Multicomponent perfect fluid with variable parameters in $n$ Ricci-flat spaces

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Abstract

$D$-dimensional cosmological model describing the evolution of a multicomponent perfect fluid with variable barotropic parameters in $n$ Ricci-flat spaces is investigated. The equations of motion are integrated for the case, when each component possesses an isotropic pressure with respect to all spaces. Exact solutions are presented in the Kasner-like form. Some explicit examples are given: 4-dimensional model with an isotropic accelerated expansion at late times and $(4 + d)$-dimensional model describing a compactification of extra dimensions.

1 Introduction

The necessity of studying multidimensional models of gravitation and cosmology [1, 2, 3] is motivated by several reasons. First, the main trend of modern physics is the unification of all known fundamental physical interactions: electromagnetic, weak, strong and gravitational ones. During the recent decades there has been a significant progress in unifying weak and electromagnetic interactions, some more modest achievements in GUT, supersymmetric, string and superstring theories.

Now, theories with membranes, $p$-branes and more vague M- and F-theories are being created and studied. Having no definite successful theory of unification now, it is desirable to study the common features of these theories and their applications to solving basic problems of modern gravity and cosmology. Moreover, if we really believe in unified theories, the early stages of the Universe evolution and black hole physics, as unique superhigh energy regions, are the most proper and natural arena for them.

Second, multidimensional gravitational models, as well as scalar-tensor theories of gravity, are theoretical frameworks for describing possible temporal and range variations of fundamental physical constants [4, 5, 6, 7].

Lastly, applying multidimensional gravitational models to basic problems of modern cosmology and black hole physics, we hope to find answers to such long-standing problems as: singular or nonsingular initial states, creation of the Universe, its flatness, creation of matter and its entropy, acceleration, coincidence and cosmological constant, origin of inflation and

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specific scalar fields which may be necessary for its realization, isotropization and graceful exit problems, stability and nature of fundamental constants [5], possible number of extra dimensions, their stable compactification etc.

The discovery of the accelerated expansion of the Universe and the fact that the flat Friedmann model with the cosmological constant or quintessence now fits best the set of different observational data, created the problems of dark matter and dark energy. This is a real revolution in modern physics as we do not know now what really the dark matter (0.20 of 0.30) and what is the dark energy (0.70) of the total energy are. Even attempts to explain it via the cosmological constant or quintessence seem to change one puzzle with another one as necessary vacuum properties or exotic scalar fields with or without strange potentials are still waiting to find their place in rigorous theories, not speaking about their real experimental confirmation.

In [9] we showed that the cosmic acceleration and coincidence problems may be solved by using an x-fluid as a quintessence and a viscous fluid as a normal matter. Viscosity of the normal matter can be explained by its own multicomponent structure. As is known, a mixture of different fluids admits a description as a single viscous fluid. We adopted the "second equations of state" in the form of some special metric dependence of the bulk and shear viscosity coefficients. These "second equations of state" generalize the so called "linear dissipative regime" in FRW world model, when the bulk viscosity coefficient is linearly proportional to the Hubble parameter. We studied $D$-dimensional homogeneous anisotropic cosmology, which allows to describe the dynamical compactification of the extra dimensions (see, for instance, our paper [10] on viscous cosmology). Other 2-component models in many dimensions also having the acceleration were found: with the cosmological constant in [11], with a perfect fluid in [8], with 2 non-Ricci-flat spaces [12], with p-branes and static internal spaces in [13], with scalar fields having exponential potentials in [17] and in four dimensions with a perfect fluid and a scalar field with the exponential potential in [15, 16] using methods developed in our multidimensional approach.

Here with the same aim the $D$-dimensional cosmological model describing the evolution of a multicomponent perfect fluid with variable barotropic parameters in $n$ Ricci-flat spaces is investigated. The equations of motion are integrated for the case, when each component possesses an isotropic pressure with respect to all spaces. The exact solutions are presented in the Kasner-like form. Some examples are given. The first example is 4-dimensional model with the Kasner-like behavior near the initial singularity and an isotropic accelerated expansion at late times. The other example is $(4 + d)$-dimensional model describing a contraction of the internal space accompanied by the expansion of the external space at early times.

