OPTIMAL STRONG APPROXIMATION FOR QUADRATIC FORMS

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Abstract. For a non-degenerate integral quadratic form $F(x_1, \ldots, x_d)$ in 5 (or more) variables, we prove an optimal strong approximation theorem. Let $\Omega$ be a compact subset of the affine quadric $F(x_1, \ldots, x_d) = 1$ over the real numbers. Take a small ball $B$ of radius $0 < r < 1$ inside $\Omega$, and an integer $m$. Further assume that $N$ is a given integer which satisfies $N \gg \epsilon, \Omega \left( r^{-1}m \right)^{1+\epsilon}$ for any $\epsilon > 0$. Finally assume that an integral vector $(\lambda_1, \ldots, \lambda_d) \pmod{m}$ is given. Then we show that there exists an integral solution $x = (x_1, \ldots, x_d)$ of $F(x) = N$ such that $x_i \equiv \lambda_i \pmod{m}$ and $\sqrt{N} \in B$, provided that all the local conditions are satisfied. We also show that 4 is the best possible exponent.

Moreover, for a non-degenerate integral quadratic form $F(x_1, \ldots, x_4)$ in 4 variables we prove the same result if $N \gg \epsilon, \Omega \left( r^{-1}m \right)^{6+\epsilon}$ and some non-singular local conditions for $N$ are satisfied. Based on our numerical experiments on the diameter of LPS Ramanujan graphs, we conjecture that the optimal strong approximation theorem holds for any quadratic form $F(X)$ in 4 variables with the optimal exponent 4.

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1. Introduction

1.1. Statement of results. Before stating our main theorem, we discuss three applications. The first addresses the question of approximating real matrices by integral matrices. More precisely, let $A = [a_{i,j}]$ be a $2 \times 2$ matrix of determinant 1 and $m$ be a positive integer. How well can one approximate $A$ by \[ m^{-1/2} B, \]
where $B = [b_{i,j}]$ is an integral matrix of determinant $m$? Tijdeman [Tij86], showed that there exists $B$ such that

\begin{equation}
\max_{i,j} |m^{-1/2} b_{ij} - a_{ij}| < C m^{-18} (\log m)^{7/9},
\end{equation}

where $C$ is a constant depending on $\max a_{i,j}$. Later, Harman [Har90] improved the exponent $-1/18$ to $-1/8$ and showed this exponent cannot be smaller than $-1/4$. Harman remarked that if Hooley’s $R^*$ Conjecture [Hoo78] were true, then the exponent drops to $-1/6$. Subsequently, Chiu [Chi95] remarked that an estimate of $|a(p)| < 2p^{r}$ for Maass cusp forms would yield an exponent of $([r + 1/2] - 1)/3$. Therefore, if one assumes the Ramanujan Conjecture for $SL(2, \mathbb{Z})$ Maass cusp forms, then the exponent drops to $-1/6$. The best known bound toward the Ramanujan conjecture is $7/64$ [Kim03] and this yields $-1/8 - 1/192$, which slightly improves the Harman’s bound. It is interesting that under the most favorable conditions, the two different methods give the same exponent $-1/6$. Our theorem for the quadratic form $q(x_1, \ldots, x_4) = x_1 x_4 - x_2 x_3$ and the archimedean place $\infty$ shows that the matrix approximation is possible with the exponent $-1/6$ without any assumptions.

**Corollary 1.1.** Fix $w$ a compact subset of $SL(2, \mathbb{R})$ and any $0 < \epsilon$. Then for every matrix $A \in w$ and $m \in \mathbb{Z}$, there exists an integral matrix $B \in M_{2 \times 2}[\mathbb{Z}]$ such that $\det(B) = m$ and

\begin{equation}
\| A - m^{-1/2} B \| \ll m^{-1/6 + \epsilon},
\end{equation}

where $\ll$ only depends on $\epsilon$ and $w$. Moreover, we cannot replace $-1/6$ in the exponent with a number smaller than $-1/4$.

**Remark 1.2.** We conjecture that our theorem, which is optimal for 5 or more variables, holds with the same optimal exponent 4 for quadratic forms in 4 variables. As a result, the matrix approximation is possible with the optimal exponent $-1/4$.

As the second application of our main theorem, we prove an asymptotic formula for the number of lattice points in a small cap on a sphere, i.e. the intersection of a ball with a small radius and the sphere. In the appendix of [BRI12], an upper bound for the number of lattice points in such small caps is given. Following a suggestion of Bourgain [Sar13a Page 23] we promote this to a lower bound and at the same time we prove this more general result, namely an asymptotic formula for the number of lattice points in a small cap. We illustrate our main theorem in the following application:

**Corollary 1.3.** Let $S^d(R)$ be any $d$-dimensional sphere of radius $R$ such that $N := R^2$ is an integer and $d \geq 4$. Suppose we are given a spherical cap of diameter $Y$ where $Y \gg R^{1/2+\delta}$ for any $\delta > 0$. Then the number of integral lattice points inside this spherical cap is given by the following formula

\begin{equation}
\sigma_{\infty} \mathcal{S}(N) \frac{Y^d}{R} + O_{\epsilon, \delta}(\frac{Y^d}{R^{1+\epsilon}}),
\end{equation}

where
where $\sigma_{\infty}$ and $\sigma(N)$ are the singular integral and series (see below), and $O$ and $\epsilon$ depend on $\delta$ not on the spherical cap or $N$. Moreover, $Y \gg R^{1/2+\delta}$ is necessary. For $d = 4$, if $Y \geq R^{2/3+\delta}$ for any $\delta > 0$ and $N$ is not divisible by $2^k$ where $2^k \gg N^\epsilon$ for any $\epsilon$, then we have the same formula.

**Remark 1.4.** $\mathcal{S}(N) = \prod_p \sigma_p(N)$ is the singular series. The singular series is nonzero if and only if $\sigma_p(N) \neq 0$ for every $p$. The condition $\sigma_p(N) \neq 0$ is equivalent to having local points over the $p$-adic integers $\mathbb{Z}_p$. If we have local points over every prime $p$ we say that all the local conditions are satisfied. For $d \geq 5$, if $\mathcal{S}(N) \neq 0$ then $\mathcal{S}(N)$ is bounded form below and above by two positive constants $c_1$ and $c_2$ where do not depend on $N$. Moreover, $Y \gg R^{1/2+\delta}$ is necessary.

**Remark 1.5.** Sarnak in his letter [Sar15a] to Aaronson and Pollington, showed that $4/3 \leq K \leq 2$ for $d = 4$. To show that $K \leq 2$ he appealed to the Ramanujan bound on the Fourier coefficients of weight $k$-modular forms, while the lower bound $4/3 \leq K$ is a consequence of an elementary number theory argument. Furthermore, Sarnak states some open problems [Sar15a, Page 24]. The first one is to show that $K < 2$ or even that $K = 4/3$. We conjecture that the optimal strong approximation holds for quadratic forms in $4$ variables which would imply that indeed $K = 4/3$.

Another application of the theorem 1.8 recovers the best known upper bound for the diameter of the Ramanujan graphs constructed by Lubotzky, Phillips and Sar- nak [LPS88] without appealing to the Ramanujan bound on the Fourier coefficients of weight two modular forms [Ele54].
Corollary 1.6. Let $q(x_1, \ldots, x_d) = \sum_{i=1}^{d} x_i^2$ where $5 \leq d$. Suppose $m$ and $N$ are given integers where $N \gg m^{4+\delta}$ for any $\delta > 0$, and assume that $(\lambda_1, \ldots, \lambda_d)$ is any integral vector where $q(\lambda_1, \ldots, \lambda_d) \equiv N \mod m$. Then we have integral solutions for

$$q(x_1, \ldots, x_d) = N \text{ with } x_i \equiv \lambda_i \mod m,$$

provided that all the local conditions are satisfied. For $d = 4$, we prove the same result when $N \gg m^{6+\delta}$ and $N$ is not divisible by $2^k$ where $2^k \gg N^\epsilon$ for any $\epsilon$. In particular, we recover the best known upper bound on the diameter of LPS Ramanujan graphs $X_{p,q}$ by choosing $N = p^n$ and $m = q$ where $q^{6+\delta} \ll p^n$ for any $\delta > 0$.

Remark 1.7. In [TS15] we demonstrated some numerical experiments on the diameter of the LPS Ramanujan graphs that the diameter is asymptotically

$$\text{diam} \approx \frac{4}{3} \log_{q-1}(n).$$

The numerical experiments for the diameter of LPS Ramanujan graphs support our conjecture for the optimal strong approximation for quadratic forms in 4 variables.

The first form of our main theorem is the following.

Theorem 1.8. Fix $F(X)$ a non-degenerate quadratic form in 5 or more variables and let $w$ be a compact subset of the affine quadric $F(X) = 1$ over the real numbers. Take a small ball $B$ of radius $0 < r < 1$ inside $\Omega$, and an integer $m$. Further assume that $N$ is a given integer which satisfies

$$N \gg (r^{-1}m)^{4+\delta},$$

for any $\delta > 0$ where the constant for $\gg$ only depends on $\Omega$ and $\delta$. Finally assume that an integral vector $(\lambda_1, \ldots, \lambda_d) \mod m$ is given. Then there exists an integral solution $F(x_1, \ldots, x_d) = N$ such that

$$x_i \equiv \lambda_i \mod m \text{ and } \frac{(x_1, \ldots, x_d)}{\sqrt{N}} \in B,$$

provided that all the local conditions are satisfied. The exponent 4 is optimal in $N \gg (r^{-1}m)^{4+\delta}$. For a quadratic form $F(X)$ in 4 variables the same result holds provided that

$$N \gg (r^{-1}m)^{6+\delta},$$

and some non-singular local conditions are required on $N$ which imply the singular series satisfies $S(N) \gg N^\epsilon$ for any $\epsilon > 0$. Note that the singular series measures the density of the local solutions. For example $\gcd(\Delta, N) = 1$ is sufficient where $\Delta$ is the discriminant of my quadratic form $F$. The exponent 6 in the theorem cannot be replaced with a number smaller than 4.

Based on our numerical experiments on the diameter of the Ramanujan graph [TS15] and the existence of the optimal lift for the case of $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/q\mathbb{Z})$ which is shown by an elementry method by Sarnak [Sar15b], we make the following conjecture.

Conjecture 1.9. We conjecture that for a quadratic form with 4 variables Theorem 1.8 holds for

$$N \gg (r^{-1}m)^{4+\delta}.$$
Therefore, the exponent 4 is the optimal exponent for every quadratic form with 4 or more variables.

In what follows, we state a stronger theorem and as a result deduce our main theorem. We give a formula for the number of integral points in an open neighborhood which is defined in adelic topology. For every prime number \( p \), there is a natural norm \( \| \cdot \|_p \) on the vector space \( \mathbb{Q}_p^d \) given by

\[
\| x - y \|_p := \max_{1 \leq i \leq n} |x_i - y_i|^p.
\]

For the archimedean place \( \infty \), we fix an arbitrary norm \( \| \cdot \|_\infty \) on \( \mathbb{R}^d \). For a place \( v \) of \( \mathbb{Q} \) and a point \( b \in \mathbb{Q}_v^d \), we define the ball \( B_v(b,r) \) centered at \( b \) and radius \( r \) as follows:

\[
B_v(b,r) := \{ x \in \mathbb{Q}_v^d : \| x - b \|_v \leq r \}.
\]

Note that for a non-archimedean place \( v \), the radius \( r \) takes discrete values \( q^n_v \), where \( q_v \) is the order of the residue field and \( n \in \mathbb{Z} \). We define the integral ring of adele as the product

\[
\mathbb{A}^d_{\mathbb{Z}} := \mathbb{R}^d \times \prod_p \mathbb{Z}_p^d,
\]

and equip it with the product metric. We define a global ball \( B \) in \( \mathbb{A}^d_{\mathbb{Z}} \) to be the product of local ones such that the radius of local balls is 1 for almost all places, i.e.

\[
B = B(b,r) \times \prod_p B(a(p),p^{-\nu_p}),
\]

such that \( \nu_p = 0 \) for almost all primes. Note that \( B(a(p),1) = \mathbb{Z}_p^d \). We define the norm of the global ball \( B \) to be

\[
|B| := r \prod_p p^{-\nu_p}.
\]

Since \( \nu_p = 0 \) for almost all primes, the norm is well defined for global balls. The following theorem is the most general form of our main theorem.

**Theorem 1.10.** Fix \( F(X) \) a non-degenerate quadratic form in \( d \geq 5 \) variables and let \( \Omega \) be a compact subset of the affine quadric \( F(X) = 1 \) over the real numbers. Consider the following compact topological space:

\[
\Theta := \Omega \times \prod_p V_N(\mathbb{Z}_p).
\]

Take a global ball \( B := B(b,r) \times \prod_p B(a(p),p^{-\nu_p}) \) inside \( \Theta \) with the following lower bound on the norm

\[
|B| \gg N^{-\frac{1}{4} + \delta},
\]

where \( b \times \prod_p a(p) \in \Theta \) and \( \gcd(a(p),p) = 1 \), i.e. \( |a(p)|_p = 1 \). Let \( |V_N(\mathbb{Z}) \cap B| \) be the number of integral points inside \( B \), then

\[
|V_N(\mathbb{Z}) \cap B| = \sigma_\infty(F,b) \prod_p \sigma_p(a(p),N)|B|^{d-1}N^{\frac{d-2}{2} + O(|B|^{d-1}N^{\frac{d-2}{2} - \epsilon})},
\]
for some $\epsilon > 0$ where $\epsilon$ and $O$ only depend on $\Omega$ and $\delta$ and not on the global ball $B$ or $N$. For a non-degenerate quadratic form $F(X)$ in 4 variables the same formula holds provided that

$$|B| \gg N^{-\frac{1}{6} + \delta}.$$  

and we require non-singular local conditions on $N$ which imply the singular series $\mathcal{G}(N) \gg N^\epsilon$ for any $\epsilon > 0$. On the other hand, fix any primitive integral vector $q := (q_1, \ldots, q_d)$, i.e. $\gcd(q_1, \ldots, q_d) = 1$ where $F(q) > 0$. Let

$$B := B(a_\infty q, r) \times \prod_p B(a_p q, p^{-\nu_p}),$$

where $a_p \in \mathbb{Z}_p$ and $a_p q \in V_N(\mathbb{Z}_p)$ if $\nu_p \neq 0$ and $a_\infty = \frac{1}{\sqrt{F(q)}}$. Note that there exists $a_p \in \mathbb{Z}_p$ such that $a_p^2 = \frac{N}{F(q)}$ if $\left(\frac{N}{p}\right) = \left(\frac{F(q)}{p}\right)$, where $\left(\frac{\ast}{\ast}\right)$ is the Legendre symbol. Hence, for the finite set of places where $\nu_p \neq 0$, we require that $\left(\frac{N}{p}\right) = \left(\frac{F(q)}{p}\right)$. Assume that

$$|B| \geq c_1 \frac{N^{-\frac{1}{4}}}{\sqrt{|q|}},$$

then there exists a global $B'$ inside $B$ with no integral points and a norm greater than

$$|B'| > c_2 \frac{N^{-\frac{1}{4}}}{\sqrt{|q|}},$$

where $c_1$ and $c_2$ are universal constants depends only on $\Omega$ and $F(X)$.

Remark 1.11. We have the following lower bounds on the product of local densities \cite[Remark 7 Page 73]{Mal62}

$$c_\epsilon N^{-\epsilon} \leq \sigma_\infty(F, b) \prod_p \sigma_p(a(p), N),$$

where $c_\epsilon$ only depends on $F(x)$ and $\epsilon$. Therefore, the error term is smaller by a factor of $N^{-\epsilon}$. If $\gcd(p^{\nu_p}, a_1(p), \ldots, a_d(p)) = p^{b_p} \neq 1$ for some $p$, we can divide $N$ by $p^{2b_p}$ and replace the local balls at the place $p$ by $B(\frac{a(p)}{p^{b_p}}, p^{-(\nu_p - b_p)})$ and use the above theorem. Since the power of $N$ is $\frac{d-2}{2}$ and the power of $|B|$ is $d-1$ in the formula \cite[11]{T1} the main term would be multiplied by $\prod_p p^{b_p}$.

