Toeplitz operators with special symbols on Segal-Bargmann spaces

Jotsaroop K. and S. Thangavelu

Abstract. We study the boundedness of Toeplitz operators on Segal-Bargmann spaces in various contexts. Using Gutzmer’s formula as the main tool we identify symbols for which the Toeplitz operators correspond to Fourier multipliers on the underlying groups. The spaces considered include Fock spaces, Hermite and twisted Bergman spaces and Segal-Bargmann spaces associated to Riemannian symmetric spaces of compact type.

1. Introduction

Given a domain $\Omega$ in $\mathbb{C}^n$ let $\mathcal{H}(\Omega, d\mu)$ stand for a weighted Bergman space of holomorphic functions contained in $L^2(\Omega, d\mu)$. Let $g$ be a Lebesgue measurable function on $\Omega$ such that $gF \in L^2(\Omega, d\mu)$ for all $F$ from a dense subspace of $\mathcal{H}(\Omega, d\mu)$. We can then define the Toeplitz operator $T_g$ by $T_gF = P(gF)$ where $P : L^2(\Omega, d\mu) \to \mathcal{H}(\Omega, d\mu)$ is the natural orthogonal projection. Such Toeplitz operators have been studied extensively in the literature.

Suppose now that $\Omega$ is invariant under the action of a Lie group $G$. The group $G$ has a natural action on $\mathcal{H}(\Omega, d\mu)$. Let us further assume that there is an isometric isomorphism $B$ between $L^2(G)$ and $\mathcal{H}(\Omega, d\mu)$. Using this, we can transfer the Toeplitz operator $T_g$ into the operator $B^{-1}T_gB$ acting on $L^2(G)$. Then the boundedness of $T_g$ becomes equivalent to that of this transferred operator which might turn out to be easier to study using harmonic analysis on the group $G$. The simplest bounded operators on $L^2(G)$ are given by Fourier multipliers and hence

1991 Mathematics Subject Classification. 47B35, 43A85, 22E30.

Key words and phrases. Segal-Bargmann transform, weighted Bergman spaces, Toeplitz operators, Fourier multipliers, Gutzmer’s formula, Hermite and Laguerre functions, symmetric spaces.
it is natural to ask which Toeplitz operators give rise to such multiplier transformations.

In this article we are interested in the case where $B$ is the Segal-Bargmann transform on the group $G$ and $\mathcal{H}(\Omega, d\mu)$ is the image of $L^2(G)$ under $B$. It turns out that we can identify a large class of symbols $g$ for which $B^{-1}T_g B$ reduces to Fourier multipliers. The groups for which such results can be proved include $\mathbb{R}^n$, Heisenberg groups and compact Lie groups. An important role is played by the so called Gutzmer’s formula. Thus for Fock spaces, Hermite and twisted Bergman spaces and Segal-Bargmann spaces associated to compact Lie groups and symmetric spaces we have identified special classes of symbols for which $T_g$’s correspond to multiplier transforms.

The plan of the paper is as follows. In the next section we look at Toeplitz operators on the classical Fock spaces. In section 3 we study Toeplitz operators on Hermite-Bergman spaces which give rise to Hermite multipliers when conjugated with the Hermite semigroup. In section 4 we characterise all Toeplitz operators on the twisted Bergman spaces that correspond to Weyl multipliers. Finally, in the last section we consider Toeplitz operators on Segal-Bargmann spaces associated to compact Lie groups and symmetric spaces. For results closely related to the theme of this paper we refer to [2], [10], [11] and [12].

2. Toeplitz operators on Fock spaces

In this section we look at Toeplitz operators on Fock spaces which have been studied by several authors, see [2] and the references there. First we consider Toeplitz operators with radial symbols and obtain a necessary and sufficient condition for $T_g$ to be bounded. Toeplitz operators with radial symbols on $\mathcal{F}(\mathbb{C})$ with a different assumption have been studied by Grudsky and Vasilevski [9]. The condition involves the heat flow $g * q_{1/4}$ and under a mild decay assumption we prove boundedness when $g$ is radial. Later we consider symbols $g(x + iy)$ which depend only on $y$ and show that they correspond to Fourier multipliers. For such symbols we show that the conjecture of Berger and Coburn [2] is true.

2.1. Radial symbols. In this subsection we consider Toeplitz operators associated to radial symbols on the Fock space $\mathcal{F}(\mathbb{C}^n)$.

$$\mathcal{F}(\mathbb{C}^n) := \{ f \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |f(z)|^2 d\mu(z) < \infty \},$$

where $d\mu(z) = (2\pi)^{-n} e^{-\frac{1}{2}|z|^2}$. It is known that $\mathcal{F}(\mathbb{C}^n)$ is a Hilbert space with the reproducing kernel explicitly given by $K(z, w) = e^{\frac{-|z-w|^2}{2}}$. Recall
that the Toeplitz operator with symbol $g$ is given by

$$T_g f(z) = P(fg)(z) = \int_{\mathbb{C}^n} g(w)f(w)e^{z \cdot \bar{w}}/2 d\mu(w).$$

An orthonormal basis for the Fock space $\mathcal{F}(\mathbb{C}^n)$ is given by

$$\zeta_\alpha(z) = \frac{z^\alpha}{2^{|\alpha|}(\alpha!)^{1/2}}.$$

As $\langle T_g \zeta_\alpha, \zeta_\beta \rangle = \langle g \zeta_\alpha, \zeta_\beta \rangle$ we can easily check that

$$\langle T_g \zeta_\alpha, \zeta_\beta \rangle = \delta_{\alpha \beta} \langle T_g \zeta_\alpha, \zeta_\alpha \rangle$$

whenever $g$ is radial. This leads to the following result for Toeplitz operators with radial symbols. In what follows we let

$$q_t(z) = (4\pi t)^{-n} e^{-\frac{1}{4} |z|^2}$$

stand for the heat kernel on $\mathbb{C}^n$ associated to the standard Laplacian and

$$\varphi_k(z) = L_n^{k-1}(\frac{1}{2} |z|^2) e^{-\frac{1}{4} |z|^2}$$

for the Laguerre functions of type $(n-1)$.

**Theorem 2.1.** Let $g$ be a measurable function on $\mathbb{C}^n$ such that $g \zeta_\alpha \in L^2(\mathbb{C}^n, d\mu)$ for all $\alpha \in \mathbb{N}^n$. Then the Toeplitz operator $T_g$ is bounded on $\mathcal{F}(\mathbb{C}^n)$ if and only if the sequence

$$\frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} g * q_{1/4}(w) \varphi_k(2w) dw$$

is bounded.

**Proof.** Using the result (see Lemma 3.2.6 in [26])

$$\int_{S^{2n-1}} \zeta_\alpha(\omega) \overline{\zeta_\beta(\omega)} d\sigma(\omega) = \delta_{\alpha \beta} \frac{(n-1)!}{(|\alpha| + n - 1)!} 2^{-|\alpha|}$$

we easily calculate that whenever $g$ is radial

$$\langle T_g \zeta_\alpha, \zeta_\beta \rangle = \delta_{\alpha \beta} \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty g(r) r^{2k+2n-1} 2^k k! e^{-\frac{1}{2} r^2} dr$$

where $k = |\alpha|$. Therefore,

$$T_g F(z) = \sum_{\alpha \in \mathbb{N}^n} R_{|\alpha|}(g) \langle F, \zeta_\alpha \rangle \zeta_\alpha(z)$$

where

$$R_k(g) = \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty g(r) r^{2k+2n-1} 2^k k! e^{-\frac{1}{2} r^2} dr.$$

The theorem now follows from the following lemma. \qed
Lemma 2.2. Let \( g \) be a radial function as in the theorem. Then for any \( k \in \mathbb{N} \),
\[
R_k(g) = c_n \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} g * q_{1/4}(w) \varphi_k(2w) dw.
\]

Proof. We make use of the following formula satisfied by Laguerre functions (see Szego [20])
\[
e^{-x^2} L_k^\alpha(x^2) = \frac{1}{k!} \int_0^\infty e^{-t^2} t^{2k+\alpha} \frac{J_\alpha(2tx)}{t^{\alpha} x^\alpha} t^\alpha dt,
\]
which can be rewritten as
\[
e^{-2x^2} L_k^{\alpha}(2x^2) = \frac{1}{k!} \int_0^\infty e^{-t^2} \left( \frac{t^2}{8} \right)^k \frac{J_\alpha(tx)}{t^{\alpha} x^\alpha} t^{2\alpha+1} dt.
\]
Inverting the Hankel transform and making a change of variables we get
\[
(2.1.1) \quad \frac{1}{k!} e^{-\frac{1}{2}t^2} \left( \frac{t^2}{2} \right)^k = \int_0^\infty e^{-2x^2} L_k^{\alpha}(2x^2) \frac{J_\alpha(2tx)}{(2tx)^\alpha} x^{2\alpha+1} dx.
\]
As both sides are holomorphic in \( t \) the above equation remains true when \( t \) is replaced by \( it \).

Under the assumption that \( g \) is radial we observe that
\[
g * q_{1/4}(z) = \int_{\mathbb{C}^n} g(w)e^{-|z-w|^2} dw
\]
reduces to a constant multiple of
\[
e^{-|z|^2} \int_0^\infty g(r)e^{-r^2} \frac{J_{n-1}(2irs)}{(2irs)^{n-1}} r^{2n-1} dr.
\]
Therefore,
\[
\int_{\mathbb{C}^n} g * q_{1/4}(w) \varphi_k(2w) dw
\]
\[
= c_n \int_0^\infty \int_0^\infty e^{-r^2} g(r)e^{-s^2} \frac{J_{n-1}(2irs)}{(2irs)^{n-1}} L_k^{n-1}(2s^2)e^{-s^2} r^{2n-1} s^{2n-1} dr ds.
\]
Using Fubini, which is justified by our assumptions on \( g \), and making use of the above identity (2.1.1) satisfied by Laguerre functions, we obtain
\[
\int_{\mathbb{C}^n} g * q_{1/4}(w) \varphi_k(2w) dw = c_n \int_0^\infty g(r) \frac{r^{2k+2n-1}}{2^{k+1} k!} e^{-\frac{1}{2}r^2}.
\]
This completes the proof of the lemma.

Corollary 2.3. Let \( g \) be a radial function as in the previous theorem. Further assume that \(|g * q_{1/4}(z)| \leq C|z|^{-1}\), for all \( z \neq 0 \). Then \( T_g \) is bounded on \( \mathcal{F}(\mathbb{C}^n) \).
Proof. In view of the theorem we only need to check that the sequence
\[
\frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty h(r)L_k^{n-1}(2r^2)e^{-r^2}r^{2n-1}dr
\]
is bounded where \( h(z) = g * q_{1/4}(z) \). Under the assumption on \( g * q_{1/4} \) this can be easily verified using the following estimates on integrals of Laguerre functions.

\[
\left| \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty h(r)L_k^{n-1}(2r^2)e^{-r^2}r^{2n-1}dr \right| \leq c_n \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty r^{-1}|L_k^{n-1}(r^2)|e^{-r^2/2}r^{2n-1}dr.
\]

We define \( L_k^{n-1}(r^2), r \in \mathbb{R} \) by
\[
L_k^{n-1}(r^2) = \left( \frac{k!(n-1)!}{(k+n-1)!} \right)^{1/2} L_k^{n-1}(r^2)r^{n-1}e^{-r^2/2}.
\]

It follows from Lemma 1.5.4 in [26] that
\[
\int_0^\infty |L_k^{n-1}(r^2)|r^{-\beta}rdr \sim k^{1/2-\beta/2}
\]
when \( k \) is large. By Stirling’s formula for large \( k \), \( \frac{k!(n-1)!}{(k+n-1)!} \sim k^{-(n-1)} \).

