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Fractal-fractional order mathematical vaccine model of COVID-19 under non-singular kernel

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A R T I C L E   I N F O

Article history:
Received 8 May 2021
Revised 30 May 2021
Accepted 7 June 2021
Available online 12 June 2021

Keywords:
COVID-19
Existence result
Atangana-Baleanu fractal-fractional derivative
Fractal-fractional Adam-Bashforth method

A B S T R A C T

In this paper, the severe acute respiratory syndrome coronavirus (SARS-CoV-2) or COVID-19 is researched by employing mathematical analysis under modern calculus. In this context, the dynamical behavior of an arbitrary order p and fractal dimensional q problem of COVID-19 under Atangana Baleanu Caputo (ABC) operator for the three cities, namely, Santos, Campinas, and Sao Paulo of Brazil are investigated as a case-study. The considered problem is analyzed for at least one solution and unique type by the applications of the theorems of fixed point and non-linear functional analysis. The Ulam-Hyers stability condition via nonlinear functional analysis for the given system is derived. In order to perform the numerical simulation, a two-step fractional type, Lagrange polynomial (Adams Bashforth technique) is utilized for the present system. MATLAB simulation tools have been used for testing different fractal fractional orders considering the data of aforementioned three regions. The analysis of the results finally infer that, for all these three regions, the smaller order values provide better constraints than the larger order values.

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1. Introduction

In January 2020, Wuhan Municipal Health Commission of China discovered the Severe Acute Respiratory Syndrome Corona Virus 2 (SARS-CoV-2), or Covid-19. The World Health Organization (WHO) announced a public health emergency of international concern. A few weeks later, this Covid-19 was declared as a global pandemic [11,12]. At the initial onset, it was quite challenging for the global health care agencies to overcome the newly arose COVID-19 pandemic. This rapidly spreading disease is a big challenge for the academic institutions and industries to develop and deploy efficient drug treatments and vaccination. Several drugs have been tried in patients with Covid-19 disease. As a result, scientists have discovered that drugs have some or no effect on the overall mortality, steps taken off ventilation, times in hospital, or viral clearance even for patients with chronic use of some of the tried drugs [3–7].

On the other hand, non-pharmaceutical interventions (NPI), such as physical distancing, use of face masks and eye protection, reduce virus transmission [8,9]. Worldwide populations have been compliant with NPI, however, in some countries, there is a controversy over its effectiveness [10]. In Brazil, the closure of non-essential activities lasted only for a short time, and the cities lifted NPI in an uncoordinated manner [9]. As a result, at the beginning of November 2020, the downward trend reversed and the number of Covid-19 cases started to rise in the second wave of infection.

Discovering the vaccines for such contagions counts years of continuous struggle, even decades, and furthermore, it is also not cost-effective. The fast development of Covid-19 vaccines was only possible since researchers had worked for years on vaccines for similar viruses, such as SARS (Severe Acute Respiratory Syndrome) and MERS (Middle East Respiratory Syndrome). In addition, the experience gained with the Ebola vaccine showed that the development of new vaccines can be accelerated with a worldwide effort, including academic institutions, industry and health care regulatory agencies, without compromising safety [11,12]. The Covid-19 pandemic, and the consequent urge for vaccination, started a race, with countries trying to vaccinate populations as fast as possible. Moreover, at the current pace, and for most countries, a successful vaccination strategy consists of a long journey ahead. In this paper, we reconsider the vaccine model [13] and use fractal-fractional analysis to describe its dynamics. The considered model is divided into four classes namely susceptible class S, infected class I, infected symptomatic positive tested Sk and a recovered class R.

\[
\begin{align*}
\frac{dS}{dt} &= \mu N - \alpha I - \frac{\alpha (1 - \theta) S(t) I(t)}{N} - \mu S(t) + \gamma R(t), \\
\frac{dI}{dt} &= \alpha (1 - \theta) S(t) I(t) - (q_1 + q_2) I(t) - \mu I(t), \\
\frac{dSk}{dt} &= q_2 I(t) - q_3 Sk(t) - \mu Sk(t).
\end{align*}
\]
\[ \frac{dR}{dt} = q_i(t) + q_3S_{sk}(t) - \gamma R(t) - \mu R(t). \]  

(1)

In system (1), the effect of social distancing measure, which is a form of NPI, is introduced by the factor \( \theta \in (0, 1) \). Where \( \theta = 0 \) means that no social distancing measure is under consideration, and \( \theta = 1 \) means a complete lockdown. Normalization with respect to the population size \( N \) is performed as:

\[ I = S(t) + I(t) + S_{sk}(t) + R(t). \]  

(2)

where \( S(t) = S(t)/N \), \( I(t) = I(t)/N \), \( S_{sk}(t) = S_{sk}(t)/N \) and \( R(t) = R(t)/N \).

Replacing (2) into model (1), we obtain

\[ \frac{ds}{dt} = \mu - \alpha (1 - \theta) s - \mu s + \gamma r \]

\[ \frac{di}{dt} = \alpha (1 - \theta) s - (q_1 + q_2)i - \mu i \]

\[ \frac{ds_{sk}}{dt} = q_2 i - q_3 S_{sk} - \mu s_{sk} \]

\[ \frac{dr}{dt} = q_1 i + q_3 S_{sk} - \gamma r - \mu r. \]  

(3)

Analyzing the proposed model in fractional order derivative along with the fractal dimension of Caputo sense because it gives many more good results than integer-order. Providing such type of results makes fractional calculus superior to integer order calculus. Therefore, the physical and biological problems may be investigated with the general fractional derivative operators on every and very low order having more degree of choice. The scientists are interested to work on fractional derivatives as compared to classical differentiation. So for the analysis is composed of a theory of existence and uniqueness, stability, feasibility and approximate solution of fractional order equations [14–18].

