ON COHEN–MACAULAY MODULES OVER NON-COMMUTATIVE SURFACE SINGULARITIES

YURIY A. DROZD AND VOLODYMYR S. GAVRAN

Abstract. We generalize the results of Kahn about a correspondence between Cohen–Macaulay modules and vector bundles to non-commutative surface singularities. As an application, we give examples of non-commutative surface singularities which are not Cohen–Macaulay finite, but are Cohen–Macaulay tame.

Contents

Introduction 1
1. Preliminaries 2
2. Kahn’s reduction 5
3. Good elliptic case 13
4. Examples 14
References 16

Introduction

Cohen–Macaulay modules over commutative Cohen–Macaulay rings have been widely studied. A good survey on this topic is the book of Yoshino [14]. In particular, for curve, surface and hypersurface singularities criteria are known for them to be Cohen–Macaulay finite, i.e. only having finitely many indecomposable Cohen–Macaulay modules (up to isomorphism). For curve singularities and minimally elliptic surface singularities criteria are also known for them to be Cohen–Macaulay tame, i.e. only having 1-parameter families of non-isomorphic indecomposable Cohen–Macaulay modules [4, 5]. Less is known if we consider non-commutative Cohen–Macaulay algebras. In [6] a criterion was given for a primary 1-dimensional Cohen–Macaulay algebra to be Cohen–Macaulay finite. In [1] (see also [5]) a criterion of Cohen–Macaulay finiteness is given for normal 2-dimensional Cohen–Macaulay algebras (maximal orders). As far as we know, there are no examples of 2-dimensional Cohen–Macaulay algebras which are not Cohen–Macaulay finite but are Cohen–Macaulay tame.

2010 Mathematics Subject Classification. Primary 16G50, Secondary 16G60, 16S38.

Key words and phrases. Cohen–Macaulay modules, vector bundles, non-commutative surface singularities.
In this paper we use the approach of Kahn [10] to study Cohen–Macaulay modules over normal non-commutative surface singularities. Just as in [10], we establish (in Section 2) a one-to-one correspondence between such modules and vector bundles over some, in general non-commutative, projective curves (Theorem 2.13). In Sections 3 and 4 we apply this result to a special case, which we call “good elliptic.” It is analogous to the minimally elliptic case in [10], though seems somewhat too restrictive. Unfortunately, we could not find more general conditions which ensure such analogy. As an application, we present two examples of Cohen–Macaulay tame non-commutative surface singularities (Examples 4.1 and 4.2). We hope that this approach shall be useful in more general situations too.

1. Preliminaries

We fix an algebraically closed field $k$, say algebra instead of $k$-algebra, scheme instead of $k$-scheme and write Hom and $\otimes$ instead of $\text{Hom}_k$ and $\otimes_k$. We call a scheme $X$ a variety if $k(x) = k$ for every closed point $x \in X$.

Definition 1.1. A non-commutative scheme is a pair $(X, A)$, where $X$ is a scheme and $A$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. If $X$ is a variety, $(X, A)$ is called a non-commutative variety. We say that $(X, A)$ is affine, projective, excellent, etc. if so is $X$.

A morphism of non-commutative schemes $(X, A) \to (Y, B)$ is their morphism as ringed spaces, i.e. a pair $(\varphi, \varphi^\#)$, where $\varphi : X \to Y$ is a morphism of schemes and $\varphi^\# : \varphi^{-1}A \to B$ is a morphism of sheaves of algebras. A morphism $(\varphi, \varphi^\#)$ is said to be finite, projective or proper if so is $\varphi$. We often omit $\varphi^\#$ and write $\varphi : (X, A) \to (Y, B)$.

For a non-commutative scheme $(X, A)$ we denote by $\text{Coh} A$ (Qcoh $A$) the category of coherent (quasi-coherent) sheaves of $A$-modules. Every morphism $\varphi : (X, A) \to (Y, B)$ induces functors of direct image $\varphi_* : \text{Qcoh} A \to \text{Qcoh} B$ and inverse image $\varphi^* : \text{Qcoh} B \to \text{Qcoh} A$, where $\varphi^* F = A \otimes_{\varphi^{-1}B} \varphi^{-1} F$. Note that this inverse image does not coincide with the inverse image of sheaves of $\mathcal{O}_X$-modules. The latter (when used) will be denoted by $\varphi_X^*$. Note also that $\varphi^*$ maps coherent sheaves to coherent. The pair $(\varphi^*, \varphi_*)$ is a pair of adjoint functors, i.e. there is a functorial isomorphism $\text{Hom}_A(\varphi^* F, G) \simeq \text{Hom}_B(F, \varphi_* G)$ for any sheaf of $B$-modules $F$ and any sheaf of $A$-modules $G$.

We call a coherent sheaf of $A$-modules $F$ a vector bundle if it is locally projective, i.e. $F_p$ is a projective $A_p$-module for every point $p \in X$. We denote by $\text{VB}(A)$ the full subcategory of $\text{Coh} A$ consisting of vector bundles.

A non-commutative scheme $(X, A)$ is said to be regular if $\text{gl. dim} A_p = \text{dim}_p X$ for every point $p \in X$ (it is enough to check this property at the closed points).

We say that $(X, A)$ is reduced if $X$ is reduced and neither stalk $A_p$ contains nilpotent ideals. Then, if $\mathcal{K} = \mathcal{K}_X$ is the sheaf of rational functions on $X$, $\mathcal{K}(A) = A \otimes_{\mathcal{O}_X} \mathcal{K}$ is a locally constant sheaf of semisimple $\mathcal{K}$-algebras. We
call it the *sheaf of rational functions* on \((X, A)\). In this case each stalk \(A_p\) is an *order* in the algebra \(\mathcal{K}(A)_p\), i.e. an \(O_{X,p}\)-algebra finitely generated as \(O_{X,p}\)-module and such that \(\mathcal{K}_p A_p = \mathcal{K}(A)_p\). We say that \((X, A)\) is *normal* if \(A_p\) is a maximal order in \(\mathcal{K}(A)_p\) for each \(p\). Note that a regular scheme is always reduced, but not necessarily normal.

A morphism \((\varphi, \varphi^\sharp) : (X, A) \to (Y, B)\) of reduced non-commutative schemes is said to be *birational* if \(\varphi : X \to Y\) is birational and the induced map \(\mathcal{K}(B) \to \mathcal{K}(A)\) is an isomorphism.

A *resolution* of a non-commutative scheme \((X, A)\) is a proper birational morphism \((\pi, \pi^\sharp) : (\hat{X}, \hat{A}) \to (X, A)\), where \((\hat{X}, \hat{A})\) is regular and normal.

**Remark 1.2.** Let \((X, A)\) be a non-commutative scheme and \(C = \text{cen}(A)\) be the center of \(A\). (It means that \(C_p = \text{cen}(A_p)\) for every point \(p \in X\).) Let also \(X' = \text{Spec} C\). The natural morphism \(\varphi : X' \to X\) is finite and \(A' = \varphi^{-1} A\) is a sheaf of \(O_{X'}\)-modules, so we obtain a morphism \((\varphi, \varphi^\sharp) : (X', A') \to (X, A)\), where \(\varphi^\sharp\) is identity. Moreover, the induced functors \(\varphi_*\) and \(\varphi_*\) define an equivalence of \(\text{Qcoh} A\) and \(\text{Qcoh} A'\). So, while we are interesting in study of sheaves, we can always suppose that \(A\) is a sheaf of *central* \(O_X\)-algebras. Note that if \((X, A)\) is normal and \(A\) is central, then \(X\) is also normal.

Given a non-commutative scheme \((X, A)\) and a morphism of schemes \(\varphi : Y \to X\), we can consider the non-commutative scheme \((Y, \varphi^\sharp Y, A)\) and uniquely extend \(\varphi\) to the morphism \((Y, \varphi^\sharp Y, A) \to (X, A)\) which we also denote by \(\varphi\). Especially, if \(\varphi\) is a blow-up of a subscheme of \(X\), we call the morphism \((Y, \varphi^\sharp Y, A) \to (X, A)\) the blow-up of \((X, A)\).

