COMBINATORIAL CATEGORICAL EQUIVALENCES OF DOLD-KAN TYPE

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Abstract. We prove a class of equivalences of additive functor categories that are relevant to enumerative combinatorics, representation theory, and homotopy theory. Let \( \mathcal{X} \) denote an additive category with finite direct sums and split idempotents. The class includes (a) the Dold-Puppe-Kan theorem that simplicial objects in \( \mathcal{X} \) are equivalent to chain complexes in \( \mathcal{X} \); (b) the observation of Church, Ellenberg and Farb [9] that \( \mathcal{X} \)-valued species are equivalent to \( \mathcal{X} \)-valued functors from the category of finite sets and injective partial functions; (c) a result T. Pirashvili calls of “Dold-Kan type”; and so on. When \( \mathcal{X} \) is semi-abelian, we prove the adjunction that was an equivalence is now at least monadic, in the spirit of a theorem of D. Bourn.

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1. Introduction

The intention of this paper is to prove a class of equivalences of categories that seem of interest in enumerative combinatorics as per [22], representation theory as per [9], and homotopy theory as per [2]. More specifically, for a suitable category \( \mathcal{P} \), we construct a
category \( \mathcal{D} \) with zero morphisms (that is, \( \mathcal{D} \) has homs enriched in the category 1/Set of pointed sets) and an equivalence of categories of the form

\[
[\mathcal{P}, \mathcal{X}] \simeq [\mathcal{D}, \mathcal{X}]_{\text{pt}}. \tag{1.1}
\]

On the left-hand side we have the usual category of functors from \( \mathcal{P} \) into any additive category \( \mathcal{X} \) which has finite direct sums and split idempotents. On the right-hand side we have the category of functors which preserve the zero morphisms. We prove (1.1) as Theorem 6.7 below, under the Assumptions 2.1-2.6 listed in Section 2.

One example has \( \mathcal{P} = \Delta_{\perp, \top} \), the category whose objects are finite ordinals with first and last element, and whose morphisms are functions preserving order and first and last elements. Then \( \mathcal{D} \) is the category with non-zero and non-identity morphisms

\[
0 \xleftarrow{\partial} 1 \xleftarrow{\partial} 2 \xleftarrow{\partial} \ldots
\]

such that \( \partial \circ \partial = 0 \). Since there is an isomorphism of categories

\[
\Delta_{\perp, \top} \cong \Delta_{\perp}^{\text{op}}, \tag{1.2}
\]

where the right-hand side is the algebraist’s simplicial category (finite ordinals and all order-preserving functions), our result (1.1) reproduces the Dold-Puppe-Kan Theorem [13, 14, 25]. Our arguments equally apply to the full subcategory \( \Delta_{\perp, \top} \neq \top \) of \( \Delta_{\perp, \top} \) obtained by removing the object 1; functors \( \Delta_{\perp, \top} \neq \top \rightarrow \mathcal{X} \) are the traditional simplicial objects in \( \mathcal{X} \).

Cubical sets also provide an example of our setting; see Example 3.3. We conclude that cubical abelian groups are equivalent to semi-simplicial abelian groups.

For a category \( \mathcal{A} \) equipped with a suitable factorization system \((\mathcal{E}, \mathcal{M})\) [15], write \( \text{Par}_{\mathcal{A}} \) (strictly the notation should also show the dependence on \( \mathcal{M} \)) for the category with the same objects as \( \mathcal{A} \) and with \( \mathcal{M} \)-partial maps as morphisms. We identify \( \mathcal{E} \) with the subcategory of \( \mathcal{A} \) having the same objects but only the morphisms in \( \mathcal{E} \). Assume each object of \( \mathcal{A} \) has only finitely many \( \mathcal{M} \)-subobjects. Let \( \mathcal{X} \) be any additive category with finite direct sums and split idempotents. Our main equivalence (1.1) includes as a special case an equivalence of categories

\[
[\text{Par}_{\mathcal{A}}, \mathcal{X}] \simeq [\mathcal{E}, \mathcal{X}] \tag{1.3}
\]

(Here \( \mathcal{D} \) is obtained from \( \mathcal{E} \) by freely adjoining zero morphisms.)

Let \( \mathcal{G} \) be the groupoid of finite sets and bijective functions. Let \( \text{FI}_\ast \) denote the category of finite sets and injective partial functions. Let \( \text{Mod}_R \) denote the category of left modules over the ring \( R \). Our original motivation was to understand and generalize the classification theorem for \( \text{FI}_\ast \)-modules appearing as Theorem 2.24 of [9], which provides an equivalence

\[
[\text{FI}_\ast, \text{Mod}_R] \simeq [\mathcal{G}, \text{Mod}_R]
\]

between the category of functors \( \text{FI}_\ast \rightarrow \text{Mod}_R \) and the category of functors \( \mathcal{G} \rightarrow \text{Mod}_R \). This is the special case of (1.3) above in which \( \mathcal{A} \) is the category \( \text{FI} \) of finite sets and injective functions, and \( \mathcal{M} \) consists of all the morphisms. For us, this result has provided a new viewpoint on representations of the symmetric groups, and a new viewpoint on Joyal species [22, 23].

In order to consider stability properties of representations of the symmetric groups, the authors of [9] also consider \( \text{FI} \)-modules: that is, \( R \)-module-valued functors from the category \( \text{FI} \). Each \( \text{FI}_\ast \)-module clearly has an underlying \( \text{FI} \)-module, so their Theorem 2.24 shows how symmetric group representations become \( \text{FI} \)-modules. Many \( \text{FI} \)-modules naturally extend to \( \text{FI}_\ast \)-modules; in particular, projective \( \text{FI} \)-modules do. Inspired by results of Putman [37], Church et al.[10] have generalized their stability results to \( \text{FI} \)-modules over Noetherian rings and make it clear how the stability phenomenon is about the \( \text{FI} \)-module being finitely generated. One application they give is a structural version of the Murnaghan Theorem.
[33, 34], a problem which has its combinatorial aspects [43]. We are reminded of the way in which Mackey functors [30, 35] give extra freedom to representation theory.

Another instance of an equivalence of the form (1.3) is when $\mathcal{A}$ is the category of finite sets with its usual (surjective, injective)-factorization system. Then $\text{Par}\mathcal{A}$ is equivalent to the category of pointed finite sets, which is equivalent to Graeme Segal’s category $\Gamma$ [40]. After completing this work, we were alerted to Teimuraz Pirashvili’s interesting paper [36] which gives this finite sets example, makes the connection with Dold-Puppe-Kan, and discusses stable homotopy of $\Gamma$-spaces.

We are very grateful to Aurelien Djament and Ricardo Andrade for alerting us to the existence of Jolanta Słomińska’s [41] and Randall Helmstutler’s [18]. These papers cover many of our examples and have many similar ideas. In particular, a three-fold factorization is part of the setting of [41]; while the additivity of $\mathcal{X}$ is replaced in [18] (now revised and published as [19]) by a semi-stable homotopy model structure, and the equivalence by a Quillen equivalence. We believe our approach makes efficient use of established categorical techniques and is complementary to these other papers. Of course, we should have discovered those papers ourselves.

Christine Vespa also emailed us suggesting connections with her joint paper [17]. We have added Example 7.8 (d) of type (1.3) about the PROP for monoids as a step in that direction.

As a further example of type (1.3), we have Example 7.6 about the category of representations of the general linear groupoid $\text{GL}(q)$ over the field $\mathbb{F}_q$. A recent development in this area is the amazing equivalence

\[ [\text{GL}(q), \text{Mod}^R] \simeq [\text{FL}(q), \text{Mod}^R] \]

proved by Kuhn [28]; here $\text{FL}(q)$ is the category of finite vector spaces over $\mathbb{F}_q$ and all linear functions, provided $q$ is invertible in $R$. Another interesting development is the work of Putman-Sam [38] on stabilization results for representations of the finite linear groupoid $\text{GL}(q)$, for example, in place of $\mathfrak{G}$. Also see Sam-Snowden [39].

Consider the basic equivalence (1.1). It is really about Cauchy (or Morita) equivalence of the free additive category on the ordinary category $\mathcal{P}$ and the free additive category on the category with zero morphisms $\mathcal{X}$. By the general theory of Cauchy completeness (see [44] for example), to have (1.1) for all Cauchy complete additive categories $\mathcal{X}$, it suffices to have it when $\mathcal{X}$ is the category of abelian groups. Cauchy completeness amounts to existence of absolute limits (see [45]) and, for additive categories, amounts to the existence of finite direct sums and split idempotents.

The strategy of proof of our main equivalence (1.1) is as follows. In Appendix A we provide general criteria for deducing that a $\mathcal{V}$-functor between $\mathcal{V}$-functor $\mathcal{V}$-categories is an equivalence given that some other $\mathcal{V}$-functor between “smaller” $\mathcal{V}$-functor $\mathcal{V}$-categories is known to be an equivalence. In Section 2 we describe the setting for our main result and then, in Section 3 show that the partial map, Dold-Puppe-Kan, and cube examples satisfy our assumptions. We show in Section 4 that the setting of Section 2 leads to an adjunction between functor categories of a particularly combinatorial kind. Section 5 reduces the proof that the adjunction is an equivalence to a particular case, then Section 6 treats that case. Section 7 provides some more examples. In the spirit of Bourn’s version [6] of Dold-Kan, Section 8 shows, under an extra assumption on our basic setting, that if $\mathcal{X}$ is semi-abelian then, instead of the equivalence (1.1), we have a monadic functor. The extra assumption is related to Stanley [42] Theorem 3.9.2 as used in Hartl et al. [17] Theorem 3.5. We feel the extra assumption should not be necessary and that there should be a proof for the semi-abelian case closer to that of the additive case but so far have been unable to find it.
2. The setting

Let $\mathcal{P}$ be a category with a subcategory $\mathcal{M}$ containing all the isomorphisms.

Assume we have a functor $(-)^*: \mathcal{M}^{\text{op}} \to \mathcal{P}$ which is the identity on objects and is such that $m^* \circ m = 1$ for all $m \in \mathcal{M}$. In particular, the morphisms in $\mathcal{M}$ are split monomorphisms (coretractions) in $\mathcal{P}$.

We write Sub$A$ for the (partially) ordered set of isomorphism classes of morphisms $m: U \to A$ into $A$ in $\mathcal{M}$. We will use the term subobject rather than “$\mathcal{M}$-subobject” for these elements. The order on Sub$A$ is the usual one. Abusing notation for this order, we simply write $U \preceq V$ when there exists an $f: U \to V$ such that $m = n \circ f$, where $m: U \to A$ and $n: V \to A$ in $\mathcal{M}$. There can be confusion when $U = V$ as objects, so we assure the reader that we will take care. A subobject of $A$ is proper when it is represented by a non-invertible $m: U \to A$ in $\mathcal{M}$; we write $U \prec A$ or $U \preceq_m A$.

Define $\mathcal{B}$ to be the class of morphisms $r \in \mathcal{P}$ with the property that, if $r = m \circ x \circ n^*$ with $m, n \in \mathcal{M}$, then $m$ and $n$ are invertible.

Define $\mathcal{D}$ to be the category with zero morphisms (that is, $1/$Set-enriched category) obtained from $\mathcal{B}$ by adjoining zero morphisms. Composition in $\mathcal{D}$ of morphisms in $\mathcal{B}$ is as in $\mathcal{P}$ if the result is itself in $\mathcal{B}$, but zero otherwise.

Define $\mathcal{I}$ to be the class of morphisms in $\mathcal{P}$ of the form $r \circ m^*$ with $m \in \mathcal{M}$ and $r \in \mathcal{B}$.

**Assumption 2.1.** Every morphism $f \in \mathcal{P}$ factors as $f = n \circ r \circ m^*$ for $m, n \in \mathcal{M}$ and $r \in \mathcal{B}$, and these $m, n,$ and $r$ are unique up to isomorphism.

**Assumption 2.2.** If $r, r' \in \mathcal{B}$ are composable then $r' \circ r \in \mathcal{I}$.

**Assumption 2.3.** If $r, m^* \circ r \in \mathcal{B}$ and $m \in \mathcal{M}$ then $m$ is invertible.

**Assumption 2.4.** The class $\mathcal{M} \circ \mathcal{M}^*$ of morphisms of the form $m \circ n^*$ with $m, n \in \mathcal{M}$ is closed under composition.

