Nonequilibrium Phase Transitions in Epidemics and Sandpiles

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Abstract

Nonequilibrium phase transitions between an active and an absorbing state are found in models of populations, epidemics, autocatalysis, and chemical reactions on a surface. While absorbing-state phase transitions fall generically in the DP universality class, this does not preclude other universality classes, associated with a symmetry or conservation law. An interesting issue concerns the dynamic critical behavior of models with an infinite number of absorbing configurations or a long memory. Sandpile models, the principal example of self-organized criticality (SOC), also exhibit absorbing-state phase transitions, with SOC corresponding to a particular mode of forcing the system toward its critical point.

Keywords: Nonequilibrium phase transitions; Critical phenomena; Self-organized criticality; Absorbing states

I. INTRODUCTION

Phase transitions arise not only in thermodynamic equilibrium, but in nonequilibrium situations such as the onset of convection [1], and in models of populations, epidemics, catalysis, cooperative transport, and markets, to cite but a few examples. The simplest models of spatially extended, nonequilibrium systems are lattice Markov processes, or interacting particle systems [2]. There is usually no Hamiltonian; rather, the model is defined by a set of transition rates, that do not satisfy detailed balance with respect to a ‘reasonable’ energy function.

Despite these differences from equilibrium statistical mechanics, phase transitions (which mark a singular dependence of macroscopic properties on control parameters), scaling, and universality, retain their fundamental importance. Such concepts transcend the equilibrium/nonequilibrium boundary. In fact, familiar universality classes, such Ising or Potts, still appear in the nonequilibrium context, along with others not encountered in equilibrium. While new
classes are still being identified, and the general principles determining critical behavior far from equilibrium have yet to be specified completely, it has become clear that symmetry and conservation laws play a fundamental role, just as in equilibrium. Some useful reviews of various aspects of nonequilibrium phase transitions can be found in Refs. [3–6]. Here I discuss some recent work on absorbing-state phase transitions and their relation to self-organized criticality.

II. ABSORBING-STATE PHASE TRANSITIONS

Absorbing-state phase transitions arise from a conflict between the spread of activity, and a tendency for this activity to die out [4,5]. The activity may represent an epidemic [7], a reaction proceeding on the surface of a catalyst [8], or spatiotemporal chaos [9,10], for example. When the system is translation-invariant, with interactions of finite range, and there are no conserved quantities, the critical behavior is expected to fall generically in the directed-percolation (DP) universality class [12,13], as has been verified in studies of a wide variety of systems.

DP and the closely related contact process [4] exhibit well known scaling properties, for example the stationary order parameter (activity density) grows $\propto (\lambda - \lambda_c)^\beta$, where $\lambda$ is the spreading or reproduction rate for activity. At the critical point, $\lambda_c$, starting from a localized active region (e.g., a single active site), the survival probability $P(t)$ decays $\propto t^{-\delta}$, while the mean number of active sites $n(t)$ grows $\propto t^\eta$; these scaling laws describe avalanches of activity [11]. For the DP class, a field theory, framed in terms of a nonnegative, scalar order parameter density $\rho(x, t)$ yields an upper critical dimension of four, and an $\epsilon$-expansion for the critical exponents [12,14]:

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho - a \rho - b \rho^2 + \eta(x, t).$$

Here $\eta(x, t)$ is zero-mean Gaussian noise with autocorrelation

$$\langle \eta(x, t)\eta(x', t') \rangle = \Gamma \rho(x, t)\delta(x - x')\delta(t - t').$$

Some other universality classes associated with absorbing-state phase transitions are: compact DP, in which activity is bounded by random walks [15,16]; the parity-conserving or directed-Ising class, exemplified by branching-annihilating random walks with an even number of offspring [17–19]; and the “conserved nondiffusing field” class [20]. The criteria for a model to belong to the parity-conserving universality class are still not completely clear.

