GENERALIZED HAAR CONDITION-BASED PHASELESS RANDOM SAMPLING FOR COMPACTLY SUPPORTED FUNCTIONS IN SHIFT-INVARIANT SPACES

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ABSTRACT. It is proved that the phase retrieval (PR) in the linear-phase modulated shift-invariant space (SIS) $V(e^{i\alpha \cdot \phi})$, $\alpha \neq 0$, is impossible even though the real-valued $\phi$ enjoys the full spark property (so does $e^{i\alpha \cdot \phi}$). Stated another way, the PR in the complex-generated SISs is essentially different from that in the real-generated ones. Motivated by this, we first establish the condition on the complex-valued $\phi$ such that the PR of compactly supported and nonseparable (CSN) functions in $V(\phi)$ can be achieved by random phaseless sampling. The condition is established from the perspective of the Lebesgue measure of the zero set of a related function system, or more precisely from the generalized Haar condition (GHC). Based on the proposed reconstruction approach, it is proved that if the GHC holds, then the PR of CSN functions in the complex-generated SISs can be achieved with probability 1, provided that the phaseless random sampling density (SD) $\geq 3$. For the real-generated case we also prove that, if the GHC holds then the PR of real-valued CSN functions can be achieved with the same probability if the random SD $\geq 2$. Recall that the deterministic SD for PR depends on Haar condition (measured in terms of the cardinality of the corresponding zero set). Compared with deterministic sampling, the proposed random sampling enjoys not only the greater sampling flexibility but the lower SD. For the lower SD, the highly oscillatory signals such as chirps can be efficiently reconstructed. To verify our results, numerical simulations were conducted to reconstruct CSN functions in the chirp-modulated SISs.
1. Introduction

Phase retrieval (PR) is a nonlinear sampling problem that seeks to reconstruct a signal \( f \), up to a unimodular scalar, from the intensities of the linear measurements

\[
\begin{align*}
    b_k := |\langle f, a_k \rangle|.
\end{align*}
\]

(1.1)

It has been widely applied in engineering problems such as coherent diffraction imaging (\([27, 30]\)), quantum tomography (\([19]\)) and inverse scattering problem (\([6]\)), and also in frame theory (\([2, 16, 23, 24]\)).

A concrete PR problem corresponds to the specific signal class and type of measurement vectors (c.f. \([13, 14, 25]\)). For example, when the target signal \( f \) is time-continuous and every measurement vector \( a_k \) is a shift of the Dirac distribution, then the corresponding PR is the phaseless sampling. Phaseless sampling has recently received much attention (e.g.\([9, 28, 33, 34, 35]\)). As other PR problems, phaseless sampling is challenging due to the ambiguousness induced by some reasons including conjugation (c.f.\([33]\)) and separability (c.f.\([9]\)). By collecting additional information beyond phaseless sampling, the ambiguousness can be eliminated and then the target can be reconstructed. For example, if \( f \) is a bandlimited function, then it follows from Thakur\([35]\) and Pohl, Yang and Boche\([28]\) that it can be reconstructed, up to a unimodular scalar, by sufficiently many phaseless samplings. Note that the spaces of bandlimited functions are shift-invariant and the corresponding generators (sinc or its dilations) are infinitely supported. Chen, Cheng, Sun and Wang\([9]\) and Sun\([34]\) established the PR for nonseparable real-valued functions in the general shift-invariant space (SIS for short) and in the shift-invariant B-spline spaces, respectively. Herein a function \( f \in H \) is referred to as being separable if there exist \( f_1, f_2 \in H \) such that \( f_1f_2 = 0 \) and \( f = f_1 + f_2 \).

In many practical applications, one needs to process the functions in the SISs generating from complex-valued generators (e.g.\([3, 29, 31]\)). In this paper, we investigate the PR of compactly supported functions in such SISs. To the best of our knowledge, there are few results in the literature on this topic. Next we introduce our motivation. Firstly, as will be implied in Theorem\([14]\) the PR in complex-generated SISs is essentially different from that in the real-generated ones. Some denotations are helpful for introducing the distinction. Throughout this paper the SIS generated by \( \phi \) is defined as

\[
    V(\phi) := \left\{ \sum_{l \in \mathbb{Z}} c_l \phi(\cdot - l) : \{c_l : l \in \mathbb{Z}\} \in \ell^2(\mathbb{Z}) \right\},
\]

(1.2)

and without loss of generality, suppose that \( \text{supp}(\phi) \subseteq (0, s) \) with the integer \( s \geq 2 \). It was proved in\([9]\) that for a real-valued \( \varphi \) with \( \text{supp}(\varphi) \subseteq (0, s) \), if the matrix

\[
    (\varphi(x_k + n))_{1 \leq k \leq 2s - 1, 0 \leq n \leq s - 1}
\]

(1.3)

is full spark (c.f.\([17, 18]\)) for arbitrary and distinct points \( \{x_k\}_{k=1}^{2s-1} \subseteq (0, 1) \), then the nonseparable functions in \( V(\varphi) \) can be reconstructed by sufficiently many phaseless
Define the linear phase-modulation of $\phi$ defined in (1.3) is full spark for any sequence of distinct points $\{x_k\}_{k=1}^{2s-1} \subseteq (0,1)$. Let $\lambda$ be such that the system $\{\varphi(\cdot + k) : k = 0, \ldots, s-1\}$ is linearly independent. Let the sequence $\{c_k\}_{k=0}^N$ be such that $\{c_k\}_{k=0}^N \neq e^{i\tilde{\theta}}\{e^{i2\alpha k}\}_{k=0}^N$ for any $\tilde{\theta} \in [0,2\pi)$. By the above linear independence, we have $\sum_{k=0}^N c_k \varphi(\cdot - k) \neq e^{i\tilde{\theta}} \sum_{k=0}^N c_k e^{i2\alpha k} \varphi(\cdot - k)$ for the above mentioned $\tilde{\theta}$. However, it is easy to check that $|\sum_{k=0}^N \overline{c_k} e^{i2\alpha k} \varphi(\cdot - k)| = |\sum_{k=0}^N c_k \varphi(\cdot - k)|$. Stated another way, the PR in $V(\varphi)$ can not be achieved despite $\varphi$ also satisfies the full spark property.

Proof. Clearly $\varphi$ inherits the full spark property of $\varphi$. It follows from the property that the system $\{\varphi(\cdot + k) : k = 0, \ldots, s-1\}$ is linearly independent. Let the sequence $\{c_k\}_{k=0}^N$ be such that $\{c_k\}_{k=0}^N \neq e^{i\tilde{\theta}}\{e^{i2\alpha k}\}_{k=0}^N$ for any $\tilde{\theta} \in [0,2\pi)$. By the above linear independence, we have $\sum_{k=0}^N c_k \varphi(\cdot - k) \neq e^{i\tilde{\theta}} \sum_{k=0}^N c_k e^{i2\alpha k} \varphi(\cdot - k)$ for the above mentioned $\tilde{\theta}$. However, it is easy to check that $|\sum_{k=0}^N \overline{c_k} e^{i2\alpha k} \varphi(\cdot - k)| = |\sum_{k=0}^N c_k \varphi(\cdot - k)|$. Stated another way, the PR in $V(\varphi)$ can not be achieved despite $\varphi$ also satisfies the full spark property.

Motivated by Theorem 1.1, it is necessary to establish a condition on the complex-valued generator such that the PR in its SIS can be achieved. Our condition will be given from the perspective of zero distribution. Before proceeding further, let us reveal the essential condition for PR, shared by the generators in [34]. Clearly, the full spark property of the matrix in (1.3) is equivalent to that the function system

\[
\Lambda_R := \{\varphi, \ldots, \varphi(\cdot + s - 1)\}
\]

satisfies the $(s - 1)$-Haar condition (HC for short) on $(0,1)$ (c.f. [11, 37] for HC). Specifically, $\Lambda_R$ is linearly independent and

\[
\eta := \sup_{0 \neq g \in \text{span}\{\Lambda_R\}} \#(Z_g \cap (0,1)) \leq s - 1.
\]

Herein $\#(E)$ is the cardinality of the set $E$, and $Z_g$ is the zero set of $g$. Motivated by the above HC, from the perspective of zero distribution we will establish the condition on $\phi := \phi_R + i\phi_3$ such that the PR in $V(\phi)$ can be achieved. Inspired by Theorem 1.1 the zero distribution should not correspond to functions in $\text{span}\{\phi, \ldots, \phi(\cdot + s - 1)\}$. Instead our PR condition will be derived from the zero distribution of the functions in $\text{span}(\Lambda)$, where

\[
\Lambda := \{\phi_R \phi_R(\cdot + k) + \phi_3 \phi_3(\cdot + k), \phi_R \phi_3(\cdot + k) - \phi_3 \phi_R(\cdot + k)\}_{k=1}^{s-1} \cup \{\phi_R^2 + \phi_3^2\}.
\]

