Approximation of Excessive Backlog Probabilities of Two Tandem Queues

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Abstract

Let $X$ be the constrained random walk on $\mathbb{Z}_+^2$ with increments $(1,0)$, $(-1,1)$ and $(0,-1)$; $X$ represents the lengths of two queues in tandem where arrivals are Poisson to the first queue with rate $\lambda$, and the service times are exponentially distributed with rates $\mu_1$ and $\mu_2$; we assume $\lambda < \mu_1, \mu_2$, i.e., $X$ is assumed stable and $\mu_1 \neq \mu_2$ (the case $\mu_1 = \mu_2$ can be handled by allowing $\mu_1$ converge to $\mu_2$). Let $\tau_n$ be the first time $X$ hits the line $\partial A_n = \{x : x(1)+x(2) = n\}$, i.e., when the sum of the components of $X$ equals $n$ for the first time. For $x \in \mathbb{Z}_+^2$, $x(1)+x(2) < n$, the probability $p_n(x) = P_n(\tau_n < \tau_0)$ is one of the key performance measures for the queueing system represented by $X$ (if the queues share a common buffer, $p_n(x)$ is the probability that this buffer overflows during the system’s first busy cycle). Let $Y$ be the random walk on $\mathbb{Z} \times \mathbb{Z}_+$ with increments $(-1,0)$, $(1,1)$ and $(0,-1)$ that is constrained to be positive only on its second component. Let $\tau$ be the first time that the components of $Y$ equal each other. Let $\rho_i = \lambda/\mu_i$, $i = 1,2$, denote the utilization rates of the nodes. We derive the following explicit formula for $P_\rho(\tau < \infty)$:

$$P_\rho(\tau < \infty) = W(y) = \rho_2^{y(1)-y(2)} + \frac{\mu_2 - \lambda}{\mu_2 - \mu_1} \rho_1^{y(1)-y(2)} \rho_1^{y(2)} + \frac{\mu_2 - \lambda}{\mu_1 - \mu_2} \rho_1^{y(1)-y(2)} \rho_2^{y(2)},$$

$y \in \mathbb{Z} \times \mathbb{Z}_+$, $y(1) > y(2)$, and show that $W(n-x_n(1),x_n(2))$ approximates $p_n(x_n)$ with relative error exponentially decaying in $n$ for $x_n = [nx]$, $x \in \mathbb{R}_+^2$, $0 < x(1) + x(2) < 1$. Our analysis consists of the following steps: 1) with an affine transformation, move the origin of the coordinate system to the point $(n,0)$ on the exit boundary $\partial A_n$; let $n \to \infty$ to remove the constraint on the $x(2)$ axis. 2) this step gives the limit unstable/transient constrained random walk $Y$ that is constrained only on the $x(1)$ axis, and reduces $P_\rho(\tau_n < \tau_0)$ to $P_\rho(\tau < \infty)$. 2) construct a basis of harmonic functions of $Y$ and use this basis to apply the classical superposition principle of linear analysis to compute $P_\rho(\tau < \infty)$. The construction of basis functions involve the use of conjugate points on a characteristic surface associated with the walk $X$. The proof that the relative error decays exponentially in $n$ uses a sequence of subsolutions of a related Hamilton Jacobi Bellman equation on a manifold; the manifold consists of three copies of $\mathbb{R}_+^2$, the zeroth glued to the first along $\{x : x(1) = 0\}$ and the first to the second along $\{x : x(2) = 0\}$. We indicate how the ideas of the paper can be generalized to more general processes and other exit boundaries.

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1 Introduction and definitions

Let $X$ be a random walk with independent and identically distributed increments $\{I_1, I_2, I_3, \ldots\}$, constrained to remain in $\mathbb{Z}_+^2$:

$$X_0 = x \in \mathbb{Z}_+^2, \quad X_{k+1} = X_k + \pi(X_k, I_k), k = 1, 2, 3, \ldots$$

$$\pi(x, v) = \begin{cases} v, & \text{if } x + v \in \mathbb{Z}_+^2 \\ 0, & \text{otherwise,} \end{cases}$$

$I_k \in \{(1, 0), (-1, 1), (0, -1)\}$, $P(I_k = (1, 0)) = \lambda, P(I_k = (-1, 1)) = \mu_1, P(I_k = (0, -1)) = \mu_2$.

Let $\tau_i \doteq \{x \in \mathbb{Z}^2 : x(i) = 0\}, i = 1, 2$, denote the constraining boundaries of the process and let $\sigma_i \doteq \inf\{k : X_k \in \tau_i\}$, $i = 1, 2$, denote the first time $X$ hits these boundaries. The components of $X$ represents the number of customers at jump times of a Jackson network consisting of two tandem queues.

We assume that $X$ is stable, i.e., $\lambda < \mu_1, \mu_2$. We also assume $\mu_1 \neq \mu_2$; subsection 7.1 comments on the case $\mu_1 = \mu_2$. Define

$$A_n = \{x \in \mathbb{Z}_+^2 : x(1) + x(2) \leq n\} \quad (1)$$

and its boundary

$$\partial A_n = \{x \in \mathbb{Z}_+^2 : x(1) + x(2) = n\}. \quad (2)$$

Let $\tau_n$ be the first time $X$ hits $\partial A_n$:

$$\tau_n \doteq \inf\{k : X_k \in \partial A_n\}. \quad (3)$$

Define $p_n \doteq P_x(\tau_n < \tau_0)$, i.e., the probability that, starting from an initial state $x \in A_n$, the total number of customers in the system reaches $n$ before the system empties. The set $A_n$ models a systemwide shared buffer of size $n$. If we measure time in the number of independent cycles that restart each time $X$ hits 0, $p_n$ is the probability that the current cycle finishes successfully (i.e., without a buffer overflow). One can change the domain $A_n$ to model other buffer structures, e.g., $\{x \in \mathbb{Z}_+^2 : x(i) \leq n, i = 1, 2\}$ models separate buffers of size $n$ for each queue in the system. The present work focuses on the domain $A_n$. The basic ideas of the paper apply to other domains, and we comment on this in Section 7. Let $Y$ be the random walk on $\mathbb{Z} \times \mathbb{Z}_+$ with increments $(-1, 0), (1, 1)$ and $(0, -1)$ that is constrained to be positive only on its second component. Let $\tau$ be the first time that the components of $Y$ equal each other (the relation between $X$ and $Y$ is explained in the paragraphs below). In Section 8 we derive the following explicit formula for $P_y(\tau < \infty)$:

$$P_y(\tau < \infty) = W^*(y) \doteq \left( \mu_2 - \frac{\mu_2 - \lambda \mu_1}{\mu_2 - \lambda} \rho_1^y \right) \left( \frac{\mu_2}{\mu_2 - \lambda} \rho_1^{y(1)-y(2)} \rho_1^{y(2)} + \frac{\mu_2}{\mu_2 - \lambda} \rho_1^{y(1)-y(2)} \rho_1^{y(2)} \right), \quad (4)$$

$y \in \mathbb{Z}_+^2, y(1) > y(2)$. Fix $x \in \{x \in \mathbb{R}_+^2 : 0 < x(1) + x(2) < 1\}$ and define $x_n = \lfloor nx \rfloor$. In Section 4 we show that $W^*(n - x_n(1), x_n(2))$ approximates $p_n(x_n)$, with relative error exponentially vanishing in $n$ (see Proposition 4.1). The following paragraphs note prior literature and results relating to the approximation of $p_n$ and summarize the analysis that lead to the results summarized above.

For a stable $X$, the event $\{\tau_n < \tau_0\}$ rarely happens and, conditioned on a fixed initial point $x$, its probability $p_n$ decays exponentially with buffer size $n$. Because $X$ is Markov, $p_n$,
as a function of the initial point \( x \), satisfies a system of linear equations, see (22). As \( n \) gets large, the number of unknowns grow like \( n^2 \) and it becomes infeasible to solve the system exactly. [29, 12] compute the large deviation limit of \( p_n(x) \), for \( x = (1,0) \), as

\[
\lim_{n \to \infty} -\frac{1}{n} \log p_n((1,0)) = \min(-\log \rho_1, -\log \rho_2),
\]

where \( \rho_i = \lambda/\mu_i \). Because \( p_n \) is a small probability, i.e., the probability of a rare event, a natural idea is to use importance sampling to approximate it via simulation. To the best of our knowledge, the article [62] is the first to study the optimal IS simulation of the two tandem walk model for the boundary \( \partial A_n \); it was observed in [62] that static changes of measure implied by optimal large deviation sample paths may not lead to optimal IS changes of measure because of the constraining boundaries of the process. [62] introduced boundary layers to the problem and allowed the change of measure to depend on whether the process is in these layers. It was observed in [39] that a simple change of measure implied by LD analysis (exchange the arrival rate with the smaller of the service rates) could perform poorly for the exit boundary \( \partial A_n = \{x : x(1) + x(2) = n\} \) for a range of parameter values. An asymptotically optimal change of measure for this boundary was developed in [29] using subsolution of a limit Hamilton Jacobi Bellman (HJB) equation; similar to the heuristic constructions in [62], the change of measure developed in [29] is dynamic, i.e., it depends on the position of the process \( X \); [69, 71, 32, 28] treats higher dimensions, more general dynamics and different exit boundaries using the subsolution approach. Let \( \tau_0 \) denote the first return time to the origin. The work [56] proposes an alternative approximation approach to probabilities of the type \( P_0(\tau_n < \tau_0) \) for a class of models under a number of assumptions; the approximation idea in [56] is to replace \( \tau_0 \) with \( \tau_\Delta \), and the initial position \( 0 \) with a random initial point on \( \Delta \) with distribution \( \pi_\Delta \), where \( \Delta \) are the constraining boundaries corresponding to a set of “non-super-stable” nodes, \( \tau_\Delta \) is the first nonzero time when one these nodes become empty, and \( \pi_\Delta \) is the stationary measure of the underlying process conditioned on \( \Delta \); [56] and its approach are further reviewed in Section 6. There is a vast literature on the analysis and simulation of rare events of constrained random walks, in particular, and on the analysis of constrained random walks in general [1, 2, 3, 5, 4, 8, 9, 10, 7, 12, 13, 15, 17, 18, 19, 21, 22, 23, 24, 20, 26, 34, 36, 37, 31, 20, 43, 44, 46, 45, 63, 28, 48, 49, 50, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 67, 70, 71, 74, 32]. Section 6 reviews a number of these works in relation to the results and the techniques of the current work.

One way to think about the LD analysis is as follows. \( p_n \) itself decays to 0, which is trivial. To get a nontrivial limit transform \( p_n \) to \( V_n \equiv -\frac{1}{n} \log p_n \); using convex duality, one can write the \( -\log \) of an expectation as an optimization problem involving the relative entropy [25] and thus \( V_n \) can be interpreted as the value function of a discrete time stochastic optimal control problem. The LD analysis consists of the law of large numbers limit analysis of this control problem; the limit problem is a deterministic optimal control problem whose value function satisfies a first order Hamilton Jacobi Bellman equation (see (55) of Section 4). Thus, LD analysis amounts to the computation of the limit of a convex transformation of the problem.

We will use another, an affine, transformation of \( X \) for the limit analysis. The proposed transformation is very simple: observe \( X \) from the exit boundary. For the two tandem walk the most natural vantage point on the exit boundary \( \partial A_n \) turns out to be the corner \( (n,0) \). Therefore, we transform the process thus

\[
Y^n \equiv T_n(X), \quad T_n : \mathbb{R}^2 \to \mathbb{R}^2, T_n(x) = y, \quad y(2) = x(2), y(1) = n - x(1). \tag{5}
\]
$T_n$ is affine and its inverse equals itself. $Y^n$, i.e., the process $X$ as observed from the corner $(n,0)$, is a constrained process on the domain $\Omega_Y^n = (n - \mathbb{Z}_+) \times \mathbb{Z}_+$. $T_n$ maps the set $A_n$ to $B_n \subset \Omega_Y^n$, $B_n = T_n(A_n)$, the corner $(n,0)$ to the origin of $\Omega_Y^n$; the exit boundary $\partial A_n$ to $\partial B_n \equiv \{y \in \Omega_Y^n, y(1) = y(2)\}$ and finally the constraining boundary $\{x \in \mathbb{Z}_+^2, x(1) = 0\}$ to $\{y \in \mathbb{Z}_+^2 : y(i) = n\}$.

As $n \to \infty$ the last boundary vanishes and $Y^n$ converges to the limit process $Y$ on the domain $\Omega_Y = \mathbb{Z} \times \mathbb{Z}_+$ and the set $B_n$ to

$$B \equiv \{y \in \Omega_Y, y(1) \geq y(2)\}. \tag{6}$$

The exit boundary for the limit problem is

$$\partial B = \{y \in \Omega_Y, y(1) = y(2)\}; \tag{7}$$

the limit stopping time

$$\tau = \inf\{k : Y_k \in \partial B\} \tag{8}$$

is the first time $Y$ hits $\partial B$. The stability of $X$ and the vanishing of the boundary constraint on $\partial_1$ implies that $Y$ is unstable / transient, i.e., with probability 1 it wanders off to $\infty$. Therefore, in our formulation, the limit process is an unstable constrained random walk in the same space and time scale as the original process but with less number of constraints and the limit problem is whether this unstable process ever hits the fixed boundary $\partial B$.

