Algebras of Functions with Fourier Coefficients in Weighted Orlicz Sequence Spaces

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Abstract. We prove that the set of all integrable functions whose sequences of negative (resp. nonnegative) Fourier coefficients belong to \( \ell_1 \cap \ell_{\Phi,\psi,w} \) (resp. to \( \ell_1 \cap \ell_{\Psi,\psi,\varrho} \)), where \( \ell_{\Phi,\psi,w} \) and \( \ell_{\Psi,\psi,\varrho} \) are two-weighted Orlicz sequence spaces, forms an algebra under pointwise multiplication whenever the weight sequences \( \varphi = \{ \varphi_n \} \), \( \psi = \{ \psi_n \} \), \( w = \{ w_n \} \), \( \varrho = \{ \varrho_n \} \) increase and satisfy the \( \Delta_2 \)-condition.

1. Introduction

Let \( T \) be the unit circle. For a complex-valued function \( f \in L^1(T) \), let \( \{ f_n \}_{n \in \mathbb{Z}} \) be the sequence of the Fourier coefficients of \( f \),

\[ f_n := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-i n \theta} d\theta. \]

Let \( W \) be the Wiener algebra of all functions \( f \) on \( T \) for which

\[ \| f \|_W := \sum_{k=-\infty}^{\infty} |f_k| < \infty. \]

Let \( F_{\alpha,\beta}^{p,r} \), where \( 1 \leq p, r < \infty \) and \( 0 \leq \alpha, \beta < \infty \), denote the set of all functions \( f \) on \( T \) for which

\[ \| f \|_{F_{\alpha,\beta}^{p,r}} := \left( \sum_{k=1}^{\infty} |f_{-k}|^p (k+1)^{\alpha p} \right)^{1/p} + \left( \sum_{k=0}^{\infty} |f_k|^r (k+1)^{\beta r} \right)^{1/r} < \infty. \]

The following result was conjectured by A. Böttcher and B. Silbermann and proved by Detlef Horbach. The proof is in [11] Section 6.54.

Theorem 1.1. If \( 1 \leq p, r < \infty \) and \( 0 \leq \alpha, \beta < \infty \), then \( W \cap F_{\alpha,\beta}^{p,r} \) is an algebra under pointwise multiplication.

1991 Mathematics Subject Classification. Primary 46J10; Secondary 46B45.
Key words and phrases. Wiener algebra, weighted Orlicz sequence space.
The author is supported by F.C.T. (Portugal) grant SFRH/BPD/11619/2002.
Note that a stronger fact is actually proved in [1] Section 6.54: the inequality
\[
\|f^2\|_{F^{p,r}_{\alpha,\beta}} \leq C_1 \|f\|_{W} \|f\|_{F^{p,r}_{\alpha,\beta}}
\]
is fulfilled for some \(C_1 > 0\) and any \(f \in W \cap F^{p,r}_{\alpha,\beta}\).
From the latter inequality and \(\|fg\|_W \leq \|f\|_W \|g\|_W\) one can easily get
\[
\|fg\|_W + \|fg\|_{F^{p,r}_{\alpha,\beta}} \leq C \left( \|f\|_W + \|f\|_{F^{p,r}_{\alpha,\beta}} \right) \left( \|g\|_W + \|g\|_{F^{p,r}_{\alpha,\beta}} \right)
\]
for some \(C > 0\) and all \(f, g \in W \cap F^{p,r}_{\alpha,\beta}\), that is, \(W \cap F^{p,r}_{\alpha,\beta}\) is a Banach algebra. This result has important applications in the theory of Toeplitz determinants and the
theory of Toeplitz operators on Lebesgue spaces \(l_p(1 < p < \infty)\) with Khvedelidze
weights \(\rho\) (see [1] Ch. 7 and Ch. 10). The aim of the present note is to generalize
this result by replacing in the definition of \(F^{p,r}_{\alpha,\beta}\)
(1) norms in weighted Lebesgue sequence spaces by norms in two-weighted
Orlicz sequence spaces;
(2) canonical weight sequences \((n+1)^\lambda\) for \(\lambda \in \{\alpha, \beta\}\) by general
increasing weight sequences satisfying the \(\Delta_2\)-condition.

