Gradient flows of time-dependent functionals in metric spaces and applications for PDEs

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Abstract

We develop a gradient-flow theory for time-dependent functionals defined in abstract metric spaces. Global well-posedness and asymptotic behavior of solutions are provided. Conditions on functionals and metric spaces allow to consider the Wasserstein space \( P_2(\mathbb{R}^d) \) and apply the results for a large class of PDEs with time-dependent coefficients like confinement and interaction potentials and diffusion. Our results can be seen as an extension of those in Ambrosio-Gigli-Savaré (2005)[2] to the case of time-dependent functionals. For that matter, we need to consider some residual terms, time-versions of concepts like \( \lambda \)-convexity, time-differentiability of minimizers for Moreau-Yosida approximations, and a priori estimates with explicit time-dependence for De Giorgi interpolation. Here, functionals can be unbounded from below and satisfy a type of \( \lambda \)-convexity that changes as the time evolves.

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1 Introduction

We consider the gradient flow equation

\[ u'(t) = -\nabla E(t, u(t)), \quad t > 0, \]
\[ u(0) = u_0, \]

where \( E : [0, \infty) \times X \to (-\infty, \infty] \) is a time-dependent functional and \((X, d)\) is a complete separable metric space. Our aim is to construct a general theory in metric spaces that can be applied for PDEs with time-dependent coefficients. In fact, with this theory in hand, we obtain global-in-time existence and asymptotic behavior of solutions in the Wasserstein space \( P_2(\Omega) \) for a number of PDEs with density of internal energy \( U \), confinement potential \( V \) and interaction potential \( W \) depending on the time-variable. That space consists of probability measures on \( \Omega \) with finite second moment endowed with the so-called Wasserstein metric \( d_2(\mu, \nu) \). Here we will focus on the whole space \( \Omega = \mathbb{R}^d \).

Gradient flows theory has been successfully developed for the case of time-independent functionals \( E(u) \) in general metric spaces \((X, d)\) (see \([7],[8],[2],[3],[15]\)). Two basic tools in the theory are the concept of curves of maximal slopes (see \([8],[15]\)) and a time-discrete approximation scheme (see \([7],[3]\)). The latter is based on the implicit variational scheme

\[ U^n_\tau \in \text{Argmin}_{v \in X} \left\{ \frac{1}{2\tau} d_2(U^{n-1}_\tau, v) + E(v) \right\}, \]  

where \( \tau > 0 \) is a time step. Notice that (1.3) consists in finding minimizers for interactive values of the Moreau-Yosida approximation \( E_\tau(u) := \inf_{v \in X} \left\{ \frac{1}{2\tau} d_2(u, v) + E(v) \right\} \) of \( E \) in \((X, d)\). Speak generally, basic hypotheses assumed on \( E \) are lower semicontinuity and some type of convexity and coercivity (see \([2]\)). For the analysis of PDEs as a gradient flows, a suitable metric space is \( P_2 \) in which the above theory has demonstrated to be particularly very fruitful. The idea of using the above discrete scheme in \( P_2 \) goes back to the work \([11]\) for the linear Fokker-Plank equation and \([20]\) for the porous medium equation. Subsequently, several authors extended this approach to a general class of continuity equations (see \([2],[1],[6]\)) with velocity field given by the gradient of the variational derivative of a time-independent functional, namely

\[ \frac{\partial u}{\partial t} = \text{div} \left( u \nabla \frac{\delta E}{\delta u} \right), \quad \text{in} \ (0, +\infty) \times \mathbb{R}^d, \]

where \( E \) is the free energy associated to PDE dealt with. Under some basic assumptions, they considered \( E \) with the form

\[ E[u] := \int_{\mathbb{R}^d} U(u(x)) \, dx + \int_{\mathbb{R}^d} u(x) V(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x - y) \, u(x) u(y) \, dx \, dy, \]

where \( U : \mathbb{R}^+ \to \mathbb{R} \) is the density of internal energy, \( V : \mathbb{R}^d \to \mathbb{R} \) is a confinement potential and \( W : \mathbb{R}^d \to \mathbb{R} \) is an interaction potential. The functional (1.5) has the classical form given
by the sum of the internal energy, potential energy and interaction energy functionals that is verified by a wide number of physical models. Beside existence of global-in-time flows, the literature contains results on uniqueness, global contraction, regularity, and asymptotic stability of solutions (see e.g. [2]). We also quote the paper [4] where a 1D non-local fluid mechanics model with velocity coupled via Hilbert transform was analyzed by using gradient flow theory in $\mathcal{P}_2$.

In [13], the authors dealt with nonlinear diffusion equations in the form

$$\partial_t u - \text{div}(A(\nabla(f(u)) + u\nabla V)) = 0,$$

where $A$ is a symmetric matrix-valued function of the spatial variables satisfying a uniform elliptic condition and $f$, $V$ are functions satisfying suitable hypotheses. They also analyzed the contraction property for solutions.

On the other hand, from a theoretical and applied point of view, it is natural to consider a time-dependence on the coefficients of some equations. For instance, a version of the stochastic Fokker-Plank equation (the one considered in [11]) is

$$dX_t = -\nabla V(t, X_t) dt + \sqrt{2\kappa(t)} dB_t, \quad (1.6)$$

where the term $\sqrt{2\kappa(t)}$ is known as the diffusion coefficient and $B_t$ stands for the classical Brownian motion. For (1.6), it is well-known that the law of processes is modeled by the PDE

$$\partial_t u = \kappa(t) \Delta u + \nabla \cdot (\nabla V(t,x) u).$$

Another example is the version of the Mckean-Vlasov equation [24]

$$dX_t = b(t, \mu_t, X_t) dt + \sqrt{2\kappa(t)} dB_t, \text{ with } b(t, \mu, x) = -\nabla W(t, \cdot) * \mu,$$

where $\mu_t$ is the law of the processes $X_t$ that obeys the PDE

$$\partial_t u = \kappa \Delta u - \nabla \cdot (b(t, u, x) u)$$

with $\kappa$ depending on the time $t$. The term $b(t, u, x) u$ corresponds to an interaction between particles with time-dependent potential.

For a bounded convex domain $\Omega \subset \mathbb{R}^d$ and $0 < T < \infty$, Petrelli and Tudorascu [21] considered the non-homogeneous Fokker-Plank equations

$$u_t - \nabla_x \cdot (u \nabla_x \psi(t,x)) - \Delta_x (P(t, u)) = g(t,x,u) \text{ in } \Omega \times (0,T) \quad (1.7)$$

with Neumann boundary conditions and nonnegative $u_0 \in L^\infty(\Omega)$ such that $\int u_0 dx = 1$. They proved existence of nonnegative bounded weak solutions by constructing approximate solutions via time-interpolants of minimizers arising from Wasserstein-type implicit schemes. Let us point out that, when $P(t,z) = \kappa(t) z$, the conditions in [21] require that the viscosity $\kappa$ is bounded away from zero, while here we allow $\kappa$ to be arbitrarily near zero (see Theorem 6.9 in subsection 6.2).
In [23], Rossi, Mielke and Savaré analyzed the doubly nonlinear evolution equation

$$\partial \psi(u'(t)) + \partial_u \mathcal{E}(t, u(t)) \geq 0 \text{ in } B', \text{ a.e. } t \in (0, T), \quad (1.8)$$

where $B$ is a separable Banach space, $0 < T < \infty$, and $u(0) = u_0$. They proposed a formulation for (1.8) in a separable metric space $(X, d)$ that extends the notion of curve of maximal slope for gradient flows in metric spaces. Existence of solutions is proved by means of a time-discrete approximation scheme in $(X, d)$ defined as

$$U^n_\tau \in \text{Argmin}_{v \in X} \{ \tau \psi(d(U^{n-1}_\tau, v)/\tau) + \mathcal{E}(t_n, v) \}, \quad (1.9)$$

where $\tau$ is a partition for $[0, T]$ and $\tau = |\tau|$ is the time step. Among others, the authors of [23] assumed that $\mathcal{E}$ satisfies the chain rule, is locally (in time) uniformly bounded from below, and differentiable in the $t$-variable with the derivative satisfying the condition

$$|\partial_t \mathcal{E}(t, u)| \leq C(\mathcal{E}(t, u) + d(u^*, u) + 2C_0), \quad (1.10)$$

for some $u^* \in X$, where

$$C_0 = -\inf_{t \in [0, T], u \in X} \mathcal{E}(t, u).$$

In fact, functionals in [23] are the sum of two time-dependent functionals $\mathcal{E}_1$ and $\mathcal{E}_2$ where $\mathcal{E}_1$ is bounded from below and $\lambda_0$-convex (uniformly with respect to $t \in [0, T]$), and $\mathcal{E}_2$ is a dominated concave perturbation of $\mathcal{E}_1$. For a bounded domain $\Omega$ and $u_0 \in H^1_0(\Omega)$, using the above approach for $\partial \psi(u'(t)) = u'(t)$ (gradient flow case), they also analyzed (1.1) with $\partial_u \mathcal{E}(t, u) = -\Delta u + F'(u) - l(t)$ in the $L^1(\Omega)$-metric. These results were improved in [18] by considering more general dissipation $\psi$. Moreover, in [18] the condition (1.10) was relaxed to $|\partial_t \mathcal{E}(t, u)| \leq C\mathcal{E}(t, u)$. We also refer the reader to [19, 22] for stability results for doubly nonlinear equations in Banach spaces.

Since our functionals are not bounded from below neither satisfies a estimate like (1.10), we can not to apply the theory from [23] and [18]. Here we assume the conditions E1, E2, E3, E4 and E5 given in Section 2 (see pages 6 and 7). Notice that E4 gives some local-in-time control from below for $\mathcal{E}$ but allows it to be unbounded from below at each $t > 0$. In (1.10) it is required some control of the time-derivative of $\mathcal{E}$ in terms of the functional itself. Instead of such estimate, we work with a condition on the difference of $\mathcal{E}$ in two different times (see E3). In order to recover the contraction property, inspired by the convexity used in [2], we propose a type of $\lambda$-convexity that changes as the time evolves (see E5). Thus, functionals could “lose convexity” in a such way that the approximation between two solutions for large times still holds, because the contraction property depends only on the mass accumulated by $\lambda$, i.e. $\int_0^t \lambda(s)ds$. In general the function $\lambda(t)$ can be unbounded both from above and below in $[0, \infty)$ but, for the contraction, it is assumed to be continuous. In Section 6, we show how to extend results for the case of $\mathcal{E}(t, u)$ having a more general density of internal energy $U(t, u)$ and viscous term $-\Delta_x(P(t, u))$ (see Theorem 6.10 and Remark 6.11 in subsection 6.3). There, the conditions on potentials prevent $\mathcal{E}(t, \rho)$ to satisfy E3. In the case $P(t, z) = \kappa(t)z$, the diffusion coefficient $\kappa$ is non-increasing. This condition
is necessary in order to have the uniform limit of the approximate solutions (4.8) in all finite interval $[0, T]$.

Another application of time-dependent gradient flows appears in the context of pursuit-evasion games. Jun [12] considered gradient flows in suitable playing fields and investigated existence and uniqueness of continuous pursuit curves that are downward gradient curves for the distance from a moving evader, i.e. a time-dependent gradient flows. In fact, his result works well in $CAT(K)$-spaces (with $K = 0$) that are complete metric spaces such that no triangle is fatter than the triangle with same edge lengths in the model space of constant curvature $K$. Also, he assumed that $E(t, u)$ is Lipschitz in $t$, locally Lipschitz in $u$, and $\lambda_0$-convex for all $t > 0$ where $\lambda_0$ is a fixed constant (i.e. $\lambda_0$-convex uniformly in $t$). Another basic hypothesis used by him is that $E_t,\tau(u)$ given by

$$
E_{t,\tau}(u) := \inf_{v \in X} \left\{ \frac{1}{2\tau} d^2(u, v) + E(t, v) \right\}
$$

is $C\tau$-Lipschitz in $t$, for all $u \in X = CAT(0)$ and $\tau > 0$, where $C > 0$ is a constant. For the time-independent case $E(u)$, we refer the reader to [16] for $X = CAT(0)$ (see also [2]) and [14] for a geometric approach in $X = CAT(K)$.

In this paper we follow the program in the book [2] that contains a relatively complete gradient flows theory in general metric spaces and its applications for the non-vectorial space $P^2$ by using optimal transport tools. So, our results can be seen as an extension of those in [2] in order to consider time-dependent functionals. For that matter, due to time-dependence of $E$, we need to handle some residual terms (see e.g. (4.14) and the estimate (4.19)) and to consider time-versions of concepts like $\lambda$-convexity (see E5) and interpolation functions as (4.3) to (4.7). One of these functions is the interpolation (4.4) that corresponds to the time-dependent convexity parameter $\lambda(t)$. Thus, some adaptations from arguments in [2] made here is not a straightforward matter and involves certain care. Also, the time-differentiability of the minimizer for the Moreau-Yosida approximation of $E$ needs to be analyzed (see Proposition 3.4) and, in order to get the convergence of the approximate solutions, a priori estimates with explicit dependence on the $t$-variable are performed in Proposition 5.2 for which the aforementioned condition E3 plays a key role.

The plan of this paper is the following. In Section 2 we recall some concepts such as proper functional and local slope, and some results on gradient flow theory in metric spaces. Also, we give the metric formulation for (1.1)-(1.2) and the basic assumptions for the functional $E$. In Section 3, we construct the approximate solutions, provide some properties for the minimizer of the Moreau-Yosida approximation, and give estimates for approximate solutions. In Section 4, we derive a priori estimates for the approximate solutions and show their locally uniform convergence in $[0, \infty)$. In Section 5, we show that the curve, which is the limit of the approximate solutions, is in fact a solution of (1.1)-(1.2) in the sense of Section 2 and obtain the contraction property for solutions. Section 6 is devoted to applying the general theory in the Wasserstein space for PDEs with time-dependent functionals as those mentioned above.
2 Metric formulation and implicit scheme

Let \((X, d)\) be a complete separable metric space and consider the functional \(E : X \to (-\infty, +\infty]\). Recall that \(E\) is said to be proper whether there is \(u_0 \in X\) such that \(E(u_0) < \infty\), and its domain is defined by

\[
\text{Dom}(E) = \{ u \in X : E(u) < \infty \}. \tag{2.1}
\]

Thus, a functional \(E\) is proper when \(\text{Dom}(E) \neq \emptyset\). Let \(f^+\) and \(f^-\) denote the positive and negative parts of an extended real-valued function \(f\). The following concept is crucial in the theory of gradient flows, and we will use it for the case of time-dependent functionals.

**Definition 2.1.** Let \(E\) be a proper functional in a metric space \(X\). The *local slope* \(\partial E\) of \(E\) at the point \(u \in X\) is defined as

\[
|\partial E|(u) = \limsup_{v \to u} \frac{(E(u) - E(v))^+}{d(u, v)}. \tag{2.2}
\]

In what follows, we recall a technical lemma that will be useful in our calculations.

**Lemma 2.2** ([2, Lemma 2.2.1]). Let \(E : X \to (-\infty, \infty]\) be a functional such that there is \(\tau^* > 0\) and \(u^* \in X\) with

\[
E_{\tau^*}(u^*) := \inf_{v \in X} \left\{ E(v) + \frac{d^2(v, u^*)}{2\tau^*} \right\} > -\infty.
\]

Then

\[
E_{\tau}(u) \geq E_{\tau^*}(u^*) - \frac{1}{\tau^* - \tau}d^2(u^*, u), \text{ for all } 0 < \tau < \tau^* \text{ and } u \in X,
\]

and

\[
d^2(u, v) \leq \frac{4\tau^\tau}{\tau^* - \tau} \left( E(v) + \frac{d^2(v, u)}{2\tau} - E_{\tau^*}(u^*) + \frac{1}{\tau^* - \tau}d^2(u^*, u) \right).
\]

In particular, the sub-levels of the map \(v \to E(v) + \frac{d^2(u,v)}{2\tau}\) are bounded.

2.1 Metric formulation

Let \(E : [0, +\infty) \times X \to (-\infty, +\infty]\) be a time-dependent functional. It is well known that the problem (1.1)-(1.2) admits a metric reformulation by using the concept of local slope (see [23]). This is given by the variational inequality

\[
\frac{d}{dt}(E(t, u(t))) \leq \partial_t E(t, u(t)) - \frac{1}{2} |\partial E(t)|^2(u(t)) - \frac{1}{2} |u'|^2(t), \tag{2.3}
\]

where \(|\partial E(t)|\) stands for the local slope of the functional \(u \to E(t, u)\), for each fixed \(t > 0\), and

\[
|u'|(t) \lim_{s \to t} \frac{d(u(s), u(t))}{|t - s|}.
\]
stands for the metric derivative of an absolutely continuous curve $u$.

