Using propagation for solving complex arithmetic constraints

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Abstract. Solving a system of nonlinear inequalities is an important problem for which conventional numerical analysis has no satisfactory method. With a box-consistency algorithm one can compute a cover for the solution set to arbitrarily close approximation. Because of difficulties in the use of propagation for complex arithmetic expressions, box consistency is computed with interval arithmetic. In this paper we present theorems that support a simple modification of propagation that allows complex arithmetic expressions to be handled efficiently. The version of box consistency that is obtained in this way is stronger than when interval arithmetic is used.

1 Introduction

One of the most important applications of constraint programming is to solve a system of numeric inequalities:

\[ g_1(x_1, x_2, \ldots, x_m) \leq 0 \]
\[ g_2(x_1, x_2, \ldots, x_m) \leq 0 \]
\[ \ldots \ldots \ldots \ldots \ldots \]
\[ g_k(x_1, x_2, \ldots, x_m) \leq 0 \]

Such systems often appear as conditions in optimization problems. Inequalities are regarded as intractable in conventional numerical analysis. The Kuhn-Tucker conditions allow these to be converted to equalities. The continuation method is a fairly, but not totally, dependable method for solving such equalities. Moreover, it is restricted to polynomials.

An important contribution of constraint programming is the box-consistency method \[3,7\], which improves on the continuation method in several ways. It applies not only to polynomials \(g_1, \ldots, g_k\) but to any functions that can be defined by an expression that can be evaluated in interval arithmetic. It computes a cover for the set of solutions and can make it fit arbitrarily closely. In this way, all solutions are found and are approximated as closely as required. The performance of the box-consistency method compares favorably with that of the continuation method \[7\].
2 Preliminaries

In this section we provide background by reviewing some basic concepts.

2.1 Constraint satisfaction problems

In a constraint satisfaction problem (CSP) one has a set of constraints, each of which is an instance of a formula. Each of the variables in the formula is associated with a domain, which is the set of values that are possible for the variable concerned. A solution is a choice of a domain element for each variable that makes all constraints true.

With each type of constraint, there is an associated domain reduction operator; DRO for short. This operator may remove from the domains of each of the variables in the constraint certain values that do not satisfy the constraint, given that the other variables of the constraint are restricted to their associated domains. Any DRO is contracting, monotonic, and idempotent.

When the DROs of the constraints are applied in a “fair” order, the domains converge to a limit or one of the domains becomes empty. A sequence of DROs activations is fair if every one of them occurs an infinite number of times [10]. The resulting cartesian product of the domains becomes the greatest common fixpoint of the DROs [10]. If one of the domains becomes empty, it follows that no solutions exist within the initial domains. This, or any variant that leads to the same result, is called a constraint propagation algorithm.

In practice, we are restricted to the domains that are representable in a computer. As there are only a finite number of these, constraint propagation terminates.

2.2 Constraint propagation

A Generic Propagation Algorithm (GPA in the sequel) is a fair sequence of DROs. A GPA maintains a pool of DROs, called active set, that still need to be applied. No order is specified for applying these operators. Even though many variants of GPA exist (see Apt [1] and Bartak [2]), they are all similar to the pseudo-code given in Figure 1. Notice that the active set A is initialized to contain all constraints.

```
put all constraints into the active set A
while ( A ≠ ∅) {
    choose a constraint C from the active set A
    apply the DRO associated with C
    if one of the domains has become empty then stop
    add to A all constraints involving variables whose domains have changed, if any
    remove C from A
}
```

Fig. 1. A pseudo-code for GPA.
2.3 Interval constraint satisfaction problems

The constraint programming paradigm is very general. It applies to domains as different as booleans, integers, finite symbolic domains, and reals. In this paper we consider *interval CSPs*, which are CSPs where there is only one type and it is equal to the set $\mathcal{R}$ of real numbers. In such CSPs domains are restricted to intervals, as reviewed below.

