Measure equivalence rigidity of the handlebody groups

Sebastian Hensel and Camille Horbez

November 22, 2021

Abstract

Let $V$ be a connected 3-dimensional handlebody of finite genus at least 3. We prove that the handlebody group $\text{Mod}(V)$ is superrigid for measure equivalence, i.e. every countable group which is measure equivalent to $\text{Mod}(V)$ is in fact virtually isomorphic to $\text{Mod}(V)$. Applications include a rigidity theorem for lattice embeddings of $\text{Mod}(V)$, an orbit equivalence rigidity theorem for free ergodic measure-preserving actions of $\text{Mod}(V)$ on standard probability spaces, and a $W^*$-rigidity theorem among weakly compact group actions.

Introduction

A central quest in measured group theory is to classify countable groups up to measure equivalence, a notion coined by Gromov in $\text{Gro93}$ as a measurable analogue to the geometric notion of quasi-isometry between finitely generated groups.

The definition is as follows: two infinite countable groups $\Gamma_1$ and $\Gamma_2$ are measure equivalent if there exists a standard infinite measure space $\Omega$ equipped with an action of $\Gamma_1 \times \Gamma_2$ by measure-preserving Borel automorphisms, such that for every $i \in \{1, 2\}$, the action of $\Gamma_i$ on $\Omega$ is free and has a fundamental domain of finite measure. The typical example is that any two (possibly non-uniform) lattices in the same locally compact second countable group $G$ are always measure equivalent, by considering the left and right multiplications on $G$ equipped with its Haar measure.

Dye proved in $\text{Dye1, Dye2}$ $\text{Dye59, Dye63}$ that all countably infinite abelian groups are measure equivalent. This was famously generalized by Ornstein and Weiss to all countably infinite amenable groups $\text{OW80}$, and in fact these form a class of the measure equivalence relation on the set of all countably infinite groups. At the other extreme of the picture, some groups satisfy very strong rigidity properties. A first striking example is the following: building on earlier work of Zimmer $\text{Zim1, Zim2}$ $\text{Zim80, Zim91}$, Furman proved that every countable group which is measure equivalent to a lattice in a higher rank Lie group, is virtually a lattice in the same Lie group $\text{Fur99a}$. In $\text{MS00}$, Monod and Shalom proved superrigidity type results for direct products of groups that satisfy an analytic form of negative curvature, phrased in terms of a bounded cohomology criterion. Later, Kida proved that, with the exception of some low-complexity cases, mapping class groups $\text{Mod}(\Sigma)$ of finite-type surfaces are $ME$-superrigid, i.e. every countable group that
is measure equivalent to \( \text{Mod}(\Sigma) \), is in fact commensurable to \( \text{Mod}(\Sigma) \) up to a finite kernel \( \text{Kid} \). This led to further strong rigidity results, for certain amalgamated free products \( \text{Kid} \), certain subgroups of \( \text{Mod}(\Sigma) \) such as the Torelli group \( \text{CK} \), some infinite classes of Artin groups of hyperbolic type \( \text{HoHu} \). Very recently, Guirardel and the second named author established that \( \text{Out}(F_N) \), the outer automorphism group of a finitely generated free group of rank \( N \geq 3 \), is also ME-superrigid \( \text{GH} \).

In the present paper, we establish a superrigidity theorem for handlebody groups, defined as mapping class groups \( \text{Mod}(V) \) of connected 3-dimensional handlebodies \( V \), i.e. \( V \) is a disk-sum of finitely many copies of \( D^2 \times S^1 \). These groups are of particular importance in 3-dimensional topology, and most notably in the theory of Heegaard splittings, see e.g. the discussion in \( \text{Hen} \) Section 4. They are also important in geometric group theory due to their direct connections to both mapping class groups of surfaces and outer automorphism groups of free groups. Notice indeed that \( \partial V \) is a closed orientable surface of finite genus \( g \geq 0 \), and \( \text{Mod}(V) \) embeds as a (highly distorted) subgroup of \( \text{Mod}(\partial V) \); it also surjects onto \( \text{Out}(F_g) \) via the action at the level of the fundamental group (with non-finitely generated kernel \( \text{McCul} \)). Recently, the geometry of handlebody groups has been shown to share many features with outer automorphism groups of free groups rather than surface mapping class groups (e.g. concerning the growth of isoperimetric functions \( \text{HH} \) or the subgroup geometry of stabilisers \( \text{HenStab} \)).

Handlebody groups are known to satisfy some algebraic rigidity properties: Korkmaz and Schleimer proved in \( \text{KS} \) that their outer automorphism group is trivial, and the first named author further proved in \( \text{Hen} \) that the natural map from the handlebody group to its abstract commensurator is an isomorphism. To our knowledge, the question of the quasi-isometric rigidity of handlebody groups (which are finitely generated by work of Suzuki \( \text{Suz} \), in fact finitely presented by work of Wajnryb \( \text{Waj} \)) is still widely open. Our main theorem establishes their superrigidity from the viewpoint of measured group theory.

**Theorem 1.** Let \( V \) be a connected 3-dimensional handlebody of finite genus at least 3. Then \( \text{Mod}(V) \) is ME-superrigid.

**Consequences.** The techniques used in the proof of Theorem 1 have several other consequences. First, we recover (with a different argument) the commensurator rigidity statement established by the first named author in \( \text{Hen} \), see Remark \( \text{rk:commensurator} \).

Second, using ideas of Furman \( \text{Fur} \) and Kida \( \text{Kid} \), we can derive that handlebody groups cannot embed as lattices in second countable locally compact groups in any interesting way.

**Corollary 2.** Let \( V \) be a connected 3-dimensional handlebody of finite genus at least 3. Let \( G \) be a locally compact second countable group, equipped with its Haar measure. Let \( \Gamma \) be a finite index subgroup of \( \text{Mod}(V) \), and let \( \sigma : \Gamma \to G \) be an injective homomorphism whose image is a lattice.

Then there exists a homomorphism \( \theta : G \to \text{Mod}(V) \) with compact kernel such that for every \( f \in \Gamma \), one has \( \theta \circ \sigma(f) = f \).

2
If $S$ is a finite generating set of $\text{Mod}(V)$, then $\text{Mod}(V)$ naturally embeds as a lattice in the automorphism group of the Cayley graph $\text{Cay}(\text{Mod}(V), S)$, defined as the simplicial graph whose vertices are the elements of $\text{Mod}(V)$, with an edge between two distinct vertices $g, h$ whenever $gh^{-1} \in S \cup S^{-1}$ (this convention excludes for instance loop-edges when $S$ contains the identity of $\text{Mod}(V)$, or multiple edges if $S$ contains an element and its inverse). The above rigidity statement about lattice embeddings has the following consequence (which can also be viewed as a very weak form of the conjectural quasi-isometry rigidity statement).

**Corollary 3.** Let $V$ be a connected $3$-dimensional handlebody of finite genus at least $3$, and let $S$ be a finite generating set of $\text{Mod}(V)$. Then every graph automorphism of $\text{Cay}(\text{Mod}(V), S)$ is at bounded distance from the left multiplication by an element of $\text{Mod}(V)$.

If $\Gamma$ is a torsion-free finite-index subgroup of $\text{Mod}(V)$, and if $S'$ is a finite generating set of $\Gamma$, then the automorphism group of $\text{Cay}(\Gamma, S')$ is countable, and in fact embeds as a subgroup of $\text{Mod}(V)$ containing $\Gamma$.

The torsion-freeness assumption is crucial in the second part of the statement: for every finitely generated group $G$ containing a nontrivial torsion element, there exists a finite generating set $S$ of $G$ such that the automorphism group of $\text{Cay}(G, S)$ is uncountable, as was observed by de la Salle and Tessera in [dlST19, Lemma 6.1].

Thanks to work of Furman [Fur99], the measure equivalence rigidity statement given in Theorem 1 can also be recast in the language of orbit equivalence rigidity of probability measure-preserving ergodic group actions. We reach the following corollary, analogous to a theorem of Kida [Kid11] for mapping class groups – see Section 4.2 for all definitions.

**Corollary 4.** Let $V$ be a connected $3$-dimensional handlebody of finite genus at least $3$. Let $\Gamma$ be a countable group. Let $\text{Mod}(V) \curvearrowright X$ and $\Gamma \curvearrowright Y$ be two free ergodic measure-preserving group actions by Borel automorphisms on standard probability spaces.

If the actions $\text{Mod}(V) \curvearrowright X$ and $\Gamma \curvearrowright Y$ are stably orbit equivalent, then they are virtually isomorphic.

Finally, our work also yields strong rigidity statements for von Neumann algebras associated (via a celebrated construction of Murray and von Neumann [MvN36]) to probability measure-preserving ergodic group actions of handlebody groups. By combining Corollary 4 with the proper proximality of handlebody groups in the sense of Boutonnet, Ioana and Peterson [BIP21] (established in [HHL20]), we reach the following corollary – see Section 4.2 for definitions, and work of Ozawa and Popa [OP10, Definition 3.1] for the notion of a weakly compact group action (as an important example, the action of a residually finite group on its profinite completion is weakly compact).

**Corollary 5.** Let $V$ be a connected $3$-dimensional handlebody of finite genus at least $3$. Let $\Gamma$ be a countable group. Let $\text{Mod}(V) \curvearrowright X$ and $\Gamma \curvearrowright Y$ be two free ergodic measure-preserving group actions by Borel automorphisms on standard probability spaces, and assume that $\Gamma \curvearrowright Y$ is weakly compact.
If the von Neumann algebras $L^\infty(X) \rtimes \text{Mod}(V)$ and $L^\infty(Y) \rtimes \Gamma$ are isomorphic, then the actions $\text{Mod}(V) \acts X$ and $\Gamma \acts Y$ are virtually conjugate.

**Proof strategy.** The general strategy of our proof of Theorem \ref{intro:main} follows Kida’s approach for mapping class groups \cite{Kid10}. General techniques from measured group theory, originating in the work of Furman \cite{Fur99a}, reduce the proof of Theorem \ref{intro:main} to a cocycle rigidity theorem (Theorem \ref{main-2}) for actions of $\text{Mod}(V)$ on standard probability spaces. In order to avoid some finite-order phenomena, it is in fact useful for us to work in a finite-index rotationless subgroup $\text{Mod}_0(V)$ (see Section \ref{rotationless} for its precise definition).

More precisely, we are given a measured groupoid $G$, which comes from restricting two actions of $\text{Mod}_0(V)$ on standard finite measure spaces to a positive measure Borel subset $Y$ on which their orbits coincide. The groupoid $G$ is thus equipped with two cocycles $\rho_1, \rho_2 : G \to \text{Mod}_0(V)$, given by the two actions: whenever two points $x, y \in Y$ are joined by an arrow $g \in G$, there is an element $\rho_1(g)$ sending $x$ to $y$ for the first action, and an element $\rho_2(g)$ sending $x$ to $y$ for the second action. Our goal is to build a canonical map $\varphi : Y \to \text{Mod}(V)$ such that $\rho_1$ and $\rho_2$ are cohomologous through $\varphi$: this means that whenever $x, y \in Y$ are joined by an arrow $g \in G$, then $\rho_2(g) = \varphi(y) \rho_1(g) \varphi(x)^{-1}$. In fact, using a theorem of Korkmaz and Schleimer which identifies $\text{Mod}(V)$ to the automorphism group of the disk graph $D$ of $V$, our goal is to build a (canonical) map $Y \to \text{Aut}(D)$. Recall that the disk graph is the graph whose vertices are the isotopy classes of meridians in $\partial V$ (i.e. essential simple closed curves that bound a properly embedded disk in $V$), and two vertices are joined by an edge if the corresponding meridians have disjoint representatives in their respective isotopy classes.

In order to build the desired map $Y \to \text{Aut}(D)$, the main step is to characterize subgroupoids of $G$ that arise as stabilizers of Borel maps $Y \to D$ in a purely groupoid-theoretic way, i.e. with no reference to the cocycles (so that a vertex stabilizer for $\rho_1$ is also a vertex stabilizer for $\rho_2$).

In the surface mapping class group setting (where the disk graph is replaced by the curve graph of the surface $\Sigma$), the important observation made by Kida is the following: curve stabilizers inside $\text{Mod}(\Sigma)$ are characterized as maximal nonamenable subgroups of $\text{Mod}(\Sigma)$ which contain an infinite amenable normal subgroup (namely, the cyclic subgroup generated by the twist about the curve). This has a groupoid-theoretic analogue, through notions of amenable and normal subgroupoids.

The situation is more complicated for handlebodies, and the above algebraic statement does not give a characterization of meridian stabilizers any longer, for several reasons that we will now explain; for simplicity we will sketch the group-theoretic version of our arguments, but in reality everything has to be phrased in the language of measured groupoids. Our most challenging task, which occupies a large part of Section \ref{me} is in fact to characterize stabilizers of nonseparating meridians. Inspired by the surface setting, we want to start with a maximal nonamenable subgroup $H$ of $\text{Mod}_0(V)$ which contains an infinite amenable normal subgroup $A$. A first bad situation we encounter is the following: $A$ could be generated by a partial pseudo-Anosov, supported on a subsurface $S \subseteq \partial V$, and $H$ be its normalizer. In the surface setting considered by Kida \cite{Kid10}, such an $H$
is not maximal, as it is contained in the stabilizer $H'$ of the boundary multicurve $\gamma$ of $S$. But for us, the group of multitwists about $\gamma$ could intersect $\text{Mod}^0(V)$ trivially; in this case $H'$ may not contain any infinite normal amenable subgroup, so $H'$ may not violate the maximality of $H$. We resolve this first difficulty by further imposing that $H$ should not be contained in a subgroup containing two normal nonamenable subgroups that centralize each other (typically, the stabilizers of a subsurface and its complement); this is why we need to exclude separating meridians from our analysis at first. With a bit more work, we manage to reduce to the case where the pair $(H, A)$ is given by the following situation: there is a multicurve $X$, together with a (possibly empty) collection $\mathfrak{A}$ of complementary components of $X$ labeled active, $H$ is the stabilizer of $X$, and $A$ is exactly the active subgroup of $(X, \mathfrak{A})$, i.e. the subgroup of the stabilizer of $X$ acting trivially on all inactive subsurfaces, and it is amenable. This still includes several possibilities: $X$ could be a nonseparating meridian and $\mathfrak{A} = \emptyset$ (in which case $A$ is the twist subgroup). But (still with $\mathfrak{A} = \emptyset$), the multicurve $X$ could also be of the form $\alpha_1 \cup \alpha_2$, where $\alpha_1$ and $\alpha_2$ together bound an annulus in $V$ (see Figure 1): the cyclic subgroup generated by the product of twists $T_{\alpha_1} T_{\alpha_2}^{-1}$ is then normal in the handlebody group stabilizer of the annulus. To exclude annuli (and in fact only retain nonseparating meridians), we use a combinatorial argument: roughly, we can always complete a nonseparating meridian to a collection of $3g - 3$ such, while doing this with annulus pairs will introduce redundancy, as the same curves will be used more than once. Combinatorially, in a collection of $3g - 3$ annuli, it is always possible to remove one without changing the link of the collection in an appropriate graph of disks and annuli.

Once we have characterized nonseparating meridians, we actually have enough information to also recover the separating ones, exploiting that these can be completed to a pair of pants decomposition by adding $3g - 4$ nonseparating meridians. Finally, a characterization of adjacency in the disk graph comes from observing that two meridians are disjoint up to isotopy if the corresponding twists commute, or in other words if these twists together they generate an amenable subgroup of $\text{Mod}(V)$.

Acknowledgments. The first named author is partially supported by the DFG as part of the SPP 2026 “Geometry at Infinity”. The second named author acknowledges support from the Agence Nationale de la Recherche under Grant ANR-16-CE40-0006 DAGGER.

1 Handlebody and mapping class group facts

In this section, we collect a few facts about handlebody groups that will be useful in the paper. The reader is refered to [Joh95, Hen] for general information about handlebody groups.

1.1 General background

Handlebodies. By a handlebody of (finite) genus $g \geq 0$, we mean a connected orientable 3-manifold which is a disk-sum of $g$ copies of $D^2 \times S^1$, where $D^2$ is a closed disk and $S^1$ is a
Meridians and annuli. Let $V$ be a handlebody. Recall that a simple closed curve on $\partial V$ is essential if it is homotopically nontrivial, i.e. it does not bound a disk on $\partial V$. An essential simple closed curve on $\partial V$ is a meridian (represented in blue in Figure 1) if it bounds a properly embedded disk in $V$.

If $c \subseteq \partial V$ is a meridian, then the Dehn twist $T_c$ associated to $c$ belongs to $\text{Mod}(V)$, viewed as a subgroup of $\text{Mod}(\partial V)$ – and this is in fact a characterisation of meridians, as follows from [McC06, Theorem 1] or [Oer02, Theorem 1.11].

For multitwists, there is another possibility. Namely, a pair $\{\alpha_1, \alpha_2\}$ of disjoint nonisotopic essential simple closed curves on $\partial V$ is an annulus pair (represented in red in Figure 1) if neither $\alpha_1$ nor $\alpha_2$ is a meridian, and there exists a properly embedded annulus $A \subseteq V$ such that $\partial A = \alpha_1 \cup \alpha_2$. An annulus twist is a mapping class of the form $T_{\alpha_1}T_{\alpha_2}^{-1}$ for some annulus pair $\{\alpha_1, \alpha_2\}$. Annulus twists belong to $\text{Mod}(V)$ ([McC06, Theorem 1] or [Oer02, Theorem 1.11]).

Lemma 1.1. Let $c$ be a meridian. Then every connected component of $\partial V \setminus c$ supports two handlebody group elements which both restrict to a pseudo-Anosov mapping class of $\partial V \setminus c$ and together generate a nonabelian free subgroup.

Proof. Let $X$ be a connected component of $\partial V \setminus c$ which is not a once-holed torus, and denote by $c_1$ a boundary component of $X$ (corresponding to one of the sides of $c$). Then, for any essential simple closed curve $\alpha \subseteq X$ which is not boundary parallel in $X$, we can (and shall) choose an essential simple closed curve $\alpha' \subseteq X$ which is not isotopic to $\alpha$ and bounds a pair of pants on $X$ together with $c_1, \alpha$ (here, we are using that $X$ is not a once-holed torus). Since $c$ is a meridian, $\alpha, \alpha'$ are either both meridians, or form an...
annulus pair. Thus, in either case, the multitwist \( f_\alpha = T_\alpha T_\alpha^{-1} \) is a handlebody group element supported in \( X \).

By choosing curves \( \alpha, \beta \) which fill \( X \) we can thus find \( f_\alpha, f_\beta \) so that no essential simple closed curve in \( X \) is fixed by both up to isotopy. This implies that the group generated by \( f_\alpha, f_\beta \) contains a pseudo-Anosov \( \psi \) ([Iva92], see also the discussion in [Man13, Section 2.4]). Conjugating \( \psi \) by \( f_\alpha \) yields a second one, and sufficiently high powers of \( \psi \) and \( f_\alpha \psi f_\alpha^{-1} \) generate a nonabelian free subgroup.

