A WEAK TYPE (1, 1) INEQUALITY
FOR MAXIMAL AVERAGES OVER CERTAIN SPARSE SEQUENCES

MICHAEL CHRIST

1. INTRODUCTION

To any strictly increasing sequence of nonnegative integers \((n_\nu : \nu \in \mathbb{N})\) is associated a maximal operator \(M\), which maps functions \(f \in \ell^1(\mathbb{Z})\) to \(Mf : \mathbb{Z} \to [0, +\infty]\), defined by

\[
Mf(x) = \sup_N N^{-1} \left| \sum_{\nu=1}^{N} f(x + n_\nu) \right|.
\]

The most fundamental example is the Hardy-Littlewood maximal function for \(\mathbb{Z}\), for which \(n_\nu \equiv \nu\). In this note we construct sequences which satisfy

\[
n_\nu \asymp \nu^m
\]

for arbitrary integer exponents \(m \geq 2\), and which have a certain algebraic character, for which the associated maximal operator is of weak type \((1, 1)\).

**Notation 1.1.** For any positive integer \(N\) and any \(n \in \mathbb{Z}\), \([n]_N\) denotes the unique element of \(\{0, \cdots, N-1\}\) congruent to \(n\) modulo \(N\). For \(n = (n_1, \cdots, n_d) \in \mathbb{Z}^d\), \([n]_N = ([n_1]_N, \cdots, [n_d]_N)\).

If \(n_\nu, c_\nu\) are sequences of positive integers which tend to infinity, we write \(n_\nu \asymp c_\nu\) to indicate that the ratio \(\frac{n_\nu}{c_\nu}\) is bounded above and below by strictly positive finite constants, independent of \(\nu \in \mathbb{N}\).

Let \((p_k)\) be a lacunary sequence of primes which satisfies

\[
\begin{align*}
p_{k+1} &\geq (1 + \delta)p_k \\
p_{k+1} &\leq C p_k
\end{align*}
\]

for some \(\delta > 0\) and \(C < \infty\). Let \(m \geq 2\) be a positive integer. Let \((a_k)\) be an auxiliary sequence of positive integers satisfying

\[
\begin{align*}
a_k &\leq C p_k^m \\
a_{k+1} &> a_k + p_k^m
\end{align*}
\]

Define \(S_k \subset \mathbb{Z}\) to be

\[
S_k = \{a_k + \sum_{r=1}^{m} p_k^{r-1}[j^r]_{p_k} : 0 \leq j < p_k\}.
\]

Thus \(S_k\) has cardinality

\[
|S_k| = p_k
\]
and $S_k \subset [a_k, a_k + p_k^m]$. Let $S = \bigcup_{k=1}^{\infty} S_k$, and define the sequence $(n_{\nu} : \nu \in \mathbb{N})$ to be the elements of $S$, listed in increasing order. The condition (1.5) ensures that every element of $S_{k+1}$ is strictly greater than every element of $S_k$.

**Theorem 1.1.** Let $m \geq 2$ be an integer. Let $(n_{\nu} : \nu \in \mathbb{N})$ be the subsequence of $\mathbb{N}$ constructed via the above recipe from a lacunary sequence of primes $p_k$ and a sequence of positive integers $a_k$ satisfying (1.3), (1.5), and (1.4). Then $n_{\nu} \sim \nu^m$, and the maximal function associated to $(n_{\nu})$ is of weak type $(1, 1)$ on $\mathbb{Z}$.

These sequences are closely related to examples given by Rudin [11] of $\Lambda(p)$ sets for $p = 4, 6, 8, \ldots$.

Bourgain [1], [2], [3] proved that the maximal operator associated to the sequence $a_{\nu} = \nu^m$ is bounded on $l^q(\mathbb{Z})$ for all $q > 1$, for arbitrary $m \in \mathbb{N}$, but the situation for the endpoint $q = 1$ was left unresolved. There have recently been several works concerned with weak type (1, 1) inequalities for maximal operators associated to sparse subsequences of integers. Buczolich and Mauldin [4], [5] have shown that the maximal operator associated to the sequence of all squares $(\nu^2 : \nu \in \mathbb{Z})$ is not of weak type (1, 1). LaVictore [9] has extended their method to show that the same holds for $(n^m : n \in \mathbb{Z})$ for all positive integer exponents $m$. In the positive direction, Urban and Zienkiewicz [12] have shown that for any real exponent $\alpha > 1$ sufficiently close to 1, the maximal function associated to the sequence $[\nu^{\alpha}]$ is of weak type (1, 1). LaVictoire [8] has shown that certain random sequences satisfying $n_{\nu} \sim \nu^m$ almost surely give rise to maximal operators which are of weak type (1, 1) for arbitrary real exponents $m \in (1, 2)$; it remains an open question whether the conclusion holds for these random sequences when $m \in [2, \infty)$.

