Abstract

Given a finite set $\sigma$ of the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and a holomorphic function $f$ in $\mathbb{D}$ which belongs to a class $X$, we are looking for a function $g$ in another class $Y$ (smaller than $X$ or incomparable with $X$) which minimizes the norm $\|g\|_Y$ among all functions $g$ such that $g|_{\sigma} = f|_{\sigma}$.

The problem considered is the following: given two Banach spaces $X$ and $Y$ of holomorphic functions on the unit disc $\mathbb{D}$, or another class $Y$ (smaller than $X$ or incomparable with $X$), two first are “individual”, in the sense of [N1] p.158, on the other hand, are of this nature. Two first are “individual”, in the sense of [N1] p.158, on the other hand, are of this nature.

Résumé

Etant donné un ensemble fini $\sigma$ du disque unité $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ et une fonction $f$ holomorphe dans $\mathbb{D}$ appartenant à une certaine classe $X$, on cherche $g$ dans une autre classe $Y$ (plus petite que $X$ ou incomparable avec $X$) qui minimise la norme de $g$ dans $Y$ parmi toutes les fonctions $g$ satisfaisant la condition $g|_{\sigma} = f|_{\sigma}$. On montre que dans le cas $Y = H^\infty$, la constante d’interpolation correspondante $c(\sigma, X, H^\infty)$ admet une majoration $c(\sigma, X, H^\infty) \leq a \varphi_X(1 - \frac{1 - |z|}{n})$ où $n = \# \sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ et $\varphi_X(t)$ est la norme de la fonctionnelle d’évaluation $f \mapsto f(t)$ sur l’espace $X$. La majoration est exacte sur l’ensemble des $\sigma$ avec $n$ et $r$ donné.

Introduction

(1) General framework. Let $\text{Hol}(\mathbb{D})$ be the space of holomorphic functions on the unit disc $\mathbb{D}$. The problem considered is the following: given two Banach spaces $X$ and $Y$ of holomorphic functions on the unit disc $\mathbb{D}$, $X, Y \subset \text{Hol}(\mathbb{D})$, and a finite set $\sigma \subset \mathbb{D}$, to find the least norm interpolation by functions of the space $Y$ for the traces $f|_{\sigma}$ of functions of the space $X$, in the worst case of $f$. The case $X \subset Y$ is of no interests, and so one can suppose that either $Y \subset X$ or $X, Y$ are incomparable.

The classical interpolation problems- those of Nevanlinna-Pick (1916) and Carathéodory-Schur (1908) (see [N2] p.231 for these two problems), on the one hand and Carleson’s free interpolation (1958) (see [N1] p.158) on the other hand- are of this nature. Two first are “individual”, in the sens that one looks simply to compute the norms $\|f\|_{H^\infty}$ or $\|f\|_{H^\infty/\lambda^\sigma H^\infty}$ for a given $f$, whereas the third one is to compare the norms $\|a\|_{D(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$ and

$$\inf \left( \|g\|_\infty, g(\lambda) = a_\lambda, \lambda \in \sigma \right).$$

Let us first explain that our problem assemblies the ones of Nevanlinna-Pick and Carathéodory-Schur.

(i) Nevanlinna-Pick interpolation problem

Given $\Lambda = (\lambda_1, ..., \lambda_n)$ in $\mathbb{D}^n$ and $W = (w_1, ..., w_n) \in \mathbb{C}^n$, to find

$$C(\Lambda, W) = \inf \left\{ \|f\|_\infty : f(\lambda_i) = w_i, i = 1..n \right\}.$$
where for any \( n \times n \) matrix \( M \), \( M >> 0 \) means that \( M \) is positive definite.

(ii) Carathéodory-Schur interpolation problem

Given \( \mathcal{A} = (a_0, ..., a_n) \in \mathbb{C}^{n+1} \), to find

\[
C(\mathcal{A}) = \inf \{ f(z) : f(0) = a_0 + a_1 z + ... + a_n z^n + ... \}.
\]

The classical answer of Schur is the following:

\[
C(\mathcal{A}) = \| (T_\varphi)_n \|,
\]

where \( T_\varphi \) is the Toeplitz operator associated with a symbol \( \varphi \), \( (T_\varphi)_n \) is the compression of \( T_\varphi \) on \( \mathcal{P}_n \), the space of analytic polynomials of degree less or equal than \( n \), and \( \varphi \) is the polynomial \( \sum_{k=0}^{n} a_k z^k \).

Notice that the Carathéodory-Schur interpolation theorem can be seen as a particular case of the famous commutant lifting theorem of Sarason and Sz-Nagy-Foias (1968) see [N2] p.230, Theorem 3.1.11.

From a modern point of view, those two interpolation problems (i)&(ii) are unified through the following mixed problem: given

- \( \sigma = \{ \lambda_1, ..., \lambda_n \} \subseteq \mathbb{D} \), the finite Blaschke product \( B_\sigma = \prod b_{\lambda_j} \), where \( b_\lambda = \frac{1 - \lambda z}{1 - \overline{\lambda} z} \), \( \lambda \in \mathbb{D} \),
- \( f \in \text{Hol}(\mathbb{D}) \),

to compute or estimate

\[
\| f \|_{H^\infty/\text{BH}^\infty} = \inf \{ \| g \|_\infty : f - g \in B_\sigma \text{Hol}(\mathbb{D}) \}.
\]

The classical Nevanlinna-Pick problem corresponds to the case \( X = \text{Hol}(\mathbb{D}) \), \( Y = H^\infty \), and the one of Carathéodory-Schur to the case where \( \lambda_1 = \lambda_2 = ... = \lambda_n = 0 \) and \( X = \text{Hol}(\mathbb{D}) \), \( Y = H^\infty \).

Here and everywhere below, \( H^\infty \) stands for the space (algebra) of bounded holomorphic functions in the unit disc \( \mathbb{D} \) endowed with the norm \( \| f \|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \). Looking at this comparison problem, say, in the form of computing/estimating the interpolation constant

\[
c(\sigma, X, Y) = \sup_{f \in X, \| f \|_X \leq 1} \inf \{ \| g \|_Y : g|_\sigma = f|_\sigma \},
\]

which is nothing but the norm of the embedding operator \( (X|_\sigma, \| . \|_{X|_\sigma}) \rightarrow (Y|_\sigma, \| . \|_{Y|_\sigma}) \), one can think, of course, on passing (after) to the limit- in the case of an infinite sequence \( \{ \lambda_j \} \) and its finite sections \( \{ \lambda_j \}_{j=1}^n \) in order to obtain a Carleson type interpolation theorem \( X|_\sigma = Y|_\sigma \). But not necessarily. In particular, even the classical Nevanlinna-Pick theorem (giving a necessary and sufficient condition on a function \( a \) for the existence of \( f \in H^\infty \) such that \( \| f \|_\infty \leq 1 \) and \( f(\lambda) = a_\lambda, \lambda \in \sigma \)), does not lead immediately to Carleson’s criterion for \( H^\infty = l^\infty(\sigma) \). (Finally, a direct deduction of Carleson’s theorem from Pick’s result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for \( c(\sigma, X, Y) \) is of interest in its own. It is a kind of “effective interpolation” because we are looking for sharp estimations or a computation of \( c(\sigma, X, Y) \) for a variety of norms \( \| . \|_X, \| . \|_Y \). For this paper, the following partial case was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]) : given a set \( \sigma \subseteq \mathbb{D} \), how to estimate \( c(\sigma, H^2, H^\infty) \) in terms of \( n = \text{card}(\sigma) \) and \( \max_{\lambda \in \sigma} |\lambda| = r \) only? (\( H^2 \) being the standard Hardy space of the disc).

Here, we consider the case of \( H^\infty \) interpolation \( (Y = H^\infty) \) and the following scales of Banach spaces \( X \):
(a) $X = H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the standard Hardy spaces on the disc $\mathbb{D}$ (see [N2] p.31-p.57) of all $f \in \text{Hol}(\mathbb{D})$ satisfying

$$
sup_{0 \leq r < 1} \left( \int_{\mathbb{T}} |f(re^\theta)|^p dm(e^\theta) \right)^{1/p} < \infty,
$$

$m$ being the Lebesgue normalized measure on $\mathbb{T}$.

(b) $X = L^2_a(1/\sqrt{k} + 1)$, the Bergman space of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$
\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k + 1} < \infty.
$$

An equivalent description of this space is: $X = L^2_a$, the space of holomorphic functions such that

$$
\int_{\mathbb{D}} |f(z)|^2 dxdy < \infty.
$$

For spaces of type (a)&(b), we show

$$
c_1 \varphi_X \left( 1 - \frac{1-r}{n} \right) \leq \sup \{ c(\sigma, X, H^\infty) : \# \sigma \leq n, |\lambda| \leq r, \lambda \in \sigma \} \leq c_2 \varphi_X \left( 1 - \frac{1-r}{n} \right),
$$

where $\varphi_X(t)$, $0 \leq t < 1$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space $X$.

In order to prove the right-hand side inequality, we first use a linear interpolation:

$$
f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,
$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \bar{\hat{g}}(k)$, and $(e_k)_{k=1}^n$ is the Malmquist basis (effectively constructible) of the space $K_B = H^2 \Theta BH^2$, $B = \Pi_{i=1}^n \beta_i$ being the corresponding finite Blaschke product, $\beta_i = \frac{1-\bar{\lambda}_i}{1-\lambda_i}$ (see N. Nikolski, [N1] p. 117)). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [Tr] Theorem 1.9.3-(a) p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$ (in terms of $\|f\|_X$), the most important is a Bernstein-type inequality $\|f\|_p \leq c_p \|B\|_\infty^p \|f\|_p$ for a (rational) function $f$ in the star-invariant subspace $K_B^p := H^p \cap Bz\overline{HF}$, $1 \leq p \leq \infty$ (for $p = 2$, $K_B^2 = K_B$), generated by a (finite) Blaschke product $B$, (K. Dyakonov [Dya1] & [Dya2]). For $p = 2$, we give an alternative proof of the Bernstein-type estimate we need and the constant $c_2$ we obtain is slightly better, see Section 4.

The lower bound problem is treated by using the “worst” interpolation $n$-tuple $\sigma = \sigma_{n, \lambda} = \{\lambda, \ldots, \lambda\}$, a one-point set of multiplicity $n$ (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to $\lambda$. We notice that spaces $X$ of (a)&(b) satisfy the condition $X \circ b_\lambda \subset X$ which makes the problem of upper/lower bound easier.

(2) Principal results. Theorems A,C&D below in this paragraph, were already announced in the note [Z1].

