A COMPARISON OF SPH ARTIFICIAL VISCOSITIES AND THEIR IMPACT ON THE KEPLERIAN DISK

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ABSTRACT

Hydrodynamical simulations of rotating disks play important roles in the field of astrophysical and planetary science. Smoothed particle hydrodynamics (SPH) has been widely used for such simulations. However, it has been known that when using SPH, a cold and thin Kepler disk breaks up due to the unwanted angular momentum transfer. Two possible reasons have been suggested for this breaking up of the disk: the artificial viscosity (AV) and the numerical error in the evaluation of pressure gradient in SPH. Which one is dominant is still unclear. In this paper, we investigate the reason for this rapid breaking up of the disk. We implemented most of the popular formulations of AV and switches, and measured the angular momentum transfer due to both AV and the error of SPH’s estimate of the pressure gradient. We found that the angular momentum transfer due to AV at the inner edge triggers the breaking up of the disk. We also found that the classical von Neumann–Richtmyer–Landshoff type AV with a high-order estimate for $\nabla \cdot v$ can maintain the disk for $\sim$100 orbits even when used with the standard formulation of SPH.

Key words: hydrodynamics – methods: numerical

1. INTRODUCTION

Because astronomical objects are formed by gravitational collapse and by the conservation of the initially imprinted tiny angular momentum made by rotating disks during the collapsing process, e.g., galactic disks, accretion disks, protoplanetary disks, and post-impact debris disks, hydrodynamical simulations of rotating disks play important roles in the field of astrophysical and planetary science. The smoothed particle hydrodynamics (SPH, Gingold & Monaghan 1977; Lucy 1977; Monaghan 1992; Springel 2010) method has been widely used for these simulations.

However, it has been long known that SPH cannot follow the long-term evolutions of thin and cold disks (e.g., Maddison et al. 1996, Imaeda & Inutsuka 2002; Okamoto et al. 2003). There are two possible reasons for the decay of the disk. One is the error in the discretized hydrodynamical force (pressure gradient), which is often referred to as the “$E_0$ error” or “zeroth order error” in the SPH discretization (e.g., Dilts 1999; Imaeda & Inutsuka 2002; Okamoto et al. 2003; Read et al. 2010). Okamoto et al. (2003) pointed out that hydrodynamical torque leads to the disruption of the galactic disk embedded in a hot halo.

The other possible reason is the artificial viscosity (AV). AV is a numerical dissipation term that is required to capture the shock, first proposed by Richtmyer (1948) and von Neumann & Richtmyer (1950). Since AV does not exist in the original governing equations, AV should act only on the shock. In SPH, there are two well-known formulations of AV. One is based on the “$\nabla \cdot v$” term where $v$ means velocity, and the other is based on the relative velocities of neighbor particles (see Sections 2.1 and 2.2 for detail). In the following, we call the former one “$\nabla \cdot v$” AV and the latter one “pairwise” AV. In almost all recent works, the latter “pairwise” AV has been used. It has been pointed out, however, that AV, particularly the latter formulation, operates not only on the shock, but also on the velocity shears. This spurious shear viscosity causes unwanted angular momentum transfer in rotating systems. To suppress this effect, several “switches” to suppress AV in shear flow have been proposed (e.g., Balsara 1995; Cullen & Dehnen 2010). Beck et al. (2016) have compared their implementation of SPH with some of these switches and the “standard” SPH (SSPH) in the context of galaxy formation.

Recently, Gaburov & Nitadori (2011) and Hopkins (2015) developed a new scheme, weighted particle hydrodynamics (WPH). Unlike SPH, WPH adopts the Riemann solver to deal with the shock, which introduces sufficient viscosity at the shock. In addition, they argued that WPH resolved the zeroth order error. Hopkins (2015) reported that WPH could maintain the thin and cold rotational disk for a much longer time than SPH could.

Despite these efforts, the main reason for the rapid disruption of the disk is still unclear. In this paper, to clarify the reason of the disruption of the disk, we compare most of the popular formulations of AV, their switches, and their derivative operators. We performed a systematic survey of all possible combinations of different formulations. We then investigate the angular momentum transfer due to both AV and SPH errors.

This paper is organized as follows. In Section 2, we overview the idea of AV, its numerical formulations, and switches. In Section 3, we show the results of the comparison test. In Section 4, we summarize the reasons for the disruption of the disk.

2. THE ARTIFICIAL VISCOSITY

In this section, we describe the formulation and implementation of AV that is used in this paper. The formulae for the pressure gradient evaluation used in the SPH method are given in Appendix A.

