Warped product pointwise bi-slant submanifolds in metallic Riemannian manifolds

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Abstract: In this paper, we study some properties of warped product pointwise bi-slant submanifolds in locally metallic Riemannian manifolds and we construct some examples in Euclidean spaces.

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1 Introduction

Metallic Riemannian manifolds and their submanifolds were defined and investigated by C. E. Hretcanu, M. Crasmareanu and A. M. Blaga in ([21], [24]), as a generalization of Golden Riemannian manifolds studied in ([14], [23], [25]). The authors of the present paper obtained some properties of invariant, anti-invariant and slant submanifolds ([5]), semi-slant submanifolds ([20]) and, respectively, hemi-slant submanifolds ([19]) in metallic and Golden Riemannian manifolds and they provided some integrability conditions for the distributions involved in these types of submanifolds. Moreover, properties of metallic and Golden warped product Riemannian manifolds were presented in some previous works of the authors ([4], [7], [26]). In the last years, the study of submanifolds in metallic Riemannian manifolds has been continued by many authors ([1], [15], [16]), which introduced the notion of lightlike submanifold of a metallic semi-Riemannian manifold.

2 Preliminaries

The name of metallic number is given to the positive solution of the equation $x^2 - px - q = 0$, which is $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$ ([29]), where $p$ and $q$ are positive integer values. The metallic structure is a particular case of polynomial structure on a manifold, which was generally defined in ([17], [18]).
Let \( \overline{M} \) be an \( m \)-dimensional manifold endowed with a tensor field \( J \) of type \((1,1)\). Then \( J \) is called a \textit{metallic structure} if it satisfies:

\begin{equation}
J^2 = pJ + qI,
\end{equation}

for \( p, q \in \mathbb{N}^* \), where \( I \) is the identity operator on \( \Gamma(T\overline{M}) \). If a Riemannian metric \( \overline{g} \) is \( J \)-compatible, i.e.: \( \overline{g}(JX, Y) = \overline{g}(X, JY) \), for any \( X, Y \in \Gamma(T\overline{M}) \), then \((\overline{M}, \overline{g}, J)\) is called a \textit{metallic Riemannian manifold} \([24]\). In this case, \( \overline{g} \) verifies:

\begin{equation}
\overline{g}(JX, JY) = \overline{g}(J^2X, Y) = p\overline{g}(JX, Y) + q\overline{g}(X, Y),
\end{equation}

for any \( X, Y \in \Gamma(T\overline{M}) \).

For \( p = q = 1 \) one obtain the \textit{Golden structure} \( J \) which satisfies \( J^2 = J + I \). If \((\overline{M}, \overline{g})\) is a Riemannian manifold endowed with a Golden structure \( J \) such that the Riemannian metric \( \overline{g} \) is \( J \)-compatible, then \((\overline{M}, \overline{g}, J)\) is called a \textit{Golden Riemannian manifold} \([14]\).

Let \( M \) be an isometrically immersed submanifold in the metallic Riemannian manifold \((\overline{M}, \overline{g}, J)\). The tangent space \( T_x\overline{M} \) of \( \overline{M} \) in a point \( x \in M \) can be decomposed into the direct sum \( T_x\overline{M} = T_xM \oplus T_x^\perp M \), for any \( x \in M \), where \( T_x^\perp M \) is the normal space of \( M \) in \( x \). Let \( i_* \) be the differential of the immersion \( i : M \to \overline{M} \). Then the induced Riemannian metric \( g \) on \( M \) is given by \( g(X, Y) = \overline{g}(i_*X, i_*Y) \), for any \( X, Y \in \Gamma(TM) \). In all the rest of the paper, we shall denote by \( X \) the vector field \( i_*X \), for any \( X \in \Gamma(TM) \).

For any \( X \in \Gamma(TM) \), let \( TX := (JX)^T \) and \( NX := (JX)^\perp \) be the tangential and normal components, respectively, of \( JX \) and for any \( V \in \Gamma(T^\perp M) \), let \( tV := (JV)^T \), \( nV := (JV)^\perp \) be the tangential and normal components, respectively, of \( JV \). Then we have:

\begin{equation}
JX = TX + NX,
\end{equation}

\begin{equation}
JV = tV + nV,
\end{equation}

for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).

The maps \( T \) and \( n \) are \( \overline{g} \)-symmetric \([5]\):

\begin{equation}
\overline{g}(TX, Y) = \overline{g}(X, TY),
\end{equation}

\begin{equation}
\overline{g}(nU, V) = \overline{g}(U, nV)
\end{equation}

and

\begin{equation}
\overline{g}(NX, V) = \overline{g}(X, tV),
\end{equation}

for any \( X, Y \in \Gamma(TM) \) and \( U, V \in \Gamma(T^\perp M) \).
We also obtain ([19]):

\begin{align*}
(2.9) & & T^2X = pTX + qX - tNX, \\
(2.10) & & pNX = NTX + nNX, \\
(2.11) & & n^2V = pnV + qV - NtV, \\
(2.12) & & ptV = TtV + tnV,
\end{align*}

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Let $\overline{\nabla}$ and $\nabla$ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and on its submanifold $(M, g)$, respectively. The Gauss and Weingarten formulas are given by:

\begin{align*}
(2.13) & & \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \\
(2.14) & & \overline{\nabla}_X V = -A_V X + \nabla^\perp_X V,
\end{align*}

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where $h$ is the second fundamental form and $A_V$ is the shape operator, which satisfy

\begin{equation}
(2.15) \quad \overline{g}(h(X, Y), V) = \overline{g}(A_V X, Y).
\end{equation}

For any $X, Y \in \Gamma(TM)$, the covariant derivatives of $T$ and $N$ are given by:

\begin{align*}
(2.16) & & (\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \\
(2.17) & & (\nabla_X N)Y = \nabla^\perp_X NY - N(\nabla_X Y).
\end{align*}

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, the covariant derivatives of $t$ and $n$ are given by:

\begin{align*}
(2.18) & & (\nabla_X t)V = \nabla_X tV - t(\nabla_X^\perp V), \\
(2.19) & & (\nabla_X n)V = \nabla_X^\perp nV - n(\nabla_X^\perp V).
\end{align*}

From (2.1) we obtain:

\begin{equation}
(2.20) \quad \overline{g}((\nabla_X J)Y, Z) = \overline{g}(Y, (\nabla_X J)Z),
\end{equation}

for any $X, Y, Z \in \Gamma(T\overline{M})$, which implies (6):
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\[ g((\nabla_X T)Y, Z) = g(Y, (\nabla_X T)Z), \]

\[ g((\nabla_X N)Y, V) = g(Y, (\nabla_X t)V), \]

for any \( X, Y, Z \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).

