Thermodynamics of k-essence

Neven Bilić
Rudjer Bošković Institute, 10002 Zagreb, Croatia
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Abstract

We discuss thermodynamic properties of dark energy using the formalism of field theory at finite temperature. In particular, we apply our formalism to a purely kinetic type of k-essence. We show quite generally that the entropy associated with dark energy is always equal or greater than zero. Hence, contrary to often stated claims, a violation of the null energy condition (phantom dark energy) does not necessarily yield a negative entropy. In addition, we find that the thermal fluctuations of a k-essence field may be represented by a free boson gas with an effective number of degrees of freedom equal to \(c^{-3}\).

1 Introduction

In a number of recent papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], thermal properties of dark energy have been discussed based on the assumption that the dark energy substance is a thermalized ensemble at a certain temperature with an associated thermodynamical entropy. It is usually assumed that this temperature is an intrinsic property of dark energy (DE) rather than the temperature of the heat bath fixed by surrounding matter. A popular model of DE is the so-called k-essence [14] which was originally introduced as a model for inflation [15]. The purpose of this paper is to analyze thermal properties of a grand canonical system described by a purely kinetic k-essence at nonzero temperature. Our aim is to extend and analyze classical solutions to field equations at nonzero temperature.

Dark energy is usually assumed to be barotropic, i.e., described by an equation of state (EOS) in the form \(p = p(\rho)\). Equivalently, one defines a field theory Lagrangian, in which DE is described in terms of a classical self-interacting field coupled to gravity (for a recent review, see [16]). Then, the EOS may be deduced from the energy-momentum tensor obtained from the variational principle. In order to explain an accelerated expansion the DE fluid must violate the strong energy condition [17, 18], which requires \(3p + \rho \geq 0\) together with \(p + \rho \geq 0\). The so-called phantom models of DE have \(p + \rho < 0\), thus violating even the null energy condition.

*E-mail: bilic@thphys.irb.hr
Clearly, from the EOS alone it is not possible to uniquely determine the thermodynamic properties of a system. One simple example is the EOS $p = \rho/3$ which may describe a massless boson gas at $T \neq 0$ (hence $S \neq 0$) but also a massless degenerate Fermi gas at $T = 0$ (hence $S = 0$). A similar situation occurs for any barotropic EOS.

A consistent grand canonical description of DE involves the thermodynamic equations with two variables: the temperature $T$ and chemical potential $\mu$. The chemical potential is associated with a conserved particle number $N$ related to the shift symmetry $\theta \rightarrow \theta + \text{const}$. As we will demonstrate, the resulting thermodynamic equations do not require negative entropy even in the phantom regime when the null energy condition is violated. We show that if there exist a nontrivial, stable configuration which we call condensate characterized by the pressure $p_{cd}$ and the density $\rho_{cd}$, then the $p_{cd} + \rho_{cd}$ term in the expression for entropy is precisely canceled out by the particle-number term. The only nonvanishing contribution to the entropy is due to thermal fluctuations that yield a thermal ensemble similar to a boson gas at nonzero temperature.

We organize the paper as follows: In section 2 we recapitulate the basic hydrodynamics of purely kinetic k-essence. In section 3 we introduce the chemical potential corresponding to the conserved charge which is related to the shift symmetry. Basic thermodynamics is discussed in section 4. The grand canonical and canonical partition functions are derived in section 5 where we also include a brief comment concerning a general k-essence. Discussion and conclusions are given in section 6.

## 2 Purely kinetic k-essence

Consider the action

$$S = \int d^4x \sqrt{-g} \left[ - \frac{R}{16\pi} + \mathcal{L}(X) \right],$$

(1)

where $R$ is the curvature scalar and

$$\mathcal{L} = m^4W(X) ; \quad X \equiv g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}$$

(2)

is the Lagrangian for the scalar field $\theta$ of dimension $m^{-1}$. The dimensionless function $W$ depends only on the dimensionless quantity $X$. Such theories have been exploited as models for inflation and dark matter/energy, for example a purely kinetic k-essence [14, 15, 19, 20] or ghost condensate [21, 22, 23]. The simplest nontrivial example of purely kinetic k-essence is a ghost condensate Lagrangian [21]

$$\mathcal{L}_{gh} = m^4(1 - X)^2,$$

(3)

which has been also studied in the context of dark matter/energy unification [20]. Another example originates from the Dirac-Born-Infeld description of a $d$-brane in string theory with the scalar Born-Infeld type Lagrangian

$$\mathcal{L}_{DBI} = -m^4\sqrt{1 - X}.$$  

(4)

This Lagrangian was derived from the Nambu-Goto action for a $d$-brane moving in the $d+2$-dimensional bulk [24, 25] (for a simple derivation, see also [26]). It may be easily seen that
\[ (4) \] yields the EOS \( p \propto -\rho^{-1} \) of the Chaplygin gas, an exotic fluid which has been suggested as a model for unification of dark energy and dark matter \[27, 28, 29\]. The generalization to \( p \propto -\rho^{-\alpha} \) \((0 \leq \alpha \leq 1)\), was suggested \[30\] and shown to derive from the Lagrangian

\[
\mathcal{L}_{\text{gen}} = -m^4 \left(1 - X^{(1+\alpha)/2\alpha}\right)^{\alpha/(1+\alpha)}
\]

which represents yet another example of purely kinetic k-essence. Subsequently, the term “quartessence” was invented \[31\] to denote unified dark matter/energy models.

