PROPERTIES OF HIGH-REDSHIFT 
LYMAN ALPHA CLOUDS
II. STATISTICAL PROPERTIES OF THE CLOUDS

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ABSTRACT

Curve of growth analysis, applied to the Lyman series absorption ratios deduced in our previous paper, yields a measurement of the logarithmic slope of distribution of Lyman α clouds in column density $N$. The observed exponential distribution of the clouds’ equivalent widths $W$ is then shown to require a broad distribution of velocity parameters $b$, extending up to 80 km s$^{-1}$. We show how the exponential itself emerges in a natural way. An absolute normalization for the differential distribution of cloud numbers in $z$, $N$, and $b$ is obtained. By detailed analysis of absorption fluctuations along the line of sight (including correlations among neighboring spectral frequency bins) we are able to put upper limits on the cloud-cloud correlation function $\xi$ on several megaparsec length scales. We show that observed $b$ values, if thermal, are incompatible, in several different ways, with the hypothesis of equilibrium heating and ionization by a background UV flux. Either a significant component of $b$ is due to bulk motion (which we argue against on several grounds), or else the clouds are out of equilibrium, and hotter than is implied by their ionization state, a situation which could be indicative of recent adiabatic collapse.

Subject headings: cosmology: observations – quasars – intergalactic medium
1. Introduction

In a previous paper (Press, Rybicki, and Schneider 1993, hereafter Paper I) we analyzed the statistics of the Lyman forest absorption, as it is seen against the UV continuum of the background quasars constituting the Schneider-Schmidt-Gunn high-redshift sample, whose redshifts \( z \) range from 2.5 to 4.3 (Schneider, Schmidt, and Gunn, 1991; hereafter SSG). We obtained an empirical fit for mean transmission as a function of redshift that agrees well with previous determinations at somewhat lower redshifts, and were additionally able to measure the mean ratios of absorption due to the lines Lyman \( \alpha \), Lyman \( \beta \), Lyman \( \gamma \), Lyman \( \delta \), and possibly Lyman \( \epsilon \). We also developed some formalism for interpreting fluctuations in the transmission about its mean, and indicated that these fluctuations were close to, but perhaps somewhat larger than can be explained by a random distribution of clouds.

Paper I restricted itself to conclusions which followed directly from the SSG data, essentially without the introduction of modeling assumptions. This paper broadens the analysis in several respects:

First, we will take as input to the analysis not only the SSG data as studied in Paper I, but also some other published data, in particular the statistical distribution of equivalent widths in the redshift range \( 1.50 < z < 3.78 \) reported by Murdoch et al. (1986), and the statistical distribution of velocity parameters \( b \) shown as preliminary data by Carswell (1989; see his review for further attribution of this published and unpublished material).

These additional data sets derive from high resolution studies that involve the identification, counting, and detailed fitting of model line profiles to the Lyman \( \alpha \) lines; we will call these results, generically, “line counting” measurements. Our results in Paper I, by contrast, did not depend on the identification of any single line, but only on aggregate statistical properties of the Lyman forest. In attempting to derive uniform conclusions from a merger of these two very different kinds of data, our challenge is to rely minimally on features of a data set that are most susceptible to systematic error.

Second, we will eventually make some standard assumptions about the microscopic state of the gas in the clouds. In particular, we will assume that the gas consists of hydrogen and helium with a primordial cosmological abundance, and that its temperature and ionization state are determined by local interaction (though not necessarily equilibrium) with a posited background UV flux, conventionally, though not necessarily, taken to be due to the continuum emission of high-redshift quasars.

The plan of this paper is as follows: In Section 2, we apply standard curve-of-growth
analysis to the measured ratios of equivalent widths determined in Paper I. We are able to determine (in the context of certain parametric models) the number distribution of cloud neutral column densities $N$ without any use of line counting measurements. This determination is highly insensitive to the assumed distribution of velocity parameters $b$; thus, while we cannot say anything about $b$ from the data of Paper I alone, our characterization of the $N$ distribution is almost completely independent of any assumptions about $b$, which is not the case in line counting measurements.

In Section 3, we show that the line counting data of Murdoch et al. (1986) can be fit, virtually perfectly, by the column density distribution that we determined in Section 2, if it is independently combined with any one of several simple model distributions for $b$. We then compare the measured $b$ distribution given by Carswell (1989) to the class of acceptable model distributions. At this point in the analysis we are able to determine an absolute normalization on the density of clouds, which we compare to previous published values. We are also able to explain the origin of the the empirical exponential distribution of equivalent widths introduced by Sargent et al. (1980).

In Section 4 we revisit the question of fluctuations in the absorption around its mean, last addressed by us in Paper I. With the detailed number distribution in both $N$ and $b$ of the previous section, we use Monte Carlo techniques to generalize the analytic calculations of Paper I. We examine both the variance and the autocorrelation (in redshift) of the absorption for signs of an underlying two point correlation function $\xi$ in the cloud distribution.

Sections 5 and 6 ask the question, “Where are all the baryons?” That is, we investigate whether, with the distribution of $N$ and $b$ already determined, it is possible to hide all of a plausible cosmological density of baryons in the population of clouds that is already known to exist at high redshifts (e.g., in the SSG sample’s Lyman forest). We find that, with almost no model assumptions (that is, without assuming an internal structure, or even a physical size, for the clouds), it is possible to do so. Indeed, the problem is how to avoid cosmologically too many baryons in the clouds: if the clouds are in thermal and ionization equilibrium, and if the measured $b$ values represent thermal velocities, then neutral hydrogen fractions are so small that the observed neutral absorption implies an overfilled universe.

The most likely solutions are either that a significant component of the velocity parameter $b$ is not thermal ("bulk motion hypothesis"), or that the clouds are in an out-of-equilibrium thermal state with respect to their ionization level, perhaps due to adiabatic heating from recent collapse ("non-UV heating hypothesis"). We discuss pros and cons of the two solutions. In either case, there is no cosmological necessity for any nonzero
Gunn-Peterson (1965) effect.

Section 7 summarizes our conclusions.

2. Curve of Growth Analysis

Although curve of growth analysis is standard textbook fare, we are applying it in an unusual range of parameters, so it is worth stating the basic equations used. For a line with central wavelength $\lambda_0$ and central frequency $\nu_0$, the Doppler widths in wavelength and frequency are defined by

$$
\Delta \lambda_D = \frac{b}{c} \lambda_0; \quad \Delta \nu_D = \frac{b}{c} \nu_0
$$

(1)

where $c$ is the speed of light and $b$ is the velocity parameter, given, in the case of thermal velocities only, by

$$
b = \left(\frac{2kT}{m_H}\right)^{1/2}
$$

(2)

For a given Lyman line of interest, in terms of the atomic physics parameters $f$ (the oscillator strength or $f$-value of the transition) and $\Gamma$ (the natural decay rate of the line, $= \sum_n A_{nn'}$) one defines a Voigt parameter,

$$
a \equiv \frac{\Gamma}{4\pi \Delta \nu_D} = \frac{\Gamma \lambda_0}{4\pi b}
$$

(3)

and an “integrated line optical depth,”

$$
\tau_0 = \frac{\pi e^2}{m_e c \Delta \nu_D} N f = \frac{\pi e^2}{m_e c} \left(\frac{N \lambda_0 f b}{b}\right)
$$

(4)

where $N$ is the column density of neutral hydrogen (assumed to be principally in its ground level). In the usual case where $a \ll 1$, the integrated line optical depth $\tau_0$ is $\sqrt{\pi}$ times the line center optical depth.

Lyman alpha clouds along the line of sight to the quasar produce absorption lines by pure extinction, so the residual intensity is given by

$$
r(\lambda) = \frac{F_c - F(\lambda)}{F_c} = 1 - e^{-\tau_0 U(x,a)},
$$

(5)
where $F(\lambda)$ is the measured flux, $F_c$ is the continuum flux, and $x \equiv (\lambda - \lambda_0)/\Delta \lambda_D$. The normalized Voigt function is defined by

$$U(x, a) = \frac{a}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{\exp(-t^2)/(x-t)^2 + a^2}{(x-t)^2 + a^2} dt$$

This particular normalization makes $\int_{0}^{\infty} U(x, a) \, dx = 1$.

The equivalent width $W$ of the line in wavelength is then given by

$$W = \int_{0}^{\infty} r(\lambda) \, d\lambda = \frac{\lambda_0 b}{c} \int_{-\infty}^{\infty} \left(1 - e^{-\tau_0 U(x, a)}\right) \, dx,$$

which may be computed by direct numerical quadrature.

Figure 1 shows the result of applying this prescription to the first five lines of the Lyman series ($\alpha, \beta, \gamma, \delta, \epsilon$) over the range $10^{13}\,\text{cm}^{-2} < N < 10^{22}\,\text{cm}^{-2}$, and for three values of $b$ that will turn out to be of interest (30, 50, 70 km s$^{-1}$). At the very lowest (unsaturated) and highest (damped) column densities, one sees that the equivalent widths have, essentially, the ratios of their respective oscillator strengths, and they are independent of $b$. In the intermediate (saturated) region, the dependence on Doppler width is significant and approximately linear in $b$. The dependence on $N$ is linear in the unsaturated regime, slowly varying ($\sim \log^{1/2} N$) in the saturated regime, and varying as $N^{1/2}$ in the damped regime. (See Spitzer 1978, §3.4, for further details.)

