Let $X$ be a complex variety or an analytic space and $x \in X$ a point. A formal arc through $x$ is a morphism $\phi : \text{Spec} \mathbb{C}[[t]] \to X$ such that $\phi(0) = x$. The set of formal arcs through $x$ – denoted by $\hat{\text{Arc}}(x \in X)$ – is naturally a (non-Noetherian) scheme.

A preprint of Nash, written in 1968 but only published later as [Nas95], describes an injection – called the Nash map – from the irreducible components of $\hat{\text{Arc}}(x \in X)$ to the set of so called essential divisors. These are the divisors whose center on every resolution $\pi : X' \to X$ is an irreducible component of $\pi^{-1}(x)$. The Nash problem asks if this map is also surjective or not. Surjectivity fails in dimension $s \geq 3$ [IK03, dF12] but holds in dimension 2 [FdBP12b].

In all dimensions, the most delicate cases are singularities whose resolutions contain many rational curves. For example, while it is very easy to describe all arcs and their deformations on Du Val singularities of type $A$, the type $E$ cases have been notoriously hard to treat [PS12, Per12].

The first aim of this note is to determine the irreducible components of the arc space of $cA$-type singularities in all dimensions. In Section 1 we prove the following using quite elementary arguments.

**Theorem 1.** Let $f(z_1, \ldots, z_n)$ be a holomorphic function whose multiplicity at the origin is $m \geq 2$. Let $X := (xy = f(z_1, \ldots, z_n)) \subset \mathbb{C}^{n+2}$ denote the corresponding $cA$-type singularity. Assume that $\text{dim } X \geq 2$.

1. $\hat{\text{Arc}}(0 \in X)$ has $(m-1)$ irreducible components $\hat{\text{Arc}}_i(0 \in X)$ for $0 < i < m$.

2. There are dense, open subsets $\hat{\text{Arc}}_i(0 \in X) \subset \hat{\text{Arc}}_i(0 \in X)$ such that

$$(\psi_1(t), \psi_2(t), \phi_1(t), \ldots, \phi_n(t)) \in \hat{\text{Arc}}_i(0 \in X)$$

iff $\text{mult} \psi_1(t) = i$, $\text{mult} \psi_2(t) = m - i$ and $\text{mult} f(\phi_1(t), \ldots, \phi_n(t)) = m$.

We found it much harder to compute the set of essential divisors and we have results only if $\text{mult}_0 f = 2$. If $\text{dim } X = 3$ then, after a coordinate change, we can write the equation as $(xy = z^2 - u^m)$. Already [Nas95] proved that these singularities have at most 2 essential divisors: an easy one obtained by blowing-up the origin and a difficult one obtained by blowing-up the origin twice. In Section 2 we use ideas of [dF12] to determine the cases when the second divisor is essential. The following is obtained by combining Theorem 1 and Proposition 9.

**Example 2.** For the singularities $X_m := (xy = z^2 - u^m) \subset \mathbb{C}^4$ the Nash map is not surjective for odd $m \geq 5$ but surjective for even $m$ and for $m = 3$.

Thus the simplest counter example to the Nash conjecture is the singularity

$$(x^2 + y^2 + z^2 + t^5 = 0) \subset \mathbb{C}^4.$$
In higher dimensions our answers are less complete. We describe the situation for the divisors obtained by the first and second blow-ups as above, but we do not control other exceptional divisors. Using Theorem 1 and Proposition 24 we get the following partial generalization of Example 2.

**Example 3.** Let \( g(u_1, \ldots, u_r) \) be an analytic function near the origin. Set \( m = \text{mult}_0 g \) and let \( g_m \) denote the degree \( m \) homogeneous part of \( g \). If \( m \geq 4 \) and the Nash map is surjective for the singularity

\[
X_g := (xy = z^2 - g(u_1, \ldots, u_r)) \subset \mathbb{C}^{r+3}
\]

then \( g_m(u_1, \ldots, u_r) \) is a perfect square.

Since we do not determine all essential divisors, the cases when \( g_m(u_1, \ldots, u_r) \) is a perfect square remain undecided.

On the one hand, this can be interpreted to mean that the Nash conjecture hopelessly fails in dimensions \( \geq 3 \). On the other hand, the proof leads to a reformulation of the Nash problem and to an approach that might be feasible, at least in dimension 3; see Section 5.

In Section 4 we observe that the deformations constructed in Section 1 also lead to an enumeration of the irreducible components of the space of short arcs – introduced in [KN13] – for \( cA \)-type singularities.

**Question 4** (Arcs on \( cDV \) singularities). It is easy to see that Theorem 1 is equivalent to saying that the image of every general arc on \( X \) is contained in an \( A \)-type surface section of \( X \).

It is natural to ask if this holds for all \( cDV \) singularities. That is, let \( (0 \in X) \subset \mathbb{C}^n \) be a hypersurface singularity such that \( X \cap L^3 \) is a Du Val singularity for every general 3-dimensional linear space (or smooth 3–fold) \( 0 \in L^3 \subset \mathbb{C}^n \).

Let \( \phi \) be a general arc on \( X \). Is it true that there is a 3–fold \( L^3 \subset \mathbb{C}^n \) containing the image of \( \phi \) such that \( X \cap L^3 \) is a Du Val singularity?

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1. **Arcs on \( cA \)-type singularities**

**Definition 5** (\( cA \)-type singularities). In some coordinates write a hypersurface singularity as

\[
X := (f(x_1, \ldots, x_{n+1}) = 0) \subset \mathbb{C}^{n+1}.
\]

Assume that \( X \) is singular at the origin and let \( f_2 \) denote the quadratic part of \( f \).

If \( \text{mult}_0 f = 2 \) then \( (f_2 = 0) \) is the tangent cone of \( X \) at the origin. We say that \( X \) has \( cA \)-type if rank \( f_2 \geq 2 \) and \( cA_1 \)-type that rank \( f_2 \geq 3 \). By the Morse lemma, if rank \( f_2 = r \) then we can choose local analytic or formal coordinates \( y_i \) such that

\[
f = y_1^2 + \cdots + y_r^2 + g(y_{r+1}, \ldots, y_{n+1}) \quad \text{where} \quad \text{mult}_0 g \geq 3.
\]

In the sequel we also use other forms of the quadratic part if that is more convenient.

Note that by adding 2 squares in new variables we get a map from hypersurface singularities in dimension \( n-2 \) (modulo isomorphism) to \( cA \)-type hypersurface singularities in dimension \( n \) (modulo isomorphism). This map is one-to-one and
onto; see [AGZV85, Sec.11.1]. Thus cA-type singularities are quite complicated in large dimensions.

We rename the coordinates and write a cA-type singularity as

\[ X := (xy = f(z_1, \ldots, z_n)) \]

Thus an arc through the origin is written as

\[ t \mapsto (\psi_1(t), \psi_2(t), \phi_1(t), \ldots, \phi_n(t)) \]

where \( \psi_i, \phi_j \) are power series such that \( \text{mult} \psi_i, \text{mult} \phi_j \geq 1 \) for \( i = 1, 2 \) and \( j = 1, \ldots, n \). We set \( \vec{\phi}(t) = (\phi_1(t), \ldots, \phi_n(t)) \).