The analogue concept of locally product manifold is considered in the context of metallic geometry, having the name of \textit{locally metallic manifold} (\cite{T}). Thus, we say that the metallic Riemannian manifold \((M, g, J)\) is \textit{locally metallic} if \( J \) is parallel with respect to the Levi-Civita connection \( \nabla \) on \( M \) (i.e. \( \nabla J = 0 \)).

\textbf{Remark 2.1.} In \cite{24} we obtained that any almost product structure \( F \) on \( M \) induces two metallic structures on \( M \):

\[ J_1 = \frac{2\sigma - p}{2} F + \frac{p}{2} I, \]

\[ J_2 = -\frac{2\sigma - p}{2} F + \frac{p}{2} I, \]

where \( \sigma = \frac{p+\sqrt{p^2+4q}}{2} \), with \( p, q \) positive integer numbers.

Also, for an almost product structure \( F \) and for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \), the decompositions into the tangential and normal components of \( FX \) and \( FV \) are given by:

\[ FX = fX + \omega X, \]

\[ FV = BV + CV, \]

where \( fX := (FX)^T \), \( \omega X := (FX)^\perp \), \( BV := (FV)^T \) and \( CV := (FV)^\perp \).

Moreover, the maps \( f \) and \( C \) are \( \bar{g} \)-symmetric (\cite{27}):

\[ \bar{g}(fX, Y) = \bar{g}(X, fY), \]

\[ \bar{g}(CU, V) = \bar{g}(U, CV), \]

for any \( X, Y \in \Gamma(TM) \) and \( U, V \in \Gamma(T^\perp M) \).

\textbf{Remark 2.2.} (\cite{20}) If \( M \) is a submanifold in the almost product Riemannian manifold \((M, \bar{g}, F)\) and \( J \) is a metallic structure induced by \( F \) on \( M \), then:

\[ TX = \frac{p}{2} X \pm \frac{2\sigma - p}{2} fX, \]

\[ NX = \pm \frac{2\sigma - p}{2} \omega X, \]

for any \( X \in \Gamma(TM) \).
3 Pointwise slant submanifolds in metallic Riemannian manifolds

B.-Y. Chen studied CR-submanifolds of a Kähler manifold which are warped products of holomorphic and totally real submanifolds, respectively ([9], [10], [11]). Also, in his new book ([8]), he presents a multitude of properties for warped product manifolds and submanifolds, such as: warped product of Riemannian and Kähler manifolds, warped product submanifolds of Kähler manifolds (with the particular cases: warped product CR-submanifolds, warped product semi-slant or hemi-slant submanifolds of Kähler manifolds), CR-warped products in complex space forms and so on.

We shall state the notion of pointwise slant submanifold in a metallic Riemannian manifold, following Chen’s definition ([12], [13]) of pointwise slant submanifold of an almost Hermitian manifold.

**Definition 3.1.** A submanifold $M$ of a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ is called **pointwise slant** if the angle $\theta_x(X)$ between $JX$ and $T_xM$ (called the Wirtinger angle) is independent of the choice of the tangent vector $X \in T_xM \setminus \{0\}$, but it depends on $x \in M$. The Wirtinger angle is a real-valued function $\theta$ (called the Wirtinger function), verifying

\[ \cos \theta_x = \frac{\|TX\|}{\|JX\|}, \]

for any $x \in M$ and $X \in T_xM \setminus \{0\}$.

A pointwise slant submanifold of a metallic Riemannian manifold is called **slant submanifold** if its Wirtinger function $\theta$ is globally constant.

In a similar manner as in ([12]) we obtain:

**Proposition 3.2.** If $M$ is an isometrically immersed submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then $M$ is a pointwise slant submanifold if and only if

\[ T^2 = (\cos^2 \theta)(pT + qI), \]

for some real-valued function $\theta$.

From (2.9) and (3.2) we have:

**Proposition 3.3.** Let $M$ be an isometrically immersed submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If $M$ is a pointwise slant submanifold with the Wirtinger angle $\theta$, then:

\[ \overline{g}(NX, NY) = (\sin^2 \theta)[p\overline{g}(TX, Y) + q\overline{g}(X, Y)] \]
and

\[(3.4) \quad tNX = (\sin^2 \theta)(pTX + qX),\]

for any \(X, Y \in \Gamma(TM)\).

From (3.2), by a direct computation, we obtain:

**Proposition 3.4.** Let \(M\) be an isometrically immersed submanifold in the metallic Riemannian manifold \((\overline{M}, \overline{g}, J)\). If \(M\) is a pointwise slant submanifold with the Wirtinger angle \(\theta\), then:

\[(3.5) \quad (\nabla_X T^2)Y = p(cos^2 \theta)(\nabla_X T)Y - \sin(2\theta)X(\theta)(pTY + qY),\]

for any \(X, Y \in \Gamma(TM)\).

### 4 Pointwise bi-slant submanifolds in metallic Riemannian manifolds

In this section we introduce the notion of pointwise bi-slant submanifold in the metallic context.

**Definition 4.1.** Let \(M\) be an immersed submanifold in a metallic Riemannian manifold \((\overline{M}, \overline{g}, J)\). We say that \(M\) is a pointwise bi-slant submanifold of \(\overline{M}\) if there exists a pair of orthogonal distributions \(D_1\) and \(D_2\) on \(M\) such that

(i) \(TM = D_1 \oplus D_2\);
(ii) \(J(D_1) \perp D_2\) and \(J(D_2) \perp D_1\);
(iii) the distributions \(D_1, D_2\) are pointwise slant.

If \(\theta_1\) and \(\theta_2\) are the slant functions of \(D_1\) and \(D_2\), respectively, then the pair \(\{\theta_1, \theta_2\}\) is called the bi-slant function.

A pointwise slant submanifold \(M\) is called proper if \(\theta_{1x}, \theta_{2x} \neq 0; \frac{\pi}{2}\), for any \(x \in M\) and both \(\theta_1, \theta_2\) are not constant on \(M\).

In particular, if \(\theta_1 = 0\) and \(\theta_2 \neq 0; \frac{\pi}{2}\), then \(M\) is called a pointwise semi-slant submanifold; if \(\theta_1 = \frac{\pi}{2}\) and \(\theta_2 \neq 0; \frac{\pi}{2}\), then \(M\) is called a pointwise hemi-slant submanifold.

