Thermodynamic Properties, Phases and Classical Vacua of Two Dimensional $R^2$-Gravity

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Abstract

Two dimensional quantum $R^2$-gravity is formulated in the semiclassical method. The thermodynamic properties, such as the equation of state, the temperature and the entropy, are explained. The topology constraint and the area constraint are properly taken into account. A total derivative term and an infrared regularization play important roles. The classical solutions (vacua) of $R^2$-Liouville equation are obtained by making use of the well-known solution of the ordinary Liouville equation. The positive and negative constant curvature solutions are 'dual' each other. Each solution has two branches ($\pm$). We characterize all phases. The topology of a sphere is mainly considered.

1 Introduction

In the recent analysis of two dimensional (2d) quantum gravity (QG), the conformal approach or the matrix model approach have been intensively done. Those approaches are nonperturbative ones and it is expected that some non-perturbative features are important to understand the theory. At the same time, however, it is known that an orthodox perturbative approach, the semiclassical approach, is also useful in 2d QG[1, 2]. We present a close examination of the latter approach. A key point in this treatment is how to treat the area constraint and the topology constraint. The regularization of infrared divergence (and ultraviolet divergence in the quantum evaluation) is also important. The semiclassical treatments of 2d QG so far are insufficient in these points. We present a new formalism.

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Despite of the long period of research, the physical picture of the Liouville theory, in relation to 2d QG, seems obscure. It shows the delicacy or the subtlety of the theory and requires some other proper formalism and regularization. As far as the popular formalism based on the conformal field theory is taken, the barrier $c_m = 1$ does not seem to be overcome. Whereas the computer simulations seem to no special difficulty for the prohibited region of $c_m [3, 4]$. This conflict seems gradually serious. Although the problem is examined from different approaches, it is fair to say the true situation is not known at present. Recently it has been shown that the semiclassical results nicely explain the simulation data [5, 6].

It is well known, in the lattice QG, that a higher-derivative term, $\beta R^2$, regularize the theory very well. It has the effect of smoothing the surface (if we take a proper coupling sign). The importance of the term is also stressed in the continuum context [7]. From the simple power-counting we see the ultra-violet behaviour becomes well regularized. We can take two standpoints about the $R^2$-term: 1) We are considering the 2d $R^2$-QG as one gravitational model; 2) We regard $R^2$-term as a regularization to define the $\beta = 0$ theory and expect its effect disappears when some limit is taken. Although 1) is mainly taken in the present paper, both standpoints are important at this time of development.

In the semiclassical analysis of 2d $R^2$-QG in [5, 6] one of the classical solutions (positive curvature solution) is analysed. In the present paper we present the full structure of the classical vacua. It is quite interesting that the positive and negative solutions are symmetric with respect to a reflection in the coupling $\beta$-space. The explanation is self-contained.

It is well known that the global quantities in the gravitational system, such as entropy, volume and temperature of the total universe, obey the laws of thermodynamics. It says those quantities can be regarded as thermodynamic state variables of an equilibrium state. We will find those properties in the present 2d model of QG. We can characterize all phases appearing in the theory by the $\beta$-dependence of the temperature.

We present a general formalism in Sec.2, where some thermodynamic func-
tions are introduced. In Sec.3 the classical solutions are obtained. They are $R^2$-gravity version of the Liouville solution and describe positive and negative constant-curvature manifolds. The analytic expressions of some physical quantites are given and analysed. We characterize all asymptotic regions of the solutions in Sec.4. In Sec.5 an integral about a parameter $\lambda$, which appears in the general formalism in connection with the area costraint, is done. The analytic expressions of cross-over points in the theory are obtained in Sec.6. It shows the essetial behaviour of the theory is controled by a toatl derivative (global) term. In Sec.7 we characterize each phase and obtain the equation of state. The expressions for temperature and entropy are obtained. We conclude in Sec.8.

2 Semiclassical Quantization and Thermodynamic Functions

Before the concrete evaluation, we describe here the present new formalism. We take the Euclidean action,

$$S_{tot} = S_{gra} + S_{m}, \quad S_{gra}[g;G,\beta,\mu] = \int d^2x \sqrt{g}(\frac{1}{G}R - \beta R^2 - \mu) ,$$

$$S_{m}[g,\Phi;c_m] = -\int d^2x \sqrt{g}(\frac{1}{2} \sum_{i=1}^{c_m} \partial_a \Phi_i \cdot g^{ab} \cdot \partial_b \Phi_i) , \quad (a, b = 1, 2 ) , \quad (1)$$

under the fixed area condition $A = \int d^2x \sqrt{g}$. Here $G$ is the gravitaional coupling constant, $\mu$ is the cosmological constant, $\beta$ is the coupling strength for $R^2$-term and $\Phi$ is the $c_m$- components scalar matter fields.

The 2 dim quantum gravity can be treated in the way similar to the flat theory by taking the conformal-flat gauge $(a, b = 1, 2)$,

$$g_{ab} = e^{\varphi} \delta_{ab} , \quad (\delta_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad (2)$$

the action $(1)$ gives us,after integrating out the matter fields and Faddeev-Popov ghost, the following partition function:

$$\int \frac{DgD\Phi}{V_{G}} \{ \exp \frac{1}{\hbar} S_{tot} \} \delta(\int d^2x \sqrt{g} - A) = \exp \frac{1}{\hbar}(\frac{8\pi(1-h)}{G} - \mu A) \times Z[A] \times (3)$$

$$Z[A] \equiv \int D\varphi \exp \frac{1}{\hbar} S_{0}[\varphi] \delta(\int d^2x \ e^{\varphi} - A) \times (4)$$

$$S_{0}[\varphi] = \int d^2x \ (\frac{1}{2\gamma}\varphi \partial^2 \varphi - \beta \ e^{-\varphi}(\partial^2 \varphi)^2 + \frac{\xi}{2\gamma} \partial_a(\varphi \partial_a \varphi) ) , \quad \frac{1}{\gamma} = \frac{1}{48\pi}(26 - c_m)$$
where the relations for Einstein term and the cosmological term: \( \int d^2x \sqrt{g} R = 8\pi(1-h) \), \( h \) = number of handles, \( \int d^2x \sqrt{g} = A \), are used. \( V_{GC} \) is the gauge volume due to the general coordinate invariance. \( \xi \) is a free parameter. The total derivative term generally appears when integrating out the anomaly equation \( \delta S_{\text{ind}}[\varphi]/\delta \varphi = \frac{1}{\gamma} \partial^2 \varphi \). This term turns out to be very important. We consider the manifold of a fixed topology of the sphere, \( h = 0 \), and with the finite area \( A \). Furthermore we consider the case \( \gamma > 0 \) \((c_m < 26)\). \( \hbar \) is Planck constant.

The Laplace transform of (3) is given by

\[
\hat{Z}[\lambda] = \int_0^\infty Z[A] e^{-\lambda A/\hbar} dA = \int \mathcal{D}\varphi \exp\left[ + \frac{1}{\hbar} \{ S_0[\varphi] - \lambda \int d^2x e^\varphi \} \right].
\]

As the arguments of \( Z,(3) \), and \( \hat{Z},(3) \), we do not write explicitly \( \beta, \gamma \)-dependence. (3) is the micro canonical distribution for the fixed area \( A \), whereas (3) is the grand canonical distribution (variable area) with the chemical potential \( \lambda \). From (3), we obtain the expectation value for the area as

\[
<A_{op}> = \frac{1}{Z} \frac{d}{d(-\lambda/\hbar)} \hat{Z}[\lambda] \equiv \int d^2x e^\varphi >_Z, \quad A_{op} \equiv \int d^2x e^\varphi.
\]

By inverting this equation we obtain \( \lambda = \bar{\lambda}(<A_{op}>). \) Equivalently we also obtain it in terms of the Legendre-transformed generating function \( \Gamma[<A_{op}>]. \)

\[
\Gamma[<A_{op}>] \equiv \ln \hat{Z}[\lambda] + \frac{1}{\hbar} \lambda \times <A_{op}>. \quad \bar{\lambda}(<A_{op}>) = \hbar \frac{d\Gamma[<A_{op}>]}{d<A_{op}>}.
\]

(7)

\( Z[A] \) can be obtained from \( \hat{Z}[\lambda] \) by the inverse Laplace transformation.

\[
Z[A] = \int d\lambda \hat{Z}[\lambda] e^{+\lambda A/\hbar} \equiv \int d\lambda Y[A, \lambda], \quad Y[A, \lambda] \equiv \int \mathcal{D}\varphi \exp \frac{1}{\hbar} \left[ S_0[\varphi] - \lambda(f d^2x e^\varphi - A) \right], \quad \Gamma^{eff}[A, \lambda] \equiv \ln Y[A, \lambda],
\]

where \( \lambda \)-integral should be carried out along an appropriate contour parallel to the
imaginary axis of the complex $\lambda$-plane (Fig.1).

\[ \hat{Z} = \int D\psi \exp \left\{ \frac{1}{\hbar}S_{\lambda}[\varphi] \right\} \]

\[ \hat{Z}[\lambda] \text{ is defined by (5). The partition function (8) can be evaluated semiclassically in the following two steps (i) and (ii). In the evaluation we will relate above thermodynamic functions.} \]