Non-integer order calculus has provided the information of the spectrum at a fractional or rational value between the integer orders [19,20]. The representation of different real phenomena have formulated by fractional order integral or differential equation like mathematical fractional order model for microorganism population, a logistic non-linear model for the human population, tuberculosis model, dirung population, hepatitis B, C models and the basic Lotka-Volterra models being the basis of all infectious problems [21–26]. The aforesaid problems have been analyzed for qualitative analysis with help of some well-known theorems of fixed point theory [27–30]. The feasibility and stability analyses have also been done through various theorems. Further, the FODEs have been checked for an analytical, semi-analytical and approximate solution using different techniques. Some of the well-known techniques were of Euler, Taylor, Adams-Bashforth, predictor-corrector, various transforms, etc [14,22,31–34]. Here we are going to analyze our proposed fractal fraction order model in sense of ABC derivative, existence, uniqueness of the solution and approximate solution by theory of fixed point and Adams-Bashforth techniques.

Atanagica presented a new concept of arbitrary order and fractal dimensional integral and differential operator and proved the interaction between fractional calculus and fractal calculus [35]. This new operator includes two orders one is fractal dimension and the other one is fractional order. The researcher studied and investigated that the fractal-fractional order operator is better to look at the mathematical model for real-world data. Several mathematical models have been investigated under this new operator such as transmission behavior of malaria disease, dynamics of the polluted lakes system, COVID-19 mathematical model, etc [36–41].

This paper aims at analyzing the newly constructed system (4) by the application of the derivative of fractal dimension and arbitrary order in forms of ABC, due to its much well result than integer-order. We will show the existence, uniqueness and stability results for the considered model, by using fixed point theory.

A mathematical model introduced in [13], is studied in fractal-fractional order derivative under ABC sense with fractal dimension \( q \) and fractional order \( p \) as

\[
\begin{align*}
\text{ABC}^D_0^p S(t) & = -\alpha (1 - \theta) s(t) - q_3 S_{sk}(t) - \mu S(t) + \gamma r(t), \\
\text{ABC}^D_0^p I(t) & = \alpha (1 - \theta) s(t) - (q_1 + q_2) i(t) - \mu i(t), \\
\text{ABC}^D_0^p S_{sk}(t) & = q_2 i(t) - q_3 S_{sk}(t) - \mu S_{sk}(t), \\
\text{ABC}^D_0^p R(t) & = q_1 i(t) + q_3 S_{sk}(t) - \gamma r(t) - \mu r(t).
\end{align*}
\]

(4)

with initial condition:

\[ S(0) = s_0, \quad i(0) = i_0, \quad s(0) = s_{sk}(0), \quad r(0) = r_0. \]

The rest of the paper we organized as follows. In Section 2, some fundamental results are recalled. We present theoretical results related to the existence, uniqueness of the solution and stability in Section 3. Section 4 is about numeric and plots for the considered model with the help of Matlab. In the last, a brief concluding is presented in part Section 5.

2. Fundamental results

**Definition 1.** [42] Suppose a defined mapping \( \phi(t) \) on \((a, b)\) having \( 0 < q \leq 1 \) and \( 0 < q \leq 1 \) be non-integer order and fractal dimension in ABC form is

\[
\text{ABC}^D_0^p \phi(t) = \frac{d}{dt} \left( \int_0^t \frac{\sigma(t)}{1 - p} \left( t - s \right)^{p-1} \phi(s) ds \right).
\]

where \( M(p) \) is called a normalizing constant and \( M(0) = M(1) = 1 \).

**Definition 2.** [42] Consider a continuous function \( \phi(t) \) in \((a, b)\) with \( 0 < q, p \leq 1 \) dimensional and arbitrary order, then integral of \( \phi(t) \) in ABC form is:

\[
\text{ABC}^D_0^p \left( \int_0^t \phi(s) ds \right) = \frac{1 - p}{M(p)} \phi(t) + \frac{q p}{M(p) 1 - p} \int_0^t (t - s)^{p-1} \phi(s) ds.
\]

**Lemma 1.** [43] The antiderivative of fractional order for the problem having \( 0 < p, q \leq 1 \) is as

\[
\text{ABC}^D_0^p \phi(t) = q t^{q-1} W(t, \phi(t)), \quad t \in [0, T],
\]

\[ \phi(0) = \phi_0, \quad 0 < p, q \leq 1, \]

is provided by

\[
\phi(t) = \phi_0 + \frac{(1 - p)}{M(p)} t^{q-1} W(t, \phi(t)) + \frac{q p}{M(p) M(p)} \int_0^t (t - s)^{p-1} s^{q-1} W(s, \phi(s)) ds.
\]

**Note:** For at least one and unique solution for the proposed problem, close norm space may be taken

\[
\psi = W = V([0, T] \times \mathbb{R}^4, \mathbb{R})
\]

where \( W = V[0, T] \) with norm in space is

\[
\|U\| = \|\phi\| = \max_{i=0}^T \left[ |s(t)| + |i(t)| + |S_{sk}(t)| + |r(t)| \right]
\]

**Theorem 1.** [44] Let \( A \subset Z \), which is convexed and chose the two functions \( F_1, F_2 \) with

1. \( F_1(w) + F_2(w) \in A \) for each \( w \in A \);
2. \( F_1 \) contraction;
3. A compact and continuous set is \( F_2 \).

having the operator equations \( F_1 w + F_2 w = w \) has one or more than one solution.
3. Theoretical approach