**Definition 1.3.** A reduced excellent non-commutative variety \((X, A)\) is called a *non-commutative surface* if \(X\) is a surface, i.e. \(\dim X = 2\). If \(X = \text{Spec} R\), where \(R\) is a local complete noetherian algebra with the residue field \(k\) (then it is automatically excellent), we say that \((X, A)\) is a *germ of non-commutative surface singularity* or, for short, a *non-commutative surface singularity*. In what follows, we identify a non-commutative surface singularity \((X, A)\) with the \(R\)-algebra \(\Gamma(X, A)\) and the sheaves from \(\text{Qcoh} A\) with modules over this algebra (finitely generated for the sheaves from \(\text{Coh} A\)).

If \((X, A)\) is a non-commutative surface, there always is a normal non-commutative surface \((X', A')\) and a finite birational morphism \(\nu : (X', A') \to (X, A)\). We call \((X', A')\), as well as the morphism \(\nu\), a *normalization* of \((X, A)\). Note that, unlike the commutative case, such normalization is usually not unique.

Let \((X, A)\) be a connected central non-commutative surface such that \(X\) is normal, \(C \subset X\) be an irreducible curve with the general point \(g\), \(\mathcal{K}_C(A) = A_g / \text{rad} A_g\) and \(\text{cen} \mathcal{K}_C(A) = \text{cen} \mathcal{K}_C(A)\). \(A\) is normal if and only if it is Cohen–Macaulay (or, the same, reflexive) as a sheaf of \(O_X\)-modules, \(\mathcal{K}_C(A)\) is a simple algebra and \(\text{rad} A_g\) is a principal left (or right) \(A_g\)-ideal for every
More precisely, we can use the following procedure of Chan–Ingalls [3].

ϕ of central normal non-commutative surfaces, we set

F set non-commutative surface (X, A φism when restricted onto CM(A)

shortly “Cohen–Macaulay module.” Obviously, VB(A) ⊆ CM(A) and these categories coincide if and only if A is regular. For a sheaf F ∈ CohA we denote by Fν the sheaf HomA(F, A). It always belongs to CM(A). We also set F† = Fνν. There is a morphism of functors Id → †, which is isomorphism when restricted onto CM(A). If ϕ : (X, A) → (Y, B) is a morphism of central normal non-commutative surfaces, we set ϕ†F = (ϕ∗F)†.

It is known that every non-commutative surface has a regular resolution. More precisely, we can use the following procedure of Chan–Ingalls [3]. The non-commutative surface (X, A) is said to be terminal [3] Definition 2.5] if the following conditions hold:

1. X is smooth.
2. All irreducible components of D = D(A) are smooth.
3. D only has normal crossings (i.e. nodes as singular points).
4. At a node p ∈ D, for one component C1 of D containing this point, the field kA(C1) is totally ramified over k(C1) of degree e = eC1(A) = eC1,p(A), and for the other component C2 also eC2,p(A) = e.

It is shown in [3] that every terminal non-commutative surface is regular and every non-commutative surface (X, A) has a terminal resolution π : (X, A) → (X, A). Moreover, such resolution can be obtained by a sequence of morphisms πi, where each πi is either a blow-up of a closed point or a normalization. Then π is a projective morphism. If (X, A) is a normal non-commutative surface singularity, ˜X = X \ {o}, where o is the unique closed point of X, the restriction of π onto π−1( ˜X) is an isomorphism and we always identify π−1( ˜X) with ˜X. The subscheme E = π−1(o)red is a connected (though maybe reducible) projective curve called the exceptional curve of the resolution π.

1 Note that the term "normal" is used in [3] in more wide sense, but we only need it for our notion of normality.
Recall also that, for a normal non-commutative surface singularity $A$, the category $\text{CM}(A)$, as well as the ramification data of $A$, only depends on the algebra $\mathcal{K}(A)$ [11 (1.6)]. If $A$ is central and connected, i.e. indecomposable as a ring, $\mathcal{K}(A)$ is a central simple algebra over the field $\mathcal{K}$, so the category $\text{CM}(A)$ is defined by the class of $\mathcal{K}(A)$ in the Brauer group $\text{Br}(\mathcal{K})$, and this class is completely characterized by its ramification data.

We also use the notion of non-commutative formal scheme, which is a pair $(\mathfrak{X}, \mathfrak{A})$, where $\mathfrak{X}$ is a “usual” (commutative) formal scheme and $\mathfrak{A}$ is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. If $(\mathfrak{X}, \mathfrak{A})$ is non-commutative scheme and $Y \subset X$ is a closed subscheme, the completion $(\hat{X}, \hat{A})$ of $(X, A)$ along $Y$ is well-defined and general properties of complete schemes and their completions, as in [7, 9], hold in non-commutative case too.

2. Kahn’s reduction

From now on we consider a normal non-commutative surface singularity $(X, A)$ and suppose $A$ central. We fix a resolution $\pi : (\tilde{X}, \tilde{A}) \to (X, A)$, where $\tilde{A}$ is also supposed central. Then $\text{CM}(\tilde{A}) = \text{VB}(\tilde{A})$ and we consider $\pi^\dagger$ as a functor $\text{CM}(A) \to \text{VB}(\tilde{A})$. A vector bundle $F$ is said to be full if it is isomorphic to $\pi^\dagger M$ for some (maximal) Cohen–Macaulay $A$-module $M$. We denote by $\text{VB}^f(\tilde{A})$ the full subcategory of $\text{VB}(\tilde{A})$ consisting of full vector bundles. We also set $\omega_{\tilde{A}} = \text{Hom}_{\tilde{X}}(\tilde{A}, \omega_{\tilde{X}})$, where $\omega_{\tilde{X}}$ is a canonical sheaf over $\tilde{X}$, and call $\omega_{\tilde{A}}$ the canonical sheaf of $\tilde{A}$. It is locally free, i.e. belongs to $\text{VB}(\tilde{A})$.

Given a coherent sheaf $F \in \text{Coh} \tilde{A}$, we denote by $\text{ev}_F$ the natural map $\Gamma(\tilde{X}, F) \otimes \tilde{A} \to F$. We say that $F$ is globally generated if $\text{Im} \text{ev}_F = F$ and generically globally generated if $\text{supp}(F / \text{Im} \text{ev}_F)$ is discrete, i.e. consists of finitely many closed points.

**Theorem 2.1** (Cf. [10, Proposition 1.2]).

1. The functor $\pi^\dagger$ establishes an equivalence between the categories $\text{CM}(A)$ and $\text{VB}^f(\tilde{A})$, its quasi-inverse being the functor $\pi_*$.  
2. A vector bundle $F \in \text{VB}(\tilde{A})$ is full if and only if the following conditions hold:
   a. $F$ is generically globally generated.
   b. The restriction map $\Gamma(\tilde{X}, F) \to \Gamma(\hat{X}, F)$ is surjective, or equivalently, using local cohomologies,
      b’ The map $\alpha_{\pi} : H^1_{\mathfrak{X}}(\tilde{X}, F) \to H^1(\hat{X}, F)$ is injective.

Under these conditions $F \simeq \pi^\dagger \pi_* F$.

**Proof.** Note that there is an exact sequence

$$0 \to \text{tors}(\pi^* M) \to \pi^* M \xrightarrow{\gamma M} \pi^\dagger M \to \overline{M} \to 0,$$

where $\text{tors}(M)$ denotes the periodic part of $M$ and the support of $\overline{M}$ consists of finitely many closed points. Since $\pi^* M$ is always globally generated, so

2 Recall that $\Gamma(\tilde{X}, F) \simeq \text{Hom}_{\tilde{A}}(\tilde{A}, F)$. 


is also $\text{Im} \gamma_M$. Therefore, $\pi^! M$ is generically globally generated. If $M$ is Cohen–Macaulay, the restriction map $\Gamma(X, M) \to \Gamma(\tilde{X}, \pi^* M)$ is an isomorphism. Since $M$ naturally embeds into $\Gamma(\tilde{X}, \pi^! M)$ and hence into $\Gamma(\tilde{X}, \pi^1 M)$, the restriction $\Gamma(\tilde{X}, \pi^1 M) \to \Gamma(\tilde{X}, \pi^! M)$ is surjective.