(i) $\varphi$-integral
First we define some quantities.

\[ S_{\lambda}[\varphi] \equiv S_0[\varphi] - \lambda \int d^2x \ e^{\varphi} \]

\[ = \int d^2x \left( \frac{1}{2\gamma} \varphi \partial^2 \varphi - \beta e^{-\varphi} (\partial^2 \varphi)^2 + \frac{\xi}{2\gamma} \partial_a (\varphi \partial_a \varphi) - \lambda e^{\varphi} \right) \]

\[ \hat{Z}[\lambda] = \int D\varphi \ e^{\frac{1}{\hbar}S_{\lambda}[\varphi]} \equiv \exp \frac{1}{\hbar} \hat{\Gamma}[\lambda] \]

\[ \hat{\Gamma}(\lambda) \text{ is the effective action corresponding to } S_{\lambda}[\varphi]. \hat{\Gamma}(\lambda) \text{ can be evaluated loop-wise [13] by the semiclassical expansion.} \]

\[ \varphi(x) = \varphi_c(x; \lambda) + \sqrt{\hbar} \psi(x) \]

where we take the 'mean field' (or 'background field') as the solution of the classical field equation for $S_{\lambda}[\varphi].$

\[ \left. \frac{\delta}{\delta \varphi} S_{\lambda}[\varphi] \right|_{\varphi_c} = 0 \]

Then $\hat{\Gamma}[\lambda] = \hbar \ln \hat{Z}[\lambda; \varphi]$ can be evaluated as

\[ \hat{Z}[\lambda] = \int D\psi \ e^{\frac{1}{\hbar}S_{\lambda}[\varphi_c + \sqrt{\hbar}\psi]} \]

\[ = \exp \frac{1}{\hbar}S_{\lambda}[\varphi_c] \times \int D\psi \ e^{\left\{ \frac{1}{2} \frac{\delta^2 S_{\lambda}}{\delta \varphi^2} |_{\varphi_c} \psi \psi + O(\sqrt{\hbar}\psi^3) \right\}} \]
\[
\exp\left\{ \frac{1}{\hbar} \hat{\Gamma}^0[\lambda] + \hat{\Gamma}^1[\lambda] + O(\text{higher-than 1-loop, } \hbar) \right\} = \exp\left\{ \hat{\Gamma}^0[\lambda] + \lambda A + O(\text{higher-than 1-loop, } \hbar^2) \right\},
\]

where \( \hat{\Gamma}^n[\lambda], (n \geq 1) \), is the n-loop quantum effects. In (12), the effect up to 1-loop order is explicitly written.

(ii) \( \lambda \)-integral

Using the result of (i), \( Z[A] \) can be written as

\[
Z[A] = \int d\lambda \exp\left\{ \frac{1}{\hbar} \hat{\Gamma}[\lambda] + \lambda A \right\}. \tag{13}
\]

The \( \lambda \)-integral of (13) can be again evaluated in the semiclassical way as follows. The dominant value \( \lambda_c \) is defined by,

\[
\frac{d}{d\lambda} (\hat{\Gamma}[\lambda] + \lambda A) \big|_{\lambda_c} = \frac{d\hat{\Gamma}[\lambda_c]}{d\lambda_c} + A = 0,
\]

\[
\hat{\Gamma}[\lambda] = \hat{\Gamma}^0[\lambda] + \hbar \hat{\Gamma}^1[\lambda] + \cdots, \quad \lambda_c = \lambda_c^0 + \hbar \lambda_c^1 + \cdots, \tag{14}
\]

where \( \lambda_c^n \) is the n-loop effect of \( \psi \)-integration and is recursively obtained. The \( \lambda \)-integral is approximately obtained by evaluating the fluctuation around \( \lambda_c \) perturbatively as follows.

\[
\lambda = \lambda_c + \omega,
\]

\[
Z[A] = \exp\left\{ \frac{1}{\hbar} \hat{\Gamma}[\lambda_c] + \lambda_c A \right\} \times \int d\omega \exp\left\{ \frac{1}{\hbar} \left( \frac{1}{2} \frac{d^2\hat{\Gamma}[\lambda]}{d\lambda^2} \right) \big|_{\lambda_c} \omega^2 + O(\omega^3) \right\}. \tag{15}
\]

This integral will be evaluated in Sec.5.

Here we note that the equation (14), by which \( \lambda_c \) is defined, is re-written as

\[
A = -\frac{d\hat{\Gamma}[\lambda_c]}{d\lambda_c} = \frac{1}{\hat{Z}[\lambda] \hbar} \frac{d\hat{Z}[\lambda]}{d(-\lambda/\hbar)} \big|_{\lambda_c}, \tag{16}
\]

which is exactly the same as (3) by identifying \( A \) above with \( <A_{\text{op}}/> \) in (3). Therefore \( \lambda_c(A) = \bar{\lambda}(A) \). And \( \exp\left\{ \frac{1}{\hbar} \hat{\Gamma}[\lambda_c] \right\} = \hat{Z}[\lambda_c] \), in the second equation of (13), is exactly the same quantity as \( \hat{Z}[\bar{\lambda}(A)] \) in (7). Furthermore \( \Gamma^{\text{eff}}[A, \lambda_c] \) exactly coincides with \( \Gamma[A] \) of (7).

\[
\ln Z[A] \approx \Gamma^{\text{eff}}[A, \lambda_c] = \frac{1}{\hbar} (\hat{\Gamma}[\lambda_c] + \lambda_c A) = \Gamma[A]. \tag{17}
\]
We have introduced some thermodynamic functions. For their comparison they are listed in Table 1.

| Indep. param. | Integral var. | Special func. | Partition Func. | Effective action |
|---------------|---------------|---------------|-----------------|------------------|
| $A$           | $\varphi(x)$  | $\varphi_c(x,\lambda)$, ($\lambda$) real, ($\lambda$) complex | $\bar{Z}[\lambda](A)$ | $\bar{Z}[\lambda](A)$ |
| $\lambda$    | $\varphi(x)$  | $\bar{\lambda}(<A_\text{op}>)$, ($\bar{Z}[\lambda](A)$) | $\Gamma[\lambda] = \hbar \ln \bar{Z}[\lambda]$, ($\bar{Z}[\lambda](A)$) | $\Gamma[<A_\text{op}>]$, ($\bar{Z}[\lambda](A)$) |
| $<A_\text{op}>$ | $\lambda$    | $\varphi(x)$ | $\gamma\partial^2 \varphi + \beta \{ e^{-\varphi}(\partial^2 \varphi)^2 - 2\partial^2 (e^{-\varphi} \partial^2 \varphi) \} - \lambda e^\varphi = 0$ | $\Gamma[<A_\text{op}>]$, ($\bar{Z}[\lambda](A)$) |

Table 1  Some Thermodynamic Functions.

We notice the area constraint, which was expressed as the delta function in ($\delta$), is replaced by the $\lambda$-integral in the present formalism using $Y[A, \lambda]$. This is the key point for correctly treating the area constraint in the semiclassical method[14]. The quantum effect is systematically evaluated loop-wise[15]: the renormalization of parameters involved in the theory, due to the quantum interaction of the Weyl mode $\varphi$, is done in eq.(12).

In the following, we will evaluate the leading order, i.e. order of $\hbar^0$.

3 Classical Vacua of $R^2$-Gravity

The classical configuration (solution) of the special case $,\beta = 0, was well-known as the Liouville solutions. (See [2] for a recent review.) Furthermore, in the context of 2 dim quantum gravity (or the string theory ), the special case was already examined by [16] and [17]. We consider the general case: $\beta$ is the arbitrary real number with the dimension of (Length)$^2$.

The classical field equation (11) is given by

$$\frac{\delta S[\varphi]}{\delta \varphi} = \frac{1}{\gamma} \partial^2 \varphi + \beta \{ e^{-\varphi}(\partial^2 \varphi)^2 - 2\partial^2 (e^{-\varphi} \partial^2 \varphi) \} - \lambda e^\varphi = 0 .$$ (18)
We make the following assumption of constant-curvature for the solution.

\[- R|_{\varphi_c} = e^{-\varphi_c} \partial^2 \varphi_c = \text{const} \equiv \frac{-\alpha}{A}, \quad (19)\]

where \(\alpha\) is a dimensionless constant. The above equation is the Liouville equation. It is easy to find that the solution (19) satisfies (18) for such real \(\alpha\) that satisfies the following equation:

\[\text{COND.1} \quad \alpha^2 \beta' - \frac{1}{\gamma} \alpha - \lambda A = 0 \quad , \quad \beta' \equiv \frac{\beta}{A} . \quad (20)\]

We may safely assume the spherical symmetry (in \((x,y)\)-plane) for a stable solution: \(\varphi_c = \varphi_c(r)\), \( r = \sqrt{x^2 + y^2} \). Then (19) reduces to

\[
\frac{1}{r} \frac{d}{dr} r \frac{d\varphi_c}{dr} + \alpha A e^{\varphi_c} = 0 \quad . \quad (21)
\]

Before further analysis, we comment on the eq.(19). The Liouville equation (19) corresponds to the ordinary gravity \((\beta = 0)\) :

\[S = \int d^2 x \left( \frac{1}{2} \gamma \partial^2 \varphi - \mu_1 e^\varphi \right) , \]

with the cosmological constant \(\mu_1 = -\frac{1}{\gamma} \frac{\alpha}{A}\), which is negative for \(\alpha > 0\) and positive for \(\alpha < 0\).

