MATHAI-QUILLEN FORMALISM

Suye Wu
Department of Mathematics, University of Colorado,
Boulder, CO 80309-0395, USA

1. Introduction

Characteristic classes play an essential role in the
study of global properties of vector bundles. Par-

2. Mathai-Quillen’s Construction

2.1 Berezin integral and supertrace

Let $\mathcal{V}$ be an oriented real vector space of
dimension $n$ with a volume element $\nu \in \wedge^n \mathcal{V}$
compatible with the orientation. The Berezin integral
of a form $\omega \in \wedge^n \mathcal{V}^*$ on $\mathcal{V}$, denoted by
$\int_B \omega$, is the pairing $(\nu, \omega)$. Clearly,
only the top degree component of $\omega$ contributes. For
example, if $\sigma \in \wedge^2 \mathcal{V}^*$ is a 2-form, then
$$\int_B e^\sigma = \begin{cases} (\nu, \frac{\omega^{(n/2)}}{(n/2)!}), & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

If $\mathcal{V}$ has a Euclidean metric $(\cdot, \cdot)$, then $\nu$ is chosen to be
of unit norm. If $\Sigma \in \text{End}(\mathcal{V})$ is skew-symmetric, then
$$\frac{1}{2} (\cdot, \Sigma \cdot)$$
is a 2-form and, if $n$ is even, the Pfaffian of $\Sigma$ is
$$\text{Pf}(\Sigma) = \int_B e^{\frac{1}{2} (\cdot, \Sigma \cdot)}.$$

The Berezin integral can be defined on elements in
a graded tensor product $\wedge^n (\mathcal{V}^* \otimes A)$, where $A$ is any
$\mathbb{Z}_2$-graded commutative algebra. For example, if we
consider the identity operator $x = \text{id}_\mathcal{V}$ as a $V$-valued func-
tion on $\mathcal{V}$, then $dx$ is a 1-form on $\mathcal{V}$ valued in $\mathcal{V}$, and
$(dx, \cdot)$ is a 1-form valued in $\mathcal{V}^*$. Let $\{e_1, \ldots, e_n\}$ be
an orthonormal basis of $\mathcal{V}$ and write $x = x^i e_i$, where
$x^i$ are the coordinate functions on $\mathcal{V}$. We let
$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_B e^{-\frac{1}{2} (\mathcal{V}, x) - (dx, \cdot)}.$$ 

The integrand is in $\mathcal{O}^*(\mathcal{V}) \otimes \wedge^n \mathcal{V}^*$. The result is
$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_B e^{-\frac{1}{2} (\mathcal{V}, x) dx^1 \wedge \cdots \wedge dx^n}, \quad (2.1)$$
a Gaussian $n$-form whose (usual) integration on $\mathcal{V}$ is
Let $\text{Cl}(\mathcal{V})$ be the Clifford algebra of $\mathcal{V}$. For any
orthonormal basis $\{e_i\}$, let $\gamma^i$ be the corresponding
generators of $\text{Cl}(\mathcal{V})$ and let $\gamma = e_i \otimes \gamma^i \in \mathcal{V} \otimes
\text{Cl}(\mathcal{V})$. For any $\omega \in \wedge^k \mathcal{V}^*$, we have $\omega(\gamma, \ldots, \gamma) =
\frac{1}{k!} \omega_{i_1 \ldots i_k} \gamma^{i_1} \cdots \gamma^{i_k} \in \text{Cl}(\mathcal{V})$. If $n$ is even,
the Clifford algebra has a unique $\mathbb{Z}_2$-graded irreducible spinor
representation $S(\mathcal{V}) = S^+(\mathcal{V}) \oplus S^-(\mathcal{V})$. For any
element $a \in \text{Cl}(\mathcal{V})$, the supertrace is $\text{str} a = \text{tr} s^+(\mathcal{V}) a -
\text{tr} s^-(\mathcal{V}) a$. If $\Sigma \in \text{End}(\mathcal{V})$ is skew-symmetric, then
$$\text{str} e^{\frac{1}{2} (\gamma, \Sigma \gamma)} = \hat{A}(\Sigma)^{-\frac{1}{2}} \text{Pf}(\Sigma),$$
where $\hat{A}(\Sigma) = \text{det} \left( \frac{\Sigma/2}{\sinh(\Sigma/2)} \right)$. More generally,
supertrace can be defined on $\text{Cl}(\mathcal{V}) \otimes A$ for any $\mathbb{Z}_2$-graded
commutative algebra $A = A^+ \oplus A^-$. If $\Sigma$ is skew-
symmetric and $\alpha \in \mathcal{V}^* \otimes A^-$, then
$$\text{str} e^{\frac{1}{2} (\gamma, \Sigma \gamma) + (\alpha_{\mathcal{V}})^\sharp} \alpha(\gamma) = \hat{A}(\Sigma)^{-\frac{1}{2}} \int_B e^{\frac{1}{2} (\cdot, \Sigma \cdot) + \alpha}.$$ 

(2.2)
2.2 Representatives of the Euler and Thom classes

Let $M$ be a smooth manifold and let $\pi: E \to M$ be an oriented real vector bundle of rank $r$. Suppose $E$ has a Euclidean structure $(\cdot, \cdot)$ and $\nabla$ is a compatible connection. The curvature $R \in \Omega^2(M, \End(E))$ is skew-symmetric, and hence $(\cdot, R \cdot) \in \Omega^2(M, \wedge^2 E^*)$.

A de Rham representative of the Euler class of $E$ is

$$e_\nabla(E) = \frac{1}{(2\pi)^r} \int e^{\frac{1}{2}(\cdot, R \cdot)} = \text{Pf}(R) \cdot. (2.3)$$

Here the Berezin integration is fiberwise in $E$: it is the pairing between the integrand and the unit section $\nu$ of the trivial line bundle $\wedge E$ that is consistent with the orientation of $E$. The de Rham cohomology class of (2.3) is independent of the choice of $(\cdot, \cdot)$ or $\nabla$.

Let $s$ be a section of $E$. Following Berline, Getzler and Vergne (1992) and Zhang (2001), we consider

$$s_\nabla, s = \frac{1}{r}(s, R \cdot) + \frac{1}{r}(\cdot, R \cdot), (2.4)$$

a differential form on $M$ valued in $\wedge^* E^*$. Mathai-Quillen’s representative of the Euler class is

$$e_{\nabla, s}(E) = \frac{(-1)^{r(r+1)}}{(2\pi)^r} \int e^{-s_\nabla, s}. (2.5)$$

One can show that $e_{\nabla, s}(E)$ is closed and that as $\beta$ varies, the cohomology class of $e_{\nabla, s}(E)$ does not change. By taking $\beta \to 0$, the de Rham class of $e_{\nabla, s}(E)$ is equal to that of $e_\nabla(E)$ when $r$ is even. The form $e_{\nabla, \beta}(E)$ provides a continuous interpolation between (2.3) and the limit as $\beta \to \infty$, when the form is concentrated on the zero locus of the section $s$. In fact, the Euler class is the Poincaré dual to the homology class represented by $s^{-1}(0)$. Hence if $n \geq m$ and if $\omega \in \Omega^{n-m}(M)$ is closed, we have

$$\int_M \omega \wedge e_{\nabla, s}(E) = \int_{s^{-1}(0)} \omega. (2.6)$$

when $s$ intersects the zero section transversely.

