A Faster Subquadratic Algorithm for the Longest Common Increasing Subsequence Problem

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Abstract

The Longest Common Increasing Subsequence (LCIS) is a variant of the classical Longest Common Subsequence (LCS), in which we additionally require the common subsequence to be strictly increasing. While the well-known “Four Russians” technique can be used to find LCS in subquadratic time, it does not seem applicable to LCIS. Recently, Duraj [STACS 2020] used a completely different method based on the combinatorial properties of LCIS to design an $O(n^2 \log \log n^2 / \log^{1/6} n)$ time algorithm. We show that an approach based on exploiting tabulation can be used to construct an asymptotically faster $O(n^2 \log \log n / \sqrt{\log n})$ time algorithm. As our solution avoids using the specific combinatorial properties of LCIS, it can be also adapted for the Longest Common Weakly Increasing Subsequence (LCWIS).

1 Introduction

In the well-known Longest Common Subsequence problem we aim to find the length of the longest subsequence common to two strings $A[1..n]$ and $B[1..n]$. A textbook exercise is to find it in $O(n^2)$ time [10], and using the so-called “Four Russians” technique this has been brought down to $O(n^2 / \log^2 n)$ for constant alphabets [10] and $O(n^2 \log \log n / \log^2 n)$ for general alphabets [3]. Recently, there was some progress in providing explanation for why a strongly subquadratic $O(n^{2-\epsilon})$ time algorithm is unlikely [1, 4], and in fact even achieving $O(n^2 / \log^{7+\epsilon} n)$ would have some exciting unexpected consequences [2]. In this paper we consider a related problem defined as follows:

**Problem:** Longest Common Increasing Subsequence (LCIS)

**Input:** integer sequences $A[1..n]$ and $B[1..n]$

**Output:** largest $\ell$ such that there exist indices $i_1 < \ldots < i_\ell$ and $j_1 < \ldots < j_\ell$ with the property that (i) $A[i_k] = B[j_k]$, for every $k = 1, \ldots, \ell$, and (ii) $A[i_1] < \ldots < A[i_\ell]$.

While this is less obvious than for LCS, LCIS can be also solved in $O(n^2)$ time [11] (and in linear space [9]), and it can be proved that a strongly subquadratic algorithm would refute SETH [6] (although faster algorithms are known for some special cases [7]). However, as opposed to LCS, the usual “Four Russians” approach, that roughly consists in partitioning the DP table into blocks of size $\log n \times \log n$, doesn’t seem directly applicable to LCIS. Very recently, Duraj [5] used a completely different approach based on some nice combinatorial properties specific to LCIS to design a subquadratic $O(n^2 (\log \log n)^2 / \log^{1/6} n)$ time algorithm.
Our contribution. We design a faster subquadratic $O(n^2 \log \log n / \sqrt{\log n})$ time algorithm for LCIS. Interestingly, instead of using the combinatorial properties of LCIS as in the previous work we apply a technique based on exploiting tabulation (but differently than in the classical “Four Russians” approach). This allows our algorithm to be modified to solve the Longest Common Weakly Increasing Subsequence (LCWIS) problem (for which an $O(n^2)$ time algorithm is also known to refute SETH [8]). This doesn’t seem to be the case for Duraj’s approach based on bounding the number of so-called significant symbol matches, that for LCWIS might be $\Omega(n^2)$. Throughout the paper we assume that $A$ and $B$ are of the same length, and the goal is to calculate the length of LCIS. However, the algorithm can be easily modified to avoid this assumption and recover the subsequence itself.

Overview of the paper. Our algorithm is based on combining two different procedures. By appropriately selecting the parameters, the overall complexity becomes $O(n^2 \log \log n / \sqrt{\log n})$ as explained in Section 5. The first procedure described in Section 3 works fast when there are only few distinct elements in both sequences. We start with a solution based on dynamic programming working in $O(t \cdot n^2)$ time, where $t$ is the number of distinct elements in both sequences. Then, we exploit tabulation to decrease its running time to $O(t \cdot n^2 / \log n)$.

The second procedure described in Section 4 is efficient when there are not too many matching pairs, that is, pairs $(i, j)$ such that $A[i] = B[j]$. The main idea is to calculate, for every such pair, LCIS of $A[1..i]$ and $B[1..j]$ that ends with $A[i] = B[j]$. This is done by applying an appropriate dynamic predecessor structure. This roughly follows the ideas of Duraj, except that instead of using van Emde Boas trees we notice that, in fact, one can plug in any balanced search trees with efficient split/merge.

In Section 6 we explain the necessary modification required to adapt our solution for LCWIS.

2 Preliminaries

We work with sequences consisting of integers. For such a sequence $A$, we write $A[i]$ to denote the $i$-th element, and $A[1..i]$ to denote the prefix of length $i$. $|A|$ is the length of $A$. Let $\sigma$ be the sequence consisting of all distinct integers present in $A$ and $B$, arranged in the increasing order, and $\text{cnt}(v)$ be the total number of occurrences of $\sigma[v]$ in $A$ and $B$.