Let us look now at the implications of Figure 1 to the Lyman $\alpha$ clouds. Sargent (1988) states straightforwardly an observational consensus that (i) the typical column density of the clouds is $N \approx 10^{15}\,\text{cm}^{-2}$, and (ii) there is a power law distribution of column densities of the form

$$f(N) \, dN \propto N^{-\beta} \, dN$$

with $\beta \approx 1.5$. Indeed, there is a lively debate (into which we are about to put an oar) on the value, and possible non-constancy, of $\beta$ as $N$ varies over 10 orders of magnitude, from $10^{12} - 10^{22}\,\text{cm}^{-2}$ (see, e.g., Tytler 1987, Bechtold 1988, Carswell et al. 1984, Carswell et al. 1987, Hunstead 1988, Rauch et al. 1992, Petitjean et al. 1993). Before proceeding, however, we need to take note of the obvious fact that consensus points (i) and (ii) above, if they are taken as fundamental features of the clouds rather than as artifacts of observational selection, are mutually incompatible: a pure power law has no typical value!

Figure 1, on the other hand, shows that the value $N \sim 10^{14}\,\text{cm}^{-2}$ is a special value observationally – it is where the Lyman $\alpha$ line becomes saturated (for reasonable values of $b$). Integrating the pure power law of equation (8) with the equivalent width shown in Figure 1, one sees that the total absorption will diverge for $N \gg 10^{14}\,\text{cm}^{-2}$ if $\beta < 1$ and will
diverge for \( N \ll 10^{14}\text{cm}^{-2} \) if \( \beta > 2 \). For \( \beta \) in the allowed range \( 1 < \beta < 2 \), the “typical” cloud responsible for absorption will have \( N \sim 10^{14}\text{cm}^{-2} \). In this paper, and in Paper III of this series, we will several times focus on the question of whether there is any good evidence for a change in the properties of the clouds at \( N \sim 10^{14} - 10^{15}\text{cm}^{-2} \). In general, we will see that the evidence is weak.

We can thus turn with renewed interest to the question of determining the exponent \( \beta \). In Paper I, we obtained values, and a full covariance matrix, for the ratios

\[
\frac{W_\beta/\lambda_\beta}{W_\alpha/\lambda_\alpha}, \quad \frac{W_\gamma/\lambda_\gamma}{W_\alpha/\lambda_\alpha}, \quad \frac{W_\delta/\lambda_\delta}{W_\alpha/\lambda_\alpha}, \quad \frac{W_\epsilon/\lambda_\epsilon}{W_\alpha/\lambda_\alpha}
\]

(9)

We now ask whether there is any single value of \( \beta \) that, convolved with Figure 1, becomes statistically consistent with the measured ratios. The figure of merit is the \( \chi^2 \) value,

\[
\chi^2 = \sum_{i,j=1}^{4} (R_i^* - R_i^0)C_{ij}^{-1}(R_j^* - R_j^0)
\]

(10)

where \( R_i \equiv (W_i/\lambda_i)/(W_\alpha/\lambda_\alpha) \), \( i = \beta, \gamma, \delta, \epsilon \), is a measured ratio from equation (9), \( C_{ij} \) is the measured covariance matrix (see Paper I), and \( R_i^0 \) is the theoretical value

\[
R_i^0 = \frac{\int dNN^{-\beta}W_i(N,b)/\lambda_i}{\int dNN^{-\beta}W_\alpha(N,b)/\lambda_\alpha}
\]

(11)

For a correct model, the value of \( \chi^2 \) should be distributed as the standard chi-square function with 4 degrees of freedom.

Figure 2 shows the result, for three different values of \( b \). (In Section 3 we will introduce distributions of \( b \) instead of single values.) One sees that the exponent \( \beta \) is determined to lie in the same narrow range, approximately independent of \( b \). The values of \( \chi^2 \) at the minima are quite consistent with 4 degrees of freedom, lending credibility both to the power law model and to Paper I’s resampling estimates of the covariance matrix (which had no knowledge of the atomic physics embodied in Figure 1). Recalling that a 1-\( \sigma \) error estimate corresponds to a change \( \Delta \chi^2 = 1 \), we can summarize the measured value and uncertainty (including both statistical uncertainty and uncertainty due to the value of \( b \)) as

\[
\beta = 1.43 \pm 0.04
\]

(12)

In the next section we will be more explicit about the range of \( N \) that contributes to this determination, but it should be clear from inspecting Figure 1, and from the fact that \( \beta \sim 1.5 \), that the range \( N \sim 10^{14} - 10^{15}\text{cm}^{-2} \) contributes most.

It is possibly a debatable question as to whether value in equation (12) is consistent with previous determinations that rely on the counting of individual clouds. Hunstead
et al. (1987a; see also Hunstead 1988) obtain $\beta = 1.57 \pm 0.05$ fitting to a range $13.25 < \log_{10} N < 16$, and find that a single power law is a good fit. This contrasts with Carswell et al. (1987), who found evidence of a break at $\log_{10} N \approx 14.35$. Petitjohn et al. (1992) report $\beta = 1.49 \pm 0.02$ as the best single power law fit over the range $13.7 < \log_{10} N < 21.8$. However, they obtain a somewhat steeper value $\beta = 1.83 \pm 0.06$ for a fit restricted to $13.7 < \log_{10} N < 16$.

Considering the completely different nature of our method (not counting clouds, and using Lyman lines higher than $\alpha$), the fact that our method is weighted differently in $N$, and the somewhat different redshift ranges, we think that all the above quoted errors should be taken with a grain of salt. In favor of our method is the fact that it is not susceptible to observational biases in the efficiency with which one identifies, in noise, the very different profiles of a weak line (at $N = 10^{13}\text{cm}^{-2}$, say) and a saturated line (at $N = 10^{16}\text{cm}^{-2}$, say).

It must quickly be said, however, that Paper I’s measured ratios of equivalent widths, taken by themselves, do not rule out broken power law models à la Carswell et al. (1987). On the contrary, Figure 3 shows the results of fitting a broken power law model

$$f(N)dN \propto \begin{cases} 
(N/N_0)^{-1.14}dN & N \leq N_0 \\
(N/N_0)^{-1.82}dN & N > N_0
\end{cases}$$

for the column density of the break $N_0$. (The exponents in this model come from a specific theoretical construction that will be detailed in Paper III. In this paper we will refer to this model simply as “Model X”.) One finds that, for any given value of $b$, Model X fits the data embarrassingly well, with $\chi^2 \approx 1$ for 4 degrees of freedom. (This is bound to happen about 1 time in 20 by chance. The different values of $b$ are highly correlated and are not independent embarrassments.) The value of the break $N_0$ is in the range $0.8 - 2 \times 10^{15}\text{cm}^{-2}$.

Figure 3 also shows, as the three “narrow” (approximate) parabolas, the result of fitting to the (completely unrealistic) delta-function model in which all clouds have a single column density $N_0$. One sees that, for each assumed $b$, there is a narrow window in $N_0$ that is marginally allowed. We consider this an artifact, a kind of “$\chi^2$ leak”, and expect that any slightly better (smaller statistical errors) determination of the equivalent width ratios would eliminate this – and hopefully not eliminate the power law or broken power law models.

To summarize: the equivalent width ratios measured in Paper I are fairly powerful at fixing one parameter within a model (e.g., either a single exponent, or the position of a single break), but not powerful at distinguishing among different models. In search of such distinctions we move, in the next section, to consider other data.
3. The Distribution of Equivalent Widths

A fundamental observational feature of the Lyman α clouds, first noted by Sargent et al. (1980), is that their equivalent widths $W$ are exponentially distributed (at least for $W > 0.3\,\text{Å}$ or so) as

$$p(W)dW = \exp\left(-\frac{W}{W^*}\right) \frac{dW}{W^*}$$

(14)

Murdoch et al. (1986) give a compilation of data, including some from Carswell et al. (1984) and Atwood, Baldwin, and Carswell (1985), all normalized to a fiducial redshift $z = 2.44$, and obtain a fitted value $W^* = 0.278 \pm 0.018\,\text{Å}$. Figure 4 shows as filled circles the full set of data given in Murdoch et al. (Pay no attention to the curves in the figure, for now.)

One should particularly take note of the distinct departure of the data from a pure exponential at small equivalent widths $W \lesssim 0.2$. Jenkins and Ostriker (1991), among others, note that the apparent upturn in the number of clouds with small equivalent widths can be largely explained by the transition from unsaturated to saturated lines (Figure 1). We will want to see if “largely” can also mean “completely”.

3.1. The Distribution in $b$

The quantities $N$ (column density) and $b$ (velocity parameter) are physically more fundamental than $W$, which follows from them (as Figure 1 indicates). Since we have in hand, from Section 2, a determination of the distribution of $N$ that is independent of all line counting techniques, we can now try to invert the data in Figure 4 and obtain a distribution function for $b$. That distribution can then be compared to the distributions that are obtained by line profile fitting (which, as we shall see, is a somewhat controversial subject). This method of determining $b$ is not really independent of line counting methods, since we must use the equivalent width data of Murdoch et al. (or some other similar data set); but it does provide an useful consistency check on some diverse data reduction methods.