Let $\sigma$ denote surface measure on the unit sphere $S^{d-1} \subset \mathbb{R}^d$. It remains an open question whether the maximal operator $Mf(x) = \sup_{k \in \mathbb{Z}} \int_{S^{d-1}} |f(x - 2^k y)| \, d\sigma(y)$ is of weak type (1, 1) in $\mathbb{R}^d$ for $d \geq 2$. Discrete analogues of continuum problems are often more delicate, but our analysis will exploit two discrete phenomena which lack obvious continuum analogues. For any strictly increasing sequence $(n_{\nu})$, $f \mapsto Mf$ is of weak type (1, 1) if and only if the same goes for $f \mapsto M(|f|)$, so it suffices to restrict attention to nonnegative functions. For any strictly increasing sequence $(n_{\nu})$, $c \sup_k |f * \mu_k| \leq Mf \leq C \sup_k |f * \mu_k|$ for all nonnegative functions $f$, where $\mu_k$ is the measure $\mu_k = 2^{-k-1} \sum_{n = 2^k (2^k + 1)}^{2^{k+1}} \delta_{n_{\nu}}$, and $\delta_n$ denotes the Dirac mass at $n$. Therefore $f \mapsto Mf$ is of weak type (1, 1), if and only if the same goes for $f \mapsto \sup_k |f * \mu_k|$.

The following is a sufficient condition, of Tauberian type, for such a maximal operator to be of weak type (1, 1). Although our application will be to operators on $\ell^1(\mathbb{Z})$, this result makes sense for any discrete group and is no more complicated to prove in that setting, so we give the general formulation.

**Theorem 1.2.** Let $G$ be a discrete group. Let $\gamma > 0$. Let $\mu_k, \nu_k : G \to \mathbb{C}$ satisfy
\begin{equation} \tag{1.8}
\text{The maximal operator } \sup_k |f| * |\nu_k| \text{ is of weak type (1, 1) on } G,
\end{equation}
each $\mu_k$ satisfies
\begin{equation} \tag{1.9}
|\text{support}(\mu_k)| \leq C 2^{k\gamma},
\end{equation}
and
\begin{equation} \tag{1.10}
\|f * (\mu_k - \nu_k)\|_{\ell^2} \leq C 2^{-k\gamma/2} \|f\|_{\ell^2} \quad \forall f \in \ell^2(\mathbb{Z}^d).
\end{equation}
Then the maximal operator $\sup_{k \in \mathbb{N}} |f * \mu_k|$ is of weak type (1, 1) on $G$. 
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For \(G = \mathbb{Z}^d\), the last hypothesis can be equivalently restated as

\[ ||\hat{\mu}_k - \hat{\nu}_k||_{L^\infty(\mathbb{T}^d)} \leq C 2^{-k\gamma/2}, \]

where \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\). In our applications, \(\mu_k\) will be a probability measure and hence \(\hat{\mu}_k(0)\) cannot be small. \(\nu_k\) will be a simpler measure, constructed in order to correct \(\hat{\mu}_k(\theta)\) for small \(\theta\).

The author is indebted to Steve Wainger for generous and essential advice concerning trigonometric sums, and for supplying reference \([10]\), and to Patrick LaVictoire for advice concerning the exposition.

2. ANALYSIS OF MAXIMAL OPERATORS

**Notation 2.1.** Let \(G\) be a discrete group. For any subsets \(A, B \subset G\), \(A + B = \{ab : a \in A \text{ and } b \in B\}\), where \(ab\) denotes the product of two group elements.\(^1\)

There is the simple inequality

\[ |A + B| \leq |A| \cdot |B|, \]

which has no straightforward analogue in continuum situations. Following Urban and Zienkiewicz \([12]\), we will make essential use of (2.1).

**Proof of Theorem [12]** Let \(f \in \ell^1\) and let \(\alpha > 0\). We seek an upper bound for \(\{x : \sup_k |f * \mu_k(x)| > \alpha\}\). Decompose \(f = \sum_{j=-\infty}^{\infty} f_j\) where

\[ f_j(x) = \begin{cases} f(x) & \text{if } |f(x)| \in [2^j, 2^{j+1}), \\ 0 & \text{otherwise}. \end{cases} \]

The functions \(f_j\) have pairwise disjoint supports, so \(\sum_j \|f_j\|_{\ell^1} = \|f\|_{\ell^1}\).

