Thermodynamics of Friedmann Equation and Masslike Function in General Braneworld

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Using the generalized procedure proposed by Wu et al \cite{23} recently, we construct the first law of thermodynamics on apparent horizon in a general braneworld model with curvature correction terms on the brane and in the bulk, respectively. The explicit entropy formulay of apparent horizon in the general braneworld is worked out. We also discuss the masslike function which associated with a new type first law of thermodynamics of the general braneworld in detail. We analyze the difference between the conventional thermodynamics and the new type thermodynamics on apparent horizon. At last, the discussions about the physical meanings of the masslike function have also been given.

Keywords: First law of thermodynamics, Braneworld, Friedmann equation, Masslike function

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1. Introduction

The four thermodynamics laws of black hole, which were originally derived from the classical Einstein Equation, provide deep insight into the connection between thermodynamics and Einstein Equation \cite{1,4}. In Jacobson’s pioneer paper \cite{5}, this connection has been extended into a general spacetime. In Jacobbian’s set-up, the Einstein equation can be derived from the proportionality of entropy to the horizon area, together with the Clausius relation $\delta Q = T dS$. Here $\delta Q$ and $T$ are the energy flux and Unruh temperature detected by an accelerated observer just inside the local Rindler causal horizons through spacetime point. From the viewpoint of thermodynamics, the Einstein equation can be regarded as the equation of state of spacetime. Since Jacobson’s work, many physicists have extended the connection between thermodynamics and gravity beyond the Einstein gravity, including the so-called $f(R)$ gravity \cite{6,9} and the scalar-tensor gravity \cite{9,13}. Recently, the connection between thermodynamics of apparent horizon and Friedmann Equation in Friedmann-Robertson-Walker (FRW) universe has been shown \cite{7,10,12}. This connection has also been extended to braneworld cosmology, for related discussions see Refs. \cite{14,19}. On the other hand, the thermodynamics of FRW universe has also...
been studied in term of holographic principle\textsuperscript{20,21}. More recently, Eling et al\textsuperscript{6} found that it is impossible to derive the field equation of $f(R)$ gravity from the Clausius relation $\delta Q = T dS$, in terms of equilibrium thermodynamics. In order to get the field equation in $f(R)$ gravity, an entropy production term has to be added to the Clausius relation which is then changed to $\delta Q = T dS + T d_i S$. Similar cases have also occurred in the scalar-tensor theory\textsuperscript{13}. The thermodynamics behaviors in $f(R)$ gravity and scalar-tensor gravity show that we have to treat these system with non-equilibrium thermodynamics, which are different with the equilibrium thermodynamics in Einstein gravity. In Ref.\textsuperscript{22} by introducing a masslike function, the authors showed that the equilibrium thermodynamics on apparent horizon of FRW universe can exist for more general theories of gravity, even including $f(R)$ gravity, scalar-tensor gravity, and Brans-Dicke theory. The masslike function provides a new type first law of thermodynamics on apparent horizon of FRW universe, which is obvious different with the conventional thermodynamical treatment of FRW spacetime. Then Wu et al\textsuperscript{23} proposed a general procedure to construct the first law of thermodynamics on apparent horizon in generalized gravity theories, and discussed a more general formulay for the masslike function. The validity of their procedure has been shown in $f(R)$ gravity, Lovelock gravity, scalar-tensor gravity, and also the Randal-Sundrum braneworld with bulk matter.

However, the universality of the procedure in a more general braneworld model has not been discussed. In this paper, we employ this procedure to study the connection between thermodynamics and the general braneworld model with correction terms, such as a 4D scalar curvature from the induced gravity on the brane, and a 5D Gauss-Bonnet curvature term. The connection between thermodynamics and this general braneworld model have also been investigated in Ref.\textsuperscript{19}. They have derived the entropy expression of the apparent horizon even though the exact analytic black hole solution is absent so far. We expect that the entropy formulay derived from the general procedure that proposed by Wu et al should be consistent with the entropy expression in Ref.\textsuperscript{19}.

