An output-sensitive algorithm for the minimization of 2-dimensional String Covers

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Abstract. String covers are a powerful tool for analyzing the quasi-periodicity of 1-dimensional data and find applications in automata theory, computational biology, coding and the analysis of transactional data. A cover of a string $T$ is a string $C$ for which every letter of $T$ lies within some occurrence of $C$. String covers have been generalized in many ways, leading to $k$-covers, $\lambda$-covers, approximate covers and were studied in different contexts such as indeterminate strings. In this paper we generalize string covers to the context of 2-dimensional data, such as images. We show how they can be used for the extraction of textures from images and identification of primitive cells in lattice data. This has interesting applications in image compression, procedural terrain generation and crystallography.

1 Motivation

1.1 1-dimensional String Covers

Redundancy is an ubiquitous phenomenon in engineering and computer science [16,17]. Periodicity is the most common and useful form of redundancy. Periodicity is a key phenomenon when analyzing physical data such as an analogue signal. Natural data is very redundant or repetitive and exhibits some patterns or regularities [10,21,22] which we may assert to be the intended information [19]. Periodicity itself has been thoroughly studied in various fields such as Signal Processing [20], Bioinformatics [6], Dynamical Systems [12] and Control Theory [4], each bringing its own insights.

However, natural data is imperfect. It is highly unlikely that natural data can ever be periodic. In fact, the data is almost or quasi-periodic [3]. This has been firstly studied over strings, the most general representation of digital data [15].

For example, assume that we want to send the word $aba$ over a noisy channel as a digital signal where letters are modulated using amplitude shift keying [15]. Since, the simple transmission is unlikely to yield the result due to the imperfect transmission channel, we add redundancy and thus send the word $aba$ multiple times. However, when errors occur the received signal only partially retains its periodicity.
1.2 Towards Multi-Dimensional Pattern Recognition

Natural data can be represented over various spaces, among which the most relevant are the 3-dimensional Euclidean space, $\mathbb{E}^3$, the (3+1)-dimensional Minkowski space-time and the complexified Hilbert spaces of observables from systems governed by Quantum Mechanics, $C^*$-algebras over some $\mathcal{B}(\mathcal{H})$ via the Gelfand-Naimark-Segal construction [9]. The entire point of String Theory is that all these representations are equivalent to some string-representation over a large-enough space [14]. It follows that the patterns exhibited by $\mathbb{E}^n$ data can be analyzed via “String Covers”.

We are looking for a Lift function that exhibits behavior akin to String Cover, except over higher-dimensional data. Note that we are interested in patterns that correlate dimensions, not just a vectorial String Cover which can be obtained by setting the alphabet to be a vector space.

In this paper we attempt to obtain such a lift for discrete 2-dimensional data which can be considered a function $I : 0, W - 1 \times 0, H - 1 \rightarrow \Sigma$, or, graphically, an image that is $W$-long, $H$-tall and with a color spectrum in bijection with $\Sigma$. 
2 Our Results

In this paper we study the lift of the String Cover operator on finite-dimensional images.

Definition 1 (Image Covers). A cover of an image \( T \) is an image \( C \) for which every element of \( T \) lies within some occurrence of \( C \).

In Section 3, we find two alternative ways of formalizing it, by using masks (as some of the 1-dimensional definitions themselves go) and prove their equivalence. We then turn our attention towards the decision problem:

Problem 1 (Image Cover Decision). Given two images \( T \) and \( C \), does one cover the other?

We give an \( O(WH) \) algorithm based upon Bird’s 1977 \cite{5} 2-dimensional matching algorithm and thus prove that it is \( \Theta(WH) \). Then, using this algorithm we study the minimization problem (Section 4):

Problem 2 (Weak Minimal Image Cover). Given an image \( T \in \text{Mat}_{H,W}(\Sigma) \) and an evaluation function \( \text{eval} : \mathbb{Z}^2 \times \mathbb{Z} \rightarrow \mathbb{R} \) which induces an order onto the covers, which is the cover \( C \in \text{Mat}_{h,w}(\Sigma) \) of \( T \) minimal with respect to \( \text{eval}(w, h) \)?

We give an \( O(W^2H^2) \Theta(\text{eval}) \) algorithm since the minimization problem is actually \( \Omega(WH) \Theta(\text{eval}) \) we look for a better algorithm. Using sorting of the input candidates according to \( \text{eval} \) we obtain \( O(nWH) \Theta(\text{eval}) \) where the \( n \)-th entry in the vector of \( (w, h) \in \mathbb{Z}^2 \times \mathbb{Z} \) sorted by \( \text{eval} \) that determines a cover of \( T \), where the time bound does not include the sort. Note that this assumption is not in general realistic, so a more honest complexity bound is \( O(WH(n + \log(WH) \Theta(\text{eval}))) \). However, there is a very important optimization criterion where the sorting is very cheap, namely the size.

