ON EXISTENCE OF NEW FAMILIES OF 2-DESIGNS AROSE FROM SUZUKI-TITS OVOID

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Abstract. A recent classification of flag-transitive 2-designs with parameters 
\((v, k, \lambda)\) whose replication number \(r\) is coprime to \(\lambda\) gives seven possible infinite families of 2-designs, three of which are new with no known generic constructions.

In this paper, we construct one of these families of 2-designs arose from Suzuki-Tits ovoid, and show that for a given positive integer \(q = 2^{2n+1} \geq 8\), there exists a 2-design with parameters \((q^2 + 1, q, q - 1)\) and the replication number \(q^2\) admitting the Suzuki group \(Sz(q)\) as its automorphism group. We also construct a family of 2-designs with parameters \((q^2 + 1, q(q - 1), (q - 1)(q^2 - q - 1))\) and the replication number \(q^2(q - 1)\) admitting the Suzuki groups \(Sz(q)\) as their automorphism groups.

1. Introduction

An ovoid in \(PG_3(q)\) with \(q > 2\), is a set of \(q^2 + 1\) points such that no three of which are collinear. The classical example of an ovoid in \(PG_3(q)\) is an elliptic quadric. If \(q\) is odd, then all ovoids are elliptic quadrics, see [4, 11], while in even characteristic, there is only one known family of ovoids that are not elliptic quadrics in which \(q \geq 8\) is an odd power of 2. These were discovered by Tits [15], and are now called the Suzuki-Tits ovoids since the Suzuki groups naturally act on these ovoids. The main

aim of this paper is to introduce two infinite families of 2-designs arose from Suzuki-Tits ovoid whose automorphism groups are the Suzuki groups. A 2-design \(D\) with parameters \((v, k, \lambda)\) is a pair \((\mathcal{P}, \mathcal{B})\) with a set \(\mathcal{P}\) of \(v\) points and a set \(\mathcal{B}\) of \(b\) blocks such that each block is a \(k\)-subset of \(\mathcal{P}\) and each two distinct points are contained in \(\lambda\) blocks. The number of blocks incident with a given point is a constant number \(r := bk/v\) called the replication number of \(D\). If \(v = b\) (or equivalently, \(k = r\)), then \(D\) is called symmetric. Further definitions and notation can be found in [5, 7, 12].

Our motivation comes from a recent classification of flag-transitive 2-designs whose replication number is coprime to \(\lambda\), [3]. Excluding thirteen sporadic examples, we have found seven possible infinite families of 2-designs with this property, three of which are new and the rest are well-known structures, namely, point-hyperplane designs, Witt-Bose-Shrikhande spaces [6], Hermitian Unital spaces [9] and Ree Unital spaces [10]. These new possible 2-designs arose from studying 2-designs admitting finite almost simple exceptional automorphism groups of Lie type, see [1]. Although, we have provided examples of these new 2-designs with smallest possible parameters [1, Section 2], we have not been aware of any generic constructions of these incidence structures. In [2], Alavi and Daneshkhah have constructed two of these infinite families of 2-designs admitting Ree groups as their automorphism groups, and so for the remaining possibility, one may ask the following question:

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Question 1. For a given prime power \( q = 2^{2n+1} \geq 8 \), does there exist an infinite family of 2-designs with parameters \( (q^2 + 1, q, q - 1) \) with replication number \( q^2 \) admitting the Suzuki group \( Sz(q) \) as their automorphism groups?

In this paper, we aim to give a positive answer to Question 1. Indeed, in Theorem 2 below, we construct an infinite family of 2-designs using the natural action of Suzuki groups on Suzuki-Tits ovoid. We also construct a 2-design with parameters \( (q^2 + 1, q(q - 1), (q - 1)(q^2 - q - 1)) \) admitting the Suzuki group \( Sz(q) \) as its flag-transitive automorphism group. We call such 2-designs Suzuki-Tits ovoid designs as they arose from Suzuki-Tits ovoid. We note that Suzuki-Tits ovoid designs have \( q(q^2 + 1) \) number of blocks, and so it is not symmetric.

2. Preliminaries

The Suzuki groups were discovered by Suzuki [13], and a geometric construction of these groups was given by Tits [15]. We mainly follow the description of these groups from [8, Section XI.3] with a few exceptions in our notation, see also [13, 14, 15].