Remark that if \(M\) is a pointwise bi-slant submanifold of \(\overline{M}\), then the distributions \(D_1\) and \(D_2\) on \(M\) verify \(T(D_1) \subseteq D_1\) and \(T(D_2) \subseteq D_2\).

**Example 4.2.** Let \(\mathbb{R}^6\) be the Euclidean space endowed with the usual Euclidean metric \((\cdot, \cdot)\). Let \(i : M \to \mathbb{R}^6\) be the immersion given by:

\[i(u, v) := (\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v, \sin v, \cos v),\]

where \(M := \{(u, v) \mid u, v \in (0, \frac{\pi}{2})\}\).
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A local orthogonal frame on $TM$ is given by:

$$Z_1 = -\sin u \cos v \frac{\partial}{\partial x_1} - \sin u \sin v \frac{\partial}{\partial x_2} + \cos u \cos v \frac{\partial}{\partial x_3} + \cos u \sin v \frac{\partial}{\partial x_4},$$

$$Z_2 = -\cos u \sin v \frac{\partial}{\partial x_1} + \cos u \cos v \frac{\partial}{\partial x_2} - \sin u \sin v \frac{\partial}{\partial x_3} + \sin u \cos v \frac{\partial}{\partial x_4} + \cos u \frac{\partial}{\partial x_5} - \sin v \frac{\partial}{\partial x_6}.$$  

We define the metallic structure $J : \mathbb{R}^6 \to \mathbb{R}^6$ by:

$$J(X_1, X_2, X_3, X_4, X_5, X_6) := (\sigma X_1, \overline{\sigma} X_2, \sigma X_3, \overline{\sigma} X_4, \sigma X_5, \overline{\sigma} X_6),$$

where $\sigma := \frac{p + \sqrt{p^2 + 4q}}{2}$ is a metallic number ($p, q \in \mathbb{N}^*$) and $\overline{\sigma} = p - \sigma$.

We remark that $J$ verifies $J^2X = pJ + qI$ and $\langle JX, JY \rangle = \langle X, JY \rangle$, for any $X, Y \in \mathbb{R}^6$.

Also, we have

$$JZ_1 = -\sigma \sin u \cos v \frac{\partial}{\partial x_1} - \overline{\sigma} \sin u \sin v \frac{\partial}{\partial x_2} + \sigma \cos u \cos v \frac{\partial}{\partial x_3} + \overline{\sigma} \cos u \sin v \frac{\partial}{\partial x_4},$$

$$JZ_2 = -\sigma \cos u \sin v \frac{\partial}{\partial x_1} + \overline{\sigma} \cos u \cos v \frac{\partial}{\partial x_2} - \sigma \sin u \sin v \frac{\partial}{\partial x_3} + \overline{\sigma} \sin u \cos v \frac{\partial}{\partial x_4} + \sigma \cos u \frac{\partial}{\partial x_5} - \overline{\sigma} \sin v \frac{\partial}{\partial x_6}.$$  

We remark that $\langle JZ_1, Z_2 \rangle = \langle JZ_2, Z_1 \rangle = 0$, $\langle JZ_1, Z_1 \rangle = \sigma \cos^2 v + \overline{\sigma} \sin^2 v$ and $\langle JZ_2, Z_2 \rangle = p$.

On the other hand we get:

$$\|Z_1\| = 1, \quad \|Z_2\| = \sqrt{2},$$

$$\|JZ_1\| = \sqrt{\sigma^2 \cos^2 v + \overline{\sigma}^2 \sin^2 v} = \sqrt{p(\sigma \cos^2 v + \overline{\sigma} \sin^2 v) + q},$$

$$\|JZ_2\| = \sqrt{\sigma^2 + \overline{\sigma}^2} = \sqrt{p^2 + 2q}.$$  

We denote by $D_1 := \text{span}\{Z_1\}$ the pointwise slant distribution with the slant angle $\theta_1$, where $\cos \theta_1 = \frac{f(u,v)}{\sqrt{p^2(u,v) + q}}$, for $f(u,v) := \sigma \cos^2 v + \overline{\sigma} \sin^2 v$ a real function on $M$.

Also, we denote by $D_2 := \text{span}\{Z_2\}$ the slant distribution with the slant angle $\theta_2$, where $\cos \theta_2 = \frac{p}{\sqrt{2(p^2 + 2q)}}$.

The distributions $D_1$ and $D_2$ satisfy the conditions from Definition 4.1.

If $M_1$ and $M_2$ are the integral manifolds of the distributions $D_1$ and $D_2$, respectively, then $M := M_1 \times \sqrt{2} M_2$ with the Riemannian metric tensor

$$g := du^2 + 2dv^2$$

is a pointwise bi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$. 

Example 4.3. In particular, if \( f \) is a metallic function (i.e. \( f^2 = pf + q \)), then \( \cos \theta_1 = 1 \) and we remark that \( M \) is a semi-slant submanifold in the metallic Riemannian manifold \((\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)\), with the slant angle \( \theta = \theta_2 \).

Example 4.4. On the other hand, if \( f = 0 \) (i.e. \( \tan v = \sqrt{-\frac{p}{q}} = \sqrt{\frac{p^2 + 4q + 4}{q}} \)), then \( \cos \theta_1 = 0 \) and we remark that \( M \) is a hemi-slant submanifold in the metallic Riemannian manifold \((\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)\), with the slant angle \( \theta = \theta_2 \).

If we denote by \( P_i \) the projections from \( T_M \) onto \( D_i \), for \( i \in \{1, 2\} \), then \( X = P_1X + P_2X \), for any \( X \in \Gamma(TM) \). In particular, if \( X \in D_i \), then \( X = P_iX \), for \( i \in \{1, 2\} \).

