Characters of the BMS Group in Three Dimensions

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Abstract

Using the Frobenius formula, we evaluate characters associated with certain induced representations of the centrally extended BMS$_3$ group. This computation involves a functional integral over a coadjoint orbit of the Virasoro group; a delta function localizes the integral to a single point, allowing us to obtain an exact result. The latter is independent of the specific form of the functional measure, and holds for all values of the BMS$_3$ central charges and all values of the chosen mass and spin. It can also be recovered as a flat limit of Virasoro characters.

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Introduction

The BMS$_3$ group is the asymptotic symmetry group of three-dimensional Einstein gravity with a vanishing cosmological constant at null infinity [1–3]. It is an infinite-dimensional extension of the Poincaré group in three dimensions. On account of the semi-direct product structure of BMS$_3$, it was recently argued in [4,5] that its irreducible unitary representations are induced representations [6–11]. The latter turn out to be classified by coadjoint orbits of the Virasoro group [12–15], which appear in this context as infinite-dimensional generalizations of the usual hyperboloidal orbits of the Poincaré group. Owing to the analogy between BMS$_3$ and Poincaré, it is tempting to call “supermomentum” the infinite-dimensional vector that generalizes the usual Poincaré momentum. In particular, the orbits of generic constant supermomenta are diffeomorphic to $\text{Diff}^+(S^1)/S^1$ while the orbit of the BMS$_3$ vacuum is the universal Teichmüller space $\text{Diff}^+(S^1)/\text{PSL}(2,\mathbb{R})$ [15,16]. The corresponding representations may be called “BMS$_3$ particles”, generalizing the notion of particle defined by Poincaré symmetry.

The purpose of this paper is to compute the characters of rotations and supertranslations in certain induced representations of the (centrally extended) BMS$_3$ group. Specifically, we will focus on representations based on orbits of constant supermomenta, which include the BMS$_3$ vacuum and massive BMS$_3$ particles. We will compute these characters using the Frobenius formula [17–19], which roughly states that the character of a group element in an induced representation is a certain integral over the corresponding orbit. Since the orbits relevant for BMS$_3$ are infinite-dimensional Virasoro coadjoint orbits, the integral required by the Frobenius formula is in that case a functional one, a priori leading to two complications: the first is the definition of the functional integral measure, and the second is the practical calculation of the integral.

Remarkably, for non-zero rotation angle, both of these complications simply disappear. Indeed, the Frobenius formula contains a delta function that localizes the functional integral to a single point — the unique point invariant under rotations on the orbit. Provided we find local coordinates on the orbit in a neighbourhood of that point, we can write down a local expression for the measure, up to an unknown prefactor. Since induced representations are independent (up to unitary equivalence) of the choice of a measure on the orbit, the contribution of the prefactor vanishes upon integrating the delta function. This provides an exact expression for the resulting character. In contrast to the Virasoro group, whose characters (in unitary representations) depend heavily on the values of the central charge and the highest weight [20–24], the BMS$_3$ characters derived here are valid for all values of the central charges and (almost) any value of the constant supermomentum on the selected orbit. Given that the BMS$_3$ group can be seen as a high-energy, high central charge limit of two Virasoro groups, this simplification should not come as a surprise.

The paper is organized as follows. In section 1, we briefly review the theory of induced representations for semi-direct products and the resulting Frobenius formula for characters, before applying these considerations to the Poincaré group in three dimensions. In section 2, we then use this technique to compute BMS$_3$ characters and discuss their relation to standard Virasoro characters at large central charge. The conclusion is devoted to certain open issues and possible extensions of this work.
1 Induced representations and the Frobenius formula

This section begins with a lightning review of the theory of induced representations (subsection 1.1), which is then used to derive the Frobenius formula for the corresponding characters (subsection 1.2). Subsection 1.3 contains an application of this formula to massive representations of the Poincaré group in three dimensions. For the sake of generality, we will not assume that an invariant measure exists on each orbit, and we will therefore use quasi-invariant measures instead. Up to this slight difference, our notations and conventions match those of [4]. For the record, we will make no attempt at mathematical rigour. In particular, in this section, we will assume that all standard regularity assumptions for the objects involved (such as local compactness or separability) are satisfied. We refer to [10] for details.

1.1 Induced representations of semi-direct products

Here we first define the notion of quasi-invariant measures (and introduce, in particular, the associated Radon-Nikodym derivative), before using it in the framework of induced representations.

Quasi-invariant measures

Let $G$ be a topological group, acting continuously and transitively on a topological space $O$, with the action of $f \in G$ on $O$ denoted by $q \mapsto f \cdot q$ for all $q \in O$. Let $\mu$ be a (Borel) measure on $O$. We say that $\mu$ is quasi-invariant under the $G$ action if, for any $f \in G$, the measure $\mu_f$ defined by

$$\int_A d\mu_f(q) = \int_{f \cdot A} d\mu(q)$$

for any measurable set $A \subseteq O$ is equivalent to $\mu$, that is, if $\mu$ and $\mu_f$ have the same sets of measure zero. By virtue of the Radon-Nikodym theorem [10, 25], this amounts to saying that, for each $f \in G$, there exists a positive function $\rho_f$ on $O$, called the Radon-Nikodym derivative of $\mu_f$ with respect to $\mu$, such that $d\mu_f(q) = \rho_f(q) d\mu(q)$ for all $q \in O$. This relation is written as

$$\rho_f(q) = \frac{d\mu_f(q)}{d\mu(q)} = \frac{d\mu(f \cdot q)}{d\mu(q)}.$$  \hspace{1cm} (1)

Since $\mu_f$ is related to $\mu$ by a group action, the Radon-Nikodym derivative satisfies the following property:

$$\rho_{fg}(q) = \rho_f(g \cdot q) \rho_g(q), \quad \forall f, g \in G, \forall q \in O.$$  \hspace{1cm} (2)

If $\rho_f(q) = 1$ for all $f \in G$ and any $q \in O$, we say that the measure $\mu$ is invariant under the action of $G$. For example, the usual Lebesgue measure $d^n x$ in $\mathbb{R}^n$ is invariant under translations and rotations, but it is quasi-invariant under diffeomorphisms: under $f : x \mapsto f(x)$, the measure transforms into $d^n f(x) = |\det(\partial f/\partial x)| d^n x$, so the corresponding Radon-Nikodym derivative $\rho_f(x) = |\det(\partial f/\partial x)|$ is the absolute value of the Jacobian determinant.

Although our presentation in the following pages will take into account the possibility of quasi-invariant measures, such subtleties will play no role once we turn to characters. Indeed, we will see that the Radon-Nikodym derivative does not contribute to the Frobenius formula (cf. eq. (12) below).
**Induced representations**

The theory of representations of semi-direct products of the form $G \ltimes \sigma A$, where $A$ is an Abelian vector group, is well known\(^1\). Under suitable regularity assumptions, it turns out that all irreducible unitary representations of such groups are so-called induced representations \([7–10]\).

We begin by fixing some notation \([4]\). Let $A^*$ be the dual of $A$. For any $p \in A^*$, define the action of $f \in G$ on $p$ by $\langle f \cdot p, \alpha \rangle \equiv \langle p, \sigma_f^{-1} \alpha \rangle$ for all $\alpha \in A$. Let then

$$\mathcal{O}_p \equiv \{ f \cdot p | f \in G \} \subset A^*$$

be the orbit of $p \in A^*$ under this action, and let

$$G_p \equiv \{ f \in G | f \cdot p = p \}$$

be the corresponding little group. We will assume that there exists a measure $\mu$ on $\mathcal{O}_p$ that is quasi-invariant under the action of $G$. Finally, seeing the group $G$ as a principal $G_p$-bundle over $\mathcal{O}_p$, introduce a continuous section $g : \mathcal{O}_p \to G : q \mapsto g_q$ such that $g_q \cdot p = q$ for all $q \in \mathcal{O}_p$. In writing this, we are assuming that $g$ is a global section, i.e. that the $G_p$-bundle $G \to \mathcal{O}_p$ is trivial. This property is not true in general but it will hold in all cases of interest below, so we will stick to it.

