Generalization of the Fourier Problem on Fluctuations in the Temperature of the Earth's Crust

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Abstract. The problem of asymptotic fluctuations of temperature and moisture content in a half-space is solved by the method of complex amplitudes. The material filling the half-space consists of a solid base (capillary-porous body) and water. The well-known Fourier solution for temperature fluctuations in half-space in the absence of moisture and under the boundary conditions of heat exchange of the first kind is generalized to the case of a wet material under the boundary conditions of Newton for temperature and Dalton for moisture content. The results of the work can be used in geocryology to model seasonal fluctuations in the thermophysical characteristics of frozen rocks and soils.

1. Introduction
The article deals with the classical problem of finding fluctuations in the temperature field in the upper layer of the earth's crust, provided that the temperature of the Earth's surface experiences diurnal or seasonal fluctuations. A partial solution of this problem, obtained by Fourier, can be found in the following work [1]. The Fourier formulas given in this work, as well as the Fourier laws based on them, are among the fundamental theoretical facts in permafrost science (geocryology) and are an important tool in solving problems of meteorology, climatology and environmental protection, as well as in the construction of buildings and the development of agriculture in the field of permafrost distribution [2, 3]. A significant disadvantage of the Fourier formulas, which limit their application to the study of geocryology problems, is that they do not take into account the presence of moisture in the soil, its movement under the influence of temperature gradients and moisture content, and evaporation in the thickness of the material and from its surface. Applying the theory of heat and mass transfer of A.V. Lykov to the problem with periodically changing boundary conditions, we obtain formulas for the thermophysical state of the material, in which the movement of moisture and its transformations will be correctly taken into account.

2. Mathematical model of heat and mass transfer of a half-space with an air flow
Bearing in mind the problem of temperature and moisture content fluctuations in the surface layer of the earth's crust, let us consider the homogeneous half-space \( x>0 \) shown in Figure 1, the boundary of which \( x=0 \) is blown by air having a temperature \( T_a \) and humidity \( \varphi \) outside the boundary layer. The material of the half-space will be considered to consist of a solid base (a capillary-porous body) and water. We also assume that the heat transfer intensity \( Q \) and the mass transfer intensity \( J \) of the surface \( x=0 \) with the air medium vary slightly along this surface, i.e. these values depend only on the time \( \tau \). In
the described situation, the distributions of temperature $T$ and moisture content $U$ will depend only on $x$ and $\tau$, i.e. the desired functions will be $T(x, \tau)$ and $U(x, \tau)$. The system of equations and boundary conditions for these functions will have the following form (1 - 6):

\[
\frac{\partial T}{\partial \tau} = a_w \frac{\partial^2 T}{\partial x^2} + \frac{r_f \gamma}{c} \frac{\partial U}{\partial \tau} \quad 0 < x < \infty; \tag{1}
\]

\[
\frac{\partial U}{\partial \tau} = a_m \frac{\partial^2 U}{\partial x^2} + a_m \delta \frac{\partial T}{\partial x} \quad 0 < x < \infty; \tag{2}
\]

\[
Q(\tau) + r(1 - \gamma) \cdot J(\tau) = \lambda \frac{\partial T}{\partial x}(0, \tau); \quad x = 0; \tag{3}
\]

\[
J(\tau) = a_m \rho \left[ \frac{\partial U}{\partial x}(0, \tau) + \delta \frac{\partial T}{\partial x}(0, \tau) \right]; \quad x = 0; \tag{4}
\]

\[
Q(\tau) = a_w \left[ T(0, \tau) - T_a \right]; \tag{5}
\]

\[
J(\tau) = a_n \left[ P(T(0, \tau)) - \varphi \cdot P(T_0) \right]; \quad P(T) = 6,03 \cdot 10^{-3} \cdot \exp \frac{17,3 \cdot T}{T + T_1}; \tag{6}
\]