Figure 1 sketches these transformations.

Fix an initial point $y \in B$ in the new coordinates; our first convergence result is Proposition 2.1 which says

$$p_n = P_{x_n}(\tau_n < \tau_0) \to P_y(\tau < \infty), \tag{9}$$

where $x_n = T_n(y)$. The proof uses the law of large numbers and LD lowerbounds to show that the difference between the two sides of (9) vanishes with $n$. With (9) we see that the limit problem in our formulation is to compute the hitting probability of the unstable $Y$ to the boundary $\partial B$.

The convergence statement (9) involves a fixed initial condition for the process $Y$. In classical LD analysis, one specifies the initial point in scaled coordinates as follows: $x_n = [nx] \in A_n$ for $x \in \mathbb{R}_+^d$. Then the initial condition for the $Y^n$ process will be $y_n = T_n(x_n)$ (thus we fix not the $y$ coordinate but the scaled $x$ coordinate). When $x_n$ is defined in this way,
(9) becomes a trivial statement because its both sides decay to 0. For this reason, Section 4 studies the relative error
\[
\frac{|P_x(\tau_n < \tau_0) - P_y(\tau < \infty)|}{P_x(\tau_n < \tau_0)};
\] (10)
Proposition 4.1 says that this error converges exponentially to 0 for the case of two-dimensional tandem walk (i.e., the process \(X\) shown in Figure 1). The proof rests on showing that the probability of the intersection of the events \(\{\tau_n < \tau_0\}\) and \(\{\tau < \infty\}\) dominate the probabilities of both as \(n \to \infty\). For this we calculate bounds in Proposition 4.3 on the LD decay rates of the probability of the differences between these events using a sequence of subsolutions of a Hamilton Jacobi Bellman equation on a manifold; the manifold consists of three copies of \(\mathbb{R}^2_+\), zeroth copy glued to the first along \(\partial_1\), and the first to the second along \(\partial_2\), where \(\partial_i = \{x \in \mathbb{R}^2_+: x(i) = 0\}\). Extension of this argument to more complex processes and domains remains for future work.

The convergence results (9) and (10) reduce the problem of calculation of \(P_x(\tau_n < \tau_0)\) to that of \(P_y(\tau < \infty)\). This constitutes the first step of our analysis and we expect it to apply more generally; see subsection 7.4.

Computation of \(P_y(\tau < \infty)\) is a static linear problem and can be attacked with a range of ideas and methods. Section 3 applies the principle of superposition of classical linear analysis to the computation of \(P_y(\tau < \infty)\). The key for its application is to construct the right class of efficiently computable basis functions to be superposed. The construction of our basis functions goes as follows: the distribution of the increments of \(Y\) is used to define the characteristic polynomial \(p: \mathbb{C}^2 \to \mathbb{C}\). \(p\) can be represented both as a rational function and as a polynomial. We call the 1 level set of \(p\), the characteristic surface of \(Y\) and denote it with \(H\), see (28). \(H\) is, more precisely, a 1 dimensional complex affine algebraic variety of degree 3. Each point on the characteristic surface \(H\) defines a log-linear function (see Proposition 3.1) that satisfies the interior harmonicity condition of \(Y\) (i.e., defines a harmonic function of the completely unconstrained version of \(Y\)); similarly, each boundary of the state space of \(Y\) has an associated characteristic polynomial and surface. \(p\) can be written as a second order polynomial in each of its arguments; this implies that most points on \(H\) come in conjugate pairs. The keystone of the approach developed in Section 3 is the following observation: log-linear functions defined by two points on \(H\) satisfying a given type of conjugacy relation can be linearly combined to get nontrivial functions which satisfy the corresponding boundary harmonicity condition (as well as the interior one); see Figure 3.1 and Proposition 3.3. We show that any solution to a harmonic system gives a harmonic function for \(Y\) in the form of linear combinations of log-linear functions (each vertex defines a log-linear function).

There is a direct connection between the computations given in the present paper and the Balayage operator \[64\], we point out this connection in subsection 3.4 Remark 2. Section 5 gives a numerical example. The conclusion (Section 7) discusses several directions for future research. Among these is the application of the approach of the present paper to constrained diffusion processes and the associated elliptic equations with Neumann boundary conditions (subsection 7.2).

2 Derivation of the limit problem

This section derives the limit problem resulting from the affine transformation \(T_n\). The derivation is simple enough and therefore will be given for a more general setup: for the
purposes of the present section we will assume $X$ to be the embedded random walk of a $d$ dimensional stable Jackson network; let, as before, $I_k$ denote the unconstrained iid increments of $X$. Define

$$I_1 \in \mathbb{R}^{d \times d}, \ I_1(j, k) = 0, j \neq k, \ I_1(j, j) = 1, j \neq 1, \ I_1(1, 1) = -1. \quad (11)$$

$I_1$ is the identity operator on $\mathbb{R}^d$ except that its first diagonal term is $-1$ rather than 1. The affine change of coordinate map will be

$$T_n = n e_1 + I_1,$$

where $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^d$. Define the sequence of transformed increments

$$J_k = I_1(I_k). \quad (13)$$

The domain of the limit $Y$ process will be $\Omega_Y = \mathbb{Z} \times \mathbb{Z}_+^{d-1}$ and the limit process will have dynamics

$$Y_{k+1} = Y_k + \pi_1(Y_k, J_k),$$

where

$$\pi_1(x, v) = \begin{cases} v, & \text{if } x + v \in \Omega_Y, \\ 0, & \text{otherwise}. \end{cases}$$

Let $A_n = \{ x \in \mathbb{Z}_+^d : x(1) + x(2) + \cdots + x(d) \leq n \}$, $\tau_n$ is the first time $X$ hits $\partial A_n = \{ x \in \mathbb{Z}_+^d : x(1) + x(2) + \cdots + x(d) = n \}$. The limit exit boundary will be $\partial B = \{ y \in \Omega_Y, y(1) \geq \sum_{i=2}^d y(i) \}$, $\tau$ is the first time $Y$ hits $\partial B$. Set $\hat{c}_1 = \{ z \in \mathbb{Z}^d : z(1) = 0 \}$ and $\sigma_1$ will be the first time $X$ hits $\hat{c}_1$.

Denote by $\mathcal{X}$ the law of large numbers limit of $X$, i.e., the deterministic function which satisfies

$$\lim_{n} P_{x_n} \left( \sup_{k \leq t \leq t_0} |X_k/n - \mathcal{X}_k/n| > \delta \right) = 0 \quad (14)$$

for any $\delta > 0$ and $t_0 > 0$ where $x_n \in \mathbb{Z}_+^d$ is a sequence of initial positions satisfying $\frac{x_n}{n} \to \chi \in \mathbb{R}_+^d$ (see, e.g., Proposition 9.5 or Theorem 7.23]). The limit process starts from $X_0 = \chi$, is piecewise affine and takes values in $\mathbb{R}_+^d$; then $s_t \equiv \sum_{i=1}^d \mathcal{X}_i(i)$ starts from $\sum_i \chi(i)$ is also piecewise linear and continuous (and therefore differentiable except for a finite number of points) with values in $\mathbb{R}_+$. The stability and bounded iid increments of $X$ imply that $s$ is strictly decreasing and

$$c_1 > -\dot{s} > c_0 > 0 \quad (15)$$

for two constants $c_1$ and $c_0$. These imply that $\mathcal{X}$ goes in finite time $t_1$ to $0 \in \mathbb{R}_+^d$ and remains there afterward.

Fix an initial point $y \in \Omega_Y$ for the process $Y$ and set $x_n = T_n(y)$; it follows from the definition of $T_n$ that

$$\frac{x_n}{n} \to e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^d. \quad (16)$$

**Proposition 2.1.** Let $y$ and $x_n$ be as above. Then

$$\lim_{n \to \infty} P_{x_n} (\tau_n < \tau_0) = P_y (\tau < \infty).$$
Proof. Note that \( x_n \in A_n \) for \( n > y(1) \). Define

\[
M_k = \max_{l < k} Y_l(i), \quad M_k^X = \min_{l < k} X_l(i).
\]

\( M \) is an increasing process and \( M_\tau \) is the greatest that the first component of \( Y \) gets before hitting \( \partial B \) (if this happens in finite time). The monotone convergence theorem implies

\[
P_y(\tau < \infty) = \lim_{n \to \infty} P_y(\tau < \infty, M_\tau < n).
\]

Thus

\[
P_y(\tau < \infty) = P_y(\tau < \infty, M_\tau < n) + P_y(\tau < \infty, M_\tau \geq n)
\]

and the second term goes to 0 with \( n \). Decompose \( P_{x_n}(\tau_n < \tau_0) \) similarly using \( M^X \):

\[
P_{x_n}(\tau_n < \tau_0) = P_{x_n}(\tau_n < \tau_0, M^X_{\tau_n} > 0) + P_{x_n}(\tau_n < \tau_0, M^X_{\tau_n} = 0).
\]

On the set \( \{M^X_{\tau_n} > 0\} \), the process \( X \) cannot reach the boundary \( \partial \bar{A} \) before \( \tau_n \), therefore over this set 1) the events \( \{\tau_n < \tau_0\} \) and \( \{\tau < \infty\} \) coincide (remember that \( X \) and \( Y \) are defined on the same probability space) 2) the distribution of \( \langle T_n(X), n - M^X \rangle \) is the same as that of \( \langle Y, M \rangle \) up to time \( \tau_n \). Therefore,

\[
P_y(\tau < \infty, M_\tau < n) + P_{x_n}(\tau_n < \tau_0, M^X_{\tau_n} = 0).
\]

The first term on the right equals the first term on the right side of (17). We know that the second term in (17) goes to 0 with \( n \). Then to finish our proof, it suffices to show

\[
\lim_{n} P_{x_n}(\tau_n < \tau_0, M^X_{\tau_n} = 0) = 0.
\]

(18) \( M^X_{\tau_n} = 0 \) means that \( X \) has hit \( \partial \bar{A} \) before \( \tau_n \). Then the last probability equals

\[
P_{x_n}(\sigma_1 < \tau_n < \tau_0),
\]

which, we will now argue, goes to 0 (\( \sigma_1 \) is the first time \( X \) hits \( \partial \bar{A} \)): (16) implies \( \mathcal{X}_0 = e_1 \). Define \( t^1 = \inf \{ t : X_1 = 0 \} \) and \( t^0 = \inf \{ t : X_1 = 0 \in \mathbb{R}^d \} \). By definition \( t^1 \leq t^0 < \infty \) Now choose \( l_0 \) in (14) to be equal to \( t^0 \), define \( \mathcal{C}_n \) \( \{ \sup_{k \in \mathbb{Z}^n} \mathcal{X}_k \} > \delta \} \) and partition (19) with \( \mathcal{C}_n \):

\[
P_{x_n}(\sigma_1 < \tau_n < \tau_0) = P_{x_n}(\{ \sigma_1 < \tau_n < \tau_0 \} \cap \mathcal{C}_n) + P_{x_n}(\{ \sigma_1 < \tau_n < \tau_0 \} \cap \mathcal{C}_n^c).
\]

The first of these goes to 0 by (14). The event in the second term is the following: \( X \) remains at most \( n\delta \) distance away from \( \mathcal{X} \) until its \( nt^0 \) step, hits \( \partial \bar{A} \) and then 0. These and (16) imply that, for \( n \) large enough, any sample path lying in this event can hit \( \partial \bar{A} \) only after time \( nt^0 \). Thus, the second probability on the right side of (20) is bounded above by

\[
P_{x_n}(\{ nt^0 < \tau_n < \tau_0 \} \cap \mathcal{C}_n^c).
\]

The Markov property of \( X \), \( \{ \sigma_1 < \tau_n < \tau_0 \} \subset \{ \tau_n < \tau_0 \} \) and (14) imply that the last probability is less than

\[
\sum_{x : |x| \leq n\delta} P_x(\tau_n < \tau_0)P_{x_n}(X_{nt^0} = x).
\]

For \( |x| \leq n\delta \), the probability \( P_x(\tau_n < \tau_0) \) decays exponentially in \( n \) (see Theorem 2.3); then, the above sum goes to 0. This establishes (18) and finishes the proof of the proposition. \( \square \)
3 Analysis of the limit problem

In this section and the rest of the paper we will be focusing on the two tandem queue process and its limit defined in Section 1. The analysis of the previous section suggests that we approximate

\[ P(x, \tau_n < \tau_0) \]

with \( W(T_n(x)) \) where

\[ W(y) \doteq P_y(\tau < \infty) = \mathbb{E}_y[1_{\{\tau < \infty\}}] . \]

The goal of this section is to develop a framework in which we will derive the following explicit formula for \( W \):

\[ W_p(y) = P_p(\tau < \infty) = \rho_2^{{y(1)-y(2)}} + \frac{\mu_2 - \lambda}{\mu_2 - \mu_1} \rho_1^{{y(1)-y(2)}} x(2) + \frac{\mu_2 - \lambda}{\mu_1 - \mu_2} \rho_2^{{y(1)-y(2)}} \rho_1^y, \]

\( y \in \mathbb{Z}_+^2, y(1) \geq y(2) \); the proof of this formula is given in as the final result (Proposition 3.6) of this section.

