INEQUALITIES FOR CASORATI CURVATURES OF SUBMANIFOLDS IN REAL SPACE FORMS

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Abstract. By using T. Oprea’s optimization methods on submanifolds, we give another proof of the inequalities relating the normalized $\delta$–Casorati curvature $\hat{\delta}_c(n-1)$ for submanifolds in real space forms.

1. Introduction

In the theory of submanifolds, the following problem is fundamental: to establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds [1]. The basic relationships discovered until now are inequalities. Recently the study of this topic has attracted a lot of attention [4, 12, 13, 14, 16, 19, 22, 23, 25, 27, 34].

On the other hand, it is well-known that the Casorati curvature of a submanifold in a Riemannian manifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form and it was preferred by Casorati over the traditional Gauss curvature because corresponds better with the common intuition of curvature [17, 31]. Later, S. Decu, S. Haesen and L. Verstraelen introduced the normalized $\delta$–Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ and established inequalities involving $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ for submanifolds in real space forms [32]. It should be noted that the normalized $\delta$–Casorati curvatures $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ vanish trivially for $n = 2$. Afterwards, some inequalities involving Casorati curvatures were proved in [20, 33, 38, 39]. Moreover, in [12], C.W. Lee, D.W. Yoon and J.W. Lee established optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections.

In [32], S. Decu, S. Haesen and L. Verstraeelen proved

Theorem 1.1. ([32], Theorem 1) Let $M^n$ be an $n$–dimensional submanifold in an $(n+p)$–dimensional real space form $N^{n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$, then we have

\[ \rho \leq \hat{\delta}_c(n-1) + \tilde{c}, \]

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where $\rho$ is the normalized scalar curvature of $M$. Moreover, the equality case holds if and only if $M^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $N^{n+p}(\tilde{c})$, such that with respect to suitable frames, the shape operators $A_r = A_{e_r}$, $r \in \{n+1, \cdots, n+p\}$, take the following forms:

$$
A_{n+1} = \begin{pmatrix}
2a & 0 & 0 & \cdots & 0 & 0 \\
0 & 2a & 0 & \cdots & 0 & 0 \\
0 & 0 & 2a & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2a & 0 \\
0 & 0 & 0 & \cdots & 0 & a
\end{pmatrix}, \quad A_{n+2} = \cdots = A_{n+p} = 0.
$$

The proof of Theorem 1.1 is based on an optimization procedure by showing that the quadratic polynomial in the components of the second fundamental form is parabolic. In Section 3 of this paper, we present another proof of Theorem 1.1 by using T. Oprea’s optimization method \cite{35,36,37}.

2. Preliminaries

Let $(M^n, g)$ be an $n-$dimensional submanifold in an $(n + p)-$dimensional real space form $(N^{n+p}(\tilde{c}), \overline{g})$ of constant sectional curvature $\tilde{c}$. The Levi-Civita connections on $N$ and $M$ will be denoted by $\overline{\nabla}$ and $\nabla$, respectively. For all $X, Y \in C^\infty(TM)$, $Z \in C^\infty(T^\perp M)$, the Gauss and Weingarten formulas can be expressed by

$$
\overline{\nabla}_XY = \nabla_XY + h(X,Y), \quad \overline{\nabla}_XZ = -A_ZX + \nabla^\perp_XZ,
$$

where $h$ is the second fundamental form of $M$, $\nabla$ is the normal connection and the shape operator $A_Z$ of $M$ is given by

$$
g(A_ZX,Y) = \overline{g}(h(X,Y),Z).
$$

We denote by $\overline{R}$ and $R$ the curvature tensors associated to $\overline{\nabla}$ and $\nabla$, then the Gauss equation is given by

$$
R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)). \tag{2.1}
$$

In $N^{n+p}$ we choose a local orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$, such that, restricting to $M^n$, $e_1, \cdots, e_n$ are tangent to $M^n$. We write $h_{rj}^i = g(h(e_i,e_j),e_r)$. Then the mean curvature vector $H$ is given by

$$
H = \sum_{r=n+1}^{n+p} \left( \frac{1}{n} \sum_{i=1}^{n} h_{ri}^i \right) e_r
$$