In this paper we discuss two well-known formulations of AV. In Section 2.1, we present the classical von Neumann–Richtmyer–Landshoff (vNRL) AV, which is based on the discretized estimate of $\nabla \cdot v$ (Richtmyer 1948; von Neumann & Richtmyer 1950; Landshoff 1955; Lucy 1977). In Section 2.2, we present the more widely used form of AV, based on the pairwise relative velocity of particles (Monaghan & Gingold 1983; Monaghan 1997). In Sections 2.3 and 2.4, we discuss the schemes proposed to reduce...
the strength of AV in regions without shocks. In Section 2.3, we discuss time-dependent AVs (Morris & Monaghan 1997; Cullen & Dehnen 2010; Read & Hayfield 2012; Rosswog 2015), and in Section 2.4, we discuss so-called shear switches (Balsara 1995; Cullen & Dehnen 2010).

2.1. vNRL AV to SPH

2.1.1. Formulation

In the vNRL AV term, an artificial “pressure” term, \( p^{AV} \), is added to the pressure in SPH equations (Equations (29) and (30)). The artificial pressure \( p^{AV} \) is given by

\[
p^{AV}_i = \begin{cases} 
- \alpha^{AV}_i \rho_i c_i h_i (\nabla \cdot v_i) \\
+ \beta^{AV}_i \rho_i h_i^2 (\nabla \cdot v_i)^2 & \text{if } (\nabla \cdot v_i) < 0, \\
0 & \text{otherwise}, 
\end{cases} 
\]

where \( \rho, c \) and \( h \) are the density, the sound speed, and the smoothing length, respectively. The parameters \( \alpha^{AV} \) and \( \beta^{AV} \) determine the strength of the viscosity. Typically \( \alpha^{AV} = 1 \) and \( \beta^{AV} = 2 \alpha^{AV} \) are used. This approach has been tested by Monaghan & Gingold (1983) and Hernquist & Katz (1989).

The corresponding timestep \( \Delta t \) for this AV is given by (Hernquist & Katz 1989):

\[
\Delta t_{CFL} = \frac{C_{CFL} h_i}{|\nabla \cdot v_i| + c_i + 1.2 (\alpha_i c_i + \beta_i h_i \min(|\nabla \cdot v_i|, 0))},
\]

where \( C_{CFL} \) is a CFL coefficient that is set to 0.3 in this paper.

2.1.2. The Discretization of the \( \nabla \cdot v \) Term

In this section we discuss the methods used to calculate the \( \nabla \cdot v \) term. To evaluate \( \nabla \cdot v \), the following scheme is widely used (e.g., Monaghan 1992):

\[
\nabla \cdot v_i = \frac{1}{\rho_i} \sum_j m_j (v_j - v_i) \cdot \nabla W(x_j - x_i; h_i),
\]

where \( m \) and \( x \) are the mass and the position vector.

Recently, García-Senz et al. (2012) proposed a discretization of derivative operators that is more accurate than those used in the SSPH discretization. In García-Senz et al. (2012), however, they did not apply their derivative to vector fields. Here, we extend their derivative estimate to \( \nabla \cdot v \) as

\[
\nabla \cdot v_i = \sum_j \frac{m_j}{\rho_j} (v_j - v_i) \cdot [M_j^{-1}(x_j - x_i)] W(x_j - x_i; h_i),
\]

\[
M_j = \sum_j \frac{m_j}{\rho_j} (x_j - x_i) \otimes (x_j - x_i) W(x_j - x_i; h_i).
\]

The derivation of Equation (4) is given in Appendix B.

Note that Price (2005) also derived a derivative operator similar to Equation (4). We have also tested the Price’s (2005) derivative operator. However, the results were similar to those obtained with Equation (4). Thus, we do not show the results with Price’s (2005) derivative operator.

Note that we also need \( \nabla \times v \) and/or \( \nabla \otimes v \) in some cases (see Sections 2.3 and 2.4). By replacing the operator \( \cdot \) in Equations (3) and (4) with \( \times \) or \( \otimes \), we can easily obtain the discretized expression for \( \nabla \times v \) and \( \nabla \otimes v \).

2.2. The Pairwise AV

The “pairwise” formulation of AV (Monaghan & Gingold 1983; Monaghan 1997) is based on the relative velocities between neighboring particles. This AV has been the most widely used form of AVs in SPH, e.g., GASOLINE (Wadsley et al. 2004), Gadget2 (Springel 2005), MAGMA (Rosswog & Price 2007), VINE (Nelson et al. 2009; Wetzstein et al. 2009), Evol (Merlin et al. 2010), and SEREN (Hubber et al. 2011).

