Strong decays of strange quarkonia in a corrected $^3P_0$ model

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Extensively applied to both light and heavy meson decay and standing as one of the most successful strong decay models is the $^3P_0$ model, in which $qar{q}$ pair production is the dominant mechanism. In this paper we evaluate strong decay amplitudes and partial widths of strange $S$ and $D$ state mesons, namely $\phi(1020), \phi(1680), \phi(2050), \phi_1(1850), \phi_2(1850)$ and $\phi_3(1850)$, in the bound-state corrected $^3P_0$ decay model ($C^3P_0$). The $C^3P_0$ model is obtained in the context of the Fock-Tani formalism, which is a mapping technique.

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I. INTRODUCTION

The study of strangeonia should enter a new era with the advent of the new Hall D photoproduction facility GlueX at Jefferson Lab [1, 2]. The main goal of the GlueX experiment is to search for and study hybrid and exotic mesons which will provide the ideal laboratory for testing QCD in the confinement regime. Another top goal of GlueX is the exploration of the light meson spectrum, where interactions of hadrons with a photon beam can be regarded as a superposition of vector mesons with an important $s\bar{s}$ component. In this sense, studies of strange final states at GlueX should lead to considerable improvement in our knowledge of the $s\bar{s}$ spectrum.

Strange quarkonia are light $(u,d,s)$ mesons with at least one strange quark or antiquark in their dominant $q\bar{q}$ valence component. These are known as kaonia if the dominant valence basis state is $n\bar{s}$ (where $n$ should be understood as $u,d$), antikaonia if $s\bar{n}$, and strangeonia if $s\bar{s}$. A principal goal of light meson spectroscopy is the identification of exotica, which are resonances that are not dominantly $q\bar{q}$ states. These include glueballs, hybrids, and multiquark systems.

In this sense, a great variety of quark-based models are known that describe with reasonable success single-hadron properties. A natural question that arises is to what extent a model which gives a good description of hadron properties is, at the same time, able to describe the complex hadron-hadron interaction or by the same principles hadron decay. In the direction of clarifying these issues is the successful decay model, the $^3P_0$ model, which considers only OZI-allowed strong-interaction decays. This model was introduced over thirty years ago by Micu [3] and applied to meson decays in the 1970 by LeYaouanc et al. [4]. This description is a natural consequence of the constituent quark model scenario of hadronic states. Since the $^3P_0$ model precedes QCD and has no clear relation to it, one might expect that a description of decays in terms of allowed QCD processes such as OGE might be more realistic. There is strong experimental evidence that the $q\bar{q}$ pair created during the decay does have spin one as is assumed in the $^3P_0$ decay model.

T. Barnes et al. [5]-[8] have made an extensive survey of meson states in the light of the $^3P_0$ model. Two basic parameters of their formulation are $\gamma$ (the interaction strength) and $\beta$ (the wave function’s extension parameter). Although they found the optimum values near $\gamma = 0.5$ and $\beta = 0.4$ GeV, for light $1S$ and $1P$ decays, these values lead to overestimates of the widths of higher-$L$ states. In this perspective a modified $q\bar{q}$ pair-creation interaction, with $\gamma = 0.4$ was preferred. The spectrum of meson resonances up to 2 GeV is only moderately well determined. For strangeonia the they calculated a set of strong decays of a total of 43 resonances into 525 two-body modes, with 891 numerically evaluated amplitudes for all energetically allowed open-flavor two-body decay modes of all $n\bar{s}$ and $s\bar{s}$ strange mesons in the $1S, 2S, 3S, 1P, 2P, 1D$ and $1F$ multiplets [9].

In the present work, we shall concentrate on the $\phi$ mesons, which are the strange $S$ and $D$ states, predicted in the quark model, and a probable $s\bar{s}$ resonances expected up to 2.2 GeV. We employ a mapping technique in order to obtain an effective interaction for meson decay. A particular mapping technique long used in atomic physics [10], the Fock-Tani formalism (FTf), has been adapted, in previous publications [13]-[18], in order to describe hadron-hadron scattering interactions with constituent interchange. Now this technique has been extended in order to include meson decay [19, 20]. Starting with a microscopic $q\bar{q}$ pair-creation interaction, in lower order, the $^3P_0$ results are reproduced. An additional and interesting feature appears in higher orders of the formalism: corrections due to the bound-state nature of the mesons and a natural modification in the $q\bar{q}$ interaction strength.

In section II we review the basic aspects of the formalism. Section III is dedicated to obtain an effective decay Hamiltonian for a $\phi$ meson, where in subsection III A the general amplitudes and decay widths are obtained with numerical analysis in subsection III B. In section V are the conclusions.

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II. MESON MAPPING AND THE \( C^3P_0 \) MODEL

A. Review of the Fock-Tani Formalism

This section reviews the formal aspects of the mapping procedure and how it is implemented to quark-antiquark meson states [13]. In the Fock-Tani formalism one starts with the Fock representation of the system using field operators of elementary constituents which satisfy canonical (anti)commutation relations. Composite-particle field operators are linear combinations of the elementary-particle operators and do not generally satisfy canonical (anti)commutation relations. “Ideal” field operators acting on an enlarged Fock space are then introduced in close correspondence with the composite ones. Next, a given unitary transformation, which transforms the single composite states into single ideal states, is introduced. Application of the unitary operator on the microscopic Hamiltonian, or on other hermitian operators expressed in terms of the elementary constituent field operators, gives equivalent operators which contain the ideal field operators. The effective Hamiltonian in the new representation has a clear physical interpretation in terms of the processes it describes. Since all field operators in the new representation satisfy canonical (anti)commutation relations, the standard methods of quantum field theory can then be readily applied.

The starting point is the definition of single composite bound states. We write a single-meson state in terms of a meson creation operator \( M^\dagger_\alpha \) as

\[
|\alpha\rangle = M^\dagger_\alpha |0\rangle,
\]

where \( |0\rangle \) is the vacuum state. The meson creation operator \( M^\dagger_\alpha \) is written in terms of constituent quark and antiquark creation operators \( q^\dagger \) and \( \bar{q}^\dagger \),

\[
M^\dagger_\alpha = \Phi^\mu_\alpha q^\dagger_\mu q^\dagger_\nu, \tag{2}
\]

\( \Phi^\mu_\alpha \) is the meson wave function and \( q^\dagger_\mu |0\rangle = \bar{q}^\dagger_\nu |0\rangle = 0 \). The index \( \alpha \) identifies the meson quantum numbers of space, spin and isospin. The indices \( \mu \) and \( \nu \) denote the spatial, spin, flavor, and color quantum numbers of the constituent quarks. A sum over repeated indices is implied. It is convenient to work with orthonormalized amplitudes,

\[
\langle \alpha | \beta \rangle = \Phi^\mu_\alpha \Phi^\mu_\beta = \delta_{\alpha \beta}. \tag{3}
\]

The quark and antiquark operators satisfy canonical anticommutation relations,

\[
\{ q^\dagger_\mu, q^\dagger_\nu, \bar{q}^\dagger_\mu, \bar{q}^\dagger_\nu \} = \delta_{\mu \nu},
\]

\[
\{ q^\dagger_\mu, q^\dagger_\nu \} = \{ q^\dagger_\mu, \bar{q}^\dagger_\nu \} = \{ q^\dagger_\nu, q^\dagger_\mu \} = \{ q^\dagger_\nu, \bar{q}^\dagger_\mu \} = 0. \tag{4}
\]

Using these quark anticommutation relations, and the normalization condition of Eq. (3), it is easily shown that the meson operators satisfy the following non-canonical commutation relations

\[
[M^\dagger_\alpha, M^\dagger_\beta] = \delta_{\alpha \beta} - M^\dagger_{\alpha \beta}, \quad [M_\alpha, M^\dagger_\beta] = 0, \quad [M_\alpha, M_\beta] = 0. \tag{5}
\]

where

\[
M^\dagger_{\alpha \beta} = \Phi^\rho_\alpha \Phi^\rho_\beta q^\dagger_\rho q^\dagger_\sigma + \Phi^\rho_\alpha \Phi^\rho_\beta \bar{q}^\dagger_\rho \bar{q}^\dagger_\sigma. \tag{6}
\]

A transformation is defined such that a single-meson state \( |\alpha\rangle \) is redescribed by an ("ideal") elementary-meson state by

\[
|\alpha\rangle \rightarrow U^{-1} |\alpha\rangle = m^\dagger_\alpha |0\rangle, \tag{7}
\]

where \( m^\dagger_\alpha \) an ideal meson creation operator. The ideal meson operators \( m^\dagger_\alpha \) and \( m_\alpha \) satisfy, by definition, canonical commutation relations

\[
[m^\dagger_\alpha, m^\dagger_\beta] = \delta_{\alpha \beta}, \quad [m^\dagger_\alpha, m_\beta] = 0. \tag{8}
\]

The state \( |0\rangle \) is the vacuum of both quark and meson degrees of freedom in the new representation. In addition, in the new representation the quark and antiquark operators \( q^\dagger, q, \bar{q}, \bar{q}^\dagger \) and \( \bar{q} \) are kinematically independent of the \( m^\dagger_\alpha \) and \( m_\alpha \)

\[
[q^\dagger_\mu, m^\dagger_\alpha] = [\bar{q}^\dagger_\mu, m^\dagger_\alpha] = [\bar{q}^\dagger_\mu, m_\alpha] = [\bar{q}^\dagger_\mu, m^\dagger_\alpha] = 0. \tag{9}
\]

The unitary operator \( U \) of the transformation is

\[
U(t) = \exp \{ t \mathcal{F} \}, \tag{10}
\]

where \( \mathcal{F} \) is the generator of the transformation and \( t \) a parameter which is set to \( -\pi/2 \) to implement the mapping. The generator \( \mathcal{F} \) of the transformation is

\[
F = m^\dagger_\alpha \tilde{M}_\alpha - \tilde{M}^\dagger_\alpha m_\alpha \tag{11}
\]

where

\[
\tilde{M}_\alpha = \sum_{i=0}^n \tilde{M}^{(i)}_\alpha, \tag{12}
\]

with

\[
[M^\dagger_\alpha, \tilde{M}^\dagger_\beta] = \delta_{\alpha \beta} + \mathcal{O}(\mathcal{F}^{n+1}),
\]

\[
[M^\dagger_\alpha, \tilde{M}_\beta] = [\tilde{M}^\dagger_\alpha, \tilde{M}^\dagger_\beta] = 0. \tag{13}
\]

It is easy to see from (11) that \( F^\dagger = -F \) which ensures that \( U \) is unitary. The index \( i \) in (12) represents the order of the expansion in powers of the wave function \( \Phi \). The \( \tilde{M}_\alpha \) operator is determined up to a specific order \( n \) consistent with (13).

The next step is to obtain the transformed operators in the new representation. The basic operators of the model are expressed in terms of the quark operators. Therefore, in order to obtain the operators in the new representation, one writes

\[
q(t) = U^{-1} q U, \quad \bar{q}(t) = U^{-1} \bar{q} U. \tag{14}
\]

Once a microscopic interaction Hamiltonian \( H_I \) is defined, at the quark level, a new transformed Hamiltonian can be obtained. This effective interaction, the Fock-Tani Hamiltonian \( \mathcal{H}_{FT} \), is obtained by the application of the unitary operator \( U \) on the microscopic Hamiltonian.
Hamiltonian describes all possible processes involving mesons and quarks. The general structure of $H_{FT}$ is of the following form

$$H_{FT} = H_q + H_m + H_{mq},$$

(15)

where the first term involves only quark operators, the second one involves only meson operators, and $H_{mq}$ involves quark and meson operators. In $H_{FT}$ there are higher order terms that provide bound-state corrections to the lower order ones. The basic quantity for these corrections is the bound-state kernel $\Delta$ defined as

$$\Delta(\rho|\lambda\nu) = \Phi_\rho^{\alpha\nu} \Phi_\lambda^{\nu\alpha}.$$  

(16)

The physical meaning of the $\Delta$ kernel becomes evident, in the sense that it modifies the quark-antiquark interaction strength \[13, 19\]. The following two examples can clarify the physical interpretation.

