Kähler-Einstein Metrics and Eigenvalue Gaps

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Abstract

The existence of Kähler-Einstein metrics on a Fano manifold is characterized in terms of a uniform gap between 0 and the first positive eigenvalue of the Cauchy-Riemann operator on smooth vector fields. It is also characterized by a similar gap between 0 and the first positive eigenvalue for Hamiltonian vector fields. The underlying tool is a compactness criteria for suitably bounded subsets of the space of Kähler potentials which implies a positive gap.

1 Introduction

Starting with the works of Calabi [4] and Yau [29], a central problem in Kähler geometry has been determining when a complex manifold admits a constant scalar curvature Kähler metric in a given Kähler class. One of the first obstructions to the existence of a cscK Kähler metric is the vanishing of the Futaki invariant, which is a character defined on the Lie algebra of holomorphic vector fields. The Yau-Tian-Donaldson conjecture [30, 27, 13] (see also [24] for a review) asserts that the existence of a Kähler metric with constant scalar curvature should be equivalent to the algebro-geometric notion of K-stability. Two recent major advances on this conjecture have been the solution of X.X. Chen, S. Donaldson, and S. Sun [8, 9, 10] of the case of Kähler-Einstein metrics on Fano manifolds, and the more recent works by X.X. Chen and J.R. Cheng [5, 6, 7] which established the equivalence between the existence of a Kähler metric with constant scalar curvature and an analytic notion of K-stability.

The K-stability condition of a Kähler class is the requirement that the generalized Futaki invariant attached to a test configuration be non-negative, and vanish only if the test configuration is a product. It is only one possible characterization of the existence of a canonical metric, and for both geometric and analytic reasons, it may be useful to have other characterizations as well. In the case of Kähler-Einstein metrics, which are the focus of the present paper, a notion of $\delta$-invariant has been proposed by Fujita-Odaka [14] and Blum and Jonsson [3], and it has been shown by R. Berman, S. Boucksom, and M. Jonsson [1] that the existence of a Kähler-Einstein metric is equivalent to the $\delta$-invariant being greater or equal to 1. In a different and even earlier direction, it had been shown in [24, 22, 32] that the Kähler-Ricci flow converges if the lowest strictly positive eigenvalue of the $\bar{\partial}$ operator on vector fields remains bounded uniformly away from 0 along the flow. It was suggested there

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that it may be possible to characterize the existence of a Kähler-Einstein metric in terms of lower bounds for this eigenvalue, and this is the problem which we solve in the present paper.

More precisely, let $X$ be a compact Kähler manifold with $c_1(X) > 0$. Fix a reference metric $\omega_0 \in c_1(X)$. For any $\omega \in c_1(X)$, let $K^{\omega_0}(\omega)$ be the $K$-energy of $\omega$ with respect to the reference metric $\omega_0 \in c_1(X)$, and $u_\omega$ be the normalized Ricci potential of $\omega$, as defined in \cite{21} below. We define $\lambda_\omega$ to be the lowest strictly positive eigenvalue of the $\bar{\partial}$ operator on vector fields, i.e.,

$$
\lambda_\omega = \inf_{V \in T^{1,0}(X), V \perp \omega, H^{0}(X, T^{1,0})} \frac{||\bar{\partial}V||^2_\omega}{||V||^2_\omega} \tag{1.1}
$$

where the subindex denotes the $L^2$ norms taken with respect to the metric $\omega$, and $\perp \omega$ indicates the perpendicularity condition with respect to $\omega$. Let $R_\omega$ be the scalar curvature of $\omega$. For each $A > 0$, we introduce the following subset of the space of Kähler metrics in $c_1(X)$,

$$
c_1(X; A) = \{ \omega \in c_1(X); ||u_\omega||_{C^0} + ||\nabla_\omega u_\omega||_{C^0} + ||R_\omega||_{C^0} \leq A, K^{\omega_0}(\omega) \leq A \}, \tag{1.2}
$$

and the corresponding eigenvalue gap for the set $c_1(X; A)$ by

$$
\lambda(X; A) = \inf_{\omega \in c_1(X; A)} \lambda_\omega. \tag{1.3}
$$

Then we have the following characterizations of the existence of a Kähler-Einstein metric:

**Theorem 1** Let $X$ be a compact Kähler manifold with $c_1(X) > 0$ and vanishing Futaki invariant. Then $X$ admits a Kähler-Einstein metric if and only if $\lambda(X, A) > 0$ for any $A > 0$.

Note that although the definition of the $K$-energy requires a choice of reference metric $\omega_0$, under a change of reference metric, it just shifts by a constant. Thus the above condition is invariant under a change of reference metric, as it should be.

To explain the second characterization, we recall the following observations due by Futaki \cite{15} (see also \cite{22}, Lemma 2). For any metric $\omega \in c_1(X)$, the differential operator operating on smooth functions $L_\omega f = -g^{\bar{i}\bar{j}} \partial_{\bar{i}} \partial_{\bar{j}} f + g^{\bar{i}\bar{j}} \partial_i u_\omega \partial_j f - f$ is non-negative, and its kernel is the space of functions $f$ with $\nabla f$ a holomorphic vector field. Let $\mu_\omega$ be the smallest positive eigenvalue of $L_\omega$. Then the corresponding eigenfunctions $f$ satisfy the identity

$$
\int_X |\nabla \nabla f|^2 e^{-u_\omega \omega} = \mu_\omega \int_X |\nabla f|^2 e^{-u_\omega \omega}. \tag{1.4}
$$
Moreover
\[ \mu_\omega = \inf_{f \in C^\infty(X)} \int_X \frac{\lvert \nabla \nabla f \rvert^2 e^{-u_\omega} \omega^n}{\int_X \lvert \nabla f \rvert^2 e^{-u_\omega} \omega^n} \]  
\hspace*{1cm} (1.5)

We introduce, in analogy with (1.3), the eigenvalue gap for Hamiltonian vector fields by
\[ \mu(X;A) = \inf_{\omega \in c_1(X;A)} \mu_\omega. \]  
\hspace*{1cm} (1.6)

**Theorem 2** Let \( X \) be a compact Kähler manifold with \( c_1(X) > 0 \) and vanishing Futaki invariant. Then \( X \) admits a Kähler-Einstein metric if and only if \( \mu(X,A) > 0 \) for any \( A > 0 \).

For each \( A > 0 \), we have the easy bound \( \mu(X;A) \geq c_A \lambda(X;A) \) for some positive constant \( c_A \). Thus the condition \( \lambda(X;A) > 0 \) in Theorem 1 implies the condition \( \mu(X;A) > 0 \) in Theorem 2. However, there does not appear to be a direct way to show that they are equivalent.

