CHARACTERIZING SERRE QUOTIENTS WITH NO SECTION FUNCTOR AND APPLICATIONS TO COHERENT SHEAVES

MOHAMED BARAKAT AND MARKUS LANGE-HEGERMANN

ABSTRACT. We prove an analogon of the fundamental homomorphism theorem for certain classes of exact and essentially surjective functors of Abelian categories \( \mathcal{D} : \mathcal{A} \to \mathcal{B} \). It states that \( \mathcal{D} \) is up to equivalence the Serre quotient \( \mathcal{A} \to \mathcal{A}/\ker \mathcal{D} \), even in cases when the latter does not admit a section functor. For several classes of schemes \( X \), including projective and toric varieties, this characterization applies to the sheafification functor from a certain category \( \mathcal{A} \) of finitely presented graded modules to the category \( \mathcal{B} = \mathcal{Coh} X \) of coherent sheaves on \( X \). This gives a direct proof that \( \mathcal{Coh} X \) is a Serre quotient of \( \mathcal{A} \).

1. INTRODUCTION

An essentially surjective exact functor \( \mathcal{D} : \mathcal{A} \to \mathcal{B} \) of Abelian categories induces an essentially surjective exact embedding \( \overline{\mathcal{D}} : \mathcal{A}/\mathcal{C} \to \mathcal{B} \), where \( \mathcal{C} \) is the kernel of \( \mathcal{D} \) and \( \mathcal{A}/\mathcal{C} \) the Serre quotient. It is natural to ask if or when \( \mathcal{D} \) is an equivalence of categories, i.e., if or when the fundamental homomorphism theorem is valid for exact functors of Abelian categories.

Grothendieck mentioned this question in [Gro57] but did not provide an answer:

Thus \( \mathcal{A}/\mathcal{C} \) appears as an abelian category; moreover the identity functor \( \mathcal{D} : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) is exact (and, in particular, commutes with kernels, cokernels, images, and coimages), \( \mathcal{D}(A) = 0 \) if and only if \( A \in \mathcal{C} \), and any object of \( \mathcal{A}/\mathcal{C} \) has the form \( \mathcal{D}(A) \) for some \( A \in \mathcal{A} \). These are the facts (which essentially characterize the quotient category) which allow us to safely apply the “mod \( \mathcal{C} \)” language, since this language signifies simply that we are in the quotient abelian category.

As Grothendieck indicated by the word “essentially” all the above conditions yet do not characterize Serre quotients: In Appendix A we give an explicit example of an exact faithful functor between Abelian categories which is not full on its image.

Gabriel proved in his thesis (cf. Proposition 3.1) the following characterization of Serre quotients under a stronger assumption: An exact functor \( \mathcal{D} : \mathcal{A} \to \mathcal{B} \) induces an equivalence between the Serre quotient category \( \mathcal{A}/\ker \mathcal{D} \) and \( \mathcal{B} \) if \( \mathcal{D} \) admits a right adjoint for which the counit of the adjunction is an isomorphism. Such a right adjoint is called a section functor.

One goal of this paper is to prove two propositions which, under weaker assumptions, allow us to recognize Serre quotients not necessarily admitting section functors. In Proposition 3.2 we were able to prove an analogon of the second isomorphism theorem for Abelian categories relating two Serre quotients modulo thick torsion subcategories. In Proposition 3.3 we give, under
an additional assumption, the following analogon of the fundamental homomorphism Theorem. If \( Q \) is a certain restriction of an ambient functor \( Q' \) satisfying the assumptions of Gabriel’s characterization then \( Q \) still induces the desired equivalence of categories \( A / \ker Q \cong B \).

Our original motivation is to establish a constructive setup for coherent sheaves on several classes of schemes \( X \). This setup begins with describing quotient categories constructively by the 3-arrow formalism of so-called the Gabriel morphisms [BLH14b]. Then, the methods in this paper allow to recognize \( B = \text{Coh} X \) as a Serre quotient of the Abelian category \( A = S\text{-grmod} \) of finitely presented graded modules over some graded coherent ring \( S \). This gives an alternative and simpler proof than the one which can be derived from [Kra97, Theorem 2.6] using a result in [Len69]. The last step in this setup is modeling the category \( A = S\text{-grmod} \) over a computable ring \( S \) constructively through a presentation matrix [BLH11]. This setup turns out to be suitable and computationally efficient for a computer implementation.

In Section 2 we collect some preliminaries about Serre quotients. In Section 3 we prove Propositions 3.2 and 3.3 which serve to identify certain categories as Serre quotient. In Section 4 we apply the two propositions to categories of coherent sheaves.

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2. PRELIMINARIES ON SERRE QUOTIENTS

In this section we recall some results about Serre quotients. From now on \( A \) is an Abelian category. See [Gab62] for proofs and [BLH13] for detailed references.

