A perturbative approach to inelastic collisions in a Bose–Einstein condensate

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Abstract

It has recently been discovered that for certain rates of mode-exchange collisions analytic solutions can be found for a Hamiltonian describing the two-mode Bose–Einstein condensate. We proceed to study the behaviour of the system using perturbation theory if the coupling constants only approximately match these parameter constraints. We find that the model is robust to such perturbations. We study the effects of degeneracy on the perturbations and find that the induced changes differ greatly from the non-degenerate case. We also model inelastic collisions that result in particle loss or condensate decay as external perturbations and use this formalism to examine the effects of three-body recombination and background collisions.

1. Introduction

A Bose–Einstein condensate (BEC) is a state of matter in which a large number of bosons occupy the same quantum mechanical ground state. As such, BECs present the opportunity to study quantum systems which display large-scale (macroscopic) collective behaviour. Multi-component BECs are most often formed in a multi-well potential in which the components are spatially separated [2, 4]. Alternatively, the multi-component formalism can be used to model a single-component BEC that possesses several internal degrees of freedom, such as varying amounts of spin [3]. Recently, multi-component BECs have found numerous applications in quantum optics [1–4].

Because of the difficulties involved in treating many-body systems exactly, numerical or approximate methods are often used. Two common approximate models are the Bose–Hubbard model of quantum optics [4] and closely related to it, the Lipkin–Meshkov–Glick model of nuclear physics [5], both used to describe the two-body interactions of spin-\(J\) systems. A family of exactly solvable many-body systems was introduced in [6] and studied in greater depth in [7]. These models describe a two-component BEC where both elastic and mode-exchange inelastic collisions occur. Mode-exchange collisions are known as general nearest-neighbour interactions in the context of multi-well BECs [9] and as inelastic collisions in the case of a single condensate consisting of particles in two hyperfine levels [10]. The family of models is parameterized by a positive integer \(n\) and hence are referred to as \(n\)-models. The terminology is used to indicate that the \(n\)-model contains interactions involving at most \(n\) bodies. When the strengths of the various particle interactions obey specific constraints, analytic solutions exist for the \(n\)-model. In this paper, we will focus mainly on the case \(n = 2\). Microscopic calculations show that mode-exchange collisions occur in BECs as a result of the interaction of a laser field with the system [10]. The 2-model includes the usual Josephson-type interactions but also allows the effects of mode-exchange collisions to be studied analytically. As such, the 2-model provides a more realistic framework for studying two-mode BECs than the canonical Josephson Hamiltonian [11].

The analytic solution found in [6] applies only when the strengths of the various interactions meet certain constraints. While in some experimental situations the rates of inelastic and...
2. A model for two-mode Bose–Einstein condensates with mode-exchange collisions

The 2-model studied in [6, 7] is governed by the Hamiltonian

\[ H_2 = A_0 + \omega(a^\dagger a - b^\dagger b) + \lambda e^{i\phi} a^\dagger b + e^{-i\phi} ab^\dagger + \mu e^{i\phi}(a^\dagger a' a b - a b^\dagger a') + \text{h.c.} \]  

(1)

where h.c. denotes the Hermitian conjugate of the preceding term. The two modes a and b are independent Bose operators satisfying [a, a^\dagger] = 1 = [b, b^\dagger] with the commutators of all remaining pairs vanishing. The term \(\omega(a^\dagger a - b^\dagger b)\) is the free energy of the a-mode and \(\hat{n}_a\) a particles in the a mode and \(\hat{n}_b\) b particles in the b mode, with frequency difference \(\omega\) between modes. This frequency difference arises because the model describes atoms in different hyperfine levels, or alternatively asymmetric spatially separated condensates [14]. A Josephson-type or spin flip interaction is included with strength \(\lambda\) and phase \(\phi\). In practice, such an interaction is induced by an external field, such as a laser [8]. This interaction may also be interpreted as modelling the tunneling of particles between modes with probability proportional to \(\lambda\). Also included are terms corresponding to number-preserving elastic and mode-exchange collisions, namely those terms containing four Bose operators. The interaction with strength \(\mu\) is a single dispersive process. The interaction with strength \(\Lambda\) describes a collision where two particles exchange their modes. The elastic interaction has strength \(\mathcal{U}\) and models the collision of a particle from each mode in which the number of particles in each mode is conserved. We remark that the mode-exchange collisions preserve the total particle number but not the relative particle number.

The Hamiltonian given in equation (1) can be efficaciously studied by first introducing the simpler Hamiltonian [6]

\[ H_0 = A_1(a^\dagger a - b^\dagger b) + A_2(a^\dagger a - b^\dagger b)^2 \]  

(2)

with real constants \(A_1\) and \(A_2\). Such a Hamiltonian models a two-mode condensate with energy difference \(\Delta\) between modes and elastic scattering probability proportional to \(A_2\). As is clear from the form of \(H_0\) the relative number operator \(\hat{n} = a^\dagger a - b^\dagger b\) is a commuting observable, and hence the number of particles in each mode is conserved. In particular, there is no probability of spin flip or tunnelling between modes. Since the total number operator \(\hat{N} = a^\dagger a + b^\dagger b\) commutes with \(H_0\) we may take the eigenvalues \(N\) and \(m\) of \(\hat{N}\) and \(\hat{m}\) as labels of the eigenstates, for which we write \(|N, m\rangle\).

For a certain choice of parameters in \(H_2\) the solutions \(|N, m\rangle\) of \(H_0\) can be used to obtain analytic solutions of \(H_2\). To see this, define a unitary two-mode displacement operator by \(U(\xi) = \exp(\xi a^\dagger b - \xi^* a b^\dagger)\) with displacement parameter \(\xi = \frac{1}{2} \theta e^{i\phi}\). It can then be shown that if the parameters in \(H_2\) satisfy

\[ A_0 = A_2 \left( N^2 \cos^2 \theta + N \sin^2 \theta \right) \]  

(3a)

\[ \omega = A_1 \cos \theta \]  

(3b)

\[ \lambda = A_1 \sin \theta \]  

(3c)

\[ \mathcal{U} = 2A_2 \left( 1 - 3 \cos^2 \theta \right) \]  

(3d)

\[ \Lambda = A_2 \sin^2 \theta \]  

(3e)

\[ \mu = 2A_2 \cos \theta \sin \theta \]  

(3f)

then \(H_2 = U^\dagger H_0 U\), a result shown by computing

\[ U^\dagger \begin{pmatrix} a & b \\ a^\dagger & b^\dagger \end{pmatrix} U = \begin{pmatrix} \cos \frac{1}{2} \theta & e^{i\phi} \sin \frac{1}{2} \theta \\ -e^{-i\phi} \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{1}{2} \theta & -e^{i\phi} \sin \frac{1}{2} \theta \\ e^{-i\phi} \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta \end{pmatrix} \times \begin{pmatrix} a & b \\ a^\dagger & b^\dagger \end{pmatrix} \]  

(4)

Observe that \(U(a^\dagger b^\dagger) U^\dagger\) may be computed by setting \(\theta \leftrightarrow -\theta\) in equation (4). The above constraints reduce the number of free parameters in the model to 3: \(A_1, A_2\) and \(\theta\).

