One of the most important concepts of baryon spectroscopy is the classification of the resonances according to the representations of the non–relativistic group SU(6)$_{sf}$⊗O(3)$_l$. Through this group the trivial spin-flavor (sf) correlation between three quarks in the 1s-shell has been assumed to apply to arbitrary internal orbital angular momenta (l). In doing so, states like, say, the positive parity resonances P$_{11}$(1720), F$_{15}$(1680), F$_{35}$(1905), and F$_{37}$(1950), are viewed to belong to a 56(2$^+$)–plet, the P$_{11}$(1710) excitation is treated as a member of a 70(0$^+$)–plet, while the negative parity baryons S$_{11}$(1535), D$_{13}$(1520), S$_{11}$(1650), D$_{13}$(1700), and D$_{15}$(1675) are assigned to a 70(1$^-$)–plet. The above examples clearly illustrate how states separated by only few MeV, like the D$_{15}$(1675), F$_{15}$(1680), and P$_{11}$(1700) states, are distributed over three different SU(6)$_{sf}$⊗O(3)$_l$ representations, whereas, on the other hand, resonances separated by about 200 MeV like the D$_{15}$(1520) and the D$_{13}$(1700) ones, are assigned to the same multiplet $^L3$. The basic idea of the multiplets as well separated families of particles of different internal but identical space–time properties, appears quite inappropriate here, where the spacing between the SU(6)$_{sf}$⊗O(3)$_l$ multiplets is much smaller as compared to the maximal mass splitting within them. In addition, the SU(6)$_{sf}$⊗O(3)$_l$ baryon classification scheme predicts a substantial excess of resonances, called ‘missing’, which yet have not been observed. Nonetheless, the SU(6)$_{sf}$⊗O(3)$_l$ symmetry predictions on the mass spectrum have been considered as quite satisfactory so far, with the excuse that the deviations from the observed masses of about ±150 MeV are small on the scale of 1500–2500 MeV.

On the other side, speed plot analysis of the pole positions on the complex energy plane of various baryon resonances ($L_{21,2J}$) with masses below ~ 2500 MeV performed by Höhler and Sabba-Stefanescu [12] have revealed a well–pronounced partial-wave clustering in baryon spectra, so called Höhler poles. As a representative example, the grouping of the S$_{11}$, P$_{11}$, P$_{13}$, D$_{13}$, D$_{15}$ and F$_{15}$ states around the pole (1665±25) - i (55±15) may be mentioned. This is quite a surprising result as it was not anticipated by any hadron model. In view of the Höhler clustering, it appears timely to question the SU(6)$_{sf}$⊗O(3)$_l$ classification and search for a new scheme for baryons which matches better with the observed spin–and parity grouping of the excited states and contains a much smaller number of unobserved (‘missing’) resonances.

To the best of our knowledge, the problem of Höhler’s poles was challenged only recently in a series of papers [13] where it was shown that they can be identified in a natural way with four–dimensional hyperspherical O(4) partial waves, here denoted by $\sigma_{2l,n}$, with $\sigma = 2, 4,$ and 6, and $n = \pm 1$ (see Fig. 1). This means that nature seems to favor the O(4) partial wave decomposition of the $\pi N$ scattering amplitude over the O(3) one. The O(4) partial waves from above are well known from the Coulomb problem, where they correspond to the (even) principal quantum numbers $n = \sigma$. They join (approximately) mass degenerate O(3) states of integer internal angular momenta, l, with $l = 0, ..., \sigma - 1$. All O(3) partial waves, $\sigma_{2l,n,l,m}$, contributing to a given O(4) pole, have either natural ($n = +1$), or...
unnatural ($\eta = -1$) parities. In other words, they transform with respect to the space inversion operation $\mathcal{P}$ as

\[ \mathcal{P} \sigma_{2l,\eta;l,m} = \eta e^{i\pi l} \sigma_{2l,\eta;l-m}, \]

\[ l = 0^-, 1^-, ..., (\sigma - 1)^-, \quad m = -l, ..., l. \]  

In coupling a Dirac spinor to the O(4) multiplets from above, the spin ($J$) and parity ($\pi$) quantum numbers of the baryon resonances are created as

\[ J^\pi = \frac{1}{2} \eta \, \frac{1}{2} \eta \, \frac{3}{2} \eta \, ..., (\sigma - 1)^{-\eta}. \]  

