Mapping properties of Fourier transforms, II

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Abstract
This is the direct continuation of the paper [2] using the same notation as there without further explanations. It deal with continuous and compact mappings of the Fourier transform $F$ between some weighted function spaces on $\mathbb{R}^n$.

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1 Basic properties

As already mentioned in [2, Problem 5.4] it is natural to deal with mapping properties of the Fourier transform $F$ in the context of the weighted spaces

$$B_{p,q}^s(\mathbb{R}^n, w_\alpha), \quad s \in \mathbb{R} \quad \text{and} \quad 0 < p, q \leq \infty,$$

(1.1)
as introduced in [2, Definition 4.3] where again

$$w_\alpha(x) = (1 + |x|^2)^{\alpha/2}, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}. \quad (1.2)$$

In addition to the isomorphic mapping $f \mapsto w_\alpha f$,

$$\|w_\alpha f |B_{p,q}^s(\mathbb{R}^n)| \sim \|f |B_{p,q}^s(\mathbb{R}^n, w_\alpha)\|$$

(1.3)
according to [2] (4.6) the lifting

$$\|w_\alpha \hat{f} |B_{p,q}^s(\mathbb{R}^n, w_\beta)| \sim \|f |B_{p,q}^{s+\alpha}(\mathbb{R}^n, w_\beta)\|, \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad (1.4)$$

will be of some use for us, [1, Theorem 6.5, pp. 265–266] and the references given there. We concentrate again on

$$B_{p,q}^s(\mathbb{R}^n, w_\alpha) = B_{p,q}^s(\mathbb{R}^n, w_\alpha), \quad s \in \mathbb{R}, \quad \alpha \in \mathbb{R}, \quad 1 < p < \infty,$$

(1.5)
and its special case
\[ H^s(\mathbb{R}^n, w_\alpha) = B^s_2(\mathbb{R}^n, w_\alpha) = B^s_{2,2}(\mathbb{R}^n, w_\alpha), \quad s \in \mathbb{R}, \quad \alpha \in \mathbb{R}. \quad (1.6) \]

**Proposition 1.1.** Let \( s \in \mathbb{R} \) and \( \alpha \in \mathbb{R} \). Then the Fourier transform
\[ F : H^s(\mathbb{R}^n, w_\alpha) \hookrightarrow H^\alpha(\mathbb{R}^n, w_\alpha) \quad (1.7) \]
is an isomorphic mapping,
\[ FH^s(\mathbb{R}^n, w_\alpha) = H^\alpha(\mathbb{R}^n, w_\alpha). \quad (1.8) \]

**Proof.** Let \( f \in H^s(\mathbb{R}^n, w_\alpha) \). Then it follows from (1.4) (with \( L^2_2(\mathbb{R}^n, w_\alpha) \) in place of \( B^s_{p,q}(\mathbb{R}^n, w_\beta) \)), \( FL_2(\mathbb{R}^n) = L_2(\mathbb{R}^n) \), and (1.3) that
\[ \| \hat{f} \|_{H^\alpha(\mathbb{R}^n, w_\alpha)} \sim \| (w_\alpha f) \hat{\,} \|_{L_2(\mathbb{R}^n, w_\alpha)} \]
\[ \sim \| w_\alpha (w_\alpha f) \hat{\,} \|_{L_2(\mathbb{R}^n)} \]
\[ \sim \| w_\alpha f \|_{H^s(\mathbb{R}^n)} \]
\[ \sim \| f \|_{H^s(\mathbb{R}^n, w_\alpha)}. \quad (1.9) \]

Conversely, for any \( g \in H^\alpha(\mathbb{R}^n, w_\alpha) \) there is an \( f \in H^s(\mathbb{R}^n, w_\alpha) \) with \( \hat{f} = g \) and a counterpart of (1.9). This proves the proposition. \( \square \)

**Remark 1.2.** In particular,
\[ FH^s(\mathbb{R}^n, w_\alpha) = H^s(\mathbb{R}^n, w_\alpha), \quad s \in \mathbb{R}, \quad (1.10) \]
may be considered as the weighted extension of
\[ FL_2(\mathbb{R}^n) = L_2(\mathbb{R}^n), \quad L_2(\mathbb{R}^n) = H^0(\mathbb{R}^n, w_0). \quad (1.11) \]

Now it is quite clear that the role played by \( L_2(\mathbb{R}^n) \) and \( L_2(\mathbb{R}^n, w_\alpha), \alpha > 0, \) in the theory of compact mappings of \( F \) between unweighted spaces as developed in [2] is now taken over by \( H^s(\mathbb{R}^n, w_\alpha), s \in \mathbb{R}, \) and \( H^s(\mathbb{R}^n, w_{s+\alpha}) \) where again the degree of compactness is measured in terms of entropy numbers as recalled in [2, Definition 4.1] including related referenced to the literature.

