Continuous functions with impermeable graphs

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Funding information
Austrian Science Fund, Grant/Award Number: F5508-N26; H2020 European Research Council, Grant/Award Number: 741420; Nemzeti Kutatási Fejlesztési és Innovációs Hivatal, Grant/Award Number: 124749

Abstract
We construct a Hölder continuous function on the unit interval which coincides in uncountably (in fact continuum) many points with every function of total variation smaller than 1 passing through the origin. We conclude that this function has impermeable graph—one of the key concepts introduced in this paper—and we present further examples of functions both with permeable and impermeable graphs. Moreover, we show that typical (in the sense of Baire category) continuous functions have permeable graphs. The first example function is subsequently used to construct an example of a continuous function on the plane which is intrinsically Lipschitz continuous on the complement of the graph of a Hölder continuous function with impermeable graph, but which is not Lipschitz continuous on the plane. As another main result, we construct a continuous function on the unit interval which coincides in a set of Hausdorff dimension 1 with every function of total variation smaller than 1 which passes through the origin.

KEYWORDS
Hausdorff dimension of zeros, intrinsic metric, permeable graph, permeable sets, uncountable zeros

MSC (2020)
26A16, 28A78, 26A21, 26B30, 26B35, 54C05, 54E40

1 | INTRODUCTION

Although continuous functions on compact intervals are among the most prominent mathematical objects studied in undergraduate analysis courses, they continue to offer many interesting questions and results on an advanced level. One message of historical importance is that general continuous functions differ significantly from differentiable ones, in that they may be nowhere differentiable and nowhere locally of bounded variation.

The differences become less expressed if one compares continuous functions with Lipschitz- or Hölder continuous ones. For example, it is well-known that with respect to the Wiener measure, almost every continuous function is Hölder continuous, but is also almost nowhere differentiable and of infinite total variation, see [10, Theorem 9.25, Theorem 9.18, Exercise 5.21] or [19, Chapter I, Theorem 2.1, Exercise 2.9, Corollary 2.5]. In this paper, we introduce a new property of functions, namely, that of permeability of its graph. This notion derives from a geometric notion of permeability of subsets...
of metric spaces, introduced in [13], which we will recall in Section 3.3, and which is related to concepts of removability of subsets, cf. [9, 24].

Loosely speaking, a function has a permeable graph, if its graph can be avoided by the graph of some bounded variation function to the degree that the graphs have at most countably many intersection points.

For a function \( f : [a, b] \to \mathbb{R} \) let \( V_b^a(f) \) denote its (possibly infinite) total variation over the interval \([a, b]\).

**Definition 1.1.** Let \( g : [a, b] \to \mathbb{R} \) be a function. We say that \( g \) has a **permeable graph**, if for all \( y \in \mathbb{R} \) and all \( \delta > 0 \) there exists a function \( f : [a, b] \to \mathbb{R} \) with \( f(a) = f(b) = y \), \( V_b^a(f) < \delta \) and such that the topological closure of the set

\[
\{ t \in [a, b] : f(t) = g(t) \}
\]

is at most countable. If a function does not have a permeable graph, we say that it has an **impermeable graph**.

If \( g \) has an impermeable graph then there exist \( y \in \mathbb{R} \) and \( \delta > 0 \) such that for any \( f : [a, b] \to \mathbb{R} \) with \( f(a) = f(b) = y \), \( V_b^a(f) < \delta \), the topological closure \( \{ t \in [a, b] : f(t) = g(t) \} \) is uncountable. Then, this set is an uncountable Borel set and hence by the perfect set theorem for Borel sets [11, 13.6 Theorem] it contains a Cantor set and therefore is of cardinality continuum.

For example, it is immediate that polynomials have permeable graphs: let \( g \) be a polynomial of degree \( n \), and let \( \delta > 0 \). If \( n \geq 1 \), then \( g \) has at most \( n \) intersections with the graph of the function \( f \) with \( f(t) = y \), \( t \in [a, b] \), and \( V_b^a(f) = 0 < \delta \). Thus, \( g \) has a permeable graph. If \( n = 0 \) or \( g \equiv 0 \), we distinguish two cases. If \( g(0) \neq y \) set \( f(t) = y \), \( t \in [a, b] \), so that \( V_b^a(f) = 0 < \delta \). Thus, \( g \) has a permeable graph. If \( n = 0 \) or \( g \equiv 0 \), we distinguish two cases. If \( g(0) \neq y \) set \( f(t) = y \), \( t \in [a, b] \), so that \( V_b^a(f) = 0 < \delta \). Therefore, the graph of \( g \) is permeable.

In the preceding example, we were even able to find a function \( f \) with only finitely many intersections with the graph of \( g \). In that case, we say that \( g \) has a **finitely permeable graph**. There is a significant generalization of this example:

**Theorem 1.** Let \( g : [a, b] \to \mathbb{R} \) be of bounded variation. Then, \( g \) has a finitely permeable graph.

**Proof.** Let \( y \in \mathbb{R} \) and \( \delta > 0 \). By Banach’s indicatrix theorem [1, Théorème 2 or Corollaire 1], \( g^{-1}((z)) \) is finite for Lebesgue-a.e. \( z \in \mathbb{R} \). Therefore, we can find \( y_0 \in \left( y - \frac{\delta}{2}, y + \frac{\delta}{2} \right) \) such that \( n := #g^{-1}(y_0) \) is finite.

Define \( f : [a, b] \to \mathbb{R} \) by

\[
f(x) := \begin{cases} 
y & \text{if } x \in \{a, b\}, 
y_0 & \text{else.} \end{cases}
\]

Then, \( V_b^a(f) < \delta \) and, by construction, \( \{ x : f(x) = g(x) \} \leq n + 2 \), since it might be that \( g(a) = y \) and/or \( g(b) = y \).

The question about permeability of graphs stands in a long tradition of studying the zero and level sets of continuous functions. For example, in [2] it was shown that for the “typical” continuous function on \([0,1]\), which has at least one zero, the set of zeros is uncountable but has Lebesgue measure 0. Banach [1] studied the cardinality of the intersection of the graph of a bounded variation function with horizontal lines. This led to the famous Banach’s indicatrix theorem. Čech [6] proved that a continuous function on a compact interval for which the intersection with every horizontal line is finite, must be monotone on some subinterval. In [4], a number of extensions of Čech’s theorem are shown. In [3], it is shown that every continuous function is either convex or concave on some subinterval, or there are two nonempty perfect sets such that the restriction of the function on the first set is strictly convex and that on the second set is strictly concave. In [16], it is shown that almost every horizontal line that intersects the graph of a continuous nowhere
differentiable function \( g \) does so in an uncountable set. That statement obviously transfers to the intersection with (non-vertical) lines or, more generally, with the graph of a differentiable function. Another result in this direction is [3, Theorem 12], which states that for a nowhere monotone continuous function \( g \), the level sets \( g^{-1}(\{y\}) \) are uncountable if \( y \in (\min(g), \max(g)) \). Our result fits into this line of research in that we present a continuous function which has the property that every continuous function of sufficiently small total variation which intersects the first function, does so in uncountably many points.

In this paper, it is shown that:

- Every function of bounded variation has a permeable graph (Theorem 1).
- Typical (in the sense of Baire category) continuous functions have permeable graphs (Theorem 2).
- There exists a Hölder continuous function with an impermeable graph (Theorem 3).
- There are examples of continuous functions \( g \) with impermeable graphs, which are “extreme” in that their points of intersection with every \( f \) with \( V_0^1(f) < 1 \), \( f(0) = 0 \) has Hausdorff dimension 1 (Theorem 4).

Using Theorem 1, we further learn:

- Every Lipschitz function has a permeable graph. In particular, every \( C^1 \) function has a permeable graph.
- Every absolutely continuous function has a permeable graph.

Thus, we give an incomplete but still very informative first study of the permeable graph property. Another result shows that permeable functions can still be rather wild:

**Theorem** [3, Theorem 8]. If \( M \) and \( N \) are first category subsets of \([0, 1]\), then there exists a function \( g : [0, 1] \to \mathbb{R} \) with range \([0,1]\) and such that for all \( m \in \mathbb{R}, t \mapsto g(t) + mt \) is not monotone on any subinterval of \([0,1]\) and

1. for every \( y \in N \), \( g^{-1}(y) \) is finite and
2. for every \( y \in \mathbb{R} \), \( g^{-1}(y) \cap M \) is finite.

Thus, choosing \( N = \mathbb{Q} \cap [0, 1] \) (or another dense set of first category), and considering the function \( g \) given by the cited theorem, we can, for each \( \delta > 0 \), construct a function \( f \) with \( V_0^1(f) < \delta \) meeting \( g \) in at most finitely many points in the same way as in the proof of Theorem 1. Therefore, the graph of \( g \) is permeable.

As another “wild” example, the square root of the modulus of the so-called Volterra-function, see, for example, [17], provides us with a function of unbounded total variation on every open interval that intersects the Smith–Volterra–Cantor-set (a.k.a. fat Cantor set), which has Lebesgue measure 1/2. Nevertheless, this function has a finitely permeable graph.

In Section 3, we apply the earlier results to a question from metric topology. First, we recall the permeability of sets, cf. Definition 3.5, which was first introduced in [13], where the following “removability-theorem” was also proven (the definition of intrinsically \( L \)-Lipschitz continuous functions will be given in Definition 3.3):

**Theorem** [13, Theorem 15]. Let \( M, Y \) be metric spaces and let \( \Theta \subseteq M \) be permeable. Then, every continuous function \( f : M \to Y \), which is intrinsically \( L \)-Lipschitz continuous on \( E = M \setminus \Theta \), is intrinsically \( L \)-Lipschitz continuous on the whole of \( M \).

A priori it is not clear why one needs a condition on \( \Theta \) besides having empty interior. It turns out, however, that in the case where \( M = \mathbb{R} \), the reverse conclusion in the above theorem, cf. [13, Theorem 23] also holds.

In the case of \( M = \mathbb{R}^2 \), there is an example (see [13, Proposition 26]) of an impermeable subset \( \Theta \) such that every continuous function \( f \) on \( \mathbb{R}^2 \) which is intrinsically Lipschitz continuous on \( \mathbb{R}^2 \setminus \Theta \) is also Lipschitz continuous on \( \mathbb{R}^2 \), but with a different Lipschitz constant.

This example is not a manifold and not a closed subset of \( \mathbb{R}^2 \). It turns out that it is considerably more difficult to construct counterexamples \( \Theta \) that are closed (as subsets of \( \mathbb{R}^d \)) topological submanifolds. It is shown in [13], that closed Lipschitz topological submanifolds are permeable, and thus [13, Theorem 15] applies. In Theorem 6, using the function constructed in Theorem 3, we construct an example of an impermeable closed topological (even Hölder) submanifold of \( \mathbb{R}^2 \), for which the conclusion of [13, Theorem 15] does not hold.
The original motivation for the theory laid out in this article comes from the study of stochastic differential equations (SDEs), whose coefficient functions are discontinuous along a manifold, but differentiable with bounded derivative elsewhere. Such SDEs are studied in [14], where a twice differentiable transform of the state space is devised so that the drift coefficient of the transformed SDE is continuous and intrinsically Lipschitz continuous. Using [14, Lemma 3.6.], which is a special case of [13, Theorem 15], one can conclude that the drift of the transformed equation is Lipschitz continuous. This makes it possible to use classical results for proving existence and uniqueness of a solution, as well as convergence of numerical methods for the transformed equation. Using that the inverse of the transform is also Lipschitz continuous, one can transfer these results to the original SDE. This “transformation method” from [14] has been widely used since, see, for example, [15, 20, 22, 23], and our study here provides insights into the limitations and possible generalizations of that method applied to SDEs or related PDEs.

We conclude the introduction with an outline of the paper. Section 2 contains the “real functions” results, where we first recall some definitions and notations in Section 2.1 and show in Section 2.2 that the set of continuous functions with permeable paths contains a dense $G_δ$-set. Then, in Section 2.3, we construct our Hölder continuous example function with an impermeable graph using nested sequences of chains of rectangles. In Section 2.4, we modify the earlier construction to obtain a still continuous (but not Hölder continuous) function such that its points of intersection with every $f$ satisfying $V(f) < 1$ and $f(0) = 0$ has Hausdorff dimension 1.

Section 3 contains the application of the earlier results in metric topology. First, Section 3.1 recalls the notions of intrinsic metric and intrinsically Lipschitz continuous functions in general metric spaces as well as the concept of a permeable subset of a metric space, as originally introduced in [13]. In Section 3.2, we show that a continuous real function has a permeable graph if and only if its graph is a permeable subset of $\mathbb{R}^2$. This is the assertion of the central Theorem 5, which thus connects the notion of a permeable set with that of a function with permeable graph. Finally, in Section 3.3, we present the construction of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is intrinsically Lipschitz continuous on $\mathbb{R}^2 \setminus \Theta$, where $\Theta$ is the graph of a Hölder continuous function $\theta : \mathbb{R} \to \mathbb{R}$, but $f$ is not Lipschitz continuous with respect to the Euclidean metric on $\mathbb{R}^2$ (Theorem 6).

2 | MAIN EXAMPLES

2.1 | Definitions and notation

Before we start the construction of our example, we recall some concepts and notation: let $(M, d)$ denote a general metric space.

- For $x \in M$ and $r > 0$, let
  $$B_r(x) := \{ y \in M : d(x, y) < r \}$$
  denote the open ball with radius $r$ and center $x$ in $M$.

- For a subset $A \subseteq M$, we denote by $\overline{A}$ the topological closure and by $\partial A$ the topological boundary of $A$.

- For a set $U \subseteq M$, let $|U| := \sup\{d(x, y) : x, y \in M\}$ denote the diameter of the set $U$.

- Given $s \geq 0$, the $s$-dimensional Hausdorff measure of the set $A \subseteq M$ is defined by
  $$\mathcal{H}^s(A) := \lim_{\varepsilon \to 0^+} \left\{ \sum_{i=1}^{\infty} |U_i|^s : \forall i \geq 1 : |U_i| < \varepsilon \text{ and } A \subseteq \bigcup_{i=1}^{\infty} U_i \right\}.$$  

- Let
  $$\dim_H(A) := \inf\{ s \geq 0 : \mathcal{H}^s(A) = 0 \}$$
  $$= \inf \left\{ s \geq 0 : \exists C_s > 0 : \forall \varepsilon > 0 : \mathcal{H}(U_i)_{i \geq 1} : \forall i \geq 1 : |U_i| < \varepsilon, \text{ } A \subseteq \bigcup_{i=1}^{\infty} U_i, \text{ and } \sum_{i=1}^{\infty} |U_i|^s < C_s \right\},$$
  denote the Hausdorff-dimension of $A$. 

