A DECOMPOSITION FORMULA FOR THE WEIGHTED COMMUTATOR

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Dedicated to George Janelidze on the occasion of his sixtieth birthday

Abstract. We decompose the weighted subobject commutator of M. Gran, G. Janelidze and A. Ursini as a join of a binary and a ternary commutator.

In their article [6], M. Gran, G. Janelidze and A. Ursini introduce a weighted normal commutator which, depending on the chosen weight, captures classical commutators such as the Huq commutator [8, 3, 1] and the Smith commutator [15, 14, 4, 1]. It is constructed as the normal closure of a so-called weighted subobject commutator. We show how this latter commutator may be decomposed as a join of a binary and a ternary commutator [8, 7] defined in terms of co-smash products [5]. We moreover explain that the corresponding concept of weighted centrality of arrows can be expressed in terms of the admissibility of certain diagrams in the first author’s sense [12].

The weighted subobject commutator. In a finitely cocomplete homological category [1, 10], a weighted cospan is a triple of morphisms

\[
\begin{array}{ccc}
W & \xrightarrow{w} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{x} & D & \xleftarrow{y} & Y \\
\end{array}
\]

in which \((x, y)\) plays the role of cospan and \(w\) is the weight. Consider the pullback

\[
\begin{array}{ccc}
W + Y & \xrightarrow{\pi_2} & \langle 1_w \rangle \\
\downarrow & \searrow & \downarrow \\
(W + X) \times_W (W + Y) & \xrightarrow{\pi_1} & W + X & \xleftarrow{\langle 1_w \rangle} \\
\end{array}
\]

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and the induced outer diagram

\[
\begin{array}{ccc}
W + X & \xrightarrow{\langle \iota_W \circ \iota_W, \iota_W \rangle} & (W + X) \times_W (W + Y) \\
\langle \iota_W \circ \iota_W, \iota_W \rangle & & \langle \iota_W \circ \iota_W, \iota_W \rangle \\
\langle \iota_W \circ \iota_W, \iota_W \rangle & \xrightarrow{\phi} & W + Y
\end{array}
\]

In [6] the morphisms \( x \) and \( y \) are said to **commute over** \( w \) if and only if there exists a dotted arrow \( \phi \) (called an **internal multiplication**) such that the above diagram is commutative.

As explained in [6], taking \( W = 0 \) captures commuting pairs in the Huq sense (\( x \) and \( y \) commute over 0 if and only if they Huq-commute), and \( w = 1_D \) captures centralising equivalence relations in the Smith sense (the respective normalisations \( x \) and \( y \) of two equivalence relations \( R \) and \( S \) on \( D \) commute over \( 1_D \) if and only if \( R \) and \( S \) Smith-commute).

Now consider the canonical comparison morphism

\[
\langle \iota_W \circ \iota_W, \iota_W \rangle : W + X + Y \rightarrow (W + X) \times_W (W + Y)
\]

which, being a regular epimorphism [6] as the comparison between a sum and a product in the category of points over an object \( W \) in a regular Mal'tsev category, induces a short exact sequence

\[
0 \rightarrow K \xrightarrow{\langle \iota_W \circ \iota_W, \iota_W \rangle} W + X + Y \xrightarrow{\langle \iota_W \circ \iota_W, \iota_W \rangle} (W + X) \times_W (W + Y) \rightarrow 0. \quad (B)
\]

The \((W, w)\)-**weighted subobject commutator** \( \kappa : [(X, x), (Y, y)]_{(W, w)} \rightarrow D \) of \( x \) and \( y \) is the direct image of \( K \) along the induced arrow to \( D \) as in

\[
(K) \xrightarrow{\kappa} (X, x), (Y, y)]_{(W, w)} \xrightarrow{\kappa} D.
\]

It is clear from the exactness of the above sequence that \( x \) and \( y \) commute over \( w \) if and only if \([(X, x), (Y, y)]_{(W, w)} \) vanishes.

The normal closure of \( \kappa \) is called the \((W, w)\)-**weighted normal commutator** of \( x \) and \( y \) and denoted by \( N[(X, x), (Y, y)]_{(W, w)} \).