Formulas (1) and (2) represent the equations of heat and moisture propagation in the region occupied by the material; equations (3) and (4) define the boundary conditions at the boundary $x=0$; formulas (5) and (6) determine the intensities of heat and mass transfer at this boundary (heat transfer according to Newton's law and mass transfer according to Dalton's law). In the given relations: $c, \rho, \lambda, \gamma, a_w, \delta$ — the thermophysical characteristics of the material, namely, the specific heat capacity, the density in the dry state, the coefficient of thermal conductivity, the evaporation criterion, the coefficient of moisture diffusion, the relative coefficient of thermal diffusion of moisture; $a_w = \lambda/(c\rho)$ — heat diffusion coefficient (coefficient of thermal conductivity); $r$ — specific heat of water vaporization; $a_w$ and $a_m$ — coefficients of heat and mass transfer of the sample surface with the air medium; $P(T)$ — function of G. K. Filonenko, modeling the dependence of the relative partial pressure of saturated water vapor on its temperature $T$ at general normal pressure; $T_1=238^\circ C$ — constant.

3. **Statement of the problem on the asymptotics of heat and mass transfer fields**

We assume that at $\tau<0$, the temperature of the material and its moisture content had constant values $T_0$ and $U_0$ over the entire half-space, the air temperature $T_a$ was equal to the material temperature $T_0$, and the air humidity $\varphi$ was equal to 1. We see that the system can be in this state indefinitely, because all the above equations are satisfied, and for the heat and mass transfer intensities we will have $Q=0$ and $J=0$. 

![Figure 1. Air-blown half-space](image-url)
Let now, starting from the moment \( t = 0 \), the air temperature \( T_a \) begins to make small fluctuations near the temperature \( T_0 \). For small deviations of the surface temperature \( T(0, t) \) and the air temperature \( T_a \) from the fixed temperature \( T_0 \), the dependence (6) can be linearized by decomposing the function \( P(T) \) into a Taylor series in the vicinity of the point \( T_0 \). Having done this, and taking \( \varphi = 1 \), instead of the formula (6) for the mass transfer intensity, we get an approximate formula for the mass transfer coefficient for the temperature difference:

\[
J_t = \alpha_0 \left[ T(0, t) - T_0 \right], \quad \alpha_0 = \alpha_m \frac{dT}{dT}(T_0).
\]  

(7)

The representation of the function \( J_t \) in the form of (7) turns the system of equations we have introduced into a linear system.

Next, we will consider the case when small changes in air temperature occur according to the harmonic law:

\[
T_a(t) = T_0 + \Delta T_a \cdot \sin(\omega \tau + \psi_\omega).
\]  

(8)

where \( \Delta T_a, \omega, \psi_\omega \) are the set values. We can offer a solution to the system (1)-(5), (7), (8) in the following form:

\[
\begin{align*}
T(x, \tau) &= T_0 + T(x) \cdot \sin(\omega \tau + \psi_t(x)), \\
U(x, \tau) &= U_0 + U(x) \cdot \sin(\omega \tau + \psi_u(x)),
\end{align*}
\]  

(9)

where both the temperature and the moisture content of the material, as well as the air temperature, make small harmonic oscillations near the equilibrium values of \( T_0 \) and \( U_0 \) at each fixed \( x \). Indeed, by calculating the difference using (9) and (8), and substituting it into formula (7), we get

\[
J_t = \alpha_0 \left[ T(0) \cdot \sin(\omega \tau + \psi_t(0)) - \Delta T_a \cdot \sin(\omega \tau + \psi_\omega) \right].
\]  

(10)

Thus, the function \( J_t \) turns out to be harmonic. Similarly, on the basis of the formula (5), the harmony of the function \( Q(t) \) is proved. But after all, all the other terms in equations (1)-(4), in accordance with (9), will also be harmonic, and therefore you can try to satisfy these equations by properly selecting the dependencies in formulas (9):

\[
T(x), U(x), \psi_t(x), \psi_u(x)
\]

In making such a selection, we must take into account the obvious physical meaning of the condition at infinity:

\[
T(x) \to 0 \quad \text{and} \quad U(x) \to 0 \quad \text{if} \quad x \to \infty.
\]  

(11)

The formulated function selection problem (9) is one of the problems without initial data; its solution gives the asymptotics of the fields \( T \) and \( U \) at \( \tau \to \infty \).