For a function $f : [a, b] \rightarrow \mathbb{R}$ let $V^b_a(f) \in [0, \infty]$ denote the total variation of $f$ over the interval $[a, b]$, that is,

$$V^b_a(f) = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a \leq x_0 < \cdots < x_n \leq b \right\}.$$  

For a function $f : [a, b] \rightarrow \mathbb{R}$ let $\ell^b_a(f) \in [0, \infty]$ denote the length of the graph of $f$ over the interval $[a, b]$, that is,

$$\ell^b_a(f) = \sup \left\{ \sum_{k=1}^{n} \sqrt{|x_k - x_{k-1}|^2 + |f(x_k) - f(x_{k-1})|^2} : n \in \mathbb{N}, a \leq x_0 < \cdots < x_n \leq b \right\}.$$  

For $m \in \mathbb{N}$ and $1 \leq i \leq m$, let $\text{pr}_i : \mathbb{R}^m \rightarrow \mathbb{R}$ denote the projection onto the $i$th component.

For a mapping $\gamma : [a, b] \rightarrow \mathbb{R}^2$ let $\gamma_1 := \text{pr}_1 \circ \gamma$ and $\gamma_2 := \text{pr}_2 \circ \gamma$ be the coordinate functions.

For a subset $A \subseteq \mathbb{R}^d$, we write $\text{conv}(A)$ for the convex hull of $A$.

### 2.2 Typical continuous functions have permeable graphs

In this subsection, we show that a typical continuous function $g \in C[0,1]$ has a finitely permeable graph. “Typical” here means that the set of functions with finitely permeable graph contains a dense $G_\delta$ set. On $C[0,1]$ we use the topology induced by the sup-norm, $||f|| = \sup_{x \in [0,1]} |f(x)|$. The precise formulation is given in the next theorem.

**Theorem 2.** There is a dense $G_\delta$ set $\mathcal{G} \subseteq C[0,1]$ such that for all $g \in \mathcal{G}$, for all $y \in \mathbb{R}$ and all $\delta > 0$ there is a function $f : [0,1] \rightarrow \mathbb{R}$ with finitely many jumps such that $f(0) = y$, $V^1_0(f) < \delta$ and $\{t : f(t) = g(t)\} \subseteq \{0\}$.

**Proof.** Let $(g_n)_{n \geq 1}$ be an enumeration of a dense set $D$ of piecewise linear functions which are nonconstant on any interval. For all $n$ let $v_n$ be the minimal number of intervals on which $g_n$ is linear and which partition $[0,1]$. Then, define the numbers $\rho_{n,k} := \frac{1}{2k(v_n+1)}$ and the sets

$$G_k := \bigcup_{n=0}^{\infty} B_{\rho_{n,k}}(g_n), \quad \mathcal{G} = \bigcap_{k=1}^{\infty} G_k.$$  

Clearly, the sets $G_k$ are open and $\mathcal{G}$ is a dense $G_\delta$ set in $C[0,1]$. Let $g \in \mathcal{G}$ and $\delta > 0$ be given. Choose $k > 0$ such that $\frac{1}{k} < \delta$. Then, there is some $n_k$ and a function $g_{n_k} \in D$ such that $g \in B_{\rho_{n_k,k}}(g_{n_k})$.

Take now $y \in \mathbb{R}$ arbitrary. We will construct a function $f$ with $V^1_0(f) < \delta$ and $\{t : f(t) = g(t)\} \subseteq \{0\}$.

This intersection set is nonempty only in the case when $g(0) = y$. Let $t_0 := 0$,

$$f_0 := \begin{cases} y, & \text{if } y \notin [g_{n_k}(0) - \rho_{n_k,k}, g_{n_k}(0) + \rho_{n_k,k}], \\ y + 2\rho_{n_k,k}, & \text{if } y \in [g_{n_k}(0), g_{n_k}(0) + \rho_{n_k,k}], \\ y - 2\rho_{n_k,k}, & \text{if } y \in [g_{n_k}(0) - \rho_{n_k,k}, g_{n_k}(0)), \end{cases}$$

and, inductively (see Figure 1), for $j = 1, \ldots, v_{n_k} + 1$ define

$$t_j := \begin{cases} \inf\{t > t_{j-1} : f_{j-1} = g_{n_k}(t) + \rho_{n_k,k} \}, & \text{if defined and } t_{j-1} \neq 1, \\ 1, & \text{otherwise.} \end{cases}$$

and, inductively
FIGURE 1. An open ball of radius $\rho_{n,k}$ around a piecewise linear continuous function $g_{n,k}$ together with a piecewise constant function which avoids the ball.

and

$$f_j := \begin{cases} f_{j-1} + 2\rho_{n,k}, & \text{if } t_j > t_{j-1} \text{ and } g_{n,k} \text{ decreases at } t_j, \\ f_{j-1} - 2\rho_{n,k}, & \text{if } g_{n,k} \text{ increases at } t_j, \\ f_{j-1}, & \text{if } t_j = t_{j-1} \text{ or } g_{n,k} \text{ has a local extremum at } t_j. \end{cases}$$

Observe that $t_{v_n+1} = 1$ and it may happen that $t_j = 1$ for $j$ less than $v_n + 1$. Assume that $0 < t_1 < \ldots < t_{v_n} < t_{v_n+1} = 1$, all other cases are similar but easier to treat. We define $f$ so that it is piecewise constant on the intervals $[t_{j-1}, t_j], j = 1, \ldots, v_n + 1$ with values $f(t) = f_{j-1}$ and $f(0) = y$.

As a locally constant function, $f$ obtains its variation only by means of its jumps, which may occur at $t_0, \ldots, t_{v_n}$ and have jump heights at most $2\rho_{n,k}$. Hence, $V_0^1(f) \leq \sum_{j=0}^{v_n} 2\rho_{n,k} = 2(v_n + 1)\rho_{n,k} = \frac{1}{k} < \delta$ by the definition of $\rho_{n,k}$. □

**Corollary 2.1.** There is a dense $G_\delta$ set $\mathcal{G} \subseteq C[0,1]$ such that for all $g \in \mathcal{G}$, $g$ has a permeable graph.

**Proof.** Let $\mathcal{G}$ be as in the proof of Theorem 2 and let $g \in \mathcal{G}$. Let $y \in \mathbb{R}$ and $\delta > 0$. Then, there exists $\tilde{f} : [0,1] \to \mathbb{R}$ with $\tilde{f}(0) = y$, and $V_0^1(\tilde{f}) < \frac{\delta}{2}$ so that $\{ t : \tilde{f}(t) = g(t) \} \subseteq \{0\}$. Define

$$f(t) := \begin{cases} \tilde{f}(t), & \text{if } t < 1, \\ y, & \text{if } t = 1. \end{cases}$$

Then, $V_0^1(f) \leq V_0^1(\tilde{f}) + |\tilde{f}(1) - y| \leq 2V_0^1(\tilde{f}) < \delta$ and $\{ t : f(t) = g(t) \} \subseteq \{0, 1\}$. □

### 2.3 Continuous functions with impermeable graph

In this subsection, we are going to construct a function $g_{H} : [0,1] \to \mathbb{R}$ with an impermeable graph. This function will be Hölder continuous. In Section 2.4, we will modify this construction, which will yield a continuous, non Hölder continuous function $g_{C}$, and with the property that the intersection set with every $f$ satisfying $f(0) = 0$ and $V_0^1(f) < 1$ has Hausdorff dimension equal to 1. At a heuristic level, it is clear that the price we need to pay to obtain larger intersection sets is that our function $g_{C}$ should become wilder.
Both functions $g_H$ and $g_C$ will be constructed as a limit of a nested sequence of chains of rectangles, a concept that we are going to define now.

**Definition 2.2.**

1. Let $(R_n)_{n=1}^N$ be a finite collection of axes-parallel rectangles contained in $[0,1] \times \mathbb{R}$, with common width $w$ and height $h$. Let $(x_n, y_n)$ be the lower left corner of $R_n = [x_n, x_n + w] \times [y_n, y_n + h]$. We call $(R_n)_{n=1}^N$ a **connected chain of rectangles**, if
   (a) $x_1 = 0, x_N + w = 1$;
   (b) $x_{n+1} = x_n + w$ for $n = 1, \ldots, N - 1$;
   (c) $|y_{n+1} - y_n| \leq h$ for $n = 1, \ldots, N - 1$.

2. Let $\{R_{n,k} : k \in \mathbb{N}_0, n = 1, \ldots, N_k\}$ be a finite collection of rectangles contained in $[0,1] \times \mathbb{R}$. Let $(x_{n,k}, y_{n,k})$ be the lower left corner of $R_{n,k} = [x_{n,k}, x_{n,k} + w_k] \times [y_{n,k}, y_{n,k} + h_k]$. We call $(R_{n,k})$ a **nested sequence of chains of rectangles**, if
   (a) for every $k$, $(R_{n,k})_{n=1}^{N_k}$ is a connected chain of rectangles with corresponding widths $w_k$ and heights $h_k$;
   (b) $N_{k+1} = d_{k+1}N_k$ for some $d_{k+1} \in \mathbb{N} \setminus \{1\}$;
   (c) $y_{j,k} \leq y_{(j-1)d_{k+1}+n,k+1} \leq y_{j,k} + h_k - h_{k+1}$ for all $j \in \{1, \ldots, N_k\}$, $n \in \{1, \ldots, d_{k+1}\}$, $k \in \mathbb{N}_0$.

See Figures 2 and 3.

Note that, in a nested sequence of chains of rectangles $\{R_{n,k} : k \in \mathbb{N}_0, n = 1, \ldots, N_k\}$, every rectangle $R_{n,k+1}$ is contained in precisely one rectangle $R_{m,k}$.

Nested sequences of chains of rectangles can be used to represent continuous functions:
Proposition 2.3.

1. Let \( \{ R_{n,k} : k \in \mathbb{N}_0, n = 1, ..., N_k \} \) be a nested sequence of chains of rectangles. Then for every \( t \in [0, 1] \) one can select a sequence \( (m(k, t))_{k=0}^{\infty} \) such that \( t \in [m(k, t)w_k, (m(k, t) + 1)w_k] \) for all \( k \in \mathbb{N}_0 \). The sequence is unique for all but countably many \( t \).

2. If \( g : [0, 1] \to \mathbb{R} \) is a continuous function, and \( (N_k)_{k=1}^{\infty} \) is a sequence of natural numbers with \( N_{k+1}/N_k \in \mathbb{N} \setminus \{1\} \), then there exists a nested sequence of chains of rectangles \( \{ R_{n,k} : k \in \mathbb{N}_0, n = 1, ..., N_k \} \) with corresponding sequence of heights \( (h_k)_{k=0}^{\infty} \) such that \( \lim_{k \to \infty} h_k = 0 \) and

\[
g(t) = \lim_{k \to \infty} y_{m(k,t),k}
\]

for all \( t \in [0, 1] \). Moreover, \( y_{m(k,t),k} \leq g(t) \leq y_{m(k,t),k} + h_k \) for every \( t \in [0, 1] \) and \( k \in \mathbb{N}_0 \).

3. If \( \{ R_{n,k} : k \in \mathbb{N}_0, n = 1, ..., N_k \} \) is a nested sequence of connected chains of rectangles with a corresponding sequence of heights \( (h_k)_{k=0}^{\infty} \) such that \( \lim_{k \to \infty} h_k = 0 \), then

\[
g(t) := \lim_{k \to \infty} y_{m(k,t),k}
\]

defines a continuous function \( g : [0, 1] \to \mathbb{R} \) and \( y_{m(k,t),k} \leq g(t) \leq y_{m(k,t),k} + h_k \) for every \( t \in [0, 1] \) and \( k \in \mathbb{N}_0 \), and

\[
\text{graph}(g) = \bigcap_{k=0}^{\infty} \bigcup_{n=1}^{N_k} R_{n,k}.
\]

Proof. The proof is left to the reader. \( \square \)

Lemma 2.4. Let \( (R_{n,k})_{n,k} \) be a nested sequence of chains of rectangles. Assume further that there exist \( j \in \mathbb{N} \setminus \{1\} \) and \( \beta \in (0, 1) \) with \( w_k = j^{-k} \) and \( h_k = \beta^k \). Then, \( g \) as defined in Equation (2.1) is Hölder-continuous with exponent \( \alpha := \frac{\log(\beta)}{\log(j)} \).

Proof. By continuity of \( g \) it is enough to show the Hölder property on the dense subset \( \{ nj^{-k} : k \in \mathbb{N}, 0 \leq n \leq j^k \} \). First, we note that

\[
|g((n+1)j^{-k}) - g(nj^{-k})| \leq h_k = \beta^k = j^k \cdot \frac{\log(\beta)}{\log(j)} = (j^{-k})^{\alpha}.
\]

(2.2)

Now, let \( x, y \in \{ nj^{-k} : k \in \mathbb{N}, 0 \leq n \leq j^k \} \), \( x \neq y \). Consider the \( j \)-adic representations

\[
x = \sum_{k=1}^{K} x_k j^{-k}, \quad y = \sum_{k=1}^{K} y_k j^{-k},
\]

with \( x_k, y_k \in \{0, ..., j-1\} \) for all \( k \). Choose \( k_0 \) such that the following inequality is satisfied:

\[
j^{-k_0} < |x - y| \leq j^{-k_0+1}.
\]

(2.3)

Let \( x^* = \sum_{\ell=0}^{k_0} x_\ell j^{-\ell} \) and \( y^* = \sum_{\ell=0}^{k_0} y_\ell j^{-\ell} \). Then, by Equation (2.2)

\[
\max\{|g(x) - g(x^*)|, |g(y) - g(y^*)|\} \leq (j-1) \sum_{\ell=k_0+1}^{K} j^{-\ell,\alpha} \leq \frac{(j-1)j^{-(k_0+1,\alpha)}}{1 - j^{-\alpha}}.
\]

(2.4)

Also, by Equation (2.2) we have

\[
|g(x^*) - g(y^*)| \leq j \cdot j^{-k_0,\alpha}.
\]
By using Equations (2.3) and (2.4) and the triangle inequality, we obtain that

\[ |g(x) - g(y)| < \left( 2 \left( \frac{(j-1)j^{-\alpha}}{1-j^{-\alpha}} + j \right) \right)^j |x - y|^\alpha. \]

This finishes the proof. \( \square \)

In the following, we construct a concrete example of a Hölder continuous function \( g_H \). From now on, we will always assume \( h_k := \frac{10^{-k}}{10} \) for all \( k \in \mathbb{N}_0 \).