From \[2\] the equation of motion for \( \theta \) follows

\[
(\mathcal{L}_X g^{\mu\nu} \theta_{,\mu})_{,\mu} = 0,
\]

where \( \mathcal{L}_X \) denotes the partial derivative with respect to \( X \). Equation \[6\] implies the existence of a conserved current

\[
j^\mu = \frac{2}{m} \mathcal{L}_X g^{\mu\nu} \theta_{,\nu}.
\]

related to the invariance under the constant shift \( \theta \to \theta + \text{const} \) of the scalar field \( \theta \). The shift symmetry reflects a correspondence between a purely kinetic k-essence and a U(1) symmetric complex scalar field theory in the so-called Thomas-Fermi approximation \[28, 32\]. It may be shown \[33\] that the Lagrangian \[2\] is equivalent to

\[
\mathcal{L} = \eta g^{\mu\nu} \Phi^*_{,\mu} \Phi_{,\nu} - m^4 U(\eta|\Phi|^2/m^2),
\]

if the amplitude of the complex scalar field \( \Phi = |\Phi| e^{i\theta} \) varies sufficiently slowly. Here \( \eta = 1 \) for a canonical scalar field and \( \eta = -1 \) for a phantom. The potential \( U(Y) \) is related to \( W(X) \) in \[2\] by a Legendre transformation

\[
W(X) + U(Y) = XY,
\]

with \( X \) and \( Y \) satisfying

\[
X = \frac{dU}{dY},
\]

\[
Y = \frac{dW}{dX}.
\]

Then the current \[7\] corresponds to the Klein-Gordon current

\[
j^\mu_{\text{KG}} = ig^{\mu\nu} (\Phi^* \Phi_{,\nu} - \Phi \Phi^*_{,\nu}),
\]

and hence, the conserved quantity \( N \) corresponds to the usual U(1) charge of the complex scalar field.

Assuming \( X > 0 \) the field \( \theta \) may be regarded as a velocity potential for the fluid 4-velocity

\[
u^\mu = g^{\mu\nu} \theta_{,\nu}/\sqrt{X}
\]

satisfying the normalization condition \( u_{\mu} u^\mu = 1 \). As a consequence, the energy-momentum tensor

\[
T_{\mu\nu} = 2\mathcal{L}_X \theta_{,\mu} \theta_{,\nu} - \mathcal{L} g_{\mu\nu}
\]
derived from the Lagrangian $\mathcal{L}$ in (1) takes the perfect fluid form,

$$T_{\mu \nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu \nu},$$

with the parametric equation of state

$$p = \mathcal{L},$$

$$\rho = 2X \mathcal{L}_X - \mathcal{L},$$

and the speed of sound

$$c_s^2 = \frac{p_X}{\rho_X}.$$  

As before, the subscript $X$ denotes the partial derivative with respect to $X$. A perfect fluid description applies only for $X > 0$. Furthermore, equations (16) and (17) imply that the domains where $\mathcal{L}_X > 0$ correspond to a canonical scalar field ($p + \rho > 0$) and those where $\mathcal{L}_X < 0$ to a phantom. In particular, if in the neighborhood of $X = 0$, $\mathcal{L} \sim \eta X$, the kinetic term is canonical for $\eta = 1$ and is of phantom type for $\eta = -1$. The field that behaves as phantom near $X = 0$ is also known under the name ghost.

The conserved particle number associated with the current (7) is

$$N = \int j^\mu d\Sigma_\mu = \int \Sigma n u^\mu d\Sigma_\mu,$$

where the integration goes over an arbitrary spacelike hyper-surface $\Sigma$ that contains the “particles”. Using the definition (13) for the velocity, we obtain the particle-number density

$$n = \frac{2}{m} \sqrt{X} \mathcal{L}_X.$$

A similar expression, with the right-hand side differing only in a dimensionful constant factor, has been derived also in [12].

### 3 Effective Lagrangian

In this section we construct the effective Lagrangian as a function of the chemical potential $\mu$ associated with the conserved particle-number (19). In the Hamiltonian formulation [34] we choose the hyper surface $\Sigma$ at constant time so that the total number of particles (19) becomes a volume integral

$$N = \frac{2}{m} \int_V \mathcal{L}_X g^{\mu \nu} \theta_\mu dV = \frac{1}{m} \int_V \frac{\partial \mathcal{L}}{\partial \theta_0} dV.$$