One may ask for entropy numbers of compact mappings
\[ F : B^{s_1}_p(\mathbb{R}^n, w_{\alpha_1}) \hookrightarrow B^{s_2}_p(\mathbb{R}^n, w_{\alpha_2}), \quad (1.12) \]
for fixed weights, which means \( \alpha_1 = \alpha_2, \) for fixed smoothness, which means \( s_1 = s_2, \) or for a mixture of both. But this will not be done here in detail. We add now a comment to the case of fixed weights and shift the more interesting task for fixed smoothness to the next section.
Theorem 1.3. Let $-\infty < s_2 < s < s_1 < \infty$. Then
\[
F : H^{s_1}(\mathbb{R}^n, w_s) \hookrightarrow H^{s_2}(\mathbb{R}^n, w_s) \tag{1.13}
\]
is compact and
\[
e_k(F) \sim \begin{cases} k^{-\frac{s_2}{n}} & \text{if } \sigma_2 < \sigma_1, \\
\left(\frac{k}{\log k}\right)^{-\frac{\sigma_2}{n}} & \text{if } \sigma_2 = \sigma_1, \\
k^{-\frac{s_1}{n}} & \text{if } \sigma_2 > \sigma_1,
\end{cases} \tag{1.14}
\]
for $2 \leq k \in \mathbb{N}$, where $s_1 = s + \sigma_1$ and $s_2 = s - \sigma_2$.

Proof. By (1.8) one has
\[
FH^{s_1}(\mathbb{R}^n, w_s) = H^{s}(\mathbb{R}^n, w_{s_1}) \tag{1.15}
\]
This extends [2, (4.37)] from $s = 0$ to $s \in \mathbb{R}$. Then one can argue as there, relying now on [2, Proposition 4.5]. \hfill \Box

Remark 1.4. This extends [2, Theorem 4.8(ii)] from $s = 0$ to $s \in \mathbb{R}$. It is quite clear that there are related counterparts of [2, Theorem 4.8(iii),(iv)] for the compact mappings
\[
F : B^{s+\sigma_1}_p(\mathbb{R}^n, w_s) \hookrightarrow B^{s-\sigma_2}_p(\mathbb{R}^n, w_s), \quad 1 < p < \infty, \quad s \in \mathbb{R} \tag{1.16}
\]
with
\[
\begin{cases} \sigma_1 > d_p^n, \quad \sigma_2 > 0 & \text{if } 1 < p \leq 2, \\
\sigma_1 > 0, \quad \sigma_2 > |d_p^n| & \text{if } 2 \leq p < \infty,
\end{cases} \tag{1.17}
\]
where as there
\[
d_p^n = 2n(\frac{1}{p} - \frac{1}{2}), \quad n \in \mathbb{N}, \quad 1 < p < \infty. \tag{1.18}
\]

2 Main assertions

So far we dealt in Theorem 1.3 and in the indicated generalizations in (1.16) with the same weight both in the source spaces and in the target spaces. The outcome is apparently a rather straightforward generalization of corresponding assertions in [2] for the unweighted spaces. The question arises what happens if the weights in the source spaces and in the target spaces are different. For this purpose it seems to be reasonable to fix first not only the
integrability parameter $p$ with $1 < p < \infty$, but also the smoothness $s \in \mathbb{R}$ and to ask, suggested by (1.10), for compact mappings

$$F : \quad B_{p}^{s}(\mathbb{R}^{n}, w_{s+\alpha}) \hookrightarrow B_{p}^{s}(\mathbb{R}^{n}, w_{s-\beta}), \quad (2.1)$$

$1 < p < \infty$, $s \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$. First we deal with the case $p = 2$ using the notation recalled in (1.6).

