Abstract

Let $G$ be a group. We define the \textit{coprime graph of subgroups} of $G$, denoted by $\mathcal{P}(G)$, is a graph whose vertex set is the set of all proper subgroups of $G$, and two distinct vertices are adjacent if and only if the order of the corresponding subgroups are coprime. In this paper, we study some connections between algebraic properties of a group and graph theoretic properties of its coprime graph.

\textbf{Keywords:} Coprime graph, finite groups, connectedness, independence number, clique number, planar.

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1 Introduction

Graph theory provide tools to study the algebraic properties of algebraic structures. In particular, there are several graphs associated with groups to study some specific properties of groups, for instance, intersection graph of subgroups of groups, prime graph of groups, non-commuting graphs of groups and permutability graph of subgroups of groups (See [1], [4], [8], [10] and the references therein). In [11], Sattanathan and Kala defined the order prime graph of a group $G$, which is a graph having the set of all elements of $G$ as its vertices, and two distinct vertices are adjacent if and only if the orders of the corresponding subgroups are coprime. They have studied some properties of this graph. This graph was further investigated by Xuanlong Ma et al [7] and Hamid Reza Dorbidi [3]. They called the order prime graph of a group as the coprime graph of a group.

The relation of coprimeness of the orders of the subgroups of a group plays a significant role in the determination of the structural properties of that group. Also the relation of coprimeness is not transitive on the set of all proper subgroups of a group. In this paper, we define the \textit{coprime graph of subgroups} of $G$, denoted by $\mathcal{P}(G)$. It is a graph having all the proper subgroups of $G$ as its vertices, and two distinct vertices $H$ and $K$ are adjacent if and only if $|H|$ and $|K|$ are coprime.
Now we recall some basic definitions and notations of graph theory. We use the standard terminology of graphs (e.g., see [5]). Let $G$ be a simple graph. $G$ is said to be $k$-partite if the vertex set of $G$ can be partitioned to $k$ sets such that no two vertices in same partitions are adjacent. A complete $k$-partite graph, denoted by $K_{n_1,n_2,\ldots,n_k}$, is a $k$-partite graph having partition sizes $n_1, n_2, \ldots, n_k$ such that every vertex in each partition is adjacent with all the vertices in the remaining partitions. In particular, $K_{1,n}$ is called a star. A graph whose edge set is empty is called a null graph or totally disconnected graph. $K_n$ denotes the complete graph on $n$ vertices. $P_n$ and $C_n$ respectively denotes the path and cycle with $n$ edges. We denote the degree of a vertex $v$ in $G$ by $\deg_G(v)$. A graph is said to be connected if any two vertices of it can be joined by a path.

The diameter of a connected graph is the maximum of the length of the shortest path between any pair of vertices. A tree is a connected graph with out cycles. $G$ is said to be $H$-free if $G$ has no subgraph isomorphic to $H$. The girth of $G$, denoted by $girth(G)$, is the length of its shortest cycle, if it exist; other wise $girth(G) = \infty$. An independent set of $G$ is a subset of $V(G)$ having no two vertices are adjacent. The independence number of $G$, denoted by $\alpha(G)$, is the cardinality of the largest independent set. A clique of $G$ is a complete subgraph of $G$. The clique number $\omega(G)$ of $G$ is the cardinality of a largest clique in $G$. The chromatic number $\chi(G)$ of $G$ is the smallest number of colours needed to colour the vertices of $G$ such that no two adjacent vertices gets the same colour. $G$ is said to be weakly $\chi$-perfect if $\omega(G) = \chi(G)$. A graph is said to be planar, if it can be drawn in the plane, so that no two lines intersect except at the vertices; otherwise it is said to be nonplanar. A graph is called unicyclic, if it contains exactly one cycle.

For any integer $n > 1$, $\pi(n)$ denotes the set of all prime divisors of $n$. If $G$ is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The number of Sylow $p$-subgroups of a group $G$ is denoted by $n_p(G)$. We denote the order of an element $a \in \mathbb{Z}_n$ by $ord_n(a)$. Moreover, through out this paper, $p$, $q$, $r$, $s$ denotes the distinct primes.

Since the only groups having no proper subgroups are the trivial group, and the groups of prime order, it follows that, we can define $\mathcal{P}(G)$ only when the group $G$ is neither the trivial group nor the group of prime order. So, unless otherwise mentioned, throughout this paper we consider only groups other than the trivial group, and the groups of prime order.

We use only elementary methods. In Section 2, we classify all the finite groups whose coprime graph of subgroups are one of totally disconnected, bipartite, connected, complete, complete bipartite, tree, star or path, and show that the coprime graph of subgroups of a finite group can not be a cycle. For a finite group $G$, we obtain the independence number, clique number, chromatic number, diameter, girth of $\mathcal{P}(G)$, and show that $\mathcal{P}(G)$ is weakly $\chi$-perfect. Moreover, we obtain the degrees of vertices of $\mathcal{P}(\mathbb{Z}_n)$, and we show that every simple graph is an induced subgraph of $\mathcal{P}(\mathbb{Z}_n)$, for some $n$.

In Section 3, we classify all the finite groups whose coprime graph of subgroups of groups are one of planar, $K_{2,3}$-free, $K_{1,4}$-free, $K_{1,3}$-free, $K_{1,2}$-free, unicyclic.
2 Some results on coprime graph of subgroups of groups

Theorem 2.1. Let $G_1$ and $G_2$ be two groups. If $G_1 \cong G_2$, then $\mathcal{P}(G_1) \cong \mathcal{P}(G_2)$.

Proof. Let $f : G_1 \to G_2$ be a group isomorphism. Define a map $\psi : V(\mathcal{P}(G_1)) \to V(\mathcal{P}(G_2))$ by $\psi(H) = f(H)$, for every $H \in V(\Gamma(G_1))$. Since a group isomorphism preserves the order of subgroups, so it follows that $\psi$ is a graph isomorphism. \qed

Remark: The converse of the above Theorem 2.1 is not true, for if $G_1 \cong \mathbb{Z}_{p^3}$ and $G_2 \cong \mathbb{Q}_8$, then the number of proper subgroups of $G_1$ is four and their orders are $p$, $p^2$, $p^3$, $p^4$; the number of proper subgroups of $G_2$ is four and their orders are 4, 4, 4, 2. Here $\mathcal{P}(G_1) \cong \mathcal{K}_4 \cong \mathcal{P}(G_2)$, but $G_1 \not\cong G_2$.

Theorem 2.2. Let $G$ be a group of order $p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, where $p_i$’s are distinct primes, $\alpha_i \geq 1$. Then

1. $\mathcal{P}(G)$ is $k$-partite;
2. $\alpha(\mathcal{P}(G)) = \max_i |\mathcal{B}_i|$, where for each $i = 1, 2, \ldots, k$, $\mathcal{B}_i$ is the set of all proper subgroups of $G$ whose order is divisible by $p_i$;
3. $\omega(\mathcal{P}(G)) = k = \chi(\mathcal{P}(G))$;

In particular, $\mathcal{P}(G)$ is weakly $\chi$-perfect.