The author hopes that such an extension will be useful because the scale of Orlicz
spaces is much wider and sensitive than the scale of Lebesgue spaces.

Note also that similar questions were considered by P. L. Ul’yanov [5] for the
set \(E\) of functions \(f \in C(T)\) with
\[
\|f\|_E := \sum_{k \in \mathbb{Z}} \omega(|f_k|) \tau_n < \infty,
\]
where \(\omega\) is a function with certain properties (a so-called modulus of continuity)
and \(\{\tau_n\}_{n \in \mathbb{Z}}\) is a weight sequence. Under some natural assumptions he proved that \(E\)
is an algebra under pointwise multiplication.

This note is organized as follows. In Section 2 we remind the notion of two-weighted
Orlicz sequence spaces and define a related class \(F^{\varphi,\psi}_{\varphi,w,\psi,\rho}\) that generalizes
the class \(F^{p,r}_{\alpha,\beta}\). Further we formulate the main result: the set \(W \cap F^{\varphi,\psi}_{\varphi,w,\psi,\rho}\) is
a Banach algebra under pointwise multiplication whenever the weight sequences
\(\varphi, w, \psi, \rho\) increase and satisfy the \(\Delta_2\)-condition. Finally we state a corollary about
factorization of nondegenerate functions in this algebra. Section 3 contains the
proofs.

2. Preliminaries and the main result

2.1. Weighted Orlicz sequence spaces. An Orlicz function \(\Phi\) is a continuous
non-decreasing and convex function defined for \(t \geq 0\) such that \(\Phi(0) = 0\) and
\(t \to \infty\) (see [3] Definition 4.4.1]). Let \(\mathbb{I}\) be either \(\mathbb{N} := \{1, 2, \ldots\}\) or
\(\mathbb{Z}_+ := \mathbb{N} \cup \{0\}\) and let \(\{\Phi_n\}_{n \in \mathbb{I}}\) be a sequence of Orlicz functions. The set \(\ell^{\Phi,\psi}(\mathbb{I})\)
of all sequences of complex numbers \(c = \{c_n\}_{n \in \mathbb{I}}\) with
\[
\sum_{n \in \mathbb{I}} \Phi_n \left( \frac{|c_n|}{\lambda} \right) < \infty
\]
for some \(\lambda = \lambda(c) > 0\) is a Banach space when equipped with the norm
\[
\|c\|_{\ell^{\Phi,\psi}(\mathbb{I})} = \inf \left\{ \lambda > 0 : \sum_{n \in \mathbb{I}} \Phi_n \left( \frac{|c_n|}{\lambda} \right) \leq 1 \right\}.
\]
The space $\ell^{\Phi_n}(I)$ is called a modular sequence space (see [3] Definition 4.d.1) or a Musielak-Orlicz sequence space (see, e.g., [4]).

Any sequence $\{\nu_n\}_{n \in I}$ of positive numbers is called a weight sequence. Let $\Phi$ be an Orlicz function and $\varphi = \{\varphi_n\}_{n \in I}$ and $w = \{w_n\}_{n \in I}$ be weight sequences. Clearly

$$\Phi_n(x) := \Phi(x\varphi_n)w_n \quad (x \geq 0)$$

is an Orlicz function for every $n \in I$. The special case of a Musielak-Orlicz sequence space generated by the sequence of Orlicz functions $I$ is denoted by $\ell^p_{\varphi,w}(I)$ and is called the two-weighted Orlicz space generated by the Orlicz function $\Phi$ and the weight sequences $\varphi = \{\varphi_n\}_{n \in I}$ and $w = \{w_n\}_{n \in I}$.