1) **First example:** consider that the starting point is a two-body microscopic quark-antiquark Hamiltonian of the form

$$H_{2q} = T(\mu) q_\mu^\dagger q_\mu + T(\nu) \bar{q}_\nu^\dagger \bar{q}_\nu + V_{qq}(\mu\nu; \sigma\rho) q_\mu^\dagger q_\nu^\dagger q_\rho q_\sigma + \frac{1}{2} V_{q\bar{q}}(\mu\nu; \sigma\rho) q_\mu^\dagger q_\nu^\dagger q_\rho q_\sigma + \frac{1}{2} V_{q\bar{q}}^*(\mu\nu; \sigma\rho) q_\mu^\dagger q_\nu^\dagger q_\rho q_\sigma.$$  

(17)

The transformation $H_{FT} = U^{-1} H_{2q} U$ is implemented again by transforming each quark and antiquark operator in Eq. (17), where a similar structure to Eq. (15) is obtained. In free space, the wave function $\Phi$ of Eq. (2) satisfies the following equation

$$H(\mu\nu; \sigma\rho) \Phi^{\mu\nu}_{\alpha} = \epsilon_{[\alpha]} \Phi^{\mu\nu}_{\alpha},$$  

(18)

where $H(\mu\nu; \sigma\rho)$ is the Hamiltonian matrix

$$H(\mu\nu; \sigma\rho) = \delta_{[\mu]}^{[\sigma]} \delta_{[\nu]}^{[\rho]} \left[ T(\sigma|) + T(\rho|) \right] + V_{qq}(\mu\nu; \sigma\rho),$$  

(19)

$\epsilon_{[\alpha]}$ is the total energy of the meson. There is no sum over repeated indices inside square brackets.

The effective quark Hamiltonian $H_{2q}$ has an identical structure to the microscopic quark Hamiltonian, Eq. (17), except for inclusion of a term corresponding to a modification in the quark-antiquark interaction as follows

$$H_{2q} = H_{2q} + \tilde{H}_{q\bar{q}},$$  

(20)

with

$$\tilde{H}_{q\bar{q}} = (-H \Delta - \Delta H + \Delta H \Delta) q_\mu^\dagger q_\nu^\dagger q_\rho q_\sigma$$  

(21)

where the contractions are $H \Delta \equiv H(\mu\nu; \tau\xi) \Delta(\tau\xi; \sigma\rho)$ and $\Delta H \Delta \equiv \Delta(\mu\nu; \zeta\eta) H(\zeta\eta; \tau\xi) \Delta(\tau\xi; \sigma\rho)$. An important property of the bound-state kernel is

$$\Delta(\mu\nu; \sigma\rho) \Phi^{\mu\nu}_{\alpha} = \Phi^{\mu\nu}_{\alpha},$$  

(22)

which follows from the wave function's orthonormalization, Eq. (3). In the case that $\Phi$ is a solution of Eq. (18), equation (20), reduces to

$$H_{2q} = H_{2q} - \epsilon_{\beta} N_{\beta}$$  

(23)

where $N_{\beta} = M_{\beta}^\dagger M_{\beta}$ is the number operator and the following property holds: $N_{\beta}[\alpha] = |\beta\rangle$.

The spectrum of the modified quark Hamiltonian, $H_{2q}$, is positive semi-definite and hence has no bound-states \[10\]. To show this, consider an arbitrary state $|\alpha\rangle$ formed from a pair quark and antiquark:

$$|\alpha\rangle = \Psi^{\mu\nu}_{\alpha} q_\mu^\dagger q_\nu^\dagger |0\rangle.$$  

(24)

The action of the Hamiltonian (23) on this state results in

$$H_{2q}|\alpha\rangle = \left( H_{2q} - \epsilon_{\beta} N_{\beta} \right) |\alpha\rangle = H_{2q}|\alpha\rangle - \epsilon_{\beta} \Phi^{\mu\nu}_{\alpha} \Psi^{\nu\mu}_{\beta} |\beta\rangle$$  

(25)

If $|\alpha\rangle$ is one of the bound eigenstates of the microscopic Hamiltonian then

$$H_{2q}|\alpha\rangle = \left( \epsilon_{[\alpha]} - \epsilon_{[\alpha]} \right) |\alpha\rangle = 0.$$  

(26)

On the other hand, if $\Psi^{\mu\nu}_{\alpha}$ is orthogonal to all bound states $\Phi^{\mu\nu}_{\alpha}$ then (25) reduces to

$$H_{2q}|\alpha\rangle = H_{2q}|\alpha\rangle.$$  

(27)

Let $\psi_{i\alpha}$ be the continuum (unbound, positive energy) eigenstates of $H_{2q}$, with energies $\epsilon_{i\alpha} \geq 0$. One can expand any $\Psi^{\mu\nu}_{\alpha}$ in the form

$$\Psi^{\mu\nu}_{\alpha} = \sum_{i} c_i \Phi^{\mu\nu}_{i\alpha} + \sum_{i} c_i \psi^{\mu\nu}_{i\alpha},$$  

(28)

where $(\Phi_{\alpha}, \psi_{i\alpha}) = 0$. Then by (26) and (27)

$$H_{2q}|\alpha\rangle = \sum_{i} \epsilon_{i\alpha} c_i \psi^{\mu\nu}_{i\alpha} |\mu\nu\rangle.$$  

(29)

Therefore,

$$\langle \alpha|H_{2q}|\alpha\rangle = \sum_{i} \epsilon_{i\alpha} |c_i|^2,$$  

(30)

where it is evident that $H_{2q}$ is semi-definite positive and therefore does not have quark-antiquark bound states.

2) **Second example:** consider in the ideal meson sector $H_{fm}$ of equation (15), many approaches similar to the Fock-Tani formalism \[13\] have obtained, for example, the meson-meson scattering interaction in the Born approximation: Resonating Group Method (RGM) \[21\], Quark Born Diagram Formalism (QBDF) \[22\],

$$H_{mm} = T_{mm} + V_{mm},$$  

(31)

where $T_{mm}$ is the kinetic term and $V_{mm}$ is the meson-meson interaction potential with constituent interchange. This potential is given by

$$V_{mm} = V^0_{mm} + V^1_{mm} + V^2_{mm},$$  

(32)
where $V^{\text{dir}}$ is the direct potential (no quark interchange), $V^{\text{exc}}_{\text{me}}$ the quark exchange term and $V^{\text{int}}_{\text{me}}$ the intra-exchange term. As shown in Ref. [13] and [23], if one extends the Fock-Tani calculation to higher orders a new meson-meson Hamiltonian is obtained

$$H_{\text{mm}} = H_{\text{mm}} + \delta H_{\text{mm}}$$

where $\delta H_{\text{mm}}$ is the bound-state correction Hamiltonian,

$$\delta H_{\text{mm}} = \frac{1}{2} \Phi_{\alpha}^{\mu} \Phi_{\beta}^{\nu} \Phi_{\gamma}^{\rho} \Phi_{\delta}^{\sigma} H(\mu \nu \lambda \tau) \Delta(\lambda \tau ; \mu \nu \rho \sigma) \Phi_{\delta}^{\rho} \Phi_{\gamma}^{\sigma}$$

with $\Phi_{\alpha}^{\mu}$, $\Phi_{\beta}^{\nu}$, $\Phi_{\gamma}^{\rho}$, $\Phi_{\delta}^{\sigma}$ denoting the quark fields.

The pair creation potential $V_{\text{mm}}$ is the direct potential (no quark interchange), where $C_{3}P_{0}$ is a condensed notation for $C_{3}P_{0} = \int | \Phi_{\alpha}^{\mu} \Phi_{\beta}^{\nu} \Phi_{\gamma}^{\rho} \Phi_{\delta}^{\sigma} |^{2} d\Omega$. The pair production strength $H_{\text{mm}}$ is calculated to leading order. In the ideal meson space the initial and final states involve only ideal meson operators $| i \rangle = | m_{i}^\dagger \rangle | 0 \rangle$ and $| f \rangle = | m_{f}^\dagger m_{f} \rangle | 0 \rangle$.

The C$^{3}$P$^{0}$ amplitude is obtained by the following matrix element,

$$\langle f | H_{\text{C}3P0} | i \rangle = \delta(\vec{P}_{\gamma} - \vec{P}_{\alpha} - \vec{P}_{\beta}) h_{fi}$$

The $h_{fi}$ decay amplitude is combined with relativistic phase space, resulting in the differential decay rate

$$d\Gamma_{\gamma \rightarrow \alpha \beta} = 2 \pi \frac{E_{\gamma} E_{\alpha} E_{\beta}}{M_{\gamma}} | h_{fi} |^{2}$$

which, after integration in the solid angle $\Omega$, a usual choice for the meson momenta is made: $\vec{P}_{\gamma} = 0$ ($P = | \vec{P}_{\alpha} | = | \vec{P}_{\beta} |$).

### III. $\phi$ Meson Decay

The previous section has outlined the essential aspects of the $C^{3}P^{0}$ model and how it is obtained from the Fock-Tani formalism, where the decay Hamiltonian $H_{\text{C}3P0}$ was deduced. In this section the phenomenological Hamiltonian $H_{\text{C}3P0}$ will be used in order to evaluate the $n^{3}S_{1}$ decays $\phi(1020)$, $\phi(1860)$, $\phi(2050)$ with $n = 1, 2, 3$ and the $1^{3}D_{J}$ decays $\phi_{1}(1850)$, $\phi_{2}(1850)$ and $\phi_{3}(1850)$ mesons.

#### A. Amplitudes and decay widths

In the following decay channels, that shall be studied, some have been observed, with no available data, while others are only theoretical [9, 24]:

(a) $\phi(1020) \rightarrow KK$;

(b) $\phi(1860) \rightarrow KK, KK^\ast, \eta\phi$;

(c) $\phi(2050) \rightarrow KK, KK^\ast, KK^\ast, KK_{1}(1270), KK_{1}(1400), KK_{2}(1340), KK^\ast(1410), KK(1460), \eta\phi, \eta'\phi, \eta_{h1}(1380)$;

(d) $\phi_{1}(1850) \rightarrow K^\ast K^\ast$;

(e) $\phi_{2}(1850) \rightarrow KK, KK^\ast, K^\ast K^\ast, \eta\phi$;

(f) $\phi_{3}(1850) \rightarrow KK, KK^\ast, K^\ast K^\ast, KK_{1}(1270), \eta\phi$, ...
where the $\frac{1}{2}$ symbol indicates an unobserved state, $\phi_1(1850)$ is sometimes referred only as $\phi(1850)$ and $K^*$ is actually $K^*(892)$ from the Particle Data Group [24].

In the calculation of the decay amplitudes, the matrix element is given by (41) where the decay Hamiltonian (38) can be split into two parts: $H^{C_{PD}} = H_m + \delta H_m$.

The matrix element of first term, containing $H_m$, the term without the bound-state correction, is given by

$$
(f | H_m | i) = -d_1 - d_2 \quad (43)
$$

where

$$
d_1 = \Phi^{*\nu\sigma}\Phi^{*\mu\lambda}\Phi^{*\alpha\gamma}V_{\mu\nu}
$$

$$
d_2 = \Phi^{*\mu\nu}\Phi^{*\sigma\epsilon}\Phi^{*\alpha\gamma}V_{\mu\nu}.
$$

The matrix element of the bound-state correction $\delta H_m$, is written as

$$
(f | \delta H_m | i) = - \sum_{k=1}^{3} \sum_{j=1}^{2} d_j^k
$$

where we introduce the following notation

$$
d_1^1 = \frac{1}{4} \Phi^{*\mu\sigma}\Phi^{*\nu\tau}\Delta(\rho; \lambda \nu)\Phi^{*\lambda\sigma}V_{\mu\nu}
$$

$$
d_1^2 = \frac{1}{4} \Phi^{*\mu\sigma}\Phi^{*\tau\nu}\Delta(\rho; \lambda \nu)\Phi^{*\lambda\sigma}V_{\mu\nu}
$$

$$
d_1^3 = \frac{1}{4} \Phi^{*\nu\lambda}\Phi^{*\sigma\epsilon}\Phi^{*\alpha\gamma}V_{\mu\nu}
$$

$$
d_2^1 = \frac{1}{2} \Phi^{*\nu\lambda}\Phi^{*\mu\sigma}\Delta(\rho; \lambda \nu)\Phi^{*\lambda\sigma}V_{\mu\nu}
$$

$$
d_2^2 = \frac{1}{2} \Phi^{*\nu\lambda}\Phi^{*\lambda\sigma}\Delta(\rho; \lambda \nu)\Phi^{*\lambda\sigma}V_{\mu\nu}
$$

$$
d_2^3 = \frac{1}{4} \Phi^{*\nu\lambda}\Phi^{*\lambda\sigma}\Delta(\rho; \lambda \nu)\Phi^{*\lambda\sigma}V_{\mu\nu}.
$$

In $d_j^k(2)$, the index $k = 1, 2, 3$ represents the first, second and third term of the correction, respectively. As can be seen in equations (43)-(46) the matrix elements depend directly on the wave functions $\Phi^{*\mu\nu}$ and the potential $V_{\mu\nu}$. Considering as the fundamental degrees of freedom color $C$, flavor $f$, spin $\chi$ and space $\Phi$, the mesons wave function can be written as product

$$
\Phi^{*\mu\nu} = C^{*\nu\mu} f^{f}_{f_a} \chi_{S_a} \Phi \eta_{\mu}(\bar{P}_a - \bar{P}_f - \bar{P}_\nu), \quad (47)
$$

allowing to calculate color, flavor and spin-space separately. Details of (47) can be found in the appendix B. This factorization of the wave function implies that equations (44) and (46) can also be put in a direct product form of color, flavor and spin-space:

$$
d_1 = d_1^1 d_1^2 d_1^{1-s} \quad ; \quad d_2 = d_2^1 d_2^2 d_2^{1-s} \quad (48)
$$

and

$$
d_1^1 = \frac{1}{4} d_1^{1c} d_1^{1f} d_1^{1-s} \quad ; \quad d_1^2 = \frac{1}{4} d_2^{1c} d_2^{1f} d_2^{1-s}
$$

$$
d_1^3 = -\frac{1}{4} d_1^{2c} d_1^{2f} d_1^{2-s} \quad ; \quad d_2^2 = -\frac{1}{2} d_2^{2c} d_2^{2f} d_2^{2-s}
$$

$$
d_1^3 = \frac{1}{4} d_1^{2c} d_1^{2f} d_1^{3-s} \quad ; \quad d_3^2 = \frac{1}{4} d_2^{2c} d_2^{2f} d_2^{3-s} \quad (49)
$$

It is essential to note that the bound-state kernel definition, Eq. (16), has an implicit contraction in the $\xi$ index, which physically implies a sum over all species condition. In practice, this means that one should sum over intermediate meson bound-states. Any of the respective meson multiplet members can be considered in this sum.