The first four-dimensional hyperspherical partial wave is always $2_{2l,+}$. From Eqs. (1) and (2) follows that it unites the first spin-particle $\frac{1}{2}^+$, $\frac{3}{2}^-$ and $\frac{5}{2}^-$ resonances. Indeed, the relative $\pi N$ momentum $L$ takes for $l = 0^+$ the value $L = 1^+$ and corresponds to the $2_{2l,1}$ state, while for $l = 1^-$ it takes the two values $L = 0^-$ and $L = 2^-$ describing in turn the $S_{2l,1}$ and $D_{2l,3}$ resonances. The isospin quantum number $I$ takes the three different values $I = 1/2, 3/2,$ and 0. The natural parity of the first O(4) partial waves reflects the arbitrary selection of a scalar vacuum through the spontaneous breaking of chiral symmetry. Therefore, up to the three lowest $N, \Delta$, and $\Lambda$ excitations, chiral symmetry is still in the Nambu–Goldstone mode. All the remaining non–strange baryon resonances have been shown in [4] to belong to either $4_{2l,-}$, or $6_{2l,-}$. For example, one finds all the seven $\Delta$–baryon resonances $S_{31}, P_{31}, P_{33}, D_{33}, D_{45}, F_{35}$ and $F_{37}$ from the $4_{3,-}$ partial wave to be squeezed within the narrow mass region from 1900 MeV to 1950 MeV, while the $l=1/2$ resonances paralleling them, of which only the $F_{17}$ state is still ‘missing’ from the data, are located around 1700$^{+20}_{-50}$ MeV. Therefore, the $F_{17}$ resonance is the only non–strange state with a mass below 2000 MeV which is ‘missing’ in the new scheme. This one ‘missing’ resonance has to be compared to at least 10 resonances considered as ‘missing’ within the traditional SU(6) $\otimes$ O(3) schemes (see [1,2] for reviews). In continuing by paralleling baryons from the third nucleon and $\Delta$ clusters with $\sigma = 6$, one finds in addition the four states $H_{11,11}$, $P_{31}$, $P_{33}$, and $D_{33}$ with masses above 2000 MeV to be ‘missing’ from the new scheme. The $H_{11,11}$ state is needed to parallel the well established $H_{3,11}$ baryon, while the $\Delta$-states $P_{31}$, $P_{33}$, and $D_{33}$ are required as partners to the (less established) $P_{11}(2100)$, $P_{13}(1900)$, and $D_{13}(2080)$ nucleon resonances. The second and third non–strange baryon clusters have been shown in [4] to be built upon a pseudoscalar vacuum and are of unnatural parities. The latter circumstance signals chiral symmetry restoration here and fixes the mass scale of the chiral phase transitions for baryons. It is remarkable, that the approximately equidistant cluster spacing of about 200 MeV to 300 MeV between the mass centers of the O(4) multiplets appearing now, is by a factor 3 to 6 larger as compared, for example, to the maximal mass splitting of 50–70 MeV within the $2_{1,+}, 2_{3,+}$, $4_{1,-}$, and $4_{3,-}$ partial waves (see Table 1).

The degeneracy of the states $(1^- - 3^-)$, and $(\frac{1}{2}^+ - \frac{3}{2}^-)$ from the region around 1700 MeV was noticed in [2] where it was explained by means of a quark model with a deformed harmonic oscillator, denoted by DOQ. However, several states from that region like the $\frac{1}{2}^+$ and $\frac{5}{2}^-$ ones, don’t fit into DOQ scheme and have been left out of consideration. We here stress that the degeneracy of the resonances considered in [2] is the direct consequence of their belonging to the O(4) clusters. The big advantage of our classification scheme is that there are no states out of the O(4) partial wave systematics.

To illustrate the substantial reduction of the number of the ‘missing’ resonances within the O(4) classification scheme of the internal orbital angular momenta in [4], it is quite instructive to consider as an example how the quantum numbers of the three lowest baryon excitation $P_{2l,1}$, $S_{2l,1}$, and $D_{2l,3}$, can emerge from the minimal O(4) symmetric quark-diquark configuration space spanned by the $1s$, $1p$, and $2s$– single–particle shells with an approximate $1p-2s$ degeneracy. The one-particle–one-hole configurations in this space give rise to the following orbitally excited (active) quark–diquarks (in standard shell–model notations) of both natural and unnatural parities:

\[ [1s_{\frac{1}{2}} \otimes 2s_{\frac{1}{2}}]|l=0^+, 1^+, \]  

\[ [1s_{\frac{1}{2}} \otimes 1p_{\frac{1}{2}}]|l=0^-, 1^-, 2^-, \]  

Note that we consider the quark-diquark model in the $j-j$ rather that in the $LS$ coupling exploited in the traditional SU(6)$\otimes$O(3) quark models.