Next, we define the grand canonical partition function

$$Z = \text{Tr} e^{-\beta(H - \mu N)} = \int [d\pi] \int_{\text{periodic}} [d\theta] \exp \int_0^\beta d\tau \int dV \left( i\pi \frac{\partial \theta}{\partial \tau} - \mathcal{H}(\pi, \theta, i\pi) + \frac{\mu}{m} \pi \right),$$

where $\tau$ denotes Euclidean time. The Lorentzian and Euclidean times are related by

$$t = -i\tau,$$
as usual. In equation (22) we employed (21) to express the particle number $N$ in terms of the conjugate momentum field

$$\pi = \frac{\partial L}{\partial \theta,0}. \quad (24)$$

The Hamiltonian density $\mathcal{H}$ is defined as usual by the Legendre transformation

$$\mathcal{H}(\pi, \theta, i) = \pi \theta,0 - L(\theta,0, \theta, i), \quad (25)$$

with

$$\theta,0 = \frac{\partial \mathcal{H}}{\partial \pi}. \quad (26)$$

A formal functional integration of (22) over $\pi$ yields the partition function expressed in terms of the effective Euclidean Lagrangian

$$Z = \int [d\theta] \exp - \int_0^\beta d\tau \int dV L_E(\theta, \mu). \quad (27)$$

Since the exact functional integration over $\pi$ is possible only if $\mathcal{H}$ is at most a quadratic function of $\pi$, we apply the saddle point approximation. In this approximation, the path integral is just the integrand evaluated for the field $\pi$ which solves the saddle point condition

$$i \frac{\partial \theta}{\partial \tau} - \frac{\partial \mathcal{H}}{\partial \pi} + \frac{\mu}{m} = 0. \quad (28)$$

Using (26) we find

$$\theta,0 = i \frac{\partial \theta}{\partial \tau} + \frac{\mu}{m}, \quad (29)$$

and with (25) we obtain

$$- L_E = i \pi \frac{\partial \theta}{\partial \tau} - \pi \theta,0 + \frac{\mu}{m} \pi + L(\theta,0, \theta, i) = L(i \frac{\partial \theta}{\partial \tau} + \frac{\mu}{m}, \theta, i). \quad (30)$$

Hence, the effective Euclidean Lagrangian is obtained from the Lagrangian [2] by replacing the derivatives of the field $\theta$ by

$$\theta,\nu \to \theta,\nu + \frac{\mu}{m} \delta^{\nu}_0. \quad (31)$$

Note the difference and similarity with the usual Euclidean field theory prescription [3]

$$\frac{\partial}{\partial \tau} \to \frac{\partial}{\partial \tau} \pm \mu, \quad (32)$$

for a canonical complex scalar field $\Phi$ where the $+$ or $-$ sign is taken when the derivative acts on $\Phi^*$ or $\Phi$, respectively.
4 K-essence thermodynamics

In a grand canonical ensemble the thermodynamic quantities such as pressure \( p \), energy density \( \rho \) and entropy density \( s \) are functions of the temperature \( T \) and chemical potentials \( \mu_i \) associated with conserved particle numbers \( N_i \). For simplicity, we consider only the case of a single conserved particle number \( N \) and its associated chemical potential \( \mu \).

We start from the standard thermodynamical equation

\[
sT = p + \rho - \mu n.
\]

(33)

Clearly, if \( \mu = 0 \) the positivity of entropy requires \( p + \rho \geq 0 \). Hence, ignoring \( \mu \) one could conclude that a phantom field must necessarily yield a fluid with negative entropy. However, this conclusion is incorrect since generally \( \mu \neq 0 \) and the entropy density given by (33) need not be negative. In fact, as we will shortly demonstrate, given arbitrary temperature \( T \) it is always possible to find a range of \( \mu \) such that \( s \geq 0 \).

The entropy and particle-number densities may be expressed as partial derivatives of \( p \)

\[
s = \left. \frac{\partial p}{\partial T} \right|_{\mu}, \quad n = \left. \frac{\partial p}{\partial \mu} \right|_{T}.
\]

(34)

Using this and (33) we also find

\[
p + \rho = T \left. \frac{\partial p}{\partial T} \right|_{\mu} + \mu \left. \frac{\partial p}{\partial \mu} \right|_{T}.
\]

(35)

We now apply these general thermodynamic considerations to purely kinetic k-essence. Equations (16) and (17) define a parametric EOS which is essentially barotropic. In other words, by eliminating parametric dependence, the EOS may be put in the form \( p = p(\rho) \). Obviously, with a barotropic EOS alone one can uniquely determine neither \( T \) nor \( \mu \). However, the k-essence relation

\[
p + \rho = 2X \mathcal{L}_X,
\]

(36)

which follows from (16) and (17), may be used to reduce the arbitrariness in functional dependence on \( T \) and \( \mu \). From (35) combined with (36) it follows that the variable \( X \) as a function of \( T \) and \( \mu \) satisfies a partial differential equation

\[
T \frac{\partial X}{\partial T} + \mu \frac{\partial X}{\partial \mu} = 2X.
\]

(37)

The most general solution to this equation is a homogeneous function of 2nd degree which may be written as

\[
X = \frac{\mu^2}{m^2} f(T/\mu).
\]

(38)

Here \( f \) is an arbitrary positive dimensionless function of \( x \equiv T/\mu \). However, the consistency with (20) places further restrictions on \( f \).

