SELF-SIMILAR SOLUTIONS TO NONLINEAR DIRAC EQUATIONS AND AN APPLICATION TO NONUNIQUENESS

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Abstract. Self-similar solutions to nonlinear Dirac systems (1) and (2) are constructed. As an application, we obtain nonuniqueness of strong solution in super-critical space $C([0, T); H^s(\mathbb{R}))$ ($s < 0$) to the system (1) which is $L^2(\mathbb{R})$ scaling critical equations. Therefore the well-posedness theory breaks down in Sobolev spaces of negative order.

1. Introduction. We are interested in the initial value problem for the following nonlinear Dirac equations

$$i(\partial_t U_1 + \partial_x U_1) = U_2(|U_1|^2 + |U_2|^2) + 2\text{Re}(\bar{U}_1 U_2)U_1,$$
$$i(\partial_t U_2 - \partial_x U_2) = U_1(|U_1|^2 + |U_2|^2) + 2\text{Re}(\bar{U}_2 U_1)U_2,$$

and

$$i(\partial_t U_1 + \partial_x U_1) = |U_2|^2 U_1,$$
$$i(\partial_t U_2 - \partial_x U_2) = |U_1|^2 U_2,$$

with the initial data $U_j(x, 0) = u_j(x)$. Here $U_j : \mathbb{R}^{1+1} \to \mathbb{C}$ for $j = 1, 2$ and $\bar{U}$ is a complex conjugate of $U$.

The systems (1) and (2) have the charge conservation

$$\int_{\mathbb{R}} (|U_1|^2 + |U_2|^2) (x, t) \, dx = \int_{\mathbb{R}} (|u_1|^2 + |u_2|^2) (x) \, dx.$$

Another important property of the systems (1), (2) is invariance under the scaling

$$U_j^\lambda (x, t) = \lambda U_j (\lambda^2 x, \lambda^2 t),$$

from which we deduce a scale invariant Sobolev space $L^2(\mathbb{R})$. We study the initial value problem of (1) and (2) in Sobolev space $H^s(\mathbb{R})$. We call $H^s$ as sub-critical space for $s > 0$, critical space for $s = 0$ and super-critical space for $s < 0$.

The system (1) occurs in the context of a nonlinear refractive index [1] and has been studied in [7] where local existence for $H^s$ ($s > 1/2$) has been proved.

The system (2) is called the Thirring model and the associated Cauchy problem has been studied by several authors [2, 4, 6, 8]. The global existence of solutions to the Thirring equations was studied in [4] in terms of Sobolev space $H^s$ ($s \geq 1$). Low regularity well-posedness was discussed in [2, 6, 8] showing that there exist a

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time $T > 0$ and solution $U_j \in C([0, T], H^s(\mathbb{R}))$ ($s \geq 0$). Furthermore, in [2], it is proven that a local solution in $L^2$ can be extended to a global one by excluding concentration of the $L^2$ norm at a point.

In this study we construct self-similar solutions to (1), (2) and establish the ill-posedness of the initial value problem to the nonlinear Dirac equations (1) in super-critical space $C([0, T]; H^s(\mathbb{R}))$ ($s < 0$). Furthermore, in [2], it is proven that a local solution in $L^2$ can be extended to a global one by excluding concentration of the $L^2$ norm at a point.

In this study we construct self-similar solutions to (1), (2) and establish the ill-posedness of the initial value problem to the nonlinear Dirac equations (1) in super-critical space $C([0, T]; H^s(\mathbb{R}))$ ($s < 0$) by showing nonuniqueness of solutions. Therefore the well-posedness theory of (1) breaks down in Sobolev spaces of negative order. We could not prove nonuniqueness of (2) because the cubic terms $|U_1|^2 U_2$, $|U_2|^2 U_1$ are not integrable. We refer to Remark 1 in section 3 for details.

Nonuniqueness of solution for Burgers’ equation has been shown in [5] by constructing a nontrivial solution which converges to zero in the $H^{s/2 - 1/4}$ topology as $t \to 0^+$. Nonuniqueness of solutions of the one-dimensional nonlinear Schrödinger equation has been studied in [3] where the author showed that there exist nonzero weak solutions varying continuously in the $H^s$ ($s < 0$) norm as $t \to 0^+$. To construct proper self-similar solutions, special algebraic structure of nonlinear terms should be considered crucially. Our main result is as follows.