In the on going part, the idea for at least one and unique solution will be derived by the application of the fixed point theory for (4). As the integral is differentiable so the proposed model (4) can be rewritten as

\[
\begin{align*}
\text{ABC}_D^p(s(t)) &= q t^{q-1} Y_1(s, i, s_{ick}, r), \\
\text{ABC}_D^p(i(t)) &= q t^{q-1} Y_2(s, i, s_{ick}, r), \\
\text{ABC}_D^p(s_{ick}(t)) &= q t^{q-1} Y_3(s, i, s_{ick}, r), \\
\text{ABC}_D^p(r(t)) &= q t^{q-1} Y_4(s, i, s_{ick}, r),
\end{align*}
\]

(6)

where

\[
\begin{align*}
Y_1(s, i, s_{ick}, r) &= \mu - \alpha (1 - \theta) s - \mu s + \gamma r, \\
Y_2(s, i, s_{ick}, r) &= \alpha (1 - \theta) s - (q_1 + q_2) s - \mu_i, \\
Y_3(s, i, s_{ick}, r) &= q_2 s - q_3 s_{ick} - \mu s_{ick}, \\
Y_4(s, i, s_{ick}, r) &= q_4 s_{ick} - q_5 s_{ick} - \gamma r - \mu r.
\end{align*}
\]

With the help of (6) and for \( t \in \phi \), the considered model (7) can be written as

\[
\begin{align*}
\text{ABC}_D^p(Y(t)) &= q t^{q-1} W(t, \phi(t)), \quad t \in [0, T], \\
\phi(0) &= 0, \quad 0 < p, q \leq 1,
\end{align*}
\]

(8)

with solution

\[
\begin{align*}
\phi(t) &= \psi_0 + \left( \frac{1 - p}{M(p)} \right) t^{q-1} W(t, \phi(t)) \\
&\quad + \frac{q p}{M(p) \Gamma(p)} \int_0^t (t - s)^{p-1} q s^{q-1} W(s, \phi(s)) \, ds,
\end{align*}
\]

(9)

where

\[
\begin{align*}
\phi(t) &= (s(t), i(t), s_{ick}(t), r(t))^T, \\
\psi_0(t) &= (s_0(t), i(t), s_{ick}(0), r_0(t))^T, \\
i(t, \phi(t)) &= P(t, i, s_{ick}(t))^T, \quad i = 1, 2, 3, 4.
\end{align*}
\]

(10)

Here, variate (4) into the unique point and define an operator \( T : Q \to Q \) as follows:

\[
T \phi(t) = \psi_0 + \left( \frac{1 - p}{M(p)} \right) t^{q-1} W(t, \phi(t)) \\
&\quad + \frac{q p}{M(p) \Gamma(p)} \int_0^t (t - s)^{p-1} q s^{q-1} W(s, \phi(s)) \, ds.
\]

(11)

Assume the operators as

\[
T = X + N,
\]

where

\[
X(\phi) = \psi_0 + \left( \frac{1 - p}{M(p)} \right) t^{q-1} W(t, \phi(t)),
\]

\[
N(\phi) = \frac{q p}{M(p) \Gamma(p)} \int_0^t (t - s)^{p-1} q s^{q-1} W(s, \phi(s)) \, ds.
\]

(12)

Next, we have to prove at least one and unique solution for problem (4).

(J1) \( \exists \) a constants \( \gamma_W, h_W, \ \exists \ \gamma_W \) such that \( W(t, \phi(t)) \leq \gamma_W |\phi| + h_W \).

(J2) \( \exists \) constants \( L_W \geq 0 \) and for every \( \phi, \tilde{\phi} \in \phi \)

\[
|W(t, \phi) - W(t, \tilde{\phi})| \leq L_W |\phi - \tilde{\phi}|.
\]

Theorem 2. The system (9) has at least one solution if (J1) and (J2) holds, then the considered system (4) also has unique solution if

\[
\left( \frac{1 - p}{M(p)} \right) t^{q-1} \leq 1.
\]

Proof. First we will prove that \( X \) is contraction by satisfying the Banach contraction theorem. For this, let \( \phi \in A \), where \( A = \{ \phi \in \phi : \| \phi \| \leq \psi, \psi > 0 \} \) is convex bounded set. The defined mapping in (12) is used, we get

\[
\begin{align*}
|X(\phi) - X(\tilde{\phi})| &= \left( \frac{1 - p}{M(p)} \right) t^{q-1} \max_{t \in [0, T]} |W(t, \phi(t)) - W(t, \tilde{\phi}(t))| \\
&\leq \left( \frac{1 - p}{M(p)} \right) t^{q-1} L_W |\phi - \tilde{\phi}|.
\end{align*}
\]

(13)

Therefore, the mapping \( X \) is bounded and hence contracted.

Next, we have to derive that the operator \( N \) is bounded and continuous (compactness) in comparing version, we also derive that \( N \) is compact. It is verified that the mapping \( N \) is continuous on \( W \) for any \( \phi \in A \), we have

\[
\begin{align*}
\|N(\phi)\| &= \max_{t \in [0, T]} \left\{ q p M(p) \Gamma(p) \int_0^t (t - s)^{p-1} q s^{q-1} W(s, \phi(s)) \, ds \right\} \\
&\leq q p M(p) \Gamma(p) \int_0^t (s)^{p-1} (1 - s)^{-1} W(s, \phi(s)) \, ds \\
&\leq q p M(p) \Gamma(p) \frac{q p}{M(p) \Gamma(p)} \left[ B(p, q) \right].
\end{align*}
\]

(14)

So by (14) the mapping \( N \) having bounds so closed, for equi-continuity take \( t_1 > t_2 \in [0, T] \),

\[
\begin{align*}
\|N(\phi(t_2)) - N(\phi(t_1))\| &= \left\{ q p M(p) \Gamma(p) \int_0^t (t_2 - x)^{p-1} x^{q-1} W(x, \phi(x)) \, dx \right. \\
&- \left. \int_0^t (t_1 - x)^{p-1} x^{q-1} W(x, \phi(x)) \, dx \right\} \\
&\leq q p B(p, q) \left( t_2^p - t_1^p \right).
\end{align*}
\]

(15)

As \( t_2 \to t_1 \), right hand side of (15) tends to zero. Also the defined mapping \( N \) so

\[
\|N(\phi(t_2)) - N(\phi(t_1))\| \to 0, \quad \text{as} \quad t_2 \to t_1.
\]

Hence we proved that \( N \) is continuous and bounded so \( N \) is uniform continuous and having bounds. Thus by the result of Arzelà-Ascoli, \( N \) shows relatively compactness and implies complete continuity. From (9) and (4) we derived that consider problem has at least one solution. \( \square \)

In the next theorem we will show the uniqueness.