To obtain Mathai-Quillen’s representative of the Thom class, we consider the pull-back of $E$ to $E$ itself. The bundle $\pi^* E \to E$ has a tautological section $x$. Applying the (2.5) to this setting, we get

$$\tau_\nabla(E) = \frac{(-1)^{r(r+1)}}{(2\pi)^r} \int e^{-\frac{1}{2}(x, x) - (\nabla x, \cdot) - \frac{1}{2}(\cdot, R \cdot)}, (2.7)$$

where $(\cdot, \cdot), \nabla$ and $R$ are understood to be the pullbacks to $\pi^* E$. This is a closed form on the total space of $E$. Moreover, its restriction to each fiber is the Gaussian form (2.1). The cohomology groups of differential forms with exponential decay along the fibers are isomorphic to those with compact vertical support or the relative cohomology groups $H^*(E, E \setminus M)$. Here $M$ is identified with its image under the inclusion $i: M \to E$ by the zero section. Under the above isomorphism, the cohomology class represented by $\tau_\nabla(E)$ coincides with the Thom class $\tau(E) = i_* [H^*(E, E \setminus M)]$ defined topologically. For any section $s \in \Gamma(E)$, we have $e_{\nabla, s}(E) = s^* \tau_\nabla(E)$.

2.3 Character form of the Thom class in K-theory

Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded vector bundle over $M$. The spaces $\Omega^*(M, E), \Gamma(\End(E))$ and $\Omega^*(M) \otimes \Gamma(\End(E))$ are also $\mathbb{Z}_2$-graded. The action of $\alpha \otimes T \in \Omega^*(M) \otimes \Gamma(\End(E))$ on $\beta \otimes s \in \Omega^*(M, E)$ is

$$\alpha \otimes T: \beta \otimes s \mapsto (-1)^{|T|/2}(\alpha \otimes \beta) \otimes (Ts).$$

The supertrace of $A \in \Gamma(\End(E))$ is $\text{str} A = \text{tr}_{E^+} A - \text{tr}_{E^-} A$; it extends $\Omega^* (M) \otimes \Gamma(\End(E)) \to \Omega^*(M)$, Let $\nabla$ be a connection on $E$ preserving the grading. $\nabla$ is an odd operator on $\Omega^*(M, E)$. If $L \in \Gamma(\End(E)^{-1})$ is odd, then $D = \nabla + L$ is called a superconnection on $E$; the curvature $D^2 = R + \nabla L + L^2 \in (\Omega^*(M) \otimes \Gamma(\End(E))^+/2$ is even. With the superconnection, the Chern character of the virtual vector bundle $E^+ \oplus E^-$ can be represented by

$$\text{ch}_{\nabla L}(E^+, E^-) = \text{str} e^{-\frac{1}{2}(\nabla + \beta L))^2} \cdot D \cdot (2.8)$$

It is a closed form on $M$ and its de Rham cohomology class is independent of the choice of $\nabla$ or $L$. If $L$ is invertible everywhere on $M$ and the eigenvalues of $\sqrt{-1} L^2$ are negative, then (2.8) is exact:

$$\text{ch}_{\nabla L}(E^+, E^-) = \frac{1}{2\pi i} \int_1^\infty \text{str} \left( e^{-\frac{1}{2}(\nabla + \beta L))^2} \cdot D \cdot \beta \right).$$

Now let $E$ be an oriented real vector bundle of rank $r = 2m$ over $M$ with a Euclidean structure $(\cdot, \cdot)$. Suppose further that $E$ has a spin structure. The associated spinor bundle $S(E) = S^+(E) \oplus S^-(E)$ is a graded complex vector bundle over $M$. For any section $s \in \Gamma(E)$, let $c(s) \in \Gamma(\End(E)^{-1})$ be the Clifford multiplication on $E$. Then for any $s, s' \in \Gamma(E)$, we have $\{c(s), c(s')\} = -2i(s, s')$. Given a connection $\nabla$ on $E$ preserving $(\cdot, \cdot)$, the induced spinor connection $\nabla S^+$ on $S^+$ preserves the grading. If $R$ is the curvature of $\nabla$, then $\sqrt{-1} R^S = -\frac{1}{2} (\gamma, R \gamma)$, where $\gamma$ is now a section of $E \otimes \text{Cl}(E)$. For any $s \in \Gamma(E)$, consider the superconnection $D_s = \nabla S^+ + \left( -\frac{1}{2\sqrt{2}} \right) c(s)$.

The Chern character form (2.8) of $S^+(E) \oplus S^-(E)$ is, using (2.2),

$$\text{ch}_{\nabla, s}(S^+(E), S^-(E)) = (-1)^m \left( \frac{R}{2\pi^r} \right)^{-\frac{1}{2}} e_{\nabla, s}(E). (2.9)$$
where $e_{\nabla s}(E)$ is given by (2.5). In cohomology groups, (2.9) reduces to
\[ \text{ch} (S^+(E)) - \text{ch} (S^-(E)) = (-1)^m \hat{A}(E)^{-\frac{m}{2}} e(E). \]

If $M$ is non-compact and the norm of $s$ increases rapidly away from $s^{-1}(0)$, then both sides of (2.9) are differential forms that decay rapidly away from $s^{-1}(0)$ and can represent cohomology classes of such. As before, we take the pull-back $\pi^* E$ with the tautological section $x$. Then (2.9) becomes
\[ \text{ch} (\pi^* S^+(E), \pi^* S^-(E)) = (-1)^m \pi^* \hat{A}(E)^{-\frac{m}{2}} \tau_{\pi}(E), \]
where $\tau_{\pi}(E)$ is given by (2.7). Both sides of (2.10) are forms on $E$ that decays exponentially in the fiber directions, hence it descends to an equality in $H^*(E; E/M)$. In the relative K-group $K(E, E; M)$, the pair $\pi^* S^\pm(E)$ with the isomorphism $c(x)$ away form the zero section is, up to a factor of $(-1)^m$, the K-theoretic Thom class $\iota_1 \in K(E, E; M)$. Therefore (2.10) reduces to the well-known formula
\[ \text{ch} (\iota_1) = \pi^* \hat{A}(E)^{-\frac{m}{2}} \iota_d 1 \]
in cohomology groups $H^*(E, E\setminus M)$. The refinement (2.10) as an equality of differential forms is due to Mathai and Quillen (1986). In fact, this is how (2.7) was derived originally.

3. Equivariant Cohomology and Equivariant Vector Bundles

3.1 Equivariant cohomology

Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$. Fixing a basis $\{ e_a \}$ of $\mathfrak{g}$, the structure constants are given by $[e_a, e_b] = t_{ab}^c e_c$. Let $\{ \vartheta^a \}$ and $\{ \varphi^a \}$ be the dual bases of $\mathfrak{g}^*$ generating the exterior algebra $\wedge (\mathfrak{g}^*)$ and the symmetric algebra $S(\mathfrak{g}^*)$, respectively. The Weil algebra is $W(\mathfrak{g}) = \wedge (\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$. We define a grading on $W(\mathfrak{g})$ by specifying $\deg \vartheta^a = 1$, $\deg \varphi^a = 2$. The contraction $\iota_a$ and the exterior derivative $d$ are two odd derivations on $W(\mathfrak{g})$ defined by
\[ \iota_a \vartheta^b = \eta^b_a, \quad \iota_a \varphi^c = 0, \quad d \vartheta^a = -\frac{1}{2} t_{bc}^a \vartheta^b \varphi^c + \varphi^a, \quad d \varphi^a = -t_{bc}^a \vartheta^b \varphi^c. \]

The Lie derivative is $L_a = \{ \iota_a, d \}$. These operators satisfy the usual (anti-)commutation relations
\[ d^2 = 0, \quad L_a = \{ \iota_a, d \}, \quad [L_a, L_b] = 0, \quad [\iota_a, \iota_b] = t_{ab}^c \iota_c, \quad [L_a, L_b] = t_{ab}^c L_c. \]
The cohomology of $(W(\mathfrak{g}), d)$ is trivial.

