Meromorphic solutions to the $q$-Painlevé equations around the origin

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Abstract. We study meromorphic solutions to the $q$-Painlevé equations with type VI, V, III around the origin. Since the origin of $q$-PVI, $q$-PV and $q$-PIII are singular points of the $q$-Briot-Bouquet type, they have holomorphic solutions for generic parameters. The $q$-PVI, $q$-PV and $q$-PIII have four, three and two meromorphic solutions around the origin respectively. We can solve linear connection problems for holomorphic solutions.

1. Introduction
A $q$-analogue of the Painlevé equations has been studied for a quarter of a century. Many discrete Painlevé equations are found in 1980s, but Grammaticos, Ramani, Papageorgiou and Hietarinta gave a systematic study on discrete Painlevé equations in 1991 [5], [14]. They discovered many discrete equations with singularity confinement property, which is a discrete analogue of the Painlevé property (no movable branch points). There are also many studies on the $q$-Painlevé equations, such as bilinear forms, the Bäcklund transformations, determinant formula for hypergeometric solutions and so on. Sakai has describes a degeneration diagram of discrete Painlevé equations from the viewpoint of spaces of initial conditions [16]. His theory elegantly explains the reason why many different discrete analogues of Painlevé equations exist for each differential Painlevé equations. Different types of discrete Painlevé equations can live on the same space of initial conditions. They correspond to directions of translation on the affine Weyl group symmetries, which act on the spaces of initial conditions. The reader may consult two good survey papers on discrete Painlevé equations [4][11].

But there exist few studies on transcendental solutions to $q$-Painlevé equations, although it is believed that generic solutions for $q$-Painlevé equations are transcendental. Here “transcendental” solutions mean solutions neither of the hypergeometric type nor algebraic.

We study some examples of special solutions to $q$-Painlevé equations, which are holomorphic around the fixed singular points. These solutions are $q$-analogue of meromorphic solutions around the fix points to differential Painlevé equations [8]. In [8], we have studied meromorphic solutions around the origin to the third, fifth and sixth Painlevé differential equations. These equations have a singular point of the Briot-Bouquet type at the origin. In general the Painlevé transcendentals have a wild singularity around the fixed singular points. But such tempered solutions play an important role in mathematical physics. Moreover, we can easily determine monodromy or the Stokes data of the linearized equations (the Lax pair) of the Painlevé equations. In this paper, we classify all of meromorphic solutions around the origin to $q$-PVI.
q-$P_V$, q-$P_{III}$ and give connection matrices of the linearized equations. The case of q-$P_{VI}$ has been studied in [12].

Isomonodromic methods gives a powerful tool in the analytic study of the Painlevé equations. For q-Painlevé equations, we consider \textit{connection preserving deformations} instead of monodromy preserving deformations. We take a linear q-difference equation $Y(qx) = A(x)Y(x)$, where $A(x)$ is a matrix valued polynomial. We choose local solutions $Y_0(x)$ around $x = 0$ and $Y_\infty(x)$ around $x = \infty$. The connection matrix $P(x)$ is given by $Y_\infty(x) = Y_0(x)P(x)$. We introduce an extra parameter $t$ and study $Y(qx) = A(x,t)Y(x)$. When the connection matrix $P(x,t)$ is a pseudo-constant in $t$, i.e. $P(x,t) = P(x,qt)$, we say that the equation $Y(qx) = A(x,t)Y(x)$ has a connection preserving deformation on $t$. This is a discrete analogue of monodromy preserving deformations. Jimbo and Sakai found a q-analog of the sixth Painlevé equation using a connection preserving deformation [6]. Murata constructed Lax forms of the q-Painlevé equations for the type I to V [10].

We can take a limit $t \to 0$ in the Lax form for our holomorphic solutions of q-Painlevé equations around $t = 0$. Since the Lax form goes to q-hypergeometric equations by taking the limit, we can determine the connection matrix exactly. Since our holomorphic solutions exist for generic parameters of the q-Painlevé equations, our solutions are different from the hypergeometric type solutions (particular solutions) shown in [7], which exist only on special parameters of the q-Painlevé equations.

