YANG-MILLS FIELDS ON $B$-BRANES

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Abstract. Considering the $B$-branes over a complex manifold $Y$ as objects of the bounded derived category $D^b(Y)$, we define holomorphic gauge fields on $B$-branes and the Yang-Mills functional for these fields. These definitions are a generalization to $B$-branes of concepts that are well known in the context of vector bundles. Given $\mathcal{F} \in D^b(Y)$, we show that the Atiyah class $a(\mathcal{F}) \in \text{Ext}^1(\mathcal{F}, \Omega^1(\mathcal{F}))$ is the obstruction to the existence of gauge fields on $\mathcal{F}$. We determine the $B$-branes over $\mathbb{C}P^n$ that admit holomorphic gauge fields. We prove that the set of Yang-Mills fields on the $B$-brane $\mathcal{F}$, if it is nonempty, is in bijective correspondence with the points of an algebraic subset of $\mathbb{C}^m$ defined by $m \cdot s$ polynomial equations of degree $\leq 3$, where $m = \dim \text{Hom}(\mathcal{F}, \Omega^1(\mathcal{F}))$ and $s$ is the number of non-zero cohomology sheaves $H^i(\mathcal{F})$. We show sufficient conditions under them any Yang-Mills field on a reflexive sheaf of rank 1 is flat.

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1. Introduction

In this article, we extend the well-known concepts of gauge field and Yang-Mills field on vector bundles to $B$-branes. From the mathematical point of view, a $B$-brane over a compact connected complex $n$-manifold $Y$ is an object of $D^b(Y)$, the bounded derived category of coherent analytic sheaves over $Y$ [1 Sect. 5.4 [2 Sect. 5.3].

A holomorphic vector bundle $V$ over $Y$ is a particular case of $B$-brane over $Y$. A holomorphic gauge field on $V$, in mathematical terms a holomorphic connection on $V$ [3], allows us to define a derivative of the holomorphic sections of $V$ along any “direction” in $Y$, giving rise to holomorphic sections. That is, the holomorphic gauge field gives identifications between the “infinitesimally close” fibers of $V$. Conversely, such consistent identifications determine a gauge field.

Not every holomorphic vector bundle supports holomorphic connections, unlike what happens in the smooth category; for example, the vanishing of the Chern class is a necessary and sufficient condition for

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the existence holomorphic connections on a line bundle (see Appendix). Moreover, if the set of these gauge fields is non-empty, it is an affine space associated to a finite dimensional vector space. These properties will hold in the extension of the holomorphic gauge fields to more general B-branes.

Gauge fields on coherent sheaves. Given a holomorphic vector bundle $V$ over the manifold $Y$, a holomorphic gauge field on $V$ can be regarded as a right inverse of the projection $\pi : J^1(V) \to V$, where $J^1(V)$ is the 1-jet bundle of $V$. For details see [28, Sect. 3.1].

That approach admits a natural translation to the context of the coherent sheaves. If $F$ is a coherent sheaf over the compact analytic manifold $Y$, we define a holomorphic gauge field on $F$ as a right inverse of the natural morphism $J^1(F) \to F$, where $J^1(F)$ is the corresponding 1-jet sheaf. Denoting by $\Omega^p$ the sheaf of the holomorphic $p$-forms on $Y$, that inverse determines a morphism of abelian sheaves $\nabla : F \to \Omega^1(F) := \Omega^1 \otimes_O F$, which satisfies the Leibniz’s rule. Conversely, such a morphism defines a holomorphic gauge field in the above sense.

The obstruction to the existence of a holomorphic connection on the sheaf $\mathcal{F}$ is an element of the group $\text{Ext}^1(F, \Omega^1(\mathcal{F}))$. Furthermore, when the set of holomorphic gauge fields on $\mathcal{F}$ is nonempty, it is an affine space associated to the finite dimensional vector space $\text{Hom}(\mathcal{F}, \Omega^1(\mathcal{F}))$.

The curvature of the holomorphic gauge field $\nabla$ is defined in the usual way and turns out to be an element $K_\nabla \in \text{Hom}(\mathcal{F}, \Omega^2(\mathcal{F}))$. When $\mathcal{K}_\nabla = 0$, we say that $\nabla$ is flat.

If $\nabla$ is a flat holomorphic connection on $\mathcal{F}$, then one has the complexes $(A^{\bullet,0}(\mathcal{F}) := A^{\bullet,0} \otimes_O \mathcal{F}, \nabla)$, where $A^{p,q}$ is the sheaf of $C^\infty$ differential forms of type $(p,q)$ on $Y$, the complex $(\Omega^\bullet(\mathcal{F}), \nabla)$ and $(A^\bullet(\mathcal{F}), \nabla + \bar{\partial})$.

If on every fibre $\mathcal{F}_x$ is defined an Hermitian metric $\langle , \rangle_x$ such that $\{\langle , \rangle_x\}_x$ is a “smooth” family (see Definition [2]), we say that $\mathcal{F}$ is a Hermitian sheaf. When $\mathcal{F}$ is a locally free sheaf, this definition coincides with the usual on holomorphic vector bundles [30, Chap. III].

Using the index formula, we prove the following theorem, which asserts that the Euler characteristic of the above complexes, when $\mathcal{F}$ is a locally free Hermitian sheaf, is determined by certain characteristic classes of $Y$ and the rank of $\mathcal{F}$.

**Theorem 1.** Let $\mathcal{F}$ be a Hermitian locally free sheaf of rank $r$ over the Kähler $n$-manifold $Y$. If $\nabla$ is a holomorphic flat gauge field on $\mathcal{F}$, then

$$\chi(\Omega^\bullet(\mathcal{F})) = r \text{e}(Y) = \chi(A^\bullet(\mathcal{F})), \quad \chi(A^{\bullet,0}(\mathcal{F})) = (-1)^n r \text{td}_C(\hat{Y}),$$
Where \( \chi(\mathcal{C}^*) = \sum_i (-1)^i \dim \Gamma(Y, \mathcal{C}^i) \), \( e(Y) \) is the Euler class of \( Y \), and \( \text{td}_c(Y) \) is the complexified Todd class of manifold conjugated of \( Y \) (i.e., if the complex structure of \( Y \) is defined by the endomorphism \( J \), the one of \( \bar{Y} \) is defined by \( -J \)).

**Yang-Mills fields on sheaves.** If \( Y \) is a Kähler manifold, on the set of holomorphic connections over the Hermitian sheaf \( \mathcal{F} \), one defines the Yang-Mills functional \( \mathcal{YM} \). The value of \( \mathcal{YM} \) at a connection \( \nabla \) is the squared norm \( \|K_{\nabla}\|_2^2 \) of its curvature. The stationary points of functional \( \mathcal{YM} \) are the Yang-Mills gauge fields on \( \mathcal{F} \). The set of such critical points will be denoted by \( \mathcal{YM}(\mathcal{F}) \). We will prove the following theorems.

**Theorem 2.** If \( \mathcal{F} \) admits a holomorphic gauge field and 
\[ m = \dim \text{Hom}(\mathcal{F}, \Omega^1(\mathcal{F})). \]
Then the set \( \mathcal{YM}(\mathcal{F}) \), of holomorphic Yang-Mills fields on \( \mathcal{F} \), is in bijective correspondence with the points of an algebraic set in \( \mathbb{C}^m \) defined by \( m \) algebraic equations of degree \( \leq 3 \). In particular, if \( m = 2 \) and the cardinal of \( \mathcal{YM}(\mathcal{F}) \) is finite, then \( \# \mathcal{YM}(\mathcal{F}) \leq 9 \).

**Theorem 3.** Let \( \mathcal{F} \) be a coherent reflexive Hermitian sheaf with rank 1 over a Hodge manifold, such that \( c_1(\mathcal{F}) = 0 \). Then a holomorphic gauge field on \( \mathcal{F} \) is Yang-Mills, iff it is flat.

**Yang-Mills fields on a \( B \)-brane.** As we said, a \( B \)-brane on \( Y \) is a complex \( (\mathcal{F}^*, \delta^*) \) of analytic coherent sheaves on \( Y \). According to the preceding paragraphs, it is reasonable to define a gauge field on this brane as an element of \( \text{Hom}_{D^b(Y)}(\mathcal{F}^*, J^1(\mathcal{F}^*)) \) which lifts the identity on \( \mathcal{F}^* \).

Using that the derived category \( D^b(\mathbb{P}^n) \) is generated by the family
\[ E = \{ \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n} \}, \]
we prove the following theorems.

**Theorem 4.** The cardinal of the set of holomorphic gauge fields on any \( B \)-brane over \( \mathbb{P}^n \) is \( \leq 1 \).

Particular \( B \)-branes over \( \mathbb{P}^n \) are the complexes consisting of direct sum of copies of \( \mathcal{O}_{\mathbb{P}^n} \)
\[ \cdots \to \bigoplus_{i \in S_p} \mathcal{O}_{\mathbb{P}^n} \xrightarrow{d^p} \bigoplus_{i \in S_{p+1}} \mathcal{O}_{\mathbb{P}^n} \to \cdots \]
where the \( S_p \) are finite sets and the coboundary operators are constant matrices. We will prove the following theorem.
**Theorem 5.** A $B$-brane on $\mathbb{P}^n$ admits a holomorphic gauge field iff it is isomorphic to a brane of the form (1.2).

When $Y$ is a Hodge manifold, it is a projective smooth variety and by the GAGA correspondence the analytic coherent sheaves on $Y$ can be considered as algebraic ones. Moreover, the category $D^b(Y)$ is equivalent to the homotopy category $K^b(Y)_{\text{coh}}$ of complexes of injective sheaves with coherent and bounded cohomology (see Subsection 3.1.2). Considering $\mathcal{F}^*$ as an object of $K^b(Y)_{\text{coh}}$, we set $\overline{\Omega^1(\mathcal{F}^*)}$ for an object of $K^b(Y)_{\text{coh}}$ quasi-isomorphic to $\Omega^1(\mathcal{F}^*)$. In this way, a gauge field $\psi$ on $\mathcal{F}^*$ is a homotopy class of morphisms between the complexes $\mathcal{F}^*$ and $\overline{\Omega^1(\mathcal{F}^*)}$, considered as complex of abelian sheaves. Thus, that homotopy class determines for each $j$ a unique morphism $\vartheta^j : \mathcal{H}^j(\mathcal{F}^*) \to \mathcal{H}^j(\Omega^1(\mathcal{F}^*))$, between the corresponding cohomology sheaves. In summary, the gauge field on $\mathcal{F}^*$ defines a family of connections $\vartheta^j$ on the cohomology sheaves $\mathcal{H}^j(\mathcal{F}^*)$.

When the cohomology sheaves $\mathcal{H}^j$ are Hermitian, in which case we say that $\mathcal{F}^*$ is Hermitian, we define the value of the Yang-Mills functional on the above gauge $\psi$ as $\sum_i (-1)^i \|K_{\psi^i}\|^2$. Thus, the Yang-Mills functional is a kind of Euler characteristic of the gauge field. This definition of the Yang-Mills functional on branes generalizes the one given for coherent sheaves, obviously.

The gauge field $\psi$ is a Yang-Mills field if it is a stationary point of Yang-Mills functional. We will also prove the following result.

**Theorem 6.** Let $(\mathcal{F}^*, \delta^*)$ be a Hermitian $B$-brane on the Hodge manifold $Y$, such that sheaves $\mathcal{H}^i(\mathcal{F}^*)$ are reflexive. A gauge field $\psi$ on the brane is a Yang-Mills field iff $\vartheta^i$ is a Yang-Mills field on $\mathcal{H}^i(\mathcal{F}^*)$ for all $i$.

In Theorem 7, we generalize the result given in Theorem 2, about the cardinal of the set of Yang-Mills fields on a sheaf, to a general brane.

The article is organized in two sections. In Section 2, we define the gauge fields on a coherent sheaf. In the first subsections of this section, we revise the definition of the 1-jet sheaf of a coherent sheaf $\mathcal{F}$ because, although this is well known in algebraic geometry, it is not so well known in the community of mathematical physicists. We prove also Theorem 1, above mentioned. The Yang-Mills functional is introduced in Subsection 2.2, where Theorems 2 and 3 are proved. We also describe some properties of the Yang-Mills fields on reflexive sheaves.

In Section 3 are considered the gauge fields on a general $B$-brane. In Subsection 3.1, we give the definition of gauge field on a $B$-brane and prove Theorems 4 and 5 about the existence of gauge fields on branes.
over $\mathbb{P}^n$. We define in Subsection 3.2 the Yang-Mills functional, showing
the reasons on which this definition is based and prove Theorems 6 and 7.

