Effective potential for the massless KPZ equation

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Abstract: In previous work we have developed a general method for casting a classical field theory subject to Gaussian noise (that is, a stochastic partial differential equation—SPDE) into a functional integral formalism that exhibits many of the properties more commonly associated with quantum field theories (QFTs). In particular, we demonstrated how to derive the one-loop effective potential. In this paper we apply the formalism to a specific field theory of considerable interest, the massless KPZ equation (massless noisy vorticity-free Burgers equation), and analyze its behaviour in the ultraviolet (short-distance) regime. When this field theory is subject to white noise we can calculate the one-loop effective potential and show that it is one-loop ultraviolet renormalizable in 1, 2, and 3 space dimensions, and fails to be ultraviolet renormalizable in higher dimensions. We show that the one-loop effective potential for the massless KPZ equation is closely related to that for $\lambda\phi^4$ QFT. In particular we prove that the massless KPZ equation exhibits one-loop dynamical symmetry breaking (via an analog of the Coleman–Weinberg mechanism) in 1 and 2 space dimensions, and that this behaviour does not persist in 3 space dimensions.

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I. INTRODUCTION

In a companion paper \cite{1}, we discussed classical field theories subject to stochastic noise $\eta(\vec{x}, t)$, described by the equation

$$D\phi(\vec{x}, t) = F[\phi(\vec{x}, t)] + \eta(\vec{x}, t).$$

(1)

Here $D$ is any linear differential operator, involving arbitrary time and space derivatives, that does not explicitly involve the field $\phi$. The function $F[\phi]$ is any forcing term, generally nonlinear in the field $\phi$. These stochastic partial differential equations (SPDEs) can be studied using a functional integral formalism which makes manifest the deep connections with quantum field theories (QFTs). Provided the noise is translation-invariant and Gaussian, it is possible to split its two-point function into an amplitude $A$ and a shape function $g_2(x, y)$, as follows

$$G_\eta(x, y) \overset{\text{def}}{=} A g_2(x - y),$$

(2)

with the convention that

$$\int d^d\vec{x} \, dt \, g_2^{-1}(\vec{x}, t) = 1 = \tilde{g}_2^{-1} (\vec{k} = \vec{0}, \omega = 0).$$

(3)

We showed that the one-loop effective potential for homogeneous and static fields is

$$\mathcal{V}[\phi; 0] \overset{\text{def}}{=} \frac{i}{2} F^2[\phi] + \frac{i}{2} A \int \frac{d^d\vec{k} \, d\omega}{(2\pi)^{d+1}} \ln \left[ 1 + \frac{\tilde{g}_2 (\vec{k}, \omega) F[\phi]}{\frac{\delta^2 F}{\delta \phi^2}} \left( D^1(\vec{k}, \omega) - \frac{\delta F}{\delta \phi} \right) \left( D^1(\vec{k}, \omega) - \frac{\delta F}{\delta \phi} \right) \right] \left( \phi \rightarrow \phi_0 \right) + O(A^2).$$

(4)

Here $\phi_0$ is any convenient background field. In the absence of symmetry breaking it is most convenient to pick $\phi_0 = 0$, but we reserve the right to make other choices when appropriate. Equation (4) is qualitatively similar to the one-loop effective potential for a self-interacting scalar QFT \cite{1, 2}:

$$\mathcal{V}[\phi; 0] = V(\phi) + \frac{i}{2} \hbar \int \frac{d^d\vec{k} \, d\omega}{(2\pi)^{d+1}} \ln \left[ 1 + \frac{\delta^2 V}{\delta \phi^2} \frac{\delta^2 V}{\delta \phi^2} \right] \left( \phi \rightarrow \phi_0 \right) + O(\hbar^2).$$

(5)

The effective potential is not only a formal mathematical tool, but it also has a deep physical meaning. In fact, in previous work \cite{1, 2} we demonstrate that the effective potential for SPDEs inherits many of the physical features and
information content of the effective potential for QFTs, and that searching for minima of the SPDE effective potential provides information about “ground states” of such SPDEs. These ground states play an important role in the study of symmetry breaking and the onset of pattern formation and structure. In this paper we apply this formalism to the specific case of the massless KPZ equation (equivalent to the massless noisy vorticity-free Burgers equation) [4–8]

\[
\left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) \phi = F_0 + \frac{\lambda}{2} (\nabla \phi)^2 + \eta, \tag{6}
\]

and use the formalism to investigate the ultraviolet (short-distance) properties of the system.

In the fluid dynamical interpretation (i.e., the vorticity-free Burgers equation) the fluid velocity is taken to be \( \nu = -\nabla \phi \). In this representation the KPZ equation is used as a model for turbulence [3], structure development in the early universe [3], driven diffusion, and flame fronts [3]. In the surface growth interpretation, \( \phi(x,t) \) is taken to be the height of the surface (typically defined over a two-dimensional plane) [3]. In this interpretation the massless KPZ equation is a natural nonlinear extension of the Edwards–Wilkinson (EW) model [10].

An explicit tadpole term \( (F_0) \) is included as we will soon see that it is a necessary ingredient in completing the renormalization program. (In QFTs a dynamically generated field-independent constant term in the equations of motion is often called a tadpole, though if such a term arises from the tree-level physics it is called an external current. In this KPZ context we prefer to use the word “tadpole” since \( F_0 \) will have to be renormalized, which makes the nomenclature “external current” inappropriate.) After renormalization, we will fix the value of the renormalized value of \( F_0 \) by using the symmetries of the KPZ equation.

Some additional comments regarding the raison-d’être of the tadpole may prove helpful. It is well known that both the Edwards–Wilkinson and KPZ equations may be written with or without such a constant (the tadpole) and that a finite constant term simply represents a change in the average velocity of the surface with respect to the laboratory frame of reference, and so may be chosen at will. On the other hand, as we will see below, it is crucial to include a bare tadpole in the KPZ equation in order to be able to consistently carry out the ultraviolet renormalization (at one-loop). Nevertheless, the renormalized tadpole may be chosen at will, and can be set to any convenient value after renormalization. (The convenient value we will finally adopt will be based on symmetry arguments and will be chosen to eliminate any spurious motion of the background field.) The need for a “bare” tadpole is not just an artifact of the effective potential renormalization but (as we will demonstrate elsewhere) is also required to carry out the one-loop ultraviolet renormalization of the full effective action.

