HODGE DECOMPOSITION FOR COUSIN GROUPS AND FOR OELJEKLAUS-TOMA MANIFOLDS

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Abstract. We compute the Dolbeault cohomology of certain domains contained in Cousin groups which satisfy a strong dispersiveness condition. As a consequence we obtain a description of the Dolbeault cohomology of Oeljeklaus-Toma manifolds and in particular the fact that the Hodge decomposition holds for their cohomology.

1. Introduction

A Cousin group $X$ is a quotient $\mathbb{C}^n/\Lambda$, where $\Lambda$ is a discrete subgroup of rank $n + m$, with $1 \leq m \leq n$, such that the global holomorphic functions on $X$ are constant. They are named after P. Cousin and introduced in [Cou10]. In [Vog83] it is shown that a Cousin group has finite dimensional Dolbeault cohomology groups provided the discrete subgroup $\Lambda$ satisfies a certain dispersiveness condition, which we shall describe in the paper and call weak dispersiveness. Moreover, Hodge decomposition is proven by Vogt to hold on $X$ under this same condition.

The aim of our paper is twofold. Firstly, we extend the result of Vogt to open sets $U$ in $\mathbb{C}^n/\Lambda$, whose inverse image in $\mathbb{C}^n$ are convex domains, see Theorem 3.1. For this we need to impose a new condition on the discrete subgroup $\Lambda$, which we shall call strong dispersiveness. We show that this condition is actually equivalent to the finite generation of the Dolbeault cohomology of such domains, see Theorem 3.4. Secondly, we use the aforementioned extension to show the Hodge decomposition and to compute the Dolbeault cohomology of Oeljeklaus-Toma manifolds, see Theorem 4.5. These are compact complex manifolds associated to number fields allowing a positive number of real embeddings as well as a positive number of complex (non-real) embeddings, see Section 4. Their construction and first properties are described in [OT05]. As a consequence, we also obtain a new way of computing the Dolbeault cohomology of Inoue-Bombieri surfaces, which are obtained as Oeljeklaus-Toma manifolds of complex dimension 2, without using powerful tools like the Riemann-Roch theorem or Serre duality and providing instead a more complex-analytical proof.

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2. Preliminary facts on Cousin groups

We present in this section basic definitions and results about Cousin groups and introduce the notions of weak and strong dispersiveness, see Definition 2.4.

**Definition 2.1.** A connected complex Lie group $X$ admitting no non-constant global holomorphic functions is called a Cousin group or a toroidal group.

Cousin groups of complex dimension $n$ are shown to appear as quotients $X = \mathbb{C}^n / \Lambda$, where $\Lambda$ is a discrete subgroup of $\mathbb{C}^n$ of rank $n + m$, with $1 \leq m \leq n$, cf. [AK] Proposition 1.1.2. Moreover $\Lambda$ may be assumed to be generated by the columns of a matrix of the form:

$$
P = \begin{pmatrix} O_{m,n-m} & T_{m,2m} \\ I_{n-m} & R_{n-m,2m} \end{pmatrix},$$

which we shall call the normal form, where $I_{n-m}$ is the $n - m$ identity matrix, $T_{m,2m}$ is a basis of the lattice of an $m$-dimensional complex torus and $R$ has real entries. Furthermore, one can arrange $T$ such that the normal form is:

$$
P = \begin{pmatrix} O_{m,n-m} & I_m & M + iN \\ I_{n-m} & R_l & R_2 \end{pmatrix},$$

where $M$ and $N$ have real entries and $N$ is invertible, see [Vog82] Proposition 2, [Vog83] Proposition 1. In the above situation we will say that $P$ is the period matrix of $\Lambda$.

**Proposition 2.2** ([Vog82] Proposition 2). Suppose that $X = \mathbb{C}^n / \Lambda$ with $\Lambda$ generated by the columns of a matrix $P$ in normal form (2.1). Then $X$ is a Cousin group if and only if for any $\sigma \in \mathbb{Z}^{n-m} \setminus \{0\}$, \( \sigma R \notin \mathbb{Z}^{2m} \).

**Proposition 2.3.** Let $X = \mathbb{C}^n / \Lambda$ be a Cousin group and let $U \subset X$ be a non-empty open subset whose inverse image $\tilde{U}$ in $\mathbb{C}^n$ is convex. Then any global holomorphic function on $U$ is constant.

**Proof.** We use essentially that $\mathbb{C}^n / \Lambda$ is a Cousin group, a similar argument as in [OT05, Lemma 2.4] and the fact that $\tilde{U}$ is convex. We may and will assume that $\Lambda$ is generated by the columns of a matrix $P$ in normal form (2.1).

For $(z_0^1, \ldots, z_0^n) \in \tilde{U}$, the set $\text{conv}((z_0^1, \ldots, z_0^n) + \Lambda)$ is a real affine $(n + m)$-dimensional plane in $\mathbb{C}^n$, where by $\text{conv}(S)$ we mean the convex hull of $S$. It is also a subset of $\tilde{U}$ by the convexity and $\Lambda$-invariance of $\tilde{U}$. Since the functions $\text{Im} z_{m+1}, \ldots, \text{Im} z_n$ are $\Lambda$-invariant we get $\text{conv}((z_0^1, \ldots, z_0^n) + \Lambda) = \mathbb{C}^m \times ((z_0^1, \ldots, z_0^n) + \mathbb{R}^{n-m})$. Therefore

$$
\tilde{U} = \bigcup_{(z_0^1, \ldots, z_0^n) \in \tilde{U}} \mathbb{C}^m \times ((z_0^1, \ldots, z_0^n) + \mathbb{R}^{n-m}) = \mathbb{C}^m \times \bigcup_{(z_0^1, \ldots, z_0^n) \in \tilde{U}} ((z_0^1, \ldots, z_0^n) + \mathbb{R}^{n-m}).
$$

Thus, $\tilde{U} = \mathbb{C}^m \times W$, where $W \subset \mathbb{C}^{n-m}$ is a convex domain, hence Stein, and moreover $\mathbb{Z}^{n-m}$-invariant.

Let now $f$ be a holomorphic function on $U$, $\hat{f}$ its lift to $\tilde{U}$ and choose arbitrarily $w \in W$. Since $\mathbb{C}^m \times (w + \mathbb{R}^{n-m}) / \Lambda$ is diffeomorphic to $(S^1)^{n+m}$, $\hat{f}$ is bounded on $\mathbb{C}^m \times (w + \mathbb{R}^{n-m})$ and therefore constant on $\mathbb{C}^m \times \{w\}$. Using the fact that $\mathbb{C}^n / \Lambda$ is a Cousin group and Proposition 2.2 we get $\hat{f} \sigma R \notin \mathbb{Z}^{2m}$ for all $\sigma \in \mathbb{Z}^{n-m} \setminus \{0\}$, hence the group generated by the column vectors $(I_{n-m} R)$ is dense in $\mathbb{R}^{n-m}$. Consequently, the image of $\mathbb{C}^m \times \{w\}$ is dense
in $\mathbb{C}^m \times (w + \mathbb{R}^{n-m})/\Lambda$ and thus, $f$ is constant on $\mathbb{C}^m \times (w + \mathbb{R}^{n-m})/\Lambda$ and $\tilde{f}$ is constant on $\mathbb{C}^m \times (w + \mathbb{R}^{n-m})$. By the identity principle, $\tilde{f}$ has to be constant on $\tilde{U}$. 

We now introduce two notions of dispersiveness which will play an important role in this paper.

**Definition 2.4.** A discrete subgroup $\Lambda$ in normal form (2.2) is said to be strongly dispersive, (respectively weakly dispersive) if

$$\forall a \in (0, 1), \ (\text{respectively } \exists a \in (0, 1)), \ \exists C(a) > 0, \ \forall \sigma \in \mathbb{Z}^{n-m} \setminus \{0\}, \ \forall \tau \in \mathbb{Z}^{2m}$$

$$||^t \sigma R + ^t \tau|| \geq C(a) a |\sigma|.$$  

(2.4)

In [Vog82] the following example of a discrete subgroup $\Lambda_\alpha$ is considered with period basis $P_\alpha$ in normal form:

$$P_\alpha = \begin{pmatrix} 0 & 1 & i \\ 1 & \alpha & 0 \end{pmatrix},$$

where $\alpha$ is a real number. By Proposition 2.2 $\mathbb{C}^2/\Lambda_\alpha$ is a Cousin group if and only if $\alpha$ is irrational.

Vogt shows in [Vog82] that for $\alpha = \sum_{j=1}^{\infty} \frac{1}{10^j}$ the discrete subgroup $\Lambda_\alpha$ is not weakly dispersive.