For the sake of completeness we give in the Appendix a simple proof,
in the context of the Čech cohomology, of the following well-known fact
in algebraic geometry and which has been mentioned above: A neces-
sary and sufficient condition for the existence of holomorphic connec-
tions on a line bundle $L$ is the vanishing of $c_1(L)$. This result can
certainly be generalised to bundles of arbitrary rank \[3\], but perhaps
such a simple proof might be of interest to some mathematical physi-
cists.

2. Gauge fields on coherent sheaves

2.1. Gauge fields on a sheaf. As explained in the Introduction, the
existence of a gauge field on a coherent sheaf $\mathcal{F}$ over a compact con-
ected complex manifold $Y$ should define an isomorphism between the
stalks of $\mathcal{F}$ at any “infinitesimally close” points of $Y$.

The idea of being infinitesimally close can formulated by means of
the first infinitesimal neighborhood $Y^{(1)}$ of the diagonal of $Y$ [10] page
698. If $R$ is a $\mathbb{C}$-algebra, $\text{Hom}(\text{Spec } R, Y)$ is the set of points of $Y$ with
values in $R$. Two points $x_1$, $x_2$ are infinitesimally close if the morphism
$(x_1, x_2) : \text{Spec } R \to Y \times Y$ factorizes through $Y^{(1)}$ [6] page 6]. In this
case, there is a morphism $h : \text{Spec } R \to Y^{(1)}$ such that $\pi_i h = x_i$, where
$Y \xrightarrow{\pi_1} Y^{(1)} \xrightarrow{\pi_2} Y$ are the projections. Given a sheaf $\mathcal{F}$ on $Y$, each element
$\alpha$ of the set
\[ (\alpha) \] gives rise to the morphism $h^*(\alpha) : x_1^* \mathcal{F} \to x_2^* \mathcal{F}$. Therefore, following
Deligne, one can consider a holomorphic gauge field on $\mathcal{F}$ as an element
of (2.1) which is the identity on $Y$.

On the other hand, each element of (2.1) determines, via the adjun-
tion isomorphism, a morphism $\mathcal{F} \to \pi_1^* \pi_2^* \mathcal{F}$, and conversely.

As $\pi_1^* \pi_2^* \mathcal{F}$ is the first jet sheaf $\mathcal{J}(\mathcal{F})$ of the coherent sheaf $\mathcal{F}$, in the
following paragraphs we review the definition of the jet sheaf and also
that of first neighborhood of the diagonal.

2.1.1. Neighborhood of the diagonal. We denote by $i : \Delta \hookrightarrow Y \times Y$ the
embedding of the diagonal. As a closed subvariety, $\Delta$ is defined by an
ideal $J$ of $\mathcal{O} := \mathcal{O}_{Y \times Y}$. We will consider the following ringed spaces
\[(Y, \mathcal{O}), \quad Y^{(1)} = \left( \Delta, \mathcal{O}_{Y^{(1)}} := (\hat{\mathcal{O}}/J^2)|_{\Delta} \right), \quad (Y \times Y, \hat{\mathcal{O}}).\]
$Y^{(1)}$ is the first infinitesimal neighborhood of $\Delta$. 
The $\mathcal{O}$-$\mathcal{O}$-bimodule structure in $\mathcal{O}_Y^{(1)}$. We set $p_1, p_2 : Y \times Y \to Y$ for the corresponding projection morphisms. The natural morphisms between the above topological spaces are shown in the following commutative diagram

\[ Y \xrightarrow{k} \Delta \xrightarrow{i} Y \times Y \xrightarrow{p_1} Y \]

Given $f \in \mathcal{O}$ and $l \in \hat{\mathcal{O}}$, the product

\[ (l + \mathcal{J}^2) \cdot f = l \circ p_2 + \mathcal{J}^2 \]

defines a right $\mathcal{O}$-module structure on $\hat{\mathcal{O}}/\mathcal{J}^2$. More explicitly, $(l(x, y) + \mathcal{J}^2) \cdot f = l(x, y)f(y) + \mathcal{J}^2$. Analogously, $f \cdot (l + \mathcal{J}^2) = (f \circ p_1)l + \mathcal{J}^2$ gives to $\hat{\mathcal{O}}/\mathcal{J}^2$ a left $\mathcal{O}$-module structure. However, the restrictions to $\mathcal{J}/\mathcal{J}^2$ of these left and right $\mathcal{O}$-module structures are equivalent.

The isomorphism $\mathcal{O} \oplus \Omega^1 \simeq \mathcal{O}_Y^{(1)}$. The cotangent sheaf $\Omega^1 := \Omega^1_Y$ can be identified with the pullback $j^{-1}(\mathcal{J}/\mathcal{J}^2)$ [7, p. 407]. This identification is defined by the correspondence

\[ \xi : j^{-1}(\mathcal{J}/\mathcal{J}^2) \to \Omega^1, \quad g + \mathcal{J}^2 \mapsto (d_x g)|_Y, \]

where $d_x g$ is the exterior derivative of $g$ with respect to the variables $x$; i.e. considering $g$ as function of the variables $x$ and keeping the variables $y$ constant.

If $l \in \hat{\mathcal{O}}$, the correspondence $l + \mathcal{J}^2 \mapsto l \circ j \oplus (d_x l)|_Y$ defines an isomorphism of right $\mathcal{O}$-modules

\[ j^{-1}(\hat{\mathcal{O}}/\mathcal{J}^2) \overset{\sim}{\simeq} \mathcal{O} \oplus \Omega^1. \]

When the abelian sheaf $\mathcal{O} \oplus \Omega^1$ is endowed with the left $\mathcal{O}$-action

\[ f \cdot (h \oplus \alpha) := fh \oplus (h df + f \alpha), \]

where $\alpha \in \Omega^1$, then $m$ in (2.5) is an isomorphism of left $\mathcal{O}$-modules. We summarize those well-known results in the following proposition.

**Proposition 1.** With the notations above introduced:

1. The $\mathcal{O}$-modules $j^{-1}(\mathcal{J}/\mathcal{J}^2)$ and $\Omega^1$ are canonically isomorphic.
2. The correspondence (2.5) defines an isomorphism between the right $\mathcal{O}$-modules $j^{-1}(\mathcal{O}/\mathcal{J}^2)$ and $\mathcal{O} \oplus \Omega^1$.
3. Equipped $\mathcal{O} \oplus \Omega^1$ with the left $\mathcal{O}$-structure defined in (2.6), the correspondence (2.5) is also an isomorphism of left $\mathcal{O}$-modules.

The exact sequence of right $\hat{\mathcal{O}}$-modules

\[ 0 \to \mathcal{J}/\mathcal{J}^2 \to \hat{\mathcal{O}}/\mathcal{J}^2 \to \hat{\mathcal{O}}/\mathcal{J} \to 0 \]
gives rise, by means the functor $j^{-1}$, the exact sequence of right $\mathcal{O}$-modules
\begin{equation}
0 \to j^{-1}(\mathcal{J}/\mathcal{J}^2) \to j^{-1}(\hat{\mathcal{O}}/\mathcal{J}^2) \to j^{-1}(\hat{\mathcal{O}}/\mathcal{J}) \to 0,
\end{equation}
or in other terms $0 \to \Omega^1 \to \mathcal{O} \oplus \Omega^1 \to \mathcal{O} \to 0$.

2.1.2. The first jet sheaf. We denote by $\pi_a = p_a \circ i : \Delta \to Y$, for $a = 1, 2$. One has the following morphism of sheaves rings over $\Delta$ (see Subsection 2.1.1)
\begin{equation}
\pi_a^{-1}\mathcal{O} \longrightarrow \mathcal{O}_{Y(1)} = \hat{\mathcal{O}}/\mathcal{J}^2, \quad h \mapsto h \circ \pi_a + \mathcal{J}^2.
\end{equation}
In particular, one can consider $\mathcal{O}_{Y(1)}$ as a right $\pi_a^{-1}\mathcal{O}$-module.

Given $\mathcal{F}$ a left $\mathcal{O}$-module on $Y$, its inverse image by $\pi_a$ is left $\mathcal{O}_{Y(1)}$-module
\begin{equation}
\pi_a^{-1}\mathcal{O} \longrightarrow \mathcal{O}_{Y(1)} = \hat{\mathcal{O}}/\mathcal{J}^2.
\end{equation}
And the first jet sheaf $\mathcal{J}^1(\mathcal{F})$ of $\mathcal{F}$ is left $\mathcal{O}$-module defined by
\begin{equation}
\mathcal{J}^1(\mathcal{F}) = \pi^*_a \pi^*_a(\mathcal{F}).
\end{equation}

Since $\pi^*_a$ is the left adjoint of $\pi_a$, one has
\begin{equation}
\text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{J}^1(\mathcal{F})) = \text{Hom}_{\mathcal{O}_{Y(1)}}(\pi^*_a \mathcal{F}, \pi^*_a \mathcal{F}).
\end{equation}

If $\mathcal{H}$ is a left $\mathcal{O}_{Y(1)}$-module, the $\mathcal{O}$-structure on $\pi^*_a \mathcal{H}$ is defined by $f \cdot s = (f \circ \pi_1) s$, where $f \in \mathcal{O}$ and $s \in \mathcal{H}$. On the other hand, the $\mathcal{O}$ action on
\begin{equation}
k^* \mathcal{H} = \mathcal{O} \otimes_{k^{-1}\mathcal{O}_{\Delta}} k^{-1}\mathcal{H}
\end{equation}
is determined by $f \cdot (1 \otimes s) = f \otimes s = 1 \otimes gs = 1 \otimes (f \circ \pi_1) s$, where $f = g \circ k$. Hence, $k^* \mathcal{H}$ and $\pi^*_a \mathcal{H}$ are isomorphic $\mathcal{O}$-modules. Thus, by
\begin{equation}
\mathcal{J}^1(\mathcal{F}) = k^{-1}(\mathcal{O}_{Y(1)}) \otimes_\mathcal{O} \mathcal{F} = j^{-1}(\hat{\mathcal{O}}/\mathcal{J}^2) \otimes_\mathcal{O} \mathcal{F}.
\end{equation}

By Proposition 1, $\mathcal{J}^1(\mathcal{F})$ is de abelian sheaf $\mathcal{F} \oplus \Omega^1(\mathcal{F})$ endowed with the following left $\mathcal{O}$-module structure
\begin{equation}
f(\sigma \oplus \beta) = f\sigma \oplus (f\beta \oplus df \otimes \sigma).
\end{equation}

One has the morphism of abelian sheaves
\begin{equation}
\eta : \mathcal{F} \to \mathcal{J}^1(\mathcal{F}) = \mathcal{F} \oplus \Omega^1(\mathcal{F}), \quad \sigma \mapsto \sigma \oplus 0.
\end{equation}

And from (2.11), it follows
\begin{equation}
f \eta(\sigma) = \eta(f\sigma) + df \otimes \sigma.
\end{equation}

Summarizing, one has the exact sequence of $\mathcal{O}$-modules
\begin{equation}
0 \to \Omega^1(\mathcal{F}) \to \mathcal{J}^1(\mathcal{F}) \to \mathcal{F} \to 0,
\end{equation}
where $\oplus$ is the direct sum in the category of abelian sheaves and the left $\mathcal{O}$-action on the central term is defined according to (2.11).

Thus, taking into account (2.1) and (2.9), we give the following definition.

**Definition 1.** A holomorphic gauge field on the coherent sheaf $F$ is an element $\text{Hom}(F, J^1(F))$ that is a right inverse of the $\mathcal{O}$-module morphism $\pi : J^1(F) \to F$.

2.1.3. *The Atiyah class.* The exact sequence (2.13), defines an element $a(F)$ of $\text{Ext}^1(F, \Omega^1(F))$, called the Atiyah class of $F$. The exact sequence (2.13) does not split, in general, in the category of $\mathcal{O}$-modules. Thus, there will exist gauge fields on $F$ iff the Atiyah class $a(F)$ vanishes.

If there exists a right inverse $\psi$ of $\pi$; then $\pi(\psi - \eta) = 0$; and thus $\nabla := \psi - \eta$ takes its values in $\Omega^1(F)$. Moreover, by (2.12), for $f \in \mathcal{O}$ and $\sigma \in F$

$$\nabla(f\sigma) = f\psi(\sigma) - f\eta(\sigma) + df \otimes \sigma = f\nabla(\sigma) + df \otimes \sigma;$$

that is, for $\nabla$ holds the Leibniz’s rule.

