LÉVY MODELS AMENABLE TO EFFICIENT CALCULATIONS

SVETLANA BOYARCHENKO AND SERGEI LEVENDORSKII

Abstract. In our previous publications (IJTAF 2019, Math. Finance 2020), we introduced a general class of SINH-regular processes and demonstrated that efficient numerical methods for the evaluation of the Wiener-Hopf factors and various probability distributions (prices of options of several types) in Lévy models can be developed using only a few general properties of the characteristic exponent \( \psi \). Essentially all popular Lévy processes enjoy these properties. In the present paper, we define classes of Stieltjes-Lévy processes (SL-processes) as processes with completely monotone Lévy densities of positive and negative jumps, and signed Stieltjes-Lévy processes (sSL-processes) as processes with densities representable as differences of completely monotone densities. We demonstrate that 1) all crucial properties of \( \psi \) are consequences of the representation

\[
\psi(\xi) = (a_+^2 \xi^2 - ia_+ \xi)ST(G_+)(-i\xi) + (a_-^2 \xi^2 + ia_- \xi)ST(G_-)(i\xi) + \left(\frac{\sigma^2}{2}\right)\xi^2 - i\mu \xi,
\]

where \( ST(G) \) is the Stieltjes transform of the (signed) Stieltjes measure \( G \) and \( a_+^2 \geq 0 \); 2) essentially all popular processes other than Merton’s model and Meixner processes are SL-processes; 3) Meixner processes are sSL-processes; 4) under a natural symmetry condition, essentially all popular classes of Lévy processes are SL- or sSL-subordinated Brownian motion.

Key words: Stieltjes-Lévy processes, sinh-acceleration, SINH-regular Lévy processes, hyper-exponential jump-diffusion model, KoBoL, CGMY, Normal inverse Gaussian processes, Normal Tempered Stable Lévy processes, Variance Gamma processes, Meixner processes, beta-model, meromorphic processes, Hyperbolic processes, Generalized Hyperbolic distributions, subordinated Brownian Motion

MSC2010 codes: 6051, 60G52, 60-08, 65C05, 91G20

Contents

1. Introduction 2
2. SINH-regular Lévy processes 4
2.1. Definitions 4
2.2. Spectrally positive and spectrally negative SINH-processes 6
2.3. Examples 7
2.4. SINH-regular infinitely divisible distributions and related Lévy processes 10
3. Stieltjes-Lévy and signed Stieltjes-Lévy processes 11
3.1. Representations of the Lévy measure via the characteristic exponent 11
3.2. Representations of the Lévy measure in the case \( \mathbb{C} = \mathbb{C} \setminus i\mathbb{R} \) 12
3.3. Stieltjes measures and functions. Stieltjes transform 13
3.4. Definition of SL and sSL processes and examples 14
3.5. Representations of sSL-measures in terms of the characteristic exponent 17

S.B.: Department of Economics, The University of Texas at Austin, 2225 Speedway Stop C3100, Austin, TX 78712–0301, sboyarch@eco.utexas.edu
S.L.: Calico Science Consulting. Austin, TX. Email address: levendorskii@gmail.com.
1. **Introduction**

The Fourier/Laplace transform and the Wiener-Hopf factorization technique allow one to express probability distributions of Lévy processes, joint probability distributions of a Lévy process and its extrema, as well as prices of wide classes of options, as integrals and repeated integrals. Wiener-Hopf factors can also be expressed as exponentials of integrals. In many cases, straightforward evaluation of these integrals using Fast Fourier transform and/or Hilbert transform is inefficient, hence additional tricks are needed. Efficient calculations are possible if the technique of conformal deformations is applied to each integral, and then the simplified trapezoid rule is used. The deformations in all integrals must be in a certain agreement. See [16, 36, 15, 24, 8, 39, 17, 19, 20] for details and comparison with other methods; note that in [19], triple and quadruple integrals are efficiently calculated.

The conformal deformation technique relies on several general properties of the characteristic exponent $\psi$; specific properties of different classes of processes are essentially irrelevant for efficient calculations. The two most important properties are: (1) $\psi$ is analytic in a union of...
an open strip and a cone around or adjacent to $\mathbb{R}$; (2) $\text{Re } \psi(\xi) \to +\infty$ as $\xi \to \infty$ in an open sub-cone. In [17], we used properties (1) and (2) to define a class of SINH-regular processes, and mentioned that all popular classes of Lévy processes are SINH-regular. If $X$ is SINH-regular, then flat contours of integration in the Fourier inversion formula and formulas for the Wiener-Hopf factors can be deformed using conformal maps of the form $\xi \mapsto i\omega_1 + b \sin(i\omega_1 + \xi)$, where a proper choice of $\omega_1, \omega \in \mathbb{R}$ and $b > 0$ depends on the properties of the integrand. The corresponding change of variables $\xi = i\omega_1 + b \sin(i\omega_1 + y)$ (sinh-acceleration) reduces calculations to an integral over a horizontal line, with the integrand decaying at infinity much faster than the initial one. In the case of Laplace inversion, we use changes of variables of the form $q = \sigma + ib \sin(i\omega_1 + y)$, where $\sigma, b, \omega > 0$.

The new integrand remains analytic in a strip around the line of integration, hence the infinite trapezoid rule is very efficient. The error of the infinite trapezoid rule decays as $\exp[-2\pi d/\zeta]$, where $d$ is the half-width of the strip of analyticity around the line of integration and $\zeta$ is the step (see, e.g., Thm. 3.2.1 in [51]). Therefore, if the strip of analyticity is not too narrow, it is relatively easy to satisfy a very small error tolerance for the discretization error. Sinh-acceleration greatly increases the rate of decay at infinity, and thus a moderate or even small number of terms in the simplified trapezoid rule suffices to satisfy a small error tolerance $\epsilon$.

Typically, the complexity of the scheme is of the order of $E \ln E$, where $E = \ln(1/\epsilon)$.

Let $q$ be the dual variable in the Laplace inversion formula. When first passage probabilities are calculated and barrier and lookback options are priced, the Wiener-Hopf factors must be calculated for all points $q$ on a deformed contour in the Bromwich integral, hence zeros of $q + \psi(\xi)$ must be avoided or, if some of the zeros are crossed, these zeros must be accurately calculated and the residue theorem needs to be applied. Therefore, it is necessary to know where these zeros are located. If the Gaver-Stehfest method is used, then only $q > 0$ are used. Almost all popular Lévy models have two additional properties: (3) the cone of analyticity of $\psi$ is $\mathbb{C} \setminus i\mathbb{R}$, and (4) for $q > 0$, equation $q + \psi(\xi) = 0$ has no solutions in $\mathbb{C} \setminus i\mathbb{R}$. Hence, the problem of locating the zeros becomes trivial. Conditions (1)-(4) are used in [37] to define a class of strongly regular Lévy processes of exponential type.

In the case of stable Lévy processes, there is no strip of analyticity of $\psi$ (formally, the strip degenerates into $\mathbb{R}$), but $\psi$ enjoys properties (2)-(4) in the cone $\mathbb{C} \setminus i\mathbb{R}$, and efficient modifications of the procedures developed for SINH-processes are possible. See [18], where efficient procedures for the evaluation of stable distributions are developed and the relative efficiency of several families of conformal deformations is discussed.

Popular classes of Lévy processes are constructed either by defining the Lévy measure directly or via subordination of the Brownian motion. In the present paper, we suggest a general construction of a wide class of processes, which enjoy properties (1)-(4) and contain all popular models except for Merton’s model. We construct the Laplace exponents of the positive and negative jumps generalizing the definition of the Bernstein functions, and demonstrate that the crucial properties of $\psi$ are consequences of the representation

\begin{equation}
\psi(\xi) = (a_2^+ \xi^2 - ia_1^+ \xi)ST(G_0^0)(-i\xi) + (a_2^- \xi^2 + ia_1^- \xi)ST(G_0^0)(i\xi) + (\sigma^2/2)\xi^2 - i\mu\xi,
\end{equation}
where $\text{ST}(\mathcal{G})$ is the Stieltjes transform of the (signed) Stieltjes measure $\mathcal{G}$, $a_j^+ \geq 0$, and $\sigma^2 \geq 0$, $\mu \in \mathbb{R}$. If $\mathcal{G}_0^+ \geq 0$, we call $X$ a Stieltjes-Lévy process (SL-process). SL-processes enjoy properties (1)-(4), and the Lévy densities of positive and negative jump components are completely monotone. If at least one of $\mathcal{G}_\pm$ is a signed measure, then (1)-(3) continue to hold but (4) may fail; the Lévy density of the corresponding jump component is the difference of two completely monotone functions, thus it can be non-monotone. We say that $X$ is a signed Stieltjes-Lévy process (sSL-process).

The rest of the paper is organized as follows. In Section 2, we give the definition of SINH-regular processes. We slightly change the definition given in [17], calculate the strips and cones of analyticity for essentially all popular classes of Lévy processes, and derive upper and lower asymptotic bounds for $|\psi|$ and $\text{Re}\psi$, respectively. These bounds are necessary ingredients for the conformal acceleration method. In Section 3, we derive a representation of the Lévy density in terms of the characteristic exponent, which naturally leads to the definitions of sSL- and SL-processes. We establish several useful properties of these processes, derive a representation of the measures associated with sSL- and SL-processes in terms of the characteristic exponent, and determine sufficient conditions for sSL- and SL-processes to be SINH-regular. These conditions can be used to find good approximations of KoBoL [10] and other processes defined by absolutely continuous measures with hyperexponential Lévy processes and more general meromorphic processes defined by discrete measures. SL-processes defined by discrete measures with accumulation points at $0$ and $+\infty$ can be used to approximate stable Lévy processes. Non-existence of solutions of the equation $q + \psi(\xi)$ on $\mathbb{C} \setminus i\mathbb{R}$ (when $q > 0$) for SL-processes is proved in Section 4. In Section 5, we prove that, under additional conditions, (i) mixtures of SINH-regular, sSL- and SL-processes are SINH-regular, sSL- and SL-processes, respectively; (ii) SINH-regular processes subordinated by SINH-processes are SINH-regular; and (iii) SL-processes subordinated by sSL-subordinator (resp., by SL-subordinator) are sSL-processes (resp., SL-processes). As a byproduct, we prove that if, for some $\beta \in \mathbb{R}$, $\psi(\xi - i\beta) = \psi(i\beta - \xi)$, $\forall \xi \in \mathbb{R}$, an SL-process is a Brownian motion (BM) subordinated by an SL-process, and derive explicit formulas for the latter (in the case of sSL-processes, an additional condition is needed). Thus, under the symmetry condition above, essentially all popular classes of Lévy processes are SL- or sSL-subordinated BM. This result unifies and generalizes the well-known constructions of the Variance Gamma processes (VGP) [42], Normal Inverse Gaussian (NIG) processes [11] and Normal Tempered Stable (NTS) processes [5] by subordination, and representations of a symmetric KoBoL (CGMY) and Meixner Lévy processes as subordinated BM [43]. In Section 6, we summarize the results of the paper and outline possible extensions. The technical details are relegated to Section A. For the reader’s convenience, in Section B we outline applications of the conformal acceleration method to several basic situations.

2. SINH-REGULAR LÉVY PROCESSES

2.1. Definitions. Let $X$ be the Lévy process on $\mathbb{R}$, let $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}_{t \geq 0})$ be the filtered probability space generated by $X$, and let $Q$ be a probability measure on $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}_{t \geq 0})$. $E = \mathbb{E}^Q$ denotes the expectation operator under $Q$. We use the definition in [10, 12] of the characteristic exponent $\psi(\xi) = \psi^Q(\xi)$ of a Lévy process $X$ under $Q$, which is marginally different from the

\footnote{Note that the Laplace exponent of an SL-subordinator is a complete Bernstein function.}
definition in [17]. Namely, $\psi$ is definable from $\mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}$. The Lévy-Khintchine formula is

\begin{equation}
\psi(\xi) = \frac{\sigma^2}{2} \xi^2 - i\mu \xi + \int_{\mathbb{R}\setminus 0} (1 - e^{i\xi x} + 1_{(-1,1)}(x)ix\xi)F(dx),
\end{equation}

where $\sigma^2 \geq 0, \mu \in \mathbb{R}$ and the measure $F(dx)$ satisfies $\int (x^2 \wedge 1)F(dx) < \infty$. $X$ is of finite variation iff $\int (|x| \wedge 1)F(dx) < \infty$, and then the term $1_{(-1,1)}(x)ix\xi$ can (and will) be omitted.

**Remark 2.1.** Throughout the paper, $\mu, \gamma, \gamma', \gamma''$ are reals satisfying $\mu_+ = \mu, \mu_- < \mu_+, -\pi/2 \leq \gamma_- \leq 0 \leq \gamma_+ \leq \pi/2$, $\gamma_- < \gamma'_- < \gamma'_+ < \gamma_+$, and $\gamma \in (0, \pi)$. We define cones $C_{\gamma_-\gamma_+} = \{e^{i\rho} | \rho > 0, \varphi \in (\gamma_-, \gamma_+) \cup (\pi - \gamma_+, \pi - \gamma_-)\}$, $C_{\gamma} = \{e^{i\rho} | \rho > 0, \varphi \in (-\gamma, \gamma)\}$, and the strip $S_{(\mu_-\mu_+)} = \{\xi | \Im \xi \in (\mu_-\mu_+)\}$.

As in [16, 37, 17], we represent the characteristic exponent in the form

\begin{equation}
\psi(\xi) = -i\mu \xi + \psi^0(\xi),
\end{equation}

and impose conditions on $\psi^0$.

**Definition 2.1.** (1) We say that $X$ is a SINH-regular Lévy process (on $\mathbb{R}$) of type $(\mu_-\mu_+); \mathcal{C}; \mathcal{C}_+)$ and order $\nu \in (0, 2]$, iff the following conditions are satisfied:

(i) $\mu_- < 0 \leq \mu_+ \text{ or } \mu_- \leq 0 < \mu_+$;
(ii) $\mathcal{C} = C_{\gamma_-\gamma_+}, \mathcal{C}_+ = C_{\gamma'_-\gamma'_+}$, where $\gamma_- < 0 < \gamma_+, \gamma_- \leq \gamma'_- \leq 0 \leq \gamma'_+ \leq \gamma_+$, and $|\gamma_-| + \gamma'_+ > 0$;
(iii) $\psi^0$ admits analytic continuation to $i(\mu_-\mu_+) + (\mathcal{C} \cup \{0\})$;
(iv) for any $\varphi \in (\gamma_-, \gamma_+)$, there exists $c_\infty(\varphi) \in \mathbb{C} \setminus (-\infty, 0]$ s.t.

\begin{equation}
\psi^0(\rho e^{i\varphi}) \sim c_\infty(\varphi)\rho^\nu, \quad \rho \to +\infty;
\end{equation}

(v) the function $(\gamma_-, \gamma_+) \ni \varphi \mapsto c_\infty(\varphi) \in \mathbb{C}$ is continuous;

(vi) for any $\varphi \in (\gamma'_-, \gamma'_+)$, $\Re c_\infty(\varphi) > 0$.

(2) We say that $X$ is a SINH-regular Lévy process (on $\mathbb{R}$) of type $(\mu_-\mu_+); \mathcal{C}; \mathcal{C}_+)$ and order $\nu = 1+$, iff the conditions above bar (2.3) are satisfied, and (2.3) is replaced with

\begin{equation}
\psi^0(\rho e^{i\varphi}) \sim c_\infty(\varphi)\rho \ln \rho, \quad \rho \to +\infty.
\end{equation}

(3) We say that $X$ is a SINH-regular Lévy process (on $\mathbb{R}$) of type $(\mu_-\mu_+); \mathcal{C}; \mathcal{C}_+)$ and order $\nu = 0+$, iff the conditions (i)-(iii) are satisfied, and, as $\xi \to \infty$ remaining in $i(\mu_-\mu_+) + \mathcal{C}$,

\begin{equation}
\psi^0(\xi) \sim c \ln |\xi|,
\end{equation}

where $c > 0$.

**Remark 2.1.** (1) Conditions for $\varphi \in (\pi - \gamma_-, \pi - \gamma_+)$ and $\varphi \in (\pi - \gamma'_-, \pi - \gamma'_+)$ follow from the conditions for $\varphi \in (\gamma_-, \gamma_+)$ and $\varphi \in (\gamma'_-, \gamma'_+)$ because $\psi(-\xi) = \overline{\psi}(\xi)$.

(2) In Definition 2.1 conditions are imposed on $\psi^0$ whereas in [17], the same conditions are imposed on $\psi$. If either $\nu \in (1, 2]$ or $\mu = 0$, then the conditions on $\psi^0$ and $\psi$ are equivalent, and the process is an elliptic SINH-process of order $\nu$ in the terminology of [17].

\textsuperscript{2}The name elliptic is natural from the point of view of the theory of PDO: if (2.3) holds, then the infinitesimal generator $L^0 = -\psi^0(D)$ is an elliptic PDO.
(3) If \( \nu < 1 \) and \( \mu \neq 1 \), then the process \( X \) with the characteristic exponent \(-i\mu \xi + \psi(\xi)\) is elliptic of order 1, in the terminology of [17]. Furthermore, according to the definition in [17], \( C_+ \) is determined by the drift term if \( \mu \neq 0 \): \( C_+ \) is the intersection of \( C \) with the upper (resp., lower) half-plane if \( \mu > 0 \) (resp., \( \mu < 0 \)).

**Definition 2.2.** For \( \nu = 0^+ \) and \( \rho > 1 \), set \( \rho' = \ln \rho \). For \( \nu = 1^+ \) and \( \rho > 1 \), set \( \rho'' = \rho \ln \rho \). A linear ordering in the set \( (0, 2] \cup \{0^+\} \cup \{1^+\} \) is defined as follows: 1) for \( \nu_1, \nu_2 \in (0, 2) \), the usual ordering; 2) \( 0^+ < \nu \) for any \( \nu \in (0, 2] \cup \{1^+\} \); 3) \( 1 < 1^+ \); 4) \( 1^+ < \nu \) for any \( \nu \in (1, 2] \).

We see that we can write (2.4) and (2.5) in the form (2.3) with \( \psi \) and order \((\nu', \nu)\) ([2.2]). In basic examples that we consider and construct, \( \nu' = \nu \). Examples with \( \nu' < \nu \) can be constructed mixing processes of different orders. See also Remark 2.2.

**Definition 2.3.** We say that \( X \) is a SINH-regular Lévy process (on \( \mathbb{R} \)) of type \((\mu_-, \mu_+); C; C_+\) and order \((\nu', \nu)\) (lower order \( \nu' \) and upper order \( \nu \)), where \( \nu', \nu \in (0, 2] \cup \{0^+, 1^+\} \), iff conditions (i)-(iii) of Definition 2.3 are satisfied, and

\[
\begin{align*}
& \text{(i) } \forall \gamma_{1-} \in (\gamma_{-}, 0) \text{ and } \gamma_{1+} \in (0, \gamma_{+}), \exists C, R > 0 \text{ s.t. } \forall \varphi \in [\gamma_{1-}, \gamma_{1+}] \text{ and } \rho > R, \\
& \quad |\psi(\rho e^{i\varphi})| \leq C \rho^\nu; \\
& \text{(ii) } \forall \gamma'_{1-} \in (\gamma_{-}', \gamma_{+}') \text{ and } \gamma_{1+} \in (\gamma_{1-}', \gamma_{1+}'), \exists c, R > 0 \text{ s.t. } \forall \varphi \in [\gamma_{1-}', \gamma_{1+}] \text{ and } \rho > R, \\
& \quad |\psi(\rho e^{i\varphi})| \leq c \rho'^\nu.
\end{align*}
\]

**Remark 2.2.** If \( \mathbb{R} \setminus \{0\} \not\subset C_+ \), hence, \( C_+ \) is adjacent to the real line but does not contain \( \mathbb{R} \setminus \{0\} \), then, in order to ensure the convergence of integrals in pricing formulas and justify conformal deformations of the contours of integration, we need to use lower bounds for \( \Re \psi(\xi) \) on \( \mathbb{R} \cup C_+ \). In all examples constructed in the paper, if \( C_+ \) is adjacent to \( \mathbb{R} \) but does not contain \( \mathbb{R} \setminus \{0\} \), \( X \) is of order \( 1^+ \) and the lower bound on \( C_+ \cup (\mathbb{R} \setminus \{0\}) \) is valid with \( |\xi| \) instead of \( |\xi| \ln(2 + |\xi|) \). The process is of order \((1, 1^+)\).

### 2.2. Spectrally positive and spectrally negative SINH-processes.

Domains of analyticity of spectrally one-sided SINH-processes are wider than the ones in Definition 2.1.

i) if there are no negative jumps, the strip of analyticity is of the form \( S_{(\mu_-, +\infty)} \), where \( \mu_- \leq 0 \), and \( C = i\mathbb{C}_\gamma \), where \( \gamma \in (\pi/2, \pi) \).

ii) if there are no positive jumps, the strip of analyticity is of the form \( S_{(-\infty, \mu_+)} \), where \( \mu_+ \geq 0 \), and \( C = -i\mathbb{C}_\gamma \), where \( \gamma \in (\pi/2, \pi) \).

**Proposition 2.4.** (a) A spectrally positive SINH-process of the upper order \( \nu \in \{0^+\} \cup (0, 1) \), with non-negative drift, is a subordinator.

(b) A spectrally negative SINH-process of the upper order \( \nu \in \{0^+\} \cup (0, 1) \), with non-positive drift, is the dual process to a subordinator.
Proof. (a) Let $x < 0$ and $\mu \geq 0$. Then, in the formula for the pdf
\begin{equation}
 p_t(x) = \frac{1}{2\pi} \int_{\text{Im}\xi = \omega} e^{-ix\xi - t(-i\mu\xi + \psi^0(\xi))}d\xi,
\end{equation}
where $\omega > \mu_-$, we can push the line of integration up: $\omega \to +\infty$, and, in the limit, obtain $p_t(x) = 0$. (b) is immediate from (a).

\begin{remark}
One-sided stable processes are SINH-regular since $\mu_- = 0$ or $\mu_+ = 0$ are allowed. For general stable Lévy processes, the strip of analyticity does not exist. Formally, $\mu_- = \mu_+ = 0$.
\end{remark}

2.3. Examples. Essentially all Lévy processes used in quantitative finance are SINH-regular.

2.3.1. The Brownian motion (BM). BM is of order 2; since $\psi^0(\xi) = \frac{\sigma^2}{2}\xi^2$ is an entire function, $C = \mathbb{C}$, $\mu_- = -\infty$, $\mu_+ = +\infty$. For any $\varphi \in [0, 2\pi)$,
\begin{equation}
 \psi^0(\rho e^{i\varphi}) \sim \frac{\sigma^2}{2}\rho^2 e^{2i\varphi}, \rho \to +\infty,
\end{equation}
hence, $c_{\infty}(\varphi) = \frac{\sigma^2}{2}e^{2i\varphi}$ has a positive real part iff $\cos(2\varphi) > 0$. It follows that $C_+ = C_{-\pi/4,\pi/4}$.

2.3.2. Merton model \[44\]. The characteristic exponent is given by
\begin{equation}
 \psi^0(\xi) = \frac{\sigma^2}{2}\xi^2 + \lambda \cdot (1 - e^{im\xi - \frac{\xi^2}{2}}),
\end{equation}
where $\sigma, s, \lambda > 0$ and $\mu, m \in \mathbb{R}$. As far as the analytical properties formulated in the definition of SINH-processes are concerned, the difference with BM is that $C = C_+ = C_{-\pi/4,\pi/4}$, and $C = C_{\gamma_-\gamma_+}$ with either $\gamma_- < -\pi/4$ or $\gamma_+ > \pi/4$ cannot be used.

2.3.3. Lévy processes with rational characteristic exponents and non-trivial BM component. The order is 2, and an admissible strip of analyticity $S_{(\mu_- \mu_+)}$ around the real axis may not contain poles of $\psi^0$. Explicit formulas for the Wiener-Hopf factors are easy to derive (see, e.g., \[12\]) in terms of poles of $\psi^0$, zeros of the function $q + \psi(\xi)$, the multiplicities of zeros and poles being taken into account. After $\mu_- , \mu_+$ are chosen, $C$ is the maximal cone around the real axis such that $\mathcal{U} = i(\mu_- , \mu_+) + (C \cup \{0\})$ contains no poles, and $C_+ = C \cap C_{-\pi/4,\pi/4}$. Since the efficiency of SINH-acceleration depends, mostly, on the “width” of $C_+$, it is advisable to choose small (in absolute value) $\mu_- \text{ and } \mu_+$ so that $C_+$ can be chosen “wider”. Lévy processes of the phase type \[11\] have rational characteristic exponents, hence, the recommendations above are applicable.