The entropy and particle-number densities may be calculated from \( p \) using (16) and (34). With help of (38) we find

\[
s = \frac{\mu}{m^2} f' \mathcal{L}_X; \quad n = \left(\frac{2\mu}{m^2} f - \frac{T}{m^2} f' \right) \mathcal{L}_X.
\]

(39)
Combining the second equation with (20) we obtain a simple differential equation for $f$

$$xf' - 2f + 2\sqrt{f} = 0,$$

(40)

with the solution

$$f = (Cx + 1)^2,$$

(41)

where $C$ is a constant. Now we require $S = 0$ at $T = 0$, which implies $f'(0) = 0$ and hence $C = 0$. This in turn implies $f(x) = 1$,

$$X = \frac{\mu^2}{m^2},$$

(42)

and $s = 0$. Hence, we come to the conclusion that the thermodynamic quantities, such as pressure and density, derived from the classical kinetic $k$-essence field theory are not temperature dependent and can only depend on the chemical potential $\mu$. Besides, we find that the corresponding entropy is zero. This result is not unexpected since it is well known that classical solutions to a scalar field theory (e.g., nontopological solitons) correspond to saddle point solutions of the Euclidean path integral at nonzero $\mu$ and zero temperature [36, 37]. The existence of stable, nontopological solitons (Q-balls) that have a nonzero value of the conserved charge was proven [38] for a complex canonical scalar field theory of the type (8) for a class of “acceptable” potentials $U$. Hence, based on the Thomas-Fermi equivalence mentioned in section 2, we infer the existence of similar stable classical configurations in a corresponding class of purely kinetic $k$-essence. Otherwise, if such configurations do not exist, then the only stable configuration at zero temperature is trivial, i.e., with $p = \rho = n = 0$, in which case the entropy $S$ is also zero.

However, if DE interacts with thermalized particles from the surroundings, its thermal fluctuations about the classical solutions would be in equilibrium with the heat bath at nonzero temperature. In this case we will have $S > 0$ as usual. In the next section we derive the corresponding thermal contribution to the partition function.

5 Thermal fluctuations

In this section we derive the grand canonical thermodynamic potential and canonical free energy for a self-gravitating kinetic $k$-essence fluid contained in a sphere of large radius in equilibrium at nonzero temperature $T = 1/\beta$. Although the equilibrium assumption implies static metric, our analysis may be also applied to an expanding cosmology in which case the thermodynamic parameters, such as temperature and chemical potential, are functions of the scale parameter $a$. In this case the background metric $g_{\mu \nu}$ generated by the mass distribution is assumed to be a slowly varying function of time on the inverse temperature scale. More precisely, we assume

$$\frac{\partial g_{\mu \nu}}{\partial t} \ll \frac{g_{\mu \nu}}{\beta}.$$  

(43)

We neglect the influence of matter and radiation assuming that their interaction with DE is small and serves only to provide a heat bath at the temperature $T$. 

7
5.1 Grand canonical ensemble

In a grand canonical ensemble we introduce the chemical potential \( \mu \) associated with the conserved particle number \( N \) as in section 3. The partition function is given by

\[
Z = \text{Tr} e^{-\beta(H-\mu N)} = \int [dg][d\theta] e^{-S_g - S_k},
\]

with the Euclidean actions \( S_g \) and \( S_k \) for the gravitational and the k-essence fields, respectively. The gravitational part may be put in the form \([39, 40]\)

\[
S_g = -\frac{1}{16\pi} \int_Y d^4x \sqrt{g_E} R - \frac{1}{8\pi} \int_{\partial Y} d^3x \sqrt{h}(K - K_0),
\]

where \( h \) is the determinant of the induced metric on the boundary, and \( K - K_0 \) is the difference in the trace of the second fundamental form of the boundary \( \partial Y \) in the metric \( g_E \) and the flat metric. The boundary is a timelike tube which is periodically identified in the imaginary time direction with period \( \beta \). Thus, the functional integration assumes the periodicity in imaginary time and the asymptotic flatness of the metric fields. The k-essence Euclidean action is

\[
S_k = \int_Y d^4x \sqrt{g_E} \mathcal{L}_E = -\int_Y d^4x \sqrt{-g} \mathcal{L},
\]

with the Euclidean Lagrangian \( \mathcal{L}_E \) given by \([30]\). The path integral is taken over asymptotically vanishing fields which are periodic in imaginary time \( \tau \) with period \( \beta \). The dominant contribution to the path integral comes from metric and k-essence fields, which are near the classical fields. The metric \( g_{\mu\nu} \) is assumed to be static spherically symmetric, and asymptotically flat, and is positive definite with the Euclidean signature due to the substitution \([23]\).

