Solving the Bethe-Salpeter equation for fermion-antifermion pseudoscalar bound state in Minkowski space

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Abstract

The new method of solution of the Bethe-Salpeter equation for quark-antiquark pseudoscalar bound state is proposed. With the help of integral representation the results are directly obtained
in Minkowski space. Dressing of Greens functions is naturally considered providing thus the correct
inclusion of the running coupling constant and the quark propagators as well as. The first numerical
results are presented for a simplified ladder approximation.

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I. INTRODUCTION

Among various approaches used in meson physics the formalism of Bethe-Salpeter and Dyson-Schwinger equations (DSE’s) plays the traditional and indispensable role. Bethe-Salpeter equation (BSE) provides quantum field theoretical starting point to describe hadrons as relativistic bound states of quarks (antiquarks). DSE’s and BSE’s framework has been widely used in order to obtain the nonperturbative information about the spectra and decays of the whole lowest pseudoscalar nonet with emphasizing attention to the QCD pseudo-Goldstone bosons - the pions \[1\]. With certain success the formalism offers a satisfactory window into the ‘next scale’ meson sector including also excited and vector, scalar mesons \[2\]. Within formalism, the mesons electromagnetic form factors have been calculated at spacelike regime of momenta \[3\].

When dealing with bound states composed from the light quarks then there is unavoidable necessity of the full use of BSE covariant framework. The nonperturbative knowledge of the Greens function composing the kernel of BSE is required. Very often, the problem is solved in the Euclidean space where it is more tractable since there is no singularities in Green functions there. The physical amplitudes can be then obtained by continuation to the Minkowski space. Note, the extraction of the mass spectra is already a complicated task \[4\], while there is a practical absence of an analytical continuation of Euclidean space form factors.

When dealing with the heavy quarkonia or with the mesons composed from the mix like \(B_{c}\) meson (the state measured in the Fermilab Tevatron by CDF Collaboration) some simplifying approximations are possible. Different approaches have been developed to reduce computational complexity of full 4-dim BSE. The so called instantaneous approximation \[5\] or quasi-potential approximation \[6\] can reduce the 4-dim BSE to the 3-dim equation in a Lorentz covariant manner. In practise, such 3-dim equations are more tractable since the solution is less involved, especially when one tricky exploits a considerable freedom in performing three dimensional reduction. Note, contrary to ladder BSE, these equations also reduce to the Schrodinger equation of non-relativistic Heavy Meson Effective Theory and nonrelativistic QCD \[7\]. However, the interaction kernels of the reduced equations often represents the input based on economical model phenomenology and the connection to the underlying theory (QCD) is less clear (if not abandoned from the beginning).

In this paper, we extend the method of solution of the full 4-dim BSE, originally developed for the pure scalar theories \[8\],\[10\],\[11\], to the theories with non-trivial spin degrees of freedom. Within a certain assumptions on the functional form of Greens functions we develop the method of direct Minkowski space BSE solution in its original manifestly Lorentz covariant (four-dimensional) manner. In order to make this paper as self-contained as possible, we will supply some basic facts concerning the BSE approach to relativistic meson bound states bellow here.

The crucial step to derive the homogeneous BSE for bound state is the assumptions that the bound state reflects itself in a pole in the four-point Green’s function for an on-shell momentum \(P, P^2 = M_j^2\)

\[
G^{(4)}(p,p',P) = \sum_j \frac{-i}{(2\pi)^4} \frac{\psi_j(p,P_{os})\bar{\psi}_j(p',P_{os})}{2E_{p_j}(P^0 - E_{p_j} + i\epsilon)} + \text{reg. term} \quad (1.1)
\]

where \(E_{p_j} = \sqrt{p^2 + M_j^2}\), \(M_j\) is a positive mass of the bound state characterized by BS wave function \(\psi_j\) within a set of quantum numbers \(j\). Then the BSE can be conventionally
written in momentum space like:

\[
S_1^{-1}(p_+, P)\psi(p, P)S_2^{-1}(p_-, P) = -i \int \frac{d^4k}{(2\pi)^4} V(p, k, P)\psi(p, P),
\]

\[
p_+ = p + \alpha P,
\]
\[
 p_- = p - (1 - \alpha)P,
\]
or equivalently written in the term of BS vertex function \( \Gamma \):

\[
\Gamma(p, P) = -i \int \frac{d^4k}{(2\pi)^4} V(p, k, P)S_1(k_+, P)\Gamma(p, P)S_2(k_-, P),
\]

where we suppress all the Dirac, flavor and Lorentz indices and \( \alpha \in (0, 1) \). The function \( V \) represents two body irreducible interaction kernel and \( S_i, i = 1, 2 \) are the dressed propagators of the constituents. The free propagators reads

\[
S_0^i(p) = \frac{\not{p} + m_i}{p^2 - m_i^2 + i\epsilon},
\]

Concerning the solution of BSE (1.3) for pseudoscalar meson, it has the generic form \[12\]:

\[
\Gamma(q, P) = \gamma_5 [\Gamma_A + \Gamma_B q.P \not{q} + \Gamma_C \not{P} + \Gamma_D \not{q} P + \Gamma_E \not{P} \not{q}]
\]

where \( \Gamma_i, i = A, B, C, D, E \) are the scalar functions of their arguments \( P, q \). If the bound state has well defined charge parity, say \( \mathcal{C} = 1 \) then they are even function of \( q.P \) and \( \Gamma_D = -\Gamma_E \).

As was already discussed in the paper \[13\], the dominant contribution to the BSE calculation for pseudoscalar meson comes namely from the first term in Eq. (1.5). Within 15 percentage accuracy it is already true for the light mesons (\( \pi, K, \eta \)) while in the case of heavy meson pseudoscalar ground state (\( \eta_c, \eta_b \)) the contributions due to the other tensor components in Eq. (1.5) are even more negligible. Hence, in this stage of our Minkowski calculation we also approximate our solution by taking \( \Gamma = \gamma_5 \Gamma_A \).

