Abstract. For any positive integer $k$, let $X_k$ be a projective irreducible nodal curve with $k$ nodes $\{y_i\}_{i=1}^k$. Let $X_0$ denote the normalisation of $X_k$ and $(x_i, z_i)$ denote the two pre-images of $y_i$ for each $i = 1, \ldots, k$. Let us fix a point $p_0 \in X_0 \setminus \{x_i, z_i\}_{i=1}^k$ and define $B^0_k := \{(b_i)_{i=1}^k \in (X_0)^k : b_i \notin \{b_j, z_j, p_0\} \text{ for } 1 \leq i, j \leq k \text{ and } i \neq j\}$.

In this article, we construct a family of projective varieties $f_k : J_k \to B^0_k$ such that

1. the "general fiber" is the compactified Jacobian of an irreducible nodal curve with $k$ nodes,
2. the fiber over $(z_1, \ldots, z_k)$ is isomorphic to $J_0 \times \mathbb{P}^k$, where $J_0$ denotes the Jacobian of $X_0$ and $\mathbb{P}$ denotes the projective irreducible rational nodal curve with only one node.

We further show that this family is topologically locally trivial over the base $B^0_k$, and the higher direct image sheaves $R^i f_k^* \mathbb{Q}$ forms a variation of mixed Hodge structures. As an application, we obtain a combinatorial description of the singular cohomologies and the mixed Hodge numbers of the cohomology groups of the compactified Jacobian of any projective irreducible nodal curve.

1. Introduction

Let $S$ be a discrete valuation ring over the field of complex numbers $\mathbb{C}$. Let us denote its generic point by $\eta$ and the closed point by $\varnothing$. Let $\mathcal{X} \to S$ be a family of curves over the discrete valuation ring whose generic fiber $\mathcal{X}_\eta$ is a smooth irreducible projective curve of genus $g \geq 2$, and the special fiber $\mathcal{X}_\varnothing$ is a projective irreducible nodal curve. Then there exists a flat family of varieties $\mathcal{Y}_S \to S$ whose generic fiber $\mathcal{Y}_\eta$ is the Jacobian of the generic curve and the special fiber $\mathcal{Y}_\varnothing$ is the compactified Jacobian of the nodal curve [16]. The objects of the compactified Jacobian are rank 1 torsion-free sheaves of degree 0 on the nodal curve. The Jacobian of a smooth projective curve has been studied extensively, and its cohomologies and Hodge structures are well-known.

Notation: For any positive integer $k$, let $X_k$ denote any irreducible nodal curve of genus $g$. Let us denote its normalisation by $q_k : X_0 \to X_k$. Let us denote the nodes of $X_k$ by $\{y_1, \ldots, y_k\}$ and the inverse image of the node $y_i$ under the normalisation map by $\{x_i, z_i\}$ for every $i = 1, \ldots, k$. We fix such a nodal curve $X_k$. We denote its compactified Jacobian by $\overline{J}_k$ and its normalisation by $\tilde{J}_k$.

Problem: In this paper, we compute singular cohomologies and mixed Hodge structures of the compactified Jacobian $\overline{J}_k$ of a projective irreducible nodal curve $X_k$. 

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Our solution: We show that there exists a flat family of projective varieties over a suitable base whose
one fibre is isomorphic to $J_k$ and another fibre is isomorphic to $J_0 \times R^k$. Moreover, the family is topolog-
ically locally trivial, and it gives a variation of Hodge structures in the sense of Deligne. Therefore, the
cohomologies and the mixed Hodge numbers of $J_k$ and $J_0 \times R^k$ are the same.

Definition 1.1. A specialisation of a projective variety $Z$ to another projective variety $Z_0$ is a proper flat
family of varieties $Z \to B$, where $B$ is an irreducible variety such that

1. $Z_{b_1}$ is isomorphic to $Z$ for some $b_1 \in B$ and
2. $Z_{b_2}$ is isomorphic to $Z_0$ for some $b_2 \in B$.

Outline of the paper: Throughout this article, we will assume that all the curves are irreducible and
defined over the field of complex numbers. This article is organised as follows:

In §3, we recall a few definitions and constructions necessary for further discussion. The results in this
section should be well-known; we include these results here for the convenience of the reader.

1. torsion-free sheaves of rank one on a nodal curve and generalised parabolic bundles of rank one,
2. the compactified Jacobian of an irreducible nodal curve, its singularities, and description of the
   normalisation map,
3. the construction of the so-called $\Theta$ bundle on a Jacobian of a smooth curve and a compactified
   Jacobian of an irreducible nodal curve, and
4. certain push-outs in the category of algebraic spaces and descent of line bundles.
5. mixed Hodge numbers and variation of Hodge structures.

In §4, we carry out the construction of the specialisation (definition (1.1)) in the case when the nodal
curve has only one node. We outline the strategy of constructing the specialisation in one node case in
the following steps.

1. Step 1. Consider the family of projective bundles

$$
\tilde{\mathcal{F}}_1 \xrightarrow{\tilde{\tau}_1} X_0
$$

where

(a) $\tilde{\mathcal{F}}_1 := \mathbb{P}(\mathcal{P} \oplus p_2^* \mathcal{P}_{z_1})$, we call it the total space,
(b) $\mathcal{P}_{z_1}$ denotes the restriction line bundle $\mathcal{P}$ on $z_1 \times J_0$, and
(c) $p_2 : X_0 \times J_0 \to J_0$ is the projection map.

We call the variety $\tilde{\mathcal{F}}_k$ the total space.

2. Step 2. There are two natural divisors $\mathcal{D}_1$ and $\mathcal{D}_1'$ given by the natural quotients $\mathcal{P} \oplus p_2^* \mathcal{P}_{z_1} \to \mathcal{P}$
and $\mathcal{P} \oplus p_2^* \mathcal{P}_{z_1} \to p_2^* \mathcal{P}_{z_1}$, respectively.

3. Step 3. There is a "twisted" isomorphism $\tau_1 : \mathcal{D}_1 \to \mathcal{D}_1'$. We call it a "twisted" isomorphism because
it does not commute with the projection morphism $\tilde{\tau}_1 : \tilde{\mathcal{F}}_1 \to X_0 \times J_0$.

4. Step 4. Using a push-out construction by Artin, we construct a family of algebraic spaces
\( \mathcal{J}_1 \)
\[ \downarrow \]
\[ X_0 \]

where \( \mathcal{J}_1 \) is the algebraic space constructed by identifying the divisors \( \mathcal{D}_1 \) and \( \mathcal{D}'_1 \) by the isomorphism \( \tau_1 \).

(5) **Step 6.** We show that \( \mathcal{J}_1 \) has only normal crossing singularity. In fact, analytically locally at a singular point it is formally smooth to \( \mathbb{C}[[u,v]]/uv \).

(6) **Step 7.** We fix a point \( p_0 \) different from \( x_1 \) and \( z_1 \). Using the choice of the point, we define a line bundle \( \Theta_1 \) on \( \mathcal{J}_1 \) which is relatively ample with respect to the map \( \tilde{f}_1 : \mathcal{J}_1 \to X_0 \).

(7) **Step 8.** Here define and henceforth work with a new base \( B^0_1 := X_0 \setminus \{p_0\} \) instead of \( X_0 \). We show that the line bundle \( \Theta_1 \) descends to \( \mathcal{J}_1|_{B^0_1} \). Therefore, it follows that the morphism \( \tilde{f}_1 : \mathcal{J}_1|_{B^0_1} \to B^0_1 \) is projective.

In §5, we generalise this construction for any irreducible nodal curve with \( k \) nodes, for any positive integer \( k \). The construction is very similar to the construction in the one node case. In this case we start with the following variety as the total space

\[ \tilde{\mathcal{J}}_k \]
\[ \downarrow \tilde{f}_k \]
\[ X_0^k \]

where

1. \( \tilde{\mathcal{J}}_k := \mathbb{P}(p^*_i, k + 1, \mathcal{D} \oplus p^*_i, k + 1, \mathcal{D}^i) \times x_0 \]
2. \( p_{i,k+1} : X_0^k \times J_0 \to X_0 \times J_0 \) denotes the projection to the product of the \( i \)-th copy of \( X_0 \) and \( J_0 \).
3. \( p_{k+1} \) denotes the projection \( X_0^k \times J_0 \to J_0 \).

There are \( k \) pairs of divisors \( (\mathcal{D}_i, \mathcal{D}'_i)^{k}_{i=1} \), where \( \mathcal{D}_i \) and \( \mathcal{D}'_i \) are the two natural divisors pulled back from \( \mathbb{P}(p^*_i, k + 1, \mathcal{D} \oplus p^*_i, k + 1, \mathcal{D}^i) \). There are \( k \) natural "twisted" isomorphisms \( \tau_i : \mathcal{D}_i \to \mathcal{D}_i^i \) (see Lemma 5.8 for details). Intuitively, \( \tau_i \) is the pullback of the twisted isomorphism between the two natural divisors in \( \mathbb{P}(p^*_i, k + 1, \mathcal{D} \oplus p^*_i, k + 1, \mathcal{D}^i) \). Unlike the single node case though, here the isomorphisms \( \tau_i \) are not defined everywhere. The isomorphisms exists only when we focus on the following new base

\[ B_k := \{(b_1, b_2, \ldots, b_k) \in X_0^k | b_i \neq b_j \text{ and } b_i \neq z_j \text{ for } 1 \leq i, j \leq k \text{ and } i \neq j \} \]

We, therefore, restrict our attention over \( B_k \).

We construct a quotient space \( \mathcal{J}_k \) over \( B_k \), inductively, by identifying the divisors in every pair via the isomorphisms between them. Repeated application of proposition 3.17 shows that the quotient space is an algebraic space. It is in fact the desired family in the multinode case. We denote it by

\[ \mathcal{J}_k \]
\[ \downarrow \tilde{f}_k \]
\[ B_k \]

We further show that the algebraic space \( \mathcal{J}_k \) has only product of normal crossing singularity. More precisely, the analytic local ring at a singular point is formally smooth to \( \mathbb{C}[[u_1, v_1, \ldots, u_k, v_k]]/u_i v_i \) for some \( 1 \leq i \leq k \). We choose and fix a point \( p_0 \in X_0 \setminus \{x_1, \ldots, x_k, z_1, \ldots, z_k\} \). With this choice of point, we define a line bundle
Theorem 1.4. \( \Theta \) is topologically locally trivial.

\( B_1^0 \) forms a variation of mixed Hodge structures over \( B_0^\circ \).

In §7, we discuss applications of the construction of the specialisation. As a corollary of the above theorem 1.4, we see that the Betti numbers of \( J_k \) and the mixed Hodge numbers of the cohomology groups of \( J_k \) are the same as the Betti numbers of \( J_0 \times R^k \) and the mixed Hodge numbers of the cohomology groups of \( J_0 \times R^k \), respectively. We compute the Betti numbers and mixed Hodge numbers of the cohomology groups of the latter using the Kunneth formula.

Corollary 1.5. (1) The \( i \)-th betti number of \( J_k \)

\[
h^i(J_k) = h^i(J_0 \times R^k) = \sum_{0 \leq j \leq \min(i/2k)} \binom{2(g-k)}{i-j} \cdot \sum_{0 \leq j \leq \min(j/2k)} \binom{k}{j} \cdot \binom{j}{2j-i}.
\] (1.1)

(2) The dimension of \( gr^{W}_i(H^i(J_k)) \) is

\[
dim_Q gr^{W}_i(H^i(J_k)) = \sum_{0 \leq i \leq l} \binom{2(g-k)}{i} \cdot \binom{k}{i-\frac{i-l}{2}} \cdot \binom{i-\frac{i-l}{2}}{i-l+t}.
\] (1.2)
(3) for \( p, q \geq 0 \) such \( p + q = l \),

\[
\dim_C \text{gr}_F \text{gr}_P \left( \text{gr}^W_F \left( H^i(J_k) \right) \right) = \sum_{0 \leq r \leq l} \left( \begin{array}{c} g - k \\ p - \frac{l}{2} \\ q - \frac{l}{2} \\ i - \frac{l - t}{2} \end{array} \right) \left( \begin{array}{c} k \\ i - l + t \end{array} \right)
\]

**Speculation on a possible generalisation.** There are two compactifications of moduli stack of vector bundles on a nodal curve, namely the moduli stack of torsion-free sheaves [18] and the moduli stack of Gieseker-vector bundles [10] and [15]. It might be possible to generalise our strategy to the higher rank case as well in order to compute the singular cohomologies and the mixed Hodge numbers of these moduli spaces.

2. **Notation and Convention**

We fix some notation and convention which will be used freely in this paper.

- We will work on the field of complex numbers.
- Let \( n \) be an integer and \( X_1, \ldots, X_n \) are \( n \) varieties. For any ordered subset \( i_1 < \cdots < i_l \) of \( \{1, \ldots, n\} \), we denote by \( p_{i_1 \cdots i_l} \) the obvious projection morphism \( X_{i_1} \times \cdots \times X_{i_l} \to \prod_{j=1}^l X_{i_j} \).
- Let \( \mathbb{R} \) denote the rational nodal curve with a single node.
- For any positive integer \( k \), we define \( X_0^k := X_0 \times \cdots \times X_0 \).
  - Let \( k \) be a positive integer. Let \( X_0 \) be a smooth projective curve. Let us choose and fix \( k \) different points \( \{z_1, \ldots, z_k\} \) of \( X_0 \). Then for any general point \( (x_1, \ldots, x_k) \in X_0^k \), we denote by \( X(x_1, \ldots, x_k) \) the irreducible nodal curve such that we have a finite morphism \( X_0 \to X(x_1, \ldots, x_k) \), which identifies \( x_i \) with \( z_i \) for every \( 1 \leq i \leq k \) and isomorphism on the complement of \( \{x_1, \ldots, x_k\} \cup \{z_1, \ldots, z_k\} \).
- For an irreducible nodal curve \( X_k \) with \( k \) number of nodes we will denote the normalisation morphism \( X_0 \to X_k \) by \( q_k \).
- Let

\[
B_k := X_0^k \setminus \bigcup_{1 \leq i, j \leq k} \{\Delta_{i,j} \cup \Psi_{i,j}\},
\]

where

\[
\Delta_{i,j} := \{(x_1, x_2, \ldots, x_k) : x_i \in X_0 \text{ and } x_i = x_j\},
\]

\[
\Psi_{i,j} := \{(x_1, x_2, \ldots, x_k) : x_i \in X_0 \text{ and } x_i = z_j\}.
\]

- The Specialisation \( \mathcal{J}_k \) constructed in section 4 and 5 as a successive push-out construction on \( \mathcal{J}_k \) over \( B_k \), in the category of algebraic spaces. We will denote this particular composite pushout by \( v_k : \mathcal{J}_k \to \mathcal{J}_k \). We will denote the projection morphism \( \mathcal{J}_k \to B_k \) by \( \pi_k \) and the projection morphism \( \mathcal{J}_k \to B_k \) by \( \pi_k \).
- For \( 0 \leq r \leq n \), we define

\[
\binom{n}{r} := \frac{n!}{r!(n-r)!} \quad \text{and} \quad \binom{r}{n} := 0.
\]

3. **Preliminaries**

The results in this section should be well-known; we include these results here for the convenience of the reader.
3.1. Torsion free sheaves of rank one and generalised parabolic bundles. Let \( X_k \) be an irreducible projective nodal curve of arithmetic genus \( g \) with exactly \( k \) nodes \( \{y_1, \ldots, y_k\} \). Let us denote by \( q_k : X_0 \to X_k \) the normalisation of \( X_k \). Let \( \{x_i, z_i\} \) denote the inverse images of the node \( y_i \) for each \( i = 1, \ldots, k \). Note that the genus of \( X_0 \) is \( (g - k) \).

By a torsion-free sheaf of rank one over \( X_k \), we mean a torsion-free \( \mathcal{O}_{X_k} \)-module of rank 1. If \( \mathcal{F} \) is a torsion-free sheaf of rank 1 over \( X_k \) which is not locally free at a node \( y_i \), the localisation \( \mathcal{F}_{y_i} \) at the node \( y_i \) is isomorphic to \( m_{y_i} \), where \( m_{y_i} \) denotes the localisation of the ideal sheaf of the node \( y_i \) ([18, Proposition 2, Page 164]).

**Definition 3.2.** A generalised parabolic bundle (GPB) of rank one over \( X_0 \) is a tuple \( (E, Q_1, \ldots, Q_k) \), where \( E \) is a line bundle on \( X_0 \) and \( E_{x_i} \oplus E_{z_i} \to Q_i \) is a quotient of dimension 1 for each \( i = 1, \ldots, k \). By the abuse of notation, we will also denote the quotient maps by \( Q_i \).

3.2.1. *Torsion-free sheaf corresponding to a GPB*. Given a GPB \( (E, Q_1, \ldots, Q_k) \) of rank one there is the following canonical rank 1 torsion-free sheaf induced by the GPB.

\[
0 \to \mathcal{F} \to (q_k)_* E \to \oplus_{i=1}^k Q_i \to 0, \tag{3.1}
\]

where \( q_k \) is the normalisation map. We will refer to \( \mathcal{F} \) as the torsion-free sheaf induced by the GPB \( (E, Q_1, \ldots, Q_k) \). The following proposition is a converse of this association.

