Detecting an Odd Restless Markov Arm with a Trembling Hand

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Abstract

In this paper, we consider a multi-armed bandit in which each arm is a Markov process evolving on a finite state space. The state space is common across the arms, and the arms are independent of each other. The transition probability matrix of one of the arms (the odd arm) is different from the common transition probability matrix of all the other arms. A decision maker, who knows these transition probability matrices, wishes to identify the odd arm as quickly as possible, while keeping the probability of decision error small. To do so, the decision maker collects observations from the arms by pulling the arms in a sequential manner, one at each discrete time instant. However, the decision maker has a trembling hand, and the arm that is actually pulled at any given time differs, with a small probability, from the one he intended to pull. The observation at any given time is the arm that is actually pulled and its current state. The Markov processes of the unobserved arms continue to evolve. This makes the arms restless.

For the above setting, we derive the first known asymptotic lower bound on the expected stopping time, where the asymptotics is of vanishing error probability. The continued evolution of each arm adds a new dimension to the problem, leading to a family of Markov decision problems (MDPs) on a countable state space. We then stitch together certain parameterised solutions to these MDPs and obtain a sequence of strategies whose expected stopping times come arbitrarily close to the lower bound in the regime of vanishing error probability. Prior works dealt with independent and identically distributed (across time) arms and rested Markov arms, whereas our work deals with restless Markov arms.

Index Terms

Multi-armed bandits, restless bandits, odd arm identification, Markov decision process, trembling hand.

I. INTRODUCTION

The problem of odd arm identification deals with identifying an anomalous (or odd) arm in a multi-armed bandit as quickly as possible, while keeping the probability of decision error small. Here, the term anomaly simply means that the law, say $\psi_1$, of one of the arms is different from the common law, say $\psi_2$, of each of the other arms. We assume that the arms are independent of each other. A decision maker, who may or may not have prior knowledge of $\psi_1$ and $\psi_2$, and whose goal it is to identify the index of the odd arm, samples the arms in a sequential manner, one at a time. The process of sampling the arms continues until the decision maker is sufficiently confident of which arm is odd, at which time he stops further sampling and declares the index of the odd arm. In forming his decision about the odd arm, it is important for the decision maker to ensure that his error probability is low (below a pre-specified threshold). It is natural to expect that smaller the pre-specified error probability threshold, longer the decision maker will have to wait before declaring the odd arm location. The main objective

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of this paper is to identify the asymptotic growth rate of the decision time as a function of the error probability, as the error probability goes to zero.

Prior works on odd arm identification consider the cases when either each arm yields independent and identically distributed (iid) observations [1]–[3], or when each arm yields Markov observations from a common finite state space [4]. When each arm yields iid observations, \( \psi_1 \) refers to the law of a random observation coming from the odd arm, while \( \psi_2 \) refers to the law of a random observation coming from any of the non-odd arms. When each arm yields Markov observations, \( \psi_1 \) refers to the transition law of the Markov process of the odd arm, while \( \psi_2 \) refers to the transition law of the Markov process of each of the non-odd arms. When the state space is discrete, the transition laws \( \psi_1 \) and \( \psi_2 \) may be specified equivalently by the respective transition probability matrices, say \( P_1 \) and \( P_2 \), where \( P_1 \neq P_2 \). We use ‘observations’ in place of the commonly used ‘rewards’ because our focus is on early identification of the odd arm in contrast to reward maximisation or regret minimisation.

An important feature of the setting in [4] is that the Markov process of any given arm evolves by one time step only when the arm is selected, and does not evolve otherwise; this is known as the setting of rested arms. In this paper, we partially extend the results of [4] to the more difficult restless arms setting in which the Markov process of each arm continues to evolve whether or not the arm is selected. The continued evolution of the Markov process of each arm makes it necessary for the decision maker to keep a record of (a) the time elapsed since each arm was previously selected (called the arm’s delay), and (b) the state of each arm as observed at its previous selection time (called the last observed state of the arm). Notice that the notion of arm delays is superfluous when the arms are rested as in [4] since the unobserved arms remain frozen at their previously observed states. It is also superfluous in the special case of the restless setting when each arm yields iid observations (as in [1]–[3]) because the last observed state of each arm is independent of the arm’s current state. Therefore, the notions of arm delays and last observed states are striking features of the setting of general restless Markov arms.

For the rest of this paper, we assume that the transition matrices \( P_1 \) and \( P_2 \) of the odd arm and the non-odd arm Markov processes are known to the decision maker. All the essential conceptual difficulties related to restless arms are retained despite this simplification. The new tools needed to overcome the difficulties are clearly brought out (Section I-C). The case when \( P_1 \) and \( P_2 \) are unknown is beyond the scope of this paper and is currently under study.

A. Motivation and The Notion of a Trembling Hand

Our motivation to study the restless odd Markov arm problem comes from the desire to extend, to more general settings, the decision theoretic formulation of a certain visual search experiment conducted by Vaidhiyan et al. [1]. In this experiment, human subjects were shown a number of images at once, with one oddball image in a sea of distracter images. The goal of the experiment was to understand the relationship between (a) the average time taken by the human subject to identify the oddball image, and (b) the dissimilarity between the oddball and distracter images as perceived by the human subject. The images used in the above experiment were static images. Vaidhiyan et al. also conducted experiments with dynamic drifting-dots images (movies), similar to the ones conducted by Krueger et al. [5], in which the dots in each movie location executed Brownian motions with identical drifts. Further, the drifts were identical in all the distracter movie locations, and were different from the drift in the oddball movie location. In this context, what are optimal strategies to identify the oddball movie? A systematic analysis of this question, along the lines of [1], requires an understanding of the restless odd Markov arm problem which forms the main subject of this paper.

It is often the case in such visual search experiments that though the subject (or decision maker) intends to focus his attention at a certain location, the actual focus location differs from the intended focus location with a small probability. We model this
in our multi-armed bandit setting as a *trembling hand* for the decision maker: with probability $1 - \eta$, the decision maker pulls the intended arm, but with probability $\eta$, the decision maker pulls a uniformly randomly chosen arm (we use the phrases ‘arm pulls’ and ‘arm selections’ interchangeably). Up to Section VI we assume that $\eta > 0$, as is often the case in visual search experiments such as those described above. The case when $\eta = 0$ is dealt with separately in Section VII.

**B. Prior Works on Restless Markov Arms**

The topic of restless Markov arms has been studied extensively in the literature in the context of reward maximisation (or equivalently, regret minimisation). In such works, each arm is assumed to yield, upon being sampled, an immediate ‘reward’ based on the arm’s current state, and regret is defined as the difference between the expected sum of rewards obtained under an arm selection scheme in relation to that obtained by a scheme that knows which arm yields the highest reward on the average. Whittle [6] refined and extended the results of Gittins [7] on the optimality, in the setting of rested arms, of a certain index-based policy. Whittle [6] demonstrated that Gittins’s policy in [7] is not necessarily optimal in the context of restless arms, introduced a new index (the Whittle’s index) which could be computed if each arm satisfied an indexability condition, demonstrated that the new index coincides with Gittins’s index in the rested setting. Yet, as Whittle showed, the new index-based policy is not necessarily optimal for the setting of restless arms.

Whittle’s results require the Markov transition laws of each of the arms to be known beforehand. Extensions of Whittle’s results to the case when the laws are not known beforehand appear in Liu et al. [8]. Ortner et al. [9] provide a policy that, when the transition laws of the arms are unknown, gives a regret of the order $O(\sqrt{T})$ after $T$ time steps in relation to a policy that knows the Markov transition laws of all the arms. Works [9], [10] deal with general state spaces (i.e., not necessarily finite or countable) and address the associated technical challenges.

In contrast to all the works mentioned above, this paper focuses on the *stopping* problem of identifying the index of the odd arm as quickly as possible. It is worth noting here, as also noted in [11], that policies which are optimal for the problem of minimising regret may not necessarily be optimal in the context of this paper.

For applications of the restless odd Markov arm problem, see [4]. For a related problem of best arm identification instead of odd arm identification, see [12], [13].

**C. A Brief Overview of the Results and Our Contributions**

We now provide a brief overview of our results and highlight our contributions.

1) We show that given a pre-specified error probability threshold $\epsilon > 0$, the expected time taken by the decision maker to identify the index of the odd arm with probability of error at most $\epsilon$ grows as $\Theta(\log(1/\epsilon))$. We give a precise characterisation of the best (smallest) constant multiplying $\log(1/\epsilon)$, which we call $R^*(P_1, P_2)$, in terms of the Markov transition probability matrices $P_1$ and $P_2$. This is the first known characterisation of this constant for the setting of restless Markov arms. See Section IV for an exact mathematical expression. We prove this by first showing a lower bound in Section IV and then a matching asymptotic upper bound in Section V.

2) In order to derive the constant $R^*(P_1, P_2)$, we use the fact that the arm delays and the last observed states form a *controlled Markov process*, with the arm selections playing the role of *controls*. This approach of ours takes into account the delays and the last observed states of *all* the arms jointly. In contrast, the approaches of [1]–[4] suggest dealing with the delays and the last observed states of each of the arms separately, which we view as a ‘local’ perspective of the arm delays and the last observed states. In Section IV-A we show that this local perspective of arm delays and last observed
states leads to an infinite dimensional, constrained, linear programming problem (LPP). The drawback of this approach is that it is not easy to find the tightest set of constraints for the LPP. As a consequence, the constant multiplier obtained as the solution to the LPP may not necessarily be the best (smallest). On the other hand, our ‘lift’ approach, which considers the delays and the last observed states of all the arms jointly, provides the necessary perspective to arrive at the best constant multiplier \( R^*(P_1, P_2) \).

3) Our lift approach, which takes into account the delays and the last observed states of all the arms, leads us to a family of Markov decision problems (MDPs) on a countable state space. While the central objective in traditional MDPs is to design policies for maximising a reward of some sort, our objective is to design policies for identifying the odd arm as quickly as possible. Towards this, we stitch together certain parameterised solutions to the MDPs and obtain a sequence of strategies whose growth rates (of expected time to decision as a function of error probability) come arbitrarily close to \( R^*(P_1, P_2) \) in the limit of vanishing error probabilities. The details may be found in Section \[V\].

4) A key ergodicity property: We show that under a stationary arm selection policy, the controlled Markov process of arm delays and last observed states is indeed a Markov process. Additionally, we show that this Markov process is ergodic when the trembling hand parameter \( \eta > 0 \) (see Lemma \[I\]). It is this ergodicity property, together with the strict positivity of the trembling hand parameter \( \eta \), that plays a crucial role in our analysis of the lower and the upper bounds. The case of no trembling hand \((\eta = 0)\) demands a careful examination since, in this case, the aforementioned ergodicity property is not readily available. See point 6 below.

5) We show that for every policy of the decision maker, stationary or otherwise, what enters into the analyses of the lower and the upper bounds is, for each possible value of arm delays \( d \), last observed states \( i \) and arm \( a \), the long-term fraction of times the aforementioned controlled Markov process visits the state \((d, i)\) and arm \( a \) is selected subsequently at each such time. This fact, together with [15, Theorem 8.8.2], enables us to restrict attention only to stationary arm selection policies in arriving at the best constant multiplier \( R^*(P_1, P_2) \). See Section \[IV\] for more details.

In spite of the above simplification, the computability of \( R^*(P_1, P_2) \) remains an issue since it involves a search over all the space of all stationary policies. One must resort to \( Q\)-learning or other such simulation strategies to find optimal stationary policies and subsequently evaluate \( R^*(P_1, P_2) \).

6) The case of no trembling hand: The trembling hand model (with \( \eta > 0 \)) may be viewed as a regularisation that gives stability of the aforementioned controlled Markov process for free. If \( \eta = 0 \), one could deliberately add some regularisation parameterised by \( \eta \), re-label the constant \( R^*(P_1, P_2) \) as \( R^*_\eta(P_1, P_2) \) for each \( \eta > 0 \), and analyse the limiting behaviour of \( R^*_\eta(P_1, P_2) \) as \( \eta \downarrow 0 \). We show that in this case, the upper bound is governed by \( \lim_{\eta \downarrow 0} R^*_\eta(P_1, P_2) \), while the lower bound is governed by \( R^*_0(P_1, P_2) \) (obtained by simply plugging \( \eta = 0 \) in the expression for \( R^*_\eta(P_1, P_2) \)). So, the question then is, do these lower and the upper bounds match? The answer, in general, is no, as we demonstrate in Section \[VII\].

7) However, in the iid and rested Markov settings of the prior works \([1]–[4]\), the upper and the lower bounds match. Thus, the upper and the lower bounds match when either (a) \( \eta > 0 \), or (b) \( \eta = 0 \) and the observations come from either iid or rested Markov arms.

The rest of this paper is organised as follows. In Section \[II\] we set up the notations that we use throughout the paper. In Section \[III\] we provide some preliminaries on MDPs. In Section \[IV\] we present the lower bound on the expected time to identify the odd arm as a function of the error probability for the setting of restless Markov arms. In the same section, we also show that by following the conventional approaches available in the prior works for deriving the lower bound, we arrive
at an infinite-dimensional linear programming problem with countably infinitely many constraints. In Section [V] we present a sequence of strategies whose expected times to identify the odd arm approach that of the lower bound in the limit of vanishing error probabilities, following which we state the main result of this paper in Section [VI]. We discuss the case of absence of trembling hand for the decision maker in Section [VII]. We present the proofs of all the results in Appendices A-E. We provide concluding remarks in Section [VIII].

II. Notations and Problem Formulation

We consider a multi-armed bandit with $K \geq 3$ arms, and define $A := \{1, \ldots, K\}$ to be the set of arms. We associate with each arm an ergodic and discrete time Markov process on a finite state space $S$. Further, we assume that the Markov process of any given arm is independent of those of the other arms. The Markovian evolution of states on one of the arms (known as the odd arm) is governed by a transition probability matrix $P_1$, and that the evolution of states on each of the non-odd arms is governed by $P_2$, where $P_2 \neq P_1$. We denote by $\mu_i$ the unique stationary distribution of $P_i$, $i = 1, 2$.

For any integer $d \geq 1$ and a transition probability matrix $P$ on $S$, let $P^d$ denote the transition probability matrix obtained by multiplying $P$ with itself $d$ times. For $i, j \in S$ and $d \geq 1$, we write $P_1^d(i|j)$ and $P_2^d(i|j)$ to denote the $(i, j)$th element of the matrices $P_1^d$ and $P_2^d$ respectively (the case $d = 1$ corresponds to $P_1$ and $P_2$ respectively). We assume that for all $i, j \in S$, (a) $P_1(j|i) > 0$ if and only if $P_2(j|i) > 0$, and (b) $\mu_1(i) > 0$ if and only if $\mu_2(i) > 0$. This assumption ensures that the decision maker cannot infer whether or not a given arm is the odd arm merely by observing certain specific state(s) on the arm. For $h \in A$, we denote by $\mathcal{H}_h$ the hypothesis that $h$ is the odd arm location.

We assume that $P_1$ and $P_2$ are known to a decision maker, whose goal it is to identify the index of the odd arm as quickly as possible, subject to an upper bound on the probability of error. In order to do so, the decision maker devises a sequential arm selection strategy in which, at each discrete time instant $t \in \{0, 1, \ldots\}$, the decision maker first identifies an arm to pull; call this $B_t$. The decision maker however has a trembling hand and, as a consequence, the intended arm $B_t$ gets pulled with probability $1 - \eta$ and a uniformly random arm gets pulled with probability $\eta$. The parameter $\eta$, which is fixed and strictly positive, governs the error in translating the decision maker’s intention into an action. Write $A_t$ for the arm that is actually pulled. The decision maker observes $A_t$, therefore knows whether or not his hand made an error in pulling the intended arm. Further, the decision maker observes the state of arm $A_t$, denoted by $\bar{X}_t$. The unobserved arms continue to undergo state evolution, making the arms restless. Thus, for each $t \geq 0$, $B_t, A_t$ and $\bar{X}_t$ denote respectively the intended arm, the selected arm, and the observed state of the selected arm at time $t$. We use the shorthand notation $(B^t, A^t, \bar{X}^t)$ to denote the collection $(B_0, A_0, \bar{X}_0, \ldots, B_t, A_t, \bar{X}_t)$.

A. Policy

A policy prescribes one of the following two actions at each time $t$: Based on the history $(B^{t-1}, A^{t-1}, \bar{X}^{t-1})$,

- choose to pull arm $B_t$ according to a deterministic or a randomised rule, or
- stop and declare the index of the odd arm.

We use $\pi$ to denote a generic policy, and let $\tau(\pi)$ denote the stopping time of policy $\pi$, where throughout this paper, all stopping times are defined with respect to the filtration $\mathcal{F}_t := \sigma(B^{t-1}, A^{t-1}, \bar{X}^{t-1})$, $t \geq 1$ and $\mathcal{F}_0 := \{\Omega, \emptyset\}$. Let $\theta(\pi)$ denote the index of the odd arm declared by the policy $\pi$ at stoppage.
Let \( P^\pi_h(\cdot) \) and \( E^\pi_h[\cdot] \) denote probabilities and expectations computed under policy \( \pi \). For ease of notation, we drop the superscript \( \pi \), and ask the reader to bear the dependence on \( \pi \) in mind. Given a target probability of error \( \epsilon > 0 \), we define \( \Pi(\epsilon) \) as the set

\[
\Pi(\epsilon) := \{ \pi : P^\pi_h(\theta(\pi) \neq h) \leq \epsilon \text{ for all } h \in \mathcal{A} \}
\]

of all policies whose probability of error at stoppage is below \( \epsilon \) for all possible odd arm locations (since a policy does not know the true odd arm location, it has to work for all possible odd arm locations). We anticipate from similar results in the prior works that

\[
\inf_{\pi \in \Pi(\epsilon)} E_h[\tau(\pi)] = \Theta(\log(1/\epsilon)).
\]

Our interest is in characterising the constant factor multiplying \( \log(1/\epsilon) \). For simplicity, we assume\(^1\) that, for each \( \epsilon > 0 \), all policies in \( \Pi(\epsilon) \) select each of the \( K \) arms in the first \( K \) instants \( t = 0, \ldots, K - 1 \). In particular, arm 1 is selected at time \( t = 0 \), arm 2 at time \( t = 1 \) and so on until arm \( K \) at time \( t = K - 1 \). This does not affect the asymptotic analysis as \( \epsilon \downarrow 0 \).

### B. Delays and Last Observed States

Recall that at each time \( t = \{0, 1, \ldots\} \), the decision maker observes only one of the arms, while the unobserved arms continue to undergo state evolution. Therefore, the probability of the observation \( X_t \) on the selected arm \( A_t \) is a function of (a) the time elapsed since the previous time instant of selection of arm \( A_t \) (called the delay of arm \( A_t \)), and (b) the state of arm \( A_t \) at its previous selection time instant (called the last observed state of arm \( A_t \)). Notice that when the arms are rested, the notion of arm delays is absent since each arm remains frozen at its previously observed state until its next selection time instant. Also, the notion of arm delays is redundant in the setting of iid observations since, in this special case, the current state of the arm selected is independent of its state at its previous selection. Thus, the notion of arm delays is a key distinguishing feature of the setting of restless arms.

We now define a new and more convenient notion of a state, based on the delays and the last observed states of the arms. As we demonstrate below, this new notion of state results in a Markov decision problem that is easier to comprehend.