We now describe briefly our approach. One direction in Theorem 1 and Theorem 2 is known, by combining the work of Perelman on the Kähler-Ricci flow with the convergence results of [22] and [32]. The main problem is to establish the other direction, namely that the existence of a Kähler-Einstein metric on \( X \) implies that the gaps \( \lambda(X;A) \) and \( \mu(X;A) \) are strictly positive for any \( A > 0 \). For each fixed \( \omega \), the eigenvalues \( \lambda_\omega \) and \( \mu_\omega \) are positive by definition. So the desired statement can be interpreted as a compactness statement with respect to a suitable topology. Our strategy for such a statement is to view the Kähler potential \( \phi \) of a metric \( \omega \in c_1(X,A) \) as the solution of a Monge-Ampère equation with right hand side depending on the Ricci potential \( u_\omega \). The \( C^\alpha \) estimates are derived by combining the theorem of Skoda-Zeriahi [33] with that of Kolodziej [17] following the idea of Guedj [2]. Then the \( C^{3,\alpha} \) priori estimates can be obtained by combining methods for the Monge-Ampère equation together with the recent techniques introduced by Chen-Cheng [5] for the constant scalar curvature problem. Next, the \( C^{2,\alpha} \) bounds imply the uniform equivalence of the metrics. This implies in turn uniform estimates of the corresponding eigenvalues on vector fields, using the arguments of [20] to handle the orthogonality condition with different metrics to holomorphic vector fields. The desired theorems follow.

2 \( C^{1,\alpha} \) estimates for metrics in \( c_1(X;A) \)

First we set up the equation. Let \( n \) be the dimension of \( X \). If \( \omega \) is any metric in \( c_1(X) \), we define its Ricci potential \( u_\omega \) by
\[ \text{Ric}(\omega) - \omega = -i \partial \bar{\partial} u_\omega, \quad \int_X e^{-u_\omega} \omega^n = \int_X \omega^n \]  
\hspace*{1cm} (2.1)
where \( \text{Ric}(\omega) = -i\partial\bar{\partial}\log \omega^n \) is its Ricci curvature form. Fix now a reference metric \( \omega_0 \in \text{c}_1(X) \), and let \( \text{Ric}(\omega_0) \) and \( u_0 \) are its Ricci form and Ricci potential, respectively. We can then write \( \omega = \omega_0 + i\partial\bar{\partial}\varphi \), where \( \varphi \) is normalized to satisfy \( \sup_X \varphi = 0 \). Since

\[
-i\partial\bar{\partial}(u_\omega - u_0) + i\partial\bar{\partial}\varphi = \text{Ric}(\omega) - \text{Ric}(\omega_0) = -i\partial\bar{\partial}\log \frac{\omega^n}{\omega_0^n}
\]

we find that \( \varphi \) satisfies the following complex Monge-Ampère equation

\[
(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{u_\omega - u_0 - \varphi + c_\varphi} \omega_0^n
\]

where \( c_\varphi \) is a specific constant, which is determined because \( \varphi, u_\omega, \) and \( u_0 \) have all been normalized. It follows from the normalization of \( u_\omega \) and \( \varphi \) that \( c_\varphi \leq 0 \).

2.1 The \( C^\alpha \) estimates on potential

The first step is the following \( C^\alpha \) estimate.

**Lemma 1** Assume that \( X \) admits a Kähler-Einstein metric \( \omega_{KE} \), which we take as reference metric \( \omega_0 = \omega_{KE} \). Then there exists \( \alpha > 0 \) with the following property. For any \( \omega \in \text{c}_1(X; A) \), there exists an automorphism \( g \) of \( X \) such that

\[
\|\psi - \sup \psi\|_{C^\alpha(\omega_{KE})} \leq C(A), \quad \text{where } g^*\omega = \omega_{KE} + i\partial\bar{\partial}\psi.
\]

**Proof.** Since \( X \) admits a Kähler-Einstein metric, we can apply the Moser-Trudinger inequality. An early form of this inequality was first proved in [27], a sharp version subsequently in [21] in the case of manifolds without holomorphic vector fields, and the full sharp and general version in [12]. In this form, it asserts that for any \( \omega \in \text{c}_1(X) \), there exists \( g \in G \) (here \( G \) is the automorphism group of \( X \)) and \( \epsilon > 0 \) depending on \( X \) such that

\[
A \geq K_{\omega_{KE}}(\omega_\phi) \geq \epsilon J_{\omega_{KE}}(g^*\omega_\phi) - \frac{1}{\epsilon}
\]

where \( J_{\omega_{KE}} \) is the Aubin-Yau functional with reference metric \( \omega_{KE} \). Thus if we write \( g^*\omega_\phi = \omega_{KE} + i\partial\bar{\partial}\psi \) we have

\[
\psi - \sup \psi \in S_{A_3} = \{ \theta \in \mathcal{E}_1(X, \omega_{KE}) \subset \text{PSH}(X, \omega_{KE}) : \sup \theta = 0 \text{ and } J_{\omega_{KE}}(\omega_\theta) \leq A_3 \}
\]

where \( \text{PSH}(X, \omega_{KE}) \) is the space of plurisubharmonic functions and \( \mathcal{E}_1(X, \omega_0) \) is the space of finite energy potentials. We now claim:

1. \( S_{A_3} \) is compact with respect to the weak \( L^1(\omega_0^n) \) topology on \( \text{PSH}(X, \omega_0) \)

2. Every element of \( S_{A_3} \) has zero Lelong number at \( z \) for all \( z \in X \).
3. For every $p \geq 1$ there exists $C(p, \omega_0, A)$ such that
\[
\int_X e^{-p\theta} \omega_0^n \leq C(p, \omega_{KE}, A) \quad \text{for all } \theta \in S_{A_3}.
\] (2.5)

These follow as in [2] respectively from Lemma 4.13, Proposition 2.13 and Theorem 4.15 (due to Skoda and Zeriahi) of [11]. Next, applying (2.3) with $\varphi = \psi - \sup_X \psi$, we obtain for $\omega \in c_1(X; A)$,
\[
(\omega_{KE} + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{-\psi - \sup \psi + \psi} e^{c \psi} \omega_{KE}^n \leq C(A) e^{\sup \psi} \omega_{KE}^n
\] (2.6)
where we have used the fact that $|u_\psi| \leq A$ and $c_\psi \leq 0$. Now if we apply (2.5) to (2.6) we obtain that $\|\psi - \sup \psi\|_{C^\alpha(\omega_{KE})} \leq C(A)$ for some $\alpha = \alpha(n, p) \in (0, 1)$ by the theorem of Kolodziej [17]. Q.E.D.

