Darboux transformation of boundary conditions of regular Dirac Sturm—Liouville problem

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Abstract
It is shown that boundary conditions of the Darboux transformed Dirac Sturm—Liouville problem are always zero-valued independently on boundary conditions of initial problem.

1. Introduction

The study of the properties of the Darboux transformation [1, 2, 3, 4, 5] (or SUSY) attracts more and more attention in the past two decades. Recent the developments in this area are reflected in the special issue of J. Phys. A 34 where the references to the early paper can be found.

Despite in depth studies, many problems in given field have yet to be solved. In particular relationship between Green functions of two Hamiltonians being SUSY partners have been studied only in few paper [6, 7, 8, 9, 10]. All these paper deal with the Green functions of Schrödinger Hamiltonians.

The attempts to generalize some of these results to the case of Dirac problem have been only recently [11, 12, 13, 14]. They lead to the necessity of consideration of the Darboux transformation of boundary conditions of the initial Sturm—Liouville problem.

In the paper [14] this problem have been considered for some particular cases, that admit the analytical solution of the Sturm—Liouville problem. Considered in [14] the boundary values of the Darboux transformed solutions of these problem are always equal zero.

The last means that both components of the spinorial solution on the ends of an interval, on which leads to the natural assumption that this conclusion is also valid in general case, but not only in particular cases, considered in [14].

The proving of this statement is the aim of the present paper. In Section 2 we derive two different forms of the relation between the solutions of the initial and

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Darboux transformed Dirac Sturm—Liouville problems. In Section 3 we use these relation to prove the main statement of this paper. In Section 4 we analyze the consequence of obtained results for the relation between the Green functions of the two SUSY Dirac partners.

2. Two representations of Darboux transformed solutions of the Dirac equation

We consider the solutions of the one-dimensional two-component Dirac equation

\[ h_0 \psi(x) = E \psi(x), \quad h_0 = \gamma \partial_x + V(x), \quad \gamma = i \sigma_2, \]

\[ V^+(x) = V(x), \quad \psi(x) = (\psi_1(x), \psi_2(x)) \]

that satisfy the homogenous boundary conditions on the ends of a close interval \([a, b]\)

\[ \psi_1(a) \sin(\alpha) + \psi_2(a) \cos(\alpha) = 0, \]

\[ \psi_1(b) \sin(\beta) + \psi_2(b) \cos(\beta) = 0. \]

It is evidently that the eigenvalue spectrum of this problem is discrete.

Let \(u(x)\) and \(v(x)\) (with the eigenvalues \(\lambda_1, \lambda_2\) correspondingly) are the solutions of this boundary problem such that matrix

\[ u(x) = \begin{pmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{pmatrix} \]

is nonsingular inside the open interval \((a, b)\).

Consider the Darboux transformation

\[ \tilde{\psi} = L \psi, \quad L = \partial_x - u_x u^{-1}, \quad u_x = \partial_x u. \]

If \(\psi\) in (5) satisfy the equation

\[ h_0 \psi = E \psi, \]

then \(\tilde{\psi}\) in (5) satisfy the equation

\[ h_1 \tilde{\psi} = E \tilde{\psi}, \]

where

\[ h_1 = h_0 + [\gamma, u_x u^{-1}]. \]

Thus, \(h_0\) and \(h_1\) are SUSY partners.

The Green function of an initial Sturm—Liouville problem is the 2 \(\times\) 2 matrix that satisfy an inhomogeneous equation

\[ (h_0 - E) G(x, t; E) = \delta(x - y) I, \]

\[ (h_0 - E) G(x, t; E) = \delta(x - y) I, \]
where \( I \) is the 2 × 2 unity matrix.