For \( t \geq K \), we denote by \( d_a(t) \) and \( i_a(t) \) respectively the delay and the last observed state of arm \( a \) at time \( t \). Write \( \bar{d}(t) := (d_1(t), \ldots, d_K(t)) \) and \( \bar{i}(t) := (i_1(t), \ldots, i_K(t)) \) for the delays and the last observed states, respectively, of the arms at time \( t \). Note that arm delays and last observed states are defined only for \( t \geq K \) since these quantities are well-defined only when at least one observation is available from each arm. We set \( \bar{d}(K) = (K, K - 1, \ldots, 1) \), and follow the convention that \( d_a(t) \geq 1 \) for all \( t \geq K \), and that \( d_a(t) = 1 \) if and only if arm \( a \) is selected at time \( t - 1 \).

We follow the rule below for updating the arm delays and last observed states: if \( A_t = a' \), then

\[
\begin{align*}
d_{a}(t+1) &= \begin{cases} 
d_{a}(t) + 1, & a \neq a', \\
i_a(t) & a = a', \\
1, & \end{cases} \\
i_{a}(t+1) &= \begin{cases} 
i_{a}(t), & a \neq a', \\
\bar{X}_t, & a = a', \\
\end{cases}
\end{align*}
\]

where \( \bar{X}_t \) is the state of the arm \( A_t = a' \) at time \( t \).

One thus has the sequence of intended arm pulls, actual arm pulls, observations, and states as follows. At each \( t \geq K \), based on \( (\bar{d}(t), \bar{i}(t)) \), choose to pull \( B_t \); due to the trembling hand, observe that \( A_t \) is pulled; see the state \( \bar{X}_t \) of arm \( A_t \); then form

\(^1\)Let us note here that this may not always be the case, and the intended arm selections may differ from the actual arms selected. If this is the case, the decision maker may exercise arm selections uniformly at random until he observes that each arm is selected at least once. By virtue of the fact that the trembling hand parameter \( \eta > 0 \), the probability of selecting any arm is strictly positive at each time instant. As a result, it can be shown that the above exercise of pulling arms randomly till each arm is selected will only take finite time almost surely.
\((d(t + 1), \bar{z}(t + 1))\). This repeats until stoppage, at which time we have the declaration \(\theta(\pi)\) (under policy \(\pi\)) as the candidate odd arm.

**C. Controlled Markov Process and the Resulting Markov Decision Problem**

From the update rule in (2), it is clear that the process \(\{(d(t), \bar{z}(t)) : t \geq K\}\) takes values in a subset \(S\) of the countable set \(\mathbb{N}^K \times S^K\), where \(\mathbb{N} = \{1, 2, \ldots\}\) denotes the set of natural numbers. The subset \(S\) is formed based on the constraint that at any time \(t \geq K\), exactly one of the components of \(d(t)\) is equal to 1, and all the other components are strictly greater than 1. Note that for all \((d, \bar{z}) \in S\) and \(t \geq K\),

\[
P(d(t + 1) = d, \bar{z}(t + 1) = \bar{z} \mid (d(s), \bar{z}(s)), B_s, K \leq s \leq t) = P(d(t + 1) = d, \bar{z}(t + 1) = \bar{z} \mid (d(t), \bar{z}(t)), B_t).
\]

On account of (3) being satisfied, we say that the evolution of the process \(\{(d(t), \bar{z}(t)) : t \geq K\}\) is controlled by the sequence \(\{B_t\}_{t \geq 0}\) of intended arm selections under policy \(\pi\). Alternatively, we say that \(\{(d(t), \bar{z}(t)) : t \geq K\}\) is a controlled Markov process, with \(\{B_t\}_{t \geq 0}\) as the sequence of controls; the terminology used here follows that of Borkar [14]. Thus, we are in a Markov decision problem (MDP) setting. We now make precise the state space, the action space, the transition probabilities and our objective.

The state space of the MDP is \(S\), with the state at time \(t\) denoted \((d(t), \bar{z}(t))\). The action space of the MDP is \(A\), with action \(B_t\) at time \(t\) possibly depending on the previous actions \(B^{t-1}\) and the previous states \(\{(d(s), \bar{z}(s)), K \leq s \leq t\}\). (It is easy to see that this is equivalent to taking an action based on \((B^{t-1}, A^{t-1}, \bar{X}^{t-1})\).) The transition probabilities for the MDP are given by

1) the trembling hand rule

\[
P(A_t = a | B_t) = \frac{\eta}{K} + (1 - \eta) \mathbb{I}_{\{B_t = a\}}, \quad \forall a \in A,
\]

2) the law associated with arm \(A_1\), and

3) the update rule (2).

In (4), \(I\) denotes the indicator function. In order to write the transition probabilities of the MDP precisely, let us introduce some notations. Given \(h, a \in A\), let \(P^a_h\) denote the transition probability matrix of the Markov process of arm \(a\) under the hypothesis \(H_h\). That is,

\[
P^a_h = \begin{cases} 
P_1, & a = h, \\
0, & a \neq h.
\end{cases}
\]

Furthermore, for any integer \(d \geq 1\), let \((P^a_h)^d\) denote the transition probability matrix obtained by multiplying \(P^a_h\) with itself \(d\) times. Then, given any \((d, \bar{z}), (d', \bar{z}') \in S\) and \(b \in A\), the transition probabilities for the MDP are given by

\[
P(d(t + 1) = d', \bar{z}(t + 1) = \bar{z}' \mid (d(t) = d, \bar{z}(t) = \bar{z}, B_t = b)) \]

\[
= \begin{cases} 
\left(\frac{\eta}{K} + (1 - \eta) \mathbb{I}_{\{b\}}\right) (P^b_h)^{d(t)}(\bar{z}_b' | \bar{z}_b(t)), & \text{if } d'_b = 1 \text{ and } d'_a = d_a(t) + 1 \text{ for all } a \neq b, \\
\bar{z}_a' = i_a(t) \text{ for all } a \neq b, & \text{otherwise},
\end{cases}
\]

where \(d'_b\) and \(\bar{z}'_a\) in (6) denote the component corresponding to arm \(b\) in \(d'\) and \(\bar{z}'\) respectively. Note that the transition probabilities defined in (6) are stationary and independent of time. Also, we have

\[
P(d(t + 1) = d', \bar{z}(t + 1) = \bar{z}' \mid (d(t) = d, \bar{z}(t) = \bar{z}, A_t = b))
\]
\[
\begin{align*}
P_B^\eta(d(t), i_b(t)) = \\
\begin{cases}
(P^\eta B^t)_{i_b(t)}(i'_b | i_b(t)), & \text{if } d'_b = 1 \text{ and } d'_a = d_a(t) + 1 \text{ for all } a \neq b, \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\] 

The left hand sides of (6) and (7) differ in that \( B_t \) in (6) is replaced by \( A_t \) in (7). We shall write \( Q(d', i' | d, i, b) \) to denote the quantity in (7).

Our objective, however, is nonstandard in the context of MDPs, and more in line with what information theorists study. We are interested in determining, for each hypothesis \( \mathcal{H}_b \), the following:

\[
\lim_{\epsilon \downarrow 0} \inf_{\pi \in \Pi(\epsilon)} \frac{E_k[\tau(\pi)]}{\log(1/\epsilon)}.
\]

In the next section, we provide some preliminaries on MDPs. The terminologies used follow that of Borkar [14].

### III. Preliminaries on MDPs

Let \( \pi \) be an arbitrary policy. Consider the controlled Markov process \( \{(d(t), i(t)) : t \geq K\} \), with the corresponding sequence of controls \( \{B_t\} \), under the policy \( \pi \). Note that for all \( t \geq K \),

\[
P(d(t + 1) = d, i(t + 1) = i | B^{t-1}, \{(d(s), i(s)) : K \leq s \leq t\})
\]

\[
= \sum_{b=1}^{K} P(B_t | B^{t-1}, \{(d(s), i(s)) : K \leq s \leq t\}) \cdot P(d(t + 1) = d, i(t + 1) = i | B^{t}, \{(d(s), i(s)) : K \leq s \leq t\})
\]

\[
= \sum_{b=1}^{K} P(B_t | B^{t-1}, \{(d(s), i(s)) : K \leq s \leq t\}) \cdot P(d(t + 1) = d, i(t + 1) = i | (d(t), i(t)), B_t),
\]

where the last line above follows from (3). From (9), it is evident that the policy \( \pi \) may be described completely by specifying \( P(B_t | B^{t-1}, \{(d(s), i(s)) : K \leq s \leq t\}) \) for all \( t \geq K \). We say that a policy \( \pi \) is a stationary randomised strategy (SRS) if there exists a Cartesian product \( \lambda \) of controls \( \lambda_{(d,i)} \) being a probability measure on \( \mathcal{A} \), such that for all \( t \geq K \) and \( b \in \mathcal{A} \), under the policy \( \pi \),

\[
P(B_t = b | B^{t-1}, \{(d(s), i(s)) : K \leq s \leq t\}) = \lambda_{(d(t), i(t))(b)}.
\]

Such an SRS \( \pi \) will be denoted \( \pi^\lambda \). Note that \( \{(d(t), i(t)) : t \geq K\} \) is a Markov process under the SRS \( \pi^\lambda \). This follows from the relation (9) where the first probability term inside the summation in (9) is now a function only of \( (d(t), i(t)) \). Let \( \Pi_{\text{SRS}} \) denote the set of all SRS policies.

For convenience, we write \( \lambda_{(d,i)}(\cdot) \) as \( \lambda(\cdot | d,i) \) so that we may write \( \lambda \) itself in the more familiar form \( \lambda(\cdot) \).

An immediate and important property of any \( \pi^\lambda \in \Pi_{\text{SRS}} \) is the following.

**Lemma 1.** Let \( \eta \in (0, 1] \). For every \( \pi^\lambda \in \Pi_{\text{SRS}} \), the controlled Markov process \( \{(d(t), i(t)) : t \geq K\} \) under the policy \( \pi^\lambda \) is irreducible, aperiodic, positive recurrent, and hence ergodic.

**Proof:** See Appendix A

A key ingredient of the proof of Lemma 1 is that the trembling hand parameter \( \eta > 0 \).

As a consequence of Lemma 1, it follows that under every \( \pi^\lambda \in \Pi_{\text{SRS}} \), a unique stationary distribution therefore exists for the Markov process \( \{(d(t), i(t)) : t \geq K\} \). Let us call this stationary distribution \( \mu^\lambda \).
With the above ingredients in place, we state in the next section the first main result of this paper – an asymptotic lower bound on the expected time to identify the odd arm.

IV. LOWER BOUND

We now present a lower bound for (8).

Given two probability distributions \( \mu \) and \( \nu \) on the finite state space \( S \), the Kullback-Leibler (KL) divergence between \( \mu \) and \( \nu \) is defined as

\[
D(\mu || \nu) := \sum_{i \in S} \mu(i) \log \frac{\mu(i)}{\nu(i)},
\]

(11)

where, by convention, \( 0 \log \frac{0}{0} = 0 \).

**Proposition 1.** Let \( \eta \in (0, 1] \) and \( h \in \mathcal{A} \) be fixed. Assume that \( \mathcal{H}_h \) is the true hypothesis. Let \( P_1 \) be the transition probability matrix of the Markov process of arm \( h \), and for each \( a \neq h \), let \( P_2 \) be the transition probability matrix of the Markov process arm \( a \). Then,

\[
\liminf_{\epsilon \downarrow 0} \inf_{\pi \in \Pi(\epsilon)} E_h[\tau(\pi)] \log(1/\epsilon) \geq \frac{1}{R^*(P_1, P_2)},
\]

(12)

where \( R^*(P_1, P_2) \) is given by

\[
R^*(P_1, P_2) := \sup_{\pi^\lambda \in \Pi_{SRS}} \min_{h' \neq h} \sum_{(d, i) \in S} \nu^\lambda(d, i, a) k(d, i, a),
\]

(13)

with

\[
k(d, i, a) := \begin{cases} 
D(P_1^{da}(\cdot||i_a) || P_2^{da}(\cdot||i_a)), & a = h, \\
D(P_2^{da}(\cdot||i_a) || P_1^{da}(\cdot||i_a)), & a = h', \\
0, & a \neq h, h', 
\end{cases}
\]

(14)

and

\[
\nu^\lambda(d, i, a) := \mu^\lambda(d, i) \left( \frac{\eta}{K} + (1 - \eta) \lambda(a | d, i) \right), \quad \forall (d, i, a) \in S \times \mathcal{A}.
\]

(15)

**Proof:** See Appendix B.

The proof of the lower bound follows the outline in [4], with necessary modifications for the setting of restless arms. The key ingredients are the data processing inequality for relative entropies, a Wald-type lemma for Markov processes, and a recognition that, for any \( (d, i) \), the long-term fraction of exits from the state \( (d, i) \) match the long-term fraction of entries into that state. This forces the long-term probability of seeing the controlled Markov process in state \( (d, i) \) to be that under its unique stationary distribution, by ergodicity (Lemma 1). These observations lead to (12).

Observe that the left hand side of (12) is evaluated by taking into consideration all policies, including those that are not necessarily SRS policies, whereas the supremum in (13) is only over SRS policies. This is a consequence of [15, Theorem 8.8.2]; see Appendix B for more details. Also, the constant \( R^*(P_1, P_2) \) in (13) is not a function of the odd arm location \( h \), i.e., the location of the odd arm has no effect on the amount of time required to identify it. This is due to symmetry in the structure of the arms.
A. An Infinite-Dimensional Linear Programming Problem

It may be a little surprising to the reader as to why the inner summation on the right hand side of \( R^*(P_1, P_2) \) in (13) is over the delays and the last observed states of all the arms when the function \( k(d, i, a) \), as given in (14), is a function only of \( d_a \) and \( i_a \), the delay and the last observed state of arm \( a \). In order to better appreciate the usefulness of taking into account the arm delays and last observed states of all the arms in deriving the lower bound, we present below a sketch of an alternative proof of the lower bound in which we first fix an arm \( a \) and consider only its delay and last observed state in the subsequent calculations. Fix arm \( a \in \mathcal{A} \). Given an integer \( d \geq 1 \), \( i, j \in \mathcal{S} \) and a policy \( \pi \), let

\[
N(\tau(\pi), d, i, a) := \sum_{t=\tau(\pi)}^{\tau(\pi) + 1} \mathbb{I}_{\{d_a(t) = d, i_a(t) = i, A_t = a, X_t^a = j\}}.
\]

(16)

Recall that \( \tau(\pi) \) denotes the stopping time of policy \( \pi \). Following the chain of equalities leading up to (78) in Appendix B with the inner summation over \((d, i)\) now replaced by \((d, i) \in \{1, 2, \ldots \} \times \mathcal{S} \), it can be shown that we arrive at the following expression in place of (78):

\[
\sum_{a=1}^{K} \sum_{d=1}^{\infty} \sum_{i \in \mathcal{S}} E_h[N(\tau(\pi), d, i, a)] D((P_h^a)^d(\cdot | i))(P_h^a)^d(\cdot | i)),
\]

(17)

where \( N(\tau(\pi), d, i, a) \) in (17) is simply the summation over all \( j \in \mathcal{S} \) of the right hand side of (16).

From the exposition in Section II we know that at any given time \( t \geq K \), the vector \( d(t) \) must satisfy the following constraint: exactly one component of \( d(t) \) is equal to 1, and all the other components are strictly greater than 1. Let us now express this constraint mathematically. Recall the assumption that the policy \( \pi \) selects, without loss of generality, arm 1 at time \( t = 0 \), arm 2 at time \( t = 1 \) and so on until arm \( K \) at time \( t = K - 1 \). From time \( t = K \) onwards, arm \( a \) may or may not be selected at all time instants, and whenever it is not selected, some arm \( b \neq a \) is selected. It is this observation (that some arm is selected at every time instant until the stopping time of the policy) that must be modelled as a constraint mathematically. Figure [1] depicts the selection of arms at various time instants for the case when \( K = 3 \).

Assume without loss of generality that under the policy \( \pi \), arm \( a \) is selected at time \( t = \tau(\pi) \). Then, it follows that

\[
(a - 1) + \sum_{i \in \mathcal{S}} \sum_{d=1}^{\infty} d N(\tau(\pi), d, i, a) + 1 = \tau(\pi) + 1;
\]

(18)

in (18), the term \((a - 1)\) on the left hand side denotes the number of time instants that have passed before arm \( a \) is selected for the first time. The second term on the left hand side of (18) denotes the total number of time instants that have passed, starting from time \( t = K \), until the final selection time instant of arm \( a \). The last term on the left hand side counts the final selection instant of arm \( a \). Thus, the total value of the left hand side of (18) is equal to the total number of time instants that have passed from \( t = 0 \) to \( t = \tau(\pi) \) (both inclusive), which is precisely the quantity on the right hand side of (18). Applying \( E_h[\cdot] \) to both sides of (18), and using the monotone convergence theorem, we arrive at the following relation after rearrangement:

\[
\sum_{i \in \mathcal{S}} \sum_{d=1}^{\infty} \frac{E_h[N(\tau(\pi), d, i, a)]}{E_h[\tau(\pi)]} + \frac{a - 1}{E_h[\tau(\pi)]} = 1.
\]

(19)
In fact, it is easy to see that (13), and therefore (19), holds for every arm, whether or not the arm is selected at time \( t = \tau(\pi) \).

We now use (17) in place of (78), along with the constraint in (19), and mimic the steps in Appendix B. It can be shown that by doing so, we arrive at the following relation in place of (83):

\[
d(\epsilon, 1 - \epsilon) \leq \sup_{\kappa} \min_{h' \neq h} \left\{ E_h \left[ \sum_{a=1}^{K} \log \frac{P_h(X_a^d)}{P_{h'}(X_a^{d-1})} \right] + \left( E_h[\tau(\pi) - K + 1] \right) \cdot \sum_{a=1}^{K} \sum_{d=1}^{\infty} \sum_{i \in S} \kappa(d, i, a) D((P_h^d)^d(\cdot | i) || (P_{h'}^d)^d(\cdot | i)) \right\},
\]

(20)

where the supremum in (20) is over all probability distributions \( \kappa \) on \( \{1, 2, \ldots\} \times S \times A \) that satisfy the constraint

\[
\sum_{i \in S} \sum_{d=1}^{\infty} d \kappa(d, i, a) = 1 \quad \text{for all } a \in A.
\]

(21)

The constraint in (21) may be obtained from (19) by letting \( E_h[\tau(\pi)] \to 0 \) (which is the same as \( \epsilon \downarrow 0 \)) and replacing the fractional term on the left hand side of (19) by \( \kappa(d, i, a); \) here, \( \kappa(d, i, a) \) represents the long-term joint probability of observing arm \( a \) to be in delay \( d \) and last observed state \( i \), and subsequently selecting arm \( a \).

Dividing both sides of (20) by \( d(\epsilon, 1 - \epsilon) \), and using the fact that \( d(\epsilon, 1 - \epsilon)/\log(1/\epsilon) \to 1 \) as \( \epsilon \downarrow 0 \), we arrive at

\[
\liminf_{\epsilon \downarrow 0} \inf_{\pi \in \Pi_\epsilon} E_h[\tau(\pi)] \geq \frac{1}{R_1^*(P_1, P_2)},
\]

(22)

where \( R_1^*(P_1, P_2) \) is the solution to the following constrained optimisation problem:

\[
R_1^*(P_1, P_2) = \sup_{\kappa} \min_{h' \neq h} \left\{ \sum_{a=1}^{K} \sum_{d=1}^{\infty} \sum_{i \in S} \kappa(d, i, a) D((P_h^d)^d(\cdot | i) || (P_{h'}^d)^d(\cdot | i)) \right\}
\]

subject to

\[
\sum_{i \in S} \sum_{d=1}^{\infty} d \kappa(d, i, a) = 1 \quad \text{for all } a \in A,
\]

\[
\sum_{a=1}^{K} \sum_{d=1}^{\infty} \sum_{i \in S} \kappa(d, i, a) = 1,
\]

\[
\kappa(d, i, a) \geq 0 \quad \text{for all } d \in \{1, 2, \ldots\}, i \in S, a \in A.
\]

(23)

Notice that (23) constitutes an infinite-dimensional linear programming problem with linear constraints. It is not clear if a solution to (23) exists. Also, it is not clear if the constraints in (23) constitute the tightest set of constraints.