We denote by $\mathcal{C}$ the set consisting of all morphisms of $\mathcal{C}_Y$-modules $\nabla : F \to \Omega^1(F)$ satisfying (2.14). An element $\nabla \in \mathcal{C}$ determines the morphism of $\mathcal{O}$-modules

$$\kappa : F \oplus \Omega^1(F) \to \Omega^1(F), \quad (\sigma, \beta) \mapsto \beta - \nabla(\sigma).$$

$\kappa$ is a left inverse of $i$. Hence, $\kappa$ defines a splitting of the extension (2.13) and thus it determines a right inverse of $\pi$; i.e. a holomorphic gauge field $\psi$ on $F$. We have proved the following proposition.

**Proposition 2.** If $a(F) = 0$, then the map $\nabla \mapsto \nabla + \eta$ defines a bijective correspondence between $\mathcal{C}$ and the set of holomorphic gauge fields on the coherent sheaf $F$.

**Proposition 3.** The set of holomorphic gauge fields on the coherent sheaf $F$, if it is nonempty, is an affine space associated to the finite dimensional vector space $\Gamma(Y, \text{Hom}(F, \Omega^1(F)))$.

**Proof.** The exact sequence (2.13) gives rise to corresponding Ext-sequence

$$0 \to \text{Hom}(F, \Omega^1(F)) \xrightarrow{\mu} \text{Hom}(F, J^1(F)) \xrightarrow{\lambda} \text{Hom}(F, F) \xrightarrow{\nu} \text{Ext}^1(F, \Omega^1(F)).$$

Since $\mu(\phi) = \pi \circ \phi$, the existence of a holomorphic gauge field $\psi$ on $F$, is equivalent to $1_F \in \text{im}(\mu) = \ker(\nu)$. In fact, the Atiyah class $a(F)$ is the image of $1_F$ by $\nu$. 
If $\psi$ and $\psi_1$ are gauge fields on the coherent sheaf $\mathcal{F}$, then $\mu(\psi_1 - \psi) = 0$; i.e., $\psi_1 - \psi \in \text{im}(\lambda)$. Thus, the set of holomorphic gauge fields on $\mathcal{F}$, if nonempty, is an affine space with vector space $\text{Hom}(\mathcal{F}, \Omega^1(\mathcal{F})) = \Gamma(Y, \mathcal{H}om(\mathcal{F}, \Omega^1(\mathcal{F})))$. As $\mathcal{H}om(\mathcal{F}, \Omega^1(\mathcal{F}))$ is a coherent sheaf, the space of holomorphic gauge fields on $\mathcal{F}$, is a finite dimensional affine space.

\[ \square \]

2.1.4. Flat gauge fields. Given a holomorphic connection on $\mathcal{F}$, it defines a morphism of $C$-modules $\nabla : \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})$ in the usual way. The composition $\mathcal{K}_\nabla := \nabla(1) \circ \nabla : \mathcal{F} \to \Omega^2(\mathcal{F})$ is the curvature of $\nabla$; furthermore,

\[ (2.15) \quad \mathcal{K}_\nabla \in \text{Hom}(\mathcal{F}, \Omega^2(\mathcal{F})) = \Gamma(Y, \mathcal{H}om(\mathcal{F}, \Omega^2(\mathcal{F}))). \]

The connection is said to be flat if $\mathcal{K}_\nabla = 0$. In this case, one has the complex

\[ (2.16) \quad \Omega^\bullet(\mathcal{F}) : \Omega^0(\mathcal{F}) \xrightarrow{\nabla} \Omega^1(\mathcal{F}) \xrightarrow{\nabla(1)} \Omega^2(\mathcal{F}) \to \]

A homomorphic connection (not necessarily flat) on a locally free sheaf $\mathcal{F}$ determines a $C$-linear map

\[ \nabla : \mathcal{A}(\mathcal{F}) \to \mathcal{A}^{1,0}(\mathcal{F}), \]

satisfying $\nabla(f \tau) = \partial(f) \tau + f \nabla \tau$, for $f$ a smooth function and $\tau$ a section of $\mathcal{A}(\mathcal{F})$. In fact, given $\varphi$ is a section of $\mathcal{A}(\mathcal{F})$, let $s = \{s_a\}_a$ be a local local frame for $\mathcal{F}$, then $\varphi = \sum_a f^a s_a$, where the $f^a$ are smooth functions. We set $\nabla(\varphi) = \sum_a (\partial(f^a)s_a + f^a \nabla s_a)$. This is a well-defined section of $\mathcal{A}^{1,0}(\mathcal{F})$ (independent of the chosen frame $s$).

If $\nabla$ is flat, the extension of $\nabla$ allows us to define the complex

\[ (2.17) \quad \mathcal{A}^\bullet(\mathcal{F}) : \mathcal{A}^{0,0}(\mathcal{F}) \xrightarrow{\nabla} \mathcal{A}^{1,0}(\mathcal{F}) \xrightarrow{\nabla(1)} \mathcal{A}^{2,0}(\mathcal{F}) \to \]

Analogously, by means of $\nabla + \bar{\partial}$, one can construct the following complex, assumed that $\nabla$ is flat,

\[ (2.18) \quad \mathcal{A}^\bullet(\mathcal{F}) : \mathcal{A}^0(\mathcal{F}) \xrightarrow{\nabla + \bar{\partial}} \mathcal{A}^1(\mathcal{F}) \xrightarrow{\nabla + \bar{\partial}} \mathcal{A}^2(\mathcal{F}) \to \]

Theorem 1 gives the values of the Euler-Poincaré characteristic of the complexes (2.16), (2.17) and (2.18), defined from a holomorphic flat gauge field on the locally free sheaf $\mathcal{F}$.

**Proof of Theorem 1.** As the dimension is an Euler-Poincaré mapping, one has

\[ (2.19) \quad \chi(C^\bullet) = \sum_i (-1)^i \dim H^i(\Gamma(Y, C^\bullet)). \]
The \( p \)-column of the following commutative diagram is a fine resolution of \( \Omega^p(F) \).

\[
\begin{array}{ccccccc}
\Omega^0(F) & \xrightarrow{\nabla} & \Omega^1(F) & \xrightarrow{\nabla} & \Omega^2(F) & \xrightarrow{\nabla} \\
\downarrow i & & \downarrow i & & \downarrow i & \\
\mathcal{A}(F) & \xrightarrow{\nabla} & \mathcal{A}^{1,0}(F) & \xrightarrow{\nabla} & \mathcal{A}^{2,0}(F) & \xrightarrow{\nabla} \\
\downarrow \partial & & \downarrow -\bar{\partial} & & \downarrow \partial & \\
\mathcal{A}^{0,1}(F) & \xrightarrow{\nabla} & \mathcal{A}^{1,1}(F) & \xrightarrow{\nabla} & \mathcal{A}^{2,1}(F) & \xrightarrow{\nabla} \\
\downarrow \partial & & \downarrow -\bar{\partial} & & \downarrow \partial & \\
\end{array}
\]

Hence, the total complex \((\mathcal{A}^\bullet(F), D := \nabla + \bar{\partial})\) defined by the double complex \((\mathcal{A}^{p,q}(F), \nabla, (-1)^p \bar{\partial})\) is \( q \)-isomorphic to the complex in the top row of diagram, i.e., the complex (2.16). But that total complex is just (2.18). Hence,

\[ (2.20) \quad H^k(\Gamma(Y, \Omega^\bullet(F))) \simeq H^k(\Gamma(Y, \mathcal{A}^\bullet(F))). \]

The complex \( \Gamma(Y, \mathcal{A}^\bullet(F)) \), of global sections of (2.18) is elliptic, since the principal symbol of the operator \( D = \nabla + \bar{\partial} \) is equal to the one of the exterior derivative, \( d \).

The Kähler metric and the Hermitian structure determine, in the usual way, an inner product \( \cdot \cdot \) in the spaces \( \Gamma(Y, \mathcal{A}^k(F)) \). We denote by \( D^\dagger \) the adjoint to \( D \) with respect to this inner product. We can apply the index formula to the operator operator

\[ P = D + D^\dagger : \Gamma(Y, \mathcal{A}^{even}(F)) \to \Gamma(Y, \mathcal{A}^{odd}(F)). \]

The characteristic classes of \( F \) vanish, since it admits a holomorphic gauge field [3, Theorem 6]; thus the Chern character \( \text{ch}(F) = r \). Then, by the index formula (see [27, page 21])

\[ (2.21) \quad \text{Index } P = r \cdot e(Y). \]

On the other hand, the operator \( \Delta := P^\dagger P = PP^\dagger = D^\dagger D + DD^\dagger \),

\[ \Delta : \bigoplus_k \Gamma(Y, \mathcal{A}^k(F)) \to \bigoplus_k \Gamma(Y, \mathcal{A}^k(F)) \]

is self-adjoint. Thus,

\[ \bigoplus_k \Gamma(Y, \mathcal{A}^k(F)) = \ker(\Delta) \oplus \text{im}(\Delta), \]

and this decomposition is orthogonal [22, Theorem 5.5, Chap III]. It is easy to check that \( \text{im}(\Delta) \), in turn admits the orthogonal decomposition
im(D) ⊕ \text{im}(D^\dagger). Reasoning as in Hodge decomposition, it is shown that
\begin{equation}
    H^i(\Gamma(Y, \mathcal{A}^\bullet(\mathcal{F}))) \simeq \{ \varphi \in \Gamma(Y, \mathcal{A}^i(\mathcal{F})) \mid \Delta \varphi = 0 \}. \tag{2.22}
\end{equation}

If \( P^\dagger P \varphi = 0 \), then 0 = \((P^\dagger P \varphi) \diamond \varphi = (P \varphi) \diamond (P \varphi)\); so \( P \varphi = 0 \). That is, \( \ker(P^\dagger P) = \ker(P) \). Thus, by (2.22)
\begin{equation}
    \ker(P) = \{ \varphi \in \Gamma(Y, \mathcal{A}_{\text{even}}(\mathcal{F})) \mid \Delta \varphi = 0 \} \simeq \bigoplus_k H^{2k}(\Gamma(Y, \mathcal{A}^\bullet(\mathcal{F}))).
\end{equation}

Analogously, \( \text{coker}(P) = \ker(P^\dagger) \simeq \bigoplus_k H^{2k+1}(\Gamma(Y, \mathcal{A}^\bullet(\mathcal{F}))) \). From (2.19) together with (2.21) and (2.20), it follows the first assertion of theorem.

For the case of complex (2.17), we set \( Q \) for the operator
\[ Q = \nabla + \nabla^\dagger : \Gamma(Y, \mathcal{A}_{\text{even}, 0}(\mathcal{F})) \to \Gamma(Y, \mathcal{A}_{\text{odd}, 0}(\mathcal{F})). \]

By de index formula \cite[page 28]{27} \cite[page 258]{22}
\begin{equation}
    \text{Index } Q = (-1)^n \left( \frac{\text{ch}(\sum_i (-1)^i \mathcal{A}^{i,0}(\mathcal{F})) \, e(T)}{e(T)} \right)[Y]. \tag{2.23}
\end{equation}

The Euler classes of the holomorphic and antiholomorphic tangent bundles to \( Y \) satisfies \( e(T) = (-1)^n e(\bar{T}) \). Furthermore, \( \mathcal{A}^{i,0} \simeq \Lambda^i T \) and (see \cite[page 242]{22})
\[ \text{ch} \left( \sum_i (-1)^i \Lambda^i T \right) = (-1)^n e(T) \, \text{td}(T)^{-1}. \]

Thus, (2.23) reduces to \( \text{Index } Q = (-1)^n r \, \text{td}(\bar{Y}) \).

As in the case of the above operator \( P \), one has
\[ \ker(Q) = \ker(Q^\dagger Q) = \bigoplus_k H^{2k}(\Gamma(Y, \mathcal{A}^{\bullet,0}(\mathcal{F}))), \]
and similarly for \( \text{coker}(Q) \). Then the theorem follows. \( \square \)

2.2. Holomorphic Yang-Mills fields. In general, the fiber at \( x \in Y \) of a coherent \( \mathcal{O} \)-module \( \mathcal{G} \) will be denoted by \( \mathcal{G}(x) := \mathcal{G}_x / \mathfrak{m}_x \mathcal{G}_x \), where \( \mathfrak{m}_x \) is the maximal ideal of \( \mathcal{O}_x \). If \( \mathcal{Z} \) is a section of \( \mathcal{G} \) the corresponding vector in \( \mathcal{G}(x) \) is denoted by \( Z(x) \).

The singularity set \( \mathcal{S} \) of \( \mathcal{G} \) is an analytic subset of \( Y \) whose codimension is greater or equal to 1. Moreover, \( \mathcal{G} \) is locally free on \( Y \setminus \mathcal{S} \). We set \( \mathcal{G} \) for the vector bundle over \( Y \setminus \mathcal{S} \) with fibers \( \mathcal{G}(x) := \mathcal{G}(x) \), determined by the locally free sheaf \( \mathcal{G}|_{Y \setminus \mathcal{S}} \).