### 3.1 Positive Curvature Solution

For the case:

\[\alpha > 0 \quad , \quad (22)\]

the solution of (21) is given by (cf. [16, 17, 2]),

\[\varphi_c(r; \alpha) = - \ln \left\{ \frac{\alpha}{8} (1 + \frac{r^2}{A})^2 \right\} . \quad (23)\]

This solution satisfies \( \int d^2 x \sqrt{g} |_{\varphi_c} = \int d^2 x \ e^{\varphi_c} = \frac{8\pi}{\alpha} A \),

\[- \int d^2 x \sqrt{g} R |_{\varphi_c} = \int d^2 x \ \partial^2 \varphi_c = -8\pi . \]

It means the manifold described by (23) has the area \(\frac{8\pi}{\alpha} A\) and is topologically the sphere. The equations (20 -23) constitute a solution of (18).

\[S_\lambda[\varphi_c] \quad \text{is given as} \quad \]

\[S_\lambda[\varphi_c] \left( = \tilde{\Gamma}^0(\lambda) \right) = (1 + \xi) \frac{8\pi}{\gamma} \ln \frac{\alpha}{8} - 16\pi \alpha \beta' + C(A) \quad , \]

\[C(A) = \frac{8\pi (2+\xi)}{\gamma} + \frac{8\pi \xi}{\gamma} \{ \ln(1 + L^2/A) - (L^2/A)/(1 + (L^2/A)) \} \quad (24)\]

\[= \frac{8\pi (2+\xi)}{\gamma} + \frac{8\pi \xi}{\gamma} \{ \ln \frac{L^2}{A} - 1 \} + O\left( \frac{A}{L^2} \right) \quad , \quad \frac{L^2}{A} \gg 1 \quad , \]
where $L$ is the infrared cut-off ($r^2 \leq L^2$) introduced for the divergent volume integral of the total derivative term ($\xi$-term). See Fig.2. The integral is log-divergent at $r \to \infty$ for the classical solution (23). $\alpha$ (or $\lambda$) is rewritten in terms of $\lambda$ (or $\alpha$) by use of (20) and $C(A)$ does not depend on $\alpha$ and $\beta$ but depends on $A$ and $\gamma$. In the above derivation, formula in App.A are useful.

Fig.2 Infra-red cut-off $L$ in the flat coordinates and the sphere manifold. For simplicity, the picture is for $\alpha = 8$. For general $\alpha$, $(x, y, r, \sqrt{A}, L)$ is substituted by $\sqrt{8/\alpha} \times (x, y, r, \sqrt{A}, L)$.

The eq. (14) at the classical level is written as,

\[
\frac{dS_\lambda[\varphi_c]}{d\lambda} + A = \left(\frac{4\pi}{\gamma} \frac{1}{\alpha} (1 + \xi) - 16\pi \beta'\right) \frac{d\alpha}{d\lambda} + A \\
= \left\{ \frac{4\pi}{\gamma} \frac{1}{\alpha} (1 + \xi) - \left(16\pi \beta' + \frac{1}{\gamma}\right) + 2\beta' \alpha \right\} \frac{d\alpha}{d\lambda} = 0 \quad ,
\]

where we have used a relation : $1 = \frac{d\alpha}{d\lambda} \frac{d\lambda}{d\alpha} = \frac{1}{A} \left(2\alpha\beta' - \frac{1}{\gamma}\right) \frac{d\alpha}{d\lambda}$, which is derived from (20). For the case $\frac{d\alpha}{d\lambda} \neq 0$ (20) says

\text{COND.2} \quad \Xi(\alpha; \beta', \gamma, \xi) \equiv 2\beta' \alpha^2 - (16\pi \beta' + \frac{1}{\gamma})\alpha + (1 + \xi) \frac{4\pi}{\gamma} = 0 \quad , \\
\alpha_c^\pm = \frac{1}{4\beta'} \left\{ 16\pi \beta' + \frac{1}{\gamma} \pm \sqrt{D} \right\} \quad ,
\]

\[
D \equiv (16\pi \beta')^2 + \frac{1}{\gamma} - \xi \frac{32\pi \beta'}{\gamma} = (16\pi \beta' - \frac{\xi}{\gamma})^2 + \frac{1 - \xi^2}{\gamma} \quad .
\]

The relation (20) gives $\lambda_c^\pm(\beta) \equiv \lambda(\beta, \alpha_c^\pm(\beta))$. Note that the determinant of the above quadratic equation, $D$, is positive definite for all real $\beta$ if the following condition is satisfied.

\[
-1 \leq \xi \leq +1 \quad .
\]
We consider this case in the following.

In summary, 2 unknown variables $\alpha$ and $\lambda$, are fixed by two conditions COND.1 and 2 and they are expressed by three physical parameters $\beta, \gamma, A$ and one free parameter $\xi$. We list here the obtained result of important physical quantities.

Curvature \[ \alpha_\pm = \frac{4\pi}{w} \{ w + 1 \pm \sqrt{w^2 + 1 - 2\xi w} \} \]

Classical Action \[ S_\lambda[\varphi_c] = (1 + \xi) \frac{4\pi}{\gamma} \ln \frac{w}{8} - \frac{w}{\gamma} \alpha_c + C(A) \]

String Tension \[ \lambda_c A = \frac{1}{16\pi\gamma}(\alpha_c^2 w - 16\pi\alpha_c) \quad (28) \]

An Expect. Value \[ -A < \int d^2x \sqrt{g}R^2 > |_c = \frac{\partial \Gamma_{\text{eff}}[\varphi_c]}{\partial \beta'} \]
\[ = -16\pi\alpha_c + \alpha_c^2 + \Xi \times \frac{1}{\alpha_c} \frac{d\alpha_c}{d\beta'} \]

Free Energy \[ -\Gamma_{\text{eff}}|_c = -S_\lambda[\varphi_c] - \lambda_c A \]

where $w \equiv 16\pi\beta'\gamma$ and $C(A)$ is given in \[ [24] \]. Note that $\Xi = 0$.

The curvature $\times A = R(\varphi_c) \times A = \alpha(w)$, the classical value of $A < \int d^2x \sqrt{g}R^2 > = -\frac{\partial \Gamma_{\text{eff}}[\varphi_c]}{\partial \beta'}$, the string tension $\times \gamma A = \gamma \lambda_c A$ and the total free energy $\times \gamma ( = -\gamma \Gamma_{\text{eff}}[A] = -\gamma(S_\lambda + \lambda_c A))$ are plotted, for the $-$branch solution, in Fig.3,4,5 and 6 respectively. In the figures we take $\xi = 0.99$ whose meaning will be explained in Sec.5, and the curves for the negative curvature solution (see Sec.3.2) are also plotted. The asymptotic behaviours of these quantities will be listed in Sec.3.3. We also plot the area $\times \frac{1}{A}$ ( $= \frac{1}{A} \int d^2x e^{\varphi_c} = \frac{8\pi}{\alpha}$ ) as the function of $w$ in Fig.7. It shows, as far as the classical configuration is concerned, the $\delta$-function condition in \[ [3] \] is not satisfied except for the $\alpha_+^\pm$-solution in the $\beta$ (or $w$) $\rightarrow +\infty$ region where $\lambda \rightarrow +\infty$, and for the $\alpha_-^\pm$-solution in the $\beta$ (or $w$) $\rightarrow -\infty$ region where $\lambda \rightarrow -\infty$. This has happened because we are approximating the quantumly fluctuating manifold by the simple classical sphere whose configuration is specified only by the effective area $\frac{1}{\alpha}$ and the string tension $\lambda$. This characteristically shows the present effective action
approach using $Y[A, \lambda]$.

Fig. 3  $A \times \text{Curvature} = \alpha(w)$, Positive and Negative Curv. Sols., $-$Branch

Fig. 4a Log-Log Plot of $A < \int d^2x \sqrt{g}R^2 > |c, w > 0$, Positive and Negative Curv. Sols., $-$Branch

Fig. 4b Linear Plot of $A < \int d^2x \sqrt{g}R^2 > |c, w > 0$, Positive and Negative Curv. Sols., $-$Branch
Fig. 5  \[ \gamma A \times (\text{String Tension}) = \gamma \lambda(w) A, \text{ Positive and Negative Curv. Sols.,} \]

- Branch

Fig. 6  \[ \gamma \times (\text{Total Free Energy}) = -\gamma \Gamma^{ff}, \text{ Positive and Negative Curv. Sols.,} \]

- Branch, The \( w \)-independent terms \((C(A), \tilde{C}_2(A))\) are omitted.

