NOVEL COMPARISON OF NUMERICAL AND ANALYTICAL METHODS FOR FRACTIONAL BURGER–FISHER EQUATION

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Abstract. In this paper, we investigate some analytical, numerical and approximate analytical methods by considering time-fractional nonlinear Burger–Fisher equation (FBFE). (1/G′)-expansion method, finite difference method (FDM) and Laplace perturbation method (LPM) are considered to solve the FBFE. Firstly, we obtain the analytical solution of the mentioned problem via (1/G′)-expansion method. Also, we compare the numerical method solutions and point out which method is more effective and accurate. We study truncation error, convergence, Von Neumann’s stability principle and analysis of linear stability of the FDM. Moreover, we investigate the L2 and L∞ norm errors for the FDM. According to the results of this study, it can be concluded that the finite difference method has a lower error level than the Laplace perturbation method. Nonetheless, both of these methods are totally settlement in obtaining efficient results of fractional order differential equations.

1. Introduction. In the last decades, many authors have produced excellent results in mathematical modeling and solving the models which have constructed in different fields of sciences and engineering. Because most of the models and systems are nonlinear, solution methods and determining of their stability regions have an important role. Moreover, fractional calculus theory, which is well-known as the generalization of the integer–order calculus, it is a rapidly advancing field of mathematics, physics, engineering, finance and artificial intelligence. It has been attracting the interest of many researchers because of the results got when it is applied to the real world models [33, 6, 13, 5, 38, 41, 42, 57, 28, 40, 24, 37, 32, 52, 16].

Since there are several types of fractional derivative and integral operators [11, 4, 46, 12, 47, 29], the researchers can choose the most comfortable operator to produce the better results of the real-life model problems. This appears in the literature as one of the illustrative advantages of the fractional calculus. For example, some researchers [31, 36, 18, 54, 66, 7, 19, 58, 61, 39, 20, 21, 48] have pointed out that which fractional derivative operator is most suitable in modeling. Moreover, in

2020 Mathematics Subject Classification. Primary: 26A33, 35R11, 65M06; Secondary: 65H20.

Key words and phrases. Nonlinear time–fractional Burger–Fisher equation, finite difference method, Laplace perturbation method, linear stability, analytical solution, Caputo fractional derivative.

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the theory of fractional operators with singular and non-singular kernels and their applications [24, 8, 35, 3, 51, 55, 49, 23, 34, 2, 56, 26, 1, 64] and the references stated in these articles.

A number of studies have been investigated in the literature to point out which numerical or analytical methods are more effective, easily-applicable and have lower error rates. Especially, in the last decades, the number of these-type efficient studies have been attracting effect. For more detailed analysis regarding both of numerical and analytical results, readers can see [30, 10, 59, 43, 65, 27, 60, 63, 50].

In this study, we consider the numerical and approximate-analytical solutions of fractional nonlinear Burger–Fisher (BF) equation in the frame of fractional derivative in the Caputo setting. In general case it is difficult to obtain the exact solutions of fractional order nonlinear differential equations, and then numerical solution methods are used to get their solutions. For this reason, theory of numerical and approximate-analytical solution methods play an important role in solving these mentioned problems.

Burger–Fisher equation is very important in many areas such as fluid dynamics, physics, engineering, etc. The study of the BF model has been investigated by many authors both for conceptual understanding of physical laws and testing various numerical methods. Also, it has been found some applications in fields as gas dynamics, number theory, heat conduction, elasticity, etc. It is a highly nonlinear equation because it is a combination of reaction, convection, and diffusion mechanisms, this equation is called Burger–Fisher because it has the properties of a convective phenomenon from Burgers equation and having diffusion transport as well as reactions kind of characteristics from Fisher equation.

Some researchers have studied on Burger–Fisher equation in the sense of numerical or analytical. Among them, Wazwaz [45] formally derived a variety of exact travelling wave solutions of distinct physical structures. In [14], Chandraker et al. investigated the numerical treatment of Burger–Fisher equation. Zhu et al. [67] obtained the numerical solution of BFE via cubic B-spline quasi-interpolation (BSQI) method. They pointed out that at each time step of the method they suggested, the complexity of the BSQI was lower than the other methods [45, 22]. In [25], Kaya and El-Sayed made an effective numerical simulation and obtained an explicit solutions of the generalized BFE. Bratsos and Khaliq [9] developed an exponential time differencing scheme by considering the method of lines for the numerical solution of the generalized BFE.

This study addresses to the comparing three different solution methods in finding the solutions of the FBFE numerically or analytically. This study is organized as follows: In Sect. 2, we give some definitions with respect to the study. In Sect. 3, we give the methodology of the solution methods. In Sect. 4, we analyze the truncation error, convergence and the linear stability of the numerical method. In Sect. 5, $L_2$ and $L_\infty$ errors norms are studied. In Sect. 6, we obtain the exact solution of the FBFE by using the $(1/G')$-Expansion method. In Sect. 7, we give numerical simulations with respect to the suggested numerical and approximate-analytical methods. In Sect. 8, we present the results and discussions. In Sect. 9, we give the concluding remarks that we obtained in this study.

