Quantum aspects of the gravitational–gauge vector coupling in the Hořava–Lifshitz
theory at the kinetic conformal point

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This work presents the main aspects of the anisotropic gravity–vector gauge coupling at all energy scales, i.e., from the IR to the UV point. This study is carry out starting from the 4+1 dimensional Hořava–Lifshitz theory, at the kinetic conformal point. The Kaluza–Klein technology is employed as a unifying mechanism to couple both interactions. Furthermore, by assuming the so–called cylindrical condition, the dimensional reduction to 3+1 dimensions leads to a theory whose underlying group of symmetries corresponds to the diffeomorphisms preserving the foliation of the manifold and a U(1) gauge symmetry. The counting of the degrees of freedom shows that the theory propagates the same spectrum of Einstein–Maxwell theory. The speed of propagation of tensorial and gauge modes is the same, in agreement with recent observations. This point is thoroughly studied taking into account all the $z = 1, 2, 3, 4$ terms that contribute to the action. In contrast with the 3+1 dimensional formulation, here the Weyl tensor contributes in a non–trivial way to the potential of the theory. Its complete contribution to the 3+1 theory is explicitly obtained. Additionally, it is shown that the constraints and equations determining the full set of Lagrange multipliers are elliptic partial differential equations of eighth–order. To check and assure the consistency and positivity of the reduced Hamiltonian some restrictions are imposed on the coupling constants. The propagator of the gravitational and Hamilton sectors are obtained showing that there are not ghost fields, what is more they exhibit the $z = 4$ scaling for all physical modes at the high energy level. By evaluating the superficial degree of divergence and considering the structure of the second class constraints, it is shown that the theory is power–counting renormalizable.

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I. INTRODUCTION

At present it has been extremely difficult to obtain a quantum description of gravity. In this direction, more than decade ago, P. Hořava proposed a candidate theory for a quantum description of gravitational interaction [1]. The fundamental feature of this theory is the anisotropic scaling law between spatial and temporal coordinates [2], hence the theory breaks the relativistic symmetry. This anisotropic scaling is realized in the framework by the introduction of the so–called critical dynamic exponent $z$, which is determined in a way that the overall coupling of the action becomes dimensionless (in general one has $z = d$ the dimension of the leaves of the foliation). In general terms, the main ingredients that this proposal gathers to address such a problem can be summarized as follows: i) the geometric framework is given by the Arnowitt–Deser–Misner (ADM) variables [3]: the three metric $\gamma_{ij}$ of the Riemannian leaves, the lapse function $N$ and the shift vector $N_i$. This establishes a particular foliation of space–time with a privileged temporal direction. ii) The group of symmetries associated with this structure corresponds to the foliation–preserving diffeomorphisms (FDiff). iii) The theory introduces higher order spatial derivatives interaction terms, up to $2z$ derivatives, compatible with the FDiff symmetry. The highest order derivative terms are the dominant ones at the UV regime and they improve the quantum behaviour compared to the $z = 1$ formulation. So, in the UV regime the theory in 3+1 dimensions, $z = 3$, is power–counting renormalizable in the perturbative scheme, in addition to preserve unitarity [1]. The Hořava–Lifshitz proposal is a general framework which allows the formulation of different theories, namely, the projectable and the non–projectable theories. The main difference between both formulations is due to the fact that the lapse function $N$ can depend only on time (projectable) or on both space and time (non–projectable). This implies that the projectable theory does not propagate the full gravitational degrees of freedom. On the other hand, compared to the projectable case, the non–projectable version accepts a broader class of terms to constitute the interaction potential of the theory [4]. These terms constructed with the lapse function $N$, specifically $a_i = \partial_i \ln N$, have several consequences when quantizing the theory perturbatively. The non–projectable theories may propagate different physical degrees of freedom. In one case it propagates a scalar field in addition to

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the gravitational modes. In the other case, it only propagates the gravitational degrees of freedom. The latter case has been called the Hořava–Lifshitz theory at the kinetic conformal point (KC point from now on) [5]. The non-projective theory with a propagating scalar field has been thoroughly discussed. In particular, the strong coupling problem of the scalar field [6–8]. The presence of the scalar field imposes strong restrictions to the couplings constants of the theory. The compatibility of the dipole radiation generated by the scalar field with the data recently obtained from the detection of gravitational waves imposes further restrictions to the couplings [9]. A way to avoid several of the restrictions to the coupling, is to consider the Hořava–Lifshitz theory at the KC point [5]. This theory only propagates the gravitational degrees of freedom, with the same quadrupole radiation formula as in GR [10]. In this case, the theory has second-class constraints not present in GR. Precisely, these second-class constraints and their preservation in time are consistently solved to eliminate the additional mode and find out the Lagrange multipliers associated with these constraints. This is a consequence of introducing the terms $a_i$, which improve the mathematical structure of the second-class constraints, which become elliptical partial differential equations, a relevant property in order to define the physical submanifold of the theory and to have a well-posed initial value problem. Finally, the physical degrees of freedom propagated by the theory, at all energy levels, is exactly the transverse–traceless tensorial modes, just as in GR. From the quantum point of view, the structure of terms compatible with the FDiff (which are around 100), improves the behavior of the propagators of the physical modes, rendering (as previously stated) the theory to be renormalizable at least by power-counting and preserving at the same time the unitarity.

This consistent behavior at the quantum level, is the consequence of including in the FDiff–Lagrangian terms with higher spatial derivatives (up to order six which corresponds to $z = 3$), since no higher time derivatives are included. In contrast with what occurs in GR, where the inclusion of higher spatial derivatives necessarily involve higher time derivatives, in order to preserve the relativistic symmetry, however ghost fields then arise [11]. In recent years the consistency of this theory has been extensively studied. For example, the group of symmetries in the projectable case, was extended by adding and extra U(1) gauge symmetry in order to eliminate the extra degree of freedom [12]. On the other hand, the non–projectable version has been related to the Einstein–aether theory [13], although truncated at second order in derivatives [7, 14, 15]. Also, both the projectable and non–projectable versions have been reproduced from the torsional Newton–Cartan geometry, when this one becomes dynamical [16]. As stated before, the non–projectable case constitutes a better scenario to face the quantum gravity issue. In this concern the Hamiltonian formulation has been addressed and exhaustively analyzed in [17–20]. Moreover, in references [5, 21] the theory at the KC point at the classical and quantum level was studied. The power counting renormalizability was shown. It is worth mentioning that, the KC point occurs when the independent dimensionless coupling constant $\lambda$, appearing in the kinetic term of the theory, takes the critical value $\lambda = 1/d$ (being $d$ the spatial dimension). Under this consideration, the kinetic term of the theory acquires a conformal invariance not present in the whole theory [1]. In a broader context, this theory has motivated research in various fields, for example in the framework of the so–called $\lambda–R$ model [22], spherically symmetric solutions and the initial value problem have been explored [23–25]. Furthermore, studies on the cosmological setting and its properties have been analyzed in the following recent works [26–29]. Also progress has been reported in the standard model domain [30], four–fermion Gross–Neveu like models [31] and the quantization in 2+1 dimensions [32]. However, the feasibility of this interesting theory goes beyond the theoretical point of view. In this concern, after the detection of GW170817 and GRB170817A [33, 34], it was shown that the propagation speeds of gravitational and electromagnetic waves match within one part in $10^{-15}$ [35]. In [9, 36, 37] they determined restrictions on the low energy coupling constants of the theory, they must be $\{\beta \sim 1, \alpha \sim 0\}$, in requirement to the relativistic observational constraints at low energies. Following the theoretical studies developed in [38, 39] where it was proved that in the limit at low energies the gravitational and electromagnetic waves propagate at the same speed $\sqrt{\beta}$, in [40] by using the GRB170817A data within a cosmological FLRW background and placing the theory at the KC point the authors showed that in the most ideal case $|1 - \sqrt{\beta}| < (10^{-19} - 10^{-18})$, thus $\beta \sim 1$.

Motivated by this good background, in this work, the quantization of the 4+1 dimensional non–projectable Hořava–Lifshitz gravity at the KC point is thoroughly analyzed. Posing the theory at the KC point, being in this case $\lambda = 1/4$, we study the classical Hamiltonian formulation and its consistency, presenting the full set of first and second class constraints. Taking advantage of previous works [38, 39], the 4+1 dimensional non–relativistic space–time formulation, is reduced to 3+1 dimensions after applying the well–known Kaluza–Klein mechanism, subject to the so–called cylindrical condition i.e., no fields depend on the extra dimension. As it is well–known, the Kaluza–Klein technology allows to unify fundamental interactions. In this case, the dimensional reduction provides an anisotropic gravity–vector gauge field (an anisotropic electromagnetism–like interaction) coupling. The problem to couple matter fields or fundamental interactions to the anisotropic gravity theory, constitutes an open branch to be investigated. In [41], it was shown that when matter fields are coupled through the minimum coupling principle, the theory loses its power–counting renormalizability at the quantum level, due to the appearance of quadratic divergences in the matter sector[55]. In order to preserve the power–counting renormalizability of the theory, in [43, 44] a new classes of terms respecting the FDiff symmetry were proposed. Mainly, these terms were built using the extrinsic curvature tensor, involving time derivatives of the three metric. These objects were called mixed derivative terms. Although, the
resulting theory remains power–counting renormalizable, keeping the Lorentz violation in the anisotropic gravity sector only, terms with negative kinetic energy (ghost fields) appeared due to the introduction of higher derivatives in time. Then, the unitarity of the theory is violated. In contrast, in the approach we follow, the coupling of the anisotropic gravity interaction with the electromagnetism–like interaction arises from an intrinsic Hořava–Lifshitz formulation, hence the unitarity is guaranteed. We will show that our proposal is indeed power–counting renormalizable. We emphasize that the 4+1 theory and Kaluza–Klein reduction is only a geometrical setting to get the anisotropic 3+1 dimensional theory [38, 39]. This approach splits the 4+1 dimensional degrees of freedom corresponding to the transverse–traceless tensorial modes into the 3+1 dimensional transverse–traceless tensorial modes plus the transverse–vector–gauge degrees of freedom. In our formulation the dilaton field is considered to be in its ground state (for further details see [39]). After dimensional reduction, the diffeomorphisms preserving the foliation (FDiff) of the 4+1 dimensional manifold, break down into the U(1) group of symmetry for the vector–gauge sector and the 3+1 dimensional FDiff for the anisotropic gravitational field. Nevertheless, the original shift function \( N_i \) does not transform properly under the infinitesimal transformation laws in 3+1 dimensions, then it is necessary to redefine this object in order to have a consistent symmetry group. With this information at hand, we study in details the reduced theory, determining its full set of constraints and the consistency of the Hamiltonian formulation. Furthermore a perturbative analysis in the so–called transverse gauge is done to determine the set of canonical variables. On the other hand the vector–gauge field presents the same structure as the Maxwell electromagnetism field i.e., the propagating degrees of freedom (two transverse vector modes) and the first class constraint coincide, being the underlying group of symmetry just the U(1) gauge symmetry. In this theory the field equations of the propagating physical modes are non–relativistic. A comparison between the anisotropic field equations and Maxwell equations was performed in [39].