2.2 $C^{3, \alpha}$ estimates on potentials

We return to the study of the equation (2.3), for a general compact Kähler manifold $X$ and reference metric $\omega_0$, not necessarily Kähler-Einstein. The goal of the present subsection is to establish the following lemma:

**Lemma 2** Let $\varphi$ be a smooth solution of the Monge-Ampère equation (2.3). Assume that $\|\varphi\|_{C^0} \leq A$, $\|u_\omega\|_{C^0} + \|\nabla u_\omega\|_{C^0} + \|\Delta u_\omega\|_{C^0} \leq A$. Then for any $\alpha \in (0, 1)$, there exists a constant $C = C(n, A, \omega_0, \alpha) > 0$ so that
\[
\|\varphi\|_{C^{3, \alpha}(X, \omega_0)} \leq C.
\] (2.7)

It is convenient to set $F = -\varphi + u_\omega - u_0 + c_\varphi$, so the equation can be written as
\[
(\omega_0 + \partial \bar{\partial} \varphi)^n = e^F \omega_0^n, \quad \sup_X \varphi = 0,
\] (2.8)
and note that $F$ depends on the Kähler potential of $\varphi$. To simplify the notation, we shall denote $u_\omega$ by just $u$. Under the assumptions of the lemma, both $\varphi$ and $u$ are bounded, so it follows from the fact that $\omega$ and $\omega_0$ have the same volume that $|c_\varphi|$ is bounded by a constant $C(A, \omega_0)$ as well. Thus we have
\[
0 < \frac{1}{C(A, \omega_0)} \leq e^F \leq C(A, \omega_0).
\] (2.9)

We divide the proof of the lemma into the following steps:
1. Apply Chen-Cheng’s argument [5] to show $\Delta_{\omega_0} \varphi$ is in $L^p(X, \omega_0)$ for any $p > 0$, hence $\varphi \in C^{1,\beta}(X, \omega_0)$ for any $\beta \in (0, 1)$ by elliptic estimates. Here $\Delta_{\omega_0}$ is the Laplacian with respect to the reference metric $\omega_0$.

2. The Hölder continuity of $\varphi$ and the assumption $\|\nabla u\|_{C^0(X,\omega)}^2 \leq C(A)$ implies that $u \in C^{0,\alpha'}(X, \omega_0)$ (see Lemma 6 below).

3. By a theorem of Li-Li-Zhang [31] (which is an improvement of a result of Yu Wang [28]), we get the $C^{2,\alpha'}(X, \omega_0)$ bound for $\varphi$.

4. After we show $u \in C^{1,\alpha''}(X, \omega_0)$ by elliptic estimates, we get the $C^{3,\alpha}$ estimate for $\varphi$ by differentiating the Monge-Ampère equation (2.3).

We begin by modifying the arguments in Chen-Cheng [5] to derive the following estimates:

**Lemma 3** There exists a constant $C = C(A, n, \omega_0) > 0$ with

$$\sup_X \|\nabla \varphi\|_{C^0(X,\omega_0)}^2 \leq C. \quad (2.10)$$

**Proof.** Denote $\Phi = -F - \lambda \varphi + \frac{1}{2} \varphi^2$ with a constant $\lambda > 0$ to be chosen later. We calculate

$$\Delta_{\omega}(e^\Phi(|\nabla \varphi|_{\omega_0}^2) + 3) = (|\nabla \varphi|_{\omega_0}^2 + 3)\Delta_{\omega}e^\Phi + 2e^\Phi Re(\nabla \Phi, \nabla |\nabla \varphi|_{\omega_0}^2) + e^\Phi \Delta_{\omega} |\nabla \varphi|_{\omega_0}^2 \quad (2.11)$$

We consider the first term in (2.11).

$$\Delta_{\omega}e^\Phi = e^\Phi(\Delta_{\omega} \Phi + |\nabla \Phi|_{\omega}^2)$$

$$= e^\Phi(\lambda - 1)\Delta_{\omega}(-\varphi) - \Delta_{\omega}u + \Delta_{\omega}u_0 + \varphi \Delta_{\omega} \varphi + |\nabla \varphi|_{\omega}^2 + |\nabla \Phi|_{\omega}^2$$

$$\geq e^\Phi((\lambda - 1 - \varphi - C_0)\text{tr}_{\omega} \omega_0 - C + |\nabla \varphi|_{\omega}^2 + |\nabla \Phi|_{\omega}^2)$$

where we used the assumption that $|\Delta_{\omega} u| \leq C(A)$, and $C_0 = C_0(\omega_0) > 0$ is a constant satisfying $-C_0 \omega_0 \leq \partial_i \partial_j u_0 \leq C_0 \omega_0$.

To deal with the third term in (2.11), we introduce a normal coordinates system for $\omega_0$ at the maximum point $x_0 \in X$ of $e^\Phi(3 + |\nabla \varphi|_{\omega_0}^2)$ such that $g_0 = (\bar{g}_{ij}) = (\delta_{ij})$ and $dg_0 = 0$ at the point. Moreover, $\omega = (g_{ij}\delta_{ij})$ is diagonal at $x_0$. We calculate at $x_0$,

$$\Delta_{\omega} |\nabla \varphi|_{\omega_0}^2 = g^{pp} \frac{\partial^2}{\partial z_p \partial \bar{z}_p}(\bar{g}^{ij} \varphi_j \varphi_i)$$

$$= g^{pp} \frac{\partial^2}{\partial z_p \partial \bar{z}_p} \varphi_j \varphi_i + g^{pp} \varphi_{ip} \varphi_{ip} + g^{pp} \varphi_{ip} \varphi_{ip} + g^{pp} \partial_i \varphi_{ji} \varphi_{ji} + g^{pp} \varphi_{ji} \partial_i \varphi_{ji}$$

$$= \bar{R}_{jkpp}g^{pp} \varphi_k \varphi_j + g^{pp} \varphi_{ip} \varphi_{ip} + g^{pp} \varphi_{ip} \varphi_{ip} + g^{pp} \varphi_{ii} \partial_p \varphi_{ii} + g^{pp} \varphi_{ii} \partial_p \varphi_{ii}$$

$$\geq -C_1 \text{tr}_{\omega} \omega_0 |\nabla \varphi|_{\omega_0}^2 + g^{pp} \varphi_{ip} \varphi_{ip} + g^{pp} \varphi_{ip} \varphi_{ip} + 2Re(\varphi_i F_i)$$.
Therefore, we get

\[
\Delta_\omega \left( e^\Phi (|\nabla \varphi|_{\omega_0}^2 + 3) \right) \geq e^\Phi \left\{ (|\nabla \varphi|_{\omega_0}^2 + 3) \left( (\lambda - 1 - \varphi - C_0) \text{tr}_\omega \omega_0 - C + |\nabla \varphi|_\omega^2 + |\nabla \Phi|_\omega^2 \right) - C_1 \text{tr}_\omega \omega_0 |\nabla \varphi|_{\omega_0}^2 + g^{ij} \varphi_{ij} \varphi_{ij} + g^{pq} \varphi_{ip} \varphi_{pq} \right. \\
\left. + 2 \text{Re}(\varphi F_i) + 2 \text{Re} \left( g^{ij} \Phi_i (\varphi_j \varphi_{ji} + \varphi_{jj} \varphi_{ji}) \right) \right\}.
\]