Calculation of the rational characteristic exponent is straightforward if the Lévy densities of positive and negative jumps are mixtures of exponential polynomials. Furthermore, all poles are on $i\mathbb{R}$, hence, $C = \mathbb{C} \setminus i\mathbb{R}$ \[34\]. The factorization of $q + \psi(\xi)$ (calculation of the Wiener-Hopf factors) simplifies if all the roots of the characteristic equation $\psi(\xi) + q = 0$ are on the imaginary axis. Then the roots can be easily calculated, and explicit formulas for the Wiener-Hopf factors as sums or products derived. See, e.g., \[34\]. A popular special case is the
hyperc–hyper-exponential jump-diffusion model (HEJD model) introduced in [31] and studied in detail in [34, 35]). The characteristic exponent is given by

\begin{equation}
F(dx) = \mathbf{1}_{(-\infty,0)}(x) \sum_{j=1}^{n^-} p_j^- \alpha_j^- e^{\alpha_j^- x} + \mathbf{1}_{(0,\infty)}(x) \sum_{j=1}^{n^+} p_j^+ \alpha_j^+ e^{-\alpha_j^+ x},
\end{equation}

where \( n^+ \) are positive integers, and \( \alpha_j^+, p_j^+ > 0 \) are reals. The characteristic exponent is

\begin{equation}
\psi^0(\xi) = \frac{\alpha^2}{2} \xi^2 - i \mu \xi + \sum_{j=1}^{n^+} p_j^+ \frac{-i \xi}{\alpha_j^+ - i \xi} + \sum_{k=1}^{n^-} p_k^- \frac{i \xi}{\alpha_k^- + i \xi},
\end{equation}

Double-exponential jump diffusion model introduced to finance in [31] (and well-known for decades) is a special case of hyper-exponential jump-diffusion models with \( n^+ = n^- = 1 \). The order is 2, \( \mu_+ = \min \alpha_k^- \), \( \mu_- = -\min \alpha_k^+ \), and \( C, C_+ \) are as in the BM model.

In [21], a class of processes with the Lévy measure of the form (2.11) with some of \( p_j^+ \) being negative is introduced, and the name mixed exponential jump diffusion model (MEJD) is suggested. Sufficient conditions for \( p_j^+ \) and \( \alpha_j^+ \) to define the non-negativity of the densities are \( p_j^+ > 0 \) and \( \sum_{j=1}^{k} p_j^+ \alpha_j^+ \geq 0 \), \( k = 1, 2, \ldots, n^\pm \). An important qualitative difference between HEJD and MEJD is that in HEJD models, the Lévy densities of positive and negative jumps are monotone (in fact, completely monotone), whereas in MEJD, the densities may be non-monotone. Note that the Lévy densities given by mixtures of exponential polynomials [34, 35] are typically non-monotone, and qualitative properties of MEJD densities can be easily reproduced by exponential polynomials. As the simplest example, the reader can compare the following two functions on \( \mathbb{R}_+ \): \( f_1(x) = e^{-\lambda_1 x} - e^{-\lambda_2 x}, \) where \( 0 < \lambda_1 < \lambda_2 \), and \( f_2(x) = x e^{-\lambda_1 x} \).

2.3.4. Variance Gamma processes (VGP). VG model was introduced to Finance in [32]. The characteristic exponent can be written in the form

\begin{equation}
\psi^0(\xi) = c[\ln(\alpha^2 - (\beta + i \xi)^2) - \ln(\alpha^2 - \beta^2)],
\end{equation}

where \( \alpha > |\beta| \geq 0, c > 0 \). VGP is SINH-regular of type \((-\alpha + \beta, \alpha + \beta); \mathbb{C} \setminus i \mathbb{R}, \mathbb{C} \setminus i \mathbb{R} \) and order 0+ because \( \forall \varphi \in (-\pi/2, \pi/2) \),

\begin{equation}
\psi^0(\rho e^{i \varphi}) = c(\ln \rho + i \varphi) + O(1), \ \rho \to +\infty.
\end{equation}

2.3.5. NIG and NTS. Normal inverse Gaussian (NIG) processes, and the generalization: Normal Tempered Stable (NTS) processes are constructed in [34, 35], respectively. The characteristic exponent is given by

\begin{equation}
\psi^0(\xi) = \delta[(\alpha^2 - (\beta + i \xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}],
\end{equation}

where \( \nu \in (0,2), \ \delta > 0, |\beta| < \alpha \); NIG obtains with \( \nu = 1 \). This is a SINH-regular process of order \( \nu \) and type \((-\alpha + \beta, \alpha + \beta); \mathbb{C} \setminus i \mathbb{R}, \mathbb{C}^{-\gamma_\nu, \gamma_\nu} \), where \( \gamma_\nu = \min\{1, 1/\nu\} \pi/2 \). Indeed, for \( \varphi \in (-\pi/2, \pi/2) \),

\begin{equation}
\psi^0(\rho e^{i \varphi}) = \delta e^{i \varphi} \rho^{\nu} + O(\rho^{\nu-1}) + O(1), \ \rho \to +\infty.
\end{equation}
2.3.6. The Meixner process. For the background, see, e.g., [49, 16, 13]. The Lévy density of the Meixner process $X$ is

$$f(x) = \frac{\exp(bx/a)}{x \sinh(\pi x/a)}$$

where $\delta, a > 0$ and the asymmetry parameter $b \in (-\pi, \pi)$. The characteristic exponent is

$$\psi^0(\xi) = 2\delta \lbrack \ln[\cosh((a\xi - ib)/2)] \rbrack - \ln \cos(b/2).$$

The formula $\ln \cosh(z) = z + \ln(1 + e^{-2z}) - \ln 2$ defines a function analytic in the right half-plane, hence, $\psi^0(\xi)$ admits analytic continuation to $\mathbb{C} \setminus i\mathbb{R}$. Set $\mu_- = (-\pi + b)/a, \mu_+ = (\pi + b)/a$. Since $\cosh((a\xi - ib)/2)) > 0$ for $\xi \in i(\mu_-, \mu_+)$, $\psi^0$ is analytic in $\mathbb{C} \setminus i((-\infty, \mu_-) \cup [\mu_+, +\infty))$. Let $\gamma \in (0, \pi/2)$. As $\xi \to \infty$ in $\mathbb{C}^+$, $\psi(\xi) \sim a\delta \xi$, therefore, $X$ is SINH-regular of order 1 and type $(((-\pi + b)/a, (\pi + b)/a), \mathbb{C} \setminus i\mathbb{R}, \mathbb{C} \setminus i\mathbb{R})$.

2.3.7. KoBoL processes. A generic process of Koponen’s family [10] was constructed as a mixture of spectrally negative and positive pure jump processes, with the Lévy measure

$$F(dx) = c_+ e^{\lambda_-^+ x} x^{-\nu_- - 1} 1_{(0, +\infty)}(x) dx + c_- e^{\lambda_-^- x} |x|^{-\nu_- - 1} 1_{(-\infty, 0)}(x) dx,$$

where $c_+ > 0, \nu_+ \in (0, 2), \lambda_- < 0 < \lambda_+$. In this paper, we allow for $c_+ = 0$ or $c_- = 0, \lambda_- = \lambda_+ = 0 < \lambda_+$. This generalization is almost immaterial for evaluation of probability distributions and expectations because for efficient calculations, the first crucial property, namely, the existence of a strip of analyticity of the characteristic exponent, around or adjacent to the real line, holds if $\lambda_- < \lambda_+$ and $\lambda_- \leq \lambda_+$. Furthermore, the Esscher transform allows one to reduce both cases $\lambda_- = 0 < \lambda_+$ and $\lambda_- < 0 \leq \lambda_+$.

(Formulas in the case $\nu_\pm = 0, 1$ are in Sect. A.1.2 and A.1.3.) A subclass with $\nu_+ = \nu_- = \nu \in (0, 2)$ and $c_+ = c_-$ was labelled KoBoL in [12] and called a process of order $\nu$. To simplify the name, we will call a pure jump process with the Lévy measure (2.19) a KoBoL process as well. As in [10], we allow for $\nu_- \neq \nu_+$ and $c_+ \neq c_-$. If either $c_- = 0$ or $c_+ = 0$, we say that the process is a one-sided KoBoL. The formula (A.10) for one-sided KoBoL of order 1 is different. One-sided KoBoL processes were used in [8] to price CDSs. Mixing one-sided processes of order $\nu \in (0, 2), \nu \neq 1$, and order 1, one can obtain more exotic characteristic exponents.

Note that a specialization $\nu \neq 1, c = c_+ > 0$, of KoBoL was named CGMY model in [22] (and the labels were changed: letters $C, G, M, Y$ replace the parameters $c, \nu, \lambda_-, \lambda_+$ of KoBoL):

$$\psi^0(\xi) = c_+ \Gamma(\nu) \lbrack (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu + \lambda_+^\nu - (\lambda_+ + i\xi)^\nu \rbrack.$$

Evidently, $\psi^0$ given by (2.21) is analytic in $\mathbb{C} \setminus i\mathbb{R}$, and $\forall \varphi \in (-\pi/2, \pi/2)$, (2.23) holds with

$$c_\infty(\varphi) = -2c\Gamma(-\nu) \cos(\nu\pi/2) e^{i\nu\varphi}.$$

Hence, $X$ is SINH-regular of type $((\lambda_-, \lambda_+), \mathbb{C} \setminus \{0\}, C^{-\gamma_\nu, \gamma_\nu})$, where $\gamma_\nu = \min\{1, 1/\nu\} \pi/2$, and order $\nu$. For the calculation of order and type for a generic KoBoL, see Section A.1.

The property does not hold if there is no such a strip (formally, $\lambda_- = 0 = \lambda_+$). The classical example are stable Lévy processes. The conformal deformation technique can be modified for this case as well [13].
2.3.8. The $\beta$-class \[32\]. The characteristic exponent is of the form

\begin{equation}(2.23)\quad \psi^0(\xi) = \frac{\sigma^2}{2} \xi^2 + \frac{c_1}{\beta_1} \left\{ B(\alpha_1, 1 - \gamma_1) - B(\alpha_1 - \frac{i \xi}{\beta_1}, 1 - \gamma_1) \right\} \\
+ \frac{c_2}{\beta_2} \left\{ B(\alpha_2, 1 - \gamma_2) - B(\alpha_2 + \frac{i \xi}{\beta_2}, 1 - \gamma_2) \right\}, \end{equation}

where $c_j \geq 0$, $\alpha_j, \beta_j > 0$ and $\gamma_j \in (0, 3) \setminus \{1, 2\}$, and $B(x, y)$ is the Beta-function. It was shown in \[32\], that all poles of $\psi^0$ are on $i \mathbb{R} \setminus \{0\}$. Hence, $(\mu_-, \mu_+)$ is the maximal interval containing 0 and no poles of $\psi^0$, and $\mathcal{C} = \mathbb{C} \setminus i \mathbb{R}$. For calculation of the order of the process and $\mathcal{C}_+$ as functions of the parameters in \(2.23\), see Section \[A.2\]. The number of parameters is larger than in the case of KoBoL but the variety of different cases (order and type) is essentially the same as in the case of KoBoL.

2.3.9. Meromorphic Lévy processes \[33\]. The Lévy measures and characteristic exponents of the meromorphic Lévy processes introduced in \[33\] are defined by almost the same formulas as in HEJD model. The difference is that the sums in \(2.11\) and \(2.12\) are infinite. A natural condition $\alpha_j^+ \to +\infty$ as $j \to +\infty$ is imposed, and the requirement that the infinite sum defines a Lévy measure is equivalent to $\sum_{j \geq 0} p_j^+ (\alpha_j^+)^{-2} < +\infty$. The poles of $\psi(\xi)$ are on $\mathbb{R} \setminus \{0\}$. If $\sigma^2 > 0$, meromorphic processes are SINH-regular of order 2 and type $((\mu_-, \mu_+), \mathcal{C} \setminus i \mathbb{R}, \mathcal{C}_{2,-\pi/4, \pi/4})$, where $(\mu_-, \mu_+)$ is the maximal interval containing 0 and no poles of $\psi^0$. If $\sigma^2 = 0$, additional conditions on the asymptotics of $p_j^+$ and $\alpha_j^+$ as $j \to +\infty$ need to be imposed to obtain a SINH-regular process. See Section \[3.6\].

2.4. SINH-regular infinitely divisible distributions and related Lévy processes. Probability distributions of Lévy processes are infinitely divisible, and any infinitely divisible distribution gives rise to a Lévy processes \[17\].

**Definition 2.5.** A distribution $\rho$ on $\mathbb{R}$ is called SINH-regular of order $\nu$ (resp., $(\nu', \nu)$) and type $((\mu_-, \mu_+), \mathcal{C}, \mathcal{C}_+)$ if the characteristic function of $\rho$ is of the form $e^{\mu \xi - \psi^0(\xi)}$, where $\psi^0$ satisfies the conditions of Definition \[2.1\] (resp., \[2.3\]).

**Example 2.6.** The characteristic function $F(\xi)$ of a generalized hyperbolic distribution constructed in \[3\] depends on parameters $\alpha, \beta, \delta, \lambda$, where $\alpha > |\beta|$, $\delta \geq 0$ and $\lambda, \mu \in \mathbb{R}$:

\begin{equation}(2.24)\quad F(\xi) = e^{i\mu \xi} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + i\xi)^2} \right)^{\lambda/2} \frac{K_\lambda(\delta \sqrt{\alpha^2 - (\beta + i\xi)^2})}{K_\lambda(\delta \sqrt{\alpha^2 - \beta^2})}. \end{equation}

Here, $K_\lambda$ is the modified Bessel function of the third kind. From the integral representation

\begin{equation}(2.25)\quad K_\lambda(z) = \int_0^\infty e^{-z \cosh t} \cosh(\lambda t) dt \quad (|\arg z| < \pi/2) \end{equation}

and the asymptotic expansion

\begin{equation}(2.26)\quad K_\lambda(z) = \frac{(\pi)}{2z}^{1/2} e^{-z} \sum_{s=0}^\infty \frac{A_s(\lambda)}{z^s} \quad (z \to \infty \text{ in } |\arg z| < 3\pi/2 - \epsilon), \end{equation}

where $\epsilon > 0$ is arbitrarily small (see \[45\] Eq. (8.03),(8.04)), it follows that, if $\delta > 0$, the distribution is SINH-regular of the same type and order as NIG.
Lemma 3.2. Let the characteristic exponent \( F(x) \) using the inverse Fourier transform \( (3.1) \) for the sign \(-\) or \((3.2) \) that \( \psi(x) \) differentiates under the integral sign in the Lévy-Khintchine formula is justified. We obtain \( \mu \) where \( \psi(x) \) \( L \) upwards, and the wings of \( \phi_\mu \实用文语种

Example 2.7. The conditional distributions in the Heston model are of order 1 and type \( ((\mu_-, \mu_+), \mathbb{C} \setminus i\mathbb{R}, \mathbb{C} \setminus i\mathbb{R}) \), where \( \mu_- < 0 < \mu_+ \) can be calculated. See [36].

Example 2.8. The conditional distributions in a wide class of stochastic volatility models with stochastic interest rates are of order \( \nu \in (0, 1) \) and type \( ((\mu_-, \mu_+), \mathcal{C}_{-\pi/4, \pi/4}, \mathcal{C}_{-\pi/4, \pi/4}) \), where \( \mu_- < 0 < \mu_+ \). See [39].

3. Stieltjes-Lévy and signed Stieltjes-Lévy processes

In this Section, we do not need the lower bound on \( \text{Re} \psi^0 \).

3.1. Representations of the Lévy measure via the characteristic exponent. Using the Cauchy integral theorem (for details, see Section [A.3]), we derive

Lemma 3.1. Let \( \psi \) be analytic in \( i(\mu_- + \mu_+) + (\mathbb{C} \setminus \{0\}) \), where \( \mu_- \leq 0 \leq \mu_+, \mu_- < \mu_+, \mathbb{C} \setminus \{0\} \subset \mathbb{R} \), and let \( (\mu, \mu_+, \mu_-) \) hold for some \( \nu \in (0, 2) \). Then, for any closed cone \( \mathcal{C}' \subset \mathcal{C} \cup \{0\} \) and interval \( [\mu_-', \mu_+]' \subset (\mu_-, \mu_+) \), there exist \( C, c > 0 \), such that, for \( j = 0, 1, \ldots, \) and \( \xi \in [\mu_-', \mu_+]' + \mathcal{C}' \),

\[
|\psi_j(\xi)| \leq Cj!e^{-j} \left( \prod_{k=0}^{j} (\nu - k) \right) (1 + |\xi|)^{\nu-j}.
\]

Using Lemma 3.1, we derive representations for the Lévy measure \( F(dx) = f(x)dx \) in the form of integrals over regular contours \( \mathcal{L}^\pm \subset i(\mu_- + \mu_+) + (\mathcal{C} \cup \{0\}) \) with the wings of \( \mathcal{L}^+ \) pointing upwards, and the wings of \( \mathcal{L}^- \) pointing downwards, for instance, \( \mathcal{L}^\pm = i\theta + (e^{i(\pi - \omega)} \mathbb{R}^+ \cup e^{i\omega} \mathbb{R}^+), \) or \( \mathcal{L}^- = i\omega_1 + b \sinh(i\omega + \mathbb{R}), \) where \( b > 0, \theta \in (\mu_- + \mu_+), \omega_1 + b \sinh \omega \in (\mu_- + \mu_+) \) and \( \omega > 0 \) for the sign \(-\) (resp., \( \omega < 0 \) for the sign \("+\" \)) is sufficiently small in absolute value.

Lemma 3.2. Let the characteristic exponent \( \psi^0 \) satisfy conditions of Lemma 3.1. Then the Lévy measure is absolutely continuous: \( F(dx) = f(x)dx, \ f \in C^\infty(\mathbb{R} \setminus \{0\}) \), and

\[
\begin{align*}
\text{f}(x) &= -\frac{1}{2\pi} \int_{L^-} e^{-ix\xi} \psi(\xi)d\xi, \ x > 0, \\
\text{f}(x) &= -\frac{1}{2\pi} \int_{L^+} e^{-ix\xi} \psi(\xi)d\xi, \ x < 0.
\end{align*}
\]

Proof. Let \( \mu_- < 0 < \mu_+. \) Then the Lévy density exponentially decays at infinity, and the differentiation under the integral sign in the Lévy-Khintchine formula is justified. We obtain \( (3.4) \)

\[
\psi''(\xi) = \int_{\mathbb{R}} e^{\imath x\xi} x^3 F(dx),
\]

where \( \psi'' \) is the characteristic function of the pure jump component. It follows from \( (3.4) \) and \( (3.1) \) that \( F(dx) = f(x)dx \) is absolutely continuous, \( f \in C^\infty(\mathbb{R} \setminus \{0\}) \), and \( f \) can be recovered using the inverse Fourier transform

\[
ix^3 f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} \psi''(\xi)d\xi, \ x \neq 0.
\]
If $x > 0$, we deform the line of integration downward

$$\int_{\mathcal{L}_-} e^{-ix\xi} \psi''_f(\xi) d\xi, \quad x > 0. \tag{3.6}$$

For any polynomial $P(\xi)$ and $x > 0$, $\int_{\mathcal{L}_-} e^{-ix\xi} P(\xi) d\xi = 0$, therefore, integrating in $(3.6)$ by parts, we obtain $(3.2)$. The proof of $(3.3)$ is by symmetry.

If either $\mu_+ = 0 < \mu_+$ or $\mu_- > 0 = \mu_+$, we take $\alpha \in (\mu_- - \mu_+, \mu_+ + \alpha)$, and consider the Esscher transform of $X$, with the characteristic exponent $\psi(\xi - i\alpha) - \psi(-i\alpha)$ and the Lévy measure $e^{\alpha x} F(dx)$. We have

$$e^{\alpha x} f(x) = -\frac{1}{2\pi} \int_{\mathcal{L}_-} e^{-ix\xi} \psi(\xi - i\alpha) d\xi, \quad x > 0, \tag{3.7}$$

where $\mathcal{L}_\alpha = \{\xi \in \mathbb{C} : i(\mu_+ - \alpha \mu_+ - \alpha) + (C \cup \{0\})$ is a sufficiently regular contour with the wings pointing down. Shifting the contour and changing the variable $\xi = \xi' + i\alpha$, we obtain $(3.2)$. Eq. $(3.3)$ is proved similarly.

\[\square\]

### 3.2. Representations of the Lévy measure in the case $C = \mathbb{C} \setminus i\mathbb{R}$

If $C = \mathbb{C} \setminus i\mathbb{R}$, then, under additional conditions, we can express $f_+ = 1_{(0, +\infty)} f$ and $f_- = 1_{(-\infty, 0)} f$ in terms of integrals over the cuts $i(\mu_- - \alpha \mu_+ + \alpha)$ and $i(\mu_+, +\infty)$, respectively, w.r.t. to certain measures $\mathcal{G}_\pm(dt) = \mathcal{G}_\pm(\psi; dt)$ (possibly, signed):

$$f_+(x) = \int_{(0, +\infty)} e^{-ix\xi} \mathcal{G}_+(dt), \quad x > 0, \tag{3.8}$$

$$f_-(x) = \int_{(0, +\infty)} e^{ix\xi} \mathcal{G}_-(dt), \quad x < 0, \tag{3.9}$$

where $\text{supp} \mathcal{G}_+ \subseteq [-\mu_+, +\infty)$ (if $\mu_+ = 0$, $\mathcal{G}_+$ has no atom at 0), and $\text{supp} \mathcal{G}_- \subseteq [\mu_+, +\infty)$ (if $\mu_- = 0$, $\mathcal{G}_-$ has no atom at 0). In terms of the Laplace transforms $\hat{\mathcal{G}}_\pm(dt)$ of measures $\mathcal{G}_\pm(dt)$

$$f_+(x) = \hat{\mathcal{G}}_+(dt)(x), \quad f_-(x) = \hat{\mathcal{G}}_-(dt)(-x). \tag{3.10}$$

**Example 3.3.** In the case of HEJD, the $\beta$-model and meromorphic processes in general, we can derive the representations $(3.8)$, $(3.9)$ moving the contour of integration in $(3.3)$ up and in $(3.2)$ down, and, on crossing each simple pole on the corresponding imaginary half-axis, apply the residue theorem. For calculation of the residues in the $\beta$-model, see Section A.3. The measures $\mathcal{G}_\pm(dt) = \mathcal{G}_\pm(\psi; dt)$ are discrete:

$$\mathcal{G}_\pm(dt) = \sum_{\alpha \in \mathcal{A}_\pm} g_\alpha^\pm \delta_\alpha, \tag{3.11}$$

where $\mathcal{A}_\pm$ are finite or discrete sets with the only accumulation point at $+\infty$. The set $-\mathcal{A}_+$ (resp., $\mathcal{A}_-$) is the set of (simple) poles of $\psi$ on $(-\infty, 0)$ (resp., $(0, +\infty)$), and $g_\alpha^+ = \text{Res}(i\psi, -i\alpha)$, $g_\alpha^- = -\text{Res}(i\psi, i\alpha)$ are positive.

If we relax the conditions on the parameters of HEJD and meromorphic model so that some of $g_\alpha^+$ are negative (but the process is a Lévy process), then we obtain signed SL-processes. MEJD processes are obtained in this way, and one can generalize the class of meromorphic processes in a similar way. For further generalizations, see Example 3.18.
Example 3.4. Let \( X \) be the one-sided stable Lévy process of index \( \alpha \in (0, 2), \alpha \neq 1 \), with the Lévy density \( f_+(x) = x^{-\alpha-1}1_{x>0} \) and the characteristic exponent \( \psi_+(\xi) = -\Gamma(-\alpha)(0 - i\xi)^\alpha \).

We reduce the integral (3.8) to the cut \( i(-\infty, 0) \):

\[
f_+(x) = \frac{\Gamma(-\alpha)}{2\pi} \left( \int_{-\infty}^{0} - \int_{0}^{\infty} \right) (0 - i\xi)^\alpha e^{-ix\xi} d\xi
= \frac{\Gamma(-\alpha)}{\pi} \int_{0}^{+\infty} e^{i\pi\alpha} - e^{-i\pi\alpha} 2i t^\alpha e^{-tx} dt.
\]

Thus, (3.8) holds with \( G_+(dt) = \Gamma(-\alpha) \sin(-\pi\alpha)\pi^{-1}t^\alpha dt \).

Definition 3.5. \( X^\pm = X_{G^\pm} \) denotes the one-sided Lévy processes given by the generating triplets \( (0, 0, f_\pm(x)dx) \), where \( f_\pm(x) := f_\pm(G^\pm, x) \) are defined by (3.8)-(3.9). The characteristic exponents of \( X^\pm \) are denoted \( \psi_\pm(\xi) := \psi_\pm(G^\pm, \xi) \).