By using the estimates above after putting \( \beta = -(n-2) \) we have
\[
\left| \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty h(r)L_k^{n-1}(2r^2)e^{-r^2}r^{2n-1}dr \right| \leq \left( \frac{k!(n-1)!}{(k+n-1)!} \right)^{1/2} \int_0^\infty |L_k^{n-1}(r^2)|r^{n-2}rdr
\]
\[
\sim k^{-(n-1)/2}k^{1/2+(n-2)/2} = 1.
\]

This proves the lemma. \( \square \)

2.2. Toeplitz operators and Fourier multipliers. A conjecture of Berger and Coburn [2] says that \( T_g \) is bounded on \( \mathcal{F}(\mathbb{C}^n) \) if and only if \( g * q_{1/4} \) is bounded. In this subsection we verify this conjecture when the symbol \( g(x+iy) \) depends only on \( y \). In such a case the problem reduces to checking if a certain Fourier multiplier is bounded on \( L^2(\mathbb{R}^n) \).

As the Fock space is closely related to the weighted Bergman space associated to the Segal-Bargmann/heat kernel transform we consider Toeplitz operators on the space \( \mathcal{B}_t(\mathbb{C}^n) \) consisting of entire functions that are square integrable with respect to \( q_{t/2}(y)dxdy \) where \( q_t \) is the standard heat kernel on \( \mathbb{R}^n \). By the results of Segal and Bargmann [1]
we know that $F \in \mathcal{B}_t(\mathbb{C}^n)$ if and only if $F = f \ast q_t$ for some $f \in L^2(\mathbb{R}^n)$ and
\[
\int_{\mathbb{R}^{2n}} |F(x + iy)|^2 q_{t/2}(y) dxdy = c_n \int_{\mathbb{R}^n} |f(x)|^2 dx.
\]
Let $g$ be a measurable function on $\mathbb{C}^n$ such that $gF$ belongs to $L^2(\mathbb{C}^n, q_{t/2}(y) dy)$ whenever $F \in \mathcal{B}_t(\mathbb{C}^n)$ and let $T_g$ be the associated Toeplitz operator.

**Theorem 2.4.** Let $g(x + iy) = g_0(y)$ be as above. Then $T_g$ is bounded on $\mathcal{B}_t(\mathbb{C}^n)$ if and only if $g_0 \ast q_{t/2}$ is bounded where the convolution is on $\mathbb{R}^n$.

**Proof.** When $F_j = f_j \ast q_t \in \mathcal{B}_t(\mathbb{C}^n), j = 1, 2$ Plancherel’s theorem leads to
\[
\int_{\mathbb{R}^n} F_1(x + iy) \overline{F_2(x + iy)} dx = c_n \int_{\mathbb{R}^n} \hat{f}_1(\xi) \overline{\hat{f}_2(\xi)} e^{-2t|\xi|^2} e^{-2y \cdot \xi} d\xi.
\]
Integrating the above with respect to $g(y)q_{t/2}(y)dy$ we see that
\[
\int_{\mathbb{C}^n} T_g F_1(x + iy) \overline{F_2(x + iy)} q_{t/2}(y) dxdy = c_n \int_{\mathbb{R}^n} m_t(\xi) \hat{f}_1(\xi) \overline{\hat{f}_2(\xi)} d\xi
\]
where
\[
m_t(\xi) = e^{-2t|\xi|^2} \int_{\mathbb{R}^n} e^{-2y \cdot \xi} g_0(y)q_{t/2}(y)dy.
\]
From this it is clear that $T_g$ is bounded if and only if $m_t$ defines a bounded Fourier multiplier on $L^2(\mathbb{R}^n)$ which happens precisely when $m_t$ is a bounded function. An easy calculation shows that $m_t(\xi) = g_0 \ast q_{t/2}(\xi)$ which proves the theorem. □

**Remark 2.1.** We can read out properties of Fourier multipliers $m_t(\xi)$ that correspond to Toeplitz operators from the work of Hille [14]. Indeed, when $t = 1/2$ which corresponds to the Fock space, the multiplier $m$ and the symbol $g$ are related via
\[
m(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g_0(y) e^{-|\xi - y|^2} dy.
\]
Assuming $n = 1$, let
\[
\int_{-\infty}^{\infty} |g_0(y)| e^{-y^2 + \alpha |y|} dy < \infty
\]
for some $\alpha > 0$. Then if $g_0(y) = \sum_{k=0}^{\infty} a_k H_k(y)$ is the expansion of $g_0$ in terms of the Hermite polynomials $H_k$, Hille [14] has proved that $m(z) = \sum_{k=0}^{\infty} a_k (2z)^k$ for all $z \in \mathbb{C}$ with $|z| < \alpha$. 
3. Toeplitz operators on Hermite-Bergman spaces

In this section we study Toeplitz operators on Hermite-Bergman spaces which are Segal-Bargmann spaces associated to the Hermite semigroup \( e^{-tH} \). As in the case of Fock spaces we show that the transferred operator \( e^{tH} T g e^{-tH} \) is a pseudo-differential operator whose Weyl symbol is related to the heat flow of \( g \). This leads to a conjecture similar to that of Berger and Coburn. By making use of Gutzmer’s formula for Hermite expansions we identify certain special symbols \( g \) which lead to Hermite multipliers.

3.1. Hermite-Bergman spaces. On \( \mathbb{R}^{2n} \) consider the weight function \( U_t \) given by

\[
U_t(x, y) = 4^n (\sinh(4t))^{-n/2} e^{t \tanh(2t)x^2 - \coth(2t)y^2}.
\]

The Hermite Bergman space \( \mathcal{H}_t(\mathbb{C}^n) \) is the space of all entire functions \( F \) which are square integrable with respect to \( U_t(x, y) dx dy \). It is known that \( F \in \mathcal{H}_t(\mathbb{C}^n) \) if and only if \( F = e^{-tH} f \) for some \( f \in L^2(\mathbb{R}^n) \) where \( e^{-tH} \) is the Hermite semigroup, see [3]. Moreover,

\[
\int_{\mathbb{R}^{2n}} |F(x + iy)|^2 U_t(x, y) dx dy = c_n \int_{\mathbb{R}^n} |f(x)|^2 dx
\]

whenever \( F = e^{-tH} f \). In the above the Hermite semigroup is defined by

\[
e^{-tH} f = \sum_{\alpha \in \mathbb{N}^n} e^{-(2|\alpha|+n)t} \langle f, \Phi_\alpha \rangle \Phi_\alpha
\]

where \( \Phi_\alpha \) are the normalised Hermite functions which are eigenfunctions of the Hermite operator \( H = -\Delta + |x|^2 \) with eigenvalues \((2|\alpha|+n)\). See [25] for more about Hermite functions.

An important tool in studying the above space is an analogue of Gutzmer’s formula for Hermite expansions which we now proceed to state. Let \( \pi(x, u) \) be the family of unitary operators defined on \( L^2(\mathbb{R}^n) \) by

\[
\pi(x, u) \varphi(\xi) = e^{i(x \cdot \xi + \frac{1}{2}x \cdot y)} \varphi(\xi + y).
\]

These are related to the Schrödinger representation of the Heisenberg group, see [21] and [7]. It is clear \( \pi(z, w) F(\xi) \) makes sense even for \((z, w) \in \mathbb{C}^n \times \mathbb{C}^n\) whenever \( F \) is holomorphic. However, the resulting function need not be in \( L^2(\mathbb{R}^n) \) unless further assumptions are made on \( F \). When \( F = \Phi_\alpha \) (or any finite linear combination of the Hermite functions) \( \pi(z, w) F(\xi) \) is indeed in \( L^2(\mathbb{R}^n) \) and using Mehler’s formula
for the Hermite functions we can prove that
\[
\int_{\mathbb{R}^n} |\pi(z, w)\Phi_{\alpha}(\xi)|^2 d\xi = (2\pi)^\frac{1}{2} e^{(u \cdot y - v \cdot x)} \Phi_{\alpha, \alpha}(2iy, 2iv)
\]
where \(\Phi_{\alpha, \alpha}\) are the special Hermite functions which are expressible in terms of Laguerre functions. Gutzmer’s formula says that a similar result is true for \(\pi(z, w)F(\xi)\) under some assumptions on \(F\).

In order to state Gutzmer’s formula we need to introduce one more notation. Let \(Sp(n, \mathbb{R})\) stand for the symplectic group consisting of \(2n \times 2n\) real matrices that preserve the symplectic form \([\langle x, u \rangle, \langle y, v \rangle] = (u \cdot y - v \cdot x)\) on \(\mathbb{R}^{2n}\) and have determinant one. Let \(O(2n, \mathbb{R})\) be the orthogonal group and we define \(K = Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R})\). Then there is a one to one correspondence between \(K\) and the unitary group \(U(n)\).

Let \(\sigma = a + ib\) be an \(n \times n\) complex matrix with real and imaginary parts \(a\) and \(b\). Then \(\sigma\) is unitary if and only if the matrix \(A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\) is in \(K\). For these facts we refer to Folland [7]. By \(\sigma.(x, u)\) we denote the action of the corresponding matrix \(A\) on \((x, u)\). This action has a natural extension to \(\mathbb{C}^n \times \mathbb{C}^n\) denoted by \(\sigma.(z, w)\) and is given by \(\sigma.(z, w) = (a \cdot z - b \cdot w, a \cdot w + b \cdot z)\) where \(\sigma = a + ib\).

**Theorem 3.1.** For a holomorphic function \(F\) we have the following formula for any \(z = x + iy, w = u + iv \in \mathbb{C}^n:\)
\[
\int_{\mathbb{R}^n} \int_K |\pi(\sigma.(z, w))F(\xi)|^2 d\sigma d\xi = e^{(u \cdot y - v \cdot x)} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(2iy, 2iv) \|P_k f\|^2_2
\]
where \(f\) stands for the restriction of \(F\) to \(\mathbb{R}^n\).

In the above formula \(P_k\) are the spectral projections of the Hermite operator defined by
\[
P_k f(x) = \sum_{|\alpha| = k} \langle f, \Phi_\alpha \rangle \Phi_\alpha(x)
\]
and
\[
\varphi_k(z, w) = L_k^{n-1} \left( \frac{1}{2} (z^2 + w^2) \right) e^{-\frac{1}{4} (z^2 + w^2)}
\]
are the holomorphically extended Laguerre functions of type \((n-1)\).

The above formula means that if either the integral or the sum is finite then they are equal. Note that the sum is clearly finite when \(f = e^{-tH} g\) for some \(g \in L^2(\mathbb{R}^n)\). We refer to [24] for a proof of the above formula. The characterisation of \(\mathcal{H}_t(\mathbb{C}^n)\) as the image of \(L^2(\mathbb{R}^n)\) under
the Hermite semigroup $e^{-tH}$ can be proved using Gutzmer’s formula, see [24]. The only other ingredient needed is the formula

$$\frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} p_{2t}(2y,2v)\varphi_k(2iy,2iv)dydv = e^{2(2k+n)t}$$

where $p_t(y,v)$ stands for the heat kernel associated to the special Hermite operator, see Section 4.