**Assumption 2.5.** For all objects $A \in \mathcal{P}$, the set Sub$A$ is finite.

**Assumption 2.6.** For all objects $A \in \mathcal{P}$, the relation $R_A$ on the set Sub$A$, defined by $mR_AN$ if and only if $m^* \circ n \in \mathcal{M}$, is contained in an antisymmetric transitive relation on Sub$A$.

**Proposition 2.7.** If $t \circ s = m \circ r$ with $s, t \in \mathcal{I}$, $r \in \mathcal{B}$ and $m \in \mathcal{M}$ then both $s$ and $t$ are in $\mathcal{B}$.

**Proof.** First note that, by uniqueness in Assumption 2.1, if $x \circ n^* = m' \circ r'$ in obvious notation, then $n^*$ is invertible. Now, with $s, t, m, r$ as in the Proposition, we can put $s = r_1 \circ m_1^*$ and $t = r_2 \circ m_2^*$. Then $r_2 \circ m_2^* \circ r_1 \circ m_1^* = t \circ s = m \circ r$. So $m_1^*$ is invertible and we conclude that $s \in \mathcal{B}$. Using Assumption 2.1, we have $m_2^* \circ s = m_3 \circ r_3 \circ n_1^*$. Then $r_2 \circ m_3 \circ r_3 \circ n_1^* = m \circ r$ implies $n_1^*$ invertible. So we may suppose $m_2^* \circ s = m_3 \circ r_4$. Then $(m_2 \circ m_3)^* \circ s = m_5^* \circ m_2^* \circ s = m_5 \circ m_3 \circ r_4 = r_4$. By Assumption 2.3, $m_2 \circ m_3$ is invertible. It follows that $m_2$ is invertible, so $t \in \mathcal{B}$. □

Recall that the limit of a diagram consisting of a family of morphisms into a fixed object $A$ is called a wide pullback; the morphisms in the limit cone are called projections. The dual is wide pushout. The following result helps in checking examples.

**Proposition 2.8.** Assume that the pullback of each morphism in $\mathcal{M}$ along any morphism in $\mathcal{M}$ exist and is in $\mathcal{M}$. Assume wide pullbacks of families of morphisms in $\mathcal{M}$ exist, have projections in $\mathcal{M}$, and become wide pushouts under $m \mapsto m^*$. Then Assumptions 2.1 and 2.2 hold.
Proof. For Assumption 2.1, take \( f: A \to B \) in \( \mathcal{P} \). Let \( n: Y \to B \) be the wide pullback of all those morphisms \( V \to B \) in \( \mathcal{M} \) through which \( f \) factors. Then \( n \in \mathcal{M} \) and there exists a unique \( f_1 \) with \( f = n \circ f_1 \). Let \( m: X \to A \) be the wide pullback of those \( p: U \to A \) in \( \mathcal{M} \) such that the morphism \( f_1 \) factors through \( p^* \). Then \( m \in \mathcal{M} \) and \( f_1 = r \circ m^* \) for a unique \( r \). Clearly \( r \in \mathcal{R} \) and we have uniqueness by a familiar argument.

Assumption 2.2 is proved as follows. Take \( t \circ s \) with \( s, t \in \mathcal{R} \). We already know \( t \circ s = m \circ u \) with \( m \in \mathcal{M} \) and \( u \in \mathcal{I} \). Form the pullback \( (n, u) \) of \((m, t)\). Then there exists \( b \) with \( a \circ b = u, n \circ b = s \). From the latter, \( n \) is invertible. So \( t = m \circ a \circ n^{-1} \), and this implies \( m \) invertible. So \( t \circ s \in \mathcal{I} \). \( \square \)

While Assumption 2.6 is used only in the proof of Theorem 6.5, it holds in all our examples because they satisfy the hypothesis of the following proposition.

**Proposition 2.9.** Suppose each hom-set of \( \mathcal{P} \) is equipped with a reflexive, transitive, antisymmetric relation \( \leq \) respected by composition on either side; thus we have a locally posetal 2-category \( \mathcal{P} \) with underlying category \( \mathcal{P} \). Suppose further that, for all \( m \in \mathcal{M} \), \( m^* \) is right adjoint to \( m \) with identity unit. Then Assumption 2.6 is redundant. Dually, the same is true if instead each \( m^* \) is left adjoint to \( m \) with identity counit.

Proof. For \( m \) and \( n \) in \( \mathcal{M} \) with codomain \( A \), let \( m \leq n \) mean that there exists \( \ell \in \mathcal{M} \) with \( m \ell \leq n \) in \( \mathcal{P} \). We claim \( \leq \) is transitive, antisymmetric, and contains the relation \( R_A \) of Assumption 2.6.

Suppose \( m_1 \leq m_2 \leq m_3 \). Then there are \( m \) and \( n \) in \( \mathcal{M} \) with \( m_1 m \leq m_2 m \) and \( m_2 n \leq m_3 n \). So \( m_1 m n \leq m_2 n \leq m_3 \) yielding transitivity.

Suppose \( m_1 \leq m_2 \leq m_1 \). Then there are \( m \) and \( n \) in \( \mathcal{M} \) with \( m_1 m \leq m_2 m \) and \( m_2 n \leq m_1 n \), and so \( m_1 m n \leq m_1 \); but \( m_1 \) is fully faithful, so \( m_1 n \leq 1 \). This gives a descending chain

\[ \cdots \leq (mn)^3 \leq (mn)^2 \leq (mn) \leq 1 \]

in \( \mathcal{P}(A, A) \). All terms of the chain are in the finite set \( \text{Sub}(A) \) so they cannot be distinct. So \((mn)^a = (mn)^b \) for some natural numbers \( a > b \). Since \( mn \) is a monomorphism, \( (mn)^{a-b} = 1 \); so \( m \) is a retraction and a monomorphism, hence invertible. Thus \( m_1 = m_2 \), proving \( \leq \) antisymmetric.

If \( m R_A n \) then there is an \( \ell \in \mathcal{M} \) with \( \ell = m^* \circ n \). By the adjointness \( m \vdash m^* \), it follows that \( m \ell \leq n \). So \( m \leq n \). This proves \( R_A \) is contained in \( \leq \). \( \square \)

For each \( u: A \to B \) in \( \mathcal{P} \), there exist \( m_u \in \mathcal{M} \) and \( s_u \in \mathcal{I} \) as in the triangle

\[
\begin{array}{ccc}
A & \overset{u}{\rightarrow} & B \\
\downarrow s_u & & \downarrow m_u \\
S_u & \rightarrow & B
\end{array}
\]

by Assumption 2.1.

**Proposition 2.10.** Suppose \( u: A \to B \) and \( v: B \to C \) in \( \mathcal{P} \).

(a) If \( s_{vu} \in \mathcal{R} \) then \( s_u \in \mathcal{R} \).

(b) If \( s_{vu} \in \mathcal{R} \) and \( u \in \mathcal{I} \) then \( u, s_u \in \mathcal{R} \).

Proof. We prove (b) first. We have \( v \circ u = m_v \circ s_v \circ u = m_v \circ m_{s_v u} \circ s_{s_v u} \). By uniqueness of factorization, we may take \( s_{vu} = s_{s_v u} \). So \( s_v \circ u = m_{s_v u} \circ s_{s_v u} \). By Proposition 2.7, if \( s_{vu} \in \mathcal{R} \) and \( u \in \mathcal{I} \) then \( s_v, u \in \mathcal{R} \). Now we prove (a). We have \( v \circ u = v \circ m_u \circ s_u \). By (b), \( s_{vu} s_u = s_{vu} \in \mathcal{R} \) implies \( s_u \in \mathcal{R} \). \( \square \)
3. Basic examples

Example 3.1. Take a category $\mathcal{A}$ with a factorization system $(\mathcal{E}, \mathcal{M})$ in the sense of [15]. Assume that the pullback of any morphism in $\mathcal{M}$ along any morphism exists and is in $\mathcal{M}$. Assume every morphism in $\mathcal{M}$ is a monomorphism and every object of $\mathcal{A}$ has only finitely many $\mathcal{M}$-subobjects. A span $f = (X \xleftarrow{f_0} U \xrightarrow{f_1} Y)$ is called a partial map $f: X \to Y$ when $f_0$ is in $\mathcal{M}$. There is a morphism from such an $f$ to $g = (X \xleftarrow{g_0} V \xrightarrow{g_1} Y)$ just when there is a (necessarily unique) morphism $U \to V$ making the two evident triangles commute. Let $\mathcal{P} = \text{Par}\mathcal{A}$ denote the category whose objects are all those of $\mathcal{A}$ and whose morphisms are isomorphism classes $[f]$ of partial maps. This underlies a locally ordered 2-category with 2-cells as above. Composition is that of spans: that is, by pullback. We identify $f: X \to Y$ in $\mathcal{A}$ with the morphism $[1_X, X, f]: X \to Y$ in $\mathcal{P}$. In this way, we have the $\mathcal{M}$ we require for $\mathcal{P}$ as the one in $\mathcal{A}$. For $m: U \to X$ in $\mathcal{M}$, a right adjoint in $\text{Par}\mathcal{A}$ is defined by $m^* = [m, U, 1_U]: X \to U$, and clearly $m^* \circ m = 1$. This gives our functor $(-)^*: \mathcal{M}^{op} \to \mathcal{P}$.

Every partial map $f = (X \xleftarrow{f_0} U \xrightarrow{f_1} Y)$ has

$$[f] = f_1 \circ f_0$$

where $f_0 \in \mathcal{M}$. Furthermore, $f_1 = m \circ e$ uniquely up to isomorphism for $m \in \mathcal{M}$ and $e \in \mathcal{E}$. It follows therefore that $\mathcal{R} = \mathcal{E}$ and that $[f] \in \mathcal{I}$ if and only if $f_1 \in \mathcal{E}$.

Now we look at our Assumptions. For Assumption 2.1, we make use of Proposition 2.8. First note that since we are assuming the $\mathcal{M}$-subobjects form a finite set, finite wide pullbacks can be obtained from pullbacks. By assumption, pullbacks of $\mathcal{M}$'s exist in $\mathcal{A}$ and a pullback of an $\mathcal{M}$ is an $\mathcal{M}$ in a factorization system. It is a pleasant exercise to see that these pullbacks remain pullbacks in $\mathcal{P}$ and become pushouts in $\mathcal{P}$ on taking left adjoints. Assumption 2.2 also follows from this but it is clear anyway since $\mathcal{R}$ is closed under composition because $\mathcal{E}$ is.

For Assumption 2.3, take $r = [1_A, A, e]$ with $e \in \mathcal{E}$ and $m = [1_V, V, m]$ in $\mathcal{M}$. To have $m^* \circ r = [n, P, u] \in \mathcal{R}$, we must have $n$ invertible and $u \in \mathcal{E}$. Then $e = m \circ u \circ n^{-1}$ implies $m \in \mathcal{E}$; so $m$ is invertible.

The class of partial maps in Assumption 2.4 are those of the form $[m, U, n]$ with $m, n \in \mathcal{M}$; these are closed under composition since pullbacks of $\mathcal{M}$'s exist and are in $\mathcal{M}$.

Assumption 2.5 was one of our assumptions on the factorization system on $\mathcal{A}$.

Notice that $\mathcal{P}$ is not closed under composition unless each pullback of an $\mathcal{E}$ along an $\mathcal{M}$ is an $\mathcal{E}$. This is true in many examples.