Proof. Let $\mathcal{A}_1$ be the set of all proper subgroups of $G$ whose order is divisible by $p_1$. For each $i \in \{2, 3, \ldots, k\}$, let $\mathcal{A}_i = \{H \mid H$ is a proper subgroup of $G$ such that $p_i$ divides $|H|\} - \bigcup_{j=1}^{i-1} \mathcal{A}_j$. Then clearly the collection $\{\mathcal{A}_i\}_{i=1}^k$ forms a partition of the vertex set of $\mathcal{P}(G)$. Also no two vertices in a same partition are adjacent in $\mathcal{P}(G)$. Moreover, $k$ is the minimal number such that a $k$-partition of the vertex set of $\mathcal{P}(G)$ is having this property, since $\pi(G) = k$. It follows that $\mathcal{P}(G)$ is $k$-partite.

Now for each $i = 1, 2, \ldots, k$, let $\mathcal{B}_i$ be the set of all proper subgroups of $G$ whose order is divisible by $p_i$. Clearly each $\mathcal{B}_i$ is a maximal independent set of $\mathcal{P}(G)$. Thus $\alpha(\mathcal{P}(G)) = \max_i |\mathcal{B}_i|$. For each $i = 1, 2, \ldots, k$, $G$ has a subgroup of order $p_i$. Then the set having one subgroup from each of these orders forms a clique in $\mathcal{P}(G)$. Since $\mathcal{P}(G)$ is $k$-partite, it follows that $\omega(\mathcal{P}(G)) = k$. Obviously, $\chi(\mathcal{P}(G)) = k$. Weakly $\chi$-perfectness of $\mathcal{P}(G)$ follows from the definition. \qed

Corollary 2.1. Let $G$ be a group with $\pi(G) = k$. Then

1. $\mathcal{P}(G)$ is totally disconnected if and only if $k = 1$;
2. $\mathcal{P}(G)$ is bipartite if and only if $k = 1, 2$.

Theorem 2.3. Let $G$ be a finite group. Then
(1) \( \mathcal{P}(G) \) is complete bipartite if and only if \( G \) is isomorphic to one of \( \mathbb{Z}_{pq}, \mathbb{Z}_q \rtimes \mathbb{Z}_p \), \( (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q \) or \( A_4 \);

(2) The following are equivalent:

(a) \( G \cong \mathbb{Z}_{pq} \) or \( \mathbb{Z}_q \rtimes \mathbb{Z}_p \);
(b) \( \mathcal{P}(G) \) is a tree;
(c) \( \mathcal{P}(G) \) is a star.

(3) The following are equivalent:

(a) \( G \cong \mathbb{Z}_{pq} \);
(b) \( \mathcal{P}(G) \) is complete;
(c) \( \mathcal{P}(G) \) is a path.

Proof. In view of part (2) of Corollary 2.1, to prove parts (1), (2), and (a) ⇔ (c) of (3) of this theorem, it is enough to consider the groups of order \( p^a \) and \( p^a q^3 \).

If \( |G| = p^a \), then by Corollary 2.1(1), \( \mathcal{P}(G) \) is totally disconnected and so it is neither complete bipartite nor a tree.

Let \( |G| = pq \) with \( p < q \). Then \( G \cong \mathbb{Z}_{pq} \) or \( \mathbb{Z}_q \rtimes \mathbb{Z}_p \). If \( G \cong \mathbb{Z}_{pq} \), then \( \mathcal{P}(G) \cong K_2 \) and so \( \mathcal{P}(G) \) is a path. If \( G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p \), then \( G \) has an unique subgroup of order \( q \), and \( q \) subgroups of order \( p \); also these are the only proper subgroups of \( G \). It follows that \( \mathcal{P}(G) \cong K_{1,q} \) and so \( \mathcal{P}(G) \) is complete bipartite, star; but not a path.

Let \( |G| = p^2 q \). Suppose \( G \) is abelian, then \( G \) has a subgroup of order \( pq \) and so \( \mathcal{P}(G) \) is disconnected. It follows that \( \mathcal{P}(G) \) is neither complete bipartite nor a tree. Now assume that \( G \) is non-abelian. Here we use the classification of groups of order \( p^2 q \) given in [2, p. 76-80].

Case 1: \( p < q \):

Case 1a: \( p \nmid (q - 1) \). By Sylow’s Theorem, it is easy to see that there is no non-abelian group in this case.

Case 1b: \( p \mid (q - 1) \) but \( p^2 \nmid (q - 1) \). In this case, there are two non-abelian groups.

The first group is \( G_1 := \mathbb{Z}_q \times \mathbb{Z}_p^2 = \langle a, b | a^q = b^p = 1, bab^{-1} = a^i, ord_q(i) = p \rangle \). Here \( G_1 \) has an unique subgroup of order \( q \); unique subgroup of order \( pq \); \( q \) subgroups of order \( p^2 \) and unique subgroups of order \( p \); also these are the only proper subgroups of \( G_1 \). Therefore, \( \mathcal{P}(G_1) \cong K_{1,q+1} \cup K_1 \). So \( \mathcal{P}(G_1) \) is disconnected. Hence \( \mathcal{P}(G_1) \) is neither complete bipartite nor a tree.

The second group is \( G_2 := \langle a, b, c | a^q = b^p = c^p = 1, bab^{-1} = a^i, ac = ca, bc = cb, ord_q(i) = p \rangle \). Here \( G_2 \) has a subgroup \( \langle a, c \rangle \) of order \( pq \) and so \( \mathcal{P}(G_2) \) is disconnected. Hence \( \mathcal{P}(G_2) \) is neither complete bipartite nor a tree.

Case 1c: \( p^2 \mid (q - 1) \). In this case, we have both groups \( G_1 \) and \( G_2 \) from Case 1b together with the group \( G_3 := \mathbb{Z}_q \times_2 \mathbb{Z}_p^2 = \langle a, b | a^q = b^{p^2} = 1, bab^{-1} = a^i, ord_q(i) = p^2 \rangle \). Here \( G_3 \) has a unique subgraph of order \( q \); unique subgroup of order \( pq \); \( q \) subgroups of order \( p^2 \), and \( q \) subgroups of order \( p \). Also
there are three non-abelian groups of order 12: $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z}$. Therefore, $\mathcal{P}(G_3) \cong (K_1 + K_2) \cup K_1$ and so $\mathcal{P}(G_3)$ is disconnected. Hence $\mathcal{P}(G_3)$ is neither complete bipartite nor a tree.

Case 2: $p > q$

Case 2a: $q \not| (p^2 - 1)$. Then there is no non-abelian subgroups.

Case 2b: $q | (p - 1)$. In this case, we have two groups.

The first one is $G_4 := \langle a, b | a^2 = b^q = 1, bab^{-1}, ord_p(i) = q \rangle$. Here $G_4$ has a unique subgroup of order $p^2$; unique subgroup of order $p$; $p$ subgroups of order $pq$; $p^2$ subgroups of order $q$. Also these are the only proper subgroups of $G_4$. Therefore, $\mathcal{P}(G_4) \cong (K_2 + K_p) \cup \overline{K}_p$ and so $\mathcal{P}(G_4)$ is disconnected. Hence $\mathcal{P}(G_4)$ is neither complete bipartite nor a tree.

Next we have the family of groups $\langle a, b, c | a^p = b^q = c^q = 1, cac^{-1} = a^i,cbc^{-1} = b^j, ab = ba, ord_p(i) = q \rangle$. There are $(q + 3)/2$ isomorphism types in this family (one for $t = 0$ and one for each pair $\{x, x^{-1}\}$ in $F_p^\times$). We will refer to all of these groups as $G_{5(t)}$ of order $p^2q$. Here $G_{5(t)}$ has a subgroup $\langle a, c \rangle$ of order $pq$ and so $\mathcal{P}(G_{5(t)})$ is disconnected. Hence $\mathcal{P}(G_{5(t)})$ is neither complete bipartite nor a tree.