**Remark 2.5.** Set $u_0 := 1$, $u_{j+1} := 10^{u_j}$ for all $j \in \mathbb{N}$, and

$$\alpha := \sum_{j=1}^{\infty} \frac{1}{u_j}.$$  

Then the discrete subgroup $\Lambda_\alpha$ generated by the columns of the matrix $P_\alpha$ given by (2.5) is weakly dispersive but not strongly dispersive.

**Proof.** The strong (respectively weak) dispersiveness condition for $\Lambda_\alpha$ is rephrased as

$$\forall a \in (0, 1), \ (\text{respectively } \exists a \in (0, 1)), \ \exists C(a) > 0, \ \forall q \in \mathbb{Z} \setminus \{0\}, \ \forall p \in \mathbb{Z}$$

(2.6)

$$|q \alpha - p| \geq C(a) a |q|.$$  

For $q = u_k$, $k \geq 1$, we get

$$\inf_{p \in \mathbb{Z}} |q \alpha - p| = \sum_{j=k+1}^{\infty} \frac{u_k}{u_j} < 2 \frac{u_k}{u_{k+1}} < 2 \frac{u_k}{u_k} = \frac{1}{5^q},$$

hence $\Lambda_\alpha$ cannot be strongly dispersive for our choice of $\alpha$.

We now check the weak dispersiveness of $\Lambda_\alpha$. For a real number $\beta$ we denote by $\{\beta\}$ its fractional part. Then for $u_k \leq q < u_{k+1}$, $k > 0$ we get

$$\{q \alpha\} > \frac{1}{10^u_k} \geq \frac{1}{10^q}.$$  

It remains to estimate $1 - \{q \alpha\}$. But it is clear that the $u_k + 1$-st decimal digit of $\{q \alpha\}$ is 0, hence the $u_k + 1$-st decimal digit of $1 - \{q \alpha\}$ is 9. Thus

$$1 - \{q \alpha\} \geq \frac{9}{10^{10^u_k}} \geq \frac{9}{10^{10^q}}.$$
which proves weak dispersiveness of $\Lambda_\alpha$ by taking $a = \frac{1}{10}$. ■

Examples of strongly dispersive discrete subgroups are provided by the following

**Proposition 2.6.** If $\Lambda$ is a discrete subgroup defining a Cousin group and such that all the entries of some period matrix are algebraic numbers, then $\Lambda$ is strongly dispersive.

*Proof.* By using a generalization of Liouville’s Theorem on the approximation of algebraic numbers ([FdN98, Theorem 1.5]) it is proved in [BO15, Theorem 4.3] that the discrete subgroup $O_K$ is weakly dispersive, see Section 4 for notations.

More precisely in the proof of [BO15, Theorem 4.3] it is shown that if $R$ is a $k \times l$ matrix with elements algebraic numbers, then there exist constants $C > 0$ and $A < 0$ such that for any $\sigma \in \mathbb{Z}^k \setminus \{0\}$ and every $\tau \in \mathbb{Z}^l$, $||\sigma R + \tau|| \geq C|\sigma|^A$. But this condition is stronger than strong dispersiveness, since clearly for any $a \in (0, 1)$, there exists a constant $C(a)$ such that $|\sigma|^A \geq C(a)|\sigma|^a$ for all $\sigma \in \mathbb{Z}^k \setminus \{0\}$. ■

The following result proved by Chr. Vogt in [Vog82], [Vog83] will be extended in Section 3 to the case of open sets in $\mathbb{C}^n/\Lambda$, whose inverse image in $\mathbb{C}^n$ are convex domains.

**Theorem 2.7 ([Vog82], [Vog83]).** If $X = \mathbb{C}^n/\Lambda$ is a Cousin group, then $H^1(X, \mathcal{O})$ is finite dimensional if and only if $\Lambda$ is weakly dispersive. Moreover, in this situation all the Dolbeault cohomology groups $H^{p,q}_\partial(X)$ are finite dimensional and $X$ satisfies the Hodge decomposition.

Additionally, Vogt gives several equivalent conditions for the finite dimensionality of $H^1(X, \mathcal{O})$ in terms of the discrete subgroup $\Lambda$, the holomorphic line bundles on $X$ and the generators of $H^1(X, \mathcal{O})$.

### 3. Dolbeault cohomology of “convex” domains in Cousin groups.

In this section we will prove analogous results to those of Theorem 2.7 for open subsets $U$ in Cousin groups $X = \mathbb{C}^n/\Lambda$, whose inverse images in $\mathbb{C}^n$ are convex domains. Occasionally we will call such open sets $U$ in $X$ “convex” by abuse of terminology. By [AK, Proposition 1.1.8] the definition of a “convex” open set in a Cousin group $X$ does not depend on the chosen presentation $\mathbb{C}^n/\Lambda$ for $X$.

**Theorem 3.1.** Let $U$ be a domain of a Cousin group $X = \mathbb{C}^n/\Lambda$, whose inverse image $\tilde{U}$ in $\mathbb{C}^n$ is a convex domain. If $\Lambda$ is strongly dispersive then $H^q(U, \Omega^p)$ is finitely generated and moreover,

$$\left\{ [dz_I \wedge d\bar{z}_J] \mid I \subseteq \{1, \ldots, n\}, J \subseteq \{1, \ldots, m\}, |I| = p, |J| = q \right\}$$

is a basis and thus, $\dim \mathbb{C} H^q(U, \Omega^p) = \binom{n}{p} \cdot \binom{m}{q}$.

We follow the lines of the proofs of Proposition 4 and Proposition 5 in [Vog83] and adapt them to the new setting.

We start with the following lemma:

**Lemma 3.2.** Let $q \geq 1$. Any $\Lambda$-invariant $(0, q)$-form $\omega$ on $\tilde{U}$ is $\bar{\partial}$-cohomologous to a $\Lambda$-invariant $(0, q)$-form on $\tilde{U}$, whose coefficients depend holomorphically on $z_{m+1}, \ldots, z_n$.

*Proof.* We notice first that $U$ is the total space of a locally trivial holomorphic fibration over a complex torus with fibre a Stein set in $(\mathbb{C}^*)^{n-m}$. As in the proof of Proposition 2.3 we
remark that \( \tilde{U} = \mathbb{C}^m \times W \), where \( W \) is a convex \( \mathbb{Z}^{n-m} \)-invariant domain. Since \( U = \tilde{U}/\Lambda \), the map

\[
\pi : U \to \mathbb{C}^m/T,
\]
given by \( \pi((z_1, \ldots, z_m)) = (z_1, \ldots, z_m) \), is well-defined, where \([ \cdot ]\) and \( \tilde{\cdot} \) are classes with respect to taking quotients by \( \Lambda \) and by the lattice generated by the columns of \( T \), respectively. Clearly, \( \pi \) is a holomorphic map and in fact, \( (3.1) \) is a fibration with fibre isomorphic to \( F := W/\mathbb{Z}^{n-m} \). Via the map \( \exp(2\pi i \cdot) \), \( \mathbb{C}^{n-m}/\mathbb{Z}^{n-m} \) is biholomorphic to \( (\mathbb{C}^*)^{n-m} \), so we regard \( F \) directly as an open subset of \( (\mathbb{C}^*)^{n-m} \). In fact, seen in this way, \( F \) is a logarithmically convex Reinhardt domain in \( (\mathbb{C}^*)^{n-m} \) and is therefore a domain of holomorphy in \( (\mathbb{C})^{n-m} \) and thus Stein, \( \text{[GF76]} \). Then \( U \) is the fibre bundle associated to \( \rho : \pi_1(\tilde{T}) \to \text{Aut}(F) \) given by

\[
\rho(t_k) = \text{diag}(\exp(2\pi ir_{1,k}), \ldots, \exp(2\pi ir_{n-m,k}))
\]
Here \( \tilde{T} := \mathbb{C}^m/T \) and thus the fundamental group of \( \tilde{T} \) is generated by the column vectors \( t_k \) of the matrix \( T \).

The proof continues now in the same steps of Proposition 4 in Vogt's paper \( \text{[Vog83]} \). For the sake of completeness, we sketch it.

We consider \( U = (U_i)_{i=1,\ldots,k} \) a finite covering of \( \tilde{T} \), which is acyclic for the sheaf \( \mathcal{O}_{\tilde{T}} \) and moreover trivializing for \( (3.1) \). This further implies together with \( F \) being Stein that the covering of \( U \), given by \( V = (V_i)_{i=1,\ldots,k} \), \( V_i = \pi^{-1}(U_i) \simeq U_i \times F \), is acyclic for \( \mathcal{O}_U \) and its associated Čech complex computes the cohomology groups of \( \mathcal{O}_U \).

