Remark on formation of colored black holes via fine tuning

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In a recent paper (gr-qc/9903081) Choptuik, Hirschmann, and Marsa have discovered the scaling law for the lifetime of an intermediate attractor in the formation of \(n=1\) colored black holes via fine tuning. We show that their result is in agreement with the prediction of linear perturbation analysis. We also briefly comment on the dependence of the mass gap across the threshold on the radius of the event horizon.

Recently, Choptuik, Hirschmann, and Marsa have discovered a new type of critical behavior within the black hole regime of the spherically symmetric Einstein-Yang-Mills model \(^2\). Using the horizon excision technique, they were able to follow the evolution of supercritical data beyond the formation of an event horizon. This allowed them to determine the ultimate behavior of the Yang-Mills hair which remains outside the black hole after the formation of an event horizon. In accordance with the no-hair property it is expected that this hair is lost either via radiation to infinity or by collapse to the black hole. Choptuik, Hirschmann, and Marsa found that the mechanism of loosing hair is different for black holes formed via two generalized (in the sense explained in \(^3\)) types of collapse established in \(^4\). For a generalized supercritical Type II collapse most of the hair is radiated off to infinity leaving the original black hole virtually unchanged. In contrast, for a generalized supercritical Type I collapse most of the hair collapses and consequently the original black hole gets bigger. These two kinds of behavior are separated by the \(n=1\) colored black hole (the static spherically symmetric black hole solution of Einstein-Yang-Mills equations \(^3\)) which plays the role of an intermediate attractor for near-critical solutions. Similarly to the Type I collapse, the lifetime \(T\) of a near-critical solution staying in the vicinity of the intermediate attractor was found to satisfy the scaling law

\[ T \sim -\lambda \ln |p - p^*|, \tag{1} \]

where the coefficient \(\lambda\) is the characteristic time scale for the decay of the \(n=1\) colored black hole, and \(|p - p^*|\) is the distance from the threshold \((p^*)\) along a one-parameter family of interpolating initial data.

As described in \(^3\), the coefficient \(\lambda\) can be computed in two independent ways: either directly from the nonlinear evolution, or perturbatively via linear stability analysis. Since Choptuik, Hirschmann, and Marsa calculated \(\lambda\) only in the first manner, here we would like to compare their result with the predictions of linear stability analysis. We think that this comparison not only validates the results of Choptuik, Hirschmann, and Marsa but, in addition, provides a useful test of the precision of their numerical estimates.

We recall that according to the picture of critical solution as a codimension-one attractor, the coefficient \(\lambda\) is a reciprocal of the eigenvalue \(\sigma\) corresponding to the single unstable mode, \(i.\ e.\), \(\lambda = 1/\sigma\). The single mode instability (within the magnetic ansatz) of the \(n=1\) colored black holes was established almost a decade ago \(^5\), however there are no (as far as we know) published results on how the eigenvalue of the unstable mode depends on the radius of the horizon (in \(^5\) one of us determined this relationship in reference with the issue of regularity of the unstable mode at the horizon but no numerical values were given there). We fill in this gap in the literature with Fig. 1, where the eigenvalue \(\sigma\) is plotted as a function of the radius of the horizon \(r_h\) of \(n = 1\) colored black hole. This plot was generated by numerically solving, after Straumann and Zhou, their eigenvalue equation for small perturbations \(^4\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{The eigenvalue \(\sigma\) corresponding to the single unstable mode about the \(n=1\) colored black hole is plotted against the radius of the event horizon \(r_h\). As \(r_h \to 0\), \(\sigma \to 0.2292\), the eigenvalue of the unstable mode about the Bartnik-Mckinnon solution. In order to fix the scale, the time coordinate is normalized to the proper time at infinity.}
\end{figure}
Choptuik, Hirschmann, and Marsa estimated that $\lambda \approx 4.88$ for $r_h \approx 0.55$ which compares well with our calculation of $1/\sigma(0.55) = 5.0736$. Unfortunately, this is the only value of $r_h$ for which we can make the comparison because Choptuik, Hirschmann, and Marsa did not estimate $\lambda$ for other values of $r_h$. It is worth noting that since the eigenvalue $\sigma$ (and eo ipso the coefficient $\lambda$) varies with $r_h$, it is universal only among those interpolating families of initial data which share the same critical solution (i.e., have the same $r_h$).

Notice that, as follows from Eq.(1) and Fig. 1, for a given distance from the threshold, the lifetime $T$ increases with $r_h$, so colored black holes with large $r_h$ should be more clearly pronounced as intermediate attractors. However, this effect is partly offset by the fact that, in the limit of large $r_h$, colored black holes become almost indistinguishable from the Schwarzschild black hole because the energy of the Yang-Mills hair rapidly decreases with $r_h$. In other words, the mass gap across the threshold is expected to decrease with $r_h$. The upper bound for the mass gap is given by

$$
\Delta M(r_h) = M_{cbh}(r_h) - \frac{r_h}{2},
$$

which is the difference between the mass $M_{cbh}(r_h)$ of the $n = 1$ colored black hole and the mass of the Schwarzschild black hole with the same radius of event horizon $r_h$. This maximal mass gap would be achieved if all hair remaining outside the horizon of near-critical solutions would either completely disperse, or completely collapse. Since in the real collapse there is always a fraction of hair which does otherwise, the actual mass gap is smaller than $\Delta M$. The maximal mass gap given by Eq.(2) is plotted in Fig. 2.

FIG. 2. The maximal mass gap across the threshold as a function of the radius of the horizon. For $r_h = 0.55$, Choptuik, Hirschmann, and Marsa obtain the mass gap $\sim 0.57$ which is approximately 95% of the maximal mass gap.

It would be interesting to compare Fig. 2 with the actual mass gap. In particular, one might wonder whether the mass gap goes to zero for a large but finite $r_h$. If this really happened, then the line of colored black holes on the phase diagram shown in Fig. 4 in [1], beginning at the triple point for $r_{bh} = 0$, would terminate at a finite distance, in an amusing similarity to the gas-liquid boundary on phase diagrams for typical substances. We realize that the numerical verification of this speculation would be rather difficult.

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