Monaghan (1997) derived this pairwise form of AV from an analogy to the Riemann Solver. In this paper, we use Monaghan’s (1997) AV as the representative for the pairwise AV:

\[
\Pi_{ij} = \begin{cases} 
-\alpha^{AV}_i v_{ij} \cdot w_{ij} & (w_{ij} < 0), \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
\alpha^{AV}_i = \frac{\alpha^{AV}_i^1 + \alpha^{AV}_i^2}{2},
\]

\[
v_{ij} = \frac{(r_j - r_i) \cdot (v_j - v_i)}{|r_j - r_i|},
\]

\[
\rho_{ij} = \frac{\rho_i + \rho_j}{2},
\]

where \( \Pi_{ij} \) is the pairwise viscosity term. The acceleration and heating due to this AV term are expressed as

\[
a^{AV}_i = - \sum_j m_j \Pi_{ij} \left( \frac{\nabla W(x_i - x_j; h_i)}{\Omega_i} + \frac{\nabla W(x_j - x_i; h_i)}{\Omega_j} \right),
\]

\[
a^{AV}_i = \frac{1}{2} \sum_j m_j \Pi_{ij} (v_i - v_j) \cdot \frac{\nabla W(x_i - x_j; h_i)}{\Omega_i},
\]

where \( a, u, \) and \( p \) are the acceleration, specific internal energy, and pressure for the particle \( i \). The function \( W \) is the kernel function and \( h \) is the smoothing length, which determines the spread of a particle and \( \Omega \) is the so-called “grad-h” term (Springel & Hernquist 2002; Hopkins 2013). The timestep \( \Delta t \) corresponding to this AV is (Monaghan 1997):

\[
\Delta t_{CFL} = \frac{2 h_i}{\max_j v_{ij}^{\text{sig}}}. 
\]

In addition to the timestep determined from AV, we also need to incorporate the timestep determined from the acceleration itself:

\[
\Delta t_{\text{Acc}} = C_{\text{Acc}} \frac{h_i}{|a_i|}. 
\]
We set $C_{\text{Acc}} = 0.3$. Then, the actual timestep for a particle $i$ is given by

$$\Delta t_i = \min(\Delta t_i^{\text{CFI}}, \Delta t_i^{\text{Acc}}).$$  \hspace{1cm} (15)

### 2.3. Shock Indicator

Morris & Monaghan (1997) proposed to vary $\alpha^{AV}$ so that AV works only on the shock. The basic idea of this approach is to increase $\alpha^{AV}$ when the shock is approaching, and gradually reduce $\alpha^{AV}$ otherwise. Rosswog et al. (2000) suggested the following time derivative for $\alpha^{AV}$:

$$\frac{d\alpha^{AV}}{dt} = (\alpha^{AV}_{\max} - \alpha^{AV}) \max(-\nabla \cdot \mathbf{v}, 0) - \frac{\alpha^{AV} - \alpha^{AV}_{\min}}{\tau},$$

$$\tau \sim \frac{h_i}{c_s}.$$  \hspace{1cm} (16)

If $\nabla \cdot \mathbf{v}$ is negative, $\alpha^{AV}$ is increased with the timescale inversely proportional to $\nabla \cdot \mathbf{v}$, up to the maximum value of $\alpha^{AV}_{\max}$. In this paper, we set $\alpha^{AV}_{\max} = 2$ and $\alpha^{AV}_{\min} = 0.1$.

Cullen & Dehnen (2010) suggested a higher order shock indicator. In their approach, the time derivative of $\nabla \cdot \mathbf{v}$ is used as the shock indicator:

$$\alpha^{AV}_i = \max\left[ \alpha^{AV}_{\max} \frac{A_i}{A_i + \max_j (u^{ij}_{\text{av}})^2 / h_i^2}, 0 \right],$$  \hspace{1cm} (18)

$$A = \max \left[ -f \frac{d(\nabla \cdot \mathbf{v})}{dt}, 0 \right],$$  \hspace{1cm} (19)

$$\frac{d(\nabla \cdot \mathbf{v})}{dt} = \text{tr}\left[ \nabla \otimes \mathbf{a} - (\nabla \otimes \mathbf{v})^2 \right],$$  \hspace{1cm} (20)

$$\frac{d\alpha^{AV}}{dt} = \frac{\alpha^{AV} - \alpha^{AV}_{\min}}{\tau},$$  \hspace{1cm} (21)

where $f$ is the shear switch described in the next section.

Other shock indicators have also been proposed. Read & Hayfield (2012) used $\nabla \cdot (\nabla \cdot \mathbf{v})$ for the shock indicator. Rosswog (2015) combined both the Cullen & Dehnen’s (2010) and the Read & Hayfield’s (2012) approaches. In this paper, for simplicity, we only show the results for the Rosswog et al. (2000) and Cullen & Dehnen (2010) shock indicators.