Problem 3 (Strong Minimal Image Cover). Given an image \( T \in \text{Mat}_{H,W}(\Sigma) \) which is the cover \( C \in \text{Mat}_{h,w}(\Sigma) \) of \( T \) minimal with respect to its \( \ell_2(\ell_1, \ell_{\infty}) \) norm, \( wh(w + h, \max(w, h)) \)?

For this problem we augment the general minimization algorithm with a preprocessing routine, based on the optimal 1-dimensional Minimal String Cover algorithm, which reduces the number of candidate pairs that need to be checked from \( \Theta(WH) \) to \( O(1) \) on the average case, reducing the complexity to \( \Theta(WH) \) on the average case and, particularly, \( O(W) \) in the worst-case for \( H = 1 \). We argue that the use of this routine never hinders performance and offers the same boost for the general case of an unknown \( \text{eval} \) function.

We conclude the article with a few very interesting applications of other generalizations of the Minimal String Cover Problem (Sections 5 and 6) such as \( k \)-covers and the Approximative String Cover Problem introduced by Amir et al. \cite{2,1} to lattice unit-cell recognition from generic images, detection of the unit cells of some quasicrystals \cite{23}, extraction of the elementary set of tiles in a
Wang Tiling, recognizing the minimal (quasi)periodic Wang Tile pattern in an image and the minimal modification required of an image for the existence of a non-trivial minimal (quasi)periodic Wang Tile pattern.

3 Image Covers

The simplest class of images is that of binary images, i.e. \( \Sigma \cong \{0, 1\} \). Binary images can be thought of as the characteristic functions of sets, with which their support is identified. Thus a binary image can be thought of as a set over \( \mathbb{Z}^2 \).

**Definition 2.** A mask of an image \( T \) with respect to an image \( C \) is a binary image \( M \) which marks the first position of some occurrences of \( C \) in \( T \).

Formally if \( T \in \text{Mat}_{H, W} (\Sigma) \) and \( C \in \text{Mat}_{h, w} (\Sigma) \) then \( M \in \text{Mat}_{H, W} (\{0, 1\}) \) is a mask of \( T \) with respect to \( C \) if

\[
M^t_j = 1 \Rightarrow T^t_{j+x-1} = C^t_{j} \quad \forall t \in [1, H], j \in [1, W], y \in [1, h], x \in [1, w]
\]

By the correspondence between binary images and sets, there exists a maximal mask with respect to cardinality and it identifies all occurrences of an image in another.

**Definition 3.** The maximal mask of an image \( T \) with respect to an image \( C \) is a binary image \( M^* \) which marks the first position of all occurrences of \( C \) in \( T \).

Formally if \( T \in \text{Mat}_{H, W} (\Sigma) \) and \( C \in \text{Mat}_{h, w} (\Sigma) \) then \( M^* \in \text{Mat}_{H, W} (\{0, 1\}) \) is the maximal mask of \( T \) with respect to \( C \) if

\[
M^*_j = 1 \iff T^t_{j+x-1} = C^t_{j} \quad \forall t \in [1, H], j \in [1, W], y \in [1, h], x \in [1, w]
\]

**Definition 4 (Weak Image Covers).** A cover of an image \( T \) is an image \( C \) for which every element of \( T \) lies within some occurrence of \( C \).

Formally, if \( T \in \text{Mat}_{H, W} (\Sigma) \) and \( C \in \text{Mat}_{h, w} (\Sigma) \), then \( C \) covers \( T \) if there exists some mask \( M \) of \( T \) with respect to \( C \) such that:

\[
\forall Y \in [1, H], X \in [1, W] \exists i \in [Y-h+1, Y], j \in [X-w+1, X] \quad M^t_j = 1
\]

Equivalently, we may define Image Covers with respect to the maximal mask:

**Definition 5 (Strong Image Covers).** A cover of an image \( T \) is an image \( C \) for which every element of \( T \) lies within some occurrence of \( C \).

Formally, if \( T \in \text{Mat}_{H, W} (\Sigma) \) and \( C \in \text{Mat}_{h, w} (\Sigma) \), then \( C \) covers \( T \) if the maximal mask \( M^* \) of \( T \) with respect to \( C \) such that:

\[
\forall Y \in [1, H], X \in [1, W] \exists i \in [Y-h+1, Y], j \in [X-w+1, X] \quad M^*_j = 1
\]

**Remark 1.** By these definitions a cover \( C \in \text{Mat}_{h, w} (\Sigma) \) of an image \( T \in \text{Mat}_{H, W} (\Sigma) \) can be identified with the \((w, h)\) pair.

