EXISTENCE OF BOUNDED UNIFORMLY CONTINUOUS
MILD SOLUTIONS ON $\mathbb{R}$ OF EVOLUTION
EQUATIONS AND SOME APPLICATIONS.

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Abstract. We prove that there is $x_0 \in X$ for which (*) $\frac{du(t)}{dt} = Au(t) + \phi(t)$, $u(0) = x$ has on $\mathbb{R}$ a mild solution $u \in C_{ub}(\mathbb{R}, X)$ (that is bounded and uniformly continuous) with $u(0) = x_0$, where $A$ is the generator of a holomorphic $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$ with $\sup_{t \geq 0} ||T(t)|| < \infty$, $\phi \in L^\infty(\mathbb{R}, X)$ and $\text{isp}(\phi) \cap \sigma(A) = \emptyset$.

As a consequence it is shown that if $F$ is the space of almost periodic AP, almost automorphic AA, bounded Levitan almost periodic LAPb, certain classes of recurrent functions REC, and $\phi \in L^\infty(\mathbb{R}, X)$ such that $M_h \phi := (1/h) \int_0^h \phi(s) \, ds \in F$ for each $h > 0$, then $u \in F \cap C_{ub}$. These results seem new and generalize and strengthen several recent Theorems.

§1. Introduction, Definitions and Notation

In this paper we study solutions of the inhomogeneous abstract Cauchy problem

(1.1) $\frac{du(t)}{dt} = Au(t) + \phi(t)$, $u(t) \in X$, $t \in J \subseteq \{\mathbb{R}^+, \mathbb{R}\}$,

(1.2) $u(0) = x$,

where $A : D(A) \to X$ is the generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on the complex Banach space $X$ and $\phi \in L^1_{\text{loc}}(J, X)$, $x \in X$.

By [15, Corollary 2.5, p. 5], it follows that $D(A)$ is dense in $X$ and $A$ is a closed linear operator.

By a classical solution of (1.1) we mean a function $u : J \to D(A)$ such that $u \in C^1(J, X)$ and (1.1) is satisfied.

By a mild solution of (1.1), (1.2) we mean a $\omega \in C(J, X)$ with $\int_0^t \omega(s) \, ds \in D(A)$ for $t \in J$ and

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\[ \omega(t) = x + A \int_0^t \omega(s) \, ds + \int_0^t \phi(s) \, ds, \quad t \in J. \]

For \( J = \mathbb{R}_+ \) this is the usual definition [1, p. 120].

A classical solution is always a mild solution (see [1, (3.1), p. 110], for \( J = \mathbb{R}_+ \), \( \phi = 0 \)). Conversely, a mild solution with \( \omega \in C^1(J, X) \) and \( \phi \in C(J, X) \) is a classical solution (as in [1, Proposition 3.1.15]).

With translation and the case \( J = \mathbb{R}_+ \) [1, Proposition 3.1.16] one can show, for \( J, A, T, \phi \) as in (1.1) and after (1.2)

**Lemma 1.1.** \( \omega : J \rightarrow X \) is a mild solution of (1.1) if and only if

\[ \omega(t) = T(t - t_0)\omega(t_0) + \int_{t_0}^t T(t - s)\phi(s) \, ds, \quad t \geq t_0, \quad t_0 \in J. \]

When \( J = \mathbb{R} \) mild solutions of (1.1) may not exist. In this paper we establish the existence of a bounded uniformly continuous mild solution on \( \mathbb{R} \) to (1.1), (1.2) for \( \phi \in L^\infty(\mathbb{R}, X) \), \( x = x(\phi) \in X \) and \( i sp(\phi) \cap \sigma(A) = \emptyset \) when \( A \) is the generator of a holomorphic \( C_0 \)-semigroup \( (T(t)) \) satisfying \( \sup_{t \in \mathbb{R}_+} ||T(t)|| < \infty \). (Theorem 3.5).

In Theorem 3.6, we show that all mild solutions of (1.1), (1.2) are bounded and uniformly continuous when \( J = \mathbb{R}_+ \) and \( \phi \) and \( A \) as in Theorem 3.5. Theorem 3.6(i) should be compared with the "Non-resonance Theorem 5.6.5 of [1]" : the result of [1] is more general since the half-line spectrum is used (\( \subset \) Beurling spectrum).

However, our main result (Theorem 3.5) can not be deduced from Theorem 5.6.5 of [1]. Moreover, in the important cases of almost periodic, almost automorphic and recurrent functions, the half-line spectrum coincides with the Beurling spectrum (see [8, Example 3.8] and [10, Corollary 5.2]).

By mild admissibility of \( F \) with respect to equation (1.1) we mean that for every \( \phi \in F \), where \( F \subset L^\infty(\mathbb{R}, X) \) is a subspace with (4.2)- (4.5) below equations (1.1), (1.2) have a unique mild solution on \( \mathbb{R} \) which belongs to \( F \) (see [14, Definition 3.2, p. 248]). In the case \( F \subset C_{ub}(\mathbb{R}, X) \), several methods have been used to prove admissibility for (1.1), (1.2) (see [16], [4], [14], [12] and references therein). The
method of sums of commuting operators was introduced in [14], and then extended to the case \( \mathcal{F} = AA(\mathbb{R}, X) \) in [12], to prove admissibility for (1.1). As consequences of Theorem 3.5 we generalize and strengthen some recent results on admissibility. Namely, we prove that if \( \phi \in \mathcal{M}F \cap L^\infty(\mathbb{R}, X) \) (see (4.1)) and \( \imath \mathrm{sp}(\phi) \cap \sigma(A) = \emptyset \), then there is a mild solution \( u_\phi \in \mathcal{F} \cap C_{ub}(\mathbb{R}, X) \) of (1.1), (1.2). For example if \( \phi \in \mathcal{M}AP \cap L^\infty(\mathbb{R}, X) \), then \( u_\phi \in AP \) or if \( \phi \in \mathcal{MAA} \cap L^\infty(\mathbb{R}, X) \), then \( u_\phi \in AA \cap C_{ub} \).

Further notation and definitions. \( \mathbb{R}_+ = [0, \infty) \), \( L(X) = \{ B : X \to X, B \text{ linear bounded} \} \) with norm \( \| B \| \), \( S(\mathbb{R}) \) contains Schwartz’s complex valued rapidly decreasing \( C^\infty \)-functions, \( C_{ub}(\mathbb{R}, X) = \{ f : \mathbb{R} \to X : f \text{ bounded uniformly continuous} \} \), \( AP = AP(\mathbb{R}, X) \) almost periodic functions, \( AA = AA(\mathbb{R}, X) \) almost automorphic functions [19], \( BAA = BAA(\mathbb{R}, X) \) Bochner almost automorphic functions [21, p. 66], [12]; for \( f \in L^1_{loc}(J, X) \) \( (Pf)(t) := \text{Bochner integral } \int_0^t f(s) \, ds \), \( \hat{f}(\lambda) = \text{Fourier transform } \int_{\mathbb{R}} f(t) e^{-i\lambda t} \, dt \), \( f_a(t) = \text{translate } f(a + t) \) where defined, \( a \) real, \( \mathrm{sp} = \text{Beurling spectrum (2.3), Proposition 2.3.} \)

## §2. Preliminaries

In this section we collect some lemmas and propositions needed in the sequel.

