The Hamiltonian Approach to Yang-Mills (2+1): An Update and Corrections to String Tension

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Abstract. Yang-Mills theories in 2+1 (or 3) dimensions are interesting as nontrivial gauge theories in their own right and as effective theories of QCD at high temperatures. I shall review the basics of our Hamiltonian approach to this theory, emphasizing symmetries with a short update on its status. We will show that the calculation of the vacuum wave function for Yang-Mills theory in 2+1 dimensions is in the lowest order of a systematic expansion. Expectation values of observables can be calculated using an effective interacting chiral boson theory, which also leads to a natural expansion as a double series in the coupling constant (to be interpreted within a resummed perturbation series) and a particular kinematical factor. The calculation of the first set of corrections in this expansion shows that the string tension is modified by about $-0.3\%$ to $-2.8\%$ compared to the lowest order value. This is in good agreement with lattice estimates.

1. Introduction

As many of you know, Yang-Mills theory in 2+1 dimensions has been the focus of research for my collaborators and myself for many years. In the first part of this talk, I shall give an update on the status of our attempts to understand this theory. In the second part, I shall explain our recent calculations of corrections to the string tension. This latter part is more than just a calculation of specific results; it alsoformulates a systematic calculational framework which may prove useful for further calculations, such as glueball masses.

Let me begin by recalling why this theory is of interest to us. Yang-Mills theories have, of course, been a great puzzle for many decades. Since the realistic case of 3+1 dimensions is highly nontrivial and difficult, physicists have often turned to the time-honored path of seeking guidance from lower dimensions. Yang-Mills theories in 1+1 dimensions are exactly solvable, but they have no propagating degrees of freedom and, as a consequence, are a little too trivial for us. In 2+1 dimensions, there are propagating degrees of freedom, the theory has nontrivial dynamics, yet seems to be within reach of some level of analytical investigation. This is aided by the fact that the theory has a dimensional coupling constant and, as a related fact, because it is super-renormalizable.

2. A short review and status report

Our approach to this problem has been to use Hamiltonian analysis and to solve the Schrödinger equation for the vacuum wave function [1, 2]. Generally, Hamiltonian methods are not easy to implement in a field theory with its infinite number of degrees of freedom and regularization.
gauge-invariant quantities can be constructed from the Wilson loop variable (by suitable choices where, in the second step, we have simply used the matrix parametrization for spatial components of the gauge potential can be combined as

\[ A = A_x = \frac{1}{2} (A_1 + i A_2), \]

\[ \bar{A} = A_x = \frac{1}{2} (A_1 - i A_2). \]

The crucial first step for us is the parametrization of these potentials as

\[ A = - \partial M M^{-1}, \quad \bar{A} = M^1 \partial M^1. \]

Here \( M \) is an \( SL(N, \mathbb{C}) \)-matrix for an \( SU(N) \)-gauge theory. More generally, \( M \) is a complex matrix which is an element of \( G^C \), the complexification of \( G \) which is the Lie group in which the gauge transformations take values.

Time-independent gauge transformations act on \( M \) via \( M(\bar{x}) \to g(\bar{x})M(\bar{x}), \ g(\bar{x}) \in SU(N) \). The hermitian matrix \( H = M^\dagger M \) is thus gauge-invariant. The fact that the gauge transformations are made homogeneous is why the matrix parametrization for \( A, \bar{A} \) is so useful.

Wave functions \( \Psi(H) \) are gauge-invariant and depend only on \( H = M^\dagger M \); this is basically the Gauss law. Further, the volume element for the space of gauge-invariant configurations can be explicitly evaluated and the inner product for the wave functions obtained as

\[ \langle 1 | 2 \rangle = \int d\mu(H) \exp(2c_A S_{wzw}(H)) \Psi_1^* \Psi_2 \]  

where \( S_{wzw}(H) \) is the Wess-Zumino-Witten action for the hermitian field \( H \) and \( d\mu(H) \) is the Haar measure for \( H \) viewed as an element of \( SL(N, \mathbb{C})/SU(N) \). The number \( c_A \) is the quadratic Casimir value for the adjoint representation; it is defined in terms of the structure constants \( f^{abc} \) (of the Lie algebra) as \( c_A \delta^{ab} = f^{amn} f^{bmn} \) and is equal to \( N \) for \( SU(N) \). The WZW action is given by

\[ S_{wzw}(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \partial \bar{H}^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_\mu H H^{-1} \partial_\nu H H^{-1} \partial_\alpha H) \]  