If $G$ acts smoothly on a manifold $M$ on the left, let $V_a$ be the vector field generated by the Lie algebra element $-e_a \in \mathfrak{g}$. Then $[V_a, V_b] = t_{ab}^c V_c$. Denote $\iota_a = \iota_{V_a}$ and $L_a = L_{V_a}$, acting on $\Omega^*(M)$. In the Weil model of equivariant cohomology, one considers the graded tensor product $W(\mathfrak{g}) \otimes ^G \Omega^*(M)$, on which the operators
\[ \iota_a = \iota_{\varphi} \varpi 1 + 1 \varpi \iota_a, \quad d = d \varpi 1 + 1 \varpi d, \quad L_a = L \varpi 1 + 1 \varpi L_a \]
act and satisfy the same relations (3.2) and (3.3). An element $\omega \in W(\mathfrak{g}) \otimes ^G \Omega^*(M)$ is basic if it satisfies $\iota_a \omega = 0$, $L_a \omega = 0$ for all indices $a$. Let $\Omega^*_G(M) = (W(\mathfrak{g}) \otimes ^G \Omega^*(M))_{\text{bas}}$ be the set of such. Elements of $\Omega^*_G(M)$ are equivariant differential forms on $M$. The operator $\tilde{d}$ preserves $\Omega^*_G(M)$ and its cohomology groups $H^*_G(M)$ are the equivariant cohomology groups of $M$. They are isomorphic to the singular cohomology groups of $EG \times_G M$ with real coefficients.

The BRST model of Kalkman (1993) is obtained by applying an isomorphism $\sigma = e^{\vartheta^a \varphi_a}$ of $W(\mathfrak{g}) \otimes ^G \Omega^*(M)$. The operators become
\[ \sigma \circ \iota_a \circ \sigma^{-1} = \iota_{\varphi} \varpi 1, \quad \sigma \circ \tilde{d} \circ \sigma^{-1} = \tilde{d} - \varphi^a \varpi \iota_a + \varphi^a \varpi L_a, \quad \sigma \circ L_a \circ \sigma^{-1} = L_a. \]
The subspace of basic forms in the Weil model becomes
\[ \sigma (\Omega^*_G(M)) = \{ S(\mathfrak{g}^*) \otimes ^G \Omega^*(M) \} G. \]

This is precisely the Cartan model of equivariant cohomology, in which the exterior differential is
\[ \tilde{d} = 1 \varpi d - \varphi^a \varpi \iota_a. \]

If $P$ is a principal $G$-bundle over a base space $B$, we can form an associated bundle $P \times_G M \to B$. Choose a connection on $P$ and let $\Theta = \Theta^a e_a \in \Omega^1(P) \otimes \mathfrak{g}$, $\Phi = \Phi^a e_a \in \Omega^2(P) \otimes \mathfrak{g}$ be the connection, curvature forms, respectively. The components $\Theta^a$, $\Phi^a$ satisfy the same relations (3.1). Replacing $\vartheta^a$, $\varphi^a$ by $\Theta^a$, $\Phi^a$, we have a homomorphism that maps $\omega \in W(\mathfrak{g}) \otimes ^G \Omega^*(M)$ to $\hat{\omega} \in \Omega^*(P \times M)$. If $\omega$ is basic, then so is $\hat{\omega}$, and the latter descends to a form $\omega$ on $P \times_G M$. Furthermore, the operator $\hat{d}$ on $\Omega^*_G(M)$ descends to $d$ on $\Omega^*(P \times_G M)$. Thus we get the Chern-Weil homomorphisms $\Omega^*_G(M) \to \Omega^*(P \times_G M)$ and $H^*_G(M) \to H^*(P \times_G M)$. For example, the vector space $\mathbb{R}^r$ has an obvious $SO(r)$ action. The Gaussian $r$-form (2.1) is invariant under $SO(r)$ and can be extended to an $SO(r)$-equivariant closed $r$-form, called the universal Thom form. Let $E$ be an orientable real vector bundle $E$ of rank $r$ with a Euclidean structure. $E$ determines a principal $SO(r)$-bundle $P$; the associated bundle $P \times_{SO(r)} \mathbb{R}^r$ is $E$ itself. By applying the Chern-Weil homomorphism to this setting, we get a closed $r$-form on $E$. This is another construction of...
the Thom form (2.7) by Mathai and Quillen (1986). Further information of equivariant cohomology can be found there and in Guillemin (1999).

### 3.2 Equivariant vector bundles

Recall that a connection on a vector bundle $E \to M$ determines, for any $k \geq 0$, a differential operator

$$\nabla: \Omega^k(M, E) \to \Omega^{k+1}(M, E).$$

The curvature $R = \nabla^2 \in \Omega^2(M, \text{End}(E))$ satisfies the Bianchi identity $\nabla R = 0$. If the connection preserves a Euclidean structure on $E$, then $R$ is skew-symmetric.

If a Lie group $G$ acts on $M$ and the action can be lifted to $E$, then $G$ also acts on the spaces $\Gamma(E)$ and $\Omega^*(M, E)$. As before, the Lie derivatives $L_a$ on these spaces are the infinitesimal actions of $a \in \mathfrak{g}$. We choose a $G$-invariant connection on $E$. The moment of the connection $\nabla$ under the $G$-action is $\mu_a = L_a - \nabla_{\mathfrak{g}} a$ acting on $\Gamma(E)$. In fact, $\mu_a$ is a section of $\text{End}(E)$, or $\mu \in \Gamma(\text{End}(E)) \otimes \mathfrak{g}^*$. If a Euclidean structure on $E$ is preserved by both the connection and the $G$-action, then $\mu_a$ is skew-symmetric. On $\Omega^*(M, E)$, we have

$$L_a = \{\iota_a, \nabla\} + \mu_a.$$

On the graded tensor product $W(\mathfrak{g}) \otimes \Omega^*(M, E)$, the contraction $\iota_a$ and the Lie derivative $L_a$ act and satisfy (3.3). In the Weil model, equivariant differential forms on $M$ with values in $E$ are the basic elements in $W(\mathfrak{g}) \otimes \Omega^*(M, E)$, which form a subspace $\Omega_G^*(M, E) = (W(\mathfrak{g}) \otimes \Omega^*(M, E))_{\text{bas}}$. The equivariant covariant derivative is

$$\tilde{\nabla} = d \otimes 1 + 1 \otimes \nabla + \varphi^a \otimes \mu_a. \quad (3.4)$$

One checks that $\{\iota_a, \tilde{\nabla}\} = \tilde{L}_a$ and hence $\tilde{\nabla}$ preserves the basic subspace $\Omega_G^*(M, E)$. The equivariant curvature $\tilde{R}$ is

$$\tilde{R} = R - \varphi^a \nabla \mu_a + \varphi^a \mu_a + \frac{1}{2} \varphi^a \varphi^b R_{ab}, \quad (3.5)$$

where $R_{ab} = R(V_a, V_b) \in \Gamma(\text{End}(E))$. It satisfies the equivariant Bianchi identity $\tilde{\nabla} \tilde{R} = 0$. Equivariant characteristic forms are invariant polynomials of $\tilde{R}$. They are equivariantly closed and their equivariant cohomology classes do not depend on the choice of the $G$-invariant connection. Hence they represent the equivariant characteristic classes of $E$ in $H_G^*(M)$.

For the BRST model, we use a similar isomorphism $\sigma = e^{\varphi^a \otimes \iota_a}$ on $W(\mathfrak{g}) \otimes \Omega^*(M, E)$. The operators become

$$\sigma \circ \iota_a \circ \sigma^{-1} = \iota_a \otimes 1,$$

$$\sigma \circ \nabla \circ \sigma^{-1} = \nabla - \varphi^a \otimes \iota_a + \varphi^a \otimes L_a,$$

$$\sigma \circ \tilde{L}_a \circ \sigma^{-1} = \tilde{L}_a$$

and the basic subspace turns into

$$\sigma(\Omega_G^*(M, E)) = (S(\mathfrak{g}^*) \otimes \Omega^*(M, E))^G.$$

This is the Cartan model, which can be found in Berline, Getzler and Vergne (1992). The equivariant connection is

$$\tilde{\nabla}' = 1 \otimes \nabla - \varphi^a \otimes \iota_a.$$

The equivariant curvature is $\tilde{R}' = (\tilde{\nabla}')^2 = \tilde{R} - \varphi^a \mu_a$ and the characteristic forms are defined similarly.

