Nonlinear time series analysis of Hyperion’s rotation: photometric observations and numerical simulations

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ABSTRACT
The case of Hyperion has been studied excessively due to the fact it is the largest known celestial body of a highly aspherical shape. It also has a low mass density and remains in a 4:3 orbital resonance with Titan. Its lightcurve, obtained through photometric observations by Klavetter (1989a,b), was initially used to show that Hyperion’s rotation exhibits no periodicity. Herein it is analyzed in the means of time series analysis. The Hurst Exponent was estimated to be \( H = 0.87 \), indicating a persistent behaviour. The largest Lyapunov Exponent \( \lambda_{\text{max}} \) unfortunately could not be given a reliable estimate because of the shortness of the dataset, consisting 38 observational points. These results are compared with numerical simulations, which gave a value \( H = 0.88 \) for the chaotic zone of the phase space. The Lyapunov time \( T_{\text{Lyap}} = \frac{1}{\lambda_{\text{max}}} \) is about 30 days, which is roughly 1.5 times greater than the orbital period. By conducting observations over a longer period an insight in the dynamical features of the present rotational state is possible.

Key words: chaos – techniques: photometric – celestial mechanics – planets and satellites: individual: Hyperion

1 INTRODUCTION

Hyperion was discovered in the XIX century by Bond (1848) and Lassel (1848), but it took more than 100 years to obtain its images due to Voyager 2 (Smith et al. 1982) and Cassini (Thomas 2010) missions. Wisdom, Peale & Mignard (1984) predicted Hyperion to remain in a chaotic rotational state due to its high oblateness. Although numerical simulations, based on this finding, are present in later literature (Boyd et al. 1994; Celletti & Chierchia 2000; Kouprianov & Shevchenko 2003), there have been only one dataset of ground-based photometric observations (Klavetter 1989a,b) sampled sufficiently to make conclusions on the rotational state. The observations were performed over a time-span of more than 50 days with 38 nights with photometric weather that resulted in high-quality data. Hyperion’s brightness was constant over a time period of each night at the level of 0.01 mag, so the observations were averaged nightly giving in summary 38 Johnson R Magnitude values. The determination of the rotational period was performed using the phase-dispersion-minimization (Stellingwerf 1978) and showed that no period from 1 hour to 50 days fits the lightcurve. Having excluded the synchronous 1:1 resonance as well as the 3:2 and 2:1 ones, one arrives at the conclusion that the observations strongly support the hypothesis of a chaotic rotation.

The nonlinear parameter in the case of an oblate body’s rotation is the so called ellipticity,

\[
\omega^2 = \frac{3(B - A)}{C},
\]

where \( A < B < C \) are the principal moments of inertia. The eccentricity \( e \) of the orbit also determines the dynamical properties of the motion. In both cases for small values of the parameters, regular motion is observed.

The properties of the obtained time series may be investigated using some tools of nonlinear analysis, i.e. Hurst Exponent (HE) and largest Lyapunov Exponent (LE). The results may be compared with numerical simulations to give a further insight in the properties of Hyperion’s chaotic rotation. The HE, denoted \( H \), is a quantitative measure of the persistence of the signal, i.e. it reveals, if present, a trend in the data. The HE is bounded in the interval from 0 to 1 and is equal to 0.5 for a random walk. For persistent data \( H > 0.5 \), while for anti-persistent \( H < 0.5 \). Periodic or quasi-periodic time series posses \( H = 1 \). It is also related to the fractal dimension of the time series \( D = 2 - H \), providing information about its self-similarity. This means that in a persistent process the increments are persistent themselves (Clegg 2006) with the same HE. A random walk has independent increments.
The LE is a measure of the sensitivity of a dynamical system to initial conditions. Although while investigating experimental data one possesses only a single realisation of the initial conditions, the algorithm of Wolf et al. (1986) allows to regain the largest LE from this single time series.

This paper is organized in the following manner. In Section 2 the photometric data are presented and the dynamical model used for numerical simulations is introduced. A review of the methods used for nonlinear analysis is presented in Section 3. Results of the dataset analysis as well as those of numerical experiments are given in Section 4. Conclusions are given in Section 5.

