VERIFICATION OF A NEW NON-LINEAR IV-EXponent:
SIMULATION OF THE 2D COULOMB GAS WITH LANGEVIN DYNAMICS.

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ABSTRACT

It has recently been suggested from scaling arguments that the non-linear IV-exponent $a$, $V \propto I^a$, for a two-dimensional superconductor is different from the exponent originally suggested by Ambegaokar et al. (AHNS). The relation between the new and the old exponent is $a = 2a_{AHNS} - 3$. The new scaling behaviour is linked to the logarithmic vortex interaction and the long range time tail which this gives rise to. Consequently one may expect that the scaling behavior is generic for models which have these basic features. The simplest model of this type is the two-dimensional Coulomb gas model with Langevin dynamics. We here explicitly verify, through computer simulations, that the IV-characteristics of this model indeed scales according to the new scaling exponent $a$.

The transition from the resistive state to the superconducting state is for two dimensional (2D) superconductors of the Kosterlitz-Thouless (KT) type. In the superconducting state the current-voltage (IV) characteristics is non-linear, $V \sim I^a$, where the exponent $a$ is larger than one. An expression for $a$ was obtained by Ambegaokar et al. (AHNS) in ref. However, it has recently been suggested that the exponent $a$ is linked to a critical dynamics of the vortex fluctuations in the low temperature phase and that, as a consequence, it is different from the AHNS-value. The relation between the new and the old value of the exponent is $a = 2a_{AHNS} - 3$. The correctness of the new exponent was corroborated by simulations of the 2D RSJ-model. Since the new exponent is linked to a critical dynamics and a critical scaling argument, it is interesting to know what ingredients are necessary to get this critical dynamics. In the present paper we find a candidate for the simplest generic model that has the required critical property.

It is well-known that the KT-transition is due to unbinding of thermally created vortex-antivortex pairs and that the voltage is due to flux flow of free vortices. The interaction between the vortices is logarithmic with distance. Our hypothesis is that the critical dynamics is linked to the logarithmic vortex interaction. The Coulomb gas interaction is logarithmic in two dimensions and consequently the thermally created vortices and antivortices can be described as a two dimensional gas of Coulomb charges. The vortices are quantized and can have vorticity $s = \pm 1$, which corresponds to positive and negative Coulomb gas charges. The total vorticity is always zero which means that the Coulomb gas always has equally many positive and negative charges. The vortices of a 2D superconductor obey a particular dynamics.

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Consequently the vortex part of a 2D superconductor can be modelled by a Coulomb gas with a particular dynamics for the Coulomb gas charges. However, if the critical dynamics is strongly linked to the logarithmic interaction, one may suspect that it is relative insensitive to the dynamical equation of the moving particles. This suggests that a 2D Coulomb gas with a simple dynamics would fall in the class of models which have the required critical property. In the present paper we show, through computer simulations, that a 2D Coulomb gas with conventional Langevin dynamics indeed belongs to this class of models.

The equation of motion for the 2D Coulomb gas with Langevin dynamics is given by

$$\frac{dr(t)}{dt} = \frac{D}{T} F_{\text{tot}}(t) + \eta(t)$$

where $r$ is the position and $F_{\text{tot}}$ is the total force acting on it due to all the other particles as well as any externally imposed force, $D$ is the diffusion constant, $T$ is the temperature (unit system such that the Boltzmann constant $k_B = 1$), and $\eta$ is a random force obeying

$$\langle \eta^\alpha(t) \eta^\beta(t') \rangle = 2D \delta_{\alpha\beta} \delta(t - t')$$

where $\alpha$ and $\beta$ denote the Cartesian components. The charges are circular disks with diameter $r_0$ such that the force acting between two particles $i$ and $j$ with charges $s_i$ and $s_j$ respectively (units such that the charge is $s = \pm 1$) and separated by the distance $r$ is given by

