Plug-and-play Solvability of the Power Flow Equations for Interconnected DC Microgrids with Constant Power Loads

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Abstract—In this paper we study the DC power flow equations of a purely resistive DC power grid which consists of interconnected DC microgrids with constant-power loads. We present a condition on the power grid which guarantees the existence of a solution to the power flow equations. In addition, we present a condition for any microgrid in island mode which guarantees that the power grid remains feasible upon interconnection. These conditions provide a method to determine if a power grid remains feasible after the interconnection with a specific microgrid with constant-power loads.

Although the presented condition are more conservative than existing conditions in the literature, its novelty lies in its plug-and-play property. That is, the condition gives a restriction on the to-be-connected microgrid, but does not impose more restrictions on the rest of the power grid.

I. INTRODUCTION

The increasing use and demand of electricity is pushing our power grids to their limits. In addition, the incorporation of renewable energy sources is steadily introducing more uncertainty into our energy sources. It has therefore become necessary to better understand the fundamental limits of our power grid, as well as to come up with smart ways to generate and distribute power.

To address these issues, there has been an increasing interest in the study of microgrids. A microgrid is a small power grid within a greater power grid. One of their key features is their plug-and-play capability: a microgrid can disconnect from the main grid to operate in island mode, and reconnect whenever necessary. This allows for leveling out the fluctuations of power generation and consumption, as well as for a microgrid to disconnect when the main grid suffers from a power outage.

There has been an increasing interest in DC (direct current) microgrids. A large portion of the day-to-day energy consumption is converted to DC. Together with the rise of photo-voltaic cells and advances in battery storage, it makes more and more sense to implement DC power grids on a larger scale.

The matching of supply and demand of power in a power grid is governed by its power flow equations. If these equations have no solution, long-term voltage stability is lost and phenomena such as blackouts and voltage collapse occur [4]. The introduction of constant-power loads leads to non-linearities for which the power flow equations may not be solvable. Conditions which guarantee their solvability are known, but are considered conservative [12], [3], [14].

There has been some research on the control of DC microgrids. A distributed control scheme was proposed in [16], and a plug-and-play control scheme was proposed in [13], although neither paper consider constant-power loads. To the best of our knowledge, no theoretical advances have been made towards power flow solvability where new sources and/or constant-power loads are introduced, and the best available approach is to recalculate the known conditions for the altered power grid.

To this end, the goal of this paper is to give conditions on the power grid and a to-be-attached microgrid, not assuming any control scheme, which guarantees that (i) the power flow equations of the power grid are solvable, (ii) the condition for (i) also holds for the interconnection of the power grid with the microgrid which makes the power flow equations of the interconnection solvable, and (iii) the conditions for (i) still hold for the original power grid after the microgrid is connected. The main advantage of this approach is that, as new microgrids are attached to the power grid, reverification of condition (i) is not necessary. We refer to this as the plug-and-play property.

The novelty of the presented approach is that it uses the (block) Cholesky decomposition as a theoretical tool to analyze the power flow equations. Commonly, whenever a new load is introduced to a power grid, conditions for the feasibility of the power grid should be re-evaluated. To suppress this effect we restrict to a “directional” power flow condition, which is based on the block Cholesky decomposition, where each block represents a microgrid.

A. Organization of the Paper

This paper is organized as follows. Section II establishes notation, lists a number of graph theoretic notions and reviews M-matrices. Section III deals with the feasibility of a DC power grid. In Section IV we analyze the interconnection of multiple microgrids and review the block Cholesky decomposition. From this we derive a sufficient condition for solvability of the power flow equations. In Section V
we consider the process of interconnecting a microgrid to a power grid, and state our main result. Section VI concludes the paper.

II. PRELIMINARIES

A. Notation

All vectors and matrices in this paper are real-valued. We denote a diagonal matrix by \( [d] := \text{diag}(d_1, \ldots, d_n) \) for a vector \( d \). Note that \( [v]w = [w]v \) for any pair of vectors \( v, w \). Inequalities between matrices such as \( A > 0 \) and \( A \geq 0 \) are meant element-wise, and the same holds for vectors. The notation \( \emptyset \) and \( \mathbb{1} \) are used for all-zeros and all-ones vectors, respectively. We let \( I_n \) denote the \( n \times n \) identity matrix.

**Definition 1.** We say \( A \) is order-preserving if \( x < y \) implies \( Ax < Ay \), for all vectors \( x, y \).

It can be shown that \( A \) is order-preserving if and only if \( A \geq 0 \) and \( A \mathbb{1} > 0 \). Any nonnegative invertible matrix is order-preserving, and its inverse is known in the literature as a monotone matrix [17]. The sum and product of two order-preserving matrices is again order-preserving.

B. Graph Theory

We let it be understood that we mean an undirected weighted graph with no self-loops. That is, a graph \( \Gamma \) is the tuple \( (V, E, w) \) with node set \( V = \{1, \ldots, |V|\} \), edge set \( E \subseteq \{(i, j) \mid i, j \in V, i \neq j \} \), and a map \( w : E \rightarrow \mathbb{R}_{>0} \) of edge weights.