A non-empty full subcategory \( C \) of an Abelian category \( A \) is called \textbf{thick} if it is closed under passing to subobjects, factor objects, and extensions. In this case the \textbf{(Serre) quotient category} \( A / C \) is a category with the same objects as \( A \) and \( \text{Hom}\)-groups

\[
\text{Hom}_{A/C}(M, N) := \lim_{M' \to M, N' \to N} \text{Hom}_A(M', N'/N).
\]

The \textbf{canonical functor} \( Q : A \to A / C \) is defined to be the identity on objects and maps a morphism \( \varphi \in \text{Hom}_A(M, N) \) to its image in the direct limit \( \text{Hom}_{A/C}(M, N) \). The category \( A / C \) is Abelian and the canonical functor \( Q : A \to A / C \) is exact and fulfills the following universal property. If \( G : A \to D \) an exact functor of Abelian categories, and \( G(C) \) is zero then there exists a unique functor \( H : A / C \to D \) with \( G = H \circ Q \).

Let \( C \subset A \) be thick. An object \( M \in A \) is called \textbf{\( C \)-torsion-free} if \( M \) has no nonzero subobjects in \( C \). Denote by \( A_C \subset A \) the pre-Abelian full subcategory of \( C \)-torsion-free objects. If every object \( M \in A \) has a maximal subobject \( H_C(M) \in C \) then \( (C, A_C) \) is a \textbf{hereditary torsion theory} of \( A \), i.e., \( C \) and \( A_C \) are additive and full subcategories, \( C \) is closed under subobjects, \( M / H_C(M) \) is \( C \)-torsion-free, i.e., lies in \( A_C \),

\[
\text{Hom}_A(C, A) = 0 \text{ for all } C \in C \text{ and } A \in A_C,
\]

\[
\text{Hom}_A(C, A) = 0 \text{ for all } C \in C \text{ implies } A \in A_C \text{ for all } A \in A, \text{ and}
\]

\[
\text{Hom}_A(C, A) = 0 \text{ for all } A \in A_C \text{ implies } C \in C \text{ for all } C \in A.
\]

In this case we call \( C \) a \textbf{thick torsion} subcategory.
Remark 2.1. For $C \subset A$ thick torsion the description of $\text{Hom}$-groups in $A/C$ simplifies to

$$\text{Hom}_{A/C}(M, N) = \lim_{M'/M \in C} \text{Hom}_A(M', N/H_C(N)).$$

An object $M \in A$ is called $C$-saturated if it is $C$-torsion-free and every extension of $M$ by an object $C \in C$ is trivial. Denote by $\text{Sat}_C(A) \subset A$ the full subcategory of $C$-saturated objects. We say that $A$ has enough $C$-saturated objects if for each $M \in A$ there exists a $C$-saturated object $N$ and a morphism $\eta_M : M \to N$ such that $\ker \eta_M \in C$. Any thick subcategory $C \subset A$ is called a localizing subcategory if the canonical functor $\mathcal{Q} : A \to A/C$ admits a right adjoint $\mathcal{S} : A/C \to A$, called the section functor of $\mathcal{Q}$. The category $C \subset A$ is localizing if and only if $A$ has enough $C$-saturated objects. The section functor $\mathcal{S} : A/C \to A$ is left exact and preserves products, the counit of the adjunction $\delta : \mathcal{Q} \circ \mathcal{S} \cong \text{Id}_{A/C}$ is a natural isomorphism, and an object $M$ in $A$ is $C$-saturated if and only if $\eta_M : M \to (\mathcal{S} \circ \mathcal{Q})(M)$ is an isomorphism, where $\eta$ is the unit of the adjunction. Finally, a localizing $C \subset A$ is a thick torsion subcategory with $H_C(M) = \ker \eta_M$.

3. Recognition of Serre Quotients

We say that an exact functor $\mathcal{Q}' : A' \to B'$ of Abelian categories admits a section functor if $\mathcal{Q}'$ admits a right adjoint $\mathcal{S}'$ such that the counit of the adjunction $\delta' : \mathcal{Q}' \circ \mathcal{S}' \cong \text{Id}_{B'}$ is a natural isomorphism. It follows that $\mathcal{Q}'$ is essentially surjective. The next proposition characterizes Serre quotients $A'/C'$ for $C'$ a localizing subcategory of $A'$.

Proposition 3.1 ([Gab62, Proposition III.2.5], [GZ67, Chap. 1.2.5.d]). Let $\mathcal{Q}' : A' \to B'$ be an exact functor of Abelian categories admitting a section functor $\mathcal{S}'$. Then $C' := \ker \mathcal{Q}'$ is a localizing subcategory of $A'$ and the adjunction $\mathcal{Q}' \dashv (\mathcal{S}' : B' \to A')$ induces an adjoint equivalence of categories $A'/C' \cong B'$.

The aim of this section is to formulate a characterization of certain Serre quotients $A \to A/C$ where the thick subcategory $C$ is not necessarily localizing. The following two propositions are analogous to the second isomorphism Theorem and to the fundamental homomorphism Theorem. However, they need some additional assumptions.