Actually we can strengthen the statements given above a little further and argue that the wave functions \( \Psi \) and the Hamiltonian can be taken to be functions of a scaled version of the current of the WZW action, namely, of \( J = (2/e) \partial H H^{-1} \). One way to see this is to consider the Wilson loop variable,

\[ W(C) = \text{Tr} \mathcal{P} e^{-\frac{i}{e} \oint_C J} = \text{Tr} \mathcal{P} \exp \left( \frac{e}{2} \oint_C J \right) \]  

where, in the second step, we have simply used the matrix parametrization for \( A, \bar{A} \). Since all gauge-invariant quantities can be constructed from the Wilson loop variable (by suitable choices of the contour of integration \( C \)), we may restrict attention to functions of \( J \).

Taking the wave functions to be functions of the current \( J \), functional derivatives with respect to \( A, \bar{A} \) may be obtained via the chain rule of differentiation. This leads to the expression for the Hamiltonian operator in terms of the \( J \) as \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \) with

\[ \mathcal{H}_0 = m \int_{\bar{z}, z} \frac{\delta}{\delta J_a(\bar{z})} \frac{\delta}{\delta J_A(\bar{z})} + \frac{2 \pi}{\delta J_a(\bar{w}) \delta J_a(\bar{z})} + \frac{1}{2} \int_{\bar{z}, z} : \bar{\partial} J^a(z) \bar{\partial} J^a(z) : \]  

\[ \mathcal{H}_1 = i \epsilon \int_{\bar{w}, z} f_{abc} \frac{\delta}{\delta J^c(\bar{w})} \frac{\delta}{\delta J_a(\bar{w}) \delta J_b(\bar{z})} \]  

where \( m = e^2 c_A / 2\pi = e^2 N / 2\pi \). Notice that the first term of the Hamiltonian is counting powers of \( J \) in the wave function on which it acts, giving an energy contribution \( m \) for each \( J \). This will turn out to be the essence of the mass gap in the theory.

The Hamiltonian operator involves products of \( J \)'s, \( \delta/\delta J \)'s and nonlocal functions such as \((z - w)^{-1}\). As in any field theory, all calculations have to be done with proper regularization.
Table 1. Comparison of magnetic mass calculations

| \(m/e^2\) | Method |
|-----------|--------|
| 0.25      | Resummation of perturbation theory [4] |
| 0.35      | Lattice, common factor for glueball masses [5] |
| 0.51      | Lattice, maximal abelian gauge [6] |
| 0.52      | Lattice, Landau gauge [6] |
| 0.44      | Lattice, \(\lambda_3 = 2\) gauge [6] |
| 0.38      | Resummation of perturbation theory [7] |
| 0.28      | Resummation of perturbation theory [8, 9] |
| 0.37      | Gauge-invariant lattice definition [10] |
| 0.32      | Calculation via our Hamiltonian method |

We will not go into this matter here, except to say that all the results presented here have been checked using regularized expressions, with a single regulator from beginning to end, to avoid possible conflicts between regularizations. For more details, see [1, 2, 3].

One interesting result can be read off from what we have done so far. By comparison with resummed perturbation theory, we can see that \(m\) serves as the propagator mass for gluons (which are generated by \(J\)). This can then be identified with the magnetic screening mass if we interpret the theory as the high temperature limit of QCD. For the gauge group \(SU(2)\), \(m = e^2/\pi \approx 0.32 e^2\). There are a number of other calculations for the magnetic screening mass, so it is interesting to compare these values. Table 1 shows that while there is some variation, the values are not outrageously different from \(e^2/\pi\).

