Existence and Nonexistence of Nontrivial Doubly Periodic Solutions of Nonlinear Telegraph Equations

Nan Deng

1 School of Applied Science, Beijing Information Science and Technology University, Beijing, 100192, PR China.

Author’s contribution

The calculation, proof and writing of this article are completed by the author independently.

Article Information

DOI: 10.9734/ARJOM/2021/v17i630311

Editor(s):
(1) Dr. Nikolaos D. Bagis, Aristotle University of Thessaloniki, Greece.

Reviewers:
(1) M. Kalamani, India.
(2) Mihaela Andrei, Dunarea de Jos University of Galati, Romania.

Complete Peer review History: http://www.sdiarticle4.com/review-history/73511

Received: 09 July 2021
Accepted: 09 September 2021

Original Research Article

Abstract

Aims/ Objectives: We discuss the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for nonlinear telegraph equations

\[ u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, u), \]

where \( c > 0 \) is a constant, \( \lambda > 0 \) is a positive parameter, \( a \in C(\mathbb{R}^2, \mathbb{R}^+) \), \( f \in C(\mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R}^+) \), and \( a, f \) are 2\(\pi\)-periodic in \( t \) and \( x \). The proof is based on a known fixed point theorem due to Schauder. In previous articles, a single telegraph equation or telegraph system have been widely studied, but there is relatively little research on nonlinear telegraph equations with a parameter and the nonlinearities are nonnegative. We would like do some research on this topic. We give new conclusions on the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for nonlinear telegraph equations under sublinear assumptions.

Study Design: Study on the existence and nonexistence of nontrivial nonnegative doubly periodic solutions.

Place and Duration of Study: School of Applied Science, Beijing Information Science & Technology University, September 2020 to present.

*Corresponding author: E-mail: 18810392077@163.com;
**Methodology:** We prove the existence and nonexistence of nontrivial nonnegative doubly periodic solutions by the results of Schauder’s fixed point theorem.

**Results:** We give new conclusions of existence and nonexistence of nontrivial nonnegative doubly periodic solutions for the equations.

**Conclusion:** We prove the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for nonlinear telegraph equations

\[ u_{tt} - u_{xx} + a(t, x)u = \lambda f(t, x, u), \]

and give new conclusions.

**Keywords:** Doubly periodic solution; telegraph equation; fixed point theorem; existence and nonexistence.

**2010 Mathematics Subject Classification:** 53C25; 83C05; 57N16.

## 1 Introduction

In this paper, we are concerned with the existence and nonexistence of nonnegative doubly periodic solutions for the nonlinear telegraph equation

\[ u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, u), \quad (t, x) \in \mathbb{R}^2, \tag{1.1} \]

and they are doubly periodic in the following sense

\[ u(t + 2\pi, x) = u(t, x + 2\pi) = u(t, x), \quad (t, x) \in \mathbb{R}^2, \]

where \( c \) is a constant and \( c > 0 \), \( \lambda \) is a positive parameter, \( a \in C(\mathbb{R}^2, \mathbb{R}^+) \), \( f \in C(\mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R}^+) \), and \( a \) and \( f \) are \( 2\pi \)-periodic in \( t \) and \( x \).

The results concerning the existence and multiplicity of nontrivial doubly periodic solutions for a single telegraph equation or telegraph system, see [1-20] and the references therein. By using weak force conditions, Wang [20] constructed some existence results for the following periodic boundary value problems:

\[
\begin{align*}
  u_{tt} - u_{xx} + c_1 u_t + a_{11}(t, x)u + a_{12}(t, x)v &= f_1(t, x, u, v) + \chi_1(t, x), \\
  v_{tt} - v_{xx} + c_2 v_t + a_{21}(t, x)u + a_{22}(t, x)v &= f_2(t, x, u, v) + \chi_2(t, x).
\end{align*}
\tag{1.2}
\]