**Admissibility.** In order to analyse the weighted subobject commutator in terms of the binary and ternary commutators considered in [8, 7], we pass via an intermediate notion from [12]. An **admissibility diagram** is a diagram of shape

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\alpha & & \beta \\
\gamma & & \gamma
\end{array}
\]
with \( f \circ r = 1_B = g \circ s \) and \( \alpha \circ r = \beta = \gamma \circ s \). Note that by taking the pullback of \( f \) with \( g \), any admissibility diagram such as \( \text{(C)} \) may be extended to

\[
\begin{array}{ccc}
A \times_B C & \xrightarrow{e_2} & B \\
\downarrow \pi_1 & \searrow f & \downarrow g \\
A & \xrightarrow{e_1} & C
\end{array}
\]

in which the pullback square is a double split epimorphism.

The triple \( \alpha, \beta, \gamma \) is said to be admissible with respect to \( p, f, r, g, s \) if there is a (necessarily unique) morphism \( \varphi: A \times B C \to D \) such that \( \varphi \circ e_1 = \alpha \) and \( \varphi \circ e_2 = \gamma \).

**Commuting pairs in terms of admissibility.** It is immediately clear from the definitions that the morphisms \( x \) and \( y \) commute over \( w \) if and only if the triple \( \langle \langle w^x \rangle, w, \langle w^y \rangle \rangle \) is admissible with respect to \( \langle \langle 1_w^0 \rangle, \iota_W, \langle 1_w^0 \rangle, \iota_W \rangle \) as in the diagram

\[
\begin{array}{ccc}
W + X & \xrightarrow{\langle 1_w^0 \rangle_0} & W \\
\downarrow \langle w \rangle & \searrow \langle w \rangle & \downarrow \langle w \rangle \\
W & \xrightarrow{w} & Y
\end{array}
\] \( \text{(D)} \)

**Admissibility in terms of commuting pairs.** Consider a diagram \( \text{(C)} \) and the induced weighted cospan

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & D \\
\downarrow \alpha & \searrow \gamma & \downarrow \gamma \circ \ker(g) \\
X = \ker(f) & \xrightarrow{\alpha \circ \ker(f)} & \ker(g) = Y.
\end{array}
\]

We claim that the triple \( \alpha, \beta, \gamma \) is admissible with respect to \( (f, r, g, s) \) if and only if \( x = \alpha \circ \ker(f) \) and \( y = \gamma \circ \ker(g) \) commute over \( w = \beta: W = B \to D \). To see this, it suffices to compare Diagram \( \text{(C)} \) with the induced Diagram \( \text{(D)} \). In fact there is a regular epimorphism of admissibility diagrams from the latter to the former which keeps \( D \) fixed and makes

\[
\begin{array}{ccc}
B + X & \xrightarrow{\langle 1_n^0 \rangle_0} & B \\
\downarrow \langle f \rangle & \searrow \langle g \rangle & \downarrow \langle \ker(g) \rangle \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}
\]

commute. This already proves the “only if” in our claim. For the “if” suppose that \( x \) and \( y \) commute over \( \beta \). For the induced arrow

\( \varphi: (B + X) \times_B (B + Y) \to D \)

to factor over the regular epimorphism

\( \langle f \rangle \times_B \langle \ker(g) \rangle: (B + X) \times_B (B + Y) \to A \times_B C, \)
we only need that it vanishes on \( \ker(\langle \ker(r) \rangle) \times \ker(\langle \ker(s) \rangle) \). This does indeed happen, as
\[
\varphi(\ker(\langle \ker(r) \rangle) \times \ker(\langle \ker(s) \rangle)) \cdot (1, 0) = \varphi(\langle 1_B + x, p_B \rangle \cdot \ker(\langle \ker(r) \rangle)) = \langle \beta \rangle \cdot \ker(\langle \ker(r) \rangle) = \alpha \cdot \ker(\langle \ker(f) \rangle) \]
is trivial. Similarly, one can check that the arrow
\[
\varphi(\ker(\langle \ker(r) \rangle) \times \ker(\langle \ker(s) \rangle)) \cdot (0, 1)
\]
is trivial.

