Minimal Length Uncertainty Relations and New Shape Invariant Models

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This paper identifies a new class of shape invariant models. These models are based on extensions of conventional quantum mechanics that satisfy a string-motivated minimal length uncertainty relation. An important feature of our construction is the pairing of operators that are not adjoints of each other. The results in this paper thus show the broader applicability of shape invariance to exactly solvable systems.

PACS: 03.65.Fd, 02.30.Hq, 11.30.Pb

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1. Introduction

Shape invariance has proven to be a useful paradigm for understanding exact solvability in quantum mechanics [1][2]. It is of interest to understand how broadly shape invariance can be applied, and in this paper, I will identify new arenas in which it can be used effectively.

In the past few years, there has been some consideration of extensions of quantum mechanics designed to comport well with string theory [3]. In particular, string theory predicts that there can be a modification in the familiar Heisenberg uncertainty relationship [4], so that it takes the form

\[ \Delta x \geq a \Delta p^2 + b \Delta p \] (1.1)

This relationship is sometimes termed a “minimal length uncertainty relationship,” since it implies a lower bound for \( \Delta x \), namely \( \Delta x \geq \sqrt{a/b} \). Of course, in order for such a relation to appear in the non-relativistic limit, the Schrödinger equation must be modified. It is precisely such modifications that are considered in [3], in which some exactly solvable models are found that satisfy the minimal length uncertainty relationship (1.1).

It is the point of this letter to show that the exact solvability of these models is no accident. These models turn out to be representatives of a new class of shape invariant models. In this letter, I will identify these new shape invariant models, and use the shape invariance to obtain solutions. One will see that these models include the models considered by [3], as well as some spectral problems that generalize the Schrödinger equation in ways beyond the modification (1.1). An explicit consequence of this result is to re-affirm the close connection between shape invariance and exact solvability for spectral problems in general, and not simply in the context of the Schrödinger equation. We also make technical progress, demonstrating that shape invariance can be applied productively in a setting in which second-order differential operators can be written as the product of pairs of first-order differential operators that are not adjoints of each other.

2. One-dimensional generalized harmonic oscillator

We begin by displaying the simplest model of [3]. To begin, [3] considers a modified commutation relation

\[ [\hat{x}, \hat{p}] = i\hbar(1 + \beta p^2) \] (2.1)
Such a commutation relation leads to the uncertainty relationship

\[ \Delta x \geq \frac{\hbar}{2} \left( \frac{1}{\Delta p} + \beta \Delta p \right) . \]  

(2.2)

This commutation relation can be realized in the momentum representation by operators

\[ \hat{x} = i\hbar \left[ (1 + \beta p^2) \frac{d}{dp} \right] + \gamma p \]

\[ \hat{p} = p \]  

(2.3)

We note that the parameter \( \gamma \) is a trivial addition here; its presence does not serve to modify the conventional commutation relation, and it is easily set to zero by a canonical transformation. It is included here not simply for making contact with [3], but more importantly to lay the groundwork for one of the considerations needed for the multi-dimensional oscillator considered in the next section.

Quantizing the classical one-dimensional harmonic oscillator

\[ H = \frac{1}{2\mu} p^2 + \frac{1}{2} \mu \omega^2 x^2 \]  

(2.4)

according to (2.1) produces the momentum space Schrödinger-like time-independent equation

\[ \left[ -\mu \hbar \omega \left\{ \left(1 + \beta p^2\right) \frac{\partial}{\partial p} \right\}^2 + 2\gamma p \left(1 + \beta p^2\right) \frac{\partial}{\partial p} \right] + \frac{\gamma (\beta + \gamma) p^2 + \gamma}{\mu \hbar \omega} \right] \Psi(p) = \frac{2E}{\hbar \omega} \Psi(p) . \]  

(2.5)

Remember, too, that the physics cannot depend on \( \gamma \).

We now wish to demonstrate that this equation can be solved exactly via shape invariance techniques. This provides a simpler understanding of the exact solvability (and a simpler calculation of the energy levels and wavefunctions) demonstrated by analytical means in [3].

A quick review of shape invariance is in order here; more detailed treatments are found in the literature, including a review [2] and a recent article revealing the underlying algebraic structure of shape invariance as a BPS phenomenon associated with centrally extended supersymmetry [3].