### 3.2. Toeplitz operators on $\mathcal{H}_t(\mathbb{C}^n)$.

Let $P : L^2(\mathbb{C}^n) \to \mathcal{H}_t(\mathbb{C}^n)$ be the orthogonal projection which is explicitly given by

$$PF(z) = \int_{\mathbb{R}^{2n}} F(u,v)K_t(z,u+iv)U_t(u,v)dudv.$$ 

Here $K_t(z,w)$ is the reproducing kernel of $\mathcal{H}_t(\mathbb{C}^n)$ defined by

$$K_t(z,w) = \sum_{\alpha \in \mathbb{N}^n} e^{-2(2|\alpha|+n)t}\Phi_\alpha(z)\Phi_\alpha(w).$$

Using Mehler’s formula we can show that

$$K_t(z,w) = (\sinh(4t))^{-\frac{n}{2}} e^{-\frac{1}{2}\coth(4t)(z^2+w^2)} e^{\frac{1}{2}\sinh(4t)\langle z,w \rangle},$$

where $\langle z,w \rangle$ is the standard Hermitian inner product on $\mathbb{C}^n$ and $z^2 = z_1^2 + \ldots + z_n^2$ etc. For a measurable function $g$ on $\mathbb{C}^n$ such that $gK_t(.,w)$ belongs to $L^2(\mathbb{C}^n,d\mu_t)$ for all $w$ (we will refer to this condition as $*$), we define the Toeplitz operator $T_g$ on $\mathcal{H}_t(\mathbb{C}^n)$ by

$$T_gf(z) = \int_{\mathbb{C}^n} g(w)f(w)K_s(z,w)d\mu_s(w).$$

By the condition ($*$), it is easy to see that $T_g$ is a densely defined operator on $\mathcal{H}_t(\mathbb{C}^n)$. Another important consequence of ($*$) is that $g*q_s$ is well defined for $0 < s < \frac{1}{2}\sinh 4t$, where $q_s(x) = (4\pi s)^{-\frac{n}{2}} e^{-\frac{1}{4s}|x|^2}$ is the heat kernel corresponding to the standard Laplacian on $\mathbb{R}^n$. In fact, it is a $C^\infty$ function on $\mathbb{C}^n$. By using the semigroup property we get

$$g * q_{s+r} = (g * q_r) * q_s,$$

when $0 < s + r < \frac{1}{2}\sinh 4t$. Now we find some necessary and sufficient conditions on $g$ such that $T_g$ is a bounded operator. These conditions are given in terms of $g * q_s$ for $0 < s < \frac{1}{2}\sinh 4t$. In order to do this we transfer $T_g$ to $L^2(\mathbb{R}^n)$ and find the corresponding Weyl symbol of the resulting operator.
Following Folland \[7\] we define the Weyl pseudo-differential operator on \(L^2(\mathbb{R}^n)\) with symbol \(\sigma \in \mathcal{S}'(\mathbb{R}^{2n})\) by
\[
(3.2.1) \quad \sigma(D, X)f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\frac{1}{2}(x + y), \xi)e^{-i(x-y) \cdot \xi} f(y) dy d\xi.
\]
We recall that (see \[7\]) \(\sigma(D, X) = W(\hat{\sigma})\), where \(W\) is the Weyl transform and \(\hat{\sigma}\) is the Fourier transform of a tempered distribution. We define for \(\sigma \in \mathcal{S}'(\mathbb{R}^{2n})\)
\[
(3.2.2) \quad \sigma_t(x, \xi) = \sigma(\cosh(2t)x, -\sinh(2t)\xi).
\]
Note that \(\sigma \to \sigma_t\) is an isomorphism on \(\mathcal{S}'(\mathbb{R}^{2n})\).

**Theorem 3.2.** Let \(T_g\), defined as above, be bounded. Then we have
\[
\|g * q_s\|_{\infty} \leq c(s)\|T_g\| \quad \text{for all } s \in (\frac{1}{8}\sinh 4t, \frac{1}{2}\sinh 4t).
\]
Conversely, if we assume that \(\|g * q_s\|_{\infty} < \infty\) for some \(0 < s < \frac{1}{8}\sinh 4t\), then \(T_g\) is bounded. Moreover, we have
\[
\|T_g\| \leq c(s)\|g * q_s\|_{\infty}.
\]
**Proof.** First let us assume that \(T_g\) is bounded. For \(\frac{1}{4}\sinh 4t \leq s < \frac{1}{2}\sinh 4t\) the proof is trivial. We look at the Berezin Transform of \(T_g\) defined by (see \[7\])
\[
(3.2.3) \quad \tilde{T}_g(z) = \langle T_g k_z, k_z \rangle_{\mathcal{H}_t}.
\]
It is easy to check that \(\tilde{T}_g(z) = g * q_{\frac{1}{4}\sinh 4t}(z)\). Here \(k_z(w) = \frac{K_t(w, z)}{\sqrt{K_t(z, z)}}\) is the normalized reproducing kernel. In fact, even if \(T_g\) is not bounded the Berezin transform is well defined because of the condition (*) and it is the same as above. By applying Cauchy-Schwarz inequality to \(3.2.3\) we get
\[
(3.2.4) \quad |g * q_{\frac{1}{4}\sinh 4t}(z)| \leq \|T_g\|, \quad z \in \mathbb{C}^n.
\]
So, by the semigroup property, when \(0 < s < \frac{1}{2}\sinh 4t\) we get \(g * q_{s+\frac{1}{4}\sinh 4t}(z) = (g * q_{\frac{1}{4}\sinh 4t}) * q_s(z)\) and
\[
(3.2.5) \quad \|g * q_{s+\frac{1}{4}\sinh 4t}\|_{\infty} \leq c(s)\|T_g\|,
\]
where \(c(s)\) is independent of \(g\). For proving the estimate for the other half of the interval in the statement of the theorem, we make use of the boundedness of the operator \(e^{tH}T_g e^{-tH}\) on \(L^2(\mathbb{R}^n)\). Let \(e^{tH}T_g e^{-tH} = W(\hat{\sigma}_t)\) for some \(\sigma \in \mathcal{S}'(\mathbb{R}^{2n})\).

In order to find the explicit form of \(\sigma_t\) we calculate the Berezin transform of \(T_g\) in terms of \(\sigma\). By using \(3.2.1\) an easy computation shows that
\[
\langle T_g k_z, k_z \rangle_{\mathcal{H}_t} = \langle e^{-tH}\sigma_t(D, X)e^{tH} k_z, k_z \rangle_{\mathcal{H}_t},
\]
where
\( (3.2.6) \quad \langle \sigma_t(D, X)e^{itH}k_z, e^{itH}k_z \rangle_{L^2(\mathbb{R}^n)} = \sigma \ast q_{\frac{1}{8}\sinh 4t}(z) \).

By equating (3.2.3) and (3.2.6) we get
\( (3.2.7) \quad g \ast q_{\frac{1}{8}\sinh 4t}(z) = \sigma \ast q_{\frac{1}{8}\sinh 4t}(z), \quad z \in \mathbb{C}^n. \)

Given that \( g \) satisfies (*) and \( \sigma \in \mathcal{S}'(\mathbb{R}^{2n}) \), for a fixed \( z \in \mathbb{C}^n \) it is easy to check the following two facts: (i) \( s \rightarrow g \ast q_s(z) \) extends as a holomorphic function to the domain
\[
D_1 = \{ \zeta \in \mathbb{C} : |\zeta - \frac{1}{4}\sinh 4t| < \frac{1}{4}\sinh 4t \}
\]
and (ii) \( s \rightarrow \sigma \ast q_s(z) \) extends as a holomorphic function to \( D_2 = \{ \zeta \in \mathbb{C} : \Re \zeta > 0 \} \). By using the above two facts we get that \( g \ast q_{\frac{1}{8}\sinh 4t} * q_s = \sigma \ast q_s \) for all \( 0 < s < \frac{3}{8}\sinh 4t \). Now taking the limit \( s \rightarrow 0 \) we get
\[
\sigma_t(x, y) = g \ast q_{\frac{1}{8}\sinh 4t}(\cosh(2t)x, -\sinh(2t)y).\]

Using the fact that \( \mathcal{B}(L^2(\mathbb{R}^n)) \) is the dual of the space of all trace class operators, we get the following:
\( (3.2.8) \quad |\text{tr}(W(f)W(\hat{\sigma}_t))| \leq \|T_g\|\|W(f)\|_{tr} \)
for all \( f \in L^2(\mathbb{C}^n) \) such that \( W(f) \) is trace class. In particular, (3.2.8) holds for all \( f \) in Schwartz class. It is easy to compute that
\( (3.2.9) \quad \text{tr}(W(\bar{z})W(f)W(\bar{z})^*W(\hat{\sigma}_t)) = \hat{f} \ast \sigma_t(z) \)
for all \( z \) when \( f \in \mathcal{S}(\mathbb{R}^{2n}) \) and \( \sigma \in \mathcal{S}'(\mathbb{R}^{2n}) \). If we choose \( f \) in (3.2.9) such that \( \hat{f}(w) = q_{t_1}(u)q_{t_2}(v), w = u + iv \) where \( t_1 = s \cosh 2t, t_2 = s \sinh 2t \) and \( z = (\cosh 2t)^{-1}x + i(\sinh 2t)^{-1}y \) we get
\[
\text{tr}(W(\bar{z})W(f)W(\bar{z})^*W(\hat{\sigma}_t)) = \sigma \ast q_s(x + iy)
\]
for all \( s > 0 \). By (3.2.8)
\[
|\sigma \ast q_s(x + iy)| \leq c(s)\|T_g\|,
\]
where \( \sigma = g \ast q_{\frac{1}{8}\sinh 4t} \) and this implies
\( (3.2.10) \quad |(g \ast q_{\frac{1}{8}\sinh 4t}) \ast q_s(x + iy)| \leq c(s)\|T_g\| \)
for all \( z \in \mathbb{C}^n \). Finally, the boundedness of \( T_g \) implies that
\[
\|g \ast q_s\|_{\infty} \leq c(s)\|T_g\|
\]
whenever \( s > \frac{1}{8}\sinh 4t \).

Conversely, let \( \|g \ast q_s\|_{\infty} < \infty \) for some \( 0 < s < \frac{1}{8}\sinh 4t \) then proceeding as in Berger and Coburn [2]
\[
\|\sigma_t\|_* \equiv \Sigma_{|\mu| + |\beta| \leq 2n + 1}\|D_{\mu}^\alpha D_{x}^\beta \sigma_t\|_{\infty} < \infty,
\]
where $\sigma = g \ast q_1^8 \sinh 4t$. Now we can appeal to Theorem 2.73 in [7] by which $\sigma_t(D, X)$ is bounded with $\|\sigma_t(D, X)\| \leq \|\sigma_t\|_*$. The Berezin symbol of $e^{-tH} \sigma_t(D, X)e^{tH}$ (see (3.2.6) and (3.2.3)) is given by

$$\left(e^{-tH} \sigma_t(D, X)e^{tH}\right)(z) = \sigma \ast q_1^8 \sinh 4t(z) = g \ast q_1^8 \sinh 4t(z)$$

which implies that

$$\tilde{T}_g \equiv \left(e^{-tH} \sigma_t(D, X)e^{tH}\right).$$

Hence by the uniqueness of the Berezin transform $T_g = e^{-tH} \sigma_t(D, X)e^{tH}$. Therefore, the boundedness of $\sigma_t(D, X)$ implies that $\|T_g\| \leq \|\sigma_t\|_*$. As shown in [2] we have $\|\sigma_t\|_* \leq c(n, s)\|g \ast q_s\|_\infty$. Hence the theorem is proved. □

REMARK 3.1. The above theorem is the analogue of Theorems 11 and 12 in [2]. As in [2] we conjecture that $T_g$ is bounded if and only if $g \ast q_1^8 \sinh 4t$ is bounded. We have a class of symbols supporting this conjecture, see Section 3.3

3.3. Hermite multipliers and Toeplitz operators. In this subsection we are interested in finding a necessary and sufficient condition on the symbol $g$ so that $e^{tH} T_g e^{-tH}$ is a Hermite multiplier. Using the fact that $W(\sigma)$ is a function of the Hermite operator if and only if the symbol $\sigma$ is a radial distribution we get the following result.

THEOREM 3.3. Given $T_g$ on $\mathcal{H}_t(\mathbb{C}^n)$ the operator $e^{tH} T_g e^{-tH}$ is a Hermite multiplier if and only if $g \ast q_1^8 \sinh(4t)(\cosh(2t)y, -\sinh(2t)v)$ is a radial function on $\mathbb{R}^{2n}$.

COROLLARY 3.4. Let $g$ be as in the theorem. Then $T_g$ is bounded on $\mathcal{H}_t(\mathbb{C}^n)$ if and only if the sequence

$$\frac{k!(n - 1)!}{(k + n - 1)!} \int_{\mathbb{R}^{2n}} g \ast q_1^8 \sinh(4t)(\cosh(2t)y, -\sinh(2t)v) \varphi_k(2y, 2v)dydv$$

is bounded.

