Well-posedness of first order semilinear PDEs by stochastic perturbation

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Abstract

We show that first order semilinear PDEs by stochastic perturbation are well-posedness for globally Holder continuous and bounded vector field, with an integrability condition on the divergence. This result extends the linear case presented in [2]. The proof is based on in the stochastic characteristics method and a version of the commuting Lemma.

Key words: Stochastic characteristic method, First order stochastic partial differential equations, Stochastic perturbation, commuting Lemma.

MSC2000 subject classification: 60H10 , 60H15.

1 Introduction

This work is motivated by the paper [2] where the linear equation

$$\begin{cases}
    du(t, x) + b(t, x)\nabla u(t, x)dt + \nabla u(t, x) \circ dB_t = 0, \\
    u(0, x) = f(x) \in L^\infty(\mathbb{R}^d),
\end{cases}$$

(1)

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has been studied, was proved existence and uniqueness of $L^\infty$-solutions for a globally Holder continuous and bounded vector field, with an integrability condition on the divergence, and where $B_t = (B^1_t, ..., B^d_t)$ is a standard Brownian motion in $\mathbb{R}^d$.

The aim of this paper is to investigate parts of this theory under the effect of nonlinear terms. Namely, we consider the semilinear SPDE

$$
\begin{cases}
    du(t, x) + b(t, x)\nabla u(t, x) \, dt + F(t, x, u) \, dt + \nabla u(t, x) \circ dB_t = 0, \\
    u(0, x) = f(x) \in L^\infty(\mathbb{R}^d).
\end{cases}
$$

(2)

We shall prove the existence and uniqueness of weak $L^\infty$-solutions for a globally Holder continuous and bounded vector field, with an integrability condition on the divergence. Moreover, we obtain a representation of the solution via stochastic flows. This is an example of nonlinear SPDE where the stochastic perturbation makes the equation well-posed.

The fundamental tools used here is the stochastic characteristics method (see for example [1], [5] and [7]) and the version of the commuting Lemma presented in [2]. That is, we follow the strategy given in [2] in combination with the stochastic characteristics method.

The article is organized as follows: Section 2 we shall define the concept of weak $L^\infty$-solutions for the equation (2) and we shall prove existence for this class of solutions. In section 3, we shall show a uniqueness theorem for weak $L^\infty$-solutions.

Through of this paper we fix a stochastic basis with a $d$-dimensional Brownian motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$.

2 Existence of weak $L^\infty$-solutions

Let $T > 0$ be fixed. For $\alpha \in (0, 1)$ define the space $L^\infty([0, T], C^\alpha(\mathbb{R}^d))$ as the set of all bounded Borel functions $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ for which
\[ [f]_{\alpha,T} = \sup_{t \in [0,T]} \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|} \]

We write the \( L^\infty([0,T], C^\alpha(\mathbb{R}^d, \mathbb{R}^d)) \) for the space of all vector fields having components in \( L^\infty([0,T], C^\alpha(\mathbb{R}^d)) \).

We shall assume that

\[ b \in L^\infty([0,T], C^\alpha(\mathbb{R}^d, \mathbb{R}^d)), \quad (3) \]

\[ \text{Div} \ b \in L^p([0,T] \times \mathbb{R}^d) \text{ for } p > 2. \quad (4) \]

\[ F \in L^1([0,T], L^\infty(\mathbb{R}^d \times \mathbb{R})) \quad (5) \]

and

\[ F \in L^\infty([0,T] \times \mathbb{R}^d, LIP(\mathbb{R})). \quad (6) \]

### 2.1 Definition of weak \( L^\infty \)–solutions

**Definition 2.1** We assume \( (3), (4), (5) \) and \( (6) \). A weak \( L^\infty \)–solution of the Cauchy problem \( (2) \) is a stochastic process \( u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d) \) such that, for every test function \( \varphi \in C_0^\infty(\mathbb{R}^d) \), the process \( \int u(t,x)\varphi(x)dx \) has a continuous modification which is a \( \mathcal{F}_t \)-semimartingale and satisfies

\[
\int u(t,x)\varphi(x)dx = \int f(x)\varphi(x) \, dx + \int_0^t \int b(s,x)\nabla \varphi(x)u(s,x) \, dx \, ds + \int_0^t \int \text{div} \ b(s,x)\varphi(x)u(s,x) \, dx \, ds \\
+ \int_0^t \int F(s,x,u)\varphi(x) \, dx \, ds + \sum_{i=0}^d \int_0^t \int D_i\varphi(x)u(s,x) \, dx \, dB_i^s
\]
Remark 2.1 We observe that a weak $L^\infty$ solution in the previous Stratonovich sense satisfies the Itô equation

