On Measures and Measurements: a Fibre Bundle approach to Contextuality

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Contextuality is the failure of “local” probabilistic models to become global ones. In this paper we introduce the notions of measurable fibre bundles, probability fibre bundles, and sample fibre bundle which capture and make precise the former statement. The central notions of contextuality are discussed under this formalism, examples worked out, and some new aspects pointed out.

1. Outlook and Motivations

Many Escher images, like Waterfall or Ascending and Descending [1] illustrate vividly the problem of contextuality. Each piece of the image is totally consistent, the pieces even connect smoothly, however, when we try to glue them all together some “obstruction” appears. Mathematically, the notion of gluing pieces together is also present. Probably, the best known example are differential surfaces, where each piece is identified with an open set of the Euclidean plane $\mathbb{R}^2$, but when the pieces are glued together, they can make very different objects like a sphere, a torus or even some objects which do not fit in $\mathbb{R}^3$, like the Klein Bottle [2]. Another example are vector bundles. If one uses a differentiable manifold (the $d$-dimensional analogue of differentiable surfaces), $M$, as the basis space and attach to each point $p \in M$ a vector space, $V_p$, with some rule about how to connect the vectors of neighbour points, one gets another differentiable manifold, $E$, with points given by pairs $(p, v)$, with $p \in M$ and $v \in V_p$. Well known to many physicists and essential for differential geometers is the example of the tangent bundle, $TM$, when $V_p = T_pM$, the tangent space of $M$ on $p$, where Lagrangian mechanics is developed [3].
Probability theory, however, is developed over the concept of a single $\sigma$-algebra, where all the relevant events are defined. This is very natural, classically, where given two events $A$ and $B$, it also makes sense to consider the event $A \cap B$, logically connected to the conjunction of the conditions for $A$ and $B$. It inherently brings the notion of a “global order” and that images like the ones from Escher are really impossible in our world. The notion of contextuality brings another status to those “impossible figures” [4]. Moreover, the notion of quantum contextuality [5] says that those impossible figures somehow appear in Nature.

In this contribution we develop the notions of measurable bundle and probability bundle as “Escherian” generalisations of the basic notions of measurable space and probability space, where Kolmogorovian probability theory is developed [6]. In other words, we set the basis for the development of a probability theory where topology plays a crucial rôle.

Despite being inspired by quantum theory, the notions developed here are totally independent of it. The examples worked out here are the most simple, yet surprising ones, and do not demand any knowledge of quantum theory. After reviewing (in sec. 2) the basic notions of probability theory which we are generalising, we introduce the central notions of contextuality and the central objects of our contribution in sec. 3. The fibre bundle approach to contextuality is developed in sec. 4, with examples and the translations of Fine-Abramsky-Brandeburger theorem for characterisation of contextuality and Budroni-Morchio results on scenarios with no contextuality to this language. The notion of contextuality subscenarios guide the sec. 5, emphasising even more the connection of contextuality and topology. Connections to other approaches are discussed in sec. 6, while some more speculative points are raised in sec. 7 before the closing in sec. 8.

2. Classical Probability Theory in a nutshell

(a) The basic notions

A famous dictum attributed to M. Kac say that probability theory is measure theory with a soul. This section is to fix concepts and notations for sample spaces, measurable spaces and probability spaces. For a deeper introduction, we recommend ref. [6], among many other textbooks. The central notion is

**Definition 2.1.** Given a set $S$, a $\sigma$-algebra over $S$, denoted $\Sigma$, is a collection of subsets of $S$ with $\emptyset \in \Sigma$, closed under complementation and countable unions.

**Remark 2.2.** The examples to be raised will use finite sets $S$. However, some of the questions and the central target of generalising usual probability theory demand the more general definition.

**Definition 2.3.** A measurable space is a pair $(\Omega, \Sigma)$, where $\Omega$ is a set and $\Sigma$ a $\sigma$-algebra on it. The set $\Omega$ is called the sample space.

The next step is to measure the sets in $\Sigma$. This demands the notion of a measure. We will jump straight to the definition of probability measure:

**Definition 2.4.** A probability measure, or simply a probability, on a measurable space $(\Omega, \Sigma)$ is a function $\mu: \Sigma \rightarrow [0, 1]$ such that

- $\mu(\emptyset) = 0$;
- If $A_i$ are pairwise disjoint sets in $\Sigma$, then $\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$;
- $\mu(\Omega) = 1$.

Again, this property is demanded on all countable unions and countable sums.
The triple \((\Omega, \Sigma, \mu)\) is called a probability space.

Another important notion in what follows is that if we fix a sample space \(\Omega\), the \(\sigma\)-algebras on it obey a partial order coming from inclusion:

**Definition 2.5.** Given two \(\sigma\)-algebras, \(\Sigma\) and \(\Sigma'\), over the same set we say that \(\Sigma\) is a coarse graining of \(\Sigma'\), or that \(\Sigma'\) refines \(\Sigma\) if \(\Sigma \subseteq \Sigma'\).

Intuitively, this means that any atom of \(\Sigma\) can be obtained as a (countable) union of atoms of \(\Sigma'\).