We call a pair of indices $(x, y)$ a matching pair when $A[x] = B[y]$. Further, we call it a $\sigma[i]$-pair when $A[x] = B[y] = \sigma[i]$.

We write $LCIS(i, j)$ to denote $LCIS(A[1..i], B[1..j])$, that is, the longest increasing common subsequence of $A[1..i]$ and $B[1..j]$. We write $LCIS^-(i, j)$ to denote the longest strictly increasing subsequence of $A[1..i]$ and $B[1..j]$ which includes both $A[i]$ and $B[j]$ (so in particular, $A[i] = B[j]$).

Throughout the paper, $\log x$ denotes $\log_2 x$.

3 First Solution

In this section we describe an algorithm for finding LCIS in $O(|\sigma| \cdot n^2 / \log n)$ time.

Let $dp_v[i][j]$ denote the largest possible length of a sequence $C$ such that:

1. $C$ is an increasing common subsequence of $A[1..i]$ and $B[1..j]$,

2. $C$ consists of elements not larger than $\sigma[v]$.
Then, our goal is to compute \( dp_v[i][n][n] \).

All \(|\sigma| \cdot n^2\) entries in \( dp \) can be calculated in \( O(1) \) time each using the following recurrence:

\[
dp_{v+1}[i][j] = \begin{cases} 
\max\{\dp_v[i][j], \dp_v[i-1][j-1] + 1\}, & \text{if } A[i] = B[j] = \sigma[v + 1], \\
\max\{\dp_v[i][j], \dp_{v+1}[i-1][j], \dp_{v+1}[i][j-1]\}, & \text{otherwise}.
\end{cases}
\]

In order to decrease the time we will speed up calculating \( dp_{v+1} \) from \( dp_v \). Because calculating \( dp_{v+1} \) only requires the knowledge of \( dp_v \), we will only keep the current \( dp_v \) and update all of its entries to obtain \( dp_{v+1} \).

**Lemma 3.1.** \( 0 \leq dp_v[i][j] - dp_v[i][j - 1] \leq 1 \) and \( 0 \leq dp_v[i][j] - dp_v[i - 1][j] \leq 1 \).

**Proof.** A subsequence of \( B[1..(j - 1)] \) is still a subsequence of \( B[1..j] \), so \( dp_v[i][j - 1] \leq dp_v[i][j] \). Consider a sequence \( C \) corresponding to \( dp_v[i][j] \), and let \( C' \) be \( C \) without the last element. Because \( C \) is a subsequence of \( B[1..j] \), \( C' \) is a subsequence of \( B[1..(j - 1)] \). So, \( C' \) is an increasing subsequence of \( A[1..i] \) and \( B[1..(j - 1)] \), hence \( |C'| \leq dp_v[i][j - 1] \). As \( |C| = |C'| + 1 \), we conclude that \( dp_v[i][j] \leq dp_v[i][j - 1] + 1 \). The second part of the lemma follows by a symmetrical reasoning. \( \square \)

Instead of maintaining \( dp_v \), we keep another table \( dp'_v[i][j] = dp_v[i][j] - dp_v[i][j - 1] \) (where \( dp_v[i][j] = 0 \) for \( j < 1 \)). Due to Lemma 3.1, each entry of \( dp'_v \) is either 0 or 1. This allows us to store each row of \( dp'_v \) by partitioning it into \( O(n/B) \) blocks of length \( B \), with every block represented by a bitmask of size \( B \) saved in a single machine word, where \( B = \alpha \log n \) for some constant \( \alpha \) to be fixed later. By definition, \( dp_v[i][j] = \sum_{k=1}^{j} dp'_v[i][k] \). In addition to \( dp'_v \), we store the value of \( dp_v[i][j] \) for every block boundary, so \( O(n^2/B) \) values overall. This will allow us later to recover any \( dp_v[i][j] \) in constant time by retrieving the value at the appropriate block boundary and adding the number of 1s in a prefix of some bitmask. We preprocess such prefix sums for every possible bitmask in \( O(2^B \cdot B) \) time and space.

