DUALITY FOR $K$-ANALYTIC COHOMOLOGY

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Abstract. We prove a duality result for the analytic cohomology of Lie groups over non-archimedean fields acting on locally convex vector spaces.

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Introduction

Let $K$ be a non-archimedean complete field and $G$ a $K$-analytic group, i.e., a group object in the category of $K$-analytic manifolds. Let furthermore $V$ be an analytic representation of $G$ and $C^\bullet(G, V)$ the complex of analytic inhomogeneous cochains of $G$ with coefficients in $V$. Its cohomology is called the $K$-analytic cohomology of $G$ with coefficients in $V$. If $K = \mathbb{Q}_p$, then Lazard showed that this is just continuous cohomology, cf. [Laz65, V.(2.3.10)]. But if $K \neq \mathbb{Q}_p$ then this is no longer the case and the $K$-analytic cohomology differs from the $\mathbb{Q}_p$-analytic cohomology.

Assume for a moment that $V$ is of finite dimension over $K$. Then for $d = \dim G$, we show the existence of a quasi-isomorphism

$$C^\bullet(G, V') \xrightarrow{\cong} C^\bullet(G, V)^{[-d]},$$

where $(\cdot)' = \text{Hom}_K(\cdot, K)$. If $V$ is not of finite dimension, then the functional analysis regrettably gets more complicated: We still get a morphism

$$C^\bullet(G, V_0') \longrightarrow \text{Hom}_K(C^\bullet(G, V), K)^{[-d]}$$

where $V_0'$ denotes the strong dual of $V$, but we need additional requirements for this morphism to be a quasi-isomorphism. For example, the Hahn-Banach theorem only holds for certain subclasses of non-archimedean fields – so taking the continuous dual is not always an exact functor. The precise statement of our main theorem includes more assumptions for these kinds of reasons, which we will explain in the first few sections.

Strategically, the proof of the duality result is charmingly straightforward: Using [Tam15], we compare analytic cohomology with Lie algebra cohomology. Hazewinkel
(cf. [Haz70]) showed a duality result for Lie algebra cohomology and plucking both results together then yields the result.

Technically, things are more complicated. The van Est comparison between analytic cohomology and Lie algebra cohomology only yields an isomorphism of cohomology groups when the underlying group is sufficiently connected – something that of course isn’t the case for non-archimedean ground fields. Showing that the duality of the Lie algebra cohomology correctly identifies the subspaces stemming from analytic cohomology is one issue, taking care of multiple topological subtleties not present in the archimedean world another.

The main application we had in mind developing these results concerns Lubin-Tate $(φ, Γ)$-modules as they appear in Iwasawa theory and especially the Herr complex used to express its cohomology. Dualising this complex should (at least morally) yield another Herr complex – but we run into all sorts of topological issues. Therefore, we can only construct a natural comparison morphism. Whether it is a quasi-isomorphism is unknown to us.

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1. Some Functional Analysis

We want to briefly recall some notions of non-archimedean functional analysis. We refer the reader to [Bou67; Sch02; Eme17] for details. An excellent overview can also be found in [Cre98]. In this section, we fix a complete non-archimedean field $K$ with valuation ring $\mathcal{O}_K$.

1.1. Foundations.

Definition 1.1. $K$ is called spherically complete, if every decreasing sequence of closed balls has a non-empty intersection.

Example 1.2. Every locally compact field is spherically complete. $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$, is not spherically complete.

Definition 1.3. A lattice $L$ in a $K$-vector space $V$ is an $\mathcal{O}_K$-submodule of $V$, which satisfies

$$V = \bigcup_{\lambda \in K} \lambda L.$$  

Definition 1.4. We call a topological $K$-vector space locally convex (or an LCVS), if it has a neighbourhood basis of lattices.

Remark 1.5. Note that a subset $M$ of a $K$-vector space is an $\mathcal{O}_K$-module if and only if for all $m, m' \in M$ and all $\lambda, \mu$ with $|\lambda|, |\mu| \leq 1$ also $\lambda m + \mu m' \in M$. This is the analogy to the usual notion of convexity. Requiring $\lambda m + (1 - \lambda)m' \in M$ regretfully does not suffice.

Remark 1.6. Let $V$ be a $K$-vector space. For every lattice $L$ in $V$, there is an attached seminorm $p_L$ defined by

$$p_L(v) = \inf_{\lambda \in K, m \in L} |\lambda|.$$  

Conversely, for a seminorm $p: V \rightarrow \mathbb{R}$ and $\varepsilon > 0$ we can define a lattice

$$V_p(\varepsilon) = \{v \in V \mid p(v) < \varepsilon\}.$$  

These constructions are inverse to one another in the following sense: For a family of seminorms $(p_i)_i$, the coarsest topology on $V$ such that all $p_i$ are continuous is the locally convex topology generated by the lattices $(V_{p_i}(\varepsilon))_{i, \varepsilon}$. Conversely, if $V$ is
locally convex, the topology on $V$ is the coarsest topology, such that all $(p_L)_L$ are continuous, where $L$ ranges over the open lattices in $V$. We refer to [Sch02, section 1.4] for details.

**Definition 1.7.** A subset $B$ of an LCVS $V$ is called bounded, if for any open lattice $L$ in $V$ there is a $\lambda \in K$ such that $B \subseteq \lambda L$.

**Proposition 1.8.** Every quasi-compact subset $C$ of an LCVS $V$ is bounded.

**Proof.** Let $L$ be an open lattice. By assumption, $V = \bigcup_{\lambda \in K} \lambda L$, so finitely many $\lambda_1 L, \ldots, \lambda_n L$ cover $C$. We can assume that none of the $\lambda_i$ lie in $O_K$. Then $C \subseteq \lambda_1 \cdots \lambda_n L$. □

**Remark 1.9.** If $K$ is not locally compact, an LCVS over $K$ does not have non-trivial compact $O_K$-submodules.

**Definition 1.10.** Let $V$ be an LCVS. We call $V$ bornological, if a $K$-linear map $V \to W$ of LCVS is continuous if and only if it respects bounded subsets. $V$ is called barrelled, if every closed lattice is open.

1.2. Dual spaces.

**Definition 1.11.** Let $V, W$ be LCVS. We denote the set of continuous $K$-linear maps from $V$ to $W$ by $L(V, W)$. For bounded subsets $B \subseteq V$ and open subsets $U \subseteq W$ we denote by $L(B, U) \subseteq L(V, W)$ those continuous linear maps which map $B$ into $U$. The families

- $\{L(S, U) \mid S \subseteq V$ a single point, $U \subseteq W$ open\}
- $\{L(C, U) \mid C \subseteq V$ compact, $U \subseteq W$ open\}
- $\{L(B, U) \mid B \subseteq V$ bounded, $U \subseteq W$ open\}

generate locally convex topologies on the space $L(V, W)$ of continuous linear maps from $V$ to $W$, which are called the weak, compact-open, and strong topology respectively. The corresponding LCVS will be denoted by $L_w(V, W), L_c(V, W)$, and $L_b(V, W)$.

**Remark 1.12.** The weak topology is coarser than the compact-open topology, which in turn is coarser than the strong topology.

**Remark 1.13.** Denote by $\mathbf{T}$ the category of Hausdorff topological spaces. (A variant of this remark also holds in the non-Hausdorff case.) For topological spaces $X, Y$ we denote by $[X, Y]$ the set $\text{Hom}_\mathbf{T}(X, Y)$ endowed with the compact-open topology. It is an easy exercise to check that for topological spaces $X, Y, Z$ there is a well-defined map

$$\text{Hom}_\mathbf{T}(X \times Y, Z) \to \text{Hom}_\mathbf{T}(X, [Y, Z])$$

sending $f$ to

$$x \mapsto (y \mapsto f(x, y)).$$

However, the obvious candidate for an inverse

$$\text{Hom}_\mathbf{T}(X, [Y, Z]) \to \text{Hom}_{\text{Set}}(X \times Y, Z),$$

sending $f$ to

$$(x, y) \mapsto f(x)(y),$$
in general does not yield continuous maps! Formally speaking, not every topological space is exponentiable. In our setting, we would have a bijection if $Y$ was locally compact, and locally compact spaces are the largest class for which this holds for all spaces $X$ and $Z$. As LCVS are only locally compact if they are finite dimensional, we cannot use the adjointness properties of the compact-open topology. In fact,
there is mostly no reason to look at the compact-open topology at all. Considering
linear maps, the strong topology plays the same role, but better.

**Proposition 1.14 (Hahn-Banach).** If $K$ is spherically complete, $V$ a LCVS and $W$
linear subspace of $V$ endowed with the subspace topology. Then every continuous
linear map $W \rightarrow K$ extends to a continuous linear map $V \rightarrow K$.

*Proof.* [Sch02, proposition 9.2, corollary 9.4]. □

There is also the following version of the Hahn-Banach theorem for LCVS of
countable type.

**Definition 1.15.** An LCVS $V$ is said to be of countable type, if for every continuous
seminorm $p$ on $V$ its completion $V_p$ at $p$ has a dense subspace of countable algebraic
dimension.

**Proposition 1.16.** Let $V$ be an LCVS of countable type and $W$
sub-vector space endowed with the subspace topology. Then every continuous linear map
$W \rightarrow K$ extends to a continuous linear map $V \rightarrow K$.

*Proof.* [PS10, corollary 4.2.6] □

**Definition 1.17.** We say that Hahn-Banach holds for an LCVS $V$, if $K$
is spherically complete or $V$ is of countable type.

**Remark 1.18.** Spaces of countable type are stable under forming subspaces, linear
images, projective limits, and countable inductive limits, cf. [PS10, theorem 4.2.13].

**Proposition 1.19.** Let $f : V \times W \rightarrow X$ be a (jointly) continuous bilinear map
of LCVS. Then it induces a continuous map $f : V \rightarrow \mathcal{L}_b(W,X)$.

*Proof.* As a jointly continuous map is also separately continuous, we have a well-
defined map $f : V \rightarrow \mathcal{L}(W,X)$. We only need to show that it is continuous with
respect to the strong topology. For this, let $B \subseteq W$ be bounded and $M \subseteq X$ an open
lattice. We need to show that the set $T$ of those $w \in W$ such that $f(w,B) \subseteq M$
is open. Let $w \in T$ and $b \in B$. By separate continuity we get open lattices
$w \in L, b \in L'$ such that $f(L \times L') \subseteq M$. As $M$ is bounded, there exists $\lambda \in K$ with
$B \subseteq \lambda L'$. Then

$$f(w + \lambda^{-1} L, B) = f(w, B) + f(L, \lambda^{-1} B) \subseteq f(w, B) + f(L, L') \subseteq M.$$

□

**Proposition 1.20 (Banach-Steinhaus).** Let $V,W$ be LCVS. If $V$ is barrelled, then
every bounded subset $H \subseteq \mathcal{L}_b(V,W)$ is equicontinuous, i. e., for every open lattice
$L' \subseteq W$ there exists an open lattice $L \subseteq V$ such that $f(L) \subseteq L'$ for every $f \in H$.