(2.12)

The last two terms are equal to (note that at \( x_0 \), \( \varphi_{ji} = \varphi_{ii} \delta_{ij} = (\delta_{ii} - 1) \delta_{ij} \))

\[
2 \text{Re} \left( - \Phi_i \varphi_i - (\lambda - \varphi) |\nabla \varphi|_{\omega_0}^2 + g^{ij} \Phi_i (g_{ii} - 1) \varphi_i \right)
= - 2(\lambda - \varphi) |\nabla \varphi|_{\omega_0}^2 - 2 \text{Re} (\langle \nabla \Phi, \nabla \varphi \rangle_\omega)
\geq - 2(\lambda - \varphi) |\nabla \varphi|_{\omega_0}^2 - |\nabla \varphi|_\omega^2 - |\nabla \Phi|_\omega^2,
\]

the last two terms on the RHS can be absorbed by the corresponding terms in the first line on the right hand side in (2.12), while

\[
2 \text{Re} (g^{ii} \Phi_i \varphi_{ji} \varphi_{ij}) \geq - g^{ii} \Phi_i \varphi_{ji} \varphi_{ij} - g^{ii} \varphi_{ij} \varphi_{ij} = - |\nabla \varphi|_{\omega_0}^2 |\nabla \Phi|_\omega^2 - g^{ij} \varphi_{ij} \varphi_{ij},
\]

and the right hand side above can also be absorbed by terms in the first and second lines of the right hand side in (2.12). So we get by combining the above that at \( x_0 \)

\[
0 \geq \Delta_\omega \left( e^\Phi (|\nabla \varphi|_{\omega_0}^2 + 3) \right)
\geq e^\Phi \left\{ (|\nabla \varphi|_{\omega_0}^2 + 3) \left( (\lambda - 1 - \varphi - C_0 - C_1) \text{tr}_\omega \omega_0 - C + \frac{1}{2} |\nabla \varphi|_\omega^2 \right) - 2(\lambda - \varphi) |\nabla \varphi|_{\omega_0}^2 \right\}
\geq e^\Phi \left( |\nabla \varphi|_{\omega_0}^2 (\text{tr}_\omega \omega_0 + \frac{1}{2} |\nabla \varphi|_\omega^2) - C |\nabla \varphi|_{\omega_0}^2 - C \right)
\geq e^\Phi \left( c(n, A) |\nabla \varphi|_{\omega_0}^{2(\frac{1}{2} + \frac{1}{2})} - C |\nabla \varphi|_{\omega_0}^2 - C \right)
\]

(2.13)

where we choose \( \lambda = 2 + \| \varphi \|_{L^\infty} + C_0 + C_1 \). In the last step we apply the inequality below which follows from Young’s inequality (i.e. \( a^{\frac{1}{n}} b^{\frac{n-1}{n}} \leq c(n) \left( \frac{1}{2} a + b \right) \) for some \( c(n) > 0 \))

\[
|\nabla \varphi|_{\omega_0}^2 \leq |\nabla \varphi|_\omega^2 \text{tr}_\omega \omega_0 \leq |\nabla \varphi|_{\omega_0}^2 (\text{tr}_\omega \omega_0)^{n-1} \left( \frac{\omega_0^m}{\omega_0^m} \right)
= e^F \left( \frac{2}{\omega_0^m} (\text{tr}_\omega \omega_0)^{\frac{n-1}{2}} \right)^n \leq c(n) e^F \left( \frac{1}{2} |\nabla \varphi|_\omega^2 + \text{tr}_\omega \omega_0 \right)^n
\leq C(n, A) \left( \frac{1}{2} |\nabla \varphi|_\omega^2 + \text{tr}_\omega \omega_0 \right)^n.
\]
From (2.13) we conclude that at $x_0$, $|\nabla \varphi|^2_{\omega_0} \leq C(n, A)$. Since $x_0$ is a maximum point of $e^\varphi (|\nabla \varphi|^2_{\omega_0} + 3)$, we see that $\sup_X |\nabla \varphi|^2_{\omega_0} \leq C(n, A)$. The lemma is proved.

We next apply the argument in the proof of Theorem 3.1 of Chen-Cheng [5]. In our case the functions $F$ and $\varphi$ are bounded so we can simplify the proof a little bit.

**Lemma 4** For any $p > 0$, there exists a constant $C_p = C(n, A, \omega_0, p) > 0$ such that

$$\int_X (\text{tr}_{\omega_0} \omega)^p \omega_0^n \leq C_p.$$  

**Proof.** We fix a constant $\alpha \geq 1$ which will be determined later. For notational simplicity, we write $\Psi = -\alpha F - \lambda \alpha \varphi$, and calculate $\Delta_\omega (e^\Psi \text{tr}_{\omega_0} \omega)$,

$$\Delta_\omega (e^\Psi \text{tr}_{\omega_0} \omega) = e^\Psi \text{tr}_{\omega_0} \omega (\Delta_\omega \Psi + |\nabla \Psi|^2_\omega) + 2e^\Psi \text{Re} (\nabla \Psi, \bar{\nabla} \text{tr}_{\omega_0} \omega)_\omega + e^\Psi \Delta_\omega \text{tr}_{\omega_0} \omega. \quad (2.14)$$

We use a normal coordinates system of $\omega_0$, so that $\omega_0 = (\delta_{ij})$, $d g_0 = 0$ and $g$ (i.e. $\omega$) is diagonal at a given point. By the standard calculations as in Yau [29], the last term in (2.14) satisfies

$$e^\Psi \Delta_\omega \text{tr}_{\omega_0} \omega \geq e^\Psi ( - C_2 \text{tr}_{\omega_0} \omega \text{tr}_{\omega_0} \omega + g^{ij} g^{j\bar{k}} \varphi_{ij \bar{k}} \varphi_{\bar{j} \bar{k}} + \Delta_\omega F - R_{\omega_0})$$

where $-C_2$ is a lower bound of the bisectional curvature of $\omega_0$, $\varphi_{ij \bar{k}}$ denotes the covariant derivative of $\varphi$ under $\nabla_{\omega_0}$ and $R_{\omega_0}$ is the scalar curvature of $\omega_0$. We cannot apply the usual maximum principle here because a priori $\Delta_\omega F$ is not bounded.