**Proposition 2.1.** Let \( F \in L^1(\mathbb{R}, L(X)) \) and \( \phi \in L^\infty(\mathbb{R}, X) \) respectively \( F \in L^1(\mathbb{R}, X) \) and \( \phi \in L^\infty(\mathbb{R}, C) \) respectively \( F \in L^1(\mathbb{R}, C) \) and \( \phi \in L^\infty(\mathbb{R}, X) \).

(i) \( F \ast \phi(t) := \int_{-\infty}^{\infty} F(s)\phi(t - s) \, ds \)

exists as a Bochner integral for \( t \in \mathbb{R} \) and \( F \ast \phi \in C_{ub}(\mathbb{R}, X) \).

(ii) If moreover \( P\phi \in L^\infty(\mathbb{R}, X) \) respectively \( P\phi \in L^\infty(\mathbb{R}, C) \), then

\[
(F \ast (P\phi))(t) = (F \ast \phi)' = \frac{d}{dt} (F \ast \phi),
\]

\[
(F \ast (P\phi))(0) = \int_{\mathbb{R}} F(s) \phi(s) \, ds.
\]

**Proof.** (i) The integrand of (i) is measurable (approximate \( F \) and \( \phi \) by continuous functions a.e), it is dominated by \( \| \phi \|_\infty \| F \|_1 \in L^1(\mathbb{R}, \mathbb{R}) \), \( |F|(t) := \| F(t) \|_1 \), so this
Lemma 2.3. Let \( \lambda \notin \mathcal{V} \) be a neighborhood with \( \hat{\mathcal{V}} \). Then, if \( h \in h_{1.1.5} \) and \( p \in \mathcal{P} \), there is \( C \) such that the integrand \( \in L^1(\mathbb{R}, L(X)) \) for any fixed \( t \in \mathbb{R} \). To \( F \) there is a sequence \( (H_n) \) of \( L(X) \)-valued step-functions with \( ||F - H_n||_{L^1} \to 0 \). It follows \( ||F \ast \phi - H_n \ast \phi||_\infty \to 0 \), so with the \( H_n \ast \phi \) also \( F \ast \phi \) is bounded and uniformly continuous. Similarly in the other cases.

(ii) By part (i) we conclude that \( F \ast P \phi, F \ast \phi \in C_{ub}(\mathbb{R}, X) \); [1, Proposition 1.2.2] and the Lebesgue convergence theorem give existence and \( (F \ast P \phi)' = F \ast \phi, \in C_{ub}(\mathbb{R}, X) \), so \( (F \ast P \phi) \in C^1(\mathbb{R}, X) \). It follows \( F \ast P \phi = P(F \ast \phi) + F \ast P \phi(0) \). \( \Box \)

In the following \( sp \) denotes the Beurling spectrum, \( sp(\phi) = sp_{[0]}(\phi) \) as defined for example in [8, (3.2), (3.3)] case \( S = \phi \in L^\infty(\mathbb{R}, X), V = L^1(\mathbb{R}, \mathbb{C}), A = \{0, \mathbb{R}\} \):

\[
(2.3) \quad sp_{[0]}(\phi) := \{ \omega \in \mathbb{R} : f \in L^1(\mathbb{R}, \mathbb{C}), \phi \ast f = 0 \text{ imply } \hat{f}(\omega) = 0 \}. 
\]

\( sp_B \) is defined in [1, p. 321], \( sp_C \) is the Carleman spectrum [1, p.293/317].

**Proposition 2.2.** If \( \phi \in L^\infty(\mathbb{R}, X) \), then

\[
(2.4) \quad sp(\phi) = sp_{[0]}(\phi) = sp_B(\phi) = sp_C(\phi).
\]

See also [8, (3.3), (3.14)].

**Proof.** \( sp(\phi) \subseteq sp_B(\phi) \): Assuming \( \lambda \in sp(\phi) \) and \( h \in L^1(\mathbb{R}, \mathbb{C}) \) with \( \hat{h}(\lambda) \neq 0 \), we conclude \( \phi \ast h \neq 0 \). This implies \( \lambda \notin sp_B(\phi) \).

\( sp_B(\phi) \subseteq sp(\phi) \): Assume \( \lambda \in sp_B(\phi) \) but \( \lambda \notin sp(\phi) \). Then there is \( h \in L^1(\mathbb{R}, \mathbb{C}) \) with \( \hat{h}(\lambda) \neq 0 \) and \( \phi \ast h = 0 \). With Wiener’s inversion theorem [11, Proposition 1.1.5 (b), p. 22], there is \( h_\lambda \in L^1 \) such that \( k = h \ast h_\lambda \) satisfies \( \hat{k} = 1 \) on some neighbourhood \( V = (\lambda - \varepsilon, \lambda + \varepsilon) \) of \( \lambda \). Now let \( g \in L^1(\mathbb{R}, \mathbb{C}) \) be such that \( \text{supp} \hat{g} \subseteq V \). Then \( k \ast g = g \) and \( 0 = \phi \ast h \ast k = \phi \ast (k \ast g) = \phi \ast g \). This implies \( \lambda \notin sp_B(\phi) \) by [1, p. 321]. This is a contradiction.

\( sp_B(\phi) = sp_C(\phi) \): See [1, Proposition 4.8.4, p. 321]. \( \Box \)

**Lemma 2.3.** Let \( \phi \in L^\infty(\mathbb{R}, X) \).

(i) If \( P \phi \in L^\infty(\mathbb{R}, X) \), then \( sp(\phi) \subseteq sp(P \phi) \subseteq sp(\phi) \cup \{0\} \).

(ii) If \( F \in L^1(\mathbb{R}, \mathbb{C}) \) or \( F \in L^1(\mathbb{R}, L(X)) \), then \( sp(F \ast \phi) \subseteq sp(\phi) \cap \text{supp} \hat{F} \).
(iii) If \( f \in L^1(\mathbb{R}, \mathbb{C}) \), then \( \text{sp}(\phi - \phi * f) \subset \text{sp}(\phi) \setminus U \), where \( U \) is the interior of the set \( \{ \lambda \in \mathbb{R} : \hat{f}(\lambda) = 1 \} \).

See also [3, Theorem 4.1.4], [14, Theorem 2.1], [17, Proposition 0.4].

**Proof.** (i) If \( f \in L^1(\mathbb{R}, \mathbb{C}) \) with \((P\phi) * f = 0\), then \((2.4)\) yields \( \phi * f = 0 \) and so the first inclusion follows from \((2.3)\) (see also [17, Proposition 04 (iv), p. 20]). The second inclusion follows by [9, Proposition 1.1. (f)] valid also for \( X \)-valued functions.

