Abstract

The local normal zeta functions of a finitely generated, torsion-free nilpotent group $G$ of class 2 depend on the geometry of the Pfaffian hypersurface associated to the bilinear form induced by taking commutators in $G$. The smallest examples of zeta functions which are not finitely uniform arise from groups whose associated Pfaffian hypersurfaces are plane curves. In this paper we study groups whose Pfaffians define singular curves, illustrating that the local normal zeta functions may indeed invoke all the degeneracy loci of the Pfaffian.

1 Introduction and statement of results

In [6] Grunewald, Segal and Smith defined the concept of the zeta function of a group. For a finitely generated, torsion-free nilpotent group $G$ - a $\mathcal{T}$-group, in short - they set

$$a_n^G := |\{H \triangleleft G | |G : H| = n\}|.$$

The group’s normal zeta function is the formal Dirichlet series

$$\zeta_G^G(s) := \sum_{i=1}^{\infty} a_n^G n^{-s}.$$

Here $s$ is a complex variable, and $\zeta_G^G(s)$ defines an analytic function in some right half-plane $\{s \in \mathbb{C} | \Re(s) > \alpha\}$ for some $\alpha \in \mathbb{Q}$. Similarly, the local
normal zeta function of $G$ at the prime $p$ counts normal subgroups of $p$-power index in $G$:

$$\zeta\vartriangleleft_{G,p}(s) := \sum_{n=0}^{\infty} a^\vartriangleleft_{p^n, p^{-ns}}.$$ 

As $G$ is nilpotent, the local and global zeta functions are related by an *Euler product*

$$\zeta\vartriangleleft_G(s) = \prod_{p \text{ prime}} \zeta\vartriangleleft_{G,p}(s).$$

The local normal zeta functions are rational functions in $p^{-s}$. The global normal zeta function is called *finitely uniform* if there are finitely many rational functions $W_1(X,Y), \ldots, W_n(X,Y) \in \mathbb{Q}(X,Y)$ such that for every prime $p$ there is an $r$ such that $\zeta\vartriangleleft_{G,p}(s) = W_r(p, p^{-s})$, and *uniform* if it is finitely uniform with $n = 1$. (Finite) uniformity is not typical for zeta functions, however. The question of the dependence of the local factors on the prime $p$ was linked to the classical problem of counting points on varieties mod $p$ by du Sautoy and Grunewald. In [5] they give a presentation of the local zeta functions as finite sums of products of rational functions in $p$ and $p^{-s}$ (which we will call the *uniform components* of the presentation) and functions counting the number of $\mathbb{F}_p$-rational points of certain (boolean combinations of) algebraic varieties which are irreducible, smooth, and intersect normally. For an arbitrary collection of such varieties one would not expect these functions to be polynomials in $p$. The description of varieties arising in this context has been an ongoing project ever since. We refer the reader to [3] for a survey on zeta functions of groups.

The first example of a group whose zeta function is not finitely uniform was given by du Sautoy ([1],[2]). The local normal factors of this class-2-nilpotent group depend on the number of $\mathbb{F}_p$-points of the elliptic curve

$$E = (y^2 + x^3 - Dx = 0), \quad D \in \mathbb{N}.$$ 

He defined a nilpotent group

$$G(E) = \langle x_1, \ldots, x_6, y_1, y_2, y_3 | [x_i, x_j] = M(y)_{ij} \rangle,$$ 

where $M(y) := \begin{pmatrix} 0 & R(y) \\ -R(y)^t & 0 \end{pmatrix}$, and

$$R(y) = \begin{pmatrix} Dy_3 & y_1 & y_2 \\ y_1 & y_3 & 0 \\ y_2 & 0 & y_1 \end{pmatrix}.$$
The curve \( E \) appears as the Pfaffian hypersurface \( P_G := (\det(M(y)) = 0) \) in \( \mathbb{P}^2(\mathbb{Q}) \) associated to this group. In [10] we proved

**Theorem 1** For almost all primes \( p \)

\[
\zeta_{G(E),p}(s) = W_1(p, p^{-s}) + |E(\mathbb{F}_p)|W_2(p, p^{-s}),
\]

(2)

where \( |E(\mathbb{F}_p)| \) is the number of \( \mathbb{F}_p \)-rational points of \( E \) and

\[
W_1(X, Y) = \frac{(1 + X^6Y^7 + X^7Y^7 + X^{12}Y^8 + X^{13}Y^8 + X^{19}Y^{15})}{\prod_{i=0}^{6}(1 - X^iY) \cdot (1 - X^9Y^{18})(1 - X^8Y^7)(1 - X^{14}Y^8)},
\]

\[
W_2(X, Y) = \frac{(1 - Y)(1 + Y)X^6Y^5(1 + X^{13}Y^8)}{\prod_{i=0}^{6}(1 - X^iY) \cdot (1 - X^9Y^{18})(1 - X^8Y^7)(1 - X^7Y^5)(1 - X^{14}Y^8)}.
\]

Non-uniformity and the shape of the local zeta functions, i.e. the existence of non-zero rational \( W_i(X, Y) \) in (2), were already established by du Sautoy in [2].

Moreover, we showed that in this manner (the number of \( \mathbb{F}_p \)-points of) every smooth plane curve defined over \( \mathbb{Q} \) could be realised in local normal zeta functions of nilpotent groups. These improvements became feasible once we interpreted the local zeta function of a nilpotent group as a generating function associated to a weight function defined on the vertex set of the Bruhat-Tits building for \( \text{Sl}_n(\mathbb{Q}_p) \). In [9] we generalised this approach to deal with \( \Sigma \)-groups \( G \) of nilpotency class 2 - let us call them \( \Sigma_2 \)-groups - whose Pfaffians \( P_G \) are smooth hypersurfaces containing no lines. This is the generic case for \( \Sigma_2 \)-groups \( G \) with ‘small centre and sufficiently large abelianization’, or, more precisely,

\[
h(G') \leq 6 \text{ and } h(G/G') > 4h(G') - 10.
\]

(3)

(Here \( G' = [G, G] \) is the group’s derived group and \( h(G) \) denotes the Hirsch length of the polycyclic group \( G \), i.e. the number of infinite cyclic factors in a decomposition series of \( G \).)

