Bananas: multi-edge graphs and their Feynman integrals

Dirk Kreimer

Received: 20 January 2023 / Revised: 2 March 2023 / Accepted: 13 March 2023 / Published online: 1 April 2023
© The Author(s) 2023

Abstract
We consider multi-edge or banana graphs $b_n$ on $n$ internal edges $e_i$ with different masses $m_i$. We focus on the cut banana graphs $\Im(\Phi_R(b_n))$ from which the full result $\Phi_R(b_n)$ can be derived through dispersion. We give a recursive definition of $\Im(\Phi_R(b_n))$ through iterated integrals. We discuss the structure of this iterated integral in detail. A discussion of accompanying differential equations, of monodromy and of a basis of master integrals is included.

Keywords  Feynman integrals · Analytic structure · Dispersion relations

Mathematics Subject Classification  81T15 · 81Q30 · 57T05

1 Introduction

We define a banana graph $b_n$ by two vertices $v_1, v_2$ connected by $n$ edges forming a multi-edge. Furthermore, $v_1, v_2$ are both $n + 1$ valent vertices so that $b_n$ has an external edge at each vertex.

1.1 General considerations

We study associated banana integrals $\Phi_R^D(b_n)$. The case $n = 3$ has been intensively studied and initiated a detailed analysis of elliptic integrals in Feynman amplitudes, see, for example, [1–11]. Evaluation at masses $m_i^2 \in [0, 1] \ni k_2^2$ was recognized to provide a rich arena for an analysis of periods in Feynman diagrams [12] including the appearance of elliptic trilogarithms at sixth root of unity in the evaluation of $b_4$ [8].

Let us pause and put the problem into context.

1 Often $b_2$ is called a bubble, $b_3$ a sunset and $b_4$ a banana graph. We call all $b_n$, $2 \leq n < \infty$ banana graphs.
1.1.1 Recursion and splitting in phase-space integrals

The imaginary part \( \Im(\Phi_D R(b_n)) \) of \( \Phi_D R(b_n) \) has been a subject of interest for almost seventy years at least [13–15]. This imaginary part has the interpretation of a phase space integral. Our attempt below to express it recursively by an iterated integral can be traced back to this early work. In fact, computing \( \Im(\Phi_D R(b_n)) \) by identifying an imaginary part \( \Im(\Phi_D R(b_{n-1})) \) as a subintegral amounts to a split in the phase-space integral and this recurses over \( n \).

1.1.2 Banana integrals and monodromy

In the more recent literature, the graphs \( b_n \) were studied in an attempt to interpret the monodromies of the associated functions depending on momenta and masses \( \Phi_D R(b_n)(s, s_0, \{m_i^2\}) \) as a generalization of the situation familiar from the study of polylogarithms. This role of elliptic functions was prominent already in the historical work cited in Sect. 1.1.1 and continued to give insights into the structure of phase space systematically [5, 9]. Recently, the aim shifted to explore it in the spirit of modern mathematics. This brought concepts developed in algebraic geometry—motives, Hodge theory, co-actions, symbols and such—to the forefront [7, 8, 11, 16–19]. For us, the focus is less on elliptic integrals and elliptic polylogarithms prominent in recent work. Rather, we focus on the recursive structure of \( \Im(\Phi_D R(b_n)) \) as it has a lot to offer still for mathematical analysis.

1.2 Iterated integral structure for \( b_n \)

Our task is to find iterated integral representations for \( \Im(\Phi_D R(b_n)) \) which give insight into their structure for all \( n \). We will use \( \Im(\Phi_D R(b_2)) \) as a seed for the iteration. \( \Im(\Phi_D R(b_3)) \) which has \( \Im(\Phi_D R(b_2)) \) as a subintegral then gives a complete elliptic integral as expected, see Sect. 2.3. Already, the computation of \( b_4 \) indicates more subtle functions to appear as Sect. 2.5 and Eq. (2.11) demonstrate. Nevertheless, it turns out that such functions are very nicely structured as we explore in Sect. 2.6.

We want to understand the function \( \Phi_D R(b_n) \) obtained from applying renormalized Feynman rules \( \Phi_D R \) in \( D \) dimensions

\[
\Phi_D R(b_n) = S_R^\Phi \Phi^D(b_n)(s, s_0),
\]

to the graph \( b_n \).

We will study in particular the imaginary part \( \Im(\Phi_D R(b_n)) \) having in mind that \( \Phi_D R(b_n) \) can be obtained from \( \Im(\Phi_D R(b_n)) \) by a dispersion integral.

We will mostly work with a kinematic renormalization scheme in which tadpole integrals evaluate to zero. This is particularly well suited for the use of dispersion. Indeed, \( \Im(\Phi_D R(b_n)) \) is free of short-distance singularities as the \( n \) constraints putting \( n \) internal propagators on-shell fix all non-compact integrations.

This reduces renormalization of \( b_n \) to a mere use of sufficiently subtracted dispersion integrals. Correspondingly, in kinematic renormalization we can work in a Hopf
algebra $H_R = H/I_{\text{tad}}$ of renormalization which divides by the ideal $I_{\text{tad}}$ spanned by tadpole integrals rendering the graphs $b_n$ primitive:

$$\Delta H_R(b_n) = b_n \otimes I + I \otimes b_n.$$ 

Therefore,

$$S_R^D \Phi^D(b_n) = \Phi^D(b_n)(s) - T^{(j)} \Phi^D(b_n)(s, s_0).$$

$\Phi^D$ are the unrenormalized Feynman rules in dimensional regularization and $T^{(j)}$ is a suitable Taylor operator.

Nevertheless, there is no necessity to regulate Feynman integrals in our approach as we can subtract on the level of integrands. Indeed, $T^{(j)}$ can be chosen to subtract in the integrand. We implement it in Eq. (1.1) using the dispersion integral. Our conventions for Feynman rules are in “App. A”.

Our interest lies in a compact formula for

$$\Im \left( S_R^D \Phi^D(b_n)(s, s_0, \{m_i^2\}) \right) = \int_{M_n} I_{\text{cut}}(b_n),$$

with $I_{\text{cut}}(b_n)$ given in Eq. (A.1). We will succeed by giving it as an iterated integral in Eq. (2.14) which is part of Thm. (2.2).

Results for $\Phi^D_R(b_n)(s, s_0, \{m_i^2\})$ then follow by (subtracted at $s_0$) dispersion which implements $T(D^2 - 1)(n - 1)$:

$$\Phi^D_R(b_n)(s, s_0, \{m_i^2\}) = \frac{(s - s_0)(D^2 - 1)(n - 1)}{\pi} \int_{\sum_j m_j^2}^{\infty} \int_{M_n} I_{\text{cut}}(b_n)(x) \frac{dx}{(x - s)(x - s_0)(D^2 - 1)(n - 1)}.$$

Note that in the Taylor expansion of $\Phi^D_R(b_n)(s, s_0, \{m_i^2\})$ around $s = s_0$, the first $(D^2 - 1)(n - 1)$ coefficients vanish. These are our kinematic renormalization conditions.

For example, $\Phi^D_R(b_2)(s_0, s_0) = 0$. On the other hand, $\Phi^2_R(b_2)(s, s_0) = \Phi^2_R(b_2)(s)$ as it does not need subtraction at $s_0$ as it is ultraviolet convergent. So, $s_0$ disappears from its definition and the dispersion integral is unsubtracted as $(D^2 - 1)(n - 1) = 0$ and for $D = 6$, $\Phi^6_R(b_2)(s_0, s_0) = 0 = \partial_s \Phi^6_R(b_2)(s, s_0)|_{s = s_0}$.

1.3 Normal and pseudo-thresholds for $b_n$

To understand possible choices for $s_0$, define a set $\text{thresh}$ of $2^{n-1}$ real numbers by

$$\text{thresh} = \left\{ (\pm m_1 \pm \cdots \pm m_n)^2 \right\},$$

and set

$$s_{\min} := \min\{x \in \text{thresh}\}.$$
Note that the maximum is achieved by \( s_{\text{normal}} := \left( \sum_{j=1}^{n} m_j \right)^2 \). Our requirement for \( s_0 \) is
\[
s_0 \preceq s_{\text{min}}. \tag{1.2}\]
This ensures that the renormalization at \( s_0 \) does not produce contributions to the imaginary part of the renormalized \( \Phi_R^D(b_n)(s, s_0) \) as \( \Im(\Phi_R^D(b_n)(s_0)) = 0 \).

We call \( s_{\text{normal}} \) normal threshold and the \( 2^{n-1} - 1 \) other elements of \( \text{thresh} \) pseudo-thresholds.

Also we call \( m_{\text{normal}}^n := \sum_{j=1}^{n} m_j \) the normal mass of \( b_n \) and any of the other \( 2^{n-1} - 1 \) numbers \(| \pm m_1 \cdots \pm m_n |\) a pseudo-mass of \( b_n \). For any ordering \( o \) of the edges of \( b_n \), we get a flag \( b_2 \subset \cdots \subset b_{n-1} \subset b_n \) such that
\[
m_{\text{normal}}^{j+1} = m_{\text{normal}}^j + m_{j+1}, \quad j \leq n - 1.
\]

On the other hand, for any chosen fixed pseudo-mass there exists at least one ordering \( o \) of edges of \( b_n \) for which the pseudo-mass is \( m_1 - m_2 \pm \cdots \).

**Remark 1.1** By the Coleman–Norton theorem [20] (or by an analysis of the second Symanzik polynomial \( \psi(b_n) \), see Eq. (D.1) in “App. D”), the physical threshold of \( b_n \) is when the energy \( \sqrt{s} \) of the incoming momenta \( k_n = (k_n; 0, 0)^T \) equals the normal mass
\[
\sqrt{s} = m_{\text{normal}}^n.
\]
The imaginary part \( \Im(\Phi_R^D(b_n)) \) is then given by the monodromy associated with that threshold and is supported at \( s \geq m_{\text{normal}}^n \).

In this paper, we are mainly interested in the principal sheet monodromy of \( b_n \) and hence in the monodromy at \( \sqrt{s} = m_{\text{normal}}^n \) which gives \( \Im(\Phi_R^D(b_n)) \). Pseudo-masses are needed to understand monodromy from pseudo-thresholds off the principal sheet.

They can always be expressed as iterated integrals starting possibly from a pseudo-threshold of \( \Phi_R^D(b_2) \). Such non-principal sheet monodromies need to be studied to understand the mixed Hodge theory of \( \Phi_R^D(b_n) \) as a multi-valued function in future work. See [21] for some preliminary considerations.

In preparation to such future work, we note that iterated integral representations can also be obtained for pseudo-thresholds in quite the same manner as in Eq. (2.14) by changing signs of masses (not mass squares) in Eq. (2.13) as given in Eq. (D.2) and correspondingly in the boundaries of the dispersion integral. This dispersion will then reconstruct variations on non-principal sheets. We collect these integral representations in “App. D” (Fig. 1).
Fig. 1 Banana graphs \( b_n \) on \( |b_n| = (n - 1) \) loops. We indicate momenta at internal edges \( e_1, \ldots, e_n \) labelling from top to bottom. We assign mass square \( m_i^2 \) to edge \( e_i \). A positive infinitesimal imaginary part is understood in each propagator. Both vertices have an external edge with incoming momenta \( k_n \) and \( -k_n \). Note that edges \( e_1, \ldots, e_j, n > j \geq 2 \) constitute a banana graph \( b_j \) with external momentum \( k_j \) flowing through. It is a \( (j - 1) \)-loop subgraph of \( b_n \). In particular, we have a sequence \( b_2 \subset b_3 \subset \cdots \subset b_n \) of graphs which gives rise to an iterated integral.

Fig. 2 The bubble \( b_2 \). It gives rise to a function \( \Phi^D_R(b_2)(k_2^2, m_1^2, m_2^2) \). We compute its imaginary part \( \Im(\Phi^D_R(b_2)(k_2^2, m_1^2, m_2^2)) \) below. It starts an induction leading to the desired iterated integral for \( \Im(\Phi^D_R(b_n)) \). The edges \( e_1, e_2 \) are given in red or blue. Shrinking one of them gives a tadpole integral \( \Phi^D_R(t_1)(m_1^2) \) (red) or \( \Phi^D_R(t_2)(m_2^2) \) (blue) (colour figure online).

2 Banana integrals \( \Im(\Phi^D_R(b_n)) \)

2.1 Computing \( b_2 \)

We start with the two-edge banana \( b_2 \), a bubble on two edges with two different internal masses \( m_1, m_2 \), indicated by two different colours in Fig. 2.
The incoming external momenta at the two vertices of $b_2$ are $k_2, -k_2$ which can be regarded as momenta assigned to leaves at the two three-valent vertices.

We discuss the computation of $b_2$ in detail as it gives a start of an induction which leads to the computation of $b_n$. The underlying recursion goes long way back as discussed in Se. (1.1.1) above, see [15] in particular. More precisely, it allows to express $\zeta(\Phi^D_R)(b_n)$ as an iterated integral with the integral $\zeta(\Phi^D_R)(b_2)$ as the start so that $b_n$ is obtained as a $(n-2)$-fold iterated one-dimensional integral.

For the Feynman integral $\Phi^D_R(b_2)$, we implement a kinematic renormalization scheme by subtraction at $s_0 := \mu^2 \leq (m_1 - m_2)^2$ in accordance with Eq. (1.2). This implies that the subtracted terms do not have an imaginary part, as $\mu^2$ is below the pseudo-threshold $(m_1 - m_2)^2$. For example, for $D = 4$

$$\Phi^4_R(b_2)(s, s_0, m_1^2, m_2^2) = \int d^D k_1 \left( \frac{1}{k_2^2 - m_1^2} \frac{1}{Q_1} \right) \left( \frac{1}{(k_2 - k_1)^2 - m_2^2} \frac{1}{Q_2} \right) - \{k_2 \to \mu^2\}.$$

We have $s := k_2^2$. For $D = 6, 8, \ldots$, subtractions of further Taylor coefficients at $s = \mu^2$ are needed.

As the $D$-vector $k_2$ is assumed timelike (as $s > 0$), we can work in a coordinate system where $k_2 = (k_{2,0}, 0)^T$ and get

$$\Phi^D_R(b_2) = \omega_D \frac{1}{\pi} \int_{-\infty}^{\infty} dk_{1,0} \int_0^{\infty} \sqrt{t_1}^{D-3} dt_1 \frac{1}{k_{2,0}^2 - t_1 - m_1^2} \frac{1}{(k_{2,0} - k_{1,0})^2 - t - m_2^2} - \{s \to s_0\}.$$

We define the Källen function, actually a homogeneous polynomial,

$$\lambda(a, b, c) := a^2 + b^2 + c^2 - 2(ab + bc + ca),$$

and find by explicit integration, for example, for $D = 4$,

$$\Phi^4_R(b_2)(s, s_0; m_1^2, m_2^2) = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2s} \ln \frac{m_1^2 + m_2^2 - s - \sqrt{\lambda(s, m_1^2, m_2^2)}}{m_1^2 + m_2^2 - s + \sqrt{\lambda(s, m_1^2, m_2^2)}} - \frac{m_1^2 - m_2^2}{2s} \ln \frac{m_1^2}{m_2^2} - \{s \to s_0\} \frac{w_2(s)}{w_2(s_0)}.$$

The principal sheet of the above logarithm is real for $s \leq (m_1 + m_2)^2$ and free of singularities at $s = 0$ and $s = (m_1 - m_2)^2$. It has a branch cut for $s \geq (m_1 + m_2)^2$. 

\copyright Springer
See, for example, [5, 21] for a discussion of its analytic structure and behaviour off the principal sheet.

The threshold divisor defined by the intersection \( L_1 \cap L_2 \) where the zero loci

\[ L_i : Q_i = 0, \]

of the two quadrics meet is at \( s = (m_1^2 + m_2^2)^2 \). This is an elementary example of the application of Picard–Lefschetz theory [22].

Off the principal sheet, we have a pole at \( s = 0 \) and a further branch cut for \( s \leq (m_1^2 - m_2^2)^2 \).