The second term in (2.14) satisfies

$$2e^\Psi \text{Re} (\nabla \Psi, \bar{\nabla} \text{tr}_{\omega_0} \omega)_\omega \geq -2e^\Psi |\nabla \Psi|^2_\omega |\nabla \text{tr}_{\omega_0} \omega|_\omega$$

$$\geq -e^\Psi \text{tr}_{\omega_0} \omega |\nabla \Psi|^2_\omega - e^\Psi \frac{|\nabla \text{tr}_{\omega_0} \omega|^2_{\omega}}{\text{tr}_{\omega_0} \omega}$$

$$\geq -e^\Psi \text{tr}_{\omega_0} \omega |\nabla \Psi|^2_\omega - e^\Psi g^{ij} g^{j\bar{k}} \varphi_{ij \bar{k}} \varphi_{\bar{j} \bar{k}},$$

where in the last step we use the inequality below as in [29]

$$|\nabla \text{tr}_{\omega_0} \omega|^2_{\omega} = \sum_i g^{ii} \sum_k \varphi_{k \bar{k} i} \leq \text{tr}_{\omega_0} \omega \sum_i g^{ii} \sum_j g^{j\bar{k}} \varphi_{j \bar{k} i} \varphi_{\bar{j} \bar{k}} \leq \text{tr}_{\omega_0} \omega g^{ii} g^{j\bar{k}} \varphi_{ij \bar{k}} \varphi_{\bar{j} \bar{k}}.$$

The first term in (2.14) is

$$e^\Psi \text{tr}_{\omega_0} \omega \Delta_\omega \Psi = e^\Psi \text{tr}_{\omega_0} \omega (\alpha \Delta_\omega \varphi - \alpha \Delta_\omega u + \alpha \Delta_\omega u_0 - \lambda \alpha \Delta_\omega \varphi)$$

$$\geq e^\Psi \text{tr}_{\omega_0} \omega (\lambda \alpha - \alpha - C_0) \text{tr}_{\omega_0} \omega_0 - C(n, A) \alpha), \quad (2.15)$$
where as before $C_0 > 0$ satisfies $-C_0 \omega_0 \leq i \partial \bar{\partial} u_0 \leq C_0 \omega_0$. Combining the above inequalities we get

$$\Delta_{\omega}(e^{\Psi} tr_{\omega_0}) \geq e^{\Psi} \left( (\lambda \alpha - \alpha - C_0 - C_2) tr_{\omega_0} \omega tr_{\omega_0} - C(n, A) \alpha tr_{\omega_0} \omega + \Delta_{\omega_0} F - R_{\omega_0} \right)$$
$$\geq e^{\Psi} \left( \alpha \left( tr_{\omega_0} \omega \right)^{\frac{\alpha}{\alpha - 1}} e^{-\frac{\epsilon}{\alpha - 1}} - C(n, A) \alpha tr_{\omega_0} \omega + \Delta_{\omega_0} F - R_{\omega_0} \right)$$
$$\geq e^{\Psi} \left( c_0 \alpha \left( tr_{\omega_0} \omega \right)^{\frac{\alpha}{\alpha - 1}} + \Delta_{\omega_0} F - C(n, A) \alpha \right) \tag{2.16}$$

where we choose $\lambda = C_0 + C_2 + 2$, $c_0 = c_0(n, A, \omega_0) > 0$ depends on the lower bound of $e^{-\frac{\epsilon}{\alpha - 1}}$, and in the last step we apply Young’s inequality $tr_{\omega_0} \omega \leq \epsilon \left( tr_{\omega_0} \omega \right)^{\frac{\alpha}{\alpha - 1}} + C(\epsilon)$ for a suitable choice of small $\epsilon > 0$.

We denote $v := e^{\Psi} tr_{\omega_0} > 0$ and for any $p \geq 1$ we have by (2.16)

$$\Delta_{\omega} v^p = p v^{p-1} \Delta_{\omega} v + p(p - 1) v^{p-2} |\nabla v|^2$$
$$\geq p v^{p-1} e^{\Psi} \left( c_0 \alpha \left( tr_{\omega_0} \omega \right)^{\frac{\alpha}{\alpha - 1}} + \Delta_{\omega_0} F - C(n, A) \alpha \right) + p(p - 1) v^{p-3} e^{\Psi} |\nabla v|^2 \omega_0, \tag{2.17}$$

where in the inequality we have applied the observation that $v |\nabla v|^2 = e^{\Psi} tr_{\omega_0} \omega |\nabla v|^2 \geq e^{\Psi} |\nabla v|^2 \omega_0$. Integrating the inequality (2.17) over $X$ against the volume form $\omega^n = e^F \omega_0^n$, we obtain

$$\int_X \left( v^{p-1} e^{\Psi + F} \left( c_0 \alpha \left( tr_{\omega_0} \omega \right)^{\frac{\alpha}{\alpha - 1}} + \Delta_{\omega_0} F \right) + (p - 1) v^{p-3} e^{\Psi + F} |\nabla v|^2 \right) \omega_0^n$$
$$\leq C(n, A) \alpha \int_X v^{p-1} e^{\Psi + F} \omega_0^n \omega_0^n. \tag{2.18}$$

To deal with the term involving $\Delta_{\omega_0} F$, we will apply the integration by parts. We calculate

$$\int_X v^{p-1} e^{\Psi + F} \Delta_{\omega_0} F \omega_0^n = \int_X v^{p-1} e^{-\alpha (1-F) - \lambda \alpha \varphi} \Delta_{\omega_0} F \omega_0^n$$
$$= \int_X \left( - (p - 1) v^{p-2} e^{-\alpha (1-F) - \lambda \alpha \varphi} \langle \nabla v, \bar{\nabla} F \rangle \omega_0 \right.$$
$$+ v^{p-1} e^{-\alpha (1-F) - \lambda \alpha \varphi} (\alpha - 1) |\nabla F|^2 \omega_0$$
$$+ v^{p-1} e^{-\alpha (1-F) - \lambda \alpha \varphi} \lambda \alpha \langle \nabla \varphi, \nabla F \rangle \omega_0^n \). \tag{2.19}$$
The second term in the right hand side of (2.19) is good. The first term in (2.19) satisfies

\[
\int_X -(p - 1)v^{p - 2}e^{-(\alpha - 1)F - \lambda_0 \alpha} \langle \nabla v, \nabla F \rangle_{\omega_0} \\
\geq -\int_X (p - 1)v^{p - 2}e^{\Psi + F} |\nabla v|_{\omega_0} |\nabla F|_{\omega_0} \\
\geq -\int_X \frac{\alpha - 1}{4}v^{p - 2}e^{-(\alpha - 1)F - \lambda_0 \alpha} |\nabla F|_{\omega_0}^2 - \int_X \frac{(p - 1)^2}{4}v^{p - 3}e^{\Psi + F} |\nabla v|_{\omega_0}^2 \\
\geq -\int_X \frac{\alpha - 1}{4}v^{p - 2}e^{-(\alpha - 1)F - \lambda_0 \alpha} |\nabla F|_{\omega_0}^2 - \int_X v^{p - 3}e^{\Psi + F} |\nabla v|_{\omega_0}^2
\]