We end this section with a remark that by taking into account the delays and the last observed states of all the arms in deriving the lower bound, as done in Appendix B, the constraint in (18) is automatically captured since any vector \( d = (d_1, \ldots, d_K) \) of arm delays belongs, by definition, to the subset \( S \) which obeys the constraint in (18). Thus, the viewpoint of controlled Markov processes and the family of MDPs that arise from this viewpoint greatly simplify the analysis of the lower bound.

V. Achievability

The question of whether the supremum in (13) is attained is still under study; see our remarks in the concluding section. Recall that this supremum is over all \( \pi^\lambda \in \Pi_{SRS} \) for \( \lambda(\cdot | \cdot) \) which are conditional probability distributions on the arms, conditioned on the arm delays and the last observed states. This is in contrast with the works [1]–[4], where the corresponding supremum is over all unconditional probability distributions on the arms. This is because, in those works, the arm delays are nonexistent. The unconditional probability measures are elements of the probability simplex on \( A \), whereas the conditional probability measures are, however, more complex due to the countably many possible values for the arm delays. In spite of
this added complexity, we can come arbitrarily close to the supremum in (13). We shall use this fact in our achievability result, which is the topic of this section.

We begin with some notations. Given \( h, h' \in \mathcal{A} \), with \( h \neq h' \), and a policy \( \pi^h \in \Pi_{\text{SRS}} \), let \( Z_{hh'}(n) \) denote the log-likelihood ratio (LLR), under the policy \( \pi^h \), of all intended arm pulls, actual arm pulls, and observations up to time \( n \) under hypothesis \( \mathcal{H}_h \) with respect to that under \( \mathcal{H}_{h'} \). Then, \( Z_{hh'}(n) \) may be expressed as, using \( P_h \) for \( P_h^\pi \),

\[
Z_{hh'}(n) = \log \frac{P_h(B^n_n, A^n_n, X^n_n)}{P_{h'}(B^n_n, A^n_n, X^n_n)} = \log \frac{P_h(B_0)}{P_{h'}(B_0)} + \log \frac{P_h(A_0|B_0)}{P_{h'}(A_0|B_0)} + \log \frac{P_h(X_0|B_0, A_0)}{P_{h'}(X_0|B_0, A_0)} \tag{24}
\]

\[
+ \sum_{t=1}^{n} \log \left( \frac{P_h(B_t|B^{t-1}_1, A^{t-1}_1, X^{t-1}_1)}{P_{h'}(B_t|B^{t-1}_1, A^{t-1}_1, X^{t-1}_1)} \right) \tag{25}
\]

\[
+ \sum_{t=1}^{n} \log \left( \frac{P_h(A_t|B^t_1, A^{t-1}_1, X^{t-1}_1)}{P_{h'}(A_t|B^t_1, A^{t-1}_1, X^{t-1}_1)} \right) \tag{26}
\]

\[
+ \sum_{t=1}^{n} \log \left( \frac{P_h(X_t|A_t, B^t_1, A^{t-1}_1, X^{t-1}_1)}{P_{h'}(X_t|A_t, B^t_1, A^{t-1}_1, X^{t-1}_1)} \right). \tag{27}
\]

We now note that the probability of choosing arm \( B_t \) at time \( t \), based on the history up to time \( t \), cannot be a function of the underlying odd arm location (which is unknown), and must therefore be the same under hypotheses \( \mathcal{H}_h \) and \( \mathcal{H}_{h'} \). Thus, the first term in (24) and the expression in (25) are 0. Also, we note that \( P_h(A_0|B_0) = P_{h'}(A_0|B_0) \), and for each \( t \),

\[
P_h(A_t|B_t, A^{t-1}_1, X^{t-1}_1) = P_{h'}(A_t|B_t, A^{t-1}_1, X^{t-1}_1)
\]

since \( A_t \), the arm that is actually pulled at time \( t \), is a function only of \( B_t \) and is related to \( B_t \) through (4). Therefore, given the history, the choice of \( A_t \) is not a function of the odd arm location, and is the same under hypotheses \( \mathcal{H}_h \) and \( \mathcal{H}_{h'} \), implying that the second term in (24) and the expression in (26) are 0. Finally, the probabilities in (27) do not depend on the intended arm pulls \( \{B_t\} \) since the state \( X_t \) observed on arm \( A_t \) is a function only of the delay and the last observed state of arm \( A_t \). Letting \( X^n_t \) denote the state of arm \( A_t = a \), and defining

\[
N(n, d, \bar{d}, a) := \sum_{t=K}^{n} \mathbb{1}_{\{d(t) = d, \bar{d}(t) = \bar{d}, A_t = a\}}, \tag{28}
\]

\[
N(n, d, \bar{d}, a, j) := \sum_{t=K}^{n} \mathbb{1}_{\{d(t) = d, \bar{d}(t) = \bar{d}, A_t = a, X^n_t = j\}}, \tag{29}
\]

for all \( (d, \bar{d}, a) \in \mathcal{S} \times \mathcal{A} \), it can be shown that

\[
Z_{hh'}(n) = \sum_{a=1}^{K} \log \frac{P_h(X^a_{n-1})}{P_{h'}(X^a_{n-1})} + \sum_{(d, \bar{d}) \in \mathcal{S}} \sum_{j \in \mathcal{S}} \left[ N(n, d, \bar{d}, h, j) \log \frac{P_{\text{LLR}}(j|\pi_h)}{P_{\text{LLR}}(j|\pi_{h'})} + N(n, d, \bar{d}, h', j) \log \frac{P_{\text{LLR}}(j|\pi_{h'})}{P_{\text{LLR}}(j|\pi_h)} \right]. \tag{30}
\]

To describe our policy, we first fix constants \( \delta > 0 \) and \( L > 1 \). These will be the parameters of our policy. Recall that the supremum in (13) is over all policies in \( \Pi_{\text{SRS}} \). For a fixed hypothesis \( \mathcal{H}_h \), by the definition of this supremum, we know that given \( \delta > 0 \), there exists \( \lambda(\cdot | \cdot) \) such that under the corresponding SRS policy \( \pi^\lambda \), we have

\[
\min_{h' \neq h} \sum_{(d, \bar{d}) \in \mathcal{S}} \sum_{a=1}^{K} \nu^\lambda,d,h(\bar{d}, \bar{a}) k(d, \bar{d}, a) > \frac{R^*(P_1, P_2)}{1 + \delta}. \tag{31}
\]

Notice that such a \( \lambda \) is, in general, a function of \( \delta \) and the true hypothesis \( \mathcal{H}_h \), although \( R^*(P_1, P_2) \) itself is not a function of the true hypothesis; let us denote this \( \lambda \) as \( \lambda_{h,\delta} \).

Our policy, \( \pi^*(L, \delta) \), is then as below.
Policy \(\pi^*(L, \delta)\):

Fix \(L > 1\) and \(\delta > 0\). Let the parameter of the trembling hand be \(\eta \in (0, 1)\). Assume\(^2\) that \(A_0 = 1, A_1 = 2, \) and so on until \(A_{K-1} = K\). Let \(M_h(n) = \min_{h' \neq h} Z_{hh'}(n)\). Follow the below mentioned steps for each \(n \geq K\).

1. Let \(\theta(n) = \arg \max_{h \in A} M_h(n)\); resolve ties at random.
2. If \(M_{\theta(n)}(n) \geq \log((K-1)L)\), stop further arm selections and declare \(\theta(n)\) as the true index of the odd arm.
3. If \(M_{\theta(n)}(n) < \log((K-1)L)\), decide to pull arm \(B_n\) according to the distribution \(\lambda_{\theta(n),\delta}(\cdot | d(n), \hat{Z}(n))\).

In item (1) above, \(\theta(n)\) denotes the guess of the odd arm at time \(n\). In item (2), we check if the LLR of hypothesis \(\mathcal{H}_{\theta(n)}\) with respect to the true hypothesis is confirmed sufficiently (\(\geq \log(K-1)L\)). If this is the case, then the policy is confident that the true odd arm location is \(\theta(n)\). The policy then terminates and outputs the index \(\theta(n)\). If the condition in item (2) fails, then the policy picks the next arm to pull.

Recall that the supremum in (13) is only over SRS policies. However, the policy \(\pi^*(L, \delta)\) described above is not an SRS policy since the distribution in item (3) is a function of \(\theta(n)\) that could potentially depend on the entire history of arm selections and observations up to time \(n\). Yet, as we show below, its performance comes arbitrarily close to that of the lower bound.

We now present results on the performance of our policy.

Lemma 2. Fix \(L > 1, \delta > 0\) and \(h \in A\), and suppose that \(\mathcal{H}_h\) is the true hypothesis. Consider the non-stopping version of the policy \(\pi^*(L, \delta)\) which runs indefinitely (i.e., if item (2) is true, it moves to item (3)). Under this policy, for every \(h' \neq h\),

\[
\liminf_{n \to \infty} \frac{Z_{hh'}(n)}{n} > 0 \quad \text{a.s.}.
\] (32)

Proof: See Appendix E.

Thanks to Lemma 2, we have \(\liminf_{n \to \infty} M_h(n)/n > 0\) under the true hypothesis \(\mathcal{H}_h\). This implies that \(M_h(n) \geq \log((K-1)L)\) a.s. for all sufficiently large values of \(n\), thus proving that the policy \(\pi^*(L, \delta)\) stops in finite time, almost surely.

Next, we show that the probability of error of our policy may be controlled by setting the parameter \(L\) suitably.

Lemma 3. Fix error probability \(\epsilon > 0\). If \(L = 1/\epsilon\), then for every \(\delta > 0\), \(\pi^*(L, \delta) \in \Pi(\epsilon)\). Here, \(\Pi(\epsilon)\) is as defined in [1].

Proof: The proof uses the fact that the policy stops in finite time a.s.. See Appendix E for the details.

With the above ingredients in place, we state the main result of this section, which is that the expected stopping time of our policy satisfies an asymptotic upper bound that comes arbitrarily close to the lower bound in (12).

Proposition 2. Fix \(h \in A\) and \(\delta > 0\), and let \(\mathcal{H}_h\) be the true hypothesis. For \(\pi = \pi^*(L, \delta)\), the stopping time \(\tau(\pi)\) satisfies

\[
\limsup_{L \to \infty} \frac{E_h[\tau(\pi)]}{\log L} \leq \frac{1 + \delta}{R^*(P_1, P_2)}.
\] (33)

Proof: In the proof, which we provide in Appendix E, we first show that as \(L \to \infty\) (or equivalently \(\epsilon \downarrow 0\)), the ratio \(\tau(\pi)/\log L\) satisfies an a.s. upper bound that matches with the right hand side of (33). We then show that the family \(\{\tau(\pi)/\log L : L > 1\}\) is uniformly integrable. Combining the a.s. convergence with uniform integrability yields (33).

Since (33) holds for any arbitrary choice of \(\delta > 0\), we have

\[
\limsup_{\delta \downarrow 0} \limsup_{L \to \infty} \frac{E_h[\tau(\pi^*(L, \delta))]}{\log L} \leq \frac{1}{R^*(P_1, P_2)}.
\] (34)

We are now ready to state the main result of this paper.

\(^2\)If this is not the case, exercise arm pulls uniformly at random until each arm is selected at least once. It can be shown that this will only take finite time a.s., and does not affect the asymptotic analysis of our policy.
VI. MAIN RESULT

Theorem 1. Consider a multi-armed bandit with $K \geq 3$ arms in which each arm is a time homogeneous and ergodic Markov process on the finite state space $S$. Fix $h \in A$, and suppose that $h$ is the odd arm. Let $P_1$ be the transition probability matrix of the Markov process of arm $h$. Also, for all $a \neq h$, let the transition probability matrix of arm $a$ be $P_2$, where $P_2 \neq P_1$. Fix $\eta \in (0, 1]$, and suppose that a decision maker who wishes to identify the odd arm has a trembling hand with parameter $\eta$. Assuming that $P_1$ and $P_2$ are known to the decision maker, the expected time required by the decision maker to identify the odd arm satisfies the asymptotic relation

$$\lim_{\epsilon \downarrow 0} \inf_{\pi \in \Pi(\epsilon)} \frac{E_\eta[\tau(\pi)]}{\log(1/\epsilon)} = \lim_{\delta \downarrow 0} \lim_{L \to \infty} \frac{E_\eta[\tau(\pi^*(L, \delta))]}{\log L} = \frac{1}{R^*(P_1, P_2)}.$$  

We thus see that the policy $\pi^*(L, \delta)$ is asymptotically optimal. As noted in Lemma 3, the parameter $L$ may be set appropriately so as to ensure that the policy meets the desired error probability at stoppage. Furthermore, the parameter $\delta$ may be set so as to ensure that the upper bound in (33) is within a desired accuracy from the lower bound in (12). Finally, we emphasise here that our analysis of the lower and upper bounds crucially relies on the trembling hand parameter $\eta$ being strictly positive.

VII. THE CASE $\eta = 0$

We now investigate the question of whether the results of this paper extend directly to the case when the decision maker does not possess a trembling hand, i.e., $\eta = 0$. While, in principle, we may consider plugging $\eta = 0$ in (12) and treat the resulting expression as the lower bound for the case when $\eta = 0$, it is not clear if this new lower bound may be approached asymptotically through a sequence of strategies (policies) in the sense of (34). It is worth noting here that the settings of (a) iid observations from the arms studied in [1]–[3], and (b) rested Markov arms studied in [4], are merely special cases of the setting of restless Markov arms. Indeed, the iid process of each arm may trivially be treated as a Markov process, and in this case, the notion of arm delays and last observed states is redundant, as pointed out in Section I. Additionally, the setting of rested Markov arms may be realised by fixing $d_a(t) = 1$ for all $a \in A$ and for all $t \geq K$, which is the same as saying that the unobserved arms do not undergo state evolution.

Thus, an affirmative answer to the above question will imply that the results (lower bound and asymptotically optimal policies) of the prior works may be recovered as special cases of the answer to the above question (albeit some technicalities such as the countably infinite alphabet of the Poisson random variables in [1], [3] versus the finite state space of the Markov process of each arm as considered in this paper).

We now bring to light the following observations.

1) Writing $R^*(P_1, P_2)$ of (13) more explicitly as $R^*_\eta(P_1, P_2)$ for $\eta \in (0, 1]$, we show that $\lim_{\eta \downarrow 0} R^*_\eta(P_1, P_2)$ exists. This is based on a key monotonicity property which we describe in Section VII-A.

2) Writing $R^*_0(P_1, P_2)$ to denote the constant obtained by plugging $\eta = 0$ in (12), we demonstrate that

$$\lim_{\eta \downarrow 0} R^*_\eta(P_1, P_2) \leq R^*_0(P_1, P_2).$$  

It is not clear if, in general, the inequality in (36) is an equality.

3) We show in Section VII-B and Section VII-C that the lower bounds for the setting when either (a) each arm yields iid observations from a common finite alphabet, or (b) each arm yields Markov observations from a common finite state space and the arms are rested, may be recovered from (12) by plugging $\eta = 0$ in (12). Furthermore, we show that for
each of the above settings, the inequality in (36) is an equality, thus implying that the lower bounds for these settings may be approached asymptotically through a sequence of “trembling-hand” based policies similar to that presented in this paper; the policies of [1] Section II.B] and [4] Section IV] are example cases in point.

A key ingredient that goes into the proofs of the results in point 3 above is the Envelope Theorem [16, Theorem 2]. A formal verification of this result for the setting of restless Markov arms still remains open.

A. A Key Monotonicity Property

Fix $\eta \in (0, 1]$, and assume that the decision maker possesses a trembling hand with parameter $\eta$. Let $\lambda = \lambda(\cdot | \cdot)$ be any conditional probability distribution on the set $A$ of arms, conditioned on the arm delays and last observed states, as described in Section [III] and let $\Lambda$ denote the set of all such conditional distributions. Define

$$\Lambda^\eta := \left\{ \frac{\eta}{K} + (1 - \eta) \lambda(\cdot | \cdot) : \lambda(\cdot | \cdot) \in \Lambda \right\}. \quad (37)$$

Note that for any $\lambda(\cdot | \cdot) \in \Lambda$, the corresponding element of $\Lambda^\eta$ is the probability distribution according to which arms are actually selected, when the decision maker intends to pull arms according to $\lambda(\cdot | \cdot)$. The following Lemma shows that $\Lambda^\eta$ is a non-decreasing in $\eta$.

**Lemma 4.** $\Lambda^\eta \subseteq \Lambda^{\eta'}$ for all $0 < \eta' < \eta \leq 1$.

**Proof:** Fix $0 < \eta' < \eta \leq 1$, and let $\frac{\eta'}{K} + (1-\eta) \lambda(\cdot | \cdot) \in \Lambda^\eta$ for some $\lambda(\cdot | \cdot) \in \Lambda$. Then, for all $(d, \hat{z}, a) \in S \times A$,

$$\frac{\eta}{K} + (1 - \eta) \lambda(a|d, \hat{z}) = \frac{\eta'}{K} + \frac{\eta - \eta'}{K} + (1 - \eta) \lambda(a|d, \hat{z}) = \frac{\eta'}{K} + (1 - \eta') \left[ \frac{\eta - \eta'}{1 - \eta'} \cdot \frac{1}{K} + \frac{1 - \eta}{1 - \eta'} \lambda(a|d, \hat{z}) \right] = \frac{\eta'}{K} + (1 - \eta') \left[ \frac{\eta''}{K} + (1 - \eta'') \lambda(a|d, \hat{z}) \right] \in \Lambda^{\eta'}, \quad (39)$$

where (39) follows by noting that the term inside the square brackets in (38) is a valid element of $\Lambda$. The relation in (39) implies that every element of $\Lambda^\eta$ is also an element of $\Lambda^{\eta'}$ whenever $\eta' < \eta$. This completes the proof.

From (37), it follows that $\Lambda^\eta \subseteq \Lambda$ for all $\eta \in (0, 1]$. Plugging $\eta = 0$ in (37), and denoting the resulting set as $\Lambda^0$, we see that $\Lambda = \Lambda^0$. Thus, it follows from Lemma 4 that

$$\bigcup_{\eta \downarrow 0} \Lambda^\eta \subseteq \Lambda^0. \quad (40)$$

Let us now turn attention to (15), and note that the right hand side of (15) represents the long-term probability of seeing the state $(d, \hat{z})$ and selecting arm $a$ subsequently with probability $\frac{\eta}{K} + (1 - \eta) \lambda(a|d, \hat{z})$. Defining $\lambda^\eta(\cdot | \cdot) := \frac{\eta}{K} + (1 - \eta) \lambda(\cdot | \cdot)$, and writing $\nu^\lambda$ in (15) as $\nu^{\lambda^\eta}$ with a slight abuse of notation, we may express the right hand side of (15) equivalently as

$$\sup_{\lambda^\eta(\cdot | \cdot) \in \Lambda^\eta} \min_{\eta' \neq \eta} \sum_{a=1}^K \sum_{(d, \hat{z}) \in S} \nu^{\lambda^\eta}(d, \hat{z}, a) k(d, \hat{z}, a). \quad (41)$$

Denoting the expression in (41) by $R^\ast_{\eta}(P_1, P_2)$, it follows from Lemma 4 that $R^\ast_{\eta}(P_1, P_2)$ is non-decreasing in $\eta$, thus implying that $\lim_{\eta \downarrow 0} R^\ast_{\eta}(P_1, P_2)$ exists.