Consider now a unitary representation $\mathcal{R} : G_p \to \text{GL}(\mathcal{E}) : f \mapsto \mathcal{R}[f]$ of a given little group $G_p$ in a complex Hilbert space $\mathcal{E}$, equipped with a scalar product $\langle \cdot | \cdot \rangle$. Let also $\mathcal{H}$ denote the Hilbert space of “wavefunctions” $\Psi : \mathcal{O}_p \to \mathcal{E}$ that are square-integrable with respect to the measure $\mu$, endowed with the scalar product

$$\langle \Phi | \Psi \rangle \equiv \int_{\mathcal{O}_p} d\mu(q) \langle \Phi(q) | \Psi(q) \rangle.$$  \hspace{1cm} (5)

The induced representation $\mathcal{T}$ associated with $\mathcal{R}$ then acts in $\mathcal{H}$ according to \([10, 11]\)

$$\left( \mathcal{T} \left[ (f, \alpha) \right] \Psi \right)(q) \equiv [\rho_{f^{-1}}(q)]^{1/2} e^{i\langle q, \alpha \rangle} \mathcal{R}_{g_q^{-1} f g_{f^{-1} q}} \Psi \left( f^{-1} \cdot q \right)$$

for all $q \in \mathcal{O}_p$ and any wavefunction $\Psi$ in $\mathcal{H}$; here $\rho$ denotes the Radon-Nikodym derivative of the measure $\mu$, as defined in (1). It is easily verified, using the above definitions (and in particular relation (2)), that this expression indeed defines a representation of $G \ltimes A$. In fact, this would be true even without the prefactor $[\rho_{f^{-1}}(q)]^{1/2}$; the latter is needed, however, to make this representation unitary\(^2\) with respect to the scalar product (5). Note that different choices of quasi-invariant measures lead to unitarily equivalent induced representations: if $\mu_1$ and $\mu_2$ are two such measures on $\mathcal{O}_p$, each defining a scalar product of the form (5) in the Hilbert spaces of $\mathcal{E}$-valued square integrable wavefunctions $\mathcal{H}_1$ and $\mathcal{H}_2$ (respectively), then the map

$$U : \mathcal{H}_1 \to \mathcal{H}_2 : \Psi \mapsto U \Psi \quad \text{with} \quad \left( U \Psi \right)(q) \equiv \left[ \frac{d\mu_1(q)}{d\mu_2(q)} \right]^{1/2} \Psi(q)$$

\hspace{1cm} (7)

---

\(^1\)Henceforth we will systematically drop the subscript $\sigma$ in $G \ltimes \sigma A$.

\(^2\)In \([4]\) we limited ourselves to invariant measures, so the Radon-Nikodym derivative never showed up.
is an isometry that intertwines the induced representations $\mathcal{T}_1$ and $\mathcal{T}_2$. (Here $d\mu_1/d\mu_2$ is the Radon-Nikodym derivative of $\mu_1$ with respect to $\mu_2$.)

When the vector group $A$ is seen as a group of translations, its dual $A^*$ consists of “momenta”. Then the $G$-orbits defined by (3) are the usual momentum orbits, familiar for instance from the Poincaré group. In that case the map (4) defines the “standard boost” $g_q$ for each $q$ on the orbit, and the measure $\mu$ is a Lorentz-invariant momentum measure on $O_p$. For example, for the orbit of a particle with mass $m$ living in $(n + 1)$-dimensional Minkowski space-time, one has $d\mu(q) = d^n q/\sqrt{m^2 + q^2}$, where $q$ denotes the spatial momentum. In that context, the space $\mathcal{E}$ of the representation $\mathcal{R}$ of the little group is the space of “internal” degrees of freedom, and the label specifying $\mathcal{R}$ is called “spin”. From now on, we will freely use the terminology of “momenta” and “spin” for any semi-direct product $G \ltimes A$ (with $A$ a vector group), as this terminology is also appropriate for the BMS$_3$ group.

Formula (6) can be conveniently re-expressed in terms of a basis of delta functions, physically representing “plane waves” or particles with definite momentum. To define this basis, introduce the Dirac delta distribution $\delta_\mu$ associated with the measure $\mu$, such that
\[
\int_{O_p} d\mu(q) \delta_\mu(k, q) \varphi(q) = \varphi(k)
\]
for any test function $\varphi$ on $O_p$ and any $k \in O_p$. Since in general $\mu$ transforms non-trivially under the $G$ action, the corresponding delta function transforms as
\[
\delta_\mu(f \cdot k, f \cdot q) = [\rho_f(q)]^{-1} \delta_\mu(k, q).
\]
Now, if $\{e_m|m = 1, 2, 3, \ldots\}$ is an orthonormal basis of $\mathcal{E}$, define the wavefunction of definite momentum $k$ and polarization $m$ to be
\[
\Psi_{k,m}(q) = \delta_\mu(k, q)e_m.
\]
The scalar product (5) of such wavefunctions is
\[
\langle \Psi_{k,m}|\Psi_{q,n} \rangle = \delta_{mn} \delta_\mu(k, q).
\]
In other words, the set $\{\Psi_{k,m}|k \in O_p, m = 1, 2, 3, \ldots\}$ forms an orthonormal basis of $\mathcal{H}$. In this basis, the action (6) of the induced representation becomes
\[
\mathcal{T}[(f, \alpha)] \Psi_{k,m} = [\rho_f(k)]^{1/2} e^{i(f \cdot k, \alpha)} \left(\mathcal{R} \left[g_{f,k}^{-1} f g_k\right]\right)_{nm} \Psi_{f,k,n},
\]
where summation over repeated indices is implicit, and where $\left(\mathcal{R} \left[g_{f,k}^{-1} f g_k\right]\right)_{nm}$ denotes the matrix element of $\mathcal{R} \left[g_{f,k}^{-1} f g_k\right]$ between the vectors $e_n$ and $e_m$.

### 1.2 Characters: the Frobenius formula

The character of a representation is the map that associates, with each group element, the trace of the operator that represents it. In the present case, we want to compute the character of the induced representation $\mathcal{T}$,
\[
\chi : G \ltimes A \to \mathbb{C} : (f, \alpha) \mapsto \text{Tr}(\mathcal{T}[(f, \alpha)]).
\]
Since the carrier space of $\mathcal{T}$ is a Hilbert space $\mathcal{H}$, the trace of $\mathcal{T}[(f, \alpha)]$ can be written as a sum of scalar products between vectors of an orthonormal basis of $\mathcal{H}$ and their images under $\mathcal{T}[(f, \alpha)]$. For the representation based on $\mathcal{O}_p$, with spin $\mathcal{R}$, a convenient orthonormal basis is provided by plane waves $\Psi_{k,n}$, as defined in (9). The “sum” over scalar products then becomes an integral over $\mathcal{O}_p$ [19]:

$$\text{Tr}(\mathcal{T}[(f, \alpha)]) \equiv \chi[(f, \alpha)] = \int_{\mathcal{O}_p} d\mu(k) \sum_{n=1}^{+\infty} \langle \Psi_{k,n} | \mathcal{T}[(f, \alpha)] \Psi_{k,n} \rangle,$$

where $\mu$ is the quasi-invariant measure on $\mathcal{O}_p$ used to define $\mathcal{T}$. By virtue of the action (11) of the induced representation on plane waves, this character can be expressed as

$$\chi[(f, \alpha)] = \int_{\mathcal{O}_p} d\mu(k) [\rho_f(k)]^{1/2} e^{i(f,k,\alpha)} \sum_{m,n=1}^{+\infty} (\mathcal{R} [g_{f,k}^{-1} f g_k])_{mn} \langle \Psi_{k,n} | \Psi_{f-k,m} \rangle.$$