The interaction kernel is approximated by the dressed gluon propagator with the interaction gluon-quark-antiquark vertices taken in their bare forms, thus we can write

\[
V(p, q, P) = g^2(\kappa)D_{\mu\nu}(p - q, \kappa)\gamma^\nu \otimes \gamma^\mu,
\]

where the full gluon propagator is renormalized at the scale \( \kappa \) and the effective strong running coupling are related through the following equations:

\[
g^2(\kappa)D_{\mu\nu}(l, \kappa) = \alpha_s(l, \kappa)\frac{P^T_{\mu\nu}(l)}{l^2 + i\epsilon} - \xi g^2(\kappa)\frac{l_\mu l_\nu}{l^4 + i\epsilon},
\]

\[
\alpha_s(q, \kappa) = \frac{g^2(\kappa)}{1 - \Pi(q^2, \kappa)},
\]
\[
P^T_{\mu\nu}(l) = -g_{\mu\nu} + \frac{l_\mu l_\nu}{l^2}.
\]

From the class of \( \xi \) linear covariant gauges the Landau gauge \( \xi = 0 \) is employed thorough the presented paper.
In the next Section we will derive the solution for the dressed ladder approximation of BSE, i.e. the all propagators are considered as dressed ones and no crossed diagrams are taken into the calculation. The BSE for quark-antiquark states has been many times considered in Euclidean space even beyond ladder approximation. Notably, the importance of dressing the proper vertices in the light quark sector has been already stressed in \[14\], so our approximations is known to be certainly limited. Going beyond rainbow (\(\gamma_\mu\) approximation) is straightforward but rather involved (for comparison, see the SDE Minkowski study published in \[15\] and in \[16\], wherein the later study includes the minimal gauge covariant vertex instead of bare one). In this paper we prefer to explain the computational method instead of making a BSE study with the most sophisticated kernel known in the literature.

The layout of the papers is as follows: In the next section we describe the method of the solution. As an exhibition of the method, the numerical results are presented in the Section 3. We conclude in Section 4.

II. INTEGRAL REPRESENTATION AND THE SOLUTION OF BSE

In this section we describe our method of the BSE solution in Minkowski space. It basically assumes that the various Greens functions that appears in the interaction kernel of BSE can be written as a weighted integral over the various spectral functions (the real distribution) \(\rho\).

More explicitly stated, the full quark and gluon (in Landau gauge) propagators are assumed to satisfy the standard Lehmann representation, which reads

\[
S(l) = \int_{0}^{\infty} d\omega \frac{\rho_v(\omega) I + \rho_s(\omega)}{l^2 - \omega + i\epsilon},
\]

\[
G_{\mu\nu}(l) = \int_{0}^{\infty} d\omega \frac{\rho_g(\omega)}{l^2 - \omega + i\epsilon} P_{\mu\nu}(l),
\]

The representations \[2.1\] \[2.2\] are strictly related with the singularity structure and analytical properties of functions used for. Until now, within a certain limitations, these integral representations have been used for nonperturbative evaluation of Greens functions in various models \[17\], noting here, the true analytical structure of QCD Greens functions is not answered (see also \[19, 20\]) with reliability.

Alternatively, the integral representation can be more complicated and generalized to the product of such weighted integrals. In fact, our treatment shall counts more general integral representation for quark propagator like the one recently considered in the paper \[22\]. Also the following alternative but notable possibility for gluon propagator, which read now:

\[
G_{\mu\nu}(l) = \frac{1}{l^2 + i\epsilon} \int_{0}^{\infty} d\omega \frac{\hat{\rho}_g(\omega)}{l^2 - \omega + i\epsilon} P_{\mu\nu}^T(l),
\]

will be automatically included in our treatment. Note, this is effectively the gluon form factor rather then gluon propagator itself which satisfies the appropriate integral representation in this case. Such expression, motivated by analyticized running coupling method \[18\], has been recently considered in the studies of dynamical chiral symmetry breaking in various approximations \[21, 22, 23\].

Further, here we generalize the idea of Perturbation Theory Integral Representation (PTIR) \[8\] to our case. The PTIR represents unique integral representaion for n-point
Green function defined by n-leg Feynman integral. The inclusion of dressed propagator representing by the internal lines of (skeleton) Feynman diagram is self-consistently achieved by their own IR (see (2.2, srfgluon)).

The generalized PTIR formula for n-point function in theory involving fields with arbitrary spin is exactly the same as in the original scalar theory considered in [8] but the spectral function receives nontrivial tensor structure. Let us denote \( \rho(\alpha, x_i) \) such generalized weight function. Then, it can be clearly decomposed into the following sum

\[
\rho(\alpha, x_i)_{\text{scal.th.}} \rightarrow \sum_j \rho_j(\alpha, x_i) P_j
\]

where \( \alpha, x_i \) represent the set of spectral variables and where \( j \) runs over the all possible independent combinations of Lorentz tensors and Dirac matrices \( P_j \). The function \( \rho_j(\alpha, x_i) \) represents just the PTIR weight function of \( j \)-form factor (the scalar function by its definition) since it can clearly written as a suitable scalar Feynman integral. Leaving the question of (renormalization) scheme dependence, we refer the reader to Nakanishi book [8] for a detailed derivation of PTIR.

Let us apply our idea to the pseudoscalar bound state vertex function. Recognizing that the singularity structure (given by denominators) of the rhs. of the BSE is the same as in the scalar models studied in [10, 11] the appropriate IR for pseudoscalar bound state vertex function \( \Gamma_A(q, P) \) has to read

\[
\Gamma_A(q, P) = \int_0^\infty d\omega \int_{-1}^1 dz \frac{\rho_A^{[N]}(\omega, z)}{[F(\omega, z; P, q)]^N},
\]

where we have introduced useful abbreviation for the denominator of the IR (2.5)

\[
F(\omega, z; P, q) = \omega - (q^2 + q.Pz + P^2/4) - i\epsilon,
\]

where \( N \) is an integer free parameter. The same IR could apply for the other components of the full function \( \Gamma \) which however will not be considered furthermore.

Substituting IR’s (2.3), (2.2), (2.1) into the rhs. of BSE (1.3) one can analytically integrate over the loop momenta. Assuming Theorem of uniqueness [8] we should arrive to the same IR (2.5) because of the rhs. of BSE (1.3). The derivation is presented in the Appendix A for the cases \( N = 1, 2 \).

In the other words, we have converted the momentum BSE (with singular kernel) to the homogeneous two dimensional integral equation for the real weight function \( \rho_A^{[N]}(\omega, z) \)

\[
\rho_A^{[N]}(\bar{\omega}, \bar{z}) = \int_0^\infty d\omega \int_{-1}^1 dz V^{[N]}(\bar{\omega}, \bar{z}; \omega, z) \rho_A^{[N]}(\omega, z)
\]

where the kernel \( V^{[N]}(\bar{\omega}, \bar{z}; \omega, z) \) is the regular multivariable function.

The kernel \( V^{[N]} \) also automatically supports the domain \( \Omega \) where the function \( \rho_A^{[N]}(\omega, z) \) is nontrivial. This domain is always smaller then the infinity strip \( 0, \infty \times -1, 1 \) as is explicitly assumed by the integral boundaries of \( \omega, z \) integrals. For instance, for simplest kernel parametrized by the free gluon propagator and with constituent quarks of mass \( m \), then for the flavor singlet meson we have \( \rho_A^{[N]}(\omega, z) \neq 0 \) only for \( \omega > m^2 \).