To perform and study the quantum program we use the techniques described in [45, 46, 51] to obtain the superficial degree of divergence of the theory at the UV point. It is worth mentioning that the power–counting renormalization is guaranteed only if higher order spacelike derivatives of the electromagnetic–like sector are included in the potential of the theory. In this concern, the dimensional reduction of the 4+1 dimensional Weyl tensor plays and important role. As it well–known, the Weyl tensor is null on a three dimensional Riemannian leaves. However, after dimensional reduction coming from 4+1 dimensions, relevant terms survive, contributing at all order i.e., \( z = 1, 2, 3, 4 \), both to the gravitational and electromagnetic sectors. This fact is very important, since not only the power–counting is assured, also the unitarity of the theory is saved.

The article is organized as follows: In Sec. II the Hamiltonian formulation, its consistency and the dimensional reduction a la Kaluza–Klein are revisited. Sec. III is devoted to the quantum formulation. Finally, Sec. IV concludes the present investigation.

II. THE HAMILTONIAN FORMULATION AND ITS CONSISTENCY REVISITED

A. The action and the geometric background

The 4+1 dimensional theory for an arbitrary \( \lambda \) was developed in [38] following the original Hořava–Lifshitz [1] gravity in 3+1 dimensions and taking into account the so–called healthy extension [4]. The starting point is a foliated 5–dimensional manifold in terms of 4–dimensional Riemannian hypersurfaces along with a privileged time direction. So, the underlying symmetry group corresponds to the FDiff. In this case, the infinitesimal generators of this symmetry are given by

\[
\delta t = f(t), \quad \delta x^\mu = \zeta^\mu(t, x^\nu),
\]

where greek indices run over \( \mu, \nu = 1, 2, 3, 4 \). Following [1], the gravitational sector is modeled by the Arnowitt–Deser–Misner (ADM hereinafter) variables, namely the spatial metric \( g_{\mu\nu} \), the lapse function \( N \) and the shift function \( N_\mu \).

The transformation laws of these objects under FDiff are given by

\[
\delta N \quad = \quad \zeta^\rho \partial_\rho N + f \dot{N} + \dot{f} N,
\]

\[
\delta N_\mu \quad = \quad \zeta^\rho \partial_\rho N_\mu + N_\rho \partial_\rho \zeta^\mu + \zeta^\rho g_{\mu\rho} + f \dot{N}_\mu + \dot{f} N_\mu,
\]

\[
\delta g_{\mu\nu} \quad = \quad \zeta^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\mu} \partial_{\nu)} \zeta^\rho + f \dot{g}_{\mu\nu}.
\]

So, the most complete and general action of the theory is

\[
S = \int dt dx \sqrt{g} N \left( \frac{1}{2\kappa} G^{\mu\nu;\rho\sigma} K_{\mu\nu} K_{\rho\sigma} - V \right),
\]
where

\[
K_{\mu\nu} = \frac{1}{2N} \left( \dot{g}_{\mu\nu} - 2 \nabla_{(\mu} N_{\nu)} \right), \\
G^{\mu\nu\rho\sigma} = \frac{1}{2} \left( g^{\mu\sigma} g^{\nu\rho} + g^{\mu\rho} g^{\nu\sigma} \right) - \lambda g^{\mu\nu} g^{\rho\sigma},
\]

(6)

are the extrinsic curvature tensor and the hypermatrix, respectively. At this point some comments are pertinent: i) Following Hořava’s proposal, one could consider the lapse function \( N \) to be a purely time dependent field [56] i.e., \( N = N(t) \) (projectable case) or allow it depends on both space and time coordinates that is \( N = N(t, x^\mu) \) (non-projectable case). From now on we shall focus on the latter one. Also following Hořava’s proposal, ii) in order to obtain a power–counting renormalizable theory we consider an anisotropic scaling law between time and space variables [1]

\[
t \rightarrow b^z t, \quad x^\mu \rightarrow b x^\mu,
\]

(7)

where the parameter \( z \) measures the anisotropy degree. The power–counting argument needs to consider \( z = 4 \) in 4+1 dimensions. In distinction with the case in 3+1 dimensions, where the minimal degree of anisotropy to render the theory to be power–counting renormalizable is \( z = 3 \) [1]. So, the dimensionality in momentum powers of the variables and fields are

\[
[t] = -z, \quad [x^\mu] = -1, \quad [g_{\mu\nu}] = [N] = 0, \quad [N_\mu] = z - 1.
\]

(8)

Under this scheme, when \( z \) is equal to the spatial dimension of the theory i.e., \( z = d = 4 \), the coupling constant \( \kappa \) becomes dimensionless. In contrast, in the relativistic limit \( \lambda = 1, \beta = 1, \alpha = 0 \) and \( z = 1 \), the coupling \( \kappa \) is dimensionful. iii) Regarding the potential of the theory \( V \), the building block to obtain it, is to consider any FDiff scalar made up with the spatial metric \( g_{\mu\nu} \), the vector [4]

\[
a_\mu = \partial_\mu \ln N,
\]

(9)

and their FDiff–covariant derivatives (including spatial Riemann tensor, Ricci tensor and Ricci scalar). It is worth mentioning that each term making the potential \( V \), has its proper independent coupling constant. This is because, we have not placed any coupling constant in front of \( V \) in Eq. (5). For example at lower energies \( z = 1 \) the potential becomes

\[
V(z=1) = -\beta R - \alpha a_\mu a^\mu,
\]

(10)

being \( \beta \) and \( \alpha \) coupling constants. The potential does not depend neither in time derivatives nor in \( N_\mu \), which is not a vector field on the leaves of the foliation. Initially, as stated before in this study we will consider the so–called KC point. In general the KC point corresponds to set the coupling constant \( \lambda \) at its critical value i.e, \( \lambda = 1/d \). In the present case, the critical value is given by \( \lambda = 1/4 \) [38, 39] (see [5] for 3+1 dimensional case). As we will discuss shortly, setting \( \lambda \) to its critical value has several consequences.

B. The Hamiltonian and its constraints

It is clear from the action (5) that the fields \( N \) and \( N_\mu \) do not have dynamic i.e, there are not temporal derivative associated to these fields. However, the fact that the potential \( V \) depends on the lapse function \( N \) in a non–trivial way through the vector \( a_\mu \), means that \( N \) and its conjugate pair \( P_N \) are part of the canonical fields of the theory. The role of the shift function \( N_\mu \), is different, it is a Lagrange multiplier. In this concern, \( P_N \) arises as the first (primary) constraints of the theory. Thus

\[
P_N = 0.
\]

(11)

Then, the complete set of canonical variables is spanned by the conjugate pairs \((g_{\mu\nu}, \pi^{\mu\nu})\) and \((N, P_N)\). These variables constitute the non–reduced phase space of the theory. On the other hand, as occurs in GR [5], the theory presents a first class constraint, associated with time–dependent spatial diffeomorphisms on the variables \((g_{\mu\nu}, \pi^{\mu\nu})\), whose Lagrange multiplier is precisely \( N_\mu \). Adopting the same nomenclature of GR, we shall call this constraint the momentum constraint and is given by

\[
\mathcal{H}^\mu \equiv -2 \partial_\nu \pi^{\mu\nu} = 0.
\]

(12)
The conjugate momentum to $g_{\mu\nu}$ satisfies
\[ \pi^{\mu\nu} = \frac{1}{2\kappa} G^{\mu\nu\rho\sigma} K_{\rho\sigma}. \] (13)

At this point the framework splits into two different theories: $\lambda = 1/4$ or $\lambda$ different from $1/4$. The Hamiltonian for $\lambda$ different from $1/4$ reads
\[ H = \int d^4x \left[ N \left( \frac{2\kappa}{\sqrt{g}} \pi^{\mu\nu} \pi_{\mu\nu} + \frac{2\kappa}{(1-4\lambda)} \pi^2 \right) + N\sqrt{g} V + N_{\mu} \dot{H}^{\mu} + \sigma P_{N} \right], \] (14)

The theory for $\lambda = 1/4$, the Hořava–Lifshitz at the KC point, satisfies from Eq. (13) the following restriction
\[ \frac{g_{\mu\nu} \pi^{\mu\nu}}{\sqrt{g}} = \frac{\pi}{\sqrt{g}} = \frac{1}{2\kappa} g_{\mu\nu} G^{\mu\nu\rho\sigma} K_{\rho\sigma} \Rightarrow g_{\mu\nu} G^{\mu\nu\rho\sigma} = (1-4\lambda) g^{\rho\sigma} = 0, \] (15)

where the hypermatrix becomes degenerated. Therefore, this implies that
\[ \pi = 0. \] (16)