Below we state the principal assumptions on the family of functionals $\mathcal{E}(t, \cdot)$ on $X$, for $t \in [0, \infty)$:

**E1.-** For each $t \geq 0$, $\mathcal{E}(t, \cdot)$ is proper and lower semicontinuous with respect to the metric $d(\cdot, \cdot)$.

**E2.-** The domain of the functionals, $D := \text{Dom}(\mathcal{E}(t, \cdot))$, is time-independent.

**E3.-** There exist $u^* \in X$ and a function $\beta : [0, \infty) \to [0, \infty)$ with $\beta \in L^1_{\text{loc}}([0, \infty))$ such that, for each $u \in D$, the function $t \to \mathcal{E}(t, u)$ satisfies

$$|\mathcal{E}(t, u) - \mathcal{E}(s, u)| \leq \int_s^t \beta(r) \ dr (1 + d^2(u, u^*)).$$  \quad (2.4)

Note that if the condition (2.4) is valid for some $u^* \in X$ then it is in fact valid for all $u^* \in X$. Also, for each $u \in D$, the function $t \to \mathcal{E}(t, u)$ is differentiable a.e. in $[0, \infty)$ and its set of differentiability points may depend on $u$.

Now we are ready to give the notion of solution for (1.1)-(1.2) that we deal with.

**Definition 2.3.** Let $u_0 \in X$ and $\mathcal{E} : [0, +\infty) \times X \to (-\infty, +\infty]$ be a functional satisfying the assumptions **E1, E2 and E3**. We say that an absolutely continuous curve $u : [0, +\infty) \to X$ is a solution for (1.1)-(1.2), if $u(0) = u_0$, the function $t \to \mathcal{E}(t, u(t))$ is absolutely continuous,

$$|u'|, |\partial \mathcal{E}(\cdot)|(u(\cdot)) \in L^2_{\text{loc}}([0, \infty)),$$  \quad (2.5)

and the variational inequality (2.3) holds true.

### 2.2 Implicit variational scheme

We start by recalling the Moreau-Yosida approximation of $\mathcal{E}$. For $\tau > 0$ and $t \geq 0$, this approximation is defined as

$$\mathcal{E}_{t, \tau}(u) := \inf_{v \in X} \{ \mathcal{E}(t, \tau, u; v) \},$$  \quad (2.6)

where the functional $\mathcal{E}(t, \tau, u; \cdot)$ is given by

$$\mathcal{E}(t, \tau, u; v) := \mathcal{E}(t, v) + \frac{d^2(u, v)}{2\tau}.$$  \quad (2.7)

Next, take a partition $\tau = \{ 0 = t^0_\tau < t^1_\tau < \cdots < t^n_\tau < \cdots \}$ of $[0, \infty)$ with $\lim_{n \to \infty} t^n_\tau = \infty$. Defining the step size $\tau_n := t^n_\tau - t^{n-1}_\tau$, one can construct the sequence

$$U^n_\tau \in \text{Argmin}_{v \in X} \{ \mathcal{E}(t^n_\tau, \tau_n, U^{n-1}_\tau; v) \},$$  \quad (2.8)

for a given family of initial data $U^0_\tau \in X$. 


Since the convergence results are locally in time, we can fix $T > 0$ arbitrary and analyze the convergence in $[0, T]$. In order to analyze rigorously the problem of minimization \((2.8)\), we give two additional assumptions that will allow to obtain uniqueness and a nice behavior of the minimizers.

**E4.** For each $T > 0$, there exist a $u^* \in X$ and $\tau^*(T) = \tau^* > 0$ such that the function $t \to \mathcal{E}_{t, \tau^*}(u^*)$ is bounded from below in $[0, T]$.

**E5.** There is a function $\lambda : [0, \infty) \to \mathbb{R}$ in $L^\infty_{loc}([0, \infty))$ such that: given points $u, v_0, v_1 \in X$, there exists a curve $\gamma : [0, 1] \to X$ satisfying $\gamma(0) = v_0, \gamma(1) = v_1$ and

$$\mathcal{E}(t, \tau, u; \gamma(s)) \leq (1 - s)\mathcal{E}(t, \tau, u; v_0) + s\mathcal{E}(t, \tau, u; v_1) - \frac{1 + \tau\lambda(t)}{2\tau}s(1 - s)d^2(v_0, v_1),$$

for $0 < \tau < \frac{1}{\lambda_T}$ and $s \in [0, 1]$, where $\lambda_T = \max\{-\inf_{t \in [0, T]} \lambda(t), 0\}$.  

**Remark 2.4.** Note that by Lemma 2.2, for each $0 < \tau < \tau^*$ and $u \in X$, we have that the function $t \to \mathcal{E}_{t, \tau}(u)$ is bounded from below in $[0, T]$. In view of the assumptions E4 and E5 we assume by technical reasons that $\tau^* < \min\{\frac{1}{\lambda_T+1}, 1\}$.

**Remark 2.5.** In E5, we consider the existence of curves $\gamma : [0, 1] \to X$ for all $v_0, v_1 \in X$ and not only for elements in the domain $D$. This will be necessary for the applications in Section 6 where we will use the concept of generalized geodesics in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$. These curves exist independently of the functionals that we will analyze in that section.

### 3 Construction and properties of the implicit scheme

In this section we provide some results about the sequence defined in \((2.8)\). They can be seen as extensions of some results in [2] to the case of time-dependent functionals. We start with the following preliminary result.

**Lemma 3.1.** Suppose E1, E4 and E5 and let $u \in X$, $0 \leq t \leq T$, $0 < \tau < \frac{1}{\lambda_T}$. Then, the minimization problem

$$\min_{v \in X} \{\mathcal{E}(t, \tau, u; v)\}$$

has a unique minimizer $u^\tau_t$.

**Proof.** Let $v_n \in D$ be a minimizing sequence, i.e., \(\lim_{n \to \infty} \mathcal{E}(t, \tau, u; v_n) = \mathcal{E}_{t, \tau}(u)\). Given $m, n \in \mathbb{N}$, by the convexity property E5, there is a curve $\gamma : [0, 1] \to X$, with $\gamma(0) = v_n, \gamma(1) = v_m$, and

$$\mathcal{E}_{t, \tau}(u) \leq \mathcal{E}(t, \tau, u; \gamma(1/2))$$

$$\leq \frac{1}{2}\mathcal{E}(t, \tau, u; v_n) + \frac{1}{2}\mathcal{E}(t, \tau, u; v_m) - \frac{\tau^{-1} + \lambda(t)}{8}d^2(v_n, v_m).$$
Thus
\[
\frac{\tau^{-1} + \lambda(t)}{4} d^2(v_n, v_m) \leq (E(t, \tau, u; v_n) - E_{t, \tau}(u)) + (E(t, \tau, u; v_m) - E_{t, \tau}(u)).
\]

It follows from the above estimative that \(v_n\) is a Cauchy sequence in \(X\) and hence it converges to some \(u^*_\tau \in X\). From the lower semicontinuity, we get that \(u^*_\tau\) is a minimizer of the functional \(E(t, \tau, u; \cdot)\). The uniqueness follows from \(E_5\) and is left to the reader. \(\square\)

In the next lemma we show that \(E^{t}_\tau(u)\) and \(u^*_\tau\) depend continuously on \((\tau, t, u)\).

**Lemma 3.2.** Assume the properties \(E_1\) to \(E_5\). Then, the following statements hold true:

a) The map \((\tau, t, u) \in (0, \tau^*(T)) \times [0, T] \times X \to E^{t}_\tau(u) \in \mathbb{R}\) is continuous.

b) The map \((\tau, t, u) \in (0, \tau^*(T)) \times [0, T] \times X \to u^*_\tau \in X\) is continuous.

**Proof.** We start with item a). Let \((\tau_n, t_n, u_n)\) be a sequence converging to \((\tau_0, t_0, u_0)\) in \((0, \tau^*(T)) \times [0, T] \times X\). Denote by \(v_n = (u_n)^{\tau_n}\) the minimizer of \(E(t_n, \tau_n, u_n; \cdot)\) given in Lemma 3.1. It follows that
\[
\limsup_{n \to \infty} E_{t_n, \tau_n}(u_n) = \limsup_{n \to \infty} E(t_n, \tau_n, u_n; v_n) 
\leq \limsup_{n \to \infty} E(t_n, \tau_n, u_n; v) 
= E(t_0, \tau_0, u_0; v),
\]
for all \(v \in X\). Taking the infimum in the right hand side, we obtain \(\limsup_{n \to \infty} E_{t_n, \tau_n}(u_n) \leq E_{t_0, \tau_0}(u_0)\). In view of Lemma 2.2, the sequence \(v_n\) is bounded. So, we can estimate
\[
\liminf_{n \to \infty} E_{t_n, \tau_n}(u_n) \geq \liminf_{n \to \infty} \left\{ \frac{1}{2\tau_n} (d(u_n, u_0) - d(v_n, u_0))^2 + E(t_n, v_n) \right\} 
= \liminf \left\{ \frac{d^2(u_n, u_0)}{2\tau_n} - \frac{d(u_n, u_0)d(u_0, v_n)}{\tau_n} + \frac{\tau_0 - \tau_n}{2\tau_0\tau_n} d^2(v_n, u_0) 
+ E(t_0, \tau_0, u_0; v) + (E(t_n, v_n) - E(t_0, v_n)) \right\} 
\geq \liminf_{n \to \infty} \left\{ E_{t_0, \tau_0}(u_0) - \int_{[t_n, t_0]} \beta(r) dr (1 + d^2(u^*_\tau, v_n)) \right\} 
= E_{t_0, \tau_0}(u_0),
\]
and thus \(\lim_{n \to \infty} E_{t_n, \tau_n}(u_n) = E_{t_0, \tau_0}(u_0)\). For item b), note that
\[
E(t_0, \tau_0, u_0; v_n) - E_{t_n, \tau_n}(u_n) = \left( \frac{d^2(u_0, v_n)}{\tau_0} - \frac{d^2(v_n, u_n)}{\tau_n} \right) 
+ (E(t_0, v_n) - E(t_n, v_n)).
\]
Using $E_3$, the boundedness of $v_n$ and the convergence $(\tau_n, t_n, u_n) \to (\tau_0, t_0, u_0)$, we get
\[
\lim_{n \to \infty} (E(t_0, \tau_0, u_0; v_n) - \delta_{t_0, \tau_0}(u_n)) = 0,
\]
and, by the item a),
\[
\lim_{n \to \infty} E(t_0, \tau_0, u_0; v_n) = \delta_{t_0, \tau_0}(u_0).
\]
It follows that $v_n$ is also a minimizing sequence for $\delta_{t_0, \tau_0}(u_0)$ and then, by the same arguments in the proof of Lemma 3.1, it converges to $(u_0)_{\tau_0}$, as required.

Because of Lemma 3.1, for each family of initial data $U^0_\tau \in X$ associated to a partition $\tau$ of $[0, +\infty)$, we have that the sequence (2.8) is well-defined for each $n \in \mathbb{N}$ such that $t^n_\tau < T + \tau^*$, if $\tau^* < \frac{1}{\lambda_{T+1}}$. In what follows, we give some estimates for the minimizer of the Moreau-Yosida approximation (2.6). These will play an important role in the convergence of approximate solutions.

**Lemma 3.3.** Assume that $E$ satisfies the properties $E1$ to $E5$. Let $0 < \tau < \tau^*$, $0 \leq t \leq T$ and $u \in D$. Then
\[
d^2(u^{t + \tau}_\tau, u^*) - d^2(u, u^*) \leq \frac{\epsilon}{\tau} + \frac{d^2(u^{t + \tau}_\tau, u^*)}{\epsilon},
\]
for all $\epsilon > 0$. If $\tau \leq \tau^*/8$, we have
\[
d^2(u^{t + \tau}_\tau, u^*) \leq 4\tau^* \left( E(t, u) + \int_t^{t + \tau^*} \beta(r) \, dr \left( 1 + d^2(u, u^*) \right) - \inf_{0 \leq \tau \leq t + \tau^*} E_{t+\tau^*}(u^*) \right)
+ 4d^2(u, u^*). \tag{3.2}
\]

**Proof.** We have that
\[
d^2(u^{t + \tau}_\tau, u^*) - d^2(u, u^*) &= -2d(u^{t + \tau}_\tau, u^*)(d(u, u^*) - d(u^{t + \tau}_\tau, u^*)) \\
&= -2d(u^{t + \tau}_\tau, u^*)d(u^{t + \tau}_\tau, u) \\
&\leq 2d(u^{t + \tau}_\tau, u^*)d(u^{t + \tau}_\tau, u) \\
&\leq \epsilon \frac{d^2(u^{t + \tau}_\tau, u)}{\tau} + \frac{d^2(u^{t + \tau}_\tau, u^*)}{\epsilon} \tag{3.3}
\]
\[
&\leq 2\epsilon (E(t + \tau, u) - E(t + \tau, u^{t + \tau}_\tau)) + \frac{d^2(u^{t + \tau}_\tau, u^*)}{\epsilon} \\
&\leq 2\epsilon (E(t + \tau, u) - E_{t+\tau^*}(u^*)) \\
&\quad + \epsilon \frac{d^2(u^{t + \tau}_\tau, u^*)}{\tau^*} + \frac{d^2(u^{t + \tau}_\tau, u^*)}{\epsilon} \tag{3.4}
\]
Notice that we have already obtained (3.1) in (3.3). Now, choosing $\epsilon = \frac{\tau^*}{2}$ in (3.4), we get
\[
\frac{1}{2} d^2(u^{t + \tau}_\tau, u^*) \leq \tau^* \left( E(t, u) + \int_t^{t + \tau^*} \beta(r) \, dr \left( 1 + d^2(u, u^*) \right) - E_{t+\tau^*}(u^*) \right) \\
+ d^2(u, u^*) + \frac{2\tau}{\tau^*} d^2(u^{t + \tau}_\tau, u^*),
\]
which implies (3.2) when $\tau \leq \tau^*/8$.\qed
The next result gives a time-differentiability property for $\mathcal{E}_{t,\tau}(u)$.

**Proposition 3.4.** Assume $E1$ to $E5$. For $0 < \tau \leq \frac{s}{8}$, the function $\tau \to \mathcal{E}_{t,\tau}(u)$ is locally absolutely continuous in $(0, \frac{s}{8}]$ and then is differentiable almost everywhere in that interval. For each $u \in D$, assume further that the set of differentiability points of $t \to \mathcal{E}(t, u)$ does not depend on $u$ (e.g., when $t \to \mathcal{E}(t, u)$ is differentiable). Then

$$\frac{d}{d\tau} \mathcal{E}_{t,\tau}(u) = \partial_t \mathcal{E}(t + \tau, u_t^{t+\tau}) - \frac{d^2(u_t^{t+\tau})}{2\tau^2}$$

(3.5)

in the set of differentiability points.

**Proof.** Let $0 < \tau_0 < \tau_1 \leq \frac{s}{8}$. Recalling that $u_t^{t+\tau_0}$ minimizes $E(t + \tau_0, \tau_0, u; \cdot)$, and using $E3$, we have that

$$\mathcal{E}_{t+\tau_1,\tau_1}(u) - \mathcal{E}_{t+\tau_0,\tau_0}(u) \leq \mathcal{E}(t + \tau_1, u_{\tau_1}^{t+\tau_1}) - \mathcal{E}(t + \tau_0, u_{\tau_0}^{t+\tau_0})$$

$$+ \frac{\tau_0 - \tau_1}{2\tau_1 \tau_0} d^2(u_t^{t+\tau_1})$$

(3.6)

Similarly, but now using $u_t^{t+\tau_1}$, it follows that

$$\mathcal{E}_{t+\tau_1,\tau_1}(u) - \mathcal{E}_{t+\tau_0,\tau_0}(u) \geq \mathcal{E}(t + \tau_1, u_{\tau_1}^{t+\tau_1}) - \mathcal{E}(t + \tau_0, u_{\tau_1}^{t+\tau_1})$$

$$+ \frac{\tau_0 - \tau_1}{2\tau_1 \tau_0} d^2(u_t^{t+\tau_1})$$

(3.7)

Notice that (3.2) allows us to estimate the terms $d(u^*, u_{\tau_1}^{t+\tau_1})$ and $d(u_t^{t+\tau_1}, i = 0, 1$, by an expression independent of $\tau$, which gives the absolute continuity in each compact interval of $(0, \frac{s}{8}]$. Now take a point $\tau \in (0, \frac{s}{8}]$ where the derivative of $\tau \to \mathcal{E}_{t,\tau}(u)$ exists. Considering lateral limits, the equality (3.5) follows by using estimates (3.6)-(3.7) and that $d(u_t^{t+\tau_1}, u_{\tau_k}^{t+\tau_k}) \to 0$ as $\tau_k \to \tau$ (see Lemma 3.2 b)).

