Comment on “Density perturbations in the ekpyrotic scenario”

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In the paper “Density perturbations in the ekpyrotic scenario”, it is argued that the expected spectrum of primordial perturbations should be scale invariant in this scenario. Here we show that, contrary to what is claimed in that paper, the expected spectrum depends on an arbitrary choice of matching variable. As no underlying (microphysical) principle exists at the present time that could lift the arbitrariness, we conclude that the ekpyrotic scenario is not yet a predictive model.

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I. INTRODUCTION

Recently, a model called the ekpyrotic scenario ¶ was proposed as a possible competitor to the inflationary paradigm § on the basis that it could solve the problems of the hot big bang model and also produce an almost scale invariant spectrum of primordial scalar perturbations. Although the foundations of the model are still under debate ¶, †, ‡, § we shall comment on the controversial ¶, †, ‡, § claim, crucial for the ekpyrotic scenario, that the power spectrum is scale-invariant ¶.

In Ref. †, the complicated five-dimensional evolution of the background is modeled by an effective four-dimensional spacetime which experiences a singular bounce at \( \eta = 0 \) (\( \eta \) being the conformal time) as the effective scale factor vanishes. The calculation of the power spectrum is a controversial issue as different authors would use different matching conditions ¶, †, ‡, § at the singular bounce. There, Bardeen’s gravitational potential ¶, ¶, ¶ diverges. The authors of Ref. † suggested to work with the comoving density contrast \( \epsilon_m \) which is regular at \( \eta = 0 \) and has two linearly independent modes \( D \) and \( E \), such that

\[
\epsilon_m = \epsilon_0 D(\eta) + \epsilon_2 E(\eta).
\]

At the singular point the equality of the coefficients before and after the bounce is assumed ¶: \( \epsilon_0(0^+) = \epsilon_0(0^-) \) and \( \epsilon_2(0^+) = \epsilon_2(0^-) \). Although this rule is given without justification (recall that the standard junction conditions follow from the Einstein equations), we shall refer to such equalities as matching conditions hereafter.

In Ref. †† we criticized the matching conditions suggested in Ref. †. We showed that they lead to an ambiguity, and thus an arbitrary choice needs to be made that present-day physics cannot make. Ref. † was subsequently modified, see Ref. ‡‡, to address some critical comments that have been made on the ekpyrotic model, one of them being the above mentioned ambiguity exhibited in Ref. †. In the new version ‡‡, more details about the matching conditions are given and one claim has been added (claim 1): “One situation of special interest is ... where the potential is irrelevant at \( \phi \to -\infty \) and there is no radiation in the incoming state. In this case, \( \epsilon_2(0^-) = 0 \). In this case, we would obtain the same final result from any matching rule which set \( \epsilon_2(0^+) = A\epsilon_2(0^-) \), with any constant \( A \).” It concerns a particular case that we had not considered and for which our result supposedly does not apply. It should be noted however that the rest of Ref. ‡‡ inconsistently relies on the general case instead of this particular one, as it was the case in the preceding version †. Claim 1 clearly utilizes, in an essential manner, the ideas developed in Ref. † and represents a tentative a priori response to the points Ref. †† raised.

References ¶, ¶‡ also state (claim 2): “The prescription is invariant under redefining the independent solutions, e.g. by adding an arbitrary amount of the solution \( E(\eta) \) to \( D(\eta) \). Matching any other non-singular perturbation variable, defined to be an arbitrary linear combination of \( \epsilon_m \) and \( \epsilon_m' \) with coefficients which are non-singular background variables (defined to possess power series expansions in \( \tau \), as above) will, with the same prescription of matching the amplitudes of both linearly independent solutions, also yield precisely the same result”. This is an important point in the ekpyrotic scenario because, if true, it would support the use of the variable \( \epsilon_m \) as a tool to define matching conditions.

Below we show that both quoted claims are incorrect.

II. IS THE EKPYROTIC SPECTRUM UNIQUE?

Let us start with claim 1, i.e. the question concerning the invariance of the spectrum under rescaling of the
parameter \( \epsilon_2 \) and let us first define our notation. The authors of Ref. \[13\] consider the equation of state close to the singular bounce to be \( \omega \equiv p/\rho = \omega_1 + \omega_2 \eta + \omega_3 \eta^2 + \cdots \). Before the bounce, the kinetic energy of the scalar field dominates and the dynamics can be approximated by that of a free scalar field for which \( \omega_1 = 1 \) and \( \omega_2 = 0 \) (note that one can show that the values of \( \omega_i \), \( \forall i > 2 \), are irrelevant for this discussion). This should apply for \( \eta < 0 \) close to the singularity (\( |\eta| \ll 1 \)) and corresponds precisely to the situation that is assumed in claim 1 and to the case where the equations of Ref. \[12\] cannot be applied (as mentioned in that article), hence the argument of Ref. \[13\]. Below, we complete the arguments of Ref. \[12\] to include this particular case and show that the conclusions of Ref. \[13\] remain unchanged.