In our approach, to solve the momentum BSE in Minkowski space is equivalent to the finding of the real solution of the real integral equation (2.7). No special choice of the frame or has been used. If one needs the resulting vertex function can be obtained by the numerical integration over the \( \rho_N \) in an arbitrary reference frame.
III. NUMERICAL RESULTS

In this section we discuss a numerical solutions of the BSE with a various interacting kernels. For this purpose we will vary the coupling strength and the effective gluon mass $m_g$ as well as. We mainly concern on the range of binding energy that coincides with the heavy quarkonia - the object of future interest of us. Also we take a discrete mass $m_g$ such that it runs from zero to the value of constituent mass of quarks, these values are expected to be relevant for the case of the real gluon propagator (when $m_g$ is replaced by the continuous spectral variable $\omega$ (2.2)). Thus, in each case, the corresponding gluon density is $\rho_g(c) = N_g \delta(c - m_g^2)$, which specify the kernel of BSE to be (in Landau gauge):

$$V(q - p) = g^2 \frac{g_{\mu\nu} + \frac{(q-p)_\mu(q-p)_\nu}{(q-p)^2}}{(q-p)^2 - m_g^2 + i\epsilon} \gamma^\nu \otimes \gamma^\mu$$  \hspace{1cm} (3.1)

where the prefactor (including the trace of color matrices) is simply absorbed into the coupling constant. For our actual calculation we have used the bare constituent propagators $S_i(p_i)$ with heavy quark mass $M$ (see the Appendix A for this approximation).

Firstly, we followed the standard procedure: after fixing the bound state mass ($P^2$) we look for the solution by iterating the BSE for spectral function with fixed coupling constant $\alpha = g^2/(4\pi)$. Very similar to the scalar case [11], the choice $N = 2$ of the power of $F$ in the IR for the bound state vertex function is preferable one. It reasonably make provision for a numerical errors from one side and the computational obstacles for high $N$ from the other side, noting here that using $N = 1$ is rather unhelpful (comparing with massive Wick-Cutkosky model) since we did not find the stable solution for a wide class of input parameters $g, m_g$ for that choice. On the other hand, using the value $N = 2$, we have found the results for the all possible interaction kernels considered here. It also includes the cases with vanishing $m_g$, which means the numerical troubles originally presented in the scalar models [11] are fully overcame here. The details of our numerical treatment are presented in Appendix B.

As it is more usually in the non-relativistic case, we fix the coupling constant $\alpha = g^2/(4\pi)$ and then look for the bound state mass spectrum. We have found the same result in the both cases, whether we fixed $P$ or $\alpha$ as a first, noting that in the later case the whole integration in the kernel $K$ needs to be performed at each iteration step, which makes the problem more computer time consuming.

The obtained solutions for $\alpha'$s for given fractional binding $\eta = \sqrt{P^2}/2m = 0.95$ and various masses $m_g$ are displayed in the Table I. When we have fixed the gluon mass as $m_g = 0.5$ and varied the fractional binding binding $\eta$ we have obtained the spectra as they are displayed in the Table II.

| $m_g/m_q$ | 0.0001 | 0.1 | 0.5 |
|-----------|--------|-----|-----|
| $\alpha$  | 0.666  | 0.669| 0.745| 1.029|

TABLE 1. Coupling constant $\alpha = g^2/(4\pi)$ for several selections of $m_g/m_q$ ($m_q$ is the constituent quark mass) with given fraction of binding $\eta = \sqrt{P^2}/2m = 0.95$.

| $\eta$ | 0.8 | 0.9 | 0.95 | 0.99 |
|--------|-----|-----|------|------|
| $\alpha$ | 1.20 | 1.12 | 1.03 | 0.816 |
The main result of this paper is development of technical framework for the solution of
Bethe-Salpeter equation in Minkowski space. To obtain the spectrum, no preferred reference
frame is needed and the wave function can be obtained in an arbitrary frame (without
numerical boosting) by simple integration of the weight function.

The treatment is based on the usage Integral representation for Green's function of given
theory, including the bound state vertices themselves. The method has been explained and

FIG. 1: The rescaled weight function \( \tau = \frac{\rho^{[2]}(\omega, z)}{\omega^2} \) for the following parameters of the model:
\( \eta = 0.95, \ m_g = 0.001M_q, \ \alpha_s = 0.666, \) the small mass \( m_g \) approximates the one gluon exchange
interaction kernel.

TABLE 2. Coupling \( \alpha_s = g^2/(4\pi) \) as a function of fraction of binding \( \eta = \sqrt{P^2/2m} \) for
exchanged massive gluon with \( m_g = 0.5m_q, \ m_q \) is the constituent quark mass.

For the illustration the weight function \( \tilde{\rho}^{[2]} \) is displayed in Fig. 1.
checked numerically on the samples of fermion-antifermion pseudoscalar bound state. It was shown that the momentum space BSE can be converted to the real equation for the real weight functions $\rho$ which is easily solvable numerically. The main motivation of the author was to develop the practical tool that respect the self-consistence of Schwinger-Dyson and Bethe-Salpeter equations. Generalizing this study to the other mesons (vectors, scalars), considering more general flavor (izospin) structure with the simultaneous improvement of the approximations (correctly dressed gluon propagator, dressed vertices, etc.) will be a basic steps towards the fully Lorentz covariant description of plethora of transitions and form factors in timelike region of fourmomenta (Minkowski space).

**APPENDIX A: KERNEL FUNCTIONS**

Writing explicitly Dirac indices the BSE for quark-antiquark bound state reads

$$\Gamma(q, P)_{\omega \rho} = i \int \frac{d^4 q}{(2\pi)^4} S(q + P/2)_{\beta \gamma} \Gamma(q, P)_{\gamma \gamma'} S(q - P/2)_{\gamma' \beta} V_{\omega \beta \rho}(q, k; P)$$  \hspace{1cm} (A1)

where the Lorentz indices of the vertex function has not been specified.

In our approximation the integral representation for pseudoscalar bound state vertex is

$$\Gamma(q, P)_{\alpha \beta} = \gamma_5 \int_0^\infty d\omega \int_{-1}^1 dz \int_0^\infty dc \int_0^\infty da \int_0^\infty db \rho^N(\omega, z, g^2 \rho_g(c))$$  \hspace{1cm} (A2)

and the generalized kernel for ladder approximation of the BSE reads

$$V_{\alpha \beta \gamma \delta} = g^2 \int_0^\infty dc \frac{\rho^N(\omega, z)}{[F(\omega, z; P, q)]^N}$$  \hspace{1cm} (A3)

where $\alpha, \beta, \gamma, \delta(\mu, \nu)$ indices show up the appropriate Dirac (Lorentz) structure. In the addition we use the IR for the all functions entering the BSE, this includes (A2,A3) and the ones for propagators $S(q \pm P)$ \hspace{1cm} (2.1).