2 DATA AND MODEL

2.1 Observational data

The photometric observations were in detail described in (Klavetter 1989a). Although the whole dataset contains 38 magnitude values, the last one was obtained after an 11-day pause. Therefore this point has been declined as it leaves a vast gap between the former one. Hence the dataset used hereafter is consisted of 37 photometric observations.

The lightcurve corrected to the zero solar phase angle and mean opposition distance is presented in Fig. 1. The intervals between consecutive observations have a mean of \( 1.47 \) days, although were not distributed uniformly. In order to obtain an evenly spaced dataset a cubic spline interpolation was performed. The result, shown in Fig. 1, is qualitatively similar to those obtained by the best model fitting; see Fig. 11 and Fig. 12 in (Klavetter 1989a). The sampling was performed with a step equal to the mean interval between observations. This sampled dataset will be analyzed in the following Sections.

2.2 Dynamical model

If one assumes that:

(i) the orbit is Keplerian with the eccentricity \( e \);

(ii) Hyperion is considered a triaxial ellipsoid and its spin axis is perpendicular to the orbit plane;

(iii) the spin axis is aligned with the shortest physical axis;

(iv) no perturbations from other celestial bodies are present,

one arrives at the equation of motion in the form (Wisdom et al. 1984; Celletti & Chierchia 2000):

\[
\dot{\theta} + \frac{\omega^2}{2} \sin 2(\theta - f) = 0,
\]

where

\[
r = \frac{1 - e^2}{1 + e \cos f}
\]

is the instantaneous radius, \( f \) – the true anomaly, \( \theta - f \) is the angle between the longest axis and the direction to the planet. The units have been chosen so that the semi-major axis is equal to 1 and the orbital period is equal to \( 2\pi \). The true anomaly is governed by the equation

\[
\dot{f} = \frac{(1 + e \cos f)^2}{(1 - e^2)^{3/2}}.
\]

Hyperion’s oblateness is \( \omega^2 = 0.79 \) and the eccentricity is \( e = 0.1 \).

3 METHODS

Throughout this article, the computer algebra system Mathematica is employed.

3.1 Power spectrum

The calculations that allowed to obtain the power spectrum were performed in a usual way, using the Fast Fourier Transform (FFT) algorithm. Having the dataset of length \( N \) sampled with a time step \( \Delta t \), the FFT can resolve frequencies \( \omega_k \) from the interval \((-\omega_{Nyq}, \omega_{Nyq})\), where

\[
\omega_{Nyq} = \frac{1}{2\Delta t}
\]

is the Nyquist frequency and the frequencies are \( \omega_k = \frac{k}{T} \), where \( T \) is the length of the time series and \( k \) is an integer from the range \((-N, N)\). The negative frequencies only differ in phase from the positive ones, hence the power spectrum is an even function. Therefore it is sufficient to constrain the frequency domain to the interval \((0, \omega_{Nyq})\).

3.2 Hurst Exponent

The persistence analysis was performed using the rescaled range \( (R/S) \) algorithm from (Suval, Prasad & Singh 2009), with the Theiler window (Theiler 1986) chosen as the first zero of the autocorrelation function. The algorithm is described in detail in the Appendix. Simplified versions of the \( R/S \) method are sensitive to the sampling step, e.g. having a regular (i.e. periodic or quasi-periodic) time series one can sample it with a step too large, what leads to the loss of periodicity, resulting in a HE around the value \( H \approx 0.5 \), indicating it to be a random walk. On the other hand, if the

![Figure 1. The corrected lightcurve. The inset shows time intervals, in days, between successive observations. The mean is 1.47 days. The error bars are from (Klavetter 1989a).](image)
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0.00 0.02 0.04 0.06 0.08
-0.5 0.0 0.5 1.0 1.5 2.0

Figure 2. Power spectrum of the photometric dataset. The dominant frequency, \( \omega_0 \approx 0.216 \), corresponds to a period of \( 0.216 \times 37 \times 1.47 = 11.75 \) days.

The sampling step is smaller, i.e. the periodicity is still clearly visible in the sampled plot, the HE reaches its proper value \( H \approx 1 \). The algorithm of Suyal et al. (2009) acts well over various sampling steps (differed by a factor of 5), although tends to give an HE exceeding 1 for regular time series. Chaotic time series are less sensitive – changing the step with a factor 2 does not give significant differences in the HE.

