Topological invariance of the Collet–Eckmann condition for one-dimensional maps

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Abstract

This paper is devoted to studying the topological invariance of several non-uniform hyperbolicity conditions of one-dimensional maps. In contrast with the case of maps with only one critical point, it is known that for maps with several critical points the Collet–Eckmann condition is not in itself invariant under topological conjugacy. We show that the Collet–Eckmann condition together with any of several slow recurrence conditions is invariant under topological conjugacy. This extends and gives a new proof of a result by Luzzatto and Wang that also applies to the complex setting.

Keywords: one-dimensional map, the Collet–Eckmann condition, topological invariance, the slow recurrence condition

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1. Introduction

Let $I$ be a compact interval of $\mathbb{R}$. Recall that a non-injective continuous map $f : I \rightarrow I$ is multimodal, if there is a finite partition of $I$ into intervals on each of which $f$ is injective. For a differentiable multimodal map $f : I \rightarrow I$, a point of $I$ is critical point of $f$ if the derivative of $f$ vanishes at it. We denote by $\text{Crit}(f)$ the set of critical points of $f$.

**Definition 1.1.** We say that an interval map $f$ satisfies the Collet–Eckmann condition (abbreviated CE condition) if all the periodic points of $f$ are hyperbolic repelling, and if there...
are two constants $C > 0$ and $\lambda > 1$ such that for all positive integer $n$ and for any critical point $c$ of $f$ which does not land at another critical point of $f$ under iterations of $f$, we have

$$|Df^n(f(c))| \geq C\lambda^n.$$  

This type of condition was first introduced by Collet and Eckmann in [CE83] providing a large class of $S$-unimodal (i.e. with negative Schwarzian derivative) maps of the interval having a finite absolutely continuous invariant measure. Further investigations by Nowicki, Przytycki, van Strien and Sands proved the equivalence of those and several similar properties in the real $S$-unimodal setting, see [Now85–NS98] and see also [PR98, PRLS03, Prz98] for rational maps.

Recall that two multimodal maps $f : I \to I$ and $g : J \to J$ are topologically conjugate, if there exists a homeomorphism $h : I \to J$ such that for each $x \in I$ we have $h(f(x)) = g(h(x))$. Moreover, if both $h$ and $h^{-1}$ are Hölder continuous, then we say that $f$ and $g$ are bi-Hölder conjugate. A topological conjugacy preserves topological properties of maps such as periodic orbits.

The problem of topological invariance of the Collet–Eckmann condition for $S$-unimodal maps was posed in [vS88] as well as by Guckenheimer and Misiurewicz in the early 1980s. In [NP98] the authors proved that the Collet–Eckmann condition is invariant under topological conjugacy within the class of $S$-unimodal maps. This result does not however generalize to the multimodal setting, see for example [Mih08, PRLS03] for some counterexamples: a pair of topologically conjugate multimodal maps one of which satisfies Collet–Eckmann condition and the other does not. But in [LW06] the authors proved that a strengthened version of the Collet–Eckmann condition for multimodal maps is topologically invariant, see [Now85, main theorem], and also [Wan01], for a precise statement. See also [LS13] for a recent related result.

In this paper, we give a conceptual proof of [Now85, main theorem] that is also simpler and shorter than the proof given by Luzzatto and Wang in [LW06] that relies heavily on delicate combinatorial arguments. In fact, our method of proof allows us to prove variants of [Now85, main theorem] for different (and more natural) notions of slow recurrence. We illustrate this by stating theorems 1.1–1.3, but other variants are straightforward to obtain. The main tool is [RL12a, proposition 5.2] that states a conjugacy between two Lipschitz continuous multimodal maps that satisfy the exponential shrinking of components condition is bi-Hölder continuous. This allows us to show that for maps satisfying a slow recurrence condition, the Topological Collet–Eckmann condition implies the Collet–Eckmann condition.

Let us be more precise. Let $I$ be a compact interval of $\mathbb{R}$. A $C^1$ multimodal map $f : I \to I$ is of class $C^3$ with non-flat critical points, if the map $f$ is of class $C^3$ outside $\text{Crit}(f)$, and if for each critical point $c$ of $f$ there exists a number $\ell_c > 1$ and diffeomorphisms $\phi$ and $\psi$ of $\mathbb{R}$ of class $C^3$, such that $\phi(c) = \psi(f(c)) = 0$, and such that on a neighborhood of $c$ on $I$, we have $|\psi \circ f| = |\phi|^\ell_c$.

In what follows, we use $\mathcal{H}_S$ to denote the collection of all interval maps which are of class $C^3$ with non-flat critical points and are with all periodic points hyperbolic repelling.