Let $\sigma = \{\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_t, \ldots, \lambda_t\}$ be a finite sequence in the unit disc, where every $\lambda_s$ is repeated according its multiplicity $m_s$, $\sum_{s=1}^t m_s = n$ and $r = \max_{i=1,\ldots,t}|\lambda_i|$. Let $X, Y$ be Banach spaces of holomorphic functions continuously embedded into the space $\text{Hol}(\mathbb{D})$. In what follows, we systematically use the following conditions for the spaces $X$ and $Y$;

\begin{equation}
(P_1) \quad \text{Hol}((1+\epsilon)\mathbb{D}) \text{ is continuously embedded into } Y \text{ for every } \epsilon > 0,
\end{equation}
(P$_2$) \[ \text{Pol}_+ \subset X \text{ and Pol}_+ \text{ is dense in } X, \]

where Pol$_+$ stands for the set of all complex polynomials \( p, p(z) = \sum_{k=0}^{N} a_k z^k, \)

(P$_3$) \[ [f \in X] \Rightarrow \left[z^n f \in X , \forall n \geq 0 \text{ and } \lim z^n f \|_\infty \leq 1 \right], \]

(P$_4$) \[ [f \in X, \lambda \in \mathbb{D}, \text{ and } f(\lambda) = 0] \Rightarrow \left[ z^m f - \lambda \in X \right]. \]

We are interested in estimating the quantity

\[ c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i \}. \]

In order to simplify the notation, the condition

\[ g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i \]

will also be written as

\[ g|_{\sigma} = f|_{\sigma}. \]

Supposing \( X \) verifies property (P$_4$) and \( Y \subset X \), the quantity \( c(\sigma, X, Y) \) can be written as follows,

\[ c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g - f \in B_\sigma X \}, \]

where \( B_\sigma \) is the Blaschke product

\[ B_\sigma = \Pi_{i=1}^{n} b_{\lambda_i}, \]

corresponding to \( \sigma \), \( b_{\lambda}(z) = \frac{\lambda - z}{1 - \lambda \bar{z}} \) being an elementary Blaschke factor for \( \lambda \in \mathbb{D}. \)

The interesting case occurs when \( X \) is larger than \( Y \), and the sense of the issue lies in comparing \( \| . \|_X \text{ and } \| . \|_Y \) when \( Y \) interpolates \( X \) on the set \( \sigma \). For example, we can wonder what happens when \( X = H^p \), the classical Hardy spaces of the disc or \( X = L^p_\alpha \), the Bergman spaces, etc..., and when \( Y = H^\infty \), but also \( Y = W \) the Wiener algebra (of absolutely converging Fourier series) or \( Y = B^0_{\infty,1} \), a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition).

It is also important to understand what kind of interpolation we are going to study when bounding the constant \( c(\sigma, X, Y) \). Namely, comparing with the Carleson free interpolation, we can say that the latter one deals with the interpolation constant defined as

\[ c(\sigma, l^\infty(\sigma), H^\infty) = \sup \{ \inf \{ \| g \|_\infty : g \in H^\infty, g|_{\sigma} = a \} : a \in l^\infty(\sigma), \|a\|_{l^\infty} \leq 1 \}. \]

We also can add some more motivations to our problem :
(a) One of the most interesting cases is $Y = H^\infty$. In this case, the quantity $c(\sigma, X, H^\infty)$ has a meaning of an intermediate interpolation between the Carleson one (when $\|f\|_{X,\sigma} \approx \sup_{1 \leq i \leq n} |f(\lambda_i)|$) and the individual Nevanlinna-Pick interpolation (no conditions on $f$).

(b) There is a straight link between the constant $c(\sigma, X, Y)$ and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an $H^\infty$-calculus $\|f(A)\| \leq c \|f\|_\infty$, $f \in H^\infty$, for an arbitrary Banach space $n$-dimensional contraction $A$ with a given spectrum $\sigma(A) \subset \sigma$. The best possible constant is $c = c(\sigma, H^\infty, W)$, so that

$$c(\sigma, H^\infty, W) = \sup_{\|f\|_\infty \leq 1} \sup \{\|f(A)\| : A : (C^n, |.|) \to (C^n, |.|), \|A\| \leq 1, \sigma(A) \subset \sigma\},$$

where $W = \{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \}$ stands for the Wiener algebra, and the interior sup is taken over all contractions on $n$-dimensional Banach spaces. An interesting case occurs for $f \in H^\infty$ such that $f|\sigma = \frac{1}{2|z|^\sigma}$ (estimation of condition numbers and the norm of inverses of $n \times n$ matrices) or $f|\sigma = \frac{1}{\lambda - z|\sigma}$ (for estimation of the norm of the resolvent of an $n \times n$ matrix).

We start studying general Banach spaces $X$ and $Y$ and give some sufficient condition under which $C_{n,r}(X,Y) < \infty$, where

$$C_{n,r}(X,Y) = \sup \{c(\sigma, X, Y) : \#\sigma \leq n, \forall j = 1..n, |\lambda_j| \leq r\}.$$

In particular, we prove the following fact.

**Theorem A.** Let $X, Y$ be Banach spaces verifying properties $(P_i)$, $i = 1..4$. Then

$$C_{n,r}(X,Y) < \infty,$$

for every $n \geq 1$ and $r, 0 \leq r < 1$.

Next, we add the condition that $X$ is a Hilbert space, and give in this case a general upper bound for the quantity $C_{n,r}(X,Y)$.

**Theorem B.** Let $Y$ be a Banach space verifying property $(P_i)$ and $X = (H, (\cdot)_H)$ a Hilbert space satisfying properties $(P_i)$ for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in Hol(1 + \epsilon \mathbb{D})$ for all $|\lambda| < r$, where $k_\lambda$ stands for the reproducing kernel of $X$ at point $\lambda$, and $X \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $Hol((1 + \epsilon \mathbb{D})$-valued function. Let $\sigma = \{\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_l, ..., \lambda_l\}$ be a sequence in $\mathbb{D}$, where $\lambda_s$ are repeated according their multiplicity $m_s$, $\sum_{s=1}^l m_s = n$. Then we have,

i)  

$$c(\sigma, H, Y) \leq \left( \sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space $H$) of the sequence $k_{\lambda_1,0}, k_{\lambda_1,1}, k_{\lambda_2,0}, k_{\lambda_2,1}, k_{\lambda_2,2}, ..., k_{\lambda_2,m_2-1}, ..., k_{\lambda_l,0}, k_{\lambda_l,1}, k_{\lambda_l,2}, ..., k_{\lambda_l,m_l-1}$, and $k_{\lambda,s} = \left( \frac{d}{d\lambda} \right)^s k_\lambda$, $i \in \mathbb{N}$. 
ii) For the case $Y = H^{\infty}$, we have

$$c(\sigma, H, H^{\infty}) \leq \sup_{z \in \mathbb{D}} \|P_{B_{\sigma}}^{H} k_{z}\|_{H},$$

where $P_{B_{\sigma}}^{H} = \sum_{k=1}^{n} (\ldots, e_{k})_{H} e_{k}$ stands for the orthogonal projection of $H$ onto $K_{B_{\sigma}}(H)$,

$$K_{B_{\sigma}}(H) = \text{span} \left( k_{\lambda_{j}, i} : 1 \leq i < m_{j}, j = 1, \ldots, t \right).$$

After that, we specialize the upper bound obtained in Theorem B (ii) to the case $X = H^{2}$, the standard Hardy space of the disc, which can be equivalently defined as

$$H^{2}(\mathbb{D}) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^{k} : \sum_{k \geq 0} |\hat{f}(k)|^{2} < \infty \right\}.$$  

Among other results, we get the following (see Proposition 2.0) : for every sequence $\sigma = \{\lambda_{1}, \ldots, \lambda_{n}\}$ of $\mathbb{D}$,

$$c(\sigma, H^{2}, H^{\infty}) \leq \sup_{z \in \mathbb{D}} \left( \frac{1}{1 - |z|^{2}} \left| B_{\sigma}(z) \right|^{2} \right)^{\frac{1}{2}} \leq \sqrt{2} \sup_{|\zeta| = 1} |B'(|\zeta|)| \leq 2 \sqrt{\frac{n}{1 - r}}.$$  

Next, we present a slightly different approach to the interpolation constant $c(\sigma, H^{2}, H^{\infty})$ proving an estimate in the following form:

$$c(\sigma, H^{2}, H^{\infty}) \leq \sup_{z \in \mathbb{T}} \left( \sum_{k=1}^{n} \frac{1 - |\lambda_{k}|^{2}}{|z - \lambda_{k}|^{2}} \right)^{\frac{1}{2}} \leq \sqrt{\frac{2n}{1 - r}}.$$  

It is shown (in Section 6) that this estimate is sharp (over $n$ and $r$). This sharpness result is treated by using the “worst” interpolation $n$–tuple $\sigma = \sigma_{n, \lambda} = \{\lambda_{1}, \ldots, \lambda_{n}\}$, a one-point set of multiplicity $n$ (the Carathéodory-Schur type interpolation). More precisely, we prove the following Theorem C, which contains the result from Corollary 2.1 and extends it to the $H^{p}$ spaces, as follows.

**Theorem C.** Let $1 \leq p \leq \infty$, $n \geq 1$, $r \in [0, 1)$, and $\lambda, |\lambda| \leq r$. We have,

$$\frac{1}{32} \left( \frac{n}{1 - |\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n, \lambda}, H^{p}, H^{\infty}) \leq C_{n, r}(H^{p}, H^{\infty}) \leq A_{p} \left( \frac{n}{1 - r} \right)^{\frac{1}{p}},$$

where $A_{p}$ is a constant depending only on $p$ and the left hand side inequality is proved only for $p \in 2\mathbb{Z}_{+}$. For $p = 2$, we have $A_{2} = \sqrt{2}$.

In particular, this gives yet another proof of the fact that $C_{n, r}(H^{2}, H^{\infty}) \leq a\sqrt{n}/\sqrt{1 - r}$.

For the Bergman space $X = L_{a}^{2}$ we have the following Theorem D.

**Theorem D.** Let $n \geq 1$, $r \in [0, 1)$, and $\lambda, |\lambda| \leq r$. We have,

$$\frac{1}{32} \frac{n}{1 - |\lambda|} \leq c(\sigma_{n, \lambda}, L_{a}^{2}, H^{\infty}) \leq C_{n, r}(L_{a}^{2}, H^{\infty}) \leq 6\sqrt{2} \frac{n}{1 - r}.$$  

The paper is organized as follows. In Subsection 1.1 we prove Theorem A. Theorem B is proved in Subsection 1.2. Sections 2&3 are devoted to the proof of the upper estimate of Theorem C, and Section 6 to the proofs of the lower bounds from Theorem C&D. In Section 5 we compare the method used in Sections 1, 2, 3 and 4 with those resulting from the Carleson free interpolation. Especially, we are interested in the cases of circular and radial sequences $\sigma$ (see below).
1. Upper bounds for $c(\sigma, X, Y)$, as a kind of the Nevanlinna-Pick problem

1.1. General Banach spaces $X$ and $Y$ satisfying properties $(P_i)$, $i = 1...4$

In this Subsection, $X$ and $Y$ are Banach spaces which satisfy properties $(P_i)$ for $i = 1...4$. We prove Theorem A which shows that in this case our interpolation constant $c(\sigma, X, Y)$ is bounded by a quantity which depends only on $n = \#\sigma$ and $r = \max_{1 \leq i \leq n} |\lambda_i|$ (and of course on $X$ and $Y$). In this generality, we cannot discuss the question of sharpness of the bounds obtained. First, we prove the following lemma.