It is also interesting to explore whether and how the connection between thermodynamics associated with the masslike function and gravity theories be generalized to the braneworld scenarios. There are two motivations encourage us to address this issue. First, the new type first law of thermodynamics on apparent horizon which related to the masslike function is a geometric relation,

\begin{equation}
\frac{dE}{dS} = T dS,
\end{equation}

where $T$ and $S$ are both geometric quantities. The energy flow through the apparent horizon is defined by a masslike function. In a general braneworld, the curvature correction terms on the brane and in the bulk must affect the energy flow crossing the apparent horizon on the brane. In this case, does this geometric relation\textsuperscript{11} also hold when the curvature correction terms exist? Whether and how do the contribution of the curvature correction terms enter the expression of the masslike
function? These questions need to be answered. Second, in the braneworld scenario, the gravity on the brane is not the Einstein gravity, the extra dimension effect on the brane may also affect the masslike function. These non-trivial contributions to the masslike function provide an abundant physical meanings of the masslike function, this may give some clues to explore the physical meanings of the masslike function and the universal geometric relation \( \text{(1)} \). In this paper, we will discuss the connection between thermodynamics and the braneworld model. The new type first law of thermodynamics related to the masslike function will be investigated in detail.

The paper is organized as follows. In Section II, we give a brief introduction of the general procedure proposed by Wu et al. by generalizing it to the case of a FRW universe with any spatial curvature. In section III, we consider a FRW universe on the brane, and construct the first law of thermodynamics on apparent horizon and calculate the entropy of apparent horizon in the general braneworld. In Section IV, we investigate the universality of the geometric relation \( \text{(1)} \) in the braneworld. The masslike function has been worked out. The physical meanings of the masslike function have also been explored. We end this paper with conclusion in Section V.

2. First Law of Thermodynamics of Friedmann Equation on Apparent Horizon

In this section, we will give a brief introduction of the general procedure that proposed by Wu et al by generalizing it to the case of a FRW universe with any spatial curvature. Let us start with an \((n + 1)\)-dimensional homogenous and isotropic FRW universe, whose metric is

\[
ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega^2_{n-1},
\]

where \( x^0 = t, x^1 = r, \tilde{r} = ar \) is the radius of the sphere, \( a \) is the scale factor, and \( d\Omega^2_{n-1} \) is the \((n - 1)\)-dimensional sphere element. Here \( h_{ab} = \text{diag}(-1, a^2/(1 - kr^2)) \) is the 2-dimensional metric, in which \( k \) is the spatial curvature constant. The dynamical apparent horizon, a marginally trapped surface with vanishing expansion, is defined by \( h_{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0 \), from this relation the radius of the apparent horizon reads

\[
\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}} = \frac{1}{\mathcal{H}},
\]

where \( H \) denotes the Hubble parameter, \( H = \dot{a}/a, \) and for convenient, we define \( \mathcal{H}^2 = H^2 + k/a^2. \) Here we set the dots represent derivatives with respect to the cosmic time \( t = x^0 \). The associated temperature on the apparent horizon can be defined as

\[
T = \frac{1}{2\pi \tilde{r}_A}.
\]
In Einstein gravity, the entropy is proportional to the horizon area

\[ S_E = \frac{A}{4G}, \]  

where the horizon area \( A = n\Omega n \tilde{r}^{n-1} \), thus we have the fundamental relation

\[ \delta Q \equiv T dS_E = \frac{n(n-1)V\tilde{r}^{-3}d\tilde{r}A}{8\pi G}, \]  

where \( V = \Omega n \tilde{r}^n \) is the volume in the horizon. Using the relation (3), we can obtain

\[ T dS_E = -\frac{n(n-1)V}{16\pi G} \frac{\partial H^2}{\partial \rho} dt dt, \]  

where \( T, S, \) and \( \mathcal{H} \) are pure geometric quantities, this implies that the above relation (7) is a pure geometric relation.