**Proposition 3.3.** Let \( \mathcal{F} \) be a torsion-free sheaf of rank one on \( X_k \) which is not locally free exactly at nodes \( \{y_1, \ldots, y_r\} \). Then there are exactly \( 2^r \) of different GPBs of degree \( \deg \mathcal{F} \) which induce the same torsion-free sheaf \( \mathcal{F} \).

**Proof.** Consider the line bundle \( E' := \pi^{*} \mathcal{F} \) of degree \( \deg \mathcal{F} - r \). There are \( 2^r \) GPBs of rank one

\[
(E, Q_i, \forall i = 1, \ldots, k),
\]

where

1. \( E := E' \otimes \mathcal{O}(p_1 + \cdots + p_r) \),
2. \( p_i \in \{x_i, z_i\} \), and
3. \( Q_i \) is the quotient \( E_{x_i} \oplus E_{z_i} \to E_{p_i} \) for \( i = 1, \ldots, r \), and
4. for \( i = r + 1, \ldots, k \), the quotient \( Q_i \) is

\[
\frac{E_{x_i} \oplus E_{z_i}}{\Gamma_{\phi_i}},
\]

where \( \Gamma_{\phi_i} \) is the graph of the natural isomorphism \( \phi_i : E_{x_i} \to E_{z_i} \) induced by \( \mathcal{F} \).

Notice that any tuple \( (E, Q_i, \forall i = 1, \ldots, k) \) as above determine a short exact sequence (as in (3.1))

\[
0 \to \ker(\gamma) \to (q_k)_* E \to \oplus_{i=1}^k Q_i \to 0. \tag{3.2}
\]

We have a natural inclusion \( \mathcal{F} \to \ker(\gamma) \) of sheaves. It can be easily seen that the degrees of the two sheaves are the same. Therefore the inclusion is an isomorphism. Notice that \( \deg E = \deg \mathcal{F} \). Using (3.2), we easily see that these are the GPBs such that the induced torsion free sheaf is \( \mathcal{F} \). \( \square \)
Remark 3.4. Notice that given any two GPBs \( \mathcal{E} := E' \otimes \mathcal{O}(p_1 + \cdots + p_r), \{Q_i\}_{i=1}^k \) and \( F := E' \otimes \mathcal{O}(p'_1 + \cdots + p'_r), \{Q'_i\}_{i=1}^k \) of rank 1 over \( X_0 \), which induce the same torsion-free sheaf \( \mathcal{F} \) (as in the above proposition), the underlying bundles of the GPBs are related by the following "twist" or so called "Hecke modification"

\[
E \mapsto E \otimes \mathcal{O}(\sum_{i=1}^r (p'_i - p_i)) \cong F
\]

(3.3)

3.5. Compactified Jacobian and its normalisation. There exists a projective variety which parametrises all the torsion-free sheaves over \( X_0 \), which are not locally free exactly at one node. From [Remark page 62], it follows that the complete local ring of the variety \( J \) consists of points \( F \) which are not locally free at exactly two points. They are related by an isomorphism which is given by

\[
(E, \{Q_j\}_{j=1}^k) \mapsto (F := E \otimes \mathcal{O}(x_i - z_l), \{Q'_j\}_{j=1}^k)
\]

(3.5)

where
(1) $Q_i$ is the quotient map $E_{x_i} \oplus E_{z_i} \to E_{z_i}$, and
(2) for each $j \neq i$, $Q_j$ is a 1-dimensional quotient of $E_{x_i} \oplus E_{z_i}$, different from $E_{x_j}$ and $E_{z_j}$, and
(3) $Q'_j$ is the quotient map $F_{x_i} \oplus F_{z_i} \to F_{x_i}$, and
(4) $Q'_j = Q_j$ for all $j \neq i$.

Since $Q'_j$ and $Q_j$ are quotients of the fibres of two different line bundles $F$ and $E$, respectively, the equality in (4) requires a justification, which can be given as follows.

Consider $U := X_0 \setminus \{x_i, z_i\}$. The line bundle $F$ is a tensor product of $E$ and a degree zero line bundle $\Theta(x_i - z_i)$. Notice that any non-zero constant function on $U$ defines a section of $\Theta(x_i - z_i)$ on $U$, by definition. In other words,

$$k_U \subset H^0(U, \Theta(x_i - z_i)),$$

where $k_U$ be the set of all constant sections on $U$. Let $\lambda_U$ denote the section with a constant non-zero value $\lambda$ over $U$. For $j \neq i$, $x_j$, $z_j \in U$. One has the following isomorphism between the fibers at $x_j$ and $E$ and $F$.

$$E_{x_j} \to F_{x_j} \cong E_{x_j} \otimes \Theta(x_i - z_i)_{x_j}$$

$$\sigma \mapsto \sigma \otimes \lambda_U,$$

where $\sigma$ is an element in the fibre of $E$ at $x_j$. Similarly, one obtain the following identification of the fibres at $z_j$. For $\sigma' \in E_{z_j}$, one has

$$E_{z_j} \to F_{z_j} \cong E_{z_j} \otimes \Theta(x_i - z_i)_{z_j}$$

$$\sigma' \mapsto \sigma' \otimes \lambda_U$$

Hence we obtain the following identification $E_{x_j} \cong F_{x_j}$ and $E_{z_j} \cong F_{z_j}$ when $j \neq i$. This induces an identification between $\mathbb{P}(E_{x_j} \oplus F_{x_j})$ and $\mathbb{P}(E_{z_j} \oplus F_{z_j})$. Notice that the identification does not depend on the choice of $\lambda$ since the same $\lambda$ has been used in both the identifications. In other words, $Q'_j := Q_j$ for all $j \neq i$.

**Remark 3.6.** We will show in lemma 5.8 that these isomorphisms extend to the whole of the normalisation of $\nu^{-1}_k(S_1)$. Moreover, the map $\nu_k$ in (3.4) is the quotient by these isomorphisms. Also, it is important to notice that these isomorphisms do not commute with the projection map $\tilde{f}_k \to f_0$.

### 3.7. Determinant of cohomology and the Theta divisor

In this subsection, we will recall the construction of the line bundle which is known as the determinant of cohomology over the Jacobian or the compactified Jacobian of a curve. We will also recall the fact that these line bundles coincide with the line bundles defined by the so-called "Theta divisors".

To begin we need to recall the following theorem.

**Theorem 3.8.** [14, Theorem, Sub-chapter 5] Let $f : X \to Y$ be a proper morphism of Noetherian schemes with $Y = \text{Spec } A$ affine. Let $\mathcal{F}$ be a coherent sheaf on $X$, flat over $Y$. There is a finite complex $K^* : 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$ of finitely generated projective $A$-modules and an isomorphism of functors

$$H^p(X \times Y \text{ Spec } B, \mathcal{F} \otimes_A B) \cong H^p(K^* \otimes_A B), (p \geq 0)$$

(3.6)

on the category of $A$-algebras $B$.

We define a line bundle $\text{Det } \mathcal{F} := \otimes_{i=0}^n (\det K^i)^{-1}(-1)^{i-1}$ over $Y$. 
Definition 3.10. Let $f: X \to Y$ be a proper morphism and $\mathcal{F}$ be a coherent sheaf over $X$, flat over $Y$. We define the following line bundle $\text{Det} \mathcal{F}$ on $Y$

$$\text{Det} \mathcal{F} := \mathcal{O}_Y^n \left( \det K^i \right)^{(-1)^{i-1}}$$ (3.7)

Definition 3.11. Let $\mathcal{P}$ be a line bundle over $X_0 \times J_0$ with the following properties:

1. $\mathcal{P}$ is a flat family of line bundles of degree 0 on $X_0$ parametrized by $J_0$,
2. the morphism given by $[E] \mapsto \mathcal{P}|_{X_0 \times [E]}$ is an isomorphism between $J_0$ and the space of isomorphism classes of line bundles of degree 0 on $X_0$.

We call such a line bundle a Poincare line bundle and we denote its determinant of cohomology by $\text{Det} \mathcal{P}$.

Definition 3.12. Let $\mathcal{F}$ be a sheaf over $X_k \times \overline{J}_k$ with the following properties:

1. $\mathcal{F}$ is flat over $\overline{J}_k$,
2. $\mathcal{F}$ is a flat family of rank 1 torsion-free sheaves of degree 0 on $X_k$ parametrized by $\overline{J}_k$,
3. the morphism given by $[F] \mapsto \mathcal{F}|_{X_k \times [F]}$ is an isomorphism between $\overline{J}_k$ and the space of isomorphism classes of torsion-free sheaves of rank 1 and degree 0 on $X_k$.

We call such a sheaf a Poincare sheaf and we denote its determinant of cohomology by $\text{Det} \mathcal{F}$.

Definition 3.13. (1) Theta divisor on $J_0$: Fix a point $x_0$ on $X_0$. Then there is a unique canonical embedding

$$\phi: X_0 \to J_0$$ (3.8)

such that $x \mapsto \mathcal{O}_{X_0}(x-x_0)$. We define the Theta divisor to be the schematic image of the map

$$X_0^{g-k-1} \to J_0$$ (3.9)

given by $(x_1, \ldots, x_{g-k-1}) \mapsto \phi(x_1) \otimes \cdots \otimes \phi(x_{g-k-1})$. We denote this divisor by $\Theta_0$.

(2) Theta divisor on $\overline{J}_k$: Fix a smooth point $x_0$ on $X_k$. Then there is a unique canonical embedding

$$\phi_k: X_k \setminus \{y_1, \ldots, y_k\} \to \overline{J}_k$$ (3.10)

such that $x \mapsto I^*_x \otimes \mathcal{O}_{X_k}(-x_0)$, where $I_x$ is the ideal sheaf of the point $x$. We define the Theta divisor to be the schematic closure of the schematic image of the map

$$(X_k \setminus \{y_1, \ldots, y_k\})^{g-1} \to \overline{J}_k$$ (3.11)

given by $(x_1, \ldots, x_{g-1}) \mapsto \phi_k(x_1) \otimes \cdots \otimes \phi_k(x_{g-1})$. We denote this divisor by $\Theta_k$.

Proposition 3.14. For any Poincare line bundle $\mathcal{P}$ over $X_0 \times J_0$ and any Poincare sheaf $\mathcal{F}$ over $X_k \times \overline{J}_k$, we have:

1. $\text{Det} \mathcal{P} \otimes \mathcal{F}_{x_0}^{(-(g-k-1)} \equiv \mathcal{O}_{J_k}(\Theta_0)$,
Consider the following short exact sequence over $\overline{J}_k$

\[ 0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}(x_0) \to \mathcal{O}_{X_0}(x_0)_{x_0} + 0 \]  

(3.13)

By pulling back this short exact sequence by the map $p_{x_0} : X_0 \times J_0 \to X_0$ and tensoring with $P$ we get:

\[ 0 \to P \to P \otimes p_{x_0}^* \mathcal{O}_{x_0}(x_0) \to P_{x_0} \to 0 \]  

(3.14)

Using the short exact sequence, we get the following isomorphism of determinant of cohomologies.

\[ \text{Det}(P \otimes p_{x_0}^* \mathcal{O}_{x_0}(x_0)) \equiv \text{Det} \otimes \mathcal{O}_{x_0}^{g-1} \]  

(3.15)

By repeating this, we get the following

\[ \text{Det}(P \otimes p_{x_0}^* \mathcal{O}_{x_0}((g-k-1)x_0)) \equiv \text{Det} \otimes \mathcal{O}_{x_0}^{g-k-1} \]  

(3.16)

Now notice that $P \otimes p_{x_0}^* \mathcal{O}_{x_0}((g-k-1)x_0)$ is a Poincare family of line bundles of degree $(g-k-1)$ on $X_0$ parametrized by $J_0$. Therefore from [12, Lemma 2.4], it follows that

\[ \text{Det} \otimes \mathcal{O}_{x_0}^{g-k-1} \equiv \text{Det}(P \otimes p_{x_0}^* \mathcal{O}_{x_0}((g-k-1)x_0)) \equiv \mathcal{O}_{x_0}(\Theta_0) \]  

(3.17)

This proves the first statement.

From [19, Corollary 14], it follows that $\text{Det}(P \otimes p_{x_0}^* \mathcal{O}_{x_0}((g-1)x_0)) \equiv \mathcal{O}_{x_0}^*(\Theta_k)$, Now the statement (2) follows from similar arguments as above.

To prove (3), consider the following exact sequence over $X_k \times \overline{J}_k$

\[ 0 \to F' \to (q_k)_{x} \mathcal{O} \to \oplus_{i=1}^k Q_i \to 0 \]  

(3.18)

The torsion-free sheaf $F'$ is a family of torsion-free sheaves of rank 1 and degree 0 over $X_k$ parametrised by $\overline{J}_k$. Therefore there exists a Poincare family $F''$ of torsion free sheaves of rank 1 and degree 0 over $X_k$ parametrized by $\overline{J}_k$ such that $\nu_k^*(F'') \equiv F'$, where $\nu_k : X_k \times \overline{J}_k \to X_k \times \overline{J}_k$ is the natural map (3.1). From (2) we have that

\[ \text{Det} F \otimes \mathcal{O}_{x_0}^{g-1} \equiv \text{Det} F' \otimes \mathcal{O}_{x_0}^{g-1} \]  

(3.19)

Then by the functoriality of the determinant of cohomology we get

\[ \nu_k^*(\text{Det} F \otimes \mathcal{O}_{x_0}^{g-1}) \equiv \nu_k^*(\text{Det} F' \otimes \mathcal{O}_{x_0}^{g-1}) \equiv \text{Det} F' \otimes \mathcal{O}_{x_0}^{g-1} \]  

(3.20)

From (3.18), we get

\[ \text{Det} F' \equiv \text{Det}(\oplus_{i=1}^k Q_i) \]  

(3.21)
Now combining this with equation (3.20) we get,

\[ v_k^*(\text{Det} \mathcal{F} \otimes \mathcal{F}_{X_0}^{(g-1)}) \cong \text{Det} \mathcal{P} \otimes (\otimes_{i=1}^k Q_i) \otimes \mathcal{P}_{X_0}^{(g-1)} \]  

(3.22)

Therefore from the previous lemma it follows that

\[ v_k^*(\text{Det} \mathcal{F} \otimes \mathcal{F}_{X_0}^{(g-1)}) \cong v_k^*(\text{Det} \mathcal{F}' \otimes \mathcal{F}_{X_0}^{(g-1)}) \cong \text{Det} \mathcal{P} \otimes \mathcal{P}_{X_0}^{(g-1)} \otimes (\otimes_{i=1}^k Q_i). \]

The fact that the line bundles in (1) and (2) are ample follows from [14, Sect. 17, p. 163] and [8, Theorem 7]. Since the morphism \( v_k : \tilde{J}_k \to J_k \) is a finite morphism and

\[ v_k^*(\text{Det} \mathcal{F} \otimes \mathcal{F}_{X_0}^{(g-1)}) \cong \text{Det} \mathcal{P} \otimes \mathcal{P}_{X_0}^{(g-1)} \otimes (\otimes_{i=1}^k Q_i), \]

therefore it is ample over \( \tilde{J}_k \).

\[ \square \]

3.15. Push-out in the category of algebraic spaces.

**Definition 3.16.** Suppose we have the following diagram of algebraic spaces:

\[
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & & \downarrow \\
V & \rightarrow & Y
\end{array}
\]  

(3.23)

Then the push-out of the diagram is an algebraic space \( Y \) which completes the diagram to make it a commutative square with the following universal property.

For any algebraic space \( Y' \) that complete the diagram to make it a commutative square factor through \( Y \)

\[
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & \downarrow & \downarrow \\
V & \rightarrow & Y \\
& \searrow & \downarrow & \quad \quad \quad \quad X \rightarrow Y := \Delta
\end{array}
\]  

(3.24)

We need the following theorem which will be used for some quotient constructions in the subsequent sections.

**Proposition 3.17.** [1, Theorem 3.1] Let \( X \) be a Noetherian algebraic space over a variety \( S \). For a closed subspace \( Z \subset X \) and a finite surjection \( g : Z \rightarrow V \), there is a universal push-out diagram of algebraic spaces

\[
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & & \downarrow \nu \\
V & \rightarrow & Y := \Delta
\end{array}
\]  

(3.25)

Furthermore,

1. \( Y \) is a Noetherian algebraic space over \( S \)
2. \( V \rightarrow Y \) is a closed embedding and \( Z = \nu^{-1}(V) \),
3. the natural map \( \ker(\mathcal{O}_Y \rightarrow \mathcal{O}_V) \rightarrow \nu_*, \ker(\mathcal{O}_X \rightarrow \mathcal{O}_Z) \) is an isomorphism, and
4. if \( X \) is of finite type over \( S \) then so is \( Y \).

**Proposition 3.18.** Assume the notation of proposition 3.17.
(1) If $X$ is a scheme, the morphism $\nu : X \to Y$ is representable by a scheme.