### 2.4. The Shear Switches

The pairwise AV that we discussed in Section 2.2 has one critical drawback. Since it operates whenever two neighboring particles are approaching, it works as shear viscosity. In order to follow the evolution of a differentially rotating disk, shear viscosity should be suppressed. Balsara (1995) proposed a switch to reduce AV when the divergence of the velocity is smaller than the rotation:

$$f_i = \begin{cases} \frac{|\nabla \cdot \mathbf{v}_i|}{|\nabla \cdot \mathbf{v}_i| + |\nabla \times \mathbf{v}_i| + \varepsilon c_i / h_i}, & \text{when } \nabla \cdot \mathbf{v}_i < 0, \\ 1, & \text{otherwise}. \end{cases}$$

$$\Pi_{ij} \rightarrow f_i f_j \Pi_{ij}.$$  \hspace{1cm} (24)

Cullen & Dehnen (2010) suggested an alternative form of the shear switch:

$$f_i = \frac{[2(1 - R_i)^{\frac{1}{2}} |\nabla \cdot \mathbf{v}_i|^2]^{\frac{1}{2}}}{[2(1 - R_i)^{\frac{1}{2}} |\nabla \cdot \mathbf{v}_i|^2 + \text{tr}(\mathbf{S}_i \mathbf{S}_i^T)]^{\frac{1}{2}}},$$

$$R_i = \frac{1}{\rho_i} \sum_j \text{sign}(\nabla \cdot \mathbf{v}_j) m_j W(x_i - x_j; h_i),$$

$$\mathbf{S} = \frac{1}{2} \left( \nabla \otimes \mathbf{v} + (\nabla \otimes \mathbf{v})^T \right) - \frac{1}{\nu} (\nabla \cdot \mathbf{v}) \mathbf{I},$$

where $\nu$ is the number of dimensions and $\mathbf{I}$ is the identity matrix. Note that since the Cullen & Dehnen (2010) shear switch compares the shear part of the velocity gradient matrix with the divergence part, this switch is really a shear switch.

### 3. KEPLERIAN DISK TEST

As shown in Table 1, we have two options for the form of AV, three for the shock indicator, three for the shear switch, and two for the formula for the discretization of $\nabla \cdot \mathbf{v}$. Therefore, we have $2 \times 3 \times 3 \times 2 = 36$ possible combinations of different schemes. In this section, we show the results of 2D Keplerian tests for all 36 AV implementations. Each run is labeled as (vNRL, M97)-(No, B95, CD10)-(No, R+00, CD10)-(SPH, G+12). The first, second, third, and fourth options are AV form, shock indicator, shear switch, and derivative operators, respectively. Note that once we select the derivative operator, all $\nabla \cdot \mathbf{v}$ terms in the shock indicator and the shear switch are replaced with the selected discretized expression. In this paper, we tested two SPH formulations, the SSPH (for review, see Monaghan 1992) and density -independent SPH (DISPH; Hopkins 2013; Saitoh & Makino 2013). However, we note that the results and conclusions are the same for both methods. Thus, in the following we only show the results with DISPH.

The Keplerian disk consists of cold gas orbiting around the central massive object. The surface density and pressure of the disk are set to uniformly 1.0 and $10^{-6}$, which are the same as those used in Hopkins (2015). The centripetal force $a_{\text{ext}}$ is added to each particle and given by the following equation:

$$a_{\text{ext}} = -\frac{GM}{(r^2 + \varepsilon^2)^{3/2}} x.$$  \hspace{1cm} (28)
In this test, we set $G = 1$ and $M = 1$ and $\varepsilon$ is the softening length that prevents numerical overflow due to the particles that fall down close to the center of the disk. We set $\varepsilon = 0.25$, if and only if $|x| < 0.25$.

We constructed the initial particle distributions as concentric rings (see Appendix C for detail). The inner and outer cutoff radii of the disk are 0.5 and 2.0. We employ 46560 particles in total. Note that we also tested the Cartesian grid initial distribution, similar to Hopkins (2015). The main mechanism for the decay of the disk is the same. In this paper, therefore, we do not show the results for the Cartesian grid case.

In order to carry out a quantitative comparison of the results, we introduce a rough estimate of the disk lifetime. Let us consider the root mean square of the radius of the particles initially located at the inner edge of the disk. When this value exceeds 10% of the radius of the inner edge, we regard the disk as disrupted.

Figures 1 and 2 show the lifetime and snapshots of the Keplerian disk for each AV implementation. The differences among the lifetimes of the disks obtained with different implementations are very large. In the case of vNRL-B95-CD10-G+12, the disk survived for a time more than 10 times longer than with the standard method used today (M97-B95-R+00-SPH). Note that the longest case, vNRL-B95-CD10-G+12', indicates the results with the vNRL-B95-CD10-G+12 scheme, but with $\alpha_{\text{min}}^{AV} = 0.025$ and $\alpha_{\text{max}}^{AV} = 0.5$. These parameters in the AV switch would also play important role for the long-time evolution of the disk.