Case 2c: $q | (p + 1)$. In this case, we have only one subgroup of order $p^2q$, given by $G_6 := (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q = \langle a, b, c | a^p = b^q = c^q = 1, ab = ba, cac^{-1} = a^i,cbc^{-1} = b^j \rangle$, where $(i, j)$ has order $q$ in $GL_2(p)$. Here $G_6$ does not have a subgroup of order $pq$. But $G_6$ has an unique subgroup of order $p^2$; $p + 1$ subgroups of order $p$; $p^2$ subgroups of order $q$, also these are the only proper subgroups of $G_6$. Hence $\mathcal{P}(G_6) \cong K_{p+2} + K_{p^2}$, which is complete bipartite; but which is not a tree.

Note that if $(p, q) = (2, 3)$, then $\mathcal{P}(G_5)$ and $\mathcal{P}(G_6)$ are both disconnected. Hence $\mathcal{P}(G)$ is neither complete bipartite nor a tree.

If $|G| = p^\alpha q^\beta$, $\alpha, \beta \geq 2$, then $G$ has a subgroup with prime index, since $G$ is solvable and so $\mathcal{P}(G)$ is disconnected. Hence $\mathcal{P}(G)$ is neither complete bipartite nor a tree.

Combining all the above cases together, the proof of parts (1), (2), and (a) $\iff$ (c) of (3) of this theorem follows.

Now, we prove (a) $\iff$ (b) of part (3): Clearly (a) $\Rightarrow$ (b). So assume that $\mathcal{P}(G)$ is complete. Then by Theorem 2.2 each partition $\mathcal{A}_i$, $i = 1, 2, \ldots, k$ of $\mathcal{P}(G)$ must contain exactly one subgroup of distinct prime order and so these subgroups are normal in $G$. If $k > 3$, then $G$ must contain a subgroup whose order is a product of $k$ distinct primes, so this subgroup is an isolated vertex in $\mathcal{P}(G)$, which is not possible. Hence $k = 2$ and so by part (1) of this theorem, it turns out that $G \cong \mathbb{Z}_{pq}$. This completes the proof.

\[ \square \]

**Theorem 2.4.** If $G$ is a finite group, then $\mathcal{P}(G) \not\cong C_n$, for $n \geq 3$.

**Proof.** Suppose $\mathcal{P}(G)$ is the cycle $H_1 - H_2 - \cdots - H_n - H_1$ of length $n$. Since $(|H_1|, |H_2|) = 1 = (|H_2|, |H_3|)$, so without loss of generality, we assume that, $|H_1| = p$, $|H_2| = q$ and $|H_3| = r$ or
Suppose $Sylow$ $q$-subgroups of $G$ partition contains only the $Sylow$ $q$-subgroup $H$. Let $P$ be a group of order $p^a$. If $|H_3| = r$, then $H_1$, $H_2$, $H_3$ are adjacent and so $P(G)$ is complete, which is not possible, by Theorem 2.3(3). If $|H_3| = p^a$, then $|H_3|, |H_4| = 1$ implies that $|H_4| = q^β$ or $r$. If $|H_4| = r$, then $H_1$, $H_2$, $H_4$ are adjacent, which is not possible. So we have $|H_4| = q^β$. Then $(|H_1|, |H_4|) = 1$ and so $H_1$ and $H_4$ are adjacent in $P(G)$. It follows that $n = 4$ and $|G| = p^a q^β$, $α, β ≥ 1$. Now we check the existence of such a group. If $α + β ≥ 4$, then $G$ has at least five proper subgroups, which is not possible. If $α + β ≤ 3$, then $|G| = p^2 q$ or $pq$. In this case, we have shown in the proof of Theorem 2.3 that $P(G)$ can not be a cycle. This completes the proof.

**Theorem 2.5.** Let $G$ be a finite group. Then $P(G)$ is connected if and only if $G$ does not have a proper subgroup $H$ with $π(H) = π(G)$. In this case, $diam(P(G)) ∈ \{1, 2, 3\}$.

**Proof.** Suppose $G$ has a subgroup, say $H$ with $π(H) = π(G)$. Then $|H|$ is not relatively prime to any other subgroups of $G$. Therefore, $P(G)$ is disconnected. Conversely, assume that $G$ does not have a subgroup $H$ with $π(H) = π(G)$. Suppose $P(G)$ is complete, then $P(G)$ is connected and $diam(P(G)) = 1$. Now assume that $P(G)$ is not complete. Let $H$ and $K$ be two non-adjacent vertices in $P(G)$. Then by assumption, $π(H) ≠ π(G)$ and $π(K) ≠ π(G)$, and so there exist $p_i, p_j ∈ P(G)$ such that $p_i /∈ π(H)$ and $p_j /∈ π(K)$. If $p_i = p_j$, then there is a path $H − H_1 − K$, where $H_1$ is a subgroup of $G$ of order $p_i$. If $p_i ≠ p_j$, then there is a path $H − H_1 − H_2 − K$, where $H_1, H_2$ are subgroups of $G$ of orders $p_i, p_j$ respectively. It follows that $P(G)$ is connected and $diam(P(G)) ≤ 3$. Note that $diam(P(Z_{pq})) = 1$, $diam(P(A_4)) = 2$ and $diam(P(Z_{pqr})) = 3$, so it shows that the diameter of $P(G)$ takes all the values in $\{1, 2, 3\}$. This completes the proof.

From the above theorem, for a given finite group $G$, if $P(G)$ is disconnected, then $G$ has a proper subgroup $H$ with $π(H) = π(G)$. It turns out that such a subgroup $H$ will be an isolated vertex of $P(G)$. As a consequence, we have the following result.

**Corollary 2.2.** Let $G$ be a finite group. If $P(G)$ is disconnected, then $P(G) ≅ G ∪ K_r$, where $G$ is a connected component of $P(G)$, and $r$ is the number of proper subgroups $H$ of $G$ with $π(H) = π(G)$.

**Theorem 2.6.** If $G$ is a finite group, then $girth(P(G)) ∈ \{3, 4, ∞\}$.

**Proof.** Let $G$ be a group of order $p_1^{α_1} p_2^{α_2} \cdots p_k^{α_k}$, where $p_i$'s are distinct primes and $α_i ≥ 1$. If $k ≥ 3$, then any three subgroups of $G$ of distinct prime orders are mutually adjacent in $P(G)$ and so $P(G)$ contains $C_3$ as a subgraph. It follows that $girth(P(G)) = 3$. If $k ≤ 2$, then by Corollary 2.1(2), $P(G)$ is bipartite and so $P(G)$ can not contain an odd cycle. Now we consider the following cases:

**Case a:** $|G| = p^α q^β$, $α, β ≥ 2$. Here $G$ has subgroups of orders $p, p^2, q, q^2$, let them be $H_1, H_2, H_3, H_4$ respectively. Then $P(G)$ contains the cycle $H_1 − H_3 − H_2 − H_4 − H_1$ and so $girth(P(G)) = 4$.

**Case b:** $|G| = p^α q$, $α ≥ 2$. Suppose Sylow $q$-subgroup of $G$ is not unique, then $G$ has at least two Sylow $q$-subgroup, let them be $H_1, H_2$ and $G$ has subgroups of order $p, p^2$, let them be $H_3, H_4$ respectively. Then $P(G)$ contains the cycle $H_1 − H_3 − H_2 − H_4 − H_1$ and so $girth(P(G)) = 4$.