If we denote by \( T_i = P_i \circ T \), for \( i \in \{1, 2\} \), then, from (2.4), we obtain:

\[
JX = T_1X + T_2X + NX.
\]

In a similar manner as in ([13]), we get:

Lemma 4.5. Let \( M \) be a pointwise bi-slant submanifold of a locally metallic Riemannian manifold \((\overline{M}, \overline{g}, J)\) with pointwise slant distributions \( D_1 \) and \( D_2 \), having slant functions \( \theta_1 \) and \( \theta_2 \). Then

(i) for any \( X, Y \in D_1 \) and \( Z \in D_2 \), we have:

\[
(\sin^2 \theta_1 - \sin^2 \theta_2)(\overline{\nabla}_X Y, pT_2Z + qZ) = \]

\[
= p[\overline{g}(\overline{\nabla}_X Y, T_2Z) + \overline{g}(\overline{\nabla}_X Z, T_1Y)] + p(\cos^2 \theta_1 + 1)\overline{g}(A_{NZ}Y + A_{NY}Z, X) - \]

\[
- \overline{g}(A_{NT_1}Y + A_{NT_2}Z, X) - \overline{g}(A_{NZT_1}Y + A_{NZT_2}Z, X);
\]

(ii) for any \( X \in D_1 \) and \( Z, W \in D_2 \), we have:

\[
(\sin^2 \theta_2 - \sin^2 \theta_1)(\overline{\nabla}_Z W, pT_1X + qX) = \]

\[
= p[\overline{g}(\overline{\nabla}_Z W, T_1X) + \overline{g}(\overline{\nabla}_Z X, T_2W)] + p(\cos^2 \theta_2 + 1)\overline{g}(A_{NX}W + A_{NW}X, Z) - \]

\[
- \overline{g}(A_{NT_2}W + A_{NT_1}X, Z) - \overline{g}(A_{NW}T_1X + A_{NX}T_2W, Z).
\]

Proof. From (2.1) we have:

\[
q\overline{g}(\overline{\nabla}_X Y, Z) = q\overline{g}(\overline{\nabla}_X Y, Z) = \overline{g}(J^2\overline{\nabla}_X Y, Z) - p\overline{g}(J\overline{\nabla}_X Y, Z),
\]

for any \( X, Y \in D_1 \) and \( Z \in D_2 \).

By using (2.2) and \((\overline{\nabla}_X J)Y = 0\), we obtain:

\[
q\overline{g}(\overline{\nabla}_X Y, Z) = \overline{g}(\overline{\nabla}_X J^2 Y, Z) - p\overline{g}(\overline{\nabla}_X JY, Z).
\]
From (4.1) we get $JX = T_1X + NX$, $JY = T_1Y + NY$ and $JZ = T_2Z + NZ$, for any $X, Y \in D_1$ and $Z \in D_2$ and from here we obtain:

$$q\tilde{g}(\nabla_X Y, Z) = \tilde{g}(\nabla_X J T_1 Y, Z) + \tilde{g}(\nabla_X J NY, Z) - p\tilde{g}(\nabla_X (T_1 Y + NY), Z) =$$

$$= \tilde{g}(\nabla_X T_1^2 Y, Z) + \tilde{g}(\nabla_X NT_1 Y, Z) + \tilde{g}(\nabla_X NY, J Z) - p\tilde{g}(\nabla_X T_1 Y, Z) +$$

$$+ p\tilde{g}(A_{NY} X, Z) =$$

$$= \tilde{g}(\nabla_X (\cos^2 \theta_1 (p T_1 Y + q Y)), Z) - \tilde{g}(A_{NT_1 Y} X, Z) + \tilde{g}(\nabla_X NY, T_2 Z + NZ) +$$

$$+ p\tilde{g}(T_1 Y, \nabla_X Z) + p\tilde{g}(A_{NY} X, Z).$$

Thus, we get:

$$q\tilde{g}(\nabla_X Y, Z) = \cos^2 \theta_1 \tilde{g}(\nabla_X (p T_1 Y + q Y), Z) - \sin 2\theta_1 X(\theta_1)\tilde{g}(p T_1 Y + q Y, Z) -$$

$$- \tilde{g}(A_{NT_1 Y} X, Z) - \tilde{g}(A_{NY} X, T_2 Z) - \tilde{g}(\nabla_X NZ, JY) + \tilde{g}(\nabla_X NZ, T_1 Y) +$$

$$+ p\tilde{g}(T_1 Y, \nabla_X Z) + p\tilde{g}(A_{NY} X, Z).$$

By using $\tilde{g}(p T_1 Y + q Y, Z) = 0$, we obtain:

$$q \sin^2 \theta_1 \tilde{g}(\nabla_X Y, Z) = p \cos^2 \theta_1 \tilde{g}(\nabla_X T_1 Y, Z) - \tilde{g}(A_{NT_1 Y} Z + A_{NY} T_2 Z, X) +$$

$$+ \tilde{g}(J NZ, \nabla_X Y) - \tilde{g}(A_{NZ} X, T_1 Y) + \tilde{g}(T_1 Y, \nabla_X Z) + p\tilde{g}(A_{NY} X, Z).$$

Using (2.10) and (3.4), we find:

$$\tilde{g}(J NZ, \nabla_X Y) = \tilde{g}(tNZ + nNZ, \nabla_X Y) =$$

$$= \sin^2 \theta_2 \tilde{g}(\nabla_X Y, q Z + p T_2 Z) + \tilde{g}(p Nz - NT_2 Z, \nabla_X Y) =$$

$$= q \sin^2 \theta_2 \tilde{g}(\nabla_X Y, Z) + p \sin^2 \theta_2 \tilde{g}(\nabla_X Y, T_2 Z) -$$

$$- p\tilde{g}(\nabla_X NZ, Y) + \tilde{g}(\nabla_X NT_2 Z, Y)$$

and from

$$\tilde{g}(\nabla_X T_1 Y, Z) = - \tilde{g}(J Y - NY, \nabla_X Z) = \tilde{g}(\nabla_X Y, J Z) - \tilde{g}(\nabla_X NY, Z) =$$

$$= \tilde{g}(\nabla_X Y, T_2 Z) - \tilde{g}(Y, \nabla_X NZ) - \tilde{g}(\nabla_X NY, Z) =$$

$$= \tilde{g}(\nabla_X Y, T_2 Z) + \tilde{g}(Y, A_{NZ} X) + \tilde{g}(A_{NY} X, Z)$$

we have:

$$q(\sin^2 \theta_1 - \sin^2 \theta_2)\tilde{g}(\nabla_X Y, Z) = p(1 - \sin^2 \theta_1)\tilde{g}(\nabla_X Y, T_2 Z) +$$

$$+ p \cos^2 \theta_1 \tilde{g}(A_{NZ} Y + A_{NY} Z, X) + \sin^2 \theta_2 \tilde{g}(\nabla_X Y, T_2 Z) -$$

$$- \tilde{g}(A_{NT_1 Y} Z + A_{NY} T_2 Z + A_{NZ} T_1 Y + A_{NT_2} Y, X) -$$

$$- p\tilde{g}(Y, \nabla_X NZ) + p\tilde{g}(T_1 Y, \nabla_X Z) + p\tilde{g}(A_{NY} X, Z)$$

and from here we get (4.2).

In the same manner we find (4.3).
Proposition 4.6. Let $M$ be a pointwise semi-slant submanifold in a locally metallic Riemannian manifold $(M, \mathbf{g}, J)$ with pointwise slant distributions $D_1$ and $D_2$, having slant functions $\theta_1$ and $\theta_2$.