At leading order, the behavior of the scale factor is a simple power-law, \( a(\eta) = \ell_0 (\eta - \eta_0)^{1/2}/(2 \sqrt{2}) \), in agreement with the notation of Eq. (24) in Ref. \[12\]. The comoving Hubble rate and the sound speed are \( \cal H = a'/a = 1/2(\eta) \) and \( c_s^2 = 1 \). The solution for the Bardeen potential \[11\] in the long wavelength limit reads

\[
\Phi = \frac{3}{\ell_0^2 \eta^2} B_1 + \frac{3}{8} B_2,
\]

with \( \theta \equiv 1/(a \sqrt{T + \omega}) \) and the integral sign stands for the primitive of the integration kernel. The coefficients \( B_1 \) and \( B_2 \) are functions of the comoving wavenumber \( k \), to be determined by means of a matching with the initial vacuum condition \[11\]. In the case at hand,\n
\[
\Phi = -\frac{3}{\ell_0^2 \eta^2} B_1 + \frac{3}{8} B_2. \tag{2}
\]

Let us note at this point that, with the choice we made of the normalization of the integral in Eq. \[1\], the limit \( \omega_1 \to 0 \) is singular, as can be seen from Eqs. \(6\) - \(7\) and as already mentioned in Ref. \[12\]. Another choice is possible as one is free to add an arbitrary constant amount of \( B_2 \) into \( B_1 \), simply by making the integral a definite one. In particular, one can arrange that the constant factor \( \ln(2) \) in the above equations be made to vanish from the outset, and it is easy to convince oneself that such a choice does not modify anything if the correct equations are used.

In the ekpyrotic scenario, and in the long wavelength limit, the coefficient \( B_2^\infty \propto k^{-3/2} \) is dominant with respect to \( B_2^\infty \propto k^{-1/2} \). This means that the relation \( \epsilon_2 \sim 9 \omega_1^2 \epsilon_0/64 \) holds true if \( \omega_1 \neq 0 \), whereas it does not if \( \omega_1 = 0 \). In the latter case, one has \( \epsilon_2 = -\ell_0^2 B_2 \epsilon_0/(8 B_1) \propto k \epsilon_0 \neq 0 \), see Eq. \(6\). The authors of Ref. \[13\] missed that the limit \( \omega_1 \to 0 \) does in no way imply \( \epsilon_2 = 0 \) exactly, as implied by claim 1, in which this value is said to be used to perform the matching. Moreover, as demonstrated below, the fact that \( \epsilon_2 \) becomes subdominant does not remove the ambiguity noticed in Ref. \[13\].

This is also related to the excessive claim in \[13\] that “there is no long wavelength contribution to \( \zeta \) in the collapsing phase”. Inserting \(6\) into the gauge-invariant variable \( \zeta \equiv (2/3)(\cal H^{-1} \Phi')/(1 + \omega) + \Phi \) (see e.g. \[9\] \[11\] \[14\]), gives

\[
\zeta = \frac{1}{2} B_2 j_0(-k \eta) + \frac{\pi k^2}{4 \ell_0^2} B_1 N_0(-k \eta) \tag{8}
\]

which, in the limit \( \eta \to 0 \), yields

\[
\zeta \sim \frac{1}{2} B_2 - \frac{1}{2 \ell_0^2} B_1 \ln(-k \eta) + \gamma_E, \tag{9}
\]

where \( \gamma_E \) is Euler constant coming from the expansion of the Bessel function. Since the second term is singular the first term is subdominant, but that does not imply that it vanishes altogether.

After the singular bounce (\( \eta > 0 \)), \( \omega_1^\infty \neq 0 \) and one recovers the relations \[12\]

\[
\epsilon^0_0(k) = \frac{8k^2}{\ell_0^2} B_1^\infty, \quad \epsilon^\infty_0(k) = -k^2 B_2^\infty, \tag{5}
\]

showing that the two “modes” decouple. It is interesting to compare with the case \( \omega_1^\infty \neq 0 \), for which one has \[12\] (here we assume \( \omega_2^\infty = 0 \) for the sake of simplicity; this does not change in any way the conclusion)

\[
\epsilon^0_0(k) = \frac{8k^2}{\ell_0^2} B_1^\infty - \frac{128 \ln(2) k^2}{9 \omega_1^\infty} B_2^\infty, \tag{6}
\]

\[
\epsilon^\infty_0(k) = -\frac{9k^2 \omega_1^\infty}{8 \ell_0^2} B_1^\infty - (1 + 2 \ln(2)) k^2 B_2^\infty. \tag{7}
\]

