Fermionic Determinant of the Massive Schwinger Model

M.P. Fry
School of Mathematics
Trinity College
Dublin 2
Ireland

Abstract

A representation for the fermionic determinant of the massive Schwinger model, or $QED_2$, is obtained that makes a clean separation between the Schwinger model and its massive counterpart. From this it is shown that the index theorem for $QED_2$ follows from gauge invariance, that the Schwinger model’s contribution to the determinant is canceled in the weak field limit, and that the determinant vanishes when the field strength is sufficiently strong to form a zero-energy bound state.

PACS numbers: 11.15.Tk, 12.20.Ds
Quantum electrodynamics in two dimensional space time ($QED_2$), otherwise known as the massive Schwinger model, is defined in Euclidean space by the action

$$S[A, \bar{\Psi}, \Psi] = \frac{1}{2} \int d^2x B^2 + \int d^2x \bar{\Psi}(\not{\partial} + m)\Psi,$$

(1)

where $\not{D} = \gamma \cdot (-i\nabla - e\vec{A})$ and $B = F_{01} = \partial_0 A_1 - \partial_1 A_0$. Our designation of $F_{01}$ as a magnetic field is consistent with regarding $S$ as the Hamiltonian for a charged, massive fermion confined to a plane in the presence of a static magnetic field perpendicular to the plane. For definiteness we set $\gamma_0 = -i\sigma_1, \gamma_1 = -i\sigma_2$, where $\sigma_{1,2}$ are the Pauli matrices. The model is super-renormalizable, requiring no infinite renormalization other than a trivial renormalization of the zero-point energy. Hence $e$ and $m$ are finite parameters.

The case when $m=0$, known as the Schwinger model [1], is exactly soluble and has become an important tool for gaining insight into gauge field theories. It continues to generate enormous interest with some fifty preprints per annum connected to the model and variations of it. The literature for the case $m \neq 0$ is sparse, the classic references remaining those in [2]. It is not thought to be exactly soluble. As might be suspected by our interpretation of the massive model’s action, its fermionic determinant determines (after integrating over the fermion mass) the one-loop effective action for $QED_4$ in the presence of smooth, polynomial-bounded, unidirectional, static magnetic fields with fast decrease at infinity [3]. Therefore, $QED_2$ contains information on physics in four dimensions and should not be regarded as just a model.

In this Letter we wish to consider $QED_2$’s gauge invariant fermionic determinant. It will appear in the computation of the theory’s n-point functions as a result of integration over the fermionic degrees of freedom. The first problem is to make sense out of the formal expressions

$$\det^2(1 - S \not{A}) = \frac{\det[(\not{p} - \vec{A})^2 - \sigma_3 B + m^2]}{\det[p^2 + m^2]},$$

(2)

on a Euclidean manifold. Here $S$ is the free (Euclidean) fermion-propagator, and $e$ has been absorbed into $A_\mu$. There are several ways to define determinants of Dirac operators [4], but one of these definitions seems more suited than others to grasp the known simplifications presented by $QED_2$, namely,
the ”propertime regularization“ definition \[5\]. It defines the determinant as

\[
\ln \det(\mathcal{D}^\dagger \mathcal{D} + m^2) = -\int_\epsilon^\infty \frac{ds}{s} Tr[\exp(-s \mathcal{D}^\dagger \mathcal{D})] e^{-sm^2},
\]  

(3)

where \(\epsilon\) is an ultraviolet cutoff which, due to super-renormalizability, can be set to zero later. Because we will always assume \(m^2 > 0\), we feel assured that potential infrared divergences due to the zero modes of \(\mathcal{D}^\dagger \mathcal{D}\) are regulated.