Let $P \to B$ be a principal $G$-bundle with a connection $\Theta$. Following (3.4), the bundle $P \times E \to P \times M$ has a connection

$$\tilde{\nabla} = d \otimes 1 + 1 \otimes \nabla + \Theta^a \otimes \mu_a.$$

It descends to a connection $\nabla$ on the vector bundle $P \times_G E \to P \times_G M$. The map $\nabla \mapsto \tilde{\nabla}$ can be considered as the analog of the Chern-Weil homomorphism for connections. There is also a homomorphism $\Omega_G^*(M, E) \to \Omega^*(P \times_G M, P \times_G E)$, which commutes with the covariant derivatives $\nabla, \tilde{\nabla}$. The curvature $\tilde{R} = \tilde{\nabla}^2$ is the image of the equivariant curvature $\tilde{R}$. Consequently, the equivariant characteristic forms descend to those of $P \times_G E \to P \times_G M$ by the usual Chern-Weil homomorphism.

Now let $E = E^+ \oplus E^-$ be a graded vector bundle over $M$ with a $G$-action preserving all the structures. We have the $\Omega_G^*(M)$-linear supertrace map $\text{str}: \Omega_G^*(M) \otimes \Gamma(\text{End}(E)) \to \Omega_G^*(M)$. If $\nabla$ is a $G$-invariant connection on $E$ preserving the grading and if $L \in \Gamma(\text{End}(E)^{-1})$ is odd and $G$-invariant, then $\tilde{D} = \nabla + L$ is an equivariant superconnection. The equivariant counterpart of (2.8) is

$$\text{ch}_{\tilde{D}}(E^+, E^-) = \text{str} e^{\frac{2\pi i}{\hbar} \tilde{D}^2} \in \Omega_G^*(M),$$

representing the equivariant Chern character of $E^+ \oplus E^-$ in $H_G^*(M)$.

### 3.3 Representatives of the equivariant Euler and Thom classes

Consider an oriented real vector bundle $E \to M$ of rank $r$ with a Euclidean structure $(\cdot, \cdot)$. Choose a connection $\nabla$ on $E$ preserving $(\cdot, \cdot)$. We assume that a Lie group $G$ acts on $M$ and that the action can be lifted to $E$ preserving all the structures on $E$. We use the Weil model; the constructions in the Cartan model are similar. For any $\alpha \in \Omega_G^*(M, E)$ and $\beta \in \Omega_G^*(M, E)$, we obtain $(\alpha, \wedge \beta) \in \Omega_G^{*+1}(M)$ by taking the wedge product of forms as well as the pairing in $E$. The Berezin integral of $\omega \in \Omega_G^*(M, \wedge^r E^*)$ along the fibers of $E$ is

$$\int^B \omega = (\nu, \omega) \in \Omega_G^r(M).$$

Here $\nu$ is the unit section of the canonically trivial determinant line bundle $\wedge^r E$.
compatible with the orientation of $E$. The equivariant Euler form
\begin{equation}
    e_{\varphi}(E) = \frac{1}{(2\pi)^2} \int_B e(Q(\cdot,\cdot)) = \text{Pf} \left( \frac{\mathcal{A}}{2\pi} \right) \quad (3.6)
\end{equation}
is equivariantly closed. It represents the equivariant Euler class $e_G(E) \in H^G(M)$.

Given a $G$-invariant section $s \in \Gamma(E)^G$, the equivariant counterpart of (2.4) is
\begin{equation}
    d_{\varphi,s} = \frac{1}{4}(s, s) + \left( \nabla s, \cdot \right) + \frac{1}{2}(\cdot, \mathcal{R} \cdot) \quad (3.7)
\end{equation}
and that of Mathai-Quillen’s Euler form (2.5) is
\begin{equation}
    e_{\varphi,s}(E) = \frac{(-1)^{r(r+1)}}{(2\pi)^{2}} \int_B \varepsilon^s \varphi, \cdot. \quad (3.8)
\end{equation}
It is also equivariantly closed, and its equivariant cohomology class is $e_G(E)$. The equivariant extension of Mathai-Quillen’s Thom form (2.7) is
\begin{equation}
    \tau_{\varphi}(E) = \frac{(-1)^{r(r+1)}}{(2\pi)^{2}} \int_B \varepsilon^{\frac{1}{2}(s, x) - \left( \nabla s, \cdot \right) - \frac{1}{2}(\cdot, \mathcal{R} \cdot)}, \quad (3.9)
\end{equation}
where $x$ is the ($G$-invariant) tautological section of $\pi^*E \to E$.

Finally, $G$ acts on the (graded) spinor bundle $S(E)$. Using the equivariant superconnection $\mathcal{D}_s = \nabla_s + \left( \frac{\mathcal{A}}{\sqrt{\nabla}} \right) \varepsilon(s)$, (2.9) generalizes to
\begin{equation}
    \chi_{\varphi,s}(S^+(E), S^-(E)) = (-1)^m \mathcal{A} \left( \frac{\mathcal{A}}{\sqrt{\nabla}} \right) e_{\varphi,s}(E). \quad (4.1)
\end{equation}
Now apply the construction to the bundle $\pi^*E \to E$ and its tautological section $x$. The pair $\pi^*S^\pm(E)$ with an odd bundle map $\varepsilon(x)$ determines, up to a factor of $(-1)^m$, the Thom class $i_1G$ in the equivariant $K$-group $K_G(E, E \setminus M)$. The equivariant analog of (2.10) descends to
\begin{equation}
    \chi_G(i_1G) = \pi^*\mathcal{A}_G(E)^{-\frac{1}{2}} i_1G \quad (4.2)
\end{equation}
in equivariant cohomology.

4. Superspace Formulation

4.1 Mathai-Quillen formalism and the superspace $\mathbb{R}^{0|1}$

Let $\mathbb{R}^{0|1}$ be the superspace with one fermionic coordinate $\theta$ but no bosonic coordinates. The translation on $\mathbb{R}^{0|1}$ is generated by $D = \frac{\partial}{\partial \theta}$, which satisfies $[D, D] = 0$. We consider a sigma model on $\mathbb{R}^{0|1}$ whose target space is an (ordinary) smooth manifold $M$ of dimension $n$. A map $X: \mathbb{R}^{0|1} \to M$ can be written as $X(\theta) = x + \sqrt{-1}i\theta$. Here $x = X|_{\theta = 0} \in M$ and $\psi = -\sqrt{-1}DX|_{\theta = 0} \in T_xM$; the latter is fermionic. Under the translation $\theta \mapsto \theta + \epsilon$, $x$ and $\psi$ vary according to the supersymmetry transformations
\begin{align}
    \delta x &= e^{\epsilon DX} = \sqrt{-1}\epsilon \psi, \\
    \delta \psi &= e^{D(DX)} = 0. \quad (4.1)
\end{align}
Clearly, $\delta^2 = 0$, which is also a consequence of $D^2 = 0$.