Lemma 1.1.0. Under $(P_2)$, $(P_3)$ and $(P_4)$, $B_\sigma X$ is a closed subspace of $X$ and moreover,

$$B_\sigma X = \{ f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)} \}.$$ 

Proof. Since $X \subset \text{Hol}(\mathbb{D})$ continuously, and evaluation functionals $f \mapsto f(\lambda)$ and $f \mapsto f^{(k)}(\lambda)$, $k \in \mathbb{N}^*$, are continous on $\text{Hol}(\mathbb{D})$, the subspace

$$M = \{ f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)} \},$$

is closed in $X$.

On the other hand, $B_\sigma X \subset X$, and hence $B_\sigma X \subset M$. Indeed, properties $(P_2)$ and $(P_3)$ imply that $h.X \subset X$, for all $h \in \text{Hol}((1 + \epsilon)\mathbb{D})$ with $\epsilon > 0$; we can write $h(z) = \sum_{k \geq 0} \hat{h}(k)z^k$ with $|\hat{h}(k)| \leq Cq^n$, $C > 0$ and $q < 1$. Then $\sum_{n \geq 0} \left\| \hat{h}(k)z^k f \right\|_X < \infty$ for every $f \in X$. Since $X$ is a Banach space we get $hf = \sum_{n \geq 0} \hat{h}(k)z^k f \in X$.

In order to see that $M \subset B_\sigma X$, it suffices to justify that

$$[f \in X \text{ and } f(\lambda) = 0] \implies [f/b_\lambda = (1 - \overline{\lambda} z)f/(\lambda - z) \in X].$$

But this is obvious from $(P_4)$ and the previous arguments. \hfill \Box

In Definitions 1.1.1, 1.1.2, 1.1.3 and in Remark 1.1.4 below, $\sigma = \{\lambda_1, ..., \lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$, $B_\sigma = \Pi_{i=1}^n b_{\lambda_i}$ is the finite Blaschke product corresponding to $\sigma$, where $b_{\lambda} = \frac{\lambda - z}{1 - \overline{\lambda} z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$.

Definition 1.1.1. Malmquist family. For $k \in [1, n]$, we set $f_k(z) = \frac{1}{1 - \overline{\lambda}_k z}$, and define the family $(e_k)_{k=1}^n$, (which is known as Malmquist basis, see [N1] p.117), by

\begin{equation}
    e_1 = \frac{f_1}{\|f_1\|_2} \quad \text{and} \quad e_k = \left(\Pi_{j=1}^{k-1} b_{\lambda_j}\right) \frac{f_k}{\|f_k\|_2},
\end{equation}

for $k \in [2, n]$, where $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$.

Definition 1.1.2. The model space $K_{B_\sigma}$. We define $K_{B_\sigma}$ to be the $n$-dimensional space :

\begin{equation}
    K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \Theta B_\sigma H^2.
\end{equation}

Definition 1.1.3. The orthogonal projection $P_{B_\sigma}$ on $K_{B_\sigma}$. We define $P_{B_\sigma}$ to be the orthogonal projection of $H^2$ on its $n$-dimensional subspace $K_{B_\sigma}$.

Remark 1.1.4. The Malmquist family $(e_k)_{k=1}^n$ corresponding to $\sigma$ is an orthonormal basis of $K_{B_\sigma}$. In particular,
\[(1.1.4) \quad P_{B_{\sigma}} = \sum_{k=1}^{n} (., e_k)_{H^2} e_k,\]

where \((., .)_{H^2}\) means the scalar product on \(H^2\).

**Lemma 1.1.5.** Let \(\sigma = \{\lambda_1, ..., \lambda_n\}\) be a sequence in the unit disc \(\mathbb{D}\) and \((e_k)_{k=1}^{n}\) the Malmquist family (see 1.1.1) corresponding to \(\sigma\). The map \(\Phi : \text{Hol}(\mathbb{D}) \to Y \subset \text{Hol}(\mathbb{D})\) defined by

\[
\Phi : f \mapsto \sum_{k=1}^{n} \left( \sum_{j \geq 0} \hat{f}(j)\overline{e_k(j)} \right) e_k,
\]

is well defined and has the following properties.

(a) \(\Phi|_{H^2} = P_{B_{\sigma}}\),

(b) \(\Phi\) is continuous on \(\text{Hol}(\mathbb{D})\) for the uniform convergence on compact sets of \(\mathbb{D}\),

(c) Let \(\Psi = Id|_X - \Phi|_X\), then \(\text{Im} (\Psi) \subset B_{\sigma}X\).

**Proof.** Indeed, the point (a) is obvious since \((e_k)_{k=1}^{n}\) is an orthonormal basis of \(K_{B_{\sigma}}\) and

\[
\sum_{j \geq 0} \hat{f}(j)\overline{e_k(j)} = \langle f, e_k \rangle,
\]

where \(\langle ., . \rangle\) means the Cauchy sesquilinear form \(\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k)\overline{g(k)}\). In order to check point (b), let \((\hat{f}_l)_{l \in \mathbb{N}}\) be a sequence of \(\text{Hol}(\mathbb{D})\) converging to 0 uniformly on compact sets of \(\mathbb{D}\). We need to see that \((\Phi(\hat{f}_l))_{l \in \mathbb{N}}\) converges to 0, for which it is sufficient to show that \(\lim_l \left| \sum_{j \geq 0} \hat{f}_l(j)\overline{e_k(j)} \right| = 0\), for every \(k = 1, 2, ..., n\). Let \(\rho \in ]0, 1[\), then \(\hat{f}_l(j) = (2\pi)^{-1} \int_{\rho}^{1} f_l(w)w^{-j-1}dw\), for all \(j, l \geq 0\). As a result,

\[
\left| \sum_{j \geq 0} \hat{f}_l(j)\overline{e_k(j)} \right| \leq \sum_{j \geq 0} \left| \hat{f}_l(j)\overline{e_k(j)} \right| \leq (2\pi\rho)^{-1} \|f_l\|_{\rho} \sum_{j \geq 0} |\overline{e_k(j)}| \rho^{-j}.
\]

Now if \(\rho\) is close enough to 1, it satisfies the inequality \(1 < \rho^{-1} < r^{-1}\), which entails \(\sum_{j \geq 0} |\overline{e_k(j)}| \rho^{-j} < +\infty\) for each \(k = 1..n\). The result follows.

We now prove point (c). Using point (a), since \(\text{Pol}_+ \subset H^2\) (\(\text{Pol}_+\) standing for the set of all complex polynomials \(p, p(z) = \sum_{k=0}^{N} a_k z^k\)), we get that \(\text{Im} (\Psi|_{Pol_+}) \subset B_{\sigma}H^2\). Now, since \(\text{Pol}_+ \subset Y\) and \(\text{Im}(\Phi) \subset Y\), we deduce that

\[
\text{Im} (\Psi|_{Pol_+}) \subset B_{\sigma}H^2 \cap Y \subset B_{\sigma}H^2 \cap X,
\]

since \(Y \subset X\). Now \(\Psi (p) \in X\) and satisfies \(\langle (\Psi (p))_{\sigma}, (\lambda) \rangle = 0\) (that is to say \(\langle (\Psi (p))_{\sigma}, (\lambda) \rangle = 0\), \(\forall \lambda \in \sigma\) (including multiplicities)) for all \(p \in \text{Pol}_+\). Using Lemma 1.1.0, we get that \(\text{Im} (\Psi|_{Pol_+}) \subset B_{\sigma}X\). Now, \(\text{Pol}_+\) being dense in \(X\) (property \((P_2)\)), and \(\Psi\) being continuous on \(X\) (point (b)), we can conclude that \(\text{Im} (\Psi) \subset B_{\sigma}X\). \(\square\)

**Proof of Theorem A.** Let \(\sigma = \{\lambda_1, ..., \lambda_n\}\) be a sequence in the unit disc \(\mathbb{D}\) and \((e_k)_{k=1}^{n}\) the Malmquist family (1.1.1) associated to \(\sigma\). Taking \(\hat{f} \in X\), we set
\[ g = \sum_{k=1}^{n} \left( \sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k, \]

where the series \( \sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \) are absolutely convergent. Indeed,

\[
\hat{e}_k(j) = (2\pi i)^{-1} \int_{RT} e_k(w) w^{-j-1} dw,
\]

for all \( j \geq 0 \) and for all \( R, 1 < R < \frac{1}{2} \). For a subset \( A \) of \( \mathbb{C} \) and for a bounded function \( h \) on \( A \), we define \( \|h\|_A := \sup_{z \in A} |h(z)| \). As a result,

\[
|\hat{e}_k(j)| \leq (2\pi R^{j+1})^{-1} \|e_k\|_{RT} \quad \text{and} \quad \sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi R)^{-1} \|e_k\|_{RT} \sum_{j \geq 0} \left| \hat{f}(j) \right| R^{-j} < \infty,
\]

since \( R > 1 \) and \( f \) is holomorphic in \( \mathbb{D} \).

