A Lloyd-model generalization: Conductance fluctuations in one-dimensional disordered systems

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We perform a detailed numerical study of the conductance $G$ through one-dimensional (1D) tight-binding wires with on-site disorder. The random configurations of the on-site energies $\epsilon$ of the tight-binding Hamiltonian are characterized by long-tailed distributions: For large $\epsilon$, $P(\epsilon) \sim 1/\epsilon^{1+\alpha}$ with $\alpha \in (0, 2)$. Our model serves as a generalization of 1D Lloyd’s model, which corresponds to $\alpha = 1$. First, we verify that the ensemble average $\langle - \ln G \rangle$ is proportional to the length of the wire $L$ for all values of $\alpha$, providing the localization length $\xi$ from $\langle - \ln G \rangle = 2L/\xi$. Then, we show that the probability distribution function $P(G)$ is fully determined by the exponent $\alpha$ and $\langle - \ln G \rangle$. In contrast to 1D wires with standard white-noise disorder, our wire model exhibits bimodal distributions of the conductance with peaks at $G = 0$ and 1. In addition, we show that $P(\ln G)$ is proportional to $G^\beta$, for $G \to 0$, with $\beta \leq \alpha/2$, in agreement to previous studies.

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I. INTRODUCTION AND MODEL

The recent experimental realizations of the so-called Lévy glasses [1] as well as “Lévy waveguides” [2] has refreshed the interest in the study of systems characterized by Lévy-type disorder (see for example Refs. [3–15]). That is, disorder characterized by random variables $\{\epsilon\}$ whose density distribution function exhibits a slowly decaying tail:

$$P(\epsilon) \sim \frac{1}{\epsilon^{1+\alpha}},$$

for large $x$, with $0 < \alpha < 2$ (this kind of probability distributions are known as $\alpha$-stable distributions [16]). In fact, the study of this class of disordered systems dates back to Lloyd [17], who studied spectral properties of a three-dimensional (3D) lattice described by a 3D tight-binding Hamiltonian with Cauchy-distributed on-site potentials [which corresponds to the particular value $\alpha = 1$ in Eq. (1)]. Since then, a considerable number of works have been devoted to the study of spectral, eigenfunction, and transport properties of Lloyd’s model in its original 3D setup [18–27] and in lower dimensional versions [26–43].

Of particular interest is the comparison between the one-dimensional (1D) Anderson model (1DAM) [44] and the 1D Lloyd’s model, since the former represents the most prominent model of disordered wires [45]. Indeed, both models are described by the 1D tight-binding Hamiltonian:

$$H = \sum_{n=1}^{L} \epsilon_n | n \rangle \langle n | - \nu_{n,n+1} | n \rangle \langle n + 1 | - \nu_{n,n-1} | n \rangle \langle n - 1 |;$$

where $L$ is the length of the wire given as the total number of sites $n$, $\epsilon_n$ are random on-site potentials, and $\nu_{n,m}$ are the hopping integrals between nearest neighbors (which are set to a constant value $\nu_{n,n\pm 1} = \nu$). However, while for the standard 1DAM (with white-noise on-site disorder $\langle \epsilon_n \epsilon_m \rangle = \sigma^2 \delta_{nm}$ and $\langle \epsilon_n \rangle = 0$) the on-site potentials are characterized by a finite variance $\sigma^2 = \langle \epsilon_n^2 \rangle$ (in most cases the corresponding probability distribution function $P(\epsilon)$ is chosen as a box or a Gaussian distribution), in the Lloyd’s model the variance $\sigma^2$ of the random on-site energies $\epsilon_n$ diverges since they follow a Cauchy distribution.

It is also known that the eigenstates $\Psi$ of the infinite 1DAM are exponentially localized around a site position $n_0$ [45]:

$$|\Psi_n| \sim \exp \left( - \frac{|n-n_0|}{\xi} \right);$$

where $\xi$ is the eigenfunction localization length. Moreover, for weak disorder ($\sigma^2 \ll 1$), the only relevant parameter for describing the statistical properties of the transmission of the finite 1DAM is the ratio $L/\xi$ [46], a fact known as single parameter scaling. The above exponential localization of eigenfunctions makes the transmission or dimensionless conductance $G$ exponentially small, i.e., [47]

$$\langle - \ln G \rangle = \frac{2L}{\xi};$$

thus, this relation can be used to obtain the localization length. Remarkably, it has been shown that Eq. (4) is also valid for the 1D Lloyd’s model [41] implying a single parameter scaling, see also [38].