if we take \(\alpha = \alpha(p) \geq p + 2\). These negative terms will be cancelled by the positive terms from (2.19) and (2.18). Next we look at the third term on the right hand side of (2.19). By Lemma 3, we have a bound on \(\sup_X |\nabla \phi|_{\omega_0}\), and thus

\[
\int_X v^{p - 1}e^{-(\alpha - 1)F - \lambda_0 \alpha} \langle \nabla \phi, \nabla F \rangle_{\omega_0} \\
\geq -C\lambda \alpha \int_X v^{p - 1}e^{-(\alpha - 1)F - \lambda_0 \alpha} |\nabla F|_{\omega_0} \\
\geq -\frac{\alpha - 1}{4}v^{p - 1}e^{-(\alpha - 1)F - \lambda_0 \alpha} |\nabla F|_{\omega_0}^2 - \frac{C\lambda^2}{\alpha - 1} \int_X v^{p - 1}e^{-(\alpha - 1)F - \lambda_0 \alpha} \\
\geq -\frac{\alpha - 1}{4}v^{p - 1}e^{-(\alpha - 1)F - \lambda_0 \alpha} |\nabla F|_{\omega_0}^2 - C\alpha \int_X v^{p - 1}e^{-(\alpha - 1)F - \lambda_0 \alpha}.
\]

Plugging the above inequalities into (2.18) and re-organizing, it follows that

\[
\int_X c_0\alpha v^{p - 1}e^{\Psi + F} (\text{tr}_{\omega_0} \omega)_{\omega_0}^{\frac{n}{n - \alpha}} \omega_0^n \leq C(n, A)\alpha \int_X v^{p - 1}e^{\Psi + F} \omega_0^n.
\]

Note that \(\Psi\) and \(F\) are both bounded by \(C(n, A)\), so we conclude that there exists a constant \(C_p = C(n, A, \omega_0, p) > 0\) such that

\[
\int_X (\text{tr}_{\omega_0} \omega)^{p - 1 + \frac{n}{n - 1}} \omega_0^n \leq C_p \int_X (\text{tr}_{\omega_0} \omega)^{p - 1} \omega_0^n. \quad (2.20)
\]

When \(p = 2\)

\[
\int_X \text{tr}_{\omega_0} \omega_0^n = \int_X (n + \Delta \omega_0 \phi) \omega_0^n = n \int_X \omega_0^n
\]

is clearly bounded. Now we define a sequence \(\{p_k\}\) with \(p_0 = 2\) and \(p_k = 2 + \frac{n}{n - 1}k\). Then (2.20) implies that

\[
\int_X (\text{tr}_{\omega_0} \omega)^{p_k} \omega_0^n \leq C_k \int_X (\text{tr}_{\omega_0} \omega)^{p_k - 1} \omega_0^n.
\]
Since \( p_k \to \infty \) as \( k \to \infty \), iterating the inequality above gives that there exists a constant \( C_k = C(n, A, \omega_0, k) > 0 \) such that
\[
\int_X (\text{tr}_{\omega_0} \omega)^{p_k} \omega_0^n \leq C_k.
\]
Lemma 4 then follows from this inequality and the Hölder inequality.

**Lemma 5** For any \( \beta \in (0, 1) \), there exists a constant \( C_\beta = C(n, A, \omega_0, \beta) > 0 \) such that
\[
\|\varphi\|_{C^{1,\beta}(X, \omega_0)} \leq C_\beta.
\]

**Proof.** By Lemma 4, \( f := \Delta_{\omega_0} \varphi \in L^p(X, \omega_0^n) \) for any \( p > 0 \). By the \( W^{2,p} \)-estimates for linear elliptic equations (c.f. Theorem 9.11 in [16]), we have
\[
\|\varphi\|_{W^{2,p}(X, \omega_0)} \leq C(\|\varphi\|_{L^p(X, \omega_0^n)} + \|f\|_{L^p(X, \omega_0^n)}) \leq C_p.
\]
The \( C^{1,\beta}(X, \omega_0) \) bound of \( \varphi \) then follows from the Sobolev embedding theorem (c.f. Corollary 7.11 in [16]) by taking \( p > 1 \) sufficiently large.

**Lemma 6** The Ricci potential \( u \) of \( \omega = \omega_0 + i \partial \bar{\partial} \varphi \) satisfies
\[
\|u\|_{C^{\alpha}(X, \omega_0)} \leq C_\alpha(n, A, \omega_0),
\]
for any \( \alpha \in (0, 1) \).

**Proof.** Observe that
\[
|\nabla u|_{\omega_0}^2 \leq |\nabla u|_{\omega}^2 \text{tr}_{\omega_0} \omega \leq A \text{tr}_{\omega_0} \omega.
\]
By Lemma 4, it follows that \( |\nabla u|_{\omega_0} \in W^{1,p}(X, \omega_0) \) for any \( p > 1 \). The lemma then follows from the Sobolev embedding theorem by taking \( p > 1 \) sufficiently large.

To prove the \( C^{2,\alpha} \)-estimate of \( \varphi \), we need the following recent result of Li-Li-Zhang, which weakens the condition of Y. Wang’s result [28] on the regularity assumption of \( \varphi \).

**Lemma 7 ([31] Theorem 1.2)** Let \( B_2 \subset \mathbb{C}^n \) be the Euclidean ball with radius 2 and center 0. Suppose \( \varphi \in PSH(B_2) \cap C(B_2) \) solves the complex MA equation
\[
\det \varphi_{ij} = f, \text{ in } B_2
\]
with \( f \geq \lambda > 0 \) for some positive \( \lambda \in \mathbb{R} \) and \( f \in C^{\alpha}(B_2) \) for some \( \alpha \in (0, 1) \). If \( \varphi \in C^{1,\beta}(B_2) \) for some \( \beta > 1 - \frac{\alpha}{n(2+\alpha) - 1} \), then \( \varphi \in C^{2,\alpha}(B_1) \) and the \( C^{2,\alpha}(B_1) \)-norm of \( \varphi \) depends only on \( n, \alpha, \beta, \lambda, \|\varphi\|_{C^{1,\beta}(B_2)} \) and \( \|f\|_{C^{\alpha}(B_2)} \).
We arrive now at the $C^{2,\alpha}$ estimates for $\varphi$:

**Lemma 8** Under the conditions spelled out in the statement of Lemma 2, there exists $\alpha > 0$ with

$$\|\varphi\|_{C^{2,\alpha}} \leq C(n, A, \alpha)$$

(2.21)

for some constant $C(n, A, \omega_0)$.

*Proof.* We note that by Lemma 1 and Lemma 6 the function on the right hand side of (2.3) has uniform $C^{0,\alpha'}(X, \omega_0)$ estimate. Lemma 5 provides the $C^{1,\beta}(X, \omega_0)$ estimates of the Kähler potential $\varphi$. Then Lemma 7 proves the $C^{2,\alpha}(X, \omega_0)$ estimates of $\varphi$. Q.E.D.