Now, the quantum numbers of the first P$_{2l,1}$, S$_{2l,1}$ and D$_{2l,3}$ excitations are determined through the coupling of the spectator $1s_{\frac{1}{2}}$ quark to natural parity diquarks, such like

\[ \left([1s_{\frac{1}{2}} \otimes 2s_{\frac{1}{2}}]|l=0^+ \otimes 1s_{\frac{1}{2}}\right)^{\frac{1}{2}^+}, \quad \left([1s_{\frac{1}{2}} \otimes 1p_{\frac{1}{2}}]|l=2^- \otimes 1s_{\frac{1}{2}}\right)^{\frac{3}{2}^-; \frac{5}{2}^-}. \]  

None of the remaining unnatural parity configurations

\[ 2 \]
\[
\left( |s_{1/2}^{-1} \otimes 1p_{1/2}^{-1} \right)^{l=0^{-},2^-} \otimes 1s_{1/2}^1, \quad \left( |s_{1/2}^{-1} \otimes 2s_{1/2}^{-1} \right)^{l=1^+} \otimes 1s_{1/2}^1 \right)^{2^+},
\]

has been observed in the mass region around 1500 MeV so far. These configurations rather occur as members of the unnatural parity \(4_{2L_{-}}\) and \(6_{2L_{-}}\) clusters. Such a radical truncation of the \(3q\)–configuration space is understandable provided, to some approximation, the diquarks

i) behave as pointlike bosonic subbaryon degrees of freedom,

ii) their parities are selected to be either natural or all unnatural.

To satisfy the parity selection rule one has to assume that the diquarks are created as one–particle states within a given Fock space (to be denoted by \(\mathcal{F}\)) built upon a vacuum (denoted by \(|0^0\rangle\)) of either positive (\(\eta = +\)), or negative (\(\eta = -\)) parity. For example, the particle-hole creation operators \(A_{a,\eta,lm}^\dagger\), with \(l = 0^+, 1^-,\) which describe in turn the scalar and vector diquarks from the \(O(4)\) partial wave \(2_{1,^+}\),

\[
A_{2_{1,^+};lm}^\dagger = \sum_{m_1m_2} (-1)^{\frac{1}{2}m_2} \left(\frac{1}{2}m_1 \frac{1}{2} - m_2\right) b_{2_{1,^+};2s_{1/2}^1m_1}^l b_{2_{1,^+};1s_{1/2}^1m_2}^l,
\]

have to act onto the baryon ground state, \(|(1s_{1/2}^1)^3\rangle\), in the same way as a fundamental bosonic operator, here denoted by \(a_{2_{1,^+};lm}^\dagger\), acts onto its Fock vacuum, so that the following mappings hold:

\[
A_{2_{1,^+};lm}^\dagger \|(1s_{1/2}^1)^3\rangle \simeq a_{2_{1,^+};lm}^\dagger \|0^+\rangle, \quad \mathcal{P}a_{2_{1,^+};lm}^\dagger |0^+\rangle = e^{i\pi l_1} a_{2_{1,^+};lm}^\dagger |0^+\rangle.
\]

The mapping in Eq. (6) is nothing but the expression for the superselection rule from above allowing only diquarks having all either natural or unnatural parity and grouped to \(O(4)\) families, called hyperquarks (HQ) in the following, as relevant degrees of freedom for the structure of the excited baryons. From this stage on we will ignore the constituent character of the \(O(4)\) boson operators and consider in the following the hyperquarks as fundamental degrees of freedom. The idea of the pointlike character of the diquarks has been exploited in the literature to reduce the three-quark Faddeev equations to a two-body quark-diquark Bethe-Salpeter equation (see, for example [5]). The essential difference between the present quark-hyperquark model (QHM) and the customary quark-diquark models (QDM) (see [6] for a digest) is the assumed \(O(4)\) clustering of the diquarks and the parity selection rule leading to the observed clustering of baryons. To be specific, the operator \(D_{2_{1,^+}}^1\) which creates the lightest hyperquark is defined as the following linear combination of fundamental one-boson states

\[
D_{2_{1,^+}}^1 |0^+\rangle = \sum_{lm} c_{lm} a_{2_{1,^+};lm}^\dagger |0^+\rangle, \quad \sum_{lm} |c_{lm}|^2 = 1, \quad \langle 0^+ | D_{2_{1,^+}}^1 |0^+\rangle = \sum_{lm} c_{lm} \mathcal{R}_{2_{1,^+}} Y_{2lm}(\alpha, \theta, \phi).
\]