1.1. q-Painlevé equations

We list up three q-Painlevé equations: q-$P_{VI}$, q-$P_V$, q-$P_{III}$. We denote $y = y(t)$, $z = z(t)$, $\bar{y} = y(qt)$, $\bar{z} = z(qt)$. We take $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ as parameters of equations.

\begin{align*}
q-P_{VI}: \quad & \frac{\bar{y}}{a_3a_4} \frac{\bar{y}}{a_3a_4} = \frac{(\bar{z} - b_t)(\bar{z} - b_2t)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \quad \frac{z\bar{z}}{b_3b_4} = \frac{(y - a_1t)(y - a_2t)}{(y - a_3)(y - a_4)} \frac{b_1b_2}{b_3b_4} = \frac{a_1a_2}{a_3a_4}, \\
q-P_V: \quad & \frac{\bar{y}}{a_3a_4} = \frac{(\bar{z} - b_t)(\bar{z} - b_2t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = \frac{(y - a_1t)(y - a_2t)}{a_4(y - a_3)} \frac{b_1b_2}{b_3} = \frac{a_1a_2}{a_3a_4}, \\
q-P_{III}: \quad & \frac{\bar{y}}{a_3a_4} = \frac{(\bar{z} - b_2t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = \frac{y(y - a_1t)}{a_4(y - a_3)}
\end{align*}

Taking a limit $q \to 1$, q-Painlevé equations go to differential Painlevé equations of the given type.

We have few applications of the q-Painlevé equations to mathematical sciences, but our new solutions would be useful in other fields in the future.

1.2. Basic notations

We set a q-shifted factorial

\[ (a_1, \ldots, a_r; q)_n = \prod_{i=1}^{n} (a_i; q)_n, \quad (a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a). \]

A basic hyper geometric series [3] is defined by

\[ \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left[ (-1)^n q^{n(n-1)/2} \right]^{1+s-r} z^n. \]
We use Jacobi’s theta function as follows:

$$\theta_q(x) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} x^n = (q, -x, -q/x; q)_\infty.$$ 

In this notation, \( \theta_q(x) \) satisfies the equations \( x \theta_q(xq) = \theta_q(x) \) and \( \theta(1/x) = \frac{1}{x} \theta(x) \).

We set

$$e_c(x) := \frac{\theta(x)}{\theta(cx)},$$

for \( c \in \mathbb{C}^\times \). We have \( e_c(xq) = ce_c(x) \).

2. \( q \)-linear equations and connections

We review a \( q \)-connection and the \( q \)-Stokes phenomenon. Adams gave a pioneer work on irregular singular points of linear \( q \)-difference equations [1]. Modern study on irregular singular points of linear \( q \)-difference equations are explained in [15].

We consider a \( q \)-difference equation of size \( r \):

$$Y(qx) = A(x)Y(x), \quad A(x) = A_0 + xA_1 + \cdots + x^N A_N. \quad (1)$$

Here \( A_j (j = 0, 1, \ldots, N) \) and \( Y(x) \) are \( r \times r \) matrices.

2.1. Connection matrix

When all of the eigenvalues of \( A_0 \) are not zero, \( x = 0 \) in (1) is called a regular singular point. When all of the eigenvalues of \( A_N \) are not zero, \( x = \infty \) in (1) is called a regular singular point.

In the following, we set the eigenvalues of \( A_0 \) is \( \rho_1, \ldots, \rho_r \) and

$$A_N = \text{diag}(\kappa_1, \ldots, \kappa_r).$$

We assume that \( \rho_j/\rho_k \not\in q\mathbb{Z} \) and \( \kappa_j/\kappa_k \not\in q\mathbb{Z} \) \((j \neq k)\).

