Higher-Rank Tensor Field Theory
of Non-Abelian Fracton and Embeddon

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Abstract

We formulate a new class of tensor gauge field theories in any dimension that is a hybrid class between symmetric higher-rank tensor gauge theory (i.e., higher-spin gauge theory) and anti-symmetric tensor topological field theory. Our theory describes a mixed unitary phase interplaying between gapless and gapped topological order phases (which can live with or without Euclidean, Poincaré or anisotropic symmetry, at least in ultraviolet high or intermediate energy field theory, but not yet to a lattice cutoff scale). The “gauge structure” can be compact, continuous, abelian or non-abelian. Our theory sits outside the paradigm of Maxwell electromagnetic theory in 1865 and Yang-Mills isospin/color theory in 1954. We discuss its local gauge transformation in terms of the ungauged vector-like or tensor-like higher-moment global symmetry. The non-abelian gauge structure is caused by gauging the non-commutative symmetries: a higher-moment symmetry and a charge conjugation (particle-hole) symmetry. Vector global symmetries along time direction may exhibit time crystals. We explore the relation of these long-range entangled matters to a non-abelian generalization of Fracton order in condensed matter, a field theory formulation of foliation, the spacetime embedding and Embeddon that we newly introduce, and possible fundamental physics applications to dark matter or dark energy.

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Dedicated to
90 years of Gauge Principle since Hermann Weyl [Elektron und Gravitation, Zeit. für Physik 56, 330-352 (1929)]
and 65 years of Yang-Mills theory [Phys. Rev. 96, 191 (1954)] in 2019.

敬致楊振寧·李爾斯理論
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1 Introduction

Gauge theory is a powerful tool in quantum field theory. One of the earliest gauge theories is the renown James Clerk Maxwell’s dynamical U(1) gauge theory of electromagnetism in 1865 [1]. Gauge principle is the underlying principle of gauge theory, proposed by Weyl [2] and pondered by Pauli and many others [3]. Chern introduced Chern characteristic classes laying the foundation of non-abelian gauge structure in terms of fiber bundles and connections in 1940s [4]. Earlier pioneer works do make an impact, leading to another famous gauge theory attempting for a physics realization: Yang-Mills theory in 1954 [5] with a non-abelian Lie gauge group (e.g. SU(N)). Yang-Mills theory incorporates the gauge principle over the isospin — by promoting the internal global symmetry of isospin (such as flavor symmetry) to a local symmetry. While its local symmetry transformation, say of matter fields, is compensated by that of the gauge field. Thus gauge theory is an ideal framework to mathematically formulate the interactions between matters, where the gauge field stands for the force mediator.

In this work, we develop a new type of gauge theory, obtained from a hybrid through the marriage between anti-symmetric tensor topological field theory (TQFT) and symmetric higher-rank tensor gauge theory. We find that the gauge structure \(^1\) can be non-abelian. Our work is inspired by the fracton order in condensed matter system (see review articles [6, 7]). To the best of our understanding, our theory is the first new example of higher-rank tensor compact non-abelian gauge theory previously unknown to the literature. It is also a first example (in fact, we construct a web of families of many examples) to have a continuous non-abelian gauge structure in any dimension in the fracton order literature.

Anti-symmetric tensor field theory has a long history as early as Kalb-Ramond work’s on higher differential form gauge fields [8]. The later development of higher form continuum field theory can be found in [9, 10] and References therein, including higher-form gauge theory and higher-form global symmetries. The anomalies of theories with ordinary global symmetries and higher-form global symmetries can also be systematically obtained by a cobordism theory [11] and a generalized version of cobordism theory with higher classifying spaces [12]. We will focus on a particular continuum anti-symmetric tensor TQFT developed in Ref. [13–23]

\(^1\)Here the gauge structure is similar to the usual concept of the gauge group. But the gauge structure for our theories is not precisely the same as the familiar Lie group framework of the well-known gauge theory. There is still a notion of commutative-ness or non-commutative-ness generator of the gauge structure, which we shall still call the commutative case as abelian, and the non-commutative case as non-abelian.
and references therein: A continuum gauge theory formulation [21,22] of group cohomology or higher group cohomology type of TQFT such as Dijkgraaf-Witten theory [24], with a discrete finite gauge group. We term this finite gauge group for TQFT as $G_{\text{TQFT}}$.

Symmetric higher-rank tensor field theory also has a long history dating back to the study of arbitrary-integer-spin bosonic fields, including massive and massless boson fields, see [25–28] and References therein. Singh-Hagen [27] studies the Fierz-Pauli theory [26] for massive arbitrary-integer-spin $s$ boson fields. Fronsdal [28] studies their theory [26,27] in the massless limit, then finds that the massless arbitrary-integer-spin $s$ boson fields can be understood in terms of symmetric tensor bosonic field of rank $s$, with certain constraints on the trace, double trace, and divergence, etc. In this work, we will only focus on a certain higher-spin theory in terms of a class of specific symmetric higher-rank tensor gauge theories.

Our inspiration for symmetric higher-rank tensor gauge theory comes from the condensed matter systems studying the quantum spin liquids described by emergent higher-rank symmetric tensor gauge fields, say a rank-$n$ tensor $A_{ijk...,}$ where $n$-indices are symmetrized. If the symmetric tensor gauge field is compact and abelian, then it is commonly referred this compact abelian theory as the “higher-rank U(1) symmetric tensor gauge theory,” or “higher-rank U(1) spin liquids” in the condensed matter literature. However, we will refrain from using this terminology: the U(1) may be an accidental misnomer. We will see that there is no precise occurrence of an ordinary U(1) gauge group in the theory, and the ungauged global symmetry is not simply a U(1) global symmetry group. (In facts, see the later Eq. (2.15), there is only a group-analogous structure that is abelian, instead of the familiar group structure.) Thus, we instead name this type theory as

\[ a \text{ higher-rank symmetric tensor compact abelian gauge theory.} \] (1.1)

Ref. [29] finds that the compact abelian symmetric tensor gauge theory is unstable in 2+1D, but it becomes stable to be gapless-ness with deconfinement in 3+1D. In contrast, the anti-symmetric tensor gauge theory with a continuous gauge group is unstable and flows to a gapped phase with confinement in 2+1d and 3+1d [30,31]. We will closely follow a version of compact abelian higher-rank symmetric tensor gauge theory developed and pioneered by Pretko and others, see for instance Ref. [32–37], where time and space indices are treated in an unequal and non-interchangeable footing. We name this type of theory as:

\[ \text{Anisotropic-type higher-rank symmetric tensor gauge theory.} \] (1.2)

We also develop another generalization of two kinds of higher-rank symmetric tensor gauge theory:

\[ \text{Euclidean-type higher-rank symmetric tensor gauge theory, or} \] (1.3)

\[ \text{Lorentzian-type higher-rank symmetric tensor gauge theory.} \] (1.4)

Let us give a quick overview and outlines where we are heading in this work. Throughout this article, we only focus on the field theory living on a flat Euclidean or Minkowski spacetime $\mathbb{R}^{d+1}$ or $\mathbb{R}^{1,d}$ with Cartesian coordinates. It will become clear soon that we can formulate non-abelian gauge structures (for example $\mathbb{Z}_2^G \ltimes (U(1))_{x(d+1)}$) introduced in Sec. 2) to a higher-rank tensor gauge theory (see later Eq. (2.30)).

- The first ingredient to obtain our non-abelian tensor gauge theory is by gauging the higher-moment global symmetry\(^3\) and a discrete $\mathbb{Z}_2^G$ charge conjugation symmetry together, relatively with a non-commutative semi-direct product ($\ltimes$) structure. This is done in Sec. 2.1.4.

\(^2\)We denote $n + 1d$ for the $n + 1$ spacetime dimensions, with $n$ spatial and 1 time dimensions. We denote $md$ for the $m$ spacetime dimensions. We denote $n + 1D$ for the $n$ spatial and 1 time dimensions.

\(^3\)For example, a vector global symmetry $U(1)_{a(d+1)}$ in spacetime or $U(1)_{a(d)}$ in space, see Sec. 2.1.1 for more details.
• The second ingredient to obtain our non-abelian gauge structure is by coupling several copies of $(\mathbb{Z}_2^C)^N$ to a twisted gauge theory of TQFT. The twisted-$(\mathbb{Z}_2^C)^N$ TQFT can be equivalent (strong and exactly dual) to a TQFT with a non-abelian gauge group with non-abelian anyonic excitations. For example, certain twisted-$(\mathbb{Z}_2^C)^3$-TQFT in 3d (2+1D) can be dual to a non-abelian gauge theory of an order 8 dihedral $D_8$ or quaternion $Q_8$ gauge group $[13,38]$. Similar non-abelian natures of field theories hold in 4d (3+1D) and higher dimensions $[21,22]$.

In the remaining of Sec. 1, we will review the familiar gauge theories (U(1) Maxwell and non-abelian Yang-Mills theories), in contrast of the new non-abelian higher-rank tensor gauge theories we develop in Sec. 2. Our gauge field theories in Sec. 2 include the following properties:

1. Unitary (thus directly applicable to quantum matter and condensed matter systems),
2. Abelian (e.g. $\prod_{J=1}^N U(1)_{x_{(d+1)}}$) or non-abelian (e.g. $\prod_{J=1}^N [\mathbb{Z}_2^C \ltimes U(1)_{x_{(d+1)}}]$),
3. Compact gauge structure (e.g. $\prod_{J=1}^N [\mathbb{Z}_2^C \ltimes U(1)_{x_{(d+1)}}]$) is compact,\(^4\)
4. Continuous gauge structure (e.g. thanks to $U(1)_{x_{(d+1)}}$ or $U(1)_{x_{(d)}}$).

In Sec. 3, we examine the non-abelian tensor gauge theory interplayed between gapped and gapless phases, including the topological degeneracy from the zero modes, the gapless gapless degrees of freedom and dispersion relations. In Sec. 4, we discuss how our theories can be related to a non-abelian generalization of Fracton order in condensed matter, the spacetime embedding and the Embeddon that we newly introduce, and a field theory formulation of foliation. In Sec. 5, we conclude with discussions and future directions on the field theory quantization (either canonical quantization, or path integral), Feynman diagrams, and quantum Hamiltonian lattice models, and applications of our theories to time crystal and dark matter.

