Kolmogorov–Sinai entropy and black holes

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Received 20 December 2007, in final form 3 July 2008
Published 11 September 2008
Online at stacks.iop.org/CQG/25/195005

Abstract

It is shown that stringy matter near the event horizon of a Schwarzschild black hole exhibits chaotic behavior (the spreading effect) which can be characterized by the Kolmogorov–Sinai entropy. It is found that the Kolmogorov–Sinai entropy of a spreading string equals the half of the inverse gravitational radius of the black hole. The Kolmogorov–Sinai entropy of a spreading string measures the rate at which information about the state of a string collapsing into the black hole is lost with time as it spreads over the horizon. A possible relation between the Kolmogorov–Sinai and Bekenstein–Hawking entropies is discussed.

PACS number: 04.70.Dy

1. Introduction

It is well known that some general relativistic systems described by the Einstein equations can exhibit chaotic behavior [1, 2]. But it is still not clear whether general relativistic systems such as black holes can do it. In series of insightful papers, Susskind [3, 4, 6] and Mezhlumian, Peet and Thorlacius [5] showed that a string falling toward a black hole spreads over the stretched horizon. In doing so, it exhibits chaotic behavior and the rate of spreading is determined by the inverse gravitational radius of the black hole.

In this paper I propose to use the conceptions of chaos theory to describe the behavior of stringy matter near the event horizon of a Schwarzschild black hole. In the following sections we will introduce the main conceptions of chaotic dynamics, demonstrate chaotic behavior of stringy matter near the event horizon of a black hole and find its Kolmogorov–Sinai entropy. In the end we discuss a possible relation between the Kolmogorov–Sinai (KS) and Bekenstein–Hawking (BH) entropies.

2. The main conceptions of chaotic dynamics

We begin with definitions. According to chaos theory [7, 8] the chaotic behavior of dynamical systems that is, systems whose state evolves with time, has its origin in the so-called local instability, when a small change in initial conditions leads to an exponential divergence of
phase-space trajectories. As a result of this sensitivity to initial conditions, the behavior of dynamical systems appears unpredictable. Suppose that two nearby trajectories in the phase space of a system start off with a separation \( d(0) \) at time \( t = 0 \). Then, if there exists a local instability, it follows that there exists a direction in phase space along which the trajectories diverge exponentially:

\[
d(t) = d(0) \, e^{ht}.
\]

(1)

The parameter \( h \) is called the Lyapunov exponent for the trajectories. If \( h \) is positive, then we say the behavior of the system is chaotic. The sum of all the positive Lyapunov exponents is called the KS entropy \( h_{KS} \). It describes the rate of change of information about the phase-space trajectories as a system evolves. Following Zaslavsky [7], we can define the KS entropy in not so formal way and connect it with thermodynamical entropy. Suppose that the phase space of a dynamical system is finite (in at least one phase-space direction). Then, since there exists the local instability (1), it follows that an initial volume in phase space \( \Delta \Gamma(0) \) becomes very complicated like a fractal. Thus we should perform coarse graining, and if we coarse grain the region, it will appear that it is growing:

\[
\Delta \Gamma(t) = \Delta \Gamma(0) \, e^{ht},
\]

(2)

where \( h \) is the averaged over phase space Lyapunov exponent. As a result, the coarse-grained Boltzmann entropy increases

\[
S(t) = ht.
\]

(3)

The quantity \( \Delta \Gamma(0) \) can be taken as simply equal to a volume of the coarse-graining. Then the expression

\[
h_{KS} = \lim_{t \to \infty} \lim_{\Delta \Gamma(0) \to 0} \frac{1}{t} \ln \left( \frac{\Delta \Gamma(t)}{\Delta \Gamma(0)} \right)
\]