Definition 2. A Hermitian metric on the coherent sheaf \( \mathcal{G} \) is a set \( \{ \langle \cdot, \cdot \rangle_x \}_{x \in Y} \) of Hermitian metrics on the fibers of \( \mathcal{G} \), such that, for \( \mathcal{Z}_1, \mathcal{Z}_2 \) sections of \( \mathcal{F} \) on an open \( U \) of \( Y \), the map \( x \in U \mapsto (Z_1(x), Z_2(x))_x \)
is $C^\infty$. A sheaf endowed with a Hermitian metric is called a Hermitian sheaf.

If $Y$ is a Kähler manifold and $Z_1, Z_2 \in \Gamma(Y, \mathcal{S})$, we set
\[(2.24)\]
\[
(Z_1, Z_2) = \int_Y \langle Z_1(x), Z_2(x) \rangle_x \, \text{dvol} = \int_{Y \setminus S} \langle Z_1(x), Z_2(x) \rangle_x \, \text{dvol}.
\]

Let $\mathcal{F}$ be a coherent sheaf on $Y$. For each $x \in Y$, we denote by $\alpha_x$ and $\lambda_x$ the natural morphisms
\[
(Hom_{\mathcal{O}}(F, \Omega^k(F)))_x \xrightarrow{\alpha_x} \text{Hom}_{\mathcal{O}_x}(\mathcal{F}_x, \Omega^k_x \otimes \mathcal{F}_x) \xleftarrow{\lambda_x} \Omega^k_x \otimes \mathcal{F}_x \xrightarrow{\text{End}_{\mathcal{O}_x}(\mathcal{F}_x)}.
\]
As $F$ is coherent, $\alpha_x$ is isomorphism [9, page 239]. Furthermore, if $F_x$ is free, then $\lambda_x$ is bijective. Hence, for each point $x$ outside of the singularity set $S$ of $F$, the fibre of $Hom_{\mathcal{O}}(\mathcal{F}, \Omega^k(\mathcal{F}))$ at $x$ can be identified with the vector space $\Omega^k(x) \otimes \text{End}(F(x))$.

In particular, the curvature $\mathcal{K} := \mathcal{K}_\nabla$ of a holomorphic connection $\nabla$, defined in (2.15), determines the vector
\[(2.25)\]
\[K(x) \in \Omega^2(x) \otimes \text{End}(F(x))\]
for each $x \in Y \setminus S$. That is, $K$ is a 2-form $\text{End}(F)$-valued.

If $\mathcal{F}$ is a Hermitian sheaf. The Kähler structure on $Y$ and metric Hermitian on $\mathcal{F}$ induce a metric on $\Omega^2 \otimes \text{End}(F)$, which will also denoted $\langle \cdot, \cdot \rangle$. According to (2.24), one defines
\[(2.26)\]
\[\|\mathcal{K}_\nabla\|^2 = \int_{Y \setminus S} \langle K_\nabla, K_\nabla \rangle \, \text{dvol} = \int_{Y \setminus S} |K_\nabla \wedge \ast K_\nabla|,
\]
where $|\cdot|$ is the corresponding norm on $\text{End}(F)$ and $\ast$ is the Hodge star operator.

More concretely, if locally $K_\nabla$ can be expressed as $\alpha \otimes A$, with $\alpha$ a 2-form and $A$ a local section of $\text{End}(F)$, then the integrand in (2.26) is $(\alpha \wedge \ast \alpha) \langle A \circ A \rangle$. In a local unitary frame of $\text{End}(F)$, if the connection is compatible with the metric, the matrix $\check{A}$ associated to $A$ is antihermitian and $\langle A \circ A \rangle = -\text{tr}(\check{A} \check{A})$. That is,
\[(2.27)\]
\[|K_\nabla \wedge \ast K_\nabla| = -\text{tr}(K_\nabla \wedge \ast K_\nabla).
\]
Assumed the set of holomorphic gauge fields on $\mathcal{F}$ is nonempty, the correspondence
\[
\nabla \in \{\text{holomorphic gauge fields on } \mathcal{F}\} \mapsto \mathcal{YM}(\nabla) = \|\mathcal{K}_\nabla\|^2
\]
is the Yang-Mills’ functional [11, page 417] [23, page 44] [24, page 357]. The fields on which $\mathcal{YM}$ vanishes are the vacuum states of the corresponding Yang-Mills theory [11, page 447]. The gauge fields, where
the functional takes a stationary value are the \textit{holomorphic Yang-Mills fields}.

If $\nabla$ is a vacuum state, by (2.26) it follows $K_\nabla = 0$, and from the Nakayama’s lemma one deduces $K_\nabla = 0$; that is, $\nabla$ is a flat connection.

Let us assume that the sheaf $\mathcal{F}$ admits a holomorphic gauge field $\nabla_0$. By Proposition 3, given $E_1, \ldots, E_m$, a basis of $\text{Hom}(\mathcal{F}, \Omega^2(\mathcal{F}))$, any holomorphic gauge field can be written $\nabla = \nabla_0 + \sum \lambda_i E_i$, with $\lambda_i \in \mathbb{C}$. The curvature

$$K_\nabla = \nabla \circ \nabla = K_{\nabla_0} + \sum_i \lambda_i B_i + \sum_{ij} \lambda_i \lambda_j B_{ij},$$

where the $B$’s are elements of $\text{Hom}(\mathcal{F}, \Omega^2(\mathcal{F}))$. Thus,

$$\|K_\nabla\|^2 = (K_\nabla, K_\nabla) = P(\lambda_1, \ldots, \lambda_m),$$

where $P$ is a polynomial of degree $\leq 4$ in the variables $\lambda_i$.

\textbf{Proof of Theorem 2.} The Yang-Mills fields are those $\nabla$ defined by constants $\lambda_i$ which satisfy the algebraic equations of degree $\leq 3$

$$\frac{\partial P}{\partial \lambda_i} = 0, \quad i = 1, \ldots, m. \quad (2.28)$$

The case $m = 2$ is a consequence of Bézout’s theorem. \hfill $\square$

2.2.1. \textit{Reflexive sheaves.} When $Y$ is a Hodge manifold, then it is a smooth projective variety, according to a well-known Kodaira’s theorem. By the GAGA correspondence, the coherent analytic sheaves on $Y$ can be identified with the algebraic ones. From now on in this Section 2.2, we assume that $Y$ is a Hodge manifold.

On the other hand, the reflexive sheaves on an algebraic variety might be thought as “vector bundles with singularities” [13, page 121]. The following properties show that these singularities may be in some cases “irrelevant”. If $\mathcal{G}$ is a reflexive sheaf on the algebraic variety $X$, then the codimension of the singularity set $\text{Sing}$ of $\mathcal{G}$ is greater than $2$ [13, Cor. 1.4]. Hence, the restriction $\Gamma(X, \mathcal{G}) \to \Gamma(X \setminus \text{Sing}, \mathcal{G})$ is an isomorphism [15, Prop. 1.11].

Let us assume that $\mathcal{F}$ is reflexive sheaf on the Hodge manifold $Y$, endowed with a Hermitian metric. Then $\mathcal{E}nd(\mathcal{F})$ and $\mathcal{H}om(\mathcal{F}, \Omega^k \otimes \mathcal{F})$ are also reflexive sheaves [21, Chapter V, Proposition (4.15)]. On the other hand, if $\mathcal{S}$ is the singularity locus of $\mathcal{F}$, one has the isomorphism

$$\mathcal{H}om(\mathcal{F}, \Omega^k \otimes \mathcal{F})|_{Y \setminus \mathcal{S}} \simeq (\Omega^k \otimes \mathcal{E}nd(\mathcal{F}))|_{Y \setminus \mathcal{S}}$$

Thus, we have the isomorphisms

$$\Gamma(Y, \mathcal{H}om(\mathcal{F}, \Omega^k \otimes \mathcal{F})) \simeq \Gamma(Y \setminus \mathcal{S}, \Omega^k \otimes \mathcal{E}nd(\mathcal{F})) \simeq \Gamma(Y, \Omega^k \otimes \mathcal{E}nd(\mathcal{F})).$$
Moreover, these finite dimensional vector spaces are also isomorphic to the space of global sections \( \Gamma(Y \setminus S, \Omega^k \otimes \text{End}(F)) \) of the vector bundle \( \Omega^k \otimes \text{End}(F) \). In particular, \( \mathcal{K}_\nabla \), the curvature of a holomorphic connection \( \nabla \) on \( \mathcal{F} \), is determined by the 2-form \( \text{End}(F) \)-valued \( \mathcal{K}_\nabla \) defined over \( Y \setminus S \).

We denote

\[
(2.29) \quad (p) \nabla : \Gamma(Y \setminus S, \Omega^p \otimes O \text{End}(F)) \to \Gamma(Y \setminus S, \Omega^{p+1} \otimes O \text{End}(F)),
\]

the operator defined by the connection \( \nabla \). In this notation Bianchi's identity is read as

\[
(2) \quad \nabla \mathcal{K}_\nabla = 0.
\]

If \( \nabla \) is a holomorphic gauge field on \( \mathcal{F} \), according to Proposition 3, any other field is of the form \( \nabla + \varepsilon \), with \( \varepsilon \in \text{Hom}(\mathcal{F}, \Omega^1 \otimes O \mathcal{F}) \). By the above identifications \( \varepsilon \) is determined by the corresponding section \( E \in \Gamma(Y \setminus S, \Omega^1 \otimes \text{End}(F)) \).

Considering a "variation" \( \nabla_\varepsilon = \nabla + \varepsilon \) of \( \nabla \), then \( \mathcal{K}_{\nabla_\varepsilon} = \mathcal{K}_\nabla + \varepsilon \nabla E + O(\varepsilon^2) \), where \( \nabla E \) is the covariant derivative of \( E \).

\[
(2.30) \quad (1/2) \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} ||\mathcal{K}_{\nabla_\varepsilon}||^2 = \int_{Y \setminus S} \langle \mathcal{K}_\nabla, \nabla E \rangle \, d\text{vol} =: (\mathcal{K}_\nabla, \nabla E).
\]

Therefore, \( \nabla \) is a Yang-Mills field if for any "variation" \( \varepsilon \) of \( \nabla \)

\[
(2.31) \quad (\mathcal{K}_\nabla, \nabla E) = 0.
\]

In particular, the flat holomorphic gauge fields are Yang-Mills.

On the other hand, the orthogonality condition (2.31) which satisfy the Yang-Mills fields gives rise to the following proposition.

**Proposition 4.** The holomorphic gauge field \( \nabla \) on the reflexive sheaf \( \mathcal{F} \) is a Yang-Mills field iff its curvature \( \mathcal{K}_\nabla \in \Gamma(Y \setminus S, \Omega^2 \otimes \text{End}(F)) \) is orthogonal to the vector space \( \text{im}((1) \nabla) \).

**Corollary 1.** If \( H^2(\Gamma(Y, \Omega^* \otimes \text{End}(F))) = 0 \), then any Yang-Mills field on \( \mathcal{F} \) is flat.

**Proof.** By Bianchi’s identity \( \mathcal{K}_\nabla \in \text{ker}((2) \nabla) \). By the hypothesis \( \mathcal{K}_\nabla \in \text{im}((1) \nabla) \). If \( \nabla \) is a Yang-Mills field, then \( \mathcal{K}_\nabla \) is a vector orthogonal to \( \text{im}((1) \nabla) \), according to Proposition 4. Thus, \( \mathcal{K}_\nabla = 0 \), and by Nakayama lemma \( \mathcal{K}_\nabla = 0 \).

A flat connection on a vector bundle defines a \( D \)-module structure on the corresponding \( O \)-module. Thus, by the Riemann-Hilbert correspondence, one has the following corollary.
Corollary 2. Let $\mathcal{F}$ be a locally free sheaf such that $H^2(\Gamma(Y, \Omega^\bullet \otimes \text{End}(F))) = 0$. If $\mathcal{F}$ admits a holomorphic Yang-Mills field, then it is defined by a representation of $\pi_1(Y)$.

Under hypotheses very different from ours, other authors have shown that vector bundles that support holomorphic connections are actually flat vector bundles (see [4, 5]).

The orthogonality condition (2.31) implies $(1)\nabla^\dagger K \nabla = 0$, where 
$$(1)\nabla^\dagger : \Gamma(Y \setminus S, \Omega^2 \otimes_\mathcal{O} \text{End}(F)) \to \Gamma(Y \setminus S, \Omega^1 \otimes_\mathcal{O} \text{End}(F))$$
is the adjoint of $(1)\nabla$. By the Bianchi’s identity, if $\nabla$ is a Yang-Mills field, then
$$(2)\nabla K \nabla = 0, \quad (1)\nabla^\dagger K \nabla = 0,$$
and conversely.