In order to extract the classical contribution, we decompose \( \theta \) as

\[
\theta(x) = \Theta(x) + m^{-2}\varphi(x),
\]

where \( \Theta \) is a solution to the classical equation of motion which we will call \textit{condensate} and \( \varphi \) describes quantum and thermal fluctuations around \( \Theta \). The factor \( m^{-2} \) is introduced for convenience so that the field \( \varphi \) has the canonical dimension one. The action \( S_k \) splits in two parts \( S_k = S_{cl}[\Theta] + S_{th}[\varphi] \) and if we neglect quantum fluctuations of the metric, the partition function factorizes as

\[
Z = Z_g Z_{cl} Z_{th},
\]

where \( Z_g = e^{-S_g} \) and \( Z_{cl} = e^{-S_{cl}} \) represent the saddle-point gravitational and classical contributions, respectively. The last factor is the contribution due to thermal fluctuations

\[
Z_{th} = \int [d\varphi] e^{-S_{th}[\varphi]}.
\]

First, we calculate the classical contribution. The classical part of the action is given by

\[
S_{cl} = -\int_Y d^4x \sqrt{-g} \mathcal{L}(X).
\]

Here and below

\[
X = g^{\mu\nu}\Theta_{\mu}\Theta_{\nu},
\]
where
\[ \Theta_0 = i \frac{\partial \Theta}{\partial \tau} + \frac{\mu}{m} = \frac{\partial \Theta}{\partial t} + \frac{\mu}{m}, \]  
(52)
\[ \Theta_i = \frac{\partial \Theta}{\partial x_i}. \]  
(53)

By making use of the field equation (6) in the comoving frame, i.e., in the reference frame where the 4-velocity components are
\[ u^\mu = \frac{\delta^\mu_0}{\sqrt{g_{00}}}; \quad u_\mu = \frac{g_{\mu 0}}{\sqrt{g_{00}}}, \]  
(54)

it follows
\[ \Theta = \omega t + \vartheta = -i \omega \tau + \vartheta, \]  
(55)
where \( \omega \) and \( \vartheta \) are constants. Furthermore, using (51) with (52) and comparing with (42) we conclude that \( \omega = 0 \). Thus, the background field \( \Theta \) is constant. However, since \( \Theta_0 \) is given by (52), the quantity \( X \) may still be a function of \( \vec{x} \) and a slowly varying function of time. Equations (51)-(52) give
\[ X = g_{00} \frac{\mu^2}{m^2} = \bar{\mu}^2 \frac{m^2}{m^2}, \]  
(56)
where \( \bar{\mu} = \mu/\sqrt{g_{00}} \) is the local chemical potential. This equation is consistent with equation (42) obtained from purely thermodynamic considerations.

In a spherically symmetric static geometry, regular solutions to equation (6) coupled with Einstein field equations describe dark energy stars [41, 42] at zero temperature. A particular example of Born-Infeld type k-essence stars have recently been studied in [43]. In the context of cosmology, the background is spatially homogeneous isotropic configuration with \( X \) depending on time through cosmological scale dependence.

According to our assumption (43) the variation of \( X \) is small on the time scale comparable with the inverse temperature, i.e., \( \partial X/\partial t \ll X/\beta \). Then, the condensate contribution to the partition function may be written as
\[ \ln Z_\chi = \beta \int_\Sigma d^3x \sqrt{-g} L(X) \]  
(57)
where \( \Sigma \) is a spacelike hyper-surface that contains the condensate. Using this equation we find the net number of particles in the condensate
\[ N = \frac{1}{\beta} \frac{\partial \ln Z_\chi}{\partial \mu} = \int_\Sigma d^3x \sqrt{g_{(3)}} \frac{2\bar{\mu}}{m^2} L_X \]  
(58)
where \( g_{(3)} = \det(-g_{ij}), i, j = 1, 2, 3 \). Alternatively, the covariant definition (19) yields
\[ N = \int_\Sigma d^3x \sqrt{g_{(3)}} n, \]  
(59)
where \( n \) is the particle-number density in the condensate. Therefore, we identify the particle-number density due to the condensate as
\[ n = 2\frac{\bar{\mu}}{m^2} L_X \]  
(60)
which coincides with (20) as it should.

Next, we calculate the thermal contribution to $Z$ starting from equation (49) where the action $S_{\text{th}}[\varphi]$ is derived from $\mathcal{L}(X)$ by keeping only the quadratic term in the expansion of $\mathcal{L}$ in powers of $\varphi$. We find

$$S_{\text{th}} = -m^{-4} \int d^4x \sqrt{-g} f^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi,$$

(61)

where

$$f^{\mu\nu} = \mathcal{L}_X g^{\mu\nu} + 2\mathcal{L}_{XX} g^{\mu\alpha} g^{\nu\beta} \Theta_{\alpha\beta}.$$

(62)

The symbol $\partial_\mu$ in equation (61) denotes the partial derivative with respect to a Lorentzian coordinate $x_\mu$. In particular, $\partial_0 = \partial/\partial t = i\partial/\partial \tau$. The $\mu$ dependence is absorbed in $f^{\mu\nu}$ through the prescription (52). The action describes a massless scalar propagating in an effective (or emergent) acoustic geometry [44, 45, 46, 47] provided the field equation for $\varphi$

$$f^{\mu\nu} \varphi_{;\mu\nu} = 0,$$

(63)

is hyperbolic, i.e., provided the effective metric tensor $f^{\mu\nu}$ has the Lorentzian signature. This holds if and only if

$$\det f^{\mu\nu} = \mathcal{L}_X g^{-1} c_s^{-2} < 0,$$

(64)

where

$$c_s^{-2} = \frac{\mathcal{L}_X + 2\mathcal{L}_{XX}}{\mathcal{L}_X},$$

(65)

is the inverse speed of sound squared. It may be easily verified that the same equation is obtained from the standard hydrodynamic definition [48]

$$c_s^2 = \frac{\partial p}{\partial \rho} \bigg|_{s/n}. $$

(66)

Since $g < 0$, $f^{\mu\nu}$ is Lorentzian if and only if $c_s^2 > 0$. Hence, the hyperbolicity condition (64) is equivalent to the requirement of hydrodynamic stability.

