Dynamic Complexity of Expansion

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Abstract

Dynamic Complexity was introduced by Immerman and Patnaik [PI97] (see also [DST95]). It has seen a resurgence of interest in the recent past, see [DHK14, ZS15, MVZ16, BJI17, Zca17, DMV18, BRZ18, SSV+20, DKM+20] for some representative examples. Use of linear algebra has been a notable feature of some of these papers. We extend this theme to show that the gap version of spectral expansion in bounded degree graphs can be maintained in the class DynAC⁰ (also known as DynFO, for domain independent queries) under batch changes (insertions and deletions) of \( O(\frac{\log n}{\log \log n}) \) many edges.

The spectral graph theoretic material of this work is based on the paper by Kale-Seshadri [KS11]. Our primary technical contribution is to maintain up to logarithmic powers of the transition matrix of a bounded degree undirected graph in DynAC⁰.

1 Introduction

Computational complexity conventionally deals with problems in which the entire input is given to begin with and does not change with time. However, in practice, the input is not always static and may undergo frequent changes with time. For instance, one may want to efficiently update the result of a query under insertion or deletion of tuples into a database. In such a scenario, recomputing the solution from scratch after every update may be unnecessarily computation intensive. In this work, we deal with problems whose solution can be maintained by one of the simplest possible models of computation: polynomial size boolean circuits of bounded depth.
The resulting complexity class $\text{DynAC}^0$ is equivalent to Pure SQL in computational power when we think of graphs (and other structures) encoded as a relational database. It is also surprisingly powerful as witnessed by the result showing that $\text{DynAC}^0$ is strong enough to maintain transitive closure in directed graphs [DKM+18]. The primary idea in that paper was to reformulate the the problem in terms of linear algebra. We follow the same theme to show that expansion in bounded degree graphs can be maintained in $\text{DynAC}^0$.

1.1 The model of dynamic complexity

In the dynamic (graph) model we start with an empty graph on a fixed set of vertices. The graph evolves by the insertion/deletion of a single edge in every time step and some property which can be periodically queried, has to be maintained by an algorithm. The dynamic complexity of the algorithm is the static complexity for each step. If the updates and the queries can be executed in a static class $\mathcal{C}$ the dynamic problem is said to belong to $\text{Dyn}\mathcal{C}$.

In this paper, $\mathcal{C}$ is often a complexity class defined in terms of bounded depth circuits such as $\text{AC}^0$, $\text{TC}^0$, where $\text{AC}^0$ is the class of polynomial size constant depth circuits with AND and OR gates of unbounded fan-in; $\text{TC}^0$ circuits may additionally have MAJORITY gates. We encourage the reader to refer to any textbook (e.g. Vollmer [Vol99]) for precise definitions of the standard circuit complexity classes. The model was first introduced by Immerman and Patnaik [PI97] (see also Dong, Su, and Topor [DST95]) who defined the complexity class $\text{DynFO}$ which is essentially equivalent\(^2\) to the uniform version of $\text{DynAC}^0$. The circuit versions $\text{DynAC}^0$, $\text{DynTC}^0$ were also investigated by Hesse and others [Hes03, DHK14].

The archetypal example of a dynamic problem is maintaining reachability (“is there a directed path from $s$ to $t$”), in a digraph. This problem has recently [DKM+18] been shown to be maintainable in the class $\text{DynAC}^0$ - a class where edge insertions, deletions and reachability queries can be maintained using $\text{AC}^0$ circuits. This answers an open question from [PI97].

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1 We will have occasion to refer to the (dlogtime-)uniform versions of these circuit classes and we adopt the convention that, whenever unspecified, we mean the uniform version.

2 More precisely, the two classes are equivalent for all domain independent queries i.e. queries for which the answer to the query is independent of the size of the domain of the structure under question. We will actually conflate $\text{DynFO}$ with $\text{DynFO}(<,+,$ $\times)$ in which class order $<$ and the corresponding addition $(+)$ and multiplication $(\times)$ are built in relations. We do so because we need to deal with multiple updates where the presence of these relations is particularly helpful – see the discussion in [DMVZ18]. We do not need to care about these subtler distinctions when we deal with $\text{DynAC}^0$ in any case.
Even more recently this result has been extended to batch changes of size $O(\frac{\log n}{\log \log n})$ (see [DMVZ18]). In this work, we study expansion testing under batch changes of size similar to above.

### 1.2 Expansion testing in dynamic graphs

In this paper we study the dynamic complexity of checking the expansion of a bounded-degree graph under edge updates. A bounded-degree graph $G$ is an expander if its second-largest eigenvalue $\lambda_G$ is bounded away from 1. Expanders are a very useful class of graphs with a variety of applications in algorithms and computational complexity, for instance in derandomization. This arises due to the many useful properties of an expander such as the fact that an expander has no small cuts and that random walks on an expander mixes well.

Our aim is to dynamically maintain an approximation of the second largest eigenvalue of a dynamically changing graph in DynAC$^0$. We show that for a graph $G$, we can answer if the second largest eigenvalue of the graph is less than a parameter $\alpha$ (meaning that $G$ is a good expander) or if $\lambda_G > \alpha'$ where $\alpha'$ is polynomially related to $\alpha$. The study of a related promise problem of testing expansion was initiated in the sparse model of property testing by Goldreich and Ron [GR11], and testers for spectral expansion by Kale and Seshadri [KS11] and vertex expansion by Czumaj and Sohler [CS10] are known.

Our algorithm is borrowed from the property testing algorithm of [KS11] where it is shown that if $\lambda_G \leq \alpha$, then random walks of logarithmic length from every vertex in $G$ will converge to the uniform distribution. On the contrary if $\lambda_G \geq \alpha'$ then this is not the case for at least one vertex in $G$. The key technical contribution in the paper is a method to maintain the logarithmic powers of the normalized adjacency matrix of a dynamic graph when there are few edge modifications.

### 1.3 Overview of the Algorithm

The Kale-Seshadri algorithm [KST11] estimates the collision probability of several logarithmically long random walks using the lazy transition matrix from a small set of randomly chosen vertices. It uses these to give a probabilistically robust test for the gap version of conductance. We would like to extend this test to the dynamic setting where the graph evolves slowly by insertion/deletion of small number of edges. Moreover, in our dynamic complexity setting the metric to measure the algorithm is not the sequential
time but parallel time using polynomially many processors. Thus it suffices
to maintain the collision probabilities in constant parallel time with polyno-
mially many processors to be able to solve the gap version of conductance
in DynAC$^0$.

This brings us to our main result:

**Theorem 1.** (*Dynamic Expansion test*) Given the promise that the
graph remains bounded degree (degree at most $d$) after every round of up-
dates, *Expansion testing*\(^3\) can be maintained in DynAC$^0$ under $O(\frac{\log n}{\log \log n})$
changes.

In other words, we need to maintain the generating function of at most
logarithmic length walks of a transition matrix when the matrix is changed
by almost logarithmically many edges\(^4\) in one step. The algorithm is based on
a series of reductions, from the above problem ultimately to two problems –
integer determinant of an almost logarithmic matrix modulo a small prime
and interpolation of a rational polynomial of polylogarithmic degree. Each
reduction is in the class AC$^0$. Moreover if there are errors in the original
data the errors do not increase after a step. On the other hands the entries
themselves lengthen in terms of number of bits. To keep them in control
we have to truncate the entries at every step increasing the error. We can
continue to use these values for a number of steps before the error grows
too large. Then we use a matrix that contains the required generating func-
tion computed from scratch. Unfortunately this “from scratch” computation
takes logarithmically many steps. But by this time $O(\frac{\log^2 n}{\log \log n})$
changes have accumulated. Since we deal with almost logarithmic many changes in log-
arithmically many steps by working at twice the speed in other words the
AC$^0$ circuits constructed will clear off two batches in one step and thus are
of twice the height. Using this, we catch up with the current change in
logarithmically many steps. Hence, we spawn a new circuit at every time
step which will be useful logarithmically many steps later.