(i) If $\theta_1 = 0$ and $\theta_2 = \theta$, we obtain:

$$\sin^2 \theta \mathbf{g}(\nabla_X Y, p T_2 Z + q Z) = -p \mathbf{g}(\nabla_X Y, T_2 Z) + \mathbf{g}(\nabla_X Z, T_1 Y) - 2p \mathbf{g}(A_{NZ} Y, X) + \mathbf{g}(A_{NZ} T_1 Y + A_{NT_2} Z Y, X),$$

for any $X, Y \in D^T$ and $Z \in D^\theta$, and

$$\sin^2 \theta \mathbf{g}(\nabla_Z W, p T_1 X + q X) = p \mathbf{g}(\nabla_Z W, T_1 X) + \mathbf{g}(\nabla_Z T_2 W, X) + p(\cos^2 \theta + 1) \mathbf{g}(A_{NW} Z Y, X) - \mathbf{g}(A_{NT_2} W Y + A_{NW} T_1 X, Z),$$

for any $X, Y \in D^\theta$ and $Z, W \in D^T$.

(ii) If $\theta_1 = \theta$ and $\theta_2 = 0$, we obtain:

$$\sin^2 \theta \mathbf{g}(\nabla_X Y, T_2 Z) = p \mathbf{g}(\nabla_X Y, T_2 Z) - \mathbf{g}(\nabla_X T_1 Y, Z) + p(\cos^2 \theta + 1) \mathbf{g}(A_{XY} Z Y, X) - \mathbf{g}(A_{NT_2} Z Y + A_{NT_2} T_1 Z, X),$$

for any $X, Y \in D^T$ and $Z \in D^\theta$, and

$$\sin^2 \theta \mathbf{g}(\nabla_Z W, p T_1 X + q X) = -p \mathbf{g}(\nabla_Z W, T_1 X) + p \mathbf{g}(\nabla_Z T_2 W, X) - 2p \mathbf{g}(A_{NX} Z, W) + \mathbf{g}(A_{NT_2} Z, W + \mathbf{g}(A_{NT_1} Z W, X) + \mathbf{g}(A_{NX} T_2 W, Z),$$

for any $X \in D^\theta$ and $Z, W \in D^T$.

Proposition 4.7. Let $M$ be a pointwise hemi-slant submanifold in a locally metallic Riemannian manifold $(M, \mathbf{g}, J)$ with pointwise slant distributions $D_1$ and $D_2$, having slant functions $\theta_1$ and $\theta_2$.

(i) If $\theta_1 = 0$ and $\theta_2 = \theta$, we obtain:

$$\cos^2 \theta \mathbf{g}(\nabla_X Y, p T_2 Z + q Z) = p \mathbf{g}(\nabla_X Y, T_2 Z) + \mathbf{g}(A_{NZ} Y + A_{NY} Z, X) - \mathbf{g}(A_{NT_2} Z Y + A_{NY} T_2 Z, X),$$

for any $X, Y \in D^T$ and $Z \in D^\theta$, and

$$p \cos^2 \theta \mathbf{g}(\nabla_Z W, X) = -p \mathbf{g}(\nabla_Z W, T_2 W) - p(\cos^2 \theta + 1) \mathbf{g}(A_{NX} W + A_{NW} X, Z) + \mathbf{g}(A_{NT_2} W X + A_{NX} T_2 W, Z),$$

for any $X \in D^T$ and $Z, W \in D^\theta$. 
(ii) If $\theta_1 = \theta$ and $\theta_2 = \frac{\pi}{2}$, we obtain:

\begin{equation}
q \cos^2 \theta g(\nabla_X Y, Z) = -p g(\nabla_X T_1 Y) - \\
-p(\cos^2 \theta + 1) g(A_{NZ} Y + A_{NY} Z, X) + g(A_{NT_1 Y} Z + A_{NZ} T_1 Y, X),
\end{equation}

for any $X, Y \in D^\theta$ and $Z \in D^\perp$, and

\begin{equation}
\cos^2 \theta g(\nabla_Z W, p T_1 X + q X) = p g(\nabla_Z T_1 X) + \\
+ p g(A_{NX} W + A_{NW} X, Z) - g(A_{NT_1 X} W + A_{NW} T_1 X, Z),
\end{equation}

for any $X \in D^\theta$ and $Z, W \in D^\perp$.

5 Warped product pointwise bi-slant submanifolds in metallic Riemannian manifolds

In (\cite{1}), the authors of this paper introduced the Golden warped product Riemannian manifold and provided a necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden. Moreover, the subject was continued in the papers (\cite{7}, \cite{22}), where the authors characterized the metallic structure on the product of two metallic manifolds in terms of metallic maps and provided a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic.

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds (of dimensions $n_1 > 0$ and $n_2 > 0$, respectively) and let $\pi_1, \pi_2$ be the projection maps from the product manifold $M_1 \times M_2$ onto $M_1$ and $M_2$, respectively. We denote by $\tilde{\varphi} := \varphi \circ \pi_1$ the lift to $M_1 \times M_2$ of a smooth function $\varphi$ on $M_1$. Then $M_1$ is called the base and $M_2$ is called the fiber of $M_1 \times M_2$. The unique element $\tilde{X}$ of $\Gamma(T(M_1 \times M_2))$ that is $\pi_1$-related to $X \in \Gamma(TM_1)$ and to the zero vector field on $M_2$ will be called the horizontal lift of $X$ and the unique element $\tilde{V}$ of $\Gamma(T(M_1 \times M_2))$ that is $\pi_2$-related to $V \in \Gamma(TM_2)$ and to the zero vector field on $M_1$ will be called the vertical lift of $V$. We denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on $M_1$ and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on $M_2$.

For $f : M_1 \to (0, \infty)$ a smooth function on $M_1$, we consider the Riemannian metric $g$ on $M := M_1 \times M_2$:

\begin{equation}
g := \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2.
\end{equation}

**Definition 5.1.** The product manifold of $M_1$ and $M_2$ together with the Riemannian metric $g$ is called the warped product of $M_1$ and $M_2$ by the warping function $f$ (\cite{3}).

A warped product manifold $M := M_1 \times_f M_2$ is called trivial if the warping function $f$ is constant. In this case, $M$ is the Riemannian product $M_1 \times M_{2f}$, where $M_{2f}$ is the manifold $M_2$ equipped with the metric $f^2 g_2$ (which is homothetic to $g_2$) (\cite{5}).
In the next considerations, we shall denote by \((f \circ \pi_1)^2 =: f^2, \pi_1^* g_1 =: g_1 \) and \(\pi_2^* g_2 =: g_2\), respectively.