It follows from the Markov property of \( Y \) that \( W \) is a harmonic function of \( Y \) (or \( Y \)-harmonic) i.e., it satisfies:

\[ V_p(y) = \mathbb{E}_y[V(Y_1)] = \sum_{v \in \mathcal{Y}} V(y + \pi_1(y,v))p(v), y \in B, \]

where

\[ \mathcal{Y} \doteq \{(-1,0),(1,1),(0,-1)\}, \]

\[ \pi_1(x,v) \doteq \begin{cases} v, & \text{if } x + v \in \mathbb{Z} \times \mathbb{Z}_+ \\ 0, & \text{otherwise} \end{cases} \]  

(23)

\[ W(y) = P_y(\tau < \infty) = 1 \text{ for } y \in \partial B \text{ implies that } W \text{ also satisfies the boundary condition} \]

\[ V|_{\partial B} = 1. \]  

(24)

A \( Y \)-harmonic function \( h \) is said to be \( \partial B \)-determined if it is of the form

\[ h(y) = \mathbb{E}[f(Y)1_{\{\tau < \infty\}}], y \in \mathbb{Z} \times \mathbb{Z}_+, y(1) \geq y(2). \]

By its definition, \( W \) is \( \partial B \)-determined. Then \( W \) is the unique \( \partial B \)-determined solution of (22,24).

Let \( Z \) denote the ordinary unconstrained random walk in \( \mathbb{Z}^2 \) with the same increments as \( Y \). The unconstrained version of (22) is

\[ V(z) = \mathbb{E}_z[V(Z_1)] = \sum_{v \in \mathcal{Y}} V(z + v)p(v), \quad z \in \mathbb{Z}^2. \]  

(25)

A function is said to be a harmonic function of the unconstrained random walk \( Z \) if it satisfies (24).

Our idea for solving (22,24) (and hence obtaining a formula for \( P_y(\tau < \infty) \)) is this:

1. Construct a class \( \mathcal{F}_Y \) of “simple” harmonic functions for the process \( Y \) (i.e., a class of solutions to (22)) For this
(a) Construct a class \( \mathcal{F}_Z \) of harmonic functions for the unconstrained process \( Z \) of the form \( z \mapsto \beta z^{(1)} \alpha z^{(2)}, (\beta, \alpha) \in \mathbb{C}^2 \).

(b) Use linear combinations of elements of \( \mathcal{F}_Z \) to find solutions to (22).

2. Represent the boundary condition (21) by linear combinations of the boundary values of the \( \partial B \)-determined members of the class \( \mathcal{F}_Y \).

The definition of the class \( \mathcal{F}_Z \) is given in (31) and that of \( \mathcal{F}_Y \) is given in (43).

A remark about uniqueness: We have assumed that \( X \) is stable; this implies that \( Y \) is unstable and therefore, the Martin boundary of this process has points at infinity. Then one cannot expect all harmonic functions of \( Y \) to be \( \partial B \)-determined and in particular the system (22,24) will not have a unique solution. In particular, the constant function \( 1 \) solves this system, but as we will see below, \( 1 \) is not \( \partial B \)-determined.

hence, once we find a solution of (22, 24) that we believe equal to \( \tau \), we will have to prove that it is \( \partial B \)-determined.

3.1 The characteristic polynomial and surface

Let us call

\[ p(\beta, \alpha) \doteq \sum_{v \in \mathcal{V}} p(v) \beta^{v^{(1)} - v^{(2)}} \alpha^{v^{(2)}} = \lambda \frac{1}{\beta} + \mu_1 \alpha + \mu_2 \frac{\beta}{\alpha}, \quad (\beta, \alpha) \in \mathbb{C}^2, \] (26)

the interior characteristic polynomial of the process \( Y \);

\[ p(\beta, \alpha) = 1 \] (27)

the interior characteristic equation of \( Y \) and

\[ \mathcal{H} = \{ (\beta, \alpha) : p(\beta, \alpha) = 1 \} \] (28)

the interior characteristic surface of \( Y \). We borrow the adjective “characteristic” from the classical theory of linear ordinary differential equations; the development below parallels that theory. \( p \) is a rational function, not a polynomial, but it obviously becomes polynomial in \( \alpha [\beta] \) when multiplied by \( \beta [\alpha] \) or a polynomial in \( \beta \) and \( \alpha \) when multiplied by \( \beta \alpha \); these polynomial representations are useful when we solve \( p(\beta, \alpha) = 1 \), but the rational representation is simpler, more flexible and natural. For this reason, we use the rational representation whenever possible, and switch to the polynomial representations when needed.

Figure 3.1 depicts the real section of the characteristic surface of the walk for \( \lambda = 0.1, \mu_1 = 0.5 \) and \( \mu_2 = 0.4 \). \( \mathcal{H} \) is an affine algebraic curve of degree 3 [40], Definition 8.1, page 32]. The characteristic equation \( p = 1 \) becomes a quadratic equation in \( \alpha \) when one multiplies it by \( \alpha \); the discriminant of this quadratic equation is

\[ \Delta(\beta) = \left( \frac{\lambda}{\beta} - 1 \right)^2 - 4 \mu_1 \mu_2 \beta. \]

Therefore, for \( \beta \in \mathbb{C}, \Delta(\beta) \neq 0 \) and \( \beta \neq 0 \), points on \( \mathcal{H} \) come in conjugate pairs \( (\beta, \alpha_1) \) and \( (\beta, \alpha_2) \), satisfying

\[ \alpha_i = \frac{1}{\alpha_{3-i}} \frac{\mu_2 \beta}{\mu_1}, i \in \{1, 2\}. \] (29)

These conjugate pairs will be central to the construction of \( Y \)-harmonic functions in subsection 3.2.2 below.

Any point on \( \mathcal{H} \) defines a harmonic function of \( Z \):
Figure 2: The real section of the characteristic surface $\mathcal{H}$ for $\lambda = 0.1$, $\mu_1 = 0.5$, $\mu_2 = 0.4$; the end points of the dashed line are an example of a pair of conjugate points $(\beta, \alpha_1)$ and $(\beta, \alpha_2)$; together they define the Y-harmonic function $h_\beta(y) = \beta^{y(1) - y(2)} \left( C(\beta, \alpha_2) \alpha_1^{y(2)} - C(\beta, \alpha_1) \alpha_2^{y(2)} \right)$, see Proposition 3.3. Each horizontal line intersecting the curve $\mathcal{H}$ twice gives a pair of conjugate points defining a Y-harmonic function.

**Proposition 3.1.** For any $(\beta, \alpha) \in \mathcal{H}$, $z \mapsto \beta^{z(1) - z(2)} \alpha^{z(2)}$, $z \in \mathbb{Z}^2$, is an harmonic function of $Z$; in particular, it satisfies (22) for $y \in \mathbb{Z}^2$, $y(1), y(2) > 0$.

*Proof.* Condition $Z$ on its first step and use $p(\beta, \alpha) = 1$. $\Box$

For $(\beta, \alpha) \in \mathbb{C}^2$, define

$$[(\beta, \alpha), \cdot] : \mathbb{Z}^2 \to \mathbb{C}, \quad [(\beta, \alpha), z] \doteq \beta^{z(1) - z(2)} \alpha^{z(2)}. \tag{30}$$

The last proposition gives us the class of harmonic functions

$$\mathcal{F}_Z \doteq \{ [(\beta, \alpha), \cdot], \ (\beta, \alpha) \in \mathcal{H} \} \tag{31}$$

for $Z$.  

10
3.2 \ log-linear harmonic functions of $Y$

Define $B^o \doteq \{ y \in \mathbb{Z}^2_+ : y(1) > y(2) \}$. Let us rewrite (22) separately for the boundary $\partial_2$ and the interior $B^o - \partial_2$:

\begin{align*}
V(y) &= \sum_{v \in \mathcal{V}} V(y + v)p(v), y \in B^o - \partial_2, \\
V(y) &= V(y)\mu_2 + \sum_{v \in \mathcal{V}, v \neq -1} V(y + v)p(v), y \in \partial_2 \cap B^o.
\end{align*}

Any $g \in \mathcal{F}_Z$ satisfies (32) (because (32) is the restriction of (25) to $B^o - \partial_2$); (32) is linear and so any finite linear combination of members of $\mathcal{F}_Z$ continues to satisfy (32). In the next two subsections we will show that appropriate linear combinations of members of $\mathcal{F}_Z$ will also satisfy the boundary condition (33) and define harmonic functions of the constrained process $Y$.

3.2.1 $Y$-harmonic function defined by a single point on $\mathcal{H}$

Remember that members of $\mathcal{F}_Z$ are of the form $\rho_1, \rho_2 : (\beta, \alpha) \to \beta^{z(1)} - \alpha^{z(2)}$ and $\rho_1, \rho_2 \in \mathcal{H}$; these define harmonic functions for $Z$ and they therefore satisfy (32). The simplest way to construct a $Y$-harmonic function is to look for $\rho_1, \rho_2 : (\beta, \alpha) \to \beta^{z(1)} - \alpha^{z(2)}$ which satisfies (22), i.e., which satisfies (32) and (33) at the same time. Substituting $\beta^{z(1)} - \alpha^{z(2)} - \rho_1, \rho_2$ in (33) we see that it solves (33) if and only if $\beta^{z(1)} - \alpha^{z(2)} - \rho_1, \rho_2$ also satisfies

\begin{equation}
\begin{split}
\rho_2(\beta, \alpha) &= 1
\end{split}
\end{equation}

where

\begin{equation}
\begin{split}
\rho_2(\beta, \alpha) &= \frac{1}{\beta} + \mu_1 \alpha + \mu_2;
\end{split}
\end{equation}

note

\begin{equation}
\begin{split}
\rho_2(\beta, \alpha) &= \rho_2(\beta, \alpha) - \mu_2 \left( \frac{\beta}{\alpha} - 1 \right).
\end{split}
\end{equation}

Let us call (34) “the characteristic equation of $Y$ on $\partial_2$” and $\rho_2$ its characteristic polynomial on the same boundary. Define the boundary characteristic surface of $Y$ for $\partial_2$ as

\begin{equation}
\begin{split}
\mathcal{H}_2 \doteq \{ (\beta, \alpha) \in \mathbb{C}^2 : \rho_2(\beta, \alpha) = 1 \}.
\end{split}
\end{equation}

For $[(\beta, \alpha), \cdot]$ to $Y$-harmonic, $(\beta, \alpha)$ must lie on

\begin{equation}
\begin{split}
\mathcal{H} \cap \mathcal{H}_2 = \{ (0,0), (1,1), (\rho_1, \rho_1) \} \subset \mathbb{C}^2;
\end{split}
\end{equation}

the third of these points gives us our first nontrivial $Y$-harmonic function:

**Proposition 3.2.** The function

\begin{equation}
\begin{split}
[(\rho_1, \rho_1), \cdot] : y \to \rho_1^{y(1)} - \rho_1^{y(2)}
\end{split}
\end{equation}

is $Y$-harmonic.