(4)

just defines the KS entropy. As is easily seen, if \( \Delta \Gamma(0) \to 0 \) that is, if \( S(t) = ht \) at \( t \to \infty \), the Boltzmann entropy does not reach a maximum on a compact phase space. But the situation changes if we fix a finite volume of coarse-graining. In this case the exponential separation of trajectories (1) and the consequent increase in volume (2) can last only on a finite time scale. Just that very case will be discussed below. In general, the calculation of the KS entropy offers great mathematical difficulties. Our consideration will be rather physical. Our task is facilitated by the fact that all our relevant quantities have exponential dependence on time. It should be kept in mind that the KS entropy \( h_{KS} \) is not really an entropy but an entropy per unit time or entropy rate, \( dS/dt \). Note also that by order of magnitude \( h_{KS} \sim h \). On the other hand, the BH entropy of a black hole

\[
S_{BH} = \frac{\pi R_g^2}{4G} = \frac{\pi}{16} \frac{R_g^2}{l_p^2} = 4\pi GM^2,
\]

(5)

is obtained from the thermodynamical relation \( dE = T dS \), where the energy of the black hole is its mass \( M \), the temperature is given by \( T_H = 1/8\pi GM \) and the area of the event horizon \( A \) is related to the gravitational radius \( R_g \), \( R_g = 2GM \), in the usual way \( A = 4\pi R_g^2 \). The BH entropy is defined in the reference frame of an external distant observer at a fixed static position above the horizon (an external observer). It is well established that, from her/his point of view, the classical physics of a quasistationary black hole can be described in terms of a ‘stretched horizon’ which is a membrane placed near the event horizon and endowed with certain mechanical, electrical and thermal properties [9]. The exact distance of this membrane above the event horizon is somewhat arbitrary. In the context of string theory—the subject of our research—the stretched horizon is most naturally thought of as lying at the string scale
above the event horizon. In what follows, we will deal with this conception (sometimes referred to simply as horizon). Our purpose is to find a kinematic effect caused by the black hole geometry with respect to which a system evolves like (1), (2). For this purpose we repeat, for completeness, some well-known facts from [10] concerning the behavior of matter near the horizon without proofs, thus making our exposition self-contained.

3. Chaotic behavior of a relativistic string collapsing into a black hole

My proposal rests on stringy matter having unusual kinematic properties near the event horizon of a black hole. According to string theory [11], the most promising candidate for a fundamental theory of matter, all particles are excitations of a one-dimensional object—a string. String theory is characterized by two fundamental parameters: the string scale \( l_s \) and the string coupling constant \( g \); if \( l_P \) is the Planck length then \( l_P = g l_s \). An important fact is that the behavior of a string in the weakly coupled string theory is very precisely described in terms of the random-walk model [11–14]. We can imagine a string as simply built by joining together bits of the string, each of which is of length \( l_s \). Suppose that the total length of the string is \( L \) and each string bit can point in any of \( n \) possible directions. Then the number of bits is \( L/l_s \) and the number of states of the string is

\[ N_s \sim n^{L/l_s} \sim \exp(L \ln n/l_s). \]  

(6)

For notational simplicity the factor \( \ln n \) will be omitted henceforward. There is no loss of generality in doing so because we can always redefine \( g, l_s \) and \( L \). We can also define the mean squared radius of the string \( \langle R^2_s \rangle = L l_s \). Then

\[ N_s \sim \exp \left( \frac{\langle R^2_s \rangle}{l_s^2} \right), \]  

(7)

and the entropy of the string is

\[ S_s \sim \frac{L}{l_s} \approx \frac{\langle R^2_{\parallel} \rangle}{l_s^2}. \]  

(8)

The mass of the string is given by \( M_s \sim T L \), where \( T \sim 1/l_s^2 \) is the string tension. So in terms of the mass

\[ S_s \sim M_s l_s. \]  

(9)

Note that in fundamental contrast to the black hole entropy (5), the entropy of a string goes like the mass.