2.2.2. The case rank $\mathcal{F} = 1$. Let us assume that $\mathcal{F}$ is a locally free sheaf of rank 1. It is known that a necessary and sufficient condition for $\mathcal{F}$ to admit a connection is that $F$ is flat (see Proposition 12 in Appendix).

On the other hand, $\mathcal{E}nd(\mathcal{F})$ is the sheaf associated to the trivial line bundle $\mathbb{C} \times Y \to Y$. Let $s$ be a local frame of the corresponding line bundle $F$. A holomorphic connection $\nabla$ on $\mathcal{F}$ is locally determined by a holomorphic 1-form $A$, $\nabla s = As$. In this frame $\nabla(\beta) = \partial \beta + A \wedge \beta - (-1)^p \beta \wedge A = \partial \beta$, for any $\text{End}(F)$-valued $p$-form $\beta$. Thus, the operator (2.29) reduces to
$$(p)\nabla = \partial : \Gamma(Y, \Omega^p) \to \Gamma(Y, \Omega^{p+1}).$$
The curvature of this connection is given by the holomorphic 2-form $\partial A$, and the Bianchi’s identity reduces to the obvious relation $\partial K_\mathcal{F} = 0$.

In this case, the complex $(\Omega^\bullet \otimes \text{End}(F), \nabla)$ is the holomorphic de Rham complex $(\Omega^\bullet, \partial)$ [29, Sect 8.2.1]. Denoting by $\mathcal{A}^{p,q}$ the sheaf of the smooth $(p,q)$-forms on $Y$, then $(\mathcal{A}^{p,\bullet}, \partial)$ is a fine resolution of $\Omega^p$. Thus, as in the proof of Theorem 1, the total complex associated to the double complex complex $(\mathcal{A}^{p,q}, \partial, (-1)^p \bar{\partial})$ is quasi-isomorphic to the holomorphic de Rham complex. Since this total complex is precisely the usual de Rham complex, one has
$$(2.34) \quad H^j(\Gamma(Y, \Omega^\bullet)) = H^j(Y, \mathbb{C}).$$

Proof of Theorem 3. We can assume that it is a line bundle [13, Prop. 1.9]; thus, the singularity locus $S$ is the empty set. Since $c_1(\mathcal{F}) = 0$, the Atiyah class vanishes (see Proposition 12 in Appendix) and $\mathcal{F}$ admits holomorphic connections.
Let $\nabla$ be a holomorphic gauge field on $\mathcal{F}$. By the Bianchi's identity, $K_\nabla$ defines a cohomology class in the space \((2.34)\) with $j = 2$. Any other gauge field $\tilde{\nabla}$ is an element of $\nabla + \text{Hom}(\mathcal{F}, \Omega^1 \otimes \mathcal{O})$. As $\mathcal{F}$ is a locally free sheaf with rank 1 \((2.35)\)

$$\text{Hom}(\mathcal{F}, \Omega^1 \otimes \mathcal{O}) \simeq \Gamma(Y, \Omega^1 \otimes \text{End}(\mathcal{F})) \simeq \Gamma(Y, \Omega^1).$$

By (2.33), the curvature of $\tilde{\nabla}$ has the form $K_{\tilde{\nabla}} = K_\nabla + \partial E$, with $E \in \Gamma(Y, \Omega^1)$ a holomorphic 1-form. As $\bar{\partial} E = 0$, the curvatures $K_\nabla$ and $K_{\tilde{\nabla}}$ determine the same cohomology class. We will denote by $c$ this cohomology class, defined by curvature of any holomorphic gauge field on $\mathcal{F}$.

If $\hat{\nabla}$ is an arbitrary Yang-Mills field, then $K_{\hat{\nabla}}$ satisfies (2.32); that is, $K_{\hat{\nabla}}$ is $\partial$-harmonic. As $Y$ is a Kähler manifold, $K_{\hat{\nabla}}$ is also $d$-harmonic. Hence, the norm of $K_{\hat{\nabla}}$ minimizes the corresponding norm in its cohomology class. That is,

\[(2.36)\]

$$\mathcal{YM}(\hat{\nabla}) = \|K_{\hat{\nabla}}\|^2 = \min\{\|\beta\|^2 \mid \beta \in c\}.$$  

On the other hand, if $\nabla_0$ is a holomorphic gauge field, \(\frac{1}{2\pi}[K_{\nabla_0}] = c_1(\mathcal{F}) = 0\). That is,

$$K_{\nabla_0} = (\partial + \bar{\partial})(B^{1,0} + B^{0,1}).$$

Since $K_{\nabla_0}$ is a $(2,0)$-form, it follows $\bar{\partial} B^{1,0} = 0$ and $dB^{0,1} = 0$. That is, $K_{\nabla_0} = \partial B^{1,0}$. The holomorphic connection $\nabla := \nabla_0 - B^{1,0}$ has curvature zero, hence it satisfies (2.31); that is $\nabla$ is a Yang-Mills field. Therefore the cohomology class $c = 0$. It follows from (2.36), that \(|K_{\nabla}| = 0\), for any holomorphic Yang-Mills field. \hfill \square

**Proposition 5.** Let $\mathcal{L}$ be a line bundle with $c_1(\mathcal{L}) = 0$. If the Hodge number $h^{1,0}(Y) = 1$, then either the cardinal $\# \text{YM}(\mathcal{L}) = 1$, or any gauge field on $\mathcal{L}$ is Yang-Mills. The latter case occurs when $H^0(Y, \Omega^1) = \mathbb{C}$.

**Proof.** The vanishing of the Chern class implies that there exist homolorphic gauge fields on $\mathcal{L}$. Let $\nabla_0$ denote a holomorphic gauge field. According to (2.35), any other gauge field is an element of $\nabla_0 + \Gamma(Y, \Omega^1)$

Since $h^{1,0}(Y) = 1$, any holomorphic gauge field on $\mathcal{L}$ is of the form $\nabla = \nabla_0 + \lambda E$, with $\lambda \in \mathbb{C}$ and $0 \neq E \in H^0(Y, \Omega^1)$. The curvature

$$K_\nabla = K_{\nabla_0} + \lambda \nabla_0(E).$$

Hence the polynomial

$$P(\lambda) \equiv \|K_\nabla\|^2 = \|K_{\nabla_0}\|^2 + 2\lambda(K_{\nabla_0}, \nabla_0(E)) + \lambda^2\|\nabla_0(E)\|^2.$$
If $\nabla_0(E) \neq 0$ the equation $\frac{dP}{dx} = 0$ has only one solution, so $\#\text{YM}(\mathcal{L}) = 1$. By contrast, when $\nabla_0(E) = 0$, it follows that $P(\lambda) = \|K_{\nabla_0}\|^2$, for all $\lambda$. Hence, in this case, the Yang-Mills functional is constant; thus, every holomorphic gauge field is a Yang-Mills field. On the other hand, since $0 = \nabla_0(E) = \partial E$, then $E$ is constant. \hfill $\square$

3. Fields on a brane

3.1. Gauge fields on a $B$-brane. Let $(\mathcal{F}^\bullet, \delta^\bullet)$ be a $B$-brane on the complex manifold $Y$; that is, $\mathcal{F}^\bullet$ is an object of the category $D^b(Y)$, the bounded derived category of coherent sheaves over $Y$. According to the observation at the beginning of Subsection 2.1 (see (2.1)), we define a gauge field on the brane $\mathcal{F}^\bullet$ as an element of $\text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, J^1(\mathcal{F}^\bullet))$ which lifts the automorphism identity of $(\mathcal{F}^\bullet, \delta^\bullet)$. This condition will be explained below in the rigorous definition of this concept.

As $\pi_i$ is flat, $L\pi_i^*$ is the usual inverse image $\pi_i^*$. By the adjunction relation (3.1) $\text{Hom}_{D^b(Y)}(\pi_1^*\mathcal{F}^\bullet, \pi_2^*\mathcal{F}^\bullet) \simeq \text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, J^1(\mathcal{F}^\bullet))$, where

$$J^1(\mathcal{F}^\bullet) := R\pi_1^*\pi_2^*\mathcal{F}^\bullet \simeq \mathcal{O}_{Y(i)} \otimes L\mathcal{F}^\bullet.$$ 

Therefore, the gauge fields on $\mathcal{F}^\bullet$ are elements of the group

$$\text{Ext}^0(\mathcal{F}^\bullet, J^1(\mathcal{F}^\bullet));$$

i.e., open strings between $\mathcal{F}^\bullet$ and $J^1(\mathcal{F}^\bullet)$ with ghost number 0 [1, Sect. 5.2], [20].

As $\mathcal{O}_{Y(i)}$ is the locally free module $\mathcal{O} \oplus \Omega^1$, then $J^1(\mathcal{F}^\bullet) \simeq \mathcal{F}^\bullet \oplus \Omega^1(\mathcal{F}^\bullet)$, with $\mathcal{O}$-structure given by (see (2.11))

$$f \ast (\sigma^\bullet \oplus \beta^\bullet) = f\sigma^\bullet \oplus (f\beta^\bullet + df \otimes \sigma^\bullet).$$

The exact sequence of complexes of $\mathcal{O}$-modules

$$0 \to \Omega^1(\mathcal{F}^\bullet) \xrightarrow{i} J^1(\mathcal{F}^\bullet) \xrightarrow{\pi} \mathcal{F}^\bullet \to 0$$

determines a distinguished triangle

$$\Omega^1(\mathcal{F}^\bullet) \xrightarrow{i} J^1(\mathcal{F}^\bullet) \xrightarrow{\pi} \mathcal{F}^\bullet \xrightarrow{+1}$$

in the category $D^b(Y)$ [19, page 46], [3, page 157]. As $\text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \cdot)$ is a cohomological functor, it follows that

$$\text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \Omega^1(\mathcal{F}^\bullet)) \to \text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, J^1(\mathcal{F}^\bullet)) \xrightarrow{\pi_0} \text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet) \to \text{Ext}^1(\mathcal{F}^\bullet, \Omega^1(\mathcal{F}^\bullet)) \to$$
is an exact sequence. The Atiyah class of $\mathcal{F}^\bullet$ is the image of $1 \in \text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$ in $\text{Ext}^1(\mathcal{F}^\bullet, \Omega^1(\mathcal{F}^\bullet))$. Thus, we give the following definition.

**Definition 3.** A gauge field on $\mathcal{F}^\bullet$ is an element $\psi \in \text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$, such that $\pi \circ \psi = 1 \in \text{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$.

From the above exact sequence, it follows the following proposition.

**Proposition 6.** The vanishing of the Atiyah class $\mathcal{F}^\bullet$ is a necessary and sufficient condition for the existence of gauge fields on this brane. Furthermore, the set of gauge fields on $\mathcal{F}^\bullet$, if is nonempty, is an affine space over the finite dimensional vector space $\text{Ext}^0(\mathcal{F}^\bullet, \Omega^1(\mathcal{F}^\bullet))$.

3.1.1. **Gauge fields on $B$-branes over $\mathbb{P}^n$.** According to the Beilinson spectral sequence [25, Chap. 2, Sect. 3.1] the set (1.1) is a strong complete exceptional sequence in the derived category $D^b(\mathbb{P}^n)$ [17, Sect 8.3]. Thus, $D^b(\mathbb{P}^n)$ is equivalent to the smallest triangulated subcategory that contains this exceptional family.

As usual, we denote with $\mathcal{F}^\bullet[l]$, with $l \in \mathbb{Z}$, the complex $\mathcal{F}^\bullet$ shifted $l$ to the left. Given $\mathcal{A}, \mathcal{B}$ elements of the generating set (1.1), let us consider morphisms $h$ between $\mathcal{A}' := \mathcal{A}[l]$ and $\mathcal{B}' := \mathcal{B}[l']$. We denote by $\text{Cone}(h) = \mathcal{A}'[1] \oplus \mathcal{B}'$ the mapping cone of $h$ [8, page 154]. We define $E^{(1)}$ the set obtained adding to $E$ the elements of the form $\text{Cone}(h)$. Hence, an element of $E^{(1)}$ is a complex whose term at a position $p$ is either 0, or $\mathcal{O}(k)$, or a direct sum of $\mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$, with $-n \leq k, k_1, k_2 \leq 0$.