1.2. Further motivations and techniques.

In several papers, Wright proved various results about the representation of a number $N$ as a sum of squares “almost proportional” to assigned positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_d$. For example in \cite{Wri33}, he improved his previous result and showed in the case of sum of five squares, if

$$\lambda_1 + \cdots + \lambda_5 = 1, \text{ where } 0 < \lambda_i, \text{ and } 0 < \epsilon,$$

then a large enough integer $N$ depending on $\epsilon$ is representable by a sum of five squares satisfying the following conditions

$$N = n_1^2 + \cdots + n_5^2,$$

$$|n_1^2 - \lambda_1 N| < U, \text{ and } N^{1-\frac{1}{4} + \epsilon} \leq U.$$
Note that the inequality $N^{1-\frac{1}{8}+\varepsilon} \leq U$ is not sharp and can be improved to $N^{1-\frac{1}{4}+\varepsilon} \leq U$. He also showed that the number of representation is

$$N^{1-\frac{1}{4}+\varepsilon} \leq U.$$

By an entirely elementary method Chowla and Auluck in [AC37] proved a sharp result for the special point $(\lambda_1, \ldots, \lambda_k) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and sum of four squares. They showed every positive integer $N \neq 0 \mod 8$ is expressible in the form

$$N = n_1^2 + \cdots + n_k^2,$$

where $n_i$ are integers satisfying

$$N^4 - n_i^2 = O(N^\frac{1}{8}).$$

It follows from Wright’s result [Wri37] that $O(N^\frac{1}{8})$ in this theorem cannot be replaced by $o(N^\frac{1}{8})$. More generally, he considered the sum of $d$ $k$th powers and proved the diagonal point $(a, \ldots, a)$ repels the integral points. We will discuss this repulsion property in more detail in section 2.

In a recent paper [Dae10], Daemen gave a lower bound for the number of the integral points $(x_1, \ldots, x_d)$ close to the diagonal point $(a, \ldots, a)$ such that

$$x_1^k + \cdots + x_s^k = N.$$

He proved that for every $k$ there exists $s_k$ such that if $s \geq s_k$, if one takes a neighbourhood bigger by a large constant than the repulsion neighbourhood of Wright [Wri37], then one has a lower bound for the number of integral points which differs by a constant compared to the upper bound. For the sum of squares, his result implies that $s_2 = 9$. He remarked that by working a little harder, one can replace the number 9 for $s_2$ by 7. Our results show that this holds with 5 or more variables.

Our method is based on the Kloosterman circle method. This method has been developed by Kloosterman to prove the local to global principle for quadratic forms in 4 variables; see [Klo22]. For the purpose of representation of integers by quadratic forms, if the number of variables is 5 or more the classical circle method of “major and minor” arcs works fine. However, for 4 variables it does not work. Kloosterman introduced a new method of dissection by Farey sequence (and no minor arcs) which deals with 4 variables. The problem of the distribution of integral points on quadrics between different residue classes to a fixed integer $m$ has been studied by Malyshev; see [Mal62]. Malyshev used Kloosterman’s method and proved a result about the distribution of integral points of the quadric $F(X) = N$ with 4 or more variables between residue class of an integer $m$. An application of our main theorem 1.8 significantly improves the exponents of $m$ and $N$ in his main theorem for $F(X)$ fixed. In our work, the method proves to be decisive in 5 or more variables as it gives the optimal exponent for lifting solutions. Our optimal result for 5 variables depends crucially on the square root cancellation in the exponential sums and also relies on the refinement of the Kloosterman method developed by Duke, Friedlander and Iwaniec that is called delta method; see [DFI93]. In the delta method we use a smooth cut off function over the Farey dissections. The delta method allows us to use the rapid decay of the Fourier transform of the weight function and makes computations handier. For proving our main theorem 1.8 we are using a version of this method developed by Heath-Brown in [HB96]. In that method we first apply
the delta method and then a Poisson summation over the sum of lattice points. We briefly describe the new method here and its advantages compared to the classical circle method.

Assume that we have a smooth bounded function \(|F(x_1, \ldots, x_d)| \leq T\) defined on a bounded region \(U \subset \mathbb{R}^d\) and it takes integral values on the integral points. We are interested in counting the integral roots of \(F\) inside \(U\). We write \(N\) for the number of integral roots weighted by a smooth weight function \(w\) which is compactly supported on \(U\). Since the function is bounded by \(T\), we have the following formula for \(N\):

\[
N = \frac{1}{2T+1} \sum_{x \in \mathbb{Z}^d} w(x) \sum_{a=0}^{T-1} e\left(\frac{aF(x)}{2T+1}\right).
\]

Following the classical circle method, we collect the fractions \(aT\) which are close to the \(\frac{r}{q}\) where \(q\) is much smaller than \(T\) in to the major arcs and estimate this sum which gives us the main term. We give an upper bound for the complement of this set which is the minor arcs. In the new circle method, the main idea is that we can expand the delta function on the integral interval \([-T, -(T-1), \ldots, T-1, T]\) with smoother functions than \(e\left(\frac{ax}{2T+1}\right)\). The dimension of the vector space of functions on this interval is \(2T+1\). If we take the following set of function

\[
\{e\left(\frac{ax}{q}\right) : \text{gcd}(a, q) = 1 \text{ and } q \leq Q := 10\sqrt{T}\}.
\]

Then this set spans the vector space of functions on the integral interval \([-T, -(T-1), \ldots, T-1, T]\). We can write the delta function as a linear combination of these functions and by an averaging argument we can assume without lost of generality that the coefficients \(c_{q,n}\) don’t depend on \(a \mod q\). Hence for \(-T \leq n \leq T\),

\[
\delta(n) = \sum_{q=1}^{Q} \sum_{a \mod q}^* c_{q,n} e\left(\frac{an}{q}\right).
\]

Note that \(q\) is now bounded by \(10\sqrt{T}\) compared to \(2T+1\) by the previous expansion, where we used \(e\left(\frac{aF(x)}{2T+1}\right)\). However, we need to control the smoothness of the coefficients \(c_{q,n}\) as a function of \(n\). Intuitively, we want to pay a higher cost for smaller value of \(q\) since it is multiplied by a smoother function \(e\left(\frac{aq}{Q}\right)\) and we want the whole function \(c_{q,n}e\left(\frac{an}{q}\right)\) be smooth. In other words, we want \(c_{k_1,n}\) be smoother than \(c_{k_2,n}\) if \(k_1 \geq k_2\). The existence of this expansion for the delta function has been established by Iwaniec and Kowalski; see [IK04 Chapter 20]. They proved that there exists an appropriate smooth function \(h(x, y)\), such that

\[
c_{q,n} = \frac{C_Q}{Q^2} h\left(\frac{q}{Q}, \frac{n}{Q^2}\right), \text{ where } Q^2 \approx T,
\]

\[
\delta(n) = \sum_{q=1}^{Q} \sum_{a \mod q}^* e\left(\frac{an}{q}\right) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right).
\]

We state the main properties of \(h(x, y)\) in Section 3.1. Using this expansion, we have the following formula for \(N\)

\[
N = \sum_{x \in \mathbb{Z}^d} \sum_{q=1}^{Q} \sum_{a \mod q}^* e\left(\frac{aF(x)}{q}\right) h\left(\frac{q}{Q}, \frac{F(x)}{Q^2}\right) w(x).
\]
We change the order of the summation and apply the Poisson summation on the sum over \( x \in \mathbb{R}^d \). We obtain

\[
N = \frac{C_Q}{Q^2} \sum_{q=1}^Q \sum_{a \mod q}^* q^{-d} \sum_{c \in \mathbb{Z}^d} S_q(a,c) I_q(c),
\]

where

\[
I_q(c) := \int_{\mathbb{R}^d} h(\frac{q}{Q} F(x) - (c,x)/q) dx,
\]

\[
S_q(a,c) := \sum_{a \mod q} e(\frac{aF(x) + (x,c)}{q}).
\]

From now on, we assume that the function \( F(x_1, \ldots, x_d) \) is a non-degenerate quadratic form. We sum the Gaussian sums \( S_q(a,c) \) over \( \sum a \mod q \) and obtain

\[
(1.18) \quad S_q(c) := \sum_{a \mod q} S_q(a,c).
\]

\( S_q(c) \) is a Kloosterman’s sum for even \( d \) or Salie’s sum for odd \( d \). In either cases, we have square root cancellation. Hence,

\[
(1.19) \quad N = \frac{C_Q}{Q^2} \sum_{q=1}^Q q^{-d} \sum_{c \in \mathbb{Z}^d} S_q(c) I_q(c).
\]

We separate the nonzero terms and write

\[
(1.20) \quad N = \frac{C_Q}{Q^2} \sum_{q=1}^Q q^{-d} S_q(0) I_q(0) + \frac{C_Q}{Q^2} \sum_{q=1}^Q q^{-d} \sum_{c \neq 0} S_q(c) I_q(c).
\]

We get the main term of the counting from the zero terms and bound the sum over the nonzero terms by putting absolute value inside the sum. We give upper bounds on

\[
(1.21) \quad \frac{C_Q}{Q^2} \sum_{q=1}^Q q^{-d} \sum_{c \neq 0} |S_q(c)||I_q(c)|.
\]

**Remark 1.12.** Note that in the sum (1.21), we are using different upper bounds by algebraic tools on \( S_q(c) \) and from analysis on \( I_q(c) \). The norm of \( c \) is not important in the sum \( S_q(c) \), it only depends on the congruence class of the integral vector \( c \mod q \). On the other hand, the fact that \( c \) is an integral vector is not important in \( I_q(c) \), the absolute value of \( I_q(c) \) depends on the position of this vector as a vector in \( \mathbb{R}^d \) and the support of the weight function \( w \).

For the integral \( I_q(c) \), we have

\[
(1.22) \quad I_q(c) = \int_{\mathbb{R}^d} h(\frac{q}{Q} F(x) - (c,x)/q) dx = \int_{\mathbb{R}^d} p_r(\xi) \int_{\mathbb{R}^d} w(x)e(\frac{F(x)}{Q^2} - \frac{(c,x)}{q}) dx,
\]

where \( p_r(\xi) := \int_{\mathbb{R}} h(r,x)e(-\xi x) dx \), is the Fourier transform of \( h(r,x) \) in variable \( x \). We consider the function \( I_q(kc) \) as a function of \( k \in \mathbb{Z} \) in the direction of the vector \( c \). We prove that, we have either an exponential decay or a polynomial decay in \( k \). If we don’t have a critical point in the oscillatory function \( \xi \frac{F(x)}{Q^2} - \frac{(c,x)}{q} \) on the
support of \( w(x) \), then we have exponential decay in \( k \) and this is the content of the Lemma 6.1. Otherwise, we have polynomial decay and this is the content of the Lemma 6.2. Following this strategy, we prove our main theorem 1.8 in section 7.

In a series of lectures and notes; see [Sar15a], Sarnak revisited the Solovay-Kitaev theorem. The general question comes from theoretical quantum computing. We briefly state the problem here and mention its relevance to our work. Assume that we have a compact group \( G \), (for example \( SL_2(\mathbb{Z}_p) \) or \( PU(2) \)). We want to find a finite set \( g := \{ s_1, s_2, \ldots, s_v \} \) such that the group \( \Gamma := \langle s_1, \ldots, s_v \rangle \) generated by \( g \) is topologically dense in \( G \) and covers \( G \) with a fast mixing rate. Sarnak defined the covering exponent \( K(g) \) in [Sar15a] and proved for example the following special case:

**Example 1.13.** Let \( G = PU(2) \) and

\[
g = \{ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2i \\ 2i & 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \}.
\]

then we have the following bound for the covering exponent

\[
(1.23) \quad \frac{4}{3} \leq K(g) \leq 2.
\]

The inequalities (1.23) is another presentation of our main theorem 1.8 for the diophantine equation \( x_1^2 + \cdots + x_2^2 = 5^h \). The lower bound \( \frac{4}{3} \leq K(g) \) comes from the repulsion of the integral points, in another words the fact that the lower bound \( r^{-4} \ll 5^h \) is optimal in our main theorem 1.8. The upper bound \( K(g) \leq 2 \) comes from the special case \( d = 4 \) in our main theorem 1.8 and the lower bound \( r^{-6} \ll 5^h \) in this case. In a letter to Scott Aaronson and Andy Pollington, Sarnak wrote [Sar15a, Page 23]:

"For \( n \geq 3 \) the determination of \( K \) (the covering exponent) for \( S \)-arithmetic points on \( S^n \) becomes easier. As pointed out to me by Bourgain, an application of the circle method as done in the Appendix [BR12], can be promoted to an asymptotic (and not just an upper bound) when one has this many variables, for counting these points in small caps. As a consequence we have together with the repulsion that for \( n \geq 3 \)

\[
(1.24) \quad K(S^n, \mathbb{Z}[\frac{1}{5}]) = 2 - \frac{2}{n}.
\]

"This was our main motivation to use the circle method which allowed us to prove our main theorem 1.8 We are very grateful for this suggestion.

1.3. **Outline of the paper.**

In section 2, we prove that the exponent 4 in Theorem 1.8 is the best possible exponent, i.e., we construct an example where \( N \gg r^{-4} \prod_p r_p^{-4} \) and for which we do not have any integral points on \( F(x) = N \) in the corresponded local neighborhood. We explain briefly the main idea here. We take a point \( a \in F(x) = N \), so that the tangent space of this point is defined by a hyperplane with integral coefficients. We consider those integral points which satisfy the mod \( m \) congruence condition \( F(a_1, \ldots, a_d) \equiv l \mod m \). Then if we slice our variety with the hyperplanes passing through these integral points and are parallel to this tangent space, the distance of those parallel hyperplanes is of order \( m \). However, if we restrict our integral point to those on the variety \( F(x) = N \), then the congruence condition \( F(a_1, \ldots, a_d) \equiv l \mod m \)
mod \( m \), impose a congruence condition \( \mod m^2 \) and as a result, the distance of those parallel hyperplanes is of order \( m^2 \). Hence, we don’t have any integral points with the given congruence condition between these slices and this gives us the repulsion that we need.

In section (3), we describe the delta method. We cite the main properties of the weight function \( h(x, y) \) in Lemma (3.4) and Lemma (3.5). These are standard lemma’s and can be find in the literature [HB96]. Lemma (3.5) is crucial in the upper bound that we give on the integral \( I_{mq}(c) \).

In section (4), we give an upper for the quadratic exponential sums \( S_{mq}(c) \). First in Lemma (4.1), we show that this exponential sum is nonzero for certain congruence class \( c \mod m \). We reduce the sum \( S_{mq}(c) \) to the Kloostermann’s sum (Salie’s sum) for even (odd) dimension. We use the well known bounds on the Kloostermann’s sum (Salie’s sum) to prove the main Lemma (4.4).

In section 5, we construct a smooth weight function \( w \) with compact support that we use in the delta method. We introduce an appropriate coordinate system for constructing \( w \). We construct \( w \) so that it satisfies the required conditions (3.9) and (3.14). Finally, we give upper bounds on the partial derivative of \( w \) that we use in section 6.