Suppose now that the conditions (a) and (b) hold. Set $M = \pi_* \mathcal{F}$. Since $\pi$ is projective, $M \in \text{Coh} \Lambda$. The condition (b) implies that $M \in \text{CM}(A)$. Note that $\Gamma(\tilde{X}, \pi^! M)$ is canonically isomorphic to $\text{Im} \gamma_M$. As $\mathcal{F}$ is generically globally generated, it implies that the natural map $\pi^! M \to \mathcal{F}$ is an isomorphism. It proves (2).

Obviously, the functors $\pi^! : \text{CM}(A) \to \text{VB}^f(\tilde{A})$ and $\pi_* : \text{VB}^f(\tilde{A}) \to \text{CM}(A)$ are adjoint. Moreover, if $M = \pi_* \mathcal{F}$, where $\mathcal{F} \in \text{VB}^f(\tilde{A})$, there are functorial isomorphisms

$$\text{Hom}_A(M, M) \simeq \text{Hom}_{\tilde{A}}(\pi^* \pi_* \mathcal{F}, \mathcal{F}) \simeq \text{Hom}_{\tilde{A}}(\pi^! \pi_* \mathcal{F}, \mathcal{F}) \simeq \text{Hom}_{\tilde{A}}(\mathcal{F}, \mathcal{F}).$$

It proves (1). \hfill \Box

**Remark 2.2.** A full vector bundle over $\tilde{A}$ need not be generically globally generated as a sheaf of $\mathcal{O}_{\tilde{X}}$-modules. Moreover, examples below show that even the sheaf $\tilde{A} = \pi^* A = \pi^! A$ need not be generically globally generated as a sheaf of $\mathcal{O}_{\tilde{X}}$-modules.

**Definition 2.3.** From now on we consider a sheaf of ideals $\mathcal{I}$ in $\tilde{A}$ such that $\text{supp}(\tilde{A}/\mathcal{I}) \subseteq E$, $\Lambda = \tilde{A}/\mathcal{I}$ and $Z = \text{Spec}(\text{cen} \Lambda)$. Then $(Z, \Lambda)$ is a projective non-commutative variety of dimension 1 (maybe non-reduced). We set $\omega_Z = \mathcal{E}xt^1_{\tilde{X}}(\mathcal{O}_Z, \omega_{\tilde{X}})$ and

$$\omega_{\Lambda} = \mathcal{E}xt^1_{\tilde{A}}(\Lambda, \omega_{\tilde{A}}) \simeq \mathcal{E}xt^1_{\tilde{X}}(\Lambda, \omega_{\tilde{X}}) \simeq \text{Hom}_Z(\Lambda, \omega_Z).$$

The sheaves $\omega_Z$ and $\omega_{\Lambda}$, respectively, are canonical sheaves for $Z$ and $\Lambda$. It means that there are Serre dualities

$$\mathcal{E}xt^i_Z(\mathcal{F}, \omega_Z) \simeq \mathcal{D}^1 i^{-i}(E, \mathcal{F}) \text{ for any } \mathcal{F} \in \text{Coh } Z,$$

$$\mathcal{E}xt^i_{\Lambda}(\mathcal{F}, \omega_{\Lambda}) \simeq \mathcal{D}^1 i^{-i}(E, \mathcal{F}) \text{ for any } \mathcal{F} \in \text{Coh } \Lambda,$$

where $DV$ denotes the vector space dual to $V$.

**Definition 2.4.** We say that an ideal $I$ of a ring $R$ is bi-principal if $I = aR = Ra$ for a non-zero-divisor $a \in R$. A sheaf of ideals $\mathcal{I} \subset \tilde{A}$ is said to be locally bi-principal if every point $x \in X$ has a neighbourhood $U$ such that the ideal $\Gamma(U, \mathcal{I})$ is bi-principal in $\Gamma(U, \tilde{A})$.

**Lemma 2.5.** If the sheaf of ideals $\mathcal{I}$ is locally bi-principal, then

$$\omega_{\Lambda} \simeq \text{Hom}_{\tilde{A}}(\mathcal{I}, \omega_{\tilde{A}}) \otimes_{\tilde{A}} \Lambda.$$

**Proof.** Let $\mathcal{I}' = \text{Hom}_{\tilde{A}}(\mathcal{I}, \omega_{\tilde{A}})$. Consider the locally free resolution $0 \to \mathcal{I} \xrightarrow{\tau} \tilde{A} \to \Lambda \to 0$ of $\Lambda$. Since $\omega_{\tilde{A}}$ is locally free over $\tilde{A}$, it gives an exact sequence

$$0 \to \omega_{\tilde{A}} \xrightarrow{\tau} \mathcal{I}' \to \mathcal{E}xt^1_{\tilde{A}}(\Lambda, \omega_{\tilde{A}}) \to 0.$$
On the other hand, tensoring the same resolution with $\mathcal{I}'$ gives an exact sequence

$$0 \to \mathcal{I}' \otimes \mathcal{A} \mathcal{I} \xrightarrow{1 \otimes \tau} \mathcal{I}' \to \mathcal{I}' \otimes \mathcal{A} \Lambda \to 0.$$ 

Since $\mathcal{I}$ is locally bi-principal, the natural map $\mathcal{I}' \otimes \mathcal{A} \mathcal{I} \to \omega_{\mathcal{A}}$ is an isomorphism, and, if we identify $\mathcal{I}' \otimes \mathcal{A} \mathcal{I}$ with $\omega_{\mathcal{A}}$, $1 \otimes \tau$ identifies with $\tau^*$. It implies the claim of the Lemma. \hfill $\Box$

**Definition 2.6.** Let $\mathcal{I} \subset \mathcal{A}$ be a bi-principal sheaf of ideals such that $\text{supp}(\mathcal{A}/\mathcal{I}) = E$, $\Lambda = \mathcal{A}/\mathcal{I}$ and $I = \mathcal{I}/\mathcal{I}^2$. (Note that $I \in V\text{B}(\Lambda)$.) $\mathcal{I}$ is said to be a weak reduction cycle if

1. $I$ is generically globally generated as a sheaf of $\Lambda$-modules.
2. $H^1(E, I) = 0$.

If, moreover,

3. $\omega_{\Lambda}^\vee = \mathcal{H}\text{om}_{\Lambda}(\omega_{\Lambda}, \Lambda)$ is generically globally generated over $\Lambda$,

$\mathcal{I}$ is called a reduction cycle.

For a weak reduction cycle $\mathcal{I}$ we define the *Kahn’s reduction functor* $R_{\mathcal{I}} : \text{CM}(A) \to \text{VB}(\Lambda)$ as

$$R_{\mathcal{I}}(M) = \Lambda \otimes \mathcal{A} \pi^\dagger M.$$ 

We fix a weak reduction cycle $\mathcal{I}$ and keep the notation of the preceding Definition. We also set $\Lambda_n = \mathcal{A}/\mathcal{I}^n$, $I_n = \mathcal{I}^n/\mathcal{I}^{n+1}$, $\mathcal{I}^{-n} = (\mathcal{I}^n)^\vee$ and $I_{-n} = \mathcal{I}^{-n}/\mathcal{I}^{1-n}$. In particular, $\Lambda_1 = \Lambda$ and $I_1 = I$. One easily sees that $I_n \cong I \otimes_{\Lambda} I \otimes_{\Lambda} \ldots \otimes_{\Lambda} I$ (n times) and $I_{-n} \cong I_n^\vee = \mathcal{H}\text{om}_{\Lambda}(I_n, \Lambda)$.

**Proposition 2.7.** If a coherent sheaf $F$ of $\Lambda$-modules is generically globally generated, then $H^1(E, I \otimes_{\Lambda} F) = 0$. In particular, $H^1(E, I_n) = 0$.