*Proof.* [Sch02, proposition 6.15] □

**Proposition 1.21.** Let $G$ be a locally compact topological group and $V$ a barrelled
LCVS. Assume that $G$ acts via linear maps on $V$. Then

$$G \times V \rightarrow V$$
is continuous if and only if it is separately continuous.

*Proof.* It is clear that a continuous group action is separately continuous.
Let $U \subseteq V$ be an open lattice and $g \in G, v \in V, gv \in U$. Let $H$ be a compact
neighbourhood of $g$ with $Hv \in U$, which exists by local compactness of $G$ and
separate continuity of the group action.
Consider the set \( M = \{ h \cdot - | h \in H \} \) of continuous linear maps \( V \rightarrow V \).

We want to show that it is bounded in the topology of pointwise convergence on \( \text{Hom}_{\text{cts}}(V,V) \). For this matter, take \( w \in V \), \( S \subset V \) an open lattice, and denote by \( L \) those continuous linear maps \( V \rightarrow V \) which map \( w \) into \( S \). We need to show that there exists \( \lambda \in K \) such that \( Hw \subseteq \lambda S \), i.e., \( M \subseteq \lambda L \).

Proposition 1.20 now shows the existence of an open lattice \( L' \) such that \( HL' \subseteq U \), or in other words, \( H \times L' \subseteq \text{mult}^{-1}(U) \). \( \square \)

**Definition 1.22.** The dual space of an LCVS \( V \) is the vector space of continuous \( K \)-linear functions \( V \rightarrow K \) and will be denoted by \( V' \).

We denote by \( V'_s \) the dual space equipped with the weak topology, which is the topology of pointwise convergence.

\( V'_c \) will denote the dual space equipped with the compact-open topology, which is also the topology of uniform convergence on compact subsets.

The strong dual will be denoted by \( V'_b \) and is defined as the topology of uniform convergence on bounded subsets of \( V \).

**Remark 1.23.** Note that for both the weak and strong duals, the dual of a direct sum of LCVS is the product of its duals. However, only for the strong dual is the dual of a product of LCVS the sum of its duals.

Note that by [Sch02, lemma 6.4], \( V'_c \) can be defined as the coarsest topology on \( V' \) such that for every quasi-compact \( K \subset V \) the map

\[
\begin{align*}
V' &\rightarrow \mathbb{R} \\
v' &\mapsto \sup_{v \in C} |v'(v)|_K
\end{align*}
\]

is continuous.

1.3. **Analyticity.**

**Definition 1.24.** Let \( E \) be a normed \( K \)-vector space and \( V \) a LCVS. A formal sum

\[
f = \sum_{n \in \mathbb{N}_0} f_n
\]

of continuous functions \( f_n: E \rightarrow V \) which are homogeneous of degree \( n \) (i.e., \( f_n(\lambda x) = \lambda^n f_n(x) \) for all \( n \in \mathbb{N}_0, \lambda \in K, x \in E \)) is called a convergent power series, if there exists an \( R > 0 \) such that for every continuous seminorm \( p: V \rightarrow K \) the following holds:

\[
\|(f_n)_n\|_{p,R} = \sup_{n \in \mathbb{N}_0} \sup_{x \in E, \|x\| \leq 1} R^n p(f_n(x)) < \infty.
\]

The supremum over all \( R \) such that for every continuous seminorm \( p \) we have \( \|(f_n)_n\|_{p,R} < \infty \) is called the radius of convergence of \( f \).

A map \( \tilde{f}: E \rightarrow V \) is called analytic in \( x \in E \), if there exists a convergent power series \( f_x \) such that for all \( h \in E \) close enough to zero, we have an equality

\[
\tilde{f}(x + h) = f_x(h).
\]

It is called analytic, if it is analytic at every point.

Let \( M \) be an analytic Banach manifold over \( K \) (i.e., \( M \) is locally isomorphic to \( K \)-Banach spaces with analytic transition maps). A map \( \tilde{f}: M \rightarrow V \) with values in \( V \) is called locally analytic, if it is analytic in charts. The radius of convergence of \( \tilde{f} \) at \( x \) is the radius of the power series development at a local chart. It might be larger than the chart itself.
Lemma 1.25. Let $f : E \longrightarrow V$ be an analytic map from a normed vector space to a Hausdorff LCVS $V$. The map $r_f : E \longrightarrow \mathbb{R}_{>0} \cup \{\infty\}$, mapping a point to the radius of convergence of the power series development of $f$ at that point, is lower semi-continuous, i.e., for every $x \in E$ we have
\[
\liminf_{x' \to x} r_f(x') \geq r_f(x).
\]
Consequently, if $C \subseteq E$ is compact, then
\[
\inf_{x \in C} r_f(x) > 0.
\]
Proof. A power series is analytic within its ball of convergence. It follows that the radius of convergence can at most increase. As lower semi-continuous maps attain their infimum in compact sets, the claim follows. □

Proposition 1.26. The development as a power series is unique, i.e., if $\tilde{f} : E \longrightarrow V$ is an analytic map between a normed $K$-vector space $E$ and a Hausdorff LCVS $V$ and if
\[
\tilde{f}(x + h) = \sum_{n \in \mathbb{N}_0} f_{x,n}^{(1)}(h) = \sum_{n \in \mathbb{N}_0} f_{x,n}^{(2)}(h)
\]
for sufficiently small $h$ with $f_{x,n}^{(i)}$ continuous and homogeneous of degree $n$, then $f_{x,n}^{(1)} = f_{x,n}^{(2)}$ for all $n$.

Proof. It suffices to show that if $\sum_n f_n$ is the zero function with $f_n$ continuous and homogeneous of degree $n$ and $(f_n)_n$ convergent close to zero, then all $f_n = 0$. Assume that $f_k \neq 0$. We can assume that $k$ is minimal with this property. Let $p$ be a continuous seminorm on $V$ with $p(f_k(x)) > 0$. By replacing $x$ with $\lambda x$ for some $\lambda$ close to zero, we can assume that $p(f_k(x)) > p(f_{k+n}(x))$ for all $n > 0$. By convergence of the power series, $\{p(f_{k+n}(x)) \, | \, n\} \subseteq \mathbb{R}$ is bounded from above by some $R \in \mathbb{R}$. Choose now $\lambda \in K$ with $|\lambda| < \max\{1, p(f_k(x))/R\}$, then it is easy to see that indeed $p(f_k(\lambda x)) \neq 0$, $k$ is minimal with this property, and $p(f_k(\lambda x)) > p(f_{k+n}(\lambda x))$ for all $n > 0$. But then $p(\sum_n f_n(x)) = p(f_k(x)) > 0$, so $\sum_n f_n$ is not the zero function. □

1.4. Strictness.

Definition 1.27. A linear map $V \longrightarrow W$ of LCVS is called strict, if the induced map
\[
V/\ker f \longrightarrow \text{im } f
\]
with the quotient topology on $V/\ker f$ and the subspace topology on $\text{im } f$ is an isomorphism.

Remark 1.28. Open linear maps are clearly strict, but strictness is remarkably bad behaved in general: Neither the sum nor the composition of strict maps needs to be strict again.

Definition 1.29. A sequence of LCVS
\[
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\]
is called exact if it is exact as a sequence of vector spaces and if the involved maps are all strict.

Proposition 1.30. Let
\[
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\]
be an exact sequence of LCVS. If Hahn-Banach holds for \( B \), then the induced sequence of abelian groups

\[
0 \longrightarrow C' \longrightarrow B' \longrightarrow A' \longrightarrow 0
\]

is also exact.

Proof. Let

\[
0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0
\]

be exact. It is clear that

\[
0 \longrightarrow C' \xrightarrow{\pi^*} B'
\]

is exact. Let \( f: B \longrightarrow K \) be in the kernel of \( \iota^* \), i.e., \( \iota A \subseteq \ker f \). This induces a map \( B/\iota A \longrightarrow K \), which by strictness is a map \( C \longrightarrow K \).

It remains to show surjectivity of \( \iota^* \), i.e., the existence of a map \( \tilde{f} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\iota A & \xrightarrow{\cong} & A \\
\downarrow{\iota} & & \downarrow{f} \\
B & \xrightarrow{\tilde{f}} & K
\end{array}
\]

But this extension exists by proposition 1.14 or proposition 1.16.

\[\blacksquare\]

Lemma 1.31. Let \( V \) be an LCVS and \( A: K^n \longrightarrow K^m \) a linear map. Then the induced map

\[A \otimes_K: V^\otimes n \longrightarrow V^\otimes m\]

is strict.

Proof. Note that finite direct sums of LCVS coincide with their product. It is then clear that every component map \( (A \otimes_K V)_i: V^\otimes n_i \longrightarrow V \) is open, so \( A \otimes_K V \) is open as well.

\[\blacksquare\]

Definition 1.32. A Fréchet space is an LCVS which is isomorphic to the projective limit of Banach spaces.

Remark 1.33. A space is Fréchet if and only if it is a complete LCVS whose topology is induced by a translation-invariant metric if and only if it is a Hausdorff topological \( K \)-vector space whose topology is induced by a countable family of semi-norms for which every Cauchy sequence converges.

Proposition 1.34 (Open-mapping theorem). Let \( f: V \longrightarrow W \) be a continuous surjective linear map from a Fréchet space to a barrelled Hausdorff LCVS. Then \( f \) is open.

Proof. [Sch02, proposition 8.6] \[\blacksquare\]

Definition 1.35. An LCVS is called an LF-space, if it is the direct limit of a countable family of Fréchet spaces, the limit being formed in the category of locally convex vector spaces.

Remark 1.36. LF-spaces are Hausdorff.

Proposition 1.37 (Open-mapping theorem for LF-spaces). Every continuous surjective linear map between LF-spaces is open.

Proof. [Sch02, proposition 8.8] \[\blacksquare\]

Proposition 1.38. If a continuous linear map \( f: V \longrightarrow W \) between LF spaces has finite-dimensional cokernel, it is strict.
Proof. Note that this does not follow immediately from proposition [1.37], as we do not know that $\text{im } f$ is again LF.

Take finitely many independent vectors whose projection to the cokernel form a basis of the cokernel. Their span in $W$ will be called $X$. As $X$ is finite dimensional, it is especially also LF and hence so is $V \oplus X$. The map

\[ V \oplus X \xrightarrow{f \oplus \text{id}} W \]

is then bijective, linear and continuous; thus it is an isomorphism by proposition [1.34] and $f$ hence an isomorphism onto $f(V)$. □

2. Analytic Actions of Lie Groups

We continue with a fixed non-archimedean field $K$.

**Definition 2.1.** A group object in the category of (finite-dimensional analytic) $K$-manifolds is called a Lie group over $K$.

**Definition 2.2.** Let $G$ be a Lie group over $K$ and $V$ a separated LCVS. A continuous action $G \times V \rightarrow V$ by continuous linear maps is called analytic, if every orbit map $g \mapsto gv$ is analytic. It is called equi-analytic, if it is analytic and the contragradient action on the dual space $G \times V' \rightarrow V'$ is analytic with respect to the strong topology on $V'$.