It is also relevant to mention that studies of transport quantities through 1D wires with Lévy-type disorder, different from the 1D Lloyd’s model, have been reported. For example, wires with scatterers randomly
spaced along the wire according to a Lévy-type distribution were studied in Refs. [3, 4, 48, 49]. Concerning the conductance of such wires, a prominent result reads that the corresponding probability distribution function \( P(G) \) is fully determined by the exponent \( \alpha \) of the power-law decay of the Lévy-type distribution and the average (over disorder realizations) \( \langle -\ln G \rangle [48, 49] \); i.e., all other details of the disorder configuration are irrelevant. In this sense, \( P(G) \) shows universality. Moreover, this fact was already verified experimentally in microwave random waveguides [2] and tested numerically using the tight-binding model of Eq. (2) with \( \epsilon_n = 0 \) and off-diagonal Lévy-type disorder [50] (i.e., with \( \nu_{n,m} \) in Eq. (2) distributed according to a Lévy-type distribution).

It is important to point out that 1D tight-binding wires with power-law distributed random on-site potentials, characterized by power-laws different from \( \alpha = 1 \) (which corresponds to the 1D Lloyd’s model), have been scarcely studied; for a prominent exception see [41]. Thus, in this paper we undertake this task and study numerically the conductance though disordered wires defined as a generalization of the 1D Lloyd’s model as follows. We shall study 1D wires described by the Hamiltonian of Eq. (2) having constant hopping integrals, \( \nu_{n,n+1} = \nu = 1 \), and random on-site potentials \( \epsilon_n \) which follow a Lévy-type distribution with a long tail, like in Eq. (1) with \( 0 < \alpha < 2 \). We name this setup the 1DAM with Lévy-type on-site disorder. We note that when \( \alpha = 1 \) we recover the 1D Lloyd’s model.

Therefore, in the following section we shall show that (i) the conductance distribution \( P(G) \) is fully determined by the power-law exponent \( \alpha \) and the ensemble average \( \langle -\ln G \rangle \); (ii) for \( \alpha \leq 1 \) and \( \langle -\ln G \rangle \sim 1 \), bimodal distributions for \( P(G) \) with peaks at \( G \sim 0 \) and \( G \sim 1 \) are obtained, revealing the coexistence of insulating and ballistic regimes; and (iii) the probability distribution \( P(\ln G) \) is proportional to \( G^{\beta} \), for vanishing \( G \), with \( \beta \leq \alpha/2 \).

II. RESULTS AND DISCUSSION

Since we are interested in the conductance statistics of the 1DAM with Lévy-type on-site disorder we have to define first the scattering setup we shall use: We open the isolated samples described above by attaching two semi-infinite single channel leads to the border sites at opposite sides of the 1D wires. Each lead is also described by a 1D semi-infinite tight-binding Hamiltonian. Using the Heidelberg approach [51] we can write the transmission amplitude through the disordered wires as \( t = -2i \sin(k) W^T (E - \mathcal{H}_{\text{eff}})^{-1} W \), where \( k = \arccos(E/2) \) is the wave vector supported in the leads and \( \mathcal{H}_{\text{eff}} \) is an effective non-hermitian Hamiltonian given by \( \mathcal{H}_{\text{eff}} = H - e^{ik} WW^T \). Here, \( W \) is a \( L \times 1 \) vector that specifies the positions of the attached leads to the wire. In our setup, all elements of \( W \) are equal to zero except \( W_{11} \) and \( W_{L1} \) which we set to unity (i.e., the leads are attached to the wire with a strength equal to the inter-site hopping amplitudes: \( \nu = 1 \)). Also, we have fixed the energy at \( E = 0 \) in all our calculations, although the same conclusions are obtained for \( E \neq 0 \). Then, within a scattering approach to the electronic transport, we compute the dimensionless conductance as \( G = t[2] \).