We assume that each rectangle \( R_{n,k} \) comes with an entry point \( a_{n,k} \in [y_{n,k}, y_{n,k} + h_k] \) and an exit point \( b_{n,k} \in [y_{n,k}, y_{n,k} + h_k] \) such that \( a_{n+1,k} = b_{n,k} \) for \( n = 1, \ldots, N_k - 1 \). It is easy to see that the \( R_{n,k}, n \in \{1, \ldots, N_k\} \) form a connected chain if such entry/exit points exist and conversely, given \( u_k, h_k \) and \( y_{0,k} \), the entry/exit points (relative to the position on the according rectangle's vertical line segment) determine the chain.

Finally, we require \( a_{md,1,k+1} = a_{m+1,k} \) and \( b_{md+1,k+1} = b_{m+1,k} \) for all \( m = 0, \ldots, N_k - 1 \).

We start with \( R_{1,0} : = [0,1] \times [-5,5] \), such that \((x_{1,0}, y_{1,0}) = (0, -5)\), \( w_0 = 1\), \( h_0 = 10 \).

Further, we set \( a_{1,0} : = \frac{5}{2}, b_{1,0} : = -\frac{5}{2} \).

We build a nested sequence of connected chains of rectangles \( \{R_{n,k} : k \in \mathbb{N}_0, 1 \leq n \leq N_k\} \) from two kinds of rectangles: “ascending” and “descending” ones:

**Convention.** We call a rectangle \( R_{n,k} \) with entry point \( a_{n,k} \) and exit point \( b_{n,k} \)

1. **ascending** if \( a_{n,k} = y_{n,k} + \frac{1}{4}h_k \) and \( b_{n,k} = y_{n,k} + \frac{3}{4}h_k \),
2. **descending** if \( a_{n,k} = y_{n,k} + \frac{3}{4}h_k \) and \( b_{n,k} = y_{n,k} + \frac{1}{4}h_k \).

Thus, in the sense of this convention, \( R_{1,0} \) is descending.

Now in level \( k = 1 \), start with connecting 14 descending rectangles, such that \( b_{14,1} = \frac{5}{2} - 14 \cdot \frac{5}{10} = -\frac{45}{10} \) (recall that \( h_1 = 1 \)). Continue by connecting 18 ascending rectangles, such that \( b_{14+18,1} = -\frac{45}{10} + 18 \cdot \frac{5}{10} = \frac{45}{10} \). We proceed with connecting another 11 groups of each 18 descending and ascending rectangles such that the interval \([-4,4]\) is crossed 12 times. We end with four ascending rectangles such that \( b_{N_1,1} = -\frac{5}{2} \) (So, \( N_1 = 14 + 12 \cdot 18 + 4 = 234 = \frac{101 - 1}{2} \)). This procedure is repeated for every \( k \), where for the descending rectangles the nesting is precisely as for \( k = 1 \), such that in each rectangle \( R_{n,k-1} \) we cross 12 times the interval \([y_{n,k-1}, y_{n,k-1} + \frac{1}{10^k}, y_{n,k-1} + \frac{1}{10^k} - \frac{1}{10^k}]\), yielding \( 14 + 12 \cdot 18 + 4 \) rectangle descendants with width \( w_k = 234^{-k} \) for \( k \geq 1 \). For ascending rectangles, we start with 14 ascending rectangles and continue in the analogous way.

Together with \( h_k = 10^{-1-k} \), the construction of our function \( g_H \) is thus finished by Proposition 2.3, Item 3, and we need to show that \( g_H \) has an impermeable graph.

For that purpose, take an arbitrary function \( f : [0,1] \rightarrow \mathbb{R} \) with \( f(0) \leq \frac{5}{2} \) and \( f(1) \geq -\frac{5}{2} \) and \( V_0(f) < 1 \). We will show that \( t \in [0,1] : f(t) = g_H(t) \) is uncountable.

Since \( V_0(f) < 1 \), the graph of \( f \) is contained in \([0,1] \times \left[ -\frac{7}{2}, \frac{7}{2} \right] \).

Since the chain \( R_{n,1}, n = 1, \ldots, N_1 \), alternates 12 times between \(-\frac{45}{10} \) and \( \frac{45}{10} \) it must cross the interval \([\frac{7}{2}, \frac{7}{2}]\) at least that many times, and so must the graph of \( g_H \).

We say that the graph of \( f \) is **upcrossing** a decreasing rectangle \( R_{n,k} \), if \( f(x_{n,k}) \leq g_H(x_{n,k}) = a_{n,k} \) and \( f(x_{n,k} + w_k) \geq g_H(x_{n,k} + w_k) = b_{n,k} \). We say an upcrossing is **good**, if \( V_{x_{n,k} + w_k}^x(f) < 10^{-k} \). In an analogous way, we may define (good) downcrossings of an ascending rectangle. A **good crossing** is a good up- or downcrossing.

The graph of \( f \) is contained in the strip \([0,1] \times \left[ -\frac{7}{2}, \frac{7}{2} \right] \) which is crossed 12 times by the chain of rectangles \( (R_{n,1}) \). We must therefore have at least six upcrossings and six downcrossings. Since the total variation of \( f \) is smaller than 1, we need to have at least 2 up- or down crossings that are good. Indeed, proceeding towards a contradiction, suppose that more than ten crossings were not good, say those at rectangles \( R_{n_{1,1}}, \ldots, R_{n_{11,1}} \). Then, \( V_0(f) \geq V_{x_{n_{1,1}}}^{x_{n_{1,1}}+w_{1,1}}(f) + \cdots + V_{x_{n_{11,1}}}^{x_{n_{11,1}}+w_{11,1}}(f) > 10 \cdot \frac{1}{10} = 1 \), which poses a contradiction.
For rectangles with good crossings, we may repeat the argument. So, we get at least $12 - 10 = 2$ good crossings on the next level for every good crossing at the present level. In particular, there are at least $2^{N_0}$-many $t \in [0,1]$ which are contained in a sequence of nested rectangles with good crossings. For all but countably many $t \in [0,1]$ the corresponding sequence $(m(k,t))_{k \geq 0}$ is unique, so that $(R_{m(k,t),k})_{k \geq 0}$ is a nested sequence of rectangles with good crossings and $x_{m(k,t),k} < t < x_{m(k,t),k} + w_k$. Consider one such $t$. Then, $f(x_{m(k,t),k}) \in [y_{m(k,t),k}, y_{m(k,t),k} + h_k]$, and $g_H(t) = \lim_{k \to \infty} y_{m(k,t),k}$, so if $f$ is continuous in $t$, then $f(t) = g_H(t)$.

Since $f$ is of bounded variation, $f$ has at most countably many points of discontinuity. Therefore, we have still $2^{N_0}$-many $t \in [0,1]$ with $g_H(t) = f(t)$.

Using Lemma 2.4 and the constructed example, we have proven the following theorem:

**Theorem 3.** There exists a Hölder continuous function $g_H : [0,1] \to \mathbb{R}$ with Hölder exponent $\frac{\log(10)}{\log(234)} \approx 0.42$ and with an impermeable graph. Specifically, for every function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $V_0^1(f) < 1$ the set $\{ t \in [0,1] : g_H(t) = f(t) \}$ is uncountable.

An interesting question is how large the Hölder exponent of a Hölder continuous function with impermeable graph may be. Can the exponent be arbitrarily close to 1? Or does it even need to be smaller than $\frac{1}{2}$? Interesting examples are paths of the Brownian motion, for which it is known that for every $\alpha < \frac{1}{2}$ they are almost surely $\alpha$-Hölder continuous (see, e.g., [10, Chapter 2, Remark 2.12]).

### 2.4 The Hausdorff-dimension of the intersection set

We will now modify the construction from Section 2.3 to get a graph which is even “more impermeable” at the price of losing Hölder continuity.

**Theorem 4.** There exists a continuous function $g_C : [0,1] \to \mathbb{R}$ which has impermeable graph and, in addition, the set $\{ t \in [0,1] : f(t) = g_C(t) \}$ has Hausdorff dimension 1 for every bounded variation function $f$ for which $f(0) = 0$ and $V_0^1(f) < 1$.

Before proving the theorem, we recall the mass distribution principle:

**Theorem (Mass distribution principle, [7, Chapter 4]).** Let $F \subseteq \mathbb{R}^d$ and let $\nu$ be a mass distribution on $F$, that is, a measure on $\mathcal{M}(F)$. Suppose that for some $s \geq 0$ there are numbers $c > 0$ and $\delta > 0$ such that $\nu(U) \leq c |U|^s$ for all sets $U$ with $|U| \leq \delta$. Then $\mathcal{H}^s(F) \geq \nu(F)/c$ and $s \leq \dim_H(F)$.

**Proof of Theorem 4.** First, we will construct a measure $\nu$ on $[0,1]$ having support on the set of points $t$ which are contained in a nested sequence of projections of rectangles with good crossings.

Let $g_C$ be the function defined by the above construction via a scheme of nested, connected chains of rectangles as above, with $(R_{m(k,t),k})_{k \geq 0}$ having height $h_k = 10^{1-k}$ at level $k$ and $w_k = \prod_{l=1}^k (1 + 10^l \cdot 18^4)^{-1}$. That is, at level $k$ we have, for each rectangle $R_{n,k-1}$, at least $10^k$ new chains of descending/ascending rectangles, crossing the strip $[y_{n,k} + 15 \cdot 10^{-k-1}, y_{n,k} + 10^{1-k} - 15 \cdot 10^{-k-1}]$ $10^k$ times. By the same reasoning as in Section 2.3, for every good crossing at level $k - 1$, we have at least $10^k - 10$ good crossings at level $k$.

Let $F_{n,k}$ denote the projection of the rectangle $R_{n,k}$ to the $x$-axis, that is, $F_{n,k} := [n w_k, (n+1) w_k] = [x_{n,k}, x_{n,k} + w_k]$. Recall that for any $t \in (0,1)$ there is a sequence $(m(k,t))_{k \geq 0}$ such that $t \in F_{m(k,t),k}$ for all $k \geq 0$ and that this sequence is unique except when $t \in (0,1) \cap \partial F_{n,k}$ for some $(n,k)$. For definiteness, we set $m(k,t) := \max \{ 1 \leq n \leq N_k : t \in F_{n,k} \}$.

This admits the following mappings (a.k.a. $F_{n,k}$-adic tree representation),

$$c : [0,1] \to \prod_{k \geq 0} \{ F_{n,k} : 1 \leq n \leq N_k \}, \quad t \mapsto (F_{m(k,t),k})_{k \geq 0}$$

and

$$c^* : c([0,1]) \to [0,1], \quad (F_{n,k})_{k \geq 0} \mapsto \inf_{k \geq 0} F_{n,k}.$$
Note that the second mapping can be extended to the domain of all nested sequences

$$S := \left\{ \varphi \in \prod_{k \geq 0} \{ F_{n,k} : 1 \leq n \leq N_k \} : \varphi_{k+1} \subseteq \varphi_k \text{ for all } k \in \mathbb{N}_0 \right\},$$

which differs from the image of $c$ only by countably many elements. We now endow $\prod_{k \geq 0} \{ F_{n,k} : 1 \leq n \leq N_k \}$ with the product $\sigma$-algebra

$$\Sigma := \bigotimes_{k \geq 0} P\left( \{ F_{n,k} : n \in \{1, ..., N_k \} \} \right).$$

Here, $P(X)$ denotes the power set of a set $X$. The function $c$ is $\mathcal{B}([0,1]) - \Sigma$-measurable and $c^*$ is $\tilde{\Sigma} - \mathcal{B}([0,1])$-measurable, where $\tilde{\Sigma}$ is the trace $\sigma$-algebra of $\Sigma$ on $c([0,1])$. We will now define a measure $\mu$ on $\Sigma$ which will have support in $S$ only. In fact, it will have support only on those sequences consisting of projections of good crossed rectangles. To that end let $\mu_0(F_{1,0}) := 1$ and, for every product set $A_k := \prod_{n_0 \leq \cdots \leq n_k} F_{n_0,0} \times \cdots \times F_{n_k,k}$ define recursively

$$\mu_{0,...,k}(A_k) := \begin{cases} 0, & \text{if } \neg (R_{n_1,1} \supseteq \cdots \supseteq R_{n_k,k}), \\ 0, & \text{if } \exists l, 1 \leq l \leq k \text{ s.t. } R_{n_l,l} \text{ has no good crossing} \\ \mu_{0,...,k-1}(F_{n_0,0} \times \cdots \times F_{n_{k-1},k-1}) \xi(n,k-1), & \text{otherwise,} \end{cases}$$

where

$$\xi(n,k) := \# \{ m \in \{(n-1)d_{k+1} + 1, ..., nd_{k+1}\} : R_{m,k+1} \text{ has a good crossing in } R_{n,k} \}$$

for $n = 1, ..., N_k$.

As the family of measures $(\mu_{0,...,k})_{k \geq 1}$ clearly satisfies the compatibility relations of the Kolmogorov extension theorem [12, Theorem 14.35], there exists a measure $\mu$ on $\Sigma$ such that $\mu$ restricted to the trace-$\sigma$-algebra of $\Sigma$ on $\prod_{0 \leq k \leq k} \{ F_{n,k} : 1 \leq n \leq N_k \}$ yields $\mu_{0,...,k}$. (Actually, already the Ionescu–Tulcea theorem [12, Theorem 14.32], which is a “smaller gun” than Kolmogorov’s theorem, suffices for this purpose.)

By means of the mapping $c^*$, we may transport the measure $\mu$ to $[0,1]$. The resulting pushforward measure $\nu := c^*(\mu)$ then has support on

$$F := \left\{ t \in [0,1] : t \in \bigcap_{k \geq 0} F_{m(k,t),k}, \text{ all } R_{m(k,t),k} \text{ have good crossings} \right\}.$$ 

By the definition of $\mu$ and $\nu$, and since for all rectangle projections $R_{n,k-1}$ with good crossings there are at least $100^k - 10$ descendant rectangles with good crossings, it follows that $\nu([t]) = 0$ for all $t \in [0,1]$.