Repeating the process with the elements of $E^{(1)}$, one obtains $E^{(2)}$, etc. The objects of the triangulated subcategory generated by the family $E$ are elements which belong to some $E^{(m)}$. Therefore, an object of the triangulated subcategory of $D^b(\mathbb{P}^n)$ generated by (1.1) is a complex $(\mathcal{G}^\bullet, d^\bullet)$, where $\mathcal{G}^p$ is a sheaf of the form

$$\mathcal{G}^p = \bigoplus_{i \in S_p} \mathcal{O}(k_{pi}),$$

with $-n \leq k_{pi} \leq 0$ and $i$ varying in a finite set $S_p$. (When $i$ “runs over the empty set”, the direct sum is taken to be 0).

**Proof of Theorem 4.** In general, given two bounded below complexes $A^\bullet$ and $B^\bullet$ in an abelian category $\mathfrak{A}$, the complex $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ is defined by (see [18, page 17])

$$\text{Hom}^m(A^\bullet, B^\bullet) = \prod_{p \in \mathbb{Z}} \text{Hom}_\mathfrak{A}(A^p, B^{p+m}),$$

(3.4)
with the differential $\delta_H$.

$$ (\delta_H g)^p = \delta_B^{m+p} g^p + (-1)^{m+1} g^{p+1} \delta_A. \tag{3.5} $$

As the complex $\mathcal{G}^\bullet$ defined in (3.3) consists of locally free $\mathcal{O}$-modules, then [14, Chap III, 6.5.1]

$$ \text{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) = H^0\text{Hom}^\bullet(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) = \{ g \in \text{Hom}^0(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) \mid \delta_H g = 0 \}, $$

where $\delta_H$ is the operator defined in (3.5). Hence, according to (3.4), it follows

$$ \text{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) \subset \text{Hom}^0(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) = \prod_p \text{Hom}(\mathcal{G}^p, \Omega^1(\mathcal{G}^p)). $$

By the additivity of the functor $\text{Hom}(\mathcal{O}, \mathcal{O})$, it follows

$$ \text{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) \subset \prod_p \bigoplus_{i,j} \text{Hom}(\mathcal{O}(k_{pj}), \Omega^1(k_{pj})), $$

where $\Omega^1(k)$ is the twisted sheaf $\Omega^1 \otimes_{\mathcal{O}} \mathcal{O}(k)$.

The summand $\text{Hom}(\mathcal{O}(k_{pj}), \Omega^1(k_{pj}))$ is equal to

$$ \text{Hom}(\mathcal{O}, \Omega^1(k_{pj} - k_{pi})) = \Gamma(\mathbb{P}^n, \Omega^1(k_{pj} - k_{pi})) = 0, $$

since $H^0(\mathbb{P}^n, \Omega^1(k)) = 0$, for any $k$ [25, page 4]. Therefore,

$$ \text{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{G}^\bullet, \Omega^1(\mathcal{G}^\bullet)) = 0. $$

From Proposition 6 it follows Theorem 4. \qed

**Proof of Theorem 5.** Let $\psi$ be a holomorphic gauge field on the above $B$-brane $\mathcal{G}^\bullet$. Then

$$ \psi \in \text{Hom}_{D^b(\mathbb{P}^n)}(\mathcal{G}^\bullet, \mathcal{I}(\mathcal{G}^\bullet)) = H^0\text{Hom}^\bullet(\mathcal{G}^\bullet, \mathcal{I}(\mathcal{G}^\bullet)) \subset \prod_p \text{Hom}(\mathcal{G}^p, (\Omega^1(\mathcal{G}^p) \oplus \mathcal{G}^p)). $$

Thus, $\psi$ determines a family $\{ \psi^p : \mathcal{G}^p \to \Omega^1(\mathcal{G}^p) \oplus \mathcal{G}^p \}$ of morphisms of $\mathcal{O}$-modules. As $\psi$ is a right inverse of $\pi$ (Definition 3), $\psi^p = \nabla^p \oplus \text{id}^p$, where $\nabla^p : \mathcal{G}^p \to \Omega^1(\mathcal{G}^p)$. The property $\psi(f \sigma) = f \ast \psi(\sigma)$ implies that $\nabla(f \sigma) = df \sigma + f \nabla(\sigma)$, for $f \in \mathcal{O}$. Hence, $\nabla^p$ is a holomorphic connection on $\mathcal{G}^p$, for any $p$.

The trace of the curvature of $\nabla^p$ is a holomorphic 2-form on $\mathbb{P}^n$; as $H^0(\mathbb{P}^n, \Omega^2) = 0$, that trace vanishes. Hence, the first Chern class of vector bundle associated to the locally free sheaf $\mathcal{G}^p$ vanishes.

On the other hand, the first Chern class of $\mathcal{G}^p$ is the sum

$$ \sum_i c_1(\mathcal{O}(k_{pi})). $$
This class is 0 iff $k_{pi} = 0$ for all $i$, since the $k_{pi} \leq 0$.

Therefore, the existence of a holomorphic gauge field on the brane $G^\bullet$ defined in (3.3) implies that $G^\bullet$ is a sequence of direct sum of copies of $O$

\[ \cdots \to \bigoplus_{i \in S_p} O \xrightarrow{d_p} \bigoplus_{i \in S_{p+1}} O \to \cdots \]

Since $\text{Hom}_O(O, O) \simeq \mathbb{C}$, the map $d_p$ is given by a constant complex matrix.

On the other hand, given the brane (3.6), if the set of indices $S_p$ has $m_p$ elements, on $ \oplus_{1}^{m_{p}} O$ we define the map $\varphi_p^p$,

\[ \varphi_p^p(\sigma_1 \oplus \cdots \oplus \sigma_{m_{p}}) = \partial \sigma_1 \oplus \cdots \oplus \partial \sigma_{m_{p}}. \]

It is a holomorphic connection on $ \oplus_{1}^{m_{p}} O$. Moreover, the family $\{\varphi_p\}$ is compatible with the “constant” differentials $d_p$. Thus, this family is a holomorphic gauge field on the brane defined by the complex (3.6). □

3.1.2. The homotopy category. Let $X$ be a smooth projective variety. The category $\text{Coh}(X)$ of coherent sheaves over $X$ has not enough injectives, for this reason it is convenient to regard $D^b(X)$ as a subcategory of $D^b(\mathcal{O}_X)$, the bounded derived category of the $\mathcal{O}_X$-modules. In fact, $D^b(X)$ is equivalent to $D^b(\mathcal{O}_X)_{\text{coh}}$, the full subcategory of $D^b(\mathcal{O}_X)$ consisting of the complexes with coherent cohomology [12, Chapter II] [26, Exp II]. Thus, $D^b(X)$ can be identified with the homotopy category whose objects are the complexes $G^\bullet$ of injective $\mathcal{O}_X$-modules, such that its cohomogy is bounded and coherent; i.e. $H^i(G^\bullet)$ is coherent and vanishes for $|j| >> 0$. We will denote this homotopy category by $K^b(X)_{\text{coh}}$.

Henceforth, we assume that $Y$ is a Hodge manifold; in this way we can identify coherent analytic sheaves on $Y$ with algebraic ones. Hence, the $B$-branes on $Y$ can be considered as objects of $K^b(Y)_{\text{coh}}$.

In accordance with the preceding paragraph, one can assume that the brane $(\mathcal{F}^\bullet, \delta^\bullet)$ is a complex of injective $\mathcal{O}$-modules with coherent cohomology modules satisfying $H^i(\mathcal{F}^\bullet) = 0$ for $|i| >> 0$. Moreover,

\[ \text{Hom}_{D^b(Y)}(\pi_1^*\mathcal{F}^\bullet, \pi_2^*\mathcal{F}^\bullet) \simeq \text{Hom}_{K^b(Y)_{\text{coh}}}(\mathcal{F}^\bullet, \mathcal{J}^1(\mathcal{F}^\bullet)), \]

where $\mathcal{J}^1(\mathcal{F}^\bullet)$ is an object of $K^b(Y)_{\text{coh}}$ $q$-isomorphic to $\mathcal{J}^1(\mathcal{F}^\bullet)$.

Thus, the elements of the space (3.1) can be identified with morphisms in the homotopy category $K^b(Y)_{\text{coh}}$.

From now on in this subsection, we delete the bullets in the notation for the complexes and set $\mathcal{J} := \mathcal{J}^1(\mathcal{F}^\bullet)$. Then there is a morphism
The gauge field $\psi$ can be regarded as a morphism in $K^b(Y)_{\text{coh}}$, $\hat{\psi} : \mathcal{F} \to \widehat{\mathcal{J}}$, such that $\hat{\pi} \circ \hat{\psi} = 1$.

On the other hand, $\hat{\psi}$ as a morphism of a homotopy category, determines a well-defined morphism of $\mathcal{O}$-modules between the cohomologies, that will be denoted in bold,

$$\psi^j : \mathcal{H}^j(\mathcal{F}) \to \mathcal{H}^j(\widehat{\mathcal{J}}) = \mathcal{H}^j(\mathcal{J}).$$

Similarly, one has the canonical projection

$$\pi^j : \mathcal{H}^j(\mathcal{J}) = \mathcal{H}^j(\mathcal{F}) \oplus (\Omega^1 \otimes_\mathcal{O} \mathcal{H}^j(\mathcal{F})) \to \mathcal{H}^j(\mathcal{F}),$$

satisfying $\pi^j \psi^j = 1$.

We set $\eta^j$ for the morphism of abelian sheaves defined by the inclusion in the direct sum

$$\eta^j : \mathcal{H}^j(\mathcal{F}) \to \mathcal{H}^j(\mathcal{J}) = \mathcal{H}^j(\mathcal{F}) \oplus (\Omega^1 \otimes_\mathcal{O} \mathcal{H}^j(\mathcal{F})), \quad \text{Hence, } \pi^j(\psi^j - \eta^j) = 0, \text{ and thus } \psi^j - \eta^j \text{ defines a morphism of abelian sheaves}$$

$$(3.7) \quad \vartheta^j : \mathcal{H}^j(\mathcal{F}) \to \Omega^1 \otimes_\mathcal{O} \mathcal{H}^j(\mathcal{F}),$$

which, by (3.2), satisfies the Leibniz’s. That is,

Proposition 7. The gauge field $\psi$ on the brane $\mathcal{F}$ determines on each sheaf $\mathcal{H}^j(\mathcal{F})$ a connection $\vartheta^j$.

Definition 4. The gauge field $\psi$ is called flat, if the curvature of the connection $\vartheta^j$ vanishes, for all $j$.

Remark 1. Let $\psi, \phi$ be two gauge fields on $\mathcal{F}$, we set

$$\xi := \phi - \psi \in \text{Hom}_{K^b(Y)}(\mathcal{F}, \mathcal{J}(\mathcal{F})), \quad \text{and denote by } \hat{\xi} \text{ the corresponding morphism } \mathcal{F} \to \mathcal{J}(\mathcal{F}) \text{ in the category } K^b(Y)_{\text{coh}}. \text{ Thus, } \hat{\xi} \text{ determines a well defined morphism of } \mathcal{O}\text{-modules between the cohomologies, } \xi^j : \mathcal{H}^j(\mathcal{F}) \to \mathcal{H}^j(\mathcal{J}). \text{ On the other hand, as } \phi \text{ and } \psi \text{ are gauge fields, } \hat{\pi} \hat{\xi} = 0. \text{ Hence, } \xi^j \text{ defines morphisms of } \mathcal{O}\text{-modules } \zeta^j(\xi) : \mathcal{H}^j(\mathcal{F}) \to \Omega^1 \otimes_\mathcal{O} \mathcal{H}^j(\mathcal{F}). \text{ We denote by } \hat{\vartheta}^j \text{ and } \chi^j \text{ the connections on } \mathcal{H}^j(\mathcal{F}) \text{ determined by } \hat{\psi} \text{ and } \hat{\phi}, \text{ respectively. Since } \xi^j = (\phi^j - \eta^j) - (\psi^j - \eta^j), \text{ it follows that } \chi^j = \vartheta^j + \zeta^j(\xi). \text{ In short, } \zeta^j(\xi) \text{ is the “variation” on the connection } \vartheta^j \text{ induced by the “variation” } \xi \text{ of the gauge field } \psi.$$

3.2. Yang-Mills fields on a brane. The result deduced in the following paragraph gives us a suggestion for the definition of the Yang-Mills functional over the gauge fields on a brane.
3.2.1. An Euler-Poincaré mapping. Let $\mathcal{A}$ be a coherent sheaf on the Hodge manifold $Y$, and $\alpha : \mathcal{A} \to \Omega^1(\mathcal{A})$ a holomorphic connection on $\mathcal{A}$. Denoting by $\mathcal{S}_\mathcal{A}$ the singularity set of $\mathcal{A}$, on $Y \setminus \mathcal{S}_\mathcal{A}$ we define differential form

$$\Phi(\mathcal{A}, \alpha) := \text{tr}(K_\alpha \wedge \ast K_\alpha) \in \Gamma(Y \setminus \mathcal{S}_\mathcal{A}, \Omega^{1\text{top}}),$$

$K_\alpha$ being the curvature of $\alpha$ considered as an $\text{End}(\mathcal{A})$-valued 2-form.

