NAKED SINGULARITIES IN DUST COLLAPSE AS AN EXISTENCE PROBLEM FOR O.D.E. AT A SINGULAR POINT

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ABSTRACT. The final state of the gravitational collapse of a marginally bound dust cloud is formulated in terms of an existence problem for the non-linear differential equation governing radial null geodesics near the singular point. Rigorous results are proved, covering the complete spectrum of the possible initial data.

1. INTRODUCTION

It is well known that stable, non singular states of superdense matter can exist only if the mass of the final object is less than a physical limit, namely the Chandrasekar limit (about $1.4M_\odot$) in the case of white dwarfs or the neutron star limit (of the order of $3M_\odot$) in the case of neutron stars. For collapsing objects which are unable to radiate away a sufficient amount of mass to fall below such limits, no final stable state is available and therefore singularities are formed.

A famous conjecture, first formulated by Roger Penrose [10] and known as the Cosmic Censorship conjecture states that a blackhole is always formed in complete gravitational collapse of reasonable matter fields. However, if stated without any further mathematical assumption, the conjecture is false, since several examples of naked singularities, i.e. solutions of the Einstein field equations describing singularities not hidden behind an absolute event horizon, are known. It is, therefore, of primary importance to understand the mathematical structure of such singularities, with the final aim of reformulating the conjecture as a theorem and hopefully prove it.

Examples of focussing naked singularities in gravitational collapse firstly arose from numerical investigations by Eardley [4] and Eardley and Smarr [5], while the first to perform a formal investigation was Christodoulou [1]. In his paper, Christodoulou used a fixed point technique to show that the equation of radial null geodesics for a collapsing dust ball starting form rest and having a parabolic density profile has a solution meeting the singularity in the past, the latter being thus “visible” to nearby observers. Since then, a technique has been developed which makes use of L’Hopital theorem to identify existence of solutions with finite tangent near the singularity (“root equation” approach, see e.g. [3]). In particular, all the possible endstates of the gravitational collapse of spherically symmetric dust have been obtained in this way [7], as well as the final states of gravitating systems of rotating particles known as Einstein clusters [6]. The root equation technique, however, proves useful only if the exact explicit solution of the Einstein field equations is known for the case at hand. As a consequence, we are still very far from a complete understanding of the Censorship problem even in the simple case of spherical symmetry, since very few exact solutions are known. In addition, the root equation approach is essentially related to the existence of solutions of a specific kind, that is not a–priori guaranteed.

In this paper, we give a o.d.e. approach to the nature of the singularities in marginally bound dust collapse. Using classical techniques we make rigorous, by explicit construction, the results obtained previously with the root equation technique.

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Thus, singularities can be of two different kinds: constant and, without loss of generality, we have put in the literature. In any case, in most physically interesting situations such singularities do not occur, so that we shall concentrate attention here only on the shell focussing case.

The relation

\[ R^2 = 2F(r)/R \]

where \( k(r) = (3/2)\sqrt{2F(r)/r^3} \). In the above formulae, \( F(r) \) is the initial distribution of mass of the cloud (and thus is a positive function). The energy density is given by

\[ \epsilon(r, t) = \frac{F''}{4\pi R^2 R'} \]

at \( t = 0 \) one has \( \epsilon(r, 0) = \frac{F''}{4\pi r^3} \) and therefore regularity of the Cauchy data at \( r = 0 \) implies \( F \approx r^3 \) as \( r \) tends to zero. We assume (as usual) the function \( F(r) \) to be Taylor-expandable near \( r = 0 \) (all our results actually hold true also if \( F \) is only of class \( C^3 \)). Therefore we put

\[ F(r) = F_0 r^3 + F_n r^{n+3} + \Gamma(r) \]

where \( \Gamma(r) \) is infinitesimal of order greater than or equal to \( n+4 \). The physical requirement that the density has to be positive and decreasing outwards further imply that \( F_0 \) is positive and \( F_n \) is negative. It follows easily that

\[ (2.2) \quad k(r) = 1 - a r^n + \gamma(r), \]

where \( \gamma(r) \) is infinitesimal of order greater than or equal to \( n+1 \), \( a \) is some positive constant and, without loss of generality, we have put \( k(0) = 1 \).