It is particularly interesting to compute the variation—the imaginary part—of \( \Phi_{R}(b_2) \) using Cutkosky’s theorem [22]. For all \( D, \)

\[
\Im(\Phi_{R}^D(b_2)) = \frac{\omega_{D}}{2} \int_{0}^{\infty} \sqrt{t_1}^{D-3} dt \int_{-\infty}^{\infty} dk_{1:0} \delta_+ \left( k_{1:0}^2 - t_1 - m_1^2 \right) \\
\delta_+ \left( (k_{2:0} - k_{1:0})^2 - t_1 - m_2^2 \right).
\]

We have

\[
\delta_+ \left( (k_{2:0} - k_{1:0})^2 - t_1 - m_2^2 \right) = \Theta(k_{2:0} - k_{1:0}) \delta \left( (k_{2:0} - k_{1:0})^2 - t_1 - m_2^2 \right),
\]

and

\[
\delta \left( (k_{2:0} - k_{1:0})^2 - t_1 - m_2^2 \right) = \frac{1}{2|k_{2:0} - k_{1:0}|_{k_{1:0}=k_{2:0}+\sqrt{t_1+m_2^2}}} \times \delta \left( k_{1:0} - k_{2:0} - \sqrt{t_1 + m_2^2} \right) \\
+ \frac{1}{2|k_{2:0} - k_{1:0}|_{k_{1:0}=k_{2:0}-\sqrt{t_1+m_2^2}}} \times \delta \left( k_{1:0} - k_{2:0} + \sqrt{t_1 + m_2^2} \right).
\]

In summary,

\[
\delta_+ \left( (k_{2:0} - k_{1:0})^2 - t_1 - m_2^2 \right) = \Theta(k_{2:0} - k_{1:0}) \delta \left( (k_{2:0} - k_{1:0})^2 - t_1 - m_2^2 \right) \\
= \frac{1}{2|k_{2:0} - k_{1:0}|_{k_{1:0}=k_{2:0}-\sqrt{t_1+m_2^2}}} \delta \left( k_{1:0} - k_{2:0} + \sqrt{t_1 + m_2^2} \right),
\]

and therefore,

\[
\Im(\Phi_{R}(b_2)) = \frac{\omega_{D}}{2} \int_{0}^{\infty} \sqrt{t_1}^{D-3} dt \delta \left( s - 2\sqrt{s}\sqrt{t_1 + m_2^2 + m_2^2 - m_1^2} \right) \frac{1}{\sqrt{t_1 + m_2^2}}.
\]
We have from the remaining $\delta$-function,
\[
\delta \left( s - 2\sqrt{s}\sqrt{t_1 + m_2^2 + m_2^2 - m_1^2} \right) = \frac{\sqrt{t_1 + m_2^2}}{\sqrt{s}} \delta \left( t_1 - \frac{\lambda(s, m_1^2, m_2^2)}{4s} \right),
\]
and hence,
\[
0 \leq t_1 = \frac{\lambda(s, m_1^2, m_2^2)}{4s},
\]
whenever the Källen function $\lambda(s, m_1^2, m_2^2)$ is positive, so for $s > (m_1 + m_2)^2$ (normal threshold, on the principal sheet) or for $0 < s < (m_1 - m_2)^2$ (pseudo-threshold, off the principal sheet).

The integral then gives
\[
\mathcal{I} \left( \Phi^D_R(b_2) \right) (s, m_1^2, m_2^2) = \omega_D \left( \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2s} \right)^{D-3} \times \Theta \left( s - (m_1 + m_2)^2 \right),
\]
with $\omega_D$ given in Eq. (A.2). We emphasize that $V^D_2$ has a pole at $s = 0$ with residue $|m_1^2 - m_2^2|/2$ and note $\lambda(s, m_1^2, m_2^2) = (s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)$.

We regain $\Phi^D_R(b_2)$ from $\mathcal{I} \left( \Phi^D_R(b_2) \right)$ by a subtracted dispersion integral, for example, for $D = 4$:
\[
\Phi^4_R(b_2)(s, s_0) = \frac{s - s_0}{\pi} \int_0^\infty \frac{\mathcal{I} \left( \Phi^4_R(b_2) \right)(x)}{(x - s)(x - s_0)} \, dx.
\]
Here, the renormalization condition implemented in the once-subtracted dispersion imposes $\Phi^D_R(b_2)(s_0, s_0) = 0$ for $D = 4$.

Finally, we note that for on-shell edges $(k_2 - k_1)^2 = m_2^2$ so
\[
\begin{align*}
k_2 \cdot k_1 &= \frac{k_2^2 - m_2^2 + m_1^2}{2}, \\
k_1^2 &= m_1^2.
\end{align*}
\]

2.2 Computing $b_3$

We now consider the three-edge banana $b_3$ on three different masses.
We start by using the fact that we can disassemble \( b_3 \) in three different ways into a \( b_2 \) subgraph, with a remaining edge providing the co-graph. Using Fubini, the three equivalent ways to write it in accordance with the flag structure \( b_2 \subset b_3 \) are:

\[
\Phi^D_R(b_3) = \int d^Dk_2 \Phi^D_R(b_2) (k_2^2, m_i^2, m_j^2) \delta_+(k_3 - k_2)^2 - m_k^2), \quad (2.1)
\]

\[
\Phi^D_R(b_3) = \int d^Dk_2 \Phi^D_R(b_2) (k_2^2, m_i^2, m_3^2) \delta_+(k_3 - k_2)^2 - m_1^2), \quad (2.2)
\]

\[
\Phi^D_R(b_3) = \int d^Dk_2 \Phi^D_R(b_2) (k_2^2, m_3^2, m_i^2) \delta_+(k_3 - k_2)^2 - m_3^2) \quad (2.3)
\]

In any of these cases for \( \Phi^D_R(b_3) \), we integrate over the common support of the distributions

\[
\Phi^D_R(b_2) (k_2^2, m_i^2, m_j^2) \sim \Theta(k_2^2 - (m_i + m_j)^2) \text{ and } \delta_+(k_3 - k_2)^2 - m_k^2).
\]

generalizing the situation for \( \Phi^D_R(b_2) \) where we integrated over the common support of

\[
\delta_+(k_1^2 - m_1^2) \text{ and } \delta_+(k_2 - k_1)^2 - m_2^2).
\]

The integral Eqs. (2.1, 2.2, 2.3) are well defined and on the principal sheet they are equal and give the variation (and hence imaginary part) \( \Phi^D_R(b_3) \) of \( \Phi^D_R(b_3) \).

\( \Phi^D_R(b_3) \) itself can be obtained from it by a sufficiently subtracted dispersion integral which reads for \( D = 4 \)

\[
\Phi^4_R(b_3)(s, s_0) = \frac{(s - s_0)^2}{\pi} \int_0^{\infty} \frac{\Phi^4_R(b_3)(x)}{(x - s)(x - s_0)^2} dx.
\]
For general $D$, $\Phi^D_R(b_3)$ is well-defined no matter which of the two edges we choose as the subgraph, and Cutkosky’s theorem defines a unique function $V^D_3(s)$,

$$\Im(\Phi^D_R(b_3)(s)) =: V^D_3(s)\Theta(s - (m_1 + m_2 + m_3)^2).$$

**Remark 2.1** Below when we discuss master integrals for $b_n$, we find that by breaking symmetry through a derivative $\partial_{m^2_i}$, we obtain four master integrals for $b_3$. $\Phi^D_R(b_3)$ itself, and by applying $\partial_{m^2_i}$ to any of Eqs. (2.1, 2.2, 2.3).

Let us compute $V^D_3$ first. We consider edges $e_1, e_2$ as a $b_2$ subgraph with an external momentum $k_2$ flowing through.

We let $k_3$ be the external momentum of $\Im(\Phi^D_R(b_3))$, $0 < k_3^2 =: s$. For the $k_2$-integration, we put ourselves in the restframe $k_3 = (k_3;0,0)^T$.

Consider then

$$\Im\left(\Phi^D_R(b_3)\right)(s) = \int d^Dk_2 \Theta(k_2^2 - (m_1 + m_2)^2)\delta_+((k_3 - k_2)^2 - m_3^2)$$

$$V^D_2(k_2^2, m_1^2, m_3^2).$$

The $\delta_+$-distribution demands that $k_{3;0} - k_{2;0} > 0$, and therefore, we get

$$\Im\left(\Phi^D_R(b_3)\right)(s) = \omega_D \int_{-\infty}^{k_{3;0}} dk_{2;0} \int_0^\infty dt_2 \sqrt{t_2} D - 3 \Theta(k_{2;0}^2 - t_2 - (m_1 + m_2)^2)$$

$$\times V^D_2(k_{2;0}^2 - t, m_1^2, m_3^2)\delta((k_{3;0} - k_{2;0})^2 - t_2 - m_3^2).$$

As a function of $k_{2;0}$, the argument of the $\delta$-distribution has two zeros:

$$k_{2;0} = k_{3;0} \pm \sqrt{t_2 + m_3^2}.$$

As $k_{3;0} - k_{2;0} > 0$, it follows $k_{2;0} = k_{3;0} - \sqrt{t_2 + m_3^2}$. Therefore, $k_{2;0}^2 - t_2 = k_{3;0}^2 + m_3^2 - 2k_{3;0}\sqrt{t_2 + m_3^2}$.

For our desired integral, we get

$$\Im\left(\Phi^D_R(b_3)\right)(s) = \omega_D \int_0^\infty dt_2 \sqrt{t_2} D - 3 \Theta\left(k_{3;0}^2 + m_3^2 - 2k_{3;0}\sqrt{t_2 + m_3^2} - (m_1 + m_2)^2\right)$$

$$\times \frac{V^D_2\left(k_{3;0}^2 + m_3^2 - 2k_{3;0}\sqrt{t_2 + m_3^2}, m_1^2, m_3^2\right)}{\sqrt{t_2 + m_3^2}}.$$
Solving for \( t_2 \), we get
\[
0 \leq t_2 \leq \frac{\lambda(s, m_3^2, (m_1 + m_2)^2)}{4s}.
\]
As \( t_2 \geq 0 \), we must have for the physical threshold \( s > (m_3 + m_1 + m_2)^2 \) which is indeed completely symmetric under permutations of 1, 2, 3, in accordance with our expectations for \( \Im(\Phi_R^{(b_3)}(s)) \). We then have
\[
\Im(\Phi_R^{(b_3)}(s)) = \Theta\left(s - (m_1 + m_2 + m_3)^2\right) \omega D \int_0^{\frac{\lambda(s, m_3^2, (m_1 + m_2)^2)}{4s}} V_2^D \left( s + m_3^2 - 2\sqrt{s} \sqrt{t_2 + m_3^2, m_1^2, m_2^2} \right) \sqrt{t_2} dt_2.
\]

There is also a pseudo-threshold off the principal sheet at \( s < (m_3 - m_1 - m_2)^2 \), see Sect. 2.

Note that the integrand vanishes at the upper boundary \( \frac{\lambda(s, m_3^2, (m_1 + m_2)^2)}{4s} \) as
\[
\lambda\left(s + m_3^2 - 2\sqrt{s} \sqrt{t_2 + m_3^2, m_1^2, m_2^2}\right) \bigg|_{t_2 = \frac{\lambda(s, m_3^2, (m_1 + m_2)^2)}{4s}} = \lambda\left((m_1 + m_2)^2, m_1^2, m_2^2\right) = 0.
\]

Let us now transform variables.
\[
y_2 := \sqrt{t_2 + m_3^2},
\]
\[
t_2 = y_2^2 - m_3^2,
\]
\[
dt_2 = 2y_2 dy_2,
\]
\[
\int_0^{\frac{\lambda(s, m_3^2, (m_1 + m_2)^2)}{4s}} V_2^D \left( s + m_3^2 - 2\sqrt{s} y_2, m_1^2, m_2^2\right) \sqrt{y_2 - m_3^2} d y_2.
\]

We get
\[
\Im(\Phi_R^{(b_3)}(s)) = \Theta\left(s - (m_1 + m_2 + m_3)^2\right) \times \omega D \int_{m_3}^{s + m_3^2 - (m_1 + m_2)^2} V_2^D \left( s + m_3^2 - 2\sqrt{s} y_2, m_1^2, m_2^2\right) \sqrt{y_2 - m_3^2} dy_2.
\]

\[
(2.4)
\]
Had we chosen \( e_2, e_3 \) or \( e_3, e_1 \) instead of \( e_1, e_2 \) for \( b_2 \), we would find in accordance with Eqs. (2.1, 2.2, 2.3)

\[
\Im \left( \Phi_R^D(b_3)(s) \right) = \Theta \left( s - (m_1 + m_2 + m_3)^2 \right) \\
\times \omega \frac{D}{2} \int_{m_1}^{s + m_1^2 - (m_2 + m_3)^2} 2\sqrt{s} \left\{ \begin{array}{c}
\frac{s_1(y_2,m_2^3)}{s + m_1^2 - 2\sqrt{s}y_2, m_2^3, m_3^2} \\
V_2^D \left( s + m_1^2 - 2\sqrt{s}y_2, m_2^3, m_3^2 \right) \sqrt{y_2 - m_1^2} \\
 V_3^D(s,m_1^2,m_2^3) \end{array} \right\} dy_2,
\]

(2.5)

or

\[
\Im \left( \Phi_R^D(b_3)(s) \right) = \Theta \left( s - (m_1 + m_2 + m_3)^2 \right) \\
\times \omega \frac{D}{2} \int_{m_2}^{s + m_2^2 - (m_3 + m_4)^2} 2\sqrt{s} \left\{ \begin{array}{c}
\frac{s_1(y_2,m_2^3)}{s + m_2^2 - 2\sqrt{s}y_2, m_2^3, m_3^2} \\
V_2^D \left( s + m_2^2 - 2\sqrt{s}y_2, m_2^3, m_3^2 \right) \sqrt{y_2 - m_2^2} \\
 V_3^D(s,m_1^2,m_2^3) \end{array} \right\} dy_2.
\]

(2.6)

with three different \( s_1^3(y_2) = s_1^3(y_2,m_2^3) \).

We omit this distinction in the future as we will always choose a fixed order of edges and call the edges in the innermost bubble \( b_2 \) edges \( e_1, e_2 \).

Finally, we note

\[
k_{2,0} = k_{3,0} - y_2,
\]
\[
k_2^2 = k_{3,0}^2 - 2k_{3,0}y_2 + m_3^2,
\]
\[
|\vec{k}_2| = \sqrt{y_2^2 - m_3^2}.
\]

Written in invariants this is

\[
k_3 \cdot k_2 = \sqrt{s}(\sqrt{s} - y_2),
\]
\[
k_2^2 = s - 2\sqrt{s}y_2 + m_3^2,
\]
\[
|\vec{k}_2| = \sqrt{y_2^2 - m_3^2}.
\]

### 2.3 \( b_3 \) and elliptic integrals

Note that for \( D = 2 \) (the case \( D = 4 \) can be treated similarly as in [5]) and using Eq. (2.4),
\[
V_3^2(s) = \omega_1 \int_{m_3}^{s + m_3^2 - (m_2 + m_1)^2} \frac{1}{\sqrt{U(y_2)}} \, dy_2,
\]

with
\[
U(y_2) = \lambda \left( s + m_3^2 - 2\sqrt{s} y_2, m_2^2, m_1^2 \right) (y_2^2 - m_3^2)
= s(y_2 - m_3)(y_2 + m_3)(y_2 - y_+)(y_2 - y_-),
\]
a quartic polynomial so that \(V_3^2\) defines an elliptic integral following, for example, [5]. Here,
\[
y_\pm = \frac{(s + m_1^2 - m_2^2) \pm 2\sqrt{m_1^2 m_2^2}}{2\sqrt{s}}.
\]

So, indeed
\[
V_3^2(s) = \frac{2\omega_1}{(y_+ + m_3)(y_- - m_3)} K \left( \frac{(y_- + m_3)(y_+ - m_3)}{(y_- - m_3)(y_+ + m_3)} \right),
\]
with \(K\) the complete elliptic integral of the first kind. Finally,
\[
\Phi_R^2(b_3)(s) = \frac{1}{\pi} \int_{(m_1 + m_2 + m_3)^2}^{\infty} \frac{V_3^2(x)}{(x - s)} \, dx,
\]
gives the full result for \(b_3\) in terms of elliptic dilogarithms in all its glory [6, 7, 16] for \(D = 2\). For arbitrary \(D\), we get
\[
\Phi_R^D(b_3)(s, s_0) = \frac{(s - s_0)^{D-2}}{\pi} \int_{(m_1 + m_2 + m_3)^2}^{\infty} \frac{V_3^D(x)}{(x - s)(x - s_0)^{D-2}} \, dx.
\]

To compare our result Eq. (2.7) with the result in [5] say, note that we can write
\[
U(y_2) = \frac{1}{4} \lambda \left( s, s_3^1, m_3^2 \right) \lambda \left( s_3^1, m_1^2, m_2^2 \right),
\]
as
\[
\lambda \left( s, s_3^1, m_3^2 \right) = \left( s_3^1 - (\sqrt{s} - m_3)^2 \right) \left( s_3^1 - (\sqrt{s} + m_3)^2 \right) = 4s \left( y_2^2 - m_3^2 \right),
\]
with \(s_3^1 = s - 2\sqrt{s} y_2 + m_3^2\), and use \(b = s_3^1, db = -2\sqrt{s} dy_2\) to compare.
2.4 Computing $b_4$

Above we have expressed $V_3^D$ as an integral involving $V_2^D$. We can iterate this procedure.

Let us compute $V_4^D$ next repeating the computation which led to Eq. (2.4). We consider edges $e_1, e_2, e_3$ as a $b_3$ subgraph with an external momentum $k_3$ flowing through.

We let $k_4$ be the external momentum of $\Im(\Phi_R^D(b_4))$, $0 < k_4^2 = s$. We put ourselves in the restframe $k_4 = (k_4; 0)^T$ for the $k_3$-integration.