The constraint given by (16) is absents when $\lambda$ has any other value. What is more, this constraint generates anisotropic conformal transformations
\[ g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}, \quad N \to e^{4\Omega} N, \quad N_{\mu} \to e^{2\Omega} N_{\mu}, \quad \Omega = \Omega(t, x^\mu). \] (17)

It is remarkable to note that the kinetic term of $\sqrt{\mathcal{G}} N G^{\mu\nu\rho\sigma} K_{\mu\rho} K_{\nu\sigma}$ is invariant under (17). However, in general the theory itself is not conformally invariant [1]. From the physical point of view, the main aim in considering the KC theory is to eliminate the extra mode, therefore the theory acquires the same degrees of freedom of GR [5]. The Hamiltonian of the theory for $\lambda = 1/4$ is given by
\[ H = \int d^4x \left[ \frac{2\kappa}{\sqrt{g}} N \pi^{\mu\nu} \pi_{\mu\nu} + \sqrt{g} N V + N_{\mu} \dot{H}^{\mu} + \sigma P_{N} + \mu \pi \right], \] (18)

where $\sigma$ and $\mu$ are Lagrange multipliers. On the other hand, preservation in time of the primary constrains (11) and (16) yields to
\[ \mathcal{H} = \frac{2\kappa}{\sqrt{g}} \pi^{\mu\nu} \pi_{\mu\nu} + \sqrt{g} \mathcal{U} = 0, \] (19)
\[ \mathcal{C} = \frac{4\kappa}{\sqrt{g}} \pi^{\mu\nu} \pi_{\mu\nu} - \sqrt{g} \mathcal{W} = 0, \] (20)

where $\mathcal{U}$ and $\mathcal{W}$ are defined as follows
\[ \mathcal{U} \equiv \frac{1}{\sqrt{g}} \frac{\delta}{\delta N} \int d^4y \sqrt{g} N V = V + \frac{1}{N} \sum_{r=1}^{z} (-1)^{r} \nabla_{\mu_1...\mu_r} \left( N \frac{\partial V}{\partial (\nabla_{\mu_1...\mu_2} a_{\mu_1})} \right) \] (21)
\[ \mathcal{W} \equiv g_{\mu\nu} \mathcal{W}^{\mu\nu}, \quad \mathcal{W}^{\mu\nu} \equiv \frac{1}{\sqrt{g} N} \frac{\delta}{\delta g_{\mu\nu}} \int d^4y \sqrt{g} N V, \] (22)

where at $z = 1$ the explicit form of Eqs. (21) and (22) are given by
\[ \mathcal{U}^{(z=1)} = -\beta^{(4)} R + \alpha a_{\mu} a^{\mu} + 2a \nabla_{\mu} a^{\mu}, \] (23)
\[ \mathcal{W}^{(z=1)} = -\beta^{(4)} R - (\alpha - 3\beta) a_{\mu} a^{\mu} + 3\beta \nabla_{\mu} a^{\mu}, \] (24)

respectively. Constraints (11) and (16) end up being second class constraints. In this concern, the constraints (19)–(20) also are second class constraints and its preservation in time leads to determine the Lagrange multipliers $\sigma$ and $\mu$ [5]. Besides, as mentioned before, the constraint (19) (the Hamiltonian constraint) is of second class, in contrast with what happens in GR where the Hamiltonian constraint is of first class. On the other hand it should be noted that the bulk part of the Hamiltonian (18) does not correspond to a sum of constraints like in the GR case. In order to
address the differentiability argument first raised in [47], we assume the following asymptotic behavior of the canonical variables

\[ g_{\mu\nu} - \delta_{\mu\nu} = \mathcal{O}(1/r), \quad \partial_\mu g_{\mu\nu} = \mathcal{O}(1/r^2), \quad \pi^{\mu\nu} = \mathcal{O}(1/r^2), \quad N - 1 = \mathcal{O}(1/r), \quad \nabla_\mu N = \mathcal{O}(1/r^2), \]

consequently,

\[ K_{\mu\nu} = \mathcal{O}(1/r^2), \quad R = \mathcal{O}(1/r^3), \quad a_\mu = \mathcal{O}(1/r^2). \]

Under these asymptotic conditions a boundary term must be present in the Hamiltonian (14) and (18), which in GR can be shown to be the ADM energy [3]

\[ E_{\text{ADM}} = \oint d\Sigma_\mu \left( \partial_\nu g_{\mu\nu} - \partial_\mu g_{\nu\nu} \right). \]

The ADM energy arises only from the \( z = 1 \) terms in the potential, from the contribution \(-\beta R\) which asymptotically falls off as \( \mathcal{O}(1/r^3) \) (see Eq. (26)). We can re-express the Hamiltonian (18) as a sum of constraints plus a boundary term \(-2\alpha \Phi_N\)

\[ H = \int d^4x \left( N \mathcal{H} + N_\mu \mathcal{H}^\mu + \sigma P_N + \mu \pi \right) + \beta E_{\text{ADM}} - 2\alpha \Phi_N, \]

where \( 2\alpha \Phi_N \) is given by [17–19]

\[ \int d^4x \sqrt{g} N \left( \mathcal{U} - \mathcal{V} \right) = 2\alpha \oint d\Sigma_\mu \partial_\mu N \equiv 2\alpha \Phi_N. \]

The canonical equations of motion associated to Hamiltonian \( H \) are

\[ \dot{N} = \sigma, \]

\[ \dot{g}_{\mu\nu} = \frac{4\kappa N}{\sqrt{g}} \pi_{\mu\nu} + 2\nabla_\mu N_\nu + \mu g_{\mu\nu}, \]

\[ \dot{\pi}^{\mu\nu} = -\frac{4\kappa N}{\sqrt{g}} \left( \pi^{\mu\rho} \pi_\rho^{\nu} - \frac{1}{4} g^{\mu\nu} \pi^{\rho\sigma} \pi_{\rho\sigma} \right) - \sqrt{g} N \mathcal{W}^{\mu\nu} - 2\nabla_\rho N^{(\mu} \pi^{\nu)}_{\rho} + \nabla_\rho \left( N^{(\mu} \pi^{\nu)}_{\rho} \right) - \mu \pi^{\mu\nu}. \]

The higher dimensional formulation, provides an ideal scenario to study the coupling between fundamental interactions in nature. To do that, the starting point is to reduce the theory from 4+1 dimensions to 3+1 dimensions employing the Kaluza–Klein approach. In going from 4+1 dimensions to 3+1 dimensions throughout the work the cylindrical condition is assumed i.e, the fields do not depend on the extra coordinate \( x^4 \), thus \( \partial_4 = 0 \) (see [38, 39] for further details in the \( z = 1 \) case).

C. Dimensional reduction to 3+1 dimensions

The higher dimensional formulation, provides an ideal scenario to study the coupling between fundamental interactions in nature. To do that, the starting point is to reduce the theory from 4+1 dimensions to 3+1 dimensions employing the Kaluza–Klein approach. In going from 4+1 dimensions to 3+1 dimensions throughout the work the cylindrical condition is assumed i.e, the fields do not depend on the extra coordinate \( x^4 \), thus \( \partial_4 = 0 \) (see [38, 39] for further details in the \( z = 1 \) case).

1. The reduced theory for \( \lambda \neq 1/4 \)

Before in going into the 3+1 dimensional theory at KC point, it is instructive to present the reduced Hamiltonian, for \( \lambda \neq 1/4 \), including all fields i.e, \( \{ \gamma_{ij}, A_i, \phi \} \), where \( \gamma_{ij} \) is the \( \text{tri–dimensional metric of the space–like Riemannian leaves, } A_i \text{ a vector field and } \phi \text{ a scalar field (the so–called dilaton field) introduced by the Kaluza–Klein mechanism.} \)

These fields come from the following ansatz

\[ g_{\mu\nu} = \begin{pmatrix} \gamma_{ij} + A_i A_j & \phi A_j \\ \phi A_i & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \gamma^{ij} & -A^j \\ -A^i & \frac{1}{\phi} + A_k A^k \end{pmatrix}. \]
where $\gamma^{ij}$ are the components of the inverse of $\gamma_{ij}$ and $A^i = \gamma^{ij}A_j$. The reduced Hamiltonian is then the following,

$$H = \int d^3x \left\{ \frac{2\kappa N}{\sqrt{-\phi}} \left[ \phi^2 p^2 + p^i p_i + \frac{p^i p_i}{2\phi} - \frac{\lambda}{4\lambda} \left( p^i \gamma_{ij} + p \phi \right)^2 \right] - N \sqrt{-\phi} \left[ \beta R + \alpha a_i a^i \right] + \Lambda \tilde{H} + \Lambda_j \mathcal{H}^j \right\} + \beta E_{\text{ADM}}, \quad \tag{34}$$

where $p^{ij}$, $p^i$ and $p$ are the conjugate momenta associated to $\gamma_{ij}$, $A_i$ and $\phi$, respectively. To obtain the above result the following decomposition for the four dimensional Ricci scalar $R$ was used

$$R = R - \frac{\phi}{4} F_{ij} F^{ij} - \frac{2}{\sqrt{-\phi}} \nabla_i \nabla^i \sqrt{-\phi}, \quad \tag{35}$$

and the field strength $F_{ij}$ is defined as follows $[57]

