Abstract

It is known that the Cabibbo-Kobayashi-Maskawa (CKM) $n \times n$ matrix can be represented by a real matrix iff there is no CP-violation, and then the Jarlskog invariants vanish. We investigate sufficient conditions for the opposite statement to hold, paying particular attention to degenerate cases. We find that higher Jarlskog invariants are needed for $n \geq 4$. One generic sufficient condition is provided by the existence of a so-called echelon cross.
1 The Setting

1.1 Generalized CKM matrix

In the standard model, Cabibbo-Kobayashi-Maskawa (CKM) matrices \([1, 2]\) are unitary square matrices.\(^1\) We will sometimes restrict to this physical case below, but it is useful to consider a more general setting.

Let \(n_u, n_d \in \mathbb{N}\) be the number of up- and down-type quarks, respectively. We will consider (possibly rectangular, not necessarily unitary) CKM \(n_u \times n_d\) matrices

\[
V = (V_{ij}^a) \in \text{Mat}_{n_u \times n_d}(\mathbb{C}),
\]

\(^1\)There is an analogous story for the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrices \([3, 4, 5]\) in the leptonic sector.
where \( i \in \{1, \ldots, n_u\} \) and \( a \in \{1, \ldots, n_d\} \).

### 1.2 Quark Masses

Let the up and down mass matrices be (possibly degenerate) diagonal matrices

\[
M_u = \text{diag}(m^u_i) \in \text{Mat}_{n_u \times n_u}(\mathbb{R}),
M_d = \text{diag}(m^d_a) \in \text{Mat}_{n_d \times n_d}(\mathbb{R}).
\]

The pertinent terms in the standard model Lagrangian density are\(^2\)

\[
\Delta \mathcal{L} = \frac{g}{\sqrt{2}} u^L_i V^a_i W^+ a d^a_L - m^u_i u^L_i u^R_i - m^d_a d^L_a d^R_a + \text{h.c.}
\]

in the mass basis.

### 1.2.1 Disclaimer

Note that in physics true degeneracy can usually only happen if it is caused by a corresponding symmetry in the theory. In this article, we will not speculate about such underlying causes. Rather we will treat degeneracy as an idealized well-defined mathematical statement, as opposed to a fuzzy experimental fact with corresponding error bars, areas of unitarity triangles, etc. The point of view of this paper is that the pure mathematical matrix problem is interesting in its own right.

### 1.3 Residual Global Flavor Symmetry

There is a residual global flavor symmetry group\(^3\) \( G \equiv G_u \times G_d \) defined via the commutant/isotropy/stabilizer groups

\[
G_u := \{ U_u \in U(n_u) \mid [U_u, M_u] = 0 \},
G_d := \{ U_d \in U(n_d) \mid [U_d, M_d] = 0 \}.
\]

It acts as\(^4\)

\[
V' = U_u V U_d^\dagger \iff V'^a_i \equiv (U_u)^i_j V^j b (U_d^\dagger)^b a,
\]

\[
u'^L/R = U_u u^L/R \iff u'^L/R = (U_u)^i_j u^L/R.
\]

\[
d'^L/R = U_d d^L/R \iff d'^L/R = (U_d)^a b d^L/R.
\]

If the quark masses are non-degenerate, then \( G_u = U(1)^{n_u} \) and \( G_d = U(1)^{n_u} \).

If masses are, say, \( r \)-fold degenerated, then there will be a corresponding \( U(r) \)-factor enhancement in the symmetry group, and so forth. In general, the groups \( G_u, G_d \) are products of unitary groups.

**Definition 1.1** Define an equivalence relation among CKM matrices

\[
V \sim V' \iff \exists U_u \in G_u, U_d \in G_d : V' = U_u V U_d^\dagger,
\]

so that CKM matrices are equivalent iff they belong to the same \( G \)-orbit.

It turns out that the model has no CP-violation iff the CKM matrix \( V \) is equivalent to a real matrix.

\(^2\)See e.g. eq. (29.56) in Ref. [6]. Recall that \( \bar{\psi} := \psi^\dagger \gamma_0 \) and \( \psi^R/L := \frac{1}{2} (1 \pm \gamma_0) \psi \).

\(^3\)A general CKM matrix \( V \) commutes with at least a \( U(1) \)-factor, cf. Schur’s Lemma, so the effective symmetry group has 1 DOF less.

\(^4\)We restrict to subgroups of unitary (rather than general linear) groups in order to be able to define generalized Jarlskog invariants with the help of Hermitian adjoint rather than inverse matrix operation (which doesn’t exist for rectangular matrices). See also a related discussion in subsection 6.4.
1.4 Unitary Matrix Decomposition

We’re in the business of trying to make complex CKM matrices real by acting with unitary matrices. It is often useful to parametrize the unitary matrices as

\[ U(n) \ni U = e^{iS} O \],

where the matrix \( O \in O(n) \) is orthogonal and the matrix \( S \) belongs to the \( \frac{1}{2}n(n+1) \)-dimensional vector space

\[ s(n) := \{ S \in \text{Mat}_{n \times n}(\mathbb{R}) \mid S^T = S \} \]

of real symmetric matrices. Also note that

\[ \forall O \in O(n) : Os(n)O^{-1} \subseteq s(n). \]

Since the orthogonal matrices \( O \) preserve real matrices, they are not that useful to us. The important role is instead played by the real symmetric matrices \( S \).

1.5 Double Commutant

Define the double commutants

\[ V_u := \{ D_u \in \text{Mat}_{n_u \times n_u}(\mathbb{R}) \mid \forall U_u \in G_u : [U_u, D_u] = 0 \}, \]

\[ V_d := \{ D_d \in \text{Mat}_{n_d \times n_d}(\mathbb{R}) \mid \forall U_d \in G_d : [U_d, D_d] = 0 \}. \]

Note that the double commutants \( V_u, V_d \) are finite dimensional real vector spaces. The elements \( D_u, D_d \) consist of only diagonal matrices. The eigenvalues/diagonal elements are degenerate if the corresponding quark masses are degenerated.