Suppose Sylow $q$-subgroup of $G$ is unique, then in the bipartition of the vertex set of $P(G)$, one partition contains only the Sylow $q$-subgroup of $G$ and another partition contains the remaining subgroups of $G$. It follows that $P(G)$ does not contains a cycle, so $girth(P(G)) = ∞$. 

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Case c: $|G| = pq$. By Theorem 2.3(3), $\mathcal{P}(G)$ is a path and so $girth(\mathcal{P}(G))$ is $\infty$.

Case d: $|G| = p^a$. By Corollary 2.1(1), $\mathcal{P}(G)$ is totally disconnected and so $girth(\mathcal{P}(G))$ is $\infty$.

Proof follows by combining all the above cases together.

Theorem 2.7. Let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i$’s are distinct primes and $\alpha_i \geq 1$. If $H$ is a proper subgroup of $\mathbb{Z}_n$ of order $p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then $deg_{\mathcal{P}(\mathbb{Z}_n)}(H) = \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (\alpha_j + 1) - 1$.

Proof. It is well known that for every divisor $d$ of $n$, $\mathbb{Z}_n$ has a unique subgroup of order $d$. Let $K$ be a subgroup of $\mathbb{Z}_n$ which is adjacent with $H$ in $\mathcal{P}(\mathbb{Z}_n)$. Then $(|H|, |K|) = 1$ and $|K| = \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} p_j^{\alpha_j}$, with $j_1, j_2, \ldots, j_s \notin \{i_1, i_2, \ldots, i_r\}$. But for each $j \notin \{i_1, i_2, \ldots, i_r\}$, the power of $p_j$ can be chosen in $(\alpha_j + 1)$ ways. It follows that, such a subgroup $K$ can be chosen in $\prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (\alpha_j + 1)$ ways.

Excluding the trivial subgroup, we have $\prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (\alpha_j + 1) - 1$ subgroups in $\mathbb{Z}_n$ which are adjacent with $H$ in $\mathcal{P}(\mathbb{Z}_n)$. This completes the proof.

Theorem 2.8. If $G$ is a simple graph on $m$ vertices, then there exist $m' \in \mathbb{N}$ such that $G$ is an induced subgraph of $\mathcal{P}(\mathbb{Z}_{m'})$.

Proof. Let $n$ be the number of maximal independent sets of $G$. Now assign $n$ distinct primes for each of these maximal independent sets. Let $v$ be a fixed vertex of $G$. If $v$ belongs to $t$ maximal independent sets of $G$, then label to $v$, the product of primes which are assigned to these $t$ maximal independent sets. Similarly we can label the other vertices of $G$. If all these labeling are distinct, then keep them as it is. Otherwise, in order to make the labeling distinct, we relabel the vertices by using different powers of these primes. Now let $m'$ be the least common multiple of the labels assigned to vertices of $G$. Again relabel each of these labels by subgroup of $\mathcal{P}(\mathbb{Z}_{m'})$ whose order is the same label. Then it turns out that $G$ is an induced subgraph of $\mathcal{P}(\mathbb{Z}_{m'})$. Hence the proof.

Now we illustrate Theorem 2.8 in the following example.

Example 2.1. Consider the graph $G$ as shown in figure 1.

![Figure 1: The graph $G$](image)

Here $I_1 := \{v_1, v_3, v_4\}$, $I_2 := \{v_2, v_5\}$ are the only maximal independent subsets of $G$. First we assign prime $p_i$ to $I_i$ for each $i = 1, 2$. Here $v_1 \in I_1$, so label $v_1$ by $p_1$; $v_2 \in I_2$, so label $v_2$ by...
Let \( \text{Theorem 3.1.} \) plane embedding will be given when the coprime graph of subgroups of a group is planar. This theorem to show the non-planarity of the coprime graph of subgroups of groups. The explicit relabel each of these labels by subgroup of \( \mathbb{Z} \) whose order is the same label. It follows that \( G \) is the induced subgraph of \( \mathbb{Z}_m \).

3 Coprime graph of subgroups of groups which are planar, unicyclic, with forbidden subgraphs

The famous Kuratowski’s Theorem (see [3 Theorem 11.13]) characterized planar graphs as those graphs which does not contain subgraphs homeomorphic to either \( K_5 \) or \( K_{3,3} \). We repeatedly use this theorem to show the non-planarity of the coprime graph of subgroups of groups. The explicit plane embedding will be given when the coprime graph of subgroups of a group is planar.

The main aim of this section is to prove the following result.

\textbf{Theorem 3.1. Let} \( G \) be a finite group. Then

\begin{enumerate}
\item \( \mathcal{P}(G) \) is planar if and only if \( G \) is one of a \( p \)-group, \( \mathbb{Z}_q \times P, \mathbb{Z}_q \times P, \) where \( P \) is a \( p \)-group, \( \langle a, b, c \mid a^q = b^p = c^p = 1, ac = ca, bc = cb, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle, \mathbb{Z}_q \times_2 \mathbb{Z}_{p^2}, \mathbb{Z}_{p^2} \times \mathbb{Z}_q, D_{12}, \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}, \mathbb{Z}_{p^2} \times (\mathbb{Z}_q \times \mathbb{Z}_q), \mathbb{Z}_9 \times \mathbb{Z}_4, D_{18}, \langle G \rangle = p^\alpha q^\beta (\alpha \geq 3) \) with \( G \) has unique cyclic Sylow \( q \)-subgroup, \( \mathbb{Z}_{pqr}, \mathbb{Z}_{p^2q^r} \) or \( \mathbb{Z}_{pqrs} \);
\item \( \mathcal{P}(G) \) is \( K_{2,3} \)-free if and only if \( G \) is one of a \( p \)-group, \( \mathbb{Z}_q \times P, \mathbb{Z}_q \times P, \) where \( P \) is a \( p \)-group, \( \langle a, b, c \mid a^q = b^p = c^p = 1, ac = ca, bc = cb, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle, \mathbb{Z}_q \times_2 \mathbb{Z}_{p^2}, D_{12}, \mathbb{Z}_{p^2q^2} \) or \( \mathbb{Z}_{pqr} \);
\item \( \mathcal{P}(G) \) is \( K_{2,2} \)-free if and only if \( G \) is one of a \( p \)-group, \( \mathbb{Z}_q \times P, \mathbb{Z}_q \times P, \) where \( P \) is a \( p \)-group, \( \langle a, b, c \mid a^q = b^p = c^p = 1, ac = ca, bc = cb, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle, \mathbb{Z}_q \times_2 \mathbb{Z}_{p^2}, D_{12} \) or \( \mathbb{Z}_{pqr} \);
\item \( \mathcal{P}(G) \) is \( K_{1,4} \)-free if and only if \( G \) is one of a \( p \)-group, \( \mathbb{Z}_{p^\alpha q^\beta}(\alpha = 1, 2, 3), \mathbb{Z}_{p^2q^2}, \mathbb{Z}_{p^3q^2}, \mathbb{Z}_{p^3q^3}, S_3 \) or \( \mathbb{Z}_{pqr} \);
\item \( \mathcal{P}(G) \) is \( K_{1,3} \)-free if and only if \( G \) is one of a \( p \)-group, \( \mathbb{Z}_{p^\alpha q^\beta}(\alpha = 1, 2), \mathbb{Z}_{p^2q^2} \) or \( \mathbb{Z}_{pqr} \);
\item \( \mathcal{P}(G) \) is \( K_{1,2} \)-free if and only if \( G \) is either a \( p \)-group or \( \mathbb{Z}_{pq} \);
\item \( \mathcal{P}(G) \) is unicyclic if and only if \( G \) is either \( \mathbb{Z}_{p^2q^2} \) or \( \mathbb{Z}_{pqr} \).
\end{enumerate}

First we start with the following result:

\textbf{Proposition 3.1. If} \( G \) is a group whose order has at least five distinct prime factors, then \( \mathcal{P}(G) \) contains \( K_5 \).