\(^4\) Here let us clarify the meaning of compactness of gauge group or gauge structure. First, for the ordinary Maxwell’s U(1) gauge theory, see Sec. 1.1.1, the gauge field $A_a$ is a gauge connection of some principle-$G$ bundle, therefore $A_a$ lives in the Lie algebra value, not the gauge group itself. In this case, there is no difference between $U(1)$-valued Lie algebra or $\mathbb{R}$-valued Lie algebra for the local gauge connection $A$. However, there are differences for the structure group (i.e., the gauge group) $G$ of the principle-$G$ bundle to be $G = U(1)$ or $G = \mathbb{R}$.

\[
\begin{align*}
\text{If the } G &= U(1) \simeq \mathbb{R}/\mathbb{Z}, \text{ then the holonomy operator } &\exp(i q \oint A) \text{ has its } \oint A \text{ being compact in } U(1), \\
\text{i.e., } &\oint A \sim \oint A + 2 \pi \text{ is identified, while the charge } q \in \mathbb{Z} \text{ is quantized.} \\
\text{If the } G &= \mathbb{R}, \text{ then the holonomy operator } &\exp(i q \oint A) \text{ has its } \oint A \text{ being non-compact in } \mathbb{R}, \\
\text{while the charge } q \in \mathbb{R} \text{ is also continuous and un-quantized.}
\end{align*}
\]

The large gauge transformations for $A$ exist in $G = U(1)$, but do not exist in $G = \mathbb{R}$. For the U(1) or $\mathbb{R}$ gauge theory, the holonomy $\oint A = \int \text{d} A$ can be arbitrary value in $U(1)$ or $\mathbb{R}$ by Stokes theorem. (In contrast, for a discrete gauge theory, such as the $\mathbb{Z}_n$-gauge theory, the holonomy $\oint A$ is zero for contractible cycles; the holonomy $\oint A \in 2 \pi n \mathbb{Z}$ can be nonzero only winding around a non-contractible cycle [an element in a nontrivial homology group $H_1(M,\mathbb{Z})$].)

Second, for the higher-rank tensor gauge theory, the compactness is more subtle. One way to justify the compactness is through the effect of the compact $\theta$-topological term and analogous Witten’s effect $[34]$. Another way to justify the compactness is to modify the previous condition $\oint A \sim \oint A + 2 \pi$ for the ordinary U(1) gauge theory to the symmetric tensor case: such that $A \sim A + 2 \pi$ under a certain modified contour integration, presumably compatible with a lattice cut-off. The reason that we need to modify the contour integration, say in the rank-2 tensor case, is that the $\oint A$, which integrates over the anti-symmetric area or volume form, cannot be paired up with the symmetric tensor $A_{ij}$ in a conventional way. See also Remark 4 in Sec. 5 for the required mathematical components of our field theory formulation.
1.1 Overview of the familiar gauge theories

Gauge field allows a local transformation, depending on the spacetime coordinate. This local transformation is known as the gauge transformation. The whole partition function $Z$ or the path integral, defined through integrating out the gauge inequivalent configuration phase space of gauge fields, is set to be invariant under the gauge transformation. We first review the familiar gauge theories and meanwhile set up our notations. For simplicity, we focus on the 4 dimensions.

1.1.1 Abelian U(1) Maxwell gauge theory

In the modern formulation of Maxwell’s U(1) gauge theory [1], we have a gauge field $A(x^\mu) = A_\mu(x^\mu)\,dx^\mu$ (locally as a 1-form as a differential form, but globally should be viewed as a U(1) gauge 1-connection), which under an infinitesimal gauge transformation becomes

$$A_\mu(r) \rightarrow A_\mu(r) + \delta A_\mu(r) = A_\mu(r) + \frac{1}{g}\partial_\mu \eta(r), \quad (1.5)$$

with the spacetime coordinate $r = (r^\mu)(dx^\mu)$ where $x^\mu := (x^0, x^i) := (t, \vec{x})$. Locally $dx^\mu$ is a differential 1-form, the $\mu$ runs through the indices of coordinate of spacetime $M$. The coupling $g$ can be related to electromagnetic coupling $e$ as $g = -e$. The $\eta(r)$ is locally 0-form with a spacetime dependence. Just like the relation of a local curvature to the 1-connection, we have the field strength 2-form to the 1-form U(1) gauge field via

$$F = dA = \frac{1}{2}F_{\mu\nu}(dx^\mu \wedge dx^\nu) = \frac{1}{2}(\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha)(dx^\mu \wedge dx^\nu), \quad (1.6)$$

with the exterior derivative $d$ and the wedge product $\wedge$. The field strength is gauge invariant:

$$F_{\mu\nu}(r) = F_{\mu\nu}(r) + \delta F_{\mu\nu}(r) \rightarrow F_{\mu\nu}(r). \quad (1.7)$$

The whole gauge invariant partition function of U(1) gauge theory is meant to be written systematically as:

$$Z_{U(1),\text{EM}} = \int [DA]e^{i\frac{1}{2}F \wedge \ast F} = \int [DA]^\mu e^{i\int d^4x \frac{1}{2}F_{\mu\nu}F_{\mu\nu}. \quad (1.8)$$

The $[DA]$ is the path integral measure, for a configuration of the gauge field $A$. Here $\ast F$ is $F$’s Hodge dual. We integrated over all allowed gauge inequivalent configurations $\int [DA]$, while gauge redundancy is mod out. The integration is under a weight factor $\exp(iS)$. In 4d (3+1D), we take the spacetime metric $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(+,-,-,-)$, and $\epsilon^{\mu_1\mu_2\mu_3\mu_4} = -\epsilon_{\mu_1\mu_2\mu_3\mu_4}$, while $\epsilon_{ijk} = \epsilon^{ijk} = \epsilon_{ijk} = \epsilon_{ijk}$ where raising or lowering indices does not affect. Generally we take $\mu, \nu, \mu_1, \ldots$ are spacetime coordinates, while $i, j, k$ are space coordinates only. This path integral may not be precisely mathematically well-defined, however it can be physically sensibly well-defined, for example being regularized via a higher energy cutoff such as a lattice cutoff scale. Our present work only focus on the physics side of rigor.

Indeed the above U(1) gauge theory follows the gauge principle [2] where the global U(1) symmetry of a point operator $\Phi$ is promoted to a local symmetry variation that can be absorbed by local symmetry variation of the 1-form gauge field. Then the 1-form gauge field plays the role of making the charged matter (namely the matter carrying gauge charge) interacting with each other. Thus the 1-form gauge field behaves as a force mediator between matter fields. The force mediator is known as a spin-1 gauge boson in physics.
1.1.2 Non-abelian SU(N) Yang-Mills gauge theories

For the non-abelian Yang-Mills theory (YM) \([5]\), the gauge field is locally a 1-form or a 1-connection

\[
A = A_\mu dx^\mu = A_\mu^\alpha T^\alpha dx^\mu,
\]

obtained from parallel transporting the SU(N) fiber of the principal principal-SU(N) bundle over the spacetime base manifold \(M\). Here \(T^\alpha\) is the generator of Lie algebra \(g\) for the gauge group (say SU(N), with \(N\) is called the number of color for gauge theory, or the number of isospin for global symmetry in physics), with the commutator \([T^\alpha, T^\beta] = if^{\alpha\beta\gamma}T^\gamma\), where \(f^{\alpha\beta\gamma}\) is chosen to be a fully anti-symmetric structure constant. Then \(A_\mu = T^\alpha A_\mu^\alpha\) is the Lie algebra valued gauge field, in the adjoint representation of the Lie algebra. In physics, \(A_\mu\) can represent the gluon vector field of quantum chromodynamics. The non-abelian field strength is

\[
F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu}^\alpha T^\alpha (dx^\mu \wedge dx^\nu) \equiv \frac{1}{2} (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + i f^{\beta\gamma\alpha} A_\mu^\beta A_\nu^\gamma) T^\alpha (dx^\mu \wedge dx^\nu).
\]

For the convenience to bridge to the physics convention, we rescale and redefine \(A = A_\mu^\alpha T^\alpha g dx^\mu = A_\mu^\alpha (-igT^\alpha) dx^\mu = A_\mu^\alpha (-ig) dx^\mu,\) thus we also redefine the field strength

\[
F = \frac{1}{2} F_{\mu\nu}^\alpha (-igT^\alpha)(dx^\mu \wedge dx^\nu) = \frac{1}{2} (\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + g f^{\beta\gamma\alpha} A_\mu^\beta A_\nu^\gamma) T^\alpha (dx^\mu \wedge dx^\nu).
\]

The \(g\) is YM coupling constant. We also have the covariant derivative:

\[
D_\mu = \partial_\mu - ig a_\mu^\alpha T^\alpha.
\]

The field strength \(F_{\mu\nu}^\alpha\) is not gauge invariant under the gauge transformation with \(V = \exp(i\eta^\alpha T^\alpha)\):

\[
A_\mu^\alpha T^\alpha \rightarrow V (A_\mu^\alpha T^\alpha + \frac{i}{g} \partial_\mu V) V^\dagger \simeq (A_\mu^\alpha + \frac{1}{g} \partial_\mu \eta^\alpha - f^{\beta\gamma\alpha} \eta^\beta A_\mu^\gamma) T^\alpha + \ldots,
\]

since the field strength transforms to

\[
F_{\mu\nu}^\alpha \rightarrow (F_{\mu\nu}^\alpha) - (f^{\beta\gamma\alpha} \eta^\beta F_{\mu\nu}^\gamma T^\alpha) + \ldots.
\]

There are color-electric field and color-magnetic field related to the field strength

\[
E_\mu^\alpha := E_\mu^{\alpha j} := F_{\mu j}^\alpha = \partial_\mu A_j^\alpha - \partial_j A_\mu^\alpha + gf^{\beta\gamma\alpha} A_j^\beta A_j^\gamma = -\partial_\mu A_j^{\alpha j} - \nabla_j A_\mu^\alpha - gf^{\beta\gamma\alpha} A_\mu^\beta A^\gamma j.
\]

\[
B_\mu^\alpha := B^{\alpha j} := -\frac{1}{2} \varepsilon^{jkl} F_{\mu kl} = -\frac{1}{2} \varepsilon^{jkl} F_{\mu kl} = \frac{1}{2} \varepsilon^{jkl} \partial_\mu A_{kl} - \partial_k A^{\alpha j} - gf^{\beta\gamma\alpha} A^\beta k A^{\gamma l}.
\]

Both color-electric field and color-magnetic field are not gauge-invariant

\[
E_\mu^\alpha T^\alpha \rightarrow (E_\mu^\alpha T^\alpha) - (f^{\beta\gamma\alpha} \eta^\beta E_\mu^\gamma T^\alpha) + \ldots, \quad B_\mu^\alpha T^\alpha \rightarrow (B_\mu^\alpha T^\alpha) - (f^{\beta\gamma\alpha} \eta^\beta B_\mu^\gamma T^\alpha) + \ldots.
\]

But the Yang-Mills action and its path integral \(Z_{YM}\) are gauge invariant (we set a normalization factor \(\kappa_{YM} = \frac{1}{2} c(G)^{-1}\) for the convenience to match the standard convention):

\[
Z_{YM} = \int [DA] e^{i \int -Tr(\kappa_{YM} F \wedge F)} = \int [DA] e^{i \int d^4x Tr((-\frac{1}{2}\kappa_{YM} F_{\mu\nu} F^{\mu\nu})} = \int [DA] e^{i \int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}} = \int [DA] e^{i \int d^4x \frac{1}{4} (F^{\alpha\beta} A_\alpha^{\beta j} + F^{\alpha\beta} A_\alpha^{\beta j})} = \int [DA] e^{i \int d^4x (-\frac{1}{2} ) ((E_j^\alpha)^2 - (B_j^\alpha)^2)}.
\]

In the next section, we move on to develop our new type of exotic higher-rank tensor gauge theories.
2 Class of New Non-Abelian Higher-Rank Tensor Gauge Theories

2.1 Euclidean or Lorentz Invariant Non-Abelian Higher-Rank Tensor Gauge Theory

Below we can formulate theories in both the Euclidean spacetime $\mathbb{R}^{d+1}$ or in the Minkowski (or Lorentzian) spacetime $\mathbb{R}^{1,d}$. The two versions of theories have the same form of path integral and the action, thus we will present them together. We denote the space and time coordinates as $(t, \vec{x}_j) = (x_{\mu})$. Our formulation below can be regarded as modifying Pretko’s spacetime anisotropic theory [36] to a spacetime isotropic theory (up to the signature of Minkowski metric).