Evidently, \( \psi_-(G; \xi) = \psi_+(G, -\xi) \), therefore, it suffices to derive the condition for \( G \) to define the Lévy densities for \( X^+_G \) and formula for \( \psi_+(G, \xi) \); the condition for \( X^-_G \) is the same, and the formula for \( \psi_-(G; \xi) \) obtains by symmetry.

Lemma 3.6. (a) Let \( G_+ \geq 0 \). Then (3.8) defines a Lévy density if and only if

\[
\int_{(0, +\infty)} \frac{G_+(dt)}{t + t^m} < \infty,
\]

where \( m = 3 \); the density is completely monotone.

(b) If

\[
\int_{(0, +\infty)} \frac{G(dt)}{t + t^m} < \infty,
\]

where \( m = 3 \), and \( f_+ \) given by (3.8) is non-negative, then \( f_+ \) is a Lévy density.

(c) The pure jump process \( X^+ \) with the Lévy density \( f_+ \) is of finite variation iff (3.13) holds with \( m = 2 \).

Proof. Substitute (3.8) into integrals \( \int_{(0, 1]} x^2 f_+(x)dx, \int_{(0, 1]} x f_+(x)dx, \int_{[1, +\infty)} x^2 f_+(x)dx \), and use Fubini’s theorem. \( \square \)

Measures \( G(dt) \) satisfying conditions of Lemma 3.6 admit natural representations in terms of Stieltjes measures, and inherit several important properties of the latter.

3.3. Stieltjes measures and functions. Stieltjes transform.

Definition 3.7. A non-negative measure \( G \) on \((0, +\infty)\) is a Stieltjes measure iff

\[
\int_{(0, +\infty)} (1 + t)^{-1}G(dt) < \infty.
\]

We write \( G \in SM_0 \). The Stieltjes transform \( ST(G) \) of \( G \) is given by

\[
ST(G)(z) = \int_{(0, +\infty)} (z + t)^{-1}G(dt).
\]
The definition of the Stieltjes transform is standard - see, e.g., ([18 Defin. 2.1]). We introduce the notation $ST(G)$ to shorten the formulation of the definitions, statements and proofs below.

**Definition 3.8.** If $G \in SM_0$ and supp $G \subset [\mu, +\infty)$, where $\mu > 0$, we write $G \in SM_\mu$.

If $G \in SM_\mu$, $\mu \geq 0$, and $f = ST(G_\mu)$, we write $f \in S_\mu$.

Evidently, for any $\mu' \in [0, \mu]$, $SM_\mu \subset SM_{\mu'}$, $S_{\mu'} \subset S_{\mu}$.

The following proposition is immediate from the definition.

**Proposition 3.9.** Let $G \in SM_\mu$. Then $ST(G)$ is analytic in $\mathbb{C} \setminus (-\infty, -\mu]$.

**Lemma 3.10.** Let $G \in SM_0$. Then, for any $\gamma \in (0, \pi)$, $ST(G)(\rho e^{i\varphi}) \to 0$ as $\rho \to +\infty$, uniformly in $\varphi \in [-\gamma, \gamma]$.

**Proof.** There exists $C = C(\gamma) > 0$ such that, uniformly in $\rho > 1$ and $\varphi \in [-\gamma, \gamma]$,

$$\left| \int_{(0, +\infty)} (t + \rho e^{i\varphi})^{-1} G(dt) \right| \leq C \int_{(0, +\infty)} (t + \rho)^{-1} G(dt) \leq C \int_{(0, +\infty)} (t + 1)^{-1} G(dt) < \infty,$$

and it remains to apply the dominated convergence theorem. $\square$

The following lemma is a straightforward reformulation of ([18 Cor. 6.3]

**Lemma 3.11.** Let $G \in SM_0$. Then for all positive continuity points $u < v$ of $t \mapsto G((-\infty, t])$,

$$G((u, v]) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{[-v, -u]} ST(G)(s - i\epsilon)ds.$$  

**Lemma 3.12.** Let $G \in SM_\mu$ have no atom at $\mu$. Then

$$\lim_{\epsilon \to 0^+} \int_{|z+\mu|=\epsilon, \text{Re} z > -\mu} ST(G)(z)dz = 0.$$  

**Proof in Section A.5**

### 3.4. Definition of SL and sSL processes and examples

The class $S$ of Stieltjes functions (see, e.g., ([18 Defin. 2.1])) is wider than $S_0$: $f \in S$ if there exist $a_0, a_1 \geq 0$ and $G \in SM_0$ such that $f(z) = a_0/z + a_1 + ST(G)(z)$. For construction of spectrally one-sided Lévy processes, $S$ is appropriate. Indeed, any $G \in SM_0$ is the Stieltjes measure of a complete Bernstein function $g$, which appears in the Stieltjes representation of $g$:

$$g(z) = a_0 + a_1 z + z ST(G)(z)$$

(see, e.g., ([18 Thm 6.2, Corr. 6.3 and Remark 6.4]). Evidently, if $a_0 = 0$, $\psi(\xi) = g(-i\xi)$ is the characteristic exponent of a subordinator.

For construction of more general Lévy processes, class $S_0$ is more convenient. We start with the following evident proposition.

**Proposition 3.13.** Measure $G$ satisfies (3.12) with $m = 3$ (resp., with $m = 2$) iff there exist $a_2, a_1 \geq 0$ and $G^0 \in SM_0$ such that $G(dt) = (a_2 t^{2} + a_1 t) G^0(dt)$ (resp., $G(dt) = t G^0(dt)$).
**Definition 3.14.** Let $\mu \geq 0$. We say that the measure $\mathcal{G}$ on $[\mu, +\infty)$ is a Stieltjes-Lévy measure (SL measure) of class $\text{SLM}_\mu$ if there exists $a_2, a_1 \geq 0$, $a_2 + a_1 > 0$, such that $(a_2 t^2 + a_1 t)^{-1} \mathcal{G}(dt) \in \text{SM}_\mu$.

We say that $\mathcal{G}$ is a signed Stieltjes-Lévy measure (sSL measure) of class $\text{sSLM}_\mu$ if $\mathcal{G}$ admits the representation $\mathcal{G} = \mathcal{G}_1 - \mathcal{G}_2$, where $\mathcal{G}_j \in \text{SLM}_\mu$, and the Laplace transform $\hat{\mathcal{G}}$ of $\mathcal{G}$ is non-negative on $(0, +\infty)$.

The following evident proposition will be used in the study of Meixner processes in Example 3.29. The proposition admits natural generalizations using maps more general than translations. See Example 5.9.

**Proposition 3.15.** Let $\mathcal{G}_1 \in \text{SLM}_0, \mathcal{G}_2 \in \text{SLM}_\mu$, where $\mu > 0$, and let there exist $A \in (0, \mu)$ such that, for any $0 \leq u < v$, $\mathcal{G}_1((u - A, v - A)) \geq \mathcal{G}_2((u, v))$. Then $\mathcal{G}_1 - \mathcal{G}_2 \in \text{sSLM}_0$.

**Definition 3.16.** Let $\mathcal{G}^0 \in \text{SM}_\mu, \mathcal{G}(dt) = (a_2 t^2 + a_1 t)\mathcal{G}^0(dt)$, and $X^\pm = X^\pm_\mathcal{G}$. Then we write $X^\pm \in SL^\pm_\mu$. If $a_1 = 0$ (resp., $a_2 = 0$), we write $X^\pm \in SL^{2;\pm}_\mu$ (resp., $X^\pm \in SL^{1;\pm}_\mu$).

Evidently, if $\mu > 0$, one can use a simpler definition of classes $SL^{2;\pm}_\mu$ instead of the general definition of classes $SL^{1;\pm}_\mu$; the statement $X \in SL^{1;\pm}_\mu$ is useful if we wish to indicate that the jump component of $X$ is of finite variation. The following proposition demonstrates that $SL^\pm_0 \neq SL^{1;\pm}_0 \cup SL^{2;\pm}_0$.

**Proposition 3.17.** Let $X^\pm$ be the one-sided stable Lévy process of index $\alpha \in (0, 2)$, with the Lévy density $f^\pm(x) = \Gamma(\alpha + 1)|x|^{-\alpha - 1}, \pm x > 0$. Then

(a) if $\alpha \in (0, 1)$, then $X^\pm$ is in $SL^{1;\pm}_0$ but not in $SL^{2;\pm}_0$;

(b) if $\alpha \in (1, 2)$, then $X^\pm$ is in $SL^{2;\pm}_0$ but not in $SL^{1;\pm}_0$;

(c) if $\alpha = 1$, then $X^\pm$ is in $SL^\pm_0$ but not in $SL^{1;\pm}_0 \cup SL^{2;\pm}_0$.

**Proof.** Since $\Gamma(\alpha + 1)x^{-\alpha - 1} = \int_0^{+\infty} e^{-tx}t^\alpha dt$, the Stieltjes-Lévy measure is $\mathcal{G}_+(dt) = t^\alpha dt$.

If $\alpha \in (0, 1)$, $t^{\alpha - 1}dt \in \text{SM}_0$; however, $t^{\alpha - 2}dt$ is not in $\text{SM}_0$.

If $\alpha \in (1, 2)$, $t^{\alpha - 2}dt \in \text{SM}_0$; however, $t^{\alpha - 1}dt$ is not in $\text{SM}_0$.

If $\alpha = 1$, $(t + t^2)^{-1}dt \in \text{SM}_0$; however, neither $t^{\alpha - 1}dt$ nor $t^{\alpha - 2}dt$ are in $\text{SM}_0$. \qed

**Example 3.18.** The natural generalization of one-sided meromorphic processes is the class of $SL^\pm_0$ processes defined by atomic measures of the form

$$\mathcal{G}(dt) = \sum_{\alpha \in \mathcal{A}} g_\alpha \delta_\alpha,$$

where $\mathcal{A} \subset (0, +\infty)$ is a countable set and, for some $a_2, a_1 \geq 0$,

$$\sum_{\alpha \in \mathcal{A}} \frac{g_\alpha}{(a_2 \alpha^2 + a_1 \alpha)(1 + \alpha)} < \infty.$$

If $\mathcal{A}$ is finite, we obtain one-sided HEJD processes. If the only point of accumulation of $\mathcal{A}$ is $+\infty$, then we obtain meromorphic processes, the $\beta$-model in particular. In the $\beta$-model, the sequence $\mathcal{A}$ is uniformly spaced. Meromorphic model can be used to approximate KoBoL (albeit rather inefficiently, especially if the atoms are uniformly spaced - more efficient approximations can be obtained using non-uniform sequences of atoms). If we allow for $\mathcal{A}$ to have an accumulation point at $0$, then we can approximate stable Lévy processes (also inefficiently).
Example 3.19. In Section [3.5] we prove that stable Lévy processes, KoBoL, VGP, NIG, and NTS are SL-processes, whereas Meixner processes are sSL processes, and derive explicit formulas for the corresponding SL- and sSL-measures.

Theorem 3.20. Let \( \mathcal{G}(dt) = (a_2t^2 + a_1t)\mathcal{G}^0(dt) \in SLM_\mu \) and \( X^\pm = X^\pm_\mu \). Then

(a) the characteristic exponent of \( X^\pm \) is of the form

\[
\psi_{\pm}(\xi) = (a_2\xi^2 \mp ia_1\xi)ST(\mathcal{G}^0)(\mp i\xi) \pm i\mu \xi,
\]

where \( \mu = \mu(a_2, a_1, \mathcal{G}^0) \in \mathbb{R} \). If \( a_2 = 0 \), and the Lévy-Khintchine formula for processes of finite variation is used, then \( \mu = 0 \);

(b) \( X^\pm \in SL_{\mu,\pm}^1 \) are finite variation processes;

(c) if \( a_2 = 0 \) and \( t\mathcal{G}^0(dt) \in L_1 \), then \( X^\pm \in SL_{\mu,\pm}^1 \) are of finite activity;

(d) if \( \mu > 0 \), then \( SL_{\mu,\pm}^1 = SL_{\mu,\pm}^2 \);

(e) if \( X^\pm \in SL_{\mu,\pm}^1 \) and \( t\mathcal{G}^0(dt) \in SM_\mu \), then \( X^\pm \in SL_{\mu,\pm}^1 \).

Proof. To prove (a), we use the Lévy-Khintchine formula and Fubini’s theorem. For details, see Section A.6. (b) follows from Lemma 3.6, and (c) can be proved in the same fashion. (d) \( X^\pm \) can be defined as an element of \( SL_{\mu,\pm}^1 \) by the measure \( (a_2 + a_1t^{-1})\mathcal{G}^0(dt) \in SM_\mu \). (e) \( X^\pm \) can be defined as an element of \( SL_{\mu,\pm}^1 \) by the measure \( (a_2t + a_1)\mathcal{G}^0(dt) \in SM_\mu \).

In the following Lemma, by a slight abuse of notation, we use the same label \( \psi_{\pm} \) for the characteristic exponent defined by the generating triplet \((\sigma^2, b, f_+(x))dx\), where \( f_+(x) = f_+(\mathcal{G}; x) \) are defined by (3.3) with \( \mathcal{G}_{\pm} = \mathcal{G}(a_2, a_1, \mathcal{G}^0) \). In the case \( a_2 = 0 \), we use the Lévy-Khintchine formula for jump component of finite variation. The proof of the lemma is immediate from Theorem 3.20, Lemma 3.10 and Proposition 3.9.

Lemma 3.21. (a) \( \psi_+ \) is analytic in \( \mathbb{C} \setminus [i(-\infty, -\mu], \psi_- \) is analytic in \( \mathbb{C} \setminus [i\mu, +\infty) \);

(b) \( \forall \gamma \in (0, \pi), \psi_+(\xi) \sim \frac{2\xi^2}{\gamma} \) as \( (\pm i\gamma, \pm \gamma) \xi \to \infty \), uniformly in \( \arg \xi \in [-\gamma, \gamma] \).

(c) If \( \sigma^2 = a_2 = 0 \), then, \( \forall \gamma \in (0, \pi), \psi_+(\xi) \sim -ib\xi \) as \( (\pm i\gamma, \pm \gamma) \xi \to \infty \), uniformly in \( \arg \xi \in [-\gamma, \gamma] \).

Definition 3.22. Let \( \mu_- \leq 0 \leq \mu_+ \). A Lévy process \( X \) is called a signed Stieltjes-Lévy process (sSL-process) of class \( sSL_{\mu_- \mu_+} \) if the Lévy density of \( X \) is of the form

\[
f(x) = 1_{(-\infty, 0)}(x) \int_{(0, +\infty)} e^{tx} \mathcal{G}^-(dt) + 1_{(0, +\infty)}(x) \int_{(0, +\infty)} e^{-tx} \mathcal{G}^+(dt),
\]

where \( \mathcal{G}^- \in SLM_{\mu_-} \) and \( \mathcal{G}^+ \in SLM_{\mu_+} \). If \( \mathcal{G}^- \in SLM_{\mu_-} \) and \( \mathcal{G}^+ \in SLM_{\mu_+} \), \( X \) is called a Stieltjes-Lévy process (sSL-process) of class \( sSL_{\mu_- \mu_+} \).

The following theorem is immediate from Definition 3.22 and Lemma 3.21.

Theorem 3.23. Let \( X \in sSL_{\mu_- \mu_+} \). Then

(i) the characteristic exponent \( \psi \) of \( X \) is of the form (1.1), where \( \mathcal{G}^0_+ \in SM_{-\mu_-}, \mathcal{G}^0_- \in SM_{\mu_+}; \)

(ii) \( \psi \) admits analytic continuation to \( \mathbb{C} \setminus [i(-\infty, -\mu_-] \cup [\mu_+, +\infty)) \);

(iii) \( \forall \gamma \in (0, \pi/2), \psi(\xi) \sim \frac{2\xi^2}{\gamma} \) as \( \xi \to \infty \), uniformly in \( \arg \xi \in [-\gamma, \gamma] \cup [\pi - \gamma, \pi + \gamma] \);

(iv) if \( \sigma^2 = a_2^+ = a_2^- = 0 \), then, for any \( \gamma \in (0, \pi/2) \), \( \psi(\xi) \sim -ib\xi \) as \( \xi \to \infty \), uniformly in \( \arg \xi \in [-\gamma, \gamma] \cup [\pi - \gamma, \pi + \gamma] \).
3.5. Representations of sSL-measures in terms of the characteristic exponent. Let $G_+ = G_{+,+} - G_{+,+}$, $G_{+,±} \in \text{SLM}_{τ+}$ and $G_- = G_{-,+} - G_{-,+}$, $G_{-,±} \in \text{SLM}_{τ+}$ be the Jordan decompositions of measures $G_+ \in G$ in (3.21) Denote by $U_{+,±}$ the set of points of continuity of $G_+(-\infty, t)$ and set $U_+ = U_{+,+} \cap U_{+,+}$. Similarly, define $U_-$. 

**Theorem 3.24.** Let $X$ be of class sSL$_{τ-τ+}$, with the Lévy density (3.21) and characteristic exponent $ψ$. Then

(a) for any $θ \in (τ-,τ+)$ and $x > 0$,

$$f_0(±x) = \frac{1}{π} \lim_{ε \to 0+} \int_0^∞ e^{-tx} \text{Im} \psi(±it + ε)dt;$$

(b) for any $u, v \in U_+$,

$$G_0((u, v)) = \frac{1}{π} \int_u^v \text{Im} \psi(±it + ε)dt;$$

(c) if $G_0(±μ_±) = 0$, then (3.22) holds with $±μ_±$ in place of $θ$.

**Proof.** We assume that $G_+ \in \text{SLM}_{τ+}$; the statements for $G_+ \in \text{SLM}_{τ+}$ follow by linearity. By symmetry, it suffices to consider $f_0$ and $G_+$. To prove (3.22), we use the representation $G_0(dt) = (a_2 t^2 + a_1 t)G_0(dt)$, where $G_0 \in \text{SLM}_{τ+}$. Since the characteristic exponent $ψ_-$ of negative jumps and the one of the Brownian motion component are analytic in $\{\text{Im} ξ < θ\}$ and polynomially bounded, we can use (3.2) with $ψ_+$ given by (3.20) in place of $ψ$. We fix $γ \in (0, π/2)$, $A > τ_-$, and $ε > 0$, and deform $L^-_2$ into the union of the following contours: $L^-_2 = \{e^{i(π+γ)}ρ | ρ ≥ A/\sin γ\}$; $L^-_2 = -i A + (-A \cot γ, -ε); L^-_2 = \{ξ \mid \text{dist} (ξ, i[-A, θ]) = ε, \text{Im} ξ ≥ -A\}; L^-_4 = -i A + (ε, A \cot γ); L^-_5 = \{ξ = e^{iγ}\rho | ρ ≥ A/\sin γ\}$. On the strength of Theorem 3.23 the integrals over $L^-_2$ and $L^-_5$ tend to 0 as $A \to +∞$ (recall that $x > 0$).

Next, we note that there exists $C > 0$ independent of $R$ and $ε$ such that

$$\left| \left( \int_{L_2^-} + \int_{L_5^-} \right) e^{-ixξ} \psi(ξ)dξ \right| ≤ Ce^{-xA} \int_{(τ-,+∞} \text{G}_0^0(dt)(a_2 t^2 + a_1 t) \times \left| \left( \int_{-A \cot γ}^{x} + \int_{ε}^{A \cot γ} \right) \frac{dy}{t - A + iy} \right|$$

Straightforward calculations prove that $\left( \int_{-A}^{x} + \int_{ε}^{A} \right) \frac{dy}{t - A + iy}$ is uniformly bounded w.r.t. $t, A, ε$, and bounded by $C/(A + t)$ on $(-τ_-, +∞)$, uniformly in $ε > 0$. Taking into account that $G_0^0 \in \text{SM}_0$ and $x > 0$, we conclude that the sum of integrals over $L^-_2$ and $L^-_5$ tends to 0 as $A \to +∞$, uniformly in $ε > 0$. 

It remains to consider the limit of the integral over $L^+_e = L^+_e(A, ε)$ as $ε → 0$ and $A$ fixed. Since $ψ(ξ) = ψ(-ξ)$ and $ψ$ is real and continuous at any point of $i(μ_-, θ)$, we have

$$-rac{1}{2π} \lim_{ε→0+} \int_{L^+_e(A, ε)} e^{-ixξ} ψ_+(ξ)dξ$$

$$= \frac{1}{2π} \lim_{ε→0+} \left( \int_{-A}^{θ} e^{(t+it)x}(-ψ_+(it-ε))idt + \int_{θ}^{-A} e^{(t-it)x}ψ_+(it+ε))idt \right)$$

$$= \lim_{ε→0+} \int_{θ}^{A} e^{-tx}(1 + g_1(t, ε)) Im ψ_+(-it - ε)dt$$

$$= \lim_{ε→0+} \int_{θ}^{A} e^{-tx}(1 + g(t, ε))(a_t^2 + a_t) Im \ ST(\mathcal{G}_0^+(−it - ε))dt,$$

where $g_j(t, ε), j = 1, 2$, are continuous, and $g_j(t, x) = o(ε), j = 1, 2$, as $ε → 0$, uniformly in $t ≥ 0$. We can choose $A$ to be a point of continuity of $t → \mathcal{G}_0^+(θ, t)$. Next, if $μ_- < 0$, we can choose $θ ∈ (μ_-, 0)$. Then, by Lemma 3.11 the limit above is

$$\int_{θ}^{A} e^{-tx}(a_t^2 + a_t) \mathcal{G}_0^+(dt) = \int_{θ}^{A} e^{-tx} \mathcal{G}_0^+(dt).$$

We have proved that $f_+(x) - \int_{θ}^{A} e^{-tx} \mathcal{G}_0^+(dt) → 0$ as $(U_+ \ni)A → +∞$, hence, (3.22) holds. Using the Esscher transform, we can reduce the case $μ_- = 0$ to the case $μ_- < 0$. The argument above demonstrates that the Laplace transform of $\mathcal{G}_0^+$ and the Laplace transform of the measure defined by the RHS of (3.23) coincide, which proves (3.23). If $\mathcal{G}_0^+(\{-μ_-\}) = 0$, equivalently, $ψ$ does not have a pole at $iμ_-$, then we use (3.17) to conclude that we can pass to the limit $θ ↓ μ_-$ in the proof of (3.22) and (3.23).

\[\square\]

The following verification theorem allows us to prove that stable Lévy processes, KoBoL, VGP, NIG, NTS and Hyperbolic processes are SL-processes whereas Meixner processes are sSL-processes but not SL-processes; the conditions of the theorem can be relaxed.

**Theorem 3.25.** Let the characteristic exponent $ψ$ of a Lévy process satisfy the following conditions:

(i) $ψ$ is analytic in $C \setminus i((-∞, μ_-) \cup [μ_+, +∞))$, where $μ_- ≤ 0 ≤ μ_+$, and $μ_- < μ_+$;

(ii) $∃ \nu < 2, δ > 0$ and $C > 0$ s.t. $∀ ξ ∈ C \setminus i((-∞, μ_-) \cup [μ_+, +∞))$, the characteristic exponent of the jump component admits the bound

$$|ψ_j(ξ)| ≤ C(|ξ|^\nu + |ξ - iμ_-|^{1+δ} + |ξ - iμ_+|^{1+δ});$$

(iii) $∀ \beta ∈ (-∞, μ_-) \cup (μ_+, +∞)$, the limit $\lim Im ψ(iβ + 0)$ exists, a.e.

Then

(a) $\mathcal{G}_±(dt) = \frac{1}{2π} Im ψ(±(it + 0))dt$ are sSL-measures;

(b) $X$ is a sSL-process, with the Lévy density given by (3.21);

(c) if $Im ψ(it+0) ≥ 0$ for $t > μ_+$, and $Im ψ(-it-0) ≥ 0$ for $t > -μ_-$, then $\mathcal{G}_±$ are SL-measures, and $X$ is an SL-process.
Proof. (a) It suffices to notice that $\overline{\psi(\xi)} = \psi(-\bar{\xi})$, and functions $t \mapsto (1/\pi) \text{Im} \, \psi(\pm(it + 0))$ are measurable and satisfy the bound (3.24). (b) The proof is a straightforward simplification of the proof of Theorem 3.24, the bound (3.24) and the dominated convergence theorem being used. (c) is immediate from (a) and the definition of sSL- and SL-measures. □

Example 3.26. Let $\psi = \psi^0$ be given by (2.15) (the case of NIG and NTS processes). Clearly, $\psi(\xi)$ admits the bound (3.24) with $\delta = 1$. Next, we represent $\psi$ in the form

$$\psi(\xi) = \delta[(\alpha - \beta - i\xi)^{\nu/2}(\alpha + \beta + i\xi)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}],$$

and derive

$$\text{Im} \, \psi(it + 0) = \delta \sin(\pi \nu/2)((t - \beta)^2 - \alpha^2)^{\nu/2}, \quad t > \mu_+ := \alpha + \beta,$$

$$\text{Im} \, \psi(it - 0) = \delta \sin(\pi \nu/2)((t + \beta)^2 - \alpha^2)^{\nu/2}, \quad t < \mu_- := -\alpha + \beta.$$

Hence, $X$ is an SL process, which is a mixture of two independent one-sided SL-processes.