(ii) \( F \in L^1(\mathbb{R}, L(X)) \): to \( \lambda \not\in \text{sp}(\phi) \) exists \( f \in L^1(\mathbb{R}, \mathbb{C}) \) with \( \phi * f = 0 \) and \( \hat{f}(\lambda) = 1 \), then \( (F * \phi) * f = F * (\phi * f) = 0 \), so \( \lambda \not\in \text{sp}(F * \phi) \), yielding \( \text{sp}(F * \phi) \subset \text{sp}(\phi) \). (The associativity of convolution used here can be shown as in [1, Proposition 1.3.1, p. 22].) If \( \lambda \not\in \text{supp} \hat{F} \), there exists \( f \in \mathcal{S}(\mathbb{R}) \) with \( (\text{supp} \hat{f}) \cap \text{supp} \hat{F} = \emptyset \) and \( \hat{f}(\lambda) = 1 \), then \( \hat{F} \hat{f} = \hat{F} f = 0 \), and so \( F * f = 0 \); this gives \( 0 = (F * f) * \phi = F * (f * \phi) = F * (\phi * f) = (F * \phi) * f \), and so \( \lambda \not\in \text{sp}(F * \phi) \), yielding \( \text{sp}(F * \phi) \subset \text{supp} \hat{F} \). (Existence of \( F * f, F * f \in L^1(\mathbb{R}, L(X)) \), and \( \hat{F} \hat{f} = \hat{F} f \) follows with Fubini-Tonelli similar as for Proposition 2.1(i).)

The proof for the case \( F \in L^1(\mathbb{R}, \mathbb{C}) \) is similar.

(iii) If \( \lambda \in \text{sp}(\phi) \cap U \), then there is \( f_\lambda \in \mathcal{S}(\mathbb{R}) \) with \( \text{supp} \hat{f}_\lambda \subset U \) such that \( \hat{f}_\lambda(\lambda) \neq 0 \) and \( f * f_\lambda = f_\lambda \). It follows \( \lambda \not\in \text{sp}(\phi - \phi * f) \). \( \Box \)

In the following if \( U, V \subset \mathbb{R} \), then \( d(U, V) := \inf_{u \in U, v \in V} |u - v| \). If \( U, V \) are compact and \( U \cap V = \emptyset \), then \( d(U, V) > 0 \).

**Lemma 2.4.** Let \( \psi \in C_b(\mathbb{R}, X) \). Then there exists a sequence of \( X \)-valued trigonometric polynomials \( \pi_n \) such that

(i) \( \sup_{n \in \mathbb{N}} ||\pi_n||_\infty < \infty \).

(ii) \( \pi_n(t) \to \psi(t) \) as \( n \to \infty \) locally uniformly on \( \mathbb{R} \).

(iii) If the Beurling spectrum \( \text{sp}(\psi) \cap [\alpha, \beta] = \emptyset \), then there exists \( \delta > 0 \) such that the Fourier exponents of \( \pi_n \) can be selected from \( \mathbb{R} \setminus (\alpha - \delta, \beta + \delta) \).
(iv) If $K_1, K_2$ are compact subsets of $(\alpha, \beta)$, where $K_1 = \text{sp}(\psi)$ is the Beurling spectrum of $\psi$ and $K_1 \cap K_2 = \emptyset$ and $\eta = d(K_1, K_2)$, then there exist $W, \varphi$ so that the Fourier exponents of $\pi_n$ can be selected from $W$, where $W$ is an open set containing $K_1$, and $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi = 1$ on $W$ and $d(K_2, W) \geq d(K_2, \text{supp } \varphi) \geq \eta/2$.

Proof. (i), (ii): Similar to the proof of Lemma 3.1 (3.1) of [4] with [1, Theorem 4.2.19].

(iii) Choose $(\alpha_1, \beta_1) \subset \mathbb{R}$ such that $[\alpha, \beta] \subset (\alpha_1, \beta_1)$ and $\text{sp}(\psi) \subset \mathbb{R} \setminus (\alpha_1, \beta_1)$. Let $f \in \mathcal{S}(\mathbb{R})$ be such that $\text{supp } \hat{f} \subset (\alpha_1, \beta_1)$ and $\hat{f} = 1$ on $[\alpha - \delta, \beta + \delta]$ for some $\delta > 0$. Let $(T_n)$ satisfy (i), (ii). Then $\pi_n = T_n - T_n \ast f$ satisfy (i), (ii) since by Lemma 2.3 (ii) $\text{sp}(\psi \ast f) \subset (\mathbb{R} \setminus (\alpha_1, \beta_1)) \cap (\alpha_1, \beta_1) = \emptyset$ implies $\psi \ast f = 0$ by [1, Theorem 4.8.2 (a)] and Proposition 2.2, and the Fourier exponents of $\pi_n$ belong to $\mathbb{R} \setminus (\alpha - \delta, \beta + \delta)$.

(iv) Let $\Gamma = (I_t)_{t \in K_1}, \Omega = \cup_{t \in K_1} I_t$ be a system of open intervals with the properties

$$I_t = (t - \delta, t + \delta), I_t \subset (\alpha, \beta) \text{ and } 0 < \delta < \eta/2 \text{ for each } t \in K_1.$$ 

With a partition of unity [18, Theorem 6.20, p. 147] there is a sequence $(\psi_i) \subset \mathcal{D}(\Omega)$ with $\psi_i$ supported in some $I_t \subset \Gamma, \psi_i \geq 0$ and $\sum_{i=1}^{\infty} \psi_i(t) = 1, t \in \Omega$. Moreover, for $K_1$ there is a positive integer $m$ and an open set $W \supset K_1$ such that

$$\varphi := \sum_{i=1}^{m} \psi_i(t) = 1 \text{ for each } t \in W.$$ 

Since $\text{supp } \psi_i \subset I_t, d(K_2, W) \geq d(K_2, \text{supp } \varphi) \geq \eta/2$. Now consider the system $\Gamma_W := (J_t)_{t \in K_1}$, where $J_t = I_t \cap W$ for each $t \in K_1$. As above there is an open set $U \supset K_1$ and $\varphi_W \in \mathcal{D}(\mathbb{R})$ such that $\varphi_W = 1$ on $U$ and $\text{supp } \varphi_W \subset W$. It follows $\varphi_W = \hat{f_w}$ with $f_w \in \mathcal{S}(\mathbb{R})$. Let $(T_n)$ be a sequence of $X$-valued trigonometric polynomials satisfying (i), (ii). Take $\pi_n = T_n \ast f_w, n \in \mathbb{N}$. Then $sp(\pi_n) \subset W$, Lemma 2.3 (iii) gives $sp(\psi - \psi \ast f_W) = \emptyset$ and so $\psi = \psi \ast f_W$ as in (iii). \qed