Under these assumptions, (1) has local solutions around \( x = 0 \) and \( x = \infty \) in the following forms [2]

$$Y_0(x) = L(x) \text{ diag}(e_{\rho_1}, \ldots, e_{\rho_r}),$$

$$Y_\infty(x) = \theta(x)^{-N} R(x) \text{ diag}(e_{\kappa_1}, \ldots, e_{\kappa_r}).$$

Here

$$L(x) = \sum_{n=0}^{\infty} L_n x^n, \quad R(x) = \sum_{n=0}^{\infty} R_n x^{-n}.$$

We assume that \( R_0 = I_r \) and \( \det L_0 = 1 \). Moreover \( L_0^{-1} A_0 L_0 = \text{diag}(\rho_1, \ldots, \rho_r) \). Such solutions are unique up to the action of diagonal matrices \( D_0, D_\infty \).

**Definition.** We define the connection matrix \( P(x) \) as

$$Y_\infty(x) = Y_0(x) P(x).$$

Matrix elements of \( P(x) \) are elliptic functions, not constants.
2.2. Basic hypergeometric series

Heine’s hypergeometric series \( _2\varphi_1(a, b; c; x) \) satisfies the equation

\[
(c - abqx)u(xq^2) - [c + q - (a + b)qx]u(qx) + q(1 - x)u(x) = 0.
\]

Local solutions around \( x = 0 \) are

\[
u_1 = _2\varphi_1(a, b; c; x), \quad u_2 = e_{q/c}(x)_2\varphi_1(qa/c, qb/c; q^2/c; x).
\]

Local solutions around \( x = \infty \) are

\[
v_1 = \frac{1}{e_{a}(-x)}_2\varphi_1(a, aq/c; aq/b; cq/abx), \quad v_2 = (a \leftrightarrow b).
\]

The connection matrix of (2) is determined by Watson [17]. Since our local solutions are different from Watson’s local solutions, the connection matrix is also different.

**Theorem 1** The connection formula between \( (u_1, u_2) \) around \( x = 0 \) and \( (v_1, v_2) \) around \( x = \infty \) is given by

\[
u_1 = \frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}v_1 + \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}v_2,
\]

\[
u_2 = \frac{(q^2/c,b/a;q)_\infty}{(q^2/c,b/a;q)_\infty}e_{q/c}(-x)_{q/c}(-x)v_1 + \frac{(qa/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}e_{q/c}(x)_{q/c}(x)v_2.
\]

2.3. \( q \)-confluent hypergeometric equation

When some exponents \( \rho_j \) (or \( \kappa_i \)) are zero, \( x = 0 \) (or \( x = \infty \)) is irregular singular. In cases of confluent functions of \( r+1\varphi_r \), we may determine the Stokes coefficients.

The \( q \)-confluent hypergeometric equation is given by

\[
(1 - abqx)u(q^2x) - \{1 - (a + b)qx\} u(qx) - qxu(x) = 0.
\]

Local solutions around \( x = 0 \) are

\[
u_1(x) = 2\varphi_0(a, b; -q, x), \quad \nu_2(x) = \frac{(abx;q)_\infty}{\theta(-qx)}_2\varphi_1\left(\frac{a}{a}, \frac{q}{b}; 0; q, abx\right).
\]

Local solutions around \( x = \infty \) are

\[
v_1(\lambda, x) = \frac{\theta\left(\frac{qax}{x}\right)}{\theta\left(\frac{q^2x}{x}\right)}_2\varphi_1\left(a, 0; \frac{aq}{b}; q, \frac{q}{abx}\right), \quad v_2(\lambda, x) = (a \leftrightarrow b)
\]

For (3), \( x = 0 \) is an irregular singular point and \( u_1 \) is a divergent series. Since \( x = \infty \) is a regular singular point, the power series in \( v_1(\lambda, x) \) and \( v_2(\lambda, x) \) are convergent.

We explain the \( q \)-Borel-Laplace method to obtain a resummation of \( u_1 \) following Zhang [18].