The following lemma is the key lemma that we shall need later for the proof of Theorem 1 and Theorem 2. It is an immediate consequence of the $C^{2,\alpha}(X, \omega_0)$-estimates of $\varphi$, and the fact that the right hand side $e^F$ of the Monge-Ampère equation (2.3) is bounded above and below:

**Lemma 9** There exists a constant $C = C(n, A, \omega_0) \geq 1$ such that

$$C^{-1} \omega_0 \leq \omega \leq C \omega_0,$$

and $u \in C^{1,\alpha}(X, \omega_0)$ for any $\alpha \in (0, 1)$.

Finally, we can complete the proof of Lemma 2. By Lemma 8 the metric $g_{\bar{j}i}$ has uniform $C^{2,\alpha}$ norm. By Lemma 6 and Cramer’s rule, its inverse $g^{\bar{j}i}$ also has uniform $C^{0,\alpha}(X, \omega_0)$ norm. The equation that $R_\omega - n = \Delta_\omega u$ can be written locally in holomorphic coordinates as

$$g^{\bar{j}i} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = R_\omega - n \in L^\infty.$$

Then the $C^{1,\alpha}(X, \omega_0)$-norm of $u$ follows from the $W^{2,p}$-estimates and Sobolev embedding theorem (c.f. [16]).

Finally, once we have the $C^{1,\alpha}(X, \omega_0)$-norm of $u$, we can take $\frac{\partial}{\partial z_i}$ on both sides of the equation (2.3) and apply local Schauder estimates to conclude that

$$\|\varphi\|_{C^{3,\alpha}(X, \omega_0)} \leq C(n, A, \omega_0, \alpha).$$

(2.22)

The proof of Lemma 2 is complete.
3 Proof of Theorem 1

One direction in Theorem 1 is a direct consequence of known results. Assume that \( \lambda(X; A) > 0 \) for any \( A \). By the work of Perelman (see \cite{26} for a detailed account), for any given initial data in \( c_1(X) \), the orbit of the Kähler-Ricci flow lies in a set \( c_1(X; A) \) for some \( A > 0 \). Thus a positive lower bound for \( \lambda(X; A) \) implies a positive lower bound for the eigenvalue \( \lambda(\omega) \) along the Kähler-Ricci flow. By the results of \cite{22, 32}, the flow converges then to a Kähler-Einstein metric.

The main issue in the present paper is to establish the other direction, namely that \( \lambda(X; A) > 0 \) for any \( A > 0 \) if a Kähler-Einstein metric is assumed to exist. But then the conditions of Lemma 1 are satisfied, and for any fixed \( A > 0 \), the metrics \( \omega \in c_1(X; A) \) have potentials which are uniformly bounded in \( C^\alpha \)-norm for some fixed \( \alpha > 0 \). By Lemma 9, they are all equivalent. The desired bound for \( \lambda(X; A) \) is then a consequence of the following lemma, which was essentially proved in \cite{20}, Lemma 1:

**Lemma 10** Let \( \omega, \bar{\omega} \) be two metrics in \( c_1(X) \) which are equivalent, in the sense that

\[
\kappa^{-1} \omega \leq \bar{\omega} \leq \kappa \omega
\]

for some constant \( \kappa > 0 \). Let \( \lambda_\omega \) and \( \lambda_{\bar{\omega}} \) be the corresponding eigenvalues, as defined in (1.1). Then

\[
c(\kappa, n)^{-1} \lambda_\omega \leq \lambda_{\bar{\omega}} \leq c(\kappa, n) \lambda_\omega
\]

for some constant \( c(\kappa, n) > 0 \) depending only on \( \kappa \) and the dimension \( n \).

**Proof.** Since this lemma is essential for our considerations and since its proof is short, we include the proof for the reader’s convenience. In the definition (1.1) for \( \lambda_\omega \) and \( \lambda_{\bar{\omega}} \), the norms \( \| \partial V \|_\omega \) and \( \| \partial V \|_{\bar{\omega}} \) as well as the norms \( \| V \|_\omega \) and \( \| V \|_{\bar{\omega}} \) are already equivalent, since the metrics \( \omega \) and \( \bar{\omega} \) are equivalent, and so are their volume forms \( \omega^n \) and \( \bar{\omega}^n \). The main issue is the difference in the orthogonality conditions \( \perp_\omega \) and \( \perp_{\bar{\omega}} \). To address this issue, consider any vector field \( V \) with \( V \perp_\omega H^0(X, T^{1,0}) \) and decompose it as

\[
V = \tilde{V} + E
\]

with \( \tilde{V} \perp_{\bar{\omega}} H^0(X, T^{1,0}) \) and \( E \in H^0(X, T^{1,0}) \). Taking inner products with respect to the metric \( \omega \) gives

\[
0 = \langle \tilde{V}, E \rangle_\omega + \langle E, E \rangle_\omega
\]

and hence by the Cauchy-Schwarz inequality,

\[
\| E \|_\omega \leq \| \tilde{V} \|_\omega.
\]
We can now write for some constant $c_1(\kappa, n)$
\[
\|\bar{\partial} V\|_\omega^2 = \|\bar{\partial} \tilde{V}\|_\omega^2 \geq c_1(\kappa, n) \|\bar{\partial} \tilde{V}\|_\omega^2
\] (3.6)
because $\omega$ and $\tilde{\omega}$ are equivalent, and at the same time, by the same equivalence and the triangle inequality,
\[
\|V\|_\omega^2 \leq 2\|\tilde{V}\|_\omega^2 + 2\|E\|_\omega^2 \leq 4\|\tilde{V}\|_\omega \leq c_2(\kappa, n) \|\tilde{V}\|_{\tilde{\omega}}. \quad (3.7)
\]
It follows that
\[
\frac{\|\bar{\partial} V\|_\omega^2}{\|V\|_\omega^2} \geq \frac{c_1(\kappa, n)}{c_2(\kappa, n)} \frac{\|\bar{\partial} \tilde{V}\|_\omega^2}{\|\tilde{V}\|_{\tilde{\omega}}^2} \geq \frac{c_1(\kappa, n)}{c_2(\kappa, n)} \lambda_{\tilde{\omega}} \quad (3.8)
\]
and hence $\lambda_\omega \geq \frac{c_1(\kappa, n)}{c_2(\kappa, n)} \lambda_{\tilde{\omega}}$. Reversing the roles of $\omega$ and $\tilde{\omega}$ gives the inequality in the opposite direction. The lemma is proved, completing the proof of Theorem 1.

