Interacting gauge fields and the zero-energy eigenstates in two dimensions

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Abstract

Gauge fields are formulated in terms of the zero-energy eigenstates of 2-dimensional Schrödinger equations with central potentials $V_a(\rho) = -a^2 g_a \rho^{2(a-1)}$ ($a \neq 0, \ g_a > 0$ and $\rho = \sqrt{x^2 + y^2}$). It is shown that the zero-energy states can naturally be interpreted as a kind of interacting gauge fields of which effects are solved as the factors $e^{ig_c \chi_A}$, where $\chi_A$ are complex gauge functions written by the zero-energy eigenfunctions. We see that the gauge fields for $a = 1$ are nothing but tachyons that have negative squared-mass $m^2 = -g_1$. We also find out U(1)-type gauge fields for $a = 1/2$ and SU(3)-type gauge fields for $a = 3/2$. Massive particles with internal structures described by the zero-energy states are also studied.

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It has been shown that 2-dimensional Schrödinger equations with the central potentials $V_a(\rho) = -a^2 g_a \rho^{2(a-1)}$ ($a \neq 0$ and $\rho = \sqrt{x^2 + y^2}$, i.e., $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$) have zero-energy eigenstates which are infinitely degenerate \[1, 2, 3, 4\]. Let us briefly repeat the argument for driving the zero-energy states. The Schrödinger equations for the energy eigenvalue $\mathcal{E}$, which are written as

$$[-\frac{\hbar^2}{2m} \Delta (x, y) + V_a(\rho)] \psi(x, y) = \mathcal{E} \psi(x, y)$$

where $\Delta (x, y) = \partial^2/\partial x^2 + \partial^2/\partial y^2$, have zero-energy ($\mathcal{E} = 0$) eigenstates. Note here that in this equation the mass $m$ and the coordinate vector $(x, y)$ can represent not only those of the single particle but those of the center of mass for a many particle system as well \[4\]. It has also been shown that, as far as the zero-energy eigenstates ($\psi_0$) are concerned, the Schrödinger equations for all $a$ can be reduced to the following equation in terms of the conformal transformations $\zeta = z^a$ with $z = x + iy$ \[2, 3\]:

$$[-\frac{\hbar^2}{2m} \Delta_a - g_a] \psi_0(u_a, v_a) = 0,$$

where $\Delta_a = \partial^2/\partial u_a^2 + \partial^2/\partial v_a^2$, using the variables defined by the relation $\zeta = u_a + iv_a$, where $u_a = \rho^a \cos a \varphi$ and $v_a = \rho^a \sin a \varphi$. That is to say, the zero-energy eigenstates for all the different numbers of $a$ are described by the same plane-wave solutions in the $\zeta_a$ plane. Furthermore it is easily shown that the zero-energy states degenerate infinitely. Let us consider the case for $a > 0$ and $g_a > 0$. Putting the function $f_n^\pm(u_a; v_a)e^{\pm ik_a u_a}$ with $k_a = \sqrt{2mg_a}/\hbar$ into Eq. \[2\], where $f_n^\pm(u_a; v_a)$ are polynomials of degree $n$ ($n = 0, 1, 2, \cdots$), we obtain the equations for the polynomials

$$[\Delta_a \pm 2ik_a \frac{\partial}{\partial u_a}] f_n^\pm(u_a; v_a) = 0.$$ 

Note that from the above equations we can easily see the relation $(f_n^-(u_a; v_a))^* = f_n^+(u_a; v_a)$ for all $a$ and $n$. General forms of the polynomials have been obtained by using the solutions in the $a = 2$ case (2-dimensional parabolic potential barrier) \[2, 3\]. Since all the solutions have the factors $e^{\pm ik_a u}$ or $e^{\pm ik_a v_a}$, we see that the zero-energy states describe stationary flows \[1, 2, 3\]. Taking account of the direction of incoming flows in the $\zeta_a$ plane that is expressed by the angle $\alpha$ to the $u_a$ axis, the general eigenfunctions with zero-energy are written as arbitrary linear combinations of eigenfunctions included in two infinite series of \[\psi_{\pm n}^{(u)}(u_a(\alpha); v_a(\alpha))\} \text{ for } n = 0, 1, 2, \cdots\text{, where}

$$\psi_{\pm n}^{(u)}(u_a(\alpha); v_a(\alpha)) = f_n^\pm(u_a(\alpha); v_a(\alpha))e^{\pm ik_a u_a(\alpha)},$$

$u_a(\alpha) = u_a \cos \alpha + v_a \sin \alpha$, and $0 \leq \alpha < 2\pi$. (For the details, see the sections II and III of Ref. \[3\].) It has been also pointed out that the motions of the $z$ direction perpendicular to the $xy$ plane can be introduced as free motions represented by $e^{\pm ik_z}$. In this case the
total energies \( E_T \) of the states are given by \( E_T = E_z \), where \( E_z = (\hbar k_z)^2/2m \) are the energies of the free motions in the \( z \) direction. Note that the zero-energy eigenfunctions cannot be normalized as same as those in scattering processes [5]. Actually it has been shown that all the zero-energy states for \( g_a > 0 \) are eigenstates in the conjugate spaces of the Gel’fand triplets [1, 2, 3, 6]. How we can understand the infinite degeneracy of the zero-energy states is the problem discussed in this letter.