The energy density becomes singular whenever \( R \) or \( R' \) vanish during the evolution. Thus, singularities can be of two different kinds: shell crossing, at which \( R' \) vanishes while \( R \) is non-zero, and shell focusing at which \( R \) vanishes. The shell crossing singularities have been frequently considered as “weak” although no proof of extensibility is as yet available in the literature. In any case, in most physically interesting situations such singularities do not occur, so that we shall concentrate attention here only on the shell focussing case.

The locus of the zeroes of the function \( R(r, t) \) defines the singularity curve \( t_s(r) \) by the relation \( R(r, t_s(r)) = 0 \). Due to formula (2.1), we have \( t_s(r) = 1/k(r) \). Physically, \( t_s(r) \) is that comoving time at which the shell of matter labeled by \( r \) becomes singular. The singularity forming at \( r = 0, t = t_s(0) \) is called central and, in dust clouds, is the unique singularity that can be naked. To see this, we recall that a singularity cannot be naked if it occurs after the formation of the apparent horizon. The apparent horizon (\( t_h(r) \), say) is the boundary of the region of trapped surfaces and is defined by the equation \( R(r, t_h(r)) = 2F(r) \), that is

\[ (2.3) \quad t_h(r) = t_s(r) - \frac{8}{27}k(r)^2 r^3 \]

so that \( t_s(r) > t_h(r) \) for any \( r > 0 \).

To analyze the causal structure of the central singularity, observe that, if the singularity is visible, at least one outgoing null geodesic must exist, that meets the singularity in the past. Such a geodesic will be a solution of

\[ \frac{dt(r)}{dr} = \varphi(r, t) \]
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where

\[ \varphi(r, t) := \sqrt{-g_{rr} / g_{00}} = 1 - k(r) t - \frac{2}{3} r k'(r) t \]

with initial datum \( t(0) = t_s(0) = 1 \). For a problem of this kind, in which the initial point is singular (the function \( \varphi \) is not defined at \( (0, t(0)) \)) no general results of existence/nonexistence are known. As a consequence, in the literature, an approach has been developed [3] which makes use of l’Hopital theorem to identify the possible values of the tangent of the geodesic curve at the singularity. What turns out is the following:

- For \( n = 1 \) or \( n = 2 \) the singularity is naked;
- For \( n = 3 \) the singularity is naked if \( a \geq a_c \) where

\[ a_c = \frac{2(26 + 15\sqrt{3})}{27} \]

Therefore, \( a_c \) is a “critical parameter”: at \( a = a_c \) a “phase transition” occurs and the endstate of collapse turns from a naked singularity to a blackhole.

- If \( n > 3 \) the singularity is covered.

This approach, however, strictly depends on the form of the solution of (2.4), that must be of the form \( 1 + x r^\alpha \) with \( x \) constant. Anyway, the root equation can be as well recovered following our approach for proving nakedness, where we will look for solutions of the form \( 1 + x(r) r^\alpha \), and impose a continuity condition on the unknown function \( x(r) \).

3. NON-EXISTENCE

We begin by stating the non-existence result. The argument covers the case \( n \geq 4 \) and gives a partial answer in the case \( n = 3 \), (the remaining part is given in Section 5).

3.1. Theorem. If \( n \geq 4 \) the singularity is covered.

To prove the above statement we need the following:

3.2. Lemma. There exists \( r_* > 0 \) such that the apparent horizon \( t_h(r) \) is a subsolution of (2.4) for \( r \in (0, r_*) \).

