Scaling of disordered recursive scale-free networks

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Abstract – In this paper, we present a solvable model of disordered recursive scale-free networks. The structure, fractality, and dimensionality are studied theoretically and numerically, which are shown to be totally different from those in ordered recursive networks. The transfinite fractal dimension, which is recently introduced to distinguish the structural differences between the networks with infinite dimension, exhibits an interesting scaling behavior. We also investigate the diffusion process on this family of networks, and it is found that the transfractal dimension can identify detailed scaling behavior of diffusion dynamics on transfractal networks.

Complex networks have been studied extensively owing to their relevance to many real systems such as the worldwide web, the Internet, energy landscapes and biological and social networks [1]. Empirical studies indicate that the networks in various fields exhibit some common topological characteristics: a logarithmically growing average distance $L$ or diameter $D$ with network size (small-world property) and a power law degree distribution (scale-free property). Recently, Song et al. have studied naturally occurring scale-free networks that seem to be fractal and small-world at the same time [2]. They applied a box covering algorithm which enabled them to demonstrate the existence of self-similarity and fractality in these real networks. In order to characterize such a behavior, a family of ordered recursive lattices [3,4] are generalized and studied by Rozenfeld and ben-Avraham (RA) [5]. By the application of a renormalization procedure, they exactly found the fractality, dimensionality, and scaling in them. Importantly, a useful measure of transfractal dimension $\tilde{d}_f$ is defined and can be used to study the self-similarity and fractality of the networks with small-world structure (infinite fractal dimension). However, the determinism and some intrinsic disadvantages in this model prevent it from describing the diversity and evolution in the natural networks. In this paper, we present a disordered model of recursive networks [6], which can overcome the shortcomings in traditional models and better characterize the real-life networks with self-similarity and fractality. Both by theoretical analysis and simulations, the structure and dimensionality are studied, which are found to be completely different from those in previous recursive networks. We also investigate the diffusion process on this family of networks, and the scaling behaviors of relaxation time are observed, which maintain in other realistic dynamics.

Firstly, we give a brief introduction of ordered recursive network of RA. A special class of their model is $(u,v)$ flowers, where every link in generation $n$ is replaced by two parallel paths consisting of $u$ and $v$ links $(u \leq v)$, to yield generation $n+1$ (fig. 1). A natural choice for the genus at generation $n=1$ is a ring consisting of $u+v=w$ links and nodes. The pseudo-fractal scale-free network (PSFN) proposed by Dorogovtsev, Goltsev, and Mendes [7] is just the case of $(1,2)$ flowers. The disordered version of this model is constructed in the same iterative manner, except that in each generation not every but only one random chosen link is updated. We denote this family of network by $(u,v)$ disordered flower (D-flower). Examples of $(1,3)$, $(2,2)$ flowers and $(1,3)$, $(2,2)$ D-flowers are shown in fig. 1. We can imagine that the $(1,2)$ D-flower is the same as the random PSFN proposed by Dorogovtsev, Mendes and Samukhin [8], though it is not displayed here.

It should be noted that in the traditional recursive networks [5,6], there exists a fatal disadvantage that in each generation the number of incoming nodes is not fixed and increases in a power law manner of $w^n$, which prevents it from characterizing the evolution of the real-life networks. This weakness is overcome in our model where the system size grows linearly with the time. On
the other hand, we point out that the disordered recursive network model proposed by Hinczewski and Berker [6] can be generated by ours. In their model, long-range bonds are introduced with probability of $p$. It can be imagined that, the network constructed with $p = 0$ is the same as $(2, 2)$ flowers, and that with $p = 1$ is a special case of a $(1, 2)$ D-flower. The structure of the network with medium $p$ is just the superposition of the $(2, 2)$ flowers phase and the $(1, 2)$ D-flower phase, which has no other new properties.

According to the recursive algorithm of construction, it is easily found that the number of links and nodes of a $(u, v)$ D-flower in generation $n$ is separately $M_n = (w - 1)n + 1$ and $N_n = (w - 2)n + 2$.