It will be enough to consider bases \( (P^n_i)_i \) and \( (P^d_a)_a \) of the appropriate projection matrices for the double commutants \( V_u, V_d \). If the mass matrices are of the form

\[ M_u = \sum_i m_i^u P^n_i, \quad M_d = \sum_a m_a^d P^d_a, \]

then the double commutants are spanned by the corresponding projections

\[ V_u = \text{span}_\mathbb{R}\{P^n_i\}, \quad V_d = \text{span}_\mathbb{R}\{P^d_a\}, \]

respectively.

1.6 Jarlskog Invariants

Definition 1.2 Given a CKM matrix \( V \) the Jarlskog invariants\(^6\) are a multi-linear map

\[ \bigwedge^2 V_u \otimes \bigwedge^2 V_d \xrightarrow{J} \mathbb{R} \]

\(^5\)Here we are slightly misusing the notation by not introducing a new summation index in degenerate cases where the number of summands are smaller. Hopefully it does not lead to confusion.

\(^6\)Technically, what is called Jarlskog invariants in this paper generalizes the usual quartic Jarlskog invariants [7, 8, 9, 10, 11]. The quadratic Jarlskog invariants

\[ J(I; A) := \text{Im}(\text{tr}(IVAV^\dagger)) = 0 \]

vanish identically, and are hence not useful. There is a straightforward generalization to higher Jarlskog invariants

\[ J(I, J, K, \ldots, P; A, B, C, \ldots, H) := \text{Im}(\text{tr}(IVAV^\dagger JV BV^\dagger KVCV^\dagger \cdots V^\dagger PV HV^\dagger))) \]

\[ = J(J, K, \ldots, P; I, B, C, \ldots, H, A) \]

\[ = -J(I, P, \ldots, K, J; H, \ldots, C, B, A), \]

of even order, which we will only discuss sporadically in footnotes. It is natural to speculate that if Jarlskog invariants of all orders vanish then the CKM matrix \( V \) is equivalent to a real matrix.
defined as
\[
J(I, J; A, B) := \text{Im}(\text{tr}(IVAV^\dagger JVBV^\dagger)) \\
= -\text{Im}(\text{tr}(IVAV^\dagger JVBV^\dagger)) \\
= -\text{Im}(\text{tr}(VBV^\dagger JVAV^\dagger)) \\
= -\text{Im}(\text{tr}(IVBV^\dagger JVAV^\dagger)) \\
= -J(I, J; B, A) \\
= -J(J, I; A, B),
\]
where \(I, J \in V_u\) and \(A, B \in V_d\).

It is enough to specify the Jarlskog invariants \(J(P^u_i, P^d_j; P^d_a, P^d_b)\) on a basis of projection matrices for \(V_u, V_d\). These are typically labelled by via corresponding row and column indices as a shorthand notation.

The Jarlskog invariants are \(G\)-invariant, and they vanish \(J = 0\) if \(V\) is equivalent to a real matrix. The main purpose of this paper is to investigate the opposite relationship.

1.6.1 Unitary Case: Linear Relations

Then \(J(I, J; A, B)\) vanishes if one of its four entries \(I, J, A, B\) is an identity matrix. Since the sum of the basis of projection matrices is the identity matrix, this leads to identities [9, 13] among the Jarlskog invariants:
\[
0 = J(1_{n_u \times n_u}, J; A, B) = \sum_i J(P^u_i, J; A, B), \\
0 = J(I, 1_{n_u \times n_u}; A, B) = \sum_j J(I, P^u_j; A, B), \\
0 = J(I, J; 1_{n_d \times n_d}, B) = \sum_a J(I, J; P^d_a, B), \\
0 = J(I, J; A, 1_{n_d \times n_d}) = \sum_b J(I, J; A, P^d_b).
\]

2 A first look

2.1 Maximally degenerated Case \(G = U(n_u) \times U(n_d)\)

This case has no Jarlskog invariants. The CKM matrix has \(n_u n_d\) imaginary numbers, while the effective symmetry action has dimension
\[
\dim s(n_u) + \dim s(n_d) - \frac{1}{\text{Schur}} = n_u n_d + \frac{(n_u - n_d)^2}{2} + \frac{n_u}{2} + \frac{n_d}{2} - 1, \geq 0
\]
which is always bigger. In fact, singular value decomposition (SVD)
\[
V = U_u \begin{pmatrix}
\geq 0 \\
\geq 0 \\
\ddots \\
\geq 0
\end{pmatrix} U_d^\dagger
\]
shows that \(V\) is equivalent to a real matrix.

2.1.1 Unitary case

If \(V\) is unitary it is enough if either the up-masses or the down-masses are totally degenerate.
2.2 Counterexample: $2n \times 2n$ matrix with $G = U(n)^2 \times U(n)^2$

The $2n \times 2n$ CKM matrix has $4n^2$ imaginary numbers minus 1 Jarlskog invariant, while the effective symmetry action has dimension

$$\frac{4 \dim s(n)}{= \frac{4n(n+1)}{\text{Schur}}} - 1.$$ \hspace{1cm} (2.3)

Note that the dimension of the symmetry action becomes too small to render a generic CKM matrix real if $n \geq 2$.

2.3 Counterexample: $3n \times 3n$ unitary matrix with $G = U(n)^3 \times U(n)^3$

A $3n \times 3n$ unitary CKM matrix has heuristically $\dim s(3n) = \frac{1}{2}3n(3n + 1)$ imaginary DOF minus 1 Jarlskog invariant, while the effective symmetry action has dimension

$$\frac{6 \dim s(n)}{= \frac{4n(n+1)}{\text{Schur}}} - 1.$$ \hspace{1cm} (2.4)

Note that the dimension of the symmetry action becomes too small to render a generic unitary CKM matrix real if $n \geq 2$.

2.4 Example: $G_u = U(n_u)$ and $G_d = U(1)^2$

Theorem 2.1 If $G_u = U(n_u)$ and $G_d = U(1)^2$, then the $n_u \times 2$ CKM matrix $V$ is equivalent to a real matrix.

Proof: Write the $n_u \times 2$ CKM-matrix as

$$V = \begin{pmatrix} a & b \\ \vec{c} & \vec{d} \end{pmatrix},$$ \hspace{1cm} (2.5)

where $a$ and $b$ constitute the first row, and $\vec{c}$ and $\vec{d}$ are column $(n_u - 1)$-vectors.