A deformation of \( \vec{\phi}(t) \) is given by power series \( (\Phi_1(t,s), \ldots, \Phi_n(t,s)) \). Then we compute

\[ f(\Phi_1(t,s), \ldots, \Phi_n(t,s)) \in \mathbb{C}[[t,s]] \]

and try to factor it to obtain

\[ \Psi_1(t,s)\Psi_2(t,s) = f(\Phi_1(t,s), \ldots, \Phi_n(t,s)) \in \mathbb{C}[[t,s]]. \]

Usually this factoring is not possible, but Newton’s method of rotating rulers says that

\[ f(\Phi_1(t,s^r), \ldots, \Phi_n(t,s^r)) \]

factors for some \( r \geq 1 \).

**6 (Proof of Theorem H).** After a linear change of coordinates we may assume that \( z_1^m \) appears in \( f \) with nonzero constant coefficient.

Set \( D := \text{mult}_f f(\phi_1(t), \ldots, \phi_n(t)) \). Assume first that \( D < \infty \) and consider

\[ F(t,s) := f(\phi_1(t) + st, \phi_2(t), \ldots, \phi_n(t)) = \sum_i \frac{\partial f}{\partial z_1^i} \cdot (st)^i. \]

We know that \( t^m \) divides \( F(s,t) \) (since \( \text{mult}_0 f = m \)) and \( (st)^m \) appears in \( F \) with nonzero coefficient (since \( z_1^m \) appears in \( f \) with nonzero coefficient). Thus \( t^m \) is the largest \( t \)-power that divides \( F(s,t) \).

Furthermore, \( t^D \) is the smallest \( t \)-power that appears in \( F \) with nonzero constant coefficient. Thus, by Lemma [4] there is an \( r \geq 1 \) such that

\[ F(t,s^r) = u(t,s) \prod_{i=1}^D (t - \sigma_i(s)) \]

where \( u(0,0) \neq 0 \) and \( \sigma_i(0) = 0 \). Furthermore, exactly \( m \) of the \( \sigma_i \) are identically zero.

For \( j = 1, 2 \) write \( \psi_j(t) = t^a_j v_j(t) \) where \( v_j(0) \neq 0 \). Note that \( a_1 + a_2 = D \) and \( u(t,0) = v_1(t)v_2(t) \).

Divide \( \{1, \ldots, D\} \) into two disjoint subsets \( A_1, A_2 \) such that \( |A_j| = a_j \) and they both contain at least 1 index \( i \) such that \( \sigma_i(t) \equiv 0 \). Finally set

\[ \Psi_1(t,s) = v_1(t) \cdot \prod_{i \in A_1} (t - \sigma_i(s)) \quad \text{and} \quad \Psi_2(t,s) = u(t,s) \cdot \frac{v_1(t)}{v_1(t)} \cdot \prod_{i \in A_2} (t - \sigma_i(s)). \]

Then

\[ (\Psi_1(t,s), \Psi_2(t,s), \psi_1(t) + st, \phi_2(t), \ldots, \phi_n(t)) \]
is a deformation of \( (\psi_1(t), \psi_2(t), \phi_1(t), \ldots, \phi_n(t)) \) whose general member is in the \( r \)th irreducible component as in [12] iff exactly \( r \) of the \( \{\sigma_i : i \in A_1\} \) are identically zero.

(This also shows that arcs with \( \text{mult} \, \psi_1(t) \geq m-1 \) and \( \text{mult} \, \psi_2(t) \geq m-1 \) constitute the intersection of all of the irreducible components.)

If \( D = \infty \), that is, when \( f(\phi_1(t), \ldots, \phi_n(t)) \) is identically zero, we need to perform some similar preliminary deformations first.

First, if both \( \psi_1(t) \), \( \psi_2(t) \) are identically zero then we can take

\( (st, 0, \phi_1(t), \phi_2(t), \ldots, \phi_n(t)) \).

Hence, up-to interchanging \( x \) and \( y \), we may assume that \( d := \text{mult} \, \psi_1(t) < \infty \). Again assuming that \( z_i^n \) appears in \( f \) with nonzero coefficient, we see that

\[ F(t, s) := f(\phi_1(t) + st^{d+1}, \phi_2(t), \ldots, \phi_n(t)) \]

is not identically zero and divisible by \( t^{d+1} \). Thus \( F(t, s)/\psi_1(t) \) is holomorphic and divisible by \( t \). Therefore

\[ \left( \psi_1(t), \frac{F(t, s)}{\psi_1(t)} \phi_1(t) + st^{d+1}, \phi_2(t), \ldots, \phi_n(t) \right) \]

is a deformation of \( (\psi_1(t), 0, \phi_1(t), \phi_2(t), \ldots, \phi_n(t)) \) such that

\[ \text{mult}_t f(\phi_1(t) + st^{d+1}, \phi_2(t), \ldots, \phi_n(t)) < \infty \]

for \( 0 < |s| \ll 1 \). \qed

We used Newton’s lemma on Puiseux series solutions in the following form.

**Lemma 7.** Let \( g(x, y) \in \mathbb{C}[[x, y]] \) be a power series. Assume that \( m := \text{mult}_0 g(x, 0) < \infty \). Then there is an \( r \geq 1 \) such that one can write \( g(x, z^r) \) as

\[ g(x, z^r) = u(x, z) \prod_{i=1}^{m} (x - \sigma_i(z)) \]

where \( u(0, 0) \neq 0 \) and \( \sigma_i(0) = 0 \) for every \( i \). The representation is unique, up-to permuting the \( \sigma_i(z) \).

Furthermore, if \( g(x, y) \) is holomorphic on the bidisc \( \overline{D}_x \times D_y \) then \( u(x, z) \) and the \( \sigma_i(z) \) are holomorphic on the smaller bidisc \( \overline{D}_x \times D_\epsilon(z) \) for some \( 0 < \epsilon \leq 1 \). \qed

2. Essential divisors on cA1-type 3-fold singularities

In dimension 3, the only cA1-type singularities are \( X_m := (xy = z^2 - t^m) \) for \( m \geq 2 \). Already [Nas93] p.37 proved that they have at most 2 essential divisors. We use the method of [EF12] 4.1 to determine the precise count.

**Definition 8.** Let \( X \) be a normal variety or analytic space and \( E \) a divisor over \( X \). That is, there is a birational or bimeromorphic morphisms \( p : X' \to X \) such that \( E \subset X' \) is an exceptional divisor. The closure of \( p(E) \subset X \) is called the center of \( E \) on \( X \); it is denoted by centery \( E \). If centery \( E = \{x\} \), we say that \( E \) is a divisor over \( (x \in X) \).

We say that \( E \) is an essential divisor over \( X \) if for every resolution of singularities \( \pi : Y \to X \), centery \( E \) is an irreducible component of \( \pi^{-1}(\text{centery } E) \). (Note that \( \pi^{-1} \circ p : X' \to Y \) is regular on a dense subset of \( E \), hence centery \( E \) is defined.)