Proof: Recall that \( t_h(r) = \frac{1}{k(r)} - \frac{8}{27} k(r)^2 r^3 \), and \( k(r) \equiv 1 - a r^n \), where in last relation \( a > 0 \) and infinitesimal of order greater than \( n \) has been dropped. We must show that \( t_h(r) \leq \varphi(r, t_h(r)), \forall r \in (0, r_*), \) with \( r_* \) sufficiently small. One gets

\[ \frac{dt_h}{dr} = -\frac{k'}{k^2} - \frac{8}{27}(2k k' r^3 + 3k^2 r^2), \]

and

\[ \varphi(r, t_h(r)) = -\frac{k'}{k^2} + \frac{4}{9} k^2 r^2 + \frac{8}{27} k k' r^3, \]

where the relation

\[ 1 - k t_h = \frac{8}{27} k^3 r^3 \]

has been used. It follows

\[ \frac{dt_h}{dr} - \varphi(r, t_h(r)) = -\frac{4}{3} k r^2 \left( \frac{2}{3} k' r + k \right), \]

that is negative for \( r \) sufficiently small and positive. \( \square \)

Proof of Theorem 3.1. Let \( t_p(r) \) the solution of \( t'(r) = \varphi(r, t(r)) \) such that \( t_p(0) = 1 \). By contradiction we suppose the existence of \( r_1 > 0 \) such that \( t_p(r_1) < t_h(r_1) \) and
sufficiently small, and of a parameter $(4.2)$

$t$ is an infinitesimal differentiable function of order greater than or equal to

Proof. and

infinitesimal for $(3.7)$

and the denominator has a fixed sign. Using (3.6) we have

$$
\partial \phi (3.6)
$$

where infinitesimal of order greater than 1

(3.5)

Hence

$$
\partial \phi (3.5)
$$

Combining 3.4 and (3.5) one gets a contradiction if $\partial \phi (3.5)$ is non-negative. Now:

$$
\partial \phi (3.6)
$$

Hence $\partial \phi (3.6)$ is negative for small values of $r$. Showing that $\partial \phi (3.6)(r, t, t(r)) \leq 0$ we have that $\partial \phi (3.6)(\xi, \theta) \leq 0$, since we can observe that the numerator in last term of (3.6) is linear in $t$ and the denominator has a fixed sign. Using (3.6) we have

$$
\partial \phi (3.7)
$$

where infinitesimal of order greater than $n$ has been dropped out in last quantity, so that the sign of the right hand side in (3.7) depends on $n$, and is strictly negative if $n \geq 4$. \hfill \Box

The above argument provides only a sufficient condition for nakedness. Indeed, it does not exhaust all cases for the singularity to be covered (see Section 5).

4. Existence

In this section we establish rigorously existence of naked singularities in the cases $n = 1, 2$.

4.1. Theorem. If $n = 1, 2$ there exists a geodesics of the form

$$
t_p(r) = 1 + x(r)r^\alpha, \quad r \in [0, r^*],
$$

where $\alpha = 1 + \frac{2}{3}n$ and $x(r)$ is a differentiable function in $[0, r^*]$ such that $x(0) > 0$.

Proof. We will show the existence of a function $x(r) \in H^{1,p}[0, r^*]$ with $p > 1$ and $r^* > 0$ sufficiently small, and of a parameter $\alpha \geq 1$ such that $x(0) > 0$ and $t = 1 + x r^\alpha$ solves equation $t'(r) = \phi(r, t(r))$, where $\phi(r, t)$ is given by (2.5):

$$
t'(r) = \frac{1 - k(r)t(r) - (t'(r))t(r)}{(1 - k(r))^{1/3}}, \quad k(r) = 1 - a x^\alpha + \gamma(r).
$$