The proof is based on Schauder’s fixed point theorem. Wang [8] pays attention to the existence and multiplicity of double periodic solutions for the nonlinear telegraph system

\[
\begin{align*}
  u_{tt} - u_{xx} + c_1 u_t + a_{11}(t, x)u + a_{12}(t, x)v &= \lambda f_1(t, x, u, v), \\
  v_{tt} - v_{xx} + c_2 v_t + a_{21}(t, x)u + a_{22}(t, x)v &= \lambda f_2(t, x, u, v),
\end{align*}
\tag{1.3}
\]

where \( f_i(t, x, u, v) \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and is bounded below. The proof is based on a well known fixed theorem in a cone. Using fixed-point theorem of a cone mapping, Wang [14] pays attention to the multiplicity of positive doubly periodic solutions of (1.1), where \( f(t, x, u) \) may change sign and is singular at \( u = 0 \).

If the nonlinearities are nonnegative, we show that the problem (1.1) admits existence and nonexistence of doubly periodic solutions. In this paper, we will use of the Schauder’s fixed point theorem to prove that the problem (1.1) admits one nontrivial nonnegative solution for small \( \lambda > 0 \) if one of \( \lim_{u \to 0^+} \frac{f(t, x, u)}{u} \) is infinity. In addition, all \( \lim_{u \to \infty} \frac{f(t, x, u)}{u} \) is zero, we show that the problem (1.1) admits a nontrivial solution for all \( \lambda > 0 \).
We also provide a result that there is nonexistence of doubly periodic solution.

The paper is organized as follows: In Section 2, we make some preliminaries; In section 3, we prove the existence and nonexistence of nontrivial nonnegative doubly periodic solutions for problem (1.1).

2 Preliminaries

Let $\mathbb{T}^2$ be the torus defined as\[ \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \].

Doubly $2\pi$-periodic functions will be defined on $\mathbb{T}^2$. We use the notations\[ L^p(\mathbb{T}^2), \quad C(\mathbb{T}^2), \quad C^\infty(\mathbb{T}^2), \quad D(\mathbb{T}^2) \] to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space $D'(\mathbb{T}^2)$ denotes the space of distributions on $\mathbb{T}^2$.

By a doubly periodic solution of (1.1) we mean that $u \in L^1(\mathbb{T}^2)$ satisfies (1.1) in the distribution sense, that is
\[
\int_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} - c\phi_t + a(t,x)\phi) dt \ dx = \int_{\mathbb{T}^2} \lambda f(t, x, u(t, x)) \phi \ dt \ dx, \quad \forall \phi \in D(\mathbb{T}^2)
\] (2.1)

with $f(\cdot, \cdot) \in L^1(\mathbb{T}^2)$.

In this section, we consider the positivity estimation for the linear equation
\[
u_{tt} - \nu_{xx} + cu_t + a(t, x)u = h(t, x), \quad \text{in} \quad D'(\mathbb{T}^2).
\] (2.2)

Let $c$ is a constant and $c > 0$. The linear differential operator is defined as
\[ Lu = u_{tt} - u_{xx} + cu_t \]
acting on functions on the torus $\mathbb{T}^2$, $u : \mathbb{T}^2 \to \mathbb{R}$. We define the formal adjoint operator
\[ L^*u = u_{tt} - u_{xx} - cu_t. \]

Given $\sigma \in \mathbb{R}$ and $h \in L^1(\mathbb{T}^2)$, we consider the linear problem
\[ Lu - \sigma u = h(t, x), \quad \text{in} \quad D'(\mathbb{T}^2). \] (2.3)

A solution of (2.3) is a function $u \in L^1(\mathbb{T}^2)$ satisfying
\[ \int_{\mathbb{T}^2} u(L^*\phi - \sigma \phi) dt \ dx = \int_{\mathbb{T}^2} h \phi dt \ dx, \quad \forall \phi \in D(\mathbb{T}^2). \]