4. Problem statement for harmonic field complexes

The functions of our problem, \( T(x, \tau) \) and \( U(x, \tau) \), are defined by formulas (9). We introduce instead the new desired functions, \( T^*(x, \tau) \) and \( U^*(x, \tau) \), and, turning to the method of complex amplitudes, we compare these new functions with their complexes \( \tilde{T}(x), \tilde{U}(x) \), according to the following rules:

\[
\begin{align*}
T^*(x, \tau) &= T(x, \tau) - T_0 = T(x) \cdot \sin(\omega \tau + \psi_t(x)) \leftrightarrow \tilde{T}(x) = T(x) \cdot \exp(i\psi_t(x)), \\
U^*(x, \tau) &= U(x, \tau) - U_0 = U(x) \cdot \sin(\omega \tau + \psi_u(x)) \leftrightarrow \tilde{U}(x) = U(x) \cdot \exp(i\psi_u(x)).
\end{align*}
\]  

(12)
Obviously, the functions $T^*$ and $U^*$ will satisfy the same equations (1) and (2) as the functions $T$ and $U$. Based on this, and using the rules for working with complexes, instead of equations (1) and (2) for $T^*$ and $U^*$, we get the equations for their complexes $\dot{T}$ and $\dot{U}$:

\[
\begin{align*}
\dot{i}o\dot{T}(x) &= a_w \frac{d^2\dot{T}(x)}{dx^2} + \frac{r_i}{c} \dot{i}oU(x); \\
\dot{i}o\dot{U}(x) &= a_m \frac{d^2\dot{U}(x)}{dx^2} + a_m \delta \frac{d^2\dot{T}(x)}{dx^2}. \\
\end{align*}
\tag{13}
\]

Here we have a system of two second-order ordinary differential equations with respect to two unknown functions. The system is linear, homogeneous, with constant coefficients, and indeterminate. To identify the only solution to the system, we turn to the remaining equations of the problem. In the domain of complexes, equations (3) and (4) will look like this:

\[
\dot{Q} + r(1-\gamma)\dot{J} = \lambda \frac{dT}{dx}(0); \quad \dot{J} = a_m \rho \left[ \frac{d\dot{U}}{dx}(0) + \delta \frac{d\dot{T}}{dx}(0) \right].
\tag{14}
\]

Forming complexes from both parts (10), and equating them, we get a given complex number

\[
\dot{J} = \alpha_i \left[ \dot{T}(0) - \Delta T_a \right], \text{ where } \Delta T_a = T_a \cdot \exp(i \psi_a)
\]

Similarly, referring to the formula (5), for the heat transfer intensity complex we will have

\[
\dot{Q} = \alpha_i \left[ \dot{T}(0) - \Delta \dot{T}_a \right]
\]

Substituting the obtained expressions in (14), after the transformations we obtain a system of two equations:

\[
\begin{align*}
\dot{T}(0) - \frac{\lambda}{\alpha_i} \frac{dT}{dx}(0) &= \Delta \dot{T}_a; \\
\dot{T}(0) - \frac{a_m \rho}{\alpha_i} \frac{d\dot{U}}{dx}(0) + \delta \frac{d\dot{T}}{dx}(0) &= \Delta \dot{T}_a.
\end{align*}
\tag{15}
\]

Here $\alpha_i = \alpha_w + r(1-\gamma)\alpha_t$ is the effective heat transfer intensity.

Having found the general solution of system (13), i.e., the general expressions for functions $\dot{T}$ and $\dot{U}$, we then, from system (15), which plays the role of boundary conditions, find the arbitrary constants included in these general expressions.