Suppose one has a Hamiltonian \( H = \tilde{A}A \) for which the spectrum is bounded from below by zero. In the familiar applications, \( \tilde{A} = A^\dagger \), which enforces the positive semi-definiteness of the spectrum, but as long as the spectrum is positive semi-definite, \( \tilde{A} \) need not be the adjoint of \( A \). The eigenfunctions for such Hamiltonians with eigenvalue zero are
those annihilated by $A$. The partner Hamiltonian $\tilde{H} = A\bar{A}$ will have the same spectrum of non-zero eigenvalues as $H$, since if $H\phi = \Lambda \phi$, then $\tilde{H}(A\phi) = \Lambda(A\phi)$. Typically, $A\phi = 0$ has solutions and $\bar{A}\tilde{\phi} = 0$ does not, and the spectral degeneracy does not extend to the zero eigenvalues.

Shape invariance exists when $H$ and $\tilde{H}$ have the same mathematical form, so that $\tilde{H}(g) = H(g') + c$, where $g$ denotes the coupling constants of the theory, $g'$ are the transformed coupling constants which are functions of the $g$, and $c$ is some $c$-number, which also generally depends on the parameters of the theory. This means that the energies of the two Hamiltonians not only exhibit the degeneracy referred to in the preceding paragraph ($E_{k+1} = \tilde{E}_k$), but also satisfy the relation $\tilde{E}_k(g) = E_k(g') + c$. These results enable us to find the energy eigenvalues algebraically.

The wavefunctions are also found algebraically, through standard raising and lowering operator techniques. The ground state of $H$ is the solution of $A\phi = 0$; the states $\tilde{\phi}_k$ of $\tilde{H}$ are related to the states $\phi_k$ of $H$ by $\tilde{\phi}_k(g) = \phi_k(g')$ as well as by $\phi_{k+1} = \bar{A}\tilde{\phi}_k$, which allows the eigenstates to be built up algebraically from the ground state of $H$.

We now demonstrate the solution of (2.5) via shape invariance. To begin, define the operators

$$A = (a + bp^2)\frac{d}{dp} + cp$$
$$\bar{A} = -(a + bp^2)\frac{d}{dp} + cp$$.

The real numbers $a$, $b$, and $c$ are all positive. Note that with this convention, there is a normalizable state annihilated by $A$ but not one annihilated by $\bar{A}$.

Consider the equation $\mathcal{H}\phi = \Lambda\phi$ where $\mathcal{H} = \bar{A}A$. Then

$$\mathcal{H}(a, b, c) = -\left[(a + bp^2)\frac{d}{dp}\right]^2 + c(c-b)p^2 - ac$$.

Similarly, $\tilde{\mathcal{H}} = A\bar{A}$ is

$$\tilde{\mathcal{H}}(a, b, c) = -\left[(a + bp^2)\frac{d}{dp}\right]^2 + c(c+b)p^2 + ca$$.

Consequently, $\mathcal{H}$ and $\tilde{\mathcal{H}}$ form a shape invariant pair, since

$$\tilde{\mathcal{H}}(a, b, c) = \mathcal{H}(a, b, c + b) + 2ca + ba$$.
Applying the principles of shape invariance, one readily obtains the eigenvalues for \(\mathcal{H}(a, b, c)\phi_k(p) = \Lambda_k \phi_k(p)\) and \(\tilde{\mathcal{H}}(a, b, c)\phi_k(p) = \tilde{\Lambda}_k \tilde{\phi}_k(p)\). One finds \(\Lambda_0 = 0, \tilde{\Lambda}_0 = a(b + 2c) = \Lambda_1, \tilde{\Lambda}_1 = a(b + 2(c + b)) + a(b + 2c) = \Lambda_2,\) and so forth, resulting in the general expression

\[
\Lambda_k = a(k^2b + 2kc) \quad k = 0, 1, 2, 3, \ldots .
\]

We can easily make contact with previous results. Note that, as pointed out above, the parameter \(\gamma\) has no physical content, and thus may readily be set to zero. Once this is done, the modified harmonic oscillator equation (2.5) of [3] is equivalent to \(m\mathcal{H}\phi(p) = \Lambda\phi(p)\), with the identifications

\[
m = \mu \hbar \omega \quad a = 1 \quad b = \beta \\
c(c - b) = \frac{1}{(\mu \hbar \omega)^2} \quad \Lambda + mca = \frac{2E}{\hbar \omega}
\]

This leads to the energy eigenvalues for (2.5)

\[
E_k = \hbar \omega \left[ \frac{1}{2} \mu \hbar \omega \beta (k^2 + k + \frac{1}{2}) + (k + \frac{1}{2}) \sqrt{1 + \frac{\beta^2 \mu^2 \hbar^2 \omega^2}{4}} \right]
\]

which agrees with the past results. It is a simple exercise, too, to obtain the wavefunctions.

