THE CARDINALITY OF THE SUBLATTICE OF CLOSED IDEALS OF OPERATORS BETWEEN CERTAIN CLASSICAL SEQUENCE SPACES

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Abstract. Theorem A and Theorem B of [1] state that for $1 < p < \infty$ the lattice of closed ideals of $\mathcal{L}(\ell_p, \ell_0)$, $\mathcal{L}(\ell_p, \ell_\infty)$ and of $\mathcal{L}(\ell_1, \ell_p)$ are at least of cardinality $2^\omega$. Here we show that the cardinality of the lattice of closed ideals of $\mathcal{L}(\ell_p, \ell_0)$, $\mathcal{L}(\ell_p, \ell_\infty)$ and of $\mathcal{L}(\ell_1, \ell_p)$, is at least $2^{2^\omega}$, and thus equal to it.

In [1] we construct $2^\omega$ operators from $\ell_p$ to $\ell_0$ which generate distinct closed operator ideals in $\mathcal{L}(\ell_p, \ell_0)$. Here we show that we can naturally choose a subset of those operators of size $2^\omega$ such that not only does each operator generate a distinct closed operator ideal, but each subset of these operators also generates a distinct closed operator ideal. Hence there are in fact $2^{2^\omega}$ distinct closed operator ideals in $\mathcal{L}(\ell_p, \ell_0)$.

Let $1 \leq p < \infty$. For appropriately chosen sequences $(u_n)$ and $(v_n)$ in $\mathbb{N}$ we constructed a uniformly bounded sequence $(T_n)$ of operators $T_n : \ell_2^{u_n} \to \ell_\infty^{v_n}$ given by appropriately scaled RIP matrices, and defined for $M \subset \mathbb{N}$ the operator $T_M : U = (\oplus_{n \in \mathbb{N}} \ell_2^{u_n})_{\ell_p} \to V = (\oplus_{n \in \mathbb{N}} \ell_\infty^{v_n})_{\ell_0}$, $(x_n) \mapsto Q_M(T_n(x_n) : n \in \mathbb{N})$, where $Q_M : V \to V$ is the canonical projection onto the coordinates in $M$.

For Banach spaces $X, Y, W, Z$, and for a set of operators $T \subset \mathcal{L}(W, Z)$, we let $\mathcal{J}^T(X, Y)$ be the closed ideal of $\mathcal{L}(X, Y)$ generated by $T$. Thus $\mathcal{J}^T(X, Y)$ is the closure in $\mathcal{L}(X, Y)$ of

$$\left\{ \sum_{j=1}^n A_j S_j B_j : n \in \mathbb{N}, (S_j)_{j=1}^n \subset T, (A_j)_{j=1}^n \subset \mathcal{L}(Z, Y), (B_j)_{j=1}^n \subset \mathcal{L}(X, W) \right\}.$$ 

We write $\mathcal{J}^T(X, Y)$ instead of $\mathcal{J}^T(X, Y)$ if $T = \{T\}$.

For infinite subsets $M, N$ of $\mathbb{N}$, [1] Theorem 1] states that:

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(i) If \( M \setminus N \) is infinite then \( T_M \not\in \mathcal{J}^{T_N}(U, V) \).
(ii) if \( N \setminus M \) is finite then \( \mathcal{J}^{T_N} \subset \mathcal{J}^M(U, V) \).

Moreover, the proof of [1, Theorem 1] shows that if \( M \setminus N \) is infinite, then there is a functional \( \Phi \in \mathcal{L}(U, V)^* \), \( \|\Phi\| \leq 1 \), with \( \Phi(T_M) = 1 \) and \( \Phi|_{\mathcal{J}^{T_N}(U, V)} = 0 \), which means that \( \operatorname{dist}(T_M, \mathcal{J}^{T_N}(U, V)) \geq 1 \). It follows from the proof of [1, Theorem 6] (see also the subsequent remark) that if \( W \) is a Banach space containing \( c_0 \) and \( J : V \to W \) is an isomorphic embedding, then we also have \( \operatorname{dist}(J \circ T_M, \mathcal{J}^{T_N}(U, W)) \geq c \), where \( c = \|J^{-1}\|^{-1} \).

Using now the same approach as in [2], we can easily deduce the following corollary.