In addition, it seems physically reasonable to require $c_s^2 < 1$ in order to avoid possible problems with causality. However, it has been shown [49, 50] that if k-essence is to solve the coincidence problem there must be an epoch when perturbations in the k-essence field propagate faster than light. In cosmology and astrophysics, it is usually assumed on the basis of causality that the speed of sound cannot exceed the speed of light [17, 34, 51]. Hence, it has been argued [49] that k-essence models which solve the coincidence problem are ruled out as realistic physical candidates for dark energy. In contrast to this, it has been argued [47, 50, 52] that superluminal sound speed propagation in generic k-essence models does not necessarily lead to causality violation and hence, in spite of the presence of superluminal signals on nontrivial backgrounds, the k-essence theories are not less legitimate than General Relativity.

As we are not presently interested in discussing or solving the coincidence problem we stick to $0 < c_s^2 < 1$. This condition is fulfilled if both $\mathcal{L}_X$ and $\mathcal{L}_{XX}$ are simultaneously either positive or negative. In the latter case from (36) we have $p + \rho < 0$ and hence,
our thermodynamic analysis is not restricted only to models that satisfy the null energy condition.

Assuming $0 < c_s^2 < 1$, equation (61) may be put in the form

$$S_{th} = -\frac{1}{2} \int_Y d^4x \sqrt{-G} G^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi.$$  \hspace{1cm} (67)

where

$$G^{\mu\nu} = \frac{2 c_s m^4}{L_X} [g^{\mu\nu} - (1 - \frac{1}{c_s^2}) u_\mu u_\nu].$$ \hspace{1cm} (68)

The matrix $G^{\mu\nu}$ is the inverse of the acoustic metric tensor defined as [44, 45]

$$G_{\mu\nu} = \frac{1}{2} \frac{L_X}{c_s m^4} [g_{\mu\nu} - (1 - c_s^2) u_\mu u_\nu],$$ \hspace{1cm} (69)

where

$$u_\mu = \frac{\Theta_{,\mu}}{\sqrt{X}}; \hspace{1cm} u^\mu = g^{\mu\nu} u_\nu$$ \hspace{1cm} (70)

is the velocity of the fluid.

The determinant $G$ is given by

$$G = \det G_{\mu\nu} = \frac{16 L_X^4}{m^4 c_s^4} g.$$ \hspace{1cm} (71)

Since the metric is static by assumption, i.e., $g_{\mu\nu}$ is independent of $t$ and $g_{0i} = 0$, the same is true for the acoustic metric $G_{\mu\nu}$ and the determinant factorizes as $G = -G_{00} G_{(3)}$ where $G_{(3)} = -\det G_{ij}; i, j = 1, 2, 3$. By making use of the substitution $\tilde{\tau} = \tau \sqrt{G_{00}}$ we obtain

$$S_{th} = \int_{\Sigma} d^3x \sqrt{G_{(3)}} \int_0^{\tilde{\beta}} d\tilde{\tau} \mathcal{L}(\tilde{\tau}, x),$$ \hspace{1cm} (72)

where

$$\mathcal{L}(\tilde{\tau}, x) = \frac{1}{2} [(\partial_\tau \varphi)^2 - G^{ij} \partial_i \varphi \partial_j \varphi].$$ \hspace{1cm} (73)

Here we have introduced the effective inverse temperature

$$\tilde{\beta} = \sqrt{G_{00}} \beta = \sqrt{\frac{G_{00}}{g_{00}}} \beta,$$ \hspace{1cm} (74)

where the quantity $\tilde{\beta} = \sqrt{g_{00}} \beta$ is the usual local inverse temperature. Equation (74) is the well-known Tolman condition for thermal equilibrium in curved space [53, 48]. The parameter $\beta$ retains its usual interpretation as the asymptotic value of the inverse temperature [54].

Using the approach developed in [37] the path integral in (49) may be easily calculated. We obtain the thermal part of the partition function in the form

$$\ln Z_{th} = - \int_{\Sigma} d^3x \sqrt{G_{(3)}} \int d^3q \frac{1}{(2\pi)^3} \ln(1 - e^{-\tilde{\beta} q}).$$ \hspace{1cm} (75)
This expression may be regarded as a proper volume integral

$$\ln Z_{\text{th}} = -\int_{\Sigma} d^3x \sqrt{g(3)} \frac{1}{V} \ln z$$

of the local partition function

$$\ln z = -V \sqrt{\frac{G(3)}{g(3)}} \int \frac{d^3q}{(2\pi)^3} \ln(1 - e^{-\beta q}) = V\pi \sqrt{\frac{G(3)}{g(3)}} \beta^{-3}$$

from which the pressure, energy density, and entropy density may be derived in the usual way:

$$p_{\text{th}} = \frac{1}{V\beta} \ln z = \frac{\pi}{360} c_s^{-3} \beta^{-4},$$

$$\rho_{\text{th}} = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln z = \frac{\pi}{120} c_s^{-3} \beta^{-4},$$

$$s = \beta (p_{\text{th}} + \rho_{\text{th}}) = \frac{\pi}{90} c_s^{-3} \beta^{-3}.$$  