Consider then

$$\Im \left( \Phi_R^D(b_4) \right) (s) = \int d^Dk_3 \Theta \left( k_3^2 - (m_1 + m_2 + m_3)^2 \right)$$

$$\delta_+ \left( (k_4 - k_3)^2 \right) - m_4^2) V_3^D \left( k_3^2, m_1^2, m_2^2, m_3^2 \right).$$

The $\delta_+ $ distribution demands that $k_4; 0 - k_3; 0 > 0$, and therefore, we get

$$\Im \left( \Phi_R^D(b_4) \right) (s) = \omega_2 \int_{-\infty}^{k_4; 0} dk_3; 0 \int_0^\infty dt_3 \sqrt{t_3}^{D-3} \Theta \left( k_3^2; 0 - t_3 - (m_1 + m_2 + m_3)^2 \right)$$

$$V_3^D \left( k_3^2; 0 - t_3, m_1^2, m_2^2, m_3^2 \right) \delta \left( (k_4; 0 - k_3; 0)^2 - t_3 - m_4^2 \right).$$

As a function of $k_3; 0$, the argument of the $\delta$-distribution has two zeros: $k_3; 0 = k_4; 0 \pm \sqrt{t_3 + m_4^2}.$

As $k_4; 0 - k_3; 0 > 0$, it follows $k_3; 0 = k_4; 0 - \sqrt{t_3 + m_4^2}$. Therefore, $k_3^2; 0 - t_3 = k_4^2; 0 + m_4^2 - 2k_4; 0\sqrt{t_3 + m_4^2}.$

For our desired integral, we get

$$\Im \left( \Phi_R^D(b_4) \right) (s) = \omega_2 \int_0^\infty dt_3 \sqrt{t_3}^{D-3} \Theta \left( k_4^2; 0 + m_4^2 - 2k_4; 0\sqrt{t_3 + m_4^2} - (m_1 + m_2 + m_3)^2 \right)$$

$$V_3^D \left( k_4^2; 0 + m_4^2 - 2k_4; 0\sqrt{t_3 + m_4^2}, m_1^2, m_2^2, m_3^2 \right) \sqrt{t_3 + m_4^2}.$$

The $\Theta$-distribution requires

$$k_4^2; 0 + m_4^2 - (m_1 + m_2 + m_3)^2 \geq 2k_4; 0\sqrt{t_3 + m_4^2}.$$ 

Solving for $t_3$, we get

$$0 \leq t_3 \leq \frac{\lambda(s, m_4^2, (m_1 + m_2 + m_3)^2)}{4s}.$$
As $t_3 \geq 0$, we must have for the physical threshold $s > (m_4 + m_3 + m_1 + m_2)^2$. We then have

$$\Im\left( \Phi_R^D(b_4)(s) \right) = \Theta\left( s - (m_1 + m_2 + m_3 + m_4)^2 \right) \omega_D \frac{\lambda(s, m_2^2, (m_1 + m_2 + m_3)^2)}{4s} \int_0^s \frac{V_3^D(s + m_4^2 - 2\sqrt{s}\sqrt{t_3 + m_1^2}, m_1^2, m_2^2, m_3^2)}{\sqrt{t_3 + m_2^2}} dt_3.$$ 

Let us now transform variables again.

$$y_3 := \sqrt{t_3 + m_4^2},$$
$$t_3 = y_3^2 - m_4^2,$$
$$dt_3 = 2y_3 dy_3,$$
$$\int_0^{y_3} \rightarrow \int_{m_4}^{y_3^2 - (m_1 + m_2 + m_3)^2} \frac{s_4(y_3)}{4\sqrt{s}} \frac{V_3^D(s + m_4^2 - 2\sqrt{s}y_3, m_1^2, m_2^2, m_3^2)}{\sqrt{y_3 - m_4^2}} dy_3.$$

We get

$$\Im(\Phi_R^D(b_4)(s)) = \Theta\left( s - (m_1 + m_2 + m_3 + m_4)^2 \right) \omega_D \frac{\lambda(s, m_2^2, (m_1 + m_2 + m_3)^2)}{4s} \int_{m_4}^{s_4(y_3) + m_2^2 - (m_1 + m_2)^2} \frac{s_4(y_3) + m_2^2 - (m_1 + m_2)^2}{2\sqrt{s_4(y_3)}} \frac{V_3^D(s + m_4^2 - 2\sqrt{s}y_3, m_1^2, m_2^2)}{\sqrt{y_3 - m_4^2}} dy_3.$$

We have thus expressed $V_4^D$ as an integral involving $V_3^D$. As we can express $V_3^D$ by $V_2^D$, we get the iterated integral,

$$V_4^D\left(s, m_1^2, m_2^2, m_3^2, m_4^2\right) = \omega_D^2 \int_{m_4}^{s + m_1^2 + m_2^2 + m_3^2} \frac{\lambda(s, m_2^2, (m_1 + m_2 + m_3)^2)}{4s} \left( \int_{m_3}^{s_4(y_2, y_3)} \frac{s_4(y_2, y_3) + m_2^2 - (m_1 + m_2)^2}{2\sqrt{s_4(y_3)}} \frac{V_2^D\left(s_4(y_2, y_3), m_1^2, m_2^2\right)}{\sqrt{y_2 - m_3^2}} dy_2 \right)$$
$$\times \sqrt{y_3 - m_4^2} dy_3.$$

We abbreviated

$$s_4(y_2, y_3) := s_4(y_3) - 2\sqrt{s_4(y_3)}y_2 + m_3^2$$
$$= s_4^0 - 2\sqrt{s_4^0}y_3 + m_2^2 - 2\sqrt{s_4^0} - 2\sqrt{s_4^0}y_3 + m_2^2y_2 + m_3^2,$$
\( s_4^0 := s \).

### 2.5 Beyond elliptic integrals for \( b_4 \)

Note that \( V_4^2 \) cannot be read as a complete elliptic integral of any kind. It is a double integral over the inverse square root of an algebraic function. \( V_3^2 \) was in contrast a single integral over the inverse square root of a mere quartic polynomial. Concretely, the relevant integrand is

\[
\frac{1}{\sqrt{(y_3^2 - m_4^2)^2(y_2^2 - m_3^2)v_4(y_2, y_3)}}.
\]

In fact, the innermost \( y_2 \) integral can still be expressed as a complete elliptic integral of the first kind as in Eq. (2.7), as \( v_4 \) is a quadratic polynomial in \( y_2 \) so that

\[
(y_2^2 - m_3^2)v_4 = (y_2 - m_3)(y_2 + m_3)(y_2 - y_{2,+})(y_2 - y_{2,-})
\]

is a quartic in \( y_2 \) albeit with coefficients \( y_{2,\pm} \) which are algebraic in \( y_3 \). We have

\[
y_{2,\pm}(y_3) = \frac{(m_1^2 + m_2^2 - m_3^2 - s_4^4(y_3)) \pm 2\sqrt{m_1^2m_2^2}}{2s_4^4(y_3)}.
\]

We get the more than elliptic integral over an elliptic integral of the first kind,

\[
V_4^2(s) = \omega_1 \int_{m_4}^\infty \frac{2\omega_1}{\sqrt{s^2 - (m_3^2 + m_2^2 + m_3^2)^2}} \frac{1}{(y_{2,-}(y_3) + m_4)(y_{2,-}(y_3) - m_4)}
\times K \left( \frac{(y_{2,-}(y_3) + m_4)(y_{2,-}(y_3) - m_4)}{(y_{2,-}(y_3) - m_4)(y_{2,+}(y_3) + m_4)} \right) dy_3. \tag{2.11}
\]

### 2.6 Computing \( b_n \) by iteration

Iterating the computation which led to Eq. (2.10), we get

**Theorem 2.2** Let \( b_n \) be the banana graph on \( n \) edges and two leaves (at two distinct vertices) with masses \( m_i \) and momenta \( k_n \), \(-k_n \) incoming at the two vertices in \( D \) dimensions.

(i) it has an imaginary part determined by a normal threshold as

\[
\Im \left( \Phi_R^D(b_n) \right)(s) = \Theta \left( s - \left( \sum_{j=1}^n m_j \right)^2 \right) V_n^D(s, \{m_i^2\}),
\]
and with a recursion \((n \geq 3)\)

\[
V_n^D(s; \{m_i^2\}) = \frac{\omega_D}{2} \int_{m_n}^{s + m_n^2 - \left(\sum_{i=1}^{n-1} m_i^2\right)^2} V_{n-1}^{D}(s_n^0 - 2\sqrt{s_n^0 y_{n-1} + m_n^2, m_1^2, \ldots, m_{n-1}^2}) \times \sqrt{y_{n-1}^2 - m_n^2} \, dy_{n-1}. 
\]

**Remark (i)** This imaginary part is the variation in \(s\) of \(\Phi_R^D(b_n)(s)\) in the principal sheet. Variations on other sheets are collected in “App. D”. See [21] for an introduction to a discussion of the role of such pseudo-thresholds.

**Theorem (ii)** Define for all \(n \geq 2\), \(0 \leq j \leq n - 2\),

\[
s_n^0 := s,
\]

and for \(n - 2 \geq j \geq 1\), \(s_n^j = s_n^j(y_{n-j}, \ldots, y_{n-1}; m_n, \ldots, m_{n-j+1})\),

\[
s_n^j = s_n^{j-1} - 2\sqrt{s_n^{j-1} y_{n-j} + m_n^2 - j+1}. \tag{2.12}
\]

Define

\[
up_n^j := \frac{s_n^j + m_n^2 - \left(\sum_{i=1}^{n-j-1} m_i\right)^2}{2\sqrt{s_n^j}}, \tag{2.13}
\]

then \(V_n^D\) is given by the following iterated integral:

\[
V_n^D(s, m_1^2, \ldots, m_n^2) := \frac{\omega_D}{2} \int_{m_n}^{s + m_n^2 - \left(\sum_{i=1}^{n-1} m_i^2\right)^2} \left( \int_{m_{n-1}}^{up_n^1(y_{n-1})} \left( \int_{m_{n-2}}^{up_n^2(y_{n-1}, y_{n-2})} \cdots \left( \int_{m_2}^{up_n^{n-2}(y_{n-1}, \ldots, y_1)} V_2^D(s_n^{n-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2) \times \sqrt{y_2^2 - m_2^2} \, dy_2 \right) \cdots \sqrt{y_{n-2}^2 - m_{n-1}^2} \, dy_{n-2} \right) \times \sqrt{y_{n-1}^2 - m_n^2} \, dy_{n-1}. \tag{2.14}
\]

Here, \(V_2^D(a, b, c) = \frac{\lambda(a, b, c)}{a^{\frac{D-3}{2}} - 1}\), so that

\[
V_2^D(s_n^{n-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2) = \omega_D \frac{\lambda(s_n^{n-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2)^{\frac{D-3}{2}}}{(s_n^{n-2}(y_2, \ldots, y_{n-1}) - 1)^{\frac{D-1}{2}}}.
\]
Remark (ii) We solve the recursion in terms of an iteration of one-dimensional integrals. $V_2^D(b_2)$ serves as the seed, $V_2^D = \omega D \lambda (s_n^{-2}, m_1^2, m_2^2)/s_D^{D-1}$ and $s_n^{-2} = s_n^{-2}(y_{n-1}, \ldots, m_3^2, m_2^2)$ depends on integration variables $y_j$ and on mass squares $m_{j+1}^2, j = 2, \ldots, n - 1$. For $b_3$, we need a single integration; for $b_n$, we need to iterate $(n - 2)$ integrals. Note that we could always do the innermost $y_2$-integral in terms of a complete elliptic integral (replacing $s^4 \rightarrow s_n^{n-3}$ in Eq. (2.11), etc.) and use that as the seed.

Theorem (iii) We have the following identities:

\[ V_n^D \left( \left( \sum_{j=1}^{n} m_j \right)^2 ; \{ m_j^2 \} \right) = 0, \quad (2.15) \]

\[ v^1_n(y_{n-1})_{y_{n-1}=v^0_n} = m_{n-1}, \quad (2.16) \]

\[ v^j_n(y_{n-j}, \ldots, y_{n-1})_{y_{n-j}=v^{j-1}_n} = m_{n-j}, \quad (2.17) \]

\[ v^{n-3}_n(y_3, \ldots, y_{n-1})_{y_3=v^{n-4}_n} = m_3, \quad (2.18) \]

\[ V_2^D(s_n^{-2}, m_1^2, m_2^2)_{y_2=v^{n-3}_n} = 0. \quad (2.19) \]

Remark (iii) Equation (2.15) ensures that the dispersion integrand vanishes at the lower boundary $x = (m_1 + \cdots + m_n)^2$ (the normal threshold) as it should. Following Eqs. (2.16–2.18) for any $y_j$-integation but the innermost integration the integrand vanishes at the lower and upper boundaries. By Eq. (2.19) for the innermost $y_{n-1}$ integral this holds for $D \geq 2$.

At $D = 2$, the result can be achieved by considering

\[ \lim_{\eta \rightarrow 0} \int_{m_3 + \eta}^{v^{n-3}_n - \eta} \cdots dy_{n-1}. \]

In the limit $\sqrt{s} \rightarrow m_{\text{normal}}^n$ for which $v^{n-3}_n \rightarrow m_3$, one confirms the analysis in [5] that a finite value at threshold remains.

Summarizing for any $D$ this amounts to compact integration as we have in any $y_j$ integration a resurrection of Stokes formula

\[ \int_{m_{j+1}}^{v^{j+1}_n} \partial y_j f(y_j) \cdots dy_j = 0, \quad (2.20) \]

for any rational function $f(y_j)$ inserted as a coefficient of $V_2^D$. The dots correspond to the other iterations of integrals in the $y_j$ variables. These are integration-by-parts identities.

This reflects the fact that the $n \delta$-functions in a cut banana $b_n$ constrain the $(n - 1)$ integrations of $k_{j;0}, j = 1, \ldots, n - 1$ and also the total integration over $r = \sum_{j=1}^{n-1} |\vec{k}_j|$. Here, we can set $|\vec{k}_j| = ru_j$, and the $u_j$ parameterize a $(n - 1)$-simplex and hence a
compactum. Angle integrals are over compact surfaces $S^{D-2}$. Only integrations over boundaries remain.

\textbf{Theorem (iv)} We have

$$
\partial_{y_j} s_n^j = -2 \sqrt{s_n^{n-k-1}} \partial_{m_{k+1}}^j s_n^j, \forall (n - j) \leq k \leq (n - 1), \quad (2.21)
$$

if all masses are different. The case of some equal masses is left to the reader.

Also,

$$
\left( \prod_{j=0}^{i-1} \sqrt{s_n^j} \right) \partial_s s_n^i = 2 \prod_{j=0}^{i-1} \left( \sqrt{s_n^j - y_{n-j-1}} \right), \quad (2.22)
$$

For derivatives with respect to masses, we have for $0 \leq r \leq k - 1$,

$$
\partial_m s_n^k = \prod_{j=r+1}^{k-1} \frac{\sqrt{s_n^j - y_{n-(j+1)}}}{\sqrt{s_n^j}}, \quad (2.23)
$$

while $\partial_m s_n^k = 1$. Furthermore, for $1 \leq i \leq n - 2 - r$, $0 \leq r \leq n - 3$,

$$
\partial_{y_{n-i}} s_n^{n-2-r} = -2 \sqrt{s_n^{i-1}} \prod_{j=2+r}^{n-1-i} \frac{s_n^{n-j-1} - y_j}{s_n^{n-j-1}}, \quad (2.24)
$$

\textbf{Remark (iv)} These formulae allow to trade $\partial_{y_j}$ derivatives with $\partial_{m_{j+1}}$ derivatives and to treat $\partial_s$ derivatives. This is useful below when discussing differential equation, integration-by-parts and master integrals for $\Phi_D^R(b_n)$.

\textbf{Theorem (v)} Dispersion. Let $|[n, v]| - 1$ (see Eq. (C.2)) be the degree of divergence of $\Phi_D^R(b_n)_v$. Then,

$$
\Phi_D^R(b_n)_v(s, s_0) = \frac{(s - s_0)^{|[n, v]|}}{\pi} \int_0^\infty \frac{V_D^R(x, \{m_i^2\})}{(x - s)(x - s_0)^{|[n, v]|}} \, dx,
$$

is the renormalized banana graph with renormalization conditions

$$
\Phi_D^R(b_n)_v^{(j)}(s_0, s_0) = 0, \ j \leq |[n, v]| - 1,
$$

where $\Phi_D^R(b_n)_v^{(j)}(s_0, s_0)$ is the $j$th derivative of $\Phi_D^R(b_n)_v(s, s_0)$ at $s = s_0$.

\textbf{Remark (v)} This gives $\Phi_D^R(b_n)_v$ from $V_D^R$ in kinematic renormalization. See “App. C” for notation. For a result in dimensional integration with MS, use an unsubtracted dispersion

$$
\Phi_{MS}^D(b_n)_v(s) = \frac{1}{\pi} \int_0^\infty \frac{V_D^R(x, \{m_i^2\})}{(x - s)^2} \, dx,
$$
and then renormalize by Eq. (B.1) as tadpoles do not vanish in MS.