(b) The simplest case and its geometry

Once one is concerned with a finite collection of (classical) random variables, \(X = \{X_i\}\), each of them taking values in finite sets \(S_i\), there is a minimal and sufficient (Boolean) \(\sigma\)-algebra to hold such situation. For only one variable, taking values in the finite set \(S\), the powerset, \(\mathcal{P}(S)\), plays this rôle. In other words, one can identify \(S\) with the sample space \(\Omega\) and use the measurable space \((S, \mathcal{P}(S))\). For the finite set of variables,\

\[
\Sigma = \prod_X \mathcal{P}(S_i) .
\]

Here, \(\prod\) is taken as the \(\sigma\)-algebra product, analogous to the set-theoretical Cartesian product. Interestingly, the simple set-theoretical identification \(\prod \mathcal{P}(S_i) \equiv \mathcal{P}(\prod S_i)\) brings something that will have deep consequences when contextuality comes to play: classically, we could change this set of random variables for just one random variable, \(X\), taking values in \(\prod S_i\). Anticipating some notions (see Definition 3.8), this means that all (classical) random variables are jointly realisable.

Whenever one wants to attach a probability measure to a \(\sigma\)-algebra \(\mathcal{P}(S)\), it is necessary and sufficient to attach a non-negative value \(p(s)\) for each \(s \in S\), such that \(\sum_S p(s) = 1\). This brings a very basic, concrete, and geometrical notion:

**Definition 2.6.** Given a finite set, \(S\), the canonical vectors of \(\mathbb{R}^{|S|}\) are the vertices of the probability simplex. The probability simplex is the convex hull of its vertices.

Indeed, this means that the usual picture of a segment, a triangle, a tetrahedron, \ldots, generalises and that the only important feature of \(S\) is its cardinality. Each point in the simplex works as a probabilistic model for the set of variables studied and this is a bijection.

(c) An Ontology for Classical Probability Theory

Classical probability theory, as introduced by Kolmogorov, allows for a very natural ontology. It is, by no means, necessary, and the Occam’s razor must be the reason for its (complete?) absence in textbooks. The sample space, \(\Omega\), can be considered as an ontological space, where every question has an answer, or, equivalently, every property is well defined. Those questions or properties are given by random variables, each of them assuming well defined values for each point \(\omega \in \Omega\). Their randomness only enters in the game under the idea that \(\omega\) is unknown (or, “hidden”). To some sense, the epistemological space is given by \(\Sigma\), or by some coarsening \(\Sigma' \subset \Sigma\), such that all random variables defined in the model has measurable sets attached to each possible value.

As already stated, such ontology is unnecessary. However, it can be considered as the classical trait to be consistent with such ontology. It is under this paradigm that space shuttles are sent to Mars, that GPS localises mobiles up to small uncertainties, and even classical statistical mechanics
was developed\(^2\). Contextuality appears whenever it fails. This is usually from where adjectives like weird, counterintuitive, or paradoxical come from.

3. Central Definitions

(a) Measurements

The most primitive notion of measurement is that it gives results. This is what we will use in order to characterise them:

Definition 3.1. A measurement, \( M \), is a (countable\(^3\)) set of labels for the possible results.

Definition 3.2. Given a measurement, \( M \), a realisation in a measurable space \((\Omega, \Sigma)\) is a partition of \( \Omega \) into elements of \( \Sigma \) subordinated to \( M \), i.e.: with elements indexed by the measurement outcomes.

Remark 3.3. At this level, all (finite) measurements are realisable. Just consider \( \Omega = M \), as sets, and \( \Sigma = P(\Omega) \).

Remark 3.4. Equivalently, one can define a realisation of \( M \) in \((\Omega, \Sigma)\) as a measurable function \( m : (\Omega, \Sigma) \rightarrow (M, P(M)) \).

Remark 3.5. Note that by this definition, whenever a measurement is done, one and only one of the outcomes appear. This is consistent with the notion of exclusive outcomes.

Whenever the measurable space is upgraded to a probability space, a realisation of \( M \) defines probabilities for all its outcomes:

Definition 3.6. Given a measurement \( M \) with a realisation in \((\Omega, \Sigma)\), if \((\Omega, \Sigma, \mu)\) is a probability space, then \( \mu \) is a (probability) measure for \( M \).

Remark 3.7. If we go along with remark 3.4, we can also see that pushing forward \( \mu \) along \( m \) gives a (probability) measure on \((M, P(M))\).

Before jumping into contextuality, we need to workout the notions related to joint measurability.

Definition 3.8. Given two measurements, \( M \) and \( N \), the joint measurement is given by \( M \land N = M \times N \) (as sets). A joint realisation in a measurable space \((\Omega, \Sigma)\) is a partition of \( \Omega \) into elements of \( \Sigma \) subordinated to \( M \land N \). If \((\Omega, \Sigma, \mu)\) is a probability space, then \( \mu \) is a joint (probability) measure for \( M \) and \( N \).

Remark 3.9. If \( M \) is realised in \((\Omega, \Sigma)\) and \( N \) is realised in \((\Omega', \Sigma')\), then \( M \land N \) is realised in \((\Omega \times \Omega', \Sigma \times \Sigma')\). This anticipates that the central notion of contextuality does not appear when considering only two measurements. This very simple observation has deep consequences: contextuality can never appear on trivial topologies \([9]\).

Definition 3.10. Given a set \( M \) and a partition \( P \) of \( M \) into disjoint nonempty subsets, the measurement \( P \) associated to \( P \) is a coarse graining of the measurement \( M \) associated to \( M \).