In section (6), we give an upper bound on the integral \( I_{mq}(c) \). In the subsection (6.1), we prove an upper bound on the integral \( I_{mq}(c) \) from the decay of the Fourier transform of \( h(x, y) \) given in Lemma (3.5). In subsection (6.2), we use the stationary phase theorem to give an upper bound on the integral \( I_{mq}(c) \). We improve the exponent \( 1 - \frac{n}{2} \) in [HB96] Lemma 22; Page 39 of the Heath-Brown paper to \( -\frac{n-1}{2} \). This improvement is essential in our treatment to get an optimal strong approximation result and this is the subject of Lemma (6.2). Finally, in the section (7) we prove Theorem (1.10).

1.4. Acknowledgements. I would like to thank my Ph.D. advisor Peter Sarnak for several insightful and inspiring conversations during the course of this work. In fact the starting point of this work was his letter [Sar15a] and in particular the key observation of Bourgain noted there as well as above. Furthermore, I am grateful for his suggestion to look at the diameter of LPS Ramanujan graphs and for pointing out to me the papers of Bourgain and Rudnick [BR12] and Heath-Brown [HB96] which turned out to be very useful. I would like to thanks Professor Jianya Liu for suggesting me to look at the relevant papers of Wright. I am also very thankful to Professor Sarnak and Professor Browning for their comments on the earlier versions of this work.

2. Repulsion of integral points

In this section we prove that the exponent 4 in our main theorem cannot be replaced by a smaller number. In fact we prove the second part of Theorem (1.10) which is a stronger result.

Let \( F(X) = X^TAX \) be our non-degenerate quadratic form with 4 or more variables and discriminante \( \Delta = \det(A) \). We recall that

\[
\Theta := \Omega \times \prod_p \mathcal{V}_n(Z_p).
\]
For every primitive integral vector $q$, a positive real number $0 < r < 1$ and an integer $m$ where
\[ \gcd(m, F(q)N\Delta) = 1, \]
and
\[ \left( \frac{N}{p} \right) = \left( \frac{F(q)}{p} \right) \]
for every prime $p|m$, where $\left( \cdot \right)$ is the Legendre symbol and $\left( \frac{\cdot}{\infty} \right)$ is the sign function. We define a global ball $B$ inside $\Theta$ as
\[ B := B(a_\infty q, r) \times \prod_p B(a_p q, p^{-\ord_p(m)}), \]
where $a_p \in \mathbb{Z}_p$ such that $a_p q \in V_N(\mathbb{Z}_p)$ if $p|m$ and $a_\infty = \frac{1}{\sqrt{F(q)}}$. Note that $|B| = rm^{-1}$.

**Remark 2.1.** Note that $a_p q \in V_N(\mathbb{Z}_p)$ implies that $a_p^2 F(q) = N$ in $\mathbb{Z}_p$. Hence, $a_p^2 = \frac{N}{F(q)}$. Since $\left( \frac{N}{p} \right) = \left( \frac{F(q)}{p} \right)$, we have two solutions $\pm a_p$ for $a_p^2 = \frac{N}{F(q)}$ in $\mathbb{Z}_p$ and for every choice of square root at any prime divisor of $m$ we have a different global ball $B$.

**Theorem 2.2.** Assume that
\[ |B| \geq c_1 \frac{N^{-\frac{1}{2}}}{\sqrt{|q|}}, \]
then there exists a global $B'$ inside $B$ with no integral points and a norm greater than
\[ |B'| > c_2 \frac{N^{-\frac{1}{2}}}{\sqrt{|q|}}, \]
where $c_1$ and $c_2$ depend only on $w$ and $F(X)$.

Proof: First, we consider the integral points inside $B_f$, the finite part of the global ball $B$, namely
\[ B_f := \prod_p B(a_p q, p^{-\ord_p(m)}). \]
We assume that $x$ is an integral point such that $x \in B_f$. Hence,
\[ x \equiv a_p q \mod p^{-\ord_p(m)}, \]
for every $p|m$.

By Chinese remainder theorem there exists $\alpha$ such that
\[ x \equiv \alpha q \mod m, \]
where $\alpha \equiv a_p \mod p^{-\ord_p(m)}$. Hence, $x = mt + \alpha q$ for some integral vector $t$. We write the Taylor expansion of $F(mt + \alpha q)$ at $\alpha q$, i.e.
\[ F(tm + \alpha q) = m^2 F(t) + 2mt^T A\alpha q + F(\alpha q) = N. \]
Since $\gcd(\alpha, m) = 1$, we deduce that
\[ t^T A\alpha \equiv \frac{N - F(\alpha q)}{2\alpha} \mod m. \]
Therefore, there exists a fixed number $l_0 \mod m^2$ such that
\[ \langle x, A\alpha \rangle \equiv \langle mt + \alpha q, A\alpha \rangle \mod m^2 \]
(2.1)
Now consider the infinite part of the global ball $B$, namely
\[ B_\infty := B(a_\infty q, r). \]
Our strategy is to find a shell inside this ball which avoids the integral points $V_N(\mathbb{Z})$.
Inside $B_\infty$ we let $\bar{q}$ be a real point in $V_N(\mathbb{R})$ such that
\[ \langle \bar{q}, Aq \rangle = \frac{m^2}{2} + l_0 + km^2 \text{ for some } k \in \mathbb{Z}, \]
and $|\bar{q} - a_\infty q|$ is minimal.
Note that $\bar{q}$ lies on the nearest parallel hyperplanes to $a_\infty q$ given by
\[ \langle x, Aq \rangle = \frac{m^2}{2} + l_0 + km^2, \]
when we vary $k \in \mathbb{Z}$. Assume that $x = \bar{q} + \xi$ is an integral point inside the global ball $B$ with $F(x) = N$. We write the Taylor expansion of of $F(x)$ at $\bar{q}$
\[ F(\bar{q} + \xi) = F(\bar{q}) + 2\bar{q}^T A\xi + F(\xi). \]
Since $F(x) = F(\bar{q}) = N$, we deduce that
\[ 2\bar{q}^T A\xi + F(\xi) = 0. \]
We give a lower bound on $|2\bar{q}^T A\xi| = \langle 2\xi, A\bar{q} \rangle$. Note that
\[ |\langle 2\xi, A\bar{q} \rangle| \geq |\langle 2\xi, a_\infty Aq \rangle| - |\langle 2\xi, A(\bar{q} - a_\infty q) \rangle|. \]
By the definition of $\bar{q}$ given in (2.2), we have $\langle 2\xi, A\bar{q} \rangle \geq m^2$ and also use the Cauchy inequality to get $|\langle 2\xi, A(\bar{q} - a_\infty q) \rangle| \leq 2|\xi||A||(\bar{q} - a_\infty q)|$. Therefore,
\[ |2\bar{q}^T A\xi| \geq m^2a_\infty - 2|\xi||A||(\bar{q} - a_\infty q)|. \]
Since $2\bar{q}^T A\xi + F(\xi) = 0$ then
\[ |F(\xi)| \geq m^2a_\infty - 2|\xi||A||(\bar{q} - a_\infty q)|. \]
By the minimality condition in (2.3), we give an upper bound on $|\bar{q} - a_\infty q|$. We have
\[ F(\bar{q}) = N, \]
\[ [(\bar{q} - a_\infty q) + \bar{q}]^T A[(\bar{q} - a_\infty q) + a_\infty q] = N. \]
Hence
\[ (\bar{q} - a_\infty q)^T A(\bar{q} - a_\infty q) + 2(\bar{q} - a_\infty q)^T Aa_\infty q = 0. \]
From the minimality condition, one can check that $\bar{q} - a_\infty q$ is the maximal eigenvector for $A$, hence
\[ (\bar{q} - a_\infty q)^T A(\bar{q} - a_\infty q) = |A||(\bar{q} - a_\infty q)|^2, \]
and moreover
\[ |2(\bar{q} - a_\infty q)^T Aa_\infty q| \leq 2a_\infty m^2. \]
Hence,
\[ |A||(\bar{q} - a_\infty q)|^2 \leq 2a_\infty m^2. \]
Therefore, (2.4)
\[ |A||(\bar{q} - a_\infty q)| \leq ma_\infty^{1/2} \sqrt{2|A|}. \]
We plug in this upper bound in the inequality (2.3), and obtain
\[ |F(\xi)| \geq m^2a_\infty - 2|\xi|ma_\infty^{1/2} \sqrt{2|A|}. \]
Since $|F(\xi)| \leq |\xi|^2|A|$, we deduce that

$$|\xi|^2 \geq \frac{m_a}{|A|} - \frac{2\sqrt{2}|ma|}{\sqrt{|A|}}. \tag{2.5}$$

From the above inequality, it is easy to check that

$$|\xi| > \frac{ma^{1/2}}{4\sqrt{|A|}}. \tag{2.6}$$

Since $\sqrt{N} = a^\infty |F(q)|^{1/2}$, there exists a constant $c_2$ depending on $A$ such that

$$\frac{|\xi|}{\sqrt{N}} \geq c_2 a^\infty |F(q)|^{1/2}.$$ 

We deduce that we don’t have any integral point inside the following global ball

$$B' := B\left(\frac{\hat{q}}{\sqrt{N}}, \frac{|\xi|}{\sqrt{N}}\right) \prod_p B\left(a_p q, p^{-\ord_p(m)}\right). \tag{2.7}$$

We also note that

$$|B'| = \frac{|\xi|}{\sqrt{N}} m^{-1} \geq c_2 a^\infty |F(q)|^{-1/2} \geq c_2 N^{-1/4} |F(q)|^{-1/4} \geq \frac{c_2 N^{-1/4}}{|A|\sqrt{|q|}}.$$

Therefore, we conclude the second part of Theorem (1.1). □

3. THE DELTA METHOD

In this section, we give a formula for the weighted number of the integral points in a small neighborhood of a point $b$ on the quadric $F(X) = N$ subjected to some congruence conditions. We use a smooth weight function $w$ which is supported in a neighborhood of the point $b$ with radius $r\sqrt{N}$. We also impose some congruence conditions mod $m$ on the integral points. We use the additive character $e\left(\frac{2\pi i}{m}\right)$ to cut these congruence conditions. We begin with the following expansion of the delta function which is developed by Duke, Friedlander and Iwaniec. We cite this theorem from [HB96, Theorem 1].

**Theorem 3.1.** For any integer $n$ let

$$\delta_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Then for any $Q > 1$ there is a positive constant $C_Q$, and an infinitely differentiable function $h(x, y)$ defined on the set $(0, \infty) \times \mathbb{R}$, such that

$$\delta_n = C_Q Q^{-2} \sum_{q=1}^{\infty} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} e\left(\frac{an}{q}\right) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right). \tag{3.2}$$

The constant $C_Q$ satisfies

$$C_Q = 1 + O_N(Q^{-N}). \tag{3.3}$$
for any $N > 0$. Moreover $h(x, y) \ll x^{-1}$ for all $y$, and $h(x, y)$ is non-zero only for $x \leq \max(1, 2|y|)$.

We assume that $F(X) = X^TAX$ is a non-degenerate quadratic form and we impose the following congruence conditions mod $m$ on the integral points $X = (x_1, \ldots, x_d)$ such that $F(X) = N$:

\begin{equation}
\label{eq:3.4}
x_i \equiv \lambda_i \mod m,
\end{equation}

for $1 \leq i \leq d$. We let

\begin{equation}
\label{eq:3.5}
x_i = t_i m + \lambda_i,
\end{equation}

where $-\frac{m-1}{2} \leq \lambda_i \leq \frac{m-1}{2}$. Then,

\begin{align*}
F(t_1 m + \lambda_1, \ldots, t_d m + \lambda_d) &= N, \\
m^2 F(t_1, \ldots, t_d) + 2m \lambda^T A t = N - F(\lambda).
\end{align*}

We define

\begin{equation}
\label{eq:3.6}
k_m := N - F(\lambda).
\end{equation}

Then,

\begin{equation}
\label{eq:3.7}
F(t_1, \ldots, t_d) + \frac{1}{m} (2\lambda^T A t - k) = 0.
\end{equation}

We also define

\begin{equation}
\label{eq:3.8}
G(t_1, \ldots, t_d) := \frac{F(mt + \lambda) - \frac{N}{m^2}}{m^2} = F(t_1, \ldots, t_d) + \frac{1}{m} (2\lambda^T A t - k).
\end{equation}

We let $w$ be a smooth compactly supported weight function such that

\begin{equation}
\label{eq:3.9}
w(x) = 0 \text{ if } |x - b| \geq r\sqrt{N}.
\end{equation}

We write $N(w, \lambda)$ for the weighted number of the integral points on the quadric $F(X) = N$ subjected to the congruence condition \eqref{eq:3.4}. We have the following formal expansions for $N(w, \lambda)$

\begin{equation}
\label{eq:3.10}
N(w, \lambda) = \sum_{t \in \mathbb{Z}^d} w(t) \delta(G(t)),
\end{equation}

\begin{equation}
\label{eq:3.11}
N(w, \lambda) = \frac{1}{m} \sum_{l \mod m} \sum_{t \in \mathbb{Z}^d} e\left(\frac{1}{m} (2\lambda^T A t - k)\right) w(t) \delta(G(t_1, \ldots, t_d)).
\end{equation}

**Remark 3.2.** Note that the summation on $l \mod m$ is non-zero only if

\begin{equation}
\label{eq:3.12}
2\lambda^T A t - k \equiv 0 \mod m.
\end{equation}

Hence, if $G(t_1, \ldots, t_d)$ has non-integral values then this summation is zero. For the integral values of $G(t)$, we use the delta method \eqref{eq:3.9} to cut out the non-zero integral values of $G(t)$.

From the assumption of the theorem \ref{thm:1.8}, we have $N = (mr^{-1})^{4+2\delta}$ for some $\delta > 0$. We take $Q = (mr^{-1})^{1+\delta}$ in the delta method \eqref{eq:3.10}. Therefore, from Remark \ref{rem:3.2}, we have the following expression for $N(w, \lambda)$
\[ N(w, \lambda) = \frac{CQ}{mQ^2} \sum_{l} \sum_{q} \sum_{a}^{*} \sum_{t} e^{\left( (lq + a)(2\lambda^T At - k) + amF(t) \right)} \]
\[ h\left( \frac{q}{Q} \frac{F(mt + \lambda) - \mu}{m^2Q^2} \right) w(t). \]

We apply the Poisson summation formula on the \( t \) variable to obtain
\[ N(w, \lambda) = \frac{CQ}{mQ^2} \sum_{l} \sum_{q} \sum_{a}^{*} \sum_{c} (mq)^{-d} S_{mq}(a, l, c) I_{mq}(c), \]
where
\[ I_{mq}(c) = \int h\left( \frac{q}{Q} \frac{F(t_1, \ldots, t_d) - \mu}{m^2Q^2} \right) w(t) e^{-\left( \frac{(c, t)}{mq} \right)} dt_1 \ldots dt_d, \]
\[ S_{mq}(a, l, c) = \sum_{b \in \mathbb{Z}^d} e^{\left( (lq + a)(2\lambda^T At - k) + amF(b) + (c, b) \right)}. \]

We define the sum \( S_{mq}(c) \) as:
\[ S_{mq}(c) := \sum_{l \mod m} \sum_{a \mod q}^{*} S_{mq}(a, l, c). \]

Therefore
\[ N(w, \lambda) = \frac{CQ}{mQ^2} \sum_{q} \sum_{c \in \mathbb{Z}^d} (mq)^{-d} S_{mq}(c) I_{mq}(c). \]

By the assumptions, \( Q = (r^{-1}m)^{1+\delta} \). In the section [4], we construct a smooth function \( w \) with compact support such that
\[ w(t) \neq 0 \] only if \( \frac{G(t_1, \ldots, t_d)}{Q^2} \ll 1. \]

Since the support of the smooth function \( h(x, y) \) is inside the interval \( 0 \leq x \leq \max(2y, 1) \). Hence, the summation on \( q \) in formula [3.13] is restricted to \( 1 \leq q \ll Q \).