There is controversy about $b$ on at least two issues, one of which we regard as observational, the other theoretical. The observational issue is whether (as claimed by Pettini et al., 1990) the distribution in $b$ is peaked at “small” values, with a median $b$ value of 17 km/s, and essentially no values of $b$ greater than 30 km/s, or whether (as claimed by
Carswell, 1989, Rauch et al., 1992, and others) the distribution in $b$ is much broader, with a mean value in the range 30-40 km/s and a tail extending significantly higher in velocity. The theoretical issue, which is especially important if the $b$ distribution is in fact broad, is whether the measured values of $b$ are indicative of thermal Doppler velocities, from which conclusions can be drawn about the microphysical state of the clouds, or whether $b$ includes a significant component due to bulk motion (in which case a host of other theoretical questions are raised).

As with many inversion problems, it is better to fit a parametric model than to try to invert ab initio. To better understand the sensitivities of the inversion, we will try two models. The first is a simple truncated Gaussian,

$$p(b)db \propto \begin{cases} \exp \left[ -\frac{(b - b_0)^2}{2b_*^2} \right] db & b > 0 \\ 0 & b < 0 \end{cases}$$

(15)

Here $b_0$ is the mode of the distribution (because of the truncation, not the mean), while $b_*$ parametrizes the standard deviation. The second model is a gamma distribution of the form

$$p(b)db \propto b^{(b_0/b_*)-1} \exp(-b/b_*)db, \quad b > 0$$

(16)

where $b_0$ is the mean of the distribution, $b_*$ again parametrizes the standard deviation.

We at this point make the assumption that $b$ and $N$ are independently distributed (this important point will be discussed further in Section 5, below) and we assume that $N$ is distributed as a pure power law with $\beta$ given by equation (12). Writing the transformation of probabilities in the form

$$p(W) = \int dN \int db p(N, b) \delta[W - W(N, b)]$$

$$= \int dN \left[ \frac{p(N, b)}{\partial W(N, b)/\partial b} \right]_{b=b(N,W)}$$

$$= \int db \left[ \frac{p(N, b)}{\partial W(N, b)/\partial N} \right]_{N=N(b,W)}$$

(17)

we perform the integral (either one of the last two forms) using the theoretical curve of growth function $W(N, b)$, and we compute a $\chi^2$ goodness of fit measure to the data in Murdoch et al., using the error bars given there. We then minimize $\chi^2$ for the two parameters $b_0$ and $b_*$ in each of the two models (15) and (16). The results, along with some values that characterize different moments of the fitted distributions (useful later) are given in Table 1. The long- and short-dashed lines in Figure 4 show the result of computing equation (17) for the best fitting models. The agreement is seen to be excellent for both
models. (The $\chi^2$ per degree of freedom is significantly less than unity, suggesting that at least some of the error bars shown in Murdoch et al. 1986 are overestimated.)

Figure 5 shows (again as long and short dashes) the shapes of the best-fitting model distributions. Also shown in that figure, as a histogram, are the data for the distribution of $b$ given in Carswell (1989). While Carswell cautions that the data (derived from several sources) is preliminary, and rightly warns of the difficulty of deriving such data from line profile fitting, we see that the basic shape and parameter values of the Carswell data are in good agreement with our fitted models that derive from the equivalent width distribution of Murdoch et al. (1986). We therefore include the Carswell data as an additional line in Table 1. The last line of that Table gives mean values over the previous lines (two fitted models plus one actual data set) and the spread of the models, which can be taken as an indication of the model uncertainty. (The standard errors of the fitted estimates for $b_0$ and $b^*$ are also comparable, around 2 km/s.)

We have also integrated Carswell’s data on $b$, via equation (17) and the $\beta$ value of equation (12), to get its predicted distribution of $W$. This is shown as the solid line in Figure 4. Note that there are no new adjustable parameters in this calculation, except for the overall normalization (moving the curve up or down). One sees that the result is in excellent agreement with the Murdoch data, virtually as good as the other two models.

As is obvious by Figure 1, it is the large $b$ tail of the distribution that produces the large $W$ exponential tail. While it is conceivable that the Carswell and Murdoch data sets, both obtained by line counting techniques, share a common tendency to misidentify blends, or lines from metal-line clouds, as Lyman $\alpha$ lines of large $W$, it seems quite unlikely that the general shape of the exponential tail in $W$, independently verified measured by many observers from Sargent et al. (1980) to the present and in several different redshift ranges, could be wholly an artifact of such misidentifications. We conclude that the high-$b$

| Distribution   | best $b_0$ | best $b^*$ | $\langle b \rangle$ | $\sigma_b$ | $\langle b^{0.54} \rangle^{1/0.84}$ |
|----------------|------------|------------|---------------------|-----------|---------------------------------|
| Gaussian       | 32         | 23         | 36                  | 20        | 61                              |
| Gamma          | 38         | 14         | 39                  | 23        | 77                              |
| Carswell data  | 36         | 18         | 70                  |           |                                 |
| Mean and Spread| 37 ± 2     | 20 ± 3     | 69 ± 8              |           |                                 |

Table 1: Fitted parameters for the distribution of velocity parameters $b$ (see text).
tails of the distributions in Figure 5, with \( b \)'s in the range 40 to 80 km/s, are real and observationally mandated by the exponential distribution of \( W \)'s.

Thus, as regards the observational controversy mentioned above, we must be firmly on the side of a broad \( b \) distribution. In this, our conclusion agrees with Rauch et al. (1992), who discuss possible sources of discrepancy in the Pettini et al. data.

Even if we exclude from consideration the long exponential tail in the \( W \) distribution, our data do not support entirely small \( b \) values: In Figure 4 we have plotted, as a dotted line, the best-fitting curve that derives from a single value (delta function distribution) of \( b \). The best-fitting value is 33 km/s (cf. Table 1). One sees that the equivalent width distribution for \( W < 0.4 \AA \) is reproduced tolerably well (though not as well as any of the broader distributions), but that there is a severe shortage of large \( W \) values.

In fact, the larger tail values for \( W \) are produced, we find, by clouds that have both larger \( b \) values and larger \( N \) values. Figure 6 shows results from the same integrations that produced the curves in Figure 4. At each equivalent width \( W \), we show the mean \( N \) of the clouds that contributed to that value of \( W \), independent of their values \( b \). One sees that values \( N \) up to \( \sim 10^{16} \text{cm}^{-2} \) are directly probed by the data points in Figure 4. Thus, Figure 4 yields strong consistency check on the value, and constancy of \( \beta \) in the power law distribution of \( N \): Its value, derived from equivalent width ratios of the Lyman series, came primarily from the saturation bend in Figure 1, at \( N \sim 10^{14} \text{cm}^{-2} \). Now, we see (Figures 4 and 6) that the correct number of clouds at \( N \sim 10^{16} \text{cm}^{-2} \) is obtained well within a factor of 2.

To emphasize this point, we have plotted, as the final curve in Figure 4, the result of integrating the broken power law Model X (equation 13) with the observed Carswell distribution of \( b \). One sees that it falls significantly below the data both at small equivalent widths and at large equivalent widths, demonstrating the apparent relative constancy of \( \beta \), at least over three orders of magnitude in \( N \).

Now to answer another question raised at the beginning of this section, we can see that the upward bend in the number of clouds at small equivalent width (Figure 4) can be explained completely as the effect of folding the standard curve of growth (Figure 1) into a constant \( \beta \) (pure power law) model. (This, incidentally, suggests caution in approximating that part of the distribution as another exponential, as in Jenkins and Ostriker 1991, since it is actually a power law that diverges at zero.) To summarize, we find no feature in any of the data sets examined that conflicts with a pure power law distribution for \( N \); no data set requires, or even unambiguously indicates, a “typical” \( N \) value of \( 10^{15} \text{cm}^{-2} \), or any other value, for the underlying cloud population.
Nothing yet in our discussion takes a position on what we have above labeled the theoretical controversy surrounding $b$, i.e., indicates whether the observed $b$’s derive purely from thermal Doppler broadening, as opposed to an additional component due to bulk motions. We will return to this point below.

### 3.2. The Distribution of Clouds in $N, b, z$

We can make explicit at this point a “standard” working model for the distribution of the Lyman $\alpha$ clouds in $N$, $b$, and redshift $z$ at high redshift: Let $N_{14}$ denote the value $N/10^{14}\text{cm}^{-2}$, and let $n(N, b, z)\, dN_{14} \, db \, dz$ denote the mean number of clouds of in an interval $dN_{14}, db, dz$. Then,

$$n(N, b, z) = N_0 (1 + z)^\gamma N_{14}^{-\beta} p(b)$$

(18)

where $p(b)$, now assumed to be normalized to unity,

$$\int p(b) db = 1$$

(19)

has the shape of any of the three distributions in Figure 5 (for example, equations (15) or (16) with parameters from Table 1). Our best estimate of $\beta$ (equation 12) is 1.43. Our best estimate of $\gamma$ (Paper I) was 2.46. We can estimate $N_0$ by equating the observed mean transmission (Paper I) to the integral of equation (18) weighted by the appropriate equivalent widths, namely

$$0.0037(1 + z)^{1+\gamma} = \int \frac{(1 + z)W(N, b)}{\lambda_0} N_0 (1 + z)^\gamma N_{14}^{-\beta} p(b) \, dN \, db$$

(20)

Doing the integral numerically for the three $b$ distributions in Figure 5, we obtain the absolute normalizations

$$N_0 = (2.57, 2.54, 2.63)$$

(21)

where the three values are using the observed (Carswell), gamma, and Gaussian distributions, respectively. The values are not, of course, as accurate as the number of significant figures shown, but are given to illustrate how little uncertainty is due to our remaining ignorance of the $b$ distribution. The units of equation (21) are “clouds per unit redshift per unit $\Delta N = 10^{14}\text{cm}^{-2}$ at $N = 10^{14}\text{cm}^{-2}$”. Most of the error in determining $N_0$ comes from the highly correlated errors in $\gamma$ and the value 0.0037, as described in Paper I. For redshifts in the range $3 < z < 4$, we estimate the overall error of equation (21) to be on the order of $\pm 10\%$. 
To compare with previous results by other investigators, we have also computed numerically the number of clouds with equivalent widths \( W > 0.32 \text{Å} \) (that value being conventional in the literature) per unit redshift, a quantity conventionally denoted \( N_0 \). We obtain
\[
N_0 = (4.2 \pm 0.5)(1 + z)^\gamma
\]  
(22)
(for redshifts \( 3 < z < 4 \)) which can be compared to the estimate of Murdoch et al. (1986) of 4.06 (no error given), and Jenkins and Ostriker’s (1991) estimate of 9 ± 3, obtained from Murdoch’s data for \( n(W) \) (the data points in our Figure 4) and their own measurements of the Lyman \( \alpha \) broadband decrement \( D_A \).