Fig. 7  \[ \frac{1}{A} \times \text{Area} = \frac{8\pi}{\alpha(w)}, \text{ Positive and Negative Curv. Sols.,} \quad -\text{Branch} \]

3.2 Negative Curvature Solution

For the case:

\[
\alpha < 0, \quad (29)
\]
the solution of (19) is given by (cf. [2]),

$$\varphi_n(r; \alpha) = -\ln \left\{ \frac{-\alpha}{8} \left( 1 - \frac{r^2}{A} \right)^2 \right\} . \quad (30)$$

The equations (20-21) and (29-30) constitute another solution of (18). It is singular at \( r = \sqrt{A} \), which means the manifold is open. There exist two independent regions: the inner region \( 0 \leq r < \sqrt{A} \) and the outer region \( r > \sqrt{A} \).

We consider only the inner region [20]. We should carefully treat the singularity by introducing a proper regularization. We regularize the inner region by \( 0 \leq r < \sqrt{A} - \epsilon \) where \( \epsilon \to +0 \). See Fig.8.

Fig.8 Regularization of singularity at \( r = \sqrt{A} \) of (30).

Then various terms in the action are evaluated as

$$\int d^2 x \, e^{\varphi_n} = \lim_{\epsilon \to +0} \int_{0 \leq r^2 < A - \epsilon} d^2 x \, e^{\varphi_n} = \frac{8\pi}{-\alpha} A \lim_{\epsilon \to +0} \left( \frac{A}{\epsilon} - 1 \right) \quad ,$$

$$\int d^2 x \, \varphi_n \varphi_n = 8\pi \lim_{\epsilon \to +0} \left\{ 2 - \left( \frac{A}{\epsilon} - 1 \right) \ln \left( \frac{-\alpha}{8} \right) + \frac{2A}{\epsilon} \left( \ln \frac{A}{\epsilon} - 1 \right) \right\} \quad , \quad (31)$$

$$\int d^2 x \, e^{-\varphi_n} (\partial^2 \varphi_n)^2 = \frac{-8\pi\alpha}{A} \lim_{\epsilon \to +0} \left( \frac{A}{\epsilon} - 1 \right) \quad ,$$

$$\int d^2 x \, \sqrt{g} R \bigg|_{\varphi_n} = \int d^2 x \, \partial_a \varphi_n \partial_a \varphi_n = 8\pi \lim_{\epsilon \to +0} \left( \frac{A}{\epsilon} - 1 \right) \quad ,$$

$$\int d^2 x \partial_a \varphi_n \partial_a \varphi_n = 16\pi \lim_{\epsilon \to +0} \left( \ln \frac{\epsilon}{A} + \frac{A}{\epsilon} - 1 \right) \quad ,$$

where \( \epsilon \) is introduced as a regularization parameter and is a positive infinitesimally-small constant with the dimension of area [21]. The divergence of the total area \( \int d^2 x \, e^{\varphi_n} \), at this stage, says the manifold considered is not closed. The singular point \( r = \sqrt{A} \) corresponds to the boundary of the open manifold.
Using the above results, the Euclidean action (4) and eq. (14), at the
classical level, are given by

$$S_\lambda[\varphi_n] = -\frac{4\pi}{\gamma}(1 + \xi)\Lambda \ln \left(\frac{-\alpha}{8}\right) + 8\pi\alpha\beta'\Lambda + \frac{8\pi}{\alpha}\lambda\Lambda + C_2(A)$$

$$C_2(A) = \frac{4\pi(1 + \xi)}{\gamma}\{2 + 2(\Lambda + 1)(\ln(\Lambda + 1) - 1)\} + \frac{8\pi\xi}{\gamma}(-\ln(\Lambda + 1) + \Lambda)$$

$$\Lambda \equiv \lim_{\epsilon \to +0}\left(\frac{A}{\epsilon} - 1\right) > 0$$

$$\frac{dS_\lambda[\varphi_n]}{d\lambda} + A = \left\{-\frac{4\pi\Lambda}{\gamma\alpha}(1 + \xi) + 8\pi\beta'\Lambda - \frac{8\pi}{\alpha^2}\lambda\Lambda\right\}\frac{d\alpha}{d\lambda} + \frac{8\pi}{\alpha}\Lambda+ A$$

$$= \left\{-\frac{4\pi\Lambda}{\gamma\alpha}(1 + \xi) + 8\pi\beta'\Lambda - \frac{8\pi}{\alpha^2}\lambda\Lambda + (2\alpha\beta' - \frac{1}{\gamma})(\frac{8\pi\Lambda}{\alpha} + 1)\right\}\frac{d\alpha}{d\lambda} = 0$$

The above equations are divergent and are not well-defined for \(\epsilon \to +0\). We
can, however, absorb the divergence by the rescaling of the coupling \(\beta'\), the
curvature parameter \(\alpha\), the cosmological parameter \(\lambda\) and some physical operators
to be evaluated.

$$\tilde{\alpha} \equiv \frac{\alpha}{\Lambda}, \quad \tilde{\beta}' \equiv \Lambda\beta' (\tilde{\beta} \equiv \Lambda\beta)$$

where \(\Lambda\) is the positive divergent constant introduced in (32). Note that \(\alpha\) does
not change its sign by this transformation: \(\tilde{\alpha} < 0\). The corresponding
transformation of \(\lambda\) is obtained by the requirement of keeping the form of (20).

$$\text{COND.1} \quad \tilde{\lambda}A \equiv \frac{\lambda}{\Lambda}A = \tilde{\alpha}^2\tilde{\beta}' - \frac{\tilde{\alpha}}{\gamma}$$

In terms of rescaled quantities, we can rewrite (32) as

$$\tilde{S}_\lambda[\varphi_n] \equiv \frac{S_\lambda[\varphi_n]}{\Lambda} = -\frac{4\pi}{\gamma}(1 + \xi)\ln \left(\frac{-\tilde{\alpha}}{8}\right) + 16\pi\tilde{\alpha}\tilde{\beta}' + \tilde{C}_2(A)$$

$$\tilde{C}_2(A) = \frac{8\pi(1 + \xi)}{\gamma}\ln(\Lambda + 1) - \frac{8\pi}{\gamma}$$

$$\frac{d\tilde{S}_\lambda[\varphi_n]}{d\tilde{\lambda}} + A = \frac{d\tilde{S}_\lambda[\varphi_n]}{d\tilde{\lambda}} + A = \frac{d\tilde{\alpha}}{d\tilde{\lambda}}\left[-\frac{4\pi}{\gamma\tilde{\alpha}}(1 + \xi) + 16\pi\tilde{\beta}' + 2\tilde{\alpha}\tilde{\beta}' - \frac{1}{\gamma}\right] = 0$$

This result shows the rescaled action is finite except a constant term
(log-divergent) and the equation for \(\tilde{\lambda}_n\) or \(\tilde{\alpha}_n\) (i.e. eq.(14)) , at the classical level,
is completely free from divergence.

$$\text{COND.2} \quad \tilde{E}(\tilde{\alpha}; \tilde{\beta}', \gamma, \xi) \equiv 2\tilde{\beta}'\tilde{\alpha}^2 + (16\pi\tilde{\beta}' - \frac{1}{\gamma})\tilde{\alpha} - \frac{4\pi}{\gamma}(1 + \xi) = 0$$
\[ \tilde{\alpha}_n^\pm = \frac{1}{4\beta'} \left\{ -16\pi \bar{\beta}' + \frac{1}{\gamma} \pm \sqrt{D} \right\}, \quad (36) \]

\[ D \equiv (16\pi \bar{\beta}')^2 + \frac{1}{\gamma^2} + 32\pi \frac{\tilde{\alpha}_n}{\gamma} \xi = (16\pi \bar{\beta}' + \frac{\xi}{\gamma})^2 + \frac{1}{\gamma^2}(1 - \xi^2). \]

COND. \( \tilde{\alpha} (\tilde{\beta}) = \tilde{\lambda}(\tilde{\alpha}_n, \tilde{\beta})(\tilde{\beta}) \). We consider, again, the following region of \( \xi \), in order to guarantee \( D \geq 0 \) for all real \( \tilde{\beta} \).

\[ -1 \leq \xi \leq 1. \quad (37) \]

The physical quantity of \( < \int d^2x \sqrt{g}R^2 > |_n \) is given by

\[ -\frac{A}{\Lambda^2} < \int d^2x \sqrt{g}R^2 > |_n = \frac{1}{\Lambda^2} \frac{d\tilde{\Gamma}^{eff}[\varphi_n]}{d\beta'} |_n = \frac{d\tilde{\Gamma}^{eff}[\varphi_n]}{d\beta'} |_n \]

\[ = +16\pi \tilde{\alpha}_n + \tilde{\alpha}_n^2 + \tilde{\Xi} \times \frac{1}{\tilde{\alpha}_n} \frac{d\tilde{\alpha}_n}{d\beta'} \quad (38) \]

where \( \tilde{\Xi} = 0 \).