5. An asymptotic solution for a mathematical model of heat and mass transfer with $\delta=0$ and $\gamma=0$

Here we will limit ourselves to calculations within the framework of one of the simplest mathematical models, in which we assume $\delta=0$ (neglect thermal diffusion, i.e., the movement of moisture to the surface occurs only due to the difference in moisture content) and $\gamma=0$ (neglect internal vaporization, i.e., the transformation of water into steam occurs only on the surface). The conditions for the applicability of such a simplified model to the problems of the theory of heat and mass transfer and the analysis of the solutions obtained with the approximations made can be found in [7-9]. As can be seen from (1) and (2), the equations for temperature and moisture content in this case are independent (the relationship between the functions $T$ and $U$ is carried out in this model only through boundary conditions), which makes significant simplifications in the algorithm for studying the process. The equations for complexes (13) under the conditions of our problem can be represented as:

\[
\dot{i}o\dot{T}(x) = a_w \frac{d^2\dot{T}(x)}{dx^2}; \quad \dot{i}o\dot{U}(x) = a_m \frac{d^2\dot{U}(x)}{dx^2}.
\]
Using the Euler method, we find general solutions to these equations that satisfy the condition at infinity (11). They will look like this:

$$
\begin{align*}
T(x) &= C_1 \cdot \exp(\mu_1 x), \quad U(x) = C_2 \cdot \exp(\mu_2 x),
\end{align*}
$$

where

$$
\mu_1 = -\frac{\omega}{a_w} \left(1 + \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right), \quad \mu_2 = -\frac{\omega}{a_m} \left(1 + \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right).
$$

$C_{1,2}$ are arbitrary constants. To find them, we use the boundary conditions (15). Putting $\delta=0$ there and substituting the general solutions found above $\hat{T}$ and $\hat{U}$, we will have:

$$
\begin{align*}
C_1 - \lambda C_1 \mu_1 / \alpha = \Delta T; \\
C_1 - a_m \mu_2 / \alpha = \Delta T.
\end{align*}
$$

Solving this system, after the transformations we find:

$$
\begin{align*}
C_1 &= \frac{\Delta T}{1 - \mu_1 / \alpha}, \quad C_2 = k C_1, \quad \text{where} \quad k = \frac{\alpha_\lambda}{\alpha_\rho \sqrt{a_w a_m}}.
\end{align*}
$$

Now we must return from the complexes (images) to the original harmonic functions of time (originals). In the method of complex amplitudes, this stage corresponds to finding the inverse Fourier transform when solving physical problems by the spectral method. We will first find the original for the complex $\hat{T}(x)$. We will first introduce a new designation. The constant $\mu_1$ from formulas (17) is written in the form $\mu_1 = -\beta_w (1 + i)$, where coefficient of attenuation of heat waves:

$$
\beta_w = \sqrt{\omega^2 (2a_w)}.
$$

The inverse value of $\Delta_w = 1/\beta_w$ is called the depth of penetration of heat waves. These names are borrowed from the theory of electromagnetic waves. The validity of the use of such terms will be justified below. The transition to the original is based on the rule of formation of complexes and is carried out according to the following scheme:

$$
\begin{align*}
\hat{T}(x) &= C_1 \cdot \exp(\mu_1 x) = |C_1| \cdot \exp(i \cdot \arg C_1) \cdot \exp[-\beta_w (1 + i)x] \leftrightarrow \\
&\leftrightarrow |C_1| \cdot \exp(-\beta_w x) \cdot \sin(\omega t + \arg C_1 - \beta_w x).
\end{align*}
$$

The main difficulty here is the reduction to the exponential form of the complex constant $C_1$, i.e., the calculation of the modulus of this constant $|C_1|$ and its argument $\arg C_1$. After performing such calculations, and adding, in accordance with (12), to the resulting harmonic function, the constant $T_0$, we obtain the desired asymptotic solution for the temperature field:

$$
\begin{align*}
T(x,t) &= T_0 + \frac{\Delta T_n \cdot \alpha_c}{\sqrt{(\alpha_c + \lambda \beta_w)^2 + (\beta_w)^2}} \exp(-\beta_w x) \times \\
&\times \sin \left[ \omega t + \psi_n - \arctg \left( \frac{\lambda \beta_w}{\alpha_c + \lambda \beta_w} \right) - \beta_w x \right].
\end{align*}
$$

Having done similar calculations, for the asymptotic field of moisture content we obtain:

$$
\begin{align*}
U(x,t) &= U_0 + \frac{k \cdot \Delta T_n \cdot \alpha_c}{\sqrt{(\alpha_c + \lambda \beta_w)^2 + (\beta_w)^2}} \exp(-\beta_m x) \times \\
&\times \sin \left[ \omega t + \psi_n - \arctg \left( \frac{\lambda \beta_w}{\alpha_c + \lambda \beta_w} \right) - \beta_m x \right].
\end{align*}
$$
Here $\beta_m = \sqrt{\omega / (2a_m)}$ is the attenuation coefficient for the moisture content waves.