We recall that $\gamma(r)$ is infinitesimal of order greater than $n$ as $r \to 0^+$. Substituting in (4.2) the expressions for $t'(r)$ and $k(r)$, and using that $n \geq 1$ one gets

$$
x' = r^{2n+1-\alpha} \left[ \frac{a + \frac{2}{3}an + rb(r) + xc(r)r^\alpha - d(r)x r^\alpha - n}{(a + ax r^\alpha - x r^\alpha - n - \delta(r)(1 + x r^\alpha))^{1/3}} \right] - \alpha x,
$$

where $b(r), c(r)$ are continuous functions differentiable in $r = 0$, $d(r) = 1 + \gamma(r), \delta(r)$ is an infinitesimal differentiable function of order greater than equal to 1 for $r \to 0^+$, and $\alpha$ is a positive parameter to be determined below. We search for $x$ such that $r x'$ is infinitesimal for $r \to 0^+$. Then the right hand side of (4.3) must be infinitesimal for $r \to 0^+$. Since the quantity in square brackets is bounded for $n = 1, 2$, it must be $\frac{2}{3}n + 1 - \alpha \geq 0$. But if the strict inequality held, the limit of the left hand side would be $-\alpha x(0)$ which
by hypothesis is strictly negative. So the only possible situation is $\alpha = 1 + \frac{2}{3}n$ from which one gets

$$x(0) = a^{\frac{2}{3}}$$

using the infinitesimal behaviour of the left hand side of (4.3).

Having chosen the values of $\alpha$ and the initial condition $x(0)$, one has actually to show the existence of a solution of

$$r x' = \left[ a \left( 1 + \frac{2}{3} n \right) + r b(r) + xc(r) \right] r^{1 + \frac{2}{3}n} - d(r) x r^{1 - \frac{2}{3}n} - \left( 1 + \frac{2}{3} n \right) x G(x, r) G^{-1}(x, r),$$

(4.5)

where

$$G(x, r) = a^{1/3} \left[ 1 + \left( \frac{-x r^{1 + \frac{2}{3}n}}{a} + x r^{1 + \frac{2}{3}n} + \frac{\delta}{a} + \frac{\delta}{a} x r^{1 + \frac{2}{3}n} \right) \right] =$$

$$= a^{1/3} \left[ 1 + \frac{1}{3} \left( \frac{-x r^{1 + \frac{2}{3}n}}{a} + x r^{1 + \frac{2}{3}n} + \frac{\delta}{a} + \frac{\delta}{a} x r^{1 + \frac{2}{3}n} \right) + A(r) r^{2(1 - \frac{2}{3}n)} \right].$$

We observe that last relation has been written using Taylor expansion of the quantity in round bracket in the first row of (4.6), and in view of this the continuous function $A(r)$ has been introduced. Using (4.6) in (4.5), and collecting terms with the same power of $x$ the differential equation becomes

$$r x' = G(x, r)^{-1} \left[ a \left( 1 + \frac{2}{3} n \right) - a^{1/3} \left( 1 + \frac{2}{3} n \right) x + r b(r) + (e(r) x + f(r) x^2) r^{1 - \frac{2}{3}n} \right],$$

(4.7)

where $e(r), f(r)$ are continuous functions, differentiable in $r = 0$. It is a straightforward calculation that $e(0) = -1, f(0) > 0$.

With the positions

$$y = x - a^{2/3}, \quad \beta = \alpha - n = 1 - \frac{1}{3}n,$$

one recovers a differential equation of the form

$$r y' = A(r, y) y + B(r, y) r^\beta,$$

(4.8)

where $A(r, y)$ and $B(r, y)$ are continuous functions such that $A(0, 0) < 0$ and $B(0, 0) > 0$.

