On matroid modularity and the coefficients of the inverse Kazhdan-Lusztig polynomial of a matroid

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Abstract
Following the work of Gao and Xie in [2], we state some properties of the inverse Kazhdan-Lusztig polynomial of a matroid. We also give partial answers to a conjecture that states that regular connected matroids are non-degenerate. Firstly, we show that the Conjecture holds for modular matroids, by proving that degenerate modular matroids are not regular; then, we also reduce the problem to proving that the statement holds on odd-rank matroids.

1 Introduction
Kazhdan-Lusztig polynomials for matroids were firstly introduced in 2016 by Elias, Proudfoot and Wakefield in [4], in analogy with the classical Kazhdan-Lusztig polynomials for Coxeter groups ([10]). While the classical version gives rise to a very broad class of polynomials (every polynomial with non-negative integer coefficients and constant term equal to 1 is the Kazhdan-Lusztig polynomial of some Coxeter group [6]), the matroidal Kazhdan-Lusztig polynomials have a very rigid structure: Braden et al. proved in [1] that their coefficients are always non-negative, and it is conjectured that they form a log-concave sequence with no internal zeros. In 2020, Gao and Xie introduced in [2] the analogue of the inverse Kazhdan-Lusztig polynomial for a matroid, employing the general framework of Kazhdan-Lusztig-Stanley theory. These polynomials were also proven to have non-negative coefficients in [1], and they are conjectured as well to be log-concave with no internal zeros. The properties of these polynomials, which will be called $P_M(t)$ and $Q_M(t)$ in accordance to all the present literature, are usually not understandable at first glance, since their definitions are given through a quite intricate recursion. Another Conjecture, found in [3], gives a statement about the degree of these polynomials. We say that a matroid $M$ of rank $r$ is degenerate if the degree of $P_M(t)$ is strictly less than \( \lfloor \frac{r+1}{2} \rfloor \). Then one has the following

Conjecture 1.1. Every connected regular matroid is non-degenerate.

Our main aim is to develop new results for $P_M(t)$ and $Q_M(t)$ in a purely combinatorial way, to give partial answers to Conjecture 1.1. In Section 3, we first give an explicit definition of the coefficient of degree 1 of $Q_M(t)$ (Theorem

\[ Q_M(t) = \sum_{F \

\text{(maximal proper component of } M \text{)}} P_M(t)^{r(F)} \]
and we link the degeneracy of a matroid $M$ to the degree of $Q_M(t)$ (Theorem 3.4). We then move on to consider the class of modular matroids, which are proven to be the only matroids with a zero-degree Kazhdan-Lusztig polynomial. In a sense, in this setting, modularity is equivalent to maximal degeneracy, since, even for high-rank matroids, $P_M(t)$ remains constant. In Section 4, Theorem 4.1 settles the Conjecture for this class of matroids, by showing that rank-3 or higher modular matroids are either not connected or not regular. Lastly, we are able to prove that the Conjecture needs to be checked only for odd-rank matroids, since if the statement holds for them then it holds for every matroid (Theorem 4.2).

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2 Background and notation

We begin our work by setting all the notation needed in the rest of the paper. Many invariants defined on matroids can actually be calculated employing an order-theoretic approach by working on their family of flats, which form a lattice by inclusion. To start, we recall the definition of a geometric lattice.

Definition 2.1. A lattice $\mathcal{L}$ is said to be atomic if each non-minimal element can be obtained as a join of its atoms. It is semimodular if it has a rank function $\rho$ such that

$$\rho(F) + \rho(G) \geq \rho(F \vee G) + \rho(F \wedge G),$$

for all $F, G \in \mathcal{L}$. If $\mathcal{L}$ is atomic and semimodular it is said to be geometric.