(2) If $X$ is a proper variety, the $\nu : X \to Y$ is a finite morphism.

(3) Suppose $X$ is a projective variety and $L$ is an ample line bundle over $X$, which descends to $Y$. Then $Y$ is a projective variety.

Proof. For any scheme $S$ over $Y$, the fiber product $X \times_Y S$ is a subspace of the scheme $X \times S$. Therefore $X \times_Y S$ is a scheme [11, Extension 3.8, Page 108]. Hence the morphism $X \to Y$ is representable.

To prove the second statement, notice that the morphism $X \to Y$ is a proper map because $X$ itself is proper. For any affine scheme $S \to Y$, the base-change $X \times_Y S \to S$ is a proper and quasi-finite morphism of schemes. Therefore $X \times_Y S \to S$ is a finite morphism. Hence the morphism $X \to Y$ is finite ([20, Lemma 45.1, Integral and finite morphisms]).

Since the morphism $X \to Y$ is finite, the descended line bundle is ample on $Y$. Therefore $Y$ is a projective variety [17, Proposition 1.2]. □

**Proposition 3.19.** Let $X$ be a Noetherian algebraic space. Suppose that $D$ and $D'$ are two disjoint Weil divisors on $X$ and $\sigma : D \to D'$ is an isomorphism of algebraic spaces. Consider the morphism $g : D \coprod D' \to D'$, where $g|_{D'} = \text{Identity}$ and $g|_{D} = \sigma$. Consider the pushout in the category of algebraic spaces

$$
\begin{array}{ccc}
D \coprod D' & \to & X \\
\downarrow g & & \downarrow \nu \\
D' & \to & Y := \overline{X}
\end{array}
$$

Let $L$ be a line bundle on the algebraic space $X$ such that there exists an isomorphism $\phi : L|_D \to L|_{D'}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
L|_D & \xrightarrow{\phi} & L|_{D'} \\
\downarrow & & \downarrow \\
D & \xrightarrow{\sigma} & D'
\end{array}
$$

Then the line bundle $L$ descends to the push-out $Y$.

Proof. We have a short exact sequence

$$0 \to I_D \cap I_{D'} \to \mathcal{O}_X \to \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \to 0,$$

where $I_D$ and $I_{D'}$ denotes the ideal sheaves of the divisors.

We have the diagonal inclusion $\mathcal{O}_{D'} \to g_* (\mathcal{O}_D \oplus \mathcal{O}_{D'}) = g_* \mathcal{O}_D \oplus g_* \mathcal{O}_{D'} \to \sigma_* \mathcal{O}_D \oplus \mathcal{O}_{D'}$ given by $f \mapsto (\sigma^*(f), f)$. So we have a short exact sequence

$$0 \to \mathcal{O}_{D'} \to g_* (\mathcal{O}_D \oplus \mathcal{O}_{D'}) \to \mathcal{O}_{D'} \to 0$$
Consider the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & Q & \rightarrow & \mathcal{O}_D & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & v_*(I_D \cap I_D') & \rightarrow & v_*\mathcal{O}_X & \rightarrow & g_*(\mathcal{O}_D \oplus \mathcal{O}_D') & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & I_{D'} & \rightarrow & \mathcal{O}_Y & \rightarrow & \mathcal{O}_D' & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & & 0 & & 
\end{array}
\]

(3.30)

where \(I_{D'}\) denotes the ideal sheaf of \(D'\) in \(\mathcal{O}_Y\).

From the statement (3) of Proposition 3.17, it follows that \(I_{D'} \equiv v_*(I_D \cap I_D')\). Therefore we have \(Q \equiv \mathcal{O}_{D'}\).

We have the following short exact sequence

\[
0 \rightarrow \mathcal{O}_Y \rightarrow v_*\mathcal{O}_X \rightarrow \mathcal{O}_{D'} \rightarrow 0
\]

(3.31)

Consider the short exact sequence

\[
0 \rightarrow (I_D \cap I_{D'}) \otimes L \rightarrow L \rightarrow L|_{D} \oplus L|_{D'} \rightarrow 0
\]

(3.32)

Taking push-forward we get

\[
0 \rightarrow v_*((I_D \cap I_{D'}) \otimes L) \rightarrow v_*L \rightarrow v_*L|_{D} \oplus v_*L|_{D'} \rightarrow 0
\]

(3.33)

Now consider the restriction of the short exact sequence to \(D'\) in \(Y\)

\[
v_*(L|_{D} \oplus L|_{D'}) \rightarrow (v_*L)|_{D'} \rightarrow v_*L|_{D'} \rightarrow 0
\]

(3.34)

Since the morphism \(v_*(L|_{D} \oplus L|_{D'}) \rightarrow (v_*L)|_{D'}\) is 0, therefore we have

\[
(v_*L)|_{D'} \equiv v_*L|_{D'}
\]

(3.35)

Notice that \(v_*(L|_{D} \oplus L|_{D'}) \equiv v_*L|_{D} \oplus v_*L|_{D'} \equiv (\sigma^{-1})^*(L|_{D}) \oplus L|_{D'}\). The commutative square 3.27 is equivalent to an isomorphism \((\sigma^{-1})^*(L|_{D}) \rightarrow L|_{D'}\) of line bundles over \(D'\). By abuse of notation we denote it by \(\phi\).

Consider the graph \(\Gamma_{\phi}\) of the morphism \(\phi\). It is a line subbundle of \((\sigma^{-1})^*(L|_{D}) \oplus L|_{D'}\). Consider the short exact sequence of sheaves:

\[
0 \rightarrow \mathcal{L} \rightarrow v_*L \rightarrow \frac{v_*L|_{D} \oplus L|_{D'}}{\Gamma_{\phi}} \rightarrow 0
\]

(3.36)

Since the morphisms \(v_*L \rightarrow v_*L|_{D} \oplus v_*L|_{D'}\) and \(v_*L|_{D} \oplus v_*L|_{D'} \rightarrow \frac{v_*L|_{D} \oplus L|_{D'}}{\Gamma_{\phi}}\) are \(\mathcal{O}_Y\) module homomorphism therefore their composite is also \(\mathcal{O}_Y\)-module homomorphism. Hence \(\mathcal{L}\) is an \(\mathcal{O}_Y\)-module. By computing fiber at every point of \(Y\) we see that \(\mathcal{L}\) is a line bundle on \(Y\).

By the adjunction formula, we have a morphism of line bundles \(v^*\mathcal{L} \rightarrow L\) over \(X\). It is an isomorphism outside the union of the two divisors \(D\) and \(D'\). Therefore if we show that the restrictions of this morphism to the two divisors are also isomorphisms, we can conclude that the morphism is isomorphism everywhere.
By restricting the short exact sequence to $D'$ we get

$$0 \to \mathcal{L}|_{D'} - (\nu_* L)|_{D'} - \frac{\nu_*(L|_D \oplus L|_{D'})}{\Gamma_\phi} \to 0.$$  \hspace{1cm}(3.37)

Notice that $\Gamma_\phi \cong \mathcal{L}|_{D'}$. Moreover, the morphism $\mathcal{L}|_{D'} - (\nu_* L)|_{D'}$ is the same as the morphism $\Gamma_\phi \to (\sigma^{-1})^*(L|_D) \oplus L|_{D'}$. From this it follows that $(\nu^* \mathcal{L})|_D \to \mathcal{L}|_D$ and $(\pi^* \mathcal{L})|_{D'} - L|_{D'}$ are both isomorphisms. Hence $\nu^* \mathcal{L} \cong \mathcal{L}$.

\[\square\]

**Remark 3.20.** The following is an example of a push-out constructed using proposition 3.17 such that the resulting quotient is not a projective variety. Consider the Hirzebruch surface $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. It has two natural divisors $\mathcal{D}$ and $\mathcal{D}'$ given by the two sections $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1)$.

These two divisors are isomorphic via the projection map $X \to \mathbb{P}^1$. Let $Y$ be the push out which identifies the two divisors. Let us denote the quotient map by $\nu : X \to Y$. Let $L$ be a line bundle on $Y$. Then $\nu^* L$ corresponds to the divisor $a \cdot \mathcal{D} + b \cdot \mathcal{H}$, where $\mathcal{H}$ denote some fiber of the Hirzebruch surface $X \to \mathbb{P}^1$. Since the line bundle is pulled back from $Y$ we have $(a \cdot \mathcal{D}' + b \cdot \mathcal{H}) \cdot \mathcal{D} = (a \cdot \mathcal{D}' + b \cdot \mathcal{H}) \cdot \mathcal{D}'$ i.e., $a \cdot 0 + b = -a + b$ i.e., $a = 0$. This implies $\nu^* L \cong \mathcal{O}(b \cdot \mathcal{H})$. But then $\nu^* L$, as well as $L$, cannot be ample. Hence the quotient $Y$ is not a projective variety.

### 3.21. Mixed Hodge numbers and variation of Hodge structures.

**Definition 3.22.** A pure Hodge structure of integral weight $l$ consists of a finite dimensional $\mathbb{Q}$-vector space $H_Q$ and a finite decreasing filtration $F^p$ of $H := H_Q \otimes \mathbb{C}$

$$\cdots \subset F^p \subset \cdots \subset H$$ \hspace{1cm}(3.38)

satisfying $H = F^p \oplus F^{1-p+1}$ for all $p$. The filtration is known as the Hodge filtration.

**Definition 3.23.** A mixed Hodge structure consists of

1. a finite dimensional $\mathbb{Q}$-vector space $H_Q$,
2. an increasing filtration $W_l$ of $H_Q$, called the weight filtration

$$\cdots \subset W_q \subset \cdots \subset H$$ \hspace{1cm}(3.39)

3. a Hodge filtration $\{F^p\}$ of $H$,

satisfying the following condition.

For every integer $l$, define

$$Gr^W_l H_Q := (W_l H_Q/W_{l-1} H_Q) \otimes \mathbb{C} = W_l H/W_{l-1} H.$$ \hspace{1cm}(3.40)

The filtration $\{F^p\}_p$ induces a filtration on $Gr^W_l H_Q$ by simply taking intersection with the filtration. More precisely,

$$F^p Gr^W_l := (W_l H \cap F^p+ W_{l-1} H)/W_{l-1} H$$ \hspace{1cm}(3.41)

for every $l$.

The condition is that the filtration defines a pure Hodge structure weight $l$ on $Gr^W_l H_Q$. 


Given a mixed Hodge structure and integers \( p \) and \( l \), define

\[
Gr^p_{F^l} W^r := F^p Gr^r_{F^l} W^r / F^{p+1} Gr^r_{F^l} W^r
\]

(3.42)

**Definition 3.24.** Let \((H_q, F_*, W_*)\) be a mixed Hodge structure. The mixed Hodge numbers of \(H\) are defined to be

\[
h^{p,q}(H) := \dim Gr^p_{F^l} W^r_{p+q}.
\]

(3.43)

Let \(X\) be any complex algebraic variety. Deligne proved that the singular cohomology groups \(H^k(X,\mathbb{Q})\) carry a mixed Hodge structure. ([4], [5] and [6])

**Definition 3.25.** For a nonsingular variety \(B\), variation of \(\mathbb{Q}\)-mixed Hodge structure \(\mathcal{V}\) (VMHS) consists of following data

1. A local system (locally constant sheaf) \(\mathcal{L}\) of finite type on the analytic manifold \(B(\mathbb{C})\)
2. A family of sub-local system \(\{W_\alpha\}\) of \(\mathcal{L}\) which defines an increasing filtration
3. A system of holomorphic sub-bundle \(\{\mathcal{F}\}\) of \(\mathcal{L} \otimes \mathcal{O}_B\) defines a decreasing filtration and Griffith transversality with respect to a flat connection \(\nabla\) satisfies on the sections of \(\mathcal{L} \otimes \mathcal{O}_B\)

\[
\nabla(\mathcal{F}) \subset \Omega^1_B \otimes \mathcal{F}^{-1}
\]

4. the fiber at \(b \in B\), \((\mathcal{L} \otimes \mathcal{O}_B(b), W(b), \mathcal{F}(b))\) defines a mixed Hodge structure.

Given a family of algebraic varieties \(f : X \to V\) one can consider the higher direct image sheaves \(R^l f_* \mathbb{Q}\). It is a natural question to ask under what condition these sheaves form a local system. Moreover, if it forms a local system, is there a natural variation of mixed Hodge structures on \(V\)? Proposition 3.27 answers this question. To state the proposition we need to recall another result, which can be thought of as the proof of existence of a relative version of a Whitney stratification for a family of algebraic varieties.

**Proposition 3.26.** [2, 8.1.3.2 Thom-Whitney’s stratifications] Let \(f : X \to V\) be an algebraic morphism. There exist finite Whitney stratifications \(\mathcal{X}\) of \(X\) and \(S = \{S_i\}_{i \leq d}\) of \(V\) by locally closed subsets \(S_i\) of dimension \(l\) \((d = \dim V)\), such that for each connected component \(S\) (a stratum) of \(S_i\)

1. \(f^{-1}(S)\) is a topological fibre bundle over \(S\), union of connected components of strata of \(X\), each mapped submersively to \(S\).
2. Local topological triviality: for all \(v \in S\), there exist an open neighborhood \(U(v)\) in \(S\) and a stratum preserving homeomorphism \(h : f^{-1}(U) \cong f^{-1}(v) \times U\) s.t. \(f|_U = \pi_U \circ h\), where \(\pi_U\) is the projection on \(U\).

**Proposition 3.27.** (geometric VMHS) [2, Proposition 8.1.16 ]

Let \(f : X \to V\) be an algebraic morphism and \(S \subset V\) a Thom-Whitney smooth strata over which the restriction of \(f\) is locally topologically trivial. Then

1. For all integers \(i \in \mathbb{N}\), the restriction to \(S\) of the higher direct image cohomology sheaves \((R^i f_* \mathbb{Z}_X)/S\) (resp. \((R^i f_* \mathbb{Z}_X/\text{Torsion})/S\)) are local systems of \(\mathbb{Z}_S\)-modules of finite type (resp. free).
2. The weight filtration \(W\) on the cohomology \(H^i(X_t,\mathbb{Q})\) of a fiber \(X_t\) at \(t \in S\) defines a filtration \(W\) by sub-local systems of \((R^i f_* \mathbb{Q}_X)/S\).
3. The graded objects \(Gr^r_m (R^i f_* \mathbb{C}_X)/S\) with the induced filtration by \(F\), are variations of Hodge structure.
Remark 3.28. By semicontinuity, it follows that given a VMHS, as above, the mixed Hodge numbers of $(H^i(X_t, \mathbb{C}), W_{*,t}, F_{*,t})$ are constant over $S$. Here $W_{*,t}$ and $F_{*,t}$ denotes the restrictions of the filtrations $W_*$ and $F_*$, respectively to the fibre over $t \in S$.

4. Specialisation of the compactified Jacobian of a nodal curve with a single node

Let $X_0$ be a smooth projective curve. Let us choose and fix a point $z \in X_0$. By a general point $x \in X_0$ we mean that $x \neq z$. In this section, we will construct an algebraic family $\mathcal{J}_1$ over $X_0$ such that the fibre over a general point $x \in X_0$ is isomorphic to $\mathcal{J}_{x_0}$ (See 2) and the fiber over $z$ is isomorphic to $J_{x_0} \times R$. By definition, it is therefore a specialisation of $\mathcal{J}_{x_0}$ to $J_{x_0} \times R$.

4.1. The construction of the total space. We will construct the family as a push-out of the following $\mathbb{P}^1$-bundle.

$$\mathcal{J}_1 := \mathbb{P}(\mathcal{P} \oplus p_2^* \mathcal{P}), \text{ over } X_0 \times J_0, \quad (4.1)$$

where

1. $\mathcal{P}$ is a Poincare line bundle over $X_0 \times J_0$,
2. $p_2 : X_0 \times J_0 \to J_0$ is the projection morphism,
3. $\mathcal{P}_0$ denotes the line bundle over $J_0$ obtained by restricting $\mathcal{P}$ to the closed subscheme $z \times J_0$ and by identifying $z \times J_0$ with $J_0$.

Remark 4.2. The variety $\mathcal{J}_1$ parametrises tuples $(x, L, L_x \oplus L_z \to Q)$, where

1. $x$ is a point of $X_0$,
2. $L$ is a line bundle of degree 0 over $X_0$,
3. $L_x \oplus L_z \to Q$ is a 1-dimensional quotient.

4.3. Two natural divisors on the total space. The $\mathbb{P}^1$ bundle has two natural sections $\mathcal{D}_1$ and $\mathcal{D}_1'$, which corresponds to the natural quotients $\mathcal{P} \oplus p_2^* \mathcal{P} \to \mathcal{P}$ and $\mathcal{P} \oplus p_2^* \mathcal{P} \to p_2^* \mathcal{P}$ respectively. Being sections these two divisors are isomorphic to $X_0 \times J_0$ (the isomorphism is given by the restrictions of the projection morphism).