**Theorem 1.1.** There exist strong solutions to (1) satisfying

$$U_j \in C([0, \infty); H^s(\mathbb{R})) \text{ with } s < 0,$$

not identically vanishing, with initial data $u_j(x) \equiv 0$.

For the precise meaning of strong solution, we refer to section 3. We will reduce (1) to a simpler equation (3) and construct self-similar solutions inside the null cone. The self-similar solutions are constructed in two separate regions and glued continuously. To construct a self-similar solution of (2), we will consider the special algebraic structure of nonlinearity which was used in [6].

In section 2, the self-similar solutions of (1) and (2) are constructed. In section 3, we introduce the definition of strong solutions and prove Theorem 1.1. We use the standard Sobolev spaces $H^s(\mathbb{R})$ with the norm $\|f\|_{H^s} = \|(1 - \Delta)^{s/2} f\|_{L^2}$.

2. **Self similar solutions.** In this section, we construct self similar solutions of equations (1) and (2). We reduce PDEs (1) and (2) to ODEs of the self-similar variable $y = x/t$. Then the ODEs are solved, considering the special algebraic structure of nonlinearity, inside the null cone.

2.1. **Self similar solution of (1).** We consider the change of variables $U_1 = U$, $U_2 = iV$ and suppose that $U$, $V$ are real valued functions. Then the system (1) reduces to

$$\partial_t U + \partial_x U = V(U^2 + V^2),$$
$$\partial_t V - \partial_x V = -U(U^2 + V^2).$$

(3)

Then any solution to (3) becomes a solution of (1). We construct self-similar solutions of the form

$$U(x, t) = \frac{1}{\sqrt{t}} u \left( \frac{x}{t} \right) \quad \text{and} \quad V(x, t) = \frac{1}{\sqrt{t}} v \left( \frac{x}{t} \right).$$
Then the system (3) reduces to
\[(y - 1)u' + \frac{1}{2}u = -v(u^2 + v^2),\]
\[(y + 1)v' + \frac{1}{2}v = u(u^2 + v^2),\]
where we use the variable \(y = x/t\). We can check that \(\frac{d}{dy} (1 + y) v^2 - (1 - y) u^2) = 0\). Assuming that the constant of integration is zero, we derive
\[(1 + y) v^2 = (1 - y) u^2.\]
(5)
From now on, we consider the case \(-1 < y < 1\). Making use of (5), we have from (4)
\[(y + 1)v' + \frac{1}{2}v = \pm \frac{2}{1 - y} \sqrt{\frac{1 + y}{1 - y}} v^2|v|\]
which becomes
\[(\sqrt{y + 1}v')' = \pm \frac{2}{1 - y} \sqrt{\frac{1}{1 - y}} v^2|v|\].
Let \(H(y) = \sqrt{1 + y} v(y)\). Then we have
\[H' = \pm \frac{2}{(1 - y)^{3/2}} H^2[H].\]
We start with the choice of +, in a region \(-1 < y \leq 0\), which leads to
\[H^2 = \frac{1}{2y + a \sqrt{1 - y^2}} \quad \text{for} \quad -1 < y \leq 0,\]
where \(a > 0\) is a constant of integration. With the choice of -, in a region \(0 \leq y < 1\), we have
\[H^2 = \frac{1}{2y - a \sqrt{1 - y^2}} \quad \text{for} \quad 0 \leq y < 1.\]
Note that \(H(0) = \frac{1}{2a}\) and \(H(-1) = 0 = H(1)\). Taking the relation \(V(x, t) = \frac{1}{\sqrt{t}} H(y)\) into account, we define in the region \(t \geq 0\)
\[V(x, t) = \begin{cases} \left(\frac{1}{2} \sqrt{\frac{1 - x}{2 + x}} \frac{1}{a \sqrt{1 - x^2 - 2x}}\right)^{\frac{1}{2}} & \text{for} \quad -t < x \leq 0, \\ \left(\frac{1}{2} \sqrt{\frac{1 - x}{2x + a \sqrt{1 - x^2}}}ight)^{\frac{1}{2}} & \text{for} \quad 0 \leq x < t, \\ 0 & \text{for} \quad |x| \geq t. \end{cases}\]
Through the similar process to \(u\), we have
\[(y - 1)u' + \frac{1}{2}u = \pm \frac{2}{1 + y} \sqrt{\frac{1 - y}{1 + y}} u^2|u|\]
and derive
\[U(x, t) = \begin{cases} \left(\frac{1}{2} \sqrt{\frac{1 + x}{2 + x}} \frac{1}{a \sqrt{1 + x^2 - 2x}}\right)^{\frac{1}{2}} & \text{for} \quad -t < x \leq 0, \\ \left(\frac{1}{2} \sqrt{\frac{1 + x}{2x + a \sqrt{1 - x^2}}}ight)^{\frac{1}{2}} & \text{for} \quad 0 \leq x < t, \\ 0 & \text{for} \quad |x| \geq t. \end{cases}\]
Note that \(U\) and \(V\) are continuous across the axis \(\{(x, t)| x = 0 \text{ and } t > 0\}\).
2.2. Self similar solution of (2). Let us try to find solutions to (2) of the form