EXAMPLE 3.1. An example of symbol satisfying the condition given in Theorem 3.3 is provided by $g(y, v) = e^{\alpha|y|^2 + \beta|v|^2}$ under suitable conditions on $\alpha$ and $\beta$. A simple calculation shows that

$$g \ast q_1^8 \sinh(4t)(\cosh(2t)y, -\sinh(2t)v) = e^{\frac{\alpha \coth(2t) \sinh(4t)}{2 - \alpha \sinh(4t)}|y|^2} e^{\frac{\beta \tanh(2t) \sinh(4t)}{2 - \beta \sinh(4t)}|v|^2}$$

and hence $g \ast q_1^8 \sinh(4t)(\cosh(2t)y, -\sinh(2t)v)$ is radial if and only if

$$\frac{\alpha \coth(2t)}{2 - \alpha \sinh(4t)} = \frac{\beta \tanh(2t)}{2 - \beta \sinh(4t)}.$$
After simplification we get the condition
\[ \alpha \coth(2t) - \beta \tanh(2t) = \alpha \beta \]
which is necessary and sufficient for the radiality of the function
\[ g * q_k \sinh(4t) (\cosh(2t)y, - \sinh(2t)v), \quad g(y,v) = e^{\alpha |y|^2 + \beta |v|^2}. \]

When the above condition is verified, by Corollary 3.3 the operator \( T_g \) is bounded on \( \mathcal{H}_t(\mathbb{C}^n) \) if and only if the sequence
\[
\frac{k! (n - 1)!}{(k + n - 1)!} \int_{\mathbb{R}^{2n}} \frac{e^{\alpha \coth(2t) \sinh(4t)}}{2 - \alpha \sinh(4t)} (|y|^2 + |v|^2)^{k/2} \phi_k(2y, 2v) dy dv
\]
is bounded. Again, by repeating the method in the Theorem 1.1 this is equivalent to the boundedness of the sequence
\[
\frac{k! (n - 1)!}{(k + n - 1)!} \int_{\mathbb{C}^n} e^{\alpha |z|^2} \phi_k(2z) dz
\]
\[
= c_n \int_{\mathbb{C}^n} e^{\lambda |z|^2} \left| \frac{2k}{k!} \frac{1 + \lambda}{1 - \lambda} \right|^k
\]
where \( \lambda = \frac{\alpha \coth(2t) \sinh(4t)}{2 - \alpha \sinh(4t)} \). Thus the condition for the boundedness of \( T_g \) reduces to \( |1 + \frac{\lambda}{1 - \lambda}| \leq 1 \) or \( \Re \lambda \leq 0 \). In terms of \( \alpha \) the condition reads as \( |\alpha|^2 \sinh(4t) - 2 \Re \alpha \geq 0 \). So, the necessary and sufficient condition for \( T_g \) to be bounded is the boundedness of \( g * q_k \sinh(4t) \). It is worth comparing this example with a similar example given in [2].

The condition in Corollary 3.3 on \( g \) is not easy to check. However, using Gutzmer’s formula we can get a sufficient condition in a more convenient form for certain special class of symbols. Consider radial functions \( h(y,v) \) on \( \mathbb{R}^{2n} \) for which
\[
\int_{\mathbb{R}^{2n}} h(y,v) e^{\alpha |y|^2 + |v|^2} (|y|^2 + |v|^2)^{k/2} < \infty
\]
for all \( k \in \mathbb{N} \). Define a function \( g \) by the equation
\[
g(\xi,v) U_t(\xi,v) = \int_{\mathbb{R}^n} e^{-2y \cdot \zeta} h(y,v) dy.
\]

**Theorem 3.5.** Suppose \( g \) is given by (3.3.12) where \( h \) satisfies (3.3.11). Then we have \( e^{tH} T_g e^{-tH} = m_t(H) \) where
\[
m_t(2k + n) = e^{-2(2k+n)t} \frac{k! (n - 1)!}{(k + n - 1)!} \int_{\mathbb{R}^{2n}} h(y,v) \phi_{2k+2n} dy dv.
\]
Consequently, \( T_g \) is bounded on \( \mathcal{H}_t(\mathbb{C}^n) \) if and only if \( |m_t(2k + n)| \leq C \) for all \( k \in \mathbb{N} \).
Proof. As we mentioned we prove this theorem by using Gutzmer’s formula. Indeed, polarising Gutzmer’s formula we obtain
\[
\int_K \int_{\mathbb{R}^n} \pi(k.(iy,iv)) F_1(\xi) \overline{\pi(k.(iy,iv))} F_2(\xi) d\xi dk \\
= \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} e^{-2t(2k+n)} \langle P_k f_1, f_2 \rangle \varphi_k(2iy, 2iv)
\]
where \( F_j = e^{-tH} f_j, j = 1, 2 \) are from \( \mathcal{H}_t(\mathbb{C}^n) \). Integrating the above identity with respect to \( h(y, v) \) we obtain
\[
\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \pi(k.(iy, iv)) F_1(\xi) \overline{\pi(k.(iy, iv))} F_2(\xi) h(y, v) d\xi dk dy dv \\
= \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} e^{-2t(2k+n)} \langle P_k f_1, f_2 \rangle \int_{\mathbb{R}^{2n}} h(y, v) \varphi_k(2iy, 2iv) dy dv.
\]
When the function \( h \) is \( K \) invariant,
\[
\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \pi(k.(iy, iv)) F_1(\xi) \overline{\pi(k.(iy, iv))} F_2(\xi) h(y, v) d\xi dk dy dv \\
= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} \pi(iy, iv) F_1(\xi) \overline{\pi(iy, iv)} F_2(\xi) h(y, v) d\xi dy dv.
\]
Recalling the definition of \( \pi(iy, iv) \) the above integral can be rewritten as
\[
\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} F_1(\xi + iv) F_2(\xi + iv) e^{-2y \xi} h(y, v) d\xi dv.
\]
Suppose now \( g(\xi, v) \) satisfies the equation
\[
g(\xi, v) U_t(\xi, v) = \int_{\mathbb{R}^n} e^{-2y \xi} h(y, v) dy
\]
and \( m_t(2k+n) \) is defined by
\[
m_t(2k+n) = e^{-2t(2k+n)} \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} h(y, v) \varphi_k(2iy, 2iv) dy dv.
\]
Then it is clear that we have obtained
\[
\int_{\mathbb{R}^{2n}} F_1(\xi + iv) F_2(\xi + iv) g(\xi, v) U_t(\xi, v) d\xi dv \\
= \sum_{k=0}^{\infty} m_t(2k+n) \langle P_k f_1, f_2 \rangle
\]
which simply means that \( (T_g F_1, F_2)_{\mathcal{H}_t} = \langle m_t(H) f_1, f_2 \rangle_{L^2} \) where
\[
m_t(H) f = \sum_{k=0}^{\infty} m_t(2k+n) P_k f
\]
TOEPLITZ OPERATORS

is the Hermite multiplier. Thus the boundedness of the Toeplitz operator \( T_g \) on \( \mathcal{H}_t(\mathbb{C}^n) \) is equivalent to the boundedness of \( m_t(H) \) on \( L^2(\mathbb{R}^n) \).

**Remark 3.2.** In the above proof of sufficiency we have not used Theorem 3.3 but the condition stated in that theorem can be verified. Indeed, when \( g \) satisfies the equation (3.3.12) a simple calculation shows that

\[
g * q_{\frac{3}{2}} \sinh(4t) (\cosh(2t)x, -\sinh(2t)y) e^{\tanh(2t)(|x|^2 + |y|^2)} = \int_{\mathbb{R}^{2n}} h(\xi, v) e^{\tanh(2t)(|\xi|^2 + |v|^2)} e^{-\frac{2}{\cosh(2t)}(x, \xi + v) d\xi d\nu}
\]

from which it is clear that \( g * q_{\frac{3}{2}} \sinh(4t) (\cosh(2t)x, -\sinh(2t)y) \) is radial whenever \( h(\xi, v) \) is radial. The above equation also suggests a relation between \( g \) and \( h \).

**Remark 3.3.** The radiality of the function

\[
g * q_{\frac{3}{2}} \sinh(4t) (\cosh(2t)x, -\sinh(2t)y)
\]

is not equivalent to the factorisation given in (3.3.12). Indeed, consider the symbol \( g(x, y) = e^{\alpha|x|^2 + \beta|y|^2} \) considered earlier with the conditions \( \alpha \coth(2t) - \beta \tanh(2t) = \alpha \beta \) and \( \Re(\alpha) < \frac{1}{2}(\sinh(2t))^{-1} \). If there exists a function \( h \) such that

\[
g(\xi, v) U_t(\xi, v) = \int_{\mathbb{R}^n} e^{-2y \cdot \xi} h(y, v) dy,
\]

then we have the relation

\[
g(\frac{i}{2} \xi, v) U_t(\frac{i}{2} \xi, v) = \int_{\mathbb{R}^n} e^{-i\xi \cdot y} h(y, v) dy.
\]

This leads to the equation

\[
e^{-\frac{1}{4}(\tanh(2t) + \alpha)|y|^2} e^{-(\coth(2t) - \beta)|v|^2} = \int_{\mathbb{R}^n} e^{-i\xi \cdot y} h(y, v) dy.
\]

By Fourier inversion we see that

\[
h(y, v) = c e^{-\frac{1}{4\tanh(2t) + \alpha}|y|^2} e^{-(\coth(2t) - \beta)|v|^2}
\]

which is not radial in general.

**Remark 3.4.** Since

\[
U_t(\xi, v) = c_t \int_{\mathbb{R}^n} p_{2t}(2y, 2v) e^{-2y \cdot \xi} dy
\]

the equation (3.3.12) is equivalent to

\[
g(\xi, v) = \int_{\mathbb{R}^n} g_1(y, v) e^{-2y \cdot \xi} dy.
\]
Indeed, if \( g \) satisfies the above equation, then the function
\[
h(y, v) = \int_{\mathbb{R}^n} g_1(y - u, v) p_{2t}(2u, 2v) du
\]
satisfies
\[
\int_{\mathbb{R}^n} h(y, v) e^{-2y \cdot \xi} dy = g(\xi, v) U_t(\xi, v)
\]
as can be easily verified. Thus for such symbols Theorem 3.5 is valid.

4. Toeplitz operators on Twisted Bergman spaces

In this section we take up the study of Toeplitz operators on twisted Bergman spaces which are Segal-Bargmann spaces associated to the special Hermite semigroup \( e^{-tL} \). These spaces arise naturally in the study of Segal-Bargmann transform on the Heisenberg group, see [15].

We show that \( e^{tL} T_g e^{-tL} \) is a Weyl multiplier if and only if the symbol \( g(x + iy, u + iv) \) depends only on \((y, v)\). By means of Gutzmer’s formula we study boundedness of \( T_g \) which correspond to multipliers for special Hermite operators.

4.1. Twisted Bergman spaces. By the term twisted Bergman spaces we mean the Hilbert space of entire functions \( F(z, w) \) on \( \mathbb{C}^{2n} \) which are square integrable with respect to the weight function
\[
W_t(z, w) = e^{(w - y - v - x) p_{2t}(2y, 2v)}
\]
where
\[
p_t(y, v) = c_n (\sinh(2t))^{-n} e^{-\frac{1}{4} \coth(2t)(|y|^2 + |v|^2)}
\]
is the heat kernel associated to the special Hermite operator \( L \), see [23]. Thus the special Hermite semigroup \( e^{-tL} \) is given by \( e^{-tL} f = f \times p_t \), the twisted convolution of \( f \) with \( p_t \). These spaces, denoted by \( \mathcal{B}_t^*(\mathbb{C}^{2n}) \), arise naturally in the study of Segal-Bargmann transform on the Heisenberg group [15]. The following result proved in [15] characterises \( \mathcal{B}_t^*(\mathbb{C}^{2n}) \).

**Theorem 4.1.** An entire function \( F \) on \( \mathbb{C}^{2n} \) belongs to \( \mathcal{B}_t^*(\mathbb{C}^{2n}) \) if and only if its restriction to \( \mathbb{R}^{2n} \) is of the form \( e^{-tL} f(x, u) \) for some \( f \in L^2(\mathbb{R}^{2n}) \). Moreover, the norm of \( F \) in \( \mathcal{B}_t^*(\mathbb{C}^{2n}) \) is the same as the norm of \( f \) in \( L^2(\mathbb{R}^{2n}) \).