Example 3.27. Let $\psi = \psi^0$ be given by (2.20) (the case of KoBoL with the positive and negative densities of order $\nu_{\pm} \in (0,2) \setminus \{1\}$). Essentially the same argument as in Example 3.26 proves that $\psi$ is the characteristic exponent of an SL process, and

$$\text{Im} \, \psi(it + 0) = -c_- \Gamma(-\nu_-) \sin(\pi \nu_-)(t - \lambda_+)^{\nu_-}, \quad t > \lambda_+,$$

$$\text{Im} \, \psi(-it - 0) = -c_+ \Gamma(-\nu_+) \sin(\pi \nu_+)(t + \lambda_-)^{\nu_+}, \quad t > -\lambda_-.$$

If $\nu_{\pm} = 1$, then, using (A.10)-(A.12), we derive

$$\text{Im} \, \psi(it + 0) = \pi c_-(t - \lambda_+), \quad t > \lambda_+,$$

$$\text{Im} \, \psi(-it - 0) = \pi c_+(t + \lambda_-), \quad t > -\lambda_-.$$

Example 3.28. Let $\psi = \psi^0$ be given by (A.19). We have

$$\text{Im} \, \psi(it \pm 0) = \pi c_{\mp}, \quad \pm t > \pm \lambda_{\pm},$$

and essentially the same argument as in Example 3.26 proves that $\psi$ is the characteristic exponent of an SL-regular process. The only difference is that $\psi$ has the logarithmic singularities at $\lambda_{\pm}$, hence, the bound (3.24) holds with $\delta \in (0,1)$.

Example 3.29. Let $\psi^0$ be given by (2.18). For $\epsilon \neq 0$, and $t \in \mathbb{R}$, we have

$$\text{Im} \, \psi^0(it + \epsilon) = 2\delta \text{Im} \ln(\cosh(\epsilon c/2 + i(\epsilon a - \epsilon b)/2))$$

$$= 2\delta \text{Im} \ln \left[\frac{\cosh(\epsilon b/2 + i(\epsilon a - \epsilon b)/2)}{\cosh(\epsilon b/2)} \cos \frac{\epsilon a - \epsilon b}{2} + i \frac{\cosh(\epsilon b/2) - \cosh(\epsilon a - \epsilon b)/2}{2} \sin \frac{\epsilon a - \epsilon b}{2}\right].$$

If $\cos \frac{\epsilon a - \epsilon b}{2} > 0$, $\text{Im} \, \psi(it \pm 0) = 0$. If $\cos \frac{\epsilon a - \epsilon b}{2} < 0$ and $\pm \sin \frac{\epsilon a - \epsilon b}{2} > 0$, $\text{Im} \, \psi(it + 0) = \mp 2\delta \pi$. Hence, Meixner processes are sSL-processes but not SL-processes. The sSL-measures are $\mathcal{G}_{\pm}(dt) = g_{\pm}(t)dt$, where, for $t > 0$,

(3.33) $g_{\pm}(t) = 2\delta \sum_{k=0}^{\pm\infty} \left(1_{a^{-1}((4k+1)\pi + b,(4k+1)\pi + b)}(t) - 1_{a^{-1}((4k+1)\pi + b,(4k+1)\pi + b)}(t)\right),$

(3.34) $g_{\pm}(t) = 2\delta \sum_{k=-\infty}^{-1} \left(-1_{a^{-1}((4k+1)\pi + b,(4k+1)\pi + b)}(-t) + 1_{a^{-1}((4k+1)\pi + b,(4k+1)\pi + b)}(-t)\right).$
The proof of the following theorem is essentially the same as the one of Theorem 3.25. The difference is that we drop an assumption that $\psi$ is a characteristic exponent of a Lévy process.

**Theorem 3.30.** Let function $\psi$ satisfy the following conditions:

(i) $\psi$ is analytic in $\mathbb{C} \setminus i(-\infty, \mu_-] \cup [\mu_+ , +\infty)$, where $\mu_- \leq 0 \leq \mu_+$, and $\mu_- < \mu_+$;

(ii) $\psi(0) = 0$ and $\psi(\xi) = \overline{\psi(-\xi)}$, for all $\xi \in i(\mu_- , \mu_+) + (\mathbb{C} \setminus i\mathbb{R})$;

(iii) conditions (i)-(iii) of Theorem 3.25 hold;

(iv) the Laplace transforms of the measures $G_{\pm}(dt) = \frac{1}{\pi} \text{Im} \psi(\mp(it + 0))dt$ are non-negative.

Then the conclusions of Theorem 3.25 are valid.

**Remark 3.1.** Theorem 3.30 allows one to prove that the functions $\psi$ in the examples above are characteristic exponents without knowing this fact in advance. In the case of the Meixner processes, condition (iv) follows from (3.33)-(3.34) and Proposition 3.15; in the other examples, the measures are non-negative, hence, (iv) is satisfied.

3.6. Regular SL and sSL processes and distributions.

**Definition 3.31.** An sSL-process (resp., SL-process) $X$ is called regular sSL-process (resp., regular SL-process) if $X$ is SINH-regular.

**Proposition 3.32.** Let $X \in sSL_{\mu_- , \mu_+}$ and (2.7) hold. Then $X$ is SINH-regular of type $((\mu_- , \mu_+); \mathbb{C} \setminus i\mathbb{R}; C_+)$, and there exists $\nu \in [\nu', 2]$ such that the order of $X$ is $(\nu', \nu)$.

**Proof.** By Lemma 3.21 all conditions of Definition 2.3 are satisfied bar (2.7).

**Example 3.33.** KoBoL, VGP, NIG and NTS are regular SL-processes whereas Meixner processes are regular sSL-processes.

**Definition 3.34.** A distribution $\rho$ on $\mathbb{R}$ is called an SL (resp., sSL; SL-regular; sSL-regular) distribution if characteristic function of $\rho$ is of the form $e^{i\mu \xi - \psi^0(\xi)}$, where $\psi^0$ is the characteristic exponent of SL (resp., sSL; SL-regular; sSL-regular) process.

**Example 3.35.** In Section A.7, we prove

**Proposition 3.36.** Let $\sigma(dx)$ a Generalized Hyperbolic distribution with the characteristic function given by (2.27). Then

(i) $\sigma(dx)$ is a regular sSL distribution;

(ii) if $\lambda \in [-2, 1]$, $\sigma(dx)$ is a regular SL distribution;

(iii) if $\lambda > 1$, $\sigma(dx)$ is not an SL distribution;

(iv) if $\lambda < -2$, then, for sufficiently small $\delta$ and/or $\alpha - |\beta|$, $\sigma(dx)$ is not an SL distribution.

In particular, hyperbolic distributions ($\lambda = 1$) and processes are SL-regular.

In the general case, verification of (2.7) for $\psi$ is reducible to verification of (2.7) for a spectrally one-sided pure jump SL process $X$. We formulate sufficient conditions on the measure $G \in SM_{\mu}$ in the representation $(a_2t^2 + a_1t)G(dt)$ of the SLM measure of $X$.

If $G(dt)$ is absolutely continuous: $G(dt) = g(t)dt$, we assume that there exists $\alpha < 2$ and $c,C > 0$ such that

$$ct^\alpha \leq g(t) \leq Ct^\alpha, \ t > 0.$$
If $\mathcal{G}(dt)$ is has atoms and/or the density is unbounded, then an analog of (3.35) is more complicated. We assume that there exist $C, c > 0$ and $\rho_0 \geq 1$ such that for $k = 0, 1$, and any $\rho \geq \rho_0$

$$c \rho^{\alpha+1} \int_{\mu/\rho}^{+\infty} \frac{t^{k+\alpha} dt}{t^2 + 1} \leq \int_{\mu/\rho}^{+\infty} t^k \mathcal{G}(\rho dt) \frac{dt}{t^2 + 1} \leq C \rho^{\alpha+1} \int_{\mu/\rho}^{+\infty} \frac{t^{k+\alpha} dt}{t^2 + 1}, \quad k = 0, 1, \rho \geq \rho_0.$$  

Clearly, (3.35) implies (3.36).

**Theorem 3.37.** Let there exist $\alpha \in [-1,0)$ such that (3.36) holds. Then

(a) if $\psi(\xi) = \xi^2 ST(\mathcal{G})(\mp i\xi) \mp i\mu \xi$ and $\alpha \in (-1,0)$, then $X^\pm$ is SINH-regular of order $(\nu,\nu)$, where $\nu = \alpha + 2$, and the interior of the cone $C_+$ contains $\mathbb{R} \setminus 0$;
(b) if $\psi(\xi) = \xi^2 ST(\mathcal{G})(\mp i\xi) \mp i\mu \xi$ and $\alpha = -1$, then $X^\pm$ is SINH-regular of order $(1,1)$.

The cone $C_+$ is adjacent to $\mathbb{R}$. For $\psi_+, C_+ \subset \{ \text{Im} \xi > 0 \}$, and for $\psi_-, C_+ \subset \{ \text{Im} \xi < 0 \}$;
(c) if $\psi(\xi) = \mp i\xi ST(\mathcal{G})(\mp i\xi)$ and $\alpha \in (-1,0)$, then $X^\pm$ is SINH-regular of order $(\nu,\nu)$, where $\nu = \alpha + 1$, and the interior of the cone $C_+$ contains $\mathbb{R} \setminus 0$;
(d) if $\psi(\xi) = \mp i\xi ST(\mathcal{G})(\mp i\xi)$ and $\alpha = -1$, then $X^\pm$ is SINH-regular of order $(0+,0+)$, and the interior of the cone $C_+$ contains $\mathbb{R} \setminus 0$.

**Proof.** Proof in Section A.8.

A natural class of discrete measures $\mathcal{G} = \sum_j p_j \delta_{t_j}$ can be defined by sequences $\{t_j\}$ that accumulate to $+\infty$ (examples are meromorphic processes and the $\beta$-model) and/or 0. The measures that accumulate both to 0 and $+\infty$ can be used to approximate $\mathcal{G}(dt)$ of stable Lévy processes, hence, approximate the characteristic exponents of stable Lévy processes. One can formulate explicit conditions on sequences $p_j, t_j, j \in \mathbb{Z}_{++}$ or $j \in \mathbb{Z}$ which imply (3.36).

**Example 3.38.** Let there exist positive constants $c_1, c_2, C_1, C_2$ such that, for $j \in \mathbb{Z}_{++}$, $c_1 \leq t_{j+1} - t_j \leq C_1$ and $c_2 \leq p_j \leq C_{2}$. Then $\mathcal{G} = \sum_{j=1}^{+\infty} p_j \delta_{t_j}$ satisfies (3.36).

**Example 3.39.** Let there exist positive constants $c_1, c_2, C_1, C_2$ such that, for $j \in \mathbb{Z}_{++}$, $c_1 t_j \leq t_{j+1} - t_j \leq C_1 t_j$ and $c_2 \leq p_j \leq C_{2}$. Then $\mathcal{G} = \sum_{j=1}^{+\infty} p_j \delta_{t_j}$ satisfies (3.36).

**Example 3.40.** Let there exist positive constants $c_1, c_2, C_1, C_2$ such that, for $j \in \mathbb{Z}$, $c_1 t_j \leq t_{j+1} - t_j \leq C_1 t_j$ and $c_2 \leq p_j \leq C_{2}$. Then $\mathcal{G} = \sum_{j=1}^{+\infty} p_j \delta_{t_j}$ satisfies (3.36).

4. SL-processes: Absence of Solutions of Equation $q + \psi(\xi) = 0$ on $\mathbb{C} \setminus i\mathbb{R}$

In this section, we prove that SL-processes enjoy the following property:

$$q + \psi(\xi) = 0$$

for any $q > 0$, equation $q + \psi(\xi) = 0$ has no solution in $\mathbb{C} \setminus i\mathbb{R}$.

The reader can easily construct examples for which this important property does not hold. The simplest example is a spectrally one sided Lévy process with $f(x) = xe^{-x}1_{(0,\infty)}(x)$ and the characteristic exponent $\psi(\xi) = -i\mu \xi + 1 - (1 - i\xi)^{-2}$: for a given $q > 0$, we can choose $\mu$ so that (4.1) fails. However, this process is not an sSL-process.
4.1. A counter-example: Meixner processes. The following counter-example shows that the property (4.1) may fail if $X$ is an sSL-process.

**Example 4.1.** Let $\psi$ be given by (2.18), and let $q > 0$. Set $z = \exp((a\xi - ib)/2)$, $2Q = \exp(\cos(b/2) - q/(2\delta))$, and consider the equation $z + 1/z - 2Q = 0$. The solutions are $z_{\pm} = Q \pm (Q^2 - 1)^{1/2}$, hence, for any $k \in \mathbb{Z}$, $\xi_{k,\pm} = (2/|a|)\ln z_{\pm} + i((b/2 + 2k\pi)]$ are solutions of the equation $\psi(\xi) + q = 0$. Thus, the equation $\psi(\xi) + q = 0$ has (infinitely many) solutions in $\mathbb{C} \setminus i\mathbb{R}$, whereas in the case of SL-processes, there are none.

We do not know if (4.1) fails for any sSL-process which is not an SL-process.

**Hypothesis SLzeros.** If the jump component $\psi_1(\xi)$ of the characteristic exponent of a Lévy process $X$ admits analytic continuation to $(\mathbb{C} \setminus \mathbb{R}) \cup \{0\}$ and there exist $q > 0$, $\sigma^2 \geq 0$ and $b \in \mathbb{R}$ such that the equation $q + \sigma^2/2\xi^2 - ib\xi + \psi_1(\xi) = 0$ has a solution in $\mathbb{C} \setminus i\mathbb{R}$, then $X$ is not a SL process.

4.2. Main theorem.

**Theorem 4.2.** Let $\psi$ be the characteristic exponent of a Stieltjes-Lévy process. Then

(a) the property (4.1) holds;

(b) if the SL-measures $\mathcal{G}_{\pm}$ defining $\psi$ are supported on $[-\mu_-, +\infty)$ and $[\mu_+, +\infty)$, respectively, and $\mu_- < \mu_+$, then, on each interval $i(\mu_-), 0), i(0, \mu_+)$, the equation $q + \psi(\xi) = 0$ has either 0 or 1 solution;

(c) the solution on $i(\mu_-, 0)$ exists iff $\mu_- < 0$ and $\psi(i(\mu_+-0)) + q < 0$;

(d) the solution on $i(0, \mu_+)$ exists iff $\mu_+ > 0$ and $\psi(i(\mu_+-0)) + q > 0$.

**Remark 4.1.** To obtain one-sided analogs, set either $\mu_- = -\infty$ or $\mu_+ = +\infty$. In Section A.9 we give simple sufficient conditions for zeros to exist for any $q$.

To prove (b)-(d), it suffices to notice that if $\mu_- < \mu_+$, the function $(\mu_-, \mu_+) \ni t \mapsto \psi(it) \in \mathbb{R}$ is concave, and $q + \psi(0) = q > 0$. To prove (a), we note that the natural domain for a Stieltjes function $ST(\mathcal{G}) \in \mathcal{S}_\mu$ is $\mathbb{C} \setminus (-\infty, -\mu]$, and rewrite the equation $\psi(\xi) + q = 0, \xi \not\in i\mathbb{R}$, in the form $F(z) = F(\sigma^2, \mu, a_2^\pm, a_1^\pm, a_2, a_1, \mathcal{G}_+, \mathcal{G}_-; z) = 0, z \not\in \mathbb{R}$, where $\sigma^2 \geq 0, \mu \in \mathbb{R}, q > 0, a^\pm_2 \geq 0, a^\pm_1 \geq 0, \mathcal{G} \pm \in \mathcal{S}M_0$, and

\[
F(z) = -\frac{\sigma^2}{2}z^2 + \mu z + (-a_2^\pm z^2 + a_1^\pm z)ST(\mathcal{G}_+)(z) + (-a_2^- z^2 - a_1^- z)ST(\mathcal{G}_-)(-z) + q.
\]

This, (a) is equivalent to the following theorem.

**Theorem 4.3.** Equation $F(z) = 0$ has no solution in $\mathbb{C} \setminus \mathbb{R}$.

4.3. Proof of Theorem 4.3: preliminaries. We use the trivial observation

\[
-a_2^\pm z^2 \pm a_1^\pm z < 0, \pm z < 0,
\]

and auxiliary propositions, which list several properties of the Stieltjes measures. The following simple technical lemma is proved in Section A.10.

**Lemma 4.4.** Let $\mathcal{G} \in \mathcal{S}M_\mu$, where $\mu \geq 0$, and $b \geq \mu$ satisfy the following condition: $\exists \delta > 0$ and $\delta \in (0, b)$ such that for any measurable non-negative $g$,

\[
\int_{(b-b, b+\delta)} g(t)\mathcal{G}(dt) \geq c \int_{(b-b, b+\delta)} g(t)dt.
\]
Then for any $\delta' \in (0, \delta)$, there exist $\epsilon_0 > 0$ and $C > 0$ s.t. for all $b' \in (b - \delta', b + \delta')$ and $\epsilon \in (0, \epsilon_0)$, $\text{Im} \ ST(G)(-b' + i\epsilon) \leq -\pi c + C\epsilon$ and $\text{Im} \ ST(G)(-b' - i\epsilon) \geq \pi c - C\epsilon$.

Let $D = \text{supp} \ G$. The function $ST(G)$ below is the analytic continuation of $ST(G)$ from either $\{\text{Im} \ \xi \leq 0\} \backslash (-D)$ or $\{\text{Im} \ \xi \geq 0\} \backslash (-D)$ to a sufficiently large simply connected open set. The following two lemmas can be formulated and proved in a somewhat more general form, replacing the equality $G(dt) = c dt$ with the inequality as in Lemma 4.3.

**Lemma 4.5.** Let $G \in \text{SM}_\mu$, where $\mu \geq 0$, $\sigma > \mu$, and $\delta \in (0, a)$ satisfy the following conditions:

(i) $\text{supp} \ G \cap (a - \delta, a) = \emptyset$;

(ii) the restriction of $G(dt)$ on $(a, a + \delta)$ is cdt, where $c > 0$.

Then, uniformly in $\varphi \in [-\pi/2, \pi/2]$, as $\epsilon \downarrow 0$,

$$ST(G)(-a + \epsilon e^{i\varphi}) = c \ln(1/\epsilon) + O(1).$$

**Proof.** We have

$$\int_{(0, +\infty)} (t - a + \epsilon e^{i\varphi})^{-1} G(dt) = c \int_{a}^{a+\delta} (t - a + \epsilon e^{i\varphi})^{-1} dt + O(1) = c \ln \frac{\delta + \epsilon e^{i\varphi}}{\epsilon e^{i\varphi}} + O(1).$$

\hfill \Box

**Lemma 4.6.** Let $G \in \text{SM}_\mu$, where $\mu \geq 0$, $b > \mu$, and $\delta \in (0, b)$ satisfy the following conditions:

(i) $\text{supp} \ G \cap (b, b + \delta) = \emptyset$;

(ii) the restriction of $G(dt)$ on $(b - \delta, b)$ is cdt, where $c > 0$.

Then, uniformly in $\varphi \in [-\pi/2, \pi/2]$, as $\epsilon \downarrow 0$,

$$ST(G)(-b - \epsilon e^{i\varphi}) = -c \ln(1/\epsilon) + O(1).$$

**Proof.** We have

$$\int_{(0, +\infty)} (t - b - \epsilon e^{i\varphi})^{-1} G(dt) = c \int_{b-\delta}^{b} (t - b - \epsilon e^{i\varphi})^{-1} dt + O(1) = c \ln \frac{-\epsilon e^{i\varphi}}{-\delta - \epsilon e^{i\varphi}} + O(1).$$

\hfill \Box

### 4.4. Proof of Theorem 4.3.

The proof is by contradiction. Assume that a solution $z_0 \in \mathbb{C} \backslash \mathbb{R}$ exists; since $ST(G)(z) = ST(G)(\bar{z})$, we may assume that $\text{Im} \ z_0 < 0$. Since $z_0$ is in the domain of analyticity, the solution does not disappear after a sufficiently small perturbation of $\sigma^2$ and sufficiently small perturbations of $G_{\pm}$ in the norm $\|G\| = \int_{(0, +\infty)} (1 + t)^{-1} G(dt)$. Hence, we may assume that the following three conditions hold: 1) $\sigma^2 > 0$; 2) each of $G_{\pm}$ is an absolutely continuous measure $\sum_{j=1}^{N_{\pm}} a_{\pm j}^{\pm} 1_{(a_{\pm j}^{\pm}, b_{\pm j}^{\pm})}(t) dt$, where $[a_j, b_j] \subset (0, +\infty)$, $j = 1, 2, \ldots, N_{\pm}$, are non-intersecting intervals; 3) $\forall k, \beta_k^\pm \not\in \bigcup_j [a_j^\pm, b_j^\pm]$. Evidently, $F$ admits analytic continuation from $\{\text{Im} \ \xi < 0\}$ to a simply connected open set containing $\{\text{Im} \ \xi \leq 0\} \backslash (D_+ \cup D_-)$, where $D_+ = \bigcup_k [-b_k^+, -a_k^+]$ and $D_- = \bigcup_{m} [a_m^-, b_m^-]$.

It follows from Lemmas 4.4 4.6 and the equality $F(z) = F(\bar{z})$ that, if $\epsilon > 0$ is sufficiently small, $F$ has no zeros in the $\epsilon$-neighborhood of $D_+ \cup D_-$. Since $\sigma^2 > 0$, $F(z) \to -\infty$ as $\mathbb{R} \ni z \to \pm\infty$, therefore, the number of intervals and zeros of $F$ in $\mathbb{R} \backslash (D_+ \cup D_-)$ is finite. Let $\{-\alpha_j^+\}$ (resp., $\{\alpha_j^-\}$) be the set of all zeros on $(-\infty, 0)$ (resp., $(0, +\infty)$). For a sufficiently small $\epsilon > 0$, we define the contour $\mathcal{L}_\epsilon \subset \{\text{Im} \ z \leq 0\}$ as the union of the following sets:
(1) \( \mathcal{L}_{e, \infty} = \{ z = -i(1/\epsilon) e^{i\varphi} \mid -\pi/2 < \varphi < \pi/2 \} \);

(2) \( \mathcal{L}_{e, 0} = \{ z \in \mathbb{R} \mid |z| \leq 1/\epsilon, \text{dist} (z, D) \geq \epsilon \} \);

(3) \( \mathcal{L}_{e, 1} = (\cup_j [-b^+_j, -a^+_j] - i\epsilon) \cup (\cup_{\ell} [b^-_{\ell}, b^-_{\ell}] - i\epsilon) \);

(4) semi-circles in the lower half-plane, of radius \( \epsilon \), around each point \(-\alpha_{n, \ell}, \alpha_{n, \ell} \);

(5) quarter-circles in the lower half-plane, of radius \( \epsilon \), with centers at \(-a^+_j, -b^+_j, a^-_{\ell}, b^-_{\ell} \), connecting \( \mathcal{L}_{e, 0} \) and \( \mathcal{L}_{e, 1} \).

We will prove that is \( \epsilon > 0 \) is sufficiently small, then the winding number \( \text{ind}(\mathcal{L}, 0, 0; F) \) of the curve \( \{ F(z) \mid z \in \mathcal{L}_e \} \), where \( z \) runs along \( \mathcal{L}_e \) in the positive direction, is 0. The winding number being equal to the number of zeros in the domain bounded by \( \mathcal{L}_e \), and \( \epsilon > 0 \) being arbitrary small, the number of zeros of \( F \) in the lower half-plane is 0 as well, contradiction.

Recall that for \( z_1, z_2 \) on \( \mathcal{L}_e \), and \( \mathcal{L}_e(z_1, z_2) \) the part of \( \mathcal{L}_e \) with \( z_1, z_2 \) being the end points,

\[
\text{ind}(\mathcal{L}_e, z_1, z_2; F) = \frac{1}{2\pi} \int_{\mathcal{L}_e(z_1, z_2)} d\arg F(z).
\]

As \( \rho \to +\infty \), \( F(-i\rho e^{i\varphi}) \sim \frac{\rho^2}{2} e^{2i\varphi} \), hence, if \( \epsilon > 0 \) is sufficiently small so that \( D_+ \cup D_- \subset \mathbb{R} \setminus \{ |z| \geq 1/\epsilon \} \), \( \arg F(z) \) increases from \(-\pi \) to \( \pi \) as \( z \) moves along \( \mathcal{L}_{e, \infty} \). Thus, \( \text{ind}(\mathcal{L}_e, -1/e, 1/e; F) = 1 \). Next, functions \( \text{ST}(\mathcal{G}_-)(-\pi), \text{ST}(\mathcal{G}_-)(\pi) \), \( z \) and \( z^2 \) are continuous in a neighborhood of \((\infty, 0)\), and real-valued on \((\infty, 0)\). Hence, it follows from (4.3) and Lemmas 4.4-4.6 that there exist \( \epsilon_0, \epsilon > 0 \) such that, for each \( \epsilon \in (0, \epsilon_0) \),

(a) the \( \epsilon \)-neighborhoods of all points \(-a^+_j, -b^+_j, -\beta^+_k, -\alpha^+_k \) do not intersect;

(b) for all \( \beta \in \cup_j [-b^+_j, -a^+_j] \), \( \text{Im}(\beta - i\epsilon) \leq -c \);

(c) for all \(-b^+_j \) and \( \varphi \in (-\pi/2, \pi/2) \), \( F(-b^+_j - e^{i\varphi}) = -C_j \ln \epsilon + O(1) \), where \( C_j > 0 \);

(d) for all \(-a^+_j \) and \( \varphi \in (-\pi/2, \pi/2) \), \( F(-a^+_j + e^{i\varphi}) = C'_j \ln \epsilon + O(1) \), where \( C'_j > 0 \).

