Stability for stochastic neutral integro-differential equations with infinite delay and Poisson jumps

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ABSTRACT

This paper investigates a kind of stochastic neutral integro-differential equations with infinite delay and Poisson jumps in the concrete-fading memory-phase space $C_p$. We suppose that the linear part has a resolvent operator and the nonlinear terms are globally Lipschitzian. We introduce sufficient conditions that ensure the existence and uniqueness of mild solutions by using successive approximation. Moreover, we target exponential stability, including moment exponential stability in $q$-th ($q \geq 2$) and almost surely exponential stability of solutions and their maps. An example illustrates the potential of the main result.

1. Introduction

Research on stochastic differential equations with delay has received attention over the last few decades because of their appropriateness to describe physical systems subject to delays, such as the ones found in biology, medicine, epidemiology, chemistry, physics, and economics (see Helge et al., 2010; Intissar., 2020; Kostikov & Romanenkov, 2020; Mao, 2007; Trung, 2020 for a brief overview). The qualitative and quantitative properties of solutions of stochastic differential equations with delay, such as the existence, uniqueness, controllability, and stability, have been considered by several authors (see Bouzahir et al., 2017; Dieye et al., 2017; Diop et al., 2014; Taniguchi et al., 2002; Zouine et al., 2020). One point, in particular, has received a lot of attention: the study of existence and asymptotic behavior of mild solutions of some stochastic differential equations on Hilbert spaces, such as the semigroup approach (Taniguchi et al., 2002), comparison theorem (Govindan, 2003), Razumikhin-type theorem (Kai & Yufeng, 2006), analytic technique (Taniguchi, 1998), and Banach fixed-point principle (Diop et al., 2014).

The literature shows many dynamical systems modeled by neutral stochastic partial differential equations 2 (Chen et al., 2014; Cui et al., 2011) [9]. For these equations, some contain the derivatives of delayed states, which differ from stochastic partial differential equations with delays that depend on the present and past states only (for more details on this theory and its applications, see Mao et al., 2017; Yue, 2014). Stochastic integro-differential equations have been intensively studied, with special attention paid to qualitative properties, such as stability, regularity, periodicity, control problems, and optimality conditions (see Dieye et al., 2019; Diop et al., 2014). Due to the existence of an integral term in the equations, we use here the theory of the resolvent operator instead of the strongly continuous semigroups operator (see Grimmer, 1982 for further details).

Yet, most of the researchers dealing with exponential stability have limited their research to finite delay (see Dieye et al., 2017; Diop et al., 2014). Regarding the infinite delay, most investigations have been done for the case of continuous dependence of solutions on the initial value, considering exponential and asymptotic estimates (see, for instance, the papers Cui & Yan, 2012; Mao et al., 2017; Ren & Xia, 2009; Yue, 2014 for an account on phase spaces). It is noteworthy that few contributions exist for characterizing the exponential stability of stochastic equations with infinite delays (see Jiang et al., 2016; Wu et al., 2017). Jiang et al. (2016) showed the exponential stability for a class of second-order neutral stochastic partial differential equations with infinite delays and impulses using the integral inequality technique. Wu et al. (2017) showed boundedness in the mean square and convergence for both solutions and their maps in the phase space $C_p$ by using the Itô formula.
Motivated by the above discussion, consider the following neutral stochastic integro-differential equations with infinite delay and Poisson jumps given on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\):

\[
d[u(t) + y(t, u_t)] = A[u(t) + y(t, u_t)]dt + f_1(t, u_t)dt + \int_0^t Y(t - s)[u(s) + y(s, u_s)]dsdt + g_1(t, u_t)dw(t) + \int_{h_1}(t, u_t, z)\tilde{N}(dt, dz), \quad \forall t \geq 0.
\]

System (1) holds with \(u_0(\cdot) = \phi \in C_{\mu}\), where \(\phi\) is \(\mathcal{F}_0\) - measurable, and the definition of the concrete fading memory-phase space \(C_{\mu}\) is detailed in the next section. \(A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}\) is a closed linear operator on a separable Hilbert space \(\mathbb{H}\) and \(Y(t)\) represents a closed linear operator such that \(Y(t)\) has \(D(Y(t))\) as a domain, where \(D(A) \subset D(Y(t))\), for all \(t \geq 0\).

\(y, f_1 : [0, +\infty) \times C_{\mu} \rightarrow \mathbb{H}\),

\(g_1 : [0, +\infty) \times C_{\mu} \rightarrow L^2(\mathbb{K}, \mathbb{H}), \quad h_1 : [0, +\infty) \times C_{\mu} \rightarrow \mathbb{H}\) are appropriate functions, the history \(u_t : (-\infty, 0] \rightarrow \mathbb{H}\), \(t \geq 0\), such that \(u_t(\theta) = u(t + \theta)\) belongs to the phase space \(C_{\mu}\). The process \(w(t)\) represents a Wiener process on a separable Hilbert space \(\mathbb{K}\) and \(\tilde{N}\) is a compensated Poisson random measure.

To the best of the authors’ knowledge, this paper is the first to present a study of the existence and exponential stability of neutral stochastic integro-differential equations with infinite delay and Poisson jumps. The main contribution of this paper is to find conditions to ensure existence, uniqueness, exponential stability in \(q\)th moment for \(q \geq 2\) and almost surely exponential stability of solutions and their maps of (1). We show the result using stochastic techniques and the resolvent operator theory, as defined in Grimmer (1982). It is worth mentioning that Diop et al. (2014) studied the system (1) with finite delay. They focused only on the existence of mild solutions and their exponential stability in mean square (Diop et al., 2014). For this reason, our approach can be seen as an extension of the result of Diop et al. (2014) for the infinite delay case.

The organization of this paper is as follows. Some notations and preliminary results are presented in Section 2. The existence and uniqueness of mild solutions for neutral stochastic integro-differential equations with infinite delay are shown in Section 3. Conditions assuring moment exponential stability in the \(q\)-th \((q \geq 2)\) and almost surely exponential stability of the solution \(u(t)\), and the solution maps \(u_t, t \geq 0\) are shown in Section 4. Finally, an example that illustrates our results is presented in Section 5.

2. Notations and preliminary results

Let \(\mathbb{H}\) and \(\mathbb{K}\) be two real separable Hilbert spaces (c.f. Taniguchi et al., 2002). We let also \(L(\mathbb{K}, \mathbb{H})\) be the space of bounded linear operators from \(\mathbb{K}\) into \(\mathbb{H}\) associated with \(\| \cdot \|\) to represent the norm operator in \(\mathbb{H}, \mathbb{K}\), and \(\mathcal{L}(\mathbb{K}, \mathbb{H})\). We assume that System (1) is equipped with a normal filtration \(\{\mathcal{F}_t\}_{t \geq 0}\).

Denote by \(N\), the Poisson random measure induced by the \(\sigma\) - finite stationary \(\mathcal{F}_t\) - adapted Poisson point process \(\tilde{p}(\cdot)\) taking values in a measurable space \((\Omega, \mathcal{B}(\Omega))\), and define the compensated Poisson random measure \(\tilde{N}\) as

\[
\tilde{N}(dt, dy) = N(dt, dy) - \pi(dy)dt,
\]

where \(N((0, t] \times \Delta) : = \sum_{\tau \in [0, t]} 1_\Delta(\tilde{p}(\tau))\) for \(\Delta \in \pi\) is the characteristic measure of \(N\).