Proceed as follows:

1. Use a $U(n_u)$-transformation to make the column vector $\begin{pmatrix} a \\ \vec{c} \end{pmatrix}$ on the form $\begin{pmatrix} |a| \\ 0 \end{pmatrix}$ (with possibly a different $a$).
2. Use a $U(1)$-rotation to make $b$ real.
3. Use a $U(n_u)$-transformation of the form $\begin{pmatrix} 1 & 0 \\ 0 & \ast \end{pmatrix}$ to make $\vec{d}$ real.

3 Non-degenerate Case $G = U(1)^{n_u} \times U(1)^{n_d}$

Introduce polar coordinates

$$V^i_a = r^i_a \exp(i\theta^i_a)$$ \hspace{1cm} (3.1)

for the matrix elements of the CKM matrix.

In the non-degenerate case, the Jarlskog invariant reads

$$J(i, j; a, b) = \text{Im}[V^i_a (V^j_a)^* V^j_b (V^i_b)^*]$$

$$= r^i_a r^j_a r^j_b r^i_b \sin (\theta^i_a - \theta^j_a + \theta^j_b - \theta^i_b).$$ \hspace{1cm} (3.2)

Definition 3.1 Given a CKM matrix $V$ in the non-degenerate case, an echelon cross is a row and column with non-zero entries only, c.f. Table 1.
Table 1: Example of an echelon cross (marked in red) for a $8 \times 7$ CKM matrix with only $U(1)$ factors (corresponding to no mass-degeneracies).

\[
\begin{array}{cccccccc}
U(1) & U(1) & U(1) & U(1) & U(1) & U(1) & U(1) \\
U(1) & * & & & & & \\
U(1) & * & & & & & \\
U(1) & * & * & * & * & * & * \\
U(1) & * & * & & & & \\
U(1) & * & & & & & \\
U(1) & * & & & & & \\
U(1) & * & & & & & \\
\end{array}
\]

Theorem 3.2 [13] In the non-degenerate case, if a CKM matrix $V$ with vanishing Jarlskog invariants $J = 0$ contains an echelon cross, then $V$ is equivalent to a real matrix.

**Proof**: Say that the echelon cross has row number $i$ and column number $a$. Use residual $U(1)^{n_u} \times U(1)^{n_d}$ symmetry to make the echelon cross real, i.e. the corresponding $\theta$-angles $\in \pi \mathbb{Z}$. (Start by making the intersection element $(i, a)$ of the cross real. Next make the $n_u + n_d - 2$ elements in the 4 arms of the cross real.) Finally consider an arbitrary element $(j, b)$ outside the cross. The Jarlskog invariant $J(i, j; a, b) = 0$ is zero. This implies that

$$r^j_b = 0 \vee \theta^j_b \in \pi \mathbb{Z},$$

i.e. the CKM matrix element $V^j_b \in \mathbb{R}$ is real.

\[
\square
\]

3.1 Non-degenerate case $n_u \leq 2 \vee n_d \leq 2$

Interestingly, we don’t need an echelon cross for the following theorem 3.3.

**Theorem 3.3** In the non-degenerate case, if a CKM matrix $V$ with vanishing Jarlskog invariants $J = 0$ has $n_u \leq 2 \vee n_d \leq 2$, then $V$ is equivalent to a real matrix.

**Proof**: Consider e.g. the case $n_u = 2$. Consider first the largest submatrix of columns that doesn’t contain any zeros. From Theorem 3.2, we can assume that this submatrix is real. The remaining columns contain at least 1 zero, and can hence be made real by a corresponding $U(1)^{n_d}$-rotation.

\[
\square
\]

3.2 Counterexample: $3 \times 3$ matrix with zero diagonal

A $3 \times 3$ CKM-matrix without an echelon cross is of the form

\[
V = \begin{pmatrix}
0 & A & \gamma \\
\beta & 0 & B \\
C & \alpha & 0
\end{pmatrix}
\]

up to row and/or column permutations. Here $A, B, C, \alpha, \beta, \gamma \in \mathbb{C}$. The Jarlskog invariants vanish $J = 0$. It contains 6 complex phases, but the effective residual symmetry group

$$U(1)^3_L \times U(1)^3_R$$

(3.5)

contains only 5 complex phases, so $V$ is generically not $U(1)$ equivalent to a real matrix.

\[
^7\text{However, if e.g. the sextic Jarlskog invariant}
\]

$$J(1, 2, 3; 3, 1, 2) = \text{Im}(\alpha\beta\gamma(ABC)^*)$$

vanish, then one can get rid of all complex phases.
3.2.1 Unitary case

Since \( \det(V) \neq 0 \), we conclude that

\[ ABC \neq 0 \lor \alpha \beta \gamma \neq 0. \]  

(3.7)

From the fact that column vectors should be orthogonal it then follows that

\[ (A, B, C) = (0, 0, 0) \lor (\alpha, \beta, \gamma) = (0, 0, 0), \]  

(3.8)

i.e. there are actually only 3 complex phases, which may easily be removed.

3.3 Counterexample: \( 4 \times 4 \) unitary matrix with zero off-diagonal

Consider a \( 4 \times 4 \) unitary CKM-matrix of the form

\[
V = \begin{pmatrix}
* & * & * & 0 \\
* & * & 0 & *
\end{pmatrix}
\]  

(3.9)

Only 6 of the 36 Jarlskog invariants are not manifestly zero from the outset:

\[
J(1, 2; 1, 2), J(1, 3; 1, 3), J(2, 4; 2, 4), J(3, 4; 3, 4), J(1, 4; 2, 3), J(2, 3; 1, 4).
\]  

(3.10)

Nevertheless, the remaining 6 are also zero because of linear relations among the Jarlskog invariants, c.f. sub-subsection 1.6.1. Let us consider a unitary CKM-matrix of the form

\[
V = e^{iS}
\]  

where

\[
S = \begin{pmatrix}
* & * & * & 0 \\
* & * & 0 & *
\end{pmatrix}
\]  

(3.11)

is an infinitesimal real symmetric matrix. One may check that \( V \) contains 12 infinitesimal imaginary entries, hereof 8 independent. However the effective residual symmetry group

\[
\frac{U(1)_L \times U(1)^4_R}{U(1)}
\]  

(3.12)

contains only 7 complex phases, so \( V \) is generically not equivalent to a real matrix.