The equation discussed in this study to compare exact and numerical solutions is the time-fractional Burger–Fisher equation which is given as [44]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} + \beta u(x,t) + \delta u^2(x,t) = 0,$$  (1)
with initial condition
\[ u(x, 0) = \frac{2\mu}{\lambda} - \frac{x}{\lambda} + \cosh |\lambda x| - \sinh |\lambda x|. \quad (2) \]

2. Some preliminaries. For the understanding and correctly applying the concept of fractional calculus, we will select the classical Caputo fractional derivative and its Laplace transformation. Although this suggested fractional derivative has a singular kernel, many scientists use it to model their own real-life problems. And it is well defined and has some advantages when it is considered with initial value problems in which the initial conditions are given.

**Definition 2.1.** The Caputo time fractional derivative (CF) is given by [33]
\[
_{0}^C D^\zeta_t \{g(t)\} = \frac{1}{\Gamma(m-\zeta)} \int^t_0 (t-k)^{m-\zeta-1} g^m(k)dk, \quad t > 0, \tag{3}
\]
where \( g \in L^1(a,b) \) and \( m < \zeta \leq m+1 \).

**Definition 2.2.** The Laplace transform (LT) of the Caputo derivative \(_{0}^C D^\zeta_t \{g(t)\}\) is given by [33]
\[
\mathcal{L}\{_{0}^C D^\zeta_t \{g(t)\}\}(p) = \frac{1}{p^{m-\zeta}} \left[p^m \mathcal{L}\{g(x, t)\}(p) - p^{m-1}g(x, 0) - \cdots - g^{(m-1)}(x, 0)\right]. \tag{4}
\]

3. Methodology. In this part of the study, we will present the fundamental methodology which has been used in this study.

3.1. Methodology of \((1/G')\)-expansion method. This section will give information about the operation of the \((1/G')\)-expansion technique [61, 64]. The nonlinear partial differential equation in the general form is given as follows:
\[
\Upsilon (u, u_x, u_t, u_{xx}, u_{tt}, \cdots) = 0. \tag{5}
\]
After the classical wave transformation \( u(x, t) = U(\xi), \xi = x - \frac{vt}{\Gamma(\alpha+1)}, \) that is applied to the Eq. (5), the following ordinary differential equation is obtained:
\[
\Omega (U, U', U'', \cdots) = 0. \tag{6}
\]
Then the solution of the Eq. (6) is accepted in the form below:
\[
U(\xi) = a_0 + \sum_{i=1}^{m} a_i \left(\frac{1}{G'(\xi)}\right)^i, \tag{7}
\]
where the \( G(\xi) \) function is the solution of the
\[
G''(\xi) + \lambda G'(\xi) + \mu = 0. \tag{8}
\]
Furthermore, we can take the general solution of second order ordinary differential equation given by (8) as the following
\[
G = G(\xi) = -\frac{\xi \mu}{\lambda} - \frac{e^{-\xi \lambda} c_1}{\lambda} + c_2, \tag{9}
\]
here, \( c_1 \) and \( c_2 \) are constants. If the derivative of the solution function given by (9) according to the \( \xi \) variable is taken for once and necessary arrangements are made, we get
\[
\frac{1}{G'} = \frac{1}{\frac{-\mu}{\lambda} + c_1 e^{-\lambda \xi}}, \tag{10}
\]
If we convert the algebraic expression given by (9) to trigonometric function, where \( c_1 = A \), we can write as the following

\[
\frac{1}{G'} = -\mu + \lambda A(\cosh(\lambda \xi) - \sinh(\lambda \xi)).
\] (11)

Also \( \lambda, \mu, a_i \) \((i = 1, 2, 3, ..., m)\) are constants and \( m \) is a positive integer to be obtained by the balancing principle. Eq. (7) into the Eq. (6) new polynomial equation is obtained. In this polynomial, \((1/G')_i\), \((i = 0, 1, 2, ..., m)\) coefficients are equalized to zero to obtain an algebraic equation system. This system is solved with the help of computer program and these solutions are placed in Eq. (7). The result is the wave solution of the Eq. (5).

3.2. Analysis of finite difference method. The basic principle of finite difference methods consists of algebraically approaching the differential operator. Derivatives in the structure of differential equations are calculated at the points of space and time that they are defined. When the derivative operator is algebraically expressed, an error occurs consciously. This error is known as truncation error. The truncation error in the structure of the numerical methods is the difference between the exact solution and the numerical solution. The principal source of this interrupt error is the use of the finite portion of the Taylor series when discrete the derivative operator. This error will be examined in the next section.