In this paper we treat classes of examples of \( \Sigma_2 \)-groups whose Pfaffian \( P_G \) is singular. Our main Theorem 2 illustrates that in general the formulae

\[
(Here we adopted additive notation for words in the \( y_i \).)
for the group’s local zeta functions may invoke not only (the number of $\mathbb{F}_p$-points of) the Pfaffian hypersurface, but indeed all the strata of its rank stratification.

For the rest of this paper $G$ shall denote a $\mathbb{T}_2$-group. Only for simplicity we assume that $G' = Z(G)$, the group’s centre, and $G/G' \cong \mathbb{Z}^d$, $d = 2r$, and $G' \cong \mathbb{Z}^{d'}$. Indeed, in general $Z(G)/G'$ is a finitely generated abelian group. As we are looking to prove results about almost all primes, we may assume it to be torsion-free. And as

$$\zeta_{G \times \mathbb{Z}^r}(s) = \zeta_G^*(s) \cdot \zeta(s - n) \zeta(s - (n + 1)) \ldots \zeta(s - (n + r - 1))$$

where $h(G) = n$ and $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ is the Riemann zeta function, we may assume $Z(G) = G'$. The condition “$d$ even” has also just been added for the convenience of presentation. We will assume that $G$ is presented as in (1) above, where $M(y)$ is an anti-symmetric $(d \times d)$-matrix of $\mathbb{Z}$-linear forms in $y = (y_1, \ldots, y_{d'})$ - but not necessarily in block form as in the particular example (i). We may then consider $\mathcal{P}_G$ as a hypersurface of degree $r$ in $\mathbb{P}^{d'-1}(\mathbb{Q})$. We say that $G$ is regular if

$$\forall' p \text{ prime } \forall \alpha \in \mathbb{P}^{d'-1}(\mathbb{Q}_p) : \operatorname{rk}(M(\alpha)) \geq 2(r - 1).$$

(Here by $\forall'$ we mean “for almost all”, i.e. for all but finitely many.)

**Lemma 1** If $\mathcal{P}_G \subseteq \mathbb{P}^{d'-1}(\mathbb{Q})$ is smooth then $G$ is regular.

**Proof.** If $\mathcal{P}_G$ is smooth, the reduction $\overline{\mathcal{P}_G}$ is smooth for almost all primes $p$. If $G$ were not regular there were infinitely many primes $p$ for which there exists a point $\mathbf{x} = (x_{ij}) \in \mathbb{P}^{d'-1}(\mathbb{F}_p)$ for which

$$\operatorname{rk}(M(\mathbf{x})) < 2(r - 1).$$

It is thus enough to show that for almost all primes such a point must be a singular point of $\overline{\mathcal{P}_G}$.

Let $\overline{i}$ be the reduction mod $p$ of the linear embedding of $\mathbb{P}^{d'-1}(\mathbb{Q})$ into the projective space $\mathbb{P}(\mathcal{S}_r)(\mathbb{Q})$ of anti-symmetric $(2r \times 2r)$-matrices over $\mathbb{Q}$ up to scalars given by

$$i : \mathbb{P}^{d'-1}(\mathbb{Q}) \rightarrow \mathbb{P}(\mathcal{S}_r)(\mathbb{Q})$$

$$\alpha \mapsto (\ldots : L_{ij}(\alpha) : \ldots)$$
where the linear forms $L_{ij}$ are given by the $ij$-th entry of the matrix $M(\alpha)$. Ignoring finitely many primes $p$ as we may we assume that $\tilde{I}$ is injective. Let $U := i(\mathbb{P}^{d-1}(\mathbb{F}_p))$ and $\mathcal{X}_r$ denote the universal Pfaffian hypersurface in $\mathbb{P}(S_r)(\mathbb{Q})$ of singular anti-symmetric $(2r \times 2r)$-matrices up to scalars. Its singular locus $\text{Sing}(\mathcal{X}_r)$ consists of matrices of rank strictly less than $2(r-1)$. We have $\mathcal{P}_G = U \cap \mathcal{X}_r$. It is enough to show that

$$U \cap \text{Sing}(\mathcal{X}_r) \leq \text{Sing}(\mathcal{P}_G).$$

But that is clear since if $x = (x_{ij}) \in U \cap \text{Sing}(\mathcal{X}_r)$ then $T_x(\mathcal{P}_G) = U$. Thus $x$ is a singular point of $\mathcal{P}_G$. $\square$

The converse of Lemma 1 does not hold, as the following example shows:

**Example.** Consider the (singular) plane curve

$$C := (y^2 - x^3 + x^2 = 0).$$

We define two groups $G_1(C)$ and $G_2(C)$ as in (1) above, with

$$R_1(y) = \begin{pmatrix} y_1 & y_2 & 0 \\ 0 & y_1 & y_2 \\ -y_3 & 0 & y_1 - y_3 \end{pmatrix}, \quad R_2(y) = \begin{pmatrix} y_1 & y_2 & 0 \\ 0 & y_1 - y_3 & y_2 \\ -y_3 & 0 & y_1 \end{pmatrix}.$$ 

Note that $C = \mathcal{P}_{G_1}(C) = \mathcal{P}_{G_2}(C)$, but that only $G_2(C)$ is regular. As we will see, this distinction will be picked up by the local normal zeta functions.