We now suppose that \( \|f\|_X \leq 1 \) and \( g = \Phi(f) \), where \( \Phi \) is defined in Lemma 1.1.5. Since \( \text{Hol}(r^{-1}\mathbb{D}) \subset Y \), we have \( g \in Y \) and using Lemma 1.1.5 point (c) we get

\[ f - g = \Psi(f) \in B_\sigma X, \]

where \( \Psi \) is defined in Lemma 1.1.5, as \( \Phi \). Moreover,

\[ \|g\|_Y \leq \sum_{k=1, n} \|f, e_k\| \|e_k\|_Y. \]

In order to bound the right hand side, recall that for all \( j \geq 0 \) and for \( R = 2/(r + 1) \in ]1, 1/r[ \),

\[ \sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi)^{-1} \|e_k\|_{2(r+1)^{-1}T} \sum_{j \geq 0} \left| \hat{f}(j) \right| \left( 2^{-1}(r + 1) \right)^j. \]

Since the norm \( f \mapsto \sum_{j \geq 0} \left| \hat{f}(j) \right| \left( 2^{-1}(r + 1) \right)^j \) is continuous on \( \text{Hol}(\mathbb{D}) \), and the inclusion \( X \subset \text{Hol}(\mathbb{D}) \) is also continuous, there exists \( C_r > 0 \) such that

\[ \sum_{j \geq 0} \left| \hat{f}(j) \right| \left( 2^{-1}(r + 1) \right)^j \leq C_r \|f\|_X, \]

for every \( f \in X \). On the other hand, \( \text{Hol}(2(r + 1)^{-1}\mathbb{D}) \subset Y \) (continuous inclusion again), and hence there exists \( K_r > 0 \) such that

\[ \|e_k\|_Y \leq K_r \sup_{|z| < 2(r+1)^{-1}} |e_k(z)| = K_r \|e_k\|_{2(r+1)^{-1}T}. \]

It is more or less clear that the right hand side of the last inequality can be bounded in terms of \( r \) and \( n \) only. Let us give a proof to this fact. It is clear that it suffices to estimate

\[ \sup_{1 < |z| < 2(r+1)^{-1}} |e_k(z)|. \]

In order to bound this quantity, notice that

\[
(1.1.6) \quad |b_\lambda(z)|^2 \leq \left| \frac{\lambda - z}{1 - \lambda z} \right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \lambda z|^2},
\]
for all \( \lambda \in \mathbb{D} \) and all \( z \in |\lambda|^{-1}\mathbb{D} \). Using the identity (1.1.6) for \( \lambda = \lambda_j, \ 1 \leq j \leq n \), and \( z = \rho e^{it}, \rho = 2(1 + r)^{-1} \), we get

\[
|e_k(\rho e^{it})|^2 \leq \left( \prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \frac{1}{1 - \lambda_k \rho e^{it}} \leq \left( \prod_{j=1}^{k-1} \left( 1 + \frac{\rho^2 - 1}{1 - |\lambda_j|^2 \rho^2} \right) \right) \frac{1}{1 - |\lambda_k| \rho},
\]

for all \( k = 2..n \). Expressing \( \rho \) in terms of \( r \), we obtain

\[
\|e_k\|_{2(r+1)^{-1}T} \leq \frac{1}{1 - \frac{2r}{r+1}} \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2}{1 - r^2 \frac{1}{(r+1)^2}} \right) \right) : = C_1(r, n),
\]

and

\[
\sum_{j \geq 0} \left| \hat{f}(j) \bar{e_k}(j) \right| \leq (2\pi)^{-1} C_r \|e_k\|_{2(r+1)^{-1}T} \|f\| \|X\| \leq (2\pi)^{-1} C_r C_1(r, n) \|f\| \|X\|
\]

On the other hand, since

\[
\|e_k\|_Y \leq K_r \|e_k\|_{2(r+1)^{-1}T} \leq K_r C_1(r, n),
\]

we get

\[
\|g\|_Y \leq \sum_{k=1}^n (2\pi)^{-1} C_r C_1(r, n) \|f\| \|X\| K_r C_1(r, n) = (2\pi)^{-1} n C_r K_r (C_1(r, n))^2 \|f\| \|X\|
\]

which proves that

\[
c(\sigma, X, Y) \leq (2\pi)^{-1} n C_r K_r (C_1(r, n))^2
\]

and completes the proof of Theorem A. \( \square \)

### 1.2. The case where \( X \) is a Hilbert space

We suppose in this Subsection that \( X \) is a Hilbert space and both \( X, Y \) satisfy properties \((P_i)\) for \( i = 1..4 \). We prove Theorem B and obtain a better estimate for \( c(\sigma, X, Y) \) than in Theorem A (see point (i) of Theorem B). For the case \( Y = H^\infty \), (point (ii) of Theorem B), we can considerably improve this estimate. We omit an easy proof of the following lemma.

**Lemma 1.2.0.** Let \( \sigma = \{\lambda_1, ..., \lambda_n, \lambda_{n+1}, ..., \lambda_t\} \) be a finite sequence of \( \mathbb{D} \) where every \( \lambda_s \) is repeated according to its multiplicity \( m_s \), \( \sum_{s=1}^t m_s = n \). Let \((H, (.)_H)\) be a Hilbert space continuously emebedded into \( \text{Hol}(\mathbb{D}) \) and satisfying properties \((P_i)\) for \( i = 2, 3, 4 \). Then

\[
K_{B_\sigma}(H) =: H \Theta B_\sigma H = \text{span} \{k_{\lambda_j, i} : 1 \leq j \leq t, \ 0 \leq i \leq m_j - 1\},
\]

where \( k_{\lambda, i} = \left( \frac{d}{d\lambda} \right)^i k_\lambda \) and \( k_\lambda \) is the reproducing kernel of \( H \) at point \( \lambda \) for every \( \lambda \in \mathbb{D} \), i.e. \( k_\lambda \in H \) and \( f(\lambda) = (f, k_\lambda)_H, \forall f \in H \).

**Proof of Theorem B. i).** Let \( f \in X, \|f\|_X \leq 1 \). Lemma 1.2.0 shows that

\[
g = P_{B_\sigma}^H f = \sum_{k=1}^n (f, e_k)_H e_k
\]

is the orthogonal projection of \( f \) onto subspace \( K_{B_\sigma} \). Function \( g \) belongs to \( Y \) because all \( k_{\lambda_j, i} \) are in \( \text{Hol}(1 + \epsilon)\mathbb{D} \) for a convenient \( \epsilon > 0 \), and \( Y \) satisfies \((P_1)\).
On the other hand, \( g - f \in B_\sigma H \) (again by Lemma 1.2.0). Moreover, using Cauchy-Schwarz inequality,

\[
\|g\|_Y \leq \sum_{k=1}^{n} |(f, e_k)_H| \|e_k\|_Y \leq \left( \sum_{k=1}^{n} |(f, e_k)_H|^2 \right)^{1/2} \left( \sum_{k=1}^{n} \|e_k\|_Y^2 \right)^{1/2} \leq \|f\|_H \left( \sum_{k=1}^{n} \|e_k\|_Y^2 \right)^{1/2},
\]

which proves i).

ii). If \( Y = H^\infty \), then

\[
|g(z)| = |(P_{B^H} f, k^z)_H| = |(f, P_{B^H} k^z)_H| \leq \|f\|_H \|P_{B^H} k\|_H,
\]

for all \( z \in \mathbb{D} \), which proves ii). 

\[\square\]

2. Upper bounds for \( C_{n, r} \left( H^2, H^\infty \right) \)

In this Section, we specialize the upper estimate obtained in point (ii) of Theorem B for the case \( X = H^2 \), the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences \( \sigma \) (see Section 6). We also develop a slightly different approach to the interpolation constant \( c(\sigma, H^2, H^\infty) \) giving more estimates for individual sequences \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) of \( \mathbb{D} \). We finally prove the right-hand side inequality of Theorem C for the particular case \( p = 2 \).

**Proposition 2.0.** For every sequence \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) of \( \mathbb{D} \) we have

\[
(I_1) \quad c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left( \frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},
\]

\[
(I_2) \quad c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|z|=1} |B'(z)|^{1/2} = \sqrt{2} \sup_{|z|=1} \left| \sum_{i=1}^{n} \frac{1 - |\lambda_i|^2}{1 - \lambda_i \bar{z}} B_\sigma(\zeta) \right|^{1/2}.
\]

**Proof.** We prove \((I_1)\). Applying point (ii) of Theorem B for \( X = H^2 \) and \( Y = H^\infty \), and using

\[
k^z(\zeta) = \frac{1}{1 - \bar{z} \zeta} \quad \text{and} \quad (P_{B_\sigma} k^z)(\zeta) = \frac{1 - B_\sigma(z) B_\sigma(\zeta)}{1 - \bar{z} \zeta},
\]

(see [N1] p.199), we obtain

\[
\|P_{B_\sigma} k\|_{H^2} = \left( \frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},
\]

which gives the result.

We now prove \((I_2)\), using \((I_1)\). The map \( \zeta \mapsto \|P_{B} (k^z)\| = \sup \{|f(\zeta)| : f \in K_B, \|f\| \leq 1\} \), and hence the map

\[
\zeta \mapsto \left( \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},
\]

is a subharmonic function so

\[
\sup_{|\zeta|<1} \left( \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2} \leq \sup_{|w|=1} \lim_{r\to 1} \left( \frac{1 - |B(rw)|^2}{1 - |rw|^2} \right)^{1/2}.
\]
Now apply Taylor’s Formula of order 1 for points \( w \in \mathbb{T} \) and \( u = rw, 0 < r < 1 \). (It is applicable because \( B \) is holomorphic at every point of \( \mathbb{T} \)). We get

\[
(B(u) - B(w))(u - w)^{-1} = B'(w) + o(1),
\]

and since \( |u - w| = 1 - |u| \),

\[
|(B(u) - B(w))(u - w)^{-1}| = |B(u) - B(w)|(1 - |u|)^{-1} = |B'(w) + o(1)|.
\]

Now,

\[
|B(u) - B(w)| \geq |B(w)| - |B(u)| = 1 - |B(u)|,
\]

\[
(1 - |B(u)|)(1 - |u|)^{-1} \leq (1 - |u|)^{-1} |B(u) - B(w)| = |B'(w) + o(1)|,
\]

and

\[
\lim_{r \to 1} ((1 - |B(rw)|)(1 - |rw|)^{-1})^{1/2} \leq \sqrt{|B'(w)|}.
\]

Moreover,

\[
B'(w) = -\sum_{i=1}^{n} (1 - |\lambda_i|^2) (1 - \bar{\lambda}_i w)^{-2} \Pi_{j=1, j \neq i}^{n} b_{\lambda_j}(w),
\]

for all \( w \in \mathbb{T} \). This completes the proof since

\[
\frac{1 - |B(rw)|^2}{1 - |rw|^2} = \frac{(1 - |B(rw)|)(1 + |B(rw)|)}{(1 - |rw|)(1 + |rw|)} \leq 2 \frac{1 - |B(rw)|}{1 - |rw|}.
\]

\[\square\]

**Corollary 2.1.** Let \( n \geq 1 \) and \( r \in [0, 1] \). Then,

\[
C_{n,r}(H^2, H^\infty) \leq 2 \left( n(1 - r)^{-1} \right)^{1/2}.
\]

Indeed, applying Proposition 2.0 we obtain

\[
|B'(w)| \leq \left| \sum_{i=1}^{n} \frac{1 - |\lambda_i|^2}{(1 - |\lambda_i|)^2} \right| \leq n \frac{1 + r}{1 - r} \leq 2n \frac{1}{1 - r}.
\]

\[\square\]

Now, we develop a slightly different approach to the interpolation constant \( c(\sigma, H^2, H^\infty) \).