2.4 Intervals

A *floating-point number* is any element of $F \cup \{-\infty, +\infty\}$, where $F$ is a finite set of reals. A *floating-point interval* is a closed connected set of reals, where the bounds, in so far as they exist, are floating-point numbers. When we write “interval” without qualification, we mean a floating-point interval. A *canonical interval* is a non-empty interval that does not properly contain an interval. For every finite non-empty interval $X$, $lb(X)$ and $rb(X)$ denote the left and right bound of $X$ respectively. For an unbounded $X$, the left or right bound is defined as $-\infty$ or $+\infty$, or both. Thus, $X = [lb(X), rb(X)]$ is a convenient notation for all non-empty intervals, bounded or not.

If a real $x$ is not a floating-point number, then there is a unique canonical interval containing it. Otherwise, there are three. Either way, there is a unique least canonical interval containing $x$, and it is denoted $x$.

A *box* is a cartesian product of intervals.

2.5 Box consistency

In [7], box consistency is computed by a relaxation algorithm implemented in interval arithmetic. The algorithm takes as input certain intervals $X_1, \ldots, X_m$ for the variables $x_1, \ldots, x_m$. It uses each of the functions $g_1, \ldots, g_k$ in the way that is explained by a generic instance that we temporarily call $g$. We assume that the function $g$ is defined by an expression $E$ containing no variables other than $x_1, \ldots, x_n$. We call the algorithm in [7] a *relaxation algorithm* because it improves the intervals for the variables one at a time while keeping the intervals for all the other variables fixed. This is similar to the relaxation algorithms of numerical analysis.

For simplicity of notation, let us assume that the interval for $x_1$ is to be improved on the basis of the fixed intervals $X_2, \ldots, X_m$ for the variables $x_2, \ldots, x_m$. This is done by means of a function $g_{X_2,\ldots,X_m}(x)$ that is defined by evaluating in interval arithmetic the expression $E$ with $x_1$ substituted for $x_1$ and $X_2, \ldots, X_m$ substituted for $x_2, \ldots, x_m$, respectively. Thus, $g_{X_2,\ldots,X_m}$ maps a real to an interval.

To improve the interval $X_1 = [lb(X_1), rb(X_1)]$ for $x_1$, suppose that for some $a < rb(X_1)$ we have that

$$lb(g_{X_2,\ldots,X_m}([a, rb(X_1)])) > 0.$$ (2)

In that case the interval for $x_1$ can be improved from $X_1$ to $[lb(X_1), a]$.

A bisection algorithm is used to find the least floating-point $a$ for which (2) holds, for fixed intervals $X_2, \ldots, X_m$. A similar bisection is used to improve the lower bound of $X_1$ using $g$ and the fixed intervals $X_2, \ldots, X_m$. This exhausts what can be done with $g$ and $X_2, \ldots, X_m$. In general, repeating this process with the other arguments and with
the other functions among $\{g_1, \ldots, g_k\}$ causes reductions of $X_2, \ldots, X_m$ and further reductions of $X_1$.

If one of the intervals becomes empty, it is shown that no solution exists within the original intervals $X_1, \ldots, X_m$. Otherwise, the box-consistency algorithm terminates with no interval reduction possible. Let us call the resulting state of the domains functional box consistency, to distinguish it from the type of box consistency described below.

As the criterion for $a$ being an improved upper bound for $X_1$ is (2) with the left-hand side evaluated in interval arithmetic, this box consistency algorithm can be improved by means of interval constraints, as was pointed out in [11]. Here it was proposed that instead of such interval arithmetic evaluation, one applies propagation to the interval CSP containing as constraints

$$g(x_1, \ldots, x_m) \leq 0$$

$$x_1 > a$$

$x_2 \in X_2, \ldots, x_m \in X_m$  \hspace{1cm} (3)

The result is called relational box consistency. In [11] it is shown that the resulting intervals are contained in those obtained by functional box consistency.

To apply propagation, one needs to decompose the arbitrarily complex expression for $g$ into multiple primitive arithmetic constraints ($x + y = z, x \times y = z$, as well as those involving trigonometric or logarithmic constraints), as explained in section 3 so that the corresponding efficient DROs can be applied. In this way the structure of $E$ is lost. As a result, GPA will activate DROs that have no effect, even though, on eventual termination of GPA, the result is the correct one: the unique and consistent state, or failure.