\[ U_j(x, t) = \frac{1}{\sqrt{t}} V_j \left( \frac{x}{t} \right), \]

where \( V_j \) are complex-valued functions. Then the Thirring equations (2) reduce to

\[
\begin{align*}
(1 - y)V_1' - \frac{1}{2}V_1 &= -i|V_2|^2 V_1, \\
(y + 1)V_2' + \frac{1}{2}V_2 &= i|V_1|^2 V_2,
\end{align*}
\]

where we use the variable \( y = x/t \). For \(-1 < y < 1\), the system (8) can be rewritten as

\[
\begin{align*}
\frac{d}{dy}((1 - y)^\frac{1}{2} V_1) &= \frac{-i}{1 - y} \frac{|(1 + y)^\frac{1}{2} V_2|^2}{(1 + y)}, \\
\frac{d}{dy}((1 + y)^\frac{1}{2} V_2) &= \frac{i}{1 + y} \frac{|(1 - y)^\frac{1}{2} V_1|^2}{(1 - y)}.
\end{align*}
\]

Then we have

\[
\begin{align*}
(1 - y)^\frac{1}{2} V_1(y) &= V_1(0) \exp \left( -i \int_0^y \frac{1}{1 - s} \frac{|(1 + s)^\frac{1}{2} V_2|^2}{(1 + s)} ds \right), \\
(1 + y)^\frac{1}{2} V_2(y) &= V_2(0) \exp \left( i \int_0^y \frac{1}{1 + s} \frac{|(1 - s)^\frac{1}{2} V_1|^2}{(1 - s)} ds \right).
\end{align*}
\]

Observing \( |e^{\int_0^y f(s)ds}| = 1 \) for a real-valued function \( f \) which was used in [6], we deduce

\[
\begin{align*}
|(1 - y)^\frac{1}{2} V_1(y)| &= |V_1(0)|, \\
|(1 + y)^\frac{1}{2} V_2(y)| &= |V_2(0)|.
\end{align*}
\]

Then the system (9) reads as

\[
\begin{align*}
(1 - y)^\frac{1}{2} V_1(y) &= V_1(0) \exp \left( -i |V_2(0)|^2 \int_0^y \frac{1}{1 - s^2} ds \right), \\
(1 + y)^\frac{1}{2} V_2(y) &= V_2(0) \exp \left( i |V_1(0)|^2 \int_0^y \frac{1}{1 - s^2} ds \right).
\end{align*}
\]