Let us give some notations for the construction of finite difference method:

a) Let spatial step represent \( \Delta x \).
b) Let time step represent \( \Delta t \).
c) \( x_i = a + i\Delta x, \) \( i = 0, 1, 2, ..., N \) points, which are the coordinates of mesh and \( N = \frac{b - a}{\Delta x}, \) \( j = 0, 1, 2, ..., M \) and \( M = \frac{T}{\Delta t} \).
d) The potential function \( u(x, t) \) is the value of the grid points at \( u(x_i, t_j) \sim u_{ij} \), where \( u_{ij} \) represent the numerical approximations at the points \( (x_i, t_j) \).

Finite difference operators as follows

\[
H_t u_{i,j} = u_{i,j} - u_{i,j-1}, \tag{12}
\]
\[
H_x u_{i,j} = u_{i+1,j} - u_{i,j}, \tag{13}
\]
\[
H_{xx} u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}. \tag{14}
\]

The discretization of partial derivatives as follows

\[
\left. \frac{\partial u}{\partial x} \right|_{i,j} = \frac{H_x u_{i,j}}{2\Delta x} + O(\Delta x^2), \tag{15}
\]
\[
\left. \frac{\partial^2 u}{\partial x^2} \right|_{i,j} = \frac{H_{xx} u_{i,j}}{(\Delta x)^2} + O(\Delta x^3), \tag{16}
\]

where \( O(\Delta x^2) \) and \( O(\Delta x^3) \) are truncation errors. These numerical values will be ignored when calculating the numerical solution. For the first time in 2010, the discretization of the Caputo derivative that Chen and Sun has presented in the study is as follows [15]:

\[
\left( \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \right) \cong \begin{cases} 
\frac{h^{-\alpha}}{\Gamma(2 - \alpha)} H_t u_{i,j+1} + \frac{h^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{j} f(k) H_t u_{i,j+1-k}, & j \geq 1, \\
\frac{h^{-\alpha}}{\Gamma(2 - \alpha)} H_t u_{i,0}, & j = 0.
\end{cases} \tag{17}
\]
In the finite difference method, inserting Eqs. (15), (16) and (17) into Eq. (1), we have the indexed term

\[
u_{i+1,j} = -\frac{1}{\Gamma[2-\alpha](-1+(\Delta x)u_{i,j})} \left[ (\Delta t)^{-\alpha} - (\Delta t)^{\alpha}\Gamma[2-\alpha]u_{i+1,j} \right. \\
- (\Delta x)^2 u_{i,j} + 2(\Delta t)^{\alpha}\Gamma[2-\alpha]u_{i,j} \\
+ \beta(\Delta t)^{\alpha}(\Delta x)^2 \Gamma[2-\alpha]u_{i,j} \\
- (\Delta t)^{\alpha}(\Delta x)^2 \Gamma[2-\alpha]u_{i,j}^2 \\
+ (\Delta x)^2 u_{i,1+j} \\
\left. + (\Delta x)^2 \sum_{k=1}^{j} f(k)H_{1}u_{i,j-k} \right], \quad (18)
\]

where \(f(k) = (k + 1)^{1-\alpha} - k^{1-\alpha}\) and the initial values \(u_{i,0} = u_{0}(x_{i})\).

3.3. Laplace perturbation approximate-analytical method. To investigate the fundamental solution method we take the following general form of fractional nonlinear PDE [31]:

\[
\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) + L[u(x, t)] + N[u(x, t)] = \theta(x, t), \quad (x, t) \in [0,1] \times [0, T], \quad m-1 < \alpha \leq m,
\]

with initial condition

\[
\frac{\partial}{\partial t} u(x, 0) = \mu_z(x), \quad z = 0, 1, ..., m - 1, \quad (20)
\]

and the boundary conditions

\[
u(0, t) = \gamma_0(t), \quad u(1, t) = \gamma_1(t), \quad t \geq 0,
\]

where \(\mu_z, \theta, \gamma_0\) and \(\gamma_1\) are known functions. In Eq. (19), we indicate the linear part of the equation with \(L[]\), the nonlinear part with \(N[]\) and \(\frac{\partial^\alpha}{\partial t^\alpha}\) shows the Caputo fractional derivative. We have considered the value of \(m\) as 1 when constructing the homotopy due to the nature of the problem we used in the study.