**Theorem (vi) Tensor integrals (see “App. C”).** We have

\[
k_{j+1} \cdot k_j = m_{j+1}^2 - s_n^{n-j-1} - s_n^{n-j} \tag{2.25}
\]

\[
= -\sqrt{s_n^{n-j-1}} \left( \sqrt{s_n^{n-j-1}} - y_{n-j} \right), \quad j \geq 2,
\]

\[
k_2 \cdot k_1 = \frac{k_2^2 - m_2^2 + m_1^2}{2}, \tag{2.26}
\]

\[
k_j^2 = s_n^{n-j}, \quad \text{in particular } k_2^2 = s_n^{n-2}, \tag{2.27}
\]

\[
k_j \cdot k_l = \frac{k_l \cdot k_{l+1}k_{l+1} \cdot k_{l+2} \cdot \cdots \cdot k_{j-1} \cdot k_j}{k_{l+1}^2 \cdots k_{j-1}^2}
\]

\[
= \sqrt{s_n^{n-j-1}} \prod_{i=l+1}^{j} \left( \sqrt{s_n^{n-i}} - y_{i+1} \right), \quad j - l \geq 1, \quad j > l, l \geq 1, \tag{2.28}
\]

\[
k_j \cdot k_1 = \frac{k_1 \cdot k_{1+1}k_{1+1} \cdot k_{1+2} \cdot \cdots \cdot k_{j-1} \cdot k_j}{k_{1+1}^2 \cdots k_{j-1}^2}
\]

\[
= \sqrt{s_n^{n-j-1}} \frac{s_n^{n-2} - m_1^2 + m_2^2}{\sqrt{s_n^{n-2}}} \prod_{i=2}^{j} \left( \sqrt{s_n^{n-i}} - y_{i+1} \right), \quad j - 1 \geq 1. \tag{2.29}
\]

Furthermore, \( V_{[n,v]}^D \) is obtained by using Eqs. (2.25–2.29) to insert tensor powers as indicated by \( v \) in the integrand of \( V_2^D(s_n^{n-2}, m_1^2, m_2^2) \) and apply derivatives with respect to mass squares accordingly.

**Remark (vi)** We first give in Fig. 3 with \( k_j^2 = s_n^{n-j} \) also the irreducible squares of internal momenta (there is no propagator \( k_j^2 - m_j^2 \) in the denominator of \( b_n \)).

Equation (2.26) is needed as Eq. (2.25) cannot cover the case \( j = 1 \), due to the fact that for the \( b_2 \) integration \( d^D k_1 \) both edges are constrained by a \( \delta_+ \)-function, while each other loop integral gains only one more constraint, giving us a \( y_j \) variable.

Equations (2.25–2.29) allow to treat tensor integrals involving scalar products of irreducible numerators. Irreducible as there is no propagator \( 1/(k_j^2 - m_{j+1}^2) \) in our momentum routing for \( b_n \), see Fig. 3.

Equations (2.28, 2.29) for irreducible scalar products follow by integrating tensors in the numerator in the order of iterated integration. For example, for the case of \( b_3 \),

\[
\int \int k_1 \cdot k_3 \frac{1}{\cdots} d^D k_1 d^D k_2 = \int A(k_2^2)k_2 \cdot k_3 \frac{1}{\cdots} d^D k_2 = C(k_3^2).
\]
Fig. 3 We indicate momenta and masses at internal edges from top to bottom. We now also indicate momentum $s_j^k$ for edges $e_2, \ldots, e_n$. The mass-shell conditions encountered in the computation of $V_{n}^{D}$ enforce $k_{j}^{2} = s_{n-j}^{n}$ for $2 \leq j \leq n$. Equation (2.25) simply expresses the fact that $-2k_{j} \cdot k_{j+1} = (k_{j+1} - k_{j})^{2} - k_{j+1}^{2} - k_{j}^{2}$ with $(k_{j+1} - k_{j})^{2} = m_{j+1}^{2}$.

and

\[
\int \int \frac{k_{1} \cdot k_{2} k_{2} \cdot k_{3}}{k_{2}^{2}} \cdots d^{D}k_{1} d^{D}k_{2} = \int \frac{A(k_{2}^{2}) k_{2} k_{2} \cdot k_{3}}{k_{2}^{2}} \cdots d^{D}k_{2} = C(k_{3}^{2}),
\]

using

\[
\int \frac{k_{1} \mu}{\cdots} d^{D}k_{1} = A(k_{2}^{2}) k_{2} \mu,
\]

and dots $\cdots$ correspond to the obvious denominator terms.

**Proof** (i) and (ii) follow from the derivation of Eq. (2.10) upon setting $4 \rightarrow n$, $3 \rightarrow n - 1$ in an obvious manner.

(iii) follows from inspection of Eq. (2.6): For example,

\[
\text{up}_{n}^{0} = \frac{s + m_{n}^{2} - (m_{1} + \cdots + m_{n-1})^{2}}{2 \sqrt{s}},
\]

\[
\text{up}_{n}^{1}(y_{n-1}) = \frac{s_{1}^{1} + m_{n-1}^{2} - (m_{1} + \cdots + m_{n-2})^{2}}{2 \sqrt{s_{1}^{1}}},
\]

with

\[
s_{n}^{1}(y_{n-1}) = s - 2 \sqrt{s} y_{n-1} + m_{n}^{2}.
\]
Then,
\[
\up_1^\up_0 n = \frac{(m_1 + \cdots + m_{n-1})^2 + m_{n-1}^2 - (m_1 + \cdots + m_{n-2})^2}{2(m_1 + \cdots + m_{n-1})} = m_{n-1},
\]
and so on.

(iv) straight from the definition Eq. (2.12) of \( s_j \). For example,
\[
\partial m^2 s_3 = \frac{\left( \sqrt{s_1} - y_{n-2} \right) \left( \sqrt{s_2} - y_{n-3} \right)}{\sqrt{s_1} \sqrt{s_2}}
\]

(v) This is the definition of dispersion in kinematic renormalization conditions.

(vi) For tensor integrals, we collect variables \( k_j ; 0 \) and \( t_j \) in any step of the computation in terms of \( y_j = \sqrt{t_j + m_{j+1}^2} \).

\[\square\]

2.6.1 \( s_n \): iterating square roots

Choose an order \( o \) of the edges which fixes
\[
b_2 \subset b_3 \subset \cdots \subset b_{n-1} \subset b_n.
\]

Here, we label
\[
E_{b_2} = \{ e_1, e_2 \}, \ E_{b_3} = \{ e_1, e_2, e_3 \}, \ldots, \ E_{b_n} = \{ E_{b_{n-1}} \cup e_n \}.
\]

Then,
\[
s_1(y_{n-1}) = s - 2\sqrt{s} y_{n-1} + m_n^2,
\]
\[
s_2(y_{n-1}, y_{n-2}) = s - 2\sqrt{s} y_{n-1} + m_n^2 - 2s - 2\sqrt{s} y_{n-1} + m_n^2 y_{n-2} + m_{n-1}^2,
\]
\[
s_3(y_{n-1}, y_{n-2}, y_{n-3}) = s - 2\sqrt{s} y_{n-1} + m_n^2 - 2s - 2\sqrt{s} y_{n-1} + m_n^2 y_{n-2} + m_{n-1}^2
\]
\[
- 2\sqrt{s} y_{n-1} + m_n^2 - 2\sqrt{s} y_{n-1} + m_n^2 y_{n-2} + m_{n-1}^2
\]
\[
\times y_{n-3} + m_{n-2}^2,
\]
\[
\ldots,
\]
\[
s_n^{n-2}(y_{n-1}, \ldots, y_3, y_2) = s^{n-3}(y_{n-1}, \ldots, y_3) - 2\sqrt{s^{n-3}(y_{n-1}, \ldots, y_3)} y_2 + m_3^2.
\]

**Remark 2.3** The iteration of square roots in particular for \( s_n^{n-2} \) which is the crucial argument in \( V_n^D(s_n^{n-2}, m_1^2, m_2^2) \) is hopefully instructive for a future analysis of periods which emerge in the evaluation of that function [11]. This iteration of square roots points to the presence of a solvable Galois group with successive quotients \( \mathbb{Z}/2\mathbb{Z} \).
reflecting iterated double covers in momentum space. Thanks to Spencer Bloch for pointing this out.

3 Differential equations and related considerations

This section collects some comments with respect to the results above with regard to:

- **Dispersion.** We want to discuss in some detail why raising powers of propagators is well defined in dispersion integrals even if a higher power of a propagator constitutes a product of distributions with coinciding support.
- **Integration by parts (ibp) [23].** We do not aim at constructing algorithms which can compete with the established algorithms in the standard approach [24]. But at least we want to point out how ibp works in our iterated integral set-up.
- **Differential equations.** Here, we focus on systems of linear first-order differential equations for master integrals [25]. We also add a few comments on higher-order differential equation for assorted master integrals which emerge as Picard–Fuchs equations [6, 7, 10, 19].
- **Master integrals.** Master integrals are assumed independent by definition with regard to relations between them with coefficients which are rational functions of mass squares and kinematic invariants [26, 27]. We will remind ourselves that such a relation can still exist for their imaginary parts [5]. We trace this phenomenon back to the degree of subtraction needed in dispersion integrals to construct their real part from their imaginary parts. Furthermore, we will offer a geometric interpretation of the counting of master integrals for graphs $b_n$.

3.1 Dispersion and derivatives

As we want to obtain full results from imaginary parts by dispersion, we have to discuss the existence of dispersion integrals in some detail. There are subtleties when raising powers of propagators. It is sufficient to discuss the example of $b_2$.

With $\Phi^D_R(b_2)$ given, consider a derivative with respect to a mass square such that a propagator is raised to second power,

$$\Phi^D_R(b_2)_{2,1} := \partial_{m^2_1}\Phi^D_R(b_2)(s, m^2_1, m^2_2).$$

Similar to the imaginary part,

$$\Im\left(\Phi^D_R(b_2)_{2,1}\right) := \partial_{m^2_1}\Im\left(\Phi^D_R(b_2)\right)(s, m^2_1, m^2_2).$$

We have (for $D = 4$ say)

$$\Im\left(\Phi^4_R(b_2)_{2,1}\right) = \frac{s - s_0}{\pi} \int_0^\infty \frac{\partial_{m^2_1}(\Theta(x - (m_1 + m_2)^2)V^4_2(x, m^2_1, m^2_2))}{(x - s)(x - s_0)} \, dx.$$
There is an issue here. It concerns the fact that to a propagator, itself a distribution,

\[ Q(r, m) = \frac{1}{r^2 - m^2} = \text{P.V.} \frac{1}{r^2 - m^2} + i\pi \delta(r^2 - m^2), \]

(using Cauchy’s principal value and the \(\delta\)-distribution) we can associate a well-defined distribution by ‘cutting’ the propagator:

\[ \frac{1}{Q(r, m)} \to \delta_+(Q(r, m)) = \Theta(r_0)\delta(r^2 - m^2). \]

The expression

\[ 2\frac{\delta_+(Q(r, m))}{Q}, \]

obtained from cutting any one of the two factors in the squared propagator,

\[ -\partial_{m^2} \frac{1}{Q(r, m)} = \frac{1}{Q^2(r, m)} \to 2\frac{\delta_+(Q(r, m))}{Q}, \]

is ill defined as the numerator forces the denominator to vanish. Hence, higher powers of propagators are subtle when it comes to cuts on any one of their factors (Fig. 4).

Remarkably, dispersion still works despite the fact that derivatives like \(\partial_{m^2}\) do just that: generating such higher powers.

We have

\[ \partial_{m_1^2} \Im(\Phi_R^D(b_2)) = \delta(s - (m_1 + m_2)^2)V_2^D(s, m_1^2, m_2^2) \left(1 + \frac{m_2}{m_1}\right) \]

\[ + \Theta(s - (m_1 + m_2)^2)\partial_{m_1^2}V_2^D(s, m_1^2, m_2^2), \]

where

\[ \left(1 + \frac{m_2}{m_1}\right) = \partial_{m_1^2}(m_1 + m_2)^2. \]
Using

\[ V_2^D (s, m_1^2, m_2^2) = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}^D}{s^{D-1}}, \]

the above is singular at \( s = (m_1 + m_2)^2 \). Indeed, both terms on the rhs are ill defined, but their sum can be integrated in the dispersion integral

\[
\partial_{m_1^2} \Phi^D_R (b_2) = \frac{(s - s_0)}{\pi} \int_0^\infty \left( \delta(x - (m_1 + m_2)^2) V_2^D (x, m_1^2, m_2^2) \left(1 + \frac{m_2}{m_1}\right) \\
\Theta(x - (m_1 + m_2)^2) \partial_{m_1^2} V_2^D (x, m_1^2, m_2^2) \right) \frac{1}{(x - s)(x - s_0)} dx,
\]

so that the singularity drops out for all \( D \) by Taylor expansion of

\[ \partial_{m_1^2} \lambda(x, m_1^2, m_2^2) = \partial_{m_1^2} \left((x - (m_1 + m_2)^2)(x - (m_1 - m_2)^2)\right), \]

near the point \( x = (m_1 + m_2)^2 \).

We are not saying that it is meaningful to replace

\[ \frac{1}{Q^2} \rightarrow \frac{\delta_+(Q)}{Q}, \]

to come to dispersion relations.

Instead, we can exchange either:

(i) Taking derivatives wrt masses on an imaginary part \( \Im (\Phi^D_R (b_n),_\nu) \) first and then doing the dispersion integral, or,

(ii) Doing the dispersion integral first and then taking derivatives.

3.2 Integration-by-parts

Integration-by-parts (ibp) is a standard method employed in high energy physics computations.

It starts from an incarnation of Stoke’s theorem in dimensional regularization

\[ 0 = \int d^D k \frac{\partial}{\partial k_\mu} v_\mu F([k \cdot r]), \]

where \( F \) is a scalar function of loop momentum \( k \) and other momenta and \( v_\mu \) is a linear combination of such momenta employing a suitable definition of \( D \)-dimensional integration for \( D \in \mathbb{C} \).

We want to discuss ibp and Stokes theorem from the viewpoint of the \( \gamma_i \)-integrations in our iterated integral.
We let \( \text{Int}_{b_n} \) be the integrand in Eq. (2.14). It is made from three factors:

\[
\text{Int}_{b_n} = Y_n^{D-3} \times 3_n^{D-3} \times \sigma_n^{1-D/2},
\]

with \( Y_n, 3_n, \sigma_n \) defined by,

\[
Y_n^{D-3} = \prod_{j=2}^{n-1} \sqrt{y_j^2 - m_{j+1}^2}^{D-3},
\]

\[
3_n^{D-3} = \sqrt{\lambda(s_n^{-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2)^{D-3}},
\]

\[
\sigma_n^{1-D/2} = \frac{1}{(s_n^{-2}(y_2, \ldots, y_{n-1}))^{D/2-1}}.
\]

We have the following identities which allow to trade derivatives with respect to \( y_j \) with derivatives with respect to \( m_{j+1}^2 \) or \( s \),

\[
\partial_{y_j} Y_n = y_j \frac{1}{y_j - m_{j+1}^2} - Y_n = -2y_j \partial_{m_{j+1}^2} Y_n, \quad (3.1)
\]

\[
\partial_{y_j} 3_n = \frac{s_n^{-2} - m_1^2 - m_2^2}{\lambda(s_n^{-2}(y_2, \ldots, y_{n-1}), m_1^2, m_2^2)} \left( \partial_{y_j} s_n^{-2} \right) 3_n
\]

\[
= (\partial_s 3_n) \left( \frac{-2\sqrt{s}}{\sqrt{s} - y_{n-1}} \prod_{k=1}^{n-1-j} \frac{s_k}{s_n - y_{n-k-1}} \right) \quad (3.2)
\]

\[
= -2\sqrt{s_n^{-j-1}} \partial_{m_{j+1}^2} 3_n,
\]

\[
\partial_{y_j} \sigma_n = \partial_{y_j} s_n^{-2} = -2\sqrt{s_n^{-j-1}} \prod_{l=2}^{n-j-1} \frac{s_{n-l-1} - y_l}{s_n^{-n-l-1}}
\]

\[
= -2\sqrt{s_n^{-j-1}} \partial_{m_{j+1}^2} \sigma_n
\]

\[
= \partial_s \sigma_n \left( \frac{-2\sqrt{s}}{\sqrt{s} - y_{n-1}} \prod_{k=1}^{n-1-j} \frac{s_k}{s_n - y_{n-k-1}} \right). \quad (3.3)
\]

We also note that

\[
\partial_{m_{j+1}^2} 3_n = (\partial_{m_{j+1}^2} s_n^{-2}) \frac{1}{m_1^2 - m_2^2} \left( m_1^2 \partial_{m_1^2} - m_2^2 \partial_{m_2^2} \right) 3_n, \quad (3.4)
\]

and

\[
\partial_s 3_n = (\partial_s s_n^{-2}) \frac{1}{m_1^2 - m_2^2} \left( m_1^2 \partial_{m_1^2} - m_2^2 \partial_{m_2^2} \right) 3_n. \quad (3.5)
\]
Furthermore, insertion of tensor structure given by $\nu$ following Sect. 1 and Eqs. (2.25–2.29) define an integrand $\text{Int}_{b_n, \nu}$.