Note that if we wish to use shape invariance to solve (2.5) when the term linear in \(\frac{d}{dp}\) is explicitly present, it is still possible to do so, as we now demonstrate.

Suppose we define modified raising and lowering operators

\[
A = F(p) \frac{d}{dp} + W(p) + \Omega(p) \\
\tilde{A} = -F(p) \frac{d}{dp} + W(p) - \Omega(p)
\]

Then

\[
\mathcal{H} = \tilde{A}A = \left[ F(p) \frac{d}{dp} \right]^2 - F(p) \left( \frac{dW}{dp} + \frac{d\Omega}{dp} \right) - 2F(p)\Omega(p) \frac{d}{dp} + W^2(p) - \Omega^2(p)
\]

\[
\tilde{\mathcal{H}} = A\tilde{A} = \left[ F(p) \frac{d}{dp} \right]^2 + F(p) \left( \frac{dW}{dp} - \frac{d\Omega}{dp} \right) - 2F(p)\Omega(p) \frac{d}{dp} + W^2(p) - \Omega^2(p)
\]

To make contact with the problem at hand, we set

\[
F(p) = (a + bp^2) \quad W(p) = c_1p \quad \Omega(p) = c_2p
\]
in which case

\[ \mathcal{H} = - \left( (a + bp^2) \frac{d}{dp} \right)^2 - 2(a + bp^2)c_2 p \frac{d}{dp} + (c_1^2 - c_1b - c_2^2 - c_2b)p^2 - c_1a - c_2a \quad , \quad (2.16) \]

which now includes the \( \frac{d}{dp} \) term. Computing \( \tilde{\mathcal{H}} = A\bar{A} \), we find

\[ \tilde{\mathcal{H}}(a, b, c_1, c_2) = \mathcal{H}(a, b, c_1 + b, c_2) + 2ac + ba \quad . \quad (2.17) \]

This result is identical in form to the shape invariance result \( \mathbb{2.9} \); the addition of \( \Omega = c_2p \) thus has no bearing on the eigenvalue spectrum. Even in considering the wavefunctions, there is no physical modification associated with including this term proportional to \( \gamma \), simply a change of variables.

3. Multi-dimensional Oscillator

We now generalize our results to another shape invariant eigenvalue equation. This will turn out to be the relevant equation for the “radial” (in momentum space) piece of the generalized Schrödinger equation that describes the \( D \)-dimensional harmonic oscillator with a minimal length uncertainty relation, the second of the two models discussed in \( \mathbb{3} \).

To obtain our new shape invariant model, we define

\[ A = (a + bp^2) \frac{d}{dp} + cp + \frac{g}{p} \]
\[ \bar{A} = -(a + bp^2) \frac{d}{dp} + cp + \frac{g}{p} \quad . \quad (3.1) \]

Then we can compare \( \mathcal{H} = \bar{A}A \) and \( \tilde{\mathcal{H}} = A\bar{A} \). One finds

\[ \mathcal{H}(a, b, c, g) = - \left[ (a + bp^2) \frac{d}{dp} \right]^2 + \frac{g(g + a)}{p^2} + c(c - b)p^2 + c(g - a) + g(c + b) \]
\[ \tilde{\mathcal{H}}(a, b, cg) = - \left[ (a + bp^2) \frac{d}{dp} \right]^2 + \frac{g(g - a)}{p^2} + c(c + b)p^2 + c(g + a) + g(c - b) \quad . \quad (3.2) \]

The shape invariance is obvious, with

\[ \tilde{\mathcal{H}}(a, b, c, g) = \mathcal{H}(a, b, c + b, g - a) + 4a(b + c) - 4bg \quad . \quad (3.3) \]

The eigenvalues for the equation \( \mathcal{H}\phi_k(p) = \Lambda_k\phi_k(p) \) are readily obtained algebraically. They are given by

\[ \Lambda_k = 4abk^2 + 4(ac - bg)k \quad . \quad (3.4) \]
As it stands, this equation is not quite the momentum-space radial equation for the modified $D$-dimensional harmonic oscillator. That equation, given in [3], is

\[-\mu \bar{\hbar} \omega \left\{ \left[ 1 + (\beta + \beta')p^2 \right] \frac{d}{dp} \right\}^2 + \left\{ \frac{D-1}{p} + (D-1)\beta + 2\gamma \right\} \left\{ \left[ 1 + (\beta + \beta')p^2 \right] \frac{d}{dp} \right\} \]

\[-\frac{L^2}{p^2} + (\gamma D - 2\beta L^2) + \left\{ (\gamma (\beta D + \beta' + \gamma) - \beta^2 L^2) \right\} p^2 \right\} R(p) + \frac{1}{\mu \bar{\hbar} \omega} p^2 R(p) = \frac{2E}{\bar{\hbar} \omega} R(p) \]