For Einstein gravity theories, one can write the Friedmann equations in the form

\[ H^2 = H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)} \rho_{\text{eff}}. \]  

Though we do not know the exact form of \( \rho_{\text{eff}} \) (and \( p_{\text{eff}} \)), we know that there must be ordinary matter density \( \rho \) in \( \rho_{\text{eff}} \) and also other quantities \( \rho_i \), such as matter or energy components besides the ordinary matter. For all gravity theories, the Friedmann equation can be expressed in a generalized form

\[ f(H^2, \rho, \rho_1, \cdots \rho_i, \cdots) = 0. \]  

It is obvious shown that \( H^2 \) is not only dependent on ordinary matter density \( \rho \), but also other quantities \( \rho_i \), i.e.,

\[ H^2 = \mathcal{H}^2(\rho, \rho_1, \cdots \rho_i, \cdots). \]  

In the general braneworld models, we will show in the next section that \( H^2 \) only dependent on the ordinary matter density \( \rho \), therefore, in the following discussions, we will restrict to consider this case only for simplicity. Then the relation (7) can be changed to

\[ T dS_E = -\frac{n(n-1)V}{16\pi G} \frac{\partial H^2}{\partial \rho} \dot{\rho} dt. \]  

The expression of \( \frac{\partial H^2}{\partial \rho} \) can be got from the Friedmann Equation (9). To construct the first law of thermodynamics \( dE = TdS \), we need to know the energy flux \( dE \) and entropy \( S \). In the general gravity theory, they are not specified. The energy flux of ordinary matter can be expressed as \( dE = V \dot{\rho} dt \). Multiplying \( \frac{16\pi G n}{n(n-1)} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \) on both sides of (11), we have

\[ \frac{16\pi G n}{n(n-1)} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} T dS_E = -V \dot{\rho} dt. \]  

In the general case, the conservation of the ordinary matter density can be written as

\[ \dot{\rho} + nH(\rho + p) = 0. \]  

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Substituting $\dot{\rho}$ into Eq. (12), we can get

$$T \frac{16\pi G}{n(n-1)} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} dS_E = n V H(\rho + p)dt. \quad (14)$$

The entropy form can be got by integrating (14). If there is just ordinary matter $\rho$ in the space, $\frac{\partial H^2}{\partial \rho}$ can be rewritten as a function of $\tilde{r}_A$. Then the entropy can be obtained by the integration

$$S = \int \frac{16\pi G}{n(n-1)} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} dS_E = \int 4\pi \tilde{r}_A^{-2} \Omega_\rho \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} d\tilde{r}_A, \quad (15)$$

so the entropy formulay is obviously dependent on $\frac{\partial H^2}{\partial \rho}$. This is the crucial result which can be used to determine the exact entropy formula for general braneworld models. Then the relation (14) can be written as

$$TdS = dE, \quad (16)$$

where $dE = V \dot{\rho} dt = n V H(\rho + p)dt$. It is the first law of thermodynamics for the gravity theories with only freedom $\rho$ in the first Friedmann equation.

When $H^2$ is not only dependent on ordinary matter density $\rho$, such as in $f(R)$ gravity, scalar-tensor gravity, and also Brans-Dicke Theory, the general expression of the first law of thermodynamics in the Friedmann equation reads

$$TdS + Td_iS = dE, \quad (17)$$

where $d_iS$ is interesting since it relates to the entropy production in the non-equilibrium thermodynamics.