Let us start our discussion from the equation in three dimensional spaces

\[
[- \Delta (x, y, z) + V_a(\rho)] \psi(r) = Q \psi(r),
\]

where \( \Delta (x, y, z) = \Delta (x, y) + \partial^2/\partial z^2 \). Note that the dimension of the potentials \( V_a \) are different from those in Eq. (1), and that of \( Q \) is also different from the energy dimension.

We consider the zero-energy solutions for the \( xy \) plane and the free motion with the non-zero momentum \( p_z \) in the \( z \) direction. This means that \( Q = (p_z/\hbar)^2 \). We notice that the solutions have the specific direction \( z \) and then the wave functions should be expressed by the \( z \) component of a vector such that

\[
\psi_z(r) = A_z(x, y)e^{ip_zz/\hbar}.
\]

Note that in general we can write Eq. (5) in a rotationally symmetric form in terms of the totally anti-symmetric symbols \( \epsilon_{ijk} \) defined by \( \epsilon_{123} = 1 \) in three dimensions as follows;

\[
[-(\epsilon_{ijk}\hat{\rho}_k\epsilon_{ilm}\hat{\rho}_l + (\hat{\rho} \cdot \hat{\rho})^2) + V_a(\rho)] \psi_\hat{\rho}(r) = Q \psi_\hat{\rho}(r),
\]

where \( \hat{\rho} = p/|p| \) stands for the unit vector of the direction of the non-zero momentum \( p \), and \( \rho^2 = \epsilon_{ijk}\hat{\rho}_k\epsilon_{ilm}\hat{\rho}_l\hat{\rho}_m \) should be taken. Note also that Eq. (6) depends only on the direction of the momentum but not on the magnitude of the momentum. Hereafter we choose the \( z \) direction as the specific direction of the momentum.

Let us introduce the vector fields \( A^\pm_n(\rho) = (\epsilon_{ijk}\hat{\rho}_k\epsilon_{ilm}\hat{\rho}_l + (\hat{\rho} \cdot \hat{\rho})^2) + V_a(\rho) \) which have the specific direction \( z \) are the eigenstates of Eq. (4) for \( Q = (p_z/\hbar)^2 \), where \( A^\pm_n = (0, 0, A^\pm_z(x, y)) \) and \( A^\pm_z(x, y)_n = \psi^\pm_{0n} \) for \( n = 0, 1, 2, \cdots \) and \( \psi^\pm_{0n} \) are obtained by replacing \( k_a = \sqrt{2mg_a/\hbar} \) with \( k_a = \sqrt{g_a} \) in the zero-energy eigenstates given by Eq. (4). Therefore the relations \( A^-_n^* = A^+_n \) hold. We study the property of the vector fields \( A^\pm_n \)

\[
A^\pm_n = (\partial_x\chi^\pm(xy)_n, \partial_y\chi^\pm(xy)_n, A^\pm_z(x, y)_n),
\]

where the functions \( \chi_n \) are defined by

\[
\chi^\pm(x, y)_n = zA^\pm_z(x, y)_n.
\]

Note that \( \chi_n^\pm \) is generally expressed in rotational invariant forms \( r \cdot A^\pm_n \), and we have a relation

\[
A^\pm_n = \partial\chi_n^\pm.
\]
In order to derive the last relation the fact that $A_z(x,y)$ do not depend on $z$ at all is important. We easily see that $\chi_n$ satisfy the same equation as $A_{zn}^\pm$

$$\left[\Delta (x,y) - V_a(\rho)\right] \chi_n^\pm = 0. \quad (9)$$

It is obvious that the fields also satisfy the equation

$$\left[-\Delta (x,y) + V_a(\rho)\right] A(x,y)^\pm = 0. \quad (10)$$

Hereafter we take off the $\pm$ symbols from $A_{zn}^\pm$, $\chi_n^\pm$ and so on. Now we introduce scalar functions $\chi_{nm} = \chi_m - \chi_n$ for $n, m = 0, 1, 2, 3, \cdots$. We easily see that