Let $L_\sigma$ be the differential operator
\[ L_\sigma u = Lu - \sigma u = u_{tt} - u_{xx} + cu_t - \sigma u \]
acting on functions on $\mathbb{T}^2$. By [3,4], if $\sigma < 0$, then $L_\sigma$ has the resolvent $R_\sigma$
\[ R_\sigma : L^1(\mathbb{T}^2) \to C(\mathbb{T}^2), \quad h \mapsto u, \]
where $u$ is the unique solution of Eq. (2.3), and the restriction of $R_\sigma$ on $L^p(\mathbb{T}^2)(1 < p < \infty)$ or $C(\mathbb{R})$ is compact. In particular, $R_\sigma : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ is a completely continuous operator.
For $\sigma = \frac{c^2}{4}$, the Green’s function of the differential operator $L_\sigma$ can be explicitly expressed, which has been obtained in [3]. We denote it by $G(t, x)$. By [Lemma 5.1 in [3]], $G$ is doubly $2\pi$-periodic, and given $h \in L^1(T^2)$, the unique solution of (2.1) can be represented by convolution product

$$u(t, x) = (R_\sigma h)(t, x) = \int_{T^2} G(t - s, x - y) h(s, y) ds dy. \quad (2.4)$$

The expression of $G(t, s)$ will be given in the following.

Let $D = \mathbb{R}^2 \setminus C$, where $C$ is the family of lines

$$x \pm 2k\pi, \quad k \in \mathbb{Z}.$$ 

Let $D_{ij}$ denote the connected component of $D$ with center at $(i\pi, j\pi)$, where $i + j$ is an odd number. By periodicity, the value of $G$ on $D_{10}$ and $D_{01}$ completely determines the value of $G$ on the whole set $D$. In $D_{10}$ and $D_{01}, G(t, x)$ is explicitly given by

$$G(t, x) = \begin{cases} \gamma_{10} e^{-ct/2}, & (t, x) \in D_{10}, \\ \gamma_{01} e^{-ct/2}, & (t, x) \in D_{01}, \end{cases} \quad (2.5)$$

where

$$\gamma_{10} = \frac{1}{2} \frac{1 + e^{-c\pi}}{(1 - e^{-c\pi})^2}, \quad \gamma_{01} = \frac{1}{2} \frac{1 + e^{-c\pi}}{(1 - e^{-c\pi})^2}.$$ 

See [3], Lemma 5.2.

From (2.5), we have

$$\underline{G} := \text{ess inf} \ G(t, x) = \frac{e^{-3\pi/2}}{(1 - e^{-c\pi})^2}, \quad (2.6)$$

$$\overline{G} := \text{ess sup} \ G(t, x) = \frac{1}{2} \frac{1 + e^{-c\pi}}{(1 - e^{-c\pi})^2}. \quad (2.7)$$

We will use the following two lemmas to simplify the proofs of our existence theorems. More importantly, the monotonicity assumptions of the nonlinearities can be relaxed.

Let $\delta > 0, f : T^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous. We define two new functions: $f^{\min}(t, x, z) : T^2 \times [0, \delta) \to \mathbb{R}^+$ and $f^{\max}(t, x, z) : T^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f^{\min}(t, x, z) = \min \{ f(t, x, u) : u \in \mathbb{R}^+, \ z \leq u(t, x) \leq \delta \text{ and } (t, x) \in T^2 \}$$

and

$$f^{\max}(t, x, z) = \max \{ f(t, x, u) : u \in \mathbb{R}^+, \ u(t, x) \leq z \text{ and } (t, x) \in T^2 \}.$$ 

It is clear that both $f^{\min}$ and $f^{\max}$ are nondecreasing.

The proof of the following two lemmas can be found in Hai [21].