Lemma \ref{lem:pa-in-complement} implies in particular that when \( V \) has genus at least 3, the stabilizer in \( \text{Mod}(V) \) of every meridian \( c \) in \( \partial V \) contains a nonabelian free subgroup (because at least one connected component of \( \partial V \setminus c \) is not a one-holed torus). The requirement of having genus at least 3 is necessary, as the following shows.

**Lemma 1.2.** Suppose that \( c \) is a separating meridian, and suppose that \( X \) is a component of \( \partial V \setminus c \) which is a one-holed torus. Then \( X \) contains a unique (nonseparating) meridian \( d_X \) which is not peripheral in \( X \) up to isotopy, and therefore

\[
\text{Stab}_{\text{Mod}(V)}(c) \subsetneq \text{Stab}_{\text{Mod}(V)}(d_X).
\]

If the genus of \( V \) is at least 3, then \( d_X \) is the only other meridian whose stabiliser contains \( \text{Stab}_{\text{Mod}(V)}(c) \) (or even a finite-index subgroup of \( \text{Stab}_{\text{Mod}(V)}(c) \)).

**Proof.** The subsurface \( X \) is the boundary of a once-spotted genus 1 handlebody \( V_1 \). Hence, there is a nonseparating meridian \( d_X \) contained in \( X \). We claim that it is the only one up to isotopy. Namely, recall that in a one-holed torus any two isotopically distinct essential simple closed curves have nonzero algebraic intersection number. However, any two meridians have algebraic intersection number zero.

In particular \( \text{Stab}_{\text{Mod}(V)}(c) \subsetneq \text{Stab}_{\text{Mod}(V)}(d_X) \). This inclusion is strict, because there exists a handlebody group element \( \varphi \) which fixes \( d_X \) and restricts to a pseudo-Anosov homeomorphism on the complementary subsurface, in particular \( \varphi \) does not fix the isotopy class of \( c \).

To show the final claim, recall from Lemma \ref{lem:pa-in-complement} that there are elements in \( \text{Stab}_{\text{Mod}(V)}(c) \) restricting to pseudo-Anosov elements on any component of \( \partial V \setminus c \) which is not a one-holed torus. If the genus of \( V \) is at least 3, the complement of \( X \) will be such a component. Hence, \( d_X \) is the unique other meridian fixed by \( \text{Stab}_{\text{Mod}(V)}(c) \) (or any finite-index subgroup).

### 1.2 Rotationless mapping classes

In order to avoid finite-order phenomena, it will be useful to work in certain finite index subgroups. We say that a mapping class \( f \) is rotationless (or pure) if the following holds: if a power of \( f \) fixes the isotopy class of a simple closed curve \( c \), then \( f \) actually fixes the oriented isotopy class of \( c \).

Let \( \Sigma \) be a surface obtained from a closed, connected, orientable surface by removing at most finitely many points. We denote by \( \text{Mod}^0(\Sigma) \) the (finite index) subgroup of
Lemma 1.3. Denote by $p : X \rightarrow \Sigma$ the mod-2-homology cover of the surface $\Sigma$. Let $\text{Mod}^1(\Sigma)$ be the subgroup of those mapping classes which admit a lift to $X$ which acts trivially on $H_1(X; \mathbb{Z}/3\mathbb{Z})$

Then $\text{Mod}^1(\Sigma)$ is a finite index subgroup of $\text{Mod}^0(\Sigma)$, and if $h \in \text{Mod}^1(\Sigma)$ is any element preserving a connected subsurface $S \subset \Sigma$, then the restriction $h|_S$ is an element of $\text{Mod}^0(S)$.

The proof uses the following covering argument.

Lemma 1.4. Let $S \subset \Sigma$ be an essential connected subsurface. Denote by $p : X \rightarrow \Sigma$ the mod-2-homology cover, and let $X_S \subset X$ be a connected component of $p^{-1}(S)$. Then the map

$$H_1(X_S; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z})$$

induced by the inclusion is injective, and the same is true with $\mathbb{Z}$ replaced with $\mathbb{Z}/n$ for any $n$.

Proof. Choose a subsurface $Y \subset X_S$ with one boundary component, so that $X_S \setminus Y$ is a bordered sphere. Denote by $\delta_0, \ldots, \delta_k$ the boundary components of $X_S$ (which are all contained in $Y$). We then have that

$$H_1(X_S; \mathbb{Z}) = H_1(Y; \mathbb{Z}) \oplus \mathbb{Z}^k,$$

where the latter summand is $\langle [\delta_1], \ldots, [\delta_k] \rangle$. The first summand injects into $H_1(X; \mathbb{Z})$, since $Y$ is a subsurface of $X$ with one boundary component.

We now aim to show that for all $i > 0$ there is a curve $\beta_i$ which is disjoint from $Y$, intersects $\delta_0, \delta_i$ each in a single point, and is disjoint from all other $\delta_j$ (i.e. that the complement of $X_S$ is connected). This will show that $[\delta_1], \ldots, [\delta_k]$ are linearly independent from each other and from $H_1(Y; \mathbb{Z})$ in $H_1(X; \mathbb{Z})$ thus showing the lemma.

For simplicity of notation, we will perform the construction only for $i = 1$. Choose a basepoint $\tilde{q}$ in $Y$, and let $q = p(\tilde{q})$ be its image in $\Sigma$. Since the mod-2 homology cover is normal (Galois), the preimage $p^{-1}(q)$ is exactly the orbit of $\tilde{q}$ under the deck group $D = H_1(\Sigma; \mathbb{Z}/2)$.

To describe the intersection $p^{-1}(q) \cap X_S$, first observe that since $X_S$ is connected, a point $\tilde{q}' \in p^{-1}(q)$ is contained in $X_S$ exactly if there is a path $\tilde{\gamma}$ connecting $\tilde{q}$ to $\tilde{q}'$ contained in $X_S$. Such paths are exactly the lifts of loops $\gamma$ based at $q$ which are contained in $S$. So $\tilde{q}'$ is contained in $X_S$ if and only if the deck group element $\gamma$ mapping $\tilde{q}$ to $\tilde{q}'$ is the image of some $\gamma \in \pi_1(S, q) \subseteq \pi_1(\Sigma, q)$. The image of $\pi_1(S, q)$ in the deck
group is exactly the subgroup \( D_S = \text{im}(H_1(S; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2)) \). Together this shows that \( p^{-1}(q) \cap X_S = D_S \).

Similarly, the components of \( p^{-1}(S) \) can be identified with the cosets of the subgroup \( D_S \subseteq D \).

To describe the cover more precisely, we choose curves \( \gamma_i \) based at \( q \) in the following way:

1. The homology classes \([\gamma_i] = x_i\) form a basis \( x_1, \ldots, x_N \) of \( H_1(\Sigma; \mathbb{Z})\).
2. \( x_1, \ldots, x_k \) is a basis of \( \text{im}(H_1(S; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z}))\), and the curves \( \gamma_i \) are contained in \( S \).
3. The curves \( \gamma_i \) for \( i = k + 1, \ldots, N \) intersect \( \partial S \) in exactly two points.

To see that these curves exist, we argue as follows. Denote by \( S_1, \ldots, S_r \) the components of \( \Sigma \setminus S \). Choose a curve \( \alpha_i \subset \partial S_i \). The connectivity of \( S \) implies that for every \( i \in \{1, \ldots, r\} \), the curve \( \alpha_i \) is homologically nontrivial (in \( H_1(\Sigma) \)) exactly if \( \partial S_i \) has more than one component. For each boundary curve \( \beta \subset \partial S_i \setminus \alpha_i \) we can find a loop \( \gamma_\beta \) based at \( q \) which intersects \( \partial S \) in two points, one on \( \beta \) and one on \( \alpha_i \). We can thus choose independent homology classes \( z_i \) defined by curves intersecting \( \partial S \) in at most two points, so that for any \( x \in H_1(\Sigma) \) there is a linear combination \( z \) of the \( z_i \), so that \( x + z \) has algebraic intersection number 0 with all curves in \( \partial S \). Any such class \( x + z \) can be realised by a multicurve disjoint from \( \partial S \). Since every homology class defined by a curve (without specified basepoint) in \( S \) can be realised by a loop based at \( q \) which intersects \( \partial S \) in two points, and every curve in \( S \) can be realised by a loop disjoint from \( \partial S \) the desired existence follows.

Lifting a curve of the type in (2) at a point \( h\tilde{q} \) stays in the same connected component \( hX_S \), while lifting a curve of the type in (3) joins \( hX_S \) to \( h'X_S \) and intersects \( \partial hX_S \) in a single point. To see that last claim observe that a lift of a curve as in (3) cannot join two points of \( hX_S \), as the image of that curve in \( H_1(\Sigma; \mathbb{Z}/2) \) would then be contained in \( \text{im}(H_1(S; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2)) \), contradicting (1) and (2).

For every \( i \in \{0, 1\} \), denote by \( Z_i \) the component of \( X \setminus p^{-1}(S) \) adjacent to \( \delta_i \). There are \( h_i \notin \text{im}(H_1(S; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2)) \), so that \( h_iX_S \) are surfaces adjacent to \( Z_i \). Namely, either \( p(Z_i) \) has genus (which is automatically the case if \( p(\delta_i) \) is separating), and contains a curve defining one of the \( x_j \) (of the second type), or \( p(Z_i) \) is a punctured sphere so that for the boundary component \( p(\delta_i) \) there is some \( x_j, j > k \) (of the third type) which intersects it once (namely, if all \( x_j \) would intersect \( p(\delta_i) \) in an even number of points, the \( x_i \) could not be a basis of \( H_1(\Sigma; \mathbb{Z}) \), since \( p(\delta_i) \) is nonseparating). In both cases the desired component is \( \pm [x_j]X_S \). Choose paths \( c_i, i = 0, 1 \) joining \( \tilde{q} \) to \( h_i\tilde{q} \) which intersect only \( \delta_i \) among the \( \delta_j \).

Since \( \text{im}(H_1(S; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2)) = (\mathbb{Z}/2)^N \) is a subgroup of \( H_1(\Sigma; \mathbb{Z}/2) = (\mathbb{Z}/2)^N \) generated by a subset of the generators, there is a path in the Cayley graph of \( H_1(\Sigma; \mathbb{Z}/2) \) from \( h_0 \) to \( h_1 \) which is disjoint from the Cayley graph of the subgroup \( \text{im}(H_1(S; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2)) \). Each edge in such a path corresponds to a right multiplication \( h \mapsto hx_i \), and we can choose a corresponding path joining \( h\tilde{q} \) to \( hX_i\tilde{q} \) which is
disjoint from $X_S$. By concatenating these paths with $c_0, c_1$ (in the right order) we then find the desired path $\beta_1$.

Lemma 1.3 is now an immediate consequence of the following corollary.

**Corollary 1.5.** Suppose that $f$ is a mapping class so that

1. $f$ admits a lift $\tilde{f}$ to the mod-$2$-homology-cover $X$, which acts trivially on $H_1(X; \mathbb{Z}/3)$
2. $f$ preserves a subsurface $S \subset \Sigma$

Then the restriction $f|_S$ acts trivially on $H_1(S; \mathbb{Z}/3)$.

**Proof.** Let $\alpha \subset S$ be a simple closed curve which is part of a basis for $H_1(S; \mathbb{Z}/3)$. Then there is a power $N = 2^n$ so that $\alpha^N$ lifts to a curve $\tilde{\alpha} \subset X_S$ (with notation as in the previous lemma).

Denote by $p_S : X_S \to S$ the restriction of the covering map (which is then also a covering). We have $(p_S)_\ast[\tilde{\alpha}] = N[\alpha]$. Since $N$ is invertible mod $3$, there is a multiple $k$ so that $(p_S)_\ast k[\tilde{\alpha}] = [\alpha] \mod 3$.

By Lemma 1.4, $H_1(X_S; \mathbb{Z}/3)$ is a subspace of $H_1(X; \mathbb{Z}/3)$. Since $\tilde{f}$ acts trivially on $H_1(X; \mathbb{Z}/3)$, this implies that the restriction $\tilde{f}_{|X_S}$ acts trivially on $H_1(X_S; \mathbb{Z}/3)$. Hence, we have $(\tilde{f}_{|X_S})_\ast k[\tilde{\alpha}] = k[\tilde{\alpha}]$. Since $\tilde{f}_{|X_S}$ is a lift of $f|_S$ this implies $(f|_S)_\ast [\alpha] = [\alpha]$. 

In the sequel of the paper, we will always let $\text{Mod}^0(V) = \text{Mod}(V) \cap \text{Mod}^1(\partial V)$, where $\text{Mod}^1(\partial V)$ is as in Lemma 1.3.

### 1.3 Infinite conjugacy classes

A countable group $G$ is said to be ICC (standing for infinite conjugacy classes) if the conjugacy class of every nontrivial element of $G$ is infinite.

**Lemma 1.6.** Let $V$ be a handlebody of genus at least $2$, and let $\varphi \in \text{Mod}(V)$ be a handlebody group element. Then either the conjugacy class of $\varphi$ is infinite, or $\varphi$ fixes the isotopy class of every meridian.

In particular, when the genus of $V$ is at least $3$, the group $\text{Mod}(V)$ is ICC.

We remark that in genus 2, the hyperelliptic involution fixes the isotopy class of every essential simple closed curve on $\partial V$, and its conjugacy class is finite in $\text{Mod}(V)$.

**Proof.** Suppose that $\varphi$ is an element with finite conjugacy class. For any meridian $c$, consider the elements $T_c^i \varphi T_c^{-i}$ for $i \in \mathbb{N}$. By finiteness of the conjugacy class, two of these have to be equal, and thus there is some $N > 0$ so that

$$T_c^N \varphi T_c^{-N} = \varphi,$$

or equivalently,

$$T_c^N = \varphi T_c^N \varphi^{-1} = T_c^N \varphi(c).$$

10
This implies $c$ is isotopic to $\varphi(c)$, see e.g. [FM12, Section 3.3]. The first part of the lemma follows since $c$ was arbitrary. The fact that $\text{Mod}(V)$ is ICC when the genus is at least 3 follows because every element fixing the isotopy class of every meridian is then trivial [KS09, Theorem 9.4].

2 Background on measured groupoids

The reader is referred to [AD13, Section 2.1], [Kid-survey] or [GH21, Section 3] for general background on measured groupoids.

Recall that a standard Borel space is a measurable space associated to a Polish space (i.e. separable and completely metrizable). A standard probability space is a standard Borel space equipped with a Borel probability measure.

A Borel groupoid is a standard Borel space $G$ (whose elements are thought of as being arrows) equipped with two Borel maps $s, r : G \to Y$ towards a standard Borel space $Y$ (giving the source and range of an arrow), and coming with a measurable composition law and inverse map and with a unit element $e_y$ per $y \in Y$. The Borel space $Y$ is called the base space of the groupoid $G$. All Borel groupoids considered in the present paper are assumed to be discrete, i.e. there are countably many arrows in $G$ with a given range (or source). It follows from a theorem of Lusin and Novikov (see e.g. [Kec95, Theorem 18.10]) that a discrete Borel groupoid $G$ can always be written as a countable disjoint union of bisections, i.e. Borel subsets $B$ of $G$ on which $s$ and $r$ are injective (in which case $s(B)$ and $r(B)$ are Borel subsets of $Y$, see [Kec95, Corollary 15.2]). A Borel groupoid $G$ with base space $Y$ is trivial if $G = \{ e_y | y \in Y \}$.

A finite Borel measure $\mu$ on $Y$ is quasi-invariant for the groupoid $G$ if for every bisection $B \subseteq G$, one has $\mu(s(B)) > 0$ if and only if $\mu(r(B)) > 0$. A measured groupoid is a Borel groupoid together with a quasi-invariant finite Borel measure on its base space $Y$.

An important example of a measured groupoid to keep in mind is the following: when a countable group $G$ acts on a standard probability space $Y$ by Borel automorphisms in a quasi-measure-preserving way, then $G \times Y$ has a natural structure of a measured groupoid over $Y$, denoted by $G \rtimes Y$: the source and range maps are given by $s(g, y) = y$ and $r(g, y) = gy$, the composition law is $(g, h)(y) = (gh, y)$, the inverse of $(g, y)$ is $(g^{-1}, gy)$ and the units are $e_y = (e, y)$.

A Borel subset $H \subseteq G$ which is stable under composition and inverse and contains all unit elements of $G$ has the structure of a measured subgroupoid of $G$ over the same base space $Y$. Given a Borel subset $U \subseteq Y$, the restriction $G|_U$ is the measured groupoid over $U$ defined by only keeping the arrows whose source and range both belong to $U$. Given two subgroupoids $H, H' \subseteq G$, we denote by $\langle H, H' \rangle$ the subgroupoid of $G$ generated by $H$ and $H'$, i.e. the smallest subgroupoid of $G$ containing $H$ and $H'$ (it is made of all arrows obtained as finite compositions of arrows in $H$ and arrows in $H'$).

A measured groupoid $G$ with base space $Y$ is of infinite type if for every positive measure Borel subset $U \subseteq Y$ and almost every $y \in U$, there are infinitely many elements of $G|_U$ with source $y$. Observe that if $G$ is of infinite type, then for every Borel subset
$U \subseteq Y$ of positive measure, the restricted groupoid $\mathcal{G}_U$ is again of infinite type.

Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, and let $G$ be a countable group. A strict cocycle $\rho : \mathcal{G} \to G$ is a Borel map such that for all $g_1, g_2 \in \mathcal{G}$, if the source of $g_1$ is equal to the range of $g_2$ (so that the product $g_1 g_2$ is well-defined), then $\rho(g_1 g_2) = \rho(g_1) \rho(g_2)$. The kernel of a cocycle $\rho$ is the subgroupoid of $\mathcal{G}$ made of all $g \in \mathcal{G}$ such that $\rho(g) = 1$. We say that $\rho$ has trivial kernel if its kernel is equal to the trivial subgroupoid of $\mathcal{G}$, i.e., it only consists of the unit elements of $\mathcal{G}$. We say that a strict cocycle $\mathcal{G} \to G$ is action-type if $\rho$ has trivial kernel, and whenever $H \subseteq G$ is an infinite subgroup, and $U \subseteq Y$ is a Borel subset of positive measure, then $\rho^{-1}(H)\big|_U$ is a subgroupoid of $\mathcal{G}_U$ of infinite type. An important example is that given a measure-preserving $G$-action on a standard probability space $Y$, the natural cocycle $\rho : G \ltimes Y \to G$ is action-type \cite[Proposition 2.26]{Kid-survey}. We warn the reader that in the latter example, it is important that the $G$-action on $Y$ preserves the measure, as opposed to only quasi-preserving it.