**Lemma 3.2.** \( 0 \leq dp_{v+1}[i][j] - dp_v[i][j] \leq 1 \).

**Proof.** Because allowing using more elements cannot decrease the length, \( dp_{v+1}[i][j] \leq dp_{v+1}[i][j] \). Let \( C \) be a sequence corresponding to \( dp_{v+1}[i][j] \), and let \( C' \) be \( C \) without the last element. Because \( C \) is strictly increasing and \( \sigma \) consists of all distinct elements, the elements of \( C' \) are not larger than \( \sigma(v) \), so \( |C'| \leq dp_v[i][j] \). Then, using \( |C'| + 1 = |C| \) we obtain that \( dp_{v+1}[i][j] - 1 \leq dp_v[i][j] \). \( \square \)

We now describe how to calculate \( dp'_{v+1} \). We start with describing an approach that works in \( O(n^2) \) time and then explain how to accelerate it to \( O(n^2/\log n) \). We use the recursion for \( dp_{v+1}[i][j] \) to update the rows of \( dp'_{v+1} \) one-by-one. While updating the entries in a row going from left to right we are no longer guaranteed that \( dp_{v+1}[i][j] \leq dp_{v+1}[i][j + 1] \), so \( dp'_{v+1}[i][j] \) can become negative. To overcome this issue, we immediately propagate each value to the right: after increasing \( dp_{v+1}[i][j] \) (by one due to Lemma 3.2) we also increase every \( dp_{v+1}[i][k] \) equal to the original value of \( dp_{v+1}[i][j] \), for all \( k > j \). This translates into setting \( dp'_{v+1}[i][j] \) to 1 and setting \( dp'_{v+1}[i][k] \) to 0, for the smallest \( k > j \) such that \( dp'_{v+1}[i][k] = 1 \). To implement this efficiently, we maintain \( k \) while considering \( j = 1, 2, \ldots, n \) in \( O(n) \) overall time. The details of this procedure are shown in Algorithm 1.
Algorithm 1 Calculate the \(i\)-th row of \(dp'_{v+1}\)

1: \begin{procedure} \textbf{CALCULATEROW}(v, i) \end{procedure}
2: \hspace{1em} \texttt{ptr} \leftarrow 1
3: \hspace{1em} \texttt{cur.value} \leftarrow 0
4: \hspace{1em} \texttt{prv.value} \leftarrow 0
5: \hspace{1em} \texttt{prv.phase} \leftarrow 0
6: \hspace{1em} \textbf{for} \ j = 1..n \textbf{do}
7: \hspace{2em} \texttt{dp'_{v+1}[i][j]} = \texttt{dp'_v[i][j]}
8: \hspace{1em} \textbf{for} \ j = 1..n \textbf{do}
9: \hspace{2em} \textbf{if} \ \texttt{ptr} \leq i \textbf{then} \texttt{ptr} \leftarrow i + 1
10: \hspace{3em} \textbf{while} \ \texttt{ptr} \leq n \textbf{and} \ \texttt{dp'_{v+1}[i][ptr]} = 0 \textbf{do}
11: \hspace{4em} \texttt{ptr} \leftarrow \texttt{ptr} + 1
12: \hspace{2em} \texttt{cur.value} \leftarrow \texttt{cur.value} + \texttt{dp'_{v+1}[i][j]}
13: \hspace{2em} \triangleright \texttt{cur.value} = \sum_{j' = 1}^{j} \texttt{dp'_{v+1}[i][j']} = \max\{\texttt{dp_v[i][j]}, \texttt{dp'_{v+1}[i][j - 1]}\}
14: \hspace{2em} \triangleright \texttt{prv.phase} = \texttt{dp_v[i - 1][j - 1]}
15: \hspace{2em} \textbf{if} \ A[i] = B[j] = \sigma[v + 1] \textbf{and} \ \texttt{cur.value} = \texttt{prv.phase} \textbf{then}
16: \hspace{3em} \texttt{dp'_{v+1}[i][j]} \leftarrow 1
17: \hspace{2em} \texttt{cur.value} \leftarrow \texttt{cur.value} + 1
18: \hspace{2em} \textbf{if} \ \texttt{ptr} \leq n \textbf{then} \texttt{dp'_{v+1}[i][ptr]} \leftarrow 0
19: \hspace{2em} \texttt{prv.phase} \leftarrow \texttt{prv.phase} + \texttt{dp'_{v+1}[i - 1][j]}
20: \hspace{2em} \texttt{prv.value} \leftarrow \texttt{prv.value} + \texttt{dp'_{v+1}[i - 1][j]}
21: \hspace{2em} \triangleright \texttt{prv.value} = \texttt{dp_v[i - 1][j]}
22: \hspace{2em} \textbf{if} \ \texttt{cur.value} < \texttt{prv.value} \textbf{then}
23: \hspace{3em} \texttt{cur.value} \leftarrow \texttt{prv.value}
24: \hspace{2em} \texttt{dp'_{v+1}[i][j]} \leftarrow 1
25: \hspace{2em} \textbf{if} \ \texttt{ptr} \leq n \textbf{then} \texttt{dp'_{v+1}[i][ptr]} \leftarrow 0

We speed up Algorithm 1 by a factor of \(B\) by considering whole blocks of \(dp'_{v+1}\) instead of single entries. Consider a single block of \(dp'_{v+1}\) consisting of the values of \(dp'_{v+1}[i][j], dp'_{v+1}[i][j + 1], ..., dp'_{v+1}[i][j + B - 1]\), and assume that they have been already partially updated by propagating the maximum. To calculate their correct values we need the following information:

1. \(dp'_v[i - 1][j], dp'_v[i - 1][j + 1], ..., dp'_v[i - 1][j + B - 1],\)
2. \(dp'_{v+1}[i - 1][j], dp'_{v+1}[i - 1][j + 1], ..., dp'_{v+1}[i - 1][j + B - 1],\)
3. \(dp'_{v+1}[i][j], dp'_{v+1}[i][j + 1], ..., dp'_{v+1}[i][j + B - 1],\)
4. \(dp_v[i - 1][j - 1],\)
5. \(dp_v[i - 1][j - 1],\)
6. \(dp_v[i - 1][j - 1],\)
7. for which indices \(j, j + 1, ..., j + B - 1\) we have \(A[i] = B[j] = \sigma[v + 1].\)

In fact, we can rewrite the procedure so that instead of the values \(dp_v[i - 1][j - 1], dp_v[i - 1][j - 1], dp_{v+1}[i][j - 1]\) only the differences \(dp_v[i - 1][j - 1] - dp_v[i - 1][j - 1]\) and \(dp_{v+1}[i][j - 1] - dp_{v+1}[i - 1][j - 1]\) are needed. By Lemma 3.1 and Lemma 3.2, both differences belong to \(\{0, 1\}\), so the whole
information required for calculating the correct values consists of \(4B + 2\) bits. Blocks \(dp'\) are already stored in separate machine words, and we can prepare, for every \(v\), an array with the \(j\)-th entry set to 1 when \(B[j] = v\), partitioned into \(n/B\) blocks of length \(B\), where each block is saved in a single machine word, in \(\mathcal{O}(|\sigma| \cdot n)\) time. This allows us to gather all the required information in constant time and use a precomputed table of size \(\mathcal{O}(2^{4B+2})\) that stores a single machine word encoding the correct values in a block for every possible combination. Additionally, the table stores the number of 1s to the right of the block that should be changed to 0. The table can be prepared in \(\mathcal{O}(2^{4B+2} \cdot B)\) time by a straightforward modification of Algorithm 1. Now we can update a whole block in constant time by retrieving the precomputed answer, but then we still might need to remove some 1s on its right. Instead of removing them one-by-one we work block-by-block. In more detail, we maintain a pointer to the nearest block that might contain a 1. Let the number of 1s there be \(4\). Second Solution

We use a BST that allows split and merge in \(\mathcal{O}(\log s)\) time, where \(s\) is the number of stored

4 Second Solution

In this section we describe an algorithm for solving LCIS in \(\mathcal{O}(\sum_{v=1}^{|\sigma|} (\text{cnt}(v))^2 (1 + \log^2(n/\text{cnt}(v))))\) time.

For every matching pair \((x, y)\), we will compute \(\text{LCIS}^{-1}(x, y)\), called the result for \((x, y)\). The algorithm proceeds in phases corresponding to the elements of \(\sigma\), and in the \(v\)-th step computes the results for all \(\sigma[v]\)-pairs. During this computation we maintain, for every \(x, y\), if there exists an already processed matching pair \((x', y')\) with result \(r\) such that \(x' < x\) and \(y' < y\). Each \(D(r)\) is implemented using the following lemma.

**Lemma 4.1.** We can maintain a set of points \(S \subseteq [n] \times [n]\) under inserting a batch of \(u \leq n\) points in \(\mathcal{O}(u(1 + \log \frac{n}{u}))\) time and answering a batch of \(q \leq n\) queries of the form “given \((x, y)\), is there \((x', y') \in S\) such that \(x' < x\) and \(y' < y\)” in \(\mathcal{O}(q(1 + \log \frac{n}{q}))\) time.

**Proof.** We observe that if the current \(S\) contains two distinct points \((x_i, y_i)\) and \((x_j, y_j)\) with \(x_i \leq x_j\) and \(y_i \leq y_j\) then there is no need to keep \((x_j, y_j)\). Thus, we keep in \(S\) only points that are not dominated. Let \((x_1, y_1), \ldots, (x_k, y_k)\) be these points arranged in the increasing order of \(x\) coordinates (observe that we cannot have two non-dominated points with the same \(x\) coordinate). So, \(x_1 < x_2 < \ldots < x_k\), where \(k \leq n\), and because the points are not dominated also \(y_1 > y_2 > \ldots y_k\). We store the \(x\) coordinates in a BST. This clearly allows us to answer a single query \((x, y)\) in \(\mathcal{O}(\log n)\) time by locating the predecessor of \(x\). To insert a point \((x, y)\), we first check that it is not dominated by locating the predecessor of \(x\). Then, we might need to remove some of the subsequent \(x\) coordinates that correspond to points that are dominated by \((x, y)\). This can be efficiently implemented by maintaining a doubly-linked list of all points, and linking each \(x\) coordinate with its corresponding point. Insertion takes \(\mathcal{O}(\log n)\) time plus another \(\mathcal{O}(\log n)\) for every removed point, so \(\mathcal{O}(\log n)\) amortised time, and a query concerning \((x, y)\) reduces to finding the predecessor of \(x\) among the \(x_is\), which is still too slow.