Proposition 3.2 (Second isomorphism theorem). Let $\tilde{A}$ be an Abelian category and $\tilde{C} \subset \tilde{A}$ a thick torsion subcategory. Then for each thick subcategory $A \subset \tilde{A}$ the intersection $C := \tilde{C} \cap A$ is a thick torsion subcategory of $A$. Furthermore the restricted canonical functor $A \to \tilde{A}/C$ is essentially surjective then it induces an equivalence of categories $A/C \cong \tilde{A}/\tilde{C}$.

Proof. From the thickness of $A \subset \tilde{A}$ it follows that for each object $N \in A$ the maximal $\tilde{C}$-subobject $H_{\tilde{C}}(N)$ lies in $C$. Furthermore $N/H_{\tilde{C}}(N)$ is $C$-free since it is $\tilde{C}$-free. Summing up, $H_C(N) := \tilde{H}_{\tilde{C}}(N)$ is the maximal $C$-subobject of $N \in A$ establishing the first assertion. By the

1These adjunctions $\mathcal{Q}' \dashv (\mathcal{S}' : B' \to A')$ are the exact reflective localization of Abelian categories.
universal property of \( \mathcal{D} \) there exists a functor \( A/\mathcal{C} \to \tilde{A}/\tilde{C} \) which is essentially surjective by our assumption. We will now show that it is fully faithful. Let \( M, N \in \mathcal{A} \). We can replace \( \tilde{N} \) by its \( \mathcal{C} \)-free factor \( N/H_\mathcal{C}(N) \) and without loss of generality assume that \( N \) is \( \mathcal{C} \)-free. Because of the thickness of \( A \subset \tilde{A} \) and the definition of \( \mathcal{C} \), the \( \tilde{A} \)-subobjects \( M' \) of \( M \) with \( M/M' \in \tilde{C} \) are exactly the \( A \)-subobjects with \( M/M' \in \mathcal{C} \) and we obtain

\[
\operatorname{Hom}_{A/\mathcal{C}}(M, N) = \lim_{M' \to M \text{ in } A, \ M/M' \in \mathcal{C}} \operatorname{Hom}_A(M', N) \quad \text{by Remark 2.1 and } N \in \mathcal{A}_\mathcal{C}
\]

\[
= \lim_{M' \to M \text{ in } \tilde{A}, \ M/M' \in \tilde{C}} \operatorname{Hom}_{\tilde{A}}(M', N)
\]

\[
= \operatorname{Hom}_{\tilde{A}/\tilde{C}}(M, N) \quad \text{by Remark 2.1 and } N \in \tilde{A}_\tilde{C}.
\]

We say that an exact and essentially surjective functor \( \mathcal{D} : A \to B \) of Abelian categories admits a sections functor up to extension if there exists an exact functor \( \mathcal{D}' : A' \to B' \) admitting a section functor with \( B \subset B' \) a replete and full Abelian subcategory, \( A \subset A' \) thick, and \( \mathcal{D} = \mathcal{D}'|_A : A \to B \). Now we can formulate the analogon of the fundamental homomorphism Theorem.

**Proposition 3.3** (Fundamental homomorphism theorem). Let \( \mathcal{D} : A \to B \) be an exact and essentially surjective functor of Abelian categories which admits a section functor up to extension. Then \( \mathcal{D} \) induces an equivalence of categories \( A/\mathcal{C} \simeq \tilde{B} \), where \( C := \ker \mathcal{D} \) is a thick torsion subcategory of \( A \).

**Proof.** By assumption there exists an exact functor \( \mathcal{D}' : A' \to B' \) of Abelian categories admitting a section functor \( \mathcal{J}' \) with \( B \subset B' \) a replete and full Abelian subcategory, \( A \subset A' \) thick, and \( \mathcal{D} = \mathcal{D}'|_A : A \to B \). First note that \( C = \ker \mathcal{D} = \tilde{C} \cap A \) for \( \tilde{C} := \ker \mathcal{D}' \). Define \( \tilde{A} \subset A' \) as the preimage\(^2\) of \( B \) under \( \mathcal{D}' \), i.e., the full subcategory of \( A' \) with object class \( \operatorname{Obj} \tilde{A} = \{ M' \in A' \mid \mathcal{D}'(M') \in B \} \). \( \tilde{A} \) is a full replete Abelian subcategory of \( A' \). The section functor \( \mathcal{J}' \) maps objects in \( B \) to objects in \( \tilde{A} \) since the counit \( \delta' : \mathcal{D}' \circ \mathcal{J}' \to \operatorname{Id}_B \) is an isomorphism. Hence, the adjunction induces a restricted adjunction \( \tilde{\mathcal{D}} \dashv (\tilde{\mathcal{J}} : B \to \tilde{A}) \) and the counit \( \tilde{\delta} : \tilde{\mathcal{D}} \circ \tilde{\mathcal{J}} \to \operatorname{Id}_B \) is still an isomorphism. Proposition 3.1 implies that \( \tilde{A}/\tilde{C} \simeq \tilde{B} \) since \( \tilde{C} = \ker \tilde{\mathcal{D}} = \ker \mathcal{D}' \subset \tilde{A} \).