Given a Polish space $\Delta$ equipped with a $G$-action by Borel automorphisms, we say that a measurable map $\varphi : Y \to \Delta$ is $(\mathcal{G}, \rho)$-equivariant if there exists a conull Borel subset $Y^* \subseteq Y$ such that for every $g \in \mathcal{G}_{Y^*}$, one has $\varphi(r(g)) = \rho(g) \varphi(s(g))$. We say that an element $\delta \in \Delta$ is $(\mathcal{G}, \rho)$-invariant if the constant map with value $\delta$ is $(\mathcal{G}, \rho)$-equivariant (equivalently, there exists a conull Borel subset $Y^* \subseteq Y$ such that $\rho(\mathcal{G}_{Y^*}) \subseteq \text{Stab}_G(\delta)$). The $(\mathcal{G}, \rho)$-stabilizer of $\delta$ is the subgroupoid of $\mathcal{G}$ made of all elements $g$ such that $\rho(g) \in \text{Stab}_G(\delta)$. A measurable map $\varphi : Y \to \Delta$ is stably $(\mathcal{G}, \rho)$-equivariant if one can partition $Y$ into at most countably many Borel subsets $Y_i$ such that for every $i$, the map $\varphi|_{Y_i}$ is $(\mathcal{G}_{Y_i}, \rho)$-equivariant.

Given two measured subgroupoids $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{G}$, we say that $\mathcal{H}$ is stably contained in $\mathcal{H}'$ if there exist a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \sqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, one has $\mathcal{H}|_{Y_i} \subseteq \mathcal{H}'|_{Y_i}$. We say that $\mathcal{H}$ and $\mathcal{H}'$ are stably equal if there exist a conull Borel subset and a partition as above such that for every $i \in I$, one has $\mathcal{H}|_{Y_i} = \mathcal{H}'|_{Y_i}$. We say that $\mathcal{H}$ is stably trivial if it is stably equal to the trivial subgroupoid of $\mathcal{G}$.

Let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$, and $B \subseteq \mathcal{G}$ be a bisection. We say that $\mathcal{H}$ is $B$-invariant if there exists a conull Borel subset $Y^* \subseteq Y$ such that for every $g_1, g_2 \in B \cap \mathcal{G}_{Y^*}$ and every $h \in \mathcal{G}_{Y^*}$ such that the composition $g_2 h g_1^{-1}$ is well-defined, we have $h \in \mathcal{H}|_{Y^*}$ if and only if $g_2 h g_1^{-1} \in \mathcal{H}|_{Y^*}$. Let now $\mathcal{H}'$ be another measured subgroupoid of $\mathcal{G}$. The groupoid $\mathcal{H}$ is normalized by $\mathcal{H}'$ if $\mathcal{H}'$ can be covered by countably many bisections $B_n \subseteq \mathcal{G}$ in such a way that $\mathcal{H}$ is $B_n$-invariant for every $n \in \mathbb{N}$. The subgroupoid $\mathcal{H}$ is stably normalized by $\mathcal{H}'$ if one can partition $Y$ into at most countably many Borel subsets $Y_i$ in such a way that for every $i$, the groupoid $\mathcal{H}|_{Y_i}$ is normalized by $\mathcal{H}'|_{Y_i}$. When $\mathcal{H} \subseteq \mathcal{H}'$, we will simply say that $\mathcal{H}$ is stably normal in $\mathcal{H}'$.

There is a notion of amenability of a measured groupoid, generalizing Zimmer’s notion of amenability of a group action, for which we refer to \cite{Kid-survey}; here we only list the properties of amenable groupoids we will need. First, if $\mathcal{G}$ is amenable and comes with a cocycle $\rho : \mathcal{G} \to G$ towards a countable group $G$, and if $G$ acts by homeomorphisms on a compact metrizable space $K$, then there exists a $(\mathcal{G}, \rho)$-equivariant Borel map...
\[ Y \to \text{Prob}(K), \text{ see Kid-survey Proposition 4.14}. \] Here Prob\((K)\) denotes the set of Borel probability measures on \(K\), equipped with the weak-* topology coming from the duality with the space of real-valued continuous functions on \(K\) given by the Riesz–Markov–Kakutani theorem. Second, whenever \(\rho : G \to G\) is a cocycle with trivial kernel, and \(A \subseteq G\) is an amenable subgroup of \(G\), then \(\rho^{-1}(A)\) is an amenable subgroupoid of \(G\) (see e.g. [GH21 Corollary 3.39]). Amenity is stable under subgroupoids and restrictions. Furthermore, if there exists a conull Borel subset \(Y^{*} \subseteq Y\) and a partition \(Y^{*} = \bigsqcup_{i \in I} Y_{i}\) into at most countably many Borel subsets such that for every \(i \in I\), the groupoid \(G|_{Y_{i}}\) is amenable, then \(G\) is amenable (this is immediate with the definition of amenability given in [GH21 Definition 3.33], see also [GH21 Remark 3.34] for the comparison to equivalent definitions).

A groupoid \(G\) over a standard probability space \(Y\) is everywhere nonamenable if for every Borel subset \(U \subseteq Y\) of positive measure, the groupoid \(G|_{U}\) is nonamenable.

### 3 Measure equivalence rigidity of the handlebody group

In this section, we prove the main theorem of the present paper.

**Theorem 3.1.** Let \(V\) be a handlebody of genus at least 3. Then \(\text{Mod}(V)\) is ME-superrigid.

Using the fact that \(\text{Mod}(V)\) is ICC (Lemma 1.6), Theorem 3.1 is a consequence of the following statement combined with [GH21 Theorem 4.5] (which builds on earlier works of Furman [Fur99a, Fur99b] and Kida [Kid10]).

**Theorem 3.2.** Let \(V\) be a handlebody of genus at least 3. Let \(G\) be a measured groupoid over a standard probability space \(Y\) (with source map \(s\) and range map \(r\)), and let \(\rho_{1}, \rho_{2} : G \to \text{Mod}^{0}(V)\) be two strict action-type cocycles.

Then there exist a Borel map \(\theta : Y \to \text{Mod}(V)\) and a conull Borel subset \(Y^{*} \subseteq Y\) such that for all \(g \in G|_{Y^{*}}\), one has \(\rho_{1}(g) = \theta(r(g))^{-1} \rho_{2}(g) \theta(s(g))\).

**Remark 3.3.** The case where \(Y\) is reduced to a point is actually already relevant: if \(f\) is an automorphism of \(\text{Mod}^{0}(V)\), then the group \(\text{Mod}^{0}(V)\), viewed as a groupoid over a point, comes equipped with two (action-type) cocycles towards \(\text{Mod}^{0}(V)\), given by the identity and \(f\). The conclusion in this case is that every automorphism of \(\text{Mod}^{0}(V)\) is a conjugation. More generally, a consequence of Theorem 3.2 is that the natural map from \(\text{Mod}(V)\) to its abstract commensurator is an isomorphism (using that \(\text{Mod}(V)\) is ICC for its injectivity). Our work therefore recovers the commensurator rigidity statement from [Hen18].

The rest of the section is devoted to the proof of Theorem 3.2. Starting from a measured groupoid \(G\) with two action-type cocycles \(\rho_{1}, \rho_{2}\) towards \(\text{Mod}^{0}(V)\), we ultimately aim to show that subgroupoids of \(G\) corresponding to meridian stabilizers for \(\rho_{1}\) - in the precise sense that they are of meridian type as in Definition 3.4 below - are also of meridian type with respect to \(\rho_{2}\). Additionally, we will prove that the property that two subgroupoids stabilize disjoint meridians is also independent of the action-type cocycle.
we choose. This will be used to build a canonical map $\theta$ from the base space $Y$ of the groupoid $G$ to the group of all automorphisms of the disk graph. We will finally appeal to the theorem of Korkmaz and Schleimer [KNS09] saying that the automorphism group of the disk graph is precisely $\text{Mod}(V)$ to conclude. We make the following definition.

**Definition 3.4** (Subgroupoids of meridian type). Let $G$ be a measured groupoid over a standard probability space $Y$, and let $\rho : G \to \text{Mod}^0(V)$ be a strict cocycle. A measured subgroupoid $H$ of $G$ is of meridian type with respect to $\rho$ if there exists a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, the groupoid $H|_{Y_i}$ is equal to the $(G|_{Y_i}, \rho)$-stabilizer of the isotopy class of a meridian $c_i$.

When $H$ can be written as in Definition 3.4, we say that the map $\varphi$ sending every $y \in Y_i$ to the isotopy class of the meridian $c_i$ is a meridian map for $(H, \rho)$. The essential uniqueness of this map (i.e. the fact that, up to measure 0, it does not depend on the choice of a partition and meridians $c_i$ as above) will follow from Lemmas 3.13 and 3.14.

Likewise, we define the notions of subgroupoids of nonseparating meridian type, and of separating meridian type, by respectively requiring $c_i$ to be nonseparating, or separating. Before completing our characterisation of subgroupoids of meridian type in Proposition 3.36, we will go through successive characterisations of subgroupoids of nonseparating-meridian type (Section 3.9) and of separating-meridian type (Section 3.10).

### 3.1 Groupoids with cocycles to a free group, after Adams [Ada94], Kida [Kid10]

Throughout the paper, we will work with the following definition.

**Definition 3.5** (Strongly Schottky pairs of subgroupoids). Let $G$ be a measured groupoid over a standard probability space $Y$. A strongly Schottky pair of subgroupoids of $G$ is a pair $(A^1, A^2)$ of amenable subgroupoids of $G$ of infinite type such that for every Borel subset $U \subseteq Y$ of positive measure, there exists a Borel subset $V \subseteq U$ of positive measure such that every normal amenable subgroupoid of $(A^1|_V, A^2|_V)$ is stably trivial.

We observe that this notion is stable under restrictions: if $(A^1, A^2)$ is a strongly Schottky pair of subgroupoids of $G$, then for every Borel subset $U \subseteq Y$ of positive measure, the pair $(A^1|_U, A^2|_U)$ is a strongly Schottky pair of subgroupoids of $G|_U$. In addition, this notion is stable under stabilization: given a pair $(A^1, A^2)$ of subgroupoids of $G$, and a partition $Y = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets, if $(A^1|_{Y_i}, A^2|_{Y_i})$ is a strongly Schottky pair of subgroupoids of $G|_{Y_i}$ for every $i \in I$, then $(A^1, A^2)$ is a strongly Schottky pair of subgroupoids of $G$.

Notice that the last conclusion implies in particular that $(A^1|_V, A^2|_V)$ is nonamenable. So the existence of a strongly Schottky pair of subgroupoids of $G$ forces $G$ to be everywhere nonamenable.

Definition 3.5 is a strengthening of the notion of a *Schottky pair of subgroupoids* from [GH21] Definition 13.1, which only required the groupoid $(A^1|_U, A^2|_U)$ to be nonamenable.
Lemma 3.6. Let $G$ be a countable group, and let $g, h \in G$ be two elements that generate a nonabelian free subgroup of $G$. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to G$.

Then $(\rho^{-1}(\langle g \rangle), \rho^{-1}(\langle h \rangle))$ is a strongly Schottky pair of subgroupoids of $\mathcal{G}$ (in particular $\mathcal{G}$ is everywhere nonamenable).

In the following proof, whenever $\Delta$ is a Polish space, the set $\text{Prob}(\Delta)$ of all Borel probability measures on $\Delta$ is equipped with the topology generated by the maps $\mu \mapsto \int_X f \, d\mu$, where $f$ varies over the set of all real-valued bounded continuous functions. When $\Delta$ is compact, this is nothing but the weak-$\ast$ topology coming from the duality given by the Riesz–Markov–Kakutani theorem. When $\Delta$ is a countable discrete space, this is nothing but the topology of pointwise convergence. The reader is referred to [Kec95, Section 17.E] for more information and basic facts regarding the Borel structure on $\text{Prob}(\Delta)$ which justify the measurability of all maps in the following proof.

Proof of Lemma [5.6] As $\langle g \rangle$ and $\langle h \rangle$ are amenable subgroups of $G$ and $\rho$ has trivial kernel, the subgroupoids $\rho^{-1}(\langle g \rangle)$ and $\rho^{-1}(\langle h \rangle)$ are amenable. As $\langle g \rangle$ and $\langle h \rangle$ are infinite and $\rho$ is action-type, the subgroupoids $\rho^{-1}(\langle g \rangle)$ and $\rho^{-1}(\langle h \rangle)$ are of infinite type.

Now it is enough to prove that if $U \subseteq Y$ is a Borel subset of positive measure, and $A$ is a normal amenable subgroupoid of $\langle \rho^{-1}(\langle g \rangle), \rho^{-1}(\langle h \rangle) \rangle$ then $A$ is stably trivial.

Let $T$ be the Cayley tree of the free group $F = \langle g, h \rangle$, with respect to the generating set $\{g, h\}$. The $F$-action on $T$ by isometries extends to an $F$-action on $\partial_\infty T$ by homeomorphisms. As $A$ is amenable and $\partial_\infty T$ is compact and metrizable, there exists an $(A, \rho)$-equivariant Borel map $U \to \text{Prob}(\partial_\infty T)$, see [Kid09, Proposition 4.14].

We claim that we can find a (possibly null) Borel subset $U_1 \subseteq U$ of maximal measure such that there exists a Borel map $\mu : U_1 \to \text{Prob}(\partial_\infty T)$ which is stably $(A|_{U_1}, \rho)$-equivariant and such that for every $y \in U_1$, the support of the measure $\mu(y)$ has cardinality at least 3. Indeed, let $\alpha$ be the supremum of the measures of all Borel subsets of $U$ having the above property, and let $(U_{1,n})_{n \in \mathbb{N}}$ be a measure-maximizing sequence of such sets – in particular, for every $n \in \mathbb{N}$, we have a Borel map $\mu_n : U_{1,n} \to \text{Prob}(\partial_\infty T)$ as above. Then the countable union $U_{1,\infty}$ of all subsets $U_{1,n}$ has measure at least $\alpha$, and we claim that it also satisfies the above property. To see this, inductively define $V_{1,0} = U_{1,0}$ and $V_{1,n} = U_{1,n} \setminus (U_{1,n} \cap V_{1,n-1})$. Then the map $\mu_{\infty} : U_{1,\infty} \to \text{Prob}(\partial_\infty T)$, defined to coincide with $\mu_n$ when restricted to $V_{1,n}$, is stably $(A|_{U_{1,\infty}}, \rho)$-equivariant, and for every $y \in U_{1,\infty}$, the support of the measure $\mu(y)$ has cardinality at least 3. This concludes the proof of our claim.

From now on, we choose a Borel subset $U_1 \subseteq U$ as in the above claim. We first prove that $A|_{U_1}$ is stably trivial. Up to partitioning $U_1$ into at most countably many Borel subsets, we can assume that the map $\mu_1$ is $(A|_{U_1}, \rho)$-equivariant (and not just stably equivariant). For every $y \in U_1$, the probability measure $\mu(y) \otimes \mu(y) \otimes \mu(y)$ on $(\partial_\infty T)^3$ gives positive measure to the $F$-invariant subset $(\partial_\infty T)^{(3)}$ made of pairwise distinct
triple. Thus, after restricting this measure to $(\partial_{\infty} T)^{(3)}$ and renormalizing to turn this restricted measure into a probability measure, we get an $(A_{U_1}, \rho)$-equivariant Borel map $U_1 \to \text{Prob}((\partial_{\infty} T)^{(3)})$. Now, denoting by $V(T)$ the vertex set of $T$, there is a natural $F$-equivariant barycenter map $(\partial_{\infty} T)^{(3)} \to V(T)$. By pushing the probability measures through this map, we get an $(A_{U_1}, \rho)$-equivariant Borel map $U_1 \to \text{Prob}(V(T))$. Let $\mathcal{P}_{\infty}(V(T))$ be the set of all nonempty finite subsets of $V(T)$. As $V(T)$ is countable, there is also a natural $F$-equivariant Borel map $\text{Prob}(V(T)) \to \mathcal{P}_{\infty}(V(T))$, sending a probability measure $\nu$ to the finite subset of $V(T)$ made of all vertices that have maximal $\nu$-measure. We thus derive an $(A_{U_1}, \rho)$-equivariant Borel map $\phi : U_1 \to \mathcal{P}_{\infty}(V(T))$.

As $\mathcal{P}_{\infty}(V(T))$ is countable, we can then find a Borel partition $U_1 = \sqcup_{i \in I} U_{1,i}$ into at most countably many Borel subsets such that for every $i \in I$, the map $\phi|_{U_{1,i}}$ is constant, with value a nonempty finite set $F_i$ of vertices of $T$. In other words, there exists a count Borel subset $U_{1,i}^* \subseteq U_{1,i}$ such that $\rho(A_{U_{1,i}})$ is contained in the $F$-stabilizer of $F_i$. As this stabilizer is trivial and $\rho$ has trivial kernel, it follows that $A_{U_1}$ is stably trivial.

We will now prove that $U_2 = U \setminus U_1$ is a null set, which will conclude the proof of the lemma. So assume towards a contradiction that $U_2$ has positive measure. We know that there exists an $(A_{U_2}, \rho)$-equivariant Borel map $\mu : U_2 \to \text{Prob}(\partial_{\infty} T)$, and that for every such map and almost every $y \in U_2$, the support of $\mu(y)$ has cardinality at most 2. Let $\mathcal{P}_{\leq 2}(\partial_{\infty} T)$ be the set of all nonempty subsets of $\partial_{\infty} T$ of cardinality at most 2. As in [Ada94, Lemma 3.2], we can thus find an $(A_{U_2}, \rho)$-equivariant Borel map $\theta_{\max} : U_2 \to \mathcal{P}_{\leq 2}(\partial_{\infty} T)$ which is maximal in the sense that for every other $(A_{U_2}, \rho)$-equivariant Borel map $\theta : U_2 \to \mathcal{P}_{\leq 2}(\partial_{\infty} T)$ and a.e. $y \in Y$, one has $\theta(y) \subseteq \theta_{\max}(y)$. Being canonical, the map $\theta_{\max}$ is then equivariant under the groupoid $(\rho^{-1}(\langle g \rangle)_{|U_2}, \rho^{-1}(\langle h \rangle)_{|U_2})$ which normalizes $A_{U_2}$. Recall that the groupoid $\rho^{-1}(\langle g \rangle)_{|U_2}$ is amenable and of infinite type. Therefore, repeating the argument from the present proof shows that there exists a maximal $(\rho^{-1}(\langle g \rangle)_{|U_2}, \rho)$-equivariant Borel map $U_2 \to \mathcal{P}_{\leq 2}(\partial_{\infty} T)$, and this must then be the constant map with value $\{g^{-\infty}, g^{+\infty}\}$. Likewise, the constant map with value $\{h^{-\infty}, h^{+\infty}\}$ is the maximal $(\rho^{-1}(\langle h \rangle)_{|U_2}, \rho)$-equivariant Borel map $U_2 \to \mathcal{P}_{\leq 2}(\partial_{\infty} T)$. As $\{g^{-\infty}, g^{+\infty}\} \cap \{h^{-\infty}, h^{+\infty}\} = \emptyset$, we have reached a contradiction. This completes our proof.

\[ \square \]

### 3.2 Canonical reduction sets, after Kida

In this section, we review work of Kida regarding groupoids with cocycles towards a surface mapping class group. Since our terminology slightly differs from Kida’s, we recall proofs for the convenience of the reader. We also mention that the results in this section can also be viewed as a special case of those in Section 3.6, applied by taking for $\mathcal{P}$ the set of all elementwise stabilizers of collections of curves on the surface, but we believe it is useful to have the arguments specified in our context. In the whole section, we let $\Sigma$ be a (possibly disconnected) orientable surface of finite type, i.e. $\Sigma$ is obtained from the disjoint union of finitely many closed connected orientable surfaces by removing at most finitely many points. We define $\text{Mod}^0(\Sigma)$ as the group of all isotopy classes of orientation-preserving diffeomorphisms of $\Sigma$ that do not permute the
connected components of $\Sigma$, and act trivially on the homology mod 3 of each connected component; in other words $\text{Mod}^0(\Sigma) = \text{Mod}^0(\Sigma_1) \times \cdots \times \text{Mod}^0(\Sigma_k)$, where $\Sigma_1, \ldots, \Sigma_k$ are the connected components of $\Sigma$.