Another important fact is that strings behave very differently from the ordinary particles. The crucial difference is that the size and shape of a string are sensitive to the time resolution. It is a smearing time over which the internal motions of the string are averaged. Susskind showed [3] that zero-point fluctuations of a string make the size of the string depend on a time resolution; the shorter the time over which the oscillations of a string are averaged the larger is its spatial extent. In low energy physics, resolution times are always large and this phenomenon is not important. Let, for example, an observation of a string last a time \( \tau_{\text{res}} \). Then Susskind argued that the contributions of modes with frequency \( \gg 1/\tau_{\text{res}} \) in the normal mode expansion for the size of the string should be averaged out. As a result of such a coarse-graining or time-smoothing, Susskind found that the mean squared radii of the string in the transverse and longitudinal directions in Planck units are \( \langle R^2_{\perp} \rangle = l_s^2 \ln(1/\tau_{\text{res}}) \) and \( \langle R^2_{\parallel} \rangle = l_s^2 / \tau_{\text{res}} \), respectively. Similar calculations can be performed for the total length of the string, and Susskind found that \( L = l_s / \tau_{\text{res}} \). Now consider a string falling toward a black hole. As is well known [10], the proper time in the frame of the string \( \tau \) and the Schwarzschild time
of an external observer $t$ are related through $\tau \sim \exp(-t/2R_g)$ due to the redshift factor. This means that the transverse size of the string will increase linearly:

$$\langle R^2_\perp \rangle = l_s^2 \frac{t}{2R_g},$$

(10)

while its longitudinal size and total length—exponentially: $\langle R^2_\parallel \rangle = l_s^2 \exp(t/2R_g)$, $L = l_s \exp(t/2R_g)$. But the longitudinal growth is rapidly canceled by the Lorentz longitudinal contraction. Thus the string approaching the horizon spreads only in the transverse directions (in this connection the subscript ‘$\perp$’ at the mean squared radius of the string will be replaced by the ‘$s$’ henceforward). The spreading process begins to occur when the string reaches the horizon at the distance of the order of the string scale $l_s$ from the horizon in a thin layer $\sim l_s$ that is, on the stretched horizon.

In (10) we can immediately recognize the linear dependence of the squared displacement of a Brownian particle from the origin on time. The theory of Brownian motion is closely related to that of random walks. One normally associates diffusion with the Brownian motion of a particle. The Schrödinger equation describing the string has the same mathematical structure as the diffusion equation of a Brownian particle. In fact, the spreading appears to behave as if the string were diffusing away from its original transverse location. As is well known, the Brownian motion is a chaotic process (moreover, it appears that one can even infer a positive KS entropy from the Brownian motion (see paragraph 19.9 in Dorfman [8] and references therein)). Thus a spreading string exhibits chaotic behavior. Indeed, Susskind [3, 4, 6] and Mezhlumian, Peet and Thorlacius [5] showed that the string configuration becomes chaotic and very complicated like a fractal during the spreading process (especially it is clear with the help of the Monte Carlo simulation of the probability functional of a string [15]). As mentioned above, during the spreading the total length of a string increases exponentially that is, the number of bits increases exponentially too:

$$N_{\text{bit}} = N_{\text{bit}}(0) \exp(t/2R_g).$$

(11)

The authors interpreted this as a branching diffusion process, where every string bit diffuses independently of others over the whole horizon and bifurcates into two bits and so on. In this connection it is relevant to remark the following. First the diffusion is a distinctive random process. But in our case there are no real random forces. The behavior of a string near the horizon is very well described by the Hamilton dynamics. If there are exact equations of motion no true randomness is possible. Second the string is a fundamental object. It is not a dissipative system. In the spreading process no points of a string should be lost or gained. The irreversible character of the spreading effect arises, as has been shown above, exceptionally from the coarse-grained or time-smoothed procedure, in which the fine details of the string motion (over the Planck time scale) have been wiped out. As an ordinary classical body, a string undergoes Lorentz transformations. As is well known [16], under the Lorentz transformations the phase-space volume of a classical body and, consequently, its entropy does not change. But due to the coarse-grained procedure the phase-space volume of a string can increase. We can give the following interpretation of (11). Initially phase points of a string occupy one cell in the phase space of the string. In the course of time, the number of bits increases. This means an increase in the number of phase-space dimensions. So the phase-space volume increases and we can interpret (11) as an increase in the number of occupied cells:

$$N_{\text{cell}}(t) = N_{\text{cell}}(0) \exp(t/2R_g).$$

(12)
In turn, this number is proportional to the distance between the trajectories of phase points that all initially occupy one cell (1), as required. To estimate the number of states of a spreading string we can also use the random-walk model. Substituting (10) into (7) we obtain

$$N_s \sim \exp\left(\left\langle R_s^2 \right\rangle / l_s^2\right) \sim \exp(t/2R_g).$$