For a power series \( f(x) = \sum_{n \geq 0} a_n \), the \( q \)-Borel transformation is given by

\[
(B_q^+ f)(\xi) := \sum_{n \geq 0} a_n q^{(n-1)} \xi^n.
\]
We fix \( \lambda \in \mathbb{C}^* \). The \( q \)-Laplace transformation of \( \varphi(\xi) \) is given by

\[
\left( \mathcal{L}_{q,\lambda}^+ \varphi \right)(x) := \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q \left( \frac{\lambda q^n}{x} \right)}.
\]

If \( f(x) \) is a convergent series, \( f = (\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}^+_{q}) f \). But we may apply the \( q \)-Borel-Laplace transformation to divergent series. We set \( \tilde{u}_1(\lambda, x) = (\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}^+_{q}) u_1 \). The resummation \( \tilde{u}_1(\lambda, x) \) on is holomorphic on \( \mathbb{C}^* \setminus (-\lambda)q^2 \) and \( u_1(x) \) is an asymptotic series of \( \tilde{u}_1(\lambda, x) \) [18]. The Stokes region is an open dense subset \( (-\lambda)q^2 \) in \( \mathbb{C}^* \). We remark that the series in \( u_2(x) \) is convergent even though \( x = 0 \) is an irregular singular point. The reader may consult [15] and its references on resummation process of divergent \( q \)-series.

In irregular case, the connection matrix depends on the resummation parameter \( \lambda \):

**Theorem 2** The connection formula of (3) is as follows:

\[
\tilde{u}_1(\lambda, x) = \frac{(b; q)_\infty}{(b; a; q)_\infty} \theta(a \lambda) v_1(\lambda, x) + \frac{(a; q)_\infty}{(a; b; q)_\infty} \theta(\lambda) v_2(\lambda, x),
\]

\[
u_2(x) = \frac{(q/a; q)_\infty}{(b/a; q)_\infty} v_1(-1, x) + \frac{(q/b; q)_\infty}{(a/b; q)_\infty} v_2(-1, x).
\]

**Remark.** The first formula is given by Zhang [18], and the second is given by Morita [9].

3. Meromorphic solutions to \( q-P_{V1}, q-P_V \) and \( q-P_{III} \)

For differential Painlevé equations \( P_{V1}, P_V \) and \( P_{III} \), generic solutions are multi-valued around the fixed singular point \( x = 0 \) and have essential singularities. But there exist finite number of single-valued meromorphic solutions around \( x = 0 \). Such solutions are transcendental in general. We have shown that we can determine exactly monodromy or the Stokes data of linearized equations of the Painlevé equations for such meromorphic solutions of \( P_{V1}, P_V \) and \( P_{III} \) around the origin [8].

In this section we list up all of meromorphic solutions \( q-P_{V1}, q-P_V \) and \( q-P_{III} \) around the origin:

**Theorem 3** For generic parameters, all of meromorphic solutions to \( q-P_{V1}, q-P_V \) and \( q-P_{III} \) around the origin are holomorphic.

(1) For generic parameters, there exist four holomorphic solutions to \( q-P_{V1} \) around the origin:

\[
\begin{align*}
I) & \quad y(t) = \frac{a_3 b_3 - a_4 b_4}{b_3 - b_4} + O(t), & z(t) = \frac{a_3 b_3 - a_4 b_4}{a_3 - a_4} + O(t), \\
II) & \quad y(t) = \frac{a_4 b_3 - a_3 b_4}{b_3 - b_4} + O(t), & z(t) = \frac{a_3 b_3 - a_4 b_4}{a_3 - a_4} + O(t), \\
III) & \quad y(t) = \frac{a_1 a_2 (b_1 - b_2)}{a_2 b_1 - a_1 b_2} t + O(t^2), & z(t) = -\frac{b_1 b_2 (a_1 - a_2)}{(a_2 b_1 - a_1 b_2) q} t + O(t^2), \\
IV) & \quad y(t) = \frac{a_1 a_2 (b_1 - b_2)}{a_1 b_1 - a_2 b_2} t + O(t^2), & z(t) = -\frac{b_1 b_2 (a_1 - a_2)}{(a_1 b_1 - a_2 b_2) q} t + O(t^2).
\end{align*}
\]
(2) There exist three holomorphic solutions to \( q \)-\( P_V \) around the origin:

I) \( y(t) = (a_3 - a_4b_3) + O(t), \quad z(t) = \left( b_3 - \frac{a_3}{a_4} \right) + O(t) \),

II) \( y(t) = \frac{a_1a_2(b_1 - b_2)}{a_2b_1 - a_1b_2} t + O(t^2), \quad z(t) = \frac{b_1b_2(a_1 - a_2)}{(a_1b_2 - a_2b_1)q} t + O(t^2) \),

III) \( y(t) = \frac{a_1a_2(b_1 - b_2)}{a_1b_1 - a_2b_2} t + O(t^2), \quad z(t) = \frac{b_1b_2(a_1 - a_2)}{(a_1b_1 - a_2b_2)q} t + O(t^2) \).

(3) There exist two holomorphic solutions to \( q \)-\( P_{III} \) around the origin:

I) \( y(t) = (a_3 - a_4b_3) + O(t), \quad z(t) = \left( b_3 - \frac{a_3}{a_4} \right) + O(t) \),

II) \( y(t) = \frac{a_1a_2a_4b_2^3}{a_3a_4b_2^2 - a_1^2b_3q} t + O(t^2), \quad z(t) = \frac{a_1^2b_2b_3}{-a_3a_4b_2^2 + a_1^2b_3q} t + O(t^2) \).

Proof. 1) By direct calculations, we can easily checked that \( q \)-\( P_J \) (\( J=III, V, VI \)) has formal solutions shown in the Theorem.

2) We can show the formal solutions are convergent by a \( q \)-analogue of the Briot-Bouquet theorem shown by Poincaré [13]:

**Lemma 4 (Poincaré)** For \( f_j(t; y) = f_j(t; y_1, y_2, \ldots, y_n) \) (\( j = 1, 2, \ldots, n \)), we assume that

a) \( f_j \) is holomorphic around \( (t, y) = (0, 0) \) and \( f_j(0; 0) = 0 \)

b) the eigenvalues of matrix

\[
\left( \frac{\partial f_j}{\partial y_k}(0; 0) \right)_{1 \leq j, k \leq n}
\]

are not \( q^n \) (\( n = 1, 2, 3, \ldots \)).

Then \( q \)-difference equation

\[
y_j(qt) = f_j(t; y) \quad (j = 1, 2, \ldots, n) \tag{4}
\]

has a unique convergent solution of the type

\[
y_j = \sum_{k=1}^{\infty} a_{jk} t^k \quad (j = 1, 2, \ldots, n).
\]

We call a system (4) with the properties (a) and (b) ‘\( q \)-Briot-Bouquet type’.

Since we can easily modify \( P_{V1}, P_V \) and \( P_{III} \) to \( q \)-Briot-Bouquet type, the solutions in the theorem are convergent. \( \square \)

**Remark.** In differential cases, \( P_{V1}, P_V \) and \( P_{III} \) have the same number of meromorphic solutions around the origin as in Theorem 3 (2.3,4).
4. Connection Preserving Deformation

A $q$-analogue of the sixth Painlevé equation is obtained as a connection preserving deformation by Jimbo and Sakai [6]. Connection preserving deformations of other types of $q$-analogue of the Painlevé equations are shown by Murata [10].

A connection preserving deformation is a compatibility condition of the following two difference equations:

$$Y(qx,t) = A(x,t)Y(x,t),$$

$$Y(x,qt) = B(x,t)Y(x,t).$$

In the cases of $q$-Painlevé equations, $A(x,t)$ and $B(x,t)$ are rank two matrices and

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2.$$  

Moreover

$$B(x,t) = \frac{x}{(x-a_1t)(x-a_2t)}(xI + B_0(t)), \quad q-P_{VI},$$

$$B(x,t) = \frac{x}{x-a_1t}(xI + B_0(t)), \quad q-P_{VIII}.$$  

We call (5) a linearized equation of the $q$-Painlevé equation.