The \((0,q)\)-form \( \omega \) is represented in the Čech cohomology by a \( q \)-cocycle \((\xi_I)_I \in \mathcal{Z}(V,\mathcal{O}_U)\), where \( I \) runs through all subsets of length \( q + 1 \) of \( \{1, \ldots, k\} \). Via the trivialization biholomorphisms \( V_I \xrightarrow{\tilde{\xi}} U_I \times F \), we regard each \( \xi_I \) as a holomorphic function on \( U_I \times F \) and develop it as a Laurent series in the fiber coordinate \( w_I \):

\[
\xi_I = \sum_{\alpha \in \mathbb{Z}^{n-m}} \xi_{I,\alpha} w_I^\alpha.
\]

Since it depends on the choice of \( \varphi \), \( \xi_{I,\alpha} \) does not define an element of \( \mathcal{O}(U_I) \), but by \( (3.2) \) it defines a section of \( \mathcal{O}_{U_I} \otimes L^{-\alpha} \), where \( L^{-\alpha} \) is the topologically trivial line bundle over \( \tilde{T} \) associated to \( \chi : \pi_1(\tilde{T}) \to \mathbb{C}^* \), given by \( \chi(t_k) = \exp(-2\pi i \sum_{j=1}^{n-m} \alpha_j r_{j,k}) \). Since \( |\chi(t_k)| = 1 \), \( |\xi_{I,\alpha}| \) is a well defined function on \( U_I \). For a fixed \( \alpha \in \mathbb{Z}^{n-m} \), one can check that \((\xi_{I,\alpha})_I \in \mathcal{Z}(U,\mathcal{O}_{\tilde{T}} \otimes L^{-\alpha})\), because \((\xi_I)_I \) is a cocycle. We consider the bi-complex which has as \( l \)-th line the complex:

\[
0 \to C^l(U,\mathcal{O}_{\tilde{T}} \otimes L^{-\alpha}) \xrightarrow{i} C^l(U,\mathcal{C}_{\tilde{T}}^\infty \otimes L^{-\alpha}) \xrightarrow{\partial} C^l(U,\Omega_{\tilde{T}}^{0,1} \otimes L^{-\alpha}) \to \ldots
\]

for \( m \geq l \geq 0 \) and as line \(-1\) the following complex:

\[
0 \to \Gamma(\mathcal{O}_{\tilde{T}} \otimes L^{-\alpha}) \xrightarrow{i} \Gamma(\mathcal{C}_{\tilde{T}}^\infty \otimes L^{-\alpha}) \xrightarrow{\partial} \Gamma(\Omega_{\tilde{T}}^{0,1} \otimes L^{-\alpha}) \xrightarrow{\partial} \ldots
\]

For \( 0 \leq r \leq m \), the \( r \)-th column is the Čech complex of the covering \( U \) and the sheaf \( \Omega_{\tilde{T}}^{0,r} \otimes L^{-\alpha} \) and the \(-1\) column is the Čech associated to \( U \) and \( \mathcal{O}_{\tilde{T}} \otimes L^{-\alpha} \). Apart from the \(-1\) column and \(-1\) line, all the other complexes are acyclic.

By a zig-zag diagram chasing we get that for \((\xi_{I,\alpha})_I \in \mathcal{Z}(U,\mathcal{O}_{\tilde{T}} \otimes L^{-\alpha})\), we find an element \( \omega_\alpha \) in \( \Gamma(\Omega_{\tilde{T}}^{0,q} \otimes L^{-\alpha}) \), representing the class of \((\xi_{I,\alpha})_I \) in Čech cohomology. We briefly describe
the procedure of finding \( \omega_{\alpha} \). Seen as an element in \( C^q(U, \mathcal{C}_T^\infty \otimes L^{-\alpha}) \), one can easily check that \( (\xi_{I,\alpha})_I \) satisfies:

\[
(\xi_{I,\alpha})_I = \delta_{q-1} \left( (\xi_{I,\alpha}^{(1)})_I \right),
\]

where \( \xi_{\{j_0,\ldots,j_q-1\},\alpha}^{(1)} := \sum_{i=1}^k \eta_i \cdot \xi_{\{i,j_0,\ldots,j_q-1\},\alpha} \), for \( (\eta_i)_i \) a partition of unity associated to the covering \( (U_i)_i \). Since \( \xi_{I,\alpha} \) are holomorphic, \( \bar{\partial} \xi_{\{j_0,\ldots,j_q-1\},\alpha}^{(1)} = \sum_{i=1}^k \xi_{\{j_0,\ldots,j_q-1\},\alpha}^{(1)} \cdot \partial \eta_i \). The induction step is to define \( \xi_{\{j_0,\ldots,j_q-p\},\alpha}^{(p)} := \sum_{i=1}^k \eta_i \cdot \bar{\partial} \xi_{\{i,j_0,\ldots,j_q-p\},\alpha} \). In this way, \( \omega_{\alpha} = \xi^{(q)} = \sum_{|K|=q} f_{K,\alpha} \partial \Omega_K \), where the coefficients \( f_{K,\alpha} \) are products of \( \xi_{I,\alpha} \) and the resulting coefficients from \( \partial \eta_i \).

Since \( \eta_i \) are compactly supported in \( U_i \), and the covering is finite, there is a constant \( C \) that bounds all the coefficients of \( \partial \eta_i \). Therefore \( |f_{K,\alpha}(x)| \leq (kC)^{\max_j} |\xi_{I,\alpha}(x)| \), for any \( x \in \tilde{T} \) and by the convergence of \( \xi_I \), we obtain the convergence of \( \sum_{\alpha \in \mathbb{Z}_{n-m}} \omega_{\alpha} \omega^\alpha \). Each \( \omega_{\alpha} \) is global \((0,q)\)-form on \( \tilde{T} \) which is \( L^{-\alpha} \)-valued. We denote by \( p : \mathbb{C}^m \rightarrow \tilde{T} \) and observe that \( \Omega := \sum_{\alpha} p^\ast \omega_{\alpha} \omega^\alpha \) is a well-defined \( \partial \)-closed \((0,q)\)-form on \( \mathbb{C}^m \times F \), invariant to the action of \( \pi_1(\tilde{T}) \), given by \( t_k \cdot (\underline{z},f) = (\underline{z} - t_k, \rho(t_k)f) \). Therefore, \( \Omega \) descends to \( U \), \( [\Omega]_T = [\omega]_T \) and moreover its coefficients clearly depend holomorphically on \( z_{m+1}, \ldots, z_n \).

\[ \square \]

We now proceed to the proof of Theorem 3.1.

**Proof.** We divide the proof in two steps.

**Step 1:** We will show that any \( \partial \)-closed, \( \Lambda \)-periodic \((p,q)\)-form \( \omega \) on \( \tilde{U} \) is \( \partial \)-cohomologous to a form \( \sum_{I,J} c_{I,J} dz_I \wedge d\bar{z}_J \) with constant coefficients \( c_{I,J} \in \mathbb{C} \).

If \( \omega \) is a \((p,0)\)-form, the statement is obvious, as \( \omega \) has to be of type \( \sum_I f_I dz_I \), with \( f_I \) holomorphic \( \Lambda \)-invariant functions on \( \tilde{U} \), but these are constant by Proposition 2.3.

Let now \( q \geq 1 \). Once we prove the statement for \((0,q)\)-forms, it will immediately follow for \((p,q)\)-forms as well, since \( \omega = \sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J = \sum dz_I \wedge (\sum_{I,J} f_{I,J} d\bar{z}_J) \) and each \( \sum_{I,J} f_{I,J} d\bar{z}_J \) is \( \partial \)-closed.

Therefore, take \( \omega = \sum_{J} f_{J} d\bar{z}_J \) on \( \tilde{U} \), \( \partial \)-closed, \( \Lambda \)-periodic. By Lemma 3.2, we may assume that \( f_J \) depend holomorphically on \( z_{m+1}, \ldots, z_n \) and \( J \subseteq \{1,2,\ldots,m\} \).