The normalization (10) then allows us to rewrite this as

$$\chi[(f, \alpha)] = \int_{\mathcal{O}_p} d\mu(k) [\rho_f(k)]^{1/2} \delta_\mu(k) e^{i(f,k,\alpha)} \sum_{n=1}^{+\infty} (\mathcal{R} [g_{f,k}^{-1} f g_k])_{nn}$$

$$= \int_{\mathcal{O}_p} d\mu(k) [\rho_f(k)]^{1/2} \delta_\mu(k) e^{i(k,\alpha)} \chi_{\mathcal{R}} [g_k^{-1} f g_k],$$

where $\chi_{\mathcal{R}}$ denotes the character of $\mathcal{R}$. The delta function restricts the integration to the subset of $\mathcal{O}_p$ consisting of points $k$ such that $f \cdot k = k$. In particular, this allows us to set the Radon-Nikodym derivative to one, since $\rho_f(k) = d\mu(f \cdot k)/d\mu(k) = d\mu(k)/d\mu(k) = 1$ on that subset. Our final formula for the character of $\mathcal{T}$ is thus

$$\chi[(f, \alpha)] = \int_{\mathcal{O}_p} d\mu(k) \delta_\mu(k) e^{i(k,\alpha)} \chi_{\mathcal{R}} [g_k^{-1} f g_k].$$

(12)

This is the Frobenius formula for characters of induced representations of $G \rtimes A$ [19].

From now on, we will no longer need to take care of Radon-Nikodym derivatives, since they do not contribute to characters. In fact, we should have expected this simplification: even though different quasi-invariant measures have different Radon-Nikodym derivatives in general, they lead to unitarily equivalent induced representations (cf. the discussion around (7)). Since the characters of equivalent representations are identical, the quantity (12) cannot depend on the measure, and must therefore be independent of its Radon-Nikodym derivative. Note also that (12) is a class function, as it should: it depends only on the conjugacy class of the group element $(f, \alpha)$ at which it is evaluated.

The Frobenius formula (12) states, roughly speaking, that the character $\mathcal{T}$ is a “sum” of characters of $\mathcal{R}$ [17, 18]. More precisely, we may recognize

$$e^{i(k,\alpha)} \chi_{\mathcal{R}} [g_k^{-1} f g_k]$$

(13)

as the character of an irreducible unitary representation of $G_p \rtimes A$, evaluated at $(g_k^{-1} f g_k, \alpha)$. If $g_k^{-1} f g_k$ does not belong to $G_p$, expression (13) as such does not make sense, but the delta function $\delta_\mu(k, f \cdot k)$ ensures that such $k$’s do not contribute to the integral: whenever $k \neq f \cdot k$, the integrand of (12) vanishes. (By contrast, when $k = f \cdot k$, then $g_k^{-1} f g_k = g_{f-k}^{-1} f g_k$ automatically belongs to $G_p$.) More generally, when $f$ is not conjugate to an element of $G_p$, $\chi[(f, \alpha)]$ vanishes.
1.3 Characters of the Poincaré group in three dimensions

The characters of induced representations of the Poincaré group in four dimensions were computed in [19, 26]. Here we apply formula (12) to perform an analogous computation for the Poincaré group in three dimensions. For simplicity, we will focus on the case of a massive particle. Our goal is both to give a concrete illustration of the Frobenius formula and to use this example later, as a guide for the characters of BMS3.

A relativistic massive particle (in three space-time dimensions) is an induced representation of the Poincaré group SL(2, R) ⋊ sl(2, R) based on the orbit of momenta \( q = (q_0, q_1, q_2) \) with positive energy satisfying

\[
-(q_0)^2 + (q_1)^2 + (q_2)^2 \equiv q_\mu q^\mu = -m^2,
\]

where \( m \) is a positive constant. We take the orbit representative to be \( p = (m, 0, 0) \), so the orbit \( \mathcal{O}_p \) is the one-sheeted hyperboloid in momentum space going through \( p \) and defined by (14). The corresponding little group is \( U(1) \), and therefore the spin of the particle is \textit{a priori} an integer. However, the fundamental group of the Poincaré group in three dimensions is \( \mathbb{Z} \) so that projective representations need to be taken into account [27]. Hence the spin may actually take any real value and an appropriate (irreducible, unitary, projective) representation \( \mathcal{R} \) of the little group is given by

\[
\mathcal{R}[\text{rotation by } \theta] = e^{ij\theta},
\]

where \( j \) is any real number. The corresponding space \( \mathcal{E} \) is just \( \mathbb{C} \).

As explained above, the character (12) vanishes whenever \( f \) is not conjugate to an element of the little group. In the present case, this means that \( \chi[(f, \alpha)] = 0 \) whenever \( f \in \text{SL}(2, \mathbb{R}) \) is not conjugate to an element of the \( U(1) \) subgroup of \( \text{SL}(2, \mathbb{R}) \) leaving \( p = (m, 0, 0) \) fixed. Thus, in order to obtain a non-trivial result, we must assume that \( f \) is conjugate to a rotation by some (possibly vanishing) angle \( \text{Rot}(f) \equiv \theta \). In that case, for any \( k \) in \( \mathcal{O}_p \) such that \( f \cdot k = k \), the character \( \chi_{\mathcal{R}} \) appearing in (12) takes the value \( \chi_{\mathcal{R}}[g_f^{-1}fg_k] = e^{ij\theta} \). Other \( k \)'s do not contribute to the integral (12) because of the delta function \( \delta_\mu(k, f \cdot k) \), so we are free to pull \( \chi_{\mathcal{R}} \) out of the integral:

\[
\chi[(f, \alpha)] = e^{ij\theta} \int_{\mathcal{O}_p} d\mu(k) \delta_\mu(k, f \cdot k) e^{i(k, \alpha)}.
\]

To compute the integral as such, we may choose the momentum measure \( \mu \) to be the Lorentz-invariant volume form

\[
d\mu(q) = \frac{d^2q}{\sqrt{m^2 + q^2}} = \frac{dq_1 dq_2}{\sqrt{m^2 + q_1^2 + q_2^2}},
\]

the associated delta function being

\[
\delta_\mu(p, q) = \sqrt{m^2 + q_1^2 + q_2^2} \delta(p_1 - q_1) \delta(p_2 - q_2),
\]

in accordance with the definition (8). The delta functions appearing on the right-hand side are the standard Dirac delta functions on the real line. Plugging these expressions in (15), the prefactors involving \( \sqrt{m^2 + q^2} \) cancel and we find

\[
\chi[(f, \alpha)] = e^{ij\theta} \int_{\mathbb{R}^2} dk_1 dk_2 e^{i(k, \alpha)} \delta(k_1 - [f \cdot k][1]) \delta(k_2 - [f \cdot k][2]),
\]

where
where the subscripts 1 and 2 denote the corresponding spatial components. Since by assumption \( f \) is conjugate to a rotation (by an angle \( \theta \)), and since the character is a class function, we are free to replace \( f \) by a pure rotation \( f_{\theta} \) in this equation:

\[
\chi[(f, \alpha)] = e^{ij\theta} \int_{\mathbb{R}^2} dk_1 dk_2 e^{i(k,\alpha)} \delta(k_1 - k_1 \cos \theta + k_2 \sin \theta) \delta(k_2 - k_1 \sin \theta - k_2 \cos \theta).
\]