**Proposition 2.3.** If a Lie group $G$ acts continuously on an LCVS $V$ and if the evaluation map $V' \times V \rightarrow K$ is continuous, then the contragradient action $G \times V' \rightarrow V'$ is also continuous.

Proof. Consider the following maps:

\[ G \times V \times V' \xrightarrow{((\cdot)^{-1}, \text{id}, \text{id})} G \times V \times V'_b \xrightarrow{(\text{mult}, \text{id})} V \times V'_b \rightarrow K \]

The last map is just the evaluation function. The composite is now clearly continuous and by proposition [1.19], so is the induced map

\[ G \times V'_b \rightarrow V'_b, \]

which is the contragradient action. □

We will spell out the following proposition in more detail than necessary to show where the name equi-analytic stems from.

**Proposition 2.4.** An analytic action $G \times V \rightarrow V$ is equi-analytic, if $V$ is of finite dimension.

Proof. By proposition [2.3] we only need to show that every orbit map

\[ g \mapsto v'(g^{-1} \cdot -) \]

is analytic. Fix $v' \in V'$.

Considering a chart $\text{coord}: U \rightarrow K^d$ of a neighbourhood $U$ of $g$ and $h$ close to the neutral element,

\[ (gh)^{-1}v = \sum_{n \in \mathbb{N}_0} F_{g,v,n}(\text{coord}(h)) \]

with $F_{g,v,n}: K^d \rightarrow V$ continuous and homogeneous of degree $n$. Define

\[ F'_{g,v',n}(x)(v) = v'(F_{g,v,n}(x)). \]

It suffices to show that

\[ F'_{g,v',n}: U \rightarrow V'_d \]
is well-defined, continuous, homogeneous of degree \( n \), and gives rise to a convergent power series, as then

\[
\sum_n F'_{g,v',n}(\text{coord}(h))(v) = \sum_n v'(F_{g,v,n}(\text{coord}(h)))
\]

\[
= v'((\sum_n F_{g,v,n}(\text{coord}(h)))
\]

\[
= v'((gh)^{-1}v)
\]

\[
= ((gh)v')(v).
\]

Note first that by linearity of \( v' \), indeed \( F'_{g,v',n} \) is homogeneous of degree \( n \). Using proposition \([1.26]\), we see that \( F'_{g,v',n} \) is \( K \)-linear. The same argument that resulted in proposition \([2.3]\) also shows that \( F'_{g,v',n} \) is continuous. It remains to show that \( (F'_{g,v',n})_n \) is convergent with respect to the strong topology, i.e., we need to show that there exists an \( R > 0 \) such that for every bounded set \( B < V \) we have that

\[
\sup_{n \in K'} \sup_{x \in K', ||x|| \leq 1} \sup_{v \in B} R^n |F'_{g,v',n}(x)(v)| < \infty,
\]

which by definition of \( F'_{g,v',n} \) is equivalent to

\[
(*): \sup_{n \in K'} \sup_{x \in K', ||x|| \leq 1} \sup_{v \in B} R^n |v'(F_{g,v,n}(x))| < \infty.
\]

Analyticity of the group action on the other hand yields that for fixed \( g \in G, v \in V \) we have an \( R_{g,v} > 0 \) such that

\[
\sup_{n \in K'} \sup_{x \in K', ||x|| \leq 1} \sup_{v \in B} R^n |v'(F_{g,v,n}(x))| < \infty.
\]

If \( B \subseteq V \) is compact, lemma \([1.25]\) yields the existence of \( R_{g,B} > 0 \) such that

\[
(\dagger): \sup_{n \in K'} \sup_{x \in K', ||x|| \leq 1} \sup_{v \in B} R^n |v'(F_{g,v,n}(x))| < \infty.
\]

We cannot directly deduce \((*)\) from \((\dagger)\), as we have no means of controlling the radius of convergence across different compact (or bounded) subsets. This homogeneity is what \emph{equi-analytic} alludes to.

By proposition \([1.26]\) for any \( u, w \in V \) and \( \lambda \in K \)

\[
F_{g,u+\lambda w,n}(x) = F_{g,u,n}(x) + \lambda F_{g,w,n}(x).
\]

It follows that if \( u \in V \) is in the linear subspace generated by \( w_1, \ldots, w_k \in V \), then for the radii of convergence of the orbit maps \( -\cdot u \) and \( -\cdot w_i \) we have the following estimate:

\[
r_{u-w_i}(g) \geq \min_i r_{-w_i}(g).
\]

If \( V \) is generated by \( v_1, \ldots, v_n \) and \( R = \min_i r_{-v_i}(g) \), then \( R > 0 \) and

\[
\sup_{n \in K'} \sup_{x \in K', ||x|| \leq 1} \sup_{v \in B} R^n |v'(F_{g,v,n}(x))| < \infty,
\]

which is more than enough to show \((*)\).

\[\square\]

**Lemma 2.5.** Let \( \varphi \) be a continuous endomorphism of \( V \). Then it induces a continuous map \( \varphi': V' \to V' \).

**Proof.** We need to show that if \( B \subseteq V \) is bounded and \( U \subseteq K \) is open, then also \( (\varphi')^{-1}(L(B,U)) = L(\varphi(B),U) \) is open. But a continuous map clearly maps bounded sets to bounded sets.

\[\square\]

**Proposition 2.6.** Let \( M \) be a Banach manifold and \( V, W \) separated LCVS. If \( f: M \to V \) is analytic and \( \varphi: V \to W \) continuous and linear, then \( \varphi \circ f: M \to W \) is also analytic.
Proof. We can assume that $M$ is a Banach space. Let $x \in M$ be arbitrary and let $f_{x,n} : M \rightarrow V$ be continuous maps, homogeneous of degree $n$, such the family $(f_{x,n})$ is a convergent power series and that for all $h$ sufficiently close to zero we have an equality

$$f(x + h) = \sum_{n} f_{x,n}(h).$$

It suffices to show that the family $(\varphi \circ f_{x,n})$ is a convergent power series. By continuity of $\varphi$, for every continuous seminorm $p$ on $W$ we can find a $\lambda_p \in \mathbb{R}$ such that

$$\rho(\varphi(y)) \leq \lambda_p \|y\|.$$

Let $R$ be the radius of convergence of $(f_{x,n})$ and $p$ a continuous seminorm on $W$. Then

$$\sup_{n \in \mathbb{N}_0} \sup_{h \in E, \|h\| \leq 1} R^n p(\varphi(f_{x,n}(h))) \leq \lambda_p \sup_{n \in \mathbb{N}_0} \sup_{h \in E, \|h\| \leq 1} R^n \|f_{x,n}(h)\| < \infty \square$$

3. Duality for Lie Algebras

For the general theory of Lie algebras and Lie groups we refer to [Ser92; Bou89]. In this section, we fix a complete non-archimedean field $K$ of characteristic zero and a Lie group $G$ over $K$. We also consider its attached Lie algebra $g$ with Lie bracket $[\cdot, \cdot]$. The adjoint action of $G$ on $g$ by differentiating conjugation maps will be denoted by $\text{Ad}(-)$, the adjoint action of $g$ on itself given by $x \mapsto [x, -]$ will be denoted by $\text{ad}(-)$.

**Definition 3.1.** For a $g$-module $M$ we define the Chevalley-Eilenberg complex

$$C^\bullet(g, V) = \text{Hom}(\Lambda^\bullet g, V)$$

concentrated in non-negative degrees by considering the differential

$$d : C^n(g, V) \rightarrow C^{n+1}(g, V)$$

given by

$$df(x_1 \wedge \cdots \wedge x_{n+1}) = \sum_i (-1)^{i+1} x_i f(x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x + n + 1)$$

$$+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{n+1}).$$

As usual, $\widehat{x_i}$ means omitting $x_i$ etc.

**Definition 3.2.** Let $V$ be a $g$-module. Define $V^{tw}$ as the vector space $V$ with a $g$-action given by

$$x^{tw} v = xv - \text{Tr}(\text{ad}(x))v,$$

where $\text{Tr}$ is the trace map.

**Proposition 3.3.** For $V \neq 0$, $V^{tw} = V$ if and only if $H^{\dim g}(g, K) \neq 0$. If $g$ is abelian or nilpotent, $V^{tw} = V$.

**Proof.** [Haz70, corollary 2] \square

In applications, this is very often the case.

**Proposition 3.4.** If $G$ is compact, then $V^{tw} = V$. 

Proof. As for a compact group $G$, the left and right Haar measures coincide, \cite[section III.3.16]{Bou89} implies that
\[
\det \text{Ad}_g = 1
\]
for all $g \in G$. By \cite[section III.4.5]{Bou89}, we see that for all $x$ in a neighbourhood of zero of $g$
\[
\text{Ad}(\phi(x)) = \exp(\text{ad} x)
\]
where $\phi$ is a local exponential map from this neighbourhood into $G$. Here, $\exp$ is the usual exponential map of $K$ extended to matrices. Applying the determinant, we see that
\[
1 = \det \exp(\text{ad} x) = \exp(\text{Tr} \text{ad} x)
\]
so $\text{Tr} \text{ad} x = 0$ in a neighbourhood of the identity. Choosing a basis of $g$ in this
neighbourhood, we see that indeed
\[
V^{\text{tw}} = V.
\]
\hspace{1cm} $\square$

Remark 3.5. The argument of proposition 3.4 shows that if $\text{Tr} \text{ad} x = 0$ for all $x \in g$, then $\det \text{Ad}(g) = 1$ for all $g$ in a neighbourhood of the identity. If $G$ is a connected Lie group over $\mathbb{R}$ or $\mathbb{C}$, then $\det \text{Ad} g = 1$ for all $g \in G$. \cite[section III.3.16]{Bou89} then implies that the left and right Haar measures of $G$ coincide.

Definition 3.6. For any natural number $n$ we denote by $[n]$ the ordered set $[n] = \{1, \ldots, n\}$. For an injective morphism of ordered sets $\phi: [k] \longrightarrow [d]$ there exists a unique morphism of ordered sets
\[
\phi^*: [d - k] \longrightarrow [d]
\]
such that $[d] = \text{im} \phi \cup \text{im} \phi^*$. We then define
\[
\text{sgn} \phi = (-1)^{\sum_{i=1}^{d-k} \phi^*(i)}.
\]

Remark 3.7. It is easy to see that for an injective morphism of ordered sets $\phi: [k] \longrightarrow [d]$ the following holds:
\[
\text{sgn}(\phi) \cdot \text{sgn}(\phi^*) = (-1)^{k(d-k)},
\]
 cf. \cite[lemma 10.17]{Tho20}.

