SINGULAR LOCI OF BRUHAT-HIBI TORIC VARIETIES

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Abstract. For the toric variety $X$ associated to the Bruhat poset of Schubert varieties in a minuscule $G/P$, we describe the singular locus in terms of the faces of the associated polyhedral cone. We further show that the singular locus is pure of codimension 3 in $X$, and the generic singularities are of cone type.

INTRODUCTION

Let $K$ denote the base field which we assume to be algebraically closed of arbitrary characteristic. Given a distributive lattice $\mathcal{L}$, let $X(\mathcal{L})$ denote the affine variety in $\mathbb{A}^{#\mathcal{L}}$ whose vanishing ideal is generated by the binomials $X_\tau X_\phi - X_{\tau \lor \phi} X_{\tau \land \phi}$ in the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$ (here, $\tau \lor \phi$ (resp. $\tau \land \phi$) denotes the join - the smallest element of $\mathcal{L}$ greater than both $\tau, \phi$ (resp. the meet - the largest element of $\mathcal{L}$ smaller than both $\tau, \phi$)). These varieties were extensively studied by Hibi in [12] where Hibi proves that $X(\mathcal{L})$ is a normal variety. On the other hand, Eisenbud-Sturmfels show in [7] that a binomial prime ideal is toric (here, “toric ideal” is in the sense of [19]). Thus one obtains that $X(\mathcal{L})$ is a normal toric variety. We shall refer to such a $X(\mathcal{L})$ as a Hibi toric variety.

For $\mathcal{L}$ being the Bruhat poset of Schubert varieties in a minuscule $G/P$, it is shown in [9] that $X(\mathcal{L})$ flatly deforms to $\widehat{G/P}$ (the cone over $G/P$), i.e., there exists a flat family over $\mathbb{A}^1$ with $\widehat{G/P}$ as the generic fiber and $X(\mathcal{L})$ as the special fiber. More generally, for a Schubert variety $X(w)$ in a minuscule $G/P$, it is shown in [9] that $X(\mathcal{L}_w)$ flatly deforms to $\widehat{X(w)}$, the cone over $X(w)$ (here, $\mathcal{L}_w$ is the Bruhat poset of Schubert subvarieties of $X(w)$). In a subsequent paper (cf. [10]), the authors of loc.cit., studied the singularities of $X(\mathcal{L}), \mathcal{L}$ being the Bruhat poset of Schubert varieties in the Grassmannian; further, in loc.cit., the authors gave the following conjecture on the singular locus of $X(\mathcal{L})$:

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Conjecture of [10].
\[ \text{Sing} X (\mathcal{L}) = \bigcup_{(\alpha, \beta)} Z_{\alpha, \beta}, \]
where \((\alpha, \beta)\) is an (unordered) incomparable pair of join-meet irreducibles in \(\mathcal{L}\), and \(Z_{\alpha, \beta} = \{P \in X (\mathcal{L}) \mid P (\theta) = 0, \forall \theta \in [\alpha \wedge \beta, \alpha \vee \beta]\}\).

(Here, for a \(P \in X (\mathcal{L}) \subset \mathbb{A}^{\# \mathcal{L}}\), and \(\theta \in \mathcal{L}\), \(P (\theta)\) denotes the \(\theta\)-th co-ordinate of \(P\).)

The sufficiency part of the above conjecture for the Bruhat poset of Schubert varieties in the Grassmannian is proved in [10], using the Jacobian criterion for smoothness, while the necessary part of the conjecture is proved in [1], using certain desingularization of \(X (\mathcal{L})\).

In [5], the authors gave a simple proof of the above conjecture for the Bruhat poset of Schubert varieties in the Grassmannian using just the combinatorics of the polyhedral cone associated to \(X (\mathcal{L})\).

It turns out that the above conjecture does not extend to a general Hibi toric variety \(X (\mathcal{L})\) (see §10 of [5] for a counter example). In [5], the authors conjectured that the above conjecture holds for other minuscule posets. The main result of this paper is the proof of the above conjecture for \(\mathcal{L}\) being the Bruhat poset of Schubert varieties in a minuscule \(G/P\) (cf. Theorem 5.16); we refer to the corresponding \(X (\mathcal{L})\) as a Bruhat-Hibi toric variety. In fact, we show (cf. Theorem 4.13) that the above conjecture holds for more general \(X (\mathcal{L})\), namely, \(\mathcal{L}\) being a distributive lattice such that \(J (\mathcal{L})\) (the poset of join irreducibles) is a grid lattice (see §3 for the definition of a grid lattice). We further prove (cf. Theorem 4.13) that the singular locus of \(X (\mathcal{L})\) is pure of codimension 3 in \(X (\mathcal{L})\), and that the generic singularities are of cone type (more precisely, the singularity type is same as that at the vertex of the cone over the quadric surface \(x_1 x_4 - x_2 x_3 = 0\) in \(\mathbb{P}^3\)).

**Sketch of proof of the above conjecture for Bruhat-Hibi toric varieties:** Let \(\mathcal{L}\) be the distributive lattice of Schubert varieties in a minuscule \(G/P\), or more generally, a distributive lattice such that the poset of join irreducibles is a grid lattice. Let \(T\) denote the torus acting on the toric variety \(X (\mathcal{L})\). Let \(M\) be the character group of \(T\). Let \(\sigma\) be the polyhedral cone associated to the toric variety \(X (\mathcal{L})\). If \(\sigma^\vee\) is the cone dual to \(\sigma\) and \(S_\sigma = \sigma^\vee \cap M\), then \(K [X (\mathcal{L})]\) is the semigroup algebra \(K [S_\sigma]\). For a face \(\tau\) of \(\sigma\), let \(D_\tau = \{\alpha \in \mathcal{L} \mid P_\tau (\alpha) \neq 0\}\), where \(P_\tau\) (cf. §1.4) is the center of the orbit \(O_\tau\). Now \(X_\tau\), the toric variety associated to the cone \(\tau\), is open in \(X_\sigma (= X (\mathcal{L}))\). Thus \(X_\sigma\) is smooth at \(P_\tau\) if and only if \(X_\tau\) is smooth at \(P_\tau\); further, \(X_\tau\) is smooth at \(P_\tau\) if and only if \(X_\tau\) is non-singular.
For \( \tau \) such that \( D_\tau = \mathcal{L}_{\alpha,\beta} = \mathcal{L} \setminus [\alpha \wedge \beta, \alpha \vee \beta] \), where \((\alpha, \beta)\) is an incomparable pair of join-meet irreducibles in \( \mathcal{L} \), we first determine a set of generators for \( \tau \) as a cone, and show that \( X_\tau \) is a singular variety. Conversely, if \( \tau \) is such that \( D_\tau \) is not contained in any \( \mathcal{L}_{\alpha,\beta} \), we show that \( X_\tau \) is non-singular. Thus the above conjecture is proved. As a consequence, we obtain that \( \text{Sing} \ X (\mathcal{L}) \) is pure of codimension 3 in \( X (\mathcal{L}) \).

It should be remarked that the Hibi toric varieties are studied in [20] also where the author proves that the singular locus of a Hibi toric variety has codimension at least three.

The sections are organized as follows: In \( \S 1 \) we recall some generalities on affine toric varieties. In \( \S 2 \) we introduce the Hibi toric varieties, and recollect some of the results (cf. [16]) on Hibi toric varieties required for our discussion. In \( \S 3 \) we introduce grid lattices and prove some preliminary results on a distributive lattice whose poset of join irreducibles is a grid lattice. In \( \S 4 \) we determine the singular locus of \( X(\mathcal{L}) \), \( \mathcal{L} \) being as above. In \( \S 5 \) we apply the results of \( \S 4 \) to Bruhat-Hibi toric varieties and determine the singular loci of these varieties.

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### 1. Generalities on toric varieties

Since our main object of study is a certain affine toric variety, we recall in this section some basic definitions on affine toric varieties. Let \( T = (K^*)^m \) be an \( m \)-dimensional torus.

**Definition 1.1.** (cf. [8], [14]) An *equivariant affine embedding* of a torus \( T \) is an affine variety \( X \subseteq \mathbb{A}^l \) containing \( T \) as a dense open subset and equipped with a \( T \)-action \( T \times X \to X \) extending the action \( T \times T \to T \) given by multiplication. If in addition \( X \) is normal, then \( X \) is called an *affine toric variety*.

