TROPICAL BOUNDS FOR EIGENVALUES OF MATRICES

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Abstract. Let \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of a \( n \times n \) matrix, ordered by nonincreasing absolute value, and let \( \gamma_1 \geq \ldots \geq \gamma_n \) denote the tropical eigenvalues of an associated \( n \times n \) matrix, obtained by replacing every entry of the original matrix by its absolute value. We show that for all \( 1 \leq k \leq n \),

\[
|\lambda_1 \ldots \lambda_k| \leq C_{n,k} \gamma_1 \ldots \gamma_k,
\]

where \( C_{n,k} \) is a combinatorial constant depending only on \( k \) and on the pattern of the matrix. This generalizes an inequality by Friedland (1986), corresponding to the special case \( k = 1 \).

1. Introduction

1.1. Motivation: bounds of Hadamard, Ostrowski, and Pólya for the roots of polynomials. In his memoir on the Graeffe method [Ost40], Ostrowski proved the following result concerning the location of roots of a complex polynomial of degree \( n \), \( p(z) = a_n z^n + \cdots + a_1 z + a_0 \). Let \( \zeta_1, \ldots, \zeta_n \) denote the complex roots of \( p \), ordered by nonincreasing absolute value, and let \( \alpha_1 \geq \ldots \geq \alpha_n \) denote what Ostrowski called the inclinaisons numériques (“numerical inclinations”) of \( p \). These are defined as the exponentials of the opposites of the slopes of the Newton polygon obtained as the upper boundary of the convex hull of the set of points \( \{(k, \log |a_k|) \mid 0 \leq k \leq n\} \). Then, the inequalities

\[
\frac{1}{k} \alpha_1 \cdots \alpha_k \leq |\zeta_1 \cdots \zeta_k| \leq \sqrt{(k+1) \alpha_1 \cdots \alpha_k} \leq \sqrt{e(k+1)\alpha_1 \cdots \alpha_k}
\]

hold, for all \( k \in [n] := \{1, \ldots, n\} \). The upper bound in (1) originated from a work of Hadamard [Had93], who proved an inequality of the same form but with the multiplicative constant \( k + 1 \). Ostrowski attributed to Pólya the improved multiplicative constant in the upper bound of (1), and obtained among other results the lower bound in (1).

The inequalities of Hadamard, Ostrowski and Pólya can be interpreted in the language of tropical geometry [Vir01, IMS07, RGST05]). Let us associate to \( p \) the tropical polynomial function, obtained by replacing the sum by a max, and by taking the absolute value of every coefficient,

\[
p^*(x) = \max_{0 \leq k \leq n} |a_k| x^k.
\]

The tropical roots of \( p^* \), or for short, of \( p \), are defined to be the nondifferentiability points of the latter function, counted with multiplicities (precise definitions will be given in the next section). These roots coincide with the \( \alpha_i \). Then, the log-majorization type inequalities (1) control the distance between the multiset of roots of \( p \) and the multiset of its tropical roots.
The interest of tropical roots is that they are purely combinatorial objects. They can be computed in linear time (assuming every arithmetical operation takes a unit time) in a way which is robust with respect to rounding errors.

1.2. Main result. It is natural to ask whether tropical methods can still be used to address other kinds of root location results, like bounding the absolute values of the eigenvalues of a matrix as a function of the absolute values of its entries. We show here that this is indeed the case, by giving an extension of the Hadamard-Ostrowski-Pólya inequalities (1) to matrix eigenvalues.

Thus, we consider a matrix $A \in \mathbb{C}^{n \times n}$, and bound the absolute values of the eigenvalues of $A$ in terms of certain combinatorial objects, which are tropical eigenvalues. The latter are defined as the tropical roots of a characteristic polynomial equation [ABG04, ABG05]. They are given here by the nondifferentiability points of the value function of a parametric optimal assignment problem, depending only on the absolute values of the entries of the matrix $A$. They can be computed in $O(n^3)$ time [GK10], in a way which is again insensitive to rounding errors.

Our main result (Theorem 5) is the following: given a complex $n \times n$ matrix $A = (a_{i,j})$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, ordered by nonincreasing absolute value, the inequality

$$|\lambda_1 \cdots \lambda_k| \leq \rho(\Lambda^k_{\text{per}}(\text{pat} A)) \gamma_1 \cdots \gamma_k$$

holds for all $k \in [n]$. Here $\gamma_1 \geq \cdots \geq \gamma_n$ are the tropical eigenvalues of $A$, $\rho$ denotes the Perron root (spectral radius), the pattern of $A$, pat $A$ is a 0/1 valued matrix, depending only on the position of non-zero entries of $A$, and $\Lambda^k_{\text{per}}(\cdot)$ denotes the $k$th permanental compound of a matrix. Note that $\rho(\Lambda^k_{\text{per}}(\text{pat} A)) \leq n!/(n-k)!$ for every matrix $A$.

Unlike in the case of polynomial roots, there is no universal lower bound of $|\lambda_1 \cdots \lambda_k|$ in terms of $\gamma_1 \cdots \gamma_k$. However, we establish a lower bound under additional assumptions (Theorem 14).

1.3. Related work. The present inequalities generalize a theorem of Friedland, who showed in [Fri86] that for a nonnegative matrix $A$, we have

$$\rho_{\text{max}}(A) \leq \rho(A) \leq \rho(\text{pat} A) \rho_{\text{max}}(A)$$

where $\rho_{\text{max}}(A)$ is the maximum cycle mean of $A$, defined to be

$$\rho_{\text{max}}(A) = \max_{i_1, \ldots, i_\ell} \left( a_{i_1, i_2} a_{i_2, i_3} \cdots a_{i_\ell, i_1} \right)^{1/\ell},$$

the maximum being taken over all sequences $i_1, \ldots, i_\ell$ of distinct elements of $[n]$. Since $\gamma_1 = \rho_{\text{max}}(A)$, the second inequality in (3) corresponds to the case $k = 1$ in (2).

This generalization is inspired by a work of Akian, Bapat and Gaubert [ABG04, ABG05], dealing with matrices $A = (a_{i,j})$ over the field $\mathbb{K}$ of Puiseux series over $\mathbb{C}$ (and more generally, over fields of asymptotic expansions), equipped with its non-archimedean valuation $v$. It was shown in [ABG04, Th. 1.1] that generally, the valuations of the eigenvalues of the matrix $A$ coincide with the tropical eigenvalues of the matrix $v(A)$. Moreover, in the non-generic case, a majorization inequality still holds [ABG05, Th. 3.8]. Here, we replace $\mathbb{K}$ by the field $\mathbb{C}$ of complex numbers, and $v$ by the archimedean absolute value $z \mapsto |z|$. Then, the majorization inequality is replaced by a log-majorization inequality, up to a modification of each scalar inequality by a multiplicative combinatorial constant, and the generic equality is replaced by a log-majorization-type lower bound, which requires restrictive conditions. These results can be understood in the light of tropical geometry, as the notion of tropical roots used here is a special case of an
amoeba [GKZ94, PR04, EKL06]. Recall that if \( K \) is a field equipped with an absolute value \( |·| \), the amoeba of an algebraic variety \( V \subset (\mathbb{K}^*)^n \) is the closure of the image of \( V \) by the map which takes the log of the absolute value entrywise. Kapranov showed (see [EKL06, Theorem 2.1.1]) that when \( K \) is a field equipped with a non-archimedean absolute value, like the field of complex Puiseux series, the amoeba of a hypersurface is precisely a tropical hypersurface, defined as the set of non-differentiability points of a certain convex piecewise affine map. The genericity result of [ABG04, ABG05] can be reobtained as a special case of Kapranov theorem, by considering the hypersurface of the characteristic polynomial equation.