The above definition of the determinant respects gauge invariance. Therefore we should be able to calculate in the Lorentz gauge \(\partial_\mu A_\mu = 0\) which, in two dimensions, allows us to set \(A_\mu = \epsilon_{\mu\nu} \partial_\nu \phi\), with \(B = -\partial^2 \phi\) and \(A = i\sigma_3 \partial \phi\). The antisymmetric tensor \(\epsilon_{\mu\nu}\) is normalized as \(\epsilon_{01} = 1\). Following Alvarez \[5\], we consider the operator

\[
\mathcal{D}_t = -i \partial_t - t A = -ie^{-t\sigma_3 \phi} \partial e^{-t\sigma_3 \phi},
\]

(4)

where \(t\) is a real parameter. Differentiating with respect to \(t\),

\[
\dot{\mathcal{D}} = -\sigma_3 \partial \mathcal{D}_t - \mathcal{D}_t \sigma_3 \phi,
\]

(5)

we calculate

\[
\frac{d}{dt} \ln \det(\mathcal{D}_t^\dagger \mathcal{D}_t + m^2) = 4 \int_\epsilon^\infty ds Tr(\sigma_3 \phi \mathcal{D}_t^2 e^{s\mathcal{D}_t^2}) e^{-sm^2}
\]

\[
= -4Tr(\sigma_3 \phi e^{s\mathcal{D}_t^2}) e^{-cm^2}
\]

\[
+ 4m^2 \int_\epsilon^\infty ds Tr(\sigma_3 \phi e^{s\mathcal{D}_t^2}) e^{-cm^2}.
\]

(6)

Noting that for small \(\epsilon\),

\[
<x|e^{\epsilon\mathcal{D}_t^2}|x> = \frac{1}{4\pi \epsilon}(1 - \epsilon t \sigma_3 \partial^2 \phi + O(\epsilon^2)),
\]

(7)

we obtain our definition of the fermionic determinant,

\[
\ln \left[\frac{\det(\mathcal{D}_t^\dagger \mathcal{D}_t + m^2)}{\det(p^2 + m^2)}\right]^{1/2} = \frac{1}{2\pi} \int d^2 x \phi \partial^2 \phi
\]

\[
+ 2m^2 \int_0^1 dt Tr\{(H_+^{(t)} + m^2)^{-1} - (H_-^{(t)} + m^2)^{-1}\} \phi,
\]

(8)
where \( H_{\pm}^{(t)} = (\vec{P} - t\vec{A})^2 \pm tB \). Note that this definition makes a clean separation between the Schwinger model, the first term, and its massive counterpart. Its perturbative expansion in powers of \( e \) is consistent with known results. Thus, it reproduces the \( O(e^2) \) result for the vacuum polarization graph

\[
\ln \det = \frac{1}{2\pi} \int \phi \partial^2 \phi + 2m^2 \int d^2x < x | \phi \frac{1}{p^2 + m^2} B \frac{1}{p^2 + m^2} | x > 
\]

\[
= -\frac{1}{2\pi} \int \frac{d^2q}{(2\pi)^2} |\hat{B}(q)|^2 \int_0^1 dz \frac{z(1-z)}{q^2z(1-z) + m^2},
\]

(9)

where \( \hat{B} \) is the Fourier transform of \( B \). In addition, graphs of \( O(e^4) \) and higher vanish order by order in the limit \( m^2 = 0 \), in accordance with Schwinger’s original result [1].

We have not integrated by parts in the first term of Eq.(8) as is usually done. In the Lorentz gauge the auxiliary potential \( \phi(x) = -\int d^2y ln|x - \vec{y}|B(\vec{y})/2\pi \) and, assuming that the flux \( \Phi = \int d^2xB \neq 0 \), integration by parts is not justified here.

It is by now evident that we are assuming our potentials are sufficiently smooth with enough fall off at infinity so that everything we have done makes mathematical sense. But note: if \( \Phi \neq 0 \), \( A_\mu \) in the Lorentz gauge behaves like a “winding” field with a \( 1/|x| \) fall off. This will have some consequences below. It might be objected that since \( A_\mu \) is to be integrated over, it should be a random field. Our strategy is to first calculate the determinant in an external field in a convenient gauge, the Lorentz gauge, assuming nice potentials, then switching to whichever gauge and potentials are best suited for making sense out of the remaining integration over \( A_\mu \). Of course, any gauge-invariant constraints imposed on the determinant required, say, to make it non-vanishing, have to be honored by the functional integral.