Example 3.2. Take $\mathcal{P}$ to be the category $\Delta_{\lambda, \top}$ of finite non-empty ordinals $n = \{0, 1, \ldots, n-1\}$ with morphisms those functions which preserve first element, last element and order. This underlies a locally ordered 2-category in which there is a 2-cell from $\xi$ to $\zeta$ just when $\xi(i) \leq \zeta(i)$ for each $i$ in the domain. Functors out of $\mathcal{P}$ are augmented simplicial objects because of the isomorphism (1.2). Take $\mathcal{M}$ to consist of all the injective functions in $\Delta_{\lambda, \top}$; each such injection $\partial$ has a left adjoint $\partial^*$ (and also a right adjoint for that matter) when regarded as a functor between ordered sets; clearly $\partial^*$ is surjective with $\partial^* \circ \partial = 1$ since $\partial$ is a fully faithful functor. A surjection $\sigma$ in $\mathcal{P}$ is of the form $\partial^*$ if and only if $\sigma(i) = 0$ implies $i = 0$. We write $\sigma_k: m + 1 \to m$ for the order-preserving surjection which takes the value $k$ twice. We write $\partial_k: m \to m + 1$ for the order-preserving injection which does not have $k$ in its image. Note that $\partial_0 \notin \mathcal{P}$, while $\sigma_k \downarrow \partial_k \downarrow \sigma_{k-1}$ as functors and $\sigma_k$ is a $\partial^*$ if and only if $k > 0$. Every $\xi \in \mathcal{P}$ factors uniquely as

$$\xi = \partial_{i_s} \circ \cdots \circ \partial_{i_1} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_l}$$

for $0 < i_1 < \cdots < i_s < n - 1$ and $0 \leq j_1 < \cdots < j_l < m - 1$. 
We claim $\mathcal{R}$ consists of the identities and the surjections $\sigma_0 : m + 1 \to m$. The only invertible morphisms in $\mathcal{P}$ are identities. Since members of $\mathcal{R}$ factor through no proper injection, they must be surjective. Every surjection $\tau$ is either of the form $\partial^* \circ \rho$ or uniquely of the form $\sigma_0 \circ \partial^*$. Neither of these forms is permissible for $\tau \in \mathcal{R}$ unless the injection $\partial$ is an identity. This proves our claim. It is also clear then that $\mathcal{I}$ consists of all the surjections in $\mathcal{P}$.

To prove the Assumptions, we make use of the simplicial identities (see page 24 of [16] for example) which, apart from $\sigma_i \circ \partial_i = 1 = \sigma_{i-1} \circ \partial_i$, say that, for all $i < j$,

$$\partial_j \circ \partial_i = \partial_i \circ \partial_{j-1}, \quad \sigma_{j-1} \circ \sigma_i = \sigma_i \circ \sigma_j, \quad \sigma_j \circ \partial_i = \partial_i \circ \sigma_{j-1}, \quad \sigma_i \circ \partial_{j+1} = \partial_j \circ \sigma_i.$$

The hypotheses of Proposition 2.8 hold since the pullback of any two monomorphisms in $\mathcal{P}$ exists and is absolute (that is, preserved by all functors); see page 27 of [16] on the Eilenberg-Zilber Theorem. Alternatively, notice that the squares

$$\begin{array}{ccc}
n - 1 & \xrightarrow{\partial_{j-1}} & n \\
\downarrow \sigma_i & & \downarrow \sigma_i \\
n & \xrightarrow{\partial_i} & n + 1
\end{array} \quad \begin{array}{ccc}
n - 1 & \xleftarrow{\sigma_{j-1}} & n \\
\downarrow \sigma_i & & \downarrow \sigma_i \\
n & \xleftarrow{\sigma_j} & n + 1
\end{array}$$

are respectively a pullback and pushout in $\mathcal{P}$ for $0 < i < j < n$. Then the general result follows by stacking these squares vertically and horizontally.

In the present example, $\mathcal{I}$ is closed under composition, making Assumption 2.2 clear.

For Assumption 2.3, suppose we have $\partial^* \circ \rho \in \mathcal{R}$ and $\rho \in \mathcal{R}$. Since $\rho$ is surjective, $\partial^* \circ \rho = 1$ implies $\rho = 1$ and hence $\partial = 1$, as required. Otherwise $\partial^* \circ \rho = \sigma_0$. Yet, if $\rho = 1$ this contradicts the lack of right adjoint for $\sigma_0$. So $\rho = \sigma_0$. Thus $\partial^* \circ \sigma_0 = \sigma_0$, and we can cancel $\sigma_0$ to obtain again $\partial = 1$.

For Assumption 2.4, the class $\mathcal{M} \circ \mathcal{M}^*$ of morphisms consists of those which reflect 0. That is, $\xi = \mu \circ \partial^*$ with $\mu \in \mathcal{M}$ if and only if $\xi(i) = 0$ implies $i = 0$. This class is clearly closed under composition.

Finally, Assumption 2.5 is clear.

Notice that the arguments above equally apply to the full subcategory $\Delta_{\leq n} \neq \top$ of $\Delta_{\leq n} \top$ obtained by removing the object 1. Functors $\Delta_{\leq n} \neq \top \to \mathcal{X}$ are the traditional simplicial objects in $\mathcal{X}^n$.

**Example 3.3.** This example is about the cubical category $\mathbb{I}$ as used by Sjoerd Crans [11] and Dominic Verity [46, 47]. Functors with domain $\mathbb{I}$ are cubical objects in the codomain category. Verity constructed $\mathbb{I}$ as the free monoidal category containing a cointerval.

For each natural number $k$, define a poset $\langle k \rangle = \{-1, 1, 2, \ldots, k, +\}$ by adjoining a bottom element $-$ and a top element $+$ to the discrete poset $\{1, 2, \ldots, k\}$. Any function $f : \langle k \rangle \to \langle h \rangle$ which preserves top and bottom is order-preserving. Thus we get a locally partially ordered 2-category with objects the $\langle k \rangle$, with morphisms the top-and-bottom-preserving functions, and with the pointwise order. Let $\mathbb{I}$ be the locally full sub-2-category consisting of those $f : \langle k \rangle \to \langle h \rangle$ for which, if $f(i), f(j) \notin \{-, +\}$ then $i < j$ if and only if $f(i) < f(j)$.

Let $\mathcal{P}$ be the underlying category of this $\mathbb{I}$. Let $\mathcal{M}$ consist of the morphisms in $\mathcal{P}$ which are injective as functions. Given such an $m : \langle k \rangle \to \langle h \rangle$ in $\mathcal{M}$, define $m^* : \langle h \rangle \to \langle k \rangle$ to send each $m(i)$ in the image of $m$ to $i$ and everything else to $+$. Clearly $m^* \in \mathcal{P}$ and $m^* \circ m = 1$. Furthermore, $mm^*(j)$ is equal to $j$ if $j = m(i)$ for some $i$, and $+$ otherwise. Therefore $1 \leq m \circ m^*$ showing $m^*$ to be left adjoint to $m$ with identity counit.

We can characterize morphisms of the form $m^*$ as those which are surjective as functions and reflect the bottom element $-$. Consequently $\mathcal{R}$ consists of the morphisms which are surjective as functions and reflect the top element $+$. 
Assumption 2.5 and 2.2 are clear.

\[
\begin{array}{ccc}
\langle \ell \rangle & \xrightarrow{q} & \langle v \rangle \\
p & & m \xleftarrow{n} \langle k \rangle \\
\langle u \rangle & & \langle h \rangle
\end{array}
\quad \begin{array}{ccc}
\langle k \rangle & \xrightarrow{n^*} & \langle v \rangle \\
m^* & & p^* \xleftarrow{n} \langle k \rangle \\
\langle u \rangle & & \langle f \rangle
\end{array}
\]

\begin{equation}
\tag{3.5}
\end{equation}

For Assumption 2.1, again we use Proposition 2.8. The existence of intersections is obvious. However, we must show that taking left adjoints gives cointersections. Take a pullback as in the left-hand diagram of (3.5) and consider the right-hand diagram. Assume \( f \circ m^* = g \circ n^* \). It suffices to show \( f \circ p \circ p^* = f \). Now \( fpp^*(i) \) is equal to \( fp(j) \) if \( i = p(j) \) for some \( j \), and equal to \(+\) otherwise. In the first case, we have \( fpp^*(i) = fp(j) = f(i) \), as required. In the second case, if \( i \) does not have the form \( p(j) \) then \( m(i) \) is not in the image of \( n \), and so \( gn^*m(i) = + \); thus \( f(i) = fm^*m(i) = + \) and we again have \( fpp^*(i) = f(i) \).

For Assumption 2.3, suppose \( m \in \mathcal{M} \) and \( r, m^* \circ r \in \mathcal{R} \). Suppose \( m^*(i) = + \). Since \( r \) is surjective, we have \( i = r(j) \) so that \( m^*r(j) = + \). However \( m^* \circ r \) reflects \(+\). So \( j = + \), yielding \( i = r(+) = + \). This proves \( m^* \) reflects \(+\) and therefore must be invertible.

For Assumption 2.4, the composites of the form \( m \circ n^* \) are clearly the (not necessarily surjective) morphisms which reflect \(-\). Clearly these are closed under composition.

There is another possible characterization of \( \mathcal{R} \), namely as the category of right adjoints to the morphisms in \( \mathcal{M} \). Thus in fact \( \mathcal{R} \) is dual to \( \mathcal{M} \). Now \( \mathcal{M} \) is really just the category \( \Delta_{\text{inj}} \) of finite ordinals and injective order-preserving maps, and \( \mathcal{R} \cong \mathcal{M}^{\text{op}} \). Also the category \( \mathcal{M}^* \), with morphisms the \( m^* \in \mathcal{M} \), is dual to \( \mathcal{M} \). The factorization of Assumption 2.1 in this case shows the category \( \mathcal{P} \) is a composite

\[ \mathbb{I} = \Delta_{\text{inj}} \circ \Delta_{\text{inj}}^{\text{op}} \circ \Delta_{\text{inj}}^{\text{op}} , \]

relative to suitably defined distributive laws.

An alternative viewpoint is that \( \mathbb{I} \) is \( \text{ParPar} \Delta_{\text{inj}} \), where in each case partial maps are defined relative to the morphisms in \( \Delta_{\text{inj}} \).

### 4. The Kernel Module

We write \( 1/\text{Set} \) for the category of pointed sets. We write \( X \land Y \) for the monoidal tensor (= smash product) on pointed sets. Let \( p\Lambda \) denote the free pointed set \( 1 + \Lambda \) on the set \( \Lambda \). This defines the value on objects of a strong monoidal functor \( p: \text{Set} \to 1/\text{Set} \) whose right adjoint is the forgetful functor.

In the setting of Section 2, we shall define a functor

\[ M: \mathcal{P}^{\text{op}} \times \mathcal{P} \rightarrow 1/\text{Set} \quad \tag{4.6} \]

which preserves zeros in the first variable.

Using the notation of (2.4), define \( M \) on objects by

\[ M(A, B) = p\{u \in \mathcal{P}(A, B): s_u \in \mathcal{R}\} . \quad \tag{4.7} \]

Suppose \( r: A_1 \to A \) in \( \mathcal{P} \) and \( f: B \to B_1 \) in \( \mathcal{P} \). Define \( M \) on morphisms by

\[ M(r, f)u = \begin{cases} f \circ u \circ r & \text{for } s_{fur} \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases} . \quad \tag{4.8} \]

To verify that this is a functor we need to see that

\[ M(r_1, f_1)M(r, f)u = M(r \circ \varnothing, r_1 \circ f_1 \circ f)u \]
holds. This amounts to showing that the left-hand side is non-zero if and only if the right-hand side is, because then both sides equal \( f_1 \circ f \circ u \circ r \circ r_1 \). In other words, we need to see that \( s_{fur}, s_{f_{1}fur_{1}} \in \mathcal{R} \) if and only if \( r \circ r_1, s_{f_{1}fur_{1}} \in \mathcal{R} \). In fact, \( s_{f_{1}fur_{1}} \in \mathcal{R} \) implies both \( s_{fur} \in \mathcal{R} \) and \( r \circ r_1 \in \mathcal{R} \). For we know by Assumption 2.2 that \( r \circ r_1 \in \mathcal{R} \); so, by Proposition 2.10(b), \( s_{f_{1}fur_{1}} \in \mathcal{R} \) implies \( r \circ r_1 \in \mathcal{R} \). By Proposition 2.10(a), we also conclude that \( s_{fur} \in \mathcal{R} \); then Proposition 2.10(b) gives \( s_{fur} \in \mathcal{R} \).

Let \( p_{\#} \mathcal{P} \) denote the free 1/\text{Set}-enriched category on the category \( \mathcal{P} \). For any locally pointed (that is, 1/\text{Set}-enriched) category \( \mathcal{X} \), we have an isomorphism of categories
\[
[p_{\#} \mathcal{P}, \mathcal{X}]_{\text{pt}} \cong [\mathcal{P}, \mathcal{X}] ,
\]
where, for emphasis, we write the subscript “pt” for the pointed-set-enriched functor category.