Proposition 2.5. If $f, \hat{f} \in L^1(\mathbb{R}, X) \cap C_b(\mathbb{R}, X)$, then ([1, Theorem 1.8.1 d), p.
\begin{align*}
45) \quad f(t) &= (1/2\pi) \int_{\mathbb{R}} \hat{f}(\lambda) e^{i\lambda t} \, d\lambda, \quad t \in \mathbb{R}.
\end{align*}

\section{Holomorphic $C_0$-Semigroups}

In this section we study (1.1) when $A$ is the generator of a holomorphic $C_0$-semigroup $(T(t))$ in the sense [1, Definition 3.7.1]. By [1, Corollary 3.7.18], it follows that there is $a > 0$ such that

\begin{align*}
(3.1) \quad \{is, s \in \mathbb{R}: |s| > a\} \subset \rho(A) \quad \text{and} \\
(3.2) \quad \sup_{|s| > a} ||s R(is, A)|| < \infty.
\end{align*}

**Lemma 3.1.** Let $b > 0$, $m \in \mathbb{N}$. There are $f_k \in C^m([0, b])$, $k = 0, 1, \ldots, n$ such that

\begin{align*}
f_k^{(k)}(b) &= 1, \quad f_k^{(j)}(b) = 0, \quad k \neq j, \quad k, j = 0, 1, \ldots, m, \\
f_k^{(j)}(0) &= 0, \quad k, j = 0, 1, 2, \ldots, m, \quad (f^{(0)} = f).
\end{align*}

**Proof.** The case $m = 2$: Let $f(t) = (\pi/2) \sin(\pi(t/b)^3/2)$, $t \in [0, b]$. Then

\begin{align*}
f_0(t) &= \sin f(t), \quad f_1(t) = (t - b)f_0(t), \quad f_2(t) = (1/2)(t - b)^2 f_0(t)
\end{align*}

satisfy all the requirements of the statement.

The general case follows by the Lagrange Interpolation Theorem ([2, p. 395]).

**Lemma 3.2.** Let $A : D(A) \to X$ be the generator of a holomorphic $C_0$-semigroup $(T(t))$.

(i) There exists $a > 0$ such that $R(i\lambda, A) = (i\lambda I - A)^{-1}$ is infinitely differentiable on $\mathbb{R} \setminus (-b, b)$ for each $b > a$. Moreover, $R(i\lambda, A)(\mathbb{R} \setminus (-b, b))$ can be extended to a function $H_m \in C^m(\mathbb{R}, L(X))$ for any $m \in \mathbb{N}$ with $H_m^{(j)}(0) = 0$, $0 \leq j \leq m$, $H_m(t)X \subset D(A)$ for $t \in \mathbb{R}$.

(ii) $H_m^{(k)} \in L^1(\mathbb{R}, L(X))$, $1 \leq k \leq m$.

**Proof.** (i) By (3.1) $\sigma(A) \cap i\mathbb{R} \subset i[-a, a]$ for some $a > 0$. It follows $R(i\lambda, A) = (i\lambda I - A)^{-1}$ is infinitely differentiable on $\mathbb{R} \setminus (-b, b)$ for each $b > a$. Fix one such
Then $H$ and let $b$ and $f_k(t)$ satisfy the conditions of Lemma 2.1 and $g_k(-t) = f_k(t)$, $t \in [0, b]$, $k = 0, 1, \ldots, m$. Set

$$F(t) = f_0(t)R(ib, A) + f_1(t)R'(ib, A) + \cdots + f_m(t)R^{(m)}(ib, A), \ t \in [0, b],$$

$$F(t) = g_0(t)R(-ib, A) + g_1(t)R'(-ib, A) + \cdots + g_n(t)R^{(n)}(-ib, A), \ t \in [-b, 0].$$

Then $F \in C^m([-b, b], \mathcal{L}(X))$. Set

$$(3.3) \quad H_m(\lambda) = R(i\lambda, A) \text{ on } \mathbb{R} \setminus (-b, b) \text{ and } H_m = F \text{ on } [-b, b].$$

Then $H_m \in C^m(\mathbb{R}, \mathcal{L}(X))$. Since $R(\lambda, A)X \subset D(A)$ by definition ([1, p. 462]) and $D(A)$ is linear, $H_m(\mathbb{R}, X) \subset D(A)$ with Lemma 3.1.

(ii) By (3.2) there is $M = M(m) > 0$ with

$$||R(i\lambda, A)||^k \leq M/|\lambda|^k \text{ on } \mathbb{R} \setminus (-b, b), \ 1 \leq k \leq m.$$  

Since ([1, p. 463, Corollary B.3])

$$H_m^{(k)}(\lambda) = k!(-i)^k(R(i\lambda, A))^{k+1} \text{ if } b \leq |\lambda|, \ 1 \leq k \leq m,$$

with (3.4), (3.5) and part (i) one has $H_m^{(k)} \in L^1(\mathbb{R}, \mathcal{L}(X)), \ 1 \leq k \leq m.$

**Proposition 3.3.** Let $m \geq 2$ and $H_m$ be as in (3.3).

(i) The function $G_m$ defined by

$$(3.6) \quad G_m(t) := (1/2\pi) \int_{\mathbb{R}} H_m(\lambda)e^{it\lambda} \, d\lambda,$$

exists as an improper Riemann integral for each $t \neq 0$.

(ii) $t^k G_m \in L^1(\mathbb{R}, \mathcal{L}(X)), \ k = 1, 2, \ldots, m - 2$.

**Proof.** With Lemma 3.2 (ii), one has $H'_m, H''_m \in L^1(\mathbb{R}, \mathcal{L}(X))$. $H_m$ being continuous on $\mathbb{R}$, for $t \neq 0$ and $T > 0$ one can use partial integration and gets

$$\int_0^T H_m(\lambda)e^{it\lambda} \, d\lambda = (1/it)H_m(T)e^{itT} - (1/(it)) \int_0^T H_m'(\lambda)e^{it\lambda} \, d\lambda,$$

similarly for the integral from $-t$ to 0. This shows that $G_m(t)$ given by (3.6) exists as an improper Riemann integral and is given by

$$(i/(2t\pi))\int_{-\infty}^{\infty} H'_m(\lambda)e^{it\lambda} \, d\lambda, \ t \neq 0.$$

Induction gives with Lemma 3.2 (ii), for $1 \leq k \leq m$,

$$(3.7) \quad G_m(t) = (1/2it) (i/t)^k \int_{\mathbb{R}} H_m^{(k)}(\lambda)e^{it\lambda} \, d\lambda, \ t \neq 0.$$
Lemma 3.4. (i) \( I(t) = \int_0^\infty R(i\lambda, A) \cos \lambda t \, d\lambda \) exists as improper Riemann integral for each \( t > 0 \). Moreover, \( \|I(t)\| = O(\ln |t|) \) for all \( 0 < t < \pi/2b \).

(ii) \( J(t) = \int_0^\infty R(i\lambda, A) \sin \lambda t \, d\lambda \) exists as improper Riemann integral for each \( t > 0 \). Moreover, \( J(t) \) is bounded for all \( t > 0 \).