We take variables $y = y(t)$, $z_i = z_i(t)$ ($i = 1,2$) such that

$$A_{12}(y,t) = 0, \quad A_{11}(y,t) = \kappa_1z_1, \quad A_{22}(y,t) = z_2,$$  

We set a variable $z$ as

$$z_2 = \kappa_1\kappa_2qz(y - a_3).$$

(1) Connection preserving deformation of $q$-$P_{V_1}$

We take a normalization of $A_2$: $A_2 = \text{diag}(\kappa_1, \kappa_2)$. We assume that eigenvalues of $A_0(t)$ are $\rho_1 = \theta_1t, \rho_2 = \theta_2t$, and $\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)(x-a_4)$. We have

$$\kappa_1\kappa_2a_1a_2a_3a_4 = \theta_1\theta_2.$$  

$$A(x,t) = \frac{\kappa_1((x-y)(x-a\alpha) + z_1)}{\kappa_1w^{-1}(\gamma x + \delta)} \quad \frac{\kappa_2w(x-y)}{\kappa_2((x-y)(x-\beta) + z_2)}$$

Here

$$\alpha = \frac{[y^{-1}(\rho_1 + \rho_2)t - \kappa_1z_1 - \kappa_2z_2) - \kappa_2((a_1 + a_2)t + a_3 + a_4 - 2y)]}{\kappa_1 - \kappa_2},$$

$$\beta = \frac{[-y^{-1}(\rho_1 + \rho_2)t - \kappa_1z_1 - \kappa_2z_2) + \kappa_1((a_1 + a_2)t + a_3 + a_4 - 2y)]}{\kappa_1 - \kappa_2},$$

$$\gamma = z_1 + z_2 + (y + \alpha)(y + \beta) + (\alpha + \beta)y - a_1a_2t^2 - (a_1 + a_2)a_3 + a_4 - \gamma a_4,$$

$$\delta = y^{-1}(a_1a_2a_3a_4t^2 - (\alpha y + z_1)(\beta y + z_2)),$$

$$b_1 = \frac{a_1a_2}{\rho_1}, \quad b_2 = \frac{a_1a_2}{\rho_2}, \quad b_3 = \frac{1}{\kappa_1q}, \quad b_4 = \frac{1}{\kappa_2}.$$  

(2) Connection preserving deformation of $q$-$P_V$

We take a normalization of $A_2$: $A_2 = \text{diag}(\kappa_1, 0)$. We assume that eigenvalues of $A_0(t)$ are $\rho_1 = \theta_1t, \rho_2 = \theta_2t$, and that $\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)$. We have

$$-\kappa_1\kappa_2a_1a_2a_3 = \theta_1\theta_2.$$
\[ z_1 z_2 = \kappa_2(y - a_1 t)(y - a_2 t)(y - a_3). \]  
\( \text{(9)} \)

Therefore \( z_1 \) is represented by \( z \):

\[ z_1 = \frac{(y - a_1 t)(y - a_2 t)}{\kappa_1 q z} \]

\( \text{(10)} \)

\[ A(x, t) = \begin{pmatrix} \kappa_1((x - y)(x - \alpha) + z_1) & w(x - y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2(x - y) + z_2 \end{pmatrix}. \]

\( \text{(11)} \)

Here

\[ \alpha = \frac{1}{\kappa_1} \left[ y^{-1}((\theta_1 + \theta_2)t - \kappa_1 z_1 - z_2) + \kappa_2 \right], \]

\[ \gamma = z_2 - \kappa_2((2y + \alpha) - (a_1 + a_2)t - a_3), \]

\[ \delta = y^{-1}(-\kappa_2 a_1 a_2 a_3 t^2 - (ay + z_1)(-\kappa_2 y + z_2)). \]