The assertion \( B \simeq \tilde{A}/\tilde{C} \simeq A/\mathcal{C} \) now follows from Proposition 3.2 once we have shown that \( A \to \tilde{A}/\tilde{C} \) is essentially surjective. As \( \text{Sat}_{\tilde{C}}(\tilde{A}) \to \tilde{A}/\tilde{C} \) is essentially surjective (even an equivalence) we need to show that for every \( M \in \text{Sat}_{\tilde{C}}(\tilde{A}) \) there exists an \( M \in A \) and \( M \to M \).

\(^2\)Note that we didn’t need the preimage \( \tilde{A} \) in the statement of the proposition.
with kernel and cokernel in \( \tilde{\mathcal{C}} \). Let \( M \in \mathcal{A} \) be a preimage of \( \mathcal{D}(M) \in \mathcal{B} \) under the essentially surjective restriction \( \mathcal{D} = \mathcal{D}_A = \mathcal{D}|_A : A \to B \). Then

\[
(\tilde{\mathcal{F}} \circ \mathcal{D})(M) \xrightarrow{\cong} (\tilde{\mathcal{F}} \circ \mathcal{D})(M) \xrightarrow{\tilde{\eta}_M^{-1} \cong} M.
\]

Furthermore, \( M \xrightarrow{\tilde{\eta}_M} (\tilde{\mathcal{F}} \circ \mathcal{D})(M) \) has kernel and cokernel in \( \tilde{\mathcal{C}} \).

\( \square \)

4. APPLICATIONS TO COHERENT SHEAVES

4.1. Coherent sheaves on projective schemes. Let \( A \) be a commutative unital ring and \( S = A[x_0, \ldots, x_n] \) the \( \mathbb{Z} \)-graded polynomial ring over \( A \) with \( \deg x_i = 1 \) for all \( i \). Let \( \mathbb{P}_A^n = \text{Proj} S \) the \( n \)-dimensional projective space \( A \). Denote by \( \text{Coh} \mathbb{P}_A^n \) the category of coherent sheaves over \( \mathbb{P}_A^n \).

The category \( S\text{-qfgrmod} \) of quasi finitely generated graded \( S \)-modules is the full subcategory of the category of (not necessarily finitely generated) graded \( S \)-modules \( M \) where the truncated submodule \( M_{\geq d} \) is finitely generated for \( d \in \mathbb{Z} \) high enough. Further, denote by \( S\text{-qfgrmod}^0 \) its thick subcategory of \( S \)-modules \( M \) with \( M_{\geq d} = 0 \) for \( d \in \mathbb{Z} \) high enough.

**Theorem 4.1** ([Ser55, GD61] ³). The sheafification functor \( \text{Sh} : S\text{-qfgrmod} \to \text{Coh} \mathbb{P}_A^n \) is exact with kernel \( S\text{-qfgrmod}^0 \). The functor \( \Gamma_* : \text{Coh} \mathbb{P}_A^n \to S\text{-qfgrmod} \) is right adjoint to \( \text{Sh} \) and the counit \( \tilde{\delta} : \text{Sh} \circ \Gamma_* \to \text{Id}_{\text{Coh} \mathbb{P}_A^n} \) is an isomorphism. The adjunction \( \text{Sh} \dashv \Gamma_* \) induces by Proposition 3.1 an adjoint equivalence of categories

\[
\text{Coh} \mathbb{P}_A^n \xrightarrow{\sim} \frac{S\text{-qfgrmod}}{S\text{-qfgrmod}^0}.
\]

Let \( S\text{-grmod} \) denote the category of finitely presented graded \( S \)-modules and \( S\text{-grmod}^0 \) its thick subcategory of \( S \)-modules \( M \) with \( M_{\geq d} = 0 \) for all \( d \) large enough.

**Corollary 4.2.** Let \( A \) be a Noetherian ring ⁴. The exact and essentially surjective sheafification functor \( \text{Sh} : S\text{-grmod} \to \text{Coh} \mathbb{P}_A^n \) induces an equivalence of categories

\[
\frac{S\text{-grmod}}{S\text{-grmod}^0} \sim \text{Coh} \mathbb{P}_A^n.
\]

where \( S\text{-grmod}^0 \) coincides with the kernel of the sheafification functor.