### 3.3. Upper Limits to Size Derived from Spacing

Knowing an absolute normalization on the distribution in \( N, b, z \), we immediately get some upper limits to the size of the clouds: Since the number of clouds decreases rapidly with increasing \( N \), the average size of clouds with column density \( \gtrsim N \) must be less than the average spacing along the line of sight of such clouds. There is, of course, no reason to think that this bound lies close to the true cloud sizes.

Distance along the line of sight is variously parametrized by the redshift separation \( dz \), the separation in observed wavelength \( d\lambda_{\text{obs}} \), the separation in comoving distance \( dr \), or the separation in physical (proper) distance \( dr_p \). The relation among these quantities is
\[
(1 + z)dr_p = dr = \frac{3000\text{Mpc} h^{-1}}{(1 + z)(1 + \Omega z)^{1/2}} \frac{\Delta\lambda_{\text{obs}}}{\lambda_0} = \frac{3000\text{Mpc} h^{-1}}{(1 + z)(1 + \Omega z)^{1/2}} dz
\]  
(23)
where \( h \) parametrizes the Hubble constant \( H_0 \) by \( h \equiv H_0/100 \text{ km s}^{-1}\text{Mpc}^{-1} \). Integrating equation (18) in both \( b \) and \( N \), the number of clouds with column densities greater than \( N \) in a redshift interval \( dz \) is
\[
dn_{>N} = \frac{N_0}{(\beta - 1)N_{14}^{\beta - 1}}(1 + z)^\gamma dz
\]  
(24)
Equations (23) and (24) imply a limit on the physical size \( r_{\text{phys}} \) of clouds
\[
r_{\text{phys}} < \frac{dr_p}{dn_{<N}} = \frac{3000 h^{-1} \text{Mpc} (\beta - 1)N_{14}^{\beta - 1}}{N_0} \frac{1}{(1 + z)^{\gamma + 2}(1 + \Omega z)^{1/2}}
\]  
(25)
Evidently the tightest limit is obtained by applying the argument at the highest redshift and to the smallest-\( N \) clouds. In Paper I, we saw that the value \( \gamma \approx 2.46 \) was justified up
to a redshift of at least $z \approx 4.2$, while Figure 6, above, shows that the observed equivalent width distribution of clouds justifies the value $\beta \approx 1.43$ down to at least $N = 10^{13}$ cm$^{-2}$, i.e., $N_{14} = 0.1$. Taking these values, and $N_0$ from equation (21) above, we get, for $N \sim 10^{13}$ cm$^{-2}$ clouds at $z \sim 4.2$,

$$r_{\text{phys}} < \begin{cases} 120 \, h^{-1} \, \text{kpc} & \text{if } \Omega = 0 \\ 50 \, h^{-1} \, \text{kpc} & \text{if } \Omega = 1 \end{cases}$$

(26)

Sargent (1988) summarizes the more direct evidence for the size of the clouds that comes from studies of close (or lensed) lines of sight at somewhat lower redshifts. It is believed that “typical” clouds (which we would infer to be at $N \sim 10^{14-15}$ cm$^{-2}$) have sizes in the range $\sim 8$ to $\sim 20$ kpc; however, these values are fairly uncertain. We return to the issue of cloud size in §5.

### 3.4. Where Does the Exponential Come From?

We have already seen in the three best-fitting curves of Figure 4 that the “standard” model of equation (18), with the parameter values given, is able to reproduce the long-observed exponential tail in the distribution of $W$, equation (14). One might well wonder how this manages to be true, since (if we momentarily exclude the gamma distribution model for $b$) there are apparently no linear exponentials anywhere in the model: the distribution in $N$ is power law, the distribution in $b$ is (or can successfully be taken to be) Gaussian, and the curve of growth varies in the saturated regime as $\ln \frac{1}{2} N$. The most true, but least illuminating, “explanation” is simply that the numerics of integrating together these three functional forms happens, readily, to produce something very close to – though not exactly – a linear exponential in $W$.

A more illuminating, though necessarily crude, explanation can be found in a toy analytical calculation which actually gives about the right value for the exponential scale $W^*$ in equation (14): From equation (6), we note that for small values of $\tau_0$ the equivalent width is approximately $W = \Delta \lambda_D \tau_0$; this is the “linear” part of the curve of growth, where $W$ increases linearly with $N$. This linear growth saturates when the line optical depth becomes of order unity, or when $\tau_0 = 1$. This implies a saturation value

$$W_{\text{sat}} = \Delta \lambda_D$$

(27)

which occurs at the saturation column depth of

$$N_{\text{sat}} = \frac{m_e c}{\pi e^2 \lambda_0 f}$$

(28)
For \( N \) larger than \( N_{\text{sat}} \), we shall approximate the “flat” part of the curve of growth literally by taking the equivalent width to be the constant \( W = W_{\text{sat}} \). We shall ignore the so-called “square root” part of the curve of growth, which occurs for much larger values of \( N \), as it appears from our numerical work that this part does not influence the results for the \( W \)’s we are concerned with. Our approximate relation for the equivalent width is then

\[
W(N, b) = \Delta \lambda_D \min\left(\frac{N}{N_{\text{sat}}}, 1\right) = A \min(N, Bb)
\]  

(29)

where the quantities \( A \) and \( B \) are defined by

\[
A = \frac{\pi e^2}{m_e c^2} \lambda_0^2 f; \quad B = \frac{m_e c}{\pi e^2 \lambda_0 f}.
\]

(30)

Now starting with the first form of equation (17), we have

\[
p(W) \propto \int dN \int db \delta[W - A \min(N, Bb)] N^{-\beta} p(b)
\]

\[
= \int_0^\infty db \delta(W - ABb) p(b) \int_0^\infty dN N^{-\beta} + \int_0^\infty dN \delta(W - A N) N^{-\beta} \int_{N/B}^\infty db p(b)
\]

\[
= \frac{1}{AB^\alpha} \int_{W/AB}^\infty dN N^{-\beta} + \frac{1}{A} \left(\frac{W}{A}\right)^{-\beta} \int_{W/AB}^\infty db p(b)
\]

(31)

We next choose the Gamma distribution (19) to represent \( p(b) \), along with the numerical values for the constants \( b_0 \) and \( b^* \) given in Table 1. Then

\[
p(W) \propto \frac{1}{(\beta - 1)AB^\alpha} \left(\frac{W}{A}\right)^{-\alpha-\beta} e^{-W/W^*} + \frac{b_0}{A} \left(\frac{W}{A}\right)^{-\beta} \Gamma(\alpha, W/W^*)
\]

(32)

where \( \alpha \equiv b_0/b^* = 2.7 \) and \( \Gamma \) denotes the incomplete gamma function. We have also defined the quantity

\[
W^* \equiv ABb^* = \frac{\lambda_0 b^*}{c}
\]

(33)

For Lyman alpha (assuming a mean redshift factor of \((1 + z) \approx 4.5\)) this implies \( W^* = 0.18b_6 \) Å, where \( b_6 \) is the value of \( b^* \) in units of \( 10^6 \) cm s\(^{-1} \) = 10 km s\(^{-1} \).