Theorem 3. With the assumption (U2), the integral (9) has a unique solution, for this, let us consider the model (4) also has a unique solution if

\[
\left( \frac{1 - p}{M(p)} \right) t^{q-1} \leq \left( \frac{q p}{M(p) \Gamma(p)} \right) \left[ B(p, q) \right] < 1.
\]

Proof. Suppose an operator \( T : \phi \to \Phi \) by

\[
\begin{align*}
T \phi(t) &= \psi_0(t) + \left[ W(t, \phi(t)) - W_0(t) \right] \left( \frac{1 - p}{M(p)} \right) t^{q-1} \\
&\quad + \frac{q p}{M(p) \Gamma(p)} \int_0^t (t - x)^{p-1} x^{q-1} W(x, \phi(x)) \, dx, \quad t \in [0, T].
\end{align*}
\]

(16)

Let \( \phi, \tilde{\phi} \in \phi \), then

\[
\begin{align*}
\|T \phi - T \tilde{\phi}\| &\leq \left( \frac{1 - p}{M(p)} \right) t^{q-1} \max_{t \in [0, T]} \left| W(t, \phi(t)) - W(t, \tilde{\phi}(t)) \right| \\
&\quad + \frac{q p}{M(p) \Gamma(p)} \int_0^t (t - x)^{p-1} t^{q-1} W(x, \phi(x)) \, dx \\
&\quad - \int_0^t (t - x)^{p-1} t^{q-1} W(x, \tilde{\phi}(x)) \, dx \\
&\leq \Theta \| \phi - \tilde{\phi} \|.
\end{align*}
\]

(17)
and
\[
\Theta = \left[ \frac{(1-p) r^{q-1} L_w}{M(p)} + \frac{q [L_w T^{p+q-1}] B(p, q)] L_w}{M(p) p(p)} \right].
\]

From (17), if is contracted, so (9) has one root. Therefore, the model (4) has one root. □

3.1. Ulam-Hyer (UH) stability

In this section we have to derive the stability for system (4) by taking a small variation \( \Phi(t) \in C[0, T] \) and fulfil \( \Phi(0) = 0 \) as

- \( |\Psi(t)| \leq \varepsilon \) for \( \varepsilon > 0 \);
- \( ABC^\alpha\phi(t) = \Psi(t, \phi(t)) + \Psi(t) \).

Lemma 2. The change system solution is \( ABC^\alpha\phi(t) = \Psi(t, \phi(t)) + \Phi(t) \),
\[
\phi(0) = \phi_0,
\]
which satisfies
\[
\left| \phi(t) - \left[ \phi_0(t) + \left[ \Psi(t, \phi(t)) - \Psi_0(t) \right] \left[ (1-p) \frac{M(p)}{M(p)} r^{q-1} + \frac{q M(p) p(p)}{M(p)} \right] \int_0^t (t-x)^p x^{q-1} \Psi(x, \phi(x)) dx \right] \right|
\leq \Gamma(p) s^{q-1} + q T^{p+q-1} \frac{1}{M(p) p(p)} B(p, q) \varepsilon.
\]

Theorem 4. By result (U2) along with (20), root of (9) is Ulam-Hyer's stable and consequently, the root for the system is Ulam-Hyer's stable if \( \Theta < 1 \).

Proof. Consider a unique solution be \( \phi \in \Phi \) and \( \tilde{\phi} \in \Phi \) be root of (9), then
\[
|\phi(t) - \tilde{\phi}(t)| = \left| \phi(t) - \left[ \phi_0(t) + \left[ \Psi(t, \phi(t)) - \Psi_0(t) \right] \left[ (1-p) \frac{M(p)}{M(p)} r^{q-1} + \frac{q M(p) p(p)}{M(p)} \right] \int_0^t (t-x)^p x^{q-1} \Psi(x, \phi(x)) dx \right] \right|
\leq \Delta_x + \frac{1}{M(p) p(p)} \int_0^t \left| \left( \phi_0(t) + \left[ \Psi(t, \phi(t)) - \Psi_0(t) \right] \left[ (1-p) \frac{M(p)}{M(p)} r^{q-1} + \frac{q M(p) p(p)}{M(p)} \right] \int_0^t (t-x)^p x^{q-1} \Psi(x, \phi(x)) dx \right) \right|
\leq \Delta_x + \Theta \psi \| \phi - \tilde{\phi} \|
\leq \Delta_x + \Theta \psi \| \phi - \tilde{\phi} \|.
\]

From (21), we can write as
\[
\| \phi - \tilde{\phi} \| \leq \psi \varepsilon_{\psi \psi}.
\]

where \( \psi_{\psi \psi} = \frac{1}{\Theta} \theta_{\psi \psi} \). From (22), we conclude that the root of (9) is Ulam-Hyer's stable and by their implication is the generalized Ulam-Hyer's stable by applying \( \psi_{\psi \psi}(\varepsilon) = \varepsilon_{\psi \psi} \varepsilon, \psi_{\psi \psi}(0) = 0 \), which shows that the solution of the proposed problem is Ulam-Hyers stable and also generalized UH stable. □