3.1. The properties of the smooth function \( h(x, y) \). In this section we cite the basic properties of the smooth function \( h(x, y) \) from [HB96]. The first lemma gives an upper bound on the \( L^1 \) norm of \( x^k y^l \partial^k h(x, y) \partial x^k \), as a function of \( y \).

**Lemma 3.3.** We have
\[ \int \left| x^k y^l \partial^k h(x, y) \partial x^k \right| dy \ll x^l, \]
for any \( l \geq 0 \) and \( k \in \{0, 1\} \).

**Proof:** This is an easy consequence of [HB96, Lemma 5].

The following lemma shows \( h(x, y) \) converges to the delta function rapidly as \( x \to 0 \). For the proof we refer the reader to [HB96, Lemma 9].
Lemma 3.4. Let $f$ be a smooth function with compact support. Then if $x \ll 1$ we have

$$\int f(y)h(x, y)dy = f(0) + O_M(x^M).$$

The following lemma shows the smooth property of the $h(x, y)$ in the $y$ variable. Since, it converges to the delta function from the previous lemma, as $x \to 0$ the decay rate of the Fourier transform is slower as $x \to 0$. For the proof, we refer the reader to [HB96, Lemma 17].

Lemma 3.5. Let $w$ be a smooth compactly supported weight function and $p(t, x)$ be the Fourier transform of $w(y)h(x, y)$ in the $y$ variable, i.e.,

$$p(t, x) = \int_{-\infty}^{\infty} w(y)h(x, y)e(-ty)dy.$$

Then, $p(t, x)$ decays faster than any polynomial in variable $xt$, i.e. for every $N \geq 0$

$$p(t) \ll_{w, N} (xt)^{-N}.$$

4. Quadratic Exponential Sums $S_{mq}(c)$

In this section, we give upper bounds for the quadratic exponential sums $S_{mq}(c)$. We show that this exponential sum is nonzero for certain congruence class $c \mod m$. By Chinese remainder theorem, we write the sum $S_{mq}(c)$ as a product of two summation:

$$S_{mq}(c) = S_1S_2.$$

In Lemma 4.2, we give an upper bound on $S_1$ from the generalized Kloostermann’s sum (Salie’s sum) for even (odd) dimension. In Lemma 4.3, by applying a Cauchy inequality and next computing the square norm, we obtain an upper bound on $S_2$. Finally, we show in Lemma 4.4 that we have square root cancellation on average for the exponential sum $S_{mq}(c)$. In the following lemma we show that $S_{mq}(a, l, c) = 0$ unless

$$c_i \equiv -2(lq + a)\lambda_i \mod m,$$

for every $1 \leq i \leq d$.

Lemma 4.1. We let $F(X) = X^TAX$ be a non degenerate quadratic form and

$$S_{mq}(a, l, c) = \sum_{b \in (\mathbb{Z}/mq\mathbb{Z})^d} e\left(\frac{(lq + a)(2\lambda^TAb - k) + amF(b) + \langle c, b \rangle}{mq}\right).$$

Then

$$S_{mq}(a, l, c) = 0,$$

unless $c \equiv -2(lq + a)\lambda \mod m$. As a result $S_{mq}(c) = 0$, unless

$$c \equiv \alpha A\lambda \mod m,$$

where $\lambda = (\lambda_1, \ldots, \lambda_d) \mod m$.

Proof: We write vector $b$ in the summation (4.1) as

$$b = qb_1 + b_2,$$
where \( b_1 \) is a vector mod \( m \) and \( b_2 \) is a vector mod \( q \). Then we write \( S_{mq}(a,l,c) \) as a summation over \( b_1 \) and \( b_2 \). We obtain

\[
S_{mq}(a,l,c) = \sum_{b_2} e\left( \frac{(lq + a)(2\lambda^T A b_2 - k) + amF(b_2) + \langle c, b_2 \rangle}{mq} \right) \sum_{b_1} e\left( \frac{(lq + a)2\lambda^T A b_1 + \langle c, b_1 \rangle}{m} \right).
\]

It is easy to check that the summation on \( b_1 \) is zero unless

\[
c \equiv -2(lq + a)A\lambda \mod m.
\]

Since

\[
S_{mq}(c) := \sum_{l \mod m} \sum_{a \mod q} S_{mq}(a,l,c),
\]

if \( S_{mq}(c) \neq 0 \), then for some \( a \mod q \) and \( l \mod m \), \( S_{mq}(a,l,c) \neq 0 \). Therefore,

\[
c \equiv -2(lq + a)A\lambda \mod m.
\]

Hence, we conclude the lemma \( \square \).

In the rest of this section, we give an upper bound on \( S_{mq}(c) \) when this summation is nonzero. We let \( \Delta := \det A \) where \( A \) is the symmetric matrix associated to our non-degenerate quadratic form \( F(X) = X^T AX \). We note that

\[
S_{mq}(c) = \sum_{l \mod m} \sum_{a \mod q} S_{mq}(a,l,c).
\]

Since the summation over \( l \) is nonzero only if \( m|2\lambda^T A b - k \) and it is \( m \) when \( m|2\lambda^T A b - k \), then we have

\[
S_{mq}(c) = m \sum_{b,a} e\left( \frac{a(2\lambda^T A b - k) + amF(b) + \langle c, b \rangle}{mq} \right).
\]

where the summation is over \( a \in (\mathbb{Z}/q\mathbb{Z})^* \) and vectors \( b \in (\mathbb{Z}/mq\mathbb{Z})^d \) such that

\[
m|2\lambda^T A b - k.
\]

We assume that

\[
q = q_1 q_2,
\]

where \( \gcd(q_1, 2\Delta m) = 1 \) and the set of primes which divide \( q_2 \) is a subset of prime divisors of \( \Delta m \). As a result \( \gcd(q_1, mq_2) = 1 \) and by Chinese remainder theorem we can write

\[
k = mq_2 k_1 + q_1 k_2,
\]

for some integers \( k_1 \) and \( k_2 \), and also

\[
b = mq_2 b_1 + q_1 b_2,
\]

where \( b_1 \in (\mathbb{Z}/q_1\mathbb{Z})^d \) and \( b_2 \in (\mathbb{Z}/mq_2\mathbb{Z})^d \) such that \( m|2\lambda^T A b - k \). We can also write

\[
a = q_2 a_1 + q_1 a_2,
\]
where $a_1 \in (\mathbb{Z}/q_1 \mathbb{Z})^*$ and $a_2 \in (\mathbb{Z}/q_2 \mathbb{Z})^*$. We substitute these values of $k = mq_2 k_1 + q_1 k_2$, $a = q_2 a_1 + q_1 a_2$ and $b = mq_2 b_1 + q_1 b_2$ in the formula (4.3) for $S_{mq}(c)$ to obtain

$$S_{mq}(c) = \sum_{a_1, b_1} e\left(\frac{2q_2 a_1 \lambda^T A b_1 + a_1 (mq_2)^2 F(b_1) + \langle c, b_1 \rangle - q_2 a_1 k_1}{q_1}\right) \times \sum_{a_2, b_2} e\left(\frac{2q_1 a_2 \lambda^T A b_2 + a_2 mq_2^2 F(b_2) + \langle c, b_2 \rangle - q_1 a_2 k_2}{mq_2}\right),$$

(4.7)

where $m|2\lambda^T A b_2 - k_2$. We define

$$S_1 := \sum_{a_1, b_1} e\left(\frac{2q_2 a_1 \lambda^T A b_1 + a_1 (mq_2)^2 F(b_1) + \langle c, b_1 \rangle - q_2 a_1 k_1}{q_1}\right),$$

(4.8)

and

$$S_2 := m \sum_{a_2, b_2} e\left(\frac{2q_1 a_2 \lambda^T A b_2 + a_2 mq_2^2 F(b_2) + \langle c, b_2 \rangle - q_1 a_2 k_2}{mq_2}\right),$$

(4.9)

where $m|2\lambda^T A b_2 - k_2$. In the following lemma we give upper bounds for $S_1$.

**Lemma 4.2.** Let $G(X) := X^T B X$ be a non-degenerate quadratic form ($\Delta := \det(B) \neq 0$). Assume that $q$ is an odd integer such that $\gcd(q, 2\Delta) = 1$ and let

$$S(G, c, c', t) := \sum_{a, b} e\left(\frac{a(G(b) + \langle c', b \rangle + t) + \langle c, b \rangle}{q}\right),$$

(4.10)

where $a$ varies in units $(\mathbb{Z}/q \mathbb{Z})^*$ and $b$ varies in $(\mathbb{Z}/q \mathbb{Z})^d$. Then

$$S(G, c, c', t) = \left(\frac{\Delta}{q}\right) \tau_q^d Kl(G, c, c', t),$$

(4.11)

where $\tau_q := \sum_{a} e\left(\frac{a^2}{q}\right)$ is the Gauss sum, $\left(\frac{\cdot}{q}\right)$ is the Jacobi symbol and $Kl(G, c, c', t)$ is either a Kloosterman’s sum or a Salie's sum. As a result, we obtain the following upper bound on $S_1$ that is defined in equation (4.7)

$$S_1 \leq q_1^{d+1} \tau(q_1) \gcd(q_1, N)^{1/2}.$$

(4.12)

Proof: Since $q$ is odd, we can diagonalize our quadratic form $G(x) \mod q$. Therefore, without loss of generality we assume that our quadratic form is a diagonal quadratic form

$$G(x) = \sum_{i=1}^{d} \alpha_i x_i^2.$$

Hence,

$$S(G, c, c', t) := \sum_{a} e\left(\frac{a t}{q}\right) \prod_{j=1}^{d} \sum_{b \in \mathbb{Z}/q \mathbb{Z}} e\left(\frac{a(\alpha_j b^2 + c'_j b) + c_j b}{q}\right).$$
We complete the square to obtain
\[ S(G, c, c', t) := \sum_a e \left( \frac{at}{q} \right) \prod_{j=1}^d \sum_{b \in \mathbb{Z}/q\mathbb{Z}} e \left( \frac{a \alpha_j (b + \frac{ac_j' + c_j}{2\alpha_j})^2}{4\alpha_j} - \frac{(ac_j' + c_j)^2}{4\alpha_j} \right) \]
\[ = \sum_a e \left( \frac{at}{q} \right) \prod_{j=1}^d e \left( \frac{(ac_j' + c_j)^2}{4\alpha_j} \right) \sum_{b \in \mathbb{Z}/q\mathbb{Z}} e \left( \frac{a \alpha_j (b + \frac{ac_j' + c_j}{2\alpha_j})^2}{q} \right). \]

We note that
\[ \sum_{b \in \mathbb{Z}/q\mathbb{Z}} e \left( \frac{a \alpha_j (b + \frac{ac_j' + c_j}{2\alpha_j})^2}{q} \right) = \left( \frac{a \alpha_j}{q} \right) \tau_q, \]
where \( \tau_q := \sum_x e \left( \frac{x^2}{q} \right) \) is quadratic Gauss sum and \( \left( \frac{a \alpha_j}{q} \right) \) is Jacobi symbol. Hence,
\[ S(G, c, c', t) = \tau_q^d \left( \frac{\Delta}{q} \right) e \left( -\sum_j \frac{c_j c_j'}{2\alpha_j} \right) \sum_a \left( \frac{a}{q} \right) \prod_j e \left( \frac{a (t - \sum_j \frac{c_j^2}{4\alpha_j}) - a^{-1} \sum_j \left( \frac{c_j^2}{4\alpha_j} \right)}{q} \right). \]

Next, we analyze the sum over \( a \). We define
\[ Kl(G, c, c', t) = e \left( -\sum_j \frac{c_j c_j'}{2\alpha_j} \right) \sum_a \left( \frac{a}{q} \right) \prod_j e \left( \frac{a (t - \sum_j \frac{c_j^2}{4\alpha_j}) - a^{-1} \sum_j \left( \frac{c_j^2}{4\alpha_j} \right)}{q} \right). \]

Note that \( \left( \frac{a}{q} \right)^d = \left( \frac{a}{q} \right) \) for odd \( d \). So, \( Kl(G, c, c', t) \) is a Salie’s sum for odd \( d \) and from standard bounds on Salie’s sum
\[ (4.13) \quad |Kl(G, c, c', t)| \leq \tau(q) \sqrt{q}. \]

Similarly, \( \left( \frac{a}{q} \right)^d = 1 \) for even \( d \). So, \( Kl(G, c, c', t) \) is a Kloosterman’s sum for even \( d \) and by Weil’s bound on Kloosterman’s sum we obtain
\[ (4.14) \quad Kl(G, c, c', t) \leq \tau(q) \sqrt{q} \gcd \left( q, \left( t - \sum_j \frac{c_j^2}{4\alpha_j}, \sum_j \left( \frac{c_j^2}{4\alpha_j} \right) \right)^{1/2}. \right. \]

This concludes the first part of our lemma. Finally, we analyze \( S_1 \). Recall that
\[ S_1 := \sum_{a_1, b_1} e \left( \frac{2q_2 a_1 X^T A b_1 + a_1 (mq_2)^2 F(b_1) + \langle c, b_1 \rangle - q_2 a_1 k_1}{q_1} \right). \]

We note that by a change of variables \( S_1 = S(G, c, c', t) \) where
\[ q = q_1, \]
\[ G = (mq_2)^2 F, \]
\[ t = -q_2 k_1, \]
\[ c' = 2q_2 A \lambda, \]
\[ c = c. \]

Recall that \( F(X) = X^T A X, \gcd(q_1, 2mq_3 \Delta) = 1 \) and \( G = (mq_2)^2 F \) is diagonalizable with eigenvalues \( \{\alpha_i\} \) then
\[ \sum_j \frac{c_j^2}{4\alpha_j} \equiv (2mq_2)^{-2} c'^T A^{-1} c' \mod q_1. \]
We apply formula (4.11) and obtain

\[ \sum_j \frac{c_j^2}{4\alpha_j} = \frac{\lambda^TA\lambda}{m^2} = \frac{F(\lambda)}{m^2} \mod q_1. \]

We substitute (4.15) and obtain

\[ |S_1| = q_1^{d/2}|K\ell(G,c,c',t)|. \]

If \( d \) is odd then by inequality (4.13) we obtain

\[ |S_1| \leq q_1^{d+1} \tau(q_1). \]

If \( d \) is even then by inequality (4.14) we obtain

\[ |S_1| \leq q_1^{d+1} \tau(q_1) \gcd(q_1, (t - \sum_j \frac{c_j^2}{4\alpha_j}))^{1/2}. \]

We substitute \( t = -q_2k_1 \) and \( \sum_j \frac{c_j^2}{4\alpha_j} = \frac{F(\lambda)}{m^2} \mod q_1 \) by congruence relation (4.15) and obtain

\[ |S_1| \leq q_1^{d+1} \tau(q_1) \gcd(q_1, (q_2k_1 + \frac{F(\lambda)}{m^2}))^{1/2}. \]

Since \( \gcd(m,q_1) = 1 \) then \( \gcd(q_1, (q_2k_1 + \frac{F(\lambda)}{m^2})) = \gcd(q_1, m^2q_2k_1 + F(\lambda)) \). Recall equation (3.5)

\[ k = mq_2k_1 + q_1k_2. \]

This implies

\[ m^2q_2k_1 + F(\lambda) \equiv km + F(\lambda) \mod q_1. \]

Recall equation (4.7)

\[ km = N - F(\lambda). \]

Hence,

\[ \gcd(q_1, (q_2k_1 + \frac{F(\lambda)}{m^2})) = \gcd(q_1, N), \]

\[ S_1 \leq q_1^{d+1} \tau(q_1) \gcd(q_1, N)^{1/2}. \]

This concludes our lemma. □

In this lemma we give an upper bound on \( S_2 \).