\(^2\)Actually, it was such a completion that Einstein, Podolsky, and Rosen were looking for in their criticism to quantum theory \([7]\) and this is what Bell rules out with his inequalities \([8]\).

\(^3\)For measurements with uncountable possible results some adaptations will be necessary, but in this contribution we want to keep things simple.
Definition 3.11. Given a joint measurement $M \land N$, the measurements $M$ and $N$ are called its marginal measurements.

Lemma 3.1. Given a joint measurement $M \land N$, the marginal measurements can be identified with coarse grainings of the joint measurement.

Proof. Given the Cartesian product $M \times N$, the partition $m \times N$ for $M \times N$ is naturally identified with the set $M$, by $m \times N \mapsto m$. Analogously, for the partition $M \times n$.

This leads to another very important relation between measurements:

Definition 3.12. Measurements $M$ and $N$ are compatible if there exist a common coarse graining, from which both can be obtained as marginals.

Remark 3.13. Given two measurements and no other restriction, one can always define the joint measurement. The strength of the notion of compatibility only shows up when we work on some nontrivial contextuality scenario, which brings us to the next subsection.

(b) Contextuality

Now we set the stage for the questions we are treating. The central ingredient is a collection, $X$, of possible measurements, $M_k$, some of them can be compatible, while other not.

Definition 3.14. Given a collection of measurements, $X$, a context for the measurement $M \in X$ is a subset $C \subset X$ with $M \in C$ in which all measurements are compatible. A maximal context for $M \in X$, $C$, is a context that cannot be enlarged by including elements of $X$. In other words, if $C$ is a maximal context in $X$, for any $N \in X \setminus C$ there is some $M' \in C$ incompatible with $N$.

As already stated, in this construction, compatibility is not an inherent property of a set of measurements. This demands the central notion of scenarios:

Definition 3.15. A measurement scenario, or a contextuality scenario, or a compatibility scenario (which we will call simply a scenario) is given by a pair $(X, C)$, where $X$ is a collection of measurements and $C$ a compatibility cover for $X$. A compatibility cover, $C$, for the measurement collection $X$ is a collection of maximal contexts in $X$ such that each $M \in X$ belongs to at least one context in $C$.

Remark 3.16. The choice of demanding maximal contexts for the cover is technical. One great advantage is economy when describing scenarios. The other possible choice would be to impose that whenever $C \in C$, then $C' \in C$ for every $C' \subset C$. The later has the topological advantage of making simplicial complexes explicit.

In usual experiments, for each context every measurements in the context can be performed and the results sampled allowing the inference of some probability distribution for the joint results of the measurements in each context.

Definition 3.17. Given a scenario $(X, C)$, an empirical model, $\mathcal{E}$, is the association of each context $C \in C$ to a probability distribution of the results of the measurements in $C$, i.e.: $p_C : \prod_{M \in C} M \rightarrow R$, with $p_C (m) \geq 0$ and $\sum p_C (m) = 1$.

Given an empirical model (also called a behaviour), it also defines probability distributions for subsets of the (maximal) contexts, via marginalisation. If $S \subset C$, then $p^S_C : \prod_{M \in S} M \rightarrow R$ is given
by
\[ p^C_S(s) = \sum_{m | \text{m}|S = s} p_C(m), \quad (3.1) \]

where \( m | S = s \) means that the restriction\(^4\) of the outcomes \( m \) for the measurements in \( C' \) to those on \( S \) gives the respective outcomes \( s \).

One important condition in those problems is the following:

**Condition 3.2.** If two contexts, \( C \) and \( C' \), overlap, the marginal condition demands
\[ p^C_{C \cap C'} = p^{C'}_{C \cap C'}. \]

Whenever this condition holds, we may drop the symbol from the context out and work with \( p_{C \cap C'} \). In other words, this is the necessary condition for a probability distribution to be defined on \( C \cap C' \), and, consequently, in all of its subsets.

**Definition 3.18.** An empirical model, \( \mathcal{E} \), in a scenario, \( (X, C) \), is non-disturbing if for every \( C, C' \in C \) the marginal condition 3.2 holds.

After defining a non-contextual model, let us present two different definitions for an empirical model to be non contextual. These definitions are shown equivalent by the Fine-Abramsky-Brandenburger Theorem\(^{10,11}\).

**Definition 3.19.** Given a scenario, \( (X, C) \), a non-contextual model is a collection of probability distributions \( p^A_C : M \times A \rightarrow \mathbb{R} \), with \( p^A_C(m, \lambda) \geq 0 \) and
\[ \sum_{m \in M} p^A_C(m, \lambda) = p^A_C(\lambda), \text{ for all } M \in X \text{ and } \lambda \in A, \quad (3.2) \]

\( \text{such that for every } C \in C, \)
\[ p_C(m) = \sum_{\lambda \in A} \prod_{M \in C} p^A_M(m, \lambda). \]

**Remark 3.20.** The existence of a non-contextual model can be read as the existence of a probability space \( (\Lambda, \Upsilon, p^A) \) from which one can obtain independent joint probabilities for each random variable in the scenario and lambda. Following the discussion in subsection 2(c), this is consistent with the idea that \( \lambda \) determines\(^5\) the outcome of each random variable and correlations are established since one cannot access the variable \( \lambda \). This is essentially the reasoning for such additional (latent, in the language of causal structures\(^{12}\)) variable \( \lambda \) be usually referred to as hidden.