If \( X \) is an analytic space, then \( Y \) is allowed to be any analytic resolution. If \( X \) is algebraic, one gets slightly different notions depending on whether one allows \( Y \)
to be a quasi-projective variety, an algebraic space or an analytic space; see [dF12].

We believe that for the Nash problem it is natural to allow analytic resolutions.

**Proposition 9.** Set $X_m := (xy = z^2 - t^m) \subset \mathbb{C}^4$.

(1) If $m \geq 5$ is odd, there are 2 essential divisors.

(2) If $m \geq 2$ is even or $m = 3$, there is 1 essential divisor.

Even in dimension 3, it seems surprisingly difficult to determine the set of essential divisors. A basic invariant is given by the discrepancy.

**Definition 10.** Let $X$ be a normal variety or analytic space. Assume for simplicity that the canonical class $K_X$ is Cartier. (This holds for all hypersurface singularities.) Let $\pi : Y \to X$ be a resolution of singularities and write

$$K_Y \sim \pi^* K_X + \sum_i a(E_i, X)E_i$$

where the $E_i$ are the $\pi$-exceptional divisors. The integer $a(E_i, X)$ is called the **discrepancy of $E_i$**. (See [KM98, Sec.2.3] for basic references and more general definitions.)

For example, let $X$ be smooth and $Z \subset X$ a smooth subvariety of codimension $r$. Let $\pi_Z : B_Z X \to X$ denote the blow-up and $E_Z \subset B_Z X$ the exceptional divisor. Then $a(E_Z, X) = r - 1$ and easy induction shows that $a(F, X) \geq r$ for every other divisor whose center on $X$ is $Z$.

We say that $X$ is **canonical** (resp. **terminal**) of $a(E_i, X) \geq 0$ (resp. $a(E_i, X) > 0$) for every resolution and every exceptional divisor.

For instance, normal $cA$-type singularities are canonical and a $cA$-type singularity is terminal iff its singular set has codimension $\geq 3$; see [Rei83] for a proof that applies to all $cDV$ singularities or [Kol13, 1.42] for a simpler argument in the $cA$ case.

**11 (Resolving $X_m$).** Blow up the origin to get $\pi_1 : X_{m,1} := B_0 X_m \to X_m$. The exceptional divisor is the singular quadric $E_1 \cong (xy - z^2 = 0) \subset \mathbb{P}^3(x, y, z, t)$. $B_0 X_m$ has one singular point, visible in the chart

$$(x_1, y_1, z_1, t) := \frac{(x, y, z, t)}{t},$$

where the local equation is $x_1 y_1 = z_1^2 - t^m - 2$. We can thus blow up the origin again and continue. After $r := \lfloor \frac{m}{2} \rfloor$ steps we have a resolution

$$\Pi_r : X_{m,r} \to X_{m,r-1} \to \cdots \to X_{m,1} \to X_m.$$  

We get $r$ exceptional divisors $E_r, \ldots, E_1$. For $1 \leq c \leq r$ the divisor $E_c$ first appears on $X_{m,c}$. At the unique singular point one can write the local equation as

$$X_{m,c} = (x_c y_c = z_c^2 - t^{m-2c}) \quad \text{and} \quad E_c = (t = 0).$$

where $(x_c, y_c, z_c, t) := (x/t^c, y/t^c, z/t^c, t)$.

We thus need to decide which of the divisors $E_1, \ldots, E_{\lfloor \frac{m}{2} \rfloor}$ are essential. It is easy to see that $E_1$ is essential and a direct computation [10] shows that $E_3, \ldots, E_{\lfloor \frac{m}{2} \rfloor}$ are not. (This is actually not needed in order to establish Example 2) The hardest is to decide what happens with $E_2$.

**Lemma 12.** Notation as above. Then

$$(1) \ a(E_c, X_m) = c \quad \text{for every} \ c.$$
(2) $E_1$ is the only exceptional divisor whose center is the origin and whose discrepancy is 1.

(3) $E_1$ appears on every resolution of $X_m$ whose exceptional set is a divisor.

(4) Let $p : Y \dasharrow X_m$ be any (not necessarily proper) birational map from a smooth analytic space $Y$ such that center$_Y E_1 \subset Y$ is not empty. Then center$_Y E_1$ is an irreducible component of the exceptional set Ex($p$).

Proof. The first claim follows from the formula

$$
\Pi^* \left( \frac{dx \wedge dy \wedge dt}{z} \right) = t^{-c} \cdot \frac{dx \wedge dy \wedge dt}{z}.
$$

Let $F$ be any other exceptional divisor whose center is the origin. Then center$_X F$ lies on one of the $E_i$, thus $a(F, X) > a(E_i, X) \geq 1$. (This also proves that $X_m$ is terminal.)

To see (3) set $W_1 := \text{center}_Y E_1 \subset Y$. Let $F_1 \subset Y$ be the exceptional divisors and note that, as in [KM98, 2.29],

$$
a(E_1, X_m) \geq (\text{codim}_Y W_1 - 1) + \sum_i \text{mult}_{W_i} F_i \cdot a(F_i, X_m). \tag{12.5}
$$

Note that $a(E_1, X_m) = 1$ and $a(F_i, X_m) \geq 1$ for every $i$. If $W_1$ is not an irreducible component of Ex($p$) then $W_1 \subset F_i$ form some $i$ and then both terms on the right hand side of (12.5) are positive, a contradiction. \hfill \square

**Lemma 13.** If $m \in \{2, 3\}$ then $B_0X$ is smooth, hence the only essential divisor is $E_1$. \hfill \square

**14** (Small resolutions and factoriality of $X_m$). If $m = 2a$ is even, then $X_m$ has a small resolution obtained by blowing up either $(x = z - t^a = 0)$ or $(x = z + t^a = 0)$. The resulting blow-ups $Y_{2a}^\pm \subset \mathbb{C}^4_{xzt}$ are defined by the equations

$$
Y_{2a}^\pm := \text{rank} \left( \begin{array}{ccc} x & z \pm t^a & u \\ z & y & v \end{array} \right) \leq 1 \tag{14.1}
$$

By contrast, $X_m$ does not have small resolutions if $m$ is odd. More generally, let

$$
X_f := (xy = f(z, t)) \subset \mathbb{C}^4
$$

be an isolated $ca$-type singularity. Write $f = \prod_j f_j$ as a product of irreducibles. The $f_j$ are distinct since the singularity is isolated. Set $D_j := (x = f_j = 0)$. By [Kol91 2.2.7] the local divisor class group is

$$
\text{Div}(0 \in X_f) = \left( \sum_j \mathbb{Z}[D_j] \right)/\sum_j [D_j]. \tag{14.2}
$$

In particular, $X_f$ is factorial iff $f$ is irreducible.

This formula works both algebraically and analytically. If we are interested in the affine variety $X_f$, then we consider factorizations of $f$ in the polynomial ring. If we are interested in the complex analytic germ $X_f$, then we consider factorizations of $f$ in the ring of germs of analytic functions. Thus, for example,

$$
(xy = z^2 - t^2 - t^3) \subset \mathbb{C}^4
$$

is algebraically factorial, since $z^2 - t^2 - t^3$ is an irreducible polynomial, but it is not analytically factorial, since

$$
z^2 - t^2 - t^3 = (z - t\sqrt{1+t})(z + t\sqrt{1+t}).
$$

Thus if $m$ is odd then $X_m$ is factorial (both algebraically and analytically) and it does not have small resolutions; see Lemma 18 for stronger results.
Lemma 15. If $m$ is even then there is a divisorial resolution whose sole exceptional divisor is birational to $E_1$. Thus the only essential divisor is $E_1$.