For purpose of brevity we will use the following abbreviation for the prefactor:

$$\int_S \equiv -3 \int_0^\infty d\omega \int_{-1}^{+1} dz \int_0^\infty dc \int_0^\infty da \int_0^\infty db \rho^N(\omega, z, g^2 \rho_g(c))$$  \hspace{1cm} (A4)

Within this convention the BSE can be written like

$$\int d\omega d\bar{\omega} \int_0^\infty d\omega \int_0^\infty d\bar{\omega} \rho^N(\omega, \bar{\omega}, c) = \int_S \int \frac{d^4 q}{(2\pi)^4} \frac{\rho(a)\rho(b)(q^2 - P^2/4) - \rho(a)\rho(b)}{[F(\omega, z; q, P)]^N D_1 D_2 D_3},$$  \hspace{1cm} (A5)

where the trace was taken over the Dirac indices (after multiplying by $\gamma_5$) and

$$D_1 = (q + P/2)^2 - a + i\epsilon,$$

$$D_2 = (q - P/2)^2 - b + i\epsilon,$$

$$D_3 = (q - P)^2 - c + i\epsilon.$$  \hspace{1cm} (A6)

In what follows, we will transform the rhs. of BSE (A5) into the integral representation, ie. lhs. of BSE (A5).
As a first we use the following algebraic identities:

\[
\frac{q^2}{F(\omega, z; q, P)} = \frac{\omega - q.Pz - P^2/4}{F(\omega, z; q, P)} - 1, \quad (A7)
\]

\[
\frac{q.P}{D_1 D_2} = \frac{1}{2} \left( \frac{1}{D_1} - \frac{1}{D_2} + \frac{b - a}{D_1 D_2} \right), \quad (A8)
\]

which gives us for the rhs. of \((A5)\):

\[
i \int \int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{-\tilde{z} \rho_{\nu}(a) \rho_{\nu}(b)}{[F(\omega, z; q, P)]^3 D_3} \left( \frac{1}{D_1} - \frac{1}{D_2} \right) \right.
\]

\[
+ \frac{(\omega - P^2/2 + \frac{a-b}{2} z) \rho_{\nu}(a) \rho_{\nu}(b) - \rho_s(a) \rho_s(b)}{[F(\omega, z; q, P)]^3 D_1 D_2 D_3} - \frac{\rho_{\nu}(a) \rho_{\nu}(b)}{[F(\omega, z; q, P)]^{3-N} D_1 D_2 D_3}, \quad (A9)
\]

In addition we will use Feynman parametrization, starting with the first term in the Rel. \((A9)\) we can get:

\[
\frac{1}{[F(\omega, z; q, P)]^N} \left( \frac{1}{D_1} - \frac{1}{D_2} \right) = (-)^N \int_0^1 dx \frac{\Gamma(N+1)x^{N-1}}{\Gamma(N)}
\]

\[
\left\{ \left[ q^2 + q.P(zx - (1-x)) + P^2/4 - \omega x - a(1-x) \right]^{-N-1}
\]

\[- \left[ q^2 + q.P(zx + (1-x)) + P^2/4 - \omega x - b(1-x) \right]^{-N-1} \right\}. \quad (A10)
\]

Substituting \((A10)\) back to the relation \((A9)\), using the Feynman variable \(y\) for matching with the scalar propagator \(D_3\) and integrating over the four-momentum \(q\) we get for the first line in \((A9)\) the following expression:

\[
(-)^{N-1} \int \frac{\rho_{\nu}(a) \rho_{\nu}(b)}{2(4\pi)^2} \int_0^1 dx \int_0^1 dy y^N x^{1-N} z
\]

\[
\left\{ \left[ \frac{P^2}{4} \left[y - y^2(x(1+z) - 1)^2\right] + q^2(1-y)y + P.Py(1-y)(x(1+z) - 1) - U(a) \right]^{-N}
\]

\[- \left[ \frac{P^2}{4} \left[y - y^2(x(z-1) + 1)^2\right] + q^2(1-y)y + P.Py(1-y)(x(z-1) + 1) - U(b) \right]^{-N} \right\}. \quad (A11)
\]

where we have labeled

\[
U(a) = [\omega x + a(1-x)] y + c(1-y). \quad (A12)
\]

Taking the substitution \(x \to \tilde{z} \) such that \(\tilde{z} = x(1+z) - 1 \) in the firts line inside the braces we can write (including prefactors):

\[
(-)^{\frac{1}{\pi}} \int \frac{dy}{(4\pi)^2} \int_{-1}^1 d\tilde{z} \left( 1 + \frac{\tilde{z}}{1+\tilde{z}} \right)^{N-1} \frac{z}{1+\tilde{z}} \frac{\rho_{\nu}(a) \rho_{\nu}(b)}{2(1-y)^N \left[ F(\tilde{\Omega}, \tilde{z}; p, P) \right]^N}, \quad (A13)
\]

where \(F\) is defined by \((2.6)\) and

\[
\tilde{\Omega} = \frac{\left( \omega + \frac{\tilde{z}}{1+\tilde{z}} \right) y + c(1-y) - P^2/4 y^2(1-\tilde{z}^2)}{y(1-y)}. \quad (A14)
\]
Introducing the identity
\[ 1 = \int_0^\infty d\tilde{\omega} \delta(\tilde{\omega} - \Omega) \] (A15)
into the Rel. (A14), changing the integral ordering and integrating over the Feynman variable \( y \) we can obtain the desired expression:
\[
\int_0^\infty d\tilde{\omega} \int_{-1}^1 d\tilde{z} \frac{\chi_1(\tilde{\omega}, \tilde{z})}{[F(\tilde{\omega}, \tilde{z}; p, P)]^N};
\]
\[
\chi_1(\tilde{\omega}, \tilde{z}) = -\int_S \frac{T^{N-1}_+(z\rho_v(a)\rho_v(b)\theta(z - \tilde{z})}{2(4\pi)^2 (1 + z)} \sum_{j=\pm} y_{aj} \theta(D_a) \theta(y_{aj}) \theta(1 - y_{aj}) \frac{1}{(1 - y_{aj})^{N-1} \sqrt{D_a}},
\]
(A16)
where \( y_{a\pm} \) are the roots of the quadratic equation
\[ y^2 A + y B_a + c = 0 \] (A17)
with the functions \( A, B_a, D_a \) defined like
\[ A = \tilde{\omega} - S; \quad B_a = (\omega - a)T_+ + a - c - \tilde{\omega}; \quad D_a = B_a^2 - 4Ac \]
\[ S = (1 - \tilde{z}^2) \frac{P^2}{4}; \quad T_{\pm} = \frac{1 \pm \tilde{z}}{1 \pm z}. \] (A18)
Repeating the similar procedure in the second term in rel. (A11) we can write for this:
\[
\chi_2(\tilde{\omega}, \tilde{z}) = \int_S \frac{T^{N-1}_+(z\rho_v(a)\rho_v(b)\theta(z - \tilde{z})}{2(4\pi)^2 (1 + z)} \sum_{j=\pm} y_{bj} \theta(D_b) \theta(y_{bj}) \theta(1 - y_{bj}) \frac{1}{(1 - y_{bj})^{N-1} \sqrt{D_b}},
\]
y_{b\pm} = \frac{-B_b \pm \sqrt{D_b}}{2A}; \quad T_+ = (\omega - b)T_+ + b - c - \tilde{\omega}; \quad D_b = B_b^2 - 4Ac \quad (A19)
where we used the label \( \chi_2 \) instead of \( \chi_1 \).