We perform now the same computations as in Proposition 5 from [Vog83] and outline the differences from our setting. Let \( (z_1, \ldots, z_m) := x + iy \) and \( (z_{m+1}, \ldots, z_n) := w \). For any \( \pi, \rho \in \mathbb{Z}^m \) and \( \sigma \in \mathbb{Z}^{n-m} \), we define the following function on \( \tilde{U} \):

\[
\gamma_{\pi,\rho,\sigma}(z_1, \ldots, z_n) := \exp \left( 2\pi i \cdot (i^t \pi - t \sigma R_1)x + (l^t \rho - t \pi M + l^t \sigma (R_1 M - R_2)) \cdot N^{-1} y + t^t \sigma w \right)
\]

Each \( \gamma_{\pi,\rho,\sigma} \) is \( \Lambda \)-invariant and thus, we develop \( f_J \) in Fourier series on \( \tilde{U} \):

\[
f_J = \sum_{\pi,\rho,\sigma} f_{J,\pi,\rho,\sigma} \gamma_{\pi,\rho,\sigma},
\]

where \( f_{J,\pi,\rho,\sigma} \in \mathbb{C} \). We define the following:

\[
a_{\pi,\rho,\sigma} := \frac{1}{2} \left( \omega(R_1) + i^t \rho - t \pi M + l^t \sigma (R_1 M - R_2) \right) \in \mathbb{C}^m
\]
\[ B := \left\{ \sum_{j=1}^{m} b_j d\bar{z}_j \mid b_j \in \mathbb{C} \right\} \]

\[ \lambda_{\pi, \rho, \sigma} : B \to \mathbb{C}, \quad \lambda_{\pi, \rho, \sigma} \left( \sum_{j=1}^{m} b_j \cdot d\bar{z}_j \right) := \frac{\sum_{j=1}^{m} b_j a_{\pi, \rho, \sigma, j}}{2\pi i ||a_{\pi, \rho, \sigma}||^2}, \]

where \( a_{\pi, \rho, \sigma, j} \) is the \( j \)-th component of \( a_{\pi, \rho, \sigma} \). We will see at a further point in the proof that \( ||a_{\pi, \rho, \sigma}|| \) does not vanish.

We extend \( \lambda_{\pi, \rho, \sigma} \) to a homomorphism:

\[ \lambda_{\pi, \rho, \sigma} \downarrow : \Lambda^k B \to \Lambda^{k-1} B \]

\[ \lambda_{\pi, \rho, \sigma} \downarrow (\alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{p=1}^{k} (-1)^{k-p} \lambda_{\pi, \rho, \sigma} (\alpha_p) \alpha_1 \wedge \ldots \wedge \hat{\alpha}_p \wedge \ldots \wedge \alpha_k \]

and define the \( \Lambda \)-periodic \((0, q - 1)\)-form on \( \tilde{U} \).

\[ \eta = \sum_{(\pi, \rho, \sigma) \neq 0} (-1)^{q-1} \left( \lambda_{\pi, \rho, \sigma} \left( \sum_{j} f_{I, \pi, \rho, \sigma} d\bar{z}_j \right) \right) \gamma_{\pi, \rho, \sigma}. \]

By a straightforward, but lengthy, computation, presented in [Vog83], one gets that \( \partial \eta = \omega - \sum_{j} f_{I, 0, 0, 0} d\bar{z}_j \). The crucial part of it is to prove that \( \eta \) is a convergent series and then the statement of Step 1 is clear. It is at this point that the strong dispersiveness of \( \Lambda \) will play an essential role.

The rest of Step 1 is devoted to the proof of the convergence of the series \( \eta \). We show first by using \( (2.4) \) that for any \( a \in (0, 1) \), there exists a constant \( C_1(a) > 0 \) such that for any \((\pi, \rho, \sigma) \neq 0\), \( ||a_{\pi, \rho, \sigma}|| \geq C_1(a) a |\sigma| \).

Take \( k_1 := ||MN^{-1}||, k_2 := \frac{1}{||N||} \).

Then clearly

\[ ||\alpha MN^{-1}|| \leq k_1 ||\alpha||, \quad \forall \alpha \in \mathbb{R}^m \]

\[ ||\alpha N^{-1}|| \geq k_2 ||\alpha||, \quad \forall \alpha \in \mathbb{R}^m \]

Let \( k := \frac{k_2}{1+k_1+k_2} \). If \((\pi, \rho, \sigma) \neq 0\) is such that \( 2||\text{Re}(a_{\pi, \rho, \sigma})|| = ||t^\rho - t^\sigma R_1|| \leq k C(a) a |\sigma| \), then by \( (2.4) \), we get \( ||t^\rho - t^\sigma R_2|| \geq (1 - k) C(a) a |\sigma| \) and therefore by \( (3.14) \) and \( (3.15) \):

\[ (3.16) \]

\[ 2||\text{Im}(a_{\pi, \rho, \sigma})|| = ||(t^\rho - t^\sigma R_2) N^{-1} - (t^\rho - t^\sigma R_1) MN^{-1}|| \geq (k_2(1-k) C(a) - k_1 k C(a)) a |\sigma| = k C(a) a |\sigma|. \]

This means that for \( C_1(a) := \frac{1}{2} k C(a) \) we get

\[ (3.17) \]

\[ ||a_{\pi, \rho, \sigma}|| \geq C_1(a) a |\sigma|, \forall (\pi, \rho, \sigma) \neq 0. \]

In particular \( ||a_{\pi, \rho, \sigma}|| \) does not vanish for \((\pi, \rho, \sigma) \neq 0\).
By the expression in (3.13), we deduce that:

\[ \eta = \sum_{|K|=q-1} \left( \sum_{(\pi,\rho,\sigma)\neq 0} t_{\pi,\rho,\sigma}^g \gamma_{\pi,\rho,\sigma} \right) d\varpi_K, \]

where \( t_{\pi,\rho,\sigma} \) is a finite sum of terms of type \( \pm f_{K\cup\{j\},\pi,\rho,\sigma} a_{\pi,\rho,\sigma,j} \). We need to show that \( h_K := \sum_{(\pi,\rho,\sigma)\neq 0} t_{\pi,\rho,\sigma} \gamma_{\pi,\rho,\sigma} \) is convergent on \( \bar{U} \).

We prove that for each \( |J|=q \), \( h_J := \sum_{(\pi,\rho,\sigma)\neq 0} f_{J,\pi,\rho,\sigma} a_{\pi,\rho,\sigma,j} \gamma_{\pi,\rho,\sigma} \) is convergent on \( \bar{U} \) and this will suffice.

Fix \( z_0 = x_0 + iy_0 \in \mathbb{C}^m \). Then

\[ h_J(z_0, w) = \sum_{\sigma} \left( \sum_{(\pi,\rho)} f_{J,\pi,\rho,\sigma} \frac{a_{\pi,\rho,\sigma,j}}{|a_{\pi,\rho,\sigma}|^2} \exp(2\pi i a(z_0)) \right) \exp(2\pi i \sigma \cdot w), \]

where \( a(z) := ((\pi - t \sigma R_1)x + (\rho - t \pi M + t \sigma (R_1 M - R_2)) \cdot N^{-1} y \in \mathbb{R} \), for any \( z \in \mathbb{C}^m \).

Recall that \( \bar{U} = \mathbb{C}^m \times W \), where \( W \) is a \( \mathbb{Z}^{n-m} \)-invariant convex Stein domain in \( \mathbb{C}^{n-m} \). We prove now that \( \sum_{\sigma} \left( \sum_{\rho,\pi} f_{J,\pi,\rho,\sigma} \frac{a_{\pi,\rho,\sigma,j}}{|a_{\pi,\rho,\sigma}|^2} \exp(2\pi i (a(z))) \right) \) is uniformly absolutely convergent on \( \mathbb{C}^m \times W_1 \), where \( W_1 = \exp(2\pi i W) \). Note that \( W_1 \) is a Reinhardt domain, therefore, \( W_1 = T^{n-m} \cdot S \), where \( S = \{(w_1, \ldots, w_{n-m}) | (w_1, \ldots, w_{n-m}) \in W_1 \} \).

Choose \( w^0 = (w_1^0, \ldots, w_{n-m}^0) \in W_1 \cap \mathbb{R}^{n-m}_+ \), a neighbourhood \( U_{z_0} \) of \( z_0 \) in \( \mathbb{C}^m \) and \( S_{w^0} = (w_1^0 - \varepsilon, w_1^0 + \varepsilon) \times \ldots \times (w_{n-m}^0 - \varepsilon, w_{n-m}^0 + \varepsilon) \), for a small \( \varepsilon > 0 \), such that \( S_{w^0} \subset W_1 \).