We define the recursive approximations for solving the suggested problems (19)-(21). Using the Laplace transform of the Caputo derivative in Def. 2.2, we define the \(L\{u(x, t)\}(\omega) = \tilde{U}(x, \omega)\) for Eq. (19). Then we can obtain the transformed functions for the Caputo fractional derivative

\[
\tilde{U}(x, \omega) = \frac{1}{\omega^\alpha} L\{L[u(x, t)] + N[u(x, t)]\} + \frac{1}{\omega} u(x, 0) + \frac{1}{\omega^\alpha} \tilde{\theta}(x, \omega), \quad (22)
\]

where \(L\{\theta(x, t)\} = \tilde{\theta}(x, \omega)\). Also considering the Laplace transforms of the boundary conditions we get

\[
L\{\gamma_0(t)\} = \tilde{U}(0, \omega), \quad L\{\gamma_1(t)\} = \tilde{U}(1, \omega), \quad \omega \geq 0.
\]

Then, if we apply the suggested method we achieve the solution of Eqs. (19)-(21) as

\[
\tilde{U}(x, \omega) = \sum_{\varepsilon=0}^{\infty} \rho^\varepsilon \tilde{U}_{\varepsilon}(x, \omega), \quad \varepsilon = 0, 1, 2, \ldots.
\]

The nonlinear part in Eq. (19) can be computed from

\[
N[u(x, t)] = \sum_{\varepsilon=0}^{\infty} \rho^\varepsilon \Phi_{\varepsilon}(x, t), \quad (25)
\]
and the components $\Phi_{\varepsilon}(x,t)$ are given in [17] as

$$
\Phi_{\varepsilon}(u_0, u_1, ..., u_{\varepsilon}) = \frac{1}{\varepsilon!} \frac{\partial^\varepsilon}{\partial \omega^\varepsilon} \left[ N \left( \sum_{i=0}^{\infty} \omega^i u_i \right) \right] \bigg|_{\omega=0}, \quad \varepsilon = 0, 1, 2, \ldots.
$$

(26)

Substituting Eqs. (24) and (25) into Eq. (22), we have the generator of the solution for the Caputo operator:

$$
\sum_{\varepsilon=0}^{\infty} \rho^\varepsilon \hat{U}_{\varepsilon}(x,\omega) = -\frac{\rho}{\omega^\alpha} \left( \mathcal{L} \left\{ L \left[ \sum_{\varepsilon=0}^{\infty} \rho^\varepsilon u_{\varepsilon}(x,t) \right] + \sum_{\varepsilon=0}^{\infty} \rho^\varepsilon \Phi_{\varepsilon}(x,t) \right\} \right)
$$

$$
+ \frac{1}{\omega} u(x,0) + \frac{1}{\omega^2} \tilde{\theta}(x,\omega).
$$

(27)

Then, by solving Eq. (27) with respect to $\rho$, we identify the following Caputo homotopies:

$$
\rho^0 : \hat{U}_0(x,\omega) = \frac{1}{\rho} u(x,0) + \frac{1}{\omega} \tilde{\theta}(x,\omega),
$$

$$
\rho^1 : \hat{U}_1(x,\omega) = -\frac{1}{\rho \omega} \mathcal{L} \{ L [u_0(x,t)] + \Phi_0(x,t) \},
$$

$$
\rho^2 : \hat{U}_2(x,\omega) = -\frac{1}{\rho \omega} \mathcal{L} \{ L [u_1(x,t)] + \Phi_1(x,t) \},
$$

$$
\vdots
$$

$$
\rho^{n+1} : \hat{U}_{n+1}(x,\omega) = -\frac{1}{\rho \omega} \mathcal{L} \{ L [u_n(x,t)] + \Phi_n(x,t) \}.
$$

(28)

Considering the case of $\rho \to 1$, we obtain that Eq. (28) shows the approximate solution for problem (27), then the solution is

$$
\Delta_n(x,\omega) = \sum_{\sigma=0}^{n} u_\sigma(x,\omega).
$$

(29)

Finally, by applying the inverse Laplace transform of Eq. (29), we obtain the approximate solution of Eq. (1),

$$
u(x,t) \equiv u_n(x,t) = \mathcal{L}^{-1} \{ \Delta_n(x,\omega) \}.
$$

(30)

4. **Truncation error, convergence and analysis of linear stability.** In this part of the study, the error analysis of the finite difference method is examined. The principle of stability is to determine the arguments that cause the error to enlarge or reduce. In addition, the reason for the small deviation in the numerical solution is limited.

**Theorem 4.1.** Finite difference method is convergence for time–fractional Burger–Fisher equation.

**Proof.** The equations given by (15), (16) and (17) are written in Eq. (1).

$$
\frac{h^{-\alpha}}{\Gamma(2-\beta)} H_i u_{i,j} + \frac{h^{-\alpha}}{\Gamma(2-\beta)} \sum_{k=1}^{j} H_i u_{i,j-k} f(k) + O(\Delta t^{2\alpha}) +
$$

$$
u_{i,j} \left( \frac{H_x u_{i,j}}{\Delta x} + O(\Delta x^2) \right) - \frac{H_x u_{i,j}}{\Delta x} - O(\Delta x^3) + \beta u_{i,j} + \delta u_{i,j} = 0.
$$

(31)

