Multi-level Qualitative Reasoning Logic

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Abstract. Qualitative reasoning is a very efficient method that people often use to solve problems. Recently, the literature about qualitative reasoning as a causal analysis and decision-making tool has been emerging. However, the existing qualitative reasoning methods are mainly used for the modeling of imprecise problems. There are not many studies on formalization tools of qualitative "inference". This paper proposes a logical system for qualitative reasoning (QRL). The main research contents include: the grammar and semantic structure of QRL are given; the meaning of formula logic truth value in QRL and its assignment rules are discussed; and the reliability and completeness of QRL are proved. Compared with fuzzy logic, probability logic and other uncertain reasoning methods, the advantage of QRL method is that it can use axiomatic reasoning method, and does not need to construct membership functions and collect a large number of samples.

1. Introduction

Qualitative reasoning is the most common and most effective method of reasoning. In the 1980s, the field of artificial intelligence began a wave of qualitative reasoning research. Typical methods include de Kleer's Qualitative Model Approach (QMA) [1,2], Forbus's Qualitative Process Theory (QPT) [3,4], and Kuipers' Qualitative Simulation (QSIM) [5,6]. Since then, qualitative reasoning has been further developed. For example, in the task planning of autonomous robots, simulation-based projection is used to predict the qualitative effects of robot manipulation. This qualitative reasoning method based on simulation-based prediction can explain the actions and their effects. By translating a qualitative physics problem into a parameterized simulation problem and performing a detailed physics-based simulation of a robot plan, the robot can infer what might happen when a task is performed in a certain way[7].

In recent years, qualitative reasoning has been used as a causal analysis and decision-making tool in social, ecological, educational, medical, business, economic and other fields. For example, Zitek et al.[8] discussed the application potential of qualitative reasoning models in sustainable management of river basins, and develops two models to capture important issues related to the sustainable...
development of Austrian River landscape. Scheer et al.[9] explored the public's perception of energy technology and investment portfolio in Germany by qualitative method, investigates the public's preference for power technology and investment portfolio in Germany, and qualitatively analyzes the views of non-professionals on informed risks and returns, reasoning patterns and judgments. Fernando et al.[10] established a qualitative reasoning model using non numerical knowledge and explicit representation of causality to reveal the impact of intensive management and traditional management on the bird community in a savanna forest in Celadon ecological region, Brazil. Kansou et al.[11] used qualitative reasoning to explain the impact of macroevolution of plants on long-term carbon cycle, and proposed a qualitative reasoning model to explain how plant growth in the Middle Paleozoic (450-300 million years ago) changed the long-term carbon cycle, leading to a sharp decline in global atmospheric carbon dioxide levels. Lee et al.[12] used the think-aloud method to qualitatively study the clinical reasoning strategies and reasoning processes used by nurses in solving the problems of patients with complex chronic diseases, to explore the clinical reasoning ability of nurses, and to determine the reasoning process. The reasoning process for a registered nurse includes the stages of assessment, analysis, diagnosis, planning, implementation, and evaluation. Hohensee et al.[13] examined the extent to which social factors influence students' reasoning about previously encountered concepts in the new learning process. This research can be used as a model to enhance the teaching of other mathematical topics and to obtain equally effective effects in the way of reasoning about previously encountered concepts.

From the above literature, it is easy to see that most of the existing qualitative reasoning methods are used in the qualitative analysis of causal relationship in modeling, while the qualitative reasoning after modeling is less, especially for the formal qualitative reasoning.

To this end, this paper proposes a multi-level qualitative reasoning logic for formalizing the process of human qualitative reasoning. The research of this paper will make up for the lack of formal research on qualitative reasoning.

2. The logical framework of QRL

2.1. Several basic concepts of QRL

**Definition1** Assuming that P is a declarative sentence describing the qualitative evaluation of something, and that P takes a value in a finite set of truth values T, then P is called a qualitative proposition.