### 1.2. The Cone Associated to a Toric Variety

Let \( M \) be the character group of \( T \), and \( N \) the \( \mathbb{Z} \)-dual of \( M \). Recall (cf. [8], [14]) that there exists a strongly convex rational polyhedral cone \( \sigma \subset N_\mathbb{R}(= N \otimes_\mathbb{Z} \mathbb{R}) \) such that

\[
K[X] = K[S_\sigma],
\]

where \( S_\sigma \) is the subsemigroup \( \sigma^\vee \cap M \), \( \sigma^\vee \) being the cone in \( M_\mathbb{R} \) dual to \( \sigma \), namely, \( \sigma^\vee = \{ f \in M_\mathbb{R} \mid f(v) \geq 0, v \in \sigma \} \). Note that \( S_\sigma \) is a finitely generated subsemigroup in \( M \).
1.3. Orbit Decomposition in Affine Toric Varieties. We shall denote $X$ also by $X_\sigma$. We may suppose, without loss of generality, that $\sigma$ spans $N_\mathbb{R}$ so that the dimension of $\sigma$ equals $\dim N_\mathbb{R} = \dim T$. (Here, by dimension of $\sigma$, one means the vector space dimension of the span of $\sigma$.)

1.4. The distinguished point $P_\tau$. Each face $\tau$ determines a (closed) point $P_\tau$ in $X_\sigma$, namely, it is the point corresponding to the maximal ideal in $K[X](= K[S_\sigma])$ given by the kernel of $e_\tau : K[S_\sigma] \to K$, where for $u \in S_\sigma$, we have

$$e_\tau(u) = \begin{cases} 1, & \text{if } u \in \tau^\perp \\ 0, & \text{otherwise} \end{cases}$$

(here, $\tau^\perp$ denotes $\{u \in M_\mathbb{R} | u(v) = 0, \forall v \in \tau\}$)

1.5. Orbit Decomposition. Let $O_\tau$ denote the $T$-orbit in $X_\sigma$ through $P_\tau$. We have the following orbit decomposition in $X_\sigma$:

$$X_\sigma = \bigcup_{\theta \leq \sigma} O_\theta$$

$$\overline{O_\tau} = \bigcup_{\theta \geq \tau} O_\theta$$

$$\dim \tau + \dim O_\tau = \dim X_\sigma$$

See [8], [14] for details.

2. The toric variety associated to a distributive lattice

We shall now study a special class of toric varieties, namely, the toric varieties associated to distributive lattices. We shall first collect some definitions as well as some notation. Let $(\mathcal{L}, \leq)$ be a poset, i.e, a finite partially ordered set. We shall suppose that $\mathcal{L}$ is bounded, i.e., it has a unique maximal, and a unique minimal element, denoted $\hat{1}$ and $\hat{0}$ respectively. For $\mu, \lambda \in \mathcal{L}, \mu \leq \lambda$, we shall denote

$$[\mu, \lambda] := \{\tau \in \mathcal{L}, \mu \leq \tau \leq \lambda\}$$

We shall refer to $[\mu, \lambda]$ as the interval from $\mu$ to $\lambda$.

Definition 2.1. The ordered pair $(\lambda, \mu)$ is called a cover (and we also say that $\lambda$ covers $\mu$ or $\mu$ is covered by $\lambda$) if $[\mu, \lambda] = \{\mu, \lambda\}$. 
2.2. Distributive lattices.

Definition 2.3. A lattice is a partially ordered set \((\mathcal{L}, \leq)\) such that, for every pair of elements \(x, y \in \mathcal{L}\), there exist elements \(x \lor y\) and \(x \land y\), called the join, respectively the meet of \(x\) and \(y\), defined by:

\[
x \lor y \geq x, \quad x \lor y \geq y,
\]

and if \(z \geq x\) and \(z \geq y\), then \(z \geq x \lor y\),

\[
x \land y \leq x, \quad x \land y \leq y,
\]

and if \(z \leq x\) and \(z \leq y\), then \(z \leq x \land y\).

Definition 2.4. Given a lattice \(\mathcal{L}\), a subset \(\mathcal{L}' \subset \mathcal{L}\) is called a sublattice of \(\mathcal{L}\) if \(x, y \in \mathcal{L}'\) implies \(x \land y \in \mathcal{L}'\); \(\mathcal{L}'\) is called an embedded sublattice of \(\mathcal{L}\) if \(\tau, \phi \in \mathcal{L}, \tau \lor \phi, \tau \land \phi \in \mathcal{L}'\) \(\Rightarrow \tau, \phi \in \mathcal{L}'\).

It is easy to check that the operations \(\lor\) and \(\land\) are commutative and associative.

Definition 2.5. A lattice is called distributive if the following identities hold:

\[
\begin{align*}
x \land (y \lor z) &= (x \land y) \lor (x \land z) \quad (1) \\
x \lor (y \land z) &= (x \lor y) \land (x \lor z). \quad (2)
\end{align*}
\]

Definition 2.6. An element \(z\) of a lattice \(\mathcal{L}\) is called join-irreducible (respectively meet-irreducible) if \(z = x \lor y\) (respectively \(z = x \land y\)) implies \(z = x\) or \(z = y\). The set of join-irreducible (respectively meet-irreducible) elements of \(\mathcal{L}\) is denoted by \(J(\mathcal{L})\) (respectively \(M(\mathcal{L})\)), or just by \(J\) (respectively \(M\)) if no confusion is possible.

Definition 2.7. An element in \(J(\mathcal{L}) \cap M(\mathcal{L})\) is called irreducible.

In the sequel, we shall denote \(J(\mathcal{L}) \cap M(\mathcal{L})\) by \(JM(\mathcal{L})\), or just \(JM\) if no confusion is possible.

Definition 2.8. A subset \(I\) of a poset \(P\) is called an ideal of \(P\) if for all \(x, y \in P\),

\[
x \in I \text{ and } y \leq x \text{ imply } y \in I.
\]

Theorem 2.9 (Birkhoff). Let \(\mathcal{L}\) be a distributive lattice with 0, and \(P\) the poset of its nonzero join-irreducible elements. Then \(\mathcal{L}\) is isomorphic to the lattice of ideals of \(P\), by means of the lattice isomorphism

\[
\alpha \mapsto I_\alpha := \{\tau \in P \mid \tau \leq \alpha\}, \quad \alpha \in \mathcal{L}.
\]

The following Lemma is easily checked.

Lemma 2.10. With the notations as above, we have

(a) \(J = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\tau, \lambda)\}\).

(b) \(M = \{\tau \in \mathcal{L} \mid \text{there exists at most one cover of the form } (\lambda, \tau)\}\).
Lemma 2.11 (cf. [16]). Let $(\tau, \lambda)$ be a cover in $\mathcal{L}$. Then $I_{\tau}$ equals $I_\lambda \cup \{\beta\}$ for some $\beta \in J(\mathcal{L})$.

2.12. The variety $X(\mathcal{L})$. Consider the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$; let $a(\mathcal{L})$ be the ideal generated by \{ $X_\alpha X_\beta - X_{\alpha \lor \beta} X_{\alpha \land \beta}$, $\alpha, \beta \in \mathcal{L}$ \}. Then one knows (cf. [12]) that $K[X_\alpha, \alpha \in \mathcal{L}] / a(\mathcal{L})$ is a normal domain; in particular, we have that $a(\mathcal{L})$ is a prime ideal. Let $X(\mathcal{L})$ be the affine variety of the zeroes in $K^l$ of $a(\mathcal{L})$ (here, $l = \# \mathcal{L}$). Then $X(\mathcal{L})$ is an affine normal variety defined by binomials. On the other hand, by [7], we have that a binomial prime ideal is toric (here, “toric ideal” is in the sense of [19], Chapter 4). Hence $X(\mathcal{L})$ is a toric variety for the action by a suitable torus $T$.

In the sequel, we shall denote $R(\mathcal{L}) := K[X_\alpha, \alpha \in \mathcal{L}] / a(\mathcal{L})$. Further, for $\alpha \in \mathcal{L}$, we shall denote the image of $X_\alpha$ in $R(\mathcal{L})$ by $x_\alpha$.

Definition 2.13. The variety $X(\mathcal{L})$ will be called a Hibi toric variety.

Remark 2.14. An extensive study of $X(\mathcal{L})$ appears first in [12].

We have that $\dim X(\mathcal{L}) = \dim T$.

Theorem 2.15 (cf. [16]). The dimension of $X(\mathcal{L})$ is equal to $\# J(\mathcal{L})$. Further, $\dim X(\mathcal{L})$ equals the cardinality of the set of elements in a maximal chain in (the graded poset) $\mathcal{L}$.