4 Allowing degeneracy

4.1 Example: \( G_u = SU(2) \) and \( G_d = U(1)^{n_d} \)

Theorem 4.1 Let\(^8\) \( G_u = SU(2) \) and \( G_d = U(1)^{n_d} \). Then the \( 2 \times n_d \) CKM matrix \( V \) is equivalent to a real matrix if \( n_d \leq 2 \), but generically not\(^9\) if \( n_d \geq 3 \).

Proof: This case has no Jarlskog invariants. Let us write the \( 2 \times n_d \) CKM matrix

\[
V = (\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3 \ \ldots \ \vec{c}_{n_d}),
\]  

(4.2)

\(^8\)Here we have cut \( G_u \) down to an effective subgroup \( SU(2) \subseteq U(2) \).

\(^9\)Theorem 4.1 holds for all \( n_d \) if additionally all the sextic Jarlskog invariants of the form

\[
J(12, 12; 1, 2; a) = \text{Im}((V^\dagger V)_{12}(V^\dagger V)_{2a}(V^\dagger V)_{a1})
\]

\[
= \text{Im}(c_1^a c_2 \bar{c}_2 c_3 \bar{c}_3 
\]

\[
= c_1^a c_2 \text{Im}((A_2 A_a + B_2 B_a)(A_a^* A_1 + B_a^* B_1))
\]

\[
= \frac{c_1^a c_2}{\neq 0} (A_2 B_2 - A_2 B_1) \text{Im}(A_a^* B_a)
\]

vanish. Here it is implicitly implied that the column 2-vectors have been prepared as indicated in the main proof. We conclude that \( \arg(B_a) = -\arg(A_a) \in \frac{\pi}{2} \mathbb{Z} \), i.e. the column 2-vector \( \vec{c}_a \) can be made real by a \( U(1) \)-rotation.
in terms of column 2-vectors. Moreover, let us use the notation \( \vec{c} = \begin{pmatrix} A \\ B \end{pmatrix} \) for an arbitrary column 2-vector.

Preparations:

- In the case \( n_d \geq 3 \), arrange if possible by column permutations, so that \( \vec{c}_1 \) and \( \vec{c}_2 \) are neither orthogonal nor parallel. (The opposite case goes as follows: Then all column 2-vector can be split into two orthogonal sets of parallel column 2-vectors. After an \( SU(2) \) transformation, we may assume that each column 2-vector has a zero component, i.e. they can all be made real by \( U(1) \)-rotations.)

- By column permutations, we may assume that the first column 2-vector \( \vec{c}_1 \) is non-zero.

- After a \( SU(2) \) transformation, we may assume that \( \vec{c}_1 \) is of the form \( \begin{pmatrix} |A| \\ 0 \end{pmatrix} \), \( A \neq 0 \).

- For the other column 2-vectors \( \vec{c} = \begin{pmatrix} A \\ B \end{pmatrix} \), we \( U(1) \)-rotate so that \( A \) and \( B \) have opposite arguments, i.e. \( AB \geq 0 \).

Let us now study the effect of an \( SU(2) \) transformation

\[
G_u = SU(2) = \left\{ \begin{pmatrix} x \\ -y^* \\ y \\ x^* \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{C})\bigg| x, y \in \mathbb{C}, \ |x|^2 + |y|^2 = 1 \right\}.
\]

(4.3)

on a column 2-vector:

\[
\begin{pmatrix} x \\ -y^* \\ y \\ x^* \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} xA + yB \\ -y^*A + x^*B \end{pmatrix}.
\]

(4.4)

A necessary and sufficient condition to achieve a real 2-vector by a \( U(1) \)-rotation is that the 2 components on the RHS of eq. (4.4) must have the same phase (modulo \( \pi \)), i.e.

\[
\mathbb{R} \ni (xA + yB)(-y^*A + x^*B)^* = xy(|B|^2 - |A|^2) + x^2 AB^* - y^2 BA^*.
\]

(4.5)

From the first column 2-vector, we conclude that \( x \) and \( y \) have opposite phases (modulo \( \pi \)). In particular \( x^2 \) and \( y^2 \) have from now on opposite arguments.

In the following we will implicitly assume that \( x \neq 0 \). (For the case \( x = 0 \), one can instead give an argument using \( y \neq 0 \) in a very similar fashion.)

Consider now another column 2-vector (different from the first). We may assume that \( AB \neq 0 \), because else the necessary and sufficient condition is already satisfied. The imaginary part of the RHS of eq. (4.5) becomes

\[
0 = \text{Im}(RHS) = (|x|^2 + |y|^2)|AB| \sin \arg(x^2 AB^*) \iff \arg(x^2 AB^*) \in \pi \mathbb{Z}.
\]

(4.6)

By choosing \( \arg(x) \) this is always possible to satisfy for a given second column 2-vector, but generically impossible for more column vectors.

\[
\square
\]

### 4.2 \( 2 \times 2 \) CKM matrix

**Theorem 4.2** A \( 2 \times 2 \) CKM matrix where all Jarlskog invariants vanish is equivalent to a real matrix.

**Proof:**

- Non-degenerate case: Use theorem 3.3.