Finally, let us denote by $R^0_{\eta}(P_1, P_2)$ the quantity obtained by plugging $\eta = 0$ in (41). Then, it follows from (40) that (36) holds.
Remark 1. It is important to note that the term \( \nu^\lambda \) in (41) has the product form given by the right hand side of (15) for all \( \eta > 0 \). This is a consequence of the ergodicity property established in Lemma 7, the proof of which relies crucially on the fact that \( \eta > 0 \). However, when \( \eta = 0 \), writing \( \nu^\lambda \) as simply \( \nu^\lambda \), a product form expression as in (15) may not exist for \( \nu^\lambda \) since it is not a priori clear if the analogue of Lemma 7 holds when \( \eta = 0 \). Thus, \( \nu^\lambda(d, i, a) \) only represents the long-term probability of seeing the state \((d, i)\) and selecting arm \(a\) subsequently with probability \( \lambda(a|d, i) \).

B. IID Observations From Arms

We now show that when each arm yields iid observations coming from a finite alphabet that is common across the arms, the inequality in (36) is indeed an equality. Fix \( h \in A \), and suppose that \( \mathcal{H}_h \) is the true hypothesis. Let arm \( h \) be associated with an iid process whose underlying law is \( \nu_1 \). Further, for all \( h' \neq h \), let arm \( h' \) be associated with an iid process whose law is \( \nu_2 \), where \( \nu_2 \neq \nu_1 \). Assume that the iid process of any given arm is independent of the iid process of each of the remaining arms. Let \( \nu_h^a \) denote the law of the iid process of arm \( a \) under the hypothesis \( \mathcal{H}_h \), i.e.,

\[
\nu_h^a = \begin{cases} 
\nu_1, & a = h, \\
\nu_2, & a \neq h.
\end{cases}
\]  

(42)

Since any iid process is trivially a Markov process, with the state space of the Markov process being the alphabet of the iid process, we may let \( P_1 \) denote the transition probability matrix of arm \( h \) and \( P_2 \) the transition probability matrix of each of the non-odd arms \( h' \neq h \). Then, for all \( i, j \in S \) and \( d \geq 1 \), we have

\[
P_1^d(j|i) = \nu_1(j), \quad P_2^d(j|i) = \nu_2(j).
\]  

(43)

Thus, when each arm yields iid observations, the function \( k(d, i, a) \) in (14) may be expressed as

\[
k(d, i, a) = \begin{cases} 
D(\nu_1||\nu_2), & a = h, \\
D(\nu_2||\nu_1), & a \neq h.
\end{cases}
\]  

(44)

In other words, the function \( k \) does not depend on either the arm delays or the last observed states. Noting that the right hand side of (44) may be written compactly as \( D(\nu_h^a||\nu_h^a) \), and plugging this in (41), we get

\[
R_\alpha^\lambda(P_1, P_2) = \sup_{\lambda^\eta(\cdot) \in \Lambda^\eta} \min_{h' \neq h} \sum_{a=1}^{K} \sum_{(d, i) \in S} \nu_h^\eta(d, i, a) D(\nu_h^a||\nu_h^a)
\]

\[
= \sup_{\lambda^\eta(\cdot) \in \Lambda^\eta} \min_{h' \neq h} \sum_{a=1}^{K} \sum_{(d, i) \in S} \mu_h^\eta(d, i) \left[ \frac{\eta}{K} + (1 - \eta) \lambda(a|d, i) \right] D(\nu_h^a||\nu_h^a)
\]

\[
= \sup_{\lambda^\eta(\cdot) \in \Lambda^\eta} \min_{h' \neq h} \frac{\eta}{K} \sum_{a=1}^{K} \sum_{(d, i) \in S} D(\nu_h^a||\nu_h^a) + (1 - \eta) \sum_{a=1}^{K} \sum_{(d, i) \in S} \mu_h^\eta(d, i) \lambda(a|d, i) D(\nu_h^a||\nu_h^a)
\]

\[
= \sup_{\lambda \in \mathcal{P}(A)} \min_{h' \neq h} \frac{\eta}{K} \sum_{a=1}^{K} D(\nu_h^a||\nu_h^a) + (1 - \eta) \sum_{a=1}^{K} \lambda(a) D(\nu_h^a||\nu_h^a),
\]  

(45)

where in (a) above, \( \mu_h^\eta \) is the long-term probability of observing the state \((d, i)\) when the arms are selected according to the distribution \( \lambda^\eta(\cdot|\cdot) \), and (b) above follows by using the fact that \( \nu^\lambda \) is a probability distribution on \( S \). Finally, in (45), the term \( \lambda(a) \) is given by

\[
\lambda(a) = \sum_{(d, i) \in S} \mu_h^\eta(d, i) \lambda(a|d, i), \quad a \in A,
\]

and \( \mathcal{P}(A) \) in (45) denotes the set of all probability distributions on the set \( A \).
We now note that for all \( \eta \in [0, 1] \), the objective function in (45) is linear (and therefore absolutely continuous) in the variable \( \eta \). Using the Envelope Theorem [16, Theorem 2], we deduce that \( R^*_\eta(P_1, P_2) \) is absolutely continuous for all \( \eta \in [0, 1] \). As a consequence, it follows that (36) holds with equality.

C. Rested Markov Arms

We now show that when each arm is a Markov process on a finite state space that is common across the arms, and the arms are rested, the inequality in (36) is indeed an equality. Fix \( h \in A \), and suppose that \( \mathcal{H}_h \) is the true hypothesis. Let each arm be associated with a time-homogeneous and ergodic discrete time Markov process on a common, finite state space \( S \). Let \( P_1 \) be the transition probability matrix of the odd arm, and let \( P_2 \) be the transition probability matrix of each of the non-odd arms. Let \( \mu_1 \) and \( \mu_2 \) denote the unique stationary distributions of \( P_1 \) and \( P_2 \) respectively. Assume that the Markov process of any given arm is independent of the Markov process of each of the remaining arms.

Let \( P^a_h \) denote the transition probability matrix of arm \( a \) under the hypothesis \( \mathcal{H}_h \), and let \( \mu^a_h \) be the stationary distribution of \( P^a_h \). It then follows that

\[
P^a_h = \begin{cases} P_1, & a = h, \\
P_2, & a \neq h, \\
\end{cases}
\]

\[
\mu^a_h = \begin{cases} \mu_1, & a = h, \\
\mu_2, & a \neq h. \\
\end{cases}
\]

(46)

When the arms are rested, as noted earlier at the beginning of this section, the delay parameter for every arm is equal to 1 for all times, i.e., \( d_a(t) = 1 \) for all \( a \in A \) and \( t \geq K \). Thus, we may omit the summation over \( d \) in (41). Writing \( \lambda(a|\cdot) \) in place of \( \lambda(a|\bar{d}, \cdot) \) and \( \nu^a(\cdot) \) in place of \( \nu^a(\cdot, \bar{d}, \cdot) \), and the following the steps presented earlier for the case of iid observations from the arms, we have

\[
R^*_{\eta}(P_1, P_2) = \sup_{\lambda^a(\cdot) \in \Lambda^a} \min_{h' \neq h} \sum_{a=1}^{K} \sum_{i \in S^K} \nu^a(\cdot, i) D(P^a_h(\cdot|i_a)||P^a_{h'}(\cdot|i_a))
\]

\[
= \sup_{\lambda^a(\cdot) \in \Lambda^a} \min_{h' \neq h} \sum_{a=1}^{K} \sum_{i \in S^K} \mu^a_h(\cdot) \left[ \frac{\eta}{K} + (1 - \eta) \lambda(a|\cdot) \right] D(P^a_h(\cdot|i_a)||P^a_{h'}(\cdot|i_a))
\]

\[
= \sup_{\lambda^a(\cdot) \in \Lambda^a} \min_{h' \neq h} \left[ \frac{\eta}{K} \sum_{a=1}^{K} \sum_{i \in S^K} \mu^a_h(\cdot) D(P^a_h(\cdot|i_a)||P^a_{h'}(\cdot|i_a))
+ (1 - \eta) \sum_{a=1}^{K} \sum_{i \in S^K} \mu^a_h(\cdot) \lambda(a|\cdot) D(P^a_h(\cdot|i_a)||P^a_{h'}(\cdot|i_a)) \right]
\]

\[
\overset{(a)}{=} \sup_{\lambda^a(\cdot) \in \Lambda^a} \min_{h' \neq h} \left[ \frac{\eta}{K} \sum_{a=1}^{K} \sum_{i_a \in S} \mu^a_h(i_a) D(P^a_h(\cdot|i_a)||P^a_{h'}(\cdot|i_a))
+ (1 - \eta) \sum_{a=1}^{K} \sum_{i_a \in S} \mu^a_h(i_a) \lambda(a|i_a) D(P^a_h(\cdot|i_a)||P^a_{h'}(\cdot|i_a)) \right].
\]

(47)

where in (a) above, we write \( \mu^a_h(i_a) \) to denote the marginal of \( \mu^a_h(\cdot) \) corresponding to arm \( a \). We now note that \( \mu^a_h(i_a) \) denotes the long-term probability of observing arm \( a \) in state \( i_a \) when the arms are selected according to the distribution \( \lambda^a(\cdot, \cdot) \). Clearly, this is equal to \( \mu^a_h(i_a) \). Furthermore, in (47), the product \( \mu^a_h(i_a) \lambda(a|i_a) \) denotes the long-term joint probability of observing arm \( a \) in state \( i_a \) and subsequently selecting arm \( a \). As remarked in [4], as a consequence of the rested nature of
the arms, this joint probability is equal to that of first selecting arm $a$ and subsequently observing it in state $i_a$. Hence, we may write (47) as

$$
R_\eta(P_1, P_2) = \sup_{\lambda \in \mathcal{P}(A)} \min_{h \neq h'} \left[ \frac{\eta}{K} \sum_{a=1}^K \sum_{i_a \in S} \mu_h^a(i_a) D(P_h^a(\cdot | i_a) \| P_h^a(\cdot | i_a)) + (1 - \eta) \sum_{a=1}^K \lambda(a) \mu_h^a(\cdot | i_a) D(P_h^a(\cdot | i_a) \| P_h^a(\cdot | i_a)) \right]
$$

where in (b) above,

$$
D(P_h^a(\cdot | i_a) \| P_h^a(\cdot | i_a)) \equiv \sum_{i \in S} \mu_h^a(i) D(P_h^a(\cdot | i_a) \| P_h^a(\cdot | i_a)).
$$

Finally, we note that for all $\eta \in [0, 1]$, the objective function in (48) is linear (and hence absolutely continuous) for all $\eta \in [0, 1]$. Using [16, Theorem 2], we get that $R_\eta^*(P_1, P_2)$ is absolutely continuous for all $\eta \in [0, 1]$. As a consequence, it follows that (36) holds with equality.

VIII. Conclusion

We make several concluding remarks to end the paper.

1. Since $\delta > 0$ is arbitrary, combining the assertions in Propositions 1 and 2, we see that

$$
\lim_{\epsilon \downarrow 0} \inf_{\pi \in \Pi(\epsilon)} \frac{E_{\epsilon}[\tau(\pi)]}{\log(1/\epsilon)} = \frac{1}{R^*(P_1, P_2)}.
$$

We have thus provided an answer for the minimum growth rate of the expected stopping time, as $\epsilon \downarrow 0$ (see [8]).

2. The ergodicity of the Markov process $(\tilde{d}(t), \tilde{z}(t))$ under any policy $\pi^k \in \Pi_{SRS}$ ensures that time averages approach the ensemble averages. This is crucial to show achievability. Note also the use of uniqueness of the stationary distribution to show the converse. The trembling hand model may be viewed as a regularisation that gives stability of the aforementioned Markov process for free. If the trembling hand parameter $\eta$ were 0, one could deliberately add some regularisation parameterised by $\eta$, and let this parameter $\eta \downarrow 0$. With $R_\eta^*(P_1, P_2)$ redefined as the growth rate with trembling hand parameter $\eta$, see [13], then $R_\eta^*(P_1, P_2)$ governs the lower bound, whereas $R_\eta^*(P_1, P_2)$ governs the upper bound. The resulting lower and upper bounds on the growth rate may have a gap.

3. The asymptotically optimal $\lambda(\cdot | \cdot)$ in the restless case may depend on history unlike the cases in [1], [4] where $\lambda(\cdot)$ did not depend on history, even in the rested Markov case. At first glance, this is surprising for the rested Markov case, but in retrospect, these features are apparent from an examination of the optimisation problem (13) in these special cases.

4. Computability of $R^*(P_1, P_2)$ may be an issue, and one must usually resort to $Q$-learning or other such simulation strategies to arrive at good policies. The fact that $D(P_k^{d_a}(\cdot | i_a) \| P_l^{d_a}(\cdot | i_a)), k, l \in \{1, 2\}$, converges as $d_a \to \infty$ could enable restriction of the countable state space $S$ to a finite set, and could lead to good approximations.

5. Other open questions: we have not studied the case when $\eta = 0$, except in the special cases of iid and rested Markov settings. Naive extensions, based on the work in this paper, lead to upper and lower bounds with a possible gap, as highlighted in the second point above. Another open question is the setting when $P_1$ and $P_2$ are unknown and have to be learnt along the way.
Appendix A

Proof of Lemma I

Let \( \eta \in (0, 1] \) be the parameter of the trembling hand. Fix \( \pi^h \in \Pi_{\text{SRS}} \) and \( h \in \mathcal{A} \), and let \( \mathcal{H}_h \) be the true hypothesis. Recall that the controlled Markov process \( \{(d(t), \bar{z}(t)) : t \geq K\} \) is indeed a Markov process under the SRS policy \( \pi^h \).

Proof of Irreducibility: Consider any two states \((d, \bar{z}) \in \mathcal{S} \) and \((d', \bar{z}') \in \mathcal{S}\), and suppose that the Markov process \(\{(d(t), \bar{z}(t)) : t \geq K\} \) is in the state \((d, \bar{z})\) at some time \( t = T_0 \). We shall now demonstrate that there exists \( N \) such that the state \((d', \bar{z}')\) may be reached starting from the state \((d, \bar{z})\) in \( N \) steps under the SRS policy \( \pi^h \). Recall that at any time \( t \), the arm that is intended to be pulled under the policy \( \pi^h \) is \( B_t \), while the actual arm that is pulled at time \( t \) is the trembled version \( A_t \); the arms \( A_t \) and \( B_t \) are related through the trembling hand relation in (4). In particular, for any \( a \in \mathcal{A} \), we have

\[
P(A_t = a \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1}) = \sum_{b=1}^{K} P(B_t = b, A_t = a \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1})
\]

\[
= \sum_{b=1}^{K} P(B_t = b \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1}) \cdot P(A_t = a \mid B_t = b, B^{t-1}, A^{t-1}, \bar{X}^{t-1})
\]

\[
= (a) \sum_{b=1}^{K} \lambda(b \mid d(t), \bar{z}(t)) \cdot \left( \frac{\eta}{K} + (1 - \eta) (1_{a = b}) \right)
\]

\[
= \frac{\eta}{K} + (1 - \eta) \lambda(a \mid d(t), \bar{z}(t))
\]

\[
\geq \frac{\eta}{K},
\]

where \((a)\) above follows from (4) and the fact that under the policy \( \pi^h \in \Pi_{\text{SRS}} \), the intended arm \( B_t \) is selected based on the history \( (B^{t-1}, A^{t-1}, \bar{X}^{t-1}) \) according to the distribution \( \lambda(\cdot \mid \cdot) \).

Assume without loss of generality that the vector \( d' \) of arm delays is such that \( d'_1 > d'_2 > \cdots > d'_K = 1 \). From [17 Proposition 1.7] for finite state Markov processes, we get that there exists an integer \( M \) such that for all \( m \geq M \),

\[
P_{1}(j|i) > 0 \text{ for all } i, j \in \mathcal{S}, \quad P_{2}(j|i) > 0 \text{ for all } i, j \in \mathcal{S}.
\]

(51)

Consider the sequence of actions and observations as follows: starting from the state \((d, \bar{z})\) at time \( t = T_0 \), let the Markov process \(\{(d(t), \bar{z}(t)) : t \geq K\} \) evolve for \( M - 1 \) time instants. Thereafter, let arm 1 be selected at the \( (T_0 + M) \)th time instant and let the state observed on arm 1 be \( i'_1 \); let arm 2 be selected at the \( (T_0 + M + d'_1 - d'_2) \)th time instant and let the state observed on arm 2 be \( i'_2 \), and so on. Finally, let arm \( K \) be observed at the \( (T_0 + M + d'_1 - d'_K) \)th time instant, and let the state observed on arm \( K \) be \( i'_K \). Additionally, let arm 1 not be selected for all \( T_0 + M < t < T_0 + M + d'_1 \); let arm 2 not be selected for all \( T_0 + M + d'_1 - d'_2 < t < T_0 + M + d'_2 \), and so on.

Clearly, the above sequence of actions and observations leads to the state \((d', \bar{z}')\) after \( M + d'_1 - d'_K \) time instants. Thus, the probability of starting from the state \((d, \bar{z})\) and reaching the state \((d', \bar{z}')\) may be lower bounded by the probability that the above sequence of actions and observations occur under the policy \( \pi^h \) which, under the hypothesis \( \mathcal{H}_h \), is given by

\[
\left( \prod_{a=1}^{K-1} P(A_{T_0+M+d'_1-d'_a} = a \mid B^{T_0+M+d'_1-d'_a-1}, A^{T_0+M+d'_1-d'_a-1}, \bar{X}^{T_0+M+d'_1-d'_a-1}) \right) \cdot \left( \prod_{a=1}^{K} (P_h)_{M+d'_1-d'_a}^{(i'_a|\pi_a)} \right)
\]

\[
\cdot \left( \prod_{a=1}^{K-1} P(A_t \notin \{1, \ldots, a\} \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1}) \right)
\]

\[
\geq \left( \frac{\eta}{K} \right)^{K-1} \cdot \left[ \prod_{a=1}^{K} (P_h)_{M+d'_1-d'_a}^{(i'_a|\pi_a)} \right] \cdot \left[ \prod_{a=1}^{K-1} P(A_t \notin \{1, \ldots, a\} \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1}) \right]
\]

\[
\geq \left( \frac{\eta}{K} \right)^{K-1} \cdot \left[ \prod_{a=1}^{K} (P_h)_{M+d'_1-d'_a}^{(i'_a|\pi_a)} \right] \cdot \left[ \prod_{a=1}^{K-1} \frac{\eta(K-a)}{K} \right]
\]
where (a) above follows from the observation that the right hand side of (50), for each \( t \), is \( \geq \eta/K \) and the fact that

\[
P(A_t \notin \{1, \ldots, a\} \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1}) = \sum_{a'=a+1}^{K} P(A_t = a' \mid B^{t-1}, A^{t-1}, \bar{X}^{t-1}) \geq \frac{\eta(K-a)}{K},
\]

and (b) follows by noting that \( K-a \geq 1 \) for \( a \in \{1, \ldots, K-1\} \). Setting \( N = M + d'_1 - d'_K \), we see that the Markov process \( \{(d(t), \bar{z}(t)) : t \geq K\} \) is in the state \((d', \bar{z}')\) at time \( t = T_0 + N \). This establishes irreducibility.

**Proof of Aperiodicity:** Fix an arbitrary \((d, \bar{z}) \in \mathbb{S}\). We shall now demonstrate that starting from the state \((d, \bar{z})\), there is a strictly positive probability for the Markov process \( \{(d(t), \bar{z}(t)) : t \geq K\} \) to return back to the state \((d, \bar{z})\) in \( M' \) steps as well as in \((M' + 1)\) steps, where \( M' \) is sufficiently large and such that \([51]\) holds for all \( m \geq M' \). This will then establish the desired aperiodicity property since the period of the state \((d, \bar{z})\) is equal to the gcd of \( M' \) and \( M' + 1 \), which is 1.