In our earlier work [1, 2], we solved the Schrödinger equation \(H \Psi_0 = 0\) for the vacuum wave function \(\Psi_0\) to the leading order in a strong-coupling expansion to obtain \(\Psi_0 = \exp(-\frac{1}{2} S)\), where

\[
S(H) = \int \partial J^a \left[ \frac{1}{(m + \sqrt{m^2 - \nabla^2})} \right] \partial J^a + O(J^3)
\]

\[
\approx \frac{1}{2m} \int \partial J^a \partial J^a = \frac{1}{4g^2} \int d^2 x \ F_{ij}^a F_{ij}^a
\]

(5)

where the second line is the approximation to modes of \(J\) with momentum \(\ll e^2\). Thus the computation of expectation values reduces in this limit to a calculation in a Euclidean two-dimensional Yang-Mills theory with a coupling \(g^2 = me^2\). The expectation value of the Wilson loop variable for the representation \(R\) then takes the form

\[
\langle W(C) \rangle \sim \exp(-\sigma_R A_C), \quad \sqrt{\sigma_R} = e^2 \sqrt{\frac{E_{ACR}}{4\pi}}
\]

(6)

where \(A_C\) is the area enclosed by the curve \(C\) and \(\sigma_R\) is the string tension. This compares favorably, to within 1 – 3%, with the lattice calculations [11, 12], see table 2.

I have not described in detail the process of solving the Schrödinger equation; this is because there is another argument which can be used to obtain the results to this order [2]. For this purpose, absorb the factor \(\exp(2c_A S_{wzw}(H))\) from the inner product into the wave function by \(\Psi = e^{-c_A S_{wzw}(H)} \Phi\). Then the Hamiltonian acting on \(\Phi\) is given by \(H \rightarrow H_\Phi = e^{c_A S_{wzw}(H)} H e^{-c_A S_{wzw}(H)}\). We shall now consider this expression when \(H\) can be approximated as \(H = e^{\varphi} \varphi + \cdots\); this small \(\varphi\) limit is appropriate for perturbation theory. The new Hamiltonian is then

\[
H_\Phi \approx \frac{1}{2} \int \left[ -\frac{\delta}{\delta \phi^a} \delta \phi^a + \phi^a (m^2 - \nabla^2) \phi^a \right] + \cdots
\]

(7)
Table 2. Comparison of predictions from (6) with lattice calculations

| Group | Representations | k=1 | k=2 | k=3 | k=2 | k=3 | k=3 |
|-------|-----------------|-----|-----|-----|-----|-----|-----|
|       |                 |     |     |     |     |     |     |
| SU(2) | 0.345           | 0.335 |     |     |     |     |     |
| SU(3) | 0.564           | 0.553 |     |     |     |     |     |
| SU(4) | 0.772 0.891     | 1.196 | 0.759 | 0.883 | 1.110 |
| SU(5) | 0.977 0.966     |     |     |     |     |     |     |
| SU(6) | 1.180 1.493 1.583 | 1.784 | 2.318 | 1.985 | 1.921 |
| SU(N) | 0.1995 N        |     |     |     |     |     |     |
| N→∞  | 0.1976 N        |     |     |     |     |     |     |

The upper entries follow from (6), the lower entries are the lattice values. k is the rank of the representation.

where \( \phi^a = \sqrt{c_A}(-\nabla^2)/8\pi m \phi^a \). We see that the leading term in \( \mathcal{H}_\Phi \) corresponds to a free field of mass \( m \) (actually \( \text{dim } G \) fields, counting the multiplicity due to the index \( a \).) The field \( \phi \) may be taken as representing the gluon (actually a gauge-invariant version with some resummations involved) and this is consistent with what we said about the gluon mass being \( m \).

The vacuum wave function in this approximation may be trivially calculated as \( \Phi_0 = \exp(-\frac{1}{2}S) \), with \( S = \int \phi^a \sqrt{m^2 - \nabla^2} \phi^a \). If this is transformed back to the \( \Psi \)-version, we find

\[
\Psi_0 \approx \exp \left[ -\frac{c_A}{\pi m} \int (\bar{\partial} \partial \phi^a) \left[ \frac{1}{m + \sqrt{m^2 - \nabla^2}} \right] (\bar{\partial} \partial \phi^a) + \cdots \right] \tag{8}
\]

This was obtained in an approximation of small \( \varphi \). However, we do know that the full wave function must be a function of the current \( J \). So we can ask the question: What function of \( J \) can we write such that it reduces to (8) in the small \( \varphi \) approximation when \( J \approx (2/e)\partial \varphi \)? The answer, to quadratic order in \( J \), is clearly the solution (5). Notice that this argument only depends on the form of integration measure which, in turn, is determined by a two-dimensional anomaly calculation. So it is rather robust; for more details, see [2].