Proposition 3.8. Let $M$ be a finite dimensional vector space with basis $e_1, \ldots, e_d$. For an injective morphism of ordered sets $\phi: [k] \longrightarrow [d]$ define
\[
e_\phi = e_{\phi(1)} \wedge \cdots \wedge e_{\phi(k)} \in \wedge^k M.
\]
Also define the $K$-linear isomorphism
\[
*: \wedge^k M \longrightarrow \wedge^{d-k} M
\]
given by
\[
*e_\phi = \text{sgn}(\phi^*) e_\phi^*.
\]
Then for any invertible endomorphism $A$ of $M$ the following holds:
\[
\det A \cdot ((A^{-1})^t \circ *) = * \circ A.
\]

Proof. This is a straight-forward piece of linear algebra, but we could not find a reference for $k \neq 1$.
Let $\phi, \psi: [k] \longrightarrow [d]$ be injective maps of ordered sets. For a matrix $A$ denote by $A_{\phi, \psi}$ the matrix with entries $(a_{\phi(i), \psi(j)})_{i,j \in [k]}$. Now a straight forward calculation (or \cite[proposition 9 in III.8.5]{Bou98}) shows that for fixed $\psi$, we have
\[
(\star) \quad A e_\psi = \sum_{\phi} (\det A_{\phi, \psi}) e_\phi,
\]
where \( \phi \) ranges over the injective maps of ordered sets \([k] \to [d]\). We hence also get

\[
(A^{-1})^t \ast \psi = \text{sgn}(\psi^*) \sum_{\phi^*} \det((A^{-1})^t_{\phi^*, \psi^*}) e_{\phi^*},
\]

where \( \phi^* \) ranges over the injective maps of ordered sets \([d - k] \to [d]\). Applying \( \ast \) to \((\bullet)\), we are reduced to showing

\[
\text{sgn}(\psi^*) \cdot \det(A) \cdot \det(A^{-1})^t_{\phi^*, \psi^*} = \text{sgn}(\phi^*) \cdot \det A_{\phi, \psi}.
\]

For \( k = 1 \), this is precisely Cramer’s rule. Generally, for a matrix \( B \), the submatrix \( B_{\phi, \psi} \) can be considered as a linear map from the span of \( e_{\phi(1)}, \ldots, e_{\phi(k)} \) to the span of \( e_{\psi(1)}, \ldots, e_{\psi(k)} \). Denote this linear map by \( B_{\phi, \psi}^{\text{ext}} \). Define \( B_{\phi, \psi}^{\text{ext}} \) via

\[
B_{\phi, \psi}^{\text{ext}} e_{\phi(i)} = B_{\phi, \psi}^{\text{res}} e_{\phi(i)},
\]

\[
B_{\phi, \psi}^{\text{ext}} e_{\psi(i)} = e_{\psi^*(i)}.
\]

It is clear that \( \det B_{\phi, \psi}^{\text{ext}} = \varepsilon \det B_{\phi, \psi} \), with \( \varepsilon = (-1)^\sum_i \phi^*(i) + \psi^*(i) \) so

\[
\varepsilon \det(B_{\phi, \psi}) \cdot e_1 \wedge \cdots \wedge e_n = B_{\phi, \psi}^{\text{ext}} e_1 \wedge \cdots \wedge B_{\phi, \psi}^{\text{ext}} e_n = \text{sgn}(\phi^*) \cdot B_{\phi, \psi}^{\text{res}} e_{\phi} \wedge e_{\psi^*}.
\]

By anticommutativity of the exterior algebra, we see that indeed

\[
B_{\phi, \psi}^{\text{res}} e_{\phi} \wedge e_{\psi^*} = B e_{\phi} \wedge e_{\psi^*}.
\]

If \( B \) is invertible, we can apply \( B^{-1} \) and get

\[
\varepsilon \det(B_{\phi, \psi}) \cdot \det(B^{-1}) \cdot e_1 \wedge \cdots \wedge e_n = \text{sgn}(\phi^*) \cdot e_{\phi} \wedge B^{-1} e_{\psi^*} = \text{sgn}(\phi^*) (-1)^{(d-k)B^{-1}} e_{\psi^*} \wedge e_{\phi}.
\]

Applying \((\bullet)\) to \((B^{-1})_{\psi^*, \phi^*}\), we find that

\[
B^{-1} e_{\psi^*} \wedge e_{\phi} = \varepsilon' \cdot \text{sgn}(\psi^*) \cdot \det((B^{-1})_{\psi^*, \phi^*}) \cdot e_1 \wedge \cdots \wedge e_n
\]

with \( \varepsilon' = (-1)^{\sum_i \phi^*(i) + \psi^*(i)} \). As clearly \( \varepsilon \varepsilon' = 1 \), we get

\[
\det(B_{\phi, \psi}) \cdot \det(B^{-1}) = \text{sgn}(\phi^*) (-1)^{(d-k)} \cdot \text{sgn}(\psi^*) \cdot \det((B^{-1})_{\psi^*, \phi^*}),
\]

and using remark 3.7, this becomes

\[
\text{sgn}(\phi^*) \cdot \det(B_{\phi, \psi}) \cdot \det(B^{-1}) = \text{sgn}(\psi^*) \cdot \det((B^{-1})_{\psi^*, \phi^*}),
\]

which is exactly what we needed to show. \( \square \)

**Theorem 3.9.** Let \( V \) be a LCVS with a continuous action by \( g \). If \( g \) if of dimension \( d \), and if \( \text{Tr}(\text{ad}(x)) = 0 \) for all \( x \in g \), then there is a \( G \)-equivariant functorial isomorphism of complexes

\[
\mathfrak{c}^*(g, V^t) \cong \mathfrak{c}^*(g, V)^t[-d].
\]

**Proof.** In [Haz70], Hazewinkel shows (without the restriction \( \text{Tr} \circ \text{ad} = 0 \)) that as abstract vector spaces

\[
\mathfrak{c}^*(g, (V^{\text{tw}})^*) \cong \mathfrak{c}^*(g, V)^*[-d],
\]

where \(( - )^* = \text{Hom}_K(-, K) \). While it is easy to check that the isomorphism respects continuous maps, it is not immediate at all that it is \( G \)-equivariant. The proof itself is a brutal calculation.
Choose a basis \((e_i)\) of \(\mathfrak{g}\) and define the star operator as in proposition 3.8.

Hazewinkel’s isomorphism stems from the following pairing:

\[
\langle - , - \rangle \colon \text{Hom}_K(\Lambda^k \mathfrak{g}, V') \times \text{Hom}_K(\Lambda^{d-k} \mathfrak{g}, V) \longrightarrow K
\]

\[
(a, b) \longmapsto (a, b) = \sum_{\phi} a(e_{\phi})(b(\ast e_{\phi}))
\]

We need to show that

\[
\langle g a, b \rangle = \langle a, g^{-1} b \rangle
\]

for all \(g \in G\). Write \(A\) for \(\text{Ad}(g)\). Then

\[
\langle g x, y \rangle = \sum_{\phi} (g x)(e_{\phi})(y(\ast e_{\phi})) = \sum_{\phi} (g x(A^{-1} e_{\phi}))(y(\ast e_{\phi})) = \sum_{\phi} x(A^{-1} e_{\phi})(g^{-1} y(\ast e_{\phi}))
\]

and

\[
\langle x, g^{-1} y \rangle = \sum_{\phi} x(e_{\phi})(g^{-1} y(A \ast e_{\phi})).
\]

In both cases, \(\phi\) runs over the injective increasing maps \([k] \longrightarrow [d]\).

By considering the finite dimensional subspace of \(V\) generated by all \(y(\ast e_{\phi})\) and their images under \(g^{-1}\), we can consider a finite dimensional vector space instead, i.e.,

\[
\langle g x, y \rangle = \sum_{i} (A^{-1} e_i')^t X^t G^{-1} Y \ast e_i'
\]

and

\[
\langle x, g^{-1} y \rangle = \sum_{i} e_i'' Y A \ast e_i'
\]

for appropriate matrices \(X, G^{-1}, Y, \ast\) and \((e_i')\) the canonical basis of \(K^{(d)}\). (The matrix \(G^{-1}\) will not be invertible in general, even though the notation does suggest this.) As \(A \ast = \ast (A^{-1})^t\) by propositions 3.4 and 3.8 equality follows, as the trace is invariant under cyclic permutations. □

4. Tamme’s Comparison Results

We will summarise the results from [Tam15] which we need as follows:

**Theorem 4.1.** Let \(K\) be a complete non-archimedean field of characteristic zero. Let \(G\) be a Lie group over \(K\) and \(V\) a barrelled LCVS with an analytic action of \(G\). Then there is a functorial morphism

\[
C^\ast(G, V) \longrightarrow \mathfrak{c}^\ast(\mathfrak{g}, V)
\]

from the analytic cochains of \(G\) with coefficients in \(V\) to the Chevalley-Eilenberg complex of the Lie algebra \(\mathfrak{g}\) of \(G\) with coefficients in \(V\).

For an open subgroup \(U \leq G\), we denote its Lie algebra by \(\mathfrak{g}(U)\). Above morphism induces for all \(n\) isomorphisms

\[
\lim_{U \leq G} H^n(U, V) \cong \lim_{U} H^n(\mathfrak{g}(U), V) = H^n(\mathfrak{g}, V).
\]

The adjoint action of \(G\) on \(\mathfrak{g}\) together with the action of \(G\) on \(V\) induce an action of \(G\) on the Chevalley-Eilenberg complex and on the Lie algebra cohomology groups. If \(G\) is compact, then above morphism of complexes induces an isomorphism

\[
H^n(G, V) \cong H^n(\mathfrak{g}, V)^G
\]

for all \(n\).

**Proof.** [Tam15], sections 3-5] □
5. The Duality Theorem

Lemma 5.1. Let \( G \) be a finite group acting linearly on an \( L \)-vector space \( V \). If the order of \( G \) is invertible in \( L \), the composition of the canonical inclusion and projection

\[
\begin{array}{ccc}
V^G & \longrightarrow & V \\
& \longrightarrow & V_G \\
\end{array}
\]

is an isomorphism.

Proof. By Maschke’s theorem, \( L[G] \) is a semisimple ring. Therefore there exists an \( L[G] \)-submodule \( W \) of \( V \) with \( V = V^G \oplus W \). Without loss of generality we can assume that \( W \) is irreducible. Denote by \( I \) the augmentation ideal in \( L[G] \). Then

\[
V_G = V^G/IV^G \oplus W/IW = V^G \oplus W/IW
\]

and as \( W \) is irreducible, \( IW \) is either 0 or \( W \). If \( IW = 0 \), then \( W \subseteq V^G \) and hence \( W = 0 \) by assumption, so \( W/IW = 0 \) in any case. \( \square \)

Fix now a complete non-archimedean field \( K \) of characteristic zero and a Lie group \( G \) over \( K \), which acts equi-analytically on an LCVS \( V \).

Lemma 5.2. Let \( R \) be a \( K \)-algebra. Assume that \( V \) carries the structure of an \( R \)-module, such that the operation of \( G \) on \( V \) is \( R \)-linear. If \( \text{H}^1(\mathfrak{g}, V) \) is finitely generated over \( R \), then there is an open subgroup of \( G \) which acts trivially on \( \text{H}^1(\mathfrak{g}, V) \).