Let us now define four constants that bound $A$ and $B$ in a small neighborhood $\mathcal{U} = [0, r^*] \times [-\epsilon, \epsilon]$ of $(r, y) = (0, 0)$:

$$A_0 \leq A(r, y) \leq A_1 < 0, \quad 0 < B_0 \leq B(r, y) \leq B_1,$$

(4.9)

Let us also define the two positive functions

$$z_0(r) = \frac{B_0}{\beta - A_0} r^\beta, \quad z_1(r) = \frac{B_1}{\beta - A_1} r^\beta, \quad r \in [0, r^*]$$

respectively solutions of the Cauchy problems

$$\begin{cases} z' = \frac{1}{r} A_0 z + B_0 r^{\beta - 1}, \\ z(0) = 0 \end{cases} \quad \begin{cases} z' = \frac{1}{r} A_1 z + B_1 r^{\beta - 1}, \\ z(0) = 0 \end{cases}$$

(4.10)

It is readily observed that $z_0(r) < z_1(r) \forall r \in [0, r^*]$. Hence, $\forall n \in \mathbb{N}$, let $y_n$ denote the solution of the ODE in (4.8) with the initial condition $y(\frac{1}{n}) = y_0$, such that

$$z_0(\frac{1}{n}) \leq y_n \leq z_1(\frac{1}{n}).$$
From comparison theorems in ODE one gets
\begin{equation}
0 < z_0(r) \leq y_n(r) \leq z_1(r), \quad r \in [1/n, r^*],
\end{equation}
and then extending \( y_n \) to \([0, r^*]\) setting \( y_n = y_0n \) in \([0, 1/n]\) we have that \(|y_n|\) are equibounded by \( K r^\beta \) with \( K \) constant. Moreover, using the ODE in (4.8) \(|y_n'|\) are equibounded by \( K r^{\beta - 1} \) which is \( L^p, p > 1 \). So, up to subsequences, \( y_n \) converges uniformly to a function \( y \) in \( H^{1,p} \), which is easily shown to be a differentiable solution of (4.8) using the ODE in (4.8) and Lebesgue theorem. □

5. The critical case

The analysis so far shows existence of naked singularities if \( n = 1, 2 \), and non–existence if \( n > 3 \). When \( n = 3 \) a partial answer is contained in Theorem 3.1. Indeed, the key point in the proof is the study of the sign of (3.7) for small values of \( r \). In the case \( k = 1 - ar^3 \), omitting infinitesimal of order greater that 3, direct substitution in (3.7) yields
\begin{equation}
\frac{\partial \varphi}{\partial t}(r, t_h(r)) = - \left( \frac{2}{3} \right)^{-3} \left( \frac{8}{27} - a \right) r^{-1},
\end{equation}
Then we must impose the condition \( \frac{8}{27} - a \geq 0 \) in order to recover the same situation as in Theorem 3.1. In other words, we have shown the following

5.1. Proposition. If \( n = 3 \) and \( a \leq \frac{8}{27} \) the singularity is covered.

Sufficient conditions to ensure existence of naked singularity can now be given, with a repetition of the argument used in Theorem 4.1. In this case one can show the existence of a solution of the kind \( t(r) = 1 + x(r)r^3 \), with \( x(r) \in H^{1,p}[0, r^*] \), \( p > 1 \) and \( x(0) > 0 \). Since \( \alpha - n = 0 \) we must be careful in treating the infinitesimal terms in the differential equation (4.3), which now takes the form
\begin{equation}
r'x' = \left[ \frac{3a - d(r)x + rb(r) + xc(r)r^3}{(a - x + ax^3 - \delta(r)(1 + x^3))^{1/3}} \right] - 3x,
\end{equation}
where \( b(r), c(r) \) and \( d(r) \) have the same meaning as in (4.3). In order to ensure the infinitesimal behaviour of the right hand side of (5.2), we must then require
\begin{equation}
\frac{3a - x(0)}{(a - x(0))^{1/3}} - 3x(0) = 0.
\end{equation}
Since we want \( x(0) > 0 \), this implies that \( a \) must be such that the algebraic equation
\begin{equation}
27x^3(a - x) - (3a - x)^3 = 0.
\end{equation}
has real positive roots. It is a simple exercise to check that this is true only if \( a \leq a_0 \) or \( a \geq a_c \), where \( a_0 = (2/27)/(26 + 15\sqrt{3})^{-1} \) while \( a_c \) is defined in (2.6). The first case however must be excluded since the solution would not live below the apparent horizon \( t_h(r) \). Indeed, we know from Proposition 5.1 that the singularity is covered if \( a < \frac{8}{27} \). We must instead accept the second interval, and the same arguments of Theorem 4.1 can be used with some slight modifications here, in order to ensure the following