Remark 4.4. The variety $\mathcal{D}_1$ parametrises tuples $(x, L, L_x \oplus L_z \to L_x)$ and The variety $\mathcal{D}_1'$ parametrises tuples $(x, L, L_x \oplus L_z \to L_z)$. Notice when $x = z$, there is an ambiguity about the quotients $L_x \oplus L_z \to L_x$ and $L_x \oplus L_z \to L_z$. To resolve this, we refer to $L_x \oplus L_z \to L_x$ as the first quotient and to $L_x \oplus L_z \to L_z$ as the second quotient. We see that the varieties $\mathcal{J}_1, \mathcal{D}_1$ and $\mathcal{D}_1'$ have universal properties because they parametrises the tuples, described above.

Lemma 4.5. $\mathcal{D}_1 \cap \mathcal{D}_1' = \emptyset$.

Proof. Since the two natural sections of $\mathbb{P}(\mathcal{P} \oplus p_2^* \mathcal{P})$ given by the two natural quotients $\mathcal{P} \oplus p_2^* \mathcal{P} \to \mathcal{P}$ and $\mathcal{P} \oplus p_2^* \mathcal{P} \to p_2^* \mathcal{P}$ are disjoint therefore $\mathcal{D}_1 \cap \mathcal{D}_1' = \emptyset$. \qed

4.6. Twisted isomorphism between the divisors $\mathcal{D}_1$ and $\mathcal{D}_1'$. Notice that $\mathcal{D}_1$ and $\mathcal{D}_1'$ are abstractly isomorphic to $X_0 \times J_0$ because they are sections of the morphism $\mathcal{J}_1 \to X_0 \times J_0$. Proposition 4.8 shows that there is another natural isomorphism (“twisted isomorphism”) between these two divisors (3.5).

Before stating the proposition, we need to fix some notation and a preparatory lemma.
Proof. We have the following diagram

where

1. \( \pi_1 \) is the projection \( \bar{\mathcal{J}}_1 \rightarrow X_0 \times J_0 \), and \( Id \times \pi_1 : X_0 \times \bar{\mathcal{J}}_1 \rightarrow X_0 \times (X_0 \times J_0) \) is the product of the identity morphism on the first factor and \( \pi_1 \) on \( \bar{\mathcal{J}}_1 \),
2. \( q : X_0 \times X_0 \times J_0 \rightarrow X_0 \times J_0 \) is the projection \( (x_1, x_2, L) \rightarrow (x_1, L) \),
3. \( r : X_0 \times X_0 \times J_0 \rightarrow X_0 \times X_0 \) is the projection \( (x_1, x_2, L) \rightarrow (x_1, x_2) \),
4. \( s_1 : X_0 \times X_0 \times J_0 \rightarrow X_0 \) and \( s_2 : X_0 \times X_0 \times J_0 \rightarrow X_0 \) are the projections onto the first and second \( X_0 \), respectively,
5. \( \bar{q} := q \circ (I \times \pi_1), \bar{r} := r \circ (I \times \pi_1) \) and \( \bar{s}_i := s_i \circ (I \times \pi_1) \).

Lemma 4.7. (1) \( \bar{r}^{-1}(\Delta) \equiv \bar{\mathcal{J}}_1 \),

(2) Let us denote by \( j : \mathcal{D}_1 \rightarrow \bar{\mathcal{J}}_1 \) and \( j' : \mathcal{D}'_1 \rightarrow \bar{\mathcal{J}}_1 \) the natural inclusion maps. Then \( \bar{r}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1) \equiv (I \times j)^{-1} \circ \bar{r}^{-1}(\Delta) \equiv \mathcal{D}_1 \), and \( \bar{r}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}'_1) \equiv (I \times j')^{-1} \circ \bar{r}^{-1}(\Delta) \equiv \mathcal{D}'_1 \).

Proof. We have the following diagram

Notice that the two squares are Cartesian. Therefore, the composite of the two squares is also Cartesian. Now the first statement follow from the observation that the composite map \( r^{-1}(\Delta) \rightarrow X_0 \times J_0 \) is an isomorphism.

Since \( r^{-1}(\Delta) \rightarrow X_0 \times J_0 \), therefore \( \bar{r}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1) \equiv (I \times j)^{-1} \circ \bar{r}^{-1}(\Delta) \equiv \mathcal{D}_1 \). The other statement follows similarly. \( \square \)

Proposition 4.8. There is a natural isomorphism \( \tau : \mathcal{D}_1 \rightarrow \mathcal{D}'_1 \) given by

\[
(x, L, L_x \oplus L_z \rightarrow L_x) \mapsto (x, L' := L \oplus \Theta_X (z - x), L'_x \oplus L'_z \rightarrow L'_x).
\]

Here for a line bundle \( M \) over \( X_0 \), we denote by \( M_x \oplus M_z \rightarrow M_x \) the first projection and by \( M_x \oplus M_z \rightarrow M_z \) the second projection.

Proof. Consider the line bundle

\[
\mathcal{D}' := \bar{q}^* \mathcal{P} \oplus \bar{r}^* \Theta_X (\Delta) \oplus \bar{s}_1^* \Theta_X (z),
\]

where \( \Delta \) is the subvariety \( \{(x,x) | x \in X_0 \} \subset X_0 \times X_0 \).
The Poincare line bundle $\mathcal{D}'$ is, by definition, a family of degree 0 line bundles over $X_0$ parametrized by $X_0 \times J_0$. For $x \in X_0$ and $L \in J_0$,

$$\mathcal{D}'|_{X_0 \times X \times L} \equiv L \otimes \Theta X_0(z-x). \quad (4.6)$$

Over $\mathcal{F}_1$ we have a universal quotient

$$(\tilde{q}^* \mathcal{D})|_{\tilde{f}^{-1}(\Delta)} \oplus (\tilde{q}^* \mathcal{D})|_{z \times \mathcal{F}_1} \to L \quad (4.7)$$

Notice that by lemma 4.7, $\tilde{f}^{-1}(\Delta)$ and $z \times \mathcal{F}_1$ both can be identified with $\mathcal{F}_1$.

Over $\mathcal{D}_1$, the quotient (4.7) becomes

$$(\tilde{q}^* \mathcal{D})|_{\tilde{f}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1)} \oplus (\tilde{q}^* \mathcal{D})|_{z \times \mathcal{D}_1} \to (\tilde{q}^* \mathcal{D})|_{\tilde{f}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1)} \quad (4.8)$$

Notice that by lemma 4.7, $\tilde{f}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1)$ and $z \times \mathcal{D}_1$ both can be identified with $\mathcal{D}_1$. We have another natural quotient line bundle over $\mathcal{D}_1$ which is the following.

$$\mathcal{D}'|_{\tilde{f}^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1)} \oplus \mathcal{D}'|_{z \times \mathcal{D}_1} \to \mathcal{D}'|_{z \times \mathcal{D}_1}. \quad (4.9)$$

By the universal properties (remark 4.4) of $\mathcal{D}_1$ and $\mathcal{D}'_1$, the above modified quotient (4.9) induces an isomorphism $\tau: \mathcal{D}_1 \to \mathcal{D}'_1$. It is straightforward to check that this isomorphism has the desired property (4.4).

4.9. The construction of the specialisation by push-out and its singularities. Consider the following topological quotient space

$$\mathcal{F}_1 := \frac{\mathcal{F}_1}{\mathcal{D}_1 \sim \mathcal{D}'_1}. \quad (4.10)$$

where $\mathcal{D}_1 \sim \mathcal{D}'_1$ means that $\mathcal{D}_1$ and $\mathcal{D}'_1$ are identified via the automorphism $\tau$ (4.4). We denote by $v_1: \mathcal{F}_1 \to \mathcal{F}_1$ the quotient map. We denote by $V$ the image of $\mathcal{D}_1$, or equivalently the image of $\mathcal{D}'$ under the quotient map $v_1$.

**Theorem 4.10.** $\mathcal{F}_1$ is an algebraic space with only normal crossing singularity.

**Proof.** From proposition 3.17, it follows that $\mathcal{F}_1$ is an algebraic space. To determine the singularity of $\mathcal{F}_1$, consider the following exact sequence of sheaves (3.31)

$$0 \to \Theta \mathcal{F}_1 \to (v_1)_* \Theta \mathcal{F}_1 \twoheadrightarrow \Theta V \to 0, \quad (4.11)$$

Let $v$ be a point of $V$ and let $v_1 \in \mathcal{D}_1$ and $v_2 \in \mathcal{D}'_1$ denote the pre-images of $v$. Then (4.11) induce the following short exact sequence of analytic local rings

$$0 \longrightarrow \hat{\Theta} \mathcal{F}_1,v \longrightarrow \hat{\Theta} \mathcal{F}_1,v_1 \oplus \hat{\Theta} \mathcal{F}_1,v_2 \longrightarrow \hat{\Theta} V,v \longrightarrow 0$$

where

$$0 \longrightarrow \hat{\Theta} \mathcal{F}_1,v \longrightarrow k[[x_1, \ldots, x_{n-1}, x_n]] \oplus k[[x_1, \ldots, x_{n-1}, x_{n+1}]] \longrightarrow k[[x_1, \ldots, x_{n-1}]] \longrightarrow 0$$

The morphism

$$k[[x_1, \ldots, x_{n-1}, x_n]] \oplus k[[x_1, \ldots, x_{n-1}, x_{n+1}]] \longrightarrow k[[x_1, \ldots, x_{n-1}]]$$
is given by

\[(f, g) \mapsto f(x_n) - g(x_{n+1}).\]

Hence we obtain \(\hat{O}_{J_1, V} \simeq \frac{\mathcal{O}(X_1, \ldots, X_{n-1}, X_n, X_{n+1})}{x_n x_{n+1}}\). Therefore, the algebraic space \(J_1\) has normal crossing singularity along \(V\).

We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{D}_1 \cup \mathcal{D}_1' & \longrightarrow & \mathcal{J}_1 \\
\downarrow & & \downarrow \\
\mathcal{V} & \longrightarrow & \mathcal{J}_1 \\
\end{array}
\]

(4.12)

Therefore we have a projection morphism \(\mathcal{J}_1 \rightarrow X_0\) from the push-out to \(X_0\). Let us denote it by \(\bar{j}_1\).

**4.11. Construction of a line bundle relatively ample with respect to the morphism \(\bar{j}_1 : J_1 \rightarrow X_0\).** Consider the following diagram

\[
\begin{array}{ccc}
\mathcal{J}_1 & \longrightarrow & \mathcal{J}_1 \\
\downarrow \bar{\pi}_1 & & \downarrow \\
X_0 \times J_0 & \overset{p_1}{\longrightarrow} & X_0 \\
\end{array}
\]

(4.13)

Let us define \(\bar{\pi}_1 := p_1 \circ \pi_1\) and \(\bar{\pi}_2 := p_2 \circ \pi_1\). Let us now choose a point \(p_0\) in \(X_0\) such that \(p_0 \neq z\). We can define an ample line bundle \(\Theta_0\) on \(J_0\) as in proposition 3.14. We will show that the line bundle \(\bar{\pi}_2^* \Theta_0 \otimes \mathcal{L}\) is a relatively ample line bundle with respect to the morphism \(\bar{\pi}_1 : \mathcal{J}_1 \rightarrow X_0\). From proposition 3.14, we have \(\bar{\pi}_2^* \Theta_0 \otimes \mathcal{L} = \bar{\pi}_2^* \text{Det} \mathcal{P} \otimes \bar{\pi}_2^* \mathcal{P}_0 \otimes \mathcal{L}\).

Consider the commutative square

\[
\begin{array}{ccc}
X_0 \times \mathcal{J}_1 & \longrightarrow & \mathcal{J}_1 \\
\downarrow \bar{q} & & \downarrow \bar{\pi}_2 \\
X_0 \times J_0 & \overset{p_2}{\longrightarrow} & J_0 \\
\end{array}
\]

(4.14)

From the above diagram we easily see the following

1. \(\text{Det} \bar{q}^* \mathcal{P} \equiv \bar{\pi}_2^* \text{Det} \mathcal{P}\),
2. \(\left(\bar{q}^* \mathcal{P}\right)|_{p_0 \times \mathcal{J}_1} \equiv \bar{\pi}_2^* \mathcal{P}_0\).

This leads to the following definition.

**Definition 4.12.** We define a line bundle over \(\mathcal{J}_1\)

\[
\mathcal{O}_1 := \text{Det} \bar{q}^* \mathcal{P} \otimes \mathcal{L} \otimes \left(\bar{q}^* \mathcal{P}\right)|_{p_0 \times \mathcal{J}_1} \otimes (\bar{q}^* \mathcal{P})|_{p_0 \times \mathcal{J}_1}^{\circ (g-1)}.
\]

(4.15)
where Det denotes the determinant of cohomologies. We refer to this line bundle as the Theta bundle over $\widetilde{J}_1$.

Notice that by (1) and (2) above we have

$$\Theta_1 \cong p_2^* \text{Det } P \otimes L \otimes p_2^* P_{p_0}$$

(4.16)

We will now show that the line bundle $\Theta_1$ is relatively ample with respect to the morphism $\widetilde{J}_1 \to X_0$. Moreover, there is an open set $B_1^0$ (yet to be defined) of $X_0$ such that the line bundle $\Theta_1$ descends to the base change of $\widetilde{J}_1$ over the open set. But before that we list out few results which determine the pull-backs by the isomorphism $\tau$ of several natural line bundles on $\widetilde{J}_1$.

4.13. Some properties of the isomorphism $\tau : \mathcal{D} \to \mathcal{D}'$.

Lemma 4.14. Let $p_{23} : X_0 \times X_0 \times J_0 \to X_0 \times J_0$ denote the projection onto the product of the second and the third factors. Let us consider the cartesian square

$$
\begin{array}{ccc}
(x_1, x_2, L) & \to & (x_1, x_2, L \otimes \mathcal{O}_{X_0}(-x_2 + z)) \\
X_0 \times X_0 \times J_0 & \xrightarrow{l \times \tau} & X_0 \times X_0 \times J_0 \\
p_{23} \downarrow & & \downarrow p_{23} \\
X_0 \times J_0 & \xrightarrow{\tau} & X_0 \times J_0
\end{array}
$$

(4.17)

Let $\mathcal{P}$ be a Poincare line bundle on $X_0 \times J_0$. Then we have the following.

(1) $(l \times \tau)^* (q^* \mathcal{P}) \cong q^* \mathcal{P} \otimes r^* \mathcal{O}_{X_0}(-\Delta) \otimes s^* \mathcal{O}_{X_0}(z)$ over $X_0 \times X_0 \times J_0$,

(2) for any point $p \in X_0$, $\tau^* (p_2^* \mathcal{P}) \cong p_2^* \mathcal{P} \otimes p_0^* \mathcal{O}_{X_0}(-p)$ over $X_0 \times J_0$,

(3) $\tau^* \text{Det } q^* \mathcal{P} \equiv \text{Det } (l \times \tau)^* (q^* \mathcal{P})$ over $X_0 \times J_0$,

(4) $\text{Det } (l \times \tau)^* (q^* \mathcal{P}) \equiv \text{Det } (q^* \mathcal{P}) \otimes (q^* \mathcal{P})_{|\Gamma = \Delta} \otimes p_2^* \mathcal{P}^{-1} \otimes p_0^* \mathcal{O}_{X_0}(z)$, over $X_0 \times J_0$.

Proof. The statement (1) follows from the universal property of $J_0$ and the definition of the map $\tau$, as in the diagram (4.17).

To prove (2), consider the following diagram

$$
\begin{array}{ccc}
X_0 \times J_0 & \xrightarrow{\tau} & X_0 \times J_0 \\
\downarrow & & \downarrow p_2 \\
J_0 & & J_0
\end{array}
$$

(4.18)

Since we want to compute $\tau^* \circ p_2^* \mathcal{P}$, we consider the following diagram instead of (4.18).

$$
\begin{array}{ccc}
X_0 \times X_0 \times J_0 & \xrightarrow{l \times \tau} & X_0 \times X_0 \times J_0 \\
\downarrow & & \downarrow q \\
X_0 \times J_0 & & X_0 \times J_0
\end{array}
$$

(4.19)
Then $\tau^* \circ p_2^* \mathcal{P}_p$ is isomorphic to $((I \times \tau)^* \circ q^* \mathcal{P})|_{p \times X_0 \times J_0}$. From (1), we have $(I \times \tau)^* (q^* \mathcal{P}) \equiv q^* \mathcal{P} \otimes r^* \Theta_{X_0 \times X_0}(-\Delta) \otimes \tilde{s}_1^* \Theta_{X_0}(z)$. Therefore,

$$\tau^* \circ p_2^* \mathcal{P}_p \equiv (q^* \mathcal{P})|_{p \times X_0 \times J_0} \otimes (r^* \Theta_{X_0 \times X_0}(-\Delta))|_{p \times X_0 \times J_0} \otimes (\tilde{s}_1^* \Theta_{X_0}(z))|_{p \times X_0 \times J_0} \equiv p_2^* \mathcal{P}_p \otimes p_1^* \Theta_{X_0}(-p) \quad (4.20)$$

The statement (3) also follows from the commutative square 4.17.

To see (4), define $\mathcal{P}'' := q^* \mathcal{P} \otimes r^* \Theta_{X_0 \times X_0}(-\Delta)$. Hence, $\mathcal{P}' = \mathcal{P}'' \otimes s_1^* \Theta_{X_0}(z)$.