- Degenerate case: There are no Jarlskog invariants. By symmetry it is enough to consider the degenerate case where 2 up-quarks have the same mass. Use theorem 4.1.
4.3 $3 \times 3$ unitary CKM matrix

This is the standard model case.\textsuperscript{10}

**Theorem 4.3** A $3 \times 3$ unitary CKM matrix where all Jarlskog invariants vanish is equivalent to a real matrix.

**Proof:** For the non-degenerate case, use theorem 3.2 if $V$ has an echelon cross, and subsubsection 3.2.1 if it doesn’t. Next let’s consider the degenerate cases. Then there are no Jarlskog invariants. For this reason it is enough to consider the smallest degenerate symmetry group, e.g.

$$G_u = U(2) \times U(1) \quad \text{and} \quad G_d = U(1)^3.$$  \hfill (4.10)

Let the unitary CKM matrix be

$$V = \begin{pmatrix} A & C & E \\ B & D & F \\ \alpha & \beta & \gamma \end{pmatrix}.$$  \hfill (4.11)

Proceed as follows:

1. We can assume (after possibly permuting columns) that the column 2-vector $\begin{pmatrix} A \\ B \end{pmatrix} \neq \vec{0}$ is non-zero.
2. Use $U(1)_R$-rotations to make the 3rd row $\alpha, \beta$ and $\gamma$ real.
3. Use a $U(2)_L$-transformation to make the column 2-vector $\begin{pmatrix} A \\ B \end{pmatrix}$ on the form $\begin{pmatrix} |A| \\ 0 \end{pmatrix}$.
4. Use orthogonality of the column vectors to conclude that $C$ and $E$ are real.
5. Use orthogonality of the column vectors to conclude that $D$ and $F$ have the same phase (modulo $\pi$).
6. Use a $U(2)_L$-transformation of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ to make $D$ and $F$ real.

\[ \Box \]

5 Case of at most 2-fold mass-degeneracies, i.e. only $U(1)$ and $U(2)$ factors

By permuting rows and columns, we may assume that all $U(2)$-factors are ordered before the $U(1)$-factors, i.e.

$$G_u = U(2)^{m_u} \times U(1)^{n_u-2m_u}, \quad m_u \in \{1, 2, \ldots, \left\lfloor \frac{n_u}{2} \right\rfloor \},$$  \hfill (5.1)

$$G_d = U(2)^{m_d} \times U(1)^{n_d-2m_d}, \quad m_d \in \{1, 2, \ldots, \left\lfloor \frac{n_d}{2} \right\rfloor \}. $$  \hfill (5.2)

**Definition 5.1**

\textsuperscript{10}The $3 \times 3 = 9$ non-degenerate Jarlskog invariants

$$J(i, j; a, b) = \sum_{k,c} \epsilon_{ijk} \epsilon_{abc}$$  \hfill (4.7)

are alternating versions of a single invariant $J$. The invariant $[7, 8, 9, 10, 11, 12]$

$$\text{Im} \det[Y_u, Y_d] \propto J \Delta(m_u^a) \Delta(m_d^d)$$  \hfill (4.8)

takes mass-degeneracy into account. Here $Y_u, Y_d$ are Yukawa matrices, and

$$\Delta(m_i) = \prod_{i<j} (m_i - m_j)$$  \hfill (4.9)

is the Vandermonde determinant.
Table 2: Example of an echelon cross (marked in red) for a $11 \times 10$ CKM matrix with only $U(1)$ and $U(2)$ factors. The echelon partners and echelon copartners are marked in blue, while the echelon children are marked in green.

|       | $U(2)$ | $U(2)$ | $U(1)$ | $U(1)$ | $U(1)$ | $U(1)$ | $U(1)$ |
|-------|--------|--------|--------|--------|--------|--------|--------|
| $U(2)$ |        | *      |        | *      |        | *      |        |
| $U(2)$ |        | *      |        | *      |        |        |        |
| $U(2)$ |        | *      |        | *      |        |        |        |
| $U(1)$ |        | *      |        | *      | *      | *      | *      |
| $U(1)$ |        |        |        | *      | *      |        |        |
| $U(1)$ |        |        |        | *      |        | *      |        |
| $U(1)$ |        |        |        |        | *      |        |        |

- A singlet-singlet $1 \times 1$ matrix element is called **echelon** if it is non-zero.
- A doublet-singlet $2 \times 1$ submatrix $\vec{v}$ is called **echelon** if it has an echelon partner. An **echelon partner** is another doublet-singlet $2 \times 1$ submatrix $\vec{w}$ within the same doublet-row such that $\vec{v}$ and $\vec{w}$ are neither parallel nor perpendicular, i.e. $\det[\vec{v}, \vec{w}] \neq 0$ and $\vec{v}^\dagger \vec{w} \neq 0$.
- A singlet-doublet $1 \times 2$ submatrix $\vec{v}^T$ is called **echelon** if it has an echelon co-partner. An **echelon co-partner** is another singlet-doublet $1 \times 2$ submatrix $\vec{w}^T$ within the same doublet-column such that $\vec{v}$ and $\vec{w}$ are neither parallel nor perpendicular.
- A doublet-doublet $2 \times 2$ submatrix is never **echelon**.

**Definition 5.2** For each pair of echelon (partner, co-partner), the singlet-singlet $1 \times 1$ matrix element in the same column as the partner, and in the same row as the co-partner, is called an **echelon child**.

**Definition 5.3** Given a CKM matrix $V$ of the singlet-deplete type, an **echelon cross** is a singlet row and a singlet column with echelon block entries only, and such that all echelon children are non-zero.

**Theorem 5.4** If a CKM matrix $V$ of the singlet-deplete type with vanishing Jarlskog invariants $J = 0$ contains an echelon cross, then $V$ is equivalent to a real matrix.

**Sketched proof:** Use repeatedly the following examples 5.1-5.3.

\[\square\]

**5.1 Example:** $G_u = U(2) \times U(1)$ and $G_d = U(1)^2$

Assume that the 3rd row and 1st column constitute an echelon cross.

\[V = \begin{pmatrix} \hat{a} & b \\ c & \hat{d} \end{pmatrix}.\]  
(5.3)

The Jarlskog invariant is

\[0 = J(12;3,1,2) = \text{Im}(\hat{b}^\dagger \hat{a} c^* \hat{d}).\]  
(5.4)

---

\(^{11}\)Echelon partners are often marked in blue in this article, c.f. Table 2. The reader may wonder why we don’t allow an echelon partner to be a doublet-doublet $2 \times 2$ submatrix. The short answer is that it turns out to not be practical/useful. See also the analysis in subsection 6.3.