**Theorem 1.** The weak and strong definitions are equivalent.
Proof. Consider the set \( S = 1, W \times 1, H \). There exists a bijection between its power set, \( \mathcal{P}(S) \), and the W-long, H-tall binary images \( \text{Mat}_{H,W}(\{0,1\}) \):

\[ \mathcal{P}(S) \ni S \leftrightarrow f(S) \in \text{Mat}_{H,W}(\{0,1\}) : f(S)^i_j = \chi_S((i,j)) \forall i \in 1, H, j \in 1, W \]

However, the image of the boolean algebra \( (\mathcal{P}(S), \cup, \cap, \setminus, \emptyset, S) \) is thus by \( f \) onto \( \text{Mat}_{H,W}(\{0,1\}) \). The new structure can be verified to be

\[ (\text{Mat}_{H,W}(\{0,1\}), \text{max}, \text{min}, M \rightarrow 1 - M, 0, 1) \]

Thus the image of the inclusion order \( \subseteq \) is the order \( \leq \) and so, if there exists a mask \( M \) such that:

\[ \forall Y \in 1, H, \ X \in 1, W \ \exists i \in Y - h + 1, Y, j \in X - w + 1, X \ M^i_j = 1 \]

then since \( M \leq M^* \) we also have:

\[ \forall Y \in 1, H, \ X \in 1, W \ \exists i \in Y - h + 1, Y, j \in X - w + 1, X \ M'^i_j = 1 \]

and vice versa: if \( M^* \) satisfies the later, then there exists at least one such mask \( M \) (precisely \( M^* \)) which satisfies the former. Thus, the two definitions are indeed equivalent. \( \triangleleft \)

While from a formal standpoint the two definitions are equivalent, from a computational standpoint it is more convenient for us to work with the strong definition, since we do not have to consider all masks.

**Lemma 1.** Given two images \( T \in \text{Mat}_{H,W}(\Sigma) \) and \( C \in \text{Mat}_{h,w}(\Sigma) \) the construction of the maximal mask of \( T \) with respect to \( C \) takes \( \Theta(WH) \) time.

**Proof.** Since the size of the output is \( WH \) we have the lower bound \( \Omega(WH) \). We effectively only have to prove the upper bound of \( O(WH) \).

Adapt now Bird’s algorithm and take \( M^* \lesssim \text{stage} (i + h - 1, j + w - 1) \). This yields the maximal mask in \( O(WH + wh) = O(WH) \) time. \( \triangleleft \)

**Theorem 2 (Image Cover Decision).** Given two images \( T \in \text{Mat}_{H,W}(\Sigma) \) and \( C \in \text{Mat}_{h,w}(\Sigma) \) checking if \( C \) is a cover of \( T \) takes \( \Theta(WH) \) time (Algorithm 7).

**Proof.** Note that we can instantly disqualify images \( C \) having \( h > H \) or \( w > W \). Moreover, in absence of other information we must check all cells of \( T \). Thus, the decision problem is at least \( \Omega(HW) \). Only the upper \( O(WH) \) is yet to be proven.

Consider now that we have computed \( M^* \), which as we have shown takes \( O(WH) \) time. We now compute the function:

\[ N(x, y) = \arg \min_{x'x} \{ (x - x', y - y') \mid (x', y') \in D(x, y) \} \]

\[ D(x, y) = \{ (x - w + 1, y - h + 1) \leq (x', y') \leq (x, y) \mid M^x_{y} = 1 \} \]
where \( D(x, y) \) is called the admissibility domain for \((x, y)\).

Note that if ever \( N(x, y) = \infty \) then \( T^y_x \) belongs to no occurrence of \( C \) in \( T \) and vice versa.

We begin by remarking the following facts:

If the optimal solution for the western neighbor is inadmissible then the only candidate that is unknown to the northern neighbor is \((x, y)\):

\[
N(x-1, y) \notin D(x, y) \Rightarrow M^y_x = 0 \forall x' \in x-w+1, x-1, y' \in y-h+1, y \Rightarrow D(x, y) \subseteq D(x, y-1) \cup \{(x, y)\}
\]

If the optimal solution for the northern neighbor is inadmissible then the only candidate that is unknown to the western neighbor is \((x, y)\):

\[
N(x, y-1) \notin D(x, y) \Rightarrow M^y_x = 0 \forall y' \in y-h+1, y-1 \Rightarrow D(x, y) \subseteq D(x-1, y) \cup \{(x, y)\}
\]