Within the Lorentz gauge we still have the freedom to shift \( \phi \) by a constant: \( \phi \rightarrow \phi + c \). By definition, the determinant depends on \( A_\mu \) and so is invariant under this shift. Referring to Eq.(8), we have consistency provided

\[
e^2\Phi/2\pi = 2m^2e \int_0^1 dt Tr[(H_+^{(et)} + m^2)^{-1} - (H_-^{(et)} + m^2)^{-1}],
\]

(10)

where we have temporarily restored the coupling \( e \). We can get rid of the \( t \)-integration by setting \( \lambda = et \) and differentiating both sides with respect to
The result is
\[ \Phi/2\pi = m^2 Tr[(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}], \]  
(11)
where we have again absorbed \( \epsilon \) into \( A_\mu \) and \( B \) and set \( H_\pm = (\vec{P} - \vec{A})^2 \mp B \). But the right-hand side of Eq.(11) is independent of \( m^2 \) [6]. One way to see this is to rewrite the right-hand side as
\[ m^2 \int_0^{\infty} ds e^{-sm^2} Tr(e^{-sH_+} - e^{-sH_-}), \]
and appeal to the supersymmetry of the operator pair \( H_\pm \) [7] so that only the zero modes of \( H_\pm \) contribute. This way regulating the trace in Eq.(11) in fact follows from our definition of the determinant [see last term in Eq.(6)] and serves as a reminder of how to deal with any doubt about the trace operation. Thus, gauge invariance leads to the condition
\[ \Phi/2\pi = Tr[P_+(0) - P_-(0)] = n_+ - n_- + \frac{1}{\pi} \sum_l [\delta_+^l(0) - \delta_-^l(0)], \]  
(12)
where \( P_\pm(0) \) are projection operators into the subspace of zero-energy modes of \( H_\pm \); \( n_\pm \) denote the number of zero-energy bound states of \( H_\pm \), and \( \delta_\pm^l(0) \) are the zero-energy phase shifts for scattering by the Hamiltonians \( H_\pm \) in a suitable angular momentum basis \( l \). Equation (12) is just the index theorem for a two-dimensional Euclidean manifold [8,9]. By the Aharonov-Casher theorem [7,10] we know that \( n_+(n_-) \) is \( \{|\Phi|/2\pi\} \), all with \( \sigma_3 = 1(\sigma_3 = -1) \) if \( \Phi > 0(\Phi < 0) \). Here \( \{x\} \) denotes the largest integer strictly less than \( x \) and \( \{0\} = 0 \). This is our first result, that the index theorem for QED\(_2\) follows from gauge invariance.

Let us now write Eq.(8) in the form
\[
\ln \det = -\frac{1}{2\pi} \int d^2 x \phi B + 2 \int_0^1 dt Tr\{[P_+(t)(0) - P_-(t)(0)]\phi\} \\
+ 2m^2 \int_0^1 dt Tr'\{[H_+^{(t)} + m^2]^{-1} - (H_-^{(t)} + m^2)^{-1}]\phi\}, \]  
(13)
where \( P_\pm^{(t)}(0) \) are projection operators into the subspace of zero-energy modes of \( H_\pm^{(t)} \). The prime on the second trace symbol indicates that zero modes are
omitted. Now consider magnetic fields such that $|\Phi|/2\pi \leq 1$ so that there are no bound states. According to Musto et al. [9] we can write the second term in Eq.(13) as

$$\frac{2}{\pi} \int_0^1 dt \text{Tr} \{[\delta^+_t(0) - \delta^-_t(0)]\phi\},$$

where the trace is over scattering states, in the limit of zero energy, of the free Hamiltonian $H_0$ defined by $H_\pm^t = H_0 + V_\pm^t$. The operators $\delta^\pm_t$ are calculated from the S-matrix $S(\lambda) = \exp(2i\delta(\lambda))$ as $\lambda \downarrow 0$. Let us assume further that the magnetic field is sufficiently weak to justify the first Born approximation

$$\delta^+_t(0) - \delta^-_t(0) = -\pi \delta(H_0)(V_+^t - V_-^t) = 2\pi t \delta(H_0).$$

