Two-photon exchange corrections to the pion form factor

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Abstract

We compute two-photon exchange corrections to the electromagnetic form factor of the pion, taking into account the finite size of the pion. Compared to the soft-photon approximation for the infrared divergent contribution which neglects hadron structure effects, the corrections are found to be $\lesssim 1\%$ for small $Q^2$ ($Q^2 < 0.1$ GeV$^2$), but increase to several percent for $Q^2 \gtrsim 1$ GeV$^2$ at extreme backward angles.

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As the lightest bound state of quarks, the pion plays a unique role in QCD. On the one hand, its anomalously small mass leads to the identification of the pion with the pseudo-Goldstone mode of dynamical chiral symmetry breaking in QCD. On the other, scattering experiments reveal a rich substructure which can be best described in terms of its quark constituents. The most basic observable which characterizes the structure of the pion is its electromagnetic form factor, \( F_\pi(q^2) \), where \( q^2 \) is the four-momentum transfer squared.

The extractions of the pion form factor in the space-like region (\( Q^2 = -q^2 > 0 \)) from measurements of the pion electroproduction reaction \( ep \rightarrow e\pi^+n \) have recently provided data on the \( Q^2 \) dependence of \( F_\pi \) up to values of \( Q^2 \approx 2.5 \text{ GeV}^2 \), and higher-\( Q^2 \) measurements are planned to \( Q^2 \approx 6 \text{ GeV}^2 \). The main uncertainty in the extraction of \( F_\pi \) is the need to use models to extrapolate the longitudinal electroproduction cross section to the physical pion mass assuming pion pole dominance of the \( t \)-channel process [1]. The pion electroproduction experiments complement low-\( Q^2 \) data obtained by scattering pions from the electrons of a hydrogen target.

As is standard in most electromagnetic scattering analyses, the pion form factor has been obtained from data assuming the validity of the one-photon exchange, or Born, approximation. Recently the accuracy of one-photon exchange approximation has been called into question by the observation of a large discrepancy between the proton electric to magnetic form factor ratio in measurements using Rosenbluth separation and polarization transfer [2]. A number of detailed studies have demonstrated that these can be mostly understood once radiative corrections arising from two-photon exchange are included, in particular those associated with hadron finite-size effects. These findings have prompted exploration of the significance of two-photon exchange in other reactions (see Refs. [3] for reviews).

In this paper we investigate the role of two-photon exchange (TPE) in electromagnetic scattering from the pion. We use the methodology developed for the application of TPE to scattering from the nucleon [4, 5, 6, 7], suitably modified to the scalar case. The analysis of TPE from a spin-0 target is, in fact, considerably simpler than that for spin-\( \frac{1}{2} \) targets.

For the elastic electron–pion scattering process, we follow the notation of Refs. [4, 5] and define the momenta of the initial electron and pion as \( p_1 \) and \( p_2 \), and of the final electron and pion as \( p_3 \) and \( p_4 \), respectively, \( e(p_1) + \pi(p_2) \rightarrow e(p_3) + \pi(p_4) \). The matrix element of
the pion current is given by
\[
\langle \pi(p_4)|J^\mu(0)|\pi(p_2) \rangle = (p_2 + p_4)^\mu F_\pi(q^2),
\] (1)
where \( q^2 = (p_4 - p_2)^2 \). In the one-photon approximation the scattering amplitude is given by
\[
M_0 = \frac{e^2}{q^2} \bar{u}_e(p_3) \gamma_\mu u_e(p_1) (p_2 + p_4)^\mu F_\pi(q^2).
\] (2)

In the target rest frame of the pion the Born cross section can then be written simply as
\[
\frac{d\sigma_0}{d\Omega} = \sigma_{\text{Mott}} F_\pi^2(q^2),
\] (3)
where
\[
\sigma_{\text{Mott}} = \frac{\alpha^2 E_3 \cos^2(\theta/2)}{4E_1^3 \sin^4(\theta/2)}
\] (4)
is the Mott cross section for electron scattering from a point particle, with \( E_1 \) and \( E_3 \) the initial and final electron energies, and \( \alpha = e^2/4\pi \) the electromagnetic fine structure constant.