As a consequence, we have the following corollary.

**Corollary 3.5.** Assume the same hypotheses of Proposition 3.4. Then, for $u \in D$, we have the identity

$$\frac{d^2(u_t^{t+\tau_1})}{2\tau^2} + \int_0^\tau \frac{d^2(u_t^{t+\tau_1})}{2r^2} dr = \int_0^\tau \partial_t \mathcal{E}(t + r, u_t^{t+\tau_1}) dr + \mathcal{E}(t, u) - \mathcal{E}(t, u_t^{t+\tau_1})$$

(3.8)

**Proof.** By integrating (3.5) from $\tau_0$ to $\tau \leq \frac{s}{8}$, it follows that

$$\mathcal{E}_{t+\tau,\tau}(u) - \mathcal{E}_{t+\tau_0,\tau_0}(u) + \int_{\tau_0}^\tau \frac{d^2(u_t^{t+\tau_1})}{2r^2} dr = \int_{\tau_0}^\tau \partial_t \mathcal{E}(t + r, u_t^{t+\tau_1}) dr.$$
In view of the definitions of $\partial_{t,\tau}(u)$ and $u^{j}_{\tau}$, and since the above integrals are finite as $\tau_{0} \to 0$, the remainder of the proof is to show that $\partial_{t+\tau_{0},\tau_{0}}(u) \to \mathcal{E}(t, u)$ as $\tau_{0} \to 0$, for each fixed $t > 0$. In fact, note that

$$\partial_{t+\tau_{0},\tau_{0}}(u) \leq \mathcal{E}(t + \tau_{0}, u),$$

and so

$$\limsup_{\tau_{0} \to 0} \partial_{t+\tau_{0},\tau_{0}}(u) \leq \mathcal{E}(t, u). \quad (3.9)$$

Also, we can conclude from (3.9) and Lemma 2.2 that $d(u, u^{t+\tau}_{\tau_{0}}) \to 0$, as $\tau_{0} \to 0$. Using the lower semicontinuity of $\mathcal{E}$, we get

$$\mathcal{E}(t, u) \geq \limsup_{\tau_{0} \to 0} \partial_{t+\tau_{0},\tau_{0}}(u)$$

$$\geq \liminf_{\tau_{0} \to 0} \left( \mathcal{E}(t, u^{t+\tau_{0}_{0}}) - \int_{t}^{t+\tau_{0}} \beta(r) \, dr + d^{2}(u, u^{t+\tau_{0}}) \right)$$

$$\geq \mathcal{E}(t, u),$$

as desired. \hfill \Box

Remark 3.6. In the last proof, we have showed in particular that $u^{t+\tau}_{\tau} \to u$ as $\tau \to 0$, when $u \in D$.

Now, we recall a discrete Gronwall lemma.

Lemma 3.7 ([2] Lemma 3.2.4). Let $A, \alpha \in [0, \infty)$ and, for $n \geq 1$, let $a_{n}, \beta_{n} \in [0, \infty)$ satisfy

$$a_{n} \leq A + \alpha \sum_{j=1}^{n} \beta_{j}a_{j}, \ \forall n \geq 1,$$

with $m = \sup_{n \in \mathbb{N}} \alpha \beta_{n} < 1$.

Then, denoting $B := A/(1 - m)$, $\theta := \alpha/(1 - m)$ and $\beta_{0} = 0$, we have that

$$a_{n} \leq Be^{\theta \sum_{i=0}^{n-1} \beta_{i}}, \ \forall n \geq 1.$$  

The variational scheme (2.8) will be the base for constructing approximate solutions for (1.1)-(1.2). The below lemma can be seen as a version of [2, Lemma 3.2.2] for the case of time-dependent functionals and gives a first set of estimates in order to control approximations.

Lemma 3.8. Assume E1 to E5. Let $\tau = \{0 = t_{0}^{0} < t_{1}^{1} < \cdots < t_{1}^{N} < \cdots \}$ be a partition of $[0, \infty)$, $\tau_{j} := t_{j}^{j} - t_{j-1}^{j-1}$, and $|\tau| = \sup_{j} |\tau_{j}|$. For $T > 0$ and $\tau^{*} < \frac{1}{\lambda_{T+1}}$, choose $N \in \mathbb{N}$ such that $T \in [t_{N-1}^{N}, t_{N}^{N})$. Suppose that there is a constant $S > 0$ satisfying

$$\mathcal{E}(0, U_{\tau}^{0}) \leq S \text{ and } d^{2}(u^{*}, U_{\tau}^{0}) \leq S. \quad (3.10)$$

Then, there exists a constant $C = C(S, T, \tau^{*}, \mathcal{E}) > 0$ such that

$$d^{2}(u^{*}, U_{\tau}^{n}) \leq C, \sum_{j=1}^{n} \frac{d^{2}(U_{j}^{j}, U_{j-1}^{j})}{2\tau_{j}} \leq \sum_{j=1}^{n} \left( \mathcal{E}(t_{j}^{j}, U_{j-1}^{j}) - \mathcal{E}(t_{j}^{j}, U_{j}^{j}) \right) \leq C, \quad (3.11)$$

for all $1 \leq n \leq N$ and $|\tau|$ sufficiently small.
Proof. By the minimizer property of $U^j_\tau$ and E3, we get
\[
\sum_{j=1}^n \frac{d^2(U^j_\tau, U^{j-1}_\tau)}{2\tau_j} \leq \sum_{j=1}^n \left( \mathcal{E}(t^j_\tau, U^{j-1}_\tau) - \mathcal{E}(t^j_\tau, U^j_\tau) \right)
\leq \mathcal{E}(0, U^0_\tau) - \mathcal{E}(t^n_\tau, U^n_\tau)
+ \sum_{j=1}^n \int_{t^j_\tau}^{t^n_\tau} \beta(r) \, dr (1 + d^2(u^*, U^{j-1}_\tau)),
\]
for $1 \leq n \leq N$. Using the first estimate in Lemma 3.3 with $u = U^{j-1}_\tau$, $u^{t+\tau}_\tau = U^j_* \tau$ and $\varepsilon = \frac{\tau^*}{2}$, we obtain
\[
\frac{1}{2} d^2(u^*, U^n_\tau) - \frac{1}{2} d^2(u^*, U^0_\tau) = \sum_{j=1}^n \frac{1}{2} d^2(u^*, U^j_\tau) - \frac{1}{2} d^2(u^*, U^{j-1}_\tau)
\leq \frac{\tau^*}{2} \left( \mathcal{E}(0, U^0_\tau) - \inf_{0 \leq t \leq T^*} \mathcal{E}_{t^*,\tau^*}(u^*) \right) + \frac{d^2(u^*, U^n_\tau)}{4}
+ \sum_{j=1}^n \frac{\tau^* d^2(U^j_/\tau, u^*)}{\tau^*} + \left( \frac{\tau^*}{2} \int_{t_\tau}^{t^n_\tau} \beta(r) \, dr \right) d^2(u^*, U^{j-1}_\tau)
+ \frac{\tau^*}{2} \int_0^{T^*+\tau^*} \beta(r) \, dr.
\]
Rearranging terms, it follows that
\[
d^2(u^*, U^n_\tau) \leq 2\tau^* \left( S - \inf_{0 \leq t \leq T^*} \mathcal{E}_{t^*,\tau^*}(u^*) \right) + 2 \left( 1 + \tau^* \int_0^{\tau^*} \beta(r) \, dr \right) S
+ 2\tau^* \int_0^{T^*+\tau^*} \beta(r) \, dr + 4 \sum_{j=1}^n \left( \frac{\tau^*}{\tau^*} + \frac{\tau^*}{2} \int_{t^j_\tau}^{t^{j+1}_\tau} \beta(r) \, dr \right) d^2(u^*, U^j_\tau)
\leq A(S, T, \tau^*, \mathcal{E}) + 4 \sum_{j=1}^n \beta_j d^2(u^*, U^j_\tau),
\]
for some constant $A = A(S, T, \tau^*, \mathcal{E}) > 0$, where $\beta_j := \frac{\tau^j_\tau}{\tau^*} + \frac{\tau^*}{2} \int_{t^j_\tau}^{t^{j+1}_\tau} \beta(r) \, dr$. By using an argument of absolute continuity, we have that $\max_{1 \leq n \leq N} 4\beta_j < 1$, for $|\tau|$ small enough. Then, the first estimate in (3.11) follows by using Lemma 3.7 in (3.13). For the second one, we use (3.12) and observe that
\[
\mathcal{E}(0, U^0_\tau) - \mathcal{E}(t^n_\tau, U^n_\tau) \leq S - \inf_{0 \leq t \leq T^*} \mathcal{E}_{t^*,\tau^*}(u^*) + \frac{d^2(u^*, U^n_\tau)}{2\tau^*},
\]
which is bounded. This concludes the proof. \qed
4 A priori estimates

It is well known that, under convexity hypotheses, the problem (1.1)-(1.2) admits a formulation based in a differential inequality. In fact, in the case when $X$ is a Euclidean space and the functional $E(t, \cdot)$ is $\lambda(t)$–convex, the curve solution $u(t)$ satisfies

$$\frac{1}{2} \frac{d}{dt} \|u(t) - v\|^2 + \frac{\lambda(t)}{2} \|u(t) - v\|^2 + E(t, u(t)) \leq E(t, v),$$

(4.1)

for all $v \in X$. Assuming the hypothesis of convexity $E_5$, one can derive a discrete version of (4.1). In fact, for each fixed $t > 0$, we have (see [2, Theorem 4.1.2])

$$\frac{1}{2\tau} d^2(u^t, v) = \frac{1}{2\tau} d^2(u, v) + \frac{\lambda(t)}{2} d^2(u^t, v) \leq E(t, v) - \mathcal{E}_{t\tau}(u).$$

(4.2)

Now we define a set of interpolating functions that will be useful in the convergence of approximate solutions. In comparison with [2], the time-dependence of $E$ generates new residual terms in the estimates and leads us to define the interpolations $T_\tau$ and $\lambda_\tau(t)$ in (4.3)-(4.4) below. The function $\lambda_\tau(t)$ is necessary in order to deal with the time-dependence on the parameter $\lambda$.

Let $\tau = \{0 = t^0_\tau < t^1_\tau < \cdots < t^n_\tau < \cdots\}$ be a partition of $[0, \infty)$ and $\tau_n := t^n_\tau - t^{n-1}_\tau$. Consider $T > 0$, $N \in \mathbb{N}$ such that $T \in (t^N_\tau - t^N_\tau]$, and the following functions defined on the interval $[0, T]$:

$$T_\tau(t) := t^n_\tau, \text{ for } t \in (t^{n-1}_\tau, t^n_\tau],$$

(4.3)

$$\lambda_\tau(t) := \lambda(t^n_\tau), \text{ for } t \in (t^{n-1}_\tau, t^n_\tau],$$

(4.4)

$$l_\tau(t) := \frac{t - t^{n-1}_\tau}{\tau_n}, \text{ for } t \in (t^{n-1}_\tau, t^n_\tau],$$

(4.5)

$$d^2_\tau(t, V) := (1 - l_\tau(t))d^2(U^{n-1}_\tau, V) + l_\tau(t)d^2(U^n_\tau, V), \text{ for } t \in (t^{n-1}_\tau, t^n_\tau],$$

(4.6)

$$E_\tau(t) := (1 - l_\tau(t))E(t^{n-1}_\tau, U^{n-1}_\tau) + l_\tau(t)E(t^n_\tau, U^n_\tau), \text{ for } t \in (t^{n-1}_\tau, t^n_\tau],$$

(4.7)

$$U_\tau(t) := U^{n-1}_\tau, \text{ for } t \in (t^{n-1}_\tau, t^n_\tau].$$

(4.8)

Also, we consider $\overline{U}_\tau(0) = U^n_\tau(0) := U^n_\tau$. The functions in (4.8) are called approximate solutions for (1.1) corresponding to the data $U^n_\tau$.

Taking $u^t = U^n_\tau$, $u = U^{n-1}_\tau$ and $v = V$, we can rewrite (4.2) as

$$\frac{1}{2} \frac{d}{dt} d^2_\tau(t, V) + \frac{\lambda_\tau(t)}{2} d^2_\tau(t, V) + E_\tau(t) - E(T_\tau(t), V) \leq$$

(4.9)

$$\frac{1}{2} \mathcal{R}_\tau(t) + (1 - l_\tau(t))(E(t^{n-1}_\tau, U^{n-1}_\tau) - E(t^n_\tau, U^n_\tau)),$$

for $t \in (t^{n-1}_\tau, t^n_\tau]$, where

$$\frac{1}{2} \mathcal{R}_\tau(t) := (1 - l_\tau(t))(E(t^n_\tau, U^n_\tau) - E(t^n_\tau, U^n_\tau)) - \frac{1}{2\tau_n} d^2(U^{n-1}_\tau, U^n_\tau).$$

(4.10)

With this notation, we have the next estimate.
Lemma 4.1. Assume EI to E5. For a partition $\tau$ with $|\tau| < \tau^*$, define the residual term
\[\mathcal{D}_\tau(t) := (1 - l_\tau(t))d(\overline{U}_\tau(t), \underline{U}_\tau(t)).\] (4.11)

We have that
\[
\frac{1}{2} \frac{d}{dt} d^2_\tau(t; V) + \frac{\lambda_\tau(t)}{2} d^2_\tau(t; V) - \left( \lambda^+_\tau(t) d(\overline{U}_\tau(t), \underline{U}_\tau(t)) + \lambda^-_\tau(t) \mathcal{D}_\tau(t) \right) d_\tau(t; V) \\
+ \mathcal{E}_\tau(t) - \mathcal{E}(\mathcal{T}_\tau(t), V) \leq \frac{1}{2} \mathcal{H}_\tau + \frac{\lambda_\tau(t)}{2} \mathcal{D}^2_\tau + (1 - l_\tau)(\mathcal{E}(t^{-1}_\tau, \overline{U}_\tau) - \mathcal{E}(t^1_\tau, \underline{U}_\tau)), \quad (4.12)
\]
for all $V \in D$ and almost every point $t \in [0, T]$.

Proof. A detailed proof for the case $\lambda_\tau(t) < 0$ can be found in [2, pg.88]. Let us explicit the proof for $\lambda_\tau(t) > 0$. For that, we can suppose that
\[d(U^n_\tau, V) < d(U^{n-1}_\tau, V),\]
and estimate
\[d^2(\overline{U}_\tau(t), V) - d^2_\tau(t; V) = (1 - l_\tau) \left( d^2(\overline{U}_\tau(t), V) - d^2(\underline{U}_\tau(t), V) \right).\]
Thus, we have
\[
d^2(\overline{U}_\tau(t), V) - d^2_\tau(t; V) \geq -d(U^n_\tau, U^{n-1}_\tau)(d(U^n_\tau, V) + d(U^{n-1}_\tau, V)) \\
+ l_\tau d(U^n_\tau, U^{n-1}_\tau)(d(U^{n-1}_\tau, V) - d(U^n_\tau, V)) \\
= -d(U^n_\tau, U^{n-1}_\tau) \left( (1 - l_\tau)d(U^{n-1}_\tau, V) \\
+ l_\tau d(U^n_\tau, V) \right) \\
\geq -2d(U^n_\tau, U^{n-1}_\tau) \left( (1 - l_\tau)d(U^{n-1}_\tau, V) + l_\tau d(U^n_\tau, V) \right) \\
\geq -2d(U^n_\tau, U^{n-1}_\tau) d_\tau(t; V),
\]
which together with (4.9) gives the desired result. \(\square\)

The next result is a slightly modified version of the Gronwall Lemma in [2, Lemma 4.1.8]. The proof is the same and we omit it.

Lemma 4.2. Let $x : [0, \infty) \to \mathbb{R}$ be a locally absolutely continuous function and let $a, b, \tilde{\lambda} \in L^1_{loc}([0, \infty))$ be such that
\[
\frac{d}{dt} x^2(t) + 2\tilde{\lambda}(t)x^2(t) \leq a(t) + 2b(t)x(t) \text{ a.e. } t \geq 0. \quad (4.13)
\]
For $T > 0$, we have that
\[
e^{\alpha(T)}|x(T)| \leq \sqrt{\left( x^2(0) + \sup_{t \in [0,T]} \int_0^t e^{2\alpha(s)}a(s) \, ds \right)^++ 2 \int_0^T e^{\alpha(s)}|b(t)| \, dt},
\]
where $\alpha(t) = \int_0^t \tilde{\lambda}(s) \, ds$. 

