The Dirichlet problem for discontinuous perturbations of the mean curvature operator in Minkowski space

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Abstract
Using the critical point theory for convex, lower semicontinuous perturbations of locally Lipschitz functionals, we prove the solvability of the discontinuous Dirichlet problem involving the operator $\leftarrow \frac{\nabla u}{\sqrt{1-|\nabla u|^2}}$.

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1 Introduction

Let $\Omega$ be an open bounded set in $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial \Omega$ of class $C^2$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying the growth condition

$$|f(x, s)| \leq C(1 + |s|^{q-1}), \quad \text{a.e. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

with some $q \in (1, \infty)$ and $C$ a positive constant. For a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, we denote

$$\underline{f}(x, s) := \lim_{\delta \to 0} \inf_{t \neq s} \{f(x, t) : |t - s| < \delta\}$$

and

$$\overline{f}(x, s) := \lim_{\delta \to 0} \sup_{t \neq s} \{f(x, t) : |t - s| < \delta\}.$$ 

In this paper we consider the discontinuous Dirichlet problem with mean curvature operator in Minkowski space:

$$\mathcal{M}(u) := \text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) \in [\underline{f}(x, u), \overline{f}(x, u)], \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0. \quad (1.2)$$

We assume that

$$\underline{f} \text{ and } \overline{f} \text{ are } N\text{-measurable} \quad (1.3)$$

(recall, a function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is called $N\text{-measurable}$ if $h(\cdot, v(\cdot)) : \Omega \to \mathbb{R}$ is measurable whenever $v : \Omega \to \mathbb{R}$ is measurable $[3]$).

By a solution of (1.2) we mean a function $u \in W^{2,p}(\Omega)$ for some $p > N$, such that $|\nabla u|_{\infty} < 1$, which satisfies

$$\mathcal{M}(u)(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))], \quad \text{a.e. } x \in \Omega$$

and vanishes on $\partial \Omega$. At our best knowledge, this type of solutions, but for differential inclusions was firstly considered by A.F. Filippov $[8]$. Also, for partial differential inclusions we refer the reader to the pioneering works of I. Massabo and C.A. Stuart $[12]$, J. Rauch $[14]$, C.A. Stuart and J.F. Toland $[17]$.

This work is motivated by the recent advances in the study of boundary value problems involving the operator $\mathcal{M}$ (see $[2], [6]$ and the references therein) and by the seminal paper of K.-C. Chang $[4]$ where the classical critical point theory is extended to locally Lipschitz functionals in order to study the problem

$$\Delta u \in [\underline{f}(x, u), \overline{f}(x, u)], \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0.$$ 

It is worth to point out that the operators $\mathcal{M}$ and $\Delta$ have essentially different structures and the theory developed in $[4]$ appears as not being applicable to problem (1.4). Thus, we shall use a more general critical point theory, namely the one concerning convex, lower semicontinuous perturbations of locally Lipschitz functionals, which was developed by D. Motreanu and P.D. Panagiotopoulos $[13]$ (also, see $[10], [11]$). It should be noticed that, using this theory, various...
existence results concerning Filippov type solutions for Dirichlet, periodic and Neumann problems involving the "p-relativistic" operator

\[ u \mapsto \left( \frac{|u'|^{p-2}u'}{(1 - |u'|^p)^{1/p}} \right) \]

were obtained in the recent paper \[9\].

A first existence result for the Dirichlet problem involving the operator \( M \) was obtained by F. Flaherty in \[7\], where it is shown that problem

\[ M(u) = 0 \quad \text{in} \; \Omega, \quad u|_{\partial \Omega} = \varphi, \]

has at least one solution, provided that \( \partial \Omega \) has non-negative mean curvature and \( \varphi \in C^2(\Omega) \) with \( \|\nabla \varphi\|_{\infty} < 1 \). The result was generalized in \[1\] by R. Bartnik and L. Simon, proving that problem

\[ M(u) = g(x, u), \quad \text{in} \; \Omega, \quad u|_{\partial \Omega} = 0 \quad (1.4) \]

is solvable, provided that the Carathéodory function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is bounded. More general, if \( g \) satisfies the \( L^\infty \)-growth condition:

for each \( \rho > 0 \) there is some \( \alpha_\rho \in L^\infty(\Omega) \) such that

\[ |g(x, s)| \leq \alpha_\rho(x) \quad \text{for a.e.} \; x \in \Omega, \; \forall \; s \in \mathbb{R} \text{ with } |s| \leq \rho, \]

it is shown in \[2\] Theorem 2.1] that \[12\] is still solvable. The approach in \[2\] relies on Szulkin’s critical point theory \[16\]. The aim of the present paper is to obtain a similar result for the discontinuous problem \[12\]. Precisely, we show in the main result (Theorem 4.1] that under assumptions \[1\] and \[3\] problem \[12\] always has at least one solution.