2.1.1 Matter field theory and higher-moment vector global symmetry

We can start from a matter field theory in $(d + 1)d$ with the global symmetry including the spacetime symmetry and the internal symmetry. We focus on first the scalar charge matter theory with a vector global symmetry. Let the complex charge matter field called

$$\Phi(t, \vec{x}) = \Phi(x) \in \mathbb{C}, \quad (2.1)$$

Figure 1: The ordinary (0-form) U(1) global symmetry transformation acts on the complex charged matter $\Phi(x) \in \mathbb{C}$, e.g. the rotor field, distributed on the spacetime coordinate $x$. For demonstration, here we show various $\Phi(x)$ fields sitting on discretized lattice points on $x$. Thus we get Eq. (2.2): $\Phi \rightarrow e^{i\eta}\Phi$.

1. Internal symmetry:

The usual ordinary (0-form) U(1) global symmetry transformation acts on the charged matter as,

$$\Phi \rightarrow e^{i\eta}\Phi, \quad (2.2)$$

where $\eta$ is a global parameter associated to the U(1) scalar charge, independent of spacetime $(t, \vec{x})$. See Fig. 1.
The vector global symmetry belongs to a generalized class of higher-moment symmetry. The vector $U(1)$ global symmetry transformation acts on the complex charged matter $\Phi(x)$, e.g. the rotor field, following the set-up in Fig. 1. Thus we get Eq. (2.3): $\Phi \rightarrow e^{i\eta(x)} \Phi := e^{i\Lambda \cdot x} \Phi$. The angle $\Lambda \cdot x$ depends on a reference point (say $x = 0$) and the distance $x$ away from the reference point.

In addition, we like to impose an additional vector global symmetry over the scalar charge matter

$$\Phi \rightarrow e^{i\eta(x)} \Phi := e^{i\Lambda \cdot x} \Phi,$$

Here $\Lambda$ is a $(d+1)$-vector on the spacetime, so the $\Lambda \cdot x$ takes the inner product with the spacetime coordinate. We term this symmetry as $U(1)_{x(d+1)}$ vector global symmetry

due to the involvement of the $d + 1$ spacetime vectors, $x$ and $\Lambda$. The vector global symmetry belongs to a generalized class of higher-moment global symmetry. See Fig. 2 for illustration of a symmetry transformation.

In fact, the $U(1)_{x(d+1)}$ involves several symmetry transformations under any $(d+1)$ linear-independent choice of $\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_d)$. So Eq. (2.3) can be chosen to be $e^{i\Lambda_0 \cdot x_0}, e^{i\Lambda_1 \cdot x_1}, \ldots, e^{i\Lambda_d \cdot x_d}$. Thus Eq. (2.4) contains

$$\Phi \rightarrow \Phi e^{i\Lambda_0 \cdot x_0} = \Phi e^{i\Lambda_0 \cdot t},$$

while $x_0 = t$ is time coordinate. So the field configuration at a certain periodic time $t \simeq t + \frac{2\pi}{\Lambda_0} Z$ is constrained. We will comment more in the Conclusion in Sec. 5.

Now let us write down the matter field theory Lagrangian. To recall, in order to have a Lagrangian kinetic term invariant under only the ordinary global symmetry, we have the familiar kinetic term $\partial^\mu \Phi^\dagger \partial_\mu \Phi$.

Figure 2: The vector global symmetry belongs to a generalized class of higher-moment symmetry. The vector $U(1)$ global symmetry transformation acts on the complex charged matter $\Phi(x)$, e.g. the rotor field, following the set-up in Fig. 1. Thus we get Eq. (2.3): $\Phi \rightarrow e^{i\eta(x)} \Phi := e^{i\Lambda \cdot x} \Phi$. The angle $\Lambda \cdot x$ depends on a reference point (say $x = 0$) and the distance $x$ away from the reference point.
In order to have a Lagrangian kinetic term invariant under the vector global symmetry (alternatively, under both the ordinary and vector global symmetry), we should abandon the familiar kinetic term \( \partial^\mu \Phi^\dagger \partial_\mu \Phi \) but design a new Lagrangian term:\(^5\)

\[
(\Phi^\dagger \partial^\mu \partial^\nu \Phi^\dagger - \partial^\mu \Phi^\dagger \partial^\nu \Phi)(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi).
\] (2.7)

It is easy to check that under Eq. (2.3) transformation,

\[
(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi) \rightarrow e^{i \eta_\nu(x)} (\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi + (i \partial_\mu \partial_\nu \eta_\nu) \Phi^2),
\] (2.8)

thus this term is covariant up to a factor; if \( \eta_\nu \) is in terms of a linear polynomial of \( x \), namely \( \eta_\nu = \Lambda \cdot x \), rather than higher-order polynomial, then \( \partial_\mu \partial_\nu \eta_\nu = 0 \). Hence Eq. (2.7) is invariant under both the ordinary and vector global symmetry Eq. (2.2) and Eq. (2.3).

There is also a discrete \( \mathbb{Z}_2^C \) charge conjugation symmetry:

\[
\Phi \rightarrow \Phi^\dagger.
\] (2.9)

It is easy to see that the ordinary \( U(1) \) and \( \mathbb{Z}_2^C \) forms a non-abelian \( U(1) \times \mathbb{Z}_2^C = SO(2) \times \mathbb{Z}_2^C = O(2) \)-symmetry. The full internal symmetry in both Euclidean/Minkowski signature is:

- Euclidean/Minkowski internal (including ordinary and higher-moment) global symmetry:

\[
\mathbb{Z}_2^C \ltimes \left( U(1) \times U(1)_{x(d+1)} \right).
\]

2. Spacetime symmetry:

- Euclidean version’s Poincaré symmetry: \( \mathbb{R}^{d+1} \rtimes O(d + 1) \), which includes Euclidean spacetime translation, rotation, and reflection (say \( R \)-symmetry, for Euclidean spacetime, this includes the parity \( P \) and time reversal \( T \) as the same symmetry). Note that the reflection \( R \)-symmetry is a \( \mathbb{Z}_2^R \) discrete symmetry, related to the 0-th homotopy group \( \pi_0 O(d + 1) = \mathbb{Z}_2 \), so the reflection \( R \) flips between two disconnected components of \( O(d + 1) \).

- Minkowski version’s Poincaré symmetry: \( \mathbb{R}^{1,d} \rtimes O(1,d) \), which includes Minkowski spacetime translation, boost, rotation, and the parity \( P \) and time reversal \( T \). Note that the \( P \) and \( T \) symmetry are both \( \mathbb{Z}_2 \) discrete symmetries, with \( \mathbb{Z}_2^P: x_j \rightarrow -x_j \) for some \( j \), and with \( \mathbb{Z}_2^T: t \rightarrow -t \) over the spacetime coordinates. They are related to the two \( \mathbb{Z}_2 \) generators of \( \pi_0 O(1,d) = (\mathbb{Z}_2)^2 \), so the \( P \) and \( T \) flip between four disconnected components of \( O(1,d) \).

The \( U(1)_{x(d+1)} \) has a nontrivial action on the Poincaré symmetry, thus we can define a semi-direct product structure: \( (\mathbb{R}^{d+1} \rtimes O(d + 1)) \ltimes U(1)_{x(d+1)} \). Combine the internal and spacetime symmetry together, we obtain the full global symmetry.

- Euclidean version’s symmetry:

\[
(\mathbb{R}^{d+1} \rtimes O(d + 1)) \ltimes \left( \mathbb{Z}_2^C \ltimes U(1)_{x(d+1)} \right).
\] (2.10)

- Minkowski version’s symmetry is similar:

\[
(\mathbb{R}^{1,d} \rtimes O(1,d)) \ltimes \left( \mathbb{Z}_2^C \ltimes U(1)_{x(d+1)} \right).
\] (2.11)

Note that the 0-form symmetry \( \mathbb{Z}_2^C \ltimes U(1) \) in fact commute with the Poincaré symmetry, thus we have the structure \( (\mathbb{R}^{d+1} \rtimes O(d + 1)) \times (\mathbb{Z}_2^C \ltimes U(1)) \) if we disregard the vector global symmetry \( U(1)_{x(d+1)} \).

---

\(^5\)In the fracton literature, this is firstly propose by Pretko [36] for the vector global symmetry. See a generalized formalism for field theory of any higher-moment global symmetry in [41, 42].
2.1.2 Symmetric higher-rank tensor gauge theory and the gauging procedure

Now we follow the gauge principle to gauge the higher-moment symmetry $U(1)_{(d+1)}$ in $(U(1) \times U(1))_{(d+1)}$. To recall the standard gauging procedure of the ordinary 0-form global symmetry, we promote the global symmetry transformation parameter $\eta$ in Eq. (2.2) to a spacetime-dependent local transformation $\eta(x)$. Thus under $\Phi \to e^{i\eta(x)}\Phi$, the derivative term $\partial_\mu \Phi$ is not covariant unless we revise it to a covariant derivative term

$$(\partial_\mu - ig_\mu)\Phi.$$ 

It is covariant under the familiar gauge transformation:

$$\Phi \to e^{i\eta(x)}\Phi, \quad a_\mu \to a_\mu + \frac{1}{g} \partial_\mu(\eta(x)).$$

(2.12)

See Fig. 3 for a demonstration of the gauge fluctuation away from the 0-form global symmetry.

| Figure 3: A demonstration of the gauge fluctuation (the dark gray color) deviates away from the 0-form global symmetry transformation of Fig. 1 (the light gray color). See Eq. (2.12). |

A major difference of ours apart from previous work is that we can also treat spacetime indices isotropically and equally (up to the metric signature) instead of an-isotropically as in [36]. After gauging $U(1)_{(d+1)}$
Figure 4: A demonstration of the gauge fluctuation (the dark gray color) deviates away from the vector global symmetry of Fig. 2 (the light gray color, which belongs to a generalized class of higher-moment symmetry). See Eq. (2.14).