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i. \quad \tag{36}$$

Furthermore, $\tilde{H}$ and $\mathcal{H}^i$ are given by

$$\mathcal{H}^i \equiv -2\nabla_j p^{ij} - p^j \gamma^{kl} F_{jk} + p \gamma^{ik} \partial_k \phi = 0, \quad \tilde{H} \equiv \partial_i p^i = 0, \quad \tag{37}$$

which correspond to the dimensional reduction of the constraints (12) As can be seen, the momentum constraint in 3+1 dimensions $\mathcal{H}^i$ acquires a new term coming from the vector–gauge field. On the other hand, the constraint given by $\tilde{H}$ is a first class constraint, generator of local gauge symmetries on $A_i$ (the equivalent 3+1 dimensional Gauss's law). Taking into account the above considerations, the infinitesimal transformation laws (2)–(4) break down into

$$\begin{align*}
\delta \gamma_{ij} &= \partial_i \xi^k \gamma_{jk} + \partial_j \xi^k \gamma_{ik} + \xi^k \partial_k \gamma_{ij} + f \gamma_{ij}, \\
\delta A_i &= \partial_i \xi^k A_k + \xi^k \partial_k A_i + \xi^k \gamma_{ki} + f A_i + f A_i, \\
\delta N &= \xi^k \partial_k N + f N + f N, \\
\delta A_i &= \partial_i \xi^k A_k + \xi^k \partial_k A_i + f A_i + \partial_i \xi^4, \\
\delta \Lambda &= \xi^i \partial_i \Lambda + \xi^i A_i + \xi^4 + f \Lambda + f \Lambda, \\
\delta \phi &= \xi^i \partial_i \phi + f \phi. \quad \tag{38}\tag{39}\tag{40}\tag{41}\tag{42}\tag{43}
\end{align*}$$

As can be observed, the Eqs. (38)–(43) are exactly the same infinitesimal transformation laws as for the 3+1 dimensional Hořava–Lifshitz gravity theory. However, the function $N_i$ is no longer the proper shift function after the reduction of the theory from 4+1 to 3+1 dimensions, since it does not transform properly under FDiff. In fact, from (39) and (40) one can deduce that the correct shift function, which together with the lapse $N$ describes the embedding of the leaves in the $3 + 1$ foliation is $[58]

$$\Lambda_i \equiv N_i - N_4 A_i, \quad \tag{44}$$

where $N_i$ and $N_4$ arise from the dimensional reduction of $N_\mu$. On the other hand, the expression (41) shows that the vector field $A_i$, transforms just as a vector–gauge field. This fact is encoded in the last term of (41). Finally, $\Lambda \equiv N_4$ is a Lagrange multiplier associated with the first class constraint, generator of the local gauge symmetries on $A_i$. So, it is clear that the dimensional reduction allows us to recover the 3+1 dimensional FDiff plus a local gauge symmetry. Besides, from (43) it is evident that $\phi$ transforms as a scalar under space–like diffeomorphisms and time reparametrizations.

2. The reduced theory at the K. C. point: $\lambda = 1/4$

Now, for our purpose we shall consider from now on that the dilaton field is in its ground i.e, $\phi = 1$ ($p = 0$), thus we are left with the remaining degrees of freedom, the 3–dimensional metric $\gamma_{ij}$ of the space–like Riemannian leaves and the vector field $A_i$. So, under these assumptions the Kaluza–Klein ansatz (33) becomes

$$g_{\mu\nu} = \begin{pmatrix} \gamma_{ij} + A_i A_j & A_j \\ A_i & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \gamma^{ij} & -A^j \\ -A^i & 1 + A_k A^k \end{pmatrix}. \quad \tag{45}$$
The infinitesimal transformation law \((38)-(43)\) remains the same, but in this case it is clear that \((43)\) is absent. So, the Hamiltonian is given by

\[
H = \int d^3x \left\{ \frac{N}{\sqrt{g}} \left[ 2\kappa \left( p^i p_{i\ell} + \frac{p^i p_{i\ell}}{2} \right) + \gamma V \right] + \Lambda H + \Lambda_j \mathcal{H}^j + \sigma P_N + \mu P \right\} + \beta \bar{E}_{ADM}, \tag{46}
\]

subject to the constraints

\[
\mathcal{H}^i = -2\nabla_i p^j - p^i \gamma^j k F_{jk} = 0, \tag{47}
\]
\[
\bar{H} = \partial_t p^i = 0, \tag{48}
\]
\[
P = \gamma_{ij} p^{ij} = 0, \tag{49}
\]

where \((49)\) comes from the dimensional reductions of \((16)\). As mentioned before, this primary second class constraint arises when the theory is placed at the KC point. It should be noted that the constraint given by Eq. \((49)\) is just the same constraint appearing in pure anisotropic gravity formulation in 3+1 dimensions at the KC point (in that case the KC point is given by \(\lambda = 1/3\)). This fact indicates that if we turn off the vector–gauge \(A_i\), the original Hamiltonian formulation at the KC point is regained \([5, 21]\). Next, the second class constraints \((19)\) and \((20)\) have the following reduced form

\[
\bar{\mathcal{H}} = \frac{2\kappa}{\sqrt{g}} \left( p^i p_{i\ell} + \frac{p^i p_{i\ell}}{2} \right) + \sqrt{g} \bar{U} = 0, \tag{50}
\]
\[
\bar{C} = \frac{3\kappa}{\sqrt{g}} \left( p^i p_{i\ell} + \frac{p^i p_{i\ell}}{6} \right) - \sqrt{g} \bar{W} = 0. \tag{51}
\]

where at \(z = 1\) the explicit form of \(\bar{U}\) and \(\bar{W}\) in Eqs. \((50)\) and \((51)\) are given by

\[
\bar{U}^{(z=1)} = -\beta R + \frac{\beta}{4} F_{ij} F^{ij} + \alpha a_i a^i + 2\alpha \nabla_i a^i, \tag{52}
\]
\[
\bar{W}^{(z=1)} = -\frac{1}{2} \beta R - \frac{1}{8} \beta F^{lm} F_{lm} - \left( \frac{\alpha}{2} - 2\beta \right) a^i a_i + 2\beta \nabla^i a_i, \tag{53}
\]

respectively. It should be noted that the barred objects represent the dimensional reduced ones. Finally, the \(E_{ADM}\) and the potential \(V\) become

\[
V = \bar{V}(\gamma_{ij}, A_i, N), \tag{54}
\]
\[
\bar{E}_{ADM} = \oint d\Sigma_I (\partial_I g_{ij} - \partial_j g_{ij}), \tag{55}
\]

where roman indices run over \(i, j = 1, 2, 3\). Along the above lines we have kept the potential of the theory unspecified. We now introduce the terms of the potential. We give explicitly the terms quadratic on the Riemann tensor and on the vector field \(A_\mu\), we will comment on the the complete set of terms of the potential later on. The most general and complete potential of the theory at all energy scales, quadratic on the Riemann tensor and the vector field \(A_\mu\) is given by (for the 3+1 dimensional case see \([43, 44]\))

\[
V^{(z=1)} = -\beta R - \alpha a_\mu a^\mu, \tag{56}
\]
\[
V^{(z=2)} = -\beta_2 R^2 - \beta_3 R_{\mu\nu} R^{\mu\nu} - \alpha_2 \nabla^\mu a^\nu \nabla_\mu a_\nu - \alpha_3 R\nabla_\mu a^\mu - \kappa_1 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}, \tag{57}
\]
\[
V^{(z=3)} = -\beta_4 \nabla_\mu R \nabla^\nu R - \beta_3 \nabla_\mu R_{\nu\rho} \nabla^\nu R^{\mu\rho} - \alpha_4 \nabla^2 a_\mu \nabla_\nu a^\nu - \alpha_3 \nabla^2 R \nabla_\mu a^\mu - \kappa_2 \nabla_\theta C_{\mu\nu\rho\sigma} \nabla^\theta C^{\mu\nu\rho\sigma}, \tag{58}
\]
\[
V^{(z=4)} = -\beta_6 \nabla^2 R \nabla^2 R - \beta_5 \nabla^2 R_{\mu\nu} \nabla^2 R^{\mu\nu} - \alpha_6 \nabla^2 \nabla_\mu a_\nu \nabla^2 \nabla_\mu a^\nu - \alpha_5 \nabla^4 R \nabla_\mu a^\mu - \kappa_3 \nabla_\theta \nabla_\phi C_{\mu\nu\rho\sigma} \nabla^\theta C^{\mu\nu\rho\sigma}, \tag{59}
\]

where \(\beta\)'s, \(\alpha\)'s and \(\kappa\)'s are coupling constants and \(\nabla^2 = \nabla_\mu \nabla^\mu\). These expressions include only the independent quadratic terms. There are terms, for instance \(R \nabla_\mu \nabla_\nu R^{\mu\nu}\), where after an integration by parts and using the contracted Bianchi’s identity \(\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R\), becomes \(\nabla_\mu R^{\mu\nu}\). There are also boundary terms which do not contribute to the action at the quadratic level and the procedure to detect them and to remove them from the above list is by employing integration by parts and some identities. We observe that in order to introduce the gauge vector field interaction via the Kaluza–Klein procedure, the highest order in spatial derivatives must be increased up to order eight. On the other hand, we have the 3+1 dimensional pure anisotropy gravity formulation \([1]\) where the
maximum order in spatial derivatives is six. As was mentioned, this fact is necessary to assure a power–counting renormalizable theory at the UV fixed point. Another interesting point to be stressed here, is the presence of the conformal Weyl tensor \( C_{\mu\nu\rho\sigma} \) absent in the 3+1 dimensional formulation. As it is well–known, the conformal Weyl tensor is identically zero in three dimensions, so the Riemann tensor can be expressed solely in terms of the Ricci tensor. It should be noted that the reduced potential \( \phi \) contains up to 8 spatial derivatives and exhausts all marginal and relevant operators for both, the gravitational and the gauge vector sector. As we shall see in the next sections, it turns out that the Kaluza–Klein mechanism we have introduced, is the correct approach to couple the gauge vector interaction to the Hořava–Lifshitz gravity since a free ghost and power–counting renormalizable theory is achieved.