**Corollary 1.** Let \( W \) be a Banach space containing \( c_0 \). Then the cardinality of the lattice of closed ideals of \( \mathcal{L}(U, W) \) is at least \( 2^{2\omega} \). In particular, for \( 1 < p < \infty \), the cardinality of the lattice of closed ideals of \( \mathcal{L}(\ell_p, c_0) \) and of \( \mathcal{L}(\ell_p, \ell_\infty) \) is \( 2^{2\omega} \).

**Proof.** Since \( V \cong c_0 \) and \( U \sim \ell_p \) when \( p > 1 \), we need only to prove the first statement. Let \( \mathcal{C} \) be a family of infinite subsets of \( \mathbb{N} \) whose cardinality is \( 2^\omega \) and whose elements are pairwise almost disjoint, i.e., if \( M \neq N \) are in \( \mathcal{C} \) then \( N \cap M \) is finite. For \( \mathcal{A} \subset \mathcal{C} \) we put \( \mathcal{T}(\mathcal{A}) = \{T_M : M \in \mathcal{A}\} \) and \( \mathcal{J}(\mathcal{A}) = \mathcal{J}^{\mathcal{T}(\mathcal{A})}(U, W) \). We claim that for any two nonempty subsets \( \mathcal{A} \) and \( \mathcal{B} \) of \( \mathcal{C} \), we have \( \mathcal{J}(\mathcal{A}) \neq \mathcal{J}(\mathcal{B}) \). Indeed, without loss of generality we can assume that \( \mathcal{A} \setminus \mathcal{B} \neq \emptyset \). Let \( M \in \mathcal{A} \setminus \mathcal{B} \) and let \( J : V \to W \) be an isomorphic embedding. We show that \( \operatorname{dist}(J \circ T_M, \mathcal{J}(\mathcal{B})) \geq c \) where \( c = \|J^{-1}\|^{-1} \). Since \( J \circ T_M \in \mathcal{J}(\mathcal{A}) \), this shows that \( \mathcal{J}(\mathcal{A}) \neq \mathcal{J}(\mathcal{B}) \).

Let \( n \in \mathbb{N} \), \( (N_j)_{j=1}^n \subset \mathcal{B} \), \( (A_j)_{j=1}^n \subset \mathcal{L}(V, W) \) and \( (B_j)_{j=1}^n \subset \mathcal{L}(U) \). Put \( N = \bigcup_{j=1}^n N_j \). We have \( A_j T_N B_j = A_j Q_N T_N B_j \) for \( j = 1, 2, \ldots, n \), and hence \( \sum_{j=1}^n A_j T_N B_j \in \mathcal{J}^{T_N}(U, W) \). Since \( M \setminus N \) is infinite, it follows that

\[
\|J \circ T_M - \sum_{j=1}^n A_j T_N B_j\| \geq \operatorname{dist}(J \circ T_M, \mathcal{J}^{T_N}(U, W)) \geq c .
\]

Since \( \mathcal{J}(\mathcal{B}) \) is the closure of the set of operators of the form \( \sum_{j=1}^n A_j T_N B_j \), the proof is complete. \( \square \)

**Remark.** A very simple duality argument (see [1, Proposition 7] and [1, Theorem 8]) shows that for \( 1 < q < \infty \), the lattice of closed ideals of \( \mathcal{L}(\ell_1, \ell_q) \) is also of cardinality \( 2^{2\omega} \). The same is true in \( \mathcal{L}(\ell_1, (\bigoplus_{n \in \mathbb{N}} \ell_2)^{c_0}) \).

In [2] it was shown that the cardinality of the set closed ideals of \( \mathcal{L}(L_p) \), \( 1 < p < \infty \), is \( 2^{2\omega} \). Note that the Hardy space \( H_1 \) and its
The predual VMO can be seen as the “well behaved” limit cases of the $L_p$-spaces. For example $\ell_2$ is complemented in both spaces, and $H_1$ contains a complemented copy of $\ell_1$ and VMO a complemented copy of $c_0$ (cf. [3] and [1 page 125]), and thus we deduce the following corollary.

**Corollary 2.** The cardinality of the lattice of closed ideals of $L(VMO)$ and $L(H_1)$ is $2^{2^\omega}$.

**References**

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