The crucial reductions are as follows:

- (Lemma 21) Dynamically maintaining the aforesaid generating func-
tion reduces to powering an almost logarithmic matrix of univariate
polynomials to logarithmic powers by adapting (the proof of) a method
by Hesse [Hes03].

\(^3\)See Section 3 for a formal definition.
\(^4\)O(\frac{\log n}{\log \log n}) to be precise – for us the term “almost logarithmic” is a shorthand for this.
• (Lemma 24) Logarithmically powering an almost logarithmically sized matrix reduces to powering a collection of similar sized matrices but to only an almost logarithmic power using the Cayley-Hamilton theorem along. This further requires the computation of the characteristic polynomial via an almost logarithmic sized determinant and interpolation.

• (Lemma 25) To compute $M^i$ for $i$ smaller than the size of $M$, we consider the power series $(I - zM)^{-1}$ and show that we can use interpolation and small determinants (of triangular matrices) to read off the small powers of $M$ from it.

• (Lemma 28) We reduce rational determinant to integer determinant modulo $p$. We invoke a known result from [DMVZ18] to place this in AC$^0$.

Since interpolation of polylogarithmic degree polynomials is in AC$^0$, this rounds off the reductions and the outline of the proof of:

**Theorem 2. (Main technical result: informal)** Let $T$ be an $n \times n$ dynamic transition matrix, in which, there are at most $O\left(\frac{\log n}{\log \log n}\right)$ changes in a step. Then we can maintain in DynAC$^0$, a matrix $\tilde{T}$ such that $|\tilde{T} - T^{\log n}| < \frac{1}{n^{\omega(1)}}$.

### 1.4 Motivation

The conductance of a graph, also referred to as the uniform sparsest cut in many works, is an important metric of the graph. Many algorithms have been designed for approximating the uniform sparsest cut in the static setting [ACL07, She09, ST13, Mad10, KRV09]. This naturally raises the question of maintaining an approximate value of the conductance in a dynamic graph subject to frequent edge changes.

In the Ph.d Thesis of Goranci [Gor19], a sequential dynamic incremental algorithm (only edge insertions allowed) with polylogarithmic approximation and sublinear worst-case update time. In [GRST20], the authors give a fully dynamic algorithm (both edge insertions and deletions allowed) with slightly sublinear approximation and polylogarithmic amortized update time. On the other hand, our work gives a fully dynamic algorithm in a parallel setting.

Another difference is that the algorithm in [Gor19, GRST20] outputs an approximate value of the conductance while our algorithm only solves the gap version.
There has also been significant related work on investigating the dynamic complexity of problems like rank, reachability and matching under single edge changes [Hes03, DHK14, DKM+18] and under batch changes [DMVZ18, DKM+20].

2 Preliminaries

We start by putting down a convention we have already been using. We refer by almost logarithmic (in $n$) a function that grows like $O\left(\frac{\log n}{\log \log n}\right)$.

2.1 Dynamic Complexity

The primary circuit complexity class we will deal with is $\text{AC}^0$, consisting of languages recognisable by a Boolean circuit family with $\land, \lor$-gates of unbounded fan-in along with $\neg$-gates of fan-in one where the size of the circuit is a polynomial in the length of the input and crucially the depth of the circuit is a constant (independent of the input). Since we are more interested in providing $\text{AC}^0$-upper bounds our circuits will be $\text{Dlogtime}$-uniform. There is a close connection between uniform $\text{AC}^0$ and the formal logic class $\text{FO}$ – to the extent that [BIS90] show that the version $\text{FO}(\leq, +, \times)$ is essentially identical to $\text{AC}^0$. We will henceforth not distinguish between the two.

The goal of a dynamic program is to answer a given query on an input graph under changes that insert or delete edges. We assume that the number of vertices in the graph are fixed and initially the number of edges in the graph is zero. Of course, we assume an encoding for the graph as a string and any natural encoding works.

The complexity of the dynamic program is measured by a complexity class $\mathcal{C}$ (such as $\text{AC}^0$) and those queries which can be answered constitute the class $\text{Dyn}\mathcal{C}$. In other words the (circuit) class $\mathcal{C}$ can handle each update given some polynomially many stored bits.

Traditionally the changes under which this can be done was fixed to one but recently [DMVZ18, DKM+20] this has been extended to batch changes. In this work we will allow nonconstantly many batch changes (of cardinality $O\left(\frac{\log n}{\log \log n}\right)$).

One technique which has proved important for dealing with batch changes is a form of pipelining suited for circuit/logic classes called “muddling” [DMS+19, SVZ18]. Suppose we have a static parallel circuit $A$ of non-constant depth that can process the input to a form from where the query is answerable easily and in addition we have a dynamic program $\mathcal{P}$ consisting
of constant depth circuits that can maintain the query but the correctness of
the results is guaranteed for only a small number of batches. This situation
may arise if e.g. the dynamic program uses an approximation in computing
the result and the ensuing errors add up across several steps making the re-
sults useless after a while. On the other hand the static circuit does precise
computation always but takes too much depth.

We need to construct a circuit that is of depth constant per batch of
changes but is promised to work for arbitrarily many batches. The idea is
to use a copy of the circuit \( A \) to process the current input to a form where
queries can be answered easily. However, by the time this happens the input
is stale in that \( d(A) \) (the depth of the circuit class) times the batch size many
changes are not included. Now we use the dynamic program \( P \) to handle
a leftover batch and the currently arriving batch in one unit of time over
the next \( d(A) \) time steps. This will allow the program to catch up with the
backlog and allow it to deliver the 2\( d(A) \) result in the 2\( d(A) \)-th time step.
Since the total depth of the circuit involved in this is \( O(d(A)) \), the average
depth remains constant. By starting a new static circuit at every time step,
that will deliver the then correct result after 2\( d(A) \)-steps, we are done. To
summarise in our particular case (an adapted and modified version of the
“muddling” lemmas from \([DMS+19, DMVZ18]\)) we have the following:

**Lemma 3.** Let \( M \) be a matrix with \( b = O(\log^2 n) \)-bit rational entries. Sup-
pose we have two routines available:

- An algorithm \( A \) that can compute \( M^{\log n} \) by an \( \text{AC}^1 \) circuit.
- A dynamic program \( P \) specified by an \( \text{AC}^0 \)-circuit that can approxi-
mately maintain \( M^{\log n} \) under batch changes of size \( l = O\left(\frac{\log n}{\log \log n}\right) \) for
\( \Omega(\log n) \) batches.

Then, we have an \( \text{AC}^0 \)-circuit that will approximately maintain \( M^{\log n} \) under
batch changes of size \( l \) for arbitrarily many batches.

**Proof.** Suppose the circuit for \( A \) has depth \( c_A \log n \) and the circuit for \( P \)
has depth \( c_P \). Then we will show how to construct a circuit \( C_t \) at time \( t \)
of depth \( d = (c_A + 2c_P) \log n = c \log n \) that will compute the value \( M^{\log n} \)
which is correct at the time \( t + \log n \). At a time there are \( \log n \) circuits
extant viz. \( C_{t-\log n+1}, C_{t-\log n+2}, \ldots, C_t \) which will deliver the correct value
of \( M^{\log n} \) at times \( t + 1, t + 2, \ldots, t + \log n \) respectively. Since the size of
each circuit \( C_t \) is polynomial in \( n \) so is the size \( s_c(n) \) of \( c \) layers of \( C_t \). Thus,
we can think that each layer of the overall circuit consists of \( c \) layers of
each of \( C_{t-\log n+1}, \ldots, C_t \) of total size \( s_c(n) \log n \) per layer i.e. it is an \( \text{AC}^0 \)
circuit.
Next we describe the algorithm $A$ that works in $\text{AC}^1$. Notice that two $n \times n$ matrices with entries that are rationals with at most polynomial in $n$ bits each can be multiplied in $\text{TC}^0$ (see e.g. [HAB02, Vol99]). Hence raising a matrix $A$ to the log $n$-th power can be done by $\text{TC}$-circuits of depth $O(\log \log n)$ (by repeated squaring). We also know that $\text{TC}^0$ is a subset of $\text{NC}^1$ [Vol99] which in turn has $\text{AC}$-circuits of depth $O(\log \log n \log \log n)$ (just cut up the circuit into $\text{NC}$-circuits of depth $\log \log n$ and expand each subcircuit into a DNF-formula of size $2^{2^{\log \log N}} = n$ – thus overall we get a depth reduction by a factor of $\log \log n$ at the expense of a linear blowup in size). Now by substituting these $\text{AC}$-circuit in the $\text{TC}$-circuits of depth $\log \log n$ we get an $\text{AC}^1$ circuit. Thus we get:

**Lemma 4.** Let $A$ be an $n \times n$ matrix with rational entries. The entries are represented with $n$ bits of precision each. Then computing $A^\ell$, where $\ell = O(\log n)$, is in $\text{AC}^1$.

