Singularities of the susceptibility of a Sinai–Ruelle–Bowen measure in the presence of stable–unstable tangencies†

BY DAVID RUELLE1,2,*

1Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA
2IHES, Route de Chartres, 91440 Bures-sur-Yvette, France

Let \( r \) be a Sinai–Ruelle–Bowen (SRB or ‘physical’) measure for the discrete time evolution given by a map \( f \), and let \( r(A) \) denote the expectation value of a smooth function \( A \). If \( f \) depends on a parameter, the derivative \( \delta r(A) \) of \( r(A) \) with respect to the parameter is formally given by the value of the so-called susceptibility function \( \Psi(z) \) at \( z = 1 \). When \( f \) is a uniformly hyperbolic diffeomorphism, it has been proved that the power series \( \Psi(z) \) has a radius of convergence \( r(\Psi) > 1 \), and that \( \delta r(A) = \Psi(1) \), but it is known that \( r(\Psi) < 1 \) in some other cases. One reason why \( f \) may fail to be uniformly hyperbolic is if there are tangencies between the stable and unstable manifolds for \((f, r)\). The present paper gives a crude, non-rigorous, analysis of this situation in terms of the Hausdorff dimension \( d \) of \( r \) in the stable direction. We find that the tangencies produce singularities of \( \Psi(z) \) for \( |z| < 1 \) if \( d < 1/2 \), but only for \( |z| > 1 \) if \( d > 1/2 \). In particular, if \( d > 1/2 \), we may hope that \( \Psi(1) \) makes sense, and the derivative \( \delta r(A) = \Psi(1) \) thus has a chance to be defined.

Keywords: susceptibility singularities; SRB measure; stable–unstable tangencies

1. Introduction

Let \( f \) be a diffeomorphism of the compact manifold \( M \), and \( r \) a Sinai–Ruelle–Bowen (SRB) measure for \( f \) (see [1–3] and references therein; for SRB measures for a class of non-invertible maps, see [4]). The derivative \( \delta_X r(A) \) of the map \( f \mapsto r \) in the direction\(^1\) of the smooth vector field \( X \), evaluated

*ruelle@ihes.fr

†An early version of this work was presented at the France–Brazil mathematical conference at IMPA in September 2009, at the LAWNP conference in Buzios in October 2009 and at the meeting Progress in Dynamics at the IHP in November 2009.

\(^1\)If we replace \( f : x \mapsto fx \) by \( x \mapsto fx + X(fx) \), then \( r \) is replaced by \( r + \delta_X r \) to first order in \( X \). The derivative of \( f \mapsto r \) in the direction of \( X \), evaluated at \( A \), is \( \delta_X r(A) \).

One contribution of 17 to a Theme Issue ‘Nonlinear dynamics in meso and nano scales: fundamental aspects and applications’.
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at the smooth real function $A$, can be formally computed to be the value at $z = 1$ of

$$\Psi(z) = \sum_{n=0}^{\infty} z^n \int \rho(dx) X(x) \cdot \partial_x (A \circ f^n). \quad (1.1)$$

We shall call $\Psi$ the susceptibility. Note that the physical susceptibility is defined for a continuous time dynamical system, and is a function of the frequency $\omega$. For the discrete time dynamics considered here, the susceptibility would be $\omega \mapsto \Psi(e^{i\omega})$, but for simplicity we call $\Psi$ the susceptibility.

In the uniformly hyperbolic case (i.e. if the support of $\rho$ is a mixing axiom $A$ attractor for $f$), $\Psi$ has a radius of convergence $r(\Psi) > 1$. One can furthermore prove that $f \mapsto \rho$ is differentiable [5] and that its derivative is given by $\Psi(1)$. Actually, the inequality $r(\Psi) > 1$ and the identity $\delta_x \rho(A) = \Psi(1)$ are proved by Ruelle [6,7]. There are corresponding results for hyperbolic flows [8,9], and generalizations to partially hyperbolic systems [10].

In non-uniformly hyperbolic situations, these assertions may fail: one may have $r(\Psi) < 1$, and $f \mapsto \rho$ is presumed to be non-differentiable. The inequality $r(\Psi) < 1$ has been proved for certain (non-invertible) unimodal maps of the interval ([11,12]; see also [13] and work in progress by Baladi & Smania). The analysis by Ruelle [13] strongly suggests that, for a certain class of unimodal maps, the function $f \mapsto \rho$ is non-differentiable, even in the (weak) Whitney sense. There is also numerical evidence [14] that $r(\Psi) < 1$ for the classical Hénon attractor. For recent work on Hénon-like diffeomorphisms, see the study of Palis & Yoccoz [15].