The HE should not exceed the value 1, although in numerical calculations this behavior is often observed for regular trajectories. There are many reasons of this meaningful result, e.g. non-stationarity of the process, additional trends (Clegg 2006), poor sampling or too short time series.

3.3 Lyapunov Exponent

The largest LE was estimated using the algorithm of Benettin et al. (1980a,b) and Wolf et al. (1986). For the observational dataset the algorithm of Wolf et al. (1984) was used as described in Kodba, Perc & Marhl (2003), while for numerical solutions of Eq. (2) the package ice.m of Sandri (1996), based on Benettin et al. (1980a,b), was used.

4 RESULTS

4.1 Photometric data

4.1.1 Power spectrum

First, the power spectrum of the dataset was calculated using the FFT. For simplification, the 37-point long time series was assumed to have a time-step \( \Delta t = 1 \); the real time-step will be obtained by multiplying by the sampling step \( t_s = 1.47 \). This means that FFT constrains the frequencies to the interval \((0, \frac{1}{2})\), with the Nyquist frequency equal to \( \frac{1}{2} \).

The power spectrum is shown in Fig. 2.

The 37-point long time series appears to be too short to give any reliable estimates, but it will be useful in interpreting the results of the Hurst analysis.

4.1.2 Hurst Exponent

The HE was estimated using the R/S method. Fig. 3 shows the log–log plot of the R/S dependence on the temporal window \( w \). The HE is the slope of the linear regression fitted to the linear region of this dependence. The obtained value is \( H = 0.87 \).

The first bending of the (approximately) linear dependence is at \( \ln w \approx 2 \), which corresponds to the period of 15.75 days (see Suyal et al. 2009), which is not exactly the same as the one obtained using the FFT, but in the context of the sparse and short time series is an agreement fine enough.

4.1.3 Lyapunov Exponent

The largest LE was estimated for a few values of the embedding dimension \( m \) and the delay time \( \tau \) gathered with corresponding LEs in Table 1. The shortness of the time series was a serious hindrance in the calculation – it allowed only for 3 to 7 iterations of the procedure. Moreover, both of the delay times were equally justified due to the course of the autocorrelation function (\( \tau = 1 \) was chosen as the \( 1/e \) breakdown, while \( \tau = 2 \) as the first zero of the autocorrelation function). This was also the reason to choose \( m = 2 \) as one of the embedding dimensions. The higher values correspond to the model described in Section 2.2 (\( m = 3 \)) and the full, three-dimensional model derived from the Euler equations (\( m = 6 \)) (Wisdom et al. 1984). The choice of \( \tau \) affected the sign of the LE, precluding from determining the character of the dataset.

Table 1. Largest LE for the observational dataset.

| \( m \) | \( m \) | \( m \) |
|-------|-------|-------|
| 2     | 3     | 6     |

\( \tau = 1 \) | 0.61 | 0.48 | 0.43 |
\( \tau = 2 \) | 0.26 | -0.99 | -0.16 |

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4.2 Numerical simulation – the chaotic case

4.2.1 Power spectrum

The solution obtained from integrating Eq. (2) is displayed in Fig. 4 and its power spectrum, in a semi–log plot, is shown in Fig. 5. This is a typical, broad-band spectrum of a purely chaotic solution of the equation of motion. The low–frequency components are more significant than large ones, however there is no frequency that could be called characteristic for this time series.

4.2.2 Hurst Exponent

The log–log plot of the \( R/S \) dependence on the temporal window \( w \) for the chaotic solution of the equation of motion is displayed in Fig. 6. The HE, estimated as the slope of the linear regression, is equal to \( H = 0.88 \).

4.3 Numerical simulation – the quasi–periodic case

4.3.1 Power spectrum

Part of the solution of the equation of motion with initial conditions that lead to a quasi–periodic trajectory is shown in Fig. 7. The power spectrum of the solution is presented in Fig. 8 in a semi–log plot. Only a part of the spectrum is displayed because the region with high frequencies is not interesting. The dominant component yields a frequency of \( \approx 0.021 \), which corresponds to the period of \( \approx 21 \) days – this is the orbital period of Hyperion. The peaks are spaced uniformly, so that the satellite exhibits a rotation with harmonic components of the frequencies being an integer multiple of the dominant one. There are components related to frequencies 2, 3 and 4 times smaller than the dominant one.