This problem was addressed in [4,11]. The remedy described there is to create a tree data structure for $E$ and perform propagation based on the tree structure. Such a structured propagation algorithm avoids superfluous activations of DROs by following the tree from the bottom to the top and then from the top to the bottom and to repeat these two traversals as a cycle.

In this paper we show that by a simple modification of the GPA one can get the effect of an optimized version of a structured propagation algorithm, yet without maintaining a tree data structure. The downward part of this algorithm is enhanced by the absence of multiple occurrences of variables. In interval arithmetic this would be an unacceptable limitation. However constraint programming allows us to translate multiple occurrences to equality constraints. This may be a welcome improvement compared to the usual way of representing a system such as in Equation (1).

In section 3 we describe our translation of a system such as in Equation (1) to a CSP consisting of primitive constraints.

3 Generating a CSP from a system of nonlinear numerical inequalities

The functions $g_1, \ldots, g_k$ in Equation (1) are defined by expressions that can be evaluated in interval arithmetic and hence can be translated to CSPs containing only primitive
constraints. Any of these expressions may have multiple occurrences of some of the variables. As there are certain advantages in avoiding multiple occurrences of variables in the same expression, we rewrite without loss of generality the system in Equation 1 to the system shown in Figure 2.

\[
\begin{align*}
g_1(x_1, x_2, \ldots, x_m) & \leq 0 \\
g_2(x_1, x_2, \ldots, x_m) & \leq 0 \\
\vdots & \\
g_k(x_1, x_2, \ldots, x_m) & \leq 0 \\
\text{allEq}(v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}) & \\
\vdots & \\
\text{allEq}(v_{p,1}, v_{p,2}, \ldots, v_{p,n_p}) &
\end{align*}
\]

Fig. 2. A system of non-linear inequalities without multiple occurrences. The variables \(\{x_1, \ldots, x_m\}\) are partitioned into equivalence classes \(V_1, \ldots, V_p\) where \(V_j\) is a subset \(\{v_{j,1}, \ldots, v_{j,n_j}\}\) of \(\{x_1, \ldots, x_m\}\), for \(j \in \{1, \ldots, p\}\).

In Figure 2, the expressions for the functions \(g_1, \ldots, g_k\) have no multiple occurrences of variables. As a result, they have up to \(m\) rather than \(n\) variables, where \(m \geq n\). This special form is obtained by associating with each of the variables \(x_j\) in Equation 1 an equivalence class of the variables in Figure 2. In each expression each occurrence of a variable is replaced by a different element of the corresponding equivalence class. This can be done by making each equivalence class as large as the largest number of multiple occurrences. The predicate \(\text{allEq}\) is true if and only if all its real-valued arguments are equal.

An advantage of this translation is that evaluation in interval arithmetic of each expression gives the best possible result, namely the range of the function values. At the same time, the \(\text{allEq}\) constraint is easy to enforce by making all intervals of the variables in the constraint equal to their common intersection. This takes information into account from all \(k\) expressions. If the system in its original form with multiple occurrences, would be translated to a CSP, then only multiple occurrences in a single expression would be exploited at one time.

The translation of a complex arithmetic expression to a CSP containing primitive arithmetic constraints is an obvious variant of the procedure that has been familiar since FORTRAN compilers parsed a complex arithmetic expression and generated code from the parse. The CSP variant of this procedure is at least as old as BNR Prolog [5], which was first implemented in the late 1980s. For a formal description of the translation we refer to [11]. We give a brief informal description here.

For the purpose of the translation, one regards an expression as a tree with operators as internal nodes and constants or variables as external nodes. We associate with each internal node a unique variable. Each internal node now generates a primitive constraint. For example, if an internal node has “/” as an operator, \(x\) as a variable, and \(y\) and \(z\)
as variables associated with left and right child nodes, respectively, then the ternary constraint \( x \times z = y \) is generated.