It will be clear in subsection 2.3.2 that the zero distribution corresponding to $\Lambda$ is natural for phaseless sampling scheme in $V(\phi)$. On the other hand, unlike that in (1.3), the condition on the zero distribution is given in the sense of measure but not
cardinality. Specifically, we only require that $\Lambda$ is linearly independent and
\[
\sup_{0 \neq g \in \text{span}\{\Lambda\}} \mu(\mathcal{Z}_g \cap (0,1)) = 0,
\]
(1.7)
where $\mu$ is the Lebesgue measure. The condition in (1.7) is termed as generalized Haar condition (GHC for short). Compared with HC, GHC is much weaker and more easy to be verified. More details about the verification are given in subsection 2.4. Moreover, we prove in Theorem 2.4 that if the GHC holds, then any compactly supported and nonseparable (CSN) function in $V(\phi)$ can be reconstructed with probability 1 provided that the random sampling density (SD) $\geq 3$. We also prove that the above result for real-valued CSN functions in real-generated SIS still holds if the random SD $\geq 2$. Stated another way, the amount of samplings in our proposed approach is independent of
$\sup_{0 \neq g \in \text{span}(\tilde{\Lambda})} \#(\mathcal{Z}_g \cap (0,1))$, where $\tilde{\Lambda}$ takes $\Lambda_{\mathbb{R}}$ and $\Lambda$ for the real-valued and complex-valued cases, respectively. The independence renders the (real-valued) random sampling different from the existing deterministic sampling since it follows from [9] and [34, Theorem 1.2] that, the greater the index $\eta$ in (1.5) is the larger deterministic SD we need to do the PR of CSN functions. Recall that chirps arise in a great number of scientific disciplines such as the investigation of atmospheric whistlers ([20]) and detection of gravitational waves ([11, 36]). Many chirps are highly oscillatory ([3, 4, 8, 29, 31]). Therefore for the CSN functions in highly oscillatory chirp-generated SIS, their zero numbers are great and so is $\sup_{0 \neq g \in \text{span}(\tilde{\Lambda})} \#(\mathcal{Z}_g \cap (0,1))$. In such SISs, great deterministic SD is required for achieving the PR of the CSN functions. Instead, we can profit from the above mentioned independence since it enables us to reconstruct such functions with lower SD. This will be witnessed in subsections 2.4 and 3.3, where the target chirps sit in the chirp-modulated SIS (given in Bhandari and Zayed [3]) and its related real-generated SISs, respectively.

The paper is organized as follows. In section 2 the phaseless random sampling is established for the CSN functions in the complex-generated $V(\phi)$, where $\phi$ satisfies the GHC. The phaseless random sampling is derived from the proposed reconstruction approach: \textit{phase decoding-coefficient recovery} (PD-CR). Based on PD-CR, it was proved that when the sampling points obey the uniform distribution and the random SD $\geq 3$, then with probability 1 any CSN function in $V(\phi)$ can be reconstructed up to a unimodular scalar. In section 3 the PD-CR is modified such that it is more adaptive to the real-generated SISs. By the modified PD-CR, the CSN functions in the real-generated SIS can be reconstructed with probability 1 provided that the random SD $\geq 2$. To confirm our results numerical simulations are conducted in subsection 2.4 and subsection 3.3. We conclude in section 4.
2. Phaseless random sampling in complex-generated SISs

This section starts with some necessary denotations. For \(0 \neq a \in \mathbb{C}\), it can be denoted by \(|a|e^{i\theta(a)}\) where \(\theta(a)\) is referred to as its phase. Traditionally, the phase of zero can be assigned arbitrarily. We say that two phases \(\theta(a)\) and \(\theta(b)\) are identical if \(\theta(a) = \theta(b) + 2k\pi\) for some \(k \in \mathbb{Z}\). The conjugate of \(a\) is denoted by \(\bar{a}\). The random variable \(t\), which obeys the uniform distribution on \((0, 1)\), is denoted by \(t \sim U(0, 1)\). Its observed value is denoted by \(\hat{t}\). For an event \(\mathcal{E}\) its probability is denoted by \(P(\mathcal{E})\).

2.1. Preliminary on GHC. The preliminaries in this subsection will be helpful in subsection 2.2 for employing GHC to the phaseless random sampling.

**Proposition 2.1.** Let \(\phi = \phi_R + i\phi_3\) be such that the associated system \(\Lambda\) defined in (1.6) satisfies the GHC on \((0, 1)\). Then both \(\Lambda_1 := \{\phi(\cdot + k) : k = 0, \ldots, s - 1\}\) and \(\Lambda_2 := \{\phi(\cdot + k) : k = 0, \ldots, s - 1\}\) also satisfy the GHC on \((0, 1)\).

**Proof.** The proof can be easily concluded by the GHC of \(\Lambda\). \(\square\)

**Remark 2.1.** Compared with those of \(\Lambda\), the components of \(\Lambda_1\) or \(\Lambda_2\) are much simpler formally. However it follows from Theorem 1.1 that the GHC of \(\Lambda_1\) (or \(\Lambda_2\)) is not sufficient for achieving the PR in \(V(\phi)\).

Next we address how to check GHC. Recall that many commonly used complex-valued generators are closely related with analytic functions. Employing the relationship, GHC can be easily checked. For example, let

\[
\phi(x) = \alpha_1 e^{-ia_2 x^2} e^{-ia_3 x} \cos^2(\alpha_4 (x - \gamma)) B(x)
\]

where \(\alpha_1 \neq 0\) and \(B(x)\) is a \(B\)-spline being compactly supported on \((0, m)\). Note that the generator

\[
\phi(x) := \frac{2}{3} \sqrt{2\pi |b|} e^{-\frac{a(x-2)^2}{2b}} e^{-\frac{p(x-2)^2}{b}} \cos^2 \left(\frac{\pi(x-2)}{4}\right) \chi_{[0,4]}(x), \quad a \neq 0
\]

for the chirp-modulated SIS in [3, section 6.3] takes the form of (2.1). Clearly, for \(\phi\) in (2.1) the components in the corresponding system \(\Lambda\) are essentially the restrictions of analytic functions on \((0, 1)\). Recall that the zero set of any analytic function has zero Lebesgue measure (c.f. [15]). Hence, if the \(2m - 1\) components \(g_i \in \Lambda\) are linearly independent on \((0, 1)\) then the GHC holds, where \(i = 1, \ldots, 2m - 1\). The linear independence can be achieved if there exists \((x_1, \ldots, x_{2m-1}) \in (0, 1)^{2m-1}\) such that the determinant

\[
|g_1(x_1) g_2(x_1) \ldots g_{2m-1}(x_1) \\
g_1(x_2) g_2(x_2) \ldots g_{2m-1}(x_2) \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
g_1(x_{2m-1}) g_2(x_{2m-1}) \ldots g_{2m-1}(x_{2m-1})|
\neq 0.
\]
Example 2.1. Let $\phi$ be as in (2.2). Then the corresponding $m = 4$. Let $a, p \in \{1, 2, \ldots, 100\}$, and $b \sim U(-5, 5)$. Uniformly choosing $(x_1, \ldots, x_7)$ from $(0, 1)$, we have that (2.3) holds with probability 1. Then by the above analysis, the corresponding GHC holds.

2.2. PD-CR scheme for phaseless sampling in complex-generated SIS. In this subsection, we provide a reconstruction scheme for compactly supported functions in $V(\phi)$ by phaseless sampling. Since a compactly supported function is the shift of a casual function taking the form

$$f = \sum_{k=0}^{N} c_k \phi(\cdot - k),$$

(2.4) then the phaseless sampling for casual functions can be easily generalized to compactly supported functions. Therefore, for simplicity we just need to address the phaseless sampling for compactly supported and casual functions. In what follows we sketch our reconstruction scheme. Clearly if the phases of the samples of $|f|$ have been decoded, then the reconstruction of $f$ will be linear and can be easily conducted. Motivated by this, we will establish an alternating scheme termed as phase decoding-coefficient recovery (PD-CR). Suppose that $\{t_{0,1}\} \cup \{t_{n,j} : j = 1, \ldots, L_n\} \subseteq (0, 1)$. PD-CR is depicted as follows:

| (2.5) Initialize $\theta(f(t_{0,1})) = \theta_0$; $c_0 = \frac{e^{i\theta_0|f(t_{0,1})|}}{\phi(t_{0,1})} \rightarrow \cdots \rightarrow |f(n + t_{n,j})| \rightarrow$ phase decoder $\rightarrow f(n + t_{n,1}) \rightarrow$ coefficient recovery $\rightarrow c_n \cdots \rightarrow f$. |

Clearly the phase decoder is the key ingredient for PD-CR. In what follows, we address what conditions the decoder needs to satisfy such that the phases can be successfully decoded. Note that in (2.5) the phase of $f(t_{0,1})$ is set to be an arbitrary value $\theta_0$. To the end of reconstructing $f$ (or equivalently the sequence $\{c_n\}_{n=0}^{N}$), up to a unimodular scalar, the phase decoder is required to provide the linear phase feedback on the initial value $\theta_0$. Specifically, if we set $\theta(f(t_{0,1})) = \tilde{\theta}_0$, then after conducting (2.5) what we obtain is $e^{i(\tilde{\theta}_0 - \theta_0)} f$. The decoder enjoying the above feedback property will be designed in Remark 2.2 (2.19). The design will be derived from Theorem 2.3.