$$\int u(t, x)\varphi(x)dx = \int f(x)\varphi(x) \, dx$$

$$+ \int_0^t \int b(s, x)\nabla\varphi(x)u(s, x) \, dxds + \int_0^t \int \text{div} \, b(s, x)\varphi(x)u(s, x) \, dxds$$

$$+ \int_0^t \int F(s, x, u)\varphi(x) \, dxds + \sum_{i=0}^d \int_0^t \int D_i\varphi(x)u(s, x) \, dxdB^i_s + \frac{1}{2} \int_0^t u(s, x)\triangle\varphi(x) \, dxds$$

for every test function $\varphi \in C_0^\infty(\mathbb{R}^d)$. The converse is also true.

2.2 Existence of weak $L^\infty$-solutions

Lemma 2.1 Let $f \in L^\infty(\mathbb{R}^d)$. We assume (3), (4), (5) and (6). Then there exits a weak $L^\infty$-solution $u$ of the SPDE (2).

Proof: Step 1 Assume that $F \in L^1([0, T], C_b^\infty(\mathbb{R}^d \times \mathbb{R}))$ and $f \in C_b^\infty(\mathbb{R}^d)$. We take a mollifier regularization $b_n$ of $b$. It is known (see [1], chapter 1) that there exist an unique classical solution $u_n(t, x)$ of the SPDE (2), that written in weak Itô form is (7) with $b_n$ in place of $b$. Moreover,

$$u_n(t, x) = Z^n(t, x, f(Y^n_t))$$

where $Y^n_t$ is the inverse of $X^n_t$, $X^n_t(x)$ and $Z^n_t(x, r)$ satisfy the following equations

$$X^n_t = x + \int_0^t b_n(s, X^n_s) \, ds + B_t,$$

and

$$Z^n_t = r + \int_0^t F(s, X^n_s(x), Z^n_s) \, ds.$$
According to theorem 5 of [2], see too remark 8, we have that
\[ \lim_{n \to \infty} \mathbb{E} \left[ \int K \sup_{t \in [0,T]} |X^n_t - X_t| \, dx \right] = 0 \]
and
\[ \lim_{n \to \infty} \mathbb{E} \left[ \int K \sup_{t \in [0,T]} |DX^n_t - DX_t| \, dx \right] = 0 \]
for any compact set \( K \subset \mathbb{R}^d \), where \( X_t(x) \) verifies
\[ X_t = x + \int_0^t b(s, X_s) \, ds + B_t. \tag{10} \]

Now, we denote
\[ u(t, x) = Z_t(x, f(Y_t)), \]

\( Y_t \) is the inverse of \( X_t \),

and
\[ Z_t = r + \int_0^t F(s, X_s(x), Z_s) \, ds. \tag{11} \]

Then, we observe that
\[ |u^n(t,x) - u(t,x)| \leq |f(Y_t) - f(Y^n_t)| + \int_0^t |F(s, X^n_s, Z^n_s(f(Y^n_t)) - F(s, X_s, Z_s(f(Y_t)))| \, ds \]
\[ \leq |f(Y_t) - f(Y^n_t)| + C \int_0^t |Z^n_s(f(Y^n_t)) - Z_s(f(Y_t))| \, ds. \]

From theorem 5 of [2], see too remark 8, and the Lipchitz property of \( F \) we conclude that \( \lim_{n \to \infty} \mathbb{E} \left[ \int K \sup_{t \in [0,T]} |u_n(t, x) - u(t, x)| \right] = 0 \) and \( u(t, x) \) is a weak \( L^\infty \)-solution of the SPDE (2).

Step 2 Assume that \( F \in L^1([0,T], C_b^\infty(\mathbb{R}^d \times \mathbb{R})) \). We take a mollifier regularization \( f_n \) of \( f \). By the last step \( u_n(t, x) = Z_t(x, f_n(Y_t)) \) is a weak
$L^\infty$–solution of the SPDE (2), that written in weak Itô form is (7) with $f_n$ in place of $f$.