For any \( \mathcal{V} \subseteq V \), the subgraph induced by \( \mathcal{V} \) is the tuple \( (\mathcal{V}, \mathcal{E}, w) \) where \( \mathcal{E} = \{\{i, j\} \mid i, j \in \mathcal{V}\} \).

We say that two nodes \( i, j \in V \) are path-connected with respect to the graph \( \Gamma = (V, E, w) \) if there exists a path \( (v_1, v_2, \ldots, v_l) \) such that \( \{v_r, v_{r+1}\} \in E \) for \( r = 1, \ldots, l-1 \), \( i = v_1 \) and \( j = v_l \).

The node set \( V \) of a graph can be partitioned into connected components such that two nodes are in the same connected component if and only if they are path-connected with respect to that graph.

We let the Laplacian matrix \( L(\Gamma) \) of the graph \( \Gamma = (V, E, w) \) be defined by \( L(\Gamma)_{ij} = -w(\{i, j\}) \) if \( \{i, j\} \in E \), \( L(\Gamma)_{ij} = 0 \) if \( i \neq j \) and \( \{i, j\} \notin E \), and \( L(\Gamma)_{ii} = -\sum_{j \neq i} L(\Gamma)_{ij} \). Each graph is fully determined by its Laplacian matrix.

C. M-matrices

We define an M-matrix as follows.

**Definition 2.** A matrix \( A \) is an M-matrix if there exists a matrix \( B \geq 0 \) such that \( A = sI - B \) where \( s \geq \rho(B) \), the spectral norm of \( B \). It is invertible if \( s > \rho(B) \).

The diagonal elements of an M-matrix are nonnegative, and its off-diagonal elements are nonpositive.

If an M-matrix is invertible, its inverse is nonnegative. Any Schur complement of an invertible M-matrix is again an invertible M-matrix.\(^1\)

\(^1\)See [17], Theorem 6.2.3 and Exercise 10.6.1.

III. FEASIBILITY OF DC POWER GRIDS

We consider a purely resistive DC grid where \( \begin{pmatrix} V_L \\ V_S \end{pmatrix} > \emptyset \) and \( \begin{pmatrix} I_L \\ I_S \end{pmatrix} \) denote the voltages potentials and currents at the nodes, respectively, which are real-valued and partitioned according to nodes being loads (L) or sources (S). The power at each node is given by

\[
\begin{pmatrix} -P_L \\ P_S \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} V_L \\ V_S \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_L \\ I_S \end{pmatrix},
\]

where \( P_L, P_S > 0 \), indicating that load nodes consume power, while source nodes supply power. We consider the setting where each load is a constant-power load, and therefore take \( P_L \) constant. The voltage and current in a resistive circuit are related via the weighted Laplacian (Kirchhoff) matrix \( Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix} \), with the weights corresponding to the admittance of the lines between the nodes in the power grid.\(^2\)

The current at the load nodes flowing into the power grid is given by \( I_L = Y_{LL}V_L + Y_{LS}V_S < 0 \). To satisfy the power demand of the loads, the following equation, known as the (purely resistive) DC power flow equation, should be satisfied for \( V_L > 0 \), where \( P_L \) and \( V_S \) are given:

\[
[V_L]Y_{LL}V_L + [V_L]Y_{LS}V_S + P_L = 0. \tag{1}
\]

To simplify notation, we introduce the following definition.

**Definition 3.** Let \( A \) be a square matrix and vectors \( b \geq 0, c > 0 \). We say \((A, b, c)\) is feasible if there exists a vector \( x > 0 \) such that \( |x|Ax - |b|x + c = 0 \).

In addition, we say the power grid represented by \( Y \) is feasible if \( (Y_{LL}, -Y_{LS}V_S, P_L) \) is feasible, where \( V_S \) and \( P_L \) are given. Note that feasibility does not depend on \( Y_{SS} \).

Being a quadratic equation in the entries of \( V_L \), (1) is not always solvable. A necessary and sufficient condition for solvability is not known when multiple interconnected load nodes are considered.\(^3\)

The paper [11] gives the following sufficient condition for the existence of a solution to (1), albeit in a different form and context.

**Theorem 4** (Simpson-Porco et al.). If \( Y_{LL}^{-1}[V_L^+]^{-1}P_L < \frac{1}{2}V_L^{-1} \), then \((Y_{LL}, -Y_{LS}V_S, P_L)\) is feasible, where \( V_L^+ := -Y_{LL}^{-1}Y_{LS}V_S \) are the open-circuit voltages.

The open-circuit voltages correspond to the voltage potentials in the situation where there is no power demand and therefore \( I_L = 0 \).

\(^2\)While we do not consider shunts in our power grids, all results presented in this paper hold also for loads with shunts, for which one would use so-called grounded Laplacian matrices.

\(^3\)If there are no connections among load nodes, \( Y_{LL} \) is diagonal and each row of (1) is a quadratic equation in \( V_L \). The discriminant associated to each row is nonnegative if and only if there exists a positive solution to (1).
IV. MICROGRIDS WITH A FIXED INTERCONNECTION TOPOLOGY

Throughout the rest of this paper we consider a DC power grid consisting of multiple interconnected DC microgrids. Our aim is to study the related power flow problem, while respecting the topological structure of microgrids. In particular, we phrase a sufficient condition for (1) for such a power grid in terms of conditions for individual microgrids and according to a hierarchical structure. This hierarchical structure represents the order in which the microgrids are connected to the power grid. To clarify, microgrids which are lower in the hierarchy were added before the ones that are higher in the hierarchy.