In the asymptotic regime \( W/W^* \gg 1 \), the incomplete gamma function can be replaced by its asymptotic value, which yields (omitting some overall factors),

\[
p(W) \propto (W/W^*)^{\alpha-\beta} e^{-W/W^*} \left[ 1 + \frac{\beta - 1}{W/W^*} \right]
\]

\[
\sim (W/W^*)^{\alpha-\beta} e^{-W/W^*},
\]

(34)
the first term clearly dominating. In the opposite limit \( W/W_* \ll 1 \), the second term dominates in equation (32), since \( \Gamma(\alpha, W/W_*) \approx \Gamma(\alpha) \) and the factor \( (W/A)^{-\beta} \) becomes very large. Thus

\[
p(W) \propto (W/W_*)^{-\beta}
\]  

(35)

It can be seen that this toy model shows more or less the right behavior. The large \( W \) limit (equation [34]), dominated by the flat part of the curve of growth, is roughly exponential [with some deviation induced by the factor \( (W/W_*)^{\alpha-\beta} \approx (W/W_*)^{1.3} \)]. In the small \( W \) limit, dominated by the linear part of the curve of growth, the power law in \( N \) makes its presence felt, and the distribution has the singular behavior \( (W/W_*)^{-\beta} \) as \( W \) goes to zero. These two behaviors are clearly seen in figure 5. Furthermore, the predicted value of \( W_* \) for the preferred value of \( b_* = 14 \text{ km s}^{-1} \) is \( W_* = 0.25 \text{A} \), which, given the crude approximations made, is not too far from the value given after equation (14). It should be emphasized that the approximations made in this calculation are not well justified, and that the answer is justified by the accurate numerical integration, not by this calculation. However, this toy example does show how it is possible for an exponential like equation (14), long observed and not previously explained, to emerge from a featureless distribution like equation (18). It also explains the upturn at small equivalent widths as a consequence of the power law distribution of column densities.

4. Fluctuations in the Transmission

In Paper I we measured the variance of the transmission \( Q \) for the Lyman forest of the SSG quasar sample, and obtained

\[
\frac{\text{Var}(Q)}{Q^2} = 0.06 \pm 0.01
\]  

(36)

This value has no fundamental significance by itself – it depends in part on the particulars of SSG’s spectroscopy, for example their slit resolution. In Paper I we developed an analytic model based on a power law distribution of \( N \), like equation (18) in this paper, but with a fixed line width (not equivalent width) \( W_0 \). The result was

\[
\frac{\text{Var}(Q)}{Q^2} = \frac{2W_0}{\Delta} \left[ \frac{1}{\kappa} (e^\kappa - 1) - 1 \right]
\]  

(37)

where

\[
\kappa \equiv (2 - 2^{\beta-1}) \ln \frac{1}{Q} \equiv (2 - 2^{\beta-1}) \tau
\]  

(38)
and $\beta$ has its same meaning as in equation (18). The instrumental resolution $\Delta$ is defined in general by

$$\Delta = \left( \int w d\lambda \right)^2 \left/ \int w^2 d\lambda \right.$$  \hspace{1cm} (39)

where $w(\lambda)$ is the instrumental response to a monochromatic line as a function of wavelength (in the same frame as $W_0$ is defined, either observed or at the absorption redshift). We noted in Paper I that, for the SSG data, equation (36) and (37) were in reasonably good agreement, though with perhaps some possibility that the required $W_0$ was unreasonably large, i.e., that the variance might be larger than could be explained by Poisson statistics alone.

We are now in a position to be more quantitative about whether there is a discrepancy, and also to look not only at the variance but also at quantities like the covariance $\langle Q(\lambda)Q(\lambda + \delta\lambda) \rangle$ which contain information on the two-point spatial correlation function of the Lyman $\alpha$ clouds. Suppose that $\xi(s)$ is the two point correlation function of the clouds at an observed wavelength separation $s$. (Below, we will convert from wavelength separations to comoving cosmological distances.) That is, given one cloud, the probability of finding a second cloud separated from the first by an observed absorption wavelength difference $s$ is the mean probability times $1 + \xi(s)$.

Because of finite spectral resolution of the SSG data, and the fact that individual clouds are not counted, the quantity $\xi(s)$ is not directly accessible to measurement. Instead, we can measure the covariance of the binned values of transmission,

$$\Xi_J \equiv \frac{1}{\tau^2} \frac{\langle Q_I Q_{I+J} \rangle - \langle Q_I^2 \rangle}{\langle Q_I^2 \rangle}$$  \hspace{1cm} (40)

where the angle brackets average over $I$, the index of the 10Å-separated bins of the SSG spectroscopic data. The factor of $\tau^{-2}$ (where $\tau$ is the mean optical depth) scales $\Xi$ to measure fluctuations in optical depth (that is, to fluctuations in number of clouds) rather than fluctuations in transmission: The two scalings are related by

$$Q \sim \exp(-\tau) \quad \text{implying} \quad \frac{\delta\tau}{\tau} \sim \frac{1}{\tau} \frac{\delta Q}{Q}$$  \hspace{1cm} (41)

If $w(s)$ is the instrumental response (as in equation (39)) above, then $\Xi_J$ (observable) is related to $\xi(s)$ (underlying) by

$$\Xi_J = \int \frac{w^*(s) \xi(s + J\Delta\lambda) ds}{\left[ \int w(s)^2 ds \right]}$$  \hspace{1cm} (42)
where $\Delta \lambda$ is the bin separation (here 10Å), and $w^*w(s)$ is (a single function of $s$) the convolution of $w$ with itself,

$$w^*w(s) \equiv \int w(s+y)w(y)dy$$  \hfill (43)

There are two interesting limits for equation (42): If $\xi(s)$ is slowly varying on a scale $x \gg \Delta$ (the instrumental resolution, equation 39), then $\Xi$ measures $\xi$ directly at wavelength separations $J\Delta \lambda$,

$$\Xi_J \approx \xi(J\Delta \lambda) \frac{\int w^*w(s)ds}{\int w^2(s)ds} = \xi(J\Delta \lambda)$$  \hfill (44)

If, on the other hand, $\xi(s)$ is nonnegligable only at values of $s$ that are $\ll \Delta$ (unresolved structure), then

$$\Xi_J \approx \frac{\int \xi(s)ds}{\Delta} \frac{\int w^*w(J\Delta \lambda)ds}{\int w^2(s)ds}$$  \hfill (45)

so that, in particular, the unlagged ordinary variance is given by

$$\Xi_0 \approx \frac{\int \xi(s)ds}{\Delta}$$  \hfill (46)

while higher values $\Xi_J$ decay as the normalized convolution of the instrumental response with itself. (For a square slit, this convolution is a triangular response starting at unity and going to zero when $s$ equals the full width of the slit.)

It is sometimes useful to think of $\xi(s)$ as having two additive parts, a Poisson piece $\xi_P$ that is some multiple of a Dirac delta function $\delta(s)$ at zero lag, and which describes fluctuations due to the discreteness of the clouds, and a smooth piece $\xi_S$ that is generated by spatial variations in the underlying mean density of clouds. (See Peebles, 1980, §33–§36, especially equation 36.7.) Then equation (37), divided by $\overline{\tau^2}$, can be viewed as calculating the strength of the delta function piece $\xi_P$ in equation (18).

With this formalism in hand, we now need to upgrade equation (37) to a more realistic model of cloud line statistics, in particular to the model equation (18) above. We have done this by Monte Carlo experiment. Repeatedly realizing equation (18) with Poisson statistics along a simulated line of sight, we have accumulated statistics on how the resulting variance varies with mean optical depth (or with $\kappa$ in equation 37). We find that the functional form of equation (37) is well reproduced, but with an “effective” value for $W_0$ (in the observed frame, e.g., with $\Delta = 25\text{Å}$ for the SSG data)

$$W_0 \approx (5.5 \pm 1) \frac{\langle b \rangle}{c} (1 + z) \lambda_0$$  \hfill (47)
The scaling with \( \langle b \rangle \), \( \lambda_0 \), and \( c \) are required by dimensional analysis. The factor \((1 + z)\) puts \( W_0 \) into the observed frame, for comparison with \( \Delta \). The Monte Carlo calculations verify (at about the 20% level) the shape of the dependence on \( \tau \), and compute the overall constant.

Using the numerical values \( \langle b \rangle = 37\text{km/s} \) (Table 1), \( \lambda_0 = 1216\text{Å} \), \( \Delta = 25\text{Å} \) (SSG), \( \langle 1 + z \rangle = 4.5 \) and \( \tau = 0.67 \) (Paper I, and SSG’s mean redshift), equations (37), (38), and (47) give

\[
\text{Var}(Q)/Q^2 = 0.076
\]

which is in fairly good agreement with Paper I’s measured value (or equation 36 above), and which will prove to be in excellent agreement with a slightly different way of reducing the data, below.

It appears, then, that there is no significant clumping of the Lyman \( \alpha \) clouds at high redshifts \( z \approx 3.5 \) in the zero-lag variance: essentially all of the variance, i.e., the fluctuations in the absorption, can be explained as Poisson fluctuations in a randomly distributed population of clouds drawn from the distribution of equation (18). In the language introduced above, \( \xi_P \) is as expected from equation (18), and \( \xi_S \) is consistent with zero.

What about the covariance at nonzero lags, equation (40)? The solid circles in Figure 7 show the result of calculating this quantity (or, actually, \( \tau^2 \) times this quantity) for the SSG data. Several remarks and caveats are necessary in interpreting this Figure:

It is actually quite difficult to determine the zero point of the covariance with high accuracy. The structure function of the absorption data is found to increase slowly with increasing lag, as would be expected from small errors in fitting the the continuum slopes of the individual SSG quasars. This causes the raw correlation function (essentially a constant minus the structure function) to have an ill-defined additive constant (see e.g. §3 of Press, Rybicki, and Hewitt 1992). We fix the zero point in Figure 7 by the assumption that there should be no significant mean correlation \( \xi \) on scales of 10 to 20 bins (corresponding, we will see below, to several tens of comoving Megaparsecs): In this range, we fit for an r.m.s. continuum slope error, and extrapolate the resulting fitted model back to smaller lags.