\(^{12}\)Echelon crosses are often marked in red in this article, c.f. Table 2. The reader may wonder why we don’t define an echelon cross built from a double-row and/or a double-column of doublet type? The short answer is that it turns out to not be practical/useful. Try!
We can assume that the column 2-vectors \( \vec{a} \) and \( \vec{b} \) are real because of theorem 4.2. From the Jarlskog invariant, we see that \( c \) and \( d \) must have the same phase (modulo \( \pi \)). (Here we have used that \( \vec{a} \) and \( \vec{b} \) are not perpendicular.) We can hence apply a \( U(1)_L \)-rotation on the 3rd row to make it real.

\[ \square \]

5.2 Example: \( G_u = U(2) \times U(1) \) and \( G_d = U(1)^3 \)

Assume that the 3rd row and the 1st column is an echelon cross.

\[
V = \begin{pmatrix}
A & C & E \\
B & D & F \\
\alpha & \beta & \gamma
\end{pmatrix}
\]  

(5.5)

We can assume that the 2 first column 3-vectors are real because of example 5.1. By applying a \( U(1)_R \)-rotation we can assume that \( E \) and \( F \) have opposite arguments. The pertinent Jarlskog invariants are

\[
0 = J(12, 3; 1, 3) \propto \text{Im}((AE + BF)\gamma^*), \\
0 = J(12, 3; 2, 3) \propto \text{Im}((CE + DF)\gamma^*).
\]  

(5.6)

We conclude that \( AE + BF, CE + DF \) and \( \gamma \) must have the same phase (modulo \( \pi \)). In particular,

\[
\mathbb{R} \ni (AE + BF)(CE + DF)^*,
\]  

(5.7)

or equivalently,

\[
0 = \text{Im}(ADEF^* + BCFE^*) = (AD - BC)|EF| \sin(2 \text{arg}(E)).
\]  

(5.8)

There are 2 cases:

- \( E = 0 \lor F = 0 \).
- \( \text{arg} E \in \frac{\pi}{2} \mathbb{Z} \).

In both cases we can make \( E \) and \( F \) real by applying a \( U(1)_R \)-rotation.

There are 2 cases:

- \( (E, F) \neq (0, 0) \). Both inner products \( AE + BF \) and \( CE + DF \) cannot be zero. Hence \( \text{Im}(\gamma) = 0 \).
- \( (E, F) = (0, 0) \). Make \( \gamma \) real by applying a \( U(1)_R \)-rotation.

\[ \square \]

5.3 Example: \( G_u = U(2) \times U(1)^2 \) and \( G_d = U(2) \times U(1)^2 \)

Assume that the 3rd row and 3rd column is an echelon cross.

\[
V = \begin{pmatrix}
A & \vec{c} & \vec{d} \\
\vec{b}^T & \beta & \gamma \\
\vec{a}^T & \alpha & *
\end{pmatrix}
\]  

(5.9)

We can assume that the column 2-vectors \( \vec{a}, \vec{b}, \vec{c} \) and \( \vec{d} \) are real because of theorem 4.2.

We can assume that phases of \( \alpha, \beta \) and \( \gamma \) are the same (modulo \( \pi \)) because of example 5.1. By applying the same 2 \( U(1)_R \)-rotations (and an opposite central \( U(1)_L \)-rotation inside \( U(2)_L \)), we can make \( \alpha, \beta \) and \( \gamma \) real (without disturbing \( \vec{c} \) and \( \vec{d} \)). Now the echelon cross is real.

From the non-degenerate theory, we can make the echelon child * real.
The 4 pertinent Jarlskog invariants are

\[ 0 = J^{(12,3;12,3)} \propto \vec{c}^T \text{Im}(A) \vec{b}, \]
\[ 0 = J^{(12,4;12,3)} \propto \vec{c}^T \text{Im}(A) \vec{a}, \]
\[ 0 = J^{(12,3;12,4)} \propto \vec{d}^T \text{Im}(A) \vec{b}, \]
\[ 0 = J^{(12,4;12,4)} \propto \vec{d}^T \text{Im}(A) \vec{a}, \]

which are 4 independent linear equations for \( \text{Im}(A) \). (The 4 conditions can be solved more easily if we use \( U(2) \) transformation to make the 2nd components of \( \vec{a} \) and \( \vec{b} \) equal to zero.) We conclude that \( \text{Im}(A) = 0. \)

\[ \square \]

6 Supplementary Material

6.1 Higher Echelon Subblocks?

The strategy so far has been to divide the question [of whether a CKM matrix is equivalent to a real matrix] into (i) a generic case where an echelon cross guarantees this, and (ii) a special case of Lebesgue-measure zero where further analysis is needed.

It is therefore natural to try to generalize echelon entries to \( n \)-fold degeneracy along the following lines.

**Definition 6.1** A \( n \)-plet-singlet \( n \times 1 \) submatrix \( \vec{v} \) is called **echelon** if it has \( n-1 \) echelon partners. An **echelon partner** is another \( n \)-plet-singlet \( n \times 1 \) submatrix \( \vec{w} \) within the same \( n \)-plet-row that satisfies the following conditions: The \( n \) \( n \)-plet-singlets are linearly independent, but pairwise not perpendicular.

However, this will not be useful as we shall see below in the case \( n = 3 \).

6.2 Example: \( G_u = SU(3) \) and \( G_d = U(1)^3 \)

Let’s write the \( 3 \times 3 \) CKM matrix as

\[ V = \begin{pmatrix} \vec{a} & \vec{b} & \vec{c} \end{pmatrix}, \]

where \( \vec{a}, \vec{b}, \) and \( \vec{c} \) are linearly independent column 3-vectors that are pairwise not perpendicular.

We can wlog. assume that \( \vec{a} \) and \( \vec{b} \) are real, c.f. theorem 2.1.

Let us parametrize the \( 3 \times 3 \) special unitary matrices as \[ U = e^{iS}O \in SU(3), \]

where \( O \in SO(3) \) is a \( 3 \times 3 \) orthogonal matrix and \( S \) is a traceless real symmetric \( 3 \times 3 \) matrix, c.f. subsection 1.4.

Since orthogonal matrices does not change the fact that \( \vec{a} \) and \( \vec{b} \) are real, we can ignore them in what follows.