For any $p$-form $\omega \in \Omega^p(M)$, we have an observable $\partial_a^a(X) = \frac{1}{p!} X^+ \omega(D, \cdots, D)|_{\theta = 0}$. In local coordinates, $\omega = \frac{1}{p!} \omega_1 \cdots \omega_p(x) \, dx^1 \wedge \cdots \wedge dx^p$ and $\partial_{a}(x, \psi) = \frac{\epsilon}{p!} \omega_1 \cdots \omega_p(x) \psi^{i_1} \cdots \psi^{i_p}$. Using $C(\cdot)$ to denote the set of function(s) on a space, we can identify $C(\operatorname{Map}(\mathbb{R}^{0|1}, M))$ with $\Omega^p(M)$. Under (4.1), $\delta \partial_a^a(X) = \epsilon \partial_{a\theta}(X)$. So $\partial_a^a(X)$ is invariant under supersymmetry if and only if $\omega$ is closed. The cohomology of $\delta$ is the de Rham cohomology of $M$. Consider the measure $[dX] = [dx][d\psi]$. In local coordinates, $[dx] = dx^1 \cdots dx^n$ is the standard (bosonic) measure and $[d\psi] = d\psi_1 \cdots d\psi^n$ is a fermionic measure such that $\int [d\psi](-1)^{a(n-a)} \psi^{i_1} \cdots \psi^{i_p} = 1$. For any $\omega \in \Omega^n(M)$, the superfield integral $\int [dX] \partial_a^a(X)$ is equal to the usual integral $\int_M \omega$ if the latter exists.

Let $E \to M$ be a real vector bundle of rank $r$ with an inner product $(\cdot, \cdot)$ and let $\nabla$ be a compatible connection whose curvature is $R$. Consider a theory whose fields are $X \in \operatorname{Map}(\mathbb{R}^{0|1}, M)$ and a fermionic section $\Xi \in \Gamma(X^*E)$. Let $\mathcal{D} = (X^* \nabla)|_{E}$ be the covariant derivative along $D$ in the pull-back bundle $X^*E \to \mathbb{R}^{0|1}$. Then $\chi = \Xi|_{\theta = 0} \in E_x$ is fermionic and $f = \mathcal{D}\Xi|_{\theta = 0} \in E_x$ is bosonic.

Given a fixed section $s \in \Gamma(E)$, we write a super-space action
\begin{equation}
    S_{SQ} = \int_{\mathbb{R}^{0|1}} [d\theta] \left( \Xi, \frac{1}{2} \mathcal{D} \Xi + \sqrt{-1} \epsilon \Xi \right) \quad (4.2)
\end{equation}
\begin{equation}
    = \frac{1}{2} (f, f) + \sqrt{-1} (f, s) - (\nabla s, \chi) + \frac{1}{2} (\chi, R(\psi, \psi) \chi). \quad (4.3)
\end{equation}
It is automatically supersymmetric. Performing the Gaussian integral over $f$ and replacing $\chi$ by $\sqrt{-1}\chi$, we get
\begin{equation}
    \int [d\Xi] e^{-S_{SQ}^c}[\Xi, \Xi] = \frac{1}{(2\pi)^{2}} \int [d\chi] e^{-S_{SQ}^c}[\Xi, \chi, \chi, \chi], \quad (4.3)
\end{equation}
where
\begin{equation}
    S_{SQ}[\Xi, \chi, \chi] = \frac{1}{2} (s, s) - \sqrt{-1} (\chi, \nabla \psi) - \frac{1}{2} (\chi, R(\psi, \psi) \chi). \quad (4.3)
\end{equation}
Actually, (4.3) holds up to possible factors of $\sqrt{-1}$ which we ignore here and in subsequent partition functions and expectations values. With this caveat, (4.3) is equal to $\partial_{a\theta}(\nabla s(E))$, where $c(\nabla, s)(E)$ is given by (2.5). Furthermore, for any closed form $\omega$ on $M$, the expectation value
\begin{equation}
    \langle \partial_{a\theta}(X) \rangle = \int [dX][d\Xi] \partial_{a\theta}(X) e^{-S_{SQ}^c}[\Xi, \Xi] \quad (4.5)
\end{equation}
is equal to (2.6).
4.2 Equivariant cohomology and gauged sigma model on $\mathbb{R}^{0|1}$

Suppose $G$ is a Lie group and $P$ is a principal $G$-bundle over $\mathbb{R}^{0|1}$. Since $\theta$ is nilpotent, we can choose a “trivialisation” of $P$ such that the connection and curvature are $A \in \Omega^1(\mathbb{R}^{0|1}) \otimes \mathfrak{g}$ and $F \in \Omega^2(\mathbb{R}^{0|1}) \otimes \mathfrak{g}$, respectively. ($\mathfrak{g}$ is the Lie algebra of $G$). In components, $c = \sqrt{-1} \text{Id} A \in \mathfrak{g}$ is fermionic and $\phi = -\sqrt{-1} i \theta F \in \mathfrak{g}$ is bosonic. The space of connections $A$ is the set of pairs $(c, \phi)$. Under $\theta \rightarrow \theta + \epsilon,$

$$\delta c = \epsilon (\phi + \frac{\sqrt{-1}}{2} [c, c]),$$

$$\delta \phi = \sqrt{-1} \epsilon [c, \phi] . \quad (4.6)$$

Thus the algebra $C(A)$ is isomorphic to the Weil algebra $\mathcal{W}(\mathfrak{g})$ and $\delta$ corresponds to the differential $d$ in (3.1). This relation between gauge theory on a fermionic space and the Weil algebra can be found in Blau and Thompson (1997).

With a trivialisation of $P$, the group of gauge transformation $\mathcal{G}$ can be identified with $\text{Map}(\mathbb{R}^{0|1}, G)$. Any group element is of the form $\hat{g} = g e^{\sqrt{-1} \xi}$, with $g = \hat{g}|_{\theta=0} \in G$ and $\xi = \sqrt{-1} \text{Id} \hat{g}^* \omega$ in $\mathfrak{g}$ (fermionic), where $\omega$ is the Maurer-Cartan form on $G$. The action of $\hat{g}$ is $A \mapsto A' = A_{2g} (A - \hat{g}^* \omega A)$ or $c \mapsto c' = A_{2g} (c - \xi)$ and $\phi \mapsto \phi' = A_{2g} \phi$. By choosing $\xi = c$, we obtained a new trivialisation, called the Wess-Zumino gauge, in which $c' = 0$. The residual gauge redundancy is $G$, and $A/\mathcal{G} = \mathfrak{g}/A_{2g}$. The Wess-Zumino gauge is not preserved by the translation on $\mathbb{R}^{0|1}$ unless we define $\delta'$ by composing $\delta$ with a suitable (infinitesimal) gauge transformation. If so, then $\delta' \phi = 0$.

Suppose $M$ is a manifold with a left action. As before, let $\{e_a\}$ be a basis of $\mathfrak{g}$ and let the vector field $V_a$ be the infinitesimal action of $-e_a$. In the gauge sigma model, we include another field $X \in \Gamma(P \times_G M)$. With a trivialisation of $P$, we can identify $X$ with a map $X: \mathbb{R}^{0|1} \rightarrow M$. The covariant derivative is given by $\nabla X = dX - A^a V_a$, $\nabla DX = \nabla_D X$. Let $x = X|_{\theta=0} \in M$ and $\psi = -\sqrt{-1} \nabla DX|_{\theta=0} \in T_x M$. Then the supersymmetric transformations are

$$\delta x_i = \sqrt{-1} \epsilon (\psi^i - c^a V^i_a),$$

$$\delta \psi^i = -\epsilon (\phi^a V^i_a + \sqrt{-1} c^j V^i_{aj}). \quad (4.7)$$

In the Wess-Zumino gauge, the transformations simplify to

$$\delta x = -\sqrt{-1} \nabla \psi, \quad \delta \psi = -\epsilon \phi^a V_a.$$

The observables form the $\mathcal{G}$-invariant part of the space $C(A \times \text{Map}(\mathbb{R}^{0|1}, M))$. For any $\omega \in \Omega^p(M)$, we have

$$\partial_\omega (X, A) = \frac{1}{p!} \omega (D X, \ldots, D X)|_{\theta=0}$$

$$= \sqrt{-1} \epsilon \omega_{ij} \cdots \epsilon_i (x) \psi^i_1 \cdots \psi^p. \quad (4.8)$$

$\partial_\omega (X, A)$ is gauge covariant: $\partial_\omega (X, A) \mapsto \partial_\omega (X, A)$, and the set of gauge invariant observables is thus identified with $(S(\mathfrak{g}) \times \Omega^*(M))^G$. Moreover, since

$$\partial_\omega (X, A) = \epsilon (\partial_{d\omega} (X, A) - \sqrt{-1} c^a \partial_{\omega^a} (X, A)) \quad (4.9)$$

$\delta$ corresponds to the differential $d'$ in BRST model.