**Proof.** That $[(\rho_1, \rho_1), \cdot]$ satisfies (32) follows from the Markov property of $Y$ and $(\rho_1, \rho_1) \in \mathcal{H}$; that $[(\rho_1, \rho_1), \cdot]$ satisfies (33) follows from the Markov property of $Y$ and $(\rho_1, \rho_1) \in \mathcal{H}_2$. \qed
3.2.2 $Y$-harmonic functions via conjugate points

Define the boundary operator $D_2$ acting on functions on $\mathbb{Z}^2$ and giving functions on $\partial_2$:

$$D_2 V = g, \quad V : \mathbb{Z}^2 \rightarrow \mathbb{C},$$

$$g(y, 0) = (\mu_2 + \lambda V(y - 1, 0) + \mu_1 V(y + 1, 1)) - V(y, 0), \quad y \in \mathbb{Z};$$

$D_2$ is the difference between the left and the right sides of (33) and gives how much $V$ deviates from being $Y$-harmonic along the boundary $\partial_2$:

**Lemma 1.** $D_2 V = 0$ if and only if $V$ is $Y$-harmonic on $\partial_2$.

The proof follows from the definitions involved. For $(\beta, \alpha) \in \mathbb{C}^2$ and $\beta, \alpha \neq 0$

$$[D_2 \left(\left(\left(\beta, \alpha\right), \cdot\right)\right)] (y, 0) = (p_2(\beta, \alpha) - 1) \beta^y.$$ 

where the left side denotes the value of the function $D_2 \left(\left(\left(\beta, \alpha\right), \cdot\right)\right)$ at $(y, 0)$, $y \in \mathbb{Z}$. By definition, $p(\beta, \alpha) = 1$ for $(\beta, \alpha) \in \mathcal{H}$; this, the last display and (36) imply

$$[D_2 \left(\left(\left(\beta, \alpha\right), \cdot\right)\right)] (y, 0) = \mu_2 \left(1 - \frac{\beta}{\alpha}\right) \beta^y$$

(38)

if $(\beta, \alpha) \in \mathcal{H}$. One can write the function $(y, 0) \mapsto \beta^y$ as $[\left(\left(\beta, \alpha\right), \cdot\right)]_{\partial_2} = [\left(\left(\beta, 1\right), \cdot\right)]_{\partial_2}$; in addition, define

$$C(\beta, \alpha) = \mu_2 \left(1 - \frac{\beta}{\alpha}\right).$$

(39)

With these, rewrite (38) as

$$D_2 \left(\left(\left(\beta, \alpha\right), \cdot\right)\right) = C(\beta, \alpha) \left(\left(\left(\beta, 1\right), \cdot\right)\right)_{\partial_2}.$$ 

(40)

The key observation here is this: $D_2 \left(\left(\left(\beta, \alpha\right), \cdot\right)\right)$ is a constant multiple of $[\left(\left(\beta, 1\right), \cdot\right)]_{\partial_2}$. This and the linearity of $D_2$ imply that for

$$\alpha_1 \neq \alpha_2, \quad (\beta, \alpha_1), (\beta, \alpha_2) \in \mathcal{H},$$

(41)

i.e., when $(\beta, \alpha_1)$ and $(\beta, \alpha_2)$ are conjugate points on $\mathcal{H}$, $[\left(\left(\beta, \alpha_1\right), \cdot\right)]_{\partial_2}$ and $[\left(\left(\beta, \alpha_2\right), \cdot\right)]_{\partial_2}$ can be linearly combined to cancel out each other’s value under $D_2$. The next proposition uses these conjugate pairs and the above argument to find new $Y$-harmonic functions:

**Proposition 3.3.** Assume $\beta \in \mathbb{C}$, $\beta \neq 0$ satisfies $\Delta(\beta) \neq 0$. Then

$$h_{\beta} = C(\beta, \alpha_2) \left[\left(\left(\beta, \alpha_1\right), \cdot\right)\right] - C(\beta, \alpha_1) \left[\left(\left(\beta, \alpha_2\right), \cdot\right)\right]$$

(42)

is $Y$-harmonic.

**Proof.** By assumption $(\beta, \alpha_1), (\beta, \alpha_2)$ are both on $\mathcal{H}$ and therefore $[\left(\left(\beta, \alpha_1\right), \cdot\right)]_{\partial_2}$ and $[\left(\left(\beta, \alpha_2\right), \cdot\right)]_{\partial_2}$ are harmonic functions of $\mathbb{Z}$ and in particular, they both satisfy (32). Then their linear combination $h_{\beta}$ also satisfies (32), because (32) is linear in $V$. It remains to show that $h_{\beta}$ solves (33) as well. $\beta \neq 0$ implies $\alpha_1, \alpha_2 \neq 0, 1$. Then (10) implies

$$D_2(h_{\beta}) = C(\beta, \alpha_2)D_2(\left(\left(\beta, \alpha_1\right), \cdot\right)) - C(\beta, \alpha_1)D_2(\left(\left(\beta, \alpha_2\right), \cdot\right))$$

$$= C(\beta, \alpha_2)C(\beta, \alpha_1)\left(\left(\left(\beta, 1\right), \cdot\right)\right)_{\partial_2} - C(\beta, \alpha_1)C(\beta, \alpha_2)\left(\left(\left(\beta, 1\right), \cdot\right)\right)_{\partial_2}$$

$$= 0$$

and Lemma 1 implies that $h_{\beta}$ satisfies (33). \qed
The function $W(y) = P_y(\tau < \infty)$ takes the value 1 on $\partial B$. For this reason, the conjugate pair on $\mathcal{H}$ that is most relevant to the computation of $P_y(\tau < \infty)$ consists of $(\rho_2, 1)$ and $(\rho_2, \rho_1)$; this pair is shown in Figure 3.1, $h_{\rho_2}$, the $Y$-harmonic function defined by this pair, equals

$$h_{\rho_2}(y) = C(\rho_2, \rho_1)[(\rho_2, 1), y] - C(\rho_2, 1)[(\rho_2, \rho_1), y]$$

which, by definitions 30 and 39, equals

$$(\mu_2 - \mu_1)\rho_2 y^{(1)} - y^{(2)} - (\mu_2 - \lambda)\rho_2 y^{(1)} - y^{(2)}\rho_1 y^{(2)}.$$ 

Note that the first term in the definition (4) of $W^*$ equals $\frac{1}{\mu_2 - \mu_1} h_{\rho_2}$.

With Proposition 3.3 we define our basic class of harmonic functions of $Y$: $F_Y \doteq \{h_\beta, \beta \neq 0, \Delta(\beta) \neq 0\}$. (43)

Members of $F_Y$ consist of linear combinations of log-linear functions; with a slight abuse of language, we will also refer to such functions as log-linear.

**Remark 1.** For the purposes of computing $P_y(\tau < \infty)$ for the tandem network case treated in the present paper a single member of $\mathcal{H}_Y$ will suffice, i.e., $h_{\rho_2}$, see Proposition 3.6 below. But $\mathcal{H}_Y$ is a whole family of simple to compute $Y$-harmonic functions and they can be used to approximate other expectations or even $P_y(\tau < \infty)$ when the underlying network is not tandem, see Remark 4 below.

### 3.3 Graph representation of log-linear harmonic functions of $Y$

Figure 3 gives a graph representation of the harmonic functions developed in the last subsection. Each node in this figure represents a member of $F_Y$. The edges represent the boundary conditions; in this case there is only one, (33) of $\partial \cap 2$, and the edge label “2” refers to $\hat{\partial}_2$. A self connected vertex represents a member of $F_Y$ that also satisfies the $\hat{\partial}_2$ boundary condition (33), i.e., $z \rightarrow \beta(\rho_1)z^{(1)}r_1^{(1)} - z^{(2)}$ of Proposition 3.2; the graph on the left represents exactly this function. The “2” labeled edge on the right represents the conjugacy relation (29) between $\alpha_1$ and $\alpha_2$, which allows these functions to be linearly combined to satisfy the harmonicity condition of $Y$ on $\hat{\partial}_2$.

We call the graphs shown in Figure 3 and the system of characteristic equations they represent a harmonic system. One can define harmonic systems for $d$ dimensional constrained random walks as well (see 72, Section 5); these systems and their solutions play a key role in the generalization of the analysis of this section to higher dimensions.
3.4 $\partial B$-determined harmonic functions of $Y$

In the subsections 3.2.1 and 3.2.2 above we have constructed classes of $Y$-harmonic functions. For the purposes of computing $W(y) = P_y(\tau < \infty)$, $y \in B$, we need $\partial B$-determined $Y$-harmonic functions. Proposition 3.4 derives simple conditions that allow one check when a member of $\mathcal{F}_Y$ is $\partial B$ determined. In this, the following fact will be useful.

Lemma 2. Define

$$\zeta_n = \inf \{ k : Y_k(1) = Y_k(2) + n \}.$$  \hspace{1cm} (44)

For $y \in \mathbb{Z}_+^2$, $0 \leq y(1) - y(2) \leq n$,

$$P_y(\zeta_n \wedge \zeta_0 = \infty) = 0.$$ \hspace{1cm} (45)

Proof. The proof follows from the fact that, when in $C = \{ y \in \mathbb{Z}_+^2, y(2) < y(1) \leq y(2) + n \}$ the process $Y$ hits $\partial C = \{ y \in \mathbb{Z}_+^2 : y(1) - y(2) = n \text{ or } (1) = y(2) \}$ in at most $n$ steps with probability greater than $\lambda^n$. For a detailed version of this argument we refer the reader to [72, Proof of Proposition 2.2].

Proposition 3.4. Let $\alpha_1$, $\alpha_2$ and $\beta$ be as in Proposition 3.3. If

$$|\beta| < 1, \quad |\alpha_1|, |\alpha_2| \leq 1$$ \hspace{1cm} (46)

then $h_\beta$ of (42) is $\partial B$-determined.

Proof. By Proposition 3.3 $h_\beta$ is $Y$-harmonic; (16) and its definition (12) imply that $h_\beta$ is also bounded on $B^\alpha$. Then $M_k = h_\beta(Y_{\tau \land \zeta_n \wedge k})$ is a bounded martingale. This, Proposition 2 and the optional sampling theorem imply

$$h_\beta(y) = \mathbb{E}_y \left[ h_\beta(Y_{\tau})1_{\{\tau < \zeta_n\}} \right] + \mathbb{E}_y \left[ h_\beta(Y_{\zeta_n})1_{\{\zeta_n \leq \tau\}} \right], y \in B^\alpha.$$ \hspace{1cm} (47)

$Y_{\zeta_n}(1) = n$ for $\tau > \zeta_n$. This and (16) imply

$$\lim_{n \to \infty} \mathbb{E}_y \left[ h_\beta(Y_{\zeta_n})1_{\{\zeta_n \leq \tau\}} \right] = \lim_{n \to \infty} \beta^n = 0.$$

This, $\lim_n \zeta_n = \infty$ and letting $n \to \infty$ in (47) give

$$h_\beta(y) = \mathbb{E}_y \left[ h_\beta(Y_{\tau})1_{\{\tau < \infty\}} \right],$$

i.e., $h_\beta$ is $\partial B$-determined.

In addition, we have:

Proposition 3.5. The $Y$-harmonic function $[(\rho_1, \rho_1), \cdot]$ of Proposition 3.2 is $\partial B$-determined.

Proof. The proof is identical to that of Proposition 3.4 and follows from $0 \leq [(\rho_1, \rho_1), y] \leq 1$ for $y \in B$ and the $Y$-harmonicity of $[(\rho_1, \rho_1), \cdot]$. \hspace{1cm} \Box

Proposition 3.4 rests on the condition (16); we refer the reader to [72, Section 4], in particular Proposition 4.13 that derives conditions under which (16) hold in the context of general two node Jackson networks. For the purposes of computing $P_y(\tau < \infty)$, we only need to consider the point $(\rho_1, \rho_1)$ and the conjugate pair $(\rho_2, 1)$ and $(\rho_2, \rho_1)$; it is trivial to check the conditions (16) for these points. This gives us the main result of this section:
Proposition 3.6. Under the stability assumption $\lambda < \mu_1, \mu_2$, $h_{p_2}$ is $\partial B$-determined and we have

$$P_y(\tau < \infty) = W^*(y) = \frac{1}{C(p_2, \rho_1)} h_{p_2}(y) + \frac{C(p_2, 1)}{C(p_2, \rho_1)} [(\rho_1, \rho_1), y].$$

$y \in B$.

The definitions (30) and (39) give us the following expanded formula for $W^*$:

$$W^*(y) = \frac{1}{C(p_2, \rho_1)} h_{p_2}(y) + \frac{C(p_2, 1)}{C(p_2, \rho_1)} [(\rho_1, \rho_1), y]$$

$$= \left( p_2^{y(1) - y(2)} + \frac{\mu_2 - \lambda}{\mu_1 - \mu_2} p_2^{(1) - y(2)} \rho_1^{y(2)} \right) + \frac{\mu_2 - \lambda}{\mu_1 - \mu_2} \rho_1^{(1) - y(2)} \rho_1^{y(2)},$$

which is the one given in (4), in the introduction.