In this way, each expression generates a primitive constraint for each internal node. Let \( v \) be the variable associated with the root. As this represents the value of the entire expression, the primitive constraint \( v \leq 0 \) is generated as well. Because of the absence of multiple occurrences, the partial CSP generated is an acyclic CSP for which local consistency implies global consistency \([8]\).

Finally, as the \( \text{allEq} \) constraints in Figure 2 do not contain expressions, they need no translation to more primitive constraints; they have the obvious, optimal, and efficient domain reduction operator described earlier.

This completes our description of how to translate a system as in Equation 1 to a CSP. To solve it by propagation, we must first consider propagation in a CSP consisting only of primitive constraints generated from the tree of a single expression together with the inequality constraint involving the root variable. This we do in the next section.

Before proceeding thus, we point out that translating the system (2) to a CSP in the way just described enhances the opportunities for parallelism in propagation beyond those already present in the system (1). To investigate these would take us beyond the limits of this paper. However, it will be useful here to highlight the structure that gives rise to these opportunities by means of a hardware metaphor.

In the first place it is important to note that the sets of constraints arising from the same expression form a cluster for the purposes of propagation. For example, with the exception of the root, the internal variables only have unique occurrences. As a result, when the DRO of a constraint is activated, it usually causes constraints generated by the same expression to be added to the active set. However, the external variables may occur in all of these clusters.

For the purposes of a parallel algorithm it is useful to imagine the clusters arising from each of the expression as hardware “cards”, each connected to a “bus”, where the lines of the bus represent the external variables in common to several expressions. Whenever two external variables belong to the same equivalence class, they are connected by a “jumper” in the hardware model.

The hardware architecture suggests a parallel process for each card that asynchronously executes DROs of constraints only involving internal variables. The processes synchronize when they access one of the bus variables. The DROs of \( \text{allEq} \) constraints can also be executed by a parallel process dedicated to each.

### 4 Modifying propagation for evaluating an expression

We first show that, regardless of efficiency, GPA can be used to evaluate a single expression. Suppose that we have an expression \( E \) in variables \( x_1, \ldots, x_n \). Let \( y \) be the variable at the root of the tree representing the value of \( E \). Let \( C \) be the CSP generated by \( E \) as described before.

**Theorem 1.** Suppose the domains of \( x_1, \ldots, x_n \) are the intervals \( X_1, \ldots, X_n \). Suppose the domains of \( y \) and the other internal variables are \( [-\infty, +\infty] \). Applying the GPA to \( C \) results in the domain of \( y \) being the same interval as the one obtained by evaluating \( E \) in interval arithmetic with \( X_1, \ldots, X_n \) substituted for \( x_1, \ldots, x_n \), respectively.
Proof. According to \[10,1\], every fair sequence of DROs in GPA converges to the same limit for the domains of the variables. There is a finite sequence \(s\) of DROs that mimics the evaluation of \(E\) in interval arithmetic. At the end of this, \(y\) has the value computed by interval arithmetic and GPA terminates.

Theorem 1 is useful in showing that, in the absence of information about the value of the expression, propagation does the equivalent of interval arithmetic. But the GPA does it in a wasteful way. GPA does not specify the order of applying the DROs other than that their sequence should be a fair one. In a typical random fair sequence, many DROs will not have any effect. This inefficient behavior is the motivation for our modifications to GPA presented here.

Theorem 2. Suppose we modify GPA so that the active set is initialized to contain instead of all constraints only those containing at most one internal variable. Suppose also that the active set is a queue in which the constraints are initially ordered according to the level they occupy in the expression tree, with those that are further away from the root placed nearer to the front of the queue. Then GPA terminates after activating the DRO of every constraint at most once. On termination, \(y\) has as domain the value that the expression has in interval arithmetic.

Thus we see that GPA has exactly the right behavior for the evaluation of an expression if only we initialize the active set with the right selection of constraints. We call this propagation with selective initialization (PSI). In the sequel, the constraints that have at most one internal variable are referred to as peripheral constraints.