Some auxiliary denotations are helpful for establishing Theorem 2.3. Let the complex-valued $\phi$ be as in (2.4) such that $\text{supp}(\phi) \subseteq (0, s)$. Define the index set $I_n$ by

$$I_n := \begin{cases} \{0, 1, \ldots, n - 1\}, & 1 \leq n \leq s - 1, \\ \{n - s + 1, \ldots, n - 1\}, & n \geq s. \end{cases}$$

(2.6)
For the target function $f$ in (2.4) with the coefficient sequence $\{c_k\}_{k=0}^N$, define auxiliary functions $\{v_{n,f}(x)\}_{n=1}^N$ on $(0, 1)$ by

$$ v_{n,f}(x) := \sum_{k \in I_n} c_k \phi(n + x - k). $$

Then it follows from $\text{supp}(\phi) \subseteq [0, s]$ that $f(n + x) = v_{n,f}(x) + c_n \phi(x)$. Moreover, based on $v_{n,f}(x)$ we define two bivariate functions

$$ A_{n,f}(x, y) + B_{n,f}(x, y)i := \frac{|f(n + x)|}{|\phi(x)|} \left[ \phi(x)\phi(y)v_n(y) - |\phi(y)|^2v_n(x) \right], $$

and

$$ C_{n,f}(x, y) := |f(n + y)|^2 - |v_{n,f}(y)|^2 + 2\Re\left(\frac{v_{n,f}(x)v_{n,f}(y)\phi(y)}{\phi(x)}\right) - |\phi(y)|^2|f(n + x)|^2 + |v_{n,f}(x)|^2, $$

where $n = 1, \ldots, N$ and $x, y \in (0, 1)$ such that $\phi(x) \neq 0$. The above bivariate functions are related via the following equation w.r.t $z \in \mathbb{C}$:

$$ (A_{n,f}(x, y) + B_{n,f}(x, y)i)z^2 - C_{n,f}(x, y)z + A_{n,f}(x, y) - B_{n,f}(x, y)i = 0, $$

where $x, y \in (0, 1)$ are any fixed points. The following lemma states that the solution to the equation in (2.10) can provide a precise feedback on the global phase information of $\{c_k\}_{k \in I_n}$. The precise feedback will lead to that of the decoder in (2.19).

**Lemma 2.2.** Let $A_{n,f}(x, y) + B_{n,f}(x, y)i \neq 0$ be defined in (2.8) associated with the sequence $\{c_k\}_{k \in I_n}$. Define $\hat{A}_{n}(x, y) + \hat{B}_{n}(x, y)(x, y)i$ via (2.8) with $\{c_k\}_{k \in I_n}$ being replaced by $\{c_k\}_{k \in I_n}$ := $e^{i\theta}\{c_k\}_{k \in I_n}$. For fixed $x, y \in (0, 1)$, suppose that the two solutions to the equation (2.10) w.r.t $z$ are $z_1$ and $z_2$. Then those to

$$ (\hat{A}_{n,f}(x, y) + \hat{B}_{n,f}(x, y)i)z^2 - C_{n,f}(x, y)z + \hat{A}_{n,f}(x, y) - \hat{B}_{n,f}(x, y)i = 0 $$

are $e^{i\theta}z_1$ and $e^{i\theta}z_2$.

**Proof.** Clearly the term $\frac{v_{n,f}(x)v_{n,f}(y)\phi(y)}{\phi(x)}$ in (2.9) is unchanged when $\{c_k\}_{k \in I_n}$ being replaced by $e^{i\theta}\{c_k\}_{k \in I_n}$, and so is $C_{n,f}(x, y)$. Moreover, the solutions to the equation in (2.10) are given by $z_1, z_2 = \frac{C_{n,f}(x, y) + \sqrt{C_{n,f}(x, y)^2 - 4A_{n,f}(x, y) - B_{n,f}(x, y)i \phi(x)}}{2(A_{n,f}(x, y) + B_{n,f}(x, y)i)}$. By $\frac{A_{n,f}(x, y) + B_{n,f}(x, y)i}{A_{n,f}(x, y) + B_{n,f}(x, y)i} = e^{-i\theta}$, the proof can be easily concluded. \qed

Based on Lemma 2.2 we establish the first main theorem of this section as follows. It concerns on a necessary and sufficient condition for successfully decoding the phases $\{\theta(f(n + t_{n,j}))\}$ in (2.3). Based on the condition the phase decoder will be provided in Remark 2.2.
Theorem 2.3. Let \( f \in V(\phi) \) be as in (2.4). Assume that neither of the phaseless sampling \( \{ f(n + t_{n_j}) \} : t_{n_j} \in (0, 1), j = 1, \ldots, L_n, n = 0, 1, \ldots, N - 1 \) is zero. Then the corresponding phases \( \{ \theta(f(n + t_{n_j})) \} \) can be determined (up to a global constant) if and only if for every \( n \), there exist \( n_1, n_2, n_3 \in \{ 1, 2, \ldots, L_n \} \) such that \( \phi(t_{n_1}) \neq 0 \) and the equation system

\[
(2.11) \begin{cases} 
(A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})i)z^2 - C_{n,f}(t_{n_1}, t_{n_2})z + A_{n,f}(t_{n_1}, t_{n_2}) - B_{n,f}(t_{n_1}, t_{n_2})i = 0, \\
(A_{n,f}(t_{n_1}, t_{n_3}) + B_{n,f}(t_{n_1}, t_{n_3})i)z^2 - C_{n,f}(t_{n_1}, t_{n_3})z + A_{n,f}(t_{n_1}, t_{n_3}) - B_{n,f}(t_{n_1}, t_{n_3})i = 0,
\end{cases}
\]

has a unique solution.

Proof. For \( n = 0 \), suppose that we choose

\[
(2.12) \theta(f(t_{0_1})) = \theta_0.
\]

Then it follows from (2.1) that \( c_0 = e^{i\theta_0}|f(t_{0_1})|/|\phi(t_{0_1})| \neq 0 \). Correspondingly, \( \theta(f(t_{0_1})) = \theta(\phi(t_{0_1}))e^{i\theta_0}|f(t_{0_1})|/|\phi(t_{0_1})| \) for \( l \neq 1 \). For \( n = 1 \), we next address how to determine \( z^* := e^{i\theta(f(t_{1_1}+1))} \) in the assumption (2.12). It is easy to check that

\[
(2.13) \begin{cases} 
|v_{1,f}(t_{1_1}) + c_1\phi(t_{1_1})| = |f(1 + t_{1_1})|, \\
v_{1,f}(t_{1_2}) + c_1\phi(t_{1_2}) = |f(1 + t_{1_2})|, \\
v_{1,f}(t_{1_3}) + c_1\phi(t_{1_3}) = |f(1 + t_{1_3})|.
\end{cases}
\]

Denote

\[
(2.14) v_{1,f}(t_{1_1}) + c_1\phi(t_{1_1}) = |f(1 + t_{1_1})|z^*
\]

with \( v_{1,f}(t_{1_1}) = e^{i\theta_0}\phi(1 + t_{1_1})|f(t_{0_1})|/|\phi(t_{0_1})| \) and \( z^* \) to be determined. Since \( \phi(t_{1_1}) \neq 0 \), then

\[
(2.15) c_1 = [|f(1 + t_{1_1})|z^* - v_{1,f}(t_{1_1})]/\phi(t_{1_1}),
\]

which together with the last two equations in (2.13) leads to that

\[
(2.16) v_{1,f}(t_{1_j}) + \phi(t_{1_j})|f(1 + t_{1_j})|z^* - v_{1,f}(t_{1_1}) = |f(1 + t_{1_j})|, j = 2, 3.
\]

By direct calculation, the equation system in (2.16) are equivalent to that in (2.11) with \( n = 1 \). Since there exists a unique solution to (2.11), \( z^* \) can be determined by solving (2.11). Furthermore, the phases \( \{ \theta(f(n + t_{n_j})) : t_{n_j} \in (0, 1), j = 1, \ldots, L_n, n = 0, 1 \} \) and \( c_0, c_1 \) are determined.

