KÄHLER POTENTIALS AND RENORMALIZATION GROUP FLOWS IN N=2 LANDAU-GINZBURG MODELS

M.T. GRISARU
Physics Department, Brandeis University, Waltham, MA 02254, USA

and

D. ZANON
Dipartimento di Fisica dell’Università di Milano and INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy

ABSTRACT

We examine the conditions for superconformal invariance and the specific form of the Kähler potential for a two-dimensional lagrangian model with $N = 2$ supersymmetry and superpotential $gX^k$. Away from the superconformal point we study the renormalization group flow induced by a particular class of Kähler potentials. We find trajectories which, in the infrared, reach the fixed point with a central charge whose value is that of the $N = 2, A_{k-1}$ minimal model.

March 1994

* Work partially supported by the National Science Foundation under grant PHY-92-22318.
In recent years the Landau-Ginzburg description of $N = 2$ supersymmetric theories in two-dimensions has been studied extensively [1, 2, 3, 4, 5, 6, 7]. It is generally accepted that these models, with given superpotentials and suitable, though unspecified Kähler potentials, describe renormalization group flows toward infrared fixed points which can be identified with $N = 2$ minimal models. Along the flow trajectories the $N = 2$ nonrenormalization theorem ensures that the form of the superpotentials remains unchanged while the Kähler potentials adjust themselves in such a way that at the fixed points the resulting actions describe superconformally invariant systems. However, to the best of our knowledge, no explicit lagrangian models have been constructed which exhibit this behaviour. In this work we present such a model. (In a recent paper Fendley and Intrilligator [8] have studied $N=2$ flows in an exact S-matrix context.)

For simplicity we discuss primarily a system with a single chiral superfield $X$, and the Landau-Ginzburg superpotential $gX^k$. We first examine the situation at the fixed point and show that the condition for superconformal invariance determines the Kähler potential up to an overall constant. In fact the model is not conformal unless the supercurrent is suitably improved and the construction of such an improvement term is possible only if the Kähler potential has a specific form. (For models with more fields the condition is less restrictive.) We then extend the analysis off-criticality and consider a lagrangian whose RG trajectories admit an IR fixed point where the Kähler potential takes the above-mentioned specific form. At the IR critical point the lagrangian is the one used in [5] (and also, in its Liouville version in [6]), but in contradistinction to these references the normalization of the kinetic term is not arbitrary, but fixed by the flow equations.

Our model has the conventional appearance

$$\mathcal{S} = \int d^2x d^4\theta \ K(X, \bar{X}) + \int d^2x d^2\bar{\theta} \ W(X) + \int d^2x d^2\bar{\theta} \ \bar{W}(\bar{X})$$

(1)

We use the following notation

$$x_\pm = \frac{1}{\sqrt{2}}(x_0 + x_1), \quad \partial_\pm = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$$

$$x_- = \frac{1}{\sqrt{2}}(x_0 - x_1), \quad \partial_- = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1)$$

(2)

with

$$\Box \equiv \partial^\mu \partial_\mu = 2\partial_\pm \partial_\mp, \quad \partial_\pm \frac{1}{x_\pm} = 2\pi i\delta^{(2)}(x)$$

(3)

The superspace coordinates are $\theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-$, and the superspace covariant derivatives satisfy

$$\{D_+, \bar{D}_+\} = i\partial_+, \quad \{D_-, \bar{D}_-\} = i\partial_-$$

(4)

1
with all other anticommutators vanishing. We also freely interchange $\int d^2 \theta \leftrightarrow D_+ D_- \equiv D^2$ and $\int d^2 \bar{\theta} \leftrightarrow D_+ \bar{D}_- \equiv \bar{D}^2$. Finally, for a kinetic term $\int X \bar{X}$ the chiral field propagator is

$$< X(x, \theta) \bar{X}(0) > = -\frac{1}{2\pi} \bar{D}^2 D^2 \delta^{(4)}(\theta) \ln[m^2(2x_+ x_- + \ell^2)]$$

where $m$ and $\ell$ are infrared and ultraviolet cutoffs respectively.