If the optimal solutions for the western and northern neighbors are both inadmissible then the only possible candidate is \((x, y)\):

\[
N(x-1, y) \notin D(x, y), N(x, y-1) \notin D(x, y) \Rightarrow M^y_x = 0 \forall x' \in x-w+1, x, y' \in y-h+1, y, (x', y') \neq (x, y) \Rightarrow D(x, y) \subseteq \{x, y\}
\]

Moreover, if \((x_1, y_1) \leq (x_2, y_2)\) we have:

\[
(x_1 - x_1^*, y_1 - y_1^*) \leq_{lex} (x_1 - x_1', y_1 - y_1') \iff (x_2 - x^*, y_2 - y^*) \leq_{lex} (x_2 - x', y_2 - y')
\]

Thus, if a candidate \((x', y')\) is admissible to both the current node and one of its western or northern neighbors, but not optimal for that neighbor it is not optimal for the current node. We obtain the dynamic programming scheme:

\[
N(x, y) \in \{N(x-1, y), N(x, y-1), (x, y)\}
\]

This scheme can be implemented in \( O(WH) \) time as shown in Algorithm 1 and, as we have proven correctly decides whether the maximal mask does indeed cover the entire image, i.e. \( C \) is a cover of \( T \). We conclude that the complexity of the decision problem is indeed \( \Theta(WH) \).

\[\square\]

4 Minimal Image Covers

Among the family of covers of an image \( T \), we would like to find a “minimal” one. For this we must define the optimization criterion. It will take the form of an evaluation function: \( \text{eval} : \mathbb{I}, W \times \mathbb{I}, H \rightarrow \mathbb{R} \)
Algorithm 1 Image Cover Decision

1: procedure Check($T, w, h$)
2: Pre-process $T$ (per Bird’s algorithm)
3: for $x \in [1, W]$ do
4: for $y \in [1, H]$ do
5: $N(x, y) = (-\infty, -\infty)$
6: if $x > 1$ and $(x - w + 1, y - h + 1) \leq N(x - 1, y)$ then
7: $N(x, y) = N(x - 1, y)$
8: end if
9: if $y > 1$ and $(x - w + 1, y - h + 1) \leq N(x, y - 1)$ then
10: if $(x, y) - N(x, y - 1) \leq \sum_{i=2}^{t} N(i, y) - N(i, y - 1)$ then
11: $N(x, y) = N(x, y - 1)$
12: end if
13: end if
14: if stage $(y, x)$ (per Bird’s algorithm) then
15: $N(x, y) = (x, y)$
16: end if
17: if $N(x, y) = (-\infty, -\infty)$ then
18: return Mismatch: $(x, y)$
19: end if
20: end for
21: end for
22: return Match
23: end procedure

Proposition 1. Obtaining the minimal image cover $C$ of $T$ with respect to $\text{eval}$ takes $O(W^2H^2) \Theta(\text{eval})$ times.

Proof. Since the number of possible covers is $\Theta(WH)$, attained for example for the image $T_i^j = 0$, and since all possible covers need to be evaluated, minimization is also $\Omega(WH) \Theta(\text{eval})$. Notably, since there are at most $\Theta(WH)$ candidates, a brute-force generation followed by a decision problem for all of them and an evaluation yields an upper bound of $O(W^2H^2) \Theta(\text{eval})$. $\square$

Proposition 2. If the minimal $C$ is a posteriori the $n$-th candidate according to the order induced by $\text{eval}$, we can obtain $C$ in $O(nWH)$ if the input is already sorted according to this order.

Proof. Note however that if the ordering induced by the $\text{eval}$ function were known a posteriori, we could queue up the candidate covers in that order (by sorting for example). For instance, if the $n$-th candidate was a posteriori the first cover encountered, the runtime of the minimization algorithm described above would be $O(nWH)$. This can be achieved with a $O(WH \log(WH)) \Theta(\text{eval})$ preprocessing via sorting. $\square$
4.1 The Size Criteria

We now study minimality with respect to a natural criterion, namely the size, as given by the $\ell_1$, $\ell_2$ and $\ell_\infty$ norms.

For the $\ell_2$ norm we have:

$$\mathbb{R} \times \mathbb{R} \ni (w, h) \rightarrow \text{eval}(w, h) = wh \in \mathbb{R}$$

The inequation $xy \leq wh$ and the constraints $1 \leq x \leq W, 1 \leq y \leq H$ define an admissibility domain $y \in [\frac{wh}{x}, \min(wh, H)]$, a discretization of the intersection of the rectangle $\text{conv}((1, 1), (1, H), (W, 1), (W, H))$ with the triangle $\text{conv}((1, 1), (1, wh), (wh, 1))$ which have surface $WH$ and $w^2h^2/2$ respectively and thus $n \approx O(\min(w^2h^2, WH))$ which leads to an upper bound of $O(\min(w^2h^2, WH))$