It follows that if \( \epsilon > 0 \) is sufficiently small, then

(i) for each \( j \) and \( z \in [-b^+_j, -a^+_j] - i\epsilon \), \( \arg F(z) \in (-\pi, 0) \);

(ii) on each connected interval of \( \mathcal{L}_{e, 0} \), \( \arg F(z) \) equals either \( \pi \) or \(-\pi \);

(iii) for all \( j \), \( \arg F(-a^+_j + i\epsilon) = -\pi \), \( \arg F(-a^+_j - i\epsilon) \in (-\pi, 0) \), \( \arg F(-b^+_j - i\epsilon) = 0 \), \( \arg F(-b^+_j + i\epsilon) \in (-\pi, 0) \);

(iv) as \( z \) moves down from \(-a^+_j + \epsilon \) to \(-a^+_j - i\epsilon \), \( w := F(z) \in \{ \text{Re} w < 0 \} \) and runs from a point on \((\infty, 0)\) to a point in the quadrant \( \{ \text{Re} w < 0, \text{Im} w < 0 \} \);

(v) as \( z \) moves down from \(-b^+_j - i\epsilon \) to \(-b^+_j + i\epsilon \), \( w := F(z) \in \{ \text{Re} w > 0 \} \) and runs from a point on \((0, +\infty)\) to a point in the quadrant \( \{ \text{Im} w > 0 \} \);

(vi) as \( z \) moves down along each semi-circle, \( F(z) \) remains in \( \{ \text{Re} w \leq 0, w \neq 0 \} \) and \( \arg F(z) \) varies either from \(-\pi \) to \( 0 \) or from \( 0 \) to \(-\pi \) or from \(-\pi \) to \( 0 \).

It follows that as \( z \) runs from \(-i\epsilon \) to \(-1/\epsilon \), \( F(z) \notin \{ \text{Im} w > 0 \} \cup \{ 0 \} \). Hence, \( \text{ind}(\mathcal{L}_e, -i\epsilon, -1/\epsilon; F) = (\pi - 0)/(2\pi) = -1/2 \). Using \( \overline{F(z)} = F(z) \) and the symmetry \( z \mapsto -z \), we observe that if we move along \( \mathcal{L}_e \) in the opposite direction (denote \( \mathcal{L}'_e \) with the opposite direction by \( \mathcal{L}'_e \)), then

\[
\frac{1}{2\pi} \int_{\mathcal{L}'_e(0, 1/\epsilon)} d\arg F(z) = 1/2.
\]

Hence, \( \text{ind}(\mathcal{L}_e, 1/\epsilon, 0; F) = -1/2 \), and

\[
\text{ind}(\mathcal{L}, 0, 0; F) = \text{ind}(\mathcal{L}_e, 0, -1/\epsilon; F) + \text{ind}(\mathcal{L}_e, -1/\epsilon, 1/\epsilon; F) + \text{ind}(\mathcal{L}_e, 1/\epsilon, 0; F) = 0.
\]
This finishes the proof of Theorem 4.3.

5. MIXING AND SUBORDINATION OF SINH-, sSL- AND SL-PROCESSES

5.1. Mixing.

5.1.1. Mixing SINH-regular processes. The following theorem is immediate from the definition of SINH-regular processes.

**Theorem 5.1.** Let the following conditions hold

(i) processes $X_j^j, j = 1, 2, \ldots, N$, are independent;
(ii) each $X_j^j$ is a SINH-regular Lévy process of order $(\nu'_j, \nu_j)$ and type $((\mu'_j - \mu_j, \mu'_j + \mu_j), C_j, C_j^+);
(iii) \mu_- := \max_j \mu'_j < \mu_+ := \min_j \mu'_j$ and $C := \cap_{j=1}^N C_j \neq \emptyset$;
(iv) there exists $\nu' \in (\min_j \nu'_j, \max_j \nu_j)$ such that $C^+ := \cap_{\nu'_j \geq \nu'} C_j^+ \neq \emptyset$;

Then, for any $a_j > 0, j = 1, 2, \ldots, N, X := \sum_{j=1}^N a_j X^j$ is SINH-regular of order $(\nu', \max_j \nu_j)$ and type $((\mu_-, \mu_+), C, C^+)$. For the sake of brevity, we omit straightforward (albeit messier) generalizations to the case of infinite and integral mixing.

5.1.2. Mixing sSL- and SL-processes. sSL- and SL-regular processes are mixtures of the BM and pure jump spectrally one-sided processes, hence, it suffices to consider mixtures of spectrally positive processes (results for spectrally negative processes are by symmetry).

Since the sum of Stieltjes measures (and, under additional regularity conditions, integral of a family of Stieltjes measures $G_\alpha, \alpha \in A$) is a Stieltjes measure, the mixture of processes $X^\alpha \in sSL^{\mu_\alpha}_+, X^\alpha \in sSL^{2+\mu_\alpha}_-, X^\alpha \in sSL^{1-\mu_\alpha}_1$, where $\mu_\alpha \geq \mu > 0, \alpha \in A$, is a process of class $sSL^{\mu_\alpha}_+$ (resp., $sSL^{2-\mu_\alpha}_+, sSL^{1+\mu_\alpha}_1$), and the same is true for SL-processes. If $\inf \mu_\alpha = 0$, then sufficient conditions for the mixture to be of the same class are more involved. We omit this case for the sake of brevity.

5.1.3. Mixing regular sSL- and SL-processes. Conditions in Sections 5.1.1 and 5.1.2 must hold.

5.2. Subordination. A Lévy process taking values in $[0, +\infty)$, which implies that its trajectories are increasing a.s., is called a subordinator. Since non-decreasing paths have bounded variation, the characteristic exponent of a subordinator is of the form

$$\psi(\xi) = -ib\xi + \int_0^{+\infty} (1 - e^{ix\xi}) F(dx),$$

where $b \geq 0$. The Laplace transform of the law of a subordinator $Z$ can be expressed as

$$E[\exp(-qZ_t)] = \exp(-t\psi(q)),$$
where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called the Laplace exponent of $Z$\textsuperscript{4}. Thus, the characteristic exponent of $Z$ is $\psi(\xi) = \Psi(-i\xi)$, and

$$
\Psi(q) = \psi(iq) = bq + \int_0^{+\infty} (1 - e^{-qx}) F(dx).
$$

If $X$ is a subordinator, $\Psi(q)$ admits analytic continuation to the right half-plane $\{\text{Re } q > 0\}$, and $\psi(\xi)$ to the upper half-plane $\{\text{Im } \xi > 0\}$. If $Z$ is SINH-regular of type $((\mu_-, +\infty), iC_\gamma, iC_{\gamma'})$, where $\mu_- \leq 0$ and $\pi/2 < \gamma' < \gamma \leq \pi$, $\Psi$ admits analytic continuation to $(\mu_-, +\infty) + (C_\gamma \cup \{0\})$.

**Lemma 5.2.** Let $Z^+ \in SL^1_{\mu^+}, \mu \geq 0$, be a subordinator. Then, for any $\gamma \in [0, \pi)$, $\Psi_Z(\mu + C^+_{\gamma^+}) \subset \mu + (C^+_{\gamma^+} \cup \{0\})$, where $C^+_{\gamma^+} = \{z \mid \arg z \in [0, \gamma]\}$.

**Proof.** The Laplace exponent of an SL-subordinator is a complete Bernstein function (see [48, Defin.6.1] for the definition of the latter). Hence, the statement of Lemma is immediate from [48, Cor. 6.6] (formulated for the case $b = 0$). $\square$

**Theorem 5.3.** ([47, Thm 30.1]) Let $Z$ be a subordinator with the Laplace exponent $\Psi$, let $Y$ be a Lévy process with the characteristic exponent $\kappa$, and suppose that $Z$ and $Y$ are independent.

Define $X_t(\omega) = Y_{Z_t(\omega)}(\omega)$, $t \geq 0$. Then $\{X_t\} = \{Y_{Z_t}\}$ is a Lévy process with the characteristic exponent $\psi_Y(\xi) = \Psi(\kappa(\xi))$.

**5.2.1. Subordination of SINH-regular processes.** For the sake of brevity, we restrict ourselves to the case of SINH-regular processes of order $\nu \in (0, 2]$, with the cones $C_\gamma$ containing $\mathbb{R} \setminus \{0\}$.

**Theorem 5.4.** Let the following conditions hold

(i) $Z$ is a SINH-regular subordinator of order $\nu_Z \in (0, 1) \cup \{0\}$ and type $((\lambda_-, +\infty), iC_\gamma, iC_{\gamma'})$, where $\lambda_- < 0$, $\pi/2 < \gamma' < \gamma \leq \pi$, with the drift $\mu_Z \geq 0$;

(ii) $Y$ is a SINH-regular Lévy process of order $\nu_Y \in (0, 2]$ and type $((\mu_-, \mu_+), C, C_+)$, where $\mu_- < 0 < \mu_+$, and $C_+ \supset \mathbb{R} \setminus \{0\}$, with the characteristic exponent $\psi_Y(\xi) = -i\mu_Y \xi + \psi_Y^0(\xi)$;

(iii) $Z$ and $Y$ are independent.

Then

1. there exist $\mu_- \in (\mu_-, 0), \mu_+ \in (0, \mu_+)$ and open coni $\tilde{C} \subset C, \tilde{C}_+ \subset C_+$ such that $\mathbb{R} \setminus \{0\} \subset \tilde{C}_+ \subset \tilde{C}$, and

$$
\psi_Y(i(\mu_-, \mu_+) + (\tilde{C} \cup \{0\})) \subset (\lambda_-, +\infty) + (\tilde{C}_+ \cup \{0\}),
$$

$$
\psi_Y(i(\mu_-, \mu_+^+) + (\tilde{C}_+ \cup \{0\})) \subset (\lambda_-, +\infty) + (\tilde{C}_+ \cup \{0\}).
$$

2. $\{X_t\} = \{Y_{Z_t}\}$ is SINH-regular of type $((\mu_-^-, \mu_+^+), \tilde{C}, \tilde{C}_+)$, and the order $\nu_X$, where

(a) if $\mu_Z > 0$, $\nu_X = \nu_Y$;

(b) if $\mu_Z = 0$, $\nu_Z \in (0, 1)$, and either $\nu_Y \in [1, 2]$ or $\nu_Y \in (0, 1)$ and $\mu_Y = 0$, then $\nu_X = \nu_Z \nu_Y$;

(c) if $\mu_Z = 0$, $\nu_Z \in (0, 1)$, and $\nu_Y \in (0, 1)$, $\mu_Y \neq 0$, then $\nu_X = \nu_Z$;

(d) if $\mu_Z = 0$, $\nu_Z = 0+$, then $\nu_X = 0+$.

\textsuperscript{4}This definition is marginally different from the definition [47, Eq. (30.1)]. We change the definition in accordance to the change of the definition of the characteristic exponent that we use so that the form of the key theorem (Thm. [5.3]) does not change.
Then

Proof. (1). It suffices to note that, for \( \xi \in \mathbb{R} \), \(|e^{-t\psi(\xi)}| = |\mathbb{E}[e^{i\xi Y}]| \leq 1 \), hence, \( \text{Re } \psi(\xi) \geq 0 \); and \( \psi(\xi) \) (resp., \( \text{Re } \psi(\xi) \)) stabilizes to a positively homogeneous function as \( \xi \to \infty \) remaining in \( C \) (resp., \( C_+ \)).

(2). Let \( \varphi \in (-\pi/2, \pi/2) \) and \( \rho \to +\infty \). Set \( \xi = \rho e^{i\varphi} \) and consider

\[
\psi_X(\xi) = \mu_Z(-i\mu Y\xi + \psi_Y(\xi)) + \Psi_Z^0(-i\mu Y\xi + \psi_Y(\xi)) = -i\mu_Z\mu Y\xi + \mu Z\psi_Y(\xi) + \psi_Y(\mu Y\xi + i\psi_Y(\xi)).
\]

If \( \mu_Z \neq 0 \), then the third term increases slower than the second one, and (a) follows. If the condition in (b) is satisfied, then \( \psi_Z^0(\mu_Y\xi + i\psi_Y(\xi)) \sim \psi_Y^0(\text{ic}_Y(\varphi)\rho^\varphi) \), and since \( \psi_Y^0(\xi) \) stabilizes to a positively homogeneous function of degree \( \nu_X \) as \( \xi \to \infty \), the conclusion in (b) is proved. If the condition in (c) is satisfied, \( \psi_Z^0(\mu_Y\xi + i\psi_Y(\xi)) \sim \psi_Y^0(\mu Y\xi) \), and (c) follows.

Finally, if \( \nu_Z = 0^+ \), then we note that \( \ln((\text{c}_Y(\varphi)\rho)^\varphi) \sim \nu_Y \ln \rho \), which proves (d).

\( \square \)

5.2.2. Subordination of and by sSL- and SL-processes. Let \( Y \in sSL_{\mu_-, \mu_+}; \mu_- \leq 0 \leq \mu_+ \), \( \mu_- < \mu_+ \), and let \( Z \in sSL_{\mu_+}, \mu > 0 \), be a subordinator with the Lévy exponent \( \Psi_Z \). Assume that the characteristic function \( \psi_Y \) enjoys the property (4.1). By Theorem 4.2 (a), if \( Y \) is a SL-process, (4.1) is satisfied.

Notice that the following possibilities exist:

(1) \( \psi_Y(i(\mu_--0)) \geq -\mu \) and \( \psi_Y(i(\mu_+ - 0)) \geq -\mu \); (2) \( \psi_Y(i(\mu_--0)) < -\mu \) and \( \psi_Y(i(\mu_+ - 0)) < -\mu \); (3) \( \psi_Y(i(\mu_+ - 0)) \geq -\mu \) and \( \psi_Y(i(\mu_+ - 0)) < -\mu \); (4) \( \psi_Y(i(\mu_+ - 0)) < -\mu \) and \( \psi_Y(i(\mu_+ - 0)) \geq -\mu \).

**Lemma 5.5.**

(a) If \( \psi_Y(i(\mu_+ - 0)) < -\mu \), there exists \( \mu'_- \in (\mu_-, \mu_+) \) such that \( \psi_Y(i\mu'_-) = -\mu \), and \( t \mapsto \psi_Y(it) \) increases on \( (\mu_-, \mu'_-) \).

(b) If \( \psi_Y(i(\mu_+ - 0)) < -\mu \), there exists \( \mu'_+ \in (\mu_-, \mu_+) \) such that \( \psi_Y(i\mu'_+) = -\mu \), and \( t \mapsto \psi_Y(it) \) decreases on \( (\mu_+', \mu_+) \).

(c) If (4.1) holds, \( \psi_X(\xi) = \Psi_Z(\psi_Y(\xi)) \) is analytic in \( \mathbb{C} \setminus i((-\infty, \mu_-] \cup [\mu_+, +\infty)) \), where

in Case (1), \( \mu_-^X = \mu_- \), \( \mu_+^X = \mu_+ \); in Case (2), \( \mu_-^X = \mu'_- \), \( \mu_+^X = \mu'_+ \);

in Case (3), \( \mu_-^X = \mu_- \), \( \mu_+^X = \mu_+ \); in Case (4), \( \mu_-^X = \mu'_- \), \( \mu_+^X = \mu'_+ \).

\( \square \)

**Proof.** It suffices to note that \( (\mu_-, \mu_+) \ni t \mapsto \psi_Y(it) \in \mathbb{R} \) is concave, and that under condition (4.1), \( \psi_Y(\xi) \in (-\infty, -\mu] \) for \( \xi \in \mathbb{C} \setminus i((-\infty, -\mu] \cup [\mu_+, +\infty)) \) implies \( \xi \in i(\mu_-, \mu_+) \).

**Theorem 5.6.** Let the following conditions hold:

(i) \( Y \in sSL_{\mu_-, \mu_+} \), and \( \psi_Y \) satisfy (3.2) and (4.1);

(ii) \( Z \in sSL_{\mu_+} \), and there exist \( \nu, \delta \in (0, 1) \) and \( C > 0 \) s.t.

\[
|\Psi_Z(z)| \leq C(|z|^{\nu} + |z - \mu|^{1+\delta}), \quad z \in \mathbb{C} \setminus (-\infty, -\mu);
\]

(iii) the limits \( \psi_Y(it + 0), t \in (-\infty, \mu_-] \cup (\mu_+, +\infty) \) and \( \Psi_Z(-t + 0), t > \mu, \) exist;

(iv) \( Z \) and \( Y \) are independent.

Then
(a) \( \{X_t\} = \{Y_{Z_t}\} \in sSL_{\mu_-^X, \mu_+^X}, \) and the sSL-measures of \( X \) are absolutely continuous: \( G_{X;\pm}(dt) = g_{X;\pm}(t)dt \in sSLM_{\pm \pi^+} \), where

\[
g_{X;+}(t) = \frac{1}{\pi} \times \begin{cases} 
\Psi_Z(\psi_Y(-it - 0)), & t \in (-\mu_-^X, +\infty), \\
\Psi_Z(\psi_Y(-it) - i0), & t \in (-\mu_-^X, -\mu_-^X), \\
0, & t \in (-\infty, -\mu_-^X),
\end{cases}
\]

\[
g_{X;-}(t) = \frac{1}{\pi} \times \begin{cases} 
\Psi_Z(\psi_Y(it + 0)), & t \in (\mu_+^X, +\infty), \\
\Psi_Z(\psi_Y(it) + i0), & t \in (\mu_+^X, \mu_+^X), \\
0, & t \in (-\infty, \mu_+^X).
\end{cases}
\]

(b) If, in addition, \( Y \) and \( Z \) are SL-processes, then \( g_{X;\pm} \geq 0 \), and \( X \) is an SL-process.

**Proof.** (a) is immediate from Theorem 3.25 and Lemma 5.5. At \( t \in (-\mu_-^X, -\mu_-) \), we use the linearization \( \psi_Y(-it - \epsilon) \sim \psi_Y(-it) + i\epsilon d\psi_Y(-it)(t) \) and the inequalities \( \psi_Y(-it) < \mu \), \( d\psi_Y(-it)/dt < 0 \). The case \( t \in (\mu_+^X, \mu_+^X) \) is by symmetry.

(b) follows from Theorem 3.25 and Lemma 5.5.

\( \square \)

### 5.3. Construction of SL- and sSL-processes as SL- and sSL-subordinated Brownian motion

First, we consider the case of absolutely continuous SL and sSL measures, and then the case of discrete measures. Below, \( \sqrt{\cdot} \) denotes the standard branch of the square root, and \( W \) is the standard Wiener process.

**Theorem 5.7.** Let the following conditions hold:

(i) \( X \in SL_{\mu_-^X, \mu_+^X}, \) where \( \mu_- \leq 0 \leq \mu_+^X, \mu_- < \mu_+^X; \)

(ii) \( \psi_X(\xi - i\beta) = \psi(\xi + i\beta), \ \forall \xi \in \mathbb{C} \setminus i((\infty, \mu_-) \cup [\mu_+, +\infty)), \)

where \( \beta = (\mu_- + \mu_+^X)/2; \)

(iii) \( \lim_{A \to +\infty} \int_{|\xi| = A, \xi \in \mathbb{R}} \frac{\psi_J(\xi)}{|\xi|^3} \, d\xi = 0. \)

Then

(a) function \( \Psi_Z(q) := \psi(\sqrt{q} - i\beta) - \psi(-i\beta) \) is the Laplace exponent of a subordinator \( Z \), under the Esscher transform \( Q_\beta \) which makes the characteristic exponent of \( X \) symmetric: \( \psi_{Q_\beta}(\xi) = \psi_{Q_\beta}(-\xi); \)

(b) \( Z \in SL_{\mu_+^X}^+, \) where \( \mu = (\mu_+ - \mu_-)^2/4; \)

(c) the Stieltjes measure of the complete Bernstein function \( \Psi_Z \) is given by

\[
G_Z^0((u, v)) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{(u, v]} \frac{\Im \psi_X(i(\sqrt{q} - \beta + \epsilon))}{q} \, dq,
\]

for any \( u > \beta^2 \) and \( v > \beta \) s.t. \( \sqrt{u} - \beta \) and \( \sqrt{v} - \beta \) are points of continuity of the distribution function \( t \mapsto G_X((-\infty, t]); \)

(d) \( X_t = Y_{Z_t}, \) where, under \( Q_\beta, Y \) is given by \( dY = \sqrt{2}dW. \)
Proof. In view of the symmetry condition \((5.9)\), we can use the Esscher transform to reduce the proof to the case of the symmetric \(\psi = \psi_X\): \(\psi(\xi) = \psi(-\xi)\) and \(\beta = 0\). Then \(\mu_+ = -\mu_-\), \(\mu = \mu^2\), \(\Psi_Z\) admits analytic continuation to \(\mathbb{C} \setminus (-\infty, -\mu]\), and \(\Psi_Z(\xi) = \psi(\xi)\) for all \(\xi \in \mathbb{C} \setminus i((-\infty, \mu_-) \cup [\mu_+, +\infty))\). Hence, (d) follows from (a).

To prove (a)-(b), first, we verify that \(G^0_Z\) given by \((5.11)\) is the Stieltjes measure:

\[
\int_{(\mu_+, \infty)} G^0_Z(dq) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{\mu}^{+\infty} \frac{\text{Im} \psi_X(i \sqrt{q + \epsilon})}{q(1 + q)}dq = 2 \int_{(0, \infty)} \frac{G_X(dt)}{t(1 + t^2)}. \tag{5.11}
\]

The last equality follows from \([3, 23]\); the integral is finite since \(G_X\) is an SL-measure.

Due to the symmetry condition \(\psi(\xi) = \psi(-\xi)\), \(\psi\) is of the form \(\psi(\xi) = \frac{\sigma^2}{2} \xi^2 + \psi_J(\xi)\), where \(\psi_J(\xi) = \psi_J(-\xi)\) is the pure jump component. Hence, the drift of \(Z\) is \(\sigma^2/2\), and it remains to prove that \(\psi_J(\xi) = \xi^2 ST(G_Z^0)(\xi^2)\). In \((5.11)\), we can replace \(\psi_X\) with \(\psi_J\), and obtain, for any \(\mu_1 \in (0, \mu)\) and \(\xi^2 \notin (-\infty, -\mu]\),

\[
\xi^2 ST(G_Z^0)(\xi^2) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{(\mu_1, \infty)} \frac{\xi^2}{\xi^2 + q} \frac{\text{Im} \psi_J(i \sqrt{q + \epsilon})}{q}dq.
\]

We change the variable \(q = t^2\), and, for any \(\mu_2 \in (0, \mu)\), obtain

\[
\xi^2 ST(G_Z^0)(\xi^2) = \frac{2}{\pi} \lim_{\epsilon \to 0^+} \int_{(\mu_2, \infty)} \frac{\xi^2}{\xi^2 + t^2} \frac{\text{Im} \psi_J(it + \epsilon)}{t}dt. \tag{5.12}
\]

Let \(\mathcal{L}_\epsilon = \{z \in \mathbb{C} \mid \text{dist}(z, (\mu_+, \infty)) = \epsilon\}\). Since \(\text{Im} \psi_J(it + \epsilon) = (\psi_J(it + \epsilon) - \psi_J(it - \epsilon))/2i\), we may rewrite \((5.12)\) as

\[
\xi^2 ST(G_Z^0)(\xi^2) = \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \int_{\mathcal{L}_\epsilon} \frac{\xi^2}{\xi^2 + z^2} \frac{\psi_J(iz)}{z}dz, \tag{5.13}
\]

where \(\mathcal{L}_\epsilon\) is passed from \(+\infty + i\epsilon\) to \(+\infty - i\epsilon\). Let the contour \(-\mathcal{L}_\epsilon\) be passed from \(-\infty - i\epsilon\) to \(-\infty + i\epsilon\). Then, on the strength of \((5.10)\),

\[
\frac{1}{2\pi i} \left( \int_{\mathcal{L}_\epsilon} + \int_{-\mathcal{L}_\epsilon} \right) \frac{\xi^2}{\xi^2 + z^2} \frac{\psi_J(iz)}{z}dz
\]

plus the sum of residues of the integrand equals 0. Since \(\psi(-iz) = \psi(iz)\), the integrals over \(\mathcal{L}_\epsilon\) and \(-\mathcal{L}_\epsilon\) are equal, hence, the R.H.S of \((5.13)\) is opposite to the sum of residues. The apparent singularity at 0 is removable and \(\psi_J(\xi) = \psi_J(-\xi)\), hence, the sum of residues is

\[
\frac{\xi^2 \psi_J(-\xi)}{2i\xi} + \frac{\xi^2 \psi_J(\xi)}{-2i\xi} = -\psi_J(\xi).
\]

Thus, \(\psi_J(\xi) = \xi^2 ST(G_Z^0)(\xi^2)\), which finishes the proof.

\[\square\]

**Theorem 5.8.** Let the following conditions hold:

(i) \(X \in sSL_{\mu_-, \mu_+}\), where \(\mu_- \leq 0 \leq \mu_+, \mu_- < \mu_+, \) satisfies \((5.9)\);

(ii) conditions of Theorem \((3, 22)\) are satisfied;

(iii) the Laplace transform of the measure \(G(dq) = \pi^{-1} \text{Im} \psi_X(i(\sqrt{q} - \beta) - 0)dq\) is positive.