Consider the following short exact sequence of sheaves over $X_0 \times X_0 \times J_0$

$$0 \to \mathcal{P}' \to \mathcal{P}'' \otimes s_1^* \Theta_{X_0}(z) \to \mathcal{P}''|_{z \times X_0 \times J_0} \to 0 \quad (4.21)$$

Therefore we have,

$$\text{Det} \mathcal{P}' \equiv \text{Det} \mathcal{P}'' \otimes (\mathcal{P}''|_{z \times X_0 \times J_0})^{-1} \quad (4.22)$$

Notice

$$\mathcal{P}''|_{z \times X_0 \times J_0} \equiv (q^* \mathcal{P})|_{z \times X_0 \times J_0} \otimes (r^* \Theta_{X_0 \times X_0}(-\Delta))|_{z \times X_0 \times J_0} \equiv p_2^* \mathcal{P}_z \otimes p_1^* \Theta_{X_0}(-z). \quad (4.23)$$

Therefore,

$$\text{Det} \mathcal{P}' \equiv \text{Det} \mathcal{P}'' \otimes (p_2^* \mathcal{P}_z)^{-1} \otimes p_1^* \Theta_{X_0}(z) \quad (4.24)$$

Now let us compute $\text{Det} \mathcal{P}''$. Consider the following short exact sequence

$$0 \to \mathcal{P}'' \to q^* \mathcal{P} \to (q^* \mathcal{P})|_{r^{-1}(\Delta)} \to 0 \quad (4.25)$$

Therefore

$$\text{Det} \mathcal{P}'' \equiv \text{Det} q^* \mathcal{P} \otimes (q^* \mathcal{P})|_{r^{-1}(\Delta)} \quad (4.26)$$

and

$$\text{Det} \mathcal{P}' \equiv \text{Det} q^* \mathcal{P} \otimes (q^* \mathcal{P})|_{r^{-1}(\Delta)} \otimes (p_2^* \mathcal{P}_z)^{-1} \otimes p_1^* \Theta_{X_0}(z) \quad (4.27)$$

This completes the proof. □

**Remark 4.15.** The above statements also holds if we replace the above square by the following

$$\xymatrix{ X_0 \times \mathcal{D}_1 \ar[d] \ar[r]^{I \times \tau} & X_0 \times \mathcal{D}_1' \ar[d] \cr \mathcal{D}_1 \ar[r]^{\tau} & \mathcal{D}_1' } \quad (4.28)$$

where $\tau$ is the isomorphism defined in lemma 4.8. We list out the statements here for future use.

1. $(I \times \tau)^* (q^* \mathcal{P}) \equiv \tilde{q}^* \mathcal{P} \otimes \tilde{r}^* \Theta_{X_0 \times X_0}(-\Delta) \otimes \tilde{s}_1^* \Theta_{X_0}(z)$ over $X_0 \times \mathcal{D}_1$,
2. for any point $p \in X_0$, we have $\tau^*(\tilde{p}_2^* \mathcal{P}_p) \equiv \tilde{p}_2^* \mathcal{P}_p \otimes \tilde{p}_1^* \Theta_{X_0}(-p)$ over $\mathcal{D}_1$,
3. $\tau^*(\text{Det} \tilde{q}^* \mathcal{P}) \equiv \text{Det} (I \times \tau)^* (\tilde{q}^* \mathcal{P})$ over $\mathcal{D}_1$,
4. $\text{Det} (I \times \tau)^* (\tilde{q}^* \mathcal{P}) \equiv \text{Det} (\tilde{q}^* \mathcal{P}) \otimes (q^* \mathcal{P})|_{r^{-1}(\Delta)} \otimes \tilde{p}_2^* \mathcal{P}_z^{-1} \otimes \tilde{p}_1^* \Theta_{X_0}(z)$ over $\mathcal{D}_1$. 21
It is necessary to clarify the notation $\tilde{\Theta}$. We remind here that if $p$ is a projection map from $X_0 \times J_0$, we denote the composition of $\tilde{\mathcal{J}}_1 \to X_0 \times J_0$ with the projection $p$ by $\tilde{p}$.

4.16. **Relative ampleness of $\tilde{\Theta}_1$.**

**Proposition 4.17.** The line bundle $\tilde{\Theta}_1$ is ample relative to the morphism $\tilde{f}_1 : \tilde{\mathcal{J}}_1 \to X_0$.

**Proof.** Since $\tilde{f}_1$ is projective, it is enough to show that the restriction of $\tilde{\Theta}_1$ to the fibre over every point $x \in X_0$ is ample. For any $x \in X_0$, $\tilde{f}_1^{-1}(x) = P(\mathcal{P}_x \oplus \mathcal{P}_2)$ (4.1). Let us denote it by $P(x)$. Then the restriction of $\tilde{\Theta}_1$ to $P(x)$ is isomorphic to

$$
\text{Det} (\tilde{q}^* \mathcal{P})|_{P(x)} \otimes \mathcal{L}|_{P(x)} \otimes (\tilde{q}^* \mathcal{P})|_{p_0 \times P(x)}^{\otimes - (g-1)} \\
\cong (\text{Det} (\tilde{q}^* \mathcal{P})|_{P(x)} \otimes (\tilde{q}^* \mathcal{P})|_{p_0 \times P(x)}}^{\otimes - (g-2)} \otimes \mathcal{L}|_{P(x)} \otimes \tilde{q}^* \mathcal{P}|_{p_0 \times P(x)}^{\otimes - 1} \\
\cong \tilde{p}_{2,x}^* \Theta_0(x) \otimes \Theta_{P(x)}(1) \otimes \tilde{q}^* \mathcal{P}|_{p_0 \times P(x)},
$$

where

1. $\tilde{p}_{2,x} : P(x) \to J_0$ denotes the projection map,
2. $\Theta_0(x)$ denotes the Theta bundle (proposition 3.14) over $P(x)$,
3. $\Theta_{P(x)}(1)$ is the tautological bundle of the projective bundle of $P(\mathcal{P}_x \oplus \mathcal{P}_2)$.

It can be easily seen that $\tilde{q}^* \mathcal{P}|_{X_0 \times P(x)}$ is isomorphic to the pull-back of the Poincare bundle $\mathcal{P}$ by the map $I \times \tilde{p}_{2,x} : X_0 \times P(x) \to X_0 \times J_0$ which is identity on the first factor and $\tilde{p}_{2,x}$ on $P(x)$. Notice that

$$
E(x) := (\tilde{p}_{2,x})_* ((\tilde{p}_{2,x})^* \Theta_0(x) \otimes \Theta_{P(x)}(1)) \cong \Theta_0(x) \otimes (\tilde{p}_{2,x})_* \Theta_{P(x)}(1) \cong \Theta_0(x) \otimes (\mathcal{P}_x \oplus \mathcal{P}_2) \cong (\Theta_0 \otimes \mathcal{P}_x) \oplus (\Theta_0 \otimes \mathcal{P}_2).
$$

Since each of the direct summands are ample line bundle, the vector bundle $E(x)$ is ample. Then by [9, Theorem 3.2], it follows that $\Theta_{P(E(x))}(1)$ is ample line bundle over $P(E(x))$, which is isomorphic to $P(\Theta_0(x) \otimes \mathcal{P}_x \oplus \Theta_0(x) \otimes \mathcal{P}_2) \equiv P(\mathcal{P}_x \oplus \mathcal{P}_2)$. Therefore the line bundle $\tilde{p}_{2,x}^* \Theta_0(x) \otimes \Theta_{P(x)}(1)$ is ample over $P(x)$ for any $x \in X_0$. Now notice that the line bundle $((\tilde{q}^* \mathcal{P})|_{p_0 \times P(x)})$ in (4.29) is isomorphic to the pullback of the line bundle $\mathcal{P}_{p_0}$ by the map $P(x) \to J_{X_0}$ and hence it is algebraically equivalent to the trivial line bundle. Therefore $\tilde{\Theta}_1|_{P(x)}$ is ample for any $x \in X_0$.

4.18. **Descent of the line bundle $\tilde{\Theta}_1$.** Consider the open set $\mathcal{B}_0 := X_0 \setminus \{p_0\}$. We take the base change of $\tilde{\mathcal{J}}_1$ over the open set $\mathcal{B}_0$. By abuse of notation, we also denote it by $\tilde{\mathcal{J}}_1$. From here onwards we will similarly base change everything on $\mathcal{B}_0$ and denote them by the same notations. It will be clear during the proof of the following theorem the reason for working only over the new base.

**Theorem 4.19.**

1. The morphism $\tilde{f}_1 : \tilde{\mathcal{J}}_1 \to \mathcal{B}_0$ is projective.
2. The fibers of the morphism have the following description.

$$
\tilde{f}_1^{-1}(x) \cong \begin{cases} 
\mathcal{T}_{x,0} & \text{for } x \neq z \\
J_0 \times R & \text{for } x = z 
\end{cases}
$$

**Proof.** Notice that

$$
\tilde{\Theta}_1|_{\mathcal{B}_1} \equiv \text{Det} \tilde{q}^* \mathcal{P} \otimes (\tilde{q}^* \mathcal{P})|_{p_0 \times \tilde{\mathcal{J}}_1}^{\otimes - (g-1)},
$$

$$
\tilde{\Theta}_1|_{\mathcal{B}_1} \equiv \text{Det} \tilde{q}^* \mathcal{P} \otimes (\tilde{q}^* \mathcal{P})|_{p_0 \times \tilde{\mathcal{J}}_1}^{\otimes - (g-1)},
$$

with the projection $p_0 \times \tilde{\mathcal{J}}_1$.
Notice that the line bundle
\[(q^* \mathcal{P}|_{\tau^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_1)}) \cong (\pi_1)^* \mathcal{P},\]
where \(\pi_1: \mathcal{D}_1 \to X_0 \times J_0\) is the projection.

Therefore
\[\tilde{\Theta}_1|_{\mathcal{D}_1} \cong \text{Det } q^* \mathcal{P} \otimes (\pi_1)^* \mathcal{P} \otimes (\pi_2^* \mathcal{P}_{p_0})^{g-(g-1)},\]
and
\[\tilde{\Theta}_1|_{\mathcal{D}_1} \cong \text{Det } q^* \mathcal{P} \otimes (\pi_2^* \mathcal{P}_{x}) \otimes (\pi_2^* \mathcal{P}_{p_0})^{g-(g-1)}.\]

Then
\[\tau^*(\tilde{\Theta}_1|_{\mathcal{D}_1}) \cong \tau^*(\text{Det } q^* \mathcal{P} \otimes (\pi_2^* \mathcal{P}_{x}) \otimes (\pi_2^* \mathcal{P}_{p_0})^{g-(g-1)}),\]
\[(\ref{equation:4.35})\]
\[\cong \tau^*((\text{Det } q^* \mathcal{P} \otimes (\pi_2^* \mathcal{P}_{x}) \otimes (\pi_1^* \sigma_{X_0}(-z)) \otimes ((\pi_2^* \mathcal{P}_{p_0}))^{g-(g-1)} \otimes (\pi_1^* \sigma_{X_0}(-p_0))^{g-(g-1)}), \quad \text{(2) and (3) in remark (4.15)})\]
\[\cong (\text{Det } \mathcal{P} \otimes (\pi_2^* \mathcal{P}_{x}) \otimes (\pi_1^* \sigma_{X_0}(-z)) \otimes ((\pi_2^* \mathcal{P}_{p_0}))^{g-(g-1)} \otimes (\pi_1^* \sigma_{X_0}(-p_0))^{g-(g-1)}), \quad \text{(4) in remark (4.15)})\]
\[\cong (\text{Det } q^* \mathcal{P} \otimes (\pi_2^* \mathcal{P}_{x}) \otimes (\pi_1^* \sigma_{X_0}(-z)) \otimes ((\pi_2^* \mathcal{P}_{p_0}))^{g-(g-1)} \otimes (\pi_1^* \sigma_{X_0}(-p_0))^{g-(g-1)}), \quad \text{(4.34)})\]
\[\cong \tilde{\Theta}_1|_{\mathcal{D}_1} \otimes (\pi_1^* \sigma_{X_0}(-p_0))^{g-(g-1)} \quad \text{(4.35)}\]

Therefore, over \(B_1^0 := X_0 \setminus \{p_0\}\), we have
\[\tau^*(\tilde{\Theta}_1|_{\mathcal{D}_1}) \cong \tilde{\Theta}_1|_{\mathcal{D}_1}\]
\[(\ref{equation:4.36})\]

Since \(\mathcal{D}_1 \cap \mathcal{D}_1' = \emptyset\), by lemma 3.19, it follows that the line bundle \(\tilde{\Theta}_1\) descends to the algebraic space \(\mathcal{J}_1\). Since \(\tilde{\Theta}_1\) is ample relative to \(\mathcal{J}_1 \to B_1^0\) and \(\mathcal{J}_1 \to \mathcal{J}_1\) is a finite map, the map \(\mathcal{J}_1 \to B_1^0\) is projective. (Proposition 3.18). This proves (1).

For \(x \neq z\), the fiber \(\tilde{\mathcal{F}}_1^{-1}(x) = P(\mathcal{P}_x \oplus \mathcal{P}_z)\) contains following two disjoint divisors

1. \(\mathcal{D}_{1,x}\) fibre of \(\tilde{\mathcal{F}}_1 : \mathcal{D}_1 \to X_0\) over \(x\).
2. \(\mathcal{D}_{1,x}'\) fibre of \(\tilde{\mathcal{F}}_1' : \mathcal{D}_1' \to X_0\) over \(x\).

The restriction of the isomorphism \(\tau : \mathcal{D}_1 \to \mathcal{D}_1'\) induces an isomorphism \(\tau_x : \mathcal{D}_{1,x} \to \mathcal{D}_{1,x}'\). The fibre \(\mathcal{J}_{1,x}\) of \(\mathcal{J}_1 \to X_0\) over the point \(x\) is the quotient of \(P(\mathcal{P}_x \oplus \mathcal{P}_z)\) by the identification \(\tau_x\). Using (3.1), it is not difficult to see that there is a family of rank 1 torsion-free sheaves of degree 0 over \(\mathcal{X}(x)\) parametrized by \(\mathcal{J}_{1,x}\). In other words, \(\tilde{\mathcal{F}}_1^{-1}(x) \cong \mathcal{J}_{1,x}\).

For \(x = z\), \(\tilde{\mathcal{F}}_1^{-1}(x) = P(\mathcal{P}_x \oplus \mathcal{P}_z) \cong J_0 \times P^1\). It has two disjoint sections \(\mathcal{D}_{1,z}\) and \(\mathcal{D}_{1,z}'\) which are the fibres of \(\tilde{\mathcal{F}}_1 : \mathcal{D}_1 \to X_0\) and \(\tilde{\mathcal{F}}_1' : \mathcal{D}_1' \to X_0\) over the point \(z\). The restriction of the isomorphism \(\tau\) induces an isomorphism \(\mathcal{D}_{1,z} \to \mathcal{D}_{1,z}'\) which maps \(L \to L \oplus O_{X_0}(z - z) = L\). Therefore the fiber \(\tilde{\mathcal{F}}_1^{-1}(z) \cong J_0 \times R\), where \(R\) is the rational nodal curve constructed by identifying the two points of the projective line. \(\square\)
5. Specialisation of the compactified Jacobian of a nodal curve with \( k(>1) \) nodes

Let us begin by choosing a point \((z_1, \ldots, z_k) \in X_0^k\) such that \(z_i\)'s are pairwise distinct. Generalizing the construction in the previous section, we will now construct \( \mathcal{J}_k \), a flat family over an open set of \( X_0^k \) containing \((z_1, \ldots, z_k)\) such that the fibre over a "general" point \((x_1, \ldots, x_k) \in X_k\) is isomorphic to \( \mathcal{J}_{(x_1, \ldots, x_k)} \), the compactified Jacobian of the nodal curve \( X(x_1, \ldots, x_k) \) (see 2) and the fiber over \((z_1, \ldots, z_k)\) is isomorphic to \( J_0 \times R \times \cdots \times R \).

5.1. Construction of the total space. Let \( p_{i,k+1} : X_k^i \times J_0 \rightarrow X_0 \times J_0 \) denote the projection to the product of \( i\)-th copy of \( X_0 \) and \( J_0 \) and \( p_{k+1} : X_k^i \times J_0 \rightarrow J_0 \) denote the projection to \( J_0 \).

Convention: For any vector bundle \( E \) over a variety \( V \), we denote by \( \mathbb{P}(E) \) the projective bundle over \( V \) of quotient line bundles of \( E \). Given a quotient line bundle \( E \rightarrow L \), we denote by \( \mathbb{P}(L) \) the section of \( \mathbb{P}(E) \rightarrow V \) determined by the quotient \( E \rightarrow L \).