To clarify the effects of each option on the disk lifetime, we show the dependences of the disk lifetime for each option in Figure 3. Here, we show how the lifetime of the disk changes when we change the method in one type, while keeping the methods for the other three types unchanged. We can clearly

![Figure 1. Lifetime of the disk for each AV implementation in units of Kepler time at $r = 0.5$ (inner edge of the disk). Note that the disk lifetime is shown in the log scale.](image-url)
see the following tendencies. (i) The disk lifetime obtained with vNRL AV is much longer than that with M97 AV. (ii) Both of the two shear switches extend the disk lifetime. The CD10 switch works better than the B95 switch for M97 AV, while the B95 switch works better than the CD10 switch for the vNRL AV. (iii) Both shock indicators extend the disk lifetime. The CD10 shock indicator works better than R+00 does. (iv) The G+12 derivative operator works better than the SSPH derivative operator. This might be due to the fact that the SPH derivative operator omits the surface term (for details see Section 3.1 in Price 2008).

Figures 4–6 show the evolution of the inner part of the Keplerian disks for three representative runs (M97-No-No-SPH, M97-B95-R+00-SPH, and vNRL-B95-R+00-G+12). In Figures 7–9, we show the hydrodynamical and AV torques as a function of the initial radius for these three runs. There is theoretically no mechanism of angular momentum transfer in the disks. Hence the evolution of the disks is induced by the numerical effect. These figures clearly describe how the disks break-up.

With M97-No-No-SPH, at the very first step a huge AV torque works at the inner edge (see Figures 4 and 7). This torque is negative and thus the inner edge of the disk is decelerated. Consequently, the inner edge starts to fall off from the rest of the disk. After the innermost ring has become completely separated from the disk, the second innermost ring of particles starts to fall off and it is then followed by a third ring, and then a fourth, and so on. Finally, these isolated rings...
break-up. In Figure 7, we can clearly see that the AV torque to the innermost ring is initially very large and remains large. Since we did not use any shear switch, M97 viscosity causes large drag on particles in the innermost ring, which is rotating faster than the neighboring outer rings.

With M97-B95-R\textsuperscript{+00-SPH}, the disk survived much longer (see Figures 5 and 8). We can see that the initial AV torque in Figures 5 and 8 is more than three orders of magnitude smaller than those in Figures 4 and 7. This much smaller AV torque resulted in the disk lifetime being roughly 10 times longer for M97-B95-R\textsuperscript{+00-SPH} compared to that for M97-No-No-SPH. Apparently, the way the disk is disrupted is quite different from that for the case of M97-No-No-SPH. The inner region of the disk becomes disordered before the falling down of the innermost ring occurs, and from this disordered ring particles are eventually kicked out, and then complete a breakdown of the inner region takes place.

With vNRL-B95-R\textsuperscript{+00-G+12}, the disk survived for around 200 orbital periods at the inner edge (see Figures 6 and 9). At the beginning of the simulation, we can see from Figure 9 that the AV torque is of the order of $10^{-11}$, which is five orders of magnitude smaller than that in the case of M97-B95-R\textsuperscript{+00-SPH}. The use of switches and the use of vNRL AV instead of the M97 AV reduce the AV torque drastically, resulting in the extension of the disk lifetime.

The disk finally breaks up after about 220 orbits. The mechanism of the breaking up of the disk seems to be similar to that in M97-B95-R\textsuperscript{+00-SPH}. The particle alignment becomes disordered first, and eventually particles are kicked out from the inner edge of the disk. However, both the time it
takes to become disordered and the time it takes to eject particles are much longer. In Figure 6 we can see that disorders of particles start to grow at $t = 75$ orbits. Then, particles near the inner edge show perturbed motion, which triggers the break-up of the disk. These disorders cause large unphysical hydrodynamical forces, as can be seen in Figure 8. As a result, the disorder propagates to the whole disk. Note that this azimuthal motion seems to have a mode with a wavenumber of 6. This comes from the construction of the initial condition (see Appendix C).

Figure 10 shows the time evolution of the AV torque per mass on the particles, which are initially on the inner edge. It is apparent that in the run with M97-No-No-SPH particles feel a large torque from the beginning to the end of the simulation.

Figure 2. (Continued.)

Figure 3. The difference of the lifetime of the disk when (a) the AV scheme, (b) the shear switch, (c) the shock indicator, and (d) the derivative estimator are changed. Points connected by lines indicate the runs performed with the same method, except for that shown in panel.
Figure 4. Distribution of specific torque by hydrodynamical force, AV force, and both forces combined from left to right for the run with M97-No-No-SPH. Only particles within [0.75, 0.75] are shown. The color-code indicates the torque. The snapshot times are normalized by the orbital time at the inner edge (r = 0.5) from top to bottom.
Figure 5. Same as Figure 4, but showing the results with M97-B95-R+00-SPH. Note that the time and the color bar are different.
Figure 6. Same as Figure 4, but showing the results with vNRL-B95-R+00-G+12. Note that the time and the color bar are different.
Figure 7. Averaged torque for the run with M97-No-No-SPH plotted against the initial radius of particles. The blue points indicate the torque by AV and red points indicate those by hydrodynamical force. The times are normalized by the orbital time at the inner edge ($r = 0.5$).

