A PACKAGE FOR COMPUTATIONS WITH SPARSE RESULTANTS

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Abstract. We introduce the Macaulay2 package SparseResultants, which provides general tools for computing sparse resultants, sparse discriminants, and hyperdeterminants. We give some background on the theory and briefly show how the package works.

Introduction

The classical Macaulay resultant [Mac02] (also called the dense resultant) of a system of \( n + 1 \) polynomial equations in \( n \) variables characterizes the solvability of the system, and therefore it is a fundamental tool in computer algebra. However, it is a large polynomial, since it depends on all coefficients of the equations. If we restrict attention to sparse polynomial equations, that is, to polynomials which involve only monomials lying in a small set, then we can replace the dense resultant with the sparse resultant.

The sparse resultant generalizes not only the dense resultant but, for specific choices of the set of monomials, we can obtain other types of classical resultants, such as for instance the Dixon resultant [Dix08] and the hyperdeterminant [Cay45, GKZ92]. In the last decades, sparse resultants have received a lot of interest, both from a theoretical point of view (see e.g., [GKZ94, Stu94, CDS98, DS15]) and from more computational and applied aspects (see e.g., [EM99, CE00, Stu02, D’A02, JKSS04, CLO05, JMSW09]).

Using the computer program Macaulay2 [GS21], dense resultants can be calculated using the package Resultants [Sta18], while sparse resultants can be calculated using the new package SparseResultants. We point out that in the latter most of the algorithms implemented are based on elimination via Gröbner basis methods. The main defect of this approach is that even when the input polynomials have numerical coefficients, in the calculation all the coefficients are replaced by variables. However, this approach suffices for a number of applications, as we try to show in the following.

This short paper is organized as follows. In Section 1, we review the general theory of sparse resultants (subsections 1.1 and 1.2) and related topics as the sparse discriminants (subsection 1.3) and the hyperdeterminants (subsection 1.4). We focus on the computational aspects used in the package SparseResultants. In Section 2, we illustrate how this package works with the help of some examples.

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SparseResultants, version 1.2, available at https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2/share/doc/Macaulay2/SparseResultants/html/index.html.
1. An overview of sparse elimination

In this section we give some background on the theory of sparse resultants, sparse discriminants, and hyperdeterminants. For details and proofs we refer mainly to [GKZ94, Chapters 8, 9, 13, and 14] and [CLO05, Chapter 7]; other references are [Stu93, Ott13].

1.1. Sparse mixed resultant. Let \( R = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) be the ring of complex Laurent polynomials in \( n \) variables. The set of monomials in \( R \) is identified with \( \mathbb{Z}^n \) by associating to \( x^\omega = x_1^{\omega_1} \cdots x_n^{\omega_n} \in R \) the exponent vector \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n \). If \( \mathcal{A} \) is a finite subset of \( \mathbb{Z}^n \), we denote by \( \mathbb{C}^\mathcal{A} \) the space of polynomials in \( R \) involving only monomials from \( \mathcal{A} \), that is, of polynomials of the form \( \sum_{\omega \in \mathcal{A}} a_\omega x^\omega \).

Let \( \mathcal{A}_0, \ldots, \mathcal{A}_n \) be \( n+1 \) finite subsets of \( \mathbb{Z}^n \) satisfying the following conditions:

1. each \( \mathcal{A}_i \) generates \( \mathbb{R}^n \) as an affine space; and
2. the union of the sets \( \mathcal{A}_i \) generates \( \mathbb{Z}^n \) as a \( \mathbb{Z} \)-module.

Let \( \mathcal{Z}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \subset \prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i} \) be the Zariski closure in the product \( \prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i} \) of the set

\[
\{ (f_0, \ldots, f_n) \in \prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i} : \exists x \in (\mathbb{C}^*)^n \text{ such that } f_0(x) = \cdots = f_n(x) = 0 \},
\]

where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) and \( f_i(x) = \sum_{\omega \in \mathcal{A}_i} a_{i,\omega} x^\omega \), for \( i = 0, \ldots, n \).