\textbf{Proof.} Since \( G \) has at least five prime factors, it has at least five subgroups of distinct prime orders and so they are adjacent with each other in \( \mathcal{P}(G) \). It follows that \( \mathcal{P}(G) \) contains \( K_5 \). \( \blacksquare \)
Proposition 3.2. Let $G$ be a group whose order has four distinct prime factors. Then

(1) $\mathcal{P}(G)$ is planar if and only if $G \cong \mathbb{Z}_{pqr}$. 

(2) $\mathcal{P}(G)$ contains $K_{2,3}$ and $K_{1,4}$.

Proof. Let $|G| = p^\alpha q^\beta r^\gamma s^\delta$. Let $H_1, H_2, H_3, H_4$ be subgroups of $G$ of order $p, q, r, s$ respectively and let $H_5 := \langle a, b \rangle$ be subgroup of $G$, where $a, b$ are elements of $G$ of order $q, s$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1\}$ and $Y := \{H_2, H_3, H_4, H_5\}$. We divide the rest of the proof into two cases:

Case 1: Let one of $\alpha, \beta, \gamma$ or $\delta \geq 3$, without loss of generality we assume that $\alpha \geq 3$. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where order of $H_1, H_2, H_3, H_4, H_5, H_6$ are $p, p^2, p^3, q, r, s$ respectively.

Case 2: Let each $\alpha, \beta, \gamma$ and $\delta \leq 2$. Suppose the Sylow subgroups of $G$ is not unique, without loss of generality, we assume that the Sylow $p$-subgroup is not unique. Here $G$ has at least three subgroups, say $H_1, H_2, H_3$ of order $p^\alpha$, and subgroups $H_4, H_5, H_6$ of order $q, r, s$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$.

Suppose all the Sylow subgroups of $G$ are unique, then $G$ is abelian. Now we consider the following subcases:

Subcase 2a: $G$ is cyclic. If $G \cong \mathbb{Z}_{pqr}$, then let $H_i, i = 1, 2, \ldots, 14$ be subgroups of $G$ of orders $p, q, r, s, pq, pr, ps, qr, qs, rs, pqr, pqs, prs, qrs$ respectively. Then $\mathcal{P}(G)$ is planar and the corresponding plane embedding is shown in Figure 2 also $\mathcal{P}(G)$ contains $K_{2,3}$ and $K_{1,4}$.

![Figure 2: $\mathcal{P}(\mathbb{Z}_{pqr})$](image)

If $G \cong \mathbb{Z}_{p^2qr}$, then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1, H_2, H_3, H_4, H_5, H_6$ are subgroups of $G$ of order $p, p^2, pq, r, s, rs$ respectively. If $G \cong \mathbb{Z}_{p^2q^2rs}$, $\mathbb{Z}_{p^2q^2r^2s}$ or $\mathbb{Z}_{p^2q^2r^2s^2}$, then $G$ has $\mathbb{Z}_{p^2qrs}$ as a subgroup and so $\mathcal{P}(G)$ contains $K_{3,3}$.

Subcase 2b: $G$ is non-cyclic abelian. Here $G$ has atleast three subgroups of any one of prime order $p, q, r$ or $s$, without loss of generality, say $p$. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with
The following are equivalent:

Proposition 3.3. Let $H$ say $X$ bipartition $K$ let $bipartition X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ are subgroups of $G$ of order $p$, $p$, $q$, $r$, $s$ respectively.

Combining all the cases together the proof follows. \hfill \Box

Proposition 3.3. Let $G$ be a group whose order has three distinct prime factors. Then

1. $\mathcal{P}(G)$ is planar if and only if $G$ is isomorphic to either $\mathbb{Z}_{pqr}$ or $\mathbb{Z}_{p^2qr}$;
2. $\mathcal{P}(G)$ contains $K_{1,2}$;
3. The following are equivalent:
   a. $G \cong \mathbb{Z}_{pqr}$;
   b. $\mathcal{P}(G)$ is $K_{1,4}$-free;
   c. $\mathcal{P}(G)$ is $K_{2,3}$-free;
   d. $\mathcal{P}(G)$ is $K_{1,3}$-free;
   e. $\mathcal{P}(G)$ is $K_{2,2}$-free;
   f. $\mathcal{P}(G)$ is unicyclic.

Proof. Let $|G| = p^\alpha q^\beta r^\gamma$. We need to consider the following cases.

Case 1: Let one of $\alpha$, $\beta$ or $\gamma \ge 3$, without loss of generality we assume that $\alpha \ge 3$. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ are subgroups of $G$ of order $p$, $p^2$, $p^3$, $q$, $r$ respectively and $H_6 := \langle a, b \rangle$, $a$, $b$ are elements of $G$ of order $q$, $r$ respectively. Moreover, $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, K_2\}$ and $Y := \{H_4\}$.

Case 2: Let each $\alpha$, $\beta$ and $\gamma \le 2$. Suppose a Sylow subgroup of $G$ is not unique, without loss of generality, we assume that Sylow $p$-subgroup is not unique. Then $G$ has at least three subgroups, say $H_1$, $H_2$, $H_3$ of order $p^\alpha$, and subgroups $H_4$, $H_5$ of order $q$, $r$ respectively, and $H_6 := \langle a, b \rangle$, $a$, $b$ are elements of $G$ of order $q$, $r$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, K_2\}$ and $Y := \{H_4\}$.

Now assume that all the Sylow subgroups of $G$ are unique. Then $G$ is abelian.

(a) Let $G$ be cyclic. If $G \cong \mathbb{Z}_{pqr}$, then let $H_i, i = 1, 2, \ldots, 6$ be subgroups of $G$ of orders $p$, $q$, $r$, $pq$, $pr$, $qr$ respectively. Then $\mathcal{P}(G)$ is planar and the corresponding plane embedding is shown in Figure 3. Also $\mathcal{P}(G)$ is unicyclic; it does not contain $K_{2,2}$, $K_{1,3}$ as subgraphs; but it contains $K_{1,2}$ as a subgraph. If $G \cong \mathbb{Z}_{p^2qr}$, then let $H_i, i = 1, 2, \ldots, 10$ be subgroups of $G$ of orders $p$, $p^2$, $q$, $r$, $pq$, $pr$, $qr$, $p^2q$, $p^2r$, $pqr$ respectively. Then $\mathcal{P}(G)$ is planar and the corresponding plane embedding is shown in Figure 4. Also $\mathcal{P}(G)$ is not unicyclic; it contains $K_{2,3}$ as a subgraph and does not contain $K_{1,4}$.

If $G \cong \mathbb{Z}_{p^2qr}, \mathbb{Z}_{p^2qr^2}$, then $\mathcal{P}(G)$ contains a subdivision of $K_5$ as shown in Figure 5 with vertices $H_i, i = 1, \ldots, 7$ of order $p$, $p^2$, $q$, $q^2$, $r$, $pr$, $qr$ respectively. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$; it contains $K_{2,3}$ as a subgraph with bipartition $X := \{H_1, H_2\}$ and $Y := \{H_3, H_4, H_5\}$. 