The above results suggest two problems: (i) proving that $r(\Psi) < 1$, $\geq 1$ or $> 1$ in cases of some generality, and (ii) relating the derivative of $f \mapsto \rho$ to $\Psi(1)$ when this quantity is defined.

The present paper is about the first problem, and presents a non-rigorous study of the singularities of $\Psi$, which may occur as a result of tangencies, i.e. tangencies of stable and unstable manifolds for the system $(f, \rho)$ assumed to have no zero Lyapunov exponent (the existence of tangencies excludes uniform hyperbolicity).

Our study is not rigorous, but suggests that $r(\Psi) < 1$ if the partial Hausdorff dimension $d$ of $\rho$ in the stable direction is less than $1/2$, while $r(\Psi) \geq 1$ if $d \geq 1/2$. This opens the possibility that, for some fat tangencies ($d$ sufficiently large), $\Psi(1)$ is well defined. In that case, a derivative of $f \mapsto \rho$ may exist with applications to the physical theory of linear response (a basic physical article on linear response is by Toll [16], whereas a review of linear response for dynamical systems is given by Ruelle [17]).

2. Example: volume-preserving diffeomorphisms

Let $\ell$ be equivalent to Lebesgue measure on $M$, and let the $f$-invariant probability measure $\rho$ be the restriction of $\ell$ to a certain open set $S \subset M$ (similarly, we may also consider the situation where $f$ acts on $R^m$, and $S$ is a bounded open set in $R^m$).

If either $\text{supp } X \subset S$, or $\text{supp } A \subset S$, we may write

$$\Psi(z) = \sum_{n=0}^{\infty} z^n \int_{S} \ell(dx) X(x) \cdot \partial_x (A \circ f^n) = - \sum_{n=0}^{\infty} z^n \int_{S} \ell(dx) [\text{div}_x X(x)] A(f^n x),$$

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and therefore $r(\Psi) \geq 1$. Furthermore, if we suppose that either $\text{supp} \, X \subset S$, or $\text{supp} \, A \subset S$ and $\ell(A) = 0$, we have $(f, \rho)$ exponentially mixing $\Rightarrow r(\Psi) > 1$.

Volume-preserving Anosov diffeomorphisms satisfy this condition, and the same is true of the time 1 map of an exponentially mixing volume-preserving Anosov flow (which is not uniformly hyperbolic). Can exponential mixing happen for non-Anosov area-preserving diffeomorphisms in two dimensions? We shall now see that mixing already implies that $\Psi(1)$ is well defined, and $\delta_X \rho(A) = \Psi(1)$ when the derivative $\delta_X$ is taken along diffeomorphisms preserving a (parameter-dependent) volume.

For simplicity, we discuss the case $S = M$. Let $\rho$ be a probability measure equivalent to Lebesgue measure on the compact manifold $M$. Denote by $\tilde{\rho}$ the density of $\rho$ with respect to Lebesgue measure on some charts. Thus $(f^* \rho) = \tilde{\rho} \circ f^{-1} / J \circ f^{-1}$, where $J(x) = |\det(D_x f)|$. Suppose now that $f$ and $\tilde{\rho}$ depend smoothly on a parameter, and denote the derivative with respect to the parameter by a prime. In particular $f' = X \circ f$ and $J'(x) = J(x)[\text{div} X(fx)]$.

Writing $\rho_1 = f^* \rho$ we have $\tilde{\rho}_1 = (\tilde{\rho} / J) \circ f^{-1}$, or $\tilde{\rho}(x) = J(x)(\tilde{\rho}_1(fx))$, hence

$$
\tilde{\rho}'(x) = J(x)[\tilde{\rho}'_1(fx) + \partial_{fx} \tilde{\rho}_1 \cdot X(fx)] + J(x)[\text{div} X(fx)]\tilde{\rho}_1(fx),
$$
or

$$
\left(\frac{\tilde{\rho}'}{f} \right) \circ f^{-1} = \tilde{\rho}'_1 + \partial \tilde{\rho}_1 \cdot X + [\text{div} X] \tilde{\rho}_1 = \tilde{\rho}'_1 + \text{div}(\tilde{\rho}'_1 X) = \tilde{\rho}'_1 + [\text{div} \tilde{\rho}_1 X] \tilde{\rho}_1,
$$
thus

$$
\int dx \, \tilde{\rho}'(x) A(fx) = \int dx \, \tilde{\rho}'_1(x) A(x) + \int dx \, \tilde{\rho}_1(x)[\text{div} \tilde{\rho}_1 X(x)] A(x).
$$

Imposing the invariance condition, $\rho = f^* \rho$, we have

$$
\int dx \, \tilde{\rho}'(x) A(f^{N+1} x) = \int dx \, \tilde{\rho}'(x) A(x) + \sum_{n=0}^N \int dx \, \tilde{\rho}(x)[\text{div} \tilde{\rho}_1 X(x)] A(f^n x). \tag{2.1}
$$

Note that $\int dx \, \tilde{\rho}(x) = 1$ implies $\int \rho(dx) (\tilde{\rho}' / \tilde{\rho})(x) = \int dx \, \tilde{\rho}'(x) = 0$.