To state our main theorem, we have to introduce further notation and hypotheses. Assume for the rest of this chapter that $d' = 3$, that $\sqrt{\det(M(y))}$ is non-zero and square-free, and that the Pfaffian curve $\mathcal{P}_G$ has irreducible components $E_i$, $i = 1, \ldots, l$, all of degree $> 1$, and has at most ordinary double points (ODP’s) as singularities, all of which are rational with rational slopes. Given a prime $p$ we write

$$c_{p,i} := |E_i|$$

for the number of model points over $\mathbb{F}_p$ of $E_i$, that is the number of $\mathbb{F}_p$-points of a non-singular curve birationally equivalent to $E_i$. For $j = 1, 2$ we write

$$n_p^{(j)} := |\{\alpha \in \mathbb{P}^2(\mathbb{F}_p) | \alpha \text{ is ODP of } \overline{\mathcal{P}_G} \text{ and } rk(M(\alpha)) = 2(r-j)\}|$$

5
for the number of $\mathbb{F}_p$-rational ODP’s of rank deficit $2j$. (In the above example we had $n_p^{(1)} = 0$ and $n_p^{(2)} = 1$ for $G = G_1(C)$, $n_p^{(1)} = 1$ and $n_p^{(2)} = 0$ for $G = G_2(C)$, and $c_{p,1} = 1 + p$ for both groups and all primes $p$.) With this notation we have

**Theorem 2** Assume that $G$ is a $\Sigma_2$-group such that $\sqrt{\det(M(y))}$ is non-zero and square-free and the curve $P_G$ has at most rational ordinary double points with rational slopes as singularities and has no lines. Then, for almost all primes $p$,

$$\frac{\zeta_{G,p}(s)}{\zeta_{2,p}(s)\zeta_p((d+3)s-3d)} = W_1(p, p^{-s}) + \left(\sum_{i=1}^l c_{p,i}\right) W_2(p, p^{-s}) + n_p^{(1)} W_3(p, p^{-s}) + n_p^{(2)} W_4(p, p^{-s}), \quad (4)$$

where

$$W_1(X, Y) = \frac{1 + X^{2r}Y^{2r+1} + X^{2r+1}Y^{2r+1} + X^{4r}Y^{2r+2} + X^{4r+1}Y^{2r+2} + X^{6r+1}Y^{4r+3}}{(1 - X^{2r+2}Y^{2r+2})(1 - X^{2r+2}Y^{2r+1})}, \quad (5)$$

$$W_2(X, Y) = \frac{(1 - Y)(1 + Y)X^{2r}Y^{2r-1}(1 + X^{4r+1}Y^{2r+2})}{(1 - X^{2r+1}Y^{2r-1})(1 - X^{2r+2}Y^{2r+1})}, \quad (6)$$

$$W_3(X, Y) = \frac{(1 - Y)^2(1 + Y)^2X^{2r}Y^{2r-3}(1 + X^{2r+1}Y^{2r-1})(1 + X^{4r+1}Y^{2r+2})}{(1 - X^{2r+2}Y^{2r+1})(1 - X^{2r}Y^{2r-3})(1 - X^{2r+1}Y^{2r-1})(1 - X^{4r+2}Y^{2r+2})}, \quad (7)$$

$$W_4(X, Y) = \frac{(1 - Y)(1 + Y)X^{2r}Y^{2r-1}(1 - X^{2r+2}Y^{2r-1})(1 + X^{4r+1}Y^{2r+2})}{(1 - X^{2r+1}Y^{2r-1})^2(1 - X^{2r+2}Y^{2r+1})(1 - X^{4r+2}Y^{2r+2})}, \quad (8)$$

In $\mathbb{D}$, $\zeta_p(s) = (1 - p^{-s})^{-1}$ denotes the $p$-th local Riemann zeta function. The division on the left hand side is purely formal.

**Corollary 1** Assume that, in addition, for each $i = 1, \ldots, l$ there exist integers $B^{(i)}$ and complex numbers $\beta_{i,j}$ such that

$$c_{p,i} = |E_i| = 1 + p - \sum_{j=1}^{B^{(i)}} \beta_{i,j}.$$
and a \( 1 \rightarrow 1 \)-correspondence
\[
\left\{ \frac{p}{\beta_{i,j}} \mid 1 \leq j \leq B^{(i)} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \beta_{i,k} \mid 1 \leq k \leq B^{(i)} \right\}.
\]

Set
\[
c_{p^{-1},i} := 1 + p^{-1} - \sum_{j=1}^{B^{(i)}} \beta_{i,j}^{-1} = p^{-1} c_{p,i}.
\]
\[
n_{p^{-1}}^{(j)} := n_{p}^{(j)}.
\]

Then for all but finitely many primes \( p \) the local normal zeta function \( \zeta_{G,p}(s) \) satisfies a functional equation
\[
\zeta_{G,p}(s) \big|_{p \rightarrow p^{-1}} = -p^{(2r+3)-(4r+3)s} \zeta_{G,p}(s).
\]

(9)

**Proof** (of Corollary 1): This follows immediately from the observation that
\[
W_{i}(p, p^{-s}) \big|_{p \rightarrow p^{-1}} = \begin{cases} p^{3} W_{i}(p, p^{-s}) & \text{for } i = 1, 3, 4, \\ p^{4} W_{i}(p, p^{-s}) & \text{for } i = 2. \end{cases}
\]

\( \square \)

Note that Theorem 1 follows from Theorem 2 by setting \( r = 3, l = 1, c_{1} = |E(\mathbb{F}_{p})|, n^{(1)} = n^{(2)} = 0 \), and that we get explicit formulae for the zeta functions of the groups constructed in the example preceding Theorem 2. The conditions of Corollary 1 are satisfied in all these cases, giving us a local functional equation.

Before we prove Theorem 2 in Section 3 we give a short introduction to our method in Section 2 to make this paper self-contained.