5 Selective initialization for obtaining box consistency

Propagation, whether modified or not, obtains results that are at least as strong, and typically stronger than, box consistency as described in \[7\]. This can already be demonstrated when considering a single expression \(E\) in variables \(x_1, \ldots, x_m\) with intervals \(X_1, \ldots, X_m\) as domains. A step towards box consistency is to reduce the interval for each of the variables separately. To simplify notation we do this for \(x_1\), keeping the interval domains \(X_2, \ldots, X_m\) for \(x_2, \ldots, x_m\) fixed. Suppose that for some \(a < \text{lb}(X_1)\) we have that \(\text{lb}(g_1([a, \text{rb}(X_1)], X_2, \ldots, X_m)) > 0\). In that case the interval for \(x_1\) can be improved from \(X_1\) to \([\text{lb}(X_1), a]\).

Suppose that instead of such an interval arithmetic evaluation, one applies propagation to the interval constraint system containing as constraints \(x_1 > a, x_2 \in X_2, \ldots, x_m \in X_m\), as well as all the ones obtained by translating the expression tree of \(E\) to primitive constraints.

Theorem 3. Suppose that \(\text{lb}(E([a, \text{rb}(X_1)], X_2, \ldots, x_m \in X_m)) > 0\). When GPA is applied to this CSP, failure results.

Proof. Consider any fair sequence that starts with a segment \(s\) mimicking the interval arithmetic evaluation of \(E\). At the end of \(s\), the interval for \(y\) has a positive lower bound, by the assumption. The fair sequence can be continued by applying the DRO for \(y \leq 0\). This yields failure.
This proves the theorem. If we use GPA in this way for box consistency instead of interval arithmetic, we never obtain a worse result and we typically obtain a better result.

However, unmodified GPA will obtain the better result in an inefficient way.

Our next result is a modification of GPA that obtains the better result in a more efficient way.

Suppose we have a CSP $S$ generated by an expression $E$. Let $y$ be the variable at the root of the tree representing $E$. Suppose we apply GPA to $S$. After the termination of GPA, we have the following theorem.

**Theorem 4.** Suppose the domain for $y$ is changed to a proper subset of it. Suppose the unique constraint containing $y$ is placed in the active set as only element. Then GPA terminates with the same result as when the active set would have been initialized to contain all constraints.

**Proof.** Let $s$ be any fair sequence of DROs. If $s$ starts with the unique constraint containing $y$, then the theorem follows. If $s$ starts with a different constraint $c$, then applying the DRO associated with $c$ does not affect any domain (DROs are idempotent and the DRO of $c$ is already in its fixpoint). Thus, removing $c$ from $s$ does not affect the fixpoint of $s$. We keep removing the first element of $s$ until we reach the unique constraint containing $y$. The new $s'$ formed starts with the unique constraint containing $y$ and has the same fixpoint as $s$. Thus, Theorem 4 is proved.

This theorem shows that a complex constraint of the form $E(x_1, \ldots, x_n) \leq 0$ can be made relationally box consistent by using a modification of GPA that is more efficient without affecting the quality of the result.

The result obtained from Theorem 4 is usually better than backward evaluation [4].

The following example illustrates that.

Let us consider the following system.

$$
\begin{align*}
  x_1/x_2 &\leq 0 \\
  x_1 &\in [-1, 1] \quad x_2 \in [-1, 1]
\end{align*}
$$

Using the decomposition described in section 3 we generate the following CSP.

$$
\begin{align*}
  y &\leq 0 \\
  x_1/x_2 &= y \\
  x_1 &\in [-1, 1], x_2 \in [-1, 1], y \in [-\infty, +\infty]
\end{align*}
$$

Applying the GPA described in Theorem 4 gives the fixpoint $[-1, 1] \times [-1, 1] \times [-\infty, -1]$. The result obtained from interval arithmetic backward evaluation is $[-1, 1] \times [-1, 1] \times [-\infty, 0]$.

Pseudo-code for the PSI algorithm is given in Figure 3.