**Definition 3.7 (Irreducibility).** Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict cocycle $\rho: \mathcal{G} \to \text{Mod}^0(\Sigma)$.

We say that $(\mathcal{G}, \rho)$ is reducible if there exist a Borel subset $U \subseteq Y$ of positive measure and an essential simple closed curve $c$ on $\Sigma$ such that the isotopy class of $c$ is $(\mathcal{G}|_U, \rho)$-invariant.

Otherwise, we say that $(\mathcal{G}, \rho)$ is irreducible.

**Definition 3.8 (Canonical reduction set).** Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict cocycle $\rho: \mathcal{G} \to \text{Mod}^0(\Sigma)$. A (possibly infinite) set $C$ of isotopy classes of essential simple closed curves on $\Sigma$ is a canonical reduction set for $(\mathcal{G}, \rho)$ if

1. every $c \in C$ is $(\mathcal{G}, \rho)$-invariant, and

2. for every Borel subset $U \subseteq Y$ of positive measure, every isotopy class $c'$ of essential simple closed curves which is $(\mathcal{G}|_U, \rho)$-invariant belongs to $C$.

Note that $(\mathcal{G}, \rho)$ is irreducible if and only if $\emptyset$ is a canonical reduction set for $\mathcal{G}$. Notice also that if a canonical reduction set for $(\mathcal{G}, \rho)$ exists, then it is unique. The following statement shows that up to a countable Borel partition of the base space, canonical reduction sets always exist.

**Lemma 3.9.** Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict cocycle $\rho: \mathcal{G} \to \text{Mod}^0(\Sigma)$.

Then there exist a partition $Y = \bigsqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, $(\mathcal{G}|_{Y_i}, \rho)$ has a canonical reduction set.

**Proof.** Let $Y'_0 \subseteq Y$ be a Borel subset of maximal measure such that there exists a partition $Y'_0 = \bigsqcup_{i \in I_0} Y_i$ into at most countably many Borel subsets such that for every $i \in I_0$, the set $C_i$ of all isotopy classes of essential simple closed curves that are $(\mathcal{G}|_{Y_i}, \rho)$-invariant is nonempty. We emphasise that $C_i$ is possibly infinite, and may contain non-disjoint curves. Notice that such a Borel subset $Y'_0$ exists, because if $(Y'_0, n)_{n \in \mathbb{N}}$ is a measure-maximizing sequence of such sets, then their countable union also satisfies the same property. Let now $Y_0 = Y \setminus Y'_0$. The maximality of the measure of $Y'_0$ ensures that $(\mathcal{G}|_{Y_0}, \rho)$ is irreducible.

For every $i \in I_0$, we then let $G_C_i$ be the elementwise stabilizer of $C_i$ in $\text{Mod}^0(\Sigma)$: this is a proper subgroup of $\text{Mod}^0(\Sigma)$ (because $C_i \neq \emptyset$). Repeating the above argument, for every $i \in I_0$, there exists a Borel partition $Y_i = Y_{i,0} \sqcup Y'_{i,0}$ such that

1. for every Borel subset $U \subseteq Y_{i,0}$ of positive measure, every $(\mathcal{G}|_U, \rho)$-invariant isotopy class of essential simple closed curve belongs to $C_i$,
Lemma 3.10. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict cocycle $\rho: \mathcal{G} \to \text{Mod}^0(\Sigma)$, and let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$. Assume that $(\mathcal{H}, \rho)$ has a canonical reduction set $C$.

Then for every measured subgroupoid $\mathcal{H}'$ of $\mathcal{G}$ that normalizes $\mathcal{H}$, the set $C$ is $(\mathcal{H}', \rho)$-invariant. In other words, denoting by $\text{Stab}(C)$ the global stabilizer of $C$ in $\text{Mod}^0(\Sigma)$, there exists a conull Borel subset $Y^* \subseteq Y$ such that $\rho(\mathcal{H}_{Y^*}) \subseteq \text{Stab}(C)$.

Proof. Since $\mathcal{H}'$ normalizes $\mathcal{H}$, there exists a covering of $\mathcal{H}'$ by countably many bisections $B_n$ that all leave $\mathcal{H}$ invariant. Up to subdividing the bisections $B_n$, we will assume that for every $n \in \mathbb{N}$, the $\rho$-image of $B_n$ is a single element $\gamma_n \in \text{Mod}^0(\Sigma)$. For every $n \in \mathbb{N}$, we let $U_n$ and $V_n$ be the source and range of $B_n$.

Every isotopy class $c \in C$ is $(\mathcal{H}_{V_n}, \rho)$-invariant, so $\gamma_n c$ is $(\mathcal{H}_{V_n}, \rho)$-invariant. If $V_n$ has positive measure, the maximality condition in the definition of a canonical reduction set ensures that $\gamma_n c \in C$. By reversing the arrows in the bisection $B_n$, we also derive that $\gamma_n c \notin C$ if $c \notin C$. Let $Y^* \subseteq Y$ be a conull Borel subset which avoids each of the countably many subsets $U_n$ and $V_n$ of zero measure. Then $\rho(\mathcal{H}_{Y^*}) \subseteq \text{Stab}(C)$. This concludes our proof. \hfill \Box

Given a (possibly infinite) set $C$ of isotopy classes of essential simple closed curves on $\Sigma$, there is up to isotopy a unique essential subsurface $S \subseteq \Sigma$ such that every curve in $C$ is isotopic into $S$, and if $S'$ is another subsurface with this property, then up to isotopy $S \subseteq S'$. We call $S$ the subsurface of $\Sigma$ filled by $C$. The multicurve $X$, obtained from $\partial S$ by only keeping one curve in each isotopy class, is called the boundary multicurve of $C$.

Corollary 3.11. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict cocycle $\rho: \mathcal{G} \to \text{Mod}^0(\Sigma)$. Let $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{G}$ be measured subgroupoids. Assume that $\mathcal{H}$ is stably normalized by $\mathcal{H}'$, and that for every Borel subset $U \subseteq Y$ of positive measure, one has $\rho(\mathcal{H}_{U}) \neq \{1\}$.

If $(\mathcal{H}, \rho)$ is reducible, then so is $(\mathcal{H}', \rho)$. 

18
Proof. Since \((\mathcal{H}, \rho)\) is reducible, we can find a Borel subset \(U \subseteq Y\) of positive measure such that \((\mathcal{H}_U, \rho)\) has a nonempty canonical reduction set \(C\). As \(\rho(\mathcal{H}_U) \neq \{1\}\), the set \(C\) does not fill \(\Sigma\), so the boundary multicurve \(X\) of \(C\) is nonempty. Up to restricting to a Borel subset of \(U\) of positive measure, we can assume that \(\mathcal{H}_U\) is normalized by \(\mathcal{H}'_U\). Then Lemma \ref{lem:crs-normal} ensures that \(C\) is \((\mathcal{H}'_U, \rho)\)-invariant. In particular \(X\) is \((\mathcal{H}'_U, \rho)\)-invariant, showing that \((\mathcal{H}', \rho)\) is reducible. \(\square\)

When \(C\) is the canonical reduction set for \((\mathcal{H}, \rho)\), the boundary multicurve \(X\) of \(C\) will be called the canonical reduction multicurve of \((\mathcal{H}, \rho)\). A connected component \(S\) of \(\Sigma \setminus X\) is then called active for \((\mathcal{H}, \rho)\) if it contains an essential simple closed curve whose isotopy class does not belong to \(C\), and inactive for \((\mathcal{H}, \rho)\) otherwise (because in the latter case, every element in the elementwise stabilizer of \(C\) acts trivially on \(S\)).

We give a few examples of active and inactive subsurfaces in the case that the essential image of \(\rho\) is a cyclic subgroup generated by \(\varphi\).

i) If \(\varphi\) is a partial pseudo-Anosov supported on a connected subsurface \(Z \subset \Sigma\), possibly composed with Dehn twists about curves contained in \(\partial Z\), then the canonical reduction multicurve is \(\partial Z\), and \(Z\) is the only active complementary component.

ii) If \(\varphi\) is a Dehn twist about a curve \(\alpha\), then the canonical reduction multicurve is \(\alpha\), and all complementary components are inactive.

3.3 Exploiting amenable normalized subgroupoids, after Kida

The following result of Kida will be used extensively in the remainder of this section, applied either to \(\partial V\) or to subsurfaces of \(\partial V\). We include a proof to explain how to deal with disconnected subsurfaces.

Lemma 3.12 (Kida \cite{Kid08a}). Let \(\Sigma\) be a (possibly disconnected) surface of finite type, so that every connected component has negative Euler characteristic. Let \(G\) be a measured groupoid, equipped with a strict cocycle \(\rho : G \to \text{Mod}^0(\Sigma)\). Let \(H\) be a measured subgroupoid of \(G\) such that \(\rho|_H\) has trivial kernel.

If \(H\) stably normalizes an amenable subgroupoid \(A\) of \(G\), with \((A, \rho)\) irreducible, then \(H\) is amenable.

Proof. Up to a countable Borel partition of the base space \(Y\) of \(G\) (which does not affect the conclusion), we will assume that \(H\) normalizes \(A\).

Let \(\Sigma_1, \ldots, \Sigma_k\) be the connected components of \(\Sigma\). Then \(\text{Mod}^0(\Sigma)\) decomposes as \(\text{Mod}^0(\Sigma) = \text{Mod}^0(\Sigma_1) \times \cdots \times \text{Mod}^0(\Sigma_k)\). For \(i \in \{1, \ldots, k\}\), let \(\rho_i : G \to \text{Mod}^0(\Sigma_i)\) be the cocycle obtained by post-composing \(\rho\) with the \(i\)-th projection.

Let \(i \in \{1, \ldots, k\}\). Then \(\text{Mod}^0(\Sigma_i)\) acts on the compact metrizable space \(\text{PML}(\Sigma_i)\) of projective measured laminations on \(\Sigma_i\). As \(A\) is amenable, there exists an \((A, \rho_i)\)-equivariant Borel map \(\mu : Y \to \text{Prob}(\text{PML}(\Sigma_i))\). The space \(\text{PML}(\Sigma_i)\) has a \(\text{Mod}(\Sigma_i)\)-invariant Borel partition into the subspace \(\text{AL}_i\) made of arational laminations, and the subspace \(\text{NAL}_i\) made of non-arational laminations.

19
Lemma 3.13. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho: \mathcal{G} \rightarrow \text{Mod}^0(V)$. Let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$.

Let $c, c'$ be two meridians, with $c$ nonseparating. Assume that there exists a Borel subset $U \subseteq Y$ of positive measure such that for all $y \in U$, the measure $\mu(y)$ gives positive measure to $\text{NAL}_i$. After restricting $\mu(y)$ to $\text{NAL}_i$ and renormalizing it to get a probability measure, we obtain an $((A_U, \rho_i))$-equivariant Borel map $U \rightarrow \text{Prob}(\text{NAL}_i)$. Let $\mathcal{P}_{<\infty}(C(\Sigma_i))$ be the countable set of all nonempty finite sets of isometry classes of essential simple closed curves on $\Sigma_i$. There is a $\text{Mod}(\Sigma_i)$-equivariant map $\text{NAL}_i \rightarrow \mathcal{P}_{<\infty}(C(\Sigma_i))$, sending a lamination to the union of all simple closed curves it contains together with all boundaries of the subsurfaces it fills. We thus get an $((A_U, \rho_i))$-equivariant Borel map $U \rightarrow \text{Prob}(\mathcal{P}_{<\infty}(C(\Sigma_i)))$. As $\mathcal{P}_{<\infty}(C(\Sigma_i))$ is countable, there is also a $\text{Mod}(\Sigma_i)$-equivariant Borel map $\text{Prob}(\mathcal{P}_{<\infty}(C(\Sigma_i))) \rightarrow \mathcal{P}_{<\infty}(C(\Sigma_i))$, sending a probability measure $\nu$ to the union of all finite sets with maximal $\nu$-measure. In summary, we have found an $((A_U, \rho_i))$-equivariant Borel map $U \rightarrow \mathcal{P}_{<\infty}(C(\Sigma_i))$. Let $V \subseteq U$ be a Borel subset of positive measure where this map is constant, with value a finite set $\mathcal{F}$. As we are working in the finite-index subgroup $\text{Mod}^0(\Sigma_i)$, every curve in $\mathcal{F}$ is $(A_{|V}, \rho_i)$-invariant, contradicting the irreducibility of $(A, \rho)$.

Therefore $\mu$ determines an $((A, \rho))$-equivariant Borel map $Y \rightarrow \text{Prob}(\text{AL}_i)$. Klarrich’s description [Kid-memoir, Section 3.4.1] of the boundary $\partial_\infty C_i$ of the curve graph of $\Sigma_i$ yields a continuous $\text{Mod}(\Sigma_i)$-equivariant map $\text{AL}_i \rightarrow \partial_\infty C_i$, so we get an $((A, \rho))$-equivariant Borel map $Y \rightarrow \text{Prob}(\partial_\infty C_i)$. Denoting by $(\partial_\infty C_i)^{(3)}$ the space of pairwise distinct triples, Kida proved in [Kid-memoir, Section 4.1] the existence of a $\text{Mod}(\Sigma_i)$-equivariant Borel map $(\partial_\infty C_i)^{(3)} \rightarrow \mathcal{P}_{<\infty}(C(\Sigma_i))$. Using again the irreducibility of $(A, \rho)$, together with an Adams-type argument as in the proof of Lemma 3.6, we deduce that there exists a Borel map $Y \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty C_i)$ which is both $(A, \rho_i)$-equivariant and $(\mathcal{H}, \rho_i)$-equivariant.

Combining all these maps as $i$ varies in $\{1, \ldots, k\}$ yields an $(\mathcal{H}, \rho)$-equivariant Borel map

$$Y \rightarrow \mathcal{P}_{\leq 2}(\partial_\infty C_1) \times \cdots \times \mathcal{P}_{\leq 2}(\partial_\infty C_k).$$

For every $i \in \{1, \ldots, k\}$, the action of $\text{Mod}(\Sigma_i)$ on $\partial_\infty C_i$ is Borel amenable [Kid-memoir, Ham09], and therefore so is the action of $\text{Mod}^0(\Sigma)$ on $\mathcal{P}_{\leq 2}(\partial_\infty C_1) \times \cdots \times \mathcal{P}_{\leq 2}(\partial_\infty C_k)$ (see e.g. [H120b, Section 3.4.1] for the relevant background). As $\rho|_Y$ has trivial kernel, it then follows from [K121, Proposition 3.38] (originally due to Kida [Kid-memoir, Proposition 4.33]) that $\mathcal{H}$ is amenable.

3.4 Uniqueness statements

Lemma 3.13. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho: \mathcal{G} \rightarrow \text{Mod}^0(V)$. Let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$.

Let $c, c'$ be two meridians, with $c$ nonseparating. Assume that there exists a Borel subset $U \subseteq Y$ of positive measure such that $\mathcal{H}|_U$ is equal to the $(\mathcal{G}|_U, \rho)$-stabilizer of the isotopy class of $c$, and the isotopy class of $c'$ is $(\mathcal{H}|_U, \rho)$-invariant.

Then $c' = c$ (up to isotopy).

Proof. The stabilizer of $c$ in $\text{Mod}^0(V)$ contains an element $g$ which restricts to a pseudo-Anosov element on $\partial V \setminus c$ (Lemma 4.1). The groupoid $\rho^{-1}((g))[U]$ is contained in $\mathcal{H}|_U$, 20
and it is of infinite type since $\rho$ is action-type. Therefore $c'$ is fixed by some positive power of $g$, which implies that $c' = c$ up to isotopy. $\square$

The following is a version of Lemma 3.13 for separating meridians, whose proof is similar and left to the reader (recall that although the stabiliser $H$ of a separating meridian $c$ may fix other nonseparating meridians, $c$ is the unique separating meridian it fixes by Lemma 1.2).

**Lemma 3.14.** Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to \text{Mod}^0(V)$. Let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$.

Let $c,c'$ be two separating meridians. Assume that there exists a Borel subset $U \subseteq Y$ of positive measure such that $H|_U$ is equal to the $(\mathcal{G}|_U,\rho)$-stabilizer of the isotopy class of $c$, and the isotopy class of $c'$ is $(\mathcal{H}|_U,\rho)$-invariant.

Then $c = c'$ (up to isotopy). $\square$

### 3.5 Property (P$_{nsep}$) and subgroupoids of non-separating meridian type

We make the following definition.

**Definition 3.15** (Product-like subgroupoid). A measured groupoid $\mathcal{P}$ is product-like if there exist two subgroupoids $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}$ which are both stably normal in $\mathcal{P}$, such that for every $i \in \{1,2\}$, the groupoid $\mathcal{P}_i$ contains a strongly Schottky pair of subgroupoids $(A_1^i, A_2^i)$, with $A_1^i$ and $A_2^i$ both stably normalized by $\mathcal{P}_{3-i}$.

Notice that this notion is stable under restrictions and stabilization. In the terminology from [GH21, Definition 13.5], the subgroupoids $\mathcal{P}_1$ and $\mathcal{P}_2$ form a pseudo-product. One difference between our definition and [GH21, Definition 13.5] is that we are working with strongly Schottky pairs of subgroupoids, while [GH21, Definition 13.5] is phrased using the weaker notion of Schottky pairs of subgroupoids. Also, we are further imposing that $\mathcal{P}_1$ and $\mathcal{P}_2$ are stably normal in an ambient groupoid $\mathcal{P}$.

We now introduce the following properties, which will be useful in order to detect subgroupoids of nonseparating-meridian type.

**Definition 3.16.** Let $\mathcal{G}$ be a measured groupoid, and let $\mathcal{A}, \mathcal{H}$ be measured subgroupoids of $\mathcal{G}$, with $\mathcal{A} \subseteq \mathcal{H}$.