A. Interface representation

A recent development in this area is the application of surface-growth scaling ideas. Consider the contact process, in which, at a given instant, each site
of the lattice is either occupied ($\sigma_i = 1$), or vacant ($\sigma_i = 0$). Transitions from $\sigma_i = 1$ to $\sigma_i = 0$ occur at a rate of unity, independent of the neighboring sites. The reverse can only occur if at least one neighbor is occupied: the transition from $\sigma_i = 0$ to $\sigma_i = 1$ occurs at a rate of $\lambda m$, where $m$ is the fraction of nearest neighbors of site $i$ that are occupied; thus the state $\sigma_i = 0$ for all $i$ is absorbing. The stationary order parameter (the activity density, or fraction of occupied sites), is zero for $\lambda < \lambda_c$.

We define a set of height variables

$$h_i(t) = \int_0^t \rho_i(t')dt'$$

(3)

that represent the cumulative activity at each site. Since the $h_i$ are non-decreasing functions of time, we can think of them as describing a growing or driven interface, and introduce the width $W$ via $W^2(t, L) = \text{var}[h_i(t)]$, on a lattice of $L^d$ sites. The width exhibits Family-Vicsek scaling [21,22], but with the important difference that here it saturates only when the process falls into the absorbing state.

At the critical point, the scaling exponents for the width are related to those of DP. For example, the growth exponent (defined via $W \sim t^{\beta_W}$) satisfies $\beta_W + \delta = 1$, where $\delta$ describes the decay of activity at the critical point, starting with all sites active: $\rho \sim t^{-\delta}$. This relation follows from a scaling hypothesis for the probability distribution $P(h; t)$ of height $h$ (at any site):

$$P(h; t) = \frac{1}{\overline{h}(t)}P[h/\overline{h}(t)],$$

(4)

($P$ is a scaling function, $\overline{h}(t)$ the mean height), that has been verified in simulations [23]. Studies of the Domany-Kinzel model show that the growth exponent exhibits a sharp cusp at the critical point, making it a sensitive indicator of criticality [24].

III. HOW SANDPILES WORK

Self-organized criticality (SOC) is usually taken to mean “spontaneous” criticality, or scale-invariance in the absence of identifiable tuning of parameters [25]. In sandpiles, SOC is an absorbing-state phase transition of the same kind as in DP (though with different critical exponents), subject to an external control mechanism [27,28]. A simple example illustrates how this operates.

Consider a system of activated random walkers (ARW) on a lattice with periodic boundaries in $d \geq 1$ dimensions. $N$ walkers are initially distributed at random over the $L^d$ sites, with no restriction on the number of walkers per site. The dynamics, which conserves the number of walkers, is simply that each site with two or more walkers has a rate of 1 to liberate a pair of walkers. Each liberated walker moves independently, without bias, to one of the neighboring sites (in one dimension, from site $j$ to $j+1$ or $j-1$ with
equal likelihood). Isolated walkers are however paralyzed. The system has two kinds of configurations: active, in which at least one site has \( \geq 2 \) walkers, and absorbing, with all walkers immobile. For \( N > L^d \) only active configurations are possible, and activity continues forever.

For \( N \leq L^d \) there are both active and absorbing configurations, the latter representing a shrinking fraction of configuration space as the density \( \zeta \equiv N/L^d \to 1 \). Given an active initial configuration (a virtual certainty for a random distribution with \( \zeta > 0 \) and \( L \) large), will the system remain active indefinitely, or will it fall into an absorbing configuration? For small \( \zeta \) it should be easy for the latter to occur, but it seems reasonable that for sufficiently large densities (but \( < 1 \)), the likelihood of reaching an absorbing configuration becomes so small that the system remains active indefinitely. Mean-field theory and extensive simulations confirm the existence of sustained activity for densities greater than some critical value \( \zeta_c \), with \( \zeta_c < 1 \). The former predicts \( \zeta_c = 1/2 \) while simulation yields 0.9489 for the critical density in one dimension, 0.7170 in two dimensions. Thus the ARW model exhibits a continuous absorbing-state phase transition at a critical density \( \zeta_c \). The critical exponents are not those of DP, due to a conserved density (i.e., of walkers), which relaxes diffusively in the presence of activity, but is frozen in inactive regions.

Now we make two simple changes in the model:

i) We open the boundaries; walkers can now jump out from the edge.

ii) To compensate for the loss of walkers at the boundaries, we include a source, which inserts a new walker at a random site, if and only if the system has fallen into an absorbing configuration. (Note that the source enjoys global information regarding the state of the system.)