The presented approach assumes that sources do not supply power to load nodes which are lower in the hierarchy, and therefore leads to more conservative conditions. On the other hand, the main feature of the approach is that only considering power flow between microgrids in a “specified direction” gives conditions for which the introduction of a new microgrid does not alter the conditions on the original power grid.

The main result of this paper is derived from Theorem 4 and relies on the block Cholesky decomposition (BCD) to describe the hierarchical structure between the microgrids. The BCD is reviewed in Section IV-B.

Throughout the rest of this section we focus on a fixed topology of microgrids and study their feasibility. In Section V we consider a dynamic topology of microgrids, where the introduction and interconnection of a new microgrid to a power grid is studied.

A. System description

We consider a DC power grid as in Section III, where again \( Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix} \). Our power grid is subdivided into \( k \) microgrids. We number our microgrids and denote the \( i \)-th microgrid by \( M_i \). The numbering of the microgrids represents the order in which the microgrids were attached to the power grid, which induces a hierarchical structure on the microgrids.

We let \( \Gamma = (\mathcal{V}, \mathcal{E}, w) \) to be the graph represented by \( Y \). The nodes of a microgrid \( M_i \) induce a subgraph of \( \Gamma \). Such a subgraph is denoted by \( \Gamma_{M_i} \).

In order to state the main result of this section, we require the following assumption on the microgrids.

**Assumption 5.** For each microgrid \( M_i \), each load node in \( M_i \) satisfies at least one the following conditions:

i) The node is path-connected with respect to the graph \( \Gamma_{M_i} \) to a source node in \( M_i \);

ii) The node is path-connected with respect to the graph \( \Gamma \) to a node in \( M_j \) with \( j < i \).

These conditions can be interpreted as follows: Condition 5.i is equivalent to the condition that the node is path-connected to a source node in \( M_i \) when the microgrid operates in island mode. Condition 5.ii is equivalent to the condition that every load node is path-connected to a node lower in the hierarchy. Note that Assumption 5 only requires that the first microgrid is able to operate in island mode. In addition, note that we do not assume that the power grid is connected.

Assumption 5 prevents the case where a microgrid is connected to a power grid while some of its load nodes are not path-connected to a source node; such a case can never be feasible. More precisely, it states that for each load node there exists a path \((v_1, \ldots, v_t)\) to a source node such that it descends hierarchy of microgrids; that is, if \( v_i \in M_s \) and \( v_j \in M_t \), then \( i < j \) implies \( s \geq t \).

**Example 6.** Figure 1 represents an interconnection of two microgrids for which Assumption 5 holds: Nodes 1, 2, 4 and 5 satisfy 5.i, while nodes 3, 4 and 5 satisfy 5.ii. Note that node 3 is not path-connected with respect to \( \Gamma_{M_2} \) to a source in \( M_2 \). This implies that Assumption 5 would not hold for node 3 if the numbering of \( M_1 \) and \( M_2 \) would be interchanged. Put differently, \( M_2 \) cannot operate in island mode.

Assumption 5 implies the following Lemma.

**Lemma 7.** The matrix \( Y_{LL} \) is an invertible M-matrix.

**Proof.** It follows from Assumption 5 that every load node is path-connected to a source node. This implies that \( Y_{LL} \) is weakly chained diagonally dominant, and hence an invertible M-matrix by Corollary 4 of [12]. □

B. The block Cholesky decomposition

This section reviews the block Cholesky decomposition (BCD) of a symmetric positive definite matrix. More specifically, we consider the BCD of M-matrices.

Consider a symmetric positive definite matrix \( A \in \mathbb{R}^{n \times n} \), along with a partition of the rows and columns of \( A \) into \( k \) nonempty subsets. We obtain \( A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \) and \( a_{ij} \in \mathbb{R}^{n_i \times n_j} \) such that \( \sum_{i=1}^{k} n_i = n \). Note that \( a_{ij} = a_{ji}^\top \). We define the matrices

\[
A_i := \begin{pmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ii} \end{pmatrix} \quad \text{for } i = 1, \ldots, k;
\]

\[
\alpha_i := \begin{pmatrix} a_{11} & \cdots & a_{i,i-1} \end{pmatrix} \quad \text{for } i = 2, \ldots, k.
\]
We have $A = A_k$ and $A_i = \begin{pmatrix} A_{i-1} & \alpha_i^\top \\ \alpha_i & a_{ii} \end{pmatrix}$ for $i = 2, \ldots, k$.

Let the matrices $C_i, D_i$ for $i = 1, \ldots, k$ be iteratively defined by

$$
C_1 := I_{n_i}, \quad C_i := \begin{pmatrix} C_{i-1} & 0 \\ \alpha_i A_{i-1} \top C_{i-1} & I_{n_i} \end{pmatrix}, \\
D_1 := A_{11}, \quad D_i := \begin{pmatrix} D_{i-1} & 0 \\ \alpha_i - \alpha_i A_{i-1} \top \alpha_i \end{pmatrix}.
$$

It follows that $A_i = C_i D_i C_i \top$ for $i = 1, \ldots, k$.