4. Approximate solution

In analysis of this part, we will construct an approximate solutions by using the fractional Adam Bashforth technique with two step algebraic expression and the law of power kernel to complete the numerical scheme for the proposed model (4), we go further with (6), defined as:

\[
\begin{align*}
ABC^\alpha [s(t)] &= q t^{q-1} Y_1[s(t), t], \\
 ABC^\alpha [i(t)] &= q t^{q-1} Y_2[i(t), t], \\
 ABC^\alpha [s_{ik}(t)] &= q t^{q-1} Y_3[s_{ik}(t), t], \\
 ABC^\alpha [r(t)] &= q t^{q-1} Y_4[r(t), t].
\end{align*}
\]

where \( Y_1, Y_2, Y_3, \) and \( Y_4 \) which is given in (7) By applying fractal-fractional integration to 1st of (6) in sense of ABC, we have
\[
s(t) - s_0 = \left[ (1-p) \frac{M(p)}{M(p)} t^{q-1} \right] Y_1[s(t), t] + \frac{q M(p) p(p)}{M(p) p(p)} \int_0^t (t-x)^p x^{q-1} Y_1[s(x), x] dx.
\]
Set $t = t_{e+1}$ for $i = 0, 1, 2 \ldots$.

$$s(t_{e+1}) - s_0 = \frac{(1-p)}{M(p)} \left( t_{e+1}^{q-1} - 1 \right) \left[ Y_1(s(t_e), t_e) \right] + \frac{q p}{M(p) \Gamma(p)} \int_{0}^{t_{e+1}} (t_{e+1} - x)^{p-1} x^{q-1} Y_1(s(x), x) \, dx,$$

$$= \frac{(1-p)}{M(p)} \left( t_{e+1}^{q-1} - 1 \right) \left[ Y_1(s(t_e), t_e) \right] + \frac{q p}{M(p) \Gamma(p)} \sum_{e=0}^{c} \int_{t_e}^{t_{e+1}} (t_{e+1} - x)^{p-1} x^{q-1} Y_1(s(x), x) \, dx.$$

Now, the function $Y_1$ if we approximate it on the interval $[t_e, t_{e+1}]$ through the interpolation polynomial as follows

$$Y_1 \approx \frac{Y_1}{\Delta}(t - t_e) - \frac{R_1}{\Delta}(t - t_e),$$

which implies that

$$s(t_{e+1}) = s_0 + \frac{(1-p)}{M(p)} \left( t_{e+1}^{q-1} - 1 \right) \left[ Y_1(s(t_e), t_e) \right] + \frac{q p}{M(p) \Gamma(p)} \sum_{e=0}^{c} \left( \frac{Y_1(s(t_e), t_e)}{\Delta} \right) \int_{t_e}^{t_{e+1}} (t_{e+1} - t)^{p-1} t_{e+1}^{q-1} \, dt - \frac{R_1}{\Delta} \int_{t_e}^{t_{e+1}} (t - t_e) (t_{e+1} - t)^{p-1} t_{e+1}^{q-1} \, dt.$$

Calculating $I_{e-1,p}$ and $I_{e,p}$ we get

$$I_{e-1,p} = \int_{t_e}^{t_{e+1}} (t - t_e) (t_{e+1} - t)^{p-1} \, dt,$$

$$= \frac{1}{p} \left[ (t_{e+1} - t_e) (t_{e+1} - t_e)^{p} - (t_e - t_e) (t_{e+1} - t_e)^{p} \right]$$

$$- \frac{1}{p(p-1)} \left[ (t_{e+1} - t_e) (t_{e+1} - t_e)^{p} - (t_{e+1} - t_e)^{p+1} \right].$$

and

$$I_{e,p} = \int_{t_e}^{t_{e+1}} (t - t_e) (t_{e+1} - t)^{p-1} \, dt,$$

$$= \frac{1}{p} \left[ (t_{e+1} - t_e) (t_{e+1} - t_e)^{p} \right]$$

$$- \frac{1}{p(p-1)} \left[ (t_{e+1} - t_e) (t_{e+1} - t_e)^{p} - (t_{e+1} - t_e)^{p+1} \right].$$

put $t_e = e \Delta$, we get

$$I_{e-1,p} = -\frac{\Delta^{p+1}}{p} \left[ (e+1 - (e-1)) (c+1 - (e+1))^p - (e - (e-1))(1+c-e)^p \right]$$

$$- \frac{\Delta^{p+1}}{p (p-1)} \left[ (1+c-(r+1))^{p+1} - (1+c-e)^{p+1} \right],$$

$$= \frac{\Delta^{p+1}}{p (p-1)} \left[ -2 (p+1) (c-e)^p + (p+1) (1+c-e)^p - (c-e)^{p+1} + (1+c-e)^{p+1} \right],$$

$$= \frac{\Delta^{p+1}}{p (p-1)} \left[ -(c-e)^p (2p+1) - (c-e) + (1+c-e)^p (p+1) + 1+c-e \right],$$

$$= \frac{\Delta^{p+1}}{p (p-1)} \left[ (1+c-e)^p (c-e+2+p) - (c-e)^p (c-e+2+2p) \right].$$