Let $E \rightarrow M$ be an equivariant vector bundle and let $\nabla$ be a $G$-invariant connection with curvature $R$ and moment $\mu$. Any $s \in \Gamma(E)^G$ defines a section of $P \times_G E \rightarrow P \times_G M$, still denoted by $s$. Consider a theory with superfields $X \in \Gamma(P \times_G M)$ and $\Xi \in \Gamma(X^{*} \times_G E)$ (fermionic). Let $\mathcal{D}$ be the covariant derivative of the pull-back connection. With a trivialisation of $P$, we put $\chi = \Xi|_{\theta=0} \in E_x$ (fermionic) and $f = \mathcal{D} \Xi|_{\theta=0} \in E_x$ (bosonic). The equivariant extension of (4.2) is

$$S_{MQ}[X, \Xi, A] = \int_{\mathbb{R}^{0|1}} d\theta \left( \Xi, \frac{1}{2} \mathcal{D} \Xi + \sqrt{-1} s \circ X \right) \quad (4.9)$$

Similar to (4.3), we get, in the Wess-Zumino gauge,

$$\int [d\Xi] e^{-S_{MQ}[X, \Xi, A]} = \frac{1}{(2\pi)^2} \int [d\chi] e^{-S_{MQ}[x, \psi, \phi, \chi]}, \quad (4.10)$$

where

$$S_{MQ}[x, \psi, \phi, \chi] = \frac{1}{2} (s, s) - \sqrt{-1} (\chi, \nabla \psi s)$$

$$-\frac{1}{4} (\chi, R(\psi, \psi) \chi) - \frac{1}{2} (\chi, \phi^a \mu^a \chi). \quad (4.11)$$

$\delta$ is equal to $\partial_{\epsilon(\nabla_s) (X, A)}$, where $\epsilon(\nabla_s)$ is given by (3.8).

4.3 The Atiyah-Jeffrey formula

Given the $G$-action on $M$, for any $x \in M$, there is a linear map $C_x : \mathfrak{g} \rightarrow T_x M$ defined by $C_x(e_a) = V_a(x)$. With an invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ and an invariant Riemannian metric on $M$, the adjoint of $C_x$ is $C^*_x : T_x M \rightarrow \mathfrak{g}$, that is, $C^*_x \in \Gamma(T^* M) \otimes \mathfrak{g}$. If $G$ acts on $M$ freely, then $C_x$ is injective and $(C^*_x)^2$ is invertible for all $x \in M$. The projection $M \rightarrow \bar{M} = M/G$ is a principal $G$-bundle. It has a connection such that the horizontal subspace is the orthogonal compliment of the $G$-orbits. The connection 1-form is $\theta = (C^*_x)^{-1} C^*_x$, whereas the curvature is $\Phi = (C^*_x)^{-1} d C^*_x$ on horizontal vectors.

Let $\omega$ be an equivariant form on $M$. Suppose $G$ acts on $M$ freely, then $\omega$ descends to a form $\bar{\omega}$ on $\bar{M}$. We look for a gauge invariant, supersymmetric quantity $\mathcal{Y}(X, A)$ such that

$$\frac{1}{\text{vol}(G)} \int [dX] [dA] \partial_\omega (X, A) \mathcal{Y}(X, A) = \int [d\bar{X}] \partial_{\bar{\omega}} (\bar{X}). \quad (4.11)$$

Mathematically, $\mathcal{Y}$ corresponds to a closed equivariant form $v$ on $M$ such that

$$\frac{1}{\text{vol}(G)} \int_{\phi \in \mathfrak{g}} [d\phi] \int_M \omega(\phi) \wedge v(\phi) = \int_M \bar{\omega}$$


which is (4.11) in the Wess-Zumino gauge. \( v \) can be understood as an equivariant homology cycle, as in Austin and Braam (1995).

Let \( P \) be a \( G \)-bundle over \( \mathbb{R}^0 \) with a connection and let \( \text{Ad}P = P \times_G \mathfrak{g} \to \mathbb{R}^0 \) be the adjoint bundle. Consider a (bosonic) superfield \( \Lambda \in \Gamma(\text{Ad}P) \). Put \( \lambda = A|_{\theta=0} \) (bosonic) and \( \eta = -\sqrt{-1}\bar{\partial}A|_{\theta=0} \) (fermionic). Choosing a trivialisation of \( P \), \( \lambda \) and \( \eta \) are both in \( \mathfrak{g} \). Under \( \theta \to \theta + \epsilon \), they transform as

\[
\delta \lambda = \sqrt{-1}T(\eta + [c, \lambda]),
\delta \eta = \epsilon([\phi, \lambda] - \sqrt{-1}[c, \eta]).
\]

(4.12)

The superspace action

\[
S_{\text{CMR}}[X, A, A] = \sqrt{-1} \int_{\mathbb{R}^0} d\theta (A, C^\dagger DX)
\]

is invariant under (4.6), (4.7) and (4.12) and, under the Wess-Zumino gauge, it is

\[
S_{\text{CMR}}[x, \psi, \phi, \eta, \lambda] = -\sqrt{-1}(\eta, C^\dagger \psi)
\]

\[
-\sqrt{-1}(\lambda, dC^\dagger (\psi, \phi)) + (\lambda, C^\dagger C\phi).
\]

(4.13)

If \( G \) acts on \( M \) freely, then

\[
T(X, A) = \int [dA] e^{-S_{\text{CMR}}[X, A, A]}
\]

(4.14)

satisfies (4.11). The factor \( T(X, A) \) in (4.11) is called projection in Cordes, Moore and Ramgoolam (1996).

Let \( E \to M \) be a \( G \)-equivariant vector bundle with a fixed \( G \)-invariant connection \( \nabla \), moment \( \mu \), and an invariant section \( s \). Consider the superspace action

\[
S_{AJ}[X, \Xi, A, A] = S_{MQ}[X, \Xi, A] + S_{\text{CMR}}[X, A, A].
\]

In the Wess-Zumino gauge and after the Gaussian integral over \( f \), it becomes the Atiyah-Jeffrey action

\[
S_{AJ}[x, \psi, \phi, \chi, \eta, \lambda] = S_{MQ}[x, \psi, \phi, \chi] + S_{\text{CMR}}[x, \psi, \phi, \eta, \lambda].
\]

(4.15)

If \( s \) intersect the zero section transversely and \( G \) acts on \( s^{-1}(0) \) freely, then \( s^{-1}(0)/G \) is smooth and

\[
\int_{s^{-1}(0)/G} \bar{\omega} = \int [dx][d\psi][d\phi][d\chi][d\eta][d\lambda]
\]

\[
\bar{\omega}(x, \psi, \phi) e^{-S_{AJ}[x, \psi, \phi, \chi, \eta, \lambda]},
\]

(4.16)

for any closed equivariant form \( \omega \) on \( M \). (4.16) is the formula of Atiyah and Jeffrey (1990) and of Witten (1988a) in an infinite dimensional setting. When \( s^{-1}(0)/G \) is not smooth, the right-hand side of (4.16) can be regarded as a definition of the left-hand side.