(13)

So the entropy is

$$S_s \sim \frac{t}{2R_g},$$

(14)

and the entropy rate is

$$\frac{dS_s}{dt} = \frac{1}{2R_g}.$$  

(15)

This behavior just corresponds to the thermal properties of a black hole and the second law of thermodynamics. Since the temperature of the black hole radiation depends on the radial position, $T(r) = T_H / \chi$, where $\chi$ is the redshift factor, $\chi = (1 - R_g / r)^{1/2}$, it follows that from the viewpoint of the external observer the string falls into an increasingly hot region. A thermal interchange will take place. So the string should ‘melt’ and spread throughout the horizon. Obviously during this process the phase volume and entropy of the body should increase. Thus (13), (14) is a natural response to the hot horizon.

Now let us return to the size (10) and total length of a string. I think that these quantities are not quite consistent with one another. The mean squared radius of the string and its total length should be related by $\left\langle R^2 \right\rangle = Ll_s$. But our quantities do not satisfy this relationship: the mean squared radius of the string has linear dependence on time, the total length—exponential. As noted by Susskind himself [3], these results were obtained in the framework of free string theory. They do not take into account such a nonperturbative phenomenon as string interactions; there are indications [6, 10] that a true growth of the transverse size should be exponential:

$$\left\langle R_s^2 \right\rangle = l_s^2 \exp(t/2R_g).$$

(16)

As we will see below, the same exponential growth governs the spreading of a classical perturbation on the stretched horizon. How does the state space of a spreading string evolve in this case? Repeating the previous algebra we obtain for the number of states $N^\prime \sim \exp(\exp(t/2R_g))$, so that the entropy becomes $S_s \sim \exp(t/2R_g)$. Thus the entropy rate is $d\ln S_s/dt = 1/(2R_g)$. But we did not yet take into account the string interactions. Apparently, they will impose constraints on the total number of string states. Indeed, the self-interaction of a string reduces the huge degeneracy of free string states and we expect

$$N \sim \exp(t/2R_g).$$

(17)

In this case the entropy is

$$S_s \sim \frac{t}{2R_g},$$

(18)

and the entropy rate is

$$\frac{dS_s}{dt} = \frac{1}{2R_g}.$$  

(19)

Note that we have the same results for small times. Finally comparing (17)–(19) with (2), (3) we find

$$h_{KS} = \frac{1}{2R_g}.$$  

(20)

5
Thus we conclude that the spreading of a string (or any body made of strings) is described by the KS entropy. Moreover its value is the same for all objects near the horizon of a black hole that is, it is universal. The surface of the horizon is a compact manifold. Since there exists the finite size of coarse-graining $l_s$, a string covers the horizon in a finite time \[ t_{\text{spread}} \sim R_g \ln \frac{R_g^2}{l_P^2}. \] (Here we have taken into account the string interactions $g \sim 1, l_s \sim l_P$.) At this time a spreading string completely covers the entire horizon of a black hole. The number of states of the string becomes \[ N \sim \exp \left( \frac{R_g^2}{l_P^2} \right) \] and the entropy of the string reaches that of the black hole, \[ S \sim R_g^2 / l_P^2. \] Note that in doing so, the total length of the string becomes \[ L \sim R_g^2 / l_P^2, \] and the corresponding mass \[ M_s \sim R_g^2 / l_P^3. \] It is a huge mass. But the redshift factor reduces it to the black hole mass. Since the spreading process begins to occur when the string reaches the stretched horizon at the proper distance \( l_P \) from the event horizon, the redshift factor is \[ \chi = \left(1 - R_g / r\right)^{1/2} \approx l_P / 2 R_g, \] and we obtain for the string mass \[ M_s \sim \chi \left( \frac{R_g^2}{l_P^2} \right) \sim R_g / (2 l_P^3). \] Thus, from the string theory point of view \[ [10], \] a black hole is nothing but a single string. According to Susskind \[ [3, 4, 6] \] and Mezhlumian, Peet and Thorlacius \[ [5], \] at the time \( t_{\text{spread}} \) the string spreads over the entire horizon and can no longer expand due to the nonperturbative effects. The result is crucial for the relaxation of the string to a statistical equilibrium: to reach a statistical equilibrium in a finite time we should have the finite time of spreading. According to chaos theory, this is an average time over which the state of a string can be predicted; after the time \( t_{\text{spread}} \) all information contained in the string will be lost and we will be able only to make statistical predictions. This time is comparable to the characteristic time of a black hole \( R_g \) but is smaller than the black hole lifetime \( \sim R_g^3 \). Hence the KS entropy of a spreading string measures the rate at which information about the state of the string (or any body made of strings) collapsing into the black hole is lost with time as it spreads over the horizon.