In QRL, a qualitative proposition P contains two meanings: (1) P represents the narration of things; (2) the logical truth value \( \alpha_{i} \) of P stands for qualitative evaluation of things. Qualitative evaluation is multi-level and can be divided into two categories: positive evaluation and negative evaluation.
Definition 2 Let True = \{α₁, α₂, ⋯⋯, αₙ\}, standing for positive evaluation; False = \{β₁, β₂, ⋯⋯, βₙ\}, standing for negative evaluation; then TF = True \cup\ False represents the set of qualitative propositional truth values in QRL, and gives a ranking of elements in TF: α₁ > α₂ > ⋯⋯ > αₙ > βₙ > βₙ⁻¹ > ⋯⋯ > β₁

The logical connectives in QRL are the same as those in the classical propositional logic PL, they are: ¬, ∧, ∨, →, ↔. However, the meaning of ¬ and → in QRL is different from that in classical logic PL.

- In PL, ¬P represents the negation of P, and in QRL, ¬P represents the inverse evaluation of P.
- In PL, P → Q stands for a purely logical relationship, while in QRL, P → Q stands for a causal relationship, that is, the existence of P will cause the existence of Q. Therefore, cross-level reasoning has more abundant content.

The multi-level truth order structure of qualitative propositions in QRL is shown in Fig. 1.

2.2. Syntax and semantics of QRL

The syntax of QRL is the same as PL. The semantics of QRL are discussed below.

Definition 3 Let p be a propositional argument, assigning p a value in TF is called an assignment to the propositional argument p. The value assigned to p is called the true value of p and is denoted as V(p).

Definition 4 Let H be a set of some formulas in QRL, and S is a set of all the different propositional arguments (atomic formulas) contained in the formula in H, V: S → TF is a mapping from the set of atomic propositional formulas S to the set of truth values TF. For any compound formula A, B in H, the assignment of the formula is defined as follows:

1. First, assign a value to the propositional argument. If A ∈ S, the true value of A is V(A);
   Then assign a value to the compound formulas containing A, B in H:

   2. V(¬A) = \{β₁, \ldots, βₙ\}; V(A) = αᵢ; i = 1, 2, ⋯⋯, n
   3. V(A ∧ B) = min \{V(A), V(B)\}
   4. V(A ∨ B) = max \{V(A), V(B)\}
   5. V(A → B) = max \{V(¬A), V(B)\}
   6. V(A ↔ B) = min \{V(A → B), V(B → A)\}
2.3. Syntax and semantics of QRL

Since QRL is a multi-valued logic, some of the typical logical properties of binary logic, such as the law of excluded middle, do not hold in QRL. However, QRL still maintains some of the classic logical properties of PL.

**Theorem 1** For any formulas A, B, the following logical properties hold:

1. Double negation law: \( V(\neg\neg A) = V(A) \);
2. De Morgan Law: \( V(\neg(A \land B)) = V(\neg A \lor \neg B) \);
3. Equivalence Law of Inverse-negative Proposition: \( V(A \rightarrow B) = V(\neg B \rightarrow \neg A) \).

Now we turn to discuss some logical properties in QRL, which are is weaker than those in classical logic.

**Definition 5** Let A be a well-formed formula, then

1. A is called a generalized tautology if and only if for any assignment function V, \( V(A) \in \text{True} \);
2. A is called a generalized contradiction if and only if for any assignment function V, \( V(A) \in \text{False} \);
3. A is called a generalized satisfiability if and only if there exists at least one assignment function V such that \( V(A) \in \text{True} \).

**Theorem 2** For any formula A, if \( V(A) = \alpha_i \) or \( V(A) = \beta_i \),

1. \( V(A \lor \neg A) = \alpha_i \),
2. \( V(A \land \neg A) = \beta_i \).

As we all know, in multi-valued logic, the law of excluded middle and the law of contradiction does not hold. However, from Theorem 2 we have \( V(A \lor \neg A) \in \text{True} \) and \( V(A \land \neg A) \in \text{False} \) in QRL for any formula A. So, in QRL \( (A \lor \neg A) \) is a generalized tautology, and \( (A \land \neg A) \) is a generalized contradictory.