For the $\ell_1$ norm we have:

$$\mathbb{R} \times \mathbb{R} \ni (w, h) \rightarrow \text{eval}(w, h) = w + h \in \mathbb{R}$$

The inequation $x + y \leq w + h$ and the constraints $1 \leq x \leq W, 1 \leq y \leq H$ define an admissibility domain $y \in [\min(w + h - x, H), \max(w + h - x, H)]$, a discretization of the intersection of the rectangle $\text{conv}((1, 1), (1, H), (W, 1), (W, H))$ with the triangle $\text{conv}((1, 1), (1, w + h - 1), (w + h - 1, 1))$ which have surface $WH$ and $(w + h - 2)^2/2$ respectively and thus $n \approx O(\min(w^2 + h^2, WH))$ which leads to an upper bound of $O(\min(w^2 + h^2, WH))$

For the $\ell_\infty$ norm we have:

$$\mathbb{R} \times \mathbb{R} \ni (w, h) \rightarrow \text{eval}(w, h) = \max(w, h) \in \mathbb{R}$$

The inequation $\max(x, y) \leq \max(w, h)$ and the constraints $1 \leq x \leq W, 1 \leq y \leq H$ define an admissibility domain $y \in [\min(\max(w, h), H), \max(\max(w, h), H)]$, a discretization of the intersection of the rectangle $\text{conv}((1, 1), (1, H), (W, 1), (W, H))$ with the square $\text{conv}((1, 1), (1, \max(w, h), \max(w, h), \max(w, h)))$ which have surface $WH$ and $\max(w, h)^2$ respectively and thus $n \approx O(\min(\max(\max(w, h))^2, WH))$ which leads to an upper bound of $O(\min(\max(\max(w, h))^2, WH))$

4.2 Boosting Average Performance by Pre-Processing

We assert that not all candidates need to be verified. For instance, if the candidate $(w, h)$ were a cover, then the first and last $w$ columns and $h$ rows would have to be tileable by $T_{\frac{w}{w}, \frac{h}{h}}$. Based on this criterion we construct some pre-processing routine.

Assume we knew a priori that $h \geq h_0$ for some $h_0$. We could solve the vectorial Minimal String Cover Problem for the first and last $h$ rows and find
the Minimal Common Cover in $\Omega(Wh)$ and obtain all its possible tilings. This implies that there also exists some $w_0$ such that $w \geq w_0$. Solving now the vectorial Minimal String Cover Problem for the first and last $w$ columns we obtain a new $h_1$ such that $h \geq h_1$. Notably, if $h_1 > h_0$, $h_0$ could not have been a cover, and we have found this in $\Omega(Wh + wH)$ time instead of $\Omega(WH)$.

Let us now cache the result for this pre-processing step and consider we would like to proceed again for some $h > h_1$. Note that we do not need to solve the vectorial String Cover Problem for all the first $h$ rows. It is sufficient that we solve it for the new $h - h_1$ ones and aggregate the solutions in $\Omega(W)$ time. The same applies for the column problem. We conclude that the total cost of this pruning is $\Omega(WH)$. It can never harm our complexity. Note that we have, along the way, renounced the use of the last lines and columns.

On the other hand, consider that $m$ iterations were pruned in this way. It would mean that our actual runtime is $\Omega((wh - m)WH)$. Can we obtain some order of magnitude for this $m$? Consider the case $H = 1$. By pre-processing we directly arrive at the Minimal String Cover for the first (and only) line. Thus, for this example we have $wh - m = 1$ and our algorithm becomes optimal for the 1-dimensional case.

Since we have established that this pre-processing is effectively free we can do it entirely a priori, i.e. obtain the transitive closure of the pre-processing function. Let $S$ be the matrix of string covers returned by the optimal Minimal String Cover algorithm for each line and $S'$ for columns, i.e. $S_j^i = 1$ if the first $j$ characters on the $i$-th line cover the $i$-th line and $S_{ij}^*$ = 1 if the first $i$ characters on the $j$-th column cover the $j$-th column. The current pre-processing is equivalent to computing the Hadamard product of the matrices:

$$S_{ij}^* = \min (S_j^i, S_{ij}^{-1}) ; \quad S_{ij}^* = \min (S_{ij}, S_{ij-1}^i) ; \quad S^* = \min (S_1, S_1^i) = S_1 \odot S_1^i$$

Notably, the number of elements that are not pruned is the number of non-zero elements of $S^*$. However:

$$S_{ij}^* = \prod_{i'=1}^{i} S_j^{i'} ; \quad S_{ij}^* = \prod_{j'=1}^{j} S_{i}^{j'} ; \quad S^* = S_{ij}^* S_{ij}^* = \prod_{i'=1}^{i} \prod_{j'=1}^{j} S_j^{i'} S_{ij}^{i'}$$

**Proposition 3.** Computing the matrix $S^*$ reduces the number of candidates that need to be checked to $\Theta(c)$ for the average case and $\Theta(1)$ for the $H = 1$ case.