**Definition 3.21.** An empirical model, \( \mathcal{E} \), is non contextual by model if it can be obtained from a non-contextual model.

**Definition 3.22.** An empirical model, \( \mathcal{E} \), is non contextual by marginals if there is a joint probability distribution \( p^X_C : \prod_{M \in X} \rightarrow \mathbb{R} \) such that for all \( C \in C \), \( p_C = p^X_C \).

As already anticipated:

**Theorem 3.3** (Fine, Abramsky, Brandenburger). An empirical model, \( \mathcal{E} \), is non contextual by model if, and only if, it is non contextual by marginals.

Given this equivalence, whenever the distinction is irrelevant we will just call those empirical models non contextual. A proof can be made very constructive: given a non-contextual model,

\(^4\)Some readers may consider more elegant to consider a projection \( \pi : m \rightarrow s \) and the sum runs over the fibre over \( s \), \( \pi^{-1}(s) \). The same projection can also be used to say that the marginal \( p^C_S \) is the result of pushing forward \( p_C \) through \( \pi \).

\(^5\)Explicitly, given a model (3.2), fixing \( \lambda \) only determines conditional probabilities \( p^A_M(m|\lambda) \), however, as again those are classical probabilities, this means that we could refine more and obtain \( (\Lambda', \Upsilon', p'^A) \) such that all those conditional probabilities be deterministic.
by multiplying all $p^A_M$ and then marginalising on $A$, one obtains the collective distribution. On
the other way around, the collective distribution can always be made as a convex combination
of deterministic distributions, which assign deterministic values for each random variable; one
simply needs to use the variables of the convex combination to act as $\lambda$. We hope our work will
shed some new light on this important result, by stressing the rôle played by topology in it.

(c) Measurable Fibre Bundles

Now we start to glue concepts. Given a scenario, $(X, C)$, by construction for each $C \in C$, one can
choose a measurable space $(\Omega^C, \Sigma^C)$ which realises the joint measurement $\bigwedge_{M \in C} M$. Each $M \in C$
$C$ gives a partition of $\Sigma^C$ and defines a coarsening of $\Sigma^C$, which we will denote $\Sigma^C_M$. Whenever
$C \cap C' \neq \emptyset$, the joint measurement of all elements of $C \cap C'$ will define the coarsenings $\Sigma^C_{C \cap C'}$
and $\Sigma^C'_{C \cap C'}$. The labelling given by such coarsening defines a bijection $\Sigma^C_{C \cap C'} \leftrightarrow \Sigma^C'_{C \cap C'}$. By
identifying the sets related by such bijections, we have an abstract $\sigma$-algebra for each $M \in X$ given
by the equivalence class of all $\Sigma^C_M$, for all $C \ni M$, which we denote by $\Sigma_M$. This construction
prepares the following:

Definition 3.23. Given a scenario $(X, C)$, a Measurable Fibre Bundle is the attachment of a
measurable space $(\Omega^C, \Sigma^C)$ to each context. Whenever $C \cap C' \neq \emptyset$, the identification $\Sigma^C_{C \cap C'} \leftrightarrow \Sigma^C'_{C \cap C'}$
is considered, leading to the fibre over the measurement $M$, $\Sigma_M$. The projection, $\pi$, is defined only for
the sets $S \in \Sigma^C$ associated to a set in $\Sigma^C_M$. Naturally, in those cases $\pi(S) = M$.

It is quite important to note that this is done on the level of the $\sigma$-algebra but not necessarily
on the level of the sample space. Let us discuss a situation where one can go up until the level of sample spaces.

Definition 3.24. A measurable fibre bundle, $(\Omega^C, \Sigma^C)$ for a scenario, $(X, C)$ is a Sample Fibre Bundle
if the bijections $\Sigma^C_{C \cap C'} \leftrightarrow \Sigma^C'_{C \cap C'}$ can consistently be obtained from relations $\Omega^C \leftrightarrow \Omega^{C'}$.

Remark 3.25. It is important to note that those relations are not necessarily functions, since multiple
associations are allowed in both directions, however, their images must generate partitions subordinated to
$C \cap C'$.

Example 3.26. Given a scenario, $(X, C)$, for each $M \in X$ choose a measurable space, $(\Omega^M, \Sigma^M)$. For
each $C \in C$, take

$$\langle \Omega^C, \Sigma^C \rangle = \prod_{M \in C} \langle \Omega^M, \Sigma^M \rangle. \quad (3.3)$$

Whenever $M \in C \cap C'$, the identification $\Sigma^C_{C \cap C'} \leftrightarrow \Sigma^C'_{C \cap C'}$ must fix the $\Sigma^M$ component in the product.
This can be naturally extended to $\Omega^M$, also acting as the identity map on it. The Measurable Bundle just
constructed is a Sample Bundle.