Now, using Eq. (2.20) we have for any such integrand,

$$\int_{m_{j+1}}^{u_{j+1}} \partial y_j (\text{Int}_{b_n, \nu}) dy_j = 0, \forall j, 2 \leq j \leq (n-1).$$

**Proposition 3.1** The above evaluates to an identity of the form,

$$\sum_j \text{Int}_{b_n, \nu_j} = 0,$$

between tensor integrals $\text{Int}_{b_n, \nu_j}$ for some tensor structures $\nu_j$.

**Proof** Derivatives with respect to $y_j$ can be traded for derivatives with respect to masses and with respect to the scale $s$ using Eqs. (3.1, 3.3, 3.1). Starting with $\nu$, this creates suitable new tensor structures $\nu_j$. Homogeneity of $\lambda$ allows to replace the $\partial_s$ derivatives by $\text{Int}_{b_n, \tilde{\nu}_j}$ with once-more modified tensor structures $\tilde{\nu}_j$. \qed

### 3.3 Differential equations

Functions $\Phi^D_R(G)(\{k_i \cdot k_j\}, \{m^2_e\})$ for a chosen Feynman graph $G$ fulfill differential equations with respect to suitable kinematical variables [25]. Those variables are given by scalar products $k_i \cdot k_j$ of external momenta. For $G = b_n$, these are differential equations in the sole scalar product $s = k_n \cdot k_n$ of external momenta.

$\Phi^D_R(b_n)(s, \{m^2_e\})$ is a solution to an inhomogeneous differential equation, and the imaginary part $\Im(\Phi^D_R(b_n)) (s, \{m^2_e\})$ solves the corresponding homogeneous one.

More precisely, there is a set of master integrals $\{b_n\}_M$ defined as a class of Feynman graphs such that any given graph $b_n$, giving rise to integrals $\Phi^D_R(b_n)(s, s_0, \{m^2_e\})$—so with all its corresponding tensor integrals and arbitrary integer powers of propagators—can be expressed as linear combinations of elements of $\{b_n\}_M$.

Let us consider the column vector $S_{b_n}$ formed by the elements of $\{b_n\}_M$. One searches for a first-order system

$$\partial_s S_{b_n}(s) = A S_{b_n}(s) + T,$$

with $A = A(s, \{m^2_e\})$ a matrix of rational functions and $T = T(\{m^2_e\})$ the inhomogeneity provided by the minors of $b_n$. Those are $(n-1)$-loop tadpoles $t_e$ obtained from shrinking an edge $e$, $t_e = b_n/e$.

One then has

$$\partial_s \Im(S_{b_n})(s) = A \Im(S_{b_n})(s),$$

where $\Im(S_{b_n})$ is formed by the imaginary parts of entries of $S_{b_n}$ and $\Im(\Phi^D_R(t_e)) = 0$. 

\[ \square \] Springer
For $b_3$, for example, one has $S_{b_3} = (F_0, F_1, F_2, F_3)^T$, with $F_0 = \Phi^D_R(b_3)$, $F_i = \partial_{m_i^2}\Phi^D_R(b_3)$, $i \in \{1, 2, 3\}$.

The $4 \times 4$ matrix $A$ and the four-vector $T$ for that example are well-known, see [10].

From such a first-order system for the full set of master integrals, one often derives a higher-order differential equation for a chosen master integral. For $b_3$ or $b_4$, it is a Picard–Fuchs equation [10].

For banana graphs $b_n$, it is a differential equation of order $(n - 1)$:

$$\sum_{j=0}^{n-1} \left( Q_{b_n}^{(j)} \frac{\partial}{\partial s} \right) \Phi^D_R(b_n)(s) = T_n(s), \quad (3.6)$$

where $Q_{b_n}^{(j)}$ are rational functions in $s$, $\{m_i^2\}$ and one can always set $Q_{b_n}^{(n-1)} = 1$. It has been studied extensively [6, 7, 10, 11, 19].

We want to outline how our iterated integral approach relates to such differential equations, to master integrals and to the integration-by-parts (ibp) identities which underlay such structures.

Our first task is to remind ourselves how to connect the homogeneous and inhomogeneous differential equations, and we turn to $b_2$ for some basic considerations.

### 3.3.1 Differential equation for $b_2$

We set $D = 2$ for the moment. Consider the imaginary part of the bubble

$$\Im(\Phi^2_R(b_2))(s) = \frac{1}{\sqrt{\lambda(s, m_1^2, m_2^2)}} \Theta(s - (m_1 + m_2)^2).$$

We can recover $\Phi^2_R(b_2)$ by dispersion which reads for $D = 2$,

$$\Phi^2_R(b_2)(s) = \frac{1}{\pi} \int_{(m_1 + m_2)^2}^{\infty} \frac{\Im(\Phi^2_R(b_2))(x)}{(x - s)} \, dx.$$

We now use this representation to analyse the well-known differential equation [6] for $b_2$ given in

**Proposition 3.2**

$$(\lambda(s, m_1^2, m_2^2) \frac{\partial}{\partial s} + (s - m_1^2 - m_2^2)) \Phi^2_R(b_2)(s) = \frac{1}{\pi}, \quad (3.7)$$

and for the imaginary part

$$(\lambda(s, m_1^2, m_2^2) \frac{\partial}{\partial s} + (s - m_1^2 - m_2^2)) \Im \left( \Phi^2_R(b_2)(s) \right) = 0. \quad (3.8)$$
Note that Eq. (3.8) is the homogeneous equation associated with Eq. (3.7) as it must be [25].

The following proof aims at deriving Eq. (3.7) from the dispersion integral.

Proof Let us first prove Eq. (3.8).

\[
\lambda \left( s, m_1^2, m_2^2 \right) \frac{\partial}{\partial s} \frac{1}{\sqrt{\lambda \left( s, m_1^2, m_2^2 \right) \Theta \left( s - (m_1 + m_2)^2 \right)}} = -\left( s + m_1^2 + m_2^2 \right) \Theta \left( s - (m_1 + m_2)^2 \right)
\]

\[
= -\frac{(s + m_1^2 + m_2^2)}{\sqrt{\lambda \left( s, m_1^2, m_2^2 \right)}} \Theta \left( s - (m_1 + m_2)^2 \right)
\]

\[
= -\left( s + m_1^2 + m_2^2 \right) \Theta \left( s - (m_1 + m_2)^2 \right)
\]

\[
= -\left( s + m_1^2 + m_2^2 \right) \Im \left( \frac{\Phi_R^2(b_2)}{\Phi_1} \right) (s),
\]

as desired. We use \( \lambda((m_1 + m_2)^2, m_1^2, m_2^2) = 0. \)

Now, for Eq. (3.7). Evaluating the lhs gives

\[
\text{LHS} = \lambda(s, m_1^2, m_2^2) \frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{1}{\sqrt{\lambda(x, m_1^2, m_2^2)(x-s)^2}} \, dx \quad (3.9)
\]

\[
+ \frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{(s - m_1^2 - m_2^2)}{\sqrt{\lambda(x, m_1^2, m_2^2)(x-s)}} \, dx. \quad (3.10)
\]

A partial integration in the first term (3.9) delivers

\[
\text{LHS} = -\lambda(s, m_1^2, m_2^2) \frac{1}{2\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{\partial_x \lambda(x, m_1^2, m_2^2)}{\sqrt{\lambda(x, m_1^2, m_2^2)^3} (x-s)} \, dx
\]

\[
+ \frac{1}{\pi} \int_{(m_1+m_2)^2}^{\infty} \frac{(s - m_1^2 - m_2^2)}{\sqrt{\lambda(x, m_1^2, m_2^2)(x-s)}} \, dx
\]

\[
- \lambda(s, m_1^2, m_2^2) \left[ \frac{1}{\pi} \frac{1}{\sqrt{\lambda(x, m_1^2, m_2^2)(x-s)}} \right]_{(m_1+m_2)^2}^{\infty}.
\]

We have

\[
\partial_x \lambda \left( x, m_1^2, m_2^2 \right) = 2 \left( x - m_1^2 - m_2^2 \right) =: \nu_1(x), \quad \nu_1(x) - \nu_1(s) = 2(x-s), \quad (3.11)
\]
and
\[ \lambda \left( s, m_1^2, m_2^2 \right) - \lambda \left( x, m_1^2, m_2^2 \right) = (s-x) \left( (s+x) - 2(m_1^2 + m_2^2) \right) =: w(x, s)(s-x). \] (3.12)
Using this the lhs of Eq. (3.7) reduces to a couple of boundary terms. We collect
\[ + \frac{1}{\pi} \left[ \frac{x}{\sqrt{\lambda_x}} \right]_{(m_1+m_2)^2}^{\infty} \]
\[ + \frac{s - 2(m_1^2 + m_2^2)}{\pi} \left[ \frac{1}{\sqrt{\lambda_x}} \right]_{(m_1+m_2)^2}^{\infty} \]
\[ + \left[ \frac{1}{\pi} \frac{\lambda_x}{\sqrt{\lambda_x}(x-s)} \right]_{(m_1+m_2)^2}^{\infty} \]
\[ = \frac{1}{\pi}, \]
as desired.
Indeed, using that
\[ w(x, s) = s + x - 2(m_1^2 + m_2^2) \]
we see that the term \( \sim w \) in the third line cancels the first and second lines. The remaining term is
\[ \left[ \frac{1}{\pi} \frac{\lambda_x}{\sqrt{\lambda_x}(x-s)} \right]_{(m_1+m_2)^2}^{\infty} = \frac{1}{\pi}, \]
as \( \sqrt{\lambda((m_1 + m_2)^2, m_1^2, m_2^2)} = 0 \) and \( \lim_{x \to \infty} \sqrt{\lambda(x, m_1^2, m_2^2)} = x. \) \( \square \)

**Remark 3.3** So, for \( b_2 \) we have by Eqs. (3.12, 3.11)
\[ Q_0(x) = \frac{2(s - m_1^2 - m_2^2)}{\lambda(s, m_1^2, m_2^2)} \]
and \( Q_1(x) = 1. \)
This is a trivial incarnation of Eq. (3.6). As \( (Q_0(x) - Q_0(s)) \sim (x-s) \), we cancel the denominator \( 1/(x-s) \) in the dispersion integral and we are left with boundary terms which constitute the inhomogeneous terms.

**Remark 3.4** The non-rational part \( \Phi_R^D(b_2)_{\text{Transc}} \) of \( \Phi_R^D(b_2) \) is divisible by \( V_2^D \) and gives a pure function in the parlance of [2]. Indeed, one wishes to identify such pure functions in the non-rational parts of \( \Phi_R^D(b_n)(s, s_0) \).
For example, for \( D = 4 \) (ignoring terms in \( \Phi_R^4(b_2)(s) \) which are rational in \( s \))
\[ \Phi_R^4(b_2)(s)_{\text{Transc}}/V_2^4(s) = \ln \frac{m_1^2 + m_2^2 - s - \sqrt{\lambda(s, m_1^2, m_2^2)}}{m_1^2 + m_2^2 - s + \sqrt{\lambda(s, m_1^2, m_2^2)}}. \]
This follows also for all \( b_n, n > 2 \), as long as the inhomogeneity \( T_n(s) \) fulfills
\[ \Im(T_n(s)) = 0, \]
which is certainly true for the case \( b_2 \) with \( T_2(s) = 1/\pi \). Indeed, for \( f(s) \) a solution of the homogeneous

\[
\left( \sum_{j=0}^{n-1} Q_j(s) \partial_j^l \right) f(s) = 0,
\]

the inhomogeneous Picard–Fuchs equation

\[
\left( \sum_{j=0}^{n-1} Q_j(s) \partial_j^l \right) g(s) = T_n(s),
\]

can be solved by setting \( g(s) = f(s)h(s) \). Using Leibniz’ rule, this determines \( h(s) \) as a solution of an equation

\[
\sum_{k=1}^{n-1} h^{(k)}(s) \left( \sum_{j=k}^{n-1} \binom{n-1}{j} a_j(s) f^{(j-k)}(s) \right) = u(s),
\]

with \( f^{(j-k)}(s) = \partial_j^{j-k} f(s) \) and similarly for \( h^{(k)}(s) \). Note \( f^{(j-k)}(s) \) are given by solving the homogeneous equation. Hence, \( g(s) \) indeed factorizes as desired.\(^2\)

This relates to co-actions and cointeracting bialgebras \([28, 29]\) and will be discussed elsewhere.

### 3.3.2 Systems of linear differential equations for \( b_n \)

To find differential equations for the iterated \( y_j \)-integrations of Eq. \((2.14)\), we first systematically shift all \( y_j \)-derivatives acting on \( \sqrt{y_j^2 - m_j^2} \) to act on \( V_2^D(s_n^2, m_1^2, m_2^2) \) using partial integration. We can ignore boundary terms by Thm. \((2.2(iii))\). We use

\[
\left( \frac{1}{\sqrt{y_{j-1}^2 - m_j^2}} \right) F = \frac{1}{2\sqrt{y_{j-1}^2 - m_j^2}} F
\]

\[
= \left( -\frac{y_{j-1}^2 - m_j^2}{2m_j^2\sqrt{y_{j-1}^2 - m_j^2}} + \frac{y_{j-1}^2}{2m_j^2\sqrt{y_{j-1}^2 - m_j^2}} \right) F
\]

\[
= \left( -\frac{1}{2m_j^2\sqrt{y_{j-1}^2 - m_j^2}} - y \left( \partial_{y_{j-1}} - \frac{1}{2m_j^2\sqrt{y_{j-1}^2 - m_j^2}} \right) \right) F
\]

\(^2\) The argument can be extended by replacing the requirement \( \Im(T_n(s)) = 0 \) by \( \text{Var}_x(T_n(s)) = 0 \) where \( \text{Var}_x \) is the variation around a given threshold divisor \( x \). For banana graphs \( b_n \), we only have to consider \( x = s_{\text{normal}} \).
\[ = -\frac{1}{2m_j^2\sqrt{y_{j-1} - m_j^2}} \partial_{y_{j-1}} F + \frac{1}{2m_j^2\sqrt{y_{j-1} - m_j^2}} (\partial_{y_{j-1}} y_{j-1} F) \]
\[ = +\frac{1}{2m_j^2\sqrt{y_{j-1} - m_j^2}} y_{j-1} (\partial_{y_{j-1}} F) \]
\[ = +\frac{1}{m_j^2\sqrt{y_{j-1} - m_j^2}} y_{j-1} \left( \sqrt{s_{j-1} - m_j^2} \partial_{m_j^2} F \right). \]

We could trade a derivative wrt \( y_{j-1} \) for a derivative wrt \( m_j^2 \) thanks to Thm. (2.2(iv)). This holds under the proviso that all masses are different. Else, we use the penultimate line as our result:

\[
\left( \frac{\partial_{m_j^2}}{\sqrt{y_{j-1} - m_j^2}} \right) F = +\frac{1}{2m_j^2\sqrt{y_{j-1} - m_j^2}} y_{j-1} (\partial_{y_{j-1}} F). \]

We can iterate this and shift higher than first derivatives

\[
\left( \frac{\partial^k_{m_j^2}}{\sqrt{y_{j-1} - m_j^2}} \right) F
\]

to derivatives on \( F. \)

We note that from the definition of \( \lambda(s_n^{n-2}, m_2^2, m_1^2) \) we have

\[
\lambda(s_n^{n-2}, m_2^2, m_1^2) = s_n^{n-2} \left( s_n^{n-2} - 2 \left( m_1^2 + m_2^2 \right) \right) + \left( m_1^2 - m_2^2 \right)^2.
\]

By Euler (\( \lambda \) is homogeneous of degree two),

\[
2\lambda(s_n^{n-2}, m_2^2, m_1^2) = \partial_{s_n^{n-2}} \lambda(s_n^{n-2}, m_2^2, m_1^2) + \partial_{m_2^2} \lambda(s_n^{n-2}, m_2^2, m_1^2) + \partial_{m_1^2} \lambda(s_n^{n-2}, m_2^2, m_1^2).
\]

Also,

\[
\partial_{m_1^2} \lambda(s_n^{n-2}, m_2^2, m_1^2) = 2 \left( m_1^2 - m_2^2 - s_n^{n-2} \right), \\
\partial_{m_2^2} \lambda(s_n^{n-2}, m_2^2, m_1^2) = 2 \left( m_2^2 - m_1^2 - s_n^{n-2} \right), \\
\partial_{m_j^2} \lambda(s_n^{n-2}, m_2^2, m_1^2) = 2 \left( s_n^{n-2} - m_1^2 - m_2^2 \right) \partial_{m_j^2} s_n^{n-2}, \forall 3 \leq j \leq n, \\
\partial_{s} \lambda(s_n^{n-2}, m_2^2, m_1^2) = 2 \left( s_n^{n-2} - m_1^2 - m_2^2 \right) \partial_{s} s_n^{n-2}.
\]
With this, Thm. (2.2) allows to derive differential equations.