In Eq. (8) the radial part of the hyperquark wave function has been denoted by \(\mathcal{R}_{2_{1,^+}}(r)\), while its angular part has been determined by the four-dimensional hyperspherical harmonics \(Y_{\sigma lm}\) defined in the standard [6] as

\[
Y_{\sigma lm}(\alpha, \theta, \phi) = i^{\sigma - 1 - l/2} l! \frac{\sigma (\sigma - l - 1)}{2\pi (\sigma + 1)} \sin^\sigma \alpha C_{\sigma - l - 1}^{l+1} \cos \alpha Y_{lm}^\sigma (\theta, \phi).
\]

Here, \(C_{\sigma - l - 1}^{l+1} (\cos \alpha)\) denote the Gegenbauer polynomials, while \(Y_{lm}^\sigma (\theta, \phi)\) are the standard three-dimensional spherical harmonics.

In general, a Lorentz covariant spin- and parity cluster \(\sigma_{1,\eta}\) is now described as

\[
\sigma_{1,\eta} = D_{2_{1,^+}}^1 b_{1s_{1/2}^1}^l |0^0\rangle [T^1 \otimes \chi_{1/2}^2]^\frac{1}{2},
\]

where \(T^1\) stands for the flavor part of the wave function of the (nonstrange) hyperquark as a symmetric isovector state, while \(\chi_{1/2}^2\) is the ordinary isospinor of the third quark. It will become clear in due course that the restriction to isovector hyperquarks in Eq. (8) ensures that the complete space-time-flavor-color wave function of the \(I = \frac{1}{2}\) clusters is totally antisymmetric, as it should be in order to respect the Fermi-Dirac statistics for quarks.

The \(O(4)\) symmetry ansatz for the quark-hyperquark model assumed in the present work is independently supported by the observed rapid convergency of the covariant diquark models in the basis of the Gegenbauer polynomials considered among others in [6]. It is worthy of being pursued especially because of the quite uncertain experimental status of the ‘missing’ resonances. The relevant spectrum generating algebra deduced in [6] is
\[ su(2)_L \otimes su(3)_c \otimes o(1,3)_{ls}, \]  
and the baryon structure acquires features similar to those of the hydrogen atom.

The apparent analogy between the spectrum of the hydrogen atom and the baryon spectra raises the question whether the positions of the Lorentz covariant spin–clusters is determined by the inverse squared of the principle quantum number of the Coulomb problem, and follow a type of Balmer-series like pattern. The answer to this question is positive. Below we give a simple empirical recursive relation which describes with quite an amazing accuracy the reported mass averages of the resonances from the Lorentz multiplets (see Table 1) with \( \sigma = 2, 4, \) and 6 only in terms of the O(4) quantum number \( \sigma \), on the one side, and the two mass parameters \( m_1=600 \text{ MeV} \), and \( m_2=70 \text{ MeV} \), on the other side,

\[ M_{\sigma'} - M_\sigma = m_1 \left( \frac{1}{\sigma^2} - \frac{1}{(\sigma')^2} \right) + \frac{1}{2} m_2 \left( \frac{\sigma^2 - 1}{2} - \frac{\sigma^2 - 1}{2} \right). \]  

The first term on the r.h.s. in Eq. (12) is the typical difference between the energies of two single particle states of principal quantum numbers \( \sigma \), and \( \sigma' \), respectively, occupied by a diquark with mass \( m_1 = 600 \text{ MeV} \) moving in a Coulomb potential. To explain the origin of the second term one needs to remember that the so(4) algebra, in being irreducible SO(4) representations can be labeled by two SU(2) indices, denoted by \( \{ j_1, j_2 \} \). For this reason, the O(4) multiplets \( \{ j_1, j_2 \} \) are eigenstates of the sum of the Casimir operators \( J_1^2 \), and \( J_2^2 \) associated in turn with the two SU(2) groups from above, and one finds [11]

\[ (J_1^2 + J_2^2)\{ j_1, j_2 \} = (j_1(j_1+1) + j_2(j_2+1))\{ j_1, j_2 \}. \]  

From this point of view, the second term on the r.h.s in Eq. (12) emerges as the eigenvalue of the O(4) partial wave considered with respect to the direct sum of the two three-dimensional rotators from Eq. (14) and is determined by

\[ \left( \frac{1}{2} J_j^2 + \frac{1}{2} J_j^2 \right)\{ j_1, j_2 \} = \frac{1}{2} J \{ j_1(j_1+1) + j_2(j_2+1) \} \{ j_1, j_2 \}, \]

for \( j_1 = j_2 = \frac{\sigma - 1}{2} \), \( j_1(j_1+1) + j_2(j_2+1) = \frac{\sigma^2 - 1}{2} \).  