There are two important symmetries of the KPZ equation that are relevant for our analysis. The first is the symmetry under the transformation

\[
\phi \rightarrow \phi + c(t), \tag{7}
\]

\[
F_0 \rightarrow F_0 + \frac{dc(t)}{dt}. \tag{8}
\]

In the fluid dynamics interpretation this symmetry amounts to a “gauge transformation” of the scalar field \( \phi \) that does not change the physical velocity (\( \nu = -\nabla \phi \)). In the surface growth interpretation this symmetry corresponds to choosing a different coordinate system that moves vertically at a speed \( dc/dt \) with respect to the initial coordinate system, and so can be thought of as a type of Galilean invariance (type I) for the KPZ equation. This transformation is a symmetry of the KPZ equation for arbitrary noise.

The second symmetry we consider holds under more restrictive conditions. Consider the transformation

\[
\vec{x} \rightarrow \vec{x}' = \vec{x} - \lambda \vec{\epsilon} t, \tag{9}
\]

\[
t \rightarrow t' = t, \tag{10}
\]

\[
\phi(\vec{x}, t) \rightarrow \phi'(\vec{x}', t') = \phi(\vec{x}, t) - \vec{\epsilon} \cdot \vec{x}. \tag{11}
\]

In the fluid dynamics interpretation this symmetry is equivalent to a Galilean transformation of the fluid velocities

\[
\nu \rightarrow \nu' = \nu - \epsilon, \tag{12}
\]

and so can also be thought of as a type of Galilean invariance (type II). In the surface growth interpretation this symmetry amounts to choosing a different coordinate system that is tilted at an angle to the vertical, with

\[
\tan(\theta) = ||\vec{\epsilon}||, \tag{13}
\]

and for this reason this transformation is often referred to as tilt invariance. While this type II Galilean invariance is an exact invariance of the zero-noise KPZ equation, it is important to keep in mind that once noise is added to the
system, this transformation will remain a symmetry only if the noise is translation-invariant and \emph{temporally white}. This can be seen by first looking at the noise two point function

\[
G_0(\vec{x}, t; \vec{t}, t) = \mathcal{A} g_2(\vec{x} - \vec{t}, t - t),
\]

then considering

\[
\vec{x} - \vec{t} \rightarrow \vec{x}' - \vec{t}' = \vec{x} - \vec{t} - \lambda \epsilon (t_1 - t_2),
\]
\[
t_1 - t_2 \rightarrow t'_1 - t'_2 = t_1 - t_2,
\]

and finally noting that the noise two-point function is invariant if and only if its support is limited by the constraint \(t_1 = t_2\), that is,

\[
G_\eta(\vec{x}, t; \vec{t}, t) = \mathcal{A} g_2(\vec{x} - \vec{t}) \delta(t_1 - t_2).
\]

But this is the \emph{definition} of translation-invariant temporally white noise.

We will use these two symmetries extensively in the body of the paper: the type I Galilean invariance is used to guarantee that the background field \(\phi_0\) is kept stationary, even in the presence of interactions and nonlinearities, and the type II Galilean invariance is similarly used to guarantee that the average slope of the background field is zero. It is best (in fact essential) to do this only after the renormalization program has been completed, so for the time being we will explicitly keep track of both the tadpole term \(F_0\) and the background field \(\phi_0\).

Applying the formalism developed in [1] to the massless KPZ equation, we demonstrate that the effective potential is one-loop ultraviolet renormalizable in 1, 2, and 3 space dimensions (and is not ultraviolet renormalizable in 4 or higher space dimensions). We will discover a formal relationship between the one-loop effective potential for the massless KPZ equation in \(d+1\) dimensions (\(d\) space and 1 time dimensions) and of the massless \(\lambda \phi^4\) QFT in \(d+2\) Euclidean dimensions, and show that the massless KPZ equation exhibits many of the properties seen in massless \(\lambda \phi^4\) QFT. In particular, we will see that the stochastic system undergoes dynamical symmetry breaking (DSB) in 1 and 2 space dimensions, and no symmetry breaking in 3 dimensions. This DSB is due to an analog of the Coleman–Weinberg mechanism of QFT. In 2 space dimensions the presence of a short-distance (ultraviolet) logarithmic divergence implies the running of the coupling constant \(\lambda\) with the energy scale, and the existence of a non-zero beta function for this coupling. We feel that the unexpected presence of DSB in 1 and 2 space dimensions is a matter of deep importance, and is something that would be very difficult to deduce by any other means.

**II. EFFECTIVE POTENTIAL: MASSLESS KPZ**

The massless KPZ equation (massless noisy vorticity-free Burgers equation) is [4][5]

\[
\left( \frac{\partial}{\partial t} - \nu \nabla^2 \right) \phi = F_0 + \lambda \phi^2 + \eta. \tag{18}
\]

We have introduced an explicit “bare” tadpole term \(F_0\) as remarked above. If we now restrict attention to homogeneous and static fields, \(\phi(\vec{x}, t) \rightarrow \phi_{\text{homogeneous--static}}\), and \(\phi_0(\vec{x}, t) \rightarrow (\phi_0)_{\text{homogeneous--static}}\), then from equation (18) it is easy to see that \(\mathcal{V}[\phi; \phi_0] \equiv 0\), and so the one-loop effective potential is uninteresting. In order to see this, note that for a homogeneous static field configuration

\[
F[\phi] \rightarrow F_0; \quad \frac{\delta F}{\delta \phi(x)} \rightarrow 0; \quad \frac{\delta^2 F}{\delta \phi(x) \delta \phi(y)} \rightarrow +\lambda \nabla_x \cdot \nabla_y \delta^d(\vec{x}, \vec{y}) \rightarrow -\lambda \nabla^2 \rightarrow +\lambda \vec{k}^2. \tag{19}
\]

Thus the integrand appearing in (18) is independent of the field \(\phi\), and \(\mathcal{V}[\phi; \phi_0]\) is zero as asserted. (This simplification does not hold for the massive noisy vorticity-free Burgers equation [massive KPZ equation] where the driving force is replaced by \(F[\phi] \rightarrow F[\phi] - m^2 \phi\). We have calculated \(\mathcal{V}[\phi; \phi_0]\) explicitly for this system and shown it is non-zero. We do not report the details here, as this is a simple exercise and the result is of limited physical interest.)