Proof. By theorem 4.1,

\[
\lim_{\rightarrow U \leq G, \text{res}} \text{H}^i(U, V) = \text{H}^i(\mathfrak{g}, V),
\]

which is \( R \)-linear by our assumptions. Taking preimages of the finitely many generators in \( \text{H}^1(\mathfrak{g}, V) \), we see that there is an open subgroup \( U \leq G \) such that \( \text{H}^i(U, V) \longrightarrow \text{H}^i(\mathfrak{g}, V) \) is surjective. This \( U \) then operates trivially on \( \text{H}^i(\mathfrak{g}, V) \). \( \square \)

Theorem 5.3. If \( G \) is compact and \( V, V'_b \) barrelled, we get a functorial (in \( V \)) morphism of complexes

\[
\begin{array}{ccc}
\mathcal{C}^\bullet(G, V'_b) & \longrightarrow & \text{Hom}_K(\mathcal{C}^\bullet(G, V), K)[-d].
\end{array}
\]

If one of the following two conditions is satisfied:

- An open subgroup of \( G \) operates trivially on the Lie algebra cohomology, the differentials in the Chevalley-Eilenberg complex are strict and Hahn-Banach holds for \( V \), or
- \( V \) is finite-dimensional,

then this morphism induces isomorphisms

\[
\text{H}^i(G, V'_b) \cong \text{H}^{d-i}(G, V)'.
\]

Proof. Note first that by lemma 5.2 an open subgroup of \( G \) operates trivially on the Lie algebra cohomology, no matter the case.

By theorem 4.1 we have morphisms

\[
\begin{array}{ccc}
\mathcal{C}^\bullet(G, V'_b) & \longrightarrow & \mathcal{C}^\bullet(\mathfrak{g}, V'_b)
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Hom}_K(\mathcal{C}^\bullet(\mathfrak{g}, V), K) & \longrightarrow & \text{Hom}_K(\mathcal{C}^\bullet(G, V), K).
\end{array}
\]
As $G$ is compact, we can employ theorem 3.9 without having to twist the Lie algebra action (cf. proposition 3.4). We therefore get a $G$-equivariant isomorphism
\[ C^\bullet(g, V'_b) \cong \text{Hom}_{K,cts}(C^\bullet(g, V), K)[-d]. \]
Composition with the inclusion
\[ \text{Hom}_{K,cts}(C^\bullet(g, V), K) \subseteq \text{Hom}_K(C^\bullet(g, V), K) \]
then yields the comparison morphism, which is clearly functorial in $V$. If the differentials in the complex $C(g, V)$ are strict and Hahn-Banach holds for $V$, we get a $G$-equivariant isomorphism on the level of cohomology:
\[ H^i(g, V'_b) \cong \text{Hom}_{K,cts}(H^{d-i}(g, V), K). \]
Especially we get the following commutative diagram:
\[
\begin{array}{ccc}
H^i(G, V'_b) & \cong & H^i(g, V'_b) \\
\cong & \uparrow & \cong \\
H^i(g, V'_b)^G & \cong & (H^{d-i}(g, V)^G)' \\
\end{array}
\]
The dashed isomorphisms are again instances of theorem 4.1. If an open subgroup of $G$ acts trivially on the Lie algebra cohomology, then the composition
\[ (H^{d-i}(g, V)^G)' \longrightarrow (H^{d-i}(g, V))' \longrightarrow (H^{d-i}(g, V)^G)' \]
is an isomorphism by lemma 5.1 and the claim follows.

**Corollary 5.4.** Let $G$ be a compact Lie group of dimension $d$ acting analytically on a finite dimensional $K$-vector space $V$. Then we have a functorial quasi-isomorphism
\[ C^\bullet(G, V') \longrightarrow C^\bullet(G, V')^*[-d]. \]

**Proof.** By proposition 2.4, we are in the setting of theorem 5.3. If $V$ is finite-dimensional, we see that $C(g, V')$ is a complex of finite dimensional vector spaces and the analytic cohomology groups are therefore finite-dimensional as well by theorem 4.1. For all cohomology groups involved, their abstract duals hence coincide with their continuous duals and the result follows.

**Remark 5.5.** Functoriality in theorem 5.3 means the following: Let $V, W$ be LCVS with equi-analytic actions of $G$ on them. Assume that $V, W, V'_b, W'_b$ are all barrelled.

Given a $G$-equivariant continuous linear map $\varphi: V \longrightarrow W$, we get a commutative diagram:
\[
\begin{array}{ccc}
C^\bullet(G, W'_b) & \longrightarrow & \text{Hom}_K(C^\bullet(G, W), K)[-d] \\
\downarrow & & \downarrow \text{Hom}_K(C^\bullet(G, \varphi), K) \\
C^\bullet(G, V'_b) & \longrightarrow & \text{Hom}_K(C^\bullet(G, V), K)[-d] \\
\end{array}
\]
That the maps involved are well-defined follows from lemma 2.5 and proposition 2.6.

**Remark 5.6.** Of course a quasi-isomorphism would be a nicer result in the setting of theorem 5.3. The obvious strategy would be topologise $C^\bullet(G, V)$ in such a manner that the differentials are strict and that the cohomology groups are topologically identical to the topology on the Lie algebra cohomology. The same argument as above would then (under the additional hypotheses on the Chevalley-Eilenberg complex and the Lie algebra cohomology) yield a quasi-isomorphism
\[ C^\bullet(G, V') \longrightarrow \text{Hom}_{K,cts}(C^\bullet(G, V), K). \]
We consider this endeavour to be rather futile, which is the reason we axiomatised and capsuled topological considerations in [Tho20] in the first place.

**Remark 5.7.** If $K = \mathbb{Q}_p$ and $V$ is a finite-dimensional $\mathbb{Q}_p$-vector space, then by [Laz65, V.(2.3.10)] analytic cohomology is just continuous cohomology. Theorem 5.3 is then a possible way to phrase Poincaré duality, which however does not coincide with Poincaré duality due to Lazard (cf. [Laz65, V.(2.5.8)]). Poincaré duality there is an integral phenomenon and the dual is given by $\text{Hom}_{\mathbb{Z},\text{cts}}(V, \mathbb{Q}_p/\mathbb{Z}_p)$.

**Example 5.8.** Let $V$ be any barrelled LCVS and $G$ a compact abelian Lie group over $K$ of dimension $d$. The trivial action of $G$ on $V$ is of course equi-analytic. The Lie algebra $\mathfrak{g}$ of $G$ then operates by zero on $V$. The differentials in the Chevalley-Eilenberg complex $\mathfrak{c}^\bullet(\mathfrak{g}, V)$ are all zero. Theorem 4.1 then yields that $H^i(G, V) \cong H^i(\mathfrak{g}, V) = \text{Hom}_K(\wedge^i \mathfrak{g}, V)$ and the isomorphism $H^i(G, V') \cong H^{d-i}(G, V')$ of theorem 5.3 stems from the pairing

\[ \wedge: \wedge^i \mathfrak{g} \times \wedge^{d-i} \mathfrak{g} \longrightarrow K. \]

6. Two Applications to $(\varphi, \Gamma)$-modules

Analytic cohomology as it appears in the theory of $(\varphi, \Gamma)$-modules has mostly had a very strong ad hoc flavour. Arguments often used crucially that $\Gamma$ is a one-dimensional Lie group and $\varphi$ a single operator. However, our framework of [Tho20] and our results of the previous section are much more flexible and easily apply themselves to higher-dimensional $\Gamma$ and multiple operators $\varphi_1, \ldots, \varphi_d$. The natural objects to look at are thus multivariable $(\varphi, \Gamma)$-modules. There is, however, no unified notion of multivariable $(\varphi, \Gamma)$-modules and to our knowledge, no definition of multivariable Lubin-Tate $(\varphi, \Gamma)$-modules has been published. Consequently many results which are well known in the univariate case are unknown to hold in the multivariable case. Our arguments, not relying on the ad hoc constructions of analytic cohomology, should easily carry over to the multivariable case as soon as the necessary category equivalences are shown. An important step towards this has recently been done by [PZ19].

6.1. An Exact Sequence of Berger and Fourquaux. We start by improving a result of Berger and Fourquaux, cf. [BF17, theorem 2.2.4], which can be stated without precisely defining $(\varphi, \Gamma)$-modules.

Let $F|\mathbb{Q}_p$ be a local field and consider the category $\mathbf{M}$ of analytic $F$-manifolds.

We fix an LF-space $A \cong \lim \lim A^{[r,s]}$ with Banach spaces $A^{[r,s]}$ for the remainder of this subsection (cf. definition 1.35). The notation $A^{[r,s]}$ will become apparent in the next subsection.

**Definition 6.1.** For $X \in \mathbf{M}$ let $f: X \longrightarrow A$ be a continuous map. We call $f$ pro-$F$-analytic, if there exists an $r$ and a factorisation

\[ f: X \longrightarrow \lim_{s \rightarrow} A^{[r,s]} \longrightarrow A, \]

such that all induced maps

\[ X \longrightarrow \lim_{s \rightarrow} A^{[r,s]} \]

\[ \Downarrow A^{[r,s]} \]




are locally $F$-analytic. We denote the set of pro-$F$-analytic maps from $X$ to $A$ by $h(X, A)$, i.e.,
\[ h(X, A) = \lim_{\longrightarrow} \lim_{\longleftarrow} \text{h\textsubscript{an}}(X, A^{[r,s]}), \]
where $\text{h\textsubscript{an}}$ denote the analytic maps in the sense of definition 1.23.

**Proposition 6.2.** An $F$-analytic map into a Fréchet space in the sense of definition 1.24 is pro-$F$-analytic.

**Proof.** Let $B = \lim_{\longleftarrow} B_n$ be a Fréchet space with all $B_n$ Banach. We need to show that if $f: M \rightarrow B$ is analytic, then so is $f: M \rightarrow B$ for each $n$. But this is precisely the content of proposition 2.6.

**Remark 6.3.** The argument in proposition 2.6 shows why not every pro-analytic map needs to be analytic: For a pro-analytic map $M \rightarrow \lim_{\longleftarrow} B_n$ (and around a fixed point in $M$), we have a positive radius of convergence $R_n$ of the power series development for every $B_n$. But there is no need for $\inf_n R_n$ to be positive, which is the natural estimate for the radius of convergence for $B$.

Let $\Gamma$ be an analytic group over $F$ and $A$ an LF-space an action from $\Gamma$ by pro-$F$-analytic maps and a continuous $L$-linear endomorphism (which we will call $\psi$) of $A$. $A$ is then a $G = \psi^{\mathbb{N}_0} \times \Gamma$-module. By setting $C^n(G, A) = h(G^n, A)$ and using the usual inhomogeneous cochain differential, we can define the pro-$F$-analytic cohomology of $G$ with coefficients in $A$ as the cohomology of this complex. This yields a well-defined cohomology theory that exhibits many of the standard features, cf. [Tho20].

We now prove the following stronger version of [BF17, theorem 2.2.4], where only the case of one-dimensional $\Gamma$ is considered. For the one-dimensional case, we can also show that the last map appearing in the exact sequence of Berger and Fourquaux is surjective.

**Theorem 6.4.** There is an exact sequence of pro-$F$-analytic cohomology groups:
\[
\begin{align*}
0 \rightarrow & H^1(\Gamma, A^{\psi=1}) \rightarrow H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \rightarrow (A/(\psi - 1)A)^\Gamma \\
& \leftarrow H^2(\Gamma, A^{\psi=1}) \rightarrow H^2(\psi^{\mathbb{N}_0} \times \Gamma, A)
\end{align*}
\]
The second to last group can be replaced by zero if $\Gamma$ is compact, has dimension one over $L$, and also operates analytically on each $\lim_{\longleftarrow} A^{[r,s]}$.