Let $C := C_k, D := D_k$, then the block Cholesky decomposition (BCD) of $A$ is given by $A = C D C \top$.

Note that

$$
C_i = \begin{pmatrix} C_{i-1} & 0 \\ \alpha_i A_{i-1} \top C_{i-1} & I_{n_i} \end{pmatrix}, \\
C_i^{-1} = \begin{pmatrix} C_{i-1}^{-1} - \alpha_i A_{i-1} \top C_{i-1}^{-1} I_{n_i} \\ -\alpha_i A_{i-1} \top D_{i-1} C_{i-1}^{-1} I_{n_i} \end{pmatrix},
$$

from which it can be observed that only the diagonal blocks of $D$ have to be inverted to obtain the BCD of $A$.

By taking $k = n$ we obtain the Cholesky decomposition of $A$, and the BCD therefore generalizes the Cholesky decomposition.

Suppose that $A$ is an invertible M-matrix, then $\alpha_i \leq 0$ and $A_{i}^{-1} \geq 0$. Since $\alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top$ is a Schur complement of the invertible M-matrix $A_i$, it is again an invertible M-matrix, and therefore $(\alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1} \geq 0$. Using this fact, it can be shown by induction on $i$ that $D_i^{-1} \geq 0$ and $C_i^{-1} \geq 0$. Since both are invertible, it follows that $D_i^{-1}$ and $C_i^{-1}$ are order-preserving for all $i$.

C. Iterative feasibility condition using the BCD

Continuing Section IV-A, we define $A := Y_{LL}$. We let $C, D$ such that $A = C D C \top$ is the BCD of $A$ where each block corresponds to a microgrid in the power grid. We will use the notation for submatrices of $A$ as in Section IV-B.

The intuition behind the BCD of $Y_{LL}$ is not straightforward, but can be seen as follows. Each diagonal block of $D$ describes the conductances between load nodes in a microgrid after all load nodes which are lower in the hierarchy are eliminated by Kron reduction. Put differently, for each block it is assumed that there is no current flow at the load nodes which are lower in hierarchy, which means that these nodes do not consume power.

The matrix $C$ describes all weighted paths between load nodes in distinct microgrids which ascend the hierarchy, and is therefore (block) lower triangular. The off-diagonal elements of $C$ are nonnegative, while $C$ has ones on its diagonal. The matrix $CD$ roughly represents the conductances between load nodes in a microgrid, under the assumption that load nodes which are lower in hierarchy draw no current, compensated by the conductances between load nodes in microgrids which are higher in hierarchy and have the same assumption. This compensation is based on paths which descend the hierarchy, giving $CD$ also a block lower triangular structure.

Lemma 8. The load nodes $k$ and $l$ in $M_i$ are path-connected with respect to the graph induced by the load nodes in $M_1, \ldots, M_i$ if and only if $(\alpha_i - \alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1} > 0$.

Proof. The connected components of the graph induced by the load nodes in $M_1, \ldots, M_i$ correspond to the irreducible components of $A_i$. We can permute the rows and columns of $A_i$ such that $A_i$ is block-diagonal. By the Perron-Frobenius theorem for irreducible matrices, the inverse of each block is positive. It follows that the $(k,l)$-entry of $A_i^{-1}$ is positive if and only if $k$ and $l$ are path-connected with respect to the graph. It can be shown that $(\alpha_i - \alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1}$ is the principal submatrix of $A_i^{-1}$ corresponding to load nodes of $M_i$.

The next lemma is the cornerstone of the main theorem in this section, and the reason why we need Assumption 5.

Lemma 9. Suppose Assumption 5 holds. The vector $C \top V_L^*$ is positive.

Proof. Recall that

$$V_L^* = -A^{-1} Y_{LS} V_S = -C^{-1} D^{-1} C^{-1} Y_{LS} V_S,$$

and therefore $C \top V_L^* = - D^{-1} C^{-1} Y_{LS} V_S$. Note that $V_S > 0$ and $-Y_{LS} \geq 0$, and therefore $-Y_{LS} V_S \geq 0$. Since $A$ is an invertible M-matrix, we have seen that $C^{-1} \geq 0$ and $D^{-1} \geq 0$. This implies that $-D^{-1} C^{-1} Y_{LS} V_S \geq 0$.

We consider the rows of $C \top V_L^*$ corresponding to the load nodes in $M_i$, which are given by

$$-(\alpha_i - \alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1} (\alpha_i A_{i-1} \top I_{n_i} 0) Y_{LS} V_S. \quad (2)$$

Consider a load node $k$ in $M_i$.

If Assumption 5.i holds for $k$, then $k$ is path-connected to a node $l$ in $M_i$ with respect to the graph $\Gamma_{M_i}$ (or coincides with $l$) such that $l$ shares an edge with a source node in $M_i$. This implies that $(\alpha_i - \alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1} (\alpha_l A_{l-1} \top I_{n_l} 0) Y_{LS} V_S > 0$ by Lemma 8 and since $-Y_{LS} V_S l > 0$ if $l$ shares an edge with a source node.