**Proof.** Define \( \mathcal{A} := S\text{-grmod} \subset S\text{-qfgrmod} =: \tilde{\mathcal{A}} \subset \tilde{\mathcal{C}} := S\text{-qfgrmod}^0 \) and \( \mathcal{C} := S\text{-grmod}^0 = \mathcal{A} \cap \tilde{\mathcal{C}} \). To apply Proposition 3.2 to the thick subcategory \( \mathcal{A} \subset \tilde{\mathcal{A}} \) we need to show that the restricted canonical functor \( \mathcal{D}|_A : A \to \tilde{\mathcal{A}}/\tilde{\mathcal{C}} \) is essentially surjective. By definition of \( S\text{-qfgrmod} \) for each \( M \in S\text{-qfgrmod} \) there exists a \( d = d(M) \in \mathbb{Z} \) high enough such that \( M_{\geq d} \) is finitely

³ The case of \( A \) a field is treated in [Ser55, Prop. III.2.3, Prop. III.2.5, Theo. III.2.2, Prop. III.2.7, Prop. III.2.8, Prop. III.3.6] and the general case in [GD61, Prop. 3.2.4, Prop. 3.4.3.ii, Prop. 3.3.5, Theorem. 3.4.4]. The adjointness is shown there by proving the zig-zag identities.

⁴ This guarantees that \( S\text{-grmod} \) is an Abelian subcategory of \( S\text{-qfgrmod} \).
generated, i.e., lies in $S$-grmod. Further, $M/M_{\geq d}$ lies in $S$-qfgrmod$^0$ by definition of the latter. Hence $M_{\geq d}$ and $M$ have isomorphic images in $S$-qfgrmod$^0$ and we obtain
\[
\frac{S\text{-qfgrmod}}{S\text{-qfgrmod}^0} \simeq \frac{S\text{-grmod}}{S\text{-grmod}^0} \simeq \mathcal{Coh}_A^n.
\]

Remark 4.3. Although the two Serre quotient categories $\frac{S\text{-qfgrmod}}{S\text{-qfgrmod}^0} \simeq \frac{S\text{-grmod}}{S\text{-grmod}^0}$ yield equivalent representations of the category $\mathcal{Coh}_A^n$ we now show that the thick torsion subcategory $S$-grmod$^0 \subset S$-grmod is not localizing, even though $S$-qfgrmod$^0 \subset S$-qfgrmod is.

Therefore, let $S = k[x, y]$ for a field $k$ and assume that there exists a functor $\mathcal{F} : \frac{S\text{-grmod}}{S\text{-grmod}^0} \to S$-grmod right adjoint to the canonical functor $\mathcal{D} : S$-grmod $\to \frac{S\text{-grmod}}{S\text{-grmod}^0}$. Define $M := S/\langle y \rangle$ and $N^n := x^{-n}M$ for each $n \in \mathbb{Z}$. The sheafification of $M$ is a skyscraper sheaf on $\mathbb{P}_k^1$. As $\mathcal{D}(N^n) \cong \mathcal{D}(M)$ there exists by the Hom-adjunction a morphism $\varphi^n : N^n \to (\mathcal{F} \circ \mathcal{D})(M)$ with kernel and cokernel in $S$-grmod$^0$. As $N^n$ and $M$ are not in $S$-grmod$^0$ the morphism $\varphi^n$ is nonzero. Thus, $\varphi^n$ must map the cyclic generator of $N^n$ of degree $-n$ to a nonzero element of degree $-n$ in $(\mathcal{F} \circ \mathcal{D})(M)$. In particular, $\dim_k((\mathcal{F} \circ \mathcal{D})(M))_{-n} > 0$ for all $n \in \mathbb{Z}$ and $(\mathcal{F} \circ \mathcal{D})(M)$ is not finitely generated. This is a contradiction.

In [BLH14a] we will prove that $\mathcal{Coh}_A$ admits for each $d \in \mathbb{Z}$ yet another representation as the Serre quotient $S$-grmod$_d/\text{S-grmod}^0_d$, where $S$-grmod$_d$ is the category of finitely presented graded $S$-modules vanishing in degrees $< d$. In this case $S$-grmod$^0_d = S$-grmod $\cap$ S-qfgrmod$^0_d \subset S$-grmod$_d$ is localizing.

Although these categories of truncated modules are computable they have the following computational disadvantage. Given a fixed $d \in \mathbb{Z}$ there exists coherent sheaves $\mathcal{F} \in \mathcal{Coh}_A^n$ for which $S$-grmod contains a vastly more efficient model for $\mathcal{F}$ than $S$-grmod$_d$. For example, the minimal number of generators of the smallest model for $\mathcal{O}_{\mathbb{P}_A^d}(k)$ in $S$-grmod$_d$ is $\max(1, \binom{n+k+d}{n})$ which is disadvantageous if $k \gg -d$. These positively twisted line bundles occur as soon as we need to dualize a locally free resolution.

4.2. Coherent sheaves on toric varieties. We refer the reader to [CLS11] for notation. Let $X_\Sigma$ be a toric variety with no torus factors and Cox ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ graded by the divisor class group $\text{Cl}_X$. We denote by $S$-grMod the category of graded $S$-modules.