### 2.2 Logarithmic space computations

In this section, we present some basic results about logarithmic space computations which will be useful to us. First, we show that reachability in graphs can be decided by bounded depth boolean circuits of subexponential size.

**Lemma 5.** Given an input graph $G$ with $|V(G)| = n$ and two fixed vertices $s$ and $t$, there is a circuit of depth $2d$ and size $n^{1/d}$ which can decide if there is a path from $s$ to $t$ in $G$.

See [COST16], pg. 613 for a proof.

In the following, we denote by $A^{\leq l}$ the words in the language $A$ that are of length at most $l$.

**Lemma 6.** Suppose $A \in \text{L}$ is a language. Then for constant $c > 0$, $A^{\leq \log^c n}$ has an $\text{AC}^0$ circuit of depth $O(1)$ and size $n^{O(1)}$.

**Proof.** Undirected reachability is $\text{L}$-hard under first order reductions by Cook and McKenzie [CM87]. Hence $A$ reduces to undirected reachability by first order reductions. Thus given $N$, we can construct an undirected graph $G_N$ of size some $N^k$, and two vertices $s, t$ thereof using first order formulas such that for all $w$ of length at most $N$, $w \in A$ iff $s, t$ are connected in $G_N$. But Lemma 5 tells us that there exists a (very-uniform) $\text{AC}$-circuit of size $N^{kN^{k/d}}$ and depth $2d$ that determines connectivity in $G_N$. Taking $N = \log^c n$, the size of the circuit becomes $2^{k c \log \log n \log^{k/d} n}$. Now pick
$d = kc + 1$ then the size becomes sublinear in $n$ (because the exponent is sublogarithmic).

## 3 Maintaining expansion in bounded degree graphs

In this section, we are interested in the problem of maintaining expansion in a dynamically updating bounded degree graph. For a degree-bounded graph $G$, let $\lambda_G$ denote the second largest eigenvalue of the normalized adjacency matrix of $G$. First, we define the problem of interest, which we call Expansion Testing:

**Definition 7. (Expansion testing)** Given a graph $G$, degree bound $d$, and a parameter $\alpha$, decide whether $\lambda_G \leq \alpha$ or $\lambda_G \geq \alpha'$ where $\alpha' = 1 - (1 - \alpha)^2/5000$.

In this section, we aim to prove the following theorem:

**Theorem 8. (Dynamic Expansion test)** Given the promise that the graph remains bounded degree (degree at most $d$) after every round of updates, Expansion testing can be maintained in DynAC$^0$ under $O(\log n \log \log n)$ changes.

Our algorithm is based on Kale and Seshadri’s work on testing expansion in the property testing model [KS11]. Our algorithm differs from theirs in that we are working on a dynamic graph, and the major technical challenge is an efficient way to maintain the powers of the normalized adjacency matrix. In this section, we will describe the algorithm and its correctness. In the subsequent sections, we will detail the method to update the power of the normalized adjacency matrix when a small number of entries change.

To prove the theorem, we will first look at the conductance of a graph $G$. For a vertex cut $(S, \overline{S})$ with $|S| \leq n/2$, the conductance of the cut is the probability that one step of the lazy random walk leaves the set $S$. We will denote by $\Phi_G(S)$ the conductance of the cut. Formally, $\Phi_G(S) = \frac{|E(S, \overline{S})|}{2d|S|}$. The conductance of the graph $\Phi_G$ is the minimum of $\Phi_G(S)$ over all vertex cuts $(S, \overline{S})$. The following inequality between the conductance of a graph and the second largest eigenvalue will be useful in our analysis (see [HLW06]).

$$1 - \Phi_G \leq \lambda_G \leq 1 - \frac{\Phi_G^2}{2}.$$

For a $d$-degree-bounded graph $G$, we will think of $G$ as a $2d$-regular graph where each vertex $v \in V$ has $2d - d(u)$ self-loops.
The main idea behind the algorithm in [KS11] is to perform many lazy random walks of length \( k = O(\log n) \) from a fixed vertex \( s \) and count the number of pairwise collisions between the endpoints of these walks. A lazy random walk on a graph from a vertex \( v \), chooses a neighbor uniformly at random with probability \( 1/2d \) and chooses to stay at \( v \) with probability \( 1 - d(v)/2d \). We can compute exactly the probability that two different random walks starting at \( s \) collide at their endpoints by computing

\[
S_s = \sum_{u \in [n]} T^\ell[s][u] \cdot T^\ell[s][u],
\]

where \( T \) is a transition matrix of the graph. Since \( T \) is symmetric, the matrix \( T^k \) must be a symmetric matrix. Then \( S_s \) is equal to the \((s, s)\) entry of the matrix \( T^{2k} \). Hence, it suffices to maintain the \((s, s)\) entry of the matrix \( T^{2k} \).

To analyze the lazy random walks in our setting, we will look at transition matrices \( T \) such that \( T[v, v] = 1 - d(v)/2d \) for every \( v \in V \) and \( T[u, v] = d(u)/2d \) for every edge \((u, v) \in G\). Notice that it is equivalent to a random-walk on a \( 2d \)-regular graph, where each vertex \( u \) with degree \( d(u) \) has \( 2d - d(u) \) self-loops, and therefore we can use the lemma stated above on the graph.

For a vertex \( v \in G \), let \( \pi^\ell_v \) denote the distribution over \( V \) of lazy random walks of length \( \ell \) starting from \( v \). The distance of this distribution from the stationary distribution (which is uniform in this case), denoted by \( D^\ell(v) \) is given by

\[
D^\ell(v)^2 = \sum_{u \in V} \left( \pi^\ell_v(u) - \frac{1}{n} \right)^2 = \sum_{u \in V} \pi^\ell_v(u)^2 - \frac{1}{n}.
\]

Observe that \( \sum_{u \in V} \pi^\ell_v(u)^2 = T^{2\ell}[v, v] \) as shown in the lemma above. We now state a technical lemma about the existence of vertex \( v \) such that \( D^\ell(v) \) is high if the graph has low conductance.

**Lemma 9 ([KS11]).** For a graph \( G(V, E) \), let \( S \subset V \) be a set of size \( s \leq n/2 \) such that the cut \((S, \overline{S})\) has conductance less than \( \delta \). Then, for any integer \( l > 0 \), there exists a vertex \( v \in S \) such that \( D^\ell(v) \) is high if the graph has low conductance.

\[
D^\ell(v) > \frac{1}{2\sqrt{s}}(1 - 4\delta)^l.
\]

We can now describe our algorithm for testing expansion. After each update, we use Theorem 35 to obtain the matrix \( \tilde{T} \) such that \( |T - T^k| \leq 1/n^3 \), where \( k = \log n/\Phi^2 \). Therefore for each \( v \), we \( \tilde{T}[v, v] \) such that \( |\tilde{T}[v, v] - \sum_{u \in V} \pi^\ell_v(u)^2| \leq 1/n^3 \). We now test if \( \tilde{T}[v, v] \leq \frac{1}{n} \left( 1 + \frac{2}{n} \right) \) for each \( v \in G \),
and reject if this is not the case even for one $v \in G$. The correctness of this algorithm follows from the two lemmas stated below.