## 2 The model

Following papers [18]-[21] we consider the metric

\[ ds^2 = -e^{2\gamma(t)}dt^2 + \sum_{i=1}^{n}\exp[2x^i(t)]ds^2_i, \]

on $D$-dimensional space-time manifold

\[ M = \mathbb{R} \times M_1 \times \ldots \times M_n, \]
where $ds^2_i$ is a metric of the Ricci-flat factor space $M_i$ of dimension $d_i$, $\gamma(t)$ and $x^i(t)$ are scalar functions of the cosmic time $t$, $a_i \equiv \exp[x^i]$ is the scale factor of the space $M_i$ and the function $\gamma(t)$ determines a time gauge. The synchronous time $t_s$ is defined by the equation $dt_s = \exp[\gamma(t)]dt$.

Under this assumption the Ricci tensor for the metric (2.1) has the following non-zero components

$$
R^0_0 = e^{-2\gamma} \left( \sum_{i=1}^n d_i (\dot{x}^i)^2 + \dot{\gamma}_0 - \dot{\gamma} \dot{\gamma}_0 \right),
$$

$$
R^m_{ni} = e^{-2\gamma} \left[ \ddot{x}^i + \dot{x}^i (\dot{\gamma}_0 - \dot{\gamma}) \right] \delta^m_{ni},
$$

with the definition

$$
\gamma_0 = \sum_{i=1}^n d_i x^i. \quad (2.3)
$$

Here indices $m_i$ and $n_i$ for $i = 1, \ldots, n$ run from $(D - \sum_{j=1}^n d_j)$ to $(\sum_{i=1}^n d_i)$, where $D = 1 + \sum_{i=1}^n d_i = \dim M$.

We consider a source of gravitational field in the form of multicomponent perfect fluid. In comoving coordinates the energy-momentum tensor of such a source reads

$$
T^M_N = \sum_{s=1}^m T^M(s)_N, \quad (2.4)
$$

$$(T^M(s)_N) = \text{diag} \left( -\rho^{(s)}(t), p^{(1)}_1(t), \ldots, p^{(s)}_1(t), \ldots, p^{(n)}_1(t), \ldots, p^{(s)}_n(t), \ldots, p^{(s)}_n(t) \right), \quad (2.5)
$$

Furthermore, we suppose that the barotropic equation of state for the perfect fluid components is given by

$$
p^{(s)}(t) = \left( 1 - h^{(s)}_i(x) \right) \rho^{(s)}(t), \quad s = 1, \ldots, m, \quad (2.6)
$$

where variable barotropic parameters are given by

$$
h^{(s)}_i(x) = \frac{1}{d_i} \frac{\partial}{\partial x^i} \Phi^{(s)}(x), \quad i = 1, \ldots, n \quad (2.7)
$$

with an arbitrary smooth function $\Phi^{(s)}(x)$ on $\mathbb{R}^n$.

The equation of motion $\nabla^M T^M_0(s) = 0$ for the perfect fluid component described by the tensor (2.5) reads

$$
\dot{\rho}^{(s)} + \sum_{i=1}^n d_i \dot{x}^i \left( \rho^{(s)} + p^{(s)}_i \right) = 0. \quad (2.8)
$$

After using equations of state (2.6) via (2.8), integrals of motion may be obtained in the form

$$
A^{(s)} = \rho^{(s)} \exp \left[ 2\gamma_0 - \Phi^{(s)}(x) \right], \quad s = 1, \ldots, m, \quad (2.9)
$$

In dimension $D$ (with gravitational constant $\kappa^2$), the set of Einstein equations $R^M_N - R^M_N / 2 = \kappa^2 T^M_N$ can be written as $R^M_N = \kappa^2 [T^M_N - T^M_N / (D-2)]$. Furthermore, like the
multidimensional geometry itself, these equations decompose blockwise to \( R_0^0 - R/2 = \kappa^2 T_0^0 \) and \( R_{ni}^m = \kappa^2 [T_{ni}^m - T\delta_{ni}/(D - 2)] \). Using the previous formulas, we obtain

\[
\frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j + V = 0, \tag{2.10}
\]

\[
\ddot{x}^i + \dot{x}^i (\dot{\gamma}_0 - \dot{\gamma}) = -\kappa^2 \sum_{s=1}^{m} A^{(s)} \left( h^{(s)}_{i} (x) - \frac{\sum_{k=1}^{n} d_k h^{(s)}_k (x)}{D - 2} \right) \times \exp \left[ \Phi^{(s)} (x) - 2(\gamma - \gamma_0) \right]. \tag{2.11}
\]

