BOUNDDED AND $L^2$-SOLUTIONS TO A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH A SQUARE INTEGRABLE FORCING TERM

ALLAN KROOPNICK

(Received 12 May 1998)

Abstract. This paper presents two theorems concerning the nonlinear differential equation $x'' + c(t)f(x)x' + a(t,x) = e(t)$, where $e(t)$ is a continuous square-integrable function. The first theorem gives sufficient conditions when all the solutions of this equation are bounded while the second theorem discusses when all the solutions are in $L^2[0,\infty)$. 

Keywords and phrases. Bounded, $L^2$-solutions, square-integrable, absolutely integrable.

1991 Mathematics Subject Classification. 34C11.

In this paper, we discuss, using standard methods, the bounded properties of the following second order nonlinear differential equation with square-integrable forcing term $e(t)$. Without loss of generality, we restrict our discussion to the nonnegative real line $[0,\infty)$. Specifically, we study the equation

$$x'' + c(t)f(x)x' + a(t,x) = e(t). \quad (1)$$

Our purpose here is to extend some previous results where $e(t)$ was an absolutely integrable but not necessarily a square integrable function. This is a somewhat more general result since previous work considered functions such as $e(t) = 1/(1 + t)^2$ but not functions like $e(t) = 1/(1 + t)$ which is not absolutely integrable though it is square integrable. The theorems presented here cover that case. Also, we develop the conditions under which all the solutions are $L^2$-solutions. By an $L^2$-solution, we mean a solution of (1) such that $\int_0^\infty x(t)^2 dt < \infty$. For some previous work covering the absolutely integrable and homogeneous cases, see [1, 2, 7, 6, 5, 4], especially, [2] for its excellent bibliography. We now turn our attention to our main results.

We see under what conditions all the solutions of (1), as well as their derivatives, are bounded. In our proof, we do not need to resort to the use of the direct method of Liapunov which is often the case. We now state and prove our first theorem.

**Theorem 1.** Given the differential equation (1) where $e(t)$ is continuous on $[0,\infty)$ and $\int_0^\infty e(t)^2 dt < \infty$. Suppose that $c(\cdot)$ is continuous on $[0,\infty)$ with $c(t) > c_\ast > 0$ and $f(\cdot)$ is continuous on $\mathbb{R}$ with $f(x) > f_\ast > 0$ where $c_\ast$ and $f_\ast$ are positive constants. Furthermore, let $a(t,x)$ be continuous on $[0,\infty) \times \mathbb{R}$ with $\int_0^\infty a(t,x)dx = \infty$ uniformly in $t$, and $x(\partial/\partial t)a(t,x) \leq 0$, then any solution $x$ to (1), as well as its derivative, is bounded as $t \to \infty$ and $\int_0^\infty x'(t)^2 dt < \infty$. 
Proof. By standard existence theory, there is a solution of (1) which exists on \([0, T]\) for some \(T > 0\). Multiply equation (1) by \(x'\) and perform an integration by parts from 0 to \(t\) on the last term of the LHS of (1) in order to obtain,

\[
\frac{x'(t)^2}{2} + \int_0^t c(s)f(x(s))x'(s)^2\,ds + \int_{x(0)}^{x(t)} a(t,u)\,du - \int_0^t \int_{x(0)}^{x(s)} \frac{\partial}{\partial s} a(s,u)\,du\,ds = \frac{x'(0)^2}{2} + \int_0^t e(s)x'(s)\,ds.
\]

(2)

Now, using the Cauchy-Schwarz inequality for integrals on the RHS of (2), we get

\[
\frac{x'(t)^2}{2} + \int_0^t c(s)f(x(s))x'(s)^2\,ds + \int_{x(0)}^{x(t)} a(t,u)\,du - \int_0^t \int_{x(0)}^{x(s)} \frac{\partial}{\partial s} a(s,u)\,du\,ds \leq \frac{x'(0)^2}{2} + \left( \int_0^t e(s)^2\,ds \right)^{1/2} \left( \int_0^t x'(s)^2\,ds \right)^{1/2}.
\]

(3)

Next, let \(H(t) = \left( \int_0^t x'(s)^2\,ds \right)^{1/2}\). Dividing both sides by \(H(t)\) yields,

\[
H(t)^{-1} \frac{x'(t)^2}{2} + \int_0^t c(s)f(x(s))x'(s)^2\,ds + \int_{x(0)}^{x(t)} a(t,u)\,du - \int_0^t \int_{x(0)}^{x(s)} a(s,u)\,du\,ds \leq H(t)^{-1} \frac{x'(0)^2}{2} + \left( \int_0^t e(s)^2\,ds \right)^{1/2}.
\]

(4)

We first need to show that \(|x|\) remains bounded. If not, then should \(|x|\) increase without bound, all terms of the LHS of equation (4) become positive by our hypotheses. Furthermore, \(H(t)^{-1} c_s f_s \int_0^t x'(s)^2\,ds = c_s f_s \left( \int_0^t x'(s)^2\,ds \right)^{1/2}\) is bounded by the RHS of equation (4). This implies that \(x'\) is square integrable and is also bounded after we examine the first term of the LHS of (4). However, the above then implies that \(|x|\) must be bounded. Otherwise, the LHS of (4) becomes infinite which is impossible. A standard argument now permits the solution to be extended on all of \([0, \infty)\), \([3, p.\ 17–18]\). Our proof is now complete.