The authors of Ref. \[13\] propose to glue the epoch after the bounce to the epoch before the bounce by imposing that \( \epsilon_0 \) and \( \epsilon_2 \) are the same before and after the singularity. Contrary to standard junction conditions, and until some more microphysics is specified, this recipe does not rest on any physical principle and one may wonder what is the rationale behind it. The reason why one is pushed to adopt a new rule in this case is linked to the fact that one is dealing with a perturbative approach around a singular background whose meaning is questionable \[4\]. Moreover, we argued in Ref. \[13\] that this prescription
is ambiguous since an arbitrary rescaling factor $f(\omega, k)$ shows up in the final result. Therefore, the scenario contains an arbitrary function in a physically measurable quantity: until this function can be calculated unambiguously by first principles, the model cannot be falsified.

As in Ref. [12], we rescale $\epsilon_2$ by the completely arbitrary factor $1/f$. This gives the constant part of the gravitational potential

$$B_2^> = B_2^< \frac{f^>}{f^<} + \frac{9\omega_1^2}{8\epsilon_0} B_1^>.$$  \hspace{1cm} (12)

This is the main equation of Sec. II. It represents the equivalent of Eq. (54) of Ref. [12] for the case $\omega_1^< = 0$. On the other hand, the decaying mode amplitude reads

$$B_1^> = -(1 + 2 \ln 2) B_1^< - \frac{16 \ln 2 \ell_0^2}{9\omega_1^2} B_2^< \frac{f^>}{f^<},$$ \hspace{1cm} (13)

given here for the sake of completeness.

Note that if the expansion of the equation of state parameter is done to higher order in $\eta$, all the relations of this section are only changed by inclusion of $\omega_2$ through the replacement $(3/8)\omega_2^2 \rightarrow w^{(2)} \equiv \omega_2 + (3/8)\omega_1^2$, in agreement with Ref. [13], while none of the $\omega_i$ for $i > 2$ does contribute.

Eq. (14) shows that the spectrum effectively acquires a scale invariant piece $\propto B_1^>$, together with an arbitrary piece $\propto B_2^> f^> / f^<$. Indeed, as discussed in Ref. [12], the functions $f^>$ and $f^<$ do depend on the background constants (e.g., $\omega$) and on the considered scale $k$ in a way which is not yet given from first principles. The choice $f^> = f^<$ fixes the $k$-dependence of the spectrum, but, without physical justification, this remains an arbitrary choice, equivalent to assume a scale invariant spectrum from the outset.

### III. AN EXAMPLE

Let us now consider claim 2 according to which the choice of $\epsilon_m$ is essentially unique. The ambiguity inherent to the method proposed in Refs. [9, 13] can be most clearly exhibited by means of an example suggested at the Euro-conference in Annecy in December 2001 by G. Veneziano [13]. This section relies completely upon his idea.

In Ref. [13] it is argued that the density contrast $\epsilon_m$ is the quantity of interest because it is finite at $\eta = 0$ and we have shown above how this quantity is used to propagate the spectrum through the singularity. However, $\epsilon_m$ is not the only finite, physically relevant, quantity. Based on the method developed in Ref. [13], we could equally well consider and use the conjugate momentum $\Pi$ to the conserved quantity $\zeta$ [9] which obeys the Hamilton like equations, valid for isentropic perturbations,

$$\Pi = z^2 \zeta', \quad \Pi' = -k^2 \epsilon_s^2 z^2 \zeta,$$ \hspace{1cm} (14)

where $z^2 \equiv a^2(1 + \omega)/c_s^2$. It is worth pointing out that since the quantity $\zeta$ diverges at the singularity, it may be more useful to combine Eqs. (14) into the single one

$$\Pi'' + [3 (\epsilon_s^2 - \omega) - 2] \mathcal{H} \Pi' + k^2 \epsilon_s^2 \Pi = 0.$$ \hspace{1cm} (15)

In the neighborhood of the singularity, $\epsilon_s^2 = \omega = 1$, and the solutions of Eq. (15) are $-k\eta J_1(-k\eta)$ and $-k\eta N_1(-k\eta)$, where $J_1$ and $N_1$ are Bessel function of the first and second kind respectively. These solutions are completely regular at $\eta = 0$. Note that Eqs. (14) and (13) apply precisely in the two cases of interest here, namely that of purely hydrodynamical perturbations and that of a free scalar field, by replacing $\epsilon_s^2 \rightarrow 1$ [13] in this last case. The quantity $\Pi$ is related to $\epsilon_m$ as

$$\Pi = a^2 \mathcal{H} \epsilon_m.$$ \hspace{1cm} (16)

Note also that the use of $\Pi$ may be argued to be more appropriate in view of the fact that in the regular bounce case [17], even though admittedly a different case as the ekpyrotic model, the variable $\epsilon_m$ ends up being an odd function of $\eta$, whereas $\Pi$ is even. In the short-duration limit, that would imply that $\epsilon_m$ experiences a jump whereas $\Pi$ is continuous.