Let us rederive, for example, the differential equation for the three-edge banana. Let us define

\[
\begin{align*}
F_0 &= \Phi_R(b_3), \\
F_1 &= \partial_{m_1^2} F_0, \\
F_2 &= \partial_{m_2^2} F_0, \\
F_3 &= \partial_{m_3^2} F_0, \\
F_s &= \partial_s F_0.
\end{align*}
\]

Then, we have

\[
(D - 3) F_0 + \sum_{j=1}^{3} m_j^2 F_j = s \partial_s F_0,
\]

(3.13)

and similarly,

\[
\left( (D - 4) + \sum_{i=1}^{3} m_i^2 \partial_{m_i^2} \right) F_j = s \partial_s F_j, \quad j \in \{1, 2, 3\}.
\]

(3.14)

The integrands \( I_i \) for \((D - 3) F_0, m_1^2 F_1, m_2^2 F_2, m_3^2 F_3, \) and \( s F_s \) can be written as

\[
I_i = \frac{\text{num}_i(y_2)}{s^{D-2}} \sqrt{y_2^2 - m_3^2} \sqrt{\lambda^{D-5} (s_1^1, m_2^2, m_1^2)}
\]

with suitable polynomials \( \text{num}_i \) in \( y_2 \). Equation (3.13) follows immediately as the corresponding numerators \( \text{num}_i(y_2) \) add to zero.

Equation (3.14) for \( F_1, F_2, F_3 \) can be proven in precisely the same manner, and many more differential equations follow from using the ibp identities Eqs. (3.1–3.3).

Furthermore, \( F_0, F_1, F_2, F_3 \) provide master integrals for the Feynman integrals \( \Phi_R(b_3)^\nu \) [10].

**Remark 3.5** Note that we can infer the independence of \( F_0, F_1, F_2, F_3 \) from the fact that the corresponding polynomials are different, in fact of different degree in \( y_2 \).

We could also use different integral representations for \( F_1, F_2, F_3 \) by setting

\[
\begin{align*}
F_3 &= \partial_{m_3^2} \text{rhs of Eq. (2.1)}, \\
F_2 &= \partial_{m_2^2} \text{rhs of Eq. (2.2)}, \\
F_1 &= \partial_{m_1^2} \text{rhs of Eq. (2.3)}.
\end{align*}
\]

and conclude from there.
3.4 Master integrals

We want to comment on two facts:

(i) A geometric interpretation of the known formula for the counting of master integrals for $b_n$,

(ii) That the independence of elements $x$ of a set $S_{b_n}$ of master integrals does not imply the independence of elements of $\Im(x)$, $x \in (S_{b_n})$.

3.4.1 A geometric interpretation: powercounting

Let us start with a geometric interpretation. We collect a well-known proposition [26, 27].

**Proposition 3.6** The number of master integrals for the $n$-edge banana with different masses is $2^n - 1$.

Let us pause. For $b_3$, we have four master integrals, $F_0$, and three possibilities to put a dot on an internal edge. Furthermore, we can shrink any of the three internal edges, giving us three two-petal roses as minors. This makes $7 = 2^3 - 1$ master integrals amounting to the fact that all tensor integrals $\Phi_R(b_n)_\nu$ can be expressed as a linear combination of those seven, with coefficients which are rational functions in the mass squares and in $s$.

Similarly, for $b_4$ we have $\Phi_R^D(b_4)$ itself, four integrals $\partial^2_{m_i^2} \Phi_R^D(b_4)$ and six $\partial^2_{m_i^2} \partial^2_{m_j^2} \Phi_R^D(b_4), i \neq j$. There are four minors as well, so that we get the desired $15 = 2^4 - 1$ master integrals.

For arbitrary $n$, there are indeed $n \choose j$ possibilities to put one dot on $j$ edges, and

$$\sum_{j=0}^{n-2} n \choose j = 2^n - n - 1,$$

possibilities to put a single dot on up to $n - 2$ edges. Furthermore, we have $n$ minors from shrinking one of the $n$ edges, so we get $2^n - 1$ master integrals.

Furthermore, it is obvious from the structure of the iterated integral in Eq. (2.14) that the two edges forming the innermost $b_2$ do not need a dot. Indeed, the corresponding loop integral in $k_1$ is fixed by two $\delta_\perp$ functions. Integration by parts then ensures that we do not need more than one dot per edge at most.

**Remark 3.7** One can analyse this from the viewpoint of powercounting. Let us choose $D = 4$ so that $b_2$ is log-divergent. Let us note that for $D = 4$

$$4(n - 1) - 2 (2n - 2) = 0,$$

(3.15)

where $\#E$ is the number of edges of a banana graph $b_n$ which has $(n - 2)$ edges with a single dot each. Equation (3.15) says that $b_n$ furnished with the maximum of $n - 2$ dots gives an overall logarithmic singular integral for any $n$. 

\[ \Delta \] Springer
Fig. 5 The graph $b_3$ and its triangular cell $C_3$. The codimension-one boundaries (sides) are given by the condition $A_i = 0$, indicated in the figure by $i = 0$, $i \in \{1, 2, 3\}$. The graph $b_3$ with two yellow leaves as external edges is put in the barycentre. All its edges are put on-shell. The cell decomposes into six sectors $m_i A_i > m_j A_j > m_k A_k$ as indicated by $i > j > k$. The lines $m_i A_i = m_j A_j$ (indicated by $i = j$) start at the midpoint $\text{mid}_{i,j}: A_k = 0$. $A_i m_i = A_j m_j$ of the codimension one boundary $A_k = 0$ and pass through the barycentre $bc : m_1 A_1 = m_2 A_2 = m_3 A_3$ towards the corner $c_k : A_i = A_j = 0$, labelled $k$. Such corners are removed. For these three lines, the three intervals $[\text{mid}_{i,j}, bc]$ from the midpoints of the sides to the barycentre of the cell form the spine. It indicated in turquoise. The bold hashed line indicated by $2 < 3$ (so $m_2 A_2 < m_3 A_3$) on the left and $2 < 1$ (so $m_2 A_2 < m_1 A_1$) on the right is an example of a fibre over one (the vertical) part (on the $1 = 3$-line) of the spine (the turquoise line from $m_1 A_1 = m_3 A_3, A_2 = 0$ to the barycentre). On the left, along the fibre the ratio $A_2 / A_3 < m_3 / m_2$ is a constant, on the right similarly. Finally, to the two yellow leaves we assign incoming four-momenta $k_3, -k_3$ with $k_3^2 = s$. The spine partitions the cell $C_3$ into three 2-cubes, boxes $\square(j)$ with four corners for any $i : \text{mid}_{i,j}, bc, \text{mid}_{i,k}, c_j$. For each such box $\square(j)$ there is a diagonal $d_j$. It is a line from a corner to the barycentre: $d_j : [c_j, bc]$ for which we have $m_i A_i = m_k A_k$. We assign to this diagonal $d_j$ a graph for which edges $e_i, e_k$ are on-shell and edge $e_j$ carries a dot. Along the diagonal $d_j$, we have $A_j m_j = (A_i m_i = A_k m_k)$ (colour figure online).

A lesser number of dots give a higher degree of divergence and hence higher subtractions in the dispersion integrals. Conceptually, higher degrees of divergence are probing higher coefficients in the Taylor expansion in $s$ which provide the needed master integrals. We see below how this interferes with counting master integrals but first our geometric interpretation as given in Fig. 5.

### 3.4.2 $b_3$ and its cell

The parametric representation of $b_3$ as given in “App. E” provides insight into the structure of its Feynman integral and the related master integrals.

**Remark 3.8** Let us note that any graph $b_n$ has a spanning tree which consists of just one of its internal edges. Hence, any associated spanning tree has length one. As $b_n$ has $n$ internal edges its associated cell $C(b_n)$ (in the sense of Outer Space [30]) is a $(n - 1)$-dimensional simplex $C_n$.

$$C(b_n) = C_n.$$
The graph $b_n$ has internal edges $e_i$. To each such edge, we assign a length $A_i$, $0 \leq A_i \leq \infty$ which we regard as a coordinate in the projective space $\mathbb{P}_{b_n} := \mathbb{P}^{n-1}(\mathbb{R}_+)$. Shrinking one edge $e_i$ to length $A_i = 0$ gives the graph $b_n/e_i$ which is associated with the codimension-one boundary determined by $A_i = 0$. It is a $(n-2)$-dimensional simplex $C_{n-1}$.

Note $b_n/e_i$ is a rose with $(n-1)$ petals. Each petal corresponds to a tadpole integral for a propagator with mass $m_j^2, j \neq i$.

Different points of $C(b_n)$ correspond to different points

$$\mathbb{P}_{b_n} \ni p : (A_1 : A_2 : \cdots : A_n).$$

We can identify $n!$ sectors $\sigma : A_{\sigma(1)} > A_{\sigma(2)} > \cdots > A_{\sigma(n)}$ for any permutation $\sigma \in S_n$ with associated sector $\sigma$.

$$\Phi^D_R(b_n)(s, s_0) = \int_{\mathbb{P}_{b_n}(\mathbb{R}_+)} \text{Int}_{b_n}(s, s_0; p) = \sum_{\sigma \in S_n} \int_{C_{n-1}} T^{(\rho^\sigma_D)} \left[ \text{Int}_{b_n}(s, s_0; p) \right],$$

(3.16)

with

$$\text{Int}_{b_n}(s, s_0; p) = \frac{\ln \Phi(b_n)(s)(p)}{\psi_{b_n}(p)} \Omega_{b_n}.$$ 

$T^{(\rho^\sigma_D)}$ is a suitable Taylor operator with subtractions at $s = s_0$ ensuring overall convergence and $\rho^\sigma_D$ the UV degree of divergence. Here,

$$\Phi(b_n)(s)(p) = \left( \prod_{j=1}^n A_j \right) \left( s - \left( \sum_{i=1}^n A_i m_i^2 \right) \left( \sum_{k=1}^n \frac{1}{A_k} \right) \right),$$

and

$$\left( \prod_{j=1}^n A_j \right) \left( \sum_{k=1}^n \frac{1}{A_k} \right).$$

Each sector allows for a rescaling according to the order of edge variables such that the singularity is an isolated pole.

Here, $TP(b_n)$ is the toric polynomial of $b_n$ as discussed in [11, 31] and prominent in the GKZ approach used there.

Such approaches with their emphasis on hypergeometrics and the role of confluence have a precursor in the study of Dirichlet measures [32]. The latter have proven their relevance for Feynman diagram analysis early on [33].
The spine of $C(b_n)$ is a $n$-star, with the vertex in the barycentre and $n$ rays from the barycentre $bc$ of $C(b_n)$ to the midpoints of the $n$ codimension-one cells $C_{n-1}$ which are $(n - 2)$-simplices.

These rays provide $n$ corresponding cubical chain complices $cc(i)$ each provided by single intervals $[0, 1]$.

For the two endpoints 0 and 1 of each $cc(i)$, we assign:

(i) to 1,—the barycentre $bc$ common to all $cc(i)$ we assign $b_n$ with internal edges removed, hence evaluated on-shell. This corresponds to $\Im(\Phi^D_R(b_n))$.

(ii) To 0, we assign the graph $b_n/e_i$ (a rose with $n - 1$ petals) with petals of equal size—hence a tadpole $\Phi^D_R(b_n/e_i)$ with $A_j m_j = A_k m_k$, $j, k \neq i$. See Fig. 5.

Figure 5 gives the graph $b_3$ and the associated cell, a 2-simplex $C_3$. It is a triangle with corners $c_1, c_2, c_3$. Points of the cell are the interior points of $C_3$ and furthermore the points in the three codimension-one boundaries $C_2(i)$, the sides of the triangle.

The corners $c_i$ are removed and do not belong to the cell. Points of the cell parameterize the edge lengths $A_i$ of the internal edges of $b_3$ as parameters in the parametric integrand, see Eq. (E.1).

The boundaries are given by $C_2(i) : A_i = 0$ and correspond to tadpole integrals for tadpoles $t_2(i) = b_3/e_i$ for which edge $e_i$ has length zero.

Corners $c_i : A_i = A_j = 0$, $i \neq j$ correspond to $b_3/e_i/e_j$ which is degenerate as it shrinks a loop.

Colours green, red, and blue indicate three different masses. It is understood that a momentum $k_3$ flows through any edge $e_i$ which is chosen to serve as a spanning tree for $b_3$.

The three edges of the graph give rise to $3!$ orderings of the edge lengths as indicated in the figure. We will split the parametric integral accordingly. See “App. E” for computational details.

To a $(i = j)$-diagonal of a box $\square(k)$, we associate a $b_3$ evaluated with edges $e_i, e_j$ on-shell and edge $e_k$ dotted, so it corresponds to $\partial_{m_3}(\Phi^D_R(b_3))$.

In the figure, there is also an arc given which is a fibre which has the diagonal $d_j$ as the base. Integrating that fibre corresponds to integrating the $b_2$ subgraph on edges $e_i, e_j$. Points $(A_i : A_j : A_k)$ on a diagonal $d_k$ fulfil

$$A_k m_k > x, x := A_i m_i = A_j m_j.$$

To the barycentre $A_i m_i = A_j m_j$, we associate $b_3$ with all three edges on-shell, a Cutkosky cut providing $\Im(\Phi^D_R(b_3))$. To the midpoints $A_i = A_j, A_k = 0$ of the edges $A_i = 0$ ($e_i = 0$ in the figure), we assign tadpole integrals. All in all we identified all seven master integrals in the figure. Note that the cell decomposition in Fig. 5 reflects the structure of the Newton polyhedron associated with $TP(b_3)$ [31].

Note that the requirement $A_i m_i = A_j m_j$ is the locus for the Landau singularity of the associated $b_2(e_i, e_j)$ and similarly for $A_1 m_1 = A_2 m_2 = A_3 m_3$ and $b_3$.

**Remark 3.9** Note that the diagonals $d_j$ can be obtained by reflecting a leg of the spine at the barycentre. The three legs and the three diagonals form the six boundaries between the sectors $A_i > A_j > A_k$. 
A similar analysis holds for any $b_n$. For example, for $b_4$ the cell is a tetrahedron with four corners $c_i$, $i \in \{1, 2, 3, 4\}$. The spine is a four-star with four lines (rays) from the barycentre $bc : m_1A_1 = m_2A_2 = m_3A_3 = m_4A_4$ to the midpoints of the four sides of the tetrahedra (triangles). Reflecting these lines at the barycentre gives four diagonals $d_j : \{bc, c_j\}$ from $bc$ to one of the four corners $c_i$.

To $bc$, we associate $\Im\left(\Phi^D_R(b_4)\right)$. To the diagonals $d_j$, we assign $\partial_{m_{ij}^2} \Im\left(\Phi^D_R(b_4)\right)$ with the edges $e_i, i \neq j$, on-shell. There are six triangles with sides $d_i, d_j, |c_i, c_j|$. To those, we assign $\partial_{m_i^2} \partial_{m_j^2} \Im\left(\Phi^D_R(b_4)\right)$ with the edges $e_k, k \neq i, j$, on-shell. See Fig. 6.

Continuing we get the expected tally: for $b_n$, we have $\binom{n}{0} = 1$ graph for the barycentre, $\binom{n}{1} = n$ graphs for the diagonals, $\binom{n}{m}, m \leq (n - 2)$ graphs for the span of $m$ diagonals, and $\binom{n}{n-1} = n$ tadpoles. It is rather charming to see how mathematics inspired by the works of Karen Vogtmann and collaborators [30] illuminates results discussed recently in terms of intersection theory [34].

### 3.4.3 Real and Imaginary Independence and Powercounting

Next, we want to compare real and imaginary parts to check that the independence of elements of $S_{b_n}$ does not necessarily imply the independence of elements of $\Im\left(S_{b_n}\right)$. We demonstrate this well-known fact [5] for $b_3$. Independence is indeed a question of the values of $D$.

For $b_3$ and $D = 2$, we need no subtraction in the dispersion integral for $F_0 = \Phi^2_R(b_3)$,

$$\Phi^2_R(b_3)(s) = \frac{1}{\pi} \int_{(m_1 + m_2 + m_3)^2}^{\infty} \frac{V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x - s)} \, dx,$$

\[ \odot \text{Springer} \]
and for $F_i = \partial_{m_i^2} F_0$ again an unsubtracted dispersion integral suffices

$$F_i(s) = \frac{1}{\pi} \int_0^\infty \frac{\partial_{m_i^2} V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)^2} dx.$$  

The four integrands $I_i$ (for the $y_2$-integration) of $\Im(F_i)$, $i \in \{0, 1, 2, 3\}$ can be expressed over a common denominator with numerators $\text{num}_i(y_2)$, and for $D = 2$ (the $(s^{n-2})^{D-1} = 1$ is absent), there is indeed a relation between the four numerators.

$$\text{num}_3(y_2) = c_0^3 \text{num}_0(y_2) + c_1^3 \text{num}_1(y_2) + c_2^3 \text{num}_2(y_2),$$

where $c_i^3$ are rational functions of $s, m_1^2, m_2^2, m_3^2$ independent of $y_2$.