**Proof.** Assume that there is a $p$ probability for any tile in $S_j^i$ and $S_{ij}^*$ to be 1 with the additional condition that $S_j^{m_i} \geq S_j^{m_i} \forall m$ and assuming that there is no single-character line nor column. Then the by the Euler approximation, the probability that $S_j^i$ be 1 is $p^{i/\log(i)}$, that $S_{ij}^*$ be 1 is $p^{j/\log(j)}$ and thus the probability that $S_{ij}^*$ be 1 is $p^{j/\log(i)+j/\log(j)}$. Thus the expected number of 1-tiles is (considering that $S_W^i = S_{ij}^H = 1$)

$$\left(1 + \sum_{i=2}^{H} p^{i/\log(i)}\right) \left(1 + \sum_{j=2}^{W} p^{j/\log(j)}\right) \leq \left(1 + \frac{p}{1-p}\right)^2 = \frac{1}{(1-p)^2}$$
We conclude that there exists a solution that is linear on the average case, \( O(cWH) \) and quadratic in the worst, with the output-sensitive complexity: \( O(whWH) \) which collapses to \( O(W) \) for the 1-dimensional case.

5 A Connection with Lattices

A lattice is an additive subgroup \( \mathcal{L} \) of \( \mathbb{R}^n \) isomorphic to \( \mathbb{Z}^n \). By definition, it is infinite and yet it is generated by \( n \) elements. Consider the isomorphism \( \phi : \mathbb{Z}^n \to \mathcal{L} \). The projection of the unit volume \( \{0, 1\}^n \) through this isomorphism \( \phi (\{0, 1\}^n) \) is called the primitive cell of the lattice and it can be tiled by translations to form the entire \( \mathcal{L} \). Note that by isomorphism we have:

\[
\phi \left( \sum_{i=1}^{n} \lambda_i e_i \right) = \sum_{i=1}^{n} \lambda_i \phi (e_i)
\]

Moreover, if \( \mathcal{L} \) is a lattice, \( R \) is a rotation and \( S \) is a scaling matrix i.e. \( S_i^j = 0 \leftrightarrow i \neq j \) then \( SRL \) is isomorphic to \( \mathcal{L} \) and thus when classifying lattices we can assume that there exists some \( \phi (e_i) = e_1 \). Moreover since \( \mathbb{Z}^n \) is isomorphic to itself by the maps \( e_i \to e_{\sigma(i)} \) for any permutation \( \sigma \in S_n \), we can assume that \( \phi (e_1) = e_1 \). Thus, all 2-dimensional lattices can be characterized by the relative phase and length of the second vector.

![Fig. 4: a grid lattice](image)

![Fig. 5: a hexagonal row lattice](image)

![Fig. 6: A mixed thing which is actually a grid lattice](image)

Given a volume in \( n \)-dimensional space and a lattice \( \mathcal{L} \subseteq \mathbb{E}^n \), we can divide it according to the lattice i.e. given \( \phi : \mathbb{Z}^n \to \mathcal{L} \) we have

\[
\mathbb{E}^n_\mathcal{L} \overset{\phi}{=} \{ C_1 \overset{\text{conv}}{=} \{ \phi (1 + \mathbf{v}) | \mathbf{v} \in \{0, 1\}^n \} \} | \mathcal{L} \subseteq \mathbb{E}^n \}
\]

Note that the translation \( \phi (1) \to \phi (1') \) maps \( C_1 \) to \( C_{1'} \) and thus the volume of any two cells is the same for a given \( \mathcal{L} \subseteq \mathbb{E}^n \). Thus we can define the quantity \( \text{vol} (\mathcal{L}) \overset{\phi}{=} \text{vol} (C_0) \) to be the unit volume of a lattice \( \mathcal{L} \).

Given some volumetric data \( \mathbb{E}^n \supseteq \mathbb{V} \ni \mathbf{x} \to \psi (\mathbf{x}) \in \mathbb{R} \) we say that a lattice \( \mathcal{L} \) is legal with respect to \( \psi \) if \( \psi \) is also translation invariant i.e.

\[
\psi (\phi (\mathbf{x})) = \psi (\phi (\mathbf{x} + \mathbf{v})) \forall \mathbf{x} \in \mathbb{Z}, \mathbf{v} \in \{0, 1\}^n, \phi (\mathbf{x}) \in \mathbb{V}, \mathbf{x} + \mathbf{v} \in \mathbb{V}
\]

Moreover, \( \mathcal{L} \) is natural with respect to \( \psi \) if it is a legal lattice, minimal with respect to the unit volume.
We would like to obtain the unit cell of the natural lattice given not the lattice points but instead a tiling of the unit cell that is cropped to a $W$-long, $H$-tall image that contains at least one copy of the unit cell i.e. volumetric 2-dimensional data.