Here,

\[
G_{ij} = d_i \delta_{ij} - d_id_j \tag{2.12}
\]

are the components of the minisuperspace metric,

\[
V = \kappa^2 \sum_{s=1}^{m} A^{(s)} \exp \left[ \Phi^{(s)} (x) - 2(\gamma - \gamma_0) \right]. \tag{2.13}
\]

The integrals (2.9) are used to replace the densities \( \rho^{(s)} \) in (2.10), (2.11) by the functions \( x^i(t) \).

It is not difficult to verify that after the gauge fixing \( \gamma = F(x^1, \ldots, x^n) \) the Einstein equations (2.11) may be considered as the Lagrange-Euler equations obtained from the Lagrangian

\[
L = e^{\gamma_0 - \gamma} \left( \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - V \right) \tag{2.14}
\]

under the zero-energy constraint (2.10).

Further we develop an integration procedure which is based on the \( n \)-dimensional Minkowsky-like geometry. Let \( \mathbb{R}^n \) be the real vector space and \( e_1, \ldots, e_n \) be the canonical basis in \( \mathbb{R}^n \) (i.e. \( e_1 = (1,0, \ldots, 0) \) etc). Let us define a symmetrical bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^n \) by

\[
\langle e_i, e_j \rangle = \delta_{ij} d_j - d_i d_j \equiv G_{ij}. \tag{2.15}
\]

The form is nondegenerate and inverse matrix to \( (G_{ij}) \) has components

\[
G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}. \tag{2.16}
\]

This form \( \langle \cdot, \cdot \rangle \) endows the space \( \mathbb{R}^n \) with a metric which signature is \((- , + , \ldots , +) \) [19]. With this in mind, a vector \( y \in \mathbb{R}^n \) is timelike, spacelike or isotropic, if \( \langle y, y \rangle \) takes negative, positive or null values respectively and two vectors \( y \) and \( z \) are orthogonal if \( \langle y, z \rangle = 0 \).

Hereafter, we use the following vectors

\[
x = x^1(t)e_1 + \ldots + x^n(t)e_n, \tag{2.17}
\]

\[
u = u^1e_1 + \ldots + u^n e_n, \quad u^i = \frac{-1}{D - 2} \quad u_i = d_i, \tag{2.18}
\]

\]
where covariant coordinates \( u_i \) of the vector \( u \) are introduced by the usual way. Moreover, we obtain

\[
\sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j = \langle \dot{x}, \dot{x} \rangle = \sum_{i=1}^{n} d_i (\dot{x}^i)^2 - \dot{\gamma}_0^2,
\]

\[
\langle u, x \rangle = \gamma_0, \quad \langle u, u \rangle = -\frac{D-1}{D-2}.
\]

### 3 Exact solutions

Now we suppose that all components are isotropic fluids, i.e. pressures \( p_1^{(s)}, \ldots, p_n^{(s)} \) in the factor spaces \( M_1, \ldots, M_n \) are equivalent for each fluid component. From the mathematical point of view it means that

\[
\Phi^{(s)}(x) = F^{(s)}(\gamma_0), \quad s = 1, \ldots, m,
\]

where \( F^{(s)} \) is an arbitrary smooth function on \( \mathbb{R} \) (\( \gamma_0 \) is defined by (2.3)) and variable barotropic parameters obtained from (2.7) have the form \( h_i^{(s)}(x) = dF^{(s)}(\gamma_0)/d\gamma_0 \) for all \( i = 1, \ldots, n \).

We use the orthogonal basis

\[
\frac{u}{\langle u, u \rangle}, f_2, \ldots, f_n \in \mathbb{R}^n,
\]

where the vector \( u \) was introduced by equation (2.18). The orthogonality property reads

\[
\langle u, f_j \rangle = 0, \quad \langle f_j, f_k \rangle = \delta_{jk}, \quad (j, k = 2, \ldots, n).
\]

Let us note that basis vectors \( f_2, \ldots, f_n \) are space-like, since they are orthogonal to the time-like vector \( u \). The vector \( x \in \mathbb{R}^n \) decomposes as follows

\[
x = \gamma_0 \frac{u}{\langle u, u \rangle} + \sum_{j=2}^{n} \langle x, f_j \rangle f_j.
\]

We need to obtain the coordinates \( \gamma_0, \langle x, f_1 \rangle, \ldots, \langle x, f_n \rangle \) as functions of the time \( t \).