**Lemma 2.1.** ([21]) If

$$f(t, x, u) > 0 \quad \text{for } 0 < u, (t, x) \in T^2,$$

and

$$\lim_{u \to 0^+} \frac{f(t, x, u)}{u} = \infty, \quad u \in \mathbb{R}^+, (t, x) \in T^2,$$

then

$$\lim_{\varepsilon \to 0^+} \frac{f^{\min}(t, x, z)}{z} = \infty.$$
Lemma 2.2. ([22,21]) Let \( u \in \mathbb{R}^+ \) and \((t, x) \in \mathbb{T}^2\), then assume \( \lim_{u \to 0^+} \frac{f(t, x, u)}{u} \) and \( \lim_{u \to 0^+} \frac{f(t, x, u)}{u} \) exist (can be infinity). Then

\[
\lim_{t \to 0^+} \frac{f_{\max}(t, x, z)}{z} = \lim_{u \to 0^+} \frac{f(t, x, u)}{u} \quad \text{and} \quad \lim_{t \to \infty} \frac{f_{\max}(t, x, z)}{z} = \lim_{u \to \infty} \frac{f(t, x, u)}{u}.
\]

To prove our results, we need the following fixed-point theorem of Schauder.

Lemma 2.3. ([23] (Schauder)) Let \( X \) be a Banach space and \( D \subseteq X \) be a bounded, convex and closed subset. Assume that \( T : D \to D \) is completely continuous, then \( T \) has a fixed point in \( D \).

3 Existence and Nonexistence

We assume the following conditions:

(H1) \( a \in C(\mathbb{T}^2), \; 0 \leq a(t, x) \leq c^2/4 \) for \((t, x) \in \mathbb{R}^2\), and \( \int_{\mathbb{T}^2} a(t, x) dtdx > 0 \).
(H2) \( f \in C(\mathbb{T}^2 \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+) \).
(H3) For all \((t, x) \in \mathbb{T}^2\), \( \lim_{u \to 0^+} \frac{f(t, x, u)}{u} = \infty \), where \( u \in \mathbb{R}^+ \).
(H4) For all \((t, x) \in \mathbb{T}^2\), \( \lim_{u \to \infty} \frac{f(t, x, u)}{u} = 0 \), where \( u \in \mathbb{R}^+ \).
(H5) For all \((t, x) \in \mathbb{T}^2\), \( \lim_{u \to 0^+} \frac{f(t, x, u)}{u} < \infty \), \( \lim_{u \to \infty} \frac{f(t, x, u)}{u} < \infty \), where \( u \in \mathbb{R}^+ \).

Let \( E \) denote the Banach space \( C(\mathbb{T}^2) \). Hereafter, we simply use \( \| \cdot \| \) to denote the norm in Banach \( E \) and \( \| \cdot \|_p \) to denote the norm in \( L^p(\mathbb{T}^2) \).

For each \( v \in E \), define \( v = A_\lambda v \) by

\[
\begin{cases}
  u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, v) & (t, x) \in \mathbb{T}^2, \\
  u(t + \pi, x) = u(t, x + \pi) = u(t, x) & (t, x) \in \mathbb{T}^2.
\end{cases}
\]

(3.1)

Then \( A_\lambda : E \to E \) is well defined, completely continuously, and fixed points of \( A_\lambda \) are solutions of (1.1) (see, e.g., [7]).

Theorem 3.1. Assume (H1)-(H3) hold. Then there is \( \lambda_0 > 0 \) such that (1.1) admits a nontrivial nonnegative doubly periodic solution for \( 0 < \lambda < \lambda_0 \).

Proof. By condition (H3), we have
\[
\lim_{u \to 0^+} \frac{f(t, x, u)}{u} = \infty,
\]
for \( u \in \mathbb{R}^+ \) and \((t, x) \in T^2\). We can choose \( \delta > 0 \) to get
\[
 f(t, x, u) > 0 \quad \text{for} \quad 0 < u(t, x) \leq \delta, \quad (t, x) \in T^2.
\]
Let \( \lambda_0 = \frac{M}{\int_{T^2} dsdy} > 0 \) and
\[
 M = \sup \{ f(t, x, u) : u(t, x) \leq \delta, \quad u \in \mathbb{R}^+, \quad (t, x) \in T^2 \} > 0. \quad (3.2)
\]