It is often convenient to add to \( S_{AJ} \) another term

\[
\Delta S[X, A, A] = -\frac{1}{4} \int_{\mathbb{R}^0} \langle [\xi_D^2 F, A], DA \rangle
\]

\[
= \frac{1}{2} \delta(\phi, [\eta, \eta]) + \frac{1}{2} [\delta(\psi, \phi), [\phi, \lambda)].
\]

(4.17)

Since (4.17) is \( \delta \)-exact and no new field is added, the integral (4.16) does not change if \( \Delta S \) is added to \( S_{AJ} \).

5. Applications to Cohomological Field Theories

We now apply the Mathai-Quillen construction formally to a number of cases in which the both the rank of the vector bundle and the dimension of the base space are infinite. Thus the (bosonic and fermionic) integrals in (4.5) or (4.16) become path integrals in quantum mechanics or quantum field theory.

5.1 Supersymmetric quantum mechanics

Let \( (M, g) \) be a Riemannian manifold and \( LM = \text{Map}(S^1, M) \), the loop space. At each point \( u \in LM \), which is a map \( u: S^1 \to M \), the tangent space is \( TuLM = \Gamma(u^*TM) \). In particular, \( \dot{u} = \frac{dt}{du} \), where \( t \) is a parameter on \( S^1 \), is a tangent vector at \( u \) and \( u \to \dot{u} \) is a vector field on \( LM \). For any Morse function \( h \) on \( M \), \( s(u) = \dot{u} + (\text{grad } h) \circ u \) is another vector field on \( LM \).

Vector fields on \( LM \) can be identified as sections of the bundle \( ev^*TM \to S^1 \times LM \), where \( ev: S^1 \times LM \to \) is the evaluation map. The Levi-Civita connection \( \nabla \) on \( TM \) pulls back to a connection on \( ev^*TM \) and the covariant derivatives along \( LM \) define a natural connection \( \nabla^{LM} \) on \( T(LM) \). For example, for any tangent vector \( \dot{V} \in TuLM = \Gamma(u^*TM) \), we have \( \nabla^{LM}_{\dot{V}} s(u) = \nabla^TM \dot{V} + (\nabla_V \text{grad } h) \circ u \), where \( \nabla^V \) is the pull-back connection on \( u^*TM \). The Riemann curvature tensor \( R \) on \( M \) determines that on \( LM \).

The (infinite dimensional) analog of (4.3) is

\[
\int [du][d\psi][d\chi] e^{-\int dt L[u, \psi, \chi]},
\]

(5.1)

where \( \psi, \chi \in TuLM = \Gamma(u^*TM) \) are fermionic and

\[
L[u, \psi, \chi] = \frac{1}{2} g(\dot{u} + \text{grad } h, \dot{u} + \text{grad } h)
\]

\[
-\sqrt{-1}g(\dot{\psi}, \nabla^TM \dot{\psi} + \nabla\psi \text{grad } h)
\]

\[
-\frac{1}{4} g(\psi, R(\psi, \chi)).
\]

(5.2)

(5.2) is, up to a total derivative, the Lagrangian of the Euclidean \( N = 2 \) supersymmetric quantum mechanics on \( M \). The partition function (5.1) is equal to Euler characteristic number of \( LM \) or \( M \), which can be confirmed by an (exact) stationary phase calculation.

5.2 Topological sigma model

Let \( \Sigma \) be a Riemann surface \( \Sigma \) with complex structure \( \varepsilon \) and let \( (M, \omega) \) be a symplectic manifold with a compatible almost complex structure \( J \). Let \( E \) be a vector bundle over \( \text{Map}(\Sigma, M) \) so that the fiber over \( u \) is \( E_u = \Gamma(u^*TM \times \Sigma) \). For any \( u \in \text{Map}(\Sigma, M) \), \( du \in E_u \) and \( u \to du \) is a section of \( E \). The pull-back of the Levi-Civita connection on \( TM \), tensored with a connection on \( T^*\Sigma \), defines a connection on \( E \).
The vector bundle to which we apply the Mathai-Quillen formalism is the anti-holomorphic part $E^{01}$ of $\mathcal{E}$. The fiber over $u \in \text{Map}(\Sigma, M)$ is $E^{01}_u = \Gamma((u^*TM \otimes T^* \Sigma)^{01})$. The sub-bundle $E^{01}$ has a connection $\nabla^{01}$ via projection from $\mathcal{E}$. $E^{01}$ has a natural section $s: u \mapsto \partial_J u = \frac{1}{2}(du + J \circ du) \circ \varepsilon$. Solutions to the equation $\partial_J u = 0$ are pseudo-holomorphic (or $J$-holomorphic) curves; let $M = s^{-1}(0)$ be the space of such. Its (virtual) dimension is
\[
\dim M = \frac{1}{2} \chi(\Sigma) + 2 c_1(u^*TM). \tag{5.3}
\]
Along any $V \in T_u \text{Map}(\Sigma, M) = \Gamma(u^*TM)$, the covariant derivative of $s = \partial_J$ is calculated in Wu (1995):
\[
\nabla^{01}_V(\partial_J) = \frac{1}{2}(\nabla^u V + J \circ \nabla^u V \circ \varepsilon) \mp 1 \nabla^V J \circ (du \varepsilon + J du), \tag{5.4}
\]
where $\nabla^u$ is the pull-back connection on $u^*TM$.

To write the Mathai-Quillen formalism for the bundle $E^{01} \to \text{Map}(\Sigma, M)$, we let $\psi \in \Gamma(u^*TM)$ and $\chi \in \Gamma((u^*TM \otimes T^* \Sigma)^{01})$ be fermionic fields. (4.4) becomes the Lagrangian
\[
L[u, \psi, \chi] = \frac{1}{2} ||du||^2 + \frac{1}{2}(du, J \circ du) \circ \varepsilon - \sqrt{-1}(\chi, \nabla^u \psi + (\nabla^u \psi, J \circ du) \circ \varepsilon) - \frac{1}{4}(\chi, (R(\psi, \varepsilon) - \frac{1}{2}(\nabla^u \psi)^2) \chi). \tag{5.5}
\]
It is precisely the Lagrangian of the topological sigma model of Witten (1988b). Here the pairing $(\cdot, \cdot)$ is induced by the Riemannian metric $\omega(\cdot, J \cdot)$ on $M$ and a metric on $\Sigma$ that is compatible with $\varepsilon$. The second term in (5.5), integrated over $\Sigma$, is equal to $\int_{\Sigma} u^*\omega = \langle [u], u_+ \rangle$.

For any differential form $\alpha \in \Omega^p(M)$, let $\Omega_\alpha(u, \psi)$ be the observable obtained from ev$^*\alpha \in \Omega^p(\Sigma \times \text{Map}(\Sigma, M))$ by identifying $\Omega^p(\Sigma \times \text{Map}(\Sigma, M))$ with $C(\text{Map}(\mathbb{R}^0, \Sigma), \text{Map}(\Sigma, M)))$. If $\alpha$ is closed and $\gamma \in H_q(\Sigma)$ is a homology cycle, then $W_{\alpha, \gamma}(u, \psi) = \int_\Sigma \Omega_\alpha(u, \psi)$ is identified with a closed $(p - q)$-form on $\text{Map}(\Sigma, M)$. For closed $\alpha \in \Omega^p(M)$ and $\gamma_i \in H_{q_i}(\Sigma)$ $(1 \leq i \leq r)$, the expectation values
\[
\left\langle \prod_{i=1}^r W_{\alpha_i, \gamma_i} \right\rangle = \int \text{d}[u] \text{d}[\psi] \text{d}[\chi] \prod_{i=1}^r W_{\alpha_i, \gamma_i}(u, \psi) e^{-S[u, \psi, \chi]} \tag{5.6}
\]
are the Gromov-Witten invariants of $(M, \omega)$. Moreover, (5.6) is non-zero only if $\sum_{i=1}^r (p_i - q_i) = \dim M$.