We are also interested in the underlying matroidal structure of $\mathcal{L}$. It is a theorem by Birkhoff [15] that every geometric lattice arises as a lattice of flats of a matroid. If $M$ is a matroid with lattice of flats isomorphic to $\mathcal{L}$ we will write $\mathcal{L} = \mathcal{L}(M)$, omitting the $M$ if the matroid is clear from the context. In this sense, we will also say that $\mathcal{L}$ is a matroid. Unless otherwise stated, we will always work with simple matroids of rank $r$. Given a simple matroid $M$ we denote with $E$ its ground set (which is equal to the set of atoms of $\mathcal{L}$), and $\chi_M(t)$ its characteristic polynomial. Moreover, we call $[\emptyset, F] \subset \mathcal{L}$ the restriction of $M$ to $F$, denoted with $M_F$, and we call $[F, E] \subset \mathcal{L}$ the contraction of $M$ by $F$ denoted by $M^F$. It is important to observe that simplicity is easily lost when performing contractions and needs to be handled carefully in recursions; if a proposition needs to be stated differently in the non-simple case we will do so; otherwise we can always suppose to work with a simplified version of matroids (removing all loops and parallel elements).

The uniform matroid of rank $k$ on a ground set $|E| = n$ will be denoted with $U_{k,n}$. The Boolean matroid, corresponding to the Boolean lattice, will be denoted with $B_n$. We denote with $w_k$ and $W_k$, respectively, the Whitney numbers of first and second kind. We recall that $\chi_M(t) = \sum_{k=0}^{r} w_k t^{r-k}$ and that $W_k$ is the number of rank-$k$ flats in the matroid. See [5] and [9] for exact definitions on matroids.
and geometric lattices.

The notion of modularity in a lattice can be stated in several different ways, but we collect in the following proposition the different characterizations we will need for our results.

**Proposition 2.2.** For a lattice $\mathcal{L}$ the following statements are equivalent

i) $\mathcal{L}$ is modular,

ii) $\mathcal{L}$ has a rank function $\rho$ such that $\rho(F) + \rho(G) = \rho(F \lor G) + \rho(F \land G)$ for any two elements $F$ and $G$,

iii) (Diamond isomorphism Theorem) For every $F$ and $G$ in $\mathcal{L}$ the intervals $[F \land G, F] \cong [G, F \lor G]$ are isomorphic.

If, moreover, $\mathcal{L}$ is geometric, then i) is also equivalent to

iv) $W_1 = W_{r-1}$,

v) $\mathcal{L}$ is the product of a boolean matroid and a projective geometry,

vi) The dual lattice $\mathcal{L}^* = (\mathcal{L}, \geq)$ is geometric.

Condition iv) is the Hyperplane Theorem and can be found for example in [9], Chapter 8. Condition v) is a Theorem by Birkhoff ([13]); we recall that a projective geometry is a geometric lattice where

- Every rank-2 flat contains at least three atoms,
- Every hyperplane intersect every rank-2 flat.

Regarding condition vi), geometric lattices are always dually atomic ([8]), hence what one has left to prove is the semimodularity of $\mathcal{L}^*$. Moreover, the Top-Heavy Conjecture, proved in [1], implies that

**Corollary 2.3.** If $\mathcal{L}$ is geometric and modular, its sequence of Whitney numbers of the second kind is symmetric, i.e. for all $k$

$$W_k = W_{r-k}.$$ 

Theorem 2.2 in [4] and Theorem 1.2 in [2] give us the definition of the Kazhdan-Lusztig polynomial and the inverse Kazhdan-Lusztig polynomial of a matroid, which we will now recall.

**Theorem 2.4.** There is a unique way to assign to each matroid $M$ two polynomials $P_M(t), Q_M(t) \in \mathbb{Z}[t]$ such that

i) If $r = 0$, then $P_M(t) = 1$.

ii) If $r > 0$, then $\deg P_M(t) < \frac{1}{2}r$. 

iii) For every matroid $M$ we have
\[ t^r P_M(t^{-1}) = \sum_{F \in \mathcal{I}(M)} \chi_{M^F}(t) P_{M^F}(t). \]

i') If $r = 0$, then $Q_M(t) = 1$.

ii') If $r > 0$, then $\deg Q_M(t) < \frac{1}{2}r$.

iii') For every matroid $M$ we have
\[ t^r \cdot (-1)^r Q_M(t^{-1}) = \sum_{F \in \mathcal{I}(M)} (-1)^{rk_{M^F}} Q_{M^F}(t) \cdot t^{rk_{M^F}} \chi_{M^F}(t^{-1}). \]