If Assumption 5.ii holds for $k$, then $k$ is path-connected to a node $l$ in $M_j$ with respect to $\Gamma$ such that $j < i$, and $l$ is the only node in the path not in $M_i$.

If $l$ is a source node, then again $(\alpha_i - \alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1} (\alpha_l A_{l-1} \top I_{n_l} 0) Y_{LS} V_S > 0$ where $l'$ is the node in the path which shares an edge with $l$.

If $l$ is a load node, then, by applying induction on $i$ in the above, Assumption 5 implies that $l$ is path-connected to a load node $m$ in $M_j$ such that $j' \leq j$, which shares an edge with a source node. Hence

$$-(\alpha_i - \alpha_i A_{i-1} \top A_{i-1} \alpha_i^\top)^{-1} (\alpha_i A_{i-1} \top I_{n_i} 0) Y_{LS} V_S m > 0$$

by Lemma 8. This implies that each row of (2) is positive.

The vector $C \top V_L^*$ is a lower bound for the open circuit voltages $V_L^*$ and roughly represents the effect of the potentials at the sources when there is no current flow at loads.

\footnote{All load nodes share an edge with a generator node if and only if $-Y_{LS} V_S > 0$.}
and where currents only flow over paths which descend the hierarchy.

The next theorem gives a sufficient condition for feasibility which is more conservative than Theorem 4, but incorporates the topological structure of the microgrids.

**Theorem 10.** Let \( Y_{LL} = CDC^T \), the block Cholesky decomposition such that the blocks (i.e., microgrids) satisfy Assumption 5. If

\[
D^{-1}C^{-1}[C^TV_s^*]^{-1}P_L < \frac{1}{4}C^TV_s^*, \tag{3}
\]

then \( Y_{LL}^{-1}[V_s^*]^{-1}P_L < \frac{1}{4}V_s^* \) and \( (Y_{LL}, -Y_{LS}V_s, P_L) \) is feasible.

**Proof.** From Lemma 9 it follows that \([C^TV_s^*]^{-1}\) is well-defined and nonnegative. The matrix \( C^{-1} \) has ones on its diagonal, and since \( C^{-1} \) is nonnegative, it follows that \( I \leq C^{-1} \). It follows that \( C^TV_s^* \leq \frac{1}{4}C^TV_s^* \) and so \( [C^TV_s^*]^{-1}P_L \leq [V_s^*]^{-1}P_L \). The matrix \( D^{-1}C^{-1} \) is order-preserving and so,

\[
D^{-1}C^{-1}[V_s^*]^{-1}P_L \leq D^{-1}C^{-1}[C^TV_s^*]^{-1}P_L < \frac{1}{4}C^TV_s^*.
\]

Multiplication by the order-preserving matrix \( C^{-1} \) yields \( C^{-1}D^{-1}C^{-1}[V_s^*]^{-1}P_L = Y_{LL}^{-1}[V_s^*]^{-1}P_L < \frac{1}{4}V_s^* \).

Theorem 4 implies that \( (Y_{LL}, -Y_{LS}V_s, P_L) \) is feasible. \( \square \)

Theorem 10 implies that if (3) holds, both \((CD, -Y_{LS}V_s, P_L)\) and \((Y_{LL}, -Y_{LS}V_s, P_L)\) are feasible.

Example 11. Consider the power grid in Figure 1 where we assume unit line conductances. We consider the microgrid \( \mathcal{M}_1 \) in island mode and let \( \hat{Y} \) represent its Laplacian matrix.

We have \( Y_{LL} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). We proceed by connecting \( \mathcal{M}_2 \) to \( \mathcal{M}_1 \) in the same fashion as in Figure 1, and let \( \hat{Y} \) represent the Laplacian matrix of the interconnected microgrids. It follows that

\[
\hat{Y}_{LL} = \begin{pmatrix} 2 & 3 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{pmatrix},
\]

where the partitioning of the matrix correspond to the two microgrids. Note that the diagonal elements corresponding to node 1 and node 2 have increased.

To formulate a plug-and-play condition as suggested in Section IV-C, it is essential for Theorem 10 that the blocks of the BCD of \( Y_{LL} \) remain unaltered when new microgrids are attached to the power grid. However, from Example 11 we see that, when new lines are introduced, the diagonal elements of the matrix \( \hat{Y}_{LL} \) change with respect to \( Y_{LL} \).

A. Virtual shunts and virtual power grids

To address the issue above, we increase the diagonal elements of the Laplacian by temporarily introducing shunts\(^5\) at load nodes. These shunts are later decreased (and potentially removed) as more lines are interconnected to a load node. We refer to these shunts as virtual shunts. The result is that the diagonal elements of the Laplacian matrix remain constant when new microgrids are attached.

Virtual shunts are not physically present in the power grid and should be seen as placeholders for prospective lines. We refer to a power grid with virtual shunts as a virtual power grid. We refer to other power grids as physical power grids, to prevent ambiguity. It should be understood that a virtual power grid does not fully capture the behavior of the associated physical power grid.