After the algebraic calculations in Eq. (31), the partial differential equation given by Eq. (1) can be expressed as an algebraic equation as follows:

$$
\frac{h^{-\alpha}}{\Gamma(2-\beta)} H_i u_{i,j} + \frac{h^{-\alpha}}{\Gamma(2-\beta)} \sum_{k=1}^{j} H_i u_{i,j-k} f(k) + u_{i,j} \frac{H_x u_{i,j}}{\Delta x} -
$$

$$
\frac{H_x u_{i,j}}{\Delta x^2} + \beta u_{i,j} + \delta u_{i,j} + O(\Delta t^{2\alpha} + \Delta x^2 - \Delta x^3) = 0.
$$

(32)
If we neglect the truncation error term in the structure of numerical analysis, the dissociation form of the Eq. (1) is as follows:

\[
\frac{h^{-\alpha}}{\Gamma(2-\beta)}(H_{i}u_{i,j} + \sum_{k=1}^{j} H_{i}u_{i,j-k}f(k) + \frac{H_{xx}u_{i,j}}{\Delta x}) - \frac{H_{xx}u_{i,j}}{(\Delta x)^{2}} + \beta u_{i,j} + \delta u_{i,j}^{2} = 0.
\]

After some arrangements, Eq. (18) can be obtained. Eq. (18) iterations can be made the numerical approximations at the points. In addition, if we refer to the truncation error \( E \), this truncation error has the following form

\[
E = \frac{-\Delta x^{2} (O(\Delta t^{2\alpha}) - O(\Delta x^{3}) + O(\Delta x^{2})u_{i,j})}{1 + \Delta xu_{i,j}}.
\]

In numerical analysis, the error is expressed as the difference between the exact solution and the approximate solution. Therefore, we can express the truncation error as follows:

\[
E = \left| U - \hat{U} \right|
\]

where \( U \) is the exact solution and \( \hat{U} \) is the approximate solution. There is a direct correlation between \( \Delta x \) and \( \Delta t \) parameters and transaction error. Considering Eq. (33), the limit of truncation error can be stated as

\[
\lim_{\Delta x \to 0, \Delta t \to 0} (E) = 0.
\]

This mathematical inequality and Eq. (34) implies the convergence of the finite difference method.

4.1. Linear stability analysis. In this section, using linearization techniques in Eq. (1) stability analysis will be examined. The linearized state of the Eq. (1) can be written as follows:

\[
\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} + \lambda \frac{\partial u(x,t)}{\partial x} - \frac{\partial^{2}u(x,t)}{\partial x^{2}} + u(x,t) + \lambda u(x,t) = 0,
\]

where \( \lambda = u(x,t) \). By considering Eqs. (12), (13) and (14), the discretization form of Eq. (35) can be written as follows:

\[
H_{i}u_{i,j} + a_{1}H_{x}u_{i,j} - a_{2}H_{xx}u_{i,j} + a_{3}u_{i,j} = 0,
\]

where \( a_{1} = \frac{\lambda \Delta t}{\Delta x} \), \( a_{2} = \frac{\Delta t}{(\Delta x)^{2}} \), \( a_{3} = \Delta t(1 + \lambda) \).

Theorem 4.2. The FDM for Eq. (1) is conditionally linear stable.

Proof. Using the Von Neumann’s stability principle (VNSP) [66] for the FDM of the Eq. (1). For that, let

\[
u_{i,j} = u(i\Delta x, j\Delta t) = u(p, q) = e^{k_{p}q}e^{i\xi p} = \varepsilon_{q} e^{i\xi p}, \quad \xi \in [-\pi, \pi],
\]

where \( e^{i\xi} = \varepsilon, \quad p = i\Delta x, \quad q = j\Delta t \) and \( i = \sqrt{-1} \). If we substitute the equality (37) into the equality (36)

\[
|\varepsilon| = \frac{1}{\Delta x^{2}} \left[ \left( -2a_{2}\Delta t + \Delta x(a_{1}\Delta t + \Delta x - a_{3}\Delta t \Delta x) \right)^{2} + a^{2}_{3}\Delta t^{2}\Delta x^{2} \sin |\xi|^{2} \right].
\]
Considering Eq. (38), if $\Delta x$ and $\Delta t$ parameters are close to zero, we can be $|\varepsilon| \leq 1$. Therefore, according to the (VNSP), finite difference methods for Eq. (1) is linear stable.