**Proof.** In lieu of [Tho20 theorem 10.26] it only remains to show the surjectivity onto $(A/(\psi - 1)A)^\Gamma$ for compact one-dimensional $\Gamma$. Note that we can’t directly use theorem 4.1 to compare this cohomology group to Lie algebra cohomology, as pro-analytic maps and analytic maps don’t necessarily coincide.

By definition 6.1 and the exactness of direct limits we can assume that $A$ is Fréchet. As every analytic map is also pro-analytic, the same proof as for [Tho20 proposition 7.1] together with [Tho20 theorem 10.26] shows that we have the following commutative diagram with exact rows:
\[
\begin{align*}
0 \rightarrow & H^1(\Gamma, A^{\psi=1}) \rightarrow H^1(\psi^{\mathbb{N}_0} \times \Gamma, A) \rightarrow (A/(\psi - 1)A)^\Gamma \rightarrow H^2(\Gamma, A^{\psi=1}) \\
& \uparrow \uparrow \uparrow \uparrow \\
0 \rightarrow & H^1_{\text{an}}(\Gamma, A^{\psi=1}) \rightarrow H^1_{\text{an}}(\psi^{\mathbb{N}_0} \times \Gamma, A) \rightarrow (A/(\psi - 1)A)^\Gamma \rightarrow H^2_{\text{an}}(\Gamma, A^{\psi=1})
\end{align*}
\]
Here, \( H^\bullet_{an} \) denotes cohomology with respect to analytic maps in the sense of definition \[ 1.2.4 \] for which the results of \[ 1.3.2 \] also hold. For these cohomology groups, we can apply theorem \[ 4.1 \] to see that

\[ H^2_{an}(\Gamma, A^{\psi=1}) = 0. \]

The exact sequence

\[ 0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^N \times \Gamma, A) \longrightarrow (A/(\psi - 1)A)^\Gamma \longrightarrow 0 \]

follows at once. \( \square \)

**Remark 6.5.** Assume that \( \Gamma \) is compact and of dimension one and that \( A = \lim_{n \to \infty} A_n \) is Fréchet. Then even if the operation on \( A \) is only pro-analytic, the same argument as in theorem \[ 6.4 \] yields exact sequences

\[ 0 \longrightarrow H^1_{an}(\Gamma, A^{\psi=1}) \longrightarrow H^1_{an}(\psi^N \times \Gamma, A_n) \longrightarrow (A_n/(\psi - 1)A_n) \longrightarrow 0 \]

for every \( n \). Assume furthermore that the image of \( A_m \) in \( A_n \) is dense for every \( n \). We then have isomorphisms

\[ H^1(\psi^N \times \Gamma, A) \cong \lim_{\chi \to \infty} H^1_{an}(\psi^N \times \Gamma, A_n) \]

by \[ 3.5.1 \] proposition 2.1.1. As taking invariants commutes with projective limits, we therefore get the following exact sequence:

\[ 0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^N \times \Gamma, A) \longrightarrow (A/(\psi - 1)A)^\Gamma \longrightarrow \lim_{\chi \to \infty} H^1_{an}(\Gamma, A^{\psi=1}) \]

Using again theorem \[ 4.1 \] we see that \( H^1_{an}(\Gamma, A^{\psi=1}) \cong H^1(\mathfrak{g}, A^{\psi=1})^\Gamma \), where \( \mathfrak{g} \) is the Lie algebra of \( \Gamma \). The action of \( \Gamma \) on \( \mathfrak{g} \) is trivial and \( \mathfrak{g} \cong L \), so \( H^1(\mathfrak{g}, A^{\psi=1})^\Gamma \) has a comparatively simple description as the \( \Gamma \)-invariants of certain quotients of \( A^{\psi=1} \), which depend on the precise group action, cf. e.g. \[ 5.5.1 \] theorem 7.4.7. For these it might in certain examples be possible to show the (topological) Mittag-Leffler condition, cf. e.g. \[ 4.6.1 \] (0.13.2.4)], and hence show that

\[ 0 \longrightarrow H^1(\Gamma, A^{\psi=1}) \longrightarrow H^1(\psi^N \times \Gamma, A) \longrightarrow (A/(\psi - 1)A)^\Gamma \longrightarrow 0 \]

is exact.

**Remark 6.6.** Considering remark \[ 6.5 \] it is natural to ask for the relationship between \( H^k(\Gamma, A) \) and \( \lim_{\chi \to \infty} H^k(\Gamma, A_n) \), where \( A = \lim_{\chi \to \infty} A_n \) is again assumed to be Fréchet.

Consider the exact sequence

\[ 0 \longrightarrow C^\bullet(\Gamma, A) \longrightarrow \prod_n C^\bullet_{an}(\Gamma, A_n) \xrightarrow{1-u} \prod_n C^\bullet_{an}(\Gamma, A_n) \longrightarrow \lim_{\chi \to \infty} C^\bullet_{an}(\Gamma, A_n) \longrightarrow 0, \]

whose middle map is given by

\[ 1 - u : (f_n)_n \longrightarrow (f_n - (A_{n+1} \to A_n) \circ f_{n+1})_n. \]

Its existence follows from very general arguments, cf. \[ 6.10.4 \] (2.4.7)] for a correct statement and proof.

If one could show that \( 1 - u \) is indeed surjective, then the long exact sequence of cohomology would yield the following short exact sequences for every \( k \):

\[ 0 \longrightarrow \lim_{\chi \to \infty} H^{k-1}_{an}(\Gamma, A_n) \longrightarrow H^k(\Gamma, A) \longrightarrow \lim_{\chi \to \infty} H^k_{an}(\Gamma, A_n) \longrightarrow 0 \]

Write \( d = \dim_{\mathbb{F}} \Gamma \). Then these short exact sequences would show that \( H^k(\Gamma, A) = 0 \) for every \( k > d + 1 \) and that \( H^{d+1}(\Gamma, A) \cong \lim_{\chi \to \infty} H^d_{an}(\Gamma, A_n) \). Especially, the considerations in theorem \[ 6.3 \] and remark \[ 6.5 \] would coincide for \( d = 1 \).
6.2. So Many Rings. Fix a complete field \( L \subseteq \mathbb{C}_p \). We mostly follow the notation of [BF17].

**Definition 6.7.** Consider the abelian group \( \text{Map}(\mathbb{Z},L) = L[[X,X^{-1}]] \). In this set, we can define the following rings. Let \( I \subset [0,1] \) be an interval. Set

\[
B^I_L = \left\{ \sum_{i \in \mathbb{Z}} a_i X^i \in L[[X,X^{-1}]] \mid \text{convergent for } z \in \mathbb{C}_p, |z| \in I \right\}.
\]

For \( I = [r,s] \), this is a Banach space over \( L \) with norm \( \| \cdot \|_{B^I_L} = \max\{\| \cdot \|_r, \| \cdot \|_s\} \), where \( \| \cdot \|_r \) and \( \| \cdot \|_s \) are defined via

\[
\left\| \sum_{i \in \mathbb{Z}} a_i X^i \right\|_t = \sup_{i \in \mathbb{Z}} |a_i| t^i.
\]

We also define

\[
B^I_L^{\dagger} = \lim_{\to} B^I_L \quad \text{for } 0 < r < 1,
\]

\[
B^I_L^{-} = \lim_{\to} B^I_L^{-} \quad \text{for } 0 < s < 1.
\]

For \( L \) finite over \( \mathbb{Q}_p \), we can also define the complete discrete valuation ring

\[
A_L = \mathcal{O}_L[[X]][X^{-1}]^\wedge
\]

with quotient field

\[
B_L = A_L[p^{-1}],
\]

where \( -^\wedge \) denotes \( p \)-adic completion. We also define

\[
B^I_L = \{ f \in B_L \mid f \text{ has a non-empty domain of convergence} \}.
\]

If \( M|L \) is a finite extension, the theory of the field of norms provides a certain ring extension \( A_{M|L} \) over \( A_L \). Its quotient field \( B_{M|L} \) is an unramified extension of \( B_L \).

We define the following complete discrete valuation ring and its quotient field:

\[
A = \bigcup_{M|L \text{ finite}} A_{M|L},
\]

\[
B = \bigcup_{M|L \text{ finite}} B_{M|L}.
\]

**Remark 6.8.** Some authors denote the rings \( A_{M|L} \) and \( B_{M|L} \) simply by \( A_M \) and \( B_M \), obfuscating the fact that these rings are relative notions: In our notation, we always have \( A_{M|M} = A_M \) but generally \( A_{M|L} \neq A_M \). We consider this abuse of notation in the literature truly abusive.

6.3. Lubin-Tate \((\varphi, \Gamma)\)-modules. Fix a finite Galois extension \( L|\mathbb{Q}_p \) for the remainder of this chapter. We denote the ring of integers of \( L \) by \( \mathcal{O}_L \) and its residue field of cardinality \( q \) by \( \kappa_L \). We also fix a uniformiser \( \pi \) of \( \mathcal{O}_L \).

6.3.1. The Lubin-Tate Case. We assume familiarity with the theory of formal multiplication in local fields, cf. e.g. [Ser67 section 3].

Denote by \( LT \) the Lubin-Tate formal \( \mathcal{O}_L \)-module attached to \( \pi \), i.e., as a set \( LT \) is the maximal ideal of the integral closure of \( \mathcal{O}_L \) in an algebraic closure of \( L \) and the addition is defined via the unique formal group law corresponding to the endomorphism

\[
[\pi](T) = T^q + \pi T.
\]
Lubin-Tate theory then yields commuting power series \([a](T) \in \mathcal{O}[[T]]\) for all \(a \in \mathcal{O}_L\), which give rise to a \(\mathcal{O}_L\)-module structure on \(LT\). It also yields a homomorphism

\[
\chi_{LT}: G_L \longrightarrow \mathcal{O}_L^\times
\]

which induces an isomorphism

\[
\chi_{LT}: \Gamma_L = G(L_\infty|L) \overset{\cong}{\longrightarrow} \mathcal{O}_L^\times,
\]

where \(L_\infty\) is the extension of \(L\) generated by all \(\pi^\infty\)-torsion points of \(LT\).

For \(f(T)\) in any of the rings \(B^{\dagger}_{\text{rig}, L}, A^\dagger_L, B^\dagger_L\) we have well-defined elements

\[
\varphi(f)(T) = f([\pi](T)),
\]

\[
(gf)(T) = f([\chi_{LT}(g)](T)), \ g \in \Gamma_L.
\]

Denote the monoid \(\varphi^\text{N0}\) by \(\Phi\). Then by construction above formula induce a continuous action of \(\Phi \times \Gamma_L\) by ring homomorphisms on each of the above rings with their respective topologies, which for \(B^{\dagger}_{\text{rig}, L}\) is even pro-\(L\)-analytic cf. e.g. [Ber16, theorem 8.1].