V. Microgrids with a dynamic interconnection topology

In this section we relate the feasibility of a power grid to an associated “virtual power grid”, such that Theorem 10 can be applied. From this, a plug-and-play condition for feasibility is obtained, which is the main result of this paper.

**Lemma 12.** Let \( A \) be an invertible \( M \)-matrix, \( E \geq 0 \) diagonal, \( b \geq 0 \) and \( c > 0 \) such that \((A+E)^{-1}b > 0\).

Suppose \((A+E)^{-1}(A+E)^{-1}b)^{-1}c < \frac{1}{4}(A+E)^{-1}b\), then

\[
A^{-1}[A^{-1}b]^{-1}c < \frac{1}{4}A^{-1}b.
\]

**Proof.** Note that \( A^{-1}(A+E) = I + A^{-1}E \geq I \) is order-preserving, which implies that \( A^{-1}b \geq (A+E)^{-1}b \), and so \([A^{-1}b]^{-1} \leq [(A+E)^{-1}b]^{-1}\). By multiplying our condition

\(^5\) A shunt is a component at a node which extracts current proportional to the voltage potential at the node.
with \( A^{-1}(A + E) \), we obtain \( A^{-1}(A + E)^{-1}b \) implies that \( A^{-1}(A + E)^{-1}c \) is a lower bound for the open-circuit voltages. This follows directly from the observation that (1) does not appear in \( Y_{SS} \). Hence, lines between sources can be altered freely, without compromising feasibility.

It is essential that the virtual shunts are chosen properly. Choosing \( E \) large leads to more conservative conditions, whereas choosing \( E \) small restricts the possible interconnections with potential microgrids.

However, from a practical point of view, it is reasonable to assume that the conductances of prospective lines are known \textit{a priori}. For example, some lines may already exist but are not in use. Also, there might be limitations on the number of lines a node can connect with, which would give an upper bound for the virtual shunt. In particular, some nodes might never connect to nodes outside their microgrid and do not require a virtual shunt.

### B. Main Theorem: Plug-and-play solvability

In this section we present the main result of the paper. The main line of this section is as follows.

We consider a power grid and define a virtual counterpart in the sense of Section V-A. We define a BCD of the lines of the loads in the virtual power grid. We assume that the BCD satisfies the conditions for Theorem 10. This implies that both the power grid and its virtual counterpart are feasible, by Lemma 12.

We proceed by introducing a microgrid and again define a virtual counterpart. We interconnect both virtual grids under the assumption that the physical interconnection satisfies Assumption 5 and does not exceed the virtual shunts. We extend the BCD of the virtual power grid to this virtual interconnection and use this to state the main theorem.

The main theorem gives a sufficient condition such that this interconnected virtual power grid is feasible. Moreover, the theorem states that the physical interconnection of the power grid and the microgrid is therefore also feasible, by Lemma 12.

Consider a DC power grid described by the Laplacian matrix \( Y_{L1L1} \) (Laplacian of the virtual power grid) and \( Y_{S1S1} \) (Laplacian of the virtual microgrid). Together with the virtual shunts \( E_{L1} \) and load voltages \( V_{L1} \), we can express the power demand at the load nodes and the virtual open-circuit voltages by \( -Y_{L1L1} - E_{L1} \) and \( Y_{S1S1} \), respectively. The vector \( V'_{L1} \) is a lower bound for the open-circuit voltages of the physical power grid.

Let \( C, D \) such that \( CDC^T = Y_{L1L1} + E_{L1} \) is a BCD, where the blocks are such that they satisfy Assumption 5.

**Assumption 13.** The following inequality holds:

\[
D^{-1}C^{-1}[C^TV_{L1}']^{-1}P_L < \frac{1}{2} C^TV_{L1}'.
\]

Assumption 13 implies that \( (Y_{L1L1} + E_{L1}, Y_{S1S1}, V_{S1}, P_L) \) is feasible by Theorem 10. It follows by Lemma 12 that \( (Y_{L1L1} - Y_{L1S1}, V_{S1}, P_L) \), the physical counterpart, is feasible as well.

In addition, consider a microgrid described by the Laplacian matrix \( Y_{S2S2} \), together with the virtual shunts \( E_{L2} \) and load voltages \( V_{L2} \). Let \( P_{L2} \) be the power demand at the load nodes.

Let the physical interconnection of the microgrid and the virtual shunts be given by the Laplacian matrix

\[
\begin{pmatrix}
Y_{L1L1} + \hat{E}_{L1} & Y_{L1L2} & Y_{L1S1} & Y_{L1S2} \\
Y_{L2L1} & Y_{L2L2} + \hat{E}_{L2} & Y_{S1S1} & Y_{S1S2} \\
Y_{S1L1} & Y_{S1L2} & Y_{S1S1} + \hat{E}_{S1} & Y_{S1S2} \\
Y_{S2L1} & Y_{S2L2} & Y_{S2S1} & Y_{S2S2} + \hat{E}_{S2}
\end{pmatrix},
\]

where \( \hat{E}_{L1}, \hat{E}_{L2}, \hat{E}_{S1}, \hat{E}_{S2} \) are defined as follows:

\[
\hat{E}_{L1} := -[Y_{S1L1} + Y_{S1S1}]; \hat{E}_{S1} := -[Y_{S1L1} + Y_{S1S1}];
\]

\[
\hat{E}_{L2} := -[Y_{S2L1} + Y_{S2S1}]; \hat{E}_{S2} := -[Y_{S2L1} + Y_{S2S1}].
\]

We make the following assumption on the interconnection.