### 1.1. Statement of results

In this subsection, we state our main results. A multimodal map $f : I \to I$ in $\mathcal{H}_S$ is topologically exact, if for every open subset $U$ of $I$ there is an integer $n \geq 1$ such that $f^n(U) = I$.

**Definition 1.2.** Given $\beta \in (0, 1)$, we say that a multimodal map $f \in \mathcal{H}_S$ satisfies the stretched exponential recurrence condition with respect to exponent $\beta$ (abbreviated $f \in \text{SER}(\beta)$), if there
exists $C > 0$ such that for any two critical points $c, c' \in \text{Crit}(f)$ and any positive integer $n \geq 1$, we have
\[ |f^n(c) - c'| \geq C \exp(-n^3). \]

Moreover, we say that the map $f$ satisfies the stretched exponential recurrence condition, if $f \in \text{SER} (\beta)$ for some $\beta \in (0, 1)$.

**Theorem 1.1.** Let $f, \hat{f}$ be two multimodal maps in $\mathcal{A}_\mathbb{R}$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Assume that $f$ satisfies both the Collet–Eckmann condition and the stretched exponential recurrence condition. Then $\hat{f}$ also satisfies both the Collet–Eckmann condition and the stretched exponential recurrence condition.

**Definition 1.3.** Given $\beta > 0$, we say that a multimodal map $f \in \mathcal{A}_\mathbb{R}$ satisfies the exponential recurrence condition with respect to $\beta$, if there exists $C > 0$ such that for any two critical points $c, c' \in \text{Crit}(f)$ and any positive integer $n \geq 1$, we have
\[ |f^n(c) - c'| \geq C \exp(-\beta n). \]

Moreover, we say that $f \in \mathcal{A}_\mathbb{R}$ satisfies the subexponential recurrence condition if for every $\beta > 0$ the map $f$ satisfies the exponential recurrence condition with respect to $\beta$.

**Theorem 1.2.** Let $f, \hat{f}$ be two multimodal maps in $\mathcal{A}_\mathbb{R}$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Assume that $f$ satisfies both the Collet–Eckmann condition and the subexponential recurrence condition. Then $\hat{f}$ also satisfies both the Collet–Eckmann condition and the subexponential recurrence condition.

**Definition 1.4.** Given $\beta > 0$, we say that a multimodal map $f \in \mathcal{A}_\mathbb{R}$ satisfies the polynomial recurrence condition with respect to exponent $\beta$ (abbreviated $f \in \text{PR} (\beta)$), if there exists $C > 0$ such that for any two points $c, c' \in \text{Crit}(f)$ and any $n \geq 1$, we have
\[ |f^n(c) - c'| \geq C n^{-\beta}. \]

Moreover, $f$ is said to satisfy the polynomial recurrence condition if $f \in \text{PR} (\beta)$ for some $\beta > 0$.

**Theorem 1.3.** Let $f, \hat{f}$ be two multimodal maps in $\mathcal{A}_\mathbb{R}$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Assume that $f$ satisfies both the Collet–Eckmann condition and the polynomial recurrence condition. Then $\hat{f}$ also satisfies both the Collet–Eckmann condition and polynomial recurrence condition.

Recall that $f$ satisfies the slow recurrence condition of [LW06] if every critical point $c$ of $f$ satisfies the following property:

\[ \lim_{\delta \to 0^+} \lim_{n \to +\infty} \frac{1}{n} \sum_{x \in \mathbb{R}_n} \log d(f^n(x)) = 0, \]

where $B_\delta (c) := B(\text{Crit}(f), \delta)$ and $d(x) := \min \{|x - c| : c \in \text{Crit}(f)\}$. This condition was first introduced by Tsujii, see [Tsu93], and so sometime it is called the Tsujii’s slow recurrence.

The following is [Now85, main theorem], and we give a new proof of it in section 4.

**Theorem 1.4.** Let $f, g$ be two multimodal maps in $\mathcal{A}_\mathbb{R}$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Suppose that $f$ satisfies
both the Collet–Eckmann condition and the slow recurrence condition. Then the map g also satisfies both Collet–Eckmann condition and slow recurrence condition.

**Remark 1.1.** The strategy used by Luzzatto and Wang to prove theorem 1.4 is to define a new condition which they call the topological slow recurrence (TSR) condition, see the precise definition on page 349 in [LW06], which depends only on the combinatorics of the critical orbits. In particular TSR is invariant under topological conjugacy. Then they proved that this condition is equivalent to the simultaneous occurrence of the slow recurrence condition and the Collet–Eckmann conditions.

1.2. Organization

The paper is organized as follows. In section 2, we recall some definitions of non-uniform hyperbolicity conditions, and collect some results of non-uniform hyperbolicity conditions and Koebe distortion. In section 3, we give the proofs of our theorems 1.1 – 1.3. In section 4, we give a new proof of theorem 1.4. In appendix, we give an analog theorem in the complex setting, see theorem A.1.