We note the difference in signs between the equations in the positive-curvature case, (26-28), and those in the negative-curvature case, (34-38). Remarkably, by the following sign change,

\[ \tilde{\beta} \to -\tilde{\beta}, \tilde{\alpha} \to -\tilde{\alpha}, (\tilde{\lambda} \to -\tilde{\lambda}, \tilde{S}_\lambda - \tilde{C}_2(A) \to -(\tilde{S}_\lambda - \tilde{C}_2(A)), \) \quad (39) \]

the above negative-curvature results (\( \tilde{\alpha}_n, \tilde{S}_\lambda - \tilde{C}_2(A), \tilde{\lambda}_n \)) as the functions of \( \tilde{\beta}' \), reduce to the same forms of the positive-curvature ones (\( \alpha_c, S_\lambda - C(A), \lambda_c \)) as the functions of \( \beta' \) \footnote{22}.

We show the behaviours of \( \tilde{\alpha}_n^\pm, -\frac{\partial \tilde{\Gamma}^{eff}}{\partial \beta'}, \gamma \tilde{\lambda}_n^\pm A \) and

\[ -\gamma \tilde{\Gamma}^{eff} = -\gamma (\tilde{S}_\lambda^\pm + \tilde{\lambda}_n^\pm A) \]

in the dotted lines of Fig.3,4,5 and 6 respectively. The figures show the above reflection symmetry clearly. The asymptotic behaviours of the above physical quantities will be listed in Table 3 of Sec.3.3.

It is very interesting that we can define finite quantities in the open manifold in the above rescaling procedure (33-35). It reminds us of the renormalization in the quantum field theory. The present case is, however, a procedure to absorb the infrared divergence due to the coordinate singularity of the classical open manifold, not to absorb the ultraviolet divergence in the quantum theory. Note that the constant-curvature sign remains negative after the rescaling and the Euler
number $\int d^2x \sqrt{g} R$ is negatively divergent. These facts make us envisage Fig.9 as the rescaled manifold. It describes a sphere punctured over the surface. Each puncture absorbs the infrared divergence.

Fig.9 Punctured sphere absorbing infrared divergence

3.3 Phases and Asymptotic Behaviours

The asymptotic behaviours of the physical quantites, obtained in Sec.3.1, are listed in Table 2, where the case of $\alpha < 0$ is excluded due to the present condition (22).
Due to the ‘reflection symmetry’, each phase of the negative-curvature solution, given in Sec.3.2, is characterized in the similar way as in the positive-curvature case. We list the phase characterization in Table 3. (‘Primes’ in Table 3 mean modification due to the sign difference.)

| Phase | $w \ll -1$ | $-1 \ll w < 0$ | $0 < w \ll 1$ | $1 \ll w$ |
|-------|-------------|----------------|----------------|------------|
| $\alpha_c^+$ | $\frac{2\pi}{\sqrt{w}} \{1 + (1 - \xi)\}$ | $-\frac{2\pi}{\sqrt{w}} \{1 - (1 + \xi)\}$ | $\frac{2\pi}{\sqrt{w}} \{1 - \ln \frac{1 + \xi}{2}\}$ | $\frac{2\pi}{\sqrt{w}} \{1 + \ln \frac{1 + \xi}{2}\}$ |
| $\gamma \lambda_c^+ A$ | $\frac{2\pi}{\sqrt{w}} \{1 + 2w\}$ | $\frac{2\pi}{\sqrt{w}} \{1 - 2w\}$ | $\frac{2\pi}{\sqrt{w}} \{1 + 2w\}$ | $\frac{2\pi}{\sqrt{w}} \{1 - 2w\}$ |
| $\gamma \Gamma_{+}^{eff}$ | $\frac{2\pi}{\sqrt{w}} \{1 + 2w\}$ | $\frac{2\pi}{\sqrt{w}} \{1 - 2w\}$ | $\frac{2\pi}{\sqrt{w}} \{1 + 2w\}$ | $\frac{2\pi}{\sqrt{w}} \{1 - 2w\}$ |

Table 2 Asymptotic behaviour of physical quantities.
$R > 0, w \equiv 16\pi \beta'\gamma, \gamma = \frac{48\pi}{26 - \xi} > 0 (\xi < 26)$. $C(A)$ is given by (24).

All phases are explained in [5] using the above asymptotic behaviour. In the present paper we will characterize each phase by the field equation satisfied in each asymptotic region in Sec.4 and by the equation of state in Sec.7.
4 Asymptotic Regions

Now we consider the classical solutions of (18) in the asymptotic regions:
(a) $|w| \to \infty$; (b) $|w| \to +0$. Table 2 and 3 in Sec.3.3 say each region has two cases.

(ai) $|\gamma \lambda A| \sim O\left(\frac{1}{w}\right)$, Weak-Field Vacua, ((A),(A'))

In this region, the following parts of (18) are dominant.

$$e^{-\varphi}(\partial^2 \varphi)^2 - 2\partial^2(e^{-\varphi}\partial^2 \varphi) = 0$$

$$\varphi_{asy} = \text{const} \quad , \quad R(\varphi_{asy}) = 0 \quad . \tag{40}$$

These vacua are defined only by the 'kinetic terms' in the action. Therefore we call these vacua ((A), (A')) Weak-Field (WF-) Vacua.[23]

(aii) $\gamma \lambda A \sim O(w)$, Perfect Sphere Vacua, ((C),(C'),(D),(D'))

$$e^{-\varphi}(\partial^2 \varphi)^2 - 2\partial^2(e^{-\varphi}\partial^2 \varphi) - c e^\varphi = 0$$

$$-R|_{asy} = e^{-\varphi_{asy}}\partial^2 \varphi_{asy} = \text{const} = \pm \sqrt{c}(\neq 0) \quad . \tag{41}$$

For the positive curvature case (lower sign case), $c$ can be fixed by the condition

$$\int d^2 x \sqrt{g} R = 8\pi \quad , \quad \sqrt{c} = \frac{8\pi}{A}$$

and the total area is $\frac{8\pi}{R} = A$. These vacua are strongly restricted by the 'potential term' of $e^\varphi$ and describe a perfect sphere. We call the vacua (D) and (C’) expansive perfect sphere where the string tension positively divergent, and the vacua (C) and (D’) tensed perfect sphere where the string tension negatively divergent.

(bi) $\gamma \lambda A \sim \text{const}$, Liouville Vacua, ((B),(B'))

$$\frac{1}{\gamma} \partial^2 \varphi - \lambda(0) e^{\varphi} = 0 \quad , \quad \text{Liouville Eq.} \quad ,$$

$$R = -e^{-\varphi}\partial^2 \varphi = -\gamma \lambda(0) \quad . \tag{42}$$

This corresponds to the $\beta = 0$ theory. In Phase (B), the Euler number is properly given by

$$\int d^2 x \sqrt{g} R = -\gamma \lambda(0) \cdot \int d^2 x \sqrt{g} = -\gamma \lambda(0) A \cdot \frac{8\pi}{\alpha(0)} = 4\pi(1 + \xi) \cdot \frac{8\pi}{4\pi(1+\xi)} = 8\pi \quad (\text{for arbitrary } \xi) \quad .$$

We call these regions Liouville Vacua.
(bii) $\gamma \lambda A \sim O\left(\frac{1}{w}\right)$, Degenerate Vacua, ((E),(E'))

In these regions the curvature must depend on $w$ in order that Eq. (18) is satisfied.

$$
\frac{1}{\gamma} e^{-\varphi} \partial^2 \varphi + \beta \{(e^{-\varphi} \partial^2 \varphi)^2 - 2e^{-\varphi} \partial^2(e^{-\varphi} \partial^2 \varphi)\} - \lambda(\beta) = 0 ,
$$

$$
R = -e^{-\varphi} \partial^2 \varphi \sim \frac{1}{A} \times O\left(\frac{1}{w}\right) . \quad (43)
$$

All terms of (18) are effective. Because the total area $\frac{8\pi}{R}$ vanishes, we name these regions degenerate vacua.

We list all above asymptotic regions in Table 3 with the effective terms of (18) marked by $\bigcirc$.

| Terms of (18) | $\frac{1}{\gamma} \partial^2 \varphi$ | $+ \beta \{(e^{-\varphi} \partial^2 \varphi)^2 - 2e^{-\varphi} \partial^2(e^{-\varphi} \partial^2 \varphi)\}$ | $- \lambda e^\varphi$ |
|--------------|-----------------|---------------------------------|---------------------|
| Weak-Field   | $\bigcirc$       |                                 | (ai)                |
| Perf. Sphere | $\bigcirc$       |                                 | (aii)               |
| Liouville    | $\bigcirc$       |                                 | (bii)               |
| Degenerate   | $\bigcirc$       |                                 | non-exist           |
| Free Boson   | $\bigcirc$       |                                 | non-exist           |

Table 3 Asymptotic States

Now we have characterized all asymptotic regions. We can see, as shown in Fig.3, $\alpha_c^-$ solution connects between WF-vacuum at $w = +\infty$ and the tensed perfect sphere vacuum at $w = -\infty$. And $\alpha_n^-$ solution connects between WF-vacuum at $w = -\infty$ and the expansive perfect sphere vacuum at $\beta = +\infty$.