Since the eigenvectors of \(H_0\) are of the form \(|N, m\rangle\) the eigenvectors of \(H_2\), when satisfying equations (3), are simply

\[ |N, m\rangle = \exp(\xi a^\dagger b - \xi^* a b^\dagger) |N, m\rangle \]  

8 See appendix A for details.
For fixed \( N \in \mathbb{N} \), the ground state of \( H_2 \), labelled as \( |\Psi_{0,m}\rangle = U^1|N,m_0\rangle \), is found by minimizing the energy \( E_{m_0} = A_1m_0 + A_2m_0^2 \) with respect to \( m_0 \). First, assume \( A_2 < 0 \). In this case, if \( A_1 < 0 \) (\( A_1 > 0 \)) the minimum occurs at \( m_0 = N \) (\( m_0 = -N \)). Secondly, say \( A_2 > 0 \). If \( \frac{A_1}{2A_2} \leq N \), then \( m_0 \) is the closest allowable integer to \( -\frac{A_1}{2A_2} \); otherwise \( m_0 \) is the closest allowable integer to \( -\text{sgn}(A_1)\sqrt{N} \). In calculations we will assume \( A_2 = 1 \), so that a choice of \( m_0 \) determines \( A_1 \).

It is important to recall that in the case of a double-well BEC, the two-mode approximation must be satisfied [1]. This means that collisions taking place in the region where the wavefunctions overlap must be less probable than collisions between particles belonging to the same well. It is possible to find an exact analytic solution to this model in terms of matching one-body and two-body wavefunctions. The same process of matching \( |N\rangle \) states to \( |N\rangle \) states for the general \( n \)-mode has as its Hamiltonian \( H_n = U^1H_{0,n}U \) where \( H_{0,n} = \sum_{k=1}^{n}A_k\hat{a}_k^\dagger\hat{a}_k^\dagger \). The eigenvectors of \( H_n \) are again \( U^1|N,m\rangle \) with energy \( E_{m} \) as above. The same process of matching coefficients in the expansion of \( U^1H_{0,n}U \) that was used to obtain equations (3) can be carried out so that \( U^1|N,m\rangle \) is the exact solution of a Hamiltonian containing interactions involving up to \( n \)-bodies. There are \( n + 2 \) free variables, namely \( A_1, \ldots, A_n, \theta \) and \( \phi \), in \( UH_{0,n}U^\dagger \) whereas the number of terms in a general Hamiltonian describing such interactions is \( (n + 2)(n + 4)(n + 6)/48 \), as shown in appendix B. So while the above method continues to give analytic solutions for all \( n \), its range of applicability decreases as \( n \) grows. It should also be noted that because \( H_{0,n} \) contains only terms with the same number of creation and annihilation Bose operators, this method cannot be used to study interactions involving an odd number of Bose operators, such as the interaction \( a^\dagger b^\dagger a b \) where one particle is lost.

We remark that all of the perturbative analysis carried out below is valid for the general \( n \)-mode, not just \( n = 2 \); we only need to change the interpretation of the perturbative analysis in each case.

### 3. Effects of parameter perturbations in the 2-model

While satisfying equation (3) is sufficient to obtain an analytic solution of \( H_2 \), the method presented above fails to produce solutions if the parameters in \( H_2 \) deviate even slightly from these constraints. Since real physical systems will in general manifest such deviations, we are led to study parameter perturbations so that the constraints are only approximately satisfied. We proceed with these perturbation calculations in the section below.

We begin by perturbing each of the coupling constants in \( H_2 \) away from the conditions given by equation (3). That is, we consider perturbations of the form \( \omega \mapsto \omega + \delta_\omega \) where \( \delta_\omega \) is small. We omit the study of perturbations \( A_0 \mapsto A_0 + \delta_A \), since this simply leads to a shift in the energy of the state by \( \delta_A \). In what follows we assume that all eigenstates are non-degenerate; the degenerate case is discussed in section 7. Throughout the paper, given an operator \( \hat{O} \) we define \( \hat{O} = U \hat{O} U^\dagger \). To perform many of the calculations that follow we have made use of a coordinate transformation between the \( |N,m\rangle \) basis and the \( |n_a,n_b\rangle \) basis, \( |N,m\rangle \mapsto |n_a = \frac{N+m}{2}, n_b = \frac{N-m}{2}\rangle \). Here \( n_a \) is an eigenvalue of \( \hat{n}_a = a^\dagger a \) and similarly for \( n_b \).

#### 3.1. \( \omega \) perturbation

A change \( \omega \mapsto \omega + \delta_\omega \) results in the perturbation \( \hat{H}_\omega = \delta_\omega \left(a^\dagger a - b^\dagger b\right) \). A calculation then shows that the non-vanishing matrix elements of \( \hat{H}_\omega \) are

\[
\langle N,m|\hat{H}_\omega|N,m\rangle = \delta_\omega m \cos \theta \tag{5}
\]

and

\[
\langle N,m \pm 2|\hat{H}_\omega|N,m\rangle = -\frac{\delta_\omega}{2} e^{\pm i\phi} \sin \theta \sqrt{N(N + 2) - m(m \pm 2)}. \tag{6}
\]

#### 3.2. \( \lambda \) perturbation

A change \( \lambda \mapsto \lambda + \delta_\lambda \) results in the perturbation \( \hat{H}_\lambda = \delta_\lambda \left(e^{i\theta} a^\dagger b + e^{-i\theta} a b\right) \), and we find that

\[
\langle N,m|\hat{H}_\lambda|N,m\rangle = \delta_\lambda m \sin \theta \tag{7}
\]

and

\[
\langle N,m \pm 2|\hat{H}_\lambda|N,m\rangle = \pm \frac{\delta_\lambda}{2} \left(e^{\pm i\theta} \cos^2 \frac{1}{2} \theta - e^{\pm 2i\theta} \sin^2 \frac{1}{2} \theta\right) \times \sqrt{N(N + 2) - m(m \pm 2)}. \tag{8}
\]

#### 3.3. \( U \) perturbation

A change \( U \mapsto U + \delta_U \) results in the perturbation \( \hat{H}_U = \delta_U a^\dagger b^\dagger a b \), and we find that

\[
\langle N,m|\hat{H}_U|N,m\rangle = \frac{\delta_U}{4} \sin^2 \theta \left(\frac{N^2 + m^2}{2} + N\right)
\]

\[
+ \delta_U \cos^2 \theta \left(\frac{N^2 - m^2}{4}\right). \tag{9}
\]

\[
\langle N,m \pm 2|\hat{H}_U|N,m\rangle = \frac{\delta_U}{8} e^{\pm i\phi} \sin^2 \theta \sqrt{(N \pm m)(N \pm m + 2)} (m \pm 1) \tag{10}
\]

and

\[
\langle N,m \pm 4|\hat{H}_U|N,m\rangle = -\frac{\delta_U}{16} e^{\pm 2i\phi} \sin^2 \theta \times \sqrt{[N(N + 2) - m(m \pm 2)](N(N + 2) - (m \pm 2)(m \pm 4))}. \tag{11}
\]
A change $\Lambda \leftrightarrow \Lambda + \delta_\Lambda$ results in the perturbation $H_\Lambda = \delta_\Lambda (e^{i\phi}a^\dagger a b b + \text{h.c.})$ and we find that

$$|m\rangle \langle H_\Lambda |m\rangle = \frac{\delta_\Lambda}{2} \sin^2 \theta \left( \frac{3m^2 - N^2}{2} - N \right),$$

(12)

and

$$|m \pm 2 \rangle \langle H_\Lambda |m \pm 2 \rangle = \frac{\delta_\Lambda}{8} e^{i\pm i\phi} \sin 2\theta \sqrt{(N \mp m) (N \pm m + 2)} (m \pm 1),$$

(13)

and

$$|m \pm 4 \rangle \langle H_\Lambda |m \pm 4 \rangle = \frac{\delta_\Lambda}{8} e^{i\pm 2i\phi} (1 + \cos^2 \theta) \times \sqrt{(N \mp m + 2)(N \mp m + 4)(N \mp m + 2)}.$$  

(14)