We find it convenient to discuss questions of superconformal invariance by coupling the above system to linearized $N = 2$ supergravity. In doing so we include a chiral “dilaton” improvement term, so that the action takes the form

$$S = \int d^2 x d^4 \theta E^{-1} K[(1 + iH, \partial) X, (1 - iH, \partial) \bar{X}] + \int d^2 x d^2 \theta e^{2\sigma} W(X)$$

$$+ \int d^2 x d^2 \bar{\theta} e^{-2\bar{\sigma}} \bar{W}(\bar{X}) + \int d^2 x d^2 \bar{\theta} \bar{R} \bar{\Psi}(\bar{X})$$

where, at the linearized level,

$$E^{-1} = 1 - [\bar{D}_+, D_+] H_+ - [\bar{D}_-, D_-] H_-$$

$$R = 4\bar{D}_+ \bar{D}_- [\bar{\sigma} + D_+ \bar{D}_+ H_+ - \bar{D}_- D_- H_-]$$

$$\bar{R} = 4 D_+ D_- [\sigma - \bar{D}_+ D_+ H_- - \bar{D}_- D_- H_+]$$

Here $H$ is the supergravity potential, while $\sigma$ is the (chiral) compensator \cite{9}. 

The superconformal properties of the system are encoded in the supercurrent

$$J_+ \equiv \frac{\delta S}{\delta H_-} = 2[D_+ X \bar{D}_+ \bar{X} K_{XX} - 2 \bar{D}_+ D_+ \Psi + 2 D_+ \bar{D}_+ \bar{\Psi}]$$

and the associated supertrace

$$J \equiv \frac{\delta S}{\delta \sigma} = -2[W - 2 \bar{D}_+ \bar{D}_- \bar{\Psi}]$$

We have introduced the Kähler metric

$$K_{X\bar{X}} = \frac{\partial^2 K}{\partial X \partial \bar{X}}$$

Superconformal invariance requires the supertrace $J$ to vanish. For the superpotential $W = gX^k$ the equations of motion (with the notation $K_X = \partial_X K$, etc.)

$$\bar{D}_+ \bar{D}_- K_X + W_X = 0$$

\footnote{In ref. \cite{9} we were working in conformal gauge so that only the compensator $\sigma$ was present, but it is easy to include the $H$ field by solving the constraints at the linearized level; see also ref. \cite{10}.}
give \( W = -\frac{1}{k} \bar{D}_+ \bar{D}_- (X K_X) \). The condition \( J = 0 \) implies then
\[
X K_X = -2k \bar{\Psi}(\bar{X})
\] (12)
modulo a linear superfield (annihilated by \( \bar{D}_+, \bar{D}_- \)) which gives no contributions to the action. We have assumed that \( \bar{\Psi} \) is local, and (anti)chirality and dimensionality require it to be just a function of \( \bar{X} \). Integrating with respect to \( X \) and imposing also \( \bar{J} = 0 \) we find
\[
K = \alpha \ln X \ln \bar{X}
\]
\[
\bar{\Psi} = -\frac{\alpha}{2k} \ln \bar{X}
\] (13)
with arbitrary constant \( \alpha \). (With this solution the improvement terms in the super-current \( J_\pm \) can be rewritten as \( \frac{2}{k} [D_+ (X D_+ K_X) - D_+ (X D_+ K_X)] \).) Using the field redefinition \( X \equiv e^\Phi \) the corresponding lagrangian can be recast in Liouville form.

We note that for such a Kähler potential conformal invariance is not broken by quantum corrections since the one-loop \( \beta \)-function, proportional to the Ricci tensor
\[
R_{XX} = \partial_X \partial_{\bar{X}} \text{tr} \ln K_{XX},
\]
vanishes and all the higher-loop contributions which involve the Riemann tensor, trivially vanish as well. Moreover, while in the bosonic or in the \( N = 1 \) supersymmetric theories the dilaton term contributes to the metric \( \beta \)-function, in the \( N = 2 \) case no metric-dilaton mixing occurs due to the chirality of \( \bar{\Psi} \).

We describe briefly the situation for a model with two fields. For example, in the case of the superpotential
\[
W = X^{k+1} + XY^2
\] (14)
the construction of an improvement term is possible only if the Kähler potential satisfies
\[
X K_X + \frac{k}{2} Y K_Y = -2(k + 1) \bar{\Psi}(\bar{X}, \bar{Y})
\] (15)
As a partial differential equation in \( X \) and \( Y \) this equation has many solutions, but these are severely restricted by requiring that the resulting metric be Ricci-flat. One finds in general the Kähler potential (assumed to be symmetric in chiral and antichiral fields)
\[
K(X, Y, \bar{X}, \bar{Y}) = \frac{A}{\nu^2} \left( \ln \frac{X}{Y} \ln \frac{\bar{X}}{\bar{Y}} \right)^\nu + B \ln XY \ln \bar{X} \bar{Y}
\] (16)
where \( A \neq 0, B \neq 0, \) and \( \nu \) are arbitrary constants. (For \( \nu = 0 \) the first term is replaced by the square of the logarithm of the expression in parantheses.) For \( \nu = 1 \) the field redefinitions
\[
\frac{X}{Y} \equiv e^{\Phi_1}, \quad XY \equiv e^{\Phi_2}
\] (17)
recast the lagrangian, including the superpotential, in Toda field theory form.