We have that any compact set $K \subset \mathbb{R}^d$ and $p \geq 1$

$$\lim_{n \to \infty} \sup_{[0,T]} \int_K |f_n(X_t^{-1}) - f(X_t^{-1})|^p \, dx =$$

$$\lim_{n \to \infty} \sup_{[0,T]} \int_{X_t(K)} |f_n(x) - f(x)|^p \, JX_t(x) \, dx = 0$$

Then we have that

$$\lim_{n \to \infty} \sup_{[0,T]} \int_K |Z_t(x, f_n(Y_t)) - Z_t(x, f(Y_t))|^p \, dx = 0.$$ 

Thus $u(t, x) = Z_t(x, f(Y_t))$ is a weak $L^\infty$–solution of the SPDE (2).

Step 3 We take a mollifier regularization $F_n$ of $F$. By the step 2, we have that $u_n(t, x) = Z_t^n(x, f(Y_t))$ is a weak $L^\infty$–solution of the SPDE (2), and hold that $Z_t^n(x, r)$ satisfies the equation (11) with $F_n$ in place of $F$.

We observe that

$$|Z_t^n(x, r) - Z_t(x, r)| \leq \int_0^t |F_n(t, X_s, Z_s^n) - F(t, X_s, Z_s)| \, ds$$

$$\leq \int_0^t |F_n(t, X_s, Z_s^n) - F_n(t, X_s, Z_s)| \, ds + \int_0^t |F_n(t, X_s, Z_s) - F(t, X_s, Z_s)| \, ds$$

$$\leq C \int_0^t |Z_s^n - Z_s| \, ds + \int_0^t |F_n(t, X_s, Z_s) - F(t, X_s, Z_s)| \, ds$$

By the Gronwall Lemma we follow that

$$\lim_{n \to \infty} |Z_t^n(x, r) - Z_t(x, r)| = 0 \text{ uniformly in } t, x, r.$$ 

Then

$$\lim_{n \to \infty} |Z_t^n(x, f(Y_t)) - Z_t(x, f(Y_t)) = 0 \text{ uniformly in } t \text{ and } x.$$ 

Therefore, we conclude that $u(t, x) = Z_t(x, f(Y_t))$ is a weak $L^\infty$–solution of the SPDE (2).
3 Uniqueness of weak $L^\infty$–solutions

In this section, we shall present an uniqueness theorem for the SPDE (2) under similar conditions to the linear case, see theorem 20 of [2]. Let $\varphi_n$ be a standard mollifier. We introduced the commutator defined as

$$\mathcal{R}_n(b, u) = (b \nabla)(\varphi_n * u) - \varphi_n * ((b \nabla)u)$$

We recall here the following version of the commutator lemma which is at the base of our uniqueness theorem.

**Lemma 3.1** Let $\phi_t$ be an $C^1$–diffeomorphism of $\mathbb{R}^d$. Assume $b \in L^\infty_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$, $\text{div}b \in L^1_{\text{loc}}(\mathbb{R}^d)$, $u \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Moreover, for $d > 1$, assume also $J\phi^{-1} \in W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ Then for any $\rho \in C^\infty_0(\mathbb{R}^d)$ there exits a constant $C_\rho$ such that, given any $R > 0$ such that $\text{supp} (\rho \circ \phi^{-1}) \subset B(R)$, we have:

a) for $d > 1$

$$|\int \mathcal{R}_n(b, u)(\phi(x))\rho(x) \, dx| \leq C_\rho \|u\|_{L^\infty_{R+1}} \left[\|\text{div}b\|_{L^2_{R+1}} \|J\phi^{-1}\|_{L^\infty_{R}} + \|b\|_{L^\infty_{R+1}} (\|D\phi^{-1}\|_{L^\infty_{R}} + \|DJ\phi^{-1}\|_{L^1_{R}})\right]$$

b) for $d = 1$

$$|\int \mathcal{R}_n(b, u)(\phi(x))\rho(x) \, dx| \leq C_\rho \|u\|_{L^\infty_{R+2}} \|b\|_{W^{1,1}_{R+2}} \|J\phi^{-1}\|_{L^\infty_{R}}$$

**Proof:** See pp 28 of [2].

We are ready to prove our uniqueness result of weak $L^\infty$–solution to the Cauchy problem [2].