Since $|a^2 \mathcal{H}| \sim \ell_0^2/16$ is finite at the singularity $\eta = 0$, this relation clearly shows that the variable $\Pi$, being a linear combination of $\epsilon_m$ and $\epsilon'_m$ (with vanishing coefficient for the latter), satisfies all the requirements of claim 2. Accordingly, it seems that there is no convincing reason to use $\epsilon_m$ rather than $\Pi$. In the vicinity of the bounce, $\Pi$ reads

$$\Pi = \frac{\ell_0^2}{16} \left[ \epsilon_0 - \frac{3}{4} \omega_1^2 \epsilon_0 \eta + \left( \epsilon_2 - \frac{3w^{(2)}}{8\epsilon_0} \right) \eta^2 + \cdots \right],$$ \hspace{1cm} (17)

where $s = -1$ before the bounce and $s = +1$ after the bounce. The proposal of Ref. [9] then would consist in assuming that the coefficients in front of the constant term on the one hand and in front of the $\eta^2$ term on the other hand are the same before and after the bounce. In the present context, this reduces to

$$-\epsilon_0^< = \epsilon_0^>, \quad -\epsilon_2^< = \epsilon_2^>, \quad 3w^{(2)} > \frac{8}{8\epsilon_0} \epsilon_0^>,$$ \hspace{1cm} (18)

since $w^{(2)} < 0$ by definition. From these relations, it is easy to establish that

$$\epsilon_0 = -\frac{8k^2}{\ell_0^2} B_1^<, \quad \epsilon_2 = k^2 B_2^< - \frac{3w^{(2)} k^2}{\ell_0^2} B_1^<.$$ \hspace{1cm} (19)

Finally, the computation of $B_2^>$ shows that the spectrum no longer contains a scale invariant piece,

$$B_2^> = -B_2^< \propto k^{-1/2},$$ \hspace{1cm} (20)

but gives a spectral index equal to 3. It should be noticed that the same calculation goes through, with the same resulting spectral index if, in Eq. (16), one replaces
the factor $a^2 H$ by its magnitude $a^2 H$ which is not only regular but also symmetric across the bounce. Therefore, the choice of the variable used to apply the rule proposed in Ref. \[6\] plays a crucial role. In the absence of an underlying reason to choose a variable rather than another at the present stage, II seems to be as relevant as $\epsilon_m$, and we are led to the conclusion that claim 2 is definitely erroneous. Moreover, Π is not the only well-defined variable that leads to such a conclusion: there is an infinite number of such possible combinations.

### IV. CONCLUSIONS

To conclude, let us mention yet another possibility. In Ref. \[10\], it was suggested that the condition used in Ref. \[1\] may be cast in the form of a matching of the energy perturbation as seen by an observer comoving with the fluid, and that such a matching is “at least as natural as matching the energy in the longitudinal gauge”. Firstly, one should notice that no rigorous proof exists that the conditions of Ref. \[10\] are equivalent to the prescription utilized in Ref. \[1\]. Secondly, we have shown in Ref. \[12\] that, in the well known case of radiation to matter transition for which one can compare with the exact solution, this matching condition yields an incorrect result. A similar conclusion has been reached in Ref. \[1\] in the case of the reheating transition in which the equation of state also jumps. However, it can be argued that a bouncing situation is in no way comparable to a jump in the equation of state and that the usual matching conditions may then not apply. This is indeed what was found in Ref. \[13\] for the case of a non singular bounce. The case for the ekpyrotic scenario is still more involved, as it was also argued that the bounce must be singular if the

brane collision is to produce radiation \[18\]. In any case, matching through an unspecified bounce is still an open question as there is no geometrical or physical argument imposing some particular choice, and no conclusion can be made about the power spectrum of perturbations in general, unless some non-singular bounce is specified, as it is the case of Ref. \[17\].

Finally, we would like to insist on the fact that the overall gauge-invariant perturbation theory may be completely meaningless when a singularity is reached, since some gauge transformations that are admissible in any other situation could turn out to be singular. In such a situation, a “gauge-invariant” variable will, at the singular point, become “gauge-dependent”, and its use rather than the use of any other variable become an arbitrary, physically meaningless, choice. In fact, the transformation between the zero shear (conformal Newtonian) slicing and the comoving slicing is indeed singular at the bounce, which actually means that one or both slices are unphysical. Lyth \[1\] has shown that there is no slicing on which the density contrast and the intrinsic curvature perturbation are finite simultaneously. In such a situation, linear perturbation theory becomes meaningless.

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