### 3. Thermodynamics of Friedmann Equation and Entropy Formulary in General Braneworlds

In this section, we will use the above procedure to investigate the thermodynamics properties of Friedmann Equation and the Entropy Formulary in General Braneworld. We consider a 3-brane embedded in a 4 + 1-dimensional space-time. For convenience and without loss of generality we choose the extra dimension along the coordinates $y$ such that the brane is located at $y = 0$. Objects corresponding to the brane are written with a tilde to be distinguished from 5D objects. We begin with the action \[^{19,24,26}\]

$$S = \frac{1}{2\kappa_5} \int dx^5 \sqrt{-g} (R - 2\Lambda + \alpha \mathcal{L}_{GB}) + \frac{1}{2\kappa_4} \int dx^4 \sqrt{-\tilde{g}} \tilde{R} + \int dx^4 \sqrt{-\tilde{g}} (\mathcal{L}_m - 2\lambda), \quad (18)$$

where $\Lambda < 0$ is the bulk cosmological constant and $\mathcal{L}_{GB} = R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD}$ is the Gauss-Bonnet correction term. $g$ is the bulk metric and $\tilde{g}$, $\tilde{R}$, and $\mathcal{L}_m$ are the curvature scalar, Ricci, and Riemann tensors, respectively. $\kappa_4$ and $\kappa_5$ are the gravitational constants on the brane and in the bulk, respectively.
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\( \mathcal{L}_m \) is the Lagrangian density of the brane matter fields, and \( \lambda \) is the brane tension (or the brane cosmological constant). For convenience, we assume that the brane cosmological constant is zero. We assume that there are no sources in the bulk other than \( \Lambda \) and redefine \( \kappa_4^2 = 8\pi G_4, \kappa_5^2 = 8\pi G_5 \).

We consider homogeneous and isotropic brane at fixed coordinate position \( y = 0 \) in the bulk, the bulk metric is described by

\[
\begin{align*}
ds^2 &= -N^2(t,y)dt^2 + A^2(t,y)\gamma_{ij}dx^i dx^j + B^2(t,y)dy^2,
\end{align*}
\]

(19)

where \( \gamma_{ij} \) is a constant curvature three-metric, with curvature index \( k \). For this metric, the generalized Friedmann equation on the brane has been obtained in Refs. 19, 24–26.

\[
\begin{align*}
- \frac{2\kappa_4^2}{\kappa_5^2} \left[ 1 + \frac{8}{3} \alpha(\mathcal{H}^2 + \Phi_0^2) \right] (\mathcal{H}^2 - \Phi_0^2)^{1/2} = -\frac{\kappa_4^2}{3} \rho + \mathcal{H}^2,
\end{align*}
\]

(20)

in which \( \Phi \) is defined as

\[
\Phi = \frac{1}{\sqrt{N^2}} \frac{\dot{A}}{A} - \frac{1}{\sqrt{b^2}} \frac{\dot{A}}{A} + \frac{k}{\sqrt{A^2}},
\]

(21)

and \( \Phi_0 = \Phi(t,0) \). In order to compare our discussion with the result obtained in [19] we use the same assumption that there is no black hole in the bulk and so \( \Phi_0 = \frac{1}{4\pi \kappa_5^2} (-1 + \sqrt{1 + \frac{4\alpha}{3}}) = \text{const.} \)

Noticing now that \( k, \kappa_4, \kappa_5 \) and \( \Phi_0 \) are all constant, it is obvious that the Friedmann equation (20) is consistent with the general form

\[
\begin{align*}
f(\mathcal{H}^2, \rho) &= \frac{2\kappa_4^2}{\kappa_5^2} \left[ 1 + \frac{8}{3} \alpha(\mathcal{H}^2 + \Phi_0^2) \right] (\mathcal{H}^2 - \Phi_0^2)^{1/2} - \frac{\kappa_4^2}{3} \rho + \mathcal{H}^2 = 0.
\end{align*}
\]

(22)

It is obvious that the \( \mathcal{H}^2 \) is only dependent on \( \rho \). In order to search the expression of \( \frac{d\mathcal{H}^2}{d\rho} \), we reexpress Eq. (22) as

\[
\begin{align*}
f &= \frac{2\kappa_4^2 + 8\kappa_4^2 \alpha \Phi_0}{3\kappa_5^2} (\mathcal{H}^2 - \Phi_0^2)^{1/2} + (\mathcal{H}^2 - \Phi_0^2)
&+ \frac{16\kappa_4^2 \alpha}{3\kappa_5^2} (\mathcal{H}^2 - \Phi_0^2)^{3/2} - \frac{\kappa_4^2}{3} \rho + \Phi_0 = 0.
\end{align*}
\]