When considering a field with an archimedean absolute value, like the field of complex numbers equipped with its usual absolute value, the amoeba of a hypersurface does not coincide anymore with a tropical hypersurface, however, it can be approximated by such a hypersurface, called spine, in particular, Passare and Rullgård [PR04] showed that the latter is a deformation retract of the former. In a recent work, Avendaño, Kogan, Nisse and Rojas [AKNR13] gave estimates of the distance between a tropical hypersurface which is a more easily computable variant of the spine, and the amoeba of a original hypersurface. However, it does not seem that the present bounds could be derived by the same method.

We note that a different generalization of the Hadamard-Ostrowski-Pólya theorem, dealing with the case of matrix polynomials, not relying on tropical eigenvalues, but thinking of the norm as a “valuation”, appeared recently in [AGS13], refining a result of [GS09]. Tropical eigenvalues generally lead to tighter estimates in the case of structured or sparse matrices.

We finally refer the reader looking for more information on tropical linear algebra to the monographs [BCOQ92, But10].

The paper is organized as follows. In Section 2, we recall the definition and properties of tropical eigenvalues. The main result (upper bound) is stated and proved in Section 3. The conditional lower bound is proved in Section 4. In Section 5, we give examples in which the upper bound is tight (monomial matrices) and not tight (full matrices), and compare the bound obtained by applying the present upper bound to companion matrices with the original upper bound of Hadamard and Pólya for polynomial roots.

2. Preliminary results

2.1. The additive and multiplicative models of the tropical semiring. The max-plus semiring \( \mathbb{R}_{\text{max}} \) is the commutative idempotent semiring obtained by endowing the set \( \mathbb{R} \cup \{−\infty\} \) with the addition \( a \oplus b = \max(a, b) \) and the multiplication \( a \odot b = a + b \). The zero and unit elements of the max-plus semiring are \( 0 = −\infty \) and \( 1 = 0 \), respectively.

It will sometimes be convenient to work with a variant of the max-plus semiring, the max-times semiring \( \mathbb{T} \), consisting of \( \mathbb{R}_{\text{max}}^+ \) (the set of nonnegative real numbers), equipped with \( a \oplus b = \max(a, b) \) and \( a \odot b = a \cdot b \), so that the zero and unit elements are now \( 0 = 0 \) and \( 1 = 1 \). Of course, the map \( x \mapsto \log x \) is an isomorphism \( \mathbb{T} \cong \mathbb{R}_{\text{max}} \). For brevity, we will often indicate multiplication by concatenation, both in \( \mathbb{R}_{\text{max}} \) and \( \mathbb{T} \). The term tropical semiring will refer indifferently to \( \mathbb{R}_{\text{max}} \) or \( \mathbb{T} \). Whichever structure is used should always be clear from the context.

2.2. Tropical polynomials. We will work with formal polynomials over a semiring \( (\mathcal{S}, \oplus, \odot) \), i.e. with objects of the form

\[
p = \bigoplus_{k \in \mathbb{N}} a_k X^k, \quad a_k \in \mathcal{S}, \quad \# \{ k \mid a_k \neq 0 \} < \infty.
\]
The set $S[X]$ of all formal polynomials over $S$ in the indeterminate $X$ is itself a semiring when endowed with the usual addition and multiplication (Cauchy product) of polynomials.

The polynomial function $x \mapsto p(x), S \rightarrow S$, determined by the formal polynomial $p$, is defined by

$$p(x) = \bigoplus_{k=0}^{n} a_k \otimes x^k,$$

where $n$ is the degree of the polynomial.

When $S$ is the max-times semiring $T$ or the max-plus semiring $\mathbb{R}_{\text{max}}$, the polynomial function becomes respectively

$$p(x) = \max_{0 \leq k \leq n} a_k x^k, \quad x \in \mathbb{R}_+$$

or

$$p(x) = \max_{0 \leq k \leq n} a_k + kx, \quad x \in \mathbb{R} \cup \{-\infty\}.$$

(4)

We will call max-plus polynomial a polynomial over the semiring $\mathbb{R}_{\text{max}}$, and max-times polynomial a polynomial over $T$. Both max-plus and max-times polynomials are referred to as tropical polynomials.

Being an upper envelope of convex functions, tropical polynomial functions are convex and piecewise differentiable. In particular, max-plus polynomial functions are piecewise linear.

Different tropical polynomials can have the same associated polynomial function. This lack of injectivity can be understood with the help of convex analysis results. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an extended real-valued function. The l.s.c. convex envelope (or also convexification) of $f$ is the function $\text{cvx} f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$(\text{cvx} f)(x) = \sup \left\{ g(x) \mid g \text{ convex l.s.c., } g \leq f \right\}.$$ 

That is, $\text{cvx} f$ is the largest convex l.s.c. minorant of $f$. Analogously we define the u.s.c. concave envelope (or concavification) of $f$ as the smallest u.s.c. concave majorant of $f$, and we denote it by $\text{cav} f$.

For any (formal) max-plus polynomial we define an extended real-valued function on $\mathbb{R}$ that represents its coefficients: more precisely, to the max-plus polynomial $p = \bigoplus_{k=0}^{n} a_k X^k$ we associate the function $\text{coef} p : \mathbb{R} \rightarrow \mathbb{R}$

$$(\text{coef} p)(x) = \begin{cases} a_k & \text{if } x = k \in \mathbb{N} \\ -\infty & \text{otherwise.} \end{cases}$$

It is clear from (4) that the max-plus polynomial function $x \mapsto p(x)$ is nothing but the Legendre-Fenchel transform of the map $- \text{coef} p$.

**Definition 1.** Let $p = \bigoplus_{k=0}^{n} a_k X^k$ be a max-plus polynomial. The Newton polygon $\Delta(p)$ of $p$ is the graph of the function $\text{cav} \text{coef} p$ restricted to the interval where it takes finite values. In other terms, the Newton polygon of $p$ is the upper boundary of the two-dimensional convex hull of the set of points $\{(k, a_k) \mid 0 \leq k \leq n, a_k \neq -\infty \}$.

The values $\text{cav} \text{coef} p(0), \ldots, \text{cav} \text{coef} p(n)$ are called the concavified coefficients of $p$, and they are denoted by $\overline{a}_0, \ldots, \overline{a}_n$, or alternatively by $\text{cav} a_0, \ldots, \text{cav} a_n$. An index $k$ such that $a_k = \overline{a}_k$ (so that the point $(k, a_k)$ lies on $\Delta(p)$) will be called a saturated index. The polynomial $\overline{p} = \bigoplus_{k=0}^{n} \overline{a}_k X^k$ is called the concavified polynomial of $p$. The correspondence between a polynomial and its concavified is denoted by $\text{cav} : p \mapsto \overline{p}$.