At this point we must consider separately two distinct cases:

- If \( \theta \neq 0 \), the only point \( k \) on \( O_p \) such that \( f_{\theta} \cdot k = k \) is \( p = (m, 0, 0) \). The delta function then “localizes” the integral, giving rise to the character

\[
\chi[(f, \alpha)] = e^{ij\theta} e^{im\alpha_0} \left| \det(I - f_{\theta}) \right|^{-1},
\]

where \( \alpha_0 \) denotes the time component of the translation vector \( \alpha \) — the only component that survives after integrating the delta function. The determinant is

\[
\det(I - f_{\theta}) = \begin{vmatrix} 1 - \cos \theta & \sin \theta \\ -\sin \theta & 1 - \cos \theta \end{vmatrix} = 4 \sin^2(\theta/2),
\]

so that

\[
\chi[(f, \alpha)] = e^{ij\theta} e^{im\alpha_0} \frac{1}{4 \sin^2(\theta/2)} \quad \text{if } \text{Rot}(f) = \theta \neq 0.
\]

- If \( f = e \) is the identity, \( \theta = 0 \). In that case, the argument of the delta function in (18) is always zero; we interpret \( \delta^{(2)}(0) \) as the spatial volume \( V \) of the system, up to factors of 2\( \pi \):

\[
\delta(k_1 - k_1) \delta(k_2 - k_2) = \frac{V}{(2\pi)^2}.
\]

We thus find

\[
\chi[(e, \alpha)] = \frac{V}{(2\pi)^2} \int_{\mathbb{R}^2} dk_1 dk_2 e^{i(k,\alpha)}.
\]

We will not evaluate this integral for arbitrary \( \alpha \). In the special case where \( \alpha \) is a pure time translation \( \alpha_t \) (so that \( \alpha_0 \equiv t \) is the only non-vanishing component of \( \alpha \)), we obtain

\[
\chi[(e, \alpha_t)] = \frac{V}{(2\pi)^2} \int_{\mathbb{R}^2} dk_1 dk_2 e^{it\sqrt{m^2 + k_1^2 + k_2^2}} = \frac{V}{2\pi} \int_m^{+\infty} u du e^{i tu}.
\]

Let us take \( t > 0 \) for definiteness. Then, adding a small imaginary part \( i \epsilon \) to \( t \) to ensure convergence, the character of a pure time translation becomes

\[
\chi[(e, \alpha_t)] = \frac{V}{2\pi} \int_m^{+\infty} u du e^{i(t+i\epsilon)u} = -\frac{V}{2\pi t^2} (1 - i\epsilon t) e^{imt}.
\]

So much for the Poincaré group in three dimensions. Before going further, let us stress a point that will be crucial once we turn to BMS3: in the steps leading from (15) to (19), the actual form of the measure \( \mu \) mattered very little. Indeed, the function multiplying \( dq_1 dq_2 \) in (16) cancelled the prefactor of the delta function (17), so the only important information was that the volume form on the orbit must be proportional to \( dq_1 dq_2 \). Any other quasi-invariant momentum measure satisfying this basic requirement would have
given the same result (19) for the character, since the prefactor of \( dq_1 dq_2 \) in that measure would have been cancelled by the inverse prefactor appearing in the corresponding delta function. This cancellation is ensured by the very definition (8) of the delta function \( \delta_\mu \). From a group-theoretic viewpoint, this simplification was to be expected: as already mentioned, two induced representations built using two different quasi-invariant measures on the orbit are (unitarily) equivalent. Since characters of equivalent representations are identical, they cannot depend on the choice of a measure.

2 Characters of the BMS\(_3\) group

We now apply the procedure described above to the (centrally extended) BMS\(_3\) group: we begin by reviewing the structure of BMS\(_3\) (subsection 2.1), before applying the Frobenius formula to the computation of characters associated with massive BMS\(_3\) particles (subsection 2.2). In subsection 2.3, we then establish the relation between these characters and those of highest weight representations of the Virasoro algebra. Finally, subsection 2.4 is devoted to the character of the BMS\(_3\) vacuum representation. For more details on induced representations of BMS\(_3\) and the associated terminology, we refer to [4,5]. As before, our presentation will not be mathematically rigorous. In particular, we will assume that the theory of induced representations, as outlined in subsection 1.1, is applicable to the BMS\(_3\) group even though the latter is infinite-dimensional.

2.1 Induced representations of the BMS\(_3\) group

The BMS\(_3\) group

Let us first collect some background material on the BMS\(_3\) group and its representations. Recall the definition [4]

\[
\text{BMS}_3 \equiv \text{Diff}^+(S^1) \ltimes \text{Ad} \text{Vect}(S^1)_{ab}
\]

"superrotations" "supertranslations"

where \( \text{Diff}^+(S^1) \) denotes the group of (orientation-preserving) diffeomorphisms of the circle while \( \text{Vect}(S^1)_{ab} \) denotes the Abelian additive group of vector fields on the circle. The same structure remains valid in the centrally extended case, with \( \text{Diff}^+(S^1) \) and \( \text{Vect}(S^1)_{ab} \) replaced by the Virasoro group \( \hat{\text{Diff}}^+(S^1) \) and its algebra, \( \hat{\text{Vect}}(S^1) \); their semi-direct product is the centrally extended BMS\(_3\) group. In the following pages we will use the same notation for the BMS\(_3\) group and its central extension, as the context should make it clear enough which group we are dealing with.

The dual of the group of supertranslations (i.e. the space \( A^* \) in the notations of subsection 1.1) is the dual space of the Virasoro algebra and consists of supermomentum vectors representing an infinite-dimensional generalization of the usual Poincaré momentum. In practice, in terms of a 2\(\pi\)-periodic angular coordinate \( \varphi \) on the circle, a supermomentum \( p \) is a quadratic density \( p(\varphi) d\varphi^2 \) paired with supertranslations according to\(^3\)

\[
\langle p, \alpha \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\varphi p(\varphi) \alpha(\varphi) \quad \forall \alpha(\varphi) d\varphi \in \text{Vect}(S^1)_{ab}.
\]  

\(^3\)The definition (24) differs by a factor of 2\(\pi\) from the convention used in [4,5]. This ensures that the zeroth Fourier mode of \( p(\varphi) \) coincides with the associated energy.
Under the action of superrotations, supermomenta transform according to the coadjoint representation of the Virasoro group. Explicitly, if $f : \varphi \mapsto f(\varphi)$ is a diffeomorphism of the circle, this action is given by
\[
(f \cdot p)|_{f(\varphi)} = \frac{1}{(f'(\varphi))^2} \left[ p(\varphi) + \frac{c_2}{12} S[f](\varphi) \right].
\] (25)

Here the prime denotes differentiation with respect to $\varphi$, while
\[
S[f](\varphi) = \frac{f'''(\varphi)}{f'(\varphi)} - \frac{3}{2} \left( \frac{f''(\varphi)}{f'(\varphi)} \right)^2
\]
is the Schwarzian derivative of $f$ at $\varphi$, and $c_2$ is the (dual of the) central charge pairing generators of superrotations with generators of supertranslations in the $\text{bms}_3$ algebra (see expression (39) below). Upon taking an infinitesimal diffeomorphism $f(\varphi) = \varphi - \varepsilon X(\varphi)$, where the vector field $X(\varphi) \frac{d}{d\varphi}$ should be understood as an element of the Lie algebra of $\text{Diff}^+(S^1)$, the transformation law (25) yields (to first order in $\varepsilon$)
\[
(f \cdot p)(\varphi) - p(\varphi) \equiv \varepsilon \delta_X p(\varphi) \quad \text{with} \quad \delta_X p = Xp' + 2X'p - \frac{c_2}{12} X'''.
\] (26)

In the context of three-dimensional asymptotically flat Einstein gravity, $p(\varphi)$ is the Bondi mass aspect [5] transforming under asymptotic superrotations according to (25), and $c_2$ takes the value $3/G$, where $G$ denotes Newton’s constant [2]. As the subscript “2” indicates, there is also a central charge $c_1$, pairing superrotations with themselves as in the usual Virasoro algebra. However, $c_1$ plays virtually no role for induced representations (except for those based on the orbit of $p = 0$ and $c_2 = 0$) and vanishes in Einstein gravity, so $c_1$ will almost never appear in what follows.