We consider now a model which flows in the IR region to the superconformal theory defined above. It is described by the superpotential $gX^k$ and the Kähler metric

$$K_{X\bar{X}} = \frac{1}{1 + bX\bar{X} + c(X\bar{X})^2}$$

(18)

corresponding to the Kähler potential

$$K = \int dXd\bar{X}K_{XX} = X\bar{X} - \frac{b}{4}(X\bar{X})^2 + \frac{b^2 - c}{9}(X\bar{X})^3 + \cdots$$

(19)

The divergences of the model require renormalization of the parameters $b$, $c$, and wave-function renormalization. However, it is convenient to rescale the field, $X \to a^{-\frac{1}{2}}X$, so that the Kähler metric and superpotential become (with a redefinition of the parameter $c$)

$$K_{X\bar{X}} = \frac{1}{a + bX\bar{X} + c(X\bar{X})^2} \ , \ ga^{-\frac{1}{2}}X^k$$

(20)

(A related metric, with $a = c$, has been discussed in a bosonic $\sigma$-model context by Fateev et al [11]. The authors of ref. [8] have speculated on the relevance of such metrics for studying N=2 flows.)

The model is rendered finite in $\sigma$-model fashion by renormalizing the metric including the parameter $a$ (this is equivalent to wave-function renormalization) and therefore, because the superpotential is not renormalized, the coupling constant $g$. At the one-loop level one finds the divergent contribution, proportional to the Ricci tensor,

$$-\left(\frac{1}{2\pi}\ln m^2\ell^2\right)R_{X\bar{X}} = \left(\frac{1}{2\pi}\ln m^2\ell^2\right)\frac{ab + 4acXX + bc(X\bar{X})^2}{[a + bX\bar{X} + c(X\bar{X})^2]^2}$$

(21)

This divergence can be cancelled by expressing the original parameters in the classical lagrangian in terms of renormalized ones

$$a = Z_a a_R \ , \ b = Z_b b_R \ , \ c = Z_c c_R$$

$$g = \mu Z_g g_R$$

(22)
with $\mu$ the mass scale, and $Z_g Z_a^{-\frac{1}{2}} = 1$ as required by the $N = 2$ nonrenormalization theorem. The renormalization constants are

$$
Z_a = 1 + b\left(\frac{1}{2\pi} \ln \mu^2 \ell^2\right)
$$

$$
Z_b = 1 + \frac{4ac}{b} \left(\frac{1}{2\pi} \ln \mu^2 \ell^2\right)
$$

$$
Z_c = 1 + b\left(\frac{1}{2\pi} \ln \mu^2 \ell^2\right)
$$

$$
Z_g = 1 + \frac{bk}{2} \left(\frac{1}{2\pi} \ln \mu^2 \ell^2\right)
$$

(23)

In the following we shall drop the subscript on the renormalized parameters.

Defining $t = \ln \mu$, the renormalized parameters satisfy the following renormalization group equations

$$
\frac{da}{dt} = -\frac{1}{\pi} ab
$$

$$
\frac{db}{dt} = \frac{4}{\pi} ac
$$

$$
\frac{dc}{dt} = -\frac{1}{\pi} cb
$$

$$
\frac{dg}{dt} = -(1 + \frac{b}{2\pi} k) g
$$

(24)

These equations have two invariants, the ratio

$$
\frac{a}{c} = \rho
$$

(25)

and the combination, which we choose to make positive and parametrize suitably,

$$
b^2 - 4ac = b^2 - 4\rho c^2 = (\pi \lambda)^2
$$

(26)

In the $b$-$c$ plane we obtain two types of trajectories, hyperbolas or ellipses, depending on the sign of $\rho$, as depicted in Fig. 1.
Since we are interested in trajectories with two fixed points we write the elliptical solutions, with $\rho < 0$. (The bosonic model studied in [11], written in a different coordinate system, has $\rho = 1$.)