It is known that the Legendre-Fenchel transform of a map depends only on its l.s.c. convex envelope; therefore, we have the following elementary result.
Proposition 1 ([Roc70, Chap 12, p. 104]). Two max-plus polynomials have the same associated polynomial function if and only if they have the same concavified coefficients, or equivalently the same Newton polygons. □

Consider the isomorphism of semirings which sends a max-times polynomial \( p = \bigoplus_{k=0}^{n} a_k x^k \) to the max-plus polynomial \( \text{Log} p := \bigoplus_{k=0}^{n} \log a_k x^k \). We define the log-concavified polynomial of \( p \) as \( \hat{p} = \text{Log}^{-1} \circ \text{cav} \circ \text{Log}(p) \). We also denote it by \( \text{lcav}(p) \); its coefficients are called the log-concavified coefficients of \( p \), and they are denoted by \( \hat{a}_0, \ldots, \hat{a}_n \), or alternatively by \( \text{lcav}(a_0), \ldots, \text{lcav}(a_n) \).

2.3. Roots of tropical polynomials. The roots of a tropical polynomial are defined as the points of non-differentiability of its associated polynomial function. So, if \( p = \bigoplus_{k=0}^{n} a_k x^k \) is a max-plus polynomial, then \( \alpha \in \mathbb{R} \cup \{-\infty\} \) is a root of \( p \) if and only if the maximum

\[
\max_{0 \leq k \leq n} a_k + k\alpha
\]

is attained for at least two different values of \( k \). The multiplicity of a root is defined to be the difference between the largest and the smallest value of \( k \) for which the maximum is attained. The same definitions apply, mutatis mutandis, to max-times polynomials. In particular, if \( p \) is a max-times polynomial, the tropical roots of \( p \) are the images of the tropical roots of \( \text{Log} p \) by the exponential map, and the multiplicities are preserved.

Cuninghame-Green and Meijer [CGM80] showed that a max-plus polynomial function of degree \( n, p(x) = \bigoplus_{k=0}^{n} a_k x^k \), can be factored uniquely as

\[
p(x) = a_n (x \oplus \alpha_1) \cdots (x \oplus \alpha_n).
\]

The scalars \( \alpha_1, \ldots, \alpha_n \) are precisely the tropical roots, counted with multiplicities.

Because of the duality arising from the interpretation of tropical polynomial functions as Fenchel conjugates, the roots of a max-plus polynomial are related to its Newton polygon. The following result is standard.

Proposition 2 (See e.g. [ABG05, Proposition 2.10]). Let \( p \in \mathbb{R}_{\text{max}}[X] \) be a max-plus polynomial. The roots of \( p \) coincide with the opposite of the slopes of the Newton polygon of \( p \). The multiplicity of a root \( \alpha \) of \( p \) coincides with the length of the interval where the Newton polygon has slope \( -\alpha \).

This proposition is illustrated in the following figure:

Here, the polynomial is \( p = -1X^3 \oplus 0X^2 \oplus 2X \oplus 1 \). The tropical roots are \(-1\) (with multiplicity 1) and \(1.5\) (with multiplicity 2).

Corollary 3. Let \( p = \bigoplus_{k=0}^{n} a_k x^k \) be a max-plus polynomial, and let \( \alpha_1 \geq \cdots \geq \alpha_n \) be its roots, counted with multiplicities. Then the following relation for the concavified coefficients of \( p \) holds:

\[
\pi_{n-k} = a_n + \alpha_1 + \cdots + \alpha_k \quad \forall k \in [n].
\]

Analogously, if \( p = \bigoplus_{k=0}^{n} a_k x^k \) is a max-times polynomial with roots \( \alpha_1 \geq \cdots \geq \alpha_n \), then the following relation for its log-concavified coefficients holds:

\[
\hat{a}_{n-k} = a_n \alpha_1 \cdots \alpha_k \quad \forall k \in [n].
\]
As pointed out in the introduction, the tropical roots, in the form of “slopes of Newton polygons”, were already apparent in the works of Hadamard [Had93] and Ostrowski [Ost40].

We next associate a tropical polynomial to a complex polynomial.

Definition 2. Given a polynomial \( p \in \mathbb{C}[z] \),
\[
p = \sum_{k=0}^{n} a_k z^k ,
\]
we define its max-times relative \( p^\times \in \mathbb{T}[X] \) as
\[
p^\times = \bigoplus_{k=0}^{n} |a_k| X^k ,
\]
and its max-plus relative \( p^+ \in \mathbb{R}_{\text{max}}[X] \) as
\[
p^+ = \log p^\times .
\]

We can now define the tropical roots of an ordinary polynomial.

Definition 3 (Tropical roots). The tropical roots of a polynomial \( p \in \mathbb{C}[z] \) are the roots \( \alpha \in \mathbb{R}_+ \) of its max-times relative \( p^\times \).

2.4. Tropical characteristic polynomial and tropical eigenvalues. In this section we recall the definition of the eigenvalues of a tropical matrix [ABG04, ABG05]. We start from the notion of permanent, which is defined for matrices with entries in an arbitrary semiring \((S, \oplus, \odot)\).

Definition 4. The permanent of a matrix \( A = (a_{i,j}) \in S^{n \times n} \) is defined as
\[
\text{per}_S : S^{n \times n} \to S , \quad \text{per}_S A = \bigoplus_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} ,
\]
where \( S_n \) is the set of permutations of \([n]\).

In particular, if \( S = \mathbb{C} \),
\[
\text{per}_\mathbb{C} A = \text{per} A := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}
\]
is the usual permanent, whereas if \( S = \mathbb{T} \), we get the max-times permanent
\[
\text{per}_\mathbb{T} A = \max_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} .
\]
Computing the usual permanent is a difficult problem which was indeed proved to be \#P-complete by Valiant [Val79]. However, computing a max-times permanent is nothing but solving an optimal assignment problem (a number of polynomial time algorithms are known for this problem, including \( O(n^3) \) strongly polynomial algorithms, see [BDM09] for more background).

Definition 5. The tropical characteristic polynomial of a max-times matrix \( A \in \mathbb{T}^{n \times n} \) is the tropical polynomial
\[
q_A \in \mathbb{T}[X] , \quad q_A = \text{per}_\mathbb{T}[X](A \oplus XI) ,
\]
where \( I \) is the max-times identity matrix.

Example.
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad A \oplus XI = \begin{pmatrix} a \oplus X & b \\ c \oplus d & X \oplus X \end{pmatrix}
\]
\[
q_A = (a \oplus X)(d \oplus X) \oplus bc = X^2 \oplus (a \oplus d)X \oplus (ad \oplus bc)
\]
Note that an alternative, finer, definition \cite{AGG09} of the tropical characteristic polynomial relies on *tropical determinants* which unlike permanents take into account signs. This is relevant mostly when considering real eigenvalues, instead of complex eigenvalues as we do here.

We can give explicit expressions for the coefficients of the tropical characteristic polynomial: if \( A = (a_{ij}) \in \T^{n \times n} \) and we write \( q_A = X^n \oplus c_{n-1}X^{n-1} \oplus \cdots \oplus c_0 \), then it is not difficult to see that

\[
c_{n-k} = \bigoplus_{I \subseteq [n]} \bigoplus_{\sigma \in S_I} a_{1, \sigma(1)} \cdots a_{n, \sigma(n)} = \bigoplus_{I \subseteq [n]} \per_I A[I, I] \quad \forall k \in [n],
\]

where \( S_I \) is the group of permutations of the set \( I \), and \( A[I, J] \) is the \( k \times k \) submatrix obtained by selecting from \( A \) the rows \( i \in I \) and the columns \( j \in J \). It will be convenient to write the coefficients of \( q_A \) in terms of the exterior powers of \( A \).