2.16. Cone and dual cone of $X(\mathcal{L})$. As above, denote the poset of join-irreducibles in $\mathcal{L}$ by $J(\mathcal{L})$ or just $J$. Denote by $\mathcal{I}(J)$ the poset of ideals of $J$. For $A \in \mathcal{I}(J)$, denote by $m_A$ the monomial:

$$m_A := \prod_{\tau \in A} y_\tau$$

in the polynomial algebra $K[y_\tau, \tau \in J(\mathcal{L})]$. If $\alpha$ is the element of $\mathcal{L}$ such that $I_\alpha = A$ (cf. Theorem 2.9), then we shall denote $m_A$ also by $m_\alpha$. Consider the surjective algebra map

$$F : K[X_\alpha, \alpha \in \mathcal{L}] \to K[m_A, A \in \mathcal{I}(J)], X_\alpha \mapsto m_A, A = I_\alpha$$

Theorem 2.17 (cf. [12], [16]). We have an isomorphism

$$K[X(\mathcal{L})] \cong K[m_A, A \in \mathcal{I}(J)].$$

Let us denote the torus acting on the toric variety $X(\mathcal{L})$ by $T$; by Theorem 2.13 we have, $\dim T = \# J(\mathcal{L}) = d$, say. Identifying $T$ with $(K^*)^d$, let $\{ f_z, z \in J(\mathcal{L}) \}$ denote the standard $\mathbb{Z}$-basis for $X(T)$, namely, for $t = (t_z, z \in J(\mathcal{L}))$, $f_z(t) = t_z$. Denote $M := X(T)$; let
Let \( N \) be the \( \mathbb{Z} \)-dual of \( M \), and \( \{ e_y, y \in J(\mathcal{L}) \} \) be the basis of \( N \) dual to \( \{ f_z, z \in J(\mathcal{L}) \} \). For \( A \in \mathcal{I}(J) \), set

\[
f_A := \sum_{z \in A} f_z
\]

Let \( V = N_{\mathbb{R}}(= N \otimes_{\mathbb{Z}} \mathbb{R}) \). Let \( \sigma \subset V \) be the cone such that \( X(\mathcal{L}) = X_{\sigma} \).

As an immediate consequence of Theorem 2.17, we have

**Proposition 2.18.** The semigroup \( S_{\sigma} \) is generated by \( f_A, A \in \mathcal{I}(J) \).

Let \( M(J(\mathcal{L})) \) be the set of maximal elements in the poset \( J(\mathcal{L}) \). Let \( Z(J(\mathcal{L})) \) denote the set of all covers in the poset \( J(\mathcal{L}) \). For a cover \((y, y') \in Z(J(\mathcal{L}))\), denote

\[
v_{y,y'} := e_{y'} - e_y
\]

**Proposition 2.19** (cf. [16], Proposition 4.7). The cone \( \sigma \) is generated by \( \{ e_z, z \in M(J(\mathcal{L})), v_{y,y'}, (y, y') \in Z(J(\mathcal{L})) \} \).

**2.20. The sublattice \( D_{\tau} \).** We shall concern ourselves just with the closed points in \( X(\mathcal{L}) \). So in the sequel, by a point in \( X(\mathcal{L}) \), we shall mean a closed point. Let \( \tau \) be a face of \( \sigma \), and \( P_{\tau} \) the distinguished point (cf. §1.4). For a point \( P \in X(\mathcal{L}) \) (identified with a point in \( \mathbb{A}^l, l = \#\mathcal{L} \)), let us denote by \( P(\alpha) \), the \( \alpha \)-th co-ordinate of \( P \). Let

\[
D_{\tau} = \{ \alpha \in \mathcal{L} | P_{\tau}(\alpha) \neq 0 \}
\]

We have,

**Lemma 2.21** (cf. [16]). \( D_{\tau} \) is an embedded sublattice.

Conversely, we have

**Lemma 2.22** (cf. [16]). Let \( D \) be an embedded sublattice in \( \mathcal{L} \). Then \( D \) determines a unique face \( \tau \) of \( \sigma \) such that \( D_{\tau} \) equals \( D \).

Thus in view of the two Lemmas above, we have a bijection

\[
\{ \text{faces of } \sigma \} \leftrightarrow \{ \text{embedded sublattices of } \mathcal{L} \}
\]

**Proposition 2.23** (cf. [16]). Let \( \tau \) be a face of \( \sigma \). Then we have \( \overline{O_{\tau}} = X(D_{\tau}) \).

**3. Grid Lattices**

In this section, we restrict our attention to a specific class of distributive lattices, and show that some desirable properties hold. Give \( \mathbb{N} \times \mathbb{N} \) the lattice structure

\[
(\alpha_1, \alpha_2) \wedge (\beta_1, \beta_2) = (\delta_1, \delta_2), \quad (\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) = (\gamma_1, \gamma_2),
\]

where \( \delta_i, \gamma_i \in \mathbb{N} \).
where \( \delta_i = \min\{\alpha_i, \beta_i\}, \gamma_i = \max\{\alpha_i, \beta_i\} \).

**Definition 3.1.** Let \( J \) be a finite, distributive sublattice of \( \mathbb{N} \times \mathbb{N} \), such that if \( \alpha \) covers \( \beta \) in \( J \), then \( \alpha \) covers \( \beta \) in \( \mathbb{N} \times \mathbb{N} \) as well. Then we say \( J \) is a grid lattice.

**Remark 3.2.** For \( J \) a grid lattice, we have the following:

1. \( J \) is a distributive lattice.
2. For any \( \mu \in J \), there exist at most two distinct covers of the form \((\alpha, \mu)\) in \( J \), i.e., there are at most two elements in \( J \) covering \( \mu \).
3. For any \( \lambda \in J \), \( \lambda \) covers at most two distinct elements in \( J \).
4. If \( \alpha, \beta \) are two covers of \( \mu \) in \( J \), then \( \alpha \lor \beta \) covers both \( \alpha, \beta \); thus the interval \([\mu, \alpha \lor \beta]\) is a rank 2 subposet of \( J \).

**Example:**

\[
\begin{array}{ccc}
3, 6 & 4, 6 & 4, 5 \\
2, 6 & 3, 5 \\
2, 5 & 3, 4 \\
2, 4 & 3, 3 \\
1, 4 & 2, 3 \\
1, 3 & 1, 2 \\
\end{array}
\]

3.3. For the rest of this section, let \( J \) be a grid lattice, and let \( L \) be the poset of ideals of \( J \). From Theorem 2.9, we have that \( L \) is a distributive lattice with \( J \) as its poset of join irreducibles. Thus we will correlate join irreducible elements in \( L \) with elements of \( J \). Recall that for \( x, y \in L \), \( x \geq y \) if and only if \( I_x \supseteq I_y \) as ideals in \( J \).

**Lemma 3.4.** Given \( \gamma_1, \gamma_2 \in J \), \((\gamma_1 \land \gamma_2)_L \) belongs to \( J \) and is in fact equal to \((\gamma_1 \land \gamma_2)_J \).

**Proof.** Let \( \theta = (\gamma_1 \land \gamma_2)_J \) and \( \phi = (\gamma_1 \land \gamma_2)_L \). Clearly \( \theta \in I_{\gamma_1} \cap I_{\gamma_2} = I_{\phi} \). Therefore \( I_{\theta} \subseteq I_{\phi} \). Let now \( \eta \in I_{\phi}(\subset J) \). Then \( \eta \leq \phi \), and thus \( \eta \) is less than or equal to both \( \gamma_1 \) and \( \gamma_2 \) in \( L \), and therefore in \( J \). Hence \( \eta \leq \theta \), and thus \( I_{\phi} \subseteq I_{\theta} \). The result follows. \( \square \)
Lemma 3.5. Let $(\alpha, \beta)$ be an incomparable pair of irreducibles (cf. Definition 2.7) in $\mathcal{L}$. Then

1. $\alpha, \beta$ are meet irreducibles in $J$,
2. $(\alpha \land \beta)_\mathcal{L} = (\alpha \land \beta)_J \in J$.

Proof. Part (2) follows from Lemma 3.4 (note that $\alpha, \beta \in J$). Now say $\alpha = (\gamma_1 \land \gamma_2)_J$ for an incomparable pair $(\gamma_1, \gamma_2)$ in $J$. Lemma 3.4 implies that $\alpha = (\gamma_1 \land \gamma_2)_\mathcal{L}$, a contradiction since $\alpha$ is meet irreducible in $\mathcal{L}$. Part (1) follows.

Thus an incomparable pair $(\alpha, \beta)$ of irreducibles in $\mathcal{L}$ determines a (unique) non-meet irreducible in $J$ (namely, $(\alpha \land \beta)_\mathcal{L}$).

We shall now show (cf. Lemma 3.8 below) that conversely a non-meet irreducible element $\mu$ in $J$ determines a unique incomparable pair $(\alpha, \beta)$ of irreducibles in $\mathcal{L}$. We first prove a couple of preliminary results:

Lemma 3.6. Let $\mu$ be a non-meet irreducible element in $J$. Then $\mu$ determines an incomparable pair $(\alpha, \beta)$ of elements (in $J$) both of which are meet-irreducible in $J$.

Proof. Let $\mu = (\mu_1, \mu_2)$ (considered as an element of $\mathbb{N} \times \mathbb{N}$). Since $\mu$ is non-meet irreducible element in $J$, there exist $x = (x_1, x_2), y = (y_1, y_2)$ in $J, x, y > \mu$ such that $x_2 > \mu_2, y_1 > \mu_1$. Define $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ in $J$ as

$$
\alpha = \text{the maximal element } x > \mu \text{ in } J \text{ such that } x_1 = \mu_1,
\beta = \text{the maximal element } y > \mu \text{ in } J \text{ such that } y_2 = \mu_2.
$$

Clearly $\alpha, \beta$ are both meet-irreducible in $J$ (note that $(\mu_1 + 1, \alpha_2)$ (resp. $(\beta_1, \mu_2 + 1)$) is the unique element in $J$ covering $\alpha$ (resp. $\beta$) in $J$). Also, it is clear that $(\alpha, \beta)$ is an incomparable pair.