**Proof.** Let $H = \Gamma(E, F)$. Consider the exact sequence

$$0 \to N \to H \otimes \Lambda \to F \to T \to 0,$$

where $N = \ker ev_F$ and $\text{supp} T$ is 0-dimensional. It gives the exact sequence

$$0 \to I \otimes_{\Lambda} N \to H \otimes I \to I \otimes_{\Lambda} F \to I \otimes_{\Lambda} T \to 0.$$ 

Since $H^1(E, H \otimes I) = H^1(E, I \otimes_{\Lambda} T) = 0$, we get that $H^1(E, I \otimes_{\Lambda} F) = 0$. \hfill $\Box$

For any vector bundle $\mathcal{F}$ over $\mathcal{A}$ set $F = \Lambda \otimes_{\mathcal{A}} \mathcal{F}$ and $F_n = \Lambda_n \otimes_{\mathcal{A}} \mathcal{F}$. There are exact sequences

$$0 \to I_{n+1} \to \Lambda_{n+1} \to \Lambda_n \to 0,$$

(2.1)  

$$0 \to I_n \otimes_{\Lambda} F \to F_{n+1} \to F_n \to 0.$$ 

For $n = 1$, tensoring the second one with $I^\vee = \mathcal{H}\text{om}_{\Lambda}(I, \Lambda)$, we get

$$0 \to F \to I^\vee \otimes_{\Lambda_2} F_2 \to I^\vee \otimes_{\Lambda} F \to 0.$$ 

**Proposition 2.8.** Let $\mathcal{I}$ be a weak reduction cycle and $\mathcal{F}$ be a vector bundle over $\mathcal{A}$ such that $F$ is generically globally generated over $\Lambda$. Then $\mathcal{F}$ is also generically globally generated and $H^1(\mathcal{X}, \mathcal{I} \otimes_{\mathcal{A}} \mathcal{F}) = 0$. 

Note that if $F$ is generically globally generated and $H^1(\tilde{X}, I \otimes \tilde{A} F) = 0$, then $F$ is also generically globally generated, since the map $H^0(\tilde{X}, F) \to H^0(\tilde{X}, F)$ is surjective.

**Proof.** We first prove the second claim. Recall that, by the Theorem on Formal Functions [7, Theorem III.11.1],

$$H^1(\tilde{X}, I \otimes \tilde{A} F) \simeq \lim_{\leftarrow n} H^1(E, I/I^n \otimes \tilde{A} F).$$

(We need not use completion, since $H^1(\tilde{X}, M)$ is finite dimensional for every $M \in \text{Coh } \tilde{X}$.) Since $I/I_n$ is filtered by $I_m (1 \leq m < n)$, we have to show that $H^1(E, I_m \otimes \tilde{A} F) = H^1(E, I_m \otimes \Lambda F) = 0$ for all $m$. It follows from Proposition 2.7.

Note that $\Gamma(\tilde{X}, F) = \Gamma(X, \pi_\ast F)$ and $\pi_\ast F$ is globally generated, since $X$ is affine. Moreover, the sheaves $F$ and $\pi_\ast F$ coincide on $\tilde{X}$. Hence $\Gamma(\tilde{X}, F)$ generate $F_p$ for all $p \in \tilde{X}$. Therefore, we only have to prove that they generate $F_p$ for almost all points $p \in E$. Since $\text{supp } \Lambda = E$, it is enough to show that the global sections of $F$ generate $F_p$ for almost all $p \in E$. From the exact sequence $0 \to I \otimes \tilde{A} F \to F \to F \to 0$ and the equality $H^1(\tilde{X}, I \otimes \tilde{A} F) = 0$ we see that the restriction $\Gamma(\tilde{X}, F) \to \Gamma(E, F)$ is surjective. Since $F$ is generically globally generated, so is also $F$.

**Corollary 2.9.** A locally bi-principal sheaf of ideals $I \subset \tilde{A}$ is a weak reduction cycle if and only if

1. $I$ is generically globally generated.
2. $H^1(\tilde{X}, I) = 0$.

It is a reduction cycle if and only if, moreover, $\omega^\vee \otimes \tilde{A} I$ is generically globally generated.

**Proof.** If $I$ is a weak reduction cycle, (1) and (2) follows from Proposition 2.8. Conversely, suppose that (1) and (2) hold. Since $H^2(\tilde{X}, \Lambda) = 0$, then $H^1(E, I) = 0$. Moreover, just as in Proposition 2.7, $H^1(\tilde{X}, I \otimes \tilde{A} F) = 0$ for any generically globally generated $F$. In particular, $H^1(\tilde{X}, I^2) = 0$. Hence the map $\Gamma(\tilde{X}, I) \to \Gamma(E, I)$ is surjective, so $I$ is generically globally generated.

Now let $I$ be a weak reduction cycle. Note that, by Lemma 2.5

$$\omega^\vee = \text{Hom}_\tilde{A}(I^\vee \otimes \tilde{A} \omega_A \otimes \Lambda, \Lambda) \simeq \text{Hom}_\tilde{A}(I^\vee \otimes \tilde{A} \omega_A, \Lambda) \simeq \text{Hom}_\tilde{A}(I \otimes \tilde{A} \omega_A, \Lambda) \simeq \omega^\vee \otimes \tilde{A} I/I^2.$$

Hence, by Proposition 2.8, $\omega^\vee$ is generically globally generated if and only if so is $\omega^\vee \otimes \tilde{A} I$.

**Proposition 2.10.** A reduction cycle always exists.
Proof. Since the intersection form is negative definite on the group of divisors on $\tilde{X}$ with support $E$ \cite{11}, there is a divisor $D$ with support $E$ such that $O_{\tilde{X}}(-D)$ is ample. Therefore, for some $n > 0$, $\mathcal{I} = \mathcal{A}(-nD)$ as well as $\omega_{\mathcal{A}}^\vee(-nD)$ are generically globally generated, and moreover, $H^1(\tilde{X}, \mathcal{I}) = 0$. Obviously, $\mathcal{I}$ is bi-principal, so it is a reduction cycle. 

Now we need the following modification of the Wahl’s lemma \cite{13} Lemma B.2]

**Lemma 2.11.** If $\mathcal{F}$ is a vector bundle over $\tilde{A}$, then

$$H^1_\mathcal{E}(\tilde{X}, \mathcal{F}) \simeq \varinjlim_n H^0(\mathcal{E}, \mathcal{I}^{-n} \otimes \tilde{A} F_n)$$

Moreover, the natural homomorphisms

$$H^0(\mathcal{E}, \mathcal{I}^{-n} \otimes \tilde{A} F_n) \to H^0(\mathcal{E}, \mathcal{I}^{-n-1} \otimes \tilde{A} F_{n+1})$$

are injective.

**Proof.** Note that $H^1_\mathcal{E}(\tilde{X}, \mathcal{F}) \simeq \varinjlim_n \text{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F})$. Consider the spectral sequence $H^p(\tilde{X}, \text{Ext}^q_{\tilde{A}}(\Lambda_n, \mathcal{F})) \Rightarrow \text{Ext}^{p+q}_{\tilde{A}}(\Lambda_n, \mathcal{F})$. Since $\text{Hom}_{\tilde{A}}(\Lambda_n, \mathcal{F}) = 0$, the exact sequence of the lowest terms gives an isomorphism $\text{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F}) \simeq H^0(\mathcal{E}, \text{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F}))$. Applying $\text{Hom}_{\tilde{A}}(\mathcal{E}, \mathcal{F})$ to the exact sequence $0 \to \mathcal{I}^{-n} \to \tilde{A} \to \Lambda_n \to 0$, we get the exact sequence

$$0 \to \mathcal{F} = \tilde{A} \otimes \mathcal{F} \to \text{Hom}_{\tilde{A}}(\mathcal{I}^{-n}, \mathcal{F}) \simeq \mathcal{I}^{-n} \otimes \tilde{A} \mathcal{F} \to \text{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F}) \to 0,$$

whence $\text{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F}) \simeq (\mathcal{I}^{-n}/\tilde{A}) \otimes \mathcal{F}$. Moreover, since $\mathcal{I}^{-n}/\tilde{A} \subseteq \mathcal{I}^{-n-1}/\tilde{A}$ and $\mathcal{F}$ is locally projective, we get an embedding $(\mathcal{I}^{-n}/\tilde{A}) \otimes \mathcal{F} \hookrightarrow (\mathcal{I}^{-n-1}/\tilde{A}) \otimes \mathcal{F}$, hence an embedding of cohomologies. It remains to note that

$$(\mathcal{I}^{-n}/\tilde{A}) \otimes \mathcal{F} \simeq (\mathcal{I}^{-n}/\tilde{A}) \otimes \Lambda_n F_n \simeq \mathcal{I}^{-n} \otimes \tilde{A} F_n,$$

since $\mathcal{I}^{-n}$ annihilates $\mathcal{I}^{-n}/\tilde{A}$. \hfill $\square$