6.2.1 Infinitesimal Analysis

At this point, we assume that \( \text{Im}(\vec{c}) \) is infinitesimal. Let us imagine that we successfully perform an infinitesimal symmetry transformation \( U = e^{iS} \), where \( S \) is an infinitesimal real symmetric \( 3 \times 3 \) matrices, such that the CKM matrix becomes real after pertinent infinitesimal \( U(1)^3 \)-rotations. This implies that

\[ \vec{a}' := e^{iS} \vec{a} \quad \Rightarrow \quad \text{Im}(\vec{a}') = S\vec{a} \parallel \vec{a}, \]
\[ \vec{b}' := e^{iS} \vec{b} \quad \Rightarrow \quad \text{Im}(\vec{b}') = S\vec{b} \parallel \vec{b}, \]
\[ \vec{c}' := e^{iS} \vec{c} \quad \Rightarrow \quad \text{Im}(\vec{c}') = S\text{Re}(\vec{c}) + \text{Im}(\vec{c}) \parallel \text{Re}(\vec{c}). \]

\[ ^{13} \text{Here we have cut } G_u \text{ down to an effective subgroup } SU(3) \subseteq U(3). \]
The 3 eqs. (6.3) contain 3 proportionality constants ($\lambda_a, \lambda_b, \lambda_c$). This means that we have $3 + 3 + 3 = 9$ real equations, but only $5 + 3 = 8$ real unknowns ($S, \lambda_a, \lambda_b, \lambda_c$). This does not have solutions in general. (Since $\vec{a}$ and $\vec{b}$ are non-perpendicular eigenvectors to the traceless real symmetric matrix $S$, it follows that their eigenvalues $\lambda_a = \lambda_b$ are equal. Nevertheless, this fact does not mean that we are short of 2 DOF rather than 1 DOF.)

6.3 Next-to-maximally degenerate case

The symmetry group is assumed to be
\[ G_u = U(n_u-1) \times U(1) \quad \text{and} \quad G_d = U(n_d-1) \times U(1). \] (6.5)
We may wlog. assume that $2 \leq n_u \leq n_d$. Let’s write the $n_u \times n_d$ CKM matrix as
\[ V = \begin{pmatrix} A & \vec{b} \\ \vec{c}^T & d \end{pmatrix}, \] (6.6)
where we assume that $A$ is a $(n_u-1) \times (n_d-1)$ matrix with a right inverse, $\vec{b}$ is a non-zero column $(n_u-1)$-vector, $\vec{c}$ is a non-zero column $(n_d-1)$-vector, and $d \in \mathbb{C}\backslash\{0\}$ is a non-zero number. (In other words, we assume for simplicity that the 4 subblocks have maximal rank.)

First use SVD to make $A$ and $d$ non-negative and diagonal. By left and right diagonal $U(1)$-rotations, we can make $\vec{c}$ real. In particular,
\[ A, \vec{c}, \text{ and } d \text{ are real.} \] (6.7)
There is 1 Jarlskog invariant:
\[ J(12 \ldots , n_u; 12 \ldots , n_d) = -\text{Im}(\vec{b}^T A^* \vec{c} d^*) \propto \text{Im}(\vec{b})^T A \vec{c}. \] (6.8)
If $n_u \leq 2$ and if $A \vec{c} \neq \vec{0}$, then we can conclude that $\text{Im}(\vec{b}) = \vec{0}$, so that $V$ is real, and we’re done. Let us therefore assume that $n_u \geq 3$.

Let us parametrize the unitary matrices as
\[ U = e^{iS} O, \] (6.9)
where $O$ is an orthogonal matrix and $S$ is a real symmetric matrix, c.f. subsection 1.4. Since orthogonal matrices do not change the fact that $A, \vec{c}$ and $d$ are real, we can ignore them in what follows. We can also ignore the last $U(1)$-factor in both groups $G_u$ and $G_d$.

6.3.1 Infinitesimal Analysis

At this point, we assume that $\text{Im}(\vec{b})$ is infinitesimal. Let us imagine that we successfully perform an infinitesimal symmetry transformation
\[ U_u = \begin{pmatrix} e^{iS_u} & 0 \\ 0 & 1 \end{pmatrix}, \quad U_d = \begin{pmatrix} e^{iS_d} & 0 \\ 0 & 1 \end{pmatrix}, \] (6.10)
where $S_u$ and $S_d$ are infinitesimal real symmetric matrices, such that $V' = U_u V U_d^\dagger$ is real.
\[ 0 = \text{Im}(A') = S_u A - A S_d \quad \Leftrightarrow \quad S_u = A S_d A^{-1}, \]
\[ 0 = \text{Im}(\vec{b}') = S_u R(\vec{b}) + \text{Im}(\vec{b}) = A S_d A^{-1} R(\vec{b}) + \text{Im}(\vec{b}), \] (6.11)
\[ 0 = \text{Im}(\vec{c}') = S_d \vec{c}. \]

If it happens that $A^{-1} R(\vec{b}) \parallel \vec{c} \neq \vec{0}$ and if $R(\vec{b}) \perp \text{Im}(\vec{b}) \neq \vec{0}$ then $S_d$ does not exist. We conclude that a vanishing Jarlskog invariant does not guarantee that the CKM matrix $V$ is equivalent to a real matrix if $n_u \geq 3$.

\[^{14}\text{However the DOF matches if we take into account that there is precisely 1 independent higher Jarlskog invariant, e.g.}
\]
\[ J(123,123;12,3) = \text{Im}((V^1 V)_{12} (V^1 V)_{23} (V^1 V)_{31}) + \text{Im}(\vec{a}_1^T \vec{b} \vec{c} \vec{d} \vec{a}) - \vec{a}_1^T \text{Im}(\vec{c}) \vec{a}_1^* \text{Re}(\vec{c}) - \vec{b} \text{Im}(\vec{c}) \vec{a}_1 \text{Re}(\vec{c}). \] (6.4)
6.4 The case \( G = U(2)^2 \times U(2)^2 \) with invertible \( 2 \times 2 \) sub-blocks

Let a \( 4 \times 4 \) CKM matrix be of the form
\[
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
(6.12)
where \( A, B, C, D \) are invertible \( 2 \times 2 \) matrices. There is 1 Jarlskog invariant
\[
J(12, 34; 12, 34) = \text{Im}(\text{tr}(AC^TDB^T)).
\]
(6.13)

First use SVD to make \( B \) and \( C \) positive and diagonal. In particular,
\[
B \text{ and } C \text{ are real.}
\]
(6.14)

Let us parametrize the \( 2 \times 2 \) unitary matrices as
\[
U = e^{iS}O \in U(2),
\]
(6.15)
where \( O \in O(2) \) is a \( 2 \times 2 \) orthogonal matrix and \( S \) is a real symmetric \( 2 \times 2 \) matrix, c.f. subsection 1.4. Since orthogonal matrices does not change the fact that \( B \) and \( C \) are real, we can ignore them in what follows.