3.5. $\mu$ perturbation

A change $\mu \leftrightarrow \mu + \delta_\mu$ results in the perturbation $H_\mu = \delta_\mu (e^{i\phi}(a^\dagger a^\dagger a b b - a b^\dagger a^\dagger b b) + \text{h.c.})$ and we find that

$$|m\rangle \langle H_\mu |m\rangle = \frac{\delta_\mu}{2} \sin 2\theta \left( \frac{3m^2 - N^2}{2} - N \right),$$

(15)

and

$$|m \pm 2 \rangle \langle H_\mu |m \pm 2 \rangle = \frac{\delta_\mu}{8} e^{i\pm i\phi} \cos 2\theta \sqrt{(N \mp m)(N \pm m + 2)} (m \pm 1),$$

(16)

and

$$|m \pm 4 \rangle \langle H_\mu |m \pm 4 \rangle = -\frac{\delta_\mu}{8} e^{i\pm 2i\phi} \sin 2\theta \times \sqrt{(N \mp m + 2)(N \pm m + 4)(N \mp m)(N \mp m - 2)}.$$  

(17)

In each of the above cases the perturbation matrix elements simplify significantly in the limiting cases $\theta = 0, \pi$, where the Josephson coupling vanishes ($\lambda = 0$) and $|m\rangle = |\pm \frac{N}{2}\rangle$ in which the potential well is symmetric ($\omega = 0$).

### 4. Perturbative effects on particle distribution

In the unperturbed case the analytic solution $|\psi_{m_0}\rangle = U^\dagger |N, m_0\rangle$ for parameters satisfying equation (3) gives an explicit expression for the particle distribution:

$$P = |\langle N, m |U^\dagger |N, m_0\rangle|^2.$$  

(18)

Using the homomorphism described in appendix A to relate the Schwinger su(2) representation to the angular momentum representation we write $P = |d^N_{m,m_0}|^2$ where

$$d^N_{m,m_0} = \sum_{k=0}^{k_{\pm}} (-1)^k \left( \frac{1}{2} (N + m_0) \right) \left( \frac{1}{2} (N - m_0) \right) \cos \theta \sqrt{\sum_{k=0}^{k_{\pm}} (-1)^k}.$$  

(19)

and

$$\Sigma = \sum_{k=0}^{k_{\pm}} (-1)^k \left( \frac{1}{2} (N + m_0) \right) \left( \frac{1}{2} (N - m_0) \right) \cos \theta \sqrt{\sum_{k=0}^{k_{\pm}} (-1)^k}.$$  

(20)

The integers $k_{\pm}$ are chosen so that the arguments of the combinatorial symbols are non-negative; explicitly, $k_- = \max \{0, \frac{m_0 - m}{2}\}$ and $k_+ = \min \{\frac{N + m}{2}, \frac{N - m}{2}\}$. We plot in figure 1 the unperturbed particle distribution for $N = 1000$, $\theta = 1$ and $m_0 = 1000$, 998 and 996. Observe that the particle distribution is independent of the phase $\phi$. We will see below that this does not hold in the perturbed case. As $\theta$ grows from 0 to $\pi$ the maxima shift towards $m = \frac{N}{2}$. The canonical two-mode BEC predicts (under certain circumstances) that the ground state solution is a superposition of two peaked distributions. From figure 1 we see that for $m_0 < N - 2$ the ground state is a superposition of more than two distributions. This is due to the mode-exchange collisions not included in the canonical model [7].

In order to include perturbative effects in the particle distribution we replace the zeroth-order ground state

$^9$ $d^N_{m,m_0}$ are the Wigner rotation matrix elements under the effect of the Lie algebra homomorphism discussed in appendix A. To see why these appear, observe that $U^\dagger = e^{-i\phi J}$ where $n = (\sin \phi, \cos \phi, 0)$ and $J = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$ is the total angular momentum vector operator. That is, $U^\dagger$, and hence $U$, are rotations of the algebra generated by $a$ and $b$. 

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*Figure 1.* Unperturbed particle distributions $P$ given by equation (18) for $N = 1000$, $\theta = 1$ and (a) $m_0 = 1000$, (b) $m_0 = 998$ and (c) $m_0 = 996$. 

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wavefunction $U^{|N, m_0\rangle}$ in equation (18) with that including first-order corrections:

$$U^{|N, m_0\rangle} (^{0+1}) = U^{|N, m_0\rangle} + |N, m_0^{(1)}\rangle.$$  

Recall that in the non-degenerate case a perturbation $H'$ induces a first-order wavefunction correction given by

$$|N, m_0^{(1)}\rangle = \sum_{m \neq m_0} a_{m_0, m} |N, m\rangle$$

$$:= \sum_{m \neq m_0} \frac{\langle N, m | H' | N, m_0 \rangle}{E_m - E_{m_0}} |N, m\rangle. \quad (21)$$

The superscripts indicate to what order in the perturbation the term corresponds; when no superscript is present the result is to zeroth order. With this, we find that to first order the particle distribution is given by

$$P^{(0+1)} = |d_{m, m_0}^N|^2 + 2 |d_{m, m_0}^N|^2 \sum_{n=-N}^{N} \text{Re}(a_{m_0, n}) d_{m, n}^N. \quad (22)$$

The summation over $n$ is restricted to allowable integers, as described above. We show in figures 2–6 the particle distributions under each of the five parameter perturbations considered above. Note that in certain cases $P^{(0+1)}$ may be negative because the first-order term in equation (22) depends linearly on the perturbation coefficients $a_{m, n}$. This of course is just a reflection that only first-order corrections have been taken into account. We address the impact such terms have on our results below.

Since equation (22) is linear in $a_{m, n}$, which are proportional to the perturbation, each of the perturbations can
be used to enhance or diminish the particle distributions in the vertical direction. We also see that as $|m_0|$ decreases the number of local maxima of the particle distribution increases and the asymmetry of the perturbations becomes more significant, vertically diminishing probability amplitudes on one side of the central maxima while vertically stretching the amplitudes on the opposite side. The figures also show that the system is much less sensitive to perturbations in $\omega$ and $\lambda$ than it is to perturbations in the collision coupling constants; in the figures the perturbations of $\omega$ and $\lambda$ are between 7 and 100 times greater than those in the collision perturbations ($\Lambda, \mu$ and $U$), while the first-order corrections in all cases are of the same scale. Moreover, the system is more sensitive to perturbations in the mode-exchange collisions $\mu$ and $\Lambda$ than it is to $U$. It is thus beneficial to maximize the matching of the mode-exchange collision constraints (3) through redefinition of parameters, allowing the perturbation to have a larger effect on the $\omega, \lambda$ and $U$ terms.

The effect on the particle distributions of varying $\theta$ is to shift the position of the central maxima on the $m$-axis: for $\theta = \frac{\pi}{2}$, so that the Josephson-type interaction is maximal ($\lambda = A_1$) and the energy of each mode is equal ($\omega = 0$), the distribution is centred around $m = 0$, while for $\theta = 0$ ($\theta = \pi$), so that the Josephson-type interaction vanishes ($\lambda = 0$) and the energy difference of the modes is maximal ($|\omega| = A_1$), it is centred around $m = N$ ($m = -N$). In each of the cases, despite being centred around different values of $m$ the perturbations display the same qualitative behaviour for any choice of $\theta$. Throughout the paper we have chosen $\theta = 1$ in the figures as this lies between the extremes of $\theta = 0$ and $\theta = \frac{\pi}{2}$.

5. Perturbative effects on entanglement

Entanglement arises in many-body quantum systems because of the superposition principle and the tensor product structure of the Hilbert space, and is of utmost importance in quantum control and quantum information. We would like to determine whether or not, by careful choice of the perturbations, we can increase the entanglement from that of the unperturbed case.