(3.5)

which is obtained from quantizing the classical $D$-dimensional harmonic oscillator Hamiltonian $H = \frac{1}{2} \mu \omega^2 \vec{x} \cdot \vec{x} + \frac{1}{2\mu \bar{\hbar}} \vec{p} \cdot \vec{p}$ according to the modified commutation relation

\[[\hat{x}_i, \hat{p}_j] = i\hbar (\delta_{ij} + \beta \hat{p}^2 \delta_{ij} + \beta' \hat{p}_i \hat{p}_j) \]

(3.6)

which is done using the representations

\[\hat{x}_i = i\hbar \left( 1 + \beta p^2 \right) \frac{\partial}{\partial p_i} + \beta' p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i \]

(3.7)

\[\hat{p}_i = p_i\]

Note that, as in the previous example, there is no physical content to $\gamma$ and, it may simply be set to any value.

The problem, however, in comparing (3.2) with (3.5), is that $H$ in the model we have written down has no term linear in $\frac{d}{dp}$, yet such a term cannot be removed from the multi-dimensional oscillator equation simply by re-defining $\gamma$, as was the case in one dimension. Fortunately, the same method used to include the term linear in $\frac{d}{dp}$ in the previous section can be used productively here.

We once again define modified $A$ and $\bar{A}$ operators, namely

\[A = (a + bp^2) \frac{d}{dp} + (c_1 + c_2)p + \frac{g_1 + g_2}{p} \]

\[\bar{A} = -(a + bp^2) \frac{d}{dp} + (c_1 - c_2)p + \frac{g_1 - g_2}{p} \]

(3.8)

Defining, as usual, $\mathcal{H} = \bar{A}A$ and $\mathcal{\tilde{H}} = A\bar{A}$, we obtain

\[\mathcal{H} = -\left( (a + bp^2) \frac{d}{dp} \right)^2 - (a + bp^2)(c_1 + c_2)p + \frac{g_1 + g_2}{p^2} \]

\[-2(a + bp^2)(c_2p + \frac{g_2}{p}) \frac{d}{dp} + (c_1p + \frac{g_1}{p})^2 - (c_2p + \frac{g_2}{p})^2 \]

(3.9)

\[\mathcal{\tilde{H}} = -\left( (a + bp^2) \frac{d}{dp} \right)^2 + (a + bp^2)(c_1 - c_2)p - \frac{g_1 + g_2}{p^2} \]

\[-2(a + bp^2)(c_2p + \frac{g_2}{p}) \frac{d}{dp} + (c_1p + \frac{g_1}{p})^2 - (c_2p + \frac{g_2}{p})^2 \]
If $c_2 = g_2 = 0$, then we simply recover the operators of (3.2); however, the terms with $c_2$ and $g_2$ in (3.9) are the same in both $\mathcal{H}$ and $\tilde{\mathcal{H}}$, and thus there is a shape invariance relationship between $\mathcal{H}$ and $\tilde{\mathcal{H}}$ in this new scenario which is identical to the relationship that followed from (3.9), even when $c_2$ and $g_2$ are non-zero. Note, though, that with $c_2$ and $g_2$ non-zero, $\mathcal{H}$ and $\tilde{\mathcal{H}}$ have exactly the same mathematical form as (3.5), the equation for the minimal uncertainty relationship $D$-dimensional harmonic oscillator. Consequently, we can use the shape invariance of the new $\mathcal{H}$ and $\tilde{\mathcal{H}}$ to obtain the solutions to (3.5). In particular, we have

$$\tilde{\mathcal{H}}(a, b, c_1, c_2, g_1, g_2) = \mathcal{H}(a, b, c_1 + b, c_2, g_1 - a, g_2) + 4a(b + c_1) - 4bg_1$$

$$\Lambda_k = 4abk^2 + 4(ac - bg)k$$

(3.10)

These $\Lambda_k$ give the spectrum for the modified $D$-dimensional oscillator, with the appropriate identification of parameters. Likewise, the ground state is found by setting $A\phi = 0$, and the excited states by the application of $\tilde{A}$.

4. Additional Shape Invariant Models in this class

In this section, I will describe the extension of the shape invariance techniques developed here to models that fit into the present analysis, but that go beyond that considerations of [3] and the simple minimal length uncertainty principle.