(iii) For \( G_m \) of Proposition 3.3 and \( m \geq 2 \), one has \( \|G_m(t)\| = O(\ln |t|) \) in some neighbourhood of 0, \( G_m \in L^1(\mathbb{R}, L(X)) \) (Bochner-Lebesgue integrable functions).

(iv) \( \widehat{G_m} = H_m \) if \( m \geq 3 \) for the \( H_m \) of Lemma 3.2.

(v) \( G_m * x = 0 \) for each \( x \in X, m \geq 2 \).

Proof. (i) For \( t \geq \pi/2b \), \( I(t) \) exists similarly as in Proposition 3.3. For \( 0 < t < \pi/2b \), we have \( I(t) = (1/t) \int_{bt}^{\infty} R(i\lambda/t, A) \cos \lambda t \, d\lambda = (1/t)[a_0(t) + \sum_{k=1}^{\infty} a_k(t)], \) where

\[
a_0(t) = \int_0^{\pi/2} R(i\lambda/t, A) \cos \lambda t \, d\lambda,
\]

\[
a_k(t) = \int_{\pi/(1+2+2(k-1))}^{\pi/2} R(i\lambda/t, A) \cos \lambda t \, d\lambda = \int_0^{\pi} (B_k(\lambda, t) - A_k(\lambda, t)) \sin \lambda t \, d\lambda,
\]

with

\[
B_k(\lambda, t) = R(i(\lambda + \pi(3/2 + 2(k-1))))/t, A),
\]

\[
A_k(\lambda, t) = R(i(\lambda + \pi(1/2 + 2(k-1))))/t, A).
\]

Since \( R(\mu, A) - R(\lambda, A) = (\lambda - \mu)R(\mu, A)R(\lambda, A) \) for any \( \lambda, \mu \in \rho(A) \) [1, B.4, p. 464], we have

\[
B_k(\lambda, t) - A_k(\lambda, t) = -i(\pi/t)R(i(\lambda + \pi(3/2 + 2(k-1))))/t, A)R(i(\lambda + \pi(1/2 + 2(k-1))))/t, A).
\]

Using (3.4) we conclude that \( \|a_k(t)\| = O(\frac{1}{t}) \) for all \( t > 0, k \in \mathbb{N} \). It follows that series \((1/t) \sum_{k=1}^{\infty} a_k \) is convergent and bounded on \((0, \infty)\). Again by (3.4),

\[
(1/t)\|a_0\| \leq M \int_0^{\pi/2} \frac{\cos \lambda}{\lambda} \, d\lambda \leq M \int_0^{\pi/2} \frac{1}{\lambda} \, d\lambda = M \ln(\pi/2) - \ln(bt) = O(\ln |t|).
\]

(ii) we have \( J(t) = (1/t) \int_{bt}^{\infty} R(i\lambda/t, A) \sin \lambda t \, d\lambda = (1/t)[b_0(t) + \sum_{k=1}^{\infty} b_k(t)], \) where

\[
b_0(t) = \int_0^{\pi} R(i\lambda/t, A) \sin \lambda t \, d\lambda, \quad b_k(t) = \int_{\pi/(1+2(k-1))}^{\pi} (R(i\lambda/t, A) \sin \lambda t) \, d\lambda.
\]

Hence \( G_m \) is continuous and \( \|G_m(t)\| \leq 1/(2\pi |t|^2) \int_0^{\infty} \|H_m^{(k)}(\lambda)\| \, d\lambda \) in each point \( t \neq 0 \).

(ii) This follows by (3.7) using Lemma 3.2 (ii).
Since $\sin \frac{\lambda}{t}$ is bounded on $\mathbb{R} \setminus \{0\}$, we conclude that $b_0(t)/t$ is bounded for all $t > 0$.
Similarly as in part (i), we conclude that the series $(1/t) \sum_{k=1}^{\infty} b_k(t)$ is convergent
for all $t > 0$ and bounded on $(0, \infty)$.

(iii) Follows as in parts (i), (ii) and Proposition 3.3 noting that

$$2\pi G_m(t) = \int_{-b}^{b} H_m(\lambda) e^{i\lambda t} d\lambda + \int_{-b}^{-\infty} R(i\lambda, A) e^{i\lambda t} d\lambda + \int_{b}^{\infty} R(i\lambda, A) e^{i\lambda t} d\lambda = [I(b, t) + \int_{b}^{\infty} (R(-i\lambda, A) + R(i\lambda, A)) \cos \lambda t + (R(i\lambda, A) - R(-i\lambda, A)) \sin \lambda t) d\lambda],$$

where $I(b, t) = \int_{-b}^{b} H_m(\lambda) e^{i\lambda t} d\lambda$.

(iv) Since $H_m', tG_m \in L^1(\mathbb{R}, L(X)) \cap C(\mathbb{R}, L(X))$ by Proposition 3.3(ii) and

$$-i(tG_m)(t) = (1/2\pi) \int_{\mathbb{R}} e^{i\lambda t} H_m'(\lambda) \, d\lambda,$$

one gets existence of $\widehat{G}_m$ and by Proposition 2.5 for $\lambda \in \mathbb{R}$

$$H_m'(\lambda) = -i \int_{\mathbb{R}} e^{-i\lambda t} tG_m(t) \, dt = \widehat{G}_m' (\lambda).$$

This implies $H_m = \widehat{G}.$

(v) $G_m * x(t) = \int_{\mathbb{R}} G_m(t) x dt = \widehat{G}_m x(0) = \widehat{G}_m (0) x = H_m(0) x = 0$, with Lemma

3.4, Proposition 2.1 and Lemma 3.2. $\blacksquare$

In the following we use:

For any $G \in L^1(\mathbb{R}, L(X))$, $\lambda \in \mathbb{R}$ with $\widehat{G}(\lambda) = R(i\lambda, A)$ and $x \in X$ one has

$Gx \in L^1(\mathbb{R}, X)$ and $(\widehat{G}x) = (\widehat{G}) x$ [1, Proposition 1.1.6], so for $e_\lambda(t) := e^{i\lambda t} x$, the

following LB-integrals exist and

$$(G * e_\lambda)(t) = e^{i\lambda t} \int_{\mathbb{R}} G(s) x e^{-i\lambda s} ds = e^{i\lambda t} (\widehat{G}x)(\lambda) = e^{i\lambda t} \widehat{G}(\lambda) x = e^{i\lambda t} R(i\lambda, A)x,$$

$t \in \mathbb{R}$, with $R(i\lambda, A)x \in D(A)$.