Figure 8. Same as Figure 7, but for the run with M97-B95-R+00-SPH.
The torque in the run with M97-B95-R+00-SPH is about three orders of magnitude smaller at the beginning, but it increases exponentially after \( t \approx 4 \) until the break-up of the disk. The AV torque in the run with vNRL-B95-R+00-G+12 is much smaller than that with M97-B95-R+00-SPH. The growth of the AV force is much slower for this case compared to the run with M97-B95-R+00-SPH. It remains small until \( t \approx 60 \), and the exponential growth after \( t \approx 60 \) is also slower.
Figure 11. Difference between the angular momentum at the initial step and those at the disk lifetime are plotted against the initial radius of particles. The red squares, green triangles, and blue circles indicate the results of vNRL-B95-R and M97-No-No-SPH at $t = 12$ orbits, and M97-No-No-SPH at $t = 0.9$ orbits, respectively. The time is normalized in the orbital time at the inner edge.

Figure 11 shows the difference between the radial distributions of angular momenta at the initial state at the end time. We can see that an unphysical transfer of angular momentum took place around inner and outer edges. No such transfer can be seen in the bulk of the disk. Near the inner edge, particles lose angular momenta, while around the outer edge they gain angular momenta. The magnitude of the angular momentum transfer at the inner edge is larger than those at the outer edge. Thus, we conclude that the main reason for the break-up of the disk is this angular momentum transfer at the inner edge. Note that the transfer of angular momentum in the case of the run with M97 AV is much faster than that in the case of vNRL AV. For M97 AV, the end of the run is at $t = 0.9$, while for vNRL AV, it is at $t = 300$.

4. SUMMARY

It is well-known that SPH has difficulties when it is used to simulate the evolution of cold and thin differentially rotating disks. There are two possible reasons for this difficulty, namely, the error in the pressure gradient and spurious shear viscosity in AV.

In this paper, we present the results of a survey of the proposed implementations of AV, their switches, and derivative operators. For the simulation of cold, Keplerian disks, we found that vNRL AV gives much better results compared to M97 AV, which is widely used. Also, a recently proposed G+12 operator for divergence calculation gives much better results compared to the traditional SPH divergence operator. Note that a similar survey has already been performed done by Hu et al. (2014). However, they fixed the form of AV to the pairwise form. Their results are consistent with our results with the pairwise AV.

Though the modern switches improve the behavior of the pairwise AV significantly, we found that the vNRL formulation behaves systematically better than the pairwise AV. The main reason for this difference is that the former responds to the compression while the latter responds to an approaching pair of particles. Thus, the pairwise formulation cannot discriminate between the compression and the shear. This difference results in the difference between the results of the Keplerian tests.

One might think that vNRL AV cannot be used in the presence of the strong shock. In Appendix D, we present the results of the shock tube tests for vNRL and M97 AVs. These tests show that vNRL AV does not lead to catastrophic results, although M97 AV shows better results. Note that AV switches (e.g., Read & Hayfield 2012; Rosswog 2015) can be easily combined with vNRL AV. We conclude that vNRL AV can be a possible alternative to the pairwise AV, especially if the system includes long-time evolution and velocity shear.

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APPENDIX A

EQUATIONS OF SPH

In this section, we briefly summarize the hydrodynamical part of SPH. One of the most widely used formulations of SPH is the SSPH. In this formulation, the equations of motion and energy are discretized as:

\[ a_i = -\sum_j m_j \left[ \frac{P_i}{\Omega_i \rho_i^2} \nabla W(x_i - x_j; h_i) + \frac{P_j}{\Omega_j \rho_j^2} \nabla W(x_i - x_j; h_j) \right], \]  
\[ \dot{\rho}_i = \frac{P_i}{\Omega_i \rho_i^2} \sum_j m_j (v_i - v_j) \cdot \nabla W(x_i - x_j; h_j). \]

Note that since these equations are antisymmetric, the momentum and energy are conserved up to the machine epsilon. The density and smoothing length are given by:

\[ \rho_i = \sum_j m_j W(x_i - x_j; h_i), \]
\[ h_i = \eta (\frac{m_i}{\rho_i})^{1/\nu}, \]

where \( \nu \) is the number of dimensions. We set \( \eta = 1.2 \), unless otherwise specified. Note that we need the equation of state to obtain the pressure from the density and specific internal energy. In the following, we used the equation of state for ideal gas:

\[ p = (\gamma - 1) \rho u, \]

where \( \gamma \) is the heat capacity ratio. We set \( \gamma \) to 1.4, unless otherwise specified. For the kernel function, we used the Wendland \( C_6 \) kernel (Dehnen & Aly 2012).