**Lemma 10.** If $\lambda_G \leq \alpha$, then $\tilde{T}[v, v] \leq \frac{1}{n} \left( 1 + \frac{2}{n} \right)$ for every $v \in G$.

**Proof.** If $\lambda_G \leq \alpha$, then $\Phi_G \geq 1 - \alpha = \Phi$. Now,

$$D_\ell(v)^2 = ||\pi_\ell v - \frac{1}{n}||^2_2 \leq \frac{1}{n^2}. $$

Therefore, $T^{2\ell}[v, v] = \sum_{u \in V} \pi_\ell u^2 \leq \frac{1}{n} \left( 1 + \frac{1}{n} \right)$. Since $|\tilde{T}[v, v] - T^{2\ell}[v, v]| \leq 1/n^3$, we have $\tilde{T}[v, v] \leq \frac{1}{n} \left( 1 + \frac{2}{n} \right)$ for every $v \in V$. \hfill $\Box$

**Lemma 11.** If $\lambda_G \geq \alpha'$, then there exists a vertex $v \in G$ such that $\tilde{T}[v, v] > \frac{1}{n} \left( 1 + \frac{2}{n} \right)$.

**Proof.** If $\lambda_G \geq \alpha'$, then we know that $\Phi_G \leq k\Phi^2$. Therefore, there exists a vertex cut $(S, \overline{S})$ such that $\Phi_G(S) \leq k\Phi^2$. From Lemma 9 we can conclude that there exists a vertex $v$ such that $D_\ell(v)^2 > \frac{1}{n} \left( 1 - 4k\Phi^2 \right)^{2\ell} \geq \frac{1}{n} \left( 1 - 4k\Phi^2 \right)^{2\ell}$. For $\ell = \ln n / 8\Phi^2$, and a sufficiently small $k < 1$, we have $D_\ell(v)^2 > \frac{1}{2n^{1+\epsilon}}$ for a small constant $\epsilon > 0$. The collision probability $\sum_{u \in V} \pi_\ell u^2$ is therefore at least $\frac{1}{n} \left( 1 + \frac{1}{n^2} \right)$. From Theorem 35 we know that $\tilde{T}[v, v] \geq \frac{1}{n} \left( 1 + \frac{1}{2n^2} \right) < \frac{1}{n} \left( 1 + \frac{2}{n} \right)$. \hfill $\Box$

The key ingredient in the algorithm is a procedure to maintain the logarithmic powers of a weighted adjacency matrix when only a small number of entries change. In the next section we will describe how to do this in $\text{DynAC}^0$.

## 4 Maintaining the logarithmic power of a matrix

We are given an $n \times n$ lazy-transition matrix $T$ that varies dynamically with the batch insertion/deletion of almost logarithmically ($O(\log n / \log \log n)$) many edges per time step. We want to maintain each entry of sum of powers: $\sum_{i=0}^{\log n} (xT)^i$. Notice that the exponent $\log n$ arises from the Kale-Seshadri expansion-testing algorithm which needs the probabilities of walks of length $\log n$. On the other hand, the almost logarithmic bound on the small number of changes is a consequence of the reductions described below from the dynamic problem above to ultimately, determinants of small matrices and interpolation of small degree polynomials. Here interpolation can be done for degrees up to polylogarithmic but known techniques [DMVZ18] permit.
determinants of at most almost logarithmic size in $\text{AC}^0$ yielding this bottleneck. Another way to view this bottleneck is: while from Lemmata 5, 6, polylogarithmic length inputs of languages in $L$ (or even $\text{NL}$: see [DMVZ18]) can be decided in $\text{AC}^0$, such bounds are not known for languages reducible to determinants.

**Definition 12.** A $b$-bit rational is a pair consisting of an integer $\alpha$ a natural number $\beta$ such that $|\alpha| < \beta \leq 2^b$. Its value is $\alpha/\beta$. By a mild abuse of notation we conflate the pair $(\alpha, \beta)$ with its value $\alpha/\beta$.

**Remark 13.** First, notice that every $b$-bit rational is smaller than 1 by definition. Second, a $B$-bit approximation $\tilde{r}$ to a rational $r$ may itself be a $b$-bit rational for some $b \neq B$. This is because the two statements $|r - \tilde{r}| \leq 2^{-B}$ and $\tilde{r} = \alpha/\beta$ where $|\alpha| < \beta \leq 2^b$ are independent.

We need some definitions and begin with the definition of a dynamic matrix and the associated problems of maintaining dynamic matrix powers.

**Definition 14.** Let $l \in \mathbb{N}$. A matrix $A \in \mathbb{Q}^{n \times n}[x]$ is said to be $(n, d, b, l)$-dynamic if:

- each coefficient of the polynomials is a $b$-bit rational
- at every step there is a change in the entries of some $l \times l$ submatrix of $A$ to yield a new matrix $A'$. The change matrix $\Delta A = A' - A$

**Definition 15.** DynMatPow$(n, d, b, k, l)$ is the problem of maintaining the value of each entry of $\sum_{i=0}^{k} (xA)^i$ for a $(n, d, b, l)$-dynamic matrix.

Let DynBipMatPow$(n, d, b, k, l)$ be the special case of DynMatPow$(n, d, b, k, l)$ where the change matrix $\Delta A$ has a support that is a bipartite graph with all edges from one bipartition to another.

Next, we define problems to which the dynamic problems will be reduced to. We begin with polynomial matrix powering. The last condition in the following bounding the constant term of entries of the powered matrix is a technical one for controlling the error.

**Definition 16.** Let MatPow$(n, d, b, k)$ be the problem of determining $\sum_{i=0}^{k} (xA)^i$ for a matrix $A \in \mathbb{Q}^{n \times n}[x]$ where all the following hold:

- the degree of the polynomials is upper bounded by $d$
- each coefficient is a $b$-bit rational
The constant term of each polynomial entry is upper bounded by $(3n)^{-1}$.

The next group of definitions involve the problems we ultimately reduce the intermediate matrix powering algorithm to. These include various determinant problems, polynomial interpolation and polynomial division.

**Definition 17.** Let $\text{Det}(n, b, v)$ be the problem of computing the value of the determinant of an $n \times n$ matrix with entries that are $b$-bit rationals bounded by $v < 1$ in magnitude.

Let $\text{Det}_p(n)$ be the problem of computing the value of the determinant of an $n \times n$ matrix with entries that are from $\mathbb{Z}_p$ for a prime $p$.

Let $\text{DetPoly}(n, d, b)$ be the problem of computing the value of the determinant of an $n \times n$ matrix with entries that are degree $d$ polynomials of $b$-bit rational coefficients.

**Definition 18.** Let $\text{Interpolate}(d, b)$ be the problem of computing the coefficients of a univariate polynomial of degree $d$, where the coefficients are rationals (not necessarily smaller than one) and where $d + 1$ evaluations of the polynomial on $b$-bit rationals are given.

**Definition 19.** Let $\text{Div}(n, m, b)$ the problem of computing the quotient of a univariate polynomial $g(x)$ of degree $n$ when it is divided by a polynomial $f(x)$ of degree $m$ where both polynomials are monic with other entries being $b$-bit rationals.

In the rest of this section, we will use the following variables consistently:

- $n$ number of nodes in the graph
- $l = O\left(\frac{\log n}{\log \log n}\right)$ the number of changes in one batch
- $k = O(\log n)$ the exponent to which we want to raise the transition matrix
- $b = \log O(1) n$ the number of bits in the representation
- $d \leq \log O(1) n$ the degree of a polynomial

Let us start with the first lemma above:

**Lemma 20.** $\text{DynMatPow}(n, d, b, k, l)$ reduces to $\text{DynBipMatPow}(2n, d, b, 2k, l)$ via a local $\text{AC}^0$-reduction.