Since $I \otimes \mathcal{A} \mathcal{F} \simeq \mathcal{I} \otimes \tilde{A} \mathcal{F}$, there is an exact sequence

$$0 \to \mathcal{I} \otimes \tilde{A} \mathcal{F} \to F_2 \to F \to 0,$$

Multiplying it with $\mathcal{I}^\vee$, we get an exact sequence

$$(2.2) \quad 0 \to F \to \mathcal{I}^\vee \otimes \tilde{A} F_2 \to \mathcal{I}^\vee \otimes \tilde{A} F \to 0,$$

which gives the coboundary map $\theta_F : H^0(\mathcal{E}, \mathcal{I}^\vee \otimes \tilde{A} F) \to H^1(\mathcal{E}, F)$.

**Proposition 2.12** (Cf. \cite{10} Proposition 1.6). Let $\mathcal{I}$ be a weak reduction cycle. A vector bundle $\mathcal{F} \in \text{VB}(\tilde{A})$ is full if and only if

1. $\mathcal{F}$ is generically globally generated over $\Lambda$.
2. The coboundary map $\theta_F$ is injective.
Proof. Let $\mathcal{F}$ be generically globally generated. Since $H^1(\tilde{X}, I) = 0$, also $H^1(\tilde{X}, I \otimes \tilde{A} \mathcal{F}) = 0$. Therefore, the map $\Gamma(\tilde{X}, \mathcal{F}) \to \Gamma(E, \mathcal{F})$ is surjective, so $F$ is generically globally generated. Conversely, if $F$ is generically globally generated, so is $\mathcal{F}$ by Proposition 2.8. Hence, this condition (1) is equivalent to the condition (1) of Proposition 2.1. So now we suppose that both $\mathcal{F}$ and $F$ are generically globally generated.

Consider the commutative diagram

$$
\begin{array}{ccc}
H^1_E(\tilde{X}, \mathcal{F}) & \xrightarrow{\alpha_F} & H^1(\tilde{X}, \mathcal{F}) \\
\uparrow i & & \downarrow p \\
H^0(E, I^\vee \otimes \tilde{A} F) & \xrightarrow{\theta_F} & H^1(E, F)
\end{array}
$$

Here $i$ is an embedding from Lemma 2.11 and $p$ is an isomorphism, since $H^1(\tilde{X}, I \otimes \tilde{A} \mathcal{F}) = 0$. If $F$ is full, $\alpha_F$ is injective, hence so is $\theta_F$.

Conversely, suppose that $\theta_F$ is injective. We show that all embeddings

(2.3) $H^0(E, I^{-n} \otimes \tilde{A} F_n) \to H^0(E, I^{-n-1} \otimes \tilde{A} F_{n+1})$

from Lemma 2.11 are actually isomorphisms. It implies that $\alpha_F$ is injective, so $\mathcal{F}$ is full.

The map (2.3) comes from the exact sequence

(2.4) $0 \to I^{-n} \otimes \tilde{A} F_n \to I^{-n-1} \otimes \tilde{A} F_{n+1} \to I^{-n-1} \otimes \tilde{A} F \to 0$

obtained from the exact sequence

$$
0 \to I \otimes \tilde{A} F_n \to F_{n+1} \to F \to 0
$$

by tensoring with $I^{-n-1}$. So we have to show that the connecting homomorphism

$$
\beta_n : H^0(E, I^{-n-1} \otimes \tilde{A} F) \to H^1(E, I^{-n} \otimes \tilde{A} F_n)
$$

is injective. We actually prove that even the map

$$
\beta'_n : H^0(E, I^{-n-1} \otimes \tilde{A} F) \to H^1(E, I^{-n} \otimes \tilde{A} F),
$$

which is the composition of $\beta_n$ with the natural map $H^1(E, I^{-n-1} \otimes \tilde{A} F_n) \to H^1(E, I^{-n} \otimes \tilde{A} F)$, is injective.

Indeed, $\beta_0$ coincides with $\theta_F$. Since all sheaves $I^n$ are generically globally generated, there is a homomorphism $m \tilde{A} \to I^n$ whose cokernel has a finite support. Taking duals, we get an embedding $I^{-n} \hookrightarrow m \tilde{A}$. Tensoring this embedding with the exact sequence (2.4) for $n = 0$ and taking cohomologies, we get a commutative diagram

$$
\begin{array}{ccc}
H^0(E, I^{-n-1} \otimes \tilde{A} F) & \xrightarrow{\beta'_n} & H^1(E, I^{-n} \otimes \tilde{A} F) \\
\downarrow & & \downarrow \\
mH^0(E, I^{-1} \otimes \tilde{A} F) & \longrightarrow & mH^1(E, F)
\end{array}
$$

where the second horizontal and the first vertical maps are injective. Therefore, $\beta'_n$ is injective too, which accomplishes the proof. \qed
We call a vector bundle $F \in \text{VB}(\Lambda)$ full if $F \simeq \Lambda \otimes_{\tilde{\Lambda}} \mathcal{F}$, where $\mathcal{F}$ is a full vector bundle over $\tilde{\Lambda}$.

**Theorem 2.13** (Cf. [10] Theorem 1.4). Let $\mathcal{I}$ be a weak reduction cycle. A vector bundle $F \in \text{VB}(\Lambda)$ is full if and only if

1. $F$ is generically globally generated.
2. There is a vector bundle $F_2 \in \text{VB}(\Lambda_2)$ such that $\Lambda \otimes_{\tilde{\Lambda}} F_2 \simeq F$ and the connecting homomorphism $\theta_F : H^0(E, I_{\mathcal{I}} \otimes_{\tilde{\Lambda}} F) \to H^1(E, F)$ coming from the exact sequence (2.2) is injective.

If, moreover, $\mathcal{I}$ is a reduction cycle, the full vector bundle $\mathcal{F} \in \text{VB}(\tilde{\Lambda})$ such that $\Lambda \otimes_{\tilde{\Lambda}} \mathcal{F} \simeq F$ is unique up to isomorphism. Thus the reduction functor $R_\mathcal{I}$ induces a one-to-one correspondence between isomorphism classes of Cohen–Macaulay $\Lambda$-modules and isomorphism classes of full vector bundles over $\Lambda$.

**Proof.** Let $\mathcal{I}$ be a weak reduction cycle, $F$ satisfies (1) and (2). If $U \subset E$ is an affine open subset, there is an exact sequence

$$0 \to I_n(U) \to \Lambda_{n+1}(U) \to \Lambda_n(U) \to 0,$$

where the ideal $I_n(U)$ is nilpotent (actually, $I_n(U)^2 = 0$). Therefore, given a projective $\Lambda_n(U)$-module $P_n$, there is a projective $\Lambda_{n+1}(U)$-module $P_{n+1}$ such that $\Lambda_n(U) \otimes_{\Lambda_{n+1}(U)} P_{n+1} \simeq P_n$. Moreover, if $P'_n$ is another projective $\Lambda_n(U)$-module, $P'_{n+1}$ is a projective $\Lambda_{n+1}(U)$-module such that $\Lambda_n(U) \otimes_{\Lambda_{n+1}(U)} P'_{n+1} \simeq P'_n$, and $\varphi_n : P_n \to P'_n$ is a homomorphism, it can be lifted to a homomorphism $\varphi_{n+1} : P_{n+1} \to P'_{n+1}$, and if $\varphi_n$ is an isomorphism, so is $\varphi_{n+1}$ too.