By $\mathcal{E}$, we denote the category whose objects are pairs $(\mathcal{A}, \alpha)$. A morphism $f : (\mathcal{A}, \alpha) \to (\mathcal{B}, \beta)$ is a morphism of coherent sheaves compatible with the connections; i.e. such that $(1 \otimes f) \circ \alpha = \beta \circ f$.

**Proposition 8.** If $0 \to (\mathcal{A}, \alpha) \xrightarrow{f} (\mathcal{B}, \beta) \xrightarrow{g} (\mathcal{C}, \gamma) \to 0$, is an exact sequence in $\mathcal{E}$, then on $Y \setminus \mathcal{S}$

$$\Phi(\mathcal{B}, \beta) = \Phi(\mathcal{A}, \alpha) + \Phi(\mathcal{C}, \gamma),$$

where $\mathcal{S}$ is the union of the singularity sets of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

**Proof.** Let $y_0 \in Y \setminus \mathcal{S}$. As the exact sequence splits locally on $Y \setminus \mathcal{S}$, there exists an open neighborhood $U$ of $y_0$ such that $g|_U$, in the sequence of locally free modules $0 \to \mathcal{A}|_U \xrightarrow{f|_U} \mathcal{B}|_U \xrightarrow{g|_U} \mathcal{C}|_U \to 0$, has a right inverse $h$.

Let $a$ be a frame for $\mathcal{A}|_U$, then $\alpha(a) = A \cdot a$, where $A$ is a matrix of 1-forms on $U$. Furthermore, $a$ can be chosen so that $A(y_0) = 0$. Similarly, let $c$ be a frame for $\mathcal{C}|_U$, then $\gamma(c) = C \cdot c$ and we choose $c$ so that $C(y_0) = 0$. From the splitting, it follows that $\{f(a), h(c)\}$ is a frame for $\mathcal{B}|_U$. By the compatibility of the connections with $f$ and $g$,

$$\beta(f(a)) = (1 \otimes f)(\alpha(a)) = (1 \otimes f)(A \cdot a) = A \cdot f(a).$$

On the other hand, $\beta(h(c)) = R \cdot f(a) + S \cdot h(c)$, with $R$ and $S$ matrices of 1-forms. But,

$$C \cdot c = \gamma(c) = \gamma(gh(c)) = (1 \otimes g)(\beta(h(c))) = (1 \otimes g)(R \cdot f(a) + S \cdot h(c)).$$

As $g \circ f = 0$ and $g \circ h = 1$, it follows that $C = S$. That is, the matrix of the connection $\beta$ in the frame $\{f(a), h(c)\}$ is

$$\text{(3.8)}$$

$$M := \begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$$

Since $A(y_0) = 0$ and $C(y_0) = 0$, the matrix of $K_\alpha(y_0)$, of the curvature of $\alpha$ at the point $y_0$, is $dA$. Analogous the matrix of $K_\gamma(y_0)$ is $dC$. The one of $K_\beta(y_0)$ is the exterior derivative of $[\alpha, \gamma]$, since $M \wedge M = 0$ at $y_0$. Then

$$\text{tr}(K_\beta(y_0) \wedge \ast K_\beta(y_0)) = \text{tr}(dA \wedge \ast dA) + \text{tr}(dC \wedge \ast dC)$$

$$= \text{tr}(K_\alpha(y_0) \wedge \ast K_\alpha(y_0)) + \text{tr}(K_\gamma(y_0) \wedge \ast K_\gamma(y_0)).$$
As \( y_0 \) is an arbitrary point of \( Y \setminus \mathcal{S} \), it follows the proposition.

Let \((\mathcal{G}^\bullet, \delta^\bullet)\) be a bounded complex of coherent sheaves on the manifold \( Y \). Let \( \nabla^\bullet \) be a family of holomorphic connections, compatible with the operators \( \delta^\bullet \). That is, \( \nabla^i : \mathcal{G}^i \to \Omega^1(\mathcal{G}^i) \) is a holomorphic gauge field on the coherent sheaf \( \mathcal{G}^i \) such that

\[
(1 \otimes \delta^i)\nabla^i = \nabla^{i+1} \delta^i.
\]

Hence, \( \nabla^i(\operatorname{Ker}(\delta^i)) \subset \operatorname{Ker}(1 \otimes \delta^i) \) and a similar relation for the image \( \operatorname{Im}(\delta^i - 1) \). It follows that \( \nabla^i \) induces a connection \( \theta^i \) on the cohomology \( \theta^i : \mathcal{H}^i(\mathcal{G}^\bullet) \to \mathcal{H}^i(\Omega^1(\mathcal{G}^\bullet)) \).

Obviously, the restrictions of \( \nabla^i \) determine connections on \( \operatorname{Ker}(\delta^i) \) and \( \operatorname{Im}(\delta^i - 1) \), respectively. one has the exact sequence

\[
0 \to (\operatorname{Ker}(\delta^i), \nabla^i) \to (\mathcal{G}^i, \nabla^i) \to (\operatorname{Im}(\delta^i), \nabla^{i+1}) \to 0
\]

in the category \( \mathcal{C} \). Similarly, we have the exact sequence

\[
0 \to (\operatorname{Im}(\delta^i - 1), \nabla^i) \to (\operatorname{Ker}(\delta^i), \nabla^i) \to (\mathcal{H}^i, \theta^i) \to 0.
\]

**Corollary 3.** Denoting with \( \mathcal{S} \) the union of the singularity sets of the sheaves \( \mathcal{G}^i \), then on \( Y \setminus \mathcal{S} \)

\[
\sum_i (-1)^i \operatorname{tr}(K_{\nabla^i} \wedge *K_{\nabla^i}) = \sum_i (-1)^i \operatorname{tr}(K_{\theta^i} \wedge *K_{\theta^i}).
\]

**Proof.** From Proposition 8 together with (3.9), it follows

\[
\Phi(\mathcal{G}^i, \nabla^i) = \Phi(\operatorname{Ker}(\delta^i), \nabla^i) + \Phi(\operatorname{Im}(\delta^i), \nabla^{i+1}).
\]

From (3.10), one obtains an analogous relation. Taking the alternate sums

\[
\sum_i (-1)^i \Phi(\mathcal{G}^i, \nabla^i) = \sum_i (-1)^i \Phi(\mathcal{H}^i, \theta^i).
\]

\[\square\]

### 3.2.2. The Yang-Mills functional

We propose a definition for the Yang-Mills functional over gauge fields on a brane. This proposal is based on the following considerations:

1. It is reasonable to require that this definition generalizes the one for coherent sheaves.
2. As a gauge field is a homotopy class of a morphism of complexes, it seems convenient to move on the cohomology of these complexes.
(3) Let $E^\bullet$ be a bounded complex of Hermitian vector bundles over the Kähler manifold $Y$, and $\nabla^\bullet$ a family of connections compatible with the Hermitian metrics and the coboundary operators. Denoting by $H^i(E^\bullet)$ the cohomology bundles, there exist connections $\theta^i$ on those bundles, induced by the family $\nabla^\bullet$. By Corollary 3 together with (2.27), one has the following equality of Euler-Poincaré type.

$$\sum_i (-1)^i \|K_{\nabla^i}\|^2 = \sum_i (-1)^i \|K_{\theta^i}\|^2.$$ 

On the basis of the above considerations, it seems appropriate to define the value of the Yang-Mills functional on the gauge $\psi$ on the brane $\mathcal{F}^\bullet$ as $\sum_i (-1)^i \|K_{\theta^i}\|^2$.

More precisely, taking into account Proposition 7, we adopt the following definitions.

**Definition 5.** The brane $(\mathcal{F}^\bullet, \delta^\bullet)$ is called a Hermitian brane, if the cohomology sheaves $H^j$ are Hermitian.

Let $(\mathcal{F}^\bullet, \delta^\bullet)$ be a Hermitian brane on the Hodge manifold $Y$. Given a gauge field $\psi$ on the brane $\mathcal{F}^\bullet$, by Proposition 7 one has the family of curvatures $K_{\theta^i}$ of the connections induced on the cohomologies. We denote by $S^i$ the singularity set of the cohomology sheaf $\mathcal{H}^i$ and let $S := \cup S^i$. On $Y \setminus S$ all the $\mathcal{O}$-modules $\mathcal{H}^i$ are locally free and we denote by $H^i$ the corresponding vector bundles. One has the respective curvature 2-forms

$$K_{\theta^i} \in \Gamma(Y \setminus S, \Omega^2 \otimes_\mathcal{O} \mathcal{End}(H^\bullet)).$$

**Definition 6.** Given a gauge field $\psi$ on the Hermitian B-brane $(\mathcal{F}^\bullet, \delta^\bullet)$, we define the value of the Yang-Mills functional at $\psi$ by

$$(3.12) \quad \mathcal{YM}(\psi) = \sum_i (-1)^i \|K_{\theta^i}\|^2.$$

The Yang-Mills fields on the brane $(\mathcal{F}^\bullet, \delta^\bullet)$ are the critical points of this functional.

Note that, if $(\mathcal{F}^\bullet, \delta^\bullet)$ is an acyclic complex, then the Yang-Mills functional for this complex is identically zero.

**Example 1.** Let $\mathcal{A}^\bullet := (\mathcal{A}^\bullet, d^\bullet_A, \alpha^\bullet)$ be a complex in the category $\mathcal{C}$; i.e, a complex of coherent sheaves with a family of holomorphic connections compatible with the coboundary operator $d_A$. Let $f := (f^\bullet)$ a morphism $f^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ in $\mathcal{C}$; that is, $f$ is a morphism of complexes compatible with the connections. One can consider the mapping cone $\mathcal{E}^\bullet$ of $f$. Thus, $\mathcal{E}^\bullet = (\mathcal{A}^\bullet[1] \oplus \mathcal{B}^\bullet, d^\bullet_C, \nabla^\bullet_C)$, with $d_C(a, b) =$
\[(d(a), (-1)^{\text{degree}} f(a) + db) \text{ and } \nabla(a, b) = (\alpha(a), \beta(b)). \]

In fact, \((1 \otimes d_C) \circ \nabla = \nabla \circ d_C\) and thus \(\mathcal{C}^\bullet\) is a complex of the category \(\mathcal{C}\).

For each \(i\) one has the following exact sequence in the category \(\mathcal{C}\)
\[
0 \to \mathcal{B}^i \to \mathcal{C}^i \to \mathcal{A}^{i+1} \to 0.
\]

From Proposition 8, \(\Phi(\mathcal{B}^i) + \Phi(\mathcal{A}^{i+1}) = \Phi(\mathcal{C}^i)\). Multiplying by \((-1)^i\) and summing
\[
\sum (-1)^i \text{tr}(K_{\beta^i} \wedge *K_{\beta^i}) + \sum (-1)^i \text{tr}(K_{\alpha^{i+1}} \wedge *K_{\alpha^{i+1}})
= \sum (-1)^i \text{tr}(K_{\nabla^i} \wedge *K_{\nabla^i})).
\]

Let us assume that

- \(\mathcal{A}^i\) and \(\mathcal{B}^i\) Hermitian sheaves for all \(i\).
- \(\alpha^i\) and \(\beta^i\) are Hermitian gauge fields (i.e., compatible with the metric) on \(\mathcal{A}^i\) and \(\mathcal{B}^i\), respectively.

Then one defines on \(\mathcal{C}^i\) the metric \(\langle \langle a, b \rangle, \langle a', b' \rangle \rangle := \langle a, a' \rangle + \langle b, b' \rangle\).

The connection \(\nabla^i\) is compatible with this metric. From the equality (3.13) together with (2.27), one deduces the following proposition.

**Proposition 9.** With the above notations and under the above hypotheses, \(\alpha\) and \(\beta\) determine in a natural way a gauge field \(\nabla\) on the mapping cone of \(f^*\) satisfying
\[
\mathcal{Y}\mathcal{M}(\beta) - \mathcal{Y}\mathcal{M}(\alpha) = \mathcal{Y}\mathcal{M}(\nabla).
\]

On the other hand, in the context of the branes theory, the fact that the branes \(\mathcal{A}^\bullet, \mathcal{B}^\bullet\) and \(\mathcal{C}^\bullet\) are the members of the distinguished triangle \(\mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{C}^\bullet \to \mathcal{A}^\bullet[1]\) means that \(\mathcal{A}^\bullet\) and \(\mathcal{C}^\bullet\) can potentially bind together to form the brane \(\mathcal{B}^\bullet[1\text{ Section 6.2.1}]\). Thus, the additive nature of equation (3.14) is consistent with this interpretation.