Let now $s \in (0,1)$. Then, choose the smallest $k_s \geq 0$ such that

$$\frac{\log(100^{k_s} - 10)}{\log(18 \cdot 100^{k_s} + 18)} > s. \tag{2.5}$$

We show that

$$\nu(F_{n,l}) \leq 21^l |F_{n,l}| \text{ for } 0 \leq l < k_s. \tag{2.6}$$

The case $l = 0$ is obvious since $\nu([0,1]) = 1 \leq 21 = 21^1 |F_{n,0}|$ trivially. For $1 \leq l < k_s$, we have that

$$\nu(F_{n,l}) \leq \frac{\nu(F_{n,l-1})}{100^l - 10}.$$
as there are at least \(100^l - 10\) rectangles in \(F_{n,l-1}\) with good crossings. Using the induction hypothesis for \(l - 1\), we continue with

\[
\frac{\nu(F_{n,l-1})}{100^l - 10} \leq \frac{21^{l-1}|F_{n,l-1}|}{100^l - 10} = \frac{21^{l-1}w_{l-1}}{100^l - 10} = \frac{21^{l-1} \prod_{1 \leq j \leq l-1} \frac{1}{18 + 18 \cdot 100^j}}{100^l - 10}.
\]

Since \(l \geq 1\), the last term is smaller than

\[
21^{l} \prod_{1 \leq j \leq l} \frac{1}{18 + 18 \cdot 100^j} = 21^l|F_{n,l}|.
\]

From \(s \in (0, 1), 1 \leq l < k_s\) and Equation (2.6) it also follows that

\[
\nu(F_{n,l}) \leq 21^{k_s-1}|F_{n,l}|^s.
\]

Next, we consider \(l \geq k_s\): here we observe for \(l = k_s\)

\[
\nu(F_{n,k_s}) \leq \nu(F_{n,k_s-1}) \leq \frac{21^{k_s-1}|F_{n,k_s-1}|}{100^{k_s} - 10} = \frac{21^{k_s-1} \prod_{1 \leq j \leq k_s-1} \frac{1}{18 + 18 \cdot 100^j}}{100^{k_s} - 10},
\]

which can be bounded by

\[
21^{k_s-1} \prod_{1 \leq j \leq k_s} \left( \frac{1}{18 + 18 \cdot 100^j} \right)^s = 21^{k_s-1}|F_{n,k_s}|^s,
\]

which follows from \(s \in (0, 1)\) and Equation (2.5). This forms an induction start, such that for \(l > k_s\), we get similarly,

\[
\nu(F_{n,l}) \leq \nu(F_{n,l-1}) \leq \frac{21^{k_s-1}|F_{n,l-1}|^s}{100^l - 10} = \frac{21^{k_s-1} \prod_{1 \leq j \leq l-1} \left( \frac{1}{18 + 18 \cdot 100^j} \right)^s}{100^l - 10}
\]

\[
\leq 21^{k_s-1} \prod_{1 \leq j \leq l} \left( \frac{1}{18 + 18 \cdot 100^j} \right)^s = 21^{k_s-1}|F_{n,l}|^s.
\]

Thus, for a given \(s \in (0, 1)\), we found \(k_s\) such that for all \(l\)

\[
\nu(F_{n,l}) \leq 21^{k_s-1}|F_{n,l}|^s. \tag{2.7}
\]

Now, we show that there exists a constant \(C_s\) such that \(\nu(U) \leq C_s|U|^s\) for all measurable subsets \(U \subseteq [0, 1]\). Clearly, it is enough to show this for the case where \(U\) is an interval, which we assume from now on. There exists \(l \in \mathbb{N}\) with \(w_l < |U| \leq w_{l-1}\). Now, \(U\) can be covered by at most two projections of rectangles of width \(w_{l-1}\), say \(U \subseteq F_{m-1,l-1} \cup F_{m,l-1}\). Moreover, we can approximate \(U\) from outside by \(F_{m_1,l} \cup \ldots \cup F_{m_1+\alpha_1,l} \cup F_{m_1+\alpha_1+1,l} \cup \ldots \cup F_{m_1+\alpha_1+\alpha_2,l}\), with \(F_{m_1+j_1,l} \subseteq F_{m-1,l-1}\) and \(F_{m_1+\alpha_1+j_2,l} \subseteq F_{m,l-1}\) for all \(0 \leq j_1 \leq \alpha_1, 1 \leq j_2 \leq \alpha_2 + 1\), for an appropriate index \(1 \leq m_1 \leq N_1\), and from inside by \(F_{m_2+l_1} \cup \ldots \cup F_{m_2+\alpha_1,l} \cup F_{m_2+\alpha_1+1,l} \cup \ldots \cup F_{m_2+\alpha_1+\alpha_2,l}\). Note that from these approximations, we get

\[
a_1 + a_2 \leq 18 \cdot 100^l + 18 \tag{2.8}
\]

and

\[
\max(1, a_1 + a_2)w_l \leq |U| \leq (a_1 + a_2 + 2)w_l. \tag{2.9}
\]

Assume that the rectangles with projections \(F_{m-1,l-1}, F_{m,l-1}\) admit good crossings (all other cases are easier to handle). Let then \(B_1\) be the number of rectangles with good crossings on level \(l\), with projections in \(F_{m-1,l-1}\). Define the same number \(B_2\) with respect to \(F_{m,l-1}\) and note that \(\min\{B_1, B_2\} \geq 100^l - 10\). Denote the number of rectangle
projections with good crossings of $F_{m_1,j}, \ldots, F_{m_1+a_1,j}$ by $b_1$ and those of $F_{m_1+a_1+1,j} \cup \ldots \cup F_{m_1+a_1+a_2+1,j}$ by $b_2$. Observe that

$$b_1 + b_2 \leq a_1 + a_2 + 2. \quad (2.10)$$

We use the additivity of $\nu$ to get

$$\nu(U) \leq \sum_{i=0}^{a_1+a_2+1} \nu(F_{m_1+i,j}) = \sum_{i=0}^{a_1} \nu(F_{m_1+i,j}) + \sum_{i=a_1+1}^{a_1+a_2+1} \nu(F_{m_1+i,j}).$$

By definition of $\nu$, the right-hand side of the above equality equals

$$\frac{b_1}{B_1} \nu(F_{m-1,j}) + \frac{b_2}{B_2} \nu(F_{m,j}).$$

Now, we may use Equation (2.7) to continue with

$$\frac{b_1}{B_1} \nu(F_{m-1,j}) + \frac{b_2}{B_2} \nu(F_{m,j}) \leq \frac{b_1}{B_1} 21^{k_i-1} |F_{m-1,j}|^s + \frac{b_2}{B_2} 21^{k_i-1} |F_{m,j}|^s$$

$$= 21^{k_i-1} \left( \frac{b_1}{B_1} + \frac{b_2}{B_2} \right) \omega_i^s = 21^{k_i-1} \left( \frac{b_1}{B_1} + \frac{b_2}{B_2} \right) \prod_{l=1}^{l-1} \frac{1}{(18 \cdot 100^l + 18)^s} \omega_i^s = : *.$$

Using Equations (2.9) and (2.10) and that $\min\{B_1, B_2\} \geq 100^l - 10$, we can further estimate

$$* \leq 21^{k_i-1} \frac{b_1 + b_2}{100^l - 10} (18 \cdot 100^l + 18)^s \frac{|U|^s}{\max(1, a_1 + a_2)^s}$$

$$\leq 21^{k_i-1} \frac{a_1 + a_2 + 2}{\max(1, a_1 + a_2)^s} \frac{1}{100^l - 10} (18 \cdot 100^l + 18)^s |U|^s.$$

Since

$$\frac{a_1 + a_2 + 2}{\max(1, a_1 + a_2)^s} \leq (a_1 + a_2)^{1-s} + \frac{2}{\max(1, a_1 + a_2)^s} \leq (18 \cdot 100^l + 18)^{1-s} + 2$$

$$\leq 3 \cdot (18 \cdot 100^l + 18)^{1-s},$$

where we used Equation (2.8) in the second inequality, we have altogether

$$\nu(U) \leq \frac{3 \cdot 21^{k_i-1} 18 \cdot 100^l + 18}{100^l - 10} |U|^s \leq 3 \cdot 21^{k_i} |U|^s.$$
Since a countable set does not change the Hausdorff-dimension, also \( \dim_H(F \setminus E) = 1 \). Since \( F \setminus E \subseteq \{ t \in [0,1] : f(t) = g_C(t) \} \), also

\[
\dim_H(\{ t \in [0,1] : f(t) = g_C(t) \}) = 1.
\]

It is not difficult to see that the function \( g_C \) is not Hölder continuous for any exponent in \((0,1)\). 

The mass distribution principle applied to the Hölder continuous function \( g_H \), constructed in Section 2.3 does not deliver Hausdorff-dimension 1 of the intersection set. In this case, the rectangle width in each step is \( w_k = \left( \frac{1}{234} \right)^k \), and in each step, for a rectangle with good crossing, one obtains at least \( 12 - 10 \) good crossings. Hence, the mass distribution principle as above only works up to \( s = \frac{\log(2)}{\log(234)} \approx 0.127 \).

Recall that we have in total 14 ascending or descending chains within the first rectangle \( R_{1,0} \). A function \( f \) with \( \text{graph}(f) \) having nonempty intersection with at least three rectangles within one ascending or descending chain needs positive variation for this. If it should touch more than three rectangles, it needs to have a variation of at least \( \frac{m-3}{2} \) there. In any case, if there are \( m_1 \) smaller rectangles in \( R_{1,0} \) which have nonempty intersection with the graph of \( f \), then the function \( f \) needs a variation of at least

\[
\frac{m_1 - 3 \cdot 14}{2} = \frac{m_1 - 42}{2}.
\]

Conversely, this means \( m_1 \leq 42 + 2v_1 \), where \( v_1 = V_0^1(f) \). Denote the variation of \( f \) over the traversed rectangle \( R_{n,1} \) by \( v_{n,2} \), so that \( \sum_{n=1}^{m_1} v_{n,2} \leq v_1 \). Denote by \( m_{n,2} \) the number of smaller rectangles in \( R_{n,1} \) which have nonempty intersection with the graph of \( f \), and denote the sum of them for all \( n \) by \( m_2 \), that is, \( m_2 := \sum_{n=1}^{m_1} m_{n,2} \). Then with the same argument as above, \( v_{n,2} \geq \frac{m_{n,2} - 42}{20} \), so that \( m_{n,2} \leq 42 + 20v_{n,2} \). Summing over all \( m_{n,2} \) we get

\[
m_2 \leq \sum_{n=1}^{m_1} (42 + 20v_{n,2}) \leq 42m_1 + 20v_1 < 42^2 + (2 \cdot 42 + 20)v_1.
\]

On the third level, we get

\[
m_3 \leq \sum_{n=1}^{m_2} 42 + 200v_{n,3} \leq 42m_2 + 200v_1 \leq 42^3 + (2 \cdot 42^2 + 20 \cdot 42 + 200)v_1,
\]

and in general

\[
m_k \leq 42^k + 2v_1 \sum_{j=1}^{k} 42^{k-j}10^{j-1} = 42^k \left( 1 + \frac{1}{21}v_1 \sum_{j=1}^{k} 42^{-(j-1)}10^{j-1} \right)
= 42^k \left( 1 + \frac{1}{21}v_1 \sum_{j=0}^{k-1} \left( \frac{10}{42} \right)^j \right) \leq 42^k \left( 1 + \frac{1}{16}v_1 \right).
\]

As \( v_1 \leq 1 \), we see that the intersection set can be covered by at most \( 42^k \cdot \frac{17}{16} \) many intervals of width \( w_k = \left( \frac{1}{234} \right)^k \). The \( s \)-dimensional Hausdorff-content \( \mathcal{H}^s(\{ t : f(t) = g_H(t) \}) \) of the intersection set is thus bounded by

\[
\lim_{k \to \infty} \frac{42^k \cdot \frac{17}{16}}{234^k} = 0.
\]
for \( s > \frac{\log(42)}{\log(234)} \) Using the mass distribution principle as above, we get

\[
0.127 \approx \frac{\log(2)}{\log(234)} \leq \dim_H(\{ t : f(t) = g_H(t) \}) \leq \frac{\log(42)}{\log(234)} \approx 0.685.
\]

In the special case of the constant function \( f \equiv 0 \), we see that it intersects exactly \( 2 \cdot 13 \) rectangles out of the 234 rectangles in each step. Using the argument above, we get that \( \dim_H(\{ t : g_H(t) = 0 \}) = \frac{\log(26)}{\log(234)} \approx 0.597 \).

We have seen that for the \( \frac{\log(10)}{\log(234)} \) Hölder continuous function \( g_H \), we obtained an estimate for the Hausdorff-dimension of the intersection set \( \{ t \in [0,1] : f(t) = g_H(t) \} \) for every function \( f \) with \( f(0) = 0 \) and variation \( V_0^1(f) \leq 1 \). In the same spirit, we may ask, whether for a given Hölder exponent \( \alpha \in (0,1) \) we can adapt our construction to produce a function \( g_\alpha \) such that there are two constants \( 0 \leq d(\alpha) \leq D(\alpha) \leq 1 \) to control the Hausdorff-dimension by

\[
d(\alpha) \leq \dim_H(\{ t \in [0,1] : f(t) = g_\alpha(t) \}) \leq D(\alpha)
\]

for all functions \( f \) with variation less or equal than 1. An immediate upper bound \( 1 - \alpha \) for \( D(\alpha) \) is given in [18], where level sets of Hölder functions are investigated, corresponding to constant functions.

### 3 APPLICATION IN METRIC TOPOLOGY

In this section, we answer a question posed in [13]: given a topological submanifold \( \Theta \subseteq \mathbb{R}^d \), closed as a subset, and a continuous function \( f : \mathbb{R}^d \to \mathbb{R} \) which is Lipschitz continuous with respect to the intrinsic metric on \( \mathbb{R}^d \setminus \Theta \), now we conclude that \( f \) is Lipschitz continuous with respect to the Euclidean metric? The question is answered in the affirmative for the case of submanifolds with Lipschitz continuous charts, cf. [13, Theorems 15 and 31]. In this section, we show that in general the answer is no, even for submanifolds with Hölder charts.

We start by recalling concepts from metric topology, including the notion of permeability of subsets of metric spaces, which was also introduced in [13]. In Section 3.2, we will show that the earlier notion of permeability of the graph of a continuous one-dimensional function is in harmony with the notion for subsets. The example of a Hölder submanifold \( \Theta \subseteq \mathbb{R}^2 \) with the property that intrinsic Lipschitz continuity of a function on \( \mathbb{R}^2 \setminus \Theta \) plus continuity on \( \mathbb{R}^2 \) does not imply Lipschitz continuity, is constructed in Section 3.3.

#### 3.1 Intrinsically Lipschitz functions and permeable subsets of metric spaces

Throughout this section, let \((M, d)\) be a metric space. To begin, we recall some definitions for metric spaces.