First, we present in Fig. 1(a) the ensemble average \( \langle -\ln G \rangle \) as a function of \( L \) for the 1DAM with Lévy-type disorder for several values of \( \alpha \). It is clear from this figure that \( \langle -\ln G \rangle \propto L \) for all the values of \( \alpha \) we consider here. Therefore, we can extract the localization length \( \xi \) by fitting the curves \( \langle -\ln G \rangle \) vs. \( L \) with Eq. (4); see dashed lines in Fig. 1(a). This behavior should be contrasted to the case of 1D wires with off-diagonal Lévy-type disorder [53] which shows the dependence \( \langle -\ln G \rangle \propto L^{1/2} \) when \( \alpha = 1/2 \) at \( E = 0 \) [50].

Also, we have confirmed that the cumulants \( \langle \langle (\ln G)^k \rangle \rangle \) obey a linear relation with the wire length [41, 54], i.e.,

\[
\lim_{L \to \infty} \frac{\langle \langle (\ln G)^k \rangle \rangle}{L} = 2^k c_k ,
\]

where the coefficients \( c_k \), with \( c_1 \equiv \xi^{-1} \), characterize the Lyapunov exponent of a generic 1D tight-binding wire with on-site disorder. We have verified the above relation, Eq. (5), for \( k = 1, 2, \) and \( 3 \); as an example in Fig. 1(b) we present the results for \( \langle \langle (\ln G)^2 \rangle \rangle \) as a
The exponent $\alpha$ diagonal Lévy-type disorder

FIG. 2: (Color online) Conductance distribution $P(G)$ for the 1DAM with Lévy-type disorder (histograms). Each panel correspond to a fixed value of $\langle \ln G \rangle$: (a) $\langle \ln G \rangle = 20$, (b) $\langle \ln G \rangle = 2$, (c) $\langle \ln G \rangle = 1$, (d) $\langle \ln G \rangle = 2/3$, (e) $\langle \ln G \rangle = 1/2$, and (f) $\langle \ln G \rangle = 1/5$. In each panel we include histograms for several values of $\alpha$, where $\alpha$ increases in the arrow direction. $E = 0$ was used. Each histogram was calculated using $10^6$ disorder realizations. The red dashed lines are the theoretical predictions of $P(G)$ for the 1DAM with white noise disorder $P_{WN}(G)$ corresponding to the particular value of $\langle \ln G \rangle$ of each panel.

function of $L$ for different values of $\alpha$. The dashed lines are fittings of the numerical data (open dots) with the function $\langle \ln G \rangle \sim \ln L$, see Eq. (5), which can be used to extract the higher order coefficient $c_2$.

Now, in Fig. 2 we show different conductance distributions $P(G)$ for the 1DAM with Lévy-type on-site disorder for fixed values of $\langle \ln G \rangle$; note that fixed $\langle \ln G \rangle$ means fixed ratio $L/\xi$. Several values of $\alpha$ are reported in each panel. We can observe that for fixed $\langle \ln G \rangle$, by increasing $\alpha$ the conductance distribution evolves towards the $P(G)$ corresponding to the 1DAM with white noise disorder, $P_{WN}(G)$, as expected. The curves for $P_{WN}(G)$ are included as a reference in all panels of Fig. 2 as red dashed lines [55]. In fact, $P(G)$ already corresponds to $P_{WN}(G)$ once $\alpha = 2$.

We recall that for 1D tight-binding wires with off-diagonal Lévy-type disorder $P(G)$ is fully determined by the exponent $\alpha$ and the average $\langle \ln G \rangle$ [50]. It is therefore pertinent to ask whether this property also holds for diagonal Lévy-type disorder. Thus, in Fig. 3 we show $P(G)$ for the 1DAM with Lévy-type on-site disorder for several values of $\alpha$, where each panel corresponds to a fixed value of $\langle \ln G \rangle$. For each combination of $\langle \ln G \rangle$ and $\alpha$ we present two histograms (in red and black) corresponding to wires with on-site random potentials $\{\epsilon_n\}$ characterized by two different density distributions [57], but with the same exponent $\alpha$ of their corresponding power-law tails. We can see from Fig. 3 that for each value of $\alpha$ the histograms (in red and black) fall on the top of each other, which is an evidence that the conductance distribution $P(G)$ for the 1DAM with Lévy-type on-site disorder is invariant once $\alpha$ and $\langle \ln G \rangle$ are fixed; i.e., $P(G)$ displays a universal statistics.