$$
\begin{align*}
    b(t) & = \pi \lambda \tanh \lambda t \\
    a(t) & = \pm \frac{\pi \lambda \sqrt{-\rho}}{2} (\cosh \lambda t)^{-1} \\
    c(t) & = \mp \frac{\pi \lambda}{2 \sqrt{-\rho}} (\cosh \lambda t)^{-1} \\
    g(t) & = g_0 e^{-t}[\cosh \lambda t]^{-\frac{1}{k}}
\end{align*}
$$

Conformal invariance is achieved at the zeroes of the coupling $\beta$-functions in eq.(24). In particular we are looking for trajectories which lead to a nontrivial IR fixed point for the coupling constant $g$, i.e. such that $b(t) \to -\frac{2\pi}{k}$ as $t \to -\infty$. We achieve this by choosing

$$
\lambda = \frac{2}{k} \quad (28)
$$

In this case the superfield $a^{\frac{1}{2}} X$ acquires anomalous dimension $1/k$ in the corresponding IR conformal theory, while $a$ and $c$ flow to zero. Therefore, the effective lagrangian with Kähler potential $K(X, \bar{X}, a(t), b(t), c(t))$ and superpotential $W(X, g(t), a(t))$ has the following behaviour in the infrared,

$$
    t \to -\infty \quad , \quad K(t) \to -\frac{k}{2\pi} \ln X \ln \bar{X} \quad , \quad W(t) \to g_0 X^k \quad (29)
$$

The improvement term at the IR fixed point has $\Psi = \frac{1}{4\pi} \ln X$. 

Changing variables, $X \equiv e^\Phi$, leads to the Liouville lagrangian

$$\mathcal{L} = -\frac{k}{2\pi} \Phi \Phi + g_0 e^{k \Phi}$$

(30)

with negative kinetic term and with normalization determined by the superpotential (cf. [5, 6]).

We emphasize that imposing conformal invariance at the one-loop level, i.e.

$R_{X\bar{X}} = 0$, is sufficient to insure the absence of divergences at higher-loop orders. Thus at the conformal point we obtain exact, all-order results. (In these models there are no “nonperturbative” divergences.)

We briefly describe a generalization which allows us to discuss the stability of the IR fixed point. We consider a model with Kähler metric

$$K_{X\bar{X}} = \frac{1}{\sum a_n (XX)^n}$$

(31)

It is easy to verify that if the sum in the denominator is finite (but contains more than the first three terms considered in eq. (20)) the model is not renormalizable by a redefinition of the parameters. If the sum is infinite one computes the one-loop divergence proportional to the Ricci tensor and after renormalization one is led to the flow equations

$$\begin{align*}
\frac{dg}{dt} &= -(1 + \frac{a_1}{2\pi} k) g \\
\frac{da_0}{dt} &= -\frac{1}{\pi} a_0 a_1 \\
\frac{da_1}{dt} &= -\frac{4}{\pi} a_0 a_2 \\
\frac{da_2}{dt} &= -\frac{1}{\pi} (a_1 a_2 + 9 a_0 a_3) \\
\frac{da_3}{dt} &= -\frac{1}{\pi} (4 a_1 a_3 + 16 a_0 a_4) \\
\frac{da_4}{dt} &= -\frac{1}{\pi} (3 a_2 a_3 + 9 a_1 a_4 + 25 a_0 a_5)
\end{align*}$$

(32)

etc. (In the last equation the presence of the term $a_2 a_3$ makes it obvious why the model is not renormalizable if one considers a finite sum with more than the original three terms.)

If we want a nontrivial IR fixed point for the coupling constant $g$, all the $a_n$ must flow to zero except for $a_1 = -\frac{2\pi}{k}$, so that we recover the special case studied above. Furthermore, by linearizing the flow equations around these values it is possible to determine that this fixed point is stable under perturbations with the $a_n \neq 0$. 

7
We compute now the central charge of our original model, from the coupling to supergravity. We are looking for contributions to the supergravity effective action of the form $R \Box^{-1} \bar{R}$. They can be determined by contributions to the $H_\pm$ self-energy, from which the covariant expression can be reconstructed. For one-loop contributions the relevant vertices are obtained from the coupling to $H_\pm$ in the Kähler potential while tree-level contributions come from the direct dilaton coupling in the improvement term.