Proof. The $m = 2$ case is in [13], hence we may assume that $m = 2n ≥ 4$.

There are 2 ways to obtain such resolutions. First, we can blow up the exceptional curve in either of the $Y_{2n}$, as in (14.1).

Alternatively, we first blow up the origin to get $B_0X_m$ which has one singular point with local equation $x_1 y_1 = z_1^2 - t_1^{2a-2}$ and then blow up $D^+ := (x_1 = z_1 + t_1^{a-1} = 0)$ or $D^- := (x_1 = z_1 - t_1^{a-1} = 0)$.

Lemma 16. [Nas95, p.37] The divisors $E_3, \ldots, E_r$ are not essential.

Proof. If $m$ is even, this follows from [15], but for the proof below the parity of $m$ does not matter.

If $2b ≥ a ≥ 0$ and $m ≥ a$ then $(u, v, w, t) → (ut^{a+1}, wt^{b+1}, t) = (x, y, z, t)$ defines a birational map

$g(a, b, m) : Z_{abm} := (uv = w^2t^{2b-a} - t^{m-2a}) → X_m$.

Note that $\text{Ex}(g(a, b, m)) = (t = 0)$ is mapped to the origin and $Z_{abm}$ is smooth along the $v$-axis, save at the origin.

If $1 ≤ c ≤ m/2$ then $(x_c, y_c, z_c, t) → (x_c t^c, y_c t^c, z_c t^c, t) = (x, y, z, t)$ defines a birational map

$h(c, m) : X_{m,c} := (x_c y_c = z_c^2 - t^{m-2c}) → X_m$.

By composing we get a birational map $g(a, b, m)^{-1} ∘ h(c, m) : Y_c → Z_{abm}$ given by

$(x_c, y_c, z_c, t) → (x_c t^{c-1}, y_c t^{c-a-1}, z_c t^{c-b-1}, t) = (u, v, w, t)$

which is a morphism if $c ≥ a + 1, b + 1$. If $c = a + 1$ and $c > b + 1$ then we have

$(x_c, y_c, z_c, t) → (x_c t^{c-1}, y_c, z_c t^{c-b-1}, t) = (u, v, w, t)$

which maps $E_c$ to the $v$-axis.

If $c ≥ 3$ then by setting $a = c - 1, b = c - 2$ we get a birational morphism $p(c, m) := g(c, c-1, m)^{-1} ∘ h(c, m)$ given by

$(x_c, y_c, z_c, t) → (x_c t^{c-1}, y_c, z_c t, t) = (u, v, w, t)$.

Note that

$p(c, m) : Y_c = (x_c y_c = z_c^2 - t^{m-2c}) → (uv = w^2t^{c-2} - t^{m-c}) = Z_{c,c-1,m}$

maps $E_c$ onto the $v$-axis. Thus $E_c$ is not essential for $c ≥ 3$.

Lemma 17. If $m ≥ 5$ is odd then $E_2$ is essential.

Proof. We follows the arguments in [12, 4.1]. Let $p : Y → X_m$ be any resolution and set $Z := \text{centery}(E_2 ⊂ Y$. Since $X_m$ is factorial (here we use that $m$ is odd), $\text{Ex}(p)$ has pure dimension 2 by [18, 2].

Assume to the contrary that $Z$ is not a divisor. Using that $a(E_2, X_m) = 2$, (12.5) implies that $Z$ is a curve, there is a unique exceptional divisor $F ⊂ Y$ that contains $Z$, $F$ is smooth at general points of $Z$ and $a(F, X_m) = 1$.

If $p(F)$ is a curve then $Z$ is an irreducible component of $p^{-1}(0)$. The remaining case is when $p(F) = 0$, thus $F = E_1$ by (12.2).

Since $t$ vanishes along $E_2$ with multiplicity 1, it also vanishes along $Z$ with multiplicity 1. Since $p^*x, p^*y, p^*z, p^*t$ all vanish along $E_1$ the rational functions $p^*(x/t), p^*(y/t), p^*(z/t)$ are regular generically along $Z$. Thus $p_1 := π_1^{-1} ∘ p : Y →$
$X_{m,1}$ is a morphism generically along $Z$. Note that our $E_2$ is what we would call $E_1$ if we started with $X_{m,1}$. Applying (12.4) to $p_1 : Y \to X_{m,1}$ we see that $Z$ is an irreducible component of $\text{Ex}(p_1)$. Since $m$ is odd, $X_{m,1}$ is analytically factorial by (14), hence $Z$ is a divisor by (18.2). This is a contradiction. \hfill $\square$

**Lemma 18.** Let $X,Y$ be normal varieties or analytic spaces and $g : Y \to X$ a birational or bimeromorphic morphism. Then the exceptional set $\text{Ex}(g)$ has pure codimension 1 in $Y$ in the following cases.

1. $Y$ is an algebraic variety and $X$ is $Q$-factorial.
2. $\dim Y = 3$ and $X$ is analytically locally $Q$-factorial.

Proof. The algebraic case is well known; see for instance the method of [Sha74, Sec. II.4].

If $\dim Y = 3$ and $\text{Ex}(g)$ does not have pure codimension 1 then it has a 1-dimensional irreducible component $C \subset Y$. After replacing $X$ by a suitable neighborhood of $g(C) \in X$ we may assume that there is a divisor $D_Y \subset Y$ such that $\text{Ex}(g) \cap D_Y$ is a single point of $C$ and $g|_{D_Y}$ is proper. Thus $D_X := g(D_Y)$ is a divisor on $X$. If $mD_X$ is Cartier then so is $g^*(mD_X)$ hence its support has pure codimension 1 in $Y$. On the other hand, $\text{Supp}(g^*(mD_X)) = \text{Ex}(g) \cup D_Y$ does not have pure codimension 1. (Note that there are many possible choices for $D_Y$; the resulting $D_X$ determine an algebraic equivalence class of divisors.) \hfill $\square$

Somewhat surprisingly, the analog of (18.2) fails in dimension 4.

**Example 19.** Let $W \subset \mathbb{P}^4$ be a smooth quintic 3-fold and $C \subset W$ a line whose normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1)$. Let $X \subset \mathbb{C}^5$ denote the cone over $W$ with vertex 0; it is analytically locally factorial by [Gro68, XI.3.14].

The exceptional divisor of the blow-up $B_0X \to X$ can be identified with $W$; let $C \subset B_0X$ be our line. Its normal bundle is $\mathcal{O}(-1) + \mathcal{O}(-1) + \mathcal{O}(-1)$.

Blow up the line $C$ to obtain $B_C B_0X \to B_0X$. Its exceptional divisor is $E \cong \mathbb{P}^1 \times \mathbb{P}^2$. One can contract $E$ in the other direction to obtain $g : Y \to X$.