In what follows we transform the last term of the second line of Rel. (A9) to the desired form of integral representation (2.5). For this purpose we basically follow the derivation already presented in [11], since the procedure is exactly the same. However, due to notational difference we present all calculational details below.

Let us denote the relevant integral as:
\[
I = i \frac{d^4q}{(2\pi)^4} D_1 D_2 D_3 [F(\omega, z; q, P)]^{N-1} \] (A20)
Using the Feynman parameterization technique we first write
\[
D_1 D_2 = \frac{1}{2} \int_{-1}^1 d\eta \frac{1}{[M^2 - f(q, P, \eta) - i\epsilon]^2};
\]
\[
M^2 \equiv \frac{a + b}{2} + \frac{a - b}{2} \eta; \quad f(q, P, \eta) = q^2 + \eta q.P + P^2/4. \] (A21)
Next the denominator of IR for bound state vertex is added:
\[
\frac{D_1 D_2}{[F(\omega, z; q, P)]^{N-1}} = \frac{\Gamma(N + 1)}{2\Gamma(N - 1)} \int_{-1}^1 d\eta \int_{-1}^1 dt \frac{(1 - t)^{N-2}}{[R - f(q, P, \tilde{z}) - i\epsilon]^{N+1}};
\]
\[
R = \omega t + (1 - t)M^2, \quad (A22)
\]
where \( \tilde{z} = tz + (1 - t)\eta \). Now, we match with the function \( D_3 \) and integrate over the four momentum \( q \). Thus we get

\[
I = i \int \frac{d^4 q}{(2\pi)^4} \frac{D_1 D_2 D_3}{[F(\omega, z; q, P)]^{N-1}} \tag{A23}
\]

\[
= -i \frac{\Gamma(N+2)}{2\Gamma(N-1)} \int_{-1}^{1} d\eta \int_{0}^{1} dt \, (1-t)t^{N-2} \int_{0}^{1} dx \, x^N \, I_q,
\]

\[
I_q = \int \frac{d^4 q}{(2\pi)^4} \left[ -q^2 + q \cdot Q - (1-x)p^2 - \frac{x}{4} P^2 + (1-x)c + xR - i\epsilon \right]^{-N+2} \]

\[
= i \frac{\Gamma(N)}{(4\pi)^2 \Gamma(N+2)} \int_{0}^{1} dx \frac{1}{x^N(1-x)^N} \frac{1}{[\Omega(t) - f(q, P, \tilde{z}) - i\epsilon]^N} \tag{A24}
\]

\[
\Omega(t) \equiv \frac{R}{1-x} + \frac{c}{x} - \frac{x}{(1-x)} S, \tag{A25}
\]

where \( Q = (1-x)p - x\tilde{z}P/2 \). The function \( S \) is defined by (A18) and note \( \tilde{z} \) lies in the interval \( -1, +1 \), \( 0 \leq S < (\sqrt{a} + \sqrt{b})^2/4 \). Interchanging the integrals over \( \eta \) and \( t \) such that:

\[
\int_{-1}^{1} d\eta \int_{0}^{1} dt = \int_{-1}^{1} d\tilde{z} \left[ \int_{0}^{T_+} \frac{dt}{1-t} \Theta(z - \tilde{z}) + \int_{0}^{T_-} \frac{dt}{1-t} \Theta(\tilde{z} - z) \right],
\]

\[
T_{\pm} = \frac{1 \pm \tilde{z}}{1 \pm z} \quad \text{and} \quad \tilde{z} = tz + (1-t)\eta,
\]

we finally obtain

\[
I = \frac{N-1}{2(4\pi)^2} \int_{-1}^{1} d\tilde{z} \int_{0}^{1} dx \frac{d}{x} \sum_{s=\pm} \Theta(s(z - \tilde{z})) \int_{0}^{T_s} \frac{dt \, t^{N-2}}{[F(\Omega(t), \tilde{z}; p, P)]^N}, \tag{A26}
\]

In addition we make \( t \)-dependence of \( F(\Omega(t), \tilde{z}; p, P) \) explicit

\[
F(\Omega(t), \tilde{z}; p, P) = \frac{J(\omega, z)}{1-x} t + F(\Omega(0), \tilde{z}; p, P). \tag{A27}
\]

where

\[
\Omega(t) = \frac{R(t) - S}{1-x} + \frac{c}{x} + S,
\]

\[
R(t) = J(\omega, z)t + \frac{b + a}{2} - \frac{b - a}{2} \tilde{z},
\]

\[
J(\omega, z) = \omega - \frac{b + a}{2} - \frac{b - a}{2} z. \tag{A28}
\]

Integrate over the variable \( t \) we get

\[
\int \frac{dt \, t^{N-2}}{F(\Omega(t), \tilde{z}; p, P)^N} = \frac{t^n}{(N-1) F(\Omega(0), \tilde{z}; p, P) [F(\Omega(t), \tilde{z}; p, P)]^{N-1}}. \tag{A29}
\]
and hence

$$I = \frac{1}{2(4\pi)^2} \int_{-1}^{1} d\tilde{z} \int_{0}^{1} dx \sum_{n=\pm} \frac{\theta[s(z - \tilde{z})]T_s^{N-1}}{(1-x)^N F(\Omega(0), \tilde{z}; p, P) [F(A(T_s), \tilde{z}; p, P)]^{N-1}}.$$

(A30)

In order to separate F’s in the denominator we use the following identity

$$\frac{1}{F(\Omega(0), \tilde{z}; p, P) [F(\Omega(t), \tilde{z}; p, P)]^{N-1}} = \frac{1-x}{J(\omega, z)T_s} \left[ \frac{1}{F(\Omega(0), \tilde{z}; p, P)} - \frac{1}{F(A(T_s), \tilde{z}; p, P)} \right] \frac{1}{[F(A(T_s), \tilde{z}; p, P)]^{N-2}}.$$

(A31)

Note, for a given a given $N$ one can repeat N-1 times this algebra until the power of the last factor vanishes, which is just the reason of introducing the operation (A31). After this operation the momentum dependence of the denominators at each term turns to be formally the same as in the desired IR. Although it is possible to derive the appropriate formula for any $N$, it would lead to an unpractical expressions (probably it cannot be written in a closed form) and we rather choose one concrete value of $N$. Motivated by the success of the scalar model studies, we take $N = 2$ from now. Explicitly written we have for $I$

$$\frac{1}{2(4\pi)^2} \int_{-1}^{1} d\tilde{z} \int_{0}^{1} dx \sum_{n=\pm} \frac{\theta[s(z - \tilde{z})]}{J(\omega, z)(1-x)} \left[ \frac{1}{F(\Omega(0), \tilde{z}; p, P)} - \frac{1}{F(A(T_s), \tilde{z}; p, P)} \right]$$