In our calculations, due to the symmetry of problem, the only possible states will have the following $n^{2s+1}L_J$ and isospin quantum numbers: $|1^1S_0\rangle$ and $I = 0$ (type $\eta, \eta'$) or $|1^3S_1\rangle$ and $I = 0$ (type $\phi, \omega$). The bound-state kernel $\Delta(\mu\nu; \rho\sigma)$ will then be a sum over $\eta, \eta', \phi$ and $\omega$ intermediate states, which can be written explicitly as

$$
\Delta(\mu\nu; \rho\sigma) = \Delta_{\eta}(\mu\nu; \rho\sigma) + \Delta_{\eta'}(\mu\nu; \rho\sigma) + \Delta_{\phi}(\mu\nu; \rho\sigma) + \Delta_{\omega}(\mu\nu; \rho\sigma) \quad (50)
$$

The color amplitude factors of (48) and (49) can be calculated directly with the definition (B11) and the color part of (37), resulting in

$$
d_1^c = d_2^c = \frac{1}{3\sqrt{3}} \quad (51)
$$

Proceeding similarly, for the bound-state correction one has

$$
d_1^{c-s} = -2 a_{1i} \gamma \int d^3K \chi_i^* \left(\vec{\sigma} \cdot \vec{K}\right) \chi_j^* \phi^* \left(2\vec{K} + \vec{P}\right) \times \phi^* \left(2\vec{K} + \vec{P}\right) \phi \left(2\vec{K} + 2\vec{P}\right) \quad (52)
$$

and the three terms with the bound-state correction are

$$
d_1^{s-s} = -2 a_{1i} \gamma \int d^3K d^3q \chi_i^* \left(\vec{\sigma} \cdot \vec{K}\right) \chi_j^* \phi^* \left(2\vec{q} + \vec{P}\right) \times \phi^* \left(2\vec{q} + \vec{P}\right) \left[\phi \left(2\vec{q} + 2\vec{P}\right) \phi^* \left(2\vec{q} + \vec{K}\right) \right] \phi \left(2\vec{q}\right)
$$

$$
d_1^{2s-s} = -2 a_{2i} \gamma \int d^3K d^3q \chi_i^* \left(\vec{\sigma} \cdot \vec{K}\right) \chi_j^* \phi^* \left(2\vec{q} + \vec{P}\right) \times \phi^* \left(2\vec{q} + \vec{P}\right) \left[\phi \left(2\vec{q} + 2\vec{P}\right) \phi^* \left(2\vec{K}\right) \right] \phi \left(2\vec{q}\right)
$$

$$
d_1^{3s-s} = -2 a_{3i} \gamma \int d^3K d^3q \chi_i^* \left(\vec{\sigma} \cdot \vec{K}\right) \chi_j^* \phi^* \left(2\vec{q} - \vec{P}\right) \times \phi^* \left(2\vec{K} - \vec{P}\right) \left[\phi \left(2\vec{q} + 2\vec{P}\right) \phi^* \left(2\vec{q} + \vec{K}\right) \right] \phi \left(2\vec{q}\right) \quad (54)
$$

where $a_{ij} = \chi_{\phi}^* \chi_{\eta}^* \chi_{\eta'}^* \chi_{\omega}^* \chi_{\phi}^* \chi_{\omega}^*$ is a number resulting from the product of the mesons spin wave functions involved.
in the decay. The coefficients \(a_{1ij}, a_{2ij}, \) and \(a_{3ij}\) are obtained in a similar form and represent the first, second and third bound-state correction term, respectively. Note that the wave functions in-between brackets in (54) are related to the bound-state kernel part and therefore it is assumed there is an implicit sum over species with \(\xi\) assuming the \(\eta,\eta',\phi,\omega\) quantum numbers. The \(d_{2s}^{a-s}\) and \(d_{2s-}^{a-s}\) amplitude factors are obtained simply by the changing \(\vec{P} \to -\vec{P}\) in (53)-(54).

B. Numerical results

1. General aspects

In this section, we present the numerical results for the \(\phi f_i(M)\) decay widths. The amplitudes can be written in a general form

\[
h_{fji}^{\text{C3P0}} = \frac{\gamma}{\pi^{1/4}} M_{fji}
\]

where \(M_{fji}\) appears in appendix C. These amplitudes are inserted in (42) and integrated over the solid angle \(\Omega\). In order to calculate \(M_{fji}\), the wave function must be determined, knowing the spin and space quantum numbers to be used, which are listed in table I. The spatial wave functions are considered to be Gaussians characterized by \(\beta\) parameter, which is the Gaussian’s width. Each decay particle has its own \(\beta\). For example, \(\phi(1020)\) has the width \(\beta_\phi, \phi(1680)\) has \(\beta_{\phi(1680)}\) and so on. The mesons which are part of the bound-state kernel also have their own widths, and are distinguished from others by the notation \(\beta_{\eta}\), \(\beta_{\eta'}\), \(\beta_\phi\) and \(\beta_\omega\). To include all subprocesses in the results, it is necessary to multiply \(\Gamma\) by a multiplicity factor \(\mathcal{F}\). For example, in \(\phi \to KK\), the possible subprocesses are: \(\phi \to K^+K^-\) and \(\phi \to K^0\bar{K}^0\). Therefore, the multiplicity factor is \(\mathcal{F}=2\). The values of \(\mathcal{F}\) are listed in table II.

The \(\phi(1020)\) is a natural candidate for the \(s\bar{s}\) state with \(\phi(1680)\) as radial excitation. One must note that the decay of \(\phi(1680)\) in \(KK\) and \(KK^*\) is sometimes cited as evidence that this state is \(s\bar{s}\). The free parameters, that shall be varied, will be the wave functions width \(\beta\) and the mixing angles. The decay of \(\phi(1020) \to K^+K^-\) has a partial width of 2.08 \(\pm\) 0.04 MeV. Following the predicted values for the mixing angles [24], we varied \(\theta_p\), \(\theta_e(1680)\), \(\theta_v\), \(\beta_\eta\), \(\beta_\eta'\), \(\beta_\phi\) and \(\beta_\omega\). Where \(\theta_e(1680)\) and \(\theta_v\) are the mixing angles of \(\phi(1020), \phi(1680)\) and \(\phi(2050)\), respectively.

2. S states

The \(\phi(1020)\) is a natural candidate for the \(s\bar{s}\) state with \(\phi(1680)\) as radial excitation. One must note that the decay of \(\phi(1680)\) in \(KK\) and \(KK^*\) is sometimes cited as evidence that this state is \(s\bar{s}\). The free parameters, that shall be varied, will be the wave functions width \(\beta\) and the mixing angles. The decay of \(\phi(1020) \to K^+K^-\) has a partial width of 2.08 \(\pm\) 0.04 MeV. Following the predicted values for the mixing angles [24], we varied \(\theta_p\) between \(-20^\circ\) and \(-10^\circ\) and \(\theta_v\) between \(26^\circ\) and \(35^\circ\). The \(\beta_\Delta\)'s were varied in the range from 0.3 to 0.6 GeV.

The two best fits for this channel have values of \(\theta_p = -10^\circ, \theta_v = 35^\circ, \beta_\eta = 25^\circ, \beta_\phi = 0.6\) GeV, which we shall call parameterization \((a)\), resulting in \(\Gamma_{\phi} = 3.21\) MeV and \(\Gamma_{\phi} = 2.81\) MeV for different parameterizations, which we shall call \((b)\), has \(\theta_p = -10^\circ, \theta_v = 26^\circ, 4\beta_\Delta = 3\beta_\phi = 1.2\) GeV, resulting in \(\Gamma_{\phi} = 2.38\) MeV and \(\Gamma_{\phi} = 2.01\) MeV.

The \(\phi(1680)\) meson has a total \(\Gamma_{\phi} = 150 \pm 50\) MeV, the \(C^3\eta\) model’s best fit yields a \(\Gamma_{\phi} = 233.29\) MeV, which corresponds to the values: \(\theta_e(1680) = 35^\circ, \beta_\eta = 0.6\) GeV, \(\beta_\phi = 0.3\) GeV, \(\beta_\omega = 0.4\) GeV and \(\beta_\omega = 0.6\) GeV. The estimated channels are in table III.

From these results one can see that for \(\phi(1020)\), the
The decay width is within the experimental range. The same does not happen for \(\phi(1680)\), where the total decay width is above the experimental value. It should be noted that in the literature there are results that indicate higher experimental values: 211 \pm 14 \pm 19 \text{ MeV in} \ [27] \text{ and 322} \pm 77 \pm 160 \text{ MeV in} \ [28]. \text{ Another important experimental result are the ratios } \Gamma_{KK}/\Gamma_{KK'} = 0.07 \pm 0.01 \text{ and } \Gamma_{\eta_\phi}/\Gamma_{KK'} \approx 0.37 \ [24]. \text{ C}^3\text{P}_0 \text{ model's fit, yields } \Gamma_{KK}/\Gamma_{KK'} = 0.71 \text{ and } \Gamma_{\eta_\phi}/\Gamma_{KK'} = 0.19. \text{ The } \theta_\phi(1680) \text{ angle is a measure of strangeness content of } \phi(1680). \text{ A tentative solution to improve these ratios is to set this angle for values beyond the } 26^\circ - 35^\circ \text{ range. An increase the angle can lower the total decay width into the range of experimental values (150 \pm 50 \text{ GeV}) and } \Gamma_{KK}/\Gamma_{KK'} \text{ didn't improve. This inconsistency could be an indication that the composition of } \phi(1680) \text{ is not well described and it may be a mixture of the states } ^3D_1 \text{ and } ^2S_1, \text{ or may have hybrid components} \ [24]. \text{ The } \phi(2050) \text{ is a } ^3S_1 \bar{s}s \text{ vector state, to which an estimated mass of 2.05 GeV is assigned, although this state is actually not known at present. It should be important in future spectroscopic studies because with } 1^- \text{ quantum numbers it can be made both in diffractive production and in } e^+e^- \text{ annihilation. As a radial excitation of } \bar{s}s \text{ one can use the previous parameterizations. Due to the fact that } \theta_\phi \text{ varies between } 26^\circ \text{ and } 35^\circ, \text{ we shall consider the extreme values and consider four sets of parameters. The results of these calculations are presented in table IV and yields an average total } C^3\text{P}_0 \text{ width of } \Gamma = 182.35 \pm 3.13 \text{ MeV.}

| \Gamma (\text{MeV}) | 3P0 | C3P0 | C3P0 | Exp |
|-------------------|-----|------|------|-----|
| KK                | 89.42 | 87.42 | 87.42 |     |
| KK*               | 123.28 | 123.25 | 122.38 |     |
| \eta_\phi         | 21.63 | 26.81 | 23.49 |     |
| \eta_\phi'        | 234.33 | 237.48 | 233.29 | 150 \pm 50 |

\text{ TABLE III. Decay width of } \phi(1680) \text{ with } \theta_p = -10^\circ, \ theta_\phi = 26^\circ, \ theta_\eta(1680) = 35^\circ, \text{ (a) } 2\beta_{\eta_\Delta} = 4\beta_\eta' = 3\beta_\phi = 2\beta_{\omega} = 1.2 \text{ GeV and (b) } 2\beta_{\eta_\Delta} = 4\beta_{\eta_\phi} = 4\beta_\phi = 3\beta_{\omega} = 1.2 \text{ GeV.} 