Recall that the above determination is achieved in the assumption (2.12). We next investigate the relation between the above determination result and that when assuming

\[
(2.17) \theta(f(t_{0_1})) = \theta_1.
\]

In the assumption (2.17), by the similar analysis as previously we have

\[
(2.18) c_0 = e^{i\theta_1}|f(t_{0_1})|/|\phi(t_{0_1})| = e^{i\theta_1}|f(t_{0_1})|/|\phi(t_{0_1})| = e^{i(\theta_1 - \theta_0)}c_0.
\]
and $\hat{\theta}(f(t_0)) = \theta(\phi(t_0)) e^{i\theta_1 |f(t_0)|} / |\phi(t_0)| = \theta(f(t_0)) + \theta_1 - \theta_0$. On the other hand, it follows from (2.18) and Lemma 2.2 that the phase of $f(t_1 + 1)$ derived from (2.11) is $\theta(z^*) + \theta_1 - \theta_0$. Correspondingly, it follows from (2.14) that $c_1 = e^{i(\theta_1 - \theta_0)c_1}$. That is, \{f(n + t_{n_j}) : t_{n_j} \in (0, 1), j = 1, \ldots, L_n, n = 0, 1\} and $c_0, c_1$ can be determined, up to the unimodular scalar $e^{i(\theta_1 - \theta_0)}$. It is straightforward to check that $\hat{\phi}(x, y) = e^{i(\theta_0 - \theta_1)}$, where $\hat{A}_2(x, y) + \hat{B}_2(x, y)$ is defined by (2.8) with $[c_0, c_1]$ being replaced by $[\hat{c}_0, \hat{c}_1]$. Now the proof can be concluded by continuing the above determination procedures for $n \geq 2$.

Remark 2.2. Now based on Theorem 2.3, we are ready to explicitly design the $n$-th phase decoder ($n \geq 1$) in (2.3) as follows:

\begin{equation}
\theta(f(n + t_m)) = \arg \min_{z_{n,k} \in \{z_{n,1}, z_{n,2}\}} \{ |z_{n,k} - z_{n,3}|, |z_{n,k} - z_{n,4}| \}
\end{equation}

where

\begin{equation}
z_{n,k} = \frac{C_n f(t_{n_1}, t_{n_2}) \pm \sqrt{C_n^2(t_{n_1}, t_{n_2}) - 4|A_n f(t_{n_1}, t_{n_2}) + B_n f(t_{n_1}, t_{n_2})|^2}}{2(A_n f(t_{n_1}, t_{n_2}) + B_n f(t_{n_1}, t_{n_2}))}, \quad k = 1, 2,
\end{equation}

and

\begin{equation}
z_{n,k} = \frac{C_n f(t_{n_1}, t_{n_2}) \pm \sqrt{C_n^2(t_{n_1}, t_{n_2}) - 4|A_n f(t_{n_1}, t_{n_2}) + B_n f(t_{n_1}, t_{n_2})|^2}}{2(A_n f(t_{n_1}, t_{n_2}) + B_n f(t_{n_1}, t_{n_2}))}, \quad k = 3, 4.
\end{equation}

We shall prove in Theorem 2.10 that the above decoder can succeed in phase decoding with probability 1.

2.3. PD-CR based phaseless random sampling. This subsection starts with the definition of the maximum gap of a function. For a function $f := \sum_{k=p}^{N-1+p} c_k \phi(\cdot - k)$ with $c_p, c_{N-1+p} \neq 0$, its maximum gap $G$ is defined as

$G(f) := \max_{0 \leq s \leq N} \{ \exists i \text{ s.t. } c_i = \ldots = c_{i+\gamma-1} = 0 \}$,

where \{i, i+1, \ldots, i+\gamma-1\} $\subseteq \{p, p+1, \ldots, N-1+p\}$. It will be clear in Proposition 2.6 that $G(f) < s - 1$ is equivalent to that $f$ is nonseparable. Based on PD-CR, we will establish the phaseless random sampling for compactly supported functions with the maximum gaps less than $s - 1$.

2.3.1. Main results in this subsection.

Theorem 2.4. Let $\phi = \phi_R + i\phi_\delta$ be as in Theorem 2.3 such that supp($\phi$) $\subseteq [0, s]$ with integer $s \geq 2$. Moreover, the set $\Lambda$ defined in (1.6) satisfies the GHC. Suppose that the target function $f$ is as in (2.4) such that $G(f) < s - 1$, and the i.i.d random sampling points $\{t_0\} \bigcup \{t_{n_1}, t_{n_2}, t_{n_3} : n = 1, \ldots, N\} \sim U(0, 1)$. Then through the PD-CR, $f$ can be reconstructed (up to a unimodular scalar) with the probability 1 by the phaseless random sampling $\{|f(t_0)|\} \bigcup \{|f(n + t_{n_1})|, |f(n + t_{n_2})|, |f(n + t_{n_3})| : n = 1, \ldots, N\}$.

Proof. The proof is given in subsection 2.3.2. □
Recall that the sampling points \( \{t_{ni}\} \subseteq (0, 1) \) in Theorem 2.4 may change as \( n \) does. But when \( \text{supp}(\phi) \subseteq [0, 2] \), the following proposition states that Theorem 2.4 still holds if letting random variables \( t_{ni} = t_i \) with \( i = 1, 2, 3 \).

**Proposition 2.5.** Let \( \phi \) and \( f \) be as in Theorem 2.4. Moreover, \( \text{supp}(\phi) \subseteq [0, 2] \). Then through the PD-CR, \( f \) can be reconstructed with the probability 1 (up to a unimodular scalar) by the phaseless random sampling

\[
\{ |f(t_0)| \} \cup \{ |f(n + t_i)| : n = 1, \ldots, N; i = 1, 2, 3 \},
\]

where \( \{t_0, t_1, t_2, t_3\} \sim U(0, 1) \).

**Proof.** The proof is given in subsection 2.3.2.

**Proposition 2.6.** Let \( \phi \) be as in Theorem 2.3. Then a compactly supported function \( f := \sum_{k=p}^{N-1+p} c_k \phi(\cdot - k) \) is nonseparable if and only if the maximum gap \( G(f) < s - 1 \).

**Proof.** It follows from Theorem 2.4 that we just need to prove the necessity. Suppose that \( N \geq s \) and \( 0 = c_i = \ldots = c_{i+s-2} \). Define \( f_1 = \sum_{k=p}^{i-1} c_k \phi(\cdot - k) \) and \( f_2 = \sum_{k=i+s-1}^{N-1+p} c_k \phi(\cdot - k) \). Then it is easy to check that \( |f_1 + e^{i\gamma} f_2| = |f| \) for any \( \gamma \in \mathbb{R} \). That is, \( f \) is separable.

For the completeness, we next address the phaseless deterministic sampling for SISs.

**Proposition 2.7.** Let \( \phi \) and \( f \) be as in Theorem 2.4. Moreover, the set \( \Lambda \) satisfies the \( m \)-Haar condition, namely, for any sequence \( \{a_k, b_k\}_{k=0}^{s-1} \) which are not all zeros, the combination

\[
\sum_{k=0}^{s-1} [a_k (\phi_{R}(x) \phi_{R}(x + k) + \phi_{3}(x) \phi_{3}(x + k)) + b_k (\phi_{R}(x) \phi_{3}(x + k) - \phi_{3}(x) \phi_{R}(x + k))] \tag{2.22}
\]

has at most \( m \) zeros in \( (0, 1) \). Then \( f \) can be determined, up to a global phase, by the phaseless sampling \( \{|f(\hat{t}_{n_0})|\} \cup \{|f(n + \hat{t}_{n_1})| : n \geq 1, l = 1, 2, \ldots, 2m + 3\} \), where the \( 2m + 3 \) points \( \{\hat{t}_{n_l} : l = 1, 2, \ldots, 2m + 3\} \subseteq \{0, 1\} \) are distinct for any fixed \( n \), and \( \hat{t}_{n_l} \in (0, 1) \) is not a zero of \( \phi \).

**Proof.** The proof is given in subsection 2.3.2.

2.3.2. *Proof of the results in subsection 2.3.* We first prove Theorem 2.4. For the clearness, the key points for proving Theorem 2.4 are summarized as follows.

**Note 2.1.** (1) By Theorem 2.3, the phase \( \theta(f(n + t_{ni})) \) being successfully decoded by (2.19) depends on (2.11) having a unique solution. When \( A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2}) \mathbf{i} = \)
0, it is straightforward to check that \( C_{n,f}(t_{n_1}, t_{n_2}) = 0 \) in (2.11). Then the first equation in (2.11) is useless for decoding \( \theta(f(n + t_{n_1})) \) and the decoder in (2.19) may well fail. Therefore the condition

\[
A_{n,f}(t_{n_1}, t_{n}) + B_{n,f}(t_{n_1}, t_{n}) \neq 0, \quad i = 2, 3
\]

is necessary for the decoder in (2.19) succeeding. We first prove in Lemma 2.8 that (2.23) occurs with probability 1. (2) If

\[
\frac{A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})}{A_{n,f}(t_{n_1}, t_{n_3}) + B_{n,f}(t_{n_1}, t_{n_3})} \neq \frac{A_{n,f}(t_{n_1}, t_{n_2}) - B_{n,f}(t_{n_1}, t_{n_2})}{A_{n,f}(t_{n_1}, t_{n_3}) - B_{n,f}(t_{n_1}, t_{n_3})} i
\]

then (2.11) has a unique solution. We will also prove that (2.24) holds with probability 1 in Theorem 2.10 and therefore (2.11) has a unique solution.