Proof. The conjugate points on $H$ for $\beta = p_2$ are $(p_2, 1)$ and $(p_2, \rho_1)$; the stability assumption $\lambda < \mu_1, \mu_2$ implies that both of these points satisfy (46). It follows from Propositions 3.3 and 3.4 that $h_{p_2}$ is a $\partial B$ determined $Y$-harmonic function; similarly, it follows from Propositions 3.2 and 3.5 that $[(\rho_1, \rho_1), \cdot]$ is a $\partial B$-determined $Y$-harmonic function. It follows that their linear combination $W^*$ is also $\partial B$-determined and $Y$-harmonic, i.e.,

$$W^*(y) = E_y[1_{\{\tau < \infty\}} W^*(Y_\tau)]$$

But $W^*(y) = 1$ on $\partial B$; therefore,

$$P_y(\tau < \infty).$$

Remark 2. The Balayage operator $T$ (see [4, page 25]) for the set $\partial B$ is the operator mapping a function $f$ on $\partial B$ to the $Y$-harmonic function $g$ on $B$, defined as follows:

$$T : f \to g(x) = E_x \left[ f(X_\tau) 1_{\{\tau < \infty\}} \right].$$

Therefore, by definition, a $Y$-harmonic function $h$ is $\partial B$-determined, if and only if it is the image of some function under the Balayage operator $T$. Computing $P_y(\tau < \infty)$ amounts to computing the image of the constant function 1 on $\partial B$ under the Balayage operator. What Propositions 3.2, 3.3, 3.4 and 3.5 do is they give us a collection of basis functions for which the Balayage operator $T$ is very simple to compute; these functions play the same role for the current problem as the one which exponential functions do in the solution of linear ordinary differential equations or the trigonometric functions in the solution of the heat and the Laplace equations. Let us rewrite Proposition 3.4 more explicitly. Suppose $\alpha_1, \alpha_2$ and $\beta$ are as in Proposition 3.4. Recall that

$$h_\beta(y) = \beta^{y(1) - y(2)} \left( C(\alpha_1) \alpha_2^{y(2)} - C(\alpha_2) \alpha_1^{y(2)} \right), y \in \mathbb{Z}^2.$$ 

Then, Proposition 3.4 says

$$E_y \left[ h_\beta(Y_\tau) 1_{\{\tau < \infty\}} \right] = h_\beta(y), \text{ i.e., } T(h_\beta|_{\partial B}) = h_\beta. \quad (48)$$
Remark 3. In this article we are interested in the computation of \( P_y(\tau < \infty) = \mathbb{E}_y[1_{\tau < \infty}] \).

More generally we may be interested in computing \( g(y) = \mathbb{E}_y[f(Y_\tau)1_{\{\tau < \infty\}}] \) for some function \( f \). To approximate this expectation, one can proceed as follows. First, approximate \( f \) with a finite superposition of the form

\[
    f^* = \sum_{i=1}^{K} w_i f_i |_{\partial B},
\]

where \( w_i \in \mathbb{C} \) and \( f_i \in \mathcal{H}_Y \), i.e., a \( Y \)-harmonic function of the form

\[
    f_i = C(\beta_i, \alpha_i^*)[(\beta_i, \alpha_i), -] - C(\beta, \alpha_i)[(\beta_i, \alpha_i^*), -],
\]

and \( |\beta_i|, |\alpha_i|, |\alpha_i^*| < 1 \); then by (43)

\[
    \mathbb{E}_y[f^*(Y_\tau)1_{\{\tau < \infty\}}] = \sum_{i=1}^{K} w_i f_i(y)
\]

would give an approximation of \( \mathbb{E}_y[f(Y_\tau)1_{\{\tau < \infty\}}] \) for \( y \in B \). The error made in this approximation will be bounded by \( \max_{y \in \partial B} |f^*(y) - f(y)| \).

4 Convergence - initial condition set for \( X \)

The convergence argument of Section 2 used an initial point for the \( Y \) process. The goal of this section is to provide a convergence argument starting from an initial position specified for the \( X \) process as \( X(0) = [nx] \) for a fixed \( x \in \mathbb{R}_+^2 \) with \( x(1) + x(2) < 1 \), as is done in LD analysis. We will show that the relative error

\[
    \frac{|P_{x_n}(\tau_n < \tau_0) - P_{T(x_n)}(\tau < \infty)|}{P_{x_n}(\tau_n < \tau_0)}
\]

decays exponentially in \( n \); see Proposition 4.1 below.

For the present analysis we will also use the limit process \( Y \) expressed in the original coordinates of the \( X \) process, which is \( \hat{X} \equiv T_n(Y) \). \( \hat{X} \) is the same process as \( X \) except that it is constrained only at the boundary \( \partial_2 \).

\[
    X_{k+1} = X_k + \pi(X_k, I_k)
\]

\[
    \hat{X}_{k+1} = \hat{X}_k + \pi_1(\hat{X}_k, I_k),
\]

where \( \pi_1 \) is as in (23). We will assume that \( X \) and \( \hat{X} \) start from the same initial position

\[
    X_0 = \hat{X}_0
\]

and whenever we specify an initial position below it will be for both processes.

As before, \( \tau_n = \inf\{k : X_1(k) + X_2(k) = \partial A_n\} \) and \( \tau = \inf\{k : Y_k \in \partial B\} \); define \( \partial A_n = \{x \in \mathbb{Z} \times \mathbb{Z}_+: x(1) + x(2) = n\} \); By definition, \( \hat{X} \) hits \( \partial A_n \) exactly when \( Y \) hits \( \partial B \); therefore, \( \tau = \bar{\tau}_n \equiv \inf\{k : X_k \in \partial A_n\} \), and \( P_{T(x_n)}(\tau < \infty) = P_{x_n}(\bar{\tau}_n < \infty) \).

Proposition 4.1. For \( x \in \mathbb{R}_+^2 \), \( 0 < x(1) + x(2) < 1, x(1) > 0 \) set \( x_n = [nx] \). Then

\[
    \frac{|P_{x_n}(\tau_n < \tau_0) - P_{T(x_n)}(\tau < \infty)|}{P_{x_n}(\tau_n < \tau_0)} = \frac{|P_{x_n}(\tau_n < \tau_0) - P_{x_n}(\bar{\tau}_n < \infty)|}{P_{x_n}(\tau_n < \tau_0)}
\]

decays exponentially in \( n \).
The proof will require several supporting results on $\sigma_1 = \inf\{k : X_k \in \delta_1\}$ and

$$\sigma_{1,2} = \inf\{k : k \geq \sigma_1, X_k \in \delta_2\},$$

$$\bar{\sigma}_{1,2} = \inf\{k : k \geq \sigma_1, \bar{X}_k(1) = -\bar{X}_k(2)\}.$$  

**Proposition 4.2.**

$$X_k(1) + X_k(2) = \bar{X}_k(1) + \bar{X}_k(2)$$  

(50)

for $k \leq \sigma_{1,2}$.

**Proof.**

$$X_k = \bar{X}_k$$  

(51)

for $k \leq \sigma_1$ implies (50) for $k \leq \sigma_1$. If $\sigma_1 = \sigma_{1,2}$ then we are done. Otherwise $X_{\sigma_1}(2) = \bar{X}_{\sigma_1}(2) > 0$ and $X_k(2) > 0$ for $\sigma_1 < k < \sigma_{1,2}$; let $\sigma_1 = \nu_1 < \nu_2 < \cdots < \nu_K < \sigma_{1,2}$ be the times when $X$ hits $\delta_1$ before hitting $\delta_2$. The definitions of $\bar{X}$ and $X$ imply that these are the only times when the increments of $X$ and $\bar{X}$ differ: $X_{\nu_j + 1} - X_{\nu_j} = 0$ and $\bar{X}_{\nu_j + 1} - \bar{X}_{\nu_j} = (-1, 1)$ if $I_{\nu_j} = (-1, 1)$; otherwise both differences equal $I_{\nu_j}$. This and (51) imply

$$X_k - \bar{X}_k = s_k \cdot (-1, 1)$$  

(52)

for $k \leq \sigma_{1,2}$ where

$$s_k = \sum_{j=1}^{K} \mathbb{1}_{\{\nu_j \leq k\}} \mathbb{1}_{\{I_{\nu_j} = (-1, 1)\}}$$

and $\cdot$ denotes scalar multiplication. Summing the components of both sides of (52) gives (50). □

Define

$$\Gamma_n = \{\sigma_1 < \sigma_{1,2} < \tau_n < \tau_0\}.$$  

$\Gamma_n$ is one particular way for $\{\tau_n < \tau_0\}$ to occur. In the next proposition we find an upper-bound on its probability in terms of

$$\gamma = -(\log(\rho_1) \vee \log(\rho_2)).$$  

**Proposition 4.3.** For any $\epsilon > 0$ there is $N > 0$ such that if $n > N$

$$P_{x_n}(\Gamma_n) \leq e^{-n(\gamma - \epsilon)},$$  

(53)

where $x_n = [nx]$ and $x \in \mathbb{R}_+^2$, $x(1) + x(2) < 1$.

The proof will use the following definitions. Let $v_0 = (0, 1)$, $v_1 = (-1, 1)$, $v_2 = (0, -1)$, $p_X(v_0) = \lambda$, $p_X(v_1) = \mu_1$, $p_X(v_2) = \mu_2$ and

$$H_a(q) = -\log \left( \sum_{i \in \{0,1,2\} - a} p_X(v_i) e^{-\langle v_i, q \rangle} + \sum_{i \in a} p_X(v_i) \right), \ a \subset \{1, 2\},$$  

(54)

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$. For $x \in \mathbb{R}_+^2$, set

$$b(x) = \{i : x(i) = 0\}.$$  

We will write $H$ rather than $H_{\emptyset}$.
Let us show the gradient operator on smooth functions on \( \mathbb{R}^2 \) with \( \nabla \). The works \[65, 29\] use a smooth subsolution of

\[
H_{b(x)}(\nabla V(x)) = 0
\]

(55)

to find a lowerbound on the decay rate of the second moment of IS estimators for the probability \( P_{\tau_n}(\tau_n < \tau_0) \). \( V \) is said to be a subsolution of (55) if \( H_{b(x)}(\nabla V(x)) \geq 0 \). The event \( \Gamma_n \) consists of three stages: the process first hits \( \partial_1 \), then \( \partial_2 \) and then finally hits \( \partial_3 \) without hitting 0. To handle this, we will use a function \((s, x) \rightarrow V(s, x)\), with two variables; for the variable we will substitute the scaled position of the \( X \) process, and the discrete variable \( s \in \{0,1,2\} \) is for keeping track of which of the above three stages the process is in; \( V \) will be a subsolution in the \( x \) variable and continuous in \((s, x)\) (when \((s, x)\) is thought of as a point on the manifold \( \mathcal{M} \) consisting of three copies of \( \mathbb{R}^2 \) (one for each stage); the zeroth glued to the first along \( \partial_1 \) and the first to the second along \( \partial_2 \) and therefore one can think of \( V \) as three subsolutions (one for each stage) glued together along the boundaries of the state space of \( X \) where transitions between the stages occur. We will call a function \((s, x) \rightarrow V(s, x)\) with the above properties a subsolution of (55) on the manifold \( \mathcal{M} \).

Define

\[
\tilde{V}^\varepsilon_i(x) = \langle r_i, x \rangle + 2\gamma - (3-i)\varepsilon, \quad \tilde{V}^{\varepsilon,j} = \bigwedge_{i=0}^j \tilde{V}^\varepsilon_i,
\]

(56)

where

\[
\mathbf{r}_0 = (0,0), \quad \mathbf{r}_1 = -\gamma(1,0), \quad \mathbf{r}_2 = -\gamma(1,1).
\]

The subsolution for stage \( j \) will be a smoothed version of \( \tilde{V}^{\varepsilon,j} \); As in \[65, 29\], we will need to vary \( \varepsilon \) with \( n \) in the convergence argument; for this reason, \( \varepsilon \) will appear as the third parameter of the constructed subsolution. The details are as follows.

The subsolution for the zeroth stage is \( \tilde{V}^{0,\varepsilon} \): \( V(0, x, \varepsilon) \equiv \gamma - 3\varepsilon \), \( \nabla V(0, \cdot) = 0 \) and it trivially satisfies (55) and is therefore a subsolution.

Define the smoothing kernel

\[
\eta_\delta(x) \equiv \frac{1}{\delta^2 M} \eta(x/\delta), \quad \eta(x) \equiv 1_{\{|x| \leq 1\}}(|x|^2 - 1), \quad M \equiv \int_{\mathbb{R}^2} \eta(x) dx
\]

To construct the subsolution for the first and the second stages we will mollify \( \tilde{V}^{j,\varepsilon} \), \( j = 1, 2 \), with \( \eta \):

\[
V(j, x, \varepsilon) \equiv \int_{\mathbb{R}^2} \tilde{V}^{j,\varepsilon}(y)\eta_{c_2\varepsilon}(x - y) dy,
\]

(57)

and \( c_2 \) is chosen so that

\[
V(1, x, \varepsilon) = V(2, x, \varepsilon)
\]

(58)

for \( x \in \partial_2 \) and

\[
V(1, x, \varepsilon) = V(0, x, \varepsilon)
\]

(59)

for \( x \in \partial_1 \) (this is possible since \( V(j, \varepsilon, x) \rightarrow \tilde{V}^{j,\varepsilon} \) as \( c_2 \rightarrow 0 \) and all of the involved functions are affine; see \[65\] page 38] on how to compute \( c_2 \) explicitly). That \( V(j, \cdot, \varepsilon) \), \( j = 1, 2 \) are subsolutions follow the concavity of \( H_a \) and the choices of the gradients \( \mathbf{r}_i \); for details we refer the reader to \[65\] Lemma 2.3.2]; a direct computation gives

\[
\left| \frac{c_2^2 V(j, \cdot, \varepsilon)}{\partial^2 V(j, \cdot, \varepsilon)_{x_i x_j}} \right| \leq \frac{c_3}{\varepsilon},
\]

(60)
We will allow \( \varepsilon \) computation.