### 3.2.3. Yang-Mills fields

If \(\phi\) and \(\psi\) are gauge fields on the \(B\)-brane \((\mathcal{F}^\bullet, \delta^\bullet)\) and \(\xi = \phi - \psi\), using the notations introduced in Remark 1, the connections on the cohomologies induced by \(\phi\) and \(\psi\) satisfy
\[
\chi^j(\xi) = \partial^j + \zeta^j(\xi),
\]

with
\[
\zeta^j(\xi) \in \Gamma(Y \setminus S, \Omega^1 \otimes_\mathcal{O} \mathcal{E}nd(H^j)).
\]

With the mentioned notation, an infinitesimal variation \(\psi_\epsilon\) of \(\psi\) is given by a family \(\epsilon^j \zeta^j\), with \(\epsilon^j \in \mathbb{C}\), which defines a morphism between \(\mathcal{F}^\bullet\) and \(\mathcal{J}(\mathcal{F}^\bullet)\) in the homotopy category. In this case, for the connections on the cohomologies, one has \(\chi^j = \partial^j + \epsilon^j \zeta^j\). Furthermore, on \(Y \setminus S\) the curvatures satisfy
\[
K_{\chi^j} = K_{\partial^j} + \epsilon^j \partial^j(\zeta^j) + O((\epsilon^j)^2),
\]
\( \vartheta^j(\zeta^j) \) being the covariant derivative of \( \zeta^j \) considered as a section of \( \Omega^1 \otimes_{\mathcal{O}} \text{End}(H^j) \).

The functional \( \mathcal{Y} \mathcal{M} \) takes at the gauge field \( \psi \) a stationary value if

\[
\frac{\partial}{\partial \epsilon} \mathcal{Y} \mathcal{M}(\psi_{\epsilon}) \big|_{\epsilon = 0} = 0,
\]

for all \( i \) and any variation of \( \psi \). That is, if

\[
(3.15) \quad \langle K_{\vartheta^i}, \vartheta^i(\zeta^i) \rangle = 0,
\]

for all \( i \) and for any \( \zeta^i \) defined by a variation of \( \psi \). Therefore, by (2.31), one has the following proposition.

**Proposition 10.** Let \((F^\bullet, \delta^\bullet)\) be a Hermitian brane, such that the sheaves \( H^i(F^\bullet) \) are reflexive, and let \( \psi \) gauge field on \( F^\bullet \). If \( \vartheta^i \) is a Yang-Mills field on \( H^i \) for all \( i \), then \( \psi \) is a stationary point of the Yang-Mills functional; i.e. \( \psi \) is a Yang-Mills field on the brane.

The following proposition is a converse to Proposition 10.

**Proposition 11.** Let \( F^\bullet \) be a Hermitian B-brane as in Proposition 10. If \( \nabla^\bullet \) is a Yang-Mills field on \( F^\bullet \), then the connection \( \vartheta^i \) induced on \( H^i \) is a Yang-Mills field on this sheaf.

**Proof.** We will consider \((F^\bullet, \delta^\bullet)\) as an object of the category \( K^b_{\text{coh}}(Y) \); that is, we assume that \((F^\bullet, \delta^\bullet)\) is a complex of injective \( \mathcal{O} \)-modules such that its cohomology is bounded and coherent.

As \( \text{Ker}(\delta^i) \) is a submodule of the injective \( \mathcal{O} \)-module \( F^i \), then \( \text{Ker}(\delta^i) \) is a retract of \( F^i \) and thus it is also an injective \( \mathcal{O} \)-module. Therefore, the following short exact sequence

\[
0 \to \text{Ker}(\delta^i) \to F^i \to \text{Coim}(\delta^i) \to 0
\]

splits. That is, \( F^i \cong \text{Ker}(\delta^i) \oplus \text{Coim}(\delta^i) \).

Since \( \text{Im}(\delta^{i-1}) \) is a retract of the injective \( \mathcal{O} \)-module \( \text{Ker}(\delta^i) \), the following short exact sequence also splits

\[
0 \to \text{Im}(\delta^{i-1}) \to \text{Ker}(\delta^i) \to H^i \to 0.
\]

Thus,

\[
(3.16) \quad F^i \cong H^i \oplus G^i,
\]

where \( G^i \) is isomorphic to the direct sum of \( \text{Coim}(\delta^i) \) and \( \text{Im}(\delta^{i-1}) \). As \( H^i \) and \( G^i \) are summands in a direct sum decomposition of an injective \( \mathcal{O} \)-module, they are also injective.

On the other hand, the coboundary operator \( \delta^i : F^i \to F^{i+1} \) induces via the isomorphisms (3.16) to the morphism

\[
(3.17) \quad \delta^i : H^i \oplus G^i \to H^{i+1} \oplus G^{i+1}, \quad (a, b) \mapsto (0, \delta^i b).
\]
Given $\xi \in \text{Hom}_{K^b_{\text{coh}}(Y)}(\mathcal{F}^*, \Omega^1 \otimes_0 \mathcal{F}^*)$, according to Remark 1, it determines $\zeta^i \in \text{Hom}(\mathcal{H}^i, \Omega^1 \otimes_0 \mathcal{H}^i)$. As $\nabla^*$ is a Yang-Mills field (3.15) is satisfied.

A general “variation” of $\vartheta^j$ is defined by an element $\tau \in \text{Hom}(\mathcal{H}^j, \Omega^1 \otimes_0 \mathcal{H}^j)$. We need to prove that

$$\langle K_{\vartheta^i}, \vartheta^j(\tau) \rangle = 0,$$

for any variation $\tau$.

The morphism $\tau$ can be extended $C_i : \mathcal{H}^i \oplus \mathcal{G}^i \to \Omega^1 \otimes_0 (\mathcal{H}^i \oplus \mathcal{G}^i)$,

where

$$C^i(a, b) = \begin{cases} (\tau(a), 0), & \text{if } i = j \\ (0, 0), & \text{if } i \neq j \end{cases}$$

Moreover, the $C_i$ are compatible with the coboundaries. For example for $i = j$, by (3.17), $((1 \otimes \delta^j) \circ C^j)(a, b) = (1 \otimes \delta^j)(\tau(a), 0) = 0$; and $C^{j+1} \circ \delta^j(a, b) = 0$. Thus, by the isomorphism (3.16) the $C_i$ determine a morphism $\xi : \mathcal{F}^* \to \Omega^1 \otimes_0 \mathcal{F}^*$ in the category $K^b_{\text{coh}}(Y)$, and the corresponding $\zeta^i$ induced in the cohomologies are all 0 except when $i = j$, in which case $\zeta^j = \tau$. Hence, by (3.15),

$$\langle K_{\vartheta^i}, \vartheta^j(\tau) \rangle = 0.$$

This holds for any “variation” $\tau$ of $\vartheta^j$. That is, by (2.31), $\vartheta^j$ is a Yang-Mills field on $\mathcal{H}^i$.

**Proof of Theorem 6.** From Proposition 11 together with Proposition 10, it follows Theorem 6.

Let us assume that the set of gauge fields on the brane $\mathcal{F}^*$ is nonempty. Let $m$ be the dimension of the vector space $\text{Ext}^0(\mathcal{F}^*, \Omega^1(\mathcal{F}^*))$. We denote by $\xi_1, \ldots, \xi_m$ a basis of this vector space. According to Proposition 6, any gauge field $\psi$ on the brane can be expressed

$$\psi = \tilde{\psi} + \sum a \lambda_a \xi_a$$

$\tilde{\psi}$ being a fixed gauge field and $\lambda_a \in \mathbb{C}$. The corresponding connection $\nabla^i$ on $\mathcal{F}^i$ is of the form

$$\nabla^i = \tilde{\nabla}^i + \sum a \lambda_a \xi^i_a,$$

with $\xi^i_a \in \text{Hom}(\mathcal{F}^i, \Omega^1(\mathcal{F}^i))$. Hence, the connections on the cohomology sheaves $\mathcal{H}^i$ can be written in the form

$$\vartheta^i = \tilde{\vartheta}^i + \sum a \lambda_a \zeta^i_a.$$
The corresponding curvatures satisfy
\[ K_{\tilde{\theta}^i} = K_{\theta^i} + \sum_a \lambda_a \tilde{\theta}^i(\zeta_a^i) + \sum_{a,b} \lambda_a \lambda_b \zeta_a^i \wedge \zeta_b^i. \]

Therefore \( \|K_{\theta^i}\|^2 \) is a polynomial \( P^i(\lambda_1, \ldots, \lambda_m) \) of degree \( \leq 4 \).

By Theorem 6, the critical points of the Yang-Mills functional on \( \mathcal{F}^* \) correspond to the points \((\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m\) which satisfy the equations \( \frac{\partial P^i}{\partial \lambda_a} = 0 \) for \( a = 1, \ldots, m \) and for all \( i \). We have the following result, which generalizes Theorem 2.

**Theorem 7.** Assume the cohomology sheaves \( H^i \) of the brane \( \mathcal{F}^* \) are reflexive and the set of gauge fields on \( \mathcal{F}^* \) is nonempty. Then this set is in bijective correspondence with the points of a subvariety of \( \mathbb{C}^m \) defined by \( m \cdot s \) polynomials of degree \( \leq 3 \), where \( m := \dim \text{Ext}^0(\mathcal{F}^*, \Omega^1(\mathcal{F}^*)) \) and \( s \) the number of nontrivial sheaves \( H^i \).

### 4. Appendix

Let \( L \) be a holomorphic line bundle over the Kähler manifold \( Y \). We denote by \( \mathcal{U} = \{U_a\}_a \) a good cover of \( Y \), such that the restrictions \( L|_{U_a} \) are trivial. Let \( \{\varphi_{ab}\} \) denote the corresponding cocycle of \( Z^1(\mathcal{U}, \mathcal{O}^*) \). By the simply connectedness of each \( U_{ab} \), one can define \( \frac{1}{2\pi i} \log \varphi_{ab} \), and \( \zeta := \{\frac{1}{2\pi i} \partial \log \varphi_{ab}\} \) is a Čech cocycle in \( Z^1(\mathcal{U}, \Omega^1) \).

**Lemma 1.** The cocycle \( \zeta \) is a coboundary iff the Atiyah class \( a(L) = 0 \).

**Proof.** A holomorphic gauge field in the trivialization \( L|_{U_a} \) is of the form \( \partial + B_a \), with \( B_a \) a holomorphic 1-form on \( U_a \). The local connections \( \{\partial + B_a\} \) can be glued to form a holomorphic connection on \( L \) iff on \( U_{ab} \)

\[ B_b - B_a = \partial \log \varphi_{ab}, \]

for all \( a, b \). Equivalently, if the cocycle \( \zeta \) is a coboundary. \( \square \)

If \( \beta \in Z^0(\mathcal{U}, \Omega^1) \) satisfies \( \delta \beta = \zeta \), where \( \delta \) is the Čech coboundary operator, then \( -d\beta \) is the 2-form on \( Y \) determined by \( \zeta \). Such a \( \beta \) can be construct from a Hermitian metric on \( L \).

A Hermitian metric on \( L \) is defined by a family \( \{f_a : U_a \to \mathbb{R}_{>0}\}_a \) of \( C^\infty \) functions, such that \( f_b = \varphi_{ab} \varphi_{ab} f_a \). Letting \( \beta_a = \frac{1}{2\pi i} \partial \log f_a \), since the transition functions \( \varphi_{ab} \) are holomorphic one has

\[ (\delta \beta)_{ab} = \frac{1}{2\pi i} \partial \log (\varphi_{ab} \varphi_{ab}) = \zeta_{ab}. \]

On the other hand,

\[ -d\beta = \frac{i}{2\pi} \bar{\partial} \partial \log f_a. \]
Thus,
\[
(\delta(-d\beta))_{ab} = \frac{-1}{2\pi i} \partial \bar{\partial} \log(\varphi_{ab}\bar{\varphi}_{ab}) = 0.
\]
That is, \(-d\beta\) is a 2-form on \(Y\).

Associated to the Hermitian metric is defined the corresponding Chern connection, whose curvature form is given by \(\partial \bar{\partial} \log f_a\). Therefore, the cocycle \(\zeta\) defines the first \(c_1(L)\). From Lemma 11, it follows the following known result [3].

**Proposition 12.** \(L\) admits a holomorphic gauge field iff \(c_1(L) = 0\).

Thus, \(L\) supports a holomorphic gauge field iff \(L\) is flat.

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