It is easy to see that these innovations amount to a control mechanism that forces the system to its critical point. Suppose that \( \zeta \) is above the critical value; then there is activity (hence no addition), and loss of walkers when activity reaches the boundary, so that \( d\zeta/dt < 0 \). If \( \zeta < \zeta_c \), on the other hand, there is no activity (and no loss), and the source is actuated, making \( d\zeta/dt > 0 \). Thus the system is forced to the critical point, and, quite naturally, exhibits scale invariance in the stationary state. Thus modified, the activated random walkers model is a continuous-time version of Manna’s stochastic sandpile; the loss and addition mechanisms yield what is commonly called self-organized criticality. Their functioning depends on the conservation of walkers, allowing the control mechanism to be described in terms of insertion and loss of “sand”, without ever making reference to the underlying control parameter \( \zeta \).

Studies of criticality in sandpiles with a periodic boundaries (so-called fixed-energy sandpiles), have yielded their critical exponents and associated avalanche exponents, and have suggested a field theory for sandpiles. The theory involves two fields, the activity density \( \rho_a \) and the particle density \( \zeta \). Upon formal elimination of the latter, one obtains
\( \frac{\partial \rho_a(x,t)}{\partial t} = D_a \nabla^2 \rho_a(x,t) - r(x)\rho_a(x,t) - b\rho_a^2(x,t) \\
\quad + w\rho_a(x,t) \int_0^t dt' \nabla^2 \rho_a(x,t') + \sqrt{\rho_a} \eta(x,t). \) 

(5)

\( \eta \) is a Gaussian white noise with autocorrelation \( \langle \eta(x,t)\eta(x',t') \rangle = D\delta(x-x')\delta(t-t'); \) \( c, b \) and \( w \) are fixed parameters. The theory resembles that for DP, except for the Laplacian memory term, and the spatial dependence of the growth rate \( r \), inherited from the initial configuration. This theory has yet to yield to renormalization group analysis [32].

Simulations support the proposal [35–37] that stochastic sandpiles fall in the same universality class as that of depinning of a linear elastic interface subject to random pinning forces, at least for \( d \geq 2 \). The deterministic Bak-Tang-Wiesenfeld sandpile [25], by contrast, appears to define a universality class sui generis, and is marked by strong nonergodic effects.

We close this section with some observations on SOC itself. SOC was proposed as a paradigm for explaining the appearance of scale invariance in nature without the need to invoke an outside agent tuning the system to criticality [25,38]. We have seen, however, that the source [component ii) of the control scheme], will require a huge quantity of information (\( L^d \) bits) in order to decide whether or not to add sand, so that the riddle of scale invariance in nature seems not to have been resolved. A way out of this dilemma is to suppose that the source is does not have access to all this information, but acts on a time scale much longer than the local dynamics. Then we can expect to observe scale invariance over a considerable range of avalanche sizes and durations, large enough, in principle, to agree with observation. A similar consideration applies to strict conservation: admitting a small dissipation rate, certain sandpile models exhibit “quasi-criticality” which may again be sufficient to account for observed scale invariance [39].

IV. INFINITE NUMBERS OF ABSORBING CONFIGURATIONS

Infinite numbers of absorbing configurations arise in models of catalysis [40], and, as noted above, in sandpiles. The simplest model with an infinite number of absorbing configurations is Jensen’s pair contact process (PCP) [41,42]. In this model, a pair of particles annihilates with probability \( p \), and creates a new particle (at a vacant nearest neighbor site) with probability \( 1-p \). Any configuration devoid of nearest-neighbor pairs is absorbing.

The static critical behavior of the PCP falls in the directed percolation universality class, but in one dimension the spreading exponents \( \delta \) and \( \eta \) vary continuously with the particle density \( \phi \) in the environment [42,43]. The spreading exponents take their DP values (e.g., \( \delta = 0.16 \)), only when the initial density \( \phi \) of isolated particles is set to the “natural” value \( \phi_{\text{nat}} \approx 0.242 \), i.e., the value generated by the process itself, running at the critical point. (For \( \phi = 0 \), for example, \( \delta \approx 0.27 \).) Numerically, \( p_c, \delta + \eta, \) and \( z \) (the exponent governing
the spread of the active region: $R^2 \sim t^z$), appear to be independent of $\phi$. Exponents $\delta$, $\eta$, and $z$ satisfy a generalized hyperscaling relation \[44,45\].