The rest of the paper is organized as follows. In Section 2 we recall some notions from nonsmooth analysis which will be needed in the sequel. The variational formulation of problem \[12\] is a key step in our approach and it is given in Section 3. Section 4 is devoted to the proof of the main result.

2 Preliminaries

Let \((X, \| \cdot \|)\) be a real Banach space and \( X^* \) its topological dual. A functional \( \mathcal{G} : X \to \mathbb{R} \) is called locally Lipschitz if for each \( u \in X \), there is a neighborhood \( \mathcal{N}_u \) of \( u \) and a constant \( k > 0 \) depending on \( \mathcal{N}_u \) such that

\[ |\mathcal{G}(w) - \mathcal{G}(z)| \leq k\|w - z\|, \quad \forall \; w, z \in \mathcal{N}_u. \]

For such a function \( \mathcal{G} \), the generalized directional derivative at \( u \in X \) in the direction of \( v \in X \) is defined by

\[ \mathcal{G}^0(u; v) = \limsup_{w \to u, \; t \searrow 0} \frac{\mathcal{G}(w + tv) - \mathcal{G}(w)}{t} \]
and the generalized gradient (in the sense of Clarke \[5\]) of $G$ at $u \in X$ is defined as being the subset of $X^*$

$$\partial G(u) = \{\eta \in X^* : G^0(u; v) \geq \langle \eta, v \rangle, \ \forall \ v \in X\},$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $X^*$ and $X$. For more details concerning the properties of the generalized directional derivative and of the generalized gradient we refer to \[5\].

If $I : X \to (-\infty, +\infty]$ is a functional having the structure

$$I = \Phi + G,$$

(2.1)

with $G : X \to \mathbb{R}$ locally Lipschitz and $\Phi : X \to (-\infty, +\infty]$ proper, convex and lower semicontinuous, then an element $u \in X$ is said to be a critical point of $I$ provided that

$$G^0(u; v - u) + \Phi(v) - \Phi(u) \geq 0, \ \forall \ v \in X.$$

The number $c = I(u)$ is called a critical value of $I$ corresponding to the critical point $u$. According to Kourogenis et al. \[10\], $u \in X$ is a critical point of $I$ iff

$$0 \in \partial G(u) + \partial \Phi(u),$$

where $\partial \Phi(u)$ stands for the subdifferential of $\Phi$ at $u \in X$ in the sense of convex analysis \[15\], i.e.,

$$\partial \Phi(u) = \{\eta \in X^* : \Phi(v) - \Phi(u) \geq \langle \eta, v - u \rangle, \ \forall \ v \in X\}. $$

Also, $I$ in (2.1) is said to satisfy the Palais-Smale condition (in short, $(PS)$ condition) if every sequence $(u_n) \subset X$ for which $(I(u_n))$ is bounded and

$$G^0(u_n; v - u_n) + \Phi(v) - \Phi(u_n) \geq -\varepsilon_n \|v - u_n\|, \ \forall \ v \in X,$$

for a sequence $(\varepsilon_n) \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

**Theorem 2.1** (\[11\] Theorem 1) If $I$ is bounded from below and satisfies the $(PS)$ condition then $c = \inf_X I$ is a critical value of $I$.

3 The variational setting

In the sequel we shall give the variational formulation of problem (1.2). With this aim, we introduce the set

$$K_0 = \{v \in W^{1,\infty}(\Omega) : \|\nabla v\|_\infty \leq 1, \ v = 0 \text{ on } \partial \Omega\}.$$

Notice that since $W^{1,\infty}(\Omega)$ is continuously (in fact, compactly) embedded into $C(\overline{\Omega})$, the evaluation at $\partial \Omega$ is understood in the usual sense. According to \[2\], $K_0$ is compact in $C(\overline{\Omega})$ and one has

$$\|v\|_\infty \leq c(\Omega) \quad \text{for all } v \in K_0,$$

(3.1)
with \(c(\Omega)\) a positive constant. Also, the functional \(\Psi : C(\Omega) \rightarrow (-\infty, +\infty]\) given by
\[
\Psi(v) = \begin{cases} 
\int_{\Omega} \left[1 - \sqrt{1 - |\nabla v|^2}\right], & \text{for } v \in K_0, \\
+\infty, & \text{for } v \in C(\Omega) \setminus K_0 \end{cases} \tag{3.2}
\]
is proper, convex and lower semicontinuous \([2, \text{Lemma 2.4}].\)