The example above explores one very important characteristic of fibre bundles: it is piecewise
a product\textsuperscript{a}. However, there is a little bit more: each $(\Omega^C, \Sigma^C)$ is actually a restriction of a global
product:

$$\langle \Omega, \Sigma \rangle = \prod_{M \in X} \langle p^M, \Sigma^M \rangle. \quad (3.4)$$

In this sense, those fibre bundles are trivial. A nontrivial example comes from the $n$–cycle
scenario, bearing in mind the celebrated Möbius strip:

\textsuperscript{a}The best mathematical word is \textit{locally}, but we will not use it here in order to avoid any misunderstanding related to the
locality concept of Bell.
Example 3.27. Consider a dychothmic \( n \)-cycle scenario, \((X, C)\), where \(X = \{M_0, M_1, \ldots, M_{n-1}\}\), each \(M_i\) a binary set, and \(C = \{C_1\}_{i=0, \ldots, n-1}\), with \(C_i = \{M_i, M_{i+1}\}\), and addition understood modulo \(n\). For each \(M_i\), take \(\Omega^{M_i} = [-1, 1]\) and Borel sets as \(\Sigma^{M_i}\). For each \(C_i\), take \(\left(\Omega^{C_i}, \Sigma^{C_i}\right) = \left(\Omega^{M_i}, \Sigma^{M_i}\right) \times \left(\Omega^{M_{i+1}}, \Sigma^{M_{i+1}}\right)\) and, as identifications \(\Omega_i \leftrightarrow \Omega_{i+1}\) take identity functions \(\omega_i \mapsto \omega_{i+1}\) for all, but one values of \(i\), for which, \(\omega_i \mapsto -\omega_{i+1}\) is used. Since those identifications extend to Borel sets, here we have explicitly constructed a Sample Bundle.

In contrast to example 3.26, in example 3.27 we have nontrivial bundles.

Remark 3.28. It is important to recognise that for all values of \(c \neq 0\), the piecewise constant function \(i \mapsto c\) fails to generate a global function. On the other hand, the constant function \(i \mapsto 0\) is globally defined and this has interesting consequences to which we shall come back latter on.

(d) Probability Fibre Bundles

Now we will include probabilities and empirical models in our discussion. The measurable bundles of subsection (c) will receive probability measures in each fibre and the marginal condition will receive a new interpretation.

The construction specialises the notion of measurable bundle. Given a scenario \((X, C)\), we may attach a probability space to each context: \(\left(\Omega^C, \Sigma^C, \mu^C\right)\). Whenever \(C \cap C' \neq \emptyset\), we want not only to have the identification \(\Sigma^C_{\cap C'} \leftrightarrow \Sigma^{C'}_{\cap C'}\), but also to define a probability measure on it. This is exactly the marginal condition 3.2 applied to the restrictions \(\mu^C_{\cap C'}\) and \(\mu^{C'}_{\cap C'}\).

Definition 3.29. Given a scenario \((X, C)\), a Probability Fibre Bundle is the attachment of a probability space \(\left(\Omega^C, \Sigma^C, \mu^C\right)\) to each context. Whenever \(C \cap C' \neq \emptyset\), the identification \(\Sigma^C_{\cap C'} \leftrightarrow \Sigma^{C'}_{\cap C'}\) is considered and the marginal condition \(\mu^C_{\cap C'} = \mu^{C'}_{\cap C'}\) is demanded, leading to the fibre over the measurement \(M, (\Sigma_M, \mu_M)\). The projection, \(\pi\), is defined only for the sets \(S \in \Sigma^C\) associated to a set in \(\Sigma^C_M\). Naturally, in those cases, \(\pi(S, \mu_M(S)) = M\).

The definition of Probability Fibre Bundles, 3.29, should be compared to the definition of Empirical Model, 3.17. The latter gives the necessary data to the former, whenever the marginal condition holds for all overlapping contexts. The problem can be stated as this: given a non-disturbing empirical model on a measurement scenario \((X, C)\), one can construct a probability fibre bundle over the same scenario. Two questions appear: can we upgrade such a bundle to a sample fibre bundle? If so, is this bundle trivial?

To close this section, let us point out another aspect. In subsection 2.(b) we introduced the probability simplex. For scenarios with finite contexts of finite measurements, each context generates a probability simplex. Whenever we condition on any (subset) of the measurements in the context, we obtain a smaller simplex, related to the remaining measurements. If we consider the intersection of two contexts, there is no meaning in conditioning on incompatible variables. However, the marginals are meaningful and they represent coarse grainings which should, then, be identifiable. What is the geometry/topology behind such identifications and, specially, coming from such identifications over a given scenario?

4. Fibre Bundle approach to Contextuality

Now we have everything set and can discuss in more details how contextuality manifests in those objects. The central result is the translation of Fine-Abramsky-Brandeburguer Theorem into the question of triviality of a Probability Fibre Bundle. This brings to the arena questions on extensions and obstructions, i.e.: on possibilities and impossibilities. It also shows the centrality of the so far ignored notion of subscenarios, which we shall discuss on section 5.
(a) An Example-Oriented Discussion

The first thing to be understood is the generalisation of the situation shown in subsection 2(b). There, for a finite collection of finite sets, the sample space could be identified with the atoms of the σ-algebra of the problem. This can be put together with example 3.26 to show that any finite scenario allows for a trivial sample bundle. Let us discuss another example of non-trivial sample bundle.