In this section we perform the perturbative quantum analysis of the theory with Hamiltonian \( (46) \). We provide a detailed analysis and discussion on the solution of the second class constraints which are a system of elliptic partial differential equations of eighth–order. Besides, we obtain the propagator of the degrees of freedom and show that the theory is power–counting renormalizable.
A. Perturbative analysis

Let us start with the perturbative study. We will show that despite the full $z = 4$ potential contains many interaction terms, the power–counting renormalization approach can be rigorously performed (see [21] for the proof that the pure gravitational Hořava–Lifshitz at the KC point in $3 + 1$ dimensions is power–counting renormalizable). So, to do this we introduce the following perturbations around a background without electromagnetic interaction

$$\gamma_{ij} = \delta_{ij} + \epsilon h_{ij}, \quad p^{ij} = \epsilon \partial^{ij} A_i = \epsilon n_i, \quad \Lambda = \epsilon n_4, \quad N = 1 + \epsilon n,$$

while for the vector field $A_i$ and its conjugate momentum $p^i$ one has

$$A_i = \epsilon \xi_i, \quad p^i = \epsilon \chi^i.$$  

As it is usual we will employ the orthogonal/longitudinal decomposition [3]

$$h_{ij} = h_{ii}^{TT} + \frac{1}{2} \left( \delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) h^T + \partial_i h^T_j + \partial_j h^T_i,$$

where the traceless transverse modes $h_{ii}^{TT}$ are subject to $\partial_i h_{ii}^{TT} = h_{ii}^{TT} = 0$ and similarly for its conjugate momentum $p^{ij}$. Furthermore, we impose the transverse gauge

$$\partial_i h_{ij} = 0.$$  

So, combining Eqs. (72) and (73), it follows that the longitudinal modes of the field $h_{ij}$ vanish i.e.,

$$\partial^2 h^L_j = 0 \Rightarrow h^L_j = 0.$$  

Next, at linear order in perturbations and using the fact $P_N = \epsilon P_n = 0$ the first class constraint (47) provides

$$\partial_i \vartheta^{ij} = 0.$$  

As happened with $h_{ij}$, the condition expressed by (75) eliminates the longitudinal part of $\vartheta^{ij}$. Besides, taking advantage from the second class constraint (49) one can determine that $\vartheta^{ij}$ is traceless, that is, $\vartheta^T = 0$. Therefore, we are left with the following set of canonical variables for the gravitational sector: \{h_{ij}^{TT}, \vartheta_{ij}^{TT}, h^T, n\}. Now, from the second class constraints (50)–(51) at linear order in perturbation one can obtain valuable information about the variables $h^T$ and $n$. So, from the mentioned expressions one gets

$$\frac{1}{8} \left[ - \left( 3 \hat{\beta}_5 + 8 \hat{\beta}_6 \right) \partial^8 + \left( 3 \hat{\beta}_3 + 8 \hat{\beta}_4 \right) \partial^6 - \left( 3 \hat{\beta}_1 + 8 \hat{\beta}_2 \right) \partial^4 + \beta \partial^2 \right] h^T + \frac{1}{2} \left[ \alpha_5 \partial^8 + \alpha_3 \partial^6 + \alpha_1 \partial^4 + \beta \partial^2 \right] n = 0,$$

and

$$\left[ - \alpha_6 \partial^8 + \alpha_4 \partial^6 - \alpha_2 \partial^4 + \alpha \partial^2 \right] n + \frac{1}{2} \left[ \alpha_5 \partial^8 + \alpha_3 \partial^6 + \alpha_1 \partial^4 + \beta \partial^2 \right] h^T = 0,$$

where we have defined

$$\hat{\beta}_1 \equiv \beta_1 + 2 \kappa_1, \quad \hat{\beta}_2 \equiv \beta_2 + \frac{2}{3} \kappa_1, \quad \hat{\beta}_3 \equiv \beta_3 + 2 \kappa_2, \quad \hat{\beta}_4 \equiv \beta_4 + \frac{2}{3} \kappa_2, \quad \hat{\beta}_5 \equiv \beta_5 + 2 \kappa_3, \quad \hat{\beta}_6 \equiv \beta_6 + \frac{2}{3} \kappa_3,$$

and $\partial^{2m}$ being the Laplacian operator powered to $m$ i.e, $\partial^{2m} = (\partial^2)^m$. As it is observed, the Eqs. (76)–(77) constitute a couple system in the canonical variables $h^T$ and $n$. To solve this set of equations we shall present the results in a compact form as follows

$$\phi = \begin{pmatrix} h^T \\ n \end{pmatrix}, \quad M = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix},$$

where

$$D_1 \equiv \frac{1}{8} \left[ - \left( 3 \hat{\beta}_5 + 8 \hat{\beta}_6 \right) \partial^8 + \left( 3 \hat{\beta}_3 + 8 \hat{\beta}_4 \right) \partial^6 - \left( 3 \hat{\beta}_1 + 8 \hat{\beta}_2 \right) \partial^4 + \beta \partial^2 \right],$$

$$D_2 \equiv \frac{1}{2} \left[ \alpha_5 \partial^8 + \alpha_3 \partial^6 + \alpha_1 \partial^4 + \beta \partial^2 \right], \quad D_3 \equiv - \alpha_6 \partial^8 + \alpha_4 \partial^6 - \alpha_2 \partial^4 + \alpha \partial^2.$$
Therefore, with the potential given in (60)–(63), the constraints \(\mathcal{H}\) and \(\mathcal{C}\) at linear order become

\[ M\phi = 0. \tag{82} \]

The first row of (82) represents the constraint \(\mathcal{C}\) and the second row corresponds to \(\mathcal{H}\). It should be noted that the set of equations given by (82) turns out to be an eighth–order elliptic partial differential equations in \(h^T\) and \(n\). However, to ensure this structure it is necessary to impose some conditions on the positivity of the coupling constants matrix. To solve the above system it is desirable to decouple \(h^T\) from \(n\). To achieve it, we multiply the Eq. (82) by

\[
\begin{pmatrix}
-\mathcal{D}_2 & \mathcal{D}_3 \\
-\mathcal{D}_2 & \mathcal{D}_1
\end{pmatrix},
\tag{83}
\]

from the left, getting a diagonal matrix acting on \(\phi\) as follows

\[ L\phi = 0, \quad L \equiv \mathcal{D}_1 \mathcal{D}_3 - \mathcal{D}_2^2. \tag{84} \]

At this stage a couple of comments are in order. First, Eq. (84) represents two decoupled equations for \(h^T\) and \(n\). Secondly, this set of decoupled equations are the same, what is more both have the same boundary conditions. Now, in general to find out the solution of (84) one can consider the generic case when the operator \(L\) is an eighth–order polynomial operator in the Laplacian, thus we can factor out Eq. (84) as follows

\[ L = K (\partial^2 - z_1) Q^{(7)} (\partial^2), \tag{85} \]

being \(Q^{(7)}(u)\) a polynomial of seventh–order on \(u\) and \(z_1\) is a root of the operator \(L\) as a polynomial on the Laplacian. To solve the above problem, we can tackle it from two perspectives: i) treating the problem as an eigenvalue problem subject to suitable boundary conditions (see Eqs. (25)–(26) of the present study) or ii) by means of the resolvent of operator \(\partial^2 - z_1\). In any case, we require the following condition

\[ K = \frac{1}{8} \left[ \alpha_6 \left(8\hat{a}_6 + 3\hat{a}_5\right) - 2\alpha_5^2 \right] \neq 0, \tag{86} \]

in order to have an elliptic operator of maximum order. Although the ellipticity property is sufficient at this point, the requirement of being of maximum order will become important later on. Next, combining Eqs. (84)–(85) we arrive to

\[ \partial^2 Q^{(7)} (\partial^2) \phi = z_1 Q^{(7)} (\partial^2) \phi. \tag{87} \]

Thereby from (86) it is clear that \(Q^{(7)} (\partial^2) \phi\) is an eigenfunction of the Laplacian \(\partial^2\). As we are focused on the asymptotically flat case, the spatial domain of the above problem is given by the whole \(\mathbb{R}^3\) space subject to the following boundary condition on \(\phi\)

\[ \phi \bigg|_{+\infty} = 0. \tag{88} \]

Moreover, the Laplace operator has not non–vanishing eigenfunctions falling asymptotically to zero in all angular directions on a noncompact domain. Hence the only solution of the equation (87) satisfying the above boundary conditions is

\[ Q^{(7)} (\partial^2) \phi = 0. \tag{89} \]

Repeating the above procedure and taken into to account the mentioned arguments, it is straightforward to prove that at linear order in perturbations the unique solution of the problem (82) is given by

\[ h^T = n = 0. \tag{90} \]

On the other hand, the above solution can be obtained by analyzing the resolvent of the operator \(\partial^2 - z_1\) in Eq. (85). By assuming \(z_1 \in \mathbb{C}\) and the operator \(\partial^2 - z_1\) acting on the space of the test function \(\psi\) satisfying the asymptotically flat boundary condition, whose domain is the full \(\mathbb{R}^3\), the problem can be written in the following way

\[ (\partial^2 - z_1) \psi = 0. \tag{91} \]

As it is well–known the Laplacian operator has a continuum spectrum given by \((-\infty, 0]\) without eigenvalues associated to it. In considering the asymptotic behavior the resolvent \((\partial^2 - z_1)^{-1}\) always exist for any \(z_1\). Notwithstanding, it behaves differently depending on whether \(z_1\) belongs to the spectrum or not. Then we can distinguished between two cases

\[ P \equiv \begin{pmatrix}
\mathcal{D}_3 & -\mathcal{D}_2 \\
-\mathcal{D}_2 & \mathcal{D}_1
\end{pmatrix}, \quad Q \equiv \begin{pmatrix}
\alpha_6 & \beta_6 \\
\beta_5 & \alpha_5
\end{pmatrix}, \]

with \(\alpha_5, \beta_5, \alpha_6, \beta_6\) being real numbers. To achieve it, we multiply the Eq. (82) by

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\begin{pmatrix}
\mathcal{D}_3 & -\mathcal{D}_2 \\
-\mathcal{D}_2 & \mathcal{D}_1
\end{pmatrix},
\tag{83}
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from the left, getting a diagonal matrix acting on \(\phi\) as follows

\[ L\phi = 0, \quad L \equiv \mathcal{D}_1 \mathcal{D}_3 - \mathcal{D}_2^2. \tag{84} \]

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in order to have an elliptic operator of maximum order. Although the ellipticity property is sufficient at this point, the requirement of being of maximum order will become important later on. Next, combining Eqs. (84)–(85) we arrive to