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4.1 More two interpolation terms

In this subsection we consider two interpolation functions that depend on two partitions \(\tau\) and \(\eta\) of \([0, \infty)\) with \(|\tau|, |\eta| < \tau^*\). So far, we have define two residual terms \(R_\tau\) and \(R_\eta\) in (4.10) and (4.11), respectively. Another one that we will work with is

\[
G_{\tau\eta}(t) := 2(1 - l_\tau(t)) \left[ E(T_\eta(t), U_\tau(t)) - E(T_\tau(t), U_\eta(t)) \right] \\
+ 2l_\tau(t) \left[ E(T_\eta(t), U_\tau(t)) - E(T_\tau(t), U_\eta(t)) \right], \quad \text{for } t \in [0, T]. \tag{4.14}
\]

Define also the interpolation function \(d^2_{\tau\eta}(t, s)\) as

\[
d^2_{\tau\eta}(t, s) = (1 - l_\eta(s))d^2_{\tau}(t, U_\eta(s)) + l_\eta(s)d^2_{\tau}(t, U_\eta(s)).
\]

Taking in (4.12) a convex combination, with coefficients \((1 - l_\eta(t))\) and \(l_\eta(t)\) for \(V = U_\eta(t)\) and \(V = U_\eta(t)\) respectively, we arrive at

\[
\frac{d}{dt}d^2_{\tau\eta}(t, t) + (\lambda_\tau + \lambda_\eta)d^2_{\tau\eta}(t, t) \leq 2 \left[ \lambda_\tau^+ d(U_\tau(t), U_\eta(t)) + \lambda_\eta^+ d(U_\eta(t), U_\eta(t)) \right] \\
+ \lambda_\tau^- (t) R_\tau(t) + \lambda_\eta^- (t) R_\eta(t) + G_{\tau\eta}(t) + G_{\eta\eta}(t).
\]

Now we can use Lemma 4.2 in the last inequality in order to estimate

\[
e^{\alpha_{\tau\eta}(t)}d_{\tau\eta}(t, t) \leq \left( d^2(U_0^\tau, U_0^\eta) + \int_0^t e^{2\alpha_{\eta\eta}(s)} \sum_{\theta \in \{\tau, \eta\}} \left( R_\theta^+(s) + \lambda_\theta^-(s) D_\theta^2(s) \right) ds \right. \\
+ \left. \int_0^t e^{2\alpha_{\tau\eta}(s)} \left( G_{\eta\eta}^+(s) + G_{\tau\eta}^+(s) \right) ds \right)^{1/2} \\
+ \int_0^t e^{\alpha_{\tau\eta}(s)} \left( \lambda_\tau^+(s) d(U_\tau(s), U_\eta(s)) + \lambda_\eta^+(s) d(U_\eta(s), U_\eta(s)) \right. \\
+ \lambda_\tau^-(s) R_\tau(s) + \lambda_\eta^-(s) R_\eta(s) \left. \right) ds, \tag{4.15}
\]

for all \(t \geq 0\), where \(\alpha_{\tau\eta}(t) := \int_0^t \lambda_\tau(s) + \lambda_\eta(s) \, ds\).

4.2 Convergence of the approximate solutions

In this section, we deal with the convergence of the approximate solutions \(U_\tau\) and \(U_\eta\). Using the minimizer property of \(U_j^\tau\) and direct calculations, one can obtain

\[
\int_0^t e^{2\alpha_{\eta\eta}(s)}(R_\tau^+(s) + \lambda_\tau^-(s) D_\tau^2(s)) \, ds \leq C |\tau| \sum_{j=1}^n \left( E(t_j^\tau, U_{j-1}^\tau) - E(t_j^\tau, U_{j}^\tau) \right). \tag{4.16}
\]
\[ \left( \int_0^t e^{\alpha \tau} \mathcal{X}_{\tau}^-(s) \mathcal{D}_{\tau}(s) \, ds \right)^2 \leq C|\tau|^2 \sum_{j=1}^{n} (\mathcal{E}(t_{j+1}, U_{j+1}) - \mathcal{E}(t_j, U_j)) \]  

(4.17)

\[ \left( \int_0^t e^{\alpha \tau} \mathcal{X}_{\tau}^+(s) d(\mathcal{U}_{\tau}(s), \mathcal{V}_{\tau}(s)) \, ds \right)^2 \leq C(T)|\tau|^2 \sum_{j=1}^{n} (\mathcal{E}(t_{j+1}, U_{j+1}) - \mathcal{E}(t_j, U_j)), \]

(4.18)

for \( T \in (t_{N-1}^+, t_N^+), 0 \leq t \leq T, \) and \( n \leq N. \)

The above estimates give some control on the residual terms \( \mathcal{R}_{\tau} \) and \( \mathcal{D}_{\tau}. \) Next, we provide an explicit estimate for the residual term (4.14). This could be useful to obtain convergence rates of approximate solutions to the gradient flow solutions. Recall the standard notations \( a \land b = \min\{a, b\} \) and \( a \lor b = \max\{a, b\}. \)

**Proposition 4.3.** Assume E1 to E5 and the boundedness condition (3.10). Let \( \tau, \eta \) be two partitions of \([0, +\infty)\) with \(|\tau|, |\eta|\) small enough as in Lemma 3.8. For \( T > 0, \) choose \( N, K \in \mathbb{N}, \) such that \( T \in (t_{N-1}^+, t_1^+) \cap (t_{K-1}^+, t_K^+). \) There is a constant \( C = C(T, S, \tau^*, \mathcal{E}) > 0 \) such that

\[ \int_0^T G_{\eta}(t) \, dt \leq C(|\tau| + |\eta|). \]  

(4.19)

**Proof.** Denote \( I_{\eta}^n = (t_{n-1}^+, t_n^+) \). For \( t_1^+, \) let \( k_1 \) be the greatest integer satisfying \( t_{k_1-1}^+ < t_1^+. \) If \( t_{n-1}^+ = t_1^+ \) define \( J_1 = I_{\eta}^1 \). Otherwise, choose \( n_1 \leq N \) as the greatest integer with the property \( t_{n_1}^+ < t_{k_1}^+ \) and define \( J_1 = I_{\eta}^1 \lor \cdots \lor I_{\eta}^{n_1}. \) In both cases, we have \( \mathcal{T}_\eta(t_{\tau}^{n_1}) = t_{k_1}^+ \) and then

\[ \int_{J_\tau^1} (1 - l_\tau(t)) \int_{\mathcal{T}_\tau(t) \lor \mathcal{T}_\eta(t)} \beta(s) \, ds \, dt \leq \int_{J_\tau^1} (1 - l_\tau(t)) \int_0^{\mathcal{T}_\tau(t_{\tau}^{n_1}) \lor \mathcal{T}_\eta(t_{\tau}^{n_1})} \beta(s) \, ds \leq (|\tau| + |\eta|) \int_0^{t_{k_1}^+} \beta(s) \, ds. \]

If \( t_{n_1+1}^+ = t_{k_1}^+ \), define \( J_2^* = I_{\eta}^{n_1+1} \). Otherwise, take the greatest integer \( k_2 \in \mathbb{N} \) such that \( t_{k_2-1}^+ < t_{n_1+1}^+. \) In the case \( t_{n_2}^+ = t_{k_2}^+ \), define \( J_2 = I_{\eta}^{n_1+1} \lor I_{\eta}^{n_2}. \) Otherwise, take the greatest integer \( n_2 \leq N \) such that \( t_{n_2}^+ < t_{k_2}^+ \) and define \( J_2^* = I_{\eta}^{n_1+1} \lor \cdots \lor I_{\eta}^{n_2}. \) Noting that \( \mathcal{T}_\eta(t_{\tau}^{n_2}) = t_{k_2}^+ \) and \( \mathcal{T}_\tau(t) \geq t_{k_2}^+ + 1, \) we get

\[ \int_{J_\tau^2} (1 - l_\tau(t)) \int_{\mathcal{T}_\tau(t) \lor \mathcal{T}_\eta(t)} \beta(s) \, ds \, dt \leq \int_{J_\tau^2} (1 - l_\tau(t)) \int_0^{\mathcal{T}_\tau(t_{\tau}^{n_2}) \lor \mathcal{T}_\eta(t_{\tau}^{n_2})} \beta(s) \, ds \leq (|\tau| + |\eta|) \int_0^{t_{k_2}^+} \beta(s) \, ds. \]

Proceeding inductively, and adding estimates obtained in the process, we arrive at

\[ \int_0^T (1 - l_\tau(t)) \int_{\mathcal{T}_\tau(t) \lor \mathcal{T}_\eta(t)} \beta(s) \, ds \, dt \leq (|\tau| + |\eta|) \int_0^{T+\tau^*} \beta(s) \, ds. \]  

(4.20)
Analogously,
\[
\int_0^T l_\tau(t) \int_{T_\tau(t) \cap T_\eta(t)} \beta(s) \, ds \, dt \leq (|\tau| + |\eta|) \int_0^{T+T^*} \beta(s) \, ds.
\] (4.21)

Adding (4.20) and (4.21), we get
\[
\int_0^T \int_{T_\tau(t) \cup T_\eta(t)} \beta(s) \, ds \, dt \leq 2(|\tau| + |\eta|) \int_0^{T+T^*} \beta(s) \, ds.
\] (4.22)

Now, recalling (4.11) and the property \( E_5 \), and using the first estimate in (3.11), for \( t \in [0, T] \) it follows that
\[
G^+_{\tau, \eta}(t) \leq 2(1 - l_\tau(t)) \int_{T_\tau(t) \cap T_\eta(t)} \beta(s) \, ds (1 + d^2(u^*, \overline{U}_\tau)(t))
\] 
\[+ 2l_\tau(t) \int_{T_\tau(t) \cap T_\eta(t)} \beta(s) \, ds (1 + d^2(u^*, \overline{U}_\tau)(t)) \]
\[\leq C(S, T, \tau^*, \xi) \int_{T_\tau(t) \cap T_\eta(t)} \beta(s) \, ds. \] (4.23)

Finally, we conclude by integrating (4.23) over \([0, T]\) and using (4.22).

In the present section and in Section 3, we have obtained some properties and estimates for \( \mathcal{E}(t, u) \) and the implicit variational scheme (2.8) associated to the problem (1.1)-(1.2). After doing that, we are in position for proceeding as in [2, pag. 91-92] and showing that the approximate solutions (4.8) converge uniformly in \([0, T]\) as \( |\tau| \to 0 \).

**Theorem 4.4.** Assume \( E_1 \) to \( E_5 \) and the condition
\[
\lim_{|\tau| \to 0} d(U^0_{U, u_0}) = 0, \quad \sup_{\tau} \mathcal{E}(0, U^0_{U, \tau}) = S < \infty,
\] (4.24)

for some \( u_0 \in \mathcal{D} \). Then, the approximate solutions \( \overline{U}_\tau \) and \( \underline{U}_\tau \) converge locally uniformly to a function \( u : [0, \infty) \to X \) satisfying \( u(0) = u_0 \). Moreover, \( u \) is independent of the family \( U^0_{U, \tau} \).

**Remark 4.5.** In fact, the convergence of the approximate solutions is valid for \( u_0 \in \mathcal{D} \).

**Proof of Theorem 4.4.** The proof follows essentially the same arguments in [2] by taking care of the time-dependence. We give some steps for the reader convenience. By taking a suitable convex combination, we arrive at
\[
d^2(\overline{U}_\tau(t), \overline{U}_\eta(t)) \leq 3d^2_{\tau, \eta}(t, t) + 3C(|\tau| + |\eta|).
\]
So, using (4.16), (4.17), (4.18) joint with Lemma 3.8, and the estimate (4.15), we obtain

\[ d_{\tau,\eta}(t, t) \leq \left( d^2(0, U_\eta) + C(\|\tau\| + \|\eta\|) + \int_0^t e^{2\alpha_{\tau,\eta}(t)}(G_{\tau,\eta}^+(t) + G_{\eta,\tau}^+(t)) \, dt \right)^{1/2} 
+ C(\|\tau\| + \|\eta\|). \]

We conclude the convergence by using Proposition 4.3 and the completeness of the space X. \qed

5 Regularity

In this section we show that the function \( u \) obtained in Theorem 4.4 is, in fact, a solution for (1.1)-(1.2) in the sense of Definition 2.3. For that matter, we need to show some regularity properties for \( u \). We begin by recalling the De Giorgi interpolation.

**Definition 5.1.** Let \( (U_n^\tau) \) be a solution for the variational scheme (2.8), defined for \( t_n^\tau \leq t \leq t_n^\tau + \sigma \). Define the De Giorgi interpolation

\[ \widetilde{U}_\tau(t) = \widetilde{U}_\tau(t_n^\tau - 1 + \delta), \quad \text{for} \ t \in (t_n^\tau - 1, t_n^\tau), \text{ and } \delta = t - t_n^\tau, \]

as the unique minimizer of the functional \( v \in X \rightarrow \mathcal{E}(t_n^\tau - 1 + \delta, \delta, U_{n-1}^\tau, v) \).

We have that the De Giorgi interpolation also converges locally uniformly to the same function \( u \) in Theorem 4.4.

**Proposition 5.2.** Assume the same hypotheses of Theorem 4.4. There is a constant \( C > 0 \) independent of \( \tau \) such that

\[ d^2(U_\tau(t), \widetilde{U}_\tau(t)) \leq \|\tau\| C \left( 1 + \frac{t - t_n^\tau}{T_{\tau}(t) - t} \int_{t}^{T_{\tau}(t)} \beta(r) \, dr \right), \quad (5.1) \]

for all \( t \in (t_n^\tau - 1, t_n^\tau) \). Thus, \( \widetilde{U}_\tau \) converges to the function \( u \) given in Theorem 4.4 a.e. in \([0, T]\). Moreover, the convergence is uniform provided that the function \( \beta \) in \( \mathbf{E3} \) belongs to \( L^\infty_{loc}([0, \infty)) \).

**Proof.** Let \( N \in \mathbb{N} \) be such that \( T \in (t_N^\tau - 1, t_N^\tau) \). First, we will show that the discrete solution \( (U_n^\tau)_{n=0}^N \) satisfies the inequality

\[ \mathcal{E}(t_n^\tau, U_n^\tau) \leq \mathcal{E}(0, 0) + C, \]

for some constant \( C \) independent of \( \tau \). In fact, by using \( \mathbf{E3} \) and the minimizer property (2.8) of \( U_n^\tau \), we obtain

\[ \mathcal{E}(t_n^\tau, U_n^\tau) \leq \mathcal{E}(t_n^\tau - 1, U_{n-1}^\tau) + \int_{t_n^\tau - 1}^{t_n^\tau} \beta(r) \, dr \left( 1 + d(u^*, U_{n-1}^\tau) \right). \]
Recall that \( d(u^*, U^n_{\tau}) \) is bounded by a constant \( C \) that depends on \( T \) and is independent of \( \tau \). Proceeding inductively, it follows that

\[
\mathcal{E}(t^n_{\tau}, U^n_{\tau}) \leq \mathcal{E}(0, U^0_{\tau}) + C \int_0^{T+\tau^*} \beta(r) \, dr,
\]

from where we get (5.2). Now, estimate (3.2) in Lemma 3.3 and (5.2) give

\[
d(\tilde{U}_{\tau}(t), u^*) \leq 4\tau^* \left( \mathcal{E}(t^n_{\tau-1}, U^n_{\tau}(t)) + C(1 + d^2(u^*, U^n_{\tau}(t))) \right) \leq 4\tau^* \mathcal{E}(0, U^0_{\tau}) + C,
\]

for \( t \in (t^n_{\tau-1}, t^n_{\tau}] \). Taking \( \delta = t - t^n_{\tau-1} \) for \( t \in (t^n_{\tau-1}, t^n_{\tau}] \), we can estimate

\[
\frac{d^2(U_{\tau}(t), \tilde{U}_{\tau}(t))}{2\delta} + \mathcal{E}(t, \tilde{U}_{\tau}(t)) \leq \frac{d^2(U_{\tau}(t), U_{\tau}(t))}{2\delta} + \mathcal{E}(t, U_{\tau}(t)) \leq \frac{d^2(U_{\tau}(t), \tilde{U}_{\tau}(t))}{2\tau_n} + \mathcal{E}(t^n_{\tau}, \tilde{U}_{\tau}(t)) + \mathcal{E}(t, U_{\tau}(t)) - \mathcal{E}(t^n_{\tau}, U_{\tau}(t)) + \left( \frac{1}{2\delta} - \frac{1}{2\tau_n} \right) d^2(U_{\tau}(t), U_{\tau}(t)).
\]

Rearranging terms and using E3, it follows that

\[
\left( \frac{1}{2\delta} - \frac{1}{2\tau_n} \right) d^2(U_{\tau}(t), \tilde{U}_{\tau}(t)) \leq \left( \frac{1}{2\delta} - \frac{1}{2\tau_n} \right) d^2(U_{\tau}(t), U_{\tau}(t)) + \int_t^{t^n_{\tau}} \beta(r) \, dr \times \left( 2 + d^2(u^*, U_{\tau}(t)) + d^2(u^*, \tilde{U}_{\tau}(t)) \right).
\]

Recalling that \( \mathcal{E}(0, U^0_{\tau}) \leq S \) and using (5.2), we obtain (5.1) and then the convergence of \( \tilde{U}_{\tau}(t) \) to \( u(t) \) in the set of Lebesgue points of \( \beta \).