Another proof of this was found in [22] which is based on the following Gutzmer’s formula for the special Hermite expansion. Recall that \( \varphi_k(x, u) = \varphi_k(x + iu) \) are the Laguerre functions of type \((n - 1)\) introduced earlier. They extend to entire functions on \( \mathbb{C}^{2n} \) and are denoted by \( \varphi_k(z, w) \). The twisted convolutions \( f \times \varphi_k \) are the projections...
onto the $k$-th eigenspace of $L$ and the special Hermite expansion of an $f \in L^2(\mathbb{R}^{2n})$ is written as

$$f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k$$

where the series converges in $L^2$. The heat kernel $p_t$ associated to the special Hermite operator can also be written as

$$p_t(x, u) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} \varphi_k(x, u).$$

**Theorem 4.2.** For any $F \in \mathcal{O}(\mathbb{C}^{2n})$ we have

$$\int_{\mathbb{R}^{2n}} \int_{K} e^{(u \cdot y - v \cdot x)} |F(\sigma(x + iy, u + iv))|^2 dx dv d\sigma$$

$$= \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \| f \times \varphi_k \|_2^2 \varphi_k(2iy, 2iv),$$

where $f$ is the restriction of $F$ to $\mathbb{R}^{2n}$.

Clearly when $F = e^{-Lt}f$ the above formula holds. So, Theorem 4.1 easily follows from Gutzmer’s formula once we have the identity

$$\int_{\mathbb{R}^{2n}} p_{2t}(2y, 2v) \varphi_k(2iy, 2iv) dy dv = \frac{(k+n-1)!}{k!(n-1)!} e^{(2k+n)t}.$$ (4.1.1)

This has been proved in [22], see Lemma 6.3. The following extension of this result is needed for the study of Toeplitz operators.

**Lemma 4.3.**

$$\int_{\mathbb{R}^{2n}} e^{i \frac{(u \cdot y - v \cdot x)}{2}} p_{t}(x-y, u-v) \varphi_k(iy, iv) dy dv = \varphi_k(ix, iv) e^{(2k+n)t}. $$ (4.1.2)

**Proof.** Recall that the symplectic Fourier transform $\tilde{f}$ of a function $f$ is defined by $\tilde{f}(x, y) = \hat{f}(\frac{1}{2}(-y, x))$. We know that $\varphi_k$’s are eigenfunctions of the symplectic Fourier transform with eigenvalues $(-1)^k$, i.e.

$$(2\pi)^{-n} \int_{\mathbb{R}^{2n}} \varphi_k(\xi, \eta) e^{i \frac{(\eta \cdot y - \xi \cdot x)}{2}} d\xi d\eta = (-1)^k \varphi_k(y, v).$$

The above equation remains true even if we replace $(y, v)$ by $(iy, iv)$. So we get

$$(2\pi)^{-n} \int_{\mathbb{R}^{2n}} \varphi_k(\xi, \eta) e^{i \frac{(y \cdot \xi - v \cdot \eta)}{2}} d\xi d\eta = (-1)^k \varphi_k(iy, iv).$$ (4.1.3)
Now putting (4.1.3) in (4.1.2) and by using Fubini’s theorem we get
\[ (-1)^k \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{i(y-u-x)}{2}} e^{-\frac{\sinh t}{4} (x-y)^2 + (u-v)^2} e^{-\frac{i(y-u-x)}{2}} \varphi_k(\xi, \eta) \, dy \, dv \, d\xi \, d\eta \]
\[ = (-1)^k \int_{\mathbb{R}^{2n}} e^{\frac{n-x-u}{2}} e^{-\frac{\sinh t}{4} (u-in)^2 + (x-i\xi)^2} \varphi_k(\xi, \eta) \, d\xi \, d\eta. \]

Now look at the function
\[ F(t) = \int_{\mathbb{R}^{2n}} e^{\frac{n-x-u}{2}} e^{-\frac{\sinh t}{4} (u-in)^2 + (x-i\xi)^2} \varphi_k(\xi, \eta) \, d\xi \, d\eta. \]

If we replace \( t \) by \( z \) with \( |\Im z| < \pi/2 \), it is easy to see that the integral converges absolutely. In fact, \( F \) can be extended as a holomorphic function to the strip \( |\Im z| < \pi/2 \) containing the real line. Consider
\[ F(-t) = \int_{\mathbb{R}^{2n}} e^{\frac{n-x-u}{2}} e^{-\frac{\sinh t}{4} (u-in)^2 + (x-i\xi)^2} \varphi_k(\xi, \eta) \, d\xi \, d\eta \]
which after using
\[ e^{-\frac{\sinh t}{4} (u-in)^2 + (x-i\xi)^2} = e^{-\frac{\sinh t}{4} (\eta+iu)^2 + (\xi+ix)^2} \]
reads as
\[ F(-t) = \int_{\mathbb{R}^{2n}} e^{\frac{n-x-u}{2}} e^{-\frac{\sinh t}{4} (\eta+iu)^2 + (\xi+ix)^2} \varphi_k(\xi, \eta) \, d\xi \, d\eta. \]

This is nothing but the twisted convolution of \( \varphi_k \) with \( \tilde{p}_t \) at \((iu, ix)\). It is easy to calculate \( F(-t) \) by recalling
\[ \tilde{p}_t(\xi, \eta) = \sum_{k=0}^{\infty} e^{-(2k+n)t} (-1)^k \varphi_k(\xi, \eta). \]

Using the above along with the fact that \( \varphi_k \times \varphi_j = c_n \delta_{j,k} \varphi_k \) we get
\[ F(-t) = (-1)^k e^{-(2k+n)t} \varphi_k(ix, iu). \]

The right hand side of the above equation is also a holomorphic function of \( t \) and both sides agree on the negative real axis. Therefore, they agree everywhere and changing \( t \) into \(-t\) we get the lemma. \( \square \)

Note that when \( x = u = 0 \) this lemma reduces to (4.1.1) as \( \varphi_k(0, 0) = \frac{(k+n-1)!}{k!(n-1)!} \).
4.2. Toeplitz operators and special Hermite multipliers. In this subsection we get some necessary and sufficient conditions for the boundedness of $T_g$ on $\mathcal{B}_t^*(\mathbb{C}^{2n})$ for a special class of symbols, by making use of Gutzmer’s Formula for the special Hermite expansion. First note that by Theorem 4.1 $e^{-tL}\Phi_{\alpha,\beta}(z,w) = e^{-(2|\beta|+n)t}\Phi_{\alpha,\beta}(z,w)$ form an orthonormal basis for $\mathcal{B}_t^*(\mathbb{C}^{2n})$. We denote $e^{-tL}\Phi_{\alpha,\beta}$ by $\phi_{\alpha,\beta}$ in this section. Consider a measurable function $g$ on $\mathbb{C}^{2n}$ for which

\begin{equation}
\int_{\mathbb{C}^{2n}} |g(z, w)\phi_{\alpha,\beta}(z, w)\phi_{\mu,\nu}(z, w)|W_t(z, w)dzd\bar{w} < \infty.
\end{equation}

Now we can define a densely defined bilinear form on $\mathcal{B}_t^*(\mathbb{C}^{2n})$ by

$$\langle T_g \phi_{\alpha,\beta}, \phi_{\mu,\nu}\rangle_{\mathcal{B}_t^*(\mathbb{C}^{2n})} := \int_{\mathbb{C}^{2n}} g(z, w)\phi_{\alpha,\beta}(z, w)\bar{\phi}_{\mu,\nu}(z, w)W_t(z, w)dzd\bar{w}.$$ 

These are nothing but the matrix entries of $T_g$. We consider special symbols $g$ for which the above densely defined bilinear form becomes a diagonal form. Such symbols are provided by functions of the form $g(x + iy, u + iv) = g_0(y, v)$ where $g_0$ is a radial function on $\mathbb{R}^{2n}$.

**Theorem 4.4.** Let $g$ be as above and satisfy (4.2.4). Then $T_g$ is bounded if and only if the sequence

$$e^{-(2k+n)2t} \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} g_0(y, v)p_{2t}(2y, 2v)\varphi_k(2iy, 2iv)dydv$$

is bounded.

**Proof.** Clearly $\langle T_g \phi_{\alpha,\beta}, \phi_{\mu,\nu}\rangle_{\mathcal{B}_t^*(\mathbb{C}^{2n})}$ is well defined for all $(\alpha, \beta)$ and $(\mu, \nu)$. As done in Section 3.3 we can polarize Gutzmer’s formula to obtain

$$\int_{\mathbb{R}^{2n}} \int_K e^{(u, y-v, x)} e^{-tL}f_1(\sigma(x + iy, u + iv))e^{-tL}f_2(\sigma(x + iy, u + iv))dxdud\sigma$$

$$= \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} e^{-(2k+n+2t)} \langle f_1 \times \varphi_k, f_2 \times \varphi_k \rangle_{L^2(\mathbb{R}^{2n})} \varphi_k(2iy, 2iv).$$

When $f_1 = \Phi_{\alpha,\beta}$ and $f_2 = \Phi_{\mu,\nu}$ the above identity reduces to

\begin{equation}
\int_{\mathbb{R}^{2n}} \int_K e^{(u, y-v, x)} \phi_{\alpha,\beta}(\sigma(z, w))\bar{\phi}_{\mu,\nu}(\sigma(z, w))dxdud\sigma
\end{equation}

$$= \sum_{j=0}^{\infty} \frac{j!(n-1)!}{(j+n-1)!} e^{-(2j+n+2t)} \langle \phi_{\alpha,\beta} \times \varphi_j, \phi_{\mu,\nu} \times \varphi_j \rangle_{L^2(\mathbb{R}^{2n})} \varphi_j(2iy, 2iv)$$

$$= \delta_{\alpha,\mu}\delta_{\beta,\nu} \frac{k!(n-1)!}{(k+n-1)!} e^{-(2k+n+2t)} \varphi_k(2iy, 2iv),$$
where $|\beta| = k$. Writing the matrix coefficients explicitly
\[
\langle T_g \phi_{\alpha,\beta}, \phi_{\mu,\nu} \rangle_{B^*_1(C^{2n})} = \int_{C^{2n}} g_0(y,v) \phi_{\alpha,\beta}(z,w) \bar{\phi}_{\mu,\nu}(z,w) W_t(z,w) \, dz \, dw.
\]
The above integral converges absolutely. Now, replace $(z,w)$ by $\sigma(z,w)$ where $\sigma \in K$. Since $g_0(y,v) W_t(z,w) \, dz \, dw$ is invariant under the action of $K$ we get
\[
\langle T_g \phi_{\alpha,\beta}, \phi_{\mu,\nu} \rangle_{B^*_1(C^{2n})} = \int_{K} \int_{C^{2n}} g_0(y,v) \phi_{\alpha,\beta}(\sigma(z,w)) \bar{\phi}_{\mu,\nu}(\sigma(z,w)) W_t(z,w) \, dz \, dw \, d\sigma.
\]
The integral converges absolutely and hence by applying Fubini’s theorem and using (4.2.5) we get
\[
\langle T_g \Phi_{\alpha,\beta}, \Phi_{\mu,\nu} \rangle_{B^*_1(C^{2n})} = \delta_{\alpha,\mu} \delta_{\beta,\nu} \frac{k!(n-1)!}{(k+n-1)!} e^{-2(k+n)^2 t} \int_{\mathbb{R}^{2n}} g_0(y,v) p_{2t}(2y,2v) \varphi_k(2iy,2iv) \, dy \, dv,
\]
where $|\beta| = k$. Thus the operator $T_g$ is diagonal in the basis
\[
\{ \phi_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}^n \}
\]
and the theorem follows. \[\Box\]

Let $h$ be a radial measurable function on $\mathbb{R}^{2n}$ and assume that
\[
\int_{\mathbb{R}^{2n}} |h(y,v)| e^{s(|u|^2+|v|^2)} \, dy \, dv < \infty
\]
for all $s > 0$. Consider the symbol defined by
\[
g(x+iy,u+iv) = h \times p_{2t}(2y,2v).
\]

**Corollary 4.5.** In the above theorem let $g$ be as above with $h$ satisfying (4.2.6). Then $T_g$ is bounded if and only if the sequence
\[
\frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{R}^{2n}} h(y,v) \varphi_k(iy,iv) \, dy \, dv
\]
is bounded.