We identify (4.6) with its obvious extension to a 1/\text{Set}-functor
\[
M: \mathcal{D}^{\text{op}} \otimes p_{\#} \mathcal{P} \longrightarrow 1/\text{Set} .
\]
Now we are in the situation for a kernel adjunction of the form (A.25) with \( \mathcal{V} = 1/\text{Set} \). For any suitably complete and cocomplete 1/\text{Set}-category \( \mathcal{X} \), we have an adjunction
\[
[\mathcal{D}, \mathcal{X}]_{\text{pt}} \cong \frac{\mathcal{M}}{\mathcal{M}} [\mathcal{P}, \mathcal{X}] .
\]
More combinatorial expressions than (A.27) and (A.26) exist for \( \mathcal{M} \) and \( \mathcal{M} \) in our present situation as we shall now see.

**Theorem 4.1.** Suppose \( M \) is the kernel as in (4.9). Let \( \mathcal{X} \) be any suitably complete and
cocomplete 1/\text{Set}-category. For any zero preserving functor \( F: \mathcal{D} \to \mathcal{X} \) and any functor \( T: \mathcal{P} \to \mathcal{X} \), there are isomorphisms
\[
\mathcal{M}(F)B \cong \sum_{S_{\leq u}B} FS , \quad \mathcal{M}(T)A \cong \bigcap_{U_{\leq u}A} \ker T(m^*: A \to U) .
\]

**Proof.** We shall prove
\[
\int^{A \in \mathcal{D}} M(A, B) \wedge FA \cong \sum_{S_{\leq u}B} FS
\]
(where we are using \( \Lambda \wedge X \) to denote the pointed-set-enriched tensor of the pointed set \( \Lambda \) with \( X \in \mathcal{X} \)) by showing that the family of morphisms \( \chi_{A} \) as defined by the diagram
\[
\begin{array}{ccc}
M(A, B) \wedge FA & \xrightarrow{\chi_{A}} & \sum_{S_{\leq u}B} FS \\
\downarrow \text{in}_{u} & & \downarrow \text{in}_{mu} \\
FA & \xrightarrow{F_{su}} & FS_{u}
\end{array}
\]
for \( u \in M(A, B) \), using the notation of (2.4), is a universal extraordinary natural transformation (that is, a universal wedge). In order for a family \( \theta_{A}: M(A, B) \wedge FA \to X \) in \( \mathcal{X} \) to be extraordinary natural, we require
\[
\theta_{A} \circ \text{in}_{u} \circ Fr = \begin{cases}
\theta_{A_{1}} \circ \text{in}_{ur} & \text{for } s_{ur} \in \mathcal{R} \\
0 & \text{otherwise}.
\end{cases}
\]
for all \( r: A_{1} \to A \) in \( \mathcal{R} \) and \( u: A \to B \) in \( \mathcal{P} \) with \( s_{u} \in \mathcal{R} \). Notice that in this situation \( s_{ur} = s_{u} \circ r \) by uniqueness of factorization and Assumption 2.2.

Now for the family \( \chi \) we have \( \chi_{A} \circ \text{in}_{u} \circ Fr = \text{in}_{mu} \circ F(s_{u} \circ r) \). If \( s_{ur} \in \mathcal{R} \) then \( s_{u} \circ r = s_{u} \circ r \circ s_{ur} \), so \( \text{in}_{mu} \circ F(s_{u} \circ r) = \text{in}_{mu} \circ F_{su} = \chi_{A_{1}} \circ \text{in}_{ur} \); otherwise, \( F(s_{u} \circ r) = F(0) = 0 \), and we have our extraordinary naturality.
For universality, we must see that a general wedge \( \varphi \) factors uniquely through \( \chi \). Define \( \varphi : \sum_{S \leq m} B FS \to X \) by \( \varphi \circ \text{in}_m = \theta_S \circ \text{in}_m \). With \( s_u \in \mathcal{R} \), we have, using (4.12),

\[
\theta_A \circ \text{in}_u = \theta_A \circ \text{in}_{s_u} = \theta_{S_u} \circ \text{in}_{m_u} \circ F s_u = \varphi \circ \text{in}_{m_u} \circ F s_u = \varphi \circ \chi_A \circ \text{in}_u .
\]

Thus \( \theta = \varphi \circ \chi \). By taking \( u = m \in \mathcal{M} \), we also see that the definition of \( \varphi \) is forced.

Define \( \tilde{T} A = \bigcap_{U \leq m} \ker T(m^* : A \to U) \) with inclusion \( \iota_A : \tilde{T} A \to TA \), and, for \( r : A \to A_1 \), define \( \tilde{T} r : \tilde{T} A \to \tilde{T} A_1 \) to be the restriction of \( Tr \). For this we need to see that \( Tr \circ \iota_A \) factors through \( \iota_{A_1} \). Take a non-invertible \( n : V \to A_1 \). Using Assumption 2.1, we have \( n^* \circ r = \ell \circ r_1 \circ m^* \). So \( (n \circ \ell)^* \circ r = r_1 \circ m^* \). If \( m \) is invertible then \( (n \circ \ell)^* \circ r \in \mathcal{S} \). By Assumption 2.3, \( n \circ \ell \) is invertible. So \( n \) is invertible, a contradiction. So \( m \) is not invertible and we have \( Tn^* \circ Tr \circ \iota_A = T(\ell \circ r_1) \circ Tm^* \circ \iota_A = T(\ell \circ r_1) \circ 0 = 0 \), yielding \( \tilde{T} r \) with \( \iota_{A_1} \circ \tilde{T} r = Tr \circ \iota_A \).

Put \( \tilde{F} B = \sum_{S \leq m} B FS \) and transport the functoriality in \( B \) across the isomorphism with \( \tilde{M}(F)B \). We will prove that there is a natural isomorphism

\[
[\mathcal{P}, \mathcal{X}](\tilde{F}, T) \cong [\mathcal{P}, \mathcal{X}]_{\text{pt}}(F, \tilde{T})
\]

and hence conclude that \( \tilde{M}(T) \cong \tilde{T} \). Take a natural transformation \( \theta : \tilde{F} \Rightarrow T : \mathcal{P} \to \mathcal{X} \). The definition of \( \tilde{F}f \) involves writing \( f \circ m = m_{fm} \circ s_{fm} \) for \( m : U \to A \) and \( m_{fm} : V \to B \) then, with \( s_{fm} \in \mathcal{R} \), we have commutativity in the following diagram.

\[
\begin{array}{ccc}
FU & \xrightarrow{\text{in}_U} & \tilde{F} A \\
\downarrow F s_{fm} & & \downarrow F f \\
F V & \xrightarrow{\text{in}_V} & \tilde{F} B \\
\end{array}
\]

Consider a noninvertible \( \ell : W \to A \) in \( \mathcal{M} \). Then \( \ell^* \in \mathcal{S} \) and \( \ell^* \notin \mathcal{R} \). So \( T \ell^* \circ \theta_A \circ \text{in}_A = \theta_W \circ \tilde{F} \ell^* \circ \text{in}_A = \theta_W \circ 0 = 0 \). This implies there exists a unique morphism \( \varphi_A : FA \to TA \) such that the following square commutes.

\[
\begin{array}{ccc}
FA & \xrightarrow{\varphi_A} & \tilde{T} A \\
\downarrow \text{in}_A & & \downarrow \iota_A \\
\tilde{F} A & \xrightarrow{\theta_A} & TA \\
\end{array}
\]

Naturality of \( \varphi \) is proved as follows using (4.13) with \( f = r \in \mathcal{R} \):

\[
\begin{align*}
\varphi_B \circ \theta_B \circ Fr &= \theta_B \circ \text{in}_B \circ Fr \\
\theta_B \circ \tilde{F} r \circ \text{in}_A &= Tr \circ \theta_A \circ \text{in}_A \\
&= Tr \circ i_A \circ \varphi_A = i_B \circ \tilde{T} r \circ \varphi_A .
\end{align*}
\]

For the inverse direction, take any natural transformation \( \varphi : F \Rightarrow \tilde{T} : \mathcal{P} \to \mathcal{X} \). Define \( \theta \) by commutativity of the following diagram.

\[
\begin{array}{ccc}
FU & \xrightarrow{\text{in}_U} & \tilde{F} A \\
\downarrow \varphi_U & & \downarrow T m \\
\tilde{T} U & \xrightarrow{\iota_U} & TU \\
\end{array}
\]

We need to prove the right-hand square of (4.13) commutes when precomposed with any \( \text{in}_U \) for any \( m : U \to A \) in \( \mathcal{M} \). In the case where \( s_{fm} \in \mathcal{R} \), the desired commutativity is
consequence of the commutativity of the following three squares.

\[ \begin{array}{c|c|c|c}
 FU & \phi_U & \sim \mathrm{T}U & i_A \\
 F \mathrm{s}_{fm} & \sim \mathrm{T}s_{fm} & T \mathrm{s}_{fm} & T f \\
 F V & \phi_V & \sim \mathrm{T}V & i_B \\
 F \mathrm{s}_{fm} & \sim \mathrm{T}s_{fm} & T \mathrm{s}_{fm} & T f
\end{array} \]  

(4.16)

In the case where \( s_{fm} \notin \mathcal{R} \), we can write \( s_{fm} = r \circ \ell^* \) for some noninvertible \( \ell \in \mathcal{M} \). Then (using both \( U \preceq A \) and \( U \preceq U \)) we have \( T f \circ \theta_A \circ \mathrm{in}_U = T(f \circ m) \circ i_U \circ \varphi_U = T(n_{fm} \circ r) \circ T \ell^* \circ i_U \circ \varphi_U = T(n_{fm} \circ r) \circ 0 \circ \varphi_U = 0 = \theta_B \circ F f \circ \mathrm{in}_U \), as required.

To show that the assignments are mutually inverse, take \( \theta \) and define \( \varphi \) by (4.14). Let \( \bar{\theta} \) be as \( \theta \) in (4.15). Then \( \bar{\theta}_A \circ \mathrm{in}_U = T m \circ i_U \circ \varphi_U = T m \circ \theta_U \circ \mathrm{in}_U = \theta_A \circ F m \circ \mathrm{in}_U = \theta_A \circ \mathrm{in}_U \). So \( \bar{\theta} = \theta \).

On the other hand, take \( \varphi \) and define \( \theta \) by (4.15). Let \( \varphi' \) be as \( \varphi \) in (4.14). Then \( i_A \circ \varphi'_A = \theta_A \circ \mathrm{in}_A = i_A \circ \varphi_A \). So \( \varphi' = \varphi \).

5. Reduction of the problem

The problem being referred to in the title of this section is to show that the adjunction (4.10) is an equivalence. We wish to use the theory in Appendix A to reduce the problem to the special case where \( \mathcal{R} \) consists only of invertible morphisms. In Section 6, we will treat this special case.

Consider \( \mathcal{P}, \mathcal{M}, \mathcal{I}, \mathcal{R} \) and \( \mathcal{D} \) as in the setting of Section 2.

Let \( \mathcal{J} \) be the class of invertible morphisms in \( \mathcal{P} \). Let \( \mathcal{K} = \mathcal{M} \circ \mathcal{M}^* \) be the subcategory of \( \mathcal{P} \) as assured by Assumption 2.4. Then \( \mathcal{K}, \mathcal{M}, \mathcal{M}^*, \mathcal{I} \) and \( \text{p} \mathcal{I} \) also fit the setting of Section 2. Denote the kernel module for this situation by \( N : \mathcal{I}^\text{op} \times \mathcal{K} \rightarrow 1/\text{Set} \); on objects it is given by \( N(C, D) = \text{p} \mathcal{M}(C, D) \).

Let \( L : \mathcal{J} \rightarrow \mathcal{D} \) and \( K : \mathcal{K} \rightarrow \mathcal{P} \) be the inclusions; both are the identity on objects and so Cauchy dense.

We have the obvious inclusion

\[ \theta_{C, D} : N(C, D) \rightarrow M(LC, KD) \]  

(5.17)

taking \( m \in \mathcal{M}(C, D) \) to \( m \in \mathcal{P}(C, D) \) (using the fact that \( s_m = 1 \)).