Let \(C((-\infty, 0], \mathbb{H})\) represent the space of all continuous functions from \((-\infty, 0]\) into \(\mathbb{H}\) equipped with the norm defined by \(\parallel \phi \parallel = \sup_{\theta \geq 0} \parallel \phi(\theta) \parallel\). For a given \(\mu > 0\), consider the fading memory \(C_{\mu}\) defined as follows:

\(C_{\mu} := \{ \phi \in C((-\infty, 0], \mathbb{H}) : \lim_{\theta \rightarrow -\infty} e^{\mu \theta} \phi(\theta) \text{ exists in } \mathbb{H}\}\)

as the chosen phase space in this paper. It is a Banach space with the norm \(\parallel \phi \parallel_{\mu} = \sup_{-\infty < \theta \leq 0} e^{\mu \theta} \parallel \phi(\theta) \parallel\). To know more details, see Hino et al. (1991) and Appendix.

Supposed that \(\{\omega(t), t \geq 0\}\) represents a \(\mathbb{K}\)-valued Wiener process which is independent of the Poisson point process on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with a positive self-adjoint covariance operator \(Q\). In addition, we suppose that there exists a complete orthonormal system \(e_i\) in \(\mathbb{K}\), a bounded sequence of positive real numbers \(\lambda_i\) such that \(Qe_i = \lambda_i e_i, i = 1, 2, \ldots\), and a sequence \(\{\beta_i(t)\}_{i \geq 1}\) of independent standard Brownian motions such that \(\omega(t) = \sum_{i=1}^{+\infty} \sqrt{\lambda_i} \beta_i(t) e_i\) for \(t \geq 0\) and \(\mathcal{F}_t\) is the \(\sigma\)-algebra generated by \(\{\omega(s) : 0 \leq s \leq t\}\) (see (Taniguchi et al., 2002)). We consider the subspace \(\mathbb{K}_0 = Q^{1/2} \mathbb{K}\) of \(\mathbb{K}\), it is a Hilbert space equipped with the inner product \(\langle u, v \rangle_{\mathbb{K}_0} = \langle Q^{-1/2} u, Q^{-1/2} v \rangle_{\mathbb{H}}\). Let \(\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0, \mathbb{H})\) be the space of all Hilbert-Schmidt operators from \(\mathbb{K}_0\) to \(\mathbb{H}\). \(\mathcal{L}_2^0\) is a separable Hilbert space endowed with the norm \(\| \psi \|_{\mathcal{L}_2^0}^2 = \text{tr}(\langle \psi Q^{1/2} \psi Q^{1/2} \rangle)\) for all \(\psi \in \mathcal{L}_2^0\). Hereafter, \(A\) and \(Y(t)\) are closed linear operators on a Banach space denoted by \(\mathbb{X}\), and \(\mathbb{Y}\) is the Banach space \(D(A)\) endowed with the graph norm \(\| y \|_{\mathbb{Y}} := \| Ay \| + \| y \|\) for \(y \in \mathbb{Y}\).
The notations $C([0, +\infty); \mathbb{Y})$, $C^1([0, +\infty); \mathbb{X})$ and $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ represent the space of continuous functions from $[0, +\infty)$ into $\mathbb{Y}$, the space of continuously differentiable functions from $[0, +\infty)$ into $\mathbb{X}$ and the set of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$, respectively.

**2.1. Preliminaries on partial integro-differential equations**

We now consider the problem
\[ dv(t) = \left( Av(t) + \int_0^t Y(t - s) v(s) ds \right) dt, \quad \forall t \geq 0, \tag{2} \]
with $v(0) = v_0 \in \mathbb{X}$.

**Definition 2.1.** (Grimmer, 1982). A bounded linear operator-valued function $R(t) \in \mathcal{L}(\mathbb{X})$, $t \geq 0$ is called a resolvent operator for (2) if the next two conditions are satisfied.

(i) $R(0) = I$ and there exist two constants $\alpha \geq 1$ and $\delta$ such that $|R(t)| \leq \alpha \exp(\delta t)$ for all $t \geq 0$.

(ii) For each element $x$ in $\mathbb{X}$, the function $t \mapsto R(t)x$ is strongly continuous for each $t \geq 0$ and for $x$ in $\mathbb{Y}$, $R(.)x \in C^1([0, +\infty); \mathbb{X}) \cap C([0, +\infty); \mathbb{Y})$ and satisfies
\[ dR(t)x = \left( A R(t)x + \int_0^t Y(t - s) R(s)x ds \right) dt \]
\[ = \left( R(t)Ax + \int_0^t R(t - s) Y(s)x ds \right) dt. \]

**Remark 1.** The resolvent operator is said to be exponentially stable when Definition 2.1(i) holds with $\delta < 0$.

The following two conditions, borrowed from Grimmer (1982), are sufficient to assure the existence of solutions for equation (2).

(A$_1$) The operator $A$ is an infinitesimal generator of a $C_0$-semigroup on $\mathbb{X}$.

(A$_2$) For all $t \geq 0, Y(t)$ denotes a closed, continuous linear operator from $D(A)$ to $\mathbb{X}$ and $Y(t)$ belongs to $\mathcal{L}(\mathbb{Y}, \mathbb{X})$. For any $y \in \mathbb{Y}$, the map $t \mapsto Y(t)y$ is bounded, differentiable, and its derivative $dY(t)y/dt$ is bounded and uniformly continuous on $[0, +\infty]$.

**Lemma 2.2.** (Grimmer, 1982). Under Assumptions (A$_1$) and (A$_2$), the existence of a resolvent operator for (2) is guaranteed, and it is unique.

We now recall conditions that assure existence of solutions for the deterministic, integro-differential equation
\[ dv(t) = \left( Av(t) + \int_0^t Y(t - s) v(s) ds + m(t) \right) dt, \quad \forall t \geq 0, \tag{3} \]
with $v(0) = v_0 \in \mathbb{X}$ and $m : [0, +\infty) \to \mathbb{X}$ is a continuous function.

**Lemma 2.3.** (Grimmer, 1982). Suppose that Assumptions (A$_1$) and (A$_2$) hold, if $v$ is a strict solution of (3) (i.e., $v(t)$ satisfies (3), for $t \geq 0$ and it belongs to $C^1([0, +\infty), \mathbb{X}) \cap C([0, +\infty), \mathbb{Y})$), then
\[ v(t) = R(t)v_0 + \int_0^t R(t - s)m(s) ds, \quad \forall t \geq 0. \tag{4} \]

Let us give the concept of solutions for the stochastic system in the next definition.

**Definition 2.4.** A mild solution of (1) is an $\mathbb{H}$-valued process $\{u(t), 0 \leq t \leq T\}$ (with $T > 0$) which satisfies the next two conditions.

(i) $u(t)$ is $\mathcal{F}_t$-adapted and $\int_0^T || u(t) ||^q dt < +\infty$ almost surely.

(ii) $u(t)$ is continuous for any $t \in [0, T]$ and equals
\[ u(t) = R(t)[\phi(0) + \gamma(0, \phi)] - \gamma(t, u_t) \\
+ \int_0^t R(t - s)f_1(s, u_s) ds \\
+ \int_0^t \int_0^t R(t - s)g_1(s, u_s) dw(s) \\
+ \int_0^t \int_0^t \int_0^t (t - s)h_1(s, u_s, z) \tilde{N}(ds, dz), \tag{5} \]
with $u_0(.) = \phi \in \mathcal{C}_0$, where $R(.)$ in (5) represents the resolvent operator of (2).

To achieve our main goal, we assume the following three assumptions:

(A$_3$) The resolvent operator $R(.)$ satisfying Lemma 2.3 is exponentially stable, i.e., there exist two constants $\lambda > 0$ and $M \geq 1$ such that $|R(t)| \leq Me^{-\lambda t}$, for all $t \geq 0$.