Consider an affine open cover $E = U_1 \cup U_2$. Let $P_{2,i} = F_2(U_i)$. Iterating the above procedure, we get projective $\Lambda_n(U_i)$-modules $P_{n,i}$ such that

$$\Lambda_n(U_i) \otimes_{\Lambda_{n+1}(U_i)} P_{n+1,i} \simeq P_{n,i}$$

for all $n \geq 2$. If $U = U_1 \cap U_2$, there is an isomorphism $\varphi_2 : P_{2,1}(U) \simeq P_{2,2}(U)$. It can be lifted to $\varphi_n : P_{n,1}(U) \simeq P_{n,2}(U)$ so that the restriction of $\varphi_{n+1}$ to $P_{n,1}$ coincides with $\varphi_n$. Hence there are vector bundles $F_n$ over $\Lambda_n$ such that $\Lambda_n \otimes_{\tilde{\Lambda}} F_{n+1} \simeq F_n$. Taking inverse image, we get a vector bundle $\tilde{\mathcal{F}} = \lim_{\leftarrow n} F_n$ over the formal non-commutative scheme $(\tilde{X}, \tilde{\Lambda})$ which is the completion of $(X, \Lambda)$ along the subscheme $E$. As $X$ is projective, hence proper over $X$, $\tilde{\mathcal{F}}$ uniquely arises as the completion of a vector bundle $\mathcal{F}$ over $\tilde{\Lambda}$ such that $\Lambda_n \otimes_{\tilde{\Lambda}} \mathcal{F} \simeq F_n$ for all $n$ (see [9] Theorem 5.1.4)). If we choose $F_2$ so that the condition (2) holds, $\mathcal{F}$ is full by Proposition 2.12. Thus $\mathcal{F}$ is full as well.

Let now $\mathcal{I}$ be a reduction cycle, $F$ be a full vector bundle over $\Lambda$ and $F_n$ be vector bundles over $\Lambda_n$ such that $\Lambda_n \otimes_{\tilde{\Lambda}} F_{n+1} \simeq F_n$ for all $n$ and $F_2$ satisfies the condition (2). As we have already mentioned, all choices of $F_n$ are locally isomorphic. Therefore, if we fix one of them, their isomorphism classes are in one-to-one correspondence with the cohomology set $H^1(E, \text{Aut} F_n)$ [8].
From the exact sequence (2.1) we obtain an exact sequence of sheaves of groups

\[ 0 \to \mathcal{H} \to \text{Aut} F_{n+1} \to \text{Aut} F_n \to 0, \]

where \( \mathcal{H} = \ker \rho \simeq \text{Hom}_{\Lambda}(F_n, I_n \otimes_{\Lambda} F) \simeq \text{Hom}_{A}(F, I_n \otimes_{A} F) \). It gives an exact sequence of cohomologies

\[ 0 \to \text{Hom}_{A}(F_n, I_n \otimes_{\Lambda} F) \to \text{Aut} F_{n+1} \to \text{Aut} F_n \delta_n \to \text{Ext}^1_{\Lambda}(F, I_n \otimes_{\Lambda} F) \to H^1(E, \text{Aut} F_{n+1}) \to H^1(E, \text{Aut} F_n). \]

The isomorphism classes of liftings \( F_{n+1} \) of a given \( F_n \) are in one-to-one correspondence with the orbits of the group \( \text{Aut} F_n \) naturally acting on \( \text{Ext}^1_{\Lambda}(F, I_n \otimes_{\Lambda} F) \). [3] Proposition 5.3.1.

We write automorphisms of \( F_n \) in the form \( 1 + \varphi \) for \( \varphi \in \text{End} F_n \). Then \( \delta(1 + \varphi) = \delta_n(\varphi) \), where \( \delta_n : \text{Hom}_{\Lambda}(F_n, F_n) \to \text{Ext}^1_{\Lambda}(F, I_n \otimes_{\Lambda} F) \) is the connecting homomorphism coming from the exact sequence (2.1). We restrict \( \delta_n \) to \( \text{Hom}_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F) \) (see the same exact sequence, with \( n \) replaced by \( n - 1 \)). The resulting homomorphism \( \delta'_n : \text{Hom}_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F) \to \text{Ext}^1_{\Lambda}(F, I_n \otimes_{\Lambda} F) \) coincides with the connecting homomorphism coming from the exact sequence (2.2) tensored with \( T^{n-1} \).

Claim 1. \( \delta'_n \) is surjective.

Indeed, since \( F, I_{n-1} \) and \( \omega^\Lambda \) are generically globally generated, so is their tensor product. Hence, there is a homomorphism \( m\Lambda \to \omega^\Lambda \otimes_{\Lambda} I_{n-1} \otimes_{\Lambda} F \), thus also \( m\omega_{\Lambda} \to I_{n-1} \otimes_{\Lambda} F \) whose cokernel has discrete support. Applying \( \text{Hom}_{\Lambda}(F, \omega_{\Lambda}) \), we get a commutative diagram

\[ \begin{array}{ccc}
\text{Hom}_{\Lambda}(F, \omega_{\Lambda}) & \longrightarrow & m\text{Ext}^1_{\Lambda}(F, I \otimes_{\Lambda} \omega_{\Lambda}) \\
\downarrow & & \downarrow \eta \\
\text{Hom}_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F) & \delta'_n \longrightarrow & \text{Ext}^1_{\Lambda}(F, I_n \otimes_{\Lambda} F),
\end{array} \]

where \( \eta \) is surjective. Note that the first horizontal map here is the \( m \)-fold Serre dual \( \theta_F^* \) to the map

\[ \theta_F : \text{Hom}_{\Lambda}(I, F) \simeq H^0(E, I^\vee \otimes_{\Lambda} F) \to \text{Ext}^1_{\Lambda}(A, F) \simeq H^1(E, F), \]

which is injective. Therefore, \( \theta_F^* \) is surjective and so is also \( \delta'_n \).

If \( n > 1 \), every homomorphism \( 1 + \varphi \) with \( \varphi \in \text{Hom}_{\Lambda}(F, I_{n-1} \otimes_{\Lambda} F) \) is invertible. Hence \( \delta \) is surjective and \( F_{n+1} \) is unique up to isomorphism. If \( n = 1 \), the set \( \{ \varphi \in \text{Hom}_{\Lambda}(F, F) \mid 1 + \varphi \text{ is invertible} \} \) is open in \( \text{Aut} F \). Therefore, its image in \( \text{Ext}^1_{\Lambda}(F, I \otimes_{\Lambda} F) \) is an open orbit of \( \text{Aut} F \). If we choose another lifting \( F'_2 \) of \( F \) so that the condition (2) holds, it also gives an open orbit. Since there can be at most one open orbit, they coincide, hence \( F'_2 \simeq F_2 \). Now, if \( F \) and \( F' \) are two full vector bundles over \( A \) such that \( \Lambda \otimes_{A} F \simeq \Lambda \otimes_{A} F' \simeq F \), we can glue isomorphisms \( \Lambda_n \otimes_{A} F \tilde{\simeq} \Lambda_n \otimes_{A} F' \) into an isomorphism \( F \tilde{\simeq} F' \). \( \square \)

Claim [1] also implies the following result.
Corollary 2.14 (Cf. [10, Corollary 1.10]). If $F \in \text{VB}^f(\tilde{A})$, then $\text{Ext}^1_{\tilde{A}}(F, F) \simeq \text{Ext}^1_{A}(F, F)$.

We omit the proof since it just copies that from [10].

3. GOOD ELLIPTIC CASE

There is one special case when the conditions of Theorem 2.13 can be made much simpler. It is analogous to the case of minimally elliptic surface singularities considered in [10, Section 2]. We are not aware of the full generality when it can be done, so we only confine ourselves to a rather restricted situation. Thus the following definition shall be considered as very preliminary. It will be used in the examples studied in the next section.