We now only consider \( 0 < \lambda < \lambda_0 \). Define a function \( f_{\min}(t, x, z) : T^2 \times [0, \delta] \to [0, \infty) \) by
\[
 f_{\min}(t, x, z) = \min \{ f(t, x, u) : u \in \mathbb{R}^+, \quad z \leq u(t, x) \leq \delta \quad \text{and} \quad (t, x) \in T^2 \}.
\]

According to Lemma 2.1, condition (H3) implies
\[
 \lim_{z \to 0^+} \frac{f_{\min}(t, x, z)}{z} = \infty.
\]

Therefore, for each \( 0 < \lambda < \lambda_0 \), there exists a positive \( \epsilon_1 < \delta \) such that
\[
 f_{\min}(t, x, \alpha) \geq \frac{\lambda_1}{\lambda} \alpha \quad (3.3)
\]
if \( 0 < \alpha \leq \epsilon_1 \) and \( \lambda_1 = \frac{1}{\int_{T^2} dsdy} \).

Now choose an \( \epsilon \) such that \( 0 < \epsilon < \epsilon_1 \). We define a subset \( K \) of \( E \) by
\[
 K = \{ u \in E : \epsilon \leq u(t, x) \leq \delta, \quad \forall (t, x) \in T^2 \}
\]
for each \( 0 < \lambda < \lambda_0 \). Note that \( \epsilon < \epsilon_1 < \delta \). It is easy to verify that \( K \) is a closed, bounded, convex subset of \( E \). We show that \( A_\lambda : K \to K \), in other words, \( u = A_\lambda v \) for \( v \in K \) we can obtain \( u \in k \).

First, since \( \epsilon \leq v(t, x) \leq \delta \), (3.1) and (3.2), we have
\[
 u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, v) \leq \lambda M \quad \forall (t, x) \in T^2,
\]
by (2.4), (2.7) and (3.4), which implies,
\[
 u(t, x) = \int_{T^2} G(t - s, x - y)h(s, y)dsdy
 = \int_{T^2} G(t - s, x - y)\lambda f(s, y, v)dsdy
 \leq \lambda M \int_{T^2} G(t - s, x - y)dsdy
 \leq \lambda M \int_{T^2} dsdy
 < \lambda_0 \lambda M \int_{T^2} dsdy < \delta.
\]

Finally, by the definition of \( f_{\min} \), we obtain
\[
 u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, v) \geq \lambda f_{\min}(t, x, \epsilon).
\]

By the choice of \( \epsilon \) and (3.3), for each \( 0 < \lambda < \lambda_0 \), we have
\[
 u_{tt} - u_{xx} + cu_t + a(t, x)u \geq \lambda_1 \epsilon \geq \lambda_1 \epsilon. \quad (3.5)
\]
By (2.4), (2.6) and (3.5), so that
\[ u(t, x) = \int_{\mathbb{T}^2} G(t - s, x - y)h(s, y)dsdy \]
\[ \geq \int_{\mathbb{T}^2} G(t - s, x - y)\lambda_1 dsdy \]
\[ \geq \lambda_1 \mathcal{G} e^\int_{\mathbb{T}^2} dsdy = \epsilon. \]

Hence, \( u \in K \) and \( A : K \to K \). By standard methods and Arzelà-Ascoli theorem, the complete continuity of \( A : K \to K \) is obvious. So it is omitted. For each \( 0 < \lambda < \lambda_0 \), according to Lemma 2.3, \( A \) admits a fixed point in \( K \), which is the desired nontrivial doubly periodic solution of (1.1).

**Example 3.2.** Consider the following problem:
\[
\begin{cases}
  u_{tt} - u_{xx} + cu_t + \sin^2(t + x)u = \lambda e^u, & (t, x) \in \mathbb{T}^2, \\
  u(t + \pi, x) = u(t, x + \pi) = u(t, x), & (t, x) \in \mathbb{T}^2.
\end{cases}
\]

It is clear to see that all conditions of Theorem 3.1 hold.