and the squared norm of $h$ over dimension $n$ is denoted by $C$ and is called the Casorati curvature of the submanifold $M$. Therefore we have

$$
C = \frac{1}{n} \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{rj}^i)^2.
$$
The submanifold $M^n$ is called to be totally geodesic if $h = 0$ and minimal if $H = 0$. Besides, $M^n$ is called invariantly quasi-umbilical if there exist $p$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_{n+p}$ such that the shape operators with respect to all directions $\xi_r$ have an eigenvalue of multiplicity $n - 1$ and that for each $\xi_r$ the distinguished eigendirection is the same [39].

Let $K(e_i \wedge e_j), \ 1 \leq i < j \leq n$, denote the sectional curvature of the plane section spanned by $e_i$ and $e_j$. Then the scalar curvature of $M^n$ is given by

$$\tau = \sum_{i<j} K(e_i \wedge e_j),$$

and the normalized scalar curvature $\rho$ is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

Suppose $L$ is an $l$-dimensional subspace of $T_xM$, $x \in M$, $\ l \geq 2$ and $\{e_1, \ldots, e_l\}$ an orthonormal basis of $L$. Then the scalar curvature $\tau(L)$ of the $l$-plane $L$ is given by

$$\tau(L) = \sum_{1 \leq \mu < \nu \leq l} K(e_\mu \wedge e_\nu),$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace $L$ is defined as

$$\mathcal{C}(L) = \frac{1}{n} \sum_{r=n+1}^{n+p} \sum_{i,j=1}^{l} (h_{ij}^{-r})^2.$$

Following [32, 38], we can define the normalized $\delta$-Casorati curvature $\delta_c(n - 1)$ and $\hat{\delta}_c(n - 1)$ by

$$[\delta_c(n - 1)]_x = \frac{1}{2} \mathcal{C}_x + \frac{n+1}{2n(n-1)} \inf \{ \mathcal{C}(L) \mid L \ a \ hyperplane \ of \ T_xM \},$$

and

$$[\hat{\delta}_c(n - 1)]_x = 2 \mathcal{C}_x - \frac{2n-1}{2n} \sup \{ \mathcal{C}(L) \mid L \ a \ hyperplane \ of \ T_xM \}.$$

For later use, we provide a brief review of T. Oprea’s optimization methods on submanifolds from [35].

Let $(N_2, \bar{g})$ be a Riemannian manifold, $N_1$ be a Riemannian submanifold of it, $g$ be the metric induced on $N_1$ by $\bar{g}$ and $f : N_2 \rightarrow \mathbb{R}$ be a differentiable function.

Following [35] we considered the constrained extremum problem

$$\min_{x \in N_1} f(x),$$

then we have

**Lemma 2.1.** ([35]) If $x_0 \in N_1$ is the solution of the problem (2.2), then

i) $(\nabla f)(x_0) \in T_{x_0}N_1$;

ii) the bilinear form

$$A : T_{x_0}N_1 \times T_{x_0}N_1 \rightarrow \mathbb{R},$$
\[ A(X, Y) = Hess_f(X, Y) + \nabla(h(X, Y), (\nabla f)(x_0)) , \]

is positive semidefinite, where \( h \) is the second fundamental form of \( N_1 \) in \( N_2 \) and \( \nabla f \) is the gradient of function \( f \).

In [36], the above lemma was successfully applied to improve an inequality relating \( \delta(2) \) obtained in [2]. Later, B.-Y. Chen extended the improved inequality to the general inequalities involving \( \delta(n_1, \cdots, n_k) \) [5].