For a bipartite quantum system the von Neumann entropy $S = -\text{Tr} (\rho \log_2 \rho)$ is a standard measure of the entanglement of the system, $\rho$ being the reduced density matrix. In the case at hand this reduces to

$$S(N, m_0, \theta) = - \sum_{m=-N}^{N} P(N, m_0, m, \theta) \log_2 P(N, m_0, m, \theta)$$

where $P$ is either the perturbed or unperturbed particle distribution, depending on the situation. As noted in the above section, the first-order particle distribution given by equation (22) can be negative for certain choices of perturbations. To resolve this problem we could either replace $P^{(1)}$ with $|P^{(1)}|$ or we could include the second-order term $\sum_{m=-N}^{N} |a_{m,n}d_m| a_{m,n}^*|^2 P^{(0)}$ in $P^{(0)}$; we use the former approach to keep the calculations strictly to first order.
Figure 7. Unperturbed entanglement using equation (23) for \( N = 100 \).

We remark that a second method could be used in computing the perturbed entanglement. Namely, above we could replace \( P(\alpha + 1) \) with \( \max(P(\alpha + 1), 0) \) so that points where the perturbed particle distribution is negative are not counted towards the perturbed entanglement. Since \( P(\alpha + 1) \) is negative only for certain values of \( m \) and \( \theta \), the differences that occur by using \( \max(P(\alpha + 1), 0) \) will also be localized in \( m \) and \( \theta \). Repeating the calculation using this approach, we have checked numerically that the qualitative behaviour of the perturbed entanglement is not affected by using \( \max(P(\alpha + 1), 0) \) instead of \( P(\alpha + 1) \). For example, for a perturbation of \( \lambda \), with the same parameter values as used below, the difference between the perturbed and unperturbed entanglements for \( \delta = 0.1 \); we choose a different \( \delta \) in the latter case so that the differences are more evident. In each of the plots of \( \Delta S \) we see that the largest changes occur approximately along the diagonal lines, connecting the points \( (\theta, m_0) = (\pi, -N) \) (respectively \( (\pi, N) \) and \( (-\pi, -N) \) (respectively \( (\pi, N) \)); these lines in \( (\theta, m_0) \) space correspond to certain strengths of the coupling constants in the Hamiltonian \( H_2 \) viewed as functions of \( m_0 \). This type of behaviour also occurs along these lines in the unperturbed entanglement plots, as shown in figure 7. Comparing the magnitude of the perturbative effects we observe that perturbations to the mode-exchange collision terms (\( U, \Lambda, \mu \)) have a much greater effect, their maximum difference being about an order of magnitude larger than those for \( \lambda \) and \( \omega \). It is also interesting to observe that the coherent states, which correspond to \( U|N, N\rangle \) and \( U|N, -N\rangle \) are the states which are maximally affected by perturbations. In such states \( A_1 \) is much larger than \( A_2 \) so that the rate of collisions is relatively small. To further study this we have plotted in

When the particle distribution contains first-order corrections, \( P^{(\alpha + 1)} = P^{(0)} + P^{(1)} \), the entanglement is given by

\[
S^{(\alpha + 1)} = S^{(0)} - \sum_{m=-N}^{N} P^{(1)}(N, m_0, m, \theta) \times \left( \log_2 P^{(0)}(N, m_0, m, \theta) + \frac{1}{\ln 2} \right),
\]

(24)

For the case at hand we read off that \( P^{(1)} = 2d_m^{N} \sum_{n=-N}^{N} \text{Re}(a_{m_0,n})d_n^{N} \). Since \( P^{(1)}(N, m_0, m, \theta) \) is proportional to the perturbation strengths we can increase or decrease the entanglement by choosing the sign of the perturbation appropriately, according to the expression for \( S^{(\alpha + 1)} \).

In figure 7 we plot the unperturbed entanglement as a function of \( \theta \) and \( m_0 \). See [7] for an extensive analysis. In figures 8–12 we plot the perturbed entanglements \( S^{(\alpha + 1)} \) for \( \delta = 0.01 \) as well as the differences \( \Delta S := S^{(\alpha + 1)} - S^{(0)} \) between the perturbed and unperturbed entanglements for \( \delta = 0.1 \); we choose a different \( \delta \) in the latter case so that the differences are more evident. In each of the plots of \( \Delta S \) we see that the largest changes occur approximately along the diagonal lines, connecting the points \( (\theta, m_0) = (\pi, -N) \) (respectively \( (\pi, N) \)) and \( (-\pi, -N) \) (respectively \( (\pi, N) \)); these lines in \( (\theta, m_0) \) space correspond to certain strengths of the coupling constants in the Hamiltonian \( H_2 \) viewed as functions of \( m_0 \). This type of behaviour also occurs along these lines in the unperturbed entanglement plots, as shown in figure 7. Comparing the magnitude of the perturbative effects we observe that perturbations to the mode-exchange collision terms (\( U, \Lambda, \mu \)) have a much greater effect, their maximum difference being about an order of magnitude larger than those for \( \lambda \) and \( \omega \). It is also interesting to observe that the coherent states, which correspond to \( U|N, N\rangle \) and \( U|N, -N\rangle \) are the states which are maximally affected by perturbations. In such states \( A_1 \) is much larger than \( A_2 \) so that the rate of collisions is relatively small. To further study this we have plotted in

Figure 8. (a) Perturbed entanglement for \( \delta = 0.01 \) as a function of \( m_0 \) and \( \theta \) for \( N = 50 \) and (b) the difference \( S^{(\alpha + 1)} - S^{(0)} \) for \( N = 100 \) and \( \delta = 0.1 \).
Figure 9. (a) Perturbed entanglement for $\delta_\lambda = 0.01$ as a function of $m_0$ and $\theta$ for $N = 100$ and (b) the difference $S^{(0+1)} - S^{(0)}$ for $N = 100$ and $\delta_\lambda = 0.1$.

Figure 10. (a) Perturbed entanglement for $\delta_U = 0.01$ as a function of $m_0$ and $\theta$ for $N = 100$ and (b) the difference $S^{(0+1)} - S^{(0)}$ for $N = 100$ and $\delta_U = 0.1$.

Figure 11. (a) Perturbed entanglement for $\delta_\Lambda = 0.01$ as a function of $m_0$ and $\theta$ for $N = 100$ and (b) the difference $S^{(0+1)} - S^{(0)}$ for $N = 100$ and $\delta_\Lambda = 0.1$. 
\( \delta \mu \) and evolution of the relative population

With an analytic solution to the system we may study the evolution of relative population. Figure 13 (a) shows the unperturbed evolution of relative population for \( \delta \omega = 0.01 \) as a function of \( m_0 \) and \( \theta \) for \( N = 100 \) and (b) the difference \( S^{(0+1)} - S^{(0)} \) for \( N = 100 \) and \( \delta \omega = 1.0 \).

Figure 12. (a) Perturbed entanglement for \( \delta \omega = 0.01 \) as a function of \( m_0 \) and \( \theta \) for \( N = 100 \) and (b) the difference \( S^{(0+1)} - S^{(0)} \) for \( N = 100 \) and \( \delta \omega = 1.0 \).

Figure 13. (a)–(e) Plots of \( S^{(0+1)} - S^{(0)} \) for \( \omega, \lambda, U, \Lambda \) and \( \mu \), respectively, for \( \theta = \frac{\pi}{4} \) as a function of \( m_0 \). The perturbation strengths used are all \( \delta = 0.1 \).

Figure 13 is a two-dimensional cut of figures 8(b)–12(b), where we have fixed \( \theta = \frac{\pi}{4} \) and allowed \( m_0 \) to vary. We observe that in each of the plots \( \Delta S \) has both a local maximum and local minimum as \( m_0 \) approaches \( N \).