By construction, $\text{Ex}(g)$ is the union of $\mathbb{P}^2$ and of a 3-fold obtained from $W$ by flopping the line $C$. The two components intersect along a line.

**Remark 20.** We will need to understand in detail the proof of (18.2) for

$$X_c := \{(xy = z^2 - ct^m) \subset \mathbb{C}^4 \text{ where } c \neq 0\}.$$ 

Let $g_c : Y_c \to X_c$ be a proper birational or bimeromorphic morphism and $E_c \subset \text{Ex}(g_c)$ a 1-dimensional irreducible component.

The proof of (18.2) associates to $E_c$ an algebraic equivalence class of non-Cartier divisors on $X_c$. Thus $m$ has to be even by (14).

If $m = 2a$ is even then the divisor class group is $\text{Div}(X_c) \cong \mathbb{Z}$. The two possible generators correspond to $(x = z - \sqrt{ct^a} = 0)$ and $(x = z + \sqrt{ct^a} = 0)$. Starting with $E_c$ we constructed a divisor $D_c \subset X_c$ which is a nontrivial element of $\text{Div}(X_c)$. Thus $[D_c]$ is a positive multiple of either $(x = z - \sqrt{ct^a} = 0)$ or $(x = z + \sqrt{ct^a} = 0)$. Hence, to $E_c \subset Y_c$ we can associate a choice of $\sqrt{c}$.

This may not be very interesting for a fixed value of $c$ (since many other choices are involved) but it turns out to be quite useful when $c$ varies.

**Proposition 21.** Let $g(u_1, \ldots, u_r, v)$ be a holomorphic function for $u_i \in \mathbb{C}$ and $|v| < \epsilon$ such that $g(u_1, \ldots, u_r, 0)$ is not identically zero. Set

$$X := \{(xy = z^2 - v^m g(u_1, \ldots, u_r, v)) \subset \mathbb{C}^{r+4}\}$$
Let \( \pi : Y \to X \) be a birational or bimeromorphic morphism. Assume that there is an irreducible component \( Z \subset \text{Ex}(\pi) \) that dominates \((x = y = z = v = 0) \subset X\), has codimension \( \geq 2 \) in \( Y \) and such that \( \pi|_Z : Z \to (x = y = z = v = 0) \) has connected fibers.

Then \( m \) is even and \( g(u_1, \ldots, u_r, 0) \) is a perfect square.

Proof. For general \( c = (c_1, \ldots, c_r) \in \mathbb{C}^r \) the repeated hyperplane section
\[
X(c) := (xy = z^2 - v^m g(c, v)) \subset \mathbb{C}^4
\]
has an isolated singularity at the origin and we get a proper birational or bimeromorphic morphism
\[
\pi(c) : Y(c) \to X(c)
\]
where \( Y(c) \subset Y \) is the preimage of \( X(c) \).

Furthermore, \( Z(c) := Z \cap Y(c) \) is an irreducible component of \( \text{Ex}(\pi(c)) \) and has codimension \( \geq 2 \) in \( Y(c) \).

Thus, as we noted above, \( m = 2a \) is even and our construction gives a function
\[
(c_1, \ldots, c_r) \mapsto \text{a choice of } \sqrt{g(c_1, \ldots, c_r, 0)}.
\]
It is clear that this function is continuous on a Zariski open set \( U \subset \mathbb{C}^r \). Therefore \( g(u_1, \ldots, u_r, 0) \) is a perfect square.

Remark 22. Conversely, assume that \( m \) is even and \( g(u_1, \ldots, u_r, 0) = h^2(u_1, \ldots, u_r) \) is a square. Write the equation of \( X \) as
\[
xy = z^2 - v^m \left( h^2(u_1, \ldots, u_r) + vR(u_1, \ldots, u_r, v) \right)
\]
Over the open set \( X^0 \subset X \) where \( h \neq 0 \), change coordinates to \( w := h^{-2}v \).
(Equivalently, blow up \((v = h = 0)\) twice.) Then
\[
D := (x = z - w^{m/2}h^{m+1}\sqrt{1 + wR(u_1, \ldots, u_r, h^2w)})
\]
is a globally well defined analytic divisor. Blowing it up gives a bimeromorphic morphism \( X_D \to X \) whose exceptional set over \( X^0 \) has codimension 2.

It seems that even if \( X \) is algebraic, usually \( X_D \) is not an algebraic variety.

3. Essential divisors on \( cA_1 \)-type singularities

In higher dimensions \( cA_1 \)-type singularities are more complicated and their resolutions are much harder to understand. There is no simple complete answer as in dimension 3.

In the previous Section, the key part is to understand the exceptional divisors that correspond to the first 2 blow-ups. These are the 2 divisors that we understand in higher dimensions as well.

23 (Defining \( E_1 \) and \( E_2 \)). In order to fix notation, write the equation as
\[
X := (xy = z^2 - g(u_1, \ldots, u_r)) \subset \mathbb{C}^{r+3}.
\]
Set \( m := \text{mult}_0 g \) and let \( g_s(u_1, \ldots, u_r) \) denote the homogeneous degree \( s \) part of \( g \). In a typical local chart the 1st blow-up \( \sigma_1 : X_1 := B_0X \to X \) is given by
\[
x_1y_1 = z_1^2 - (u'_r)^{-2}g(u'_1u'_s, \ldots, u'_{r-1}u'_r, u'_r)
\]
where \( x = x_1u'_r, y = y_1u'_r, z = z_1u'_r, u_1 = u'_1u'_s, \ldots, u_{r-1} = u'_{r-1}u'_r \) and \( u_r = u'_r \).
The exceptional divisor is the rank 3 quadric
\[
E_1 := (x_1y_1 - z_1^2 = 0) \subset \mathbb{P}^{r+2}.
\]
Note also that
\[(u'_i)^{-2}g(u'_1, \ldots, u'_{r-1}, u'_r, u'_r) = (u'_i)^{-2}\left(g_m(u'_1, \ldots, u'_{r-1}, 1) + u'_r g_{m+1}(u'_1, \ldots, u'_{r-1}, 1) + \cdots \right). \tag{23.4}\]

From this we see that, for \(m \geq 4\), the blow-up \(X_1\) is singular along the closure of the linear space
\[L := (x_1 = y_1 = z_1 = u'_r = 0), \tag{23.5}\]
\(X_1\) has terminal singularities and a general 3-fold section has equation
\[x_1y_1 = z^2 - (u'_r)^m - 2 \left(g_m(c_1, \ldots, c_{r-1}, 1) + u'_r g_{m+1}(c_1, \ldots, c_{r-1}, 1) + \cdots \right). \tag{23.6}\]

Blowing up the closure of \(L\) we obtain \(X_2\) with exceptional divisor \(E_2\). As in Lemma \([12]\) we compute that
\[(6) \ a(E_1, X) = r, \quad (7) \ a(E_2, X) = r + 1, \quad (8) \ a(F, X) \geq r + 1 \text{ for every other exceptional divisor whose center on } X \text{ is the origin and} \quad (9) \text{ the pull-backs of the } u_i \text{ vanish along } E_1, E_2 \text{ with multiplicity 1.}\]

The key computation is the following.