**Theorem 2.2.** For every sequence \( \sigma = \{\lambda_1, ..., \lambda_n\} \) of \( \mathbb{D} \),

\[
c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left( \frac{\sum_{k=1}^{n} (1 - |\lambda_k|^2)}{\sum_{k=1}^{n} |z - \lambda_k|^2} \right)^{1/2}
\]

\[\text{Proof.} \] In order to simplify the notation, we set \( B = B_\sigma \). We consider \( K_B \) (see Definition 1.1.2) and the Malmquist family \( (e_k)_{k=1}^{n} \) corresponding to \( \sigma \) (see Definition 1.1.1). Now, let \( f \in H^2 \) and

\[
g = P_B f = \sum_{k=1}^{n} (f, e_k)_{H^2} e_k,
\]

(see Definition 1.1.3 and Remark 1.1.4). Function \( g \) belongs to \( H^\infty \) (it is a finite sum of \( H^\infty \) functions) and satisfies \( g - f \in BH^2 \). Applying Cauchy-Schwarz inequality we get
\[ |g(\zeta)| \leq \sum_{k=1}^{n} |(f, e_k)_{H^2}| e_k(\zeta) \leq \left( \sum_{k=1}^{n} |(f, e_k)_{H^2}|^2 \right)^{1/2} \left( \sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{1/2}, \]

for all \( \zeta \in \mathbb{D} \). As a result, since \( f \) is an arbitrary \( H^2 \) function, we obtain

\[ c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{T}} \left( \sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{1/2}, \]

which completes the proof. \( \square \)

**Corollary 2.3.** For any sequence \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) in \( \mathbb{D} \),

\[ c(\sigma, H^2, H^\infty) \leq \left( \sum_{j=1}^{n} \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{1/2}. \]

Indeed,

\[ \sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left( \sum_{k=1}^{n} \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{1/2} \]

and the result follows from Theorem 2.2. \( \square \)

**Proof of Theorem C (\( p = 2 \), the right-hand side inequality only).** Since \( 1 + |\lambda_j| \leq 2 \) and \( 1 - |\lambda_j| \geq 1 - r \) for all \( j \in [1, n] \), applying Corollary 2.3 we get

\[ C_{n,r}(H^2, H^\infty) \leq \sqrt{2n^{1/2}(1 - r)^{-1/2}}. \] \( \square \)

**Remark 2.4.** As a result, we get once more the same estimate for \( C_{n,r}(H^2, H^\infty) \) as in Corollary 2.1, with the constant \( \sqrt{2} \) instead of \( 2 \).

It is natural to wonder if it is possible to improve the bound \( \sqrt{2n^{1/2}(1 - r)^{-1/2}} \). We return to this question in Section 5 below.

### 3. Upper bounds for \( C_{n,r}(H^p, H^\infty) \), \( p \geq 1 \)

In this Section we extend Corollary 2.1 to all Hardy spaces \( H^p \): we prove the right-hand side inequality of Theorem C, \( p \neq 2 \). We first prove the following lemma.

**Lemma 3.0.** Let \( n \geq 1 \) and \( 0 \leq r < 1 \). Then,

\[ C_{n,r}(H^1, H^\infty) \leq 2n(1 - r)^{-1}. \]

**Proof.** Let \( f \in H^1 \) such that \( \|f\|_{H^1} \leq 1 \) and let

\[ g = \Phi(f) = \sum_{k=1}^{n} \langle f, e_k \rangle e_k, \]

where, as always, \( (e_k)_{k=1}^{n} \) is the Malmquist basis corresponding to \( \sigma \) (see 1.1.1), \( \Phi \) is defined in Lemma 1.1.5, and where \( \langle . , . \rangle \) means the Cauchy sesquilinear form \( \langle f, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{g(k)} \). That is to say that,
for all $\zeta \in \mathbb{D}$, which gives,

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1}^n e_k e_k(\zeta) \right\|_{H^\infty} \leq \left\| \sum_{k=1}^n e_k e_k(\zeta) \right\|_{H^\infty}.$$ 

Since Blaschke factors have modulus 1 on the unit circle, 

$$\|e_k\|_{H^\infty} \leq (1 + |\lambda_k|)^{1/2} (1 - |\lambda_k|)^{-1/2}.$$ 

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^n \|e_k\|_{H^\infty} \left| e_k(\zeta) \right| \leq \sum_{k=1}^n \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^n (1 + |\lambda_k|) (1 - |\lambda_k|)^{-1} \leq 2n(1 - r)^{-1},$$ 

for all $\zeta \in \mathbb{D}$, which completes the proof. \hfill $\square$

**Proof of Theorem C** ($p \neq 2$, the right-hand side inequality only). Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a sequence in the unit disc $\mathbb{D}$, $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, and $T : H^p \rightarrow H^\infty / B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma H^p\},$$

for every $f$. Then,

$$\| T \|_{H^p \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, H^p, H^\infty).$$

There exists $0 \leq \theta \leq 1$ such that $1/p = 1 - \theta$, and since (we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be]) $[H^1, H^\infty]_\theta = H^p$ (a topological identity : the spaces are the same and the norms are equivalent (up to constants depending on $p$ only), see [J]),

$$\| T \|_{[H^1, H^\infty]_\theta \rightarrow H^\infty / B_\sigma H^\infty} \leq (A_1 c(\sigma, H^1, H^\infty))^{1-\theta} (A_\infty c(\sigma, H^\infty, H^\infty))^\theta,$$

where $A_1, A_\infty$ are numerical constants, and using, Lemma 3.0, the fact that $c(\sigma, H^\infty, H^\infty) \leq 1$, and a known interpolation Theorem (see [Tr], Theorem 1.9.3-(a) p.59), we find

$$\| T \|_{[H^1, H^\infty]_\theta \rightarrow H^\infty / B_\sigma H^\infty} \leq (2A_1 n(1 - r)^{-1})^{1-\theta} A_\infty^\theta = (2A_1)^{1-\theta} A_\infty^\theta (n(1 - r)^{-1})^{1/2},$$

which completes the proof. \hfill $\square$

### 4. Upper bounds for $C_{n,r} \left( L^2_a, H^\infty \right)$

In this Section, we generalize Corollary 2.1 to the case of spaces $X$ which contain $H^2$: $X = l^2_a((k+1)^a), a \leq 0$, the Hardy weighted spaces of all $f(z) = \sum_{k \geq 0} \hat{f}(k) z^k$ satisfying

$$\|f\|_{X}^2 := \sum_{k \geq 0} |\hat{f}(k)|^2 (k+1)^{2a} < \infty.$$ 

Notice that $H^2 = l^2(1)$ and $L^2_a(\mathbb{D}) = l^2_a((k+1)^{-\frac{a}{2}})$. We prove the right-hand side inequality of Theorem D and the main technical tool used in its proof is a Bernstein-type inequality for rational functions.

#### 4.1. Bernstein-type inequalities for rational functions
Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [L], [BoEr], [DeLo], [Bl]). Perhaps, the stronger and closer to ours (Proposition 4.1) of all known results are due to K.Dyakonov [Dya1&[Dya2]. First, we prove Proposition 4.1 below, which tells that if $\sigma = \{\lambda_1, ..., \lambda_n\} \subset \mathbb{D}$, $r = \max_j |\lambda_j|$, and $f \in K_{B_\sigma}$, then

$$(\ast) \quad \|f\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where $\alpha_{n,r}$ is a constant (explicitly given in Proposition 4.1) depending on $n$ and $r$ only such that $0 < \alpha_{n,r} \leq \frac{5}{2} \frac{n}{1-r}$. Proposition 4.1 is in fact a partial case ($p = 2$) of the following K. Dyakonov’s result [Dya1] (which is, in turn, a generalization of M. Levin’s inequality [L] corresponding to the case $p = \infty$): it is proved in [Dya1] that the norm $\|D\|_{K_B^{p} \to H^p}$ of the differentiation operator $Df = f'$ on the star-invariant subspace of the Hardy space $H^p$, $K_B^p := H^p \cap B\overline{zH^p}$, (where the bar denotes complex conjugation) satisfies the following inequalities

$$c_p \|B\|_{\infty} \leq \|D\|_{K_B^{p} \to H^p} \leq c_p' \|B\|_{\infty},$$

for every $p$, $1 \leq p \leq \infty$ where $c_p$ and $c_p'$ are positives constants depending on $p$ only, $B$ is a finite Blaschke product and $\|\cdot\|_{\infty}$ means the norm in $L^\infty(\mathbb{T})$. For the partial case considered in Proposition 4.1 below, our proof is different and the constant is slightly better. More precisely, it is proved in [Dya1] that $c_2' = \frac{1}{3c_2}$, $c_2 = \frac{36+\pi}{2\pi}$ and $c = 2\sqrt{3\pi}$ (as one can check easily ($c$ is not precised in [Dya1])). It implies an inequality of type $(\ast)$ (with a constant about $\frac{14 \pi}{2}$ instead of $\frac{5}{2}$).

In [Z2], we discuss the “asymptotic sharpness” of our constant $\alpha_{n,r}$: we find an inequality for

$$\sup \|D\|_{K_B \to H^2} = C_{n,r} \quad \text{(sup is over all } B \text{ with given } n = \deg B \text{ and } r = \max_{\lambda \in \sigma} |\lambda|,)$$

which is asymptotically sharp as $n \to \infty$. Our result in [Z2] is that there exists a limit $\lim_{n \to \infty} C_{n,r} \frac{n}{1-r} = \frac{1+r}{1-r}$ for every $r$, $0 \leq r < 1$. Our method is different from [Dya1&[Dya2] and is based on an elementary Hilbert space construction for an orthonormal basis in $K_B$.

**Proposition 4.1.** Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order $n$), $r = \max_j |\lambda_j|$, and $f \in K_B = H^2 \Theta BH^2$. Then for every $n \geq 2$ and $r \in [0, 1)$,

$$\|f'\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where $\alpha_{n,r} = \left[1 + (1+r)(n-1) + \sqrt{n-2}\right] (1-r)^{-1}$ and in particular,

$$\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2},$$

for all $n \geq 1$ and $r \in [0, 1)$.