**Proposition-Definition 1.1** ([GKZ94], Chapter 8, §1). Under the above assumptions, the variety \( \mathcal{Z}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \) is an irreducible hypersurface in \( \prod_{i=0}^n \mathbb{C}^{\mathcal{A}_i} \) that can be defined by an integral irreducible polynomial \( \text{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \in \mathbb{Z}[(a_{i,\omega}), i = 0, \ldots, n] \) in the coefficients \( a_{i,\omega} \) of \( f_i \), for \( i = 0, \ldots, n \). Such a polynomial \( \text{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \) is unique up to sign and is called the \( (\mathcal{A}_0, \ldots, \mathcal{A}_n) \)-resultant (also known as the sparse (mixed) resultant).

The polynomial \( \text{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n} \) is homogeneous with respect to each group of variables \( (a_{i,\omega}) \), for \( i = 0, \ldots, n \). Moreover, \( \text{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}(f_0, \ldots, f_n) = 0 \) if the \( (n+1) \)-tuple \( (f_0, \ldots, f_n) \) belongs to (1.1).

**Example 1.2.** Let \( d_0, \ldots, d_n \) be positive integers. For \( i = 0, \ldots, n \), let \( \mathcal{A}_i = \{ \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{j=1}^n \omega_j \leq d_i \} \). Then the \( (\mathcal{A}_0, \ldots, \mathcal{A}_n) \)-resultant coincides with the classical (affine) resultant \( \text{Res}_{d_0, \ldots, d_n} \), also called the dense resultant. Therefore, if \( F_i \in \mathbb{C}[x_0, x_1, \ldots, x_n] \) denotes the polynomial obtained by homogenizing \( f_i \in \mathbb{C}^{\mathcal{A}_i} \) with respect to a new variable \( x_0 \), then \( \text{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}(f_0, \ldots, f_n) = 0 \) if and only if \( F_0, \ldots, F_n \) have a common non-trivial root.

1.2. Sparse unmixed resultant. Keep the notation and assumptions as above. If all the sets \( \mathcal{A}_i \) coincide with each other, that is, \( \mathcal{A}_0 = \cdots = \mathcal{A}_n = \mathcal{A} \), then the \( (\mathcal{A}_0, \ldots, \mathcal{A}_n) \)-resultant is called the \( \mathcal{A} \)-resultant (also known as the sparse (unmixed) resultant). In this case, we have a useful geometric interpretation that allows us to write down the \( \mathcal{A} \)-resultant in a compact form. Indeed, by choosing a numbering \( \omega^{(0)}, \ldots, \omega^{(k)} \) of the elements of \( \mathcal{A} \), we get a map \( \phi_\mathcal{A} : (\mathbb{C}^*)^n \to \mathbb{P}^k \) defined by \( \phi_\mathcal{A}(x) = (\omega^{(0)}(x) : \cdots : \omega^{(k)}(x)) \). Let \( X_\mathcal{A} \subset \mathbb{P}^k \) be the closure of the image of \( \phi_\mathcal{A} \), which is an irreducible toric variety of dimension \( n \). Then, by taking pull-backs we get an identification between the space of polynomials in \( \mathbb{C}^\mathcal{A} \) with the space of linear forms on \( \mathbb{P}^k \). Moreover, if \( f_0, \ldots, f_n \in \mathbb{C}^{\mathcal{A}} \) have a common root in \( (\mathbb{C}^*)^n \) then the corresponding linear forms \( l_0, \ldots, l_n \) on \( \mathbb{P}^k \) define a linear subspace that intersects \( X_\mathcal{A} \). From this, the following proposition follows directly.
Proposition 1.3 ([GKZ94], Chapter 8, §2). The polynomial \( \text{Res}_A \in \mathbb{Z}[a_0^{(i)}, \ldots, a_k^{(i)}, i = 0, \ldots, n] \) coincides with the \( X \)-resultant of \( X_A \subset \mathbb{P}^k \). More precisely, let \( W_A \subset \mathbb{G}(k - n - 1, \mathbb{P}^k) \) be the Chow hypersurface of the variety \( X_A \), and let

\[
\psi : \mathbb{P}(\mathbb{C}^{(n+1)\times(k+1)}) \rightarrow \mathbb{G}(n, k) \simeq \mathbb{G}(k - n - 1, k)
\]

be the natural projection from the projectivization of the space of complex matrices of shape \((n + 1) \times (k + 1)\) to \( \mathbb{G}(n, k) \). Then we have that \( \text{Res}_A \) is the polynomial defining the pull-back \( \psi^{-1}(W_A) \).