Having in view the growth condition \((1.1)\), we define \(\hat{F} : L^q(\Omega) \rightarrow \mathbb{R} \) by
\[
\hat{F}(v) = \int_{\Omega} F(x, v), \quad \forall \, v \in L^q(\Omega),
\]
where
\[
F(x, s) = \int_{0}^{s} f(x, \xi) d\xi \quad (x \in \Omega, \ s \in \mathbb{R})
\]
and, on account of the embedding \(C(\Omega) \subset L^q(\Omega)\), we introduce the functional
\[
\mathcal{F} = \hat{F}|_{C(\Omega)}. \tag{3.3}
\]
From \([4, \text{Theorem 2.1}]\), one has that \(\hat{F}\) is locally Lipschitz in \(L^q(\Omega)\) and
\[
\partial \hat{F}(v) \subset \left[ f(\cdot, v(\cdot)), \overline{f}(\cdot, v(\cdot)) \right], \tag{3.4}
\]
for all \(v \in L^q(\Omega)\). Then, by the continuity of the embedding \(C(\Omega) \subset L^q(\Omega)\) it is clear that \(\mathcal{F}\) is locally Lipschitz on \(C(\Omega)\). Also, since \(C(\Omega)\) is dense in \(L^q(\Omega)\), it holds (see \([5]\), p. 47):
\[
\partial \hat{F}(v) = \partial \mathcal{F}(v), \quad \forall \, v \in C(\Omega). \tag{3.5}
\]

**Lemma 3.1** Let \(v \in K_0\). If \(\ell \in \partial \mathcal{F}(v)\), then there is some \(\zeta_\ell \in L^\infty(\Omega)\) such that \(\zeta_\ell(x) \in \left[f(x, v(x)), \overline{f}(x, v(x))\right]\) for a.e. \(x \in \Omega\) and
\[
\langle \ell, w \rangle = \int_{\Omega} \zeta_\ell w \tag{3.6}
\]
for all \(w \in C(\Omega)\).

**Proof.** From \((3.5)\) and \((3.4)\) we infer that there is a function \(\zeta_\ell \in L^{q'}(\Omega)\) with \(1/q + 1/q' = 1\), such that \(\zeta_\ell(x) \in \left[f(x, v(x)), \overline{f}(x, v(x))\right]\) for a.e. \(x \in \Omega\) and \((3.4)\) holds true for all \(w \in L^q(\Omega)\). To see that \(\zeta_\ell \in L^\infty(\Omega)\), from \((1.1)\) and \((3.4)\), one gets
\[
-C_1 \leq f(x, v(x)) - \bar{f}(x, v(x)) \leq C_1, \quad \text{for a.e. } x \in \Omega,
\]
with \(C_1 = C(1 + c(\Omega)^{q^{-1}}).\) This shows that \(|\zeta_\ell(x)| \leq C_1\) for a.e. \(x \in \Omega\) and the proof is complete. \(\blacksquare\)
The functional framework of Section 2 fits the following choices: $X = C(\Omega)$, $\Phi = \Psi$ in (6.2), $G = F$ in (6.3) and 

$$I := \Psi + F.$$ 

Notice that, the compactness of $K_0 \subset C(\Omega)$ implies that $I$ satisfies the $(PS)$ condition.

4 Main result

We have the following

**Theorem 4.1** Assume that (1.1) and (1.3) hold true. If $u$ is a critical point of $I$, then $u$ is a solution of problem (1.2). Moreover, $I$ is bounded from below and attains its infimum at some $u_0 \in K_0$, which solves problem (1.2).

**Proof.** Let $u$ be a critical point of $I$. Then $u \in K_0$ and there exist $h_u \in \partial \Psi(u)$ and $\ell_u \in \partial F(u)$ such that

$$\langle h_u, w \rangle + \langle \ell_u, w \rangle = 0, \quad \forall \ w \in C(\Omega).$$

This and the fact that $h_u \in \partial \Psi(u)$ yield

$$\Psi(w) - \Psi(u) + \langle \ell_u, w - u \rangle \geq 0, \quad \forall \ w \in C(\Omega). \quad (4.1)$$

Using Lemma 3.1 we deduce that there is some $\zeta_u = \zeta(\ell_u) \in L^\infty(\Omega)$ such that

$$\zeta_u(x) \in [f(x, u(x)), \overline{f}(x, u(x))], \quad \text{a.e. } x \in \Omega \quad (4.2)$$

and

$$\langle \ell_u, w \rangle = \int_{\Omega} \zeta_u w, \quad \forall \ w \in C(\Omega). \quad (4.3)$$