Assume, without loss of generality, that \( d \) is such that \( d_1 > d_2 > \cdots d_K = 1 \). Let \( M \) be such that \([51]\) holds for all \( m \geq M \). Using arguments similar to that in the proof presented above, the probability of starting from the state \((d, \bar{z})\) at some time \( t = T_0 \) and returning back to the state \((d, \bar{z})\) after \( M + d_1 \) time instants may be lower bounded, under hypothesis \( \mathcal{H}_h \), by

\[
\left(\frac{\eta}{K}\right)^{K-1} \prod_{a=1}^{K} (P^a_{h})^{M+d'_1-d'_a} (i'_a | i_a) \cdot \prod_{a=1}^{K-1} \frac{P}{1} t = T_0 + M + d_1 - d_a + 1 \eta/K > 0.
\]

Setting \( M' = M + d_1 - d_K \) yields the desired result.

**Proof of positive recurrence:** Let

\[
p_\eta := \frac{\eta}{K} \min \left\{ \min\{P^M_1(j|i) : i, j \in \mathcal{S}\}, \min\{P^M_2(j|i) : i, j \in \mathcal{S}\} \right\},
\]

here, once again, \( M \) is such that \([51]\) holds for all \( m \geq M \). Therefore, it follows that \( p_\eta > 0 \). Let

\[
r(\pi^\lambda) := \min\{t > K : d(t) = d(K), \bar{z}(t) = \bar{z}(K)\}
\]

denote the first return time of the Markov process \( \{(d(t), \bar{z}(t)) : t \geq K\} \) to its initial state (i.e., the state at time \( t = K \)) under the SRS policy \( \pi^\lambda \). Note that

\[
r(\pi^\lambda) \leq M \cdot K \cdot \tau_\eta \quad a.s.,
\]

where \( \tau_\eta \) is a Geometric random variable with parameter \( p_\eta \). In other words, \( r(\pi^\lambda) \) is a.s. upper bounded by the first return time of \( \{(d(t), \bar{z}(t)) : t \geq K\} \) to its initial state measured only at time instants that are integer multiples of \( M \cdot K \). It then follows that

\[
E[r(\pi^\lambda)] \leq M \cdot K \cdot E[\tau_\eta] = M \cdot K \cdot \frac{1}{p_\eta} < \infty,
\]

thus implying that the Markov process \( \{(d(t), \bar{z}(t)) : t \geq K\} \) is positive recurrent under \( \pi^\lambda \). This completes the proof of positive recurrence, and also the proof of the lemma.
APPENDIX B

PROOF OF PROPOSITION 1

This proof is organised as follows. Given $\epsilon > 0$, we first obtain in Lemma 5 a lower bound for $E_h[Z_{hh'}(\tau(\pi))]$ for all $\pi \in \Pi(\epsilon)$ using a change of measure argument of Kaufmann et al. [12]. We then obtain an upper bound for $E_h[Z_{hh'}(\tau(\pi))]$ in terms of $E_h[\tau(\pi)]$. Combining the aforementioned upper and lower bounds, and letting $\epsilon \downarrow 0$, will yield the desired result. The ergodicity result of Lemma 1 plays a crucial role in deriving the final lower bound of (12).

A. A Lower Bound on $E_h[Z_{hh'}(\tau(\pi))]$ for $\pi \in \Pi(\epsilon)$

As a first step towards deriving the lower bound, we use a result of Kaufmann et al. [12] to obtain a lower bound for $E_h[Z_{hh'}(\tau(\pi))]$ in terms of the error probability parameter $\epsilon$. However, this requires a generalisation of [12] Lemma 18, a change of measure argument for arms with iid observations, to the setting of restless arms with Markov observations. This result is given in the following lemma.

Lemma 5. Fix $\pi \in \Pi(\epsilon)$, and let $\tau(\pi)$ be the stopping time of policy $\pi$. Let $\mathcal{F}_{\tau(\pi)}$ be the $\sigma$-algebra

$$\mathcal{F}_{\tau(\pi)} = \{ E \in \mathcal{F} : E \cap \{ \tau(\pi) = t \} \in \mathcal{F}_t \text{ for all } t \geq 0 \},$$

(58)

where for each $t \geq 0$, $\mathcal{F}_t = \sigma(B^t, A^t, \tilde{X}^t)$. Then, for any $h, h' \in \mathcal{A}$ such that $h' \neq h$, the relation

$$P_{h'}(E) = E_h[1_E \exp(-Z_{hh'}(\tau(\pi)))]$$

(59)

holds for all $E \in \mathcal{F}_{\tau(\pi)}$.

Proof of Lemma 5: We prove the lemma by demonstrating, through mathematical induction, that the relation

$$E_h'[g(B^t, A^t, \bar{X}^t)] = E_h[g(B^t, A^t, \bar{X}^t) \exp(-Z_{hh'}(t))]$$

(60)

holds for all $t \geq 0$ and for all measurable functions $g : \mathcal{A}^{t+1} \times \mathcal{A}^{t+1} \times \mathcal{S}^{t+1} \to \mathbb{R}$. The proof for the case $t = 0$ may be obtained as follows. For any $g : \mathcal{A} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$, we have

$$E_h'[g(B_0, A_0, \bar{X}_0)] = \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in \mathcal{S}} g(b, a, i) P_{h'}(B_0 = b, A_0 = a, \bar{X}_0 = i)$$

$$= \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in \mathcal{S}} g(b, a, i) P_{h'}(B_0 = b) P_{h'}(A_0 = a | B_0 = b) P_{h'}(\bar{X}_0 = i | A_0 = a)$$

$$\overset{(a)}{=} \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in \mathcal{S}} g(b, a, i) P_h(B_0 = b) P_h(A_0 = a | B_0 = b) P_h(\bar{X}_0 = i | A_0 = a)$$

$$= \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in \mathcal{S}} g(b, a, i) P_h(B_0 = b) P_h(A_0 = a | B_0 = b) P_{h'}(X_0^a = i),$$

(61)

where (a) follows using the facts that $P_h(B_0 = a) = P_{h'}(B_0 = b)$ and $P_h(A_0 = a | B_0 = b) = P_{h'}(A_0 = a | B_0 = b)$ (see Section V). Assuming that $X_0^a \sim \nu$, where $\nu$ is a probability distribution on $\mathcal{S}$, independent of the true hypothesis (which is not known to policy $\pi$), we have

$$E_h'[g(A_0, \bar{X}_0) | \mathcal{H}_{hh'}] = \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in \mathcal{S}} g(b, a, i) P_h(B_0 = b) P_h(A_0 = a | B_0 = b) \nu(i)$$

$$= \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in \mathcal{S}} g(b, a, i) P_h(B_0 = b) P_h(A_0 = a | B_0 = b) P_h(X_0^a = i | A_0 = a).$$

(62)
Also, we have (see Section V)\

\[ Z_{hh'}(0) = \log \frac{P_h(B_0, A_0, \bar{X}_0)}{P_{h'}(B_0, A_0, \bar{X}_0)} = 0. \]  

(64)

Combining (63) and (64), we get \( E_{h'}[g(B_0, A_0, \bar{X}_0)] = E_h[g(B_0, A_0, \bar{X}_0) \exp(-Z_{hh'}(0))] \), thus proving (60) for \( t = 0 \).

We now note that (60) is true for some \( t > 0 \), and demonstrate that it also true for \( t + 1 \). By law of iterated expectations,

\[ E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})] = E_{h'}[E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})|\mathcal{F}_t]]. \]  

(65)

Noting that \( E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})|\mathcal{F}_t] \) is a measurable function of \( (B^t, A^t, \bar{X}^t) \), by the induction hypothesis, we have

\[ E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})] = E_{h'}[E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})|\mathcal{F}_t] \exp(-Z_{hh'}(t))]. \]  

(66)

We now note that

\[ E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})|\mathcal{F}_t] \exp(-Z_{hh'}(t)) \]

\[ = \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in S} \left[ g(B^t, A^t, \bar{X}^t, b, a, i) \cdot P_{h'}(B_{t+1} = b|B^t, A^t, \bar{X}^t) \right. \\
\]

\[ \left. \cdot P_{h'}(A_{t+1} = a|B^{t+1} = b, B^t, A^t, \bar{X}^t) \cdot P_{h'}(\bar{X}_{t+1} = i|B^{t+1} = b, A_{t+1} = a, B^t, A^t, \bar{X}^t) \cdot \exp(-Z_{hh'}(t)) \right] \]

\[ \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in S} \left[ g(B^t, A^t, \bar{X}^t, b, a, i) \cdot P_h(B_{t+1} = b|B^t, A^t, \bar{X}^t) \right. \\
\]

\[ \left. \cdot P_h(A_{t+1} = a|B^{t+1} = b, B^t, A^t, \bar{X}^t) \cdot P_h(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t) \cdot \exp(-Z_{hh'}(t)) \right], \]  

(67)

where (a) above is due to the fact that \( Z_{hh'}(t) \) is a measurable function of \( (A^t, \bar{X}^t) \), and in writing (b), we use the following facts: for any \( t \),

- \( P_{h'}(B_{t+1} = b|B^t, A^t, \bar{X}^t) = P_h(B_{t+1} = b|B^t, A^t, \bar{X}^t) \),
- \( P_{h'}(A_{t+1} = a|B^{t+1} = b, B^t, A^t, \bar{X}^t) = P_h(A_{t+1} = a|B^{t+1} = b, B^t, A^t, \bar{X}^t) \), and
- \( P_{h'}(\bar{X}_{t+1} = i|B^{t+1} = b, A_{t+1} = a, B^t, A^t, \bar{X}^t) = P_h(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t) \).

See Section V for a justification of why the above facts are true. It then follows that

\[ \sum_{i \in S} P_{h'}(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t) \exp(-Z_{hh'}(t)) \]

\[ = \sum_{i \in S} P_h(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t) \exp(-Z_{hh'}(t)) P_h(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t) \]

\[ = \exp(-Z_{hh'}(t + 1, a, i)) P_h(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t). \]  

(68)

where in (68), the quantity \( Z_{hh'}(t + 1, a, i) \) is defined as

\[ Z_{hh'}(t + 1, a, i) := Z_{hh'}(t) + \log \frac{P_h(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t)}{P_{h'}(\bar{X}_{t+1} = i|A_{t+1} = a, A^t, \bar{X}^t)}. \]

Substituting (68) in (67) and simplifying, we get

\[ E_{h'}[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})|\mathcal{F}_t] \exp(-Z_{hh'}(t)) \]

\[ = \sum_{b=1}^{K} \sum_{a=1}^{K} \sum_{i \in S} \left[ g(B^t, A^t, \bar{X}^t, b, a, i) \cdot P_h(B_{t+1} = b|B^t, A^t, \bar{X}^t) \right. \\
\]

\[ \left. \cdot P_h(A_{t+1} = a|B^{t+1} = b, B^t, A^t, \bar{X}^t) \cdot P_h(\bar{X}_{t+1} = i|A_{t+1} = a, B^t, A^t, \bar{X}^t) \cdot \exp(-Z_{hh'}(t)) \right]. \]
\[ P_h(A_{t+1} = a|B_{t+1} = b, B^t, A^t, \bar{X}^t) \cdot P_h(\bar{X}_{t+1} = i|B_{t+1} = b, A_{t+1} = a, B^t, A^t, \bar{X}^t) \cdot \exp(-Z_{hh'}(t + 1, a, i)) \]

(69)

\[ = E_h[g(B^{t+1}, A^{t+1}, \bar{X}^{t+1})|F_t]. \]

(70)

Applying \( E_h[\cdot] \) to both sides of (70) and using (66) along with the law of iterated expectations, we arrive at the desired relation. This proves (69) for all \( t \geq 0 \).

Finally, for any \( E \in \mathcal{F}_{\tau(\pi)} \), we have

\[ P_{h'}(E) = E_{h'}[1_E] \]
\[ = E_{h'} \left[ \sum_{t \geq 0} 1_{E \cap \{\tau(\pi) = t\}} \right] \]
\[ = \sum_{t \geq 0} E_{h'} \left[ 1_{E \cap \{\tau(\pi) = t\}} \right] \]
\[ = \sum_{t \geq 0} E_h \left[ 1_{E \cap \{\tau(\pi) = t\}} \exp(-Z_{hh'}(t)) \right] \]
\[ = \sum_{t \geq 0} E_h \left[ 1_{E \cap \{\tau(\pi) = t\}} \exp(-Z_{hh'}(\tau(\pi))) \right] \]
\[ = E_h [1_E \exp(-Z_{hh'}(\tau(\pi)))], \quad (71) \]

where (a) is due to monotone convergence theorem, and (b) above follows from (60) and the fact that \( E \cap \{\tau(\pi) = t\} \in \mathcal{F}_t \) for all \( t \geq 0 \) since \( E \in \mathcal{F}_{\tau(\pi)} \). This completes the proof of the lemma.

Lemma [5] in conjunction with [12, Lemma 19], yields the following inequality for all policies \( \pi \in \Pi(\epsilon) \) and all \( h' \neq h \):

\[ E_h[Z_{hh'}^n(\tau(\pi))] \geq \sup_{E \in \mathcal{F}_{\tau(\pi)}} d(P_h(E), P_{h'}(E)), \quad (72) \]

where for any \( x, y \in [0, 1] \),

\[ d(x, y) := x \log(x/y) + (1 - x) \log((1 - x)/(1 - y)) \]

is the binary relative entropy function. As noted in [12], \( x \mapsto d(x, y) \) is monotone increasing for \( x < y \) and the \( y \mapsto d(x, y) \) is monotone decreasing for any fixed \( x \). Also, for any \( \pi \in \Pi(\epsilon) \), we have

\[ P_h(\theta(\pi) = h) \geq 1 - \epsilon, \quad P_h(\theta(\pi) = h') \leq \epsilon \]

for all \( h' \neq h \). Combining the aforementioned facts, we get

\[ \min_{h' \neq h} E_h[Z_{hh'}^n(\tau(\pi))] \geq d(\epsilon, 1 - \epsilon). \quad (73) \]

for all \( \pi \in \Pi(\epsilon) \).

**B. Proceeding Towards the Lower Bound**

We now obtain an upper bound for the left hand side of (73). Fix \( \pi \in \Pi(\epsilon) \) and \( h' \neq h \) arbitrarily. Then, from (30),

\[ E_h[Z_{hh'}^n(\tau(\pi))] \]

(74)

\[ = E_h \left[ \sum_{a=1}^{K} \log \frac{P_h(X_{a-1}^n)}{P_h'(X_{a-1}^n)} \right] + E_h \left[ \sum_{(d, j) \in S} \sum_{a=1}^{K} N(\tau(\pi), d, j, a) \log \left( \frac{(P_h^n)^{d_{a,j}}(j|a)}{(P_h')^{d_{a,j}}(j|a)} \right) \right]. \]

To simplify the second expectation term on the right hand side of (74), we use the following lemma.
Lemma 6. Fix $h \in \mathcal{A}$. For any policy $\pi$, let $E_h$ and $E_{h'}$ denote the expectations computed under hypothesis $\mathcal{H}_h$ and under policy $\pi$. Then, for all $(d, i) \in \mathcal{S}$, $a \in \mathcal{A}$ and $j \in \mathcal{S}$,

$$E_h[E_a[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] = E_h[E_a[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] (P_h^a)_{d_a}^{(j|i_a)}. \quad (75)$$

Proof of Lemma 6: Substituting $n = \tau(\pi)$ in (29), we have

$$E_h[E_a[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] = E_h \left[ \sum_{t=K}^{\tau(\pi)} \mathbb{1}_{d(t) = d, i(t) = i, A_t = a, X_t^a = j} \left| X_{a-1}^a \right| \tau(\pi) \right] = E_h \left[ \sum_{t=K}^{\tau(\pi)} P_h(d(t) = d, i(t) = i, A_t = a, X_t^a = j | X_{a-1}^a) \right]. \quad (76)$$

For each $t$ in the range of the summation in (76), the conditional probability term for $t$ may be expressed as

$$P_h(d(t) = d, i(t) = i, A_t = a, X_t^a = j | X_{a-1}^a) = P_h(d(t) = d, i(t) = i, A_t = a | X_{a-1}^a) \cdot P_h(X_t^a = j | A_t = a, d(t) = d, i(t) = i, X_{a-1}^a) = P_h(d(t) = d, i(t) = i, A_t = a | X_{a-1}^a) \cdot (P_h^a)_{d_a}^{(j|i_a)}. \quad (77)$$

Plugging (77) back in (76) and simplifying, we arrive at the desired relation in (75).

Using Lemma 6, the second expectation term on the right hand side of (74) can be simplified as follows.

$$E_h \left[ \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} \sum_{j \in \mathcal{S}} N(\tau(\pi), d, i, a, j) \log \frac{(P_h^a)_{d_a}^{(j|i_a)}}{(P_h^a)_{d_a}^{(j|i_a)}} \right] = E_h \left[ \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} \sum_{j \in \mathcal{S}} N(\tau(\pi), d, i, a, j) \log \frac{(P_h^a)_{d_a}^{(j|i_a)}}{(P_h^a)_{d_a}^{(j|i_a)}} \right] \approx E_h \left[ \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} \sum_{j \in \mathcal{S}} E_h[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] \log \frac{(P_h^a)_{d_a}^{(j|i_a)}}{(P_h^a)_{d_a}^{(j|i_a)}} \right] = E_h \left[ \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} \sum_{j \in \mathcal{S}} E_h[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] \cdot (P_h^a)_{d_a}^{(j|i_a)} \cdot \log \frac{(P_h^a)_{d_a}^{(j|i_a)}}{(P_h^a)_{d_a}^{(j|i_a)}} \right]$$

$$= \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} E_h[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] \cdot D((P_h^a)_{d_a}^{(j|i_a)} || (P_h^a)_{d_a}^{(j|i_a)})$$

where in the above set of equations, $(a)$ follows from Lemma 6 and (78) is due to monotone convergence theorem and the fact that

$$E_h[E_a[N(\tau(\pi), d, i, a) | X_{a-1}^a] | \tau(\pi)] = E_h[N(\tau(\pi), d, i, a)].$$

Plugging (78) back in (74), we get

$$E_h[Z_{hh'}(\tau(\pi))] = E_h \left[ \sum_{a=1}^{K} \log \frac{P_h(X_{a-1}^a)}{P_h(X_{a-1}^a)} \right] + \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} E_h[N(\tau(\pi), d, i, a) \cdot D((P_h^a)_{d_a}^{(j|i_a)} || (P_h^a)_{d_a}^{(j|i_a)})]. \quad (79)$$

Noting that

$$\sum_{a=1}^{K} \sum_{(d, i) \in \mathcal{S}} E_h[N(\tau(\pi), d, i, a)] \overset{(a)}{=} E_h \left[ \sum_{(d, i) \in \mathcal{S}} \sum_{a=1}^{K} N(\tau(\pi), d, i, a) \right]$$
where \((a)\) above is due to monotone convergence theorem, we write (79) as
\[
E_h[Z_{hh'}(\tau(\pi))]
\]
\[
= E_h\left[\sum_{a=1}^{K} \log \frac{P_h(X^a_n)}{P_{h'}(X^a_{n-1})}\right] + \left(E_h[\tau(\pi) - K + 1]\right) \cdot \sum_{(d, l) \in \mathbb{S}} \sum_{a=1}^{K} E_h[N(\tau(\pi), (d, l), a)] E_h[\tau(\pi) - K + 1] \cdot D((P^n_h)_{d,a}(\cdot|\pi)) \cdot (P^n_{h'})_{d,a}(\cdot|\pi).
\]
Combining (72) and (82), and noting that (82) holds for all \(h' \neq h\), we get
\[
d(\epsilon, 1 - \epsilon) \leq \min_{h' \neq h} \left\{ E_h\left[\sum_{a=1}^{K} \log \frac{P_h(X^a_n)}{P_{h'}(X^a_{n-1})}\right] + \left(E_h[\tau(\pi) - K + 1]\right) \cdot \sum_{(d, l) \in \mathbb{S}} \sum_{a=1}^{K} E_h[N(\tau(\pi), (d, l), a)] E_h[\tau(\pi) - K + 1] \cdot D((P^n_h)_{d,a}(\cdot|\pi)) \cdot (P^n_{h'})_{d,a}(\cdot|\pi)\right\}
\]
\[
\leq \sup_{\nu} \min_{h' \neq h} \left\{ E_h\left[\sum_{a=1}^{K} \log \frac{P_h(X^a_n)}{P_{h'}(X^a_{n-1})}\right] + \left(E_h[\tau(\pi) - K + 1]\right) \cdot \sum_{(d, l) \in \mathbb{S}} \sum_{a=1}^{K} \nu(d, l, a) D((P^n_h)_{d,a}(\cdot|\pi)) \cdot (P^n_{h'})_{d,a}(\cdot|\pi)\right\},
\]
where the supremum in (83) is over all state-action occupancy measures satisfying
\[
\sum_{a=1}^{K} \nu(d, l, a) = \sum_{(d', l') \in \mathbb{S}} \sum_{a=1}^{K} \nu(d', l', a) Q(d, l|d', l', a) \quad \text{for all} \quad (d, l) \in \mathbb{S},
\]
\[
\sum_{(d, l) \in \mathbb{S}} \sum_{a=1}^{K} \nu(d, l, a) = 1,
\]
\[
\nu(d, l, a) \geq 0 \quad \text{for all} \quad (d, l, a) \in \mathbb{S} \times \mathbb{A}.
\]
Recall that \(Q\) in (84) denotes the transition probability matrix given by (7). The left hand side of (84) represents the long-term probability of leaving the state \((d, l)\), while the right hand side of (85) represents the long-term probability of entering into the state \((d, l)\). Thus, (84) is the global balance equation for the controlled Markov process \(\{(d(t), i(t)) : t \geq K\}\). Equations (85) and (86) together imply that \(\nu\) is a probability measure on \(\mathbb{S} \times \mathbb{A}\).