Once we have found an unit cell, any translation or rotation of it is still an unit cell which describes the same geometry and thus we have no interest in selecting any particular one. We accept any unit cell of any natural lattice.

Since a legal lattice is invariant to translations, we may always fix the origin of one unit cell on $T_1$. Since it is invariant to rotations we may always fix that one of its unit vectors is along the $T^1$ row. However, it may be that the other axis is not along the $T_1$ column, as is the case for hexagonal lattices. Moreover, it may be that our image does not end after an integer number of tiles, but instead a fractional one. In this case, the end fraction has to appear in the cover. We conclude that the shortest cover may never contain more than the volume of the box-cover of 4 unit tiles. In fact, it never contains 2 entire unit tiles on any side. Moreover, it will always contain at least one unit tile or a seed of it.

Note that this approach is especially interesting in the case of quasi-periodic crystals (which do not admit a Bravais lattice) [23]. This lifts the $k$-covers problem and asks for the $k$ unit cells which have been used, for example in a Penrose tiling [7].

6 Applications in Computer Graphics

Consider the task of producing huge, unique maps for games, such as mazes or dungeons. Without procedural terrain generation this task is anything between infeasible and impossible, depending on the desired size and the available time and budget. Many games use Wang Tiles [8,13] to produce huge maps (an interesting example is the Infamous game produced by Sucker Punch). They have recently garnered around them a very large community.

Wang tiles are formal systems visually modeled by square tiles with colors on each side. Two Wang tiles may only be tiled along an edge if the colors match. The most popular problems concerning them were: whether a set of Wang tiles can cover the plane and whether this can be done in a periodic way [11].

A Wang tile can also be represented as a 3-by-3 image. Two such images may be tiled together either along an edge or a corner. The formal system isomorphism is trivial: two 3-by-3 images may be tiled together on an edge if the colors
match. This is very much like String Covers, except two such images may never be tiled one alongside another.

From a constructive perspective, this difference matters. From a destructive one, it does not. Consider the following problems:

**Problem 4 (Minimal Wang Cover).** Given a tiling of some Wang Tiles check if there exists a periodic pattern covering it.

**Problem 5 (k-Wang Covers).** Given a tiling of some Wang Tiles check if there exist k patterns which, when tiled together cover the image.

**Problem 6 (Approximate Wang Cover).** Given a tiling of some Wang Tiles find the minimal number of pixels to be changed for it to be covered by a single periodic pattern.

When given a 3-tall image the first two collapse to vectorial String Cover and vectorial k-Covers. For the last one, we must also impose that the black and gray pixels which we added ourselves are never corrupted. Thus we impose that the distance between two tilings is infinite if a black or gray pixel is corrupted. Hence it is equivalent to the Approximate String Cover of Amir et al. with an almost-Hamming metric.

**Problem 7 (Generalization to pseudo-metrics).** Given a compression palette \( \Gamma \subseteq \Sigma \) and an algorithm that is consistent with respect to the colors it replaces i.e. \( A : \Sigma \rightarrow \Gamma \) and a tiling of some Wang Tiles, check if the solutions to the above problems change.

The last problem is not important from a computational perspective; in fact it is quite trivial, but it gives substance to the pseudo-metric variations of String Cover problems.

Assume we have computationally efficient algorithms for the problem above. How can we best use them? As an explorer we can know in advance what awaits us at the next turn. As a game developer, we can use these algorithms on maps produced by designers in order to extract textures. Another interesting point is that this gives us the natural lattice of our textures.

Consider a game with hexagonal tiles that wants to make use of Perlin noise \[18\]. It is unnatural that it be used purely, since the rectangular lattice is not actually legal. On the other hand, since we can obtain \( L \), we by default have a mapping \( \phi^{-1} : L \rightarrow \mathbb{Z}^n \). In this domain, our lattice is indeed rectangular. Thus, it is here that we should apply our Perlin noise.