Let \( \mathcal{P} \) be a Poincare bundle over \( X_0 \times J_0 \). For every integer \( 1 \leq i \leq k \), we define a projective bundle

\[
\mathbb{P}_i := \mathbb{P}(p_{i,k+1}^* \mathcal{P} \oplus p_{k+1}^* \mathcal{P}_{z_i})
\]
over \( X_0^k \times J_0 \). We define

\[
\mathcal{F}_k := \mathbb{P}_1 \times (X_0^k \times J_0) \mathbb{P}_{i-1} \times (X_0^k \times J_0) \mathbb{P}(p_{i,k+1}^* \mathcal{P}) \times (X_0^k \times J_0) \mathbb{P}_{i+1} \times (X_0^k \times J_0) \cdots \times (X_0^k \times J_0) \mathbb{P}_k,
\]

(5.1)

Remark 5.2. The variety \( \mathcal{F}_k \) parametrises tuples \((x_1, \ldots, x_k, M_1, M_2, \ldots, M_k) \rightarrow L_j\), where

(1) \( x_1, \ldots, x_k \) are points of \( X_0 \),
(2) \( M \) is a line bundle of degree 0 over \( X_0 \),
(3) \( M_1, \ldots, M_k \rightarrow L_j \) is a 1-dimensional quotient for every \( i = 1, \ldots, k \).

5.3. \( k \)-pairs of natural divisors on \( \mathcal{F}_k \). For each \( i \geq 1 \), we define

\[
\mathcal{D}_i := \mathbb{P}_1 \times (X_0^k \times J_0) \cdots \times (X_0^k \times J_0) \mathbb{P}_{i-1} \times (X_0^k \times J_0) \mathbb{P}(p_{i,k+1}^* \mathcal{P}) \times (X_0^k \times J_0) \mathbb{P}_{i+1} \times (X_0^k \times J_0) \cdots \times (X_0^k \times J_0) \mathbb{P}_k,
\]

and

\[
\mathcal{D}_i' := \mathbb{P}_1 \times (X_0^k \times J_0) \cdots \times (X_0^k \times J_0) \mathbb{P}_{i-1} \times (X_0^k \times J_0) \mathbb{P}(p_{k+1}^* \mathcal{P}_{z_i}) \times (X_0^k \times J_0) \mathbb{P}_{i+1} \times (X_0^k \times J_0) \cdots \times (X_0^k \times J_0) \mathbb{P}_k,
\]

(5.2)

(5.3)

Lemma 5.4. \( \mathcal{D}_i \cap \mathcal{D}_i' = \emptyset \) for all \( 1 \leq i \leq k \).

Proof. Since the two natural sections of \( \mathbb{P}(p_{i,k+1}^* \mathcal{P} \oplus p_{k+1}^* \mathcal{P}_{z_i}) \) given by the two natural quotients \( p_{i,k+1}^* \mathcal{P} \oplus p_{k+1}^* \mathcal{P}_{z_i} \) and \( p_{i,k+1}^* \mathcal{P} \oplus p_{k+1}^* \mathcal{P}_{z_i} \) are disjoint therefore \( \mathcal{D}_i \cap \mathcal{D}_i' = \emptyset \).

Remark 5.5. The variety \( \mathcal{D}_i \) parametrises tuples \((x_1, \ldots, x_k, M_1, M_2, \ldots, M_k) \rightarrow L_j\), where \( L_1 = M_1 \) and \( L_j \) is any quotient for \( j \neq i \). The variety \( \mathcal{D}_i' \) parametrises tuples \((x_1, \ldots, x_k, M_1, M_2, \ldots, M_k) \rightarrow L_j\), where \( L_1 = M_2 \) and \( L_j \) is any quotient for \( j \neq i \). We see that the varieties \( \mathcal{F}_k, \mathcal{D}_i \) and \( \mathcal{D}_i' \) have universal properties because they parametrise the tuples, described above.

5.6. Twisted isomorphisms \( \tau_i : \mathcal{D}_i \rightarrow \mathcal{D}_i' \) for every \( i = 1, \ldots, k \). First let us define a new base which is the following open subset of \( X_0^k \).

\[
B_k := X_0^k \setminus \bigcup_{1 \leq i, j \leq k} (\Delta_{i,j} \cup \Psi_{i,j}),
\]

(5.4)
where
\[
\Delta_{i,j} := \{(x_1, x_2, \ldots, x_k) : x_j \in X_0 \text{ and } x_i = x_j\},
\]
\[
\Psi_{i,j} := \{(x_1, x_2, \ldots, x_k) : x_i \in X_0 \text{ and } x_i = x_j\}.
\]

Let us consider restrict \( \mathcal{F}_k \) over \( B_k \), obtained by the base change by \( B_k \). Since these \( \mathcal{k} \)-pairs of irreducible smooth divisors \( \{D, D_i\}_{i=1}^k \) on \( \mathcal{F}_k \) are flat over \( X_0^k \), also flat over \( B_k \). From here onwards, we work over \( B_k \) and with the base change of the families.

Consider the following line bundle
\[
\mathcal{P}' := q^* \mathcal{P} \oplus \bar{r}_i^* O(-\Delta) \oplus \bar{s}^* O(z_i) \text{ over } X_0 \times \mathcal{F}_k,
\] (5.5)

where the maps \( \bar{q}, \bar{r}_i, \bar{s} \) for \( i = 1, \ldots, k \) are composites of projection morphisms described as follows.

\[
\begin{array}{ccc}
X_0 \times \mathcal{F}_k & \xrightarrow{\bar{r}_i} & X_0 \times X_0^k \times J_0 \\
\downarrow I \times \bar{\pi}_k & & \downarrow s_i \\
X_0 \times X_0 \times J_0 & \xrightarrow{q} & X_0 \times J_0 \\
\downarrow & & \downarrow r_j \\
X_0 \times X_0 & & X_0
\end{array}
\] (5.6)

(1) \( (x, x_1, \ldots, x_k, L) \mapsto (x, L), (x, x_1, \ldots, x_k, L) \mapsto (x, x_i) \),
(2) \( (x, x_1, \ldots, x_k, L) \mapsto x_i \),
(3) \( (x, x_1, \ldots, x_k, L) \mapsto x \),
(4) \( \bar{\pi}_k : \mathcal{F}_k \rightarrow X_0^k \times J_0 \) denotes the natural projection map,
(5) \( \bar{q} := q \circ (I \times \bar{\pi}_k), \bar{r}_i := r_i \circ (I \times \bar{\pi}_k), \bar{s}_i := s_i \circ (I \times \bar{\pi}_k) \) and \( \bar{s} := s \circ (I \times \bar{\pi}_k) \).

Lemma 5.7. \hspace{1cm} (1) \( \bar{r}_j^{-1}(\Delta) \cap (X_0 \times \mathcal{F}_k) \equiv \mathcal{F}_k \) for all \( i, j \).
(2) \( \bar{r}_j^{-1}(\Delta) \cap (X_0 \times \mathcal{D}_i) \equiv \mathcal{D}_i \) for all \( i, j \).

Proof. Similar to the proof of Lemma 4.7. \hspace{1cm} \( \square \)

Since \( \mathcal{F}_k \) is a fibre product of \( \mathcal{k} \) projective bundles \( \mathcal{P}_i \) for \( i = 1, \ldots, k \), therefore there are \( \mathcal{k} \) universal quotients. Let us denote them by \( \mathcal{L}_1, \ldots, \mathcal{L}_k \).

Lemma 5.8. \hspace{1cm} (1) There are \( \mathcal{k} \) isomorphisms \( \tau_i : \mathcal{D}_i \rightarrow \mathcal{D}'_i \) for \( i = 1, \cdots, k \), which on \( \text{Spec} \mathbb{C} \)-valued points can be described as follows:
\[
\tau_i : \mathcal{D}_i \rightarrow \mathcal{D}'_i
\]
\[
(L, Q_1, \ldots, Q_k) \rightarrow (L' := L \oplus \Theta(z_i - x_i), Q'_1, \ldots, Q'_k),
\] (5.7)

where \( Q'_t := Q_t \), for \( t \neq i \), \( Q_i \) is the first projection \( L_{x_i} \oplus L_{z_i} \rightarrow L_{x_i} \) and \( Q'_i \) is the second projection \( L'_{x_i} \oplus L'_{z_i} \rightarrow L'_{z_i} \).
(2) these automorphisms are compatible in the following sense

\[
\begin{array}{c}
\mathcal{D}_i \cap \mathcal{D}_j \xrightarrow{r_j} \mathcal{D}_i' \cap \mathcal{D}_j' \\
\downarrow r_j \quad \downarrow r_j \\
\mathcal{D}_i \cap \mathcal{D}_j' \xrightarrow{r_j} \mathcal{D}_i' \cap \mathcal{D}_j'
\end{array}
\]

Proof. The restrictions of universal quotients \( (L_j)_{j=1}^k \) on the divisors \( \mathcal{D}_i \) can be expressed as the following collection of \( k \) universal quotients.

For \( j \neq i \),

\[
(q^* \mathcal{P})|_{\tilde{r}^{-1}_i(\Delta) \cap \mathcal{D}_i} + (q^* \mathcal{P})|_{\tilde{z}_i \times \mathcal{D}_i} \to L_j|_{\mathcal{D}_i} \quad \text{for } j \neq i. \tag{5.8}
\]

Otherwise,

\[
(q^* \mathcal{P})|_{\tilde{r}^{-1}_i(\Delta) \cap \mathcal{D}_i} + (q^* \mathcal{P})|_{\tilde{z}_i \times \mathcal{D}_i} \to (q^* \mathcal{P})|_{\tilde{r}^{-1}_i(\Delta) \cap \mathcal{D}_i}. \tag{5.9}
\]

We modify the quotients on \( \mathcal{D}_i \) in the following way.

\[
\mathcal{P}_i'|_{\tilde{r}^{-1}_i(\Delta) \cap \mathcal{D}_i} \oplus \mathcal{P}_i'|_{\tilde{z}_i \times \mathcal{D}_i} \to \mathcal{L}_j' \quad \text{where } j \neq i, \tag{5.10}
\]

and for \( j = i \),

\[
\mathcal{P}_i'|_{\tilde{r}^{-1}_i(\Delta) \cap \mathcal{D}_i} \oplus \mathcal{P}_i'|_{\tilde{z}_i \times \mathcal{D}_i} \to \mathcal{P}_i'|_{\tilde{z}_i \times \mathcal{D}_i}, \tag{5.11}
\]

where \( \mathcal{L}_j' \) is a quotient line bundle defined in the following way.

First, let us denote by \( U \) the complement of the of divisors \( \tilde{r}^{-1}_i(\Delta) \) and \( \tilde{s}^{-1}(z) \) in \( \tilde{\mathcal{F}}_k \). The restriction of the line bundles \( \mathcal{P}_i' \) and \( q^* \mathcal{P} \) on \( \mathcal{D}_i \) are naturally isomorphic. To see this notice that any constant function on \( U \) defines a section in \( \Gamma(U, (\tilde{r}_i^* \mathcal{O}(-\Delta) \oplus \tilde{s}^* \mathcal{O}(z_i))) \), the set of global sections of the line bundle \( \tilde{r}_i^* \mathcal{O}(-\Delta) \oplus \tilde{s}^* \mathcal{O}(z_i) \). We fix any such constant section and using it we can identify the restriction of \( \mathcal{P}_i' \) and \( q^* \mathcal{P} \) on \( \mathcal{D}_i \). Therefore, we can define

\[
\mathcal{L}_j':= L_j|_{\mathcal{D}_i}. \tag{5.12}
\]

By the universal property (remark 5.5) of \( \mathcal{P}_i' \) the modified quotients (5.10) and (5.11) induce an isomorphism

\[
\tau_j : \mathcal{D}_i \to \mathcal{D}_i'. \tag{5.13}
\]

Since \( \mathcal{L}_j' = L_j|_{\mathcal{D}_i} \) for every \( j \neq i \), we have

\[
\tau_j^*(\mathcal{L}_j|_{\mathcal{D}_i}) = L_j|_{\mathcal{D}_i}. \tag{5.14}
\]

Therefore, it follows that there are \( k \) isomorphisms \( \tau_1 \) which on \( \mathbb{C} \)-valued points has the desired properties. The second assertion follows from straightforward set-theoretic checking. \( \square \)

5.9. Construction of the quotient space and its singularities. We construct the quotient space \( \mathcal{F}_k \) inductively following 4.10. Repeated application of Theorem 3.17 shows that \( \mathcal{F}_k \) is an algebraic space.

Set \( \mathcal{F}_0 := \tilde{\mathcal{F}}_k, \mathcal{D}_0' := \mathcal{D}_1 \) and \( D_i^0 := D_i' \) for every \( 1 \leq i \leq k \). After having defined \( \mathcal{F}_{i-1} \), we define

\[
\mathcal{F}_j := \frac{\mathcal{F}_{j-1}}{\mathcal{D}_j^{-1} / \sim_{\tau_j} \mathcal{D}_j'^{-1}}, \tag{5.15}
\]

where \( \mathcal{D}_j^{-1} \) and \( \mathcal{D}_j'^{-1} \) are images of \( \mathcal{D}_j \) and \( \mathcal{D}_j' \) in \( \mathcal{F}_{j-1} \).
Lemma 5.10. $\varphi_j^{-1} \cap \varphi_{j-1}^{-1} = \emptyset$ for every $1 \leq j \leq k$.

Proof. From lemma 5.4 it follows that the statement holds for $j = 1$. Let us check it for $j = 2$ and for this purpose we can assume that $k = 2$. Then the configuration of the divisors $D_1, D'_1, D_2, D'_2$ is the following

\[
\begin{array}{ccc}
\varphi_1 & \varphi_2 \\
D_1 & D_2 & D'_1 \\
\varphi'_1 & \varphi'_2 \\
D'_2 & & \\
\end{array}
\]

Let us fix any $(x_1, x_2) \in X_0 \times X_0$. It will suffice to check the following

1. if $(L, q_1, q_2) \in \varphi_1 \cap \varphi_2$ and $(M, p_1, p_2) \in \varphi'_1 \cap \varphi'_2$ then $\tau_1(L, q_1, q_2) \neq (M, p_1, p_2)$,
2. if $(L, q_1, q_2) \in \varphi_1 \cap \varphi'_2$ and $(M, p_1, p_2) \in \varphi'_1 \cap \varphi_2$ then $\tau_1(L, q_1, q_2) \neq (M, p_1, p_2)$.

It is enough to check one of them because the proofs are the same. Let us check (1). Notice that $q_1 : L_{x_1} \oplus L_{z_1} \to L_{x_1}$ and $q_2 : L_{x_2} \oplus L_{z_2} \to L_{x_2}$ both are first projections. Therefore $\tau_1(L, q_1, q_2) := (L' := L(z_1 - x_1), q'_1 : L'_{x_1} \oplus L'_{z_1} \to L'_{x_1}, q'_2 : L'_{x_2} \oplus L'_{z_2} \to L'_{x_2})$, where $q'_1$ is the second projection and $q'_2$ is the first projection. Now notice that $p_1 : M_{x_1} \oplus M_{z_1} \to M_{z_1}$ and $p_2 : M_{x_2} \oplus M_{z_2} \to M_{z_2}$ are both second projections. Since $q'_2$ is the first projection and $p_2$ is the second projection, therefore $\tau_1(L, q_1, q_2) \neq (M, p_1, p_2)$.

The proof for the general $j$ is similar. \hfill \square

Since the isomorphisms $\tau_i$ commutes with the projection onto $B_k$ the morphism $\tilde{f}_k : \tilde{\mathcal{X}}_k \to B_k$ descends to a morphism $\tilde{f}_k : \mathcal{X}_k \to B_k$.

Proposition 5.11. The quotient space $\mathcal{X}_k$ is an algebraic space and has only $k$-th product of normal crossing singularities.

Proof. From Theorem 4.10, $\mathcal{X}_1$ has only normal crossing singularity along the image of the divisor $\varphi_1$ and smooth elsewhere. Consider the following commutative diagram

\[
\begin{array}{cccc}
Z := \varphi_2^{1} \sqcup \varphi_2^{1} & \longrightarrow & \mathcal{X}_1 \\
\downarrow f_2 & & & \downarrow \ \\
V := \varphi_2^{1} & \longrightarrow & \mathcal{X}_2 := \tilde{\mathcal{X}}_2
\end{array}
\] (5.16)

It is enough to check the singularity of $\mathcal{X}_2$ along the codimension 2 subspace $\varphi_2^{1} \cap \varphi_1^{1}$. Let $v_2' \in \varphi_2^{1} \cap \varphi_1^{1}$ and $v_2$ and $v_2'$ are the two preimages under $\tau_2$. Then we have the following
normal crossing singularities along \( V \). The bundle on \( i \)-subspace \( J \) follows that \( k_{\theta_j \cap V} \). Then we have the following

\[
\begin{array}{c}
0 \to \hat{\theta}_{j_1 \cap V} \\
\downarrow \equiv \Upsilon \to \hat{\theta}_{j_1 \cap V} \oplus \hat{\theta}_{j_1 \cap V} \\
\equiv \Upsilon \to \hat{\theta}_{j_1 \cap V} \\
0 \to \hat{\theta}_{j_1 \cap V}
\end{array}
\]

The morphism is given by

\[
\frac{k([x_1, x_2])}{x_1 \cdot x_2} \left[ [x_3, x_5, \ldots, x_n] \right] \oplus \frac{k([x_1, x_2])}{x_1 \cdot x_2} \left[ [x_4, x_5, \ldots, x_n] \right] \to \frac{k([x_1, x_2])}{x_1 \cdot x_2} \left[ [x_5, \ldots, x_n] \right]
\]

\((f, g) \mapsto f(\text{mod } x_3) - g(\text{mod } x_4)\)

It follows that \( \hat{\theta}_{j_1 \cap V} \equiv \frac{k([x_1, x_2, x_3])}{x_1 x_2 x_3} \). Therefore the algebraic space \( J_2 \) has only product of two normal crossing singularities along \( V \).