By using the rate-equation approach [9], the degree distribution $p(k)$ can be obtained analytically for $(u, v)$ D-flower. Let $N_n(k)$ be the number of nodes with degree $k$ in generation $n$, then we can write down the rate equation for the degree distribution

$$
\frac{dN_n(k)}{dn} = \frac{(k - 1)N_n(k - 1) - kN_n(k)}{M_n} + (w - 2)\delta_{k, 2}.
$$

Here the first term accounts for the new links that connect to nodes with $k - 1$ links, thus increasing their degree to $k$. The second term describes the new links connecting to nodes with $k$ links turning them into nodes with $k + 1$ links, decreasing the number of nodes with $k$ links. The third term accounts for the new nodes with two links. In the asymptotic limit $N_n(k) = N_n p(k)$ and $M_n = (w - 1)n$,

leading to

$$
p(k) = \begin{cases} 
    \frac{p(k - 1)}{k + w - 1}, & \text{for } k > 2 \\
    \frac{w - 1}{w + 1}, & \text{for } k = 2
\end{cases},
$$

leading to

$$
p(k) = \frac{(w - 1)\prod_{i=0}^{w-1} (2 + i)}{\prod_{i=0}^{w-1} (k + i)}.
$$

It should be noted that the degree distribution exponent of disordered recursive network is $\gamma = w$ for large $k$, which is different from that of ordered one [5]

$$
\gamma = 1 + \frac{\ln w}{\ln 2}.
$$

In fig. 2, we report the numerical results of degree distribution for $(u, v)$ D-flowers, which is exactly in agreement with the theoretical analysis.

In RA’s ordered recursive networks, there is a vast difference between $(u, v)$ flowers with $u = 1$ and $u > 1$. If $u > 1$, the diameter $D_n$ of the $n$-th–generation flower grows as a power of $n$, and thus it possesses a finite fractal dimension $d_f$. For $u = 1$, however, the diameter scales linearly with $n$, which indicates the existence of small-world property and the lack of finite fractal dimension. In order to distinguish between different networks of infinite dimensionality, Rozenfeld and ben-Avraham define a new measure of dimensionality, transfractal dimension $\bar{d}_f$, characterizing how mass scales with diameter:

$$
D \sim \frac{1}{\bar{d}_f} \ln N.
$$

We demonstrate that these features still hold for disordered recursive networks. The transfractal dimension $\bar{d}_f$ for a $(1, v)$ D-flower can be solved by using a mean-field approximation [10]. In the following derivation, we replace diameter $D$ by average distance $L$ in the definition of transfractal dimension (eq. (5)). It will be illustrated that
there exist no qualitative differences. We represent each node by the time that it enters the network. Let \( d(i, j) \) denotes the distance between node \( i \) and node \( j \), and thus the average distance of the model in the generation \( n \) is

\[
L(n) = \frac{2\sigma(n)}{N_n(N_n - 1)},
\]

where the total distance is

\[
\sigma(n) = \sum_{1 \leq i < j \leq N_n} d(i, j).
\]

According to the method of construction for \((1, v)\) D-flower, \( \sigma(n) \) evolves as

\[
\sigma(n + 1) = \sigma(n) + S(w - 2) + \sum_{i=1}^{N_n} d(i, N_n + j),
\]

where the second term on the right side accounts for the sum of distance between the new added \( w - 2 \) nodes, and the last term stands for the total distance between the new added nodes and the ones already existing. In current generation, we renormalize the updated link and the two marginal nodes as a single node \( r \), then

\[
\sigma(n + 1) = \sigma(n) + S(w - 2) + \sum_{i=1}^{N_n} d(i, r),
\]

where \( S(w - 2, r) \) denotes the sum of distance between node \( r \) and the new \( w - 2 \) nodes. Since the shortest path from node \( r \) to an arbitrary node \( i \) \((i \leq N_n)\) never passes the newly added nodes, we can rewrite \( \sum_{i=1}^{N_n} d(i, r) \) as \( L(n)(N_n - 1) \) by a mean-field approximation. Therefore, in the asymptotic limit

\[
\sigma(n + 1) - \sigma(n) = (w - 2)d(N)/dN,
\]

leading to

\[
\sigma(N) = \left[ \frac{S(w - 2, r)}{w - 2} \ln N + C \right] N^2 - \frac{S(w - 2)}{w - 2},
\]

where \( C \) is constant independent of \( N \). In theormodynamical limit, incorporating with \( L(N) \sim \sigma(N)/N^2 \) and \( L(N) \sim \frac{1}{d_f} \ln N \) yields

\[
d_f \sim \frac{w - 2}{S(w - 2, r)}. \tag{12}
\]