Therefore, imposing mixing gives that

$$
\int dx \, \tilde{\rho}'(x) A(f^{N+1} x) = \int \rho(dx) \left( \frac{\tilde{\rho}'}{\tilde{\rho}} \right)(x) A(f^{N+1} x)
$$
tends to 0 when $N \to \infty$. Equation (2.1) now implies that

$$
\Psi(z) = -\sum_{n=0}^\infty z^n \int dx \, \tilde{\rho}(x)[\text{div} \tilde{\rho}_1 X(x)] A(f^n x)
$$
is well defined for $z = 1$, and $\int dx \, \tilde{\rho}'(x) A(x) = \Psi(1)$.

**Conclusion.** Suppose that $\rho$ is $f$-ergodic, with density $\tilde{\rho}$, and that $(f, \rho)$ is mixing on a function space $S$ containing $\tilde{\rho}' / \tilde{\rho}$ and $A$. Then $\Psi(1)$ is well defined, and $\delta_X \rho(A) = \int dx \, \tilde{\rho}'(x) A(x) = \Psi(1)$.

*Phil. Trans. R. Soc. A* (2011)
3. Computer simulations

It is accepted that, using a computer, one can numerically approximate an SRB measure by the time average

$$\frac{1}{N_1 - N_0} \sum_{n=N_0+1}^{N_1} \delta_{f^n x}$$

for large $N_1 - N_0$ (and $N_0$ moderately large). The idea is that the computed orbit $f^n x$ is noisy because of round-off errors, and that this noisy orbit has an SRB time average (see [18] and, for example, [19]). The Lyapunov exponents, and the coefficient $L_+$ introduced below, can also in principle be determined numerically. It is therefore possible to estimate $r(\Psi)$ in particular cases, and to test the relations proposed above between the stable dimension $d$ of $\rho$ and the convergence radius $r(\Psi)$ in the presence of tangencies.

For example, let $\dim M = 2$, and let the Lyapunov exponents $\lambda_-$ and $\lambda_+$ of $(f, \rho)$ satisfy $\lambda_- < 0 < \lambda_+$, so that [20] $d = \lambda_+ / |\lambda_-|$. Does the presence of tangencies together with $\lambda_+ / |\lambda_-| < 1/2$ imply $r(\Psi) < 1$? This appears to be the case for the classical Hénon attractor. Does $\lambda_+ / |\lambda_-| \geq 1/2$ imply $r(\Psi) \geq 1$? Are there examples with tangencies and $r(\Psi) > 1$?

4. Singularities of $\Psi$ in the presence of tangencies

It is readily seen that the power series (1.1) defining the susceptibility has a radius of convergence $r(\Psi) > 0$. Tangencies between stable and unstable manifolds for $(f, \rho)$ are expected to produce singularities of $\Psi$, thus limiting $r(\Psi)$. A difficulty of the problem is that the set of points of tangency has measure zero. Note in this respect that the angle between stable and unstable manifolds is defined $\rho$-a.e. (almost everywhere), and that the a.e. range of this angle determines if tangencies are allowed or not. A similar comment can be made for higher-order contacts of the stable–unstable manifolds. It is reasonable to exclude those higher-order contacts which (given the dimension of $M$) are non-generic if the stable and unstable manifolds are regarded as independent. At a generic tangency point $O$, the unstable manifold is folded in a way that is basically two-dimensional (corresponding to variables $x$ and $y$ introduced below). Along the orbit $(f^n O)$, we have folds that are sharper and sharper as $n \to \infty$. This exponential sharpening of the folds, combined with the derivative $\partial_x$ in equation (1.1), produces the singularities of $\Psi(z)$ with $|z| < 1$, which we want to study.

One can prove that $r(\Psi) < 1$ for certain unimodal maps of the interval (see comments in the third paragraph of §1); these maps are non-invertible and give a degenerate example of tangencies that is relatively accessible to mathematical study. In what follows, we discuss a crude imitation of the one-dimensional situation for higher-dimensional diffeomorphisms. In the case of unimodal maps of the interval with an ergodic measure $\rho$ absolutely continuous with respect to Lebesgue, the density of $\rho$ has spikes $\sim |x - f^n c|^{-1/2}$ on one side of the points $f^n c$ of the postcritical orbit. These spikes are at the origin of the singularities of $\Psi(z)$ inside the unit circle. Instead of an individual postcritical point, we find for higher-dimensional diffeomorphisms a family of tangencies of stable...
and unstable manifolds: think of a pile of (local) unstable manifolds (with
tangencies) carrying part of the measure $\rho$. This means that the spikes are ‘spread
out’ or ‘smoothed’ (corresponding to integration over a measure transverse to
the unstable manifolds). This smoothing may give weaker singularities of $\Psi$
(i.e. larger $r(\Psi)$).