**Assumption 14.** The interconnection of the power grid and the microgrid is such that \( \hat{E}_{L1} \leq E_{L1}, \hat{E}_{L2} \leq E_{L2} \) and Assumption 5 holds.

The bounds on \( \hat{E}_{L1} \) and \( \hat{E}_{L2} \) imply that the virtual interconnection of both virtual grids does not exceed the virtual shunts. This will allow us to use Lemma 12 in Theorem 15.

Let \( \hat{C}, \hat{D} \) such that

\[
\hat{C}\hat{D}\hat{C}^T = \begin{pmatrix}
Y_{L1L1} + E_{L1} & Y_{L1L2} \\
Y_{L2L1} & Y_{L2L2} + E_{L2}
\end{pmatrix}
\]

is a BCD. From Section IV-B it follows that

\[
\hat{C}^{-1} = \begin{pmatrix} C^{-1} & 0 \\ -Y_{L2L1}(CDC^T)^{-1}I_n \end{pmatrix}; \hat{D}^{-1} = \begin{pmatrix} D^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix},
\]

where we define

\[
R := Y_{L2L2} - Y_{L2L1}(CDC^T)^{-1}Y_{L1L2}.
\]

Define \( V'_{L1} := -\hat{C}\hat{D}\hat{C}^T)^{-1}(Y_{S1S1}V_{S1}Y_{S2S2}V_{S2}) \), which is a lower bound for the open-circuit voltages of the interconnected power grid.

Finally, let \( (\hat{C}^TV'_{L1})_{L2} \), denote the entries of \( \hat{C}^TV'_{L1} \) corresponding to the load nodes of the original power grid, and similar for \( (\hat{C}^TV'_{L1})_{L2} \) and the microgrid.
Theorem 15 (Plug-and-Play Solvability). Suppose that Assumptions 13 and 14 are satisfied. If the condition

\[-R^{-1}Y_{L_2L_1}(C_{\text{DC}}C^T)^{-1}[(\tilde{C}^T V'_{L_1})_{L_1}]^{-1}P_{L_1}
+ R^{-1}[(\tilde{C}^T V'_{L_2})_{L_2}]^{-1}P_{L_2} < \frac{1}{4}(\tilde{C}^T V'_{L_1})_{L_1}\]

holds, then

\[
\hat{D}^{-1}\hat{C}^{-1}[\tilde{C}^T V'_{L_1}]^{-1}\left(\begin{array}{c}
P_{L_1} \\
P_{L_2}
\end{array}\right) < \frac{1}{4}\tilde{C}^T V'_{L_1}
\]

and the interconnection of the power grid and the microgrid is feasible.

Proof. Due to the block triangular structure of \(\tilde{C}\), the rows of (6) corresponding to the virtual power grid are given by

\[D^{-1}C^{-1}[(\tilde{C}^T V'_{L_1})_{L_1}]^{-1}P_{L_1} < \frac{1}{4}(\tilde{C}^T V'_{L_1})_{L_1}.
\]

For the same reason we have

\[(\tilde{C}^T V'_{L_1})_{L_1} = -D^{-1}C^{-1}(Y_{L_1S_1}V_{S_1} + Y_{L_1S_2}V_{S_2})
\geq -D^{-1}C^{-1}Y_{L_1S_1}V_{S_1} = \tilde{C}^T V'_{L_1},
\]

which implies \([(\tilde{C}^T V'_{L_1})_{L_1}]^{-1} \leq [\tilde{C}^T V'_{L_1}]^{-1}\). Assumption 13 implies that

\[D^{-1}C^{-1}[(\tilde{C}^T V'_{L_1})_{L_1}]^{-1}P_{L_1} \leq D^{-1}C^{-1}[\tilde{C}^T V'_{L_1}]^{-1}P_{L_1},
\]

\[< \frac{1}{4}\tilde{C}^T V'_{L_1} \leq \frac{1}{4}(\tilde{C}^T V'_{L_1})_{L_1}.
\]

Therefore (7) is satisfied.

Note that (5) is equivalent to the rows of (6) corresponding to the virtual microgrid. Hence if (5) holds, then so does (6).