(A$_4$) Let $q \geq 2$ be an integer, there exists a real number $K_0 > 0$, such that
\[ || \gamma(t, \xi) - \gamma(t, \eta) ||^q \leq K_0 || \xi - \eta ||^q_{C_{\mu}} \]
for all $\xi, \eta \in C_{\mu}$, and all $t \geq 0$.

(A$_5$) Let $q \geq 2$ be an integer, there exists a real number $K_1 > 0$, such that
\[ || f_1(t, \xi) - f_1(t, \eta) ||^q \leq K_1 || \xi - \eta ||^q_{C_{\mu}} \]
\[ || g_1(t, \xi) - g_1(t, \eta) ||^q \leq K_2 \]
\[ \forall \xi, \eta \in C_\mu \text{ and all } t \geq 0, \]
for all \( \xi, \eta \in C_\mu \) and all \( t \geq 0 \),
In particular, there holds \( \gamma(t, 0) = f_1(t, 0) = g_1(t, 0) = h_1(t, 0, \cdot) = 0 \).

Remark 2. We consider the assumption \( \gamma(t, 0) = f_1(t, 0) = g_1(t, 0) = h_1(t, 0, \cdot) = 0 \) for all \( t \geq 0 \), to guarantee that there exists a zero equilibrium solution to the stochastic equation (1). If this assumption does not hold, the equilibrium solution for equation (1) can always be transformed into the zero equilibrium solution of another equation.

### 3. Existence and uniqueness

In this section, we present sufficient conditions to guarantee the existence and uniqueness of mild solutions of the equation in (1). To do so, we use the method of successive approximations and some stochastic analysis techniques. Still, we have to develop some new techniques to deal with infinite delay. Hereafter, we replace \( \mathbb{X} \) by the Hilbert space \( \mathbb{H} \) in (A1) and (A2). Now, we present the following main result.

**Theorem 3.1.** Assumed that Assumptions (A1), (A2), (A4) and (A3) hold, and \( K_0 < \frac{1}{10^{q-1}} \). Then, equation (1) has a unique mild solution.

**Proof.** The proof of this theorem uses the following sequence of successive approximations that is defined for \( t \leq 0 \) by \( u^n(t) = \phi(t) \) for any \( n \in \mathbb{N} \) and for \( 0 \leq t \leq T \) by

\[
\begin{align*}
u^n(t) = & \mathcal{R}(t)[\phi(0) + \gamma(0, t, \phi)] - \gamma(t, u^n_t) \\
& + \int_0^t \mathcal{R}(t-s)f_1(s, u^n_t)ds \\
& + \int_0^t \mathcal{R}(t-s)g_1(s, u^n_t)dw(s) \\
& + \int_0^t \mathcal{R}(t-s)h_1(s, u^n_t, z)\tilde{N}(ds, dz)
\end{align*}
\]

for any \( n \geq 1 \) and \( u^n(0) = \mathcal{R}(t)\phi(0) \) when \( 0 \leq t \leq T \). Take \( M_T = \sup_{0 \leq t \leq T} \| \mathcal{R}(t) \|_{\mathbb{L}(\mathbb{H})} \), from the uniform boundedness \( M_T < \infty \). The remaining arguments are divided into three main steps.

**Step 1:** First, let us show that \( u^n(t), n \geq 0 \) is a bounded sequence. From (6), for any \( t \in (-\infty, 0] \), we have \( E \| u^n(t) \|^q = E \| \phi(t) \|^q \leq E \| \phi \|^q_\mu < \infty \). For any \( t \in [0, T] \), we obtain that \( E \sup_{0 \leq s \leq t} \| u^n(s) \|^q \leq M_T^q E \| \phi \|^q_\mu < \infty \).

If \( n \geq \hat{q} \), consider the five terms

\[
I_1 := E \sup_{0 \leq s \leq t} \| \mathcal{R}(s)[\phi(0) + \gamma(0, \phi)] \|^q,
\]
\[
I_2 := E \sup_{0 \leq s \leq t} \| \gamma(s, u^n_t) \|^q,
\]
\[
I_3 := E \sup_{0 \leq s \leq t} \| \int_0^s \mathcal{R}(r-s)f_1(r, u^n_{t-1})dr \|^q,
\]
\[
I_4 := E \sup_{0 \leq s \leq t} \| \int_0^s \mathcal{R}(r-s)g_1(r, u^n_{t-1})dw(r) \|^q,
\]
\[
I_5 := E \sup_{0 \leq s \leq t} \| \int_0^s \mathcal{R}(r-s)h_1(r, u^n_{t-1}, z)\tilde{N}(dr, dz) \|^q.
\]

Considering the definition of the sequence \( u^n(t) \), together with the elementary inequality \( \left( \sum_{i=1}^5 |a_i| \right)^q \leq 5^{q-1} \sum_{i=1}^5 |a_i|^q \), we have

\[
E \sup_{0 \leq s \leq t} \| u^n(s) \|^q \leq 5^{q-1} (I_1 + I_2 + I_3 + I_4 + I_5), \quad (7)
\]

From Assumption (A4), we obtain

\[
I_1 = E \sup_{0 \leq s \leq t} \| \mathcal{R}(s)[\phi(0) + \gamma(0, \phi)] \|^q \leq 2^{q-1} M_T^q (1 + K_0) E \| \phi \|^q_\mu, \quad (8)
\]

and

\[
I_2 = E \sup_{0 \leq s \leq t} \| \gamma(s, u^n_t) \|^q \leq K_0 E \sup_{0 \leq s \leq t} \| u^n_t \|^q_\mu. \quad (9)
\]

In addition, combining Assumption (A5) and Holder inequality yields

\[
I_3 \leq M_T^q E \sup_{0 \leq s \leq t} \left( \int_0^s \| f_1(r, u^n_{t-1}) \| dr \right)^q \leq M_T^q \nu^{q-1} K_1 \int_0^t E \sup_{0 \leq r \leq s} \| u^n_{t-1} \|^q_\mu ds. \quad (10)
\]

Now, using Lemma 6.2 (see Appendix), and by a similar reasoning on \( I_3 \), we show that

\[
I_4 \leq c_q M_T^q \nu^{q-1} K_1 \int_0^t E \sup_{0 \leq r \leq s} \| u^n_{t-1} \|^q_\mu ds, \quad (11)
\]

where

\[
c_q = \frac{q(q-1)}{2}.
\]

On the other hand, combining Assumption (A5), Lemma 6.3 (see Appendix) and Holder inequality on \( I_5 \), we find that

\[
I_5 \leq D_5 \left\{ E \left( \int_0^t M_T^q K_1 \| u^n_{t-1} \|^q_\mu dr \right)^\frac{q}{2} + E \left( \int_0^t M_T^q K_1 \| u^n_{t-1} \|^q_\mu dr \right) \right\}
\]

Thus, we obtain the boundedness of \( u^n(t) \).