Let \( F = GF(q) \) be the finite field of size \( q = 2^{2n+1} \geq 8 \), and let \( \theta \) be the automorphism of \( F \) mapping \( \alpha \to \alpha^r \), where \( r = \sqrt{2q} = 2^{n+1} \). Therefore, \( \theta^2 \) is the Frobenius automorphism \( \alpha \mapsto \alpha^2 \). Let \( e \) be the identity \( 4 \times 4 \) matrix, and let \( e_{ij} \) be the \( 4 \times 4 \) matrix with 1 in the entry \( ij \) and 0 elsewhere. For \( x, y \in F \) and \( \kappa \in F^\times \), define

\[
s(x, y) = e + xe_{21} + ye_{31} + x^\theta e_{32} + (x^{1+\theta} + xy + y^\theta)e_{41} + (x^{1+\theta} + y)e_{42} + xe_{43}; \\
m(\kappa) = e_1^{\kappa}e_{11} + e_2^{\kappa}e_{22} + e_3^{\kappa}e_{33} + e_4^{\kappa}e_{44}; \\
\tau = e_{14} + e_{23} + e_{32} + e_{41}.
\]

We know that \( s(x, y) \cdot s(z, t) = s(x + z, y + t + x^\theta z) \), and hence the set \( Q \) of the matrices \( s(x, y) \) is a group of order \( q^2 \). The set of matrices \( m(\kappa) \) forms a cyclic group \( M \cong F^\times \) of order \( q - 1 \). Since \( m(\kappa)^{-1} \cdot s(x, y) \cdot m(\kappa) = (x\kappa, y\kappa^{1+\theta}) \), the group \( H \) generated by \( Q \) and \( M \) is a semidirect product of a normal subgroup \( Q \) by a complement \( M \), and so it has order \( q^2(q - 1) \). The Suzuki group \( Sz(q) \) is a subgroup of \( GL_4(q) \) generated by \( H \) and the \( 4 \times 4 \) matrix \( \tau \) defined as in (3). In what follows, \( G \) will denote the Suzuki group \( Sz(q) \) with \( q = 2^{2n+1} \geq 8 \).

The Suzuki group \( G \) naturally acts on the projective space \( PG_3(q) \) via \( [w]^z := [w^z] \), for all \( x \in G \) and \( [w] \in PG_3(q) \). In fact, \( G \) acts as a doubly transitive permutation group of degree \( q^2 + 1 \) on the Suzuki-Tits ovoid

\[
P = \{p(\alpha, \beta) \mid \alpha, \beta \in F\} \cup \{\infty\} \subseteq PG_3(q),
\]

where \( p(\alpha, \beta) = [\alpha^{2+\theta} + \alpha\beta + \beta^\theta, \beta, \alpha, 1] \) and \( \infty := [1, 0, 0, 0] \in PG_3(q) \). In particular, the action of the matrices \( s(x, y) \) and \( m(\kappa) \) on the projective points \( p(\alpha, \beta) \) of \( P \setminus \{\infty\} \) can be explicitly written as follows

\[
p(\alpha, \beta)^{s(x, y)} = p(\alpha + x, \beta + y + \alpha x^\theta + x^{1+\theta}), \\
p(\alpha, \beta)^{m(\kappa)} = p(\alpha \kappa, \beta \kappa^{1+\theta}).
\]

Note that \( H \) fixes \( \infty \), and hence \( H \) is the point-stabilizer \( G_\infty \). The subgroup \( H \) of \( G \) acts as a Frobenius group on \( P \setminus \{\infty\} \), where \( Q \) is the Frobenius kernel of \( H \) acting regularly on \( P \setminus \{\infty\} \), and \( M = G_{\infty, \omega} \) is the Frobenius complement of \( H \) fixing the second point \( \omega := p(0, 0) = [0, 0, 0, 1] \in P \) and acting semiregularly on
\(P \setminus \{\infty, \omega\}\). Therefore, the stabilizer of any three points in \(P\) is the trivial subgroup. Moreover, the map \(Hg \mapsto \infty^g\) induces a permutational isomorphism between the \(G\)-action on the set of right cosets of \(H\) in \(G\) and the \(G\)-action on the Suzuki-Tits avoid \(P\).