**Lemma 4.3.** Recall from equation (4.7)

\[ S_2 := m \sum_{a_2,b_2} e\left(\frac{2a_1a_2\lambda^TAb_2 + a_2mq_2^2F(b_2) + \langle c,b_2 \rangle - q_1a_2k_2}{mq_2}\right), \]

where \( m|2\lambda^TAb_2 - k_2 \) and \( b_2 \in (\mathbb{Z}/mq_2\mathbb{Z})^d \). Then,

\[ (4.16) \quad S_2 \ll_\Delta m^{d}q_2^{1+d/2}. \]

**Proof:** From the Cauchy inequality on \( a_2 \) variable we have

\[ |S_2|^2 \leq m^2\phi(q_2) \sum_{a_2} \left| \sum_{b_2} e\left(\frac{2a_1a_2\lambda^TAb_2 + a_2mq_2^2F(b_2) + \langle c,b_2 \rangle - q_1a_2k_2}{mq_2}\right)\right|^2 \]

\[ \leq m^2\phi(q_2) \sum_{a_2} \sum_{b_2,b_2'} e\left(\frac{2a_1a_2\lambda^TAb_2 - b_2' + a_2mq_2^2(F(b_2) - F(b_2')) + \langle c,b_2 - b_2' \rangle}{mq_2}\right). \]
where \( m | 2 \lambda^T A b_2 - k_2 \) and \( m | 2 \lambda^T A b'_2 - k_2 \). We change the variables and write \( u = b_2 - b'_2 \). Hence, 

\[
|S_2|^2 \leq m^2 \phi(q_2) \sum_{a_2} \sum_{u, b_2} \left( 2 q_2 a_2 \lambda^T A u + a_2 m q_2^2 (2 b_2 A u + F(u)) + \langle c, u \rangle \right) \frac{2}{mq_2}.
\]

where \( m | 2 \lambda^T A b_2 - k_2 \) and \( m | 2 \lambda^T A u \). The summation over \( b_2 \) is zero unless \( q_2 | \Delta \gcd(u) \).

In other words, the summation is non-zero only if

\[
u \in \left( \frac{q_2 \mathbb{Z}}{\gcd(\Delta, q_2) m \mathbb{Z}} \right)^d \equiv \left( \mathbb{Z}/ \gcd(\Delta, q_2) m \mathbb{Z} \right)^d.
\]

Since \( b_2 \in \left( \mathbb{Z}/mq_2 \mathbb{Z} \right)^d \), then

\[
|S_2|^2 \leq m^2 \phi(q_2) \sum_{a_2} \sum_{u, b_2} \left( 2 q_2 a_2 \lambda^T A u + a_2 m q_2^2 (2 b_2 A u + F(u)) + \langle c, u \rangle \right) \frac{2}{mq_2}.
\]

Hence,

\[
|S_2| \ll \Delta m^d q_2^{1 + \frac{d}{2}}.
\]

This concludes the lemma. □

In the last lemma of this section we prove an upper bound on the average norm of the exponential sums \( S_{qm}(c) \) with respect to \( q \).

**Lemma 4.4.** We have the following upper bound

\[
\sum_{q=1}^{X} m^{-d} q^{-\frac{d-1}{4}} |S_{qm}(c)| \ll_{\Delta} m^r X^{1+\epsilon},
\]

where \( X = O(N^A) \) for fixed power \( A \).

Proof: This is a consequence of Lemma 4.2 and Lemma 4.3. We factor

\[
q = q_1 q_2,
\]

where \( \gcd(q_1, 2m\Delta) = 1 \) and prime factors of \( q_2 \) are subset of prime divisors of \( m\Delta \).

From the formula (4.17), we deduce that

\[
S_{mq}(c) = \left| S_1 \right| \left| S_2 \right|.
\]

By Lemma 4.2

\[
S_1 \leq q_1^{d+1+\epsilon} \gcd(q_1, N)^{1/2}.
\]

Lemma 4.3 implies

\[
S_2 \ll \Delta m^d q_2^{1+d/2}.
\]

Therefore, we have

\[
\sum_{q=1}^{X} m^{-d} q^{-\frac{d-1}{4}} |S_{qm}(c)| \ll \sum_{q=1}^{X} q_1^{1/2} q_2^{1/2} \gcd(q_1, N)^{1/2} \leq X^\epsilon \sum_{q=1}^{X} q_2^{1/2} \gcd(q_1, N)^{1/2}.
\]
Note that \( q = q_1q_2 \) and the prime factors of \( q_2 \) are subsets of prime divisors of \( m\Delta \), hence
\[
\sum_{q=1}^{X} q_2^{1/2} \gcd(q_1, N)^{1/2} \leq m^{\epsilon}X^{1+\epsilon}.
\]

Therefore, we conclude the lemma.

5. Construction of smooth weight function \( w \)

In this section, we construct a smooth weight function \( w \) with compact support that we use in the delta method. Our goal is to count the number of integral points on the quadric \( F(X) = N \) weighted by a smooth weight function \( w \) with support inside a small box of size \( r\sqrt{N} \). The definition of the smooth weight function \( w \) depends only on \( b, r \) and fixed real valued smooth functions \( \psi_1 \) and \( \psi_2 \) with compact support defined over \( \mathbb{R} \) and \( \mathbb{R}^{d-1} \), respectively. We introduce an appropriate coordinate system for constructing \( w \). In Lemma (5.2), we show that \( w \) satisfies the required conditions (3.6) and (3.14). Moreover, we give upper bounds on the partial derivative of \( w \) that we use in the next section.

First, we introduce the coordinate system that we use to construct \( w \). Recall that \( F(x_1, \ldots, x_d) = X^TAX \) is a non-degenerate integral quadratic form and \( b/\sqrt{N} \in \Omega \) where \( \Omega \) is a fixed compact subset of quadric \( F(X) = 1 \). We change our standard basis into a new basis given by an orthonormal basis of the \( d-1 \) dimensional tangent space at \( b \) on the hyper-surface \( F(X) = N \) and the normal vector at \( b \). The normal vector is in the direction of the gradient of \( F(x_1, \ldots, x_d) = X^TAX \) at point \( b \):
\[
\nabla F|_b = 2Ab.
\]

Let \( e_d \) be the norm 1 vector in the direction of \( \nabla F|_b \). The tangent space of the level set \( F(X) = N \) at \( b \) is the orthogonal complement of \( e_d \) and we denote it by
\[
T^b(F) := e_d^\perp.
\]

We restrict the quadratic from \( F(X) \) to the \( d-1 \) dimensional subspace \( T^b(F) \). By a standard theorem in Linear algebra we can find an orthogonal basis
\[
(5.1) \quad B_1 := \{e_1, \ldots, e_{d-1}\}.
\]

for \( T^b(F) \) such that \( F \) is diagonal in this basis over \( \mathbb{R} \). Namely,
\[
F\left( \sum_{i=1}^{d-1} u_i e_i \right) = \sum_{i=1}^{d-1} \lambda_i u_i^2.
\]

Next, we construct the smooth weight function \( w(X) \) with compact support that satisfies the following required conditions (3.6) and (3.14) on \( w \):
\[
w(X) = 0 \text{ if } |X - b| \gg r\sqrt{N},
\]
\[
w(X) = 0 \text{ if } \frac{F(X) - N}{m^2Q^2} \gg 1.
\]

Recall our notations
\[
Q = (mr^{-1})^{1+\delta},
\]
\[
\sqrt{N} = Qmr^{-1}.
\]
By these conditions, the smooth weight function \( w \) is supported inside an open box given by the fattening of the tangent space of the quadric \( F(X) = N \) at \( b \). More precisely, let

\[
y(X) := \frac{F(X) - N}{m^2Q^2}.
\]

Since \( F(b) = b^T Ab = N \neq 0 \) and \( e_i^T Ab = 0 \) for \( 1 \leq i \leq d - 1 \), then

(5.2) \( B_2 := \{ e_1, \ldots, e_{d-1}, b \} \),

is also a basis for \( \mathbb{R}^d \). Given \( X \in \mathbb{R}^d \), we express the vector \( X - b \) in two bases \( B_1 \cup \{ e_d \} \) and \( B_2 \) as follows:

\[
X - b = \sum_{i=1}^{d-1} m\tilde{u}_i e_i + m\alpha e_d,
\]

(5.3) \( X - b = \sum_{i=1}^{d-1} m\tilde{u}_i e_i + m\beta b \).

If \( b = \sum_{i=1}^d b_i e_i \) then

\[
\begin{align*}
\tilde{u}_i &= u_i + \beta b_i, \\
\alpha &= \beta b_d.
\end{align*}
\]

Let \( \psi_1 \) and \( \psi_2 \) be real value compactly supported smooth functions defined over \( \mathbb{R} \) and \( \mathbb{R}^{d-1} \), respectively. We define the smooth weight function \( w(X) \) as follows:

(5.4) \[
w(X) = \begin{cases} 
\frac{X^T Ab}{R(b,e_d)} \psi_1(y)^2 \psi_2(\tilde{u}/Q) & \text{if } 1 + m\beta > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

In Lemma (5.2), we check that this smooth weight function satisfies the required conditions (3.6) and (3.14).

Remark 5.1. The factor \( \frac{X^T Ab}{R(b,e_d)} \) is in our weight function \( w \), in order to simplify the oscillatory integral \( I_{mq}(c) \) in Lemma (6.2). Note that the support of weight function \( \psi_1(y)^2 \psi_2(\tilde{u}/Q) \) is localized around \( b \) and \(-b\). We define \( w \) such that it only has a support around \( b \).

In the following lemma we give upper bounds on the partial derivatives of \( w \). We use these bounds in Lemma (6.1).

Lemma 5.2. Let \( w, \psi_1 \) and \( \psi_2 \) be the smooth weight functions defined in equation (5.4). \( X, \tilde{u}, \beta, R \) and \( N \) are defined as above. Write \( w = w_0 \psi_1 \) where \( w_0(X) = \frac{X^T Ab}{R(b,e_d)} \psi_1(y) \psi_2(\tilde{u}/Q) \). Consider the coordinates system \((x_1, \ldots, x_d)\) where \( x_i = \tilde{u}_i/Q \) and \( x_d = \beta N/mQ^2 \). Then for every \( 0 \leq n \leq d \) and \( 1 \leq i \leq d \)

(5.5) \[
w_0(x_1, \ldots, x_d) = 0 \text{ if } \max_{1 \leq i \leq d} |x_i| > C,
\]

where \( C \) is a constant that only depends on the support of \( \psi_1 \) and \( \psi_2 \) and the symmetric matrix \( A \). Moreover,

(5.6) \[
|\frac{\partial^n w_0}{\partial x_i^n}| < C_n,
\]

where the constant \( C_n \) only depends on \( A, \) compact set \( \Omega \), \( \max_j \left( \frac{\partial^j \psi_1}{\partial x^j} \right) \) and \( \max_{i,j} \left( \frac{\partial^j \psi_2}{\partial x^j} \right) \) where \( 1 \leq i \leq d - 1 \) and \( 0 \leq j \leq n \).
Proof: First we show that
\[ w_0(x_1, \ldots, x_d) = 0 \text{ if } \max_{1 \leq i \leq d} |x_i| > C. \]

Since \( |w_0(X)| \leq \left| \frac{X^T A b}{R(b,e_d)} \right| \psi_1(y) \psi_2(x_1, \ldots, x_{d-1}) \), there exists a constant \( C' \) depending on the support of \( \psi_2 \) and \( \psi_1 \) such that if \( \max_{1 \leq i \leq d-1} |x_i| > C' \) or \( y > C' \) then \( w_0(x_1, \ldots, x_d) = 0 \). Assume that \( w_0(x_1, \ldots, x_d) \neq 0 \) then \( 1 + m\beta > 1, y \leq C' \) and \( |x_i| < C' \) for \( 1 \leq i \leq d-1 \). We express \( y \) in terms of \( (x_1, \ldots, x_{d-1}, x_d) \)
\[
y = \frac{F(X) - N}{m^2 Q^2} = \left( \sum_i m\tilde{u}_i e_i + (1 + m\beta)b \right)^T A \left( \sum_i m\tilde{u}_i e_i + (1 + m\beta)b \right) - N
\]
\[
= \sum_{i=1}^{d-1} \tilde{u}_i^2 m^2 \lambda_i + (1 + m\beta)^2 N - N
\]
\[
= \sum_{i=1}^{d-1} \lambda_i + x_d(2 + m\beta).
\]

Therefore,
\[
|y| = \left| x_d(2 + m\beta) \right| \leq C' + \sum_{i=1}^{d-1} C'^2 \lambda_i.
\]

Since \( 1 + m\beta > 1 \), then
\[
|y| < C' + \sum_{i=1}^{d-1} C'^2 \lambda_i.
\]

We define \( C := C' + \sum_{i=1}^{d-1} C'^2 \lambda_i \). Therefore,
\[
w_0(x_1, \ldots, x_d) = 0 \text{ if } \max_{1 \leq i \leq d} |x_i| > C.
\]

This concludes the first part of our lemma.

Next, we give upper bounds on the partial derivatives of \( w \). We assume that \( |x_i| < C \) for \( 1 \leq i \leq d \). We apply the Leibniz’s product rule and obtain
\[
\frac{\partial^n w_0}{\partial x_i^n} = \sum_{j_1 + j_2 + j_3 = n} \frac{\partial^{j_1} X^T A b}{R(b,e_d)} \frac{\partial^{j_2} \psi_1(y)}{\partial x_i^{j_2}} \frac{\partial^{j_3} \psi_2(x_1, \ldots, x_{d-1})}{\partial x_{i_3}^{j_3}}.
\]

Hence, it suffices to show that the partial derivative of each factor on the right hand side is \( O(1) \). First we show that
\[
\frac{\partial^{j_1} X^T A b}{\partial x_i^{j_1}} = O(1).
\]

Recall that
\[
X - b = \sum_{i=1}^{d-1} m\tilde{u}_i e_i + m\beta b
\]
\[
= \sum_{i=1}^{d-1} mQx_i e_i + (m^2 Q^2 x_d/N)b.
\]
Hence,

\[
\frac{X^T Ab}{R \langle b, e_d \rangle} = \frac{b^T Ab + (X - b)^T Ab}{R \langle b, e_d \rangle} + \frac{(X - b)^T Ab}{R \langle b, e_d \rangle}
\]

\[
= \frac{b^T Ab}{R \langle b, e_d \rangle} + \frac{d - 1}{R \langle b, e_d \rangle} \sum_{i=1}^{m} Q x_i e_i^T Ab R \langle b, e_d \rangle + \frac{(m^2 Q^2 x_d/N) b^T Ab}{R \langle b, e_d \rangle}.
\]

Since \(b^T Ab = N\) and \(e_i^T Ab = 0\) for \(1 \leq i \leq d - 1\) then

\[
\frac{X^T Ab}{R \langle b, e_d \rangle} = \frac{N}{R \langle b, e_d \rangle} + \frac{(m^2 Q^2 x_d)}{R \langle b, e_d \rangle}.
\]

This is an affine function in \(x_d\). Since \(x_d < C\) then it suffices to show that

\[
\frac{N}{R \langle b, e_d \rangle} = O(1),
\]

\[
\frac{(m^2 Q^2)}{R \langle b, e_d \rangle} = O(1).
\]

Recall that \(N = R^2 = m^2 r^{-2} Q^2\) and \(e_d = \frac{Ab}{|Ab|}\) is the unit vector in direction of \(Ab\). Then

\[
\frac{(m^2 Q^2)}{R \langle b, e_d \rangle} \leq \frac{N}{R \langle b, e_d \rangle}
\]

\[
= \sqrt{(b/\sqrt{N})^T A^2 (b/\sqrt{N})}.
\]

Note that \(b/\sqrt{N} \in \Omega\). Hence

\[
\frac{N}{R \langle b, e_d \rangle} = O(1),
\]

\[
\frac{(m^2 Q^2)}{R \langle b, e_d \rangle} = O(1).
\]

Next we show that

\[
\frac{\partial^j \psi_1 (y)}{\partial x_i^j} = O(1).
\]

By the chain rule and the Leibniz's product rule, we write \(\frac{\partial^j \psi_1 (y)}{\partial x_i^j}\) in terms of \((\frac{\partial^j \psi_1}{\partial x_i^{j_1}})\) and \((\frac{\partial^j y}{\partial x_i^{j_2}})\) where \(1 \leq j_1, j_2 \leq j\). Hence, it suffices to show that

\[
\frac{\partial^j y}{\partial x_i^j} = O(1).
\]
We express $y$ in terms of $(x_1, \ldots, x_{d-1}, x_d)$ where $x_i = \tilde{u}_i/Q$ and $x_d = \beta N/mQ^2$.

\[ y = \frac{F(X) - N}{m^2Q^2} \]
\[ = \frac{\left( \sum m\tilde{u}_i e_i + (1 + m\beta)b \right)^T A \left( \sum m\tilde{u}_i e_i + (1 + m\beta)b \right) - N}{m^2Q^2} \]
\[ = \sum_{i=1}^{d-1} \tilde{u}_i^2m^2\lambda_i + (1 + m\beta)^2N - N \]
\[ = \sum_{i=1}^{d-1} x_i^2\lambda_i + x_d(2 + x_d m^2Q^2/N). \]

Note that $y$ is a quadratic form in $(x_1, \ldots, x_{d-1}, x_d)$, so it suffices to show that the coefficients $\lambda_i$ and $m^2Q^2/N$ are $O(1)$. Note that $\lambda_i$ are bounded by $|A|$ (the norm of $A$) and $m^2Q^2/N < 1$. Therefore, we conclude our lemma $\square$.