By virtue of (4.3), inequality (4.1) becomes

$$\Psi(w) - \Psi(u) + \int_{\Omega} \zeta_u (w - u) \geq 0, \quad \forall \ w \in C(\Omega). \quad (4.4)$$

On account of Lemma 2.2 in [6], for each function $e \in L^\infty(\Omega)$, the Dirichlet problem

$$M(v) = e(x) \quad \text{in } \Omega, \quad v|_{\partial \Omega} = 0$$

has an unique solution $v_e \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. Then, from Lemma 2.3 in [2], one has that $v_e$ is the unique solution in $K_0$ of the variational inequality

$$\int_{\Omega} [\sqrt{1 - |\nabla v|^2} - \sqrt{1 - |\nabla w|^2} + e(w - v)] \geq 0, \quad \forall \ w \in K_0.$$
and hence,

\[ \Psi(w) - \Psi(v_e) + \int_{\Omega} e(w - v_e) \geq 0, \quad \forall \ w \in C(\Omega). \]

From this and (4.4), we infer that \( u = v_e \), with \( e = \zeta u \). But, on account of (1.2), this means that \( u \) solves problem (1.2).

Next, for arbitrary \( u \in K_0 \), by (1.1) and (3.1), the primitive \( F \) satisfies

\[ |F(x, u(x))| \leq C(c(\Omega) + c(\Omega)^q/q) =: C_2, \quad \text{for a.e. } x \in \Omega. \]

Hence,

\[ |F(u)| \leq \int_{\Omega} |F(x, u)| \leq C_2 \text{vol}(\Omega), \quad \forall \ u \in K_0. \]

We deduce that the functional \( I \) is bounded from below on \( C(\Omega) \). Then, using that \( I \) verifies the \((PS)\) condition and Theorem 2.1, we have that

\[ c = \inf_{C(\Omega)} I = \inf_{K_0} I \]

is a critical value of \( I \) and the proof is complete.

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**References**

[1] R. Bartnik and L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Comm. Math. Phys.* 87 (1982), 131–152.

[2] C. Bereanu, P. Jebelean and J. Mawhin, The Dirichlet problem with mean curvature operator in Minkowski space - a variational approach, *Adv. Nonlinear Stud.* 14 (2014), 315–326.

[3] K.-C. Chang, The obstacle problem and partial differential equations with discontinuous nonlinearities, *Comm. Pure Appl. Math.* XXXIII (1980), 117–146.

[4] K.-C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* 80 (1981), 102–129.

[5] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

[6] C. Corsato, F. Obersnel, P. Omari and S. Rivetti, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space, *J. Math. Anal. Appl.* 405 (2013), 227–239.
[7] F. Flaherty, The boundary value problem for maximal hypersurfaces, *Proc. Natl. Acad. Sci. USA* **76** (1979), 4765–4767.

[8] A.F. Filippov, Right hand side discontinuous differential equations, *Trans. Am. Math. Soc.* **42** (1964), 199–227.

[9] P. Jebelean and C. Şerban, Boundary value problems for discontinuous perturbations of singular $\phi$-Laplacian operator, *submitted*.  

[10] N.C. Kourogenis, J. Papadrianos and N.S. Papageorgiu, Extensions of nonsmooth critical point theory and applications, *Atti Sem. Mat. Fis. Univ. Modena* **L** (2002), 381–414.

[11] C. Lefter, Critical point theorems for lower semicontinuous functionals, *An. Științ. Univ. Al. I. Cuza Iași, Ser. Nouă, Mat.* **47** (2001), 189–198.

[12] I. Massabo and C.A. Stuart, Elliptic eigenvalue problems with discontinuous nonlinearities, *J. Math. Anal. Appl.* **66** (1978), 261–281.

[13] D. Motreanu and P.D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Kluwer Academic Publishers, Nonconvex Optimization and Its Applications, Vol. 29, Dordrecht, 1999.

[14] J. Rauch, Discontinuous semilinear differential equations and multiple valued maps, *Proc. Amer. Math. Soc.* **64** (1977), 277–282.

[15] R.T. Rockaffellar, *Convex Analysis*, Princeton University Press, 1972.

[16] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), 77–109.

[17] C.A. Stuart and J.F. Toland, A variational method for boundary value problems with discontinuous nonlinearities, *J. London Math. Soc.* **21** (1980), 319–328.