**Proposition 24.** Notation as above and assume that \(m \geq 4\).

1. \(E_1\) is an essential divisor.
2. \(E_2\) is an essential divisor iff \(g_m(u_1, \ldots, u_r)\) is not a perfect square.

Proof. By \((23.6)\) and \((23.8)\), \(E_1\) has the smallest discrepancy among all divisors over \(X\) whose center on \(X\) is the origin. Thus \(E_1\) is essential by Proposition \([20]\).

If \(E_2\) is non-essential then there is a resolution \(\pi : Y \to X\) and an irreducible component \(W \subset \text{Supp } \pi^{-1}(0)\) such that \(Z := \text{center } E_2 \subset W\). By \((23.9)\), the \(\pi^*u_i\) vanish at a general point of \(Z\) with multiplicity 1. Since the \(\pi^*u_i\) vanish along \(W\), this implies that \(\text{Supp } \pi^{-1}(0)\) is smooth at a general point of \(Z\). In particular, \(W\) is the only irreducible component of \(\text{Supp } \pi^{-1}(0)\) that contains \(Z\) and \(W\) is smooth at general points of \(Z\). Therefore the blow-up \(B_W Y\) is smooth over the generic point of \(Z\). So, if we replace \(Y\) by a suitable desingularization of \(B_W Y\), we get a situation as before where in addition \(W\) is a divisor.

The \(\pi^*u_i\) are local equations of \(W\) at general points of \(Z\) and \(\pi^*x, \pi^*y, \pi^*z\) all vanish along \(W\). Thus the rational functions
\[\pi^*(x/u_r), \pi^*(y/u_r), \pi^*(z/u_r), \pi^*(u_1/u_r), \ldots, \pi^*(u_{r-1}/u_r),\]
are all regular at general points of \(Z\). Hence the birational map \(\sigma^{-1}_1 \circ \pi : Y \to B_0 X = X_1\) is a morphism at general points of \(Z\). Furthermore, \(\sigma^{-1}_1 \circ \pi\) maps \(W\) birationally to \(E_1 \subset X_1\) and it is not a local isomorphism along \(Z\) since \(Y\) is smooth but \(X_1\) is singular along the center \(L\) of \(E_2\). Thus \(Z\) is an irreducible component of \(\text{Ex}(\sigma^{-1}_1 \circ \pi)\). Since \(E_2 \to L\) has connected fibers, all the assumptions of Proposition \([21]\) are satisfied by the equation of the blow-up
\[x_1y_1 = z^2 - (u'_r)^m - 2 \left(g_m(u'_1, \ldots, u'_{r-1}, 1) + u'_r g_{m+1}(u'_1, \ldots, u'_{r-1}, 1) + \cdots \right). \tag{23.3}\]

Thus \(m\) is even and \(g_m(u'_1, \ldots, u'_{r-1}, 1)\) is a perfect square. Since it is a dehomogenization of \(g_m(u_1, \ldots, u_{r-1}, u_r)\), the latter is also a perfect square.

The converse follows from Remark \([22]\) \(\Box\)
Definition 25. For \((x \in X)\) let \(\text{min-discrep}(x \in X)\) be the infimum of the discrepancies \(a(E, X)\) where \(E\) runs through all divisors over \(X\) such that center\(_X E = \{x\}\). (It is easy to see that either \(\text{min-discrep}(x \in X) \geq -1\) and the infimum is a minimum or \(\text{min-discrep}(x \in X) = -\infty\); cf. [KM98, 2.31]. We do not need these facts.)

Proposition 26. Let \((x \in X)\) be a canonical singularity and \(E\) a divisor over \(X\) such that

1. \(\text{center}_X E = \{x\}\) and
2. \(a(E, X) < 1 + \text{min-discrep}(x \in X)\).

Then \(E\) is essential.

Proof. Let \(F\) be any non-essential divisor over \(X\) whose center on \(X\) is the origin. Thus there is a resolution \(\pi : Y \to X\) and an irreducible component \(W \subset \text{Supp} \pi^{-1}(x)\) such that

\[
Z := \text{center}_Y F \subset W.
\]

Let \(E_W\) be the divisor obtained by blowing up \(W \subset Y\). As we noted in (10),

\[
a(E_W, Y) = \text{codim}_Y W - 1 \quad \text{and} \quad a(F, Y) \geq \text{codim}_Y Z - 1 \geq \text{codim}_Y W. \quad (26.3)
\]

Write \(K_Y = \pi^*K_X + D_Y\) where \(D_Y\) is effective since \(X\) is canonical and note that

\[
a(E_W, X) = a(E_W, Y) + \text{mult}_W D_Y\quad \text{and} \quad a(F, X) \geq a(F, Y) + \text{mult}_Z D_Y. \quad (26.4)
\]

Since \(\text{mult}_Z D_Y \geq \text{mult}_W D_Y\), we conclude that

\[
a(F, X) \geq 1 + a(E_W, X) \geq 1 + \text{min-discrep}(x \in X). \quad (26.5)
\]

Thus any divisor \(E\) with \(a(E, X) < 1 + \text{min-discrep}(x \in X)\) is essential. \(\Box\)

4. Short arcs

Let \(\mathbb{D} \subset \mathbb{C}\) denote the open unit disk and \(\overline{\mathbb{D}} \subset \mathbb{C}\) its closure. The open (resp. closed) disc of radius \(\epsilon\) is denoted by \(\mathbb{D}(\epsilon)\) (resp. \(\overline{\mathbb{D}}(\epsilon)\)). If several variables are involved, we use a subscript to indicate the name of the coordinate.

27 (Short arcs). [KN13] Let \(X\) be an analytic space and \(p \in X\) a point. A short arc on \((p \in X)\) is a holomorphic map \(\phi(t) : \overline{\mathbb{D}}_t \to X\) such that \(\text{Supp} \phi^{-1}(p) = \{0\}\).

The space of all short arcs is denoted by \(\text{ShArc}(p \in X)\). It has a natural topology and most likely also a complex structure that, at least for isolated singularities, locally can be written as the product of a (finite dimensional) complex space and of a complex Banach space; see [KN13 Sec.11] for details.

A deformation of short arcs is a holomorphic map \(\Phi(t, s) : \overline{\mathbb{D}}_t \times \mathbb{D}_s \to X\) such that \(\Phi(t, s_0) : \overline{\mathbb{D}}_t \to X\) is a short arc for every \(s_0 \in \mathbb{D}_s\). Equivalently, if \(\text{Supp} \Phi^{-1}(p) = \{0\} \times \mathbb{D}_s\).

In general the space of short arcs has more connected components than the space of formal arcs. As a simple example, consider arcs on \((xy = z^n) \subset \mathbb{C}^3\). For \(0 < i < m\) the deformations

\[
(t, s) \mapsto (t^i(t + s)^{m-i}, t^{m-i}(t + s)^i, t(t + s)) \quad (27.1)
\]

show that the arc \((t^m, t^m, t^2)\) is in the closure of the families (142), provided we work in the space of formal arcs. However, (27.1) is not a deformation of short arcs and \((t^m, t^m, t^2)\) is a typical member of a new connected component of \(\text{ShArc}(0 \in (xy = z^n))\).
By contrast, adding one more variable kills this component. For example, starting with the arc \((t^m, t^n, t^2, 0)\) on \((xy = z^n) \subset \mathbb{C}^4\), we have deformations of short arcs
\[
(t, s) \mapsto (t^i (t + s)^{m-i}, t^{m-i} (t + s)i, t(t + s), ts).
\] (27.2)

This example turns out to be typical and it is quite easy to modify the deformations in the proof of Theorem 29 to yield the following.