**Step 2:** Next, we show that \( \| u^n(t) \| \) is uniformly bounded on \( [0, T] \).

**Step 3:** Finally, we prove the uniqueness of mild solutions.
\[
\begin{align*}
\leq D_q \left( M_q^0 \int_0^t \| u^n \|_{C_{\psi}}^q \, dt + M_q^0 \int_0^t \| u^{n-1} \|_{C_{\psi}}^q \, dt \right) \\
\leq D_q M_q^0 K_1 (t K_1 \frac{q}{q-1} + 1) \int_0^t E \sup_{0 \leq r \leq s} \| u^{n-1} \|_{C_{\psi}}^q \, ds, \quad (12)
\end{align*}
\]
Substituting (8)–(12) into (7), we obtain
\[
E \sup_{0 \leq s \leq t} \| u^n(s) \|^q \leq 10^{-q} M_q^0 (1 + K_0) E \| \phi \|_{C_{\psi}}^q + 5^{q-1} K_0 E \sup_{0 \leq s \leq t} \| u^q \|_{C_{\psi}}^q \\
+ \tilde{C}_1(q, T) \int_0^t E \sup_{0 \leq r \leq s} \| u^{n-1} \|_{C_{\psi}}^q \, ds, \quad (13)
\]
where \( \tilde{C}_1(q, T) = 5^{q-1} M_q^0 K_1 (t^{q-1} + c_q t^{q-2} + D_q \left( (t K_1 \frac{q}{q-1} + 1) \right)) \). By using the definition of the norm \( \| . \|_{C_{\psi}} \) (see Appendix), we can write
\[
\sup_{0 \leq s \leq t} \| u^n \|_{C_{\psi}}^q \leq \| \phi \|_{C_{\psi}}^q + \sup_{0 \leq s \leq t} \| u^n(s) \|^q.
\]
Define
\[
\tilde{C}_2(q, T) = \frac{10^{-q} M_q^0 (1 + K_0) + 5^{q-1} K_0 + \tilde{C}_1(q, T) T}{1 - 5^{q-1} K_0} E \| \phi \|_{C_{\psi}}^q.
\]
We can write
\[
E \sup_{0 \leq s \leq t} \| u^n(s) \|^q \leq \tilde{C}_2(q, T) + \tilde{C}_3(q, T) \int_0^t E \sup_{0 \leq r \leq s} \| u^{n-1}(r) \|^q \, ds.
\]
Therefore, for any \( k \geq 1 \), one can conclude that
\[
\begin{align*}
\max_{1 \leq n \leq k} E \sup_{0 \leq s \leq t} \| u^n(s) \|^q & \leq \tilde{C}_2(q, T) + \tilde{C}_3(q, T) \int_0^t \max_{1 \leq n \leq k} E \sup_{0 \leq r \leq s} \| u^{n-1}(r) \|^q \, ds, \\
\end{align*}
\]
where \( \tilde{C}_3(q, T) = (1 - 5^{q-1} K_0)^{-1} \tilde{C}_1(q, T) \). Besides, we have
\[
\begin{align*}
\max_{1 \leq n \leq k} E \| u^{n-1}(r) \|^q & \leq E \| u^0(r) \|^q + \max_{1 \leq n \leq k} E \| u^n(r) \|^q \\
& \leq M_q^0 E \| \phi \|_{C_{\psi}}^q + \max_{1 \leq n \leq k} E \| u^n(r) \|^q.
\end{align*}
\]
It follows that
\[
\begin{align*}
\max_{1 \leq n \leq k} E \sup_{0 \leq s \leq t} \| u^n(s) \|^q & \leq \tilde{C}_2(q, T) + \tilde{C}_3(q, T) T M_q^0 E \| \phi \|_{C_{\psi}}^q + \tilde{C}_3(q, T) T M_q^0 E \sup_{0 \leq r \leq s} \| u^n(r) \|^q \, ds \\
& \leq \tilde{C}_4(q, T) + \tilde{C}_5(q, T) \int_0^t \max_{1 \leq n \leq k} E \sup_{0 \leq r \leq s} \| u^n(r) \|^q \, ds,
\end{align*}
\]
where \( \tilde{C}_4(q, T) = \tilde{C}_2(q, T) + \tilde{C}_5(q, T) T M_q^0 E \| \phi \|_{C_{\psi}}^q \).

The Gronwall inequality gives
\[
\max_{1 \leq n \leq k} E \sup_{0 \leq s \leq t} \| u^n(s) \|^q \leq \tilde{C}_4(q, T) e^{\tilde{C}_5(q, T)t} \leq \tilde{C}_4(q, T) e^{\tilde{C}_5(q, T)t} < \infty.
\]
Since \( k \) is arbitrary, we have
\[
E \| u^n(t) \|^q \leq \tilde{C}_4(q, T) e^{\tilde{C}_5(q, T)t} < \infty, \quad 0 \leq t \leq T,
\]
which assures that the sequence \( \{ u^n, n \in \mathbb{N} \} \) is bounded.

**Step 2:** Now we show that \( u^n, n \in \mathbb{N} \) is a Cauchy sequence. From the construction of successive approximations, we have \( u^n(t) = u^{n-1}(t) \) on \( (-\infty, 0) \), for \( n \geq 1 \). For \( t \in [0, T] \), we can prove that
\[
\| u^n \|_{C_{\psi}}^q \leq (1 + M_q^0) \| \phi \|_{C_{\psi}}^q.
\]
Therefore, observe from (6) that
\[
E \| u^n(t) - u^0(t) \|^q \leq 5^{q-1} \sum_{i=1}^{\infty} I_i,
\]
where \( I_1 = E \| R(t) g(t, \phi) \|_q \). Using (A4), we obtain
\[
I_1 \leq M_q^0 K_0 E \| \phi \|_{C_{\psi}}^q.
\]
To show the result for \( I_2 \), we combine (A4) with the inequality \( (a + b)^q \leq 2^{q-1} (a^q + b^q) \) to obtain
\[
\begin{align*}
I_2 = E \| g(t, u^n) \|^q & \leq K_0 E \| u^n \|_{C_{\psi}}^q \\
& \leq 2^{q-1} K_0 E \| u^1 - u^0 \|_{C_{\psi}}^q + E \| u^n \|_{C_{\psi}}^q \\
& \leq 2^{q-1} K_0 E \| u^1 - u^0 \|_{C_{\psi}}^q + 2^{q-1} K_0 (1 + M_q^0) E \| \phi \|_{C_{\psi}}^q.
\end{align*}
\]
Regarding \( I_3 \), combining (A5) and the Holder inequality produces
\[
\begin{align*}
I_3 = E \int_0^t \| R(t - s) f(s, u^n_s) \|^q ds \\
& \leq M_q^0 E \int_0^t \| f(s, u^n_s) \|^q ds \\
& \leq M_q^0 T^{q-1} K_1 E \int_0^t \| u^n \|_{C_{\psi}}^q \, ds \\
& \leq (M_T T)^q K_1 (1 + M_q^0) E \| \phi \|_{C_{\psi}}^q.
\end{align*}
\]
Similarly to Step 1, from Lemma 6.2, we have
\[ I_4 = E \left\| \int_0^t \mathcal{R}(t-s)g_1(s, u^0_t)dw(s) \right\| ^q \]
\[ \leq E \sup_{t \in [0,T]} \left\| \int_0^t \mathcal{R}(t-s)g_1(s, u^0_t)dw(s) \right\| ^q \]
\[ \leq c_q M_t^g T^{\frac{q-1}{2}} K_1 \int_0^T E \left\| u^0_t \right\| _E^q \, ds \]
\[ \leq c_q M_t^g K_1 T^2 (1 + M_t^g) E \left\| \phi \right\| _E^q. \]  

(19)

Finally, by employing (A_5) and Lemma 6.3, we can proceed similarly to obtain

\[ I_5 = E \left\| \int_0^t \int_0^t \mathcal{R}(t-s)h_1(s, u^0_t, z) \tilde{N}(ds, dz) \right\| ^q \]
\[ \leq E \sup_{t \in [0,T]} \left\| \int_0^t \int_0^t \mathcal{R}(t-s)h_1(s, u^0_t, z) \tilde{N}(ds, dz) \right\| ^q \]
\[ \leq D_q M_t^g K_1 ((TK_1)^{\frac{q-1}{2}} + 1) \int_0^T E \left\| u^0_t \right\| _E^q \, ds \]
\[ \leq D_q M_t^g K_1 T (1 + M_t^g) ((TK_1)^{\frac{q-1}{2}} + 1) E \left\| \phi \right\| _E^q. \]  