(b) Let $G$ be non-cyclic abelian. Here $G$ has at least three subgroups of any one of prime order $p$, $q$, $r$ or $s$, let it be $p$. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1, H_2, H_3, H_4, H_5, H_6$ are subgroups of $G$ of order $p, p, p, p, q, r$ respectively. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$. 

The proof follows by combining all the above cases together. \hfill \Box

**Proposition 3.4.** Let $G$ be a group of order $p^\alpha q^\beta$, $\alpha, \beta \geq 2$, $\alpha + \beta \geq 6$. Then

1. $\mathcal{P}(G)$ is planar if and only if $|G| = p^\alpha q^2$ with $G$ has unique cyclic Sylow $q$-subgroup;
2. $\mathcal{P}(G)$ is $K_{1,4}$-free if and only if $G \cong \mathbb{Z}_{p^3}q^3$;
3. $\mathcal{P}(G)$ contains $K_{2,3}$.

**Proof.** Proof is divided into two cases:

**Case 1:** $\alpha + \beta \geq 6$ and $\alpha, \beta \geq 3$. Let $H_1, H_2, H_3, H_4, H_5, H_6$ be subgroups of $G$ of order $p, p^2, p^3, q, q^2, q^3$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$.

If $\alpha \geq 4$, then $G$ has subgroups of order $p, p^2, p^3, p^4, q$, let them be $H_1, H_2, H_3, H_4, H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$.

Let $\alpha = 3$. Suppose Sylow $p$-subgroup and Sylow $q$-subgroup of $G$ are cyclic and are unique, then $G$ is cyclic and so $\mathcal{P}(G)$ is bipartite with one partition contains all subgroups whose orders has $p$ as a
divisor and another partition contains remaining proper subgroups of $G$. Thus $\mathcal{P}(G) \cong K_{3,3} \cup \overline{K}_{10}$. It follows that $\mathcal{P}(G)$ is $K_{1,4}$-free.

If Sylow $p$-subgroup or Sylow $q$-subgroup of $G$ is not cyclic, without loss of generality, we assume that Sylow $p$-subgroup is not cyclic, then $G$ has subgroups of order $p$, $p^2$, $p^3$, $p^i$, $q$, for some $i \in \{1,2\}$, let them be $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$.

If Sylow $p$-subgroup or Sylow $q$-subgroup is not unique, without loss of generality, we assume that Sylow $p$-subgroup is not unique, then $G$ has subgroups of order $p$, $p^2$, $p^3$, $q$, let them be $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$.

Case 2: $\alpha + \beta \geq 6$, $\beta = 2$. Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ be subgroups of $G$ of order $p$, $p^2$, $p^3$, $p^i$, $q^2$, $q$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$; also $\mathcal{P}(G)$ contains $K_{2,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$.

If Sylow $q$-subgroup of $G$ is not unique, then $G$ has subgroups of order $p$, $p^2$, $p^3$, $q^2$, $q$, let them be $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$.

If Sylow $q$-subgroup of $G$ is unique and isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$, then $G$ has subgroups of order $p$, $p^2$, $p^3$, $q^2$, $q$, let them be $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$.

If Sylow $q$-subgroup of $G$ is unique and cyclic, then $G$ has an unique subgroups of order $q$, $q^2$, let them be $H_1$, $H_2$ respectively. Then $\mathcal{P}(G)$ is planar, since $\mathcal{P}(G)$ is bipartite with one partition contains all the subgroups whose order has $p$ as a divisor and another partition contains $H_1$, $H_2$.

The proof follows by combining the above cases together. \qed

**Proposition 3.5.** Let $G$ be a group of order $p^3q^2$. Then

1. $\mathcal{P}(G)$ is planar if and only if $G$ has a unique cyclic Sylow $q$-subgroup;
2. $\mathcal{P}(G)$ is $K_{1,4}$-free if and only if $G \cong \mathbb{Z}_{p^3q^2}$.
3. $\mathcal{P}(G)$ contains $K_{2,3}$.

**Proof.** Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ be subgroups of $G$ of order $p$, $p^2$, $p^3$, $q$, $q^2$ respectively. Here $\mathcal{P}(G)$ contains $K_{3,2}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5\}$.

Suppose Sylow $p$-subgroup, and Sylow $q$-subgroup of $G$ are cyclic, and are unique, then $G$ is cyclic and so $\mathcal{P}(G)$ is bipartite with one partition contains all subgroups whose order has $p$ as a divisor and another partition contains remaining subgroups of $G$. Thus $\mathcal{P}(G) \cong K_{2,3} \cup \overline{K}_7$. It follows that $\mathcal{P}(G)$ is $K_{1,4}$-free.

If Sylow $p$-subgroup of $G$ is not unique, then $G$ has subgroups of order $p^2$, $p^3$, $p^i$, $q^2$, say $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$. 
If Sylow $p$-subgroup of $G$ is not cyclic, then $G$ has subgroups of order $p^3, p^2, p, q^2$, for some $i \in \{1, 2\}$, say $H_1, H_2, H_3, H_4, H_5$ respectively. Then $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$.

Suppose Sylow $q$-subgroup of $G$ is isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$, then $G$ has subgroups of order $p^3, p^2, p, q, q^2$, say $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$ and $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

Suppose Sylow $q$-subgroup of $G$ is not unique, then $G$ has subgroups of order $p^3, p^2, p, q, q^2$, $q^2, q^2$, say $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$ and it contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

If Sylow $q$-subgroup of $G$ is unique and it is cyclic, then $G$ has an unique subgroup of order $q$, $q^2$, let them be $H_1, H_2$ respectively. Then $\mathcal{P}(G)$ is planar, since $\mathcal{P}(G)$ is bipartite with one partition contains all the subgroups whose order has $p$ as a divisor and another partition contains $H_1, H_2$. It follows that $\mathcal{P}(G)$ is planar if and only if $G$ has a unique cyclic Sylow $q$-subgroup. In [9] Myron Owen Tripp showed that up to isomorphism, there are 15 such groups of order $p^3q^2$ exist. \[\square\]

**Proposition 3.6.** Let $G$ be a group of order $p^2q^2$. Then

1. $\mathcal{P}(G)$ is planar if and only if $G$ is isomorphic to one of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$, $\mathbb{Z}_{p^2} \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, $\mathbb{Z}_q \times \mathbb{Z}_4$ or $D_{18}$;
2. $\mathcal{P}(G)$ contains $K_{2,2}$;
3. The following are equivalent:
   a. $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}$;
   b. $\mathcal{P}(G)$ is $K_{1,4}$-free;
   c. $\mathcal{P}(G)$ is $K_{2,3}$-free;
   d. $\mathcal{P}(G)$ is $K_{1,3}$-free;
   e. $\mathcal{P}(G)$ is unicyclic.

**Proof.** We consider the following cases:

**Case 1:** $G$ is non-abelian. We use the classification of groups of order $p^2q^2$ given in [6]. Without loss of generality, we assume that $p > q$. Let $P$ and $Q$ denote a Sylow $p$-subgroup and Sylow $q$-subgroup of $G$ respectively. By Sylow’s theorem, $n_p = 1, q, q^2$. But $n_p = q$ is not possible, since $p > q$. If $n_p = q^2$, then $p \mid (q + 1)(q - 1)$, this implies that $p \mid (q + 1)$, which is true only when $(p, q) = (3, 2)$. When $(p, q) \neq (3, 2)$, then $G \cong P \rtimes Q$.