(25)

and

$$I_{e,p} = -\frac{\Delta^{p+1}}{p} \left[ (e+1 - e) (1+c-(e+1))^p \right] - \frac{\Delta^{p+1}}{p (p-1)} \left[ (1+c-(e+1))^{p+1} - (1+c-e)^{p+1} \right],$$

$$= \frac{\Delta^{p+1}}{p (p-1)} \left[ -(p+1) (c-e)^p - (c-e)^{p+1} + (1+c-e)^{p+1} \right],$$

$$= \frac{\Delta^{p+1}}{p (p-1)} \left[ (c-e)^p(-(e+1) - (c-e)) + (1+c-e)^{p+1} \right],$$

$$= \frac{\Delta^{p+1}}{p (p-1)} \left[ (1+c-e)^p (c-e+1+p) \right].$$

(26)
substituting the values of (25) and (26) in (24), we get

\[
s(t_{1+c}) = \left\{ \begin{array}{ll}
  s_0 + \frac{(1-p)}{M(p)} t_{1+c}^{q-1} \left[ Y_1(s(t_c), t_c) \right] + \frac{q p}{M(p)} \sum_{e=0}^{c-1} \left( \Delta^{\epsilon+1} Y_1(s(t_c), t_c) \right) \\
  \frac{t_{1+c}^{q-1} Y_1(s(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 2 + p) - (c - e)^p (c - e + 2 + 2p) \right] \\
  \frac{t_{1+c}^{q-1} Y_1(s(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 1 + p) \right] \end{array} \right. 
\]  

(27)

Similarly for the remaining three classes i, s_{ik} and r we obtain the scheme as

\[
i(t_{1+c}) = \left\{ \begin{array}{ll}
  i_0 + \frac{(1-p)}{M(p)} t_{1+c}^{q-1} \left[ Y_2(i(t_c), t_c) \right] + \frac{q p}{M(p)} \sum_{e=0}^{c-1} \left( \Delta^{\epsilon+1} Y_2(i(t_c), t_c) \right) \\
  \frac{t_{1+c}^{q-1} Y_2(i(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 2 + p) - (c - e)^p (c - e + 2 + 2p) \right] \\
  \frac{t_{1+c}^{q-1} Y_2(i(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 1 + p) \right] \end{array} \right. 
\]  

(28)

\[
s_{ik}(t_{1+c}) = \left\{ \begin{array}{ll}
  s_{ik0} + \frac{(1-p)}{M(p)} t_{1+c}^{q-1} \left[ Y_3(s_{ik}(t_c), t_c) \right] + \frac{q p}{M(p)} \sum_{e=0}^{c-1} \left( \Delta^{\epsilon+1} Y_3(s_{ik}(t_c), t_c) \right) \\
  \frac{t_{1+c}^{q-1} Y_3(s_{ik}(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 2 + p) - (c - e)^p (c - e + 2 + 2p) \right] \\
  \frac{t_{1+c}^{q-1} Y_3(s_{ik}(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 1 + p) \right] \end{array} \right. 
\]  

(29)

\[
r(t_{1+c}) = \left\{ \begin{array}{ll}
  r_0 + \frac{(1-p)}{M(p)} t_{1+c}^{q-1} \left[ Y_4(r(t_c), t_c) \right] + \frac{q p}{M(p)} \sum_{e=0}^{c-1} \left( \Delta^{\epsilon+1} Y_4(r(t_c), t_c) \right) \\
  \frac{t_{1+c}^{q-1} Y_4(r(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 2 + p) - (c - e)^p (c - e + 2 + 2p) \right] \\
  \frac{t_{1+c}^{q-1} Y_4(r(t_c), t_c)}{\Delta} \left[ (1 + c - e)^p (c - e + 1 + p) \right] \end{array} \right. 
\]  

(30)

Fig. 1. Dynamics of all four compartment s(t), i(t), s_{ik}(t), r(t) of the consider model (4) at various fractal dimension q and arbitrary order p for the city of Santos.
4.1. Numerical simulation and discussion

Here, we discuss the simulation of the obtained numerical scheme for São Paulo, Santos and Campinas, three major cities in the State of São Paulo connecting the shores with the interior of the São Paulo state without vaccination. The four compartment of the proposed model are susceptible population $s(t)$, infected population $i(t)$, sick population $s_{sk}(t)$ and recovered population $r(t)$, simulated for each of the three cities against the available data given in Tables 1–3. The social distance or isolation index parameter $\theta$ has a great impact on the infection.

Fig. 1–d are the situation of covid-19 for all the four compartments of the consider deterministic fractal-fractional model for an about ten months in the city of Santos at various fractional order without vaccination. The susceptible cases declines as infected and sick individuals increases, the recovered cases also grows up with the passage of time. All the compartmental simulation are then going toward their disease equilibrium point and the situation of covid-19 in the city may be controlled by keeping high the isolation index $\theta$. The simulation shows that keeping the values for the different parameter given in Table 1 will be suitable for controlling or stabilizing the covid-19.

Fig. 2 a, d are the representative of the dynamics of the COVID-19 for all the four compartments of the consider deterministic or idealistic fractal-fractional model for an about ten months in the city of Campinas at different arbitrary order without vaccination. The susceptible cases decreases as infected and sick individuals grows, the recovered individuals also increases with the passage of time. All the compartmental simulation are then going toward their disease equilibrium point and the situation of covid-19 in the

| Table 1 |
|---------------------------------------------|
| The parameters values Santos. |
| Parameters | Numerical values |
| $s(t)$ | 0.999754 hundred thousand |
| $i(t)$ | 0.000246 hundred thousand |
| $s_{sk}(t)$ | 0.000206 hundred thousand |
| $r(t)$ | 0.000200 hundred thousand |
| $\mu$ | 0.000027 |
| $\gamma$ | 1 |
| $\alpha$ | 0.775985 |
| $\theta$ | 0.415375 |
| $\beta_1$ | 0.2 |
| $\beta_2$ | 0.2 |
| $\beta_3$ | 0.04782 |

| Table 2 |
|---------------------------------------------|
| The parameters values Campinas. |
| Parameters | Numerical values |
| $s(t)$ | 0.999883 hundred thousand |
| $i(t)$ | 0.000206 hundred thousand |
| $s_{sk}(t)$ | 0.000196 hundred thousand |
| $r(t)$ | 0.000190 hundred thousand |
| $\mu$ | 0.000034 |
| $\gamma$ | 0.038255 |
| $\alpha$ | 0.776520 |
| $\theta$ | 0.414454 |
| $\beta_1$ | 0.2 |
| $\beta_2$ | 0.2 |
| $\beta_3$ | 0.06782 |