**Theorem 3.1** Assume (3), (4), (5) and (6). Then, for every $f \in L^\infty(\mathbb{R}^d)$ there exists an unique weak $L^\infty$–solution of the Cauchy problem [2].
Proof:

Step 1 (Itô-Ventzel-Kunita formula) Let $u, v$ be are two weak $L^\infty$-solutions and $\varphi_n$ be a standard mollifier. We put $w = u - v$, applying the Itô-Ventzel-Kunita formula (see Theorem 8.3 of [6]) to $F(y) = \int w(t, z)\varphi_n(y - z) \, dz$, we obtain that

$$\int w(t, z)\varphi_n(X_s - z) \, dz$$

is equal to

$$\int_0^t \int b(s, z)\nabla[\varphi_n(X_s - z)]w(s, z) \, dz \, ds + \int_0^t \int \text{div } b(s, z)\varphi_n(X_s - z)u(s, z) \, dz \, ds +$$

$$\int_0^t \int (F(s, z, u) - F(s, z, v))\varphi_n(X_s - z) \, dz \, ds + \sum_{i=1}^d \int_0^t \int w(s, z)D_i[\varphi_n(X_s - z)] \, dz \, dB^i_s +$$

$$\int_0^t \int (b\nabla)(w(\cdot, \cdot) \ast \varphi_n)(X_s) \, ds - \sum_{i=1}^d \int_0^t \int w(s, z)D_i[\varphi_n(X_s - z)] \, dz \, dB^i_s.$$

Then

$$\int w(t, z)\varphi_n(X_t - z) \, dz =$$

$$\int_0^t \int (F(s, z, u) - F(s, z, v))\varphi_n(X_s - z) \, dz \, ds + \int_0^t \mathcal{R}_n(w, b)(X_s(x)) \, ds,$$

where $\mathcal{R}_n$ is the commutator defined above.

Step 2 ($\lim_{n \to \infty} \int_0^t \mathcal{R}_n(w, b)(X_s) \, ds = 0$) We argue as in [2]. We observe by Lemma 3.1 and the Lebesgue dominated theorem that

$$\lim_{n \to \infty} \int_0^t \int \mathcal{R}_n(w, b)(X_s)\rho(x) \, ds = 0$$

for all $\rho \in C_0^\infty(\mathbb{R}^d)$, for details see Theorem 20 of [2].

Step 3 ($w = 0$) We observe that
\[
\lim_{n \to \infty} (w(t, \cdot) \ast \varphi_n)(\cdot) = w(t, \cdot)
\]
, where the convergence is in \( L^1([0, T], L^1_{\text{loc}}(\mathbb{R}^d)) \). From the flow properties of \( X_t \), see theorem 5 of \cite{2}, we obtain
\[
\lim_{n \to \infty} (w(t, \cdot) \ast \varphi_n)(X_t) = w(t, X_t)
\]
and
\[
\lim_{n \to \infty} ((F(t, \cdot, u) - F(t, \cdot, v)) \ast \varphi_n)(X_t) = (F(t, X_t, u(t, X_t)) - F(t, X_t, v(t, X_t)),
\]
where the convergence is \( \mathbb{P} \) a.s in \( L^1([0, T], L^1_{\text{loc}}(\mathbb{R}^d)) \). Then by steps 1, 2 we have
\[
w(t, X_t) = \int_0^t F(s, \cdot, X_s, u(t, X_s)) - F(s, \cdot, X_s, v(t, X_s)) \, ds.
\]
Thus, for any compact set \( K \subset \mathbb{R}^d \) we obtain that
\[
\int_K |w(t, X_t)| \, dx \leq \int_0^t \int_K |F(s, \cdot, X_s, u(t, X_s)) - F(s, \cdot, X_s, v(t, X_s))| \, dxds.
\]
\[
\leq C \int_0^t \int_K |w(t, X_s)| \, dxds.
\]
where \( C \) is constant related to the Lipchitz property of \( F \). It follows
\[
\int_K |w(t, X_t)| \, dx \leq C \int_0^t \int_K |w(t, X_s)| \, dxds.
\]
and thus \( w(t, X_t) = 0 \) by the Gronwall Lemma.
Remark 3.1 We observe that the unique solution $u(t, x)$ has the representation $u(t, x) = Z_t(x, f(X_t^{-1}))$, where $X_t$ and $Z_t$ satisfy the equations (10) and (11) respectively.

Remark 3.2 We mention that other variants of the theorem 3.1 can be proved. In fact, the step 2 is valid under other hypotheses, see corollary 23 of [2].

Remark 3.3 We recall that relevant examples of non-uniqueness for the deterministic linear transport equation are presented in [2] and [3]. Currently we do not get a counter-example itself of the non-linear case. An interesting future work is to study if the nonlinear case may induce new pathologies.

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