3. D states

As well known \(\phi_3(1850)\) was first reported in \(K^-p \rightarrow \phi_3\Lambda\) at \(\text{CERN's bubble-chamber experiment} \ [29]. \text{ Originally reported in } KK \text{ and } KK^*, \text{ with a total width of } 87^{+26}_{-23} \text{ MeV and a relative branching fraction of } B(KK^*/KK) = 0.55^{+0.85}_{-0.45} \ [24]. \text{ Following the same strategy as in the case of the S-states, the } \phi_3(1850) \text{ parameters that resulted in the best fit are shown in table V were: } \theta_p = -10^\circ, \ theta_\phi = 26^\circ, \ theta_\eta_3 = 35^\circ, \beta_{\eta_3} = 0.6 \text{ GeV, } \beta_{\eta_3'} = 0.4 \text{ GeV, } \beta_\phi = 0.6 \text{ GeV and } \beta_{\omega} = 0.3 \text{ GeV. The } C^3\text{P}_0 \text{ branching fraction resulted in } B(KK^*/KK) = 0.13, \text{ in accordance with the experimental limit.}

\text{ After fixing these parameters, it was possible to estimate the decay widths for the unobserved mesons } \phi_3(1850), \phi_2(1850). \text{ The estimates are shown in table VI. Again four sets of parameters were considered in these calculations. The average total } C^3\text{P}_0 \text{ width's estimates are } \Gamma(\phi_3) = 0.595 \pm 0.058 \text{ MeV and } \Gamma(\phi_2) = 182.10 \pm 15.87 \text{ MeV.} 

IV. COMPARING \(^3\text{P}_0\) AND \(^3\text{P}_0\) MODELS

In our former study, in Ref. [20], decays in the light 1S and 1P sectors were analyzed and a comparison was made with the usual \(^3\text{P}_0\) results. Specifically, the decay processes: \(\rho \rightarrow \pi\pi, b_1 \rightarrow \omega\pi, a_1 \rightarrow \rho\pi, a_2 \rightarrow \rho\pi, h_1 \rightarrow \rho\pi, f_0 \rightarrow \pi\pi \) and \(f_2 \rightarrow \pi\pi\). One of the highlights of this study was the fact that all of these channels had
experimental data. The model was adjusted in order to minimize \( R \), defined by

\[
R^2 = \sum_{i=1}^{7} \left[ a_i(\gamma, \beta) - 1 \right]^2
\]

with \( a_i(\gamma, \beta) = \frac{\Gamma^{\text{th}}(\gamma, \beta)}{\Gamma_i^{\text{exp}}} \). The comparison of the \( ^3P_0 \) model with \( C^3P_0 \) implied in obtaining a minimum value for (56) as a function of the parameters \( \gamma \) and \( \beta \). It was shown that the inclusion of the correction terms reduced the \( R \) value. A clear demonstration that the bound-state correction globally improves the fit. The average difference between the predictions of \( ^3P_0 \) and \( C^3P_0 \), in each individual channel, ranged from 1% to 14%. The higher differences were in the channels with lighter mesons in the final state. In the heavier channels, the leading order \( ^3P_0 \) is dominant and the bound-state corrections represent an actual next to leading order correction.

In the present, we studied the strange \( S \) and \( D \) states where data from individual channels are, in general, still not known. In the best situation, only the total \( \Gamma \) has an experimental value. Again a comparison was made between the theoretical predictions for \( ^3P_0 \) and \( C^3P_0 \). For example, in the decay of \( \phi(1020) \rightarrow KK \), two fits were presented with a difference of 14% between \( ^3P_0 \) and \( C^3P_0 \). Again, the impact of the correction was larger in a channel with lighter mesons. A qualitative interpretation consists in observing that the tightly bound quark-antiquark pair of lower states are affected in a larger extent by the bound-state correction, when compared to their radial excitation. In the heavier channels, as seen in tables III to VI, this effect is clear and the discrepancy falls again to about 1%. Similar to the case of the heavier mesons of \( 1S \) and \( 1P \) sectors studied in [20].

V. CONCLUSIONS

In this paper, we have concentrated on the \( \phi \) mesons, which are the strange \( S \) and \( D \) states, predicted in the quark model, as probable \( ss \) resonances expected up to 2.2 GeV. The central body of this study was to employ the Fock-Tani formalism, a field-theoretic mapping technique in order to obtain an effective interaction for meson decay. We have outlined the essential aspects of the \( C^3P_0 \) model, the bound-state corrected \( ^3P_0 \) model, and how it is obtained from the Fock-Tani formalism, where the decay Hamiltonian \( H^{C^3P_0} \) was deducted.

This work is intended as a modest guide for future experimental studies of meson spectroscopy that may become possible with the advent of the new Hall D photoproduction facility GlueX at Jefferson Lab. The main goal of the GlueX experiment is to search for and study hybrid and exotic mesons. In this sense we have studied 6 \( ss \) states, in which 3 were unobserved, presenting interesting issues for future experimental studies involving the conventional quark model states. The predicted total widths for these new, rather narrow states, that have not been identified, are \( \Gamma(\phi(2050)) = 182.35 \pm 3.13 \) MeV, \( \Gamma(\phi(1) = 0.595 \pm 0.058 \) MeV and \( \Gamma(\phi(2) = 182.10 \pm 15.87 \) MeV.

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Appendix A: Physical nature of bound-state corrections

The bound-state corrections are an essential aspect of the Fock-Tani formalism, sometimes called orthogonality corrections because they are equivalent to the description of the continuum states by orthogonalized plane waves with the projection to bound-states subtracted [10]-[12]. Consider, for example, an ideal two meson state (which shall be represented by a round ket)

\[
|\alpha \beta\rangle = m^\dagger_\alpha m^\dagger_\beta |0\rangle .
\]

The norm of (A1) can be calculated using (8):

\[
(\rho \sigma |\alpha \beta\rangle = \delta_{\alpha \rho} \delta_{\beta \sigma} + \delta_{\alpha \sigma} \delta_{\beta \rho} .
\]

The same calculation can be done for the physical two meson state

\[
|\alpha \beta\rangle = M^\dagger_\alpha M^\dagger_\beta |0\rangle .
\]

Defining the norm function as \( N(\rho \sigma ; \alpha \beta) \equiv (\rho \sigma |\alpha \beta\rangle \) and using (5), one obtains

\[
N(\rho \sigma ; \alpha \beta) = \delta_{\alpha \rho} \delta_{\beta \sigma} - N_E(\rho \sigma ; \alpha \beta) + (\rho \leftrightarrow \sigma)
\]
where
\[ N_E(\rho; \alpha \beta) = \Phi^e_{\rho} \Phi^s_{\sigma} \Phi^\lambda_{\alpha} \Phi^\tau_{\beta}. \] (A5)

The presence of \( N_E \) in (A4) has its origin in the composite nature of the meson operator \( M_\alpha \) and implies that the two meson state is not normalized as in (A2). A correctly normalized state would be written as
\[ \overline{\alpha \beta} = N^{-\frac{1}{2}}(\alpha \beta; \alpha' \beta') |\alpha' \beta'\rangle. \] (A6)

Now, consider the following state
\[ |\mu \nu \alpha \rangle = q^{\mu}_{\beta} \bar{q}^{\nu}_{\alpha} M_{\alpha} |0\rangle, \] (A7)
which by a similar procedure can be normalized, resulting in
\[ \overline{\mu \nu \alpha} = N^{-\frac{1}{2}}(\mu \nu \alpha; \mu' \nu' \alpha') |\mu' \nu' \alpha'\rangle, \] (A8)
where
\[ N_{\eta}(\mu \nu \alpha; \mu' \nu' \alpha') = \delta_{\mu \mu'} \Phi^{\xi_{\mu'}}_{\alpha} \Phi^{\xi_{\nu'}}_{\alpha} + \delta_{\nu \nu'} \Phi^{\xi_{\mu'}}_{\alpha} \Phi^{\xi_{\nu'}}_{\alpha} \]
\[ - \Phi^{\xi_{\mu}}_{\alpha} \Phi^{\xi_{\nu}}_{\alpha}. \] (A9)

A decay in which \( A \rightarrow BC \) is described by an amplitude obtained evaluating the following matrix element \( \langle BC|V|A \rangle \). In second quantization this is written as
\[ \langle BC|V|A \rangle = \langle 0 | M_\alpha M_\beta V_{\mu \nu} q^{\mu}_{\alpha} \bar{q}^{\nu}_{\alpha} M_{\alpha} |0\rangle = V_{\mu \nu} \langle \alpha \beta | \mu \nu \gamma \rangle. \] (A11)

According to what was shown, the state vectors in the decay amplitude (A11) should be replaced by a normalized version
\[ \langle BC|V|A \rangle = V_{\mu \nu} \langle \alpha \beta | \mu \nu \gamma \rangle \]
\[ = V_{\mu \nu} N^{-\frac{1}{2}}(\alpha \beta; \beta' \alpha') \langle \alpha' \beta'| \mu' \nu' \gamma' \rangle \]
\[ \times N^{-\frac{1}{2}}(\mu' \nu' \gamma'; \mu \nu \gamma). \] (A12)

Defining a potential norm function as
\[ N_V(\mu \nu \alpha' \beta'; \mu' \nu' \gamma') \equiv V_{\mu \nu} \langle \alpha' \beta'| \mu' \nu' \gamma' \rangle \]
\[ = V_1 - V_3 \] (A13)
where \( V_1 \) with one or three wave functions, is given by
\[ V_1 = \Phi^{\xi_{\mu'}}_{\alpha} \delta_{\nu \gamma} V_{\mu \nu} \]
\[ V_3 = \Phi^{\xi_{\mu}}_{\alpha} \Phi^{\xi_{\nu}}_{\alpha} \Phi^{\xi_{\gamma}}_{\alpha} V_{\mu \nu} + (\alpha' \leftrightarrow \beta'). \] (A14)

The complete evaluation of the normalized decay amplitude is reduced to the expansion of (A12):
\[ \langle BC|V|A \rangle = N^{-\frac{1}{2}}(\alpha \beta; \beta' \alpha') N_V(\mu \nu \alpha' \beta'; \mu' \nu' \gamma') \]
\[ \times N^{-\frac{1}{2}}(\mu' \nu' \gamma'; \mu \nu \gamma). \] (A15)

For example, in the lowest order of the expansion
\[ N^{-\frac{1}{2}} \approx 1 + \frac{1}{2} N_E; \quad N_{\eta}^{-\frac{1}{2}} \approx 1 + \frac{1}{2} N_{\eta} \] (A16)

one obtains
\[ \langle BC|V|A \rangle \approx -\Phi^{\xi_{\mu}}_{\rho} \Phi^{\xi_{\nu}}_{\sigma} \Phi^{\xi_{\gamma}}_{\tau} V(\rho \xi_{\lambda}; \omega \sigma) \]
\[ + (\alpha \leftrightarrow \beta'), \] (A17)
where
\[ V(\rho \xi_{\lambda}; \omega \sigma) = (\delta + \Delta_f) V_{\mu \nu}, \] (A18)
with
\[ \Delta_f = f \left[ \delta_{\sigma \xi} \delta_{\lambda \mu} \Delta(\rho \tau; \omega \nu) + \delta_{\xi \nu} \delta_{\lambda \omega} \Delta(\rho \tau; \mu \sigma) \right] \]
\[ - 2\delta_{\sigma \xi} \delta_{\lambda \omega} \Delta(\rho \tau; \mu \nu). \] (A19)

In (A19), \( f \) is combinatorial factor with the value \( f = 1/4 \) related to the truncation of (A16) in the lowest order. When higher orders are considered new contributions change this factor to the Fock-Tani value \( f = 1/4 \) of (40).

In summary, the essence of the Fock-Tani formalism is to move the bound-state information from the state vector (1), written in second quantization, into the interaction (38). As shown in the First and Second examples, this action has different impacts in the physical system. The new state vector is now the ideal state vector (7) which satisfies canonical commutation relations (8).

If one chooses not to use the Fock-Tani formalism the decay amplitude can be evaluated directly by calculating the \( \langle BC|V|A \rangle \) matrix element. As a first approximation, neglecting the meson's composite structure, the \( ^3P_0 \) gives a correct leading order contribution to \( \langle BC|V|A \rangle \). To go beyond this lowest order, one needs to calculate \( \langle BC|V|A \rangle \) and expand the kernels \( N, N_{\eta} \) and \( N_V \) in (A15), introducing the overlap effects due to the extended nature of the meson, which constitute the bound-state corrections.