Suppose we have an eigenvalue equation $H\phi = E\phi$ where $H$ is given by

$$H = -\left[(a + b \sinh^2 y) \frac{d}{dy}\right]^2 + \frac{\gamma}{\cosh^2 y}$$

(4.1)

and the parameters $a$, $b$, and $\gamma$ are all positive. Here, I have deliberately labeled the variable $y$ to emphasize that the technique described herein is not restricted to a particular physical framework. Let us define

$$A = (a + b \sinh^2 y) \frac{d}{dy} + g \tanh y$$

$$\tilde{A} = -(a + b \sinh^2 y) \frac{d}{dy} + g \tanh y$$

(4.2)

Then

$$\mathcal{H}(a, b, g) = \tilde{A}A = -\left[(a + b \sinh^2 y) \frac{d}{dy}\right]^2 - \frac{g(g + a - b)}{\cosh^2 y} + g(g - b)$$

(4.3)
where \( g \), like \( a \) and \( b \), is positive. The partner operator \( \tilde{\mathcal{H}} = A\bar{A} \) is related to \( \mathcal{H} \) by

\[
\tilde{\mathcal{H}}(a, b, g) = \mathcal{H}(-a, -b, g) = \mathcal{H}(a, b, g + b - a) + (2g - a)(a - b)
\]  \hspace{1cm} (4.4)

Thus this model exhibits shape invariance under the transformation \( g \to g + b - a \). Using the standard techniques, we see that the eigenvalues \( E_k \) for the operator \( \mathcal{H} \) are simply

\[
E_k = k(2g - b)(b - a) - k^2(a - b)^2 \hspace{1cm} k = 0, 1, 2, \ldots
\]  \hspace{1cm} (4.5)

Note that for large enough \( k \), these energies become negative, which we know is not physically allowed. This is not special to the example at hand. In fact, if we consider ordinary quantum mechanics with potential \( 1/\cosh^2(x) \), there is only a finite number of bound states; these are the states extracted by the standard shape invariance argument, and when the energy goes negative, it is the signal that we have left the realm of physical states. There is always at least one normalizable state, which is the state with vanishing energy in the supersymmetric/shape invariance framework. This entire structure persists in the problem at hand.

Note, too, that it is simple to make contact with the original operator \( H \) in (4.1) above. To be precise,

\[
\mathcal{H}(a, b, \gamma) = \mathcal{H}(a, b, \bar{\gamma}) - \bar{\gamma}(\bar{\gamma} - b)
\]  \hspace{1cm} (4.6)

where \( \bar{\gamma} = \frac{1}{2}[(b - a) + \sqrt{(b - a)^2 + 4\gamma}] \).

Finally, we note that if generalize this procedure to equations given by \( \mathcal{H} = \tilde{B}\tilde{B} \) where

\[
B = \left( 1 + f(y) \right) \frac{d}{dy} + g(y) \quad \tilde{B} = -\left( 1 + f(y) \right) \frac{d}{dy} + g(y)
\]  \hspace{1cm} (4.7)

then we can always identify models in which

\[
f(y) = \frac{\alpha g^2(y) + \beta}{g'(y)}
\]  \hspace{1cm} (4.8)

for which \( \tilde{B}\tilde{B} \) has the same mathematical form as \( BB \), and the spectra are positive semi-definite, so that the shape invariance approach is applicable.
5. Conclusions

In this paper, I have demonstrated that shape invariance can be applied effectively to exactly solvable models in extensions of quantum mechanics. These include models that possess a modified uncertainty relationship, namely the minimal length uncertainty relation, which also emerges in string theory. As usual, shape invariance provides a simpler and clearer understanding of the exact solvability of the models in question. We have found, too, that shape invariance can be effectively applied even for spectral problems in which the differential operator being studied must be written as the product of lower order differential operators that are not adjoints of each other.

We note, too, that given the connection between centrally extended supersymmetry and shape invariance demonstrated in [3], the present work implies a supersymmetric construction that can incorporate these minimal-length uncertainty principle extensions of quantum mechanics, and the use of BPS techniques to extract the states in these theories. The exploration of those issues, however, lies beyond the scope of this paper.

6. Acknowledgments

This research was supported in part by the National Science Foundation under Grants No. PHY04-57048 and PHY05-51164, by the Japan Society for the Promotion of Science. Portions of this work were conducted at RIKEN (The Institute for Chemical and Physical Research) and at the Kavli Institute for Theoretical Physics, both of which I thank for their hospitality.
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