**Definition 3.1** (Path, arc, length).

1. A *path* in \( M \) is a continuous mapping \( \gamma : [a, b] \to M \). We also say that \( \gamma \) is a path in \( M \) from \( \gamma(a) \) to \( \gamma(b) \).
2. An injective path is called an *arc*.
3. If \( \gamma : [a, b] \to M \) is a path in \( M \), then its *length* \( \ell(\gamma) \) is defined as

\[
\ell(\gamma) := \sup \left\{ \sum_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1})) : n \in \mathbb{N}, a = t_0 < \cdots < t_n = b \right\}.
\]

**Definition 3.2** (Intrinsic metric, length space). Let \( E \subseteq M \) and \( \Gamma(x, y) \) be the set of all paths of finite length in \( E \) from \( x \) to \( y \). The *intrinsic metric* \( \rho_E \) on \( E \) is defined by

\[
\rho_E(x, y) := \inf \{ \ell(\gamma) : \gamma \in \Gamma(x, y) \}, \quad (x, y \in E),
\]

with the convention \( \inf \emptyset = \infty \). The metric space \((M, d)\) is a *length space*, iff \( \rho_M = d \).
Note that $\rho_E$ is not a proper metric in that it may assume the value infinity. Of course, one could relate $\rho_E$ to a proper metric via

$$\tilde{\rho}_E(x, y) := \begin{cases} \frac{\rho_E(x, y)}{1 + \rho_E(x, y)}, & \text{if } \rho_E(x, y) < \infty, \\ 1, & \text{if } \rho_E(x, y) = \infty. \end{cases}$$

However, we stick to $\rho_E$, as it is the more natural choice and the extended co-domain does not lead to any difficulties.

It is readily checked that if $\rho_E(x, y) < \infty$ for all $x, y \in E$, then $(E, \rho_E)$ is a length space.

**Definition 3.3** (Intrinsically Lipschitz continuous function). Let $E \subseteq M$, let $(Y, d_Y)$ be a metric space and $f : M \rightarrow Y$ be a function.

1. We call $f$ **intrinsically $L$-Lipschitz continuous** on $E$ iff $f|_E : E \rightarrow Y$ is Lipschitz continuous with respect to the intrinsic metric $\rho_E$ on $E$ and $d_Y$ on $Y$ and $L$ is a Lipschitz constant for $f|_E$.
2. We call $f$ **intrinsically Lipschitz continuous** on $E$ iff $f$ is intrinsically $L$-Lipschitz continuous on $E$ for some $L$.
3. In the above cases, we call $M \setminus E$ an **exception set** (for intrinsic Lipschitz continuity) of $f$.

Note that if $\Theta \subseteq M$ is an exception set for intrinsical Lipschitz continuity that does not mean that Lipschitz continuity holds only on $E = M \setminus \Theta$. For example, if $f : M \rightarrow Y$ is intrinsically Lipschitz continuous, then every subset $\Theta \subseteq M$ is an exception set for intrinsic Lipschitz continuity of $f$.

A classical method for proving Lipschitz continuity of a differentiable function also works for intrinsic Lipschitz continuity:

**Example 1.** Let $A \subseteq \mathbb{R}^d$ open and let $f : A \rightarrow \mathbb{R}$ be differentiable with $\sup_{x \in A} \| \nabla f(x) \| < \infty$. Then, $f$ is intrinsically Lipschitz continuous on $A$ with Lipschitz constant $\sup_{x \in A} \| \nabla f(x) \|$.

A proof can be found in [14, Lemma 3.6].

**Example 2.** Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x) = \| x \| \arctan(x)$. Then, $f$ is not Lipschitz continuous with respect to the Euclidean metric, since

$$\lim_{h \rightarrow 0^+} f(\cos(\pi - h), \sin(\pi - h)) = -\pi \text{ and } \lim_{h \rightarrow 0^+} f(\cos(\pi + h), \sin(\pi + h)) = \pi.$$

It is readily checked, however, that $f$ is Lipschitz continuous on $E = \mathbb{R}^2 \setminus \{ x \in \mathbb{R}^2 : x_1 < 0, x_2 = 0 \}$ with respect to the intrinsic metric $\rho_E$. Thus, $f$ is intrinsically Lipschitz continuous with exception set $\Theta := \{ x \in \mathbb{R}^2 : x_1 < 0, x_2 = 0 \}$ in the sense of Definition 3.3.

**Remark 3.4.** It is almost obvious that every function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, which is continuous on $\mathbb{R}^2$ and intrinsically Lipschitz continuous on $\mathbb{R}^2 \setminus \{(x_1, x_2) : x_1 < 0, x_2 = 0 \}$, is Lipschitz continuous on $\mathbb{R}^2$. One can use that $\Theta := \{ (x_1, x_2) : x_1 < 0, x_2 = 0 \}$ does not pose a “hard” barrier, since every straight line connecting two points in $\mathbb{R}^2 \setminus \Theta$ has at most one intersection point with $\Theta$ and so one can conclude the Lipschitz continuity by approaching $\Theta$ from either side (we invite the reader to make this argument rigorous).

To make the elementary property of “not being a hard barrier” precise, we define at this point the notion of permeability.

**Definition 3.5.** Let $E, \Theta \subseteq M$.

1. The $\Theta$-**intrinsic metric** $\rho^\Theta_E$ on $E$ is defined by

$$\rho^\Theta_E(x, y) := \inf \{ t : \gamma \in \Gamma^\Theta(x, y) \}$$

where $\Gamma^\Theta(x, y)$ is the set of all paths $\gamma : [a, b] \rightarrow M$ of finite length in $E$ from $x$ to $y$, such that $\{ \gamma(t) : t \in [a, b] \} \cap \Theta$ is at most countable. (Again, we use the convention that $\inf \emptyset = \infty$.)
2. The $\Theta$-finite intrinsic metric $\rho_E^{\Theta, \text{FIN}}$ on $E$ is defined by

$$\rho_E^{\Theta, \text{FIN}}(x, y) := \inf \{ \ell(y) : y \in \Gamma^{\Theta, \text{FIN}}(x, y) \}$$

where $\Gamma^{\Theta, \text{FIN}}(x, y)$ is the set of all paths $\gamma : [a, b] \to M$ of finite length in $E$ from $x$ to $y$, such that $\{\gamma(t) : t \in [a, b]\} \cap \Theta$ is finite.

3. We call $\Theta$ permeable relative to $M$ iff $\rho_M = \rho_M^{\Theta}$. 
4. We call $\Theta$ finitely permeable relative to $M$ iff $\rho_M = \rho_M^{\Theta, \text{FIN}}$.

When the ambient space $(M, d)$ is understood and there is no danger of confusion, we simply say $\Theta$ is (finitely) permeable.

Note that there seems to be a difference between the definitions of a permeable set and a permeable graph. Theorem 5 will state that in fact a continuous function has a permeable graph if and only if its graph is a permeable subset of $\mathbb{R}^2$.

Remark 3.6. A set $\Theta \subseteq M$ is (finely) permeable iff for any $x, y \in M$ and every $\varepsilon > 0$ there exists a path $\gamma$ from $x$ to $y$ in $M$ with $\ell(\gamma) < \rho_M(x, y) + \varepsilon$ and such that $\{\gamma(t) : t \in [a, b]\} \cap \Theta$ is at most countable (finite). Clearly, every finitely permeable set is permeable.

We now state a result from [13].

**Theorem** [13, Theorem 15]. Let $\Theta \subseteq M$ be permeable, $(Y, d_Y)$ a metric space. Then, every continuous function $f : M \to Y$, which is intrinsically $L$-Lipschitz continuous on $E = M \setminus \Theta$, is intrinsically $L$-Lipschitz continuous on the whole of $M$. □

In particular, if $M$ is a length space, we get the following corollary.

**Corollary** [13, Corollary 20]. Let $M$ be a length space and let $\Theta \subseteq M$ be permeable. Then, every continuous function $f : M \to Y$ into a metric space $(Y, d_Y)$, which is intrinsically $L$-Lipschitz continuous on $E = M \setminus \Theta$, is $L$-Lipschitz continuous on the whole of $M$ (i.e., with respect to $d$). □

Revisiting Remark 3.4, we see that the situation is more interesting if we replace the set $\mathbb{R}^2 \setminus \{(x, 0) : x < 0\}$ by $\mathbb{R}^2 \setminus \Theta$, where $g$ is some continuous function on $(-\infty, 0]$. By the above theorem, every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, which is intrinsically Lipschitz on $\mathbb{R}^2 \setminus \Theta$ is automatically Lipschitz on $\mathbb{R}^2$, if $\Theta$ is permeable, which in turn is the case iff $g$ has permeable graph, by Theorem 5.

From Theorem 2, we know that for a typical $g$, $\Theta$ is permeable, but from Corollary 3 we also know that not every Hölder function $g$ pertains this property.

The study of permeability of subsets of $\mathbb{R}$ is very tidy: a subset $\Theta$ is permeable iff it has countable closure. Further, this is equivalent to the property that every continuous intrinsically Lipschitz continuous function with exception set $\Theta$ is Lipschitz continuous.

**Theorem** [13, Theorem 23]. Let $\Theta \subseteq \mathbb{R}$. Then $\Theta$ has countable closure if and only if for all intervals $I \subseteq \mathbb{R}$ and all functions $f : I \to \mathbb{R}$ the properties

- $f$ is intrinsically Lipschitz continuous with exception set $\Theta$,
- $f$ is continuous,

imply that $f$ is Lipschitz continuous on $I$.

Consider $\Theta = (\mathbb{R} \setminus \mathbb{Q})^2$. It has been shown in [13, Proposition 26] that $\Theta$ is impermeable, but every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is intrinsically Lipschitz continuous on $\mathbb{R}^2 \setminus \Theta$ is Lipschitz continuous. However, one cannot conclude $L$-Lipschitz continuous from $L$-intrinsically Lipschitz continuous, rather, in this particular example, the Lipschitz constant has to be multiplied with $\sqrt{2}$. 

3.2 Graphs of functions and images of paths

This subsection contains technical results about the connection between the notions of continuous functions with permeable graphs and continuous functions, whose graphs are permeable sets. The main result is Theorem 5, which states that these concepts coincide.

**Definition 3.7.** Let \( f : [a, b] \to \mathbb{R} \).

- We say \( f \) is **regulated**, if its right-sided limit \( f(x^+) := \lim_{y \searrow x} f(y) \) exists in every \( x \in [a, b) \), and its left-sided limit \( f(x-) := \lim_{y \nearrow x} f(y) \) exists in every \( x \in (a, b] \).
- If \( f \) is regulated, we define \( f(a-) := f(a) \) and \( f(b+) := f(b) \).
- If \( f \) is regulated we define its **connected graph**
  \[
  \text{cgraph}(f) := \bigcup_{x \in [a, b]} \text{conv} \{f(x), f(x+), f(x-)\}.
  \]
- We call a function **unfrayed**, if it is regulated and \( f(x) \in \text{conv} (\{f(x+), f(x-)\}) \) for all \( x \in [a, b] \).

**Proposition 3.8.** Let \( f \) be a regulated function. Then

\[
V(f) = \sup \sum_{k=1}^{n} \left( |f(x_{k-1}^+)-f(x_{k-1})| + |f(x_{k-})-f(x_{k-1}^+)| + |f(x_{k})-f(x_{k-})| \right)
\]

and

\[
\ell(f) = \sup \sum_{k=1}^{n} \left( |f(x_{k-1}^+)-f(x_{k-1})| + \sqrt{(x_{k} - x_{k-1})^2 + (f(x_{k}^-) - f(x_{k-1}^+))^2} + |f(x_{k})-f(x_{k-})| \right),
\]

where the suprema are taken over all partitions \( a = x_0 < \cdots < x_n = b \) of \([a, b]\).

**Proof.** The proof is quite straightforward and is left to the reader. \( \square \)

The next proposition states that there is a one-to-one relation between right-continuous scalar functions of bounded variation and certain paths of finite length connecting the endpoints of the function’s graph in the plane. Basically, the path traces the graph and the jumps of the function. Recall that the coordinate functions of a given function \( \gamma : [0, 1] \to \mathbb{R}^2 \) are denoted by \( \gamma_1, \gamma_2 \).

**Proposition 3.9.** Let \( a, b \in \mathbb{R}^2 \), \( a < b \). Let an arc \( \gamma : [0, 1] \to \mathbb{R}^2 \) be given whose first component \( \gamma_1 \) is nondecreasing with \( \gamma_1(0) = a \) and \( \gamma_1(1) = b \). Then, there is an unfrayed function \( f : [a, b] \to \mathbb{R} \) such that \( \text{cgraph}(f) = \gamma([0,1]) \).

Conversely, if an unfrayed function \( f : [a, b] \to \mathbb{R} \) is given, then there exists an arc \( \gamma : [0, 1] \to \mathbb{R}^2 \) whose first component \( \gamma_1 \) is nondecreasing with \( \gamma_1(0) = a \) and \( \gamma_1(1) = b \), and for which \( \gamma([0,1]) = \text{cgraph}(f) \).

In both cases, \( \ell^b_a(f) = \ell(\gamma) \) and \( V^b_a(f) = V^1_0(\gamma_2) \).

**Proof.** We define a function \( \gamma_1^+ : [a, b] \to \mathbb{R} \) by

\[
\gamma_1^+(x) := \begin{cases} 
\inf \{ t \geq 0 : \gamma_1(t) > x \}, & \text{if } x \in [a, b), \\
1, & \text{if } x = b.
\end{cases}
\]

Since \( \gamma_1 \) is continuous and non-decreasing, \( \gamma_1^+ : [a, b] \to \mathbb{R} \) is a non-decreasing function. Moreover, \( \gamma_1(\gamma_1^+(x)) = x \) for all \( x \in [a, b] \). Note that \( \gamma_1^+ \) is right-continuous on \([a, b]\) and \( \gamma(x^-) \) exists on \((a, b]\). Define

\[
f(x) := \begin{cases} 
\gamma_2(\gamma_1^+(x)), & \text{if } x \in (a, b], \\
\gamma_2(0), & \text{if } x = a.
\end{cases}
\]
Noting that $f$ has jumps precisely on intervals where $\gamma_1$ is constant, it is not difficult to see that $f$ is unfrayed with $c\text{graph}(f) = \gamma([0, 1])$.

The opposite direction of the proof is also quite simple. To obtain $\gamma$, one needs to connect the vertical jumps of the unfrayed function $f$ by vertical line segments. Further details are left to the reader. \hfill $\square$

**Proposition 3.10.** Let $a < b$, $f_a, f_b \in \mathbb{R}$, and let $\gamma : [0, 1] \to \mathbb{R}$ be a path connecting $(a, f_a)$ and $(b, f_b)$. Then, there is an arc $\tilde{\gamma} : [0, 1] \to \mathbb{R}$ connecting the same points with $\tilde{\gamma}_1$ monotonically increasing and such that $\ell(\tilde{\gamma}) \leq \ell(\gamma)$ and $V^b_0(\tilde{\gamma}_2) \leq V^b_0(\gamma_2)$. Furthermore, there are at most countably many disjoint intervals $([s_k, t_k])_{0 \leq k < K}$, $K \in \mathbb{N}_0 \cup \{\infty\}$, such that $\tilde{\gamma}_1$ is constant on each of the $[s_k, t_k]$ and $\gamma|_{[0, 1] \setminus \bigcup_{k=0}^{K-1} [s_k, t_k]} = \tilde{\gamma}|_{[0, 1] \setminus \bigcup_{k=0}^{K-1} [s_k, t_k]}$.