Now we define, in the region \( t \geq 0 \),

\[
U_1(x, t) = \begin{cases} 
\frac{V_1(0)}{\sqrt{t - x}} \exp \left( i \frac{|V_2(0)|^2}{2} \log \frac{t - x}{t + x} \right) & \text{for } -t < x < t, \\
0 & \text{for } |x| \geq t,
\end{cases}
\]

and

\[
U_2(x, t) = \begin{cases} 
\frac{V_2(0)}{\sqrt{x + t}} \exp \left( i \frac{|V_1(0)|^2}{2} \log \frac{t + x}{t - x} \right) & \text{for } -t < x < t, \\
0 & \text{for } |x| \geq t.
\end{cases}
\]
3. **Proof of Theorem 1.1.** Let us introduce the definition of strong solution to (1), (2) which can be written in the following form

\[
\partial_t U_1 + \partial_x U_1 = F_1(U_1, U_2), \\
\partial_t U_2 - \partial_x U_2 = F_2(U_1, U_2),
\]

where \( F_1, F_2 \) are cubic polynomials.

**Definition 3.1.** Consider the Cauchy problem (10) with initial data \((u_1(x), u_2(x)) \in (H^s(\mathbb{R}))^2\). It is said that \( U = (U_1, U_2) \) is a strong solution to the Cauchy problem on the time interval \([0, T]\) provided that

\[
(U_1, U_2) \in (C([0, T]; H^s(\mathbb{R})))^2
\]

satisfies the equations (10) in the following sense. For any \( \phi \in C_0^\infty(\mathbb{R} \times (-T, T)) \), we have

\[
\int_0^T \int_{\mathbb{R}} U_1 \partial_t \phi + U_1 \partial_x \phi + F_1(U_1, U_2) \phi \, dx \, dt + \int_{\mathbb{R}} u_1(x) \phi(x, 0) \, dx = 0,
\]

and

\[
\int_0^T \int_{\mathbb{R}} U_2 \partial_t \phi - U_2 \partial_x \phi + F_2(U_1, U_2) \phi \, dx \, dt + \int_{\mathbb{R}} u_2(x) \phi(x, 0) \, dx = 0,
\]

where \( F_1(U_1, U_2) \in L_{loc}^1(\mathbb{R} \times (-T, T)) \).

Now we prove Theorem 1.1. We will show that the functions \( U \) and \( V \) in section 2.1 are nontrivial strong solutions of (1) with the trivial initial data \( U(x, 0) \equiv 0 \equiv V(x, 0) \). First of all, we know that \( U(\cdot, t), V(\cdot, t) \in L^p(\mathbb{R}) \) (1 \( \leq p < 4 \)) for each \( t > 0 \). Here we just show the case of \( V \):

\[
\int_{\mathbb{R}} |V(x, t)|^p \, dx = \int_{-t}^{t} t^{-\frac{p}{2}} |v(x/t)|^p \, dx = t^{1 - \frac{p}{2}} \int_{-1}^{1} |v(y)|^p \, dy
\]

\[
= t^{1 - \frac{p}{2}} \left( \frac{1}{2} \right)^{\frac{p}{2}} \int_{-1}^{0} \left( \frac{1}{a \sqrt{1 - y^2} - 2y} \right)^{\frac{p}{2}} \left( \frac{1 - y}{1 + y} \right)^{\frac{p}{2}} \, dy
\]

\[
+ t^{1 - \frac{p}{2}} \left( \frac{1}{2} \right)^{\frac{p}{2}} \int_{0}^{1} \left( \frac{1}{2y + a \sqrt{1 - y^2}} \right)^{\frac{p}{2}} \left( \frac{1 - y}{1 + y} \right)^{\frac{p}{2}} \, dy.
\]

Let

\[
g(y) = \begin{cases} 
\frac{1}{a \sqrt{1 - y^2} - 2y} & \text{on } -1 \leq y \leq 0, \\
\frac{1}{2y + a \sqrt{1 - y^2}} & \text{on } 0 \leq y \leq 1.
\end{cases}
\]

Then we can check that \( 0 < g(y) \leq \frac{1}{y} \) for \( a \geq 2 \). Actually we have \( g(-1) = g(1) = \frac{1}{2} \).