**Definition 6.** The \( k \)-th exterior power or \( k \)-compound of a matrix \( A \in \C^{n \times n} \) is the matrix \( \Lambda^k A \in \C^{(\binom{n}{k}) \times (\binom{n}{k})} \) whose rows and columns are indexed by the subsets of cardinality \( k \) of \([n]\), and whose entries are defined as

\[
(\Lambda^k A)_{I,J} = \det A[I, J].
\]

The \( k \)-th trace of \( A \) is then defined as

\[
\tr^k A = \tr (\Lambda^k A) = \sum_{I \subseteq [n]} \det A[I, I]
\]

for all \( k \in [n] \). If we replace the determinant with the permanent in Equation (5), we get the \( k \)-th *permanental exterior power* of \( A \), denoted by \( \Lambda^k_{\per} A \).

Analogously, for a matrix \( A \in \T^{n \times n} \), we define the \( k \)-th *tropical exterior power* of \( A \) to be the matrix \( \Lambda^k_{\T} A \in \T^{(\binom{n}{k}) \times (\binom{n}{k})} \) whose entries are

\[
(\Lambda^k_{\T} A)_{I,J} = \per_I A[I, J]
\]

for all subsets \( I, J \subseteq [n] \) of cardinality \( k \). The \( k \)-th *tropical trace* of \( A \) is defined as

\[
\tr^k_{\T} A = \tr_{\T} (\Lambda^k_{\T} A) = \max_{I \subseteq [n]} \per_I A[I, I].
\]

One readily checks that the coefficients of \( q_A \) are given by \( c_{n-k} = \tr^k_{\T} A \).

**Definition 7** (Tropical eigenvalues). Let \( A \in \T^{n \times n} \) be a max-times matrix. The *(algebraic) tropical eigenvalues* of \( A \) are the roots of the tropical characteristic polynomial \( q_A \).

Moreover, we define the tropical eigenvalues of a complex matrix \( A = (a_{ij}) \in \C^{n \times n} \) as the tropical eigenvalues of the associated max-times matrix \( |A| = (|a_{ij}|) \).

**Remark.** No polynomial algorithm is known to compute all the coefficients of the tropical characteristic polynomial (see e.g. \cite{BL07}). However, the *roots* of \( q_A \), only depend on the associated polynomial function, and can be computed by solving at most \( n \) optimal assignment problems, leading to the complexity bound of \( O(n^3) \) of Burkard and Butković \cite{BB03}. Gassner and Klinz \cite{GK10} showed that this can be reduced in \( O(n^3) \) using parametric optimal assignment techniques.

Before proceeding to the statement of our main result, we need a last definition.
Definition 8. Denote by $\Omega_n$ the set of all cyclic permutations of $[n]$. For any $n \times n$ complex matrix $A = (a_{i,j})$ and for any bijective map $\sigma$ from a subset $I \subset [n]$ to a subset $J \subset [n]$, we define the weight of $\sigma$ with respect to $A$ as

$$w_A(\sigma) = \prod_{i \in I} a_{i,\sigma(i)}.$$

If $A$ is a nonnegative matrix, meaning a matrix with nonnegative real entries, $\sigma$ is a permutation of $I$, and $I$ has cardinality $\ell$, we also define the mean weight of $\sigma$ with respect to $A$ as

$$\mu_A(\sigma) = w_A(\sigma)^{1/\ell} = \left(\prod_{i \in I} a_{i,\sigma(i)}\right)^{1/\ell},$$

and the maximal cycle mean of $A$ as

$$\rho_{\text{max}}(A) = \max_{\sigma \in \Omega_n} \mu_A(\sigma).$$

If we interpret $A$ as the adjacency matrix of a directed graph with weighted edges, then $\rho_{\text{max}}(A)$ represents the maximum mean weight of a cycle over the graph.

Remark. Since any permutation can be factored into a product of cycles, we can equivalently define the maximal cycle mean in terms of general permutations instead of cycles:

$$\rho_{\text{max}}(A) = \max_{1 \leq \ell \leq n} \max_{I \subset [n], |I| = \ell} \max_{\sigma \in S_I} \mu_A(\sigma).$$

Proposition 4 (Cuninghame-Green, [CG83]). For any $A \in \mathbb{T}^{n \times n}$, the largest root $\rho_T(A)$ of the tropical characteristic polynomial $q_A$ is equal to the maximal cycle mean $\rho_{\text{max}}(A)$.

We shall occasionally refer to $\rho_T(A)$ as the tropical spectral radius of $A$.

Remark. The term algebraic eigenvalue is taken from [ABG06], to which we refer for more background. It is used there to distinguish them from the geometric tropical eigenvalues, which are the scalars $\lambda \in \mathbb{T}$ such that there exists a non-zero vector $u \in \mathbb{T}^n$ (eigenvector) such that $A \odot u = \lambda \odot u$. It is known that every geometric eigenvalue is an algebraic eigenvalue, but not vice versa. Also, the maximal geometric eigenvalue coincides with the maximal cycle mean $\rho_{\text{max}}(A)$, hence with the maximal algebraic eigenvalue, $\rho_T(A)$.

3. Main result

We are now ready to formulate our main result.

Theorem 5. Let $A \in \mathbb{C}^{n \times n}$ be a complex matrix, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues, ordered by nonincreasing absolute value (i.e., $|\lambda_1| \geq \ldots \geq |\lambda_n|$). Moreover, let $\gamma_1 \geq \ldots \geq \gamma_n$ be the tropical eigenvalues of $A$. Then for all $k \in [n]$, we have

$$|\lambda_1 \cdots \lambda_k| \leq U_k \gamma_1 \cdots \gamma_k$$

where

$$U_k = \rho(q_A^k(\text{pat } A)).$$

To prove this theorem, we shall need some auxiliary results.
3.1. **Friedland’s Theorem.** Let \( A = (a_{i,j}) \) and \( B = (b_{i,j}) \) be nonnegative matrices. We denote by \( A \circ B \) the Hadamard (entrywise) product of \( A \) and \( B \), and by \( A^{[r]} \) the entrywise \( r \)-th power of \( A \). That is:

\[
(A \circ B)_{i,j} = a_{i,j} b_{i,j}, \quad (A^{[r]})_{i,j} = a_{i,j}^r.
\]

**Theorem 6** (Friedland, [Fri86]). Let \( A \) be a nonnegative matrix. Define the limit eigenvalue of \( A \) as

\[
\rho_{\infty}(A) = \lim_{r \to +\infty} \rho(A^{[r]})^{1/r}.
\]

Then we have

\[(7) \quad \rho_{\infty}(A) = \rho_{\text{max}}(A),\]

and also

\[
\rho(A) \leq \rho(\text{pat } A) \rho_{\text{max}}(A),
\]

where \( \text{pat } A \) denotes the pattern matrix of \( A \), defined as

\[
(\text{pat } A)_{i,j} = \begin{cases} 
0 & \text{if } a_{i,j} = 0 \\
1 & \text{otherwise}
\end{cases}
\]

Friedland’s result is related to the following log-convexity property of the spectral radius.