(20)

Substituting (16)–(20) into (15) results

\[ E \left\| u^1(t) - u^0(t) \right\| ^q \leq 10^{q-1} K_0 E \left\| u^1_t - u^0_t \right\| _E^q + 5^{q-1} (M_t^g K_0 + 2^{q-1} K_0 (1 + M_t^g)) \
+ (M_t T)^q K_1 (1 + M_t^g) + c_q M_t^g K_1 T^2 (1 + M_t^g) \
+ D_q M_t^g K_1 T (1 + M_t^g) [(TK_1)^{\frac{q-1}{2}} + 1]) E \left\| \phi \right\| _E^q \
\leq \tilde{C}_5(q, T) + 10^{q-1} K_0 E \left\| u^1_t - u^0_t \right\| _E^q, \]

where

\[ \tilde{C}_5(q, T) = 5^{q-1} (M_t^g K_0 + 2^{q-1} K_0 (1 + M_t^g)) \
+ (1 + M_t^g) T^q K_1 + c_q M_t^g K_1 T^2 \
+ D_q M_t^g K_1 T [(TK_1)^{\frac{q-1}{2}} + 1]) E \left\| \phi \right\| _E^q. \]

On the other hand, note that

\[ \left\| u^1_t - u^0_t \right\| _E^q \leq \sup_{0 \leq s \leq T} \left\| u^1_t(t) - u^0(t) \right\| ^q \]
and recalling that \( 1 - 10^{q-1} K_0 > 0 \), we can deduce

\[ E \sup_{0 \leq s \leq T} \left\| u^1_t(t) - u^0_t(t) \right\| ^q \leq \frac{\tilde{C}_5(q, T)}{1 - 10^{q-1} K_0} =: \tilde{C}_6(q, T). \]

By similar arguments as above, we get

\[ E \left\| u^2(t) - u^1(t) \right\| ^q \leq 4^{q-1} \left( E \left\| u^1(t) - u^0(t) \right\| ^q + E \left\| \int_0^t \mathcal{R}(t-s)[f_1(s, u^0_t) - f_1(s, u^1_s)] dw(s) \right\| ^q \right) \
+ E \left\| \int_0^t \mathcal{R}(t-s)[g_1(s, u^1_s) - g_1(s, u^0_t)] dw(s) \right\| ^q \
+ E \left\| \int_0^t \mathcal{R}(t-s)[h_1(s, u^0_t, z) - h_1(s, u^1_s, z)] \tilde{N}(ds, dz) \right\| ^q \]
\[ \leq 4^{q-1} K_0 E \left\| u^2_t - u^1_t \right\| _E^q \
+ 4^{q-1} M_t^g K_1 [T^{q-1} + c_q T^{q-2} + D_q ((TK_1)^{\frac{q-1}{2}} + 1)] \int_0^T E \left\| u^1_s - u^0_s \right\| _E^q \, ds \]
\[ \leq 4^{q-1} K_0 E \sup_{0 \leq s \leq T} \left\| u^2(s) - u^1(s) \right\| ^q \
+ 4^{q-1} M_t^g K_1 [T^{q-1} + c_q T^{q-2} + D_q ((TK_1)^{\frac{q-1}{2}} + 1) t \tilde{C}_6(q, T)]. \]

Therefore

\[ E \sup_{0 \leq s \leq T} \left\| u^3(s) - u^2(s) \right\| ^q \leq \frac{4^{q-1} M_t^g K_1 [T^{q-1} + c_q T^{q-2} + D_q ((TK_1)^{\frac{q-1}{2}} + 1) t \tilde{C}_6(q, T)]}{1 - 4^{q-1} K_0} \tilde{C}_7(q, T), \]

where \( \tilde{C}_7(q, T) := \frac{4^{q-1} M_t^g K_1 [T^{q-1} + c_q T^{q-2} + D_q ((TK_1)^{\frac{q-1}{2}} + 1)]}{1 - 4^{q-1} K_0} \).

We can also prove that

\[ E \sup_{0 \leq s \leq T} \left\| u^3(s) - u^2(s) \right\| ^q \leq \frac{(\tilde{C}_7(q, T) t)^2}{2!} \tilde{C}_6(q, T). \]

(21)

Indeed, by repeating the iteration as in, for all \( n \geq 0 \), we obtain

\[ E \sup_{0 \leq s \leq T} \left\| u^{n+1}(s) - u^n(s) \right\| ^q \leq \frac{(\tilde{C}_7(q, T) t)^n}{n!} \tilde{C}_6(q, T). \]

Therefore, for any \( m > n \geq 0 \), we obtain

\[ E \left\| u^m(t) - u^n(t) \right\| ^q \leq \tilde{C}_6(q, T) \sum_{k=n}^{m-1} \frac{(\tilde{C}_7(q, T) t)^k}{k!} \to 0 \text{ as } n \to +\infty. \]
This argument proves that $u^n(t), n \geq 0$ is a Cauchy sequence in $L^q(\Omega,\mathbb{H})$.

**Step 3:** Now we prove the existence and uniqueness of the solution of equation (1). One has that $u^n(t) \to u(t)$ as $n \to \infty$ in $L^q$. The Borel–Cantelli lemma gives us $u^n(t)$ uniformly converge to $u(t)$ as $n \to \infty$, for $t \in (-\infty, T]$. Using Assumption $(A_4)$ and $(A_5)$, for all $t \in [0, T]$, we can prove the next inequality holds:

$$E \| y(t, u^n) - y(t, u_1) \|^q \leq K_0 E \| u^n - u_1 \|_{C^q}^q$$

$$\leq K_0 \sup_{0 \leq s \leq t} \| u(s)^q - u(s) \|^q \to 0, \text{ as } n \to \infty.$$

Note that

$$E \| \int_0^t R(t - s) [f_1(s, u^n) - f_1(s, u)] ds \|^q$$

$$\leq M_1^q T^{q-1} K_1 \int_0^t E \| u^n - u \|_{C^q}^q ds \to 0, \text{ as } n \to \infty,$$

Similarly, we have

$$E \| \int_0^t R(t - s) [g_1(s, u^n) - g_1(s, u)] ds \|^q$$

$$\leq c_q M_1^q T^{q-2} K_1 \int_0^t E \| u^n - u \|_{C^q}^q ds \to 0, \text{ as } n \to \infty,$$

and

$$E \| \int_0^t \int_\mathbb{R} (t - s) [h_1(s, u^n, z) - h_1(s, u, z)] \tilde{N}(dr, dz) \|^q$$

$$\leq D_q \| M_1^q \{ (TK_1)^{q-2} + 1 \} K_1 \int_0^t E \| u^n - u \|_{C^q}^q ds$$

$$\to 0, \text{ as } n \to \infty.$$

Therefore, we take the limits on both sides of (6) with respect to $n$ to obtain

$$u(t) = R(t) \{ \phi(0) + y(0, \phi) \} - y(t, u_1)$$

$$+ \int_0^t R(t - s) f_1(s, u) ds$$

$$+ \int_0^t R(t - s) g_1(s, u) dw(s)$$

$$+ \int_0^t \int_\mathbb{R} (t - s) h_1(s, u, z) \tilde{N}(ds, dz).$$

We can check the uniqueness of the solution by employing the Gronwall lemma, together with a similar argument as that used in the proof of Step 2. This argument completes the proof.

**Remark 3.** We point out that the local solution exists and it is unique on $(-\infty, T]$ for each real number $T > 0$, then, existence of the solution to equation (1) is global, that is, $u(t)$ is defined in $(-\infty, +\infty)$.