In particular, if $\Phi(x) = x^p, 1 \leq p < \infty$, then $\ell^p_{\varphi,w}(I)$ is the weighted Lebesgue sequence space $\ell^p(I)$.

$$\|c\|_{\ell^p_{\varphi,w}(I)} = \left( \sum_{n \in I} |c_n|^p \rho_n \right)^{1/p} \quad (\rho_n = \varphi_n^p w_n, \quad n \in I).$$

So, in this case, $\ell^p_{\varphi_1}(I) = \ell^p_{\varphi_1^{(1)}}(I)$, where $\varphi^p := \{\varphi_n^p\}_{n \in I}$. But in the general case an Orlicz function $\Phi$ is not homogeneous. Hence the weighted Orlicz sequence spaces $\ell^p_{\varphi_1}(I)$ and $\ell^p_{\varphi_1^{(1)}}(I)$ are essentially different.

Applying criteria of coincidence of Musielak-Orlicz sequence spaces and Musielak-Orlicz sequence classes (see [4] Theorem 8.13(b)) to the case of the sequence of Orlicz functions $I$, one can get the following.

**Proposition 2.1.** Suppose $\Phi$ is an Orlicz function, $\Phi(x) > 0$ whenever $x > 0$, and $\varphi = \{\varphi_n\}_{n \in I}, w = \{w_n\}_{n \in I}$ are weight sequences. Then

$$\ell^p_{\varphi,w}(I) = \left\{ c = \{c_n\}_{n \in I} : \sum_{n \in I} |c_n|^{\varphi_n} w_n < \infty \right\}$$

if and only if there exist positive numbers $\delta, K$ and a sequence of nonnegative numbers $\{d_n\}_{n \in I}$ such that for all $x \geq 0$ and $n \in I$,

$$\Phi(x\varphi_n)w_n < \delta \quad \Rightarrow \quad \Phi(2x\varphi_n)w_n \leq K\Phi(x\varphi_n)w_n + d_n$$

and $\sum_{n \in I} d_n < \infty$.

**2.2. The main result.** Let $\Phi$ and $\Psi$ be Orlicz functions and let

$$\varphi = \{\varphi_n\}_{n \in I}, \quad w = \{w_n\}_{n \in I}, \quad \psi = \{\psi_n\}_{n \in I}, \quad \varphi = \{\varphi_n\}_{n \in I}$$

be weight sequences. We denote by $F_{\varphi,w;\psi,\varphi}$ the set of all functions $f \in L^1(\mathbb{T})$ such that the sequence $\{f_{-n}\}_{n \in \mathbb{N}}$ of all negative Fourier coefficients of $f$ belongs to the two-weighted Orlicz space $\ell^p_{\varphi,w}(\mathbb{N})$ and the sequence $\{f_{n}\}_{n \in \mathbb{Z}^+}$ of all nonnegative Fourier coefficients of $f$ belongs to the two-weighted Orlicz space $\ell^p_{\varphi,\varphi}(\mathbb{Z}^+)$. The set $W \cap F_{\varphi,w;\psi,\varphi}$ is a Banach space with respect to the norm

$$\|f\|_{W \cap F} := \|f\|_w + \|f\|_\varphi.$$
\[
\|f\|_- := \left\| \{f_k\}_{k \in \mathbb{N}} \right\|_{L^\Phi,\psi,\tilde{w}(\mathbb{N})} = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \Phi \left( \frac{|f_k| \varphi_k}{\lambda} \right) w_k \leq 1 \right\},
\]
\[
\|f\|_+ := \left\| \{f_k\}_{k \in \mathbb{Z}_+} \right\|_{L^\Phi,\psi,\tilde{w}(\mathbb{Z}_+)} = \inf \left\{ \mu > 0 : \sum_{k=0}^{\infty} \Psi \left( \frac{|f_k| \psi_k}{\mu} \right) \varrho_k \leq 1 \right\}.
\]