1. We say that the pair $(\mathcal{H}, \mathcal{A})$ satisfies Property (Q$_{nsep}$) if the following conditions hold:
   
   (a) $\mathcal{H}$ is everywhere nonamenable;
   (b) $\mathcal{A}$ is amenable, of infinite type, and stably normal in $\mathcal{H}$;
   (c) if $\mathcal{B}$ is a stably normal amenable subgroupoid of $\mathcal{H}$, then $\mathcal{B}$ is stably contained in $\mathcal{A}$.
(d) if $\mathcal{H}'$ is another subgroupoid of $\mathcal{G}$ which is everywhere nonamenable and contains a stably normal amenable subgroupoid of infinite type, and if $\mathcal{H}$ is stably contained in $\mathcal{H}'$, then $\mathcal{H}$ is stably equal to $\mathcal{H}'$;

(e) for every Borel subset $U \subseteq Y$ of positive measure, the groupoid $\mathcal{H}|_U$ is not contained in any product-like subgroupoid of $\mathcal{G}|_U$.

2. We say that $\mathcal{H}$ satisfies Property (P$_{nsep}$) if there exists a measured subgroupoid $\mathcal{A} \subseteq \mathcal{H}$ such that $(\mathcal{H}, \mathcal{A})$ satisfies Property (Q$_{nsep}$).

Remark 3.17. These properties are stable under restrictions and stabilization. Also, if $\mathcal{H}$ satisfies Property (P$_{nsep}$), then a subgroupoid $\mathcal{A} \subseteq \mathcal{H}$ such that $(\mathcal{H}, \mathcal{A})$ satisfies Property (Q$_{nsep}$) is “stably unique” in the following sense: if $\mathcal{A}$ and $\mathcal{A}'$ are two such subgroupoids, there exist a conull Borel subset $Y^* \subseteq Y$ and a partition $Y^* = \bigcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, one has $\mathcal{A}|_{Y_i} = \mathcal{A}'|_{Y_i}$.

The goal of the present section is to prove that subgroupoids of nonseparating meridian type with respect to an action-type cocycle $\mathcal{G} \to \text{Mod}^0(V)$ satisfy Property (P$_{nsep}$).

Proposition 3.18. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to \text{Mod}^0(V)$. Let $c$ be a nonseparating meridian, let $\mathcal{H}$ be the $(\mathcal{G}, \rho)$-stabilizer of the isotopy class of $c$, and let $\mathcal{A} = \rho^{-1}(\langle T_\mathcal{C} \rangle)$.

Then $(\mathcal{H}, \mathcal{A})$ satisfies Property (Q$_{nsep}$).

Proposition 3.18 is the combination of our three next lemmas. The first checks Assertions (a),(b) and (c) from Definition 3.16. For later convenience, in this lemma, we also allow for separating meridians in the statement.

Lemma 3.19. Let $\mathcal{G}$ be a measured groupoid, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to \text{Mod}^0(V)$. Let $c$ be a meridian, and let $\mathcal{H}$ be the $(\mathcal{G}, \rho)$-stabilizer of the isotopy class of $c$. Let $\Sigma \subseteq \partial V$ be the union of all components of $\partial V \setminus c$ which are not once-holed tori. Let $A \subseteq \text{Stab}_{\text{Mod}^0(V)}(c)$ be the kernel of the restriction homomorphism to $\text{Mod}^0(\Sigma)$, and let $\mathcal{A} = \rho^{-1}(A)$.

Then $\mathcal{H}$ is everywhere nonamenable, $\mathcal{A}$ is a normal amenable subgroupoid of $\mathcal{H}$ of infinite type, and every stably normal amenable subgroupoid of $\mathcal{H}$ is stably contained in $\mathcal{A}$.

Proof. Notice that the subsurface $\Sigma$ is nonempty because the genus of $V$ is at least 3. Lemma 1.1 ensures that $\text{Stab}_{\text{Mod}^0(V)}(c)$ contains a nonabelian free subgroup, so Lemma 3.1 shows that $\mathcal{H}$ is everywhere nonamenable.

Normality of $\mathcal{A}$ in $\mathcal{H}$ follows from the normality of $A$ in $\text{Stab}_{\text{Mod}^0(V)}(c)$. Notice that $A$ is amenable (using Lemma 1.2 in the case where one of the complementary components of $c$ is a once-holed torus). As $\rho$ has trivial kernel, it follows that $\mathcal{A}$ is amenable. And

\[\text{notice that if } c \text{ is nonseparating, or if } c \text{ is separating and both complementary components have genus at least 2, then } A = \langle T_\mathcal{C} \rangle \cap \text{Mod}^0(V)\]
\( \mathcal{A} \) is of infinite type because \( \mathcal{A} \) is infinite (it always contains a power of \( T_c \)) and \( \rho \) is action-type.

Let now \( \mathcal{B} \subseteq \mathcal{H} \) be a stably normal amenable subgroupoid of \( \mathcal{H} \). Let \( S \subseteq \Sigma \) be a connected component of \( \Sigma \). Let \( \rho_S : \mathcal{H} \to \text{Mod}^0(S) \) be the cocycle obtained by post-composing \( \rho \) with the restriction homomorphism. Let also \( F \subseteq \text{Stab}_{\text{Mod}^0(V)}(c) \) be a nonabelian free subgroup which embeds into \( \text{Mod}^0(S) \) under the restriction homomorphism, and whose image in \( \text{Mod}^0(S) \) contains a pseudo-Anosov mapping class (this exists because \( S \) is not a once-holed torus, see Lemma 1.1).

By Lemma 3.9 we can find a partition \( Y = \bigcup_{i \in I} Y_i \) into at most countably many Borel subsets such that for every \( i \in I \), the pair \( (B_{Y_i}, \rho_S) \) has a canonical reduction set \( \mathcal{C}_i \). As \( \mathcal{B} \) is stably normal in \( \mathcal{H} \), up to refining the above partition, we can assume that for every \( i \in I \), the groupoid \( B_{Y_i} \) is normal in \( \mathcal{H}_{|Y_i} \). Lemma 3.10 thus ensures that \( \mathcal{C}_i \) is \( (\mathcal{H}_{|Y_i}, \rho_S) \)-invariant, so either \( \mathcal{C}_i = \emptyset \) or \( \mathcal{C}_i \) fills \( S \).

Assume towards a contradiction that \( \mathcal{C}_i = \emptyset \) for some \( i \in I \) such that \( Y_i \) has positive measure. In other words \( (B_{Y_i}, \rho_S) \) is irreducible. As \( \rho_S \) has trivial kernel in restriction to \( \mathcal{H}' \), and as \( \mathcal{H}' \) (which is contained in \( \mathcal{H} \)) normalizes \( \mathcal{B} \), Lemma 3.12 implies that \( \mathcal{H}'_{|Y_i} \) is amenable. But \( F \) is a nonabelian free group and \( \rho \) is action-type, so we get a contradiction.

It follows that for every \( i \in I \), there exists a conull Borel subset \( Y_i^* \subseteq Y_i \) such that \( \rho_S(B_{Y_i^*}) = \{1\} \). As \( S \) was an arbitrary connected component of \( \Sigma \), this precisely means that \( \mathcal{B} \) is stably contained in \( \mathcal{A} \).

We now check Assertion (d) from Definition 3.10.

**Lemma 3.20.** Let \( \mathcal{G} \) be a measured groupoid over a standard probability space \( Y \), equipped with a strict action-type cocycle \( \rho : \mathcal{G} \to \text{Mod}^0(V) \). Let \( c \) be a nonseparating meridian, and let \( \mathcal{H} \) be the \((\mathcal{G}, \rho)\)-stabilizer of the isotopy class of \( c \).

If \( \mathcal{H}' \) is a subgroupoid of \( \mathcal{G} \) which is everywhere nonamenable and contains a stably normal amenable subgroupoid of infinite type, and if \( \mathcal{H} \) is stably contained in \( \mathcal{H}' \), then \( \mathcal{H} \) is stably equal to \( \mathcal{H}' \).

**Proof.** Let \( \mathcal{A}' \) be an amenable subgroupoid of \( \mathcal{G} \) of infinite type which is contained in \( \mathcal{H}' \) and stably normal in \( \mathcal{H}' \). By Lemma 3.9 we can find a partition \( Y = \bigcup_{i \in I} Y_i \) into at most countably many Borel subsets such that for every \( i \in I \), the pair \( (A'_{|Y_i}, \rho) \) has a (possibly empty) canonical reduction set \( \mathcal{C}_i \). For every \( i \in I \), we let \( X_i \) be the (possibly empty) boundary multicurve of \( \mathcal{C}_i \). As \( \mathcal{A}' \) is stably normal in \( \mathcal{H}' \), up to refining the above partition, we can assume that for every \( i \in I \), the set \( \mathcal{C}_i \) is \( (\mathcal{H}'_{|Y_i}, \rho) \)-invariant (Lemma 3.10), and therefore so is the multicurve \( X_i \). As \( \mathcal{H} \) is stably contained in \( \mathcal{H}' \), we will also assume up to refining the above partition once more that for every \( i \in I \), one has \( \mathcal{H}_{|Y_i} \subseteq \mathcal{H}'_{|Y_i} \). In particular \( X_i \) is \( (\mathcal{H}_{|Y_i}, \rho) \)-invariant, which implies that either \( X_i = \emptyset \) or \( X_i = c \).

Let \( i \in I \) be such that \( Y_i \) has positive measure. If \( X_i = \emptyset \), then as \( \mathcal{A}' \) is of infinite type and \( \rho \) has trivial kernel, we deduce that \( \mathcal{C}_i = \emptyset \), i.e. \( (A'_{|Y_i}, \rho) \) is irreducible. Lemma 3.12 then implies that \( \mathcal{H}'_{|Y_i} \) is amenable, a contradiction. Therefore \( X_i = c \), so the isotopy
Lemma 3.22. Let $G$ be a measured groupoid, equipped with a strict action-type cocycle $\rho : G \to \text{Mod}^0(V)$, and let $\mathcal{H}$ be a measured subgroupoid of $G$. Let $c$ be a separating meridian, and assume that the isotopy class of $c$ is $(\mathcal{H}, \rho)$-invariant.

Then $\mathcal{H}$ does not satisfy Property $(P_{\text{nsep}})$. 

3.6 Stabilizers of separating meridians do not satisfy Property $(P_{\text{nsep}})$
Proof. We first assume that one complementary component \( \Sigma \) of \( c \) is a once-holed torus. Then \( \Sigma \) contains, up to isotopy, a unique nonseparating meridian \( \delta \) (Lemma 1.2), so \( \mathcal{H} \) is contained in the \((\mathcal{G}, \rho)\)-stabilizer \( \mathcal{H}' \) of the isotopy class of \( \delta \). In addition \( \mathcal{H}' \) is everywhere nonamenable and contains \( \rho^{-1}(\langle T_{\delta} \rangle) \) as a normal amenable subgroupoid of infinite type. Finally \( \mathcal{H}' \) is not stably contained in \( \mathcal{H} \) because \( \partial V \setminus \delta \) supports a pseudo-Anosov handlebody group element \( g \), and no nontrivial power of \( g \) preserves the isotopy class of \( c \). So Assumption (d) from Definition 3.16 fails.

We now assume that both complementary components \( \Sigma_1, \Sigma_2 \) of \( c \) have genus at least 2. Let \( \mathcal{P} \) be the \((\mathcal{G}, \rho)\)-stabilizer of \( c \). Then \( \mathcal{H} \) is contained in \( \mathcal{P} \), and we will prove that \( \mathcal{P} \) is product-like (which will imply that Assumption (e) from Definition 3.16 fails). For every \( i \in \{1, 2\} \), let \( P_i \) be the subgroup of \( \text{Mod}^0(V) \) made of elements that have a representative supported in \( \Sigma_i \), and let \( \mathcal{P}_i = \rho^{-1}(P_i) \). Then \( \mathcal{P}_i \) is normal in \( \mathcal{P} \). For every \( i \in \{1, 2\} \), let \( f_i^1 \) and \( f_i^2 \) be two elements of \( P_i \) that generate a nonabelian free subgroup of \( \text{Mod}^0(V) \). For every \( i \in \{1, 2\} \) and every \( j \in \{1, 2\} \), let \( A_i^j = \rho^{-1}(\langle f_i^j \rangle) \). Then \( A_i^j \) is normalized by \( \mathcal{P}_{3-i} \), and Lemma 3.6 ensures that \( (A_1^1, A_2^2) \) is a strongly Schottky pair of subgroupoids of \( \mathcal{G} \). This completes our proof. \( \square \)

### 3.7 Admissible decorated multicurves and their active subgroups

A decorated multicurve is a pair \((X, \mathfrak{A})\), where \( X \) is a multicurve on \( \partial V \), and \( \mathfrak{A} \) is a subset of the set of complementary components of \( X \) on \( \partial V \). We make the following definition.

**Definition 3.23.** Let \((X, \mathfrak{A})\) be a decorated multicurve. The subgroup \( A \) \( \text{of} \) \( \text{Mod}^0(V) \) \( \text{made of}\) all elements that preserve the isotopy class of \( X \) \( \text{act trivially on}\) all complementary subsurfaces \( \text{not in} \) \( \mathfrak{A} \) \( \text{is called the active subgroup of} \) \((X, \mathfrak{A})\).

The decorated multicurve \((X, \mathfrak{A})\) is admissible if its active subgroup \( A \) is amenable, \( X \) \( \text{is the boundary multicurve of} \) \( A \), and \( \mathfrak{A} \) \( \text{is its set of active complementary components.} \)

Here is an example. If \( X \) consists of a single nonseparating meridian on \( \partial V \), or a separating meridian none of whose complementary components is a once-holed torus, and \( \mathfrak{A} = \emptyset \), then \((X, \mathfrak{A})\) is admissible, and its active subgroup is the corresponding twist subgroup. In the case where one of the complementary components of a meridian \( c \) is a once-holed torus (and contains a unique nonseparating meridian \( d \) up to isotopy), then \((c \cup d, \emptyset)\) is admissible, with active subgroup the twist subgroup about \( c \cup d \). Other examples come from annulus pairs instead of meridians.

Let \( \mathcal{G} \) be a measured groupoid over a standard probability space \( Y \), equipped with an action-type cocycle \( \rho : \mathcal{G} \to \text{Mod}^0(V) \). We say that a pair \((\mathcal{H}, \mathcal{A})\) of subgroupoids of \( \mathcal{G} \) is admissible with respect to \( \rho \) if there exist a conull Borel subset \( Y^* \subseteq Y \) and a partition \( Y^* = \sqcup_{i \in I} Y_i \) into at most countably many Borel subsets, such that for every \( i \in I \), there exist a multicurve \( X_i \) on \( \partial V \), and a subset \( J_i \) of the set of all complementary components of \( X_i \) such that \((X_i, J_i)\) is admissible, \( \mathcal{H}_{|Y_i} \) is equal to the \((\mathcal{G}_{|Y_i}, \rho_{|Y_i})\)-stabilizer of the isotopy class of \( X_i \), and denoting by \( A_i \subseteq \text{Mod}^0(V) \) the active subgroup of \((X_i, J_i)\), one has \( A_{i|Y_i} = \rho^{-1}(A_i)_{|Y_i} \). Notice that, although the above partition is not unique (one
Lemma 3.24. Let $G$ be a measured groupoid over a standard probability space $X$, equipped with a strict action-type cocycle $\rho : G \to Mod^0(V)$. Let $A, H$ be measured subgroupoids of $G$, with $A \subseteq H$.

If $(H, A)$ satisfies Property $(Q_{nsep})$, then $(H, A)$ is an admissible pair.

Proof. By Assumption (b) from Definition 3.16, the groupoid $A$ is amenable, of infinite type, and stably normal in $H$. Up to a countable partition of the base space $X$, we will assume that $A$ is normal in $H$. Up to a further partition, we can also assume that $(A, \rho)$ has a canonical reduction set $C$ (Lemma 3.9). Let $X$ be the boundary multicurve of $H$, and set $\mathfrak{A}$ be the set of all active complementary components of $X$ not in $A$. Up to replacing $Y$ by a conull Borel subset, we will assume using Lemma 3.10 that $\rho(\mathfrak{A}) \subseteq \text{Stab}_{Mod^0(V)}(X)$.

We will first prove that $(X, \mathfrak{A})$ is admissible, so let us assume towards a contradiction that it is not. Let $A \subseteq Mod^0(V)$ be the active subgroup of $(X, \mathfrak{A})$. Then there exists a conull Borel subset $Y^* \subseteq Y$ such that $\rho(\mathfrak{A}|_{Y^*}) \subseteq A$. Therefore $C$ is exactly the set of all curves whose isotopy class is $A$-invariant, so $X$ is the boundary multicurve of $A$ and $\mathfrak{A}$ is its set of active complementary components. Therefore, our assumption that $(X, \mathfrak{A})$ is not admissible implies that $A$ is not amenable, so it contains two elements $f, g$ which together generate a nonabelian free group (by the Tits alternative for mapping class groups).

Let $\Sigma_1$ be the union of all subsurfaces in $\mathfrak{A}$, viewed as a (possibly disconnected) surface of finite type. Let $\rho_1 : H \to Mod^0(\Sigma_1)$ be the cocycle obtained by composing $\rho$ with the restriction to $\Sigma_1$. We now observe that for every $U \subseteq Y$ of positive measure, the restriction to $U$ of kernel of $\rho_1$ is nontrivial: otherwise $(A^1, \rho_1)$ is irreducible and $H_U$ normalizes $A_U$, so Lemma 3.12 ensures that $H_U$ is amenable, a contradiction to Assumption (a) from Definition 3.16.

Let $B$ be the kernel of $\rho_1$. The groupoid $B$ is normal in $H$. We first assume that $B$ is amenable, and reach a contradiction in this case. Assumption (c) from Definition 3.16 ensures that there exists a Borel subset $U \subseteq Y$ of positive measure such that $B_U \subseteq A_U$. But the $\rho$-image of every element of $B_U$ acts trivially on all components in $\mathfrak{A}$, while the $\rho$-image of every element of $A_U$ acts trivially on all components in $\mathfrak{A}$. It follows that for every $g \in B_U$, the element $\rho(g)$ is a multitwist around curves in $X$. As $\rho$ has trivial kernel and $B_U$ is nontrivial, it follows that the subgroup $\text{Tw} \subseteq Mod^0(V)$ consisting of all multitwists about the curves in $X$ is infinite. Let $H' = \rho^{-1}(\text{Stab}_{Mod^0(V)}(X))$. Then $H'_U \subseteq H'_U$, and $H'_U$ is everywhere nonamenable (it contains $H_U$) and contains $\rho^{-1}(\text{Tw})_U$, as a normal amenable subgroupoid of infinite type. So Assumption (d) from Definition 3.16 ensures that there exists a Borel subset $U' \subseteq U$ of positive measure such that $H'_{U'} = H_{U'}$. Now, the groupoid $\rho^{-1}((f, g))_{U'}$ is contained in $H_{U'}$, so it normalizes $A_{U'}$, and $\rho_1$ has trivial kernel in restriction to $\rho^{-1}((f, g))_{U'}$ (as otherwise $(f, g)$ would contain an infinite amenable normal subgroup). As $(A_{U'}, \rho_1)$ is irreducible, Lemma 3.12 implies that $\rho^{-1}((f, g))_{U'}$ is amenable, a contradiction to Lemma 3.6.
We now assume that \( B \) is nonamenable, and also reach a contradiction in this case. As \( \rho \) has trivial kernel, the subgroup \( P_2 \) of \( \text{Mod}^0(V) \) made of all elements that fix the isotopy class of \( X \) and act trivially on all connected components in \( \mathfrak{A} \) is nonamenable, and therefore contains a nonabelian free subgroup. Let \( P = \text{Stab}_{\text{Mod}^0(V)}(X) \), and let \( \mathcal{P} = \rho^{-1}(P) \) (i.e. \( \mathcal{P} = \mathcal{H}' \) with the notation from above). We will now reach a contradiction to Assumption (e) from Definition 3.16 by proving that \( \mathcal{P} \) is a product-like subgroupoid of \( \mathcal{G} \) (in which \( \mathcal{H} \) is contained).