3. Another proof of Theorem 1.1

From (2.1) we have
\[
R_{ijj} = \tilde{c} + \sum_{r=n+1}^{n+p} [h_{ir}^r h_{jj}^r - (h_{ij}^r)^2] ,
\]
which implies
\[
2\tau = n^2 \| H \|^2 - nC + n(n - 1)\tilde{c}.
\]
Consider the following function \( \mathcal{P} \) which is a quadratic polynomial in the components of the second fundamental form:
\[
\mathcal{P} = 2n(n - 1)C + \frac{(n - 1)(1 - 2n)}{2}C(L) - 2\tau + n(n - 1)\tilde{c}.
\]
Assuming, without loss of generality, that \( L \) is spanned by \( e_1, e_2, \cdots, e_{n-1}, e_n \), combining (3.1) it follows that
\[
\mathcal{P} = \sum_{r=n+1}^{n+p} \left\{ \frac{2n - 3}{2} \sum_{i=1}^{n-1} (h_{ii}^r)^2 + 2(n - 1)(h_{nn}^r)^2 + (2n - 1) \sum_{1 \leq i < j \leq n-1} (h_{ij}^r)^2 \right\}
+ 2(2n - 1) \sum_{i=1}^{n-1} (h_{in}^r)^2 - 2 \sum_{1 \leq i < j \leq n} h_{ii}^r h_{jj}^r \}
\geq \sum_{r=n+1}^{n+p} \left\{ \frac{2n - 3}{2} \sum_{i=1}^{n-1} (h_{ii}^r)^2 + 2(n - 1)(h_{nn}^r)^2 - 2 \sum_{1 \leq i < j \leq n} h_{ii}^r h_{jj}^r \}
\}
\]

For \( r = n + 1, \cdots, n + p \), let us consider the quadratic form
\[
f_r : \mathbb{R}^n \rightarrow \mathbb{R} ,
\]
\[
f_r(h_1^r, \cdots, h_{nn}^r) = \frac{2n - 3}{2} \sum_{i=1}^{n-1} (h_{ii}^r)^2 + 2(n - 1)(h_{nn}^r)^2 - 2 \sum_{1 \leq i < j \leq n} h_{ii}^r h_{jj}^r
\]
and the constrained extremum problem
\[
\min f_r ,
\]
subject to \( F : h_{11}^r + \cdots + h_{nn}^r = k^r \),
where \( k^r \) is a real constant.
The partial derivatives of the function \( f_r \) are

\[
\frac{\partial f_r}{\partial h^r_{11}} = (2n - 3)h^r_{11} - 2 \sum_{i=2}^{n} h^r_{ii}
\]

(3.3)

\[
\frac{\partial f_r}{\partial h^r_{22}} = (2n - 3)h^r_{22} - 2h^r_{11} - 2 \sum_{i=3}^{n} h^r_{ii}
\]

(3.4)

\[
\ldots \ldots
\]

\[
\frac{\partial f_r}{\partial h^r_{n-1,n-1}} = (2n - 3)h^r_{n-1,n-1} - 2h^r_{nn} - 2 \sum_{i=1}^{n-2} h^r_{ii}
\]

(3.5)

\[
\frac{\partial f_r}{\partial h^r_{nn}} = 4(n - 1)h^r_{nn} - 2 \sum_{i=1}^{n-1} h^r_{ii}
\]

(3.6)

For an optimal solution \((h^r_{11}, h^r_{22}, \ldots, h^r_{nn})\) of the problem in question, the vector \( \text{grad} f_r \) is normal at \( \mathcal{F} \), that is, it is colinear with the vector \((1, 1, \ldots, 1)\). From (3.3), (3.4), (3.5) and (3.6), it follows that a critical point of the considered problem has the form

\[
(h^r_{11}, h^r_{22}, \ldots, h^r_{n-1,n-1}, h^r_{nn}) = (2t^r, 2t^r, \ldots, 2t^r, t^r).
\]

(3.7)

As \( \sum_{i=1}^{n} h^r_{ii} = k^r \), by using (3.7), we have

\[
h^r_{11} = h^r_{22} = \ldots = h^r_{n-1,n-1} = \frac{2}{2n - 1} k^r, \quad h^r_{nn} = \frac{1}{2n - 1} k^r.
\]

(3.8)