Let $\mu, \alpha, \beta$ be as in the above Lemma. In particular, we have, $\mu_1 = \alpha_1 < \beta_1, \mu_2 = \beta_2 < \alpha_2$.

Lemma 3.7. With notation as in Lemma 3.6, we have,

1. $(\alpha \lor \beta)_J = (\beta_1, \alpha_2)$,
2. $\alpha$ is the maximal element of the set $\{ x = (x_1, x_2) \in J \mid x_1 = \alpha_1 \}$, and $\beta$ is the unique maximal element of the set $\{ x = (x_1, x_2) \in J \mid x_2 = \beta_2 \}$.

Proof. Assertion (2) is immediate from the definition of $\alpha, \beta$. Assertion (1) is also clear.

Lemma 3.8. Let $\mu, \alpha, \beta$ be as in Lemma 3.6. Then $\alpha$ and $\beta$ are irreducibles in $\mathcal{L}$. Thus the non-meet irreducible element $\mu$ of $J$ determines a unique incomparable pair of irreducibles in $\mathcal{L}$.
Proof. We will show the result for $\alpha$ (the proof for $\beta$ being similar). Since $\alpha \in J$, $\alpha$ is join irreducible in $L$. It remains to show that $\alpha$ is meet irreducible in $L$. If possible, let us assume that there exists an incomparable pair $(\theta_1, \theta_2)$ in $L$ such that $\theta_1 \wedge \theta_2 = \alpha$; without loss of generality, we may suppose that $\theta_1$ and $\theta_2$ both cover $\alpha$. Then there exist (cf. Lemma 2.11) $\gamma, \delta \in J$ such that

$$I_{\theta_1} = I_\alpha \cup \{\gamma\}, I_{\theta_2} = I_\alpha \cup \{\delta\}.$$  

We have

$$I_\gamma \cap I_\delta \subset I_{\theta_1} \cap I_{\theta_2} = I_\alpha.$$  

Also, $\gamma, \delta$ are either covers of $\alpha$ in $J$, or non-comparable to $\alpha$. (They cannot be less than $\alpha$ because they are not in $I_\alpha$.)

**Case 1:** Suppose $\gamma$ and $\delta$ are covers of $\alpha$ in $J$. Then $\alpha$ is not meet irreducible in $J$, a contradiction (cf. Lemma 3.5(1)).

**Case 2:** Suppose $\gamma$ covers $\alpha$ in $J$, and $\delta$ is non-comparable to $\alpha$. Let $\delta = (\delta_1, \delta_2)$, $\xi = (\xi_1, \xi_2) = (\alpha \vee \delta)_J$. Then the fact that $\xi > \alpha$ (since $\alpha, \delta$ are incomparable) implies (in view of Lemma 3.7(2)) that $\xi_1 > \mu_1$; hence $\delta_1 = (\xi_1) \geq \mu_1 + 1$, and $\delta_2 < \alpha_2$. Also, $\gamma = (\mu_1 + 1, \alpha_2)$ (cf. Lemma 3.7(2)). Therefore $\gamma \wedge \delta = (\mu_1 + 1, \delta_2)$, but this element is non-comparable to $\alpha$, and thus $I_\gamma \cap I_\delta \not\subset I_\alpha$, a contradiction to $(\ast)$. Hence we obtain that the possibility "$\gamma$ covers $\alpha$ in $J$ and $\delta$ is non-comparable to $\alpha"$ does not exist. A similar proof shows that the possibility "$\delta$ covers $\alpha$ in $J$ and $\gamma$ is non-comparable to $\alpha"$ does not exist.

**Case 3:** Suppose both $\gamma = (\gamma_1, \gamma_2)$ and $\delta = (\delta_1, \delta_2)$ are non-comparable to $\alpha = (\mu_1, \alpha_2)$. As in Case 2, we must have $\delta_2 < \alpha_2$, and thus $\delta_1 > \mu_1$. Similarly, $\gamma_2 < \alpha_2$, $\gamma_1 > \mu_1$. Thus the minimum of $\{\gamma_1, \delta_1\}$ is still greater than $\mu_1$, therefore $I_\gamma \cap I_\delta \not\subset I_\alpha$, a contradiction to $(\ast)$.

Thus our assumption that $\alpha$ is non-meet irreducible in $L$ is wrong, and it follows that $\alpha$ (and similarly $\beta$) is meet irreducible in $L$. \qed

We continue with the above notation; in particular, we denote $\mu = (\mu_1, \mu_2), \mu_1 = \alpha_1 < \beta_1, \mu_2 = \beta_2 < \alpha_2$.

**Lemma 3.9.** Let $x = (x_1, x_2) \in J$. If $x \not\in I_\alpha \cup I_\beta$, then $x > \alpha \wedge \beta$.

*Proof.* By hypothesis, we have $x \not\leq \alpha, x \not\leq \beta$.

We first claim that $x_1 > \alpha_1$; for, if possible, let us assume $x_1 \leq \alpha_1$. Since $x \not\leq \alpha$, we must have $x_2 > \alpha_2$. Thus $x \lor \alpha = (\alpha_1, x_2) > \alpha$, since $\alpha \not\leq x$; but this is a contradiction, by the property of $\alpha$ (cf. Lemma 3.7(2)). Hence our assumption is wrong, and we get $x_1 > \alpha_1$.

Similarly, we have $x_2 > \beta_2$, and the result follows (note that by our notation (and definition of $\alpha, \beta$), we have $\alpha \wedge \beta = (\alpha_1, \beta_2)$). \qed
Definition 3.10. For an incomparable (unordered) pair \((\alpha, \beta)\) of irreducible elements in \(\mathcal{L}\), define
\[
\mathcal{L}_{\alpha,\beta} = \mathcal{L} \setminus [\alpha \land \beta, \alpha \lor \beta].
\]

Proposition 3.11. \(\mathcal{L}_{\alpha,\beta}\) is an embedded sublattice.

Proof. First, we show that \(\mathcal{L}_{\alpha,\beta}\) is a sublattice. To do this, we identify \(\mathcal{L}\) with the “lattice of ideals” of \(J\). Thus, for \(x \in \mathcal{L}_{\alpha,\beta}\), either \(I_x \not\subseteq (I_\alpha \cap I_\beta)\) or \(I_x \not\subseteq (I_\alpha \cup I_\beta)\), by definition of \(\mathcal{L}_{\alpha,\beta}\). Note that \(I_\alpha \cap I_\beta = I_{\alpha \land \beta}\), and \(I_\alpha \cup I_\beta = I_{\alpha \lor \beta}\).

Case 1: Let \(x, y \in \mathcal{L}_{\alpha,\beta}\) such that \(I_x, I_y \not\subseteq I_{\alpha \land \beta}\). Then clearly \(I_x \cap I_y \not\subseteq I_{\alpha \land \beta}\); and thus \(x \land y \in \mathcal{L}_{\alpha,\beta}\). We also have (by the definition of ideals) that \(\alpha \land \beta \not\subseteq I_x, I_y\) (note that \(\alpha \land \beta \in J\) (cf. Lemma 3.5(2))), therefore \(\alpha \land \beta \not\subseteq I_x \cup I_y\), and therefore \(x \lor y \in \mathcal{L}_{\alpha,\beta}\).

Case 2: Let \(x, y \in \mathcal{L}_{\alpha,\beta}\) such that \(I_x \not\subseteq I_{\alpha \land \beta}\) and \(I_y \not\subseteq I_{\alpha \lor \beta}\). Then clearly \(I_x \cap I_y \not\subseteq I_{\alpha \land \beta}\) and \(I_x \cup I_y \not\subseteq I_{\alpha \lor \beta}\). Hence, \(x \lor y, x \land y \in \mathcal{L}_{\alpha,\beta}\).

Case 3: Let \(x, y \in \mathcal{L}_{\alpha,\beta}\) such that \(I_x, I_y \not\subseteq I_{\alpha \lor \beta}\). Clearly \(I_x \cup I_y \not\subseteq I_{\alpha \lor \beta}\); hence, \(x \lor y \in \mathcal{L}_{\alpha,\beta}\).

Claim: \(I_x \cap I_y \not\subseteq I_{\alpha \lor \beta}\).