The $\text{BMS}_3$ group being a semi-direct product, its (irreducible) unitary representations are expected to be induced from those of its little groups, in the sense of subsection 1.1. The relevant orbits $O_p$ then are coadjoint orbits of the Virasoro group [12–15]. The little group of a generic orbit is Abelian so the corresponding internal space $\mathcal{E}$ is simply $\mathbb{C}$, and the Hilbert space $\mathcal{H}$ of the associated induced representations should be a space of complex-valued, square-integrable wavefunctionals on the orbit. (We say “wavefunctionals” instead of “wavefunctions” to stress the fact that the orbits are, in this case, spaces of functions.) But in order to define what “square-integrable” means, we need a supermomentum measure on the orbit.

**The question of the measure**

The problem of defining quasi-invariant measures on Virasoro coadjoint orbits is well known; see e.g. [28–30]. In that context, a result that is especially relevant for our purposes is the theorem due to Shavgulidze [31–34] (see also [35,36]) which states that the group of $C^k$ diffeomorphisms of any compact manifold can be endowed with a Borel measure that is quasi-invariant under the action of $C^{k+\ell}$ diffeomorphisms (here $k$ and $\ell$ are any two positive integers). This suggests that there exist quasi-invariant measures on $\text{Diff}^+(S^1)$, which in turn should provide quasi-invariant measures on a Virasoro coadjoint...
orbit \( \mathcal{O}_p \) with compact little group; indeed, if \( \bar{\mu} \) is a quasi-invariant measure on \( \text{Diff}^+(S^1) \) and if \( \pi: \text{Diff}^+(S^1) \to \mathcal{O}_p \) is the natural projection, we can define a quasi-invariant measure \( \mu \) on \( \mathcal{O}_p \) by \( \mu(A) \equiv \bar{\mu}(\pi^{-1}(A)) \) for any measurable subset \( A \subseteq \mathcal{O}_p \). When the little group is non-compact, however, this naive procedure may break down.

We will not dwell on such questions here and we will not attempt to make our considerations mathematically precise. Instead, our point of view will be a practical one: the supermomentum measures that we need are functional integral measures on certain (Fréchet) manifolds consisting of functions on the circle. Such measures are encountered on a daily basis in quantum mechanics, statistical physics and field theory. Provided one is willing to define Hilbert spaces of square-integrable functions with the help of functional measures, their use in induced representations of \( \text{BMS}_3 \) is no more controversial than in quantum physics. Our hope is that techniques similar to those that led to a rigorous construction of path integral measures can be adapted to quasi-invariant measures on Virasoro coadjoint orbits. In the following pages, we will rely on this assumption to justify the use of functional integrals when computing characters.

### 2.2 Characters of massive \( \text{BMS}_3 \) particles

In this subsection we consider the induced representation based on the orbit \( \mathcal{O}_p \) of a constant supermomentum \( p = m - c^2/24 \) (with \( c^2 > 0 \)), representing a \( \text{BMS}_3 \) particle with rest mass \( m > 0 \) normalized with respect to the vacuum. The condition \( m > 0 \) ensures that energy is bounded from below on the orbit \( [14] \). We will denote by \( j \in \mathbb{R} \) the spin of the (projective) representation of the corresponding little group \( \text{U}(1) \) of rigid rotations of the circle. Our goal is to compute the character of this representation using the Frobenius formula \((12)\).

Let us pick \( (f, \alpha) \in \text{BMS}_3 \) and call \( \chi_{m,j}[(f, \alpha)] \) the corresponding character. As explained earlier, the latter vanishes if \( f \) is not conjugate to an element of the little group. To obtain a non-trivial result in the present case, we must therefore assume that \( f \) is conjugate to a pure rotation, the angle of which is given by the Poincaré rotation number of \( f \) \([15]\):

\[
\text{Rot}(f) = \lim_{n \to +\infty} \frac{f^n(\varphi) - \varphi}{n} \equiv \theta.
\]

Here \( \varphi \in [0, 2\pi) \) is arbitrary and \( f \) is seen as a diffeomorphism of \( \mathbb{R} \) satisfying \( f(\varphi + 2\pi) = f(\varphi) + 2\pi \), with \( f^n \equiv f \circ f \circ \ldots \circ f \). As in subsection 1.3, the character \( \chi_{\mathcal{R}} \) of the little group can then be pulled out of the integral \((12)\), producing an overall constant factor \( e^{i\theta} \). The character \( \chi_{m,j}[(f, \alpha)] \) thus reduces to \((15)\), except that now the integral is taken over the (infinitesimal) Virasoro coadjoint orbit \( \mathcal{O}_p \) and that \( \langle k, \alpha \rangle \) denotes the pairing \((24)\) rather than a finite-dimensional scalar product. Since the character is a class function, we are free to replace \( f \) by a pure rotation \( f_\theta \) inside the integral. Depending on whether \( \theta \) vanishes or not, we are then faced with two completely different problems.

If \( \theta = 0 \) so that \( f \) is actually the identity, the term \( \delta_{\mu}(k, f \cdot k) = \delta_{\mu}(k, k) \) leads to an infrared divergence analogous to \((21)\). Assuming that we have regularized this divergence somehow, we are left with the task of computing a genuine path integral — we must integrate the functional \( \exp[i\langle k, \alpha \rangle] \) over all quadratic densities \( k \) belonging to \( \mathcal{O}_p \). We
shall not attempt to perform this computation here, only briefly returning to this issue in the conclusion of this work.

A radically different situation occurs if $\theta \neq 0$. In that case, the delta function $\delta_\mu(k, f_\theta \cdot k)$ localizes the integral to the only point on $O_p$ that is invariant under rotations, namely the constant supermomentum $p = m - c_2/24$. This allows us to replace $k$ by $m - c_2/24$ in the term $e^{i(k, \alpha)}$ of eq. (15), which gives

$$\langle k, \alpha \rangle = \langle p, \alpha \rangle^{(24)} = \alpha^0(m - c_2/24)$$

where $\alpha^0$ denotes the zeroth Fourier mode of the supertranslation $\alpha(\varphi) = \sum_{n \in \mathbb{Z}} \alpha^ne^{-in\varphi}$. None of the higher Fourier modes of $\alpha$ contribute to the character. We can thus pull the exponential out of the integral in (15) and the character reduces to

$$\chi_{m,j}[\langle f, \alpha \rangle] = e^{ij\theta}e^{i\alpha^0(m-c_2/24)} \int_{O_p} d\mu(k)\delta_\mu(k, f_\theta \cdot k). \quad (28)$$