Before proceeding, let us recall a well-known estimate for the slope \( |\partial \mathcal{E}(t)| \). Recall that \( u^{t+\tau}_t \) stands for the minimizer of \( \mathcal{E}(t + \tau, \tau, u; \cdot) \). Then \( u^{t+\tau}_t \in \text{Dom}(\partial \mathcal{E}(t + \tau)) \) and

\[
|\partial \mathcal{E}(t + \tau)|(u^{t+\tau}_t) \leq \frac{d(u, u^{t+\tau}_t)}{\tau}. \tag{5.3}
\]

Under the convexity hypothesis E5, we have that the local slope \( |\partial \mathcal{E}(t)| \) is lower semicontinuous and

\[
|\partial \mathcal{E}(t)|(u) = \sup_{v \neq u} \left( \frac{\mathcal{E}(t, u) - \mathcal{E}(t, v)}{d(u, v)} + \frac{1}{2} \lambda(t)d(u, v) \right) + . \tag{5.4}
\]

The next lemma will be useful to show \( W^{1,1}_{loc} \)-regularity for functions with a certain type of control in their variations.
Lemma 5.3. Let $T > 0$ and $f, g, \beta \in L^1([0, T])$ be such that
\[
|f(t) - f(s)| \leq (g(t) + g(s))|t - s| + \int_s^t \beta(r) \, dr,
\]
for $s < t$. Then $f \in W^{1,1}([h, T - h])$, for all $0 < h < T/2$.

Proof. Since the function $t \to \int_0^t \beta(r) \, dr$ belongs to $W^{1,1}([0, T])$, we have the difference quotient property
\[
\sup_{0 < |\hat{h}| < h} \int_h^{T-h} \left| \frac{1}{\hat{h}} \int_t^{t+\hat{h}} \beta(r) \, dr \right| \, dt < \infty. \tag{5.5}
\]
Using the notation
\[
\Delta_{\hat{h}}(f)(t) = \frac{f(t + \hat{h}) - f(t)}{\hat{h}},
\]
we obtain
\[
\int_h^{T-h} |\Delta_{\hat{h}}(f)(t)| \, dt \leq \int_h^{T-h} g(t) + g(t + \hat{h}) + \left| \frac{1}{\hat{h}} \int_t^{t+\hat{h}} \beta(r) \, dr \right| \, dt \leq 2\|g\|_{L^1} + \int_h^{T-h} \left| \frac{1}{\hat{h}} \int_t^{t+\hat{h}} \beta(r) \, dr \right| \, dt,
\]
which gives the desired regularity by employing a difference quotient argument. $\square$

Now we are ready to show that the limit $u$ in Theorem 4.4 is a time-dependent gradient flow in the sense of Definition 2.3.

Theorem 5.4. Assume $E1$ to $E5$. The limit $u : [0, \infty) \to X$ in Theorem 4.4 is locally absolutely continuous and its metric derivative $|u'|$ belongs to $L^2_{\text{loc}}([0, \infty))$. Moreover, if the function $t \to \mathcal{E}(t, u)$ is differentiable for $u \in D$, its time-derivative is upper semicontinuous in the $u$-variable (with respect to the metric), and the property
\[
\liminf_{n \to \infty} \frac{\mathcal{E}(t_n, u_n) - \mathcal{E}(t, u_n)}{t_n - t} \geq \partial_t \mathcal{E}(t, u) \tag{5.6}
\]
holds true, then the function $t \to \mathcal{E}(t, u(t))$ is absolutely continuous and satisfies the identity
\[
\mathcal{E}(t, u(t)) - \mathcal{E}(0, u(0)) = \int_0^t \partial_t \mathcal{E}(s, u(s)) \, ds - \frac{1}{2} \int_0^t |u'|^2(s) \, ds - \frac{1}{2} \int_0^t |\partial \mathcal{E}(s)|^2(u(s)) \, ds. \tag{5.7}
\]
In particular, $u$ is a solution for (1.1)-(1.2).

Remark 5.5. Definition 2.3 does not contain (5.6). Note also that this assumption is used to prove (5.7) and, in fact, is not necessary to obtain the absolute continuity of $t \to \mathcal{E}(t, u(t))$. 21
Proof of Theorem 5.4. Let $T > 0$ and denote by
\[ |U_\tau'(t)| = \frac{d(U_{\tau}^n, U_{\tau})}{\tau_n} \] (5.8)
the discrete derivative of $U_\tau(t)$ in each interval $(t_{\tau}^{n-1}, t_{\tau}^n]$. By Lemma 3.8, we have that
\[ \int_0^t |U_\tau'(s)|^2 \, ds \leq C, \] (5.9)
for each $t \in [0, T]$. Thus, we can extract a sequence $\tau_k$ such that $|\tau_k| \to 0$ and $|U_\tau'|$ converges weakly in $L^2([0, T])$ for some function $m$. Fix $0 \leq s < t \leq T$ and choose $p = p(s)$ and $n = n(s) \in \mathbb{N}$ with $s \in (t_{\tau_k}^{p-1}, t_{\tau_k}^p]$ and $t \in (t_{\tau_k}^{n-1}, t_{\tau_k}^n]$. It follows from (5.8) and triangular inequality that
\[ d(U_{\tau_k}(s), U_{\tau_k}(t)) \leq \int_{t_{\tau_k}^{p-1}}^{t_{\tau_k}^p} |U_\tau'|(r) \, dr. \]
Letting $k \to +\infty$, and using the weak convergence, we conclude that $u$ is absolutely continuous and $|u'| \leq m$. Also, after a change of variables, we can employ the identity (3.8) to obtain
\[ \frac{1}{2} \int_0^{t_{\tau_k}^p} |U_\tau'|^2(r) \, dr + \int_0^{t_{\tau_k}^p} \frac{d^2(U_{\tau_k}(r), \bar{U}_{\tau_k}(r))}{2r^2} \, dr = \int_0^{t_{\tau_k}^p} \partial_t \mathcal{E}(r, \bar{U}_{\tau_k}(r)) \, dr + \mathcal{E}(0, U_{\tau_k}^0) - \mathcal{E}(t_{\tau_k}^n, U_{\tau_k}^n). \] (5.10)
For the above subsequence, we have
\[ \mathcal{E}(t, u(t)) \leq \liminf_{k \to \infty} \mathcal{E}(t, U_{\tau_k}(t)) = \liminf_{k \to \infty} \mathcal{E}(\tau_k(t), U_{\tau_k}(t)), \]
and so, using (5.3) and (5.4), we arrive at
\[ \frac{1}{2} \int_0^t |u'|^2(r) \, dr + \frac{1}{2} \int_0^t |\partial_t \mathcal{E}(r, u(r))|^2 \, dr + \mathcal{E}(t, u(t)) \leq \liminf_{k \to \infty} \left( \frac{1}{2} \int_0^{\tau_k(t)} |U_\tau'|^2(r) \, dr + \int_0^{\tau_k(t)} \frac{d^2(U_{\tau_k}(r), \bar{U}_{\tau_k}(r))}{2r^2} \, dr \right. \]
\[ \left. + \mathcal{E}(\tau_k(t), U_{\tau_k}(t)) \right) \leq \limsup_{k \to \infty} \int_0^{\tau_k(t)} \partial_t \mathcal{E}(r, \bar{U}_{\tau_k}(r)) \, dr + \mathcal{E}(0, u(0)) \]
\[ \leq \int_0^t \partial_t \mathcal{E}(r, u(r)) \, dr + \mathcal{E}(0, u_0), \] (5.11)
where, by convenience, we have chosen $U_{\tau}^0 = u_0$ (recall that $u$ does not depend on $U_{\tau}^0 \to u_0$). Notice that in particular $\sup_{t \in [0, T]} \mathcal{E}(t, u(t)) < \infty$. On the other hand, in view of [2, Lemma
there exist an increasing absolutely continuous function \( s : [0, T] \to [0, L] \), whose inverse \( t \) is Lipschitz, and a curve \( \hat{u} : [0, L] \to X \) such that \(|\hat{u}'(s)| \leq 1 \) and \( u(t) = \hat{u}(s(t)) \).

Considering the function \( \varphi(s) = \mathcal{E}(t(s), \hat{u}(s)) \) and using (5.4), it follows that

\[
\varphi(s_1) - \varphi(s_2) \leq (|\partial \mathcal{E}(t(s_1))|(|\hat{u}(s_1)| + \lambda_T C) |s_2 - s_1| \\
+ (1 + C^2) \int_{s_1}^{s_2} \beta(t(s)) \, dt \, ds,
\]

for \( s_1 < s_2 \), where \( C = \sup_{s \in [0, L]} d(u^*, \hat{u}(s)) \). Replacing the roles of \( s_1 \) and \( s_2 \), we obtain

\[
|\varphi(s_1) - \varphi(s_2)| \leq (|\partial \mathcal{E}(t(s_1))|(|\hat{u}(s_1)| + |\partial \mathcal{E}(t(s_2))|(|\hat{u}(s_2)| + 2\lambda_T C)|s_2 - s_1| \\
+ (1 + C^2) \int_{s_1}^{s_2} \beta(t(s)) \, dt \, ds.
\]

By using Lemma 5.3, we can conclude that \( \varphi \) is absolutely continuous and then \( \mathcal{E}(t, u(t)) \) also does so. It follows that \( \mathcal{E}(t, u(t)) \) is derivable at almost every point \( t \in [0, T] \). Let \( t_0 \in [0, T] \) be a differentiability point of \( \mathcal{E}(t, u(t)) \) for which the metric derivative \( |u'(t_0)| \) exists. Taking \( t_n \downarrow t_0 \), we get

\[
\liminf_{n \to \infty} \frac{\mathcal{E}(t_n, u(t_n)) - \mathcal{E}(t_0, u(t_0))}{t_n - t_0} \geq \frac{\mathcal{E}(t_0, u(t_0)) - \mathcal{E}(t_0, u(t_0))}{t_n - t_0} \\
= \partial_t \mathcal{E}(t_0, u(t_0)) - |\partial \mathcal{E}(t_0)| |u(t_0)| |u'(t_0)|.
\]

Integrating the above inequality, and using (5.11), we obtain (5.7).

**Corollary 5.6.** Under the hypotheses of Theorem 4.4. There exists a subsequence of partitions \( \tau_k \) such that \( \mathcal{E}_{\tau_k}(t) \) defined in (4.7) converges to \( t \to \mathcal{E}(t, u(t)) \) in \( L^1_{\text{loc}}([0, \infty)) \), and therefore a.e. in \( [0, \infty) \) (up to a subsequence), where \( u \) is as in Theorem 4.4.

**Proof.** We only need to show that, for \( T > 0 \), the functions \( f_\tau \) and \( g_\tau \) defined as

\[
f_\tau(t) := \mathcal{E}(t^n_\tau, U^n_\tau), \quad \text{for} \ t \in (t_{\tau}^{n-1}, t^n_\tau],
\]

and

\[
g_\tau(t) := \mathcal{E}(t^{n-1}_\tau, U^{n-1}_\tau), \quad \text{for} \ t \in (t^n_{\tau-1}, t^n_\tau],
\]

converge to \( t \to \mathcal{E}(t, u(t)) \) in \( L^1([0, T]) \), as \( \tau \to 0 \). First, note that for each partition \( \{0 = t_0 < t_1 < \cdots < t_L = T\} \) of \([0, T]\), we can bound the variation of \( f_\tau \) as

\[
\sum_{l=1}^{L} |f_\tau(t_l) - f_\tau(t_{l-1})| \leq \sum_{n=1}^{N} (|\mathcal{E}(t^n_\tau, U^{n-1}_\tau) - \mathcal{E}(t^n_\tau, U^n_\tau)| \\
+ C \int_0^{t^*_{\tau+1}} \beta(s) \, ds),
\]

(5.12)
where $C > 0$ is a constant independent of $\tau$. By Lemma 3.8, the summation in the right hand side of (5.13) is bounded, and therefore the total variation of $f_\tau$ in $[0, T]$ is uniformly bounded. Analogously, the total variation of $g_\tau$ in $[0, T]$ is uniformly bounded. It follows from [10, Chap. 5, Theorem 4] that there exist a subsequence $\tau_k$ and functions $A, B \in L^1([0, T])$ such that $f_{\tau_k} \to A$ and $g_{\tau_k} \to B$ in $L^1([0, T])$, as $k \to \infty$. Also, it is not hard to show that $A = B \geq \mathcal{E}(t, u(t))$ a.e in $[0, T]$. Now, the same argument used in the proof of (5.7) can be used in order to show the equality $A = \mathcal{E}(t, u(t))$ a.e. $t \in [0, T]$. In fact, if $f_{\tau_k}(t) \to A$ in $L^1([0, T])$, then

\[
\frac{1}{2} \int_0^t |u'(r)|^2 \, dr + \frac{1}{2} \int_0^t |\partial \mathcal{E}(r)|^2(u(r)) \, dr + A(t)
\]

\[
\leq \liminf_{k \to \infty} \left( \frac{1}{2} \int_0^{\tau_k(t)} |U_{\tau_k}'|^2 \, dr + \int_0^{\tau_k(t)} \frac{d^2(U_{\tau_k}, \tilde{U}_{\tau_k})}{2r^2} \, dr + \mathcal{E}(\tau_k(t), \tilde{U}_{\tau_k}) \right)
\]

\[
\leq \int_0^t \partial \mathcal{E}(r, u(r)) \, dr + \mathcal{E}(0, u_0),
\]

and, using (5.7), we are done. \hfill \Box

**Remark 5.7.** As a consequence, we have that the solution $u(t) \in \text{Dom}(|\partial \mathcal{E}(t)|)$ for almost every point $t \in (0, \infty)$.

### 5.1 Contraction property

Consider the condition

**E6.** The function $\lambda(t)$ is continuous.

Having at hand the estimates obtained in previous sections, the contraction property holds if we assume E6. Here we only sketch its proof for the reader convenience.

For $\lambda(t)$ continuous, the interpolation $\lambda_{\tau}$ defined in (4.4) converges uniformly to $\lambda$, as $|\tau| \to 0$, in each bounded interval of $[0, \infty)$. Recall the following technical lemma [25, Lemma 23.28].