Let us choose local coordinates $(x, X, y, Y) \in \mathbb{R} \times \mathbb{R}^{s-1} \times \mathbb{R} \times \mathbb{R}^{u-1}$, such that
the $s$-dimensional stable manifolds are given by $(y, Y) = \text{const.}$, and the local
unstable manifold $U$ through $O$ is given by $x = ay^2, X = 0, Y = 0$ (for definiteness
we take $a > 0$). The conditional measure of $\rho$ on $U$ is thus
\[\sim D(d x d X d y d Y) = \delta(x - ay^2)\delta(X)\delta(Y) d x d X d y d Y.\]

One can argue that the variable $Y$ does not play an important role in the
present discussion, and we shall omit it, which amounts to taking $u = 1$. Using
similar local coordinates near $fO$, we assume that the map $f$ has the form

\[(x, X, y) \mapsto (e^{L_+}x, e^{L_-}X, e^{L_+} - y),\]

where $L_- < 0, L_+ > 0$, and $e^L$ is a contraction (stronger than that given by $e^{h_-}$).

The assumption that the unstable manifolds are parallel affine manifolds is
crude, and so is the assumption that $L_-, L_+$ and $A$ are constant coefficients. In
fact, one might think of $L_-, L_+$ and $A$ as Lyapunov exponents. But $L_+$, the only
one of these coefficients to appear in the final formulae, is really the mean rate
of expansion along a forward orbit ($f^n O$) of tangencies. These crude assumptions
may be, in part, justified by the fact that we are looking for the leading singular
behaviour associated with a subset of unstable manifolds. We shall use informally
the notation $\approx$ (approximately equal to) and $\sim$ (approximately proportional to)
in trying to find the leading singular behaviour.

The contribution of the conditional measure $\Delta$ of $\rho$ on the piece $U$ of unstable
manifold is

\[\Psi^{\Delta}(z) \sim \sum_{n=0}^{\infty} z^n \int \Delta(d x d X d y) X(x, X, y) \cdot \partial_{(x, X, y)}(A \circ f^n).\]

Singularities for $|z| \leq 1$ can only come from the component $X_1$ of $X$ in the
$x$-direction, giving

\[\Psi^{\Delta}(z) \sim \sum_{n=0}^{\infty} z^n \int d x d y \delta(x - ay^2)X_1 \partial_x(A_1 \circ f^n)\]

\[\approx \sum_{n=0}^{\infty} (z e^{L_+})^n X_1(0) \int (f^{*n} \delta(x - ay^2) d x d y) A'_1(x),\]

where $A'_1(x)$ is the derivative of $A_1(x)$, which is $A$ evaluated in the coordinates
$(x, 0, 0)$ centred at $f^n O$.

One has

\[\delta(x - ay^2) = \frac{1}{2\sqrt{a}x} \left[ \delta \left( y - \sqrt{\frac{x}{a}} \right) + \delta \left( y + \sqrt{\frac{x}{a}} \right) \right].\]
we obtain the same estimate of
of
where the integration is over the variable
(such behaviour was found for the contributions of different spikes in the unimodal case). Thus, let

(*n\to*JG)

be part of the measure
so that

(*n\to*JG)

hence

\[
f^n((x - ay^2) \, dxdy) = \frac{e^{-nL_+/2}}{2\sqrt{a}} \left[ \delta \left( y - \frac{\sqrt{a}}{a_n} \right) + \delta \left( y + \frac{\sqrt{a}}{a_n} \right) \right] \, dxdy,
\]

where

\[a_n = a e^{nL_+} e^{-2nL_-}\]. Therefore

\[
\Psi^\Delta(z) \approx \sum_{n=0}^{\infty} (z e^{L_+/2})^n \frac{X_1(0)}{\sqrt{a}} \int_{\sqrt{a}}^{\text{cutoff}} \frac{dx}{\sqrt{x}} A'_1(x),
\]

so that

\[r(\Psi^\Delta) = e^{-L_+/2} < 1\]. This result is in agreement with that obtained using the spikes of the invariant density for unimodal maps in one dimension (which are limiting cases of diffeomorphisms with tangencies).