Example 4.1 (Hollow Triangle). Consider the scenario \((X, C)\) given by

\[
X = \{M_a, M_b, M_c\},
\]

\[
C = \{\{M_a, M_b\}, \{M_b, M_c\}, \{M_c, M_a\}\}, \tag{4.1a}
\]

with \(M_a = \{\uparrow, \downarrow\}, M_b = \{0, 1\}, \text{ and } M_c = \{g, r\}. \) For each context take the minimal measurable space

\[
\left(\mathcal{O}_{ab}^{ac}, \Sigma_{ab}^{ac}\right) = (M_a \times M_b, \mathcal{P}(M_a \times M_b)),
\]

\[
\left(\mathcal{O}_{bc}^{ac}, \Sigma_{bc}^{ac}\right) = (M_b \times M_c, \mathcal{P}(M_b \times M_c)), \tag{4.1b}
\]

\[
\left(\mathcal{O}_{ca}^{ac}, \Sigma_{ca}^{ac}\right) = (M_c \times M_a, \mathcal{P}(M_c \times M_a)).
\]

Now we need the coarse graining in order to define the identifications. Here we will introduce contextuality, which can be seen as a twist (topologically saying); for each context, the first variable will have the “natural” coarse graining, while the second will be “inverted”. Explicitly:

\[
\mathcal{C}_a^{ab} = \{\uparrow = \{(\uparrow, 0), (\uparrow, 1)\}, \downarrow = \{(\downarrow, 0), (\downarrow, 1)\}\},
\]

\[
\mathcal{C}_b^{ab} = \{0 = \{(\uparrow, 1), (\downarrow, 1)\}, 1 = \{(\uparrow, 0), (\downarrow, 0)\}\},
\]

\[
\mathcal{C}_c^{bc} = \{0 = \{(0, g), (0, r)\}, 1 = \{(1, g), (1, r)\}\},
\]

\[
\mathcal{C}_b^{bc} = \{g = \{(0, r), (1, r)\}, r = \{(0, g), (1, g)\}\},
\]

\[
\mathcal{C}_c^{ca} = \{g = \{(g, \uparrow), (g, \downarrow)\}, r = \{(r, \uparrow), (r, \downarrow)\}\},
\]

\[
\mathcal{C}_a^{ca} = \{\uparrow = \{(g, \downarrow), (r, \downarrow)\}, \downarrow = \{(g, \uparrow), (r, \uparrow)\}\}. \tag{4.1c}
\]

This defines a non-trivial sample bundle. We shall keep this notation that makes contextuality and the twist explicit, by using such minimal construction where \(\Sigma^C = \mathcal{P}(\mathcal{C})\) and the coarse grainings are projections or “twisted projections”; but in order to not consider this as artificially made, the reader should also consider the equivalent case where \(\mathcal{C}_a^C\) is a binary set with labels unrelated to the elements of \(\mathcal{O}^C\).

To see how contextuality is deeply related to such bundle, we should interpret what a choice like \((\uparrow, 1, g)\) would mean in terms of the measurements. In the context \(\{M_a, M_b\}\) this implies outcomes \((\uparrow, 0)\), while in the context \(\{M_b, M_c\}\) this implies \((1, r)\), which already shows the contextuality of \(M_b\). Accordingly, for the context \(\{M_c, M_a\}\) this implies \((g, \downarrow)\), making clear that those “global assignments” which try to define values for all measurements can exhibit contextuality. Generalising the choice made, any assignment \((a, b, c)\) would imply opposite answers for each measurement on its two contexts.

This example generalises for any size, \(n\), of the cycle. As we shall see later on, there is a parity issue decisive in order to distinguish intrinsic contextuality from removable contextuality.

A very beautiful phenomenon shows up when building probability bundles over sample bundles like the one in example 4.1. In those examples, contextuality is manifest in its stronger case: disturbance. Suppose one can characterise that a “state” \((a, b, c)\) was prepared. Then, when measuring \(M_a\), the result \(a\) would imply the context \((M_a, M_b)\), while the result \(\bar{a}\) would imply the context \((M_c, M_a)\).
Example 4.2. Consider the Hollow Triangle scenario of example 4.1 and the following empirical model on it:

\[ p_{ab}(\uparrow, 1) = \frac{1}{2} = p_{ab}(\downarrow, 0), \]
\[ p_{bc}(0, r) = \frac{1}{2} = p_{bc}(1, g), \]
\[ p_{ca}(g, \downarrow) = \frac{1}{2} = p_{ca}(r, \uparrow), \]

with all other probabilities null. First, by checking the marginals, see that this empirical model is nondisturbing. Now, one can easily check that this model can be considered as a balanced convex combination of the two “global assignments”: \((\uparrow, 0, g)\) and \((\downarrow, 1, r)\). Other three equally interesting sample bundle realisations of (contextual) probability bundles are from the other combinations of \((a, b, c)\) with \((\bar{a}, \bar{b}, \bar{c})\). Topologically, those non-disturbing empirical models are related to the null constant function of remark 3.28.

We have seen two examples of sample bundles, one trivial, one not, and some probability bundles which realise given empirical models.

(b) Making Topology (more) Explicit

The central question now appears: given a scenario, can all empirical models be obtained as trivial probability bundles? The interesting answer is: only the non-contextual ones! That is how Fine-Abramsky-Brandenburger Theorem 3.3 translates here:

**Theorem 4.1 (Fibre Bundle version of Fine-Abramsky-Brandenburger).** Given a scenario, \((\mathcal{X}, \mathcal{C})\), an empirical model can be realised as a trivial probability bundle iff the model is non contextual.

**Proof.** If the empirical model can be realised as a trivial probability bundle, then the sample bundle can be extended to a product bundle and the model is non contextual by marginals.