It can be construct from two solutions ("left" and "right") of the Dirac equation

\[
h_0 \psi_{L(R)}(x, E) = E \psi_{L(R)}
\]

which satisfy following boundary conditions

\[
\psi_{L1}(a) \cos(\alpha) + \psi_{L2}(a) \sin(\alpha) = 0,
\]

\[
\psi_{R1}(b) \cos(\beta) + \psi_{R2}(b) \sin(\beta) = 0.
\]

Then

\[
G_{ik} = \frac{[\psi_{Li}\psi_{Rk}(x - y) + \psi_{Ri}\psi_{Lk}(y - x)]}{W}\{\psi_L, \psi_R\},
\]

\[
W = \det \begin{vmatrix} \psi_{L1}(x) & \psi_{R1}(x) \\ \psi_{L2}(x) & \psi_{R2}(x) \end{vmatrix},
\]

\[
dW\{\psi_L, \psi_R\} = 0.
\]

The boundary conditions for components of \( G_{ik} \) are the following:

\[
G_{1k}(a, y, E) \cos(\alpha) + G_{2k}(a, y, E) \sin(\alpha) = 0,
\]

\[
G_{i1}(x, b, E) \cos(\beta) + G_{i2}(x, b, E) \sin(\beta) = 0.
\]

The expression for the Green function of SUSY partner of \( h_0 \) can be obtained from (13) by the substitution

\[
\psi_{L,R}(x) \rightarrow \tilde{\psi}_{L,R}(x) = L\psi_{L,R},
\]

where the operator \( L \) is defined by (5).

The boundary conditions of the Darboux transformed problem need be derived from boundary condition of the initial problem and properties of \( L \) the operator \( L \).

The equation (5) in components reads:

\[
\tilde{\psi}_1 = \psi_1' - \frac{a}{D} \psi_1 - \frac{b}{D} \psi_2,
\]

\[
\tilde{\psi}_2 = \psi_2' - \frac{d}{D} \psi_1 - \frac{c}{D} \psi_2,
\]

\[
a = v_2 u_1' - u_2 v_1',
\]

\[
b = u_1 v_1' - v_1 u_1',
\]

\[
d = v_2 u_2' - u_2 v_2',
\]

\[
c = u_1 v_2' - v_1 u_2',
\]

\[
D = u_1 v_2 - u_2 v_1.
\]

For the following the another expressions for the components \( \tilde{\psi}_{1,2} \), that don’t contain derivations.

If \( \psi \) is the arbitrary solution of equation \( h_0 \psi = E \psi \), then taking into account that matrix \( u \) obey the equation

\[
h_0 u = u \Lambda,
\]

\[
\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},
\]
it is easy to prove that
\[
\tilde{\psi}_1 = -E\psi_2 + \frac{\pi}{D} \psi_1 + \frac{\bar{b}}{D} \psi_2, \tag{24}
\]
\[
\tilde{\psi}_2 = E\psi_1 - \frac{\bar{a}}{D} \psi_1 - \frac{\bar{c}}{D} \psi_2, \tag{25}
\]
\[
\bar{a} = (\lambda_1 - \lambda_2)u_2v_2, \quad \bar{b} = \lambda_2v_2u_1 - \lambda_1u_1v_1, \tag{26}
\]
\[
\bar{c} = (\lambda_2 - \lambda_1)u_1v_1, \quad \bar{d} = \lambda_1u_1v_2 - \lambda_2v_1u_2, \tag{27}
\]
with the help of eqs. (18), (19), (24), (25) in the next section we prove that
\[
\tilde{\psi}_{L1}(a, E) = \tilde{\psi}_{L2}(a, E) = 0, \tag{28}
\]
\[
\tilde{\psi}_{R1}(b, E) = \tilde{\psi}_{R2}(b, E) = 0. \tag{29}
\]

### 3. Boundary conditions of the Sturm—Liouville problem after Darboux transformation

Here we prove only relations (28). The proving of (29) is similar.