If (6) is satisfied, then by Theorem 10 the virtual interconnection

\[
\begin{pmatrix}
Y_{L_1L_1} + E_{L_1} & Y_{L_1L_2} \\
Y_{L_2L_1} & Y_{L_2L_2} + E_{L_2}
\end{pmatrix}
\begin{pmatrix}
Y_{L_1S_1} & Y_{L_1S_2} \\
Y_{L_2S_1} & Y_{L_2S_2}
\end{pmatrix}
\begin{pmatrix}
V_{S_1} \\
V_{S_2}
\end{pmatrix}
\begin{pmatrix}
P_{L_1} \\
P_{L_2}
\end{pmatrix}
\]

is feasible. Since \(\hat{E}_{L_1} \leq E_{L_1}\) and \(\hat{E}_{L_2} \leq E_{L_2}\), we may use Lemma 12 to conclude that the physical interconnection

\[
\begin{pmatrix}
Y_{L_1L_1} + \hat{E}_{L_1} & Y_{L_1L_2} \\
Y_{L_2L_1} & Y_{L_2L_2} + \hat{E}_{L_2}
\end{pmatrix}
\begin{pmatrix}
Y_{L_1S_1} & Y_{L_1S_2} \\
Y_{L_2S_1} & Y_{L_2S_2}
\end{pmatrix}
\begin{pmatrix}
V_{S_1} \\
V_{S_2}
\end{pmatrix}
\begin{pmatrix}
P_{L_1} \\
P_{L_2}
\end{pmatrix}
\]

is feasible as well. \(\square\)

Note that Assumption 13 and (6) are of the same form. Hence, microgrids can be attached to a power grid in an iterative fashion by using Theorem 15 to guarantee feasibility. This provides a method to determine if a power grid remains feasible after connecting a specific microgrid.

C. Example of plug-and-play solvability

The following example illustrates the application of Theorem 15 to determine if a power grid remains feasible after interconnecting a specific microgrid.

Example 16. Consider again the power grid in Figure 1 with unit line conductance, where we connect \(M_2\) to the power grid \(M_1\). We continue in the notation of Section V-B.

Let the voltage potentials at each source be 1 volt, and consider constant-power loads such that \(P_{L_1} = \frac{1}{2^{13}} (2 \ 2)^T\) and \(P_{L_2} = \frac{1}{2^{13}} (1 \ 9 \ 7)^T\).

For microgrids \(M_1\) and \(M_2\) we have

\[
Y_{L_1L_1} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}; \quad Y_{L_2L_2} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}.
\]

We consider the virtual shunts

\[
E_{L_1} = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}; \quad E_{L_2} = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

We have \(C = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}\) and \(D = Y_{L_1L_1} + E_{L_1} = \begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}\).

The virtual open-circuit voltages for \(M_1\) are \(V'_{L_1} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3}
\end{pmatrix}^T\). It follows that \(\tilde{C}^T V'_{L_1} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3}
\end{pmatrix}\) and

\[
D^{-1}C^{-1}[\tilde{C}^T V'_{L_1}]^{-1}P_{L_1} = \begin{pmatrix}
\frac{2}{5} & \frac{1}{12}
\end{pmatrix} < \frac{1}{9} = \frac{1}{3} = \frac{1}{4}\tilde{C}^T V'_{L_1}.
\]

Hence Assumption 13 is satisfied and \(M_1\) is feasible in island mode.

We consider the interconnection in Figure 1, which satisfies \(\hat{E}_{L_2} = E_{L_2}\) and \(\hat{E}_{L_2} = E_{L_2}\). Since Assumption 5 holds for the interconnection, Assumption 14 is satisfied.

We compute \(V'_{L_2} = \begin{pmatrix}
1 & 1 & 1 & 1
\end{pmatrix}^T\) and \(\tilde{C}^T V'_{L_1} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & 1 & 1 & 1
\end{pmatrix}^T\). Note that \(Y_{L_2L_1} = \begin{pmatrix}
0 & -1 \\
0 & -1
\end{pmatrix}\).

We compute \(R^{-1} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{12}
\end{pmatrix}\). Continuing from the left-hand side of (5), we have

\[
- \begin{pmatrix}
2 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{12}
\end{pmatrix} \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3} \\
\frac{1}{12}
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{2}{5} \\
\frac{1}{12}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
2 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{12}
\end{pmatrix} \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3} \\
\frac{1}{12}
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{3} \\
\frac{1}{12}
\end{pmatrix}
\]

\[
\approx \begin{pmatrix}
0.24 \\
0.24
\end{pmatrix} < \frac{1}{3} \begin{pmatrix}
1 \\
1
\end{pmatrix} = \frac{1}{4}(\tilde{C}^T V'_{L_1})_{L_1}.
\]

Hence, the interconnection of the two microgrids is feasible, by Theorem 15.

VI. Conclusion and Future Research

In this paper we studied the interconnection of purely resistive DC microgrids with constant-power loads. We have presented a sufficient condition for the feasibility of a DC power grid. In addition, we have presented a sufficient condition for the interconnection of a microgrid such that the former sufficient condition also holds for their interconnection. This establishes a method to determine if a power grid remains feasible after connecting a specific microgrid. The method was illustrated by an example.

The main result of the paper is directly derived from paper [11] and is more conservative than the condition in [11].
However, the novelty of the presented result is its plug-and-play property. It can be shown that the implications for feasibility presented in this paper are not specific to the condition in [11] and hold in the general setting. Hence, alternative or improved sufficient conditions for feasibility may lead to other conditions with the plug-and-play property.

Other topics for future research may include more accurate models for DC power flow, improvement on voltage controller design for guaranteeing feasibility and a detailed analysis of the conservative nature of the presented result.

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