2. Preliminaries

In what follows, let I and J be two compact intervals of \( \mathbb{R} \). In this section, we collect some known results which will be used in proving our theorems.

2.1. Topological Collect–Eckmann condition

Let \( f : I \rightarrow I \) be a multimodal map and fix \( r > 0 \). Recall that given an integer \( n \geq 1 \), the criticality of \( f^n \) at a point \( x \) of \( I \) with respect to \( r \) is the number of those \( j \in \{0, \ldots, n-1\} \) such that the connected component of \( f^{-n-j}(B(f^n(x), r)) \) containing \( f^j(x) \) contains a critical point of \( f \) in \( \text{Crit}(f) \). We say that \( f \) satisfies the Topological Collet–Eckmann condition (abbreviated TCE condition), if for some choice of \( r > 0 \) there are constants \( D \geq 1 \) and \( \theta \in (0, 1) \), such that the following property holds: for each point \( x \in I \) the set \( G_x \) of all those integers \( m \geq 1 \) for which the criticality of \( f^m \) at \( x \) is less than or equal to \( D \), satisfies

\[
\liminf_{n \to +\infty} \frac{1}{n} \#(G_x \cap \{1, \ldots, n\}) \geq \theta.
\]

The TCE condition was first introduced in [PR98] for rational maps, see also [NP98, PRL14]. Clearly, the TCE condition is topologically invariant provided there are no parabolic points.

Recall that an interval map \( f : I \rightarrow I \) satisfies the exponential shrinking of components condition with respect to \( \lambda > 1 \), if there are constants \( C > 1 \) and \( \delta > 0 \) such that for every interval \( J \) contained in \( I \) that satisfies \( |J| \leq \delta \), the following holds: for every positive integer \( n \) and every connected component \( W \) of \( f^{-n}(J) \) we have the following inequality

\[
|W| \leq C\lambda^{-n}.
\]

Moreover, we say that \( f \) satisfies the exponential shrinking of components condition if there exists \( \lambda > 1 \) such that \( f \) satisfies the exponential shrinking of components condition with respect to \( \lambda \).

We will use the following fact that was proved by Rivera-Letelier in [RL12a].

**Lemma 2.1 (Corollary A and C, [RL12a]).** Let \( f : I \rightarrow I \) be a multimodal interval map in \( \mathcal{A}_d \) that is topologically exact, then the TCE condition is equivalent to the exponential
shrinking of components condition. Moreover, if $f$ satisfies the CE condition, then $f$ also satisfies the exponential shrinking of components condition, and the TCE condition.

We will also use the following lemma.

**Lemma 2.2 (Proposition 5.2, [RL12a]).** Let $I$ and $J$ be two compact intervals of $\mathbb{R}$. Let $f : I \to I$ be a Lipschitz continuous multimodal map and $g : J \to J$ a multimodal map satisfying the exponential shrinking of components condition. If $h : I \to J$ is a homeomorphism conjugating $f$ to $g$, then $h$ is Hölder continuous.

**Lemma 2.3 (Lemma A.2, [RL12b]).** Let $f : I \to I$ be an interval map in $A\mathbb{R}$ that is topologically exact. Then for every $\kappa > 0$ there is $\delta > 0$ such that for every $x$ in $I$, every integer $n \geq 1$, and every pull-back $W$ of $(\delta B)x$ by $f^n$, we have $|W| < \kappa$.

### 2.2. Distortion lemmas

Let $\tau > 0$ and let $J_1 \subset J_2$ be two intervals of $I$. We say that $J_1$ is $\tau$-well inside $J_2$, if both components of $J_2 \setminus J_1$ have length at least $\tau |J|$.

**Lemma 2.4 (Theorem A, [LS10]).** Let $f : I \to I$ be a multimodal map in $A\mathbb{R}$. Then for each $\tau > 0$ there exist $C > 1$ and $\xi > 0$ satisfying the following. Let $T \subset I$ be an open interval, $J$ a closed subinterval of $T$ and an integer $s \geq 1$ such that the following hold:

1. $f^s : T \to f^s(T)$ is a diffeomorphism;
2. $|f^s(T)| \leq \xi$;
3. $f^s(J)$ is $\tau$-well inside $f^s(T)$.

Then for each pair $x$ and $y$ of points in $J$ we have

$$\frac{|Df^s(x)|}{|Df^s(y)|} \leq C.$$  

Furthermore, $C \to 1$ as $\tau \to +\infty$.

### 3. Proof of main theorems

This section is devoted to provide the proofs of theorems 1.1–1.3, which depend on the following two propositions.