That is, changing a “small” submatrix of the input dynamic matrix results in “small” small submatrix change in the output matrix of the reduction. The notion of smallness being almost logarithmic.
Proof. Let $A$ be an $(n, d, b, l)$-dynamic matrix. Let $B$ be the following $2 \times 2$ block matrix with entries from $\mathbb{Q}^{n \times n}$:

$$
\begin{pmatrix}
0_n & A \\
I_n & 0_n
\end{pmatrix}.
$$

Here $0_n, I_n$ are respectively the $n \times n$ all zeroes, identity matrices. Then clearly,

$$
B^{2k} = \begin{pmatrix}
A^k & 0_n \\
0_n & A^k
\end{pmatrix}
$$

Notice that:

$$
B' - B = \begin{pmatrix}
0_n & A' - A \\
0_n & 0_n
\end{pmatrix},
$$

is a directed bipartite graph with all edges from the first partition of $n$ vertices to the second partition of $n$ vertices, completing the proof. \qed

4.1 Generalising Hesse’s construction

Let $G$ be a weighted directed graph with a weight function $w : E \to \mathbb{R}^+$ and weighted adjacency matrix $A$. Let $H = H^{(k)}_G(x)$ denote the weighted graph with weighted adjacency matrix $A_H = \sum_{i=0}^k (xA)^i$ where $k$ is an integer and $x$ is a formal (scalar) variable. Let $G'$ be a graph on the vertices of $G$ differing from $G$ in a “few” edges and $A'$ be its adjacency matrix. Denote by $\Delta A = A' - A$. Notice that $\Delta A$ contains both positive and negative entries. Let $\Delta_+A$ be the matrix consisting of the positive and $\Delta_-A$ of the negative entries of $\Delta A$. Let $U$ be the affected vertices i.e. the vertices on which any of the inserted/deleted edges in $\Delta A$ (i.e. the support of the edges whose adjacency matrices are $\Delta_+A$ and $-\Delta_-A$) are incident.

Lemma 21. Suppose there exists a partition of the affected vertices into two sets $U_i, U_o$ such that all inserted and deleted edges are from a vertex in $U_i$ to a vertex in $U_o$. Consider the matrices $\Delta_\sigma$, for $\sigma \in \{+, -\}$, of dimension $|U| + 2$, viewed as a weighted adjacency matrix of a graph on $U \cup \{s, t\}$ where $s, t \in V(G) \setminus U$, and whose entries are defined as below:

$$
\Delta_\sigma[u, v] = \begin{cases}
\sigma w_{uv} x & \text{if } u \in U_i \text{ and } v \in U_o \\
A_H[u, v] & \text{if } u \in U_o \text{ and } v \in U_i \\
A_H[s, v] & \text{if } u = s \text{ and } v \in U_i \\
A_H[u, t] & \text{if } v = t \text{ and } u \in U_o \\
0 & \text{otherwise}
\end{cases}
$$

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Then the number of \( s, t \) walks in \( G' \) of length \( k \leq \ell \) are given by the coefficient of \( x^k \) in \( A_H[s, t] + \Delta^k_+[s, t] \).

Proof. We will prove this separately for the cases \( \sigma = + \) and \( \sigma = - \). The proof follows the general strategy of Hesse’s proof in [Hes[3].

When \( \sigma = + \), we insert edges into the graph \( G \). When new edges are added to \( G \), the total number of walks from \( s \) to \( t \) is the sum of the number of walks that are already present and the new walks due to the insertion of the new edges. Observe that all the \( s - t \) walks in the graph on \( U \cup \{s, t\} \) must pass through the new edges and every such walk of length \( l \) is counted exactly once in \( \Delta^k_+[s, t] \).

The more interesting case is when \( \sigma = - \), and edges are deleted from \( G \). In this case we \( A_H[s, t] \) contains all walks from \( s \) to \( t \) including the deleted edges, and we need to delete only those walks that contain at least one edge that is deleted. The proof follows along the same lines as Hesse’s proof when a single edge is deleted. The idea is to show that every walk from \( s \) to \( t \) of length \( l \) is counted exactly once in \( A_H[s, t] + \Delta^k_-[s, t] \).

Let \( P \) be any \( s-t \) walk in \( G \) that contains edges that are deleted. Firstly, \( P \) is counted exactly once in \( A_H[s, t] \). Suppose that \( r \) of the deleted edges occur in \( P \) and the \( i^{th} \) edge occurs \( k_i \) times. Among the \( k_i \) occurrences of the \( i^{th} \) edge we can choose \( l_i \) occurrences, for each \( i \), and this gives a walk where these are the edges from \( U_i \) to \( U_o \) that we choose in graph on \( U \cup \{s, t\} \), and the remaining are counted in the walk from \( s \) to \( U_i \), \( U_i \) to \( U_o \) and \( U_o \) to \( t \). There are \( \prod_{i=1}^r \binom{k_i}{l_i} \) such choices, and for each choice the corresponding summand for the walk in \( \Delta^k_-[s, t] \) is \((-1)^{l_1+l_2+\ldots+l_r} \). When \( l_1 = l_2 = \ldots = l_r = 0 \), the walk is counted in \( A_H[s, t] \). Therefore, the contribution of the walk \( P \) to the sum is given by

\[
\sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \cdots \sum_{l_r=0}^{k_r} (-1)^{l_1+l_2+\ldots+l_r} \prod_{i=1}^r \binom{k_i}{l_i} = \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \cdots \sum_{l_r=0}^{k_r} \prod_{i=1}^r (-1)^{l_i} \binom{k_i}{l_i} = \prod_{i=1}^r \left( \sum_{l_i=0}^{k_i} (-1)^{l_i} \binom{k_i}{l_i} \right) = 0.
\]

Since the walks that do not pass through the deleted edges never appear in the new graph on \( U \cup \{s, t\} \) that we created and are hence counted in \( A_H[s, t] \), this completes the proof for the case \( \sigma = - \).

From the lemma above, we can conclude the following reduction.
Lemma 22. DynBipMatPow\((2n, d, b, k, l)\) reduces to MatPow\((l, d, b, k)\) via an AC\(^0\)-reduction.

Proof. Let \(A\) denote the \((2n, d, b, l)\)-dynamic matrix such that the support of the changes is a bipartite graph. Let \(A_H = \sum_{i=0}^{k} (xA)^i\). From Lemma 21, we know that if \(l\) entries of \(A\) change, then the \((s, t)\) entry in the new sum, \(A_H'[s, t]\), can be computed in two steps, first by computing \(A_H[s, t] + \Delta^k_s[s, t]\) to obtain \(A_H'[s, t]\) and then computing \(A_H'[s, t]\) as \(A_H'[s, t] + \Delta^k_s[s, t]\). The lemma follows from these observations.

In the following lemma, we analyze the error incurred in the matrix \(A_H\) due to error in the matrix \(A_H'\):

Lemma 23. Let \(\tilde{A}_H\) be a \(b\)-bit approximation of the matrix \(A_H\), then the corresponding matrix \(\tilde{A}_H'\) obtained from \(\tilde{A}_H\) is a \(b-1\)-bit approximation of \(A_H\).

Proof. Let \(\tilde{A}_H = A_H - E\) where \(E\) denotes an error-matrix with each entry a polynomial with coefficients upper-bounded by \(1/2^b\). Each entry in \(\tilde{A}_H\) is represented by a \(d\)-degree polynomial with \(b\)-bit rational coefficients.

We can compute \(\tilde{A}_H'[s, t] = \tilde{A}_H[s, t] + \tilde{\Delta}^k_s[s, t]\), where \(\tilde{\Delta}_\sigma\) can be constructed from \(\tilde{A}_H'\). We can write \(\tilde{\Delta}_\sigma[s, t] = \Delta^k_s[s, t] - E'[s, t]\), where \(E'\) is an error matrix consisting of polynomials of degree at most \(d\). We will now show that the coefficients of these polynomials are upper-bounded by \(1/2^b\).