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Thereby from (86) it is clear that \(Q^{(7)} (\partial^2) \phi\) is an eigenfunction of the Laplacian \(\partial^2\). As we are focused on the asymptotically flat case, the spatial domain of the above problem is given by the whole \(\mathbb{R}^3\) space subject to the following boundary condition on \(\phi\)

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Moreover, the Laplace operator has not non–vanishing eigenfunctions falling asymptotically to zero in all angular directions on a noncompact domain. Hence the only solution of the equation (87) satisfying the above boundary conditions is

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Repeating the above procedure and taken into to account the mentioned arguments, it is straightforward to prove that at linear order in perturbations the unique solution of the problem (82) is given by

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On the other hand, the above solution can be obtained by analyzing the resolvent of the operator \(\partial^2 - z_1\) in Eq. (85). By assuming \(z_1 \in \mathbb{C}\) and the operator \(\partial^2 - z_1\) acting on the space of the test function \(\psi\) satisfying the asymptotically flat boundary condition, whose domain is the full \(\mathbb{R}^3\), the problem can be written in the following way

\[ (\partial^2 - z_1) \psi = 0. \tag{91} \]

As it is well–known the Laplacian operator has a continuum spectrum given by \((-\infty, 0]\) without eigenvalues associated to it. In considering the asymptotic behavior the resolvent \((\partial^2 - z_1)^{-1}\) always exist for any \(z_1\). Notwithstanding, it behaves differently depending on whether \(z_1\) belongs to the spectrum or not. Then we can distinguished between two cases
• $z_1 \notin (-\infty, 0]$: In this case the resolvent $(\partial^2 - z_1)^{-1}$ is a bounded operator ensuring automatically $\psi = 0$.

• $z_1 \in (-\infty, 0]$: Here the inverse operator $(\partial^2 - z_1)^{-1}$ exists but it is an unbounded operator, however since it is applied to the zero of the space of functions, it yields the same result as in the first case i.e., $\psi = 0$.

It turns out that the formulation of the theory at KC point due to its constraint structure and by setting a suitable boundary conditions in order to have a differentiable Hamiltonian, allows us to eliminate both $h^T$ and $n$ from the set of canonical variables. On the other hand, when the coupling constant $\lambda$ is different from $1/4$ the theory presents only one second class constraint, which allows to express $n$ in terms of $h^T$, therefore the latter one is a propagating degree of freedom of the theory [38]. To ensure the uniqueness of the solution we have assumed that the operator $L$ is strictly elliptic. In this regard, the theory in the IR fixed point ($z = 1$), under particular values of the coupling constants, is equivalent to GR at linearized level [38] (see [5, 21] for the 3+1 case). This fact can be reached if and only if the fourth–order term of $L$, associated with the couplings $[59]$ of the lowest order operators ($z = 1$), $\beta$ and $\alpha$ of the theory, is nonzero

$$\beta(\alpha - 2\beta) \neq 0, \quad \beta = 1 .$$

(92)

Next, as was mentioned earlier, the reduced theory posses two first class constraints (47)–(48) and four second class constraints $P_N = 0$, (49) and (50)–(51). Then, the preservation in time of the second class constraints (50)–(51) leads to determine the Lagrange multipliers $\sigma$ and $\mu$. Thus, at linear order in perturbations the constraints (50)–(51) lead to the following problem

$$\mathcal{M} \begin{pmatrix} \mu \\ \sigma \end{pmatrix} = 0 .$$

(93)

Following the same steps as before, the unique solution of the problem given by Eq. (93) is

$$\mu = \sigma = 0 .$$

(94)

Now, from the equation of motion (31) in its 3+1 dimensional fashion and under linearized perturbative scheme, we can solve for the for $n_i$ (the perturbation of the shift function $\Lambda_i$) as follows

$$\dot{h}_{ij}^{TT} = \partial_i^{TT} + \partial_i n_j + \partial_j n_i + \mu \delta_{ij} .$$

(95)

Taking the divergence of the above equation and using (73) and (93) one gets

$$\partial^2 n_j + \partial_i \partial_j n_i = 0 ,$$

(96)

combining the previous equation with the asymptotically flat boundary conditions, provides the following solution

$$n_i \bigg|_{+\infty} = 0 .$$

(97)

Finally, the above analysis reveals that the gravitational sector is being described by the pair of conjugate canonical variables: $\{h_{ij}^{TT}, p_{ij}^{TT}\}$ at linear order in perturbations. On the other hand, for the gauge vector sector one can impose

$$\partial_i \xi_i = 0 ,$$

(98)

the equivalent of the so–called Coulomb gauge. The decomposition of the perturbative vector gauge in transverse/longitudinal parts reads

$$\xi_i = \xi_i^T + \partial_i \xi_i^L ,$$

(99)

where as usual the transverse part is subject to $\partial_i \xi_i^T = 0$. So, from Eqs. (98)–(99) it is clear that

$$\partial^2 \xi_i^L = 0 \Rightarrow \xi_i^L = 0 .$$

(100)

Next, from (49) the longitudinal modes of the momentum $\chi_i$ are eliminated leaving only the transverse modes $\chi_i^T$. Therefore, the spectrum of the vector gauge field is represented by the set: $\{\xi_i^T, \chi_i^T\}$. 


B. The Hamiltonian and the physical modes

As was shown above after solving the full set of constraints, the anisotropic gravity–vector gauge field coupling in the transverse gauge \((73)\) and \((98)\), is described by the spectrum: \(\{h_{ij}^{TT}, \vartheta_{ij}^{TT}, \xi_i^T, \chi_i^T\}\). This spectrum coincides with the Einstein–Maxwell covariant theory in the linear perturbative scheme \([38, 39]\). Of course, these modes are not the relativistic graviton and photon as the Einstein–Maxwell theory predicts, although the propagating degree of freedom are just the same two traceless transverse tensorial modes and two transverse vector gauge modes. Hence, the reduced Hamiltonian is obtained from the Hamiltonian \((46)\) after substituting in the second order Hamiltonian density the linear version of the constraints. Then, it is expressed by

\[
H_{\text{RED}} = \int d^3x \left[ 2\kappa \vartheta_{ij}^{TT} \vartheta_{ij}^{TT} + \frac{1}{4} h_{ij}^{TT} \vartheta_{ij}^{TT} + \kappa \chi_i^T \chi_i^T + \frac{1}{2} \xi_i^T \vartheta_{ij}^{TT} \right],
\]

where the selfadjoint operator \(V\) is given by

\[
V = -\beta \partial^2 - \hat{\beta}_1 \partial^4 + \hat{\beta}_3 \partial^6 - \hat{\beta}_5 \partial^8. \tag{102}
\]

We remark that both the gravitational and gauge vector sectors have the same partial differential operator \((102)\) acting on them. This property is a consequence of starting with a 4+1 formulation and the Kaluza–Klein dimensional reduction. Otherwise, the operators would have been completely different.

To ensure a strictly positive definite and well–posed Hamiltonian, it is necessary to impose some constraints on the coupling constants \(\beta, \hat{\beta}_1, \hat{\beta}_3\) and \(\hat{\beta}_5\) (we are assuming \(\kappa > 0\)), thus requiring \(V > 0\). Besides, the dominant term at the lower energy range demands \(\beta > 0\) whilst from the high energy scale it follows \(\hat{\beta}_5 < 0\). So, to meet the above requirement, it remains to analyze the sign of the constants \(\hat{\beta}_1\) and \(\hat{\beta}_3\). To do so, we will introduce the Fourier transform of the operator \((102)\). In all cases, the Fourier transform \(\tilde{V}\) is positive if and only if the operator \((102)\) is positive, that is, its spectrum is positive. If we denote the Fourier transform of the operator \(V\) as \(\tilde{V}\) and consider \(\tilde{V}\) to be negative for some positive argument, then \(\tilde{V}\) has a minimum \(-M, M < 0\) (see figure 1). On this condition, the operator \(V\) has continuous spectrum \([-M, +\infty)\). In fact, consider the equation

\[
(V + m)w(x) = G(x), \tag{103}
\]

if \(-m\) does not belong to the interval \([-M, +\infty)\) then the Fourier transform of the equation allows to obtain the transform \(\tilde{w}\)

\[
\tilde{w} = \frac{\tilde{G}}{\tilde{V} + m}, \tag{104}
\]

and assuming \(G\) on \(L^2\), \(\tilde{w}\) also belongs to \(L^2\) since the denominator is bounded away from 0 and well behaved at infinity. Consequently, \(w\) belongs to \(L^2\). \(V + m\) is then bounded and \(-m\) belongs to the resolvent set of the operator \(V\). If \(-m\) belongs to the interval \([-M, +\infty)\), \(\tilde{w}\) has isolated poles and for generic \(G\) it does not belong to \(L^2\), neither does \(w\). The inverse of \((V + m)\) is only defined on a domain of \(L^2\) and then \(-m\) belongs to the continuous spectrum of \(V\).

![FIG. 1: The potential \(\tilde{V}\) with negatives values for in the region \(u > 0\).](image)

So, we study all possibilities as follows
1. $\hat{\beta}_1 \leq 0$ and $\hat{\beta}_3 \leq 0$, which automatically ensures $V \geq 0$ in the operatorial sense. Where we are assuming $V$ acting on a space of functions with compact support.