Remember, however, that \(\Psi\) is defined with the measure \(\rho\) rather than \(\Delta\). Thus, let \(\Gamma\) be part of the measure \(\rho\), carried by a pile of unstable manifolds (with tangencies) near \(O\), and write

\[
\Gamma(dx\,dX\,dy) = \int \gamma(d\xi\,dX) \delta(x - a(\xi, X)(y - b(\xi, X))^2 - c(\xi, X)) \, dxdy,
\]

where the integration is over the variable \(\xi\), and \(\gamma(d\xi\,dX)\) is a transverse measure of \(\rho\) in the stable direction, and we assume \(a(\xi, X) > 0\). It will turn out that we obtain the same estimate of \(r(\Psi^\Gamma)\) for different \(\Gamma\), and we expect that the contributions \(\Psi^\Gamma\) to \(\Psi\) of different \(\Gamma\) will add up convergently for \(|z| < r(\Psi^\Gamma)\) (such behaviour was found for the contributions of different spikes in the unimodal case). The most singular part of \(\Psi^\Gamma\) is of the form

\[
\Psi^\Gamma_1(z) = \sum_{n=0}^{\infty} z^n \int \Gamma(dx\,dX\,dy) X_1(x, X, y) \partial_x(A_1 \circ f^n)
\]

\[
\approx \sum_{n=0}^{\infty} z^n \int \gamma(d\xi\,dX) \delta(x - a(\xi, X)(y - b(\xi, X))^2 - c(\xi, X))
\]

\[
\times X_1(x, X, y) \partial_x A_1(e^{nL_+} x) \, dxdy
\]

\[
= \sum_{n=0}^{\infty} (ze^{L_+/2})^n \int \gamma(d\xi\,dX) \delta(x - a(\xi, X)(y - b(\xi, X))^2 - c(\xi, X))
\]

\[
\times X_1(x, X, y) A'_1(e^{nL_+} x) \, dxdy.
\]

To define \(A_1(x) = A(x, 0, 0)\) and \(A'_1(x)\) we have again used coordinates \((x, 0, 0)\) centred at \(f^n O\). Note that \(A'_1(e^{nL_+} x) = A'_1([f^n(x, 0, 0)]_1)\), where \([\cdot]_1\) denotes the first component. Therefore, when \(n\) is large, the functions \(x \mapsto [f^n(x, 0, 0)]_1\) and \(A'_1(e^{nL_+} x)\) oscillate many times, with a frequency \(\sim nL_+\).

We may replace \(X_1(x, X, y)\) by \(\tilde{X}(\xi, X) = X_1(c(\xi, X), X, b(\xi, X))\), and write

\[
\int \delta(x - a(\xi, X)(y - b(\xi, X))^2 - c(\xi, X)) \, dy = \frac{1}{\sqrt{a(\xi, X)}} \frac{1}{\sqrt{x - c(\xi, X)}},
\]

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where the right-hand side is replaced by 0 if $x < c(\xi, X)$. Then

$$\psi_1^x(z) \approx \sum_{n=0}^{\infty} (z e^{L})^n \int \frac{\gamma(d\xi \, dX)}{\sqrt{a(\xi, X) \sqrt{x - c(\xi, X)}}} A_1'(e^{nL} x) c \, d\xi.$$

If we let $\tilde{\gamma}(d\tilde{\xi})$ be the image of the measure $\gamma(d\xi \, dX)/\sqrt{a(\xi, X)}$ by $(\xi, X) \mapsto \tilde{\xi} = c(\xi, X)$, we finally obtain

$$\psi_1^x(z) \approx \sum_{n=0}^{\infty} (z e^{L})^n \int h(x) A_1'(e^{nL} x) \, dx, \quad h(x) = \int \frac{\tilde{\gamma}(d\tilde{\xi})}{\sqrt{x - \tilde{\xi}}}.$$

5. Estimates when supp $\tilde{\gamma}$ has zero Lebesgue measure

We assume now that supp $\tilde{\gamma}$ has Lebesgue measure $= 0$. Given $x \notin$ supp $\tilde{\gamma}$, let

$$\xi^* = \max\{\xi \in$ supp $\tilde{\gamma} : \xi < x\},$$

and let $\gamma^*(d\eta)$ be the image, restricted to $\eta \geq 0$, of $\tilde{\gamma}(d\tilde{\xi})$ by $\tilde{\xi} \mapsto \eta = \xi^* - \tilde{\xi}$. We then have

$$h(x) = \int \frac{\gamma^*(d\eta)}{(x - \xi^*) + \eta} = \int \frac{\phi'(\eta) \, d\eta}{\sqrt{(x - \xi^*) + \eta}},$$

where $\phi(\eta) = \int_0^\eta \gamma^*(dt)$ and $\phi(\eta) \sim \eta^d$ for small $\eta$. If $0 < \alpha < 1 - d$, we let

$$h_1(x) = \int ((x - \xi^*) + \eta)^{\alpha-1} \phi'(\eta) \, d\eta = (1 - \alpha) \int_0^\infty ((x - \xi^*) + \eta)^{\alpha-2} \phi(\eta) \, d\eta,$$

where we have put an upper limit $+\infty$ to the integral because it does not need a cutoff. Therefore

$$h_1(x) \approx (1 - \alpha) \frac{(x - \xi^*)^{1+d}}{(x - \xi^*)^{2-\alpha}} \int_0^\infty \frac{\phi(t) \, dt}{(1 + t)^{1+\alpha}} = C_\alpha (x - \xi^*)^{d+\alpha-1}.$$