By imposing more stringent conditions on \(a(t,x)\), all the solutions become \(L^2\)-solutions. We now state and prove under what conditions this is true.

Theorem 2. Assume the hypotheses of Theorem 1 hold. In addition, suppose that \(a(t,x)x > a_s x^2\) for some positive constant \(a_s\), and \(c'(t) \leq 0\), then all the solutions of (1) are \(L^2\)-solutions.

Proof. In order to see that \(x\) is in \(L^2[0, \infty)\), we must first multiply equation (1) by \(x\), then integrate from 0 to \(t\), and integrate by parts the first term on the LHS in order to obtain

\[
x(t)x'(t) - \int_0^t x'(s)^2\,ds + \int_0^t c(s)f(x(s))x(s)x'(s)\,ds + \int_0^t x(s)a(s,x(s))\,ds = x(0)x'(0) + \int_0^t e(s)x(s)\,ds.
\]

(5)
Next, let \( F(x) = \int_0^x u f(u) \, du \). Now, upon another integration by parts, the above may be rewritten as

\[
\begin{align*}
  x(t)x'(t) - \int_0^t x'(s)^2 \, ds + c(t)F(x(t)) &
  \quad - \int_0^t F(x(s))c'(s) \, ds + \int_0^t x(s)a(s,x(s)) \, ds 
  \leq K,
\end{align*}
\]

(6)

where \( K = |x(0)x'(0)| + |\int_0^t e(s)x(s) \, ds| + |c(0)F(x(0))| \). Notice that the term \( M(t) = \int_0^t e(s)x(s) \, ds \) is bounded by \( (\int_0^t e(s)^2 \, ds)(\int_0^t x(s)^2)^{1/2} \) by using the Cauchy-Schwarz inequality. Dividing the LHS of (6) by \( M(t) \) and using the hypotheses of Theorem 2 immediately yields,

\[
\begin{align*}
  M(t)^{-1}x(t)x'(t) - \int_0^t x'(s)^2 \, ds + c(t)F(x(t)) &
  \quad - \int_0^t F(x(s))c'(s) \, ds + a \left( \int_0^t x(s)^2 \, ds \right)^{1/2} 
  \leq \frac{K}{M(t)}.
\end{align*}
\]

(7)

Since the RHS of (7) is bounded and all the terms on the LHS of (7) are either bounded or positive, the result follows because the LHS cannot be unbounded. Here, we need that \( x' \) is square integrable.

**Example.** Consider the second order linear differential equation,

\[
x'' + K(t)x' + l(t)x = h(t),
\]

(8)

where \( K(\cdot) \) and \( l(\cdot) \) are continuous functions defined for \( t \geq 0 \) with \( K(\cdot) \) and \( l(\cdot) \) having continuous nonpositive derivatives with \( K(t) > K_0 > 0, l(t) > l_0 > 0, \) and \( \int_0^\infty h(t)^2 \, dt < \infty \), where \( K_0 \) and \( l_0 \) are positive constants. Given these conditions, the above theorems show that all the solutions of (8), as well as their derivatives, are bounded and in \( L^2[0, \infty) \).

\( \square \)

**References**

[1] H. A. Antosiewicz, *On nonlinear differential equations of the second order with integrable forcing term*, J. London Math. Soc. **30** (1955), 64–67. MR 16,477f. Zbl 064.08404.

[2] Z. S. Athanassov, *Boundedness criteria for solutions of certain second order nonlinear differential equations*, J. Math. Anal. Appl. **123** (1987), no. 2, 461–479. MR 88d:34044. Zbl 642.34031.

[3] J. K. Hale, *Ordinary Differential Equations*, Pure and Applied Mathematics, vol. XXI, Wiley-Interscience [John Wiley & Sons], New York, London, Sydney, 1969. MR 54 7918. Zbl 186.40901.

[4] A. Kroopnick, *L^2-solutions to y'' + c(t)y' + a(t)b(y) = 0*, Proc. Amer. Math. Soc. **39** (1973), 217–218. MR 47 2135. Zbl 264.34041.

[5] A. J. Kroopnick, *Note on bounded L^p-solutions of a generalized Liouville equation*, Pacific J. Math. **94** (1981), no. 1, 171–175. MR 82h:34065. Zbl 458.34022.

[6] **, *Bounded and L^p-solutions of a generalized Liouville equation with integrable forcing term*, Missouri J. Math. Sci. **10** (1998), 15–19.

[7] A. Strauss, *Liapunov functions and L^p solutions of differential equations*, Trans. Amer. Math. Soc. **119** (1965), 37–50. MR 31#2461. Zbl 128.31401.

KROOPNICK: OFFICE OF PROGRAM BENEFITS POLICY, 3-F-26 OPERATIONS BUILDING, SOCIAL SECURITY ADMINISTRATION, 6401 SECURITY BOULEVARD, BALTIMORE, MARYLAND 21235, USA
Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

| Deadline                | Date          |
|-------------------------|---------------|
| Manuscript Due          | May 1, 2009   |
| First Round of Reviews  | August 1, 2009|
| Publication Date        | November 1, 2009|

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie

Hindawi Publishing Corporation
http://www.hindawi.com