$$A_{zn} + \partial_z \chi_{nm} = A_{zm}, \quad (11)$$

and thus we have relations

$$A_n + \partial \chi_{nm} = A_m. \quad (12)$$

It is trivial that these conditions can be extended to arbitrary suitable pairs of linear combinations of $A_n$ and $\chi_{nm}$. Thus we see that the transformation (12) in terms of $\chi$ is a kind of the gauge transformation for the vector fields $A_n$ interacting with the external potentials $V_a$. In this picture the freedom of the gauge transformations can be understood as the freedom arising from the infinite degeneracy of the zero-energy states in two dimensions. It is also quite naturally understood that the gauge fields are represented by the transverse waves perpendicular to the moving direction. Now we may say that the vector fields $A$ are a kind of gauge fields that are interacting with the external potentials $V_a$. It should be strongly noticed that the fields $A$ are not always real rather complex in general. Actually the polynomials $f_n^\pm$ given in Eq. (4) are real only for $n = 0, 1$ and complex for $n \geq 2$.\n
Let us here proceed from Schrödinger equations in 3-dimensions to Klein-Gordon equations for massless particles in 4-dimensional Lorentz spaces. To make the discussion simple, we consider the case where a particle is moving in the $z$ direction with the non-zero momentum $p_z$. We may write it as

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta (x,y,z) + V_a(\rho)\right] \psi_z(r) = 0, \quad (13)$$

where $\psi_z(r) = A_z(x,y) e^{i(p_z r - p_o t)/\hbar}$. Since $p_0^2 = p_z^2$ for the massless particle, the derivatives with respect to the $z$ and $t$ cancel out each other in Eq. (13), and thus we have the same equation for $A_z$ as that for the zero-energy eigenstates. Now we can introduce the 4-vector fields as

$$A_n = (A_0, A_n),$$

where $A_0$ should be taken as functions satisfied by $\partial_t A_0 = 0$, and actually we may put $A_0 = 0$. In general the separation of the two directions perpendicular to the non-zero 3-momentum $p$ in 4-dimensional spaces can be carried out in terms of the 4-momenta
defined by \( p^+ = (|p|, p) \) and \( p^- = (-|p|, p) \) such that \( \epsilon^{\mu\nu\lambda\sigma} p_\mu p_\nu p_\lambda p_\sigma \), where \( \epsilon^{\mu\nu\lambda\sigma} \) is the totally anti-symmetric tensor defined by \( \epsilon_{0123} = 1 \). It is trivial to write \( \rho = x^2 + y^2 \) and \( \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) in the covariant expressions in terms of these tensors. We consider the same gauge functions \( \chi_{nm} \). Now we can write the gauge covariant derivatives as

\[
D_\mu = \partial_\mu - ig_c A_\mu,
\]

and thus the gauge transformation of a field \( \phi(r) \) and \( A \) are given as usual

\[
\phi(r) \rightarrow e^{ig_c \chi} \phi(r), \quad A_\mu \rightarrow A_\mu + \partial_\mu \chi,
\]

Note here that in these gauge transformations \( A_0 \) does not change at all, because \( \partial_t \chi = 0 \).

In the present choice for \( \chi \) given in Eq. (8) we have a trivial relation

\[
\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta (x, y, z) \right] \chi = -\partial^\mu A_\mu.
\]

This fact means that by regauging \( A_\mu \) in terms of \( \chi_A = -zA_z \) we can get Lorentz condition \( \partial^\mu A'_\mu = 0 \) where \( A_\mu = A_\mu + \partial_\mu \chi_A \), but we find out \( A'_\mu = 0 \) for \( \forall \mu \) by choosing \( A_0 = 0 \). That is to say, we can eliminate the fields \( A_\mu \) from the covariant derivatives, and thus all the effects can be described by the factors \( e^{ig_c \chi A} \) that are not the phase factor because \( \chi_A \) are generally not real. \( \chi_A \) has the infinite variety arising from the degeneracy of the zero-energy states. How can we determine the gauge functions \( \chi_A \)? We can point out one possibility that the boundary conditions determine the gauge functions \( \chi_A \). In this view the improper choice of the gauge functions \( \chi_A \) for the boundary conditions produces the gauge interactions by \( A_\mu \).

Here we study the simplest case of \( a = 1 \), where the equation is expressed by

\[
\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta (x, y, z) - g_1 \right] \psi_z(r) = 0.
\]

Taking account of \( g_1 > 0 \), this equation is nothing but that for a tachyon having the negative squared-mass \( m^2 = -g_1 \). This fact means that the tachyon can be a kind of these gauge fields and all the effects due to the tachyon are expressed by the factors \( e^{ig_c \chi A} \). Now we may say that in the world occupied by tachyons the vacuum will have the tachyon factors and then the space is possibly not flat any more.