Remark 1.4. With the notation of the proposition above, in coordinates the map \( \psi \) is defined by the \((n + 1) \times (n + 1)\) minors of the generic \((n + 1) \times (k + 1)\) matrix of variables

\[
\begin{pmatrix}
(a_0^{(0)} & a_1^{(0)} & \cdots & a_k^{(0)}) \\
\vdots & \ddots & \ddots & \vdots \\
(a_0^{(n)} & a_1^{(n)} & \cdots & a_k^{(n)})
\end{pmatrix}.
\]

(1.2)

In particular, \( \text{Res}_A \) can be expressed as a homogeneous polynomial of degree \( \deg(X_A) \) in the \((n + 1) \times (n + 1)\) minors of the matrix (1.2).

Example 1.5. Let \( A = \{(\omega_1, \omega_2) \in \mathbb{Z}^2 : \omega_1 + \omega_2 \leq 2\} \), so that \( X_A \subset \mathbb{P}^5 \) is the Veronese surface. The \( A \)-resultant is a polynomial of degree 12 in 18 variables with 21,894 terms. It can be expressed as a polynomial of degree 4 in the Plücker coordinates of \( \mathbb{G}(2, 5) \) with 74 terms.

1.3. Sparse discriminant. We continue by letting \( A \subset \mathbb{Z}^n \) be a finite set of \( k + 1 \) elements that generate \( \mathbb{Z}^n \) as a \( \mathbb{Z} \)-module, and let \( \phi_A : (\mathbb{C}^*)^n \rightarrow \mathbb{P}^k \) and \( X_A \subset \mathbb{P}^k \) be defined as above. Let \( \nabla_A \subset \mathbb{C}^A \) be the Zariski closure of the set

\[
\{ f \in \mathbb{C}^A : \exists x \in (\mathbb{C}^*)^n \text{ such that } f(x) = \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0 \}.
\]

(1.3)

Proposition-Definition 1.6 ([GKZ94], Chapter 9, §1). The projectivization \( \mathbb{P}(\nabla_A) \subset \mathbb{P}^k \) of the variety \( \nabla_A \) coincides with the dual variety \( X_A^\vee \) of \( X_A \). In the case where \( X_A \) is a hypersurface, an integral irreducible polynomial \( \text{Disc}_A \) defining it (which is unique up to sign) is called the \( A \)-discriminant (also known as the sparse discriminant).

Thus the \( A \)-discriminant (when it exists) is a homogeneous polynomial \( \text{Disc}_A \in \mathbb{Z}[a_{\omega}, \omega \in A] \), and \( \text{Disc}_A(f) = 0 \) for each polynomial \( f \) belonging to (1.3).

Example 1.7. Let \( d \geq 1 \) and let \( A = \{(\omega_1, \ldots, \omega_n) \in \mathbb{Z}_n^\geq 0 : \sum_{j=1}^{n} \omega_j \leq d\} \). Then the \( A \)-discriminant coincides with the classical (affine) discriminant \( \text{Disc}_A \), also called the dense discriminant. Therefore, if \( F \in \mathbb{C}[x_0, x_1, \ldots, x_n] \) denotes the polynomial obtained by homogenizing \( f \in \mathbb{C}^A \) with respect to a new variable \( x_0 \), then \( \text{Disc}_A(f) = 0 \) if and only if the hypersurface \( \{ F = 0 \} \subset \mathbb{P}^n \) is not smooth.

Remark 1.8 ("Cayley trick", [GKZ94], Chapter 9, Proposition 1.7). Let \( A_0, \ldots, A_n \subset \mathbb{Z}^n \) be finite subsets satisfying the assumptions of subsection 1.1. Let \( A \subset \mathbb{Z}^n \times \mathbb{Z}^n \) be defined by \( A = \langle A_0 \times \{0\} \rangle \cup \langle A_1 \times \{e_1\} \rangle \cup \cdots \cup \langle A_n \times \{e_n\} \rangle \), where the \( e_i \) are the standard basis.
vectors of $\mathbb{Z}^n$. Thus a polynomial $f \in \mathbb{C}[A]$ has the form

$$f_0(x) + \sum_{i=1}^{n} y_i f_i(x) \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

where $f_i \in \mathbb{C}[A_i]$. We have the following relation (up to sign), known as the “Cayley trick”:

$$\text{Res}_{A_0, \ldots, A_n}(f_0, \ldots, f_n) = \text{Disc}_{A}(f_0(x) + \sum_{i=1}^{n} y_i f_i(x)).$$