If $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2} = \langle a, b \mid a^{p^2} = b^q = 1, bab^{-1} = a^i, i^q \equiv 1(mod\ p^2) \rangle$, then $\mathcal{P}(G)$ is planar, since it is bipartite with one partition contains all the subgroups of $G$ whose order has $q$ as a divisor, and another partition contains the subgroups of $G$ order $p, p^2$; also $G$ has $p^2$ Sylow $q$-subgroups.

It follows that $\mathcal{P}(G)$ contains $K_{2,3}$ and $K_{1,4}$ as subgraphs and so $\mathcal{P}(G)$ is not unicyclic.
If $G \cong \mathbb{Z}_p^2 \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $\mathcal{P}(G)$ is planar, since it is bipartite with one partition contains all the subgroups whose order has $q$ as a divisor, and another partition contains the subgroups of $G$ order $p$, $p^2$; also $G$ has $p^2$ Sylow $q$-subgroups. It follows that $\mathcal{P}(G)$ contains $K_{2,3}$ and $K_{1,4}$ as a subgraph and so $\mathcal{P}(G)$ is not unicyclic.

If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times (\mathbb{Z}_q \times \mathbb{Z}_q) := \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = a^i b^p, bcb^{-1} = a^k b^q \rangle$, where $(i, j, k)$ has order $q^2$ in $GL_2(p)$, then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ are subgroups of $G$ of order $p$, $p$, $q^2$, $q^2$, $q$ respectively. Also $G$ has $p^2$ Sylow $q$-subgroups, so $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph.

If $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times (\mathbb{Z}_q \times \mathbb{Z}_q)$, then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ are subgroups of $G$ of order $p$, $p$, $q$, $q$, $q$ respectively. Also $G$ has $p^2$ Sylow $q$-subgroups, so $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph.

If $(p, q) = (3, 2)$, then up to isomorphism, there are nine groups of order 36. In the following we consider each of these groups.

(a) $G \cong D_{18}$. Here $G$ has unique subgroups of order 3 and 9 respectively; the order of remaining proper subgroups have 2 as their common divisor. Therefore, $\mathcal{P}(G)$ is planar, since it is bipartite with one partition contains subgroups of order 3, 9, and another partition contains remaining proper subgroups of $G$. Since there are 18 subgroups of $D_{18}$ has order 2, so $\mathcal{P}(G)$ contains $K_{2,3}$ and $K_{1,4}$ as a subgraph.

(b) $G \cong S_3 \times S_3$. Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$, $H_7$ be subgroups of $G$ of order 3, 3, 9, 2, 2, 2, 2 respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

(c) $G \cong \mathbb{Z}_3 \times A_4$. Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$, $H_7$ be subgroups of $G$ of order 3, 3, 9, 2, 2, 2, 4 respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

(d) $G \cong \mathbb{Z}_6 \times S_3$. Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$, $H_7$ be subgroups of $G$ of order 3, 3, 9, 2, 2, 2, 2 respectively. Here $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

(e) $G \cong \mathbb{Z}_9 \times \mathbb{Z}_4 = \langle a, b \mid a^9 = b^4 = 1, bab^{-1} = a^i, i^4 \equiv 1 \pmod{9} \rangle$, then $\mathcal{P}(G)$ is planar, since it is bipartite with one partition contains all the subgroups whose order has $p$ as a divisor, and another partition contains the subgroups of order $q$, $q^2$. It follows that $\mathcal{P}(G)$ contains $K_{2,3}$ as a subgraph. Also $G$ has nine subgroups of order 4. So $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$, where $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ are subgroups of $G$ of order 4, 4, 4, 2, 3 respectively.
(f) $G \cong \mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4) = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, ac = ca, bc^{-1} = b^i, \text{ord}_2(i) = 3 \rangle$. Let $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ be subgroups of $G$ of order 3, 3, 9, 2, 4, 4, 4 respectively. Then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6, H_7\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

(g) $G \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_4 = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, cac^{-1} = a^b b^j, cb^{-1} = a^k b^j \rangle$, where $\binom{i j}{k l}$ has order 4 in $GL_2(3)$. Let $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ be subgroups of $G$ of order 3, 3, 3, 9, 2, 4, 4 respectively. Here $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_5, H_6, H_7\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$.

(h) $G \cong (\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)) \times \mathbb{Z}_2$. Let $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ be subgroups of $G$ of order 3, 3, 9, 2, 2, 2 respectively. Here $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_5, H_6, H_7\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$.

(i) $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3$. Let $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ be subgroups of $G$ of order 3, 3, 9, 2, 2, 2, 4 respectively. Here $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_5\}$.

**Case 2:** $G$ is abelian. If $G \cong \mathbb{Z}_{p^aq^2}$, then it is easy to see that $\mathcal{P}(G) \cong K_{2,2} \cup K_3$, which is planar, $K_{1,3}$-free and it contains $K_{2,2}$ as a subgraph.

If $G \cong \mathbb{Z}_{p^aq} \times \mathbb{Z}_q$, then $\mathcal{P}(G)$ is planar, since $\mathcal{P}(G)$ is bipartite with one partition contains all the subgroups whose order has $p$ as a divisor, and another partition contains the subgroups of order $q$, $q^2$. It follows that $\mathcal{P}(G)$ contains $K_{2,3}$ as a subgraph. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$, where $H_1, H_2, H_3, H_4, H_5$ are subgroups of $G$ of order $q$, $q$, $q^2$, $p$ respectively.

If $G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$, then $\mathcal{P}(G)$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ are subgroups of $G$ of order $p, p, p, q, q, q^2$ respectively. Also $\mathcal{P}(G)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_4, H_5, H_6, H_7\}$ and $Y := \{H_1\}$.

The proof follows by combining all the above cases together. \qed

**Proposition 3.7.** Let $G$ be a group of order $p^aq$. Then

1. $\mathcal{P}(G)$ is planar if and only if $G$ is one of $\mathbb{Z}_q \times P$, $\mathbb{Z}_q \rtimes P$, where $P$ is a $p$-group, $\langle a, b, c \mid a^q = b^p = c^p = 1, ac = ca, bc = cb, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle$, $\mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$ or $D_{12}$;

2. The following are equivalent:
   
   a. $G$ is one of $\mathbb{Z}_q \times P$, $\mathbb{Z}_q \rtimes P$, where $P$ is a $p$-group, $\langle a, b, c \mid a^q = b^p = c^p = 1, ac = ca, bc = cb, bab^{-1} = a^i, \text{ord}_q(i) = p \rangle$, $\mathbb{Z}_q \rtimes_2 \mathbb{Z}_{p^2}$ or $D_{12}$;
(b) \( \mathcal{P}(G) \) is \( K_{2,3} \)-free;

(c) \( \mathcal{P}(G) \) is \( K_{2,2} \)-free.

(3) \( \mathcal{P}(G) \) is \( K_{1,4} \)-free if and only if \( G \) is one of \( \mathbb{Z}_{p^\alpha q} \) \((\alpha = 1, 2, 3)\) or \( S_3 \).