**Acknowledgements.** This article was written when the author held a Postdoctoral Fellowship from the UK’s Engineering and Physical Sciences Research Council (EPSRC). It comprises a part of the author’s Cambridge PhD-Thesis, supported by the Studienstiftung des deutschen Volkes and the Cambridge European Trust. We would like to thank Marcus du Sautoy for his encouragement as supervisor, Fritz Grunewald for his continuous support and invaluable conversations, Burt Totaro for asking me about singular Pfaffians, and many others for sharing their knowledge and enthusiasm with me.
2 Counting in Bruhat-Tits buildings

Now let $G$ be again a $\mathfrak{T}_2$-group with

$$Z(G) = G', h(G/G') = d = 2r, h(G') = d'.$$

Nothing will be lost if we assume $d' \geq 2, d \geq 4$. For a fixed prime $p$, the computation of the $p$-th normal local zeta function of $G$ comes down to an enumeration of lattices (= subgroups of finite index) in the $\mathbb{Z}_p$-Lie algebra (with Lie brackets induced by taking commutators)

$$G_p := (G/Z(G) \oplus Z(G)) \otimes \mathbb{Z}_p$$

which are ideals in $G_p$. We call a lattice $\Lambda \leq \mathbb{Z}_n$ maximal (in its homothety class) if $p^{-1} \Lambda \not\leq \mathbb{Z}_n$. The key observation is the following

Lemma 2 For each lattice $\Lambda' \leq G'_p$ put $X(\Lambda')/\Lambda' = Z(G_p/\Lambda')$. Then

$$\zeta_{G,p}(s) = \zeta_{Z_p}(s) \sum_{\Lambda' \leq G'_p} |G'_p : \Lambda'|^{d-s}|G_p : X(\Lambda')|^{-s}$$

$$= \zeta_{Z_p}(s) \zeta_p((d + d')s - dd') \sum_{\Lambda' \leq G'_p, \Lambda' \text{ maximal}} |G'_p : \Lambda'|^{d-s}|G_p : X(\Lambda')|^{-s}$$

This is essentially a local version of Lemma 6.1 in [6]. One of the main theorems of [6] establishes the rationality of the generating function\footnote{Recall that it is in general not true that $A(p, p^{-s})$ is a rational function in both $p^{-s}$ and $p$, i.e. that $\zeta_{Z_p}^d(s)$ is uniform.} $A(p, p^{-s})$ in $p^{-s}$. Recall that (homothety classes of) maximal lattices are in one-to-one correspondence with vertices of the Bruhat-Tits building $\Delta_{d'}$ for $SL_{d'}(\mathbb{Q}_p)$ (e.g. [4], §19). To derive an explicit formula for $A(p, p^{-s})$ requires a quantitative understanding of two weight functions defined on the set of vertices of the building $\Delta_{d'}$. We write

$$A(p, p^{-s}) = \sum_{[\Lambda]} p^{d_{w([\Lambda'])} - sw'([\Lambda'])}, \quad (10)$$
where, for a homothety class $[\Lambda']$ of a maximal lattice $\Lambda'$ in $G'_p \cong \mathbb{Z}^{d'}_p$ we define

$$w([\Lambda']) := \log_p(|G'_p : \Lambda'|),$$

$$w'([\Lambda']) := w([\Lambda']) + \log_p(|G_p : X(\Lambda')|)$$

In [10] we explain in detail how the evaluation of $w'$ might be reduced to solving linear congruences. To recall the result in Theorem 3 below we have to introduce some more notation.

A maximal lattice $\Lambda' \leq \mathbb{Z}^{d'}_p$ is said to be of type $\nu(\Lambda') = (I, r_I)$ if $\Lambda'$ has elementary divisors

$$(p^\nu) := \left(1, \ldots, 1, p^{r_{i_1}}, \ldots, p^{r_{i_l}}, \ldots, p^{\sum_{j=1}^l r_{i_j}}, \ldots, p^{\sum_{j=1}^l r_{i_j}}\right)$$

(11)

for

$$I = \{i_1, \ldots, i_l\} \leq d' - 1, 1 \leq i_1 < \cdots < i_l \leq d' - 1,$$

(12)

and $r_I = (r_{i_1}, \ldots, r_{i_l}) \in \mathbb{N}^l_{>0}$. Here we set $n := \{1, \ldots, n\}$. By abuse of notation we may say that a maximal lattice $\Lambda'$ is of type $I$ if it is of type $(I, r_I)$ for some positive vector $r_I$, and that a (not necessarily maximal) lattice or a homothety class has type $I$ if the maximal element of the class has type $I$.

The group $\Gamma = Sl_n(\mathbb{Z}_p)$ acts transitively on the set of maximal lattices of fixed type. If we choose a basis for the $\mathbb{Z}_p$-module $G'_p$ and represent lattices as the row span of $d' \times d'$-matrices and denote by $\Gamma_{\nu}$ the stabilizer in $\Gamma$ of the lattice generated by the diagonal matrix whose entries are given by the vector (11), the orbit-stabiliser theorem gives us a one-to-one correspondence

$$\{\text{ maximal lattices of type } (I, r_I) \} \longleftrightarrow \Gamma / \Gamma_{\nu}. \quad (13)$$

The correspondence (13) allows us to describe $|G_p : X(\Lambda')|$ for a maximal lattice $\Lambda'$ in terms of $M(y)$, the matrix of commutators in a presentation for $G$ as in (11). Its entries are $\mathbb{Z}$-linear forms in generators $y_1, \ldots, y_{d'}$ for $G'$.

**Theorem 3** [11], §2.2] If $\Lambda'$ corresponds to the coset $\alpha \Gamma_{\nu}$ under (13), where $\alpha \in \Gamma$ with column vectors $\alpha^j, j = 1, \ldots, d'$. Then the index $|G_p : X(\Lambda')|$ equals the index of the kernel of the following system of linear congruences in $G_p/G'_p$:

$$\forall i \in \{1, \ldots, d'\} \quad \bar{\mathbb{M}}(\alpha^i) \equiv 0 \mod (p^\nu)_i,$$

(14)
where \( \overline{\gamma} = (\overline{\gamma_1}, \ldots, \overline{\gamma_d}) \in G_p/G'_p \cong \mathbb{Z}_p^d \) and \((p')_i\) denotes the \(i\)-th entry of the vector \((p')\) given in (11).