**Lemma 5.2.** ([3]) If \(\nabla\) denotes the Levi-Civita connection on \(M := M_1 \times_f M_2\), then:

\[
\nabla_X Z = \nabla_Z X = X(\ln f)Z,
\]

for any \(X, Y \in \Gamma(TM_1)\) and \(Z, W \in \Gamma(TM_2)\).

The warped product \(M_1 \times_f M_2\) of two pointwise slant submanifolds \(M_1\) and \(M_2\) of a metallic Riemannian manifold \((M, \bar{g}, J)\) is called a **warped product pointwise bi-slant submanifold**. Moreover, it is called **proper** if both \(M_1\) and \(M_2\) are proper pointwise slant submanifolds in \((M, \bar{g}, J)\).

In a similar manner as in ([22]), we get:

**Proposition 5.3.** Let \(M := M_1 \times_f M_2\) be a warped product pointwise bi-slant submanifold in a locally metallic Riemannian manifold \((M, \bar{g}, J)\) with slant functions \(\theta_1, \theta_2\) and warped function \(f\). Then, for any \(X, Y \in \Gamma(TM_1)\) and \(Z, W \in \Gamma(TM_2)\), we have:

\[
\begin{align*}
\bar{g}(h(X,Y), NZ) &= -\bar{g}(h(X,Z), NY), \\
\bar{g}(h(X,Z), NW) &= 0, \\
\bar{g}(h(Z,W), NX) &= T_1 X(\ln f)\bar{g}(Z,W) - X(\ln f)\bar{g}(Z, T_2 W).
\end{align*}
\]

**Proof.** For any \(X, Y \in \Gamma(TM_1)\) and \(Z \in \Gamma(TM_2)\), by using ([22], (2.4), (2.13), (5.2) and \(\nabla J = 0\) we obtain:

\[
\begin{align*}
\bar{g}(h(X,Y), NZ) &= \bar{g}(\nabla_X Y, JZ) - \bar{g}(\nabla_X Y, T_2 Z) \\
&= \bar{g}(\nabla_X T_1 Y, Z) + \bar{g}(\nabla_X NY, Z) + \bar{g}(\nabla_X T_2 Z, Y) \\
&= -\bar{g}(\nabla_X Z, T_1 Y) - \bar{g}(A_{NY} X, Z) + \bar{g}(Y, \nabla_X T_2 Z) \\
&= -X(\ln f)\bar{g}(T_1 Y, Z) - \bar{g}(h(X,Z), NY) + X(\ln f)\bar{g}(Y, T_2 Z).
\end{align*}
\]

On the other hand, \(\bar{g}(T_1 Y, Z) = \bar{g}(J Y, Z) = \bar{g}(Y, JZ) = \bar{g}(Y, T_2 Z)\) and we obtain (5.3).

For any \(X \in \Gamma(TM_1)\) and \(Z, W \in \Gamma(TM_2)\), by using ([22], (2.4), (2.13), (5.2) and \(\nabla J = 0\) we obtain:

\[
\begin{align*}
\bar{g}(h(X,Z), NW) &= \bar{g}(\nabla_X Z, JW) - \bar{g}(\nabla_X Z, T_2 W) \\
&= \bar{g}(\nabla_X T_2 Z, W) - \bar{g}(A_{NZ} X, W) - \bar{g}(\nabla_X Z, T_2 W) \\
&= X(\ln f)[\bar{g}(T_2 Z, W) - \bar{g}(Z, T_2 W)] - \bar{g}(h(X, W), NZ)
\end{align*}
\]
and using 
\[ g(T_2Z, W) - g(Z, T_2W) = g(JZ, W) - g(Z, JW) = 0, \]
we obtain 
\[ (5.6) \quad g(h(X, Z), NW) = -g(h(X, W), NZ). \]

On the other hand, after interchanging \( Z \) by \( X \), we have:
\[ g(h(Z, X), NW) = -g(\nabla_ZT_1X, W) = -g(A_{NX}Z, W) = g(\nabla_ZX, T_2W) = g(h(Z, W), NX) = g(h(X, W), NZ) \]
and using (5.6) we get (5.4).

For any \( X \in \Gamma(TM_1) \) and \( Z, W \in \Gamma(TM_2) \), by using (2.2), (2.4), (2.13), (5.2) and \( \nabla J = 0 \) we obtain:
\[ g(h(Z, W), NX) = g(\nabla_ZW, JX) - g(\nabla_ZW, T_1X) = g(\nabla_ZT_2W, X) + g(\nabla_ZNW, X) - g(\nabla_ZW, T_1X) = -g(T_2W, \nabla_ZX) - g(A_{NW}Z, X) + g(W, \nabla_ZT_1X) = \]
and we get (5.5).

**Proposition 5.4.** Let \( M := M_1 \times_f M_2 \) be a warped product pointwise bi-slant submanifold in a locally metallic Riemannian manifold \((M, g, J)\) with slant functions \( \theta_1, \theta_2 \) and warped function \( f \). Then, for any \( X \in \Gamma(TM_1) \) and \( Z \in \Gamma(TM_2) \), we have:
\[ (5.7) \quad (\nabla_XT^2)Z = p(\cos^2 \theta)(\nabla_XT)Z. \]

**Proof.** From (2.1) we have:
\[ (5.8) \quad q\bar{g}(\nabla_XZ, W) = q\bar{g}(\nabla_XZ, W) = \bar{g}(J^2\nabla_XZ, W) - p\bar{g}(J\nabla_XZ, W), \]
for any \( X \in \Gamma(TM_1) \) and \( Z, W \in \Gamma(TM_2) \).