**Lemma 2.8.** Let \( \phi \) satisfy \( \text{supp}(\phi) \subseteq [0, s] \) with integer \( s \geq 2 \). Moreover, the system \( \Lambda \) in (1.6) associated with \( \phi \) is supposed to satisfy the GHC in (0, 1). Then for any \( f \in V(\phi) \) taking the form (2.4) and satisfying \( G(f) < s - 1 \), and for the i.i.d random variables \( t_{n_1}, t_{n_2} \sim U(0, 1) \), the quantity \( A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})i \) defined in (2.8) is not zero with the probability 1.

**Proof.** Note that \( G(f) < s - 1 \). Moreover, \( \Lambda_2 \) in Proposition 2.1 (ii) satisfies the GHC on (0, 1). Then for any \( n \geq 0 \), it is easy to prove that the event

\[
\mathcal{E}_{n,0} = \{ \phi(t_{n_1})f(n + t_{n_1}) \neq 0 \}
\]

occurs with probability 1. Stated another way, the Lebesgue measure \( \mu(U_{n,0}) = 0 \), where the set

\[
U_{n,0} = \{ \hat{t}_{n_1} \in (0, 1) : \phi(\hat{t}_{n_1})f(n + \hat{t}_{n_1}) = 0 \}.
\]

Define an auxiliary function w.r.t \( t_{n_1} \) and \( t_{n_2} \) by

\[
an_{n,f}(t_{n_1}, t_{n_2}) + b_{n,f}(t_{n_1}, t_{n_2})i := \overline{\phi}(t_{n_1})\phi(t_{n_2})\overline{\tau}_n(t_{n_2}) - \overline{\tau}_n(t_{n_1})|\phi(t_{n_2})|^2.
\]

Clearly \( A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})i \neq \frac{[f(n + t_{n_1})]}{[\phi(t_{n_2})]}(a_{n,f}(t_{n_1}, t_{n_2}) + b_{n,f}(t_{n_1}, t_{n_2})i) \). It follows from \( G(f) < s - 1 \) and the GHC of \( \Lambda_2 \) in Proposition 2.1 (ii) that \( \phi(x)\overline{\tau}_n(x) \neq 0 \) and \( |\phi(x)|^2 \neq 0 \). On the other hand, by the GHC of \( \Lambda_1 \), \( \phi(x)\overline{\tau}_n(x) \) and \( |\phi(x)|^2 \) are linearly independent. Hence it follows from the probability \( P(\mathcal{E}_{n,0}) = 1 \) and the GHC of \( \Lambda \) that \( P(\mathcal{E}_{n,1}) = 1 \), where the event \( \mathcal{E}_{n,1} \) w.r.t \( t_{n_1} \) is defined by

\[
\mathcal{E}_{n,1} := \{ \overline{\phi}(t_{n_1})\phi(x)\overline{\tau}_n(x) - \overline{\tau}_n(t_{n_1})|\phi(x)|^2 \neq 0 \}.
\]

Stated another way, the measure \( \mu(U_{n,1}) = 0 \) where the set

\[
U_{n,1} := \{ \hat{t}_{n_1} \in (0, 1) : \overline{\phi}(\hat{t}_{n_1})\phi(x)\overline{\tau}_n(x) - \overline{\tau}_n(\hat{t}_{n_1})|\phi(x)|^2 \equiv 0 \}.
\]

Additionally, for any fixed sampling point \( \hat{t}_{n_1} \in (0, 1) \setminus U_{n,1} \), it follows from the GHC of \( \Lambda \) that \( \mu(U_{\hat{t}_{n_1},2}) = 0 \) as well where the set

\[
U_{\hat{t}_{n_1},2} := \{ \hat{t}_{n_2} \in (0, 1) : \overline{\phi}(\hat{t}_{n_1})\phi(\hat{t}_{n_2})\overline{\tau}_n(\hat{t}_{n_2}) - \overline{\tau}_n(\hat{t}_{n_1})|\phi(\hat{t}_{n_2})|^2 = 0 \}.
\]
Denote by $\mathcal{E}_{n,2}$ the event $\{A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})i \neq 0\}$. By direct calculation we have

$$
1 \geq P(\mathcal{E}_{n,2}) \\
\geq P(\mathcal{E}_{n,2} \cap \{t_{n_1} \notin U_n\}) \\
= P(\mathcal{E}_{n,2}|\{t_{n_1} \notin U_n\})P(\{t_{n_1} \notin U_n\}) \\
= P(\mathcal{E}_{n,2}|\{t_{n_1} \notin U_n\}) = 1,
$$

(2.29)

where $U_n = U_{n,0} \cup U_{n,1}$. The proof is concluded. \hfill \Box

It can easily follow from Theorem 1.1 that the PR in $V(\phi)$ can not necessarily be achieved only by the condition

$$
A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})i \neq 0, \quad A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_3})i \neq 0.
$$

In what follows, we investigate the probabilistic behavior of the phase $\theta(A_{n,f}(t_{n_1}, t_{n_2}) + iB_{n,f}(t_{n_1}, t_{n_2}))$. It will be helpful for proving that Note 2.23 (2.24) holds with probability 1.

**Lemma 2.9.** Let $t_{n_1}, t_{n_2} \sim U(0, 1)$, and $A_{n,f}(t_{n_1}, t_{n_2}) + iB_{n,f}(t_{n_1}, t_{n_2})$ be as in Lemma 2.8. Then for any $\alpha \in (0, 2\pi)$, it holds that

$$
P\{\theta(A_{n,f}(t_{n_1}, t_{n_2}) + iB_{n,f}(t_{n_1}, t_{n_2})) \neq \alpha\} = 1.
$$

(2.30)

**Proof.** Clearly, if $\frac{|f(n+t_{n_1})|}{|\phi(n_{n_1})|^2} \neq 0$ then it follows from (2.8) that $\theta[A_{n,f}(t_{n_1}, t_{n_2}) + B_{n,f}(t_{n_1}, t_{n_2})i] = \theta[a_{n,f}(t_{n_1}, t_{n_2}) + b_{n,f}(t_{n_1}, t_{n_2})i]$, where $a_{n,f}(t_{n_1}, t_{n_2}) + b_{n,f}(t_{n_1}, t_{n_2})i$ is defined in (2.26). By direct calculation, for $x \in (0, 1)$ and any fixed $\hat{t}_{n_1} \in (0, 1)$ we have

$$
\Re(a_{n,f}(\hat{t}_{n_1}, x) + ib_{n,f}(\hat{t}_{n_1}, x)) \\
= u_{n_1}(\phi_{\Re}(x) + \phi_{\Im}^2(x)) + \sum_{k \in I_n}[\overline{c}_{\hat{t}_{n_1}, k, \Re}(\phi_{\Re}(x)\phi_{\Re}(x + n - k) + \phi_{\Im}(x)\phi_{\Im}(x + n - k))] \\
- \sum_{k \in I_n}[\overline{c}_{\hat{t}_{n_1}, k, \Im}(\phi_{\Im}(x)\phi_{\Re}(x + n - k) - \phi_{\Re}(x)\phi_{\Im}(x + n - k))],
$$

(2.31)

and

$$
\Im(a_{n,f}(\hat{t}_{n_1}, x) + ib_{n,f}(\hat{t}_{n_1}, x)) \\
= v_{n_1}(\phi_{\Re}(x) + \phi_{\Im}^2(x)) + \sum_{k \in I_n}[\overline{c}_{\hat{t}_{n_1}, k, \Re}(\phi_{\Re}(x)\phi_{\Re}(x + n - k) + \phi_{\Im}(x)\phi_{\Im}(x + n - k))] \\
+ \sum_{k \in I_n}[\overline{c}_{\hat{t}_{n_1}, k, \Im}(\phi_{\Im}(x)\phi_{\Re}(x + n - k) - \phi_{\Re}(x)\phi_{\Im}(x + n - k))],
$$

(2.32)

where $u_{n_1}(\hat{t}_{n_1}) = u_{n_1} + iv_{n_1}$ and $c_{\hat{t}_{n_1}, k} := \phi(t_{n_1})c_k = \overline{c}_{\hat{t}_{n_1}, k, \Re} + i\overline{c}_{\hat{t}_{n_1}, k, \Im}$. Firstly, it is easy to derive from $G(f) < s - 1$ and the GHC of $\Lambda$ in (1.6) that $\Re(a_{n,f}(\hat{t}_{n_1}, x) + ib_{n,f}(\hat{t}_{n_1}, x)) \neq 0$ and $\Im(a_{n,f}(\hat{t}_{n_1}, x) + ib_{n,f}(\hat{t}_{n_1}, x)) \neq 0$. Or equivalently, for any fixed
Now we are ready to calculate the probability that \( \text{phase} \) has only one solution on the unit circle. Consequently, with the same probability the equation system in \((2.11)\) can be decoded by \((2.19)\). □

Based on Lemma \([2.9]\), we are ready to prove that the decoding in \((2.19)\) will succeed with probability 1.