So for $x \in X$ and the $\lambda$ and the $G$ above, the equation

$$(3.8) \quad u'(t) = Au(t) + xe^{i\lambda t}, \ t \in \mathbb{R}, \ u(0) = R(i\lambda, A)x$$

has a classical solution $u(t) = e^{i\lambda t} R(i\lambda, A)x = (G * e_\lambda)(t), \ t \in \mathbb{R}.$

Let $A, a, b$ be as in Lemma 3.2, $\delta = b - a$ and $c = b + 3\delta$. Choose $f \in S(\mathbb{R})$ such

that

$$(3.9) \quad \widehat{f}(\lambda) = 1 \text{ on } [-b + \delta, b + \delta] \text{ and supp } \widehat{f} \subset (-b - 2\delta, b + 2\delta).$$

Let $\phi \in L^\infty(\mathbb{R}, X)$. Set
\[(3.10) \quad \phi_1 = \phi \ast f \text{ and } \phi_2 = \phi - \phi \ast f.\]

Then (3.9) and Lemma 2.3 (ii) give

\[(3.11) \quad sp(\phi_1) \subset \text{supp } \hat{f} \subset (-b + 2\delta, b + 2\delta) \text{ and } sp(\phi_2) \cap [-b, b] = \emptyset.\]

**Theorem 3.5.** Let \((T(t))\) be a holomorphic \(C_0\)-semigroup on \(X\) with generator \(A\) such that \(\sup_{t \geq 0}|T(t)| < \infty\) and let \(\phi \in \mathcal{L}^\infty(\mathbb{R}, X)\) with Beurling spectrum \(sp(\phi)\) satisfying \(i \cdot sp(\phi) \cap \sigma(A) = \emptyset\). Then there is \(x_\phi \in X\) such that the equations (1.1), (1.2) with \(x = x_\phi\) have a bounded uniformly continuous mild solution on \(\mathbb{R}\).

**Proof.** Let \(\phi_1, \phi_2\) be given by (3.10).

Case \(\phi_1\): Let \(K_1 = sp(\phi_1), iK_2 = \sigma(A) \cap i\mathbb{R}\) and \(\eta = d(K_1, K_2)\). Then \(K_1, K_2 \subset (-c, c), \) where \(c\) is defined above (3.9). By Lemma 2.4 (iv) there exist \(W, \varphi\) and a bounded sequence of trigonometric polynomials \((\pi_n)\) such that \(\pi_n(t) \to \phi_1(t)\) as \(n \to \infty\) uniformly on each bounded interval of \(\mathbb{R}\) and the Fourier exponents of \((\pi_n)\) belong to \(W\), where \(W\) is an open set \(W \supset K_1\) with \(d(K_2, W) \geq d(K_2, \text{supp } \varphi) \geq \eta/2, \varphi \in \mathcal{D}(\mathbb{R})\) and \(\varphi = 1\) on \(W\). Set \(Q(\lambda) = \varphi(\lambda) R(i\lambda, A)\) on \(\mathbb{R} \setminus K_2\) and 0 on \(K_2\). Since \(\text{supp } \varphi \cap K_2 = \emptyset\), \(Q\) is infinitely differentiable on \(\mathbb{R}\) and has compact support, with \(\hat{Q} \in C_b(\mathbb{R}, L(X)) \cap L^1(\mathbb{R}, L(X))\) by partial integration, so with Proposition 2.5 the \(Q\) is the Fourier transform of a bounded continuous function \(F \in L^1(\mathbb{R}, L(X))\). We claim that \(u_{\phi_1}(t) = (F \ast \phi_1)(t)\) is a mild solution on \(\mathbb{R}\) of the equation \(u'(t) = Au(t) + \phi_1(t), u(0) = (F \ast \phi_1)(0)\). Using (3.8) and \(sp(\pi_n) \subset W\), one can show that \(u_n(t) := (F \ast \pi_n)(t)\) is a classical solution on \(\mathbb{R}\) of the equation \(u'(t) = Au(t) + \pi_n(t), u(0) := (F \ast \pi_n)(0)\) and therefore is a mild solution. So by Lemma 1.1, \(u_n(t) = T(t-t_0)(F \ast \pi_n(t_0)) + \int_{t_0}^t T(t-s)\pi_n(s) \, ds\) for each \(t \geq t_0 \in \mathbb{R}\), \(u_n(0) = (F \ast \pi_n)(0), n \in \mathbb{N}\). Passing to the limit when \(n \to \infty\), we get with Lemma 2.4 (i), (ii) and the dominated convergence theorem

\[(F \ast \phi_1)(t) = T(t-t_0)(F \ast \phi_1)(t_0) + \int_{t_0}^t T(t-s)\phi_1(s) \, ds\] for each \(t \geq t_0 \in \mathbb{R}\).

Now Lemma 1.1 shows that \(u_{\phi_1}(t)\) is a mild solution on \(\mathbb{R}\) of the equation (1.1)
with $\phi = \phi_1$ and $x = (F * \phi_1)(0)$.

Case $\phi_2$: By (3.11) case $\phi_2$, one has $sp(\phi_2) \cap [-b, b] = \emptyset$. It follows $P\phi_2 \in C_{ub}(\mathbb{R}, X)$ by [9, Corollary 4.4 valid also for $X$-valued functions]. By Lemma 2.3, $sp(\phi_2) \subset sp(P\phi_2) \subset sp(\phi_2) \cup \{0\}$. With $h \in S(\mathbb{R})$ such that $\hat{h}(\lambda) = 1$ on $[-a/2, a/2]$ and $supp \hat{h} \subset (-a, a)$, one has $sp((P\phi_2 \ast h) \subset \{0\}$, $(P\phi_2) \ast h = x_0 \in X$ [3, Theorem 4.2.2], so by Lemma 2.3 (ii) the spectrum of $\psi := P\phi_2 - (P\phi_2) \ast h$ satisfies $sp(\psi) \subset sp(\phi_2)$. By Lemma 2.4 (iii) there exists a sequence of trigonometric polynomials $\psi_n(t)$ such that $\pi_n(t) \to \psi(t)$ as $n \to \infty$ uniformly on each bounded interval of $\mathbb{R}$ and the Fourier exponents of $\psi_n(t)$ belong to $\mathbb{R} \setminus [-b, b]$. Using (3.8), one can show that $u_n(0) := (G \ast \psi_n)(t)$ is a classical solution on $\mathbb{R}$ of the equation $u'(t) = Au(t) + \pi_n(t)$, $u(0) := G \ast \pi_n(0)$, where $G = G_3$ defined by (3.7) case $m = 3$.

So by Lemma 1.1 (1.4), $u_n(t) = T(t - t_0)(G \ast \pi_n(t_0)) + \int_{t_0}^{t} T(t - s)\pi_n(s) \, ds$ for each $t \geq t_0 \in \mathbb{R}$, $u_n(0) = G \ast \pi_n(0)$, $n \in \mathbb{N}$. Passing to the limit when $n \to \infty$, we get with Lemma 2.4 (i), (ii), $G \in L^1(\mathbb{R}, L(X))$ by Lemma 3.4(iii)

$$(G \ast \psi)(t) = T(t - t_0)(G \ast \psi)(t_0) + \int_{t_0}^{t} T(t - s)\psi(s) \, ds$$

for each $t \geq t_0 \in \mathbb{R}$.