(23)

Operate with \( \frac{d}{d\rho} \) on the above equation, after several steps of simple calculation, we get

\[
\begin{align*}
\left( \frac{d\mathcal{H}^2}{d\rho} \right)^{-1} &= \frac{3}{8\pi G_4} + \frac{3}{8\pi G_5} \frac{\ddot{\mathcal{A}}}{1 - \Phi_0 \ddot{\mathcal{A}}} + \frac{3\alpha}{2\pi G_5} \frac{2 - \Phi_0 \ddot{\mathcal{A}}}{\sqrt{1 - \Phi_0 \ddot{\mathcal{A}}}} \frac{1}{\ddot{\mathcal{A}}}. 
\end{align*}
\]

(24)

Since \( \mathcal{H}^2 \) is only dependent on \( \rho \) in the Friedmann equation, noticing that \( n = 3 \) and \( G = G_4 \) on the 3-brane and making use of the entropy expression (14) and Eq. (24), we obtain the entropy associated with the apparent horizon on the brane as

\[
\begin{align*}
S &= \frac{3\Omega_3}{2G_4} \int \ddot{\mathcal{A}} \ddot{\mathcal{A}} + \frac{3\Omega_3}{2G_5} \int \frac{\ddot{\mathcal{A}}^2 d\ddot{\mathcal{A}}}{\sqrt{1 - \Phi_0 \ddot{\mathcal{A}}}} + \frac{6\alpha \Omega_3}{G_5} \int \frac{2 - \Phi_0 \ddot{\mathcal{A}}}{\sqrt{1 - \Phi_0 \ddot{\mathcal{A}}}} d\ddot{\mathcal{A}}.
\end{align*}
\]

(25)
Integrating the above expression, the explicit form of the entropy can be obtained as

\[ S = \frac{3Ω_3 \tilde{r}_A^2}{4G_4} + \frac{2Ω_3 r_A^2}{4G_5} \left( \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \Phi_0 r_A^2 \right) + \frac{6Ω_3 \tilde{r}_A^2}{G_5} \left[ \Phi_0 \left( \frac{3}{2}, \frac{5}{2}, \frac{1}{2}, \frac{5}{2}, \Phi_0 r_A^2 \right) + \sqrt{1 - \Phi_0 \tilde{r}_A^2} \right], \quad (26) \]

where \( \left( \frac{3}{2}, \frac{5}{2}, \frac{1}{2}, \frac{5}{2}, \Phi_0 r_A^2 \right) \) is a hypergeometric function. This expression is exactly consistent with the entropy formulay obtained by Sheykhi et al.\(^1\text{9}\). The corresponding first law of thermodynamics \( (16) \) reads

\[ T dS = dE = 3VH(\rho + p)dt. \quad (27) \]

This is just the Clausius relation in the version of black hole thermodynamics. From \( (27) \), we can see clearly that there is no additional entropy production term \( d_iS \), this implies that the thermodynamics we treated in the general braneworld is equilibrium thermodynamics.

Although the entropy formulay we obtained is the same as that obtained in Ref.\(^1\text{9}\), the expressions of the first law of thermodynamics on apparent horizon are different. In Ref.\(^1\text{9}\), Sheykhi et al obtained the entropy formulay by applying the first law of thermodynamics, \( TdS + WdV = dE \), to the apparent horizon of a FRW universe on the brane, while in this paper, the first law of thermodynamics we applied is \( TdS = dE \). We would like to point out here that this difference is not worth to worry, because the result in Ref.\(^1\text{9}\) is consistent with the one in this paper. As pointed out in Ref.\(^1\text{1}\)\), this difference is due to the different definitions of \( dE \). In Ref.\(^1\text{1}\)\), \( E \) is the matter energy inside the apparent horizon, \( E = Ω_n \tilde{r}_A^n \rho \).