The term \( \frac{\sigma^2 - 1}{2} \) in Eq. (14) is the generalization of the three-dimensional \( j(j+1) \) rule to four dimensions. In Eq. (15) \( J \) plays the role of an effective inertial moment. Comparison of Eq. (15) to (12) reveals that the parameter \( 1/m_2 = 2.82 \text{ fm} \) corresponds to the inertial moment \( J = 2/5MR^2 \) of some ‘effective’ rigid-body resonances with mass \( M = 1085 \text{ MeV} \) and a radius \( R=1.13 \text{ fm} \). Therefore, the energy spectrum in Eq. (12) can be considered to emerge from a quark-hyperquark model with a Coulomb potential (\( H_{\text{Coul}} \)) and a four–dimensional rigid rotator (\( T_{\text{rot}}^{(4)} \)). The corresponding Hamiltonian \( H^{QHM} \) is given by

\[ H^{QHM} = H_{\text{Coul}} + T_{\text{rot}}^{(4)} = \frac{\alpha_C}{r} + \frac{1}{2J} (J_1^2 + J_2^2). \]  

This Hamiltonian is diagonal in the basis of the O(4) partial waves and the parameter \( m_1/\alpha_C \) plays a role similar to that of the Rydberg constant. Note that while the splitting between the Coulomb states decreases with increasing principal quantum number \( \sigma \), the difference between the energies of the rotational states increases linearly with \( \sigma \) so that the net effect is an approximate equidistance of the baryon cluster positions. In extending the Hamiltonian in Eq. (16) to include O(4) violating terms such like \( \sim l \cdot s, \sim l^2 \), or introducing different inertial momenta \( J \) to account for possible deformation effects, the O(3) splitting of the O(4) clusters can be studied along the line of the collective models of nuclear structure [12]. Finally, the purely space-time version of chiral symmetry of our model in Eq. (11) can also be extended to include the combined space-time & flavor chiral symmetry leading to a Goldstone-boson-quark interaction in the spirit of Refs. [3].

The introduction of O(4) correlations between the diquark O(3) partial waves brings numerous advantages over treating them as independent ordinary spherical partial waves. In particular, it allows for the relativistic description of the O(4) hyperquark propagators and therefore, for the relativistic propagators of the resulting baryon clusters.
Indeed, baryons grouped to O(4) partial wave–clusters $\sigma_{2\ell,\eta}$ of momentum $p_\sigma$, and mass $M$ are nothing but the reducible Lorentz representations $\{\frac{\eta}{2}, \frac{\ell}{2}\} \otimes \{\frac{\eta}{2}, 0\} \otimes \{0, \frac{\ell}{2}\}$ known as Rarita–Schwinger (RS) spinors. They are described by totally symmetric traceless rank–$(\sigma - 1)$ Lorentz tensors with Dirac spinor components and satisfy both the Dirac and Proca equations:

$$
(p \cdot \gamma - M)\Psi_{\mu_1 \mu_2 \cdots \mu_{\sigma - 1}} = 0,
$$

$$
(g^{\mu \nu} - \frac{1}{M^2} p^\mu p^\nu)\Psi_{\mu_1 \mu_2 \cdots \mu_{\sigma - 1}} = \Psi^\nu_{\mu_2 \cdots \mu_{\sigma - 1}}.
$$

The RS spinors have been used by Weinberg in his classical work [14] for embedding the higher–spin states $J = \sigma - \frac{1}{2}$. The essential difference between Weinberg’s scheme and the one presented here is that the lower-spin states entering the RS spinors are no longer redundant components that need be eliminated, but physically observable O(3) resonances reflecting the composite character of baryons. The totally antisymmetric character of the $I = \frac{3}{2}$ cluster wave function is now ensured by the ansatz in Eq. (11) where the hyperquark was constructed as an isovector. As both the space-time and isospin parts of wave function of the $\sigma_{1, \eta}$ clusters are now totally symmetric by construction, its color part is, as usual, in the totally antisymmetric color singlet state. This structure of the cluster wave function is fully consistent with the symmetry in Eq. (11), where the SU(3)$_c$ transformations are completely independent from the space and time ones. In such a case the diquark correlation in isospin space does not necessarily imply a similar correlation in color space. For this reason, in color space the three quarks can still be treated as bound to an antisymmetric singlet. Moreover, the space-time hyperquark correlation does not necessarily require a diquark in isospin space. This allows one to construct the correct wave function for the $I = \frac{3}{2}$ clusters in considering hyperquarks as purely space–time objects, while keeping in both isospin and flavor spaces the concept of the three independent quarks bound to a symmetric isospin– and a totally antisymmetric color states, respectively.