**Theorem 2.10.** Let \( \phi \) be as in Lemma \([2.8]\) such that \( \Lambda \) in \((1.6)\) satisfies the GHC on \((0, 1)\). Suppose that the target function \( f \) is as in \((2.4)\) and the variables \( t_{n_1}, t_{n_2} \) and \( t_{n_3} \sim U(0, 1) \) for \( n = 0, \ldots, N \). Then with probability 1 the equation system in \((2.11)\) has only one solution on the unit circle. Consequently, with the same probability the phase \( \theta(f(n + t_{n_1})) \) can be decoded by \((2.19)\).

**Proof.** By Lemma \([2.8]\), \( A_{n,f}(t_{n_1}, t_{n_2}) + iB_{n,f}(t_{n_1}, t_{n_2}) \neq 0 \) with the probability 1. Define three events

\[
E_1 := \{A_{n,f}(t_{n_1}, t_{n_2}) + iB_{n,f}(t_{n_1}, t_{n_2}) \neq 0\}, \quad E_2 := \{A_{n,f}(t_{n_1}, t_{n_2}) - iB_{n,f}(t_{n_1}, t_{n_2}) \neq 0\}, \quad E_3 := \{A_{n,f}(t_{n_1}, t_{n_3}) + iB_{n,f}(t_{n_1}, t_{n_3}) \neq 0\},
\]

and

\[
E_1|E_2 := \{A_{n,f}(t_{n_1}, t_{n_2}) - iB_{n,f}(t_{n_1}, t_{n_2}) - b(t_{n_1}, t_{n_2})(A_{n,f}(t_{n_1}, t_{n_3}) + iB_{n,f}(t_{n_1}, t_{n_3})) \neq 0|E_2\},
\]

where \( b(t_{n_1}, t_{n_2}) = r_1 + ir_2 = \frac{A_{n,f}(t_{n_1}, t_{n_2}) - iB_{n,f}(t_{n_1}, t_{n_2})}{A_{n,f}(t_{n_1}, t_{n_2}) + iB_{n,f}(t_{n_1}, t_{n_2})} \neq 0 \). By Lemma \([2.9]\), we have \( P(E_1|E_2) = 1 \). Now by Theorem \([2.3]\) the proof can be concluded. □
Proof of Theorem 2.4 Note that $|f(t_0)| = |c_0\phi(t_0)| \neq 0$ with the probability 1. As in the proof of Theorem 2.3 let $c_0 := |f(t_0)|/\phi(t_0)$. That is, with the probability 1, $c_0$ can be reconstructed up to a unimodular scalar $e^{i\theta}$. Suppose that by the phaseless sampling $\{|f(k + t_k)| : k = 0, \ldots, n - 1, t_k \in (0, 1), j = 1, 2, 3\}$, the coefficients $\{c_k\}_{k=0}^{n-1}$ have been reconstructed with the probability 1 up to the scalar $e^{i\theta}$. By Theorem 2.10, for the random point $t_{n_1} \sim U(0, 1)$, it follows from the Haar condition of $\Lambda$ in (1.6) that $\phi(t_{n_1}) \neq 0$ with the probability 1. Moreover, by Theorem 2.10, the equation system in (2.11) has only one solution on the unit circle. That is, the phase $\theta(f(n + t_{n_1}))$ can be determined. Then with the probability 1, $c_n$ can be reconstructed by

$$(2.36)\quad c_n := \frac{|f(t_{n_1} + n)|e^{i\theta f(n + n)} - v_{n,f}(t_{n_1})}{\phi(t_{n_1})},$$

where the computation of $v_{n,f}(t_{n_1})$ is conducted by $\{c_k\}_{k=0}^{n-1}$. The rest of the proof can be easily concluded by the induction.

Proof of Proposition 2.5 Denote $D_f := \bigcup_{n=0}^{N}Z_{f(n)}(0, 1) = \bigcup_{n=0}^{N}\{x \in (0, 1) : f(n + x) = 0\}$. By the GHC of $\Lambda$ in (1.6) on $(0, 1)$, the Lebesgue measure $\mu(D_f) = 0$. Then with probability 1, it holds that $f(n + t_1)\phi(t_1) \neq 0$ for any $n$. It follows from $\text{supp}(\phi) \subseteq [0, 2]$ that

$$(2.37)\quad A_{n,f}(t_1, t_j) + iB_{n,f}(t_1, t_j) = c_{n-1}\frac{|f(n + t_1)|}{|\phi(t_1)|^2} [\overline{\phi(t_1)}\phi(t_j)\overline{\phi(1 + t_j)} - |\phi(t_j)|\overline{\phi(1 + t_1)}],$$

where $j = 2, 3$. It is derived from the nonseparability of $f$ that $c_{n-1} \neq 0$. By Lemma 2.8 and the proof of Theorem 2.10 for fixed $n$ it holds with with probability 1 that

$$(2.38)\quad (A_{n,f}(t_1, t_2) + iB_{n,f}(t_1, t_2))(A_{n,f}(t_1, t_3) + iB_{n,f}(t_1, t_3)) \neq 0$$

and

$$(2.39)\quad \frac{A_{n,f}(t_1, t_2) + iB_{n,f}(t_1, t_2)}{A_{n,f}(t_1, t_2) - iB_{n,f}(t_1, t_2)} \neq \frac{A_{n,f}(t_1, t_3) + iB_{n,f}(t_1, t_3)}{A_{n,f}(t_1, t_3) - iB_{n,f}(t_1, t_3)}.$$.

It follows from (2.37) that for $s = 2$, the events in (2.38) and (2.39) is independent of $n$ provided that $f(n + t_1) \neq 0$. Now the proof is concluded by Theorem 2.8.

Proof of Proposition 2.7 For any fixed $n$, it follows from the $m$-Haar condition in (2.22) of $\Lambda$ that both $f(x + n)$ and $\overline{v_{n,f}(x)}\phi(x)$ have at most $m$ zeros on $(0, 1)$. Then

$$(2.40)\quad \#\left(\{t_{n_l} : l = 1, 2, \ldots, 2m + 3\} \cap (Z_{f(n)})^c \cap (Z_{\overline{v_{n,f}}})^c\right) \geq 3,$$
where \((\mathcal{Z}_g)^c\) denotes the nonzero set of the function \(g\). Without loss of generality, we assume that \(f(\hat{t}_{n_1}) \neq 0\) and \(\overline{\nu}_{n,f}(\hat{t}_{n_1}) \phi(\hat{t}_{n_1}) \neq 0\). It follows from (2.4) that

\[
A_{n,f}(\hat{t}_{n_1}, x) + B_{n,f}(\hat{t}_{n_1}, x)i = \frac{|f(n + \hat{t}_{n_1})|}{|\phi(\hat{t}_{n_1})|^2}[a_{n,f}(\hat{t}_{n_1}, x) + b_{n,f}(\hat{t}_{n_1}, x)i],
\]

where \(a_{n,f}(\hat{t}_{n_1}, x) + b_{n,f}(\hat{t}_{n_1}, x)i = \phi(\hat{t}_{n_1})\overline{\nu}(x)\nu_{n,f}(x) + \overline{\nu}_{n,f}(\hat{t}_{n_1})|\phi(x)|^2\). It follows from the \(m\)-Haar condition of \(\Lambda\) again that

\[
\#\left(\{\hat{t}_{n_l} : l = 2, \ldots, 2m + 3\} \cap (\mathcal{Z}_{a_{n,f}(\hat{t}_{n_1}, x) + b_{n,f}(\hat{t}_{n_1}, x)i})^c\right) \geq m + 2.
\]

Without loss of generality, it is assumed that \(a_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2}) + b_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2})i \neq 0\). Therefore \(A_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2}) + B_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2})i \neq 0\). Employing the \(m\)-Haar condition of \(\Lambda\) again, there is at least one point, denoted by \(\hat{t}_{n_3}\) without loss of generality, in \(\{\hat{t}_{n_l} : l = 2, \ldots, 2m + 3\}\) such that \(A_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_3}) + B_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_3})i \neq 0\) and

\[
\frac{A_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2}) + iB_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2})}{B_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2}) - iB_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_2})} \neq \frac{A_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_3}) + iB_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_3})}{A_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_3}) - iB_{n,f}(\hat{t}_{n_1}, \hat{t}_{n_3})}.
\]

Now it follows from (2.41) and (2.43) that there is a unique solution to (2.11). Recalling that \(n\) is arbitrary, the rest of the proof can be easily concluded.