(A32)

Integrating per-partes in variable $x$ we get for (A32)

$$\frac{-1}{2(4\pi)^2} \int_{-1}^{1} d\tilde{z} \int_{0}^{1} dx \sum_{n=\pm} \frac{\theta[s(z - \tilde{z})] \ln(1-x)}{J(\omega, z)} \left[ \frac{d\Omega(0)}{dx} \frac{1}{F^2(\Omega(0), \tilde{z}; p, P)} - \frac{dA(T_s)}{dx} \frac{1}{F^2(A(T_s), \tilde{z}; p, P)} \right].$$

(A33)

Implementing the following identity into the integrand of (A33)

$$1 = \int_{0}^{\infty} \delta(\tilde{\omega} - \Omega(t))$$

(A34)

into the integrand of (A33) and changing the order of the integration and performing integration in the variable $x$, we arrive to the desired result for the second term in the second line of the Rel. (A9):

$$\int_{0}^{\infty} d\tilde{\omega} \int_{-1}^{1} d\tilde{z} \frac{\chi_3(\tilde{\omega}, \tilde{z})}{[F(\tilde{\omega}, \tilde{z}; p, P)]^N},$$

$$\chi_3(\tilde{\omega}, \tilde{z}) = \int_{S} \rho_u(a)\rho_v(b) \sum_{j=\pm} \left\{ \frac{\ln(1-x_j(0))}{J(\omega, z)} \theta[D(0)]\theta[x_j(0)]\theta[1-x_j(0)] \text{sgn} \left[ \frac{dA(x_j(0))}{dx_j(0)} \right] ight\}$$

$$- \sum_{n=\pm} \theta[s(z - \tilde{z})] \frac{\ln(1-x_j(T_s))}{J(\omega, z)} \theta[D(0)]\theta[x_j(T_s)]\theta[1-x_j(T_s)] \text{sgn} \{E[x_j(T_s)]\}$$

(A35)
where we have included omitted prefactor in \((A34)\) and \(x_j\) are the roots of the delta function argument in \((A34)\):

\[
x_{\pm}(T) = \frac{-B(T) \pm \sqrt{D(T)}}{2A}
\]

\[
D(T) = B(T)^2 - 4Ac, \quad B(T) = R(T) - \bar{\omega} - c
\]

\[
\frac{d\Omega(t)}{dx} = \frac{E(x)}{1-x}, \quad E(x) = \bar{\omega} - S - \frac{c}{x^2}.
\]

Introducing a bit compact notation for the sum over an arbitrary function \(U\) with parameter \(T\)

\[
\sum_T U(T) \equiv U(0) - \theta(z - \bar{z})U(T_+) - \theta(\bar{z} - z)U(T_-)
\]

we can rewrite the above relation \((A35)\) like

\[
\chi_3(\bar{\omega}, \bar{z}) = \int_{\mathbb{R}} \rho_v(a) \rho_v(b) \frac{1}{2(4\pi)^2} \sum_{j=\pm} \ln(1 - x_j(T)) \theta[D(T)] \theta[x_j(T)] \theta[1 - x_j(T)] \text{sgn} E[x_j(T)].
\]

The first term in the second line of Rel.\((A39)\) should be derived can treated in very similar fashion as the previous case, note only a different power of \(F\) in the denominator. Doing this explicitly and adding a correct prefactor the appropriate expression reads:

\[
\chi_4(\bar{\omega}, \bar{z}) = \int_{\mathbb{R}} \rho_v(a) \rho_v(b) \frac{1}{2(4\pi)^2} \sum_{j=\pm} \ln(1 - x_j(T)) \theta[D(T)] \theta[x_j(T)] \theta[1 - x_j(T)] \text{sgn} E[x_j(T)].
\]

Assuming the validity of Theorem o the Uniqueness then the momentum BSE is converted to the equation for the weight function. It reads

\[
\rho^{[2]}(\bar{\omega}, \bar{z}) = c \int_{0}^{\infty} ds \int_{-1}^{1} dz V^{[2]}(\bar{\omega}, \bar{z}; \omega, z) \rho^{[2]}(\omega, z),
\]

where the kernel is simply given by the sum of contributions derived above

\[
V^{[2]}(\bar{\omega}, \bar{z}; \omega, z) = -3g^2 \int_{0}^{\infty} dc \int_{0}^{\infty} da \int_{0}^{\infty} db \rho_q(c) \rho_q(b) \sum_{i=1}^{4} \chi_i(\bar{\omega}, \bar{z}; \omega, z).
\]

**Heavy quark approximation -unequal mass case**

When the quark is sufficiently heavy (say \(m_q(2\text{GeV}) >> \Lambda_{QCD}\)) then the approximation of quark mass function by a constant appears to be adequate. Neglecting the selfenergy correction is equivalent to use a free heavy quark propagator, this corresponds with the free particle spectral functions:

\[
M_1 \rho_v(a) = \rho_s(a) = M_1 \delta(a - M^2),
\]

\[
M_2 \rho_v(b) = \rho_s(b) = M_2 \delta(b - M^2),
\]
where the variable $a, b$ distinguish the type of quark from which the bound state is composed. For completeness we write down the kernel explicitly here for this case.

\[
V^{[2]}(\tilde{\omega}, \tilde{z}; \omega, z) = \frac{-3g^2}{2(4\pi)^2} \int_0^\infty dc \rho_g(c) (\chi_1 + \chi_2 + \chi_3 + \chi_4); \quad (A43)
\]

\[
\chi_1 = -T_+ \frac{z\theta(z - \tilde{z})}{1 + z} \sum_{\pm} \frac{y_{a\pm} \theta(D_a) \theta(y_{a\pm}) \theta(1 - y_{a\pm})}{(1 - y_{a\pm}) \sqrt{D_a}},
\]

\[
\chi_2 = T_+ \frac{z\theta(z - \tilde{z})}{1 - z} \sum_{\pm} \frac{y_{b\pm} \theta(D_b) \theta(y_{b\pm}) \theta(1 - y_{b\pm})}{(1 - y_{b\pm}) \sqrt{D_b}},
\]

\[
\chi_3 = \sum_T \sum_{j=\pm} \ln(1 - x_j(T)) \frac{\theta[D(T)] \theta[x_j(T)] \theta[1 - x_j(T)] \text{sgn} E[x_j(T)]}{J(\omega, z)},
\]

\[
\chi_4 = \left( \omega - \frac{P^2}{2} + \frac{M_1^2 - M_2^2}{2} z - M_1 M_2 \right)
\]

\[
\sum_T \sum_{j=\pm} \frac{\theta[D(T)] \theta[x_j(T)] \theta[1 - x_j(T)]}{J^2(\omega, z)} \left\{ \frac{T J(\omega, z)}{(1 - x_j(T))|E[x_j(T)]| - \ln(1 - x_j(T)) \text{sgn} E[x_j(T)]} \right\}
\]

and where the arguments $a, b$ in the functions $x, J, D$ and must be replaced by the quark masses $M_1, M_2$.