For each \(j \in \mathbb{Z}\) define the exceptional set

\[ E_j = \bigcup_{2^{k\gamma} < \alpha^{-1}2^j} (\text{support } (f_j) + \text{support } (\mu_k)) \]

and

\[ E = \bigcup_{j \in \mathbb{Z}} E_j. \]

Since \(\|f_j\|_1 \sim 2^j |\text{support } (f_j)|\),

\[ |E_j| \leq \sum_{k:2^{k\gamma} < \alpha^{-1}2^j} |\text{support } (f_j)| \cdot 2^{k\gamma} \leq 2 \sum_{k:2^{k\gamma} < \alpha^{-1}2^j} 2^{-j} 2^{k\gamma} \|f_j\|_1 \leq C \gamma \alpha^{-1} \|f_j\|_1 \]

and therefore

\[ |E| \leq C \alpha^{-1} \|f\|_1. \]

Set \(\lambda_k = \mu_k - \nu_k\). For \(x \notin E\),

\[ f * \mu_k(x) = \sum_{j \leq 2^{k\gamma} \alpha} f_j * \mu_k(x) \]

\(^1\)The additive notation is used for \(A + B\), even though \(G\) is not assumed to be Abelian, in order to simplify an expression below.
Therefore

\[ |f \ast \mu_k(x)| \leq \sum_{2^j \leq 2^{k+s}} |f_j| \ast |\nu_k|(x) + \sum_{2^j \leq 2^{k+s}} |f_j \ast \lambda_k(x)| \]

\[ \leq |f| \ast |\nu_k|(x) + \sum_{2^j \leq 2^{k+s}} |f_j \ast \lambda_k(x)|. \]

Therefore

\[(2.7) \quad \{x \in G : \sup_k |f \ast \mu_k(x)| > \alpha\} \]

\[ \leq \{E\} + \{x \in G : \sup_k |f| \ast |\nu_k|(x) > \frac{\alpha}{2}\} + \{x \in G : \sup_k \sum_{2^j \leq 2^{k+s}} |f_j \ast \lambda_k(x)| > \frac{\alpha}{2}\}. \]

Therefore, by hypothesis (1.8), it suffices to show that

\[(2.8) \quad \{x \in G : \sup_k \sum_{2^j \leq 2^{k+s}} |f_j \ast \lambda_k(x)| > \frac{\alpha}{2}\} \leq C \alpha^{-1} \|f\|_1. \]

For \(0 \leq s \in \mathbb{Z}\) define

\[(2.9) \quad G_s(x) = \left( \sum_k |f_j(k,s) \ast \lambda_k(x)|^2 \right)^{1/2} \]

where \(j(k,s)\) is the unique integer satisfying \(2^{k+s} \alpha \leq 2^{j(k,s)} < 2^{k+s+1} \alpha\). A generous upper bound is

\[(2.10) \quad \sup_k \sum_{2^j \leq 2^{k+s}} |f_j \ast \lambda_k(x)| \leq \sum_{s=0}^\infty G_s(x) \]

Therefore by hypothesis (1.10),

\[ \|G_s\|_2^2 = \sum_k \|f_j(k,s) \ast \lambda_k\|_2^2 \leq C \sum_k 2^{-k} \|f_j(k,s)\|_2^2 \]

\[ \leq C \sum_k 2^{-k} \|f_j(k,s)\|_\infty \|f_j(k,s)\|_1 \leq C \sum_k 2^{-k} 2^{j(k,s)} \|f_j(k,s)\|_1 \]

\[ \leq C \sum_k 2^{-k+2k-s} \alpha \|f_j(k,s)\|_1 = C 2^{-s} \alpha \sum_k \|f_j(k,s)\|_1 \leq C 2^{-s} \alpha \|f\|_1. \]

Therefore \(\|\sum_s G_s\|_2^2 \leq C \alpha \|f\|_1\) and consequently

\[(2.11) \quad \{x : \sum_{s=0}^\infty G_s(x) > \frac{\alpha}{2}\} \leq 4 \alpha^{-2} \|\sum_s G_s\|_2^2 \leq C \alpha^{-1} \|f\|_1 = C \alpha^{-1} \|f\|_1. \]

This concludes the proof of Theorem 1.2.