By Proposition 2.1 (i) we conclude that $G \ast \psi, G \ast \phi_2 \in C_{ub}(\mathbb{R}, X)$. By (2.4), $(G \ast \psi) \in C^1(\mathbb{R}, X)$. By Lemma 1.1 and [1, Proposition 3.1.15 valid also for $J = \mathbb{R}$], $G \ast \psi$ is a classical solution on $\mathbb{R}$ of the equation $u'(t) = Au(t) + \psi(t)$, $u(0) = G \ast \psi(0)$. So, $G \ast \psi(t) \in D(A)$ for each $t \in \mathbb{R}$. Since $G \ast \psi = G \ast P\phi_2 - G \ast x_0$ and $G \ast x_0 = 0$ by Lemma 3.4 (v), it follows $G \ast P\phi_2(t) \in D(A)$ for each $t \in \mathbb{R}$, with Proposition 2.1 then

$$(G \ast \psi)'(t) = G \ast \phi_2(t) = A(G \ast (P\phi_2 - x_0)(t)) + P\phi_2(t) - x_0 = A(P(G \ast \phi_2)(t)) + A(G \ast (P\phi_2)(0)) + (P\phi_2)(t) - x_0 = -x_0 + A(G \ast (P\phi_2)(0)) + AP(G \ast \phi_2(t) + P\phi_2(t),$$

with $P(G \ast \phi_2)(t) \in D(A)$ for each $t \in \mathbb{R}$. It follows that $G \ast \phi_2(0) = -x_0 + A(G \ast P\phi_2)(0))$. Hence $v = G \ast \phi_2$ is a mild solution of $v'(t) = Av(t) + \phi_2(t)$ on $\mathbb{R}$, $v(0) = G \ast \phi_2(0)$ by (1.3).

Finally, $u_\phi := F \ast \phi_1 + G \ast \phi_2$ is a mild solution of (1.1) on $\mathbb{R}$ with $x = u_\phi(0) = $
$F \ast \phi_1(0) + G \ast \phi_2(0) = x_\phi$ bounded and uniformly continuous by the above. \footnote{}

**Theorem 3.6.** Let $A, \phi$ be as in Theorem 3.5 and let $\phi_1, \phi_2$ be as in (3.10). There exist $F, G \in L^1(\mathbb{R}, L(X))$ so that

(i) For each $x \in X$, $u(t) = T(t)x + F \ast \phi_1 - F \ast \phi_1(0) + G \ast \phi_2 - G \ast \phi_2(0)$ is the unique mild solution on $\mathbb{R}_+$ of (1.1). Moreover, $u \in C_{ub}(\mathbb{R}_+, X)$.

(ii) In addition, if $J = \mathbb{R}$, $(T(t))$ is a bounded $C_0$-group ($\sup_{t \in \mathbb{R}} ||T(t)|| < \infty$), then for each $x \in X$ equations (1.1), (1.2) have a unique mild solution on $\mathbb{R}$ given by $u(t) = T(t)x + F \ast \phi_1 - F \ast \phi_1(0) + G \ast \phi_2 - G \ast \phi_2(0)$. Moreover, $u \in C_{ub}(\mathbb{R}, X)$.

**Proof.** (i) By [1, Proposition 3.1.16] the solutions of $v' = Av$ on $\mathbb{R}_+$ are given by $v(t) = T(t)x$, (i) follows by Theorem 3.5.

(ii) With Theorem 3.5 only $u' = Au, u(0) = x$, has to be considered.

Uniqueness: (1.4) gives $u(0) = T(0 - (-n))u(-n) = T(n)u(-n), u(-n) = T(-n)u(0)$, $u(t) = T(t - (-n))T(-n)u(0) = T(t)u(0), t > -n, n \in \mathbb{N}$.

Existence: $u(t) := T(t)x, t \in \mathbb{R}$, gives $u(t_0) = T(t_0)x$ or $x = T(t_0)u(t_0)$, so $u(t) = T(t)T(-t_0)u(t_0) = T(t - t_0)u(t_0)$, with Lemma 1.1 $u$ is a mild solution on $\mathbb{R}$. \footnote{4. Existence of generalized almost periodic solutions of equation (1.1) in the non-resonance case

For $\mathcal{A} \subset L^1_{loc}(\mathbb{R}, X)$ we define mean classes $\mathcal{M} \mathcal{A}$ by ([5, p. 120, Section 3])

(4.1) $\mathcal{M} \mathcal{A} := \{ f \in L^1_{loc}(\mathbb{R}, X) : \mathcal{M}_hf \in \mathcal{A}, h > 0 \}$, where

$\mathcal{M}_hf(t) = (1/h) \int_0^h f(t + s) \, ds$.

Usually $\mathcal{A} \subset \mathcal{M} \mathcal{A} \subset \mathcal{M}^2 \mathcal{A} \subset \cdots$ with the $\subset$ in general strict (see [6, Proposition 2.2, Example 2.3]).

We denote by $\mathcal{F}$ any class of functions having the following properties:

(4.2) $\mathcal{F}$ linear $\subset L^1_{loc}(J, X) \subset X^{\mathbb{R}}$.

(4.3) $(\phi_n) \subset \mathcal{F} \cap C_{ub}$ and $\phi_n \to \psi$ uniformly on $\mathbb{R}$ implies $\psi \in \mathcal{F}$.}
(4.4) $\phi \in \mathcal{F}$, $a \in \mathbb{R}$ implies $\phi_a \in \mathcal{F}$.

(4.5) $B \circ \phi \in \mathcal{F}$ for each $B \in L(X)$, $\phi \in \mathcal{F} \cap \mathcal{C}_{ab}$.

**Lemma 4.1.** If $\mathcal{F}$ satisfies (4.2)-(4.5) and $\phi \in L^\infty(\mathbb{R},X) \cap \mathcal{MF}$, $F \in L^1(\mathbb{R},L(X))$ respectively $L^1(\mathbb{R},\mathbb{C})$ then $F \circ \phi \in \mathcal{F} \cap \mathcal{C}_{ab}(\mathbb{R},X)$.