5. $L_2$ and $L_{\infty}$ error norms. In this section, with the aid of a computer package program, all calculations made assurance error norms will be given. In numerical analysis, we use $L_2$ and $L_{\infty}$ error norms to test the numerical results of the numerical results to the exact solution. The formulas of these error norms are as follows:

The $L_2$ error norm can be defined as [62]

$$L_2 = \|u^{\text{exact}} - u^{\text{numeric}}\|_2 = \sqrt{\sum_{j=0}^{N} |u_j^{\text{exact}} - u_j^{\text{numeric}}|^2},$$

and $L_{\infty}$ error norm can be defined as

$$L_{\infty} = \|u^{\text{exact}} - u^{\text{numeric}}\|_{\infty} = \max_j |u_j^{\text{exact}} - u_j^{\text{numeric}}|.$$

6. Dynamical behaviour of $(1/G')$-expansion method. In this section, new type of hyperbolic trigonometric traveling wave solution solutions of Eq. (1) will be studied. After applying the Eq. (1) to the traveling wave transformation, the following ordinary differential equation is obtained:

$$-\nu U' + UU' - U'' + \beta U + \delta U^2 = 0,$$  

(39)

according to the balancing principle, the balancing term is $m = 1$. Therefore, the solution of the Eq. (39) is as follows:

$$U(\xi) = a_0 + a_1 \left( \frac{1}{G'} \right).$$  

(40)

In accordance with the principle of the opening method described in Sect. 3.1, the following conditions are obtained.

Case 1. $\beta = 2\delta\lambda$, $\nu = 2\delta - \lambda$, $a_0 = 0$, $a_1 = 2\mu$,

$$u_1(x,t) = \frac{2\mu}{\frac{\mu}{2} + c_1 \cosh \left[ \lambda \left( x - \frac{t^{\alpha}(2\delta - \lambda)}{\Gamma[1+\alpha]} \right) \right] - c_1 \sinh \left[ \lambda \left( x - \frac{t^{\alpha}(2\delta - \lambda)}{\Gamma[1+\alpha]} \right) \right]},$$  

(41)

Figure 1. Traveling wave solution $u_1(x,t)$ of Eq. (1) by substituting the values $\mu = 1$, $\lambda = -0.5$, $c_1 = 1$, $\alpha = 0.8$, $\delta = -2$, in Eq. (41).
Figure 1 shows the hyperbolic type solution representing the stationary wave at any time.

**Case 2.** \( \lambda = v - 2\delta, \beta = -2(v\delta - 2\delta^2), a_0 = 2(v - 2\delta), a_1 = 2\mu, \)

\[ u_2(x, t) = 2(v - 2\delta) + \frac{2\mu}{-\frac{\mu}{v-2\delta} + c_1 e^{(\mu + x)(v + 2\delta)}}, \]

(42)

**Figure 2.** Traveling wave solution \( u_2(x, t) \) of Eq. (1) by substituting the values \( \mu = 1, \lambda = -0.5, c_1 = 1, \alpha = 0.8, \delta = -2, \) in Eq. (42).

Figure 2 shows the solution representing the stationary wave at any time.

**Case 3.** \( \delta = \frac{x - \lambda}{2}, \beta = -v\lambda + \lambda^2, a_0 = 2\lambda, a_1 = 2\mu, \)

\[ u_3(x, t) = 2\lambda + \frac{2\mu}{-\frac{\xi}{\lambda} + c_1 \cosh \left[ x\lambda - \frac{\xi + \lambda}{1 + \alpha} \right] - c_1 \sinh \left[ x\lambda - \frac{\xi + \lambda}{1 + \alpha} \right]}, \]

(43)

**Figure 3.** Traveling wave solution \( u_3(x, t) \) of Eq. (1) by substituting the values \( \mu = 1, \lambda = -0.5, c_1 = 1, \alpha = 0.8, v = 1, \) in Eq. (43).

Figure 3 shows the hyperbolic type solution representing the stationary wave at any time.
7. Numerical implementations.

7.1. Finite difference scheme. We discuss Eq. (1). To solve this mathematical model by the finite difference method, we need the exact solution given in Eq. (41) and the initial condition given in the form

\[ u_0(x) = \frac{2\mu}{-\lambda + \cosh |\lambda x| - \sinh |\lambda x|}. \]  

(44)

In Eqs. (41) and (44), we take the following special numerical values: \( \lambda = 0.1, \delta = -6, \alpha = 0.8, \mu = -1, \beta = 2\delta\lambda, c_1 = 1 \). Then the exact solution and the initial condition can respectively be written as

\[ u(x,t) = \frac{-2}{10 + \cosh [0.1 (12.9914t^{0.8} + x)] - \sinh [0.1 (12.9914t^{0.8} + x)]}, \]  

(45)

and

\[ u(x,0) = \frac{-2}{10 + \cosh [0.1x] - \sinh [0.1x]}. \]  

(46)

In addition, the parameters in the discretization Eq. (18) obtained by the finite difference method of the Eq. (1) are given the same values as the values \( \Delta x = \Delta t = 0.01 \). The mathematical index formula of the spatial step 2 \( \Delta x \), which is a step ahead of \( \Delta x \) representing the spatial step, can be written as:

\[
\begin{align*}
  u_{i+1,j} &= \frac{1}{-10000 + 100u_{i,j}} \left[ -43.3588 \sum_{k=1}^{j} \left( -k^{0.199} + (1 + k)^{0.199} \right) \right. \\
  &\quad \times (H_{i,j} - k) \\
  &\quad + 10000 u_{i+1,j} - 19998.8 u_{i,j} \\
  &\quad + 106 u_{i,j}^2 - 43.3588 (H_{i,j}) \right].
\end{align*}
\]  

(47)

Numerical solutions are obtained with the help of the computer program and the iterative principle considering Eq. (47). These numerical solutions will be discussed in the next section in detail.