2.4. Numerical simulation. The aim of this subsection is to confirm Theorem 2.4 by conducting the PD-CR approach in subsection 2.2. Our test SIS \(V(\varphi_{a,b,p})\) is from [3 subsection 6.3.1], where the generator is

\[
\varphi_{a,b,p}(x) = \frac{2}{3}\sqrt{\frac{2\pi}{|b|}}e^{-\frac{1}{2}(x-2)^2}e^{-\frac{1}{2}(x-2)^2} \cos^2 \frac{\pi(x-2)}{4} \chi_{[0,4]}(x).
\]

The target function

\[
f_a(x) = \sum_{n=0}^{15} c_n \phi_{a,0.8,1}(x - n),
\]

where \(a = 4, 50\). See Figure 2.1 for the graphs of two target functions \(f_4(x)\) and \(f_{50}(x)\). Recall that the PD-CR is derived from the GHC satisfied by the corresponding system \(\Lambda\) defined in (1.6). By Example 2.1 the GHC holds for both \(\phi_{4,0.8,1}\) and \(\phi_{50,0.8,1}\). For the \(m\)-Haar condition (defined in (1.5) or (2.22)), it is easy to derive from the structure of \(\phi_{a,b,p}\) that \(m\) increases as \(a\) does. To illustrate the phenomenon, for example take

\[
\text{span}\{\Lambda\} \ni h_a(x) = \phi_{a,0.8,1,\Re}(x)\phi_{a,0.8,1,\Re}(x + 3) + \phi_{a,0.8,1,\Im}(x)\phi_{a,0.8,1,\Im}(x + 3)
\]

\[
+ \phi_{a,0.8,1,\Re}(x)\phi_{a,0.8,1,\Im}(x + 3) - \phi_{a,0.8,1,\Im}(x)\phi_{a,0.8,1,\Re}(x + 3).
\]

See Figure 2.2 (a-b) for the zero distributions of \(h_4(x)\) and \(h_{50}(x)\) on \((0,1)\), respectively. Clearly, the zero amount of \(h_{50}(x)\) is much larger than that of \(h_4(x)\). That is, if we use the PD-CR by deterministic sampling to reconstruct \(f_{50}\), then Proposition 2.7 requires that the sampling density needs to be much larger than 3. However, it follows from Theorem 2.4 that \(f_a(x)\) can be reconstructed with probability 1, up to
a unimodular, by the phaseless random samplings \{\mid f_a(t_0)\mid\} \cup \{\mid f_a(n + t_{n_1})\mid, \mid f_a(n + t_{n_2})\mid, \mid f_a(n + t_{n_3})\mid : n = 1, \ldots, 15\}, where \( t_0, t_{n_1}, \ldots, t_{n_3} \sim U(0, 1) \). That is, the random sampling density 3 is sufficient for the reconstruction. In the noiseless setting, \(10^3\) trials are conducted to reconstruct \(f_4\) and \(f_{50}\), respectively. In this section the reconstruction error is defined as

\[
\text{error}(f_a) := \log_{10}(\min_{\gamma \in [0, 2\pi]} \|\{c_k\} - e^{i\gamma}\{\tilde{c}_k\}\|_2/\|\{c_k\}\|_2),
\]

where \(\{\tilde{c}_k\}\) is the coefficient sequence of the reconstruction result \(f_{a,r}(x) = \sum_{n=0}^{20} \tilde{c}_n \phi_{a,0.8,1}(x - n)\). The PD-CR is considered to reconstruct the target successfully if \(\text{error}(f_a) \leq 1.8\). Moreover, the cumulative distribution function (CDF) of \(\text{error}(f_a)\) is defined as

\[
\text{CDF}(x) = \frac{\#(\text{error}(f_a) \leq x)}{\#\text{total trials}}.
\]

Clearly, it follows from Figure 2.2 (c-d) that with probability 1, the targets are reconstructed perfectly in the noiseless setting.

In what follows we examine the robustness of PD-CR to the noise. We add the Gaussian noise \(\varepsilon \sim N(0, \sigma^2)\) to the observed noiseless samplings. That is, we employ the noisy samplings \{\mid f_a(t_0)\mid + \varepsilon\} \cup \{\mid f_a(n + t_{n_1})\mid + \varepsilon, \mid f_a(n + t_{n_2})\mid + \varepsilon, \mid f_a(n + t_{n_3})\mid + \varepsilon :...
$n = 1, \ldots, 15$ to conduct PD-CR. Following [21], the variance $\sigma^2$ is chosen such that the desired signal to noise ratio (SNR) is expressed by

$$(2.49) \quad \text{SNR} = 10 \log_{10} \left( \frac{||F_a||^2}{4\sigma^2} \right),$$

where $||F_a||^2 = |f_a(t_{01})|^2 + \sum_{k=1}^{3} \sum_{n=1}^{15} |f_a(n + \hat{t}_{nk})|^2$. In the noisy setting, $10^3$ trials are also conducted to reconstruct $f_4(x)$ and $f_{50}(x)$, respectively. Their reconstruction success rates (CDF($-1.8$)) are recorded in Table 1. Clearly the reconstruction via PD-CR is robustness to noise corruption.

| $a$ | SNR  | 50    | 60    | 70    | 80    | 90    | 100   | 110   | 120   | 130   |
|-----|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 4   |      | 0.0070| 0.0940| 0.2800| 0.4830| 0.6440| 0.7990| 0.8860| 0.9410| 0.9970|
| 50  |      | 0.0240| 0.1770| 0.4040| 0.6270| 0.7330| 0.8430| 0.9090| 0.9520| 0.9880|

**Table 1.** Success rate vs noise level (SNR).

### 3. Phaseless sampling in real-valued SISs

Let $\varphi$ be real-valued and continuous such that $\text{supp}(f) \subseteq (0, s)$ with integer $s \geq 2$. Motivated by (1.6), Lemma 2.8, Lemma 2.9 and Theorem 2.10, for the real-valued...
For the target function

\[ f \]

satisfies the GHC. As in (1.2), the SIS generated by \( \varphi \) is denoted by

\[ \Lambda := \{ \varphi(x)\varphi(x+k) : k = 0, \ldots, s-1 \} \]

where the coefficient sequence \( \text{supp}(\varphi) \) satisfies the GHC. As in (1.2), the SIS generated by (3.1)

\[ V(\varphi) := \{ \sum_{l \in \mathbb{Z}} c_l \varphi(\cdot - l) : \{ c_l : l \in \mathbb{Z} \} \in \ell^2(\mathbb{Z}) \}, \]

with \( c_0 \neq 0 \). Before establishing the first main result of this section, we give some denotations. As in (2.7) and (2.6), define index sets \( I_n \) by

\[ I_n := \begin{cases} \{0, 1, \ldots, n-1\}, & 1 \leq n \leq s-1, \\ \{n-s+1, \ldots, n-1\}, & n \geq s. \end{cases} \]

For the target function \( f \) in (3.3) with the coefficient sequence \( \{c_k\}_{k=0}^N \), define auxiliary functions \( \{v_n(x)\}_{n=1}^N \) on \((0, 1)\) by

\[ v_{n,f}(x) := \sum_{k \in I_n} c_k \varphi(n+x-k), \quad A_{n,f}(x, y) := \frac{|f(n+x)|}{|\varphi(x)|^2} \left[ \varphi(x)\varphi(y)v_{n,f}(y) - v_{n,f}(x)\varphi^2(y) \right], \]

and

\[ C_{n,f}(x, y) := |f(n+y)|^2 - |v_{n,f}(y)|^2 + \frac{2v_{n,f}(x)v_{n,f}(y)\varphi(y)}{\varphi(x)} - \frac{\varphi(y)}{\varphi(x)}^2 |f(n+x)|^2 + |v_{n,f}(x)|^2. \]

### 3.1 Phaseless sampling for real-valued signals in shift-invariant space.

**Theorem 3.1.** Let the real-valued generator \( \varphi \) be as mentioned in (3.2) such that \( \text{supp}(\varphi) \subseteq [0, s] \) with integer \( s \geq 2 \). Moreover, the function set \( \Lambda \) defined in (3.1) satisfies the GHC. Suppose that the target \( f \) is as in (3.3) such that its maximum gap \( G(f) < s-1 \). Then \( f \) can be determined with the probability 1, up to a global sign, by the phaseless random samplings \( \{|f(t_{01})|\} \cup \{|f(n+t_{n1})|, |f(n+t_{n2})| : n = 1, \ldots, N\} \), where the i.i.d sampling points \( \{t_{01}\} \cup \{t_{n1}, t_{n2} : n = 1, \ldots, N\} \sim \mathcal{U}(0, 1) \).