**Proposition 2.1.** Let $s \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$. Then

$$F : \quad H^{s}(\mathbb{R}^{n}, w_{s+\alpha}) \hookrightarrow H^{s}(\mathbb{R}^{n}, w_{s-\beta}) \quad (2.2)$$

is compact and

$$e_{k}(F) \sim \begin{cases} k^{-\frac{\alpha}{n}} & \text{if } \alpha < \beta, \\ (\frac{k}{\log k})^{-\frac{\alpha}{n}} & \text{if } \alpha = \beta, \\ k^{-\frac{\alpha}{n}} & \text{if } \alpha > \beta, \end{cases} \quad (2.3)$$

$2 \leq k \in \mathbb{N}$.

**Proof.** By [2 Corollary 4.7(ii)] one has that

$$\text{id} : \quad L_{2}(\mathbb{R}^{n}, w_{\alpha}) \hookrightarrow H^{-\sigma}(\mathbb{R}^{n}) \quad (2.4)$$

with $\alpha > 0$ and $\sigma > 0$ is compact and

$$e_{k}(F) \sim \begin{cases} k^{-\frac{\sigma}{n}} & \text{if } \sigma < \alpha, \\ (\frac{k}{\log k})^{-\frac{\sigma}{n}} & \text{if } \sigma = \alpha, \\ k^{-\frac{\sigma}{n}} & \text{if } \sigma > \alpha, \end{cases} \quad (2.5)$$

$2 \leq k \in \mathbb{N}$. Then it follows from the two isomorphisms (1.3) and (1.4) that

$$\text{id} : \quad H^{s+\alpha}(\mathbb{R}^{n}, w_{s}) \hookrightarrow H^{s}(\mathbb{R}^{n}, w_{s-\beta}), \quad s \in \mathbb{R}, \quad (2.6)$$

with $\alpha > 0$ and $\beta > 0$ is compact and

$$e_{k}(F) \sim \begin{cases} k^{-\frac{\alpha}{n}} & \text{if } \alpha < \beta, \\ (\frac{k}{\log k})^{-\frac{\alpha}{n}} & \text{if } \alpha = \beta, \\ k^{-\frac{\alpha}{n}} & \text{if } \alpha > \beta, \end{cases} \quad (2.7)$$

$2 \leq k \in \mathbb{N}$. Now one obtains (2.3) from (2.6), (2.7) and

$$FH^{s}(\mathbb{R}^{n}, w_{s+\alpha}) = H^{s+\alpha}(\mathbb{R}^{n}, w_{s}) \quad (2.8)$$

according to Proposition 1.1.
We extend Proposition 2.1 and ask for conditions ensuring that \( F \) in (2.1) is compact. Let \( d_p^n = 2n(\frac{1}{p} - \frac{1}{2}) \) be as in (1.18).

**Theorem 2.2.** (i) Let \( 1 < p \leq 2, \ s \in \mathbb{R} \) and \( \alpha > 0, \beta > 0 \). Then

\[
F : \ B^s_p(\mathbb{R}^n, w_{s+\alpha}) \hookrightarrow B^s_p(\mathbb{R}^n, w_{s-d_p^n-\beta}) \quad (2.9)
\]

is compact and

\[
e_k(F) \leq c \begin{cases} 
  k^{-\frac{\alpha}{n}} & \text{if } \alpha < \beta, \\
  \left( \frac{k}{\log k} \right)^{-\frac{\alpha}{n}} (\log k)^{\frac{1}{p} - \frac{1}{2}} & \text{if } \alpha = \beta, \\
  k^{-\frac{\alpha}{n}} & \text{if } \alpha > \beta,
\end{cases} \quad (2.10)
\]

for some \( c > 0 \) and all \( 2 \leq k \in \mathbb{N} \).

(ii) Let \( 2 \leq p < \infty, \ s \in \mathbb{R} \) and \( \alpha > 0, \beta > 0 \). Then

\[
F : \ B^s_p(\mathbb{R}^n, w_{s+|d_p|+\alpha}) \hookrightarrow B^s_p(\mathbb{R}^n, w_{s-\beta}) \quad (2.11)
\]

is compact and

\[
e_k(F) \leq c \begin{cases} 
  k^{-\frac{\alpha}{n}} & \text{if } \alpha < \beta, \\
  \left( \frac{k}{\log k} \right)^{-\frac{\alpha}{n}} (\log k)^{\frac{1}{p} - \frac{1}{2}} & \text{if } \alpha = \beta, \\
  k^{-\frac{\alpha}{n}} & \text{if } \alpha > \beta,
\end{cases} \quad (2.12)
\]

for some \( c > 0 \) and all \( 2 \leq k \in \mathbb{N} \).