Fig. 2. Dynamics of all four compartment $s(t)$, $i(t)$, $s_{sk}(t)$, $r(t)$ of the consider model (4) at various fractal dimension $q$ and arbitrary order $p$ for the city of Campinas.
Fig. 3. Dynamics of all four compartment $s(t)$, $i(t)$, $s_{re}(t)$, $r(t)$ of the consider model (4) at various fractal dimension $q$ and arbitrary order $p$ for the city of São Paulo.

### Table 3

| Parameters | Numerical values |
|------------|------------------|
| $s(t)$ | 0.999800 hundred thousand |
| $i(t)$ | 0.000200 hundred thousand |
| $s_{re}(t)$ | 0.000176 hundred thousand |
| $r(t)$ | 0.00010600 hundred thousand |
| $\mu$ | 0.000036 |
| $\gamma$ | 0.032755 |
| $\alpha$ | 0.811520 |
| $\theta$ | 0.444654 |
| $\beta_1$ | 0.2 |
| $\beta_2$ | 0.2 |
| $\beta_3$ | 0.05872 |

City may be controlled by keeping well isolation index $\theta$. The simulation shows that keeping the values for the different parameter given in Table 2 will be suitable for controlling or stabilizing the covid-19.

Fig. 3 a–d shows the dynamics of covid-19 for all the four quantities of the proposed deterministic or idealistic fractal-fractional order model for an about ten months in the city of Campinas at different arbitrary order before vaccination. The susceptible population decreases as infected and sick individuals grows, the recovered class also increases with the passage of time. All the compartmental simulation are then going toward their disease equilibrium point and the situation of covid-19 in the city may be controlled by keeping high the value of an isolation index $\theta$. The simulation confirms that keeping the values for the various parameter given in Table 3 will be suitable for controlling or stabilizing the covid-19.

### 5. Conclusions

The methodology adopted in this research has helped in successfully investigating the COVID-19 model by employing fractal fractional derivative. The considered system was processed with the help of the Atangana Bleanu Caputo (ABC) fractal-arbitrary order derivative. By utilizing the Banach fixed point theorems, we have exploited the uniqueness of the solution. For at least one solution for the considered system we have applied the Krasnosilki’s theorem. Further, the system examined for the derivation of Ulam-Hyer’s stability and the generalized Ulam-Hyer’s stability via nonlinear functional analysis concepts. Various fractal fractional order derivatives have been used for numerical simulation/scheme of the considered system. All the compartments have been tested for three different cities of the São Brasil state against very good available fitted data. Moreover, from the analysis of the covid-19 model, it can be realized that fractal-fractional order derivative of the mathematical model of the real-life problem gives better performance than classical calculus. The entire investigation deals with whole spectrum for the dynamical behavior of the proposed system at different fraction order and fractal independent variable dimension $t$ lying between 0 and 1.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
CRediT authorship contribution statement

Ebrahim A. Alghyne: Conceptualization, Formal analysis, Project_administration, Writing - review & editing. Muhammed Ibrahim: Conceptualization, Software, Software, Validation, Writing - original draft.