**Theorem 3** In QRL, the following formulas are generalized tautology:

1. \( A \land B \rightarrow A \);
2. \( A \rightarrow A \lor B \);
3. \( (A \rightarrow B) \land A \rightarrow B \);
4. \( (A \rightarrow B) \land (B \rightarrow C) \rightarrow (A \rightarrow C) \).

3. The Minimal Axiom System QRL₀

In this section, we will present the minimal axiom system of our logic, and prove its soundness and completeness.

3.1. Axioms and reasoning rules

(1) The following three axioms constitute the minimal axiom system QRL₀ of QRL:

\[
\begin{align*}
&\text{(Ax1)} \quad A \rightarrow (B \rightarrow A); \\
&\text{(Ax2)} \quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)); \\
&\text{(Ax3)} \quad (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A).
\end{align*}
\]

(2) The reasoning rule of QRL₀ is MP (Modus Ponens) reasoning rule:

If A and A → B, then B.

3.2. Some Logical Properties in QRL₀

**Definition 6** Let \( \Gamma \) be a set of formulas in QRL₀, a formula A is said to be provable from \( \Gamma \), denoted as \( \Gamma \vdash \text{QRL₀} \ A \), if A can be deduced from \( \Gamma \) with axioms or reasoning rules.

**Definition 7** Let \( \Gamma \) be a set of formulas, A be a formula in QRL₀. A is called a theorem, if \( \Gamma \vdash \text{QRL₀} \ A \) and \( \Gamma \) is empty. In this case, it is abbreviated as \( \vdash A \).

**Theorem 4** Axioms (Ax1), (Ax2), and (Ax3) in QRL₀ are generalized tautologies.

**Theorem 5** In QRL₀ if A and A → B are generalised tautologies, then B is also generalised tautologies.
3.3. Reliability of QRL₀
Although QRL₀ does not have the same reliability theorem as the classical propositional logic PL, QRL₀ still has a slightly weaker reliability theorem.

**Theorem 6** (Soundness) Any theorem in QRL₀ is generalized tautologies.

Proof: Let A be a theorem in QRL₀, and its proof is A₁, A₂, ⋯, Aₘ, where Aₘ=A. Applying mathematical induction on m, our derivation is as follows. When m=1, we know that A=A₁ is an axiom. Thus, by Theorem 4 we know that A is generalized tautologies.

Now assume that A₁, A₂, ⋯, Aₘ₋₁ are generalized tautologies. Then, by Definition 4, Aₘ is either an axiom or provable from previous formulas A₁, Aₖ (j, k<m) by using MP reasoning rule. If Aₘ is an axiom, by Theorem 4, Aₘ is generalised tautologies; if Aₘ is derived from previous two formulas, by Theorem 5, Aₘ is a generalized tautology.

3.4. Consistency of QRL₀
**Definition 8** supposes that QRL₀* is an axiomatic system containing all the axioms of QRL₀. If all the theorems in QRL₀ are still QRL₀* theorems, then QRL₀* is an extension of QRL₀.

**Definition 9** An axiom system QRL₀* is consistent, if there is no formula A in QRL₀*, such that both A and (¬A) are theorems of QRL₀*.

**Definition 10** Let QRL₀* be an extension of QRL₀. If there is no formula A of QRL₀ such that A and (¬A) both are theorem of QRL₀*, then QRL₀* is a consistent extension of QRL₀.

**Theorem 7** QRL₀ is consistent.

**Theorem 8** Let QRL₀* be a consistent extension of QRL₀, and let A be the formula of QRL₀ but not a theorem of QRL₀*. If (¬A) is added as an axiom to QRL₀* to get a new extension QRL₀**, then QRL₀** is also consistent.

3.5. Completeness of QRL₀

**Definition 11** Let QRL₀* be an extension of QRL₀. If for each formula A, either A or (¬A) is a theorem of QRL₀*, then QRL₀* is a complete extension.

**Theorem 9** Let QRL₀* be a consistent extension of QRL₀, then there is a consistent complete extension of QRL₀*.

Proof: Let A₀, A₁, A₂, ⋯ be an enumeration of all the formulas of QRL₀. We construct an extended sequence of QRL₀*, J₀, J₁, J₂, ⋯, as follows:

Let J₀=QRL₀*.