The issue of variable spreading exponents remains unresolved and controversial. While evidence for variable exponents has been found in many simulations \[10,42,43,46–48\], there is as yet no complete theory. A field theory \[49\] again involves the order parameter $\rho$ coupled to a second field (which may represent the density of isolated particles), that is frozen in regions with $\rho = 0$:

$$\frac{\partial \rho}{\partial t} = D_\rho \nabla^2 \rho - a_\rho \rho - b_\rho \rho^2 + w_\rho \phi \rho + \eta_\rho \tag{6}$$

$$\frac{\partial \phi}{\partial t} = D_\phi \nabla^2 \rho - a_\phi \rho - b_\phi \rho^2 + w_\phi \phi \rho + \eta_\phi \tag{7}$$

(Here both noise autocorrelations are $\propto \rho$.) Eliminating $\phi$, one obtains an equation for $\rho$ with a memory term; the stationary properties have been shown to be the same as DP. While it has not been possible to determine the spreading exponents analytically, numerical integration of the field theory yields $\delta$ and $\eta$ that vary with the coupling $w_\rho$ between the two fields \[50\].

The difficulty of understanding nonuniversal spreading in models with an infinite number of absorbing configurations has motivated the study of simplified descriptions. Noting that anomalous spreading behavior is associated with a long memory of the initial configuration, Grassberger et al. proposed a generalized epidemic process, in which DP (with a unique absorbing configuration), is modified to have long memory \[51\]. Specifically, the probability $p'$ for a virgin site to become infected is different from that ($p$) for used sites (those that have already been infected and have recovered). These authors find that in one dimension, at the critical point $p_c$ (which is independent of $p'$), the model exhibits variable exponents $\delta$ and $\eta$ for $p' > p_c$, but faster than power-law decay for $p' < p$, that is, no scaling.

Since the very existence of scaling in a model with a long memory of the initial condition appears to be in doubt, it is helpful to have some simple examples in which a variable exponent can be established analytically. A continuously variable survival probability exponent, $\delta$, has been demonstrated for DP confined to a parabola \[52\], and for compact DP in a similar geometry \[53\]. In these cases the space-time boundary is fixed, but recently a similar result was shown for a random walk with a long memory in the form of a movable partial reflector \[54\].

Consider an unbiased, discrete-time random walk $x_t$ on a one-dimensional lattice, with $x_0 = 1$, and the origin absorbing. On visiting a virgin site, the walker is pushed back to its previous position with probability $r$ (analogous to $p' < p$ in the generalized epidemic process). One may think of this as being due to a reflector that sits just beyond the maximal site that has yet been visited. (Initially the reflector is at $x = 2$.) Asymptotic analysis of the master equation, and exact iteration of transition probabilities for finite times, yield a variable survival exponent $\delta$. One may distinguish two kinds of reflector. A
soft reflector moves forward one step at every encounter with the walker, even if the latter is reflected back. In this case we find

$$\delta = \frac{1 + r}{2}. \quad (8)$$

A hard reflector, on the other hand, moves only if the walker succeeds in occupying the virgin site, analogous to the spread of activity in the generalized epidemic process. Here the decay exponent can be much larger:

$$\delta = \frac{1}{2(1 - r)}. \quad (9)$$

An interesting feature of the hard-reflector case is that for \( r \) close to 1, decay of \( P(t) \) looks faster than power-law over many decades: the asymptotic power law only becomes evident at very long times. Variable exponents \( \delta \) and \( \eta \) are also found in the case of compact DP with movable reflectors. If and how these findings apply to the more difficult question of spreading in models with an infinite number of absorbing configurations, and in the generalized epidemic process, remains to be seen.

V. SUMMARY

Absorbing-state phase transitions are attracting increasing attention as connections to SOC, spatiotemporal chaos, and interface dynamics are uncovered. At the same time, various puzzles regarding universality and the nature of scaling in models with an infinite number of absorbing configurations remain unsolved, and can be expected stimulate analysis and the search for improved theoretical and computational methods.
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