We note that \( S(w - 2, r) \) just represents the total distance from a node to all the other \( w - 2 \) ones in a 1-dimension ring of length \( w - 1 \). Therefore, it is easy to obtain \( S(w - 2, r) \sim (w - 2)(w - 1) \) for large \( w \), which leads to a scaling behavior of transfractal dimension of \((1, v)\) D-flower

\[
d_f \sim \frac{1}{w - 1}. \tag{13}
\]

Figure 3 shows the results for the transfractal dimension \( d_f \) of a \((1, v)\) D-flower, which confirms our analytical ones. It should be noticed that the scaling behavior of transfractal dimension (eq. (13)) holds for both definitions of \( d_f \) with diameter \( D \) and average distance \( L \).

For the case of \((u, v)\) D-flowers with \( u > 1 \), the total distance \( \sigma(n) \) evolves as

\[
\sigma(n + 1) = \sigma(n) + S(w - 2) + \sum_{i=1}^{N_n} d(i, N_n + j)
\]

\[+(u - 1) \times P \frac{N_n(N_n - 1)}{2}. \tag{14}\]

The parameter \( P \) accounts for the fraction of shortest paths of all pairs of nodes whose length increase by \( u - 1 \) due to the updated link. It should be noted that this term does not exist in the total distance for \( u = 1 \). Therefore, in the asymptotic limit we obtain

\[
\sigma(N) > \frac{P(u - 1)}{6(w - 2)} N^3. \tag{15}\]

Incorporating with \( L(N) \sim \sigma(N)/N^2 \) yields

\[
L(N) > \frac{P(u - 1)}{6(w - 2)} N^2, \tag{16}\]

which means that a finite fractal dimension \( d_f \) exists in \((u, v)\) D-flowers with \( u > 1 \). One should notice that
cannot theoretically calculate the fractal dimension \( d \) laid out. All the data is obtained by averaging over 10\(^3\) previous ordered recursive network of RA (dashed line) is also size measured fractal dimension linear fit of the data in the main panel corresponding to the network realizations.

To give an analytical expression of the breakdown of mean-field approximation, it is difficultly to topology of the current network. Moreover, as a result of depends on the position of the updated link and the parameter \( P \) is a nonzero variable, which directly depends on the position of the updated link and the topology of the current network. Moreover, as a result of the breakdown of mean-field approximation, it is difficultly to give an analytical expression of \( P \). Therefore, we cannot theoretically calculate the fractal dimension \( d_f \) of a \((u, v)\) D-flower with \( u > 1 \).

We investigate their \( d_f \) numerically and find some novel properties. The D-flower is limited to a simple case with \( u = v \), and the numerical result is displayed in fig. 4. The \( d_f \) of the \((u, u)\) D-flower decreases monotonically with \( u \), and approaches to infinity on the left side in the limit of \( u \to 1 \), which recovers the previous result. It should be noticed that the quantity of fractal dimension for a disordered recursive network is completely different from that of an ordered one [5]. For a D-flower, the fractal dimension approaches 3 in the large limit of \( u \), while in the ordered case \( d_f \) approaches unity in the same limit. We argue that this phenomenon is due to the introduction of disorder in the construction procedure, which always enlarges the effective dimension [11] of the network (see the inset of fig. 4).