We now consider the subgroup \(Q_0\) of \(Q\) consisting of all matrices \(s(0, y)\). Note that the matrices of the form \(s(0, y)\) are the only involutions in \(Q\). Thus \(Q_0\) is a normal subgroup of \(Q\) of order \(q\). Moreover, \(Q_0 = Q' = \mathbb{Z}(Q)\). Let \(K\) be the subgroup of \(H\) generated by \(Q_0\) and \(M\), that is to say,

\[K = \langle s(0, y), m(\kappa) \mid y \in \mathbb{F} \text{ and } \kappa \in \mathbb{F}^\times \rangle,\]

where \(s(0, y)\) and \(m(\kappa)\) are as in (1) and (2), respectively. Then \(K\) is a Frobenius group of order \(q(q - 1)\) whose Frobenius kernel and Frobenius complement are \(Q_0\) and \(M\), respectively. Let now

\[\Delta_1 = \{\infty\}, \Delta_2 = \{p(0, \beta) \mid \beta \in \mathbb{F}\}, \text{ and } \Delta_3 = \{p(\alpha, \beta) \mid \alpha \in \mathbb{F}^\times, \beta \in \mathbb{F}\}.\]

Then \(|\Delta_1| = 1, |\Delta_2| = q\) and \(|\Delta_3| = q(q - 1)\). Assuming these notation, we prove the following key lemma.

**Lemma 1.** The subsets \(\Delta_1, \Delta_2\) and \(\Delta_3\) are the only orbits of \(K\) in its action on the ovoid \(P\). Moreover, \(K\) is the setwise-stabilizer \(G_{\Delta_i}\), for \(i = 2, 3\).

**Proof.** Since \(K\) is a subgroup of \(H\) fixing \(\infty\), it follows that \(\Delta_1\) is an orbit of \(K\). It is easily followed by (6) that the \(\Delta_i\), for \(i \in \{2, 3\}\), are \(M\)-invariant subsets of \(P\). Moreover, by (5), we have that \(p(\alpha, \beta)^{s(0, y)} = p(\alpha, \beta + y)\), and so the \(\Delta_i\) are also \(Q_0\)-invariant. Therefore, by (7), we conclude that the \(\Delta_i\) are \(K\)-invariant subsets of \(P\). Further, if \(\alpha \in \mathbb{F}^\times\), then it follows from (5) and (6) that \(p(0, 0)^{s(0, \beta)} = p(0, \beta)\) and \(p(1, 0)^{m(\alpha)s(0, \beta)} = p(\alpha, \beta)\). This implies that \(K\) is transitive on each \(\Delta_i\), and since the set of \(\Delta_i\), for \(i = 1, 2, 3\), forms a \(K\)-invariant partition of \(P\), we conclude that the \(\Delta_i\) are all distinct \(K\)-orbits on \(P\).

We now prove that \(G_{\Delta_2} = K\). Obviously, \(K\) is a subgroup of \(G_{\Delta_2}\). Recall that \(G\) is generated by \(s(x, y), m(\kappa)\) and \(\tau\) defined as in (1), 2 and (3), respectively. The fact that \(p(0, 0)^\tau = \infty\) implies that \(\tau\) does not fix \(\Delta_2\). Hence, \(G_{\Delta_2}\) is a subgroup of \(H = \langle s(x, y), m(\kappa) \mid x, y \in \mathbb{F} \text{ and } \kappa \in \mathbb{F}^\times \rangle\). We have also proved that \(K = Q_0 M\) fix \(\Delta_2\), and so \(K \subseteq G_{\Delta_2} \leq H\). If a generator \(g := s(x, y)\) fixes \(\Delta_2\), then (5) yields \(p(\alpha, \beta + y + x^{1+q^2}) = p(0, \beta)^{s(x, y)} \in \Delta_2\), and so \(x = 0\), that is to say, \(g = s(0, y) \in Q_0\). Recall that \(M\) fixes \(\Delta_2\), and hence we conclude that \(G_{\Delta_2} = K\). We finally show that \(G_{\Delta_3} = K\). By the same argument as in the previous case, \(G_{\Delta_3}\) is a subgroup of \(H\) containing \(K\). Let \(s(x, y) \in H\) fix \(\Delta_3\). Then \(p(\alpha, \beta)^{s(x, y)} \in \Delta_3\) for all \(\alpha \neq 0\), and so by (5), this is equivalent to \(x \neq \alpha\), for all \(\alpha \neq 0\). Thus, \(x = 0\), and hence \(G_{\Delta_3}\) is a subgroup of \(K = Q_0 M\) implying that \(G_{\Delta_3} = K\).