**Lemma 5.8.** Let $F = F(t, s)$ be a function $[0, \infty) \times [0, \infty) \to \mathbb{R}$ locally absolutely continuous in the variable $t$ and uniformly continuous in $s$, and locally absolutely continuous in $s$ and uniformly in $t$; that is, there exists a nonnegative $m \in L^1_{\text{loc}}([0, \infty))$ such that

\[
|F(t, s) - F(t', s)| \leq \int_t^{t'} m(r) \, dr \quad \text{and} \quad |F(t, s) - F(t, s')| \leq \int_s^{s'} m(r) \, dr,
\]

where $m$ does not depend on $s$ in the first inequality and on $t$ in the second one. Then, the function $\delta(t) := F(t, t)$ is locally absolutely continuous and, for almost every point $t_0 \in [0, \infty)$, we have

\[
\frac{d}{dt} \delta(t) \leq \limsup_{t \to t_0} \left( \frac{F(t_0, t) - \delta(t_0)}{t - t_0} \right) + \limsup_{t \to t_0} \left( \frac{F(t, t_0) - \delta(t_0)}{t - t_0} \right). \tag{5.14}
\]
Integrating (4.9) from $s$ to $t$ with $0 \leq s < t \leq T$ and taking the subsequence $\tau_k$ given in Corollary 5.6, we can pass the limit and use $E_6$ in order to obtain the inequality
\[
\frac{1}{2}d^2(u(t), V) - \frac{1}{2}d^2(u(s), V) + \int_s^t \frac{\lambda(r)}{2}d^2(u(r), V) + E(r, u(r)) \, dr \leq \int_s^t E(r, V) \, dr.
\] (5.15)

Let $u, v$ be two solutions given by Theorem 4.4 with initial data $u_0, v_0 \in D$, respectively. Recall that, by Lemma 3.8, both curves $u$ and $v$ are locally bounded. Also,
\[
|d^2(u(t), v(s)) - d^2(u(t'), v(s))| \leq d(u(t), u(t'))(d(u(t), v(s)) + d(u(t'), v(s)))
\leq C(T) \int_t^t |u'(r)| \, dr.
\]

where $C(T) = 2 \sup_{0 \leq t \leq T} (d(u(t), u^*) + d(v(t), u^*))$. Similarly, one can show the local absolute continuity in the variable $s$ for the function $F(t, s) = d^2(u(t), v(s))$. It follows that $d^2(u(t), v(s))$ verifies the hypotheses in Lemma 5.8. Next, using (5.15), a direct computation gives
\[
\frac{d}{dt}d^2(u(t), v(t)) + 2\lambda(t)d^2(u(t), v(t)) \leq 0,
\] (5.16)
for almost every point $t \in [0, \infty)$, which implies
\[
d(u(t), v(t)) \leq e^{-\int_0^t \lambda(s) \, ds}d(u_0, v_0).
\] (5.17)

Remark 5.9. The time-dependent functional $E$ can be “weakly” convex ($\lambda(t) < 0$) at a certain $t = t_0$. In fact, we could have $\int_0^{t_0} \lambda(s) \, ds < 0$ and solutions distance themselves. However, according to the behavior of $\lambda(t)$, the convexity could be improved ($\lambda(t) > 0$ and $\int_0^t \lambda(s) \, ds > 0$) as $t$ increases. In this case, we would recover the time-exponential approximation between the solutions $u$ and $v$.

6 Applications for PDEs in the Wasserstein space

In this section we apply the theory developed in previous ones for time-dependent functionals associated to PDEs in the Wasserstein space. This space has a very nice geometric structure and is suitable to address gradient flow equations.

We start by recalling some definitions and properties of that space. We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of Borel probability measures in $\mathbb{R}^d$ with finite second order moment, i.e. $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ if $\mu$ is a positive Borel measure,
\[
\mu(\mathbb{R}^d) = 1 \quad \text{and} \quad M_2(\mu) := \int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < \infty.
\]

We can endow $\mathcal{P}_2(\mathbb{R}^d)$ with the weak-topology or the so-called narrow topology by considering the following notion of convergence:
\[
\mu_k \rightharpoonup \mu \quad \text{as} \quad k \to \infty \iff \lim_{k \to \infty} \int_{\mathbb{R}^d} f(x) \, d\mu_k(x) = \int_{\mathbb{R}^d} f(x) \, d\mu(x),
\] (6.1)
for all $f \in C^0_1(\mathbb{R}^d)$, where $C^0_1(\mathbb{R}^d)$ stands for the set of bounded continuous functions. On the other hand, $\mathcal{P}_2(\mathbb{R}^d)$ endowed with the Wasserstein distance is a complete metric space. This metric is defined by means of the Monge-Kantorovich problem and reads as

$$d^2_2(\mu, \nu) = \min \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x, y) : \gamma \in \Gamma(\mu, \nu) \right\}, \quad (6.2)$$

where $\Gamma(\mu, \nu) = \{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \gamma(A \times \mathbb{R}^d) = \mu(A), \gamma(\mathbb{R}^d \times B) = \nu(B) \}$. In fact, there exists at least one probability measure in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ that reaches the minimum in $(6.2)$. This is called the optimal transport plane and is supported in the graphic of the sub-differential of a convex lower semicontinuous function (see [26]). We denote by $\mathcal{P}_{2,ac}(\mathbb{R}^d)$ the set of probability measures in $\mathcal{P}_2(\mathbb{R}^d)$ that are absolutely continuous with respect to the Lebesgue measure. If $\mu$ does not give mass to sets with Hausdorff-dimension less than $d - 1$ (e.g. if $\mu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$), there exists a map $t^\mu_\nu : \mathbb{R}^d \to \mathbb{R}^d$ that coincides with the gradient of a convex lower semicontinuous function, such that $\nu = t^\mu_\nu \# \mu$ and the optimal transport plane $\gamma_0$ is given by the push-forward of $\mu$ via the map $Id \times t^\mu_\nu$, i.e. $\gamma_0 = (Id \times t^\mu_\nu) \# \mu$. Thus, we have that (see [26, theorem 2.12])

$$d_2^2(\mu, \nu) = \int_{\mathbb{R}^d} |x - t^\nu_\mu(x)|^2 \, d\mu(x). \quad (6.3)$$

We also recall the concept of generalized geodesics [2].

**Definition 6.1.** Let $\sigma, \mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ and let $\gamma_0, \gamma_1 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be two optimal plans that reach the minimum in $(6.2)$ for $d(\sigma, \mu_0)$ and $d(\sigma, \mu_1)$, respectively. Let $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be a $3$-plane such that $P_{1,2} \# \gamma = \gamma_0$ and $P_{1,3} \# \gamma = \gamma_1$ where $P_{i,j}$ denotes the projections on the coordinates $x_i$ and $x_j$. A generalized geodesic with base point $\sigma$ connecting $\mu_0$ to $\mu_1$ is defined by $\mu_t = ((1 - t)P_2 + tP_3) \# \gamma$, for $t \in [0, 1]$.

Although the theory in previous sections can be used to analyze general functionals in $\mathcal{P}_2(\mathbb{R}^d)$, we shall concentrate our attention in the following cases:

**The time-dependent potential energy**

$$\mathcal{V}(t, \mu) = \int_{\mathbb{R}^d} V(t, x) \, d\mu(x), \quad (6.4)$$

where $V : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is a time-dependent potential and the **time-dependent interaction energy**

$$\mathcal{W}(t, \mu) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(t, x, y) \, d(\mu \times \mu)(x, y), \quad (6.5)$$

where $W : [0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is an interaction potential. We also are interested in the case of **time-dependent diffusion coefficient** in the internal energy functional

$$\mathcal{U}(t, \mu) = \kappa(t)\mathcal{U}(\mu) = \kappa(t) \int_{\mathbb{R}^d} \rho \log(\rho) \, dx, \quad (6.6)$$

where $\kappa : [0, \infty) \to (0, \infty)$ and $d\mu = \rho \, dx$ is an absolutely continuous measure with respect to the Lebesgue one. For singular measures, we set $\mathcal{U}(t, \mu) = +\infty$. 

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Remark 6.2. The tools developed in previous sections are not directly applicable for time-dependent \( \kappa \). The reason is that the condition \( E_3 \) is not satisfied for arbitrary \( \kappa \), but only for \( \kappa \) constant. So, we postpone the case of \( \kappa \) depending on \( t \) for later.

### 6.1 The case with constant diffusion

We consider the functionals

\[
E_1(t, \mu) = \kappa \mathcal{U}(\mu) + \mathcal{V}(t, \mu)
\]

and

\[
E_2(t, \mu) = \mathcal{W}(t, \mu),
\]

where \( \mathcal{U} \) is defined in (6.6) and \( \kappa \geq 0 \) is a constant. In order to apply the theory, we assume some conditions on the potentials.

V1.- For each fixed \( t \geq 0 \), \( V(t, \cdot) \) is \( \lambda(t) \)–convex, for some function \( \lambda : [0, \infty) \to \mathbb{R} \) in \( L^\infty_{\text{loc}}([0, \infty)) \), that is, \( V(t, x) - \frac{\lambda(t)}{2} |x|^2 \) is convex.

V2.- Let \( \partial^\circ V(t, x) \) denote the element of minimal norm in the subdifferential of \( V(t, \cdot) \) at the point \( x \in \mathbb{R}^d \). We assume that \( |\partial^\circ V(t, 0)| \) is locally bounded and \( t \to V(t, 0) \) is locally bounded from below.

V3.- There exists a function \( \beta \in L^1_{\text{loc}}([0, +\infty)) \) such that

\[
|V(s, x) - V(t, x)| \leq \int_s^t \beta(r) \, dr (1 + |x|^2), \text{ for } 0 \leq s < t. \tag{6.7}
\]

We consider V2 for \( x = 0 \) only for simplicity. Indeed, this condition can be assumed for any (fixed) \( x_0 \in \mathbb{R}^d \). Moreover, it is not necessary to choose the element of minimal norm in the subdifferential. In fact, it would be enough to make a measurable choice (in \( t \)) in the subdifferential.

We start with the following result.

**Proposition 6.3.** Assume the hypotheses V1 to V3. If there exists \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) such that \( d\mu = \rho dx \),

\[
\int_{\mathbb{R}^d} \rho \log(\rho) \, dx < \infty, \text{ and } \int_{\mathbb{R}^d} V(0, x) \, d\mu(x) < \infty, \tag{6.8}
\]

then the functional \( E_1 \) satisfies E1 to E5.

**Proof.** Taking \( s = 0 \) in V3, it follows that

\[
|V(t, x) - V(0, x)| \leq \int_0^t \beta(s) \, ds (1 + |x|^2). \tag{6.9}
\]
Next, we show the estimate
\[ V(t, x) \geq -A(t) - B_T |x|^2 \] (6.10)
where \( A(t) = -V(t, 0) + \frac{1}{2} |\partial^0 V(t, 0)|^2 \) and \( B_T = \frac{1}{2}(1 + \lambda_T^-) \) for all \( t \in [0, T] \). In fact, by the definition of subdifferential, we have
\[
V(t, x) \geq V(t, 0) + \langle \partial^0 V(t, 0), x \rangle - \frac{\lambda_T^-}{2} |x|^2 \\
\geq V(t, 0) - \frac{1}{2} |\partial^0 V(t, 0)|^2 - \frac{1}{2}(1 + \lambda_T^-)|x|^2;
\]
and so (6.10) follows. This estimate implies that the functional \( V \) is lower semicontinuous with respect to the Wasserstein metric, for each fixed \( t \geq 0 \). Since the internal energy functional \( U \) is also lower semicontinuous (see [26]), we obtain E1. Using (6.9), the second condition in (6.8), and (6.10), it follows that \( \text{Dom}(V(t, \cdot)) \) is nonempty and time-independent, which gives E2. The property E3 is a direct consequence of V3 by taking \( u^* = \delta_0 \in \mathcal{P}_2(\mathbb{R}^d) \). E5 follows by using the convexity of the function \( \mu \to d^2_2(\sigma, \mu) \) along generalized geodesics with base point \( \sigma \) and the convexity of the potential \( V(t, \cdot) \). Next, we turn to E4. Recall the estimate [11]
\[
\int_{\mathbb{R}^d} \rho \log(\rho) \, dx \geq -C(1 + M_2(\mu))^{\alpha},
\] (6.11)
where \( \alpha \in (0, 1) \) and \( C > 0 \) are constants depending only on the dimension \( d \), and \( d\mu = \rho dx \in \mathcal{P}_{2,ac}(\mathbb{R}^d) \). Thus, we obtain from (6.10) that
\[
\frac{d^2_2(\delta_0, \mu)}{2\tau^*} + E_1(t, \mu) \geq \frac{1}{2\tau^*} M_2(\mu) - \kappa C(1 + M_2(\mu))^{\alpha} - A(t) - BM_2(\mu) \\
= \left( \frac{1}{2\tau^*} - B \right) M_2(\mu) - \kappa C(1 + M_2(\mu))^{\alpha} - A(t).
\]
Choosing \( \tau^*(T) > 0 \) such that \( \frac{1}{\tau^*(T)} > 1 + \lambda_T^- \), and using V2, the last expression is bounded from below by a constant depending on \( \alpha, \kappa, d, \lambda_T^-, \tau^*, T \), and so E4 follows. \( \square \)

In view of the hypotheses in Theorem 5.4, we need to impose one more condition on \( V \) in order to obtain the needed regularity for the functional \( V \), as expected.

**Lemma 6.4.** Let \( \kappa \geq 0 \) and \( D_1 = \text{Dom}(E_1) \). If, in addition to V1, V2 and V3, we assume that \( t \to V(t, x) \) is differentiable for each \( x \in \mathbb{R}^d \), then the function \( t \to V(t, \mu) \) is differentiable for each \( \mu \in D_1 \). Moreover, for each sequence \( t_n \to t \) and \( d_2(\mu_n, \mu) \to 0 \), we have that
\[
\lim_{n \to \infty} \frac{V(t_n, \mu_n) - V(t, \mu_n)}{t_n - t} = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} V(t, x) \, d\mu(x). \tag{6.12}
\]

**Proof.** We take \( \sigma \in \mathcal{P}_{2,ac}(\mathbb{R}^d) \) and the maps \( t_{\sigma}^{\mu_n} \) and \( t_{\sigma}^{\mu} \) that realize the optimal transports from \( \sigma \) to \( \mu_n \) and from \( \sigma \) to \( \mu \), respectively. Then,
\[
\frac{V(t_n, \mu_n) - V(t, \mu_n)}{t_n - t} = \int_{\mathbb{R}^d} \frac{V(t_n, t_{\sigma}^{\mu_n}) - V(t, t_{\sigma}^{\mu})}{t_n - t} \, d\sigma(x). \tag{6.13}
\]
By [26, pag. 71], we have that \( t^n_\sigma(x) \rightarrow t^\mu_\sigma(x) \) a.e. in \( \mathbb{R}^d \) with respect to \( \sigma \). Using a version of the dominated convergence theorem, we can take the limit in (6.13), as \( n \rightarrow \infty \), and obtain (6.12).

The metric space \( \mathcal{P}_2(\mathbb{R}^d) \) and functionals addressed here present more structure than those in previous sections, where an abstract theory has been developed. So, it is natural to wonder if gradient flow solutions as in Definition 2.3 is related to other senses of solutions in \( \mathcal{P}_2(\mathbb{R}^d) \). In this direction, we show that the solution \( u \) associated to the functional \( \mathcal{E}_1 \) is in fact a distributional solution for the Fokker-Planck equation. In the next result, we state precisely this fact and give some properties for \( u \).

**Theorem 6.5.** Consider the functional \( \mathcal{E}_1 \) with \( \kappa \geq 0 \) and potential \( V \) satisfying the assumptions \( V1 \) to \( V3 \) and the differentiability condition in Lemma 6.4. Then, given \( \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \), the curve \( \mu : [0, \infty) \rightarrow \mathcal{P}_2(\mathbb{R}^d) \) given in Theorem 4.4 is a distributional solution for the Fokker-Planck equation

\[
\partial_t \rho = \kappa \Delta \rho + \nabla \cdot (\nabla V(t,x) \rho),
\]

with \( \lim_{t \to 0^+} \mu(t) = \mu_0 \) weakly as measure. If \( \kappa > 0 \), such curve is absolutely continuous with respect to the Lebesgue measure, i.e. \( d\mu_t(x) = \rho(t,x)\,dx \), and \( \rho(t, \cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^d) \). Also, \( \mu \) satisfies the energy identity

\[
\mathcal{E}_1(s, \mu(s)) = \mathcal{E}_1(t, \mu(t)) + \int_s^t \int_{\mathbb{R}^d} |\Psi_1(r,t)|^2 - \partial_t V(r,x) \right) \,d\mu_r(x) \,dr
\]

for \( s < t \), where \( \Psi_1 : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a vector field satisfying the identity

\[
\rho(t,x)\Psi_1(t,x) = \kappa \nabla \rho(t,x) + \rho(t,x) \nabla_x V(t,x), \quad \text{for } \kappa > 0,
\]

and \( \Psi_1 = \partial^\kappa V(t,x) \) for \( \kappa = 0 \). Moreover, if the function \( \lambda \) satisfies \( E6 \), and \( \mu_1, \mu_2 \) are two solutions, we have the contraction property

\[
d_2(\mu_1(t), \mu_2(t)) \leq e^{-\int_0^t \lambda(s) \,ds} \,d_2(\mu_0, \mu_1).
\]

**Proof.** First we calculate the variation of \( \mathcal{E}_1(t, \mu(t)) \). We have that

\[
\mathcal{E}_1(s, \mu(s)) - \mathcal{E}_1(t, \mu(t)) = \kappa (\mathcal{U}(\mu(s)) - \mathcal{U}(\mu(t))) + \int_{\mathbb{R}^d} (V(s,x) - V(t,x)) \,d\mu_t(x)
\]

\[
+ \int_{\mathbb{R}^d} V(t,x) \,d\mu_s - \int_{\mathbb{R}^d} V(t,x) \,d\mu_t.
\]

Dividing (6.19) by \( s - t \), using Lemma 6.4, and recalling that the function \( \mathcal{E}_1(t, \mu(t)) \) is absolutely continuous, we get

\[
\frac{d}{dt} \mathcal{E}_1(t, \mu(t)) = - \int_{\mathbb{R}^d} \langle \Psi_1(t,x), v(t,x) \rangle \,d\mu_t(x) + \int_{\mathbb{R}^d} \partial_t V(t,x) \,d\mu_t(x),
\]

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where \( v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is the vector field associated to the absolutely continuous curve \( \mu_t, \|v(t, \cdot)\|_{L^2(\mu_t; \mathbb{R}^d)} = \|\mu'_t\|(t) \), and \( \Psi_1 \) is the vector field satisfying \( \|\Psi_1(t)\|_{L^2(\mu_t; \mathbb{R}^d)} = |\partial E(t)|(\mu_t) \). Moreover, \( v \) verifies the continuity equation

\[
\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0,
\]

in the distributional sense, with \( v_t(x) = v(t, x) \). Using (5.7) together with (6.20), we obtain

\[-\Psi_1(t, x) = v(t, x) \quad \text{for} \quad \mu_t \text{-a.e.} \ x \in \mathbb{R}^d \quad \text{and the identity (6.15)}. \]

Similar results hold true for the functional \( E_2 \). In what follows, we state the hypotheses for \( W \) and \( \mathcal{W} \) in order to treat \( E_2 \) in light of the abstract Theorem 5.4 in metric spaces.