The astute reader might notice that the value at zero lag,

\[
\text{Var}(Q)/Q^2 = 0.073 \pm 0.008
\]

is different from Paper I’s value, quoted in equation (36) above. The reason is the different method, just described, used to determine the zero point. (See Paper I for a description of the moving average method there used to remove systematic errors due to incorrect
extrapolations of the continuum spectra of the SSG quasars.) One would be correct in concluding that there may well be on the order of 0.01 in unremoved systematic errors in the data shown in Figure 7. The error bars shown are determined by resampling of the SSG quasars, and should accurately reflect the statistical errors, but not the systematic errors, of the points.

The solid line in Figure 7 is the result of predicting $\text{Var}(Q)/\overline{Q}^2$ from equation (42), a square slit response function of width 25Å (SSG’s value), and assuming Poisson distributed clouds (equations 37 and 47) with no additional correlation function, i.e., $\zeta_S = 0$. There are no new free parameters in this prediction, i.e., it is not normalized to the data in any way. We see that not only is the agreement excellent at zero lag, but also the expected slit response is accurately reproduced at lags 1 and 2. In bins 3 through 7 or 8, one sees a very interesting “shoulder” decaying from about 0.01 (or $\zeta = 0.01 \tau^{-2} \approx 0.02$), to about half this value. While there may be some diffraction widening of the slit response present in bin 3, it is quite unlikely that it should extend to bin 7 (70Å out). Further, the shoulder is highly significant in terms of the statistical errors measured by resampling. Unfortunately, the shoulder is at about the amplitude where we do not have confidence in our removal of systematics. We consider it suggestive as a marginal, but not believable, detection of a correlation function at high redshifts and large scales.

With greater confidence we can assert a value of twice the shoulder value as a good upper limit to the value of the correlation function $\xi$ on scales from 3 to 7 bins. That is, at redshift $z \approx 3.5$,

\[
\begin{align*}
\xi &< 0.04 \quad \text{for} \quad \Delta \lambda_{\text{obs}} = 30 \text{ Å} \\
\xi &< 0.02 \quad \text{for} \quad \Delta \lambda_{\text{obs}} = 70 \text{ Å}
\end{align*}
\]

(50)

The relation between $\Delta \lambda_{\text{obs}}$ and comoving distance scale $\Delta r$ was already given in equation (23). It follows that the 30Å and 70Å scales in equation (50) correspond to 7.8 and 18 $h^{-1}$ Mpc, respectively, if $\Omega \approx 1$; or to 16 and 38 $h^{-1}$ Mpc, respectively, if $\Omega \approx 0$.

It is tempting to identify the apparent shoulder as the progenitor, in the gravitational instability picture, of correlation structure that is seen today. In an $\Omega = 1$ universe, the growth of perturbations in the linear regime would increase $\xi$ to the present by a factor $(1 + z)^2 \approx 20$, so that the scaled values of equation (50) are within a plausible factor of today’s observed values (e.g., the IRAS galaxies in Saunders et al. 1992) on the corresponding spatial scales. Indeed, one might hope to determine the value of $\Omega$ from such an analysis, since the growth factor of perturbations to the present is suppressed by a known factor in lower $\Omega$ cosmologies.

The problems with carrying out this program include (i) the uncertain systematic
errors in Figure 7; (ii) the fact that, if simplistic comparison with today’s correlation function is justified at all, then one predicts much larger covariances at lags 0 to 2 than are seen; (iii) the fact that the Lyman α clouds are particularly likely, in most simple theoretical pictures, to be “ionized away” when they are nearest to density peaks where quasars (or other ionization sources) may have formed. In fact, item (iii) may be the explanation of item (ii); but it is clear that no simple analysis can carry any degree of credibility. We therefore defer these issues to a later paper.

Here, sticking more closely to the data, it is of interest to see if any evolution with redshift can be detected in the correlation properties of the SSG sample. Figure 7 also plots, as smaller open symbols, the results of dividing the sample into 3 redshift ranges, \( z < 3 \), \( 3 < z < 3.5 \), and \( 3.5 < z \). The corresponding dashed curves are the predictions of the pure Poisson model, equation (18), with no adjusted parameters and with the assumption of a Poisson distribution, \( \xi_S = 0 \). In general, the prediction for higher redshifts is towards larger fluctuations, due both to the \( \kappa \) dependence in equation (37) and to the explicit dependence on \( 1 + z \) in equation (47). One sees that, to the level of possible systematic errors previously described, there is good agreement for the two higher redshift samples, with mean redshifts \( \langle z \rangle = 3.3 \) and \( \langle z \rangle = 3.8 \). For, the lowest-redshift sample, with mean redshift \( \langle z \rangle = 2.8 \), there is some indication (with large statistical error bars, as shown in the Figure) of an excess variance – the triangles (highest) are about 2-\( \sigma \) from the dotted (lowest) curve. The excess has the triangular instrumental profile, suggesting that it could be explained as an unresolved correlation function (equation 46) with

\[
\int \xi(\lambda_{\text{obs}})d\lambda_{\text{obs}} \approx (18 \pm 9) \text{ Å}
\]  

This value is several times larger than the value that we infer from the data of Webb (1986) as shown in Carswell (1989); however the error bars are large, so the discrepancy may not be real.

Qualitatively, our data support previous suggestions, e.g., by Carswell on the basis of higher-redshift data from Atwood et al. (1985) and Carswell et al. (1987), that the correlation function grows from an undetectable to a significant level in the fairly narrow redshift range between \( z \approx 3.5 \) to \( z \approx 2.5 \). (See also Shaver, 1988.) This redshift range also marks the epoch where (i) there is a change in the value of \( \gamma \), the exponent relating cloud density and redshift (see, e.g., Figure 9 in SSG), and (ii) metal line clouds become important. We think that it is fair game for theorists to infer a connection among these phenomena. In particular, it may be that the spatially correlated population of Lyman α clouds that appears at redshifts \( z \lesssim 2.8 \), perhaps associated with galaxy formation, is a completely distinct population from the more rapidly disappearing (towards low redshifts) population of primordial, spatially uncorrelated, clouds.
5. Where Are All the Baryons?

Models for light element production in the hot big bang give good limits on $\Omega_b H_0^2$, where $\Omega_b$ is the present fraction of critical density due to baryonic matter. Recent analyses (Olive et al. 1990, Walker et al. 1991) give the value and uncertainty

$$\Omega_b h^2 = 0.013 \pm 0.003$$  (52)

Obviously it is of interest to know whether this density of baryons can be accommodated, at high redshift, completely in the observed density of Lyman $\alpha$ clouds. By far the greatest uncertainty, as we shall now see, lies in the ionization state of the clouds and, in turn, in the question of whether the ionization state is consistent with the interpretation of observed $b$ values as thermal, and/or with an equilibrium thermal and ionization model.

The clouds are predominantly ionized. Measured, or deduced, column densities $N$ refer in the first instance to neutral hydrogen (whose corresponding volume number density we will denote $n$), while the value of $\Omega_b$ is determined by the corresponding column or volume densities of total hydrogen, which we denote $N_H$ and $n_H$. Let $f_0$ denote the hydrogen neutral fraction; and let $\rho$ denote the total cosmological baryon density (including an assumed helium mass fraction $\approx 0.25$); so we have the relations

$$N = f_0 N_H \quad n = f_0 n_H \quad \rho = 1.33 m_p n_H$$  (53)

We can compute the mean (smoothed) density of hydrogen $n_H$ at a given redshift $z$ along the line of sight by knowing, as we do, the mean density of clouds with each possible column density $N$ (equation 18), as follows:

$$n_H = \frac{dN_H}{dr_p} = \frac{dz}{dr_p} \frac{dN_H}{dz} = \frac{dz}{dr_p} \int \left( \frac{N}{f_0(N)} \right) N_0(1 + z)^\gamma N_1^{-\beta} dN_{14}$$

$$= 2.8 \times 10^{-14} h \text{ cm}^{-3} (1 + z)^{\gamma+2} (1 + \Omega z)^{1/2} \int \frac{N_{14}^{-\beta+1}}{f_0(N)} dN_{14}$$  (54)

Here $r_p$ is physical (proper) distance (see equation 23), and $f_0(N)$ denotes the harmonic mean value of $f_0$ for all clouds with neutral column density between $N$ and $N + dN$, averaging over any other internal parameters (notably $b$). That is, $f_0(N)$ is the value defined by $N_H = N/f_0(N)$. It is an important point that, if $f_0(N)$ is averaged correctly, then equation (54) is independent of the physical size, shape, and distribution of the clouds,
e.g., whether they are spherical, or thin sheets, clumped or unclumped, etc. Equation (54) requires only that a random line of sight be a fair sample of the universe.

We can rewrite equation (54) as a direct measurement of $\Omega_b h^2$,

$$
\Omega_b h^2 = \frac{1.33 \rho n_H}{1+z} \frac{8 \pi G}{3 H_0^2} = 3.4 \times 10^{-9} h(1+z)^{-1}(1+\Omega z)^{1/2} \int \frac{N_{14}^{-\beta+1}}{f_0(N)} dN_{14}
$$

(55)

where $\tilde{H}_0 \equiv 100 \text{ km s}^{-1}$.

5.1. Inconsistency of Equilibrium Model with $\Omega_b$

Let us consider first the baseline hypothesis that the clouds are in thermal and ionization equilibrium with a background UV flux, and that the measured $b$ velocity widths are thermal. (We will see, in fact, that this baseline hypothesis leads to an observational reductio ad absurdum.)