By using (2.2), (5.2) and \( \nabla_XJ = 0 \), we obtain:
\[ (5.9) \quad qX(\ln f)\bar{g}(Z, W) = \bar{g}(\nabla_XJZ, JW) - p\bar{g}(\nabla_XJZ, W) \]
and using \( JZ = T_2Z + NZ \), for any \( Z \in \Gamma(TM_2) \), we have:
\[ qX(\ln f)\bar{g}(Z, W) = \bar{g}(\nabla_XT_2Z, T_2W) + \bar{g}(\nabla_XT_2Z, NW) + \bar{g}(\nabla_XJNZ, W) - p\bar{g}(\nabla_XT_2Z, W) - p\bar{g}(\nabla_XNZ, W). \]
Thus, from (2.10) and (5.2) we get:
\[ qX(\ln f)\overline{g}(Z, W) = X(\ln f)\overline{g}(T_2 Z, T_2 W) + \overline{g}(h(X, T_2 Z), NW) + \]
\[ + \overline{g}(\nabla_X tNZ, W) + \overline{g}(\nabla_X nNZ, W) - pX(\ln f)\overline{g}(T_2 Z, W) + p\overline{g}(A_{NZ} X, W). \]

From (5.4) we obtain \( \overline{g}(h(X, T_2 Z), NW) = 0 \) and \( \overline{g}(A_{NZ} X, W) = 0. \)

Thus, by using (2.6), (3.2) and (3.4), we have:
\[ qX(\ln f)\overline{g}(Z, W) = X(\ln f)\overline{g}(\cos^2 \theta_2 (pT_2 Z + qZ), W) + \]
\[ + \overline{g}(\nabla_X (\sin^2 \theta_2 (pT_2 Z + qZ)), W) + \overline{g}(\nabla_X (pNZ - NT_2 Z), W) - \]
\[ - pX(\ln f)\overline{g}(T_2 Z, W) \]

which implies
\[ \sin(2\theta_2)X(\theta_2)\overline{g}(pT_2 Z + qZ, W) = p\overline{g}(h(X, W), NZ) - \overline{g}(h(X, W), T_2 Z) = 0. \]

Example 5.5. Let \( \mathbb{R}^6 \) be the Euclidean space endowed with the usual Euclidean metric \( \langle \cdot, \cdot \rangle \). Let \( i : M \rightarrow \mathbb{R}^6 \) be the immersion given by:
\[ i(u, v) := (u \sin v, u \cos v, u, u \cos v, u \sin v, v), \]
where \( M := \{(u, v) \mid u > 0, v \in (0, \frac{\pi}{2})\} \).

A local orthogonal frame on \( TM \) is given by:
\[ Z_1 = \sin v \frac{\partial}{\partial x_1} + \cos v \frac{\partial}{\partial x_2} + \cos v \frac{\partial}{\partial x_3} + \sin v \frac{\partial}{\partial x_5} \]
\[ Z_2 = u \cos v \frac{\partial}{\partial x_1} - u \sin v \frac{\partial}{\partial x_2} - u \sin v \frac{\partial}{\partial x_4} + u \cos v \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}. \]

We define the metallic structure \( J : \mathbb{R}^6 \rightarrow \mathbb{R}^6 \) by:
\[ J(X_1, X_2, X_3, X_4, X_5, X_6) := (\sigma X_1, \sigma X_2, \sigma X_3, \overline{\sigma} X_4, \overline{\sigma} X_5, \overline{\sigma} X_6), \]
where \( \sigma := \sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2} \) is a metallic number \( (p, q \in \mathbb{N}^*) \) and \( \overline{\sigma} = p - \sigma \).

We remark that \( J \) verifies \( J^2 X = pJ + qI \) and \( \langle JX, Y \rangle = \langle X, JY \rangle \), for any \( X, Y \in \mathbb{R}^6 \). Also, we have:
\[ JZ_1 = \sigma \sin v \frac{\partial}{\partial x_1} + \sigma \cos v \frac{\partial}{\partial x_2} + \sigma \frac{\partial}{\partial x_3} + \overline{\sigma} \cos v \frac{\partial}{\partial x_4} + \overline{\sigma} \sin v \frac{\partial}{\partial x_5} \]
\[ JZ_2 = \sigma u \cos v \frac{\partial}{\partial x_1} - \sigma u \sin v \frac{\partial}{\partial x_2} - \overline{\sigma} u \sin v \frac{\partial}{\partial x_4} + \overline{\sigma} u \cos v \frac{\partial}{\partial x_5} + \overline{\sigma} \frac{\partial}{\partial x_6}. \]
We remark that \( \langle JZ_1, Z_2 \rangle = \langle JZ_2, Z_1 \rangle = 0, \langle JZ_1, Z_1 \rangle = 2\sigma + \bar{\sigma} \) and \( \langle JZ_2, Z_2 \rangle = u^2(\sigma + \bar{\sigma}) + \bar{\sigma} \).

On the other hand we get:

\[
\|Z_1\| = \sqrt{3}, \quad \|Z_2\| = \sqrt{2u^2 + 1},
\]
\[
\|JZ_1\| = \sqrt{2\sigma^2 + \bar{\sigma}^2}, \quad \|JZ_2\| = \sqrt{u^2(\sigma^2 + \bar{\sigma}^2) + \bar{\sigma}^2}.
\]

We denote by \( D_1 := \text{span}\{Z_1\} \) the slant distribution with the slant angle \( \theta_1 \), where \( \cos \theta_1 = \frac{2\sigma + \bar{\sigma}}{\sqrt{3(2\sigma^2 + \bar{\sigma}^2)}} \). Also, we denote by \( D_2 := \text{span}\{Z_2\} \) the pointwise slant distribution with the slant angle \( \theta_2 \), where \( \cos \theta_2 = \frac{u^2(\sigma + \bar{\sigma}) + \bar{\sigma}}{\sqrt{(2u^2 + 1)(u^2(\sigma^2 + \bar{\sigma}^2) + \bar{\sigma}^2)}} \).

The distributions \( D_1 \) and \( D_2 \) satisfy the conditions from Definition 5.1.

If \( M_1 \) and \( M_2 \) are the integral manifolds of the distributions \( D_1 \) and \( D_2 \), respectively, then \( M := M_1 \times \sqrt{2u^2 + 1} M_2 \) with the Riemannian metric tensor

\[
g := 3du^2 + (2u^2 + 1)dv^2
\]

is a warped product pointwise bi-slant submanifold in the metallic Riemannian manifold \( (\mathbb{R}^6, \langle \cdot, \cdot \rangle, J) \).

6 Warped product pointwise semi-slant or hemi-slant submanifolds in metallic Riemannian manifolds

In this section we get some properties of pointwise semi-slant and pointwise hemi-slant submanifolds in locally metallic Riemannian manifolds.

**Definition 6.1.** Let \( M := M_1 \times_f M_2 \) be a warped product bi-slant submanifold in a metallic Riemannian manifold \( (\overline{M}, \overline{g}, J) \) such that one of the components \( M_i \) (\( i \in \{1, 2\} \)) is an invariant submanifold (respectively, anti-invariant submanifold) in \( \overline{M} \) and the other one is a pointwise slant submanifold in \( \overline{M} \) with the Wirtinger angle \( \theta_x \in [0, \frac{\pi}{2}] \). Then we call the submanifold \( M \) **warped product pointwise semi-slant submanifold** (respectively, **warped product pointwise hemi-slant submanifold**) in the metallic Riemannian manifold \( (\overline{M}, \overline{g}, J) \).