What \emph{is} physically interesting on the other hand, is to study the \emph{massless} KPZ equation and to consider a linear and static field configuration

\[
\phi = -\vec{v} \cdot \vec{x}, \tag{20}
\]

where \(\vec{v}\) is now a constant vector. Notice that for this choice one has \(D \phi = 0\). In the hydrodynamic interpretation of the massless KPZ equation this corresponds to a constant velocity flow: \(\vec{v} = -\nabla \phi\). In the surface growth interpretation,
\[ |\vec{v}| \] corresponds to a constant slope of the surface \( \vec{V} \). There is an instructive analogy with QED here: taking a constant vector potential in QED is relatively uninteresting, it corresponds to zero electromagnetic field strength and can be gauged away (modulo topological constraints). On the other hand, a constant electromagnetic field strength can be described by a linear vector potential, and leads to such useful quantities as the Euler–Heisenberg effective potential and the Schwinger effective Lagrangian for QED \( \mathcal{L}_S \).

Inspection of the derivation contained in reference \( [1] \) reveals that although the effective potential \( [4] \) was originally defined for homogeneous and static fields, for the massless KPZ equation \( [15] \), it continues to make sense for linear and static fields. The fact that for the massless KPZ equation \( F[\phi] \) is position independent for these linear static fields is essential to this observation. For such a field configuration, \( \phi = -\vec{v} \cdot \vec{x} \), we get

\[
F[\phi] \rightarrow F_0 + \frac{\lambda}{2} \nu^2; \quad \delta F \rightarrow -\lambda \vec{v} \cdot \vec{\nabla} x; \quad \delta^2 F \rightarrow +\lambda \vec{\nabla} x \cdot \vec{\nabla} y \rightarrow -\lambda \vec{\nabla}^2 x \rightarrow +\lambda \vec{k}^2. \tag{21}
\]

The zero-loop effective potential is

\[
\mathcal{V}_{\text{zero-loop}} [\nu; \nu_0] = \frac{1}{2} \left[ (F_0 + \frac{1}{2} \lambda \nu^2)^2 - (F_0 + \frac{1}{2} \lambda \nu_0^2)^2 \right], \tag{22}
\]

where the background field is \( \phi_0 = -\vec{v}_0 \cdot \vec{x} \). Note that this zero-loop effective potential is formally equivalent to that of \( \lambda \phi^4 \) QFT—with the velocity \( \nu \) playing the role of the quantum field \( \phi_{\text{QFT}} \). Even at zero loops (tree level) we see that if we were to have \( F_0 < 0 \) the effective potential would take on the “Mexican hat” form, so that the onset of spontaneous symmetry breaking (SSB) would not be at all unexpected \( [4, 3] \). However, as previously mentioned, the renormalized value of \( F_0 \) is not physically relevant and can always be changed by a type I Galilean transformation. We will soon see that SSB is not a feature of the KPZ equation. Instead we encounter a much more subtle effect: the onset of dynamical symmetry breaking (DSB) which we have detected via a one-loop computation.

We start the one-loop calculation by noting that

\[
D - \frac{\delta F}{\delta \phi} = \partial_t + \lambda \vec{v} \cdot \vec{\nabla} - \nu \vec{\nabla}^2 \quad \rightarrow \quad -i\omega + i\lambda \vec{v} \cdot \vec{k} + \nu \vec{k}^2, \tag{23}
\]

while for the adjoint quantity

\[
D^\dagger - \frac{\delta F^\dagger}{\delta \phi} = -\partial_t - \lambda \vec{v} \cdot \vec{\nabla} - \nu \vec{\nabla}^2 \quad \rightarrow \quad +i\omega - i\lambda \vec{v} \cdot \vec{k} + \nu \vec{k}^2, \tag{24}
\]

so that

\[
\left( D^\dagger - \frac{\delta F^\dagger}{\delta \phi} \right) \left( D - \frac{\delta F}{\delta \phi} \right) = -(\partial_t + \lambda \vec{v} \cdot \vec{\nabla})^2 + \nu^2 (\vec{\nabla}^2)^2 \quad \rightarrow \quad (\omega - \lambda \vec{v} \cdot \vec{k})^2 + \nu^2 (\vec{k}^2)^2. \tag{25}
\]

Using this we specialize equation \( [10] \) to

\[
\mathcal{V}[v; v_0] = \frac{1}{2} (F_0 + \frac{1}{2} \lambda \nu^2)^2 + \frac{\lambda}{4} \int \frac{d^dk}{(2\pi)^d+1} \ln \left[ 1 + \frac{\tilde{g}_2(\vec{k}, \omega) \lambda (F_0 + \frac{1}{2} \lambda \nu^2) \vec{k}^2}{(\omega - \lambda \vec{v} \cdot \vec{k})^2 + \nu^2 (\vec{k}^2)^2} \right] - (\vec{v} \rightarrow \vec{v}_0) + O(A^2). \tag{26}
\]

Equivalently

\[
\mathcal{V}[v; v_0] = \frac{1}{2} (F_0 + \frac{1}{2} \lambda \nu^2)^2 + \frac{\lambda}{4} \int \frac{d^dk}{(2\pi)^d+1} \ln \left[ \frac{(\omega - \lambda \vec{v} \cdot \vec{k})^2 + \nu^2 (\vec{k}^2)^2 + \tilde{g}_2(\vec{k}, \omega) \lambda (F_0 + \frac{1}{2} \lambda \nu^2) \vec{k}^2}{(\omega - \lambda \vec{v} \cdot \vec{k})^2 + \nu^2 (\vec{k}^2)^2} \right] \] \[ - (\vec{v} \rightarrow \vec{v}_0) + O(A^2). \tag{27}
\]