In certain situations (fully described in [12]), Theorem 5 stated below, can be used instead of Theorem 4.

In addition to the assumptions of Theorem 4, we suppose that the expression $E$ has no multiple occurrences of any variable.
put only peripheral constraints into the active set $A$
while ($A \neq \emptyset$) {
    choose a constraint $C$ from the active set $A$
    apply the DRO associated with $C$
    if one of the domains is empty then stop
    add to $A$ all constraints involving variables whose domains have changed, if any
    remove $C$ from $A$
}

Fig. 3. Pseudo-code for Propagation by Selective Initialization.

Theorem 5. Suppose the domain for $y$ is changed to a proper subset of it. Suppose the unique constraint containing $y$ is placed in the active set as only element. Then GPA terminates after having activated the DRO of each constraint at most once.

Moreover, the result is the same as when the active set would have been initialized to contain all constraints.

This theorem is based on the fact that once the fixpoint is obtained, reducing the domain of one variable may cause the other domains to be reduced but not itself. As shown in [12], this is true when DROs have certain properties. For example, one could allow domains to be the union of disjoint intervals, as in the systems of Hyvönen or Havens [8, 6]. But when DROs are those described in this paper, reducing the domain of a variable can affect its own domain as shown in the example above. Even though $[-1, 1] \times [-1, 1] \times [-\infty, +\infty]$ is a fixpoint of the CSP

$$\frac{x_1}{x_2} = y$$
$$x_1 \in [-1, 1], x_2 \in [-1, 1], y \in [-\infty, +\infty]$$

reducing the domain of $y$ to $[-\infty, 0]$ leads to a further reduced domain of $y$ that is equal to $[-\infty, -1]$.

6 Conclusion

In this paper, we presented a slight modification to propagation to get the same or better results than structured propagation. Such a modification is used in a box-consistency algorithm to solve systems of non-linear inequalities but it can also be used when solving other CSPs obtained by translating complex expressions. In fact, most CSPs of practical importance seem to be derived from complex expressions.

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References

1. K.R. Apt. The essence of constraint propagation. *Theoretical Computer Science*, 221(1-2):179–210, 1999.
2. Roman Bartak. Theory and practice of constraint propagation. In *Proceedings of the 3rd Workshop on Constraint Programming in Decision and Control (CPDC 2001)*, pages 7–14. J. Figwer(Editor), 2001.
3. F. Benhamou, D. McAllester, and P. Van Hentenryck. CLP(Intervals) revisited. In *Logic Programming: Proc. 1994 International Symposium*, pages 124–138, 1994.
4. Frédéric Benhamou, Frédéric Goualard, Laurent Granvilliers, and Jean-François Puget. Revising hull and box consistency. In *Proceedings of the 16th International Conference on Logic Programming*, pages 230–244. MIT Press, 1999.
5. BNR. BNR Prolog user guide and reference manual. Version 3.1 for Macintosh, 1988.
6. W.S. Havens, S. Sidebottom, G. Sidebottom, J. Jones, and R. Ovans. Echidna: A constraint logic programming shell. Technical report, Simon Fraser University, Canada, 1992.
7. Pascal Van Hentenryck, Laurent Michel, and Yves Deville. *Numerica: A Modeling Language for Global Optimization*. MIT Press, 1997.
8. E. Hyvönen. Constraint reasoning based on interval arithmetic: The tolerance propagation approach. *Artificial Intelligence*, 58:71–112, 1992.
9. A.P. Morgan. *Solving Polynomial Systems Using Continuation for Scientific and Engineering Problems*. Prentice-Hall, 1987.
10. M.H. van Emden. Value constraints in the CLP Scheme. *Constraints*, 2:163–183, 1997.
11. M.H. van Emden. Computing functional and relational box consistency by structured propagation in atomic constraint systems. In *Proc. 6th Annual Workshop of the ERCIM Working Group on Constraints; deposited at CoRR*, 2001.
12. M.H. van Emden and B. Moa. Enhancing constraint propagation by selective initialization. In preparation, 2003.