If the empirical model, \(\mathcal{E}\), is non contextual there is a global probability distribution for the measurements in \(\mathcal{X}\) and this is an empirical model \(\mathcal{E}\) in the scenario \((\mathcal{X}, \mathcal{X})\), giving a trivial bundle whose restriction to \((\mathcal{X}, \mathcal{C})\) gives a trivial probability bundle realising \(\mathcal{E}\).

In the proof above we already used one instance of a very important topological result: not all basis (in our case, scenarios) can support non-trivial bundles\(^7\). This explains and extends a lot the remark 3.9, and put in a topologically broader context the result on ref. \([9]\). This deserves a little more discussion. In a very lousy way, we could say that trivial basis can only generate trivial bundles. To make it precise, one should define what a trivial basis mean. One very good example of “trivial basis”\(^7\) is any contractible hypergraph. Since a cycle always make a (hyper-)graph non-Contractible, in graph language, the trivial basis/scenarios are trees.

**Corollary 4.2 (Budroni-Morchio).** If a scenario \((\mathcal{X}, \mathcal{C})\) is free of cycles, than it is also free of contextuality.

Topologically, cycles allow for non-trivial homology\(^8\). In refs. \([13–16]\), the authors point out the influence of cohomology in contextuality and nonlocality. We hope this bundle approach will reinforce this relation. As characterised in Thm. 4.1, the central question in contextuality is about extending a probability fibre bundle into a product one. In topology, it is usual to work with obstructions, e.g.: a nontrivial homology class implies the nontriviality of some bundle and forbids such extension. In this sense, a nontrivial first homology group says that a scenario allows for contextuality, while a nontrivial cohomology class is a way of witnessing contextuality \([13–16]\).

\(^7\)In the proof we explicitly use that a trivial scenario \((\mathcal{X}, \mathcal{X})\) supports only trivial bundles.

\(^8\)They are closed “curves” which are not boundaries.
(c) Some comments on models

It is interesting to come back to the example of the Hollow Triangle, 4.1, and its generalisations for the n-cycle. We naturally should contrast it with the product bundle, example 3.26, build on the same scenario. A very natural question is: are there other interesting sample bundles in this scenario? The answer is: essentially no! One could guess: the product bundle has no twist while in example 4.1 we made three twists; what about one or two twists? Interestingly, these cases correspond to relabelings of the previous two cases: if we take the product bundle but “flip” one of its variables, it will generate a bundle with two twists, which, however, is something we can call removable contextuality. Such a bundle is not explicitly a product, but it is isomorphic to one, so it can not support contextuality. Analogously, if we flip on variable from the example 4.1, we will “eliminate two flips”, but we still get a non-trivial bundle, as one should guess.

Another interesting point to call is that while trivial sample bundles allow for “realistic models” and classical interpretations, those non-trivial sample bundles can be considered as “realistic” models for “classical” interpretations on topologically richer ontological spaces\(^9\).

5. Contextuality Subscenarios

When we interpret noncontextuality as the possibility of describing an empirical model using a trivial probability bundle, i.e.: as an extension problem, another concept pops up: what about other extensions? In order to make this question more precise, we need a partial order relation for scenarios:

**Definition 5.1.** A scenario \((X', C')\) is a subscenario of \((X, C)\) if \(X' \subseteq X\) and \(C' \subseteq C\), where this last condition means that for every \(C' \in C\) there is \(C \in \mathcal{C}\) such that \(C' \subseteq C\).

We will denote \((X', C') \preceq (X, C)\) when \((X', C')\) be a subscenario of \((X, C)\).

**Remark 5.2.** The classical scenarios are \((X, X)\), for any \(X\), and for every scenario it is true that \((X, C) \preceq (X', X)\).

There are two canonical ways of creating subscenarios, and any subscenario can be obtained using any or both of them combined.

**Definition 5.3.** \((X', C') \preceq (X, C)\) is an induced subscenario if \(C' = \mathcal{C} \cap \mathcal{P}(X')\), where this last symbol is used in the sense that \(C' \in \mathcal{C} \cap \mathcal{P}(X')\) iff one of two situations happen: either \(C' \in \mathcal{C} \cap \mathcal{P}(X')\) or \(C' = \overline{C}_{X'|}\) for some \(C \in \mathcal{C}\).

**Definition 5.4.** \((X', C') \preceq (X, C)\) is a (context-)restricted subscenario if \(X' = X\).

By remark 5.2, every scenario is a context-restricted subscenario of the classical scenario for measurements \(X\).

Definition 3.22 can now be extended:

**Definition 5.5.** Given \((X', C') \preceq (X, C)\), an empirical model \(\mathcal{E}'\) in the scenario \((X', C')\) extends to the scenario \((X, C)\) if there is an empirical model \(\mathcal{E}\) in \((X, C)\) such that for all \(C' \in C\), \(p_{\mathcal{E}'}^{C'} = p_{\mathcal{E}}^{C'}\).

Simply putting those definitions together one gets:

**Theorem 5.1.** An empirical model \(\mathcal{E}\) in \((X, C)\) is noncontextual iff it extends to the classical scenario \((X, X)\).

\(^9\)Again, no one has to adhere to such “ontology”, neither the author. But is seems very beautiful and deserves more study, at least from the mathematical viewpoint.
A beautiful mathematical structure appears when we build a sequence of nested scenarios, and ask about the possible extensions:

**Definition 5.6.** A sequence of scenarios is an ordered collection of scenarios \( \{(X, C)_i\} \) where \( (X, C)_i \preceq (X, C)_{i+1} \). A sequence of scenarios is complete if in every step \( (X, C)_i \) is a proper subscenario of \( (X, C)_{i+1} \) such that there is no proper subscenario in between them.