**Proof.** Using Remark 1.1.4, $f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k$, $\forall f \in K_B$. Noticing that,

$$e_k' = \sum_{i=1}^{k-1} \frac{b_{\lambda_i}}{b_{\lambda_k}} e_k + \frac{1}{(1-\lambda_k z)} e_k,$$

for $k \in [2, n]$, we get

$$f' = (f, e_1)_{H^2} e_1' + \sum_{k=2}^n (f, e_k)_{H^2} e_k' =$$

$$= (f, e_1)_{H^2} \frac{\lambda_1}{(1-\lambda_1 z)} e_1 + \sum_{k=2}^n (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b_{\lambda_i}}{b_{\lambda_k}} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \lambda_k \frac{1}{(1-\lambda_k z)} e_k,$$
which gives

\[ f' = (f, e_1)_{H^2} \frac{\lambda_1}{(1 - \lambda z)} e_1 + \sum_{k=2}^{n} \sum_{i=1}^{n-1} (f, e_k)_{H^2} b_{\lambda_i} b_{\lambda_i}^* e_k \chi_{[1, k-1]}(i) + \sum_{k=1}^{n} (f, e_k)_{H^2} \frac{1}{(1 - \lambda k z)} e_k = \]

\[ = (f, e_1)_{H^2} \frac{\lambda_1}{(1 - \lambda_1 z)} e_1 + \sum_{i=1}^{n} b_{\lambda_i} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=1}^{n} (f, e_k)_{H^2} \frac{1}{(1 - \lambda k z)} e_k, \]

where \( \chi_{[1, k-1]} \) is the characteristic function of \([1, k - 1]\). Now,

\[ \left\| (f, e_1)_{H^2} \frac{\lambda_1}{(1 - \lambda_1 z)} e_1 \right\|_{H^2} \leq \left\| (f, e_1)_{H^2} \right\|_{H^2} \leq \frac{1}{1 - r} \left\| e_1 \right\|_{H^2} \leq \frac{1}{1 - r}, \]

using Cauchy-Schwarz inequality and the fact that \( e_1 \) is a vector of norm 1 in \( H^2 \). By the same reason, we have

\[ \left\| \sum_{k=2}^{n} \frac{\lambda_k}{(1 - \lambda k z)} e_k \right\|_{H^2} \leq \sum_{k=2}^{n} \left\| (f, e_k)_{H^2} \right\|_{H^2} \left\| \frac{1}{(1 - \lambda k z)} \right\|_{\infty} \left\| e_k \right\|_{H^2} \leq \left( \sum_{k=2}^{n} \left| (f, e_k)_{H^2} \right|^2 \right)^{1/2} \sqrt{n - 2} \leq \frac{1}{1 - r} \left\| f \right\|_{H^2} \sqrt{n - 2}. \]

Further,

\[ \left\| \sum_{i=1}^{n-1} b_{\lambda_i} b_{\lambda_i}^* \sum_{k=i+1}^{n} (f, e_k)_{H^2} e_k \right\|_{H^2} \leq \sum_{i=1}^{n-1} \left\| b_{\lambda_i} b_{\lambda_i}^* \right\|_{\infty} \sum_{k=i+1}^{n} \left| (f, e_k)_{H^2} \right|^2 \right\|_{H^2} = \]

\[ = \left( \max_{1 \leq i \leq n-1} \left\| b_{\lambda_i} b_{\lambda_i}^* \right\|_{\infty} \right) \sum_{i=1}^{n-1} \left( \sum_{k=i+1}^{n} \left| (f, e_k)_{H^2} \right|^2 \right)^{1/2} \leq \max_{i=1}^n \left\| b_{\lambda_i} b_{\lambda_i}^* \right\|_{\infty} \left\| f \right\|_{H^2}. \]

Now, using

\[ \left\| b_{\lambda_i} b_{\lambda_i}^* \right\|_{\infty} = \left\| \frac{|\lambda_i|^2 - 1}{(1 - \lambda_i z) (\lambda_i - z)} \right\|_{\infty} \leq \frac{1 + |\lambda_i|}{1 - |\lambda_i|} \leq \frac{1 + r}{1 - r}, \]

we get

\[ \left\| \sum_{i=1}^{n-1} b_{\lambda_i} b_{\lambda_i}^* \sum_{k=i+1}^{n} (f, e_k)_{H^2} e_k \right\|_{H^2} \leq (1 + r) \frac{n - 1}{1 - r} \left\| f \right\|_{H^2}. \]

Finally,

\[ \left\| f' \right\|_{H^2} \leq \left[ 1 + (1 + r)(n - 1) + \sqrt{n - 2} \right] (1 - r)^{-1} \left\| f \right\|_{H^2}. \]

In particular,

\[ \left\| f \right\|_{H^2} \leq (2n - 1 + \sqrt{n - 2}) (1 - r)^{-1} \left\| f \right\|_{H^2} \leq 5.2^{-1} n(1 - r)^{-1} \left\| f \right\|_{H^2}, \]

for all \( n \geq 2 \) and for every \( f \in K_B \). (The case \( n = 1 \) is obvious because \( \left\| f' \right\|_{H^2} \leq (1 - r)^{-1} \left\| f \right\|_{H^2} \) for every \( f\) of the form \( f = (1 - \lambda z)^{-1}, \lambda \in \mathbb{D} \)). \( \square \)

**4.2. AN UPPER BOUND FOR** \( c \left( \sigma, L^2_{\alpha}, H^\infty \right) \)
Corollary 4.2. Let $\sigma$ be a sequence in $\mathbb{D}$. Then,
\[ c(\sigma, l^2_\alpha ((k+1)^{-1}), H^\infty) \leq 6\sqrt{2} (n(1-r)^{-1})^{3/2}. \]
Indeed, let $H = l^2_\alpha ((k+1)^{-N})$ and $B = B_\sigma$ the finite Blaschke product corresponding to $\sigma$. Let $\widetilde{P}_B$ be the orthogonal projection of $H$ onto $K_B = K_B(H^2)$. Then $\widetilde{P}_B|_{H^2} = P_B$, where $P_B$ is defined in 1.1.4. We notice that $\widetilde{P}_B : H \to H$ is a bounded operator and the adjoint $\widetilde{P}_B^* : H^* \to H^*$ of $\widetilde{P}_B$ relatively to the Cauchy pairing $\langle \cdot, \cdot \rangle$ satisfies $\widetilde{P}_B^* \varphi = \widetilde{P}_B \varphi = P_B \varphi$, $\forall \varphi \in H^* \subset H^2$, where $H^* = l^2_\alpha ((k+1)^N)$ is the dual of $H$ with respect to this pairing. If $f \in H$, then $|\widetilde{P}_B f(\zeta)| = |\langle \widetilde{P}_B f, k_\zeta \rangle| = |\langle f, \widetilde{P}_B^* k_\zeta \rangle|$, where $k_\zeta = (1 - \zeta z)^{-1} \in H^2$ and
\[ |\widetilde{P}_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K \left(\|P_B k_\zeta\|_{H^2} + \|P_B k_\zeta\|_{H^2} \right), \]
where
\[ K = \max \{1, \sup_{k \geq 1} (k+1)^{-1}\} = 2 \]
Since $P_B k_\zeta \in K_B$, Proposition 4.1 implies
\[ |\widetilde{P}_B f(\zeta)| \leq \|f\|_H K \left(\|P_B k_\zeta\|_{H^2} + 5.2^{-1} (n(1-r)^{-1}) \|P_B k_\zeta\|_{H^2} \right) \leq A (n(1-r)^{-1})^{3/2} \|f\|_H, \]
where $A = \sqrt{2} K (1/2 + 5/2) = 6\sqrt{2}$, since $\|P_B k_\zeta\|_2 \leq \sqrt{2} (n(1-r)^{-1})^{1/2}$, and since we can suppose $n \geq 2$, (the case $n = 1$ being obvious).

\[ \square \]

Proof of Theorem E (the right-hand side inequality only). The case $\alpha = 0$ corresponds to $X = H^2$ and has already been studied in Section 1 (we can choose $A(0) = \sqrt{2}$). We now suppose $\alpha < 0$. Let $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ and $T : l^2_\alpha ((k+1)^{\alpha}) \longrightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by
\[ T f = \{g \in H^\infty : f - g \in B_\sigma l^2_\alpha ((k+1)^{\alpha}) \}, \]
for every $f$. Then,
\[ \|T\|_{l^2_\alpha ((k+1)^{\alpha}) \to H^\infty/B_\sigma H^\infty} = c(\sigma, l^2_\alpha ((k+1)^{\alpha}), H^\infty). \]

Setting $\theta = -\alpha$ with $0 < \theta \leq 1$, we have (as in Theorem D, we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])
\[ [l^2_\alpha ((k+1)^0), l^2_\alpha ((k+1)^{-1})]_{\theta,2} = l^2_\alpha \left(\left((k+1)^0\right)^{2\frac{1-\theta}{2}} \left((k+1)^{-1}\right)^{2\frac{\theta}{2}}\right) = l_\alpha \left((k+1)^{\alpha}\right), \]
which entails, using Corollary 4.2 and (again) [Tr] Theorem 1.9.3-(a) p.59,
\[ \|T\|_{l^2_\alpha ((k+1)^{\alpha}) \to H^\infty/B_\sigma H^\infty} \leq \left(c(\sigma, l^2_\alpha ((k+1)^0), H^\infty)\right)^{1-\theta} \left(c(\sigma, l^2_\alpha ((k+1)^{-1}), H^\infty)\right)^{\theta} \leq A(0)^{1-\theta} A(1)^{\theta} \left(n(1-r)^{-1}\right)^{\frac{1-\theta}{2} + \frac{\theta}{2}}. \]
It remains to use $\theta = -\alpha$ and set $A(\alpha) = A(0)^{1-\theta} A(1)^{\theta}$. In particular, for $\alpha = -1/2$ we get $(1-\theta)/2 + 3\theta/2 = 1$ and
\[ A(-1/2) = A(0)^{-1/2} A(1)^{1/2} = \sqrt{2} \left(6\sqrt{2}\right)^{1/2} = 2\sqrt{3}. \]

\[ \square \]
5. ABOUT THE LINKS WITH CARLESON INTERPOLATION

Recall that given a (finite) set $\sigma = \{\lambda_1, ..., \lambda_n\} \subset \mathbb{D}$, the Carleson interpolation constant $C_1(\sigma)$ is defined by

$$C_1(\sigma) = \sup_{\|a\|_\infty \leq 1} \inf \left( \| g \|_\infty : g \in H^\infty, \ g|_\sigma = a \right).$$

We introduce the evaluation functionals $\varphi_\lambda$ for $\lambda \in \mathbb{D}$, as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, ...$)

$$\varphi_\lambda(f) = f(\lambda), \ f \in X, \ \text{and} \ \varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \ f \in X.$$

**Theorem 5.1.** Let $X$ be a Banach space, $X \subset Hol(\mathbb{D})$, and $\sigma = \{\lambda_1, ..., \lambda_n\}$ be a sequence of distinct points in the unit disc $\mathbb{D}$. We have,

$$\max_{1 \leq i \leq n} \| \varphi_{\lambda_i} \| \leq c(\sigma, X, H^\infty) \leq C_1(\sigma) \max_{1 \leq i \leq n} \| \varphi_{\lambda_i} \|,$$

where $C_1(\sigma)$ stands for the Carleson interpolation constant.