We denote by \(W^+\) (resp. by \(W^-\)) the collection of all weight sequences \(\{\nu_n\}_{n=0}^\infty\) (resp. \(\{\nu_n\}_{n=1}^\infty\)) such that

(i) \(\nu_0 > 0\) (resp. \(\nu_1 > 0\));
(ii) \(\nu_n \leq \nu_{n+1}\) for \(n \in \mathbb{Z}_+\) (resp. for \(n \in \mathbb{N}\));
(iii) \(\{\nu_n\}_{n=0}^\infty\) (resp. \(\{\nu_n\}_{n=1}^\infty\)) satisfies the \(\Delta_2\)-condition, that is, there exists a constant \(C_\nu \in (0, \infty)\) such that \(\nu_{2n} \leq C_\nu \nu_n\) for \(n \in \mathbb{N}\).

From (ii) and (iii) it follows that \(C_\nu \geq 1\).

**Theorem 2.2.** If \(\Phi, \Psi\) are arbitrary Orlicz functions, \(\varphi = \{\varphi_n\}_{n=1}^\infty, w = \{w_n\}_{n=1}^\infty\) are weight sequences in \(W^-,\) and \(\psi = \{\psi_n\}_{n=0}^\infty, \varrho = \{\varrho_n\}_{n=0}^\infty\) are weight sequences in \(W^+,\) then for every \(f, g \in W \cap F^\Phi,\Psi_{\varphi,w;\psi,\varrho},\)

\[
\|fg\|_{W \cap F} \leq C \|f\|_{W \cap F} \|g\|_{W \cap F}
\]

where \(C := 1 + 2(1 + C_w)C_\varphi + 2(1 + C_\varrho)C_\psi.\)

This theorem will be proved in the next section.

Clearly, the weight sequences \(\tilde{\varphi} = \{(n+1)^\alpha\}_{n=1}^\infty\) and \(\tilde{\psi} = \{(n+1)^\beta\}_{n=0}^\infty\) belong to \(W^-\) and \(W^+\), respectively, whenever \(0 \leq \alpha, \beta < \infty.\)

If

\[
\tilde{\Phi}(x) := x^p, \quad \tilde{\Psi}(x) := x^r, \quad x \geq 0, \quad 1 \leq p, r < \infty,
\]

then \(F^\Phi,\Psi_{\varphi,1;\psi,1}\) with the norm \(\|f\|_F := \|f\|_- + \|f\|_+\) is isometrically isomorphic to \(F^\tilde{\Phi},\tilde{\Psi}_{\tilde{\varphi},1;\tilde{\psi},1}\). Hence Theorem 1 follows from Theorem 2.2.

**Corollary 2.3.** Under the assumptions of Theorem 2.2 we have the following.

(a) \(W \cap F^\Phi,\Psi_{\varphi,w;\psi,\varrho}\) is a commutative Banach algebra under pointwise multiplication. The maximal ideal space of this algebra coincides with \(\mathbb{T}\).

(b) If \(b \in W \cap F^\Phi,\Psi_{\varphi,w;\psi,\varrho}\) does not vanish on \(\mathbb{T}\) and the Cauchy index of \(b\) vanishes, then \(b\) has a logarithm in \(W \cap F^\Phi,\Psi_{\varphi,w;\psi,\varrho}\).

(c) If we let

\[
G(b) := \exp \left( (\log b)_0 \right), \quad b_\pm(t) := \exp \left( \sum_{n=1}^{\infty} (\log b)_{\pm n} t^{\pm n} \right) \quad (t \in \mathbb{T}),
\]

then \(b = G(b)b_-b_+\) and \(b_{\pm 1} \in W \cap F^\Phi,\Psi_{\varphi,w;\psi,\varrho}\).