Further study of the linear congruences (14) will enable us to formulate conditions on the group \(G\) under which the explicit computation of \(A(p, p^{-s})\) - and thus \(\zeta_{G,p}(s)\) - is feasible. Given \(\alpha \in \Gamma\) and \(I\) as in (12), let \(\overline{\alpha}\) denote the reduction mod \(p\) and define vector spaces

\[ V_{i_j} := \langle \overline{\alpha_{i_j} + 1}, \ldots, \overline{\alpha_{d'}} \rangle < \mathbb{P}^{d'}(\mathbb{F}_p), \quad i_j \in I. \]

Clearly \(\text{codim}(V_{i_j}) = i_j - 1\). We will call the nested sequence of subspaces

\[ V_{i_1} > \cdots > V_{i_l} \]

the flag of type \(I\) associated to \(\alpha\) or indeed to the maximal lattice \(\Lambda'\) if \(\nu(\Lambda') = I\) and \(\Lambda'\) corresponds to \(\alpha \Gamma_\nu\) under (13).\(^2\) Given a fixed point \(x \in \mathbb{P}^{d' - 1}(\mathbb{F}_p)\), we call a lattice \(\Lambda'\) a lift of \(x\) if its associated flag contains \(x\) as 0-dimensional member. Note that then necessarily \(d' - 1 \in \nu(\Lambda')\).

Given \(I \leq d' - 1\) and \(\overline{x} \in \mathbb{P}^{d' - 1}(\mathbb{F}_p)\), we set

\[ A_I(p, p^{-s}) := \sum_{\nu(\Lambda') = I} p^{d_w([\Lambda']) - s w'([\Lambda'])}, \]

\[ A_{\overline{x}}(p, p^{-s}) := \sum_{\Lambda' \text{ lifts } \overline{x}} p^{d_w([\Lambda']) - s w'([\Lambda'])}. \]

Clearly

\[ A(p, p^{-s}) = \sum_{I \leq d' - 1} A_I(p, p^{-s}) \quad (15) \]

\[ = \sum_{\overline{x} \in \mathbb{P}^{d' - 1}(\mathbb{F}_p)} A_{\overline{x}}(p, p^{-s}) + \sum_{I \not\leq d' - 1} A_I(p, p^{-s}) \quad (16) \]

Formula (16) gives us an inkling how the combinatorial object \(A(p, p^{-s})\) is linked to the geometry of the Pfaffian \(\mathcal{P}_G\), a hypersurface in projective \((d' - 1)\)-space. Theorem 3 tells us that the number \(|G_p : X(\Lambda')|\) depends in the first instance on the way the flag associated to \(\Lambda'\) intersects with \(\mathcal{P}_G\).

\(^2\)It is indeed straightforward to show that this is well-defined, i.e. independent of the coset representative \(\alpha\).
Assume from now on that its defining polynomial $\sqrt{\det(M(y))}$ is non-zero and square-free, and that $\mathcal{P}_G$ contains no lines. This last condition will be extremely useful and is generically satisfied if $r > 2d' - 5$ (see Beauville’s Appendix to \cite{9} for details).

In the easiest case $\mathcal{P}_G$ has no $\mathbb{F}_p$-rational points. Thus $\det(M(\alpha))$ is a $p$-adic unit for all $\alpha \in \mathbb{Z}_p^{d'} \setminus (p\mathbb{Z}_p^{d'})$ and (14) is equivalent to the single congruence

$$g \equiv 0 \mod p^{\sum_{j=1}^{r_j}}.$$ 

Hence

$$|G_p : X(\Lambda'|) = p^{d\sum_{j=1}^{r_j}}.$$ 

(17)

In particular, both weight functions $w$ and $w'$ only depend on the maximal lattices’ type. Such functions have been studied in \cite{9}:

**Theorem 4** \cite{9} Assume that the weight functions $w([\Lambda'])$ and $w'([\Lambda'])$ only depend on the elementary divisor type of $\Lambda'$. Then there exist integers $a_i, b_i$ for $i \in \{1, \ldots, d' - 1\}$, depending only on $d, d'$ and $i$, such that, for all $I \leq d' - 1$

$$A_I(p, p^{-s}) = b_I(p^{-1}) \prod_{i \in I} \left. \frac{X_i}{1 - X_i} \right|_{X_i \rightarrow p^{b_i - a_i s}},$$

(18)

where $b_I(p) \in \mathbb{Z}[p]$ denotes the number of $\mathbb{F}_p$-rational points of the projective variety of flags of type $I$ (and $b_{\emptyset}(p) = 1$). In particular

$$A_\pi(p, p^{-s}) = \left. \left( \sum_{I \in \mathcal{P}^{d-2}} b_I(p^{-1}) \prod_{i \in I} \left. \frac{X_i}{1 - X_i} \right|_{X_i \rightarrow p^{b_i - a_i s}} \right) \right|_{I = A^{(1)}(p, p^{-s})},$$

(19)

independently of $\pi \in \mathbb{P}^{d-1}(\mathbb{F}_p)$. The following local functional equation holds

$$A(p, p^{-s})|_{p \rightarrow p^{-1}} = (-1)^{d' - 1} p^{-\binom{d'}{2}} A(p, p^{-s}).$$

In this special case $A(p, p^{-s})$ is an instantiation of a type of functions first studied by Igusa \cite{7}. The polynomials $b_I(p)$ are easily computed in terms of $q$-binomial polynomials (cf. \cite{3}). The proof of the functional equation relies on a symmetry of the (poset) lattice of Schubert cells for the complete
flag variety. We note that this symmetry is independent of the ‘numerical data’ \( a_i, b_i \). See [9], §4.2.1, for details. There it is also proved that

\[
a_i := d + d' - i, \quad b_i := (d + i)(d' - i), \quad 1 \leq i \leq d' - 1
\]

(20)
is the right numerical data for the case that \( P_G \) has no \( \mathbb{F}_p \)-points. Formula (5) may then be retrieved from (15), (18) and (20).