We have found the KS entropy for a fundamental string spreading over the event horizon of a black hole. It is widely believed, however, that the spreading effect is not a peculiar feature of a special (still hypothetical) kind of matter. In the framework of the so-called infrared/ultraviolet connection \[ [10], \] it is a general property of all matter at energies above the Planck scale. If we want to study progressively smaller and smaller objects we must, according to conventional physics, use higher and higher energy probes. But once gravity is involved that rule is changed radically. Since at energies above the Planck scale black holes are created, it follows \[ [10], \] that as we raise the energy we probe larger and larger distances. In other words a very high frequency is related to a large size scale, \( \Delta \nu \Delta \tau \sim l_P^2 \). Then, taking into account the redshift factor, we can obtain the exponential growth of the transverse size of matter similar to \( [10], \) as required.

4. Is there a relation between the KS and BH entropies?

As mentioned above, a falling string spreads over the stretched horizon until its entropy becomes equal to the black hole entropy:

\[ S_s = S_{\text{BH}} = \frac{\pi R_s^2}{l_P^2}. \] (22)

The spreading ends and only a new falling string or any other perturbation can start it again. The next string falling toward the horizon interacts with a previous one lying on the horizon in such a way that the formation of a single (new) string is thermodynamically favored, etc. So the stretched horizon is a single string made out of all strings whenever fallen into it. From
the string theory point of view \cite{10}, a black hole is nothing but a single string lying on the sphere of the radius $R_g$. But since a black hole absorbs a string, its gravitational radius must increase. According to the teleological nature of the event horizon (see chapter VI in Thorne et al \cite{9}), before a fall of the next string, the gravitational radius and the horizon area increase like $\exp(t/2R_g)$. This means that the spreading effect takes place. In this case the entropy rate is
\[
\frac{d\ln S_{BH}}{dt} = \frac{1}{2R_g}.
\] (23)
Thus we can conclude that
\[
h_{KS} = \frac{d\ln S_{BH}}{dt}.
\] (24)
In my opinion, it is just a required relationship between the KS and BH entropies. I make no claim of having a rigorous mathematical proof of this statement. But our physical arguments are completely based on the proved conclusions of string theory and dynamics of the stretched horizon. From the physical point of view, the KS entropy of a black hole measures the rate at which information about the black hole (or a string lying on the stretched horizon) is lost during a perturbation.

5. Conclusions

In this paper we have shown that stringy matter near the event horizon of a black hole with the gravitational radius $R_g$ exhibits instability (the spreading effect) and chaotic behavior, which can be characterized by the Kolmogorov–Sinai entropy $h_{KS}$. We have found that for a spreading string $h_{KS} = 1/2R_g$. The KS entropy of a spreading string measures the rate at which information about the string (or any body made of strings) collapsing into a black hole is lost as the string (the body) spreads over the horizon. We have also discussed a possible relation between the Kolmogorov–Sinai and Bekenstein–Hawking entropies and suggested that $h_{KS} = d\ln S_{BH}/dt$.

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