**Proposition 3.1.** Let $f : I \to I$ and $\hat{f} : J \to J$ be multimodal maps in $A\mathbb{R}$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Assume that $f$ satisfies both the exponential shrinking of components condition and $SER(\beta)$ for some $\beta \in (0, 1)$. Then $\hat{f}$ satisfies the TCE condition, and there is $\beta' > 0$ such that $\hat{f} \in SER(\beta')$.

**Proof.** By lemma 2.1 and the topological invariance of the TCE condition, we only need to prove that the map $f$ satisfies the stretched exponential recurrence condition. Combining lemmas 2.1 and 2.2, we know that there is a bi-Hölder continuous $h : I \to J$ such that for each $x \in I$ we have $h(f(x)) = \hat{f}(h(x))$. Let $K > 0$ and $\alpha \in (0, 1]$ be the constants such that for each $x, y \in J$ we have

$$\frac{|Dh(x)|}{|Dh(y)|} \leq K.$$  

Furthermore, $C \to 1$ as $\tau \to +\infty$.
Let $C > 0$ such that for any two points $c, c' \in \text{Crit}(f)$ and any integer $n \geq 1$, we have
\[
|f^n(c) - c'| \geq C \exp(-n^\beta).
\]

Let $\hat{c}$ and $\hat{c}'$ be any two critical points of $\hat{f}$, and $n \geq 1$. Notice that $h^{-1}(\hat{c})$ and $h^{-1}(\hat{c}')$ are critical points of $f$. It follows that
\[
K|\hat{f}^n(\hat{c}) - \hat{c}'| \geq |h^{-1}(\hat{f}^n(\hat{c})) - h^{-1}(\hat{c}')| = |f^n(h^{-1}(\hat{c})) - h^{-1}(\hat{c}')| \geq C \exp(-n^\beta).
\]
This gives us
\[
|\hat{f}^n(\hat{c}) - \hat{c}'| \geq \left(\frac{C}{K}\right)^{1/\alpha} \exp\left(-\frac{n^\beta}{\alpha}\right).
\]

Therefore, $\hat{f}$ satisfies SER($\beta'$) for some $\beta' > \beta$ depending only on $\alpha$ and $\beta$, and the proof is complete. $\square$

Following the proof of proposition 3.1, we can obtain the following two lemmas. The details are left to the interested reader to check.

**Lemma 3.2.** Let $f : I \to I$ and $\hat{f} : J \to J$ be multimodal maps in $\mathcal{A}_R$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Assume that $f$ satisfies both the exponential shrinking of components condition and the subexponential recurrence condition. Then $\hat{f}$ satisfies both of the TCE condition and the subexponential recurrence condition.

**Lemma 3.3.** Let $f : I \to I$ and $\hat{f} : J \to J$ be multimodal maps in $\mathcal{A}_R$ that are topologically exact, and that are topologically conjugate by a conjugacy preserving critical points. Assume that $f$ satisfies both the exponential shrinking of components condition and the polynomial recurrence condition of $\beta > 0$. Then $\hat{f}$ satisfies the TCE condition, and there is $\beta' > 0$ such that $\hat{f} \in \text{PR}(\beta')$.

**Proposition 3.4.** Let $f : I \to I$ be an interval map in $\mathcal{A}_R$ that satisfies the exponential shrinking of components condition with respect to some $\lambda > 1$, then there is $\beta_0 > 0$ such that the following holds. If $f$ satisfies the exponential recurrence condition with respect to $\beta \in (0, \beta_0)$, then $f$ satisfies the Collet–Eckmann condition.

**Proof.** Putting $M := \sup |Df|$, by the exponential shrinking of components condition we have that $M > 1$. Moreover, reducing $\lambda$ if necessary, we assume that $\lambda$ is in $(1, M)$, and let $\delta_0 > 0$ be given by the exponential shrinking of components condition with respect to $\lambda$. Setting
\[
\beta_0 = \frac{(\log \lambda)^2}{2(\log M - \log \lambda)},
\]
we prove the proposition with $\beta_0$. To do this, fix $\beta \in (0, \beta_0)$, and assume that $f$ satisfies the exponential recurrence condition with respect to $\beta$. Then there exists $C_\beta > 0$ such that for any two critical points $c, c' \in \text{Crit}(f)$ and any positive integer $n$, we have
\[
|f^n(c) - c'| \geq C_\beta \exp(-\beta n). \quad (1)
\]

For any critical value \( v \) of \( f \) and every positive integer \( n \), let \( m = 0 \) be the smallest integer such that
\[
\lambda^{-m} \leq C_\beta \exp(-\beta(n + 1)).
\]