Note that under the assumption that \( P_G \) contains no higher-dimensional linear subspaces, the formulae for the \( A_I(p, p^{-s}) \), \( d' - 1 \not\in I \), are given by (18) regardless if \( P_G \) has \( \mathbb{F}_p \)-rational points, at least for almost all primes \( p \). This follows from inspection of the linear congruences (14). It can also be shown ([9], Prop. 3) that under this assumption, for \( x \in \mathbb{P}^{d' - 1}(\mathbb{F}_p) \), for almost all primes,

\[
A_x(p, p^{-s}) = \left( \sum_{\Lambda' \text{ lifts } x} p^{d'w(\Lambda') - s(w(\Lambda'))} \right) \left( \sum_{I' \in d' - 2} b_{I'}(p^{-1}) \prod_{i \in I'} \frac{X_i}{1 - X_i} \right)_{X_i \rightarrow p^{b_i - a_i}}
\]

(21)

If \( x \not\in \mathbb{P}_{\overline{G}} \) and the numerical data is as in (20) above, this formula coincides with the one given in (19). If \( x \) is a rational point of \( \mathbb{P}_{\overline{G}} \), however, equation (21) tells us that the mistake we have made by choosing (18) instead is localised in the first factor. To correct it we have to compute \( A_{x'}(p, p^{-s}) \), a sum which extends just over lattices lifting the (highly incomplete) flag consisting just of the point \( x \). This generating function will depend on the structure of \( P_G(\mathbb{Q}_p) \) near \( x \). The easiest case - that \( x \) is a smooth point of \( \mathbb{P}_{\overline{G}} \) - was dealt with in [9] and will be recalled in the remainder of this chapter. The main point of Theorem 2 is to illustrate that to deal with even the tamest singularities requires additional information (here on the rank filtration) and intricate computations. A satisfactory machinery to compute \( A_{x'}(p, p^{-s}) \) in general would be highly desirable but seems out of reach by far. It would have to implement - among other things - a resolution of singularities for \( P_G \). We will study the effects of dropping the simplifying assumption “\( P_G \) contains no lines” in a future paper.

If \( x \) is a rational point of \( \mathbb{P}_{\overline{G}} \), the linear congruences (14) to be solved in order to compute \( A_{x'}(p, p^{-s}) \) are

\[
\overline{E}M(\alpha^{d'}) \equiv 0 \mod p^{r_{d' - 1}}, \quad (22)
\]
where \( \alpha'' \in \mathbb{P}^{d'-1}(\mathbb{Z}/(p^{r'i})) \) is a lift of \( \overline{\alpha} \). If \( \overline{\alpha} \) is a smooth point of \( \mathcal{P}_G \) we can find local coordinates such that (22) reads like

\[
\mathfrak{g} \ \text{diag} \left( \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, J_2, \ldots, J_2 \right) \equiv 0 \mod p^{r''d'-1}, \tag{23}
\]

where \( J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( x \in p\mathbb{Z}_{p}/(p^{d'-1}). \) Thus

\[
|G_p : X(A')| = p^d \sum_{j=1}^{r} r_j - 2 \min\{r_{d'-1}, v_p(x)\}, \tag{24}
\]

with \( p \)-adic valuation \( v_p. \)

A quantitative version of Hensel’s Lemma describes how \( v_p(x) \) varies as \( A' \) runs over the maximal lattices lifting \( \overline{\alpha} \) and allows us to compute the generating function

\[
A'_x(p, p^{-s}) = \frac{p^{2r-s(2r-1)}(1 - p^{2r+1-s(2r+1)})}{(1 - p^{2r+1-s(2r-1)})(1 - p^{2r+2-s(2r+1)})} = A^{(2)}(p, p^{-s}), \text{ say,}
\]

and thus the “correction term”

\[
W_2(p, p^{-s}) = \left( A^{(2)}(p, p^{-s}) - A^{(1)}(p, p^{-s}) \right) \left( \sum_{r' \in d'-2} b_{r'}(p^{-1}) \prod_{i \in r'} \frac{X_i}{1 - X_i} \right) \Bigg|_{X_i \to p^{2r'-s} - a_i},
\]

giving us (6) explicitly. See [9], §4.2, for details. A variant of this Lemma will be derived in Section 3 (formula (27)) to deal with singular points.

### 3 Proof of Theorem 2

To prove Theorem 2 we have to compute the two ‘correction terms’ \( W_3(p, p^{-s}) \) and \( W_4(p, p^{-s}) \). We have seen that it suffices to compute \( A'_x(p, p^{-s}) \) for a fixed point of rank deficit 2 and 4, respectively. Note that \( d' = 3. \)
3.1 \( W_3(p, p^{-s}) \)

Let \( \pi \) be a fixed ODP of \( P_G \) of rank deficit 4. Dismissing finitely many primes if necessary, we may assume that we can find local affine coordinates such that condition \((22)\) reads like

\[
\mathbf{g} \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 \\ -x_2 & 0 \end{pmatrix}, J_2 \equiv 0 \mod p^a, \tag{25}
\]

\((x_1, x_2 \in p\mathbb{Z}/(p^a), a = r_2 \geq 1)\). We compute the weight function as

\[
w'(\Lambda') = (d + 1)a - 2(\min\{a, v_p(x_1)\} + \min\{a, v_p(x_2)\}).
\]

We shall factorize the map

\[
[\Lambda'] \mapsto p^{dw(\Lambda') - sw'(\Lambda')}
\]

over the set \( N := \{(a, b, c) \in \mathbb{N}^3_0 | a \geq b, a \geq c\} \) as \( \psi \circ \phi\) where

\[
\begin{align*}
\phi : [\Lambda'] &\mapsto (a, v_p(x_1), v_p(x_2)) \\
\psi : (a, b, c) &\mapsto p^{2r a - s((2r+1)a - 2 \min(a,b) - 2 \min(a,c))} \tag{26}
\end{align*}
\]