Another interesting line of development has been about glueball masses. The authors of [13] have followed the general line of reasoning we have outlined, but used a somewhat different wave function given by

\[
\Psi_0 = \exp \left[ -\frac{1}{4m} \int \bar{\partial}J K[L] \bar{\partial}J \right], \quad K[L] = \frac{J_2(4\sqrt{L})}{\sqrt{L} J_1(4\sqrt{L})} \tag{9}
\]

where \( L = D\bar{\partial}/m^2 \) and \( J_1, J_2 \) are Bessel functions of orders 1 and 2 respectively. The kernel \( K \) is, despite appearances, very close to the kernel in (5), as is clear from the graphical comparison in figure 1 (I thank A. Yelnikov for this picture). Glueball masses were obtained by calculating the two-point function for different color-singlet composite operators, characterized by spin, parity and charge conjugation properties \( (j^{PC}) \) notation. We will not display the results here for lack of space, but they are given in [13] and are in reasonable agreement with the lattice data of reference [11, 14].

There have been a number of other developments as well. The formalism has been extended to the Yang-Mills-Chern-Simons theory and issues related to dynamical mass generation and screening of Wilson loops have been analyzed [15].
Figure 1. Comparison of KKN and LMY kernels for the Gaussian term of the vacuum wave function as a function of the momentum $p$

Another interesting question has been about the screening of $Z_N$-invariant representations. The energy of a glue-lump state made of a heavy scalar field and the gluons can be estimated in the limit of large $e^2$ and large $N$ [3]. This gives the energy at which an adjoint string breaks as $E_s \approx 7.92 \, m$, for $SU(2)$, to be compared with the lattice value of $E_s \approx 8.68 \, m$ [16]; we are off by about 8.8%. Considering the difficulties in calculating both in our method and the lattice, this may be taken as not too bad. The calculation is more a qualitative demonstration of feasibility: it is possible to incorporate screening in our formalism.

We have also worked out the extension of the formalism to the case of space being $S^2$, i.e., for $\mathbb{R} \times S^2$ [17], as a prelude to calculations on the torus which may be used to analyze finite-temperature effects. In this regard, I also mention the recent paper by Abe, where some of the torus formalism is developed and arguments on deconfinement are presented [18].

3. Corrections to string tension

We now turn to the string tension formula (6) [19]. This is important for several reasons. First, we want to see the vacuum wave function (5) and the corresponding string tension (6) as the lowest order in a systematic expansion scheme. Secondly, notice that, in the large $N$ limit, our formula differs from the lattice value by about 1%. A different lattice calculation shows the difference to be about 1.55% [20]. In both cases, the lattice calculations are considered accurate enough that the deviations are statistically significant. So we need a systematic method which has the potential to calculate corrections.

There are two types of corrections possible. For the first set, we can calculate the corrections to the propagators (or the coefficient of $\partial J \partial J$ in the wave function) and then use the result in the calculation of the Wilson loop. This set of corrections will be independent of the representation $R$ of the Wilson line. The second set would involve vertex type corrections and can be sensitive to the representation of the Wilson line. We will leave out these for now, focusing on the first set. The corrections are expected to be purely numerical, so, at first glance, seem to be impossible to control in any way. However, we shall see that a systematic expansion is possible along the following lines.

We rewrite the derivation of the vacuum wave function as a recursive procedure for the solution of the Schrödinger equation from which it will be clear that (5) is the lowest order result in a systematic expansion. For this purpose, we will treat $m$ and $e$ as independent parameters, setting $m = e^2 c_a / 2\pi$ only at the end of all calculations. For $\Psi_0^* \Psi_0 = e^F$, this gives $F$ as a power series in $e$, of the form

$$ F = \int f^{(2)}_{a_1a_2}(x_1, x_2) \, J^{a_1}(x_1) J^{a_2}(x_2) + \frac{e}{2} f^{(3)}_{a_1a_2a_3}(x_1, x_2, x_3) \, J^{a_1}(x_1) J^{a_2}(x_2) J^{a_3}(x_3) $$

$$ + \frac{e^2}{4} f^{(4)}_{a_1a_2a_3a_4}(x_1, x_2, x_3, x_4) \, J^{a_1}(x_1) J^{a_2}(x_2) J^{a_3}(x_3) J^{a_4}(x_4) + \cdots $$