Using the time gauge of the type

\[
\gamma = f(\gamma_0),
\]

we come to the Lagrangian (2.14) in the terms of the coordinates in such basis as follows

\[
L = \frac{1}{2} e^{\gamma_0 - f(\gamma_0)} \left( \frac{\dot{\gamma}_0^2}{\langle u, u \rangle} + \sum_{j=2}^{n} (\dot{x}^j)^2 \right) - \kappa^2 \sum_{s=1}^{m} A^{(s)} e^{\gamma_0 + F^{(s)}(\gamma_0)}.
\]

The Lagrange-Euler equations for the coordinates \( \langle x, f_1 \rangle, \ldots, \langle x, f_n \rangle \)

\[
\frac{d}{dt} [e^{\gamma_0 - f(\gamma_0)} \langle \dot{x}, f_j \rangle] = 0
\]
give immediately the following integrals of motion

\[ e^{\gamma_0 - f(\gamma_0)} \langle \dot{x}, f_j \rangle = a^j, \quad j = 2, \ldots, n, \]  

(3.8)

where \( a^j \) is an arbitrary constant. Using (3.8) we may present the zero-energy constraint in the form

\[ \dot{\gamma}_0^2 + \langle u, u \rangle e^{2[f(\gamma_0) - \gamma_0]} \left( \sum_{j=2}^{n} (a^j)^2 + 2\kappa^2 \sum_{s=1}^{m} A(s) e^{F(s)(\gamma_0)} \right) = 0. \]  

(3.9)

The last equation admits obtaining of the unknown function \( \gamma_0 \) in quadratures for arbitrary functions \( f(\gamma_0), F^{(1)}(\gamma_0), \ldots, F^{(m)}(\gamma_0) \), then the model is integrable.

To present exact solutions in the Kasner-like form we introduce the following vector

\[ s = \sum_{j=2}^{n} a^j f_j \equiv \sum_{i=1}^{n} s^i e_i. \]  

(3.10)

Owing to the orthogonality property given by equation (3.3) the coordinates \( s^i \) of this vector in the canonical basis \( e_1, \ldots, e_n \) satisfy the following constraints

\[ \langle s, s \rangle = \sum_{i,j=1}^{n} G_{ij} s^i s^j = \sum_{j=2}^{n} (a^j)^2, \quad \langle s, u \rangle = \sum_{i=1}^{n} d_i s^i = 0 \]  

(3.11)

Then the vector \( x \) we need to find may be presented as

\[ x = \gamma_0 \frac{u}{\langle u, u \rangle} + \frac{\text{sgn}(\dot{\gamma}_0)}{\sqrt{-\langle u, u \rangle}} \int \left( \langle s, s \rangle + 2\kappa^2 \sum_{s=1}^{m} A(s) e^{F(s)(\gamma_0)} \right)^{-1/2} d\gamma_0 \]  

(3.12)

Here we remind the multi-dimensional generalization of the well-known Kasner solution [18]. It reads (for the synchronous time \( t_s \)) as follows

\[ ds^2 = -dt_s^2 + \sum_{i=1}^{n} A_i t_s^{2\varepsilon^i} ds_i^2. \]  

(3.13)

Such a metric describes the evolution of the vacuum model under consideration. Kasner parameters \( \varepsilon^i \) satisfy the constraints

\[ \sum_{i=1}^{n} d_i \varepsilon^i = 1, \quad \sum_{i=1}^{n} d_i (\varepsilon^i)^2 = 1. \]  

(3.14)

One may easily verify that the introduced parameters \( s^i \) and the Kasner parameters \( \varepsilon^i \) are connected by

\[ \frac{\text{sgn}(\dot{\gamma}_0)}{\sqrt{-\langle u, u \rangle}} \cdot s^i = \langle s, s \rangle \left( \varepsilon^i - \frac{u^i}{\langle u, u \rangle} \right), \quad i = 1, \ldots, n. \]  

(3.15)