Remark 2.5. These polynomials also arise in the more general framework of Kazhdan-Lusztig-Stanley theory. We call $f$ the unique function that satisfies
\[ \bar{f} = \chi \ast f, \]
where $\bar{f}$ is the involution $\bar{f}(t) = t^r f(t^{-1})$ and $\ast$ denotes the convolution in the incidence algebra of $\mathcal{I}$. Then $f$ is called the right Kazhdan-Lusztig-Stanley function associated to $\chi$. In this setting then one has that
\[ P_M(t) = (f)_{\emptyset, E}(t). \]
Then, the polynomial $Q_M(t)$ can be found to be equal to
\[ Q_M(t) = (-1)^r (f^{-1})_{\emptyset, E}(t), \]
where $f^{-1}$ is the inverse of $f$ in the incidence algebra of $\mathcal{I}$. This justifies the name "inverse polynomial" for $Q_M(t)$, since it arises as the inverse of $P_M(t)$ with respect to the convolution product. In particular, this means that
\[ \sum_{F \in \mathcal{I}(M)} P_{M^F}(-1)^{rk_{M^F}} Q_{M^F} = \sum_{F \in \mathcal{I}(M)} P_{M^F}(-1)^{rk_{M^F}} Q_{M^F} = 0. \]

Remark 2.6. To ease some of the computations, we will also use this version of condition iii'),
\[ Q_M(t) = (-1)^r \sum_{F \in \mathcal{I}(M)} (-1)^{rk_{M^F}} t^{rk_{M^F}} Q_{M^F}(t^{-1}) \cdot \chi_{M^F}(t). \]

3 New results on the coefficients of $P_M(t)$ and $Q_M(t)$

We will denote the coefficient of $t^j$ in a polynomial $p(t)$ as $[t^j]p(t)$.

It is observed in Proposition 2.4 in [2] that

**Proposition 3.1.**
\[ [t^0] Q_M(t) = [t^0] \chi_M(t) = |w_r| = |\mu(M)|. \]
Our next goal is to compute the coefficient of $t$ in the same fashion of Proposition 2.12 in \cite{[4]}. To do that we recall the definition of the doubly-indexed Whitney numbers.

**Definition 3.2.** For all natural numbers $i$ and $j$, let

$$w_{i,j} := \sum_{\text{rk } F = i, \text{rk } G = j} \mu(F, G) \quad \text{and} \quad W_{i,j} := \sum_{\text{rk } F = i, \text{rk } G = j} \zeta(F, G),$$

where $\mu$ is the Moebius function of $\mathcal{L}(M)$ and $\zeta$ is its inverse in the incidence algebra of $\mathcal{L}(M)$. If $i$ is set to 0, we recover the Whitney numbers of first and second kind.

**Theorem 3.3.** For every matroid $M$,

$$[t]Q_M(t) = |w_{1,r}| - |w_{0,r-1}|.$$

**Proof.** We just need to compute the coefficient of $t$ in the right-hand side of the equation in Remark 2.6. First of all we notice that flats with rank greater than or equal to 2 will give no contribution to the coefficient of $t$ because the terms of $Q_{M'}(t^{-1})$ have degrees ranging in $(-\frac{m-1}{2}, 0]$, which means that the lowest term of $t^{\text{rk } F}Q_{M'}(t^{-1})\chi_{M'}(t)$ has degree strictly greater than 1. If we consider $F = \emptyset$, then the corresponding term in the sum is reduced to $(-1)^{\text{rk } M}\chi_M(t)$, since $Q_{M'}(t) = 1$. Then we get a contribution for the coefficient of $t$ of $(-1)^{\text{rk } M}[t]\chi_M(t)$. The sign of $[t]\chi_M(t)$ is opposite to the parity of $\text{rk } M$, which means that

$$(-1)^{\text{rk } M}[t]\chi_M(t) = -|[t]\chi_M(t)|.$$

Let us now consider a rank-1 flat, $i \in E$. Combining ii') and 3.1 we get that $Q_{M_i}(t) = Q_{B_i}(t) \equiv 1$, thus its corresponding term in the sum is

$$(-1)^{\text{rk } M - 1}t\chi_{M_i}(t).$$

This means that each of these terms gives a contribution to the coefficient of $t$ of $(-1)^{\text{rk } M - 1}[t^0]\chi_{M_i}(t)$, which, for the same reason as before, is equal to $|[t^0]\chi_{M_i}(t)|$. Putting everything together tells us that

$$[t]Q_M(t) = \sum_{i \in E} [t^0]\chi_{M_i}(t) - |[t]\chi_M(t)|.$$

A straightforward substitution using the definition of $\chi_M(t)$ gives us the desired result.