**Proof.** As $h$ and $p_t$ are both radial, so is $g_0(y,v) = g(iy,iv)$. Hence by Theorem 4.4 we know that $T_g$ is bounded if and only if
\[
\frac{k!(n-1)!}{(k+n-1)!} e^{-2(k+n)^2 t} \int_{\mathbb{R}^{2n}} g_0(y,v) p_{2t}(2y,2v) \varphi_k(2iy,2iv) \, dy \, dv.
\]
As $g(iy,iv) p_{2t}(2y,2v) = h \times p_{2t}(2y,2v)$ the above simplifies to
\[
\frac{k!(n-1)!}{(k+n-1)!} e^{-2(k+n)^2 t} \int_{\mathbb{R}^{2n}} h \times p_{2t}(2y,2v) \varphi_k(2iy,2iv) \, dy \, dv.
\]
Because of (4.2.6) we can use Fubini’s theorem to change the order of integration. By Lemma 4.3 we have

\[ \int_{\mathbb{R}^{2n}} p_{2t}(x - 2y, u - 2v) e^{i(u - y - v, x)} \varphi_k(2iy, 2iv) dy dv = e^{(2k+n)2t} \varphi_k(ix, iu). \]

Using this in (4.2.7) we obtain the corollary. \qed

From now on let us assume that \( g \) is a measurable function on \( \mathbb{C}^{2n} \) such that \( \int_{\mathbb{C}^{2n}} |g(z, w)\phi_{\alpha, \beta}(z, w)|^2 W_t(z, w) dz dw < \infty \) for all \( \alpha, \beta \). We will refer to this condition as (**). Note that the condition (**) on \( g \) implies that it belongs to

\[ L^2(\mathbb{C}^{2n}, e^{(u, y - v, x)} e^{-\frac{(|x|^2 + |u|^2)}{2}} e^{-\cotht 2t + \frac{1}{2}|y|^2 + |v|^2}) dz dw. \]

For such symbols the Toeplitz operator on \( B_t^*(\mathbb{C}^{2n}) \) is defined by \( T_g(f) := P(gf) \) where \( P \) is the orthogonal projection from \( L^2(\mathbb{C}^{2n}, W_t) \) onto \( B_t^*(\mathbb{C}^{2n}) \). We study the class of symbols \( g \) for which \( T_g \) is bounded and \( e^{tL} T_g e^{-tL} \) is a right Weyl multiplier, i.e. \( W(e^{tL} T_g e^{-tL}) = W(f) M_t \) for some \( M_t \in B(L^2(\mathbb{R}^{2n})) \).

Before proving the next theorem we prove a lemma which will be used. Let

\[ V_t(z, w) = e^{(u, y - v, x)} e^{-\frac{(|x|^2 + |u|^2)}{2}} e^{-\cotht 2t + \frac{1}{2}|y|^2 + |v|^2} \]

and consider the measure \( d\tau(z, w) = V_t(z, w) dz dw \), where \( dz dw \) is the Lebesgue measure on \( \mathbb{R}^{4n} \). Let \( P(\mathbb{R}^{4n}) \) be the set of all polynomials on \( \mathbb{R}^{4n} \). Note that \( P(\mathbb{R}^{4n}) \subset L^2(\mathbb{C}^{2n}, d\tau(z, w)). \)

**Lemma 4.6.** \( P(\mathbb{R}^{4n}) \) is dense in \( L^2(\mathbb{C}^{2n}, d\tau(z, w)). \)

**Proof.** By abuse of notation let us denote any polynomial \( p \in P(\mathbb{R}^{4n}) \) by \( p(z, w) \). As the weight function \( V_t(z, w) \) corresponding to \( d\tau \) is nowhere vanishing, it is enough to show that the linear span of \( p(z, w) (V_t(z, w))^{1/2} \) is dense in \( L^2(\mathbb{C}^{2n}) \). More precisely, if there exists \( g \in L^2(\mathbb{C}^{2n}) \) such that

\[ (4.2.8) \quad \int_{\mathbb{C}^{2n}} g(z, w) p(z, w) (V_t(z, w))^{1/2} dz dw = 0 \]

for all \( p \in P(\mathbb{R}^{4n}) \) then we need to show \( g = 0 \). Now suppose that there exists \( g \) satisfying (4.2.8). It is easy to see that by completing the square in \( V_t(z, w) \) (4.2.8) can be rewritten as

\[ (4.2.9) \quad \int_{\mathbb{C}^{2n}} g(z - v, w + y) p(z - v, w + y) e^{-\frac{1}{4}(|x|^2 + |u|^2)} e^{-\cotht 2t - \frac{1}{2}(|y|^2 + |v|^2)} dz dw = 0 \]
for all \( p \in \mathcal{P}(\mathbb{R}^{4n}) \). If we let \( \tilde{g}(z, w) = g(z + v, w - y) \) then it is clear that \( \tilde{g} \in L^2(\mathbb{C}^{2n}) \) whenever \( g \in L^2(\mathbb{C}^{2n}) \) and \( \| \tilde{g} \|_{L^2(\mathbb{C}^{2n})} = \| g \|_{L^2(\mathbb{C}^{2n})} \).

So, it is enough to show that \( \tilde{g} = 0 \). The equation (4.2.9) means that

\[
\int_{\mathbb{C}^{2n}} \tilde{g}(z, w)q(z, w)e^{-\frac{1}{4}(|x|^2 + |u|^2)e^{-\frac{\cosh 2t - 1}{2}(|y|^2 + |v|^2)}} dzdw = 0
\]

for all \( q \in \mathcal{P}(\mathbb{R}^{4n}) \). As the linear span of functions of the form

\[
q(z, w)e^{-\frac{1}{4}(|x|^2 + |u|^2)e^{-\frac{\cosh 2t - 1}{2}(|y|^2 + |v|^2)}}
\]

is dense in \( L^2(\mathbb{C}^{2n}) \) the last equation implies \( \tilde{g} = 0 \) proving the lemma.

\[ \square \]

**Theorem 4.7.** Let a Lebesgue measurable function \( g \) on \( \mathbb{C}^{2n} \) satisfy (**) and let \( T_g \) be the corresponding Toeplitz operator on \( \mathcal{B}_t^*(\mathbb{C}^{2n}) \). Then \( T_g = 0 \) if and only if \( g = 0 \) a.e.

**Proof.** When \( g = 0 \) a.e. clearly \( T_g = 0 \). Conversely, let \( T_g = 0 \). We need to prove that \( g = 0 \) a.e. By using the explicit form of the functions \( \phi_{\alpha, \beta} \) namely, \( \phi_{\alpha, \beta}(z, w) = P_{\alpha, \beta}(z, w)e^{-\frac{|x|}{2} + \frac{|u|}{2}} \), where \( P_{\alpha, \beta} \) are holomorphic polynomials on \( \mathbb{C}^{2n} \) of degree \( |\alpha| + |\beta| \) the condition (**) takes the form

\[
\int_{\mathbb{C}^{2n}} |g(z, w)P_{\alpha, \beta}(z, w)|^2 V_t(z, w) dzdw < \infty
\]

for all \( \alpha, \beta \). The above also implies that \( g \in L^2(\mathbb{C}^{2n}, V_t(z, w)dzdw) \) in particular. In view of the previous lemma, proving \( g = 0 \) a.e. is equivalent to proving that

\[
(4.2.10) \quad \int_{\mathbb{C}^{2n}} g(z, w)p(z, w)V_t(z, w)dzdw = 0
\]

for all \( p \in \mathcal{P}(\mathbb{R}^{4n}) \). The assumption \( T_g = 0 \) gives us for all \( \alpha, \beta, \mu, \nu \)

\[
\langle T_g \phi_{\alpha, \beta}, \phi_{\mu, \nu} \rangle_{\mathcal{B}_t^*(\mathbb{C}^{2n})} = \int_{\mathbb{C}^{2n}} g(y, v)\phi_{\alpha, \beta}(z, w)\overline{\phi_{\mu, \nu}(z, w)}W_t(z, w)dzdw = 0.
\]

Again by using the explicit form of \( \phi_{\alpha, \beta} \) we get

\[
\int_{\mathbb{C}^{2n}} g(y, v)P_{\alpha, \beta}(z, w)\overline{P_{\mu, \nu}(z, w)}V_t(z, w)dzdw = 0.
\]

We claim that for every \( \alpha, \beta \), the monomial \( z^\alpha w^\beta \) belongs to the linear span of \( \{ P_{\mu, \nu}(z, w) : |\mu| + |\nu| = |\alpha| + |\beta| \} \). This claim would then prove (4.2.10) which in turn would prove \( g = 0 \) a.e. In fact, once we have the claim (4.2.10) will be true for all polynomials of the form \( p(z, w) = z^\alpha w^\beta z^\mu w^\nu \) which in turn will prove (4.2.10) for all monomials \( x^\alpha y^\beta u^\mu v^\nu \) and hence for all polynomials. Returning to the claim it is
sufficient to prove it for \( z, w \) purely real. We know that the special Hermite functions \( \Phi_{\mu,\nu}(x, u) \) give all the eigenfunctions of the dilated Hermite operator \( H(1/2) = -\Delta + \frac{1}{4}(|x|^2 + |u|^2) \) on \( \mathbb{R}^{2n} \). (see [25])

More precisely,

\[
H(1/2)\Phi_{\mu,\nu} = (|\mu| + |\nu| + n)\Phi_{\mu,\nu}.
\]

If \( H_{\alpha,\beta}(x, u) \) stand for the (ordinary) Hermite polynomials on \( \mathbb{R}^{2n} \) adapted to \( H(1/2) \) then it can be written as a linear combination of \( P_{\mu,\nu}(x, u) \) with \( |\mu| + |\nu| = |\alpha| + |\beta| \). It is well known that \( x^\alpha u^\beta \) can be written as a linear combination of \( H_{\mu,\nu} \) and hence as a linear combination of \( P_{\mu,\nu} \) as well. Thus \( g \) is orthogonal to all polynomials in \( L^2(\mathbb{C}^{2n}, V_t) \) and this proves the result.

We now characterise all the symbols \( g \) for which \( e^{itL}T_g e^{-itL} \) reduces to a Weyl multiplier. For this characterisation we need to consider symbols \( g \) so that \( g_{a,b}(z, w) := g(z + a, w + b) \) satisfies condition (**) for all \( (a, b) \in \mathbb{R}^{2n} \).

**Theorem 4.8.** Let \( g_{a,b} \) satisfy (**) for all \( (a, b) \in \mathbb{R}^{2n} \) and let the corresponding \( T_g \) be a bounded operator on \( B_t^*(\mathbb{C}^{2n}) \). Then \( e^{itL}T_g e^{-itL} \) is a right Weyl multiplier if and only if \( g(z, w) = g(iy, iv) \).

**Proof.** Let us first assume that \( T_g \) is bounded and corresponds to a right Weyl multiplier \( M_t \). As proved in [4] we know that \( M_t = W(\sigma) \), for some \( \sigma \in S'(\mathbb{R}^{2n}) \). Therefore, we have

\[
e^{itL}T_g e^{-itL}f = f \times \sigma,
\]

for all \( f \in L^2(\mathbb{R}^{2n}) \). Recall that the twisted translations of functions on \( \mathbb{R}^{2n} \) are defined by

\[
\tau(a, b)f(x, u) := e^{-i/2(a.u - b.x)}f(x - a, u - b), (a, b) \in \mathbb{R}^{2n}.
\]

Clearly, \( \tau(a, b) \) is a unitary map on \( L^2(\mathbb{R}^{2n}) \). It is easy to check that \( \tau(a, b)f \times g = \tau(a, b)(f \times g) \) when \( f, g \in L^2(\mathbb{R}^{2n}) \). This implies that \( e^{itL}T_g e^{-itL} \) commutes with twisted translations (see \([4.2.11]\)). As \( e^{-itL}f = f \times p_t, e^{-itL}f \) is equivariant under twisted translations. By using the fact that \( e^{-itL} \) is a unitary map from \( L^2(\mathbb{R}^{2n}) \) onto \( B_t^*(\mathbb{C}^{2n}) \) and its equivariance under twisted translations, we get that

\[
\langle \tau(a, b)F, \tau(a, b)G \rangle_{B_t^*(\mathbb{C}^{2n})} = \langle F, G \rangle_{B_t^*(\mathbb{C}^{2n})}
\]

for all \( (a, b) \in \mathbb{R}^{2n} \) and \( F, G \in B_t^*(\mathbb{C}^{2n}) \). (Here, \( \tau(a, b) \) on \( B_t^*(\mathbb{C}^{2n}) \) is the natural extension to holomorphic functions.)