**Proof.** For the phaseless random samplings \( \{|f(t_{01})|\} \cup \{|f(n+t_{n1})|, |f(n+t_{n2})| : n = 1, \ldots, N\} \), by the similar analysis in Lemma 2.8, it is easy to prove that the followings hold with probability 1:

\[ |f(t_{01})| > 0, \quad A_{n,f}(t_{n1}, t_{n2}) \neq 0. \]

Motivated by the PD-CR in subsection 2.2 (2.5), assume that

\[ f(t_{01}) = |f(t_{01})|, \]
namely, assigning $\theta(f(t_{01})) = 0$, then $c_0 = \frac{|f(t_{01})|}{\varphi(t_{01})}$. We continue to determine $\theta(f(t_{01} + 1))$ and $c_1$. Similarly to (2.13), we have

$$
\begin{align*}
(3.8) & \\
& \begin{cases}
|v_{1,f}^R(t_{11}) + c_1\varphi(t_{11})| = |f(1 + t_{11})|,
|v_{1,f}^R(t_{12}) + c_1\varphi(t_{12})| = |f(1 + t_{12})|.
\end{cases}
\end{align*}
$$

Let $f(1 + t_{11}) = z^*|f(1 + t_{11})|$ with $z^* \in \{1, -1\}$ to be determined. Then it is easy to check that $z^*$ is the solution to

$$
(3.9) \quad A_{1,f}(t_{11}, t_{12})z^2 - C_{1,f}(t_{11}, t_{12})z + A_{1,f}(t_{11}, t_{12}) = 0.
$$

Clearly, it follows from (3.6) that there exist at most two solutions to the above equation with probability 1. Note that the product of the two solutions is 1. Then there exists a unique solution to (3.9) with probability 1. More precisely,

$$
(3.10) \quad z^* = \text{sgn} \left( \frac{C_{1,f}(t_{11}, t_{12})}{A_{1,f}(t_{11}, t_{12})} \right).
$$

Therefore with the same probability $c_1 = \frac{z^*[f(1 + t_{11})]-v_{1,f}(t_{11})}{\varphi(t_{11})}$ by the assumption (3.7). Recall that it is required $f_r = \pm f$, where $f_r$ is the reconstruction result to be obtained. Naturally $\text{sgn}(c_1)$ should be changed if assuming that

$$
(3.11) \quad f(t_{01}) = -|f(t_{01})|.
$$

It is verified in what follows. Clearly, compared with those corresponding to the assumption (3.7), $\text{sgn}(A_{1,f}^R(t_{11}, t_{12}))$ changes while $C_{1,f}^R(t_{11}, t_{12})$ does not. Then under this assumption (3.11), like that in (3.9) the sign of $f(1 + t_{11})$ is determined by

$$
-A_{1,f}^R(t_{11}, t_{12})z^2 - C_{1,f}^R(t_{11}, t_{12})z + A_{1,f}^R(t_{11}, t_{12}) = 0.
$$

Similarly to that in (3.10), the above solution is

$$
\text{sgn} \left( \frac{C_{1,f}^R(t_{11}, t_{12})}{-A_{1,f}^R(t_{11}, t_{12})} \right).
$$

For any $k \in \{1, \ldots, N\}$, suppose that we have reconstructed $c_0, c_1, \ldots, c_{k-1}$ w.r.t the initial assignment that $f(t_{01}) = |f(t_{01})|$. Next we investigate how to reconstruct $c_k$. Similarly to (3.8), we have

$$
(3.12) \quad \begin{cases}
|v_{k,f}^R(t_{k1}) + c_k\varphi(t_{k1})| = |f(1 + t_{k1})|,
|v_{k,f}^R(t_{k2}) + c_k\varphi(t_{k2})| = |f(1 + t_{k2})|.
\end{cases}
$$

Let $v_{k,f}^R(t_{k1}) + c_k\varphi(t_{k1}) = |f(1 + t_{k1})|z$ with $z \in \{1, -1\}$ to be determined as follows:

$$
A_{k,f}^R(t_{k1}, t_{k2})z^2 - C_{k,f}^R(t_{k1}, t_{k2})z + A_{k,f}^R(t_{k1}, t_{k2}) = 0.
$$

For (3.6), there exists a solution to the above equation. Then

$$
(3.13) \quad c_k = \left[ |f(1 + t_{k1})|z - v_{k,f}^R(t_{k1}) \right] / \varphi(t_{k1}).
$$

Moreover, in the assumption (3.11), by the similar analysis as previously, $\text{sgn}(f(1 + t_{k1}))$ and $\text{sgn}(c_k)$ change. The rest of the proof will be concluded by the induction. □
3.2. Reconstruction approach. Based on the proof of Theorem 3.1, in what follows we establish the reconstruction scheme for compactly supported functions in $V(\psi)$.

\[
\begin{align*}
(3.14) & \\
\text{Initialize } & \theta(f(t_0)) = 0; \\
c_0 &= \frac{|f(t_0)|}{\phi(t_0)} \rightarrow \cdots \rightarrow \{|f(k + t_j)|\}_{j=1}^2 \rightarrow \text{sgn}(f(k + t_{k_1})) = \text{sgn}\left(\frac{C_{k_1}^{\Re}(t_{k_1}, t_{k_2})}{A_{k,f}^{\Re}(t_{k_1}, t_{k_2})}\right) \\
&\rightarrow f(k + t_{k_1}) \rightarrow \text{coefficient recovery} (3.13) \rightarrow c_k \cdots \rightarrow f.
\end{align*}
\]

Since the phase of a real-valued data is either 0 or $\pi$, it is not necessary to employ the phase decoder in (2.5) in the real-valued case. Instead, in scheme (3.14) it is decoded by the relatively simple formula: $\text{sgn}(f(k + t_{k_1})) = \text{sgn}\left(\frac{C_{k_1}^{\Re}(t_{k_1}, t_{k_2})}{A_{k,f}^{\Re}(t_{k_1}, t_{k_2})}\right)$.

3.3. Numerical simulation. This subsection aims at examining the efficiency of random sampling established in Theorem 3.1. The generator $\phi$ herein is chosen as $\phi_{a,-0.238,1,\Re}$, the real part of $\phi_{a,-0.238,1}$ defined via subsection 2.4 (2.44), where $a = 10$.
\[
V(\Lambda \text{ in (3.1) satisfies the GHC, and therefore by Theorem 3.1 any CSN function is sufficient for the phase retrieval of compactly supported functions.}
\]

The reconstruction error is defined in (2.48). Clearly, it is confirmed by Figure 3.4 that.

It is easy to check that
\[
\text{(3.15)}
\]

\[
f_a(t) = \sum_{n=0}^{20} c_n \phi_{a, -0.238, 1, \sqrt{t} - n}, c_0 \neq 0.
\]

The target function to be reconstructed is
\[
\text{(3.16)}
\]

\[
\text{Error}(f_a) := \log_{10}(\min_{\gamma \in \{1, -1\}} \|\{c_k\} - \gamma \{\tilde{c}_k\}\|_2/\|\{c_k\}\|_2).
\]

As in subsection 2.4, the approach is considered to reconstruct the target successfully if \(\text{Error}(f_a) \leq -1.8\), and the as cumulative distribution function (CDF) of the reconstruction error is defined in (2.48). Clearly, it is confirmed by Figure 3.4 that \(f_{10}\) and \(f_{50}\) can be reconstructed perfectly in the noiseless setting. To check the robustness to noise, we also conduct the reconstruction of \(f_{10}\) and \(f_{50}\) for \(10^3\) trials in the noisy setting, respectively. As in subsection 2.4, we add the Gaussian noise \(\epsilon \sim N(0, \sigma^2)\) to the

**Figure 3.4.** (a) The CDF of \(\text{Error}(f_{10})\); (b) The CDF of \(\text{Error}(f_{50})\).
observed noiseless samplings \( \{ |f_a(t_0)| \} \cup \{ |f_a(n + \hat{t}_{n_1})|, |f_a(n + \hat{t}_{n_2})| : n = 1, \ldots, 20 \} \). The variance \( \sigma^2 \) is chosen via (2.49) with 46 therein replaced by 41. As in the noiseless case, \( 10^3 \) trials are also conducted to reconstruct \( f_{10} \) and \( f_{50} \), respectively. The success rates (CDF(−1.8)) are recorded in Table 2.

| a  | SNR | 80  | 85  | 90  | 95  | 100 | 105 | 110 | 115 | 120 |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 10 |     | 0.8340 | 0.8640 | 0.8680 | 0.9100 | 0.9480 | 0.9620 | 0.9740 | 0.9920 | 0.9960 |
| 50 |     | 0.6740 | 0.8020 | 0.8440 | 0.8700 | 0.9120 | 0.9220 | 0.9340 | 0.9720 | 0.9890 |

Table 2. Success rate vs noise level (SNR).

4. Conclusion

In conclusion, based on the generalized-Haar condition we establish the random phaseless sampling for the complex-generated and real-generated shift-invariant spaces. We first prove that the full spark property of the generator is not sufficient for the phase retrieval in the complex-generated shift-invariant space (Theorem 1.1). For the phaseless samplings of the compactly supported signals in a complex-generated shift-invariant space, we establish a necessary and sufficient condition for decoding the phases of the samplings (Theorem 2.3). Based on the necessary and sufficient condition, we establish a reconstruction approach: PD-CR. By the PD-CR, a compactly supported and nonseparable signals in the the complex-generated shift-invariant spaces can be reconstructed with probability 1 provided that the random sampling density is not smaller than 3 (Theorem 2.4). The PD-CR is modified such that it is more adaptive to real-generated shift-invariant spaces. By the modified PD-CR, a compactly supported and nonseparable signals in the the real-generated shift-invariant spaces can be reconstructed with probability 1 provided that the random sampling density is not smaller than 2 (Theorem 3.1). Numerical simulations are conducted on the highly oscillatory signals in chirp-modulated SISs to confirm our results.

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