Since \( f^m(B(f^n(v), \delta_0 M^{-m})) \subset B(f^{n+m}(v), \delta_0) \), the exponential shrinking of components condition implies that for each \( j \in \{0, 1, \ldots, n\} \), the connected component \( W_j \) of \( f^{-(n-j)}(B(f^n(v), \delta_0 M^{-m})) \) containing \( f(v) \) satisfies
\[
\text{diam}(W_j) \leq \lambda^{-(m+n-j)} \leq \lambda^{-(n-j)} C_\beta \exp(-\beta(n + 1)) < C_\beta \exp(-\beta j).
\]

In particular, \( \text{diam}(W_0) \leq \lambda^{-(m+n)} \). On the other hand, combining with inequality (1), we have that each \( W_j \) is disjoint from \( \text{Crit}(f) \). It follows that the map
\[ f^n : W_0 \to B(f^n(v), \delta_0 M^{-m}) \]

is a diffeomorphism. By lemma 2.4, there exists some constant \( C > 0 \) independent of \( n \) and \( v \), such that the following holds
\[
|Df^n(v)| \geq \frac{CM^{-m}}{\text{diam}(W_0)} \geq C M^{-m} \lambda^{-(m+n)} = C \lambda^{n+m} M^{-m}. \quad (2)
\]

If \( m = 0 \), then we obtain
\[
|Df^n(v)| \geq C \lambda^n.
\]

Otherwise, setting
\[
C' := \left(\lambda^{-1} C_\beta \right)^{\frac{\log M}{\log \lambda}} \log_M \lambda\^{-1}
\]

and by the minimality of \( m \) we have
\[
\lambda^m M^{-m} = \left(\lambda^{-m} \frac{\log M}{\log \lambda} \log \lambda \right)^{-1} \geq \left(\lambda^{-1} C_\beta \exp(-\beta(n + 1)) \right)^{\frac{\log M}{\log \lambda} \log \lambda \left(\lambda^{-1} \log \lambda - 1\right)}.
\]

If putting \( \beta := 2\beta \left(\frac{\log M}{\log \lambda} - 1\right) \), then we have \( \lambda := \lambda \exp(-\beta) > 1 \) by the definition of \( \beta \). Moreover, putting \( C_0 := C C' > 0 \), \( C' > 0 \), and using with (2), for every positive integer \( n \), we have
\[
|Df^n(v)| \geq CC' \lambda^n \exp\left(-\beta(n + 1) \left(\frac{\log M}{\log \lambda} - 1\right)\right) \geq C_0 \lambda^n \exp(-\beta n) = C_0 \lambda^n.
\]

This implies that the map \( f \) satisfies the Collet–Eckmann condition for some \( \lambda_0 \), and completes the proof. \( \Box \)
Proof of theorem 1.1. By lemma 2.1, we have that the map $f$ satisfies the exponential shrinking of components condition. It follows from proposition 3.1 that $\hat{f}$ satisfies the stretched exponential recurrence condition. Therefore, we have that $\hat{f}$ satisfies the subexponential recurrence condition. Moreover, by proposition 3.4, we have that $\hat{f}$ satisfies the Collet–Eckmann condition, and complete the proof.

Proof of theorem 1.2. By lemma 2.1, we have that the map $f$ satisfies the exponential shrinking of components condition. It follows from lemma 3.2 that $\hat{f}$ satisfies the subexponential recurrence condition. Moreover, by proposition 3.4, we have that $\hat{f}$ satisfies the Collet–Eckmann condition, and complete the proof.

Proof of theorem 1.3. By lemma 2.1, we have that the map $f$ satisfies the exponential shrinking of components condition. It follows from lemma 3.3 that $\hat{f}$ satisfies the polynomial recurrence condition. Therefore, we have that $\hat{f}$ satisfies the subexponential recurrence condition. Moreover, by proposition 3.4, we have that $\hat{f}$ satisfies the Collet–Eckmann condition, and complete the proof.

4. Proof of theorem 1.4

In this section, we give a new and conceptual proof of theorem 1.4. In what follows, let $f : I \to I$ be a multimodal map in $\mathcal{A}_R$, and suppose that $f$ satisfies both the Collet–Eckmann condition and the slow recurrence condition. Let $g : J \to J$ be a multimodal map in $\mathcal{A}_R$ that is topologically conjugate to $f$ by the conjugacy $h : I \to J$ preserving critical points. By lemmas 2.1 and 2.2, we know that $h$ is bi-Hölder continuous. It follows that there exist two constants $K > 1$ and $\alpha \in (0, 1)$ such that for every pair of points $x, y \in I$ we have

$$|h(x) - h(y)| \leqslant K|x - y|^{\alpha};$$

and for every pair of points $s, t \in J$ we have

$$|h^{-1}(s) - h^{-1}(t)| \leqslant K|s - t|^{\alpha}.$$ 

First, we prove the following proposition before proving the theorem.