It follows from the analysis of Black (1981) that there is a broad regime of density $\rho$ and incident UV intensity $J_\nu$ (Lyman limit intensity in units ergs cm$^{-2}$s$^{-1}$Hz$^{-1}$sr$^{-1}$) where both the temperature of the gas, and also its neutral fraction, depend only on the combination $J_\nu/\rho$. In particular, if $\rho$ and $J_\nu$ are in c.g.s. units, then

$$
T = 1690 \text{ K} \mu \left(\frac{J_\nu}{\rho}\right)^{2/7} \text{ or } b = 5.3 \text{ km s}^{-1} \mu^{1/2} \left(\frac{J_\nu}{\rho}\right)^{1/7}
$$

(56)

and

$$
f_0 = 397 \left(\frac{J_\nu}{\rho}\right)^{-1.22}
$$

(57)

(Here $\mu$ is the mean molecular weight of the ionized gas.) According to Black, these expressions are applicable at least in the range $5 \times 10^3 < T < 5 \times 10^5$K, $10^{-6} < n_H < 10^{-3}$cm$^{-3}$, $10^{-22} < J_\nu < 10^{-20}$ (c.g.s; note that Black’s $J_0$ is $4\pi$ times our $J_\nu$).

Equations (56) and (57) imply a unique relationship between the neutral fraction $f_0$ and the velocity (or temperature) parameter $b$,

$$
f_0 = \left(\frac{b}{8.56 \text{ km/s}}\right)^{-8.54} \left(\frac{\mu}{0.64}\right)^{4.27} = 3.7 \times 10^{-6} \left(\frac{b}{37 \text{ km/s}}\right)^{-8.54} \left(\frac{\mu}{0.64}\right)^{4.27}
$$

(58)

and a relationship among $n_H$, $b$, and $J_\nu$,

$$
n_H = \frac{\rho}{1.36 m_p} = 1.2 \times 10^{-4} \text{ cm}^{-3} \left(\frac{b}{37 \text{ km/s}}\right)^{-7} \left(\frac{\mu}{0.64}\right)^{7/2} \left(\frac{J_\nu}{10^{-21} \text{ c.g.s}}\right)
$$

(59)
(Hereafter, we will take \( \mu = 0.64 \) and suppress the parametric dependence on \( \mu \).) Obviously the large values of the exponents in \( b \) require that we use these relations with some caution. Various authors have estimated \( J_\nu \) as being in the range \( 10^{-22} \) to \( 10^{-21} \); see Bajtlik, Duncan, and Ostriker (1988) for discussion.

A first application of equation (55) is to assume that \( f_0 \) is independent of \( N \) and uniform throughout any given cloud. This would be true for the base case of pressure confined clouds in a uniform confining medium and with a uniform UV illumination. Then there is no subtlety in averaging \( f_0(N) \); it can come out of the integral as \( f_0 \) in equation (58). The remaining integral is weakly divergent, so we must adopt some value \( N_{\text{max}} \) for a cutoff in the column density, in terms of which we get,

\[
\Omega_b h^2 = 1.6 \times 10^{-3} h (1 + z)^{\gamma - 1} (1 + \Omega z)^{1/2} \left( \frac{b}{37 \text{ km s}^{-1}} \right)^{8.54} N_{14\text{ max}}^{0.53} \tag{60}
\]

Here the normalizing value for \( b \), namely 37 km s\(^{-1}\), has been chosen to be the observed mean value for \( b \) (Table 1).

One sees that the cosmologically observed value for \( \Omega_b h^2 \) (equation 52) is not just easy to produce, it is easy to vastly exceed: Table I shows that the appropriate average \( \langle b^{8.54} \rangle^{1/8.54} \) is apparently on the order of 70 km s\(^{-1}\); and \( N_{14\text{ max}} \) is surely not less than 100, corresponding to \( N_{\text{max}} = 10^{16}\text{cm}^{-2} \), and probably greater. These values would imply \( \Omega_b h^2 \approx 5h \), which is excluded even for a fully baryonic \( \Omega = 1 \) universe!

One might momentarily wonder whether internal structure within a single cloud, with the temperature \( T \) varying along the line of sight provides a possible resolution. However, such variation always acts in the wrong direction: It is possible to hide more baryons within a cooler, narrower, undetectable line core; but baryons implied by an observed thermal line width must always be there.

One can turn the argument around by inverting equation (60) for \( b \). (Now we are on the good side of the large exponent, and thus quite insensitive to the other assumptions made.) One finds that, for \( N_{\text{max}} \) in the plausible range \( 10^{18}\text{cm}^{-2} \) down to \( 10^{16}\text{cm}^{-2} \), the observed \( \Omega_b \) is consistent with an ionization fraction \( f_0 \) that would in equilibrium derive from thermal values of \( b \) in the range 20 to 25 km s\(^{-1}\). Looking back at Figure 5, one sees that this velocity range is where the observed \( b \) distribution is falling off rapidly to smaller \( b \) values; in other words, considering the possibility of observational errors and other sources of dispersion, the deduced equilibrium \( b \) resembles a lower bound to the observed \( b \) values.

5.2. Inconsistency of Equilibrium Model with Size and Mass of Clouds
Let us follow the previous logic to the extreme. Equations (58) and (59) give a characteristic length scale $L$ for a cloud with parameters $N$, $b$, $J_\nu$, namely,

$$
L \sim \frac{N}{n_H f_0} \sim 7.3 \times 10^5 \text{ pc} \left( \frac{N}{10^{15}\text{cm}^{-2}} \right) \left( \frac{b}{37 \text{ km/s}} \right)^{15.54} \left( \frac{J_\nu}{10^{-21} \text{ c.g.s}} \right)^{-1}
$$

There are two separate points to be made about equation (61). First, the value obtained for $L$ is implausibly large, not only by the apparent factor of $\sim 100$ for the parameters having the scaling values given (see discussion of cloud sizes at the end of §3.3), but also by an additional factor of about $2^{15.54} \sim 4 \times 10^4$ if the distribution for $b$ has the broad tail found in §3.

In view of the large exponent on $b$, a better use of equation (61) is, as before, to run it backwards: Suppose that $L$ is in the range $10^4 \pm 1 \text{ pc}$, as seems required by other observations, and that the other parameters have the scaling values given. Then (61) implies

$$
\langle b_{eq}^{15.54} \rangle^{1/15.54} \approx 28 \pm 5 \text{ km s}^{-1}
$$

where $b_{eq}$ is the thermal $b$ value that would be in equilibrium with the required neutral fraction $f_0$.

The point can be made even more forcefully if we write the characteristic mass $M$ of the cloud (more accurately, the mass of that part of the cloud whose size is on the order of the line of sight penetration),

$$
M \sim \rho L^3 \sim 1.48 \times 10^{12} M_\odot \left( \frac{N}{10^{15}\text{cm}^{-2}} \right)^3 \left( \frac{b}{37 \text{ km/s}} \right)^{39.62} \left( \frac{J_\nu}{10^{-21} \text{ c.g.s}} \right)^{-2}
$$

Here the value of the exponent in $b$ is truly worthy of awe. If we reverse the equation, and assume $M$ in the range $10^8 \pm 3 M_\odot$, which is cosmologically plausible, we get

$$
\langle b_{eq}^{39.62} \rangle^{1/39.62} \approx 29 \pm 6 \text{ km s}^{-1}
$$

An additional interesting point about equation (63) is that, because of the enormous exponent, it is simply not possible to accommodate any significant range of $b$ within the cosmologically available mass range for $M$, if $b$ is related to neutral fraction by equation (60) or anything similar. A factor of two variation in $b$ from cloud to cloud, at fixed $N$, induces a mass range of a factor $10^{12}$! Also note that the observed range of $N$, which we take conservatively in this paper to be $\sim 10^3$, itself induces a mass range of a factor $10^9$. Moreover, many observers (e.g., Petitjean, 1992) show evidence of $N$ itself extending over 8 orders of magnitude!
6. Possible Resolutions

Quite obviously, the UV-driven equilibrium thermal model for $b$ is unviable. On the one hand (§3), a broad distribution of $b$’s, extending to 80 km s$^{-1}$ or higher, is observationally required. On the other hand (§6), the neutral fractions implied by such values of $b$ are incompatible with observed limits on $\Omega_b$, the size and mass of the clouds, and the dynamic range of mass available to the clouds.

Any possible resolution of the paradox must substantially disconnect a measured value of $b$ from the ionization state of its cloud. There would seem to be two ways to do this. First, as has been long noted in the literature, the observed value $b$ could be due to bulk matter motions rather than thermal velocities.

The bulk motion hypothesis raises a range of theoretical difficulties that have been discussed by other investigators. More observationally, in the context of this paper, it is worth noting that, if $b$ includes a significant component of bulk motions, then any curve of growth analysis based on Voigt profiles (including that of §2) will be very seriously in error in the saturated region. In particular, if, within a single cloud, the velocity tail falls off less rapidly than a Gaussian (as one would expect of the broad-tailed distributions characteristic of hydrodynamic motions), then the actual value of $N$, the neutral column density, can be much less than deduced from standard curve of growth. The widely observed number distribution of $N$’s up to high values is then based on an incorrect analysis, and the fact that the observed distribution is a power law that extrapolates smoothly from smaller $N$ values becomes something of a miracle.