We use a BST that allows split and merge in \(\mathcal{O}(\log s)\) time, where \(s\) is the number of stored
elements, for example AVL trees. Additionally, we store the size of the subtree in every node. Then we have the following easy proposition.

**Proposition 4.2.** We can split BST into at most $b$ smaller BSTs containing $\Theta(s/b)$ elements each in $O(b(1 + \log \frac{s}{b}))$ time.

**Proof.** As long as there is a BST of size at least $2s/b$ we split it into two BSTs of (roughly) equal sizes. Assuming for simplicity that both $s$ and $b$ are powers of 2, this takes $O\left( \sum_{i=0}^{\log b-1} 2^i \log(s/2^i) \right)$ overall time, which can be bounded by calculating $\int_1^b \log(s/x)dx = O(b(1 + \log(s/b))).$ □

To process a batch of $b$ insertions/queries efficiently, we first sort them in $O(b(1 + \log(n/b)))$ time. Then, we split the BST into at most $b$ smaller BSTs containing $\Theta(s/b)$ elements each, where $s$ is the number of stored elements, using Proposition 4.2. Because insertions/queries are sorted, we can determine for each of them the relevant BST by a linear scan, and then insert/query the relevant BST in $O(1 + \log(s/b))$ time per operation (if there are more than $s/b$ insertions to the same smaller BST, we split it into trees containing single elements, and partition the insertions into groups of $\Theta(s/b)$). Finally, we merge the BSTs into pairs, quadruples, and so on. By the calculation from the proof of Proposition 4.2 this also takes $O(b(1 + \log(s/b)))$ time. □

Lemma 4.1 is already enough to binary search for the result of $(x, y)$ in $O(\log^2 n)$ time due to the following property.

**Lemma 4.3.** Consider any $r$ and an already processed matching pair $(x', y')$ with result $r$. Then either $r = 1$ or there exists an already processed matching pair $(x'', y'')$ with result $r - 1$ such that $x'' < x'$ and $y'' < y'$.

**Proof.** Assume that $r \geq 2$ and consider a sequence $C$ which realises the result for $(x', y')$. Then $C[1..\lceil C \rceil - 1]$ is an increasing subsequence of both $A[1..(x' - 1)]$ and $B[1..(y' - 1)]$. Let $A[x'']$ and $B[y'']$ be its last elements in $A$ and $B$, respectively. Then $x'' < x'$, $y'' < y'$, and $A[x''] = B[y'']$, so $(x'', y'')$ is a matching pair, and because $C$ is strictly increasing this matching pair must have been already processed. □

However, our goal is to spend $O(1 + \log^2(n/\text{cnt}(v)))$ time per every $(x, y)$. We exploit the following property.

**Lemma 4.4.** Consider two $\sigma[i]$ pairs $(x, y_1)$ and $(x, y_2)$, where $y_1 < y_2$. The result for $(x, y_2)$ is at least as large as for $(x, y_1)$.

**Proof.** Consider a sequence $C$ which realises $\text{LCIS}^\rightarrow(x, y_1)$. Then, replacing $y_1$ with $y_2$ we obtain a valid candidate for the value of $\text{LCIS}^\rightarrow(x, y_2)$. □

Consider all $\sigma[v]$ pairs with the same $x$ coordinate $(x, y_1), (x, y_2), \ldots, (x, y_{\text{cnt}(\sigma[v])})$. We binary search for the result of $(x, y_i)$ for $i = \text{cnt}(v), \ldots, 2, 1$. By Lemma 4.4, in the $i$-th step we can start with the result found in the $(i + 1)$-th step. Using doubling binary search, by convexity of the log function the overall complexity becomes $O(\text{cnt}(v)(1 + \log(n/\text{cnt}(v))))$. This is still too slow, as every step involves a separate invocation Lemma 4.1 and takes $O(\log n)$ time. To obtain the final speed up, we process all $x$ coordinates $x_1, x_2, \ldots, x_{\text{cnt}(v)}$ together. The high level idea is to synchronise all binary searches and exploit the possibility of asking a batch of queries.