Finally, we observe that the most significant changes occur in the region \( m_0 > 0 \), where more particles lie in the \( a \) mode. Positive \( m_0 \) implies that, for fixed \( A_1 > 0 \), the scattering length for same-mode collisions is positive and as \( m_0 \) approaches \( N \), the collision rate becomes smaller. We then conclude that condensates with negative scattering lengths are more resilient to parameter perturbations and high same-mode collision rates help stabilize the condensate.

6. Evolution of relative population

With an analytic solution to the system we may study the evolution of the relative population \( \langle \hat{\rho}(t) \rangle := \langle \psi(t) | \hat{\rho} | \psi(t) \rangle \) as a function of time, where the initial state is given by \( | \psi(t = 0) \rangle = \sum_{n,m=-N}^{N} C_n U^{|n,m} \rangle \). For simplicity we restrict our study to the case in which \( C_m \in \mathbb{R} \). To first order in the perturbation parameter we have

\[
\langle \hat{\rho}(t) \rangle^{(0+1)} = \sum_{m=-N}^{N} C_m e^{-i E_{m}^{(0)} t} U^{|n,m\rangle} + \sum_{m=-N}^{N} \sum_{k=-N}^{N} a_{m,k} |N,k\rangle \cdot
\]

We find \( \langle \hat{\rho}(t) \rangle^{(0+1)} = \langle \hat{\rho}(t) \rangle + \langle \hat{\rho}^{(1)}(t) \rangle \) where

\[
\langle \hat{\rho}(t) \rangle = \cos \theta \sum_{m=-N}^{N} m C_m^2 + \sin \theta \sum_{m=-N}^{N-2} C_m C_{m+2} \times \sqrt{N(N+2) - m(m+2)} L_m(t)
\]

with \( L_m := \cos (\phi + (E_{m+2} - E_m) t) \) and

\[
\langle \hat{\rho}^{(1)}(t) \rangle = 2 \sum_{m=-N}^{N} \sum_{l=-N}^{N} a_{m,l} C_m C_l \cos \theta \cos (E_l - E_m) t
\]

\[
+ \sin \theta \left( \sum_{l=-N}^{N-2} a_{m,l} C_m C_{l+2} \sqrt{N(N+2) - l(l+2)} \times \cos [\phi + (E_{l+2} - E_m) t]
\]

\[
+ \sum_{l=-N}^{N} a_{m,l} C_m C_{l-2} \sqrt{N(N+2) - l(l-2)} \times \cos [\phi - (E_{l-2} - E_m) t] \right).
\]

Figure 14(a) shows the unperturbed evolution of relative population given by equation (25). We see the Rabi-like oscillations with relative population collapse and revival. In figures 14(b) and (c) the evolution of relative population under the parameter perturbations \( \delta \omega = 0.005 \) and \( \delta \omega = 0.05 \) is shown. As \( \delta \omega \) increases the time-averaged value of \( \langle \hat{\rho} \rangle^{(0+1)} \) decreases; from equation (26) changing the sign of \( \delta \omega \) would have increased this average value. We see in figure 14(c)
that the perturbation has broken down; the maximum value of \(|\langle \hat{m} \rangle^{(0\pm 1)}|\) is greater than the total particle number. We also see in figure 14(c) that as \(\delta_\omega\) grows the time of population collapse significantly decreases. Because of the complexity of the correction term (equation (26)) we do not attempt an analytic study of the perturbative effects on population collapse and revival times. Finally, we note that \(\langle \hat{m} \rangle^{(0\pm 1)}(t)\) has non-trivial time dependence if and only if the initial state is entangled.

7. Degenerate perturbations

Throughout the above analysis we have assumed that the unperturbed states are non-degenerate. We proceed now to study the degenerate case, which results for specific values of the constants \(A_1\) and \(A_2\). With the total particle number \(N\) fixed, two distinct states, labelled by relative population numbers \(m_1\) and \(m_2\), have the same energy precisely when \(m_1 + m_2 = \frac{A_2}{A_1}\). From the analysis of the perturbations completed above we know that the perturbation matrix elements are non-vanishing only if \(m_1 - m_2 \in [2, 4]\), where we have assumed without loss of generality that \(m_1 > m_2\). We conclude that there are at most two pairs of degenerate states.

Consider as an example the perturbation \(\omega \mapsto \omega + \delta_\omega\) with \(A_1 = -[N + (N - 2)] A_2\). Then the only pair of degenerate states is \(|N, N\rangle\) and \(|N, N - 2\rangle\); each have energy \(A_2 N (2 - N)\). The matrix of interest is

\[
\Delta_\omega = \begin{pmatrix}
\langle N, N | \hat{H}_\omega | N, N \rangle & \langle N, N | \hat{H}_\omega | N, N - 2 \rangle \\
\langle N, N - 2 | \hat{H}_\omega | N, N \rangle & \langle N, N - 2 | \hat{H}_\omega | N, N - 2 \rangle
\end{pmatrix};
\]

(27)

its eigenvalues and eigenvectors yield the first-order energy and wavefunction corrections. Using the calculations performed above it remains to find the solutions \(\epsilon_\pm\) of the quadratic \(\det(\Delta_\omega - \epsilon I_{2\times2}) = 0\). The eigenvalues and corresponding eigenvectors of \(\Delta_\omega\) are

\[
\epsilon_\pm = (N - 1) \cos \theta \pm \sqrt{(N - 1) \cos^2 \theta}
\]

(28a)

\[
| \pm \rangle = A_\pm \langle \sqrt{N} \sin \theta | N, N - 2 \rangle \\
+ (\cos \theta \pm \sqrt{(N - 1) \cos^2 \theta} | N, N \rangle)
\]

(28b)

with \(A_\pm\) a suitable normalization constant.

Figure 15(a) plots \(\epsilon_\pm\) as a function of \(\theta\) for \(N = 1000\) and \(\delta_\omega = 0.01\). We see that the energy is lowered for all values of \(\theta\); had \(\delta_\omega\) been negative the opposite would have been true. Figures 15(b) and (c), which plot the perturbed particle distributions for \(\delta_\omega = \pm 0.01\), show that even for very small perturbations in \(\omega\) there is a noticeable change in the particle distribution. This is in contrast to non-degenerate perturbations of \(\omega\), where even for \(\delta_\omega = 15\) there was not a large change in the particle distribution. The induced change

![Figure 14](image1.png)

Figure 14. (a) The unperturbed evolution of relative probability \(\langle \hat{m} \rangle(t)\) for \(N = 50\). (b), (c) The perturbed evolution \(\langle \hat{m} \rangle^{(0\pm 1)}(t)\) given by equations (25) and (26) for \(N = 50\) with \(\delta_\omega = \frac{1}{50}\) and \(\delta_\omega = \frac{1}{500}\), respectively.

![Figure 15](image2.png)

Figure 15. (a) \(\epsilon_+\) (solid) and \(\epsilon_-\) (dot) from equation (28a) for degenerate perturbations with \(N = 1000\) and \(\delta_\omega = 1\) as a function of \(\theta\). (b), (c) The corresponding unperturbed (dot) and perturbed (solid) particle distributions \(P\) with \(N = 1000\), \(\theta = 1\) for \(\delta_\omega = 0.01\) and \(\delta_\omega = -0.01\), respectively.
in the particle distribution is also qualitatively different from that in the non-degenerate case. This arises because the correction in the degenerate case is sinusoidal with frequency much greater than that of the unperturbed particle distribution. Regardless of the sign of $\delta_\omega$, the central maximum of the particle distributions is shifted to smaller values of $m$.