Proposition 4.1. Assume that $g$ in $\mathcal{A}_R$ satisfies simultaneously the exponential shrinking of components condition and the slow recurrence condition. Then $g$ satisfies the Collet–Eckmann condition.

Proof. By the hypothesis that all critical points of $g$ are non-flat, there exist $\ell > 1$, $\kappa > 0$ and $L > 1$ such that for every critical point $c^g \in \text{Crit}(g)$ and every $x \in B(c^g, \kappa)$, we have

$$|Df(x)| \geqslant \frac{1}{L} |x - c^g|^{\ell}.$$  \hspace{1cm} (3)

Let $\eta$ be the constant given by lemma 2.3 for $\kappa = \kappa_0$, and let $C, \xi$ be the constants given by lemma 2.4 with $\tau = \frac{1}{2}$. Moreover, since $g$ satisfies the exponential shrinking of components condition, then there are constants $\tilde{C} > 2$, $\eta_2 > 0$ and $\lambda > 1$ such that for every interval $T$ contained in $J$ that satisfies $|T| \leqslant 2\eta_2$, the following holds: for every positive integer $n$ and every connected component $W$ of $g^{-n}(J)$ we have $|W| \leqslant C\lambda^{-n}$. 

Now setting $\eta_0 = \min\{\eta, \eta_2, \xi, 1\}$, and fix a critical point $c^g \in \text{Crit}(g)$ and put $v^g := g(c^g)$. Put
Since the map $g$ satisfies the slow recurrence condition, there exist $\eta_3 \in (0, \min\{\eta_1, \kappa_0\})$ and a positive integer $N_1$ such that for every $\delta \in (0, \eta_3]$ and every positive integer $n \geq N_1$, the following inequality holds.

$$\sum_{i \in n \cap \mathbb{Z} \cap \mathbb{N}} - \log d(g^{i}(c^\delta)) < \varepsilon_+.$$  \hspace{1cm} (4)

In particular,

$$\#\{1 \leq i \leq n : g^i(c^\delta) \in B_{d}(\delta)\} \leq \frac{n_{\varepsilon_+}}{-\log \eta_3} < n_{\varepsilon_+}.$$  \hspace{1cm} (5)

For every $n \geq N_1$, let us define a quasi-chain $[\hat{W}_k]_{k=0}^{n}$ by the following rules:

(i) $\hat{W}_k = B(g^i(v^\delta), \eta_3)$;

(ii) Once $\hat{W}_{k+1} \supseteq g^{k+1}(v^\delta)$ is defined, letting $\hat{W}'_{k+1}$ be the connected component of $g^{-1}(\hat{W}_{k+1})$ which contains $g^k(v^\delta)$;

(iii) If $\hat{W}'_k$ contains no critical point of $g$, then $\hat{W}_k = \hat{W}'_k$, and otherwise, let $\hat{W}_k = B(g^k(v^\delta), \eta_3)$.

Let $n_0 = n$ and let $n_1 > n_2 > \cdots > n_m$ be all integers in $\{1, \ldots, n-1\}$ such that $\hat{W}_{n_0}$ contains a critical point of $g$. Then by inequality (5) we have

$$m \leq \frac{n_{\varepsilon_+}}{-\log \eta_4} \leq n_{\varepsilon_+}.$$  \hspace{1cm} (6)

Moreover, for every $1 \leq i \leq m$, the map $g^{n_i-n_{i-1}} : \hat{W}_{n_{i-1}} \rightarrow \hat{W}_{n_i}$ is a diffeomorphism, $\hat{W}_{n_{i-1}} \subset B(c, 2\eta_3)$ for some $c \in \text{Crit}(g)$ and $g^{n_{i-1}}(v^\delta)$ is the middle point of $\hat{W}_{n_{i-1}}$. Then by lemma 2.4, and the exponential shrinking of components condition, we have

$$|Dg^{n_i-n_{i-1}}(g^{n_{i-1}}(v^\delta))| \geq C_1 \lambda^{n_i-n_{i-1}-1}.$$  \hspace{1cm} (7)

where $C_1 = 2\eta_3(\overline{C\overline{C}})^{-1} \in (0, 1)$. Similarly, we have

$$|Dg^{n_{i+1}}(v^\delta)| \geq C_1 \lambda^{n_{i}}.$$  \hspace{1cm} (8)

On the other hand, by the choice of $\eta_4$ and inequality (3), for every positive integer $i$ such that $1 \leq i \leq m$ we have

$$|Dg(g^{n_i}(v^\delta))| \geq \frac{1}{L}|g^{n_i}(v^\delta) - e^\delta| \geq \frac{(d(g^{n_i}(v^\delta)))^\delta}{L}.$$  \hspace{1cm} (9)