(a) Case A

Assuming $d < 1/2$ and taking $\alpha = 1/2$, we may thus conclude that $h(x) \approx C(x - \xi^*)^{d-1/2}$. Hence, if $I = (\xi^*, \xi^{**})$ is an interval of length $\ell$ of the complement of supp $\gamma$, we may estimate

$$\int_I |h(x)|^p \, dx \approx C^p \int_0^\ell \, dt \, t^{-(1/2-d)} = C' \ell^{1-p(1/2-d)},$$

if $1 \leq p < (1/2 - d)^{-1}$.

By scaling, we assume that the number of intervals $I$ with $|I| \approx \ell$ is $\sim \ell^{-d}$. We thus have

$$\int |h(x)|^p \, dx \sim \sum_\ell \ell^{1-d-p(1/2-d)}.$$

If $p \geq 1$, and $1 - d - p(1/2 - d) > 0$, we thus have $h \in L_p$ (and the bound $p < (1 - d)/(1/2 - d)$ appears best possible). Let $1/p + 1/q = 1$; then the Fourier

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\[ \frac{1}{q} < 1 - \frac{1/2 - d}{1 - d} = \frac{1/2}{1 - d}, \quad (5.1) \]

and this bound appears best possible. If \(|\hat{h}(s)| \sim s^{-t}\) for large \(s\), we need \(tq > 1\), i.e. \(t > 1/q\) if \(1/q\) satisfies equation (5.1), i.e.

\[ |\hat{h}(s)| \sim s^{-t} \quad \text{with} \quad t \geq \frac{1/2}{1 - d}. \]

We now come to the estimation of \(\int h(x)A'_1(e^{nL_+}x) \, dx\), where, for large \(n\), \(x \mapsto A'_1(e^{nL_+}x)\) is rapidly oscillating with frequency \(\sim nL_+\). Since we are interested in the most singular part of \(h\), we may replace it by a function with compact support. Because \(A'_1\) is a derivative, there is no zero-frequency contribution to the integral, and we have

\[ \int h(x)A'_1(e^{nL_+}x) \, dx \sim \hat{h}(e^{nL_+}), \]

with a negligible contribution of higher harmonics. Therefore

\[ \left| \int h(x)A'_1(e^{nL_+}x) \, dx \right| \sim |\hat{h}(e^{nL_+})| \sim e^{-tnL_+} \leq \exp \left[ -n \frac{1/2}{1 - d} L_+ \right], \]

and the bound again appears best possible, so that \(\Psi^T_1(z)\) converges for

\[ |z| < \exp \left[ - \left( 1 - \frac{1/2}{1 - d} \right) L_+ \right] = \exp \left[ - \frac{1/2 - d}{1 - d} L_+ \right], \]

i.e. \(r(\Psi^T_1) = \exp(-(1/2 - d) L_+/(1 - d))\). A reasonable guess would appear to be

\[ r(\Psi) = \exp \left[ - \frac{1/2 - d}{1 - d} L_+ \right], \]

so \(e^{-L_+/2} < r(\Psi) < \exp((d - 1/2)L_+) < 1\).

(b) Case B

Assuming \(1/2 \leq d < 1\), we write \(h\) (which is the convolution product \(\tilde{\gamma} * (\cdot)^{-1/2}\)) as

\[ h \sim \tilde{\gamma} * (\cdot)^{\alpha-1} * (\cdot)^{\beta-1}, \]

where \(\alpha, \beta > 0\), \(\alpha + \beta = 1/2\), \(d + \alpha < 1\) or \(\beta = 1/2 - \alpha, 0 < \alpha < 1 - d\). We thus have

\[ h = h_1 * (\cdot)^{\beta-1} \quad \text{with} \quad h_1 = \tilde{\gamma} * (\cdot)^{\alpha-1}, \]

and we have seen that \(h_1(x) \approx C_{\alpha}(x - \xi^*)^{d+\alpha-1}\). We find as in case A that we can take \(|\hat{h}_1(s)| \sim s^{-t}\) with \(t \geq \alpha/(1 - d)\), hence

\[ \hat{h}(s) \sim s^{-\alpha/(1-d)} s^{-\beta} = s^{-1/2 - \alpha d/(1-d)}, \]