4 Proof of Theorem 2

Again, one direction of the theorem follows from the results of Perelman and [21, 32]. To prove the other direction, namely that the existence of a Kähler-Einstein metric implies a strictly positive gap $\mu(X; A)$ for any $A > 0$, we argue by contradiction. Recall the operator $L_\omega$ defined for a metric $\omega$ with Ricci potential $u$ by
\[
L_\omega f = -g^{jk} \nabla_j \nabla_k f + g^{jk} \nabla_k f \nabla_j u - f
\]
and whose eigenvalues and eigenfunctions satisfy the identity (1.4).

Assume then that $X$ is Kähler-Einstein, and that there exists a sequence of metrics $\omega_j = \omega_0 + i\bar{\partial} \bar{\partial} \varphi_j \in c_1(X; A)$ such that the eigenvalues $\mu_j$ of the operator $L_{\omega_j}$ goes to 0 as $j \to \infty$. We take $f_j$ to be eigenfunctions of $L_{\omega_j}$ with eigenvalues $\mu_j$, normalized by $\|f_j\|_{L^2(\omega_0, \omega_j)} = 1$. It follows from straightforward calculation that for any holomorphic vector field $V \in H^0(X, T^{1,0}X)$
\[
\int_X \langle \bar{\partial}_\omega f_j, V \rangle_{\omega_j} e^{-u_j} \omega_j^n = 0. \quad (4.1)
\]
By Lemma 9 and Lemma 8, we can apply the elliptic estimates to $f_j$, which satisfies the linear equation
\[
-g^{pq} \nabla_p \nabla_q f_j + g^{pq} \nabla_q f_j \nabla_p u_j - f_j = \mu_j f_j
\]
to conclude that
\[
\|f_j\|_{C^{2,\alpha}(X, \omega_0)} \leq C(n, A), \quad \forall j.
\]
Up to a subsequence, we may assume the Ricci potentials $u_j$ converge in $C^{1,\alpha}$ to a function $u_\infty \in C^{1,\alpha}$, the metrics $\omega_j$ converge in $C^{1,\alpha}$ to a metric $\omega_\infty \in C^{1,\alpha}$, and the functions $f_j$
converge in $C^{2,\alpha}$ to a function $f_\infty \in C^{2,\alpha}$. In particular, we have $\|f_\infty\|_{L^2(X,e^{-u_\infty}\omega_\infty^m)} = 1$. Passing to the limit, (4.1) gives that

$$\int_X \langle \nabla_{\omega_\infty} f_\infty, V \rangle_{\omega_\infty} e^{-u_\infty}\omega_\infty^m = 0, \quad \forall V \in H^0(X,T^{1,0}X). \quad (4.2)$$

Observe that the equations (i.e. (1.4))

$$\int_X |\bar{\nabla}\nabla f_j|_{\omega_j}^2 e^{-u_j}\omega_j^m + \int_X |\nabla f_j|_{\omega_j}^2 e^{-u_j}\omega_j^m = (1 + \mu_j) \int_X |\bar{\nabla} f_j|_{\omega_j}^2 e^{-u_j}\omega_j^m$$

hold for any $j$. Since $\mu_j \to 0$, passing to limit we get

$$\int_X |\bar{\nabla}\nabla f_\infty|_{\omega_\infty}^2 e^{-u_\infty}\omega_\infty^m = 0,$$

which implies $\nabla\nabla f_\infty = 0$, i.e. $\nabla_{\omega_\infty} f_\infty$ is a holomorphic vector field. From (4.2) we conclude that $\int_X |\nabla f_\infty|_{\omega_\infty}^2 e^{-u_\infty}\omega_\infty^m = 0$. However, this contradicts the identity

$$1 = \int_X f_\infty^2 e^{-u_\infty}\omega_\infty^m = \int_X |\nabla f_\infty|_{\omega_\infty}^2 e^{-u_\infty}\omega_\infty^m.$$

The proof of Theorem 2 is complete.

We observe that this argument could have been used also for the proof of Theorem 1. However, the argument there is more direct, and provides more precise information on the bounds for $\lambda_\omega$.

5 Further remarks

We note that in Theorem 1 we cannot in general replace the gap $\lambda(X;A)$ for each $A > 0$ by the gap $\lambda(X) = \inf_{\omega \in c_1(X)} \lambda_\omega$ over all of $c_1(X)$. A simple counterexample is provided by the 2-dimensional sphere, which admits a Kähler-Einstein metric, but can be seen to have

$$\lambda(S^2) = 0 \quad (5.1)$$

as follows. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that $\eta = 0$ on $(-1,1/3]$ and $\eta = 1$ on $[2/3,1)$. Let $a,N > 0$ and let $f : [0,3N + 2] \to \mathbb{R}$ be a non-negative concave function, positive and smooth on $(0,3N + 2)$ such that

1. $f(0) = f(3N + 2) = 0$
2. $f(x) = a$ for $x \in [1,3N + 1]$
and let $X$ be the surface obtained by revolving the graph of $y = f(x)$ around the $x$ axis. Thus $X$ is a smooth manifold (if we choose $f$ so that its tangent line is vertical at 0 and $3N + 2$ and is tangent to the graph to infinite order), looks like a cigar, is flat between $x = 1$ and $x = 3N + 1$ and is diffeomorphic to $S^2$. Moreover, we can choose $a$ so that the area of $X$ is 1 (so $a$ is roughly $\frac{1}{3N^2}$). Let $g_N$ be metric obtained by restricting the euclidean metric in $\mathbb{R}^3$. Let $V_1$ be a smooth vector field on $X$ defined as follows.

$$V_1 = \eta(x-1)\eta(N+1-x)\frac{\partial}{\partial x}$$

so $V_1$ is a smooth vector field on $X$ compactly supported in $\{(x,y,z) \in M : x \in (1, N+1)\}$. Similarly we define $V_2$ supported in $(N+1, 2N+1)$ and $V_3$ supported in $(2N+1, 3N+1)$.

Next we let

$$V = c_1V_1 + c_2V_2 + c_3V_3 \quad (5.2)$$

where the $c_i \in \mathbb{R}$ are chosen so that $V$ is orthogonal to the 3-dimensional space of holomorphic vector fields and $c_1^2 + c_2^2 + c_3^2 = 1$. Now $|V_i|$ is roughly equal to 1 so $\|V_i\|_{L^2} \sim 1/3$ so $\|V\|_{L^2} \sim c_1^2\|V_1\|_{L^2}^2 + c_2^2\|V_2\|_{L^2}^2 + c_3^2\|V_3\|_{L^2}^2 \sim \frac{1}{9}$. On the other hand $\nabla V_1 = 0$ for $2 < x < N$ so $\|\nabla V_1\|_{L^2}^2 = O\left(\frac{1}{N}\right)$ which implies $\|\nabla V\|_{L^2} = O\left(\frac{1}{N}\right)$. In particular, $\lambda_{\omega_N} \leq O\left(\frac{1}{N}\right)$. This establishes our claim.

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