5 $\lambda$-integral

In this section we do the $\lambda$-integral of (13) in the lowest order. This part gives us some contribution to $Z[A]$. We consider the positive curvature solution. After splitting $\lambda$ around $\lambda_c : \lambda = \lambda_c + \omega$, the $\omega$-integral part of (15) is approximated as

$$
Z_{\omega}[A] \equiv \int d\omega \exp \left\{ \frac{1}{2} \frac{d^2 S[\lambda]}{d\lambda^2} \bigg|_{\lambda_c} \omega^2 + O(\omega^3) \right\}
= \int d\omega \exp \left\{ \frac{1}{2} \frac{d^2 S_c}{d\lambda^2} \bigg|_{\lambda_c} \omega^2 \right\} . \quad (44)
$$

From (24) and (20), we can obtain

$$
\frac{d^2 S_c}{d\lambda^2} = -\frac{4\pi \gamma}{\alpha^2} \left(\frac{d\lambda}{d\alpha}\right)^2 + (\frac{4\pi \gamma}{\alpha} - 16\pi \beta') \frac{d^2 a}{d\lambda^2}
$$
\[ = A^2 \frac{4\pi}{\alpha} \left\{ 4(1 - \xi)\alpha^2 \beta'^2 + (1 + \xi)(2\alpha\beta' - \frac{1}{\gamma})^2 \right\} \]
\[ = A^2 \frac{4\pi}{\alpha^2} \left\{ (1 - \xi)\left(\frac{\alpha w}{8\pi}\right)^2 + (1 + \xi)(\frac{\alpha w}{8\pi} - 1)^2 \right\} , \]

where we have used some relations derived from

(20) : \( \frac{d\alpha}{d\lambda} = A / (2\alpha\beta' - \frac{1}{\gamma}) \), \( \frac{d^2\alpha}{d\lambda^2} = -2\beta' A^2 / (2\alpha\beta' - \frac{1}{\gamma})^3 \). Putting \( \alpha^\pm \)-solution of (28) into the above expression, we can confirm \( \frac{d^2 S_\lambda}{d\lambda^2} \big|_{\alpha^\pm} < 0 \), \( \frac{d^2 S_\lambda}{d\lambda^2} \big|_{\alpha^\pm} > 0 \) for all \( \beta \) (or \( w \)) region. (Note that \{ \} -part of (45) is positive definite.) For the \( \alpha^\pm \)-solution, it is necessary to change the integral path from the original pure imaginary \( \omega \) in Fig.1 to the real \( \omega \) as shown in Fig.10. \( Z_\omega[A] \) is evaluated as

\[ Z_\omega[A] = \frac{1}{\sqrt{-\frac{d^2 S_\lambda}{d\lambda^2} \big|_{\alpha^\pm}}} \text{ for } -\text{branch solution} \]
\[ Z_\omega[A] = \frac{1}{\sqrt{+\frac{d^2 S_\lambda}{d\lambda^2} \big|_{\alpha^\pm}}} \text{ for } +\text{branch solution} . \]

Using the results of Table 2, the asymptotic behaviours are evaluated as

\begin{align*}
\text{Phase (A) } \ w &\gg 1 \ , \ \ln Z_\omega \sim -\ln A - \ln w , \\
\text{Phase (B) } \ |w| &\ll 1 \ , \ \ln Z_\omega \sim -\ln A + \text{const} , \\
\text{Phase (C) } \ w &\ll -1 \ , \ \ln Z_\omega \sim -\ln A + \frac{1}{2} \ln |w| , \\
\text{Phase (D) } \ w &\gg 1 \ , \ \ln Z_\omega \sim -\ln A + \frac{1}{2} \ln w , \\
\text{Phase (E) } 0 < w &\ll 1 \ , \ \ln Z_\omega \sim -\ln A + \frac{1}{2} \ln w .
\end{align*}
We notice the first term of each right-hand side contributes to the string susceptibility (see Sec.6). The second term does not have a factor of 
\[ 4\pi/\gamma = (26 - c_m)/12 \] in comparison with the \( \Gamma_{\text{eff}} \) of Table 2. Because the factor means the number of freedom in this thermodynamical system (see Sec.7), their contribution is negligible except for the case: \( c_m \approx 26. \)

6 Cross-Over Points and Determination of \( \xi \)

Let us see the \( \xi \)-dependence of the cross-over points. Because the negative constant curvature solution is obtained by the reflection symmetry from the positive one, we discuss only the latter one. As for the +branch solution, the string tension \( \lambda \) changes its sign at \( w_0 \), which is located somewhere between (D)-phase and (E)-phase of Table 2 (see Fig.5). We can obtain it as the zero of \( \lambda^+_c(w_0) = 0 \) in (28).

\[
  w_0(\xi) = \frac{4}{3 - \xi} .
\]  

(48)

The - solution has two cross-over points. The log-log plot of \( -\frac{\partial \Gamma_{\text{eff}}}{\partial \beta'}[\varphi_c] \) (Fig.4a) shows, at some point \( w_c > 0 \) between phase (A) and (B), the behaviour changes from the linearly-descending line to the constant-line as we decrease \( w \). The linear plot of \( -\frac{\partial \Gamma_{\text{eff}}}{\partial \beta'}[\varphi_c] \) (Fig.4b) shows, at some point \( w'_c < 0 \) between phase (B) and (C), the behaviour changes from the linearly-descending line to the constant-line as we decrease \( w \) in the negative region. Those straight lines can be obtained as

\[
-\frac{\partial \Gamma_{\text{eff}}}{\partial \beta'}[\varphi_c] \to 64\pi^2 \frac{1 + \xi}{w} \quad \text{as } w \to +\infty ,
\]

\[
-\frac{\partial \Gamma_{\text{eff}}}{\partial \beta'}[\varphi_c] \to 16\pi^2(1 + \xi)\{(3 - \xi) - (1 - \xi)^2w + O(w^2)\} \quad \text{as } w \to +0 ,
\]

\[
-\frac{\partial \Gamma_{\text{eff}}}{\partial \beta'}[\varphi_c] \to 64\pi^2 + \frac{0}{|w|} + O(w^{-2}) \quad \text{as } w \to -\infty .
\]

We can clearly define the changing points \( w_c \) and \( w'_c \) as the cross-point of two corresponding asymptotic lines above, and obtain as

\[
  w_c(\xi) = \frac{4}{3 - \xi} = w_0(\xi) , \quad w'_c(\xi) = -\frac{1}{1 + \xi} .
\]  

(50)
All cross-over points depend on the parameter of the total derivative term $\xi$, and which says the global term controls the essential behaviour of the theory.

What value should we take for $\xi$? It can be answered, purely within the theory, from the quantum analysis[15]. When we take $\xi = 1$, the renormalization-group beta functions have zeros for $w \geq 1$. Here, however, we fix the parameter $\xi$ by adjusting the asymptotic ($A \to \infty$) behaviour of $Z[A]$, for the case $\beta = 0$, with the KPZ (conformal) result[7]. The asymptotic behaviour of $Z[A]$ for $\beta = 0$ is given as

$$
\alpha_\gamma^- - \text{solution}
$$

$$
Z[A]|_{w=0} \sim A^{-\frac{8w\xi}{6}} \cdot A^{-1} = A^{-\frac{25-c_m}{6}} \xi^{-1},
$$

as $A \to +\infty$, \hspace{1cm} (51)

where the factor $A^{-1}$ comes from $Z_\omega$ in Sec.5. The KPZ result[7] is

$$
Z_{KPZ}[A] \sim A^{\gamma_s-3}, \quad \gamma_s = \frac{1}{12}\{c_m - 25 - \sqrt{(25 - c_m)(1 - c_m)}\} + 2. \hspace{1cm} (52)
$$

In order to adjust our result with the KPZ result in the classical limit $c_m \to -\infty: Z_{KPZ}[A] \sim A^{\frac{1}{6}c_m}$, we must take

$$
\xi = 1. \hspace{1cm} (53)
$$

Taking $\xi = 1$, the asymptotic behaviour of $Z[A]$ for the $\alpha_\gamma^-$-solution is

$$
Z[A] \sim A^{-\frac{25-c_m}{6}-1}, \quad A \to +\infty. \hspace{1cm} (54)
$$

Now we compare the KPZ result and the semiclassical result in the normalized form.

$$
Z_{norm}[A] = \frac{Z_{KPZ}[A]}{Z_{KPZ}[A]|_{c_m=0}} \sim A^{\gamma_s(c_m)-\gamma_s(c_m=0)},
$$

$$
\gamma_s(c_m) - \gamma_s(c_m = 0) = \frac{1}{12}\{c_m + 5 - \sqrt{(25 - c_m)(1 - c_m)}\} , \hspace{1cm} (55)
$$

$$
Z_{norm}[A] \equiv \frac{Z[A]}{Z[A]|_{c_m=0}} \sim A^{\frac{c_m}{6}}, \hspace{1cm} .
$$
In Fig.11, the present semiclassical result and the KPZ result are plotted.

In the following, we take $\xi = 1$.