First observe that every entry of \(\Delta_\sigma\) is a degree \(d\) polynomial with coefficients at most \(1/2\). We will bound the term corresponding to \(\Delta^k_\sigma\) and the remainder separately. Since each entry of \(\Delta_\sigma\) is at most \(1/2\), we can bound the first term by \(k^2 < 2^{2^b}\). The remainder of the sum can be upper bounded by \(\frac{k^2}{2^{2^b}}\). Therefore, each coefficient of the polynomials of this matrix is bounded by \(\frac{k^2}{2^{2^b}} + \frac{k^2}{2^{2^b}} \leq 1/2^b\).

Therefore, we can write \(\tilde{A}_H'[s, t] = \tilde{A}_H[s, t] + \tilde{\Delta}^k_s[s, t] = A_H[s, t] + \Delta^k_s[s, t] - E''[s, t]\) where \(E''\) is an error matrix with each entry bounded by \(1/2^{b-1}\). □

4.2 From powering to small determinants

We first need to reduce the exponent from logarithmic to almost logarithmic. The following lemma in fact reduces it from polylogarithmic to almost logarithmic.

Lemma 24. MatPow\((l, d, b, k)\) AC\(^0\)-reduces to the conjunction of the following:
\textbf{MatPow}(l, 0, lb, l), \textbf{DetPoly}(l, 1, lb + 2k^2d), \textbf{Interpolate}(dk, kl^2b + k^3ld) and \textbf{Div}(k, l, l^2b + 2k^2ld)

We use a trick (see e.g. [ABD14][HV06]; notice that the treatment is similar but not identical to that in [ABD14] because there we had to power only constant sized matrices) to reduce large exponents to exponents bounded by the size of the matrix for any matrix powering problem via the Cayley-Hamilton theorem (see e.g. Theorem 4, Section 6.3 in Hoffman-Kunze [HK71]).

\textbf{Proof.} Given a matrix \( M \in \mathbb{Q}^{l \times l}[x] \), let \( M_i \) be the value of the polynomial matrix \( M \) with rationals \( x_i \) substituted instead of \( x \), for \( i \in \{0, \ldots, dk\} \). Here \( x_0, \ldots, x_{dk} \) are \( dk + 1 \) distinct, sufficiently small rationals (say \( x_i = \frac{i}{(3dk)^2} \)). Let \( \chi_{M_i}(z) \) denote its characteristic polynomial \( \det(zI - M_i) \). We write \( z^k = q_i(z)\chi_{M_i}(z) + r_i(z) \) for unique polynomials \( q_i, r_i \) such that \( \deg(r_i) < \deg(\chi_{M_i}) = l \). Now, \( M_i^k = q_i(M_i)\chi_{M_i}(M_i) + r_i(M_i) = r_i(M_i) \). Here, the last equality follows from the Cayley-Hamilton theorem that asserts that \( \chi_{M_i}(M_i) = 0 \). But \( r_i(z) \) is a polynomial of degree strictly less than the dimension of \( M_i \) and each monomial in this involves powering \( M_i \) to an exponent bounded by \( l - 1 \). Finally computing \( M^k \) reduces to interpolating each entry from the corresponding entries of \( M_i^k \).

Now we analyse this algorithm. First we evaluate the matrix at \( dk + 1 \) points \( x_0, x_1, \ldots, x_{dk} \) where \( x_i = \frac{i}{(3dk)^2} \). This yields a matrix \( M_i \) whose entries are bounded by \( \frac{1}{3k} + \sum_{j=1}^{dk} j^2(3dk)^{-2j} < \frac{1}{3k} + \sum_{j=1}^{d} (3k)^{-j} < \frac{1}{3k} + \frac{1}{3k-1} < k^{-1} < (3l)^{-1} \) in magnitude.

We then compute the characteristic polynomial of \( M_i \). Notice that the value of \( \det(zI - M_i) \) is a monic polynomial with coefficient of \( z^{l-j} \) bounded by \( j!(l+j)(3l)^{-j} < 1 \) for \( j > 0 \). Suppose, teh denominator of an entry of \( M \) is bounded by \( \beta \). Then The denominator of \( M_i \) is bounded by \( \beta(3dk)^{2dk} \). Moreover, the denominator of this coefficient is further bloated to at most \( \beta^2(3dk)^{2dk} \leq 2^{lb+2k^2d} \) (where we use that \( l \log 3dk \approx k \)). Thus this corresponds to an instance of \textbf{DetPoly}(l, 1, lb + 2k^2d).

In the next step, we divide \( z^k \) by the characteristic polynomial of \( M_i \), \( \chi_{M_i}(z) \). This corresponds to an instance of \textbf{Div}(k, l, l^2b + 2k^2ld).

For computing the evaluation of the remainder polynomial on an \( M_i \), we need to power an \( l \times l \) matrix \( M_i \) with \( lb \)-bit rational entries to exponents bounded by at most \( l - 1 \). This can be accomplished by \textbf{MatPow}(l, 0, lb, l) by recalling that the each entry of \( M_i \) is bounded by \( (3l)^{-1} \).

Finally, we obtain \( M^k \) by interpolation. Every entry of \( M^k \) is a polynomial of degree at most \( dk \). Every coefficient of this polynomial is an \( kb \)-bit
rational and moreover the evaluation on entries of \( r_i(M_l^j) \) are given which are \( kl^2b + 2k^3ld \)-bit i.e. via Interpolate\((dk, kl^2b + 2k^3ld)\).

Next, we reduce almost logarithmic powers of almost logarithmic sized matrices to almost logarithmic sized determinants of polynomials.

**Lemma 25.** MatPow\((l, d, b, l)\) AC\(^0\)-reduces to the conjunctions of DetPoly\((l, 1, lb)\), Det\((l + 1, lb, (l + 1)^{-1})\), Interpolate\((dl, lb)\)

**Proof.** Let \( A^{(j)} \) be the univariate polynomial matrix \( A = A(x) \) evaluated at point \( x = x_j \) where \( x_0, \ldots, x_{dl} \) are \( dl + 1 \) distinct rationals say \( x_j = \frac{1}{(kdl)^j} \). Consider the infinite power series \( p^{(s, t, j)}(z) = (I - zA^{(j)})^{-1}[s, t]. \)

\((I - zA^{(j)})^{-1} = \sum_{i=0}^{\infty} z^i(A^{(j)})^i\). Thus \( p^{(s, t, j)}(z) \) is the generating function of \((A^{(j)})^i[s, t]\) parameterised on \( i \). \( p^{(s, t, j)}(z) \) can be also be written, by Cramer’s rule (See, for example, Section 5.4, p. 161, Hoffman-Kunze [HK71]). as the ratio of two determinants – the numerator being the determinant of the \((t, s)\)-th minor of \((I - zA^{(j)})\), say \( D^{(s, t, j)}(z) \) and the denominator being the determinant \( D^{(j)}(z) \) of \( I - zA^{(j)} \). Thus \( p^{(s, t, j)}(z) = \frac{D^{(s, t, j)}(z)}{D^{(j)}(z)} \). In other words, \( D^{(s, t, j)}(z) = p^{(s, t, j)}(z)D^{(j)}(z) \). Now let us compare the coefficients of \( z^i \) on both sides:

\[
D^{(s, t, j)}_i = \sum_{k=0}^{i} p^{(s, t, j)}_k D^{(j)}_{i-k}
\]

Letting, \( i \) run from 0 to degree of \( D^{(j)} \) which is \( l = \text{dimension of } A \), we get \( l + 1 \) equations in the \( l + 1 \) unknowns \( p^{(s, t, j)}_i \) for \( i \in \{0, \ldots, l\}, j \in \{0, \ldots, d\} \).

Equivalently, this can be written as the matrix equation: \( M^{(j)}\pi = d^{(j)} \) where \( M^{(j)} \) is an \((l + 1) \times (l + 1)\) matrix with entries \( M^{(j)}_{ik} = D^{(j)}_{i-k}, \) for \( 0 \leq k \leq i \leq l \) and zero for all other values of \( i, k \) lying in \( \{0, \ldots, l\} \).