Note that Claim implies that \(x \land y \in \mathcal{L}_{\alpha,\beta}\). If possible, let us assume that \(I_x \cap I_y \subseteq I_{\alpha \lor \beta}\). Now the hypothesis that \(I_x, I_y \not\subseteq I_{\alpha \lor \beta}\) implies that there exist \(\theta, \delta \in I_{\alpha \lor \beta}\) such that \(\theta \in I_x, \theta \not\subseteq I_{\alpha \lor \beta}\), and \(\delta \in I_y, \delta \not\subseteq I_{\alpha \lor \beta}\). Now \(I_\theta \cap I_\delta \subseteq I_x \cap I_y \subseteq I_{\alpha \lor \beta}\) (note that by our assumption, \(I_x \cap I_y \subseteq I_{\alpha \lor \beta}\)). Hence we obtain that either \(\theta \land \delta \leq \alpha\) or \(\theta \land \delta \leq \beta\); let us suppose \(\theta \land \delta \leq \alpha\) (proof is similar if \(\theta \land \delta \leq \beta\)). By Lemma 3.9 we have that both \(\theta, \delta \geq \alpha \land \beta\), and hence \(\theta \land \delta \geq \alpha \land \beta\). Thus

\[
\alpha \geq \theta \land \delta \geq \alpha \land \beta = (\alpha_1, \beta_2)
\]

(**)

Let \(\xi = (\xi_1, \xi_2) = \theta \land \delta\). Then (***) implies that \(\xi_1 = \alpha_1\); hence at least one of \(\{\theta_1, \delta_1\}\), say \(\theta_1\) equals \(\alpha_1\). This implies that \(\theta_2 > \alpha_2\) (since \(\theta \not\subseteq I_{\alpha}\)). This contradicts Lemma 3.5(2). Hence our assumption is wrong and it follows that \(I_x \cap I_y \not\subseteq I_{\alpha \lor \beta}\).

This completes the proof in Case 3. Thus we have shown that \(\mathcal{L}_{\alpha,\beta}\) is a sublattice.

Next, we will show that \(\mathcal{L}_{\alpha,\beta}\) is an embedded sublattice. Let \(x, y \in \mathcal{L}\) be such that \(x \lor y, x \land y\) are in \(\mathcal{L}_{\alpha,\beta}\). We need to show that \(x, y \in \mathcal{L}_{\alpha,\beta}\). This is clear if either \(x \land y \not\subseteq \alpha \lor \beta\) or \(x \lor y \not\subseteq \alpha \land \beta\) (in the former case, \(x, y \not\subseteq \alpha \lor \beta\), and in the latter case, \(x, y \not\subseteq \alpha \land \beta\)). Let us then suppose that \(x \land y \subseteq \alpha \lor \beta\) and \(x \lor y \subseteq \alpha \land \beta\); this implies that \(x \land y \not\subseteq \alpha \land \beta\) and \(x \lor y \not\subseteq \alpha \lor \beta\) (since, \(x \lor y, x \land y\) are in \(\mathcal{L}_{\alpha,\beta}\), i.e., \(I_x \cap I_y \not\subseteq I_{\alpha \lor \beta}\) and \(I_x \cup I_y \not\subseteq I_{\alpha \lor \beta}\)). We will now show that \(x, y \in \mathcal{L}_{\alpha,\beta}\).
Since \( \alpha \land \beta \not\in I_x \cap I_y \), we have that one of the elements \( \{x, y\} \) must not be greater than or equal to \( \alpha \land \beta \), say \( x \not\geq \alpha \land \beta \). This implies that \( x \in \mathcal{L}_{\alpha, \beta} \). It remains to show that \( y \in \mathcal{L}_{\alpha, \beta} \). If \( y \not\geq \alpha \land \beta \), then we would obtain that \( y \in \mathcal{L}_{\alpha, \beta} \). Let us then assume that \( y \geq \alpha \land \beta \); i.e. \( I_y \supset I_\alpha \cap I_\beta \). Note that for any \( \delta \in I_x \), we have \( \delta \leq x \) and thus \( \delta \not\geq \alpha \land \beta \). By Lemma 3.9, \( \delta \in I_\alpha \cup I_\beta \), and therefore \( I_x \subset I_\alpha \cup I_\beta \). Since by hypothesis \( I_x \cup I_y \not\subset I_\alpha \cup I_\beta \), we must have \( I_y \not\subset I_\alpha \cup I_\beta \). Therefore, \( y \in \mathcal{L}_{\alpha, \beta} \).

This completes the proof of the assertion that \( \mathcal{L}_{\alpha, \beta} \) is an embedded sublattice, and therefore the proof of the Proposition. \( \square \)

4. Singular locus of \( X(\mathcal{L}) \)

In this section, we determine the singular locus of \( X(\mathcal{L}) \), \( \mathcal{L} \) being as in §3. Let \( \sigma \) be the cone associated to \( X(\mathcal{L}) \). We follow the notation of §11 and §2.

**Definition 4.1.** A face \( \tau \) of \( \sigma \) is a singular (resp. non-singular) face if \( P_\tau \) is a singular (resp. non-singular) point of \( X_\sigma \).

**Definition 4.2.** Let us denote by \( W \) the set of generators for \( \sigma \) as described in Proposition 2.19. Let \( \tau \) be a face of \( \sigma \), and let \( D_\tau \) be as in §2.20. Define

\[
W(\tau) = \{ v \in W \mid f_{I_\alpha}(v) = 0, \forall \alpha \in D_\tau \}.
\]

Then \( W(\tau) \) gives a set of generators for \( \tau \).

4.3. Determination of \( W(\tau) \). Let \((\alpha, \beta)\) be an incomparable (unordered) pair of irreducible elements of \( \mathcal{L} \). By Proposition 3.11, \( \mathcal{L}_{\alpha, \beta} \) is an embedded sublattice of \( \mathcal{L} \) (\( \mathcal{L}_{\alpha, \beta} \) being as in Definition 3.10). Let \( \tau_{\alpha, \beta} \) be the face of \( \sigma \) corresponding to \( \mathcal{L}_{\alpha, \beta} \) (cf. Lemma 2.22; note that \( D_{\tau_{\alpha, \beta}} = \mathcal{L}_{\alpha, \beta} \)). Let us denote \( \tau = \tau_{\alpha, \beta} \). Following the notation of §3, let \( \mu(= (\mu_1, \mu_2)) = \alpha \land \beta, \alpha_1 = \mu_1, \beta_2 = \mu_2 \). Since \( \mu \) is not meet irreducible in \( J \), there are two elements \( A \) and \( B \) in \( J \) covering \( \mu \), namely, \( A = (\alpha_1, \beta_2 + 1), B = (\alpha_1 + 1, \beta_2) \). Also, we have that \( A \lor B \) (in the lattice \( J \) covers both \( A \) and \( B \), (cf. Remark 3.2). Let \( C = (A \lor B)_J \); then \( C = (\alpha_1 + 1, \beta_2 + 1) \).

It will aid our proof below to notice a few facts about the generating set \( W(\tau) \) of \( \tau \). First of all, \( e_1 \) is not a generator for any \( \tau_{\alpha, \beta} \); because \( i \in \mathcal{L}_{\alpha, \beta} \) for all pairs \((\alpha, \beta)\), and \( e_1 \) is non-zero on \( f_{I_1} \).

Secondly, for any cover \((y, y'), y > y' \) in \( J(\mathcal{L}) \), \( e_{y'} - e_y \) is not a generator of \( \tau \) if \( y' \in \mathcal{L}_{\alpha, \beta} \), because \( f_{I_{y'}}(e_{y'} - e_y) \neq 0 \). Thus, in determining the elements of \( W(\tau) \), we need only be concerned with elements \( e_{y'} - e_y \) of \( W \) such that \( y' \in J \cap [\alpha \land \beta, \alpha \lor \beta] \).
Lemma 4.4. $J \cap [\alpha \land \beta, \alpha \lor \beta] = \{x \in J \mid x \in [\mu, \alpha] \cup [\mu, \beta]\}$.

Proof. The inclusion $\supseteq$ is clear. To show the inclusion $\subseteq$, let $x \in J \cap [\alpha \land \beta, \alpha \lor \beta]$. If possible, assume $x \not\in [\mu, \alpha] \cup [\mu, \beta]$; the assumption implies that $x \not\in I_\alpha \cup I_\beta (= I_{\alpha \lor \beta})$. Hence we obtain that $x \not\leq \alpha \lor \beta$, a contradiction to the hypothesis that $x \in [\alpha \land \beta, \alpha \lor \beta]$. □

Lemma 4.5. The set $\{x \in J \mid x \not\in I_\alpha \cup I_\beta\}$ has a unique minimal element; moreover that element is $C$.

Proof. For any $x$ in this set, we have $x > \alpha \land \beta$ (cf. Lemma 3.9). Hence by Lemma 3.7, (2), and the hypothesis that $x \not\in I_\alpha \cup I_\beta$, we obtain that $x_1 > \alpha_1$, $x_2 > \beta_2$. Therefore,

$$\{x \in J \mid x \not\in I_\alpha \cup I_\beta\} = \{x \in J \mid x_1 > \alpha_1, x_2 > \beta_2\}.$$

This set clearly has a minimal element, namely $C = (\alpha_1 + 1, \beta_2 + 1)$. □

Theorem 4.6. Following the notation from above, we have

$$W(\tau) = \{e_\mu - e_A, e_\mu - e_B, e_A - e_C, e_B - e_C\}.$$

Proof. Claim 1: $W(\tau) \supset \{e_\mu - e_A, e_\mu - e_B, e_A - e_C, e_B - e_C\}$.