Definition 3.1. Let $\pi : (\tilde{X}, \tilde{A}) \to (X, A)$ be a resolution of a non-commutative surface singularity, $I$ be a weak reduction cycle and $\Lambda = \tilde{A}/I$. We say that the weak reduction cycle $I$ is good elliptic if $\Lambda \simeq \mathcal{O}_Z$ where $Z$ is a reduced curve of arithmetic genus 1 (hence $\omega_Z \simeq \mathcal{O}_Z$). Obviously, then $I$ is a reduction cycle. If a non-commutative surface singularity $(X, A)$ has a resolution $(\tilde{X}, \tilde{A})$ such that there is a good elliptic reduction cycle $I \subset \tilde{A}$, we say that $(X, A)$ is a good elliptic non-commutative surface singularity.

Remark 3.2. One easily sees that being good elliptic is equivalent to fulfillments of the following conditions for some resolution:

1. $h^1(\tilde{X}, \tilde{A}) = 1$.
2. There is a weak reduction cycle $I$ such that $\Lambda$ is commutative and reduced.

Then $I$ is also a reduction cycle.

For good elliptic non-commutative surface singularity we can state a complete analogue of [10, Theorem 2.1]. Moreover, the proof is just a copy of the Kahn’s proof, so we omit it.

Theorem 3.3. Suppose that $I$ is a good elliptic reduction cycle for a resolution $(\tilde{X}, \tilde{A})$ of a non-commutative surface singularity $(X, A)$, $\Lambda = \tilde{A}/I$ and $I = I/I^2$. A vector bundle $F$ over $\Lambda$ is full if and only if $F \simeq G \oplus m\Lambda$, where the following conditions hold:

1. $G$ is generically globally generated.
2. $H^1(E, G) = 0$.
3. $m \geq h^0(E, I^\vee \otimes_{\Lambda} G)^3$

If these conditions hold and $M$ is the Cohen–Macaulay $A$-module such that $F \simeq R_{I^\vee} M$, then $M$ is indecomposable if and only if either $m = h^0(E, I^\vee \otimes_{\Lambda} G)$ or $F = \Lambda$ (then $M = A$).

Now, just as in [5] (and with the same proof), we obtain the following result.

---

$^3$If we identify $\Lambda$ with $\mathcal{O}_Z$, then $I^\vee \otimes_{\Lambda} G$ is identified with $G(Z)$. 

Corollary 3.4. Suppose that \( \mathcal{I} \) is a good elliptic reduction cycle for a resolution \( (\tilde{X}, \tilde{A}) \) of a non-commutative surface singularity \( (X, A) \) and \( \Lambda = \tilde{A}/\mathcal{I} \simeq O_Z \). The non-commutative surface singularity \( (X, A) \) is Cohen–Macaulay tame if and only if \( Z \) is either a smooth elliptic curve or a Kodaira cycle (a cyclic configuration in the sense of [5]). Otherwise it is Cohen–Macaulay wild.

For the definitions of Cohen–Macaulay tame and wild singularities see [5, Section 4]. Though in this paper only the commutative case is considered, the definitions are completely the same in the non-commutative one.

4. Examples

In what follows we consider non-commutative surface singularities \( (X, A) \), where \( X = \text{Spec} R \) and \( R = \mathbb{k}[[u, v]] \). We define \( A \) by generators and relations. The ramification divisor \( D = D(A) \) is then given by one relation \( F = 0 \) for some \( F \in R \), so it is a plane curve singularity.

When blowing up the closed point \( o \), we get the subset \( \tilde{X} \subseteq \text{Proj} R[\alpha, \beta] \) given by the equation \( u\beta = v\alpha \). We cover it by the affine charts \( U_1 : \beta \neq 0 \) and \( U_2 : \alpha \neq 0 \), so their coordinate rings are, respectively, \( R_1 = R[\xi]/(u - \xi v) \) and \( R_2 = R[\eta]/(v - \eta u) \), where \( \xi = \alpha/\beta \) and \( \eta = \beta/\alpha \).

Example 4.1.

\[
A = R( x, y \mid x^2 = v, y^2 = u(u^2 + \lambda v^2), xy + yx = 2\varepsilon uv),
\]

where \( \lambda \notin \{ 0, 1 \} \) and \( \varepsilon^2 = 1 + \lambda \). Then \( F = uv(u - v)(u - \lambda v) \), so \( D \) is of type \( T_{44} \). We set \( z = xy \), so \( \{ 1, x, y, z \} \) is an \( R \)-basis of \( A \) and \( z^2 = 2\varepsilon uvz - \lambda v^2 \). One can check that \( k_C(A) \) is a field, namely a quadratic extension of \( k(C) \), for every component of \( D \). For instance, if this component is \( u = v \), and \( g \) is its general point, then, modulo the ideal \( (u - v)A_g \), \( (z - \varepsilon uv)^2 = 0 \), so \( z - \varepsilon uv \in \text{rad} A_g \). Moreover,

\[
(z - \varepsilon uv)^2 = z^2 - 2\varepsilon uvz + (1 + \lambda)u^2v^2 = -uv(u^2 + \lambda v^2) + (1 + \lambda)u^2v^2 = uv(u - v)(\lambda v - u).
\]

Since \( uv(\lambda v - u) \) is invertible in \( A_g \), \( u - v \in (z - \varepsilon uv)A_g \). One easily sees that \( (z - \varepsilon uv)A_g \) is a two-sided ideal and \( A_g/(z - \varepsilon uv)A_g \simeq k[[u]][x]/(x^2 - u) \) is a field. (Note that in this factor \( \varepsilon uvx = xx = vy \), so \( y = \varepsilon ux \).) Therefore, \( (X, A) \) is normal and its ramification index equals 2 on every component of \( D \).

After blowing up the closed point \( o \in X \), we get

\[
\pi^*A(U_1) \simeq R_1( x, y \mid x^2 = v, y^2 = \xi(\xi^2 + \lambda)v^3, xy + yx = 2\varepsilon \xi v^2, z^2 = 2\varepsilon \xi v^2z - \lambda \xi^4(\xi^2 + \lambda)).
\]

and \( z^2 = 2\varepsilon \xi v^2z - \lambda \xi^4(\xi^2 + \lambda) \). So we can consider the \( R_1 \)-subalgebra \( A_1 = \pi^*A(U_1)\langle z_1 \rangle \) of \( k(A) \), where \( z_1 = v^{-2}z - \varepsilon \xi \). Note that \( y_1 = v^{-1}y = \)
\[ xz_1 + \varepsilon \xi x \in A_1. \]

\[ \pi^*A(U_2) \simeq R_2\{x, y | x^2 = \eta u, y^2 = u^3(1 + \lambda \eta^2), xy + yx = 2\varepsilon \eta u^2 \} \]

and \( z^2 = 2\varepsilon \eta^2 z - \eta u^2(1 + \eta^2 \lambda). \) So we can consider the \( R_2 \)-subalgebra \( A_2 = \pi^*A(U_2)(y_2, z_2) \) of \( K(A), \) where \( y_2 = u^{-1}y, z_2 = u^{-2}z - \varepsilon \eta. \)

Since \( y_2 = \eta y_1 \) and \( z_2 = \eta^2 z_1, \) \( A_1(U_1 \cap U_2) = A_2(U_1 \cap U_2), \) so we can consider the non-commutative surface \( (\tilde{X}, \tilde{A}), \) where \( \tilde{A}(U_1) = A_1, \tilde{A}(U_2) = A_2. \) One can check, just as above, that it is normal. Its ramification divisor \( \tilde{D} \) is given on \( U_1 \) by the equation \( \xi v(\xi - 1)(\xi - \lambda) = 0 \) and on \( U_2 \) by \( u\eta(1 - \eta)(1 - \lambda \eta) = 0, \) so its components are projective lines and have normal crossings. Moreover, \( e_C(A) = 2 \) for every component \( C \) of \( \tilde{D}, \) and if \( x \in C \) is a node of \( \tilde{D}, \) then \( e_{C,x}(A) = 2. \) Hence \((\tilde{X}, \tilde{A})\) is a terminal resolution of \((X, A).\)