Finally, we present the logarithms of scale factors in the Kasner-like form

\[ x^i = \gamma_0 \frac{u^i}{\langle u, u \rangle} + \frac{1}{\langle s, s \rangle} \int \left( \langle s, s \rangle + 2\kappa^2 \sum_{s=1}^{m} A(s) e^{F(s)(\gamma_0)} \right)^{-1/2} d\gamma_0 \cdot \left( \varepsilon^i - \frac{u^i}{\langle u, u \rangle} \right). \]  

(3.16)
We recall that coordinates of the vector \( u \) are defined by (2.18). The function \( \gamma_0(t) \) is determined by (3.9). If we use it as the time coordinate the exact solution for the metric (2.1) looks as follows

\[
ds^2 = \frac{D - 2}{D - 1} \cdot \langle s, s \rangle + 2\kappa^2 \sum_{s=1}^{n} A(s)e^{F(s)(\gamma_0)} \cdot d\gamma_0^2 + \sum_{i=1}^{n} \exp[2x^i]ds_i^2. \tag{3.17}
\]

4 Examples

4.1 4-dimensional model

Now we consider a special case with \( D = 4 \) manifold

\[
\mathbb{M}^4 = \mathbb{R} \times M^1_1 \times M^1_2 \times M^1_3
\]

and a single perfect fluid. The dominant energy condition, applied to the energy-momentum tensor (2.5) reduces to the following inequalities for the fluid variable barotropic parameter \( h = F'_{0}(\gamma_0) \)

\[
0 \leq F'(\gamma_0) \leq 2.
\]

The exact solution for this special model looks as follows

\[
x^i = \frac{\gamma_0}{3} + \sqrt{\langle s, s \rangle} \int \left( \langle s, s \rangle + 2\kappa^2 A e^{F(\gamma_0)} \right)^{-1/2} d\gamma_0 \cdot \left( \varepsilon^i - \frac{1}{3} \right), \tag{4.1}
\]

\[
\rho(t) = A \exp [F(\gamma_0(t)) - 2\gamma_0(x)] , \quad p(t) = [1 - F'(\gamma_0(t))] \rho(t), \tag{4.2}
\]

where the function \( \gamma_0(t) = x^1(t) + x^2(t) + x^3(t) \) is the solution to the separable ordinary differential equation

\[
\frac{\exp[\gamma_0 - f(\gamma_0)]d\gamma_0}{\sqrt{\langle s, s \rangle} + 2\kappa^2 A e^{F(\gamma_0)}} = \pm \frac{2}{3} dt. \tag{4.3}
\]

We recall that function \( f(\gamma_0) \) defines a time gauge \((f(\gamma_0) \equiv 0 \) corresponds to the synchronous time). The Kasner parameters \( \varepsilon^i \) obey the relations (3.14) with \( d_1 = d_2 = d_3 = 1 \). The nonnegative parameter \( \langle s, s \rangle \) is arbitrary. When \( \langle s, s \rangle = 0 \) one gets an isotropic solution, otherwise the solution describes an anisotropic behaviour.

Let us analyze the behaviour of the exact solution assuming that the function \( F(\gamma_0) \) is monotonically increasing with \( F'(\gamma_0) > 0 \). We choose the following time gauge

\[
e^{\gamma_0 - f(\gamma_0)} = F'(\gamma_0)e^{F(\gamma_0)} > 0.
\]

Then the equation (4.3) immediately gives

\[
e^{F(\gamma_0)} = \frac{1}{3} A\kappa^2(t - t_0) \cdot \left( t - t_0 + \frac{\sqrt{6}\langle s, s \rangle}{A\kappa^2} \right). \tag{4.4}
\]
It follows from (4.4) that \( F(\gamma_0) \to +\infty \) and \( \gamma_0 \to +\infty \) as \( t \to +\infty \). Then the equation (4.1) presented in the form

\[
\frac{dx^i}{d\gamma_0} = \frac{1}{3} + \langle s, s \rangle \frac{\varepsilon^i - 1/3}{\sqrt{\langle s, s \rangle + 2\kappa^2 A e^{F(\gamma_0)}}} \tag{4.5}
\]

shows an isotropic expansion of the universe with \( x^i = \gamma_0/3 + \text{const}, \ i = 1, 2, 3 \), at late times.