**Theorem 7** (Kingman [Kin61], Elsner, Johnson and Da Silva, [EJD88]). If \( A \) and \( B \) are nonnegative matrices, and \( \alpha, \beta \) are two positive real numbers such that \( \alpha + \beta = 1 \), then \( \rho(A^{[\alpha]} \circ B^{[\beta]}) \leq \rho(A)^{\alpha} \rho(B)^{\beta} \).

**Corollary 8.** If \( A \) and \( B \) are nonnegative matrices, then \( \rho(A \circ B) \leq \rho(A) \rho_{\text{max}}(B) \).

**Proof.** Let \( p, q \) be two positive real numbers such that \( \frac{1}{p} + \frac{1}{q} = 1 \). By applying Theorem 7 to the nonnegative matrices \( A^{[\alpha]} \) and \( B^{[\beta]} \), and \( \alpha = \frac{1}{p} \), we get \( \rho(A \circ B) \leq \rho(A^{[\alpha]})^{\frac{1}{p}} \rho(B^{[\beta]})^{\frac{1}{q}} \). Then by taking the limit for \( q \to \infty \) and using the identities of Theorem 6 we obtain \( \rho(A \circ B) \leq \rho(A) \rho_{\text{max}}(B) \).

3.2. **Spectral radius of exterior powers.** The next two propositions are well known.

**Proposition 9** (See e.g. [HJ90, Theorem 8.1.18]). The following statements about the spectral radius hold:

(a) For any complex matrix \( A \) we have \( \rho(A) \leq \rho(|A|) \);

(b) If \( A \) and \( B \) are nonnegative matrices and \( A \leq B \), then \( \rho(A) \leq \rho(B) \).

**Proposition 10** (See e.g. [MM92, 2.15.12]). If \( A \in \mathbb{C}^{n \times n} \) has eigenvalues \( \lambda_1, \ldots, \lambda_n \), then the eigenvalues of \( \Lambda^k A \) are the products \( \prod_{i \in I} \lambda_i \) for all subsets \( I \subset [n] \) of cardinality \( k \).

An immediate corollary of Proposition 10 is that if \( |\lambda_1| \geq \cdots \geq |\lambda_n| \), then the spectral radius of \( \Lambda^k A \) is

\[
\rho(\Lambda^k A) = |\lambda_1 \cdots \lambda_k|.
\]

In the tropical setting we can prove the following combinatorial result, which will be one of the key ingredients of the proof of Theorem 5.

**Theorem 11.** Let \( A \in \mathbb{C}^{n \times n} \) be a complex matrix, and let \( \gamma_1 \geq \cdots \geq \gamma_n \) be its tropical eigenvalues. Then for any \( k \in [n] \) we have

\[
\rho_T(\Lambda^k_A | A|) \leq \gamma_1 \cdots \gamma_k.
\]
The proof of this theorem relies on the following result, which is a variation on classical theorems of Hall and Birkhoff on doubly stochastic matrices. Recall that a circulation matrix of size $n \times n$ is a nonnegative matrix $B = (b_{i,j})$ such that for all $i \in [n]$, $\sum_{j \in [n]} b_{i,j} = \sum_{j \in [n]} b_{j,i}$. The weight of this matrix is the maximum value of the latter sums as $i \in [n]$. We call partial permutation matrix a matrix having a permutation matrix as a principal submatrix, all the other entries being zero. The support of a partial permutation matrix consists of the row (or column) indices of this principal submatrix.

**Lemma 12.** Every circulation matrix $B = (b_{i,j})$ with integer entries, of weight $\ell$, can be written as the sum of at most $\ell$ partial permutation matrices.

**Proof.** We set $s_i = \sum_{j \in [n]} b_{i,j} = \sum_{j \in [n]} b_{j,i}$, so that $s_i \leq \ell \quad \forall i \in [n]$. If we add to $B$ the diagonal matrix $D = \text{Diag}(\ell - s_1, \ldots, \ell - s_n)$, we obtain a matrix with nonnegative integer entries in which the sum of each row and each column is $\ell$. A well known theorem (see e.g. Hall, [Ha98, Theorem 5.1.9]), allows us to write

\[ B + D = P^{(1)} + \cdots + P^{(\ell)} \]

where the $P^{(i)}$'s are permutation matrices. Furthermore we can write $D$ as a sum of diagonal matrices $D^{(1)}, \ldots, D^{(\ell)}$ such that $D^{(i)} \leq P^{(i)} \quad \forall i \in [\ell]$. In this way we have

\[ B = (P^{(1)} - D^{(1)}) + \cdots + (P^{(\ell)} - D^{(\ell)}) = B^{(1)} + \cdots + B^{(\ell)} \]

where every $B^{(m)} = (b_{i,j}^{(m)})$ is a partial permutation matrix (possibly zero). \qed

**Proof of Theorem 11.** Let $A \in \mathbb{C}^{n \times n}$ be a complex matrix. By definition, the tropical eigenvalues $\gamma_1 \geq \cdots \geq \gamma_n$ of $A$ are the roots of the tropical characteristic polynomial $q_A = \text{per}_T([A] \oplus X I)$. Recall that $\text{tr}_T^k |A|$ is the $(n-k)$-th coefficient of $q_A$, with the convention $\text{tr}_T^0 |A| = 1$. We shall denote by $\text{tr}_T^k |A|$ the $(n-k)$-th log-concavified coefficient of $q_A$.

In the following formulas we will denote by $S_{I,J}$ the set of bijections from $I$ to $J$. By Proposition 4, we have

\[
\rho_T(\Lambda_T^k |A|) = \max_{\ell \in [n]} \max_{\#I_1 = k} \left( \Lambda_T^k |A|_{I_1 I_2} \cdots \Lambda_T^k |A|_{I_\ell I_1} \right)^{1/\ell}
\]

\[
= \max_{\ell \in [n]} \max_{\#I_1 = k} \max_{\sigma_1 \in S_{I_1 I_2}} \cdots \max_{\#I_\ell = k} \max_{\sigma_\ell \in S_{I_{\ell-1} I_1}} \left( \prod_{i_1 \in I_1} |a_{i_1 \sigma_1(i_1)}| \cdots \prod_{i_\ell \in I_\ell} |a_{i_\ell \sigma_\ell(i_\ell)}| \right)^{1/\ell}.
\]

(8)

The product in parentheses is a monomial in the entries of $|A|$ of degree $k \cdot \ell$. We rewrite it as

\[
\prod_{i \in [n]} |a_{i,j}|^{b_{i,j}},
\]

where $b_{i,j}$ is the total number of times the element $|a_{i,j}|$ appears in the product. We can arrange the $b_{i,j}$ into a matrix $B = (b_{i,j})$, and observe that $\sum_{j \in [n]} b_{i,j} = \sum_{j \in [n]} b_{j,i} \quad \forall i \in [n]$, so that $B$ is a circulation matrix. In fact, for every $m \in [\ell]$, every index $i \in I_m$ contributes for 1 to the $i$-th row of $B$ (because of the presence of $|a_{i,\sigma_m(i)}|$ in the product), and also for 1 to the $i$-th column of $B$ (because of the presence of $|a_{\sigma_{m-1}(i),i}|$ in the product). By Lemma 12, we can write $B = B^{(1)} + \cdots + B^{(r)}$ with $r \leq \ell$, where $B^{(1)}, \ldots, B^{(r)}$ are partial permutation matrices.
with respective supports \( I^{(1)}, \ldots, I^{(r)} \). We set \( B^{(r+1)} = \cdots = B^{(l)} = 0 \) and \( I^{(r+1)} = \cdots = I^{(l)} = \emptyset \).