7 Phases, Thermodynamic Properties and Equation of State

In this section we examine the thermodynamic properties of the system using the obtained analytic expression. We consider the positive curvature solution. The partition function is given by

$$Z[A] = \int d\lambda \exp \{ \hat{\Gamma}[\lambda] + \lambda A \} \approx \exp \{ \hat{\Gamma}[\lambda_c] + \lambda_c A \},$$

$$\frac{d}{d\lambda}(\hat{\Gamma}[\lambda] + \lambda A)|_{\lambda = \lambda_c} = \frac{d\hat{\Gamma}[\lambda_c]}{d\lambda_c} + A = 0. \tag{56}$$

Under the variation of the total area: $A \to A + \Delta A$, $\ln Z[A]$ changes by

$$\Delta(\ln Z[A]) = \lambda_c \cdot \Delta A + \Delta \cdot \frac{d\lambda_c}{d\lambda} \cdot (\frac{d\hat{\Gamma}}{d\lambda} + A)|_{\lambda = \lambda_c} = \lambda_c \cdot \Delta A. \tag{57}$$

Because the free energy $F$ is given by $F = -\ln Z[A]$, the pressure $P$ is obtained as

$$P = -\frac{\partial}{\partial A} \ln Z[A] = \lambda_c. \tag{57}$$

The pressure is the same as the string tension.

We define the temerature $T(w)$, imitating the Boyle-Charles’ law, as follows.

$$P \cdot A \equiv \frac{4\pi}{\gamma} T(w),$$

$$T(w) = \frac{2\lambda_c A}{4\pi} = \frac{1}{64\pi^2}(\alpha_c^2 w - 16\pi \alpha_c), \tag{58}$$

$$\alpha_c(w) = \begin{cases} \frac{4\pi}{2w}(w + 1 + |w - 1|) & \text{for } + \text{ branch solution} \\ \frac{4\pi}{2w}(w + 1 - |w - 1|) & \text{for } - \text{ branch solution} \end{cases}$$
\[ N \equiv \frac{4\pi}{\gamma} = \frac{(26 - c_m)}{12} \] corresponds to the 'mol number'. The temperature is the (dimensionless) string tension per a unit mol. The final analytic form of the temperature is given by

\[ T(w) = \begin{cases} 
  -\frac{1}{w} & \text{for } 0 < w \leq 1 \\
  w - 2 & \text{for } 1 \leq w 
\end{cases} \]

The behaviour of \( T(w) \) is plotted in Fig.12.

**Fig.12** Temperature \( T = T(w) \), Pos. Curv. Sol.

The asymptotic form of temperature in each phase is given by,

Phase (A) \( w \gg 1 \) , \( P \cdot A = -\frac{4\pi}{\gamma} \frac{1}{w} \) , \( T_{(A)} = -\frac{1}{w} \)

Phase (B) \( |w| \ll 1 \) , \( P \cdot A = -\frac{8\pi}{\gamma} (1 + O(w)) \) , \( T_{(B)} \approx -2 \)

Phase (C) \( w \ll -1 \) , \( P \cdot A = \frac{4\pi}{\gamma} w (1 + O(w^{-1})) \) , \( T_{(C)} \approx w \)

Phase (D) \( w \gg 1 \) , \( P \cdot A = \frac{4\pi}{\gamma} w (1 + O(w^{-1})) \) , \( T_{(D)} \approx w \)

Phase (E) \( 0 < w \ll 1 \) , \( P \cdot A = -\frac{4\pi}{\gamma} \frac{1}{w} \) , \( T_{(E)} = -\frac{1}{w} \)

The negativeness of temperature, in Phase (A),(B) and (C), says the matter-gass particles attract each other. The small absolute value of \( T_{(A)} \) says the matter-gass particles move almost freely. We can do the same analysis for the negative curvature solution. The corresponding temperature is obtained by reflecting the graph of Fig.12 following (39). It is interesting that the matter-gass particles attracting each other on an open manifold can be regarded as the 'repulsive' particles on a regularized closed manifold.
The entropy is similarly obtained. Using the relation:
\[ \Delta w|_{A:\text{fixed}} = w \cdot \Delta \beta \], \[ \Delta T|_{A:\text{fixed}} = \frac{\partial T}{\partial w} \cdot w \Delta \beta \], it is given as
\[ S_{\text{ent}} = -\left( \frac{\partial F}{\partial T} \right)_A \]
\[ = + \frac{1}{w} \frac{\partial w}{\partial T} \cdot \beta \frac{\partial}{\partial \beta} \ln Z[A] = -\frac{1}{16\pi\gamma} \frac{\partial w}{\partial T} \cdot A < \int d^{2}x \sqrt{g}R^{2} > . \quad (61) \]

We see the entropy is related to the expectation value:
\[ A < \int d^{2}x \sqrt{g}R^{2} > \text{ considered in Sec.3, as above. Using the following results from } (28) (\xi = 1 \text{ is taken}), \]
\[ + \text{ branch solution} \quad A < \int d^{2}x \sqrt{g}R^{2} > = \left\{ \begin{array}{ll}
(8\pi)^{2}\left(\frac{2}{w} - \frac{1}{w^{2}}\right) & \text{for } 0 < w \leq 1 \\
(8\pi)^{2} & \text{for } 1 \leq w 
\end{array} \right. \]
\[ - \text{ branch solution} \quad A < \int d^{2}x \sqrt{g}R^{2} > = \left\{ \begin{array}{ll}
(8\pi)^{2} & \text{for } w \leq 1 \\
(8\pi)^{2}\left(\frac{2}{w} - \frac{1}{w^{2}}\right) & \text{for } 1 \leq w 
\end{array} \right. \quad (62) \]
we obtain the expression for the entropy.
\[ + \text{ branch solution} \quad S_{\text{ent}} = \left\{ \begin{array}{ll}
-\frac{4\pi}{\gamma}(2w - 1) & \text{for } 0 < w \leq 1 \\
-\frac{4\pi}{\gamma} & \text{for } 1 \leq w 
\end{array} \right. \]
\[ - \text{ branch solution} \quad S_{\text{ent}} = \left\{ \begin{array}{ll}
-\frac{4\pi}{\gamma} & \text{for } w \leq 1 \\
-\frac{4\pi}{\gamma}(2w - 1) & \text{for } 1 \leq w 
\end{array} \right. \quad (63) \]

The graph of \( S_{\text{ent}} \) is plotted in Fig.13.

Fig.13 Entropy per unit mol, \( S_{\text{ent}}(w)/(4\pi/\gamma) \), Pos.Curv.Sol.

The largeness of the absolute value of \( S_{\text{ent}} \) in Phase (A) shows the much amount of freedom of the system, whereas the fixed value in Phase (B) and (C) shows the possible configurations are restricted.

From the behaviours of the temperature and the entropy, the cross-over in the \( - \) solution looks to occur only at one point, \( w = 1 \), on the \( w \)-axis. The corresponding one to \( w_{c}' \) in Sec.6 does not appear.
In Fig.14, all phases above are pictorially depicted.

Fig.14 Schematic image of surface in each phase

8 Discussions and Conclusions

Among two branches, the $-\text{branch}$ (of the positive curvature) solution appears in the lattice simulation\cite{5,6}. It is consistent with the present analysis, where $-\text{branch}$ is energetically preferable to $+\text{branch}$. Some features of $+\text{branch}$ are the same as those obtained in \cite{25} using the conformal field approach\cite{5}. It seems important to analyse the relation between the present semiclassical approach and the conformal field approach.

We discuss the meaning and the possible role of the negative curvature solution. The existence of the vacua related by the reflection symmetry: $R \leftrightarrow -R$, is one of stressing points of this paper. We may say the appearance of 'dual' solutions reflects the reflection symmetry: $R \leftrightarrow -R$ in the 'induced' $R^2$-gravity $\mathcal{L}_{\text{ind}} = \frac{1}{2\gamma} R + \frac{1}{2} \beta R^2 - \mu$. The symmetry appears manifestly due to the $R^2$-term. Their topologies, however, are different: the positive curvature solution satisfies $\int d^2x \sqrt{g} \ R = 8\pi$, which means the sphere topology, whereas the negative one
satisfies: \( \int d^2 x \sqrt{g} \ R = -8\pi \frac{A}{\epsilon} = -\infty \), which means the topology of a sphere with the infinite number of punctures (Fig.9). We suppose this reflection symmetry of vacua is general for manifolds with other topologies. It means a physical quantity on a manifold can also be calculated on another manifold with a different topology. It requires further analysis for clarity.

The semiclassical approach can easily provide the physical meaning such as thermodynamic properties. The present system can be regarded as the closed thermodynamic system where many scalar-matter particles move in the gravitational potential and whose configuration is thermally in an equilibrium state. The \( R^2 \) coupling, \( w \) (or \( \beta \)), parametrises the temperature. The phase difference can be thermodynamically interpreted as the difference of \( w \)-dependence of the temperature.

The important role of the integration parameter \( \lambda \) introduced in (8) and of the total derivative term discussed in Sec.5 show the proper treatment of the area constraint and the topology constraint is so important to understand the 2d QG. In the treatment, the infrared regularizations of Fig.2 and of Fig.8 are nicely used. Evaluation of the quantum effects to the present classical results is an important work to be done. It can be taken into account perturbatively as explained in (12). The renormalization has already been analysed in [15].

The present approach can be valid for the higher-dimensional quantum gravity. The 3 dim QG has been recently ‘measured’ in the Lattice simulation with a high statistics. The semiclassical analysis of the data will soon become an urgent work to be done. The success of the perturbative 2d QG using the semiclassical method is strongly encouraging for the further progress of the perturbative quantum gravity in the realistic dimensions.