(4) \( \mathcal{P}(G) \) is \( K_{1,3} \)-free if and only if \( G \) either \( \mathbb{Z}_{pq} \) or \( \mathbb{Z}_{pq^2} \);

(5) \( \mathcal{P}(G) \) is \( K_{1,2} \)-free if and only if \( G \cong \mathbb{Z}_{pq} \);

(6) \( \mathcal{P}(G) \) is acyclic.

Proof. Proof is divided into several cases.

Case 1: \( \alpha \geq 3 \).

Suppose Sylow \( q \)-subgroup of \( G \) is not unique, then \( G \) has subgroups of order \( p, p^2, p^3, q, q \), let them be \( H_1, H_2, H_3, H_4, H_5, H_6 \) respectively. Here \( \mathcal{P}(G) \) contains \( K_{3,3} \) as a subgraph with bipartition \( X := \{H_1, H_2, H_3\} \) and \( Y := \{H_4, H_5, H_6\} \).

Suppose Sylow \( q \)-subgroup is unique, then \( G \cong \mathbb{Z}_p \times P \) or \( \mathbb{Z}_q \times P \), where \( P \) is a Sylow \( p \)-subgroup of \( G \) and so \( \mathcal{P}(G) \) is bipartite with one partition contains a subgroup whose order is \( q \) and another partition contains remaining proper subgroups of \( G \). Hence \( \mathcal{P}(G) \) is planar and \( K_{2,2} \)-free.

Now we check the \( K_{1,4} \)-freeness of \( \mathcal{P}(G) \). If \( \alpha \geq 4 \), then \( G \) has subgroups of order \( p, p^2, p^3, p^4, q \), let them be \( H_1, H_2, H_3, H_4, H_5 \) respectively. Then \( \mathcal{P}(G) \) contains \( K_{1,4} \) as a subgraph with bipartition \( X := \{H_1, H_2, H_3, H_4\} \) and \( Y := \{H_5\} \).

Let \( \alpha = 3 \). Suppose Sylow \( p \)-subgroup of \( G \) is not unique, then \( G \) has subgroups of order \( p, p^2, p^3, p^3, q \), let them be \( H_1, H_2, H_3, H_4, H_5 \) respectively. Then \( \mathcal{P}(G) \) contains \( K_{1,3} \) as a subgraph with bipartition \( X := \{H_1, H_2, H_3, H_4\} \) and \( Y := \{H_5\} \).

Suppose Sylow \( p \)-subgroup of \( G \) is unique and cyclic, then \( G \) has a unique subgroups of order \( p, p^2, p^3 \), let them be \( H_1, H_2, H_3, H_4, H_5 \) respectively. Then \( \mathcal{P}(G) \) is a bipartite graph with one partition contains \( H_1, H_2, H_3 \) and another partition contains remaining proper subgroups of \( G \). Hence \( \mathcal{P}(G) \) is \( K_{1,4} \)-free. But \( \mathcal{P}(G) \) contains \( K_{1,3} \) as a subgraph with bipartition \( X := \{H_1, H_2, H_3\} \) and \( Y := \{H_4\} \), where \( H_4 \) is a subgroup of order \( q \).

Case 2: \( \alpha = 2, \beta = 1 \).

Suppose \( G \) is cyclic, then \( G \) has only four proper subgroups and their orders are \( p^2, p, q, pq \) respectively. Therefore, \( \mathcal{P}(G) \cong K_{1,2} \cup K_1 \), which is planar and \( K_{1,2} \)-free.

If \( G \) is non-cyclic abelian, then \( G \) has \( p + 1 \) subgroups of order \( p \); unique subgroup of order \( p^2 \); unique subgroup of order \( q \); \( p + 1 \) subgroups of order \( pq \). Also these are the only proper subgroups of \( G \). It follows that \( \mathcal{P}(G) \cong K_{1,p+2} \cup K_{p+1} \) and hence \( \mathcal{P}(G) \) is planar; it contains \( K_{1,4} \) as a subgraph and it is \( K_{2,2} \)-free.

If \( G \) is non-abelian, then we need to consider the list of groups of order \( p^2 q \) used in the proof of Theorem 2.3

(a) From the structure of \( \mathcal{P}(G_1) \), it follows that \( \mathcal{P}(G_1) \) is planar, \( K_{2,2} \)-free and it contains \( K_{1,4} \).
(b) Note that $G_2$ has unique subgroup of order $q$. $\mathcal{P}(G_2)$ is planar, since $\mathcal{P}(G_2)$ is bipartite with one partition contains all the subgroups whose order has $p$ as a divisor, and another partition contains the subgroup of order $q$. So $\mathcal{P}(G_2)$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_4\}$ and $Y := \{H_5\}$, where $H_1, H_2, H_3, H_4, H_5$ are subgroups of $G$ of order $p, p, p, p^2, q$ respectively. Clearly $\mathcal{P}(G_2)$ is $K_{2,2}$-free.

(c) From the structure of $\mathcal{P}(G_3)$, it follows that $\mathcal{P}(G_3)$ is planar, $K_{2,2}$-free and it contains $K_{1,4}$.

(d) From the structure of $\mathcal{P}(G_4)$, it follows that $\mathcal{P}(G_4)$ is planar, and it contains $K_{2,3}, K_{1,4}$ as a subgraph.

(e) $G_{5(t)}$ has at least three subgroups of order $p$, and at least three subgroups of order $q$. So $\mathcal{P}(G_{5(t)})$ contains $K_{3,3}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3\}$ and $Y := \{H_4, H_5, H_6\}$, where $H_i, i = 1, 2, 3$ and $H_j, j = 4, 5, 6$ are subgroups of $G$ of order $p, p, p, q, q, q$ respectively. Also $\mathcal{P}(G_{5(4)})$ contains $K_{1,4}$ as a subgraph with bipartition $X := \{H_1, H_2, H_3, H_7\}$ and $Y := \{H_4\}$, where $H_7$ is a subgroup of $G$ of order $p^2$.

(f) From the structure of $\mathcal{P}(G_6)$, it follows that $\mathcal{P}(G_6)$ contains $K_{3,3}$ as a subgraph, so it is non-planar; also it contains $K_{1,4}$ as a subgraph.

(g) From the structure of $\mathcal{P}(D_{12})$, it follows that $\mathcal{P}(D_{12})$ is planar, $K_{2,2}$-free and it contains $K_{1,4}$ as a subgraph. Also from the structure of $\mathcal{P}(A_4)$, it follows that $\mathcal{P}(A_4)$ contains $K_{3,3}$ and $K_{1,4}$ as a subgraph.

**Case 3:** Let $\alpha = \beta = 1$. Then $G$ has a unique subgroup of order $q$, let it be $H$; it has $q$ subgroups of order $p$; also these are the only proper subgroups of $G$. It follows that $\mathcal{P}(G) \cong K_{1,q}$, which is planar, acyclic and $K_{2,2}$-free; it is $K_{1,4}$-free if and only if $q = 3$; it contains $K_{1,3}$.

The proof follows by combining all the above cases together.

**Proposition 3.8.** If $G$ is a $p$-group, then $\mathcal{P}(G)$ is planar, $K_{2,3}$-free, $K_{2,2}$-free, $K_{1,4}$-free, $K_{1,3}$-free, $K_{1,2}$-free and acyclic.

**Proof.** If $G$ is a $p$-group, then order of every subgroup of $G$ is power of $p$ and so no two subgroups are adjacent in $\mathcal{P}(G)$. Thus $\mathcal{P}(G)$ is totally disconnected and this completes the proof.

Putting together all the Propositions proved so far in this section, we obtain the main Theorem

2.6

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