mathcal{U} = \mathcal{U}_d \setminus \mathcal{U}$, where $\mathcal{U}'$ is the SOCP-relaxed feasible region and $\mathcal{U}$ is the SOCP-inexact region. We developed Algorithm 1 to get $\mathcal{U}_{\mu_{\max}}$, a convex polytopic approximation of $\mathcal{U}'$, and Algorithm 2 to get $\mathcal{U}_{\text{poly}}$, a convex polytopic approximation of $\mathcal{U}$. Algorithm 2 can run multiple times to obtain a more accurate approximation of nonconvex $\mathcal{U}$. The outputs of multiple runs of Algorithm 2 are then removed from $\mathcal{U}_{\text{poly}}$, to obtain a generally nonconvex polytopic approximation of $\mathcal{U}$. It can be expressed as a finite union of convex polytopes, which facilitates the computation of Minkowski sum, as remarked after Definition 1.

**V. Numerical results**

We conduct numerical experiments on the IEEE 33-node network model in Fig. 1. The proposed algorithms are implemented to approximate the feasible region of active power injections $(u_1, u_2)$ at nodes 13 and 29, respectively, where renewable energy sources are installed.

In particular, Fig. 2 shows the output $\mathcal{U}_{\mu_{\max}}$ by Algorithm 1 in different iterations, which approaches the actual SOCP-relaxed feasible region $\mathcal{U}'$. The actual region is found by solving SOCP $\text{FP}'(u)$ over sampled points $u$ in the $(u_1, u_2)$ space. We observe that Algorithm 1 terminates with $d_{\mu_{\max}} = 0$ in a finite number of iterations, thus returning a convex polytope $\mathcal{U}_{\mu_{\max}} = \mathcal{U}'$ with a minor error due to sampling resolution.

We run two other cases where we halve and double the current limits $\mathcal{T}$, respectively. Fig. 3 displays the decrease of $d_{\mu_{\max}}$ towards zero over the iterations of Algorithm 1 in all the three cases. Table IV shows the number of iterations and computation time required for each case to converge. We observe that a more stringent line-flow limit results in slower convergence. Overall, the computation time is acceptable considering that the feasible region is typically searched day-ahead or in a planning phase.

We run Algorithm 2 multiple times as proposed in Section IV. The obtained convex polytopes (whose union serves as an approximation of $\mathcal{U}$) are removed from the output $\mathcal{U}_{\text{poly}}$ by Algorithm 1 leaving an approximation of the feasible power injection region $\mathcal{U}$. Fig. 4 compares this approximation with $\mathcal{U}$ obtained by checking sampled points (close to the actual $\mathcal{U}$), which shows the ability of the proposed algorithms to provide a simple and relatively accurate approximation.
TABLE I
COMPUTATION TIME UNDER DIFFERENT CURRENT LIMITS

| Iteration | Time | Iteration | Time | Iteration | Time |
|-----------|------|-----------|------|-----------|------|
| 33        | 391s | 26        | 263s | 8         | 63s  |

![Graph showing SOCP-relaxed region and feasible region](image1)

**VI. CONCLUSION**

In this paper, we developed a polytopic approximation of the feasible power injection region on the AC Dist-Flow model of radial networks. To be specific, a power-injection feasibility problem ensuring both power-flow solvability and voltage and current safety was formulated and relaxed to a convex SOCP. Its dual problem, also an SOCP, was proved to attain strong duality. Built on the dual SOCP, our first algorithm established a convex polytopic approximation of the SOCP-relaxed feasible region. We proved that the approximation is exact if the algorithm terminates in a finite number of iterations. Then our second algorithm provided a polytopic approximation of the SOCP-inexact power injection region, which subtracted from the first polytope, leading to a polytopic approximation of the target feasible region.

The proposed approximation extends previous methods which explored the feasible region subject to a specific shape or direction of the power injection vector, as well as those based on linearized power flow models. Numerical experiments show that the proposed method can provide a relatively accurate approximation in a simple polytopic form with moderate computation efforts. For future work, we plan to improve accuracy and convergence of the proposed algorithms, especially the one to remove SOCP-inexact power injections.