**Appendix B: Wave functions**

The Pauli matrices are given by
\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (B1)
and the corresponding spinors are
\[ \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \chi^c_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \chi^c_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \] (B2)

The meson flavor components listed in the decay channels (a)-(f) depend on the isospin \( I \) and strangeness \( s \)

1. \( I = 0 \)
\[ \phi, \eta, \eta', h_1 \rightarrow \frac{c_1}{\sqrt{2}} (|u\bar{u}| + |d\bar{d}|) + c_2 |s\bar{s}| \] (B3)
\[ K^+ \rightarrow -|u \bar{s}\rangle ; \quad I_z = +1/2 \]
\[ K^0 \rightarrow -|d \bar{s}\rangle ; \quad I_z = -1/2 \]  
(B4)

3. \( I = 1/2, s = -1 \)
\[ \bar{K}^0 \rightarrow -|s \bar{d}\rangle ; \quad I_z = +1/2 \]
\[ K^- \rightarrow |s \bar{u}\rangle ; \quad I_z = -1/2 . \]  
(B5)

The SU(3) mixing coefficients \( c_1 \) and \( c_2 \), in (B3), for the pseudo-scalar mesons \( \eta \) and \( \eta' \) are related through the angle \( \theta_p \)
\[ |\eta\rangle = \frac{c_1(\theta_p)}{\sqrt{2}} ([u \bar{u}] + |d \bar{d}\rangle] - c_2(\theta_p)|s \bar{s}\rangle \]
\[ |\eta'\rangle = \frac{c_2(\theta_p)}{\sqrt{2}} ([u \bar{u}] + |d \bar{d}\rangle] + c_1(\theta_p)|s \bar{s}\rangle . \]  
(B6)

Similarly, the coefficients of the vector mesons \( \phi(1020) \) and \( \omega(782) \) are related through angle \( \theta_v \).
\[ |\phi\rangle = \frac{c_1(\theta_v)}{\sqrt{2}} ([u \bar{u}] + |d \bar{d}\rangle] - c_2(\theta_v)|s \bar{s}\rangle \]
\[ |\omega\rangle = \frac{c_2(\theta_v)}{\sqrt{2}} ([u \bar{u}] + |d \bar{d}\rangle] + c_1(\theta_v)|s \bar{s}\rangle . \]  
(B7)

where
\[ c_1(\theta_i) = \frac{\cos \theta_i}{\sqrt{3}} - \sqrt{\frac{2}{3}} \sin \theta_i \]
\[ c_2(\theta_i) = \sqrt{\frac{2}{3}} \cos \theta_i + \frac{\sin \theta_i}{\sqrt{3}} . \]  
(B8)

Equations (B4)-(B5) are also valid mesons to \( K_1, K^*, K_1' \) and \( K_2' \). In particular, \( K_1(1270) \) and \( K_1(1400) \) are related by mixing angle \( \theta \)
\[ |K_1(1270)\rangle = + \cos \theta \ |1^1 P_3\rangle + \sin \theta \ |1^3 P_3\rangle \]
\[ |K_1(1400)\rangle = - \sin \theta \ |1^1 P_3\rangle + \cos \theta \ |1^3 P_3\rangle . \]  
(B9)

For antikaons there is a change in the sign
\[ |\bar{K}_1(1270)\rangle = - \cos \theta \ |1^1 P_3\rangle + \sin \theta \ |1^3 P_3\rangle \]
\[ |\bar{K}_1(1400)\rangle = + \sin \theta \ |1^1 P_3\rangle + \cos \theta \ |1^3 P_3\rangle . \]  
(B10)

The color wave function is the same for all mesons, is given by
\[ C^{\mu \nu} = \frac{1}{\sqrt{3}} \delta^{\mu \nu} ; \quad c_k = 1, 2, 3 . \]  
(B11)

The spin wave functions can be singlet or triplet:

- Singlet \( (S = 0) \)
\[ \frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle) ; \quad S_z = 0 \]  
(B12)

- Triplet \( (S = 1) \)
\[ \frac{1}{\sqrt{2}} (| \uparrow \uparrow \rangle + | \downarrow \downarrow \rangle) ; \quad S_z = +1 \]
\[ | \uparrow \downarrow \rangle ; \quad S_z = 0 \]
\[ | \downarrow \uparrow \rangle ; \quad S_z = -1 . \]  
(B13)

The spatial wave functions are harmonic oscillator functions, as they describe color confinement and provide analytical amplitudes
\[ \Phi_{nl}(\vec{p}_\mu - \vec{p}_\nu) = \delta(\vec{P}_\mu - \vec{p}_\mu - \vec{p}_\nu) \phi_{nl}(\vec{p}_\mu - \vec{p}_\nu) , \]  
(B14)

where \( \phi_{nl}(\vec{p}_\mu - \vec{p}_\nu) \) is given by
\[ \phi_{nl}(\vec{p}_\mu - \vec{p}_\nu) = \left( \frac{1}{2 \beta} \right)^l \frac{n_l!}{n_0! \Gamma(n + l + 1/2)} \frac{1}{(n_0 - n + l + 1/2) \beta^l} \right)^2 Y_{lm}(\Omega_{\vec{p}_\mu - \vec{p}_\nu}) . \]  
(B15)

where \( p_{\mu \nu} \) is the internal momentum, \( Y_{lm} \) spherical harmonic and \( \beta \) the Gaussian width parameter. The momentum wave function \( \phi(\vec{p}_\mu - \vec{p}_\nu) \), the normalization constant \( N_{nl} \) and the Laguerre polynomials \( L_{n-1}^{l+1/2} (p) \), that depend on the radial \( n \) and orbital \( l \) quantum numbers, are all defined as
\[ N_{nl} = \left[ \frac{2 (n - 1)!}{\beta^l \Gamma(n + l + 1/2)} \right]^{1/2} \]
\[ L_{n-1}^{l+1/2} (p) = \sum_{k=0}^n \frac{(-k)^l \Gamma(n + l + 1/2)}{k! (n - k - 1)! \Gamma(k + l + 3/2)} \rho^k , \]  
(B16)

where \( n = 1, 2, \ldots \) and \( l = 0, 1, \ldots \)

Appendix C: Amplitudes

In this appendix, we present the results of the algebraic decay amplitudes of \( h_{fi} \), for subprocesses of (a)-(f), obtained with the \( C^3 P_0 \) model. Defining
where $\beta_A$ is the width of the Gaussian initial state of the meson, $\beta_B$ and $\beta_C$ of the final state mesons and $\beta$ for the bound-state correction mesons, one has

$$M_{f_i}^{\phi(1020)\rightarrow KK} = C_{10}^{\phi(1020)} Y_{11} (\Omega_p) \quad (C2)$$

$$M_{f_i}^{\phi(1680)\rightarrow KK} = C_{10}^{\phi(1680)} K Y_{11} (\Omega_p) \quad (C3)$$

$$M_{f_i}^{\phi(1680)\rightarrow KK^*} = C_{11}^{\phi(1680)} K K^* Y_{10} (\Omega_p) \quad (C4)$$

$$M_{f_i}^{\phi(2050)\rightarrow \eta \phi} = C_{11}^{\phi(2050)} Y_{10} (\Omega_p) \quad (C5)$$

$$M_{f_i}^{\phi(2050)\rightarrow KK} = C_{10}^{\phi(2050)} K K Y_{11} (\Omega_p) \quad (C6)$$

$$M_{f_i}^{\phi(2050)\rightarrow KK^*} = C_{11}^{\phi(2050)} K K^* Y_{10} (\Omega_p) \quad (C7)$$

$$M_{f_i}^{\phi(2050)\rightarrow K^* K^*} = C_{12}^{\phi(2050)} K^* K^* Y_{1-1} (\Omega_p) \quad (C8)$$

$$M_{f_i}^{\phi(2050)\rightarrow K K_1(1270)} = C_{10}^{\phi(2050)} K K_1(1270) Y_{00} (\Omega_p) + C_{21}^{\phi(2050)} K K_1(1270) Y_{20} (\Omega_p) \quad (C9)$$

$$M_{f_i}^{\phi(2050)\rightarrow K K_1(1400)} = C_{01}^{\phi(2050)} K K_1(1400) Y_{00} (\Omega_p) + C_{21}^{\phi(2050)} K K_1(1400) Y_{20} (\Omega_p) \quad (C10)$$

$$M_{f_i}^{\phi(2050)\rightarrow K K_1^*(1430)} = C_{20}^{\phi(2050)} K K_1^*(1430) Y_{21} (\Omega_p) \quad (C11)$$

$$M_{f_i}^{\phi(2050)\rightarrow K K_2^*(1430)} = C_{22}^{\phi(2050)} K K_2^*(1430) Y_{2-1} (\Omega_p) \quad (C12)$$

$$M_{f_i}^{\phi(2050)\rightarrow K K^*(1410)} = C_{11}^{\phi(2050)} K K^*(1410) Y_{10} (\Omega_p) \quad (C13)$$

$$M_{f_i}^{\phi(2050)\rightarrow K K(1460)} = C_{10}^{\phi(2050)} K K(1460) Y_{11} (\Omega_p) \quad (C14)$$

$$e_1(p, \beta_A, \beta_B, \beta_C) = e^{-\frac{(s_0^2 + s_1^2)}{s^2 (s_0^2 + s_1^2)}}$$

$$e_2(p, \beta_A, \beta_B, \beta_C, \beta) = e^{-\frac{(2s_0^2 + 2s_1^2 + s_2^2)}{s^2 (s_0^2 + s_1^2 + s_2^2)}}$$

$$\phi(1020) \rightarrow K^+ K^- :$$

$$M_{f_i}^{\phi(2050)\rightarrow \eta \phi} = C_{11}^{\phi(2050)} \eta\phi Y_{10} (\Omega_p) \quad (C15)$$

$$M_{f_i}^{\phi(2050)\rightarrow \eta' \phi} = C_{12}^{\phi(2050)} \eta'\phi Y_{10} (\Omega_p) \quad (C16)$$

Where $C_{LS}$ are given by
\[ C_{10}^{(φ^{(1620)})} = (c_1^φ - c_2^φ) \left\{ f_1(p_{KK}, β, β, K) e_1(p_{KK}, β, φ, β, K, β) \\
- c_1^φ c_2^φ f_2(p_{KK}, β, φ, β, K, β, η) e_2(p_{KK}, β, φ, β, K, β, η) \\
- c_1^φ c_2^φ f_2(p_{KK}, β, φ, β, K, β, η) e_2(p_{KK}, β, φ, β, K, β, η) \right\} \] (C28)

\[ f_1(p, β_A, β_B) = \frac{8 p β_A^{3/2} (β_B^2 + β_A^2)}{\sqrt{3} (β_B^2 + 2β_A^2)^{5/2}} \] (C29)

\[ f_2(p, β_A, β_B, β) = \frac{16 p β_A^{3/2} β_B^3 (β_B^2 + β_A^2) (β_B^2 + β^2)}{3\sqrt{3} (β_B^2 + 2β_A^2 β_A^2 + 2β_B^2 (β^2 + β_A^2))^{5/2}} \] (C30)

\[ \phi(1680) \rightarrow K^+K^- : \]

\[ C_{10}^{(φ^{(1680)})_{KK}} = (c_1^φ^{1680} - c_2^φ^{1680}) \left\{ f_3(p_{KK}, β, β, K) e_1(p_{KK}, β, φ, β, K, β) \\
+ c_1^φ c_2^φ f_4(p_{KK}, β, φ, β, K, β) e_2(p_{KK}, β, φ, β, K, β, η) \\
+ c_1^φ c_2^φ f_4(p_{KK}, β, φ, β, K, β, η) e_2(p_{KK}, β, φ, β, K, β, η) \right\} \] (C31)

\[ f_3(p, β_A, β_B) = \frac{4\sqrt{2} p β_A^{3/2} \left[ -3β_B^2 + β_A^2 + 20β_B^2 β_A^4 + 12β_A^6 - 2β_A^2 (β_B^2 + β_A^2) p^2 \right]}{3(β_B^2 + 2β_A^2)^{9/2}} \] (C32)

\[ f_4(p, β_A, β_B, β) = \frac{8\sqrt{2} p β_A^{3/2} β_B^2 (β_B^2 + β^2)}{9 (β_B^2 + 2β_A^2 β_A^2 + 2β_B^2 (β^2 + β_A^2))^{9/2}} \left[ (3β_B^2 (β_B^2 + 2β^2) - 7β_A^2 β_B^2 - 6 (β_B^2 + β^2) β_A^4) \right] \] (C33)

\[ \phi(1680) \rightarrow K^+K^{*-} : \]

\[ C_{11}^{(φ^{(1680)})_{KK^*}} = (c_1^φ^{1680} + c_2^φ^{1680}) \left\{ f_5(p_{KK^*}, β, β, K, β_K) e_1(p_{KK^*}, β, φ, β, K, β_K) \\
+ c_1^φ c_2^φ f_6(p_{KK^*}, β, φ, β, K, β_K, η) e_2(p_{KK^*}, β, φ, β, K, β_K, η) \\
+ c_1^φ c_2^φ f_6(p_{KK^*}, β, φ, β, K, β_K, η) e_2(p_{KK^*}, β, φ, β, K, β_K, η) \right\} \] (C34)