Similarly, \( d^{(j)} \) is a vector with entries \( d^{(j)}_k = D^{(s, t, j)}_k \) and \( \pi \) the vector with \( l + 1 \) unknowns \( \pi_k = p^{(s, t, j)}_k \) again for \( i, k \in \{0, \ldots, l\} \). Notice that specifically in this argument, indices of matrices/vectors start at 0 instead of 1 for convenience.

Next, we show that the matrix \( M \) is invertible. We make the trivial but crucial observation:

**Observation 26.** The constant term in \( D^{(j)}(z) = \det(I - zA^{(j)}) \) is 1.

This implies that:

\(^6\)Here we use the convention that \( a_i \) denotes the coefficient of \( z^i \) in \( a(z) \), where \( a(z) \) is a power series (or in particular, a polynomial).
Proposition 27. \(M^{(j)}\) is a lower triangular matrix with all principal diagonal entries equal to 1 hence has determinant 1.

Next we can interpolate the values of \(A^{i}[s, t]\) from the values of \((A^{(j)})^{i}[s, t] = [z^{i}]p^{(s, t, j)}\) for \(i \in \{0, \ldots, t\}\), \(j \in \{0, \ldots, \frac{dl}{(3dl)^2}\}\).

Now we analyse this algorithm. First, we evaluate the matrix at \(dl+1\) distinct rationals. Each entry of the \(j\)-th matrix is now bounded by \(\frac{1}{3l} + \sum_{i=1}^{dl} j^{i}(3dl)^{-2i} < (l + 1)^{-1}\). We then compute determinant of the matrix \(I - zA^{(j)}\) where \(A \in \mathbb{Q}^{l \times l}\). The number of bits in the denominators are less than \(2bl\) and moreover the values of coefficient of \(z^j\) is less than \(j!(l)^{(l+1)^{-j}} < 1\) if \(j > 0\). Thus the coefficients are \(lb\)-bit rationals. Hence, computing the determinant of \(I - zA^{(j)}\) corresponds to an instance of \(\text{DetPoly}(l, 1, lb)\).

The next step is to compute the inverse of a \((l+1) \times (l+1)\) matrix \(M^{(j)}\) above. The \((a, b)\) entry of \(M^{(j)}\) is either zero or equals the coefficient of \(z^{a-b}\) in a cofactor of \((I - zA^{(j)})\). By a logic similar to that used in the proof of Lemma \([24]\) these are all \(lb\)-bit rationals. Further, a crude upper bound on their values is \((l+1)^{-1}\) as above. Thus we get an instance of \(\text{Det}(l+1, lb, (l+1)^{-1})\).

Finally, we find the matrix \(A^{i}\) from the values of \((A^{(j)})^{i}\). This corresponds to an instance of \(\text{Interpolate}(dl, lb)\).

4.3 Working with small determinants

First we reduce the problem of computing almost logarithmic determinants of small polynomials to computing almost logarithmic determinants over small rationals.

Lemma 28. \(\text{DetPoly}(l, d, b)\) \(\text{AC}^0\)-reduces to the conjunction of \(\text{Interpolate}(dl, lb), \text{Det}(l, lb, (l+1)^{-1})\).

Proof. Here, we need to compute the determinant of an \(l \times l\) matrix with each entry a degree \(\leq d\) polynomial with \(b\) bit coefficients. Clearly, the determinant is a degree at most \(ld\) polynomial. So, we plug in \(ld+1\) different values \(x_0, x_1, \ldots, x_{ld}\), where \(x_i = \frac{i}{(3dl)^2}\) into the determinant polynomial. This yields a determinant with entries bounded by \((l + 1)^{-1}\) in magnitude as in the proof of Lemma \([24]\).

The next step is to interpolate a degree at most \(ld\) polynomial. The coefficient of \(x^m\) where \(m \leq ld\) is bounded by 1 as in the previous lemmas, and the number of bits is at most \(lb\) as well.

\(\square\)
Then we show how to compute small rational determinants by using Chinese Remaindering and the computation of determinants over small fields.

**Lemma 29.** For every $c > 0$, $\text{Det}(\frac{\log n}{\log \log n}, \log^c n, v) \in \mathsf{AC}^0$

**Proof.** Let $b = \log^c n$. The basic idea is to use the Chinese Remainder Theorem, CRT (see e.g. [HAB02]) with prime moduli that are of magnitude at most $\log^{c+1} n$ to obtain the determinant of $2^b A$ which is an integer matrix (since the entries are $b$-bit rationals of magnitude at most $2^b$). For $n^{O(1)}$ primes this problem is solvable in $\mathsf{TNC}^0$ by [HAB02] and hence in $\mathsf{L}$. Thus, by Lemma 6 it is in $\mathsf{AC}^0$ for primes of magnitude polylogarithmic. We of course need to compute the determinants modulo the small primes for which we use Lemma 31.

Notice that in the definition of $\text{Det}$, the third argument, that is $v$, is needed for error analysis which is done in the following lemma. For our purposes, $v \leq \frac{1}{l+1}$.

**Lemma 30.** Let $A$ be a $l \times l$ matrix with entries that are $b$-bit rationals smaller than $l^{-1}$. Let $\tilde{A}$ be a $l \times l$ matrix each of whose entries is a $B$-bit approximation to the corresponding entry of $A$. Then assuming $B = \Omega(l^2)$, $\det(\tilde{A})$ is a $B$-bit approximation to $\det(A)$.

**Proof.** Difference between corresponding monomials in the two determinants is easily seen to be upper bounded by $2^{-(B-l-1)}$ in magnitude. Notice that the assumption $B = \Omega(l^2)$ tacitly implies that we can neglect all monomials that contain more than one term of magnitude $2^{-B}$ and just need to consider the terms that consist of exactly one $2^{-B}$ and the rest being the actual entries. Hence the (signed) sum over all monomial (differences) is upper bounded by $2^{-B}$ in magnitude.

**Lemma 31.** (Paraphrased from Theorem 8 [DMVZ18]) If $p \in O(n^c)$ is a prime then, $\text{Det}_p(\frac{\log n}{\log \log n}) \in \mathsf{AC}^0$.

### 4.4 The complexity of polynomial division

We use a slight modification of the Kung-Sieveking algorithm as described in [ABD14][HV06]. The algorithm in [ABD14] worked over finite fields while here we apply it to divide polynomials of small heights and degrees over rationals. The algorithm and its proof of correctness follows Lemma 7 from [ABD14] in verbatim. We reproduce the relevant part for completeness (with minor emendments to accommodate for the characteristic):
Lemma 32. Let $g(x)$ of degree $n$ and $f(x)$ of degree $m$ be monic univariate polynomials over $\mathbb{Q}[x]$, such that $g(x) = q(x)f(x) + r(x)$ for some polynomials $q(x)$ of degree $(n - m)$ and $r(x)$ of degree $(m - 1)$. Then, given the coefficients of $g$ and $f$, the coefficients of $r$ can be computed in $\text{TC}^0$. In other words $\text{Div} (n,m,b) \in \text{TC}^0$ if $m < n$ and $b = n^{O(1)}$.

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{i=0}^{n} b_i x^i$, $r(x) = \sum_{i=0}^{m-1} r_i x^i$ and $q(x) = \sum_{i=0}^{n-m} q_i x^i$. Since $f, g$ are monic, we have $a_m = b_n = 1$. Denote by $f_R(x), g_R(x), r_R(x)$ and $q_R(x)$ respectively the polynomial with the $i$-th coefficient $a_{m-i}, b_{n-i}, r_{m-i-1}$ and $q_{n-m-i}$ respectively. Then note that $x^m f(1/x) = f_R(x)$, $x^n g(1/x) = g_R(x)$, $x^{n-m} q(1/x) = q_R(x)$ and $x^{m-1} r(1/x) = r_R(x)$.