(3) Connection preserving deformation of \( q\)-\( P_{\text{III}} \)

We take a normalization of \( A_2 \): \( A_2 = \text{diag}(\kappa_1, 0) \), We assume that eigenvalues of \( A_0(t) \) are \( \theta_1 t, 0 \) and that \( \det A(x, t) = \kappa_1 \kappa_2 x(x - a_1 t) / \kappa_3 (x - a_3) \). We have

\[ -\kappa_1 \kappa_2 a_1 a_2 a_3 = \theta_1 \theta_2. \]

\( \text{(12)} \)

Therefore \( z_1 \) is represented by \( z \):

\[ z_1 = \frac{y(y - a_1 t)}{\kappa_1 q z}. \]

\( \text{(13)} \)

\[ A(x, t) = \begin{pmatrix} \kappa_1((x - y)(x - \alpha) + z_1) & w(x - y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2(x - y) + z_2 \end{pmatrix}. \]

\( \text{(14)} \)

Here

\[ \alpha = \frac{1}{\kappa_1} \left[ y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2 \right], \]

\[ \gamma = z_2 - \kappa_2((2y + \alpha) - (a_1 + a_2)t - a_3), \]

\[ \delta = -y^{-1}(ay + z_1)(-\kappa_2 y + z_2). \]

For detail, see [6] and [10].

5. Connection formula for holomorphic solutions around the origin

We show that connection matrix of linearized equations of \( q\)-\( P_{\text{VI}} \), \( q\)-\( P_{\text{V}} \) and \( q\)-\( P_{\text{III}} \) by taking the limit \( t \to 0 \).

**Theorem 5** For solutions (I-IV) to \( q\)-\( P_{\text{VI}} \) and the solution (I) of \( q\)-\( P_{\text{V}} \), \( A(x, 0) \) reduces to basic hypergeometric function \( {}_2 F_1(t) \).

For solutions (I, II) of \( q\)-\( P_{\text{III}} \) and for solutions (II),(III) of \( q\)-\( P_{\text{V}} \), \( A(x, 0) \) reduces to basic hypergeometric function \( {}_1 F_1(t) \).
In this sense, we can determine the connection problem of linearized $q$-Painlevé equations because the connection formula of $2\varphi_1(t)$ is shown Watson and connection formula of $1\varphi_1(t)$ is shown by Zhang and Morita as explained in section two. Theorem 5 is just a $q$-analogue of results in [8].

The case of $q$-$P_{V_1}$ has been shown in [12].
Since we obtain other cases by similar ways, we explain the cases of $q$-$P_V$ (I) and (II)

[I] The case of the solution (I) to $q$-$P_V$:

For the linearized equation

$$Y(qx) = A(x,t)Y(x),$$

we set $x = \xi t$. Then

$$Y(q\xi t) = (A_0 + A_1\xi t + t^2A_2\xi^2)Y(\xi t).$$

We set $Y(\xi t) = \xi^q t Z(\xi)$. (Here $l_q t = \log t/\log q$)

$$Z(q\xi) = (A_0/t + A_1\xi + tA_2\xi^2)Z(\xi)$$

Taking the limit $t \to 0$, we have (Eigenvalues of $A_0(t)$ are $\theta_1 t, \theta_2 t$)

$$Z(q\xi) = (A_0/t + A_1\xi)Z(\xi)$$

$$A_0/t = \begin{pmatrix}
\frac{a_1a_3(b_1+b_2)}{b_1b_2} & a_3 \left( \frac{a_2b_1b_2}{a_1q} - 1 \right) & 0 \\
\frac{a_3b_1b_2(a_1a_2q-a_1^2b_1b_2)w'(0)}{a_3b_1b_2(a_1a_2q-a_1^2b_1b_2)w'(0)} & 0 & a_4
\end{pmatrix}$$

$$A_1 = \begin{pmatrix}
\frac{-a_1a_2}{a_3b_1b_2} & 0 & 0 \\
\frac{a_3b_1b_2(a_1+2a_1b_1-a_1b_2)}{a_3b_1b_2(1-b_3q)w'(0)} & 0 & -a_4
\end{pmatrix}$$

Therefore it reduces to hypergeometric equations $2\varphi_1\left(\frac{a_2b_2}{a_2} \cdot \frac{a_1q}{a_1b_1} \cdot \frac{b_2q}{b_1} \cdot \frac{\xi}{a_1}\right)$.