6. The integral $I_{mq}(c)$

In this section, we give upper bounds on the integral $I_{mq}(c)$ that appears in the expansion (3.10). We partition the integral vectors $c \neq 0 \in \mathbb{Z}^d$ into two sets. We call them by ordinary and exceptional vectors. In section (6.1), we give an upper bound on $I_{mq}(c)$ where $c$ is an ordinary vector. We use the rapid decay of the Fourier transform of $h(x, y)$ and bounds on the partial derivatives of $w$ that we proved in Lemma (5.2). In section (6.2), we give an upper bound on $I_{mq}(c)$ for the exceptional vectors. Our method is different in this case. We use the stationary phase theorem on the oscillatory integral $I_{mq}(c)$.

In what follows, we describe the set of ordinary and exceptional integral vectors $c$. The definition of the ordinary and exceptional vectors depend on the real point $b \in \Omega$, $m$, $r$ and $\epsilon$ that are defined in Theorem (1.3). We use the same notations as section 5. Recall that $B_1 \cup \{e_d\}$ that is defined in section 5.4 is an orthonormal basis for $\mathbb{R}^d$. We write a given integral vector $c$ in this basis

\[ c = \sum_{i=1}^{d} c_i e_i. \]

We remark that $c$ does not necessarily have integral coordinates anymore. We define two types of ordinary vectors. We call them ordinary of type I and type II. Type I ordinary vectors $c$ are the integral vectors $c$ such that their norm are large, more precisely

\[ |c| \geq (mr^{-1})^{1+\epsilon}, \]

where $|c| = \max_i c_i$. The type II vectors $c$ are the integral vectors $c$ such that there exists $1 \leq i \leq d-1$ such that

\[ c_i \gg m(r^{-1})^\epsilon, \]
\[ |c| < (mr^{-1})^{1+\epsilon}. \]

We call the complement of these integral vectors the exceptional integral vectors.

Next, we give two formulas for the volume form $dt = dt_1 \ldots dt_d$ by changing the coordinate system into $(\tilde{u}_1, \ldots, \tilde{u}_{d-1}, \beta)$ and $(\tilde{u}_1, \ldots, \tilde{u}_{d-1}, y)$. We use them to
estimate the integral $I_{mq}(c)$ in Lemma \([6.1]\) and \([6.2]\). Recall that $X = mt + \lambda$ where $t = (t_1, \ldots, t_d)$. Hence

$$dX = m^d dt.$$ 

Since $\{e_1, \ldots, e_d\}$ is an orthonormal basis then

$$dX = m^d du_1 \ldots du_{d-1} d\alpha.$$ 

Hence,

$$dt = du_1 \ldots du_{d-1} d\alpha.$$ 

Note that $(\tilde{u}_1, \ldots, \tilde{u}_{d-1}, \beta)$ is the coordinate system associated to basis $\{e_1, \ldots, e_{d-1}, b\}$. Hence,

$$(6.4) \quad dt = \langle b, e_d \rangle d\tilde{u}_1 \ldots d\tilde{u}_{d-1} d\beta.$$ 

We change the variables as in lemma \([5.2]\) to $x_i = \tilde{u}_i/Q$ and $x_d = \beta N/mQ^2$. Then

$$(6.5) \quad dt = \frac{Q^{d+1} m \langle b, e_d \rangle}{N} dx_1 \ldots dx_d.$$ 

Next, we give a formula for $dt$ in terms of $d\tilde{u}_1 \ldots d\tilde{u}_{d-1} dy$. Since $y = \frac{f(X) - N}{mQ^2}$, then

$$\frac{\partial y}{\partial \beta}(X) = \frac{X^T Ab}{mQ^2}.$$ 

Hence,

$$d\tilde{u}_1 \ldots d\tilde{u}_{d-1} d\beta = \left(\frac{\partial y}{\partial \beta}\right)^{-1} d\tilde{u}_1 \ldots d\tilde{u}_{d-1} dy,$$

$$= \frac{mQ^2}{X^T Ab} d\tilde{u}_1 \ldots d\tilde{u}_{d-1} dy.$$ 

Therefore,

$$(6.6) \quad dt = \frac{mQ^2 \langle b, e_d \rangle}{X^T Ab} d\tilde{u}_1 \ldots d\tilde{u}_{d-1} dy.$$ 

6.1. Upper bound on the integral $I_{mq}(c)$ for ordinary vectors.

In this section, we prove that $I_{mq}(c)$ rapidly decay when $|c| \to \infty$ and $c$ is an ordinary vector. Moreover, we show that the contribution of ordinary vectors $c$ in the delta method is bounded.

**Lemma 6.1.** Let $c$ be an ordinary vector and $w$ is a weight function that is defined in equation \([5.4]\). If $c$ is an ordinary type I vector, then for any $N > 0$:

$$(6.7) \quad I_{mq}(c) \ll_N \left(|c|r^{-1} m\right)^{-N},$$ 

and if $c$ is an ordinary type II vector we have

$$(6.8) \quad I_{mq}(c) \ll_N \left(r^{-1} m\right)^{-N}.$$ 

As a corollary, the contribution of ordinary vectors is bounded in the delta method, namely

$$(6.9) \quad \frac{C_Q}{mQ^2} \sum_q \sum_{c'} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \varepsilon 1.$$ 

where $\sum_{c'}$ is the sum over all ordinary vectors.
Proof: Recall that
\[ I_{mq}(c) = \int h\left(\frac{q}{Q}, \frac{F(mt + \lambda) - N}{m^2Q^2}\right)w(t)e\left(-\frac{\langle c, t \rangle}{mq}\right)dt_1 \ldots dt_d. \]

Recall from Lemma (5.2) that \(w\) is the product of two compactly supported weight functions as follows:
\[ w(t) = w_0(t)\psi_1(y), \]
where \( y = F\left(mt + \lambda\right) - N\frac{m^2Q^2}{m}. \) Let \( p_\psi(\xi) \) be the Fourier transform of \( h\left(\frac{q}{Q}, y\right)\psi_1(y) \), i.e.,
\[ p_\psi(\xi) = \int_{-\infty}^{\infty} \psi_1(y)h\left(\frac{q}{Q}, y\right)e(-\xi y) dy. \]
Therefore,
\[ I_{mq}(c) = \int_{\xi} p_\psi(\xi) \int_{t} w_0(t)e(\xi y - \frac{\langle t, c \rangle}{mq})dt_1 \ldots dt_d. \]

We change the coordinate system to \((\tilde{u}_1, \ldots, \tilde{u}_{d-1}, \beta)\) and apply formula (6.4) to obtain
\[ I_{mq}(c) = \langle b, e_d \rangle \int_{\xi} p_\psi(\xi) \int_{t} w_0(t)e(\xi y - \frac{\langle t, c \rangle}{mq})d\tilde{u}_1 \ldots d\tilde{u}_{d-1}d\beta d\xi, \]
where \( d\tilde{u} = d\tilde{u}_1 \ldots d\tilde{u}_{d-1} \). We express \( y \) in terms of \((\tilde{u}_1, \ldots, \tilde{u}_{d-1}, \beta)\)
\[ y = \frac{F(X) - N}{m^2Q^2} \]
\[ = \frac{(\sum \tilde{u}_i e_i + (1 + m\beta)b)^T A(\sum \tilde{u}_i e_i + (1 + m\beta)b) - N}{m^2Q^2} \]
\[ = \frac{\sum_{i=1}^{d-1} \tilde{u}_i^2 m^2 \lambda_i + (1 + m\beta)^2N - N}{m^2Q^2}. \]

We also write the following expansion for the inner product \( \langle t, c \rangle \)
\[ \langle t, c \rangle = \left\langle \frac{X - \lambda}{m}, c \right\rangle \]
\[ = \left\langle \frac{b - \lambda}{m}, c \right\rangle + \left\langle \frac{X - b}{m}, c \right\rangle \]
\[ = \langle t_0, c \rangle + \sum_{i=1}^{d-1} c_i \tilde{u}_i + \beta \langle c, b \rangle. \]
where \( b = mt_0 + \lambda \). We substitute the above expressions in equation (6.10) and obtain
\[ I_{mq}(c) = e\left(-\frac{\langle c, t_0 \rangle}{mq}\right) \langle b, e_d \rangle \int_{\xi} p_\psi(\xi) \int_{\beta, \tilde{u}} w_0(\tilde{u}_1, \ldots, \tilde{u}_{d-1}, \beta)e\left(\frac{\xi \beta(2/m + \beta)N}{Q^2} - \frac{\beta \langle b, c \rangle}{mq}\right) \]
\[ \times e\left(\sum_{i=1}^{d-1} \frac{\xi \lambda_i \tilde{u}_i^2}{Q^2} - \frac{c_i \tilde{u}_i^2}{mq}\right)d\beta d\tilde{u}. \]
Since we are only concerned with $|I_{mq}(c)|$, so we drop $e(-\frac{(c_i t)}{mq})$. We change the variables to $x_i = \hat{u}_i/Q$ and $x_d = \beta N/mQ^2$. By equation (6.13), we obtain

$$|I_{mq}(c)| = \frac{Q^{d+1}m(b,e_d)}{N} \left| \int_{\xi} p_\xi(\xi) \right.$$ 

(6.13)

$$\int w_0(x)e\left(x_d(2\xi + \frac{Q^2(b,c)}{qN} + \frac{\xi x_d^2m^2Q^2}{N})\right)$$

$$e\left(\sum_{i=1}^{d-1}[\xi \lambda_i x_i^2 + c_i Q x_i]\right)dx_1 \ldots dx_d d\xi.$$

At this point, we assume that $c$ is an ordinary vector of type II, i.e., there exist $1 \leq i \leq d-1$ such that

$$c_i \gg m(mr^{-1})^\epsilon.$$ 

We split the integral $I_{mq}(c)$ as a sum of two.

$$I_{mq}(c) = I_{mq}(c)_1 + I_{mq}(c)_2,$$

where $I_{mq}(c)_1$ is the integral over $\xi$ where $|\xi| < (r^{-1}m)^{\epsilon/2}Q_q^\epsilon$ and $I_{mq}(c)_2$ is the integral over the complement when $|\xi| > (r^{-1}m)^{\epsilon/2}Q_q^\epsilon$. First, we show that

(6.14) 

$$|I_{mq}(c)_2| \ll (r^{-1}m)^{-N}.$$ 

This is a consequence of the Lemma 3.5. Recall that

$$p_\xi(\xi) \ll (|\xi|Q_q^{-1})^{-N}.$$ 

Since $|\xi| > (r^{-1}m)^{\epsilon/2}Q_q^\epsilon$, we deduce the claimed upper bound on $|I_{qm}(c)_2|$. It remains to show that

$$|I_{mq}(c)_1| \ll (r^{-1}m)^{-N}.$$ 

Then from the Fubini’s theorem, we change the order of integration, and first take integration on the $x_i$ variable

$$|I_{mq}(c)| = \frac{Q^{d+1}m(b,e_d)}{N} \left| \int_{\xi} p_\xi(\xi) \right.$$ 

(6.15)

$$\int w_0(x)e\left(x_d(2\xi + \frac{Q^2(b,c)}{qN} + \frac{\xi x_d^2m^2Q^2}{N})\right)$$

$$e\left(\sum_{i=1}^{d-1}[\xi \lambda_i x_i^2 + c_i Q x_i]\right)dx_1 \ldots dx_d d\xi.$$

Note that by Lemma 3.2, the weight function $w_0(x_0, \ldots , x_i, \ldots , x_d)$ has compact support in a fixed interval $[-C, C]$ and its partial derivatives are bounded $|\partial^{\alpha}_{x_0} w_0| < C_n$. Since $|\xi| \ll (r^{-1}m)^{\epsilon/2}Q_q^\epsilon$ and $c_i \gg m(mr^{-1})^\epsilon$ we deduce that following lower
bound on the derivative of the oscillatory in the $x_i$ variable
\[
\frac{\partial (\xi \lambda_i x_i^2 + \frac{c_i Q x_i}{mq})}{\partial x_i} \Rightarrow \frac{c_i Q x_i}{mq} \Rightarrow \frac{(mr^{-1})^e Q}{q} \Rightarrow (mr^{-1})^e.
\]
(6.16)

Note that by integrating by part multiple times in the $x_i$ variable we have a fast decay for $I_{qm}(c)_1$. Because
\[
\int_{x_i} w(x)e (\xi \lambda_i x_i^2 + \frac{c_i Q x_i}{mq}) dx_i \ll_N (mr^{-1})^{-N\epsilon},
\]
for every $N > 0$. Hence, if $c$ is a type II vector, we proved the claimed upper bound on $I_{qm}(c)_1$ and therefore on $I_{qm}(c)$. In order to show that the contribution of the type II vectors is bounded in the delta method, namely
\[
\frac{C Q}{mqQ^2} \sum_{q=1}^{Q} \sum_{c} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \epsilon 1.
\]
(6.18)

Since $|c| \leq (r^{-1}m)^{1+\epsilon}$, then number of type II vectors is bounded by $(r^{-1}m)^{d(1+\epsilon)}$. Note that $Q = (r^{-1}m)^{1+\epsilon}$. Therefore by choosing $N$ large enough, we conclude our Lemma for type II vectors.