The change of energy \( dE \) inside the apparent horizon is

\[ dE = nΩ_n \tilde{r}_A^{n-1} \rho d\tilde{r}_A - nΩ_n \tilde{r}_A^n (\rho + p)Hdt. \quad (28) \]

In this paper, the definition of \( dE \) is

\[ dE = nΩ_n \tilde{r}_A^n (\rho + p)Hdt, \quad (29) \]

while the term of volume change is absent in our consideration.

So far, we have constructed the first law of thermodynamics of Friedmann equation on apparent horizon in a general braneworld. As expected, the entropy formulay of apparent horizon obtained in this section is consistent with the entropy formulay obtained by Sheykhi et al.\(^1\text{9}\). Now, we give some remarks about above discussions in order: (i) As pointed out in Ref.\(^1\text{9}\) the physical meaning of the entropy formulay \( (26) \) is obvious. The first term in \( (26) \) is Bekenstein- Hawking entropy on the brane, the second term obeys the 5-dimensional area formula in the bulk and the third term come off the contribution of the Gauss-Bonnet correction term. (ii) The Eq. \( (26) \) is a very general entropy formulay in braneworld, it can reduce to the entropy of several special braneworld models.\(^1\text{4},1\text{5},1\text{9}\).

Such as the Dvali-Gabadadze-Porrati (DGP) braneworld is the limiting case when \( \alpha = 0 \), the...
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Randall-Sundrum (RS) II braneworld in the limit \( \kappa_4 \rightarrow \infty \) and \( \alpha = 0 \), the pure Gauss-Bonnet braneworld is the case with \( \kappa_4 \rightarrow \infty \).

4. Masslike Function in General Braneworlds

In this section, we will begin to study the first law of thermodynamics associated with the masslike function and search for the expression of the masslike function in braneworld. As shown in Ref. [22], the masslike function in \((3 + 1)\)-dimensional Einstein gravity reads

\[
M = \frac{\tilde{r}}{2G} (1 + h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}).
\]  

(30)

Using this masslike function, the first law of Einstein gravity reads

\[
TdS_E = k^a \partial_a M dt = dE,
\]  

(31)

where \( k^a = (-1, H r) \) is null (approximate) generator of the apparent horizon. The above expression plays an important role in determining the exact expression of the masslike function in modified gravity. As pointed out in Ref. [23], this masslike function can be insteaded by a more generalized form. Using Eq.(14), the masslike function satisfies

\[
\frac{16\pi G}{n(n-1)} \left( \frac{\partial \mathcal{H}^2}{\partial \rho} \right)^{-1} TdS_E = \frac{16\pi G}{n(n-1)} \left( \frac{\partial \mathcal{H}^2}{\partial \rho} \right)^{-1} k^a \partial_a (M + f_1) dt = k^a \partial_a \hat{M} dt,
\]  

(32)

where \( M \) is the \((n + 1)\)-dimensional masslike function, which reads

\[
M = \frac{n(n-1) \Omega_n \tilde{r}^{n-2}}{16\pi G} (1 + h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}),
\]  

(33)

and \( f_1 \) and also \( f_2 \) (which will be defined below) are arbitrary functions satisfying

\[
k^a \partial_a f_i = 0 \quad (i = 1, 2)
\]

(34)

on the apparent horizon. From Eq.(32), we get

\[
\hat{M} = \frac{16\pi G}{n(n-1)} \left( \frac{\partial \mathcal{H}^2}{\partial \rho} \right)^{-1} (M + f_1) + f_2.
\]  

(35)

Noticing that \( n = 3 \) and \( G = G_4 \) on the brane, substituting Eq.(24) into Eq.(35), the above formulary gives the exact expression of the masslike function \( \hat{M} \) in the braneworld,

\[
\hat{M} = \left( 1 + \frac{G_4}{G_5} \frac{\tilde{r}_A}{\sqrt{1 - \Phi_0 \tilde{r}_A^2}} + \frac{4\alpha G_4}{G_5} \frac{2 - \Phi_0 \tilde{r}_A^2}{\sqrt{1 - \Phi_0 \tilde{r}_A^2}} \frac{1}{\tilde{r}_A} \right) \times \left( \frac{3\Omega_3 \tilde{r}}{8\pi G_4} (1 + h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}) + f_1 \right) + f_2.
\]  

(36)

Using this masslike function, the first law of the general braneworld now reads

\[
TdS = k^a \partial_a \hat{M} dt = dE.
\]  

(37)
This result exactly has the same form as the one that we have given in the previous section.