6.4.1 Infinitesimal Analysis

At this point, we assume that \( \text{Im}(A) \) and \( \text{Im}(D) \) are infinitesimal. Let us imagine that we successfully perform an infinitesimal symmetry transformation
\[
U_u = \begin{pmatrix} e^{iS_u} & 0 \\ 0 & e^{iR_u} \end{pmatrix}, \quad U_d = \begin{pmatrix} e^{iS_d} & 0 \\ 0 & e^{iR_d} \end{pmatrix},
\]
(6.16)
where \( S_u, R_u, S_d \) and \( R_d \) are infinitesimal real symmetric \( 2 \times 2 \) matrices, such that \( V' = U_uVU_d^\dagger \) is real.

\[
0 = \text{Im}(A') = S_u\text{Re}(A) - \text{Re}(A)S_d + \text{Im}(A), \\
0 = \text{Im}(B') = S_uB - BR_d \iff R_d = B^{-1}S_uB, \\
0 = \text{Im}(C') = R_uC - CS_d \iff S_d = C^{-1}R_uC, \\
0 = \text{Im}(D') = R_u\text{Re}(D) - \text{Re}(D)R_d + \text{Im}(D).
\]

Eliminating \( S_d \) and \( R_d \), we get
\[
-\text{Im}(A) = S_u\text{Re}(A) - \text{Re}(A)C^{-1}R_uC, \\
-\text{Im}(D) = R_u\text{Re}(D) - \text{Re}(D)B^{-1}S_uB.
\]

Eliminating \( K_u \) leads to
\[
\text{Im}(A) + S_u\text{Re}(A) = \text{Re}(A)C^{-1}[\text{Re}(D)B^{-1}S_uB - \text{Im}(D)]\text{Re}(D)^{-1}C.
\]

Multiplying from right with \( C^{-1}\text{Re}(D)B^{-1} \) leads to
\[
[\text{Im}(A) + S_u\text{Re}(A)]C^{-1}\text{Re}(D)B^{-1} = \text{Re}(A)C^{-1}[\text{Re}(D)B^{-1}S_u - \text{Im}(D)B^{-1}].
\]

Taking trace yields the following consistency condition
\[
\text{tr}(\text{Im}(A)C^{-1}\text{Re}(D)B^{-1}) + \text{tr}(\text{Re}(A)C^{-1}\text{Im}(D)B^{-1}) = 0.
\]

This is generically different from the condition that the infinitesimal Jarlskog invariant
\[
0 = J(12, 34; 12, 34) = \text{tr}(\text{Im}(A)C^T\text{Re}(D)B^T) + \text{tr}(\text{Re}(A)C^T\text{Im}(D)B^T)
\]
vanishes. We conclude that a vanishing Jarlskog invariant does not guarantee that the CKM matrix \( V \) is equivalent to a real matrix.
6.4.2 Discussion

The above consistency condition suggests that the relevant invariant uses inverse matrix operations rather than Hermitian adjoint:

\[ \tilde{J}(12,34;12,34) = \text{Im}(\text{tr}(AC^{-1}DB^{-1})). \]  

(6.23)

However, this would not work for non-invertible blocks. For a similar reason, we must restrict the symmetry group to unitary groups rather than general linear groups.

Acknowledgement: The work of K.B. is supported by the Czech Science Foundation (GACR) under the grant no. GA20-04800S for Integrable Deformations.

References

[1] N. Cabibbo, *Unitary symmetry and leptonic decays*, Phys. Rev. Lett. 10 (1963) 531

[2] M. Kobayashi and T. Maskawa, *CP violation in the renormalizable theory of weak interaction*, Prog. Theor. Phys. 49 (1973) 652

[3] B. Pontecorvo, *Mesonium and anti-mesonium*, Phys. JETP6 (1957) 429.

[4] Z. Maki, M. Nakagawa and S. Sakata, *Remarks on the unified model of elementary particles*, Prog. Theor. Phys.28 (1962) 870.

[5] B. Pontecorvo, *Neutrino Experiments and the Problem of Conservation of Leptonic Charge*, Phys. JETP26 (1968) 984.

[6] M. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2014.

[7] C. Jarlskog, *Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation*, Phys. Rev. Lett. 55 (1985) 1039; Erratum: Phys. Rev. Lett. 58 (1987) 1698.

[8] C. Jarlskog, *A basis independent formulation of the connection between quark mass matrices, CP violation and experiment*, Z. Phys. C29 (1985) 491.

[9] C. Jarlskog and R. Stora, *Unitarity polygons and CP violation areas and phases in the standard electroweak model*, Phys. Lett. B208 (1988) 268.

[10] J.D. Bjorken and I. Dunietz, *Rephasing-invariant parametrizations of generalized Kobayashi-Maskawa matrices*, Phys. Rev. D36 (1987) 2109.

[11] C. Jarlskog, *Introduction to CP violation*, in *CP Violation* (Advanced Directions in High Energy Physics), Ed: C. Jarlskog, World Scientific, (1989) 1–40.

[12] K. Fujii, *A Geometric Parametrization of the Cabibbo-Kobayashi-Maskawa Matrix and the Jarlskog Invariant*, Int. J. Geom. Meth. Mod. Phys. 6 (2009) 1057, arXiv:0901.2180.

[13] U. Cavazos Olivas, S.R. Juarez Wysozka and P. Kielenowski, *CP violation for four generations of quarks*, Int. J. Mod. Phys. A35 (2020) 2050029, arXiv:1909.09255.