At earlier times as \( t \to t_0 \) one gets \( e^{F(\gamma_0)} \to +0 \) from the equation (4.4). Then the equations (4.3), (4.5) lead to \( \exp[x^i] \sim (t_s - t_{s0})^{\varepsilon^i} \) in the main order as the synchronous time \( t_s \to t_{s0} + 0 \). This shows the Kasner-like behavior near the initial singularity.

Hereafter we consider this single perfect fluid with the variable barotropic equation of state as an associated description of two different perfect fluids: normal matter with positive energy density \( \rho_m \) and nonnegative pressure \( p_m = (1 - h_m) \rho_m \) and quintessence with positive energy density \( \rho_Q \) and negative pressure \( p_Q = (1 - h_Q) \rho_Q \). The description implies

\[
\rho = \rho_m + \rho_Q, \quad p = p_m + p_Q. \tag{4.6}
\]

If barotropic parameters \( h_m, h_Q \) and \( h = F'(\gamma_0) \) are specified, we get from the equations (4.2), (4.6)

\[
\rho_m = A \cdot \frac{h_Q - F'(\gamma_0)}{h_Q - h_m} \cdot \exp[F(\gamma_0) - 2\gamma_0],
\]

\[
\rho_Q = A \cdot \frac{F'(\gamma_0) - h_m}{h_Q - h_m} \cdot \exp[F(\gamma_0) - 2\gamma_0].
\]

Then, the energy densities ratio looks as

\[
\alpha = \frac{\rho_Q}{\rho_m} = \frac{F'(\gamma_0) - h_m}{h_Q - F'(\gamma_0)}. \tag{8}
\]

In what follows we suppose that the barotropic parameters \( h_m \) and \( h_Q \) are constant and such that

\[
0 \leq h_m \leq 1, \ 1 < h_Q \leq 2.
\]

Then the barotropic parameter of the fluid

\[
h = F'(\gamma_0) = h_Q - \frac{h_Q - h_m}{\alpha(\gamma_0) + 1}
\]

is positive and the energy densities ratio \( \alpha \) is a function of \( \gamma_0 \). As we have already shown such model describes an isotropic expansion of the universe at late times. If we suppose that the model is asymptotically coherent, i.e. the energy densities ratio is asymptotically constant

\[
\lim_{\gamma_0 \to +\infty} \alpha(\gamma_0) = \alpha_0 = \text{const},
\]
the scale factor and the density are in the main order
\[ e^{x_i} \sim t_s^{\frac{2}{3} \left(2 - h_Q + h_m - h_m \alpha_0 + 1\right)} \], \[ \rho \sim t_s^{-2} \text{ as } t_s \to +\infty, \]
where \( t_s \) is the synchronous time. This is the late times isotropic asymptotic of the solution. Further, if the following condition holds
\[ \frac{h_Q - h_m}{\alpha_0 + 1} < \frac{4}{3}, \]
the exact solution describes the power law isotropic accelerated expansion at late times. If the normal matter is dust \((h_m = 1)\) and \(\alpha_0\) according to the observations is equal to 7/3, then we obtain \(h_Q > 31/21\) or \(p_Q < -10/21 \cdot \rho_Q\).

### 4.2 Multidimensional model

Now we study the behaviour of the exact solution for the manifold
\[ \mathcal{M} = \mathbb{R} \times M_1^3 \times M_2^{d_2} \times \ldots \times M_n^{d_n}, \]
of the dimension \(D = 4 + d\), where \(M_1^3\) is the external flat 3-dimensional space and \(M_2^{d_2}, \ldots, M_n^{d_n}\) are internal spaces. Let us admit that on some early stage of the evolution the functions \(F^{(s)}(\gamma_0), s = 1, \ldots, m\), determining the barotropic parameters \(h_i^{(s)}(x) = dF^{(s)}(\gamma_0)/d\gamma_0, s = 1, \ldots, m\) and \(i = 1, \ldots, n\), of the fluid components, obey the relation
\[ \sum_{s=1}^{m} A^{(s)} e^{F^{(s)}(\gamma_0)} = \frac{A}{2\kappa^2} = \text{const} > 0. \] (4.7)

In terms of densities \(\rho^{(s)}, s = 1, \ldots, m\), the formula (4.7) looks as follows
\[ \sum_{s=1}^{m} \rho^{(s)} = \frac{A}{2\kappa^2} e^{-2\gamma_0}. \]