To integrate the delta function, we need to find coordinates on $O_p$. As a first step, note that each supermomentum $k(\varphi)$ can be expanded in Fourier series as

$$k(\varphi) = \sum_{n \in \mathbb{Z}} k_n e^{-in\varphi}, \quad (29)$$

with $(k_n)^* = k_{-n}$. Then, according to the transformation law (25), the action of a rotation $f_\theta(\varphi) = \varphi + \theta$ on $k$ is simply $(f_\theta \cdot k)|_{\varphi} = k(\varphi - \theta)$. In terms of Fourier modes, this corresponds to

$$k_n \mapsto [f_\theta \cdot k]_n = k_n e^{in\theta}.$$

We will soon see that the character obtained using this transformation is divergent, as we might expect since the group is infinite-dimensional. To regularize the result, we will therefore consider complex rotations instead of real ones: let

$$\tau \equiv \frac{1}{2\pi}(\theta + i\epsilon) \quad (30)$$

be a complex parameter with $\epsilon > 0$, and define the transformation of the supermomentum $k$ under a rotation by $2\pi\tau$ to be

$$k_n \mapsto [f_{2\pi\tau} \cdot k]_n = \begin{cases} k_n e^{2\pi in\tau} & \text{if } n > 0, \\ k_0 & \text{if } n = 0, \\ k_n e^{2\pi in\tau} & \text{if } n < 0. \end{cases} \quad (31)$$

We will show below that this seemingly ad hoc modification is related to thermodynamical considerations and to the fact that the BMS$_3$ group is a “high-energy” limit of two Virasoro groups. Note also that this prescription allows for “Euclidean” rotations (that is, rotations through an imaginary angle) while preserving the reality condition $(k_n)^* = k_{-n}$.

The question now is how to express the measure $\mu$ and the delta function $\delta_\mu$ in terms of modes $k_n$. On the orbit $O_p$, diffeomorphic to $\text{Diff}^+(S^1)/S^1$, each supermomentum $k$ is uniquely determined by its non-zero Fourier modes; in other words, the non-zero Fourier modes of $k$ determine its zero-mode, given that $k$ belongs to $O_p$. An easy way
to see this locally, in a neighbourhood of \( p \), is to consider the action (26) of infinitesimal superrotations on \( p \) [13]. Upon expanding the vector field \( X(\varphi)\frac{d}{d\varphi} \) as a Fourier series

\[
X(\varphi) = \sum_{n \in \mathbb{Z}} X_n e^{-i n\varphi}
\]

and writing \( p = m - c_2/24 \), this action becomes

\[
(\delta_X p)(\varphi) = \sum_{n \in \mathbb{Z}} 2n \left( m + \frac{c_2}{24}(n^2 - 1) \right) X_n e^{-i n\varphi} \equiv \sum_{n \in \mathbb{Z}} \delta p_n e^{-i n\varphi}.
\]

(32)

In this expression, the zero-mode \( \delta p_0 \) always vanishes, regardless of \( X(\varphi) \). By contrast, all other Fourier modes \( \delta p_n \) can be made non-zero by a suitable choice of \( X(\varphi) \). Thus, at least in a neighbourhood of \( p \), we may choose the non-zero Fourier modes of supermomenta as coordinates on \( \mathcal{O}_p \). In the notation of (26) and (29), when \( k \) is close to \( p \) so that \( k(\varphi) = p + \varepsilon(\delta_X p)(\varphi) \), the modes \( k_n \) with non-zero \( n \) reduce to \( \varepsilon \delta p_n \) (while \( k_0 \) reduces to \( m - c_2/24 \) to first order in \( \varepsilon \)). Thus, in terms of \( k_n \)'s, the supermomentum measure \( \mu \) in (28) must take the form

\[
d\mu(k) = \text{(Some } k\text{-dependent prefactor)} \times \prod_{n \in \mathbb{Z}^*} dk_n,
\]

(33)

where the prefactor is generally unknown. In the standard notation of quantum mechanics, the infinite product \( \prod_{n \in \mathbb{Z}^*} dk_n \) would be written as a path integral measure \( Dk \), being understood that the zero-mode of \( k \) must not be integrated over. By virtue of the definition (8), the delta function corresponding to \( \mu \) reads

\[
\delta_\mu(q, k) = \text{(Some } k\text{-dependent prefactor)}^{-1} \times \prod_{n \in \mathbb{Z}^*} \delta(q_n - k_n),
\]

where the \( \delta \)'s on the right-hand side are the usual one-dimensional Dirac delta functions. Thus, the prefactors of \( d\mu \) and \( \delta_\mu \) cancel out and expression (28) boils down to

\[
\chi_{m,j}[(f, \alpha)] = e^{ij\theta} e^{ia\theta(m-c_2/24)} \int_{\mathbb{R}^2} \prod_{n \in \mathbb{Z}^*} dk_n \prod_{n \in \mathbb{Z}^*} \delta(k_n - [f_{2\pi\tau} \cdot k]_n) \equiv (31) e^{ij\theta} e^{ia\theta(m-c_2/24)} \int_{\mathbb{R}^2} \prod_{n=1}^\infty dk_n \prod_{n=1}^\infty \delta(k_n(1 - e^{2\pi in\tau}))^2,
\]

(34)

where we replaced the real angle \( \theta \) by its complex counterpart \( 2\pi\tau \), defined by (30). Writing \( q \equiv \exp[2\pi i\tau] \) and evaluating the integral, the character of a massive BMS\(_3\) particle finally reduces to

\[
\chi_{m,j}[(f, \alpha)] = e^{ij\theta} e^{ia\theta(m-c_2/24)} \frac{1}{\prod_{n=1}^\infty |1 - q^n|^2} \quad \text{if } \text{Rot}(f) = \theta \neq 0.
\]

(35)

This can be rewritten as

\[
\chi_{m,j}[(f, \alpha)] = \left|q^{1/12}/|\eta(\tau)|^2\right| e^{ij\theta} e^{ia\theta(m-c_2/24)}
\]
in terms of the Dedekind Eta function

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{+\infty} (1 - q^n).$$

We stress that, at this stage, and in contrast to standard conformal field theory, the number \(\tau\) should not be interpreted as a modular parameter. The small parameter \(\epsilon\) in (30) was merely introduced to ensure convergence of the determinant arising from the integration of the delta function in (34). This being said, the occurrence of the Eta function is compatible with the modular transformations used in [37, 38] to compute the entropy of cosmological solutions.

2.3 Comparison to Poincaré and Virasoro characters

From Poincaré to BMS$_3$

Expression (35) is a natural extension of the Poincaré character (20). Indeed, taking \(\epsilon = 0\) in (35) and being careless about convergence issues, we find

$$\chi_{m,j}[\{f, \alpha\}] = e^{ij\theta} e^{ia0} \prod_{n=1}^{+\infty} \frac{1}{4 \sin^2(n\theta/2)}.$$  

Up to the normalization of energy, the term \(n = 1\) exactly reproduces (20), while the contribution of higher Fourier modes can be loosely interpreted as coming from the infinitely many Poincaré subgroups of BMS$_3$.

This phenomenon is analogous to the relation between SL($2, \mathbb{R}$) and Virasoro. Indeed, the character of a rotation \(f_{2\pi\tau}\) through a complex angle \(2\pi\tau\) with positive imaginary part in a highest weight representation of the sl($2, \mathbb{R}$) algebra is

$$\chi_h[f_{2\pi\tau}] = \text{Tr} (q^{L_0}) = q^h (1 + q + q^2 + \cdots) = \frac{q^h}{1 - q} \quad (36)$$

where \(q \equiv e^{2\pi i \tau}\) and where \(L_0\) denotes the generator of rotations in sl($2, \mathbb{R}$). The term \(q^h\) comes from the highest weight state (with weight \(h\)), while \((1 - q)^{-1}\) is the contribution of its sl($2, \mathbb{R}$) descendants. This should be compared to the character of a highest weight representation of the Virasoro algebra at central charge \(c > 1\),

$$\chi_h[f_{2\pi\tau}] = \text{Tr} (q^{L_0 - c/24}) = \frac{q^{h-c/24}}{\prod_{n=1}^{+\infty} (1 - q^n)}, \quad (37)$$

which is obviously a generalization of (36), including a contribution from higher Fourier modes reminiscent of the infinitely many SL($2, \mathbb{R}$) subgroups of the Virasoro group. Again, the denominator is interpreted as the contribution of descendant states.