Therefore, combining with inequalities (7)–(9), we have
It follows from (6) and the slow recurrence condition that
\[
\frac{\log |Dg^n(v^\delta)|}{n} \geq \log \lambda + \frac{m}{n} \log C_1 - \frac{m}{n} \log(\lambda L) + \frac{\ell}{n} \sum_{i=1}^{m} \log d(g^n(v^\delta))
\]
\[
\geq \log \lambda - \frac{n \varepsilon}{\log \eta_4} \log \left(\frac{2 \eta_4}{C C_1}\right) - \frac{n \varepsilon}{\log \eta_4} \log(\lambda L) + \frac{\ell}{n} \sum_{i=1}^{m} \log d(g^n(v^\delta))
\]
\[
= \log \lambda - \frac{\varepsilon}{\log \eta_4} \log \left(\frac{2 \eta_4}{C C_1}\right) - \varepsilon \log(\lambda L) + \frac{\ell}{n} \sum_{i=1}^{m} \log d(g^n(v^\delta))
\]
\[
\geq \log \lambda - \varepsilon \left(\frac{\log(2(C C_1)^{-1})}{\log \eta_4} + \log(\lambda L) + 1\right) + \frac{\ell}{n} \sum_{\delta \in B_\delta} \log d(g^n(v^\delta))
\]
\[
\geq \log \lambda - \varepsilon \left(\frac{\log(2(C C_1)^{-1})}{\log \eta_4} + \log(\lambda L) + 1 + \ell\right).
\]

Therefore, by the slow recurrence condition and the definition of \(\varepsilon_\alpha\), we have
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log |Dg^i(v^\delta)| \geq \log \lambda - \varepsilon \left(\frac{\log(2(C C_1)^{-1})}{\log \eta_4} + \log(\lambda L) + 1 + \ell\right)
\]
\[
= \frac{1}{2} \log \lambda,
\]
and complete the proof of the proposition.

\[\square\]

**Proof of theorem 1.4.** Let \(h, K\) and \(\alpha\) be as given in the beginning of this section. First, combining lemma 2.1 and the topological invariance of the TCE condition, we know that \(g\) satisfies the exponential shrinking of components condition. In view of proposition 4.1, it remains to prove that the map \(g\) satisfies the slow recurrence condition. In fact, it suffices to prove that for every critical point \(c^\varepsilon \in \text{Crit}(g)\) and any \(\varepsilon > 0\), there exist \(\delta_0 > 0\) and a positive integer \(N\) such that for every \(\delta \in (0, \delta_0]\) and every positive integer \(n \geq N\), the following inequality holds.

\[
\frac{1}{n} \sum_{\delta \in B_\delta} - \log d(g^i(c^\delta)) < \varepsilon.
\]

Fix a critical point \(c^\delta \in \text{Crit}(g)\) and \(\varepsilon > 0\). Putting \(c^\delta := h^{-1}(c^\varepsilon)\), then we have by the hypothesis that \(c^\delta\) is in \(\text{Crit}(f)\). Moreover, by the hypothesis on \(f\) we have that for \(\varepsilon' := \frac{\alpha \varepsilon}{1 + \log K} > 0\), there exist \(\delta_1 \in (0, \varepsilon^{-1})\) and a positive integer \(N\) such that for every \(\delta \in (0, \delta_1]\) and every positive integer \(n \geq N\), the following inequality holds.

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\[
\frac{1}{n} \sum_{\delta \leq i \leq n} - \log d(f^i(c^f)) < \varepsilon'.
\]

(11)

It follows that

\[
\# \{1 \leq i \leq n : f^i(c^f) \in B_\delta(\delta_1) \} \leq \frac{n\varepsilon'}{-\log \delta_1} < n\varepsilon'.
\]

(12)

Now setting \(\delta_0 := \left(\frac{n}{K}\right)^\frac{1}{n}\), and noting that for every point \(c_1^g \in \text{Crit}(g)\) and every positive integer \(i\) we have that \(h^{-1}(c_1^f)\) is in \(\text{Crit}(f)\), and that

\[
|f^i(c^f) - h^{-1}(c_1^f)| = |h^{-1}(g^i(c^g)) - h^{-1}(c_1^f)| \leq K|g^i(c^g) - c_1^f|.
\]

It follows that for every \(\delta \in (0, \delta_0]\), every positive integer \(i\) such that \(g^i(c^g) \in B_\delta(\delta_1)\), we have that \(f^i(c^f)\) is in \(B_\delta(\delta_1)\) and

\[
d(g^i(c^g)) \geq \left(\frac{d(f^i(c^f))}{K}\right)^\frac{1}{n}.
\]