Then
(a) $G(dq)$ is an sSL-measure;
(b) function $\Psi_Z(q) := \psi(\sqrt{q} - i\beta) - \psi(-i\beta)$ is the Laplace exponent of a subordinator $Z$;
(c) $Z \in sSL_\mu^+$, where $\mu = (\mu_+ - \mu_-)^2/4$, and $G(dq)$ is the sSL-measure of $Z$;
(d) $G(dq)$ is the sSL-measure of $Z$;
(e) $X_t = Y_{Z_t}$, where $Y$ is given by $dY = \sqrt{2}dW$ as in Theorem 3.7.

Proof. Theorem 3.25 allows us to prove (a); the only subtlety, namely, the proof of the condition (iii), is assumed away. The rest of the proof is the same as in the case of $X \in SL_{\mu_-,\mu_+}$.

Example 5.9. Let $X$ be a Meixner process. Conditions (i)-(ii) are verified in Example 3.29. In view of (3.33), it suffices to prove that, for $0 < u < v$ and $x > 0$,

\[
\left( \int_\sqrt{v+u}^{\sqrt{v+u+2v}} - \int_\sqrt{u+u+2v}^{\sqrt{u+u+2v}} \right) e^{-qx}G(dq) > 0.
\]

Since $x > 0$, it suffices to prove that $2\sqrt{v+u} - \sqrt{u+2v} > \sqrt{u+2v}$; this inequality holds since the function $u \mapsto \sqrt{u}$ is concave.

Now we consider SL processes with discrete SL-measures. Assume that the measure is symmetric; using the Esscher transform, one can generalize the result below.

Theorem 5.10. Let $X$ be a driftless SL-process with the symmetric discrete Lévy measure; equivalently,

\[ G_\pm = \sum_{\alpha \in A} p_\alpha \lambda_\alpha \delta_{\lambda_\alpha}, \]

where $A$ is a countable set, $p_\alpha, \lambda_\alpha > 0$, and there exist $a_1, a_2 > 0$ such that

\[ \sum_{\alpha \in A} \frac{p_\alpha}{(a_1 + a_2 \lambda_\alpha)(1 + \lambda_\alpha)} < \infty. \]

Then $X_t = Y_{Z_t}$, where $Z$ is an SL-subordinator with the SL measure $G_Z = 2\sum_{\alpha \in A} p_\alpha \lambda_\alpha^2 \delta_{\lambda_\alpha}$, and $Y = 2W$.

Proof. For the sake of brevity, assume that $X$ has no BM component. Then

\[ \psi(\xi) = \sum_{\alpha \in A} p_\alpha \left( \frac{-i\xi}{\lambda_\alpha - i\xi} + \frac{i\xi}{\lambda_\alpha + i\xi} \right) = 2\sum_{\alpha \in A} p_\alpha \frac{\xi^2}{\lambda_\alpha^2 + \xi^2}. \]

Hence, $X_t = Y_{Z_t}$, where the Laplace exponent of $Z$ is given by

\[ \Psi_Z(q) = 2\sum_{\alpha \in A} p_\alpha \frac{q}{\lambda_\alpha^2 + q}, \]

provided we prove that $G_Z$ is an SL-measure, equivalently,

\[ \sum_{\alpha \in A} p_\alpha \frac{\lambda_\alpha^2}{\lambda_\alpha^2(1 + \lambda_\alpha^2)} < \infty. \]

The last inequality follows from (5.15).
6. Conclusion

The conformal deformations technique (which is similar to but simpler and more flexible than the saddle point method) allows one to develop efficient numerical methods for the evaluation of the Wiener-Hopf factors and various probability distributions (prices of options of several types) in Lévy models and models with (conditional) infinitely divisible distributions. The crucial conditions are: 1) the characteristic exponent \( \psi \) admits analytic continuation to a union \( U \) of a strip and cone around or adjacent to \( \mathbb{R} \); 2) \( \text{Re} \psi(\xi) \to +\infty \) as \( (U \ni \xi) \to \infty \). We call processes satisfying 1)-2) SINH-regular because the most efficient family of conformal deformations is in terms of the function \( \text{sinh} \). We showed that essentially all popular classes of Lévy processes other than stable Lévy processes are SINH-regular, and calculated the corresponding strips and cones. Choices of appropriate conformal deformations simplify if the cone of analyticity is the set \( \mathbb{C} \setminus i\mathbb{R} \). We constructed a class of signed Stieltjes-Lévy (sSL-) processes enjoying this property, and showed that all popular classes of Lévy processes except for the Merton model are sSL-processes. The construction is based on the representation of Lévy densities of positive and negative jumps as Laplace transforms of measures of the form \( (a_2t^2 + a_1t)G_{\pm}(dt) \), where \( G_{\pm}(dt) \) are the differences of Stieltjes measures. If \( G_{\pm}(dt) \) are Stieltjes measures, then we say that the process is a Stieltjes-Lévy (SL-) process. We proved that SL-processes enjoy an additional property, namely, the absence of solutions of the equation \( \psi(\xi) + q = 0 \) for any \( q > 0 \), which simplifies calculations of the Wiener-Hopf factors, calculations of joint probability distributions of a Lévy process and its extrema, and pricing options with barrier and/or lookback features. We proved that all popular classes of Lévy processes except for the Merton model and the Meixner processes are SL-processes, and that Meixner processes are sSL-processes. We derived a representation of \( \psi \) in terms of the Stieltjes functions, and a representation of the measures \( G_{\pm}(dt) \) in terms of \( \psi \). One of the representation theorems contains a set of sufficient conditions on a function \( \psi \) to be a characteristic exponent of a Lévy process \( X \), and conditions for \( X \) to be an sSL-process or an SL-process. We showed that the theorem is applicable to all popular Lévy processes, and used the theorem to prove that, under a natural symmetry condition, all popular classes of processes are subordinated Brownian motions. We proved that, under additional weak regularity conditions, mixtures of SINH-regular processes, sSL-processes and SL-processes are processes of the same class, and that the subordination of an SL-process by an sSL- (resp., SL-) subordinator is an sSL- (resp., SL-) process.

We leave to the future the study of the further generalization (generalized SL-processes: gSL-processes), which is needed to include in the general framework Lévy processes with Lévy densities given by exponential polynomials. The generalization can be obtained allowing for derivatives of measures (in the sense of generalized functions).

The following extensions of the results of the paper seem to be of interest:

1) describe classes of infinitely divisible distributions (e.g., conditional distributions on stochastic volatility models and models with stochastic interest rates) and additive processes that lead to \( \psi \) of sSL- or SL-processes;

2) using the representation in terms of Stieltjes measures, study efficient approximations of processes with absolutely continuous measures by processes with discrete measures (e.g., approximations of KoBoL by HEJD model or meromorphic processes). Examples in the paper

\[5\] In the case of stable Lévy processes, the strip degenerates into \( \mathbb{R} \) but a modified conformal deformation technique is applicable in this case as well [18].
suggest that approximations by measures with uniformly spaced atoms could be less efficient than approximations by non-uniformly spaced measures;

3) study the relative efficiency of Monte-Carlo procedures for sSL- and SL-processes based on the subordinated BM-representation as in [33] vs the approximation of the transition probability density using the conformal deformation technique [6] [17] [18]. Numerical examples in [6] suggest that the latter is more efficient than the former;

4) generalizations for Lévy processes on $\mathbb{R}^n$.

References

[1] S. Asmussen. *Ruin Probabilities*. Number 2 in Advanced Series on Statistical Science and Applied Probability. World Scientific, River Edge, NJ, 2000.

[2] S. Asmussen, F. Avram, and M.R. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and their Applications*, 109(1):79–111, 2004.

[3] O.E. Barndorff-Nielsen. Exponentially decreasing distributions for the logarithm of particle size. *Proc. Roy. Soc. London, Ser. A.*, 353:401–419, 1977.

[4] O.E. Barndorff-Nielsen. Processes of Normal Inverse Gaussian Type. *Finance and Stochastics*, 2:41–68, 1998.

[5] O.E. Barndorff-Nielsen and S.Z. Levendorski˘ı. Feller Processes of Normal Inverse Gaussian type. *Quantitative Finance*, 1:318–331, 2001.

[6] M. Boyarchenko. Fast simulation of Lévy processes. Working paper, August 2012. Available at SSRN: http://ssrn.com/abstract=2138661.

[7] M. Boyarchenko and S. Levendorski˘ı. Prices and sensitivities of barrier and first-touch digital options in Lévy-driven models. *International Journal of Theoretical and Applied Finance*, 12(8):1125–1170, December 2009.

[8] M. Boyarchenko and S. Levendorski˘ı. Ghost Calibration and Pricing Barrier Options and Credit Default Swaps in spectrally one-sided Lévy models: The Parabolic Laplace Inversion Method. *Quantitative Finance*, 15(3):421–441, 2015. Available at SSRN: http://ssrn.com/abstract=2445318.

[9] S. Boyarchenko and S. Levendorski˘ı. Generalizations of the Black-Scholes equation for truncated Lévy processes. Working Paper, University of Pennsylvania, April 1999.

[10] S. Boyarchenko and S. Levendorski˘ı. Option pricing for truncated Lévy processes. *International Journal of Theoretical and Applied Finance*, 3(3):549–552, July 2000.

[11] S. Boyarchenko and S. Levendorski˘ı. Barrier options and touch-and-out options under regular Lévy processes of exponential type. *Annals of Applied Probability*, 12(4):1261–1298, 2002.

[12] S. Boyarchenko and S. Levendorski˘ı. *Non-Gaussian Merton-Black-Scholes Theory*, volume 9 of Adv. Ser. Stat. Sci. Appl. Probab. World Scientific Publishing Co., River Edge, NJ, 2002.

[13] S. Boyarchenko and S. Levendorski˘ı. Perpetual American options under Lévy processes. *SIAM Journal on Control and Optimization*, 40(6):1663–1696, 2002.

[14] S. Boyarchenko and S. Levendorski˘ı. New efficient versions of fourier transform method in applications to option pricing. Working paper, 2011. Available at SSRN: http://ssrn.com/abstract=1846633.

[15] S. Boyarchenko and S. Levendorski˘ı. Efficient Laplace inversion, Wiener-Hopf factorization and pricing lookbacks. *International Journal of Theoretical and Applied Finance*, 16(3):1350011 (40 pages), 2013. Available at SSRN: http://ssrn.com/abstract=1979227.

[16] S. Boyarchenko and S. Levendorski˘ı. Efficient variations of Fourier transform in applications to option pricing. *Journal of Computational Finance*, 18(2):57–90, 2014. Available at http://ssrn.com/abstract=1673034.

[17] S. Boyarchenko and S. Levendorski˘ı. Sinh-acceleration: Efficient evaluation of probability distributions, option pricing, and Monte-Carlo simulations. *International Journal of Theoretical and Applied Finance*, 22(3), 2019. DOI: 10.1142/S0219024919500110. Available at SSRN: https://ssrn.com/abstract=3129881 or http://dx.doi.org/10.2139/ssrn.3129881.
[18] S. Boyarchenko and S. Levendorskiǐ. Conformal accelerations method and efficient evaluation of stable distributions. *Acta Applicandae Mathematicae*, 169:711–765, 2020. Available at SSRN: https://ssrn.com/abstract=3206696 or http://dx.doi.org/10.2139/ssrn.3206696.

[19] S. Boyarchenko and S. Levendorskiǐ. Static and semi-static hedging as contrarian or conformist bets. *Mathematical Finance*, 3(30):921–960, 2020. Available at SSRN: https://ssrn.com/abstract=3329694 or http://arxiv.org/abs/1902.02854.

[20] S. Boyarchenko, S. Levendorskiǐ, J.L. Kirkby, and Z. Cui. SINH-acceleration for B-spline projection with option pricing applications. *International Journal of Theoretical and Applied Finance*, -( ), 2022. Available at SSRN: https://ssrn.com/abstract=3921840 or arXiv:2109.08738.

[21] N. Cai and S.G. Kou. Option pricing under a mixed-exponential jump diffusion model. *Operations Research*, 60(1):64–77, 2012.

[22] P. Carr, H. Geman, D.B. Madan, and M. Yor. The fine structure of asset returns: an empirical investigation. *Journal of Business*, 75:305–332, 2002.

[23] P. Carr and D.B. Madan. Option valuation using the Fast Fourier Transform. *Journal of Computational Finance*, 2(4):61–73, 1999.

[24] M. de Innocentis and S. Levendorskiǐ. Pricing discrete barrier options and credit default swaps under Lévy processes. *Quantitative Finance*, 14(8):1337–1365, 2014. Available at: DOI:10.1080/14697688.2013.826814.

[25] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1:281–299, 1995.

[26] F. Fang, H. Jönsson, C.W. Oosterlee, and W. Schoutens. Fast valuation and calibration of credit default swaps under Lévy dynamics. *Journal of Computational Finance*, 14(2):57–86, Winter 2010.

[27] F. Fang and C.W. Oosterlee. A novel pricing method for European options based on Fourier-Cosine series expansions. *SIAM Journal on Scientific Computing*, 31(2):826–848, 2008.

[28] F. Fang and C.W. Oosterlee. Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions. *Numerische Mathematik*, 114(1):27–62, 2009.

[29] M.de Innocentis and S. Levendorskiǐ. Calibration and Backtesting of the Heston Model for Counterparty Credit Risk. Working paper, April 2016. Available at SSRN: http://ssrn.com/abstract=2757008.

[30] M.de Innocentis and S. Levendorskiǐ. Calibration Heston Model for Credit Risk. *Risk*, pages 90–95, September 2017.

[31] S.G. Kou. A jump-diffusion model for option pricing. *Management Science*, 48(8):1086–1101, August 2002.

[32] A. Kuznetsov. Wiener-Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann.Appl.Prob.*, 20(5):1801–1830, 2010.

[33] A. Kuznetsov, A.E. Kyprianou, and J.C. Pardo. Meromorphic Lévy processes and their fluctuation identities. *Annals of Applied Probability*, 22(3):1101–1135, 2012.

[34] S. Levendorskiǐ. Pricing of the American put under Lévy processes. Research Report MaPhySto, Aarhus, 2002. Available at http://www.maphysto.dk/publications/MPS-RR/2002/44.pdf, http://www.maphysto.dk/cgi-bin/gp.cgi?publ=441.

[35] S. Levendorskiǐ. Pricing of the American put under Lévy processes. *International Journal of Theoretical and Applied Finance*, 7(3):303–335, May 2004.

[36] S. Levendorskiǐ. Efficient pricing and reliable calibration in the Heston model. *International Journal of Theoretical and Applied Finance*, 15(7), 2012. 125050 (44 pages).

[37] S. Levendorskiǐ. Method of paired contours and pricing barrier options and CDS of long maturities. *International Journal of Theoretical and Applied Finance*, 17(5):1–58, 2014. 1450033 (58 pages).

[38] S. Levendorskiǐ. Fractional-Parabolic Deformations with Sinh-Acceleration. Working paper, April 2016. Available at SSRN: http://ssrn.com/abstract=2758811.

[39] S. Levendorskiǐ. Pitfalls of the Fourier Transform method in Affine Models, and remedies. *Applied Mathematical Finance*, 23(2):81–134, 2016. Available at http://dx.doi.org/10.1080/1350486X.2016.1159918 or http://ssrn.com/abstract=2367547.

[40] S.Z. Levendorskiǐ and J. Xie. Pricing of Discretely Sampled Asian Options Under Lévy Processes. Working paper, June 2012. Available at SSRN: http://papers.ssrn.com/abstract=2088214.

[41] A. Lipton. Assets with jumps. *Risk*, pages 149–153, September 2002.

[42] D.B. Madan and F. Milne. Option pricing with V.G. martingale components. *Mathematical Finance*, 1(4):39–55, 1991.
A.1. The order and type of KoBoL processes.

A.1.1. KoBoL processes of order \( \nu \in (0,2) \), \( \nu \neq 1 \).

**Proposition A.1.** Let \( \psi^0 \) be given by (2.20). Then

(i) Spectrally positive KoBoL of order \( \nu_+ \in (0,2) \), \( \nu_+ \neq 1 \), are SINH-regular of order \( \nu_+ \) and type \( ((\lambda_-,-\infty),\mathbb{C} \setminus i(-\infty,0],\mathbb{C}_+) \), where \( \nu_+ \neq 1 \).

(1) if \( \nu_+ \in (1,2) \), \( \mathcal{C}_+ = \mathcal{C}_{\lambda_-,\gamma_+} \), \( \gamma_+ = \max\{-1, (1 - 3/\nu_+) \pi/2, (1 - 1/\nu_+) \pi/2\} \).

(2) if \( \nu_+ \in (0,1) \), \( \mathcal{C}_+ = i\mathcal{C}_{\min\{1,1/(2\nu_+)\}} \).

We have

\[
\nu_+ (\omega) = -c_+ \Gamma(-\nu_+) \exp[i(-\pi/2 + \nu_+)].
\]

(ii) If \( \nu_+ > \nu_- \) and \( c_+ > 0 \), then \( X \) is SINH-regular of order \( \nu_+ \) and type \( ((-\infty,\lambda_+),\mathbb{C} \setminus i[0,\infty),\mathbb{C}_+) \), where \( c_\infty \) is given by (A.7), and \( \mathcal{C}_+ \) is the same as in (i).

(iii) Spectrally negative KoBoL of order \( \nu_- \in (0,2) \), \( \nu_- \neq 1 \), are SINH-regular of order \( \nu_- \) and type \( ((-\infty,\lambda_+),\mathbb{C} \setminus i[\infty,\infty),\mathbb{C}_+) \), where

(1) if \( \nu_- \in (1,2) \), \( \mathcal{C}_+ = \mathcal{C}_{\gamma_,\gamma_-} \), \( \gamma_- = (1/\nu_- - 1)\pi/2, \gamma_+ = \min\{1,3/\nu_- - 1\} \pi/2 \), and

(2) if \( \nu_- \in (0,1) \), \( \mathcal{C}_+ = -i\mathcal{C}_{\min\{1,1/(2\nu_-)\}} \).

We have

\[
\nu_- (\omega) = -c_- \Gamma(-\nu_-) \exp[i(\pi/2 + \nu_-)].
\]

(iv) If \( \nu_+ > \nu_- \) and \( c_+ > 0 \), then \( X \) is SINH-regular of order \( \nu_- \) and type \( ((-\infty,\lambda_+),\mathbb{C} \setminus i[0,\infty),\mathbb{C}_+) \), where \( c_\infty \) is given by (A.7), and \( \mathcal{C}_+ \) is the same as in (i).

(v) If \( \nu_+ = \nu_- = \nu \) and \( c_+ \neq 0 \), then \( X \) is SINH-regular of order \( \nu_- \) and type \( ((-\infty,\lambda_+),\mathbb{C} \setminus i[\infty,\infty),\mathbb{C}_+) \), where \( \mathcal{C}_+ = \mathcal{C}_{\gamma_-,\gamma_+} \), \( \gamma_- = \min\{\nu, \pi/2 \} \), and \( \gamma_+ = -\gamma_- \).

We have

\[
c_\infty (\varphi) = -\Gamma(-\nu) \nu e^{-\pi i \nu/2} + c_- e^{i \pi i \nu/2} e^{i \varphi \nu}.
\]

(vi) In particular, if \( \psi^0 \) is given by (2.21), then \( \gamma_+ = \gamma_\nu := \min\{1,1/\nu\} \pi/2, \gamma_- = -\gamma_+ \), and (2.22) holds.
Proof. Evidently, $\psi^0$ is analytic in the complex plane with the cuts $i(-\infty, \lambda_-)$ and $i(\lambda_+, +\infty)$, hence, $\mu_\pm = \lambda_\pm$ and $C = C \setminus i\mathbb{R}$. Consider the asymptotics of the characteristic exponents of one-sided KoBoL processes

\begin{align}
(A.4) & \quad \psi^0_+ (\nu_+, c_+, \lambda_-; \xi) = c_+ \Gamma(-\nu_+) ((-\lambda_-)^{\nu_+} - (-\lambda_- - i\xi)^{\nu_+}), \\
(A.5) & \quad \psi^0_- (\nu_-, c_-, \lambda_+; \xi) = c_- \Gamma(-\nu_-) (\lambda_+^{\nu_-} - (\lambda_+ + i\xi)^{\nu_-})
\end{align}

as $\xi \to \infty$ in $iC_\pi$ and $-iC_\pi$, respectively:

\begin{align}
(A.6) & \quad \psi^0_+ (\nu_+, c_+, \lambda_-; \rho e^{i\varphi}) = -c_+ \Gamma(-\nu_+) \exp[i(-\pi/2 + \varphi)\nu_+] \rho^{\nu_+} + O(\rho^{\nu_+ - 1}), \\
(A.7) & \quad \psi^0_- (\nu_-, c_-, \lambda_+; \rho e^{i\varphi}) = -c_- \Gamma(-\nu_-) \exp[i(\pi/2 + \varphi)\nu_+] \rho^{\nu_-} + O(\rho^{\nu_- - 1}).
\end{align}

Let $\varphi \in (-\pi/2, \pi/2)$. We have

\begin{equation}
(A.8) \quad \text{Re}(-\Gamma(-\nu) \exp[i(-\pi/2 + \varphi)\nu]) = -\Gamma(-\nu) \cos((-\pi/2 + \varphi)\nu) > 0
\end{equation}

iff either

(i) $\nu \in (1, 2)$ (hence, $-\Gamma(-\nu) < 0$) and $-3\pi/2 < (-\pi/2 + \varphi)\nu < -\pi/2$; equivalently, $\varphi \in (\max\{-1, (1 - 3/\nu)\pi/2, (1 - 1/\nu)\pi/2\}, (1 - 1/\nu)\pi/2)$, or

(ii) $\nu \in (0, 1)$ (hence, $-\Gamma(-\nu) > 0$) and $-\pi/2 < (-\pi/2 + \varphi)\nu < \pi/2$, equivalently, $\varphi \in (\max\{-1, (1 - 1/\nu)\pi/2\}, (1 - 1/\nu)\pi/2) \subset ((1 - 1/\nu)\pi/2, (1 + 1/\nu)\pi/2)$. Hence,

$$C_+ = \{\rho e^{i\varphi} \mid \rho > 0, \max\{-1, (1 - 1/\nu)\pi/2\} \subset \gamma \subset \pi/2 - \max\{-1, (1 - 1/\nu)\pi/2\} = iC_\gamma,$$

where $\gamma = \pi/2 - \max\{-1, (1 - 1/\nu)\pi/2\} = \min\{1, 1/(2\nu)\}\pi/2$.

Similarly,

\begin{equation}
(A.9) \quad \text{Re}(-\Gamma(-\nu) \exp[i(\pi/2 + \varphi)\nu]) = -\Gamma(-\nu) \cos((\pi/2 + \varphi)\nu) > 0
\end{equation}

if and only if either

(i) $\nu \in (1, 2)$ (hence, $-\Gamma(-\nu) < 0$) and $\pi/2 < (\pi/2 + \varphi)\nu < 3\pi/2$; equivalently, $\varphi \in ((1/\nu - 1)\pi/2, \min\{1, 3/\nu - 1\}\pi/2)$, or

(ii) $\nu \in (0, 1)$ (hence, $-\Gamma(-\nu) > 0$) and $-\pi/2 < (\pi/2 + \varphi)\nu < \pi/2$, equivalently, $C_+ = -iC_\min\{1, 1/(2\nu)\}\pi/2$.