**Equal mass case**

In the case of quarkonia the kernel becomes more symmetric with respect to the variable $z$ and the formula for the kernel further simplify. The function $R$ depends on $z$'s only through the variable $T$ such that

\[
J(\omega, z) \rightarrow J = \omega - M^2; \quad R(T) \rightarrow R(T) = JT + M^2 \quad (A45)
\]

where $M$ is a common mass $M = M_1 = M_2$. The roots become

\[
y_{a\pm} \rightarrow x_{j=\pm}(T_+); \quad y_{b\pm} \rightarrow x_{j=\pm}(T_-), \quad (A46)
\]

in this case and the kernel can be written in more compact form:

\[
V^{[2]}(\tilde{\omega}, \tilde{z}; \omega, z) = \frac{-3g^2}{2(4\pi)^2} \int_0^\infty dc \rho_g(c) K^{[2]}(\tilde{\omega}, \tilde{z}; \omega, z, c) \quad (A47)
\]

\[
K^{[2]}(\tilde{\omega}, \tilde{z}; \omega, z, c) =
\]

\[
\sum_{s=\pm} \sum_{j=\pm} \theta[(s(z - \tilde{z}))] \theta[D(T_s)] \theta[x_j(T_s)] \theta[1 - x_j(T_s)] \frac{s z T_s x_j(T_s)}{(1 + s z) \sqrt{D(T_s) (1 - x_j(T_s))}}
\]

\[
\sum_T \sum_{j=\pm} \theta[D(T)] \theta[x_j(T)] \theta[1 - x_j(T)] \left\{ \frac{(1 - \frac{P^2}{2}) T}{|E[x_j(T)]|(1 - x_j(T))} + \ln(1 - x_j(T)) \text{sgn} E[x_j(T)] \frac{P^2}{2 J^2} \right\}
\]

(A48)
One gluon exchange approximation

In order to consider one gluon approximation we should take the massless limit $c \to 0$. In addition we also restrict ourselves to the equal mass case. One can easily recognize that the root $(x_\pm = 0)$ is trivial and associated contribution in (A48) vanishes.

For the purpose of brevity we label

$$x(T) \equiv x_-(T) = \frac{\omega - R(T)}{\bar{\omega} - \bar{S}}.$$  \hspace{1cm} (A49)

Taking into account the relations

$$\sqrt{D}/x = A, E = A,$$

and doing a little algebra we get for the kernel:

$$V_{OG,E}^{(2)}(\bar{\omega}, \bar{z}; \omega, z) = -\frac{3g^2}{(4\pi)^2} \left\{ \frac{P^2}{4J^2} \theta(\bar{\omega} - m^2) \ln \left( \frac{m^2 - S}{\bar{\omega} - S} \right) \right. \right.$$

$$+ \left. \sum_{s=\pm} \theta(s(z - \bar{z}))\theta(-B_s)\theta(A + B_s) \left[ \frac{T_s \left( \frac{1}{4} - \frac{P^2}{4J^2} \right)}{A + B_s} + \ln \left( 1 + \frac{B_s}{A} \right) \right] \right\}$$

For completeness we review the complete list of the functions at this place:

$$A = \bar{\omega} - S, \quad J = \omega - M^2, \quad S = (1 - \bar{z}^2)\frac{P^2}{4}$$

$$B_s = (\omega - M^2)T_s + M^2 - \bar{\omega}, \quad T_\pm = \frac{1 \pm \bar{z}}{1 \pm \bar{z}}.$$ \hspace{1cm} (A51)

Case $N=1$ for

Repeating the derivation for the parameter $N = 1$ we should obtain the following homogeneous equation

$$\rho^{[1]}(\bar{\omega}, \bar{z}) = \int_0^\infty d\omega \int_{-1}^1 dz V^{[1]}(\bar{\omega}, \bar{z}; \omega, z)\rho^{[1]}(\omega, z)$$ \hspace{1cm} (A52)

where the kernel is given by the following expression

$$V^{[1]}(\bar{\omega}, \bar{z}; \omega, z) = \int_S \frac{\sum_{i=1}^4 \chi_i}{(4\pi)^2},$$

$$\chi_1 = -\rho_v(a)\rho_v(b)\frac{z}{1 + z} \theta(z - \bar{z}) \sum_{j=\pm} \frac{y_{aj}\theta(D_a)\theta(y_{aj})\theta(1 - y_{aj})}{\sqrt{D_a}},$$

$$\chi_2 = \rho_v(a)\rho_v(b)\frac{z}{1 - z} \theta(\bar{z} - z) \sum_{j=\pm} \frac{y_{bj}\theta(D_b)\theta(y_{bj})\theta(1 - y_{bj})}{\sqrt{D_b}},$$

$$\chi_3 = \rho_v(a)\rho_v(b) \sum_{j=\pm} \theta[D(0)]\theta[x_j(0)]\theta[1 - x_j(0)] \left[ \frac{\rho_v(a)\rho_v(b) \left( \omega - \frac{P^2}{2} + \frac{a - b}{2} \right)}{E[x_j(T)]} \right] - \rho_s(a)\rho_s(A),$$

$$\chi_4 = \sum_T \sum_{j=\pm} \frac{\theta[D(T)]\theta[x_j(T)]\theta[1 - x_j(T)]}{E[x_j(T)]J(\omega, z)} \left[ \rho_v(a)\rho_v(b) \left( \omega - \frac{P^2}{2} + \frac{a - b}{2} \right) - \rho_s(a)\rho_s(A) \right].$$
where we have used the same notations and conventions as in the previous Section. The appropriate derivation is very straightforward and exactly repeats the steps made for the case with $N = 2$, hence we make only few comments here. The relations for $\chi_{1,2}$ are in fact derived in the previous section (since for general $N$). The relation for $\chi_4$ is adopted from the paper [11] and remaining function $\chi_3$ follows from the conversion of the term with $F^0 = 1$, i.e.

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{D_1D_2D_3}, \quad \text{(A54)}$$

which is presented in the momentum space BSE in that case. The appropriate derivation is in fact more easy then in the case $N = 2$ and the appropriate result represents the basic Perturbation Theory Integral Representation for a scalar triangle with one leg momentum constrained such that $P^2 < (M_1 + M_2)^2$.

