Achieving Minimum Length Scale in Heaviside-based Morphological Filters

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Abstract. Minimum length scale can fulfil the requirements for manufacturing and provide the extended robustness of design performance. This paper proposes a method to impose the minimum length scale in Heaviside-based morphological filters. With the method, the physical filter radius is first utilized to construct the element neighbourhood in density filter. Then, the density filter is embedded in the Heaviside filter and modified Heaviside filter. Finally, the morphological filters are constructed based on the principles of morphology-based restriction schemes, in which the Heaviside filter plays the role of dilation filter and the modified Heaviside filter acts as the erosion filter. Test results show that the minimum structural sizes in the final design are larger than the specified filter radius size. The characteristics of the basic filters and the embedded filters are discussed.

1. Introduction

Topology optimization is an advanced design tool which aims at distributing a specified amount of material in a given design domain by minimizing an objective function and fulfilling a set of constraints [1]. To date, many topology optimization methods have been developed, including density [2], level set [3], topological derivative [4], phase field [5], evolutionary [6] and several others.

Density method is one of the most widely used methods, which includes the popular Solid Isotropic Material with Penalization (SIMP) method [7]. Usually, the compliance is employed as objective function and constraints are placed on the amount of material used. Then, the optimization process of density method can be described as operating on a fixed domain of finite elements with the goal of minimizing the compliance within a prescribed material volume by identifying whether each element should consist of solid material or void.

It is now well-known that this discretized optimization problem lacks solutions in general. During the optimization, generating more holes will decrease the objective function, which is referred to as mesh-dependence in density method [8]. Another issue with the discretized optimization problem is that the design variables can only use discrete values of 0 (void material) or 1 (solid material), making it difficult to solve [9]. This problem can be handled by relaxing the design variables to 0 and 1. Such a relaxation, however, leads to a large number of grey transition zones between the solid and void phases in the final design. The morphological filters [10] are proposed to obtain designs close to 0/1 solutions. By applying the filters, the mesh-dependence instability can also be greatly inhibited.

In order to obtain manufacturable and well-performing designs, based on our previous work [11], the minimum length scale is imposed in the Heaviside-based morphological filters. On the one hand,
imposing minimum length scale can fulfill the requirement for manufacturing, i.e. achieving minimum length scale on the optimized design and thus ensure prototype manufacturability [12]; on the other hand, imposing minimum length scale provides the extended robustness of the performance [13].

2. Construction of Heaviside-Based Morphological Filters

Morphological filters include a family of regularization schemes that are based on image morphology operators, namely dilation filter, erosion filter, open filter, close filter, open-close filter and close-open filter [14]. Among them, dilation and erosion filters work as the principle operators. The close filter is defined as a dilation followed by an erosion, the open filter as an erosion followed by a dilation, the close-open filter as opening followed by closing, and the open-close filter as closing followed by opening.

In Heaviside-based morphological filters [11], the Heaviside filter [15] plays the role of dilation filter and the modified Heaviside filter [10] acts as the erosion filter. The Heaviside filter is defined as

\[ x_e = 1 - e^{-\beta x_e} + \tilde{x}_e e^{-\beta} \]  

where \( x_e \) represent the Heaviside filtered densities, \( \beta \) is the smoothness parameter, \( \tilde{x}_e \) are the densities obtained from the density filter.

The modified Heaviside filter is defined as

\[ \tilde{x}_e = e^{-\beta(1-x_e)} - (1-\tilde{x}_e)e^{-\beta} \]

where \( \tilde{x}_e \) represent the modified Heaviside filtered densities.

Based on the principle of morphological filters, the close filter is defined as a dilation followed by an erosion, which is expressed as

\[ \tilde{x}_e = e^{-\beta(1-x_e)} - (1-\tilde{x}_e)e^{-\beta} \]

Correspondingly, the open filter is defined as an erosion followed by a dilation, which is given as

\[ x_e = 1 - e^{-\beta x_e} + \tilde{x}_e e^{-\beta} \]

By sequentially applying the close and open filters, the open-close filter can be obtained

\[ \tilde{x}_e = 1 - e^{-\beta x_e} + \tilde{x}_e e^{-\beta} \]

The sensitivity \( \frac{\partial f}{\partial x_i} \) of a function \( f(\tilde{x}_e) \) with respect to the \( i \)-th design variable \( x_i \) is obtained by applying the chain rule

\[ \frac{\partial f}{\partial x_i} = \sum_{e=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{s=1}^{\infty} \frac{\partial f}{\partial \tilde{x}_e} \frac{\partial \tilde{x}_e}{\partial x_i} \frac{\partial \tilde{x}_e}{\partial x_p} \frac{\partial \tilde{x}_e}{\partial x_q} \frac{\partial \tilde{x}_e}{\partial x_s} \frac{\partial \tilde{x}_e}{\partial x_i} \]

3. Imposing Minimum Length Scale in Heaviside-Based Morphological Filters

As shown in Eq. (1), the Heaviside filter is built upon the density filter. In density filter, each element density is redefined as a weighted average of the neighbourhood densities of the element. This process utilizes an intrinsic length scale to regularize the problem formulation and make the topology results mesh independent. This feature offers the possibility to control the member size of the final designs.