Moreover, we want to emphasize the coexistence of insulating and ballistic regimes characterized, respectively, by the two prominent peaks of $P(G)$ at $G = 0$ and $G = 1$. This behavior, which is more evident for $\langle \ln G \rangle \sim 1$ and $\alpha \leq 1$ (see Figs. 2 and 3), is not observed in 1D wires with white-noise disorder (see for example the red dashed curves in Fig. 2). This coexistence of opposite transport regimes has been already reported in systems with anomalously localized states: 1D wires with obstacles randomly spaced according to Lévy-type density distribution [48, 50] as well as in the so-called random-mass Dirac model [58].

Finally, we study the behavior of the tail of the distribution $P(\ln G)$. Thus, using the same data of Fig. 3,
in Fig. 4 we plot \( P(\ln G) \). As expected, since \( P(G) \) is determined by \( \alpha \) and \( \langle -\ln G \rangle \), we can see that \( P(\ln G) \) is invariant once those two quantities (\( \alpha \) and \( \langle -\ln G \rangle \)) are fixed (red and black histograms fall on top of each other). Moreover, from Fig. 4 we can deduce a power-law behavior:

\[
P(\ln G) \propto G^\beta
\]

for \( G \to 0 \) when \( \alpha < 2 \). For \( \alpha = 2 \), \( P(\ln G) \) displays a log-normal tail (not shown here), expected for 1D systems in the presence of Anderson localization. Actually, the behavior (6) was already anticipated in [41] as \( P(G) \sim G^{-(2-\lambda)/2} \) for \( G \to 0 \) with \( \lambda < \alpha \); which in our study translates as \( P(\ln G) \propto G^{\lambda/2} \) (since \( P(\ln G) = G P(G) \)) with \( \lambda/2 = \beta \leq \alpha/2 \). Indeed, we have validated the last inequality in Fig. 5 where we report the exponent \( \beta \) obtained from power-law fittings of the tails of the histograms of \( P(\ln G) \). In addition, we have observed that the value of \( \beta \) depends on the particular value of \( \langle -\ln G \rangle \) characterizing the corresponding histogram of \( P(\ln G) \). Also, from Fig. 5 we note that \( \beta \approx \alpha/2 \) as the value of \( \langle -\ln G \rangle \) decreases.

III. CONCLUSIONS

In this work we have studied the conductance \( G \) through a generalization of Lloyd’s model in one dimension: We consider one-dimensional (1D) tight-binding wires with on-site disorder following a Lévy-type distribution, see Eq. (1), characterized by the exponent \( \alpha \) of the power-law decay. We have verified that different cumulants of the variable \( \ln G \) decrease linearly with the length wire \( L \). In particular, we were able to extract the eigenfunction localization length \( \xi \) from \( \langle -\ln G \rangle = 2L/\xi \). Then, we have shown some evidence that the probability distribution function \( P(G) \) is invariant, i.e., fully determined, once \( \alpha \) and \( \langle -\ln G \rangle \) are fixed; in agreement with other Lévy-disordered wire models [2, 48–50]. We have also reported the coexistence of insulating and ballistic regimes, evidenced by peaks in \( P(G) \) at \( G = 0 \) and \( G = 1 \); these peaks are most prominent and commensurate for \( \langle -\ln G \rangle \sim 1 \) and \( \alpha \leq 1 \). Additionally we have shown that \( P(\ln G) \) develops power-law tails for \( G \to 0 \), characterized by the power-law \( \beta \) (also invariant for fixed \( \alpha \) and \( \langle -\ln G \rangle \)) which, in turn, is bounded from above by \( \alpha/2 \). This upper bound of \( \beta \) implies that the smaller the value of \( \alpha \) the larger the probability to find vanishing conductance values in our Lévy-disordered wires.