**Proof.** Step 1. The case \( p = 2 \) is covered by Proposition 2.1 (even with equivalence instead of an estimate from above).

Step 2. Let \( 1 < p < 2 \). In modification of (2.9) we ask first under which conditions \( F \) in (2.1),

\[
F : \ B^s_p(\mathbb{R}^n, w_{s+\alpha}) \hookrightarrow B^s_p(\mathbb{R}^n, w_{s-\beta}) \quad (2.13)
\]

is compact. By the isomorphism (1.3) and the well-known embedding for unweighted spaces we have the continuous embedding

\[
\text{id}_1 : \ B^s_p(\mathbb{R}^n, w_{s+\alpha}) \hookrightarrow H^{s-n(\frac{1}{p} - \frac{1}{2})}(\mathbb{R}^n, w_{s+\alpha}). \quad (2.14)
\]

This shows, combined with

\[
F H^{s-n(\frac{1}{p} - \frac{1}{2})}(\mathbb{R}^n, w_{s+\alpha}) = H^{s+\alpha}(\mathbb{R}^n, w_{s-n(\frac{1}{p} - \frac{1}{2})}), \quad (2.15)
\]
covered by Proposition 1.1 that (2.13) can be reduced to the question under which conditions

\[ id_2 : \quad H^{s+\alpha}(\mathbb{R}^n, w_{s-n(\frac{1}{p} - \frac{1}{2})}) \hookrightarrow B_p^s(\mathbb{R}^n, w_{s-\beta}) \] (2.16)

is compact. For this purpose we specify [2, Proposition 4.5, (4.7)–(4.14)] to \( id_2 \). This requires in the notation used there \( \alpha > 0, \delta = \alpha + n(1 - \frac{1}{p} + \frac{n}{2}) = \alpha + \frac{1}{2}d_p^n, \quad \varrho = \frac{\delta}{n} > 0, \) (2.17)

and

\[ \frac{1}{p} < \frac{1}{p_*} = \frac{1}{2} + \frac{1}{n}(s - \frac{n}{p} + \frac{n}{2} - s + \beta) = 1 - \frac{1}{p} + \frac{\beta}{n} \] (2.18)

resulting in \( \beta > 2n(\frac{1}{p} - \frac{1}{2}) = d_p^n \). Then it follows from [2, Proposition 4.5] that \( id_2 \) in (2.16) is compact. Replacing there \( s_1 \) by \( s + \alpha, s_2 \) by \( s, \alpha \) by \( \beta - n(\frac{1}{p} - \frac{1}{2}) = \beta - \frac{1}{2}d_p^n, p_1 \) by 2 and \( p_2 \) by \( p \) one obtains for the corresponding entropy numbers

\[ e_k(\text{id}_2) \leq c \begin{cases} k^{-\frac{\alpha}{p}} & \text{if } \alpha + \frac{1}{2}d_p^n < \beta - \frac{1}{2}d_p^n, \\ (\frac{k}{\log k})^{-\frac{\alpha}{p}}(\log k)^{\frac{1}{p} - \frac{1}{2}} & \text{if } \alpha + \frac{1}{2}d_p^n = \beta - \frac{1}{2}d_p^n, \\ k^{-\frac{\alpha}{p} + \frac{1}{p} - \frac{1}{2} + \frac{1}{p} - \frac{1}{2}} & \text{if } \alpha + \frac{1}{2}d_p^n > \beta - \frac{1}{2}d_p^n. \end{cases} \] (2.19)

Then (2.10) follows from (2.14)–(2.16) and (2.19) replacing there \( \beta \) by \( \beta + d_p^n \).