**Proposition 5.1.** For \( \theta : N \Rightarrow M(L, K) \) defined at (5.17), the corresponding natural families

\[ \theta_{r, A, D} : \int_{C \in \mathcal{J}} N(C, D) \wedge \mathcal{D}(A, LC) \rightarrow M(A, KD) , \]

\[ \theta_{c, C, B} : \int_{D \in \mathcal{K}} \text{p} \mathcal{D}(KD, B) \wedge N(C, D) \rightarrow M(LC, B) , \]

as in Appendix A, are both invertible.

**Proof.** For \( r : A \rightarrow C \) in \( \mathcal{R} \) and \( m : C \rightarrow D \) in \( \mathcal{M} \), we have \( \theta_{r, A, D}[m \wedge r] = m \circ r \). Yet every \( u : A \rightarrow D \) with \( s_u \in \mathcal{R} \) factors uniquely up to a morphism in \( \mathcal{I} \) as \( u = m_u \circ s_u = \theta_{r, A, D}[m_u \wedge s_u] \). So we have our inverse bijection.

By the definition of the second coend, for all \( k : D \rightarrow D_1 \) in \( \mathcal{K} \) and \( g : D_1 \rightarrow B \), we have \( [(g \circ k) \wedge m] \) equal to \( [g \wedge (k \circ m)] \) when \( k \circ m \in \mathcal{M} \), zero otherwise. For \( m : C \rightarrow D \) in \( \mathcal{M} \) and \( f : D \ightarrow B \) in \( \mathcal{P} \), we have \( [f \wedge m] = [(f \circ m) \wedge 1] = [u \wedge 1] \) for some \( u \in \mathcal{P} \). But \( u = \ell \circ r \circ n^* \) for some \( n, \ell \in \mathcal{M} \) and \( r \in \mathcal{R} \). So \( [u \wedge 1] = [\ell \circ r \wedge n^*] \) provided \( n^* \in \mathcal{M} \), zero otherwise. However, \( n^* \in \mathcal{M} \) implies \( n \) invertible. So \( s_u \in \mathcal{R} \). In other words, \( [f \wedge m] \) is either zero or of the form \( [u \wedge 1] \) for a unique \( u \) with \( s_u \in \mathcal{R} \). By definition, \( \theta_{c, C, B}[f \wedge m] = f \circ m \) when \( s_{fm} \in \mathcal{R} \), zero otherwise. We have shown that an inverse to \( \theta_{c, C, B}^* \) is provided by \( u \mapsto [u \wedge 1]_C \). \( \square \)
Combining Proposition 5.1 and Corollary A.3, we have:

**Corollary 5.2.** If $\tilde{N}$ is an equivalence or crudely monadic then so is $\tilde{M}$.

6. **The case $\mathcal{P} = \mathcal{K}$**

Let $\mathcal{K}$ be a category with a subcategory $\mathcal{M}$ which contains all the isomorphisms. Assume we have a functor $(-^*) : \mathcal{M}^{op} \to \mathcal{K}$ such that $m^* \circ m = 1$ for all $m \in \mathcal{M}$. Assume every morphism $f$ in $\mathcal{K}$ factors as $f = m \circ \ell^*$ with $m, \ell \in \mathcal{M}$. Then $(\mathcal{M}^*, \mathcal{M})$ is a factorization system on $\mathcal{K}$ in the sense of [15]; indeed, $\mathcal{M}^*$ consists of the rejections and $\mathcal{M}$ of the coretractions. We also assume the set SubA of $\mathcal{M}$-subobjects of each object $A$ is finite. Let $\mathcal{I}$ be the groupoid of invertible morphisms in $\mathcal{K}$.

Let $J: \mathcal{I} \to \mathcal{M}$ and $I: \mathcal{M} \to \mathcal{K}$ be the inclusion functors. Both functors are the identity on objects.

Let $\mathcal{X}$ be any category admitting finite coproducts and finite products.

**Lemma 6.1.** Each functor $F: \mathcal{I} \to \mathcal{X}$ has a pointwise left Kan extension

$$\mathcal{I} \xrightarrow{J} \mathcal{M} \xrightarrow{F} \mathcal{X} \xleftarrow{\text{Lan}_J F} \mathcal{K}$$

along $J: \mathcal{I} \to \mathcal{M}$ defined on objects by:

$$(\text{Lan}_J F)X = \sum_{U \leq m X} FU .$$

For morphisms $f: X \to Y$ in $\mathcal{M}$, the following triangle commutes.

$$FU \xrightarrow{\text{in}_m} (\text{Lan}_J F)X \xrightarrow{\text{in}_f} (\text{Lan}_J F)Y$$

**Proof.** The inclusion $U \leq_m X \mapsto (U, m: JU \to X)$ of the discrete category on the set $\{U: U \leq_m X\}$ into the comma category $J/X$ has a left adjoint, taking $(V, f: JV \to X)$ to $V \leq_f X$. It is an adjoint equivalence. It follows that the colimit of $J/X \xrightarrow{\text{dom}} \mathcal{I} \xrightarrow{F} \mathcal{X}$ can be calculated by restricting along the inclusion and so is the coproduct displayed. □

**Lemma 6.2.** Each functor $T: \mathcal{M} \to \mathcal{X}$ has a pointwise right Kan extension

$$\mathcal{M} \xleftarrow{I} \mathcal{K} \xrightarrow{T} \mathcal{X} \xrightarrow{\text{Ran}_I T} \mathcal{K}$$

along $I: \mathcal{M} \to \mathcal{K}$ defined on objects by:

$$(\text{Ran}_I T)X = \prod_{U \leq_m X} TU .$$
For morphisms $k: X \to Y$ in $\mathcal{K}$, the square
\[
\begin{array}{c}
(Ran T)X \xrightarrow{pr_m} TU \\
(Ran T)k \downarrow \downarrow \downarrow T\ell \\
(Ran T)Y \xrightarrow{pr_n} TV
\end{array}
\] (6.21)

commutes, where $n^* \circ k = \ell \circ m^*$ with $n, m, \ell \in \mathcal{M}$.

Proof. The functor $U \xrightarrow{\sim} m X \mapsto \left( m^*: X \to IU, U \right)$ from the discrete category on the set $\{ U: U \xrightarrow{\sim} m X \}$ into the comma category $X/I$ has a right adjoint, taking $\left( n \circ \ell^*: X \to IA, A \right)$ to $W \xrightarrow{\sim} n X$, and so is initial. It follows that the limit of $X/I \xrightarrow{\text{colim}} \mathcal{M} \xrightarrow{T} X$ can be calculated as the product displayed. □

This gives the two adjunctions
\[
\begin{array}{c}
\left[ \mathcal{I}, \mathcal{X} \right] \xrightarrow{\text{Lan}_J} \left[ \mathcal{M}, \mathcal{X} \right] \\
\left[ \mathcal{I}, 1 \right] \xleftarrow{\text{Ran}_I} \left[ \mathcal{K}, \mathcal{X} \right]
\end{array}
\] (6.22)

Proposition 6.3. The functor $[I, 1]: [\mathcal{K}, \mathcal{X}] \to [\mathcal{M}, \mathcal{X}]$ is comonadic.

Proof. The functor $I$ is Cauchy dense since it is bijective on objects. The result follows by Proposition A.1. □

Proposition 6.4. If $\mathcal{X}$ is an additive category with finite direct sums and split idempotents, then $\text{Lan}_J: [\mathcal{I}, \mathcal{X}] \to [\mathcal{M}, \mathcal{X}]$ is comonadic.

Proof. Since the unit of the adjunction $\text{Lan}_J \dashv [J, 1]$ is a pointwise coretraction $FA \to FA \oplus \bigoplus_{U \leq A} FU$, and hence a strong monomorphism, the left adjoint $\text{Lan}_J$ is conservative, so it will be comonadic provided that it preserves certain equalizers. In fact, it preserves all finite limits. Since limits in $[\mathcal{M}, \mathcal{X}]$ are formed pointwise, it suffices to see that each
\[
ev_A \circ \text{Lan}_J: [\mathcal{I}, \mathcal{X}] \to \mathcal{X}
\]
preserves finite limits where $\ev_A: [\mathcal{M}, \mathcal{X}] \to \mathcal{X}$ is evaluation at $A \in \mathcal{M}$. By Lemma 6.1,
\[
ev_A \circ \text{Lan}_J \cong \bigoplus_{U \leq A} \ev_U.
\]
Each evaluation $\ev_U$ preserves limits and so their direct sum does so. □

Theorem 6.5. For any additive category $\mathcal{X}$ with finite direct sums and split idempotents, the functor
\[
\hat{N}: [\mathcal{I}, \mathcal{X}] \to [\mathcal{K}, \mathcal{X}]
\]
is an equivalence of categories.

Proof. By the explicit formula for $\hat{N}$ of Theorem 4.1, the diagram
\[
\begin{array}{c}
[\mathcal{I}, \mathcal{X}] \xrightarrow{N} [\mathcal{K}, \mathcal{X}] \\
\text{Lan}_J \downarrow \downarrow \downarrow [\mathcal{M}, \mathcal{X}]
\end{array}
\]
commutes up to isomorphism; thus since the downward arrows are comonadic, $\hat{N}$ is induced by a morphism $\Theta: G \to H$ of the comonads $G$ and $H$ induced by the adjunctions $\text{Lan}_J \dashv [J, 1]$ and $[I, 1] \dashv \text{Ran}_I$, respectively. Furthermore, $\hat{N}$ will be an equivalence if (and only if) $\Theta$ is
invertible. Now $\Theta$ is invertible just when each component $\Theta_F$: $\text{Lan}_J(FJ) \to \text{Ran}_J(F)J$ is invertible, and such a component is given by the composite

$$\tilde{N}(FJ)I \xrightarrow{\eta'_J(FJ)I} \text{Ran}_I(\tilde{N}(FJ))I \cong \text{Ran}_I(\text{Lan}_J(FJ))I \xrightarrow{\text{Ran}_I(\varepsilon)} \text{Ran}_I(F)I$$

where $\eta'$ is the unit of the adjunction $[I, 1] \dashv \text{Ran}_I$ and $\varepsilon$ is the counit of the adjunction $\text{Lan}_J \dashv [J, 1]$. More explicitly, this has component at $A$ given by a map

$$\sum_{U \subseteq A} FU \xrightarrow{(\Theta_F)_A} \prod_{V \subseteq A} FV$$

which can in turn be specified by giving a map $FU \to FV$ for each choice of $m: U \subseteq A$ and $n: V \subseteq A$; and this is given by $F(n^*m)$ if $n^*m \in \mathcal{M}$ and 0 otherwise.

To conclude, we use Assumption 2.6 and Assumption 2.5 to deduce that each relation $R_A$ has a linear refinement on $\text{Sub}(A)$. So we can list all the subobjects as

$$U_0 \xrightarrow{1_A=m_0} A, U_1 \xrightarrow{m_1} A, \ldots, U_n \xrightarrow{m_n} A$$

such that $m_iR_Am_j$ implies $i \leq j$. Then the components of $\Theta$ are represented by an upper-triangular matrix with identity morphisms in the main diagonal. Since $\mathcal{X}$ is additive (including existence of additive inverses), such matrices are invertible.

\begin{remark}
In case $\mathcal{X}$ is a protomodular 1/\text{Set}-category (see Section 8 for references), morphisms $f: A + B \to A \times B$, with $pr_1 \circ f \circ i_1 = 1_A$, $pr_2 \circ f \circ i_2 = 1_B$ and $pr_2 \circ f \circ i_1 = 0$, are strong epimorphisms. Even though $\text{Lan}_J$ is no longer comonadic, the comonad morphism $\Theta: G \to H$ can still be defined, and it is a strong epimorphism. We would like this to be the basis of a proof that $\tilde{N}$ (and hence $\tilde{M}$ using Corollary 5.2) is crudely monadic but we do not know how. We will give a proof that $M$ is crudely monadic in Section 8 under the extra Assumption 8.1.