References

[1] Maxmen A. Why did the world’s pandemic warning system fail when COVID hit? Nature 2021;589(7843):499–500.
[2] World Health Organization. WHO timeline - COVID-19.
[3] Repurposed antiviral drugs for COVID-19 – interim WHO solidarity trial results, N Engl J Med 2021;384(6):497–511. doi: 10.1056/NEJMc203184.
[4] Johnston C, Brown ER, Stewart J, Karita HCS, Kissinger PJ, Dwyer J, Hosek S, et al. Hydroxychloroquine with or without azithromycin for treatment of early SARS-CoV-2 infection among high-risk outpatient adults: a randomized clinical trial. EClinicalMedicine 2021;33:100773.
[5] Eftekhar SP, Kazemi S, Barary M, Javanian M, Ebrahimipour S, Zaei N. Effect of hydroxychloroquine and azithromycin on QT interval prolongation and other cardiac arrhythmias in COVID-19 confirmed patients. Cardiovasc Ther 2021:2021.
[6] Rahman AK, Purdy AG, Ender PT. COVID-19 pneumonia in patients on chronic hydroxychloroquine therapy: three cases of COVID-19 pneumonia. Case Rep Infect Dis 2020:2020.
[7] Bignardi PR, Vengrus CS, Aquino BM, Neto AC. Use of hydroxychloroquine and chloroquine in patients with COVID-19: a meta-analysis of randomized clinical trials. Pathog Global Health 2021:1–12.
[8] Chu DK, AKI EA, Duda S, Solo K, Yaacoub S, Schinemann HJ, El-hazekel A, et al. Physical distancing, face masks, and eye protection to prevent person-to-person transmission of SARS-CoV-2 and COVID-19: a systematic review and meta-analysis. Lancet 2020;395(10242):1973–87.
[9] de Souza Santos AA, Candido DS, de Souza WM, Buss L, Li SL, Pereira BHM, Wu C-H, Sabino EC, Faria NR. Dataset on SARS-CoV-2 non-pharmaceutical interventions in brazilian municipalities. Sci Data 2021;8(1):1–6.
[10] Scheid J, Lupien SP, Ford GS, West SL. Commentary: physiological and psychological impact of face mask usage during the COVID-19 pandemic. Int J Environ Res Public Health 2020;17(8):6655.
[11] Ball P. The lightning-fast quest for covid vaccines - and what it means for other diseases. Nature 2021;589(7840):16–18. doi: 10.1038/d41586-02036526-1.
[12] Wolf J, Bruno S, Eichberg M, Jannatt R, Rudo S, VanRheenen S, Coller B-A. Applying lessons from the Ebola vaccine experience for SARS-CoV-2 and other epidemic pathogens. npj Vaccines 2020;5(1):1–5.
[13] Batistela CM, Correa D.PF, Bueno A.M, Piqueira J.R.C. SIRIS-vaccine dynamic model for COVID-19 pandemic. 2021. arXiv preprint arXiv:2104.07402.
[14] Lakshmikantham V, Leela S. Nagumo-type uniqueness result for fractional differential equations. Non-linear Anal 2009;71:2886–9.
[15] Podlubny I. Fractional differential equations, mathematics in science and engineering. New York: Academic Press; 1999.
[16] Hiller R. Applications of fractional calculus in physics. Singapore: World Scientific; 2000.
[17] Rossikhin YA, Shitikova MV. Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl Mech Rev 1997;50:15–67.
[18] Rahman Mu, Arfan M, Shah K, Gömez-Aguilar JF. Investigating a nonlinear dynamical model of COVID-19 disease under fuzzy caputo, random and ABC fractional order derivative. Chaos Solitons Fractals 2020;140:110232.
[19] Singh H, Pandey R, Srivastava H. Solving non-linear fractional variational problems using Jacobi polynomials. Mathematics 2019;7(3):224.
[20] Singh H, Sahoo MR, Singh OP. Numerical method based on Galerkin approximation for the fractional advection-dispersion equation. Int J Appl Comput Math 2017;3(3):2171–87.
[21] Zhang Y. Initial boundary value problem for fractal heat equation in the semi-infinite region by yang-laplace transform. Therm Sci 2014;18(2):677–81.
[22] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York: Wiley; 1993.
[23] Elsayed H, Hassan, Kilicman A. A note on solutions of wave, Laplace's and heat equations with convolution terms by using a double laplace transform. Appl Math Lett 2008;21(12):1324–9.
[24] Spiga G, Spiga M. Two-dimensional transient solutions for crossflow heat exchangers with neither gas mixed. J Heat Transf-Trans Asme 1987;109(2):281–6.
[25] Khan T, Shah K, Khan RA, Khan A. Solution of fractional order heat equation via triple laplace transform in 2 dimensions. Math Methods Appl Sci 2018;41(2):818–25.
[26] Shah K, Khalil H, Khan RA. Analytical solutions of fractional order diffusion equations by natural transform method. Iranian J Sci Technol Trans A 2018;42(3):1479–90.
[27] Ahmad B, Sivasundaram S. On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. Appl Math Comput 2010;217:480–7.
[28] Bai Z. On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal 2010;72:916–24.
[29] Atangana A, Araż SL. New concept in calculus: piecewise differential and integral operators. Chaos Solitons Fractals 2021:145:110638.
[30] Atangana A, Araż SL. Nonlinear equations with global differential and integral operators: existence uniqueness with application to epidemiology. Results Phys 2021;20:103593.
[31] Kilbas AA, Srivastava H, Trujillo J. Theory and application of fractional differential equations. Elsevier 2006;294.
[32] Rahman Mu, Arfan M, Shah K, Kumam P, Shatuyvi M. Nonlinear fractional mathematical model of tuberculosis (TB) disease with incomplete treatment under Atangana-Baleanu derivative. Alex Eng J 2021.
[33] Dubey VP, Dubey S, Kumar D, Singh J. A computational study of fractional model of atmospheric dynamics of carbon dioxide gas. Chaos Solitons Fractals 2020;110375.
[34] Khoza MC. Modelling of groundwater flow within a leaky aquifer with fractal-fractional differential operator. University of the Free State: 2020. Phd diss.
[35] Atangana A. Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. Chaos Solitons Fractals 2017;102:396–406.
[36] Gómez-Aguilar JF, Cruz-Fraga T, Abdeljawad T, Khan A, Khan H. Analysis of fractional-fractional malaria transmission model. Fractals 2020. doi:10.1142/S021834882000411.
[37] Qureshi S, Atangana A. Fractal-fractional differentiation for the modeling and mathematical analysis of nonlinear diarrhea transmission dynamics under the use of real data. Chaos Solitons Fractals 2020;136:109812.
[38] Arfan M, Alraabiai H, Rahman Mu, Sun YL, Hashim AS, Pansera BA, Ahmadian A, Salahshour S. Investigation of fractal-fractional order model of COVID-19 in pakistan under Atangana-Baleanu caputo (ABC) derivative. Results Phys 2021:24:104046.
[39] Atangana A. Modelling the spread of COVID-19 with new fractal-fractional operators: can the lockdown save mankind before vaccination? Chaos Solitons Fractals 2020;136:109860.
[40] Owolabi KM, Atangana A, Akgul A. Modelling and analysis of fractal-fractional partial differential equations: application to reaction-diffusion model. Alex Eng J 2020;59(4):2477–90.
[41] Atangana A, Jain S. A new numerical approximation of the fractional ordinary differential equation. Eur Phys J Plus 2018;133(2):1–15.
[42] Atangana A. Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination? Chaos Solitons Fractals 2020;136:109860.
[43] Abdeljawad T, Baleanu D. Discrete fractional differences with nonsingular discrete Mittag-Leffler kernels. Adv Differ Equ 2016:11–18.
[44] Burton TA. Krauraskeksi N-typed fixed point theorem with applications to fractional nonlinear dynamical system. Adv Math Phys 2019:9. Article ID 6763842.