4.3.2 Hurst Exponent

Fig. 9 presents the log–log plot of the \( R/S \) dependence on the temporal window \( w \). The HE, estimated as the slope of the linear part is \( H = 1.04 \), which exceeds 1. The first bending is at \( \ln w \approx 4 \), which corresponds to a period of 55 days. The dominant frequency is not retained herein, although the few next bendings correspond to other harmonic components. Moreover, the oscillating part of the \( \ln(R/S) \) plot is a typical behavior for a regular trajectory.
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4.3.3 Lyapunov Exponent

The largest LE converged to the value \( \lambda_{\text{max}} = 0.003 \), although the convergence plot revealed a systematic decrease, which corroborates that in the infinite limit of the number of iterations the largest LE is equal to zero, confirming the regular character of the solution.

5 CONCLUSIONS

In this work the tools of time series analysis such as the Hurst Exponent and the Lyapunov Exponent were used to investigate the lightcurve of Hyperion. The Hurst analysis lead to the conclusion that the rotation, during the observations, was persistent. The \( R/S \) dependence on \( \omega \) also allowed to regain the period achieved through the spectral analysis. The LE could not be determined in a satisfactory way because of the extreme shortness of the dataset.

Results of the numerical simulations for the chaotic region of the phase space were consistent with observations in the case of the HE. The LE indicated a Lyapunov time \( \approx 1.5 \) times greater than the orbital period and is consistent with previous research. For the quasi–periodic solution, within the accuracy of the computations, the expected values of \( H \approx 1 \) and \( \lambda_{\text{max}} \approx 0 \) were obtained.

A more reliable estimate on the HE and LE of Hyperion’s lightcurve would allow an insight into its present rotational state. The HE could indicate complex behaviour on longer time scales while the LE would determine how strongly the current rotation is chaotic. To do this a longer time series is required, at least consisted of 500 observational points (Katsev & L’Heureux 2003; Rosenstein, Collins & De Luca 1993). Assuming an observational time step of 2 days it would take about three years to gather the required data.

The approach presented here may be applied to other solar system minor bodies, e.g. asteroids, which are of great interest due to the plans of using them as a base for deep space missions.

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APPENDIX A: THE R/S ALGORITHM

Let \( X_i \) be the experimental time series with \( i \in \{1, \ldots, N\} \). For a temporal window \( w \) such that \( w_t \leq w \leq N \), where \( w_t \) is the Theiler window, consider the subset of the original dataset \( x_j(t_0, w) = X_i \) such that now an integer \( j \in [t_0, t_0 + w - 1] \). One can rename the index \( j \) for each subset so that it is started from 1. Then the mean of each of these subsets is

\[
\bar{x}(t_0, w) = \frac{1}{w} \sum_{j=1}^{w} x_j.
\]

Similarly one calculates the standard deviation corresponding to the above means:

\[
S(t_0, w) = \left[ \frac{1}{w-1} \sum_{j=1}^{w} (x_j - \bar{x}(t_0, w))^2 \right]^{\frac{1}{2}}.
\]

Note that the Theiler window can not be equal to 1, because \( w \in \{w_t, \ldots, N\} \). Then rescale the datasets by their mean

\[
y_j(t_0, w) = x_j - \bar{x}(t_0, w)
\]

and define new variables \( Y_k \) by calculating the accumulative sum

\[
Y_k(t_0, w) = \sum_{j=1}^{k} (x_j - \bar{x}(t_0, w)).
\]

Then the range is defined as the difference between the maximal and minimal value in each set \( \{Y_k\}_{k=1, \ldots, w} \):

\[
R(t_0, w) = \max\{Y_k\} - \min\{Y_k\}.
\]

The rescaled range \( R/S \) is defined as

\[
(R/S)(t_0, w) = \frac{R(t_0, w)}{S(t_0, w)}.
\]

Now, taking \( t_0 \) to run from 1 to \( N - w + 1 \), one calculates the \( R/S \) for each temporal window \( w \) as the average of those values:

\[
(R/S)(w) = \frac{1}{N - w + 1} \sum_{t_0=1}^{N-w+1} (R/S)(t_0, w).
\]

The rescaled range varies with \( w \) as a power law, with the power \( H \) being the HE, hence the slope of the linear regression in the log–log plot of \( (R/S)(w) \) versus \( w \) is the seeked value of the HE.

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