Including \( O(\alpha) \) radiative corrections leads to a modification of the Born cross section arising from vertex corrections, vacuum polarization, inelastic bremsstrahlung, and two-photon exchange. As discussed in Refs. [4, 5], only the latter lead to a dependence on the scattering angle, or equivalently on the virtual photon polarization parameter \( \varepsilon = (1 + 2(1 + \tau) \tan^2(\theta/2))^{-1} \), where \( \tau = Q^2/4m_\pi^2 \), and \( m_\pi \) is the pion mass. While the scattering angle naturally depends on the reference frame, we can more generally express \( \varepsilon \) in terms of Lorentz invariants as
\[
\varepsilon = \frac{\nu^2 - \tau(1 + \tau)}{\nu^2 + \tau(1 + \tau)},
\] (5)
where \( \nu \equiv p_1 \cdot p_2/m_\pi^2 - \tau \). In the target rest frame we have \( p_1 \cdot p_2 = E_1 m_\pi \).

The total TPE amplitude, including the box and crossed-box diagrams, has the form
\[
M_{\gamma\gamma} = ie^4 \int \frac{d^4k_1}{(2\pi)^4} \frac{L_{\mu\nu} H^{\mu\nu}}{(k_1^2 - \lambda^2)(k_2^2 - \lambda^2)},
\] (6)
where \( k_1 \) and \( k_2 \) are the momenta of the virtual photons, with \( k_1 + k_2 = q \). The parameter \( \lambda \) is introduced as an infinitesimal photon mass in the photon propagators to regulate the infrared (IR) divergences. The leptonic tensor \( L_{\mu\nu} \) is given by
\[
L_{\mu\nu} = \bar{u}_e(p_3) \left[ \frac{\gamma_\mu (p_1 - k_1)^2 - m_e^2 \gamma_\nu}{(p_1 - k_1)^2 - m_e^2} + \frac{\gamma_\nu (p_1 - k_2)^2 - m_e^2 \gamma_\mu}{(p_1 - k_2)^2 - m_e^2} \right] u_e(p_1),
\] (7)
where $m_e$ is the electron mass. The hadronic tensor $H^{\mu\nu}$ in principle contains contributions from all excitations of the initial state. In practice we approximate this by the pion elastic contribution

$$H^{\mu\nu} = F_\pi(k_1^2) F_\pi(k_2^2) \frac{(2p_2 + k_1 + q)\mu(2p_2 + k_1)\nu}{(p_2 + k_1)^2 - m_\pi^2}. \quad (8)$$

The TPE contribution to the cross section is then given by the interference of the TPE amplitude $\mathcal{M}_{\gamma\gamma}$ and the Born amplitude $\mathcal{M}_0$. This can be parametrized in terms of a multiplicative correction $1 + \delta$, where

$$\delta = \frac{2\text{Re}\{\mathcal{M}_0^* \mathcal{M}_{\gamma\gamma}\}}{|\mathcal{M}_0|^2}. \quad (9)$$

The pion form factor is then modified according to

$$F_\pi^2(q^2) \rightarrow F_\pi^2(q^2)(1 + \delta). \quad (10)$$

Experimental analyses of electromagnetic form factor data typically use radiative corrections computed by Mo & Tsai (MT) in the soft-photon approximation \[8\], in which hadronic structure effects are neglected. The TPE corrections are approximated by taking only the IR-divergent contribution at the photon poles, setting $k_1 \rightarrow 0$ and $k_2 \rightarrow 0$ in the numerator of Eq. (6). In this approximation $\mathcal{M}_{\gamma\gamma}$ becomes proportional to the Born amplitude $\mathcal{M}_0$, and the corresponding correction $\delta$ to the Born cross section is independent of hadronic structure (or indeed of the type of hadronic target). Mo & Tsai approximate the remaining loop integration by further reducing it to a 3-point function

$$K(p_i, p_j) = p_i \cdot p_j \int_0^1 dy \ln \left(\frac{p_y^2}{\lambda^2}\right)/p_y^2,$$

where $p_y = p_i y + p_j (1 - y)$, with the total box plus crossed-box contribution given by

$$\delta_{\text{IR}}^{(MT)} = -2\frac{\alpha}{\pi} [K(p_1, p_2) - K(p_3, p_2)]. \quad (11)$$