BMS$_3$ characters as a flat limit of Virasoro characters

The divergence of the BMS$_3$ character (35) as \(\epsilon \rightarrow 0\) is identical to the divergence of the Virasoro character (37) as \(\tau\) becomes real. In this sense, the divergence of the BMS$_3$ character is not a pathology of the BMS$_3$ group, but rather a general behaviour we should
expect from any infinite-dimensional group; the divergence is cured by adding an imaginary part $i\epsilon$ to the rotation angle. The origin of this imaginary part can be traced back to the fact that the BMS$_3$ group is a flat/ultrarelativistic limit of two Virasoro groups [39, 40], as follows.

If we denote by $L_m$ and $\bar{L}_m$ the generators of two commuting copies of the Virasoro algebra with central charges $c$ and $\bar{c}$ and if $\ell$ is some length scale, one can define

$$ P_n \equiv 1/\ell \left( L_n + \bar{L}_{-n} \right), \quad J_n \equiv L_n - \bar{L}_{-n}, \quad c_1 \equiv c - \bar{c} \quad \text{and} \quad c_2 \equiv c + \bar{c}. \quad (38) $$

In the limit $\ell \to +\infty$, and provided $c_1$ and $c_2$ are finite in that limit, these generators span a centrally extended BMS$_3$ algebra:

$$ [J_n, J_m] = (n-m)J_{n+m} + \frac{c_1}{12} n^3 \delta_{n+m,0}, \quad [J_n, P_m] = (n-m)P_{n+m} + \frac{c_2}{12} n^3 \delta_{n+m,0}, \quad [P_n, P_m] = 0. \quad (39) $$

In the context of three-dimensional Einstein gravity on AdS with Brown-Henneaux boundary conditions [41], the $L_n$’s and the $\bar{L}_n$’s are surface charges associated with asymptotic symmetries. The parameter $\ell$ is related to the negative cosmological constant $\Lambda = -1/\ell^2$, and $c = \bar{c} = 3\ell/2G$. The regime of large $\ell$ then corresponds to the “flat limit” of AdS, and the BMS$_3$ central charges are finite since $c_2 = 3/G$ and $c_1 = 0$.

Now, the character of a highest weight representation of two copies of the Virasoro algebra, labelled by the (strictly positive) highest weights $h$, $\bar{h}$ and the central charges $c > 1$ and $\bar{c} > 1$, is

$$ \text{Tr} \left( q^{L_0-c/24} \bar{q}^{\bar{L}_0-\bar{c}/24} \right) = \frac{\theta \bar{\theta}}{2\pi} e^{2\pi i \tau} \prod_{n=1}^{+\infty} |1 - q^n|^2, \quad q = e^{2\pi i \tau}. \quad (40) $$

Let us see how we can recover the BMS$_3$ character (35) as a flat limit of this expression. First, we write the modular parameter as $\tau = \frac{1}{2\pi} (\theta + i\beta/\ell)$, where $\beta > 0$ is $\ell$-independent. Provided the highest weights $h$ and $\bar{h}$ scale with $\ell$ in such a way that the numbers

$$ m \equiv \lim_{\ell \to +\infty} \frac{1}{\ell} (h + \bar{h}) \quad \text{and} \quad j \equiv \lim_{\ell \to +\infty} (h - \bar{h}) - \frac{c_1}{24} \quad (41) $$

be finite, the large $\ell$ limit of the quantities appearing on the right-hand side of (40) is

$$ \tau \sim \frac{1}{2\pi} (\theta + i\epsilon), \quad q^{h-c/24} \bar{q}^{\bar{h}-\bar{c}/24} \sim e^{ij\theta} e^{-\beta(m-c_2/24)}. $$

(Here the imaginary part of $\tau$ goes to zero, but we keep writing it as $\epsilon > 0$ to reproduce the regularization used in (35).) We conclude that the flat limit of (40) is

$$ \lim_{\ell \to +\infty} \text{Tr} \left( q^{L_0-c/24} \bar{q}^{\bar{L}_0-\bar{c}/24} \right) = e^{ij\theta} e^{-\beta(m-c_2/24)} \frac{1}{\prod_{n=1}^{+\infty} |1 - q^n|^2}, $$

which coincides with the BMS$_3$ character (35) provided we consider a supertranslation whose zero-mode is a Euclidean time translation, $\alpha = i\beta$. The left-hand side of this expression can be interpreted as

$$ \lim_{\ell \to +\infty} \text{Tr} \left( q^{L_0-c/24} \bar{q}^{\bar{L}_0-\bar{c}/24} \right) \overset{(38)}{=} \text{Tr} \left( e^{i\theta J_0} e^{-\beta(P_0-c_2/24)} \right) = \chi[(f_0, \alpha)], $$

14
where $f_\theta$ denotes a rotation by $\theta$. In this form, the matching between the flat limit of the Virasoro character (37) and the BMS$\,_3$ character (35) is obvious.

By the way, the definition (41) explains why induced representations of BMS$\,_3$ can roughly be seen as an ultrarelativistic/high-energy limit of Virasoro highest weight representations: keeping $m$ finite and non-zero as $\ell$ goes to infinity requires $\hbar$ and/or $\bar{\hbar}$ to grow linearly with $\ell$, meaning that $\hbar + \bar{\hbar}$ must take an infinitely large value in the limit $\ell \to +\infty$. This observation, together with (38), also gives us an intuitive picture of why the energy spectrum of BMS$\,_3$ particles is continuous: the typical distance between two consecutive eigenvalues of $P_0 = (L_0 + \bar{L}_0)/\ell$ is $1/\ell$, which shrinks to zero when $\ell$ goes to infinity. Thus, the only way for $P_0$ to have more than one eigenvalue is that its spectrum be continuous. This is by no means a pathology, as the same situation occurs with the energy spectrum $q_0 = \sqrt{m^2 + q^2}$ in induced representations of the Poincaré group.

Further comments

Even though BMS$\,_3$ and Virasoro characters are related through the flat/ultrarelativistic limit just described, they are strikingly different in many respects. Indeed, the result (35) holds for any (positive) value of the central charge $c_2$, any (positive) value of the mass $m$, and any spin $j$. By contrast, the character of an irreducible, unitary highest weight representation of the Virasoro algebra, labelled by the values of $c$ and $\hbar$, depends heavily on those values: when $c < 1$, only certain discrete values of $c$ and $\hbar$ lead to unitary representations [20, 21], and the corresponding character is strikingly different from (37) [22–24]. From that viewpoint, induced representations of the BMS$\,_3$ group are less intricate than highest weight representations of the Virasoro algebra. Since the former are high-energy, high central charge limits of the latter, this seems reasonable: all complications occurring at small $c$ vanish when $\ell$ goes to infinity, since $c$ scales linearly with $\ell$ by assumption.