The use of an \(L^2\) bound on the complement of an exceptional set in order to obtain a weak type \((1,1)\) inequality was pioneered by Fefferman [7], and was applied to maximal functions in [6]. Exceptional sets constructed as algebraic sums of supports, adapted to the measures \(\mu_k\), were used for a continuum analogue of this problem in Theorems 3 and 4 of [6].
3. Construction of examples

3.1. The Fourier transform on a finite cyclic group. For any positive integer \( N \) let \( \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \) be the cyclic group of order \( N \). We will often identify elements of \( \mathbb{Z}_N \) with elements of \([0, 1, \cdots , N - 1]\) in the natural way.

The dual group of \( \mathbb{Z}_p^m \) may, and will, be identified with \( \mathbb{Z}_p^m \). The most convenient normalization of the Fourier transform for our purposes is

\[
\hat{f}(\xi) = \sum_k f(k)e^{-2\pi ik\cdot \xi/p} \quad \text{for } \xi \in \mathbb{Z}_p^m
\]

where \( k \cdot \xi \) denotes the usual Euclidean inner product of two elements of \([0, 1, \cdots , p-1]^m\), regarded as elements of \( \mathbb{Z}^m \). This Fourier transform satisfies

\[
\|\hat{f}\|_2 = p^{-m/2}\|f\|_2 \quad \text{and} \quad f(k) = p^{-m}\sum_\xi \hat{f}(\xi)e^{2\pi ik\cdot \xi/p}.
\]

The convolution of two functions is \( f \ast g(x) = \sum_y f(x - y)g(y) \). Products and convolutions are related by

\[
\hat{f} \ast \hat{g} = \hat{f} \cdot \hat{g} \quad \text{and} \quad \hat{f}g = p^{-m}\hat{f} \ast \hat{g}.
\]

Therefore

\[
\|\hat{f}g\|_\infty \leq p^{-m}\|\hat{f}\|_1\|\hat{g}\|_\infty
\]

and

\[
\|f \ast g\|_2 = p^{-m/2}\|\hat{f} \ast \hat{g}\|_2 \leq p^{-m/2}\|\hat{g}\|_\infty\|\hat{f}\|_2 = \|\hat{g}\|_\infty\|f\|_2.
\]

3.2. On certain exponential sums. Define the probability measure \( \sigma_{p,m} \) on \( \mathbb{Z}_p^m \) to be

\[
\sigma_{p,m} = p^{-1}\sum_{k=0}^{p-1} \delta(k,k^2,\cdots,k^m).
\]

Thus

\[
|\text{support } (\sigma_{p,m})| = p = |\mathbb{Z}_p^m|^{1/m}.
\]

This will lead to examples of Theorem \[1.2\] with exponent \( \gamma = \frac{1}{m} \).

Lemma 3.1 (Weil). Let \( m \geq 1 \), and let \( p > m \) be prime. Then

\[
|\sigma_{p,m}(\xi)| \leq (m - 1)p^{-\frac{1}{2}} \quad \text{for all } 0 \neq \xi \in \mathbb{Z}_p^m.
\]

Three comments are in order. Firstly, this illustrates a general principle that \( \mathbb{Z}_p \) has only one scale when \( p \) is prime. In contrast, the most natural example of a sparsely supported measure on \( \mathbb{R}^d \) whose Fourier transform exhibits power law decay is surface measure \( \sigma \) on the unit sphere \( S^{d-1} \) for \( d \geq 2 \), which satisfies \( |\hat{\sigma}(\xi)| \sim |\xi|^{-(d-1)/2} \) for generic large \( \xi \); the size of \( \hat{\sigma} \) is best described for most \( \xi \) by a power of \( |\xi| \), rather than by a constant.

Secondly, no weaker bound \( O(p^{-\frac{1}{2}+\delta}) \) will suffice in the construction below to yield a sequence satisfying the hypotheses of Theorem \[1.2\]