The logarithmic IR singularity in $\lambda$ is exactly canceled by a similar singularity arising from the bremsstrahlung correction involving the interference between real photon emission from the electron and from the pion.

To quantify the effect of the IR-finite, hadron structure dependent contribution, in Fig. II we show the difference between the full TPE correction $\delta$ and the MT prescription \[8\] as a function of $\varepsilon$ for various $Q^2$. In the numerical calculations we use a monopole parametrization for the “bare” pion form factor in Eq. (6),

$$F_\pi(q^2) = \frac{1}{(1 - q^2/\Lambda^2)}, \quad (12)$$
FIG. 1: (Color online) Two-photon exchange correction to the pion form factor squared as a function of $\varepsilon$ for various $Q^2$, relative to the Mo & Tsai (MT) contribution. A monopole parametrization is used for $F_\pi$ in the full calculation.

with $\Lambda = 770$ MeV corresponding to the $\rho$-meson mass. The loop integrals of Eq. (6) can then be done analytically, and expressed in terms of Passarino-Veltman 2-, 3-, and 4-point functions [9]. In the calculations we use the computer program FEYNCALC [10].

At low $Q^2$ ($Q^2 \sim 0.01$ GeV$^2$) the TPE correction is positive and of the order of 1% at backward angles (small $\varepsilon$), decreasing to zero in the $\varepsilon \to 1$ (forward angle) limit. With increasing $Q^2$ the correction becomes smaller (more negative) up to $Q^2 \sim 1 - 2$ GeV$^2$, especially in the extreme backward region ($\varepsilon \to 0$), but changes sign at intermediate $\varepsilon$. Note, however, that unlike electron-proton scattering, the electron-pion scattering cross section vanishes at the extreme backward angles limit ($\varepsilon = 0$). Above $Q^2 \sim 2$ GeV$^2$ the correction grows once again, reaching $\sim 1\%$ at $Q^2 = 10$ GeV$^2$.

The $Q^2$ dependence is more clearly illustrated in Fig. 2, where $\delta$ is shown for fixed $\varepsilon$ over the range $Q^2 = 1 - 6$ GeV$^2$. Interestingly, the correction is most positive at very small $Q^2 \ll 1$ GeV$^2$ and large $Q^2 \gg 1$ GeV$^2$, reaching its minimum values at $Q^2 \sim 1 - 2$ GeV$^2$. At small $\varepsilon$ the $Q^2$ dependence is seen to change most rapidly.

While the monopole parametrization is known to give a good description of the pion form factor data at low $Q^2$, it overestimates $F_\pi(q^2)$ at larger $Q^2$. An alternative parametrization to the monopole which fits the available data and builds in gauge invariance constraints for the $Q^2 \to 0$ limit and perturbative QCD expectations for the $Q^2 \to \infty$ behavior was given in
FIG. 2: (Color online) Two-photon exchange correction to the pion form factor squared as a function of $Q^2$ for various $\varepsilon$, relative to the Mo & Tsai (MT) \cite{8} contribution, for the monopole (left) and monopole + pQCD (right) parametrizations of $F_\pi(q^2)$.

FIG. 3: (Color online) Two-photon exchange correction to the pion form factor squared as a function of $\varepsilon$ for various $Q^2$, relative to the Maximon & Tjon (MTj) \cite{13} contribution. A monopole parametrization is used for $F_\pi$ in the full calculation.