The Lorentz-covariant hyperquark propagator $D^{HQ}_{\mu_1 \mu_2 \cdots \mu_{\sigma - 1};\nu_1 \nu_2 \cdots \nu_{\sigma - 1}}$ is easily constructed from Proca’s spin–1 propagators, on the one side, and the mass-shell condition, on the other side, and is given by

$$
D^{HQ}_{\mu_1 \mu_2 \cdots \mu_{\sigma - 1};\nu_1 \nu_2 \cdots \nu_{\sigma - 1}} = \bigotimes_{n=1}^{n=\sigma - 1} \left( g_{\mu_n \nu_n} - \frac{1}{p^2} p_{\mu_n} p_{\nu_n} \right).
$$

The relativistic propagators of the spin–parity baryon clusters are obtained as direct products of the hyperquark propagator in Eq. (19) and the Dirac projector corresponding to the spectator quark

$$
S_{\mu_1 \mu_2 \cdots \mu_{\sigma - 1};\nu_1 \nu_2 \cdots \nu_{\sigma - 1}} = \frac{\gamma \cdot p + M}{2M} D^{HQ}_{\mu_1 \mu_2 \cdots \mu_{\sigma - 1};\nu_1 \nu_2 \cdots \nu_{\sigma - 1}}.
$$

Let us consider, for concreteness, the case of the spinor-vector $\Psi_\mu$. Because Proca’s equation in (18) eliminates the spin–zero (time) component from the $\{1/2, 1/2\}$ representation (one Lorentz index) and ensures that the four–vector describes a spin-1 field, the lowest spin-1/2 state will drop out of the multi–spinor and can be described independently by the Dirac equation. With that, the $(S_{21,1}, D_{21,3})$ cluster is now described in terms of the Lorentz vector with Dirac spinor components $\Psi_\mu$ from Eq. (22) and its propagator is given by

$$
S_{\mu \nu} = \frac{(\gamma \cdot p + M)(g_{\mu \nu} - \frac{1}{p^2} p_\mu p_\nu)}{2M(p^2 - M^2)}.
$$

In noting that, say, the first $S_{11}$ and $D_{13}$ states are separated by only 15 MeV, one sees that calculating the relativistic contribution of these states to the amplitude of processes like meson photoproduction at threshold, is now straightforward. Along the line of the representation theory of the Lorentz group, both the construction of cluster propagators and interactions with external fields are also straightforward. For example, for the case of a $B \rightarrow N + V$ process, where $B$ stands for a Lorentz covariant spin- and parity cluster, while $V$ is a vector meson, a possible effective Lagrangian can be written as

$$
\mathcal{L}_{BVN} = \bar{\Psi}_1^{\mu_1 \mu_2 \cdots \mu_{\sigma - 1}} \left( \frac{f_\sigma}{m_\sigma} \partial_{\mu_2} \cdots \partial_{\mu_{\sigma - 1}} A_{\mu_1} + \frac{f'_\sigma}{m'_\sigma} \partial_{\mu_1} \cdots \partial_{\mu_{\sigma - 1}} A \right) \psi_N,
$$

where $A_\mu$ denotes the vector meson field, while $f_\sigma$ and $f'_\sigma$ can be fitted to data.

From the Lorentz–Dirac index notation for the spin- and parity clusters in Eq. (23) one directly reads off that the first resonance-cluster will predominantly couple to systems carrying each one Dirac and one Lorentz index like the pion–nucleon (or $\eta$–nucleon) systems. On the contrary, the second and third spin–clusters will prefer couplings to multipion–nucleon final states (one Dirac- and several Lorentz indices) in agreement with the empirical observations.
According to that, the reason for the observed suppression of the $S_{11}(1650) \rightarrow N + \eta$ decay channel as compared to the $S_{11}(1535) \rightarrow N + \eta$ one, can be a simple re-distribution of decay strength in favor of the new opened $S_{11}(1650) \rightarrow N + \pi + \pi$ channel.