**Proof.** Let $\tilde{\gamma}_1 : [0, 1] \to [a, b]$ be defined by $\tilde{\gamma}_1(t) := \sup_{0 \leq s \leq t} \gamma_1(s)$. Then, $\tilde{\gamma}_1$ is continuous and monotonically increasing and therefore there are at most countably many (pairwise disjoint) intervals $([s_k, t_k])_{0 \leq k < K}$ on which it is constant, and on the complement of $\bigcup_{k=0}^{K-1} [s_k, t_k]$ it is strictly increasing, for some $K \in \mathbb{N}_0 \cup \{\infty\}$. Then, we set

$$
\tilde{\gamma}_2(t) := \begin{cases} 
\gamma_2(t), & \text{if } t \notin \bigcup_{k=0}^{K-1} (s_k, t_k), \\
\gamma_2(s_k) + \frac{\gamma_2(t_k) - \gamma_2(s_k)}{t_k - s_k} (t - s_k), & \text{if } t \in (s_k, t_k).
\end{cases}
$$

Then, $\tilde{\gamma} := (\tilde{\gamma}_1, \tilde{\gamma}_2)$ is an arc, since $\tilde{\gamma}_2$ is strictly monotone on the intervals where $\tilde{\gamma}_1$ is constant, and $\tilde{\gamma}(0) = (a, f_a)$ and $\tilde{\gamma}(1) = (b, f_b)$. Using this definition, we leave to the reader the easy task to verify that $\tilde{\gamma}$ satisfies the claim of the proposition. \hfill $\square$

We give another simple result on the relation between the length of a path and the total variation of its component functions.

**Proposition 3.11.** Let $\gamma : [a, b] \to \mathbb{R}^2$ be a path in $\mathbb{R}^2$. Then for $j \in \{1, 2\}$

$$
V^b_a(\gamma_j) \leq \ell(\gamma) \leq V^b_a(\gamma_1) + V^b_a(\gamma_2) \quad \text{and} \quad V^b_a(\gamma_1) + V^b_a(\gamma_2) \leq \sqrt{2}\ell(\gamma).
$$

Let $f : [a, b] \to \mathbb{R}$ be a function. Then,

$$
V^b_a(f) \leq \ell^b_a(f) \leq b - a + V^b_a(f) \quad \text{and} \quad b - a + V^b_a(f) \leq \sqrt{2}\ell^b_a(f).
$$

**Proof.** Again, we leave the simple proof to the reader. \hfill $\square$

**Proposition 3.12.** Let $f : [a, b] \to \mathbb{R}$ have bounded variation. Then, $\tilde{f} : [a, b] \to \mathbb{R}$, $t \mapsto f(t-)$ exists, $V^b_a(\tilde{f}) \leq V^b_a(f)$ and $\ell^b_a(\tilde{f}) \leq \ell^b_a(f)$.

**Proof.** As $f$ is a function of bounded variation, the left and right limits exist, so $\tilde{f}$ can be defined and is unfrayed. By Proposition 3.8,

$$
V^b_1(f) = \sup_{k=1}^n \left( |f(x_{k-1}+) - f(x_{k-1})| + |f(x_k) - f(x_{k-1}+)| + |f(x_k) - f(x_k-1)| \right)
$$

$$
= \sup_{k=1}^{n-1} \left( |f(x_k+)-f(x_k)| + |f(x_k)-f(x_k-)| + \sum_{k=1}^n |f(x_k)-f(x_{k-1}+)| + |f(x_0+)-f(x_0)| + |f(x_n)-f(x_n-)| \right)
$$
\[
\begin{align*}
&\geq \sup_{n-1} \sum_{k=1}^{n} |f(x_k^+) - f(x_k^-)| + \sum_{k=1}^{n} |f(x_k^-) - f(x_{k-1}^+)| \\
&\quad + |f(x_0^+) - f(x_0^-)| + |f(x_n^-) - f(x_n^-)| \\
&\geq \sup_{n-1} \sum_{k=1}^{n} \left( |\tilde{f}(x_k^+) - \tilde{f}(x_k^-)| \right) + \sum_{k=1}^{n} |\tilde{f}(x_k^-) - \tilde{f}(x_{k-1}^+)| \\
&\quad + |\tilde{f}(x_0^+) - \tilde{f}(x_0^-)| + |\tilde{f}(x_n^-) - \tilde{f}(x_n^-)| \\
&= V_0^1(\tilde{f}),
\end{align*}
\]

as \( |f(x_n^-) - f(x_n^-)| \geq 0 = |\tilde{f}(x_n^-) - \tilde{f}(x_n^-)| \).

A similar argument shows the claim about the lengths of the graphs. \(\square\)

**Proposition 3.13.** Let \( g : [0, 1] \to \mathbb{R} \) be a function with permeable graph. Then for any subinterval \([a, b] \subseteq [0, 1] \), \( g|_{[a,b]} \) has a permeable graph on \([a, b]\).

**Proof.** Let \( y \in \mathbb{R} \) and \( \delta > 0 \) be given. Since \( g \) has a permeable graph, there is a function \( \hat{f} : [0, 1] \to \mathbb{R} \) such that \( \hat{f}(0) = \hat{f}(1) = y \), \( V_0^1(\hat{f}) < \frac{\delta}{3} \) and \( \{ t \in [0,1] : g(t) = \hat{f}(t) \} \) is countable. Hence, \( |\hat{f}(a) - y| < \frac{\delta}{3} \) and \( |\hat{f}(b) - y| < \frac{\delta}{3} \). Define \( f : [a, b] \to \mathbb{R} \) by

\[
\begin{align*}
t \mapsto \begin{cases} 
y, & t \in [a, b], \\
\hat{f}(t), & t \in (a, b). 
\end{cases}
\end{align*}
\]

Then, \( V_a^b(f) \leq |\hat{f}(a) - y| + V_a^b(\hat{f}) + |\hat{f}(b) - y| < \delta \) and

\[\{ t \in [a, b] : g(t) = \hat{f}(t) \} \subseteq \{ t \in [0,1] : g(t) = \hat{f}(t) \} \cup \{a, b\},\]

so \( \{ t \in [a, b] : g(t) = f(t) \} \) is countable. \(\square\)

**Theorem 5.** Let \( g : [a, b] \to \mathbb{R} \) be a continuous function. Then, \( g \) has permeable graph iff the graph of \( g \) is a permeable subset of \( \mathbb{R}^2 \).

**Proof.** We assume that \( a = 0, b = 1 \). First, we show that the permeable graph property of a continuous function \( g : [0, 1] \to \mathbb{R} \) implies the permeability of graph(\( g \)) as a subset of \( \mathbb{R}^2 \).

Any vertical line segment connecting two points in the plane has at most one intersection point with the graph of \( g \) and is trivially their shortest connection. Hence, we only concentrate on line segments between points \((x_1, y_1), (x_2, y_2)\) with \( x_1 \neq x_2 \). In particular, without loss of generality, we may assume that \((x_1, y_1) = (0, f_a), (x_2, y_2) = (1, f_b)\).

Let \( \delta > 0 \) be given. The graph of \( g \) is a zero set with respect to the two-dimensional Lebesgue-measure \( \lambda^{(2)} \), which one can see by applying Fubini’s theorem. Hence, also

\[
\int \mathbb{1}_{\text{graph}(g)}(t, \delta) \, d\lambda^{(2)} = 0.
\]

Again by Fubini’s theorem, we get that for some \( u \in \left[-\frac{\delta}{4}, \frac{\delta}{4}\right] \) (and actually \( \lambda \)-almost all \( u \)), \( \lambda\{ t \in [0,1] : u + f_a + (f_b - f_a)t = g(t) \} = 0 \). Moreover, we may choose \( u \) so that \( u + f_a \neq g(0) \) and \( u + f_b \neq g(1) \). We define \( \phi(t) := u + f_a + (f_b - f_a)t \) and \( F := \{ t \in [0,1] : \phi(t) = g(t) \} \). As \([0, 1] \setminus F \) is an open subset of \([0,1] \), it can be written as disjoint union \([0, a_0) \cup \)
Since $F$ has measure zero, we can find $K > 0$ such that

$$F \subseteq [0, 1] \setminus \left( [0, a_0) \cup (b_0, 1] \cup \bigcup_{k=1}^{K} (a_k, b_k) \right) =: \bigcup_{k=0}^{K} [c_k, d_k]$$

and $\lambda \left( \bigcup_{k=0}^{K} [c_k, d_k] \right) = \sum_{k=0}^{K} (d_k - c_k) < \frac{\delta}{4(\|f_b - f_a\|_1 - \|f_0 - f_a\|_2 + 1)}$. As $g$ is permeable on each interval $[c_k, d_k]$ by Proposition \ref{prop_width}, it follows that there is a function $f_{c_k, d_k}$ such that $V_{c_k}^{d_k}(f_{c_k, d_k}) < \frac{\delta}{4(K + 1)}$, $f_{c_k, d_k}(c_k) = f_{c_k, d_k}(d_k) = \phi(c_k)$ and $\{ t \in [0, 1] : f_{c_k, d_k}(t) = g(t) \}$ is countable. We define the function $f : [0, 1] \to \mathbb{R}$ by

$$t \mapsto \begin{cases} 
\phi(t), & t \notin \bigcup_{k=0}^{K} (c_k, d_k], \\
 f_{c_k, d_k}(t), & t \in (c_k, d_k].
\end{cases}$$

Then, $f(0) = f_a + u$, $f(1) = f_b + u$ and $\{ t \in [0, 1] : f(t) = g(t) \}$ is countable as we defined the function $f$ piecewisely on finitely many disjoint intervals.

Let now $\tilde{f}(t) := f(t-)$ for all $t \in [0, 1]$ (we can take left limits as $f$ is of bounded variation). The operation only takes effect on the intervals $(c_k, d_k], k = 0, \ldots, K$, where $\tilde{f}(t) = \tilde{f}_{c_k, d_k}(t) := \tilde{f}_{c_k, d_k}(t-)$, on the other intervals $\tilde{f} = f$ holds. By Propositions \ref{prop_derivative} and \ref{prop_width}, taking into account the possible jumps at the locations $d_k$, we get

$$\ell^1_0(\tilde{f}) \leq \ell^1_0(f) \leq \ell^1_0(\phi) + \sum_{k=1}^{K} \ell_{d_k}^{b_k}(\phi) + \sum_{k=0}^{K} \ell_{c_k}^{d_k}(f_{c_k, d_k}) + \sum_{k=0}^{K} |\phi(d_k) - \phi(c_k)| + \sum_{k=0}^{K} \left( (d_k - c_k) + V_{c_k}^{d_k}(f_{c_k, d_k}) + |\phi(d_k) - \phi(c_k)| \right),$$

where we used Proposition \ref{prop_derivative}. We continue the above estimation with

$$\ell^1_0(\tilde{f}) = \sqrt{a_0^2 + (f_a + u - \phi(a_0))^2 + (1 - b_0)^2 + (f_b + u - \phi(b_0))^2}$$

$$+ \sum_{k=1}^{K} \sqrt{(b_k - a_k)^2 + (\phi(b_k) - \phi(a_k))^2}$$

$$+ \sum_{k=0}^{K} \left( (d_k - c_k) + |\phi(d_k) - \phi(c_k)| + V_{c_k}^{d_k}(f_{c_k, d_k}) \right)$$

$$< \sqrt{a_0^2 + (f_a + u - \phi(a_0))^2 + (1 - b_0)^2 + (f_b + u - \phi(b_0))^2}$$

$$+ \sum_{k=1}^{K} \sqrt{(b_k - a_k)^2 + (\phi(b_k) - \phi(a_k))^2}$$

$$+ \sum_{k=0}^{K} \left( (d_k - c_k) + |\phi(d_k) - \phi(c_k)| + \frac{\delta}{4(K + 1)} \right).$$

By the form of $\phi$, the last sum can be written as

$$\sum_{k=0}^{K} \left( \sqrt{(d_k - c_k)^2 + (\phi(d_k) - \phi(c_k))^2} + (\|1, f_b - f_a\|_1 - \|1, f_b - f_a\|_2)(d_k - c_k) + \frac{\delta}{4(K + 1)} \right).$$
Gluing the pieces of the line segments together, we infer
\[
\ell_1^i(\tilde{f}) < \left\| (0, f_a) - (1, f_b) \right\|_2 + \frac{\delta}{2}.
\]

Since \( \tilde{f} \) is left-continuous and \( g \) is continuous, if \((t_n)_{n \geq 0} \) with \( f(t_n) = g(t_n) \) tends to \( t \in [0, 1] \) from below, then \( f(t-) = g(t) \) and therefore \( t \in \{ t \in [0, 1] : f(t) = g(t) \} \). Hence, the set \( \{ t \in [0, 1] : \tilde{f}(t) = g(t) \} \) is also countable.

By our choice of \( u, a_0 > 0 \) and \( b_0 < 1 \), thus \( \tilde{f}(0) = f_a + u \) and \( f(1) = f(1-) = f_b + u \).

Due to Proposition 3.9, there is an arc \( \gamma : [0, 1] \to \mathbb{R}^2 \) connecting \((0, f_a + u)\) and \((1, f_b + u)\) and \( \ell(\gamma) = \ell_1^i(\tilde{f}) < \left\| (0, f_a) - (1, f_b) \right\|_2 + \frac{\delta}{2} \) and \( \gamma([0,1]) = cgraph(\tilde{f}) \). Thus additionally to \( \{ t \in [0, 1] : f(t) = g(t) \} \), we get the closure of
\[
\left\{ t \in [0, 1] : \tilde{f}(t) \neq \tilde{f}(t+) \text{ and } \min(\tilde{f}(t), \tilde{f}(t+)) \leq g(t) \leq \max(\tilde{f}(t), \tilde{f}(t+)) \right\},
\]
which we call \( J_g \).