Combining Corollary 5.2 with Theorem 6.5, we obtain:
\end{remark}

\begin{theorem}
In the setting of Section 2, the adjunction (4.10) is an equivalence for any additive category $\mathcal{X}$ in which idempotents split and finite direct sums exist.
\end{theorem}

7. Examples of Theorem 6.7

\begin{example}
We begin with a baby version of the Dold-Puppe-Kan Theorem. Let $\text{Pt}\mathcal{X}$ denote the category whose objects are split epimorphisms in $\mathcal{X}$, the morphisms are morphisms of the epimorphisms which commute with the splittings; this is what Bourn [5] calls the category of points in $\mathcal{X}$. Take $\mathcal{P}$ to be underlying category of the 2-category which is the free-living adjunction $\mu^* \dashv \mu$ with identity counit $\mu^* \circ \mu = 1$. So $[\mathcal{P}, \mathcal{X}] \cong \text{Pt}\mathcal{X}$ and we have our $(-)^*$ functor choosing the left adjoint. Let $\mathcal{M}$ consist of all the monomorphisms. Then $\mathcal{R}$ contains only the identities. Theorem 6.7 yields

$$\text{Pt}\mathcal{X} \simeq \mathcal{X} \times \mathcal{X}$$

This example also shows the necessity of $\mathcal{X}$ having homs enriched in abelian groups (not merely commutative monoids). We need $\mathcal{X}$ to have kernels of split epimorphisms and finite coproducts already. If we also ask that it have finite products then considering the split epimorphism $X \times Y \to Y$ given by the projection, the counit is the canonical map $X + Y \to X \times Y$, so if this is invertible we have hom enrichment in commutative monoids. Now considering the codiagonal $X + X \to X$, split by one of the injections, it is not hard to show that $1_X$ has an additive inverse.

\end{example}
Example 7.2. ([13, 14, 25]) Applying Theorem 6.7 to Example 3.2 yields that $[\Delta_{\perp, \top}, \mathcal{X}]$ is equivalent to the category of non-negatively graded chain complexes in $\mathcal{X}$.

Example 7.3. Applying Theorem 6.7 to Example 3.3 yields $[I, \mathcal{X}] \cong [\Delta_{\text{inj}}^\text{op}, \mathcal{X}]$, the category of semi-simplicial objects in $\mathcal{X}$.

Example 7.4. A (set) species in the sense of Joyal [22] is a functor $F : \mathcal{S} \rightarrow \text{Set}$ where $\mathcal{S}$ is the category of sets and bijective functions. A pointed-set species is a functor $F : \mathcal{S} \rightarrow 1/\text{Set}$, the category where $R$ is a field is the basic situation of [23].

Following [9], we write $\mathcal{FI}$ for the category of finite sets and injective functions. We then see that $\mathcal{M} = \mathcal{FI}$ and $\mathcal{E} = \mathcal{S}$, while $\mathcal{P} = \mathcal{FI}^\#$, the category of injective partial functions.

Corollary 7.5. [9] The functor
\[ \hat{M} : [\mathcal{S}, \text{Mod}_R] \rightarrow [\mathcal{FI}^\#_R, \text{Mod}_R] \]
is an equivalence of categories.

Example 7.6. Another example relevant to [24] is the category $\mathcal{A} = \text{FLI}(q)$ of finite vector spaces over the field $\mathbb{F}_q$ of cardinality $q$ (a prime power) and injective linear functions. Let $\mathcal{E} = \text{GL}(q)$ be the category (groupoid) of finite $\mathbb{F}_q$-vector spaces and bijective linear functions. Then $\mathcal{P} = \text{FLI}(q)^\#$ is the category of finite $\mathbb{F}_q$-vector spaces and injective partial linear functions.

Corollary 7.7. The functor
\[ \hat{M} : [\text{GL}(q), \text{Mod}_R] \rightarrow [\text{FLI}(q)^\#, \text{Mod}_R] \]
is an equivalence of categories.

Example 7.8. Here are a few examples of categories $\mathcal{A}$ as in Example 3.1 to which Theorem 6.7 applies with $\mathcal{E}$ the surjections and $\mathcal{M}$ the injections:
(a) the category of finite abelian groups and group morphisms;
(b) the category of finite abelian $p$-groups and group morphisms;
(c) the category of finite sets and all functions.
(d) the category of finite sets and functions equipped with linear orders in their fibres (this is the PROP for monoids).

In example (d), not all pullbacks exist but all pullbacks along injective functions do. Theorem 6.7 also applies to $\mathcal{M}$ in place of $\mathcal{A}$ in these examples. Then $\mathcal{E}$ is replaced by the groupoid of invertible morphisms in $\mathcal{A}$. In case of example (a), the paper [20] describes the groupoid being represented in $\mathcal{X}$.

Example 7.9. Consider the “algebraic” simplicial category $\Delta_+$ whose objects are all the natural numbers and whose morphisms $\xi : m \rightarrow n$ are order-preserving functions
\[ \xi : \{0, 1, \ldots, m - 1\} \rightarrow \{0, 1, \ldots, n - 1\} \]
Put $\mathcal{A} = \Delta^\text{op}_+$. Take $\mathcal{M}$ in $\mathcal{A}$ to consist of the surjections in $\Delta^\text{op}_+$. Pushouts of surjections along arbitrary morphisms exist in $\Delta_+$. Then $\mathcal{E} = \Delta^\text{op}_{\text{inj}}_+$ and $\mathcal{P}$ is the opposite of the category whose morphisms $m \rightarrow n$ are cospans
\[ m \xrightarrow{\xi} r \leftarrow^\sigma n \]
in $\Delta_+$ with $\sigma$ surjective. We could also take the “topological” simplicial category $\Delta$ (omit the object 0) to obtain a reinterpretation of the preoperads in $\mathcal{X}$ in the sense of [2].
Example 7.10. Here is a rather trivial example involving $\Delta$. Take $\mathcal{A}$ to be the category of non-empty ordinals and morphisms the order-preserving functions which preserve first element. Let $\mathcal{M}$ be the class of morphisms which are inclusions of initial segments. This is part of a factorization system where $\mathcal{E} = \Delta \perp \not\cong$ is the category of ordinals with distinct first and last element and morphisms the order-preserving functions which preserve first and last element. Sometimes $\mathcal{E}$ is called the category of *intervals*; there is a duality isomorphism $\mathcal{E} \cong \Delta^{\text{op}}$.

In this case, not only do we have the equivalence $[\mathcal{E}, X] \simeq [\mathcal{P}, X]$ of Theorem 6.7, we actually also have an isomorphism $\mathcal{P} \cong \mathcal{E}$.

Example 7.11. Take $\mathcal{A}$ to be a (partially) ordered set with finite infima and the descending chain condition. Then every morphism is a monomorphism and the strong epimorphisms are equalities. So $\mathcal{E}$ is the discrete category $\text{ob} \mathcal{A}$ on the set of elements of the ordered set. The reader may like to contemplate the case where $\mathcal{A}$ is the set of strictly positive integers ordered by division.

8. When $\mathcal{X}$ is semiabelian

Semiabelian categories include the category $\text{Grp}$ of (not necessarily abelian) groups and group morphisms. In [6] Dominique Bourn gave a version of the Dold-Puppe-Kan Theorem (Example 7.2) for the case where the codomain category $\mathcal{X}$ was semiabelian. In that case it asserted monadicity of the right adjoint in

$$[\mathcal{P}, \mathcal{X}]_{\text{pt}} \cong [\mathcal{P}, \mathcal{X}] \quad (8.23).$$

In this section, we provide a version of this for $\mathcal{P}$ as in Section 2, not only for $\Delta^{\text{op}}$. However, we require an extra assumption related to idempotents. Such conditions occur in Lawvere [29] and Kudryavtseva-Mazorchuk [27].

Assumption 8.1. The maximal proper elements of $\text{Sub} A$ can be listed $m_1, \ldots, m_n$ such that the idempotents $e_i = m_i \circ m_i^*$ on $A$ satisfy $e_j \circ e_i \circ e_j = e_j \circ e_i$ for all $i < j$.

Throughout we assume our category $\mathcal{X}$ has zero morphisms (that is, has homs enriched in pointed sets).

We begin by providing a non-additive version of the material at the end of Appendix B on idempotents.

Proposition 8.2. Suppose the category $\mathcal{X}$ has kernels of idempotents. Let $e, f$ be idempotents on an object $A$ of $\mathcal{X}$. If $e \circ f \circ e = e \circ f$ then the intersection of the kernels of $e$ and $f$ exists.

Proof. Let $k: K \to A$ be the kernel of $e$. Since $e \circ f \circ k = e \circ f \circ e \circ k = 0$, there exists a unique $g$ with $f \circ k = k \circ g$. Then $k \circ g \circ g = f \circ k \circ g = f \circ f \circ k = f \circ k = k \circ g$ and $k$ is a monomorphism. So $g$ is idempotent. Then the kernel $\ell : L \to K$ of $g$ is easily verified to be the intersection of the kernels of $e$ and $f$. \qed

Protomodular categories were defined by Bourn [5]: a category $\mathcal{X}$ (with zero morphisms) is *protomodular* when it is finitely complete and, for each object $A$, the functor $\ker : \text{Pt} A \to \mathcal{X}$ is conservative. Here $\text{Pt} A$ is the category whose objects $(p, X, s)$ consist of morphisms
Lemma 8.3. In a protomodular category, if \((p, X, s)\) is an object of \(\text{Pt} A\) and \(k : K \to X\) is the kernel of \(p\) then \(s : A \to X, k : K \to X\) are jointly strongly epimorphic.

Proof. Suppose \(m : Y \to X\) is a monomorphism and \(m \circ u = s, m \circ v = k\) for some \(u, v\). Then \(m : (p \circ m, Y, u) \to (p, X, s)\) is a morphism of \(\text{Pt} A\). Using \(v\), we see that \(m\) induces an isomorphism between the kernel of \(p \circ m\) and \(K\). Since \(\ker : \text{Pt} A \to \mathcal{X}\) is conservative, \(m : (p \circ m, Y, u) \to (p, X, s)\) is invertible. So \(m\) is invertible.

Proposition 8.4. Let \(a_1, \ldots, a_n\) be a list of idempotents on an object \(A\) of a protomodular category \(\mathcal{X}\). Suppose \(a_i \cdot a_j \circ a_i = a_i \circ a_j\) for \(i < j\). Suppose \(a_i = m_i \circ m_i^*\) is a splitting of \(a_i\) via a subobject \(m_i : A_i \to A\) and retraction \(m_i^*\). Let \(k_i : K_i \to A\) be the kernel of \(m_i^*\) (or equally of \(a_i\)). Then the morphisms \(m_1, \ldots, m_n\) along with the inclusion \(\cap_i \ker m_i^* \to A\) are jointly strongly epimorphic.

Proof. By Lemma 8.3, for each \(i\), the morphisms \(m_i : A_i \to A\) and \(k_i : K_i \to A\) are jointly strongly epimorphic; we will loosely say “\(A_i\) and \(K_i\) cover \(A\)”.

If \(i > 1\) then \(a_1 \cdot a_i \cdot k_1 = a_1 \cdot a_i \cdot a_j \cdot k_1 = 0\), and so \(a_i \cdot k_1\) lands in \(K_1\), providing a factorization \(a_i \cdot k_1 = k_1 \cdot a_i^1\). Now \(a_i^1\) is also an idempotent, and, for \(1 < i < j\), \(k_1 \cdot a_i^1 \cdot a_j^1 = a_i \cdot a_j \cdot a_i \cdot a_i \cdot k_1 = a_i \cdot a_j \cdot k_1 \cdot a_i^1 \cdot a_j^1\) and so \(a_i^1 \cdot a_j^1 \cdot a_i = a_i^1 \cdot a_j^1\). Clearly the splitting \(A_i^1\) of \(a_i^1\) is contained in \(A_i\).

The kernel \(K_i^1\) of \(a_i^1\) is \(K_i^1 = K_i \cap K_i\) since \(a_i \cdot x = a_i \cdot x = 0\) is equivalent to \(x = k_1 \cdot y\) and \(k_1 \cdot a_i^1 \cdot y = a_i \cdot k_1 \cdot y = s_i \cdot x = 0\); so in fact \(a_i^1 \cdot y = 0\).