We must show that for any $x \in \mathcal{L}_{\alpha, \beta}$, $f_{I_x}$ is zero on these four elements of $W$. If possible, let us assume that there exists a $x \in \mathcal{L}_{\alpha, \beta}$ such that $f_{I_x}$ is non-zero on some of the above four elements. Then clearly $x \geq \mu(= \alpha \land \beta)$. Hence $x \not\leq \alpha \lor \beta$ (since $x \not\in [\alpha \land \beta, \alpha \lor \beta]$), i.e., $I_x \not\subseteq I_\alpha \cup I_\beta$. Therefore $I_x$ contains some join irreducible $\gamma$ such that $\gamma \not\leq \alpha, \beta$; hence, $I_x \not\subseteq I_\alpha \cup I_\beta$. This implies (cf. Lemma 4.5) that $\gamma \geq C$. Hence we obtain that $C \in I_x$. Therefore, $x \geq C$, and $f_{I_x}$ is zero on all of the four elements of Claim 1, a contradiction to our assumption. Hence our assumption is wrong and Claim 1 follows.

Claim 2: $W(\tau) = \{e_\mu - e_A, e_\mu - e_B, e_A - e_C, e_B - e_C\}$.

In view of §4.3 it is enough to show that for all $\theta \in J \cap [\alpha \land \beta, \alpha \lor \beta]$, the element $e_\theta - e_\delta \in W$ which is different from the four elements of Claim 1 is not in $W(\tau)$. In view of Lemma 4.4 it suffices to examine all covers in $J$ of all elements in $([\mu, \alpha] \cup [\mu, \beta])_J$. This diagram represents
the part of the grid lattice $J$ we are concerned with:

\[ \begin{array}{c}
\alpha \\
A'' \\
C'' \\
\rightarrow \\
A' \\
\rightarrow \\
C' \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
B \\
\mu \\
\end{array} \]

In the diagram above, consider $A' = (\alpha_1, \beta_2 + n)$, $A'' = (\alpha_1, \beta_2 + n + 1)$, and $C' = (\alpha_1 + 1, \beta_2 + n)$. Note that all elements of $J \cap [\mu, \alpha]$ can be written in the form of $A'$. Thus, we need to check elements $e_{A'} - e_{A''}$ and $e_{A'} - e_{C'}$ in $W$.

First, we observe that $C' \in L_{\alpha, \beta}$, and $f_{I_{C'}}$ is non-zero on $e_{A'} - e_{A''}$. Hence $e_{A'} - e_{A''} \notin W(\tau)$.

Next, let $x = (A' \lor C)_{L}$, (note that $x$ is not in $J$, and thus does not appear on the diagram above). Then $I_x = I_{A'} \cup I_C$; and we have $x \in L_{\alpha, \beta}$ (since, $C \notin I_{\alpha} \cup I_{\beta}$ and $x > C$, we have, $x \notin \alpha \lor \beta$). Moreover, $f_{I_x}$ is non-zero on $e_{A'} - e_{C'}$. Hence $e_{A'} - e_{C'} \notin W(\tau)$.

This completes the proof for the interval $[\mu, \alpha]$, and a similar discussion yields the same result for the interval $[\mu, \beta]$.

Thus Claim 2 (and hence the Theorem) follows. \hfill \square

As an immediate consequence of Theorem 4.6, we have the following

**Theorem 4.7.** Let $(\alpha, \beta)$ be an incomparable pair of irreducibles in $L$. We have an identification of the (open) affine piece in $X(\mathcal{L})$ corresponding to the face $\tau_{\alpha, \beta}$ with the product $Z \times (K^*)^{\#L-3}$, where $Z$ is the cone over the quadric surface $x_1 x_4 - x_2 x_3 = 0$ in $\mathbb{P}^3$.

**Lemma 4.8.** The dimension of the face $\tau_{\alpha, \beta}$ equals 3.

**Proof.** By Theorem 4.6, a set of generators for $\tau_{\alpha, \beta}$ is given by $\{e_{\mu} - e_A, e_{\mu} - e_B, e_A - e_C, e_B - e_C\}$. We see that a subset of three of these generators is linearly independent. Thus if the fourth generator can be put in terms of the first three, the result follows. Notice that

\[ (e_{\mu} - e_A) - (e_{\mu} - e_B) + (e_A - e_C) = e_B - e_C. \]
The following fact from [5] (Lemmas 6.21, 6.22) holds for a general toric variety.

**Fact 4.9.** Let $\tau$ be a face of $\sigma$. Then $P_\tau$ is a smooth point of $X_\sigma$ if and only if $X_\tau$ is non-singular.

Combining the above fact with Theorem 4.7, we obtain the following:

**Theorem 4.10.** $P_\tau \in \text{Sing } X_\sigma$, for $\tau = \tau_{\alpha,\beta}$. Further, the singularity at $P_\tau$ is of the same type as that at the vertex of the cone over the quadric surface $x_1x_4 - x_2x_3 = 0$ in $\mathbb{P}^3$.

Next, we will show that the faces containing some $\tau_{\alpha,\beta}$ are the only singular faces.

**Lemma 4.11.** Let $(y, y')$, $y > y'$ be a cover in $J$. Then either $e_{y'} - e_y \in W(\tau_{\alpha,\beta})$ for some incomparable pair $(\alpha, \beta)$ of irreducibles in $L$, or $y, y'$ are comparable to every other element of $J$.

**Proof. Case 1:** Let $y'$ be non-meet irreducible in $J$.

In view of the hypothesis, we can find an incomparable pair $(\alpha, \beta)$ of irreducibles in $L$ such that $y' = \alpha \wedge \beta$, as shown in Lemma 3.8 (with $\mu = y'$). Thus $e_{y'} - e_y = e_\mu - e_A$ or $e_{y'} - e_y = e_\mu - e_B$ as in Theorem 4.6.

**Case 2:** Let $y'$ be meet irreducible, but not join irreducible (in $J$).

Let $x_1$ and $x_2$ be the two elements covered by $y'$ in $J$ (cf. Remark 3.2); thus $(x_1 \vee x_2)_J = y'$.

For convenience of notation, all join and meet operations in this proof will refer to the join and meet operations in the lattice $J$.

**Claim (a):** If both $x_1$ and $x_2$ are meet irreducible (in $J$), then $y', y$ are comparable to every element of $J$.

If possible, let us assume that there exists a $z \in J$ such that $z$ is non-comparable to $y'$. We first observe that $z$ is non-comparable to both $x_1$ and $x_2$; for, say $z, x_1$ are comparable, then $z > x_1$ necessarily (since $z, y'$ are non-comparable). This implies that $x_1 \leq z \wedge y' < y'$, and hence we obtain that $x_1 = z \wedge y' < y'$ (since $(y', x_1)$ is a cover), a contradiction to the hypothesis that $x_1$ is meet irreducible. Thus we obtain that $z$ is non-comparable to both $x_1$ and $x_2$. Now, we have, $z \vee x_i \geq z \vee y'$ (note that $x_i, i = 1, 2$ being meet irreducible in $J$, $y'$ is the unique element covering $x_i, i = 1, 2$, and hence $z \vee x_i \geq y'$). Hence $(z \vee y') \geq z \vee x_i \geq z \vee y'$, and we obtain

$$z \vee x_1 = z \vee y' = z \vee x_2.$$  

On the other hand, the fact that $z \wedge y' < y'$ implies that $z \wedge y' \leq x_1$ or $x_2$. Let $i$ be such that $z \wedge y' \leq x_i$. Then $z \wedge y' \leq z \wedge x_i \leq z \wedge y'$;
therefore
\[ z \land x_i = z \land y'. \]

Now
\[ y' \land (x_i \lor z) = y' \land (y' \lor z) = y'; \quad (y' \land x_i) \lor (y' \land z) = x_i \lor (x_i \land z) = x_i. \]
Therefore \( J \) is not a distributive lattice (Definition 2.5), a contradiction. Hence our assumption is wrong and it follows that \( y' \) is comparable to every element of \( J \), and since \( y \) is the unique cover of \( y' \), \( y \) is also comparable to every element of \( J \). Claim (a) follows.

Continuing with the proof in Case 2, in view of Claim (a), we may suppose that \( x_1 \) is not meet irreducible (in \( J \)). Then by Lemma 3.6 (with \( \mu = x_1 \)), there exists a unique incomparable pair \( (\alpha, \beta) \) of meet irreducibles (in \( J \)) such that \( x_1 = \alpha \land \beta \). In view of the fact that \( y' \) is a cover of \( x_1 \), we obtain that \( y' \) is equal to \( A \) or \( B \) (\( A, B \) being as in §4.3), say \( y' = A \); this in turn implies that \( y = C \) (\( C \) being as in §4.3; note that by hypothesis, \( y \) is the unique element covering \( y' \) in \( J \)). Therefore we obtain that \( e_{x_1} - e_{y'} (= e_{x_1} - e_A), e_{y'} - e_y (= e_A - e_C) \) are in \( W(\tau_{\alpha,\beta}) \).