**Definition 6.9.** Let \(R\) be either of \(B^{\dagger}_{\text{rig}, L}, B^\dagger_L, A^\dagger_L, B^\dagger_L\). A \((\varphi, \Gamma_L)\)-module \(M\) over \(R\) is a free \(R\)-module of finite rank with a semi-linear continuous action of \(\Phi \times \Gamma_L\). It is called étale if \(\varphi(M)\) generates \(M\).

### 6.3.2. Relation to Iwasawa Cohomology

Recall the following result due to Kisin and Ren.

**Theorem 6.10** ([KR09, theorem 1.6]). The functor

\[
V \longrightarrow D_\mathcal{O}(V) = (A \otimes_{\mathcal{O}_L} V)^{G(\mathcal{O}_L|L_\infty)}
\]

establishes an equivalence between the categories of \(\mathcal{O}_L\)-linear representations of \(G_L\) and étale \((\varphi, \Gamma_L)\)-modules over \(A^\dagger_L\).

For any étale \((\varphi, \Gamma_L)\)-module \(D\) over \(A^\dagger_L\) there is an \(\mathcal{O}_L\)-linear endomorphism

\[
\psi: D \longrightarrow D
\]

satisfying

\[
\psi \circ \varphi = \frac{q}{\pi} \text{id}_D,
\]

cf. e.g. [SV16, p. 416].

There is the following relationship between \((\varphi, \Gamma_L)\)-modules and Iwasawa cohomology. While \((\varphi, \Gamma_L)\)-modules have plentiful applications, this is our main reason for studying them.

**Theorem 6.11** ([SV16 theorem 5.13]).

\[
H^1_{Iw}(L_\infty, V(\chi_cyc^{-1} \chi_{LT cyc})) = D_\mathcal{O}(V)^{\psi=1},
\]

where \(\chi_cyc\) denotes the cyclotomic character.

**Corollary 6.12.** If \(L \neq \mathbb{Q}_p\), we have \(D^{\psi=1,\Gamma=1} = 0\) for any étale \((\varphi, \Gamma_L)\)-module \(D\) over \(A^\dagger_L\).

**Proof.** Together with these two aforementioned results, this follows immediately from [TV19 theorem 8.2], as elements in \(D^{\psi=1,\Gamma=1}\) are torsion over the Iwasawa algebra. \(\square\)
6.3.3. Overconvergence. For the remainder of this chapter, we also fix a finite extension $F/L$.

**Definition 6.13.** For an $L$-linear representation $V$ of $G_F$ set

$$D(V) = (B^t \otimes_L V)^{G(\mathbb{G}_m|F^{L,\infty})}.$$  

$\Gamma = G(FL_{\infty}|F)$ is an open subgroup of $\Gamma_L$ and by the Lubin-Tate character hence isomorphic to an open subgroup of $\mathbb{O}^\times_L$. $D(V)$ is an étale $(\varphi, \Gamma)$-module over $B_L$.

**Definition 6.14.** Let $D$ be a $(\varphi, \Gamma)$-module over $B_L$. If there is a basis of $D$ such that all endomorphisms in $\Phi \times \Gamma$ have representation matrices in $B^t_L$, we call $D$ overconvergent. This basis generates a $(\varphi, \Gamma)$-module over $B^t_L$, which we will call $D^t$.

A Galois representation $V$ is called overconvergent if $D(V)$ is. We will then write $D^t(V)$ instead of $D(V)^t$.

**Definition 6.15.** Let $V$ be an overconvergent Galois representation. Set

$$D^t_{\text{rig}}(V) = B^t_{\text{rig}, F} \otimes_{B^t_F} D^t(V).$$

**Definition 6.16.** A finite dimensional $L$-linear representation $V$ of $G_F$ is called $L$-analytic, if for all embeddings $\tau: L \rightarrow \mathbb{Q}_p$ different from the fixed one,

$$C_p \otimes^\tau_L V$$

is a trivial semilinear $C_p$-representation, i.e., as a Galois module it is isomorphic to $C_p \otimes_L \tilde{V}$ for an $L$-vector space $\tilde{V}$ with trivial Galois operation.

**Lemma 6.17.** A finite dimensional $L$-linear representation $V$ is $L$-analytic if and only if $V^* = \text{Hom}_L(V, L)$ is.

**Proof.** Note that a representation is trivial if and only if its dual is. The statement then follows from the isomorphisms

$$(C_p \otimes^\tau_F V)^* \cong C_p \otimes^\tau_F V^* \cong C_p \otimes^\tau_F V^*,$$

where $-^\vee = \text{Hom}_{C_p}(-, C_p)$. \qed

**Proposition 6.18.** The action of $\Gamma$ on $D^t_{\text{rig}}(V)$ is pro-$L$ analytic.

**Proof.** This follows for example from [Ber16] theorem 8.1]. \qed

**Proposition 6.19** ([FX13, p. 2554]). Let $D$ be an étale $(\varphi, \Gamma)$-module over $B^t_{\text{rig}, F}$, which be finite generation can be written as $D = B^t_{\text{rig}, F} \otimes \mathbb{B}_{\text{rig}, F}^r$, $D^r$ for some $r$ and some $(\varphi, \Gamma)$-module $D^r$ over $B^t_{\text{rig}, F}$. Then the series

$$\log g = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (g-1)$$

converges for $g$ small enough to an operator on $D^r$. By $\mathbb{Z}_p$-linear extension, this gives rise to a well-defined action of the Lie algebra $\mathfrak{g} \cong L$ via

$$(x, d) \mapsto (\log(\exp x))(d).$$

**Definition 6.20.** A $(\varphi, \Gamma)$-module $D$ over $B^t_{\text{rig}, F}$ is called $L$-analytic, if the action of the Lie algebra of $\Gamma$ on $D$ is $L$-linear.

Berger then shows the following refinement of the category equivalence.

**Theorem 6.21** ([Ber16] theorem D]). $V \mapsto D^t_{\text{rig}}(V)$ is an equivalence of categories between $L$-analytic representations of $G_F$ and étale $L$-analytic $(\varphi, \Gamma)$-modules over $B^t_{\text{rig}, F}$. 

6.4. Towards Duality in the Herr Complex. We continue to use the notation from section 6.3.
Let $V$ be an $L$-analytic representation of $G_F$. Then the Herr complex is given by the double complex
\[ C^\bullet(\Gamma, D_{\text{rig}}^\dagger(V)) \cong \Gamma \quad \text{and} \quad C^\bullet(\Gamma, D_{\text{rig}}^\dagger(V)), \]
whose attached double complex is quasi-isomorphic to
\[ C^\bullet(\Phi \times \Gamma, D_{\text{rig}}^\dagger(V)), \]
by [Tho20, theorem 11.6]. Here $C^\bullet$ denotes the pro-$F$-analytic cochains.
It seems natural to try to apply our duality result theorem 5.3 to this setting, however, this is not immediately possible.
Starting with $D_{\text{rig}}^\dagger(V)$, there are (at least) three natural ways to dualise this object:
We can consider the $(\phi, \Gamma)$-module attached to the dual representation, the module theoretic dual over the Robba ring $B_{\text{rig},L}^\dagger$, or the topological dual $D_{\text{rig}}^\dagger(V)$. We first need to investigate how they relate to one another.

Lemma 6.22. $D_{\text{rig}}^\dagger(V^*) = \text{Hom}_{B_{\text{rig},L}^\dagger}(D_{\text{rig}}^\dagger(V), B_{\text{rig},L}^\dagger)$.

Proof. It is well known that the category equivalence over $B_{\text{L}}$ is a functor of closed monoidal categories and hence respects taking duals. As the duals of analytic representations are again analytic by lemma 6.17, the finer category equivalence theorem 6.21 also has to respect duals. □

Let $\Omega \in \mathbb{C}_p$ be the period of $LT$, cf. [Col16, §1.1.3]. If $L \neq \mathbb{Q}_p$, it is transcendent over $\mathbb{Q}_p$. Let $K$ be the complete subfield of $\mathbb{C}_p$ generated by $L$ and $\Omega$. For a $(\phi, \Gamma)$-module $D$ over $B_{\text{rig},L}^\dagger$, write $D_K$ for the respective $(\phi, \Gamma)$-module over $B_{\text{rig},K}^\dagger$ after extension of scalars.
Serre duality implies the following result:

Proposition 6.23. $\text{Hom}_{B_{\text{rig},K}^\dagger}(D_{\text{rig}}^\dagger(V)_K, B_{\text{rig},K}^\dagger)(\chi_{LT}) = (D_{\text{rig}}^\dagger(V)_K)^*_{B_{\text{rig},K}^\dagger}$, where the dual is taken over $K$.

Proof. [SV19, lemma 2.37] □

Note that as the extension $K|\mathbb{Q}_p$ is infinite, we cannot assume that $K$ is spherically complete. However, we at least have the following.

Proposition 6.24. Étale $(\phi, \Gamma)$-modules over $B_{\text{rig},K}^\dagger$ are of countable type.

Proof. It suffices to show that $B_{\text{rig},K}^\dagger$ is of countable type. The completions of $B_{\text{rig},K}^\dagger$ at the various continuous seminorms are exactly the rings $B_{\text{rig},K}^{[r,s]}$. We will show that the countable set $L(\Omega)[X, X^{-1}]$ is dense in every $B_{\text{rig},K}^{[r,s]}$, where $L$ is a number field which is dense in $L$.
Let $f = \sum_n a_n X^n \in B_{\text{rig},K}^{[r,s]}$ and $\varepsilon > 0$. Convergence of $f$ on the closed annulus of inner radius $r$ and outer radius $s$ implies
\[ \sup_{n < k} |a_n| r^n \to 0 \quad (k \to -\infty) \]
and
\[ \sup_{n > k} |a_n| s^n \to 0 \quad (k \to \infty). \]
We can therefore choose \( k \) with \( \| \sum_{n< -k} a_n X^n \|_r, \| \sum_{n> k} a_n X^n \|_s < \varepsilon \). As \( \tilde{L}(\Omega) \) is dense in \( K \), we can also choose \( \alpha_{-k}, \ldots, \alpha_k \in \tilde{L}(\Omega) \) such that
\[
\max_i |a_i - \alpha_i| < \varepsilon r^k.
\]
It follows that
\[
\left\| f - \sum_{i=-k}^k \alpha_i X^i \right\|_{\mathcal{B}^i_{rig,K}} < \varepsilon.
\]

**Proposition 6.25.** There are natural morphisms of complexes
\[
\begin{array}{rcl}
C^*(\Phi \times \Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) & \longrightarrow & \text{Hom}_K(C^*(\Phi \times \Gamma, D_{rig}^{\dagger}(V) K), K)[-2]
\end{array}
\]
and
\[
\begin{array}{rcl}
C^*(\Phi \times \Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) & \longrightarrow & \text{Hom}_K(C^*(\Phi \times \Gamma, D_{rig}^{\dagger}(V) K), K)[-2]
\end{array}
\]
stemming from a comparison of Lie algebra cohomology.