5.2. Proposition. If \( n = 3 \) and \( a \geq a_c \) the singularity is naked.

What remains to be analyzed is whether naked singularities may exist for \( a \in (\frac{8}{27}, a_c) \). Actually, such solutions represent blackholes, since we can show that the sufficient condition of Proposition 5.2 is also necessary in this case.

5.3. Proposition. If \( n = 3 \) and the singularity is naked, then \( a \geq a_c \).
Proof. Let \( t_\rho(r) \) be a solution of the differential equation \( t'_\rho(r) = \varphi(r, t_\rho(r)) \). We can write it in the form \( t_\rho(r) = 1 + x(r)r^3 \), although in this case we don’t know the behaviour of \( x(r) \) near the origin \( r = 0 \). We just know \( x \) continuous, \( x(r)r^3 \to 0 \) as \( r \to 0^+ \) and, since the singularity is naked, \( t_\rho(r) \leq t_h(r) = 1 + (a - \frac{8}{27})r^3 + o(r^3) \). Last fact implies

\[
x(r) \leq a - \frac{8}{27} + \eta,
\]

for \( r > 0 \) sufficiently small, where \( \eta \ll 1 \) is a constant. Then \( x(r) \) is bounded from above in a right neighborhood of \( r \). But it is also bounded from below. Indeed, \( t'_\rho(r) > 0 \) since \( \varphi(r, t_\rho(r)) > 0 \) for \( r > 0 \) small, and then \( x(r)r^3 \) is increasing. Thus \( x(r)r^3 \) must approach 0 from above as \( r \to 0^+ \), and then \( x(r) \) must be positive, and therefore bounded.

Now let us write (5.2) as

\[
r x' = \frac{3a - x + f(r)}{(a - x + g(r, x))^{1/3}} - 3x
\]

where \( f(r) \to 0 \) and, since \( x r^3 \to 0 \), also \( g(r, y) \to 0 \) per \( r \to 0^+ \).

Moreover, let us define

\[
Q(a, x) = \frac{3a - x}{(a - x)^{1/3}} - 3x.
\]

Recall that (5.4) ensures that \( (a - x)^{1/3} \to 0 \) for \( r \) small. Then (5.5) may be written as

\[
Q(a, x) = r x' - h(r, x),
\]

with \( h(r, x) \to 0 \) as \( r \to 0^+ \).

At this point we don’t know whether \( \lim_{r \to 0^+} x(r) \) exists or not. If it does, then \( r x'(r) \)

tends to 0. Indeed, from (5.6) we get that \( r x'(r) \) tends to \( Q(a, x(0)) \); if this quantity was not null, then \( x'(r) \) would behave like \( \frac{1}{r} \) in a right neighborhood of 0, and then \( x(r) \) would not be bounded, behaving like \( \log r \). Then \( Q(a, x(0)) = 0 \), which means that \( a \) is such that \( x(0) \) is a positive root of the equation (5.3).

If \( \lim_{r \to 0^+} x(r) \) does not exist, since \( x \) is bounded there must exists a sequence \( (r_n, x_n = x(r_n)) \) with \( r_n \to 0 \) and \( x'(r_n) = 0 \) as \( n \to \infty \). This shows that \( \{x_n\} \) is such that \( Q(a, x_n) \to 0 \). Up to subsequences, \( \{x_n\} \) converges to a positive root of (5.3).

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