By [Mus02, Theorem 1.1] (cf. also [CLS11, Prop. 6.1.3]) the global section functor
\[
\Gamma : \mathcal{Coh}_A X_\Sigma \to S$-grMod : $\mathcal{F} \mapsto \bigoplus_{\alpha \in \text{Cl}_X} \Gamma(X_\Sigma, \mathcal{F}(\alpha))
\]
is right inverse but not right adjoint to the exact sheafification functor $\text{Sh} : S$-grMod $\to \mathcal{Coh}_A X_\Sigma$. Recently Perling found the right adjoint of $\text{Sh}$, also valid in the singular case.

Theorem 4.4 ([Per14, Theorem 3.8 and Remark 3.9]). Let $X_\Sigma$ be a toric variety with no torus factor. There exists a so-called lifting functor $\Gamma$ right adjoint to the exact sheafification functor $\text{Sh} : S$-grMod $\to \mathcal{Coh}_A X_\Sigma$, where the counit of the adjunction $\delta : \text{Sh} \circ \Gamma \to \text{Id}_{\mathcal{Coh}_A X_\Sigma}$ is a natural isomorphism.

This allows us to characterize toric sheaves as quotients of finitely generated modules.
Corollary 4.5. Let $X_\Sigma$ be a toric variety with no torus factor. The exact and essentially surjective sheafification functor $\text{Sh} : S\text{-grmod} \rightarrow \text{Coh} X_\Sigma$ induces the equivalence

$$\frac{S\text{-grmod}}{\text{ker(Sh)}} \simeq \text{Coh} X_\Sigma,$$

of categories where $S\text{-grmod}^0$ is defined as the kernel of the sheafification functor.

Proof. The essential surjectivity of $\text{Sh}$ is the statement of [Mus02, Cor. 1.2] (cf. also [CLS11, Prop. 6.A.4]). Thus, the assumption of Proposition 3.3 is fulfilled with $\mathcal{Q}' \dashv (\mathcal{S}' : \mathcal{B}' \rightarrow \mathcal{A}')$ being the adjunction $\text{Sh} \dashv (\hat{\Gamma} : q\text{Coh} X_\Sigma \rightarrow S\text{-grMod})$ from Theorem 4.4. □

In fact Perling proved Theorem 4.4 in a more general setup described in [Per14] following [Hau08, ADHL]. This setup covers toric varieties over arbitrary fields, Mori dream spaces, and categories of equivariant coherent sheaves on them. Furthermore, Trautmann and Perling proved in [PT10, Proposition 5.6.(2),(4)] that the sheafification functor restricted to the subcategory of finitely generated graded modules is essentially surjective onto the category of coherent sheaves.

APPENDIX A. THE NONEXISTENCE OF A FUNDAMENTAL HOMOMORPHISM THEOREM

In this appendix we show that a naive fundamental homomorphism theorem of exact functors between Abelian categories cannot exist. Such a theorem would imply that the corestriction of an exact faithful functor to its image is full. To justify our counterexample we need the following argument.

Lemma A.1. An exact faithful functor of Abelian categories is conservative.⁶

Proof. Let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be such a functor and $\varphi : M \rightarrow N$ be a morphism in $\mathcal{A}$. Consider the exact $\mathcal{A}$-sequence $0 \rightarrow \ker \varphi \xrightarrow{i} M \xrightarrow{\varphi} N \xrightarrow{\pi} \coker \varphi \rightarrow 0$. As $\mathcal{B}$ is Abelian and $\mathcal{F}$ exact, $\mathcal{F}(\varphi)$ is an isomorphism iff the morphisms $\mathcal{F}(i)$ and $\mathcal{F}(\pi)$ are zero in $\mathcal{B}$. The faithfulness of $\mathcal{F}$ implies that then $i$ and $\pi$ are zero. Finally, since $\mathcal{A}$ is Abelian it follows that $\varphi$ is an isomorphism. □

Example A.2. Let $\mathcal{B}$ be the category of $k$-vector spaces, $G$ a nontrivial group, and $\mathcal{A}$ the category of $G$-representations with object $(V, \rho : G \rightarrow \text{GL}(V))$ and $G$-equivariant morphisms in $\mathcal{B}$. The forgetful functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}, (V, \rho) \mapsto V$ is exact and faithful with $\text{image} \mathcal{F} = \mathcal{B}$. First note that $1_V \in \text{Hom}_B(V, V)$ is in the image of $1_{(V, \rho)}$ under $\mathcal{F}$. However, $1_V \notin \text{image} \mathcal{F}$ of $\text{Hom}_A((V, \rho), (V, \rho'))$ for two inequivalent representations $(V, \rho) \neq (V, \rho')$; a preimage would be an $\mathcal{A}$-isomorphism since $\mathcal{F}$ is conservave by Lemma A.1. A contradiction.