To deduce an explicit formula for the rational generating function \( A_{\pi}(p, p^{-s}) \) we must study the cardinalities of the fibres of \( \phi \) and must brake up the cone \( N \) in order to eliminate the “min” in \((26)\). It is indeed not hard to compute

\[
|\phi^{-1}(a, b, c)| = \begin{cases} 
1 & \text{if } a = b = c, \\
(1 - p^{-1}) p^{a-b} & \text{if } a > b, a = c, \\
(1 - p^{-1}) p^{a-c} & \text{if } a > c, a = b, \\
(1 - p^{-1})^2 p^{2a-b-c} & \text{if } a > b, a > c,
\end{cases} \tag{27}
\]

and to see that the following decomposition of \( N \) suits our needs:

\[
N = N_0 + N_1 + N_2 + N_3, \text{ where} \tag{28}
\]

\[
N_0 := \{(a, b, c) \in N | a = b = c\}, \\
N_1 := \{(a, b, c) \in N | a = c > b\}, \\
N_2 := \{(a, b, c) \in N | a = b > c\}, \\
N_3 := \{(a, b, c) \in N | a > b, a > c\}.
\]
Indeed, if we set \( n_j := \dim(N_j) \), we have

\[
(a, b, c) \in N_j \Rightarrow \frac{\phi^{-1}(a, b, c) |\psi(a, b, c)|}{(1 - p^{-1})^{n_j - 1}} = m_{j, X}(p, p^{-s})^a m_{j, Y}(p, p^{-s})^b m_{j, Z}(p, p^{-s})^c
\]  

(29)

for suitable Laurent monomials \( m_{j, X}(p, p^{-s}), m_{j, Y}(p, p^{-s}) \) and \( m_{j, Z}(p, p^{-s}) \) in variables \( p \) and \( p^{-s} \).

Let \( F_j(X, Y, Z), j = 0, \ldots, 3 \) be the zeta function of the cone \( N_j \), i.e. the rational function

\[
F_j(X, Y, Z) = \sum_{(a,b,c) \in N_j} X^a Y^b Z^c.
\]

Simply substituting and summing up yields

\[
A^{(4)}(p, p^{-s}) := A^{(4)}_X(p, p^{-s}) = \sum_{\lambda' \text{ lifts } \lambda, \nu(\lambda') = 2} p^{d_{w([\lambda'])} - s \nu([\lambda'])} =
\]

\[
\sum_{j=0}^{3} \sum_{(a,b,c) \in N_j} |\phi^{-1}(a, b, c)| \psi(a, b, c) = \sum_{j=0}^{3} (1 - p^{-1})^{n_j - 1} F_j(X, Y, Z) \bigg|_{X=m_{j, X}(p, p^{-s}), Y=m_{j, Y}(p, p^{-s}), Z=m_{j, Z}(p, p^{-s})}
\]

\[
p^{2r-s}(2r-3)(1 + p^{2r+1-s(2r-1)} - 2p^{2r-s}(2r-1) - 2p^{2r-1-s(2r+1)} + p^{4r+1+4sr} + p^{2r-s(2r+1)})
\]

\[
(1 - p^{2r-s}(2r-3))(1 - p^{2r+1-s(2r-1)})(1 - p^{2r+s-2(2r+1)})
\]

Table 1 records the generating functions \( F_j(X, Y, Z) \) together with the integers \( n_j \) and the Laurent monomials \( m_{j, X}(p, p^{-s}), m_{j, Y}(p, p^{-s}), m_{j, Z}(p, p^{-s}) \).

The latter are easily read off from (26) and (27). Routine computations with rational functions now yield the desired expression (30) for

\[
W_4(p, p^{-s}) = \left( A^{(4)}(p, p^{-s}) - 2A^{(2)}(p, p^{-s}) + A^{(1)}(p, p^{-s}) \right) \left( 1 + (p^{-1} + 1) \frac{X_1}{1 - X_1} \right) \bigg|_{X_1 \to p^{4r+2-s(2r+2)}}
\]

\[
\text{for } A \text{ Laurent monomial in variables } x_1, \ldots, x_n \text{ is a power product } x_1^{r_1} \cdots x_n^{r_n} \text{ with } r_i \in \mathbb{Z}.
\]
\[ j \quad n_j \quad F_j(X,Y,Z) \quad m_j,X(p,p^{-s}) \quad m_j,Y(p,p^{-s}) \quad m_j,Z(p,p^{-s}) \]

\begin{array}{cccc}
1 & 1 & \frac{XYZ}{1-XYZ} & p^{2r-s(2r+1)} & p^{-2s} & p^{-2s} \\
2 & 2 & \frac{X^2Y^2Z^2}{(1-XYZ)(1-XZ)} & p^{2r+1-s(2r+1)} & p^{1-2s} & p^{-2s} \\
3 & 2 & \frac{X^2Y^2Z}{(1-XYZ)(1-XY)} & p^{2r+1-s(2r+1)} & p^{-2s} & p^{-1-2s} \\
4 & 3 & \frac{X^2YZ(1-X^2YZ)}{(1-XYZ)(1-XY)(1-Z)} & p^{2r+2-s(2r+1)} & p^{1-2s} & p^{-1-2s} \\
\end{array}

Table 1: Computing \( A^{(4)}(p,p^{-s}) \).