One should also mention the observational issue usually termed “the $b$–$N$ controversy”, and described in Shaver, Wampler, and Wolfe (1991). There, claims is of a positive observed correlation between $b$ and $N$ (which would be theoretically expected in most mechanisms for bulk motion) have proved quite difficult to substantiate and are thought by many to be entirely due to selection effects. In this context we should note the success of our assumption, in §3, of uncorrelated $N$’s and $b$’s in satisfying all observational constraints considered.

Finally, it is hard to reconcile most mechanisms for generating bulk motion with the observed lack of cloud clumping on small scales (§4). Hydrodynamic or gravitational processes capable of driving highly supersonic differential motions within a cloud should also be expected to act on scales comparable to intercloud distances (§3.3). Our analysis of fluctuations in §4 would easily have detected (e.g. in the zero-lag bin of Figure 7) any tendency for clouds to be clumped in groups as small as a few, or for that matter any comparable tendency for clouds to be more uniformly distributed than random. The lack
of such a detection tends to support less violent scenarios of cloud formation and evolution than those that can give large bulk motions.

An alternative to the bulk motion hypothesis (as particularly emphasized to us by M. Rees; see also Rees 1988) is the notion that observed $b$'s do represent the thermal cloud state, but that this thermal state is not in equilibrium with the ionization state determined by interaction with the UV background. In particular, it may be possible to place the clouds in a physical regime where their thermal cooling times are not short compared with the (then) Hubble time.

If such a picture is combined with the idea that we are seeing clouds that have collapsed and heated adiabatically, then a consistent picture may possibly emerge: The neutral fraction $f_0$ is determined over time by interaction with a background UV flux. There is an implied concomitant universal heating of the gas, to a temperature that one might identify with a value at or below the lower edge of the $b$ distribution in Figure 5.

Subsequently, clouds collapse by volume factors that vary somewhat from cloud to cloud, giving the observed broad, indeed thermal, distribution of $b$. Since adiabatic heating is inefficient at altering ionization, the values $f_0$ remain in a universal, narrow range, as required by the arguments of §5. In Paper III we will investigate this, and related, models.

7. Conclusions

The conclusions of this paper are as follows:

1. The equivalent width ratios for the Lyman sequence found in Paper I are completely consistent with a featureless power law distribution of Lyman $\alpha$ clouds in column density $N$, with exponent $\beta = 1.43 \pm 0.04$. Distributions with a break in the power law can also fit the equivalent width ratio data when the break is around $N = 10^{15}\text{cm}^{-2}$.

2. Assuming the above featureless power law distribution in $N$, standard curve of growth analysis is able to reproduce the detailed distribution of cloud equivalent widths (notably both the exponential tail at large equivalent widths, and the turn-up at small equivalent widths), but only for distributions in $b$ that have mean values $\sim 37\text{ km s}^{-1}$ and a significant tail extending as high as $70\text{ km s}^{-1}$. Such distributions are in fact observed by some observers.

3. The turn-up at small equivalent widths likely does not mark any change in the properties of the clouds, but is a consequence purely of the curve of growth along the
observed line of sight.

4. Broken power laws, which do fit the \( N \) distribution, do not fit the equivalent width distribution as well, but they are not wholly ruled out.

5. The long-observed exponential tail in the distribution of equivalent widths can be real without being physically fundamental: it emerges as an artifact of combining the tail of the distribution in \( b \) with the slowly rising equivalent widths characteristic of the curve of growth in the saturated region.

6. The posited distributions in \( N \) and \( b \), plus Paper I’s normalization of the absorption as a function of redshift, give an absolute normalization on the number of clouds as a function of \( N, b, \) and \( z \). Upper limits on the physical size of clouds, in the range of 50 to 120 \( h^{-1} \) kpc at \( z \sim 4.2 \), follow.

7. At the highest redshifts studied, the posited distributions in \( N \) and \( b \) are able completely to account for fluctuations in the absorption (along different lines of sight or as a function of redshift), as due to Poisson fluctuations in an uncorrelated cloud distribution. However, there are marginal (not by themselves very believable) detections of a nonzero correlation function in two different regimes: (i) At the lowest redshift accessible to this study, \( z \approx 2.8 \), there is some evidence of unresolved correlation on a scale \( \lesssim 2h^{-1} \) Mpc, possibly with amplitude somewhat greater than previously reported. (ii) In the full sample, on scales of tens of Mpc, there is some evidence of correlation at a level \( \xi \sim 0.01 \) or 0.02. Because of possible systematic errors, however, we prefer, to take twice these values as upper limits.

8. If the observed \( b \) distribution is thermal, and if the ionization state of the gas is related to its thermal state by an equilibrium model (e.g., Black 1981), then the observed broad \( b \) distribution would imply \( \Omega_b \gg 1 \) in clouds. Similarly, the broad \( b \) distribution implies clouds that are both too large and too massive.

9. On the other hand, if the ionization state is that implied by a UV-heated temperature of about 20 to 25 km s\(^{-1}\), then at high redshift the Lyman \( \alpha \) clouds contain all the baryons deduced from models of light element production in the hot big bang (i.e., probably all the baryons in the universe).

10. The broad \( b \) distribution must therefore be ascribed either to bulk motions (which raises both theoretical difficulties and likely incompatibilities both with the observed featureless power law distribution in \( N \) obtained from curve-of-growth analyses, and with the observed lack of small-scale cloud clumping), or else the observed clouds must be out of thermal equilibrium, and heated by a process that is relatively inefficient at ionization.
Recent adiabatic collapse is a plausible candidate.

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Fig. 1.— Equivalent width $W$ for Lyman series lines in hydrogen as a function of column density $N$ and thermal velocity parameter $b$. Equivalent widths grow linearly with $N$ in the unsaturated region below $N \sim 10^{14-15}$ cm$^{-2}$, until they saturate at a value set by the thermal doppler width. In the saturated regime, widths grow slowly as $\sim \log^{1/2} N$.

Fig. 2.— Goodness-of-fit $\chi^2$ values as a function of exponent $\beta$ for fitting observed Lyman series absorption ratios and a curve of growth model to a power law model for the number of clouds as a function of their column density $N$. With four measured ratios, there are four degrees of freedom. The value of the exponent is seen to be narrowly determined (note expanded abscissa) independently of the assumed value for the velocity parameter $b$.

Fig. 3.— Goodness-of-fit $\chi^2$ values as in Figure 2, but fitting for the characteristic column density $N$ associated with the broken power-law model of equation (13) [broad parabolas] or the unrealistic model of a single, universal $N$ [narrow parabolas]. One sees that, while none of these models are excluded by consideration of absorption ratios alone, the ratios are fairly powerful at fixing one scale parameter.

Fig. 4.— Relative number of Lyman $\alpha$ clouds as a function of their rest equivalent width. Filled circles: data compiled by Murdoch et al. (1986). Note the famous exponential tail extending over $> 2$ decades in cloud number density. Solid, long- and short-dashed curves: Predictions based on curve of growth analysis and the three distributions for velocity parameter $b$ that are shown in Figure 5. The exponential tail emerges naturally from $b$ distributions with $\langle b \rangle \approx 37$ km s$^{-1}$ and $\sigma_b \approx 20$ km s$^{-1}$ (see text for details). Dotted and dash-dot curves: Models which deviate from either a broad $b$ distribution or a pure power-law distribution in $N$ do not give acceptable fits to the observed equivalent width distribution.

Fig. 5.— Solid histogram: Carswell’s (1989) compilation of the distribution of $b$ values for Lyman $\alpha$ clouds. Long- and short-dashed curves: gamma law and truncated Gaussian distributions, each parametrized by a mean and width, which give best fits to the observed equivalent width distribution shown in Figure 4. One sees that the behavior of the $b$ distribution at small values of $b$ is not well constrained, but that both models, and Carswell’s data, indicate a tail of large $b$ values extending to $> 80$ km s$^{-1}$.
Fig. 6.— To determine the range of column densities $N$ that are tested by the excellent agreement of models with data in Figure 4, the mean value $N$ of clouds with a specified equivalent width is here plotted against that equivalent width, for four of the $b$ models shown in Figure 4. One sees that all the models probe the range $N \sim 10^{13}$ cm$^{-2}$ to $N \sim 10^{16}$ cm$^{-2}$.

Fig. 7.— Fluctuations in transmission along the line of sight, here shown as the fractional covariance of absorption along a single line of sight measured at two wavelengths separated by an integral number of 10Å bins. Solid dots: observed covariance for the full SSG sample of quasars. Error bars are determined by bootstrap resampling. Solid curve: prediction (with no adjustable parameters) for unclustered clouds with a distribution in $z$, $N$, and $b$ determined in this paper without reference to fluctuation statistics. The triangular shape comes from the instrumental response of an assumed square slit. The good agreement between the data and the prediction put strong limits on a cloud correlation function $\xi$. The shoulder in bins 3 through 7, if real, implies a value $\xi \sim 0.02$ on a comoving scale $\sim 10 h^{-1}$Mpc at redshift $z \approx 3.5$. However, systematic errors in the determination of the zero point are also of the same order, so the detection is suggestive only. (See text for details.) The open symbols and dotted curves repeat the analysis for low-, medium-, and high-redshift subsamples of the full sample. There is weak (about 2 $\sigma$) evidence for the appearance of a positive correlation function in the lowest-redshift sample (see text).