**Proof.** By Theorem 2.2 \(W \cap F^\Phi,\Psi_{\varphi,w;\psi,\varrho}\) is a commutative Banach algebra under pointwise multiplication. The description of its maximal ideal space is standard. Part (b) follows from part (a) and [1] Section 2.41(e). Part (c) is an immediate consequence of part (b).
3. Proofs

3.1. Auxiliary results.

Proposition 3.1. If \( \{\nu_n\}_{n=0}^{\infty} \in W^+ \) (resp. \( \{\nu_n\}_{n=1}^{\infty} \in W^- \)) and \( k \in \mathbb{Z}^+ \) (resp. \( k \in \mathbb{Z} \)), then

\[
\nu_k \leq C_{\nu} \nu_j \quad \text{for} \quad j \geq k - [k/2].
\]

Proof. If \( k < 2 \), then \( k - [k/2] = k \). Hence from (ii) and \( C_{\nu} \geq 1 \) we get (3).

If \( k \geq 2 \), then \( 2[k/2] \leq k \). Therefore \( k \leq 2(k - [k/2]) \). In that case from (ii) and (iii) it follows that

\[
\nu_k \leq \nu_{2(k-[k/2])} \leq C_{\nu} \nu_{k-[k/2]} \leq C_{\nu} \nu_j
\]

for \( j \geq k - [k/2] \), i.e., we obtain (3). \( \Box \)

Let \( \{f_n\}_{n=-\infty}^{\infty} \) and \( \{g_n\}_{n=-\infty}^{\infty} \) be the Fourier coefficient sequences of functions \( f \in L^1(T) \) and \( g \in L^1(T) \), respectively. Put \( a_n := |f_n| \) and \( b_n := |g_n| \).

Proposition 3.2. (a) If \( k \in \mathbb{N} \), then

\[
|f g|_k \leq \sum_{j=0}^{\infty} a_j b_{-k-j} + \sum_{j=1}^{[k/2]} \sum_{j=0}^{\infty} a_{-j} b_{-k+j} + \sum_{j=0}^{\infty} \sum_{j=1}^{[k/2]} b_j a_{-k+j}.
\]

(b) If \( k \in \mathbb{Z}_+ \), then

\[
|f g|_k \leq \sum_{j=1}^{\infty} a_{-j} b_{k+j} + \sum_{j=0}^{[k/2]} \sum_{j=1}^{\infty} a_j b_{k-j} + \sum_{j=0}^{\infty} \sum_{j=1}^{[k/2]} b_j a_{k-j}.
\]

Proof. (a) For \( k \in \mathbb{N} \),

\[
|f g|_k = \left| \sum_{j=-\infty}^{\infty} f_j g_{-k-j} \right| \leq \sum_{j=-\infty}^{\infty} a_j b_{-k-j} = \sum_{j=-\infty}^{\infty} a_j b_{-k-j} = \sum_{j=-\infty}^{\infty} a_j b_{-k-j}
\]

\[
= \sum_{j=-\infty}^{-k} a_j b_{-k-j} + \sum_{j=-k+1}^{-1} a_j b_{-k-j} + \sum_{j=0}^{\infty} a_j b_{-k-j} =: \sigma_1 + \sigma_2 + \sigma_3.
\]

Changing variables in \( \sigma_1 \) \( (r = -k - j) \), we obtain

\[
\sigma_1 := \sum_{j=-\infty}^{-k} a_j b_{-k-j} = \sum_{r=0}^{\infty} a_{-k-r} b_r.
\]

Obviously, \( \sigma_2 = 0 \) if \( k = 1 \). Hence (4) follows from (3) and (5) for \( k = 1 \).