2. $\hat{\beta}_1 > 0$ and $\hat{\beta}_3 > 0$. In this case we factorize $V$ as

$$\tilde{V}(k^2) = k^2 \left( |\hat{\beta}_5| k^6 - \hat{\beta}_3 k^4 - \hat{\beta}_1 k^2 + \beta \right),$$

(105)

where we have expressed (102) in the Fourier space representation i.e., $\tilde{V}(k^2)$. The Eq. (105) can be considered as a polynomial of fourth–order on $k^2$. Now, making $k^2 = u$, Eq. (105) becomes

$$\tilde{V}(u) = u \left( |\hat{\beta}_5| u^3 - \hat{\beta}_3 u^2 - \hat{\beta}_1 u + \beta \right).$$

(106)

The cubic polynomial in the right hand side of (106) has two possibilities, namely: i) 3 real roots (2 positives and 1 negative) or ii) 2 complex roots and 1 real negative root. Imposing $\Delta < 0$. Ensuring the positivity of $V$, means preventing it from taking negative values in the range $u \geq 0$. So, the desired solution is given by the case ii). It is only possible if the discriminant of the cubic equation satisfies $\Delta < 0$. Explicitly it reads

$$\Delta = \left( 18|\hat{\beta}_5| |\hat{\beta}_1 + 4|\hat{\beta}_3|^2 \right) \hat{\beta}_3 \beta + \left( \hat{\beta}_3^2 + 4|\hat{\beta}_5|\hat{\beta}_1 \right) \hat{\beta}_1^2 - 27|\hat{\beta}_5|^2 \beta^2 < 0.$$  

(107)

3. $\hat{\beta}_1 > 0$ and $\hat{\beta}_3 \leq 0$. In this case, Eq. (105) reads

$$\tilde{V}(u) = u \left( |\hat{\beta}_5| u^3 + |\hat{\beta}_3| u^2 - \hat{\beta}_1 u + \beta \right).$$

(108)

Here the cubic polynomial in (108) posses: i) 3 real roots (2 positives and 1 negative) or ii) 2 complex roots and 1 real negative root. Again of the case ii) meet the desire solution to our problem. Then, to assure this solution the discriminant

$$\Delta = - \left( 18|\hat{\beta}_5| |\hat{\beta}_1 + 4|\hat{\beta}_3|^2 \right) |\hat{\beta}_3| \beta + \left( |\hat{\beta}_3|^2 + 4|\hat{\beta}_5| \hat{\beta}_1 \right) \hat{\beta}_1^2 - 27|\hat{\beta}_5|^2 \beta^2 < 0.$$  

(109)

4. $\hat{\beta}_1 \leq 0$ and $\hat{\beta}_3 > 0$. Finally, in this case

$$\tilde{V}(u) = u \left( |\hat{\beta}_5| u^3 - \hat{\beta}_3 u^2 + |\hat{\beta}_1| u + \beta \right)$$

(110)

the same situation occurs as in 2) and 3) i.e., i) 3 real roots (2 positives and 1 negative) or ii) 2 complex roots and 1 real negative one. Imposing $\Delta < 0$ to guarantee ii), one has

$$\Delta = - \left( 18|\hat{\beta}_5| |\hat{\beta}_1 + 4|\hat{\beta}_3|^2 \right) |\hat{\beta}_3| \beta + \left( |\hat{\beta}_3|^2 - 4|\hat{\beta}_5| |\hat{\beta}_1| \right) |\hat{\beta}_1|^2 - 27|\hat{\beta}_5|^2 \beta^2 < 0.$$  

(111)

The figure 2 depicts the behavior of $\tilde{V}$ against $u$ for the mentioned case (2 complex and 1 real negative roots) discussed before. As it is observed, in this case $\tilde{V}$ has a global minimum denoted by $\tilde{V}_0$ which is always negative as it is required for our purposes. Besides, there are not global maximum. So, putting together all the above conditions on the coupling constants, the requirements in order to have a positive operator $V$ are:

$$\alpha \neq 2\beta, \quad \beta > 0, \quad \hat{\beta}_5 < 0,$$

(112)

and for the other couplings either they satisfy condition 1. or they do not satisfy 1. but the discriminant $\Delta < 0$. Under these conditions the operator $V$ is an elliptic positive operator.
C. The propagator of the Hamiltonian spectrum

Once the consistency of the set of first and second class constraints was determined, the positivity of the Hamiltonian assured and the physical degrees of freedom propagated by theory determined, we can establish the propagator of the independent physical modes. This information together with the knowledge of the general structure of the interactions will allow us the computation of the superficial degree of divergence of 1PI diagrams and consequently to determine if the theory is power–counting renormalizable.

To compute the propagator we write the path integral in terms of the reduced phase space as follows

\[ Z_0 = \int \prod_{i \leq j} D^{TT} h_{ij} \, D^{TT} \xi_k \, D^{TT} \chi_k \exp \left[ i \int dtd^3x \left( \xi_k \mathcal{H}^{TT}_{ij} \xi_k - \mathcal{H}_{RED} \right) \right], \]  

where the reduced Hamiltonian density can be obtained from Eq. (101). Performing a Gaussian integration in \( \xi_k \) and \( \chi_k \) and replacing the reduced Hamiltonian density, the path integral (113) becomes

\[ Z_0 = \int \prod_{i \leq j} D^{TT} h_{ij} \exp \left[ i \int dtd^3x \left( \frac{1}{2\kappa} h_{ij}^{TT} \xi_k^{TT} \xi_k - \mathcal{H}_{RED} \right) \right] \int \prod_k D^{TT} \chi_k \exp \left[ i \int dtd^3x \left( \frac{1}{2\kappa} \xi_k^{TT} \xi_k - \mathcal{H}_{RED} \right) \right]. \]

Finally, the full propagator (Green function) of the physical modes for the anisotropic–gravitational sector is given by

\[ \langle h_{ij}^{TT} h_{kl}^{TT} \rangle = \frac{\mathcal{P}_{ijkl}^{TT}}{\frac{\omega^2}{2\kappa} - \beta_1 k^2 + \beta_2 k^4 + \beta_3 k^6 + \beta_5 k^8}, \]

while for the vector–gauge field sector is expressed by

\[ \langle \xi_i^{TT} \xi_j^{TT} \rangle = \frac{\theta_{ij}}{\frac{\omega^2}{2\kappa} - \beta_1 k^2 + \beta_2 k^4 + \beta_3 k^6 + \beta_5 k^8}, \]

where

\[ \mathcal{P}_{ijkl}^{TT} = \frac{1}{2} (\theta_{ik} \theta_{jl} + \theta_{il} \theta_{jk} - \theta_{ij} \theta_{kl}), \quad \theta_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}. \]

At this point we notice the following facts. First, with respect to the propagator associated with the gravitational sector, its structure and behavior is just the same as the original Hořava proposal’s, except that in the denominator there is a new contribution \( \beta_5 k^8 \) inherited from the Kaluza–Klein dimensional reduction. Secondly, at high \( \omega \) and \( \hat{k} \) the theory is dominated by the \( z = 4 \) mode \( (\omega^2/2k + \beta_5 k^8)^{-1} \). Now, the main aim is to analyze the power–counting renormalizability of the theory. In this concern, we must look at the possible divergences in the UV regime...
coming from the interactions. The interaction terms compromise a pure gravitational interaction, a pure vector–gauge interaction and a mixture between them. To address this point, we shall do a qualitatively study of the structure of the interactions, being necessary to go beyond the linear order in perturbations. In turn, this requires solving the second class constraints at higher order in perturbations, since in principle the first class constraints, can be treated by the usual quantization techniques for gauge systems. The second class constraints determine \( h^T \) and \( n \). We use a perturbative approach in order to solve the constraints to all orders. The main point is to show that when solved in terms of the \( h_{ij}^{TT} \) modes, they do not introduce new derivatives to the interaction. To illustrate how it works in this case, we present the second class constraint \( \mathcal{H}_2 \) at second order in perturbations

\[
2\varepsilon (D_2 h^T + D_3 n) = \epsilon^2 \left[ -2\kappa \partial_i^{TT} \partial_i^{TT} + \frac{\beta_1}{4} \partial_i^{TT} \partial_i^{TT} h_{ij}^{TT} + \frac{\beta_3}{4} \partial_i^{TT} \partial_i^{TT} h_{ij}^{TT} + \frac{\beta_5}{4} \partial_i^{TT} \partial_i^{TT} h_{ij}^{TT} \right. \\
+ \left( \beta + \alpha_1 \delta^2 + \alpha_3 \delta^4 + \alpha_5 \delta^6 \right) \left( h_{ij}^{TT} \partial_i^{TT} h_{ij}^{TT} + \frac{3}{4} \partial_i^{TT} h_{ij}^{TT} h_{ij}^{TT} - \frac{1}{2} \partial_i^{TT} h_{ij}^{TT} \partial_i^{TT} h_{ij}^{TT} \right) \\
- \kappa \chi_i^{TT} \chi_i^{TT} + \kappa_1 \left( \partial_i \partial_j \chi_i^{TT} \partial_i \partial_j \chi_j^{TT} - 2 \partial_i \partial_j \chi_i^{TT} \partial_i \partial_k \chi_j^{TT} + \partial_i \partial_k \chi_i^{TT} \partial_i \partial_k \chi_j^{TT} + \partial_\alpha \partial_\beta \chi_i^{TT} \partial_\alpha \partial_\beta \chi_j^{TT} \right) \\
+ \kappa_2 \left( \partial_i \partial_j \chi_i^{TT} \partial_i \partial_j \chi_j^{TT} + \partial_\alpha \partial_\beta \chi_i^{TT} \partial_\alpha \partial_\beta \chi_j^{TT} \right) \\
+ \kappa_3 \left( \beta_1 \partial_i \partial_j \chi_i^{TT} \partial_i \partial_j \chi_j^{TT} + \beta_5 \partial_i \partial_k \chi_i^{TT} \partial_i \partial_k \chi_j^{TT} \right),
\]

where

\[
\hat{\beta_1} = \frac{\beta_1}{2} + \kappa_1, \quad \hat{\beta_2} = \frac{\beta_2}{2} + \kappa_2, \quad \hat{\beta_3} = \frac{\beta_3}{2} + \kappa_3,
\]

and the operators \( D_2 \) and \( D_3 \) were defined in Eq. (81). In obtaining the above result we have replaced the solution of \( h^T \) and \( n \) at first order i.e., \( h^T = n = 0 \) in all second order terms weighted by \( \epsilon^2 \). To obtain the solution at the next orders, the corresponding order solutions must be substituted on the right hand member. Hence, the second class constraints \( \mathcal{H}_2 \) and \( \mathcal{C}_2 \) yields linear equations for the variables \( h^T \) and \( n \) at any order in perturbations where the operator acting on them is just the matrix operator \( \mathbb{M} \).