This completes the proof of the assertion in Case 2.

**Case 3:** Let \( y' \) be both meet irreducible and join irreducible in \( J \).

If \( y' \) is comparable to every other element of \( J \), then \( y \) is also comparable to every other element of \( J \), since by hypothesis, \( y \) is the unique element covering \( y' \) in \( J \); and the result follows.

Let then there exist a \( z \in J \) such that \( z \) and \( y' \) are incomparable. This in particular implies that \( y' \neq 0_J \); let \( x \in J \) be covered by \( y' \) (in fact, by hypothesis, \( x \) is unique). Proceeding as in Case 2 (especially, the proof of Claim (a)), we obtain that \( x \) is non-meet irreducible. Hence taking \( \mu = x \) in Lemma 3.6 and proceeding as in Case 2, we obtain that \( e_{y'} - e_y \) is in \( W(\tau_{\alpha,\beta}) \) \( ((\alpha, \beta) \) being the incomparable pair of irreducibles determined by \( \mu \)).

This completes the proof of the Lemma. \( \square \)

**Theorem 4.12.** Let \( \tau \) be a face of \( \sigma \) such that \( D_\tau \) is not contained in any \( L_{\alpha,\beta} \), for all incomparable pair \( (\alpha, \beta) \) of irreducibles in \( \mathcal{L} \); in other words \( \tau \) does not contain any \( \tau_{\alpha,\beta} \). Then \( \tau \) is nonsingular.

**Proof.** As in Definition 4.2 let
\[ W(\tau) = \{ v \in W \mid f_{I_\alpha}(v) = 0, \forall \alpha \in D_\tau \}. \]
Then \( W(\tau) \) gives a set of generators for \( \tau \). By Remark 4.9 and §2.1 of §8, for \( \tau \) to be nonsingular, it must be generated by part of a basis for \( N \) \( (N \) being as in §11). If \( W(\tau) \) is linearly independent, then it would follow that \( \tau \) is non-singular. (Generally this is not enough to prove
that \( \tau \) is nonsingular; but since all generators in \( W \) have coefficients equal to \( \pm 1 \), any linearly independent subset of \( W \) will serve as part of a basis for \( N \).

If possible, let us assume that \( W(\tau) \) is linearly dependent. Recall that the elements of \( W \) can be represented as all the line segments in the lattice \( J \), with the exception of \( e_1 \). Therefore, the linearly dependent generators \( W(\tau) \) of \( \tau \) must represent a “loop” of line segments in \( J \). This loop will have at least one bottom corner, left corner, top corner, and right corner.

Let us fix an incomparable pair \((\alpha, \beta)\) of irreducibles in \( L \). By Theorem 4.6, we have that \( W(\tau_{\alpha, \beta}) = \{ e_\mu - e_A, e_\mu - e_B, e_A - e_C, e_B - e_C \} \) (notation being as in that Theorem). These four generators are represented by the four sides of a diamond in \( J \). Thus, by hypothesis, the generators of \( \tau \) represent a loop in \( J \) that does not traverse all four sides of the diamond representing all four generators of \( \tau_{\alpha, \beta} \). We have the following identification for \( L_{\alpha, \beta} \):

\[
L_{\alpha, \beta} = \{ x \in L \mid f_{I_\theta} \equiv 0 \text{ on } W(\tau_{\alpha, \beta}) \}.
\]

(†)

The above identification for \( L_{\alpha, \beta} \) together with the hypothesis that \( D_{\tau} \nsubseteq L_{\alpha, \beta} \) implies the existence of a \( \theta \in D_{\tau} \cap [\alpha \land \beta, \alpha \lor \beta] \); note that by (†), we have

\[
f_{I_\theta} \neq 0 \text{ on } W(\tau_{\alpha, \beta})
\]

This implies in particular that \( \theta \npreceq C \) (\( C \) being as the proof of Theorem 4.6); also, \( \theta \succeq \mu(= \alpha \land \beta) \), since \( \theta \in [\alpha \land \beta, \alpha \lor \beta] \). Based on how \( \theta \) compares to both \( A \) and \( B \), we can eliminate certain elements of \( W \) from \( W(\tau) \). There are four possibilities; we list all four, as well as the corresponding generators in \( W(\tau_{\alpha, \beta}) \) which are not in \( W(\tau) \), i.e., those generators \( v \) in \( W(\tau_{\alpha, \beta}) \) such that \( f_{I_\theta}(v) \neq 0 \):

\[
\begin{align*}
\theta \npreceq A, \theta \npreceq B & \Rightarrow e_\mu - e_A, e_\mu - e_B \notin W(\tau) \\
\theta \succeq A, \theta \npreceq B & \Rightarrow e_A - e_C, e_\mu - e_B \notin W(\tau) \\
\theta \npreceq A, \theta \succeq B & \Rightarrow e_\mu - e_A, e_B - e_C \notin W(\tau) \\
\theta \succeq A, \theta \succeq B & \Rightarrow e_A - e_C, e_B - e_C \notin W(\tau)
\end{align*}
\]

Therefore, we obtain

neither \( \{ e_\mu - e_A, e_A - e_C \} \) nor \( \{ e_\mu - e_B, e_B - e_C \} \) is contained in \( W(\tau) \) (\(*\))

for any \( \tau_{\alpha, \beta} \) ((\( \alpha, \beta \)) being an incomparable pair of irreducibles in \( L \)).

Let \( y', z' \) denote respectively, the left and right corners of our loop; let \( (y, y'), (z, z') \) denote the corresponding covers (in \( J \)) which are contained in our loop. Now \( y', z' \) are non-comparable; hence, by
Lemma 4.11 we obtain that \((y, y')\) (resp. \((z, z')\)) are contained in some \(W(\tau_{\alpha, \beta})\) (resp. \(W(\tau_{\alpha', \beta'})\)). Hence we obtain (by Theorem 4.6, with notation as in that Theorem)

\[ \{e_{\mu} - e_{y'}, e_{y'} - e_{y}\} = \{e_{\mu} - e_A, e_A - e_C\} \text{ or } \{e_{\mu} - e_B, e_B - e_C\} \]

But this contradicts \((\ast)\). Thus our loop in \(J\) that represented \(W(\tau)\) cannot have both left and right corners; therefore \(W(\tau)\) is not a loop at all, a contradiction. Hence, our assumption (that \(W(\tau)\) is linearly dependent) is wrong, and the result follows.

Combining the above Theorem with Theorem 4.10 and Lemma 4.8, we obtain our first main Theorem:

**Theorem 4.13.** Let \(L\) be a distributive lattice such that \(J(L)\) is a grid lattice. Then

1. \(\text{Sing } X(L) = \bigcup_{(\alpha, \beta)} \sigma_{\tau_{\alpha, \beta}}\), the union being taken over all incomparable pairs \((\alpha, \beta)\) of irreducibles in \(L\).
2. \(\text{Sing } X(L)\) is pure of codimension 3 in \(X(L)\); further, the generic singularities are of cone type (more precisely, the singularity type is the same as that at the vertex of the cone over the quadric surface \(x_1x_4 - x_2x_3 = 0\) in \(\mathbb{P}^3\)).

5. **Singular Loci of Bruhat-Hibi Toric Varieties**

In this section, we prove results for Bruhat-Hibi Toric Varieties. We first start with recalling minuscule \(G/P\)’s

5.1. **Minuscule Weights and Lattices.** Let \(G\) be a semisimple, simply connected algebraic group. Let \(T\) be a maximal torus in \(G\). Let \(X(T)\) be the character group of \(T\), and \(B\) a Borel subgroup containing \(T\). Let \(R\) be the root system of \(G\) relative to \(T\); let \(R^+\) (resp. \(S = \{\alpha_1, \ldots, \alpha_l\}\)) be the set of positive (resp. simple) roots in \(R\) relative to \(B\) (here, \(l\) is the rank of \(G\)). Let \(\{\omega_i, 1 \leq i \leq l\}\) be the fundamental weights. Let \(W\) be the Weyl group of \(G\), and \((,\) a \(W\)-invariant inner product on \(X(T) \otimes \mathbb{R}\). For generalities on semisimple algebraic groups, we refer the reader to [3].