As outlined in Section 8.8, the controlled Markov process \(\{(d(t), i(t)) : t \geq K\}\), together with the sequence \(\{B_t : t \geq 0\}\) of intended arm selections (or equivalently the sequence \(\{A_t : t \geq 0\}\) of actual arm selections), defines a Markov decision problem (MDP) with state space \(\mathbb{S}\) and action space \(\mathbb{A}\). Note that \(\mathbb{S}\) is a countable set. A simple extension of [15, Theorem 8.8.2] to countable state space MDPs, in conjunction with Lemma 1 implies a one-one correspondence between any feasible solution to (84)-(86) and policies in \(\Pi_{\text{SRS}}\). In other words, [15, Theorem 8.8.2] implies that for any given \(\nu\) satisfying (84)-(86), we can find a SRS policy \(\pi^\lambda \in \Pi_{\text{SRS}}\) such that \(\nu^\lambda(d, l, a) = \nu(d, l, a)\) for all \((d, l, a) \in \mathbb{S} \times \mathbb{A}\). Recall that under the SRS policy \(\pi^\lambda\), the controlled Markov process \(\{(d(t), i(t)) : t \geq K\}\) is indeed a Markov process whose transition probability matrix is ergodic (Lemma 1) and possesses \(\mu^\lambda\) as its unique stationary distribution. The associated ergodic state occupancy measure, \(\nu^\lambda\), is then defined according to (15).
On account of [15, Theorem 8.8.2], we may replace the supremum in (83) by a supremum over all SRS policies. Doing so leads us to the relation

\[
d(\epsilon, 1 - \epsilon) \leq \sup_{\pi \in \Pi_{\text{srs}}} \min_{h' \neq h} \left\{ E_h \left[ \sum_{a=1}^{K} \log \frac{P_h(X_{a-1}^n)}{P_{h'}(X_{a-1}^n)} \right] \right. \\
+ \left. \left( E_h[\tau(\pi) - K + 1] \right) \cdot \sum_{(d,i) \in S} \sum_{a=1}^{K} \epsilon(h, a) D((P_h^a)^{(d)} (\cdot| i_a)) ((P_{h'}^a)^{(d)} (\cdot| i_a)) \right\}
\]

for all \( \pi \in \Pi(\epsilon) \). Observe that the constant term multiplying \( E_h[\tau(\pi) - K + 1] \) in (87) is finite; further, it is not a function of either \( \epsilon \) or of \( \pi \in \Pi(\epsilon) \). The finiteness of \( R^*(P_1, P_2) \) follows from the following observation: denote by \( \mu_h^n \) the stationary distribution of the transition probability matrix \( P_h^n \) (i.e., \( \mu_h^n = \mu_1 \) for \( a = h \) and \( = \mu_2 \) for all \( a \neq h \)), an application of the ergodic theorem to the Markov process of arm \( a \) yields

\[
D((P_h^a)^{(d)} (\cdot| i_a)) ((P_{h'}^a)^{(d)} (\cdot| i_a)) \to D(\mu_h^n || \mu_{h'}^n) < \infty \quad \text{as} \quad d_a \to \infty,
\]

where the finiteness of the limit above follows from the assumption that for all \( i \in S \), \( \mu_1(i) > 0 \) if and only if \( \mu_2(i) > 0 \) (see Section I). Since every convergent sequence is bounded, we may write \( D((P_h^a)^{(d)} (\cdot| i_a)) ((P_{h'}^a)^{(d)} (\cdot| i_a)) \leq C \) for all \((d,i,a) \in S \times A \), where \( 0 < C < \infty \). Using (85), it follows that \( R^*(P_1, P_2) \) is bounded above by \( C \).

Let us also note that the first term inside the braces in (87) does not depend on \( \epsilon \). Since \( d(\epsilon, 1 - \epsilon) \to d(0,1) = +\infty \) as \( \epsilon \to 0 \), the boundedness of \( R^*(P_1, P_2) \) shows that \( \epsilon \to 0 \) is equivalent to \( E_h[\tau(\pi)] \to \infty \) for all \( \pi \in \Pi(\epsilon) \). Letting \( \epsilon \to 0 \), and using \( d(\epsilon, 1 - \epsilon)/\log(1/\epsilon) \to 1 \) as \( \epsilon \to 0 \), we arrive at the lower bound in (12).

**APPENDIX C**

**PROOF OF LEMMA 2**

Notice that the key ingredient in the proof of Lemma 1 is the fact that for each \( t \geq K \), the probability term in (50) is \( \geq \eta/K > 0 \) whenever the trembling hand parameter \( \eta > 0 \). This property holds true even for the policy \( \pi^*(L, \delta) \). We leverage this to first show below that for all \((d,i) \in S \),

\[
\lim_{n \to \infty} \inf_{n \to \infty} \frac{N(n,d,i)}{n} > 0 \quad a.s.
\]

under the policy \( \pi^*(L, \delta) \), where for each \( n \geq K \),

\[
N(n,d,i) := \sum_{t=K}^{n} I(d(t) = d, i(t) = i)
\]

denotes the number of times the controlled Markov process \{\( (d(t), i(t)) : t \geq K \)\} visits the state \((d,i)\). From [17, Proposition 1.7], we know that there exists an integer \( M \) such that for all \( m \geq M \),

\[
P_1^m(j|i) > 0 \quad \text{for all} \quad i,j \in S, \quad P_2^m(j|i) > 0 \quad \text{for all} \quad i,j \in S.
\]

Fix an arbitrary \((d,i) \in S \), and assume without loss of generality that \( d_1 > d_2 > \cdots > d_K = 1 \). Also assume, again without loss of generality, that the controlled Markov process \{\( (d(t), i(t)) : t \geq K \)\} starts in the state \((d,i)\), i.e., \( d(K) = d, i(K) = i \). From Appendix A, we know that the probability of the process \{\( (d(t), i(t)) : t \geq K \)\} starting in the state \((d,i)\) and returning back to the state \((d,i)\) after \( M + d_1 - d_K \) time instants, call this \( p(d,i) \), is lower bounded by the quantity in (53). Since (53) is strictly positive, it follows that \( p(d,i) > 0 \).

Clearly, then, the term \( N(n,d,i) \) may be lower bounded a.s. by the number of visits to the state \((d,i)\) measured only at times \( t = K + M + d_1 - d_K, K + 2(M + d_1 - d_K), K + 3(M + d_1 - d_K) \) and so on until time \( t = n \). Note that at each
of these time instants, the probability that the process \( \{(d(t), \dot{i}(t)) : t \geq K\} \) is in the state \((d, \dot{i})\) is equal to \(p(d, \dot{i})\). Thus, we have
\[
N(n, d, \dot{i}) \geq \text{Bin} \left( \frac{n - K + 1}{M + d_1 - d_K}, \ p(d, \dot{i}) \right) \quad \text{a.s.,} \tag{92}
\]
where the notation Bin\((m, q)\) denotes a Binomial random variable with parameters \(m\) and \(q\). It then follows that
\[
\lim \inf_{n \to \infty} \frac{N(n, d, \dot{i})}{n} \geq \lim \inf_{n \to \infty} \frac{\text{Bin} \left( \frac{n - K + 1}{M + d_1 - d_K}, \ p(d, \dot{i}) \right)}{n} = \lim \inf_{n \to \infty} \frac{\text{Bin} \left( \frac{n - K + 1}{M + d_1 - d_K}, \ p(d, \dot{i}) \right)}{n - K + 1} \cdot \frac{1}{M + d_1 - d_K} \tag{93}
\]
where \((a)\) above follows from the strong law of large numbers. This establishes \((89)\).

We now show that for all \(a \in A\),
\[
\lim \inf_{n \to \infty} \frac{N(n, d, \dot{i}, a)}{n} > 0 \quad \text{a.s.} \tag{94}
\]
Subsequently, we use \((94)\) to establish \((32)\). Fix an arbitrary \(a \in A\), and define
\[
S(n, d, \dot{i}, a) := \sum_{t=K}^{n} \left[ I_{\{A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}\}} | - P(A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1}) \right]. \tag{95}
\]
For each \(t \geq K\), since \(I_{\{A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}\}} | - P(A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1}) | \leq 2 \) \(\text{a.s.}\), and
\[
E[I_{\{A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}\}} | - P(A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1})] |A^{t-1}, \dot{X}^{t-1}] = 0,
\]
the collection \(\{I_{\{A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}\}} \} \) is a bounded martingale difference sequence. Using \([18\text{ Theorem 1.2A}]\), we get that
\[
\frac{S(n, d, \dot{i}, a)}{n} \to 0 \quad \text{a.s.} \tag{96}
\]
as \(n \to \infty\). This implies that for every choice of \(\varepsilon > 0\), there exists \(N_\varepsilon\) sufficiently large such that
\[
\frac{N(n, d, \dot{i}, a)}{n} \geq \frac{1}{n} \sum_{t=K}^{n} P(A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1}) - \varepsilon \quad \text{for all } n \geq N_\varepsilon. \tag{97}
\]
Now, for each \(t \geq K\),
\[
P(A_t = a, \dot{d}(t) = \dot{d}, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1})
= P(A_t = a|d(t) = d, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1}) \cdot P(d(t) = d, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1})
= \frac{\eta}{K} + (1 - \eta) \lambda_{\theta(t)}(a | d, \dot{i}) \cdot P(d(t) = d, \dot{i}(t) = \dot{i}|B^{t-1}, A^{t-1}, \dot{X}^{t-1})
\geq \frac{\eta}{K} \cdot I_{\{d(t) = d, \dot{i}(t) = \dot{i}\}}, \tag{98}
\]
where \((98)\) follows from the fact that \(d(t)\) and \(\dot{i}(t)\) are measurable with respect to the history \((B^{t-1}, A^{t-1}, \dot{X}^{t-1})\). Plugging \((98)\) in \((97)\), we get
\[
\frac{N(n, d, \dot{i}, a)}{n} \geq \frac{\eta}{K} \cdot \frac{N(n, d, \dot{i})}{n} - \varepsilon \quad \text{a.s.} \tag{99}
\]
Using (102) in (101), we get that for every choice of $C < i,j$
for all sufficiently large values of $n$. Setting $\varepsilon = \frac{\eta}{K} \cdot \frac{p(d,i)}{2(M+d_1-d_K)} - \varepsilon$
for all sufficiently large values of $n$. Setting $\varepsilon = \frac{\eta}{K} \cdot \frac{p(d,i)}{2(M+d_1-d_K)}$ establishes (94).

**Proof of Lemma 2** For any $h' \neq h$, we have
\[
\frac{1}{n} Z_{hh'} (n) = \sum_{(d,i) \in S} \sum_{j \in S} \frac{N(n, d, i, h, j)}{n} \log \frac{P_{1d}^h(j|i_h)}{P_{2d}^h(j|i_h)} + \frac{N(n, d, i, h', j)}{n} \log \frac{P_{2d}^{h'}(j|i_{h'})}{P_{2d}^{h'}(j|i_{h'})}.
\] (101)
Since $N(n, d, i, a) \to \infty$ a.s. as $n \to \infty$ (this follows from the fact that $\lim_{n \to \infty} N(n, d, i, a)/n > 0$ established earlier) for every $a \in A$, we apply the Ergodic theorem to deduce that
\[
\frac{N(n, d, i, a, j)}{N(n, d, i, a)} \to (P_{1d}^a)^{d_a}(j|a) \text{ as } n \to \infty \text{ a.s.}
\] (102)
Using (102) in (101), we get that for every choice of $\varepsilon$, there exists $N_\varepsilon$ sufficiently large such that for all $n \geq N_\varepsilon$,
\[
\frac{1}{n} Z_{hh'} (n) \geq \sum_{(d,i) \in S} \sum_{j \in S} \frac{N(n, d, i, h, j)}{n} (P_{1d}^h(j|i_h) + \varepsilon) \log P_{1d}^h(j|i_h)
+ \sum_{(d,i) \in S} \sum_{j \in S} \frac{N(n, d, i, h, j)}{n} (P_{1d}^h(j|i_h) - \varepsilon) \log \frac{1}{P_{2d}^h(j|i_h)}
+ \sum_{(d,i) \in S} \sum_{j \in S} \frac{N(n, d, i, h', j)}{n} (P_{2d}^{h'}(j|i_{h'}) + \varepsilon) \log P_{2d}^{h'}(j|i_{h'})
+ \sum_{(d,i) \in S} \sum_{j \in S} \frac{N(n, d, i, h', j)}{n} (P_{2d}^{h'}(j|i_{h'}) - \varepsilon) \log \frac{1}{P_{1d}^{h'}(j|i_{h'})}
= \sum_{(d,i) \in S} \frac{N(n, d, i, h, j)}{n} D(P_{1d}^h(\cdot|i_h)\|P_{2d}^{h'}(\cdot|i_{h'})) + \sum_{(d,i) \in S} \frac{N(n, d, i, h', j)}{n} D(P_{2d}^{h'}(\cdot|i_{h'})\|P_{1d}^h(\cdot|i_h))
+ \sum_{(d,i) \in S} \frac{N(n, d, i, h, j)}{n} \left( \sum_{j \in S} \log P_{1d}^h(j|i_h) P_{2d}^{h'}(j|i_{h'}) \right) + \sum_{(d,i) \in S} \frac{N(n, d, i, h', j)}{n} \left( \sum_{j \in S} \log P_{2d}^{h'}(j|i_{h'}) P_{1d}^h(\cdot|i_h) \right).
\] (103)
As a consequence of the convergence theorem for finite state Markov processes [17, Theorem 4.9], we have
\[
P_{1d}^h(j|i) \to \mu_1(j) > 0 \text{ as } d \to \infty
P_{2d}^h(j|i) \to \mu_2(j) > 0 \text{ as } d \to \infty
\] (104)
for all $i, j \in S$. This implies that the term inside the square brackets in (103) is bounded from below (say by a constant $C < 0$). We then have
\[
\frac{1}{n} Z_{hh'} (n) \geq \sum_{(d,i) \in S} \frac{N(n, d, i, h, j)}{n} D(P_{1d}^h(\cdot|i_h)\|P_{2d}^{h'}(\cdot|i_{h'})) + \sum_{(d,i) \in S} \frac{N(n, d, i, h', j)}{n} D(P_{2d}^{h'}(\cdot|i_{h'})\|P_{1d}^h(\cdot|i_h)) + C \varepsilon
\] (105)
for all $(d,i) \in S$ and for all $n \geq N_\varepsilon$. Now, fix an arbitrary $(d,i) \in S$ such that $d_1 > d_2 > \cdots > d_K = 1$. From (99), we know that there exist constants $N_h, N_{h'}$ sufficiently large such that
\[
\frac{N(n, d, i, h, j)}{n} \geq \frac{\eta}{K} \cdot \frac{p(d,i)}{2(M+d_1-d_K)} - \varepsilon, \quad \frac{N(n, d, i, h', j)}{n} \geq \frac{\eta}{K} \cdot \frac{p(d,i)}{2(M+d_1-d_K)} - \varepsilon
\] (106)
for all $n \geq \max\{N_h, N_{h'}, N_\varepsilon\}$. Combining (106) and (105), we may choose $\varepsilon > 0$ appropriately so that the right hand side of (105) is strictly positive. This establishes the desired result.
APPENDIX D
PROOF OF LEMMA \(3\)

The policy \(\pi^*(L, \delta)\) commits error if one of the following events is true:

1) The policy never stops in finite time.
2) The policy stops in finite time and declares \(h' \neq h\) as the true index of the odd arm.