At the \( i \)-th step we have

\[
Z := \mathcal{D}_{i-1} \bigcup \mathcal{D}_{i-1}^* \to J_{i-1} \quad \begin{array}{c}
\tau_i \\
\end{array}
\]

\[
V := \mathcal{D}_{i-1}^* \to J_i := \frac{J_{i-1}}{\tau_i}
\]

Although it is exactly a similar calculation we will describe the singularity of \( J_i \) along the codimension \( i \) subspace \( \mathcal{D}_{i-1}^* \mathcal{D}_{i-1}^* \mathcal{D}_{i-1}^* \). Let \( v_i' \in \mathcal{D}_{i-1}^* \mathcal{D}_{i-1}^* \mathcal{D}_{i-1}^* \) and \( v_i \) and \( v_i' \) are the two preimages under \( \tau_i \). Then we have the following

\[
0 \to \hat{\theta}_{j_i \cap V} \\
\downarrow \equiv \Upsilon \to \hat{\theta}_{j_i \cap V} \oplus \hat{\theta}_{j_i \cap V} \\
\equiv \Upsilon \to \hat{\theta}_{j_i \cap V} \\
0 \to \hat{\theta}_{j_i \cap V}
\]

\[
\begin{array}{c}
0 \to \hat{\theta}_{j_i \cap V} \quad \overset{R \otimes k([x_{2i-1}])}{\to} \quad \hat{\theta}_{j_i \cap V} \\
\end{array}
\]

\[
\begin{array}{c}
0 \to \hat{\theta}_{j_i \cap V} \quad \overset{R \otimes k([x_{2i-1}])}{\to} \\
\end{array}
\]

\[
0 \to \hat{\theta}_{j_i \cap V} \quad \overset{R \otimes k([x_{2i-1}])}{\to} \\
\]

\[
0 \to \hat{\theta}_{j_i \cap V} \quad \overset{R \otimes k([x_{2i-1}])}{\to} \\
\]

\[
0 \to \hat{\theta}_{j_i \cap V} \quad \overset{R \otimes k([x_{2i-1}])}{\to} \\
\]

where \( R := \frac{k([x_1, x_2, \ldots, x_{2i-3}, x_{2i-2}])}{x_1 x_2 \ldots x_{2i-3}} \left[ [x_{2i+1}, \ldots, x_n] \right] \) and the morphism

\[
R \otimes k([x_{2i-1}]) \to R \otimes k([x_{2i-1}]) \to R
\]

is given by \((f, g) \mapsto f(\text{mod } x_{2(i-1)}) - g(\text{mod } x_{2i}).\)

Hence

\[
\hat{\theta}_{j_i \cap V} \equiv \frac{k([x_1, x_2, \ldots, x_{2i-1}, x_{2i}])}{x_1 x_2 \ldots x_{2i-1} x_{2i}} \left[ [x_{2i+1}, \ldots, x_n] \right].
\]

Therefore the algebraic space \( J_i \) has only product of \( i \)-many normal crossing singularities along \( V \). \[ \square \]

5.12. Theta bundle on \( J_k \) and its relative ampleness.

**Definition 5.13.** We define a line bundle

\[
\Theta_k := \text{Det } \tilde{q} \ast \mathcal{P} \ast \left( \otimes_{i=1}^k \mathcal{L}_i \right) \ast \left( \otimes_{i=1}^{g-1} \mathcal{P} \right) \left| p_k \ast J_k \right.
\]

\[
\text{(5.19)}
\]

Notice that the line bundle \( \Theta_k \) is isomorphic to \( \overline{p_{k+1}} \ast \text{Det } \mathcal{P} \ast \left( \otimes_{i=1}^k \mathcal{L}_i \right) \ast \overline{p_{k+1}} \ast \mathcal{P} \).
Proposition 5.14. The line bundle $\tilde{\Theta}_k$ is relatively ample for the morphism $\tilde{f}_k : \tilde{F}_k \to X_0^k$.

Proof. Let $\tilde{x} := (x_1, \ldots, x_k) \in B_k$ be any point. The restriction of $\tilde{\Theta}_k$ to the fiber $\tilde{f}_k^{-1}(\tilde{x})$ is

$$\tilde{\Theta}_k|_{\tilde{f}_k^{-1}(\tilde{x})} = \tilde{p}_{k+1, x}^* \det \otimes \tilde{p}_{k+1, x}^* \mathcal{O}_{\tilde{F}_0} \otimes (\otimes_{i=1}^k \mathcal{L}_i)|_{\tilde{f}_k^{-1}(\tilde{x})}$$

$$\equiv (\tilde{p}_{k+1, x}^* \det \otimes \tilde{p}_{k+1, x}^* \mathcal{O}_{\tilde{F}_0} \otimes (\otimes_{i=1}^k \mathcal{L}_i)|_{\tilde{f}_k^{-1}(\tilde{x})} \otimes \tilde{p}_{k+1, x}^* \mathcal{P}_0^{k-1}$$

$$\equiv \tilde{p}_{k+1, x}^* \Theta_0 \otimes (\otimes_{i=1}^k \Theta_{(i, 1)}(1)) \otimes \tilde{p}_{k+1, x}^* \mathcal{P}_0^{k-1},$$

where

1. $\mathcal{P}_i(\tilde{x})$ denotes the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_x)$ over $f_0$.
2. $\Theta_{p_i(\tilde{x})}(1)$ denotes the pullback of the tautological bundle of the projective bundle $\mathbb{P}(\tilde{x})$ by the natural projection morphism $\mathbb{P}_i(\tilde{x}) \times_{f_i} \cdots \times_{f_j} \mathbb{P}_k(\tilde{x}) \to \mathbb{P}_i(\tilde{x}),$ and
3. $\tilde{p}_{k+1, x}$ denotes the natural projection $\tilde{f}_k^{-1}(\tilde{x}) \to f_0$.

Notice that the last equality holds because $\Theta_{p_i(\tilde{x})}(1) \equiv \mathcal{L}_i|_{\tilde{f}_k^{-1}(\tilde{x})}$ for every $i = 1, \ldots, k$. Now consider the Segre-embedding

$$\mathbb{P}_1(\tilde{x}) \times \cdots \times \mathbb{P}_k(\tilde{x}) \hookrightarrow \mathbb{P}(\otimes_{i=1}^k (\mathcal{P}_i \oplus \mathcal{O}_x))$$

Notice that

$$(\tilde{p}_{k+1, x})^* (\otimes_{i=1}^k \mathcal{P}_i \oplus \mathcal{O}_x) \equiv (\otimes_{i=1}^k \mathcal{P}_i \oplus \mathcal{O}_x)$$

Now every direct summand of the above equation is $\Theta_0 \otimes (\otimes_{i=1}^k \mathcal{P}_i)$, where $p_i \in \{x_i, z_i\}$ for all $i \in \{1, \ldots, k\}$. Since every direct summand is ample, the vector bundle is ample. Also notice that the line bundle $\mathcal{P}_0$ is algebraically equivalent to the trivial line bundle. Therefore the line bundle $\tilde{p}_{k+1, x}^* \Theta_0 \otimes (\otimes_{i=1}^k \mathcal{P}_i)$ is ample. Hence its pullback $\tilde{\Theta}_k$ is an ample line bundle on $\mathbb{P}(\mathcal{P}_0 \oplus \mathcal{O}_x)$ and $\mathbb{P}(\mathcal{P}_0 \oplus \mathcal{O}_z)$. Since the morphism $\tilde{f}_k \to X_0^k$ is projective and the line bundle $\tilde{\Theta}_k$ is ample on every fiber of the morphism $\tilde{f}_k$ is relatively ample. \hfill \Box

5.15. Descent of the line bundle $\tilde{\Theta}_k$.

Lemma 5.16. (1) $\tau_i^* (\tilde{p}_{k+1, x}^* \mathcal{P}_x) \equiv \tilde{p}_{k+1, x}^* \mathcal{P}_x \otimes s_i^* \Theta_{X_0}(-x)$, for any point $x \in X_0$, (2) $\tau_i^* ((\det \tilde{q}^* \mathcal{P})|_{\tilde{F}_0}) \equiv \tilde{p}_{k+1, x}^* ((\tilde{q}^* \mathcal{P}) \otimes (\tilde{q}^* \mathcal{P})|_{\tilde{F}_0^{-1}(\tilde{f}_i \cap \tilde{F}_0 \cap (\tilde{F}_0 \otimes \mathcal{O}_{X_0} (-z_i))) \otimes s_i^* \Theta_{X_0} (z_i))$

Proof. Similar to the proof of remark 4.15. \hfill \Box

We define a smaller open set

$$B_0^k := B_k \setminus \{(x_1, \ldots, x_k) | x_i = p_i \text{ for some } i \in \{1, \ldots, k\}\}$$

(5.22)

Theorem 5.17. (1) The morphism $\tilde{f}_k : \tilde{F}_k \to B_0^k$ is projective. (2) The fibers of the morphism $\tilde{f}_k$ can be described as follows.

$$\tilde{f}_k^{-1}(x_1, \ldots, x_k) \equiv \begin{cases} \tau_{x_1} \cdots \tau_{x_k} & \text{if } x_i \neq z_i \text{ for all } 1 \leq i \leq k \\ I_0 \times \mathbb{R}^k & \text{if } x_i = z_i \text{ for all } 1 \leq i \leq k \end{cases}$$

(5.23)

Proof. First we claim that the line bundle $\tilde{\Theta}_k$ is invariant under the isomorphisms $\tau_i$ for all $i = 1, \ldots, k$. Assuming the claim, we see that $\tilde{\Theta}_k$ descends at each of the $k$-steps of the quotient construction. Let us
denote the descended line bundle on \( J_k \) by \( \Theta_k \). Since \( \Theta_k \) is a relatively ample line bundle for the proper morphism \( f_k : J_k \to B_k^0 \), and \( \nu_k : J_k \to J_k \) is a finite morphism, the descended line bundle \( \Theta_k \) is also relatively ample for the morphism \( f_k : J_k \to B_k^0 \). Therefore \( f_k \) is a projective morphism.

The proof of the claim is similar to the proof of Theorem 4.19, except here we have to use (5.14). For every \( i = 1, \ldots, k \), we have

\[
\tilde{\Theta}_k|_{\mathcal{G}_i} \equiv (\det \tilde{q}^* \mathcal{P})|_{\mathcal{G}_i} \otimes (\otimes \circ \tilde{L}_{j \neq i} (\mathcal{G}_i)) \otimes \tilde{L}_1|_{\mathcal{G}_i} \otimes (\tilde{q}^* \mathcal{P}^\otimes (g-1))|_{p_0 \times \mathcal{G}_i} \\
\equiv \tilde{p}_{k+1}^-* (\det \mathcal{P}) \otimes (\otimes \circ \tilde{L}_{j \neq i} (\mathcal{G}_i)) \otimes \tilde{L}_1|_{\mathcal{G}_i} \otimes (\tilde{q}^* \mathcal{P})|_{\tilde{r}_i^{-1}(\Delta) \cap (X_0 \times \mathcal{G}_i)} \otimes \tilde{p}_{k+1}^-* (\mathcal{P}^\otimes (g-1))
\]

and

\[
\tilde{\Theta}_k|_{\mathcal{G}_i} \equiv (\det \tilde{q}^* \mathcal{P})|_{\mathcal{G}_i} \otimes (\otimes \circ \tilde{L}_{j \neq i} (\mathcal{G}_i)) \otimes \tilde{L}_1|_{\mathcal{G}_i} \otimes (\tilde{q}^* \mathcal{P})|_{\tilde{z}_i \times \mathcal{G}_i} \otimes \tilde{p}_{k+1}^-* (\mathcal{P}^\otimes (g-1))
\]

Therefore,

\[
\tau_1^* \tilde{\Theta}_k|_{\mathcal{G}_i} \equiv \tau_1^* (\det \tilde{q}^* \mathcal{P})|_{\mathcal{G}_i} \otimes (\otimes \circ \tilde{L}_{j \neq i} (\mathcal{G}_i)) \otimes \tilde{L}_1|_{\mathcal{G}_i} \otimes (\tilde{q}^* \mathcal{P})|_{\tilde{z}_i \times \mathcal{G}_i} \otimes \tilde{p}_{k+1}^-* (\mathcal{P}^\otimes (g-1))
\]

Therefore, over \( B_k^0 \), we have

\[
\tau_1^* \tilde{\Theta}_k|_{\mathcal{G}_i} \equiv \tilde{\Theta}_k|_{\mathcal{G}_i} 
\]

The proof of the second statement is similar to the proof of the second statement of Theorem 4.19. □

6. Local triviality of the family of \( J_k \) over \( B_k^0 \)

The main theme of this section is to prove \( J_k \) is a topologically fiber bundle over \( B_k^0 \). As \( f_k \) is not a smooth map, one can’t use Ehresmann fibration Theorem. We apply the The first isotopy lemma of Thom to conclude \( f_k \) is a locally trivial fibration. To do that we need to construct a stratification \( \mathcal{S} \) of \( J_k \) which satisfy Whitney conditions and the restriction of the map \( f_k \) on each stratum is a submersion.

**Definition 6.1.** Let \( M \) be a smooth manifold and \( N \) be a closed subset of \( M \). A collection \( \mathcal{S} := \{ X_\alpha : \alpha \in I, X_\alpha \text{ are locally closed submanifold of } M \} \) is said to be a stratification of \( N \) if \( N \equiv \bigcup_{\alpha \in I} X_\alpha \) and \( \bar{X}_\alpha \setminus X_\alpha = \bigcup X_\beta \), for some \( \beta \in I \) and \( \beta \neq \alpha \).

**6. Whitney conditions.** A stratification \( \mathcal{S} \) of \( N \) is said to be a Whitney stratification if \( \mathcal{S} \) is locally finite and satisfy the following conditions at every point \( x \in N \). Let us choose a pair \( (X_\alpha, X_\beta) \) such that \( X_\beta \subset \bar{X}_\alpha \) and \( x \in X_\beta \).
(1) Condition (a): We say that the pair \((X_a, X_β)\) satisfies Whitney condition \((a)\) at \(x\) if for any sequences \(\{x_n\} \subset X_a\) such that \(\{x_n\}\) converges to \(x\), the sequence \(\{T_{x_n}X_a\}\) of tangent planes of \(X_a\) at \(x_n\) converges to a plane \(\mathcal{F} := \lim T_{x_n}X_a \subset T_x M\) of \(\dim(X_a)\) and \(T_x X_β \subset \mathcal{F}\) where \(T_x X_β\) is the tangent plane of \(X_β\) at \(x\).

(2) Condition (b): The pair \((X_a, X_β)\) satisfies Whitney condition \((b)\) at \(x\) if for any sequences \(\{x_n\} \subset X_a\), \(\{y_n\} \subset X_β\) converging to \(x\), then \(\mathcal{F} \supset \tau\), the limit of the secants joining \(x_n\) and \(y_n\), \(\tau := \lim T_{x_n}X_β\).

Any stratification that satisfies the above conditions is called Whitney stratification. A stratification that satisfies the Condition \((b)\) of Whitney will also satisfy the Condition \((a)\) [13, Lemma 2.2].

The following lemma is well-known and we leave the proof to the reader.

**Lemma 6.3.** Let \(X\) and \(Y\) be two varieties with a given stratification \(A\) and \(B\) which are Whitney stratifications. Then their product stratification will be a Whitney stratification.