We know that \(A\) may be covered by \(A_1\) and \(K_1\). By Lemma 8.3 again, we know, for each \(i > 1\), that \(K_1\) may be covered by the splitting of \(A_i^1\) and the kernel \(K_i^1 = K_i \cap K_i\) of \(a_i^1\). Since \(A_i^1 \leq A_i\), we see that \(A\) may be covered by \(A_1, A_i,\) and \(K_1 \cap K_i\).

Now continue inductively.

Theorem 8.5. Suppose \(\mathcal{D}\) is as in Section 2 and \(\mathcal{X}\) is protomodular (with zero morphisms) with finite coproducts. Then the components of the counit of the adjunction \((8.23)\) are strong epimorphisms.

Proof. The counit has components

\[\sum_{B \leq m, A \cap B} \ker Tn^* \to TA.\]

We prove this is a strong epimorphism by induction on the number \(k\) of maximal proper subobjects \(A_1, \ldots, A_k\) of \(A\), with \(m_i : A_i \to A\). The result is clear for \(k = 1\). For \(k > 1\), consider the following diagram.

\[\begin{array}{ccc}
\sum_i \sum_{B \leq m, A \cap B} \ker Tn^* & \xrightarrow{\delta} & \sum_i T A_i \\
\sum_{B \leq m, A \cap B} \ker Tn^* & \xrightarrow{\alpha} & FA & \xleftarrow{\beta} & \sum_i \ker Tn^*
\end{array}\]

The component of the counit is strongly epimorphic if and only if \(\alpha\) and \(\beta\) are jointly strongly epimorphic. The top row is a coproduct of components of the counit already known to be
strongly epimorphic by induction. So it suffices to show that $\gamma$ and $\beta$ are jointly strongly epimorphic. Rewriting the domain of $\beta$ as
\[
\bigcap_{C \prec n, A} \ker Tm^n = \bigcap_{i=1}^k \ker Tm_i^n,
\]
we see that Proposition 8.4 applies to yield what we want. \qed

A category $\mathcal{X}$ is semiabelian [21] when it has zero morphisms, is protomodular, is Barr exact, and has finite coproducts. A category is regular [1] when it is finitely complete, and has the (strong epimorphism, monomorphism)-factorization system existing and stable under pullbacks. It follows that every strong epimorphism is regular (that is, a coequalizer); see [8] for a proof. A category is Barr exact when it is regular and every equivalence relation is a kernel pair.

We mentioned Bourn’s category $\text{Pt}\mathcal{X}$ in Example 7.1. We will use the following routine fact.

**Lemma 8.6.** If $\mathcal{X}$ is a semiabelian category then the functor $\text{Pt}\mathcal{X} \to \mathcal{X}$, sending each split epimorphism to its kernel, preserves strong epimorphisms.

**Proof.** A strong epimorphism

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow s & & \downarrow t \\
A & \xrightarrow{f} & B
\end{array}
\]

in $\text{Pt}\mathcal{X}$ has $f$ and $g$ strong epimorphisms in $\mathcal{X}$. From this it is easily verified that the square involving the downward-pointing arrows is a pushout. Factor the morphism in $\text{Pt}\mathcal{X}$ as

\[
\begin{array}{ccc}
X & \xrightarrow{g_1} & Z & \xrightarrow{f_1} & Y \\
\downarrow s & & \downarrow t_1 & & \downarrow t \\
A & \xrightarrow{1_A} & A & \xrightarrow{f} & B
\end{array}
\]

in which the right-hand square involving the downward-pointing arrows is a pullback. The induced morphism $\ker q_1 \to \ker q$ is invertible. Semiabelian categories are Mal’tsev [21]. Therefore, as a comparison morphism to the pullback in a pushout square of regular epimorphisms in a Mal’tsev exact category, $g_1$ is a strong epimorphism (see Theorem 5.7 of [7]).

The induced morphism $\ker p \to \ker q_1$ is the pullback of $g_1$ along $\ker q_1 \to Z$ and so is a strong epimorphism by regularity. \qed

**Theorem 8.7.** If $\mathcal{P}$ is as in Section 2 and $\mathcal{X}$ is semiabelian then the functor $\tilde{M}: [\mathcal{P}, \mathcal{X}] \to [\mathcal{D}, \mathcal{X}]_{\text{pt}}$ of (4.10) is (crudely) monadic.

**Proof.** By Theorem 8.5, the right adjoint $\tilde{M}$ is conservative (since this is logically equivalent to the counit being a strong epimorphism). Since $\mathcal{X}$ is semiabelian, it has coequalizers. Therefore $[\mathcal{P}, \mathcal{X}]$ has coequalizers, so, for crude monadicity [32], it suffices to show that $\tilde{M}$ preserves coequalizers of reflexive pairs.

Both $[\mathcal{P}, \mathcal{X}]$ and $[\mathcal{D}, \mathcal{X}]_{\text{pt}}$ are semiabelian. A limit-preserving functor between semiabelian categories preserves coequalizers of reflexive pairs provided it preserves strong (= regular) epimorphisms (see Lemma 5.1.12 of [3]).
Let \( q : S \to T \) be a strong epimorphism in \([\mathcal{P}, \mathcal{X}]\). Each \( q_A : SA \to TA \) is a strong epimorphism. We must show each \( \tilde{q}_A : \tilde{S}A \to TA \) is a strong epimorphism.

Recall that \( TA \) is calculated by a sequence of kernels of split epimorphisms. This sequence depends only on the object \( A \) and the category \( \mathcal{P} \), not on the particular functor \( T \). The desired result follows on repeated application of Lemma 8.6. \( \square \)

**Appendix A. A general result from enriched category theory**

Let \( \mathcal{V} \) be a symmetric monoidal closed category which is complete and cocomplete. The tensor product of \( A, B \in \mathcal{V} \) is written \( A \otimes B \) and the unit for tensor is \( I \), following [26]. For a \( \mathcal{V} \)-category \( \mathcal{X} \), we also write \( A \otimes X \) for the tensor (=copower) and write \( [A, X] \) for the cotensor (=power) of \( A \in \mathcal{V} \) and \( X \in \mathcal{X} \); we have

\[
\mathcal{X}(A \otimes X, Y) \cong \mathcal{V}(A, \mathcal{X}(X, Y)) \cong \mathcal{X}(X, [A, Y]).
\]

A \( \mathcal{V} \)-functor \( L : \mathcal{C} \to \mathcal{A} \) is Cauchy dense (as a morphism \( L_* = \mathcal{A}(1, \mathcal{L}, L) : \mathcal{C} \to \mathcal{A} \) in the bicategory \( \mathcal{V}\text{-Mod} \) when the morphism

\[
\int^C \mathcal{A}(LC, A_1) \otimes \mathcal{A}(A, LC) \to \mathcal{A}(A, A_1)
\]

induced by composition in \( \mathcal{A} \) is a strong epimorphism in \( \mathcal{V} \) for all \( A, A_1 \in \mathcal{A} \). When \( \mathcal{V} \) is the category of abelian groups this means that each \( A \in \mathcal{A} \) is a retract of a finite direct sum of objects \( LC \) in the image of \( L \). The following result is standard but we provide a proof.

**Proposition A.1.** Suppose \( \mathcal{X} \) is a cocomplete \( \mathcal{V} \)-category, \( \mathcal{C} \) is a small \( \mathcal{V} \)-category, and \( L : \mathcal{C} \to \mathcal{A} \) is a Cauchy dense \( \mathcal{V} \)-functor. Then the \( \mathcal{V} \)-functor \( [L, 1] : [\mathcal{A}, \mathcal{X}] \to [\mathcal{C}, \mathcal{X}] \) is both monadic and comonadic, and so preserves and reflects limits, colimits, and strong epimorphisms, as well as being conservative.

**Proof.** The left adjoint to \([L, 1]\) is left Kan extension \( \text{Lan}_L \) along \( L \) defined by the coend formula

\[
\text{Lan}_L(G) = \int^C \mathcal{A}(LC, -) \otimes GC.
\]

Using the coend form of the Yoneda lemma (see [12]), we see that the component at \( F \in [\mathcal{A}, \mathcal{X}] \) of the counit for the adjunction \( \text{Lan}_L \dashv [L, 1] \) is isomorphic to the morphism

\[
\int^C \mathcal{A}(LC, -) \otimes \mathcal{A}(A, LC) \otimes FA \to \int^A \mathcal{A}(A, -) \otimes FA
\]

induced by (A.24). Since \( L \) is Cauchy dense, these components are all strong epimorphisms. It follows that the right adjoint \([L, 1]\) is conservative. It is clear that \([L, 1]\) preserves colimits. Since colimits exist in \([\mathcal{A}, \mathcal{X}]\), it follows that \([L, 1]\) also reflects colimits. By an easy case of the Beck monadicity theorem [31], since \([L, 1]\) has a left (respectively, right) adjoint, is conservative and preserves coequalizers (respectively, equalizers), it is monadic (respectively, comonadic). \( \square \)

Suppose \( M : \mathcal{B} \to \mathcal{A} \) is a \( \mathcal{V} \)-module viewed as a \( \mathcal{V} \)-functor \( M : \mathcal{A}^{\text{op}} \otimes \mathcal{B} \to \mathcal{V} \). We call \( M \) the kernel of the adjunction

\[
[\mathcal{A}, \mathcal{X}] \xrightarrow{\text{Lan}_M} [\mathcal{B}, \mathcal{X}], \quad \text{M} \triangleright \mathcal{Y}
\]

where

\[
\tilde{M}(TA) = \int_B [M(A, B), TB]
\]

(A.26)
and
\[ \widehat{M}(F)B = \int^A M(A, B) \otimes FA. \]  
(A.27)

The counit and unit of (A.25) will be denoted by
\[ \varepsilon_M : \widehat{M}M \Rightarrow 1_{[\mathcal{A}, \mathcal{B}]} \quad \text{and} \quad \eta_M : 1_{[\mathcal{A}, \mathcal{B}]} \Rightarrow \widehat{MM}. \]

To say \( M : \mathcal{B} \to \mathcal{A} \) is an equivalence as a \( \mathcal{V} \)-module is the same as saying \( \widehat{M} : [\mathcal{B}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}] \) is an equivalence of \( \mathcal{V} \)-categories. We also say \( \mathcal{A} \) and \( \mathcal{B} \) are Cauchy (or Morita) equivalent when such an equivalence exists. It follows that \( \mathcal{A}^{\text{op}} \) and \( \mathcal{B}^{\text{op}} \) are also Cauchy equivalent, and that the adjunction (A.25) is an equivalence for any Cauchy complete \( \mathcal{X} \). See [44] for more detail.