Figure 16(a) shows the perturbed entanglement as a function of $\theta$ and $m_0$ for $\delta_\omega = 0.005$ and $N = 100$, while (b) plots the difference $S^{(0+1)} - S^{(0)}$ with $\delta_\omega = 0.005$ for $N = 100$.
Therefore, we find that high collision rates also help stabilize the condensate against perturbations in the degenerate case. However, in this case, condensates with positive scattering lengths are more stable.

Similarly, we find that for a perturbation \( \lambda \mapsto \lambda + \delta \lambda \), assuming again that \(|N, N\rangle\) and \(|N, N - 2\rangle\) are the degenerate states, the energy corrections are

\[
\frac{\epsilon_{\delta \lambda}}{\delta \lambda} = (N - 1) \sin \theta \pm \sqrt{N - (N - 1) \sin^2 \theta}, \tag{29}
\]

which is just the energy correction equation for perturbations in \( \omega \) with the substitution \( \cos \theta \mapsto \sin \theta \); we obtain the wavefunction corrections \(|\pm\rangle\) for \( \delta \lambda \) in the same manner. Figure 17 plots the perturbed energy and perturbed particle distributions for \( \delta \lambda = 1 \) and \( \delta \lambda = \pm 0.01 \). The same comments made above about figure 15 for perturbations \( \delta \lambda \) hold in this case as well. In figure 18 we plot the perturbed entanglement and entanglement difference \( \Delta S \) for \( \delta \lambda = 0.005 \). The figure shows similar behaviour to figure 16. However, we note that in the case of \( \lambda \) perturbations \( \Delta S \) is strictly positive (for positive \( \delta \lambda \)). There are again extrema (this time both maxima) for \( m_0 = -N \) and \( \theta \approx \frac{3\pi}{4}, \frac{5\pi}{4}, \) with \( \Delta S \) vanishing as \( m_0 \) approaches \( N \). Also as in the case above, the system is very sensitive to perturbations, with \( \Delta S \approx 10 \) at some points. Note that high collision rates help stabilize the condensate against perturbations in \( \lambda \) for both positive and negative scattering lengths.

The study of degenerate perturbations in the remaining parameters is completed in the same way as is done above, so we omit these.

### 8. External perturbations

As mentioned in the introduction it is of interest to study perturbations that model additional interaction terms not included in the original Hamiltonian (equation (1)). In what follows we will largely be interested in perturbations that do not preserve the total number of particles in the system. We begin with a general discussion of loss terms and proceed to use this formalism to discuss the effects of background collisions and TBR.

#### 8.1. General loss terms

The Hamiltonian \( H_2 \) we are studying also does not include terms that do not preserve the total particle number. In the system, the effects of particle loss terms are expected to be minimal, we may treat loss terms as perturbations to the system. A primary source of particle loss is inelastic collisions [11, 16]. In magnetic trap particle-type exchange terms dominate loss mechanisms, while in optical traps, these terms may be neglected [17].

We briefly comment on the applicability of the perturbative approach to particle loss. We suppose that a very small number of particles escape the trap by either gaining enough momentum after a collision or by forming a molecule which is not trapped by the potential. The loss of particles in BECs is commonly analysed employing a classical rate equation [11]. However, since we have at hand a solvable quantum model (in the unperturbed case) we can treat particle loss within a quantum framework. More specifically, we consider three-body collisions where most events lead to situations where molecules do not form. Such situations can be engineered experimentally; it has been shown that it is now possible to inhibit molecule TBR in atomic BECs via the application of resonant 2\( \pi \) laser pulses [18]. Therefore, most particles remain in the condensate. This three-body interaction is included in the model which has an exact analytical solution. Since the particle loss is extremely small, we assume that the system remains coherent to a very good approximation. The small amount of particle loss is then treated as a perturbation through a non-Hermitian term in the Hamiltonian. An alternative approach within the quantum formalism is to consider a master equation. Such analysis will be considered in future investigations.

The most general loss term can be written as

\[
H_{\text{loss}} = \sum_{k_a=0}^{N_a} \sum_{k_b=0}^{N_b} f_{k_a,k_b} (\hat{n}_{k_a}, \hat{n}_{k_b}) a^{k_a} b^{k_b}, \tag{30}
\]

where \( f_{k_a,k_b} \) are some functions and \( N = N_a + N_b \) is the total particle number of the condensate. By probabilistic arguments we expect \( f_{k_a,k_b} \rightarrow 0 \) as \( k_a + k_b \) grows so that the cases of primary interest are those with \( k_a \) and \( k_b \) small. However, it should be noted that higher order collisions (i.e. not just \( k_a = 1, 2, k_b = 1, 2 \)) are of physical significance, particularly when the condensate is in its coldest phase and of high particle density [11, 16]. By suitably choosing \( f_{k_a,k_b} \) we can model specific loss terms.

In order to study loss terms we must first make some adjustments to the analysis performed above. The full Hilbert space \( \mathcal{H} \) of the Hamiltonian (equation (2)) can be orthogonally decomposed as the Fock space \( \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \) where, in the \( \{N, m\} \) basis,

\[
\mathcal{H}_n = \{ |n, m\rangle \mid m \in \{0, -n + 2, \ldots, n - 2, n\} \}. \tag{31}
\]

Since the Hamiltonian and perturbations we have considered so far have all commuted with the total number operator we have been able to first choose a total particle number \( N \) for the system, or equivalently the subspace \( \mathcal{H}_N \subset \mathcal{H} \) of the total Hilbert space, and then proceed with calculations. In order to study loss terms we must enlarge the state space to be \( \mathcal{H}_A = \bigoplus_{a \in A} \mathcal{H}_a \) for \( A \subset \{0\} \cup \mathbb{N} \) is the set of all accessible total particle numbers. For example, if \( H_{\text{loss}} \propto a^2 \), then \( A = \{N, N - 2\} \).

Although the energy \( E_m \) of the state \(|N, m\rangle\) is only functionally dependent on \( m \) it has an implicit dependence on \( N \) since \( N \) restricts the values of \( m \). So while \(|N, m\rangle\) is non-degenerate as an element of \( \mathcal{H}_N \) it may be degenerate as an element of \( \mathcal{H}_A \). Considering again the Hamiltonian (equation (2)) we see that \(|N, m\rangle \in \mathcal{H}_N \oplus \mathcal{H}_{N-2} \) is non-degenerate if \( m = \pm N \) and degenerate otherwise. In general, let \( S_A(m) = \{|n, m\rangle \mid n \in A\} \) and say that \( S_A(m) \) is degenerate if it contains more than one element and say it is non-degenerate otherwise.

We examine the effects of the degeneracy \( S_A(m) \) below. Note that the degeneracy studied in this section is caused by the interactions (which determine the Hilbert space), whereas
the degeneracy studied in the previous section was caused by a specific choice of the coupling constants $A_1$ and $A_2$.

In order to study the particle distribution of the perturbed state $|N, m_0\rangle^{(0\alpha)}$ we must modify equation (18); if we were to use this formula there would be no perturbative effects on the particle distribution since $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal if $i \neq j$. A suitable generalization is given by

$$P_{\text{gen}} = \left| \sum_{n \in \mathcal{A}} (n, m|U|\langle N, m_0\rangle^{(0\alpha)}) \right|^2.$$  

(31)

In the case of no loss terms $\mathcal{A} = [N]$ and $P_{\text{gen}}$ reduces to equation (18).

Note that the state of the system after losing particles can become mixed. Since we are working perturbatively we neglect this effect and assume that the resultant state of the system after particle loss remains pure, albeit slightly modified as given in the examples we discuss below.