Then it is easy to check \( V(\cdot, t) \in L^p(\mathbb{R}) \) (1 \( \leq p < 4 \)) for each \( t > 0 \). Moreover, taking into account \( L^p(\mathbb{R}) \hookrightarrow H_{\frac{2}{3} - \frac{2}{3p}}(\mathbb{R}) \) for \( 1 < p < 2 \), we can verify (11)

\[
U, V \in C([0, T]; H_{\frac{2}{3} - \frac{2}{3p}}(\mathbb{R})),
\]

and

\[
\lim_{t \to 0^+} (\|U\|_{H_{\frac{2}{3} - \frac{2}{3p}}(\mathbb{R})} + \|V\|_{H_{\frac{2}{3} - \frac{2}{3p}}(\mathbb{R})}) \leq C \lim_{t \to 0^+} (\|U\|_{L^p(\mathbb{R})} + \|V\|_{L^p(\mathbb{R})}) = 0,
\]

which shows that the nontrivial solutions converge to zero in the \( H^s \) (\( s < 0 \)) norm as \( t \to 0^+ \).
We note that nonlinear terms $V^2 U$, $U^2 V$, $V^3$, $U^3$ are integrable. Here we just show the case of $V^2 U$. Making use of (6) and (7), we have

$$\int_{-t}^{t} V^2 U \, dx = \frac{1}{t \sqrt{t}} \int_{-t}^{t} v^2(x/t) u(x/t) \, dx = \frac{1}{t \sqrt{t}} \int_{-1}^{1} v^2(y) u(y) \, dy$$

$$= \frac{1}{\sqrt{t}} \int_{-1}^{0} \frac{1}{2} \left( \frac{1 - y}{1 + y} \frac{1}{a \sqrt{1 - y^2}} - 2y \right) \frac{1}{2} \sqrt{1 + y} \frac{1}{a \sqrt{1 - y^2}} \, dy$$

$$+ \frac{1}{\sqrt{t}} \int_{0}^{1} \frac{1}{2} \left( \frac{1 - y}{1 + y} \frac{1}{a \sqrt{1 - y^2}} - 2y \right) \frac{1}{2} \sqrt{1 + y} \frac{1}{a \sqrt{1 - y^2}} \, dy.$$

Considering $0 < g(y) \leq \frac{1}{2}$ for $a \geq 2$, we have

$$\int_{0}^{T} \int_{-t}^{t} V^2 U \, dx \, dt \leq \int_{0}^{T} \frac{1}{8 \sqrt{t}} \int_{-1}^{1} \left( \frac{1 - y}{1 + y} \right)^{\frac{1}{2}} \, dx \, dt < \infty.$$

Now we are ready to show (12). Denote, for $\varepsilon > 0$,

$$D_{\varepsilon}^{T} = \{(x, t) \mid t \geq -x + \varepsilon, x \leq 0\} \text{ and } D_{\varepsilon}^{T} = \{(x, t) \mid t \geq x + \varepsilon, x \geq 0\}.$$

Taking into account $U(x, 0) = 0$, we will show that

$$\lim_{\varepsilon \to 0^+} \int_{D_{\varepsilon}^{T} \cup D_{\varepsilon}^{T}} U \partial_t \phi + U \partial_x \phi + V(U^2 + V^2) \phi \, dx \, dt = 0,$$

for $\phi \in C_0^\infty(\mathbb{R} \times [-T, T])$. Since the integrand has a finite $L^1$ norm over $\mathbb{R} \times [-T, T]$, (12) follows from dominated convergence theorem. Considering $-\partial_t U - \partial_x U + V(U^2 + V^2) = 0$ in $D_{\varepsilon}^{T}$ and $D_{\varepsilon}^{T}$ respectively and applying Green’s Theorem, we have

$$\int_{D_{\varepsilon}^{T} \cup D_{\varepsilon}^{T}} U \partial_t \phi + U \partial_x \phi + V(U^2 + V^2) \phi \, dx \, dt$$

$$= \int_{D_{\varepsilon}^{T}} \partial_t (U \phi) + \partial_x (U \phi) \, dx \, dt + \int_{D_{\varepsilon}^{T}} \partial_t (U \phi) + \partial_x (U \phi) \, dx \, dt$$

$$= \int_{T \varepsilon}^{T} (U \phi)(0, s) \, ds + \int_{-T + \varepsilon}^{0} (-2U \phi)(s, -s + \varepsilon) \, ds$$

$$+ \int_{T \varepsilon}^{T} (U \phi)(0, s) \, ds + \int_{0}^{T - \varepsilon} (-U \phi)(s, s + \varepsilon) + (U \phi)(s, s + \varepsilon) \, ds.$$