1.4. Hyperdeterminant. An important special type of sparse discriminant is the determinant (or hyperdeterminant) of multidimensional matrices, which was introduced by Cayley \cite{Cay} (see also \cite{GKZ94}, Chapter 14 and \cite{Ott}). Let $f$ be a multilinear form in $r$ groups of variables $x_0^{(1)}, \ldots, x_{k_1}^{(1)}, \ldots, x_0^{(r)}, \ldots, x_{k_r}^{(r)}$, that is

$$f = \sum_{0 \leq i_1, \ldots, i_r \leq k} a_{i_1, \ldots, i_r} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}.$$

Let $A \subset \mathbb{Z}^{(k_1+1)+\cdots+(k_r+1)}$ denote the set of exponent vectors that can occur in such a form $f$. Notice that to give $f$ is equivalent to giving an $r$-dimensional matrix

$$M_f = (a_{i_1, \ldots, i_r})_{0 \leq i_1, \ldots, i_r \leq k}$$

of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$. The determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ is defined to be the $A$-discriminant, that is, for a form $f$ as above, we have

$$\det(M_f) = \text{Disc}_{A}(f).$$

One sees that the variety $X_A$ is the image of the Segre embedding of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$. Therefore, the hypersurface in $\mathbb{P}(\mathbb{C}^{(k_1+1)\times\cdots\times(k_r+1)})$ defined by the determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ is the dual variety of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$. Notice also that we have $\det(M_f) = 0$ if and only if the hypersurface $\{f = 0\} \subset \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$ is not smooth.

The next two basic results have been proved in \cite[Chapter 14, Theorems 1.3 and 2.4]{GKZ94}.

Theorem 1.9 (\cite{GKZ94}). The determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ exists (that is the dual variety of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}$ is a hypersurface) if and only if

$$2 \max_{1 \leq i \leq r} (k_i) \leq \sum_{j=1}^{r} k_j. \quad (1.5)$$

Theorem 1.10 (\cite{GKZ94}). Denote by $N(k_1, \ldots, k_r)$ the degree of the determinant of shape $(k_1 + 1) \times \cdots \times (k_r + 1)$ when (1.5) is satisfied, and let $N(k_1, \ldots, k_r) = 0$ otherwise. We have

$$\sum_{k_1, \ldots, k_r \geq 0} N(k_1, \ldots, k_r) z_1^{k_1} \cdots z_r^{k_r} = \frac{1}{(1 - \sum_{i=2}^{r}(i-2)e_i(z_1, \ldots, z_r))^2},$$

where $e_i(z_1, \ldots, z_r)$ is the $i$-th elementary symmetric polynomial.
Remark 1.11 ([GKZ94, Chapter 4, Propositions 1.4 and 1.8]). The determinant of shape \((k_1 + 1) \times \cdots \times (k_r + 1)\) is invariant under the action of \(\text{SL}(k_1 + 1) \times \cdots \times \text{SL}(k_r + 1)\) on the space of matrices of shape \((k_1 + 1) \times \cdots \times (k_r + 1)\). It is also invariant under permutations of the dimensions, that is, if \(M = (a_{i_1, \ldots, i_r})\) is a matrix of shape \((k_1 + 1) \times \cdots \times (k_r + 1)\) and \(\sigma\) is a permutation of indices \(1, \ldots, r\), denoting by \(\sigma(M)\) the matrix of shape \((k_{\sigma^{-1}(1)} + 1) \times \cdots \times (k_{\sigma^{-1}(r)} + 1)\), whose \((i_1, \ldots, i_r)\)-th entry is equal to \(a_{\sigma(i_1), \ldots, \sigma(i_r)}\), we have \(\det(\sigma(M)) = \det(M)\).