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**APPENDIX. CONSTANT PARAMETERS**

This appendix provides in full detail the constant matrices, vectors, and numbers used in Section IV.

**A. Equation (3)**: \( A_f, B_f, \gamma_f, A_s, \gamma_s \)

The vectors \( x = (v, \ell, P, Q) \) and \( (p, q) \) are arranged in the order explained in Section II. Let \( C \in \{-1, 0, 1\}^{N \times N+1} \) be the incidence matrix of the radial network, with its element at \( k \)-th row, \( j \)-th column:

\[
C_{kj} = \begin{cases} 
1, & \text{if } k = i \text{ for line } i \rightarrow j \\
-1, & \text{if } k = j \text{ for line } i \rightarrow j \\
0, & \text{otherwise.}
\end{cases}
\]

Removing the first row of \( C \), we get the reduced incidence matrix \( \overline{C} \in \{-1, 0, 1\}^{N \times N} \). Define diagonal matrices \( R := \text{diag}(r_{ij}, \forall i \rightarrow j) \) and \( X := \text{diag}(r_{ij}, \forall i \rightarrow j) \). Denote the \( N \times N \) all-zero matrix as \( O_N \), identity matrix as \( I_N \), and \( N \)-dimensional all-zero column vector as \( 0_N \). First define:

\[
B_f = \begin{bmatrix} I_N & O_N & O_N \\
O_N & I_N & O_N \\
O_N & O_N & O_N \end{bmatrix}, \quad B_g = \begin{bmatrix} 0_N \\
0_N \\
0_N \end{bmatrix}.
\]
Let $B'_{fd}$ be a submatrix of $B'$ containing only the columns corresponding to the known constant power injections $d$. Then $B_f$ only includes the rest of the columns, i.e., those corresponding to the variable power injections $u$. We have:

$$A_f = \begin{bmatrix} O_N & -R & -C & O_N \\ O_N & -X & O_N & -C \\ C^T & (R^2+X^2) & -2R & -2X \end{bmatrix}, \quad \gamma_f = \gamma'_f + B'_{fd}d.$$  

Define column vectors $v := (v_i, \forall i = 1,...,N)$, $\bar{v} := (\bar{v}_i, \forall i = 1,...,N)$, and $\bar{\ell} = (\bar{\ell}_{ij}, \forall i \to j)$. To write safety limits (2) as $A_s x + \gamma_s \leq 0$, we need:

$$A_s = \begin{bmatrix} I_N & O_N & O_N & O_N \\ -I_N & O_N & O_N & O_N \\ O_N & I_N & O_N & O_N \\ O_N & -I_N & O_N & O_N \end{bmatrix}, \quad \gamma_s = \begin{bmatrix} -\bar{v} \\ -\bar{v} \\ -\bar{\ell} \\ 0_N \end{bmatrix}.$$

B. Equation (4): $A_y, b_y, c_q, \gamma_q$

To make (4b)–(4c) the same as:

$$\begin{bmatrix} 2P_{ij} \\ 2Q_{ij} \\ v_i - \ell_{ij} \end{bmatrix}_2 \leq v_i + \ell_{ij} + z_{q,ij}, \quad \forall i \to j$$

we need $A_y, b_y, c_q, \gamma_q$ as follows:

- For all $i \to j$, $A_{y,ij}$ is a $3 \times (4N)$ sparse matrix with all elements zero except its element at the first row, $(2N+j)$-th column equal to 2; at the second row, $(3N+j)$-th column equal to 2; at the third row, $i$-th column equal to 1 (if $i \neq 0$), and $(N+j)$-th column equal to $-1$.  
- For all $i \to j$ except $0 \to 1$, $b_{y,ij}$ is a three-dimensional column vector of all zeros; $b_{y,01} = [0, 0, v_0]^T$.  
- For all $i \to j$, $c_{q,ij}$ is a $(4N)$-dimensional row vector of all zeros except its $i$-th (if $i \neq 0$) and $(N+j)$-th elements both equal to 1.  
- $\gamma_{q,ij} = 0$ for all $i \to j$ except $0 \to 1$; $\gamma_{q,01} = v_0$.  