Let \( J := \{ t \in [0, 1] : \tilde{f}(t) \neq \tilde{f}(t+) \} \) be the set of jump locations of \( \tilde{f} \), which is countable, as \( \tilde{f} \) is of bounded variation. Take a sequence \((t_n)_{n \geq 0} \) in \( J_g \) converging to some \( \bar{t} \). Then, there are two possibilities: Either, there is a jump at \( \bar{t} \), then \( \bar{t} \) is contained in \( J \). Or, there is no jump at \( \bar{t} \), then \( \tilde{f} \) is continuous at \( \bar{t} \). In the latter case, by the continuity of \( \tilde{f} \) and \( g \), it follows that \( \tilde{f}(\bar{t}) = g(\bar{t}) \) and hence \( \bar{t} \in \{ t \in [0, 1] : f(t) = g(t) \} \). In any case, we have that
\[
\overline{J_g} \subseteq J \cup \{ t \in [0, 1] : f(t) = g(t) \},
\]
and the right-hand side is countable. We infer that \( \overline{\gamma([0,1]) \cap \text{graph}(g)} \) is also countable.

Extending \( \gamma \) to an arc \( \gamma \) by additionally connecting vertically \((0, f_a)\) with \((0, f_a + u)\) and \((1, f_b)\) with \((1, f_b + u)\), we get, as \( u \in \left[ \frac{\delta}{4}, \frac{\delta}{4} \right] \),
\[
\left\| (0, f_a) - (1, f_b) \right\|_2 \leq \ell(\gamma) \leq \ell(\gamma) + \frac{\delta}{2} = \ell_1^i(\tilde{f}) + \frac{\delta}{2} < \left\| (0, f_a) - (1, f_b) \right\|_2 + \frac{\delta}{2} + \frac{\delta}{2} = \left\| (0, f_a) - (1, f_b) \right\|_2 + \delta,
\]
as desired.

Conversely, assume that \( \text{graph}(g) \) is a permeable set. Let \( y \in \mathbb{R} \) and \( \delta > 0 \). As for the implication before, using a Fubini argument, we infer that there is \( u \in \left[ y - \frac{\delta}{4}, y + \frac{\delta}{4} \right] \) such that \( \lambda(\{ t \in [0, 1] : u = g(t) \}) = 0 \) and \( g(0) \neq u \neq g(1) \). Set again \( F := \{ t \in [0, 1] : u = g(t) \} \). Then, we can find \( K > 0 \) such that
\[
F \subseteq [0, 1] \setminus \left( [0, a_0) \cup (b_0, 1) \cup \bigcup_{k=1}^{K} (a_k, b_k) \right) = \bigcup_{k=0}^{K} [c_k, d_k]
\]
and \( \lambda \left( \bigcup_{k=0}^{K} [c_k, d_k] \right) = \sum_{k=0}^{K} (d_k - c_k) < \frac{\delta}{4} \). As \( g \) is permeable, we can find paths \( \gamma_{c_k,d_k} : [0, 1] \to \mathbb{R} \), connecting \((c_k, u)\) and \((d_k, u)\) such that \( \ell(\gamma_{c_k,d_k}) < (d_k - c_k) + \frac{\delta}{4(K+1)} \) and \( \gamma_{c_k,d_k}([0,1]) \cap \text{graph}(g) \) is countable. By Proposition 3.10, we can find for each \( \gamma_{c_k,d_k} \) an arc \( \tilde{\gamma}_{c_k,d_k} \), connecting the same points and having monotone first component. We then set \( \gamma : [0, 1] \to \mathbb{R}, \)
\[
\gamma(t) := \begin{cases} (t, u), & \text{if } t \not\in \bigcup_{k=0}^{K} [c_k, d_k], \\ \tilde{\gamma}_{c_k,d_k}(t-c_k), & \text{if } t \in [c_k, d_k]. \end{cases}
\]
By Proposition 3.9, there exists an unfrayed function \( f \) with \( \text{cgraph}(f) = \tilde{\gamma}([0,1]) \) and \( V_1^0(f) = V_1^0(\tilde{\gamma}) \). Note that \( f \) jumps at \( t \) if and only if \( t \) is in an interval where the paths \( \tilde{\gamma}_{c_k,d_k} \) have constant first component, so \( \{ t \in [0, 1] : f(t) = g(t) \} \) is
determined by the closed sets where $\tilde{c}_k, d_k$ equals $c_k, d_k$, intersected with the graph of $g$ (which is also closed). Thus, $\{t \in [0,1] : f(t) = g(t)\}$ is countable. The variation of $f$ can be estimated by

$$V_0^1(f) = \sum_{k=0}^{K} V_{c_k}^d(f) \leq \sum_{k=0}^{K} V_{c_k}^d(y_2).$$

Proposition 3.11 implies that

$$V_{c_k}^d(y_2) = V_0^1((\tilde{c}_k, d_k)_{2}) \leq \sqrt{2} \epsilon (c_k, d_k) - V_{c_k}^d((\tilde{c}_k, d_k)_{1}) = \sqrt{2} \epsilon (c_k, d_k) - (d_k - c_k)$$

since $c_k, d_k$ has increasing first component, and by Proposition 3.10, $\epsilon (\tilde{c}_k, d_k) \leq \ell (c_k, d_k)$. As $\ell (c_k, d_k) < (d_k - c_k) + \frac{\delta}{4(k+1)}$, we get that

$$V_0^1(f) < \sum_{k=0}^{K} (\sqrt{2} - 1)(d_k - c_k) + \frac{\delta}{4} \leq \frac{\delta}{2}.$$ 

Put

$$\tilde{f}(t) := \begin{cases} y, & \text{if } t \in \{0,1\}, \\ f(t), & \text{else}. \end{cases}$$

Since $u \in \left[y - \frac{\delta}{4}, y + \frac{\delta}{4}\right]$, we get that $V_0^1(\tilde{f}) < \delta$ and the set $\{t \in [0,1] : \tilde{f}(t) = g(t)\}$ is countable, as asserted. \qed

### 3.3 A non-Lipschitz intrinsically Lipschitz function

The main goal of this section is Theorem 6. This theorem provides an example of a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is intrinsically Lipschitz continuous on $\mathbb{R}^2 \setminus \Theta$ for some impermeable subset $\Theta \subseteq \mathbb{R}^2$, but not Lipschitz continuous. This example shows that the prerequisite of permeability in [13, Theorem 15] cannot simply be dropped. Ultimately, $\Theta$ will be the graph of a continuous function $\theta : \mathbb{R} \to \mathbb{R}$, and therefore a topological submanifold of $\mathbb{R}^2$, which is closed as a set. By Theorems 1 and 5, the function $\theta$ cannot be absolutely continuous or Lipschitz continuous.

Recall that the function $g_H$ was introduced in Section 2.3. For the remainder of this section, let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} g_H(x), & x \in [\varepsilon_1, \varepsilon_2], \\ 0, & \text{else}, \end{cases} \quad (3.1)$$

where $\varepsilon_1$ and $\varepsilon_2$ are the smallest and largest zeros of $g_H$, respectively. Note that $g$ also has an impermeable graph. Indeed, if this were not the case, then one could find a function $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and $V_0^1(f) < \frac{1}{4}$ such that the set $\{t \in [0,1] : f(t) = g(t)\}$ is countable. Define

$$\tilde{f}(x) := \begin{cases} f(x), & x \in [\varepsilon_1, \varepsilon_2], \\ 1/4, & \text{else}. \end{cases}$$

Then, $V_0^1(\tilde{f}) < V_0^1(f) + |f(\varepsilon_1)| + |f(\varepsilon_2)| + 1/4 \leq 3V_0^1(f) + 1/4 < 1$ and

$$\{t \in [0,1] : \tilde{f}(t) = g(t)\} \subseteq \{t \in [0,1] : f(t) = g(t)\},$$

so the former is countable, a contradiction.
Let
\[ A := \{(1 + x, g(x)) : x \in [0, 1]\}. \]  

(3.2)

According to Theorems 5 and 3, this implies that there is \( c > 0 \) such that
\[ \ell(\gamma) \geq 1 + c \]  

(3.3)

for every path \( \gamma \) from (1,0) to (2,0) with countable \( \gamma([0,1]) \cap A \).

Next, we define a useful representation of the classical Cantor set as a subset of \( \mathbb{R}^2 \) using two affine transformations.

Let \( T_1(x, y) := \left( \frac{x}{3}, \frac{y}{2} \right) \) and \( T_0(x, y) := \left( \frac{2 + x}{3}, \frac{y}{2} \right) \). For a number \( n \in \mathbb{N} \) let \( d(k, n) \in \{0,1\} \) be the \( k \)th binary digit, \( n = \sum_{k=0}^{\infty} d(k, n)2^k \), and \( T_n := T_{d(0,n)} \circ \ldots \circ T_{d(\lfloor \log_2(n) \rfloor, n)} \).

Note that, if \( C \in [0,1] \) is the classical Cantor set, then \( C \times \{0\} = ([0,1] \times \{0\}) \setminus \bigcup_{n \in \mathbb{N}} T_n((1,2) \times \{0\}) \).

Lemma 3.14. Define \( \tilde{\Theta} := \bigcup_{n \in \mathbb{N}} T_n(A) \) and \( \Theta := \tilde{\Theta} \cup (C \times \{0\}) \). Then, \( \Theta \) is the graph of a Hölder continuous function \( \vartheta : [0,1] \to \mathbb{R} \), with Hölder exponent \( \beta := \frac{\log(10)}{\log(234)} \).

Further, let \( f : \mathbb{R} \times \{0\} \to \mathbb{R}, \)
\[ f((t, 0)) = \begin{cases} 0, & t < 0, \\ \mathcal{C}(t), & 0 \leq t \leq 1, \\ 1, & t > 1, \end{cases} \]
where \( \mathcal{C} : [0,1] \to \mathbb{R} \) is the classical Cantor (or Devil's) staircase function, cf. [21, p. 252]. Then, \( f \) is Lipschitz continuous with respect to \( \rho_{\mathbb{R}^2}^\Theta \) restricted to \( \mathbb{R} \times \{0\} \).

Proof. We first show the Lipschitz continuity of \((\text{the restriction of})\ f\) with respect to \( \rho_{\mathbb{R}^2}^\Theta \). Let \( t_1, t_2 \in \mathbb{R} \). We have to show that
\[ |f((t_2, 0)) - f((t_1, 0))| \leq \frac{2}{c} \rho_{\mathbb{R}^2}^\Theta((t_1, 0), (t_2, 0)). \]

We concentrate on the interesting case, where \( t_1, t_2 \in C \) with \( t_1 < t_2 \). Let \( k = \min\{j \geq 1 : 1 < (t_2 - t_1)3^j\} + 1 \). There is precisely one interval \( I \) of the form \( I = (j3^{-k}, (j+1)3^{-k}) \) between \( t_1 \) and \( t_2 \) and therefore there exists \( n \in \mathbb{N} \) with \( I = \text{pr}_1 \circ T_n ((1,2) \times \{0\}) \), and \( k = \lfloor \log_2(n) \rfloor + 1 \).

W.l.o.g., \( t_1 \leq \left( \frac{1}{3} \right)^k < 2 \cdot \left( \frac{1}{3} \right)^k \leq t_2 \), such that there exist \( s_1 \in [0,1] \) and \( s_2 \in [2,3] \) such that \((t_1, 0) = T_{2^k-1}(s_1, 0) \) and \((t_2, 0) = T_{2^k-1}(s_2, 0) \).

Let \( \eta : [0,1] \to \mathbb{R}^2 \) be a path from \((t_1, 0)\) to \((t_2, 0)\) with \( \ell(\eta) < \infty \) and such that \( \eta([0,1]) \cap \Theta \) has countable closure.

Write \( \tilde{\eta} := (T_{2^k-1})^{-1} \circ \eta \), that is, \( \eta = (T_k)^k(\tilde{\eta}) \). Note that \((s_1, 0) = \tilde{\eta}(0) \) and \((s_2, 0) = \tilde{\eta}(1) \). Let \( r_0 := \sup\{t \geq 0 : \tilde{\eta}(t) = 1\} \) and \( r_1 := \inf\{t \geq 0 : \tilde{\eta}(t) = 2\} \). We define a path \( \gamma \) from \((1,0)\) to \((2,0)\) as the concatenation of the straight line from \((1,0)\) to \(\tilde{\eta}(r_0)\), the path \(\tilde{\eta}|_{[r_0, r_1]}\) and the straight line from \(\tilde{\eta}(r_1)\) to \((2,0)\). Note that \(\gamma([0,1]) \cap A \subseteq \tilde{\eta}([0,1]) \cap A \) and therefore the closure of \(\gamma([0,1]) \cap A \) is countable. Using Equation (3.3), we get
\[ 1 + c \leq \ell(\gamma) = |\tilde{\eta}(r_0) - (1,0)| + \ell(\tilde{\eta}|_{[r_0, r_1]}) + |\tilde{\eta}(r_1) - (2,0)| \]
\[ = |\tilde{\eta}_2(r_0)| + \ell(\tilde{\eta}|_{[r_0, r_1]}) + |\tilde{\eta}_2(r_1)| \]
\[ \leq \|\tilde{\eta}(r_0) - (s_1,0)\| + \ell(\tilde{\eta}|_{[r_0, r_1]}) + \|\tilde{\eta}(r_1) - (s_2,0)\| \]
\[ \leq \ell(\tilde{\eta}|_{[0,r_0]}) + \ell(\tilde{\eta}|_{[r_0,r_1]}) + \ell(\tilde{\eta}|_{[r_1,1]}) = \ell(\tilde{\eta}). \]
We can find a path \( \tilde{\gamma} : [0, 1] \to \mathbb{R}^2 \) with increasing first component like in Proposition 3.10. Thus, \( \ell(\tilde{\gamma}) \leq \ell(\gamma) \) and \( V_0(\tilde{\gamma}_2) \leq V_0(\gamma_2) \). We show like in the proof of Theorem 5, that \( \tilde{\gamma}([0, 1]) \cap A \) is countable: by construction, \( \tilde{\gamma} \) has at most countable number of vertical line segments, where intersections with the set \( A \) that have not already been part of \( \gamma([0, 1]) \cap A \), may occur. Since \( A \) is the graph of a function, each vertical line segment admits at most one additional intersection point. Let now \( \tilde{x} \) be a limit point of those additional intersection points. Then, \( \tilde{x} \) is either part of a vertical segment of \( \tilde{\gamma} \), of which there are at most countably many, or \( \tilde{x} \) is not part of a vertical segment. In the latter case, \( \tilde{x} \) must be contained in \( \gamma([0, 1]) \), since on the complement of the vertical segments, the images of \( \gamma \) and \( \tilde{\gamma} \) coincide (by Proposition 3.10) and they both contain \( \tilde{x} \) because of their continuity. So, in this case, \( \tilde{x} \) is contained in \( \gamma([0, 1]) \cap A \), which is countable. So in both cases, the possible elements of \( \tilde{\gamma}([0, 1]) \cap A \) are contained in \( \gamma([0, 1]) \cap A \) or in the countable set of intersection points arising from vertical segments of \( \tilde{\gamma} \). Hence, Equation (3.3) holds with \( \tilde{\gamma} \) instead of \( \gamma \). Using Proposition 3.11, we get (note that \( T_{2^{k-1}} = (T_1)^k \))

\[
\ell(\eta) \geq V_0(\eta_2) = \left( \frac{1}{2} \right)^k V_0(\tilde{\gamma}_2) \geq \left( \frac{1}{2} \right)^k (\ell(\tilde{\gamma}) - V_0(\tilde{\gamma}_1)) \geq \left( \frac{1}{2} \right)^k (\ell(\gamma) - 1) \geq \left( \frac{1}{2} \right)^k c.
\]

It follows that \( \rho_{\mathbb{R}^2}((t_1, 0), (t_2, 0)) \geq \left( \frac{1}{2} \right)^k c \).