Suppose \( M : \mathcal{B} \to \mathcal{A} \) and \( N : \mathcal{D} \to \mathcal{C} \) are \( \mathcal{V} \)-modules, and \( L : \mathcal{C} \to \mathcal{A} \) and \( K : \mathcal{D} \to \mathcal{B} \) are \( \mathcal{V} \)-functors. Each \( \mathcal{V} \)-natural family
\[ \theta_{C,D} : N(C, D) \to M(LC, KD) \]  
(A.28)
induces \( \mathcal{V} \)-natural families \( \theta'_{A,D} : \)
\[ \int^C N(C, D) \otimes \mathcal{A}(A, LC) \xrightarrow{\int^C \theta \otimes 1} \int^C M(LC, KD) \otimes \mathcal{A}(A, LC) \xrightarrow{\text{actf}} M(A, KD) \]

and \( \theta'^\ell_{B,C} : \)
\[ \int^D \mathcal{B}(KD, B) \otimes N(C, D) \xrightarrow{\int^D 1 \otimes \theta} \int^D \mathcal{B}(KD, B) \otimes M(C, D) \xrightarrow{\text{actf}} M(LC, B). \]

We obtain three \( \mathcal{V} \)-natural transformations (A.29), all mates under appropriate adjunctions.
\[
\begin{array}{ccc}
[\mathcal{A}, \mathcal{B}] & \xrightarrow{[L, 1]} & [\mathcal{C}, \mathcal{A}] \\
\widehat{\mathcal{M}} & \leftarrow & \theta' \leftarrow \mathcal{M} \\
\mathcal{N} & \leftarrow & \mathcal{M} \\
\mathcal{B} & \leftarrow & \mathcal{D}
\end{array}
\quad
\begin{array}{ccc}
[\mathcal{B}, \mathcal{D}] & \xrightarrow{[K, 1]} & [\mathcal{D}, \mathcal{B}] \\
\mathcal{N} & \leftarrow & \mathcal{M} \\
\mathcal{A} & \leftarrow & \mathcal{C}
\end{array}
\quad
\begin{array}{ccc}
[\mathcal{D}, \mathcal{A}] & \xrightarrow{[L, 1]} & [\mathcal{C}, \mathcal{D}] \\
\mathcal{N} & \leftarrow & \mathcal{M} \\
\mathcal{B} & \leftarrow & \mathcal{D}
\end{array}
\]  
(A.29)

Notice that \( \hat{\theta} \) is invertible if and only if \( \theta' \) is invertible. Also, we have the following two commutative squares.
\[
\begin{array}{ccc}
\hat{\mathcal{N}}[L, 1] & \xrightarrow{\theta'^{\ell} \mathcal{M}} & [K, 1] \hat{\mathcal{M}} \mathcal{M} \\
\hat{\mathcal{N}} & \leftarrow & \mathcal{M} \\
\hat{\mathcal{N}} \hat{\mathcal{N}}[K, 1] & \xrightarrow{\mathcal{N}[K, 1]} & [K, 1]
\end{array}
\quad
\begin{array}{ccc}
[L, 1] & \xrightarrow{\eta_{[L, 1]}} & \mathcal{N} \hat{\mathcal{N}}[L, 1] \\
[L, 1] & \xrightarrow{\eta_{[L, 1]}} & \mathcal{N} \hat{\mathcal{N}}[L, 1] \\
[L, 1] \hat{\mathcal{M}} \mathcal{M} & \xrightarrow{\theta \mathcal{M}} & \hat{\mathcal{N}}[K, 1] \\
[L, 1] \hat{\mathcal{M}} \mathcal{M} & \xrightarrow{\eta_{[L, 1]}} & \mathcal{N} \hat{\mathcal{N}}[L, 1]
\end{array}
\]  
(A.30)

**Proposition A.2.** Suppose \( \mathcal{X} \) is a cocomplete \( \mathcal{V} \)-category, the \( \mathcal{V} \)-functors \( L : \mathcal{C} \to \mathcal{A} \) and \( K : \mathcal{D} \to \mathcal{B} \) are Cauchy dense with \( \mathcal{C} \) and \( \mathcal{D} \) small, and \( \theta : N \Rightarrow M(L, K) \) is as in (A.28).

- (a) If \( \hat{\theta} \) is a strong epimorphism and \( \hat{\mathcal{N}} \) is conservative then \( \mathcal{M} \) is conservative.
- (b) If \( \theta'^{\ell} \) is a strong monomorphism and \( \mathcal{N} \) is conservative then \( \mathcal{M} \) is conservative.
- (c) If \( \hat{\theta} \) is invertible and \( \mathcal{N} \) preserves reflexive coequalizers then \( \mathcal{M} \) preserves reflexive coequalizers.
- (d) If \( \theta'^{\ell} \) is invertible and \( \mathcal{N} \) preserves reflexive coequalizers then \( \mathcal{M} \) preserves reflexive coequalizers.
- (e) If \( \theta \) and \( \theta'^{\ell} \) are invertible and \( \mathcal{N} \) is fully faithful then \( \mathcal{M} \) is fully faithful.
- (f) If \( \hat{\theta} \) and \( \theta'^{\ell} \) are invertible and \( \mathcal{N} \) is fully faithful then \( \mathcal{M} \) is fully faithful.
Proof. For (a), contemplate the left square of (A.30). If $\tilde{\theta}$ is a strong epimorphism, so too is $\tilde{N}\tilde{\theta}$. If $\tilde{N}$ is conservative then $\varepsilon_N[K,1]$ is a strong epimorphism. It follows that $[K,1]\varepsilon_M$ is a strong epimorphism. By Proposition A.1, $\varepsilon_M$ is a strong epimorphism. So $\tilde{M}$ is conservative.

For (b), contemplate the right square of (A.30) and use the dual argument.

For (c), look at the rightmost square of (A.29) with its invertible 2-cell. Take any kind of colimit in $[\mathcal{B},\mathcal{D}]$. It is preserved by $[K,1]$ and so, if the that kind of colimit is preserved by $\tilde{N}$, then $[L,1]\tilde{M}$ preserves the colimit. By Proposition A.1, $[L,1]$ reflects the colimit and we have the result.

For (d), look at the leftmost square of (A.29) and apply the dual argument.

For (e), contemplate the left square of (A.30) and deduce from our hypotheses that $[K,1]\varepsilon_M$ is invertible. By Proposition A.1, $\varepsilon_M$ is invertible, so $\tilde{M}$ is fully faithful.

For (f), contemplate the right square of (A.30) and use the dual argument.

We shall say a $\mathcal{V}$-functor is crudely monadic when it has a left adjoint, is conservative, has reflexive coequalizers existing in the domain, and preserves reflexive coequalizers.

Corollary A.3. Suppose $L$ and $K$ are Cauchy dense and both $\theta^L$ and $\theta^R$ are invertible.

(a) If $\tilde{N} : [\mathcal{D}, \mathcal{V}] \to [\mathcal{C}, \mathcal{V}]$ is crudely monadic then $\tilde{M} : [\mathcal{B}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]$ is too.

(b) If $\tilde{N} : [\mathcal{D}, \mathcal{V}] \to [\mathcal{C}, \mathcal{V}]$ is an equivalence then $\tilde{M} : [\mathcal{B}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]$ is too.

Appendix B. Remarks on idempotents

Define a relation on any monoid $M$ by $a \sqsubseteq b$ when $ba = a$. This relation is transitive; indeed, we have a stronger property.

Proposition B.1. If $ub \sqsubseteq a$ and $vc \sqsubseteq b$ then $uvc \sqsubseteq a$.

If $u_ia_i \sqsubseteq a_{i-1}$ for $1 \leq i \leq n$ then $u_1 \cdots u_na_n \sqsubseteq a_0$.

Proof. For the first sentence, the assumptions are $aub = ub$ and $bvc = vc$. So $auvc =aubvc = ubvc = uvc$; that is, $uvc \sqsubseteq a$. The second sentence follows by induction. □

The relation is only reflexive for idempotents: clearly $a \sqsubseteq a$ is equivalent to $aa = a$.

The unit 1 of the monoid is a largest element in the sense that $a \sqsubseteq 1$ for all $a \in M$. That is, 1 is the empty meet. However, not all meets need exist. A meet for $a,b \in M$ is an element $a \land b$ with $a \land b \sqsubseteq a$ and $a \land b \sqsubseteq b$, and, if $x \sqsubseteq a$ and $x \sqsubseteq b$ then $x \sqsubseteq a \land b$. In particular, $a \land b$ must be an idempotent. Meets of lists of $n$ elements are defined in the obvious way and, for $n \geq 2$, can be constructed from iterated binary meets when they exist.

Proposition B.2. If $ab \sqsubseteq b$ and $a \sqsubseteq a$ then $a \land b = ab$.

If $a_1, \ldots, a_n$ are idempotents such that $a_ia_j \sqsubseteq a_j$ for $i \sqsubseteq j$ then $a_1 \land \cdots \land a_n = a_1 \cdots a_n$.

Proof. For the first sentence, we are told that $ab \sqsubseteq b$, while $a \sqsubseteq a$ implies $aa = a$, and so $aaab = ab$, yielding $ab \sqsubseteq a$. For the second sentence the result is clear for $n = 1$ since we suppose $a_1$ idempotent. Assume the result for $n-1$; so $a_1 \land \cdots \land a_{n-1} = a_1 \cdots a_{n-1}$. Apply the second sentence of Proposition B.1 to the inequalities $a_ia_n \sqsubseteq a_n$ to deduce $a_1 \cdots a_{n-1}a_n \sqsubseteq a_n$. So, by the first sentence, $a_1 \cdots a_n = a_1 \cdots a_{n-1} \land a_n = a_1 \land \cdots \land a_{n-1} \land a_n$, as required. □

Notice that, if $ab = ba$ and $b$ is idempotent, then $bab = abb = ab$, so $ab \sqsubseteq b$. So the proposition applies to commuting idempotents.

Now suppose we have a ring $R$. We can apply our results to the multiplicative monoid of $R$. We say idempotents $e$ and $f$ in $R$ are orthogonal when $ef = fe = 0$. A list $e_0, e_1, \ldots, e_n$ of idempotents is orthogonal when each pair in the list is orthogonal. The list is complete when $e_0 + e_1 + \cdots + e_n = 1$. An easy induction shows that a complete list of idempotents is orthogonal if and only if $e_ie_j = 0$ for $i \leq j$. 
For each \( a \in R \), put \( \bar{a} = 1 - a \). Clearly if \( a \) is idempotent, so is \( \bar{a} \).

Let \( R^o \) denote the ring obtained from \( R \) by reversing multiplication.

**Proposition B.3.**

(a) \( b \subseteq a \) in \( R^o \) if and only if \( \bar{b} \subseteq \bar{a} \) in \( R^o \).

(b) For \( b \) an idempotent, \( ab \subseteq b \) if and only if \( \bar{a} \bar{b} \subseteq \bar{b} \) in \( R^o \).

(c) If \( a \) and \( b \) are idempotents and \( ab \subseteq b \) in \( R \) then \( e_0 = ab, e_1 = \bar{a}b, e_2 = \bar{b} \) is a complete list of orthogonal idempotents.

**Proof.**

(a) \( b \subseteq a \) in \( R^o \) means \((1 - b)(1 - a) = 1 - b \) in \( R \); that is, \( ba = a \) which means \( a \subseteq b \) in \( R \).

(b) \( \bar{a} \bar{b} \subseteq \bar{b} \) in \( R^o \) means \( b \bar{a} = \bar{b} a \) in \( R \). That is, \((1 - b)(1 - a)(1 - b) = (1 - b)(1 - a)\). That is, \( 1 - a - b + ab - b + ba + bb - bab = 1 - b - a + ba \). That is, \( bab = ab \), which is \( ab \subseteq b \) in \( R \).

(c) We already know \( e_0 \) and \( e_2 \) are idempotent. They are also orthogonal: \( e_0 e_2 = ab(1 - b) = ab - ab = 0 \) and \( e_2 e_0 = (1 - b)ab = ab - bab = 0 \). Therefore \( e_0 + e_2 = ab + 1 - b = 1 - (1 - a)b = e_1 \) is idempotent. So \( e_1 \) is idempotent and \( e_0 + e_1 + e_2 = 1 \). The calculations \( e_0 e_1 = ab(1 - a)b = ab - abab = 0 \) and \( e_1 e_2 = (1 - a)b(1 - b) = b - ab - b + ab = 0 \) complete the proof. \( \square \)

We can extend part (c) inductively to obtain:

**Proposition B.4.** Suppose \( a_1, \ldots, a_n \) are idempotents such that \( a_i a_j \subseteq a_j \) for \( i \leq j \). Then \( e_i = \sum a_{i+1}a_{i+2} \ldots a_n \) for \( 0 \leq i \leq n \) (in particular, \( e_0 = a_1 a_2 \ldots a_n \) and \( e_n = a_n \)) defines a complete list of orthogonal idempotents.

Suppose \( \mathcal{X} \) is an additive category in which idempotents split. Our results apply to the endomorphism monoid \( \mathcal{X}(A, A) \) of each object \( A \in \mathcal{X} \). If \( a \) is an idempotent on \( A \), we have a splitting:

\[
\begin{array}{c}
A \\
\downarrow{r_a} \\
aA
\end{array} \quad \begin{array}{c}
aA \\
\downarrow{i_a} \\
A
\end{array} \quad \begin{array}{c}
A \\
\downarrow{r_a} \\
aA
\end{array}
\]

Yet, we also have a splitting for \( \bar{a} = 1 - a \) which incidentally provides a kernel \( \bar{a}A \) for \( a \) and so a direct sum decomposition of \( A \):

\[ A \cong \bar{a}A \oplus aA \, . \]

More generally, for any complete list \( e_0, e_1, \ldots, e_n \) of orthogonal idempotents in \( \mathcal{X}(A, A) \), we obtain a direct sum decomposition

\[ A \cong e_0A \oplus e_1A \oplus \cdots \oplus e_nA \, . \]

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