higher-moment symmetry, we have a “gauge group-analogous structure” from Eq. (2.4):

\[ \left[ U(1) \times (d+1) \right] \] ,

(2.15)

where the big bracket \([\ldots]\) stands for being dynamically gauged.\(^6\) We can define the rank-3 field strength \(F_{\mu\nu\xi}\) of the symmetric rank-2 tensor \(A_{\nu\xi}\) as:

\[ F_{\mu\nu\xi} = \partial_\mu A_{\nu\xi} - \partial_\nu A_{\mu\xi}. \]

(2.16)

\(F_{\mu\nu\xi}\) is anti-symmetric with respect to the first two indices \(\mu \leftrightarrow \nu\). And we can define the kinetic Lagrangian term for gauge fields as:

\[ \mathcal{L}_{A,\text{kinetic}} := \frac{1}{g^2} |F_{\mu\nu\xi}|^2 := \frac{1}{g^2} F_{\mu\nu\xi} F^{\mu\nu\xi} = \frac{1}{g^2} (\partial_\mu A_{\nu\xi} - \partial_\nu A_{\mu\xi})(\partial^\mu A^{\nu\xi} - \partial^\nu A^{\mu\xi}). \]

(2.17)

The equation of motion (EOM) for a pure gauge theory is

\[ \partial^\mu F_{\mu\nu\xi} = \partial^\mu(\partial_\nu A_{\mu\xi} - \partial_\mu A_{\nu\xi}) = 0. \]

(2.18)

We certainly can write down the classical field theory by giving the action, with or without matter field (which we can set \(\Phi = 0\)):

\[ \int_{M^{d+1}} d^{d+1}x \left( |F_{\mu\nu\xi}|^2 + \left| (\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi - i g A_{\mu\nu} \Phi^2) \right|^2 + V(\Phi^2) \right). \]

(2.19)

\(^6\)Again, it is only a “group-analogous structure” but not quite a group, because the 0-form symmetry \((Z_2^f)\) and higher moment symmetry \((U(1)_{x,(d+1)})\) are fundamentally different. Note that we only gauge \(U(1)_{x,(d+1)}\) but we do not gauge \(U(1)\) in \((U(1) \times U(1))_{x,(d+1)}\), but the \(U(1)\) global symmetry would be lost altogether after gauging \(U(1)_{x,(d+1)}\). Importantly, we do not introduce the usual 1-form \(U(1)\) gauge field, because we do not gauge 0-form \(U(1)\) symmetry. We introduce the symmetric rank-2 tensor \(A_{\nu\xi}\) to only gauge the \(U(1)_{x,(d+1)}\) symmetry.
Here \( V(|\Phi|^2) \) is a potential term. The \( M^{d+1} \) is a \( d + 1 \) spacetime manifold. Ideally, we hope to discuss a quantum theory. Formally, we propose a schematic path integral:

\[
Z_{\text{rk-2-sym-} \Phi} = \int [DA_{\mu\nu}] [D\Phi] [D\Phi^\dagger] \exp(i \int_{M^{d+1}} \! \! d^{d+1}x \left( |F^\mu_{\nu\xi}|^2 + |(\Phi \partial_{\mu} \partial_{\nu} \Phi - \partial_{\mu} \Phi \partial_{\nu} \Phi - ig A_{\mu\nu} \Phi^2)|^2 + V(|\Phi|^2) \right)).
\]

We shorthand “rk-2-sym-\( \Phi \)” for the rank-2 symmetric tensor gauge theory coupled to matters. Unfortunately, this field theory has no free nor quadratic term that we can start with to do a perturbative theory to extract the higher-order term effects. In fact, due to the leading order term is already highly nontrivial interacting quartic interactions, a fully fledged quantum field theory by a field quantization is still an open question. We comment more the quantum aspects of the theory in Conclusion Sec. 5.

### 2.1.3 Field strength, electric and magnetic tensors: Independent components and representations

The rank-3 field strength \( F^\mu_{\nu\xi} \) can also be interpreted as the higher-rank generalized electric tensor field \( \overline{E} \) and magnetic tensor field \( \overline{B} \). To do so, we take those \( F^\mu_{\nu\xi} \) have an odd number of time indices (0 or \( t \)) as the electric tensor field \( \overline{E} \), while those \( F^\mu_{\nu\xi} \) have an even number of time indices (0 or \( t \)) as the magnetic tensor field \( \overline{B} \). Above in this section, we have a full general discussion in any dimension. Below, we focus on the 4d (3+1D) spacetime, where \( i, j, k, \ell \in \{1, 2, 3\} \) are space coordinates only without time. Let us count independent (\( \equiv \) indpt) components. We define:

\[
\begin{align*}
\overline{E} &:= \left\{ \overline{E}_{ij} := F_{0ij} = -F_{i0j} = \partial_0 A_{ij} - \partial_i A_{0j}, \text{ with } 3^2 = 9 \text{-indpt components} \right. \\
&\quad \left. \text{where } F_{ij0} = -F_{j0i} = \partial_0 A_{ij} - \partial_j A_{0i}, \text{ thus } i \neq j \text{ for 3-indpt components} \right) \\
\overline{B} &:= \left\{ \overline{B}_{\ell k} := \frac{1}{2} \epsilon_{\ell ij} F_{ij k} = \frac{1}{2} \epsilon_{\ell ij} (\partial_i A_{jk} - \partial_j A_{ik}), \text{ with } i, j \text{ summed} \right. \\
&\quad \left. \text{where } F_{ijk} = -F_{jik} = \partial_i A_{jk} - \partial_j A_{ik}, \text{ thus } i \neq j \text{ with } 3^2 3 = 9 \text{-indpt components} \right) \\
&\quad \left. \overline{B}_j := F_{0j0} = -F_{j00} = \partial_0 A_{j0} - \partial_j A_{00}, \text{ with 3-indpt components} \right).
\end{align*}
\]

Thus both \( \overline{E} \) and \( \overline{B} \) tensor contains 12 components. Here the 9 means the spatial rank-2 tensor (say \( i, j \in \{1, 2, 3\} \)), and the 3 means spatial vector (say \( j \in \{1, 2, 3\} \)). We can understand the above in terms of representation of spatial rotation symmetry group \( \text{SO}(3) \). The \( \text{SO}(3) \) has a \( (2n + 1) \)-dimensional representation, we need that \( n = 0 \) gives the trivial representation \( 1 \), \( n = 1 \) which gives the vector representation \( 3 \), and the \( n = 2 \) gives the 5-dimensional representation \( 5 \). The \( \overline{E}_i \) and \( \overline{B}_j \) give the the vector representation \( 3 \) with 3 components. The \( \overline{E}_{ij} \) and \( \overline{B}_{\ell k} \) give the decomposition of representations \( (1 + 3 + 5) \) with 9 components. Thus, we have both tensor fields decomposed in terms of spatial rotational \( \text{SO}(3) \) representations:

\[
\begin{align*}
\overline{E} &= 3 + (1 + 3 + 5), \\
\overline{B} &= 3 + (1 + 3 + 5).
\end{align*}
\]

### 2.1.4 Gauge the \( \mathbb{Z}_2^C \)-charge conjugation symmetry: 
Non-abelian higher-moment continuous gauge structure

For both the ungauged theory (e.g. Eq. (2.7)) and the gauged theory (e.g. Eq. (2.20)), there is a discrete charge conjugation \( \mathbb{Z}_2^C \) global symmetry, or the so-called particle-hole symmetry in condensed matter
system. For both the ungauged theory and the gauged theory, the $Z_2^C$ symmetry flips the complex scalar $\Phi$ to its complex conjugation,

$$\Phi \rightarrow \Phi^\dagger,$$  \hspace{1cm} (2.24)

It is easy to see that the ordinary U(1) global symmetry and $Z_2^C$ symmetry of Eq. (2.10) and Eq. (2.11) do not commute. Take these two symmetry transformations, as $U_{U(1)}$ and $U_{Z_2^C}$ operators respectively, we see that the transformations act on $\Phi$ as:

$$U_{Z_2^C} U_{U(1)} \Phi = U_{Z_2^C} (e^{i\eta} \Phi) = e^{i\eta} \Phi^\dagger,$$

$$U_{U(1)} U_{Z_2^C} \Phi = U_{Z_2^C} (\Phi^\dagger) = e^{-i\eta} \Phi^\dagger.$$ \hspace{1cm} (2.25)

For the ordinary U(1) global symmetry, we fix $\eta$ to be a spacetime-independent parameter as a constant. Thus $U_{Z_2^C} U_{U(1)} \neq U_{U(1)} U_{Z_2^C}$, we have a non-abelian symmetry group structure the $Z_2^C \times U(1) = Z_2^C \times SO(2) = O(2)$.

For the vector global symmetry in Eq. (2.4), $e^{i\eta \nu(x)} \Phi = e^{i\Lambda \cdot x} \Phi$, so

$$U_{Z_2^C} U_{U(1)} x_{(d+1)} \Phi = U_{Z_2^C} (e^{i\eta} \Phi) = e^{i\eta} \Phi^\dagger,$$

$$U_{U(1)} x_{(d+1)} U_{Z_2^C} \Phi = U_{Z_2^C} (\Phi^\dagger) = e^{-i\eta} \Phi^\dagger.$$ \hspace{1cm} (2.26)

We still have the non-commutative structure $U_{Z_2^C} U_{U(1)} x_{(d+1)} \neq U_{U(1)} x_{(d+1)} U_{Z_2^C}$, resulting in a non-abelian symmetry structure $Z_2^C \times U(1) x_{(d+1)}$.

For the gauged theory, the $Z_2^C$ symmetry also flips the sign of the real-valued gauge field $A_{\mu\nu}$, and the vector “gauge symmetry transformation” parameter $\eta_\nu(x)$ as follows

$$A_{\mu\nu} \rightarrow -A_{\mu\nu}, \hspace{0.5cm} \eta_\nu(x) \rightarrow -\eta_\nu(x).$$ \hspace{1cm} (2.27)

Thus, the Eq. (2.13) under $Z_2^C$-symmetry transformation becomes its complex conjugation:

$$(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi - ig A_{\mu\nu} \Phi^2) \rightarrow (\Phi^\dagger \partial_\mu \partial_\nu \Phi^\dagger - \partial_\mu \Phi^\dagger \partial_\nu \Phi^\dagger + ig A_{\mu\nu} (\Phi^\dagger)^2),$$ \hspace{1cm} (2.28)

such that the action Eq. (2.19) is complex conjugation $Z_2^C$-symmetric invariant.

Below we work out the non-commutative-ness of gauge transformations from $Z_2^C$ and U(1)$_{x_{(d+1)}}$:

$$U_{Z_2^C} U_{U(1)} x_{(d+1)} A_{\mu\nu} = U_{Z_2^C} (A_{\mu\nu} + \frac{1}{g} \partial_\mu \partial_\nu \eta_\nu) = -A_{\mu\nu} + \frac{1}{g} \partial_\mu \partial_\nu \eta_\nu.$$  \hspace{1cm} (2.29)

Namely, if we gauge U(1)$_{x_{(d+1)}}$ higher-moment symmetry in Eq. (2.14), and we further gauge $Z_2^C$ ordinary 0-form symmetry, then we can obtain a gauge theory with the new “gauge structure” as:

$$\left[ Z_2^C \times \left( U(1)_{x_{(d+1)}} \right) \right].$$ \hspace{1cm} (2.30)

Here the big bracket $[\ldots]$ indicates the global symmetry inside is being dynamically gauged. So among the global symmetry of Eq. (2.10) and Eq. (2.11), the ordinary U(1)-global symmetry of Eq. (2.10) and Eq. (2.11) is not a global symmetry anymore after gauging U(1)$_{x_{(d+1)}}$ as the way we did in Eq. (2.14).