Since we have systematically investigated the dimensionality of disordered recursive networks, one of the most important issues is to check the effects of the fractal dimension \( d_f \) or transfractal dimension \( \tilde{d}_f \) on the dynamics taking place upon them. At present, we briefly consider diffusion of a particle on the \((u, v)\) D-flower. The particle is put on a randomly chosen node as the seed, and allowed to hop to one of its neighboring nodes, which are chosen randomly at each time step. Instead of the return probability [12], we consider here the number of nodes visited by the particle during time \( t \), denoted by participation ratio \( P(t) \). Therefore, it takes the values between \( P(t = 0) = 1 \) and \( P(t \to \infty) = N \). To characterize diffusion, we define the diffusion time \( \tau \) associated with the participation ratio by the condition \( P(t = \tau) = cN \) with a constant \( c \) between zero and unity.

Simulations of the diffusion are conducted on the \((u, v)\) D-flower and the diffusion time \( \tau \) is measured. The numerical factor \( c = 0.1 \) is chosen, and we confirmed that other values of \( c \) do not change the scaling behavior of \( \tau \).

Figure 5(a) displays the dependence of diffusion time \( \tau \) on the network size \( N \) of the \((1, v)\) D-flower. It is observed that the diffusion time displays a linear behavior \( \tau \sim N \), which is consistent with the infinite fractal dimension of the \((1, v)\) D-flower and the result of diffusion on small-world networks [13]. Importantly, the linear fit slope, which denotes the speed of increase of \( \tau \) with \( N \), has a scaling behavior with transfractal dimension, \( \tau \sim 1/d_fN \), shown in the upper inset of fig. 5(a). This indicates that, although the relationship of diffusion time \( \tau \sim N \) holds for all of the networks with infinite fractal dimension, the more detailed quantitative difference can be distinguished by the transfractal dimension \( \tilde{d}_f \) of those networks. This...
result provides us with a further understanding of the  
transfractal dimension. Since we have \( d_f \sim \frac{1}{u-1} \), it is  
easily obtained that \( \tau \sim (u - 1)N \), which is displayed in  
the lower inset of fig. 5(a). For the \((u, u)\) D-flower with  
\( u > 1 \), the diffusion time \( \tau \) vs. the network size \( N \) is  
plotted in fig. 5(b). The scaling behavior \( \tau \sim N^{(d_f+1)/d_f} \)  
is observed (see the upper left inset of fig. 5(b)) in spite of  
the deviation due to the finite-size effects inherent in the  
numerical calculation of \( d_f \) [14]. From the lower right inset  
of fig. 5(b), we observe that the scaling exponent obtained  
numerically approaches \( 1+1/3 \) as \( u \) increases, which is  
in agreement with the behavior of fractal dimension that \( d_f \rightarrow 3 \) in a large limit of \( u \).

Most of the real-life networks are simultaneously small-
world and fractal, whose evolution and structure can be  
well reproduced through our model. Furthermore, the  
transfinite dimension of the networks can be theoretically  
obtained, which can well distinguish the relaxation time  
of dynamics process taking place on these networks with  
infinite fractal dimension. On the other hand, the classical  
diffusion of a single particle is one of the simplest dynamics  
models. Moreover, the behavior of the relaxation time  
persists in other more realistic dynamics processes, such  
as disease propagation, evolutionary dynamics and so on.  
As we obtain above, the scaling behavior \( \tau \sim N^{(d_f+1)/d_f} \)  
of the diffusion relaxation time on networks with finite  
dimension is the same as that in evolutionary dynamics  
on \( D \)-dimensional hypercubic lattices, where the mutant  
spreading time scales with the population size \( N \) as  
\( N^{(D+1)/D} \) [15]. The temporal effect of these real-life  
dynamics processes on networks is of great importance,  
and the reason why we study the simple diffusion process  
here is to give a more clearly physical picture.

In summary, we have analytically and numerically  
investigated the scaling behaviors of the degree distribu-
tion, dimensionality and diffusion of disordered  
recursive networks. Due to the introduction of disorder  
which extensively exists in nature, many properties  
emerging in previous ordered recursive networks dramat-
ically change. We think that these findings will give a  
further understanding of the self-similarity, fractality and  
scaling in the real complex networks. Since this model  
possesses novel properties, it will be interesting to study  
many physics processes, such as synchronization and  
Anderson transition, upon the networks. Research along  
this line is in progress.

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