(10)
$f^{(2)}$, $f^{(3)}$, etc., are determined recursively in powers of $e$, with each function having an expansion in powers of $e^2$; e.g., $f^{(2)} = f^{(2)}_0 + e^2 f^{(2)}_1 + \cdots$, etc. To the lowest order, with $q, \bar{q}$ denoting the momenta,

$$f^{(2)}_{a_1a_2}(x_1,x_2) = \delta_{a_1a_2} \left[-\bar{q}^2/(m + E_q)\right]_{x_1,x_2} \approx \delta_{a_1a_2} \frac{\bar{q}^2}{2m}$$

(11)

which leads to (5). To calculate corrections to order $e^2$, we need the lowest order results for $f^{(3)}$ and $f^{(4)}$. These are given in [19].

We are interested in corrections to $f^{(2)}$ since that determines the string tension. From the recursive solution of the Schrödinger equation, we find

$$e^2 f^{(2)}_2(q) \approx \frac{\bar{q}^2}{2m} (1.1308) + \ldots$$

(12)

Seemingly, this is a 113% correction, but, as we shall see, other "loop" corrections are important.

Since the measure of integration has the WZW action, we can transform the functional integration over $\Psi^a_0\Psi_0 = e^J$ into the integration over a two-dimensional chiral boson field $\varphi, \bar{\varphi}$ (not the $\varphi$ we used in parametrizing $H$ as $e^{F_\bar{\varphi}}$). In other words, for an observable $O$,

$$\langle O \rangle = \int d\mu(H) \ e^{2c_A S_{wzw}(H)} \ e^{F(J)} \ O(J) = \int [d\varphi d\bar{\varphi}] \ e^{-S(\varphi)} \ O(\sqrt{2\pi/mc_A} \ \bar{\varphi}t^a \varphi)$$

$$S(\varphi) = \int \left( Z_2 \bar{\varphi} \partial \varphi + Z_1 \bar{\varphi} \bar{C} \varphi \right) - F(Z_1 \sqrt{2\pi/mc_A} \ \bar{\varphi}t^a \varphi$$

(13)

($Z_1, Z_2$ are renormalization constants, we will not discuss these further here; see [19].) Notice that, effectively, the current $J^a$ is replaced by $Z_1 \sqrt{2\pi/mc_A} \ \bar{\varphi}t^a \varphi$. The function $F(Z_1 \sqrt{2\pi/mc_A} \ \bar{\varphi}t^a \varphi)$ contains vertices, $F^{(2)}$ with two currents (quartic in $\varphi, \bar{\varphi}$), $F^{(3)}$ with three currents, etc. For example, we may diagrammatically represent $F^{(2)}$, with two $\varphi$'s and two $\bar{\varphi}$'s, as

$$F^{(2)} = \frac{2\pi}{mc_A} \int (\bar{\varphi} t^a \varphi)_{x} f^{(2)}(x,y)(\bar{\varphi} t^a \varphi)_{y}$$

The corrections to $F^{(2)}$, which is what we are interested in, may be viewed as Wilsonian renormalization corrections to the quartic vertex in this two-dimensional field theory.

In computing the corrections to $F^{(2)}$, we can treat the vertices $F^{(3)}, F^{(4)},$ etc., perturbatively since they carry powers of $e$. However, the lowest term in the vertex $F^{(2)}$, corresponding to $f^{(2)}_{a_0a_2}(x_1, x_2)$, has no powers of $e$ and hence its contributions must be summed up. This means that the current-current correlator becomes

$$\langle \varphi^{a} \varphi^{b}(x) \varphi^{b} \varphi^{a}(y) \rangle = \delta^{ab} c_A \int \frac{d^2k}{(2\pi)^2} \ e^{i\bar{q}(x-y)} \ k \left( \frac{m}{E_k} \right)$$

(14)