Theorem 5.1 tells us that, for $\sigma$ with a “reasonable” interpolation constant $C_1(\sigma)$, the quantity $c(\sigma, X, H^\infty)$ behaves as $\max_i \| \varphi_{\lambda_i} \|$. However, for “tight” sequences $\sigma$, the constant $C_1(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem 5.1 is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences $\sigma$, Theorem 5.1 should give a better estimate than those of Theorem C and Theorem D.

Now, how does the interpolation constant $C_1(\sigma)$ behave in terms of the characteristics $r$ and $n$ of $\sigma$? We answer this question for some particular sequences $\sigma$, see Examples 5.2, 5.3 and 5.4.

**Proof of Theorem 5.1.** Let $f \in X$. By definition of $C_1(\sigma)$, there exists $g \in H^\infty$ such that

$$f(\lambda_i) = g(\lambda_i) \ \forall i = 1..n \ \text{with} \ \| g \|_\infty \leq C_1(\sigma) \max_i |f(\lambda_i)| \leq C_1(\sigma) \max_i \| \varphi_{\lambda_i} \| \| f \|_X.$$

Now, taking the supremum over all $f \in X$ such that $\| f \|_X \leq 1$, we get the right-hand side inequality. The left-hand side one is clear since if $g \in H^\infty$ satisfies $f(\lambda_i) = g(\lambda_i) \ \forall i = 1..n$, then $\| g \|_\infty = |g(\lambda_i)| = |f(\lambda_i)| = |\varphi_{\lambda_i}(f)|, \ \forall i = 1..n$. \hfill $\Box$

Now, how does the interpolation constant $C_1(\sigma)$ behave in terms of the characteristics $r$ and $n$ of $\sigma$?

In what follows, we compare these quantities for three geometrically simple configurations: two-points sets $\sigma$, circular and radial sequences $\sigma$. The proofs of the following statements (5.2, 5.3 and 5.4) are given in [Z3].

**Example. 5.2.** Two points sets. Let $\sigma = \{\lambda_1, \lambda_2\}, \ \lambda_i \in \mathbb{D}, \ \lambda_1 \neq \lambda_2$. Then,

$$|b_{\lambda_1}(\lambda_2)|^{-1} \leq C_1(\sigma) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1},$$

and Theorem 5.1 implies

$$c(\sigma, X, H^\infty) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1} \max_{i=1,2} \| \varphi_{\lambda_i} \|,$$

whereas a straightforward estimate gives

$$c(\sigma, X, H^\infty) \leq \| \varphi_{\lambda_1} \| + \max_{|\lambda_i| \leq r} \| \varphi_{\lambda,1} \| (1 + |\lambda_1|),$$

where $r = \max (|\lambda_1|, |\lambda_2|)$ and the functional $\varphi_{\lambda,1}$ is defined in the beginning of Section 5. The difference is that the first upper bound blows up when $\lambda_1 \to \lambda_2$, whereas the second one is still well-bounded.

**Example. 5.3.** Circular sequences. Let $0 < r < 1$ and $\sigma = \{\lambda_1, \lambda_2, ..., \lambda_n\}, \ \lambda_i \neq \lambda_j, \ |\lambda_i| = r$ for every $i$, and let $\alpha = \min_i |\lambda_i - \lambda_j|/(1 - r)$. Then, $\alpha^{-1} \leq C_1(\sigma) \leq 8e^{K'(1+K\alpha^{-3})}$, where $K, K' > 0$ are absolute constants. Therefore,

$$c(\sigma, X, H^\infty) \leq 8e^{K'(1+K\alpha^{-3})} \max_{|\lambda_i| = r} \| \varphi_{\lambda} \|$$
for every $r$—circular set $\sigma$ (an estimate does not depending on $n$ explicitly). In particular, there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $n$ uniformly distributed points $\lambda_1, \ldots, \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2\sin \left(\frac{\pi}{2n}\right)$, we have
\[
(1) \quad c(\sigma, H^2, H^\infty) \leq \varphi(n(1-r)^{-1})(1-r)^{-\frac{1}{2}}, \text{ for every } n \text{ and } r, 0 < r < 1 \quad \text{and in particular, for } n \leq [r(1-r)^{-1}] \text{ we obtain}

\[
c(\sigma, H^2, H^\infty) \leq c(1-r)^{-\frac{1}{2}},
\]
whereas our specific upper bound in Theorem C, (which is sharp over all $n$ elements sequences $\sigma$), gives
\[
c(\sigma, H^2, H^\infty) \leq c(1-r)^{-1}
\]
only.

(2) $c(\sigma, L^2_a, H^\infty) \leq \varphi(n(1-r)^{-1})(1-r)^{-1}$, for every $n$ and $r$, $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain
\[
c(\sigma, L^2_a, H^\infty) \leq c(1-r)^{-1},
\]
whereas our specific upper bound in Theorem D, (which, again, is sharp over all $n$ elements sequences $\sigma$), gives
\[
c(\sigma, L^2_a, H^\infty) \leq c(1-r)^{-2}
\]
only.

**Example. 5.4. Radial sequences.** Now we consider geometric sequences on the radius of the unit disc $\mathbb{D}$, say on the radius $[0, 1)$. Let $0 < \rho < 1$, $p \in (0, \infty)$ and
\[
\lambda_j = 1 - \rho^j \rho^p, \quad j = 0, \ldots, n,
\]
so that the distances $1 - \lambda_j = \rho^j \rho^p$ form a geometric progression; the starting point is $\lambda_0 = 1 - \rho^p$. Let
\[
r = \max_{0 \leq j \leq n} \lambda_j = \lambda_k = 1 - \rho^{n+p},
\]
and $\delta = \delta(B) = \min_{0 \leq k \leq n} |B_k(\lambda_k)|$, where $B_k = b_{\lambda_k}^{-1}B$. It is known that $\delta^{-1} \leq C_1(\sigma) \leq 8\delta^{-2}$, see ([N1], p 189). So, we need to know the asymptotic behaviour of $\delta = \delta(B)$ when $n \to \infty$, or $\rho \to 1$, or $\rho \to 0$, or $p \to \infty$, or $p \to 0$.

Claim. Let $\sigma_{n, \rho, p} = \{1 - \rho^{p+k}\}^n_{k=1}, 0 < \rho < 1$, $p > 0$. The estimate of $c(\sigma, H^2, H^\infty)$ via the Carleson constant $C_1(\sigma)$ (using Theorem 5.1) is comparable with or better than the estimates from Theorem C (for $X = H^2$) and Theorem D (for $X = L^2_a$) for sufficiently small values of $\rho$ (as $\rho \to 0$) and/or for a fixed $\rho$ and $n \to \infty$. In all other cases, as for $p \to \infty$ (which means $\lambda_1 \to 1$), or $\rho \to 1$, or $n \to \infty$ and $\rho \to 1$, it is worse.

**Remark 5.5.** More specific radial sequences are studied in [Z3]: sparse sequences $\sigma$ ($\rho \to 0$, or at least $0 < \rho \leq \epsilon < 1$), condensed sequences $\sigma$ ($\rho \to 1$) and long sequences ($n \to \infty$).

### 6. Lower bounds for $C_{n,r}(X, H^\infty)$

#### 6.1. The cases $X = H^2$ and $X = L^2_a$

Here, we consider the standard Hardy and Begman spaces on the disc $\mathbb{D}$: $X = H^2 = l^2_a(1)$ and $X = L^2_a = l^2_a((k + 1)^{-1/2})$, and the problem of lower estimates for the one point special case $\sigma_{n, \lambda} = \{\lambda, \lambda, \ldots, \lambda\}$, $(n$ times) $\lambda \in \mathbb{D}$. Recall the definition of our constrained interpolation constant for this case
\[
c(\sigma_{n, \lambda}, H, H^\infty) = \sup \left\{ \|f\|_{H^\infty/\mathcal{B}H^\infty} : f \in H, \|f\|_H \leq 1 \right\},
\]
where \( \|f\|_{H^\infty/\beta^aH^\infty} = \inf \{ \|f + b^a_g\|_\infty : g \in H \} \). Our goal in this Subsection is to prove the sharpness of the upper estimate from Theorem C \((p = 2)\) and Theorem D for the quantities \( C_{n,r}(H^2, H^\infty) \) and \( C_{n,r}(L^2_a, H^\infty) \), that is to say, to get the lower bounds from Theorem C \((p = 2)\) and Theorem D.

Recall that the spaces \( l^2_a((k + 1)^\alpha) \) are defined in Section 4.

In the proof, we use properties of reproducing kernel Hilbert space on the disc \( \mathbb{D} \), see for example [N2]. Let us recall some of them adapting the general setting to special cases \( X = l^2_a((k + 1)^\alpha) \). As it is mentioned in Section 4,

\[
l^2_a((k + 1)\alpha) = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|^2 = \sum_{k \geq 0} |\hat{f}(k)|^2(k + 1)^{2\alpha} < \infty \right\}.
\]

The reproducing kernel of \( l^2_a((k + 1)^\alpha) \), by definition, is a \( l^2_a((k + 1)^\alpha) \)-valued function \( \lambda \mapsto k^\alpha_\lambda \), \( \lambda \in \mathbb{D} \), such that \( (f, k^\alpha_\lambda) = f(\lambda) \) for every \( f \in l^2_a((k + 1)^{-\alpha}) \), where \((.,.)\) means the scalar product \( (h, g) = \sum_{k \geq 0} \overline{h(k)}g(k)(k + 1)^{-2\alpha} \). Since one has \( f(\lambda) = \sum_{k \geq 0} \hat{f}(k)\lambda^k(k + 1)^{2\alpha}(k + 1)^{-2\alpha} \) (\( \lambda \in \mathbb{D} \)), it follows that

\[
k^\alpha_\lambda(z) = \sum_{k \geq 0} (k + 1)^{2\alpha}\lambda^k z^k, z \in \mathbb{D}.
\]

In particular, for the Hardy space \( H^2 = l^2_a(1) \) \((\alpha = 0)\), we get the Szegö kernel

\[
k_\lambda(z) = (1 - \lambda z)^{-1},
\]

for the Bergman space \( L^2_a = l^2_a\left((k + 1)^{-1/2}\right) \) \((\alpha = -1/2)\) - the Bergman kernel \( k^{-1/2}_\lambda(z) = (1 - \lambda z)^{-2} \).

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given the reproducing kernel \( k \) of \( H^2 \) and \( \varphi \in \{ z^N : N = 1, 2 \} \), the function \( \varphi \circ k \) is also positive definit and the corresponding Hilbert space is

\[
H_\varphi = \varphi(H^2) = l^2_a\left((k + 1)^{-1/2}\right).
\]

It satisfies the following property: for every \( f \in H^2 \), \( \varphi \circ f \in \varphi(H^2) \) and \( \|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|^2_{H^2}) \) (see [N2] p.320).