If \( k > 1 \), then changing variables in \( \sigma_2 \) \( (r = -j) \) we get

\[
\sigma_2 := \sum_{j=-k+1}^{-1} a_j b_{-k-j} = \sum_{r=1}^{k-1} a_{-r} b_{-k+r}.
\]

If \( k = 2m \) and \( m \in \mathbb{N} \), then

\[
\sigma_2 = \sum_{j=1}^{2m-1} a_{-j} b_{-2m+j} = \sum_{j=1}^{m} a_{-j} b_{-2m+j} + \sum_{j=m+1}^{2m-1} a_{-j} b_{-2m+j}.
\]
Changing variables in the second sum \( (r = 2m - j) \), we obtain

\[
\sum_{j=m+1}^{2m-1} a_{-j} b_{-2m+j} = \sum_{r=1}^{m-1} a_{-2m+r} b_{-r} \leq \sum_{j=1}^{m} b_{-j} a_{-2m+j}.
\]

Since \( [k/2] = [2m/2] = m \), from (5)–(8) we deduce that (4) holds for \( k = 2m \) and \( m \in \mathbb{N} \).

If \( k = 2m + 1 \) and \( m \in \mathbb{N} \), then

\[
\sigma_2 = \sum_{j=1}^{2m} a_{-j} b_{-(2m+1)+j} = \sum_{j=1}^{m} a_{-j} b_{-(2m+1)+j} + \sum_{j=m+1}^{2m} a_{-j} b_{-(2m+1)+j}.
\]

Changing variables in the second sum \( (r = 2m + 1 - j) \), we infer that

\[
\sum_{j=m+1}^{2m} a_{-j} b_{-(2m+1)+j} = \sum_{r=1}^{m} a_{-(2m+1)+r} b_{-r}.
\]

Since \( [k/2] = [(2m+1)/2] = m \), from (9)–(10) and (9)–(10) we conclude that (4) is satisfied for \( k = 2m + 1 \) and \( m \in \mathbb{N} \). Part (a) is proved. Part (b) is proved analogously to Part (a).

3.2. Proof of Theorem 2.2

PROOF. The idea of this proof is borrowed from [1] Theorem 6.54. Let us show that

\[
\|fg\|_{-} \leq C_{-}(\|f\|_{W}\|g\|_{-} + \|g\|_{W}\|f\|_{-}),
\]

where \( C_{-} := (1 + C_{w})C_{\varphi} \), \( C_{\varphi} \) and \( C_{w} \) are the constants in the \( \Delta_2 \)-condition for the sequences \( \{\varphi_n\}_{n=1}^{\infty} \) and \( \{w_n\}_{n=1}^{\infty} \), respectively. If \( \|f\|_{W}\|g\|_{-} + \|g\|_{W}\|f\|_{-} = 0 \), then (11) is obvious.

Assume that \( \|f\|_{W}\|g\|_{-} + \|g\|_{W}\|f\|_{-} > 0 \). Since \( \Phi \) is increasing, from Proposition 3.2(a) it follows that for \( k \in \mathbb{N} \),

\[
\Phi \left( \frac{|(fg)_{-k}|\varphi_k}{C_{-}(\|f\|_{W}\|g\|_{-} + \|g\|_{W}\|f\|_{-})} \right) \leq \Phi \left( \frac{\sum_{i=1}^{6} \sigma_i}{\sum_{i=1}^{6} \sigma_i} \right),
\]
where
\[\Sigma_1 := \sum_{j=0}^{\infty} (a_j \|g\|_{-}) \frac{b_{-k+j} \varphi_k}{C_{-} \|g\|_{-}}, \quad \sigma_1 := \sum_{j=0}^{\infty} a_j \|g\|_{-},\]
\[\Sigma_2 := \sum_{j=1}^{[k/2]} (a_{-j} \|g\|_{-}) \frac{b_{-k+j} \varphi_k}{C_{-} \|g\|_{-}}, \quad \sigma_2 := \sum_{j=1}^{[k/2]} a_{-j} \|g\|_{-},\]
\[\Sigma_3 := \sum_{j<-[k/2]} (a_j \|g\|_{-}) \cdot 0 = 0, \quad \sigma_3 := \sum_{j<-[k/2]} a_j \|g\|_{-},\]
\[\Sigma_4 := \sum_{j=0}^{\infty} (b_j \|f\|_{-}) \frac{a_{-k+j} \varphi_k}{C_{-} \|f\|_{-}}, \quad \sigma_4 := \sum_{j=0}^{\infty} b_j \|f\|_{-},\]
\[\Sigma_5 := \sum_{j=1}^{[k/2]} (b_{-j} \|f\|_{-}) \frac{a_{-k+j} \varphi_k}{C_{-} \|f\|_{-}}, \quad \sigma_5 := \sum_{j=1}^{[k/2]} b_{-j} \|f\|_{-},\]
\[\Sigma_6 := \sum_{j<-[k/2]} (b_j \|a\|_{-}) \cdot 0 = 0, \quad \sigma_6 := \sum_{j<-[k/2]} b_j \|f\|_{-}.
\]
Since Φ is convex and Φ(0) = 0, from [12] and Jensen's inequality (see, e.g., [2] Theorem 90) it follows that