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The author thanks N. Ishibashi, H. Kawai, N. Tsuda and T. Yukawa for
Appendix A. Some useful formula

Some useful formulas are collected here.

(i) Positive curvature solution

\[\varphi_c(r; \alpha) = -\ln \left\{ \frac{\alpha}{8} (1 + \frac{r^2}{A})^2 \right\} , \quad \alpha > 0 \quad . \tag{64}\]

\[e^{-\varphi_c} = \frac{\alpha}{8}(1 + \frac{r^2}{A})^2 \quad , \quad \partial^2 \varphi_c = -\frac{8A^{-1}}{(1 + A^{-1}r^2)^2} , \]

\[\int d^2x \sqrt{g} \bigg|_{\varphi_c} = \int d^2x e^{\varphi_c} = \frac{8\pi}{\alpha} A , \quad \int d^2x \varphi_c \partial^2 \varphi_c = 8\pi (2 + \ln \frac{\alpha}{8}) , \]

\[\int_{x^2 \leq L^2} d^2x \partial_a \varphi_c \partial_a \varphi_c = 16\pi \left\{ \ln \left(1 + \frac{L^2}{A}\right) - \frac{L^2/A}{1 + L^2/A} \right\} , \]

\[\int d^2x \sqrt{g} R^2 \bigg|_{\varphi_c} = \int d^2x e^{-\varphi_c} (\partial^2 \varphi_c)^2 = \frac{8\pi\alpha}{A} , \quad \int d^2x \partial^2 \varphi_c = -8\pi , \quad \tag{65}\]

(cf. For the oriented closed manifold: \(\int d^2x \sqrt{g} R = 8\pi (1 - h)\),

\[h = \text{Number of handles} \quad , \]

\[R|_{\varphi_c} = -e^{-\varphi_c} \partial^2 \varphi_c = \frac{\alpha}{A} > 0 \quad . \]

\[2A^{-1} \int_0^\infty \frac{r \, dr}{(1 + A^{-1}r^2)} = 1 \quad , \quad \int \frac{\ln (x + 1)}{(x + 1)^2} = -\ln \frac{x + 1}{x + 1} - \frac{1}{x + 1} . \quad \tag{66}\]

(ii) Negative curvature solution

\[\varphi_n(r; \alpha) = -\ln \left\{ \frac{-\alpha}{8} (1 - \frac{r^2}{A})^2 \right\} , \quad \alpha < 0 \quad . \tag{67}\]

This solution is singular at \(r = \sqrt{A}\). The following volume integrals are evaluated using two ways of regularization: \(k = 1\) \(\int d^2x = \lim_{\epsilon \to +0} \int_{0 \leq r^2 \leq A - \epsilon} \) and \(k = 2\) \(\int d^2x = \lim_{\epsilon \to +0} (\int_{0 \leq r^2 \leq A - \epsilon} + \int_{A + \epsilon \leq r^2})\).
\[ e^{-\varphi_n} = \frac{-\alpha}{8} (1 - \frac{r^2}{A})^2, \partial^2 \varphi_n = \frac{8A^{-1}}{(1 - A^{-1}r^2)^2}, \]
\[
\int d^2x \sqrt{g} \bigg|_{\varphi_n} = \int d^2x e^{\varphi_n} = \frac{8\pi}{-\alpha A} \left( \frac{kA}{\epsilon} - 1 \right),
\]
\[
\int d^2x \varphi_n \partial^2 \varphi_n = 8\pi \left( 2 - \left( \frac{kA}{\epsilon} - 1 \right) \ln \left( \frac{-\alpha}{8} \right) + \frac{2kA}{\epsilon} (\ln A - 1) \right),
\]
\[
\int d^2x \sqrt{g}R^2 \bigg|_{\varphi_n} = \int d^2x e^{-\varphi_n}(\partial^2 \varphi_n)^2 = \frac{-8\pi\alpha}{A} \left( \frac{kA}{\epsilon} - 1 \right),
\]
\[
\int d^2x \sqrt{g}R = 8\pi \left( \frac{kA}{\epsilon} - 1 \right) \ln (1 - h) \text{ for oriented closed surface },
\]
\[
R \big|_{\varphi_n} = -e^{-\varphi_n} \partial^2 \varphi_n = \frac{\alpha}{A} < 0.
\]

\[
\int \frac{1}{\pi A} \int d^2x \frac{1}{(1 - A^{-1}r^2)^2} = 2A^{-1} \int_0^\infty \frac{r \, dr}{(1 - A^{-1}r^2)}
\]
\[
= 2A^{-1} \lim_{\epsilon \to 0} \left( \int_{0 < r^2 < A - \epsilon} + \int_{r^2 > A + \epsilon} \right) \frac{r \, dr}{(1 - A^{-1}r^2)} = \frac{kA}{\epsilon} - 1.
\]
\[
\int \ln \left( \frac{1 - x}{x} \right) = -\ln \left( \frac{1 - x}{x} \right) - \frac{1}{x - 1}, \int \frac{\ln (x - 1)}{(x - 1)^2} = -\frac{\ln (x - 1)}{x - 1} - \frac{1}{x - 1}.
\]

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[9] The sign for the action is different from the usual convention as seen in (3).

[10] The uniqueness of this term, among all possible total derivatives, is shown in Discussions (sect. 6) of [5].

[11] This is for the comparison with the ‘classical limit’ $c_m \to -\infty$. We can do the same analysis for $\gamma < 0$ without any difficulty.

[12] In this section only, we explicitly write $\hbar$ (Planck constant) in order to show the perturbation structure clearly.

[13] The expansion parameter is the Planck constant $\hbar$. It is known, in the field theory, the expansion with respect to $\hbar$ is equivalent to that with respect to the number of loops in the Feynman diagrams (loop-expansion).

[14] This method is so popular that its naming is diverse depending on the applied circumstances: mean-field method, WKB-approximation, stationary phase approximation, effective action method, background-field method, etc.

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[18] This condition turns out to be satisfied in the following solution

[19] Because of the term $-\lambda \left( \int d^2 x \ e^\varphi - A \right)$ in $Y[A, \lambda]$, (8), we see $\lambda$ can be interpreted as the string (surface) tension.

[20] Adding the effect of the outer region does not change the essential part of the following content. See App.A for detail.
[21] In App.A, the quantities (31) are evaluated using two ways of regularization:
   1) inner region and 2) inner and outer regions.

[22] The infrared regularization parts, $C(A)$ and $\tilde{C}_2(A)$, does not obey the
   reflection rule. Those parts do not depend on $w$ and directly come from the
   topology of the manifold. The topology of a sphere and that of the
   infrared-regularized sphere (Fig.9) is quite different.

[23] Note that these vacua are meaningful only locally. The global constraint
   $\int d^2x \sqrt{g} R = 8\pi$ is violated in these vacua and is irrelevant for the dynamics.

[24] In the text, we understand $\xi \rightarrow +1 - 0$. Practically all numerical evaluations,
   except Fig.12 and Fig.13, are done at $\xi = 0.99$.

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Figure Captions

• Fig.1 The contour of $\lambda$-integral in the complex $\lambda$-plane

• Fig.2 Infra-red cut-off $L$ in the flat coordinates and the sphere manifold. For simplicity, the picture is for $\alpha = 8$. For general $\alpha$, $(x, y, r, \sqrt{A}, L)$ is substituted by $\sqrt{8/\alpha} \times (x, y, r, \sqrt{A}, L)$.

• Fig.3 $A \times$ Curvature $= \alpha(w)$, Positive and Negative Curv. Sols., $-$Branch

• Fig.4a Log-Log Plot of $A < \int d^2x \sqrt{g} R^2 > c$, $w > 0$, Positive and Negative Curv. Sols., $-$Branch

• Fig.4b Linear Plot of $A < \int d^2x \sqrt{g} R^2 > c$, Positive and Negative Curv. Sols., $-$Branch

• Fig.5 $\gamma A \times$ (String Tension) $= \gamma \lambda(w) A$, Positive and Negative Curv. Sols., $-$Branch

• Fig.6 $\gamma \times$ (Total Free Energy) $= -\gamma \Gamma^{eff}$, Positive and Negative Curv. Sols., $-$Branch, The $w$-independent terms $(C(A), \tilde{C}_2(A))$ are omitted.

• Fig.7 $\frac{1}{A} \times$ Area $= \frac{8\pi}{\alpha(w)}$, Positive and Negative Curv. Sols., $-$Branch

• Fig.8 Regularization of singularity at $r = \sqrt{A}$ of (30).

• Fig.9 Punctured sphere absorbing infrared divergence

• Fig.10 $\lambda$-integral path for $-$branch solution

• Fig.11 Semiclassical result versus KPZ formula for string susceptibility

• Fig.12 Temperature $T = T(w)$, Pos. Curv. Sol.

• Fig.13 Entropy per unit mol, $S_{\text{ent}}(w)/(4\pi/\gamma)$, Pos.Curv.Sol.

• Fig.14 Schematic image of surface in each phase
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