**Theorem 3.3.** Assume (H1)-(H4) hold and suppose that, for \( f(t, x, u) \) in (H3),
\[ f(t, x, u) > 0 \quad \text{for} \ 0 < u(t, x), \ (t, x) \in \mathbb{T}^2. \]
Then (1.1) admits a nontrivial nonnegative doubly periodic solution for all \( \lambda > 0 \).

**Proof.** We define a function \( f_{\max} : \mathbb{T}^2 \times [0, \infty) \to [0, \infty) \) by
\[ f_{\max}(t, x, z) = \min \{ f(t, x, u) : u \in \mathbb{R}^+, u(t, x) \leq z \text{ and } (t, x) \in \mathbb{T}^2 \}. \]

In consideration of Lemma 2.2, The theorem has
\[ \lim_{u \to \infty} \frac{f(t, x, u)}{u} = 0, \quad u \in \mathbb{R}^+ \text{ and } (t, x) \in \mathbb{T}^2, \]
then
\[ \lim_{z \to \infty} \frac{f_{\max}(t, x, z)}{z} = 0. \]

We can choose a sufficient large \( \delta > 0 \) so that
\[ \frac{f_{\max}(\delta)}{\delta} < \eta, \quad (3.6) \]
where \( \eta > 0 \) satisfying
\[ \lambda \eta \mathcal{G} e^\int_{\mathbb{T}^2} dsdy \leq 1. \quad (3.7) \]

With this \( \delta \), we define a function \( f_{\min}(t, x, z) : \mathbb{T}^2 \times [0, \delta] \to [0, \infty) \) by
\[ f_{\min}(t, x, z) = \min \{ f(t, x, u) : u \in \mathbb{R}^+, z \leq u(t, x) \leq \delta \text{ and } (t, x) \in \mathbb{T}^2 \}. \]

In view of Lemma 2.1, condition (H3) implies
\[ \lim_{z \to 0^+} \frac{f_{\min}(t, x, z)}{z} = \infty. \]

There exists a positive \( \epsilon_1 < \delta \) such that
\[ f_{\min}(\alpha) \geq \frac{\lambda_1}{\lambda} \alpha \quad (3.8) \]

76
if $0 < \alpha \leq \epsilon_1$ and $\lambda_1 = \frac{1}{\Delta_1} \int_{\Omega} dy$. 

Now choose a positive $\epsilon$ such that $0 < \epsilon < \epsilon_1 < \delta$. We now define a subset $K$ of $E$ by

$$K = \{ u \in E : \epsilon \leq u(t, x) \leq \delta, \forall (t, x) \in \Omega^2. \}$$

Then $K$ is a closed, bounded, convex subset of $E$. We claim that $u = A \lambda : K \to K$ for $v \in K$, we can have $u \in K$. First, by the definition of $f_{\min}$ and (3.6),

$$\epsilon \leq v(t, x) \leq \delta,$$  

so we have

$$u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, v) \leq \lambda f_{\max}(t, x, \delta) \leq \lambda \eta \delta \quad \forall (t, x) \in \Omega^2,$$ \tag{3.9}

by (2.4), (2.7), (3.7) and (3.9), which implies,

$$u(t, x) = \frac{1}{\Delta_1} \int_{\Omega} G(t - s, x - y)h(s, y)dsdy \leq \frac{1}{\Delta_1} \int_{\Omega} G(t - s, x - y)\lambda \eta \delta dsdy \leq \lambda \eta \delta \int_{\Omega} G(t - s, x - y)dsdy \leq \lambda \eta \delta G \int_{\Omega} dsdy \leq \delta.$$

Finally, by the definition of $f_{\min}$ and the choice of $\epsilon, \epsilon_1$, we have

$$u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda f(t, x, v) \geq \lambda f_{\min}(t, x, \epsilon) \geq \frac{\lambda}{\lambda_1 \epsilon} \geq \lambda_1 \epsilon. \tag{3.10}$$

In a similar way, we also obtain

$$u(t, x) \geq \epsilon.$$

In conclusion, $u \in K$ and $A \lambda : K \to K$. By standard methods and Arzelà-Ascoli theorem, the complete continuity of $A \lambda : K \to K$ is obvious. So it is omitted. By Lemma 2.3, $A \lambda$ admits a fixed point in $K$, which is the desired nontrivial doubly periodic solution of (1.1).