7.2. Implementation of the Laplace perturbation method. We consider the time fractional nonlinear Burger–Fisher equation (1). In order to solve this problem by using the suggested method, we apply the Laplace transform to it with its initial condition. Then we get

\[
\begin{align*}
  \hat{U}(x,\omega) &= \frac{1}{\omega^\alpha} L \left\{ u_{xx}(x,t) + \frac{6u(x,t)}{5} \right\} \\
  &\quad - \frac{1}{\omega^\alpha} L \left\{ u(x,t) u_x(x,t) - 6u^2(x,t) \right\} + \frac{1}{\omega^\alpha} L \left\{ u(x,0) \right\}.
\end{align*}
\]  

(48)

Considering the inverse LT of the last equation, we obtain

\[
\begin{align*}
  u(x,t) &= u(x,0) + L^{-1} \left\{ \frac{1}{\omega^\alpha} L \left\{ u_{xx}(x,t) + \frac{6u(x,t)}{5} \right\} \right\} \\
  &\quad - L^{-1} \left\{ \frac{1}{\omega^\alpha} L \left\{ u(x,t) u_x(x,t) - 6u^2(x,t) \right\} \right\}.
\end{align*}
\]  

(49)

Then, if we apply the LPM, we have

\[
\begin{align*}
  \sum_{\varepsilon=0}^{\infty} \rho^\varepsilon \hat{U}_\varepsilon(x,\omega) &= u(x,0) + \rho L^{-1} \left\{ \frac{1}{\omega^\alpha} L \left\{ u_{xx}(x,t) + \frac{6u(x,t)}{5} \right\} \right\} \\
  &\quad - \rho L^{-1} \left\{ \frac{1}{\omega^\alpha} L \left\{ \sum_{\varepsilon=0}^{\infty} \rho^\varepsilon \Phi_\varepsilon(u) \right\} \right\}.
\end{align*}
\]  

(50)
In Eq. (50), $\Phi_e(u)$ are the polynomials that show the nonlinear terms defined in (26). These polynomials are given in the following way:

\[
\Phi_0(u) = u_0(u_0)_x - 6u_0^2,
\]
\[
\Phi_1(u) = u_0(u_1)_x + u_1(u_0)_x - 12u_0u_1,
\]
\[
\Phi_2(u) = u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x - 12u_0u_2 - 6u_1^2,
\]

(51)

Comparing the coefficients of $\rho$ in Eq. (50), we get

\[
\rho^0 : u_0(x,t) = -\frac{2}{10 + \cosh(x^{10}) - \sinh(x^{10})},
\]
\[
\rho^1 : u_1(x,t) = \frac{121^{\alpha}}{50\Gamma(1+\alpha)\{10 + \cosh(x^{10}) - \sinh(x^{10})\}}^{\alpha},
\]
\[
\rho^2 : u_2(x,t) = \frac{121^{\alpha} \{200\Gamma(1+\alpha)\{2+19 \cosh(x^{10}) + 19 \sinh(x^{10})\}\}}{5000\Gamma(1+\alpha)\{10 + \cosh(x^{10}) - \sinh(x^{10})\}^{2\alpha}},
\]
\[
- \frac{5000\Gamma(1+\alpha)\{2+19 \cosh(x^{10}) + 19 \sinh(x^{10})\}}{5000\Gamma(1+\alpha)\{10 + \cosh(x^{10}) - \sinh(x^{10})\}^{2\alpha}},
\]
\[
+ \frac{5000\Gamma(1+\alpha)\{2+19 \cosh(x^{10}) + 19 \sinh(x^{10})\}}{5000\Gamma(1+\alpha)\{10 + \cosh(x^{10}) - \sinh(x^{10})\}^{2\alpha}},
\]

(52)

In this way, the other components of the series can be obtained by using a computer package program. Therefore the approximate solution of Eq. (1) is given in the following series

\[
u(x,t) = \sum_{m=0}^{\infty} u_m(x,t).
\]

(53)

Figure 4 shows the numerical computations obtained with the Laplace perturbation method and finite difference method for the fractional parameter $\alpha = 0.8$ in the Caputo derivative operator sense. It also represents the error values according to these mentioned methods.

8. Results and discussions. In this section, the numerical solutions obtained by implementation of the Laplace perturbation method and finite difference method will be discussed. Advantages and disadvantages of both methods will be pointed out. The results will be supported with graphs and tables.