The event in item 1 above has zero probability as a consequence of Lemma \(2\). Thus, the probability of error of policy \(\pi = \pi^*(L, \delta)\) may be evaluated as follows: suppose \(\mathcal{H}_h\) is the true hypothesis. Then,

\[
P_h(\theta(\pi) \neq h) = P_h \left( \exists \ n \text{ and } h' \neq h \text{ such that } \theta(\pi) = h' \text{ and } \tau(\pi) = n \right).
\]

We now let

\[
\mathcal{R}_{h'}(n) := \{ \omega : \tau(\pi)(\omega) = n, \theta(\pi)(\omega) = h' \}
\]

denote the set of all sample paths for which the policy stops at time \(n\) and declares \(h'\) as the true index of the odd arm. Clearly, \(\{\mathcal{R}_{h'}(n) : h' \neq h, n \geq 0\}\) is a collection of mutually disjoint sets. Therefore, we have

\[
P_h(I(\pi \neq h)) = P_h \left( \bigcup_{h' \neq h} \bigcup_{n=0}^{\infty} \mathcal{R}_{h'}(n) \right)
\]

\[
= \sum_{h' \neq h} \sum_{n=0}^{\infty} P_h(\tau(\pi) = n, \theta(\pi) = h')
\]

\[
= \sum_{h' \neq h} \sum_{n=0}^{\infty} \int_{\mathcal{R}_{h'}(n)} dP_h(\omega)
\]

\[
\overset{(a)}{=} \sum_{h' \neq h} \sum_{n=0}^{\infty} \int_{\mathcal{R}_{h'}(n)} \exp(Z_h(n)(\omega)) \ d(B^n(\omega), A^n(\omega), \bar{X}^n(\omega))
\]

\[
\overset{(b)}{=} \sum_{h' \neq h} \sum_{n=0}^{\infty} \int_{\mathcal{R}_{h'}(n)} \exp(-Z_{h'}(n)(\omega)) \exp(Z_h(n)(\omega)) \ d(B^n(\omega), A^n(\omega), \bar{X}^n(\omega))
\]

\[
\overset{(c)}{=} \sum_{h' \neq h} \sum_{n=0}^{\infty} \left\{ \int_{\mathcal{R}_{h'}(n)} \frac{1}{(K-1)\lambda} dP_{h'}(\omega) \right\}
\]

\[
= \sum_{h' \neq h} \frac{1}{(K-1)\lambda} P_{h'} \left( \bigcup_{n=0}^{\infty} \mathcal{R}_{h'}(n) \right) \leq \frac{1}{\lambda}.
\]

where in (a) above,

\[
Z_h(n) := \log P_h(B^n, A^n, \bar{X}^n)
\]
denotes the log-likelihood of all intended arm pullings, actual arm pullings and observations up to time \(n\) under hypothesis \(\mathcal{H}_h\). (b) above follows by noting that for \(h \neq h'\), \(Z_{h'}(n) = Z_h(n) - Z_{h'}(n) = -Z_{h'}(n)\), and (c) follows from the fact that when \(\mathcal{H}_{h'}\) is the true hypothesis, the condition \(M_{h'}(n) \geq \log((K-1)\lambda)\) is satisfied, which in particular implies that \(Z_{h'}(n) \geq \log((K-1)\lambda)\). It is then clear that setting \(L = 1/\epsilon\) yields the desired result.

APPENDIX E
PROOF OF PROPOSITION \(2\)

This section is organised as follows. First, we show in Proposition \(3\) that under the policy \(\pi^*(L, \delta)\), the test statistic \(M_h(n)\) has the right drift, one that comes from the ergodic occupancy measure corresponding to \(\pi^{\lambda_h, \delta}\) when \(\mathcal{H}_h\) is the true hypothesis.
We then show in Lemma 7 that the stopping time of the policy $\pi^*(L, \delta)$ grows with $L$ (i.e., lower probability of error implies more time required to stop and declare the odd arm location correctly with high confidence). More specifically, we show in Lemma 8 that ratio $\tau(\pi) / \log L$ has, in the limit as $L \to \infty$, an a.s. upper bound that matches with the right hand side of (33). Finally, we prove in Proposition 4 that the family $\{\tau(\pi) / \log L : L \geq 1\}$ is uniformly integrable. The a.s. convergence of Lemma 8 combined with uniform integrability of Proposition 4 yields the desired upper bound in (33).

**Proposition 3.** Fix an arbitrary $L > 1$, $\delta > 0$ and $h \in \mathcal{A}$, and let $\mathcal{H}_h$ be the true hypothesis. For every $h' \neq h$, under the non-stopping version of policy $\pi^*(L, \delta)$, we have

$$\lim_{n \to \infty} \frac{Z_{hh'}(n)}{n} = \sum_{(\tilde{d}, \tilde{i}) \in \mathbb{S}} \nu^{\lambda, \delta}(\tilde{d}, \tilde{i}, h) D(P_{1}^{d_{h}}(\cdot|\tilde{i}_{h}) \| P_{2}^{d_{h}}(\cdot|\tilde{i}_{h})) + \nu^{\lambda, \delta}(\tilde{d}, \tilde{i}, h') D(P_{2}^{d_{h'}}(\cdot|\tilde{i}_{h'}) \| P_{1}^{d_{h'}}(\cdot|\tilde{i}_{h})).$$

(110)

Consequently, it follows that

$$\lim_{n \to \infty} \frac{M_{h}(n)}{n} = \min_{h' \neq h} \sum_{(\tilde{d}, \tilde{i}) \in \mathbb{S}} \nu^{\lambda, \delta}(\tilde{d}, \tilde{i}, h) D(P_{1}^{d_{h}}(\cdot|\tilde{i}_{h}) \| P_{2}^{d_{h}}(\cdot|\tilde{i}_{h})) + \nu^{\lambda, \delta}(\tilde{d}, \tilde{i}, h') D(P_{2}^{d_{h'}}(\cdot|\tilde{i}_{h'}) \| P_{1}^{d_{h'}}(\cdot|\tilde{i}_{h})).$$

(111)

**Proof of Proposition 3.** From Lemma 2 it follows that when $\mathcal{H}_h$ is the true hypothesis,

$$\liminf_{n \to \infty} \frac{M_{h}(n)}{n} = \liminf_{n \to \infty} \min_{h' \neq h} \frac{Z_{hh'}(n)}{n} > 0 \text{ a.s..}$$

(112)

This in turn implies that $\liminf_{n \to \infty} M_{h}(n) > 0$ a.s.. An immediate consequence of this is that for any $h' \neq h$,

$$\limsup_{n \to \infty} M_{h'}(n) = \limsup_{n \to \infty} \min_{a \neq h'} Z_{h'a}(n) \leq \limsup_{n \to \infty} Z_{h'h}(n) \leq \limsup_{n \to \infty} -Z_{hh'}(n) = -\liminf_{n \to \infty} Z_{hh'}(n) \leq -\liminf_{n \to \infty} M_{h}(n) < 0.$$ 

(113)

The above set of inequalities imply the following important result: suppose $\mathcal{H}_h$ is the true hypothesis. Then, for any $L > 1$ and $\delta > 0$, under the non-stopping version of policy $\pi^*(L, \delta)$, we have

$$\theta(n) = h \text{ a.s. for all sufficiently large values of } n.$$ 

(114)

The condition in (114) implies that for all $(\tilde{d}, \tilde{i}, a) \in \mathbb{S} \times \mathcal{A}$,

$$\lim_{n \to \infty} P(A_{n} = a|d(n) = \tilde{d}, \tilde{i}(n) = \tilde{i}, \{(d(t), \tilde{i}(t)) : K \leq t < n\}) = \lim_{n \to \infty} \frac{\eta}{K} + (1 - \eta) \lambda_{\theta(n), \delta}(a|d, \tilde{i})$$

$$= \frac{\eta}{K} + (1 - \eta) \lambda_{h, \delta}(a|d, \tilde{i})$$

(115)

which in turn leads to the following convergences a.s. as $n \to \infty$:

$$\frac{N(n, d, \tilde{i}, a)}{\sum_{(\tilde{d}, \tilde{i}) \in D} N(n, \tilde{d}, \tilde{i})} \to \frac{\eta}{K} + (1 - \eta) \lambda_{h, \delta}(a|d, \tilde{i}),$$

(116)

$$\frac{\sum_{(\tilde{d}, \tilde{i}) \in D} N(n, \tilde{d}, \tilde{i})}{n} \to \mu^{\lambda, \delta}(d, \tilde{i}).$$

(117)

It now follows that for any $h' \neq h$,

$$\lim_{n \to \infty} \frac{Z_{hh'}(n)}{n}$$

...
implies that the limit supremum on the right hand side of (121) is equal to 0. From the convergences in (104), we note that the coefficient of $\delta$ can be bounded. We now show that the stopping time of policy $\pi^*(L, \delta)$ grows with $L$.

**Lemma 7.** Fix $h \in A$ and $\delta > 0$, and suppose that $\mathcal{H}_h$ is the true hypothesis. Then, under policy $\pi = \pi^*(L, \delta)$, we have

$$\lim_{L \to \infty} \inf \tau(\pi) = \infty \text{ a.s.}$$  \hspace{1cm} (119)

**Proof of Lemma 7.** Assume without loss of generality that the policy $\pi = \pi^*(L, \delta)$ pulls arm 1 at time $t = 0$, arm 2 at time $t = 1$ and so on until arm $K$ at time $t = K - 1$. In order to prove the lemma, we note that it suffices to prove the following statement:

for each $m \geq K$, \hspace{1cm} \lim_{L \to \infty} P_h(\tau(\pi) \leq m) = 0. \hspace{1cm} (120)

Fix $m \geq K$, and note that

$$\limsup_{L \to \infty} P_h(\tau(\pi) \leq m) = \limsup_{L \to \infty} P_h \left( \exists K \leq n \leq m \text{ and } \tilde{h} \in A \text{ such that } M_{\tilde{h}}(n) > \log((K-1)L) \right) \hspace{1cm} (121)$$

where the first inequality above follows from the union bound, and the second inequality is due to Markov’s inequality.

We now show that for each $n \in \{K, \ldots, m\}$, the expectation term inside the summation in (121) is finite. This will then imply that the limit supremum on the right hand side of (121) is equal to 0, thus proving the desired result. Note that

$$M_{\tilde{h}}(n) = \min_{h \neq \tilde{h}} Z_{hh'}(n) \leq Z_{hh'}(n) \text{ for all } h' \neq \tilde{h}. \hspace{1cm} (122)$$

Fix an arbitrary $h' \neq \tilde{h}$. Then,

$$Z_{hh'}(n) = \sum_{(d, \tilde{j}) \in \mathcal{S}} \sum_{j \in \mathcal{S}} N(n, d, i, h, j) \log \frac{P^d_{h'}(j|i)}{P^d_{h}(j|i)} + N(n, d, \tilde{i}, h', j) \log \frac{P^d_{h'}(j|i)}{P^d_{h}(j|i)} \hspace{1cm} (123)$$

From the convergences in (104), we note that the coefficient of $n$ in (123) is finite. Thus, it follows that $E[M_{\tilde{h}}(n)] \leq E[Z_{hh'}(n)] \leq nC$ for all $h' \neq \tilde{h}$, where $C < \infty$ represents the constant multiplying $n$ in (123).

Going further, let $R_{\lambda, \delta}$ denote the right hand side of (111).
Lemma 8. Fix \( h \in \mathcal{A} \) and \( \delta > 0 \), and suppose that \( \mathcal{H}_h \) is the true hypothesis. Then, under policy \( \pi = \pi^*(L, \delta) \), we have

\[
\limsup_{L \to \infty} \frac{\tau(\pi)}{\log L} \leq \frac{1}{R_{\lambda, \delta}} \text{ a.s.},
\]

(124)

Proof: Note that as a consequence of Proposition [3] and Lemma [7], we have

\[
\lim_{L \to \infty} \frac{M_h(\tau(\pi))}{\tau(\pi)} = R_{\lambda, \delta} \text{ a.s.}
\]

(125)

We now show that for any \( h' \neq h \) and \( n \geq K \), the increment \( Z_{hh'}(n) - Z_{hh'}(n - 1) \) is bounded. Towards this,

\[
Z_{hh'}(n) - Z_{hh'}(n - 1)
= \log \frac{P_h(\hat{A}^n, \hat{X}^n)}{P_{h'}(\hat{A}^n, \hat{X}^n)} - \log \frac{P_h(\hat{A}^{n-1}, \hat{X}^{n-1})}{P_{h'}(\hat{A}^{n-1}, \hat{X}^{n-1})}
= \log \frac{P_h(\hat{A}^n|\hat{X}^{n-1})}{P_{h'}(\hat{A}^n|\hat{X}^{n-1})}
= \sum_{(d,j) \in S} \sum_{a=1}^K \sum_{j \in S} \mathbb{I}_{d(n) = d, j(n) = j, A_n = a, X_n = j} \log \frac{(P^n_h)^a_{d}(j|i_a)}{(P^n_{h'})_{d}(j|i_a)}
= \sum_{(d,j) \in S} \sum_{j \in S} \left[ \mathbb{I}_{d(n) = d, j(n) = j, A_n = h, X_n = j} \log \frac{P^n_{h}(j|i_{h})}{P^n_{h'}(j|i_{h'})} + \mathbb{I}_{d(n) = d, j(n) = j, A_n = h', X_n = j} \log \frac{P^n_{h'}(j|i_{h'})}{P^n_{h'}(j|i_{h'})} \right].
\]

(126)

We now note that whenever either the numerator or the denominator of the logarithmic terms in (126) is equal to 0, then the corresponding indicator function is also equal to 0. This, together with the convergences in (104), implies that the right hand side of (126) is bounded. This, together with the collection \( \{Z_{hh'}(n) - Z_{hh'}(n - 1) : 1 \leq n \leq K - 1\} \) of finitely many terms, each of which is finite, establishes the boundedness of the increments \( \mathbb{Z}_{hh'}(n) - \mathbb{Z}_{hh'}(n - 1) \) for all \( n \geq 1 \) and all \( h' \neq h \).

When \( \mathcal{H}_h \) is the true hypothesis, we note from the definition of stopping time \( \tau(\pi) \) that \( M_h(\tau(\pi) - 1) < \log((K - 1)L) \), which implies that there exists \( h'' \neq h \) such that \( Z_{hh''}(\tau(\pi) - 1) < \log((K - 1)L) \). Using this, we have

\[
\limsup_{L \to \infty} \frac{M_h(\tau(\pi))}{\log L} = \limsup_{L \to \infty} \min_{h' \neq h} \frac{Z_{hh'}(\tau(\pi))}{\log L}
\leq \limsup_{L \to \infty} \frac{Z_{hh''}(\tau(\pi))}{\log L}
\leq \limsup_{L \to \infty} \frac{\log((K - 1)L)}{\log L}
= 1 \text{ a.s.,}
\]

(127)

where (a) above is due to boundedness of the increments established above. Then, using (125) along with (127) yields

\[
\limsup_{L \to \infty} \frac{\tau(\pi)}{\log L} = \limsup_{L \to \infty} \left\{ \left( \frac{\tau(\pi)}{M_h(\tau(\pi))} \right) \left( \frac{M_h(\tau(\pi))}{\log L} \right) \right\}
\leq \left( \lim_{L \to \infty} \frac{\tau(\pi)}{M_h(\tau(\pi))} \right) \left( \limsup_{L \to \infty} \frac{M_h(\tau(\pi))}{\log L} \right)
\leq 1 \text{ a.s.,}
\]

(128)

thus completing the proof of the lemma.

Since, by definition, \( R_{\lambda, \delta} > R^{*}(P_1, P_2) \), it follows that

\[
\limsup_{L \to \infty} \frac{\tau(\pi)}{\log L} \leq \frac{1 + \delta}{R^{*}(P_1, P_2)} \text{ a.s.}
\]

(129)
We now prove that the family \( \{ \tau(\pi)/\log L : L > 1 \} \) is uniformly integrable; here, \( \pi = \pi^*(L, \delta) \). This, along with the a.s. convergence in (129), yields the desired upper bound in (33).

**Proposition 4.** For any fixed \( \delta > 0 \), the family of random variables \( \{ \tau(\pi^*(L, \delta))/\log L : L > 1 \} \) is uniformly integrable.

**Proof of Proposition 4** Fix \( h \in \mathcal{A} \), and suppose that \( \mathcal{H}_h \) is the true hypothesis. Then, in order to establish the desired uniform integrability, it suffices to show that

\[
\limsup_{L \to \infty} E_h \left[ \exp \left( \frac{\tau(\pi)}{\log L} \right) \right] < \infty. \tag{130}
\]

Towards this, let us first define

\[
D_{hh^*} := \sum_{(d, \delta) \in \delta} \sum_{a=1}^K \nu^{h, \delta}(d, i, a) D((P^a_n)_{d+a} \cdot | i_a) \| (P^a_n)_{d+a} \cdot | i_a). \tag{131}
\]

Let

\[
\tilde{n}(L) := \frac{4 \log((K-1)L)}{D_{hh^*}} + K - 1, \tag{132}
\]

and let

\[
u(L) := \exp \left( \frac{1 + \tilde{n}(L)}{\log L} \right). \tag{133}
\]

Let \( \pi_h^* = \pi^*_h(L, \delta) \) denote the version of policy \( \pi^*(L, \delta) \) that stops only upon declaring \( h \) as the index of the odd arm. Clearly, \( \tau(\pi_h^*) \geq \tau(\pi) \) a.s.. Then,

\[
\limsup_{L \to \infty} E_h \left[ \exp \left( \frac{\tau(\pi)}{\log L} \right) \right] = \limsup_{L \to \infty} \int_0^\infty P_h \left( \frac{\tau(\pi)}{\log L} > \log x \right) dx
\leq \limsup_{L \to \infty} \int_0^\infty P_h \left( \tau(\pi_h^*) \geq \lceil (\log x)(\log L) \rceil \right) dx
\leq \limsup_{L \to \infty} \left\{ \nu(L) + \int_{\nu(L)}^\infty P_h \left( \tau(\pi_h^*) \geq \lceil (\log x)(\log L) \rceil \right) dx \right\}
= \exp \left( \frac{4}{D_{hh^*}} \right) + \limsup_{L \to \infty} \sum_{n \geq \tilde{n}(L)} \exp \left( \frac{n + 1}{\log L} \right) P_h(M_h(n) < \log((K-1)L)), \tag{134}
\]

where \((a)\) above follows by upper bounding the probability term by 1 for all \( x \leq \nu(L) \). We now show that for each \( n \geq \tilde{n}(L) \), the corresponding probability term inside the summation has an exponential upper bound. It then follows that this exponential upper bound result results in the finiteness of the right hand side of (134), thus completing the proof of the proposition.

We now demonstrate the above stated exponential upper bound property in the following lemma. We conclude the paper with the proof of the lemma.

**Lemma 9.** Fix \( \delta > 0 \) and \( h \in \mathcal{A} \), and suppose that \( \mathcal{H}_h \) is the true hypothesis. There exist constants \( B > 0 \) and \( 0 < \theta < \infty \) independent of \( L \) such for all \( n \geq \tilde{n}(L) \),

\[
P_h(M_h(n) < \log((K-1)L)) \leq Be^{-\theta n}. \tag{135}
\]

**Proof of Lemma 9** Since

\[
P_h(M_h(n) < \log((K-1)L)) = P_h \left( \min_{h' \neq h} Z_{hh^*}(n) < \log((K-1)L) \right)
\leq \sum_{h' \neq h} P_h \left( Z_{hh^*}(n) < \log((K-1)L) \right); \tag{136}
\]
the last line above follows from the union bound. In order to prove the lemma, it suffices to show that each term inside the summation in (136) is exponentially bounded. Going further, we drop the superscript $\pi$ in $P_h(\cdot)$ for ease of notation.

Fix $h' \neq h$. Recall that under the hypothesis $\mathcal{H}_h$, the transition probability matrix of arm $h$ is $P_1$, while that of arm $h'$ is $P_2$, where $P_2 \neq P_1$. The latter condition of $P_2 \neq P_1$ implies that there exists $i^* \in \mathcal{S}$ such that $P_1(\cdot|i^*) \neq P_2(\cdot|i^*)$. Equivalently, we have

$$D(P_1(\cdot|i^*)||P_2(\cdot|i^*)) > 0, \quad D(P_2(\cdot|i^*)||P_1(\cdot|i^*)) > 0.$$  

Going further, let us fix an arbitrary $(d^*, i^*) \in \mathcal{S}$ such that $d_h^* = 1$ and $i_h^* = i^*$, where $i^*$ is as defined above.

For $n \geq K$, let

$$\Delta Z_{hh'}(n) := Z_{hh'}(n) - Z_{hh'}(n - 1)$$

\begin{equation}
= \sum_{(d, i) \in \mathcal{S}} \sum_{a = 1}^K \sum_{j \in \mathcal{S}} I(d(n) = d, i(n) = i, A_n = a, X_n = j) \log \left( \frac{(P_h^a)^{da}(j|i_a)}{(P_h^a)^{da}(j|i_a)} \right) \tag{137}
\end{equation}

denote the increment of the log-likelihood process of all the intended arm pulls, actual arm pulls and observations under hypothesis $\mathcal{H}_h$ with respect to those under hypothesis $\mathcal{H}_{h'}$; note that $\Delta Z_{h'h}(n) = -\Delta Z_{hh'}(n)$. We then have the following key property satisfied by $\Delta Z_{hh'}(n)$.