Here $E_k = \sqrt{k^2 + m^2}$; the $m/E_k$ factor arises from the summation of corrections due to $F^{(2)}_0$, shown diagrammatically in figure 2. Our computational strategy is then the following: We will construct loop diagrams generated by $F^{(3)}$ (3 factors of $\bar{\varphi} t^a \varphi$) and $F^{(4)}$ (4 factors of $\bar{\varphi} t^a \varphi$), taking account of the arbitrary insertions of $F^{(2)}$'s, and the corresponding factors of $m/E_k$, as in figure 3. There are also renormalizations (due to $F^{(2)}$) we have to take into account. We then sum up $F^{(2)}$ insertions in all diagrams (of order $e^2$) generated by $F^{(3)}$ and $F^{(4)}$, using powers of $m/E_k$ to group them appropriately. We will compute corrections to order $e^2$ and up to 4 powers of $m/E_k$. Denoting the factors of $m/E_k$ by shaded circles at the vertices, the corrections to the
Figure 2. The current-current correlator including all contributions from $F^{(2)}$

Figure 3. Corrections from $F^{(2)}$ summed up as a factor of $m/E_k$ (shaded circle at vertex) and sample renormalization diagrams

low momentum limit of $f^{(2)}$ may summarized as in figure 4. We show the coefficients of $q^2/2m$, for small $q$, $\bar{q}$.

Let $C_n$ denote the partial sum of corrections up to terms with $(m/E_k)^n$, starting with $C_0 = 1.1308$ from the recursive procedure (12). We then find $C_1 = 0.5496$, $C_2 = 0.2730$, $C_3 = 0.0373$. There is a small ambiguity in an integral for the last diagram in figure 5 [19], so that $C_4$ is in the range $-0.05843$ to $-0.00583$. The partial sums are systematically decreasing in value, showing that the ordering of diagrams by powers of $m/E_k$ does constitute a sensible expansion. Also, the corrections to the order we have calculated are small. For the string tension, we find

$$\sqrt{\sigma_R} = e^2 \frac{C_{ACR}}{4\pi} \left\{ \left( 1 - 0.02799 + \cdots \right) \right\}$$

This correction, of the order of $-2.8\%$ to $-0.03\%$, is entirely consistent with lattice calculations.

$^1$ $\bar{C}$ may be set to zero at this point, it was included to show how $Z_1$, $Z_2$ appear in $S(\varphi)$. Also we may note that the transformation to $\varphi$, $\bar{\varphi}$ is analogous to the fermionization of the WZW model.

Figure 4. Corrections to the low momentum limit of the $F^{(2)}$ vertex
Terms of order $(m/E_k)^5$ are expected to contribute at the level of a fraction of 1%, and likewise for diagrams with two or more current loops.

4. Discussion

Perhaps the most important point to be mentioned is that one has a systematic expansion and a scheme for calculations. At first glance, the possibility of any expansion scheme in 2+1 dimensional Yang-Mills theory seems very remote, to say the least. The coupling constant $e^2$ does not constitute an expansion parameter. It simply tells us that modes of momenta $\ll e^2$ should be treated nonperturbatively, while modes of momenta $\gg e^2$ can be treated perturbatively; it is at best a marker for this separation. Nevertheless, from what we have discussed, a systematic calculational procedure is possible. This is done by keeping $m$ and $e^2$ as independent until the end, giving a parametric way to classify various contributions to any calculation. Further, there is a kinematic factor, of the form $(m/E_k)^0$, which helps in the further classification of corrections. The corrections to the lowest order results for the string tension then come out to be rather small.

In the language of ordinary perturbation theory, we may note that our analysis involves at least three different resummations. First, the definition of $H$ in terms of $A$, $\hat A$ is an infinite series with nonlocal terms. Second, in including $m$ in $H_0$, there is another resummation, since $m$ is really of order $e^2$. Thirdly, we also carry out the summation of contributions due to $F^{(2)}$ to get factors of $m/E_k$. An a priori attempt at these resummations, even with some guidance from the Hamiltonian analysis, is clearly very difficult.

Where do we go from here? Higher order corrections to the string tension, a better computation of glueball masses, extension to the torus and finite-temperature effects as well as the inclusion of matter fields are some of the questions we could now explore systematically.

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