\footnote{The $Z_2^C$-symmetry is an ordinary unitary 0-form global symmetry in the context of [10].}
Again \[ \mathbb{Z}^C_2 \ltimes (U(1)_x)_{(d+1)} \] is not quite a group, which is definitely not just \( \mathbb{Z}^C_2 \ltimes U(1) = O(2) \), but only as what we call a “gauge structure” or “gauge group-analogous structure.”

Let us temporarily turn off the gauged matter \( \Phi \) in Eq. (2.20), then focus on promoting the global \( \mathbb{Z}^C_2 \) symmetry to a local symmetry that can be gauged in a pure tensor gauge theory. This means that we promote the \( \mathbb{Z}^C_2 \) symmetry Eq. (2.27) to a dynamical local transformation involving a gauge parameter. It is useful to first embed \( \mathbb{Z}^C_2 \subset U(1)^C \), such that we consider an enlarged \( U(1)^C \) gauge symmetry with a gauge parameter \( \gamma_c(x) \):

\[
A_{\mu\nu} \rightarrow e^{i\gamma_c(x)} A_{\mu\nu}, \quad C_\nu \rightarrow C_\nu + \frac{1}{g_c} \partial_{\nu} \gamma_c(x). \tag{2.31}
\]

This Eq. (2.31) deserves explanations. We introduce a new \( g_c \) coupling and the new 1-form \( \mathbb{Z}^C_2 \)-gauge field \( C \) coupling to the 0-form symmetry \( \mathbb{Z}^C_2 \)-charged \( A_{\mu\nu} \). We know that \( A_{\mu\nu} \) has an odd \( \mathbb{Z}^C_2 \) charge because Eq. (2.31) says \( A_{\mu\nu} \rightarrow -A_{\mu\nu} \) under \( \mathbb{Z}^C_2 \). Note \( A_{\mu\nu} \in \mathbb{R} \) is real-valued, but a generic \( e^{i\gamma_c(x)} \) complexifies the \( A_{\mu\nu} \in \mathbb{C} \). However, what we really mean is to restrict gauge transformation so it is only \( \mathbb{Z}^C_2 \)-gauged (not \( U(1)^C \)-gauged),

\[
e^{i\gamma_c(x)} := (-1)^{\gamma_c(x)} \in \{ \pm 1 \}, \quad \gamma_c(x) \in \pi \mathbb{Z}, \quad \gamma_c'(x) \in \mathbb{Z}, \tag{2.32}
\]

so \( \gamma_c'(x) \in \mathbb{Z} \) is an integer, and \( A_{\mu\nu} \in \mathbb{R} \) stays in real. Thus \( \gamma_c'(x) \) can jumps between even or odd integers, while the \( \mathbb{Z}^C_2 \)-gauge transformation is better formulated on a lattice. We can directly rewrite the above Eq. (2.31) on a simplicial complex and a triangulable manifold, e.g. following the Dijkgraaf-Witten’s discretized topological gauge theory on a lattice [24].

We also define a new covariant derivative with respect to \( \mathbb{Z}^C_2 \):

\[
D^c_\mu := (\partial_\mu - ig_c C_\mu). \tag{2.33}
\]

We need to modify \( U(1)_{\times (d+1)} \)-gauge transformation Eq. (2.14) and combine with the \( \mathbb{Z}^C_2 \)-gauge transformation Eq. (2.31) to:\(^8\)

\[
A_{\mu\nu} \rightarrow e^{i\gamma_c(x)} A_{\mu\nu} + \frac{1}{2g} (D^c_\mu D^c_\nu + D^c_\nu D^c_\mu)(\eta_v(x))
\]

\(^8\) One may also naively consider

\[
A_{\mu\nu} \rightarrow e^{i\gamma_c(x)} A_{\mu\nu} + \frac{1}{g} D^c_\mu D^c_\nu(\eta_v(x))
\]

\[
= e^{i\gamma_c(x)} A_{\mu\nu} + \frac{1}{g} \left\{ \partial_\mu \partial_\nu (\eta_v) - ig_c (\partial_\mu C_\nu) \eta_v - ig_c (C_\mu (\partial_\nu \eta_v) + C_\nu (\partial_\mu \eta_v)) \right\} - (g_c)^2 C_\mu C_\nu \eta_v, \tag{2.34}
\]

but in this case, the \( A_{\mu\nu} \) is not symmetric under this specific gauge transformation — even for this case however does not affect the gauge covariance of

\[
F^c_{\mu\nu\zeta} \rightarrow e^{i\gamma_c(x)} F^c_{\mu\nu\zeta} - \frac{ig_c}{g} (C_\mu \partial_\nu \partial_\zeta \eta_v - C_\nu \partial_\mu \partial_\zeta \eta_v) + \frac{(-ig_c)^2}{g} (C_\mu \partial_\nu (C_\zeta \eta_v) - C_\nu \partial_\mu (C_\zeta \eta_v)) - \frac{ig_c}{g} (\partial_\mu (C_\nu C_\zeta \eta_v) - \partial_\nu (C_\mu C_\zeta \eta_v)) + \frac{ig_c}{g} (\partial_\mu C_\nu \eta_v - \partial_\nu C_\mu \eta_v) D^\zeta \eta_v
\]

\[
= e^{i\gamma_c(x)} F^c_{\mu\nu\zeta} - \frac{ig_c}{g} (\partial_\mu C_\nu - \partial_\nu C_\mu) D^\zeta \eta_v
\]

\[
= e^{i\gamma_c(x)} F^c_{\mu\nu\zeta}. \tag{2.35}
\]

Here \( (\partial_\mu C_\nu) \eta_v \) term in the variation Eq. (2.34) can be shown to be equivalent to \( \frac{(\partial_\mu C_\nu + \partial_\nu C_\mu) \eta_v}{g} \) when it is on-shell, i.e., the on-shell theory satisfies EOM: \( dC = 0 \Rightarrow (\partial_\mu C_\nu - \partial_\nu C_\mu) = 0 \) so \( C \) is locally flat — a local constraint for \( \mathbb{Z}^C_2 \)-gauge field \( C \). So Eq. (2.36) and Eq. (2.34) are equivalent on-shell. Moreover, in Eq. (2.35), we can eliminate \( (\partial_\mu C_\nu - \partial_\nu C_\mu) D^\zeta \eta_v \) via again \( (\partial_\mu C_\nu - \partial_\nu C_\mu) = 0 \) because \( dC = 0 \).

On the other hand, if we wish to make sense of the off-shell symmetric tensor gauge field \( A_{\mu\nu} \), we will use the definition Eq. (2.36) instead.
\[ F_{\mu\nu\xi}(x) A_{\mu\nu} + \frac{1}{g} (\partial_\mu \partial_\nu (\eta_\mu) - i g_c \partial_\mu (C_{\nu\eta_\mu}) + \partial_\nu (C_{\mu\eta_\nu})) \]

\[ = e^{i \gamma(x)} A_{\mu\nu} + \frac{1}{g} (\partial_\mu \partial_\nu (\eta_\mu) - i g_c \partial_\mu (C_{\nu\eta_\mu}) + \partial_\nu (C_{\mu\eta_\nu})) \]

\[ = e^{i \gamma(x)} A_{\mu\nu} + \frac{1}{g} (\partial_\mu \partial_\nu (\eta_\mu) - i g_c \partial_\mu (C_{\nu\eta_\mu}) + \partial_\nu (C_{\mu\eta_\nu})) \cdot \eta_\mu - i g_c (C_{\mu}(\partial_\nu(\eta_\mu)) + C_{\nu}(\partial_\mu(\eta_\nu))) -(g_c)^2 C_{\mu}(C_{\nu}(\eta_\nu)) \cdot (g_c)^2 C_{\nu}(C_{\mu}(\eta_\nu)) \].

\[ C_\nu \rightarrow C_\nu + \frac{1}{g_c} \partial_\nu \gamma(x). \] (2.36)

We re-define Eq. (2.16)’s \( F_{\mu\nu\xi} \) into the new gauge covariant field strength \( \hat{F}_{\mu\nu\xi} \):

\[ \hat{F}_{\mu\nu\xi} := D^c_{\mu} A_{\nu\xi} - D^c_{\nu} A_{\mu\xi} := (\partial_\mu - i g_c \partial_\mu (C_{\nu\eta_\mu}) + \partial_\nu (\eta_\mu) - i g_c (C_{\mu}(\partial_\nu(\eta_\mu)) + C_{\nu}(\partial_\mu(\eta_\nu))) -(g_c)^2 C_{\mu}(C_{\nu}(\eta_\nu)) \] (2.37)

We can show that the field strength \( \hat{F}_{\mu\nu\xi} \) is covariant under both the modified U(1)_{x(d+1)} gauge transformation and \( Z_2 \) gauge transformation Eq. (2.36):

\[ \hat{F}_{\mu\nu\xi} \rightarrow e^{i \gamma(x)} \hat{F}_{\mu\nu\xi} - i \frac{g_c}{g} (\partial_\mu (\frac{D_\xi C_{\mu}}{2}) \eta_\nu) - \partial_\nu (\frac{D_\xi C_{\mu}}{2}) \eta_\nu) \]

\[ - i \frac{g_c}{g} (\partial_\mu (C_{\nu\eta_\mu} + C_{\xi}(\partial_\nu(\eta_\mu)) - i g_c \partial_\mu (C_{\nu\eta_\mu} + C_{\xi}(\partial_\nu(\eta_\mu))) \cdot \eta_\mu + i g_c (\partial_\nu (C_{\mu}(\partial_\nu(\eta_\mu)) + C_{\mu}(\partial_\nu(\eta_\mu))) \cdot \eta_\mu \]

\[ = e^{i \gamma(x)} \hat{F}_{\mu\nu\xi} + \frac{g_c}{g} (\partial_\mu (D_\xi C_{\mu}) - \partial_\nu (D_\xi C_{\mu})) (D_\nu \eta_\mu) + \partial_\nu (\eta_\mu) \cdot \eta_\nu + \frac{g_c}{g} (\partial_\nu (D_\xi C_{\mu}) - \partial_\nu (D_\xi C_{\mu})) (D_\nu \eta_\mu) \]

\[ = e^{i \gamma(x)} \hat{F}_{\mu\nu\xi}, \] (2.38)

which is covariant (invariant up to a ± phase Eq. (2.32) \( e^{i \gamma(x)} := (-1)^{\gamma(x)} \in \{ \pm 1 \}) \). Here we can eliminate \((\partial_\mu C_{\nu} - \partial_\nu C_{\mu}) = (\partial_\nu C_{\mu} - \partial_\mu C_{\nu}) = (\partial_\mu C_{\nu} - \partial_\nu C_{\mu}) = 0\), because \( d C = 0 \) is locally flat via the local constraint for \( Z_2 \)-gauge field \( C \). Let us put things altogether in the next paragraph.