We could have guessed that such a simplification would occur on the basis of dimensional arguments. Indeed, the parameters labelling induced representations of BMS$\,_3$ are dimensionful: both $m$ and $c_2$ have dimension of mass. This is manifest in the flat limit relations (38) and (41), where the mass dimension arises due to the length scale $\ell$. Alternatively, it may be seen as a consequence of the fact that supermomenta are an infinite-dimensional generalization of momenta, hence naturally carrying a dimension of mass. Thus the values of $m$ and $c_2$ can be tuned at will by a suitable choice of units, and, in contrast to Virasoro highest weight representations, we should not expect to find sharp bifurcations in the structure of BMS$\,_3$ induced representations as $m$ and $c_2$ vary. In this sense, the character (35) is universal in the region of positive $m$ and $c_2$.

2.4 Character of the BMS$\,_3$ vacuum representation

We now turn to the character of the induced representation with vanishing spin based on the orbit of the vacuum, $p = -c_2/24$. The computation is very similar to that of subsection 2.2, save for the fact that the little group is the Lorentz group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\mathbb{Z}_2$ rather than $\text{U}(1)$. In particular, the orbit $O_{\text{vac}}$ of $p$ is diffeomorphic to the universal Teichmüller space $\text{Diff}^+(S^1)/\text{PSL}(2, \mathbb{R})$.

As before, the quantity we wish to compute is $\chi[(f, \alpha)]$, where $\alpha$ is any supertranslation. Since the little group here is larger than $\text{U}(1)$, we can obtain a non-trivial character
even when \( f \) is not conjugate to a rotation — for example if \( f \) is a boost in \( \text{PSL}(2, \mathbb{R}) \). Although this computation may be interesting for physics, we will not consider it here and we will focus on the case in which \( f \) is conjugate to a rotation \( f_0 \). We will also assume, as in subsection 2.2, that the rotation number \( \theta \) of \( f \), given by (27), is non-zero. The delta function in the Frobenius formula (12) then forces the integral to pick its only non-zero contribution from the unique rotation-invariant point on the orbit \( O_{\text{vac}} \), that is, the point \( p = -c_2/24 \). The character can thus be written in a form analogous to (28),

\[
\chi_{\text{vac}}[(f, \alpha)] = e^{-i\alpha^0 c_2/24} \int_{O_{\text{vac}}} d\mu(k) \delta_\mu(k, f_0 \cdot k), \tag{42}
\]

where \( \mu \) is now a quasi-invariant measure on \( O_{\text{vac}} \cong \text{Diff}^+(S^1)/\text{PSL}(2, \mathbb{R}) \). We can of course expand any supermomentum \( k \) belonging to \( O_{\text{vac}} \) as a Fourier series (29), and we define the transformation of Fourier modes under complex rotations to be (31). The subtlety now is that we must understand which Fourier modes appear in the measure, or equivalently which modes can be used as coordinates on the orbit.

Fortunately, we only need local coordinates in a neighbourhood of the rotation-invariant point, \( p = -c_2/24 \). We can thus rely again on an argument based on the action (32) of infinitesimal superrotations on constant supermomenta. Taking \( m = 0 \) in that expression, we now find that not only the zero-mode \( \delta p_0 \), but also the modes \( \delta p_{\pm 1} \), vanish for all choices of \( X(\phi) \). This means that, in a neighbourhood of \( p = -c_2/24 \), we can endow the orbit \( O_{\text{vac}} \) with coordinates given by the higher Fourier modes of \( k(\phi) \), that is, the modes \( k_{\pm 2}, k_{\pm 3}, \text{etc.} \) In particular, in this neighbourhood of the rotation-invariant point, the measure \( \mu \) on \( O_{\text{vac}} \) takes the form

\[
d\mu(k) = (\text{Some } k\text{-dependent prefactor}) \times \prod_{n=2}^{+\infty} dk_n dk_{-n},
\]

where, as in (33), the prefactor is unknown. Once more, the definition (8) of the corresponding delta function ensures that this prefactor cancels with its inverse appearing in \( \delta_\mu \), which reduces the vacuum character (42) to

\[
\chi_{\text{vac}}[(f, \alpha)] = e^{-i\alpha^0 c_2/24} \int_{\mathbb{R}^{2\infty}} \prod_{n=2}^{+\infty} dk_n dk_{-n} \prod_{n=2}^{+\infty} \delta(k_n - [f_{2\pi \tau} \cdot k]_n) \delta(k_{-n} - [f_{2\pi \tau} \cdot k]_{-n})
\]

\[
\overset{(31)}{=} e^{-i\alpha^0 c_2/24} \left| \int_{\mathbb{R}^{\infty}} \prod_{n=2}^{+\infty} dk_n \prod_{n=2}^{+\infty} \delta (k_n (1 - q^n)) \right|^2,
\]

where \( q \equiv \exp[2\pi i \tau] \) and \( \tau = (\theta + i\epsilon)/2\pi \). Integrating the delta functions and taking into account the determinant, we end up with

\[
\chi_{\text{vac}}[(f, \alpha)] = e^{-i\alpha^0 c_2/24} \frac{1}{\prod_{n=2}^{+\infty} [1 - q^n]^2} \quad \text{when } \text{Rot}(f) = \theta \neq 0. \tag{43}
\]

As before, this expression can be interpreted as the trace

\[
\text{Tr} \left( e^{i\theta J_0} e^{i\alpha^0 (P_0 - c_2/24)} \right)
\]

in the vacuum representation of BMS\(_3\). It can also be recovered as a flat limit of the product of two vacuum Virasoro characters. In that context, the truncated product \( \prod_{n=2}^{+\infty}(\ldots) \) arises due to \( \text{SL}(2, \mathbb{R}) \)-invariance of the Virasoro vacuum.
Conclusion and outlook

In this work we have shown that the standard Frobenius formula (12) for characters of induced representations can be applied to the BMS$_3$ group. Our key results were formulas (35) and (43), which can also be seen as flat limits of the corresponding Virasoro characters. A crucial step during the computation was the use of local coordinates on Virasoro orbits, which allowed us to write down the integration measure up to an unknown prefactor. The latter eventually turned out to be irrelevant, as it was cancelled by the prefactor of the delta function.

One important case is still missing in our considerations. Indeed, in subsection 2.2, we restricted our attention to superrotations conjugate to non-trivial rigid rotations of the circle. In doing so, we left aside the computation of BMS$_3$ characters associated with pure supertranslations. As already pointed out, such characters are necessarily infrared-divergent, as is the Poincaré character (23). If we assume that the BMS$_3$ version of this divergence can be put under control, and if we focus on pure time translations, we are left with the task of computing the BMS$_3$ analogue of the integral in (22), which is schematically of the form

$$\int_{O_p} \mathcal{D}k \exp\left[ itE[k]\right] \quad \text{or} \quad \int_{O_p} \mathcal{D}k \exp\left[ - \beta E[k]\right]. \quad (44)$$

Here $E[k]$ denotes the energy functional on $O_p$, that is, the zero-mode of $k(\varphi)$ when $k$ belongs to $O_p$. (The second expression in (44) is the Euclidean version of the first one.) We did not attempt to evaluate this integral here, but hope to address this question in the future.

Many other open issues were left aside in the present paper. One of these is the relation of BMS$_3$ characters to one-loop partition functions of three-dimensional asymptotically flat gravity, which is analogous to the matching between Virasoro characters and partition functions on AdS$_3$ [42]. Indeed, it is easily verified that the flat limit of the heat kernel computation of [42] precisely reproduces (43) as the one-loop partition function of gravitons around thermal flat space (provided we identify $\alpha^0 = i\beta$). We will turn to a thorough investigation of this relation elsewhere. Independently of this, several natural extensions of our considerations are available: one may imagine computing characters of certain higher-spin extensions of BMS$_3$ [43–45] or of the supersymmetric BMS$_3$ group [46,47]. In all those cases, the semi-direct product structure of BMS$_3$ should be essential in determining the appropriate unitary representations and the associated character formula.

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