This is as far as we can go \textit{without making further assumptions about the noise}. For instance, one very popular choice is \textit{temporally white}, which means delta function correlated in time so that \( \tilde{g}_2(\vec{k}, \omega) \rightarrow \tilde{g}_2(\vec{k}) \) is a function of \( \vec{k} \) only. We can shift the integration variable \( \omega \) to \( \omega - \lambda \vec{v} \cdot \vec{k} \). (The frequency integral is convergent.) The resulting integral becomes

\[
\mathcal{V}[v; v_0] = \frac{1}{2} (F_0 + \frac{1}{2} \lambda \nu^2)^2 + \frac{\lambda}{4} \int \frac{d^dk}{(2\pi)^d+1} \ln \left[ \frac{\omega^2 + \nu^2 (\vec{k}^2)^2 + \tilde{g}_2(\vec{k}) \lambda (F_0 + \frac{1}{2} \lambda \nu^2) \vec{k}^2}{\omega^2 + \nu^2 (\vec{k}^2)^2} \right] \] \[ - (\vec{v} \rightarrow \vec{v}_0) + O(A^2). \tag{28}
\]
This may be simplified by extracting an explicit factor of $\nu^2 \tilde{k}^2$. If we do so, the one-loop effective potential becomes
\begin{align}
\mathcal{V}[v; v_0] &= \frac{1}{2} \left[ (F_0 + \frac{4}{d} \lambda v^2)^2 - (F_0 + \frac{4}{d} \lambda v_0^2)^2 \right] \\
&\quad + \frac{4}{d} A \nu \int \frac{d^d \tilde{k}}{(2\pi)^d} \left\{ \sqrt{\nu^2(\tilde{k}^2)^2 + \tilde{g}_2(\tilde{k}) \lambda(F_0 + \frac{4}{d} \lambda v^2)\tilde{k}^2} - \sqrt{\nu^2(\tilde{k}^2)^2 + \tilde{g}_2(\tilde{k}) \lambda(F_0 + \frac{4}{d} \lambda v_0^2)\tilde{k}^2} \right\} \\
&\quad + O(A^2).
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&\quad + \frac{4}{d} A \nu \int \frac{d^d \tilde{k}}{(2\pi)^d} \left\{ \sqrt{\tilde{k}^2 + \tilde{g}_2(\tilde{k}) \frac{\lambda}{\nu^2}(F_0 + \frac{4}{d} \lambda v^2) - \sqrt{\tilde{k}^2 + \tilde{g}_2(\tilde{k}) \frac{\lambda}{\nu^2}(F_0 + \frac{4}{d} \lambda v_0^2)} \right\} \\
&\quad + O(A^2).
\end{align}

Remembering that $\lim_{\tilde{k} \to 0} \tilde{g}_2(\tilde{k}) = 1$, it is clear that there are no infrared divergences ($\tilde{k} \to 0$) to worry about, at least for this effective potential at one loop order. Furthermore, modulo possibly perverse choices for the spatial noise spectrum, the tadpole term $F_0$ is essential in renormalizing the theory. Note that the bare potential contains terms proportional to $v^d$, $v^3$, and $v^4$, whereas the one-loop contribution, when one expands it in powers of $v^2$ has terms proportional to $v^{2n}$ for $n = 1, 2, 3\ldots$. For definiteness, let us now take the spatial noise spectrum to be cutoff white (the temporal spectrum has already been chosen to be exactly white), i.e.,
\begin{align}
\tilde{g}_2(\tilde{k}) = \tilde{g}_2(|\tilde{k}|) = \Theta(\Lambda - k).
\end{align}

The effective potential is then
\begin{align}
\mathcal{V}[v; v_0] &= \frac{1}{2} \left[ (F_0 + \frac{4}{d} \lambda v^2)^2 - (F_0 + \frac{4}{d} \lambda v_0^2)^2 \right] \\
&\quad + \frac{4}{d} A \nu \int_{k < \Lambda} \frac{d^d \tilde{k}}{(2\pi)^d} \left\{ \sqrt{\tilde{k}^2 + \frac{\lambda}{\nu^2}(F_0 + \frac{4}{d} \lambda v^2) - \sqrt{\tilde{k}^2 + \frac{\lambda}{\nu^2}(F_0 + \frac{4}{d} \lambda v_0^2)} \right\} \\
&\quad + O(A^2).
\end{align}

With this choice of noise, the $v^2$ term is ultraviolet divergent and proportional to $\Lambda^d$, the $v^3$ term is proportional to $\Lambda^{d-2}$, and the $v^4$ term to $\Lambda^{d-4}$. To have any hope of absorbing the infinities into the bare action, thereby permitting us to take the $\Lambda \to \infty$ limit, we must have $d < 4$ (because there is no $v^5$ term in the bare potential). That is: the massless KPZ equation (subject to white noise) is one-loop ultraviolet renormalizable only in 1, 2, and 3 space dimensions. (In 0 space dimensions the KPZ equation is trivial.) And even so, one-loop renormalizability requires an explicit tadpole term. (Without a tadpole term there is no term proportional to $v^2$ in the zero-loop potential, and therefore there is no possibility of renormalizing the leading divergence.) Strictly speaking the claim of one-loop renormalizability also requires investigation of the wave-function renormalization. This appears in the one-loop effective action, which is beyond the scope of the present paper (here we confine attention to the one-loop effective potential), and will be discussed in future work.