This notion of sequence of scenarios allows for growing sets of questions, but a central question comes for sequences of the form \( (X, C) \prec \ldots \prec (X, C)_i \prec \ldots \prec (X, X) \), i.e. a strictly increasing sequence that finishes on the classical scenario for the same set of measurements. A very simple result is:

**Theorem 5.2.** If an empirical model in \( (X, C) \) is noncontextual, then it extends to all elements of the sequence (5.1).

The beautiful question comes: if an empirical model in \( (X, C) \) is contextual, which extensions are allowed in a sequence (5.1) and which are not? In other words: at which steps contextuality really makes its presence? We close this section with a short example.

**Example 5.7.** Let us take the 5-cycle as the starting scenario: \( X = \{X_i\}_{i=1}^{5}, C = \{\{X_i, X_{i+1}\}\}_{i=1}^{5} \). This scenario supports a bundle like the one in the Hollow Triangle example, 4.2. We can consider the coarser scenario with the same measurements and \( C = \{\{X_{i-1}, X_i, X_{i+1}\}\}_{i=1}^{5} \) and we see that the model extends to it. It also extends to the scenario given by \( C = \{\{X_{i-1}, X_i, X_{i+1}, X_{i+2}\}\}_{i=1}^{5} \). The proof of theorem 5.2 can be applied to other extensions, like here, to say that the empirical model extends to any scenario in between the cycle and this four-element-context cover. Since no proper subscenario fits in between this last one and the classical, we can say that it is this step that allows for contextuality.

Other interesting consequences of the notion of subscenarios will be presented elsewhere [18].

### 6. Connection to other approaches and previous literature

Naturally, the here proposed approach to contextuality is not isolated from other proposals. The starting point of describing each measurement in each of its contexts has some similarity with *Contextuality by Default* (CbD) [19]. The difference appears when we demand the marginal condition 3.2 to hold, while CbD relax it, in order to be able of treating empirical data which usually do not obey such condition strictly. Probably the notion of extended contextuality [20] be the link in between the two approaches, but some interesting notions can come up from the difference among original contexts and extended ones.

The central rôle played by topology is also present in the (Pre-)Sheaf approach [10], to which we have already made reference. The notion of extendability and its links to topology had been discussed previously [21] and we should always mention that the idea of gluing probability spaces and ask for its (non-)triviality was already present in ref. [22].

All those contemporary notions have some debt with the graph approach to contextuality [23] and to its variations [24].

### 7. Discussions and Future Developments

Many points of this approach must be developed elsewhere and also many questions are still open.

One very natural thing is to consider continuous measurement scenarios. A natural example comes from spin (or polarisation) measurements on a qubit. There, the Bloch (or Poincaré) sphere
serve as a parameter space for the measurements of the scenario; however, the measurement cover would be made of singletons with no transition function defined. Clearly, we need a good definition of a connection, in order to associate elements of “neighbour” fibres, even when they do not belong to the same context. A very good geometrical question is: after defining such a connection, how its curvature relates to contextuality?

A topologically interesting question comes to higher order homology. The first homology group is deeply connected with the existence of nontrivial cycles. And nontrivial first homology groups is sufficient for the existence of nontrivial probability bundles which realise contextual empirical models, including examples coming from quantum theory. Is it possible to have a measurement scenario with trivial first homology group, but nontrivial higher order homology groups, like a sphere, which also support nontrivial probability bundles? Moreover, are those contextual models obtainable from quantum theory? Or is it true that quantum contextuality really depends on the non-triviality of the first homology group?

The notion of subscenarios brings with it the idea of covers of a given bundle. Example 5.7 has discussed a collection of different subscenarios, but all with the same topology, except by the last one, where contextuality shows up. In this sense we identify that “closing the holes” is the essential step in order to reduce the possibilities of contextuality. The universal cover of a probability bundle will be deeply related to the notion of extended contextuality and this also seems to deserve more studies.

Another empty avenue to be travelled comes from the geometry of the simplest example: the probability simplex of subsec. 2.(b). We see that the marginal condition implies some interesting identifications. Moreover, whenever nontrivial probability bundles appear, the global picture becomes richer and richer. To have a good description of some examples is a nice target for the near future.

8. Conclusion

In this paper we introduced a fibre bundle approach to contextuality. Actually, three kinds of bundles play interesting rôles in this approach: measurable bundles, probability bundles, and sample bundles. Mathematically, this proposal represents a merging of topological notions with probability theory. From the viewpoint of contextuality, it is another approach, with many similarities with the (pre-)sheaf approach, but possibly more comprehensible to many physicists. Naturally, another approach can allow for new interpretations and new insights. Some interesting questions were raised in this pages and we believe that their answers will help us figuring out many interesting aspects of contextuality.

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Disclaimer. Important to say that the connection between Escher paintings and contextuality has long been used by Oxfordians like Shane Mansfield and Samson Abramsky, even during a time when I believed to have been the first to make such connection.

\[^{10}\text{By construction, cycles which are not boundaries are usually understood as folding some “hole”, from where the nontriviality comes.}\]
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