In addition, we would like to point out that, although Eq. (27) and (37) have the same form, the first law of thermodynamics expressed in Eq. (37) is a new type thermodynamics and the corresponding physical meanings are different. First, the corresponding definition of the energy flow $dE$ through apparent horizon are different. Because of the masslike function, the energy flow $dE$ defined in (37) includes the contribution of gravitational field such as the Gauss-Bonnet term and bulk contribution in the general braneworld, in addition to the matter field on the brane. But in previous section, the energy flow $dE$ is defined as the matter field energy crossing the apparent horizon within an infinitesimal of time $dt$. Second, the universality of Eq. (27) and (37) are obvious different. In $f(R)$ gravity and scalar-tensor theory, the equilibrium thermodynamics relation Eq. (27) does not hold on apparent horizon for Friedmann equation. In order to construct the thermodynamics of apparent horizon for Friedmann equation, one has to modify the thermodynamics relation (27) to nonequilibrium case by adding an entropy production term. But for the relation Eq. (37) associated with the masslike function, its validity has been verified in various gravity theory including $f(R)$ gravity, scalar-tensor theory, Gauss-Bonnet gravity and more general Loveloke Gravity. And in this section, it also hold in the general braneworld model.

The masslike function $\tilde{M}$ obtained above plays a important role in the thermodynamic description of the gravitational dynamics and determines the energy flows passing through the horizon. For a variety of theories of gravity, the masslike function reduces to the Misner-Sharp mass at the apparent horizon. Therefore, the investigation of this mass-like function may shed lights on the Misner-Sharp mass of the braneworld. In order to conveniently discuss the physical meanings of the masslike function, we set $f_1 = 0$ and $f_2 = 0$, and notice that $h^{ab}\partial_a \tilde{r} \partial_b \tilde{r} = 0$ on the apparent horizon, the masslike function $\tilde{M}$ reduces to

$$\tilde{M} = \frac{3\Omega_3 \tilde{r}^3}{8\pi G_4} + \frac{3\Omega_3 \tilde{r}^2}{8\pi G_5} \frac{1}{\sqrt{1 - \Phi_0 \tilde{r}^2}} + \frac{3\Omega_3 \alpha}{2\pi G_5} \frac{2 - \Phi_0 \tilde{r}^2}{\sqrt{1 - \Phi_0 \tilde{r}^2}}.$$  

(38)

It is obvious that this masslike function contains the contributions of the extra dimension and the Gauss-Bonnet correction term in the bulk. This means that the energy flows passing through the horizon on the brane may have some non-trivial connection with the extra dimension and the Guass-Bonnet curvature correction terms in the bulk. This non-trivial connection also has some interesting effects on the accelerated expansion of the universe.

In the context of dynamical black holes, the dynamics of the black hole spacetime can be described by its quasi-local first law of thermodynamics, this thermodynamics associated with a quasi-local mass which determines the energy (or mass) of the trapping horizon of black holes. In general relativity, the quasi-local mass usually be selected as the Misner-Sharp mass. Such quasi-local mass has also been generalized to the Einstein-Gauss-Bonnet gravity by Hideki Maeda and Asato.
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In FRW spacetime, the apparent horizon is an important trapping horizon, at this level, its thermodynamics coincides with the quasi-local thermodynamics of dynamical trapping horizons. For the models of braneworld considered in this paper, the new type first law of thermodynamics associated with masslike function is just the quasi-local first law of thermodynamics. This implies that the above masslike function (38) should coincides with the quasi-local mass of apparent horizon, which can be regarded as the generalized Misner-Sharp mass in braneworlds. Therefore, the thermodynamics of apparent horizon can provide an approach to investigate the properties of Misner-Sharp mass in the braneworlds.