1. Common points of two methods:
   (a) The initial condition is necessary to obtain a numerical solution.
   (b) A scientific computation technique is necessary for complex operations and iterations.
   (c) The solution function is in serial format.

2. Different aspects of two methods:
   (a) The finite difference method is used for the discretization of the Taylor series while the Laplace perturbation method is used for the discretization of the Laplace operator.
   (b) While the spatial and time steps are used in the finite difference method, the terms of the approximate solution series are calculated separately in Laplace perturbation method.
   (c) In finite difference method, finite difference operators are obtained by using finite difference operators. Laplace and inverse Laplace operators are used in Laplace perturbation method.
(d) The finite difference method is less compared to the Laplace perturbation method because the discretization equation can be written algebraically because it can be written algebraically.

(e) The numerical results obtained in the finite difference method are closer to the exact solution.

(f) Finite difference method is also used as balancing term, Laplace perturbation method is not used.

These are well observed in the following tables.

| $x_i$ | $t_j$ | Numerical FDM | Numerical LPM | Exact | Errors FDM | Errors LPM |
|-------|-------|----------------|---------------|-------|------------|------------|
| 0.00  | 0.01  | -0.182279      | -0.182996     | -0.182350 | 7.09713x10^{-5} | 6.45583x10^{-4} |
| 0.01  | 0.01  | -0.182296      | -0.183012     | -0.182367 | 7.09118x10^{-5} | 6.45497x10^{-4} |
| 0.02  | 0.01  | -0.182312      | -0.183027     | -0.182383 | 7.09118x10^{-5} | 6.44425x10^{-4} |
| 0.03  | 0.01  | -0.183282      | -0.183043     | -0.182399 | 7.07930x10^{-5} | 6.44366x10^{-4} |
| 0.04  | 0.01  | -0.182444      | -0.183058     | -0.182415 | 7.07337x10^{-5} | 6.4332x10^{-4} |
| 0.05  | 0.01  | -0.182600      | -0.183074     | -0.182431 | 7.06744x10^{-5} | 6.4329x10^{-4} |

*Table 1. Exact solution, numerical results and absolute error of FDM and LPM for Eq. (1) at $\Delta x = \Delta t = 0.01$."

It is observed that the method of finite difference is more damaging in Table 1. While the finite difference method sweeps the solution range with an iteration, this iteration step is taken as 0.01. Since our aim in this study is to compare both numerical methods, this 0.01 step has been taken into consideration while obtaining the approximate solutions in LPM. The purpose of the considering this value is to make the comparison of the two methods, the same points have been taken into account in order to make the comparison meaningful. The approximation of the numerical solution obtained by the finite difference technique to the exact solution is ensured by the $L_2$ and $L_\infty$ error norm in Table 2.

| $\Delta x = \Delta t$ | $L_2$ | $L_\infty$ |
|-----------------------|-------|------------|
| 0.1                   | 0.000444585 | 0.000216239 |
| 0.05                  | 0.000343115 | 0.000190017 |
| 0.02                  | 0.00018088 | 0.000114257 |
| 0.01                  | 0.000067741 | 0.000070912 |
| 0.002                 | 9.2731x10^{-6} | 0.0000210066 |
| 0.001                 | 3.82314x10^{-6} | 0.0000121997 |

*Table 2. $L_2$ and $L_\infty$ error norm when $0 \leq \Delta x = \Delta t \leq 1$."

In the finite difference method, the error depends on the choice of the values $\Delta x$ and $\Delta t$, whereas in the Laplace method depends on the number of the solution series terms. The following graphs can be looked at to better observe these errors and to provide comments.

9. **Conclusions.** In this study, firstly, we have obtained the analytical solution of the mentioned problem via $(1/G')$-expansion method. This analytical method has some advantages and disadvantages. The biggest advantage is that it is an effective method that is easily applicable, reliable and requires less mathematical computation. The main disadvantage is that it only produces a hyperbolic-type
traveling wave solution. Then we have successfully obtained the numerical results of time fractional Burger–Fisher equation via FDM and LPM methods. Moreover, we have compared the numerical method solutions and point out which method is more effective and accurate. We have also studied the truncation error, convergence analysis, Von Neumann’s stability principle and analysis of linear stability of the FDM. We have investigated the $L_2$ and $L_\infty$ norm errors for the FDM. These numerical methods are good mathematical techniques to obtain numerical/approximate solutions for non-linear models. The differences and similarities of the both numerical methods have been explored. Both analytical and numerical methods are important instruments in obtaining solutions of NLPDE. Moreover, the advantages and disadvantages of both methods have been discussed. It has been observed that the numerical solutions obtained by the finite difference method are closer to the
exact solution. In addition to this, both methods can gain different meanings depending on the field. Results and discussions have been presented with graphs and tables. Finally, the results of this study will help to select the suitable methods in various areas where numerical analysis is more important in applied sciences.

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Received June 2019; 1st revision July 2019; 2nd revision September 2020.

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