Since

$$\int_{T \varepsilon}^{T} (U \phi)(0, s) \, ds + \int_{T \varepsilon}^{T} (U \phi)(0, s) \, ds = 0,$$

$$\int_{0}^{T - \varepsilon} (-U \phi)(s, s + \varepsilon) + (U \phi)(s, s + \varepsilon) \, ds = 0,$$

we have

$$\int_{-T + \varepsilon}^{0} (-2U \phi)(s, -s + \varepsilon) \, ds = \int_{0}^{T - \varepsilon} -2(U \phi)(-t, t + \varepsilon) \, dt,$$

where we use substitute $s = -t$. Considering

$$U(-t, t + \varepsilon) = \frac{1}{\sqrt{2}} \left( \frac{\varepsilon}{\varepsilon + 2t} \right)^{\frac{1}{2}} \left( \frac{1}{a \sqrt{\varepsilon^2 + 2\varepsilon t + 2t}} \right)^{\frac{1}{2}}$$
and applying dominated convergence theorem, we obtain
\[
\lim_{\varepsilon \to 0^+} \left| \int_0^{T-\varepsilon} -\sqrt{2} (U\phi)(-t, t + \varepsilon) \, dt \right|
\leq \|\phi\|_{L^\infty} \lim_{\varepsilon \to 0^+} \int_0^{T-\varepsilon} \left( \frac{\varepsilon}{\varepsilon + 2t} \right)^{\frac{1}{4}} \left( \frac{1}{a\sqrt{\varepsilon^2 + 2\varepsilon t + 2t}} \right)^{\frac{1}{2}} \, dt = 0.
\]
Therefore we show that (12) holds. The equality (13) can be obtained in a similar way.

Remark 1. Let us consider functions \(U_j\) in section 2.2. Put \(V_1(0) = 1 = V_2(0)\) for simplicity. We can check \(U_j(\cdot, t) \in L^p(\mathbb{R}) (1 \leq p < 2)\) for each \(t > 0\). In fact, we have
\[
\int_\mathbb{R} |U_1(x, t)|^p \, dx = \int_{-t}^t \frac{1}{(t-x)^{p/2}} \, dx = t^{1-\frac{p}{2}} \int_{-1}^1 \frac{1}{(1-y)^{p/2}} \, dy,
\]
where we use change of variable \(y = x/t\). Taking into account \(L^p(\mathbb{R}) \hookrightarrow H^{s-\frac{1}{p}}(\mathbb{R})\) for \(1 < p < 2\), we can verify (11) and
\[
\lim_{t \to 0^+} \|U_j\|_{H^{s-\frac{1}{p}}(\mathbb{R})} \leq C \lim_{t \to 0^+} \|U_j\|_{L^p(\mathbb{R})} = 0,
\]
for \(1 < p < 2\). Therefore \(U_j\) converges to zero in the \(H^s (s < 0)\) norm as \(t \to 0^+\). Here we should note that the nonlinear terms \(|U_1|^2U_2, |U_2|^2U_1\) are not integrable. We have, for instance,
\[
\int_0^T \int_\mathbb{R} |U_1|^2 |U_2| \, dx \, dt = \int_0^T \int_{-1}^1 \left( \frac{1}{\sqrt{t-x}} \right)^2 \frac{1}{\sqrt{t+x}} \, dx \, dt.
\]
Therefore we have a problem in understanding \(U_j\) as a solution to (2) in the sense of distribution.

The main difference between the self-similar solutions for equations (1) and (2) is that, when one writes
\[
|U_1| = \frac{1}{\sqrt{t-x}} h(x, t),
\]
the nature of \(h\) is essentially different: for system (1), it behaves like \((t-x)^{1/4}\) as \(x \to t\); in system (2), it is a constant. This is the essential point that makes the nonlinearity integrable for (1) but not for (2).

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