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so that
\[ \left| \int h(x) A'_1(e^{nL} x) \, dx \right| \sim |\hat{h}(e^{nL})| \leq \exp \left[ -n \left( \frac{\alpha d}{1 - d} + \frac{1}{2} \right) L_+ \right], \]
and \( \Psi^T_1(z) \) converges for
\[ |z| < \exp \left[ \left( \frac{\alpha d}{1 - d} + \frac{1}{2} - 1 \right) L_+ \right] = \exp \left[ \left( \frac{\alpha d}{1 - d} - \frac{1}{2} \right) L_+ \right], \]
where we may let \( \alpha \to 1 - d \). Hence we may estimate
\[ r(\Psi^T_1) \geq \exp \left[ \left( \frac{\alpha d}{1 - d} - \frac{1}{2} \right) L_+ \right] = \exp \left[ \left( \frac{d - 1}{2} \right) L_+ \right], \]
which is greater than 1. In fact, \( r(\Psi^T_1) = \exp[(d - 1/2)L_+] \) is a reasonable guess.

The convergence radius \( r(\Psi) \) now depends on the behaviour of \( (f, \rho) \) away from tangencies, and we may expect that the derivative \( \partial_x \) in equation (1.1) plays a less important role. Therefore, \( r(\Psi) \) is expected to depend on the mixing properties of \( (f, \rho) \), over which one has some control [21,22]. One may thus hope that \( r(\Psi) \geq 1 \), or even \( r(\Psi) > 1 \), and that \( \Psi(1) \) is well defined. The situation where the set of tangencies is large \( (d > 1/2) \) reminds one of Newhouse’s study of persistent tangencies (wild hyperbolic sets, infinitely many sinks, see [23–25]). While the situation considered by Newhouse has very discontinuous topology, it is not unthinkable that the particular measure \( \rho \) behaves differentiably in some sense.

(c) Case C

It is plausible that the results of cases A and B remain true without the condition that \( \text{supp} \tilde{\gamma} \) has zero Lebesgue measure. Furthermore, if the stable dimension \( d \) of \( \rho \) is \( d \geq 1 \), one can write \( d \) as a sum of partial dimensions (see [19], section IV.D, and references therein, in particular [1]), and use arguments as above. One expects thus that the formula \( r(\Psi^T_1) \geq \exp[(d - 1/2)L_+] \) will remain correct in that case and, as argued in case B, we may then have \( r(\Psi) \geq 1 \) or even \( r(\Psi) > 1 \).

If we have a continuous time dynamical system (a flow) instead of discrete time dynamics (a diffeomorphism), we expect similar results in the presence of tangencies: a susceptibility function \( \hat{k}(\omega) \) with singularities in the upper half \( \omega \)-plane if \( d < 1/2 \), no singularity if \( d \geq 1/2 \) and \( \hat{k}(0) \) hopefully well defined if \( d > 1/2 \). The continuous time dynamical situation is that most relevant for physical applications.

6. Physical discussion

In this brief physically oriented discussion we shall, for simplicity, use the language of discrete time dynamical systems. We have made above a non-rigorous analysis of how tangencies between stable and unstable manifolds may influence the radius of convergence \( r(\Psi) \) of the susceptibility function. We have found two different regimes depending on whether the stable dimension \( d \) of the SRB measure \( \rho \) is \( d < 1/2 \) or \( d \geq 1/2 \).
If \( d < 1/2 \), we expect \( r(\Psi) < 1 \), i.e. the tangencies cause singularities of \( \Psi(z) \) with \( |z| < 1 \). Such singularities reflect the exponential growth of small periodic perturbations of the dynamics \((f, \rho)\) [13]. Experimentally, this may be visible as resonant behaviour when a physical system is excited by a weak periodic signal: it would be of particular interest to study the case of hydrodynamic turbulence.

If \( d \geq 1/2 \), we expect \( r(\Psi) \geq 1 \) and, if \( d > 1/2 \), the value \( \Psi(1) \) may be well defined. Since \( \Psi(1) \) is formally related to the derivative of \( \rho \) with respect to \( f \), we may hope that this derivative exists in some sense. This would apply to physical systems not too far from equilibrium (at equilibrium, \( \rho \) has a density, and \( d \geq 1 \) unless all Lyapunov exponents vanish), with obvious application to linear response in non-equilibrium statistical mechanics. For large physical systems (\( \dim M \) large), when there is chaos and a density of Lyapunov exponents can be defined, one also expects \( d \) to be large by the Kaplan–Yorke formula (see [19], section IV.C, and references therein, in particular [26]), provided the degrees of freedom of the large system have a sufficiently strong effective interaction.