Let \( P_1 \leq P \) be the normal subgroup made of all elements of \( P \) that act trivially on all components in \( \mathfrak{A} \), and recall that \( P_2 \leq P \) is the normal subgroup made of all elements of \( P \) acting trivially on all components in \( \mathfrak{A} \). Then \( \mathcal{P}_1 = \rho^{-1}(P_1) \) is normal in \( \mathcal{P} = \rho^{-1}(P) \) for every \( i \in \{1, 2\} \). Notice that \( P_1 \) contains the nonabelian free subgroup \( \langle f, g \rangle \), and as we saw in the previous paragraph that \( P_2 \) also contains a nonabelian free subgroup. For every \( i \in \{1, 2\} \), let \( A^1_i, A^2_i \) be two cyclic subgroups of \( P_i \) that generate a nonabelian free subgroup, and for \( j \in \{1, 2\} \), let \( A^j_i = \rho^{-1}(A^j_i) \). As \( P_1 \) and \( P_2 \) centralize each other, it follows that each \( A^j_i \) is normalized by \( P_{3-i} \). In addition, Lemma 3.16 ensures that \( (A^1_1, A^2_1) \) is a strongly Schottky pair of subgroupoids of \( \mathcal{G} \). So \( \mathcal{P} \) is a product-like subgroupoid of \( \mathcal{G} \), which is the desired contradiction.

This contradiction shows that \((X, \mathfrak{A})\) is admissible. Now, let \( \mathcal{A}' = \rho^{-1}(A) \), and let \( \mathcal{H}' \) be the \((\mathcal{G}, \rho)\)-stabilizer of the isotopy class of \( X \). Then \( \mathcal{H} \) is contained in \( \mathcal{H}' \), and \( \mathcal{H}' \) contains \( A' \) as a normal amenable subgroupoid of infinite type. So Assertion (d) from Definition 3.16 ensures that \( \mathcal{H} \) is stably equal to \( \mathcal{H}' \). And Assertion (c) then implies that \( \mathcal{A} \) is stably equal to \( \mathcal{A}' \). This proves that \((\mathcal{H}, \mathcal{A})\) is an admissible pair. \( \square \)

### 3.8 Compatibility

Two decorated multicurves \((X, \mathfrak{A})\) and \((X', \mathfrak{A}')\) are **compatible** if \( X \) and \( X' \) are disjoint up to isotopy, and given any two components \( S \in \mathfrak{A} \) and \( S' \in \mathfrak{A}' \), either \( S \) and \( S' \) are isotopic, or they are disjoint up to isotopy. We start with the following observation.

**Lemma 3.25.** Let \((X, \mathfrak{A})\) and \((X', \mathfrak{A}')\) be two decorated multicurves, with respective active subgroupoids \( \mathcal{A} \) and \( \mathcal{A}' \).

If \((X, \mathfrak{A})\) and \((X', \mathfrak{A}')\) are compatible, then \( \langle \mathcal{A}, \mathcal{A}' \rangle \) is amenable.

**Proof.** Otherwise, using the Tits alternative for mapping class groups [McCa], there exist \( \varphi, \psi \in \langle \mathcal{A}, \mathcal{A}' \rangle \) that together generate a nonabelian free group. Let \( S \) be the union of all subsurfaces in \( \mathfrak{A} \cap \mathfrak{A}' \). The conclusion is clear if \( S \) is empty (as \( A \) and \( A' \) commute in this case), so we will assume otherwise.

We observe that \( \langle \varphi \vert_S, \psi \vert_S \rangle \) is not virtually abelian: otherwise, as \( A \) and \( A' \) are amenable subgroups of \( \text{Mod}(V) \), there would exist two other virtually abelian subgroups \( B, B' \subseteq \text{Mod}(\partial V) \) (made of mapping classes supported on \( \Sigma \setminus S \) and \( \Sigma' \setminus S \), respectively), which commute and both commute with \( \langle \varphi \vert_S, \psi \vert_S \rangle \), such that every element in \( \langle \varphi, \psi \rangle \) is a product of an element in \( B \), an element in \( B' \), and an element in \( \langle \varphi \vert_S, \psi \vert_S \rangle \). This would contradict the fact that \( \langle \varphi, \psi \rangle \) is a nonabelian free group.

27
Therefore \( \langle \varphi |_S, \psi |_S \rangle \) contains a nonabelian free subgroup \( \langle f, g \rangle \), and the commutator subgroup of \( \langle f, g \rangle \) is a nonabelian free group contained in \( A \cap A' \). This contradiction completes our proof.

Let \( G \) be a measured groupoid over a standard probability space \( Y \) which admits an action-type cocycle \( \rho : G \to \text{Mod}^0(V) \). Two admissible pairs \((H, A)\) and \((H', A')\) (with respect to \( \rho \)) are \emph{compatible with respect to \( \rho \)} if, denoting by \((X, \mathfrak{A})\) and \((X', \mathfrak{A}')\) their respective decomposition maps, for a.e. \( y \in Y \), the pairs \((X(y), \mathfrak{A}(y))\) and \((X'(y), \mathfrak{A}'(y))\) are compatible. The following proposition gives a purely groupoid-theoretic characterization of compatibility (i.e. with no reference to the cocycle \( \rho \)).

**Proposition 3.26.** Let \( G \) be a measured groupoid over a standard probability space \( Y \), equipped with a strict action-type cocycle \( \rho : G \to \text{Mod}^0(V) \). Let \((H, A)\) and \((H', A')\) be two admissible pairs with respect to \( \rho \). Then the following are equivalent.

1. \((H, A)\) and \((H', A')\) are compatible with respect to \( \rho \);
2. for every Borel subset \( U \subseteq Y \) of positive measure, there exists a Borel subset \( V \subseteq U \) of positive measure such that \( \langle A_{|V}, A'_{|V} \rangle \) is amenable.

**Proof of Proposition 3.26.** Let \( Y = \bigsqcup_{i \in I} Y_i \) be a countable Borel partition of a conull Borel subset \( Y^* \subseteq Y \) such that for every \( i \in I \), there exist admissible pairs \((X_i, \mathfrak{A}_i)\) and \((X'_i, \mathfrak{A}'_i)\) such that \( H_{|Y_i} = \rho^{-1}(\text{Stab}_{\text{Mod}^0(V)}(X_i)) \) and \( H'_{|Y_i} = \rho^{-1}(\text{Stab}_{\text{Mod}^0(V)}(X'_i)) \), and letting \( A_i, A'_i \subseteq \text{Mod}^0(V) \) be the active subgroups of \((X_i, \mathfrak{A}_i)\) and \((X'_i, \mathfrak{A}'_i)\) respectively, we have \( A_{|Y_i} = \rho^{-1}(A_i) \) and \( A'_{|Y_i} = \rho^{-1}(A'_i) \).

We first prove that \( -1 \Rightarrow -2 \). If \( 1 \) fails, then there exists \( i_0 \in I \) such that \( Y_{i_0} \) has positive measure and \((X_{i_0}, \mathfrak{A}_{i_0})\) and \((X'_{i_0}, \mathfrak{A}'_{i_0})\) are not compatible. Then there exist \( g_{i_0} \in A_{i_0} \) and \( g'_{i_0} \in A'_{i_0} \) that generate a nonabelian free subgroup of \( \text{Mod}^0(V) \), as follows from [Koh12 Theorem 1.8, and the sentence following it]. Lemma \([\text{7.10}]\) ensures that for every Borel subset \( V \subseteq Y_{i_0} \) of positive measure, the groupoid \( \langle \rho^{-1}(\langle g_{i_0} \rangle)_{|V}, \rho^{-1}(\langle g'_{i_0} \rangle)_{|V} \rangle \) is nonamenable. Therefore \( \langle A_{|V}, A'_{|V} \rangle \) is also nonamenable for every Borel subset \( V \subseteq Y_{i_0} \) of positive measure, so \( 2 \) fails.

We now prove that \( 1 \Rightarrow 2 \). If \( 1 \) holds, then for every \( i \in I \) such that \( Y_i \) has positive measure, the pairs \((X_i, \mathfrak{A}_i)\) and \((X'_i, \mathfrak{A}'_i)\) are compatible, so \( \langle A_i, A'_i \rangle \) is amenable (Lemma \([\text{7.20}]\)). Let now \( U \subseteq Y \) be a Borel subset of positive measure, and let \( V \subseteq U \) of positive measure be contained in \( Y_{i_0} \) for some \( i_0 \in I \). Then \( \langle A_{|V}, A'_{|V} \rangle \) is contained in \( \rho^{-1}(\langle A_{i_0}, A'_{i_0} \rangle)_{|V} \), which is amenable because \( \langle A_{i_0}, A'_{i_0} \rangle \) is and \( \rho \) has trivial kernel.

Let \( H, H' \) be two measured subgroupoids of \( G \) of meridian type with respect to an action-type cocycle \( \rho : G \to \text{Mod}^0(V) \). We say that \( H \) and \( H' \) are \emph{compatible with respect to} \( \rho \) if, denoting by \( \varphi, \varphi' \) their respective meridian maps with respect to \( \rho \), for a.e. \( y \in Y \), the meridians \( \varphi(y) \) and \( \varphi'(y) \) are disjoint up to isotopy.

**Corollary 3.27.** Let \( G \) be a measured groupoid over a standard probability space \( Y \), equipped with two strict action-type cocycles \( \rho_1, \rho_2 : G \to \text{Mod}^0(V) \), and let \( H, H' \) be two measured subgroupoids of \( G \) of meridian type with respect to both \( \rho_1 \) and \( \rho_2 \).
Then \( H \) and \( H' \) are compatible with respect to \( \rho_1 \) if and only if they are compatible with respect to \( \rho_2 \).

**Proof.** Let \( Y^* = \bigcup_{j \in J} Y_j \) be a partition of a conull Borel subset \( Y^* \subseteq Y \) into at most countably many Borel subsets such that for every \( i \in \{1, 2\} \) and every \( j \in J \), there exist meridians \( c_{i,j}, c'_{i,j} \) such that \( H_{|Y_j}, H'_{|Y_j} \) are equal to the \((G_{|Y_j}, \rho_i)\)-stabilizers of the isotopy classes of \( c_{i,j}, c'_{i,j} \), respectively.

For every \( i \in \{1, 2\} \) and every \( j \in J \), let \( A_{i,j} \) (resp. \( A'_{i,j} \)) be the subgroup of \( \text{Mod}^0(V) \) made of all elements that act trivially in restriction to every connected component of \( \partial V \setminus c_{i,j} \) (resp. \( \partial V \setminus c'_{i,j} \)) which is not a once-holed torus. Notice that \( A_{i,j}, A'_{i,j} \) are the active subgroups of some admissible decorated multicurves \((X_{i,j}, A_{i,j}), (X'_{i,j}, A'_{i,j})\), by letting \( X_{i,j} \) and \( X'_{i,j} \) be obtained from \( c_{i,j} \) and \( c'_{i,j} \) by adding the unique nonseparating meridian in every complementary component which is a once-holed torus, and letting \( A_{i,j} = A'_{i,j} = \emptyset \). See the examples right after Definition 3.23. Notice that \( c_{i,j} \) and \( c'_{i,j} \) are disjoint up to isotopy if and only if \((X_{i,j}, \emptyset)\) and \((X'_{i,j}, \emptyset)\) are compatible.

For every \( i \in \{1, 2\} \), let \( A_i \subseteq H \) be a subgroupoid such that \((A_i)_{|Y_j} = \rho_i^{-1}(A_{i,j})_{|Y_j}\) for every \( j \in J \), and let \( A'_i \subseteq H' \) be defined in the same way, using \( A'_{i,j} \) in place of \( A_{i,j} \). Then \((H, A_i)\) and \((H', A'_i)\) are admissible pairs with respect to \( \rho_i \). Lemma 3.19 thus ensures that \( A_1 \) and \( A_2 \) are stably equal (as they are both stably maximal for the property of being a stably normal amenable subgroupoid of \( H \)), and likewise \( A'_1 \) and \( A'_2 \) are stably equal. The conclusion therefore follows from Proposition 3.26.

### 3.9 Characterizing subgroupoids of nonseparating-meridian type

The goal of this section is to prove the following proposition.

**Proposition 3.28.** Let \( G \) be a measured groupoid over a standard probability space \( Y \), equipped with two strict action-type cocycles \( \rho_1, \rho_2 : G \to \text{Mod}^0(V) \), and let \( H \subseteq G \) be a measured subgroupoid.

Then \( H \) is of nonseparating-meridian type with respect to \( \rho_1 \) if and only if it is of nonseparating-meridian type with respect to \( \rho_2 \).

A decorated multicurve \((X, A)\) is clean if it is not of the form \((c, \emptyset)\) for some separating meridian \( c \). The graph of clean admissible decorated multicurves \( M \) is the graph whose vertices correspond to isotopy classes of clean admissible decorated multicurves, where two distinct vertices are joined by an edge if the corresponding decorated multicurves are compatible. The graph of nonseparating meridians \( D^\text{nsep} \) is the graph whose vertices correspond to isotopy classes of nonseparating meridians, where two distinct vertices are joined by an edge if the corresponding meridians are disjoint up to isotopy. Notice that \( D^\text{nsep} \) is naturally a subgraph of \( M \), by sending a nonseparating meridian \( c \) to the pair \((c, \emptyset)\).

**Lemma 3.29.** Every injective graph map \( \bar{\mathcal{M}} \) from \( D^\text{nsep} \) to \( M \) takes its values in \( D^\text{nsep} \) (viewed as a subgraph of \( M \) via the natural inclusion).

\(^2\)i.e. preserving adjacency and non-adjacency
Proof. Let \( v \in V(\mathbb{D}_{\text{nsep}}) \) be a vertex. By completing \( v \) to a pair of pants decomposition made of nonseparating meridians, we can find \( 3g - 3 \) pairwise distinct, pairwise adjacent vertices \( v = v_1, \ldots, v_{3g - 3} \) such that for every \( i \in \{1, \ldots, 3g - 3\} \), one has
\[
\text{Lk}_{\mathbb{D}_{\text{nsep}}}(\{v_1, \ldots, v_{3g - 3}\}) \subseteq \text{Lk}_{\mathbb{D}_{\text{nsep}}}(\{v_1, \ldots, v_i, v_{i+1}, \ldots, v_{3g - 3}\}) \setminus \{v_i\}.
\]
So the same property should hold for their images in \( \mathbb{M} \), which correspond to decorated multicurves \((X_1, \mathfrak{A}_1), \ldots, (X_{3g - 3}, \mathfrak{A}_{3g - 3})\). For every \( i \in \{1, \ldots, 3g - 3\} \), let \( \Sigma_i \) be the subsurface of \( \partial V \) equal to the union of all subsurfaces in \( \mathfrak{A}_i \), together with all annuli around curves in \( X_i \) that are not boundary curves of any subsurface in \( \mathfrak{A}_i \). Notice that the set \( \{\Sigma_1, \ldots, \Sigma_{3g - 3}\} \) cannot contain both a subsurface \( S \) and the collar neighborhood \( A \) of one of its boundary components, as otherwise removing \( A \) from the collection does not change the link. More generally, for every \( i \in \{1, \ldots, 3g - 3\} \), one of the connected components \( \Sigma'_i \subseteq \Sigma_i \) is not a connected component of some \( \Sigma_j \) with \( j \neq i \), and is also not the collar neighborhood of a boundary curve of \( \Sigma_j \) – otherwise removing \( (X_i, \mathfrak{A}_i) \) from the collection does not change its link. So the subsurfaces \( \Sigma'_1, \ldots, \Sigma'_{3g - 3} \) are pairwise nonisotopic and \( \{\Sigma'_1, \ldots, \Sigma'_{3g - 3}\} \) does not contain a subsurface together with the collar neighborhood of one of its boundary components. For every \( i \in \{1, \ldots, 3g - 3\} \), let \( \{b_{i,1}, \ldots, b_{i,k_i}\} \) be the set of all boundary curves of \( \Sigma'_i \), and let \( \{d_{i,1}, \ldots, d_{i,k_i}\} \) be a set of isotopy classes of essential simple closed curves on \( \Sigma'_i \) that form a pair of pants decomposition of \( \Sigma'_i \) (with the convention that in the case of an annulus, the former set contains two isotopic curves, and the latter set is empty). The tuple consisting of all \( b_{i,j} \) and \( d_{i,j} \) contains at least \( 6g - 6 \) curves, each being repeated at most twice up to isotopy (and the \( d_{i,j} \) are not isotopic to any other curve in the collection). So every subsurface \( \Sigma'_i \) contributes exactly two curves that are both of the form \( c_{i,j} \), and is therefore an annular subsurface. Furthermore, since there are \( 3g - 3 \) such, and no \( \Sigma'_i \) appears as a subsurface of \( \Sigma'_j, j \neq i \), we actually have \( \Sigma'_i = \Sigma_i \) for all \( i \). Therefore \((X_i, \mathfrak{A}_i) = (c_i, \emptyset)\), where \( c_i \) is the core curve of the annulus \( \Sigma_i \). As \((X_i, \mathfrak{A}_i)\) is admissible, some power of the twist around \( c_i \) must belong to the handlebody group, so \( c_i \) is a meridian by [Oer02, Theorem 1.11] or [MCC04, Theorem 1]. As \((X_i, \mathfrak{A}_i)\) is clean, the meridian \( c_i \) is nonseparating, and the conclusion follows. \( \square \)

Proof of Proposition 3.22. Let \( v \in V(\mathbb{D}_{\text{nsep}}) \), in other words \( v \) is the isotopy class of a nonseparating meridian. Let \( H_v \) be the \((G, \rho_1)\)-stabilizer of \( y_v \), and let \( A_v = \rho_1^{-1}(T^{(v)}) \). Then \((H_v, A_v)\) satisfies Property \((Q_{\text{nsep}})\) (by Proposition 3.18) applied to the cocycle \( \rho_1 \). Lemma 3.23, applied to the cocycle \( \rho_2 \), implies that \((H_v, A_v)\) is an admissible pair with respect to \( \rho_2 \). So there exist a countable Borel subset \( Y^* \subseteq Y \) and a partition \( Y^* = \sqcup_{i \in I} Y_{v,i} \) into at most countably many Borel subsets such that for every \( i \in I \), there exists a (unique) admissible pair \((X_{v,i}, \mathfrak{A}_{v,i})\) such that \((H_v)_{|Y_{v,i}}\) is the \((G_{|Y_{v,i}}, \rho_2)\)-stabilizer of \( X_{v,i} \) and, denoting by \( A_{v,i} \) the active subgroup of \((X_{v,i}, \mathfrak{A}_{v,i})\), one has \((A_v)_{|Y_{v,i}} = \rho_1^{-1}(A_{v,i})_{|Y_{v,i}} \). In addition, Lemma 3.22 ensures that \((X_{v,i}, \mathfrak{A}_{v,i})\) is clean. For every \( y \in Y \) and every \( v \in V(\mathbb{D}_{\text{nsep}}) \), we then let \( \theta(y, v) = (X_{v,i}, \mathfrak{A}_{v,i}) \) whenever \( y \in Y_{v,i} \). This defines a Borel map \( \theta : Y \times V(\mathbb{D}_{\text{nsep}}) \to V(\mathbb{M}) \).