The normalized eigenfunctions of $H_0$ are $\psi_{El}(\vec{r}) = J_l(kr)e^{il\theta}/\sqrt{4\pi}$. Then

$$\frac{2}{\pi} \int_0^1 dt \int_0^\infty dE \sum_{l=-\infty}^{\infty} <El|\delta(H_0)B|El> = \frac{1}{2\pi} \int dE \sum_{l=-\infty}^{\infty} J_l^2(kr)B(\vec{r})\phi(\vec{r})$$

where we used the identity $\sum_{l=-\infty}^{\infty} J_l^2(x) = 1$. This result cancels the first term in Eq.(13) and is our second result, namely that the Schwinger model’s contribution to the determinant of $QED_2$ is canceled in first Born approximation by a contribution from the zero modes in the massive sector. It may be that our weak-field approximation to the second term in Eq.(13) is exact for $|\Phi|/2\pi \leq 1$, but we have not been able to prove this.

Finally, let us increase the magnetic field to $|\Phi|/2\pi > 1$ so that zero-energy bound states of $H_\pm^t$ begin to appear. These states are of the form [7,11] $\psi^{(t)}(x,y) = f_{\pm} \exp(\pm t\phi)$, where $f_{\pm}$ are t-independant polynomials in $x \pm iy$ of degree $< |\Phi|t/2\pi - 1$, and $\phi$ is the auxiliary potential defined above. These zero modes are not in general orthonormal. We define the norm matrix $N_{ij}(t) = (\psi_i^{(t)}, \psi_j^{(t)})$ and the projection kernel on the zero-mode
\(L^2\) subspace \([12]\)

\[
P^{(t)}(x,y) = \sum_{i,j=1}^{n} \psi_i^{(t)}(x)(N^{-1}(t))_{ij}(\psi_j^{(t)})^\dagger(y),
\]

with \(\text{Tr}P^{(t)} = n \equiv \{|\Phi|t/2\pi\}.\) As previously noted, the bound states all have the same chirality, depending on the sign of \(\Phi.\) Their contribution to the second term in Eq.(13) is, for \(|\Phi|/2\pi > 1,\)

\[
\pm 2 \int_0^1 dt \sum_{i,j=1}^{n} (N^{-1}(t))_{ij} \int d^2 x f_i f_j^* \phi e^{\pm 2t\phi}
\]

\[
= \pm 2 \lim_{\epsilon \downarrow 0} \left[ \int_{2\pi(1+\epsilon)/|\Phi|}^{4\pi/|\Phi|} dt \int d^2 x \phi e^{\pm 2t\phi}
\right.
\]

\[
+ \int_{4\pi(1+\epsilon)/|\Phi|}^{6\pi/|\Phi|} dt \sum_{i,j=1}^{2} (N^{-1}(t))_{ij} \int d^2 x f_i f_j^* \phi e^{\pm 2t\phi} + \cdots \right]
\]

The above integrals can be expressed in terms of the norms \(N_{ij}\) and their derivatives with respect to \(t\) after an integration by parts in \(t.\) The result is the following zero-energy bound state contribution to the second term in Eq.(13):

\[
\lim_{\epsilon \downarrow 0} \ln \left[ \frac{N(4\pi/|\Phi|)}{N(2\pi(1+\epsilon)/|\Phi|)} \right.
\]

\[
\times \left. \frac{\det N_{ij}(\frac{6\pi}{|\Phi|})}{\det N_{ij}(\frac{4\pi(1+\epsilon)}{|\Phi|})} \right]_{i,j=1}^2
\]

\[
\cdots \frac{\det N_{ij}(1)}{\det N_{ij}(2\pi \left\{ \frac{|\Phi|}{2\pi} \right\} (1+\epsilon)/|\Phi|)} \mid_{i,j=1}^{(|\Phi|/2\pi}}.
\]