These expressions are the well-known thermodynamic equations that represent a massless relativistic Bose gas in curved space. The factor $c_s^{-3}$ is the effective number of degrees of freedom which depends on the chemical potential for a particular k-essence model. This shows that a sizable fraction of radiation today can be attributed to thermal fluctuations of DE. For example, for the Born-Infeld type k-essence we find

$$c_s^{-3} = (1 - \mu^2/m^2)^{-3/2}.$$  

If we use this model for a dark matter/energy unification, the fit with homogeneous cosmology today would yield the speed of sound squared of the order $c_s^2 = \Omega_\Lambda$ with the effective number of degrees of freedom $\Omega^{-3/2} \approx 1.6$, comparable with 2 for photons.

The gravitational part of the partition function may be calculated from (45) with help of Einstein field equations. Using the result of Gibbons and Hawking for the surface term [39], we obtain

$$\ln Z_g = -\beta M + \int_Y d^4x \sqrt{\gamma} T^0_0,$$

where $M$ is the total mass and $T^\mu_\mu$ is the energy-momentum tensor of k-essence field averaged with respect to the partition function [48]. The averaged energy-momentum tensor may be split up into two parts:

$$T^\mu_\nu = T^\mu_{\text{cd} \nu} + T^\mu_{\text{th} \nu},$$

where the first term on the right-hand side is the classical part which comes from the condensate and the second term represents the thermal fluctuations.

It may be shown that both terms are of the form [15] which characterizes a perfect fluid. Hence, we have

$$T^0_0 = \rho_{\text{cd}} + \rho_{\text{th}}; \quad T^i_i = p_{\text{cd}} + p_{\text{th}},$$

where the thermal pressure and energy density are given by (78) and (79), respectively. The condensate pressure $p_{\text{cd}}$ and energy density $\rho_{\text{cd}}$ are given by (16) and (17), respectively, in
which the quantity $X$ is given by (56). Putting the condensate (57), the thermal (76), and the gravitational (82) contributions together, we find the total grand canonical thermodynamic potential as

$$\Omega(\beta, \mu) \equiv -\frac{1}{\beta} \ln Z = M - \int_{\Sigma} d^3x \sqrt{-g} \left( p_{\text{cd}} + \rho_{\text{cd}} + p_{\text{th}} + \rho_{\text{th}} \right).$$  

(85)

This is just the standard form of the thermodynamic potential

$$\Omega = E - TS - \mu N,$$  

(86)

in which we identify the energy $E$ with the total mass $M$, the $TS$ term with the thermal contribution

$$TS = \int_{\Sigma} d^3x \sqrt{-g} (p_{\text{th}} + \rho_{\text{th}}),$$  

(87)

and the $\mu N$ term with the contribution of the condensate

$$\mu N = \int_{\Sigma} d^3x \sqrt{-g} (p_{\text{cd}} + \rho_{\text{cd}}) = \mu \int_{\Sigma} d^3x \sqrt{g(3)} n.$$  

(88)

Note that the entropy $S$ is strictly positive because the right-hand side of (87) is positive and the quantity $T$ is a positive temperature of the heat bath.

### 5.2 Canonical ensemble

Now consider a self-gravitating k-essence fluid with $N$ particles contained in a two-dimensional sphere of large radius in equilibrium at nonzero temperature $T = 1/\beta$. In a canonical ensemble, instead of the chemical potential $\mu$ we fix the particle number. Thus, a canonical ensemble is subject to the constraint (59) with $N$ fixed.

The free energy of a canonical ensemble may be derived from the grand canonical partition function with the help of the Legendre transform

$$F(\beta, N) = \Omega(\beta, \mu) + \mu N.$$  

(89)

The quantity $\mu$ in this expression is an implicit function of $N$ and $T$, such that for given $N$ and $T$ the constraint (59) is satisfied. From (85) and (59) with (60) it follows that

$$F = M - \int_{\Sigma} d^3x \sqrt{-g} \left( p_{\text{th}} + \rho_{\text{th}} \right).$$  

(90)

The second term on the right-hand side is related to the entropy through (80) and the free energy may be expressed in the familiar form

$$F = M - TS,$$  

(91)

where the total entropy $S$ is defined as a proper volume integral

$$S = \int_{\Sigma} s u^\mu d\Sigma_\mu = \int_{\Sigma} d^3x \sqrt{g(3)} s$$  

(92)

of the entropy density $s$ given by (80).
### 5.3 More general k-essence

The above considerations may be similarly applied to a general class of k-essence models described by

\[ S = \int d^4x \sqrt{-g} \left[ -\frac{R}{16\pi G} + \mathcal{L}(\theta, X) \right] , \]