Recently, a novel formulation for the SPH, DISPH, was proposed by Saioto & Makino (2013) to impose the description of the hydrodynamical instability of SSPH. In this formulation,
the equations of motion and energy are discretized as

\[
\mathbf{a}_i = - \left( \gamma - 1 \right) \sum_j m_j u_j \left[ \frac{1}{\Omega_i q_i} \nabla W(x_i - x_j; h_i) \right]
+ \frac{1}{\Omega_i q_i} \nabla W(x_i - x_j; h_i),
\]

\[
\dot{u}_i = (\gamma - 1) \frac{u_i}{\Omega_i q_i} \sum_j m_j u_j (v_i - v_j) \cdot \nabla W(x_i - x_j; h_i),
\]

where

\[
q_i = \sum_j m_j u_j W(x_i - x_j; h_i).
\]

In this paper, we employ the same time integrator as Cullen & Dehnen (2010), which is the second order Runge–Kutta integrator scheme. The procedure of this timestep integrator can be summarized as follows.

**Step 1**: Drift \( r_i^{(n+1)} \): 

\[
r_i^{(n+1)} = r_i^{(n)} + u_i^{(n)} \Delta t + a_i^{(n)} \frac{\Delta t^2}{2},
\]

where the superscript \( (n) \) indicates the value of \( n \)th step.

**Step 2**: Predict the velocity and energy at the next step:

\[
\dot{v}_i^{(n+1)} = v_i^{(n)} + a_i^{(n)} \Delta t,
\]

\[
\dot{u}_i^{(n+1)} = u_i^{(n)} + \ddot{u}_i^{(n)} \Delta t.
\]
At this step, we calculate the $\nabla v_i^{(n)}$, $\nabla \times v_i^{(n)}$, and $\nabla \otimes v_i^{(n)}$ to obtain $a_i^{(n+1)}$ and $u_i^{(n+1)}$ using $v_i^{(n+1)}$ and $u_i^{(n+1)}$. We then compute $a_i^{(n+1)}$ and $u_i^{(n+1)}$ using $v_i^{(n+1)}$ and $u_i^{(n+1)}$.

Step 3: Correct $v_i$ and $u_i$:

$$ v_i^{(n+1)} = v_i^{(n)} + (a_i^{(n)} + a_i^{(n+1)}) \frac{\Delta t}{2}, \quad (40) $$

$$ u_i^{(n+1)} = u_i^{(n)} + (\dot{u}_i^{(n)} + \dot{u}_i^{(n+1)}) \frac{\Delta t}{2}. \quad (41) $$

Note that in the case in which the acceleration depends on only the position and not on the velocity (e.g., gravity), Step 2 can be omitted. In this case, this integrator results in a second order leapfrog integrator.

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APPENDIX B

THE DERIVATION OF THE DERIVATIVE OPERATOR OF G+12

Let us consider the expansion of an arbitrary physical quantity $A(x')$ around $x$:

$$ A(x') = A(x) + (x' - x) \cdot \nabla A(x) + \mathcal{O}((x' - x)^2). \quad (42) $$

Multiplying $(x' - x)$ by Equation (42) yields

$$ [A(x') - A(x)](x' - x) = (x' - x)[(x' - x) \cdot \nabla A(x)], \quad (43) $$

$$ = [(x' - x) \otimes (x' - x)] \nabla A(x). \quad (44) $$
Taking the convolution of both sides and applying SPH discretization, we obtain
\[ M_i \mathbf{A} = \sum_j (A_j - A_i) (x_j - x_i) \frac{m_j}{\rho_j} W(x_i - x_j; h_i) = M_i \nabla A_i, \quad (45) \]
\[ M_i = \sum_j (x_j - x_i) \otimes (x_j - x_i) \frac{m_j}{\rho_j} W(x_i - x_j; h_i), \quad (46) \]
and
\[ \nabla A_i = \sum_j (A_j - A_i) M^{-1}(x_j - x_i) \frac{m_j}{\rho_j} W(x_i - x_j; h_i). \quad (47) \]

It is easy and straightforward to extend this equation to a vector field. Let us consider the derivative of \( \alpha \)-component of vector \( \mathbf{v} \) by \( \beta \)-component:
\[ \nabla^\beta v_i^\alpha = \sum_j (v_j^\alpha - v_i^\alpha) G_{ji}^\beta, \quad (48) \]
\[ G_{ji} = \frac{m_j}{\rho_j} M_j^{-1}(x_j - x_i) W(x_i - x_j; h_i). \quad (49) \]
The divergence of \( \mathbf{v} \) is then given by
\[ \nabla \cdot \mathbf{v}_i = \sum_\delta \nabla^\delta v_i^\delta \quad (50) \]
\[ = \sum_\delta \sum_j (v_j^\delta - v_i^\delta) G_{ji}^\delta, \quad (51) \]
\[ = \sum_j (v_j - v_i) \cdot G_{ji}. \quad (52) \]
Figure 15. Same as Figure 13, but with DISPH.