Based on the continuum formulation of discrete gauge theories (e.g. [21]), since \( C \) is a 1-form \( Z_2 \)-gauge field, we can either (1) treat \( C \) as a \( Z_2 \)-valued 1-cochain, or (2) treat it as a U(1) gauge field but impose a locally flat condition \( d C = 0 \) and \( \oint C = m \pi \mod 2 \pi \), with \( m \in Z_2 \). For the later purpose for (2), this can be done by introducing a Lagrange multiplier \((d-1)\)-form \( B \) field (in 3+1d, we have\( d = 3 \)). We introduce the famous level-2 anti-symmetric (≡ anti) tensor BF theory,

\[ Z_{\text{level-2-BF}} = \int [DB][DC] \exp(i \int_{M^{d+1}} \frac{2}{2\pi} B dC). \] (2.39)

So far we have a \((d-1)\)-form \( Z_2 \)-gauge \( B \) field and 1-form \( Z_2 \)-gauge field \( C \), obviously we abbreviate \( B dC := B \land dC \). Note it is also a standard convention to omit the wedge product \( \land \) when we are taking the wedge product between the differential forms. Furthermore, ideally and for simplicity, by leaving out the matter \( \Phi \), we can aim for a full quantum theory for a gauge theory of a gauge-analogous structure Eq. (2.30). Formally, we propose a schematic path integral:

\[ Z_{\text{rk-2-sym-asym-BF}} := \int [DA_{\mu\nu}][DB][DC] \exp(i \int_{M^{d+1}} \left( \hat{F}_{\mu\nu\xi}^c (d^{d+1}x) + \frac{2}{2\pi} B dC \right)), \] (2.40)

which is non-trivially fully gauge-invariant under Eq. (2.36), the gauge transformation of \( B \) field via a local \((d-2)\)-form \( \eta_B \):

\[ B \rightarrow B + d\eta_B \] (2.41)
and Eq. (2.38). The term
\[ |\hat{F}^c_{\mu\nu\xi}|^2 := \hat{F}^c_{\mu\nu\xi} \hat{F}^{c,\mu\nu\xi\dagger} = ((\partial_\mu - ig_c C_\mu) A_{\nu\xi} - (\partial_\nu - ig_c C_\nu) A_{\mu\xi}) ((\partial^\mu + ig_c C^\mu) A^{\nu\xi} - (\partial^\nu + ig_c C^{\nu}) A^{\mu\xi}) \] (2.42)
pairs \( \hat{F}^c_{\mu\nu\xi} \) with its complex conjugation \( \hat{F}^{c,\mu\nu\xi\dagger} \) contracting all indices. We shorthand “rk-2-sym” for the rank-2 symmetric tensor gauge theory. We shorthand “asym-BF” for the anti-symmetric tensor BF theory.

Our theory is unitary. At this stage, we only deal with the kinematics of the theory, we discuss its possible quantum dynamics in Conclusion in Sec. 5.

2.1.5 Coupling to anti-symmetric tensor topological field theories

Another new ingredient of our approach is that we can introduce \( Z_2^C \)-gauge theory Eq. (2.40) which is an anti-symmetric tensor topological field theory. A formal way to introduce this \( Z_2^C \)-gauge theory in \((d+1)d\) is that it is a group cocycle element \( \omega_{d+1} \) of the topological gauge theory specified by the group cohomology \( H^{d+1} \) of the gauge group \( G_g \) \[24\]:
\[ \omega_{d+1} \in H^{d+1}(G_g, \mathbb{R}/\mathbb{Z}) := H^{d+1}(BG_g, \mathbb{R}/\mathbb{Z}), \] (2.43)
or the topological cohomology \( H^{d+1} \) of the classifying space of \( BG_g \) in the second expression \[24\]. The cohomology group \( H^{d+1}(G_g, \mathbb{R}/\mathbb{Z}) \) always forms an abelian group.

If we only have a single copy of an abelian symmetric higher-rank tensor gauge theory with a single gauge structure \([Z_2^C \ltimes (U(1)_{x_{(d+1)}})]\), then we can determine possible bosonic \( Z_2^C \)-gauge theory via \( H^{d+1}(Z_2^C, \mathbb{R}/\mathbb{Z}) \). More generally, we can consider \( N \) copies of such abelian symmetric higher-rank tensor gauge theory with \( N \) copies of gauge structure
\[ \prod_{J=1}^N [Z_2^C \ltimes (U(1)_{x_{(d+1)}})]. \] (2.44)
So we have the ordinary gauge group
\[ G_g = [(Z_2^C)^N] \] (2.45)
thus a \([(Z_2^C)^N]\)-gauge theory in addition to some higher-moment gauge structure (formally not a group), then we can specify a group cocycle \[24\], which is computed systematically in Ref. [13,14]
\[ \omega_{d+1} \in H^{d+1}(G_g, \mathbb{R}/\mathbb{Z}) = H^{d+1}((Z_2^C)^N, \mathbb{R}/\mathbb{Z}). \] (2.46)
See Table 1 for explicit results for a generic cohomology group.

- If the \( \omega_{d+1} \) is a trivial cocycle, namely it satisfies the coboundary relation
\[ \omega_{d+1} = \delta\beta_d, \]
then which is a coboundary term under the coboundary operator \( \delta \). The \( \beta_d \in C^d(G_g, \mathbb{R}/\mathbb{Z}) \) is a lower dimensional \( d \)-cochain. This means that \( \omega_{d+1} \simeq 1 \) is identically to the identity in the cohomology group.
Table 1: Cohomology group: Table from Ref. [14] exhibits the exponent of the $Z_{\gcd \otimes^m (N_i)}$ class in the cohomology group $H^d(G_g, \mathbb{R}/\mathbb{Z})$ for a finite Abelian group $G_g = \prod_{u=1}^k \mathbb{Z}_{N_u}$. We abbreviate the greatest common divisor $\equiv \gcd$. We define a shorthand of $Z_{\gcd (N_{1},N_{2},\ldots,N_{m})} \equiv Z_{N_{1},\ldots,N_{m}} \equiv Z_{\gcd \otimes^m (N_i)}$, etc also for other higher gcd. The Type $m$ in the top row of the Table is given by the number $(m)$ of cyclic gauge groups in the gcd class $Z_{\gcd \otimes^m (N_i)}$. The number of exponents can be systematically obtained by K inneth formula and Universal Coefficient Theorem. This table can also be independently derived by a field theory approach [14]. For example, we obtain $H^d(G_g, \mathbb{R}/\mathbb{Z}) = \prod_{1 \leq i<j<l<m<n \leq k} Z_{N_{i} \times (Z_{N_{ij}})^2 \times (Z_{N_{ijl}})^4 \times (Z_{N_{ijlm}})^3 \times (Z_{N_{ijlmn}})}$, etc. We can derive the continuum field theory from the group cohomology result, or the other way around [14,18,21]. For the purpose of present work, we simply take $G_g = [\{Z^C_2\}^N]$ as in Eq. (2.45).

- If the $\omega_{d+1}$ is a nontrivial cocycle, namely $\omega_{d+1} \neq \delta \beta_d$, which is not exact for any cochain $\beta_d$, the gauge theory is commonly called as the twisted group cohomology gauge theory, or Dijkgraaf-Witten gauge theory (\equiv DW) of gauge group $G_g$ [24]. Ref. [13,14,18,21–23] had systematically studied these topological gauge theories as (i) discrete cocycle partition functions, and, (ii) continuum TQFTs. See Table 2 and Sec. 3.1 for the overview of the continuum TQFT formulations Ref. [14,18,21,23].

Table 2 and Sec. 3.1 for the overview of the continuum TQFT formulations Ref. [14,18,21,23].

Now that $\omega_{d+1}(\{C_I\}) \in H^{d+1}(G_g, \mathbb{R}/\mathbb{Z})$ maps to the $\mathbb{R}/\mathbb{Z} = U(1)$ coefficient, a complex $U(1)$ phase, physically $\omega_{d+1}(\{C_I\})$ is the weight of the quantum amplitude in the path integral. Thus, summing over such a $\omega_{d+1}(\{C_I\})$ is also related to the orbifold construction.

$$Z_{\text{rk-2-sym}} = \sum_{I=1}^{N} \text{Tr} \left( \left[ DA_{I,\mu \nu} \right] [DB_I] [DC_I] \right) \text{exp} \left( i \sum_{I=1}^{N} \int_{M^{d+1}} \left( \sum_{I=1}^{N} \hat{F}_{\mu \nu \xi}^c I \right)^2 (d^{d+1}x) + \frac{2}{2\pi} \sum_{I=1}^{N} B_I dC_I \right) \cdot \omega_{d+1}(\{C_I\}).$$

(2.47)

where $\omega_{d+1} \in H^{d+1}(\{Z^C_2\}^N, \mathbb{R}/\mathbb{Z})$. We shorthand “asym-DW” for the anti-symmetric tensor twisted (\equiv tw) Dijkgraaf-Witten (DW) gauge theory. Adding the DW topological term, Eq. (2.41) needs to be modified to $B \to B + d\eta_B + \ldots$, some examples of the additional \ldots terms are shown in Table 2. We can apply the variation principle on the $A, B$ and $C$ fields to obtain their EOMs (i.e., Euler-Lagrange equation):

$$\frac{\delta}{\delta A^c_{\mu \xi}} \ldots \Rightarrow D^{\mu} \hat{F}_{\mu \nu \xi}^c = (\partial^\mu - i g_c C^\mu) \left( (\partial_\nu - i g_c C_\nu) A_{\mu \xi} - (\partial_\nu - i g_c C_\nu) A_{\mu \xi} \right) = 0.$$  (2.48a)

$$\frac{\delta}{\delta B} \ldots \Rightarrow \frac{2}{2\pi} dC = 0.$$  (2.49)

$$\frac{\delta}{\delta C} \ldots \Rightarrow \frac{2}{2\pi} dB = # C_I \wedge C_J \wedge \cdots \wedge dC_K - i4g_c # \star (A^c \hat{F}_{\mu \nu \xi} d\mu^\nu).$$  (2.50)

Here $#$ are some factors depending on the data $\omega_{d+1}$ in terms of the continuum TQFTs given in Table 2. In Eq. (2.48), Eq. (2.49) and Eq. (2.50), we only list down the schematic result since we do not yet precisely provide the data of the group cohomology cocycle $\omega_{d+1}$, which will be given later explicitly in Sec. 3.1. The allowed topological term (or the twisted term) $C_I \wedge C_J \wedge \cdots \wedge dC_K$ depends on the spacetime dimensions and the copies of $G_g$, see Table 1 and Table 2. Here are some comments about EOM:
• In Eq. (2.48), the dependence of $A$ and $C$ reveals the first ingredient of the non-abelian gauge structure (see Sec. 1): A higher-moment symmetry $(U(1)_{(d+1)})$ and a charge conjugation (particle-hole) symmetry $\mathbb{Z}_2^C$ form the non-commutative symmetries. Notice that this EOM is linear respect to the solutions of $A$, but nonlinear respect to the solutions of $C$.