\[ f_5(p, β_A, β_B, β_C) = -\frac{\sqrt{2} p β_B^{3/2} β_C^{3/2} β_A^{3/2}}{3 (β_B^2 β_C^2 + β_C^2 β_A^2 + β_A^2 β_C^2)^{9/2}} \left[ 12β_B^6 β_C^2 - 2β_B^4 β_C^4 (β_B^2 + β_C^2) β_A^2 - 20β_B^2 β_C^2 β_A^4 \right] \] (C35)

\[ f_6(p, β_A, β_B, β_C, β) = -\frac{2\sqrt{2} p β_B^{3/2} β_C^{3/2} β_A^{3/2}}{9 (β_B^2 β_C^2 + β_C^2 β_A^2 + β_A^2 β_C^2 + 2 β_A^2 β_B^2 β_C^2 + β_B^2 β_C^2 + β_C^2 β_B^2)^{9/2}} \left[ 2 (β_B^2 β_A^2 + β_B^2 β_C^2 + β_A^2 β_C^2 + 2 β_B^2 β_A^2 β_C^2) β_C^2 \right] \] (C36)

\[ \phi(1680) \rightarrowηφ : \]
\[
\phi(2050) \rightarrow K^+K^- :
\]

\[
C_{10}^{\phi(2050)KK} = (c_1^{\phi_{2050}} - c_2^{\phi_{2050}}) \left\{ f_7(pKK, \beta_{\phi_{2050}}, \beta_K) e_1(pKK, \beta_{\phi_{2050}}, \beta_K, \beta_K) + c_1^{*\Delta} c_2^{\Delta} f_8(pKK, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}, \beta_K, \beta_K) e_2(pKK, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}) \right\}
\]

\[
f_7(p, \beta_A, \beta_B) = \frac{2\sqrt{2}p_{\beta_A}^{3/2}}{3\sqrt{5}(\beta_B^2 + 2\beta_A^2)^{13/2}} \left[ 5(\beta_B^2 + 2\beta_A^2)^2 (3\beta_B^4 - 17\beta_B^2\beta_A^4 + 16\beta_B^2\beta_A^2 + 2\beta_B^4) - 4\beta_A^4 (\beta_B^2 + 2\beta_A^2) \times (-5\beta_B^2 + 9\beta_B^2\beta_A^2 + 10\beta_A^4) p^2 + 4\beta_A^4 (\beta_B^2 + 2\beta_A^2) p^4 \right]
\]

\[
f_8(p, \beta_A, \beta_B, \beta) = \frac{4\sqrt{2} p_{\beta_B}^{3/2} p_{\beta_A}^{3/2}}{9\sqrt{5}(\beta_B^2 + 2\beta_A^2 + 2\beta_B^2 (\beta_A^2 + \beta_B^2))^{13/2}} \left[ 5(\beta_B^2 + 2\beta_A^2)^2 - 2\beta_A^4 (\beta_B^2 + 2\beta_A^2) \times (-11\beta_B^2\beta_A^2 - 6\beta_B^2 + 2\beta_A^2) \beta_A^4 + 2\beta_B^2 + 2\beta_A^2 (\beta_B^2 + 2\beta_A^2) p^2 + 4\beta_A^2 (\beta_B^2 + 2\beta_A^2) p^4 \right]
\]

\[
\phi(2050) \rightarrow K^+K^- :
\]

\[
C_{11}^{\phi(2050)KK^-} = (c_1^{\phi_{2050}} + c_2^{\phi_{2050}}) \left\{ f_9(pKK^*, \beta_{\phi_{2050}}, \beta_K) e_1(pKK^*, \beta_{\phi_{2050}}, \beta_K, \beta_K^*) + c_1^{\Delta} c_2^{\Delta} f_{10}(pKK^*, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}) e_2(pKK^*, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}, \beta_K, \beta_{\phi_{2050}}) \right\}
\]

\[
f_9(p, \beta_A, \beta_B, \beta_C) = \frac{p_{\beta_B}^{3/2} p_{\beta_A}^{3/2} p_{\beta_C}^{3/2}}{6\sqrt{10}(\beta_B^2 + 2\beta_A^2 (\beta_B^2 + \beta_A^2))^{13/2}} \left[ 20 (6\beta_B^10\beta_{C}^{10} - 5\beta_B^2\beta_{C}^8 (\beta_B^2 + \beta_A^2)^2 - 20\beta_B^2\beta_{C}^{10} \times (\beta_B^2 + \beta_A^2)^2 (\beta_B^2 + \beta_A^2)^2 (\beta_B^2 + \beta_A^2)^3 (\beta_B^2 + \beta_A^2)^4 (\beta_B^2 + \beta_A^2)^6 + 3 (\beta_B^2 + \beta_A^2)^5 (\beta_B^2 + \beta_A^2)^5 + 5 \times (\beta_B^2 + \beta_A^2)^3 (\beta_B^2 + \beta_A^2)^4 (\beta_B^2 + \beta_A^2)^6 (\beta_B^2 + \beta_A^2)^8 \right]
\]

\[
f_{10}(p, \beta_A, \beta_B, \beta_C, \beta) = \frac{p_{\beta_B}^{3/2} p_{\beta_A}^{3/2} p_{\beta_C}^{3/2}}{9\sqrt{10}(\beta_B^2 + 2\beta_A^2 (\beta_B^2 + \beta_A^2) + 2 (\beta_B^2 + \beta_A^2)^2)^{13/2}} \left[ 20 (-\beta_B^2\beta_A^2 + \beta_B^2 (\beta_B^2 + \beta_A^2)^2 + 2 (\beta_B^2 - \beta_A^2) \beta_B^2 (\beta_B^2 + \beta_A^2)^2 \times (\beta_B^2 - \beta_A^2) - \beta_B^2 (11\beta_B^2\beta_A^2 + 6\beta_B^2\beta_A^2 + 2\beta_B^2 - 9\beta_B^2 + 11\beta_B^2\beta_A^2 + 6\beta_A^2) - 3 (4\beta_B^2\beta_A^2 + 4\beta_B^2\beta_A^2 (\beta_B^2 - \beta_A^2)) + 4\beta_B^2\beta_A^2 (\beta_B^2 + \beta_A^2)^2 (\beta_B^2 + \beta_A^2) + 2 (\beta_B^2 + \beta_A^2) \times (\beta_B^2 + \beta_A^2) \beta_B^2 (10\beta_B^2 + 9\beta_B^2\beta_A^2 - 5\beta_B^2 - 20\beta_B^2\beta_B^2 + \beta_B^2 (9\beta_B^2 - 30\beta_B^2) + 2\beta_B^2\beta_A^2 (5\beta_B^2 + 9\beta_B^2) - 5 (4\beta_B^2\beta_A^2 + 4\beta_B^2\beta_A^2 (\beta_B^2 - \beta_A^2)) p^2 + \beta_B^2 \times (\beta_B^2 + \beta_A^2 + 2\beta_B^2)^4 (\beta_B^2\beta_A^2 + \beta_B^2 (2\beta_B^2 + \beta_A^2)^2 + 2 (\beta_B^2 + \beta_A^2) (\beta_B^2 + \beta_A^2)^2) \right]
\]
\( \phi(2050) \to K^+K^- : \)

\[
C^{\phi(2050)K^+K^-}_{12} = 2 \left( c_1^{\phi2050} - c_2^{\phi2050} \right) \left\{ -f_7(pK^+K^+, \beta_{\phi2050}, \beta_{K^+}, \beta_{K^-}) e_1(pK^+K^+, \beta_{\phi2050}, \beta_{K^+}, \beta_{K^-}) \\
+2c_1^{\phi^2} c_2^{\phi} f_8(pK^+K^+, \beta_{\phi2050}, \beta_{K^+}, \beta_{K^-}) e_2(pK^+K^+, \beta_{\phi2050}, \beta_{K^+}, \beta_{K^-}) \\
+2c_1^{\phi} c_2^{\phi^2} f_8(pK^+K^+, \beta_{\phi2050}, \beta_{K^+}, \beta_{K^-}) e_2(pK^+K^+, \beta_{\phi2050}, \beta_{K^+}, \beta_{K^-}) \right\}
\]

(C44)

\( \phi(2050) \to K^K_1(1270) : \)

\[
C^{\phi(2050)K_1^K(1270)}_{01} = -\left( c_1^{\phi2050} + c_2^{\phi2050} \right) \left\{ f_11(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) e_1(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) \\
+c_1^{\eta_1} c_2^{\eta_2} f_12(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) \\
+c_1^{\eta_2} c_2^{\eta_1} f_12(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) \right\} \cos \theta \\
+\left( c_1^{\phi2050} - c_2^{\phi2050} \right) \left\{ 2 f_11(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) e_1(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) \\
-c_1^{\phi} c_2^{\phi} f_12(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) \\
-c_1^{\phi} c_2^{\phi} f_12(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) \right\} \sin \theta
\]

(C45)

\[
C^{\phi(2050)K_{12}(1270)}_{21} = -\left( c_1^{\phi2050} + c_2^{\phi2050} \right) \left\{ f_{13}(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) e_1(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) \\
+c_1^{\eta_1} c_2^{\eta_2} f_{14}(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) \\
+c_1^{\eta_2} c_2^{\eta_1} f_{14}(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\eta_1}) \right\} \cos \theta \\
-\left( c_1^{\phi2050} - c_2^{\phi2050} \right) \left\{ f_{13}(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) e_1(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}) \\
-2c_1^{\phi} c_2^{\phi} f_{14}(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) \\
-2c_1^{\phi} c_2^{\phi} f_{14}(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) e_2(pKK_1, \beta_{\phi2050}, \beta_{K}, \beta_{K_1}, \beta_{\phi}) \right\} \sin \theta
\]

(C46)

\[
f_{11}(p, \beta_A, \beta_B, \beta_C) = \frac{\beta_B^{7/2} \beta_C^{7/2} \beta_A^{3/2}}{12 \sqrt{15} (\beta_B^{2} \beta_C^{2} + \beta_B^{2} + \beta_C^{2})^{15/2}} \left[ 240 \beta_A^{3} (11 \beta_B^{4} \beta_C^{2} - 14 \beta_B^{2} \beta_C^{2} (\beta_B^{2} + \beta_B^{2}) \beta_A^{2})^{2} + 3 (\beta_B^{2} + \beta_B^{2})^{2} (\beta_A^{2} + \beta_B^{2} (\beta_B^{2} + \beta_B^{2}))^{3} + 20 (6 \beta_B^{6} \beta_C^{6} - 3 \beta_B^{4} \beta_C^{2} (\beta_B^{4} + \beta_B^{4}) \beta_A^{2} + 44 \beta_B^{2} \beta_C^{2} (\beta_B^{2} + \beta_B^{2})^{2} (\beta_B^{2} + \beta_B^{2})^{2} + 6 (\beta_B^{2} + \beta_B^{2})^{3} \beta_A^{2}) - 23 \beta_B^{4} \beta_C^{2} (\beta_B^{4} + \beta_B^{4}) \beta_A^{2} \right] 
\]