In a similar manner as in Theorem 2 from [22], we obtain:

**Theorem 6.2.** If \( M := M_T \times_f M_\theta \) is a warped product pointwise semi-slant submanifold in a locally metallic Riemannian manifold \( (\overline{M}, \overline{g}, J) \) with the pointwise slant angle \( \theta_x \in (0, \frac{\pi}{2}) \), for \( x \in M_\theta \), then the warping function \( f \) is constant on the connected components of \( M_T \).
Proof. For any $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_\theta) \setminus \{0\}$, by using (2.13) in $\nabla_Z JX = J\nabla_Z X$ and (5.2), we obtain:

$$TX(\ln f)Z + h(TX, Z) = T\nabla_Z X + N\nabla_Z X + th(X, Z) + nh(X, Z).$$

From the equality of the normal components of the last equation, it follows

$$h(TX, Z) = X(\ln f)NZ + nh(X, Z)$$

and replacing $X$ with $TX = JX$ (for $X \in \Gamma(TM_T)$) in (6.1), we obtain:

$$h(J^2X, Z) = TX(\ln f)NZ + nh(TX, Z).$$

Thus, we get:

$$TX(\ln f)\bar{g}(NZ, NZ) = \bar{g}(h(J^2X, Z), NZ) - \bar{g}(nh(TX, Z), NZ) =$$

$$= p\bar{g}(h(TX, Z), NZ) + q\bar{g}(h(X, Z), NZ) - \bar{g}(nh(TX, Z), NZ),$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$.

From (5.4) we have $\bar{g}(h(TX, Z), NZ) = \bar{g}(h(X, Z), NZ) = 0$, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$ and by using (3.3), we get:

$$TX(\ln f)\sin^2 \theta[p\bar{g}(TZ, Z) + q\bar{g}(Z, Z)] = -\bar{g}(nh(TX, Z), NZ).$$

On the other hand, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, we have $TX \in \Gamma(TM_T)$ and $TZ \in \Gamma(TM_\theta)$ and from (5.4), we obtain:

$$\bar{g}(h(TX, Z), NZ) = \bar{g}(h(TX, Z), NTZ) = 0.$$

Thus, by using (2.1) and (2.7), we have:

$$\bar{g}(nh(TX, Z), NZ) = \bar{g}(h(TX, Z), nNZ) = \bar{g}(h(TX, Z), J^2Z - JTZ) =$$

$$= p\bar{g}(h(TX, Z), NZ) + q\bar{g}(h(TX, Z), Z) - \bar{g}(h(TX, Z), NTZ) = 0$$

and using (6.2), we obtain:

$$TX(\ln f)\tan^2 \theta_x \bar{g}(TZ, TZ) = 0,$$

for any $Z \in \Gamma(TM_\theta)$ and $x \in M_\theta$.

Since $\theta_x \in (0, \frac{\pi}{2})$ and $TZ \neq 0$, we get $TX(\ln f) = 0$, for any $X \in \Gamma(TM_T)$, which implies that the warping function $f$ is constant on the connected components of $M_T$. 


Theorem 6.3. If $M := M_\theta \times_f M_\theta$ is a warped product pointwise semi-slant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the pointwise slant angle $\theta_x \in (0, \frac{\pi}{2})$, for $x \in M_\theta$, then

$$(A_{NT_1Y}X - A_{NT_1X}Y) \in \Gamma(TM_\theta),$$

for any $X, Y \in \Gamma(TM_\theta)$.

Proof. For any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\theta) \setminus \{0\}$, from (4.5) and the symmetry of the shape operator, we have:

$$\sin^2 \theta \overline{g}([X, Y], T_2^2 Z) = \overline{g}([X, Y], T_2 Z) - \overline{g}(\nabla_X T_1 Y - \nabla_Y T_1 X, Z) +$$

$$+ p(\cos^2 \theta + 1)\overline{g}(h(X, Z), NY) - \overline{g}(h(Y, Z), NX)] - \overline{g}(h(X, Z), NT_1 Y) +$$

$$+ \overline{g}(h(Y, Z), NT_1 X) + \overline{g}(h(X, Y), NT_2 Z) - \overline{g}(h(Y, X), NT_2 Z) -$$

$$- \overline{g}(h(X, T_2 Z), NY) + \overline{g}(h(Y, T_2 Z), NX).$$

Using (2.2) and (5.3), we obtain:

$$\overline{g}(\nabla_X T_1 Y - \nabla_Y T_1 X, Z) = \overline{g}(\nabla_X JY - \nabla_Y NX - \nabla_Y NX, Z) =$$

$$= \overline{g}(\nabla_X Y, JZ) - \overline{g}(\nabla_Y X, JZ) + \overline{g}(A_{NY} X, Z) - \overline{g}(A_{NX} Y, Z) =$$

$$= \overline{g}([X, Y], JZ) + \overline{g}(h(X, Z), NY) - \overline{g}(h(Z, Y), NX) = \overline{g}([X, Y], T_2 Z).$$

From (5.3) we get:

$$\overline{g}(h(X, Z), NY) = \overline{g}(h(Y, Z), NX) = - \overline{g}(h(X, Y), NZ).$$

Thus, using the symmetry of the shape operator, we have:

$$\overline{g}(h(X, T_2 Z), NY) - \overline{g}(h(Y, T_2 Z), NX) =$$

$$= - \overline{g}(h(X, Y) NT_2 Z) + \overline{g}(h(Y, X), NT_2 Z) = 0$$

and

$$\overline{g}(h(X, Z), NT_1 Y) - \overline{g}(h(Y, Z), NT_1 X) = \overline{g}(A_{NT_1 Y} X - A_{NT_1 X} Y, Z).$$

Thus, we obtain:

$$\sin^2 \theta \overline{g}([X, Y], T_2^2 Z) = \overline{g}(A_{NT_1 Y} X - A_{NT_1 X} Y, Z),$$

which implies the conclusion.

Following the same steps such in (22), we can prove that:

Theorem 6.4. If $M := M_\perp \times_f M_\theta$ (or $M := M_\theta \times_f M_\perp$) is a warped product pointwise hemi-slant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the pointwise slant angle $\theta_x \in (0, \frac{\pi}{2})$, for $x \in M_\theta$, then the warping function $f$ is constant on the connected components of $M_\perp$ if and only if

$$(6.3) \quad A_{NZ} X = A_{NX} Z,$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$ (or $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, respectively).
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