Thirdly, the bound \( O(p^{-1/2}) \) is the best that can hold for a measure \( \sigma \) whose support has cardinality \( p \), unless \( \|\sigma\|_{L^1} \ll 1 \). Indeed, let \( E \) be the support of \( \sigma \). Let \( \sigma_0 \) be the constant function
\[ \sigma_0(n) = cp^{-m} \] for all \( n \in \mathbb{Z}_p^m \), where \( c = \sum_n \sigma(n) \). Then \( \widehat{\sigma - \sigma_0(\xi)} \) vanishes at \( \xi = 0 \), and 
\[
\| \sigma - \sigma_0 \|_{E(\ell)} \leq |E|^{1/2}\| \sigma - \sigma_0 \|_{\ell_2}^{1/2} = p^{1/2}p^{-m/2}\| \sigma - \sigma_0 \|_{\ell_2} \leq p^{1/2}p^{-m/2}|\mathbb{Z}_p^m|^{1/2}\| \widehat{\sigma - \sigma_0} \|_{\ell_\infty}^{1/2} = p^{1/2}\| \widehat{\sigma - \sigma_0} \|_{\ell_\infty}^{1/2}.
\]
Since \( \| \sigma_0 \|_{E(\ell)} \leq p^{-m}\| \sigma \|_1|E| \), this implies that 
\[
\| \sigma \|_1 = \| \sigma \|_{E(\ell)} \leq p^{1/2}\sup_{\xi \neq 0} |\widehat{\sigma(\xi)}|^{1/2} + p^{1-m}\| \sigma \|_1,
\]
so 
\[
\| \sigma \|_1 \leq 2p^{1/2}\sup_{\xi \neq 0} |\widehat{\sigma(\xi)}|^{1/2}.
\]
Thus the construction is tightly constrained.

**Proof of Lemma 3.1**

\[
\widehat{\sigma_{p,m}}(\xi) = p^{-1}\sum_{k=0}^{p-1} e^{-2\pi ik(k\xi_1 + k^2\xi_2 + \cdots + k^m\xi_m)/p}.
\]
The sum, without the initial factor \( p^{-1} \), is a well studied quantity whose absolute value is \( \leq (m - 1)p^{1/2} \). An elementary proof may be found in [10], Theorem 5.38.

Denote by \( \delta_k \) the function \( \delta_k(n) = 1 \) if \( n = k \) and \( = 0 \) if \( n \neq k \). The measures \( \sigma_{p,m}^0 = |\mathbb{Z}_p^m|^{-1}\sum_{k \in \mathbb{Z}_p^m} \delta_k \) satisfy 
\[
\widehat{\sigma_{p,m}^0}(\xi) = \begin{cases} 1 = \widehat{\sigma_{p,m}^0}(0) & \text{if } \xi = 0 \\ 0 & \text{else} \end{cases}
\]
and therefore \( \sigma_{p,m}^* = \sigma_{p,m} - \sigma_{p,m}^0 \) satisfies 
\[
(3.9) \quad \| \sigma_{p,m}^* \|_{\ell_\infty} \leq (m - 1)p^{-1/2}.
\]

### 3.3. Transference to \( \mathbb{Z}^m \)

We wish to transfer \( \sigma_{p,m}^* \) to a measure on \( \mathbb{Z} \), preserving this \( L^\infty \) Fourier transform bound, in order to obtain the desired examples. The most straightforward attempt apparently does not work, but the following more roundabout procedure, combining an extension to \( \mathbb{Z}_3^m \) with cutoff functions, does the job. It will be convenient to transfer first to \( \mathbb{Z}_3^m \), then to \( \mathbb{Z} \) in a separate step.

Consider \( \mathbb{Z}_3^m \), which we identify with \( [-p, 2p - 1]^m \). Likewise we identify \( \mathbb{Z}_3p \) with \( [-p, 2p - 1] \). Assume that \( p \) is odd. The function 
\[
\kappa(k) = \sigma_{p,m}^*([k]_p)
\]
is \( p \)-periodic on \( \mathbb{Z}_3p \) and satisfies 
\[
(3.10) \quad \widehat{\kappa}(\xi) = \begin{cases} 0 & \text{unless each } \xi_j \in [-p, 2p - 1] \text{ is divisible by } 3 \\ 3^m\widehat{\sigma_{p,m}^*}(\xi_1/3, \cdots, \xi_m/3) & \text{otherwise}. \end{cases}
\]
In particular, 
\[
(3.11) \quad \| \widehat{\kappa} \|_{\ell_\infty} \leq Cp^{-1/2}.
\]
Define $\varphi : \mathbb{Z}_{3p} \to \mathbb{R}$ by
\[
\begin{align*}
\varphi(i) &= 1 \quad \text{if } i \in [0, p - 1] \\
\varphi(i) &= 0 \quad \text{if } i \in [-p, -p + (p - 1)/2] \\
\varphi(i) &= 0 \quad \text{if } i \in [p - 1 + (p - 1)/2, 2p - 1] \\
\varphi \text{ is affine} &\quad \text{on the interval } [-p + (p - 1)/2, 0] \\
\varphi \text{ is affine} &\quad \text{on the interval } [p - 1, p - 1 + (p - 1)/2].
\end{align*}
\]
(3.12)