All statements (a)-(e) are immediate from the properties of $\psi^0_\pm$ established above. \hfill \Box

A.1.2. KoBoL processes of order 1. Formulas for $\psi^0$ are derived in [10, 12]. We have to consider several sets of conditions on the parameters on the RHS of (2.19).

(1) If $c_- = 0$ and $\nu_+ = 1$, $c_+ > 0$, $\lambda_- \leq 0$, we have

\begin{equation}
(A.10) \quad \psi^0_+ (1; \lambda; \xi) = c_+ ((-\lambda_-) \ln(-\lambda_-) - (-\lambda_- - i\xi) \ln(-\lambda_- - i\xi)),
\end{equation}

and, therefore, for $\varphi \in (-\pi/2, \pi/2)$, as $\rho \to +\infty$,

\begin{equation}
(A.11) \quad \psi^0_+ (1; \lambda; \rho e^{i\varphi}) = c_+ e^{i(\pi/2 + \varphi)} \rho \ln \rho + O(\ln \rho).
\end{equation}

Hence, $X$ is SINH-regular of order 1+ and type $(\lambda_- + \infty), C \setminus i(-\infty, 0], C_{-\pi/2, 0})$. 

(2) If \( c_+ = 0 \) and \( \nu_- = 1, c_- > 0, \lambda_+ \geq 0 \), we have

\[
\psi^0_-(1; \lambda; \xi) = c_- (\lambda_+ \ln \lambda_+ - (\lambda_+ + i\xi) \ln(\lambda_+ + i\xi)),
\]

and, therefore, for \( \varphi \in (-\pi/2, \pi/2) \), as \( \rho \to +\infty \),

\[
\psi^0_-(1; \lambda; \rho e^{i\varphi}) = c_- e^{i(-\pi/2 + \varphi)} \rho \ln \rho + O(\ln \rho).
\]

Hence, \( X \) is SINH-regular of order 1+ and type \( ((-\infty, \lambda_+), \mathbb{C} \setminus i[0, +\infty), \mathcal{C}_{0, \pi/2}) \).

(3) If \( c_+, c_- > 0, \nu_- = \nu_+ = 1 \), and either \( \lambda_- < 0 \leq \lambda_+ \) or \( \lambda_- \leq 0 < \lambda_+ \), then

\[
\psi^0(\xi) = c_+ [(-\lambda_-) \ln(-\lambda_-) - (\lambda_- - i\xi) \ln(-\lambda_- - i\xi)]
\]

\[+ c_- [\lambda_+ \ln \lambda_+ - (\lambda_+ + i\xi) \ln(\lambda_+ + i\xi)].\]

Hence, for \( \varphi \in (-\pi/2, \pi/2) \), as \( \rho \to +\infty \),

\[
\psi^0_+(\rho e^{i\varphi}) = e^{i\varphi} ((c_+ - c_-) i\rho \ln \rho + (c_+ + c_-) \pi \rho) + O(\ln \rho).
\]

Therefore, \( X \) is a SINH-regular process of

- (i) order 1+ and type \( ((\lambda_-, \lambda_+), \mathbb{C} \setminus i\mathbb{R}, \mathcal{C}_{-\pi/2,0}) \), if \( c_+ > c_- \);
- (ii) order 1+ and type \( ((\lambda_-, \lambda_+), \mathbb{C} \setminus i\mathbb{R}, \mathcal{C}_{0, \pi/2}) \), if \( c_- > c_+ \);
- (iii) order 1 and type \( ((\lambda_-, \lambda_+), \mathbb{C} \setminus i\mathbb{R}, \mathbb{C} \setminus i\mathbb{R}) \), if \( c_+ = c_- > 0 \).

A.1.3. \textit{KoBoL processes of order 0+}. The formulas for \( \psi^0 \) are derived in [10, 12]. We have to consider several sets of conditions on the parameters on the RHIS of (2.19).

(1) If \( c_- = 0 \) and \( \nu_+ = 0, c_+ = c > 0, \lambda_- \leq 0 \), we have

\[
\psi^0_+(\xi) = c \ln(-\lambda_- - i\xi) - \ln(-\lambda_-),
\]

and, therefore, for \( \varphi \in (-\pi/2, 3\pi/2) \), as \( \rho \to +\infty \),

\[
\psi^0_+(\rho e^{i\varphi}) = c \ln \rho + O(1).
\]

Hence, \( X \) is SINH-regular of order 0+ and type \( ((\lambda_-, +\infty), \mathbb{C} \setminus i(-\infty, 0], \mathbb{C} \setminus i(-\infty, 0]) \).

(2) If \( c_- = 0 \) and \( \nu_- = 0, c_- = c > 0, \lambda_+ \geq 0 \), we have

\[
\psi^0_+(\xi) = c \ln(\lambda_+ + i\xi) - \ln(\lambda_+),
\]

and, therefore, for \( \varphi \in (-3\pi/2, \pi/2) \), as \( \rho \to +\infty \), [A.17] holds. Hence, \( X \) is SINH-regular of order 0+ and type \( ((-\infty, \lambda_+), \mathbb{C} \setminus i[0 + \infty), \mathbb{C} \setminus i[0 + \infty]) \).

(3) Let \( \nu_\pm = 0, c_\pm > 0 \), and either \( \lambda_- < 0 \leq \lambda_+ \) or \( \lambda_- \leq 0 < \lambda_+ \). Then

\[
\psi^0_+(\xi) = c_+ (\ln(-\lambda_- - i\xi) - \ln(-\lambda_-)) + c_- (\ln(\lambda_+ + i\xi) - \ln \lambda_+),
\]

and, therefore, for \( \varphi \in (-\pi/2, \pi/2) \), as \( \rho \to +\infty \), [A.17] holds with \( c = c_+ + c_- \). Hence, \( X \) is SINH-regular of order 0+ and type \( ((\lambda_-, \lambda_+), \mathbb{C} \setminus i\mathbb{R}, \mathbb{C} \setminus i\mathbb{R}) \). Note that if \( c_+ = c_- \), [A.19] defines the characteristic exponent of a VGP.
A.2. The order and type of processes of the $\beta$-class. To calculate the order of the process and find $C_{++}$, recall that if $z \to \infty$ in the sector $|\arg z| \leq \pi - \delta$, where $\delta > 0$, then, for any $a, b \in \mathbb{C}$,

\[(A.20) \quad \frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b}.\]

Hence, for any $\delta > 0$, as $\xi \to \infty$ in $C_{-\pi/2 + \delta, \pi/2 - \delta}$,

\[(A.21) \quad \frac{\Gamma(\pm i\xi + a)}{\Gamma(\pm i\xi + b)} \sim (\pm i\xi)^{a-b},\]

and we conclude that, as $\xi \to \infty$ in $C_{\pi/2 - \delta}$,

\[(A.22) \quad \psi^{0}(\xi) = \frac{\sigma^{2}}{2} \xi^{2} - \frac{c_{1}}{\beta_{1}} \Gamma(1 - \gamma_{1})(-i\xi)^{-1 + \gamma_{1}}(1 + o(1)) - \frac{c_{2}}{\beta_{2}} \Gamma(1 - \gamma_{2})(i\xi)^{-1 + \gamma_{2}}(1 + o(1)) + O(1).\]

Consider the following cases.

(i) $\sigma^{2} > 0$. Since $\gamma_{1}, \gamma_{2} < 3$, the first term on the RHS of (A.22) is the leading term of asymptotics, hence, $X$ is SINH-regular of order 2.

(ii) $\sigma^{2} = 0$, $\gamma_{1} > \gamma_{2}$, $c_{1} > 0$. The leading term of asymptotics is the second term on the RHS of (A.22). Fix $\varphi := \arg \xi \in (-\pi/2, \pi/2)$, and let $\rho = |\xi| \to +\infty$. We have

\[(A.23) \quad \psi^{0}(\xi) \sim c^{1}_{\infty}(\varphi)\rho^{-1 + \gamma_{1}},\]

where $c^{1}_{\infty}(\varphi) = -(c_{1}/\beta_{1})\Gamma(1 - \gamma_{1})e^{i(-\pi/2 + \varphi)(1 + \gamma_{1})}$. Since

\[\Re c^{1}_{\infty}(\varphi) = -(c_{1}/\beta_{1})\Gamma(1 - \gamma_{1})\cos((-\pi/2 + \varphi)(1 + \gamma_{1})),\]

- in the case $\gamma_{1} \in (1, 2)$, we have $-\Gamma(1 - \gamma_{1}) > 0$, hence, $\Re c^{1}_{\infty}(\varphi) > 0$ iff $-\pi/2 < (-\pi/2 + \varphi)(\gamma_{1} - 1) < \pi/2$. Thus, $X$ is of order $\gamma_{1} - 1$, and $C_{++} = C_{\gamma_{1} - \gamma_{+}}$, where $\gamma_{-} = \max\{-1, -1/(\gamma_{1} - 1) + 1\}\pi/2$ and $\gamma_{+} = \pi/2$;

- in the case $\gamma_{1} \in (2, 3)$, we have $-\Gamma(1 - \gamma_{1}) < 0$, hence, $\Re c^{1}_{\infty}(\varphi) > 0$ iff $-3\pi/2 < (-\pi/2 + \varphi)(\gamma_{1} - 1) < -\pi/2$. Thus, $X$ is of order $\gamma_{1} - 1$, and $C_{++} = C_{\gamma_{1} - \gamma_{+}}$, where $\gamma_{-} = \max\{-3/(\gamma_{1} - 1) + 1, -1\}\pi/2$ and $\gamma_{+} = (1 - 1/(\gamma_{1} - 1))\pi/2$.

(iii) $\sigma^{2} = 0$, $\gamma_{2} > \gamma_{1}$, $c_{2} > 0$. The leading term of asymptotics is the third term on the RHS of (A.22). Fix $\varphi := \arg \xi \in (-\pi/2, \pi/2)$, and let $\rho = |\xi| \to +\infty$. We have

\[(A.24) \quad \psi^{0}(\xi) \sim c^{2}_{\infty}(\varphi)\rho^{-1 + \gamma_{2}},\]

where $c^{2}_{\infty}(\varphi) = -(c_{2}/\beta_{2})\Gamma(1 - \gamma_{2})e^{i(\pi/2 + \varphi)(-1 + \gamma_{2})}$. Since

\[\Re c^{2}_{\infty}(\varphi) = -(c_{2}/\beta_{2})\Gamma(1 - \gamma_{2})\cos((\pi/2 + \varphi)(-1 + \gamma_{2})),\]

- in the case $\gamma_{2} \in (1, 2)$, we have $-\Gamma(1 - \gamma_{2}) > 0$, hence, we have $\Re c^{\infty}(\varphi) > 0$ iff $-\pi/2 < (\pi/2 + \varphi)(\gamma_{2} - 1) < \pi/2$. Thus, $X$ is of order $\gamma_{2} - 1$, and $C_{++} = C_{\gamma_{2} - \gamma_{+}}$, where $\gamma_{-} = -\pi/2$ and $\gamma_{+} = \min\{1, 1/(\gamma_{2} - 1) - 1\}\pi/2$;

- in the case $\gamma_{2} \in (2, 3)$, we have $-\Gamma(1 - \gamma_{2}) < 0$, hence, $\Re c^{\infty}(\varphi) > 0$ iff $\pi/2 < (\pi/2 + \varphi)(\gamma_{2} - 1) < 3\pi/2$. Thus, $X$ is of order $\gamma_{2} - 1$, and $C_{++} = C_{\gamma_{2} - \gamma_{+}}$, where $\gamma_{-} = (1/(\gamma_{2} - 1) - 1)\pi/2$ and $\gamma_{+} = \min\{1, (3/(\gamma_{2} - 1) - 1)\}\pi/2$. 

Calculating the residues, we find

\[ \psi^0(\xi) \sim c_{\gamma;\infty} e^{i\phi(\gamma-1)} \rho^{-1+\gamma}, \]

where \( c_{\gamma;\infty} = -\Gamma(1-\gamma)((c_1/\beta_1)e^{-i(\gamma-1)\pi/2} + (c_2/\beta_2)e^{i(\gamma-1)\pi/2}). \)

If \( \gamma \in (1, 2), \) then \(-\Gamma(1-\gamma) > 0 \) and \( \text{Re } e^{\pm i(\gamma-1)\pi/2} > 0. \) If \( \gamma \in (2, 3), \) then \(-\Gamma(1-\gamma) < 0 \) and \( \text{Re } e^{\pm i(\gamma-1)\pi/2} < 0. \) In both cases, \( \text{Re } c_{\gamma;\infty} > 0, \) \( \varphi_\gamma := \text{arg } c_{\gamma;\infty} \in (-\pi/2, \pi/2) \) and \( \text{Re}(c_{\gamma;\infty} e^{i(\varphi_\gamma_0)(\gamma-1)}) > 0 \) iff \(-\pi/2 < (\varphi_0 + \varphi)(\gamma - 1) < \pi/2. \) Therefore, \( X \) is of order \( \gamma - 1, \) and \( C_+ = C_{\gamma-\gamma_0}, \) where \( \gamma_0 = \max\{-\pi/2, -\pi/(2(\gamma - 1)) \} \) and \( \gamma_0 = \min\{\pi/(2(\gamma - 1)) - \varphi_0, \pi/2\}. \)

(v) If \( \sigma^2 = 0 \) and \( \gamma_1, \gamma_2 \in (0, 1), \) then \( X \) is a compound Poisson process, which we do not consider in this paper.

A.3. Calculation of the residues in the \( \beta \)-model. From (2.23), we see that the poles in the upper half-plane are at \( \xi_n, n = 0, 1, \ldots, \) defined by \( \alpha_2 + \frac{i\xi_2}{\beta_2} = -n. \) Hence, \( \xi_n = i(\alpha_2 + n\beta_2). \) Using the relation between the Beta and Gamma functions and reflection formula for the latter, we obtain

\[
B(\alpha_2 + \frac{i\xi_2}{\beta_2}, 1 - \gamma_2) = \frac{\Gamma(\alpha_2 + \frac{i\xi_2}{\beta_2})\Gamma(1 - \gamma_2)}{\Gamma(\alpha_2 + \frac{i\xi_2}{\beta_2} + 1 - \gamma_2)} = \frac{\pi\Gamma(1 - \gamma_2)\Gamma(-\alpha_2 - \frac{i\xi_2}{\beta_2} + \gamma_2)\sin(\pi(\alpha_2 + \frac{i\xi_2}{\beta_2} + 1 - \gamma_2))}{\sin(\pi(\alpha_2 + \frac{i\xi_2}{\beta_2}))\Gamma(1 - \alpha_2 - \frac{i\xi_2}{\beta_2})}.
\]

Letting \( \xi = \xi_n + z, \) where \( z \to 0, \) we obtain \( \alpha_2 + \frac{i\xi_2}{\beta_2} = -n + \frac{i\xi}{\beta_2}, \) and, therefore,

\[
B(\alpha_2 + \frac{i\xi}{\beta_2}, 1 - \gamma_2) = \frac{\Gamma(1 - \gamma_2)\Gamma(n - \frac{i\xi}{\beta_2} + \gamma_2)\sin(\pi(-n + \frac{i\xi}{\beta_2} + 1 - \gamma_2))}{\sin(\pi(-n + \frac{i\xi}{\beta_2}))\Gamma(1 + n - \frac{i\xi}{\beta_2})} = \frac{\Gamma(1 - \gamma_2)\Gamma(n + \gamma_2)\sin(\pi(1 - \gamma_2))}{\beta_2\Gamma(1 + n)} + O(1), \quad z \to 0.
\]

Calculating the residues, we find

\[
f_+(x) = \sum_{n=0}^{+\infty} e^{-an}a_n p_n,
\]

where \( a_n = \alpha_2 + n\beta_2 \) and

\[
a_n p_n = \frac{\beta_2\Gamma(1 - \gamma_2)\Gamma(n + \gamma_2)\sin(\pi(1 - \gamma_2))}{\Gamma(n + 1)}.
\]

As \( n \to +\infty, \) \( \Gamma(n + \gamma_2)/\Gamma(n + 1) \sim n^{\gamma_2 - 1}. \) Since \( \gamma_2 \in (0, 3), \) \( \gamma_2 \notin \{1, 2\}, \) we have \( p_n \sim Cn^{\gamma_2 - 2} \) as \( n \to +\infty, \) where \( C \) is independent of \( n. \) Thus, the \( \beta \)-model is a meromorphic process with \( a_n = a_n \) and \( p_n \) satisfying

\[
\sum_{n=0}^{+\infty} p_n(a_n)^{-2} \leq C \sum_{n=0}^{+\infty} (\alpha_n)^{\gamma_2 - 4} < +\infty.
\]
By symmetry, we calculate the densities of negative jumps are calculated and study the convergence of the series.

A.4. **Proof of Lemma 3.1** For a given domain of the form $i[\mu'_-, \mu'_+] + (C' \cup \{0\})$, there exist $c > 0, \mu''_0 \in (\mu, \mu'), \mu''_+ \in (\mu'_+, \mu_+)$ and closed cone $C'' \subset C \cup \{0\}$ such that, for each $\xi \in i[\mu'_-, \mu'_+] + (C' \cup \{0\})$, the ball $B(\xi, c(1 + |\xi|))$ of radius $c(1 + |\xi|)$, centered at $\xi$, is a subset of $i[\mu''_-, \mu''_+] + (C'' \cup \{0\})$. It follows from the definition of SINH-regular processes that there exists $C_0 > 0$ such that, for all $\xi \in i[\mu''_-, \mu''_+] + C''$ and $j = 0, \pm 0$, the bound (3.1) holds. It follows that there exists $C > 0$ s.t., if $\nu \in (0, 2]$, then, for $\xi \in i[\mu'_-, \mu'_+] + (C' \cup \{0\})$ and $j = 0$, the following modified version of (3.1) holds:

(A.28) \[
\max_{\eta \in B(\xi, c(1 + |\xi|))} |\psi(\eta)| \leq C(1 + |\xi|)^\nu.
\]

By the Cauchy integral theorem, for any $\xi \in i[\mu'_-, \mu'_+] + (C' \cup \{0\})$,

(A.29) \[
\psi(\xi) = \frac{1}{2\pi i} \int_{\partial B(\xi, c(1 + |\xi|))} \frac{\psi(\eta)}{\xi - \eta} d\eta.
\]

Differentiating under the integral sign $j$ times and using (A.28), we obtain

\[
|\psi^{(j)}(\xi)| \leq C_j j! e^{-j-1} (2 + |\xi|)^{\nu-j-1} \frac{1}{2\pi} \int_{\partial B(\xi, c(1 + |\xi|))} |d\eta| \leq C_1 j! e^{-j} (1 + |\xi|)^{\nu-j},
\]

which proves the bound (3.1).

A.5. **Proof of Lemma 3.12.** We have

\[
\int_{|x+\mu|=\epsilon, \text{Re} z>\mu} ST(G)(z) dz = \int_{\mu}^{+\infty} \ln \frac{t-\mu+it\epsilon}{t-\mu-it\epsilon} G(dt).
\]

Choose a function $\delta = \delta(\epsilon)$ such that $\delta(\epsilon) \to 0+$ and $\epsilon/\delta(\epsilon) \to 0$ as $\epsilon \to 0+$. The integrand on the RHS above is uniformly bounded and $G((\mu, \mu + \delta)) \to 0$ as $\delta \to 0$, hence, the integral over $(\mu, \mu + \delta)$ tends to 0. On $(\mu + \delta, 1)$, the integrand is $O(\epsilon/\delta)$, and $G((\mu + \delta, 1)) < \infty$, hence, the integral over $(\mu + \delta, 1)$ tends to 0 as well. Finally, on $[1, +\infty)$, the integrand is bounded by $Ce/t$, where $C$ is independent of $t, \epsilon$, and $\int_{[1, +\infty)} t^{-1} G(dt) < \infty$, hence, the integral over $[1, +\infty)$ tends to 0.

A.6. **Proof of Theorem 3.20 (a).** Let $\text{Im} \xi \geq 0$ and $a_2 = 1, a_1 = 0$. Then we calculate

\[
\psi_+(\xi) = \int_0^{+\infty} (1 - e^{ix\xi} + 1_{[0,1]}(x)ix\xi) \int_{(0, +\infty)} e^{-tx} t^2 G(dt) dx
\]

\[
= \int_{(0, +\infty)} G(dt) \left[ t^2 \int_0^{+\infty} e^{-tx} dx - t^2 \int_0^{+\infty} e^{-(t-i\xi)x} dx + i\xi t^2 \int_0^1 e^{-tx} dx \right]
\]

\[
= \int_{(0, +\infty)} G(dt) \left[ t - \frac{t^2}{t-i\xi} + i\xi (1 - e^{-i}(1 + t)) \right]
\]

\[
= \xi^2 \int_{(0, +\infty)} \frac{G(dt)}{t - i\xi} - i\xi \int_{(0, +\infty)} G(dt) e^{-t}(1 + t).
\]
(To prove that Fubini’s theorem is applicable, one represent the integral w.r.t. \( x \) as the sum of integrals over \((0, 1]\) and \([1, +\infty)\), and uses the bound \( |1 - e^{ix} + 1_{[0, 1]}(x)ix| \leq C(\epsilon)x^2, 0 < x < 1,\) to prove the absolute convergence). Similarly, if \( a_2 = 0, a_1 = 1,\) we calculate

\[
\psi_+(\xi) = \int_0^{+\infty} (1 - e^{ix}) \int_{(+, +\infty)} e^{-tx} tG(dt)dx
\]

\[
= \int_{(+, +\infty)} G(dt) \left[ t \int_0^{+\infty} e^{-tx} dx - t \int_0^{+\infty} e^{-(t-i)x} dx \right]
\]

\[
= \int_{(+, +\infty)} G(dt) \left[ 1 - \frac{t}{t-i\xi} \right] = -i\xi \int_{(+, +\infty)} \frac{G(dt)}{t-i\xi}.
\]

A.7. Calculation of the (s)SL-measures for Generalized Hyperbolic distributions. In \([2, 25]\), we change the variables \( e^t = y + \sqrt{y^2 - 1}, dt = \frac{dy}{\sqrt{y^2 - 1}};\) then \( y = y' + 1:\)

\[
K_\lambda(z) = e^{-z} \int_0^{+\infty} e^{-y'(y' + 1 + \sqrt{(y')^2 + 2y'})^\lambda + (y' + 1 + \sqrt{(y')^2 + 2y'})^{-\lambda}} \frac{dy'}{2\sqrt{(y')^2 + 2y'}}.
\]

We apply \([A.30]\) for \( z = \delta\sqrt{\alpha^2 - (\beta + i\xi)^2}\) for \( \xi = it + \epsilon, \) where \( t > \alpha + \beta \) and \( \epsilon \downarrow 0 \) or \( t < -\alpha + \beta \) and \( \epsilon \uparrow 0.\) By symmetry, it suffices to consider the case \( t > \alpha + \beta.\) We represent \( z \) in the form \( z = \rho e^{i\varphi},\) where \( \varphi \in (0, \pi/2) \) and \( \rho = \rho_+(t) = \delta\sqrt{(\alpha - \beta) + (t - \alpha - \beta)},\) fix \( \varphi,\) rotate the line of integration to \( e^{-i\varphi}u,\) and change the variable \( y' = e^{-i\varphi}u.\) The result is

\[
K_\lambda(\rho e^{i\varphi}) = e^{-\rho e^{i\varphi}} I_\lambda(\rho, \varphi),
\]

where

\[
I_\lambda(\rho, \varphi) = \int_0^{+\infty} e^{-\rho e^{i\varphi}} \frac{u + e^{i\varphi} + \sqrt{u^2 + 2ue^{i\varphi}} - \lambda}{2\sqrt{u^2 + 2ue^{i\varphi}}} du.
\]

If \( \lambda = 1 \) (the case of Hyperbolic processes), the integral simplifies:

\[
I_1(\rho, \varphi) = e^{-i\varphi} \int_0^{+\infty} e^{-\rho u} \frac{u + e^{i\varphi}}{\sqrt{u^2 + 2ue^{i\varphi}}} du.
\]

In the limit \( \varphi = \pi/2 - 0,\) we have

\[
I_\lambda(\rho, \pi/2 - 0) = \int_0^{+\infty} e^{-\rho u} \frac{e^{-i\lambda \varphi}(u + i + \sqrt{u^2 + 2ui})^\lambda + e^{i\lambda \varphi}(u + i + \sqrt{u^2 + 2ui})^{-\lambda}}{2\sqrt{u^2 + 2ui}} du,
\]

and

\[
I_1(\rho, \pi/2 - 0) = -i \int_0^{+\infty} e^{-\rho u} \frac{u + i}{\sqrt{u^2 + 2ui}} du.
\]

The density of the measure \( G_+(\lambda; dt) = 1_{(\alpha, \beta, +\infty)}(t)g_+(\lambda; t)dt \) is

\[
g_+(\lambda; t) = -\text{Im} [\ln I_\lambda(\rho_+(t), \pi/2 - 0)] + (1 - \lambda)\rho_+(t)\pi/2
\]

\[
= (1 - \lambda)\frac{\pi}{2}\rho_+(t) - \text{arg} I_\lambda(\rho(t), \pi/2 - 0).
\]

It is easy to see that in cases \( \lambda = 0 \) or \( \lambda = 1,\) \( \text{Im} I_\lambda(\rho, \pi/2 - 0) < 0,\) hence,

\[
\text{Im}(-\ln I_\lambda(\rho, \pi/2 - 0)) = -\text{arg} I_\lambda(\rho, \pi/2 - 0) > 0,
\]
LÉVY MODELS AMENABLE TO EFFICIENT CALCULATIONS

and \( g_+(\lambda; t) > 0, t > \alpha + \beta \). By symmetry, \( \mathcal{G}_-(\lambda; dt) = 1_{(\alpha-\beta, +\infty)}(t)g(\lambda; dt) \), where

\[
(A.35) \quad g_-(\lambda; t) = (1 - \lambda)\frac{\pi}{2} \rho_-(t) - \arg \lambda_+(\rho_-(t), \pi/2 - 0),
\]

where \( \rho_-(t) = \delta\sqrt{(t - \alpha + \beta)(\alpha + \beta + t)} \). Hence, if \( \lambda = 0, 1 \), the distribution \( \sigma(dx) \) is a SL distribution.