Figure 16. Radius vs. density profiles of the Noh implosion test are shown. In the left panel, the red triangles indicate the results with vNRL-No-No-G+12, while the blue circles indicate those with vNRL-No-No-SPH. In the central panel, the red triangles indicate the results with vNRL-No-R+00-G+12, while the blue circles indicate those with vNRL-No-R+00-SPH. In the right panel, the red triangles indicate the results with M97-No-R+00-G+12, the blue circles indicate those with M97-No-R+00-SPH, and the green squares indicate those with M97-No-No-SPH. The solid curve indicates the analytic solution.
Similarly, we can easily derive the expression for the rotation and dyadic:

$$\nabla \star v_i = \sum_j (v_j - v_i) \star [M^{-1}(x_j - x_i)]_{\rho_j} W(x_i - x_j; h_i),$$

(53)

where $\star$ is a placeholder operator for $\cdot \{,\} \star \delta \ll$,.

APPENDIX C
INITIAL PARTICLE PLACEMENT

For the initial distribution of particles, we used concentric rings. Shirley & Chiu (1997) made concentric rings by converting a unit square into a circle. In this section, following Shirley & Chiu (1997), to realize a uniform density disk, place particles on concentric rings. These rings are obtained by a simple coordinate transformation from regular particle placement in a regular $n$-gon. The procedure of this algorithm is as follows, for the case of a circle with a unit radius expressed by $N$ rings. Repeat the following for $1 \leq i \leq N$. Place $n \times i$ the particles in the circle of radius $r_i = i/N$, with equal spacing in the azimuthal direction. In this paper, we use $n = 6$. Note that this procedure can be extended to disks with a non-uniform surface density profile. We will report it in forthcoming paper.

We can start the calculation from this initial placement. However, we found that particles on the inner edge show radial oscillation that shortens the disk lifetime significantly. Thus, before performing the numerical simulations, we stabilize the initial conditions by adding the damping term to the radial component of acceleration:

$$a_i^{\text{damp}} = -\frac{C_{\text{damp}}}{\Delta t} \frac{x_i \cdot v_i}{|x_i|^2},$$

(54)

where $C_{\text{damp}}$ is set to 0.1. Note that in this process we ignore the AV acceleration. Time of process is set to $10 \times 2\pi$. We used the final snapshot of this process as the initial condition of the calculations done in Section 3.

APPENDIX D
1D AND 2D SHOCK WAVE TESTS

The shock tube problem is one of the most commonly used test problems to check the capability of numerical methods to handle the shocks. In this section we present the results of the shock tube test for both SSPH and DISPH. We used a 1D computational domain $[-0.5:0.5]$. Here we performed two shock tube tests; one is the standard Sod shock tube test (Sod 1979) and the other includes strong shock. The initial condition of former test is given by

$$(\rho, p, v) = \begin{cases} (1, 1, 0) & (x < 0), \\ (0.5, 0.2, 0) & \text{(otherwise)}. \end{cases}$$

The initial condition for the strong shock test is given by

$$(\rho, p, v) = \begin{cases} (1, 1000, 0) & (x < 0), \\ (1, 0.001, 0) & \text{(otherwise)}. \end{cases}$$

We place 768 equal mass particles for the standard shock tube test and 1024 equal mass particles for the strong shock test. For the strong shock test, we set $\eta = 1.6$.

Figure 12 shows the results of the standard shock tube test. All runs show good agreement with the analytic solution. However, vNRL AV produces somewhat broader shock than M97 AV (Figures 12(a) and (c)). The R+00 switch narrows the shock width for vNRL AV (Figure 12(b)). However, in the post-shock region, wiggles can be seen, especially in the velocity. The difference between SPH derivative operator and $G+12$ derivative operator are small (see blue circles and red triangles in Figure 12).

Figure 13 shows the results of the strong shock test. Similar trends with the standard shock tube test can be seen. No significant
The strong shock with DISPH. In Saitoh & Makino (2013), slightly modified DISPH, which does not smooth pressure but does smooth the power of pressure, is proposed. This can improve the accuracy of the strong shock with DISPH.

We also performed the 2D Noh cylindrical implosion test, which involves strong shock (Noh 1987). In this test, we consider the two-dimensional computational domain \((-0.5 < x < 0.5\) and \(-0.5 < y < 0.5\)) filled with a fluid. Initially, the density and pressure of the fluid in the domain have uniform values and they are set to unity and \(10^{-6}\). The radial separation between each ring is set to \(1/128\). The analytic density solutions are

\[
\rho(t) = \begin{cases} 
16 \left( \frac{r}{L} \right)^3 & (r < \frac{L}{2}) \\
1 + \frac{r}{L} & \text{(otherwise)}.
\end{cases}
\]  

(57)

Figures 16 and 17 show the results of the 2D Noh cylindrical implosion test with each AV implementation. Each AV implementation shows good agreement with the analytic solution. Around the central region, a high-density region with the pairing AV has advantages for dealing with strong shocks. On the other hand, vNRL form has advantages for cases where we need to treat shear flow correctly.

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