⁵The image of a functor $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{B}$, denoted by $\text{image} \mathcal{G}$, is the smallest subcategory of $\mathcal{B}$ which contains all image morphisms $\mathcal{G}(\varphi)$.

⁶A functor $\mathcal{F}$ is called conservative if reflects isomorphisms, i.e., if $\mathcal{F}(\varphi) \text{ iso} \implies \varphi \text{ iso}$. 
This shows that the corestriction of an exact faithful functor to its image is not necessarily full, in particular, it is not an equivalence of categories.

Any exact functor \( \mathcal{G} : \mathcal{A} \to \mathcal{B} \) of Abelian categories with kernel \( \mathcal{C} := \ker \mathcal{G} \) induces a unique faithful exact functor \( \mathcal{H} : \mathcal{A}/\mathcal{C} \to \mathcal{B} \) such that \( \mathcal{G} = \mathcal{H} \circ \mathcal{D} \). This is the universal property of the canonical functor \( \mathcal{D} : \mathcal{A} \to \mathcal{A}/\mathcal{C} \). Note that \( \im \mathcal{D} \) is in general strictly contained in \( \mathcal{A}/\mathcal{C} \) and that \( \im \mathcal{G} \) is strictly contained in \( \im \mathcal{H} \). In the rest of the appendix we want to define an image-notion for which equality holds in both cases.

We define the conservative image of a functor \( \mathcal{G} : \mathcal{A} \to \mathcal{B} \), denoted by \( \text{conim} \mathcal{G} \), to be the smallest subcategory of \( \mathcal{B} \) which contains \( \im \mathcal{G} \) and inverses of \( \mathcal{B} \)-isomorphisms in \( \im \mathcal{G} \). We call \( \mathcal{G} \) conservatively surjective if \( \text{conim} \mathcal{G} = \mathcal{B} \). The name is motivated by \( \text{conim} \mathcal{G} = \im \mathcal{G} \) for any conservative functor \( \mathcal{G} \).

**Lemma A.3.** Let \( \mathcal{D} : \mathcal{A} \to \mathcal{D} \), \( \mathcal{G} : \mathcal{A} \to \mathcal{B} \), and \( \mathcal{H} : \mathcal{H} \to \mathcal{B} \) be functors with \( \mathcal{H} \circ \mathcal{D} = \mathcal{G} \) and \( \mathcal{D} \) conservatively surjective. Then \( \text{conim} \mathcal{G} = \text{conim} \mathcal{H} \).

**Proof.** It is clear that \( \text{conim} \mathcal{H} \supset \text{conim} \mathcal{G} \). Every morphism \( \alpha \) in \( \text{conim} \mathcal{H} \) is a finite composition of morphisms of the form \( \mathcal{H}(\varphi_{i})^{\epsilon_{i}} \), with \( \epsilon_{i} \in \{ \pm 1 \} \). Since \( \text{conim} \mathcal{D} = \mathcal{D} \) each \( \varphi_{i} \) is in turn a finite composition of morphisms of the form \( \mathcal{D}(\psi_{ij})^{\sigma_{ij}} \), with \( \sigma_{ij} \in \{ \pm 1 \} \). Finally \( \alpha \) is then a finite composition of morphisms of the form \( \mathcal{G}(\psi_{ij})^{\epsilon_{i} \sigma_{ij}} \). \( \square \)

**Proposition A.4.** Let \( \mathcal{G} : \mathcal{A} \to \mathcal{B} \) be an exact functor of Abelian categories, \( \mathcal{C} := \ker \mathcal{G} \), \( \mathcal{H} : \mathcal{A}/\mathcal{C} \to \mathcal{B} \) the unique functor such that \( \mathcal{G} = \mathcal{H} \circ \mathcal{D} \). Then \( \mathcal{H} \) is an exact, faithful, and conservative functor. Furthermore \( \im \mathcal{H} = \text{conim} \mathcal{H} = \text{conim} \mathcal{G} \).

**Proof.** \( \mathcal{H} \) is exact and faithful by definition and hence conservative by Lemma A.1. Hence \( \im \mathcal{H} = \text{conim} \mathcal{H} \). The canonical functor \( \mathcal{D} \) is conservatively surjective, as every morphism \( \varphi : M \to N \) in \( \mathcal{A}/\mathcal{C} \) is of the form \( \mathcal{D}(M' \xrightarrow{\alpha} M)^{-1}\mathcal{D}(M' \xrightarrow{\beta} N/N')\mathcal{D}(N \xrightarrow{\sigma} N/N')^{-1} \), with \( M/M', N' \in \mathcal{C} \). The equality \( \text{conim} \mathcal{H} = \text{conim} \mathcal{G} \) follows from Lemma A.3. \( \square \)

**References**

[ADHL] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cox rings (project) (http://www.mathematik.uni-tuebingen.de/~hausen/CoxRings/download.php?name=coxrings).