Now suppose that the statement of Theorem 4.3 is not true, i.e., there exists \( n \) such that

\[
\tau
\]

reduces to

\[
\tau
\]

We will allow \( \varepsilon \) to depend on \( n \) so that \( \varepsilon_n \to 0 \) and \( n\varepsilon_n \to \infty \). Define \( S_k = 1_{[\sigma_k > k]} + 1_{[\sigma_{k+1} > k]} \), \( M_0 = 1 \) and

\[
M_{k+1} = M_k \exp \left( -n \left( V\left( S_{k+1}, \frac{X_{k+1}}{n}, \varepsilon_n \right) - V\left( S_k, \frac{X_k}{n}, \varepsilon_n \right) \right) - 1_{(n > \sigma)} \frac{c_3}{n\varepsilon} \right)
\]

That \( V(j, \cdot, \varepsilon_n) \), \( j = 0, 1, 2 \) are subsolutions of \( \mathcal{H} \), the relations \( \mathcal{H} \) and \( \mathcal{H} \) imply that \( M \) is a supermartingale; \( \mathcal{H} \) and \( \mathcal{H} \) allow us to replace gradients in \( \mathcal{H} \) and \( \mathcal{H} \) with finite differences and \( \mathcal{H} \) and \( \mathcal{H} \) preserve the supermartingale property of \( M \) as \( S \) passes from \( 0 \) to 1 and from 1 to 2. This and \( M \geq 0 \) imply (see \[33, Theorem 7.6\])

\[
\mathbb{E}_{\tau_n} \left[ \prod_{k=1}^{\tau_{0,n}} \exp \left( -n \left( V\left( S_{k+1}, \frac{X_{k+1}}{n}, \varepsilon_n \right) - V\left( S_k, \frac{X_k}{n}, \varepsilon_n \right) \right) - 1_{(n > \sigma)} \frac{c_3}{n\varepsilon} \right) \right] \leq 1,
\]

where \( \tau_{0,n} = \tau_n \wedge \tau_0 \). Restrict the expectation on the left to \( 1_{\Gamma_n} \) and replace \( 1_{(n > \sigma)} \) with 1 to make the expectation smaller:

\[
\mathbb{E}_{\tau_n} \left[ 1_{\Gamma_n} e^{-\frac{c_3}{n\varepsilon n} \tau_{0,n}} \exp \left( -n \sum_{k=1}^{\tau_{0,n}} V\left( S_{k+1}, \frac{X_{k+1}}{n}, \varepsilon_n \right) - V\left( S_k, \frac{X_k}{n}, \varepsilon_n \right) \right) \right] \leq 1.
\]

Over \( \Gamma_n \), \( X \) first hits \( \hat{c}_1 \) and then \( \hat{c}_2 \) and finally \( \hat{c}_A_n \). Furthermore, the sum inside the expectation is telescoping across this whole trajectory; these imply that the last inequality reduces to

\[
\mathbb{E}_{\tau_n} \left[ 1_{\Gamma_n} e^{-\frac{c_3}{n\varepsilon n} \tau_{0,n}} \exp(-n(V(2, X_{\tau_{0,n}}, \varepsilon_n) - V(0, X_0, \varepsilon_n))) \right] \leq 1.
\]

\[\tau_{0,n} = \tau_n \text{ on } \Gamma_n \text{ and therefore on the same set } X_{\tau_{0,n}} \in \hat{c}_n. \text{ This, } V(0, \cdot, \varepsilon_n) = \gamma - 3\varepsilon_n, \text{ and the previous inequality give}
\]

\[
\mathbb{E}_{\tau_n} \left[ 1_{\Gamma_n} e^{-\frac{c_3}{n\varepsilon n} \tau_{0,n}} \right] \leq e^{-n(\gamma-3\varepsilon_n)}.
\]

Now suppose that the statement of Theorem 4.3 is not true, i.e., there exists \( \varepsilon > 0 \) and a sequence \( n_k \) such that

\[
P_{\tau_{n_k}}(\Gamma_{n_k}) > e^{-n_k(\gamma-\varepsilon)}
\]
for all $k$. Let us pass to this subsequence and drop the subscript $k$. [38, Theorem A.1.1] implies that one can choose $c_4 > 0$ so that $P(\tau_{0,n} > nc_4) \leq e^{-n(\gamma+1)}$ for $n$ large. Then

$$
\mathbb{E}_{x_n} \left[ 1_{\Gamma_n} e^{-\frac{\epsilon n}{\Gamma_n^{1/2}}} \right] \geq \mathbb{E}_{x_n} \left[ 1_{\Gamma_n} e^{-\frac{\epsilon n}{\Gamma_n^{1/2}}} 1_{\{\tau_{0,n} \leq nc_4\}} \right] \\
\geq e^{-\frac{c_4}{n^{1/2}}} \mathbb{E}_{x_n} \left[ 1_{\Gamma_n} 1_{\{\tau_{0,n} \leq nc_4\}} \right]
$$

$P(E_1 \cap E_2) \geq P(E_1) - P(E_2^c)$ for any two events $E_1$ and $E_2$; this and the previous line imply

$$
\geq e^{-\frac{c_4}{n^{1/2}}} (P_{x_n}(\Gamma_n) - P_{x_n}(\tau_{0,n} > nc_4)) \\
\geq e^{-\frac{c_4}{n^{1/2}}} \left( e^{-n(\gamma - \epsilon)} - e^{-n(\gamma+1)} \right).
$$

By assumption $n \varepsilon_n \to \infty$ which implies $c_3 c_4 / n \varepsilon_n \to 0$; this and the last inequality say

$$
\mathbb{E}_{x_n} \left[ 1_{\Gamma_n} e^{-\frac{\epsilon n}{\Gamma_n^{1/2}}} \right] \text{ cannot decay at an exponential rate faster than } \gamma - \epsilon, \text{ but this contradicts (63) because } \varepsilon_n \to 0. \text{ Then, there cannot be } \epsilon > 0 \text{ and a sequence } \{n_k\} \text{ for which (63) holds and this implies the statement of Proposition 4.3.} \quad \Box
$$

Define $\tau_3 = \log(\rho_2, 1, 1)$ and $V(x) = (-\log(\rho_1) + \langle r_1, x \rangle) \wedge (-\log(\rho_2) + \langle r_3, x \rangle)$, for $x \in \mathbb{R}^2$

**Proposition 4.4.**

$$
\lim_{n \to \infty} -\frac{1}{n} \log P_{x_n}(\tau_n < \tau_0) = V(x)
$$

for $x \in \mathbb{R}^2_+$, $0 < x(1) + x(2) < 1$ and $x_n = [nx]$.

The omitted proof is a one step version of the argument used in the proof of Proposition 4.3 and uses a mollification of $V$ as the subsolution.

**Proposition 4.5.** For any $\epsilon > 0$ there is $N > 0$ such that if $n > N$

$$
P_x(\sigma_1 < \sigma_1, 2 < \tau < \infty) \leq e^{-n(\gamma - \epsilon)} \quad (66)
$$

where $x_n = [nx]$ and $x \in \mathbb{R}^2_+$, $x(1) + x(2) < 1$.

**Proof.** Write

$$
P_x(\sigma_1 < \sigma_1, 2 < \tau < \infty) = P_x(\sigma_1 < \sigma_1, 2 < \bar{\sigma}_1, 2 < \tau < \infty) + P_x(\sigma_1 < \sigma_1, 2 < \tau < \bar{\sigma}_1, 2).
$$

The definitions of $X$ and $\tilde{X}$ imply $\tau_0 \geq \bar{\sigma}_1, 2$. Then, if a sample path $\omega$ satisfies $\sigma_1(\omega) < \sigma_1, 2(\omega) < \tau(\omega) < \bar{\sigma}_1, 2$, it must also satisfy $\sigma_1(\omega) < \sigma_1, 2(\omega) < \tau_n(\omega) < \tau_0(\omega)$. This and Proposition 4.4 imply that there is an $N$ such that

$$
P_{x_n}(\sigma_1 < \sigma_1, 2 < \tau < \bar{\sigma}_1, 2) \leq e^{-n(\gamma - \epsilon)},
$$

for $n > N$. On the other hand, Proposition 4.6 and the Markov property of $\tilde{X}$ imply

$$
P_{x_n}(\sigma_1 < \sigma_1, 2 < \bar{\sigma}_1, 2 < \tau < \infty) \leq c_5 e^{-n(\gamma - \epsilon)},
$$

for some constant $c_5 > 0$. These imply (66). \quad \Box
These imply that for $c \in \mathbb{C}$ the sums of their components remain equal before time $\epsilon$.

By Propositions 4.3 and 4.5 for $P_x(\tau < x)$ and $P_{\tilde{x}}(\tilde{\tau} < \tilde{x})$, this (69) and the decompositions (67) and (68) imply

$$P_{x_n}(\tau_n < \tau_0) = P_{x_n}(\tau_n < \sigma_1 < \tau_0) + P_{x_n}(\sigma_1 < \tau_n < \tau_0) + P_{x_n}(\sigma_1 < \sigma_2 < \tau_0) + P_{x_n}(\sigma_1 < \sigma_2 < \tau < \infty).$$

By definition $X$ and $\tilde{X}$ are identical until they hit $\tilde{c}_1$; therefore $\{\tau_n < \sigma_1\} = \{\tau < \sigma_1\}$ and

$$P_{x_n}(\tau_n < \sigma_1) = P_{x_n}(\tilde{\tau} < \sigma_1).$$

The processes $X$ and $\tilde{X}$ begin to differ after they hit $\tilde{c}_1$; but Proposition 4.2 says that the sums of their components remain equal before time $\epsilon$.

By Propositions 4.3 and 4.4 for $P_{x_n}(\tau_n < \tau_0)$ is large. On the other hand, Proposition 4.4 says for $P_{x_n}(\tau_n < \tau_0) \geq e^{-n(\gamma_1 + \epsilon_0)}$ for $n$ large where $\gamma_1 \triangleq V(x) < \gamma$. Choose $\epsilon$ and $\epsilon_0$ to satisfy

$$\gamma - \gamma_1 > \epsilon + \epsilon_0.$$

These imply that for $c_6 = (\epsilon + \epsilon_0) + \gamma_1 - \gamma < 0$

$$\frac{|P_{x_n}(\tau_n < \tau_0) - P_{x_n}(\tilde{\tau} < \infty)|}{P_{x_n}(\tau_n < \tau_0)} < e^{c_6 \lambda_n}$$

when $n$ is large; this is what we have set out to prove.

It is possible to generalize Proposition 4.1 in many directions. In particular, one expects it to hold for any tandem walk of finite dimension with the same exit boundary; the proof will almost be identical but requires a generalization of Proposition 4.4, which, we believe, will involve the same ideas given in its proof. We leave this task to a future work.

5 Numerical Example

Proposition 4.1 says that for $x \in \mathbb{R}^+$ and $x_n = [n.x]$, the relative error

$$\frac{|W^*(T_n(x_n)) - P_{x_n}(\tau_n < \tau_0)|}{P_{x_n}(\tau_n < \tau_0)}$$

decays exponentially in $n$. Let us see numerically how well this approximation works. Set $\mu_1 = 0.4$, $\mu_2 = 0.5$, $\lambda = 0.1$ and $n = 60$. In two dimensions, one can quickly compute $P_{x_n}(\tau_n < \tau_0)$ by numerically iterating (22) and using the boundary conditions $V_{\tilde{c}_A n} = 1$ and $V(0) = 0$; we will call the result of this computation “exact.” Because both $W^*(T_n(x_n))$ and $P_{x_n}(\tau_n < \tau_0)$ decay exponentially in $n$, it is visually simpler to compare

$$V_n = -\frac{1}{n} \log P_{x_n}(\tau_n < \tau_0), \text{ and } W_n = -\frac{1}{n} \log W^*(T_n(x_n)).$$

(70)
Figure 4: On the left: level curves of $V_n$ (thin blue) and $W_n$ (thick red); on the right: the graph of $(W_n - V_n)/W_n$

The first graph in Figure 4 are the level curves of $W_n$ of $V_n$; they all completely overlap except for the first one along the $x(2)$ axis. The second graph shows the relative error $(W_n - V_n)/V_n$; we see that it appears to be zero except for a narrow layer around 0 where it is bounded by 0.02.