We claim that for almost every \( y \in Y \), the map \( \theta(y, \cdot) \) determines a graph embedding \( \mathbb{D}_{\text{nsep}} \to \mathbb{M} \). Let us first explain how to complete the proof of the proposition from
this claim. By Lemma 3.29, every graph embedding $D_{n\text{sep}} \to M$ sends nonseparating meridians to nonseparating meridians. Therefore, if $v$ is a nonseparating meridian, then $X_{v,i}$ is a nonseparating meridian (and $\mathfrak{A}_{v,i} = \emptyset$) whenever $Y_{v,i}$ has positive measure, and the proposition follows.

We are now left with proving the above claim. First, for almost every $y \in Y$, the map $\theta(y, \cdot)$ is injective. Indeed otherwise, as $V(D_{n\text{sep}})$ is countable, there exist a Borel subset $U \subseteq Y$ of positive measure and two non-isotopic nonseparating meridians $c,c'$ such that for every $y \in U$, one has $\theta(y, c) = \theta(y, c')$ (we denote by $(X, \mathfrak{A})$ the common image). In particular, the $(G|_U, \rho_1)$-stabilizer of $c$ is stably equal to the $(G|_U, \rho_1)$-stabilizer of $c'$, since they are both stably equal to the $(G|_U, \rho_2)$-stabilizer of $X$. This contradicts Lemma 3.13.

Second, Proposition 3.26 ensures that for almost every $y \in Y$, the map $\theta(y, \cdot)$ is a graph map, i.e. it preserves both adjacency and non-adjacency.

3.10 Characterizing subgroupoids of separating-meridian type

In this section, we establish a purely groupoid-theoretic characterization of subgroupoids of separating-meridian type with respect to a strict action-type cocycle towards $\text{Mod}^0(V)$, and derive that being of separating-meridian type is a notion that does not depend on the choice of such a cocycle.

3.10.1 Property ($P_{\text{sep}}$)

We proved in Proposition 3.28 that for a subgroupoid $H \subseteq G$, being of nonseparating meridian does not depend on the choice of an action-type cocycle $G \to \text{Mod}^0(V)$. Also, it follows from Corollary 3.27 that compatibility of two subgroupoids of nonseparating meridian type is also independent of such a choice. Thus, the following notion is a purely groupoid-theoretic property.

**Definition 3.30 (Property ($P_{\text{sep}}$)).** Let $G$ be a measured groupoid over a standard probability space $Y$ which admits a strict action-type cocycle towards $\text{Mod}^0(V)$. A measured subgroupoid $H \subseteq G$ satisfies Property ($P_{\text{sep}}$) if

1. $H$ contains a strongly Schottky pair of subgroupoids;

2. there exists a stably normal amenable subgroupoid $B \subseteq H$ of infinite type, such that for every measured subgroupoid $H' \subseteq G$ of nonseparating-meridian type, and every stably normal amenable subgroupoid $A \subseteq H'$ of infinite type, the intersection $A \cap B$ is stably trivial;

3. given any two subgroupoids $H_1, H_2 \subseteq G$ of nonseparating-meridian type, and any Borel subset $U \subseteq Y$ of positive measure, assuming that $H|_U \subseteq (H_1 \cap H_2)|_U$, then $(H_1)|_U$ and $(H_2)|_U$ are stably equal;

4. there exist $3g - 4$ measured subgroupoids $H_1, \ldots, H_{3g-4}$ of $G$ of nonseparating-meridian type, which are pairwise compatible, such that
Proposition 3.31. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho: \mathcal{G} \to \text{Mod}^0(V)$. Let $c$ be a separating meridian, and let $\mathcal{H}$ be the $(\mathcal{G}, \rho)$-stabilizer of the isotopy class of $c$.

Then $\mathcal{H}$ satisfies Property (P$_{\text{sep}}$).

Proof. For Property (P$_{\text{sep}}$)(1), Lemma 3.1 ensures that the stabilizer in Mod$^0(V)$ of any separating meridian contains a nonabelian free subgroup (recall that we are assuming that the genus of $V$ is at least 3). Lemma 3.6 thus implies that $\mathcal{H}$ contains a strongly Schottky pair of subgroupoids.

For Property (P$_{\text{sep}}$)(2), notice that the cyclic subgroup $\langle T_c \rangle$ generated by the Dehn twist about $c$ is normal in the stabilizer of $c$. Therefore $\mathcal{B} = \rho^{-1}(\langle T_c \rangle)$ is a normal subgroupoid of $\mathcal{H}$, which is amenable as $\rho$ has trivial kernel, and of infinite type because $\rho$ is action-type. Let now $\mathcal{H}'$ be a measured subgroupoid of $\mathcal{G}$ of nonseparating meridian type, and let $\mathcal{A} \subseteq \mathcal{H}'$ be a stably normal amenable subgroupoid of infinite type. By Lemma 1.19 we can find a partition $Y^* = \bigsqcup_i Y_i$ of a countable Borel subset $Y^* \subseteq Y$ into at most countably many Borel subsets such that for every $i \in I$, there exists a nonseparating meridian $d_i$ such that $\mathcal{A}|_{Y_i} \subseteq \rho^{-1}(\langle T_{d_i} \rangle)|_{Y_i}$. It follows that $(\mathcal{A} \cap \mathcal{B})|_{Y_i}$ is trivial, so $\mathcal{A} \cap \mathcal{B}$ is stably trivial.

Property (P$_{\text{sep}}$)(3) follows from the fact that for every separating meridian $c$, there is at most one nonseparating meridian $d$ such that every infinite-order element of $\text{Stab}_{\text{Mod}^0(V)}(c)$ has a power that fixes $d$ (in fact, the existence of such a $d$ occurs precisely when one of the two connected components of $\partial V \setminus c$ is a one-holed torus, in which case it contains a unique nonseparating meridian up to isotopy, and we take $d$ as such – notice that we are using the fact that the genus of $V$ is at least 3 here; see Lemma 1.2). We now prove that $\mathcal{H}$ satisfies Property (P$_{\text{sep}}$)(4). Let $\{c_1, \ldots, c_{3g-4}\}$ be a set of $3g-4$ pairwise non-isotopic nonseparating meridians which together with $c$ form a pair of pants decomposition of $\partial V$. For every $j \in \{1, \ldots, 3g-4\}$, let $\mathcal{H}_j$ be the $(\mathcal{G}, \rho)$-stabilizer of the isotopy class of $c_j$. Then $\mathcal{H}_{1}, \ldots, \mathcal{H}_{3g-4}$ are of nonseparating meridian type. Lemma 3.13 ensures that they satisfy Assertion (4.a). Finally, Lemma 3.19 ensures that every stably normal amenable subgroupoid $\mathcal{A}_j$ of $\mathcal{H}_j$ is stably contained in $\rho^{-1}(\langle T_{c_j} \rangle)$. In particular each $\mathcal{A}_j$ is stably contained in $\mathcal{H}$. □

32
3.10.2 Characterization

Our goal is now to characterise subgroups of separating meridian-type by proving the following proposition.

**Proposition 3.32.** Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to \text{Mod}^0(V)$. Let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$. The following assertions are equivalent.

1. The subgroupoid $\mathcal{H}$ is of separating-meridian type with respect to $\rho$.

2. The subgroupoid $\mathcal{H}$ satisfies Property $(P_{\text{sep}})$, and is stably maximal among all measured subgroupoids of $\mathcal{G}$ with respect to this property, i.e. if $\mathcal{H}'$ is another subgroupoid satisfying Property $(P_{\text{sep}})$, and if $\mathcal{H}$ is stably contained in $\mathcal{H}'$, then $\mathcal{H}$ is stably equal to $\mathcal{H}'$.

Before turning to the proof of Proposition 3.32, we record the following consequence.

**Corollary 3.33.** Let $\mathcal{G}$ be a measured groupoid over a base space $Y$, equipped with two strict action-type cocycles $\rho_1, \rho_2 : \mathcal{G} \to \text{Mod}^0(V)$, and let $\mathcal{H} \subseteq \mathcal{G}$ be a measured subgroupoid.

Then $\mathcal{H}$ is of separating-meridian type with respect to $\rho_1$ if and only if it is of separating-meridian type with respect to $\rho_2$.

Our goal is now to prove Proposition 3.32. Our first lemma exploits the first two assumptions of Property $(P_{\text{sep}})$ in order to derive information about the possible canonical reduction multicurves of $\mathcal{B}$.

**Lemma 3.34.** Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to \text{Mod}^0(V)$. Let $\mathcal{B}, \mathcal{H}$ be measured subgroupoids of $\mathcal{G}$, with $\mathcal{B} \subseteq \mathcal{H}$. Assume that

1. $\mathcal{H}$ contains a strongly Schottky pair of subgroupoids;

2. $\mathcal{B}$ is amenable and of infinite type, and stably normal in $\mathcal{H}$;

3. for every measured subgroupoid $\mathcal{H}' \subseteq \mathcal{G}$ of nonseparating-meridian type, and every stably normal amenable subgroupoid $\mathcal{A} \subseteq \mathcal{H}'$ of infinite type, the intersection $\mathcal{A} \cap \mathcal{B}$ is stably trivial.

Then for every Borel subset $U \subseteq Y$ of positive measure, the pair $(\mathcal{B}_U, \rho)$ cannot have a canonical reduction multicurve consisting of a single nonseparating meridian.

**Proof.** Assume towards a contradiction that $(\mathcal{B}_U, \rho)$ has a canonical reduction multicurve which is reduced to a single nonseparating meridian $c$. As $\mathcal{B}$ is stably normal in $\mathcal{H}$, up to restricting to a positive measure Borel subset of $U$, we can assume that $c$ is $(\mathcal{H}_U, \rho)$-invariant. In particular, letting $\Sigma = \partial V \setminus c$, we have a natural cocycle $\rho' : \mathcal{H}_U \to \text{Mod}^0(\Sigma)$. The kernel of $\rho'$ is contained in $\rho^{-1}(\langle T_c \rangle)_U$. As $\rho$ has trivial...
Lemma 3.35. Let $\mathcal{G}$ be a measured groupoid over a standard probability space $Y$, equipped with a strict action-type cocycle $\rho : \mathcal{G} \to \text{Mod}^{\emptyset}(V)$. Let $\mathcal{H}$ be a measured subgroupoid of $\mathcal{G}$ which satisfies Property $(P_{\text{sep}})$. Then there exists a partition $Y = \sqcup_{i \in I} Y_i$ into at most countably many Borel subsets such that for every $i \in I$, there exists an $(\mathcal{H}_{|Y_i}, \rho)$-invariant isotopy class of separating meridian.

Proof. Let $\mathcal{H}_1, \ldots, \mathcal{H}_{3g-4}$ be subgroupoids of $\mathcal{G}$ provided by Property $(P_{\text{sep}})(4)$. Up to partitioning $Y$ into at most countably many Borel subsets, we can assume that for every $j \in \{1, \ldots, 3g-4\}$, the groupoid $\mathcal{H}_j$ is equal to the $(\mathcal{G}, \rho)$-stabilizer of the isotopy class of a nonseparating meridian $d_j$.

Let $\mathcal{B} \subseteq \mathcal{H}$ be as in Property $(P_{\text{sep}})(2)$. By Lemma 3.12, we can find a partition $Y = \sqcup_{i \in I} Y_i$ into at most countably many Borel subsets of positive measure such that for every $i \in I$, the pair $(\mathcal{B}_{|Y_i}, \rho)$ has a canonical reduction set $\mathcal{C}_i$, with boundary multicurve $X_i$. As $\mathcal{B}$ is stably normal in $\mathcal{H}$, up to refining this partition, we can assume that for every $i \in I$, the isotopy class of the multicurve $X_i$ is $(\mathcal{H}_{|Y_i}, \rho)$-invariant.

We first observe that for every $i \in I$, one has $\mathcal{C}_i \neq \emptyset$. Indeed, otherwise $(\mathcal{B}_{|Y_i}, \rho)$ is irreducible. As $\mathcal{B}$ is amenable and stably normal in $\mathcal{H}$, and $\rho$ has trivial kernel, Lemma 3.12 implies that $\mathcal{H}_{|Y_i}$ is amenable, contradicting Property $(P_{\text{sep}})(1)$.

For every $j \in \{1, \ldots, 3g-4\}$, let $T_j$ be the Dehn twist about the meridian $d_j$. Then $A_j = \rho^{-1}(\langle T_j \rangle)$ is a normal amenable subgroupoid of $\mathcal{H}_j$. Property $(P_{\text{sep}})(4.b)$ thus ensures that $A_j$ is stably contained in $\mathcal{H}$. Therefore, for every curve $c$ in $X_i$, there exists a positive integer $k$ such that the isotopy class of $c$ is fixed by $T_j^k$. This implies that $X_i$ is disjoint (up to isotopy) from all meridians $d_j$.

We now claim that for every $i \in I$, the multicurve $X_i$ contains at most one of the curves $d_j$. Indeed, assume by contradiction that it contains two curves $d_{j_1}$ and $d_{j_2}$. Then $\mathcal{H}_{|Y_i}$ is contained in $(\mathcal{H}_{j_1} \cap \mathcal{H}_{j_2})_{|Y_i}$. As $\mathcal{H}_{j_1}$ and $\mathcal{H}_{j_2}$ are of nonseparating meridian type with respect to $\rho$, Property $(P_{\text{sep}})(3)$ implies that there exists a positive measure Borel subset $U \subseteq Y_i$ such that $(\mathcal{H}_{j_1})_U = (\mathcal{H}_{j_2})_U$, contradicting Property $(P_{\text{sep}})(4.a)$.

As $\{d_1, \ldots, d_{3g-4}\}$ is a set of $3g-4$ pairwise disjoint and pairwise non-isotopic non-separating simple closed curves on $\partial V$, one of the complementary components of the union of all curves $d_j$ is a 4-holed sphere $S$. Notice that every essential simple closed curve contained in $S$ is a meridian, and $X_i$ may contain such a curve. This leaves three possibilities for the canonical reduction multicurve of $(\mathcal{B}_{|Y_i}, \rho)$, namely:
1. a single nonseparating meridian (either one of the meridians \(d_j\), or else a nonseparating meridian contained in \(S\));

2. the union of a nonseparating meridian \(d_j\) and a nonseparating essential simple closed curve (in fact a meridian) contained in \(S\);

3. a separating (on \(\partial V\)) essential simple closed curve (in fact a meridian) contained in \(S\), possibly together with a meridian \(d_j\).

The first case is excluded by Lemma 3.34, the second case is excluded using Property (P_{sep}) (3) and Lemma 3.13, and the last case leads to the desired conclusion of our lemma.

Proof of Proposition 3.32. We first prove that (1) \(\Rightarrow\) (2). Let \(H\) be a measured subgroupoid of \(G\) of separating meridian type with respect to \(\rho\), and let \(Y^* = \bigcup_{i \in I} Y_i\) be a partition of a conull Borel subset \(Y^* \subseteq Y\) into at most countably many Borel subsets, such that for every \(i \in I\), the groupoid \(H|_{Y_i}\) is equal to the \((G|_{Y_i}, \rho)\)-stabilizer of the isotopy class of a separating meridian \(c_i\).

Proposition 3.31 implies that \(H\) satisfies Property (P_{sep}). We need to check that \(H\) is stably maximal among all measured subgroupoids of \(G\) that satisfy Property (P_{sep}). So let \(H'\) be a measured subgroupoid of \(G\) which satisfies Property (P_{sep}), and such that \(H\) is stably contained in \(H'\). By Lemma 3.35 up to refining the above partition of \(Y\), we can assume that for every \(i \in I\), there exists a separating meridian \(c'_i\) whose isotopy class is \((H'|_{Y_i}, \rho)\)-invariant. Lemma 3.11 implies that \(c_i = c'_i\) for every \(i \in I\). It follows that \(H'\) is stably contained in \(H\), so they are stably equal. This completes our proof of the implication (1) \(\Rightarrow\) (2).

We now prove that (2) \(\Rightarrow\) (1), so let \(H\) be a measured groupoid of \(G\) that satisfies Assertion (2). By Lemma 3.35 we can find a partition \(Y = \bigcup_{i \in I} Y_i\) into at most countably many Borel subsets such that for every \(i \in I\), there exists a separating meridian \(c_i\) whose isotopy class is \((H|_{Y_i}, \rho)\)-invariant. Let \(H'\) be a measured subgroupoid of \(G\) such that for every \(i \in I\), the groupoid \(H'|_{Y_i}\) is equal to the \((G|_{Y_i}, \rho)\)-stabilizer of the isotopy class of \(c_i\). Then \(H\) is stably contained in \(H'\). In addition, \(H'\) is of separating meridian type, so Proposition 3.31 shows that \(H\) satisfies Property (P_{sep}). The maximality assumption on \(H\) therefore implies that \(H\) is stably equal to \(H'\). Hence \(H\) itself is of separating meridian type, which concludes our proof.

3.11 Conclusion

Before concluding the proof of our main theorem, we first record the following easy consequence of Propositions 3.28 and 3.32.

Proposition 3.36. Let \(G\) be a measured groupoid, equipped with two strict action-type cocycles \(\rho_1, \rho_2 : G \to \text{Mod}^0(V)\), and let \(H \subseteq G\) be a measured subgroupoid.