**Example 3.4.** Consider the following problem:

$$\left\{ \begin{array}{ll}
 u_{tt} - u_{xx} + cu_t + \sin^2(t + x)u = \lambda u^\frac{1}{2} & (t, x) \in \Omega^2, \\
 u(t + \pi, x) = u(t, x + \pi) = u(t, x) & (t, x) \in \Omega^2.
\end{array} \right.$$  

It is clear to see that all conditions of Theorem 3.2 hold.

The following will allow us to establish a nonexistence theorem:

**Theorem 3.5.** Assume (H1)-(H2) and (H5) hold. Then there is $\lambda_0 > 0$ such that (1.1) admits no nontrivial doubly periodic solution for $0 < \lambda < \lambda_0$.

**Proof.** By conditions (H1)-(H2) and (H5), there exists a constant $C > 0$ such that

$$f(t, x, u) \leq Cu \quad u \in \mathbb{R}^+, \quad (t, x) \in \Omega^2. \tag{3.11}$$

Choose $\lambda_0 > 0$ so that

$$\lambda_0 \Delta_1 \int_{\Omega} dsdy < 1.$$

Now assume $v \in K$ is a nontrivial solution of (1.1). We will prove that this leads to a contradiction if $0 < \lambda < \lambda_0$. For (3.11), $0 < \lambda < \lambda_0$ and $(t, x) \in \Omega^2$

$$v_{tt} - v_{xx} + cv_t + a(t, x)v = \lambda f(t, x, v) \leq \lambda Cv. \tag{3.12}$$

77
By (2.4), (2.7) and (3.12), then
\begin{align*}
v(t, x) &= \int_{T^2} G(t-s, x-y) \lambda f(t, x, v) ds dy \\
&\leq \lambda C \|v\| \int_{T^2} G(t-s, x-y) ds dy \\
&\leq \lambda C \|v\| G \int_{T^2} ds dy < \alpha \|v\|,
\end{align*}
where \(\alpha = \lambda_0 CG \int_{T^2} ds dy\). Thus
\[\|v\| \leq \alpha \|v\|,\]
which is a contradiction since \(\alpha < 1\).

**Example 3.6.** Consider the following problem:
\[
\begin{cases}
  u_{tt} - u_{xx} + cu_t + \sin^2(t + x)u = \lambda ue^{-u} & (t, x) \in T^2, \\
  u(t + \pi, x) = u(t, x + \pi) = u(t, x) & (t, x) \in T^2.
\end{cases}
\]

It is clear to see that all conditions of Theorem 3.3 hold.

4 Conclusions

Assume that (H1)-(H3) hold, then there exists \(\lambda_0 > 0\) such that (1.1) admits a nontrivial nonnegative doubly periodic solution for \(0 < \lambda < \lambda_0\). Assume (H1)-(H4) hold and suppose that, for \(f(t, x, u)\) in (H3),
\[f(t, x, u) > 0 \text{ for } 0 < u(t, x), (t, x) \in T^2,\]
then (1.1) admits a nontrivial nonnegative doubly periodic solution for all \(\lambda > 0\). Assume (H1)-(H2) and (H5) hold, then there is \(\lambda_0 > 0\) such that (1.1) admits no nontrivial doubly periodic solution for \(0 < \lambda < \lambda_0\).

Acknowledgement

Thanks to my teacher’s help in research, modification and so on, I can complete this paper. At the same time, I would like to thank Beijing Information Science & Technology University for providing the good learning conditions.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Kim WS. Double-periodic boundary value problem for nonlinear dissipative hyperbolic equations. J. Math. Anal. Appl. 1990;145(1):1-16.