For any \( a \in (0, 1) \), on \( U_{z_0} \times T^{n-m} \cdot S_{w^0} \), we have by (3.17):

\[ mC_1(a)^{-1} \sum_{\sigma} \left( \sum_{(\rho,\pi)} f_{J,\pi,\rho,\sigma} |a_{\rho,\pi,\sigma,j} - |a_{\rho,\pi,\sigma}| \right) |w^0^\sigma| \leq mC_1(a)^{-1} \sum_{\sigma} \left( \sum_{(\rho,\pi)} |f_{J,\pi,\rho,\sigma} - |a_{\rho,\pi,\sigma}| \right) |w^0^\sigma|. \]

We split now the series \( \sum_{\sigma} \left( \sum_{(\rho,\pi)} |f_{J,\pi,\rho,\sigma} - |a_{\rho,\pi,\sigma}| \right) |w^0^\sigma| \) in a sum of \( 2^{n-m} \) series

\[ h_g^a := \sum_{\sigma \in D_g} \left( \sum_{(\rho,\pi)} |f_{J,\pi,\rho,\sigma} - |a_{\rho,\pi,\sigma}| \right) |w^0^\sigma|, \]

where \( g : \{1, \ldots, n-m\} \rightarrow \{-1, 1\} \) and \( D_g = \{(\sigma_1, \ldots, \sigma_{n-m}) \in \mathbb{Z}^{n-m} \setminus \{0\} | \text{sgn}(\sigma_i) = g(i)\} \).

By convention we consider \( \text{sgn}(0) = 1 \). Then on \( U_{z_0} \times T^{n-m} \cdot S_{w^0} \):
\[
\sum_{\sigma \in D_{g,\rho,\pi}} |f_{\sigma}| - |\sigma| \frac{w_{\sigma}}{w_{\sigma} + g(1)\varepsilon} |\sigma_1| \cdots \frac{w_{\sigma-m}}{w_{\sigma-m} + g(n-m)\varepsilon} |\sigma_{n-m}| |w_1^0 + g(1)\varepsilon \cdots |w_n^0 + g(n-m)\varepsilon| |\sigma_{n-m}| \leq \sum_{\sigma \in D_{g,\rho,\pi}} |f_{\sigma}| - |\sigma| \delta(w_{\sigma}^{0}, \ldots, w_{n-m}^{0}, \ldots, \varepsilon) |w_1^0 + g(1)\varepsilon| \cdots |w_n^0 + g(n-m)\varepsilon| |\sigma_{n-m}|,
\]
where \(\delta(w_{\sigma}^{0}, \ldots, w_{n-m}^{0}, \ldots, \varepsilon) = \max\left\{ \frac{w_{\sigma}^{0} + g(1)\varepsilon}{w_{\sigma}^{0} + g(1)\varepsilon} \right\} \). We can choose now \(a \in \sum D_{g,\rho,\pi} \) to be \(\delta(w_{\sigma}^{0}, \ldots, w_{n-m}^{0}, \ldots, \varepsilon) \) and thus,
\[
h_{g}^{\delta} \leq \sum_{\sigma \in D_{g,\rho,\pi}} |f_{\sigma}| - |\sigma| |w_1^0 + g(1)\varepsilon| \cdots |w_n^0 + g(n-m)\varepsilon| |\sigma_{n-m}|.
\]

But the series in the right hand side above is bounded by a constant \(C((w_{\sigma}^{0}, \ldots, w_{n-m}^{0}, \ldots, \varepsilon)) \), since \(f_{\sigma} \) is holomorphic in \(z_{m+1}, \ldots, z_{n} \) and thus, the series \(\sum_{\sigma,\pi,\rho} f_{\sigma,\pi,\rho}\varepsilon \exp(2\pi iz)w^{\sigma} \) is absolutely uniformly convergent on \(U_{z_0} \times \mathbb{T}^{n-m} \times S_{\rho,\pi,\sigma}^{0} \).

What we actually proved above is that \(\sum_{\sigma \neq 0} \sum_{(\pi,\rho)} f_{\sigma,\pi,\rho} \varepsilon |\alpha_{\sigma,\pi,\rho}||\pi,\rho,\sigma| \) is convergent. But if \(\sigma = 0 \), we observe that for \((\pi,\rho) \neq 0, |a_{\pi,\rho,0}| \geq 1 \). Indeed, it is clear by \(2|\Re a_{\pi,\rho,0}| = |\pi| \) and \(2||\pi|\pi| \neq |\pi| \). Consequently, the missing part of \(h_{J} \), which is \(\sum_{(\pi,\rho) \neq 0} f_{\pi,\rho,0} \frac{\alpha_{\pi,\rho,0}}{||\pi,\rho,0||} \gamma_{\pi,\rho,0} \), is dominated by \(\sum_{(\pi,\rho) \neq 0} f_{\sigma,\pi,\rho} \gamma_{\pi,\rho,0} \), which is convergent since \(f_{\sigma} \) is. We conclude that \(\eta \) is convergent on \(\tilde{U} \) and Step 1 is proved.

**Step 2.** We prove now that \(\{[dz_{I} \wedge d\sigma_{I}] \mid I = (1 \leq i_1 < \ldots < i_p \leq n), J = (1 \leq i_1 < \ldots < i_q \leq m) \} \) is a basis for \(H^{q}(U,\Omega^{p}) \). Step 1 tells us that \(H^{q}(U,\Omega^{p}) \) is generated by \(\{dz_{I} \wedge d\sigma_{I} \mid I = (1 \leq i_1 < \ldots < i_p \leq n), J = (1 \leq i_1 < \ldots < i_q \leq m) \} \), therefore \(\dim_{C}H^{q}(U,\Omega^{p}) \leq \binom{m}{q} \binom{n}{q} \). Since all \(H^{q}(U,\Omega^{p}) \) and \(H_{dR}^{q}(U,\mathbb{C}) \) are finitely generated, we can apply Frölicher’s inequality and get:

\[
\dim_{C}H_{dR}^{q}(U,\mathbb{C}) \leq \sum_{p+q=l} \dim_{C}H^{q}(U,\Omega^{p}) \leq \sum_{p+q=l} \binom{n}{p} \binom{m}{q} = \binom{n+m}{l}
\]

As \(U \simeq (S^{1})^{n+m} \times \mathbb{R}^{n-m} \), \(\dim_{C}H_{dR}^{l}(U,\mathbb{C}) = \binom{n+m}{l} \), therefore we have equality in (3.21) and the conclusion follows.

The fact that equality holds in (3.21) immediately implies

**Corollary 3.3.** If \(\Lambda \) is strongly dispersive, then Hodge decomposition holds for any “convex” domain \(U \) in the Cousin group \(\mathbb{C}^{n}/\Lambda \).

We next state and prove a converse of Theorem 3.1

**Theorem 3.4.** If \(H^{1}(U,\mathcal{O}) \) is finite dimensional for every open subset \(U \) of the Cousin group \(X = \mathbb{C}^{n}/\Lambda \) such that its inverse image \(\tilde{U} \) in \(\mathbb{C}^{n} \) is convex, then \(\Lambda \) is strongly dispersive.
We are led to consider the strategy of defining an infinite family of linearly independent elements in $\Lambda$. For any position $i$, it tells us that:

\[
\exists a \in (0,1), \forall C > 0, \exists \sigma(C) \in \mathbb{Z}^{n-m} \setminus \{0\}, \exists \tau(C) \in \mathbb{Z}^{2m}, \|t^t \sigma(C) R - t^t \tau(C)\| < Ca^{\sigma(C)},
\]

which by taking $C = \frac{1}{k}$, with $k \in \mathbb{N}^*$, implies that:

\[
(3.22) \quad \exists a \in (0,1), \forall k \in \mathbb{N}^*, \exists (\frac{1}{k}) \in \mathbb{Z}^{n-m} \setminus \{0\}, \exists (\frac{1}{k}) \in \mathbb{Z}^{2m}, \|t^t \sigma(\frac{1}{k}) R - t^t \tau(\frac{1}{k})\| < \frac{1}{k}a^{\sigma(\frac{1}{k})}.
\]

For convenience, we shall use the notation $\sigma(k)$ instead of $\sigma(\frac{1}{k})$. We can assume that $\sigma(k) \neq \sigma(l)$ for $k \neq l$, otherwise we can extract a subsequence $(k_i)_{i \in \mathbb{N}}$ such that $\sigma(k_i)$ are all different. Indeed, if we had a finite set of values for the sequence $(\sigma(k))_{k \in \mathbb{N}}$, we would have a subsequence $(k_j)_{j \in \mathbb{N}}$ such that $\sigma(k_j) = c \neq 0$, for all $k_j$. Then by (3.22), we get that $t^t c R \in \mathbb{Z}^{2m}$, which is impossible by Proposition 2.2. Moreover, by taking again a subsequence if needed, we can consider that for any position $i \in \{1, \ldots, n-m\}$, $\text{sgn}(\sigma(k_i))$ is constant. Therefore, (3.22) tells us that:

\[
(3.23) \quad \exists a \in (0,1), \forall k \in \mathbb{N}^*, \exists (k) \in \mathbb{Z}^{n-m} \setminus \{0\}, \exists (k) \in \mathbb{Z}^{2m}, \|t^t (k) R - t^t (k)\| < \frac{1}{k}a^{(k)} \leq a^{(k)}|_{k \to \infty}.
\]