It is also clear from the above that the ultraviolet renormalizability of the KPZ equation depends critically on the high momentum behaviour of the noise. Let us temporarily return to equation (22), and suppose that the noise is power-law distributed in the ultraviolet with $\tilde{g}_2(\tilde{k}) = \tilde{g}_2(|\tilde{k}|) \approx (k_0/k)^\theta \Theta(\Lambda - k)$. Then the $n$th term in the expansion has ultraviolet behaviour proportional to $(F_0 + \frac{4}{d} \lambda v^2)^n \Lambda^{d+2-2n-\theta}$. The massless KPZ equation is then one-loop ultraviolet renormalizable provided the $n = 3$ term (and higher terms) are ultraviolet finite, that is, for $d < 4 + 3\theta$. We will not deal with these issues any further in this paper but will instead confine ourselves to white noise. (Recall that it is only at intermediate stages that the white noise is regulated by an ultraviolet spatial cutoff. After renormalization, we will be considering noise that is exactly white.)
We can also point out an unexpected result of this calculation: There is a formal connection between the massless KPZ equation and massless $\lambda \phi^4$ QFT. Consider equation (33). This is recognizable (either from QFT or equilibrium statistical field theory) as the effective potential for $\lambda \phi^4$ Lorentzian QFT in $d+1$ space dimensions, or equivalently $\lambda \phi^4$ statistical field theory in $(d+1)$ Euclidean spacetime dimensions. To make the connection, interpret $\lambda F_0 / v^2$ as the mass term $m^2$ of the QFT, $\chi^2 / v^2$ as $\lambda Q_{\text{FT}}$ (the coupling constant of the QFT), and $v$ as the mean field $\phi_{\text{FT}}$.

Note that we have at this stage (temporarily) kept the parameters $(F_0)_{\text{renormalized}}$ and $v_0$ non-zero for clarity. We now invoke the type I and type II Galilean symmetries of the KPZ equation to fix these parameters:

—(1) In the fluid dynamical interpretation, the tadpole term does not have any physical significance. (Because of the symmetry $F_0 \to F_0 - \kappa$, $\phi \to \phi - \kappa t$, the tadpole does not affect any of the physical fluid velocities.) We are interested in a static background field configuration $\vec{v}_0$, but from the equation of motion we see that for the spatially averaged field

$$\frac{1}{\Omega} \left\langle \int \frac{\partial \phi}{\partial t} d^d \vec{x} \right\rangle = F_0 + \frac{2}{3} \lambda v_0^2 + O(A),$$

so we must set the renormalized value of $F_0$ to $-\frac{2}{3} \lambda v_0^2$. (This works for arbitrary Gaussian noise.) Secondly, the type II Galilean invariance of the fluid dynamical system allows us to set $v_0 = 0$ without loss of generality: pick a coordinate system moving with the bulk fluid velocity of the background field. (This requires translation-invariant and temporally white noise.) Combining the two symmetries yields $(F_0)_{\text{renormalized}} = 0$ and $v_0 = 0$.

—(2) In the surface growth interpretation $(F_0)_{\text{renormalized}}$ is a contribution to the spatially-averaged velocity of the interface. This average velocity term can always be scaled away by the shift $\phi \to \phi - (F_0)_{\text{renormalized}} t$, which in the surface growth interpretation results from a type I Galilean transformation that places one in an inertial frame moving with the average surface profile. Doing so again sets the renormalized value of $(F_0)_{\text{renormalized}}$ to $-\frac{2}{3} \lambda v_0^2$. Secondly, the type II Galilean invariance now corresponds to tilting the coordinate system away from the vertical. For any constant slope background field a simple tilt can then be used to set the slope to zero. Combining the two symmetries yields $(F_0)_{\text{renormalized}} = 0$ and $v_0 = 0$, so the surface growth interpretation and the fluid dynamics interpretation are in agreement as to what the physically relevant choice of parameters is.

The perhaps somewhat unusual way in which the tadpole $F_0$ is first introduced and then renormalized to zero has a direct analog in ordinary $\lambda \phi^4$ QFT: when we consider massless $\lambda \phi^4$ QFT it is well known that a bare mass must be introduced to successfully complete the renormalization program. It is only after renormalization that the renormalized mass may be set to zero, and massless $\lambda \phi^4$ QFT makes sense only in this post-renormalization fashion.

### A. Massless KPZ: $d = 1$

For $d = 1$ we are interested in the integral

$$\int_0^\Lambda dk^2 \left[ \sqrt{k^2 + a} - \sqrt{k^2 + b} \right].$$

This integral is the restriction to one spatial dimension of (33). It is divergent, and it will require renormalization to extract physical answers from this formal result. The most direct way of proceeding is via the “differentiate and integrate” trick where one considers

$$I(a) \overset{\text{def}}{=} \int_0^\infty dk^2 \left[ \sqrt{k^2 + a} - \sqrt{k^2} \right].$$

Now differentiate twice

$$\frac{d^2 I(a)}{da^2} = -\frac{1}{4} \int_0^\infty dk^2 \frac{1}{(k^2 + a)^{3/2}} = -\frac{1}{2\sqrt{a}}. \tag{37}$$

(This last integral is now well defined and finite.) If we integrate the above equation twice, we get

$$I(a) = \kappa a - \frac{2}{3} a^{3/2}. \tag{38}$$

Here $\kappa$ is a constant of integration (which happens to be infinite). There would, in principle, be a second constant of integration, but that is fixed to be zero by the condition that $I(0) = 0$. We absorb $\kappa$ into the bare action, where it renormalizes $F_0$. That is, we write
where the counterterm \([\delta F_0]\) is chosen so as to render the integral (when expressed in terms of renormalized quantities) finite. A brief calculation yields

\[
\delta F_0 = -\frac{\kappa \lambda}{4\pi \nu}.
\]

In one space dimension the only parameter that gets renormalized at order \(O(\lambda)\) is this tadpole term. In terms of renormalized quantities the effective potential is thus

\[
V[v; v_0; d = 1] = \frac{\lambda}{8} v^4 - \frac{1}{6\pi} A \frac{\lambda}{2^{3/2} \nu^{2}} |v|^3 + O(\lambda^2). \quad (41)
\]

These are now renormalized parameters at order \(O(\lambda)\). Setting the slope \(v_0\) and the renormalized tadpole \(F_0\) to their physical values of zero yields

\[
V[v; v_0 = 0; d = 1] = \frac{\lambda}{8} v^4 - \frac{1}{6\pi} A \frac{\lambda}{2^{3/2} \nu^{2}} |v|^3 + O(\lambda^2). \quad (42)
\]

Note that the potential is not analytic at zero field (a phenomenon well-known from massless QFTs). For large \(v\) the classical potential dominates, while for small \(v\) the one-loop effects dominate. The minimum of the potential is not at \(v = 0\), since the \(-|v|^3\) term is negative and dominant near \(v = 0\). Thus we encounter something very interesting — the system undergoes dynamical symmetry breaking (DSB) in a manner qualitatively similar to the Coleman–Weinberg mechanism of particle physics. The qualitative form of the effective potential is sketched in figure 1.