**Lemma 10.** For any $(d, i) \in \mathcal{S}$, $a \in \mathcal{A}$ and $0 < s < 1$, we have

$$E_h \left[ e^{s \Delta Z_{hh'}(n)} \bigg| A_n = a, d(n) = d, i(n) = i \right] \leq 1 \quad \forall n, \tag{138}$$

with strict inequality in (138) if $(d, i) = (d^*, i^*)$ and $a = h$.

**Proof of Lemma 10** Note that

$$E_h \left[ e^{s \Delta Z_{hh'}(n)} \bigg| A_n = h, d(n) = d, i(n) = i \right] = \sum_{j \in \mathcal{S}} \left( \frac{(P_h^a)^{da}(j|i_a)}{(P_h^a)^{da}(j|i_a)} \right)^s P_h(X_n = j | A_n = a, d(n) = d, i(n) = i)$$

$$= \sum_{j \in \mathcal{S}} \left( \frac{(P_h^a)^{da}(j|i_a)}{(P_h^a)^{da}(j|i_a)} \right)^s (P_h^a)^{da}(j|i_a)$$

$$= \sum_{j \in \mathcal{S}} ((P_h^a)^{da}(j|i_a))^{1-s} ((P_h^a)^{da}(j|i_a))^s$$

$$\leq \left( \sum_{j \in \mathcal{S}} (P_h^a)^{da}(j|i_a) \right)^{1-s} \cdot \left( \sum_{j \in \mathcal{S}} (P_h^a)^{da}(j|i_a) \right)^s$$

$$= 1, \tag{139}$$

where (a) above is due to Hölder’s inequality, and the last line follows from the fact that $(P_h^a)^{da}(\cdot|i_a)$ and $(P_h^a)^{da}(\cdot|i_a)$ are probability distributions on $\mathcal{S}$. When $(d, i) = (d^*, i^*)$ and $a = h$, the inequality in (a) is a strict inequality since $(P_h^a)^{da}(\cdot|i_a) = P_1(\cdot|i^*)$ and $(P_h^a)^{da}(\cdot|i_a) = P_2(\cdot|i^*)$, and since by the definition of $i^*$, $P_1(\cdot|i^*) \neq P_2(\cdot|i^*)$.

As an immediate consequence of Lemma 10 we have the following result.

**Lemma 11.** For any $(d, i) \in \mathcal{S}$, $a \in \mathcal{A}$ and $0 < s < 1$, we have

$$E_h \left[ e^{s \Delta Z_{hh'}(n)} \bigg| \mathcal{F}_{n-1} \right] I(d(n) = d, i(n) = i) \leq 1 \quad \forall n, \tag{140}$$

with strict inequality in (140) if $(d, i) = (d^*, i^*)$ and $a = h$.  

Proof of Lemma 11: We have

\[ E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | \mathcal{F}_{n-1} \right] \mathbb{1}_{\{d(n) = \bar{d}, \bar{i}(n) = \bar{i}\}} = E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1} \right] \]

\[ = \sum_{a=1}^{K} P(A_n = a | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1}) \cdot E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = a, d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1} \right] \]

\[ = P(A_n = h | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1}) \cdot E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = h, d(n) = \bar{d}, i(n) = \bar{i} \right] + \sum_{a \neq h} P(A_n = a | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1}) \cdot E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = a, d(n) = \bar{d}, i(n) = \bar{i} \right] \]

\[ \leq P(A_n = h | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1}) \cdot E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = h, d(n) = \bar{d}, i(n) = \bar{i} \right] + (1 - P(A_n = h | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1})) \]

\[ \leq P(A_n = h | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1}) \cdot \left( E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = h, d(n) = \bar{d}, i(n) = \bar{i} \right] - 1 \right) + 1 \]

(141)

(142)

where (a) above follows by noting that

\[ E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = a, d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1} \right] = E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | A_n = a, d(n) = \bar{d}, i(n) = \bar{i} \right], \]

(b) uses the result of Lemma 10 (c) follows from the fact that for any \( n \geq K \), under the policy \( \pi^*(L, \delta) \),

\[ P(A_n = h | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1}) = \frac{\eta}{K} + (1 - \eta) \lambda_{\theta(n), \delta}(h|d, i) \]

\[ \geq \frac{\eta}{K}, \]

and (d) is straightforward. Clearly, the inequalities in (b), (c) and (d) above are strict when \((\bar{d}, \bar{i}) = (d^*, i^*)\) and \( a = h \).

Going further, let \( c \) denote the constant on the right hand side of (141) when \((\bar{d}, \bar{i}) = (d^*, i^*)\). From the arguments above, we know that \( c < 1 \). We now have

\[ E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | \mathcal{F}_{n-1} \right] \]

\[ = \sum_{(\bar{d}, \bar{i}) \in S} E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | \mathcal{F}_{n-1} \right] \cdot \mathbb{1}_{\{d(n) = \bar{d}, i(n) = \bar{i}\}} \]

\[ = c \mathbb{1}_{\{d(n) = d^*, i(n) = i^*\}} + \sum_{(\bar{d}, \bar{i}) \neq (d^*, i^*)} E_h \left[ e^{s \Delta Z_{h,t}^{(n)}} | d(n) = \bar{d}, i(n) = \bar{i}, \mathcal{F}_{n-1} \right] \cdot \mathbb{1}_{\{d(n) = \bar{d}, i(n) = \bar{i}\}} \]

\[ = \begin{cases} 
  c, & d(n) = d^*, i(n) = i^*, \\
  \leq 1, & \text{otherwise.} 
\end{cases} \]

(143)

The above set of inequalities immediately lead us to the following important result.

Lemma 12. For \( 0 < s < 1 \),

\[ E_h \left[ e^{s Z_{h,t}^{(n)}} \right] \leq B_1 e^{-\theta_1 n}, \]

(144)

where \( B_1 > 0 \) and \( \theta_1 > 0 \) are constants which depend on \( h \), \( h' \) and \( s \).

Proof of Lemma 12: We have

\[ E_h \left[ e^{s Z_{h,t}^{(n)}} \right] = E_h [e^{s Z_{h,t}^{(n-1)}}] E_h [e^{s \Delta Z_{h,t}^{(n)} | \mathcal{F}_{n-1}}] \]
Towards this, fix $h$ settles at (147) Fix an arbitrary Lemma 13. Plugging (146) back in (145), and noting that (147) In the above set of equations, (a) follows from [143], the notation $E[X; A]$ in (b) stands for $E[X 1_A]$, and the last line follows by noting that $e^{n\lambda h,\delta (d^*, i^*)} \leq 1$ a.s. We now note that $\{N(n, d^*, i^*) - N(K, d^*, i^*) : n \geq K\}$ is a bounded martingale. Using the Azuma-Hoeffding inequality, we then have

$$P_h \left( N(n, d^*, i^*) \leq \frac{n\mu h,\delta (d^*, i^*)}{2} \right) = P_h \left( N(n, d^*, i^*) - N(K, d^*, i^*) \leq \frac{n\mu h,\delta (d^*, i^*)}{2} - (N(K, d^*, i^*)) \right) \leq \exp \left( -\frac{n\mu h,\delta (d^*, i^*)^2}{8} \right).$$

(146)

Plugging (146) back in (145), and noting that $c$ is a function of $s$, we arrive at (144).

As a consequence of Lemma 12 we have the following result.

**Lemma 13.** Fix an arbitrary $h \in A$, and suppose that $\mathcal{H}_h$ is the true hypothesis. Consider the non-stopping version of the policy $\pi = \pi^*(L, \delta)$. There exist constants $C_R$ and $\gamma > 0$ such that

$$P_h \left( \min_{h' \neq h} Z_{hh'}(n) < R \right) \leq C_R e^{-\gamma n}.$$  

(147)

In (147), $C_R$ is independent of $h$ but $\gamma$ depends on $h$.

**Proof of Lemma 13.** Writing $P_h(\cdot)$ for $P_h(\cdot)$ and $E_h[\cdot]$ for $E_h[\cdot]$, we have

$$P_h \left( \min_{h' \neq h} Z_{hh'}(n) < R \right) = P_h \left( \max_{h' \neq h} Z_{h'h}(n) > -R \right) \leq \sum_{h' \neq h} P_h (Z_{h'h}(n) > -R) \leq \sum_{h' \neq h} P_h (sZ_{h'h}(n) > -sR) \quad \forall \ 0 < s < 1$$

(a) $\leq \sum_{h' \neq h} e^{sR} E_h \left[ e^{sZ_{h'h}(n)} \right]$$$

(b) $\leq e^{sR} \sum_{h' \neq h} B_1 e^{-\theta n}$

$\leq e^{sR} \cdot (K - 1) \cdot \max_{h' \neq h} B_1 e^{-\theta n}$

$\leq C_R e^{-\gamma n},$

(148)

where $\max_{h' \neq h} B_1 e^{-\theta n} = e^{-\gamma}$ and $C_R = K e^{sR}$. In the above set of equations, (a) is due to Chernoff’s bound for $0 < s < 1$, and (b) is due to Lemma 12.

From (144), we know that under the non-stopping version of the policy $\pi^*(L, \delta)$, the guess of the odd arm $\theta(n)$ eventually settles at $h$ with probability 1 under the hypothesis $\mathcal{H}_h$. Indeed, we now show using Lemma 13 that something stronger holds. Towards this, fix $h \in A$, and suppose that $\mathcal{H}_h$ is the true hypothesis. Let

$$T_h := \inf \{n : \theta(n) = h \text{ for all } n' \geq n\},$$

(149)
We have the following result for \( T_h \).

**Lemma 14.** Fix an arbitrary \( h \in \mathcal{A} \), and suppose that \( \mathcal{H}_h \) is the true hypothesis. Consider the non-stopping version of the policy \( \pi^*(L, \delta) \). There exist constants \( C > 0 \) and \( b > 0 \), both finite and possibly depending on \( h \), such that

\[
P_h(T_h > n) \leq Ce^{-bn}. \tag{150}
\]

**Proof:** We have

\[
P_h(T_h > n) \leq P_h(\exists \ n' \geq n \text{ such that } \theta(n') \neq h)
\leq \sum_{n' \geq n} P_h(\theta(n') \neq h)
= \sum_{n' \geq n} P_h(\exists \ h' \neq h \text{ such that } \theta(n') = h')
\leq \sum_{n' \geq n} P_h(M_{h'}(n') > \max_{h' \neq h'} M_{h'}(n))
\leq \sum_{n' \geq n} P_h(M_h(n') - M_{h'}(n') < 0). \tag{151}
\]

We now note that

\[
M_h(n') - M_{h'}(n') = M_h(n') - \min_{h' \neq h} Z_{h' h'}(n') \\
\geq M_h(n') - Z_{h' h'}(n') \\
= M_h(n') + Z_{h' h'}(n') \\
\geq 2 \min_{h' \neq h} Z_{h' h'}(n'). \tag{152}
\]

Using \(152\) in \(151\), we get

\[
P_h(T_h > n) \leq \sum_{n' \geq n} P_h(\min_{h' \neq h} Z_{h' h'}(n) < 0). \tag{153}
\]

The result now follows from Lemma [13].

We now use the results presented above to derive the desired exponential upper bound for each term of the summation in \(136\). Note that for any \( \epsilon' > 0 \), we have

\[
P_h(Z_{hh'}(n) < \log((K - 1)L))
= P_h\left(\sum_{k=K}^n \Delta Z_{hh'}(k) < \log((K - 1)L)\right)
\leq P_h\left(\sum_{k=K}^n \Delta Z_{hh'}(k) - E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] + \epsilon'\right)
+ \sum_{k=K}^n (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon')
\leq P_h\left(\sum_{k=K}^n (\Delta Z_{hh'}(k) - E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] + \epsilon') < 0\right) + P_h\left(\sum_{k=K}^n (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon') < 0\right)
+ P_h\left((n - K + 1)(D_{hh'} - 2\epsilon') < \log((K - 1)L)\right). \tag{154}
\]
We first choose \( \epsilon' \) such that
\[
(n - K + 1) (D_{hh'} - 2\epsilon') \geq \log((K - 1)L) \quad \forall n \geq \hat{n}(L).
\]

In particular, \( \epsilon' = D_{hh'}/4 \) works. Let us fix this \( \epsilon' \) for the rest of the proof, and note that this choice of \( \epsilon' \) ensures that the third probability term in (154) is equal to 0. We now focus on the first probability term in (154), and note that each term inside the summation has strictly positive mean. Thus, from Chernoff’s bounding technique [19, Lemma 2], we get that there exists \( b(\epsilon') \) such that
\[
P_h \left( \sum_{k=K}^{n} (\Delta Z_{hh'}(k) - E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] + \epsilon') < 0 \right) \leq e^{-(n-K+1)b(\epsilon')}, \quad (155)
\]

It thus remains to show that the second probability term in (154) is bounded above exponentially. To do so, we use the proof technique of Vaidhiyan et al. [1] pp. 4793-4794 and adapt it to our setting of restless Markov observations.

Let
\[
\tilde{C} := \inf_{(d,\ell) \in \mathcal{S}, a \in A} E_h[\Delta Z_{hh'}(n) | A_n = a, \ell(n) = d, \tilde{\ell}(n) = \tilde{\ell}] - D_{hh'}
\]

where in (158),
\[
\tilde{C} = \inf_{(d,\ell) \in \mathcal{S}, a \in A} D((P_h^n)^{d_a} (\cdot | i_a))((P_{h'}^n)^{d_a} (\cdot | i_a)) - D_{hh'}. \quad (156)
\]

Note that \( \tilde{C} \leq 0 \) by the definition of \( D_{hh'} \). Choose \( \epsilon'' \) such that
\[
\tilde{\epsilon} := \epsilon' + \epsilon'' \tilde{C} > 0;
\]

here, \( \epsilon' = D_{hh'}/4 \) as chosen earlier. We may then write the second probability in (154) as follows:
\[
P_h \left( \sum_{k=K}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon') < 0 \right)
\]
\[
= P_h \left( \sum_{k=K}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon') < 0, \; T_h \leq n\epsilon'' \right)
\]
\[
+ P_h \left( \sum_{k=K}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon') < 0, \; T_h > n\epsilon'' \right)
\]
\[
\leq P_h \left( \sum_{k=K}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon') < 0, \; T_h \leq n\epsilon'' \right) + P_h \left( T_h > n\epsilon'' \right). \quad (157)
\]

From Lemma [14] the second probability term in (157) is bounded above exponentially. Concentrating on the first probability term in (157), and following the steps leading up to [1] Eq. (54)), we arrive at the relation
\[
P_h \left( \sum_{k=K}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \epsilon') < 0, \; T_h \leq n\epsilon'' \right)
\]
\[
\leq P_h \left( \sum_{k=[n\epsilon''] + 1}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \tilde{\epsilon}) < 0, \; T_h \leq n\epsilon'' \right)
\]
\[
= \tilde{P}_h \left( \sum_{k=[n\epsilon''] + 1}^{n} (E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}] - D_{hh'} + \tilde{\epsilon}) < 0 \right), \quad (158)
\]

where in (158), \( \tilde{P}_h \) is a new probability measure under which at each time instant, an arm is selected according to the policy \( \pi^*(L, \delta) \) but assuming that \( \theta_n = h \) for all \( n \).

We now note that under the measure \( \tilde{P}_h \),
\[
\tilde{E}_h[E_h[\Delta Z_{hh'}(k)|\mathcal{F}_{k-1}]] = \sum_{(d,\ell) \in \mathcal{S}} \sum_{a=1}^{K} \tilde{P}_h(d(k) = d, \tilde{\ell}(k) = \tilde{\ell}) \left( \frac{n}{K} + (1 - \eta) \lambda_{h,\delta}(a|d, \tilde{\ell}) \right) D((P_h^n)^{d_a} (\cdot | i_a))((P_{h'}^n)^{d_a} (\cdot | i_a)), \quad (159)
\]
where \( \tilde{E}_h \) in (159) denotes expectation under the measure \( \tilde{P}_h \). We claim that under the measure \( \tilde{P}_h \), the collection \( \{(d(k), \tilde{i}(k)) : k \geq \lfloor ne^\eta \rfloor + 1\} \) is a Markov process. Indeed, for all \( k \geq \lfloor ne^\eta \rfloor + 1 \),

\[
\tilde{P}_h(d(k + 1) = d', \tilde{i}(k + 1) = \tilde{i}' | (d(t), \tilde{i}(t)), \lfloor ne^\eta \rfloor + 1 \leq t \leq k)
\]

\[
= \begin{cases} 
\left( \frac{n}{R} + (1 - \eta) \lambda_h, \delta(a[d(k), \tilde{i}(k)]) \right) (P_h^a)^{d_a(k)}(\tilde{i}_a(k)), & \text{if } d'_a = 1 \text{ and } d'_b = d_b(k) + 1 \text{ for all } b \neq a, \\
0, & \text{otherwise}.
\end{cases}
\]

Let us fix \( d' = (K, K - 1, \ldots, 1) \) and \( \tilde{i}' = (1, \ldots, 1) \), where we assume without loss of generality that \( 1 \in S \). From [17, Proposition 1.7], we know that there exists an integer \( M \) sufficiently large such that each entry of the transition probability matrices \( P_1^M \) and \( P_2^M \) is strictly positive. We now use this fact to demonstrate that for any \( \{(d, \tilde{i}) \in S, \text{ if the process } \{(d(t), \tilde{i}(t)) : t \geq K \} \text{ starts in the state } (d, \tilde{i}) \text{ at some time } t_0 \geq \lfloor ne^\eta \rfloor + 1, \text{ then it has a strictly positive probability of being in the state } (d', \tilde{i}') \text{ at time } t = t_0 + M + d'_1 - d'_K. \) Indeed, following the arguments in Appendix [A] this probability may be lower bounded as

\[
\tilde{P}_h(d(T_0 + M + d'_1 - d'_K) = d', \tilde{i}(T_0 + M + d'_1 - d'_K) = \tilde{i}' | d(K) = d, \tilde{i}(K) = \tilde{i}) 
\]

\[
\geq \left( \frac{\eta}{K} \right)^{-K - 1} \prod_{a=1}^{K} (P_h^a)^{M + d'_1 - d'_K}(1|a) \prod_{t=T_0 + M + d'_1 - d'_K}^{T_0 + M + d'_1 - d'_K + 1} \frac{\eta}{K}.
\]

Denoting the right hand side of (161) by \( \alpha \), and noting that \( \alpha > 0 \), we have that

\[
(\tilde{P}_h)^{M + d'_1 - d'_K}((d', \tilde{i}') \mid d, \tilde{i}) \geq \alpha \prod_{(d'', \tilde{i}'')} \mathbf{1}_{(d''', \tilde{i}''') = (d', \tilde{i}')} \quad \text{for all } (d, \tilde{i}), (d'', \tilde{i}'') \in S.
\]

The condition in (162) is referred to as the “ Doeblin’s minorisation condition” (see [20, Eq. (5)]). Noting that (a) the Markov process \( \{(d(k), \tilde{i}(k)) : k \geq \lfloor ne^\eta \rfloor + 1\} \) is ergodic under the measure \( \tilde{P}_h \) with \( \mu^{h} \) as the unique stationary distribution, (b) \( (d', \tilde{i}') \in S \) holds, and (c) the increment \( \Delta Z_{h}(k) \) is bounded for each \( k \) as demonstrated in (126), we apply [20, Theorem 1] to deduce that the second probability term in (154) is bounded above exponentially. This completes the proof of the lemma. ■

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