**W1.** For each fixed \( t \geq 0 \), the interaction potential \( W(t, x, y) \) is symmetric and, for \( t = 0 \), it satisfies a quadratic growth condition, namely \( W(t, x, y) = W(t, y, x) \) and \( W(0, x, y) \leq C(1 + |x|^2 + |y|^2) \).

**W2.** For each fixed \( t \geq 0 \), \( W(t, \cdot) \) is \( \lambda(t) \)-convex, for some function \( \lambda : [0, \infty) \to \mathbb{R} \) as in E5. Let \( \partial^e W(t, x, y) \) denote the element of minimal norm in the subdifferential of \( W(t, \cdot) \) at the point \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \). We assume that \( |\partial^e W(t, 0, 0)| \) is locally bounded and \( t \to W(t, 0, 0) \) is locally bounded from below.

**W3.** There exists a function \( \beta \in L^1_{loc}([0, +\infty)) \) such that

\[
|W(s, x, y) - W(t, x, y)| \leq \int_s^t \beta(r) \, dr \, (1 + |x|^2 + |y|^2), \quad \text{for} \ 0 \leq s < t. \quad (6.22)
\]

The reason for assuming W1 is to obtain a quadratic growth for \( W(t, x, y) \), for each \( t > 0 \), and then one can use the results in [5]. In fact, using W1, this growth follows directly from W3. Proceeding as in Proposition 6.3, again we get that the functional \( E_2 \) satisfies E1 to E5. Assuming a differentiability property in the \( t \)-variable, we obtain the analogous of Lemma 6.4. Here we only state the results for the functional \( E_2 \). The proof is similar to that of Theorem 6.5 and is left to the reader.

**Theorem 6.6.** Consider the functional \( E_2 \) with the interaction potential \( W \) satisfying W1 to W3. Suppose also that for \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \) the function \( t \to W(t, x, y) \) is differentiable. Then, given \( \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \), the curve \( \mu : [0, \infty) \to \mathcal{P}_2(\mathbb{R}^d) \) given in Theorem 4.4 is a distributional solution for the continuity equation

\[
\partial_t \rho = \nabla \cdot (v(t, x) \rho),
\]

with \( \lim_{t \to 0^+} \mu(t) = \mu_0 \) weakly as measure, where

\[
v(t, x) = \int_{\mathbb{R}^d} \eta(t, x, y) \rho(y) \, dy, \quad \mu \text{-a.e. in } \mathbb{R}^d,
\]

\[
\text{with } \lim_{t \to 0^+} \mu(t) = \mu_0 \text{ weakly as measure, where } \]

\[
v(t, x) = \int_{\mathbb{R}^d} \eta(t, x, y) \rho(y) \, dy, \quad \mu \text{-a.e. in } \mathbb{R}^d,
\]

\[
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\]
and \( \eta(t, x, y) = \frac{1}{2}(\eta_1(t, x, y) + \eta_2(t, y, x)) \) for some Borel measurable selection \((\eta_1, \eta_2) \in \partial W(t, \cdot, \cdot)\). Moreover, \( \mu \) satisfies the energy identity

\[
E_2(s, \mu(s)) = E_2(t, \mu(t)) + \int_s^t \int_{\mathbb{R}^d} (|\nabla v(r, x)|^2 - \partial_t W(v, x)) \, d\mu(x) \, dr \tag{6.25}
\]

for \( s < t \). Furthermore, if the function \( \lambda \) satisfies \( E6 \) and \( \mu_1, \mu_2 \) are two solutions, we have the contraction property

\[
d_2(\mu_1(t), \mu_2(t)) \leq e^{-\int_s^t \lambda(s) \, ds} d_2(\mu_0, \mu_1). \tag{6.26}
\]

**Remark 6.7.** Let us observe that the vector field \( v \) in (6.23) is characterized by the form (6.24) thanks to the results of Carrillo-Lisini-Mainini [5]. They showed that, in general, the Borel measurable selection of \( \eta \) depends on the probability \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and is not necessarily given by the minimal selection in the subdifferential of \( W \). In the particular case when \( W(t, x, y) = w(t, y - x) \) is given by a symmetric function \( w : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \), the \( \lambda(t) \)-convexity of \( W(t, \cdot, \cdot) \) follows from the one of \( w \) only if \( \lambda(t) \leq 0 \) and therefore we can not use the results for any \( \lambda(t) \)-convexity of \( w \).

Of course, we can consider a functional of the type (see subsection below to the time-dependent viscosity)

\[
E(t, \mu) = \mathcal{U}(\mu) + \mathcal{W}(t, \mu)
\]

and apply the metric theory in order to obtain existence of curves satisfying the conclusions in Theorem 5.4, the contraction property and a continuity equation. On the other hand, we do not know how to describe the velocity field \( v \) in this general case, however it is expect that \( \mu \) satisfies a Mackean-Vlasov equation of the type \( \partial_t \mu = \Delta \mu + \nabla \cdot (v \mu) \) for \( v \) as in (6.24).

Finally, if we assume that \( W(t, x, y) = w(t, y - x) \) is \( \lambda(t) \)-convex with \( \lambda(t) \leq 0 \) and satisfies a doubling condition property \( w(t, x + y) \leq C_t(1 + w(x) + w(y)) \), then one can show that the curve \( \mu_t \) given in Theorem 4.4 is a distributional solution of the Mackean-Vlasov equation

\[
\partial_t \mu_t = \Delta \mu_t + \nabla \cdot ((\nabla w(t) * \mu_t) \mu_t).
\]

### 6.2 The case with time-dependent diffusion

Now we consider the case when \( \kappa : [0, \infty) \to (0, \infty) \). For the sake of simplicity, we consider the functional

\[
E(t, \mu) = \kappa(t) \mathcal{U}(\mu) + \mathcal{V}(t, \mu), \tag{6.27}
\]

where \( \mathcal{U} \) and \( \mathcal{V} \) are defined in (6.6) and (6.4), respectively, and \( \kappa \) is a positive function locally absolutely continuous. Also, we assume that \( \mathbf{V1} \) to \( \mathbf{V3} \) hold true. Thus, by assuming that \( \mathbf{V(0, \cdot)} \) satisfies (6.8), we have that the domain of \( \mathcal{E} \) is time-independent. Notice that the functional \( \mathcal{E} \) satisfies \( \mathbf{E1}, \mathbf{E2}, \mathbf{E4} \) and \( \mathbf{E5} \), but not \( \mathbf{E3} \). In fact, as observed in Remark 6.2, \( \mathbf{E3} \) holds true, if and only if, \( \kappa(t) \) is constant. Here we need to assume that \( \kappa \) is non-increasing. An important fact is the following:
Remark 6.8. Let $\mu_n \in \mathcal{P}_2(\mathbb{R}^d)$ and $t_n \in [0, \infty)$ be two bounded sequences, where $\mu_n$ is bounded with respect to the Wasserstein metric $d_2$, such that the numeric sequence $\mathcal{E}(t_n, \mu_n)$ is bounded from above. Then, the numeric sequences $\mathcal{U}(\mu_n)$ and $\mathcal{V}(t_n, \mu_n)$ are bounded. In fact, it follows from (6.11) that the sequence $\kappa(t_n)\mathcal{U}(\mu_n)$ is bounded from below and thus $\mathcal{V}(t_n, \mu_n)$ is bounded from above. Similarly, from (6.10) we have that $\mathcal{V}(t_n, \mu_n)$ is bounded from below, and then $\kappa(t_n)\mathcal{U}(\mu_n)$ is bounded from above.

Using Remark 6.8, we obtain easily the same conclusions of Lemmas 3.1 and 3.2. Let the potential $V$ be differentiable in the $t$-variable. Using the minimality of $\mu^{t+\tau}_\tau$, we have that for $\tau_0 < \tau_1$

$$\mathcal{E}_{t+\tau_1, \tau_1}(u) - \mathcal{E}_{t+\tau_0, \tau_0}(u) \leq (\kappa(t + \tau_1) - \kappa(t + \tau_0))\mathcal{U}(\mu^{t+\tau_0}_\tau) + \mathcal{V}(t + \tau_1, u^{t+\tau_0}_\tau) - \mathcal{V}(t + \tau_0, u^{t+\tau_0}_\tau) \leq (\kappa(t + \tau_1) - \kappa(t + \tau_0))\mathcal{U}(\mu^{t+\tau_0}_\tau) + \frac{\tau_0 - \tau_1}{2}\mathcal{E}_{t+\tau_0}(u, u^{t+\tau_0}_\tau) + \int_{t+\tau_0}^{t+\tau_1} \beta(r) \, dr (1 + M(u^{t+\tau_0}_\tau)),$$

where above we used the estimate (6.11). Analogously, the reverse inequality follows. In view of Remark 6.8, we can argue as in Proposition 3.4 and Corollary 3.5 in order to obtain the identity (3.8) for the functional (6.27).

Up until this point, notice that we have not needed the monotonicity hypothesis for $\kappa$. In what follows, we comment on an essential step in order to recover Lemma 3.8. In fact, recalling the notation for discrete solution $(U^j_\tau)$ of the variational scheme, and using that $\kappa(t)$ is non-increasing and (6.11), we estimate

$$\sum_{j=1}^n \kappa(t^j_\tau)(\mathcal{U}(U^{j-1}_\tau) - \mathcal{U}(U^j_\tau)) \leq \kappa(0)\mathcal{U}(U^0_\tau) - \kappa(t^0_\tau)\mathcal{U}(U^0_\tau) - C \sum_{j=1}^n (\kappa(t^j_\tau) - \kappa(t^{j-1}_\tau))(1 + M_2(U^{j-1}_\tau)).$$

From here we can repeat the arguments in order to obtain the same conclusion of Lemma 3.8. In the case when $\mathcal{V} \equiv 0$, it is not necessary to suppose the monotonicity of $\kappa$ because the difference $\mathcal{U}(U^{j-1}_\tau) - \mathcal{U}(U^j_\tau)$ is positive and a more direct estimate can be performed.

Going back to Section 4, it is easy to see that it remains only to estimate

$$\int_0^t [(1 - l_\tau(s))(\kappa(T_\eta(s)) - \kappa(T_\tau(s)))\mathcal{U}(U_\tau(s))]^+ \, ds, \quad 0 \leq t \leq T,$$

where $T > 0$ is fixed and $\tau$, $\eta$ are two partitions with small sizes. Indeed, since $\mathcal{E}(T_\tau(t), U_\tau(t))$ is bounded from above by a constant independent of $\tau$, it follows from Remark 6.8 that $\mathcal{U}(U_\tau(t))$ is bounded by a constant independent of $\tau$. Thus, the integral (6.28)
can be estimated by proceeding similarly to Proposition 4.3, and then we obtain the convergence of the approximate solutions (4.8). In this way, the functional $E$ defined in (6.27) presents properties and results contained in Sections 4 and 5. So, we have the following:

**Theorem 6.9.** Let $E$ be the functional defined in (6.27) with $\kappa : [0, \infty) \to (0, \infty)$ an non-increasing absolutely continuous function and let the potential $\kappa$ and the differentiability condition in Lemma 6.4. Then, given $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, the curve $\mu : [0, \infty) \to \mathcal{P}_2(\mathbb{R}^d)$ obtained in Theorem 4.4 is absolutely continuous with respect to the Lebesgue measure, i.e. $d\mu_t(x) = \rho(t, x) dx$, $\rho(t, \cdot) \in W^{1,1}_{loc}(\mathbb{R}^d)$ for each $t \in [0, \infty)$, and $\mu$ is a distributional solution for the Fokker-Planck equation

$$\partial_t \rho = \kappa(t) \Delta \rho + \nabla \cdot (\nabla V(t, x) \rho),$$

with $\lim_{t \to 0^+} \mu(t) = \mu_0$ weakly as measure. Also, $\mu(t)$ satisfies the energy identity

$$E_1(s, \mu(s)) = E_1(t, \mu(t)) + \int_s^t \int_{\mathbb{R}^d} \left( |\Psi_1(r, t)|^2 - \partial_t V(r, x) \right) \rho(r, x) dx dr$$

$$- \int_s^t \int_{\mathbb{R}^d} \kappa'(r) \rho(r, x) log(\rho(r, x)) dx dr, \text{ for } s < t,$$

where $\Psi_1 : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a vector field satisfying

$$\rho(t, x) \Psi_1(t, x) = \kappa(t) \nabla \rho(t, x) + \rho(t, x) \nabla V(t, x) \text{ for } \mu_t \text{-a.e. } x \in \mathbb{R}^d.$$  

Moreover, if the function $\lambda$ satisfies E6 and $\mu_1, \mu_2$ are two solutions, we have the contraction property

$$d_2(\mu_1(t), \mu_2(t)) \leq e^{-\int_0^t \lambda(s) ds} d_2(\mu_0, \mu_1).$$

### 6.3 More general internal energy

In this subsection we give the outline to construct the time-dependent gradient flow for more general internal energy functionals. Let $U : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a continuous function such that $C^1((0, \infty) \times (0, \infty))$. Consider the internal energy functional

$$U(t, \mu) = \left\{ \begin{array}{ll} \int_{\mathbb{R}^d} U(t, \rho(x)) dx, & \text{if } d\mu = \rho dx \\ +\infty, & \text{otherwise.} \end{array} \right.$$  

(6.33)

We assume the following condition on $U$.

**U1.-** There exist functions $a, A : [0, \infty) \to [0, \infty)$ with $a \in L^1_{loc}([0, \infty))$ and $A \in L^1([0, \infty))$ such that

$$-A(t) U^+(0, z) \leq \frac{\partial U}{\partial t}(t, z) \leq a(t) U^-(0, z),$$  

for all $t, z \in [0, +\infty)$, and $U(0, z)$ has superlinear growth at infinite, i.e. $\lim_{z \to +\infty} \frac{U(t, z)}{z} = +\infty$.  

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U2.- There exist $\alpha \in (0, 1)$ with $\alpha > \frac{d}{d+2}$ and positive constants $c_1, c_2 \geq 0$ such that

\[ U(0, z) \geq -c_1z - c_2z^{\alpha}. \]

U3.- $U(0, 0) = 0, z \to U(t, z)$ is convex, and $z \to z^dU(t, z^{-d})$ is convex and non-increasing on $(0, +\infty)$, for each $t > 0$.

Without loss of generality, we can assume that $\|A\|_1 = \int_0^\infty A(t) \, dt < 1$; otherwise, we can replace $U$ by $\frac{U}{\|A\|_1 + 1}$. Firstly, let us note that U1 and U2 imply

\[
U(t, z) = \int_0^t \frac{\partial U}{\partial t}(r, z) \, dr + U(0, z) \\
\geq -\left( \int_0^t A(r) \, dr \right) U^+(0, z) + U(0, z) \\
= \left( 1 - \int_0^t A(r) \, dr \right) U^+(0, z) - U^-(0, z) \tag{6.35} \\
\geq -U^-(0, z) \geq -c_1z - c_2z^{\alpha}. \tag{6.36}
\]

Then, recalling that $\alpha > \frac{d}{d+2}$, it follows from (6.36) that

\[
U(t, \mu) \geq -\left( c_1 + c_2 \int_{\mathbb{R}^d} \rho(x)^\alpha \, dx \right) \\
\geq -\left( c_1 + c_2 \left( \int_{\mathbb{R}^d} (1 + |x|^2)\rho(x) \, dx \right)^\alpha \left( \int_{\mathbb{R}^d} \frac{1}{(1 + |x|^2)^{\frac{\alpha}{1-\alpha}}} \, dx \right)^{1-\alpha} \right) \\
= -\left( c_1 + c_2C_\alpha (1 + M_2(\mu))^{\alpha} \right). \tag{6.37}
\]

Therefore, the functional in (6.33) is well-defined from $[0, +\infty) \times \mathcal{P}_2(\mathbb{R}^d)$ to $(-\infty, +\infty]$. It follows from (6.35) that $U(t, \cdot)$ has a superlinear growth, for each fixed $t \geq 0$. So, by standard arguments (see [17]), one can show that the functional $U(t, \cdot)$ is lower semicontinuous with respect to the weak topology, for each fixed $t \geq 0$. Thus, $U(t, \cdot)$ verifies E1.