![Figure 1](image.png)

**FIG. 1.** The one-loop effective potential for the KPZ equation in \(d = 1\) space dimensions. The behaviour at the origin is non-analytic in that the third derivative is discontinuous. This distorted Mexican hat potential indicates the onset of dynamical symmetry breaking. For comparison we also plot the zero-loop (“classical”) effective potential. Note that there is no symmetry breaking at tree level.

Symmetry breaking is said to be spontaneous if there is a symmetry in the potential that is not shared by the zero-loop ground states (e.g., the Higgs mechanism). If the symmetry is preserved at the classical level (zero-noise), but is broken once fluctuations are taken into account (broken by loop effects) then the symmetry breaking is said to be dynamical (e.g., massless \(\lambda \phi^4\) theory, Coleman–Weinberg mechanism).

For small \(||\vec{v}||\) the one-loop contributions drive the minimum of the potential away from zero field. To find the location of the minimum we calculate

\[
\frac{dV[v; v_0 = 0; d = 1]}{dv} = \frac{\lambda}{2} v^3 - \frac{1}{2\pi} A \frac{\lambda}{2^{3/2} \nu^{2}} \text{sign}(v) v^2 + O(\lambda^2) = 0.
\]

This permits us to estimate the shift in the expectation value of the velocity field
\[ v_{\text{min}} = \pm A \frac{\lambda}{2\pi^{1/2} \nu^2} + O(A^2). \]  

Unfortunately, the presence of the unknown \( O(A^2) \) terms renders it impossible to make any definitive statement about the precise value of \( v_{\text{min}} \), apart from the fact that it is non-zero \[^{11,12} \]. (This is a common feature in DSB, as perturbatively detecting the occurrence of DSB is easier than finding the precise location of the minimum.) To complete the specification of \( v_{\text{min}} \) one would have to calculate the \( O(A^2) \) terms in the effective potential and verify that the one-loop estimate of \( v_{\text{min}} \) occurs at values of the velocity where the \( O(A^2) \) term is negligible. This is not a trivial task, and we refer the reader to several texts where this is more fully addressed \[^{11,12} \].

This DSB is particularly intriguing in that it suggests the possibility of a noise driven pump. For example, in thin pipes where the flow is essentially one-dimensional, and provided the physical situation justifies the use of the vorticity-free Burgers equation for the fluid, this result indicates the presence of a bimodal instability leading to the onset of a bulk fluid flow with velocity dependent on the noise amplitude. In the surface growth (line growth) interpretation the onset of DSB corresponds to an initially flat line breaking up into a sawtooth pattern of domains in which the slope takes on the values \( \pm v_{\text{min}} \).

**B. Massless KPZ: \( d = 2 \)**

The present discussion is relevant to either (1) surface evolution on a two dimensional substrate, or (2) thin superfluid films (since superfluids are automatically vorticity-free, justifying the application of the zero-vorticity Burgers equation).

An immediate consequence of the analogy between the one-loop effective potential for the massless KPZ equation (for white noise) and that for the massless \( \lambda \phi^4 \) QFT is that we can write down the renormalized one-loop effective potential for \( d = 2 \) (space dimensions) by inspection, merely by recalling that for the \( d = 4 \) (spacetime dimensions) scalar field theory we have (see, e.g., \[^{2,3} \])

\[
\begin{align*}
\mathcal{V}[v; v_0; d = 2] &= \frac{1}{2} \left[ \left| F_0(\mu) + \frac{1}{4} \lambda(\mu) v^2 \right|^2 - \left| F_0(\mu) + \frac{1}{4} \lambda(\mu) v_0^2 \right|^2 \right] \\
& \quad + \frac{1}{2} A \Delta \frac{1}{(2\pi)^2} \nu^4 \left[ \left| F_0(\mu) + \frac{1}{4} \lambda(\mu) v^2 \right|^2 \ln \left( \frac{F_0(\mu) + \frac{1}{4} \lambda(\mu) v^2}{\mu^2} \right) \right] \\
& \quad - \left| F_0(\mu) + \frac{1}{4} \lambda(\mu) v_0^2 \right|^2 \ln \left( \frac{F_0(\mu) + \frac{1}{4} \lambda(\mu) v_0^2}{\mu^2} \right) + O(A^2).
\end{align*}
\]

Here \( \mu \) is the renormalization scale, as normally used in QFT \[^{2,3} \]. (In a condensed matter setting this might be thought of as a measure of the coarse-graining scale.) Its presence is a side effect of the logarithmic divergences, which happen to occur in \( d = 2 \) space dimensions for the KPZ equation.

If we now tune \( v_0 \) and the renormalized value of the tadpole \( F_0 \) to their physical values of zero we obtain

\[
\mathcal{V}[v; v_0 = 0; d = 2] = \frac{\lambda^2}{8} v^4 + \frac{1}{(2\pi)^2} \frac{\lambda^4}{4\nu^4} v^2 \ln \left( \frac{v^2}{\mu^2} \right) + O(A^2).
\]

This potential is zero at \( v = 0 \), then becomes negative, though for large enough fields \( (v > \mu) \) the potential again becomes positive. That this potential has a nontrivial minimum exhibiting DSB is exactly the analog in this massless KPZ problem of the well-known Coleman–Weinberg mechanism encountered in the extensions of the standard model of particle physics \[^{2,3} \]. As in \( d = 1 \), it is easy to see that the minimum effective potential occurs for \( v_{\text{min}} \neq 0 \), but because of the presence of unknown \( O(A^2) \) terms it is difficult to give a good estimate for the value of \( v_{\text{min}} \) \[^{11,12} \].