**Proof.** \( D_{rig}^{\dagger}(V^*) K(\chi_{LT}) \) is an étale \( (\varphi, \Gamma) \)-module over \( \mathcal{B}_{rig,K}^i \) and has the structure of an LF-space over \( K \), so
\[
D_{rig}^{\dagger}(V^*) K(\chi_{LT}) = \lim_{\leftarrow} \lim_{\rightarrow} D_{rig}^{\dagger,r,s}(V^*) K(\chi_{LT}),
\]
where \( D_{rig}^{\dagger,r,s}(V^*) \) are Banach spaces over \( K \).

We see that
\[
C^*(\Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) = \lim_{\leftarrow} \lim_{\rightarrow} C^*(\Gamma, D_{rig}^{\dagger,r,s}(V^*) K(\chi_{LT})).
\]
By theorem 4.1 we get a morphism
\[
C^*(\Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) \longrightarrow \lim_{\leftarrow} \lim_{\rightarrow} C^*(\mathfrak{g}, D_{rig}^{\dagger,r,s}(V^*) K(\chi_{LT})).
\]
Now
\[
\lim_{\leftarrow} \lim_{\rightarrow} C^*(\mathfrak{g}, D_{rig}^{\dagger,r,s}(V^*) K(\chi_{LT})) = C^*(\mathfrak{g}, D_{rig}^{\dagger}(V^*) K(\chi_{LT}))
\]
as \( \mathfrak{g} \) is finite-dimensional. Analogously we also get a morphism
\[
C^*(\Gamma, D_{rig}^{\dagger}(V) K) \longrightarrow C^*(\mathfrak{g}, D_{rig}^{\dagger}(V) K).
\]
By lemma 6.22 and proposition 6.23 we can identify \( D_{rig}^{\dagger}(V^*) K(\chi_{LT}) \) with \( (D_{rig}^{\dagger}(V) K)^* = \text{Hom}_{\text{cts}}(D_{rig}^{\dagger}(V) K, K)_b \), where the strong dual is now taken over \( K \). By theorem 3.9 we get a \( \Gamma \)-equivariant \( K \)-linear morphism
\[
C^*(\Gamma, D_{rig}^{\dagger}(V) K) \longrightarrow C^*(\mathfrak{g}, D_{rig}^{\dagger}(V) K)[1].
\]
Composing all these morphism, we get a functorial morphism of complexes
\[
C^*(\Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) \longrightarrow \text{Hom}_K(C^*(\Gamma, D_{rig}^{\dagger}(V) K), K)[-1],
\]
which we can extend to a double complex as follows:
\[
\begin{array}{rcl}
C^*(\Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) & \longrightarrow & \text{Hom}_K(C^*(\Gamma, D_{rig}^{\dagger}(V) K), K)[-1]
\end{array}
\]
\[
\begin{array}{c}
\downarrow \scriptstyle{C(\Gamma, \varphi - 1)}
\end{array}
\]
\[
\begin{array}{rcl}
C^*(\Gamma, D_{rig}^{\dagger}(V^*) K(\chi_{LT})) & \longrightarrow & \text{Hom}_K(C^*(\Gamma, D_{rig}^{\dagger}(V) K), K)[-1]
\end{array}
\]
Here, \( \varphi \) denotes the intrinsic \( \varphi \)-operator on \( D_{rig}^{\dagger}(V) K \) and \( \varphi' \) its vector space dual. Note that the dualised \( \varphi \)-operator on \( \text{Hom}_{\mathcal{B}_{rig,L}^i} (D_{rig}^{\dagger}(V), \mathcal{B}_{rig,L}^i) \) is the intrinsic
\( \psi \)-operator on \( D_{\text{rig}}^1(V^*) \) and vice versa, cf. \cite{SV16} remarks 4.7 and 5.6. The diagram can hence also be written as

\[
\begin{array}{ccc}
C^*(\Gamma, D_{\text{rig}}^1(V^*)_K(\chi_{LT})) & \to & \text{Hom}_K(C^*(\Gamma, D_{\text{rig}}^1(V)_K), K)[-1] \\
\downarrow & & \downarrow \text{Hom}_K(C(\Gamma, \psi^{-1}), K) \\
C^*(\Gamma, D_{\text{rig}}^1(V^*)_K(\chi_{LT})) & \to & \text{Hom}_K(C^*(\Gamma, D_{\text{rig}}^1(V)_K), K)[-1],
\end{array}
\]

where \( \psi \) is the intrinsic \( \psi \)-operator of \( D_{\text{rig}}^1(V^*)_K(\chi_{LT}) \).

By \cite{Tho20} theorem 11.6, this induces the first morphism of complexes as required. The second morphism can be constructed completely analogously: Instead of using the intrinsic \( \varphi \)-operator of \( D_{\text{rig}}^1(V)_K \) in diagram (\( \star \)) on the right hand side, start with its intrinsic \( \psi \)-operator. Then we get the vector space dual \( \psi' \) on the left hand side, which is the intrinsic \( \varphi \)-operator of \( D_{\text{rig}}^1(V^*)_K \).

\[ \square \]

Remark 6.26. The comparison morphism

\[
C^*(\Gamma, D_{\text{rig}}^1(V^*)_K) \to C^*(\Gamma, D_{\text{rig}}^1(V)_K)
\]

does probably not induce an isomorphism on cohomology after taking \( G \)-invariant on the right hand side. We expect \( \lim^1 \)-terms to appear. Note however that for the first cohomology group, a Mittag-Leffler argument makes a comparison possible, cf. \cite{BF17} proposition 2.1.1.

Apart from this and under the assumptions of theorem 5.3, i.e., strict differentials in the Chevalley-Eilenberg complex and an open subgroup of \( \Gamma \) operating trivially on the Lie algebra cohomology, we can follow the same argument to compare cohomology groups, as by propositions 1.30 and 6.24 taking duals is exact.

Remark 6.27. In degrees zero and one, \( \varphi \) and \( \psi \) yield the same cohomology groups, cf. \cite{BF17} corollary 2.2.3.

References

\[ \text{[Ber16]} \] L. Berger, “Multivariable \((\varphi, \Gamma)\)-modules and locally analytic vectors”, \textit{Duke Math. J.}, vol. 165, no. 18, pp. 3567–3595, 2016. \texttt{doi:10.1215/00127094-3674441}.

\[ \text{[BF17]} \] L. Berger and L. Fourquaux, “Iwasawa theory and \( F \)-analytic Lubin-Tate \((\varphi, \Gamma)\)-modules”, \textit{Doc. Math.}, vol. 22, pp. 999–1030, 2017.

\[ \text{[Bou67]} \] N. Bourbaki, \textit{Éléments de mathématique. Fasc. XXXIII. Variétés différentielles et analytiques. Fascicule de résultats (Paragraphes 1 à 7)}, ser. Actualités Scientifiques et Industrielles, No. 1333. Hermann, Paris, 1967, p. 97.

\[ \text{[Bou89]} \] N. Bourbaki, \textit{Lie groups and Lie algebras. Chapters 1–3}, ser. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989, pp. xviii+450, Translated from the French, Reprint of the 1975 edition.

\[ \text{[Bour98]} \] ——, \textit{Algebra I. Chapters 1–3}, ser. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998, pp. xxiv+709, Translated from the French, Reprint of the 1989 English translation.

\[ \text{[Col16]} \] P. Colmez, “Représentations localement analytiques de \( GL_2(Q_p) \) et \((\varphi, \Gamma)\)-modules”, \textit{Represent. Theory}, vol. 20, pp. 187–248, 2016.

\[ \text{[Crc98]} \] R. Crew, “Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve”, \textit{Ann. Sci. École Norm. Sup. (4)}, vol. 31, no. 6, pp. 717–763, 1998. \texttt{doi:10.1016/S0012-9593(99)80001-9}.

\[ \text{[Eme17]} \] M. Emerton, “Locally analytic vectors in representations of locally \( p \)-adic analytic groups”, \textit{Mem. Amer. Math. Soc.}, vol. 248, no. 1175, pp. iv+158, 2017. \texttt{doi:10.1090/memo/1175}.

\[ \text{[FX13]} \] L. Fourquaux and B. Xie, “Triangulable \( O_F \)-analytic \((\varphi, \Gamma)\)-modules of rank 2”, \textit{Algebra Number Theory}, vol. 7, no. 10, pp. 2545–2592, 2013.
REFERENCES

[Gro61] A. Grothendieck, “Éléments de géométrie algébrique. III. étude cohomologique des faisceaux cohérents. Γ”, *Inst. Hautes Études Sci. Publ. Math.*, no. 11, p. 167, 1961.

[Haz70] M. Hazewinkel, “A duality theorem for cohomology of Lie algebras”, *Mat. Sb. (N.S.),* vol. 83 (125), pp. 639–644, 1970.

[KR09] M. Kisin and W. Ren, “Galois representations and Lubin-Tate groups”, *Doc. Math.*, vol. 14, pp. 441–461, 2009.

[Laz65] M. Lazard, “Groupes analytiques p-adiques”, *Inst. Hautes Études Sci. Publ. Math.*, no. 26, pp. 389–603, 1965.

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Second. Springer-Verlag, Berlin, 2008, pp. xvi+825. doi: 10.1007/978-3-540-37889-1.

[PS10] C. Perez-Garcia and W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2010, vol. 119, pp. xiv+472. doi: 10.1017/CBO9780511729959.

[PZ19] A. Pal and G. Zábrádi, “Cohomology and overconvergence for representations of powers of Galois groups”, *Journal of the Institute of Mathematics of Jussieu*, pp. 1–61, 2019. doi: 10.1017/S1474748019000197.

[Sch02] P. Schneider, *Nonarchimedean functional analysis*, ser. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002, pp. vi+156. doi: 10.1007/978-3-602-04728-6.

[Ser67] J.-P. Serre, “Local class field theory”, in *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, Thompson, Washington, D.C., 1967, pp. 128–161.

[Ser92] J.-P. Serre, *Lie algebras and Lie groups*, Second, ser. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1992, vol. 1500, pp. viii+168, 1964 lectures given at Harvard University.

[SV16] P. Schneider and O. Venjakob, “Coates-Wiles homomorphisms and Iwasawa cohomology for Lubin-Tate extensions”, in *Elliptic curves, modular forms and Iwasawa theory*, ser. Springer Proc. Math. Stat. Vol. 188, Springer, Cham, 2016, pp. 401–468.

[SV19] ——, *Regulator maps*, personal communication, Mar. 18, 2019.

[Tam15] G. Tamme, “On an analytic version of Lazard’s isomorphism”, *Algebra Number Theory*, vol. 9, no. 4, pp. 937–956, 2015.

[Tho19] O. Thomas, “On analytic and Iwasawa cohomology”, PhD thesis, Ruprecht Karl University of Heidelberg, 2019. doi: 10.11588/heidok.00027443.

[Tho20] ——, *Cohomology of topologised monoids*, 2020. arXiv: 2008.01471 [math.GR]

[TV19] O. Thomas and O. Venjakob, “On spectral sequences for Iwasawa adjoints à la Jannsen for families”, in *Proceedings of Iwasawa 2017*, 2019, to appear.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, vol. 38, pp. xiv+450. doi: 10.1017/CBO9781139644136.

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