We fix an arbitrary point \( x \in \mathcal{F} \). The 2-form \( \mathcal{A} : T_x\mathcal{F} \times T_x\mathcal{F} \rightarrow \mathbb{R} \) has the expression

\[
\mathcal{A}(X, Y) = \text{Hess}_{f_r}(X, Y) + \langle h'(X, Y), (\text{grad} f_r)(x) \rangle,
\]

where \( h' \) is the second fundamental form of \( \mathcal{F} \) in \( \mathbb{R}^n \) and \( \langle \cdot, \cdot \rangle \) is the standard inner-product on \( \mathbb{R}^n \). In the standard frame of \( \mathbb{R}^n \), the Hessian of \( f_r \) has the matrix

\[
\begin{pmatrix}
2n - 3 & -2 & -2 & \cdots & -2 & -2 \\
-2 & 2n - 3 & -2 & \cdots & -2 & -2 \\
-2 & -2 & 2n - 3 & \cdots & -2 & -2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-2 & -2 & -2 & \cdots & 2n - 3 & -2 \\
-2 & -2 & -2 & \cdots & -2 & 4(n - 1)
\end{pmatrix}
\]
As $F$ is totally geodesic in $\mathbb{R}^n$, considering a vector $X$ is tangent to $F$ at the arbitrary point $x$ on $F$, that is, verifying the relation $\sum_{i=1}^{n} X_i = 0$, we have

$$A(X, X) = (2n - 1) \sum_{i=1}^{n-1} X_i^2 + (4n - 2)X_n^2 - 2(X_1 + X_2 + \cdots + X_n)^2$$

$$= (2n - 1) \sum_{i=1}^{n-1} X_i^2 + (4n - 2)X_n^2$$

$$\geq 0.$$ 

Consequently the point $(h_{11}^r, h_{22}^r, \cdots, h_{nn}^r)$ given by (3.8) is a global minimum point, here we used Lemma 2.1. Inserting (3.8) into $f_r$ we have $f_r = 0$. Hence we have

(3.9) $P \geq 0$.

From (3.2) and (3.9) we can derive inequality (1.1). The equality case of (1.1) holds if and only if we have the equality in all the previous inequalities. Thus the shaper operators take the desired forms. Besides, $h_{ij}^r = 0, \ i \neq j, \ \forall i, j, r$ means that the normal connection $\nabla^\perp$ is flat, or still, that the normal curvature tensor $R^\perp$, i.e., the curvature tensor of the normal connection is trivial.

4. Casorati ideal submanifolds in real space forms

The notion of ideal immersions was introduced by B.-Y. Chen in the 1990s. Roughly speaking, an ideal immersion of a Riemannian manifold into a real space form is a nice isometric immersion which produces the least possible amount of tension from the ambient space at each point. B.-Y. Chen established many inequalities in terms of $\delta-$invariants and claimed that the submanifold satisfying the equality case is called ideal submanifold. Such submanifolds are also called Chen’s submanifolds. The ideal submanifolds in real space forms and complex space forms have been characterized by B.-Y. Chen [2, 6, 7, 8, 9]. Besides, Einstein, conformally flat, semisymmetric, and Ricci-semisymmetric submanifolds satisfying Chen’s inequality in real space forms were studied by Dillen, Petrovic, Verstraelen, Özgür and Tripathi [11, 15]. More details about ideal submanifolds, we refer to see [10, 21, 26].

Submanifolds for which the equality case of inequalities for the Casorati curvatures, will be called Casorati ideal submanifolds [32, 33]. In [32], the authors proved that

**Theorem 4.1.** ([32], Corollary 4) The Casorati ideal submanifold with $n \geq 4$ for (1.1) is conformally flat submanifold with trivial normal connection.

**Theorem 4.2.** ([32], Corollary 5) The Casorati ideal submanifold for (1.1) is pseudo-symmetric manifold.

In this paper, we have
Theorem 4.3. The Casorati ideal submanifold for (1.1) is Einstein if and only if it is totally geodesic submanifold.

Proof. From the equation of Gauss, we have

\[ \text{Ric}(e_i) = (n-1)c + 2(2n-3)a^2, \quad i = 1, 2, \cdots, n-1, \]

\[ \text{Ric}(e_n) = (n-1)c + 2(n-1)a^2. \]

As \( M^n \) is Einstein, \( \text{Ric}(e_i) = \text{Ric}(e_n), \quad i = 1, 2, \cdots, n-1. \) Thus \( a = 0 \), which implies \( h = 0. \)

\[ \square \]

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