Let $\mu \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$ be such that $d\mu = \rho dx$ and $\mathcal{U}(0, \mu) < \infty$. We have

\[
U(t, \rho(x)) \leq \left( \int_0^t a(r) \, dr \right) U^-(0, \rho(x)) + U(0, \rho(x)),
\]

and then $\mathcal{U}(t, \mu) < \infty$ for all $t > 0$. On the other hand, if $d\mu = \rho dx$ is such that $\mathcal{U}(t, \mu) < +\infty$ for all $t > 0$, then, by substituting $z = \rho(x)$ in (6.35), we get

\[
\left( 1 - \int_0^t A(r) \, dr \right) U^+(0, \rho(x)) - U^-(0, \rho(x)) \leq U(t, \rho(x)). \tag{6.38}
\]

It follows by integrating (6.38) that $\mathcal{U}(0, \mu) < \infty$. So, $\mathcal{U}(t, \mu)$ verifies E2.
Denote \( \text{Dom}(U(t, \cdot)) = D \subset \mathcal{P}_{2,ac}(\mathbb{R}^d) \). Since \( U^-(0,0) = U^+(0,0) = 0 \), note that

\[
U(t,0) = \int_0^t \frac{\partial U}{\partial t}(r,0) \, dr = 0,
\]

and then \( D \) is nonempty. For \( s < t \) and \( \mu \in D \) with \( d\mu = \rho \, dx \), we have

\[
U(t, \rho(x)) - U(s, \rho(x)) \leq \left( \int_s^t a(r) \, dr \right) (c_1 \rho(x) + c_2 \rho(x)^{\alpha}).
\]

The same arguments used in (6.37) lead us to

\[
U(t, \mu) - U(s, \mu) \leq \left( \int_s^t a(r) \, dr \right) (c_1 + c_2 C_\alpha (1 + M_2(\mu))) \text{, for all } 0 \leq s < t. \tag{6.39}
\]

We are going to use (6.39) as a substitute for the condition \( \text{E3} \). Also, \( \text{E4} \) follows from (6.39).

In fact,

\[
U(t, \mu) + \frac{d^2_2(\mu, \delta_0)}{2\tau^*} \geq -(c_1 + c_2 C_\alpha (1 + M_2(\mu))) + \frac{M_2(\mu)}{2\tau^*}.
\]

Now, it is easy to see that for \( \tau^* > 0 \) small enough the last expression is bounded from below, as desired. Note that we have \( \text{E5} \) with \( \lambda \equiv 0 \) because \( \text{U3} \) implies that \( U(t, \cdot) \) is convex along of generalized geodesics.

Let us remark that Lemmas 3.1 and 3.2 can be proved by proceeding as in Section 3 (and using (6.35) for Lemma 3.2). In order to recover the differentiability property in Proposition 3.4, we recall the notation

\[
\mathcal{E}_{t,\tau}(\mu) = \inf_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ \frac{d^2_2(\mu, \nu)}{2\tau} + U(t, \nu) \right\} = \frac{d^2_2(\mu, \mu_{\tau})}{2\tau} + U(t, \mu_{\tau}).
\]

Then, by taking \( \tau_0 < \tau_1 \) and \( \mu \in D \), we have

\[
\mathcal{E}_{t+\tau_1,\tau_1}(\mu) - \mathcal{E}_{t+\tau_0,\tau_0}(\mu) \leq U(t + \tau_1, \mu_{\tau_0}^{t+\tau_0}) - U(t + \tau_0, \mu_{\tau_0}^{t+\tau_0}) + \frac{\tau_0 - \tau_1}{2\tau_0 \tau_1} d^2_2(\mu, \mu_{\tau_0}^{t+\tau_0}). \tag{6.40}
\]

Denoting \( d\mu_{\tau}^{t+\tau} = \rho_{\tau}^{t+\tau} \, dx \), we can estimate

\[
U(t + \tau_1, \mu_{\tau_0}^{t+\tau_0}) - U(t + \tau_0, \mu_{\tau_0}^{t+\tau_0}) = \int_{\mathbb{R}^d} \int_{t+\tau_0}^{t+\tau_1} \frac{\partial U}{\partial r}(r, \rho_{\tau_0}^{t+\tau_0}(x)) \, dr \, dx
\]

\[
\leq \int_{t+\tau_0}^{t+\tau_1} a(r) \, dr \int_{\mathbb{R}^d} U^-(0, \rho_{\tau_0}^{t+\tau_0}(x)) \, dx.
\]

The last integral on \( \mathbb{R}^d \) is uniformly bounded in \( \tau_0 \) on compact sets of \( (0, \tau^*) \). Replacing the roles of \( \tau_0 \) and \( \tau_1 \), we get

\[
\mathcal{E}_{t+\tau_1,\tau_1}(\mu) - \mathcal{E}_{t+\tau_0,\tau_0}(\mu) \geq U(t + \tau_1, \mu_{\tau_1}^{t+\tau_1}) - U(t + \tau_0, \mu_{\tau_1}^{t+\tau_1}) + \frac{\tau_0 - \tau_1}{2\tau_0 \tau_1} d^2_2(\mu, \mu_{\tau_1}^{t+\tau_1}). \tag{6.41}
\]
and

$$U(t + \tau_1, \mu^{t+\tau_1}) - U(t + \tau_0, \mu^{t+\tau_1}) \geq - \int_{t+\tau_0}^{t+\tau_1} A(r) \, dr \int_{\mathbb{R}^d} U^+(0, \rho^{t+\tau_1}) \, dx.$$  

Now we need a uniform estimate for $\int_{\mathbb{R}^d} U^+(0, \rho^{t+\tau_1}(x)) \, dx$. Substituting $t = t + \tau_1$ and $z = \rho^{t+\tau_1}(x)$ in (6.35), and afterwards integrating it, we arrive at

$$\left(1 - \|A\|_1\right) \int_{\mathbb{R}^d} U^+(0, \rho^{t+\tau_1}(x)) \, dx \leq \int_{\mathbb{R}^d} U^-(0, \rho^{t+\tau_1}(x)) \, dx + \varepsilon_{t+\tau_1, \tau_1}(\mu). \quad (6.42)$$

The first term in the right hand side of (6.42) is locally uniformly bounded in $(0, \tau^*)$. By the continuity of the map $\tau \rightarrow \varepsilon_{t+\tau_1, \tau_1}(\mu)$, the second term also verifies so. Therefore, we conclude that the function $\tau \rightarrow \varepsilon_{t+\tau_1, \tau}(\mu)$ is absolutely continuous in each compact subinterval of $(0, \tau^*)$. Now a version of the dominated convergence theorem leads us to the formula

$$\frac{d}{d\tau} \varepsilon_{t+\tau_1, \tau}(\mu) = \int_{\mathbb{R}^d} \frac{\partial U}{\partial r}(t + \tau, \rho^{t+\tau}(x)) \, dx - \frac{d^2_2(\mu, \mu^{t+\tau})}{2\tau^2}, \quad (6.43)$$

for each differentiability point $\tau \in (0, \tau^*)$. The identity above implies the integral equality (3.8) in Corollary 3.5.

In the sequel, we sketch the proof of Lemma 3.8 in the case of this present section. Recalling the notation for the discrete solution in (2.8), we have

$$\frac{1}{2}(M_2(U^n_\tau) - M_2(U^0_\tau)) \leq \sum_{j=1}^{n} \frac{1}{2}(M_2(U^j_\tau) - M_2(U^{j-1}_\tau))$$

$$\leq \sum_{j=1}^{n} \frac{1}{2}(\tau^* \frac{d^2(U^j_\tau, U^{j-1}_\tau)}{2\tau_j} + 2\tau_j \frac{M_2(U^j_\tau)}{\tau^*})$$

$$\leq \frac{\tau^*}{2}(U(0, U^0_\tau) - U(t^n_\tau, U^n_\tau)) + \sum_{j=1}^{n} \frac{\tau_j}{\tau^*} M_2(U^j_\tau)$$

$$+ \frac{\tau^*}{2} \sum_{j=1}^{n} (U(t^j_\tau, U^{j-1}_\tau) - U(t^{j-1}_\tau, U^{j-1}_\tau)).$$

Using the above estimate and (6.39), we can proceed as in Lemma 3.8 and reobtain the conclusions of this lemma for the functional $U(t, \mu)$.

Now we deal with the convergence of the approximate solutions. In comparison with subsection 4.2, there is only a new term that reads as

$$\int_{0}^{t} (1 - l_\tau(t)) [U(T_\tau(t), U_\tau(t)) - U(T_\eta(t), U_\eta(t))] \, dt.$$  

Notice that it is necessary to consider only the case $T_\tau(t) < T_\eta(t)$. So, we have that

$$U(T_\tau(t), U_\tau(t)) - U(T_\eta(t), U_\eta(t)) \leq \left(\int_{T_\tau(t)}^{T_\eta(t)} A(r) \, dr\right) \int_{\mathbb{R}^d} U^+(0, U_\tau(t, x)) \, dx. \quad (6.44)$$
Using (6.35) we can estimate the integral over \( \mathbb{R}^d \) in (6.44) locally uniformly in \([0, \infty)\). Now, by replacing the function \( \beta \) by \( a \) or \( A \), one can repeat the same arguments in the proof of Proposition 4.3, obtain the estimate (4.19) and afterwards the convergence of the approximate solutions (4.8) to a curve \( \mu : [0, \infty) \to \mathcal{P}_2(\mathbb{R}^d) \).

In what follows, we sketch the main arguments for the convergence of the De Giorgi interpolation (see Proposition 5.2).

Let \( \delta = t - t_n^{n-1} \), for \( t \in (t_n^{n-1}, t_n^n] \). Using the minimizer property of \( \tilde{U}_\tau \) and \( U_\tau \), we can obtain
\[
\frac{\tau_n - \delta}{2\tau_n\delta} d_2^2(U_\tau(t), \tilde{U}_\tau(t)) + U(t, \tilde{U}_\tau(t)) - U(t_n^n, \tilde{U}_\tau(t)) 
\leq \frac{\tau_n - \delta}{2\tau_n\delta} d_2^2(U_\tau(t), U_\tau(t)) + U(t, U_\tau(t)) - U(t_n^n, U_\tau(t)),
\]
and so
\[
\frac{\tau_n - \delta}{2\tau_n\delta} d_2^2(U_\tau(t), \tilde{U}_\tau(t)) \leq \frac{\tau_n - \delta}{2\tau_n\delta} d_2^2(U_\tau(t), U_\tau(t)) 
+ \int_t^{t_n^n} a(r) dr \int_{\mathbb{R}^d} U^-(0, \tilde{U}_\tau(t, x)) dx 
+ \int_t^{t_n^n} A(r) dr \int_{\mathbb{R}^d} U^+(0, U_\tau(t, x)) dx.
\]

By using these estimates, one can obtain the conclusions of Proposition 5.2. In order to recover some properties in Theorem 5.4, note that the inequality \( \geq \) in (5.7) follows from the same arguments and Fatou’s Lemma. The absolutely continuity of the map \( t \to U(t, \mu(t)) \) in each bounded interval of \([0, \infty)\) is a consequence of (5.4) with \( \lambda(t) \equiv 0 \) and the estimate (6.35).

Although we are not able to obtain the reverse inequality \( \leq \) in (5.7), and so the energy identity (5.7) (see Remark 6.12 below), it is not hard to show the estimate
\[
U(t, \mu(t)) - U(s, \mu(s)) \leq \int_s^t \int_{\mathbb{R}^d} \frac{\partial U}{\partial t}(r, \mu(r)) dx dr - \frac{1}{2} \int_s^t |\mu'|^2(r) dr 
- \frac{1}{2} \int_s^t |\partial U(r)|^2(\mu(r)) dr.
\]

Even without the energy identity and minimal selection, it is possible to show that the approximate solutions converge to a distributional solution \( \mu \). Denote \( P(t, z) = z \frac{\partial U}{\partial z}(t, z) - U(t, z) \). We summarize the results for the functional (6.33) in the theorem below.

**Theorem 6.10.** Consider the internal energy functional defined in (6.33) and assume \( U_1 \) to \( U_3 \). Then the curve \( \mu : [0, \infty) \to \mathcal{P}_2(\mathbb{R}^d) \), \( d\mu_t = \rho(x, t)dx \), given by Theorem 4.4 is a distributional solution of the equation
\[
\partial_t \rho - \nabla_x \cdot (\nabla_x P(t, \rho(t, x))) = 0
\]
with initial condition \( \mu_0 \in \mathcal{P}_2(\mathbb{R}^d) \). Moreover, the contraction property (5.17) holds.
Proof. For simplicity take a uniform step size $\tau > 0$, consider the partition $\{0 < \tau < 2\tau < 3\tau < \cdots \}$, and choose $\xi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Consider also the flow $\Phi_\delta$ associated to the field $\xi$, i.e.

$$(\Phi_\delta)' = \xi(\Phi).$$

(6.48)

Then, by the minimizer property of $U^n_{\tau}$, we have

$$U(n\tau, U_\delta) + \frac{d^2(U_{\tau}^{n-1}, U_\delta)}{\tau} - U(n\tau, U^n_{\tau}) - \frac{d^2(U_{\tau}^{n-1}, U^n_{\tau})}{\tau} \geq 0,$$

(6.49)

where $U_\delta = \Phi_\delta \# U^n_{\tau}$ is the push-forward of $U^n_{\tau}$ via $\Phi_\delta$. Then, by standard arguments (see e.g. [11]), it follows that

$$\lim_{\delta \to 0} \frac{U(n\tau, U_\delta) - U(n\tau, U^n_{\tau})}{\delta} = \int_{\mathbb{R}^d} -P(n\tau, U^n_{\tau}(x))\text{div} \xi \, dx$$

and

$$\lim_{\delta \to 0} \tau^{-1} \frac{d^2(U_{\tau}^{n-1}, U_\delta) - d^2(U_{\tau}^{n-1}, U^n_{\tau})}{\delta} = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(x - y) \cdot \xi(y)}{\tau} \, d\gamma(x, y),$$

where $\gamma \in \Gamma(U_{\tau}^{n-1}, U^n_{\tau})$ is an optimal plane for the transport from $U_{\tau}^{n-1}$ to $U^n_{\tau}$. Changing $\xi$ by $-\xi$ in (6.48) (by symmetry in (6.49)) and taking $\xi = \nabla \zeta$, we obtain that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(x - y)}{\tau} \cdot \nabla \zeta(y) \, d\gamma(x, y) - \int_{\mathbb{R}^d} P(n\tau, U^n_{\tau}(x))\Delta \zeta \, dx = 0.$$

(6.50)

Let us remark that the above calculations also allow to conclude that $P(n\tau, U^n_{\tau}) \in W^{1,2}(\mathbb{R}^d)$ is bounded uniformly. Thus, we can use an argument of weak convergence and estimates as in Lemma 3.8 in order to obtain that the curve $\mu : [0, \infty) \to \mathcal{P}_2(\mathbb{R}^d)$ solves (6.47) in the distributional sense.

Remark 6.11. The conditions U1 to U3 work well if we consider a functional as being the sum of the internal energy and another functional as in two previous subsections. In the present subsection, we have preferred to consider only the internal energy for the sake of simplicity.

Remark 6.12. Let us observe that the energy identity was not obtained in Theorem 6.10. The reason is that, in order to obtain such property in this general case, it would be necessary to handle the limit

$$\lim_{t \to t_0} \frac{U(t, \mu(t)) - U(t_0, \mu(t))}{t - t_0}.$$

(6.51)

By making a change of variable (see [26, Theorem 4.8]), the calculus of (6.51) is related to stability results for the Monge-Ampère equation. However, as far as we know, such results are available in the literature (see [9]) under restrictions stronger than the ones that we have in our context.
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