Then \(H\) is of meridian type with respect to \(\rho_1\) if and only if it is of meridian type with respect to \(\rho_2\).
We will now simply say that \( \mathcal{H} \) is of meridian type to mean that it is of meridian type with respect to any action-type cocycle \( G \to \text{Mod}(V) \). We are now in position to complete the proof of Theorem 3.2, which as we already explained at the beginning of this section yields the measure equivalence superrigidity of handlebody groups in genus at least 3.

**Proof of Theorem 3.2.** Let \( G \) be a measured groupoid over a standard probability space \( Y \), and let \( \rho_1, \rho_2 : G \to \text{Mod}(V) \) be two strict action-type cocycles. Let \( \mathbb{D} \) be the disk graph of \( V \): we recall that its vertices are the isotopy classes of meridians in \( \partial V \), and two such isotopy classes are joined by an edge if they have disjoint representatives.

Proposition 3.36 ensures that for every vertex \( v \in \mathbb{D} \), there exists a Borel map \( \phi_v : Y \to \mathbb{D} \) such that for every \( w \in \mathbb{D} \), letting \( Y_{v,w} = \phi_v^{-1}(w) \), the \((G_{Y_{v,w}}, \rho_1)\)-stabilizer of \( v \) is stably equal to the \((G_{Y_{v,w}}, \rho_2)\)-stabilizer of \( w \). Lemmas 7.13 and 7.14 ensure that the map \( \phi_v \) is essentially unique.

For every \( y \in Y \) and every \( v \in \mathbb{D} \), we then let \( \psi(y, v) = \phi_v(y) \). This defines a Borel map \( \psi : Y \times \mathbb{D} \to \mathbb{D} \).

We claim that for a.e. \( y \in Y \), the map \( \psi(y, \cdot) \) is a graph automorphism of \( \mathbb{D} \). Indeed, injectivity follows from the same argument as in the proof of Proposition 7.25, and the fact that \( \psi(y, \cdot) \) is almost everywhere a graph map follows from Corollary 7.27. We now show that for almost every \( y \in Y \), the map \( \psi(y, \cdot) \) is surjective. So let \( c \) be a meridian. By Proposition 3.30, there exists a Borel partition of a conull Borel subset \( Y^* \subseteq Y \) into at most countably many Borel subsets \( Y_i \) such that for every \( i \), the \((G_{Y_i}, \rho_2)\)-stabilizer of \( c \) coincides with the \((G_{Y_i}, \rho_1)\)-stabilizer of some \( c_i \in \mathbb{D} \). It follows that \( \psi(y, c_i) = c \) for almost every \( y \in Y_i \). Surjectivity follows.

By the main theorem of [KS09], the natural map \( \text{Mod}(V) \to \text{Aut}(\mathbb{D}) \) is an isomorphism (noting again that the genus of \( V \) is at least 3). We can thus find a Borel map \( \theta : Y \to \text{Mod}(V) \) so that for a.e. \( y \in Y \) we have that \( \psi(y, \delta) = \theta(y)(\delta) \) for all meridians.

We are left with showing that \( \theta \) satisfies the equivariance condition required in Theorem 3.2. This amounts to proving that there exists a conull Borel subset \( Y^* \subseteq Y \) such that for every \( g \in G_{Y^*} \) and every vertex \( v \in \mathbb{D} \), one has \( \psi(r(g), \rho_1(g)v) = \rho_2(g)\psi(s(g), v) \). As \( G \) is a countable union of bisections, it is enough to prove it for almost every \( g \) in a bisection \( B \) (inducing a Borel isomorphism between \( U = s(B) \) and \( V = r(B) \)). Up to further partitioning \( B \), we can assume that \( \rho_1|_B \) and \( \rho_2|_B \) are constant, with values \( \gamma_1, \gamma_2 \), and that \( \psi(\cdot, v) \) is constant, with value \( w \). We now aim to show that for almost every \( y \in V \), one has \( \psi(y, \gamma_1v) = \gamma_2w \). By definition of \( \psi \), the \((G_U, \rho_1)\)-stabilizer of \( v \) is stably equal to the \((G_U, \rho_2)\)-stabilizer of \( w \). Conjugating by the bisection, it follows that the \((G_V, \rho_1)\)-stabilizer of \( \gamma_1v \) is stably equal to the \((G_V, \rho_2)\)-stabilizer of \( \gamma_2w \), which is exactly what we wanted to show. \( \square \)
4 Applications

4.1 Lattice embeddings and automorphisms of the Cayley graph

A first consequence of our work is that handlebody groups do not admit any interesting lattice embeddings in locally compact second countable groups.

**Theorem 4.1.** Let $V$ be a handlebody of genus at least 3. Let $G$ be a locally compact second countable group, equipped with its (left or right) Haar measure. Let $\Gamma$ be a finite index subgroup of $\text{Mod}(V)$, and let $\sigma : \Gamma \to G$ be an injective homomorphism whose image is a lattice.

Then there exists a homomorphism $\theta : G \to \text{Mod}(V)$ with compact kernel such that for every $f \in \Gamma$, one has $\theta \circ \sigma(f) = f$.

**Proof.** Theorem 3.2 precisely says that $\text{Mod}(V)$ is rigid with respect to action-type cocycles in the sense of [GH21, Definition 4.1]. As $\text{Mod}(V)$ is ICC (Lemma 1.6), the theorem thus follows from [GH21, Theorem 4.7].

A theorem of Suzuki ensures that $\text{Mod}(V)$ is finitely generated (it is in fact finitely presented by work of Wajnryb). Given a finitely generated group $G$ and a finite generating set $S$ of $G$, the Cayley graph $\text{Cay}(G, S)$ is defined as the simple graph whose vertices are the elements of $G$, with an edge between distinct elements $g, h$ if $g^{-1}h \in S \cup S^{-1}$.

**Theorem 4.2.** Let $V$ be a handlebody of genus at least 3.

1. For every finite generating set $S$ of $\text{Mod}(V)$, every automorphism of $\text{Cay}(\text{Mod}(V), S)$ is at bounded distance from the left multiplication by an element of $\text{Mod}(V)$.

2. For every torsion-free finite-index subgroup $\Gamma \subseteq \text{Mod}(V)$ and every finite generating set $S'$ of $\Gamma$, the automorphism group of $\text{Cay}(\Gamma, S')$ is countable (in fact it embeds as a subgroup of $\text{Mod}(V)$ containing $\Gamma$).

**Proof.** Using the fact that $\text{Mod}(V)$ is ICC, this follows from Theorem 3.2 and [GH21, Corollary 4.8] (the idea behind the proof is to view $\text{Mod}(V)$ as a cocompact lattice in the automorphism group of its Cayley graph and apply the previous theorem).

As mentioned in the introduction, torsion-freeness of $\Gamma$ is crucial in the second conclusion in view of [HST19, Lemma 6.1].

---

3. Theorem 4.7 from [GH21] records works of Furman and Kida. The idea behind its proof is that the lattice embedding $\sigma$ determines a self measure equivalence coupling of $\Gamma$ (acting on $G$ equipped with its Haar measure), and the rigidity statement provided by Theorem 3.2 from the present paper ensures that the self coupling of $\Gamma$ on $G$ factors through the obvious coupling on $\text{Mod}(V)$ where $\Gamma$ acts by left/right multiplication. This yields a Borel map $G \to \text{Mod}(V)$, and some extra work is needed to upgrade it to a continuous homomorphism.
4.2 Orbit equivalence rigidity and von Neumann algebras

Seminal work of Furman Fur3 has shown that measure equivalence rigidity is intimately related to orbit equivalence rigidity of ergodic group actions. In fact two countable groups are measure equivalent if and only if they admit stably orbit equivalent free measure-preserving ergodic actions by Borel automorphisms on standard probability spaces, see Gab-l2 Proposition 6.2.

Orbit equivalence rigidity. Let $\Gamma_1$ and $\Gamma_2$ be two countable groups, and for every $i \in \{1, 2\}$, let $(X_i, \mu_i)$ be a standard probability space equipped with a free ergodic measure-preserving action of $\Gamma_i$.

The actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ are virtually conjugate (as in Kid-oe Kid08b Definition 1.3) if there exist finite normal subgroups $F_i \triangleleft \Gamma_i$, finite-index subgroups $Q_i \subseteq \Gamma_i/F_i$, and free ergodic measure-preserving actions $Q_i \curvearrowright Y_i$ on standard probability spaces, so that $Q_1 \curvearrowright Y_1$ and $Q_2 \curvearrowright Y_2$ are conjugate, and for every $i \in \{1, 2\}$, the action of $\Gamma_i/F_i$ on $X_i/F_i$ is induced from the $Q_i$-action on $Y_i$. This implies in particular that the groups $\Gamma_1$ and $\Gamma_2$ are virtually isomorphic (i.e. commensurable up to finite kernels).

The following is a weaker notion. The actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ are stably orbit equivalent if there exist positive measure Borel subsets $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ and a measure-scaling isomorphism $\theta : A_1 \rightarrow A_2$4 such that for almost every $x \in A_1$, one has

$$\theta((\Gamma_1 \cdot x) \cap A_1) = (\Gamma_2 \cdot \theta(x)) \cap A_2.$$ 

A free ergodic measure-preserving action of $\Gamma$ on a standard probability space $X$ is OE-superrigid if for every countable group $\Gamma'$, and every free ergodic measure-preserving action of $\Gamma'$ on a standard probability space $X'$, if the $\Gamma$-action on $X$ is stably orbit equivalent to the $\Gamma'$-action on $X'$, then the two actions are virtually conjugate (in particular $\Gamma$ and $\Gamma'$ are virtually isomorphic).

The following theorem follows from our work in the exact same way as for mapping class groups of surfaces Kid-oe Kid08b (see also Fur-survey Fur11b Lemma 4.18).

Theorem 4.3. Let $V$ be a handlebody of genus at least 3. Then every free ergodic measure-preserving action of $\text{Mod}(V)$ on a standard probability space is OE-superrigid.

Rigidity of von Neumann algebras. Let $\Gamma$ be a countable group, and let $X$ be a standard probability space equipped with a standard ergodic action of $\Gamma$. Associated to the $\Gamma$-action on $X$ is a von Neumann algebra $L^\infty(X) \rtimes \Gamma$, obtained from the Murray–von Neumann construction MvN36.

We refer the reader to the work of Ozawa and Popa OP10 Definition 3.1 for the notion of a weakly compact group action. Let us only mention here that these include profinite actions, i.e. those obtained as inverse limits of actions on finite probability spaces (see OP10 Proposition 3.2). For example, this applies to the action of any

\[ \theta \]

In other words $\theta$ induces a measure space isomorphism between the probability spaces $\frac{1}{\mu_1(A_1)} A_1$ and $\frac{1}{\mu_2(A_2)} A_2$.
residually finite countable group on its profinite completion, equipped with the Haar measure. As a subgroup of $\text{Mod}(\partial V)$, the handlebody group $\text{Mod}(V)$ is residually finite by a theorem of Grossman \cite{Gro}. A free ergodic measure-preserving action of a countable group $\Gamma$ on a standard probability space $X$ is $W^*_{wc}$-superrigid if for every countable group $\Gamma'$, and every weakly compact free ergodic measure-preserving action of $\Gamma'$ on a standard probability space $X'$, if the von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Gamma'$ are isomorphic, then the $\Gamma$-action on $X$ is virtually conjugate to the $\Gamma'$-action on $X'$.

**Theorem 4.4.** Let $V$ be a handlebody of genus at least 3. Then every free ergodic measure-preserving action of $\text{Mod}(V)$ on a standard probability space is $W^*_{wc}$-superrigid.

**Proof.** Let $X$ be a standard probability space equipped with a free ergodic measure-preserving action of $\text{Mod}(V)$, and let $X'$ be a standard probability space equipped with a weakly compact free ergodic measure-preserving action of a countable group $\Gamma'$. Assume that there exists an isomorphism $\theta : L^\infty(X) \rtimes \text{Mod}(V) \to L^\infty(X') \rtimes \Gamma'$. By \cite[Theorem 7]{HHL20}, the group $\text{Mod}(V)$ is properly proximal in the sense of Boutonnet, Ioana and Peterson \cite{BIP21}. It thus follows from \cite[Theorem 1.4]{BIP21} that up to unitary conjugacy, the isomorphism $\theta$ sends $L^\infty(X)$ to $L^\infty(X')$. This implies that the actions $\Gamma \curvearrowright X$ and $\Gamma' \curvearrowright X'$ are orbit equivalent (see \cite{Sin}), so the conclusion follows from the orbit equivalence rigidity statement provided by Theorem 4.3. \hfill $\square$

**Remark 4.5.** Beyond the weakly compact case, the only known $W^*$-superrigidity result for handlebody groups concerns their Bernoulli actions, that is, actions of the form $\text{Mod}(V) \curvearrowright X_0^{\text{Mod}(V)}$, where $X_0$ is a standard probability space not reduced to a point, and the action is by shift. More precisely, when $V$ has genus at least 3, if a Bernoulli action $\text{Mod}(V) \curvearrowright X$ and a free, ergodic, probability measure-preserving action of a countable group have isomorphic von Neumann algebras, then the actions are conjugate. This follows from \cite[Theorem A.2]{HH21b}, based on work of Ioana, Popa and Vaes \cite[Theorem 10.1]{IPV13}, applied by letting $\Gamma_0$ be the cyclic subgroup generated by a Dehn twist about a nonseparating meridian $\alpha$, letting $\Gamma_1$ be the stabilizer of the isotopy class of $\alpha$, and $\Gamma = \text{Mod}(V)$. Indeed, to check that \cite[Theorem A.2]{HH21b} applies, we only need to find an element $g \in \text{Mod}(V)$ such that $g\Gamma_1g^{-1} \cap \Gamma_1$ is infinite, and $(\Gamma_1, g)$ generates $\text{Mod}(V)$. For this, let $\beta, \gamma$ be nonseparating meridians such that $\alpha, \beta, \gamma$ are pairwise disjoint, pairwise non-isotopic, and have connected complement. Let $g \in \text{Mod}(V)$ be an element sending $\alpha$ to $\beta$ and commuting with the twist $T_\gamma$. Then $g\Gamma_1g^{-1} \cap \Gamma_1$ is infinite because it contains $T_\gamma$. And $\text{Mod}(V)$ is generated by $\Gamma_1$ and $g$ because the simplicial graph with vertices the isotopy classes of nonseparating meridians, and edges the nonseparating pairs, is connected (as easily follows from the connectivity of the disk graph) with quotient a single edge.
References

[AD13] C. Anantharaman-Delaroche. The Haagerup property for discrete measured groupoids. In Operator algebra and dynamics, volume 58 of Springer Proc. Math. Stat., pages 1–30, Heidelberg, 2013. Springer.

[Ada94] S. Adams. Indecomposability of equivalence relations generated by word hyperbolic groups. Topology, 33(4):785–798, 1994.

[BIP21] R. Boutonnet, A. Ioana, and J. Peterson. Properly proximal groups and their von Neumann algebras. Ann. Sci. Éc. Norm. Supér., 54(2):445–482, 2021.

[CK15] I. Chifan and Y. Kida. $\mathcal{O}E$ and $W^*$ rigidity results for actions by surface braid groups. Proc. Lond. Math. Soc. (3), 111(6):1431–1470, 2015.

[dlST19] M. de la Salle and R. Tessera. Characterizing a vertex-transitive graph by a large ball. J. Topol., 12(3):705–743, 2019.

[Dye59] H.A. Dye. On groups of measure preserving transformation. I. Amer. J. Math., 81:119–159, 1959.

[Dye63] H.A. Dye. On groups of measure preserving transformations. II. Amer. J. Math., 85:551–576, 1963.

[FM12] B. Farb and D. Margalit. A primer on mapping class groups, volume 49 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012.

[Fur99a] A. Furman. Gromov’s measure equivalence and rigidity of higher-rank lattices. Ann. Math., 150:1059–1081, 1999.

[Fur99b] A. Furman. Orbit equivalence rigidity. Ann. of Math. (2), 150(3):1083–1108, 1999.

[Fur11a] A. Furman. Mostow–Margulis rigidity with locally compact targets. Geom. Funct. Anal., 11(1):30–59, 2011.

[Fur11b] A. Furman. A survey of measured group theory. In Geometry, rigidity, and group actions, Chicago Lectures in Math., pages 296–374, Chicago, IL, 2011. Univ. Chicago Press.

[Gab02] D. Gaboriau. Invariants $l^2$ de relations d’équivalence et de groupes. Publ. Math. Inst. Hautes Études Sci., 95:93–150, 2002.

[GH21] V. Guirardel and C. Horbez. Measure equivalence rigidity of $\text{Out}(F_N)$. arXiv:2103.03696, 2021.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In Geometric group theory, volume 2 of London Math. Soc. Lecture Note Ser., pages 1–295, Cambridge Univ. Press, Cambridge, 1993.

[Gro75] E.K. Grossman. On the residual finiteness of certain mapping class groups. J. London Math. Soc. (2), 9:160–164, 1974/75.

[Ham09] U. Hamenstädt. Geometry of the mapping class groups I: Boundary amenability. Invent. Math., 175(3):545–609, 2009.

[Hen] S. Hensel. A primer on handlebody groups. preprint, available at http://www.mathematik.uni-muenchen.de/~hensel/papers/hno4.pdf.

[Hen18] S. Hensel. Rigidity and flexibility for handlebody groups. Comment. Math. Helv., 93(2):335–358, 2018.

[Hen21] S. Hensel. (Un)distorted stabilisers in the handlebody group. J. Topol., 14(2):460–487, 2021.

[HH12] U. Hamenstädt and S. Hensel. The geometry of the handlebody groups I: Distortion. J. Topol. Anal., 4(1):71–97, 2012.

[HH20a] C. Horbez and J. Huang. Boundary amenability and measure equivalence rigidity among two-dimensional Artin groups of hyperbolic type. arXiv:2004.09325, 2020.
[OP10] N. Ozawa and S. Popa. On a class of $\text{II}_1$ factors with at most one Cartan subalgebra. *Amer. J. Math.*, 132(3):841–866, 2010.

[OW80] D.S. Ornstein and B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980.

[Sin55] I.M. Singer. Automorphisms of finite factors. *Amer. J. Math.*, 77:117–133, 1955.

[Suz77] S. Suzuki. On homeomorphisms of a 3-dimensional handlebody. *Canadian J. Math.*, 29(1):111–124, 1977.

[Waj98] B. Wajnryb. Mapping class group of a handlebody. *Fund. Math.*, 158(3):195–228, 1998.

[Zim80] R.J. Zimmer. Strong rigidity for ergodic actions of semisimple groups. *Ann. Math.*, 112(3):511–529, 1980.

[Zim91] R.J. Zimmer. Groups generating transversals to semisimple Lie group actions. *Israel J. Math.*, 73(2):151–159, 1991.

Sebastian Hensel
Mathematisches Institut der Universität München
D-80333 München
*e-mail*: hensel@math.lmu.de

Camille Horbez
Université Paris-Saclay, CNRS, Laboratoire de mathématiques d’Orsay, 91405, Orsay, France
*e-mail*: camille.horbez@universite-paris-saclay.fr