In view of the mathematical difficulty of analysing dynamical systems with tangencies, a computer experimental study would be desirable. The situation of choice would be that of two-dimensional diffeomorphisms with an SRB measure \( \rho \), such that the Lyapunov exponents \( \lambda_+ \) and \( \lambda_- \) satisfy \( \lambda_- < 0 < \lambda_+ \). In that case, we know [20] that \( d = \lambda_+ / |\lambda_-| \), and the radius of convergence \( r(\Psi) \) is also accessible numerically. For the classical Hénon attractor, we have \( d < 1/2 \), and it appears [14] that \( r(\Psi) < 1 \). In other cases, studied by Ueda [27] and Ueda & Akamatsu [28], visual inspection of the computer plot of the attractor seems to indicate a large \( d \), and it would be desirable to estimate \( r(\Psi) \).

### 7. Infinitesimally stable ergodic measures

Consider the general situation of a diffeomorphism \( f \) of the compact manifold \( M \), and of an ergodic measure \( \rho \) for \( f \) on \( M \). We want to study formally the stability of \( \rho \) under an infinitesimal change of \( f \).

We shall use a space \( \mathcal{D} \) of smooth functions on \( M \), with dual \( \mathcal{D}^* \), and a space \( \mathcal{V} \) of smooth vector fields on \( M \). If \( X \in \mathcal{V} \), we write \( \hat{X}(A) = \int \rho(dx) X(x) \cdot \partial_x A \), so that \( \hat{X} \in \mathcal{D}^* \). Defining \( T : \mathcal{D}^* \to \mathcal{D}^* \) and \( Tf : \mathcal{V} \to \mathcal{V} \) by

\[
(T\hat{\xi})(A) = \hat{\xi}(A \circ f) \quad \text{and} \quad ((Tf)X)(fx) = (T_x f)X(x),
\]

we find that \( ((Tf)X)^\wedge = T\hat{X} \).

Consider \( \rho + \hat{X} \) as an infinitesimal perturbation of \( \rho \) (this corresponds to replacing \( \rho \) by its image under \( x \mapsto x + X(x) \)). The measure \( \rho \) is mapped to itself by \( f \), while \( \rho + \hat{X} \) is mapped to \( \rho + T\hat{X} \). We say that \( \rho \) is infinitesimally stable (or attracting) if \((T^n\hat{X})(A) \to 0 \) exponentially with \( n \) whenever \( X \in \mathcal{V} \) and \( A \in \mathcal{D} \) (we observe that an alternative (weaker) requirement would be that \( \Psi(1) = \sum_{n=0}^{\infty} \rho(dx) X(x) \cdot \partial_x (A \circ f^n) \) converges whenever \( X \in \mathcal{V} \) and \( A \in \mathcal{D} \)). It is plausible that an infinitesimally stable measure must be SRB.

We perturb \( f \) to \( \tilde{f} = f + X \circ f \), where \( X \in \mathcal{V} \). If \( \hat{\xi} \in \mathcal{D}^* \), the \( \tilde{f} \)-invariance of \( \rho + \hat{\xi} \), i.e.

\[(\rho + \hat{\xi})(A \circ (f + X \circ f)) = (\rho + \hat{\xi})(A),\]

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is then given, to first order in $X$, by

$$
\int \rho(dx) [A(fx) + X(fx) \cdot \partial_{fx}A] + \xi(A \circ f) = \rho(A) + \xi(A),
$$

or $\dot{X} + T\xi = \xi$, hence $T^n\xi - T^{n+1}\xi = T^n\dot{X}$ and hence $\xi - T^{N+1}\xi = \sum_{n=0}^{N} T^n \dot{X}$. Therefore, if $\rho$ is infinitesimally stable, we obtain $\rho + \xi$, which is $\dot{f}$-invariant to first order by taking

$$
\xi(A) = \sum_{n=0}^{\infty} (T^n \dot{X})(A),
$$

and $\xi$ is unique such that $(T^n\dot{X})(A) \to 0$ for all $A \in D$ when $n \to \infty$. This shows that the linear response $X \mapsto \xi$ is related to infinitesimal stability.

The above considerations apply to the uniformly hyperbolic situation where $\rho$ is an SRB measure on an axiom A attractor. The purpose of the present paper has been to make plausible the infinitesimal stability of SRB measures in a different situation where there are stable–unstable tangencies. Note that $\Psi(z) = \sum_{n=0}^{\infty} z^n (T^n \dot{X})(A)$, so that $r(\Psi) > 1$ is equivalent to the infinitesimal stability condition that $(T^n \dot{X})(A) \to 0$ exponentially.

I am indebted to Artur Ávila, Viviane Baladi, Bruno Cessac, Jean-Pierre Eckmann and Jean-Christophe Yoccoz for valuable remarks.

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