On the other hand, we have \( |f((t_2, 0)) - f((t_1, 0))| \leq \left( \frac{1}{2} \right)^{k-1} \), so that

\[
|f((t_2, 0)) - f((t_1, 0))| \leq \frac{2}{c} \rho_{\mathbb{R}^2}((t_1, 0), (t_2, 0)).
\]

Since \( g(0) = g(1) = 0, \Theta \) is the graph of a continuous function \( \theta : [0, 1] \to \mathbb{R} \) by construction. Now, we show the claim about the Hölder continuity of \( \theta \). For this assume that \( g \) is Hölder continuous with exponent \( \alpha \) and constant \( C_g \), and let \( t_1, t_2 \in [0, 1] \). If \( t_1, t_2 \in C \), then \( \theta(t_1) = \theta(t_2) = 0 \) and there is nothing to show. Next, consider the case where \( t_1, t_2 \) are in the same connected component of \( \mathbb{R} \setminus C \). That is, \( (t_1, 0), (t_2, 0) \in T_n((1, 2) \times \{0\}) \) for some \( n \in \mathbb{N} \). Again, we may concentrate on the case \( n = 2^{k-1} - 1 \), so that w.l.o.g. \( \frac{1}{3} \left( \frac{1}{3} \right)^{k-1} < t_1 < t_2 < \frac{2}{3} \left( \frac{1}{3} \right)^{k-1} \), and \( 1 < s_1 := \text{pr}_1(T^{-1}(t_1, 0)) < s_2 := \text{pr}_1(T^{-1}(t_2, 0)) < 2 \). By Theorem 3 and Equation (3.1), we have

\[
|g(s_1) - g(s_2)| \leq C_g |s_1 - s_2|^{\beta},
\]

so that

\[
|\theta(t_1) - \theta(t_2)| = 2^{-k}|g(s_1) - g(s_2)| \leq 2^{-k}C_g |s_1 - s_2|^{\beta} = 2^{-k}C_g (3^k|t_1 - t_2|)^{\beta} = \left( \frac{3^{\beta}}{2} \right)^k C_g |t_1 - t_2|^{\beta}.
\]

Now since \( \beta \leq \frac{\log(2)}{\log(3)} \), we have \( \frac{3^\beta}{2} \leq 1 \) and \( |\theta(t_1) - \theta(t_2)| \leq C_g |t_1 - t_2|^{\beta} \). Next, suppose that \( t_1 \in C, t_2 \in [0, 1] \setminus C \). Denote by \((u, v)\) the largest open interval with \( t_2 \in (u, v) \subseteq [0, 1] \setminus C \). If \( t_1 < t_2 \), then by the continuity of \( \theta \)

\[
|\theta(t_1) - \theta(t_2)| = |0 - \theta(t_2)| = |\theta(u) - \theta(t_2)| = |\theta(u) - \theta(t_2)| \leq C_g |u - t_2|^{\beta} \leq C_g |t_1 - t_2|^{\beta},
\]

and in the same way we can treat the case \( t_2 < t_1 \).

Next, we consider the case where \( t_1 \) and \( t_2, t_1 < t_2 \), lie in different connected components \((u_1, v_1)\) and \((u_2, v_2)\) of \([0, 1] \setminus C\), respectively. Then

\[
|\theta(t_1) - \theta(t_2)| = |\theta(t_1) - \theta(t_1) + \theta(u_2) - \theta(t_2)| \\
\leq |\theta(t_1) - \theta(t_1)| + |\theta(u_2) - \theta(t_2)|
\]

\[
|\theta(t_1) - \theta(t_2)| = 2^{-k}|g(s_1) - g(s_2)| \leq 2^{-k}C_g |s_1 - s_2|^{\beta} = 2^{-k}C_g (3^k|t_1 - t_2|)^{\beta} = \left( \frac{3^\beta}{2} \right)^k C_g |t_1 - t_2|^{\beta}.
\]
\[ C_g |t_1 - v_1|^\beta + C_g |u_2 - t_2|^\beta \leq 2^{1-\beta} C_g |t_1 - u_2 - t_2|^\beta \leq 2^{1-\beta} C_g |t_1 - t_2|^\beta, \]

where we used the special case Hölder inequality \((x + y)^q \leq 2^{q-1} (x^q + y^q)\) for all \(x, y \in [0, \infty), q \in [1, \infty)\) with \(x = |t_1 - v_1|^\beta, y = |u_2 - t_2|^\beta\) and \(q = 1/\beta\).

\[ \square \]

**Theorem 6.** There exists a continuous function \(f : \mathbb{R}^2 \to \mathbb{R}\) which is intrinsically Lipschitz continuous on \(\mathbb{R}^2 \setminus \Theta\), where \(\Theta\) is the graph of a Hölder continuous function \(\vartheta\), but \(f\) is not Lipschitz continuous with respect to the Euclidean metric.

**Proof.** The function \(f\) constructed in Lemma 3.14 is Lipschitz continuous on \(\mathbb{R} \times \{0\}\) with respect to \(\rho^\Theta_{\mathbb{R}^2}\).

By the Kirszbraun extension theorem, cf. [8, Theorem 2.10.43] (or the elementary special case [8, 2.10.44]), there exists a Lipschitz continuous extension of \(f\) on \(\mathbb{R}^2\) which we again denote by \(f\). Hereby “Lipschitz” means with respect to the metric \(\rho^\Theta_{\mathbb{R}^2}\). But since \(\rho^\Theta_{\mathbb{R}^2} \leq \rho_{\mathbb{R}^2 \setminus \Theta}\), \(f\) is also Lipschitz with respect to the intrinsic metric on \(\mathbb{R}^2 \setminus \Theta\). Restricted to the domain \([0, 1] \times \{0\}\), \(f\) equals the Cantor staircase function, which is not Lipschitz continuous with respect to the Euclidean metric. It remains to show the continuity of \(f\) with respect to the Euclidean metric:

To that end, take a sequence \((x_n)\), converging to \(x \in \mathbb{R}^2\). Note that, being the graph of a continuous function on \([0, 1], \Theta\) is closed.

Case 1: \(x \notin \Theta\).

Then, there is \(r > 0\) and \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(x_n \in B_r(x) \subseteq \mathbb{R}^2 \setminus \Theta\). It follows that in this ball \(\rho^\Theta_{\mathbb{R}^2}\) equals the Euclidean metric. So, \(f\) is Lipschitz therein with respect to both metrics and in particular \(f\) is continuous in \(x\).

Case 2: \(x \in \Theta\).

Then w.l.o.g., we can assume that all the \(x_n\) lie above or on the graph of \(\vartheta\) and \(\text{pr}_1(x_n) \not\in \text{pr}_1(x)\). Let \(u_n := (\text{pr}_1(x_n), \vartheta(\text{pr}_1(x_n)))\). Then, as \(x_n \to x\), also \(x_n - u_n \to 0\). Also, since \(\vartheta\) is continuous,

\[ M(x_n, x) := \max_{\text{pr}_1(x_n) \leq s \leq \text{pr}_1(x)} (\vartheta(s) - \vartheta(\text{pr}_1(x_n))) \to 0. \]

Now, define

\[ u_n := \left( \text{pr}_1(x_n), \max \left\{ \text{pr}_2(x_n) - \frac{1}{n} + \vartheta(\text{pr}_1(x_n)) + M(x_n, x) \right\} \right). \]

and

\[ z_n := \left( \text{pr}_1(x), \max \left\{ \text{pr}_2(x_n) - \frac{1}{n} + \vartheta(\text{pr}_1(x_n)) + M(x_n, x) \right\} \right). \]

Then, the path \(\gamma_n\), defined as the concatenation of line segments \(u_n \to w_n \to z_n \to x\) is in \(\mathbb{R}^2 \setminus \Theta\) apart from \(u_n\) and \(x\) and contains \(x_n\) in its image. Forms of such paths are illustrated in Figure 4.
Further,

$$
\ell(\gamma_n) = \max \left\{ \text{pr}_2(x_n) - \theta(\text{pr}_1(x_n)), \frac{1}{n} + M(x_n, x) \right\} + (\text{pr}_1(x) - \text{pr}_1(x_n)) \\
+ \max \left\{ \text{pr}_2(x_n) - \theta(\text{pr}_1(x)), \frac{1}{n} + \theta(\text{pr}_1(x_n)) - \theta(\text{pr}_1(x)) + M(x_n, x) \right\},
$$

and $\ell(\gamma_n) \to 0$ as all terms on the right-hand side tend to 0. Since the image of $\gamma_n$ has at most two points in common with $\Theta$, it follows that $\rho_{\| \cdot \|}^\Theta(x_n, x) \leq \ell(\gamma_n) \to 0$. By the Lipschitz continuity with respect to $\rho_{\| \cdot \|}^\Theta$ with Lipschitz constant, say $L$, we have

$$
|f(x_n) - f(x)| \leq L \rho_{\| \cdot \|}^\Theta(x_n, x) \to 0 \quad \text{as} \quad x_n \to x,
$$

and we obtain the continuity of $f$.

**ACKNOWLEDGMENTS**

Gunther Leobacher and Alexander Steinicke are grateful to Dave L. Renfro for connecting them with Zoltán Buczolich.

Z. Buczolich: The project leading to this application has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 741420). This author was also supported by the Hungarian National Research, Development and Innovation Office–NKFIH, Grant 124749 and at the time of completion of this paper was holding a visiting researcher position at the Rényi Institute. G. Leobacher is supported by the Austrian Science Fund (FWF): Project F5508-N26, which is part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications.”

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**REFERENCES**

[1] S. Banach, *Sur les lignes rectifiables et les surfaces dont l’aire est finie*, Fund. Math. 7 (1925), 225–236.

[2] T. D. Benavides, *How many zeros does a continuous function have?* Am. Math. Mon. 93 (1986), no. 6, 464–466.

[3] J. B. Brown, U. B. Darji, and E. Larsen, *Nowhere monotone functions and functions of nonmonotonic type*, Proc. Am. Math. Soc. 127 (1999), no. 1, 173–182.

[4] A. M. Bruckner, and S. H. Jones, *Behavior of continuous functions with respect to intersection patterns*, Real Anal. Exch. 19 (1993/94), no. 2, 414–432.

[5] Z. Buczolich., *Sets of convexity of continuous functions*, Acta Math. Hung. 52 (1988), no. 3, 291–303.

[6] E. Čech, *Sur les fonctions continues qui prennent chaque leur valeur un nombre fini de fois*, Fundam. Math. 17 (1931), 32–39.

[7] K. J. Falconer, *The geometry of fractal sets*, vol. 85, Cambridge University Press, Cambridge, 1985.

[8] H. Federer, *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band vol. 153, Springer-Verlag, New York, 1969.

[9] S. Kalmykov, L. V. Kovalev, and T. Rajala, *Removable sets for intrinsic metric and for holomorphic functions*, J. Anal. Math. 139 (2019), 751–772.

[10] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Graduate texts in mathematics, vol. 113, Springer, New York, 1991.

[11] A. Kechris, *Classical descriptive set theory*, Graduate texts in mathematics, Springer, New York, 1995.

[12] A. Klenke, *Probability theory: a comprehensive course*, Universitext, Springer, London, 2013.

[13] G. Leobacher and A. Steinicke, *Exception sets of intrinsic and piecewise Lipschitz functions*, J. Geom. Anal. 32 (2022), no. 4, 118.

[14] G. Leobacher and M. Szölgyenyi, *A strong order 1/2 method for multidimensional SDEs with discontinuous drift*, Ann. Appl. Probab. 27 (2017), no. 4, 2383–2418.

[15] G. Leobacher and M. Szölgyenyi, *Convergence of the Euler–Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient*, Numer. Math. 138 (2018), no. 1, 219–239.

[16] S. Minakshisundar, *On the roots of a continuous nondifferentiable function*, J. Indian Math. 4 (1940), 31–33.

[17] T. Müller-Gronbach and L. Yaroslavtseva, *A strong order 3/4 method for SDEs with discontinuous drift coefficient*, IMA J. Numer. Anal. 42 (2022), no. 1, 229–259.

[18] T. Müller-Gronbach and L. Yaroslavtseva, *On the performance of the Euler–Maruyama scheme for SDEs with discontinuous drift coefficient*, Ann. Inst. H. Poincaré Probab. Statist. 56 (1178), no. 2, 1162–1178. 2020.
[19] A. Neuenkirch, M. Szölgyenyi, and L. Szpruch, *An adaptive Euler–Maruyama scheme for stochastic differential equations with discontinuous drift and its convergence analysis*, SIAM J. Numer. Anal. 57 (2019), no. 1, 378–403.

[20] J. C. Ponce-Campuzano and M. Á. Maldonado-Aguilar, *Vito Volterra’s construction of a nonconstant function with a bounded, non-Riemann integrable derivative*, BSHM Bull. J.Br. Soc. Hist. Math. 30 (2015), 143–152.

[21] T. Rajala, A note on the level sets of Hölder continuous functions on the real line, 2008. [https://users.jyu.fi/tamaraja/publications/Raj08Hol.pdf](https://users.jyu.fi/tamaraja/publications/Raj08Hol.pdf), Online, accessed 07-26-21.

[22] D. Revuz and M. Yor, *Continuous Martingales and Brownian motion*, Grundlehren der mathematischen Wissenschaften, Springer, Berlin, 2004.

[23] B. S. Thomson, J. B. Bruckner, and A. M. Bruckner, *Elementary real analysis*, Prentice-Hall, 2001.

[24] M. Younsi, *On removable sets for holomorphic functions*, EMS Surv. Math. Sci. 2 (2015), no. 2, 219–254.

**How to cite this article:** Z. Buczolich, G. Leobacher, and A. Steinicke, *Continuous functions with impermeable graphs*, Math. Nachr. 296 (2023), 4778–4805. [https://doi.org/10.1002/mana.202200268](https://doi.org/10.1002/mana.202200268)