The qualitative form of the effective potential is sketched in figure \[^{5} \]. If we estimate the location of the minimum by differentiating the effective potential we obtain

\[
v_{\text{min}} = \pm \mu \exp \left( -\frac{(2\pi)^2}{2} \frac{v^3}{\lambda^2 A} - \frac{1}{4} + O(A) \right).
\]

Note that \( v_{\text{min}} \rightarrow 0 \) as \( A \rightarrow 0 \), as it should, to recover the tree level minimum \( v_{\text{min}} = 0 \).
FIG. 2. The one-loop effective potential for the KPZ equation in $d = 2$ space dimensions. The behaviour at the origin is non-analytic in that the fourth derivative exhibits a logarithmic singularity. This distorted Mexican hat potential indicates the onset of dynamical symmetry breaking. For comparison we also plot the zero-loop (“classical”) effective potential, note that there is no symmetry breaking at tree level.

Following the analysis of [13] we can immediately extract the one-loop beta function. In order to do so, we make use of the fact that the bare effective potential does not depend on the renormalization scale:

$$\frac{\mu}{d\mu} \mathcal{V}[v; v_0; d] = 0. \quad (48)$$

We get

$$\frac{\mu}{d\mu} \left( \frac{\lambda^2(\mu)}{8} v^4 \right) = \frac{\mathcal{A}}{16\pi^2} \frac{\lambda^4 v^4}{\nu^3} + O(A)^2. \quad (49)$$

Comparing the coefficients of the $v^4$ terms we obtain

$$\beta \lambda \overset{\text{def}}{=} \frac{\mu}{d\mu} \lambda = \frac{\mathcal{A}}{4\pi^2} \frac{\lambda^3}{\nu^3} + O(A)^2. \quad (50)$$

We cannot extract the beta function for the wavefunction renormalization of $\phi$ (or $v$) from the present analysis. This would require a calculation of the effective action for an inhomogeneous field, an issue which we postpone for the future. Equations (49) and (50) are correct because one can show [13] that to one loop there is no wavefunction renormalization for the KPZ field in this background.

C. Massless KPZ: $d = 3$

This case is of physical interest for three-dimensional fluids. For $d = 3$ we are interested in the integral

$$\int_0^\infty dk^2 k^2 \left[ \sqrt{k^2 + a} - \sqrt{k^2 + b} \right]. \quad (51)$$

Define the quantity

$$\mathcal{I}(a) \overset{\text{def}}{=} \int_0^\infty dk^2 k^2 \left[ \sqrt{k^2 + a} - \sqrt{k^2 + b} \right]. \quad (52)$$

The “differentiate and integrate” trick [8] leads to

$$\frac{d^3 \mathcal{I}(a)}{da^3} = 3 \int_0^\infty dk^2 \frac{k^2}{(k^2 + a)^{3/2}} = \frac{1}{2\sqrt{a}}. \quad (53)$$
(The integral is now well behaved and finite). We now integrate this thrice to obtain

$$I(a) = \kappa_1 a + \kappa_2 a^2 + \frac{4}{15} a^{5/2}. \quad (54)$$

Here $\kappa_1$ and $\kappa_2$ are now two constants of integration (which happen to be infinite). There would in principle be a third constant of integration, but that is fixed to be zero by the condition that $I(0) = 0$. (The sign in front of the $a^{5/2}$ term is important, since this sign is positive we will see that there is no possibility of DSB in three space dimensions.)

We absorb $\kappa_1$ and $\kappa_2$ into the bare potential, where they renormalize both $F_0$ and $\lambda$. The relevant counterterms are

$$\delta F_0 = -\frac{\kappa_1}{8\pi^2} \lambda, \quad \text{and} \quad \delta \lambda = -\frac{\kappa_2}{4\pi^2} \left( \frac{\lambda}{\nu} \right)^3, \quad (55)$$

so as to yield

$$V[v; v_0; d = 3] = \frac{1}{2} [(F_0 + \frac{1}{2} \lambda v^2)^2 - (F_0 + \frac{1}{2} \lambda v_0^2)^2] + \frac{1}{30\pi^2} A \frac{\lambda^{5/2}}{\nu^4} \left[ (F_0 + \frac{1}{2} \lambda v^2)^{5/2} - (F_0 + \frac{1}{2} \lambda v_0^2)^{5/2} \right] + O(A^2). \quad (56)$$

These are now all renormalized parameters at order $O(A)$. Setting to zero both $v_0$ and the renormalized value of $F_0$ we have

$$V[v; v_0 = 0; d = 3] = \frac{\lambda^2}{8} \nu^4 + \frac{1}{30\pi^2} A \frac{\lambda^3}{2^{5/2} \nu^4} |v|^5 + O(A^2). \quad (57)$$

At zero-loops the vacuum is symmetric at $v = 0$. Adding one-loop physics does not change this. Note that there is an all-important sign change in the one-loop contribution to the effective potential in comparing $d = 3$ with $d = 1$. The DSB that is so interesting in $d = 1$, and via the Coleman–Weinberg mechanism in $d = 2$, is now completely absent in $d = 3$. Observe also that the effective potential is non-analytic at $v = 0$. The qualitative form of the effective potential is sketched in figure 3.

III. DISCUSSION

In this paper we have explicitly calculated the renormalized one-loop effective potential for the massless KPZ equation in 1, 2, and 3 space dimensions. Although the effective potential is by definition time-independent, we
already find a very interesting structure for the static ground states of the system. There is a close analogy between
the statics of the massless KPZ system and the static behaviour of massless $\lambda \phi^4$ QFT—and much of the vacuum
structure of the massless $\lambda \phi^4$ QFT carries over into the ground state structure of the massless KPZ equation. It is
important to underscore the fact that the analysis presented here has focussed on the short distance or ultraviolet
properties of the KPZ equation, which as we have seen, reveal a complementary phenomenology to the perhaps more
common studies concerned with the infrared, or long distance properties of the KPZ equation. This distinction shows
up, among other places, in the requirement of the bare tadpole term, the structure of the ultraviolet divergences
requiring renormalization, the phenomena of dynamical symmetry breaking, and the marked dimension dependence
of the one-loop ultraviolet renormalization group equation, and its associated fixed point. The generality of these
structural and dimension-dependent features are confirmed by studying the (one-loop) effective action associated to
the KPZ equation; these general results will be presented elsewhere.