$$F_{ij} = s_i s_j \left( \frac{1}{r} - \frac{1}{r_0} K_1 \left( \frac{r}{r_0} \right) \right)$$

where $K_1$ is a modified Bessel function of order 1. Note that the force between two particles vanishes for $r = 0$. This means that the charge distribution of a particle is soft, which is in accordance with the precise vortex-Coulomb gas particle analogy. In our simulations we choose a 2D box of length $L$ with periodic boundary conditions and a constant particle density $n$. The results presented here are basically for $n = 5 \times 10^{-3} r_0^{-2}$ and $L/r_0 = 226$ (i.e. the total number of particles is $N = 256$). The simulations are done by discretizing time with a time spacing $\Delta t$ and introducing a Gaussian random noise $\eta(t)$ acting independently on each particle at each time step. The equation of motion is then solved on the computer by using a standard Euler integration method. The largest time sequences used in the present simulations are given by $t/\Delta t \approx 5 \times 10^6$. The $\Delta t$ chosen has to be small enough to ensure that the equation of motion is correctly solved yet as large as possible in order to achieve as large time sequencies as possible.

We are calculating the average charge current $I_p$ as a function of an external force $F_{\text{ext}}$. The average charge current is given by

$$I_p = \langle \sum_{i=1}^{N} s_i \frac{dr_i(t)}{dt} \rangle = \frac{D}{T} \langle \sum_{i=1}^{N} F_{\text{tot}}^{(i)}(t) \rangle$$

where the sum is over all particles $N$ and $F_{\text{tot}}^{(i)}$ is the total force acting on the particle $i$ and the brackets $\langle \rangle$ denote a time average. The voltage $V$ in a 2D superconductor...
is due to the flux flow which means that $V$ is proportional to $I_p$. The force acting on a vortex is given by the Lorentz force and is consequently proportional to the current. This means that the externally imposed current $I_p$ is proportional to the external force $F_{ex}$. Thus the $IV$-characteristics for a 2D superconductor corresponds to the $F_{ex}I_p$-characteristics of the Coulomb gas. Figure 1 shows the obtained $F_{ex}I_p$-characteristics for a sequence of temperatures plotted as $\ln I_p$ against $\ln F_{ex}$. As seen in the figure the data fall on straight lines in the limit of small $F_{ex}$ which implies that $I_p \propto F_{ex}^a$, where the slope of the line gives the corresponding exponent $a$. In this way the non-linear $F_{ex}I_p$- exponent $a$ is obtained. As also seen in figure 1 the $F_{ex}I_p$-data can be very well fitted to the functional form $AK_0(BF_{ex}) + C \ln F_{ex}$ over an appreciable range of $F_{ex}$, where $A = 1 - a$, and $B$ and $C$ are constants. This implies that $I_p = CF_{ex} \exp(AK_0(BF_{ex}))$ and this observation somewhat simplifies the determination of the exponent $a$. The obtained exponents $a$, corresponding to the data in figure 1, are shown in figure 2 together with estimated error bars. According to the critical scaling theory these exponents $a$ should be related to the static charge density correlations. The precise relation is

$$a = \frac{1}{\tilde{T}t} - 1$$

where $\tilde{t}$ is related to the Fourier transform of the dielectric function $\tilde{\epsilon}(k)$. The dielectric function is in turn related to the charge density correlations by

$$\frac{1}{\tilde{\epsilon}(k)} = 1 - \frac{2\pi L^2}{Tk^2} \langle \Delta \hat{n}(k) \Delta \hat{n}(-k) \rangle$$
where
\[ \Delta n(r) = \sum_{i=1}^{N} s_i \delta(r - r_i) \] (7)

is the charge density. The Fourier transform of the dielectric function has the leading small \( k \) dependence
\[ \frac{1}{\tilde{\epsilon}(k)} = \frac{1}{\tilde{\epsilon}} \frac{k^2}{k^2 + \lambda^{-2}} \] (8)

where \( \lambda \) is the screening length. This relation defines \( \tilde{\epsilon} \) in terms of \( \tilde{\epsilon}(k) \). The screening length \( \lambda \) is infinite for temperatures below the KT transition temperature. This means that \( \tilde{\epsilon} = \tilde{\epsilon}(k = 0) \) in the low temperature phase.