[II] The case of the solution (II) to $q$-$P_V$:

We can take a limit $t \to 0$ directly:

$$A(x,0) = \tilde{A}_1 x + \tilde{A}_2 x^2.$$ 

Here is

$$\tilde{A}_1 = \begin{pmatrix}
\frac{a_3b_1b_2+a_1a_3b_1b_2-a_2b_1-a_1b_2}{b_1b_2(1-b_3q)} & \frac{w(0)}{a_3b_1b_2(1-b_3q)w'(0)} & a_3b_1b_2f_2(1-b_3q)w(0) \\
\frac{a_3b_1b_2+a_1a_3b_1b_2-a_2b_1-a_1b_2}{a_3b_1b_2(1-b_3q)w'(0)} & 0 & a_4
\end{pmatrix}$$

$$\tilde{A}_2 = \text{diag}(\kappa_1,0)$$

Solution to $Y(qx) = A(x,0)Y(x)$ around $x = 0$ are given by

$$Y^{(0)}(x) = \frac{q^{u(u-1)/2}}{(x/a_3)^{\infty}} \begin{pmatrix}
y_{11}^{(0)} \\
y_{12}^{(0)} \\
y_{21}^{(0)} \\
y_{22}^{(0)}
\end{pmatrix} x D_0$$
Here each elements are represented by $\varphi_1$ as follows:

\[ y_{11}^{(0)} = C_{11} \cdot 1^{\varphi_1} \left( \frac{a_1 b_2}{a_2 b_3}, \frac{a_1 b_2}{a_2 b_3}, \frac{x}{a_4} \right), \quad y_{12}^{(0)} = C_{12} \cdot 1^{\varphi_1} \left( \frac{a_4 b_1}{a_1 b_3}, \frac{a_2 b_1}{a_1 b_3}, \frac{x}{a_4} \right), \]

\[ y_{21}^{(0)} = C_{21} \cdot 1^{\varphi_1} \left( \frac{a_1 b_1}{a_3 b_1}, \frac{a_1 b_2}{a_2 b_1}, \frac{x}{a_4} \right), \quad y_{22}^{(0)} = C_{22} \cdot 1^{\varphi_1} \left( \frac{a_2 b_4}{a_3 b_2}, \frac{a_2 b_1}{a_1 b_2}, \frac{x}{a_4} \right), \]

where

\[ C_{11} = b_2(a_1 a_2 - a_3 a_4 b_1 b_2), \quad C_{12} = b_1(a_1 a_2^2 - a_3 a_4 b_1 b_2)w(0), \]
\[ C_{21} = a_1(a_2 - a_3 b_2)(a_2 - a_4 b_2)/w(0), \quad C_{22} = a_2(a_1 - a_3 b_1)(a_1 - a_4 b_1). \]

which reduced to the $q$-confluent hypergeometric equations.

6. Summary and future problems

As the same as differential Painlevé equations, the origin $t = 0$ of $q$-analogue of $q$-$P_{VI}$, $q$-$P_{V}$, $q$-$P_{III}$ is a singular point of $q$-Briot-Bouquet type. For such $q$-Painlevé equations, there exist a finite number of holomorphic solutions around singular points. We can determine connection/Stokes coefficients of linear $q$-difference equations for holomorphic solutions of $q$-Painlevé equations at the fixed singular point.

We will study other $q$-Painlevé equations. There are no results on higher order cases. We should study nonlinear $q$-Stokes phenomenon around singular points of non $q$-Briot-Bouquet type. Such singularity appears at the infinity of $q$-$P_3$ except for $J=VI$.

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