On the non-uniqueness of marginally separated boundary layer flows

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A stationary or time dependent, laminar flow with a locally separated boundary layer is considered. The Navier-Stokes equations are analysed with the method of matched asymptotic expansions. The resulting integro-differential equation, known as the fundamental equation of marginal separation, is solved numerically by means of a spectral method based on Chebyshev polynomials. The critical value of the parameter controlling the magnitude of the adverse pressure gradient is associated with a bifurcation of the stability characteristics of the locally separated shear layer. The solution behaviour of the integro-differential equation in the corresponding parameter space is investigated. Special emphasis is placed on the observed non-uniqueness of solutions and the associated branch points.

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1 Introduction and problem description

We consider planar, steady or unsteady, marginally separated laminar boundary layer flows. These high Reynolds number flows can be observed for example at the leading edge suction side of a slender airfoil at a small angle of attack or in channel flows with suction and are characterized by a region of adverse pressure gradient.

As is well-known, Prandtl’s classical boundary layer equations cease to be valid near a point of vanishing skin friction. However, if the interaction between the viscous wall layer displacement and the thereby in the outer inviscid region induced pressure is accounted for at the same level of approximation (see [3] and [4]), a self-consistent description of incipient boundary layer separation can be formulated. Its fundamental equation reads, [4], [7],

\[ A^2 - X^2 + \Gamma = \lambda \int_X^\infty \frac{\partial^2 A(\xi, T)}{\sqrt{\xi - X}} d\xi - \gamma \int_{-\infty}^X \frac{\partial A(\xi, T)}{\sqrt{X - \xi}} d\xi \]

where \( A = A(X, T) \), \( X \) and \( T \) denote the displacement function or equivalently the local wall shear stress, the streamline coordinate and the time, respectively. Furthermore, \( \Gamma \) is a parameter controlling the imposed adverse pressure gradient conditions and \( \lambda, \gamma \) are positive constants.

The well-known non-uniqueness of steady solutions \( A(X) \) of eq. (1) in certain \( \Gamma \)-ranges, the corresponding bifurcation points, and the associated stability properties are of key interest. The main focus of the present study lies on the investigation of the stability behaviour of the single branches, especially at the loop, fig. 2. To this end, a perturbation analysis and a spectral method based on Chebyshev polynomials are applied.

2 Bifurcation problem and numerical method

Steady solutions of (1) are depicted in fig. 1 for characteristic values of \( \Gamma \) (blue lines). Negative values of \( A \) indicate separated flow, i.e., the existence of laminar separation bubbles. The fundamental curve of marginal separation, fig. 2, reveals the (multiple) non-uniqueness of the steady solutions within the range \( \Gamma \in [0, \Gamma_c \approx 2.66] \). The parabolic shape of the \( A(X = 0), \Gamma \)-curve near the points \( c, e \) and \( g \) suggest an expansion of the form \( A(X, T; \Delta \Gamma) \approx A_{\infty}(X) + \sqrt{\Delta \Gamma} a_1(X, T) + \Delta \Gamma a_2(X, T) + \cdots \), with the appropriate slow time scale \( \tilde{T} = \sqrt{\Delta \Gamma} T \). The deviation from the bifurcation points \( i = c, e, g \) is measured by \( \Delta \Gamma = \pm (\Gamma_i - \Gamma) \rightarrow 0 \) where the minus sign applies only for \( i = c \). Insertion of this expansion into (1) leads to a staggered system of equations for the perturbations \( a_1, a_2, \) etc., linearised about the distributions \( A_{\infty}(X) \) at the bifurcation points \( \Gamma = \Gamma_i \). As it turns out, the complete determination of \( a_1(X, T) = b(X(c(T) \approx 0)) \) requires the application of Fredholm’s alternative leading to a solvability condition for the shape function \( c(T) \), [1]. The individual eigenfunctions \( b(X) \) for the points \( c, e \) and \( g \) are depicted in fig. 1 (red lines).