(93)

with the most general Lagrangian which, in addition to \( X \), depends explicitly on the scalar field \( \theta \). The hydrodynamic quantities \( u_\mu, p, \rho, T_{\mu\nu} \), and \( c_s \) associated with (93) are, as before, defined by (13)-(18). However, unlike in purely kinetic k-essence, the equation of motion for \( \theta \)

\[ (2\mathcal{L}_X g^{\mu\nu} \partial_\mu \theta)_\nu - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \]

(94)

is no longer of the current conservation form and the analysis of sections 3 and 4 does not apply. Here, for example, we cannot establish a correspondence with the canonical complex scalar field theory. As in the case of a real scalar field, because of the absence of a conserved Noether current there exist no stable nontopological solutions [56] although time dependent solutions similar to oscillatons [57, 58], or static unstable configurations similar to unstable scalar solitons [59], are not excluded. The only stable solutions at zero temperature are trivial, i.e., those with \( p = \rho = 0 \) so that the condensate contribution to the partition function is absent.

As there is no conserved particle number the canonical and grand canonical ensembles coincide, i.e., \( F(\beta) = \Omega(\beta) \), and the free energy \( F \) is given by (91). The thermal contribution is of the form (94) where the action \( S_{th}[\varphi] \) is obtained by expanding \( \mathcal{L}(X, \theta) \) in powers of \( \varphi \) with \( \theta = \Theta + m^{-2} \varphi \), as before. Here, \( \Theta \) describes a configuration for which \( p = \rho = 0 \). Using the procedure described in section 5.1 we find the expression similar to (67)

\[ S_{th} = -\frac{1}{2} \int_Y d^4x \sqrt{-G} \left( G^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m_{\text{eff}}^2 \varphi^2 \right) , \]

(95)

where

\[ m_{\text{eff}}^2 = \frac{m^4 c_s}{4L_X^2} \left( 2X \mathcal{L}_{X\Theta\Theta} - \mathcal{L}_{\Theta\Theta} + 2 \frac{\partial f^{\mu\nu}}{\partial \Theta} \Theta_{,\mu\nu} \right) \]

(96)

is the effective mass [17], \( X = g^{\mu\nu} \Theta_{,\mu} \Theta_{,\nu} \), and the matrices \( f^{\mu\nu} \) and \( G^{\mu\nu} \) are defined as in (62) and (68), respectively. The subscripts \( X \) and \( \Theta \) in (96) denote the partial derivatives with respect to \( X \) and \( \Theta \), respectively. The quantity \( c_s \) is the sound speed defined as in (65). The action (95) describes a massive scalar propagating in an effective acoustic geometry provided the condition (64) is met. Again, using the approach developed in [37] we find the expression for the thermal partition function in the form of the proper volume integral (76) over the local partition function

\[ \ln z = -V \sqrt{G^{(3)}} g^{(3)} \int \frac{d^3q}{(2\pi)^3} \ln(1 - e^{-\beta E}) \]

(97)

where \( E = \sqrt{m_{\text{eff}}^2 + q^2} \) and \( g^{(3)} \) and \( G^{(3)} \) are the determinants of the respective spatial metrics defined as in section 5.1. From this partition function we find the usual expressions for the
pressure, energy density, and entropy density of an ideal gas of massive bosons

\[ p_{\text{th}} = \frac{g_{\text{eff}}}{\beta} \int \frac{d^3q}{(2\pi)^3} \ln(1 - e^{-\beta E}), \quad (98) \]

\[ \rho_{\text{th}} = g_{\text{eff}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{1 - e^{-\beta E}}, \quad (99) \]

\[ s = \tilde{\beta}(p_{\text{th}} + \rho_{\text{th}}) \quad (100) \]

with the effective number of degrees of freedom \( g_{\text{eff}} = 1/c_s^3 \) which depends on the model.

### 6 Conclusions

We have derived a grand canonical and canonical description of k-essence type of DE. The thermodynamic equations are generally expressed in terms of two variables: the temperature \( T \) and chemical potential \( \mu \). The chemical potential is associated with a conserved particle number \( N \) related to the shift symmetry. The derived thermodynamic equations show that the entropy does not have to be negative even in the phantom regime, contrary to the claims often stated in the recent literature (see, e.g., [8] and references therein) that a violation of the null energy condition implies negative entropy. We have demonstrated that the entropy is greater or equal to zero and is strictly zero at zero temperature. We have show that if there exist a nontrivial, stable configuration which we call condensate characterized by the pressure \( p_{\text{cd}} \) and the density \( \rho_{\text{cd}} \), then the particle-number term in the expression for entropy cancels out the contribution of \( p_{\text{cd}} \) and \( \rho_{\text{cd}} \). Furthermore, we have shown that the only nonvanishing contribution to the entropy is due to thermal fluctuation analogous to those of a massless boson field. The thermal ensemble behaves as a free massless gas at nonzero temperature with an effective number of degrees of freedom equal to \( c_s^{-3} \). Similarly, thermal fluctuations of a general k-essence field yield an effective free gas of massive bosons.

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