There are at least two important cases where determinants can be computed without resorting to elimination. We briefly recall them in 1.4.1 and 1.4.2.

1.4.1. Schläfli’s method. Let \(M\) be an \(r\)-dimensional matrix of shape \((k_1 + 1) \times \cdots \times (k_r + 1)\) corresponding to a multilinear form \(f \in \mathbb{C}[x_0^{(1)}, \ldots, x_{k_1}^{(1)}; : : :; x_0^{(r)} , \ldots, x_{k_r}^{(r)}]\). Assume that there exist both the determinants of shapes \((k_1 + 1) \times \cdots \times (k_r + 1)\) and \((k_1 + 1) \times \cdots \times (k_{r-1} + 1)\). We can interpret the \(r\)-dimensional matrix \(M\) as an \((r - 1)\)-dimensional matrix \(M(x_0^{(r)}, \ldots, x_{k_r}^{(r)})\) of shape \((k_1 + 1) \times \cdots \times (k_{r-1} + 1)\) whose entries are linear forms in the variables \(x_0^{(r)}, \ldots, x_{k_r}^{(r)}\); in other words, we can see \(f\) as a polynomial \(\tilde{f} \in \mathbb{C}[x_0^{(r)}, \ldots, x_{k_r}^{(r)}])[x_0^{(1)}, \ldots, x_{k_1}^{(1)}; : : :; x_0^{(r-1)} , \ldots, x_{k_{r-1}}^{(r-1)}]\). Let \(F_M = F_M(x_0^{(r)}, \ldots, x_{k_r}^{(r)}) = \det(M(x_0^{(r)}, \ldots, x_{k_r}^{(r)}))\), which is a homogeneous polynomial in \(x_0^{(r)}, \ldots, x_{k_r}^{(r)}\), and let \(\text{Disc}(F_M)\) be the (classical) discriminant of \(F_M\). Then we have the following:

Theorem 1.12 ([GKZ94, Sch52]). The polynomial \(\text{Disc}(F_M)\) is divisible by the determinant \(\det(M)\). Moreover if the shape of \(M\) is one of the following
\[(1.6) \quad m \times m \times 2, \quad m \times m \times 3, \quad 2 \times 2 \times 2 \times 2, \quad \text{with} \quad m \geq 2,\]
then we have \(\text{Disc}(F_M) = \det(M)\).

The method above turns out to be very effective; however it was conjectured in [GKZ94, p. 479], and later proved in [WZ96], that the shapes in (1.6) are the only ones for which the method gives the determinant exactly.

1.4.2. Determinants of boundary shape. For an \((r + 1)\)-dimensional matrix \(M\) of shape \((k_0 + 1) \times (k_1 + 1) \times \cdots \times (k_r + 1)\), we say that it is of boundary shape if the inequality (1.5) is an equality. W.l.o.g., we can assume that \(k_0 = \max_{0 \leq i \leq r}(k_i)\), so that \(k_0 = k_1 + \cdots + k_r\). Let \(f \in \mathbb{C}[x_0^{(0)}, \ldots, x_{k_0}^{(0)}; : : :; x_0^{(r)} , \ldots, x_{k_r}^{(r)}]\) be the corresponding multilinear form of such a matrix \(M\). Thinking \(f\) as a linear polynomial in \((\mathbb{C}[x_0^{(1)}, \ldots, x_{k_1}^{(1)}; : : :; x_0^{(r)} , \ldots, x_{k_r}^{(r)}])[x_0^{(0)}, \ldots, x_{k_0}^{(0)}]\), we can interpret \(M\) as a list of \(k_0 + 1\) multilinear forms \(f_0, \ldots, f_{k_0}\) in the \(r\) groups of variables \(x_0^{(1)}, \ldots, x_{k_1}^{(1)}; : : :; x_0^{(r)} , \ldots, x_{k_r}^{(r)}\).

A simple consequence of the “Cayley trick” (see [GKZ94, Chapter 3, Corollary 2.8]) gives the following:

Proposition 1.13 ([GKZ94]). The determinant of an \((r + 1)\)-dimensional matrix \(M\) of boundary shape \((k_0 + 1) \times \cdots \times (k_r + 1)\) coincides with the resultant of the multilinear forms \(f_0, \ldots, f_{k_0}\), that is, \(\det(M) = 0\) if and only if the system of multilinear equations \(f_0(x) = \cdots = f_{k_0}(x) = 0\) has a non-trivial solution on \(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}\). In other words,
the determinant of shape \((k_0 + 1) \times \cdots \times (k_r + 1)\) coincides with the X-resultant of the Segre embedding of \(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r}\).