Numerical computations have been conducted by means of a spectral method based on Chebyshev polynomials of degree at most \( n \). Due to the singular far field behaviour \( A(X, T) \sim |X| + \cdots \) as \( X \rightarrow \pm \infty \), the displacement function is written as \( A = A_1(X) + \tilde{A}(X, T) \) where \( A_1 = \sqrt{X^2 + 1} \) while the unknown part \( \tilde{A} \) is bounded and exhibits algebraic decay in the far field. The infinite domain \( X \in (-\infty, \infty) \) is mapped to the Chebyshev interval \( s \in [-1, 1] \) by means of the nonlinear transformation \( X(s) = X_0 + B \tan(\pi s/2) \). Here the numerical parameters \( X_0 \) and \( B \) can be adjusted in a certain range to account for the specific solution behaviour. The unequally spaced (clustered) Gauss-Lobatto nodes are defined by \( s_j = -\cos(j\pi/n), j = 0, \ldots, n \) and are used to ensure the interpolation process to be well-conditioned, [6].

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separation bubbles are predicted for the range $\Gamma \approx b$ (see labels in fig. 2). Red lines: eigenfunctions $b(X)$ corresponding to bifurcation points $c$, $e$ and $g$.

Fig. 2: Fundamental curve of marginal separation (parametric plot $A(X = 0)$ versus $\Gamma$) and its stability properties. Red/blue lines: stable/unstable solution branches of eq. (1). Dashed line: asymptote $A(0) = -5\Gamma^3/(108\lambda^{12/5})$ as $\Gamma \to 0^+$, [5]. Green lines: parabola approximations for the bifurcation points $c$, $e$ and $g$.

| bifurcation point | $\Gamma_1$ | $\mu$ | $c_s$ |
|-------------------|------------|-------|-------|
| $c$               | 2.6609     | 1.2968| 0.8781|
| $e$               | 1.1978     | -11.9155| 0.2897|
| $g$               | 1.3606     | 17.1276| 0.2416|

Table 1: Various parameters characterizing the bifurcation points; the normalisation $\delta = 1$ has been applied. $c_s = \pm \sqrt{\delta/\mu}$ for $c$ and $g$. $c_s = \pm \sqrt{-\delta/\mu}$ for $e$.

### 3 Stability analysis

The evolution equation for the shape function (amplitude) reads $dc/d\bar{T} + \mu c^2 = \delta = 0$, where $\mu$ and $\delta$ are real constants. Here the minus sign holds for the bifurcation points $c$ and $g$ while the positive sign in front of $\delta$ applies for $e$. By means of the transformation $c(\bar{T}) + c_s = 2c_s u(t)$ and $\bar{T} = t/(2\mu c_s)$, where $c_s = \pm \sqrt{\delta/\mu}$ (see tab. 1) denote the stationary points of $c(\bar{T})$ (i.e. the upper/lower branch solutions indicated by the green lines in fig. 2), one obtains the canonical form of the evolution equation $du/dt = u - u^2$ which is of Fisher’s type, [1]. The stability properties of the upper and lower branches $u_s = (1, 0)$ can be analysed using the ansatz $u = u_s + \Delta u e^{\omega t}$ with the amplitude $|\Delta u| \ll 1$ and the angular frequency $\omega$. We found a negative angular frequency only for $u_s = 1$, identifying the upper branches to be stable (attracting), whereas the lower branches $u_s = 0$ turned out to be unstable (repelling); see the arrows indicating the stability properties in fig. 2. As a consequence, so-called short separation bubbles are predicted in the range $\Gamma \in [\Gamma_b \approx 2.28, \Gamma_s \approx 2.66]$ where $\Gamma_b$ corresponds to incipient separation (vanishing bubble length). In agreement with the observations described in [2], so-called long separation bubbles are predicted for the range $\Gamma \in [\Gamma_s \approx 1.20, \Gamma_g \approx 1.36]$ at the upper branch of the loop, fig. 2.

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