In density filter, the element size is assumed constant and density information is evaluated in element centre. The neighbourhood of element \( e \) is specified by the elements whose center-to-center distance is smaller than a given filter radius \( R \). In Fig. 1, the design domain is a square area with an edge length of \( L \), the center-to-center distance between two adjacent elements is \( l \). The neighbourhood of element \( e \) is marked in blue.
Due to the constant element size, the center-to-center distance is identical for different meshes (Fig. 2(a)). In our method, the element size is made to change with the refinement of the mesh, making center-to-center distance follows the change of the mesh (Fig. 2(b)). Then, the filter radius is used to represent the physical length scale. By this way, the length scale of the solid elements of the final design can be represented by the filter radius, and different length scales can be achieved by specifying different filter radii.

In Fig. 2(a), the adjacent center-to-center distance \( l \) is identical, the filter radius only serves to identify the elements that affect the element \( e \). The element neighborhood does not change with the refinement of mesh. In Fig. 2(b), the adjacent center-to-center distance \( l \) changes with the refinement of mesh, and the element neighborhood varies accordingly as the mesh refines.
4. Case Study

4.1. Problem Formulation
The classic compliance minimization problem is investigated in this section. The MBB beam is used as research object, and the settings are shown in Fig. 3. The load is applied vertically in the upper left corner and there is symmetric boundary condition along the left edge $\Gamma_D$, the structure is supported horizontally in the lower right corner $\Gamma_E$.

The optimization problem for the MBB beam may be written as:

$$
\min_{\rho} f(x) = U^T K U = \sum_{e=1}^{N} u_e^T k_e u_e
$$

subject to:

$$
K U = F
$$

$$
g = V(x)/V^* - 1 = \sum_{e=1}^{N} v_e x_e / V^* - 1 \leq 0
$$

$$
0 \leq x \leq 1
$$

(7)

where $f$ is the compliance, $K$, $U$ and $F$ are the global stiffness matrix, displacement vector and force vector, respectively, $u_e$ is the element displacement vector, $k_e = x_e k_0$, is the element stiffness matrix, $k_0$ is the element stiffness matrix for unit Young’s modulus, $\rho$ is the penalty factor, $N$ is the number of elements used to discretize the design domain $\Omega$, $V$ is the material volume and $V^*$ is the material resource constraint, $v_e$ is the volume of element $e$.

4.2. Test Examples
The continuation method is applied to update the penalty factor $\rho$ and the smoothness parameter $\beta$:
the penalty factor $\rho$ is increased by 0.05 every 10 iterations until it reaches 3; the smoothness parameter $\beta$ is increased by 1.01 times per iteration until it reaches 50. Parameter settings for the test example are as follows: the design domain $\Omega$ is discretized with 120 by 40 bi-linear quadrilaterals, the material resource constraint is $V^* = 0.5$, the initial penalty factor is $\rho = 1$, the initial smoothness parameter is $\beta = 1$ and the filter radius is $r_{\text{min}} = 0.8$. Figure 4 displays the optimization results of the MBB beam.
Figure 4. Optimization results for the MBB test example. For each filter, two images are shown. The upper one shows the filtered design variable field and the lower one shows the original density variable field. (a) open filter; (b) close filter; (c) open-close filter; (d) close-open filter.

Figure 4(a) to (d) display the optimization results of open filter, close filter, open-close filter and close-open filter, respectively. As shown, all the four filters are volume preserving, i.e. the filtered structural volumes are the same as the original structure volumes. This property helps to avoid oscillations in optimization iterations when using the continuation method, thereby improving the stability and convergence of the optimization process [16].

As seen from Fig. 4(a) and (b), the optimization results of open filter and close filter exhibit different topologies, but the minimum structural sizes of both are larger than the filter radius size. Similarly, in Fig. 4(c) and (d), the minimum structural sizes of the final topologies are also greater than the filter radius size.

As for the final designs, the filtered topologies of open-close filter and close-open filter are similar, but the original topologies exhibit different characteristics. This is because the characteristics of the basic filter are passed to the embedded filters. In close-open filter, open filter is the basic filter, so the close-open filter exhibits the characteristics of open filter. Similar situations occur in the open-close filter.

5. Conclusion
According to the principles of morphology-based restriction schemes, the Heaviside-based morphological filters are constructed in this paper. This is on the basis of our previous work [11]. In the density filter, the physical filter radius is utilized to construct the element neighbourhood, so that the element neighbourhood changes with the refinement of mesh.

By this way, the length scales of the solid elements are represented by the filter radius, different length scales can be achieved by specifying different filter radii. Test results show that among the constructed four filters, the minimum structural sizes in the final design are larger than the specified filter radius size.

The constructed four filters are volume preserving, this property is helpful to the stability and convergence of the optimization process. In both open-close filter and close-open filter, the characteristics of the basic filter are passed to the embedded filters.

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