In the present situation it is enough to know the distribution of the momenta at the UV regime. Then, one can consider only those terms that contribute with the highest power of momenta in the Fourier space. At any order in perturbations the highest number of spatial derivatives contained in both \( \mathcal{H}_2 \) and \( \mathcal{C}_2 \) is exactly the same number of derivatives acting on the fields \( h^T \) and \( n \). As can be appreciated from Eq. (118) the maximum order in spatial derivatives acting on \( h^T \) and \( n \) is eight while the total number of derivatives on the right hand member acting on products of \( h_{ij}^{TT} \) and \( \xi_i^T \) is also eight. Moreover, the constraints \( \mathcal{H}_2 \) and \( \mathcal{C}_2 \) do not contain spatial derivatives of the conjugate momenta. Thereby, for the second and higher order in perturbations, one can model (schematically) the dominant part of the solutions at the UV fixed point as follows

\[
h_i^{TT}, n \sim \left( \frac{1}{(\partial_m)^{2z}} (\partial_n)^{2z} \right) (h_{ij}^{TT} \cdots h_{kl}^{TT}), \quad \left( \frac{1}{(\partial_m)^{2z}} (\partial_n)^{2z} h_{ij}^{TT} \cdots h_{kl}^{TT} \partial_{pq}^{TT} \partial_{rs}^{TT} \right), \quad \left( \frac{1}{(\partial_m)^{2z}} (\partial_n)^{2z} \right) (\xi_{i}^{T} \cdots \xi_{j}^{T}), \\
\left( \frac{1}{(\partial_n)^{2z}} \right) (\xi_{i}^{T} \cdots \xi_{k}^{T} \chi_{q}^{T} \chi_{p}^{T}), \quad \left( \frac{1}{(\partial_m)^{2z}} (\partial_n)^{2z} \right) (h_{ij}^{TT} \cdots \xi_{k}^{T}).
\]

The above entails that at the highest order in derivatives, the operator \( \mathbb{M} \) can be expressed as the differential operator \( \partial^{2z} \) times a matrix of dimensionless coupling constant with determinant (86) which must be different from zero. This condition is a fundamental one, otherwise the number of derivatives in the denominator of (120) would be less than 2z. In that case the contribution of \( h^T \) and \( n \) to the interaction terms will have positive powers of momenta and could imply the non–renormalizability of the theory.

The above argument shows that under the condition (86) the contribution of \( h^T \) and \( n \) to the interactions, in terms of \( h_{ij}^{TT} \) and \( \xi_i^T \), does not change number of derivatives of the interaction terms. From the schematic solution provided by Eq. (120) one can see (for example) that the cubic interactions at 2s order like \( h^T h_{ij}^{TT} \partial_i h_{ij}^{TT} \) and \( h^T \xi_i^T \partial_i \xi_i^T \), after substituting the solution for \( h^T \), keep the vertex contribution with 8 powers of momenta. We can now calculate
the superficial degree of divergence of the theory, determining if under the previous considerations the full theory is power–counting renormalizable or not. To tackle this point we shall follow very closely the analysis presented in references [45, 46]. Next, the power–counting renormalization of the 1PI diagrams follows from the propagators (internal lines \(I\)) expressed by Eqs. (115)–(116), loop integrals (\(L\)) and vertex (\(V\)). Both, the gravitational and vector–gauge propagating degrees of freedom almost share the same structure in considering their Green functions. We consider the conventionally normalized fields [51], that is, its dimensions are determined by the requirement that the \(\tilde{\beta}_5\) coupling constant in (102) is dimensionless. So, for internal lines one has

\[
\langle h_{ij}^T h_{kl}^{TT} \rangle_{\omega, \vec{k}} \to \Xi^{-2z}, \quad \langle \xi_i^T \xi_j^T \rangle_{\omega, \vec{k}} \to \Xi^{-2z}. \tag{121}
\]

Now, for the loop integrals we need to impose a different cutoff \(\Xi_\omega\) for the energy. The dependence of the latter on the momentum cutoff can be inferred from the dimension of the propagator yielding to

\[
\Xi_\omega \to \Xi^z. \tag{122}
\]

Finally, one obtains

\[
\int d\omega d^d k \to \Xi^{d+z}. \tag{123}
\]

On the other hand the vertices, can bring at most \(2z\) powers of momentum. Thus, the superficial degree of divergence for the diagram with \(L\) loops, \(I\) internal lines and \(V\) vertices satisfies

\[
D \leq (d + z) L + 2z (V - I) \tag{124}
\]

or equivalently

\[
D \leq (d - z) L + 2z (L + V - I). \tag{125}
\]

By using the topological relation \(L - 1 = I - V\) we obtain

\[
D \leq (d - z) L + 2z. \tag{126}
\]

We thus see that the number of loops in the diagram improves the degree of divergence. For \(L = 1\) we get

\[
D \leq d + z, \tag{127}
\]

and for \(L = 9\) in the case we are considering \((z = 4\) and \(d = 3\)) we get

\[
D < 0, \tag{128}
\]

that is those diagrams are finite. The Eq. (124) implies the power–counting renormalizability of the theory. Besides, the Hamiltonian formulation of the theory ensures its unitarity.

IV. CONCLUDING REMARKS

In this work, the quantization of the anisotropic gravity–vector gauge field coupling was analyzed. The starting point of this study is the extended 4+1 dimensional Hořava–Lifshitz theory at the KC point [39]. This higher dimensional framework, provides after a Kaluza–Klein dimensional reduction procedure (assuming that the dilaton field is at its ground state i.e., \(\{\phi = 1, p = 0\}\)), a 3+1 dimensional theory describing the pure anisotropic gravity–vector gauge field interaction. The resulting theory in 3+1 dimensions possess six constraints, two of them are of first class given by Eqs. (47)–(48) and four of second class \(P_N = 0\), (49), (50) and (51). It should be noted that, after the dimensional reduction the first class constraint (48), becomes the generator of the gauge symmetry transformations on the vector field \(A_i\). On the other hand, the second class constraint (49) is exactly the same constraint arising in the original 3+1 dimensional formulation at the KC point [21].

In order to ensure a power–counting renormalizable theory at the UV fixed point, the reduced potential \(\hat{V}\) must contain all the marginal and relevant terms at all energy scales \(z = 1, 2, 3, 4\). In this concern the marginal operators are up to order eight in spatial derivatives and the relevant deformations compromise two, four and six order spatial derivatives. As it is observed, from Eqs. (56)–(59) the Weyl tensor contributes in a non–trivial way to the potential of the theory. This is so because, the Weyl tensor is non–vanishing in a 4–dimensional manifold. The complete
list of all terms contributing to the quadratic action are displayed by Eqs. (60)–(63). As highly non–linear second class constraints are present, we have chosen to solve them, by using a perturbative approach. From them we may eliminate \( h^T \) and \( n \) in terms of the \( h^{ij}_T \) and \( \xi^T \) modes. In addition, the transverse gauge has been imposed and the transverse/longitudinal decomposition has been used. The solution of the second class constraints at linear order in perturbations, Eqs. (82) and (93), show that the auxiliary variables \( h^T \) and \( n \) as well as the Lagrange multipliers \( \sigma \) and \( \mu \) at linear order are zero, thus the theory propagates only the transverse–traceless tensorial modes for the gravitational sector and the transverse vector modes for the gauge field. This result is supported by the fact of having elliptic partial differential equations and subject to suitable boundary conditions. So, all this analysis confirms that there are not extra or ghost modes propagated by theory. With this information at hand we have obtained the propagator for both the gravitational and gauge vector fields. As can be appreciated from Eqs. (115) and (116) both interactions have exactly the same physical propagator (and consequently the same dispersion relation) at all energy scales i.e, from the IR fixed point \((z = 1)\) to the UV fixed point \((z = 4)\). Furthermore, the physical propagator at the UV regime effectively has the scaling in momenta as predicted by the original proposal [1]. This is an important property of the propagators of the Horava–Lifshitz theory at the KC point. Since, as it is well known from relativistic theories, the propagator of both interactions coincides, but not necessarily for the Horava–Lifshitz theory in the high energy regime. This property arises from the fact that the coupling constants of the quadratic terms on the fields that contribute to the propagator, are not independent. To check the power–counting renormalizable feature we have schematically determined the contribution of the interaction terms to the Feynman diagrams, as shown in Eq. (120). From this expression it is clear that the vertices do not introduce any pathological behavior in the power–counting process. Therefore, as was proposed in [45, 46, 52–54], when high order spatial derivative terms are included in the theory, the superficial degree of divergence of the Feynman diagrams is bounded by the order of the highest differential operator. The superficial degree of divergence is generically bounded by \( 2z \) but as the number of loops in the Feynman diagrams increases, the bound on \( D \) decreases in a way that for \( L = 8, D \leq 0 \). This point is a relevant one, because as previous studies have reported [41, 43, 44] when matter fields are coupled to this anisotropic gravity theory old problems such as ghost fields and divergences appear again. In contrast the analysis provided in this work the Kaluza–Klein approach assures the power–counting renormalization when all terms up to \( z = 4 \) are included.

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This issue resembles us GR, since when matter fields are coupled, the theory diverges at one loop [42].

As we will see later, this version contains less degrees of freedom than the non–projectable version of the theory.

Here we are using the usual definition for the field strength $F_{ij}$, in comparing with the work [38], where the field strength $F_{ij}$ has the opposite sign. Then, the terms involving the vector–gauge $A_i$ in the fields equations and the first class constraint, have the opposite sign with respect to the expressions given in the present article.

In the first article [38], this Lagrange multiplier was defined as: $\Lambda_i \equiv 2 (N_i - N_i A_i)$.

These constants are the coefficients of the lower order terms ($z = 1$) in the potential of the theory.