8.2. Effects of the degeneracy of $S_A(m)$

If $S_A(m)$ is degenerate, then the matrix of interest to degenerate perturbation theory $\langle n_1, m_1|H_{\text{loss}}|n_2, m_2\rangle$ is triangular with zeros along the diagonal. Indeed, $H_{\text{loss}}$ contains only annihilation terms and conjugation by $U$ maps annihilation operators to annihilation operators. So $H_{\text{loss}}|n_1, m_1\rangle$ is a sum of states with total particle number $n$ less than $n_1$. In particular, the matrix above has a spectrum consisting only of zeros and so we can learn nothing from first-order perturbation theory.

Alternatively, if $S_A(m)$ is non-degenerate we may apply the tools of non-degenerate perturbation theory. Although it is straightforward to compute the matrix elements of $H_{\text{loss}}$ given by equation (30) in general, the requirement that $S_A(m)$ be non-degenerate severely limits the usefulness of such a calculation. We instead focus on some specific choices of $f_{k\alpha}, k\alpha$ that model interactions of physical interest.

8.2.1. Background collisions. Background collisions most often occur in BECs when particles from the condensate collide with a residual background gas in the condensate chamber, or alternatively, with metastable atoms within the condensate [19]. As a simple illustration of how we may treat background collisions as perturbations, consider the case in which the initial state is $|N, N\rangle$, so that only $a$ mode particles can be ejected. The general loss term\(^{10}\) (equation (30)), after conjugation by $U$, is written as $H_{\text{loss}} = \sum_{k=1} c_k \cos k \theta d_k^+ + O(b)$ where we use $O(b)$ to denote terms with more $b$ powers than $b^0$ powers. Since any term of $O(b)$ annihilates $|N, N\rangle$ the perturbative correction can be found by neglecting all such terms. We compute the desired matrix elements to find that

$$\langle N, N|1 = \sum_{k=1}^{N} \alpha_k \cos k \theta \sqrt{\prod_{j=0}^{N-k-1} (N-j)} |N-k, N-k\rangle.$$  

(32)

which yields the generalized particle distribution

$$P_{\text{gen}} = |d_{m,N}^N|^2 + 2 \sum_{k=1}^{N} \alpha_k \cos k \theta \sqrt{\prod_{j=0}^{N-k-1} (N-j)} d_{m,N-k}^N d_{m,N-k}.$$  

(33)

where we set $d_{m,N-k}^N = 0$ if $|m| > N - k$. From this expression we see that we could have omitted terms that eject an odd number of $a$-mode particles. Indeed, if $k$ is odd, then $d_{m,N-k}^N$ is identically zero as a function of $m$ since if $d_{m,n}^N \neq 0$, then $N, m$ and $n$ all have the same parity\(^{11}\). We plot $P_{\text{gen}}$ in figure 19, considering terms that eject 2, 4 and 6 particles for $N = 1000$ and $\theta = 1$. In figure 19(a) we set $\alpha_2 = -0.1$, $\alpha_4 = -0.001$ and $\alpha_6 = -5 \times 10^{-6}$. We take $\alpha_i < 0$ since the inclusion of a loss term should decrease the energy of the system. In figures 19(b)–(d) we increase each $|\alpha_i|$ and therefore the height of the particle distribution. Also evident from the figures and the scale of the $\alpha_i$ is that as $i$ increases the perturbations have a larger effect. The fact that the perturbations do not blow up reflects the requirement that $f_{k\alpha}, k\alpha \rightarrow 0$ for large $k\alpha$. Indeed, we have found using numerical simulations that the perturbations that eject $k$ particles diverge with increasing $k$.

We remark here that as a consequence of $d_{m,N-k}^N d_{m,N-k}$ vanishing for $k$ odd we cannot learn anything about spin–flip terms from first-order perturbation theory. Indeed, such terms would be modelled by perturbations of the form $b^{\alpha}a$ and $b^{\alpha}a^\dagger$.

\(^{10}\) Explicitly, we take $f_{k\alpha}, k\alpha = 0$ if $k\alpha \neq 0$ and $f_{k\alpha}, k\alpha = \alpha_k \in \mathbb{R}$ otherwise.

\(^{11}\) We can, however, use second-order perturbation theory where terms annihilating an odd number of particles will have an effect.
a^1 aa$, assuming that particles in the $a$ mode have the greatest energy; these terms clearly reduce the total number of the system by an odd number.

Setting $P^{(1)}$ equal to the second term on the right-hand side of equation (33), we can use equation (24) to compute and increase the entanglement. Since $P^{(1)}$ depends linearly on the interaction strengths $\alpha_k$, we see the most obvious manner in which to increase the entanglement is to make $|\alpha_k|$ large; the sign of $\alpha_k$ depends on $k, \theta$ as well as $A_1$ and $A_2$.

Figure 20 plots the perturbed entanglement caused by the inclusion of background collisions as a function of $m_0$ and $\theta$. We see that the largest effect of the perturbation occurs in the region $\theta \approx \pi/4$ and large negative values of $m_0$. The region with large negative $m_0$ corresponds to the case where the scattering length between particles is negative, in which most particles lie in the $b$ mode of the condensate. Since the background collisions considered here eject particles from the $a$ mode they serve to further decrease the value of $m$. We understand the large effect of the perturbation on the aforementioned region as follows: since most particles lie in the $b$ mode, ejecting any particle from the $a$ mode has a large effect on the system, since it already has only a small number of $a$ mode particles relative to the number of $b$ mode particles. Figure 20(b) shows that for values of $m_0$ above the region in question, where the $b$ mode particles become more scarce, the perturbation has little effect on the system. Again, we understand this as being because ejecting an $a$ mode particle from a state with large $m$ value is of little significance to the system as a whole. We also see from this figure that the entanglement is decreased, regardless of $m_0$ and $\theta$, for this specific choice of $\alpha_i$.

8.2.2. Three-body recombination. Perhaps the most physically important type of inelastic collision leading to particle loss is TBR [20]. TBR occurs when three particles in a single mode collide to form a diatomic molecule and a particle of the same mode that carries off any excess energy. Depending on this energy and the energy of the potential trap the resultant particle may or may not escape the trap [20].

A perturbation modelling TBR would be most naturally treated in the $\mathfrak{su}(3)$ formalism where the extra Bose operator would correspond to the diatomic molecule. However, in the $\mathfrak{su}(2)$ formalism we have no such third mode. We thus model TBR by the Hamiltonian

$$H_{\text{TBR}} = C \left( \sigma a^1 aa + (1 - \sigma)aaa \right).$$

(34)

The parameter $\sigma$ describes the probability of the emitted particle remaining trapped in the condensate. The term proportional to $aaa$ lowers the total particle number by an
odd number and hence, as explained above, will have no effect on the perturbed particle distribution. We thus neglect this term from our analysis and absorb the constant $C$ into $\sigma$. Note that $\sigma$ can now be negative. With $H_{\text{TBR}}$ as a perturbation we have $A = \{N, N - 2, N - 3\}$. We take as our unperturbed state $|N, N\rangle$, which is non-degenerate. Proceeding, we find

$$|N, N\rangle^{(1)} = \sigma \frac{\cos^{4} \frac{\theta}{2} \sqrt{N(N-1)(N-2)}}{2A_1 + 4A_2(N-2)} |N - 2, N - 2\rangle$$

(35)

which gives the generalized particle distribution

$$P_{\text{gen}}(m) = \left|d_{m,N}^N\right|^2 + 2\sigma \frac{\cos^{4} \frac{\theta}{2} \sqrt{N(N-1)(N-2)}}{2A_1 + 4A_2(N-2)} d_{m,N}^{N-2} = 0$$

(36)

$P_{\text{gen}}$ given by equation (36) is plotted in figure 21. As expected the sign of $\sigma$ determines whether the perturbation vertically shrinks or stretches the particle distribution. We also see that the system is very sensitive to TBR terms, as a perturbation of order 0.001 causes significant changes to the particle distribution.