The product in the definition of \( \rho_T(\Lambda_A^k) \) (inside the parentheses in (8)) can thus be rewritten as

\[
\prod_{i \in [n]} |a_{i,j}|^{b_{i,j}} = \prod_{m=1}^\ell \left( \prod_{i \in [n]} |a_{i,j}|^{b_{i,j}^{(m)}} \right)
\]

\[
\leq \prod_{m=1}^\ell \tilde{\text{tr}}_T^{#I^{(m)}} |A|
\]

\[
\leq \prod_{m=1}^\ell \tilde{\text{tr}}_T^{#I^{(m)}} |A|
\]

\[
\leq \left( \tilde{\text{tr}}_T^k |A| \right)^\ell,
\]

where the last inequality follows from the log-concavity of \( k \mapsto \tilde{\text{tr}}_T^k |A| \) and from the fact that \( \frac{1}{\ell} \sum_{m=1}^\ell #I^{(m)} = k \). So, using (8), we conclude that \( \rho_T(\Lambda_A^k) \leq \tilde{\text{tr}}_T^k |A| \).

Now, \( \tilde{\text{tr}}_T^k |A| \) is the \((n-k)\)-th concavified coefficient of the tropical polynomial \( q_A \), whose roots are \( \gamma_1 \geq \ldots \geq \gamma_n \). Applying Corollary 3, and recalling that \( \text{tr}_0^0 |A| = 1 \), we obtain

\[
\tilde{\text{tr}}_T^k |A| = \gamma_1 \cdot \cdots \cdot \gamma_k,
\]

so we conclude that

\[
\rho_T(\Lambda_A^k) \leq \gamma_1 \cdot \cdots \cdot \gamma_k.
\]

\[3.3. \text{Proof of Theorem 5.} \] For all subsets \( I, J \) of \([n]\), we have

\[
|\Lambda_A^k|_{I,J} = |\det A[I,J]| \leq |A[I,J]| \leq \#\{\sigma \in S_{I,J} | w_A(\sigma) \neq 0\} \cdot \max_{\sigma \in S_{I,J}} w_A(\sigma)
\]

\[
= \left( \Lambda_{\text{per}}^k (\text{pat } A) \right)_{I,J} \left( \Lambda_A^k |A| \right)_{I,J}.
\]

Since this holds for all \( I \) and \( J \), we can write, in terms of matrices,

\[
|\Lambda_A^k | \leq \left( \Lambda_{\text{per}}^k (\text{pat } A) \right) \circ \left( \Lambda_A^k |A| \right).
\]

We have

\[
|\lambda_1 \cdots \lambda_k | = \rho(\Lambda_A^k) \quad \text{(by Proposition 10)}
\]

\[
\leq \rho(\left( \Lambda_{\text{per}}^k (\text{pat } A) \right) \circ \left( \Lambda_A^k |A| \right)) \quad \text{(by (9) and Proposition 9)},
\]

\[
\leq \rho(\Lambda_{\text{per}}^k (\text{pat } A)) \rho_T(\Lambda_A^k |A|) \quad \text{(by Corollary 8 and Proposition 4)}
\]

\[
\leq \rho(\Lambda_{\text{per}}^k (\text{pat } A)) \gamma_1 \cdots \gamma_k \quad \text{(by Theorem 11)}
\]

and the proof of the theorem is complete.

\[\square\]

4. Lower bound

We next show that the product of the \( k \) largest absolute values of eigenvalues can be bounded from below in terms of the \( k \) largest tropical eigenvalues, under some quite restrictive non-degeneracy conditions.
Lemma 13. Let $A = (a_{i,j}) \in \mathbb{C}^{n \times n}$ be a complex matrix, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues, ordered by nonincreasing absolute value (i.e., $|\lambda_1| \geq \ldots \geq |\lambda_n|$). Moreover, let $\gamma_1 \geq \ldots \geq \gamma_n$ be the tropical eigenvalues of $A$. Let $k \in [n]$ be a saturated index for the tropical characteristic polynomial $q_A$. Suppose $\text{tr}^k A \neq 0$, and let $C_k$ be any positive constant such that

$$C_k \text{tr}^k |A| \leq |\text{tr}^k A|.$$  

Then the following bound holds:

$$\frac{C_k}{(\frac{n}{k})^k} \gamma_1 \cdots \gamma_k \leq |\lambda_1 \cdots \lambda_k|.$$  

Proof. Thanks to Ostrowski’s lower bound in (1), we already have

$$\alpha_1 \cdots \alpha_k \leq \binom{n}{k} |\lambda_1 \cdots \lambda_k|,$$

where $\alpha_1 \geq \ldots \geq \alpha_n$ are the tropical roots of the ordinary characteristic polynomial

$$p_A(x) = \det(xI - A) = x^n - (\text{tr} A)x^{n-1} + (\text{tr}^2 A)x^{n-2} + \cdots + (-1)^n \text{tr}^n A.$$  

Moreover, by Corollary 3 we have

$$\gamma_1 \cdots \gamma_k = \text{lca}(\text{tr}^k |A|)$$

and since $k$ is a saturated index for $q_A$, $\text{lca}(\text{tr}^k |A|) = \text{tr}^k |A|$. Now we can use Equation (10) and write

$$\gamma_1 \cdots \gamma_k = \text{tr}^k |A| \leq \frac{1}{C_k} |\text{tr}^k A| \leq \frac{1}{C_k} \text{lca}(\text{tr}^k |A|) = \frac{1}{C_k} |\alpha_1 \cdots \alpha_k| \leq \frac{n!}{C_k} |\lambda_1 \cdots \lambda_k|.$$  

\hfill \Box

Theorem 14. Let $A$, $\lambda_1, \ldots, \lambda_n$, $\gamma_1, \ldots, \gamma_n$ be as in Lemma 13, and let $k$ be a saturated index for the tropical characteristic polynomial $q_A$. Suppose that among the subsets of cardinality $k$ of $[n]$ there is a unique subset $T_k$ for which there exists a (possibly not unique) permutation $\sigma \in S_n$ that realizes the maximum

$$\max_{I \subseteq [n]} \max_{\sigma \in S_I} \prod_{i \in I} |a_{i,\sigma(i)}|$$

(that is, $w_{|A|}(\sigma) = \text{tr}^k |A|$). Suppose $\det A[T_k, T_k] \neq 0$. Finally suppose that, for any permutation $\sigma$ of any subset of cardinality $k$ except $T_k$, $w_{|A|}(\sigma) \leq \delta_k \cdot w_{|A|}(\sigma) = \delta_k \text{tr}^k |A|$, with

$$\delta_k < \frac{|\det A[T_k, T_k]|}{\text{tr}^k |A| \left(\binom{n}{k} - 1\right) k!}.$$  

Then the inequality

$$Lk \gamma_1 \cdots \gamma_k \leq |\lambda_1 \cdots \lambda_k|$$

holds with

$$L_k = \frac{1}{\binom{n}{k}} \left(\frac{|\det A[T_k, T_k]|}{\text{tr}^k |A|} - \delta_k \left(\binom{n}{k} - 1\right) k!\right).$$  