\[\square\]

### 3. Existence of Suzuki-Tits ovoid designs

In this section, we prove our main result Theorem 2 and construct two infinite families of 2-designs admitting \(G = \text{Sz}(q)\) as their automorphism groups. In order to construct out favourite designs, we use [5, Proposition III.4.6]. In fact, if \(G\) is a doubly transitive permutation group on a finite set \(P\) of size \(v\) and \(B\) is a subset of \(P\) of size \(k \geq 2\), then the incidence structure \(D = (P, B^G)\) is a 2-design with parameters \((v, k, \lambda)\) with automorphism group \(G\), where \(B^G = \{B^x \mid x \in G\}\). The
design $\mathcal{D}$ has $b = |G : G_B|$ number of blocks and $\lambda$ is equal to $bk(k - 1)/v(v - 1)$. We now prove our main result.

**Theorem 2.** Let $G = Sz(q)$ with $q = 2^{2n+1} \geq 8$. Let also $\mathcal{P}$ be the Suzuki-Tits ovoid defined as in (4), and let $\Delta_i$ be as in (8), for $i \in \{2,3\}$. Then

(a) if $\mathcal{B}_2 = \Delta_2^q$, then $(\mathcal{P}, \mathcal{B}_2)$ is a 2-design with parameters $(q^2 + 1, q, q - 1)$ and the replication number $q^2$;

(b) if $\mathcal{B}_3 = \Delta_3^q$, then $(\mathcal{P}, \mathcal{B}_3)$ is a 2-design with parameters $(q^2 + 1, q(q - 1), (q - 1)(q^2 - q - 1))$ and the replication number $q^2(q^2 - q - 1)$;

Moreover, the Suzuki group $Sz(q)$ is a flag-transitive automorphism group of the designs in parts (a) and (b) acting primitively on the points set $\mathcal{P}$ but imprimitively on the blocks set $\mathcal{B}_i$.

**Proof.** By the fact that $G$ is doubly transitive on $\mathcal{P}$, [5, Proposition III.4.6] implies that the incidence structures $\mathcal{D}_i = (\mathcal{P}, \mathcal{B}_i)$ are 2-designs with parameters $(v, k_i, \lambda_i)$, for $i \in \{2,3\}$, admitting $G = Sz(q)$ as their automorphism groups. Recall that $v = |\mathcal{P}| = q^2 + 1$. Moreover, by Lemma 1, we have that $|G_{\Delta_i}| = |K| = q(q - 1)$, for $i \in \{2,3\}$, where $K$ is subgroup of $G$ defined as in (7). Therefore, the designs $\mathcal{D}_i$ in parts (a) and (b) have (the same) number of blocks $b = |G : G_{\Delta_i}| = q(q^2 + 1)$. Note that $k_i = |\Delta_i|$ and $\lambda_i = bk_i(k_i - 1)/v(v - 1)$, and the replication number $r_i$ is equal to $bk_i/v$, for $i \in \{2,3\}$. Therefore, $\mathcal{D}_2$ is a 2-design with parameters $(q^2 + 1, q, q - 1)$ as in part (a) and $\mathcal{D}_3$ is a 2-design with parameters $(q^2 + 1, q(q - 1), (q - 1)(q^2 - q - 1))$ as in part (b). Since $G$ is transitive on $\mathcal{B}_i$ and $\Delta_i$ is an orbit of $K = G_{\Delta_i}$, we conclude that $G$ is a flag-transitive automorphism group of $\mathcal{D}_i$, for $i \in \{2,3\}$. The group $G$ is primitive on $\mathcal{P}$ as it is doubly transitive on $\mathcal{P}$, however, $G$ is imprimitive on $\mathcal{B}_i$ as the block-stabilizer $K = G_{\Delta_i}$ is not a maximal subgroup of $G$. \[\square\]

We remark here that the design constructed in Theorem 2(a) introduces an infinite family of examples of 2-designs with $gcd(r, \lambda) = 1$ obtained in [1, Theorem 1.1(a)] and gives a positive answer to Question 1.

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