Now, it is easy to check with the help of Eqs. (1) and (2) that the three different Rarita–Schwinger spinors considered by us are distributed over two distinct Fock spaces, denoted by $\mathcal{F}_1$, and $\mathcal{F}_2$, respectively, which have opposite vacuum parities and are separated by a well pronounced gap of about 300 MeV:

\begin{align*}
\mathcal{F}_1: & \quad 2_{2I,+} : \Psi_{\mu} : P_{2I,1}; S_{2I,1}, D_{2I,3}, \quad \text{for} \quad I = 0, 1 \frac{1}{2}, 1 \frac{3}{2}, \quad \text{and} \\
\mathcal{F}_2: & \quad 4_{2I,-} : \Psi_{\mu_1 \mu_2} : S_{2I,1}; P_{2I,1} P_{2I,2}; D_{2I,3}, D_{2I,5}; F_{2I,5}, F_{2I,7}, \\
& \quad 6_{2I,-} : \Psi_{\mu_1 \ldots \mu_4} : S_{2I,1}; P_{2I,1} P_{2I,2}; D_{2I,3}, D_{2I,5}; F_{2I,5}, F_{2I,7}; \\
& \quad \Gamma_{2I,7}, \Gamma_{2I,9}; H_{2I,9}, H_{2I,11}, \quad \text{for} \quad I = 1 \frac{1}{2}, 1 \frac{3}{2}.
\end{align*}

The first Fock space, $\mathcal{F}_1$, is built upon a $0^+$ vacuum and appears in the spectra of all three baryons $N$, $\Delta$ and $\Lambda$. It always contains only the lowest $O(4)$ partial wave with $\sigma = 2$ and $\eta = +1$. In contrast to this, the second Fock space, $\mathcal{F}_2$, has a $0^-$ vacuum and contains the $\sigma = 4$, and 6 partial waves with $\eta = -1$. The positive parity of the vacuum of $\mathcal{F}_1$ reflects the realization of chiral symmetry in the hidden Nambu–Goldstone mode at low energies. This mode is well known to be characterized by a non-vanishing vacuum quark condensate. On the contrary, in $\mathcal{F}_2$ chiral symmetry must be restored because an isotriplet scalar boson of even $G$ parity, as would be required for the Goldstone boson of a hidden mode there, is absent from the spectrum. The $N \rightarrow 4_{1,-}$ and $N \rightarrow 6_{1,-}$ excitations are, therefore, chiral phase transitions. Phase transitions of that type have been studied, for example, in [13,16] by means of the change of the quark condensate with temperature within the framework of the modified $\sigma$-model with parity doubling [17]. There, the $S_{11}(1535)$ state has been considered as the chiral partner to the nucleon and the $N \rightarrow S_{11}$ excitation was shown to reveal the hysteresis behavior typical for 1st order phase transitions. From the considerations given above follows that the $S_{11}(1535)$ resonance can not be considered as the chiral partner of the nucleon as its internal orbital angular momentum is $I = 1^-$ instead of the required $I = 0^-$. The lightest spin-1/2$^-$ resonance which is built upon a pseudoscalar Fock vacuum and satisfies thereby the criteria for a parity partner to the nucleon, is the second $S_{11}(1650)$ state. It is this resonance that has to enter the calculation of the parameters of the chiral phase transitions for baryons.

Finally, special attention has to be paid to the $\Lambda$ hyperon excitations, where only the first $O(4)$ partial wave $2_{\alpha,+}$ has been found to join the $S_{01}(1405)$, $D_{03}(1520)$, and the $P_{03}(1600)$ states. Here the mass degeneracy of the resonances is not as well pronounced as compared to the non-strange sector. In addition, above 1800 MeV one finds exact parity degeneracy of states such like the $S_{01}(1800)$–$P_{01}(1810)$, and the $P_{03}(1890)$–$D_{03}(2000)$ pairs. A possible interpretation of this phenomenon could be the existence of a reflection–asymmetric hyperon shape. That such a shape can be the reason for the occurrence of parity doublets in baryon spectra was considered repeatedly over the years by different authors [13]. Such parity pairs can equally well be described relativistically as they can be mapped onto the members of the new type of Lorentz multiplets \{m, 0\} $\otimes$ \{0, m\} $\otimes$ \{[1, 2], 0\} $\oplus$ \{0, [1, 2]\}. For example, the $\Lambda$ cluster \{3/2$^-$, $-3/2^-$\} could be identified with the \{1, 0\} $\otimes$ \{0, 1\} $\otimes$ \{[1, 2], 0\} $\oplus$ \{0, [1, 2]\} space which is a totally antisymmetric 2nd rank Lorentz tensor with Dirac spinor components, $\Psi_{\mu(\nu)}$ (see [4] for details).