• In Eq. (2.49), in any case, $dC = 0$, so $C$ is always a locally flat $\mathbb{Z}_2$ gauge field.

• In Eq. (2.50), the dependence of $B$ to $A$ and $C$ reveals a second ingredient for the possible non-abelian gauge structure (see Sec. 1): By coupling the symmetric tensor gauge theory to a non-abelian TQFT. See more details in Sec. 3.1.2. Notice that this EOM is nonlinear respect to both the solutions of $A$ and of $C$.

Of course, the discussions in this whole subsection have two versions, simple by changing the spacetime metric signatures: Euclidean-type higher-rank symmetric tensor gauge theory Eq. (1.3) and Lorentzian-type higher-rank symmetric tensor gauge theory Eq. (1.4). Their physics interpretations would be different.

2.2 Anisotropic Non-Abelian Higher-Rank Tensor Gauge Theory for Space and Time

Let us consider the anisotropic-type higher-rank symmetric tensor gauge theory Eq. (1.2). Let us list down the modification for this new case Eq. (1.2) from the previous subsection.

2.2.1 Electric and magnetic tensors: Independent components and SO(3) representations

First, we need to introduce a symmetric rank-2 tensor $A_{ij}$ of $d \times d$ components in a $(d+1)d$ spacetime. We also need a scalar potential $A_0$. Instead of the $F_{\mu\nu\xi}$ in Eq. (2.16), or the electromagnetic tensor $\mathbf{E}$ in Eq. (2.21) and $\mathbf{B}$ in Eq. (2.22), we redefine the electric field and magnetic field tensors similar to Pretko’s Ref. [32–36]:

$$\tilde{E}_{ij} := -\partial_0 A_{ij} + \partial_i \partial_j A_0 = -\partial_t A_{ij} + \partial_i \partial_j A_0.$$  \hspace{1cm} (2.51)

$$\tilde{B}_{ij} := \varepsilon_{ilm} \partial_l A_{mj}.$$  \hspace{1cm} (2.52)

(Naively we can also write $\tilde{B}_{ijk} \propto \partial_j A_{ik} - \partial_k A_{ij}$ as a rank-3 tensor, but it is not necessary.) Now that $\tilde{E}_{ij}$ and $\tilde{B}_{ij}$ are rank-2 tensors. In $(3+1)d$ spacetime, we have $3 \times 3 = 9$ components for each. Let us count the independent $\tilde{E}_{ij}$ and $\tilde{B}_{ij}$ in terms of the spatial rotational SO(3) representations as we did in Eq. (2.23). Note that $\tilde{E}_{ij}$ is symmetric, thus we have $\tilde{E}_{ii}$ 3 components and $\tilde{E}_{ij \neq j}$ another 3 components. Note that $\tilde{B}_{ij}$ is neither symmetric nor anti-symmetric, thus 9 components. We have both tensor fields decomposed in terms of SO(3) representations:

$$\tilde{E} = (1 + 5),$$

$$\tilde{B} = (1 + 3 + 5).$$  \hspace{1cm} (2.53)

The 1 in $\tilde{E}$ is from the identity component $\frac{1}{3} \sum_{i=1}^{3} \tilde{E}_{ii}(I_3)$. The 5 in $\tilde{E}$ are from other components.

\footnote{This means that if $A$ is a solution of EOM and $A'$ is a solution of EOM, then their linear combination $A + A'$ is also a solution of EOM. However, if $C$ is a solution of EOM and $C'$ is a solution of EOM, then their linear combination $C + C'$ in general is not a solution of EOM.}
2.2.2 Matter field theory and symmetric higher-rank tensor gauge theory

We restrict our global symmetry transformation to the space in contrast to Eq. (2.3):

\[ \Phi \rightarrow e^{i\eta_v(x)}\Phi := e^{i\vec{A} \cdot \vec{x}}\Phi. \]  \hspace{1cm} (2.54)

Here \( \vec{A} \) is a \((d)\)-vector on the space, so the \( \vec{A} \cdot \vec{x} \) takes the inner product with the space coordinate. In contrast to Eq. (2.4), we term this

\[ \mathrm{U}(1)_{x(d)} \text{ vector global symmetry}. \]  \hspace{1cm} (2.55)

Promoting the vector global symmetry transformation to a gauge transformation, follow Sec. 2.1.1, this yields Pretko’s field theory \([34]\) with some couplings \( g_0 \) and \( \lambda \):

\[
Z'_{\text{rk-2-sym-}} := \int [DA_0][DA_{ij}][D\Phi][D\Phi^\dagger] \exp(i \int_{M^{d+1}} d^{d+1}x \left( (|E_{ij}|^2 - |\tilde{B}_{ij}|^2) + |(\partial_t - i g_0 A_0)\Phi|^2 - \lambda |(\partial_t \partial_j \Phi - \partial_i \Phi \partial_j \Phi - ig A_{ij} \Phi^2)|^2 + V(|\Phi|^2 + \ldots) \right)).
\]  \hspace{1cm} (2.56)

The theory is gauge invariant under:

\[ \Phi \rightarrow e^{i\eta_v(x)}\Phi, \quad A_0 \rightarrow A_0 + \frac{1}{g} \partial_0(\eta_v(x)) = A_0 + \frac{1}{g} \partial_t(\eta_v(x)), \quad A_{ij} \rightarrow A_{ij} + \frac{1}{g} \partial_i \partial_j(\eta_v(x)). \]  \hspace{1cm} (2.57)

There are other possible terms listed in \ldots, see, for instance, Ref. [34].

2.2.3 Gauge \( Z_C^2 \) symmetry for another non-abelian higher-moment-gauged tensor theory

Similar to Sec. 2.1.4, we aim to obtain a non-abelian tensor gauge theory by gauging the ordinary 0-form \( Z_C^2 \) charge conjugation symmetry:

\[ \Phi \rightarrow \Phi^\dagger, \quad A_0 \rightarrow -A_0, \quad A_{ij} \rightarrow -A_{ij}, \quad \eta_v(x) \rightarrow -\eta_v(x). \]  \hspace{1cm} (2.58)

Under the same \( Z_C^2 \)-covariant derivative Eq. (2.33)’s \( D_\mu^c := (\partial_\mu - ig_v C_\mu) \), similar to Eq. (2.36) and Eq. (2.32), we couple the gauged theory to DW topological terms as in Sec. 2.1.5. We have the combined gauge transformation\(^{10}\)

\[
A_0 \rightarrow e^{i\gamma_v(x)} A_0 + \frac{1}{g} D_0^c(\eta_v(x)) = e^{i\gamma_v(x)} A_0 + \frac{1}{g} D_i^c(\eta_v(x)).
\]

\[
A_{ij} \rightarrow e^{i\gamma_v(x)} A_{ij} + \frac{1}{2g} (D_i^c D_j^c + D_j^c D_i^c)(\eta_v(x))
= e^{i\gamma_v(x)} A_{ij} + \frac{1}{g} \left( \partial_i \partial_j(\eta_v) - ig_v (\partial_i C_j + \partial_j C_i) / 2 \right) \eta_v - ig_v C_i(\partial_j \eta_v) - ig_v C_j(\partial_i \eta_v) - (g_v)^2 C_i C_j \eta_v.
\]

\(^{10}\)Naively, we may define:

\[
A_{ij} \rightarrow e^{i\gamma_v(x)} A_{ij} + \frac{1}{g} D_i^c D_j^c(\eta_v(x))
= e^{i\gamma_v(x)} A_{ij} + \frac{1}{g} \left( \partial_i \partial_j(\eta_v) - ig_v \partial_i (C_j \eta_v) - ig_v \partial_j (C_i \eta_v) - (g_v)^2 C_i C_j \eta_v \right).
\]

However, this results in \( A_{ij} \) is not symmetric under this specific gauge transformation — this case however does not affect the gauge covariance of \( \hat{E}_{ij}^c \) and \( \hat{B}_{ij}^c \) tensor at least for the on-shell case.
\[ C_j \to C_j + \frac{1}{g_c} \partial_j \gamma_c(x). \]
\[ e^{i\gamma_c(x)} := (-1)^{\gamma_c(x)} \in \{\pm 1\}. \]
\[ B \to B + d\eta_B + \ldots \]

(2.59)

A \mathbb{Z}_2^C\)-gauge transformation can be much easier defined in a discretized spacetime [24]. Examples of the additional \ldots terms are shown in Table 2. We also refine the \( \tilde{E}_{ij}, \tilde{B}_{ij} \) to the covariant derivative Eq. (2.33) version’s \( \tilde{E}_{ij}^c, \tilde{B}_{ij}^c \):

\[ \tilde{E}_{ij}^c := -D_0^c A_{ij} + \frac{D_i^c D_j^c + D_j^c D_i^c}{2} A_0 = -D_i^c A_{ij} + \frac{D_i^c D_j^c + D_j^c D_i^c}{2} A_0. \]
\[ \tilde{B}_{ij}^c := \varepsilon_{i\ell m} D_\ell A_{m,j}. \]

(2.60)

(2.61)

Again, we can check the covariance of \( \tilde{E}_{ij}^c \) and \( \tilde{B}_{ij}^c \) tensor under the gauge transformation, after some long calculations and subtle cancellations:

\[ \tilde{E}_{ij}^c \to e^{i\gamma_c(x)} \tilde{E}_{ij}^c, \]
\[ \tilde{B}_{ij}^c \to e^{i\gamma_c(x)} \tilde{B}_{ij}^c, \]

up to a \( e^{i\gamma_c(x)} := (-1)^{\gamma_c(x)} \in \{\pm 1\} \) phase, as long as we can use \( dC \sim \varepsilon_{i\ell m} \partial_C m = 0 \) for the on-shell flat gauge \( C \) condition. Again we couple a symmetric tensor gauge theory to a DW term \( \omega_{d+1}(\{C_I\}) \in H^{d+1}(G_g, \mathbb{R}/\mathbb{Z}) \) shown in Ref. [14, 18, 21–23] to gain another unitary field theory with anisotropic space-time symmetry.\(^{11}\)

\[
 Z'_{gh,2-sym} := \int \prod_{I=1}^{N} [DA_I,0][DA_I,ij][DB_I][DC_I] \exp \left( i \int_{M^{d+1}} \sum_{I=1}^{N} \left( |C_{ij}^c|^2 - |\tilde{B}_{ij}^c|^2 \right) (dx^{d+1}) \right) \nonumber
 + \frac{2}{2\pi} \sum_{I=1}^{N} B_I dC_I \cdot \omega_{d+1}(\{C_I\}) \]  

(2.62)