(C47)
\[
f_{12}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{\beta^2 \beta^2 \beta^2}{18 \sqrt{15} (\beta^2 + \beta^2 + \beta^2)} \left[ 240 \beta^2 \beta^2 \beta^2 (\beta^2 + \beta^2) + 13 (\beta^2 + \beta^2 + \beta^2) + 20 (\beta^2 + \beta^2 + \beta^2) + (\beta^2 + \beta^2 + \beta^2) \right]^{15/2} \text{ (C48)}
\]

\[
f_{13}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{\beta^2 \beta^2 \beta^2}{60 \sqrt{3} (\beta^2 + \beta^2 + \beta^2 + \beta^2)} \left[ 120 \beta^0 \beta^0 \beta^0 (\beta^2 + \beta^2 + \beta^2) \beta^2 - 752 \beta^0 \beta^0 \beta^0 \right] \text{ (C49)}
\]

\[
f_{14}(p, \beta_A, \beta_B, \beta_C, \beta) = \frac{\beta^2 \beta^2 \beta^2}{90 \sqrt{3} (\beta^2 + \beta^2 + \beta^2) + \beta^2 + \beta^2 + \beta^2 + \beta^2} \left[ 4 \beta^0 (\beta^2 + \beta^2) (\beta^2 + \beta^2) + 5 (\beta^2 + \beta^2 + \beta^2 + \beta^2) + 2 (\beta^2 + \beta^2 + \beta^2 + \beta^2) \right]^{15/2} \text{ (C50)}
\]

\[
\phi(2050) \rightarrow K^+ K^- (1400) : 
\]

\[
C_{01}^{(2050)K K (1400)} = (c_1^{(2050)} + c_2^{(2050)}) \left\{ f_{11}(p KK, \beta_A, \beta_B, \beta_C, \beta) e_1(p KK, \beta_A, \beta_B, \beta_C, \beta) + c_2^{(2050)} f_{12}(p KK, \beta_A, \beta_B, \beta_C, \beta) e_2(p KK, \beta_A, \beta_B, \beta_C, \beta) \right\} \sin \theta
\]

\[
+ \frac{(c_1^{(2050)} - c_2^{(2050)})}{\sqrt{2}} \left\{ 2 f_{11}(p KK, \beta_A, \beta_B, \beta_C, \beta) e_1(p KK, \beta_A, \beta_B, \beta_C, \beta) - c_2^{(2050)} f_{12}(p KK, \beta_A, \beta_B, \beta_C, \beta) e_2(p KK, \beta_A, \beta_B, \beta_C, \beta) \right\} \cos \theta \text{ (C51)}
\]
\[ C_{21}^{(2050)K K_1(1400)} = (c_1^{2050} + c_2^{2050}) \left\{ f_{13}(p K K_1, \beta_\phi 2050, \beta_k, \beta_k) e_1(p K K_1, \beta_\phi 2050, \beta_k, \beta_k) \\
+ c_1^{\phi 2050} c_2^{\phi 2050} f_{14}(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) e_2(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) \\
+ c_1^{\eta \Delta} c_2^{\eta \Delta} f_{14}(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) e_2(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) \right\} \sin \theta \\
- \left( \frac{c_1^{2050} - c_2^{2050}}{\sqrt{2}} \right) \left\{ f_{13}(p K K_1, \beta_\phi 2050, \beta_k, \beta_k) e_1(p K K_1, \beta_\phi 2050, \beta_k, \beta_k) \\
- 2c_1^{\phi 2050} c_2^{\phi 2050} f_{14}(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) e_2(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) \\
- 2c_1^{\eta \Delta} c_2^{\eta \Delta} f_{14}(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) e_2(p K K_1, \beta_\phi 2050, \beta_k, \beta_k, \beta_\eta \Delta) \right\} \cos \theta \right) \] (C52)

\[ \phi(2050) \rightarrow K^+ K^{*-} (1430) : \]

\[ C_{20}^{(2050)K K^*_1(1430)} = (c_1^{2050} - c_2^{2050}) \left\{ f_{14}(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*, \beta_\phi \Delta) e_2(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*, \beta_\phi \Delta) \\
+ c_1^{\phi 2050} c_2^{\phi 2050} f_{14}(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*, \beta_\phi \Delta) e_2(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*, \beta_\phi \Delta) \right\} \] (C53)

\[ \phi(2050) \rightarrow K^+ K^{*-} (1430) : \]

\[ C_{22}^{(2050)K K^*(1430)} = \sqrt{3} \left( c_1^{2050} - c_2^{2050} \right) \left\{ f_{13}(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*) e_1(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*) \\
+ c_1^{\phi 2050} c_2^{\phi 2050} f_{14}(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*, \beta_\phi \Delta) e_2(p K K^*_1, \beta_\phi 2050, \beta_k, \beta_K^*, \beta_\phi \Delta) \right\} \] (C54)

\[ \phi(2050) \rightarrow K^+ K^{*-} (1410) : \]

\[ C_{11}^{(2050)K K^*(1410)} = (c_1^{2050} + c_2^{2050}) \left\{ f_{15}(p K K^*, \beta_\phi 2050, \beta_k, \beta_k) e_1(p K K^*, \beta_\phi 2050, \beta_k, \beta_k) \\
+ c_1^{\phi 2050} c_2^{\phi 2050} f_{16}(p K K^*, \beta_\phi 2050, \beta_k, \beta_k, \beta_\phi \Delta) e_2(p K K^*, \beta_\phi 2050, \beta_k, \beta_k, \beta_\phi \Delta) \right\} \] (C55)

\[ f_{15}(p, \beta_A, \beta_B, \beta_C) = \frac{\beta_A^3/2 \beta_B^3/2 \beta_C^3/2}{24 \sqrt{15} (\beta_B^2 \beta_C^2 + \beta_A^2 (\beta_B^2 + \beta_C^2))^{17/2}} \left[ 40 (\beta_B^2 \beta_C^2 + \beta_A^2 (\beta_B^2 + \beta_C^2))^3 (18 \beta_B \beta_C^3 + \beta_A \beta_B^3 \beta_C^4 \\
\times (275 \beta_B^2 - 27 \beta_C^2) (\beta_B^2 + \beta_C^2) - 11 \beta_A^2 \beta_B^2 \beta_C^2 (\beta_B^2 - 3 \beta_C^2) (\beta_B^2 + \beta_C^2)^2 - 9 \beta_A^2 (\beta_B^2 - \beta_C^2) \\
\times (\beta_B^2 + \beta_C^2)^3 - 11 \beta_A^2 \beta_B^2 \beta_C^2 (19 \beta_B^2 + 3 \beta_C^2)) - 4 (\beta_B^2 \beta_C^2 + \beta_A^2 (\beta_B^2 + \beta_C^2))^2 (30 \beta_B^2 \beta_C^2 \\
+ 2 \beta_A^2 \beta_B^2 \beta_C^2 (265 \beta_B^2 - 3 \beta_C^2) (\beta_B^2 + \beta_C^2)^2 + 30 \beta_A^2 (\beta_B^2 + \beta_C^2) (\beta_B^2 + \beta_C^2)^3 - 5 \beta_A^2 \beta_B^2 \beta_C^2 \beta_D^2 \\
\times (\beta_B^2 + \beta_C^2) (61 \beta_B^2 + 12 \beta_C^2) + \beta_A^2 \beta_B^2 \beta_C^2 (\beta_B^2 + \beta_C^2)^3 (79 \beta_B^2 + 84 \beta_C^2)) p^2 - 2 \beta_A^2 (\beta_B^2 + \beta_C^2)^2 \\
\times (20 \beta_B^2 \beta_C^2 + 3 \beta_A^2 (\beta_B^2 - \beta_C^2) (\beta_B^2 + \beta_C^2)^4 - 15 \beta_A^2 \beta_B^2 \beta_C^2 (\beta_B^2 + \beta_C^2)^2 (5 \beta_B^2 + \beta_C^2) - 4 \beta_A^2 \beta_B^2 \beta_C^2 \\
\times (\beta_B^2 + \beta_C^2)^3 (4 \beta_B^2 + 3 \beta_C^2) - 6 \beta_A^2 \beta_B^2 \beta_C^2 (6 \beta_B^2 + 7 \beta_B^2 \beta_C^2 + \beta_C^2))^p - \beta_A^2 \beta_B^2 \beta_C^2 (\beta_B^2 + \beta_C^2)^4 \\
\times (2 \beta_B^2 \beta_C^2 + \beta_A^2 (\beta_B^2 + \beta_C^2))^p \right] \] (C56)
\[
\phi(2050) \rightarrow K^+K^- (1460): \quad C_{10}^{(2050)K\bar{K}}(1460) = - (c_1^{\phi2050} - c_2^{\phi2050}) \left\{ \frac{2}{c_1^{\phi2050}} f_15(pK\bar{K}_{1460}, \beta_{\phi2050}, \beta_K, \beta_{\bar{K}_{1460}}) e_1(pK_{1460}, \beta_{\phi2050}, \beta_K, \beta_{\bar{K}_{1460}}) e_1(pK_{1460}, \beta_{\phi2050}, \beta_K, \beta_{\bar{K}_{1460}}) + (c_1^{\phi2050} - c_2^{\phi2050}) f_16(pK_{1460}, \beta_{\phi2050}, \beta_K, \beta_{\bar{K}_{1460}}, \beta_{\eta_{\Delta}}) e_2(pK_{1460}, \beta_{\phi2050}, \beta_K, \beta_{\bar{K}_{1460}}, \beta_{\eta_{\Delta}}) \right\} (C58)
\]

\[
\phi(2050) \rightarrow \eta\phi: \quad C_1^{(\phi2050)\eta\phi} = 2 \left\{ \left( c_1^{\phi2050} c_1^{\phi2050} + c_2^{\phi2050} c_2^{\phi2050} \right) f_9(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi) e_1(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi) - \left( c_1^{\phi2050} c_1^{\phi2050} (c_1^{\phi2050})^2 + c_2^{\phi2050} c_2^{\phi2050} (c_2^{\phi2050})^2 \right) f_{10}(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \phi_\Delta) e_2(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \phi_\Delta) - \left( c_1^{\phi2050} c_1^{\phi2050} (c_1^{\phi2050})^2 + c_2^{\phi2050} c_2^{\phi2050} (c_2^{\phi2050})^2 \right) f_{10}(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \omega_\Delta) e_2(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \omega_\Delta) \right\} (C59)
\]

\[
\phi(2050) \rightarrow \eta\phi: \quad C_1^{(\phi2050)\eta\phi} = 2 \left\{ \left( c_1^{\phi2050} c_1^{\phi2050} + c_2^{\phi2050} c_2^{\phi2050} \right) f_9(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi) e_1(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi) - \left( c_1^{\phi2050} c_1^{\phi2050} (c_1^{\phi2050})^2 + c_2^{\phi2050} c_2^{\phi2050} (c_2^{\phi2050})^2 \right) f_{10}(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \phi_\Delta) e_2(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \phi_\Delta) - \left( c_1^{\phi2050} c_1^{\phi2050} (c_1^{\phi2050})^2 + c_2^{\phi2050} c_2^{\phi2050} (c_2^{\phi2050})^2 \right) f_{10}(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \omega_\Delta) e_2(p\eta\phi, \beta_{\phi2050}, \beta_\eta, \beta_\phi, \omega_\Delta) \right\} (C60)
\]

\[
\phi(2050) \rightarrow \eta h_1 (1380): \quad C_{01}^{(\phi2050)\eta h_1} = 2 \left\{ - (2c_1^{\phi2050} c_1^{\phi2050} + c_2^{\phi2050} c_2^{\phi2050}) f_{11}(p\eta h_1, \beta_{\phi2050}, \beta_\eta, \beta_{h_1}) e_1(p\eta h_1, \beta_{\phi2050}, \beta_\eta, \beta_{h_1}) + (2c_1^{\phi2050} c_1^{\phi2050} (c_1^{\phi2050})^2 + c_2^{\phi2050} c_2^{\phi2050} (c_2^{\phi2050})^2) f_{12}(p\eta h_1, \beta_{\phi2050}, \beta_\eta, \beta_{h_1}, \beta_{h_1}) e_2(p\eta h_1, \beta_{\phi2050}, \beta_\eta, \beta_{h_1}, \beta_{h_1}) \right\} (C61)
\]
\[ C^{(2050)}_{21} \phi_1 = 2 \left\{ -\left( 2c_1^h c_1^\phi \phi_{2050} + c_2^h c_2^\phi \phi_{2050} \right) f_13(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta_1}, \beta_{h_1}) e_1(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta_1}, \beta_{h_1}) + \left( 2c_1^h c_1^\phi \phi_{2050} + c_2^h c_2^\phi \phi_{2050} \right) f_14(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta_1}, \beta_{h_1}, \beta_{\eta h}) + \left( 2c_1^h c_1^\phi \phi_{2050} + c_2^h c_2^\phi \phi_{2050} \right) f_14(p_{\eta h_1}, \beta_{\phi_{2050}}, \beta_{\eta_1}, \beta_{h_1}, \beta_{\eta h}) \right\} \]

(1850) \[
C^{(2050)}_{11} \phi_2 = \left\{ f_18(p, \beta, \phi_1 + \beta_{\phi_1}) e_1(p, \beta, \phi_1 + \beta_{\phi_1}) + f_19(p, \beta, \phi_1 + \beta_{\phi_1}) e_1(p, \beta, \phi_1 + \beta_{\phi_1}) + f_20(p, \beta, \phi_1 + \beta_{\phi_1}) e_1(p, \beta, \phi_1 + \beta_{\phi_1}) \right\} \]

(1950) 

(2050) 