The norm of the first bound state, occurring when \(2 \geq |\Phi|/2\pi > 1,\) diverges as \(\epsilon \downarrow 0:\)

\[
N(2\pi(1+\epsilon)/|\Phi|) = \int d^2 x \exp[\pm 4\pi(1+\epsilon)\phi/|\Phi|]
\]

\[
= 2\pi \int_{R}^\infty dr r \exp[-2(1+\epsilon)\ln r] + \text{finite at } \epsilon = 0,
\]

\[
(18)
\]
where $R$ is large compared to the range of $B$. Hence the logarithm in the expression displayed between Eqs. (17) and (18) becomes minus infinity, thereby causing a zero to appear in the fermionic determinant of $QED_2$, as seen from Eq.(13). This is our third result. Including more bound states does not improve matters. The problem seems to lie with the slow $1/|x|$ fall off of $A_\mu$ in the Lorentz gauge when $\Phi \neq 0$.

We have always kept $m^2 > 0$. If we take the limit $m^2 = 0$, it appears from the foregoing that the zero-mass limit of the fermionic determinant of $QED_2$ does not converge uniformly to that of the Schwinger model, which was calculated with $m^2 = 0$ ab initio. This statement is made subject to the proviso that the $m^2 = 0$ limit is taken before an expansion in powers of $e$ is made; otherwise, as previously noted, we do indeed regain the Schwinger model’s determinant if we take the $m^2 = 0$ limit order by order.

More questions have been raised here than answered, but our results do indicate that $QED_2$ remains a rich and relatively unexplored source of physics.

The author wishes to thank L.O’Raifeartaigh, S.Sen and J.Sexton for useful discussions.

References

[1] J.Schwinger, Phys.Rev.128, 2425(1962).

[2] S.Coleman,R.Jackiw,and L.Susskind, Ann.Phys.(N.Y)93, 267(1975); S.Coleman. *ibid.*101, 239(1976); J.Fröhlich and E.Seiler, Helv.Phys.Acta 49,889(1976).

[3] M.P.Fry, Phys.Rev. D45, 682(1992).

[4] A review of a number of possible definitions is given by E.Seiler in *Gauge Theories: Fundamental Interactions and Rigorous Results*, Proceedings of the International Summer School of Theoretical Physics, Poiana Brasov, Romania, 1981, edited by P.Dita, V.Geogescu, and R.Purice, Progress in Physics Vol.5 (Birkhäuser, Boston, 1982), p.263.

[5] O.Alvarez, Nucl.Phys. B216, 125(1983); B238, 61(1984).See also Appendix 4 of S.Blau, M.Visser and A.Wipf, Int. J. Mod. Phys. A4, 1467(1989).
[6] L.S.Brown, R.D.Carlitz, and C.Lee, Phys.Rev. D16, 417(1977); C.W.Bernard, A.H.Guth, and E.J.Weinberg, ibid.D17, 1053(1978).

[7] H.L.Cycon, R.G.Froese, W.Kirsch, and B.Simon, Schrödinger Operators (Springer, Berlin/Heidelberg/New York, 1987).

[8] D.Boyanovsky and R.Blankenbecler, Phys.Rev. D31, 3234(1985); T.Jaroszewicz, ibid. D34, 3128, (1986).

[9] R.Musto, L.O’Raifeartaigh, and A.Wipf, Phys.Lett.B175, 433(1986).

[10] Y.Aharonov and A.Casher, Phy.Rev. A19,2461(1979).

[11] R.Jackiw, Phys.Rev. D29, 2375(1984).

[12] I.Sachs and A.Wipf, ETH Zürich preprint ETH-TH/91-15.