### 4. Exponential stability

Here, we use the Gronwall lemma and the properties of the concrete-phase space $C^q$ to obtain the exponential stability for the solutions of the stochastic equation (1) and their maps. Other researchers have studied the stability as well (Dieye et al., 2017, 2019), but their results are based on the stochastic convolution, an approach completely detached from ours.

**Definition 4.1.** The mild solution of (1) is said to be $q$-th moment exponentially stable when $(q \geq 2)$ if, for any initial value $\phi \in C^q$, $F_0$ – measurable, there exist two positive real numbers $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$E \| u(t) \|^q \leq \alpha_1 E \| \phi \|^q \exp(-\alpha_2 t), \text{ for all } t \geq 0.$$

For the sake of notational simplicity, we define the function

$$\tilde{C}_q = \frac{\tilde{C}_q(q)}{1 - 5^{q-1} K_0} \cdot \frac{1}{\lambda}.$$

Now, we can introduce the main result of this paper in the following theorem.

**Theorem 4.1** Suppose that $(A_3)$ and all conditions of Theorem 3.1 hold. Suppose in addition that the next two inequalities hold: $\mu > \lambda$ and

$$\frac{\tilde{C}_q(q)}{1 - 5^{q-1} K_0} \cdot \frac{1}{\lambda} < 0. \quad (23)$$

Then, the mild solution $u(t)$ and the solution maps $u_t$ to equation (1) are $q$-th moments exponentially stable.

**Proof.** $q$-th moment exponential stability of $u(t)$:

Combining (5), $(A_3)$, $(A_4)$ and $(A_5)$, we can write

$$E \| u(t) \|^q \leq 5^{q-1} E \| R(t)(\phi(0) + y(0, \phi)) \|^q + 5^{q-1} E \| y(t, u_1) \|^q$$

$$+ 5^{q-1} E \| \int_0^t R(t - s) f_1(s, u) ds \|^q + 5^{q-1} E \| \int_0^t R(t - s) g_1(s, u) dw(s) \|^q$$

$$+ 5^{q-1} E \| \int_0^t \int_\mathbb{R} (t - s) h_1(s, u, z) \tilde{N}(ds, dz) \|^q. \quad (24)$$
Note that
\[ I_1 := E \| \mathcal{R}(t)(\phi(0) + y(0, \phi)) \|_q^q \leq 2^{q-1}(E \| \mathcal{R}(t)\phi(0)\|_q^q + E \| \mathcal{R}(t)y(0, \phi)\|_q^q) \leq 2^{q-1}M^q e^{-q\lambda t}(1 + K_0)E \| \phi \|_{C_p}^q, \] (25)
and that
\[ I_2 := E \| y(t, u_t)\|_q^q \leq K_0E \| u_t \|_{C_p}^q. \] (26)

By Holder inequality, it yields that
\[ I_3 := E \| \int_0^t \mathcal{R}(t-s)f_1(s, u_s)ds\|_q^q \leq E\int_0^t Me^{-\lambda(t-s)} \| f_1(s, u_s) \| ds\|_q^q \leq M^qE\left(\int_0^t e^{-\lambda(t-s)} \| f_1(s, u_s) \| ds\right)^q \leq M^q\left(\int_0^t e^{-\lambda(t-s)} ds\right)^{q-1}E\int_0^t e^{-\lambda(t-s)} \| f_1(s, u_s)\|_q^q ds. \]

We note that \( t e^{-\lambda(t-s)} < \lambda^{-1}; \) therefore, the last inequality becomes
\[ I_3 \leq M^q(\frac{q-2}{2(q-1)\lambda})^{q-1}K_1\int_0^t e^{-\lambda(t-s)}E \| u_s \|_{C_p}^q ds. \] (27)

Recalling Holder Lemma and Assumption (A3), as before, we use Holder inequality to get that
\[ I_4 := E\left(\int_0^t \int_{\mathbb{R}} \mathcal{R}(t-s)g_1(s, u_s)\omega(s)ds\right)^q \leq c_\omega M^qK_1\left(\frac{q-2}{2(q-1)\lambda}\right)^{q-2}K_2\int_0^t e^{-\lambda(t-s)}E \| u_s \|_{C_p}^q ds. \] (28)

Finally, from Lemma 6.3, Assumptions (A3) and (A5) and by Holder inequality, we can write
\[ I_5 := E\left(\int_0^t \int_{\mathbb{R}} \chi(s, u_s, z)\mathcal{N}(ds, dz)\right)^q \leq D_{\chi}E\left(\int_0^t e^{2\lambda(t-s)} \| u_s \|_{C_p}^q ds + E\int_0^t e^{4\lambda(t-s)} \| u_s \|_{C_p}^q ds\right) \leq D_{\chi}M^qK_1\left(\frac{q-2}{2(q-1)\lambda}\right)^{q-2}K_2\left(\int_0^t e^{-2\lambda(t-s)} \| u_s \|_{C_p}^q ds + E\int_0^t e^{-4\lambda(t-s)} \| u_s \|_{C_p}^q ds\right), \]
which implies that
\[ I_5 \leq D_{\chi}M^qK_1\left(\frac{q-2}{2(q-1)\lambda}\right)^{q-2}K_2\left(\int_0^t e^{-2\lambda(t-s)} \| u_s \|_{C_p}^q ds + E\int_0^t e^{-4\lambda(t-s)} \| u_s \|_{C_p}^q ds\right). \]

If \( q = 2, \) the last two inequalities hold true with convention \( 0^q := 1. \) Substituting (25)–(29) into (24), we obtain
\[ E \| u(t)\|_q^q \leq 10^{q-1}M^q e^{-q\lambda t}(1 + K_0)E \| \phi \|_{C_p}^q + 5^{q-1}K_0E \| u_t \|_{C_p}^q + C_{s}(q)e^{-\lambda t}\int_0^t e^{\lambda s}E \| u_s \|_{C_p}^q ds, \] (30)
where \( C_{s}(q) \) satisfies (22). Multiplying both sides of (30) by \( e^{\lambda t} \) yields
\[ e^{\lambda t}E \| u(t)\|_q^q \leq 10^{q-1}M^q e^{\lambda(t-1)}(1 + K_0)E \| \phi \|_{C_p}^q + 5^{q-1}K_0e^{\lambda t}E \| u_t \|_{C_p}^q + C_{s}(q)e^{-\lambda t}\int_0^t e^{\lambda s}E \| u_s \|_{C_p}^q ds. \] (31)

From properties of the norm \( \| . \|_{C_p} \) (see Appendix), we can write for any \( t \geq 0 \)
\[ \| u_t \|_{C_p}^q \leq e^{-q\mu t} \| \phi \|_{C_p}^q + C_{s}(q)e^{-\lambda t}\int_0^t e^{\lambda s}E \| u(s)\|_q^q ds, \]
which implies that
\[ E\sup_{0 \leq s \leq t} e^{\lambda s} \| u(s)\|_q^q \leq \frac{1}{1 - 10^{q-1}M^q(1 + K_0)} \left[ e^{\lambda(t-1)} + 5^{q-1}K_0\left(1 - e^{\lambda(t-1)}\right)\right]E \| \phi \|_{C_p}^q + C_{s}(q)e^{-\lambda t}\int_0^t e^{\lambda s}E \| u(s)\|_q^q ds. \] (32)