Ref. [11]. Using this parametrization the TPE correction $\delta$ is shown in Fig. 2 (right panel). As expected, the differences at low $Q^2$ are negligible, but become noticeable at high $Q^2$. The qualitative behavior of the corrections, however, is not affected by the specific form chosen.

We should note that the effects illustrated in Figs. 1 and 2 are not physical, but merely
reflect the accuracy with which the full result can be approximated by the particular prescription for the IR-divergent contribution. It is only relevant because the MT approximation is widely used for applications of radiative corrections in analyses of electron scattering data [1, 2, 12]. An alternative prescription was suggested by Maximon & Tjon (MTj) [13], who also approximate the TPE by the contributions at the photon poles, setting $k_1 \to 0$ and $k_2 \to 0$ in the numerator of Eq. (6). However, no further approximation is made in evaluating the remaining loop integration, which can be written in terms of 4-point functions. In the limit $Q^2 \gg m_e^2$ the 4-point functions simplify significantly, and the TPE correction $\delta_{\text{IR}}^{(\text{MTj})}$ can be written as

$$
\delta_{\text{IR}}^{(\text{MTj})} = -\frac{2\alpha}{\pi} \ln \left( \frac{E_1}{E_3} \right) \ln \left( \frac{Q^2}{\lambda^2} \right). \tag{13}
$$

As before, the logarithmic $\lambda$ singularity is cancelled exactly by the inelastic bremsstrahlung contribution.

The difference between the full, hadron structure dependent result for $\delta$ and the MTj approximation is illustrated in Fig. 4 as a function of $\varepsilon$. At low $Q^2$ ($Q^2 \lesssim 0.1$ GeV$^2$) the correction is similar to that in Fig. 4 for the MT IR prescription, but is generally smaller in magnitude at larger $Q^2$. In particular, it displays a somewhat milder $\varepsilon$ dependence for backward angles at higher $Q^2$. Qualitatively, however, the behavior of the correction as a

![Graph](image-url)
function of \( Q^2 \) is similar for the MTj and MT prescriptions, as Fig. 4 shows, with \( \delta \) most positive at low \( Q^2 \), \( Q^2 \ll 1 \text{ GeV}^2 \) (for all \( \varepsilon \)), decreasing to a minimum at \( Q^2 = 1 - 2 \text{ GeV}^2 \), before rising again for larger \( Q^2 \).

Note that in contrast to the proton form factor case, where the TPE effects give large corrections to the elastic form factors extracted from longitudinal-transverse (LT) separated cross sections at large \( Q^2 \) [14], the TPE corrections to the pion form factor are relatively small. This is related to the fact that electron scattering from a spin-0 target is described by a single form factor, with no LT separation necessary. On the other hand, since \( F_\pi \) is extracted by performing an LT separation of the pion electroproduction cross section, TPE with one photon attached to the pion and the other to the initial proton or final neutron could modify the longitudinal cross section, and may need to be considered.

The reliability of the results at high \( Q^2 \) (\( Q^2 \gg 1 \text{ GeV}^2 \)) may be more questionable since only pion elastic intermediate states are included in this analysis, and we expect excited hadronic states to play a greater role with increasing \( Q^2 \) [6, 7]. However, since the mass difference between the pion and the next excited resonant state, the \( \rho \) meson, is almost 5 times as large as the pion mass, we would not expect these contributions to be significant for the pion. The nonresonant contributions and the off-shell dependence of the pion form factor, on the other hand, may need to be examined in future analyses.

In summary, we have computed the two-photon exchange contribution to the pion electromagnetic form factor arising from the finite size of the pion. Compared with the standard infrared contribution computed in the soft-photon approximation, the hadron structure dependent corrections are \( \lesssim 1\% \) at low \( Q^2 \) (\( Q^2 \sim 0.01 \text{ GeV}^2 \)), but increase to several percent at \( Q^2 \gtrsim 1 \text{ GeV}^2 \) at backward angles. These contributions will need to be taken into account in the treatment of radiative corrections for extractions of the pion form factor from high-precision pion electroproduction data.

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