For J₁, if ⊢ₜ QRL₀* A₀, then J₁=J₀;

if A₀ is not a theorem of J₀, then adding (¬A₀) to the axioms of J₀ constitutes the axioms of J₁.

For any Jₙ (n≥1), its axioms is constructed as follows: if ⊢ₜ Jₙ₋₁ Aₙ₋₁, then Jₙ=Jₙ₋₁; otherwise, the axioms of Jₙ is obtained by adding (¬Aₙ₋₁) to the axioms of Jₙ₋₁.

By hypothesis, QRL₀* is consistent, that is, J₀ is consistent. For n ≥ 1, if Jₙ₋₁ is consistent, then according to our method of constructing Jₙ and Theorem 8, Jₙ is consistent. So by induction, each Jₙ is consistent.

Let J be an extension of QRL₀*, and all the axioms of Jₙ are included in the axioms of J.

Next we prove that J is a consistent and complete extension of QRL₀*.

1) Prove that J is consistent.

Suppose J is inconsistent. Then there exists formula A such that ⊢ₜ QRL₀ A and ⊢ₜ QRL₀ ¬A hold.

Since the proofs of A and (¬A) in J are both finite-length formula sequences, and the number of axioms that appear in the proofs is also finite, so there must be a sufficiently large n such that these formulas are included in Jₙ. Thus, we have ⊢ₜ Jₙ A and ⊢ₜ Jₙ ¬A. This contradicts the fact that J has been proved to be consistent.

2) Prove that J is complete.

Let A be a formula of QRL₀. By the construction of J, A must appear in the sequence A₀, A₁, A₂, ⋯. Without loss of generality, assume A=Aₖ. If ⊢ₜ Jₖ Aₖ, then ⊢ₜ J Aₖ, because the axioms in Jₖ are all in J.
If \( \vdash_{J} A_{k} \) is not true, then \( \vdash_{J_{k+1}} \neg A_{k} \), thus \( \vdash_{J} \neg A_{k} \). So, in any case, for any formula \( A \), either \( \vdash_{J} A \) or \( \vdash_{J} \neg A \), so \( J \) is complete.

**Theorem 10** (completeness theorem of \( \text{QRL}_0 \)) If \( A \) is a formula of \( \text{QRL}_0 \) and is a generalized tautology, then \( \vdash_{\text{QRL}_0} A \).

Proof: Let \( J \) be a completely consistent extension of \( \text{QRL}_0 \). The assignment function \( v \) is defined on the formula of \( \text{QRL}_0 \) as follows: if \( \vdash_{J} B \), then \( v(B) \in \text{True} \), otherwise \( v(B) \in \text{False} \). Let \( A \) be a formula of \( \text{QRL}_0 \) and be a generalized tautology. If \( A \) is not a theorem of \( \text{QRL}_0 \), since \( J \) is a complete extension of \( \text{QRL}_0 \), so \( (\neg A) \) is a theorem of \( J \).

According to the definition of the above assignment function \( v \), we have \( v(\neg A) \in \text{True} \). On the other hand, by the known conditions in this theorem, \( A \) is a generalized tautology, so we have \( v(A) \in \text{True} \), which contradicts that \( J \) is a consistent extension of \( \text{QRL}_0 \).

So far, we have proved the reliability and completeness of \( \text{QRL}_0 \). So we get the following conclusion: \( A \) is a theorem of \( \text{QRL}_0 \) if and only if it is a generalized tautology of \( \text{QRL}_0 \). It shows that \( \text{QRL} \) is successful in extending qualitative reasoning to a multi-level logical platform.

4. Conclusion
Based on the work in this article, we draw the following conclusions:

1. \( \text{QRL} \) is a universal qualitative reasoning logic that can be used to formalize the multi-level evaluation and reasoning of qualitative analysis.

2. \( \text{QRL} \) can be axiomatic, with logical properties of reliability and completeness, suitable for automatic reasoning.

3. \( \text{QRL} \) does not require large-scale experimental data samples to train models and membership functions in the modeling and inference process, thereby making up for some of the deficiencies of probabilistic logic and fuzzy logic.

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