We notice in particular that

\[
(6.1.0) \quad H_z = H^2 \quad \text{and} \quad H_{z^2} = L^2_a.
\]

The above relation between the weighted spaces \( l^2_a((k + 1)^\alpha) \) and the spaces \( \varphi(H^2) = H_\varphi \) leads to establish the prove of the left-hand side inequalities from Theorem C \((p = 2)\) only and Theorem D.

**Proof of Theorem C \((p = 2)\) and Theorem D, (left-hand side inequalities only) .**

1) We set

\[
Q_n = \sum_{k=0}^{n-1} (1 - |\lambda|^2)^{1/2} b^k_\lambda (1 - \lambda z)^{-1}, H_n = \varphi \circ Q_n \quad \text{and} \quad \Psi = bH_n, b > 0.
\]
Then \( \|Q_n\|_2^2 = n \), and hence by the above Aronszajn-deBranges inequality,

\[
\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi (\|Q_n\|_2^2) = b^2 \varphi(n).
\]

Let \( b > 0 \) such that \( b^2 \varphi(n) = 1 \).

2) Since the spaces \( H_\varphi \) and \( H^\infty \) are rotation invariant, we have \( c(\sigma_n, \lambda, H_\varphi, H^\infty) = c(\sigma_n, \lambda, H_\varphi, H^\infty) \) for every \( \lambda, \mu \) with \( |\lambda| = |\mu| = r \). Let \( \lambda = -r \). To get a lower estimate for \( \|\Psi\|_{H_\varphi/b^2 H_\varphi} \) consider \( G \) such that \( \Psi - G \in b^n H\text{ol}(\mathbb{D}) \), i.e. such that \( bH_n \circ b_\lambda - G \circ b_\lambda \in \mathbb{Z}^n H\text{ol}(\mathbb{D}) \).

3) First, we show that

\[
\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda
\]
is a polynomial (of degree \( n \) if \( \varphi = z \) and \( 2n \) if \( \varphi = z^2 \)) with positive coefficients. Note that

\[
Q_n \circ b_\lambda = \sum_{k=0}^{n-1} z^k (1 - |\lambda|^2)^{1/2} \left( 1 - (1 - \lambda) \sum_{k=1}^{n-1} z^k - \lambda z^n \right) = (1 - r^2)^{-1/2} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + r^2 z^n \right) =: (1 - r^2)^{-1/2} \psi_1.
\]

Hence, \( \psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ (1 - r^2)^{-1/2} \psi_1 \) and

\[
\varphi \circ \psi_1 = \psi_1^n(z), \quad N = 1, 2.
\]

4) Next, we show that

\[
\sum_{j=0}^{m} (\psi) =: \sum_{j=0}^{m} \hat{\psi}(j) \geq \begin{cases} (2\sqrt{2})^{-1} \sqrt{n(1 - r)^{-1}} if N = 1 \\ 16^{-1} n(1 - r)^{-1} if N = 2 \end{cases},
\]

where \( m = n/2 \) if \( n \) is even and \( m = (n + 1)/2 \) if \( n \) is odd.

Indeed, setting \( S_n = \sum_{j=0}^{n} z^j \), we have both for \( N = 1 \) and \( N = 2 \)

\[
\sum_{j=0}^{m} (\psi_1^N) = \sum_{j=0}^{m} \left( \left( 1 + (1 + r) \sum_{t=1}^{n-1} z^t + r z^n \right) \right)^N \geq \sum_{j=0}^{m} (S_{n-1}^N).
\]

Next, we obtain

\[
\sum_{j=0}^{m} (S_{n-1}^N) = \sum_{j=0}^{m} \left( \left( \frac{1 - z^n}{1 - z} \right) \right)^j = \sum_{j=0}^{m} \left( (1 - z)^{-N} \right) = \sum_{j=0}^{m} \left( \sum_{j=0}^{m} C_{N+j-1}^j z^j \right) = \sum_{j=0}^{m} C_{N+j-1}^j = \begin{cases} m + 1 if N = 1 \\ (m + 1)(m + 2)/2 if N = 2 \end{cases} \leq \begin{cases} n/2 if N = 1 \\ (n + 2)(n + 4)/8 if N = 2 \end{cases} \geq \begin{cases} n/2 if N = 1 \\ n^2/8 if N = 2 \end{cases}.
\]

Finally, since \( \sum_{j=0}^{m} (\psi) = b \sum_{j=0}^{m} (\varphi \circ \psi_1) = b (1 - r^2)^{-N/2} \sum_{j=0}^{m} (\psi_1^N) \) we get

\[
\sum_{j=0}^{m} (\psi) \geq \begin{cases} (2(1 - r))^{-1/2} nb/2 if N = 1 \\ (2(1 - r))^{-1} n^2 b/8 if N = 2 \end{cases},
\]
with \( b = \varphi(n) = \begin{cases} \frac{n^{-1/2}}{2} & \text{if } N = 1 \\ n^{-1} & \text{if } N = 2 \end{cases} \). This gives the result claimed.

5) Now, using point 4) and denoting \( F_n = \Phi_m + z^m \Phi_m \), where \( \Phi_k \) stands for the \( k \)-th Fejer kernel, we get

\[
\| \Psi \|_{H^\infty / \mathcal{Z}^2 H^\infty} = \| \Psi \|_{H^\infty / z^n H^\infty} \geq 2^{-1} \| \psi \ast F_n \|_\infty \geq 2^{-1} \sum_{j=0}^m \hat{\psi}(j) \geq \begin{cases} (4\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} & \text{if } N = 1 \\ 32^{-1} n(1-r)^{-1} & \text{if } N = 2 \end{cases}
\]

6) In order to conclude, it remains to use (6.1.0).

\[ \square \]

### 6.2. The case \( X = H^p \)

Here we prove the sharpness (for even \( p \)) of the upper estimate found in Theorem C. We first prove the following lemma.

**Lemma. 6.2.0** Let \( p, q \) such that \( \frac{2}{q} \in \mathbb{Z}_+ \), then \( c(\sigma, H^p, H^\infty) \geq c(\sigma, H^q, H^\infty)^{\frac{2}{q}} \) for every sequence \( \sigma \) of \( \mathbb{D} \).

**Proof.** **Step 1.** Recalling that

\[
c(\sigma, H^p, H^\infty) = \sup_{\| f \|_p \leq 1} \inf g_{\| g \|_\infty} \{ \| g \|_\infty : g \in Y, g_{|\sigma} = f_{|\sigma} \},
\]

we first prove that

\[
c(\sigma, H^p, H^\infty) = \sup_{\| f \|_p \leq 1} \inf f_{\text{outer}} \inf g_{\| g \|_\infty} \{ \| g \|_\infty : g \in Y, g_{|\sigma} = f_{|\sigma} \}.
\]

Indeed, we clearly have the inequality

\[
\sup_{\| f \|_p \leq 1} \inf f_{\text{outer}} \inf g_{\| g \|_\infty} \{ \| g \|_\infty : g \in Y, g_{|\sigma} = f_{|\sigma} \} \leq c(\sigma, H^p, H^\infty),
\]

and if the inequality were strict, that is to say

\[
\sup_{\| f \|_p \leq 1} \inf f_{\text{outer}} \inf g_{\| g \|_\infty} \{ \| g \|_\infty : g \in Y, g_{|\sigma} = f_{|\sigma} \} < \sup_{\| f \|_p \leq 1} \inf g_{\| g \|_\infty} \{ \| g \|_\infty : g \in Y, g_{|\sigma} = f_{|\sigma} \},
\]

then we could write that there exists \( \epsilon > 0 \) such that for every \( f = f_i, f_o \in H^p \) (where \( f_i \) stands for the inner function corresponding to \( f \) and \( f_o \) to the outer one) with \( \| f \|_p \leq 1 \) (which also implies that \( \| f_o \|_p \leq 1 \), since \( \| f_o \|_p = \| f \|_p \)), there exists a function \( g \in H^\infty \) verifying both \( \| g \|_\infty \leq 1 - \epsilon)c(\sigma, H^p, H^\infty) \) and \( g_{|\sigma} = f_{o|\sigma} \). This entails that \( f_{|\sigma} = (f_i g)_{|\sigma} \) and since \( \| f_i g \|_\infty = \| g \|_\infty \leq 1 - \epsilon)c(\sigma, H^p, H^\infty) \), we get that \( c(\sigma, H^p, H^\infty) \leq (1 - \epsilon)c(\sigma, H^p, H^\infty) \), which is a contradiction and proves the equality of Step 1.

**Step 2.** Using the result of Step 1, we get that \( \forall \epsilon > 0 \) there exists an outer function \( f_o \in H^q \) with \( \| f_o \|_q \leq 1 \) and such that

\[
\inf g_{\| g \|_\infty} \{ \| g \|_\infty : g \in Y, g_{|\sigma} = f_{o|\sigma} \} \geq c(\sigma, H^q, H^\infty) - \epsilon.
\]

Now let \( F = f_o^2 \in H^p \), then \( \| F \|_p^p = \| f_o \|_q^q \leq 1 \). We suppose that there exists \( g \in H^\infty \) such that \( g_{|\sigma} = F_{|\sigma} \) with

\[
\| g \|_\infty < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{2}{q}}.
\]
Then, since \( g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i) \) for all \( i = 1..n \), we have \( g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i) \) and \( g^{\frac{q}{p}} \in H^\infty \) since \( q \in \mathbb{Z}_+ \). We also have

\[
\left\| g^{\frac{q}{p}} \right\| \infty = \left\| g \right\|^{\frac{q}{p}} < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},
\]

which is a contradiction. As a result, we have

\[
\left\| g \right\| \infty \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},
\]

for all \( g \in H^\infty \) such that \( g|_\sigma = F|_\sigma \), which gives

\[
c(\sigma, H^p, H^\infty) \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},
\]

and since that inequality is true for every \( \epsilon > 0 \), we get the result. \( \square \)

**Proof of Theorem C (the left-hand side inequality for \( p \in 2\mathbb{N}, p > 2 \) only).** We first prove the lower estimate for \( c(\sigma_n, \lambda, H^p, H^\infty) \). Writing \( p = 2(p/2) \), we apply Lemma 6.2.0 with \( q = 2 \) and this gives

\[
c(\sigma_n, \lambda, H^p, H^\infty) \geq c(\sigma_n, \lambda, H^2, H^\infty)^{\frac{p}{2}} \geq 32^{\frac{1}{2}} (n(1 - |\lambda|^{-1}))^{\frac{p}{2}}
\]

for all integer \( n \geq 1 \). The last inequality is a consequence of Theorem C (left-hand side inequality) for the particular case \( p = 2 \) which has been proved in Subsection 6.1. \( \square \)

**Acknowledgement.**

I would like to thank Professor Nikolai Nikolski for all of his work, his wisdom and the pleasure that our discussions gave to me.

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