In summary, baryon resonance group to hyperspherical $O(4)_s$ spin–parity clusters rather than to ordinary $O(3)_l$ partial waves, and the non–strange baryon spectra are completely generated by the relativistic $\text{su}(2)_l \otimes \text{su}(1,3)_l \otimes \text{su}(3)_c$ group algebra rather than by the algebra of the non–relativistic $\text{SU}(6)_s \otimes \text{SU}(3)_c$ group. In other words, in performing an $O(4)$ partial wave decomposition of the $\pi N$ scattering amplitude, one would find the three Höhler poles $2_{2I,+}$, $2_{2I,-}$, and $6_{2I,-}$ on the complex energy plane rather than several dozens independent $O(3)$ partial waves. This specific of the $\pi N$ scattering may be related to the role of the pion as the Goldstone boson of the spontaneously broken chiral symmetry SU(2)$_L \otimes$ SU(2)$_R$, the space-time version of which acts as the universal covering of our $O(4)$.

The major advantage of our new relativistic spectrum generating algebra for baryons is that it reconciles such seemingly contrary ideas of the baryon structure like the constituent quark model, on the one side, and the multi–spinor representations of the Lorentz group used for structureless particles, on the other side. In case, the ‘missing’ resonances would not be found experimentally, the canonical belief about the three fermionic degrees of freedom of baryons has to be revised towards the $O(4)$ symmetric quark-hyperquark degrees of freedom, and thereby, towards a new symmetry of strong interaction of purely relativistic origin. The $O(4)$ symmetry of the internal orbital angular momenta is likely to emerge as the low mass limit of the approximate conformal SO(2,4) symmetry of the QCD lagrangian for the light-flavor quarks.

The scenario of the present work indicates that for the $\pi N$ scattering channel, the quark–hyperquark configurations decouple from the remaining 3q-states. In channels with non–Goldstone mesons, such like, say, the $\omega N$ one, these couplings may not vanish. In such a case the baryon excitations would acquire much richer structure and some of
the traditional ‘missing’ resonances could occur in the spectra as supplementary to the ruling O(4) pattern, an idea originally due to Ref. [19]. In the light of our O(4) systematics, the success of the traditional U(3) symmetric quark models in describing the dynamical properties of the resonances such like their branching ratios, form factors etc. is nonetheless understandable in so far that the physical observables are, in principle, independent on the choice of the Hilbert space basis, provided, the configuration spaces exploited are large enough. Our point here is that the O(4) symmetric quark-hyperquark model in Eq. (16) is the most economical starting point for resonance studies and a serious candidate as a guiding rule in designing the baryon spectra.

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Fig. 1 Baryon spectra in terms of the O(4) partial waves $\sigma_{2I,\eta}$. Each dashed area contains $2\sigma - 1$ different spin-states. The appearance of a second isolated $P_{11}$ state must be of dynamical origin and can contain, for example, nucleon–gluon components, a possibility considered in Ref. [5]. We expect a second $P_{11}$ resonance paralleling the hybrid $P_{33}(1600)$ state in the nucleon sector to strongly overlap with the Roper resonance from the lowest $2_{1,+}$ partial wave and be hidden there.

| $\sigma_{2I,\eta}$ | $M^b_{\sigma}$ | $M^{exp}_{\sigma}$ | $\delta^{max}_{\sigma}$ | Höhler pole             |
|-------------------|---------------|-------------------|---------------------|-------------------------|
| $2_{1,+}$         | 1441          | 1498              | 58                  |                         |
| $4_{1,-}$         | 1763          | 1689              | 31                  | (1665\pm25)-i(55\pm15) |
| $6_{1,-}$         | 2113          | 2102              | 148                 | (2110\pm50) -i(180\pm50) |
| $2_{3,+}$         | 1734          | 1690              | 70                  |                         |
| $4_{3,-}$         | 2056          | 1922              | 28                  | (1820\pm30)-i(120\pm30) |
| $6_{3,-}$         | 2406          | 2276              | 144                 |                         |
| $2_{0,+}$         | 1618          | 1508              | 103                 |                         |