Consider the ideal \( \mathcal{I} \subset \tilde{A} \) such that \( \mathcal{I}(U_1) = (x) \) and \( \mathcal{I}(U_2) = (x, y_2). \) Note that \( \eta y_2 = xz_2 - \varepsilon \eta x, \) \( z_2y_2 = (1 + \lambda \eta^2)x - \varepsilon \eta y_2 \) and \( y_2^2 = (1 + \lambda \eta^2)u. \) Therefore, if \( p \in U_2 \) and \( \eta(p) \neq 0, \) then \( y_2, \) \( u \in \tilde{A}_p = x\tilde{A}_p, \) while if \( p \in U_2 \) and \( \eta(p) = 0, \) then \( x, u \in \tilde{A}_p y_2 = y_2\tilde{A}_p. \) Hence \( \mathcal{I} \) is bi-principal.

\[ A_1/\mathcal{I}(U_1) \simeq k[\xi, z_1]/(z_1^2 - \xi(\xi - 1)(\lambda - \xi)), \]

and

\[ A_2/\mathcal{I}(U_2) \simeq k[\eta, z_2]/(z_2^2 - \eta(1 - \eta)(\lambda \eta - 1)), \]

hence \( \Lambda = \tilde{\Lambda}/\mathcal{I} \simeq O_Z, \) where \( Z \) is an elliptic curve. Moreover, \( x \) is a global section of \( \mathcal{I}, \) hence of \( I = \mathcal{I}/\mathcal{I}^2, \) and it generates \( I_p \) for every point \( p \in Z \) except the point \( \infty \) on the chart \( U_2, \) where \( \eta = 0. \) So \( \mathcal{I} \) is a good elliptic reduction cycle and \( I \simeq O_Z(\infty). \)

Now, by Theorem 3.3, Cohen–Macaulay modules over \( A \) can be obtained as follows. We identify \( Z \) with \( \operatorname{Pic}^0(Z) \) taking \( \infty \) as the zero point. Denote by \( G(r, d; p) \) the indecomposable vector bundle over \( Z \) of rank \( r, \) degree \( d \) and the Chern class \( p \in Z = \operatorname{Pic}^0(Z) \) (see [2]). It is generically globally generated if and only if either \( d > 0 \) or \( d = 0, r = 1 \) and \( p = \infty. \) In the latter case \( G(1, 0; \infty) \simeq O_Z. \) Then \( I^r \otimes_A G(r, d; p) \simeq G(r, d - r; p). \) Moreover,

\[ h^0(Z, G(r, d; p)) = \begin{cases} 0 & \text{if } 1 \leq d < 0 \text{ or } d = 0 \text{ and } p \neq \infty, \\ 1 & \text{if } d = 0 \text{ and } p = \infty, \\ d & \text{if } d > 0. \end{cases} \]

So if \( M \) is an indecomposable Cohen–Macaulay \( A \)-module and \( M \not\simeq A, \) then it is uniquely determined by its Kahn reduction \( R_{\mathcal{I}}M \) which is one of the following vector bundles:

- \( G(r, d; p), \) where \( d < r \) or \( d = r, p \neq \infty; \) then \( \text{rk } M = r. \)
- \( G(r, r; \infty) \oplus O_Z, \) where \( r > 1; \) then \( \text{rk } M = r + 1. \)
- \( G(r, d; p) \oplus (d - r)O_Z, \) where \( d > r; \) then \( \text{rk } M = d. \)

In particular, \( A \) is Cohen–Macaulay tame in the sense of [5]. Namely, for a fixed rank \( r, \) Cohen–Macaulay \( A \)-modules of rank \( r, \) except one of them, form
2(r − 1) families parametrized by Z and one family parametrized by $Z \setminus \{ \infty \}$, arising, respectively, from $G(d, r, p)$ ($1 \leq d < r$), $G(r', r, p)$ ($1 \leq r' < r$) and $G(r, r, p)$ ($p \neq \infty$).

**Example 4.2.**

$A = R\langle x, y \mid x^3 = v, y^3 = u(u - v), xy = \zeta yx \rangle$, where $\zeta^3 = 1$, $\zeta \neq 1$.

Then $F = uv(u - v)$ (the singularity of type $D_4$). Just as above, one can check that $A$ is normal and $e_c(A) = 3$ for every component $C$ of $D$. After blowing up, on the chart $U_1$ we can consider the algebra $A_1 = \pi^* A(U_1)/\langle w_1, z_1 \rangle$, where $w_1 = v^{-1} y^2$, $z_1 = v^{-1} xy$, and on the chart $U_2$ we can consider the algebra $A_2 = \pi^* A(U_2)/\langle w_2, z_2 \rangle$, where $w_2 = u^{-1} y^2$, $z_2 = u^{-1} xy$. Again $A_1(U_1 \cap U_2) = A_2(U_1 \cap U_2)$, so we can glue them into a non-commutative surface $(\tilde{X}, \tilde{A})$. One can verify that it is terminal. Let $\mathcal{I}$ be the locally bi-principal ideal in $\tilde{A}$ such that $\mathcal{I}(U_1) = (x)$ and $\mathcal{I}(U_2) = (x, w_2)$. Then $\tilde{A}/\mathcal{I} \simeq \mathcal{O}_Z$, where $Z$ is the elliptic curve given by the equation $z_1^3 = \xi (\xi - 1)$ on $U_1$ and by $z_2^3 = \eta (1 - \eta)$ on $U_2$. Again $x$ defines a global section of $\mathcal{I}$, hence of $I$, and $I \simeq \mathcal{O}_Z(\infty)$, where $\infty$ is the point on $U_2$ with $\eta = 0$. Therefore, $\mathcal{I}$ is a good elliptic reduction cycle and Cohen–Macaulay modules over $A$ are described in the same way as in Example 4.1. In particular, $A$ is also Cohen–Macaulay tame.

**References**

[1] Artin M., Maximal orders of global dimension and Krull dimension two, Invent. Math., 1986, 84, 195–222
[2] Atiyah M., Vector bundles over an elliptic curve, Proc. Lond. Math. Soc., 1957, 7, 414–452
[3] Chan D, Ingalls C., The minimal model program for orders over surfaces, Invent. Math., 2005, 161, 427–452
[4] Drozd Y, Greuel G.-M., Cohen-Macaulay module type, Compos. Math., 1993, 89, 315–338
[5] Drozd Y, Greuel G.-M., Kashuba I., On Cohen–Macaulay modules on surface singularities, Mosc. Math. J., 2003, 3, 397–418
[6] Drozd Y, Kirichenko V., Primary orders with a finite number of indecomposable representations, Izv. Akad. Nauk SSSR Ser. Mat., 1973, 37, 715–736
[7] Hartshorne R., Algebraic Geometry, Springer-Verlag, New York–Berlin–Heidelberg, 1977
[8] Grothendieck A., A General Theory of Fibre Spaces with Structure Sheaf, University of Kansas, Lawrence, 1955
[9] Grothendieck A., Éléments de géométrie algébrique: III. Étude cohomologique des faisceaux cohérents, Première partie, Publ. math. I.H.É.S., 1961, 11, 5–167
[10] Kahn C. P., Reflexive modules on minimally elliptic singularities, Math. Ann., 1989, 285, 141–160
[11] Lipman J., Rational singularities, with application to algebraic surfaces and unique factorization, Publ. math. I.H.É.S., 1969, 36, 195–279
[12] Reiner I., Maximal orders, London Math. Soc. Monogr. Ser., 1975, 5
[13] Wahl J., Equisingular deformations of normal surface singularities, I, Ann. Math., 1976, 104, 325–356
[14] Yoshino Y., Cohen–Macaulay Modules over Cohen–Macaulay Rings, London Math. Soc. Lecture Notes Ser., 1990, 146

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereschenkovska str. 3, 01601 Kyiv, Ukraine
E-mail address: y.a.drozd@gmail.com, drozd@imath.kiev.ua
URL: www.imath.kiev.ua/~drozd
E-mail address: v.gavran@yahoo.com