For $D = 2$, a second relation follows from the fact that the integrand involves the square root of a quartic polynomial ([5], App. D),

$$\frac{1}{\sqrt{y_2^2 - m_3^2}} V_3^2(y_2) = \frac{1}{\sqrt{s}(y_2 - m_3)(y_2 + m_3)(y_2 - y_+)(y_2 - y_-)},$$

where we set for the quadratic polynomial $\lambda(s_3^1(y_2), m_1^2, m_2^2)$,

$$\lambda(s_3^1(y_2), m_1^2, m_2^2) = s(y_2 - y_+)(y_2 - y_-),$$

which defines $y_\pm$. See Sect. 2.3.

Investigating

$$J_n = \int_{m_3}^{u_3^0} \frac{y_2^n}{\sqrt{s}(y_2 - m_3)(y_2 + m_3)(y_2 - y_+)(y_2 - y_-)} dy_2,$$

as in [5] delivers a further relation between the $F_i$, and we are hence left with only two independent master integrals for the imaginary parts of $b_3$ in $D = 2$.

For $b_3$ and $D = 4$, on the other hand we need a double subtraction in the dispersion integral for $F_0 = \Phi^4_R(b_3)$,

$$\Phi^4_R(b_3)(s, s_0) = \frac{(s - s_0)^2}{\pi} \int_0^\infty \frac{V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)(x-s_0)^2} dx,$$

whilst for $F_i = \partial_{m_i^2} F_0$ a once-subtracted dispersion integral suffices,

$$F_i(s) = \frac{(s - s_0)}{\pi} \int_0^\infty \frac{\partial_{m_i^2} V_3^D(x, m_1^2, m_2^2, m_3^2)}{(x-s)(x-s_0)^2} dx.$$  

The four integrands $I_i$ (for the $y_2$-integration) of $\Im(F_i)$, $i \in \{0, 1, 2, 3\}$ have to be expressed over a different common denominator $D = 4$, in particular having an extra factor $s_3^1$. There is no relation between them.
This reflects the fact that the $F_0$ dispersion
\[
\Phi_R^4(b_3)(s, s_0) = \frac{(s - s_0)}{\pi} \int_{(m_1 + m_2 + m_3)^2}^{\infty} \left( \frac{V^D_3(x, m_1^2, m_2^2, m_3^2)}{(x - s)(x - s_0)} - \frac{V^D_3(x, m_1^2, m_2^2, m_3^2)}{(x - s_0)^2} \right) \, dx,
\]
subsumes the Taylor expansion $s$ near $s_0$ to second order.

In contrast, the $F_i, i \in \{1, 2, 3\}$,
\[
\partial_{m_i^2} \Phi_R^4(b_3)(s, s_0) = \frac{1}{\pi} \int_{(m_1 + m_2 + m_3)^2}^{\infty} \left( \frac{V^D_3(x, m_1^2, m_2^2, m_3^2)}{(x - s)} - \frac{V^D_3(x, m_1^2, m_2^2, m_3^2)}{(x - s_0)} \right) \, dx,
\]
subsume the Taylor expansion in $s$ near $s_0$ to first order.

This is in agreement with the powercounting in Eq. (3.15) and forces the relation between the four $F_i$ to be $\sim s \partial_s F_0$, see Eq. (3.13). The relation Eq. (3.17) is spoiled by the extra coefficient in the Taylor expansion of $\Phi_R^4(b_3)(s, s_0)$.

We are left with four, not two, master integrals. Indeed, starting with a dotted log-divergent banana integral reducing the number of dots demands more subtractions in the dispersion integral. Any relation between imaginary parts with different numbers of dots is spoiled by the difference in degree needed for the subtractions in the dispersion integral.

Acknowledgements This is work originating from discussions with Karen Vogtmann and Marko Berghoff which are gratefully acknowledged. I thank Spencer Bloch, David Broadhurst and Bob Delbourgo for friendship and for sharing insights into the mathematics and physics of quantum field theory over the years and David for pointing out some older literature. Enjoyable discussions with Ralph Kaufmann on possible similarities of the structure of phase-space integrals and his use of singularity theory in applied quantum field theory [35] were a welcome stimulus to write these results.

Funding Open Access funding enabled and organized by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Appendix A: Feynman rules for banana graphs

Having introduced the graphs $b_n$ as our subject of interest we define Feynman rules for their evaluation. We follow the momentum routing as indicated in Fig. 1.

The graph $b_n$ gives rise to an integrand $I_{b_n}$ (setting $k_0 = (0, 0)^T$, where the $D$-vector $k_0$ is set to the zero-vector $(0, 0)^T \in \mathbb{M}^D$):
\[
I_{b_n} = \omega^{D}_{(n-1)} \prod_{j=0}^{n-1} \frac{1}{(k_{j+1} - k_j)^2 - m_{j+1}^2},
\]
and we set $Q_{j+1} = (k_{j+1} - k_j)^2 - m_{j+1}^2$, $0 \leq j \leq (n-1)$ for the $n$ quadrics $Q_{j+1}$, $j = 0, \ldots, n - 1$. Here,

$$
\omega_{(n-1)}^D := d^D k_1 \cdots d^D k_{n-1}
$$

is a $D \times (n-1)$-form in a $(n-1)$-fold product $\mathbb{M}_n$ of $D$-dimensional Minkowski spaces

$$
\mathbb{M}_n := \left( \mathbb{M}^D \right)^{(n-1)}.
$$

The function $\Phi_R^D (b_n) (s)$ is multi-valued as a function of $s := k_n^2$. It has an imaginary part given by a cut which amounts to replacing for each quadric

$$
\frac{1}{Q_{j+1}} \rightarrow \delta_+ ((k_{j+1} - k_j)^2 - m_{j+1}^2),
$$

in the integrand $I_{b_n}$. This is Cutkosky’s theorem [36] applied to $b_n$. The distribution $\delta_+$ acts as

$$
\delta_+ ((k_{j+1} - k_j)^2 - m_{j+1}^2) = \Theta(k_{j+1;0} - k_{j;0}) \delta((k_{j+1} - k_j)^2 - m_{j+1}^2),
$$

using the Heavyside distribution $\Theta$ and Dirac $\delta$-distribution.

The integrand for the cut banana is correspondingly

$$
I_{\text{cut}} (b_n) = \omega_{(n-1)}^D \prod_{j=0}^{n-1} \delta_+ \left( (k_{j+1} - k_j)^2 - m_{j+1}^2 \right). \quad (A.1)
$$

We take the external momentum $k_n$ to be timelike so that we can choose $k_n = (k_{n;0}, \vec{0})^T$ and set $k_j = (k_{j;0}, \vec{k}_j)^T$. We also set $\vec{k}_j \cdot \vec{k}_j =: t_j$ and have $k_j^2 = k_{j;0}^2 - t_j$, and finally define $\hat{k}_j = \frac{\vec{k}_j}{\sqrt{t_j}}$. Hence,

$$
d^D k_j = dk_{j;0} \sqrt{t_j} \cdot D^{D-3} dt_j \, d\hat{k}_j,
$$

with an angular measure

$$
\int_{S^{D-2}} \, d\hat{k}_j = \omega_2.
$$

Here,

$$
\omega_2 = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma \left( \frac{D-1}{2} \right)}, \quad \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{D-1}{2} \right)} \equiv \sqrt{\pi}. \quad (A.2)
$$
Fig. 7 The Hopf algebra disentangling the five-banana $b_5$. On the right, we also get roses with $n$ petals, or tadpoles in a physicists parlance. There are $5 = \binom{5}{3}$ labellings for the $b_4$ banana in the first term in the second row, and $10 = \binom{5}{2} = \binom{5}{3}$ for the next two tensorproducts. The final term in the third row has $30 = \binom{5}{2} \binom{3}{2}$ labellings, as there are $\binom{5}{2}$ possibilities to label the edges of the first $b_2$ banana, and then $\binom{3}{2}$ to label the second one.

We then have as integrations

$$\int_{M^D} d^D k_j f(k_j) = \int_{-\infty}^{\infty} d k_{j:0} \int_0^\infty \sqrt{t_j} D^{D-3} d t_j \int_{S^{D-2}} d \hat{k}_j f(k_{j:0}, t_j, \hat{k}_j).$$

Appendix B: Minimal subtraction

For the reader which likes to compare with dimensional regularization and the use of minimal subtraction as renormalization, we have kept $D$ complex in most formulae and note that in such a situation the coproduct for $b_n$ reads

$$\Delta_H(b_n) = b_n \otimes \mathbb{1} + \mathbb{1} \otimes b_n + \sum_{x, |x| \leq n} x \otimes t_{n-|x|}. \quad (B.1)$$

Here, the sum is over all monomials $x$ of banana graphs $b_j$ on less than $n$ edges. For example,

$$\Delta(b_5) = b_5 \otimes \mathbb{1} + \mathbb{1} \otimes b_5 + \binom{5}{2} b_2 \otimes t_3 + \binom{5}{3} b_3 \otimes t_2 + \binom{5}{4} b_4 \otimes t_1 + \binom{5}{5} b_2 b_2 \otimes t_1.$$
\[- \sum_{cycl} (\Phi^D(b_2(e_i, e_j))) \Phi^D(t_1(e_k)).\]

Here, \(\Phi^D\) are unrenormalized Feynman rules in \(D\) dimensions which evaluate into a Laurent series in \(D - 2n\), \(n\) a suitable integer, \(\ldots\) is the projection onto the pole part and the sum is over the three cyclic permutations of \(i, j, k\).

This MS-renormalization results \(\Phi^D_{MS}(b_n)\) can be related to \(\Phi^D_R(b_n)\) if so desired. See also the discussion with regards to MS and tadpoles in [28].

### Appendix C: Tensor structure

#### Tensor integrals

We are not interested in \(\Phi^D_R(b_n)\) alone. To satisfy the needs of computational practice, we should also raise the powers of quadrics by taking derivatives \(\partial^{k_{ij}}\) with respect to mass squares \(m^2_j\) and we should allow scalar products \(k_i \cdot k_j\) in the numerator.

For such a generalization to arbitrary powers of propagators and numerator structures, we use the notation

\[\Phi^D_R(b_n)_{\nu}(s, \{m^2_i\}),\]

where \(\nu\) is a \((n(n+1)/2 - 1)\)-dimensional row vector with integer entries (see Sect. (2.5.1.) in [10]).

- The first \(n\) entries \(\nu_i, 1 \leq i \leq n\) give the powers of the \(n\) edge propagators \(1_{Qe}\),
- the \((n - 2)\) entries \(\nu_i, (n + 1) \leq i \leq (2n - 2)\) are reserved for powers of \(k_i \cdot k_n\) \((1 \leq i \leq (n - 2))\),
- the \((n - 2)\) entries \(\nu_i, (2n - 1) \leq i \leq (3n - 4)\) are reserved for powers of \(k^2_2, \ldots, k^2_{n-1}\),
- and the remaining \((n - 2)(n - 3)/2\) entries are reserved for powers \(\nu_{ji}\) of \(k_j \cdot k_i\), \(|j - l| \geq 1, 1 \leq j, l \leq (n - 1)\) and \(3n - 3 \leq i \leq \left(\frac{n(n+1)}{2} - 1\right)\).

This is all what is needed as \(k^2_i = Q_1 + m^2_1\) and \(2k_i \cdot k_{i-1} = k^2_i + k^2_{i-1} - Q_i - m^2_i\), \(n \geq i \geq 2\).

For example,

\[
\Phi^D_R(b_4)_{\nu_1, \ldots, \nu_{13}}(s, m^2_1, \ldots, m^2_4)
= \int_{M_4} \omega_D(3) \prod_{j=0}^{3} \frac{(k_1 \cdot k_4)^{\nu_5}(k_2 \cdot k_4)^{\nu_6}(k_2^2)^{\nu_7}(k_2^3)^{\nu_8}(k_1 \cdot k_3)^{\nu_{13}}}{(k_{j+1} - k_j)^2 - m^2_{j+1}} \nu^{v_{j+1}}.
\]

For the imaginary part, we have correspondingly

\[
\Im \left(\Phi^D_R(b_4)_{\nu_1, \ldots, \nu_{13}}\right)(s, m^2_1, \ldots, m^2_4)
\]
\[= \int_{\mathcal{M}_4} \omega_3^{D} \prod_{j=0}^{3} \partial_{m_{j+1}}^{v_{j+1}} \left( \prod_{l=0}^{3} \delta_+ (k_{l+1} - k_l)^2 - m_{l+1}^2 \right) \]
\[\times (k_1 \cdot k_4)^{v_5} (k_2 \cdot k_4)^{v_6} (k_2^2)^{v_7} (k_3^2)^{v_8} (k_1 \cdot k_3)^{v_{13}} \).\]

We discuss differential equations for \( \Phi_R^D (b_n)_\nu \), as well as partial integration identities and the reduction to master integrals starting from our representation for \( \Phi_R^D (b_n)_\nu \) in Sects. (3.3, 3.4).

**Dispersion for \( \Phi_R^D (b_n)_\nu \)**

For banana graphs \( b_n \) on two vertices, dispersion for tensor integrals is rather simple:

\[\Phi_R^D (b_n)_\nu (s, s_0, \{m_j^2\}) = \frac{(s - s_0)[n, \nu]}{\pi} \int_{(m_1 + \cdots + m_n)^2}^{\infty} \frac{V_D^{[n, \nu]}}{(x - s)(x - s')^{[n, \nu]}} \, dx, \quad (C.1)\]

where \([n, \nu] - 1\) is the superficial degree of divergence of \( \Phi_R^D (b_n)_\nu \) according to \( \nu \):

\[\quad[[n, \nu]] = \left( \frac{D}{2} - 1 \right) (n - 1) + \sum_{j=1}^{n} v_j + \left[ \sum_{j=n+1}^{2n-2} \frac{v_j}{2} \right] + \sum_{j=2n-1}^{3n-4} v_j + \sum_{jl} v_{jl}. \quad (C.2)\]

This is based on

\[\Im \left( \Phi_R^D (b_n)_\nu \right) (s, s_0, \{m_j^2\}) = \Theta(s - (m_1 + \cdots + m_n)^2) V_D^{[n, \nu]}.\]

For \( V_D^{[n, \nu]} \), see Eqs. (2.25–2.29).

**Appendix D: Pseudo-thresholds**

Let us remind ourselves of a parametric analysis of the second Symanzik polynomial (with masses) \( \Phi \) for the banana graphs \( b_b \):

\[\varphi(b_n)(s) = \left( \prod_{j=1}^{n} A_j \right) \left( s - \left( \sum_{j=1}^{n} m_j^2 A_j \right) \left( \sum_{j=1}^{n} \frac{1}{A_j} \right) \right). \quad (D.1)\]

The equation

\[\varphi(b_n)(m_n^{n_{normal}}) = 0,\]

has a solution in the simplex \( A_i > 0 \) for positive \( A_i \) given by \( A_i m_i = A_j m_j \).
For \( m \) any pseudo-mass, the solution of \( \varphi(b_n)(m) = 0 \) requires at least one \( A_i \) to be negative and it hence gives no monodromy on the physical sheet.

Still, the variations associated with pseudo-masses and thresholds are needed for a full analysis of \( \Phi_R^D(b_n) \) to find their Hodge structure.

So, let \( \sigma_n \) be a sequence of the form

\[
\sigma^n := (\pm m_1 \pm m_2 \pm \cdots \pm m_n),
\]

with a sign chosen for each entry \( m_i \). Let \( p(i) \in \{\pm 1\} \) be the sign of the \( i \)-entry. A global sign change leaves the pseudo-thresholds invariant (\(|a - b| = |b - a|\)), so we have \( 2^{n-1} \) choices and adopt to the convention \( p(1) = +1 \).

For a flag

\[
(b_2 \subset b_3 \subset \cdots \subset b_n),
\]

this determines subsequences \( \sigma^2 \subset \sigma^3 \subset \cdots \sigma^n \) in an obvious manner.

Define

\[
up^j,\sigma_n := \frac{s_n^j + m_{n-j}^2 - \left( \sum_{||,i=1}^{m_{\sigma_n-j-1}} p(i)m_i \right)^2}{2\sqrt{s_n^j}},
\]

which also defines the pseudo-mass \( m_{\sigma_n-j-1} \):

\[
m_{\sigma_n-j-1} = \sum_{||,i=1}^{n-j-1} p(i)m_i = |\cdots| m_1 + p(2)m_2 | + p(3)m_3 | + \cdots | + p(n-j-1)m_{n-j-1}|.
\]

Define

\[
\Theta_{n,+} = \Theta(s - (m_n + m_{\sigma_n-1})^2), \quad \Theta_{n,-} = \Theta((m_n - m_{\sigma_n-1})^2 - s).
\]

Now, set for \( p(n-1) = +1 \):

\[
\text{Var}(b_n^\sigma) = \Theta_{n,p(n)}
\]

\[
\frac{\omega \rho}{2} \int_{\sigma_{n-1,n-1}}^{\omega n,\sigma_n} \sqrt[2]{\Phi^D_{\sigma_{n-1,n-1}} \left( \sqrt[2]{s_n^0 y_n-1 + m_n^2 m_1^2 \cdots m_{n-1}^2} \sqrt[2]{\frac{2}{y_n-1} - m_{n-1}^2} \right)^{D-3}} \, dy_{n-1}.
\]

and for \( p(n-1) = -1 \):

\[
\text{Var}(b_n^\sigma) = \Theta_{n,p(n)}
\]
× \omega \int_{0, \sigma}^{\infty} V_{\alpha_{n-1}, \sigma}^{D} \left( s_0^n - 2 \sqrt{s_0^n y_{n-1} + m_2 \sigma_n^2, \ldots, m_{n-1}^2} \right) \sqrt{y_{n-1}^2 - m_2^D} \, dy_{n-1}.