**Definition 6.** Given a lattice \( \mathbb{Z}^n \xrightarrow{\phi} L \subseteq \mathbb{E}^n \) and a noise-function appropriate for rectangular lattices \( \mathcal{P} : \mathbb{Z}^n \rightarrow \mathbb{R} \), we can lift it to \( L \):

\[
L \ni x \rightarrow \mathcal{P}_L(x) \triangleq \mathcal{P}(\phi^{-1}(x)) \in \mathbb{R}
\]

Thus we can define the Perlin noise appropriate for a given Wang system. Note that the magnitude of Perlin noise is an input parameter. Thus, without changing the game or inducing unnatural patterns, as a game developer we can easily add a diversity grade for games using Wang tiles for terrain generation.
References

1. Amir, A., A. Levy, M. Lewenstein, R. Lubin, and B. Porat: Can we recover the cover? In 28th Annual Symposium on Combinatorial Pattern Matching, CPM 2017, July 4–6, 2017, Warsaw, Poland, pp. 25:1–25:15, 2017. https://doi.org/10.4230/LIPIcs.CPM.2017.25

2. Amir, A., A. Levy, R. Lubin, and E. Porat: Approximate cover of strings. In 28th Annual Symposium on Combinatorial Pattern Matching, CPM 2017, July 4–6, 2017, Warsaw, Poland, pp. 26:1–26:14, 2017. https://doi.org/10.4230/LIPIcs.CPM.2017.26

3. Apostolico, A. and D. Breslauer: Of periods, quasiperiods, repetitions and covers. In Structures in Logic and Computer Science, A Selection of Essays in Honor of Andrzej Ehrenfeucht, pp. 236–248, 1997.

4. Bacciotti, A. and L. Rosier: Liapunov functions and stability in control theory. Springer Science & Business Media, 2006.

5. Bird, R.S.: Two dimensional pattern matching. Information Processing Letters, 6(5):168–170, 1977.

6. Brodzik, A.K.: Quaternionic periodicity transform: an algebraic solution to the tandem repeat detection problem. Bioinformatics, 23(6):694–700, 2007.

7. Bursill, L. and P.J. Lin: Penrose tiling observed in a quasi-crystal. Nature, 316(6023):50–51, 1985.

8. Derouet-Jourdan, A., M. Salvati, and T. Jonchier: Procedural wang tile algorithm for stochastic wall patterns. June 2017.

9. Frydryszak, A. and L. Jakóbczyk: Generalized gelfand-naimark-segal construction for supersymmetric quantum mechanics. letters in mathematical physics, 16(2):101–107, 1988.

10. Havlin, S., S. Buldyrev, A. Goldberger, R. Mantegna, S. Ossadnik, C.K. Peng, M. Simons, and H. Stanley: Fractals in biology and medicine. Chaos, Solitons and Fractals, 6:171 – 201, 1995, ISSN 0960-0779. Complex Systems in Computational Physics.

11. Jeandel, E. and M. Rao: An aperiodic set of 11 wang tiles. arXiv preprint arXiv:1506.06492, 2015.

12. Katok, A. and B. Hasselblatt: Introduction to the modern theory of dynamical systems, vol. 54. Cambridge university press, 1997.

13. Kopf, J., D. Cohen-Or, O. Deussen, and D. Lischinski: Recursive Wang tiles for real-time blue noise, vol. 25. ACM, 2006.

14. Larraguível, H., G. López, and J. Nieto: Nambu-goto action and classical rebits in any signature and in higher dimensions. Rev. Mex. Fis., 63:214, 2017.

15. Middlestead, R.: Digital Communications with Emphasis on Data Modems: Theory, Analysis, Design, Simulation, Testing, and Applications. Wiley, 2017, ISBN 9780470408520.

16. Ming, L. and P.M. Vitányi: Kolmogorov complexity and its applications. In Algorithms and Complexity, pp. 187–254. Elsevier, 1990.

17. Muchnik, A., A. Semenov, and M. Ushakov: Almost periodic sequences. Theoretical Computer Science, 304(1-3):1–33, 2003.

18. Perlin, K.: Improving noise. In ACM Transactions on Graphics (TOG), vol. 21, pp. 681–682. ACM, 2002.

19. Searle, J.R., F. Kiefer, M. Bierwisch, et al.: Speech act theory and pragmatics, vol. 10. Springer, 1980.
20. Sethares, W.A. and T.W. Staley: *Periodicity transforms*. IEEE transactions on Signal Processing, 47(11):2953–2964, 1999.

21. Timmermans, M., R. Heijmans, and H. Daniels: *Cyclical patterns in risk indicators based on financial market infrastructure transaction data*. 2017.

22. Tychonoff, A.: *Théorèmes d’unicité pour l’équation de la chaleur*. Matematiceskij sbornik, 42(2):199–216, 1935.

23. Wlodawer, A., W. Minor, Z. Dauter, and M. Jaskolski: *Protein crystallography for aspiring crystallographers or how to avoid pitfalls and traps in macromolecular structure determination*. The FEBS journal, 280(22):5705–5736, 2013.