**Proof.** By Proposition 2.1 (i), $F \circ \phi$ exists and $\phi \in \mathcal{F}$ for each $B \in L(X)$. To show that $F \circ \phi \in \mathcal{C}_{ab}(\mathbb{R},X)$ it is enough to show $(B \chi_I) \circ \phi \in \mathcal{F}$, $I = [\alpha, \beta)$ for each $B \in L(X)$. With Proposition 1.6] one has $(B \chi_I) \circ \phi = B(\chi_I \circ \phi)$, with (4.5) we have to show $B \chi_I \circ \phi \in \mathcal{F}$. Now $\phi \in \mathcal{MF}$ gives $\int_0^h \phi(\cdot + s) ds \in \mathcal{F}$ if $h > 0$, (4.4) for $a = -h$ then $\int_0^h \phi(\cdot - h + s) ds = - \int_0^{-h} \phi(\cdot + s) ds \in \mathcal{F}$, which gives $\phi_{\alpha, \beta} \in \mathcal{F}$. The proof of $F \in L^1(\mathbb{R},\mathbb{C})$ is the same. $\blacksquare$

Examples of $\mathcal{F}$ satisfying (4.2)-(4.5) are the spaces of almost periodic functions $\text{AP} = \text{AP}(\mathbb{R},X)$, almost automorphic functions $\text{AA}(\mathbb{R},X)$, Bochner almost automorphic functions $\text{BAA}(\mathbb{R},X)$, bounded Levitan almost periodic functions $\text{LAP}_b(\mathbb{R},X)$ [7, p. 430], linear subspaces with (4.2) of bounded recurrent functions $\text{REC}_b(\mathbb{R},X) = \text{RC}$ of [7, p. 427], Eberlein almost periodic functions $\text{EAP}(\mathbb{R},X)$ and so on.

**Theorem 4.2.** Let $A$, $\phi$ be as in Theorem 3.5 and $\phi \in \mathcal{MF}$ with $\mathcal{F}$ satisfying (4.1), (4.2)-(4.5). Then there is $x_\phi \in X$ such that the equation (1.1), (1.2) with $x = x_\phi$ has a bounded uniformly continuous mild solution on $\mathbb{R}$ which belongs to $\mathcal{F}$.

**Proof.** By Theorem 3.5, $u_\phi := F \circ \phi_1 + G \circ \phi_2$ is a mild solution of (1.1) on $\mathbb{R}$ with $x = u_\phi(0) = F \circ \phi_1(0) + G \circ \phi_2(0) = x_\phi$, bounded and uniformly continuous. Since $\phi \in \mathcal{MF} \cap L^\infty, \phi_1 = \phi \circ f \in \mathcal{F} \cap \mathcal{C}_{ab}$ by Lemma 4.1; since $\mathcal{F} \cap \mathcal{C}_{ab} \subset \mathcal{MF}$ by [6, Proposition 2.2, p.1011], $\phi_1 \in \mathcal{MF}$. Again with Lemma 4.1 one gets $F \circ \phi_1 \in \mathcal{F}$.
\[ F \cap C_{ub} \]. Similarly, \( \phi_2 = \phi - \phi_1 \in \mathcal{MF} \cap L^\infty \) and so \( G \ast \phi_2 \in F \cap C_{ub} \) by Lemma 4.1.

We should remark for example that if \( \phi \) is bounded Stepanoff \( S^p \)-almost periodic, the \( u_\phi \in AP \) [5, (3.8), p. 134]. Also, if \( \phi \) is a Veech almost automorphic function [19], then \( u_\phi \) is a uniformly continuous Bochner almost automorphic function [21, p. 66], [7, (3.3)].

**Example 4.3.** \( X = Y^n \), \( Y \) complex Banach space, \( A = \text{complex } n \times n \) matrix, \( u, \phi \) \( X \)-valued in (1.1), \( \phi \in L^\infty(\mathbb{R},X) \). Then, if \( \text{sp}(\phi) \) contains no purely imaginary eigen-value of \( A \) and \( \phi \in \mathcal{MF}, F \) with (4.1)-(4.5), then (1.1) has a mild solution on \( \mathbb{R} \) which belongs to \( F \cap C_{ub} \). This extends a well known result of Favard [13, p. 98-99].

Another example would be a result on the almost periodicity of all solutions of the inhomogeneous wave equation in the non-resonance case [20, p. 179, 181 Théorème III.2.1], [1, Proposition 7.1.1], here one has a \( C_0 \)-group, all solutions of the homogeneous equation are almost periodic.

**References**

1. Arendt W., Batty C.J.K., Hieber M. and Neubrander F.: Vector-valued Laplace Transforms and Cauchy problems, Monographs in Math., Vol. 96, Basel, Boston, Berlin: Birkhäuser, 2001.
2. Apostol, Tom M.: Calculus of Several Variables with Applications to Probability and Vector Analysis, Blaisdell Pub. Comp., New York, London, 1962.
3. Basit B.: Some problems concerning different types of vector valued almost periodic functions, Dissertationes Math. 338 (1995), 26 pages.
4. Basit, B.: Harmonic analysis and asymptotic behavior of solutions to the abstract Cauchy problem, Semigroup Forum 54 (1997), 58-74.
5. Basit, B. and Günzler, H.: Asymptotic behavior of solutions of systems of neutral and convolution equations, J. Differential Equations 149 (1998), 115-142.
6. B. Basit and H. Günzler: Generalized Esclangon-Landau results and applications to linear difference-differential systems in Banach spaces, Journal of Difference Equations and Applications, 10, No. 11 (2004), 1005-1023.
7. Basit, B. and Günzler, H.: Difference property for perturbations of vector valued Levitan almost periodic functions and their analogs, Russ. Jour. Math. Phys. 12 (4), (2005) 424-438.
8. Basit, B. and Günzler, H.: Relations between different types of spectra and spectral characterizations, Semigroup forum 76 (2008), 217-233.
9. Basit, B. and Pryde, A. J.: Polynomials and functions with finite spectra on locally compact abelian groups, Bull. Austral. Math. Soc. Vol. 51 (1995), 33-42.
10. Basit, B. and Pryde, A. J.: Equality of uniform and Carleman spectra of bounded measurable functions, Analysis Paper 122, (February 2007), 20 pages.
11. Benedetto, J.J.: Spectral Synthesis, B. G. Teubner Stuttgart, 1975.
12. Diagana, T., Nguerekata, G.M. and Minh, N.V.: Almost automorphic solutions of evolution equations, Proc. AMS., 132 (2004), 3289-3298.
13. Favard, J.: Lecons sur les Fonctions Presque-Périodique, Gauthier-Villars, Paris 1933.
14. Murakami, S. and Naito, T. and Nguyen Van Minh, N.V.: Evolution semigroups and sums of commuting Operators: A new approach to the admissibility theory of function spaces, J. Differential Equations 164 (2000), 240-285.
15. Pazy A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.
16. Phong V. Q.: The operator equation $AX - XB = C$ with unbounded operators $A$ and $B$ and related abstract Cauchy problems, Math. Z. 208 (1991), 567-588.
17. Prüss J.: Evolutionary Integral Equations and Applications, Monographs in Mathematics. Birkhäuser Verlag, Basel 1993.
18. Rudin W.: Functional Analysis, McGraw-Hill book Company, New York, 1973.
19. Veech W.: Almost automorphic functions on groups, Amer. J. Math. 87 (1965), 719-751.
20. Zaidman S.: Solutions presque-périodiques des quations hyperboliques, Ann. sci. cole norm. sup. III, Ser. 79 (1962), 151-198.
21. Zaidman S.: Almost-periodic functions in abstract spaces, Research Notes in Math. 126, Pitman Adv. Publishing Program, Boston 1985.

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