In 1 and 2 space dimensions we have exhibited the occurrence of dynamical symmetry breaking. In the hydrodynamic
interpretation symmetry breaking corresponds to instability of the zero-velocity background, leading to the onset of
bulk flows in the fluid. In the surface growth interpretation symmetry breaking corresponds to instability of the planar
interface, leading to a domain structure wherein different domains exhibit different slopes (all of the same magnitude).
Thus even the static ground state structure of the KPZ equation is surprisingly rich, considerably richer than one
might have reasonably expected. These considerations lead one to conjecture that the one-loop methods developed in
[1] and illustrated here, may be profitably applied to reveal the onset of instabilities in the Kuramoto–Sivashinkin (KS)
equation. The KS equation, used to model flame-front propagation, may be regarded as a generalization of the KPZ
equation with an additional fourth-derivative term ($\vec{\nabla}^4 \phi$). In two dimensional combustion, recent experiments have
demonstrated the existence of so-called “fingering” instabilities as a function of oxygen flow across the surface [15].
On the other hand, there is no fingering in three-dimensional combustion (because of convection). The KS equation
is also Galilean invariant. Thus, it may well exhibit a dimension dependent one-loop DSB akin to that of the KPZ
equation.

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APPENDIX A: JACOBIAN FUNCTIONAL DETERMINANT FOR BURGERS/KPZ

We are interested in evaluating the Jacobian determinant for the massive KPZ equation, with $F[\phi(\vec{x})] = F_0 -
\nu m^2 \phi + \frac{1}{2} (\vec{\nabla} \phi)^2$:

$$\mathcal{J} = \det \left( D - \frac{\delta F}{\delta \phi} \right).$$

We do so by means of the formalism developed in [1]. Thus

$$\frac{\delta F[\phi(\vec{x})]}{\delta \phi(\vec{y})} = -\nu m^2 \delta(\vec{x} - \vec{y}) + \lambda \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla} \delta(\vec{x} - \vec{y}).$$

And so

$$\text{Tr} \left[ \frac{\delta F[\phi(\vec{x})]}{\delta \phi(\vec{y})} \right] = \text{Tr} \left[ -\nu m^2 \delta(\vec{x} - \vec{y}) + \lambda \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla} \delta(\vec{x} - \vec{y}) \right]$$

$$= -\nu m^2 \delta(\vec{0}) - \lambda \delta^d(\vec{0}) \int \vec{\nabla}^2 \phi \; d\vec{x}$$

$$= -\delta^d(\vec{0}) \left\{ \nu m^2 + \lambda \int_{\mathcal{B}} \vec{\nabla} \phi \cdot d\vec{S} \; dt \right\}$$
The ante-penultimate expression is somewhat formal, but is certainly field independent, with at worst some dependence on the boundary conditions. Invoking the formalism developed in [1], this implies that \( J_{\text{KPZ}} \) is a field-independent constant. That is, for either the massless or massive KPZ equation, the functional Jacobian determinant is a field-independent constant which can be ignored. There are general formal arguments (see for example Zinn-Justin [3] pp. 373, 307, or related comments in Itzykson–Zuber [16] p. 448) to the effect that terms proportional to \( \delta^d(\vec{0}) \) can always be discarded in dimensional regularization.

**APPENDIX B: FEYNMAN RULES FOR BURGERS/KPZ FIELD THEORY**

From the massive Burgers/KPZ stochastic differential equation

\[
\left[ \frac{\partial}{\partial t} - \nu(\nabla^2 - m^2) \right] \phi = F_0 + \frac{\lambda}{2}(\nabla \phi)^2 + \eta,
\]

we deduce the partition function [1]:

\[
Z[J] = \int (D\phi) \exp \left( \int J \phi \right) \exp \left( -\frac{1}{2} \int \int \left[ \partial_t \phi - \nu(\nabla^2 - m^2)\phi - F_0 - \frac{\lambda}{2}(\nabla \phi)^2 \right] G_\eta^{-1} \left[ \partial_t \phi - \nu(\nabla^2 - m^2)\phi - F_0 - \frac{\lambda}{2}(\nabla \phi)^2 \right] \right). \tag{B2}
\]

We point out at this stage, that the functional determinant can be discarded in this case. There is only one propagator and two vertices:

**Propagator**: 

\[
G_{\text{field}}(\vec{k}, \omega) = \frac{G_\eta(\vec{k}, \omega)}{\omega^2 + \nu^2(k^2 + m^2)^2}; \tag{B3}
\]

**Three-point vertex**:

\[
[F_0 - \frac{\lambda}{2}(\vec{k}_1 \cdot \vec{k}_2)] \frac{1}{(2\pi)^d+1} \frac{(-i\omega_3 + \nu\vec{k}_3^2)}{G_\eta(\vec{k}_3, \omega_3)} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \delta(\omega_1 + \omega_2 + \omega_3); \tag{B4}
\]

**Four-point vertex**:

\[
\frac{1}{2} \left[ F_0 - \frac{\lambda}{2}(\vec{k}_1 \cdot \vec{k}_2) \right] \left[ F_0 - \frac{\lambda}{2}(\vec{k}_3 \cdot \vec{k}_4) \right] \frac{1}{(2\pi)^d+1} G_\eta(\vec{k}_1 + \vec{k}_2, \omega_1 + \omega_2) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4). \tag{B5}
\]

Feynman diagram computations (in this “direct” formalism) are now straightforward (though tedious!). If one is careful to ask only physical questions, calculations based on this “direct” formalism will yield the same answers as those extracted from Feynman diagrams based on the more common Martin–Siggia–Rose (MSR) formalism [3,17,18].

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