(2150)
\[
\begin{align*}
 f_{21}(p, \beta_A, \beta_B, \beta_C) &= \frac{2 \beta_B^{3/2} \beta_C^{3/2} (\beta^2_B + \beta^2_C) \beta_A^{7/2} p}{5 (\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2))^{9/2}} \left[ 20 (\beta^2_B \beta_C^4 \beta_A^2 + \beta_B^2 \beta_C^2 (\beta_C^2 + \beta_A^2)) - (\beta_B^2 + \beta_C^2) \right] \\
 &\times (2 \beta_B^2 \beta_C^2 + (\beta_B^2 + \beta_C^2) \beta_A^2) p^2 
\end{align*}
\]  
\( (C72) \)

\[
\begin{align*}
 f_{22}(p, \beta_A, \beta_B, \beta_C, \beta) &= -\frac{4 \beta_B^{3/2} \beta_C^{3/2} \beta_A^{7/2} p}{45 (\beta_C^2 \beta_B^2 + 2 \beta^2_B + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2 \beta^2))^{9/2}} \left[ 20 \beta_B^2 \beta_C^2 (\beta_B^2 + 2 \beta^2_B + \beta_A^2) + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2 \beta^2_B) \right] \\
&\times (\beta_B^2 + \beta_C^2 + 2 \beta^2) - (\beta_B^2 + \beta_C^2 + 2 \beta^2) (2 \beta_C^2 (\beta_B^2 + \beta^2_B + \beta_A^2 (\beta_B^2 + \beta_C^2 + 2 \beta^2_B)) p^2 
\end{align*}
\]  
\( (C73) \)

\[
\begin{align*}
 f_{23}(p, \beta_A, \beta_B, \beta_C) &= -\frac{2 \sqrt{7} \beta_B^{3/2} \beta_C^{3/2} \beta_A^{7/2} (\beta_B^2 + \beta_C^2)^2 \beta_A^{7/2} (2 \beta_B^2 \beta_C^2 + (\beta_B^2 + \beta_C^2) \beta_A^2) p^3}{15 (\beta_B^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2))^{9/2}} 
\end{align*}
\]  
\( (C74) \)

\[
\begin{align*}
 f_{24}(p, \beta_A, \beta_B, \beta_C, \beta) &= -\frac{4 \sqrt{7} \beta_B^{3/2} \beta_C^{3/2} \beta_A^{7/2} (\beta_B^2 + \beta_C^2)^2 (\beta_B^2 + \beta_C^2 + 2 \beta^2_B + (\beta_B^2 + \beta_C^2 + 2 \beta^2) p^3)}{15 (\beta_B^2 \beta_C^2 + \beta_B^2 (\beta_C^2 + \beta_A^2 + 2 \beta^2_B))^{9/2}} 
\end{align*}
\]  
\( (C75) \)

\( \phi_2(1850) \to K^+K^- : \)

\[
\begin{align*}
 C_{12}^{\phi_2 K^+K^-} &= -\sqrt{10} (c_{1}^{\phi_2} - c_{2}^{\phi_2}) \left\{ f_{17}(p_K K^+, \beta_{\phi_2}, \beta_{K^+}, \beta_{K^-}) e_{1}(p_K K^+, \beta_{\phi_2}, \beta_{K^+}, \beta_{K^-}) \\
&+ c_{1}^{\phi_2} c_{2}^{\phi_2} f_{18}(p_K K^+, \beta_{\phi_2}, \beta_{K^+}, \beta_{K^-}) e_{2}(p_K K^+, \beta_{\phi_2}, \beta_{K^+}, \beta_{K^-}) \\
&+ c_{1}^{\phi_2} c_{2}^{\phi_2} f_{18}(p_K K^+, \beta_{\phi_2}, \beta_{K^+}, \beta_{K^-}) e_{2}(p_K K^+, \beta_{\phi_2}, \beta_{K^+}, \beta_{K^-}) \right\} 
\end{align*}
\]  
\( (C76) \)

\( \phi_2(1850) \to \eta \phi : \)

\[
\begin{align*}
 C_{11}^{\phi_2 \eta \phi} &= -2 \left\{ (2 c_{1}^{\eta} c_{1}^{\phi_2} + c_{2}^{\eta} c_{2}^{\phi_2}) f_{21}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}) e_{1}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}) \\
&+ (2 c_{1}^{\eta} c_{1}^{\phi_2} (c_{1}^{\phi_2})^2 + c_{2}^{\eta} c_{2}^{\phi_2} (c_{2}^{\phi_2})^2) f_{22}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) e_{2}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) \\
&+ (2 c_{1}^{\eta} c_{1}^{\phi_2} (c_{1}^{\phi_2})^2 + c_{2}^{\eta} c_{2}^{\phi_2} (c_{2}^{\phi_2})^2) f_{22}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) e_{2}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) \right\} 
\end{align*}
\]  
\( (C78) \)

\[
\begin{align*}
 C_{31}^{\phi_2 \eta \phi} &= -2 \left\{ (2 c_{1}^{\eta} c_{1}^{\phi_2} + c_{2}^{\eta} c_{2}^{\phi_2}) f_{23}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}) e_{1}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}) \\
&+ (2 c_{1}^{\eta} c_{1}^{\phi_2} (c_{1}^{\phi_2})^2 + c_{2}^{\eta} c_{2}^{\phi_2} (c_{2}^{\phi_2})^2) f_{24}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) e_{2}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) \\
&+ (2 c_{1}^{\eta} c_{1}^{\phi_2} (c_{1}^{\phi_2})^2 + c_{2}^{\eta} c_{2}^{\phi_2} (c_{2}^{\phi_2})^2) f_{24}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) e_{2}(p_{\eta \phi}, \beta_{\phi_2}, \beta_{\eta}, \beta_{\phi}, \beta_{\phi_2}) \right\} 
\end{align*}
\]  
\( (C79) \)

\( \phi_3(1850) \to K^+K^- : \)

\[
\begin{align*}
 C_{40}^{\phi_3 K^+K^-} &= -\frac{5}{4} (c_{1}^{\phi_3} - c_{2}^{\phi_3}) \left\{ 2 f_{19}(p_{KK}, \beta_{\phi_3}, \beta_{K}) e_{1}(p_{KK}, \beta_{\phi_3}, \beta_{K}) \\
&+ c_{1}^{\phi_3} c_{2}^{\phi_3} f_{20}(p_{KK}, \beta_{\phi_3}, \beta_{K}) e_{2}(p_{KK}, \beta_{\phi_3}, \beta_{K}) \\
&+ c_{1}^{\phi_3} c_{2}^{\phi_3} f_{20}(p_{KK}, \beta_{\phi_3}, \beta_{K}) e_{2}(p_{KK}, \beta_{\phi_3}, \beta_{K}) \right\} 
\end{align*}
\]  
\( (C80) \)
\[ \phi_3(1850) \rightarrow K^+K^- : \]
\[ C_{31}^{\phi_3 KK^*} = -\sqrt{\frac{5}{6}} (c_1^{\phi_3} + c_2^{\phi_3}) \left\{ \begin{array}{l}
3 f_{23}(p_{KK^*}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_1(p_{KK^*}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{24}(p_{KK^*}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{KK^*}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{24}(p_{KK^*}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{KK^*}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0})
\end{array} \right\} \]  
(C81)

\[ \phi_3(1850) \rightarrow K^+K^+ : \]
\[ C_{12}^{\phi_3 K^+K^+} = 6 \sqrt{\frac{5}{3}} (c_1^{\phi_3} - c_2^{\phi_3}) \left\{ \begin{array}{l}
f_{17}(p_{K^+K^+}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_1(p_{K^+K^+}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{18}(p_{K^+K^+}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{K^+K^+}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{18}(p_{K^+K^+}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{K^+K^+}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0})
\end{array} \right\} \]  
(C82)

\[ \phi_3(1850) \rightarrow K^+K^-_{1270} : \]
\[ C_{21}^{\phi_3 KK^*_{1270}} = \left( c_1^{\phi_3 + \phi_3^{2050}} + c_2^{\phi_3} \right) \left\{ \begin{array}{l}
f_{25}(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_1(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{26}(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{26}(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0})
\end{array} \right\} \]  
(C83)

\[ \phi_3(1850) \rightarrow K^+K^-_{1270} : \]
\[ C_{41}^{\phi_3 KK^*_{1270}} = \left( c_1^{\phi_3 + \phi_3^{2050}} + c_2^{\phi_3} \right) \left\{ \begin{array}{l}
f_{29}(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_1(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{30}(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) \\
+ c_1^{\phi_3} c_2^{\phi_3} f_{30}(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}) e_2(p_{KK^*_{1270}}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0}, \beta_{K^0})
\end{array} \right\} \]  
(C84)

\[ f_{25}(p, \beta_A, \beta_B, \beta_C) = \frac{\sqrt{2}^{7/2} \beta_C^{7/2} (\beta_B^2 + \beta_C^2) \beta_A^{7/2} p^2}{7 \sqrt{15} (\beta_C^2 + \beta_B^2) (\beta_C^2 + \beta_A^2)^{11/2}} \left[ 28 \left( 4 \beta_B^4 \beta_C^4 + 5 \beta_B^2 \beta_C^2 (\beta_B^2 + \beta_C^2) \beta_A^4 + (\beta_B^2 + \beta_C^2)^2 \beta_A^4 - 3 (\beta_B^2 + \beta_C^2) (2 \beta_B^2 \beta_C^2 + (\beta_B^2 + \beta_C^2) \beta_A^2) p^2 \right) \right] \]  
(C86)

\[ f_{26}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{2 \sqrt{2}^{3/2} \beta_B^{3/2} \beta_C^{3/2} (\beta_B^2 + \beta_C^2 + 2 \beta_A^2)^{7/2} p^2}{21 \sqrt{15} (\beta_B^2, \beta_C^2 + 2 \beta_A^2) (\beta_B^2 + 2 \beta_A^2 + 2 \beta_A^2)^{11/2}} \left[ 28 \left( 4 \beta_C^4 (\beta_A^2 + \beta_B^2) (\beta_B^2 + 2 \beta_A^2)^2 \right) \right. \]  
(C87)
\[ f_{27}(p, \beta_A, \beta_B, \beta_C) = \frac{4\beta_B^2 \beta_C^2 \beta_A^2 (\beta_B^2 + \beta_C^2) \beta_A^2 p^2}{7\sqrt{15} (\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2))^{11/2}} \left[ 28 \left( \beta_B^2 \beta_C^2 \beta_A^2 + \beta_B^2 \beta_C^2 (\beta_C^2 + \beta_A^2) \right) - (\beta_B^2 + \beta_C^2) \right] \]
\[ (2\beta_B^2 \beta_C^2 + (\beta_B^2 + \beta_C^2) \beta_A^2 p^2) \] (C88)

\[ f_{28}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{2\beta_B^3 \beta_C^2 \beta_A^2 (\beta_B^2 + \beta_C^2 + 2\beta^2)^2 p^2}{21\sqrt{15} (\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2))^{11/2}} \left[ 28 \beta_B^2 \beta_A^2 (\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2))^{11/2} \right]
\[ + 2 (\beta_B^2 + \beta_C^2)^2 + (\beta_B^2 + \beta_C^2)^2 (\beta_B^2 + \beta_C^2)^2 (\beta_B^2 + \beta_C^2)^2 p^2 ) \] (C89)

\[ f_{29}(p, \beta_A, \beta_B, \beta_C, \beta) = \frac{\sqrt{2} \beta_B^2 \beta_C^2 \beta_A^2 (\beta_B^2 + \beta_C^2)^2 \beta_A^2}{21\sqrt{5} (\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2))^{11/2}} p^4 \] (C90)

\[ f_{30}(p, \beta_A, \beta_B, \beta_C, \beta) = -\frac{2\sqrt{2} \beta_B^3 \beta_C^2 \beta_A^2 (\beta_B^2 + 2\beta^2) (\beta_B^2 + \beta_C^2 + 2\beta^2)^2 \beta_A^2 p^4}{63\sqrt{5} (\beta_C^2 \beta_A^2 + \beta_B^2 (\beta_C^2 + \beta_A^2)(\beta_B^2 + \beta_C^2 + 2\beta^2)^2 \beta_A^2)^{11/2}} \] (C91)

\[ \phi_3(1850) \rightarrow \eta \phi : \]

\[ C_{31}^{o \phi \phi} = \sqrt{\frac{10}{3}} \{ \left[ 3 (2c_1^2 \phi_1 \phi_2 \phi_3 + c_2^2 \phi_2 \phi_3) f_{23}(p, \phi_2, \phi_3, \phi_3, \phi_3) \right] e_1(p, \phi_2, \phi_3, \phi_3, \phi_3) 
\[ + \left[ 2c_2^2 \phi_2 \phi_3 \phi_3 (\phi_2^\ast \phi_3)^2 + c_2^2 \phi_2 \phi_3 (\phi_3^\ast \phi_3) \right] f_{24}(p, \phi_2, \phi_3, \phi_3, \phi_3, \phi_3) \right] \}
\[ + \left[ 2c_1^2 \phi_1 \phi_3 (\phi_1^\ast \phi_3)^2 + c_2^2 \phi_2 \phi_3 (\phi_2^\ast \phi_3) \right] f_{24}(p, \phi_2, \phi_3, \phi_3, \phi_3, \phi_3) \} \] (C92)