[2] Kim WS. Multiple doubly periodic solutions of semilinear dissipative hyperbolic equations. J. Math. Anal. Appl. 1996;197(3):735-748.
[3] Ortega R, Robles-Perez AM. A maximum principle for periodic solutions of the telegraph equations. J. Math. Anal. Appl. 1998;221(2):625-651.

[4] Li Y. Positive doubly periodic solutions of nonlinear telegraph equations. Nonlinear Analysis. 2003;55(3):245-254.

[5] Li Y, Zhang H. Positive doubly periodic solutions of telegraph equations with delays. Boundary Value Problems. 2015;1:1-12.

[6] Jang TS. A new solution procedure for the nonlinear telegraph equation. Communications in Nonlinear Science and Numerical Simulation. 2015;29:307-326.

[7] Mollahasani N, Moghadam M, Afroz K. A new treatment based on hybrid functions to the solution of telegraph equations of fractional order. Applied Mathematical Modelling. 2016;40(4):2804-2814.

[8] Wang FL, An YK. Doubly periodic solutions to a coupled telegraph system. Nonlinear Analysis: Theory, Methods and Applications. 2012;75(4):1887-1894.

[9] Fucik S, Mawhin J. Generalized periodic solutions of nonlinear telegraph equations. Nonlinear Analysis. 1978;2(5):609-617.

[10] Li Y. Maximum principles and the method of upper and lower solutions for time-periodic problems of the telegraph equations. Journal of Mathematical Analysis and Applications. 2007;327(2):997-1009.

[11] Mawhin J, Ortega R, Robles-Perez AM. Maximum principles for bounded solutions of the telegraph equation in space dimensions two and three and applications. Journal of Differential Equations. 2005;208(1):42-63.

[12] Yao H, Lin Y. New algorithm for solving a nonlinear hyperbolic telegraph equation with an integral condition. International Journal for Numerical Methods in Biomedical Engineering. 2011;27(10):1558-1568.

[13] Zhurov AI, Polyanin AD. Symmetry reductions and new functional separable solutions of nonlinear Klein-Gordon and telegraph type equations. Journal of Nonlinear Mathematical Physics. 2020;27(2):227-242.

[14] Wang F, An Y. Multiple positive doubly periodic solutions for a singular semipositone telegraph equation with a parameter. Boundary Value Problems. 2013;2013(1):1-8.

[15] Graef JR, Kong L. Uniqueness and parameter dependence of positive doubly periodic solutions of nonlinear telegraph equations. Opuscula Mathematica. 2014;34(2):363-373.

[16] Wang FL, An YK. Three positive doubly periodic solutions of a nonlinear telegraph system. Applied Mathematics and Mechanics. 2009;30(1):81-88.

[17] Wang FL, An YK. Nonnegative doubly periodic solutions for nonlinear telegraph system. J. Math. Anal. Appl. 2008;338(1):91-100.

[18] Wang FL, An YK. Existence and multiplicity results of positive doubly periodic solutions for nonlinear telegraph system. J. Math. Anal. Appl. 2009;349(1):30-42.

[19] Wang FL, An YK. Nonnegative doubly periodic solutions for nonlinear telegraph system with twin-parameters. Appl. Math. Comput. 2009;214(1):310-317.

[20] Wang FL. Doubly periodic solutions of a coupled nonlinear telegraph system with weak singularities. Nonlinear Anal. Real World Appl. 2011;12(1):254-261.

[21] Hai DD, Wang HY. Nontrivial solutions for p-Laplacian systems. J. Math. Anal. Appl. 2007;330(1):186-194.
[22] Wang H. On the number of positive solutions of nonlinear systems. J. Math. Anal. Appl. 2003;281(1):287-306.

[23] Guo D, Lakshmikantham V. Nonlinear Problems in Abstract Cones. Academic Press. New York. 1988.

© 2021 Deng; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/73511