We know that the degeneracy of a matroid $M$ is linked to the polynomial $P_M(t)$, but the definition could be given with respect to the polynomial $Q_M(t)$, since the coefficients of the highest possible degree of the two polynomial are equal, as the next result will show.

**Theorem 3.4.** Let $M$ be a matroid of rank $r \neq 2$. Then

$$[t^{\lfloor \frac{r+1}{2} \rfloor}] P_M(t) = [t^{\lfloor \frac{r+1}{2} \rfloor}] Q_M(t).$$

In particular, a matroid $M$ of rank $r$ is non-degenerate if and only if $Q_M(t)$ has degree $\lfloor \frac{r+1}{2} \rfloor$. 

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Proof. In the case of \( r = 2k + 1 \) this becomes a special case of Corollary 8.8 in [7]. We know that
\[
\sum_{F} P_{M_F}(t)(-1)^{rk_{M_F}} Q_{M_F}(t) = 0.
\]
If \( F = \emptyset \) or \( F = E \) we get, respectively, \( P_M(t) \) and \( (-1)^{2k+1}Q_M(t) = -Q_M(t) \).
Let us show that no other term in the sum can have degree \( k \). If \( \emptyset \neq F \neq E \),
\[
\deg P_{M_F} Q_{M_F} = \deg P_{M_F} + \deg Q_{M_F} \leq \left\lfloor \frac{rk_F - 1}{2} \right\rfloor + \left\lfloor \frac{r - rk_F - 1}{2} \right\rfloor.
\]
Hence,
\[
0 = [t^k]0 = [t^k](P_M(t) - Q_M(t)) = [t^k]P_M(t) - [t^k]Q_M(t).
\]
In general, we prove it by induction on \( r = 2k \), supposing the statement holds for every matroid of rank lower than \( r \). Write
\[
P_M(t) = -\sum_{F \neq \emptyset} (-1)^{rk_F} Q_{M_F}(t)P_{M_F}(t)
\]
and
\[
Q_M(t) = -\sum_{F \neq \emptyset} (-1)^{rk_F} P_{M_F}(t)Q_{M_F}(t).
\]
Therefore,
\[
[t^{k-1}]P_M(t) = -\sum_{F \neq \emptyset} (-1)^{rk_F} [t^{k-1}]Q_{M_F}(t)P_{M_F}(t)
\]
and similarly for \([t^{k-1}]Q_M(t)\). We know that \( \deg Q_{M_F} \leq \left\lfloor \frac{rk_F - 1}{2} \right\rfloor \) and \( \deg P_{M_F} \leq \left\lfloor k - \frac{rk_F}{2} + 1 \right\rfloor \), which means that
\[
[t^{k-1}]Q_{M_F}(t)P_{M_F}(t) = [t^{\left\lfloor \frac{rk_F - 1}{2} \right\rfloor}]Q_{M_F}(t) [t^{\left\lfloor k - \frac{rk_F}{2} + 1 \right\rfloor}]P_{M_F}(t).
\]
Hence, by induction hypothesis, we get that
\[
[t^{k-1}]Q_{M_F}(t)P_{M_F}(t) = [t^{k-1}]P_{M_F}(t)Q_{M_F}(t),
\]
which, substituted back in the formula for \([t^{k-1}]P_M(t)\) and \([t^{k-1}]Q_M(t)\), completes the proof. \( \square \)

**Remark 3.5.** The only rank for which Proposition 3.4 is not true is \( r = 2 \), because the maximal degree is 0 but Proposition 3.1 gives infinite counterexamples since \([t^0]\chi_{U_{2,n}}(t) = n - 1 \). More specifically, the only rank-2 matroid for which the statement holds is the Boolean matroid \( B_2 \).