\quad V_{\alpha_{n,n}, \sigma(n-1)=-1}

Apart from the variation for the normal threshold (with \( p(i) = +1 \) for all \( 1 \leq i \leq n \)) which gives \( \text{Var}(b_n^{+m_1, +m_2, \ldots, +m_n}) = \infty \left( \Phi_R^D(b_n) \right) \), we get \( 2^{n-1} - 1 \) further variations corresponding to pseudo-masses and their pseudo-thresholds. They will be discussed elsewhere.

**Appendix E: \( b_3 \) parametrically**

Let us recapitulate \( b_3 \) in the parametric representation. We list basic considerations. A detailed analysis in the view of [37, 38] is left to future work.

**E.1. The parametric integral**

Let \( Q_{b_3} \) be the one-dimensional real vector space spanned by \( s = k_3^2 \), the square of the Minkowski four-momenta \( k_3, -k_3 \) assigned to the two vertices of \( b_3 \). Let \( \mathbb{P}_{b_3} = \mathbb{P}^2(\mathbb{R}_+) \) be a projective space given by the ratios of the nonnegative side lengths of the internal edges of \( \Theta \).

The parametric integrand function (we consider masses as implicit parameters)

\[ F_{b_3} : Q_{b_3} \times Q_{b_3} \times \mathbb{P}_{b_3} \rightarrow \mathbb{C} \]

is (see, for example, Sect. (5.2.1.) in [39])

\[ F_{b_3}(s, s_0; p) := (s - s_0) A_1 A_2 A_3 \frac{\ln \left( \frac{\Phi_\Theta(s; p)}{\Phi_\Theta(s_0; p)} \right)}{\psi_\Theta^{3}} \]

\[ + (s_0 A_1 A_2 A_3 - (m_1^2 A_1 + m_2^2 A_2 + m_3^2 A_3) \psi_{b_3}) \ln \left( \frac{\Phi_\Theta(s; p)}{\Phi_\Theta(s_0; p)} \right) \]

\[ \times \frac{\partial_s \ln \left( \frac{\Phi_\Theta(s; p)}{\Phi_\Theta(s_0; p)} \right)}{\psi_\Theta^{3}} \bigg|_{s = s_0}. \]  

(E.1)

Here,

\[ \Phi_{b_3} : Q_\Theta \times \mathbb{P}_\Theta \rightarrow \mathbb{C} \]

is

\[ \Phi_{b_3}(r; p) = r A_1 A_2 A_3 - (m_1^2 A_1 + m_2^2 A_2 + m_3^2 A_3) \psi_{b_3}, \]

\[ \psi_{b_3} = A_1 A_2 + A_2 A_3 + A_3 A_1. \]
Note $F_{b_3}(s, p)$ and $\partial_s F_{b_3}(s, p)$ both vanish at $s = s_0$ for all $p$, so these are on-shell renormalization conditions.

The parametric form is the integrand

$$\text{Int}_{b_3}(s, s_0; p) := F_\Theta(s, s_0, p)\Omega_{b_3},$$

$$\Omega_{b_3} = +A_1 dA_2 \wedge dA_3 - A_2 dA_1 \wedge dA_3 + A_3 dA_1 \wedge dA_2.$$  

We then have the renormalized value

$$\Phi^D_R(b_3)(s, s_0) = \int_{P^2(\mathbb{R}_+)} \text{Int}_\Theta(s, s_0; p), \tag{E.2}$$

from integrating out $p$ which is the parametric equivalent of Eqs. (2.1, 2.9).

### E.2. Sectors and fibrations

To study fibrations in our integrand, we start from the fact that there are six orderings of the edge lengths for the three edge variables $A_i$.

Consider, for example, the sectors $1 > 3 > 2$ and $3 > 1 > 2$ of Fig. 5 so that edge $e_2$ has the smallest length. For the choice $1 > 3 > 2$ rescale $A_2 = a_2 A_1$, $A_3 = a_3 A_1$, and in that sector $1 > 3 > 2$, we have

$$\int_{P^2(\mathbb{R}_+) \cap (1 > 3 > 2)} F_{b_3} \Omega_{b_3} = \int_0^\infty \left( \int_0^{a_3} F_{b_3}(1, a_2, a_3) da_2 \right) da_3.$$  

A further change $a_2 = a_3 b_2$ leads to a sector decomposition (in the sense of physicists)

$$\int_0^\infty \left( \int_0^1 a_3 F_{b_3}(1, b_2 a_3, a_3) db_2 \right) da_3 = \int_0^1 \left( \int_0^\infty a_3 F_{b_3}(1, b_2 a_3, a_3) da_3 \right) db_2.$$  

For any chosen $0 < b_2 < 1$, $a_3 F_{b_3}(1, b_2 a_3, a_3)$ gives points on the corresponding chosen fibre and $\text{Fib}(b_2)$ is the integral along that fibre. Integrating $b_2$ integrates all fibre integrals $\text{Fib}(b_2)$ to the two sector integrals on both sides of the spine.

In fact, for $0 < a_3 < m_1/m_3$ we are on the left of the spine and for $m_1/m_3 < a_3 < \infty$ on the right.

Let us look at $\Phi_{b_3}$ under the rescalings.

$$\Phi_{b_3}(A_1, A_2, A_3) = s A_1 A_2 A_3 - (m_1^2 A_1 + m_2^2 A_2 + m_3^2 A_3)(A_1 A_2 + A_2 A_3 + A_3 A_1))$$

---

3 Divergent subgraphs exist but do not need renormalization as the cographs are tadpoles which can be set to zero in kinematic renormalization. Accordingly $F_\Theta$ vanishes when any two of its three edge variables $A_i$ vanish.

4 $\Omega_{b_3} \rightarrow A_1^3 da_2 \wedge da_3$ under that rescaling.
\[ \begin{align*}
&\rightarrow sa_2a_3 - (m_1^2 + m_2^2a_2 + m_3^2a_3)(a_2 + a_2a_3 + a_3)) \\
&\rightarrow sb_2a_3^2 - (m_1^2 + m_2^2b_2a_3 + m_3^2a_3)(b_2a_3 + b_2a_3^2 + a_3)) \\
&= a_3(sb_2a_3 - (m_1^2 + m_2^2b_2a_3 + m_3^2a_3)(b_2 + b_2a_3 + 1)) =: \tilde{\Phi}_{b_3}(s, b_2, a_3).
\end{align*} \]

For \( \psi_{b_3} \), we find

\[ (A_1A_2 + A_2A_3 + A_3A_1) \]
\[ \rightarrow (a_2 + a_2a_3 + a_3) \]
\[ \rightarrow a_3(b_2 + b_2a_3 + 1). \]

We thus find in the region where \( e_2 \) is the smallest edge the integrand function \( \text{Int}_{b_3, 2}(b_2, a_3) \)

\[ \text{Int}_{b_3, 2}(b_2, a_3) := a_3F_{b_3}(1, b_2a_3, a_3) = (s - s_0)b_2a_3 \]

\[ \begin{align*}
&\ln \left( \frac{sa_3b_2 - (m_1^2 + m_2^2b_2a_3 + m_3^2a_3)(1 + b_2(1 + a_3))}{s_0a_3b_2 - (m_1^2 + m_2^2b_2a_3 + m_3^2a_3)(1 + b_2(1 + a_3))} \right) \\
&\times \frac{(b_2(1 + a_3) + 1)^3}{(b_2(1 + a_3) + 1)^3} \\
&\ln \left( \frac{\tilde{\Phi}_{b_3}(s; b_2, a_3)}{\tilde{\Phi}_{b_3}(s_0; b_2, a_3)} \right) - (s - s_0) \left( \partial_s \ln \left( \frac{\tilde{\Phi}_{b_3}(s; b_2, a_3)}{\tilde{\Phi}_{b_3}(s_0; b_2, a_3)} \right) \right)_{s=s_0}. 
\end{align*} \]

Note that \( \text{Int}_{b_3, 2}(0, a_3) = 0 \) as it must be as petals evaluate to zero under renormalized Feynman rules in on-shell renormalization conditions.

Finally,

\[ \text{Fib}(b_2) = \int_0^{\infty} \text{Int}_{b_3, 2}(b_2, a_3)da_3. \]

A point along the (1 = 3)-line of the spine is given by \((1, b_2, 1) \in P_{b_3}, \) for all \( 0 < b_2 < 1. \)

**Remark 3.10** Upon rescaling in each of the sectors in the three cubes of Fig. 5 accordingly and summing over sectors, we get a symmetric representation equivalent to averaging over the three possible ways of expressing Eq. (2.7) using any of \( s_3^1(\gamma_2, m_2^2) \) and similar to [9].

**References**

1. Veltman, M.: Unitarity and causality in a renormalizable field theory with unstable particles. Physica 29, 186 (1963)

\( \copyright \) Springer
2. Brödel, J., Duhr, C., Dulat, F., Penante, B., Tancredi, L.: Elliptic Feynman integrals and pure functions. J. High Energy Phys. 2019, 23 (2019). arXiv:1809.10698 [hep-th]

3. Broedel, J., Duhr, C., Dulat, F., Marzucca, R., Penante, B., Tancredi, L.: An analytic solution for the equal-mass banana graph. JHEP 09, 112 (2019)

4. Caffo, M., Czyż, H., Laporta, S., Remiddi, E.: The master differential equations for the 2-loop sunrise selfmass amplitudes. Nuovo Cim. 111(4), 365–389 (1998). arXiv:hep-th/9805118

5. Remiddi, E., Tancredi, L.: Schouten identities for Feynman graph amplitudes; the Master Integrals for the two-loop massive sunrise graph. Nucl. Phys. B 880, 343 (2014). arXiv:1311.3342 [hep-th]

6. Adams, L., Bogner, C., Weinzierl, S.: The sunrise integral and elliptic polylogarithms. PoS LL 2016, 033 (2016). https://doi.org/10.22323/1.260.0033. arXiv:1606.09457 [hep-ph]

7. Bloch, S., Kerr, M., Vanhove, P.: Local mirror symmetry and the sunset Feynman integral. Adv. Theor. Math. Phys. 21, 1373 (2017). https://doi.org/10.4310/ATMP.2017.v21.n6.a1. arXiv:1601.08181 [hep-th]

8. Bloch, S., Kerr, M., Vanhove, P.: A Feynman integral via higher normal functions. Compos. Math. 151(12), 2329–2375 (2015). https://doi.org/10.1112/S0010437X15007472. arXiv:1406.2664 [hep-th]

9. Davydychev, I., Delbourgo, R.: Explicitly symmetrical treatment of three-body phase space. J. Phys. A 37, 4871–4886 (2004). arxiv:hep-th/0405248)

10. Zayadeh, R.: Picard–Fuchs Equations of Dimensionally Regulated Feynman Integrals. Thesis Mainz University. https://openscience.ub.uni-mainz.de/bitstream/20.500.12030/3696/1/3663.pdf

11. Bönisch, K., Fischbach, F., Klemm, A., Nega, C., Safari, R.: Analytic structure of all loop banana amplitudes. J. High Energy Phys. 2021, 66 (2021). arXiv:2008.10574 [hep-th]

12. Broadhurst, D.: Feynman integrals, L-series and Kloosterman moments. Commun. Number Theory Phys. 10(3), 527–569 (2016)

13. Kersevan, B.P., Richter-Was, E.: Improved phase space treatment of massive multi-particle final states. Eur. Phys. J. C 39, 439–450 (2005). (( hep-ph/0405248))

14. Block, M.M.: Phase-space integrals for multiparticle systems. Phys. Rev. 101, 796 (1956)

15. Srivastava, P.P., Sudarshan, G.: Multiple production of pions in nuclear collisions. Phys. Rev. 110, 765 (1958)

16. Bloch, S., Vanhove, P.: The elliptic dilogarithm for the sunset graph. J. Number Theor. 148, 328–364 (2015). https://doi.org/10.1016/j.jnt.2014.09.032. arXiv:1309.5865 [hep-th]

17. Brown, F.: Invariant differential forms on complexes of graphs and Feynman integrals. SIGMA 17, 103 (2021)

18. Bloch, S., Esnault, H., Kreimer, D.: On motives associated to graph polynomials. Commun. Math. Phys. 267, 181–225 (2006)

19. Broedel, J., Duhr, C., Matthes, N.: Meromorphic modular forms and the three-loop equal-mass banana integral. J. High Energy Phys. 2022, 184 (2022). https://doi.org/10.1007/JHEP02(2022)184. arXiv:2109.15251

20. Coleman, S., Norton, R.: Singularities in the physical region. Nuovo Cim. 38, 438 (1965)

21. Kreimer, D.: Multi-valued Feynman graphs and scattering theory. In: Bluemlein, J., et al. (eds.) Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory. Texts & Monographs in Symbolic Computation. Springer, Berlin (2019)

22. Bloch, S., Kreimer, D.: Cutkosky Rules and Outer Space. arXiv:1512.01705

23. Chetyrkin, K., Tkachov, F.: Integration by parts: the algorithm to calculate $\beta$-functions in 4 loops. Nucl. Phys. B 192, 23 (1981)

24. Laporta, S.: High-precision calculation of multi-loop Feynman integrals by difference equations. Int. J. Mod. Phys. A 15, 5087 (2000)

25. Remiddi, E.: Differential equations for Feynman graph amplitudes. Nuovo Cim. A 110, 1435–1452 (1997). hep-th/9711188

26. Kalmykov, M., Kniehl, B.: Counting the number of master integrals for sunrise diagrams via the Mellin-Barnes representation. JHEP 1707, 031 (2017). arXiv:1612.06637 [hep-th]

27. Bitoun, T., Bogner, C., Klausen, R.P., Panzer, E.: Feynman integral relations from parametric annihilators. Lett. Math. Phys. 109(3), 497–564 (2019). arXiv:1712.09215 [hep-th]

28. Kreimer, D., Yeats, K.: Algebraic interplay between renormalization and monodromy. Adv. Theor. Math. Phys. (2023). In print. arXiv:2105.05948 [math-ph]

29. Kreimer, D.: Outer space as a combinatorial backbone for Cutkosky rules and coactions. https://doi.org/10.1007/978-3-030-80219-6_12. arXiv:2010.11781 [hep-th]
30. Culler, M., Vogtmann, K.: Moduli of graphs and automorphisms of free groups. Invent. Math. **84**(1), 91–119 (1986)

31. Vanhove, P.: Feynman integrals, Toric geometry and mirror symmetry. In: Blümlein, J., Schneider, C., Paule, P. (eds.) Elliptic Integrals. Elliptic Functions and Modular Forms in Quantum Field Theory. Texts & Monographs in Symbolic Computation, Springer, Berlin (2019)

32. Carlson, B.C.: Special Functions of Applied Mathematics, AP (1977)

33. Brucher, L., Franzkowski, J., Kreimer, D.: Loop integrals, R functions and their analytic continuation. Mod. Phys. Lett. A **9**, 2335–2346 (1994). arXiv:hep-th/9307055 [hep-th]

34. Mastrolia, P., Mizera, S.: Feynman integrals and intersection theory. JHEP **02**, 139 (2019). https://doi.org/10.1007/JHEP02(2019)139, arXiv:1810.03818 [hep-th]

35. Kaufmann, R.M., Khlebnikov, S., Wehefritz-Kaufmann, B.: Singularities, swallowtails and Dirac points. An analysis for families of Hamiltonians and applications to wire networks, especially the Gyroid. Ann. Phys. **327**, 2865–2884 (2012)

36. Cutkosky, R.E.: Singularities and discontinuities of Feynman amplitudes. J. Math. Phys. **1**, 429–433 (1960). https://doi.org/10.1063/1.1703676

37. Berghoff, M.: Feynman amplitudes on moduli spaces of graphs. Ann. Inst. Poincaré **D7**(2), 203 (2020). arXiv:1709.00545

38. Berghoff, M., Kreimer, D.: Graph complexes and Feynman rules. Commun. Number Theor. Phys. **17**, 103–172 (2023). https://doi.org/10.4310/CNTP.2023.v17.n1.a4, arXiv:2008.09540 [hep-th]

39. Brown, F., Kreimer, D.: Angles, scales and parametric renormalization. Lett. Math. Phys. **103**, 933–1007 (2013). https://doi.org/10.1007/s11005-013-0625-6, arXiv:1112.1180 [hep-th]

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.