Define $\rho = \rho_{p,m} : \mathbb{Z}_{3p} \to \mathbb{R}$ by
\[
\rho = \varphi \kappa.
\]
(3.13)

$\rho$ is nonnegative, and
\[
\rho(k) \equiv \sigma^*(p,m)(k) \quad \forall k \in [0, p - 1].
\]
(3.14)

Define $\rho^\sharp : \mathbb{Z}^m \to \mathbb{R}$ by
\[
\rho^\sharp(k) = \begin{cases} 
\rho([k]_{3p}) & \text{for } k \in [-p, 2p - 1]^m \\
0 & \text{else}
\end{cases}
\]
(3.15)

where $k$ is interpreted as an element of $\mathbb{Z}^m$ on the left-hand side, and as an element of $\mathbb{Z}_{3p}^m$ on the right.

**Lemma 3.2.**

\[
\|\hat{\rho}^\sharp\|_{L^\infty(\mathbb{T}^m)} \leq C p^{-1/2}.
\]
(3.16)

**Proof.** Let $\theta \in \mathbb{T}^m = [0, 1]^m$. Write $\theta = \xi/3p + \eta$ where $\xi \in \mathbb{Z}^m$ and $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$ satisfies $|\eta_j| \leq C p^{-1}$ for all $j$.

\[
\hat{\rho}^\sharp(\theta) = \sum_{k \in \mathbb{Z}^m} \rho^\sharp(k) e^{-2\pi i k \cdot \theta} = \sum_{k \in [-p, 2p - 1]^m} \varphi(k) \kappa(k) e^{-2\pi i k \cdot \eta} e^{-2\pi i \xi / 3p}.
\]

Interpret this last expression as the Fourier transform on the group $\mathbb{Z}_{3p}^m$, evaluated at $\xi$, of $\psi \kappa$ where $\psi : \mathbb{Z}_{3p}^m \to \mathbb{R}$ is defined by $\psi(k) = \varphi(k) e^{-2\pi i k \cdot \eta}$. By (3.4), since $\hat{\kappa} = O(p^{-1/2})$, it suffices to show that $\|\psi\|_{L^1} \leq C p^m$.

$\psi(k)$ factors as $\prod_{j=0}^m \varphi(k_j) e^{-2\pi i k_j \cdot \eta_j}$, so its Fourier transform likewise factors. Thus it suffices to prove that

\[
\sum_{\xi=0}^{3p-1} \sum_{n=-p}^{2p-1} | \sum_{n=-p}^{2p-1} \varphi(n) e^{-2\pi i n \xi / 3p} | \lesssim p \quad \text{provided that } |\xi| \leq p^{-1}.
\]
(3.17)

For $\xi = 0$ there is the trivial bound $O(p)$, since $\|\varphi\|_{\infty} = 1$. For $\xi \in [1, \ldots, 3p - 1]$, we employ the summation by parts formula

\[
\sum_{n=-p}^{2p-1} a_n b_n = b_{2p} A_{2p-1} - \sum_{n=-p}^{2p-1} A_n \Delta b_n \quad \text{where } A_n = \sum_{j=-p}^{n} a_j \text{ and } \Delta b_n = b_{n+1} - b_n.
\]
Define $\Phi(n) = \varphi(n)e^{-2\pi in\varepsilon}$. For convenience of notation, extend $\Phi$ to be a $3p$-periodic function on $\mathbb{Z}$. Sum by parts with $a_n = e^{-2\pi in\varepsilon/3p}$ and $b_n = \Phi(n)$ to obtain, since $\Phi(2p) = \Phi(-p) = 0$,
\[
\sum_{n=-p}^{2p-1} \Phi(n)e^{-2\pi in\varepsilon/3p} = \sum_{n=-p}^{2p-1} \Delta \Phi(n) \frac{e^{-2\pi i(n+1)\varepsilon/3p} - e^{2\pi i\varepsilon/3}}{e^{-2\pi i\varepsilon/3p} - 1} = -(e^{-2\pi i\varepsilon/3p} - 1)^{-1} \sum_{n=-p}^{2p-1} \Delta \Phi(n)(e^{-2\pi i(n+1)\varepsilon/3p} - e^{2\pi i\varepsilon/3}) = -(e^{-2\pi i\varepsilon/3p} - 1)^{-1}e^{-2\pi i\varepsilon/3p} \sum_{n=-p}^{2p-1} \Delta \Phi(n)e^{-2\pi in\varepsilon/3p}
\]
since $\sum_{n=-p}^{2p-1}(\Phi(n+1) - \Phi(n)) = 0$. A second summation by parts yields a bound
\[
(3.18) \quad \left| \sum_{n=-p}^{2p-1} \Phi(n)e^{-2\pi in\varepsilon/3p} \right| \leq C \|e^{-2\pi i\varepsilon/3p} - 1\|_{L^1}^2 \|\Delta^2 \Phi\|_{L^1}. 
\]
Clearly $\|\Delta^2 \Phi\|_{L^1} \leq Cp^{-1}$. It is straightforward to verify that
\[
\|e^{-2\pi i\varepsilon/3p} - 1\|_{L^1([0,1])} \leq Cp^2.
\]
Combining these bounds yields the required inequality (3.17). \qed