We are led to consider $\tilde{U} := \mathbb{C}^m \times \prod_{i=1}^{n-m} \mathbb{H}^{\text{sgn}(\sigma(k_i))}$ and $U := \tilde{U}/\Lambda$, where we set $\mathbb{H}^1 := \mathbb{H}$, $\mathbb{H}^{-1} := -\mathbb{H} = \{z \in \mathbb{C} | \text{Im} z < 0\}$. Here we have set $\text{sgn}(0) = +1$ by abuse of notation. In fact by applying the automorphism

\[
z \mapsto (z_1, \ldots, z_m, z_{m+1}^{\text{sgn}(\sigma(k_1))}, \ldots, z_{m}^{\text{sgn}(\sigma(k_{n-m}))})
\]

of $\mathbb{C}^n$ we reduce ourselves to the situation where all $\sigma(k)_i$ are non-negative. In the sequel we will suppose that this is the case. Thus the considered convex domain in $\mathbb{C}^n$ will be $\tilde{U} = \mathbb{C}^m \times \mathbb{H}^{n-m}$.

We are in a situation where the sheaf cohomology $H^1(U, \mathcal{O})$ may be computed as the group cohomology $H^1(\Lambda, H^0(\tilde{U}, \mathcal{O}))$, where $\Lambda$ on $H^0(\tilde{U}, \mathcal{O})$ naturally via translation on $\tilde{U}$, see [Mum70, Appendix to Section 2]. Thus

\[
H^1(U, \mathcal{O}) \cong H^1(\Lambda, H^0(\tilde{U}, \mathcal{O})) = Z^1(\Lambda, H^0(\tilde{U}, \mathcal{O}))/B^1(\Lambda, H^0(\tilde{U}, \mathcal{O})),
\]

where

\[
Z^1(\Lambda, H^0(\tilde{U}, \mathcal{O})) := \{A : \Lambda \times \tilde{U} \to \mathbb{C} | A(\lambda, \cdot) \in H^0(\tilde{U}, \mathcal{O}) \forall \lambda \in \Lambda, A(\lambda_1 + \lambda_2, z) = A(\lambda_1, z + \lambda_2) + A(\lambda_2, z), \forall \lambda_1, \lambda_2 \in \Lambda, \forall z \in \tilde{U}\},
\]

\[
B^1(\Lambda, H^0(\tilde{U}, \mathcal{O})) := \{A : \Lambda \times \tilde{U} \to \mathbb{C} | \exists g \in H^0(\tilde{U}, \mathcal{O}) A(\lambda, z) = g(z + \lambda) - g(z) \forall \lambda \in \Lambda, \forall z \in \tilde{U}\}.
\]

The strategy is to define an infinite family of linearly independent elements in $H^1(\Lambda, H^0(\tilde{U}, \mathcal{O}))$. For any $\sigma \in \mathbb{Z}^{n-m} \setminus \{0\}$ we set $\eta_{\sigma} := \max_{j} |\exp(2\pi i \sigma \cdot r_j) - 1|$, where $r_j$ for $j \in \{1, \ldots, 2m\}$ are the columns of $R$. We shall denote by $v_i$ the columns of $P$.

For $\lambda = \sum_{j=1}^{n+m} n_j v_j \in \Lambda, n_j \in \mathbb{Z}$ we will further denote by $l(\lambda) := \sum_{j=1}^{n+m} |n_j|$. 

Proof. We shall argue by contradiction, namely, we show that if $\Lambda$ is not strongly dispersive, then there exists an open “convex” $U$ such that $H^1(U, \mathcal{O})$ is infinite dimensional. Indeed, if $\Lambda$ is not strongly dispersive,

\[
\exists a \in (0,1), \forall C > 0, \exists \sigma(C) \in \mathbb{Z}^{n-m} \setminus \{0\}, \exists \tau(C) \in \mathbb{Z}^{2m}, \|t^t \sigma(C) R - t^t \tau(C)\| < Ca^{\sigma(C)},
\]
For each $x \in (0, 1)$, we define an element $A^x(\lambda) \in Z^1(\Lambda, H^0(\tilde{U}, \mathcal{O}))$ by:
\begin{equation}
A^x(\lambda, z) := \sum_{k \in \mathbb{N}} a^x|\sigma(k)| \left( \frac{\exp(2\pi i^t \sigma(k) \cdot (\lambda_{m+1}, \ldots, \lambda_n)) - 1}{\eta_\sigma(k)} \right) \exp(2\pi i^t \sigma(k) \cdot (z_{m+1}, \ldots, z_n)).
\end{equation}

Let us check the holomorphicity of $A^x(\lambda, \cdot)$ on $\tilde{U}$ for every $\lambda$, the other condition being clearly satisfied. To this aim, as $\frac{\exp(2\pi i^t \sigma(k) \cdot (\lambda_{m+1}, \ldots, \lambda_n)) - 1}{\eta_\sigma(k)}$ is bounded by $l(\lambda)$ it suffices to check that the series $S := \sum_{k \in \mathbb{N}} a^x|\sigma(k)| |w_{m+1}|^{\sigma(k)}_1 \ldots |w_n|^{\sigma(k)}_{n-m}$ is uniformly convergent on $\mathbb{D}^{n-m}$. But this is clear since $a^x < 1$.

Note that
\begin{equation}
A^x(v_i, z) = 0, \, \forall z \in \tilde{U}, \, \forall i \in \{1, \ldots, n - m\}.
\end{equation}

Take now $s > 0$ and $0 < x_1 < \ldots < x_s < 1$. We will show that the classes of $A^{(x_1)}, \ldots, A^{(x_s)}$ are $\mathbb{C}$-linearly independent in $H^1(\Lambda, H^0(\tilde{U}, \mathcal{O}))$.

Suppose that this is not the case. Then there exist $c_1, \ldots, c_s \in \mathbb{C}$, not all zero and a holomorphic function $g$ on $\tilde{U}$ such that
\begin{equation}
\sum_{i=1}^s c_i A^{(x_i)}(\lambda, z) = g(z + \lambda) - g(z).
\end{equation}

From (3.25) and (3.26) we deduce that $g$ is $(0, \mathbb{Z}^{n-m})$-periodic and therefore has a Fourier series expansion
\begin{equation}
g = \sum_{\sigma \in \mathbb{Z}^{n-m} \setminus \{0\}} g_{\sigma} \exp(2\pi i^t \sigma \cdot (z_{m+1}, \ldots, z_n)).
\end{equation}

Using now (3.24) and plugging (3.27) in (3.26), we get $g_{\sigma} = \sum_{i=1}^s c_i a^x|\sigma(k)| \frac{\eta_\sigma(k)}{\eta_\sigma(k)}$ if $\sigma = \sigma(k)$ for some $k \in \mathbb{N}$ and $g_{\sigma} = 0$ otherwise. Therefore
\begin{equation}
g = \sum_{\sigma(k) \in \mathbb{Z}^{n-m} \setminus \{0\}} \left( \sum_{i=1}^s c_i \frac{a^x|\sigma(k)|}{\eta_\sigma(k)} \right) \exp(2\pi i^t \sigma(k) \cdot (z_{m+1}, \ldots, z_n)).
\end{equation}

Since $g$ is holomorphic, the following series is absolutely uniformly convergent on $\mathbb{D}^{n-m}$:
\begin{equation}
g_1 := \sum_{\sigma(k) \in \mathbb{Z}^{n-m} \setminus \{0\}} \left( \sum_{i=1}^s c_i \frac{a^x|\sigma(k)|}{\eta_\sigma(k)} \right) w_{m+1}^{\sigma(k)} \ldots w_n^{\sigma(k)_{n-m}}.
\end{equation}

A straightforward computation shows that $|t^t\sigma(k)R - t^t \tau(k)|| < a^{\sigma(k)}$ for some $\tau(k) \in \mathbb{Z}^{2m}$ entails $\eta_\sigma < 2\pi a^{\sigma(k)}$, for all $k$. It follows that
\begin{equation}
L := \limsup_k \frac{1}{|\sigma(k)|/\eta_\sigma(k)} \geq a^{-1}.
\end{equation}

Set $\rho := \frac{1}{L}$. We have $\rho \leq a$.