Put \( \psi(x, E) = \tilde{\psi}_L(x, E) \) and introduce the following combinations of quantities \( \psi_{L1}, \psi_{L2}, u_1, u_2, v_1, v_2 \):

\[
\phi_L = \cos (\alpha)\psi_{L1} + \sin (\alpha)\psi_{L2}, \tag{30}
\]
\[
\varphi_L = -\sin (\alpha)\psi_{L1} + \cos (\alpha)\psi_{L2}, \tag{31}
\]
\[
g_u = \cos (\alpha)u_1 + \sin (\alpha)u_2, \quad h_u = -\sin (\alpha)u_1 + \cos (\alpha)u_2, \tag{32}
\]
\[
g_v = \cos (\alpha)v_1 + \sin (\alpha)v_2, \quad h_v = -\sin (\alpha)v_1 + \cos (\alpha)v_2, \tag{33}
\]
\[
\tilde{\phi}_L = \cos (\alpha)\tilde{\psi}_{L1} + \sin (\alpha)\tilde{\psi}_{L2}, \quad \tilde{\varphi}_L = -\sin (\alpha)\tilde{\psi}_{L1} + \cos (\alpha)\tilde{\psi}_{L2}. \tag{34}
\]

From here we see that
\[
\phi_L(a) = g_h(a) = g_v(a) = 0, \tag{35}
\]
\[
D = u_1v_2 - u_2v_1 = g_u h_v - g_v h_u, \quad D(a) = 0, \tag{36}
\]
\[
\psi_{L1} = \cos (\alpha)\phi_L - \sin (\alpha)\varphi_L, \quad \psi_{L2} = \sin (\alpha)\phi_L + \cos (\alpha)\varphi_L, \tag{37}
\]
\[
u_1 = \cos (\alpha)g_u - \sin (\alpha)h_u, \quad u_1 = \sin (\alpha)g_u + \cos (\alpha)h_u, \tag{38}
\]
\[
v_2 = \sin (\alpha)g_v - \cos (\alpha)h_v. \tag{39}
\]

Then combining equations from (30) to (40) it is easy derive
\[
\tilde{\phi}_L = \phi_L' - \frac{g_u'}{D}(\phi_L h_v - \varphi_L g_v) - \frac{g_v'}{D}(-\phi_L h_u + \varphi_L g_u), \tag{41}
\]
\[
\tilde{\varphi}_L = E\phi - \frac{g_u^2 \lambda_1}{D}(\phi_L h_v - \varphi_L g_v) - \frac{g_v^2 \lambda_2}{D}(\phi_L h_u - \varphi_L g_u). \tag{42}
\]
Both numerators and denominators in these expressions become zero when $x \to a$. So, we need use the L’Hospital rules to open the indefinities arising in eqs. (44), (42) when $x \to a$.

The simple calculations lead to the following results:

\[
\tilde{\phi}_L(a) = 0, \quad (43) \\
\tilde{\varphi}_L(a) = 0. \quad (44)
\]

Combination of eqs. (43), (44) leads to the result

\[
\tilde{\psi}_{L1}(a) = \tilde{\psi}_{L2}(a) = 0, \quad (45)
\]

cited in previous Section (see equation (28)).

4. Conclusion

In the paper [14] it has been proved that for arbitrary initial Dirac Sturm—Liouville problem the following integral relation is valid:

\[
\text{tr} \int_a^b (\tilde{G}(x, x, E) - G(x, x, E))dx = \frac{1}{E - \lambda_1} + \frac{1}{E - \lambda_2} - \Delta, \quad (46)
\]

\[
\Delta = \frac{\tilde{\psi}_{L1}(a)\phi_{R1}(a) + \tilde{\psi}_{L2}(a)\phi_{R2}(a) - \tilde{\phi}_{R1}(b)\psi_{L1}(b) - \tilde{\phi}_{R2}(b)\psi_{L2}(b)}{W\{\phi_L, \psi_R\}}. \quad (47)
\]

Here $G(x, y, E)$ is the Green function of the initial Sturm—Liouville problem, $\tilde{G}(x, y, E)$ is the Green function of the Darboux transformed problem.

From eqs. (28), (29) it follows that $\Delta = 0$ and the trace of difference of the Green functions of the Hamiltonians-superpartners are equal sum of two pole terms.

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