We use the Kung-Sieveking algorithm (as implemented in [ABD14]). The algorithm is as follows:

1. Compute $\tilde{f}_R(x) = \sum_{i=0}^{n-m} (1 - f_R(x))^i$ via interpolation.
2. Compute $h(x) = \tilde{f}_R(x) g_R(x) = c_0 + c_1 x + \ldots + c_{d(n-m)+n} x^{m(n-m)+n}$.
   From which the coefficients of $q(x)$ can be obtained as $q_i = c_{m(n-m)+n-i}$.
3. Compute $r(x) = g(x) - q(x) f(x)$.

The proof of correctness of the algorithm is identical to that in [ABD14]. The proof of the lemma is immediate because polynomial product is in $\text{TC}^0$ from [HAB02].

Lemma 33. $\text{Div}(k,l,b) \text{AC}^0\text{-reduces to Interpolate}(kl, kb)$

Proof. In the first step of the algorithm from the proof of Lemma 32, we need to interpolate a polynomial of degree at most $(k-l)l$. Also, the coefficients of the polynomial are rationals with rationals that are $(k-l)b$ bits long. Notice that we do not require the coefficients of the polynomial to be smaller than $1$.

4.5 Interpolation error analysis

The following lemma shows that no precision is lost during each call to Interpolate.

Lemma 34. Let $f(z)$ be a polynomial of degree $d$ with entries that are rationals not necessarily smaller than $1$. Suppose, $z_i = \frac{i}{(3d)^2}$ for $i \in \{0, \ldots, d\}$ are $d + 1$ values. If we know $B$-bit approximations $\tilde{f}_i$ to the values $f(z_i)$, then the interpolant of these values is a function $\tilde{f}$ whose coefficients are at least $B$-bit approximations of the corresponding coefficients of $f$.
Proof. (Lagrange) interpolation can be viewed as computing $V^{-1}F$ where $V$ is a $(d + 1) \times (d + 1)$ Vandermonde matrix such that $V_{ij} = z_i^j$ while $F_i = f(z_i)$ are entries of a column vector. The determinant of the Vandermonde matrix is $\prod_{0 \leq j < i \leq d} (z_i - z_j)$. This equals $\prod_{0 \leq j < i \leq d} \frac{i-j}{(3d)^2} = \left( \prod_{i=1}^{d} i! \right) \frac{1}{(3d)^{d(d-1)}}$. On the other hand, the various co-factors are upper bounded in magnitude by $d! \prod_{i=1}^{d} z_i^{d-i+1} = d! \prod_{i=1}^{d} \frac{1}{(3d)^{2(d-i+1)}} = d! \prod_{i=1}^{d} \frac{1}{(3d)^{2(d-i+1)}} \leq d! \prod_{i=1}^{d} \frac{1}{(3d)^{d(d-1)}}$ by considering the monomial with the largest magnitude.

Thus an entry of the inverse i.e. the ratio of a co-factor and the determinant is upper bounded by: $\frac{d! (3d)^{d(d-1)}}{(3d)^{d(d+1)}} = \frac{d!}{(3d)^{d+1}} < 1$.

Hence the coefficients of $V^{-1}(F - \tilde{f})$ (where $\tilde{f}$ is the column vector with entries $\tilde{f}_i$) are bounded by $2^{-B}$ completing the proof. Notice that we do not use the magnitude of $f(z_i)$ or of $\tilde{f}_i$ in the proof but only that their difference is small.

### 4.6 Putting it together with error analysis

We now reach the main theorem of this Section:

**Theorem 35.** Let $T$ be an $(n, \log n, \log^2 n, \frac{\log n}{\log \log n})$-dynamic adjacency matrix. Then we can maintain in $\text{DynAC}^0$, a matrix $\tilde{T}$ such that $|\tilde{T} - T^{\log n}| < \frac{1}{n^{\omega(1)}}$.

**Proof.** We use the reductions presented in Lemmas 20, 22, 24, 25, 28 and 27.
to prove the result.

\textbf{DynMatPow}(n, d, b, k, l) ≤ \textbf{AC}^0\textbf{DynBipMatPow}(2n, d, b, 2k, l)

\textbf{Lemma 20} ≤ \textbf{AC}^0\textbf{MatPow}(l, d, b, 2k, l)

\textbf{Lemma 22} ≤ \textbf{AC}^0\textbf{MatPow}(l, 0, lb, l) \land \textbf{DetPoly}(l, 1, lb + 8k^2d) \land \textbf{Interpolate}(2dk, 2kl^2b + 8k^3ld) \land \textbf{Div}(2k, l, l^2b + 8k^2ld)

\textbf{Lemma 24} ≤ \textbf{AC}^0\textbf{MatPow}(l, 0, lb, l) \land \textbf{DetPoly}(l, 1, lb + 8k^2d) \land \textbf{Interpolate}(2dk, 2kl^2b + 8k^3ld) \land \textbf{Div}(2k, l, l^2b + 8k^2ld)

\textbf{Lemma 25} ≤ \textbf{AC}^0\textbf{DetPoly}(l, 1, lb) \land \textbf{Det}(l + 1, lb, (l + 1)^{-1}) \land \textbf{Interpolate}(0, lb) \land \textbf{DetPoly}(l, 1, lb + 8k^2d) \land \textbf{Interpolate}(2dk, 2kl^2b + 8k^3ld) \land \textbf{Div}(2k, l, l^2b + 8k^2ld)

\textbf{Lemma 28} ≤ \textbf{AC}^0\textbf{Interpolate}(dl, ldb) \land \textbf{Det}(l, ldb, (l + 1)^{-1}) \land \textbf{Interpolate}(2dk, 2kl^2b + 8k^3ld)

\textbf{Lemma 33} \equiv \textbf{Interpolate}(2\log^2 n, \log n \log^5 n \log log n, 10 \log^4 n \log log n, \log n \log log n, (\log log n + 1)^{-1})

Each \textbf{DynMatPow} call boils down to a number of \textbf{Det}, \textbf{Interpolate} calls as above.

Though there is no loss of precision in each call to \textbf{Det} (Lemma 30) and \textbf{Interpolate} (Lemma 34), in Lemma 22 we lose \(O(1)\)-bits of precision (See Lemma 23). However, the length of the bit representation grows by a factor of \(9\frac{\log^2 n}{\log \log n}\) at every batch. Thus to keep the number of bits under control we need to truncate the matrix at \(\log^2 n\)-bits again so that now the powered matrix is now a \(\log^2 n - O(1)\)-bit approximation. This \(O(1)\) will deteriorate at every step so that we can afford to perform at least \(\Omega(\log n)\) steps before
we recompute the results from scratch, that is, do muddling.

We can do muddling by invoking Lemma 3 where we pick \( A \) to be the algorithm from Lemma 4 and with the above sequence of reductions as the dynamic program \( P \) for handling a batch (or actually two batches – one old and one new) of changes.

\[ \square \]

5 Conclusion

In this paper we solve a gap version of the expansion testing problem, wherein we want to test if the expansion is greater than \( \alpha \) or less than a \( \alpha' \). The dependence of \( \alpha' \) on \( \alpha \) in this paper is due to the approach of using random walks and testing the conductance. It is natural to ask if there is an alternate method leading to a better dependence. A more natural question is whether we can maintain an approximation of the second largest eigenvalue of a dynamic graph.

An alternative direction of work would be to improve on the number of updates allowed per round. In this paper, we show how to test expansion when almost logarithmic \( O(\frac{\log n}{\log \log n}) \) changes are allowed per round. The largest determinant that we can compute in \( \text{AC}^0 \) is at most of an almost logarithmic size and is the bottle neck that prevents us from improving this bound. We know of another way (obtained via careful adaptation of a proof in [Nis94]) to approximate the powers of the transition matrix when \( \log^{O(1)} n \) changes are allowed per round. Unfortunately, we don’t get a strong enough approximation that leads to an algorithm for approximating conductance.

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