**Binary and ternary Higgins commutators.** If \( k: K \to X \) and \( l: L \to X \) are subobjects of an object \( X \) in a finitely cocomplete homological category, then the (Higgins) commutator \( [K, L] \leq X \) is the image of the induced morphism
\[
K \circ L \xrightarrow{i_{K,L}} K + L \xrightarrow{\langle f \rangle} X,
\]
where
\[
K \circ L = \ker(\langle 1_L \rangle); K \to K \times L).
\]
As explained in \([6]\), the Higgins commutator is another special case of the weighted subobject commutator recalled above. This commutator was first introduced in \([7, 11] \). Higher-order versions of it exist and are studied in \([5, 7]\).

The object \( K \circ L \), as the \( K \circ L \circ M \) below, is an example of a co-smash product \([5]\). It is worth recalling from \([11]\) that it may be computed as the intersection \( K \cap L \times L \cap K \), where the object \( K \cap L \) from \([2]\) is the kernel in the split exact sequence
\[
0 \longrightarrow K \cap L \longrightarrow K + L \xrightarrow{i_{K,L}} K \longrightarrow 0.
\]
Furthermore, also the sequence
\[
0 \longrightarrow K \circ L \longrightarrow K \cap L \longrightarrow L \longrightarrow 0 \tag{E}
\]
is split exact.

If \( m: M \to X \) is another subobject of \( X \), then the ternary commutator \([K, L, M] \leq X \) is defined as the image of the composite
\[
K \circ L \circ M \xrightarrow{i_{K,L,M}} K + L + M \xrightarrow{\langle f \rangle} X,
\]
where \( i_{K,L,M} \) is the kernel of the morphism
\[
K + L + M \xrightarrow{\langle 1_K, 1_L, 1_M \rangle} (K + L) \times (K + M) \times (L + M).
\]
It is well known that co-smash products are not associative, in general; furthermore, ternary co-smash products or commutators need not be decomposable into iterated binary ones: see \([5, 7, 8]\).

**Theorem 1.** Consider a weighted cospan \([A]\) such that \( x \) and \( y \) are normal monomorphisms (= kernels) in a finitely cocomplete homological category. Then \( x \) and \( y \) commute over \( w \) precisely when the commutators \([X, Y] \) and \([X, Y, \text{Im}(w)] \) vanish.
Theorem 2. Given a weighted cospan $[A]$ in a finitely cocomplete homological category, the $(W, w)$-weighted subobject commutator of $x$ and $y$ decomposes as

$$[(X, x), (Y, y)]_{(W, w)} = [X, Y] \lor [X, Y, \text{Im}(w)].$$

Proof. First of all we show that $x$ and $y$ coincide with the images of $\langle w \rangle \cdot \ker(\langle 1w \rangle_0)$ and $\langle w \rangle \cdot \ker(\langle 1w \rangle_0)$, respectively, as in (D). To see this, consider the diagram with short exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & W \otimes X & \longrightarrow & W + X & \overset{\iota_X}{\longrightarrow} & W & \longrightarrow & 0 \\
0 & \longrightarrow & X & \overset{x}{\longrightarrow} & D & \overset{d}{\longrightarrow} & D_0 & \longrightarrow & 0.
\end{array}
$$

It is clear that $\langle 1w \rangle \cdot \iota_X = 0$ induces the factorisation $\eta^W_X$ of $\iota_X$ over the kernel $\kappa_{B,X}$ of $\langle 1w \rangle$. Similarly, since

$$d \cdot \langle w \rangle \cdot \kappa_{B,X} = d \cdot w \cdot \langle 1w \rangle \cdot \kappa_{B,X}$$

is trivial we obtain the dotted factorisation $\xi$. Now

$$x \cdot \xi \cdot \eta^W_X = \langle w \rangle \cdot \kappa_{B,X} \cdot \eta^W_X = \langle w \rangle \cdot \iota_X = x,$$

so $\xi \cdot \eta^W_X = 1_X$ because $x$ is a monomorphism. In particular, $\xi$ is a regular epimorphism. It follows that $x$ is the image of $\langle w \rangle \cdot \kappa_{B,X}$.