We start with modifying the proof of Lemma 4.1 to allow for more general queries: given $x$, we want to find the smallest $y$ such that there exists $(x', y') \in S$ with $x' < x$ and $y' < y$ (or detect that
there is none). The modification is straightforward and doesn’t increase the time complexity. Now we can restate processing all pairs with the same \( x \) coordinates. We start with a counter \( c \) initially set to \( n \) and \( i \) set to \( \text{cnt}(v) \). As long as \( i \geq 1 \), we use doubling binary search starting at \( c \) to find the result for \((x,y_i)\). Let \( c' \) be the found result. We use the modified Lemma 4.1 to determine the smallest \( y \) such that \( c' \) is the result for \((x,y)\) and then keep decreasing \( i \) as long as \( i \geq 1 \) and \( y_i > y \). Then, we decrease \( c' \) by 1 and repeat.

We further reformulate processing all pairs with the same \( x \) coordinate. Consider a conceptual complete binary tree on \( n \) leaves (without losing generality, \( n \) is a power of 2). Every node corresponds to an interval \([a,b]\), and by querying such a node we will understand querying structure \( D(a) \) with the current \((x,y_i)\). Consider the leaf corresponding to \( c \). Calculating \( c' \) with doubling binary search can be phrased as starting at the leaf corresponding to \( c \) and going up as long as the query at the current node fails (we only need to ask a query if the previous node was the right child of the current node; otherwise, we can immediately jump to the nearest ancestor with such property). After having reached the first ancestor for which the query succeeds, we descend from its left child to the leaf corresponding to \( c' \) by repeating the following step: if querying the right child of the current node succeeds we descend to the right child, and otherwise we descend to the left child.

Now we are able to synchronize the binary searches as follows. We traverse the conceptual complete binary tree recursively: to traverse the subtree rooted at node \( u \) with children \( u_L \) and \( u_R \), we (i) visit \( u \), (ii) recursively traverse the subtree rooted at \( u_R \), (iii) visit \( u \) again, (iv) recursively traverse the subtree rooted at \( u_L \). Thus, every node is visited twice. We claim that when visiting the nodes of the conceptual complete binary tree using this strategy, for any \( x \) coordinate we are always able to wait till we encounter the node that should be queried next. This is formalised in the following lemma.

**Lemma 4.5.** Let the result for \((x,y_{i+1})\) be \( c \) and the result for \((x,y_i)\) be \( c' < c \). All queries necessary to calculate \( c' \) can be answered during the traversal after the second visit to \( c \) and before the second visit to \( c' \).

**Proof.** The calculation consists of two phases. First, we need to ascend from the leaf corresponding to \( c \), reaching its first ancestor \( u \) at which the query fails. Recall we only need to ask queries if the previous node is the left child of the current node. For each such node \( v \) we will be able to use second visit to \( v \) in the traversal. Thus, we will process all such queries after the second visit to \( u \). Then, we need to descend from the left child of \( u \). In every step, we query the right child \( v_r \) of the current node \( v \), and continue either in the left or in the right subtree of \( v \). To this end, we use the first visit to \( v_r \) in the traversal.

For each \( x \) coordinate, by convexity of the log function, we need to query at most \( O(\text{cnt}(v)(1 + \log(n/\text{cnt}(v)))) \) nodes of the conceptual binary tree. Denoting by \( q_u \) the number of queries to a node \( u \), we thus have \( \sum_u q_u = s = O(\text{cnt}(v)^2(1 + \log(n/\text{cnt}(v)))) \). Invoking Lemma 4.1, the total time to answer all these queries is \( \sum_u q_u(1 + \log(n/q_u)) \). By convexity of the function \( f(x) = x \log(n/x) \), this is maximised when all \( q_u \)s are equal, but there are only \( n \) of them, making the total time:

\[
\sum_{u} q_u(1 + \log(n/q_u)) \leq s(1 + \log(n^2/s)) \leq s(1 + \log(n^2/\text{cnt}(v)^2)) = O(\text{cnt}(v)^2(1 + \log(n/\text{cnt}(v))))^2.
\]

**5 Combining Solutions**

Let \( c \) be a parameter to be fixed later. We call \( \sigma[v] \) frequent if \( \frac{n}{c} < \text{cnt}(v) \), and rare otherwise.

We partition the sequence \( \sigma \) into fragments. Each fragment is either a single frequent element or a maximal range of rare elements. By definition of a frequent element and maximality of fragments
consisting of rare elements, we have $O(c)$ fragments. We maintain the $dp_v$ table as in the first solution, but we only update it after having processed a whole fragment. So, when considering a fragment starting at $\sigma[v]$ we only assume that the values of $dp_{v-1}$ can be access in constant time. For a fragment consisting of a single frequent element, we proceed exactly as in the first solution. In the remaining part of the description we describe how to process a fragment consisting of rare elements $\sigma[v], \sigma[v+1], \ldots$.