We compute away from the fixed point, using an effective configuration-space propagator

$$< X(x, \theta) \bar{X}(x', \theta') > = \frac{-K^{XX}}{2 \pi} \bar{D}^2 D^2 \delta^{(4)}(\theta - \theta') \ln \{ m^2 [2(x - x')_\pm (x - x')_\pm + \ell^2] \}$$  \hspace{1cm} (33)

where $K^{XX}$ is the inverse of the Kähler metric (cf [12] eq. (3.13); additional terms, involving derivatives of the Kähler metric in the propagator do not give relevant contributions). The couplings to $H_\pm$ can be read from the action in eq. (6) or, equivalently, from the supercurrent. From the Kähler potential we have

$$2i \int d^4 \theta \ H_\pm D_+ X \bar{D}_+ \bar{X} K_{XX}$$  \hspace{1cm} (34)

giving the one-loop contribution

$$- \frac{1}{\pi^2} H_\pm \frac{\bar{D}_- D_- \bar{D}_+ D_+}{(x - x')^2_\pm} H_\pm$$  \hspace{1cm} (35)

The coupling from the dilaton term is

$$4 \int d^4 \theta (\bar{\Psi} - \Psi) \partial_\theta H_\pm$$  \hspace{1cm} (36)

and leads to the contribution

$$- \frac{16}{\pi} \bar{\Psi}_X K^{XX} \bar{X} H_\pm \frac{\bar{D}_- D_- \bar{D}_+ D_+}{(x - x')^2_\pm} H_\pm$$  \hspace{1cm} (37)

Using the relation between the Kähler potential and the improvement term at the fixed point we obtain then the total contribution

$$- \frac{1}{\pi^2} H_\pm \frac{\bar{D}_- D_- \bar{D}_+ D_+}{(x - x')^2_\pm} H_\pm \left( 1 - \frac{2}{k} \right) \Rightarrow \frac{1}{4 \pi} R \frac{1}{\Box} \bar{R} \left( 1 - \frac{2}{k} \right)$$  \hspace{1cm} (38)

and the correct central charge for the $N = 2$, $A_{k-1}$ minimal model,

$$c = 1 - \frac{2}{k}$$  \hspace{1cm} (39)
Returning to the flows in eq. (27) we note that moving away from the IR fixed point along the RG trajectory toward the UV region, we reach a value of $t$, namely

$$ t = \frac{k}{2} \ln \frac{X \bar{X}}{\sqrt{-\rho}} $$

where the effective Kähler metric becomes singular. Therefore even if the trajectories we have been considering display an UV fixed point we cannot actually reach it. Had we started past the singularity, in the UV region, we could follow the flow to the UV fixed point, $t \to +\infty$ and find

$$ K(t) \to \frac{k}{2\pi} \ln X \ln X, \quad W(t) \to g_0 X^k $$

It is interesting to observe that the central charge for this conformal field theory,

$$ c = 1 + \frac{2}{k} $$

(42)

corresponds to the value of $c$ for the analytic continuation of the N=2 minimal models.

Acknowledgment D. Zanon thanks the Physics Department of Harvard University for hospitality during the period when some of this work was done. We thank M. Bershadsky, M. Raciti and C. Vafa for useful discussions.

References

[1] D. Kastor, E. Martinec and S. Shenker, Nucl. Phys B316 (1989) 590.

[2] C. Vafa and N. Warner, Phys. Lett. 218B (1989) 51; W. Lerche, C. Vafa and N. Warner, Nucl. Phys B324 (1989) 427.

[3] P.S. Howe and P.C. West, Phys. Lett. B227 (1989) 397; B244 (1990) 270.

[4] S. Cecotti, L. Girardello and A. Pasquinucci, Nucl. Phys. B328 (1989) 701; Int. J. Mod. Phys. A6 (1991) 2427.

[5] A. Marshakov and A. Morozov, Phys. Lett. B235 (1990) 97.

[6] H.C. Liao and P. Mansfield, Phys. Lett. 255B (1991) 237.
[7] E. Witten, *On the Landau-Ginzburg description of N=2 minimal models*, IASSNS-HEP-93/10 preprint.

[8] P. Fendley and K. Intrilligator, Nucl. Phys. **B413** (1994) 653.

[9] M.T. Grisaru and D. Zanon, Phys. Lett. **184B** (1987) 209.

[10] A. Alnowaiser, Class. Quantum Grav. **7** (1990) 1033.

[11] V.A. Fateev, E. Onofri and Al.B. Zamolodchikov, Nucl. Phys. **B406** (1993) 521.

[12] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, Nucl. Phys. **B277** (1986) 388.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9403194v2