We know from the above discussion that $x$ and $y$ commute over $w$ precisely when the triple $\langle x \rangle, w, \langle y \rangle$ is admissible with respect to $\langle 1w \rangle, \iota_W, \langle 1w \rangle_0, \iota_W$. Lemma 4.5 in [S] now tells us that this happens if and only if the commutators $[X, Y]$ and $[X, Y, \text{Im}(w)]$ vanish. \[\square\]

Via Theorem 4.6 in [S] we now recover the known result that the Smith is Huq condition [13] holds if and only if, for any given cospan of normal monomorphisms $(x, y)$, the property of commuting over $w$ is independent of the chosen weight $w$ making $(x, y, w)$ a weighted cospan.

We also see that the $(W, w)$-weighted normal commutator $N[(X, x), (Y, y)]_{(W, w)}$ of $x$ and $y$ is the normal closure of $[X, Y] \lor [X, Y, \text{Im}(w)]$ in $D$, since these two normal subobjects satisfy the same universal property. We shall, however, not insist further on this, because we can obtain the following refinement (Theorem 2).

Lemma 1. If $X$, $Y$, and $W$ are objects in a finitely cocomplete homological category, then there is a decomposition

$$(X + Y) \circ W \cong ((X \circ Y \circ W) \times (X \circ W)) \times (Y \circ W).$$

More precisely, there exists an object $V$ and split short exact sequences

$$0 \longrightarrow V \longrightarrow (X + Y) \circ W \overset{\pi}{\longrightarrow} Y \circ W \longrightarrow 0$$

and

$$0 \longrightarrow X \circ Y \circ W \overset{\pi}{\longrightarrow} V \overset{\pi}{\longrightarrow} X \circ W \longrightarrow 0.$$

Proof. This is Lemma 2.12 in [S], a result which was first obtained by M. Hartl and B. Loiseau. \[\square\]

Theorem 2. Given a weighted cospan $[A]$ in a finitely cocomplete homological category, the $(W, w)$-weighted subobject commutator of $x$ and $y$ decomposes as

$$[(X, x), (Y, y)]_{(W, w)} = [X, Y] \lor [X, Y, \text{Im}(w)].$$
Proof. We decompose the kernel $K$ of the short exact sequence (E) into a join of the co-smash products $X \circ Y$ and $X \circ Y \circ W$ considered as subobjects of $K$. The result then follows from the compatibility of the ternary co-smash product with image factorisations (Corollary 2.14 in [7] and the fact that co-smash products preserve monomorphisms). Indeed, the image of the composite $X \circ Y \circ W \to W \to X \circ Y$ is a subobject of $rp_{X,Y,W}$. It is easily seen that $r_{X,Y} \subseteq rp_{X,Y,W}$ and that these two inclusions are jointly regular epic.

Consider the cube of solid split epimorphisms

which, taking kernels horizontally, yields two $3 \times 3$ diagrams (or, equivalently, a $3 \times 3$ diagram of vertical split epimorphisms). Note that the bottom one has $X \circ Y$, and the top one $K$, in its back left corner. It suffices to prove that, taking kernels vertically now, we obtain the split exact sequence

$$0 \to X \circ Y \to K \to X \circ Y \to 0$$

in the back left corner of the induced $3 \times 3 \times 3$ diagram. Taking vertical kernels of the front and middle sections of the diagram above, we already obtain a morphism

$$0 \to U \to (X + Y) \circ W \to Y \circ W \to 0$$

of short exact sequences. Using (E) we see that the sequence

$$0 \to U \to (X + Y) \circ W \to Y \circ W \to 0$$

is split exact. Noting that $V$ in Lemma [1] is the object $U$, we see that the co-smash product $X \circ Y \circ W$ must coincide with the kernel of $U \to X \circ Y$, which we already know coincides with the needed kernel of $K \to X \circ Y$. □

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