Proof. To prove the theorem it is sufficient to show that (10) holds with

$$C_k = \left(\frac{|\det A[T_k, T_k]|}{\text{tr}^k |A|} - \delta_k \left(\binom{n}{k} - 1\right) k!\right).$$
We have
\[ |\text{tr}^k A| = \left| \sum_{I \in [n]} \det A[I, I] \right| \]
\[ \geq \left| \det A[T_k, T_k] \right| - \sum_{I \neq T_k} \det A[I, I] \]
\[ \geq \left| \det A[T_k, T_k] \right| - \sum_{I \neq T_k} \per |A[I, I]| \]
\[ \geq \left( \frac{|\det A[T_k, T_k]|}{\text{tr}^k_k |A|} \right) - \delta_k \left( \binom{n}{k} - 1 \right) \text{tr}^k_k |A| \]
\[ = C_k \text{tr}^k_k |A| , \]
and the hypothesis on \( \delta_k \) guarantees that \( C_k > 0 \).

If the maximum in (11) is attained by exactly one permutation, then the statement of Theorem 14 can be slightly modified as follows.

**Theorem 15.** Let \( A, \lambda_1, \ldots, \lambda_n, \gamma_1, \ldots, \gamma_n \) and \( k \) be as in Theorem 14. Suppose that the maximum in (11) is attained for a unique permutation \( \bar{\sigma} \), and that for any other permutation \( \sigma \) of any \( k \)-subset of \([n]\) the inequality \( \frac{w_{A}(\sigma)}{w_{A}(\bar{\sigma})} \leq \eta_k \) holds for some
\[ \eta_k \leq \frac{1}{\left( \binom{n}{k} k! - 1 \right)} . \]
Then the inequality
\[ L_k \gamma_1 \cdots \gamma_k \leq |\lambda_1 \cdots \lambda_k| \]
holds with
\[ L_k = \frac{1}{(k)} \left( 1 - \eta_k \left( \binom{n}{k} k! - 1 \right) \right) . \]

**Proof.** The arguments of the proof are the same as for Theorem 14. In the present case, we have
\[ |\text{tr}^k A| = \left| \sum_{I \in [n]} \det A[I, I] \right| \]
\[ \geq \left| \frac{w_{A}(\bar{\sigma})}{\text{tr}^k_k |A|} \right| - \sum_{\sigma \neq \bar{\sigma}} \left| w_{A}(\sigma) \right| \]
\[ \geq \text{tr}^k_k |A| - \left( \binom{n}{k} k! - 1 \right) \eta_k \text{tr}^k_k |A| , \]
and we conclude applying Lemma 13. \( \square \)

5. Optimality of the upper bound and comparison with the bounds for polynomial roots

We now discuss briefly the optimality of the upper bound for some special classes of matrices. Throughout this paragraph, if \( A \) is a complex \( n \times n \) matrix, then \( \lambda_1, \ldots, \lambda_n \) will be its eigenvalues (ordered by nonincreasing absolute value), and \( \gamma_1 \geq \cdots \geq \gamma_n \) will be its tropical eigenvalues.
5.1. Monomial matrices. Recall that a monomial matrix is the product of a diagonal matrix (with non-zero diagonal entries) and of a permutation matrix. We next show that the upper bound is tight for monomial matrices.

**Proposition 16.** If $A$ is a monomial matrix, then, for all $k \in [n]$, the inequality in Theorem 5 is tight,

$$|\lambda_1 \cdots \lambda_k| = \rho(\Lambda^k_{\text{per}}(\text{pat } A)) \gamma_1 \cdots \gamma_k \quad \forall k \in [n]$$

**Proof.** We claim that if $A$ is a monomial matrix, then, the absolute values of the eigenvalues of $A$ coincide with the tropical eigenvalues of $|A|$, counted with multiplicities.

To see this, assume that $A = DC$ where $D$ is diagonal and $C$ is a matrix representing a permutation $\sigma$. If $\sigma$ consists of several cycles, then, $DC$ has a block diagonal structure, and so, the characteristic polynomial of $A$ is the product of the characteristic polynomials of the diagonal blocks of $A$. The same is true for the tropical characteristic polynomial of $|A|$. Hence, it suffices to show the claim when $\sigma$ consists of a unique cycle. Then, denoting by $d_1, \ldots, d_n$ the diagonal terms of $D$, expanding the determinant of $xI - A$ or the permanent of $xI \oplus A$, one readily checks that the characteristic polynomial of $A$ is $x^n - d_1 \cdots d_n$, whereas the tropical characteristic polynomial of $|A|$ is $x^n \oplus |d_1 \ldots d_n|$. It follows that the eigenvalues of $A$ are the $n$th roots of $d_1 \ldots d_n$, whereas the tropical eigenvalues of $|A|$ are all equal to $|d_1 \ldots d_n|^{1/n}$. So, the claim is proved.

It remains to show that $\rho(\Lambda^k_{\text{per}}(\text{pat } A)) = 1$. Note that pat $A = C$. We claim that $\Lambda^k_{\text{per}} C$ is a permutation matrix. In fact, for any fixed $k \in [n]$, let $I$ be a subset of cardinality $k$ of $[n]$. Since $C$ is a permutation matrix, there is one and only one subset $J \subseteq [n]$ such that per $C[I, J] \neq 0$: precisely, if $C$ represents the permutation $\sigma : [n] \to [n]$, then per $C[I, \sigma(I)] = 1$ and per $C[I, J] = 0 \forall J \neq \sigma(I)$. This means that each row of $\Lambda^k_{\text{per}} C$ contains exactly one 1, and the other entries are zeroes. Since the same reasoning is also valid for columns, we can conclude that $\Lambda^k_{\text{per}} C$ is a permutation matrix, and as such its spectral radius is 1.

$\square$

5.2. Full matrices. Monomial matrices are among the sparsest matrices we can think of. One may wonder what happens in the opposite case, when all the matrix entries are nonzero. We next discuss a class of instances of this kind, in which the upper bound is not tight. We only consider the case $k = n$ for brevity, although it is not the only case for which the equality fails to hold.

**Proposition 17.** Let $A = (a_{i,j})$ be a $n \times n$ complex matrix, $n \geq 3$, and suppose $|a_{i,j}| = 1$ for all $i, j \in [n]$. Then the inequality in Theorem 5 can not be tight for $k = n$.

**Proof.** For any couple $(I,J)$ of $k$-subsets of $[n]$, the $(I,J)$ element of the matrix $\Lambda^k_{\text{per}}(\text{pat } A)$ is given by the permanent of a $k \times k$ matrix of ones, that is $k!$; so $\Lambda^k_{\text{per}}(\text{pat } A)$ is a $\binom{n}{k} \times \binom{n}{k}$ matrix with all entries equal to $k!$. Its spectral radius is therefore $\binom{n}{k} k!$ (and $(1, \ldots, 1)^T$ is an eigenvector for the maximum eigenvalue). For $k = n$, $\rho(\Lambda^k_{\text{per}}(\text{pat } A))$ reduces to $n^n$, so our upper bound would be $|\lambda_1 \cdots \lambda_n| \leq n! \gamma_1 \cdots \gamma_n$. Now, the left-hand side can be thought of as $|\text{det } A|$, and on the other hand $\gamma_1 = \cdots = \gamma_n = 1$ (the tropical characteristic polynomial is $q_A(x) = x^n \oplus x^{n-1} \oplus \cdots \oplus x \oplus 1 = (x \oplus 1)^n \forall x \geq 0$). So the inequality in Theorem 5 is equivalent to $|\text{det } A| \leq n^n$. But the well-known Hadamard bound for the determinant yields in this case $|\text{det } A| \leq (\sqrt{n})^n = n^{n/2}$, and since $n^{n/2} < n! \forall n \geq 3$ the inequality of Theorem 5 can not be tight.