From Eq. (38), we can obtain the masslike function of several special braneworld models:

(i). In the limit \( \alpha \rightarrow 0 \), Eq.(38) reduces to the masslike function on the apparent horizon in the warped DGP braneworld embedded in an \( AdS_5 \) bulk

\[
\tilde{M} = \frac{3\Omega_3 r_A}{8\pi G_4} + \frac{3\Omega_3 r_A^2}{8\pi G_5} \frac{1}{\sqrt{1 - \Phi_0 r_A^2}}.
\]

(ii). When in the limit \( \alpha \rightarrow 0 \) and \( \Phi_0 \rightarrow 0 \), Eq.(38) reduces to the masslike function on the apparent horizon in pure DGP braneworld with a Minkowskian bulk

\[
\tilde{M} = \frac{3\Omega_3 r_A}{8\pi G_4} + \frac{3\Omega_3 r_A^2}{8\pi G_5}.
\]

(iii). In the limit \( \alpha \rightarrow 0 \) and \( G_4 \rightarrow \infty \), while keeping \( G_5 \) finite, the first and the last terms in Eq.(38) vanish and we obtain the masslike function on the apparent horizon in the RS II braneworld

\[
\tilde{M} = \frac{3\Omega_3 r_A^2}{8\pi G_5} \frac{1}{\sqrt{1 - \Phi_0 r_A^2}}.
\]

(iv). Finally, keeping \( \alpha \) finite, and in the limit \( G_4 \rightarrow \infty \) and \( \Phi_0 \rightarrow 0 \), we obtain the masslike function on the apparent horizon in the Gauss-Bonnet braneworld with a Minkowskian bulk

\[
\tilde{M} = \frac{3\Omega_3 r_A^2}{8\pi G_5} (1 + \frac{24\alpha}{r_A}).
\]

Although, the connections between these masslike functions and Misner-Sharp mass are not very clear, it is predictable that these masslike functions may play an important role in investigating the Misner-Sharp mass in the braneworld.

5. Conclusion and Discussions

In this paper we have studied the thermodynamics of the apparent horizon of FRW Universe in a general braneworld model with curvature correction terms on the brane and in the bulk, respectively. Using the general procedure developed by Wu et al, we have constructed the first law of thermodynamics on apparent horizon of
FRW Universe and obtained the exact entropy formula of apparent horizon in the braneworld. As expected, the entropy formula we obtained is the same as the one obtained by Sheykhi et al. We have also studied the first law of thermodynamics of apparent horizon associated with the masslike function in the braneworld. This is a new type first law of apparent horizon and it is a universality result in more generalized gravity theories. Its validity in the braneworld has been verified. The difference between this new type first law of thermodynamics and the conventional first law of thermodynamics in the braneworld have also been discussed. We have also calculated the exact expression of the masslike function in the braneworld. As expected, the masslike function contains the contributions of the extra dimension and the Guass-Bonnet curvature correction terms in the bulk. This implies that the energy flow crossing the apparent horizon on the brane should contain the contributions from bulk and curvature correction terms. The physical meanings of the masslike function have been discussed in the context of quasi-local thermodynamics in the dynamical horizon. As like in the Einstein gravity, \( f(R) \) gravity, and the LoveLock gravity the masslike function reduces to the Misner-Sharp mass on apparent horizon, we concluded that the masslike function should also reduce to the generalized Misner-Sharp mass on apparent horizon in the braneworld.

We noticed that in our construction of first law of thermodynamics in Section III, the energy conservation on the brane is assumed. This means that there is no interaction between extra dimension and the matter on the brane. When the bulk matter is assumed, the energy conservation on the brane may not hold. It is of great interest to extend the thermodynamics to the braneworld model with bulk matter content.

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