In the simulations we calculate \( \Delta \tilde{n}(k, t) \) for a long time sequence for a given fixed value of \( k \). From such a sequence we readily obtain the time average of the charge density correlations \( \langle \Delta \tilde{n}(k) \Delta \tilde{n}(-k) \rangle \) and from this \( 1/\tilde{\epsilon}(k) \). The value of \( \tilde{\epsilon} \) is then determined by using equation (8). This determination of \( \tilde{\epsilon} \) is shown in figure 3 and figure 4 gives the values obtained. The scaling exponent \( a = 1/T\tilde{\epsilon} - 1 \) is now calculated from these values of \( \tilde{\epsilon} \) and is compared to the actual exponent \( a \) determined directly from our \( I_p F_{ex} \)-simulations in figure 2. As seen in figure 2, the agreement with the scaling theory prediction is excellent. According to the AHNS-theory, the exponent \( a \) should instead be given by \( a_{AHNS} = 1/2T\tilde{\epsilon} + 1 = a/2 + 3/2 \). This prediction is also shown in figure 2. As is apparent from figure 2, the AHNS-prediction does not agree with the simulated exponents. Thus figure 2 clearly verifies that the new non-linear IV exponent gives a correct description of the 2D Coulomb gas with Langevin dynamics and that the AHNS-prediction is not correct for this model. The same conclusion was obtained in case of the 2D RSJ-model. The present simulations suggests that the new scaling exponent is quite robust and that particles

Fig. 2. The non-linear IV-exponent \( a \) as a function of temperature (dashed lines are guides to the eye).
Fig. 3. The dielectric function as a function of $|k|$ for $T=0.12, 0.14, 0.16, 0.18, 0.23, 0.26$, from top to bottom, respectively, and the corresponding fits used to obtain $\tilde{\epsilon}$.

Fig. 4. $1/\tilde{\epsilon}$ obtained by fitting equation (8) to the data in figure 3.

with a logarithmic interaction seems to be the most essential feature for obtaining this new exponent.

According to the scaling theory, the new critical exponent is linked to a long time tail in the charge density correlations. We have explicitly calculated the long time behaviour of the charge density correlations function

$$\hat{g}(k, t) = \langle \hat{n}(k, t)\hat{n}(-k, 0) \rangle$$  \hspace{1cm} (9)

From a practical point of view it is easier to converge the average for large values of $t$ if one uses a $k$-value somewhat larger than the smallest possible $k = 2\pi/L$. Figure 5 shows the result for a temperature ($T = 0.18$) somewhat below the KT transition ($T_{KT} \approx 0.22$). In this case a reasonable convergence was achieved for the $k$-vector $k = \frac{2\pi}{L}(4, 4)$. The result is plotted as $t\hat{g}(k, t)$ versus $t$. As seen in the figure $t\hat{g}(k, t)$ is constant to very good approximation for large $t$ which means that $\hat{g}(k, t) \propto 1/t$. The result given in figure 5 shows that the time correlations for a small finite $k$ fall to good approximation like $1/t$ over a very large $t$-interval. This result suggests that
\( \hat{g}(k = 0, t) \) has a true \( 1/t \) tail for large \( t \).

In summary the results displayed in figures 1-4 demonstrate that the 2D Coulomb gas with Langevin dynamics has a non-linear IV-exponent \( a \) which is distinctly different from the AHNS-prediction and is instead given by the scaling theory by Minnhagen at al.\(^\text{1}\). In addition the result given in figure 5 is indeed consistent with a \( 1/t \)-tail in the charge density correlations, as required by the scaling theory. We expect these findings to be quite general and so should apply to a 2D superconductor and a 2D Josephson coupled array.

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