We will now show that \( g(z, w) = g(iy, iv) \) a.e. In view of Theorem 4.7 and the hypothesis on \( g_{a,b} \) it is enough to show that

\[
\langle T_g \Phi_{\alpha,\beta}, \Phi_{\mu,\nu} \rangle_{B_t^*(\mathbb{C}^{2n})} = \langle T_{g_{a,b}} \Phi_{\alpha,\beta}, \Phi_{\mu,\nu} \rangle_{B_t^*(\mathbb{C}^{2n})}
\]
for all \((a, b)\) and \(\alpha, \beta, \mu, \nu\). But this can be easily shown to be true. Indeed, by making the change of variable \((z, w) \rightarrow (z + a, w + b)\) in the equation

\[
\langle T_g \tau(a, b) \phi_{\alpha, \beta}, \tau(a, b) \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})} = \\
\int_{C^{2n}} g(z, w) \tau(a, b) \phi_{\alpha, \beta}(z, w) \tau(a, b) \overline{\phi_{\mu, \nu}(z, w)} W_t(z, w) dz \, dw
\]

it is easy to see that

\[
\langle T_g \tau(a, b) \phi_{\alpha, \beta}, \tau(a, b) \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})} = \langle T_g a, b \phi_{\alpha, \beta}, \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})}.
\]

Therefore, it is enough to show that

\begin{equation}
(4.2.12) \quad \langle T_g a, b \phi_{\alpha, \beta}, \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})} = \langle T_g \tau(a, b) \phi_{\alpha, \beta}, \tau(a, b) \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})}
\end{equation}

for all \((a, b) \in \mathbb{R}^{2n}\) and multi-indices \((\alpha, \beta)\) and \((\mu, \nu)\). In other words, we need to show that \(T_g\) commutes with twisted translations, which is immediate as \(e^{tL}T_g e^{-tL}\) commutes with twisted translations and \(e^{-tL}\) is equivariant under them. This proves the first part of the theorem.

Conversely, assume that \(g(z, w) = g(iy, iv)\) and \(T_g\) is bounded. Clearly,

\[
\langle T_g a, b \phi_{\alpha, \beta}, \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})} = \langle T_g \tau(a, b) \phi_{\alpha, \beta}, \tau(a, b) \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})}
\]

for all \((a, b) \in \mathbb{R}^{2n}\). As shown earlier, this implies that

\[
\langle T_g \phi_{\alpha, \beta}, \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})} = \langle T_g \tau(a, b) \phi_{\alpha, \beta}, \tau(a, b) \phi_{\mu, \nu} \rangle_{\mathcal{B}^*_t(C^{2n})}.
\]

So, \(T_g\) commutes with twisted translations. Again, as shown before \(e^{tL}T_g e^{-tL}\) commutes with twisted translations as well. This means that there exists \(\sigma \in S'(\mathbb{R}^{2n})\) such that

\[
e^{tL}T_g e^{-tL} \sigma = \sigma \times f
\]

for all \(f \in L^2(\mathbb{R}^{2n})\). When we take the Weyl transform on both sides we get \(W(e^{tL}T_g e^{-tL} f) = W(f) W(\sigma)\). As \(T_g\) is bounded, this proves that \(e^{tL}T_g e^{-tL}\) is a right Weyl multiplier. \(\square\)

5. Toeplitz operators associated to symmetric spaces

Segal-Bargmann spaces associated to Riemannian symmetric spaces have been studied by Hall \cite{10}, Stenzel \cite{19} and Kr"otz et al \cite{16}. The situation of non-compact symmetric spaces is much more complicated whereas the compact case is well understood as a weighted Bergman spaces. In both cases we have Gutzmer’s formula using which we can study Toeplitz operators that correspond to Fourier multipliers on the underlying group. In this section we study such operators in the case of compact symmetric spaces, extending some results of Hall \cite{12}. the case of noncompact Riemannian symmetric spaces will be taken up elsewhere.
5.1. Lassalle-Gutzmer formula. Consider a compact symmetric space $X = U/K$ where $(U, K)$ is a compact symmetric pair. We may assume that $K$ is connected and $U$ is semisimple. We let $u = \mathfrak{u} + \mathfrak{p}$ stand for the Cartan decomposition of $u$ and let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$. Functions $f$ on $X$ can be viewed as right $K$-invariant functions on $U$. If $\pi \in \hat{U}$ then it can be shown that $\hat{f}(\pi) = 0$ unless $\pi$ is $K$-spherical, i.e., the representation space $V$ of $\pi$ has a unique $K$-fixed vector $u$. It then follows that $\hat{f}(\pi)v = (v, u)\hat{f}(\pi)u$ for any $v \in V$ which means that $\hat{f}(\pi)$ is of rank one. Let $\hat{U}_K$ stand for the equivalence classes of $K$-spherical representations of $U$. Then there is a one to one correspondence between elements of $\hat{U}_K$ and a certain discrete subset $\mathcal{P}$ of $a^*$ called the set of restricted dominant weights.

For each $\lambda \in \mathcal{P}$ let $(\pi_\lambda, V_\lambda)$ be a spherical representation of $U$ of dimension $d_\lambda$. Let $\{v_{\lambda}^j, 1 \leq j \leq d_\lambda\}$ be an orthonormal basis for $V_\lambda$ with $v_1$ being the unique $K$-fixed vector. Then the functions

$$\varphi_\lambda^j(g) = \langle \pi_\lambda(g)v_1, v_{\lambda}^j \rangle$$

form an orthogonal family of right $K$-invariant analytic functions on $U$ and we can consider them as functions of the symmetric space. When $x = g.o \in X$, we simply denote by $\varphi_\lambda^j(x)$ the function $\varphi_\lambda^j(g.o)$. The function $\varphi_\lambda^j(x)$ is $K$ biinvariant, called an elementary spherical function. It is usually denoted by $\varphi_\lambda$. The Fourier coefficients of $f \in L^2(X)$, are defined by

$$\hat{f}_j(\lambda) = \int_X f(x)\overline{\varphi_\lambda^j(x)}dm_0(x)$$

and the Fourier series is written as

$$f(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda \sum_{j=1}^{d_\lambda} \left( \hat{f}_j(\lambda)\varphi_\lambda^j(x) \right).$$

Then the Plancherel theorem reads as

$$\int_X |f(x)|^2dm_0(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda \left( \sum_{j=1}^{d_\lambda} |\hat{f}_j(\lambda)|^2 \right).$$

Let $U_C$ (resp. $K_C$) be the universal complexification of $U$ (resp. $K$). The group $K_C$ sits inside $U_C$ as a closed subgroup. We may then consider the complex homogeneous space $X_C = U_C/K_C$, which is a complex variety and gives the complexification of the symmetric space $X = U/K$. The Lie algebra $u_C$ of $U_C$ is the complexified Lie algebra $u_C = u + iu$. For every $g \in U_C$ there exists $u \in U$ and $X \in u$ such that $g = u \exp iX$. Let $\Omega$ be any $U$ invariant domain in $X_C$ and let $O(\Omega)$ stand for the space of holomorphic functions on $\Omega$. The group $U$ acts on
\( \mathcal{O}(\Omega) \) by \( T(g)f(z) = f(g^{-1}z) \). For each \( \lambda \in \mathcal{P} \) the matrix coefficients \( \varphi_\lambda^j \) extend to \( X_C \) as holomorphic functions. When \( f \in \mathcal{O}(\Omega) \), it can be shown that the series

\[
f(z) = \sum_{\lambda \in \mathcal{P}} d_\lambda \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \varphi_\lambda^j(z)
\]

converges uniformly over compact subsets of \( \Omega \). The above series is called the Laurent expansion of \( f \) and we have the following formula known as Gutzmer’s formula for \( X \).

**Theorem 5.1.** For every \( f \in \mathcal{O}(X_C) \) and \( H \in i\mathfrak{a} \), we have

\[
\int_U |f(g, \exp(H).o)|^2 dg = \sum_{\lambda \in \mathcal{P}} d_\lambda \left( \sum_{j=1}^{d_\lambda} |\hat{f}_j(\lambda)|^2 \right) \varphi_\lambda(\exp(2H).o).
\]

This theorem is due to Lassalle, see [17] and [18] for a proof. This formula has been used by Faraut [5] to give an elegant proof of a theorem of Stenzel [19] on the Segal-Bargmann transform for the compact symmetric space \( X \). The second author has used the same to study holomorphic Sobolev spaces in [27].

**5.2. Segal-Bargmann transform on \( X \).** Let \( \Delta \) stand for the Laplace-Beltrami operator on \( X \) suitably shifted so that its spectrum consists of \(|\lambda + \rho|^2\) where \( \lambda \in \mathcal{P} \) and \( \rho \) is the half sum of positive roots (see Faraut [5]). The solution of the heat equation associated to \( \Delta \) with initial condition \( f \in L^2(X) \) is given by the expansion

\[
u(x, t) = \sum_{\lambda \in \mathcal{P}} d_\lambda e^{-t|\lambda + \rho|^2} \left( \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \varphi_\lambda^j(x) \right).
\]

By defining the heat kernel \( \gamma_t \) by

\[
\gamma_t(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda e^{-t|\lambda + \rho|^2} \left( \sum_{j=1}^{d_\lambda} \varphi_\lambda^j(x) \right)
\]

the solution can be written as \( u(g, t) = f * \gamma_t(g) \) where the convolution is taken on \( U \). For \( f \in L^2(X) \) it can be shown that the solution \( u \) extends to \( X_C \) as a holomorphic function. The map taking \( f \) into \( u(z, t) = f * \gamma_t(g), z = g.o \) is called the Segal-Bargmann transform and has been studied by Hall [10], Stenzel [19] and others.

The image of \( L^2(X) \) under the Segal-Bargmann transform has been characterised by Stenzel [19] as a weighted Bergman space. The weight function \( w_t \) is given in terms of the heat kernel on the noncompact dual \( Y \) of \( X \). Consider the group \( G = K \exp(i\mathfrak{p}) \) whose Lie algebra
is $t + ip$. Under the assumption that $U$ is semisimple, $G$ turns out to be a real semisimple group and $K$ a maximal compact subgroup. The noncompact dual is then defined as $Y = G/K$. Let $\Delta_G$ be the Laplace-Beltrami operator on $Y$ with heat kernel defined by

$$\gamma^1_t(g) = \int_{(ia)^*} e^{-t(|\lambda|^2 + |\rho|^2)}\psi_\lambda(g)|c(\lambda)|^{-2}d\lambda.$$ 

Here $\psi_\lambda$ are the spherical functions on $Y$. Define a weight function $w_t(z)$ on $X_C$ by

$$w_t(z) = \gamma^1_t(\exp(2H)), \quad z = u \exp(H), \quad u \in U, \quad H \in ia.$$ 

Then we have the following result.

**Theorem 5.2.** The Segal-Bargmann transform is an isometric isomorphism between $L^2(X)$ and the space of all holomorphic functions on $X_C$ that are square integrable with respect to $w_t(z)dm$ where $dm$ is the invariant measure on $X_C$.

This theorem is due to Stenzel [19]; for an elegant proof using Gutzmer’s formula see Faraut [5]. The key ingredient in Faraut’s proof is Lassalle’s formula and the following relation between $\varphi_\lambda$ and $\psi_\lambda$, namely

$$\varphi_\lambda(\exp(H)) = \psi_{-i(\lambda + \rho)}(\exp(H)), \quad H \in ia.$$ 

To conclude this subsection let us recall the following integration formulas on $X_C$ and $Y$:

$$\int_{X_C} f(x)dm(x) = c \int_U \int_{ia} f(u \exp(H),o)J(H)dudH,$$

$$\int_{Y} f(x)dm_1(x) = c \int_K \int_{ia} f(u \exp(H),o)J_1(H)dudH.$$ 

Here the Jacobians $J$ and $J_1$ are defined in terms of the roots, see Faraut [5]. We need the following fact that $J(H) = J_1(2H)$.