Step 3. Let \( 2 < p < \infty \). As in Step 4 of the proof of [2, Theorem 4.8] we rely on the duality

\[ B_p^s(\mathbb{R}^n, w_\sigma)' = B_{p'}^{-s}(\mathbb{R}^n, w_{-\sigma}), \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \] (2.20)

\( s \in \mathbb{R}, \sigma \in \mathbb{R}, \) in the framework of the dual pairing \((S(\mathbb{R}^n), S'(\mathbb{R}^n))\). The isomorphism (1.3) shows that also the weighted spaces \( B_p^s(\mathbb{R}^n, w_\sigma) \) are isomorphic to \( \ell_p \). Then one can apply the duality theory for entropy numbers as described there. Using \( d_{p'}^n = -d_p^n \) one obtains part (ii) of the above theorem by the indicated duality from part (i).

**Remark 2.3.** The outcome may justify to deal not only with unweighted spaces as in the Theorems [2, Theorem 4.8] but also with the above weighted spaces for fixed \( s \in \mathbb{R} \) and \( p \) with \( 1 < p < \infty \). The typical gap \( d_p^n \) for the smoothness in the unweighted case is now shifted to the weights.
Remark 2.4. If $1 < p \leq 2$, $\alpha \neq \beta$ and $0 < q_\alpha, q_\beta \leq \infty$ then the estimate (2.10) remains valid for
\[ F : B_{p,q_\alpha}^s(\mathbb{R}^n, w_{s+\alpha}) \hookrightarrow B_{p,q_\beta}^s(\mathbb{R}^n, w_{s-d_\alpha^p-\beta}). \] (2.21)
Similarly for (2.11), (2.12). This follows by real interpolation in the same way as in [2, Corollary 4.10]. For this purpose one should first shift the $s$–dependence for the weights to the exponents of the estimates for the corresponding entropy numbers and use afterwards that the interpolation of $B_{p,q}^s(\mathbb{R}^n, w_\gamma)$ for a fixed weight $w_\gamma$ is the same as for their unweighted ancestors $B_{p,q}^s(\mathbb{R}^n)$.

3 Spectral theory

Let $K : B \hookrightarrow B$ be a linear compact operator in a complex infinitely dimensional quasi–Banach space $B$. Let $e_k(B)$ be its entropy numbers. Let $\{\lambda_k(K)\}$ be the sequence of all non–zero eigenvalues of $K$, repeated according to their algebraic multiplicity and naturally ordered by magnitude. We used in [3] Carl’s observation
\[ |\lambda_k(K)| \leq \sqrt{2e_k(K)}, \quad k \in \mathbb{N}, \] (3.1)
to study the distribution of eigenvalues for distinguished so–called Fourier operators based on corresponding assertions in [2]. Details, explanations and references may be found in [3]. This will not be repeated here. One can now use Theorem 2.2 and other assertions obtained in this note to extend this theory to further classes of operators. This will not be done. But we illustrate what can be expected by a simple example. Let as before $w_\sigma(x) = (1 + |x|^2)^{\sigma/2}, \sigma \in \mathbb{R}, x \in \mathbb{R}^n$. Recall that $H^0(\mathbb{R}^n, w_\sigma) = L_2(\mathbb{R}^n, w_\sigma)$.

Proposition 3.1. Let $\alpha > 0$. Then $K_\alpha = w_{-\alpha} \circ F \circ w_{-\alpha}$ is a compact operator in $L_2(\mathbb{R}^n)$ and for some $c > 0$,
\[ |\lambda_k(K_\alpha)| \leq c \left( \frac{k}{\log k} \right)^{-\frac{n}{\alpha}}, \quad 2 \leq k \in \mathbb{N}. \] (3.2)

Proof. One can decompose $K_\alpha$ into the two isomorphic mappings $f \mapsto w_{-\alpha}f$ from $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n, w_\alpha)$ and from $L_2(\mathbb{R}^n, w_{-\alpha})$ onto $L_2(\mathbb{R}^n)$ combined with
\[ F : L_2(\mathbb{R}^n, w_\alpha) \hookrightarrow L_2(\mathbb{R}^n, w_{-\alpha}). \] (3.3)
Then (3.2) follows from (3.1) and Proposition 2.1 with $s = 0$ and $\alpha = \beta > 0$.

\[ \square \]

Remark 3.2. As said, Proposition 3.1 may serve as an example which type of assertions can be expected if one deals more systematically with problems of this type based on [2], [3].

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