### 5.3 Topological gauge theory

Let $M$ be a compact, oriented four-manifold, $G$, a compact, semisimple Lie group, and $P \to M$, a principal $G$-bundle. Denote by $\mathcal{A}$ the space of connections on $P$ and $\mathcal{G}$, the group of gauge transformations. The Lie algebra of $\mathcal{G}$ is $\text{Lie}(\mathcal{G}) = \Gamma(AdP) = \mathcal{O}^0(M, AdP)$. At $A \in \mathcal{A}$, the tangent space is $T_A \mathcal{A} = \mathcal{O}^1(M, AdP)$.

Both spaces have inner products if we choose an invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of $G$ and a Riemannian metric $g$ on $M$. The infinitesimal action of $\mathcal{G}$ on $A = \nabla_A: \text{Lie}(\mathcal{G}) \to T_A \mathcal{A}$.

With a Riemannian metric, any 2-form on $M$ decomposes into self-dual and anti-self-dual parts: $\Omega^2(M) = \Omega^2_+ (M) \oplus \Omega^2_- (M)$. We consider a trivial vector bundle $\mathcal{E} \to A$ whose fiber is $\mathcal{O}^2_+ (M, AdP)$. $\mathcal{G}$ acts on $\mathcal{E}$ and the bundle is $\mathcal{G}$-equivariant. The trivial connection on $\mathcal{E}$ is $\mathcal{G}$-invariant; the moment is given by $\phi \in \Gamma(AdP); \chi \in \mathcal{O}^2_+ (M, AdP) \mapsto [\phi, \chi]$. The bundle $\mathcal{E}$ has a natural section $s: A \mapsto \phi \cdot \mathcal{F}_\phi$, the self-dual part of the curvature. Its derivative along $V \in \Omega^1(M, AdP) = T_A \mathcal{A}$ is $L_V s = (\nabla_A V)^+$. The section $s$ is $\mathcal{G}$-invariant, the zero set $s^{-1}(0)$ is the space of anti-self-dual connections, and the quotient $M = s^{-1}(0)/\mathcal{G}$ is the instanton moduli space. Its (virtual) dimension is
\[
\dim M = 4 \hat{h}(\mathfrak{g}) k(p) - \frac{1}{2} \dim G(\chi(M) + \sigma(M)), \tag{5.7}
\]
where $\hat{h}(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$ and $k(p) = -\frac{1}{2 \dim \mathfrak{g}} p_1(AdP), [M] \in \mathbb{Z}$ is the instanton number of $P$.

We proceed with the Mathai-Quillen interpretation of Atiyah and Jeffrey (1990). Let $\psi \in \Omega^1(M, AdP), \chi \in \mathcal{O}^2_+ (M, AdP), \eta \in \Gamma(AdP)$ be fermionic fields and $\phi, \lambda \in \Gamma(AdP)$, bosonic fields. The combination of (4.15) and (4.17) is given by the Lagrangian
\[
L[A, \psi, \phi, \chi, \eta, \lambda] = \frac{1}{2} ||F^+_A||^2 + \langle \phi, \nabla A \chi \rangle - \sqrt{-1}(\eta, \nabla A \psi) - \sqrt{-1}(\lambda, [\psi, \chi]) + \frac{1}{2} ||[\phi, \lambda]||^2. \tag{5.7}
\]
With an additional topological term proportional to $\mathcal{F}_A \wedge \mathcal{F}_A$, (5.7) is the Lagrangian of topological gauge theory of Witten (1988a). Here $(\cdot, \cdot)$ is the pairing induced by a Riemannian metric on $M$ and an invariant inner product on $\mathfrak{g}$.

There is a tautological connection on the $G$-bundle $A \times P \to A \times M$. It is invariant under the $\mathcal{G}$-action. Identifying $\mathcal{O}^1(\mathcal{A})$ with $C(\text{Map}(\mathbb{R}^0, \mathcal{A}))$ and using the Cartan model, the $\mathcal{G}$-equivariant curvature is $\mathcal{F} = \mathcal{F}_A + \sqrt{-1} \psi + \phi$. For any homology cycle $\gamma \in H_{q_i}(M)$,
\[
W_{\gamma}(A, \psi, \phi) = \frac{1}{4 \hat{h}(\mathfrak{g})} \int_{\gamma} (\mathcal{F}, \mathcal{F}) \tag{5.8}
\]
corresponds to a closed $\mathcal{G}$-equivariant form on $A$. For $\gamma_i \in H_{q_i}(M)$ $(1 \leq i \leq r)$, the expectation values
\[
\left\langle \prod_{i=1}^r W_{\gamma_i} \right\rangle = \frac{1}{\text{vol}(\mathcal{G})} \int [\mathcal{A}] [\mathcal{A}] [d\phi] [d\chi] [dn] [d\lambda] \prod_{i=1}^r W_{\gamma_i} (A, \psi, \phi) e^{-S[A, \psi, \phi, \chi, \eta, \lambda]} \tag{5.9}
\]
are, up to a factor of $|Z(G)|$, Donaldson invariants of $M$. Moreover, (5.9) is non-zero only if $\sum_{i=1}^{r}(4 - q_i) = \dim M$.

Other cohomological field theories can also be understood or constructed by the Mathai-Quillen formalism. Of such we mention only the topological field theories of Abelian and non-Abelian monopoles in Labastida and Mariño (1995), which are related to the Seiberg-Witten invariants.

See also

Topological quantum field theory: overview. Donaldson-Witten theory. Topological sigma models. Equivariant cohomology and the Weil and Cartan models. Characteristic classes.

Further Reading

Atiyah, M F and Jeffrey, L C (1990) Topological Lagrangians and cohomology. *J. Geom. Phys.* 7: 119-138.

Austin, D M and Braam, P J (1995) *Equivariant homology*. Math. Proc. Camb. Phil. Soc. 118: 125-139.

Berline, N, Getzler, E and Vergne, M (1992) *Heat kernels and Dirac operators*. Berlin: Springer-Verlag.

Blau, M and Thompson, G (1997) Aspects of $N_T \geq 2$ topological gauge theories and D-branes. *Nucl. Phys.* B 492: 545-590.

Cordes, S, Moore, M and Ramgoolam, S (1996) Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories. (Les Houches, 1994), pp. 505-682, North-Holland, Amsterdam.

Guillemin, V W and Sternberg, S (1999) *Supersymmetry and equivariant de Rham theory*. Berlin: Springer-Verlag.

Kalkman, J (1993) BRST model for equivariant cohomology and representatives for the equivariant Thom class. *Commun. Math. Phys.* 153: 447-463.

Labastida, J M F and Mariño, M (1995) A topological lagrangian for monopoles on four-manifolds. *Phys. Lett. B* 351: 146-152; Non-abelian monopoles on four-manifolds. *Nucl. Phys. B* 448: 373-395.

Mathai, V and Quillen, D (1986) Superconnections, Thom classes, and equivariant differential forms. *Topology* 25: 85-100.

Witten, E (1988a) Topological quantum field theory. *Commun. Math. Phys.* 117: 353-386.

Witten, E (1988b) Topological sigma models. *Commun. Math. Phys.* 118: 411-449.

Wu, S (1995) On the Mathai-Quillen formalism of topological sigma models. *J. Geom. Phys.* 17: 299-309.

Zhang, W (2001) *Lectures on Chern-Weil theory and Witten deformations*, Nankai Tracts in Mathematics, Vol. 4. Singapore: World Scientific.