We consider all $\sigma[v']$-pairs, for $v' = v, v+1, \ldots$. We will compute $LCIS^\rightarrow(x, y)$ for each such matching pair $(x, y)$, and store it in the appropriate structure $D(r)$ implemented as described in Lemma 4.1. To compute the values of $LCIS^\rightarrow(x, y)$ for all $\sigma[v']$-pairs, we use parallel binary search as in the second solution with the following modification. To check if $LCIS^\rightarrow(x, y_i) > r$, we need to consider two possibilities for the corresponding sequence $C$ ending at $A[x] = B[y_i] = \sigma[v']$:

1. If $C||C| - 1|$ belongs to the same fragment then it is enough to check if $D(r)$ contains a pair $(x', y')$ with $x' < x$ and $y' < y_i$.

2. Otherwise, it is enough to check if $dp_{v-1}[x][y_i] \geq r$.

Additionally, after having found $c'$ we need to keep decreasing $i$ as long as $i \geq 1$ and the answer for $(x, y_i)$ is $c'$, and this needs to be tested in constant time per each such $i$. We again need to consider two possibilities, and either compare $y_i$ with the value of $y'$ found by querying $D(c' - 1)$ with $x$, or test if $dp_{v-1}[x][y_i] \geq r$ in constant time. Overall, this incurs only additional constant time per every step of the binary search for every considered matching pair.

After having considered all $\sigma[v']$-pairs for the last element $\sigma[v']$ in the current fragment, we need to compute $dp_{v'}$ from $dp_{v-1}$ and the calculated values of $LCIS^\rightarrow$. Of course, we want to operate on $dp'_{v'}$ and $dp'_{v-1}$ instead of $dp_{v'}$ and $dp_{v-1}$. This is done row-by-row. The $i$-th row is computed in two steps.

First, we need to set $dp'_{v'}[i][j] = \max\{dp'_{v'}[i-1][j], dp_{v-1}[i][j]\}$ for every $j = 1, 2, \ldots, n$. This is done by processing whole blocks in constant time and precomputing the result for every possible combination of the following information:

1. $dp'_{v'}[i-1][j], dp'_{v'}[i-1][j+1], \ldots, dp'_{v'}[i-1][j+B-1],$
2. $dp'_{v-1}[i][j], dp'_{v-1}[i][j+1], \ldots, dp'_{v-1}[i][j+B-1],$
3. $dp_{v'}[i-1][j-1],$
4. $dp_{v-1}[i][j-1].$

This can be preprocessed in $O(4B \cdot B^2)$ time after observing that, as in the first solution, only the difference $dp'_{v'}[i-1][j-1] - dp_{v-1}[i][j-1]$ is relevant and, additionally, it can be capped at $B$ (if it is bigger than $B$ then we can set it to $B$). The time is $O(n/B)$.

Second, we need to consider the values of $LCIS^\rightarrow(i, j)$ computed for the current fragment. If the result computed for a matching pair $(i, j)$ is $r$ then we need to update $dp_{v'}[i][j'] = \max\{dp_{v'}[i][j'], r\}$, for every $j' \geq j$. This can be done by simultaneously scanning all such $j$s and the blocks. By maintaining the maximum $r$, we can update the value of $dp_{v'}[i][j]$ at the beginning of the block. Then, we consider all other $j$'s belonging to the same block, and consider its corresponding result $r'$. If $dp_{v'}[i][j'] \geq r'$ then this result is irrelevant, and otherwise we must increase some of the values in the block by 1 (as $dp_{v'}[i][j'-1]$ is assumed to have been already updated and due to Lemma 3.1). As in the first solution, this is implemented by setting $dp'_{v'}[i][j'] = 1$ and changing the nearest 1 into 0. Overall, the time is bounded by the number of considered matching pairs plus additional $O(n/B)$ time.
We set \( B = \frac{\log n}{c} \) so that the preprocessing time is \( o(n) \). For each frequent element we spend \( \mathcal{O}(n^2/B) \) time, so \( \mathcal{O}(n^2/B \cdot c) \) overall. For each fragment consisting of rare elements, the time is \( \mathcal{O}(\text{cnt}(v) \log^2(n/\text{cnt}(v))) \) for every \( v \) to compute the results, and then \( \mathcal{O}(n^2/B) \) plus the number of results. Using \( \text{cnt}(v) \leq n/c \), where \( c \) is sufficiently large, and calculating the derivative of \( f(x) = x \log^2(n/x) \) we upper bound \( \text{cnt}(v) \log^2(n/\text{cnt}(v)) \leq n/c \cdot \log^2 c \) for every rare \( v \), so the overall time is \( \mathcal{O}(n^2/B \cdot c + n/c \cdot \log^2 c \sum_v \text{cnt}(v)) = \mathcal{O}(n^2/B \cdot c + n^2/c \cdot \log^2 c) \).

Choosing \( c = \sqrt{\log n \log \log n} \) we obtain an algorithm working in \( \mathcal{O}(n^2 \log \log n / \sqrt{\log n}) \) time.