For $x = (1, 0)$, the exact value for the probability $P_x(\tau_{60} < \tau_0)$ is $1.1285 \cdot 10^{-35}$ and the approximate value given by $W^*(T_n(x))$ equals $1.2037 \cdot 10^{-35}$. Slightly away from the origin these quantities quickly converge to each other. For example, $P_x(\tau_{60} < \tau_0) = 4.8364 \cdot 10^{-35}$, $W^*(T_n(x)) = 4.8148 \cdot 10^{-35}$ for $x = (2, 0)$ and $P_x(\tau_{60} < \tau_0) = 7.8888 \cdot 10^{-31}$, $W^*(T_n(x)) = 7.8885 \cdot 10^{-31}$ for $x = (9, 0)$.

6 Literature Review

There is a vast literature related to the analysis presented in this article. Below we review a number of related works and point out the connections between them and the present work.

There is a clear correspondence between the structures which appear in the LD analysis and the subsolution approach to IS estimation of $p_n$ of [68, 29, 31, 71, 32] and those involved in the methods developed in this paper. This connection is best expressed in the following equation (in the context of two tandem walk just studied): For $q = (q_1, q_2) \in \mathbb{R}^2$ set $\beta = e^{q_1}$ and $\alpha = e^{q_1 - q_2}$; then

$$H(q) = -\log(p(\beta, \alpha)),$$

where $p$ is the characteristic polynomial defined in [26]. A similar relation exists between $H_2$ and $p_2$. In the LD analysis $H$ and $H_2$ appear as two of the Hamiltonians of the limit deterministic continuous time control problem; the gradient of the limit value function lies on their zero level sets. Parallel to our construction in subsection 3.2.1, the articles using the subsolution approach construct subsolutions to a limit HJB equation using points on or inside the 0 level curve of the hamiltonians $H$ and $H_2$ or their intersection; for example, the gradient $r_1$, defined following display (56), lies exactly on this intersection and corresponds to the point $(\rho_1, \rho_1)$ lying on $\mathcal{H} \cap \mathcal{H}_2$; an example from prior work is given in [29].
9], the point $r_1$ lying on the intersection of the 0 level sets of the Hamiltonians $H$ and $H_2$ correspond again to the point $(\rho_1, \rho_1)$ lying on $\mathcal{H} \cap \mathcal{H}_2$ identified in subsection 3.2.1. These works use subsolutions to estimate variances of IS estimators (again based on the same subsolution) for buffer overflow probabilities of the form $P_x(\tau_n < \tau_0)$ and concentrate on the initial point $x = 0$. Concentrating on the initial points $x = 0$ allows great flexibility on the choice of the exit boundary $\partial A_n$.

In the present work we have studied the probability $P_x(\tau_n < \tau_0)$, which is a natural quantity to study if one is interested in the buffer overflow events of a queueing system. Many other quantities naturally come up in the analysis of buffer overflows. The work [17], studies conditional probabilities of the following form:

$$P_0(\sup_{t \in [0,T]} |X_{\text{inf}}/n - \mathbf{x}(t)| \leq \varepsilon |\tau_{(n,0)} < \tau_0|),$$

where $X$ is the embedded random walk of a Jackson network with increments $(0,1), (-1,1), (1,-1), (0,-1)$ and $\tau_{(n,0)}$ is the first time $X$ hits the point $(n,0)$ and $\mathbf{x}(\cdot)$ is a limit deterministic process to be computed; i.e., in studying (71) one is interested in the behavior of the queueing process conditioned on the rare event $\{\tau_{(n,0)} < \tau_0\}$. The key idea in [17] and many other works studying overflow events in queueing systems (see, e.g., the list of references in [17]) is that if one chooses $\mathbf{x}$ in (71) to be the fluid limit of the time reversed process of $X$, the above conditional probabilities converge to 1 (see [17] Theorem 2). Then, in this line of analysis, the key steps are the computation of the dynamics of the reversed process and its fluid limit. In computing these one needs the stationary distribution of the $X$ process; an approximation of the stationary distribution is needed in cases when it is not known exactly.

The work [17] considers a modification of the above two dimensional system, for which the stationary distribution is not known and constructs approximations of its stationary distribution in cases when it is not known exactly.

The work [56] considers the buffer overflow of a chosen node in a given stable network. The process considered in [56] is $r + m$ dimensional: the first dimension represents the node whose overflow event is to be studied, the dimensions $2, 3, ..., r$ represent nodes that become unstable when the first node overflows, and the $m$ dimensions $r + 1, ..., r + m$, represent the “super-stable” nodes. The analysis of [56] is based on the $h$-transform of the embedded random walk of the queueing system with the modification that its constraints are removed for the non super-stable dimensions, i.e., the first $r$ dimensions (this process is denoted $W^\infty$), the $h$ function is an harmonic function of the $W^\infty$ process and is taken to be of the form $e^{\alpha x_1} d(x_2, ..., x_{r+m})$: [56] gives conditions under which such an $h$ function exists based on results from [60]. For $n > 0$, let $\tau_n$ be the first time the first component of $W$ hits $n$, i.e., $\tau_n = \inf\{k : W(k) \in F_n\}$, $F_n = \{x \in \mathbb{Z}^{r+m}_+: x_1 \geq n\}$; let $\tau_0$ denote the first time $W$ hits the origin $0$. Finally, let $\tau_\Delta$ denote the first time after time 0, one of the nodes from 1 to $r$ hits 0, i.e., $\tau_\Delta = \inf\{k : k > 0, W \in \Delta\}$, where $\Delta = \{x : x_j = 0, \text{for some } j \in \{1, 2, 3, ..., r\}\}$ is the constraining boundary of the state space for the components 1 to $r$; remember that these are the nodes that are assumed to become unstable when the first component overflows. As an intermediate step in its analysis, [56] derives the following approximation result: let $\pi_\Delta$ denote the stationary measure conditioned on $\Delta$ and $E_{\pi_\Delta}$ denote expectation conditioned on $W(0)$ having initial distribution $\pi_\Delta$. Let $\tau_0$ be the first return time to 0, i.e., $\tau_0 = \inf\{k >
0 : $W_k = 0$. \[56\] Lemma 1.8 states, under the assumptions made in the paper,

$$\lim_{n \to \infty} \frac{\pi(0)P_0(\tau_n < \tau_0) - \pi(\Delta)P_{\pi\Delta}(\tau_n < \tau_\Delta)}{\pi(0)P_0(\tau_n < \tau_0)} = 0. \quad (72)$$

\[56\] develops the following representation for $\pi(\Delta)P_{\pi\Delta}(\tau_n < \tau_\Delta)$:

$$\pi(\Delta)P_{\pi\Delta}(\tau_n < \tau_\Delta) = e^{-\alpha_n}E_{\pi\Delta}[h(W(1))\Psi(W(1))]. \quad (73)$$

$\Psi$ is defined as follows:

$$\Psi(x) = E_x[a^{-1}(\Psi(x))e^{-\alpha(\Psi(x) - \nu)}1_{\tau_n < \tau_\Delta}]. \quad (74)$$

where, $\Delta = \{x \in \mathbb{Z}_+^2 \times \mathbb{Z}^m, x_j \leq 0, j \in \{1, 2, 3, ..., r\}\}$, $\Psi(x)$ is the $h$-transform of the process $W^x$. For the computation of the expectation part of the formula (73), \[56\] suggests simulation. The seven conditions (see \[56\] page 113, introduction) that \[56\] is based on are conditions on the twisted process, the stationary distribution $\varphi$ of its last $m$ components and on the stationary distribution $\pi$ of the original process. \[56\] Section 3 treats the two dimensional constrained random walk on $\mathbb{Z}_+^2$ with increments $(-1, 0)$, $(1, 0)$, $(0, -1)$, $(0, 1)$, $(1, 1)$; for this process \[56\] constructs explicitly an $h$ function of the form $h(x) = a_1x_1 a_2 x_2$, where $(a_1, a_2) \in \mathbb{R}^2$ is a point on a curve whose definition is analogous to the definition of the characteristic surface $H$.

The work \[58\] employs the ideas of removing constraints on one of the boundaries and using points on curves associated with the resulting process to study the tail asymptotics of the stationary distribution of a two dimensional nearest neighbor random walk $L$ constrained to remain in $\mathbb{Z}_+^2$. To study the asymptotic decay rate of $\nu(n, k)$ in $n$ for a fixed $k$, \[58\] considers the random walk $L^{(1)}$, which has the same dynamics as $L$ except that it is not constrained on the vertical axis. Associated with this process, \[58\] defines two curves, whose definitions are parallel to the definition of $H$ and $H_1$ (see the definition of $D$ on \[58\] page 554]) and uses points on and inside these curves to define solutions to an eigenvalue / eigenvector problem associated with the problem (see \[58\] Theorem 3.1]); for the study of tail asymptotics along the vertical axis, \[58\] uses the same analysis but this time removing the constraint on the horizontal axis. For further works along this line of research we refer the reader to \[48, 21, 59\].

The work \[42\] develops an explicit formula for the large deviation local rate function $L(x, v)$ of a general Jackson network, starting from representations of these rates as limits derived in \[26, 0\]. For this, \[42\] employs “free processes;” these are versions of the original process obtained by removing those constraints from the original process that are not involved in a given direction $v$ at a given point $x \in \mathbb{R}^d$. The proofs in \[42\] use fluid limits for the free process under a change of measure (i.e., a twisted/h-transformed version of the free process); the changes of measures used here correspond to using $h$-functions of the form $e^{\theta(x)}$ where $\theta$ is a point on a characteristic surface (analogous to $H$ in this work or $H$ in \[29\]) associated with the process being transformed (see \[42\] Section 6). As an application of its results, \[42\] computes the limit $\lim_{n \to \infty} \frac{1}{n} \log E_0[\tau_n]$ by noting from \[62\] that this limit equals

$$\lim_{n \to \infty} \frac{1}{n} \log P_0(\tau_n < \tau_0),$$

which is the LD decay rate of the probability we have studied in this paper for general stable Jackson networks; \[42\] derives the explicit formula $\min_{1 \leq i \leq d} - \log(\rho_i)$ for the above LD rate using the explicit local rate functions developed in the same work and the explicit formulas available for the stationary distribution of the underlying process.
The Martin boundary of an unstable process is a characterization of the directions through which the process may diverge to $\infty$. The idea of using points on characteristic surfaces, and the idea of removing constraints from the process to simplify analysis, appear also in works devoted to identifying Martin boundaries of constrained or stopped processes. An example is [43], which identifies the Martin boundary of two-dimensional random walks in $\mathbb{Z}^2_+$ and which are stopped as soon as they hit the boundary of $\mathbb{Z}^2_+$. This work breaks up its analysis into three cases: 1) the directions $q \in \mathbb{R}^2_+$, where both components of $q$ are nonzero, 2) the directions $q$ such that $q(1) = 0$, and 3) directions such that $q(2) = 0$. For each of these cases, [43] work with what it calls local processes; the local process for the first case is a completely unconstrained random walk, the local process for the second case is a process keeping the horizontal axis (i.e., the vertical boundary is removed) and the third case is the reverse of the last. [43] uses LD analysis of the local processes, harmonic functions of the form

$$h_a(x) = \begin{cases} x_1 e^{(a,x)} - \mathbb{E}_x[S_1(\tau)e^{(a,x)}1_{\{\tau<\infty\}}], & \text{if } q(a) = (0, 1), \\ x_2 e^{(a,x)} - \mathbb{E}_x[S_2(\tau)e^{(a,x)}1_{\{\tau<\infty\}}], & \text{if } q(a) = (1, 0), \\ e^{(a,x)} - \mathbb{E}_x[e^{(a,x)}1_{\{\tau<\infty\}}], & \text{otherwise.} \end{cases}$$

where $S$ is the underlying process, $\tau$ is the first hitting time to the boundary of $\mathbb{Z}^2_+$, $a$ is a given point on a surface associated with $S$ (defined analogous to $\mathcal{H}$), $q(a)$ is the mean direction of $S$ under an exponential change of measure defined by $a$ (see [43], page 1108). In this connection let us also cite [50], which uses geometry and complex analysis to identify the Martin boundary of random walks on $\mathbb{Z}^2$, $\mathbb{Z} \times \mathbb{Z}_+$, and $\mathbb{Z}^2_+$.