Recall that \( K_0 < \frac{1}{10^{q-1}} \) and \( \lambda - q\mu < 0, \) hence
\[ E\sup_{0 \leq s \leq t} e^{\lambda s} \| u(s)\|_q^q \leq \frac{1}{1 - 10^{q-1}M^q(1 + K_0) + 5^{q-1}K_0}E \| \phi \|_{C_p}^q + C_{s}(q)e^{-\lambda t}\int_0^t e^{\lambda s}E \| u(s)\|_q^q ds. \] (33)
\[
\leq \frac{1}{1 - 5\lambda^{-1} K_0} \left[ 10^{q-1} M^q (1 + K_0) + 5^{q-1} K_0 + \frac{\tilde{C}_8(q)}{q\mu - \lambda} \right] E \| \phi \|_{\mathcal{C}_p}^q \\
+ \frac{\tilde{C}_9(q)}{1 - 5\lambda^{-1} K_0} \int_0^t E \sup_{0 \leq r \leq s} e^{\lambda r} \| u(r) \|_{\mathcal{C}_p}^q \, ds
\]
\[
= \tilde{C}_9(q) + \tilde{C}_{10}(q) \int_0^t E \sup_{0 \leq r \leq s} e^{\lambda r} \| u(r) \|_{\mathcal{C}_p}^q \, ds,
\]
where \( \tilde{C}_9(q) := \frac{\tilde{C}_8(q)}{1 - 5\lambda^{-1} K_0} E \| \phi \|_{\mathcal{C}_p}^q \) and \( \tilde{C}_{10}(q) := \frac{\tilde{C}_9(q)}{1 - 5\lambda^{-1} K_0} \). Using the Gronwall lemma, we get
\[
E \sup_{0 \leq s \leq t} \| u(s) \|_{\mathcal{C}_p}^q \leq \tilde{C}_9(q) e^{\tilde{C}_{10}(q)t},
\]
which implies that
\[
E \| u(t) \|_{\mathcal{C}_p}^q \leq \tilde{C}_9(q) e^{(\tilde{C}_{10}(q)-1)t}. \tag{32}
\]
Therefore, from the condition in (23), the result of the \( q \)-th moment exponential stability of solution \( u(t) \) is satisfied.

\section{Almost surely exponential stability}

\begin{definition}
The mild solution of (1) is said to be almost surely exponentially stable if the following inequality is guaranteed almost surely
\[
\lim_{t \to \infty} \sup_{t \geq 0} \frac{1}{t} \log \| u(t) \| < 0,
\]
for any \( \mathcal{F}_0 \) measurable initial value \( \phi \in \mathcal{C}_p \).
\end{definition}

We present the main result of this section.

\begin{theorem}
Assume that all the conditions of Theorem 4.1 hold true. Then
\begin{enumerate}
\item \( \lim_{t \to \infty} \sup_{t \geq 0} \frac{1}{t} \log \| u(t) \| \leq \frac{\tilde{C}_{10}(q) - \lambda}{q} \varepsilon \)
\item \( \lim_{t \to \infty} \sup_{t \geq 0} \frac{1}{t} \log \| u(t) \|_{\mathcal{C}_p} \leq \frac{\tilde{C}_{10}(q) - \lambda}{q} \varepsilon \)
\end{enumerate}
\end{theorem}

\begin{proof}
Now we show (i) for all \( n \geq 0 \). It follows from Theorem 4.1 that
\[
E \sup_{n \geq t \geq n+1} \| u(t) \|_{\mathcal{C}_p}^q \leq \tilde{C}_9(q) e^{(\tilde{C}_{10}(q)-1)(n+1)} = \tilde{C}_9(q) e^{\tilde{C}_{10}(q)-\lambda} e^{(\tilde{C}_{10}(q)-\lambda)n}.
\]
Let \( I_n \) be the interval \([n, n+1]\), for any \( \varepsilon \in (0, 1) \). Define
\[
\tilde{C}_{10}(q) = \frac{\tilde{C}_9(q)}{1 - 5\lambda^{-1} K_0}.
\]
Since, from assumption, \( 1 - \varepsilon > 0 \) and \( \tilde{C}_{10}(q) - \lambda < 0 \), we can use the Markov inequality to write
\[
\mathbb{P} \left( \sup_{t \in I_n} \| u(t) \|_{\mathcal{C}_p}^q > e^{(\tilde{C}_{10}(q)-1)n\varepsilon} \right) \leq E \| u(t) \|_{\mathcal{C}_p}^q \frac{e^{(\tilde{C}_{10}(q)-1)n\varepsilon}}{e^{(\tilde{C}_{10}(q)-1)n\varepsilon}} \tag{34}
\]
\[
\leq \tilde{C}_9(q) e^{\tilde{C}_{10}(q)-\lambda} e^{(\tilde{C}_{10}(q)-\lambda)n(1-\varepsilon)}, \tag{35}
\]
The rightmost term of (34) is bounded from above by
\[
\sum_{n=0}^{\infty} e^{(1-\varepsilon)(\tilde{C}_{10}(q)-1)n} < \infty.
\]
Therefore, the Borel-Cantelli lemma assures that there exists an integer \( n_0 \) such that, for all \( n \geq n_0 \),
\[
\sup_{t \in I_n} \| u(t) \|^{q} \leq e^{(\hat{C}_{10}(q) - \lambda)n\varepsilon} \quad \text{almost surely.}
\]

Thus, if \( t \in I_n \) and \( n \geq n_0 \), we get
\[
\frac{1}{t} \log \| u(t) \|^{q} \leq \frac{1}{n}(\hat{C}_{10}(q) - \lambda)n\varepsilon = \frac{(\hat{C}_{10}(q) - \lambda)\varepsilon}{q}
\]
almost surely.

It follows that
\[
\lim_{t \to \infty} \sup_{t \in I_n} \frac{1}{t} \log \| u(t) \| \leq \frac{(\hat{C}_{10}(q) - \lambda)\varepsilon}{q}
\]
which shows that (i) is satisfied. The argument to prove (ii) follows analogous reasoning. Namely, one can use the fact that the solution maps \( u_t \) are \( q \)-th moment exponentially stable and can conclude by the result of repeating that previous reasoning. The details are omitted.□

**Remark 4.** The author of (Grimmer, 1982) presents sufficient conditions for the exponential stability of the resolvent operator \((R(t))_{t \geq 0}\). The paper (Grimmer, 1982) shows \( \lambda \) and \( M \) from the contraction of the \( C_0 \)-semigroup \((S(t))_{t \geq 0}\) and the properties of the function \( b \) by using the infinitesimal generator of the translation semigroup.

5. Example

Set \( \mu > 0 \) and \( \theta \in C(\mathbb{R}^+, (-\infty, 0]) \). Consider the following neutral stochastic integro-differential equation with infinite delay and Poisson jumps of the form:

\[
\begin{align*}
\frac{dx(t, \xi)}{dt} &+ e^{\mu(t)}\Gamma(x(t, \theta(t)), \xi)) = -e^{\mu(t)}\varphi_t(x(t, \theta(t)), \xi)) dt \\
&+ \int_{-\infty}^{t} \Gamma_{t+s} \omega(x(t, \theta(t)), \xi) \, ds + \int_{-\infty}^{t} \omega_t(x(t, \theta(t)), \xi) \, ds
\end{align*}
\]

Here, \( \omega(t) \) denotes the standard \( \mathbb{R} \)-valued Wiener process; \( \Gamma, \varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{H}, f : \mathbb{R}^+ \times (-\infty, 0] \times \mathbb{R} \to \mathbb{R} \) and \( h : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions, \( b, \Gamma \in C(\mathbb{R}^+, \mathbb{R}) \) and \( x_0 \in C_\mu \).