[BLH11] Mohamed Barakat and Markus Lange-Hegermann, *An axiomatic setup for algorithmic homological algebra and an alternative approach to localization*, J. Algebra Appl. 10 (2011), no. 2, 269–293, (arXiv:1003.1943). MR 2795737 (2012f:18022).

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7 Enlarging the image only deepens the problem. In the following example, due to Eisele [Eis], an inclusion functor of an Abelian subcategory has the inequivalent target category as its essential image:

Let \( k \) be a field and \((R,m)\) be a finite dimensional local \( k \)-algebra with \( R/m \cong k \). Let \( \mathcal{A} \) be the category of \( R \)-modules and \( \mathcal{B} \) be the category with the same objects as \( \mathcal{A} \) but \( \text{Hom}_R(M,N) := \text{Hom}_k(M,N) \), i.e., all \( k \)-vector space homomorphisms. The forgetful (identity on objects) functor \( \mathcal{F} : \mathcal{A} \to \mathcal{B} \) is clearly exact and surjective (on objects). \( \mathcal{B} \) is equivalent to the category of all \( k \)-vector spaces as every \( k \)-vector space can be seen as an \( R \)-module via \( k \cong R/m \). The kernel of \( \mathcal{F} \) is the subcategory \( 0_\mathcal{A} \) of zero objects in \( \mathcal{A} \). But \( \mathcal{A}/0_\mathcal{A} \cong \mathcal{A} \) is not equivalent to \( \mathcal{B} \) (if \( R \neq k \)).
[BLH13] Mohamed Barakat and Markus Lange-Hegermann, *On monads of exact reflective localizations of Abelian categories*, Homology Homotopy Appl. **15** (2013), no. 2, 145–151, ([arXiv:1202.3337](https://arxiv.org/abs/1202.3337)). MR 3138372

[BLH14a] Mohamed Barakat and Markus Lange-Hegermann, *A constructive approach to coherent sheaves via Gabriel monads*, ([arXiv:1409.6100](https://arxiv.org/abs/1409.6100)), 2014.

[BLH14b] Mohamed Barakat and Markus Lange-Hegermann, *Gabriel morphisms and the computability of Serre quotients with applications to coherent sheaves*, ([arXiv:1409.2028](https://arxiv.org/abs/1409.2028)), 2014.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094)

[Eis] Florian Eisele, *Does there exist a wide but not full abelian subcategory of an abelian category?*, MathOverflow, (accessed 2012-08-21): ([http://mathoverflow.net/questions/103868](http://mathoverflow.net/questions/103868)).

[Gab62] Pierre Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448. MR 0232821

[GD61] Alexandre Grothendieck and Jean Dieudonné, *Éléments de géométrie algébrique II*, Publications Mathématiques, vol. 8, Institute des Hautes Études Scientifiques., 1961.

[Gro57] Alexander Grothendieck, *Sur quelques points d’algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221. Translated by Marcia L. Barr and Michael Barr: Some aspects of homological algebra, ([ftp://ftp.math.mcgill.ca/barr/pdffiles/gk.pdf](ftp://ftp.math.mcgill.ca/barr/pdffiles/gk.pdf)). MR MR0102537 (21 #1328)

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967. MR 0210125 (35 #1019)

[Hau08] Jürgen Hausen, *Cox rings and combinatorics. II*, Mosc. Math. J. **8** (2008), no. 4, 711–757, 847. MR 2499353 (2010b:14011)

[Kra97] Henning Krause, *The spectrum of a locally coherent category*, J. Pure Appl. Algebra **114** (1997), no. 3, 259–271. MR 1426488 (98e:18006)

[Len69] Helmut Lenzing, *Endlich präsentierbare Moduln*, Arch. Math. (Basel) **20** (1969), 262–266. MR 0244322 (39 #5637)

[Mus02] Mircea Mustaţă, *Vanishing theorems on toric varieties*, Tohoku Math. J. (2) **54** (2002), no. 3, 451–470. MR 1916637 (2003e:14013)

[Per14] Markus Perling, *A lifting functor for toric sheaves*, Tohoku Math. J. (2) **66** (2014), no. 1, 77–92, ([arXiv:1110.0323](https://arxiv.org/abs/1110.0323)). MR 3189480

[PT10] M. Perling and G. Trautmann, *Equivariant primary decomposition and toric sheaves*, Manuscripta Math. **132** (2010), no. 1–2, 103–143, ([arXiv:0802.0257](https://arxiv.org/abs/0802.0257)). MR 2609290 (2011c:14051)

[Ser55] Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278. MR MR0068874 (16,953c)

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KAIERSLAUTERN, 67653 KAIERSLAUTERN, GERMANY**

*E-mail address:* barakat@mathematik.uni-kl.de

**LEHRSTUHL B FÜR MATHEMATIK, RWTH AACHEN UNIVERSITY, 52062 AACHEN, GERMANY**

*E-mail address:* markus.lange.hegermann@rwth-aachen.de