It remains to prove the lemma for type I vectors $c$. We proceed as before and split the integral $I_{mq}(c)$ as a sum of two.

\[
I_{mq}(c) = I_{mq}(c)_1 + I_{mq}(c)_2,
\]

But this time, $I_{mq}(c)_1$ is the integral over $\xi$ where $|\xi| < (|c|r^{-1}m)^{\epsilon/2} \frac{Q}{q}$ and $I_{mq}(c)_2$ is the integral over $|\xi| > (|c|r^{-1}m)^{\epsilon/2} \frac{Q}{q}$. From the same line as in (6.14), we deduce
\[
|I_{mq}(c)_2| \ll (|c|r^{-1}m)^{-N}.
\]
(6.19)

It remains to prove $I_{mq}(c)_1 \leq (|c|mr^{-1})^{-N}$ for the type I vectors $c$. If $|c_d| \neq \max_i |c_i|$, we proceed as before. So, we assume the harder case where $c_d = |c|$, because our lower bound for the partial derivative in $x_d$ variable is worse than other variables. We proceed the same strategy and use the Fubini’s theorem. We first take integration along the $x_d$ variable and we give a lower bound for the partial derivative of the oscillatory function in $x_d$ variable.

\[
\frac{\partial (x_d(2\xi + \frac{Q^2 \langle b,c \rangle}{qN}) + \frac{\xi x_d^2 m^2 Q^2}{N})}{\partial x_i} \Rightarrow \frac{Q^2 \langle b,c \rangle}{qN} - |2\xi| - \frac{2\xi m^2 Q^2}{N}.
\]
(6.20)

Recall that $N = Q^2 m^2 r^{-2}$, and $|\xi| < (|c|r^{-1}m)^{\epsilon/2} \frac{Q}{q}$, hence
\[
\frac{\partial (x_d(2\xi + \frac{Q^2 \langle b,c \rangle}{qN}) + \frac{\xi x_d^2 m^2 Q^2}{N})}{\partial x_d} \Rightarrow \frac{Q^2 \langle b,c \rangle}{qN} \Rightarrow (|c|mr^{-1})^e.
\]
(6.21)

By integrating by part multiple times in the $x_d$ variable we deduce
\[
I_{mq}(c)_1 \ll_N (|c|r^{-1}m)^{-N}.
\]
(6.22)
Therefore we concluded the claimed upper bound \((6.7)\) for the type I vectors. In order to prove the lemma it remains to show
\[
C Q m Q^2 \sum_{q=1}^{Q} \sum_c (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll 1.
\]
where the sum is over the ordinary type I vectors. We note that the following sum is bounded over \(c \in \mathbb{Z}^d\) and as a result over the type I integral vectors.
\[
\sum_{c \in \mathbb{Z}^d} \frac{1}{|c|^{d+1}} \ll 1.
\]
Hence, by choosing \(N\) large enough in \((6.22)\), we deduce our Lemma. \(\Box\)

6.2. Upper bound on the integrals \(I_{mq}(c)\) for exceptional vectors.

In this section we prove an upper bound for \(I_{mq}(c)\) when \(c\) is an exceptional vector. Recall that exceptional vectors are the complement of the ordinary vectors, i.e., the integral vectors \(c\) with the following conditions:
\[
|c_i| < m (mr^{-1})^\epsilon, \text{ for } 1 \leq i \leq (d-1),
\]
\[
|c_d| < (m^{-1}m)^{1+\epsilon}.
\]

**Lemma 6.2.** Let \(c\) be an exceptional integral vector then we have the following upper bound on \(I_{mq}(c)\)
\[
|I_{mq}(c)| \ll \frac{Q^{d-1}}{R} \min \left[ \left( \frac{|Q|}{mr^{-1}q} \right)^{-\frac{d-1}{2}}, 1 \right].
\]
where \(R = \sqrt{N} = Q mr^{-1}\) and \(Q = (mr^{-1})^{1+\epsilon}\).

**Proof:** We apply formula \((6.6)\) and \((6.11)\) and obtain
\[
|I_{mq}(c)| = m Q^2 \int \frac{h(\frac{q}{Q}, y) w(t)}{X^T A b} e\left(\sum_{i=1}^{d-1} c_i \tilde{u}_i + \beta \langle c, b \rangle_{mq}\right) \tilde{u}_1 \ldots \tilde{u}_{d-1} dy.
\]
We use the weight function \(w\) in formula \((6.14)\):
\[
w(X) = \frac{X^T A b}{R (b, e_d)} \psi_1(y)^2 \psi_2\left(\frac{\tilde{u}}{Q}\right).
\]
Hence, \((6.27)\)
\[
|I_{mq}(c)| = \frac{m Q^2}{R} \int h(\frac{q}{Q}, y) \psi_1(y)^2 \psi_2\left(\frac{\tilde{u}}{Q}\right) e\left(\sum_{i=1}^{d-1} c_i \tilde{u}_i + \beta \langle c, b \rangle_{mq}\right) \tilde{u}_1 \ldots \tilde{u}_{d-1} dy.
\]
Next, we write \(\beta\) in terms of \(\tilde{u}_1, \ldots, \tilde{u}_{d-1}\) and \(y\) in order to separate the variables in the oscillatory integral \(I_{mq}(c)\). We give Taylor series of \(\beta\) in terms of \(\tilde{u}_1, \ldots, \tilde{u}_{d-1}, y\).
Recall that
\[
y = \frac{F(X) - N}{m^2 Q^2}
\]
\[
= \left( \sum \tilde{u}_i e_i + (1 + m \beta) b^T A \left( \sum \tilde{u}_i e_i + (1 + m \beta) b \right) \right) - N.
\]
Hence,
\[
m^2 Q^2 y = \sum_{i=1}^{d-1} \tilde{u}_i^2 m^2 \lambda_i + (1 + m \beta)^2 b^T A b - N.
\]
Since \( b^T Ab = N \), then
\[
(1 + m\beta)^2 = \frac{N + m^2Q^2y - \sum_{i=1}^{d-1} m^2u_i^2 \lambda_i}{N}.
\]
By changing the variables to \( x_i := \frac{u_i}{Q} \) for \( 1 \leq i \leq d-1 \) and using \( N = m^2r^{-2}Q^2 \), we obtain
\[
m\beta = \sqrt{1 + r^2 - \sum_{i=1}^{d-1} r^2 x_i^2 \lambda_i - 1}.
\]
By Lemma (5.2), if \( w \neq 0 \) then \( x_i < C \) and \( y < C \) for some fixed constant \( C \). Hence, by writing Taylor series
\[
\beta = \frac{1}{2} \left[ m^{-1}r^2 - \sum_{i=1}^{d-1} m^{-1}r^2 x_i^2 \lambda_i \right] + \phi(x_1, \ldots, x_{d-1}, y),
\]
where
\[
\frac{\partial^k \phi}{\partial x_1 \cdots \partial x_k} = O\left(r^{4m^{-1}}\right).
\]
for every \( k \geq 0 \) and every any point inside the support of \( w \). We change the variables to \( (x_1, x_2, \ldots, x_{d-1}, y) \) in formula (6.27) and replace \( \beta \) with formula (6.28)
\[
|L_{mq}(c)| \leq \frac{Q^{d-1}}{R} \left| \int h\left(\frac{q}{Q}, y\right) \psi_1(y)^2 e\left(\frac{\langle c, b \rangle y}{2(mr^{-1})^2 q}\right) \psi_2(x)e\left(\frac{\sum_{i=1}^{d-1} Qc_i x_i - m^{-1}r^2 \sum_{i=1}^{d-1} x_i^2 \lambda_i}{mq} \langle c, b \rangle + \phi(x, y) \langle c, b \rangle\right) dx dy \right|,
\]
where \( x = (x_1, \ldots, x_{d-1}) \). We define
\[
\xi(x_1, \ldots, x_{d-1}, y) := \frac{\sum_{i=1}^{d-1} Qc_i x_i - m^{-1}r^2 \sum_{i=1}^{d-1} x_i^2 \lambda_i}{mq} \langle c, b \rangle + \phi(x, y) \langle c, b \rangle,
\]
\[
G(y) := \int \psi_2(x)e\left(\xi(x_1, \ldots, x_{d-1}, y)\right) dx.
\]
By Fubini’s theorem
\[
|L_{mq}(c)| \leq \frac{Q^{d-1}}{R} \left| \int h\left(\frac{q}{Q}, y\right) \psi_1(y)^2 e\left(\frac{\langle c, b \rangle y}{2(mr^{-1})^2 q}\right) G(y) dy \right|,
\]
\[
|L_{mq}(c)| \leq \frac{Q^{d-1}}{R} \sup_{y \in [-C, C]} \left( \psi_1(y)^2 G(y) \right) \int |h\left(\frac{q}{Q}, y\right)| dy.
\]
By lemma (3.3) for \( l = k = 0 \), we deduce that
\[
\int |h\left(\frac{q}{Q}, y\right)| dy < C_1,
\]
where \( C_1 \) is a constant independent of \( q/Q \). Note that \( \psi_1 \) is a fixed smooth function with compact support then \( \sup_{y \in [-C, C]} \left( \psi_1(y)^2 \right) < C_2 \) where \( C_2 \) is a fixed constant.
Therefore,

\[
\left| I_{mq}(c) \right| \ll \frac{Q^{d-1}}{mQ^2} \sup_{y \in [-C,C]} (G(y)).
\]

We give an upper bound on the oscillatory integral \( G(y) \). First we deal with the cases where the phase function \( \xi \) does not have critical points. We have

\[
\frac{\partial}{\partial x_i} \xi(x_1, \ldots, x_{d-1}, y) = \frac{1}{mq} (Qc_i - \frac{\lambda_i (c,b) x_i}{mr-2} + \langle c,b \rangle \frac{\partial \phi}{\partial x_i}).
\]

Without loss of generality we assume that \( r < \epsilon_0 \) where \( \epsilon_0 \) depends only on the quadratic form \( F \) and the compact set \( \Omega \). Assume that there exists \( 1 \leq l \leq (d-1) \) such that

\[
\left| \lambda_l (c,b) \right| \frac{C}{mr-2} \leq Qc_l.
\]

where constant \( C \) is defined in Lemma (5.2). We note that \( \frac{\partial \phi}{\partial x_i} = O\left(r^4 m^{-1}\right) \) in the support of \( w \). Hence, by choosing \( \epsilon_0 \) small enough

\[
\left| \frac{\partial}{\partial x_i} \phi (c,b) \right| \leq \frac{Qc_l}{4}.
\]

By applying the inequalities (6.34) and (6.35) on equation (6.33), we obtain the following lower bound on the partial derivative \( \frac{\partial}{\partial x_i} \) of the phase function

\[
\frac{\partial}{\partial x_i} \xi(x_1, \ldots, x_{d-1}, y) \geq \frac{Qc_l}{2mq}.
\]

By integration by part we deduce that

\[
G(y) \ll_A \min \left[ \left( \frac{Q|c|}{mq} \right)^{-A}, 1 \right].
\]

Hence, by (6.32)

\[
\left| I_{mq}(c) \right| \ll \frac{Q^{d-1}}{mQ^2} \min \left[ \left( \frac{Q|c|}{mq} \right)^{-A}, 1 \right].
\]

This concludes the lemma by assuming (6.34). It remains to prove our lemma when

\[
\left| \lambda_l (c,b) \right| \frac{C}{mr-2} > \frac{Qc_l}{4}
\]

for every \( 1 \leq l \leq d-1 \). Since \( |b| \approx R = mr^{-1}Q \) then

\[
c_l \ll \frac{\langle c, \hat{b} \rangle}{r^{-1}}.
\]

where \( \hat{b} := b/R \in \Omega \) and constants in \( \ll \) depends only on the quadratic form \( F \) and the compact set \( \Omega \). Let \( \tilde{b} = \sum_{i=1}^{d} \hat{b}_i e_i \) then by choosing \( \epsilon_0 \) small enough, we deduce that

\[
1/2 \langle c, \tilde{b} \rangle \leq c_d \tilde{b}_d < 2 \langle c, \hat{b} \rangle,
\]

and \( |c_d| = \max_i c_i \).

Hence,

\[
\left| c \right| \ll \left| \langle c, \hat{b} \rangle \right|.
\]
where the constant in $\ll$ depends on $\tilde{b}_d^{-1}$ which has an upper bound depending on compact set $\Omega$. Therefore,

$$\frac{\partial^2}{\partial x_i \partial x_j} \xi(x_1, \ldots, x_{d-1}, y) = \frac{1}{mq} \left( -\delta(i, j) \frac{\lambda_i \langle c, b \rangle}{mr^{-2}} \right) + \frac{\partial^2}{\partial x_i \partial x_j} \phi \langle c, b \rangle.$$

where $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ otherwise. By applying (6.2), we obtain

$$\frac{\partial^2}{\partial x_i \partial x_j} \xi(x_1, \ldots, x_{d-1}, y) = -\delta(i, j) \frac{\lambda_i \langle c, b \rangle}{mr^{-1}} + O\left( \frac{\langle c, b \rangle}{qm^2r^{-2}} \right).$$

We substitute $b = R\tilde{b}$ and $R = mr^{-1}Q$

$$\frac{\partial^2}{\partial x_i \partial x_j} \xi(x_1, \ldots, x_{d-1}, y) = -\delta(i, j) \frac{\lambda_i Q \langle c, \tilde{b} \rangle}{mr^{-1}} + O\left( \frac{Q \langle c, \tilde{b} \rangle}{mr^{-1}q} \right).$$

Finally by inequality (6.36) and the stationary phase theorem on the oscillatory integral $G(y)$, we obtain the following upper bound:

(6.37) $$G(y) \ll \frac{Q^{d-1}}{R} \min \left[ \left( \frac{Q|c|}{mr^{-1}q} \right)^{-\frac{d-1}{2}}, 1 \right].$$

Therefore, we conclude the lemma. □

6.3. Bounding the sum over the exceptional vectors. In this section we give an upper bound on the contribution of the exceptional integral vectors $c$ in the delta method. First, we show the number of exceptional eigenvalues with a norm less than $X$ is less than $X \left( r^{-1}m \right)^r$. This upper bound on the number of exceptional vectors and the upper bound on the integral $I_{qm}(c)$ allow us to bound the contribution of the exceptional integral vectors in the delta method.

**Lemma 6.3.** The number of exceptional integral vectors $c$ such that

$$c \equiv \alpha a \mod m,$$

for $\alpha \in \mathbb{Z}/m\mathbb{Z}$ and $a \in \mathbb{Z}/m\mathbb{Z}^d$, and

$$|c| \leq X,$$

is bounded by $X \left( r^{-1}m \right)^r$.

**Proof:** The proof is based on the covering of $\mathbb{R}^d$ by boxes of volume $m^{d-1}$ around the integral points with the following congruence condition

$$c \equiv \alpha a \mod m.$$

From the definition of the exceptional vectors we know that

(6.38) $$|c_i| < m(mr^{-1})^r.$$

Therefore, all the exceptional vectors with $|c| < X$ lies in a box with size $X \times m(mr^{-1})^r \times \cdots \times m(mr^{-1})^r$.

Therefore the number of exceptional vectors is less than

$$\frac{Xm^{d-1}(mr^{-1})^{(d-1)e}}{\text{Volume of the box}} = X(mr^{-1})^{(d-1)e}.$$

By choosing $e$ small enough in the definition of the exceptional vectors we conclude the lemma. □.
We are ready to give an upper bound on the contribution of the exceptional vectors in the delta method. In order to do that we need two more auxiliary lemmas.