Figure 22 plots the perturbed entanglement for $\sigma = -0.5$. We choose $\sigma < 0$ since we expect the inclusion of a TBR term to decrease the energy of the system. As figure 22(b) shows, the TBR term decreases the entanglement of the system. The system is most significantly affected near $\theta \approx \frac{\pi}{2}$ and large negative $m_0$. This can also be seen in figure 22(a) as the perturbation causes a small local minimum of the entanglement in this region. For large positive $m_0$ we see that the perturbation has very little effect on the entanglement. The same reasoning as used in the discussion above of background collisions can be used to explain why the perturbations for large positive $m_0$ do not have a large effect.

9. Discussion

We have successfully studied the effects of a number of perturbations on the two-mode BEC model considered in [6]. We have found the corrections to the condensate wavefunctions, which in turn allowed the determination of the corrections to the particle distribution, time evolution of the relative number operator and the entanglement. In the non-degenerate case we have shown that the model of [6], in which the coupling constants of the Hamiltonian are constrained, is robust to perturbations in these constants. The system is most sensitive to perturbations in the elastic scattering length $l_A$ and mode-exchange parameters $\Lambda$ and $\mu$. In each of the parameter perturbations the entanglement of the coherent states $|N, N\rangle$ and $|N, -N\rangle$ is most affected. We also observed that for specific values of $m_0$ and $\theta$, the entanglement perturbations are especially large. It was found that when the condensate is degenerate (because of specific choices of $A_1$ and $A_2$) it is much more sensitive to perturbations, both in terms of particle distributions and entanglement. The effects on the entanglement are qualitatively different than in the non-degenerate case. In particular, the perturbations to the entanglement are mainly present in the regions in which $m_0$ is close to $-N$ which corresponds to a condensate with small negative scattering length.

We have also extended the formalism to include the analysis of interactions involving particle loss. From these we can predict corrections to the particle distribution, entanglement and evolution of the relative population. The changes induced by TBR and background collisions are qualitatively different than those induced by parameter perturbations. We find that the system is very sensitive to these external perturbations, which induce large changes in the particle distribution and entanglement from relatively small external coupling strengths, as compared to the induced changes from parameter perturbations. As with the degenerate parameter perturbation, we found that the perturbative effects on the entanglement become negligible as $m_0$ approaches $N$, i.e. when the scattering length is small and positive. In general, we can conclude that higher collision rates make the condensate more stable to perturbations.

Our results promise to be useful in the experimental realization of two-mode BECs which are stable to parameter perturbations and particle loss. We plan to extend our analysis to include many-body interactions which are present at cooler stages of the condensate and study the role of such interactions in the stability of the condensate. A numerical computation of the effects of our Hamiltonian that goes beyond
perturbation theory would be interesting, both to corroborate the results obtained here and to see how they are manifest non-perturbatively.

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Appendix A. The Schwinger $su(2)$ Boson representation

Let $J_z$ and $J_x = J_z \pm i J_y$ be the usual generators of the Lie algebra $su(2)$, satisfying

$$[J_z, J_x] = \pm J_x, \quad [J_x, J_y] = 2 J_z.$$  \hspace{1cm} (A.1)

The eigenstates are labelled as $|j, m_{\text{ang}}\rangle$ where

$$J_z |j, m_{\text{ang}}\rangle = m_{\text{ang}} |j, m_{\text{ang}}\rangle, \quad J_x |j, m_{\text{ang}}\rangle = j (j + 1) |j, m_{\text{ang}}\rangle.$$  \hspace{1cm} (A.2)

Note that for fixed $j$, $m_{\text{ang}}$ may take any of the $2j + 1$ values $-j, -j + 1, \ldots, j$. We also have $J_z |j, m_{\text{ang}}\rangle = \sqrt{j(j+1) - m_{\text{ang}}(m_{\text{ang}} + 1)} |j, m_{\text{ang}}\rangle$. The Schwinger representation of $su(2)$ defines a Lie algebra homomorphism between the angular momentum representation (generated by $J_\alpha$ and $J_\beta$) and a bosonic representation.

We may group the operators $\{a, a^\dagger\} = \{b, b^\dagger\}$ and all other pairs having vanishing commutator. Define the mapping

$$J_x \mapsto a^\dagger b, \quad J_y \mapsto ab^\dagger, \quad J_z \mapsto \frac{1}{2} (a^\dagger a - b^\dagger b).$$ \hspace{1cm} (A.3)

and extend linearly. This gives the desired homomorphism [21]. A short calculation shows that $J_x$ is mapped to $\frac{1}{2}N^x + \frac{1}{2}N^y$ where we have defined $N^x \equiv a^\dagger a + b^\dagger b$. We label the basis states in the bosonic representation as $|\frac{N}{2}, \frac{n}{2}\rangle$, with $N$ the eigenvalue of $\hat{N}$ and $m$ the eigenvalue of $\hat{m} \equiv a^\dagger a - b^\dagger b$; this is in complete analogy with the label $|j, m_{\text{ang}}\rangle$.

Now rescaled, $m$ may take on the $N + 1$ values $-N, -N + 2, \ldots, N - 2, N$. Similarly we have

$$ab^\dagger |N, m\rangle = \frac{1}{2} \sqrt{N(N+2) - m(m+2)} |N, m+2\rangle.$$ \hspace{1cm} (A.4)

Then $a^\dagger b^\dagger |N, m\rangle$ is mapped to a multiple of $|N, m+2\rangle$ which annihilates a particle in mode $b$ while creating one in mode $a$. Since $j(k - 1) + 1$ is an integer greater than or equal to $q$, we may group the monomials of $H$ into three groups according to whether there are less (more) $a$ modes than $b$ modes, or whether they are equal in number. Doing so we find

$$\chi(d = 2k) = \sum_{p=0}^{k-1} \left( \left( \left\lfloor \frac{p}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{k}{2} \right\rfloor + 1 \right).$$ \hspace{1cm} (A.5)

Since $\chi(k-1) + \left\lfloor \frac{k}{2} \right\rfloor = k - 1$, we obtain the recursion relation

$$\chi(2k) = \chi(2(k-1)) + k + 1.$$  \hspace{1cm} (A.6)

Repeating application of this relation yields $\chi(2k) = \chi(2) + \sum_{j=1}^{k+1} J$. Using $\chi(2) = 3$ gives the desired result.

Therefore we can decompose any polynomial into the sum of its homogeneous pieces, the above proposition is sufficient to count the maximum number of terms in the most general Hamiltonian of the class described above.
Corollary B.2. Let $H$ be in the desired class of Hamiltonians and suppose $H$ is of degree $n \geq 0$. Then the number of independent monomials in $H$ is at most $\frac{1}{28}n^3 + \frac{1}{2}n^2 + \frac{11}{12}n + 1$.

We see that the total number of parameters of the most general allowed Hamiltonian involving up to $n$-body interactions grows like $n^3$. We would like to compare this result to the number of terms in the $n$-model of [6, 7]. Unfortunately, we have been unable to count the number of terms in the $n$-model. In order to do so one would need to conjugate

$$H_{0,n} = \sum_{i=0}^{n} A_i (a^\dagger a - b^\dagger b)^i$$

by the displacement operator $U(\xi)$. The number of terms in this calculation grows quickly with $n$ which makes the conjecture of a formula for the number of terms difficult. Regardless, explicit calculations for $n \leq 3$, summarized in table A1, suggest that there are a significant number of terms missed by the $n$-model. Whether or not these terms are important in an experimental setting is another question. Note that the number of independent parameters in the $n$-model is $n + 1$ and is much smaller than the number of terms appearing. That is, the coefficients of the interaction terms appearing are not all independent.

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