From the physical point of view this relation means that the multicomponent perfect fluid appears as a whole as the stiff matter. In this case the exact solution for the scale factors is
\[ e^{x_i(t_s)} = a_i^{0} \cdot (t_s - t_{s0})^{\tilde{\varepsilon}^i}, \] (4.8)
where
\[ \tilde{\varepsilon}^i = \frac{1}{D - 1} + \left(\varepsilon^i - \frac{1}{D - 1}\right) \left(1 + \frac{A}{\langle s, s \rangle}\right)^{-1/2}. \]

Using the Kasner-like constraints (3.14) one easily obtains the similar constraints for the parameters \(\tilde{\varepsilon}^i\)
\[ \sum_{i=1}^{n} d_i \tilde{\varepsilon}^i = 1, \quad \sum_{i=1}^{n} d_i \left(\varepsilon^i\right)^2 = \frac{1}{D - 1} + \frac{D - 2}{D - 1} \cdot \left(1 + \frac{A}{\langle s, s \rangle}\right)^{-1} \equiv \delta. \] (4.9)
It is readily seen from (4.9) that if the parameter \( A/\langle s, s \rangle \) is small, then \( \delta \approx 1 \) and, consequently, \( \varepsilon^i \approx \tilde{\varepsilon}^i \). Then the model describes the Kasner-like behavior of the multidimensional model with the expansion of one part of the spaces \( M_1^3, M_2^d, \ldots, M_n^d \) and the contraction of another part. From the physical viewpoint the behavior with expansion of the external space \( M_1^3 \) and the contraction of internal space (or spaces) is of the most interest. Let us analyze this solution for the space-time manifold \( \mathbb{M}^{1+d} = \mathbb{R} \times M_1^3 \times M_2^d \), i.e. when there is only one \( d \)-dimensional internal space \( M_2^d \). In this case the solution to the set of equations (4.9) reads

\[
\tilde{\varepsilon}^1 = \frac{1}{d + 3} \left( 1 \pm \frac{d}{3} \sqrt{3 \cdot \frac{1 + 2/d}{1 + A/\langle s, s \rangle}} \right), \quad \tilde{\varepsilon}^2 = \frac{1}{d + 3} \left( 1 \mp \sqrt{3 \cdot \frac{1 + 2/d}{1 + A/\langle s, s \rangle}} \right) \quad (4.10)
\]

We notice that within the model the function

\[
a_1^{(JBD)} = e^{\varepsilon^1} = \text{const} \cdot (t_s - t_{s0})^\varepsilon^1
\]

is the scale factor of the external space with respect to the Jordan-Brance-Dicke frame (see, for instance, [22],[23],[24]). With respect to the Einstein frame the scale factor of the external space reads

\[
a_1^{(E)} = a_1^{(JBD)} \cdot e^{d \cdot x^2/2} = \text{const} \cdot (t_s - t_{s0})^\varepsilon^1,
\]

where

\[
\tilde{\varepsilon}^1 = \varepsilon^1 + d \cdot \varepsilon^2/2 = \frac{1}{d + 3} \left( d + 2 - \frac{d}{6} \sqrt{3 \cdot \frac{1 + 2/d}{1 + A/\langle s, s \rangle}} \right) < \varepsilon^1. \quad (4.11)
\]

In order to obtain the contraction of the internal space (i.e. \( \varepsilon^2 < 0 \)) we take the upper signs in the formulas (4.10),(4.11) with the following obvious condition

\[
\sqrt{3 \cdot \frac{1 + 2/d}{1 + A/\langle s, s \rangle}} > 1.
\]

Then \( A/\langle s, s \rangle \in (0, 2 + 6/d) \). Using the last one easily gets the following inequalities

\[
\frac{1}{3} < \tilde{\varepsilon}^1 < \frac{1 + \frac{d}{3} \sqrt{3(1 + 2/d)}}{d + 3}, \quad \frac{d + 2 - \frac{d}{3} \sqrt{3(1 + 2/d)}}{d + 3} < \tilde{\varepsilon}^1 < \frac{1}{3}.
\]

The analysis shows that if the solution describes contraction of the internal space then the external space in this solution may expand only with deceleration.

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