3.4. Transference from $\mathbb{Z}^m$ to $\mathbb{Z}$. The next step is to transfer $\rho^\dagger$ from $\mathbb{Z}^m$ to $\mathbb{Z}$. Define $F : \mathbb{Z}^m \to \mathbb{Z}$ by $F(k_1, \ldots, k_m) = \sum_{j=1}^m p^{j-1}k_j$. $F$ maps $[0, p-1]^m$ bijectively to $[0, p^m - 1]$. For $n \in \mathbb{Z}$ define $\rho^\dagger(n)$ to be the pushforward of $\rho^\dagger$ via $F$, that is,
\[
(3.19) \quad \rho^\dagger(n) = \sum_{k:F(k)=n} \rho^\dagger(k).
\]
Then
\[
\sum_{n \in \mathbb{Z}} e^{-2\pi i\theta n}\rho^\dagger(n) = \sum_{k \in [-\ell,\ell]^m} \rho^\dagger(k)e^{-2\pi i\theta \sum_{j=1}^m p^{j-1}k_j} = \hat{\rho}(\theta, p\theta, p^2\theta, \ldots, p^{m-1}\theta),
\]
so
\[
(3.20) \quad \|\rho^\dagger\|_{L^\infty(T)} \leq \|\hat{\rho}\|_{L^\infty(T^m)} \leq Cp^{-1/2}.
\]
The Fourier transform on the left-hand side is that for $\mathbb{Z}$; the one on the right is that for $\mathbb{Z}^m$. The same bound holds, of course, for any translate of $\rho^\dagger$.

We have defined a linear, positivity preserving operator $\Gamma : \ell^1(\mathbb{Z}^m) \to \ell^1(\mathbb{Z})$; $\Gamma$ extends a function to $\mathbb{Z}^m$, multiplies by the cutoff function $\phi$, transplants the result to $\mathbb{Z}^m$, then pushes it forward to $\mathbb{Z}$. $\rho^\dagger = \Gamma(\sigma_{p,m}) - \Gamma(\sigma_{p,m}^0)$ is expressed as $\mu^\dagger - \nu^\dagger$ where the summands have the following properties. $\mu^\dagger$ is nonnegative and
\[
(3.21) \quad \mu^\dagger \geq p^{-1} \sum_{j=-p}^{2p-1} \delta_{g(j)}
\]
where $g(j) = \sum_{r=1}^m p^{r-1}[j^r]_p$, while $\nu^\dagger$ is supported in $[-Cp^m, Cp^m]$ and satisfies
\[
(3.22) \quad \|\nu^\dagger\|_{L^\infty} \leq Cp^{-m}.
\]
Finally, because $\Gamma$ is linear and preserves positivity, $\phi \geq 0$, and $\phi \equiv 1$ on $[0, p-1]^m$, $\mu^\dagger \geq p^{-m} \sum_{n \in S} \delta_n$ where $S = \{g(j) : j \in [0, p-1]\}$. 
Now let \((p_k)\) be any sequence of odd primes satisfying (1.3), and let \((a_k)\) be an arbitrary sequence of natural numbers satisfying (1.5) and (1.4). Define \(\lambda_k(n) = \rho_{p_k}^m(n - a_k).\) \(\lambda_k\) decomposes as \(\lambda_k = \tilde{\mu}_k - \nu_k\) where \(\tilde{\mu}_k(n) = \mu_{k}^M(n - a_k)\) and \(\nu_k(n) = \nu_k^{M}(n - a_k)\). The pair \(\tilde{\mu}_k, \nu_k\) satisfies the hypotheses of Theorem 1.2; the maximal operator \(f \mapsto \sup_k |f * \nu_k|\) is dominated by a constant multiple of the Hardy-Littlewood maximal operator by virtue of (3.22) and (1.4). Therefore the maximal operator \(f \mapsto \sup_k |f * \tilde{\mu}_k|\) is of weak type \((1, 1)\) on \(\mathbb{Z}\).