### 3.2 \( W_3(p,p^{-s}) \)

Now let \( \overline{\mathfrak{x}} \) be a fixed ODP of \( \overline{\mathcal{P}}_G \) of rank deficit 2. Ignoring finitely many primes as we may, we choose local coordinates such that condition \( (22) \) becomes

\[
\overline{\mathfrak{g}} \text{ diag } \left( \begin{pmatrix} 0 & x_1x_2 \\ -x_1x_2 & 0 \end{pmatrix}, J_2, J_2 \right) \equiv 0 \mod p^a, \tag{31}
\]

\((x_1, x_2 \in p\mathbb{Z}/(p^a), a = r_2 \geq 1)\). We compute the weight function

\[
w'(\{\Lambda'\}) = (d + 1)a - 2 \min\{a, v_p(x_1) + v_p(x_2)\}.
\]

Again we shall factorize

\[
[\Lambda'] \mapsto p^{dw'(\{\Lambda'\}) - sw'(\{\Lambda'\})}
\]

over the set \( N := \{(a,b,c) \in \mathbb{N}_3^3 \mid a \geq b, a \geq c\} \) as \( \psi \circ \phi \) where

\[
\phi : [\Lambda'] \mapsto (s, v_p(x_1), v_p(x_2)) \\
\psi : (a,b,c) \mapsto p^{2ra-s(2r+1)a-2\min(a,b+c)}
\]

Recall the formula for \( |\phi^{-1}(a,b,c)| \) given in \( (27) \). The subdivision \( (28) \) of \( N \), however, has to be refined to

\[
N = \sum_{i=0}^{5} N_i, \text{ where } N_0, N_1, N_2 \text{ as in } (28) \text{ and } \tag{32}
\]

\[
N_3 := \{(a,b,c) \in N \mid a = b + c\}, \\
N_4 := \{(a,b,c) \in N \mid a > b + c\}, \\
N_5 := \{(a,b,c) \in N \mid a < b + c\}.
\]

16
If we set $n_j := \dim(N_j)$ for $j = 0, 1, 2$ and $n_j := 3$ for $j = 3, 4, 5$, equation (29) holds again all $j$ and suitable Laurent monomials, recorded in Table 2.

| j | $n_j$ | $F_j(X, Y, Z)$ | $m_{j,X}(p, p^{-s})$ | $m_{j,Y}(p, p^{-s})$ | $m_{j,Z}(p, p^{-s})$ |
|---|---|---|---|---|---|
| 0 | 1 | $\frac{XYZ}{1-XY Z}$ | $p^{2r-s(2r-1)}$ | 1 | 1 |
| 1 | 2 | $\frac{X^2 Y Z^2}{(1-XY Z)(1-XZ)}$ | $p^{2r+1-s(2r-1)}$ | $p^{-1}$ | 1 |
| 2 | 2 | $\frac{X^2 Y Z^2}{(1-XY Z)(1-XY)}$ | $p^{2r+1-s(2r-1)}$ | 1 | $p^{-1}$ |
| 3 | 3 | $\frac{X^2 Y Z}{(1-XY)(1-XZ)}$ | $p^{2r+2-s(2r-1)}$ | $p^{-1}$ | $p^{-1}$ |
| 4 | 3 | $\frac{X^3 Y Z}{(1-XY)(1-XZ)(1-X)}$ | $p^{2r+2-s(2r+1)}$ | $p^{-1-2s}$ | $p^{-1-2s}$ |
| 5 | 3 | $\frac{X^3 Y Z^2}{(1-XY)(1-XZ)(1-XY Z)}$ | $p^{2r+2-s(2r+1)}$ | $p^{-1}$ | $p^{-1}$ |

Table 2: Computing $A^{(3)}(p, p^{-s})$.

Substituting and summing up yields

$$A^{(3)}(p, p^{-s}) := A_2^{(3)}(p, p^{-s}) = \sum_{\substack{\mathcal{N}' \text{ lifts } \mathcal{N} \\
\nu(\mathcal{N}')=2}} p^{\nu(\mathcal{N}')} = \sum_{j=0}^{5} \sum_{(a,b,c) \in N_j} \left| \phi^{-1}(a, b, c) \right| \psi(a, b, c) = \sum_{j=0}^{5} (1 - p^{-1})^{n_j-1} \frac{F_j(X, Y, Z)}{X=m_{jX}(p, p^{-s}) \atop Y=m_{jY}(p, p^{-s}) \atop Z=m_{jZ}(p, p^{-s})}$$

$$= \left. \frac{p^{2r-s(2r-1)}(1 - 2p^{2r+1-s(2r-1)} + p^{2r+2-s(2r-1)} - p^{2r+2-s(2r+1)} + p^{4r+2-s(2r+1)})}{(1 - p^{2r+1-s(2r-1)})^2 (1 - p^{2r+2-s(2r+1)})} \right|_{X \to p^{4r+2-s(2r+2)}}$$

(33)

and finally the desired formula (34) for

$$W_3(p, p^{-s}) = \left. \left( A^{(3)}(p, p^{-s}) - 2A^{(2)}(p, p^{-s}) + A^{(1)}(p, p^{-s}) \right) \left( 1 + (p^{-1} + 1) \frac{X_1}{1-X_1} \right) \right|_{X_1 \to p^{4r+2-s(2r+2)}}$$

(34)

See Table 2 for the zeta functions of the respective cones and the corresponding Laurent monomials.
References

[1] M.P.F. du Sautoy, A nilpotent group and its elliptic curve: non-uniformity of local zeta functions of groups, Israel J. Math. 126 (2001), 269–288.

[2] _____, Counting subgroups in nilpotent groups and points on elliptic curves, J. Reine Angew. Math. 549 (2002), 1–21.

[3] M.P.F. du Sautoy and D. Segal, Zeta functions of groups, New horizons in pro-p groups, Progr. Math., Birkhäuser, Boston MA, 2000, pp. 249–286.

[4] P. Garrett, Buildings and classical groups, Chapman & Hall, 1997.

[5] F.J. Grunewald and M.P.F. du Sautoy, Analytic properties of zeta functions and subgroup growth, Ann. Math. 152 (2000), 793–833.

[6] F.J. Grunewald, D. Segal, and G.C. Smith, Subgroups of finite index in nilpotent groups, Invent. Math. 93 (1988), 185–223.

[7] J.-I. Igusa, Universal p-adic zeta functions and their functional equations, Amer. J. Math. 111 (1989), 671–716.

[8] I.G. Macdonald, Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, 1995.

[9] C. Voll, Functional equations for local normal zeta functions of nilpotent groups, with an Appendix by A. Beauville (http://arxiv.org/abs/math.GR/0305362).

[10] _____, Zeta functions of groups and enumeration in Bruhat-Tits buildings, Amer. J. Math., to appear (http://arxiv.org/abs/math.GR/0305360).