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[1] P. Barthelemy, J. Bertolotti, and D. S. Wiersma, Nature 453, 495 (2008).
[2] A. A. Fernandez-Marin, J. A. Mendez-Bermudez, J. Carbonell, F. Cervera, J. Sanchez-Dehesa, and V. A. Gopar, Phys. Rev. Lett. 113, 233901 (2014).
[3] C. W. J. Beenakker, C. W. Groth, and A. R. Akhmerov, Phys. Rev. B 79, 024204 (2009).
[4] R. Burioni, L. Caniparoli, and A. Vezzani, Phys. Rev. E 81, 060101(R) (2010).
[5] A. Eisfeld, S. M. Vlaming, V. A. Malyshev, and J. Knoester, Phys. Rev. Lett. 105, 137402 (2010).
[6] J. Bertolotti, K. Vynck, L. Pattelli, P. Barthelemy, S. Lepri, D. S. Wiersma, Adv. Functional Materials 20, 965 (2010).
[7] P. Barthelemy, J. Bertolotti, K. Vynck, S. Lepri, D. S. Wiersma, Phys. Rev. E 82, 011101 (2010).
[8] M. Burresi, V. Radhalakshmi, R. Savo, J. Bertolotti, K. Vynck, and D. S. Wiersma, Phys. Rev. Lett. 108, 110604 (2012).
[9] C. W. Groth, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. E 85, 021138 (2012).
[10] R. Burioni, S. di Santo, S. Lepri, and A. Vezzani, Phys. Rev. E 86, 031125 (2012).
[11] S. M. Vlaming, V. A. Malyshev, A. Eisfeld, and J. Knoester, J. Chem. Phys. 138, 214316 (2013).
[12] R. Burioni, E. Ubaldi, and A. Vezzani, Phys. Rev. E 89, 022135 (2014).
[13] P. Bernabo, R. Burioni, S. Lepri, and A. Vezzani, Chaos Solitons Fractals 67, 11 (2014).
[14] S. S. Zakeri, S. Lepri, and D. S. Wiersma, Phys. Rev. E 91, 032112 (2015).
[15] A. G. Ardakani and M. G. Nezhadhaghighi, J. Opt. 17, 105601 (2015).
[16] V. V. Uchaikin and V. M. Zolotarev, Chance and Stability. Stable Distributions and their Applications (VSP, Utrecht, 1999).
[17] P. Lloyd, J. Phys. C 2, 1717 (1969).
[18] M. Saitoh, Phys. Lett. A 33, 44 (1970); Progr. Theor. Phys. 45, 746 (1971).
[19] A. P. Kumar and G. Baskaran, J. Phys. C 6, L399 (1973).
[20] K. Hoshino, Phys. Lett. A 56, 133 (1976).
[21] W. R. Bandy and A. J. Glick, Phys. Rev. B, 16, 2346 (1977).
[22] S. Kivelson and C. D. Gelatt, Phys. Rev. B 20, 4167 (1979).
[23] B. Simon, Phys. Rev. B 27, 3859 (1983).
[24] D. E. Rodrigues and J. F. Weisz, Phys. Rev. B 34, 2306 (1986).
[25] E. Kolley and W. Kolley, J. Phys. C 21, 6099 (1988).
[26] R. Johnston and H. Kunz, J. Phys. C 16, 4565 (1983).
[27] D. E. Rodrigues, H. M. Pastawski, and J. F. Weisz, Phys. Rev. B 34, 8545 (1986).
[28] D. J. Thouless, J. Phys. C 5, 77 (1972).
[29] K. Ishii, Suppl. Progr. Theor. Phys. 53, 77 (1973).
[30] R. Abou-Chacra and D. J. Thouless, J. Phys. C 7, 65 (1974).
[31] D. J. Thouless, J. Phys. C 16, L929 (1983).
[32] A. MacKinnon, J. Phys. C 17, L389 (1984).
[33] M. O. Robbins and B. Koiller, Phys. Rev. B 32, 4576 (1985).
[34] D. L. Shepelyansky, Phys. Rev. Lett. 56, 677 (1986).
[35] S. Fishman, R. E. Prange, and M. Grimiasty, Phys. Rev. A 39, 1628 (1989).
[36] G. Casati, I. Guarneri, F. Izrailev, S. Fischman, and L. Molinari, J. Phys.: Condens. Matter 4, 149 (1992).