Let $X$ be the constrained random walk in $\mathbb{Z}^2_+$ with increments $(1, 0), (-1, 0), (0, 1)$, and $(0, -1)$ and let $\tau_n$ be as in [43]. A classical problem in computer science going back to [47] section 2.2.2, exercise 13 is the analysis of the following expectation:

$$\mathbb{E} \left[ \max(X_1(\tau_n), X_2(\tau_n)) \right], \quad (75)$$

i.e., the expected size of the longest queue at the time of buffer overflow. This expectation is computed in [47] for the case $P(I_k = (1, 0)) = P(I_k = (0, 1)) = 1/2$, $P(I_k = (-1, 0)) = P(I_k = (0, -1)) = 0$. Various versions of this problem has since been treated in [74], [34], [54], [19], [53], [41]. [54] treats a generalization of this problem where the dynamics of the random walk depend on its position; the approach of [54] uses large deviations techniques from [38]. [74] treats the approximation of (74) for the case when the increments have a symmetric distribution as follows: $P(I_k = (1, 0)) = P(I_k(0,1)) = (1 - p)/2$ and $P(I_k = (-1, 0)) = P(I_k(0, -1)) = p/2$; furthermore $p < 1/2$ is assumed, i.e., the process is assumed unstable. Under these assumptions, [74] develops an approximation for the expectation in (75) as $n \to \infty$. The main idea in [74] is the following: under the assumptions of the paper one can ignore both of the constraining boundaries of the process, to prove this the author uses LD bounds on iid Bernoulli sequences (see [74], Lemma 3). Then an explicit computation for the unconstrained process using elementary techniques gives the desired approximation.

7 Conclusion

In this section we point out several implications of our results, work in progress and possible extensions.
7.1 The case $\mu_1 = \mu_2$

The formula (4) for $P_y(\tau < \infty)$ (derived in Proposition 3.4) requires $\mu_1 \neq \mu_2$. The case $\mu_1 = \mu_2$ can be handled by letting $\mu_2 \rightarrow \mu_1$ in (4); this gives

$$P_y(\tau < \infty) = \rho^{y(1) - y(2)} + \frac{\mu - \lambda}{\mu} \rho^{y(1)}(y(1) - y(2)),$$

where $\rho = \lambda/\mu$ and $\mu_1 = \mu_2 = \mu$. Note that the case $\mu_1 = \mu_2$ leads to the linear term $y(1) - y(2)$.

7.2 Constrained diffusions with drift and elliptic equations with Neumann boundary conditions

Diffusion processes are weak limits of random walks. Thus, the results of the previous sections can be used to compute/approximate Balayage and exit probabilities of constrained unstable diffusions. We give an example demonstrating this possibility.

For $a, b > 0$ let $X$ be the the constrained diffusion on $\mathbb{R} \times \mathbb{R}_+$ with infinitesimal generator $L$ defined as

$$f \rightarrow Lf, Lf = \langle \nabla f, ((2a + b)(a - b)) + \frac{1}{6} \nabla^2 f : \left( \begin{array}{c}
2 \\
1 \\
2
\end{array} \right),$$

where $\nabla^2$ denotes the Hessian operator, mapping $f$ to its matrix of second order partial derivatives. On $\{x : x(2) = 0\}$ $X$ is pushed up to remain in $\mathbb{R} \times \mathbb{R}_+$ (the precise definition involves the Skorokhod map, see, e.g., [51]). $a, b > 0$ implies that, starting from $B = \{x : x(1) > x(2)\}$, $X$ has positive probability of never hitting $\partial B = \{x : x(1) = x(2)\}$. Let $\tau$ be the first time $X$ hits $\{x : x(1) = x(2)\}$. Proposition 3.6 for $d = 2$ suggests

$$P_x(\tau < \infty) = e^{-(a+2b)3(x(1)-x(2))} + \frac{a + 2b}{a - b} e^{-(a+2b)3(x(1)-x(2))} e^{-(2a+b)3x(2)} - \frac{a + 2b}{a - b} e^{-3(2a+b)x(1)}, x \in B. \quad (76)$$

One can check directly that the right side of the last display satisfies

$$LV = 0, \quad \langle \nabla V, (0, 1) \rangle = 0, x \in \partial_2.$$  

This and a verification argument similar to the proof of Proposition 3.4 will imply (76).

7.3 General Jackson networks

Multiple approximations We have seen with Proposition 4.1 that $P_{y,n}(\tau < \infty)$ approximates $P_{x,n}(\tau_n < \tau_0)$, $x_n = [nx]$ very well (i.e., with exponentially decaying relative error) for all $x \in A = \{x \in \mathbb{R}^2_+, 0 < x(1) + x(2) < 1\}$ when $n$ is large. When $X$ is the constrained random walk associated with a general two dimensional Jackson network, this will not be true in general and to get a good approximation across all $A$ we will have to use the transformation $T_n^2(x) = (x(1), (n - x(2)))$ as well as $T_n$. $T_n^2$ moves the origin of the coordinate system to the corner $(0, n)$ of $\partial A_n$. Thus, for general two dimensional $X$, we will have to construct two limit processes $Y^1$ and $Y^2$: $Y^1$ will be as above and $Y^2$ will be the limit of $Y^{2,n} = T_n^2(X)$; the limit probability will be, as before $P_y(\tau^2 < \infty)$ where $\tau^2$ is the first time $Y^2$ hits $\partial B$. In $d$, dimensions we will have $d$ possible limit processes, one for each corner of $\partial A_n$ providing precise approximations for initial points which lie away from the boundaries.
missing in the limit problem. For a numerical example see subsection 8.2 of the preprint [72].

One work in progress, based on the approach of Section 4, gives details of these ideas in the context of Jackson networks consisting of parallel queues. The same work also considers the approximation of the expectation (75) using the techniques of the present work.

Approximation of $P_y(\tau < \infty)$ in general

Second issue is the generalization of the computation of the limit probability $P_y(\tau < \infty)$. As we have seen in Proposition 3.6 in the case of two tandem queues, it is possible to compute this probability exactly as the superposition of two $Y$-harmonic functions: $[(\rho_1, \rho_1), \cdot]$ and $h_{\rho_2}$. For general two dimensional Jackson networks, superposition of these two functions will only give an approximation of $P_y(\tau < \infty)$; to construct better approximations one will proceed as indicated in Remark 3 and use a linear combination of finite number of functions in the class of $Y$-harmonic functions constructed in subsections 3.2.1 and 3.2.2 to approximate the constant function 1 on the boundary $\partial B$; the error made in this approximation on $\partial B$ will provide an upperbound for the error made in the approximation of $P_y(\tau < \infty)$ for any $y \in B$. The numerical example in [72, subsection 8.2] also demonstrates this point.

$\partial B$-determined $Y$-harmonic functions

In the above paragraph we have noted that in general, to construct improved approximations of $P_y(\tau < \infty)$, we will need to use further $Y$-harmonic functions of the form

$$h_{\beta} = \beta^{y(1) - y(2)} \left( C(\beta, \alpha_2)\alpha_1^{y(2)} - C(\beta, \alpha_1)\alpha_2^{y(2)} \right)$$

where $(\beta, \alpha_1)$ and $(\beta, \alpha_2)$ are conjugate and $\Delta(\beta) \neq 0$. We know by Proposition 3.4 that $h_\beta$ is $\partial B$-determined, if $|\alpha_1|, |\alpha_2| \leq 1$ and $|\beta| < 1$. Suppose we fix $\alpha \in \{ z \in \mathbb{C}, |z| = 1 \}$ and compute $\beta$ and $\alpha^*$ so that $(\beta, \alpha)$ and $(\beta, \alpha^*)$ are conjugate ($\beta$ and $\alpha^*$ are computed by solving the characteristic equation $p = 1$). In view of Proposition 3.4 and in view of the fact that $h_\beta$ will be used in the approximation of a $\partial B$-determined $Y$-harmonic function, a natural question is the following: under what conditions on the parameters of the model do $|\alpha^*| \leq 1$ and $|\beta| < 1$ hold? This problem is studied for the general two dimensional Jackson network in Section 4 of [72] (in particular, see Proposition 4.12 and Proposition 4.13). These propositions require simplifying conditions on the system parameters (e.g., see [72, condition (56), page 18]). Derivation of more precise conditions remains an open problem.

Harmonic systems

In subsection 3.3 we have pointed out that the classes of $Y$-harmonic functions constructed in subsections 3.2.1 and 3.2.2 have graph representations, as shown in Figure 3 we refer to these graphs and the system of equations they represent as “harmonic systems.” It is possible to generalize these graphs to walks in $d$ dimensions and corresponding to each solution to the system of equations represented by the graph one can define a $Y$-harmonic function; this is done in the preprint [72, Section 5] (see Definitions 5.1 and 5.2, Proposition 5.2, generalizing Proposition 3.3, Proposition 5.3 generalizing Proposition 3.4).

d-tandem queues

Remarkably, it turns out to be possible to define a class of harmonic systems and explicitly solve them to generalize the formula (4) for $P_y(\tau < \infty)$ to $d$ tandem queues. This is done in Section 6 of [72]. As an example, let us consider $d = 3$. To compute $P(\tau < \infty)$, one uses, in addition to the graphs given in Figure 3, the graph given in Figure 6. Proposition 6.3 of [72] implies that, for
where

\begin{equation}
\mu_i \neq \mu_j, \quad i,j \in \{1,2,3\},
\end{equation}

the following function solves the harmonic system given in Figure 5:

\begin{equation}
\hat{h}_{\rho_3}(y) = \rho_3^y - \left[ 1 - c_3 \rho_2^y - c_3 c_1 \rho_1^y \right] \rho_1^y + c_3 c_2 \rho_1^y \rho_2^y
\end{equation}

where

\[
c_2 = \frac{\mu_2 - \lambda}{\mu_2 - \mu_1}, \quad c_3 = \frac{\mu_3 - \lambda}{\mu_3 - \mu_2}, \quad c_1 = \frac{\mu_3 - \lambda}{\mu_3 - \mu_1}.
\]

The \( Y \) process for the 3-tandem queues is a random walk on \( \mathbb{Z} \times \mathbb{Z}_2^d \) with increments \((-1,0,0), (1,1,0), (0,-1,1) \) and \( (0,0,-1) \). \( h \) of \( \hat{h}_{\rho_3} \) is a \( Y \)-harmonic function. There are four terms in the sum \ref{eq:harmonic_sum} defining \( \hat{h}_{\rho_3} \), each of these terms corresponds to a node of the graph in Figure 6. None of them is \( Y \)-harmonic individually. But the particular linear combination in \ref{eq:harmonic_sum} is indeed \( Y \)-harmonic. Two further \( Y \)-harmonic functions used in the calculation of \( P_y(\tau < \infty) \) are

\begin{equation}
\hat{h}_{\rho_2} = \rho_2^y \left( \rho_2^y - c_2 \rho_1^y \rho_2^y \right), \quad \hat{h}_{\rho_1} = \rho_1^y \left( 1 - c_2 \rho_1^y \rho_2^y \right) \rho_2^y
\end{equation}

the harmonic systems for these functions are “edge-completions” of those given in Figure 3 (see Definition 5.4 of \cite{72}). The exact formula for \( P_y(\tau < \infty) \) for \( y \in \mathbb{Z} \times \mathbb{Z}_2^d \), \( y(1) \geq y(2) + y(3) \) is given in \cite{72} Proposition 6.5 as

\[ P_y(\tau < \infty) = \hat{h}_{\rho_3} + c_3 \hat{h}_{\rho_2} + c_1 c_3 \hat{h}_{\rho_1}. \]

To treat the case when \ref{eq:inhomog} doesn’t hold it suffices to take limits in the last formula, which leads to polynomial terms in \( y \).

### 7.4 Extension to other processes and domains

In the foregoing sections, we have approximated \( P_x(\tau_n < \tau_0) \) in two stages: 1) use an affine change of coordinates to move the origin to a point on the exit boundary and take limits; as a result, some of the constraints in the prelimit process disappear and one obtains as a limit process an unstable constrained random walk and as a limit problem the probability of return \( P_y(\tau < \infty) \) of the unstable process; 2) find a class of basis functions on the exit boundary on which the Balayage operator of the limit process has a simple action; then try to approximate the function 1 (i.e., the value of \( P_y(\tau < \infty) \) on the exit boundary) on the exit boundary with linear combinations of the functions in the basis class. The type of problem we have studied here is of the following form: there is a process \( X \) with a certain law of large number limit which takes \( X \) away from a boundary \( \partial A_n \) towards a stable point or a region; \( \tau_0 \) is the first time the process gets into this stable region. We are interested in the probability \( P(\tau_n < \tau_0) \).
We expect the first step to be applicable to a range of problems that fit into this scenario. The second stage obviously depends on the particular dynamics of the original process. Ongoing research considers two tandem queues with Markov modulated dynamics; optimal IS simulation for this process was developed in [70]. For Markov modulated dynamics, one needs a more general class of $Y$-harmonic functions than those constructed in Section 3 and the resulting equations are of higher degree and harder to analyze but the main ideas of Section 3 do generalize. The present work focused on the exit boundary $\partial A_n$; another natural exit boundary is $\{y : y(i) \leq |a|n\}$ for $a_i > 0$, $i = 1, 2$. We expect the analysis of this paper to generalize to this exit boundary, with the following important modification: for this boundary, there are three points on the exit boundary from which one must conduct a limit analysis: the corners $n(0,a_2)$, $n(0,a_1)$ and $n(a_1,a_2)$. For the last one the limit process will be the completely unconstrained version of the random walk. Providing the details of this and further extensions to other processes and exit boundaries remain problems for future research.

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