**APPENDIX B: NUMERICAL PROCEDURE**

In this section we describe the numerical procedure actually used for obtaining the spectra of the bound states. Because of the structure of the integral equations to be solved, the treatment is similar the procedure used in the papers [10, 11], however we have used some tricky improvements modification here which enforce the numeric stability when the mass parameter $c$ in gluon IR (or simple vector boson mass $m_c$) is to small compared to the constituent mass. We describe the technical difference in that case. As a first we describe the numerical treatment for the case $c \approx m$.

The equation (A48) represents the homogeneous linear integral equation which solution needs to be properly normalized. Adopting the following auxiliary normalization condition for this purpose:

$$1 = \int_{-1}^{1} dz \int_{0}^{\infty} d\omega \frac{\rho^{[2]}(\omega, z)}{J^2}.$$

Then we can find that the equal constituent mass BSE can be transformed into the inhomogeneous integral equation (B2)

$$\rho^{[2]}(\tilde{\omega}, \tilde{z}) = K^{[2]}_I(\tilde{\omega}, \tilde{z}) + \int_{0}^{\infty} d\omega \int_{-1}^{1} dz K^{[2]}_H(\tilde{\omega}, \tilde{z}; \omega, z) \rho^{[2]}(\omega, z) \quad \text{(B2)}$$

with the following kernels:

$$K^{[2]}_I(\tilde{\omega}, \tilde{z}) = -\frac{3g^2}{(4\pi)^2} \frac{P^2}{4} \sum_{j = \pm} \theta[D(0)]\theta[x_j(0)]\theta[1 - x_j(0)] \ln(1 - x_j(0)) \text{sgn}E[x_j(0)], \quad \text{(B3)}$$

$$K^{[2]}_H(\tilde{\omega}, \tilde{z}; \omega, z, z', c) = \frac{3g^2}{2(4\pi)^2} \sum_{j = \pm} \theta[D(0)]\theta[x_j(0)]\theta[1 - x_j(0)] \ln(1 - x_j(0)) \frac{\text{sgn}E[x_j(0)]}{J} - 1$$

$$+ \frac{3g^2}{2(4\pi)^2} \sum_{s = \pm} \sum_{j = \pm} \theta[s(z - \tilde{z})] \theta[D(T_s)]\theta[x_j(T_s)]\theta[1 - x_j(T_s)]$$

$$\left\{ \frac{szT_s x_j(T_s)}{(1 + sz}\sqrt{D(T_s)(1 - x_j(T_s))} \right\}.$$
The meaning of the all functions were established in the previous appendix, for convenience of the reader we review them also here:

\[ x \pm (T) = \frac{-B(T) \pm \sqrt{D(T)}}{2A} \quad ; \quad D(T) = B(T)^2 - 4Am_c, \]
\[ A = \bar{\omega} - S; \quad B(T) = R(T) - \bar{\omega} - m_c, \]
\[ E(x) = \bar{\omega} - \frac{m_c}{x^2} - S; \quad R(T) = JT + m^2; \quad J = \omega - m^2, \]
\[ J = \omega - m^2; \quad S = (1 - \bar{z}^2) \frac{P^2}{4}; \quad T_\pm = \frac{1 \pm \bar{z}}{1 \pm \bar{z}}. \] (B5)

The equation (B2) has been actually used for the numerical solution when \( m_c/m_q \simeq 1 \).

The kernel (B2) is free of the running singularities because of the presence of theta functions, however in the case of massless gluon kernel we would have the singularity just on the boundary. There \( J \to 0 \) as \( \omega \) approaches the quarks threshold. We have found that this instability is avoided if we generate the inhomogeneous term in a following manner: We formally add the zero in the following form

\[ (f(\tilde{\omega}, \tilde{z}) - f(\tilde{\omega}, \tilde{z})) \frac{\rho[2](\omega, z)}{\omega^2} \] (B6)

to the right hand side of the original homogeneous equation Bethe-Salpeter equation for weight function \( \rho \). Then solving the equation

\[ \rho[2](\tilde{\omega}, \tilde{z}) = f(\tilde{\omega}, \tilde{z}) + \int_0^\infty d\omega \int_{-1}^1 dz \left[ V[2](\tilde{\omega}, \tilde{z}; \omega, z) - \frac{f(\tilde{\omega}, \tilde{z})}{\omega^2} \right] \rho[2](\omega, z) \] (B7)

within the following normalization condition

\[ 1 = \int_{-1}^1 dz \int_0^\infty d\omega \frac{\rho[2](\omega, z)}{\omega^2}. \] (B8)

is equivalent to the solution of the original BSE. As a suitable function we choose

\[ f(\tilde{\omega}, \tilde{z}) = 1 \] (B9)

We have found that this method is applicable for any positive value of \( m_c^2 \), but is slowly convergent when is used for previously discussed case \( m_c \simeq m_q \). Note here, that up to the small numerical error the equation (B2) offers the same spectra for such \( \alpha's \) and \( m'_g's \) where the both equations (B2), (B7) are numerically stable.

The equation (B7) has been solved by the method of the iteration. If the iterations failed– measure being both the difference of the rhs. and lhs. of the integral equation and deviation of the auxiliary normalization integral from a predefined value– we were changing the coupling constant (in the treatment with \( P^2 \) fixed, otherwise procedure is opposite) until the solution was found. For numerical solution we discretize integration variables \( \omega \) and \( z \) using Gauss-Legendre quadratures (with suitable mapping from \( -1, +1 \rightarrow < \omega_{\text{min}}, \infty > \)
for $\omega$). Equations (17), (12) is solved on the grid of $N = N_z \ast N_\omega$ points which are spread on the rectangle $(-1, +1) \ast (\omega_{\text{min}}, \infty)$. The value $\omega_{\text{min}}$ is given by the support of the spectral function. In the all cases we take $N_\omega = 2N_z$. Examples of a numerical convergence for some cases of bound states are presented in Table III. As we can see, there is a rather weak dependence of the eigenvalue $\alpha$ on the number of mash points $N_\omega$. The last value is calculated from the weighted average (WA) with $N_\omega$ be the appropriate weight.

| $N_\omega$ | 32 | 40 | 64 | 80 | WA  |
|------------|----|----|----|----|-----|
| $\eta = 0.95; \, m_g/M = 10^{-3}$ | 0.6611 | 0.6690 | 0.6697 | 0.6734 | 0.669 |
| $\eta = 0.95; \, m_g/M = 0.5$ | 1.037 | 1.0259 | 1.0210 | 1.0229 | 1.029 |
| $\eta = 0.99; \, m_g/M = 0.5$ | 0.818 | 0.8127 | 0.8158 | 0.8155 | 0.816 |

TABLE 3. The coupling $\alpha_s = g^2/(4\pi)$ for the ladder BSE with fixed ratio $m_c/M$ as a function of the number of mesh-points.

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