[37] C. Mudry, P. W. Brouwer, B. I. Halperin, V. Gurarie, and A. Zee, Phys. Rev. B 58, 13539 (1998).
[38] L. I. Deych, A. A. Lisyansky, and B. L. Altshuler, Phys. Rev. Lett. 84, 2678 (2000); Phys. Rev. B 64, 224202 (2001).
[39] D. M. Gangardt and S. Fishman, Phys. Rev. B 63, 245106 (2001).
[40] C. Fuchs and R. v. Baltz, Phys. Rev. B 63, 085318 (2001).
[41] M. Titov and H. Schomerus, Phys. Rev. Lett. 91, 176601 (2003).
[42] D. Roy and N. Kumar, Phys. Rev. B 76, 092202 (2007).
[43] G. G. Kozlov, Theor. Math. Phys. 171, 531 (2012).
[44] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[45] 50 Years of Anderson Localization, E. Abrahams, ed. (World Scientific, Singapore, 2010).
[46] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B 22, 3519 (1980).
[47] I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, Introduction to the theory of disordered systems (Willey, New York, 1988).
[48] F. Falce and V. A. Gopar, Europhys. Lett. 92, 57014 (2010).
[49] A. A. Fernandez-Marin, J. A. Mendez-Bermudez, and V. A. Gopar, Phys. Rev. A 85, 035803 (2012).
[50] I. Amanatidis, I. Klefogianni, F. Falceto, and V. A. Gopar, Phys. Rev. B 85, 235450 (2012).
[51] C. Mahaux and H. A Weidenmuller, Shell Model Approach in Nuclear Reactions, (North-Holland, Amsterdam,1969); J. J. M. Verbaarschot, H. A. Weidenmuller, and M. R. Zirnbauer, Phys. Rep. 129, 367 (1985); I. Rotter, Rep. Prog. Phys. 54, 635 (1991).
[52] R. Landauer, IBM J. Res. Dev. 1, 223 (1957); 32, 336 (1988); M. Buttiker, Phys. Rev. Lett. 57, 1761 (1986); IBM J. Res. Dev. 32, 317 (1988).
[53] It is pertinent to remark that the dependence $\langle\ln G\rangle \propto L^{1/2}$, when $\alpha = 1/2$ at $E = 0$, reported in [50] for 1D wires with off-diagonal Lévy-type disorder was observed when the wire length $L$ was defined as the total sum of the hopping integrals $L = \sum_n \nu_{\alpha,n,n+1}$.
[54] H. Schomerus and M. Titov, Eur. Phys. J. B 35, 421 (2003); Phys. Rev. B 67, 100201(R) (2003).
[55] Using the results in Ref. [56], Eq. (2) of that work, $P_{\text{WS}}(G)$ is given by

$$
P_{\text{WS}}(G) = C \sqrt{\frac{\text{acosh}(1/\sqrt{G})}{G^{3/2}(1-G)}} \exp \left[ -\frac{1}{s} \text{acosh}^2 \left( \frac{1}{\sqrt{G}} \right) \right],
$$

where $C$ is a normalization constant and $s = L/\ell$, $\ell$ being the mean free path. The parameter $s$ can be obtained numerically from the ensemble average $\langle\ln G\rangle = -L/\ell$.

[56] I. Klefogianni, I. Amanatidis, V. A. Gopar, Phys. Rev. B 88, 205414 (2013).

[57] We have used the particular density distributions:

$$
\rho_1(\epsilon) = \frac{1}{\Gamma(\alpha)} \left( \frac{1}{2} \right)^{\alpha} \frac{1}{\epsilon^{1+\alpha}} \exp \left( -\frac{1}{2\epsilon} \right)
$$

(8)
and
\[ \rho_2(\epsilon) = \frac{\alpha}{(1 + \epsilon)^{1 + \gamma}}. \]

where \( \Gamma \) is the Euler gamma function.

[58] M. Steiner, Y. Chen, M. Fabrizio, and A. O. Gogolin, Phys. Rev. B 59, 14848 (1999).