**Lemma 6.4.** We have the following upper bound on the sum over the exceptional vectors with norm $|c| > Z$ and $1 \leq q \leq X$ for every $\epsilon > 0$

\[
\sum_{q=1}^{X} \sum_{|c| \geq Z} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} X (r^{-1} m)^{-\frac{d}{2}(d-1) + \epsilon} Z^{-\frac{d-3}{2}}.
\]

**Proof:** By Lemma 4.1 $S_{mq}(c) = 0$, unless $c \equiv \alpha a \mod m$. So, without loss of generality, we assume that $c \equiv \alpha a \mod m$. We use the upper bound on $I_{mq}(c)$ that is proved in Lemma 6.2 and derive

\[
\sum_{q=1}^{X} \sum_{|c| \geq Z} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} \sum_{q=1}^{X} \sum_{|c| \geq Z} q^{-d} m^{-d} |S_{mq}(c)| \left( \frac{|c|}{r^{-1}mq} \right)^{-\frac{d+1}{2}}.
\]

By Lemma 4.4, we have

\[
\sum_{q=1}^{X} m^{-d} q^{-\frac{d+1}{2}} |S_{qm}(c)| \ll \Delta m^{d} X^{1+\epsilon}.
\]

Therefore,

\[
\sum_{q=1}^{X} m^{-d} q^{-\frac{d+1}{2}} |S_{qm}(c)| \ll \frac{mQ^{d+1}}{R} X^{1+\epsilon} (r^{-1} m)^{-\frac{d}{2}(d-1)} \sum_{|c| \geq Z} \left( \frac{1}{|c|} \right)^{\frac{d+1}{2}}.
\]

By Lemma 6.3, we deduce that

\[
\sum_{|c| \geq Z} \left( \frac{1}{|c|} \right)^{\frac{d+1}{2}} \leq \left( r^{-1} m \right)^{\epsilon} Z^{-\frac{d-3}{2}}.
\]

Since $X < Q = (r^{-1} m)^{1+\delta}$, a small enough choice for $\epsilon$ in the definition of the exceptional vectors $c$ concludes the lemma. □

**Lemma 6.5.** We have the following upper bound on the sum over $X \leq q \leq Q$ and exceptional vectors with $|c| < Z$

\[
\sum_{q=X}^{Q} \sum_{|c| < Z} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} Z R^* X^{-\frac{d-3}{2}}.
\]

**Proof:** We use the trivial upper bound on $I_{mq}(c)$, i.e.

\[
|I_{mq}(c)| \ll \frac{mQ^{d+1}}{R}.
\]

By using the above bound and Lemma 4.4, we have

\[
\sum_{q=X}^{Q} \sum_{|c| < Z} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} Q^* X^{-\frac{d-3}{2}} \sum_{|c| < Z} 1.
\]

By Lemma 6.3, we obtain

\[
\sum_{|c| < Z} 1 \ll Z (mr^{-1})^\epsilon.
\]
Proof: We prove this lemma by induction on \( k \)
\[
\sum_{q=X}^{Q} \sum_{|c|<Z} (mq)^{-d}s_{mq}(c)I_{mq}(c) \leq \frac{mQ^{d+1}}{R} ZQ^e X^{-\frac{d+1}{2}}.
\]

Therefore, we conclude the lemma. \( \square \)

In the following lemma we apply the alternative use of the Lemma 6.4 and Lemma 6.5 to improve an upper bound on the contribution of the exceptional vectors. We make an increasing sequence of \((X_k, Z_k)\) of integers

\[ X_k < X_{k+1} \quad \text{and} \quad Z_k < Z_{k+1}, \]

such that the contribution of the exceptional vectors is negligible in the delta method if either \(|c| < Z_k\) or \(q < X_k\). Our strategy to bound the contribution of the exceptional vectors \( c \) is to find a \( k \) such that \( Q < X_k \) or \((r^{-1}m)^{1+\epsilon} < Z_k\). We show such a \( k \) exists in the following lemma.

**Lemma 6.6.** For every integer \( k \geq 0 \), there exists an \( \eta > 0 \) such that if \( X_k = (r^{-1}m)^{\frac{2}{d}-(d-1)\left[\sum_{i=0}^{k-1}(d-3)^2\right]-\eta} \) and \( Z_k = (r^{-1}m)^{\frac{2}{d}-(d-1)\left[\sum_{i=0}^{k-1}(d-3)^2\right]-\eta} \), then we have the following upper bound on the the sum over the exceptional integral vectors \( c \)
\[
\sum_{q=1}^{Q} \sum_{c} (mq)^{-d}s_{mq}(c)I_{mq}(c) - \sum_{q=X_k}^{Q} \sum_{|c|>Z_k} (mq)^{-d}s_{mq}(c)I_{mq}(c) \leq \frac{mQ^{d+1}}{R} ZQ^e.
\]

Proof: We prove this lemma by induction on \( k \). For \( k = 0 \), we let \( X_0 = (r^{-1}m)^{\frac{2}{d}-(d-1)\left[\sum_{i=0}^{0}(d-3)^2\right]-\eta} = 1 \) and \( Z_0 = 1 \). Hence, it suffices to show that
\[
\sum_{q=1}^{Q} \sum_{c} (mq)^{-d}s_{mq}(c)I_{mq}(c) \leq \frac{mQ^{d+1}}{R} ZQ^e.
\]

Note that this is implied by Lemma 6.3 for \( X = (r^{-1}m)^{\frac{2}{d}-(d-1)\epsilon} \) and \( Z = 1 \). Next, we prove the inductive step. So, we assume the lemma holds for \( k \). From the inductive hypothesis, we have
\[
\sum_{q=1}^{Q} \sum_{c} (mq)^{-d}s_{mq}(c)I_{mq}(c) - \sum_{q=X_k}^{Q} \sum_{|c|>Z_k} (mq)^{-d}s_{mq}(c)I_{mq}(c) \leq \frac{mQ^{d+1}}{R} ZQ^e,
\]

where \( X_k = (r^{-1}m)^{\frac{2}{d}-(d-1)\left[\sum_{i=0}^{k-1}(d-3)^2\right]-\eta} \) and \( Z_k = (r^{-1}m)^{\frac{2}{d}-(d-1)\left[\sum_{i=0}^{k-1}(d-3)^2\right]-\eta} \). Note that
\[
\sum_{q=X_k}^{Q} \sum_{|c|>Z_k} (mq)^{-d}s_{mq}(c)I_{mq}(c) = \sum_{q=X_{k+1}}^{Q} \sum_{|c|>Z_{k+1}} (mq)^{-d}s_{mq}(c)I_{mq}(c) + \sum_{q=X_k}^{Q} \sum_{|c|>Z_{k+1}} (mq)^{-d}s_{mq}(c)I_{mq}(c) + \sum_{q=X_k}^{Q} \sum_{Z_k < |c| < Z_{k+1}} (mq)^{-d}s_{mq}(c)I_{mq}(c).
\]
By Lemma (6.4), the middle sum (6.47) is bounded above by
\[
X_{k+1} \sum_{q=0}^{X_k} \sum_{|c| > Z_{k+1}} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} X_{k+1} (r^{-1} m)^{-\frac{d-3}{2}} Z_{k+1}^{\frac{d-3}{2}}.
\]
(6.47)

By Lemma (6.5), the last sum (6.47) is also bounded above by
\[
Q \sum_{q=X_k}^{Z_k} \sum_{|c| < Z_{k+1}} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} Z_{k+1} Q X_k^{-\frac{d-3}{2}}
\]
(6.48)

Hence the lemma holds for \( k + 1 \), and hence, we conclude the lemma \( \square \).

**Corollary 6.7.** Let \( \delta > 0 \) if the quadratic form \( F(x) \) has \( d \geq 5 \) variables, and \( \delta > 1 \) if \( d = 4 \). Then there exists an \( \epsilon > 0 \) which only depends on \( d \) and \( \delta \) such that
\[
\sum_{q=1}^{Q} \sum_{c \neq 0} (mq)^{-d} S_{mq}(c) I_{mq}(c) \ll \frac{mQ^{d+1}}{R} Q^{-\epsilon}
\]
(6.50)

**Proof:** By Lemma (6.4), it suffices to find a \( k \) such that \( X_k > Q \) or \( Z_k > (r^{-1} m)^{1+\epsilon} \).
If \( d \geq 5 \), this is trivial, since \( \frac{d-3}{2} \geq 1 \) and in fact \( X_k \) and \( Z_k \) go to infinity as \( k \to \infty \).
For \( d = 4 \), it is easy to check there exists \( k \) such that \( X_k > Q \) or \( Z_k > (r^{-1} m)^{1+\epsilon} \).

### 7. The main theorem

In this section, we derive the main term of the smooth counting and as a result prove our optimal strong approximation theorem. The main term comes from \( c = 0 \) and summing over \( 1 \leq q \leq Q \). We estimate the integral \( I_{mq}(0) \) it turns out that the main term of the integrals \( I_{mq}(0) \) is independent of \( q \) if \( 1 \leq q \leq Q^{1-\epsilon} \), and the value is equal to the singular integral which appears in the circle method. On the other hand if \( Q^{1-\epsilon} \leq q \leq Q \) the exponential sum is bounded by Lemma (6.4)

\[
\sum_{q=Q^{R^{-\epsilon}}}^{Q} (mq)^{-d} |S_q(0)| \ll Q^{\frac{d-4}{2} + \epsilon}.
\]
(7.1)

Hence, their contributions are negligible in the main term. Therefore, the main term is the product of the singular integral with the sum of \( S_{mq}(0) \) over \( 1 \leq q \leq Q^{1-\epsilon} \). This sum is converging to the singular series. As claimed, we derive a smooth counting formula with a power saving error term which is the product of the singular series with the singular integral.

#### 7.1. Singular integral

In the following lemma, we estimate the integral \( I_{mq}(0) \) for \( 1 \leq q \leq Q^{1-\epsilon} \).

**Lemma 7.1.** Suppose that \( 1 \leq q \leq Q^{1-\epsilon} \). Then
\[
I_{mq}(0) = \sigma_{\infty}(F, w) \frac{mQ^{d+1}}{R} + O_N(Q^{-N}),
\]
(7.2)
for any $N > 0$, where $\sigma_\infty(F,w)$ is the singular integral and equals to

\[
\sigma_\infty(F,w) = \int_{y(x)=0} \frac{R}{X^T A e_d} w(x) dx_1 \ldots dx_{d-1}
\]

(7.3)

\[
= \lim_{\epsilon \to 0} \int_{0 < u(x) < \epsilon} \frac{w(x) dx_1 \ldots dx_d}{\epsilon}
\]

where we normalize the variable so that $-1 < x_i < 1$.

Proof: We change the variables from $(t_1, \ldots, t_d)$ to $(u_1, \ldots, u_{d-1}, y)$. We recall the formula (6.4) and plug in $c = 0$ in the formula to obtain

\[
I_{mq}(0) = \int h\left(\frac{q}{Q}, \frac{F(mt + \lambda) - N}{m^2 Q^2}\right) w(t) dt_1 \ldots dt_d
\]

(7.4)

\[
= \frac{mQ^2}{R} \int X^T A e_d h\left(\frac{q}{Q}, y\right) w(t) du_1 \ldots du_{d-1} dy.
\]

We normalize variables again so that they lies in the interval $[-1,1]$.

\[Qx_i = u_i,\]

\[
\frac{mQ^2}{R} x_d = u_d.
\]

Therefore, we obtain

\[
I_{mq}(0) = \frac{mQ^{d+1}}{R} \int \frac{R}{X^T A e_d} h\left(\frac{q}{Q}, y\right) w(x,y) dx_1 \ldots dx_{d-1} dy.
\]

We apply the Fubini’s theorem and first take integration over $y$

\[
I_{mq}(0) = \frac{mQ^{d+1}}{R} \int \frac{R}{X^T A e_d} dx_1 \ldots dx_{d-1} \int h\left(\frac{q}{Q}, y\right) w(x,y) dy.
\]

We invoke Lemma 3.4 and estimate the integral over $y$

\[
I_{mq}(0) = \frac{mQ^{d+1}}{R} \int \frac{R}{X^T A e_d} w(x,0) dx_1 \ldots dx_{d-1} + O_N(Q^{-N}).
\]

By definition of $\sigma_\infty(w, F)$, we deduce that

\[
I_{mq}(0) = \sigma_\infty(F,w) \frac{mQ^{d+1}}{R} + O_N(Q^{-N}).
\]

Note that

\[
\frac{\partial y}{\partial (x_d)}^{-1} = \frac{R}{X^T A e_d}
\]

Therefore, by change of variable formula we conclude

\[
\sigma_\infty(F,w) = \int_{y(x)=0} \frac{R}{X^T A e_d} w(x) dx_1 \ldots dx_{d-1}
\]

(7.5)

\[
= \lim_{\epsilon \to 0} \int_{0 < u(x) < \epsilon} \frac{w(x) dx_1 \ldots dx_d}{\epsilon} \square
\]
7.2. **Singular series.** In this section, we show the sum $S_{mq}(0)$ over $1 \leq q \leq Q^{1-\epsilon}$ is convergent to the singular series $S(N, p)$. In the following lemma, we show the contribution of $Q^{1-\epsilon} \leq q \leq Q$ is negligible in the delta method.

**Lemma 7.2.** We have the following upper bound on the sum over $Q^{1-\epsilon} \leq q \leq Q$

$$\sum_{q=Q^{1-\epsilon}}^{Q} (mq)^{-d} S_{mq}(0) I_{mq}(0) \ll \frac{mQ^{d+1}}{R} Q^{\frac{3d}{2}+\epsilon}. \tag{7.6}$$

Proof: We use the trivial bound on $I_{mq}(0)$ and we obtain

$$I_q(0) \ll \frac{mQ^{d+1}}{R}. \tag{7.7}$$

By Lemma 4.4, we have

$$\sum_{q=QR^{-\epsilon}}^{Q} (mq)^{-d} S_{mq}(0) \ll Q^{3-d/2+\epsilon}. \tag{7.8}$$

Hence, we conclude the Lemma. □

We invoke the following lemma; see [HB96, Page 50; Lemma 31] which shows that the sum $S_{mq}(0)$ over $1 \leq q \leq Q^{1-\epsilon}$ is convergent to the singular series $S(N, p)$ and the singular series is the product of the local densities $\prod_p \sigma_p$.

**Lemma 7.3.** Let $F(x)$ be a non-degenerate quadratic form in $d$ variables where $d \geq 4$, then

$$\sum_{1 \leq q \leq X} (qm)^{-d} S_{mq}(0) = \prod_p \sigma_p + O(X^{3-d/2+\epsilon}). \tag{7.9}$$

As a result

$$\sum_{1 \leq q \leq Q^{1-\epsilon}} (qm)^{-d} S_{mq}(0) = \prod_p \sigma_p + O(Q^{3-d/2+\epsilon}). \tag{7.10}$$

Finally, we prove Theorem (1.10).

Proof: We write the smooth expansion of the delta function as stated in formula 3.13

$$N(a, w) = \frac{C_Q}{mQ^2} \sum_{q=1}^{Q} \sum_{c \in \mathbb{Z}^d} (mq)^{-d} S_{mq}(c) I_{mq}(c). \tag{7.11}$$

From Corollary 6.7 and Lemma 7.2 we can drop the nonzero integral vectors $c$ and restrict the sum over $q$ to $1 \leq q \leq Q^{1-\epsilon}$, hence

$$N(w, \lambda) = \frac{C_Q}{mQ^2} \sum_{q=1}^{Q^{1-\epsilon}} (mq)^{-d} S_{mq}(0) I_{mq}(0) + O\left(\frac{Q^{d-1}}{R} R^{-\epsilon}\right). \tag{7.12}$$

By Lemma 7.1 we have

$$I_{mq}(0) = \sigma_\infty(F, w) \frac{mQ^{d+1}}{R} + O\left(\frac{Q^{d-1}}{R} R^{-\epsilon}\right).$$

Therefore,

$$N(w, \lambda) = \sigma_\infty(F, w) \frac{Q^{d-1}}{R} \sum_{q=1}^{Q^{1-\epsilon}} (mq)^{-d} S_{mq}(0) + O\left(\frac{Q^{d-1}}{R} R^{-\epsilon}\right).$$
Finally, from Lemma 3.4, we obtain

\[ N(a, w) = \sigma_\infty(F, w) \prod_p \sigma_p(a, N) \frac{Q^{d-1}}{R} + O_\epsilon\left(\frac{Q^{d-1}}{R} R^{-\epsilon}\right). \]

Since we derived a smooth counting formula for the number of integral points with a power saving error term. By a standard method in analysis (majorants and minorants for the indicator function of a ball) we can find two smooth weight function \( w_1(x) \) and \( w_2(x) \) such that they approximate from below and above the indicator function \( \chi(x) \) of the open ball \( B(b, r) \), i.e.

\[ w_1(x) \leq \chi(x) \leq w_2(x). \]  

Therefore we deduce our counting formula and Theorem 1.10.

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