Remark 1.14. We also mention that the determinant of a matrix \(M\) of boundary shape \((k_0 + 1) \times \cdots \times (k_r + 1)\) can be explicitly expressed as the determinant of an ordinary square matrix of order \(\frac{(k_0+1)!}{k_1! \cdots k_r!}\) whose entries are linear forms in the entries of \(M\); see [GKZ94, Chapter 14, Theorem 3.3].

2. Sparse resultants in Macaulay2

In this section, we briefly describe some of the functions implemented in the package SparseResultants. For more details and examples, we refer to its documentation.

One of the main functions is \texttt{sparseResultant}, which via elimination techniques calculates sparse mixed resultants \(\text{Res}_{A_0, \ldots, A_n}\) (see subsection 1.1) and sparse unmixed resultants \(\text{Res}_A\) (see subsection 1.2). This function can be called in two ways. The first one is to pass a list of \(n+1\) matrices \(A_0, \ldots, A_n\) over \(\mathbb{Z}\) and with \(n\) rows to represent the sets \(A_0, \ldots, A_n \subset \mathbb{Z}^n\); see also the classical reference [Dix08] and for the rule to evaluate it at the \(n+1\) polynomials \(f_i\). The second way to call \texttt{sparseResultant} is to pass directly the polynomials \(f_i\). This is equivalent to forming the matrices \(A_i\), whose columns are given by \(\{\omega \in \mathbb{Z}^n : \text{the coefficient in } f_i \text{ of } x^\omega \neq 0\}\) (see the function \texttt{exponentsMatrix}) and then proceeding as described above.

As an example we now calculate a particular type of sparse unmixed resultant, known as the Dixon resultant (see [Stu93, Section 2.4] and [CLO05, Chapter 7, §2, Exercise 10]; see also the classical reference [Dix08]).

Example 2.1. Consider the following system of three bihomogeneous polynomials of bidegree \((2,1)\) in the two groups of variables \((x_0, x_1), (y_0, y_1)\):

\[
\begin{align*}
&c_{1,1}x_1^2y_1 + c_{1,2}x_1x_2y_1 + c_{1,3}x_1^2y_1 + c_{1,4}x_1^2y_2 + c_{1,5}x_1x_2y_2 + c_{1,6}x_1^2y_2 = 0, \\
&c_{2,1}x_2^2y_1 + c_{2,2}x_1x_2y_1 + c_{2,3}x_2^2y_1 + c_{2,4}x_1^2y_2 + c_{2,5}x_1x_2y_2 + c_{2,6}x_2^2y_2 = 0, \\
&c_{3,1}x_1^2y_1 + c_{3,2}x_1x_2y_1 + c_{3,3}x_2^2y_1 + c_{3,4}x_1^2y_2 + c_{3,5}x_1x_2y_2 + c_{3,6}x_2^2y_2 = 0.
\end{align*}
\]

Putting \(x_2 = y_2 = 1\) we get a system of three non-homogeneous polynomials in two variables \((x, y) = (x_1, y_1)\), of which we can calculate the sparse (unmixed) resultant. This polynomial is homogeneous of degree 12 in the 18 variables \(c_{1,1}, \ldots, c_{3,6}\) with 20,791 terms, which vanishes if and only if (2.1) has a nontrivial solution. The time for this computation is less than one second (on a standard laptop computer).