Let \(P\) be a maximal parabolic subgroup of \(G\) with \(\omega\) as the associated fundamental weight. Let \(W_P\) be the Weyl group of \(P\) (note that \(W_P\) is the subgroup of \(W\) generated by \(\{s_\alpha \mid \alpha \in S_P\}\)). Let \(W^P = W/W_P\). We have that the Schubert varieties of \(G/P\) are indexed by \(W^P\), and thus \(W^P\) can be given the partial order induced by the inclusion of Schubert varieties.
Definition 5.2. A fundamental weight $\omega$ is called \textit{minuscule} if $\langle \omega, \beta \rangle \leq 1$ for all $\beta \in R^+$; the maximal parabolic subgroup associated to $\omega$ is called a \textit{minuscule parabolic subgroup}.

Remark 5.3 (cf [13]). Let $P$ be a maximal parabolic subgroup; if $P$ is minuscule then $W/W_P$ is a distributive lattice.

Definition 5.4. For $P$ a minuscule parabolic subgroup, we call $L = W/W_P$ a \textit{minuscule lattice}.

Definition 5.5. We call $X(L)$ a \textit{Bruhat-Hibi toric variety} (B-H toric variety for short) if $L$ is a minuscule lattice.

In order to begin work on these B-H toric varieties, we first list all of the minuscule fundamental weights. Following the indexing of the simple roots as in [4], we have the complete list of minuscule weights for each type:

- Type $A_n$: Every fundamental weight is minuscule
- Type $B_n$: $\omega_n$
- Type $C_n$: $\omega_1$
- Type $D_n$: $\omega_1, \omega_{n-1}, \omega_n$
- Type $E_6$: $\omega_1, \omega_6$
- Type $E_7$: $\omega_7$.

There are no minuscule weights in types $E_8, F_4, \text{ or } G_2$.

Before proving that each minuscule lattice has grid lattice join irreducibles, we must introduce some additional lattice notation. For a poset $P$, let $I(P)$ represent the lattice of ideals of $P$. Thus for a distributive lattice $L$, $L = I(J(L))$ (cf. Theorem 2.9). (Notice that the empty set is considered the minimal ideal, and in Theorem 2.9 we do not include the minimal element in $P$. Therefore, in this section, $I(J)$ will have a minimal element that is not an element of $J$.)

For $k \in \mathbb{N}$, let $\mathbf{k}$ be the totally ordered set with $k$ elements. The symbols $\oplus$ and $\times$ denote the disjoint union and (Cartesian) product of posets.

Let $X_n(\omega_i)$ denote the minuscule lattice $W/W_P$ where $P$ is a parabolic subgroup associated to $\omega_i$ in the root system of type $X_n$. 
Theorem 5.6 (cf. [18] Propositions 3.2 and 4.1\footnote{Our notation differs significantly than that used in [18]; namely that where we use $I$, Proctor uses $J$; whereas we use $J$ to signify the set of join irreducibles.}). The minuscule lattices have the following combinatorial descriptions:

\[
\begin{align*}
\text{A}_{n-1} (\omega_j) &\cong I(I(j - 1 \oplus n - j - 1)) \\
\text{C}_n (\omega_1) &\cong 2n \\
\text{B}_n (\omega_n) &\cong \text{D}_{n+1} (\omega_{n+1}) \cong \text{D}_n (\omega_n) \cong I(I(1 \oplus n - 2)) \\
\text{D}_n (\omega_1) &\cong I^{n-1} (1 \oplus 1) \\
\text{E}_6 (\omega_1) &\cong \text{E}_6 (\omega_6) \cong I^4 (1 \oplus 2) \\
\text{E}_7 (\omega_7) &\cong I^5 (1 \oplus 2).
\end{align*}
\]

This theorem is very convenient in working with the faces of B-H toric varieties, because the join irreducible lattice of each of these minuscule lattices is very easy to see, simply by eliminating one $I(\cdot)$ operation. Our goal is to show that each minuscule lattice has join irreducibles with grid lattice structure.

5.7. Minuscule lattices $\text{A}_{n-1} (\omega_j)$.

Remark 5.8 (cf. [18] Proposition 4.2). The join irreducibles of the minuscule lattice $\text{A}_{n-1} (\omega_j)$ are isomorphic to the lattice $j \times n - j$. Therefore, every element of $J(\text{A}_{n-1} (\omega_j))$ can be written as the pair $(a, b)$, for $1 \leq a \leq j$, $1 \leq b \leq n - j$. This leads us to the following result,

Corollary 5.9. The minuscule lattice $\text{A}_{n-1} (\omega_j)$ has grid lattice join irreducibles.

Note that the result about the singular loci of B-H toric varieties of type $\text{A}_n (\omega_j)$ was already proved in [5], as well as more results about the multiplicities of singular points, but using the unique combinatorics of these lattices.

5.10. Minuscule lattices $\text{C}_n (\omega_1)$. This minuscule lattice is totally ordered, and the associated B-H toric variety is simply the affine space of dimension $2n$.

5.11. Minuscule lattices $\text{B}_{n-1} (\omega_{n-1}) \cong \text{D}_n (\omega_{n-1}) \cong \text{D}_n (\omega_n)$. From Theorem 5.6 we have

\[
J(\text{D}_n (\omega_n)) \cong I^2 (1 \oplus n - 3) \cong \text{A}_{n-1} (\omega_2).
\]
It is a well known result that $A_{n-1}(2)$ represents the lattice of Schubert varieties in the Grassmannian of 2-planes in $K^n$, and the Schubert varieties are indexed by $I_{2,n} = \{(i_1, i_2) \mid 1 \leq i_1 < i_2 \leq n\}$. Therefore,

$$J(D_n(\omega_n)) \cong I_{2,n}.$$ 

The lattice $I_{2,n}$ is therefore distributive, (being another minuscule lattice), and clearly a grid lattice. This leads to the following result,

**Corollary 5.12.** The minuscule lattices $B_{n-1}(\omega_{n-1})$, $D_n(\omega_{n-1})$, $D_n(\omega_n)$ have grid lattice join irreducibles.

**5.13. Minuscule lattices $D_n(\omega_1)$.** From Theorem 5.6 we have $J(D_n(\omega_1)) \cong \mathcal{I}^{n-2}(1 \oplus 1)$. This lattice of join irreducibles is isomorphic to the following sublattice of $\mathbb{N} \times \mathbb{N}$ (drawn horizontally):

```
(2, n - 2) --- (2, n - 1) --- (n, n - 1)
\        \     \        \     \\
(1, 1) --- (1, n - 2) --- (1, n - 1)
```

Clearly this is a grid lattice.

**5.14. Minuscule lattices $E_6(\omega_1) \cong E_6(\omega_6)$, and $E_7(\omega_7)$.** Let $H_6 = E_6(\omega_1) = E_6(\omega_6)$ and $H_7 = E_7(\omega_7)$. Since there are only two exceptional cases, it is best to explicitly give the grid lattice structure to the join irreducibles. Thus, we have the two join irreducible lattices below, with each lattice point given coordinates in $\mathbb{N} \times \mathbb{N}$. Coincidentally, $J(H_6) = D_5(\omega_5)$ and $J(H_7) = H_6$. 

---

**Note:** The notation $J(D_n(\omega_n)) \cong I_{2,n}$ refers to the join lattice of Schubert varieties, which is isomorphic to the lattice of Schubert varieties in the Grassmannian of 2-planes in $K^n$. The lattice $I_{2,n}$ is distributive and clearly a grid lattice, leading to the result of Corollary 5.12. The minuscule lattices $B_{n-1}(\omega_{n-1})$, $D_n(\omega_{n-1})$, and $D_n(\omega_n)$ have grid lattice join irreducibles, as stated in Corollary 5.12. For $D_n(\omega_1)$, the join lattice is represented by a grid structure with the specified coordinates. The minuscule lattices $E_6(\omega_1)$ and $E_7(\omega_7)$ are compared to $D_5(\omega_5)$ and $H_6$, respectively, with $J(H_6) = D_5(\omega_5)$ and $J(H_7) = H_6$. The notation $\mathcal{I}^{n-2}(1 \oplus 1)$ refers to the lattice structure resulting from Theorem 5.6.
This completes the individual discussion for each type of minuscule lattice, leading us to the following result.

**Corollary 5.15.** If $\mathcal{L}$ is a minuscule lattice, then $J(\mathcal{L})$ is a grid lattice.

Thus, for $\mathcal{L}$ any minuscule lattice, letting

$$\Phi = \{ (\alpha, \beta) \mid \alpha, \beta \text{ non-comparable irreducibles in } \mathcal{L} \},$$

we have completed the proof of the conjecture from [5], thanks to Theorem 4.13.

**Theorem 5.16.** For the B-H toric variety $X(\mathcal{L})$, 

$$\text{Sing } X(\mathcal{L}) = \bigcup_{(\alpha, \beta) \in \Phi} \tau_{\alpha, \beta}.$$
In other words, $X(L)\text{ is smooth at } P_\tau$ ($\tau$ being a face of $\sigma$) if and only if for each pair $(\alpha, \beta) \in \Phi$, there exists at least one $\gamma \in [\alpha \land \beta, \alpha \lor \beta]$ such that $P_\tau(\gamma)$ is non-zero.

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