Formulas (20) and (21) and represent the main result of the work. The attenuation coefficients $\beta_w$ and $\beta_m$ included in them are functions of the frequency $\omega$, and all other coefficients are constants defined above.

6. Discussion

From our calculations, we derive statements that are called Fourier laws and are widely used in solving problems of geocryology [2]:

1) the dependence of the amplitude of the temperature field fluctuations $\Delta T(\omega, x)$ on the frequency $\omega$ and depth $x$ is determined by the formula

$$\Delta T(\omega, x) = \Delta T(0) \cdot \exp \left( - \sqrt{\omega / 2a_w} x \right)$$

2) the time lag of the maxima (minima) of the temperature in the material from the corresponding moments on the surface is proportional to the depth and is determined by the formula

$$\Delta \tau = \sqrt{1/(2a_w \omega)} \cdot x$$

These known results are valid under boundary conditions of the first kind, i.e., when the given function is the temperature of the boundary with its parameters $T_0, \Delta T(0), \psi(0)$. But in our case, we have adopted more general boundary conditions of the third kind, when the given function is the air temperature (8) with its parameters $T_0, \Delta T_n, \psi_n$. In this new situation, the Fourier laws for the temperature field are transformed as follows:

1) remain in effect, taking into account that it should now be calculated by the formula (15);
2) we should now talk about the lag time relative to the corresponding moments for the air temperature, and not for the surface temperature, and the new formula will look like this:

$$\Delta \tau = \sqrt{1/(2a_w \omega)} \cdot x + \frac{1}{\omega} \arctg \left( \frac{\lambda \beta_w}{\alpha_c + \lambda \beta_w} \right)$$

Laws with a similar formulation can be formulated for the field of moisture content.

As an example, we present the results of calculating annual temperature fluctuations, provided that the surface temperature is 20 °C (Table 1).

| Depth, m | The temperature field fluctuations $\Delta T$, °C |
|----------|-----------------------------------------------|
| 1        | 11.6                                          |
| 2        | 6.9                                           |
| 3        | 4.4                                           |
| 4        | 2.7                                           |

These data are in good agreement with experimental measurements. The time lag of the maximum temperature at a depth of 4 m was also adjusted. According to our calculations, it is 4.32 months (according to classical Fourier formulas, 4 months exactly).

Based on the developed algorithm, the analysis proposed in this paper can be extended to other types of boundary conditions and equations of heat and moisture propagation, for example, to models used in the theory of two-phase filtration [10]. The article considers the harmonic steady-state mode. Using the spectral method, the proposed calculation algorithm can be easily extended to periodic modes of any kind.

7. Conclusion

The problem of asymptotic fluctuations of temperature and moisture content in a half-space is solved by the method of complex amplitudes. Heat exchange of the half-space with the air medium occurs according to Newton's law, and mass transfer - according to Dalton's law. The problem of asymptotic

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distributions of temperature and moisture content under conditions when the air temperature changes in time according to the harmonic law is solved by the method of complex amplitudes. The constructed solution and the conclusions that follow from it are generalizations of the Fourier formulas and Fourier laws known in the literature, which relate to the situation when the material filling the half-space does not contain moisture, and according to the harmonic law, not the air temperature changes, but the temperature at the boundary. Within the framework of the obtained general solution, a particular solution is constructed for the mathematical model of heat and mass transfer, which does not take into account thermal diffusion and internal vaporization. The results of the work can be used in geocryology as a theoretical tool for modeling seasonal fluctuations in the thermophysical state of the soil, which is an important task in planning economic activities in the field of permafrost distribution.

8. References
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