## 6 Longest Common Weakly Increasing Subsequence

In this section we explain how to modify the algorithm to solve the weakly increasing version of the problem. We adapt both solutions without changing their complexity as explained below, and then combine them using the same threshold for the frequent/rare elements to arrive at \( \mathcal{O}(n^2 \log \log n / \sqrt{\log n}) \) complexity.

### 6.1 First solution

We define \( dp \) as in the algorithm for LCIS. It can be calculated using the following recurrence (slightly different than for LCIS):

\[
dp_{v+1}[i][j] = \begin{cases}  
\max\{\dp_v[i][j], \dp_{v+1}[i-1][j-1] + 1\}, & \text{if } A[i] = B[j] = \sigma[v+1], \\
\max\{\dp_v[i][j], \dp_{v+1}[i-1][j], \dp_{v+1}[i][j-1]\}, & \text{otherwise.}
\end{cases}
\]

The proof of Lemma 3.1 still holds, so we can store a table \( dp' \) and retrieve any value of \( dp \) from \( dp' \) in constant time.

Algorithm 1 stays essentially the same so we skip a detailed explanation. The speed up is implemented by considering whole blocks of \( dp'_{v+1} \) instead of single entries. Consider a single block of \( dp'_{v+1} \) consisting of the values of \( dp'_{v+1}[i][j], dp'_{v+1}[i][j+1], \ldots, dp'_{v+1}[i][j+B-1], \) and assume that they have been already partially updated by propagating the maximum. To calculate their correct values we need the following information:

1. \( dp'_{v+1}[i-1][j], dp'_{v+1}[i-1][j+1], \ldots, dp'_{v+1}[i-1][j+B-1], \)
2. \( dp'_{v+1}[i][j], dp'_{v+1}[i][j+1], \ldots, dp'_{v+1}[i][j+B-1], \)
3. \( dp_{v+1}[i-1][j-1], \)
4. \( dp_{v+1}[i][j-1], \)
5. for which indices \( j, j+1, \ldots, j+B-1 \) we have \( A[i] = B[j] = \sigma[v+1]. \)

Once again we can rewrite the procedure so that instead of the values \( dp_{v+1}[i-1][j-1] \) and \( dp_{v+1}[i][j-1] \) only the difference \( dp_{v+1}[i][j-1] - dp_{v+1}[i-1][j-1] \) is needed. By Lemma 3.1, the difference belongs to \( \{0,1\} \), so the whole information required for calculating the correct values consists of \( 3B+1 \) bits. This allows us to update the whole table in \( \mathcal{O}(n^2/B) \) as for LCIS.

We set \( B = \frac{\log n}{4} \) as to make required preprocessing \( o(n) \). Overall complexity of the algorithm becomes \( \mathcal{O}(\|\sigma\|n^2/\log n) \).
6.2 Second solution

Calculating the result for each $\sigma[v]$-pair consists of two phases. In the first phase, for each $\sigma[v]$-pair $(x, y)$, we calculate the result assuming that all previous elements in the subsequence are strictly smaller than $\sigma[v]$. In the second phase, we calculate the result assuming that the previous element is also equal to $\sigma[v]$. The first phase can be implemented exactly as for LCIS in $O(cnt(v)^2(1 + \log^2(n/cnt(v))))$ time. We now focus on explaining how to implement the second phase. Let $prev_A[x]$ denote the greatest $x'$ fulfilling $A[x'] = A[x]$, if there is no such then $prev_A[x] = 0$. Similarly we define $prev_B[y]$, both array can be prepared in negligible $O(n \log n)$ time.

We analyze all $\sigma[v]$-pairs in the increasing order of rows and columns. Consequently, when analysing a pair $(x, y)$, for all other $\sigma[v]$-pairs with $x' \leq x$, $y' \leq y$ we have already correctly calculated $LCWIS^{-}(x', y')$. The proof of Lemma 4.4 still holds for LCWIS, and implies that among all other $\sigma[v]$-pairs $(x', y')$ such that $x' \leq x$ and $y' \leq y$ the pair $(prev_A[x], prev_B[y])$ has the largest result. We can calculate $LCWIS^{-}(x, y)$ as the maximum of the result computed in first phase and $LCWIS^{-}(prev_A[x], prev_B[y]) + 1$.

The second phase takes only $O(cnt(v)^2)$ time, so the overall complexity remains $O(cnt(v)^2(1 + \log^2(n/cnt(v))))$.

7 Conclusions

The $O(n^2 \log \log n/\sqrt{\log n})$ complexity doesn’t seem to be right answer yet, at least for LCIS. It seems to us that one can apply the combinatorial bound of Duraj on the number of significant pairs, and combine it with our approach, to achieve an even better complexity. However, as this doesn’t seem to result in a clean bound of (say) $O(n^2/\log n)$ yet, we leave determining the exact complexity for future work.

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