$\square$
5.3. Comparison with the Hadamard-Pólya’s bounds for polynomial roots.

Finally, we discuss the behavior of the upper bound of Theorem 5 for the case of a companion matrix. Since the eigenvalues of a companion matrix are exactly the roots of its associated polynomial, this will allow a comparison between the present matrix bounds and the upper bound of Hadamard and Pólya discussed in the introduction. We start by showing that the usual property of companion matrices remains true in the tropical setting.

Lemma 18. Consider the polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \), and let \( A \) be its companion matrix. Then the tropical eigenvalues of \( A \) are exactly the tropical roots of \( p \).

Proof. The matrix is

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{pmatrix}
\]

By definition, its tropical eigenvalues are the tropical roots of the polynomial \( q_A(x) = \bigoplus_{k=0}^{n} \text{tr}_T[A] x^{n-k} \), so to verify the claim it is sufficient to show that \( \text{tr}_T[A] = |a_{n-k}| \) for all \( k \in \{0, \ldots, n\} \). Recall that \( \text{tr}_T[A] \) is the maximal tropical principal submatrix of \( A \) (see Equation (6)). It is easy to check that the only principal submatrices with a non-zero contribution are those of the form \( |A|[I_k, I_k] \) with \( I_k = \{n - k + 1, \ldots, n\} \), and in this case \( \text{per}_T[A][I_k, I_k] = |a_{n-k}| \).

Lemma 19. If \( A \) is the companion matrix of a polynomial of degree \( n \), then,

\[
\rho(\bigwedge_{\text{per}}^k(\text{pat} A)) \leq \min(k+1, n-k+1).
\]

Proof. First, we note that nonzero entries of \( \bigwedge_{\text{per}}^k(\text{pat} A) \) can only be 1’s, because \( \text{pat} A \) is a \((0,1)\)-matrix, and the tropical permanent of any of its square submatrices has at most one non-zero term. By computing explicitly the form of \( \bigwedge_{\text{per}}^k(\text{pat} A) \), for example following the method used by Moussa in [Mou97], we see that each column of \( \bigwedge_{\text{per}}^k(\text{pat} A) \) has either one or \( k+1 \) nonzero entries, and each row has either one or \( n-k+1 \) nonzero entries. In terms of matrix norms, we have \( \|\bigwedge_{\text{per}}^k(\text{pat} A)\|_1 = k+1 \), and \( \|\bigwedge_{\text{per}}^k(\text{pat} A)\|_\infty = n-k+1 \). Since both these norms are upper bounds for the spectral radius, we can conclude that \( \rho(\bigwedge_{\text{per}}^k(\text{pat} A)) \leq \min(k+1, n-k+1) \).

Thus, by specializing Theorem 5 to companion matrices, we recover the version of the upper bound (1) originally derived by Hadamard, with the multiplicative constant \( k+1 \). By comparison, the multiplicative constant in Lemma 19 is smaller due to its symmetric nature. However, it was observed by Ostrowski that the upper bound in (1) can be strengthened by exploiting symmetry. We give a formal argument for the convenience of the reader.

Lemma 20. Let \( P = \{ p \in \mathbb{C}[z] \mid \deg p = n \} \) be the set of complex polynomials of degree \( n \). Denote the roots and the tropical roots as above. Suppose that the inequality \( |z_1 \cdots z_k| \leq f(k) \cdot \alpha_1 \cdots \alpha_k \) holds for some function \( f \), for all \( k \in [n] \) and for all polynomials \( p \in P \). Then the inequality \( |z_1 \cdots z_k| \leq f(n-k) \cdot \alpha_1 \cdots \alpha_k \) also holds for all \( k \in [n] \) and for all polynomials \( p \in P \).

Proof. Consider a polynomial \( p \in P \), \( p(z) = a_n z^n + \cdots + a_0 \) with roots \( z_1, \ldots, z_n \) (ordered by nonincreasing absolute value) and tropical roots \( \alpha_1 \geq \ldots \geq \alpha_n \). Arguing
by density, we may assume that $a_0 \neq 0$. Then, we build its reciprocal polynomial
\[ p^*(z) = z^n p(1/z) = a_0 z^n + \cdots + a_n. \]
It is clear that the roots of $p^*$ are $\zeta_1^{-1}, \ldots, \zeta_n^{-1}$. Moreover, its tropical roots are $\alpha_1^{-1} \leq \cdots \leq \alpha_n^{-1}$: this can be easily proved by observing that the Newton polygon of $p^*$ is obtained from the Newton polygon of $p$ by symmetry with respect to a vertical axis, and thus it has opposite slopes.

Since $p^* \in P$, by hypothesis we can bound its $n - k$ largest roots:
\[ \left| \frac{1}{\zeta_n} \cdots \frac{1}{\zeta_{k+1}} \right| \leq f(n-k) \frac{1}{\alpha_n} \cdots \frac{1}{\alpha_{k+1}}. \]
By applying Corollary 3 (and observing that 0 is a saturated index for the max-times relative $p^*$) we also have
\[ |a_0| = |a_n| \alpha_1 \cdots \alpha_n, \]
so we can write
\[
|\zeta_1 \cdots \zeta_k| = |\zeta_1 \cdots \zeta_n| \left| \frac{1}{\zeta_n} \cdots \frac{1}{\zeta_{k+1}} \right|
\leq \frac{|a_0|}{|a_n|} \frac{1}{\zeta_n} \cdots \frac{1}{\zeta_{k+1}} f(n-k) \frac{1}{\alpha_n} \cdots \frac{1}{\alpha_{k+1}}
= \alpha_1 \cdots \alpha_n \cdot f(n-k) \frac{1}{\alpha_n} \cdots \frac{1}{\alpha_{k+1}} = f(n-k) \cdot \alpha_1 \cdots \alpha_k.
\]

Therefore, it follows from the Pólya’s upper bound (1) that
\[ |\zeta_1 \cdots \zeta_k| \leq \min \left( \sqrt{\frac{(k+1)^{k+1}}{k^k}}, \sqrt{\frac{(n-k+1)^{n-k+1}}{(n-k)^{n-k}}} \right) \alpha_1 \cdots \alpha_k, \]
for all $k \in [n]$. This is tighter than the bound derived from Theorem 5 and Lemma 19. In the latter lemma, we used a coarse estimation of the spectral radius, via norms. A finer bound can be obtained by computing the true spectral radius of $\mathop{\text{pat}}^A$ for the companion matrix $A$, but numerical experiments indicate this still does not improve Pólya’s bound. This is perhaps not surprising as the latter is derived by analytic functions techniques (Jensen inequality and Parseval identity), which do not naturally carry over to the more general matrix case considered here.

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