**Corollary 3.6.** If \( M \) is a matroid of rank \( r = 2k \), then
\[
[t^{k-1}] P_M(t) = \frac{1}{2} \sum_{rk_F \text{ odd}} P_{M_F}(t)Q_{M_F}(t).
\]
Proof. We know that

\[ P_M(t) + Q_M(t) = -\sum_{\emptyset \neq F \subset E} P_{M_F}(t)(-1)^{r-r_F}Q_{M_F}(t). \]

Since \([t^{k-1}]P_M(t) = [t^{k-1}]Q_M(t)\), then

\[ 2[t^{k-1}]P_M(t) = [t^{k-1} \left( -\sum_{\emptyset \neq F \subset E} P_{M_F}(t)(-1)^{r-r_F}Q_{M_F}(t) \right) \];

however, if \(\rho(F)\) is even, the maximal degree of \(P_{M_F}(t)Q_{M_F}(t)\) is \(k-2\), which means that only odd-rank flats give contributions to the maximal degree, hence the result.

We now continue by investigating the behaviour of modular matroids with respect to their Kazhdan-Lusztig polynomials. Proposition 2.14 in [4] gives the following condition on \(P_M(t)\) regarding the modularity of \(\mathcal{L}(M)\).

**Proposition 3.7.** The following conditions are equivalent

i) \(P_M(t) = 1\),

ii) \([t]P_M(t) = 0\),

iii) \(\mathcal{L}(M)\) is modular.

In a sense then, modular matroids are the most degenerate ones and it should be clear by now why we are interested in studying them with regards to Conjecture [1]. We can now give necessary and sufficient conditions for modularity with respect to \(Q_M(t)\).

**Theorem 3.8.** The modularity of \(\mathcal{L}(M)\) is equivalent to \(\deg Q_M(t) = 0\).

Proof. The proof is by induction on \(r\). From the relation

\[ (-1)^r Q_M(t) = -\sum_{F \neq \emptyset} P_{M_F}(t)(-1)^{r-r_F}Q_{M_F}(t), \]

if \(\mathcal{L}(M)\) is modular, then so is the lattice of any minor of \(M\). By induction then, \((-1)^r Q_M(t)\) is constant. Conversely, suppose that \(M\) is not modular but \(\deg Q_M(t) = 0\), and that \(M\) is minimal by contraction, that is \(M\) is not modular and \(M_F\) is modular for every \(F \neq \emptyset\). Then,

\[ P_M(t) = -\sum_{F \neq \emptyset} (-1)^{r_F}Q_{M_F}(t). \]

If every \(M_F\) is modular, then \(P_M(t)\) is constant, which is a contradiction. Consider, then, a flat \(F\) such that \(M_F\) is not modular and that is minimal by restriction, that is \(M_G\) is modular for every \(G \subseteq F\). Since by induction hypothesis \(M^G\) is modular for every flat \(\emptyset \neq G \subseteq E\), then \(M^G\) is modular as well, proving that \(M_F\) is indeed modular.

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4 New results on the Conjecture of degeneracy

We now finally move on to give some partial answers to Conjecture 1.1. We first observe that the Fano matroid is a degenerate matroid, since its rank is equal to 3 but it is modular; however, it is not a valid counterexample to the Conjecture since it is not regular. With this in mind, we can actually prove that the Conjecture holds for all modular matroids.

**Theorem 4.1.** If $M$ is a connected modular matroid of rank $r \geq 3$, then it is not regular.

**Proof.** Firstly, we point out that a quick check shows that the statement is true for matroids on less than 8 elements (See for example [11]). We also see that the only rank-3 connected modular matroids on 7 elements are the Fano matroid and its dual, which are known to be representable only on fields of characteristic 2 and, therefore, not regular. Consider now a rank-3 modular connected matroid $M$ on $n$ elements; since it is connected, in particular it does not contain any coloop, hence it is a connected projective geometry, where the hyperplanes coincide with the rank-2 flats. If $M$ contains any hyperplane $H$ with $k \geq 4$ elements then $M_H \cong U_{2,k}$, which is a minor of $M$ not representable in fields of characteristic 2. Thus, we just need to check what happens if all hyperplanes of $M$ have exactly 3 elements; by Theorem 7.2.5 in [9] (See also [12]), we can compute its characteristic polynomial to be equal to

\[
\chi_M(t) = (t-1)(t-2)(t-n+3) = t^3 - nt^2 + (3n-7)t + (6-2n).
\]