Define
\begin{equation}
S_i := c_i \sum_{k \in \mathbb{N}} \left( \frac{a^x|\sigma(k)|}{\eta_\sigma(k)} \right) w_{m+1}^{\sigma(k)} \ldots w_n^{\sigma(k)_{n-m}}.
\end{equation}
We may suppose that all coefficients $c_i$ are non-zero. Restricting $g$ and the series $S_i$ to $D$ via the diagonal embedding $D \hookrightarrow D^{n-m}$ we get

\begin{equation}
S_1|_D = g_1|_D - S_2|_D - \ldots - S_s|_D.
\end{equation}

But the convergence radius of each $S_i|_D$ equals $a^{-x_1}\rho$ and is thus lower or equal to $a^{1-x_1}$ and also lower than 1. It follows that the convergence radius of the series appearing on the left hand side of equation (3.28) is strictly smaller than the convergence radius of the right hand side. This is a contradiction.

In fact the family $A^{(x)}$, $x \in (0, 1)$ provides an infinite set of linearly independent elements of $H^1(U, \mathcal{O})$.

**Corollary 3.5.** Let $X = \mathbb{C}^n/\Lambda$ be a Cousin group. Then $\Lambda$ is strongly dispersive if and only if $H^1(U, \mathcal{O})$ is finitely generated for every “convex” domain $U$ in $X$.

4. **Dolbeault Cohomology of Oeljeklaus-Toma Manifolds**

In this section we will apply Theorem 3.1 to determine the Dolbeault cohomology of Oeljeklaus-Toma manifolds. We start by a brief presentation of their construction following [OT05].

Let $\mathbb{Q} \subseteq K$ be an algebraic number field with $n$ embeddings in $\mathbb{C}$, out of which $s$ are real, $\sigma_1, \ldots, \sigma_s; K \to \mathbb{R}$, and $2t$ are complex conjugated embeddings, $\sigma_{s+1}, \ldots, \sigma_{s+t}, \sigma_{s+t+1} = \overline{\sigma_{s+1}}, \ldots, \sigma_{s+2t} = \overline{\sigma_{s+t}}; K \to \mathbb{C}$. Clearly, $n = s + 2t$.

Let $\mathcal{O}_K$ be the ring of algebraic integers of $K$, and $\mathcal{O}_K^{s,+}$ be the group of totally positive units, which is the subset of $\mathcal{O}_K$ consisting of those units with positive image through all the real embeddings.

Consider the action $\mathcal{O}_K \ni \mathbb{H}^s \times \mathbb{C}^t$ given by:

$$T_a(w_1, \ldots, w_s, z_{s+1}, \ldots, z_{s+t}) := (w_1 + \sigma_1(a), \ldots, z_{s+t} + \sigma_{s+t}(a)),$$

where $\mathbb{H}$ denotes the upper half-plane and the action $\mathcal{O}_K^{s,+} \ni \mathbb{H}^s \times \mathbb{C}^t$ given by dilatations,

$$R_u(w_1, \ldots, w_s, z_{s+1}, \ldots, z_{s+t}) := (w_1 \cdot \sigma_1(u), \ldots, z_{s+t} \cdot \sigma_{s+t}(u)).$$

In [OT05] it is shown that there always exists a subgroup $U \subseteq \mathcal{O}_K^{s,+}$ such that the action $\mathcal{O}_K \times U \ni \mathbb{H}^s \times \mathbb{C}^t$ has no fixed point, is properly discontinuous and co-compact. The **Oeljeklaus-Toma manifold** (OT, shortly) associated to the algebraic number field $K$ and to the admissible subgroup of positive units $U$ is

$$X(K, U) := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \rtimes U.$$

By construction, $X(K, U)$ is a smooth fibre bundle over $\mathbb{R}^s_{>0}/U$, which is diffeomorphic to a real $s$-dimensional torus $\mathbb{T}^s$. Moreover, the fibre is again a real torus:

\begin{equation}
\mathbb{T}^{s+2t} \to X(K, U) \to \mathbb{T}^s,
\end{equation}

but the fibration is not principal. We call $X(K, U)$ of **simple type** if there exists no proper intermediate extension $\mathbb{Q} \subseteq K' \subseteq K$ such that $U \subset \mathcal{O}_K^{s,+}$.

By [OT05] Lemma 2.4 $\mathbb{C}^{s,+}/\mathcal{O}_K$ is a Cousin group and by Proposition 2.6 one has

**Proposition 4.1.** The discrete subgroup $\mathcal{O}_K$ is strongly dispersive.
Theorem 3.1 will be applied to the “convex” open subset $(\mathbb{H}^s \times \mathbb{C}^t)/\mathcal{O}_K$ of the Cousin group $\mathbb{C}^{s+t}/\mathcal{O}_K$.

Warning: In the previous section, we denoted by $U$ a convex domain in a Cousin group, but for the rest of the exposition, $U$ shall only stand for an admissible group of positive units. Also $n$ equals now $s+2t$ and no longer denotes the dimension of the Cousin group we consider. From now on, we use the notation $X$ instead of $X(K,U)$ for the Oeljeklaus-Toma manifold and not for the Cousin group $\mathbb{C}^{s+t}/\mathcal{O}_K$.

Remark 4.2. In [MT15] the construction of Oeljeklaus-Toma manifolds was slightly generalized by replacing the discrete subgroup $\mathcal{O}_K$ by an additive subgroup $M$ of rank $s+2t$ which is stable under the action of $U$. The resulting manifolds $X(M,U)$ were shown to admit finite unramified covers of type $X(\mathcal{O}_K,U)$. All our results extend without difficulty to this larger class of compact complex manifolds. When $s=t=1$ the class of manifolds of type $X(M,U)$ coincides with the class of Inoue-Bombieri surfaces, [MT15] Remark 8].

By the Dolbeault isomorphism, $H^{p,q}_\bar{\partial}(X) \simeq H^q(X,\Omega^p)$, where $\Omega^p$ is the sheaf of germs of holomorphic $p$-forms.

We shall compute $H^q(X,\Omega^p)$ by using three instruments: the Leray-Serre spectral sequence associated to the fibration $\mathcal{U}$, Theorem 3.1 and Frölicher-type inequalities.

We denote by $pE_{i,j}$ the Leray-Serre spectral sequence associated to (4.1) and the sheaf $\Omega^p$. Then $pE_{i,j}^2 = H^j(T^s, R^j\pi_*\Omega^p)$, where $R^j\pi_*\Omega^p$ is the sheafification of the presheaf $\hat{T}_j^p$ given by:

\[
\hat{T}_j^p(W) = H^j(\pi^{-1}(W), \Omega^p_{|\pi^{-1}(W)}),
\]

for any open set $W$ of $\mathbb{T}^s$. We use the notation $\hat{T}_j^p$ from now on, instead of $R^j\Omega^p$.

Lemma 4.3. For any $0 \leq p, j \leq s+t$, $\hat{T}_j^p$ is the local system on $\mathbb{T}^s$ associated to the representation $\rho : U \to GL(N(p,j),\mathbb{C})$,

\[
\rho(u) = \text{diag}(\sigma_1(u)\sigma_j(u)),
\]

where $N(p,j) = \binom{s+t}{p}\binom{t}{j}$, $I$ runs through all the subsets of length $p$ of $\{1, \ldots, s+t\}$, $J$ through all the subsets of length $j$ of $\{1, \ldots, t\}$ and for any $K = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, s+t\}$, $\sigma_K(u) := \sigma_{i_1}(u)\cdots\sigma_{i_k}(u)$, with the convention that if $K \subseteq \{1, \ldots, t\}$, $\sigma_K(u) := \sigma_{s+i_1}(u)\cdots\sigma_{s+i_k}(u)$.

Remark 4.4. In particular, when $j > t$, $\hat{T}_j^p$ is the sheaf that vanishes on every open set of $\mathbb{T}^s$. Note that $\pi_1(\mathbb{T}^s) = U$.