\[
\sum_{k=1}^{N} \Phi \left( \frac{|(fg)_{-k} \varphi_k|}{C_{-} \|g\|_{-} \|f\|_{-} + \|g\|_{-} \|f\|_{-}} \right) w_k \leq \frac{\|g\|_{-} (\sigma_1(N) + \sigma_2(N)) + \|f\|_{-} (\sigma_3(N) + \sigma_4(N))}{\|f\|_{-} \|g\|_{-} + \|g\|_{-} \|f\|_{-}},
\]

where
\[
\sigma_1(N) := \sum_{k=1}^{N} \sum_{j=0}^{\infty} a_j \Phi \left( \frac{b_{-k+j} \varphi_k}{C_{-} \|g\|_{-}} \right) w_k, \quad \sigma_2(N) := \sum_{k=1}^{N} \sum_{j=1}^{[k/2]} a_{-j} \Phi \left( \frac{b_{-k+j} \varphi_k}{C_{-} \|g\|_{-}} \right) w_k,
\]
\[
\sigma_3(N) := \sum_{k=1}^{N} \sum_{j=0}^{\infty} b_j \Phi \left( \frac{a_{-k+j} \varphi_k}{C_{-} \|f\|_{-}} \right) w_k, \quad \sigma_4(N) := \sum_{k=1}^{N} \sum_{j=1}^{[k/2]} b_{-j} \Phi \left( \frac{a_{-k+j} \varphi_k}{C_{-} \|f\|_{-}} \right) w_k.
\]

Taking into account (i), (ii), and \(C_{\varphi} \geq 1\), we have

\[
\sigma_1(N) = \sum_{j=0}^{\infty} a_j \sum_{k=1}^{N} \Phi \left( \frac{b_{-k+j} \varphi_k}{C_{-} \|g\|_{-}} \right) w_k \leq \sum_{j=0}^{\infty} a_j \sum_{k=1}^{N} \Phi \left( \frac{b_{-k+j} \varphi_{k+j}}{C_{-} \|g\|_{-}} \right) w_{k+j}
\]
\[
\leq \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \Phi \left( \frac{b_{-k} \varphi_k}{C_{-} \|g\|_{-}} \right) w_k \leq \|f\|_{W} \sum_{k=1}^{\infty} \Phi \left( \frac{C_{\varphi} b_{-k} \varphi_k}{C_{-} \|g\|_{-}} \right) w_k.
\]
Changing variables in the second sum in $\sigma_2(N)$ ($r = k - j$), we obtain

$$
\sigma_2(N) := \sum_{k=1}^{N} \sum_{j=1}^{\lfloor k/2 \rfloor} a_{-j} \phi \left( \frac{b_{-k+j} \varphi_k}{C_- \|g\|} \right) w_k = \sum_{k=1}^{N} \sum_{r=k-[k/2]}^{k-1} a_{r-k} \phi \left( \frac{b_r \varphi_k}{C_- \|g\|} \right) w_k.
$$