Here we shall briefly comment on massive particles apart from the gauge fields. In the case for \( g_1 = 0 \) Eq. (10) represents the equation for massless particles. The addition of \( (-m^2 + m^2)/\hbar^2 \) to the potential term does not change the equation. We can, however, separate the equation into two parts for the fields \( \phi(r, t) = A_z(x, y) \psi(z, t) \) such that

\[
\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + (m/\hbar)^2 \right] \psi_z(z, t) = 0
\]

and

\[
[ - \Delta (x, y) - (m/\hbar)^2 ] A_z(x, y) = 0.
\]
One for the two dimensional $zt$ space is the usual equation for relativistic free motions for massive particles, whereas the other is nothing but the equation for the zero-energy states. Now we have found out a new freedom in the equation for massless particles in Lorentz spaces, which express free motions of massive particles. Of course, such massive particles must have internal structures which are described by the infinite degrees of freedom due to the zero-energy states. We should also note that the addition of a mass term $(m/h)^2$ to Eq. (13) derive the equation for the massive particle which has internal structures of the zero-energy states for the internal potential $V_a$.

Now we briefly study two other interesting gauges for $a = 1/2$ and $a = 3/2$.

(1) For the $a = 1/2$ case we have the potential $V_{1/2}(\rho) = -g_{1/2} \rho^{-1}$. This potential seems to represent a U(1)-type potential between two particles being in the $xy$ plane. Actually such situations will occur in the case for describing the currents in a surface. We may say that the fields represent the currents exchanged between two charged particles. Note here that the current of the gauge fields in the $xy$ plane is described by the stationary flows turning over at the origin as shown in Figs.1 and 2, where $\alpha = 0$ in Eq. (4) is taken, and the current is defined as usual $j_\mu = \text{Re}[\phi^\ast(\omega(\rho)\partial_\mu\phi)]$. These behaviors of the currents are easily understood, considering the fact that the gauge fields can be written by the solutions of Eq. (4) which are similar to the plane waves in the $u_{1/2}v_{1/2}$ plane. It is noted that the whole of the $u_{1/2}v_{1/2}$ plan is represented by two sheets of the usual $xy$ plane just as same as Riemann surface with the cut on the x axis. We can understand the reason why the cuts exist in Figs.1 and 2, because the direction of the flows in the half-plane upper than the cuts is different from that of the lower half-plane. We may say that the sources of the gauge fields stay at origin in Figs.1 and 2. The connection of the two flows given in Figs. 1 and 2 can be performed as Fig.3. In this figure the gauge fields are bounded around the finite cut, and thus we may say that this state represents a bound state of two sources having opposite charges. Note here that we can easily make standing wave states without any cuts in the $u_{a}v_{a}$ plane for arbitrary $a$, and those states can be mapped into the $xy$ plane in terms of the inverse transformations of the conformal transformations.

(2) For the case of $a = 3/2$ the potential has the linear dependence with respect to $\rho$. This situation reminds us of the SU(3) color potential introduced in quark-confinement dynamics. From the relations $\varphi_a = a \varphi$ we can generally show that the currents for the plane waves in the $u_{a}v_{a}$ plane are described by the corner flows which round the center with the angle $\pi/a$ for arbitrary surfaces of $a$. For $a = 3/2$ the current of the gauge fields in the $xy$ plane is described as Fig. 4 for $\alpha = 0$, where three different currents must exist in the $xy$ plane because the angle of the corner flows is $2\pi/3$, and the cut again appears on the $x$ axis from 0 to $\infty$. It should be noticed that the cut appears only on one direction such as on the positive part of the $x$ axis in Fig.4. In this case we have two different types of the connections among the gauge fields given in Figs.5 and 6. It is obvious that Fig.6 represents a three gauge field vertex in non-Abelian gauge theories. We may consider that Fig.5 represents a diagram related to bound states of quark-antiquark like mesons in QCD, whereas Fig.6 does those for three quarks like baryons. To study the
problems we have to understand the properties of sources that appear at the end of the cuts. Note finally that SU(N) gauge fields will appear for $a = N/2$, where $N =$ positive integers, and the cuts appear only in the cases for $N =$ odd integers.
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**Fig. 1:** Flows of the states with the factor $e^{-ik_au_a}$ for $a = 1/2$.

**Fig. 2:** Flows of the states with the factor $e^{ik_au_a}$ for $a = 1/2$. 
Fig. 3: Connection of two flows for $a = 1/2$.

Fig. 4: Flows for $a = 3/2$. 
Fig. 5: Connection of two flows for $a = 3/2$.

Fig. 6: Image for the connection of three flows for $a = 3/2$, where all the angles between two connecting vectors should be $3\pi/2$ in real connections.