Since each \(\tilde{\mu}_k\) is nonnegative, the same applies to \(\sup_k |f * \mu_k^*|\) for any sequence of functions \(0 \leq \mu_k^* \leq \tilde{\mu}_k\). Since \(\tilde{\mu}_k \geq p_k^{-m} \sum_{n \in S_k} \delta_n = \mu_k\), we may set \(\mu_k^* = \mu_k\) to deduce Theorem 1.1 \(\square\)

### 3.5. A second set of examples

The following variant produces examples which are less sparse, with \(n_\nu \asymp \nu^m\) for \(m = \frac{d + 1}{d}\), for any positive integer \(d\). Fix a positive integer \(d \geq 1\). Let \((p_k : k \in \mathbb{N})\) be any lacunary sequence of primes. Let \((a_k)\) be an auxiliary sequence of natural numbers satisfying

\[
\begin{align*}
    a_k & \leq C' p_k^{d + 1} \\
    a_{k+1} & \geq a_k + p_k^{d + 1}
\end{align*}
\]

Define

\[
n(j, p) = j_1 + pj_2 + \cdots + p^{d-1}j_d + p^d [\lfloor j \rfloor^2]_p
\]

for \(j \in [0, p - 1]^d\) and

\[
S_k = \{ n(j, p_k) + a_k : j \in [0, p - 1]^d \}.
\]

Set \(S = \cup_{k \in \mathbb{N}} S_k\), and let the sequence \((n_\nu : \nu \in \mathbb{N})\) be the elements of \(S\), listed in increasing order.

**Theorem 3.3.** Let the sequences of primes \((p_k)\) and natural numbers \((a_k)\) satisfy (1.3) and (3.23). Then the associated subsequence \((n_\nu)\) of \(\mathbb{N}\) satisfies \(n_\nu \asymp \nu^{(d+1)/d}\), and the maximal function associated to \((n_\nu)\) is of weak type \((1, 1)\) on \(\mathbb{Z}\).

The proof of Theorem 3.3 is essentially identical to that of Theorem 1.1 except that the exponential sum bound of Lemma 3.1 is replaced by the following simpler bound. For \(\xi \in \mathbb{Z}_p^{d+1}\) we write

\[
\xi = (\xi', \xi_d + 1) \in \mathbb{Z}_p^d \times \mathbb{Z}_p.
\]

**Lemma 3.4.** Let \(d\) be any positive integer, and let \(p\) be any prime.\n
\[
p^{-d} \sum_{n \in [0, p - 1]^d} e^{-2\pi i (n \cdot \xi' + |n|^2 \xi_d + 1)/p} \leq p^{-d/2} \quad \text{for all } 0 \neq \xi \in \mathbb{Z}_p^{d+1}.
\]

Here \(|n|^2 = \sum_{j=1}^d n_j^2\) where \(n = (n_1, \ldots, n_d)\).

These exponential sums factor as products of \(d\) Gauss sums, so the lemma follows from the fact that

\[
\sum_{n=0}^{p-1} |e^{-2\pi i (an + bn^2)/p}| = \begin{cases} 
  p^{1/2} & \text{if } b \neq 0 \\
  0 & \text{if } b = 0 \text{ and } a \neq 0 \\
  p & \text{if } a = b = 0.
\end{cases}
\]

Thus the measure

\[
\sigma = p^{-d} \sum_{n \in \mathbb{Z}_p^d} \delta_n([n^2]_p)
\]

satisfies \(|\hat{\sigma}(\xi)| \leq p^{-d/2}\) whenever \(\xi \neq 0\). The proof of Theorem 1.1 therefore applies. \(\square\)
Remark 3.1. Theorem 1.1 produces sequences satisfying $n_\nu \asymp \nu^m$ for $m = 2, 3, 4, \cdots$. For any prescribed rational exponent $r > 2$, an example satisfying $n_\nu \asymp \nu^r$ can be constructed by using one value of $m$ for some indices $k$ and a second value for the others; details are left to the reader. For any rational $r \in (1, 2)$, an example may be constructed in the same way using instead the construction of Theorem 3.3.

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MICHAEL CHRIST, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840, USA

E-mail address: mchrist@math.berkeley.edu