For \( \lambda \in [-2, 2] \), we verify the condition \( \text{Im} \lambda_+(\rho, \pi/2 - 0) < 0 \) numerically checking that for all \( y > 0 \), the imaginary part of the integrand is negative, hence, if \( \lambda \in [-2, 1] \), \( \sigma(dx) \) is a regular SL-distribution. If \( \lambda > 1 \), then, for large \( t \), \( g_\pm(\lambda, t) < 0 \), and \( \sigma(dx) \) is not a SL-distribution.

If \( \lambda < -2 \), the imaginary part of integrand changes sign, hence, we cannot obtain a definitive result by these simple means. We only note that if \( \lambda < -2 \), then, as \( u \to +\infty \), the leading term of asymptotics of the imaginary part of the fraction in front of \( e^{-\mu u} \) is \( -\sin(\lambda \pi/2)u^{-\lambda-1} \), hence, if \( -\lambda \pi/2 \in (1, 2) \cup (3, 4) \cup \cdots \), then, as \( \rho_\pm(t) \downarrow 0 \), \( g_\pm(\lambda, \rho_\pm) \to -\infty \). It follows that \( \sigma(dx) \) is not a SL-distribution if \( \lambda < -2 \) and either \( \alpha - \beta \) or \( \alpha + \beta \) is sufficiently close to 0 or \( \delta \) is. The question whether \( \lambda < -2 \) implies that \( \sigma(dx) \) is not a SL distribution for all admissible \( \delta, \alpha, \beta \) remains open.

A.8. Proof of Theorem 3.37. Fix \( \gamma \in (0, \pi/2) \), and let \( \xi = \rho e^{i\varphi} \), where \( \rho > 0 \) and \( |\varphi| \leq \gamma \). We have

\[
\text{Re } \psi_-(\xi) = \text{Re } \int^{+\infty}_\mu \frac{\xi^2 \mathcal{G}(dt)}{t + i\xi} = \rho^2 \text{Re} \left\{ (\cos(2\varphi) + i \sin(2\varphi)) \int^{+\infty}_\mu \frac{(t - \rho \sin \varphi - i\rho \cos \varphi) \mathcal{G}(dt)}{(t - \rho \sin \varphi)^2 + \rho^2 \cos^2 \varphi} \right\}
\]

\[
= \rho^2 (\cos(2\varphi) I_1(\mathcal{G}; \mu, \varphi; \rho) + \rho \sin \varphi I_0(\mathcal{G}; \mu, \varphi; \rho)),
\]

where

\[
I_k(\mathcal{G}; \mu, \varphi; \rho) := \int^{+\infty}_\mu \frac{t^k \mathcal{G}(dt)}{(t - \rho \sin \varphi)^2 + \rho^2 \cos^2 \varphi}.
\]

On the strength of (A.36), there exist \( C_1, c_1, \rho_0 > 0 \) such that for \( k = 0, 1, \varphi \in [-\gamma, \gamma] \), and \( \rho \geq \rho_0 \),

\[
(A.36) \quad c_1 \rho^{\nu+k-3} \int^{+\infty}_{\mu/\rho} \frac{t^{k+\nu-2} dt}{t^2 + 1} \leq I_k(\mathcal{G}; \mu, \varphi; \rho) \leq C_1 \rho^{\nu+k-3} \int^{+\infty}_{\mu/\rho} \frac{t^{k+\nu-2} dt}{t^2 + 1}.
\]

(a) If \( \nu = \alpha + 2 \in (1, 2) \),

\[
(A.37) \quad \int^{+\infty}_{\mu/\rho} \frac{t^{k+\nu-2} dt}{t^2 + 1} = c(k; \nu) + o(1), \quad \rho \to +\infty,
\]

where

\[
c(k; \nu) = \int^{+\infty}_0 \frac{t^{k+\nu-2} dt}{t^2 + 1},
\]

therefore there exists \( C_1 \) independent of \( \rho \) and \( \varphi \) s.t.

\[
\text{Re } \psi_-(\xi) \geq \rho^\nu [c_1(c_1; \nu) \cos(2\varphi) + c(0, \nu)(\sin \varphi)_] + C_1 c(0, \nu)(\sin \varphi_) + o(\rho^\nu).
\]

Evidently, there exist \( -\pi/2 < \gamma_- < 0 < \gamma_+ < \pi/2 \) s.t. for all \( \varphi \in (\gamma_-, \gamma_+) \),

\[
c_1 c(1, \nu)(\cos(2\varphi) + (\sin \varphi)_+) + C_1 c(0, \nu)(\sin \varphi) > 0,
\]
hence, (2.7) holds with $C_
u = C_{\nu', \nu''}$, for any $[\nu', \nu''] \subset (\nu, \nu')$. The upper bound $|\psi_-(\xi)| \geq C(1 + |\xi|)^\nu$ is proved similarly.

(b) If $\nu = 1$, then, for $k = 1$, (A.37) holds but
$$\int_{\mu/\rho}^{+\infty} \frac{\mu - 2 dt}{t^2 + 1} \sim \ln \rho + O(1), \rho \to +\infty,$$
hence,
$$\Re \psi_-(\xi) \geq \rho c_1 ((1; 1) \cos (2\varphi) + (\sin \varphi, 1) \ln \rho) + (\ln \rho) C_1 (\sin \varphi - 1 + o(\rho^\nu)).$$
It follows that there exist $c > 0$ and $\gamma > 0$ s.t. for $0 \leq \varphi \leq \gamma$,
$$\Re \psi_-(\xi) \geq \rho c(1 + (\sin \varphi, 1) \ln \rho)$$

(but if $\varphi < 0$, then $\Re \psi_-(\xi) \sim c(\varphi, \rho) \ln \rho$, where $c(\varphi) > 0$). The upper bound
$$|\psi_-(\xi)| \leq C(1 + |\xi|)^\nu \ln (2 + |\xi|)$$
is proved similarly, and we conclude that $X$ is SINH-regular of order $(1, 1+)$.

(c) Let $\nu = \alpha + 1 \in (0, 1)$. Then we consider
$$\Re \psi_-(\xi) = \Re \int_{\mu}^{+\infty} \frac{i \xi G(dt)}{t + i \xi}$$
$$= \rho \Re \left( (i \cos(\varphi) - \sin(\varphi)) \int_{\mu}^{+\infty} \frac{(t - \rho \sin \varphi - i \rho \cos \varphi) G(dt)}{(t - \rho \sin \varphi)^2 + \rho^2 \cos^2 \varphi} \right)$$
$$= \rho (-\sin \varphi I_1(G; \mu, \varphi; \rho) + \rho \cos(2\varphi) I_0(G; \mu, \varphi; \rho)).$$
The bound (A.36) assumes the form
$$c_1 \rho^\nu k - 2 \int_{\mu/\rho}^{+\infty} \frac{t^{\nu - k - 1} dt}{t^2 + 1} \leq I_k(G; \mu, \varphi; \rho) \leq C_1 \rho^\nu k - 2 \int_{\mu/\rho}^{+\infty} \frac{t^{\nu - k - 1} dt}{t^2 + 1},$$
and we continue trivially modifying the proof in the case $\nu \in (1, 2)$.

(d) is proved similarly.

A.9. **Sufficient conditions for the existence of zeros on $i(\mu - \mu_0)$**.

**Lemma A.2.** Let $G \in SM_\mu$, $\mu \geq 0$. Then, as $\epsilon \to 0+$, uniformly in $\varphi \in [-\pi/2, \pi/2]$,

(i) $\Re ST(G)(-\mu + e^{i\varphi}) \to +\infty$ if $A := \int_{(\mu, +\infty)} (t - \mu)^{-1} G(dt) = +\infty$;

(ii) if $A_1 := \int_{(\mu, +\infty)} (t - \mu)^{-1} G(dt) < +\infty$, then, as $\epsilon \downarrow 0$,
$$\Re ST(G)(-\mu + e^{i\varphi}) = e^{-1} G(\{\mu\}) \cos \varphi + A_1 + o(1);$$

(iii) if $G_+$ (resp., $G_-$) satisfies the condition in (i), then, for any $q > 0$, the equation $q + \psi(\xi) = 0$ has a solution on $i(\mu_-, 0)$ (resp., on $i(0, \mu_+)$).

**Proof.** An atom at $\mu$ contributes $\Re (G(\{\mu\})/(\mu - \mu_0 + e^{i\varphi})) = e^{-1} G(\{\mu\}) \cos \varphi$, hence, it remains to consider $G(dt)$ without an atom at $\mu$. For any $\epsilon > 0$, function $\varphi \mapsto ST(G)(-\mu + e^{i\varphi})$ is continuous on the compact $[-\pi/2, \pi/2]$, hence, it suffices to prove (i) and (ii) for a fixed $\varphi$. For $t > \mu$,
$$\Re \frac{1}{-\mu + e^{i\varphi} + t} = \Re \frac{t - \mu + e \cos \varphi - i e \sin \varphi}{(t - \mu + e \cos \varphi)^2 + e^2 \sin^2 \varphi} = \frac{t - \mu + e \cos \varphi}{(t - \mu + e \cos \varphi)^2 + e^2 \sin^2 \varphi}.$$
Hence,
\[ \text{Re } ST(G)(-\mu + \epsilon e^{i\varphi}) < \int_{(\mu, +\infty)} (t - \mu + \epsilon \cos \varphi)^{-1} G(dt) < \int_{(0, +\infty)} (t - \mu)^{-1} G(dt), \]
and \(\text{Re}(1/(-\mu + \epsilon e^{i\varphi} + t)) \to 1/(-\mu + t)\) as \(\epsilon \to 0^+\), for any \(t > \mu\). Hence, if \(A_1 < \infty\), then (ii) is valid on the strength of the dominated convergence theorem.

Let \(A_1 = \infty\). If \(\cos \varphi = 0\), then, for every \(t, \epsilon \mapsto \text{Re } \frac{1}{\mu + \epsilon e^{i\varphi} + t} = \frac{t - \mu}{(t - \mu)^2 + \epsilon^2}\) increases as \(\epsilon \downarrow 0\). Hence, if the integrals
\[ I(\varphi, \epsilon) = \int_{(\mu, +\infty)} \frac{1}{t - \mu + \epsilon e^{i\varphi}} G(dt) \]
are bounded by a finite constant \(B(\varphi)\) independent of \(\epsilon > 0\), then \(A \leq B(\varphi) < \infty\) by Fatou’s lemma, contradiction. If \(\cos \varphi > 0\), we use \(\epsilon^2 (\sin \varphi)^2 < (\tan \varphi)^2 (t - \mu + \epsilon \cos \varphi)^2\) to obtain the bound
\[ I(\varphi, \epsilon)(1 + \tan^2 \varphi) \geq I_1(\varphi, \epsilon) := \int_{(\mu, +\infty)} \frac{1}{t - \mu + \epsilon \cos \varphi} G(dt). \]
The same argument as in the case \(\cos \varphi = 0\) shows that \(I_1(\varphi, \epsilon) \to +\infty\), hence, \(I(\varphi, \epsilon) \to +\infty\). Part (iii) is evident.

\[ \Box \]

A.10. Proof of Lemma 4.4. We prove the first bound; the second one follows from \(ST(G)(z) = ST(G)(\bar{z})\). Let \(\epsilon > 0\). We have \(\text{Im } (-b' + i\epsilon + t)^{-1} = -\epsilon((t - b')^2 + \epsilon^2)^{-1}\), therefore
\[ \text{Im } \int_{(0, +\infty)} (-b' + i\epsilon + t)^{-1} G(dt) = -\epsilon \int_{(b-\delta,b+\delta)} ((t - b')^2 + \epsilon^2)^{-1} G(dt) + O(\epsilon) \]
\[ \leq -\epsilon \int_{(b-\delta,b+\delta)} (t - b')^2 (t - b')^2 + \epsilon^2)^{-1} dt + O(\epsilon) \]
\[ = -\epsilon \int_{(b-\delta,b+\delta)} \frac{dt}{(t^2 + \epsilon^2)} + O(\epsilon) \]
\[ = -\epsilon \int_{-\infty}^{+\infty} \frac{dt}{(t^2 + \epsilon^2)} + O(\epsilon), \]
where the constant for the \(O\)-terms can be chosen the same for all \(b' \in (b - \delta', b + \delta')\).

Appendix B. Outline of the Conformal Acceleration Method

B.1. Evaluation of probability distributions and pricing European options. Efficient evaluation of probability distributions and expectations in Lévy models is possible under certain regularity conditions on \(\psi\). The first two conditions are used in [9, 10, 12, 11] to define the class of Regular Lévy Processes of Exponential type (RLPE): (1) \(F(dx)\) is absolutely continuous, and the Lévy density \(f(x)\) exponentially decays as \(x \to \pm \infty\), equivalently, \(\psi\) admits analytic continuation to a strip \(S_{(\mu_-, \mu_+)}\), where \(\mu_- < 0 < \mu_+\); (2) \(f(x) \sim c_\pm |x|^{-\nu_\pm - 1}, x \to 0, +\infty\), where \(c_\pm > 0\) and \(\nu_\pm \in (0, 2)\). If \(\nu_\pm \not\in \{0, 1\}\), an equivalent condition is: as \(\xi \to \infty\) in \(S_{(\mu_-, \mu_+)}\), \(\psi(\xi)\) stabilizes to a positively homogeneous function (if \(\nu_\pm \in \{0, 1\}\), additional log-factor emerges). As it is remarked in op.cit., all classes of Lévy processes used in finance enjoy properties (1) and (2); several classes of Lévy processes constructed later also enjoy
these properties. The importance of properties (1)-(2) are already seen in the simplest case of calculation of expectations of the form

\( V(t, x) = \mathbb{E}^Q \left[ e^{-r(T-t)} G(X_T) \mid X_t = x \right] \).

(This is the price of the European option with the payoff \( G(X_T) \) at the maturity date \( T \), in the market with the constant riskless rate \( r \), under the measure \( \mathbb{Q} \) chosen for pricing.) Assuming that the Fourier transform

\[
\hat{G}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-ix\xi} G(x) dx
\]

is well-defined in a strip \( S_{(\mu'_-, \mu'_+)} \), where \( \mu'_- \leq \mu'_- < \mu'_+ \leq \mu'_+ \), we expand \( G \) in the Fourier integral, substitute the result into (B.1), apply Fubini’s theorem to change the order of taking expectation and integration, and obtain, for any \( \omega \in (\mu'_-, \mu'_+) \),

\[
V(x, t) = (2\pi)^{-1} \int_{\text{Im} \xi = \omega} e^{ix\xi + (T-t)(r+\psi(\xi))} \hat{G}(\xi) d\xi,
\]

where \( x' = x + \mu(T-t) \) and \( \mu \) is the one in (2.2). The simplest way to evaluate the integral on the RHS of (B.3) is to apply the simplified trapezoid rule. The error of the infinite trapezoid rule decays as \( \exp[-2\pi d/\zeta] \), where \( d \) is the half-width of the strip of analyticity around the line of integration, and \( \zeta \) is the step. See, e.g., Thm. 3.2.1 in [50]. If the strip of analyticity is not too narrow, it is easy to satisfy a very small error tolerance for the discretization error. However, one must choose the line of integration and \( \zeta \) carefully otherwise serious errors may result. Popular variations of this straightforward approach such as Carr-Madan method [23] and COS method [27, 28, 26] introduce additional errors which lead to systematic errors in practically important situations. Lewis-Lipton formula is the standard Fourier inversion formula with the prefixed line of integration; the choice of the line is non-optimal in almost all cases and increases the CPU time. See [11, 16, 8, 24, 30, 29, 31, 17, 20] for details and numerical examples.

**B.2. Conformal acceleration method.** In many cases of interest, the integrand decays slowly at infinity, and a very large number of terms of the truncated sum (simplified trapezoid rule) is needed to satisfy even a moderate error tolerance. However, in the case of standard European options, and in the case of piece-wise polynomial approximations of complicated payoffs [7, 24, 30], \( \hat{G} \) is meromorphic with a finite number of simple poles; in [20], approximations with infinite number of poles appear. If \( X \) is SINH-regular of order \( (\nu', \nu) \) with \( \nu' > 0 \), one can use an appropriate conformal deformation and the corresponding change of variables to reduce calculations to the case of an integrand which is analytic in a strip around the line of integration and decays at infinity faster than exponentially. Generally, the most efficient change of variables (sINH-acceleration) is of the form \( \xi = \chi_{\omega_1, \omega, b}(y) \), where \( \omega_1, \omega \in \mathbb{R}, b > 0 \) and

\[
\chi_{\omega_1, \omega, b}(y) = i\omega_1 + b \sinh(i\omega + y).
\]

If \( x' > 0 \), the contour of integration is deformed upwards, hence, \( \omega > 0 \) is used, if \( x' < 0 \), the contour of integration is deformed downwards, hence, \( \omega < 0 \) is used. If \( x' = 0 \), \( \omega \) of either sign can be used but, depending on the cone \( C_\pm = C_{\gamma_-, \gamma_+} \), one of the choices is better. Typically, the choice \( \omega = (\gamma_- + \gamma_+)/2 \) is optimal. In [17], the reader can find detailed general recommendation for the choice of the parameters \( \omega_1, b, \omega \) and the grid for the simplified trapezoid rule needed to satisfy the desired error tolerance.
In a number of applications, we also used fractional-parabolic deformations
\begin{equation}
\chi_{\omega; a,b}^+(\eta) = i\omega \pm ib(1 \mp i\eta)^a,
\end{equation}
where \( \omega \in \mathbb{R} \), \( a > 1 \), \( b > 0 \), and hyperbolic or sub-polynomial deformations defined by \( \xi = \eta \ln^m(1 + b\eta^2) \), where \( m \geq 1 \), \( b > 0 \). See [18] for the general discussion, applications to efficient evaluation of special functions, analogs of the three families for evaluation of stable distributions with explanation when the seemingly less efficient families are more efficient than the sinh-acceleration, and further references. In [38], the composition of fractional-parabolic and sinh changes of variables was used.

The complexity of the numerical scheme based on the sinh-acceleration [B.4] is of the order of \( E \ln E \), where \( E = \ln(1/\epsilon) \); in the case of VGP with \( \mu = 0 \), of the order of \( O(E^2) \). See [17] for details. The idea of conformal acceleration method is similar to the idea of the saddle point method. However, the universal families of conformal deformations are simpler to use, especially when deformations of several lines of integration are needed, and the deformations must be in a certain agreement [15, 37, 19].

B.3. Barrier options. Consider the no-touch barrier option on an asset \( S_t = X_t \), with the lower barrier \( H = e^h \) and maturity date \( T \). The riskless rate \( r \) is constant. Assume that under an EMM chosen for pricing, \( X \) is SINFH-regular of type \((\mu_-, \mu_+), C, C_+ \), where \( \mu_- < 0 \) and \( C_+ \) contains \( \mathbb{R} \setminus \{0\} \). In [10, 11, 12], we have derived general formulas for prices of barrier options. In the case of the no-touch option, the general formula reduces to
\begin{equation}
V(h; T; x) = \frac{e^{-rT}}{(2\pi)^2} \int_{\text{Re}q = \sigma} e^{\sigma T} q^{-1} \int_{\text{Im} \xi = \omega^0} e^{i(x-h)\xi} \frac{\phi_q^- (\xi)}{-i\xi} d\xi dq,
\end{equation}
where \( \omega^0 \in (\mu_- , 0) \) is sufficiently small in absolute value, and \( \phi_q^- (\xi) \) is the ‘minus-Wiener-Hopf factor’. Recall that if \( q > 0 \), then \( \phi_q^- (\xi) = \mathbb{E}[e^{i\xi X_T}] \), where \( T_q \) is the exponentially distributed random variable of mean \( 1/q \) independent of \( X \), and \( X_T = \inf_{0 \leq t \leq T} X_t \) is the infimum process. Since \( \psi \) admits analytic continuation to the strip \( (\mu_- , \mu_+) \), there exist \( \sigma_-(q) < 0 < \sigma_+(q) \) s.t. for any \( q' \in (\sigma_-(q), \sigma_+(q)) \) and any \( \xi \in \{\text{Im} \xi < \omega'\} \),
\begin{equation}
\phi_q^- (\xi) = \exp \left[ \frac{1}{2\pi i} \int_{\text{Im} \eta = \omega'} \frac{\xi \ln(q + \psi(\eta))}{\eta(\xi - \eta)} d\eta \right].
\end{equation}
Since \( X \) is SINFH-regular, the RHS admits analytic continuation w.r.t. \( q \) and \( \xi \) to unions of strips and coni, denote these unions \( U_1 \) and \( U_2 \), respectively, and the choice of \( U_2 \) imposes restrictions on the choice of \( U_2 \).

If the Gaver-Stehfest method is used, one needs to calculate the inner integral in (B.6) for positive \( q \)’s only. Since \( x - h > 0 \), we deform the inner contour into a contour of the form \( \mathcal{L}_1 := \chi_{\omega; b} (\mathbb{R}) \), where \( \omega \in (0, \gamma_+) \); the restrictions on the choice of other parameters depend on \( q \) and \( \psi^\alpha \): in the process of deformation, we must have \( q + \psi(\xi) \neq 0 \) because the analytic continuation of \( \phi_q^- (\xi) \) to \( C_+ \cup \{\text{Im} \xi > 0\} \) is defined using the Wiener-Hopf factorization formula
\begin{equation}
\phi_q^+ (\xi) \phi_q^- (\xi) = \frac{q}{q + \psi(\xi)},
\end{equation}
where \( \phi_q^+ (\xi) = \mathbb{E}[e^{i\xi \bar{X}_T}] \), and \( \bar{X}_t \) is the supremum process. It follows that if \( X \) is an SL-process, then any \( \omega \in (0, \pi/2) \) can be chosen, and the only restriction is that \( q + \chi_{\omega; b} > 0 \). If one of the zeros of \( q + \psi(\xi) \) on \( i\mathbb{R} \) is crossed, the residue theorem is applied. The contour in (B.7)
is deformed downward into a contour \( L_2 := \chi_{\omega'} \cdot \omega' (\mathbb{R}) \), where \( \omega' < 0 \); if \( X \) is an SL process, any \( \omega' \in (-\pi/2, 0) \) can be used. The two crucial restrictions are weaker: 1) \( L_2 \) is above \( i\mu_- \); 2) \( L_1 \) is above \( L_2 \) so that the integrand is well-defined in the process of deformation, for all \( \xi \).

If the conformal change of variables \( q = \tilde{\chi}_{\sigma;b;\omega}(y_1) = \sigma + ib \sinh(i\omega + y_1), \sigma > 0, b > 0, \omega > 0 \) in the Bromwich integral is used, then the parameters of \( L_1, L_2 \) and \( L_3 := \tilde{\chi}_{\sigma;b;\omega}(R) \) must be chosen so that, in the process of all three deformations, \( q + \psi(\eta) \neq 0 \) and \( q + \psi(\xi) \neq 0, \xi - \eta \neq 0 \). For details, see [15], where the less efficient family of fractional-parabolic deformations was used. The reformulation of the procedures in [15] to the case of sinh-deformations is straightforward.

If the payoff of the barrier option depends on \( X_T \), then the explicit formula derived in [10, 11, 12] involves the triple integral (and two Wiener-Hopf factors must be calculated); if the option pays when the barrier is crossed, then quadruple integrals appear [19]. For explicit efficient procedures for evaluation of these integrals, see [19].