**Theorem 28.** Let \(X = (xy = f(z_1, \ldots, z_n) \subset \mathbb{C}^{n+2}\) be a \(cA\)-type singularity. Assume that \(\dim X \geq 3\) and \(m := \text{mult}_0 f \geq 2\).

Then \(\text{ShArc}(0 \in X)\) has \((m - 1)\) irreducible components as in (1.2).

It is not always clear if a deformation \(\Phi(t, s)\) is short or not. There is, however, one case when this is easy, at least over a smaller disc \(D_s(\epsilon) \subset D_s\).

**Lemma 29.** Let \(\Phi(t, s) = (\Phi_1(t, s), \ldots, \Phi_r(t, s))\) be a deformation of arcs on \(X \subset \mathbb{C}^r\). Assume that \(\Phi(t, 0)\) is short and \(\Phi_i(t, s)\) is independent of \(s\) and not identically zero for some \(i\). Then \(\Phi(t, s_0) : \overline{D_t} \to X\) is short for \(|s_0| \ll 1\).

Proof. By assumption \(\Phi_*(s, s_0)^{-1}(p) \subset \Phi_1(s, s_0)^{-1}(p) = \Phi_i(s, 0)^{-1}(p)\) for every \(s_0 \in D_s\), thus there is a finite subset \(Z = \Phi_i(s, 0)^{-1}(p) \subset \overline{D_t}\) such that
\[
\Phi^{-1}(0) \subset Z \times D_s \quad \text{and} \quad \Phi^{-1}(0) \cap (s = 0) = \{(0, 0)\}.
\]
Since \(\Phi^{-1}(0)\) is closed, this implies that
\[
\Phi^{-1}(0) \cap \left[\overline{D_t} \times D_s(e)\right] \subset \{0\} \times D_s(e) \quad \text{for} \ 0 < e \ll 1. \quad \square
\]

30 (Proof of Theorem 28). At the very beginning of the proof of Theorem 1 after a linear change of coordinates we may assume that \(z^m\) appears in \(f\) with nonzero coefficient and \(\phi_2\) is not identically zero. Then the construction gives a deformation of short arcs by Lemma 29.

The deformations at the end of the proof were written to yield short arcs. \(\square\)

5. A revised version of the Nash problem

As we saw, the Nash map is not surjective in dimensions \(\geq 3\). In this section we develop a revised version of the notion of essential divisors. This leads to a smaller target for the Nash map, so surjectivity should become more likely. Our proposed variant of the Nash problem at least accounts for all known counter examples.

We start with a reformulation of the original definition of essential divisors.

31. Let \(Y\) be a complex variety and \(Z \subset Y\) a closed subset. Let \(\widehat{\text{Arc}}(Z \subset Y)\) denote the scheme of formal arcs \(\phi : \text{Spec} \mathbb{C}[[t]] \to Y\) such that \(\phi(0) \in Z\).

An easy but key observation is the following.

311. If \(Y\) is smooth, then the irreducible components of \(\widehat{\text{Arc}}(Z \subset Y)\) are in a natural one-to-one correspondence with the irreducible components of \(Z\).

We say that a divisor \(E\) over \(Y\) is essential for \(Z \subset Y\) if \(E\) is obtained by blowing up one of the irreducible components of \(Z\). (For each irreducible component \(Z_i \subset Z\), the blow-up \(\overline{B_2Y}\) contains a unique divisor that dominates \(Z_i\).)

The definition of essential divisors can now be reformulated as follows.

312. Let \((x \in X)\) be a singularity. A divisor \(E\) is essential for \((x \in X)\) if \(E\) is essential for \((\text{Supp} \pi^{-1}(x) \subset Y)\) for every resolution \(\pi : Y \to X\).
In order to refine the Nash problem, we need to understand singular spaces for which the analog of (31.1) still holds.

**Definition 32 (Sideways deformations).** Let $X$ be a variety (or an analytic space) and $\phi : \text{Spec } \mathbb{C}[t] \to X$ a formal arc such that $\phi(0) \in \text{Sing } X$. A *sideways deformation* of $\phi$ is a morphism $\Phi : \text{Spec } \mathbb{C}[[t, s]] \to X$ such that
\[
\Phi^* I_{\text{Sing } X} \subset (t, s)^m \quad \text{for some } m \geq 1
\]
where $I_{\text{Sing } X} \subset \mathcal{O}_X$ is the ideal sheaf defining $\text{Sing } X$.

If $\Phi$ comes from a convergent arc $\Phi^{an} : \mathbb{D}_t \times \mathbb{D}_s \to X$ then this is equivalent to assuming that for every $0 \neq |s_0| \ll 1$ the nearby arc $\Phi^{an}(t, s_0)$ maps $\mathbb{D}_t(\epsilon)$ to $X \setminus \text{Sing } X$ for some $0 < \epsilon \leq 1$.

We say that $(x \in X)$ is *arc-wise Nash-trivial* if every general arc in $\widehat{\text{Arc}}(x \in X)$ has a sideways deformation. (By [FdBP12a], this implies that every arc in $\widehat{\text{Arc}}(x \in X)$ has a sideways deformation.)

**Comment 33.** If $(x \in X)$ is an isolated singularity with a small resolution $\pi : X' \to X$ then every arc has a sideways deformation. We can lift the arc to $X'$ and there move it away from the $\pi$-exceptional set. This is not very interesting and the notion of essential divisors captures this phenomenon.

To exclude these cases, we are mainly interested in arc-wise Nash-trivial singularities that do not have small modifications. If arc-wise Nash-trivial singularities are log terminal then assuming $\mathbb{Q}$-factoriality captures this restriction, but in general one needs to be careful of the difference between $\mathbb{Q}$-factoriality and having no small modifications.

Also, in the few examples we know of, general arcs of every irreducible component of $\widehat{\text{Arc}}(x \in X)$ have sideways deformations. If there are singularities where sideways deformations exist only for some of the irreducible components, the following outline needs to be suitably modified.

The main observation is that, for the purposes of the Nash problem, $\mathbb{Q}$-factorial arc-wise Nash-trivial singularities should be considered as good as smooth points. The first evidence is the following straightforward analog of (31.1).

**Lemma 34.** Let $Y$ be a complex variety with isolated, arc-wise Nash-trivial singularities. Let $Z \subset Y$ a closed subset that is the support of an effective Cartier divisor. Then the irreducible components of $\widehat{\text{Arc}}(Z \subset Y)$ are in a natural one-to-one correspondence with the irreducible components of $Z$. $\square$

If $Z$ has lower dimensional irreducible components, the situation seems more complicated, but, at least in dimension 3, the following seems to be the right generalization of (31.1).