Since $\phi$ is increasing, it follows from the latter equality and Proposition 3.1 that

$$
\sigma_2(N) \leq C_w \sum_{k=1}^{N} \sum_{j=1}^{k-1} a_{-k+j} \phi \left( \frac{C_w b_{-j} \varphi_j}{C_- \|g\|} \right) w_j
$$

$$
\leq C_w \sum_{k=1}^{N} \sum_{j=1}^{\infty} a_{-k+j} \phi \left( \frac{C_w b_{-j} \varphi_j}{C_- \|g\|} \right) w_j = C_w \sum_{j=1}^{\infty} \phi \left( \frac{C_w b_{-j} \varphi_j}{C_- \|g\|} \right) w_j \sum_{k=1}^{N} a_{-k+j}
$$

$$
\leq C_w \sum_{j=1}^{\infty} \phi \left( \frac{C_w b_{-j} \varphi_j}{C_- \|g\|} \right) w_j \sum_{k=-\infty}^{\infty} a_{-k} = C_w \|f\|_W \sum_{j=1}^{\infty} \phi \left( \frac{C_w b_{-j} \varphi_j}{C_- \|g\|} \right) w_j.
$$

Combining (14) and (15), we arrive at

$$
(16) \quad \sigma_1(N) + \sigma_2(N) \leq \|f\|_W (1 + C_w) \sum_{k=1}^{\infty} \phi \left( \frac{C_w b_{-k} \varphi_k}{C_- \|g\|} \right) w_k.
$$

Since $\phi$ is an Orlicz function, $\phi(x)/x$ is a non-decreasing function (see, e.g., [3], p. 139). Thus,

$$
(1 + C_w) \phi(x) \leq \phi((1 + C_w)x), \quad x \geq 0.
$$

Applying this inequality to (16), we obtain for $N \geq 1$,

$$
(17) \quad \sigma_1(N) + \sigma_2(N) \leq \|f\|_W \sum_{k=1}^{\infty} \phi \left( \frac{b_{-k} \varphi_k}{\|g\|} \right) w_k \leq \|f\|_W
$$

Analogously one can show that for $N \geq 1$,

$$
(18) \quad \sigma_3(N) + \sigma_4(N) \leq \|g\|_W.
$$

Taking into account that $N$ is arbitrary, from (13), (17), and (18) we get

$$
\sum_{k=1}^{\infty} \phi \left( \frac{|(fg)_{-k} \varphi_k|}{C_- (\|f\|_W \|g\| - \|g\|_W \|f\|)} \right) w_k \leq 1.
$$

Therefore,

$$
(19) \quad \|fg\|_- \leq C_- (\|f\|_W \|g\|_- + \|g\|_W \|f\|_-) \leq 2C_- \|f\|_W \|g\|_W \|f\|_W.
$$

By using of Proposition 3.2 b), one can similarly prove that

$$
(20) \quad \|fg\|_+ \leq C_+ (\|f\|_W \|g\|_+ + \|g\|_W \|f\|_+) \leq 2C_+ \|f\|_W \|g\|_W \|f\|_W,
$$

where $C_+ := (1 + C_{\psi}) C_{\varphi}$, $C_{\varphi}$ and $C_{\psi}$ are the constants in the $\Delta_2$-condition for the sequences $\psi = \{\psi_n\}_{n=0}^{\infty}$ and $\varphi = \{\varphi_n\}_{n=0}^{\infty}$, respectively. Combining (14), (20), and

$$
\|fg\|_W \leq \|f\|_W \|g\|_W \leq \|f\|_W \|g\|_W \|f\|_W,
$$

we arrive at (2).

**Acknowledgment.** I would like to thank Albrecht Böttcher (Chernitz Technical University, Germany) for useful remarks on an earlier version of this paper.
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