Proof. We show first that $\hat{T}_j^p$ is locally constant, namely, that for any $x \in \mathbb{T}^s$, there exists an open set $W \ni x$ such that $\hat{T}_j^p|_W$ is constant. Indeed, let $W \ni x$ be a trivialization open set for (4.1) such that $W$ is the image of an open convex set $\hat{W} \subset \mathbb{R}^s_{>0}$. Then $\pi^{-1}(W)$ is biholomorphic to $\hat{W} \times \mathbb{R}^s \times \mathbb{C}^t/\mathcal{O}_K$. Since $\hat{W} \times \mathbb{R}^s \times \mathbb{C}^t$ is an open convex $\mathcal{O}_K$-invariant and since $\mathcal{O}_K$ is strongly dispersive by Proposition 4.1, we are in a situation where Theorem 3.1 applies.
By Theorem 3.4, applied to \( \hat{W} \times \mathbb{R}^s \times \mathbb{C}^t \) and \( \mathcal{O}_K \), we get that for any open “convex” trivialization set \( W \)

\[
(4.4) \quad \dim \mathbb{C}H^j(\pi^{-1}(W), \Omega^p_{\pi^{-1}(W)}) = \binom{s + l}{p} = N(p, j).
\]

The basis of \( H^j(\pi^{-1}(W), \Omega^p_{\pi^{-1}(W)}) \) is, therefore, given by \( \{ [dz_I \wedge d\mathfrak{z}_J] \mid |I| = p, |J| = j, I \subseteq \{1, \ldots, s + t\}, J \subseteq \{1, \ldots, t\} \} \).

Since the set of “convex” open sets \( W \) is co-final, in the sense that

\[
(T^p_j)_x = \lim_{V \ni x} T^p_j(V) = \lim_{W \ni x, W \text{ convex}} T^p_j(W),
\]

we have \( (\hat{T}^p_j)_x = (T^p_j)_x = \mathbb{C}^N(p, j) \), meaning that \( \hat{T}^p_j \) is locally constant.

In order to determine the corresponding representation of \( U \), we need to check how an element \( u \in U \) acts on the basis \( [dz_I \wedge d\mathfrak{z}_J] \). From the definition of \( OT \)-manifolds, we have:

\[
u^*(dz_I \wedge d\mathfrak{z}_J) = \sigma_I(u)\mathfrak{z}_J(u)dz_I \wedge d\mathfrak{z}_J
\]

and consequently the representation associated to \( \hat{T}^p_j \) is precisely \( \rho \). Since \( \rho \) is diagonal, we deduce moreover that

\[
(4.5) \quad \hat{T}^p_j = \bigoplus_{I,J} L_{I,J},
\]

where \( L_{I,J} \) is the flat complex line bundle over \( \mathbb{T}^s \) associated to the representation \( \rho_{I,J} : U \to \mathbb{C}^*, \rho_{I,J}(u) = \sigma_I(u)\mathfrak{z}_J(u) \).

**Theorem 4.5.** Any \( OT \)-manifold \( X \) satisfies the Hodge decomposition, in the sense that

\[
\dim \mathbb{C}H^l_{dR}(X) = \sum_{p+q=l} \dim \mathbb{C}H^q(X, \Omega^p).
\]

**Proof.** By the Frölicher inequality, we have:

\[
(4.6) \quad \dim \mathbb{C}H^l_{dR}(X) \leq \sum_{p+q=l} \dim \mathbb{C}H^q(X, \Omega^p).
\]

Since \( pE_{\mathfrak{z}^i} \Rightarrow H^*(X, \Omega^p) \), by a Frölicher type inequality, we get:

\[
(4.7) \quad \dim \mathbb{C}H^q(X, \Omega^p) \leq \sum_{i+j=q} \dim \mathbb{C}H^i(\mathbb{T}^s, \hat{T}^p_j).
\]

By (4.5), \( \dim \mathbb{C}H^l(\mathbb{T}^s, \hat{T}^p_j) = \dim \mathbb{C}H^l(\mathbb{T}^s, \bigoplus_{I,J} L_{I,J}) \) and using now Lemma 2.4 in [IO], the following holds:

\[
(4.8) \quad \dim \mathbb{C}H^l(\mathbb{T}^s, \bigoplus_{I,J} L_{I,J}) = \dim \mathbb{C}H^l(\mathbb{T}^s) \cdot \sharp \{I \subseteq \{1, \ldots, s + t\}, J \subseteq \{1, \ldots, t\} \mid |I|= p, |J| = j, \rho_{I,J} \equiv 1\}
\]

\[
(4.9) \quad \left(s \atop i\right) \cdot \sharp \{I \subseteq \{1, \ldots, s + t\}, J \subseteq \{1, \ldots, t\} \mid |I|= p, |J| = j, \sigma_I(u) \cdot \mathfrak{z}_J(u) \equiv 1\}
\]
Putting together (4.6), (4.7) and (4.8), we have:
\[
\dim_{\mathbb{C}} H^l_{dR}(X) \leq \sum_{p+q=l} \dim_{\mathbb{C}} H^q(X, \Omega^p) \leq \sum_{p+q=l} \sum_{i+j=q} \binom{s}{i} \cdot \sharp\{ I \subseteq \{1, \ldots, s+t \}, J \subseteq \{1, \ldots, t \} | |I|=p, |J|=j, \sigma_I(u) \cdot \sigma_J(u) = 1 \}
\] (4.10)
Using the fact that for any 1 \leq r \leq t, \sigma_{s+r}(u) = \sigma_{s+t+r}(u), the last term of the inequality can be rewritten as \[ \sum_{p+q=l} \binom{s}{q} \cdot \sharp\{ I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, s+2t\} | \sigma_{i_1}(u) \cdots \sigma_{i_p}(u) = 1, \forall u \in U \}. \]
By Theorem 3.1 in [IO], this is exactly \[ \dim_{\mathbb{C}} H^l_{dR}(X). \]
Hence all the inequalities above are actually equalities and we obtain Hodge decomposition. 

**Corollary 4.6.** For any OT manifold \( X \) of type \((s, t)\), \( \dim_{\mathbb{C}} H^l(X, \mathcal{O}) = s \).

**Proof.** By [OT05], we know that \( b_1 = s \) and \( H^0(X, \Omega^1) = 0 \). Applying now Theorem 4.5 for \( l = 1 \), we immediately get \( \dim_{\mathbb{C}} H^1(X, \mathcal{O}) = s \).

In [IO] it is shown that the de Rham cohomology of an OT manifold \( X \) can be easily computed if \( X \) satisfies the following condition:
Condition (C): \( \sigma_I \sigma_J \equiv 1 \) if and only if \( I = J = \emptyset \) or \( I = \{1, \ldots, s+t\} \) and \( J = \{1, \ldots, t\} \), where \( \sigma_I \) and \( \sigma_J \) are defined on \( U \) as in Lemma 4.3.

**Remark 4.7.** If \( X \) carries a locally conformally Kähler metric, Condition (C) is automatically satisfied, as it is shown in the proof of [IO, Proposition 6.4].

By a straightforward computation that results from (4.10) being an equality, we also have the following:

**Corollary 4.8.** If \( X \) satisfies Condition (C), then
\[
\dim_{\mathbb{C}} H^q(X, \mathcal{O}) = \binom{s}{q} \text{ if } q \leq s, \quad \dim_{\mathbb{C}} H^q(X, \Omega^{s+t}) = \binom{s}{q-t} \text{ if } q > t
\]
and the rest of the Dolbeault cohomology groups are trivial.

**Remark 4.9.** In [Kas13] it is shown that any OT manifold \( X \) admits a solvmanifold presentation \( \Gamma \backslash G \), in such a way that the natural complex structure on \( G \) is \( G \)-left invariant. It is well known that the Lie algebra cohomology \( H^*(\mathfrak{g}) \) injects into \( H^*_{dR}(X) \). If \( X \) satisfies Condition (C), one can check that the generators given in [IO] are \( G \)-invariant, hence the inclusion morphism \( H^*(\mathfrak{g}) \to H^*_{dR}(X) \) is an isomorphism. This and the Hodge decomposition for \( X \) gives an isomorphism at the level of Dolbeault cohomologies \( H^{*,*} \mathfrak{g} \cong H^{*,*}(X) \).

**Corollary 4.10.** If \( X \) is of simple type, then \( H^0(X, \Omega^2) = 0 = H^1(X, \Omega^1) \) and \( \dim_{\mathbb{C}} H^2(X, \mathcal{O}) = \binom{s}{2} \).

**Proof.** By the proof of [OT05, Proposition 2.3], we deduce that if \( X \) is of simple type, then for any different indices \( i_1, i_2 \in \{1, \ldots, s+2t\} \), \( \sigma_{i_1} \sigma_{i_2} : U \to \mathbb{C}^* \) is not trivial and moreover, \( b_2 = \binom{s}{2} \). Therefore, using again (4.10) for \( l = 2 \), we obtain the stated dimensions. 

Finally by Corollary 4.6 and Remark 4.2 we get

**Corollary 4.11.** For an Inoue-Bombieri surface \( X \) one has \( \dim_{\mathbb{C}} H^1(X, \mathcal{O}) = 1 \).
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