Therefore, combining with inequalities (11) and (12), for every \(\delta \in (0, \delta_0]\) and every positive integer \(n \geq N\), we have

\[
\frac{1}{n} \sum_{\delta \leq i \leq n} - \log d(g^i(c^g)) \leq \frac{1}{n} \sum_{\delta \leq i \leq n} - \log \left[\left(\frac{d(f^i(c^f))}{K}\right)^\frac{1}{n}\right]
\]

\[
= \frac{1}{n} \sum_{\delta \leq i \leq n} f'(c^f) \in B_\delta(\delta_1) - \frac{1}{\alpha} \log d(f^i(c^f)) + \frac{1}{n} \sum_{\delta \leq i \leq n} f'(c^f) \in B_\delta(\delta_1) \log K\]

\[
\leq \frac{1}{\alpha} \varepsilon' + \frac{1}{n} n\varepsilon' \cdot \log K = \frac{1 + \log K}{\alpha} \cdot \varepsilon' = \varepsilon.
\]

This proves inequality (10), and so the map \(g\) satisfies the slow recurrence condition. The proof of the theorem is completed.

\[\square\]

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Appendix. Rational maps

In this appendix, let \( f \) be a rational map of degree at least 2. Let \( \text{Crit}(f) \) denote the set of critical points of \( f \), let \( J(f) \) denote the Julia set of \( f \) and let

\[
\text{Crit}(f) = \text{Crit}(f) \cap J(f).
\]

As the definitions defined in the real setting, we can also define the Collet–Eckmann condition, the stretched exponential condition, the TCE condition and the exponential shrinking of components condition for rational maps.

More precisely, we say that \( f \) satisfies the Collet–Eckmann condition (abbreviated CE condition) if all the periodic points in \( J(f) \) are hyperbolic repelling, and if there are two constants \( C > 0 \) and \( \lambda > 1 \) such that for each critical point \( c \in \text{Crit}(f) \), we have

\[
|Df^n(c)| \geq C\lambda^n.
\]

Given \( \beta \) in \((0, 1)\), we say that the rational map \( f \) satisfies the stretched exponential recurrence condition of exponent \( \beta \) (SER(\( \beta \))), if there exists \( C > 0 \) such that for any two critical points \( c, c' \in \text{Crit}(f) \) and any \( n \geq 1 \), we have

\[
|f^n(c) - c'| \geq C \exp(-n^\beta).
\]

Moreover, the map \( f \) is said to satisfy the stretched exponential recurrence condition if \( f \in \text{SER}(\beta) \) for some \( \beta \in (0, 1) \).

Given \( \beta > 0 \), we say that \( f \) satisfies the exponential recurrence condition with respect to \( \beta \), if there exists \( C > 0 \) such that for any two critical points \( c, c' \in \text{Crit}(f) \) and any positive integer \( n \geq 1 \), we have

\[
|f^n(c) - c'| \geq C \exp(-\beta n).
\]

Moreover, we say that \( f \) satisfies the subexponential recurrence condition if for every \( \beta > 0 \) the map \( f \) satisfies the exponential recurrence condition with respect to \( \beta \).

Given \( \beta > 0 \), we say that \( f \) satisfies the polynomial recurrence condition with respect to exponent \( \beta \) (PR(\( \beta \))), if there exists \( C > 0 \) such that for any two points \( c, c' \in \text{Crit}(f) \) and any \( n \geq 1 \), we have

\[
|f^n(c) - c'| \geq Cn^{-\beta}.
\]

Moreover, \( f \) is said to satisfy the polynomial recurrence condition if \( f \in \text{PR}(\beta) \) for some \( \beta > 0 \).

We can also use the analogy proof of theorem 1.1 to prove the following analog of theorem 1.1 in the complex setting. In fact, this is a feature of the method of the proof of theorem 1.1.

**Theorem A.1.** Let \( f, \hat{f} \) be two rational maps of degree at least two, and that are topologically conjugate by a conjugacy preserving critical points. Assume that \( f \) satisfies both the Collet–Eckmann condition and the stretched exponential (resp. subexponential, polynomial) recurrence condition. Then \( \hat{f} \) also satisfies both the Collet–Eckmann condition and the stretched exponential (resp. subexponential, polynomial) recurrence condition.

**Proof.** The proof of this theorem follows the same outline as that of theorem 1.1, replacing lemmas 2.1 and 2.2 by the following two results, respectively:

- TCE condition is equivalent to the exponential shrinking of components condition for rational maps, see [PRLS03, main theorem];
The conjugacy between two rational maps of degree at least 2 is bi-Hölder continuous, see [PR99, theorem A].

The details are left to the reader.

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