### 6.4. Stratification by successive singular loci.

Given a subset \(\{i_1, \ldots, i_r\} \subset \{1, \ldots, k\}\) with \(1 \leq i_1 < \cdots < i_r \leq k\) and a map \(\phi : \{i_1, \ldots, i_r\} \rightarrow \{1, 2\}\) we define a stratification of \(\mathcal{J}_k\) as follows. Recall

\[
\widetilde{\mathcal{J}}_k := \mathbb{P}(p_{i_1,k+1}^* \mathcal{P} + p_{i_1+1,k}^* \mathcal{P}_{z_1}) \times \cdots \times \mathbb{P}(p_{i_r,k+1}^* \mathcal{P} + p_{i_r+1,k}^* \mathcal{P}_{z_1}).
\]

Consider

\[
\mathcal{W}^\phi_{i_1, \ldots, i_r} := \mathbb{P}(p_{i_1,k+1}^* \mathcal{P} + p_{i_1+1,k}^* \mathcal{P}_{z_1}) \times \cdots \times \mathbb{P}(p_{i_r,k+1}^* \mathcal{P} + p_{i_r+1,k}^* \mathcal{P}_{z_1}).
\]

where

\[
\mathcal{W}^\phi_{i_r} := \mathbb{P}(p_{i_r+1,k}^* \mathcal{P}_{z_1}) \quad \text{if} \quad \phi(i_r) = 1
\]

and

\[
\mathcal{W}^\phi_{i_r} := \mathbb{P}(p_{i_r,k+1}^* \mathcal{P}_{z_1}) \quad \text{if} \quad \phi(i_r) = 2.
\]

Define

\[
\mathcal{S}_r := \bigcup_{1 \leq i_1 < \cdots < i_r \leq k, \phi} \mathcal{W}^\phi_{i_1, \ldots, i_r}.
\]

Set \(\mathcal{S}_r := \nu_k(\mathcal{S}_r)\), where \(\nu_k : \mathcal{J}_k \rightarrow \mathcal{J}_k\). We also have

\[
\mathcal{S}_k \subset \mathcal{S}_{k-1} \subset \cdots \subset \mathcal{S}_1 \subset \mathcal{S}_0 := \mathcal{J}_k.
\]

It follows that \(\mathcal{S}_0 = \bigsqcup_{i=0}^k (S_i \setminus S_{i+1})\), where \(S_{k+1} = \emptyset\), the empty set. In particular, we obtain \(\mathcal{S} := \{S_i \setminus S_{i+1} : 0 \leq i \leq k\}\), a stratification of \(\mathcal{J}_k\).

**Remark 6.5.** From Proposition 5.11, it follows that the singular locus of every connected component of \(S_i\) lies in a unique connected component of \(S_{i+1}\) for every \(i = 0, \ldots, k\). In fact, along \(S_i \setminus S_{i+1}\) the variety \(S_0\) has a product of exactly \(i\)-many normal crossing singularities.

**Theorem 6.6.** The stratification \(\mathcal{S}\) on \(\mathcal{J}_k\) is a Whitney stratification.

**Proof.** For any \(p \in \mathcal{J}_k\), a local analytic neighborhood \(U_p\) of \(p\) is homeomorphic to

\[
U_p \cong X_0 \times X_1 \times \cdots \times X_k,
\]
where $X_0 \cong \mathbb{A}^g$, $X_i \cong \text{Spec} \left( \frac{k[x,y]}{(xy)} \right)$ and $g = \dim(\mathcal{J}_k) - k$. It is enough to prove that the restriction of the stratification $S := \{ S_i \setminus S_{i+1} : 0 \leq i \leq k \}$ to $U_p$ is a Whitney stratification on $U_p$. To do that we show the restriction of $S$ to $U_p$ is the product of a Whitney stratification of each $X_i$. Then using Lemma 6.3, it follows that $S$ is a Whitney stratification.

Let us consider the case when $X_0$ is a point. We have $U_p \cong X_1 \times \cdots \times X_k$. The product stratification on $X_1 \times \cdots \times X_k$ is the following. For each $i \geq 1$, the Whitney stratification on $X_i$ is

$$X_i = (X_i \setminus 0_i) \cup 0_i,$$

where $0_i$ is the only singular point of $X_i$. Then the zero dimensional strata on $X_1 \times \cdots \times X_k$ is

$$\mathcal{T}_0 := \{ (0_1, \ldots, 0_k) \}.$$

Let us define $T_1^i := 0_1 \times 0_2 \times \cdots \times X_i \setminus 0_i \times 0_{i+1} \times \cdots \times 0_k$. The one dimensional strata is

$$\mathcal{T}_1 := \bigcup_{1 \leq i \leq k} T_1^i.$$

Similarly a typical $j$-dimensional stratum is

$$T_{j}^{i_1, \ldots, i_j} := 0_1 \times \cdots \times (X_{i_1} \setminus 0_{i_1}) \times 0_{i_1+1} \times \cdots \times (X_{i_j} \setminus 0_{i_j}) \times \cdots \times 0_k,$$

and the $j$-dimensional strata is

$$\mathcal{T}_j := \bigcup_{1 \leq i_1 < \cdots < i_j \leq k} T_{j}^{i_1, \ldots, i_j}.$$

In particular for $j = k$, the $k$-dimensional strata is the following

$$\mathcal{T}_k := (X_1 \setminus 0_1) \times \cdots \times (X_k \setminus 0_k).$$

The product stratification $\mathcal{T} = \{ \mathcal{T}_j : 1 \leq j \leq k \}$ can also be expressed as

$$\mathcal{T}_j = \mathcal{T}_j \setminus \mathcal{T}_{j-1},$$

where

$$\mathcal{T}_j := \bigcup_{1 \leq i_1 < \cdots < i_j \leq k} \mathcal{T}_{j}^{i_1, \ldots, i_j}$$

and

$$\mathcal{T}_{j}^{i_1, \ldots, i_j} = 0_1 \times \cdots \times X_{i_1} \times 0_{i_1+1} \times \cdots \times X_{i_j} \times \cdots \times 0_k.$$

Evidently, $\mathcal{T}_{k-j}$ is the locus of points at which $U_p$ has the product of exactly $j$-many normal crossings singularities. Therefore from the Remark 6.5 it follows that $S_{j \mid U_p} = \mathcal{T}_{k-j}$. Hence $S$ is a Whitney stratification. The general case will be followed by replacing $0_i$ by $\mathbb{A}^g \times 0_i$ for $1 \leq i \leq k$.

\[ \square \]

**Theorem 6.7.**

1. The morphism $f_k : \mathcal{J}_k \to B^0_k$ is topologically locally trivial.

2. $\mathcal{R}^i f_k_* \mathcal{Q}$ forms a VMHS over $B^0_k$. 

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Proof. There is a relatively ample line bundle $\Theta_k$ on the projective variety $\mathcal{J}_k$. We can replace $\Theta_k$ by its sufficiently large power so that it is relatively very ample. Therefore we have an embedding:

$$\mathcal{J}_k \hookrightarrow \mathbb{P}(H^0(\Theta_k))$$

(6.1)

The morphism $\mathbb{P}(H^0(\Theta_k)) \to B_k^0$ is a submersion and $\mathcal{J}_k$ is a closed subset of $\mathbb{P}(H^0(\Theta_k))$ which has a Whitney stratification given by $\mathcal{J}_k = \bigcup_{i=0}^{k}(S_i \setminus S_{i+1})$ such that the projection from every strata $S_i \setminus S_{i+1} \to B_k^0$ is a submersion. Therefore from Thom's first isotropy theorem [13, Proposition 11.1] it follows that $\mathcal{J}_k \to B_k^0$ is topologically locally trivial. This proves (1).

By (1), $\mathfrak{f}_k$ is topologically locally-trivial. Hence $\mathfrak{R}^i\mathfrak{f}_k \mathbb{Q}$ is a locally constant sheaf of finite type over $B_k^0$ for all $i$. Since $B_k^0$ is nonsingular, $\mathfrak{R}^i\mathfrak{f}_k \mathbb{Q}$ forms a VMHS over $B_k^0$ with a canonical choice of $\{\mathcal{W}_n\}$ and $|\mathcal{F}^p|$ [2, Proposition 8.1.16]. This proves (2).

### 7. Applications: Betti numbers and mixed Hodge numbers of the cohomologies of a compactified Jacobian

In [3, Section 5] Bhosle and Parameswaran computed the Betti numbers of $\mathcal{J}_k$ by comparing the Betti numbers with that of the normalisation of $\mathcal{J}_k$ and using an induction on the genus of the nodal curve. Here we present much simpler way to compute the Betti numbers using the family $\mathcal{J}_k$.

**Corollary 7.1.**

1. Then $i$-th betti number of $\tilde{\mathcal{J}}_k$

$$h^i(\tilde{\mathcal{J}}_k) = h^i(J_0 \times R^k) = \sum_{0 \leq i \leq \min(1,2k)} \binom{2g}{i-l} \sum_{\frac{i}{2} \leq j \leq \min(1,k)} \binom{k}{j} \binom{j}{2j-l}.$$  

(7.1)

2. The dimension of $\text{gr}^W_i(H^i(\tilde{\mathcal{J}}_k))$ is

$$\dim \text{gr}^W_i(H^i(\tilde{\mathcal{J}}_k)) = \sum_{0 \leq i \leq l} \binom{2g}{i-l} \binom{k}{l} \binom{l-t}{i-l+t}.$$  

(7.2)

and

3. For $p, q \geq 0$ such $p + q = l$, the dimension of

$$\dim \text{gr}^P_k \text{gr}^P_j (\text{gr}^W_i (H^i(\tilde{\mathcal{J}}_k))) = \sum_{0 \leq i \leq l} \binom{2g}{p-l-t} \binom{g}{q-l-t} \binom{k-i-l-t}{i-l+t} \binom{l-t}{i-l+t}.$$  

Proof. (1). As $\mathfrak{f}_k : \mathcal{J}_k \to B_k^0$ is topologically locally trivial, the fiber over $(x_1, \ldots, x_k)$ is homeomorphic to the fiber over $(z_1, \ldots, z_k)$. Thus their Betti numbers agree i.e.,

$$h^i(\mathcal{J}_k) = h^i(J_0 \times R^k).$$  

(7.3)

Now consider the Kunneth decomposition

$$H^i(R^k) = \bigoplus_{0 \leq i \leq j \leq k} \binom{k-j}{i-j} H^0(R) \otimes H^1(R) \otimes H^2(R),$$  

(7.4)
where $2j - t = i$. Since each of the Kunneth components are one dimensional,

$$h^i(R^k) = \sum_{0 \leq t \leq j \leq k} \binom{k}{j} \binom{j}{t} = \sum_{\frac{i}{2} \leq j \leq \min(i,k)} \binom{k}{j} \binom{j}{2j-i}$$

$$h^i(J_0 \times R^k) = \sum_{0 \leq l \leq \min(2k,2)} h^{i-l}(J_0 \times R^k)$$

$$= \sum_{0 \leq l \leq \min(2k,2)} \binom{g}{i-l} \sum_{\frac{i}{2} \leq j \leq \min(i,k)} \binom{k}{j} \binom{j}{2j-i}. \quad \text{(by (7.4))}$$

Hence the proof of (7.1) follows.

proof of (2) and (3). From Theorem (6.7), $\mathcal{R}^I_{l,k}, Q$ forms a VMHS. Thus for each $j \geq 0$, $\text{gr}_j^W (\mathcal{R}^I_{l,k}, Q)$ forms a canonical variation of Hodge structures. In particular the dimension and Hodge numbers of $\text{gr}_j^W (H^I(J_k))$ and $\text{gr}_j^W (H^I(J(X_0) \times R^k))$ are equal.

For the rational nodal curve $R$, the cohomology $H^2(R)$ has pure weight 2 of type $(1,1)$ and $H^1(R)$ and $H^0(R)$ has weight 0 of type $(0,0)$. Therefore the weight of each summand in (7.4) is $2(j - t)$ and type $(j - t, j - t)$. In particular each of the summand is isomorphic to the Hodge-Tate structure $Q(t - j)$.

Hence for any $l \geq 0$,

$$\dim Q \text{gr}_{2l}^W \left( H^I(R^k) \right) = 0.$$ and $\text{gr}_{2l}^W (H^I(R^k))$ is isomorphic to direct sum of $Q(-l)$ as a mixed Hodge structures. Thus for all $i \geq 0$, $H^I(R^k)$ has a mixed Hodge-Tate structures. In order to compute the dimension of $\text{gr}_{2l}^W (H^I(R^k))$, using (7.4), one obtains

$$j - t = l \quad \text{(7.5)}$$

$$2j - t = i. \quad \text{(7.6)}$$

Solving (7.5) and (7.6) we have $t = i - 2l$ and $j = i - l$. Therefore

$$\dim Q \text{gr}_{2l}^W \left( H^I(R^k) \right) = \binom{k}{i-2l} \binom{i-l}{i-2l} \quad \text{(7.7)}$$

Now consider

$$\dim Q \text{gr}_l^W \left( H^I(J_0 \times R^k) \right) = \sum_{0 \leq t \leq l} h^t(J_0 \times R^k) \cdot \dim Q \left( \text{gr}_{l-t}^W H^{l-t}(R^k) \right). \quad \text{(7.8)}$$

Since $H^I(R^k)$ is a mixed Hodge-Tate structure, $\text{gr}_{l-t}^W H^{l-t}(R^k) = 0$ if and only if $l \neq t \mod 2$. Then from (7.8), one has

$$\dim Q \text{gr}_l^W \left( H^I(J_0 \times R^k) \right) = \sum_{0 \leq t \leq l} h^t(J_0 \times R^k) \cdot \dim Q \left( \text{gr}_{l-t}^W H^{l-t}(R^k) \right)$$

$$= \sum_{0 \leq t \leq l} \binom{2g}{t} \binom{k}{i-l-t} \binom{i-l-t}{i-l+t} \quad \text{(by (7.7))}$$

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Since $H^t(J_0)$ is a pure Hodge structure, the Hodge number of $h^{r,s}(H^t(J_0))$ is

$$h^{r,s}(H^t(J_0)) = \dim \mathcal{C} \mathcal{F}_r^g \mathcal{F}_s^g(H^t(J_0)) = \binom{g}{r} \binom{g}{s},$$

where $r + s = t$. Taking sum over all such $0 \leq t \leq l$ such that $l = t \pmod{2}$, we obtain the mixed Hodge number of type $(p, q)$ such that $p + q = l$ where $p = r + \frac{l-t}{2}$ and $q = s + \frac{l-t}{2}$. Therefore

$$\dim \mathcal{C} \mathcal{F}_r^p \mathcal{F}_s^p(\mathcal{C} W^i(J_k)) = \sum_{0 \leq t \leq l} \binom{g}{p - \frac{l-t}{2}} \binom{g}{q - \frac{l-t}{2}} \binom{k}{i - \frac{l-t}{2}} \binom{i - l - t}{i + l}.$$ 

□

References

[1] Artin, M. Algebraization of Formal Moduli: II. Existence of Modifications Annals of Mathematics Second Series, Vol. 91, No. 1 (Jan., 1970), pp. 88-135
[2] Brosnan, Patrick and El Zein, Fouad. Variations of mixed Hodge structure. Hodge theory, volume 49 of Math. Notes, pages 333–409. Princeton Univ. Press, Princeton, NJ, 2014.
[3] Bhosle, Usha N. and Parameswaran, A. J. Some result on the compactified Jacobian of a nodal curve, preprint, 2018
[4] Deligne, P. Theorie de Hodge I, Actes du Congres international des Mathematiciens (Nice, 1970), Gauthier–Villars, (1971), 425–430.
[5] Deligne, P. Theorie de Hodge II, Publications Mathematiques de IHES 40, (1971), 5–57.
[6] Deligne, P. Theorie de Hodge III, Publications Mathematiques de IHES 44, (1974), 5–77.
[7] D’Souza, Cyril. Compactification of generalised Jacobians. Proc. Indian Acad. Sci. Sect. A Math. Sci., 88(5):419–457, 1979.
[8] Esteves, Eduardo. Very ampleness for theta on the compactified Jacobian. Math. Z., 226(2):181–191, 1997.
[9] Hartshorne, Robin. Ample vector bundles. Inst. Hautes Études Sci. Publ. Math., (29):63–94, 1966.
[10] Gieseker, David. A degeneration of the moduli space of stable bundles, J. Differential Geom. 19(1): 173-206 (1984). DOI: 10.4310/jdg/1214438427
[11] Knutson, Donald. Algebraic spaces. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971.
[12] Lang, Serge. Introduction to Arakelov theory. Springer-Verlag, New York, 1988.
[13] Mather, John. Notes on topological stability. Bull. Amer. Math. Soc. (N.S.), 49(4):475–506, 2012.
[14] Mumford, David. Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.
[15] Nagaraj, D. S., Seshadri, C. S. Degenerations of the moduli spaces of vector bundles on curves II (generalized Gieseker moduli spaces), Proceedings of the Indian Academy of Sciences - Mathematical Sciences volume 109, pages 165–201 (1999)
[16] Oda, Tadao and Seshadri, C. S. Compactifications of the generalized Jacobian variety. Trans. Amer. Math. Soc., 253:1–90, 1979.
[17] Pascual-Gainza, Pere. Ampleness’ criteria for algebraic spaces. Archiv der Mathematik volume 45, pages270–274(1985).
[18] Seshadri, C. S. Fibres vectoriels sur les courbes algebraiques, Asterisque, 96, 1982. http://www.numdam.org/issue/AST_1982__96__1_0.pdf
[19] Soucaris, A. The ampleness of the theta divisor on the compactified jacobian of a proper and integral curve, Compositio Mathematica, Volume 93 (1994) no. 3, p. 231-242
[20] Stack Project, Morphisms of Algebraic Spaces, https://stacks.math.columbia.edu/download/spaces-morphisms.pdf

Department of Mathematics, Tata Institute of Fundamental Research, Mumbai

Email address: sdas@math.tifr.res.in

Department of Mathematics, Tata Institute of Fundamental Research, Mumbai

Email address: param@math.tifr.res.in