### 5.3. Toeplitz operators and Fourier multipliers.

Given a symbol $g(z)$ defined on $X_C$ we consider the Toeplitz operator $T_g$ on the Segal-Bargmann space $HL^2(X_C, w_t)$. In this subsection we are interested in finding symbols $g$ so that $e^{i\Delta T_g}e^{-t\Delta}$ is a Fourier multiplier on $L^2(X, dm_0)$. Given a bounded function $a(\lambda)$ the Fourier multiplier $a(D)$ is defined by

$$a(D)f(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda a(\lambda) \left( \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda)\varphi^\lambda_j(x) \right)$$ 

for all $f \in L^2(X, dm_0)$. It is clear that $a(D)$ is bounded if and only if $a$ is bounded. Using Gutzmer’s formula we can easily prove the following result.
THEOREM 5.3. Suppose $h$ is a $K$-biinvariant distribution on $G$ so that $h \ast \gamma_2^1$ is well defined. Let $g(z)w_i(z) = h \ast \gamma_2^1(\exp(2H))$ whenever $z = u \exp(H), u \in U, H \in i\mathfrak{a}$. Then $e^{t\Delta}T_g e^{-t\Delta} = a(D)$ where

$$a(\lambda) = \int_{i\mathfrak{a}} h(\exp(H))\psi_{-i(\lambda+\rho)}(\exp(H))J_1(H)dH.$$  

PROOF. When $F, F' \in H\mathcal{L}^2(X_C, w_i)$ we can use the polarised form of Gutzmer's formula to get

$$\int_{U} F(u \exp(H), \gamma_1) F'(u \exp(H), \gamma_1) du$$

where $F = f \gamma_1$ and $F' = f' \gamma_1$. Integrating the above with respect to $h \ast \gamma_2^1(\exp(2H))J(H)dH$ and recalling the definition of $g(z)$ we obtain

$$\int_{X_C} F(z)F'(z)g(z)w_i(z)dm(z) = \sum_{\lambda \in \mathcal{P}} d_{\lambda}e^{-2(\lambda+\rho)^2t} \left( \sum_{j=1}^{d_{\lambda}} \tilde{f}_j(\lambda) \overline{\tilde{f}'_j(\lambda)} \right) \varphi_{\lambda}(\exp(2H))$$

As $J(H) = J_1(2H)$ and $\varphi_{\lambda}(\exp(H)) = \psi_{-i(\lambda+\rho)}(\exp(H))$ the integral on the right hand side reduces to

$$\int_{i\mathfrak{a}} h \ast \gamma_2^1(\exp(H))\psi_{-i(\lambda+\rho)}(\exp(H))J_1(H)dH = e^{2t(\lambda+\rho)^2} \tilde{h}(-i(\lambda + \rho)).$$

Thus we have

$$\int_{X_C} T_g F(z)F'(z)w_i(z)dm(z) = \int_{X} a(D)f(x)f'(x)dm_0(x)$$

which proves the theorem. \qed

REMARK 5.1. When we take $h$ to be the distribution $p(\Delta)\delta_e$ where $p$ is a polynomial it follows that $a(\lambda) = p(-|\lambda+\rho|^2)$ so that $a(D) = p(i\Delta)$. Hence the differential operator $p(i\Delta)$ corresponds to the Toeplitz operator with $T_g$ with symbol $g(z) = \gamma_2(\exp(2H))^{-1}p(\Delta)\gamma_2(\exp(2H)), z = u \exp(H).o$ In the context of compact Lie groups $U$, Hall [12] has considered more general differential operators on $U$ and studied the symbols of Toeplitz operators corresponding to them using a different method.
5.4. Some remarks on compact Lie groups. Let us rewrite our theorem in the previous section as follows. Given a $K$--biinvariant function $g_0$ on $Y = G/K$ define $g(z) = g_0(\exp(2H))$, $z = u \exp(H) \cdot o, H \in i \mathfrak{a}$. Then we have

**Corollary 5.4.** Let $g$ be as above. Then the Toeplitz operator $T_g$ is bounded on $H L^2(X, \omega_tdm)$ if and only if

$$
\left| \int_{i \mathfrak{a}} g_0(\exp(H)) \gamma_{2t}^1(\exp(H)) \psi_{-i(\lambda + \rho)}(\exp(H)) J_1(H) dH \right| \leq C e^{2t|\lambda + \rho|^2}
$$

for all $\lambda \in \mathcal{P}$.

Let $U$ be a compact semisimple Lie group which can be treated as a compact symmetric space. In this case the group $G$ turns out to be a complex Lie group and hence the heat kernel $\gamma_{1t}$ is explicitly known, see Gangolli [8]. We also have explicit expressions for the spherical functions $\varphi_\lambda$ (Weyl character formula) and $\psi_\lambda$. More precisely,

$$
\psi_\lambda(\exp(H)) = \frac{\sum_{s \in W} c(s\lambda)e^{is\lambda(H)}}{\prod_{\alpha \in Q}(e^{\alpha(H)} - e^{-\alpha(H)})}
$$

where $W$ is the Weyl group, $c$ is the Harish-Chandra $c$--function and $Q$ is the set of positive roots. The heat kernel is given by

$$
\gamma_{1t}^1(\exp(H)) = C_t e^{-t|\rho|^2} \prod_{\alpha \in Q}(e^{\alpha(H)} - e^{-\alpha(H)}) e^{-\frac{t}{4}\|H\|^2}.
$$

These two results are proved in Gangolli [8]; see also Helgason [13].

Defining $\pi(\lambda) = \prod_{\alpha \in Q} \alpha(H_\lambda)$ where $H_\lambda \in i \mathfrak{a}$ corresponds to $\lambda$ we have the simple formula $c(\lambda) = \pi(\rho)/\pi(i\lambda)$.

The Jacobian factor $J_1(H)$ appearing in the integration formula for $Y = G/K$ is also expressible in terms of the roots $\alpha \in Q$. Thus it can be checked that

$$
\int_{i \mathfrak{a}} g_0(\exp(H)) \gamma_{2t}^1(\exp(H)) \psi_{-i(\lambda + \rho)}(\exp(H)) J_1(H) dH
$$

$$
= C_t e^{-t|\rho|^2} \sum_{s \in W} c(-is(\lambda + \rho)) \int_{i \mathfrak{a}} g_0(\exp(H)) e^{s(\lambda + \rho)(H)} \pi(H) e^{-\frac{t}{4}\|H\|^2} dH.
$$

Note that when $g_0 = 1$ the integral

$$
\int_{i \mathfrak{a}} \gamma_{2t}^1(\exp(H)) \psi_\lambda(\exp(H)) J_1(H) dH
$$

reduces to

$$
e^{-2t|\rho|^2} \sum_{s \in W} c(s\lambda) \int_{i \mathfrak{a}} \pi(H) e^{is\lambda(H)} e^{-\frac{t}{4}\|H\|^2} dH.
$$
\[ e^{-2t|\rho|^2} \left( \sum_{s \in W} c(s\lambda) \pi(is\lambda) \right) e^{-2t|\lambda|^2} = C_t e^{-2t(|\lambda|^2 + |\rho|^2)} \]

which is the defining relation for the heat kernel.

**Theorem 5.5.** Let \( T_g \) be a Toeplitz operator on the Segal-Bargmann space associated to a compact Lie group \( U \) where \( g(u \exp(H), o) = g_0(\exp(H)), u \in U, H \in \mathfrak{a}. \) Then \( T_g \) is bounded if and only if

\[
| \int_{\mathfrak{a}} g_0(\exp(H)) e^{(\lambda + \rho)(H)} \pi(H) e^{-\frac{1}{4t} |H|^2} dH | \leq C_t |\pi(i(\lambda + \rho))| e^{2t|\lambda + \rho|^2}. 
\]

Defining \( g_1(H) = g_0(\exp(H)) \pi(H) \) the above condition can be put in the form

\[
| \int_{\mathfrak{a}} g_1(H) e^{-\frac{1}{4t} |H|^2 - 4t(\lambda + \rho)^2} dH | \leq C_t |\pi(i(\lambda + \rho))| 
\]

for all \( \lambda \in \mathcal{P} \). This has an obvious resemblance with the sufficient condition we obtained for the Fock spaces.

**Acknowledgments**

The second author is supported in part by J. C. Bose Fellowship from the Department of Science and Technology (DST).

**References**

[1] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[2] C. A. Berger and L.A. Coburn, *Heat Flow and Berezin-Toeplitz estimates*, Amer. J. Math., **3** (1994), 563-590.

[3] D.-W. Byun, *Inversions of Hermite semigroup*, Proc. Amer. Math. Soc., **118** (1993), 437-445.

[4] I. Daubechies, *On the distributions corresponding to bounded operators in the Weyl quantisation*, Comm. Math. Phys., **75** (1980), 229-238.

[5] J. Faraut, *Espaces Hilbertiens invariant de fonctions holomorphes*, Seminaires et Congres **7** (2003), Societe Math. France, 101-167.

[6] J. Faraut, *Analysis on the crown of a Riemannian symmetric space*, pp. 99-110 in Lie groups and symmetric spaces, Amer. Math. Soc. Transl. Ser.2, **210** (2003), Amer. Math. Soc., Providence, RI.

[7] G. B. Folland, *Harmonic Analysis on Phase space*, Princeton University Press, Princeton, New Jersey (1989).

[8] R. Gangolli, *Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces*, Acta Math. **121** (1968), 151-192.

[9] S. M. Grudsky and N. L. Vasilevski, *Toeplitz operators on the Fock space: radial component effects*, Integr. equ. oper. theory **44** (2002), 10-37.

[10] B. Hall, *The Segal-Bargmann “coherent state” transform for compact Lie groups*, J. Funct. Anal. **122** (1994), no. 1,103-151.

[11] B. Hall and W. Lewkeeratiyutkul, *Holomorphic Sobolev spaces and the generalised Segal-Bargmann transform*, J. Funct. Anal. **217** (2004), 192-220.
[12] B. Hall, Berezin-Toeplitz quantization on Lie groups, J. Funct. Anal. 255 (2008) 2488-2506.
[13] S. Helgason, Groups and geometric Analysis, 83 (2002), Amer. Math. Soc., Providence, RI.
[14] E. Hille, A class of reciprocal functions, Ann. Math. 27 (1926), 427-464.
[15] B. Krötz, S. Thangavelu and Y. Xu, The heat kernel transform for the Heisenberg group, J. Funct. Anal. 225, no.2, 301-336 (2005).
[16] B. Krötz, G. Olafsson and R. Stanton, The image of the heat kernel transform on Riemannian symmetric spaces of noncompact type, Int. Math. Res. Notes, no. 22, 1307-1329 (2005).
[17] M. Lassalle, Série de Laurent des fonctions holomorphes dans la complexification d’un espace symétrique compact, Ann. Sci. 'Ecole Norm. Sup. (4) 11:2 (1978), 167-210.
[18] M. Lassalle, L’espace de Hardy d’un domaine de Reinhardt généralisé, J. Funct. Anal. 60:3 (1985), 309-340.
[19] M. Stenzel, The Segal-Bargmann transform on a symmetric space of compact type, J. Funct. Anal. 165 (1999), 44-58.
[20] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publi., Providence, RI (1967).
[21] S. Thangavelu, Harmonic analysis on the Heisenberg group, Prog. in Math. Vol. 159, Birkhäuser, Boston (1998).
[22] S. Thangavelu, Gutzmer’s formula and Poisson integrals on the Heisenberg group, Pacific J. Math. 231 (2007), 217-238.
[23] S. Thangavelu, Hermite and Laguerre semigroups: some recent developments, Seminaires et Congres (to appear).
[24] S. Thangavelu, An analogue of Gutzmer’s formula for Hermite expansions, Studia Math. 185 (2008), no.3, 279-290.
[25] S. Thangavelu, Lectures on Hermite and Laguerre expansions, Princeton University Press, Princeton, New Jersey (1993).
[26] S. Thangavelu, An introduction to the uncertainty principle: Hardy’s theorem on Lie groups, Prog. in Math. Vol. 217, Birkhäuser, Boston (1998).
[27] S. Thangavelu, Holomorphic Sobolev spaces associated to compact symmetric spaces, J. Funct. Anal. 251 (2007) 438-462.

Department of Mathematics, Indian Institute of Science, Bangalore-560 012
E-mail address: jyoti@math.iisc.ernet.in, veluma@math.iisc.ernet.in