\$ M2 --no-preload \\
Macaulay2, version 1.18 \\
i1 : needsPackage "SparseResultants";
i2 : R = ZZ[c_{-}(1,1) .. c_{-}(3,6)][x,y]; 
i3 : f = (c_{-}(1,1)*x^2*y+c_{-}(1,2)*x*y+c_{-}(1,3)*y+c_{-}(1,4)*x^2+c_{-}(1,5)*x+c_{-}(1,6),

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\[c_{(2,1)}x^2y + c_{(2,2)}xy + c_{(2,3)}y + c_{(2,4)}x^2 + c_{(2,5)}x + c_{(2,6)},
\]
\[c_{(3,1)}x^2y + c_{(3,2)}xy + c_{(3,3)}y + c_{(3,4)}x^2 + c_{(3,5)}x + c_{(3,6)});\]

\[\text{i4 : } A = \text{exponentsMatrix } f;\]
\[\text{o4 : } \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \]

\[\text{i5 : time Res = sparseResultant } A;\]
\[\text{-- used 0.241391 seconds}\]
\[\text{o5 : SparseResultant (sparse unmixed resultant associated to } \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix})\]

\[\text{i6 : time U = Res } f;\]
\[\text{-- used 0.574002 seconds}\]
\[\text{i7 : (first degree } U, \# \text{ terms } U)\]
\[\text{o7 = (12, 20791)}\]

Another function of the package is \text{sparseDiscriminant}, which calculates sparse discriminants \(\text{Disc}_A\) (see subsection \(1.3\)). This function works similar to the previous one. In particular, it accepts as input either a matrix representing the exponent vectors of a (Laurent) polynomial or just directly the polynomial.

**Example 2.2.** Using the Cayley trick (formula (1.4)), we express the dense resultant of three generic ternary forms of degrees 1, 1, 2 (which is a special type of sparse mixed resultant) as a sparse discriminant. The calculation time is less than one second.

\[\text{i8 : clearAll;}\]
\[\text{i9 : K = ZZ[a_0..a_2,b_0..b_2,c_0..c_5], Rx = K[x_1,x_2];}\]
\[\text{i10 : f = (a_0+a_1*x_1+a_2*x_2,}\]
\[\text{b_0+b_1*x_1+b_2*x_2,}\]
\[\text{c_0+c_1*x_1+c_2*x_2+c_3*x_1^2+c_4*x_1*x_2+c_5*x_2^2);}\]
\[\text{i11 : Rxy = K[x_1,x_2,y_1,y_2], f' = (sub(f_0,Rxy), sub(f_1,Rxy), sub(f_2,Rxy));}\]
\[\text{i12 : time sparseResultant(f_0,f_1,f_2) ==}\]
\[\text{--sparseDiscriminant(f'_0 + y_1*f'_1 + y_2*f'_2)}\]
\[\text{-- used 0.746274 seconds}\]
\[\text{o12 = true}\]

A derived function of \text{sparseDiscriminant} is \text{determinant} (or simply \text{det}), which calculates determinants of multidimensional matrices (see subsection \(1.4\)). However for this last one, more specialized algorithms are also available and automatically applied.

**Example 2.3.** Here we calculate the determinant of a generic four-dimensional matrix of shape \(2 \times 2 \times 2 \times 2\) (see also \cite{HSYY08}). This polynomial is homogeneous of degree 24 in the 16 variable entries of the matrix and it has \(2,894,276\) terms. The approach for this calculation is to apply Theorem 1.6 recursively. The calculation time is about 10 minutes, but it takes much less time if we specialize the entries of the matrix to be random numbers.

\[\text{i13 : M = genericMultidimensionalMatrix } \{2,2,2,2\}\]
\[\text{o13 = \{\{(a , a ), (a , a )\}, \{(a , a ), \ldots\}}\]
\[\{0,0,0,0,0,0,0,1,0,0,1,0,0,1,1,0,1,0,0,1,0,1,0,1\} \ldots\]
Example 2.4. Here we take $A$ and $B$ to be random matrices of shapes $2 \times 2 \times 2 \times 4$ and $4 \times 2 \times 2 \times 5$, respectively. We calculate the convolution $A \ast B$ (see [GKZ94, p. 449]), which is a matrix of shape $2 \times 2 \times 2 \times 2 \times 5$. Then we verify a formula proved in [DO03] for $\det(A \ast B)$, which generalizes the Cauchy-Binet formula in the multidimensional case. The approach for the calculation of the determinant of shape $4 \times 2 \times 2 \times 5$ is using Proposition 1.13, while the determinants of shapes $2 \times 2 \times 2 \times 4$ and $2 \times 2 \times 2 \times 2 \times 5$ are calculated using Remark 1.14. The calculation time is less than one second.

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