**Conjecture 35.** Let $Y$ be a 3-dimensional complex variety with isolated, $\mathbb{Q}$-factorial, arc-wise Nash-trivial singularities. Let $Z \subset Y$ be a closed subset. Then the irreducible components of $\widehat{\text{Arc}}(Z \subset Y)$ are in a natural one-to-one correspondence with the union of the following two sets.

1. Irreducible components of $Z$.
2. Irreducible components of $\text{Arc}(p \in Y)$, where $p \in Y$ is any singular point such that $p \in Z$ and $\dim_p Z \leq 1$. 

Definition 36. Assumptions as in (35). A divisor over $Y$ is essential for $Z \subset Y$ if it corresponds to one of the irreducible components of $\text{Arc}(Z \subset Y)$, as enumerated in (35.1–2).

Definition 37. Let $(x \in X)$ be a 3-dimensional, normal singularity. A divisor $E$ over $X$ is called very essential for $(x \in X)$ if $E$ is essential for $(\text{Supp} \pi^{-1}(x) \subset Y)$ for every proper birational morphism $\pi: Y \to X$ where $Y$ has only isolated, $Q$-factorial, arc-wise Nash-trivial singularities. (As in (5), it is better to allow $Y$ to be an algebraic space.)

It is easy to see that the Nash map is an injection from the irreducible components of $\text{Arc}(x \in X)$ into the set of very essential divisors. One can hope that there are no other obstructions.

Problem 38 (Revised Nash problem). Is the Nash map surjective onto the set of very essential divisors for normal 3-fold singularities?

As a first step, one should consider the following.

Problem 39. In dimension 3, classify all $Q$-factorial, arc-wise Nash-trivial singularities.

Hopefully they are all terminal and a complete enumeration is possible. The papers Hay05a, Hay05b contain several results about partial resolutions of terminal singularities.

We treat two easy cases in (40) (41). A positive solution of Question 4 would imply that all isolated, 3-dimensional cDV singularities are arc-wise Nash-trivial.

Theorem 40. Let $(0 \in X)$ be a cA-type singularity such that $\dim \text{Sing} X \leq \dim X - 3$. Then all general arcs as in (4.2) have sideways deformations.

Proof. We use the notation of the proof of Theorem 1. Since mult $f(\phi_1(t), \ldots, \phi_n(t)) = m$, we see that mult $\phi_j(t) = 1$ for at least one index $j$. We may assume that $j = 1$ and $\phi_1(t) = t$. Thus, after the coordinate change $z_i \to z_i - \phi_1(z_1)$ for $i = 2, \ldots, n$ and an additional general linear coordinate change among the $z_2, \ldots, z_n$ we may assume that

1. $\phi_1(t) = t$,
2. $\phi_j(t) \equiv 0$ for $j > 1$,
3. $(x y = g(z_1, z_2)) \subset \mathbb{C}^4$ has an isolated singularity at the origin and $g(z_1, z_2)$ is divisible neither by $z_1$ nor by $z_2$ where $g(z_1, z_2) = f(z_1, z_2, 0, \ldots, 0)$.

By Lemma 1 there is an $r \geq 1$ such that

$$g(t, s^r) = u(t, s) \prod_{i=1}^{m} (t - \sigma_i(s)).$$

Since $g(z_1, z_2)$ is not divisible by $z_1$ none of the $\sigma_i$ are identically zero. Since $g(t, s)$ has an isolated singularity at the origin and is not divisible by $s$, $g(t, s^r)$ has an isolated singularity at the origin. Thus all the $\sigma_i(s)$ are distinct.

As before, for $j = 1, 2$ write $\psi_j(t) = t^{a_j} v_j(t)$ where $v_j(0) \neq 0$. Note that $a_1 + a_2 = m$ and $u(t, 0) = v_1(t)v_2(t)$.

Divide $\{1, \ldots, m\}$ into two disjoint subsets $A_1, A_2$ such that $|A_j| = a_j$. Finally set

$$\Psi_1(t, s) = v_1(t) \cdot \prod_{i \in A_1} (t - \sigma_i(s)) \quad \text{and} \quad \Psi_2(t, s) = \frac{u(t, s)}{v_1(t)} \cdot \prod_{i \in A_2} (t - \sigma_i(s)).$$
Then
\[(\Psi_1(t, s), \Psi_2(t, s), t, s^r, 0, \ldots, 0)\]
is a sideways deformation of \((\psi_1(t), \psi_2(t), t, 0, \ldots, 0)\).

The opposite happens for quotient singularities.

**Proposition 41.** Let \((0 \in X) := \mathbb{C}^n / G\) be an isolated quotient singularity. Then arcs with a sideways deformation are nowhere dense in \(\hat{\text{Arc}}(0 \in X)\).

**Proof.** Let \(\Phi : \text{Spec} \mathbb{C}[[t, s]] \to X\) be a sideways deformation of an arc \(\phi(t) = \Phi(t, 0)\). By the purity of branch loci, \(\Phi\) lifts to an arc \(\tilde{\Phi} : \text{Spec} \mathbb{C}[[t, s]] \to \mathbb{C}^n\). In particular, \(\phi : \text{Spec} \mathbb{C}[[t]] \to \mathbb{C}^n\) lifts to \(\tilde{\phi} : \text{Spec} \mathbb{C}[[t]] \to \mathbb{C}^n\).

By [KN13], such arcs constitute a connected component of \(\text{ShArc}(0 \in X)\). We claim, however, that these arcs do not cover a whole irreducible component of \(\hat{\text{Arc}}(0 \in X)\).

It is enough to show the latter on some intermediate cover of \(X\). The simplest is to use \((0 \in Y) := \mathbb{C}^n / C\) where \(C \subset G\) is any cyclic subgroup.

Set \(r := |C|\), fix a generator \(g \in C\) and diagonalize its action as \((x_1, \ldots, x_n) \mapsto (\epsilon^{a_1} x_1, \ldots, \epsilon^{a_n} x_n)\) where \(\epsilon\) is a primitive \(r\)th root of unity. Thus \(Y\) is the toric variety corresponding to the free abelian group \(N = \mathbb{Z}^n + \mathbb{Z}(a_1/r, \ldots, a_n/r)\) and the cone \(\Delta = (Q_{\geq 0})^n\).

The Nash conjecture is true for toric singularities and by [IK03, Sec.3] the essential divisors are all toric and correspond to interior vectors of \(N \cap \Delta\) that can not be written as the sum of an interior vector of \(N \cap \Delta\) and of a nonzero vector of \(N \cap \Delta\).

In our case, all such vectors are of the form \((\overline{ca_1}/r, \ldots, \overline{ca_n}/r)\) for \(c = 1, \ldots, r - 1\)

where \(\overline{ca_i}\) denotes remainder mod \(r\).

Arcs that lift to \(\mathbb{C}^n\) correspond to the vector \((1, \ldots, 1)\), which is not minimal.

In fact
\[(1, \ldots, 1) = (\overline{a_1}/r, \ldots, \overline{a_n}/r) + (\overline{(r-1)a_1}/r, \ldots, \overline{(r-1)a_n}/r)\).

\[\square\]

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