Let \( \mathbb{H} = L^2(0, \pi) \) with the norm \( \| \cdot \| \), we define \( A : D(A) \subset \mathbb{H} \to \mathbb{H} \) by \( A = \frac{\partial}{\partial r} \), with domain \( D(A) = H^2(0, \pi) \cap H^1_0(0, \pi) \). It is well known that \( A \) is the infinitesimal generator of a strongly continuous contraction semigroup \((S(t))_{t \geq 0}\) on \( \mathbb{H} \). Thus, \((A_1)\) holds. Let \( Y : D(A) \subset \mathbb{H} \to \mathbb{H} \) be the operator defined by \( Y(t)y = b(t)Ay \) for \( t \geq 0 \) and \( y \in D(A) \). The resolvent operator \((R(t))_{t \geq 0}\) decays exponentially, i.e. \( \| R(t) \| \leq Me^{-\lambda t}, \) see Remark for more details. We now suppose that the next three conditions are valid.

(i) For \( t \geq 0, s \leq 0 \) and \( z \in \mathbb{H}, \Gamma(t, 0) = f(t, s, 0) = g(t, 0) = h(t, 0, z) = 0 \).

(ii) Let \( q \geq 2 \), there exist real numbers \( \alpha_0 \in (0, 1/q) \), \( \alpha_1 > 0 \) such that
\[
\| \Gamma(t, x_1) - \Gamma(t, x_2) \|^{q} \leq \alpha_0 |x_1 - x_2|^{q}
\]
\[
\| g(t, x_1) - g(t, x_2) \|^{q} \vee \int |h(t, x_1, z)| dz
\]
\[
\leq \alpha_1 |x_1 - x_2|^{q}
\]
for \( t \geq 0 \) and \( x_1, x_2 \in \mathbb{R} \), where \(| \cdot |\) denotes the norm of \( \mathbb{R} \).

(iii) Let \( q \geq 2 \), there exists an integrable function \( r : (-\infty, 0] \to [0, +\infty) \) such that
\[
|f(t, s, x_1) - f(t, s, x_2)|^{q} \leq r(s)|x_1 - x_2|^{q}
\]
for \( t \geq 0, s \leq 0 \) and \( x_1, x_2 \in \mathbb{R} \).

In addition, we assume that
\[
\left( \frac{q-1}{q} \right)^{q-1} \int_{-\infty}^{0} r(s) ds \leq \alpha_1.
\]

For \( t \geq 0, \xi \in [0, \pi] \) and \( \phi_1 \in C_\mu \) define the operators \( \gamma_1, \phi_1 : \mathbb{R}^+ \times C_\mu \to \mathbb{H}, g_1, \phi_1 : \mathbb{R}^+ \times C_\mu \to L^2_1(\mathbb{R}, \mathbb{H}) \) and \( h_1 : \mathbb{R}^+ \times C_\mu \to \mathbb{H} \) as follows:
\[
\gamma(t, \phi_1)(\xi) := e^{\mu(t)}\Gamma(t, \phi_1(\theta(t)))(\xi)
\]
\[
g_1(t, \phi_1)(\xi) := e^{\mu(t)}g(t, \phi_1(\theta(t)))(\xi)
\]
\[
h_1(t, \phi_1, z)(\xi) := e^{\mu(t)}h(t, \phi_1(\theta(t)), z)(\xi)
\]
which contain a variable delay, and
\[
f_1(t, \phi_1)(\xi) := \int_{-\infty}^{t} e^{\mu(s)}f(t, s, \phi_1(s)(\xi)) ds
\]
contains a distributed delay. Set \( u(t) = x(t, \xi) \) for all \( t \geq 0 \) and all \( \xi \in [0, \pi] \), and \( \phi(\theta) = x_0(\theta, \xi) \) for all \( \theta \in (-\infty, 0] \) and all \( \xi \in [0, \pi] \). Then, (36) takes the form of the system (1).

Applying (ii) together with the definition of the norm \( \| \cdot \|_{C_\mu} \), we have that
\[
\| \gamma(t, \phi_1) - \gamma(t, \phi_2) \|^{q} \leq \alpha_0 \| \phi_1 - \phi_2 \|^{q}
\]
6. Conclusion

In this paper, we have studied neutral stochastic integro-differential equations with infinite delay and Poisson jumps under global Lipschitz conditions. In this study, we have used successive approximations to show the existence of mild solutions. We also prove the exponential stability of solutions and their maps. It is uncertain whether our approach copes with weaker conditions, such as local Lipschitz and non-Lipschitz conditions. The results in this paper can be seen as an extension of the ones in (Diop et al., 2014) because we consider the infinite-delay case here; in contrast, the authors of (Diop et al., 2014) have considered the finite delay case.

PUBLIC INTEREST STATEMENT

Stochastic processes are much used to represent mathematical models for phenomena and systems that vary randomly. These phenomena and systems may include the growth of bacterial population, price changes in the stock market, extinction and persistence of diseases, movement of a gas molecule and the number of phone calls. Representing such random phenomena motivates the study of stochastic differential equations. Characterizing the stability of stochastic systems has become a central topic in systems sciences. In this paper, we focus on the stability of stochastic integro-differential equations with noise.

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Appendix

The next result is immediate from the definition of the phase space $C_{\mu}$ see (Hino et al., 1991) for more details

**Proposition 6.1.** The phase space $C_{\mu}$ satisfies the next proprieties.

(a) If $u : (-\infty, T) \rightarrow \mathbb{H}, T > 0$ is such that $u_0 \in C_{\mu}$ and $u$ is continuous on $[0, T)$, then for every $t \in [0, T)$, the conditions below hold:

(i) $u_t \in C_{\mu}$

(ii) $\| u(t) \| \leq \| u_t \|_{C_{\mu}}$
(iii) \( \| u_t \|_{C_\mu} \leq e^{-\mu t} \| u_0 \|_{C_\mu} + e^{-\mu t} \sup_{0 \leq s \leq t} e^{\mu s} \| u(t) \| \). (b)

The corresponding history \( t \rightarrow u_t \) of the function \( u \) in (a) is a \( C_\mu \)-valued continuous function in \([0, T]\). (c) The space \( C_\mu \) is complete. (d) Suppose that \( \{ \psi^n \} \) is a Cauchy sequence in \( C_\mu \), and if \( \psi^n(\theta) \) converges to \( \psi(\theta) \) for \( \theta \) on any compact subset of the interval \((-\infty, 0]\), then \( \varphi \in C_\mu \) and \( \| \varphi^n - \varphi \|_{C_\mu} \to 0 \) as \( n \to 0 \).

**Lemma 6.2.** ([20, Thm. 4.36, p. 114]). For a \( \mathcal{L}_2^0 \)-valued predictable process \( \psi \) and for any \( q \geq 1 \), we have the following inequality

\[
\sup_{0 \leq s \leq t} \mathbb{E} \left[ \int_s^t |\psi(l)\,dl|^{2q} \right] \leq (q(2q-1))^q \left( \int_0^T \mathbb{E} (|\psi(s)|^2)^{q/2} \,ds \right)^{q/2}.
\]

**Lemma 6.3.** ([18]). Let \( \psi : [0, \infty) \times \rightarrow \mathbb{H} \) and suppose that

\[
\int_0^T \int_0^T (\psi(s, z) \pi(\,dz\,ds < \infty, \text{for any } q \geq 2. \text{ Then, there exists a positive real number } D_q > 0 \text{ such that}
\]

\[
\mathbb{E} \left( \sup_{0 \leq s \leq T} \left\| \int_0^s (s, z)N(ds, dz) \right\|^q \right) \leq D_q \left\{ \mathbb{E} \left( \int_0^T (\psi(s, z) \pi(ds) |^{q/2} \right)^{q/2} + \mathbb{E} \int_0^T |\psi(s, z)\|^q(\,dz\,)ds \right\}.
\]