However, its Tutte polynomial must be of the form

\[
T_M(x, y) = \sum_{A \subseteq E} (x-1)^{3-rkA} (y-1)^{|A|-rkA}
\]

\[
= (x-1)^3 + n(x-1)^2 + \binom{n}{2}(x-1) + n(x-1)(y-1) + \left[\binom{n}{3} - n\right] + \sum_{k=4}^{n} \binom{n}{k} (y-1)^{k-3},
\]

which means that the characteristic polynomial is equal to

\[
\chi_M(t) = -T_M(1-t,0)
\]

\[
= t^3 - nt^2 + \frac{n(n-3)}{2} t - \frac{n^2 - 5n + 2}{2}.
\]

The two expressions for $\chi_M$ are equal if and only if $n = 7$, which means that $M$ is a Fano matroid (or its dual). This lets us conclude that the statement holds on rank-3 matroids. For a generic rank $r$, consider a connected projective geometry $M$ and all its rank-3 flats. If $G$ is a rank-3 flat, $M_G$ must be a modular (not necessarily connected) geometric lattice and thus must be isomorphic to one among these possibilities

- $B_3$,
- $B_1 \oplus U_{2,k}$.
• A rank-3 connected projective geometry.

However, $M$ is a connected projective geometry and its rank-2 flats need to have at least three elements, which lets us exclude the first two options. Therefore, every restriction is isomorphic to a rank-3 connected projective geometry, which is not regular.

These results let us then conclude that

• If a matroid is modular, connected and regular it is isomorphic to either $B_1$ or $U_{2,3}$, which are both non-degenerate;

• If a connected regular matroid of rank $r \leq 4$ is not modular, then it is non-degenerate.

Lastly, we show that the non-degeneracy of even-rank matroids can be inferred by the non-degeneracy of odd-rank matroids.

**Theorem 4.2.** If every connected, regular, odd-rank matroid is non-degenerate, then every connected, regular, even-rank matroid is non-degenerate.

To prove this, we need the following preliminary result.

**Lemma 4.3.** If $M$ is simple and connected, then there exists an element $i \in E$ such that $L(M_i)$ is isomorphic to the lattice of flats of a simple connected matroid.

**Proof.** The proof is by induction on $n$. If $n = r$, $M \cong B_n$ is not connected, thus we can assume $n \geq r + 1$. If $n = r + 1$, then $M \setminus i$ is a matroid on $r$ elements of rank $r$ (since $M$ is connected it does not contain coloops and the deletion of one element does not decrease the rank). This implies that $M^i$ is connected for every $i \in E$ ([13] Theorem 6.5). If now $n > r + 1$, and every $M^i$ is not connected, this implies that $M \setminus i$ is connected and simple for every $i \in E$. By induction hypothesis, for every $j \neq i$, $(M \setminus i)^j = (M^j) \setminus i$ is connected. However, since $M^i$ is not connected, this means that $M^i$ has two connected components, namely, $i$ and $E \setminus (i \cup j)$, and in particular that $i$ is a coloop in $M^j$ (it cannot be a loop because that would imply that $i$ and $j$ are parallel). This means that $i$ is not contained in any circuit of $M^j$, or, in other words, that no circuit of $M$ contains both $i$ and $j$, thus contradicting the hypothesis on $M$ being connected.

We are now ready to prove Theorem 4.2.

**Proof.** If $M$ is a matroid of rank $r = 2k$, we know from [3.6] that

\[
[r^{k-1}] P_M(t) = \frac{1}{2} \sum_{rk \text{ even}} \left( [\frac{n-1}{2}] P_M(t) \left[ [\frac{n-1}{2}] \right] Q_M(t). \right.
\]

If $M$ is connected and regular, then all its restrictions are regular; by the previous Lemma there exists an $i \in E$ such that $M^i$ is also connected (and obviously of odd rank), meaning that $M^i$ is non-degenerate by hypothesis. In particular, $\deg P_M(t) = [\frac{n-1}{2}] = k - 1 = [\frac{n-1}{2}] \deg P_M(t).

We remark that we are not concerned with the sign of the coefficients of $P_M(t)$ and $Q_M(t)$ because they are already proven to be positive; hence, in a sum with no alternating signs, if one term is non-zero then we are sure that the sum is positive (non-zero) too.
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