Again we apply the variation principle on the \( A, B \) and \( C \) fields to gain anisotropic EOM (for \( I = 1 \) only):

\[ \frac{\delta}{\delta A^0} (\ldots) \Rightarrow D^c D^c \tilde{E}_{ij}^c = D^c D^c \left( -D_i^c A_{ij} + \frac{D_i^c D_j^c + D_j^c D_i^c}{2} A_0 \right) = 0, \]
\[ \frac{\delta}{\delta A_{ij}} (\ldots) \Rightarrow D_i^c \tilde{E}_{ij}^c - \varepsilon_{i\ell m} D_\ell \tilde{B}_{m,j} = D_i^c (-D_i^c A_{ij} + \frac{D_i^c D_j^c + D_j^c D_i^c}{2} A_0) - \varepsilon_{i\ell m} D_\ell (\varepsilon_{m\ell m'} D_\ell A_{m',j}) = 0, \]
\[ \frac{\delta}{\delta B} (\ldots) \Rightarrow \frac{2}{2\pi} dC = 0, \]
\[ \frac{\delta}{\delta C_0} (\ldots) \Rightarrow \frac{2}{2\pi} dB = \#C_I \wedge C_J \wedge \ldots \wedge dC_K - i g_c \ast (\#^A_{ij} \tilde{E}_{ij}^c) dx^0, \]
\[ \frac{\delta}{\delta C_{ij}} (\ldots) \Rightarrow \frac{2}{2\pi} dB = \#C_I \wedge C_J \wedge \ldots \wedge dC_K - i g_c (\#(D^c_{ij} A^0) \tilde{E}_{ij}^c + \#(\varepsilon^i j m A^m_{ij}) \tilde{B}_{ij}^c) dx^i. \]

(2.63)

(2.64)

(2.65)

(2.66)

(2.67)

Several factors \# can be determined fully via the given data and Table 2 later. Again \( \varepsilon_{ijk} = \varepsilon^{ijk} = \varepsilon^i j k = \varepsilon^{ij} k \), raising or lowering indices does not affect its sign for spatial indices. Comments on EOMs:

- In Eq. (2.63) and Eq. (2.64), this EOM is \textit{linear} respect to solutions of \( A \), but \textit{nonlinear} respect to solutions of \( C \), see footnote 9.

\(^{11}\)Readers should beware that the definitions of \( B \) and \( B \) are fundamentally different. We use \( B \) for the any-symmetric tensor field (here \((d - 2)\)-form gauge field), here we use \( B \) to represent the magnetic tensor field in its anisotropic version.
• In Eq. (2.65), in any case, \( dC = 0 \) is always a locally flat \( \mathbb{Z}_2 \) gauge field.

• In Eq. (2.66) and Eq. (2.67), the dependence of \( B \) to \( A \) and \( C \) reveals a second ingredient for the possible non-abelian gauge structure (see Sec. 1): By coupling the symmetric tensor gauge theory to a non-abelian TQFT. See more details in Sec. 3.1.2. Notice that this EOM is nonlinear respect to both the solutions of \( A \) and of \( C \).

3 Non-Abelian Tensor Gauge Theory Interplayed Between Gapped And Gapless Phases

Now we provide evidence that our theories in Eq. (2.47) and Eq. (2.62) are in the interplay between gapped phases (see Sec. 3.1) and gapless phases (see Sec. 3.2). Let us organize and summarize our statements line by line.

1. The \textit{phases} mean that \textit{the states of matter in the phase diagram} of the quantum matter or of the condensed matter.

2. The \textit{gapped} phases mean that the excitations in the phases are massive and highly energetic expansive. In fact, in the TQFT limit, it costs infinite energy to break up the extend operators (such as line and surface operators) with open ends — only the open ends can host massive particles or heavy objects.

3. The \textit{gapless} phases mean that the dispersion of excitations is continuous and massless in the large system size limit.

4. The anti-symmetric tensor TQFTs describe gapped phases shown in Sec. 3.1. The \textit{gapped excitations} are the anyonic particles (described by the worldline of \( C \) field), or the anyonic strings/(\( d - 3 \))-branes (described by the worldsheet/volume of \( B \) field).

5. The symmetric tensor gauge theories describe gapless phases that will be shown in the following Sec. 3.2.

3.1 Anti-Symmetric Tensor Topological Field Theory and Gapped Phase

Shown in Table 2, here we quickly overview the continuum field theory formulation [18,21,22] of topological gauge theory from a cohomology group [24]: \( H^d(G_g, \mathbb{R}/\mathbb{Z}) \) for a finite Abelian group \( G_g = \prod_{n=1}^{\text{finite}} \mathbb{Z}_{N_n} \), or specifically a gauge group \( G_g = (\mathbb{Z}_2^C)^m \) of \( m \)-layers of gauged charge conjugation symmetries.

The first column of Table 2 shows the continuum anti-symmetric tensor TQFT actions and their \textit{gauge transformations} to all order in analytic exact forms. Namely, Table 2 contains a finite gauge transformations, instead of only infinitesimal gauge transformations. There is no further higher order term that we can add to the \textit{gauge transformations} beyond what we had found in Table 2.

In fact, the term in the path integral of Eq. (2.47) and Eq. (2.62) \( \int ([DB_I][DC_I]) \exp(i \int_{M^{d+1}} \frac{2}{2\pi} \sum_{I=1}^{N} B_I dC_I) \cdot \omega_{d+1} (\{C_I\}) \) can be replaced to this continuum theory [18,21,23], e.g.,

\[
\int ([DB_I][DC_I]) \exp(i \int_{M^{d+1}} \frac{2}{2\pi} \sum_{I=1}^{N} B_I \wedge dC_I + \#C_I \wedge C_J \wedge \cdots \wedge dC_K). \tag{3.1}
\]
with some coefficients #, shown in Table 2.

In Table 2, we provide the link configurations from the extended operators linked in the spacetime [18, 21, 23]. The extended operators include, for example, line operators on circles $S^1$, and the surface operators on $S^2$ or $T^2$.

Readers can find the explicit derivation of:

\[
\left\{ \begin{array}{l}
\text{link invariants and braiding statistics with extended operators from a continuum TQFT in [21].} \\
\dim(\mathcal{H}_d) = \text{GSD associated to } d\text{-dimensional space (counting zero energy modes) in [22].}
\end{array} \right.
\]

The EOMs follow Eq. (2.48), Eq. (2.49), and Eq. (2.50). The first two do not depend on the topological term $\omega_{d+1}$ while the third EOM does:

\[
\frac{\delta}{\delta C} = 0 \implies \frac{2}{2\pi} dB = +\xi_\omega - 4ig_c \star (A^{\mu\xi} F^{\alpha\xi}_{\mu\nu} dx^\nu) = 0. \tag{3.2}
\]

The $\xi_\omega$ depends on $\omega_{d+1}$. For the five cases listed in Table 2, the $\xi_\omega$ corresponds to, or, is proportional to:

1. $\frac{p_{IJ}}{2\pi} dC^J$.

2. $c_{123} C^2 \wedge C^3 + \text{permutations (1} \leftrightarrow \text{2} \leftrightarrow \text{3)}$.

3. 0.

4. $\frac{N_{IJ} N_{JK} N_{KL}}{(2\pi)^2 N_{JJ}} C^J \wedge dC^K$.

5. $c_{1234} C^2 \wedge C^3 \wedge C^4 + \text{permutations (1} \leftrightarrow \text{2} \leftrightarrow \text{3} \leftrightarrow \text{4)}$. 

23
3.1.1 Abelian TQFTs v.s. Top-Type Non-Abelian TQFTs

Table 2: Continuum field theory formulation [18, 21, 22] of topological gauge theory from cohomology group [24] of Table 1. Previously we list down a gauge group $G_g = (\mathbb{Z}_2^C)^m$ because of the limitation of our model’s charge conjugation symmetry, but we can replace it to a more genuine gauge group $(\mathbb{Z}_N)^m$, see Ref. [14, 18, 21–23]. The first column shows the proposed continuum anti-symmetric tensor TQFT actions for these theories and their gauge transformations to all order in analytic exact forms. The second column shows the group-cohomology cocycle data $\omega$ as a certain partition-function solution of Dijkgraaf-Witten theory, where $\omega$ belongs to the group-cohomology group, $\omega \in H^{d+1}(G, \mathbb{R}/\mathbb{Z}) = H^{d+1}(G, U(1))$. The notations such as the mod $N_J$ summation for $[\ldots]_{N_J}$ are introduced in Ref. [14]. The third column shows the path integral form which encodes the braiding process of particles (on the open ends of 1-worldline) and strings (on the open ends of 2-worldsheet) in the spacetime. In 2+1D (3d), $C$ and $B$ are 1-forms, while $\gamma_c$ and $\eta_B$ are 0-forms. In 3+1D (4d), $B$ is a 2-form, $C$ and $\eta_B$ are 1-forms, while $\gamma_c$ is a 0-form. The $I, J, K \in \{1, 2, 3, \ldots\}$ belongs to the gauge subgroup indices, $N_{12\ldots u} \equiv \gcd(N_1, N_2, \ldots, N_u)$ is defined as the greatest common divisor (gcd) of $N_1, N_2, \ldots, N_u$. Here $p_{IJ} \in Z_{N_{1IJ}}, p_{123} \in Z_{N_{123}}, p_{IJK} \in Z_{N_{IJK}}, p_{1234} \in Z_{N_{1234}}$ are integer coefficients. The $c_{IJ}, c_{123}, c_{IJK}, c_{1234}$ are quantized coefficients labeling distinct topological gauge theories, where $c_{12} = \frac{1}{(2\pi)^2} N_1 N_2 p_{12}$, $c_{123} = \frac{1}{(2\pi)^3} N_1 N_2 N_3 p_{123}$, $c_{1234} = \frac{1}{(2\pi)^4} N_1 N_2 N_3 N_4 p_{1234}$. The quantization labeling are described and derived in [13,14]. Be aware that we define both $p_{IJ}$ and $c_{IJ}$ as constants with fixed-indices $I, J, \ldots$ without summing over those indices. We additionally define $\bar{c}_{IJ} \equiv \epsilon_{IJ}c_{12}$ with the $\epsilon_{IJ} = \pm 1$ as an anti-symmetric Levi-Civita alternating tensor where $I, J, \ldots$ are free indices needed to be Einstein-summed over. The $c_{12}$ is fixed, not summed over.

3.1.1 Abelian TQFTs v.s. Top-Type Non-Abelian TQFTs
In fact, when we have the number of $N$-copies of charge conjugation symmetry $(Z^C_2)^N$ as the same as $N = d + 1$ of total spacetime dimensions, then we are allowed to input an exotic cocycle $\omega_{d+1}^{\text{Top}} \in \mathbb{H}^{d+1}(\mathbb{Z}^C_2)^{N=d+1}, \mathbb{R}/\mathbb{Z}$ into the path integral. In the form of continuum gauge theory, Eq. (3.1) can be written as

$$\int \left( \prod_{I=1}^{N} [DB_I][DC_I] \right) \exp(i \int_{M^{d+1}} \left( \sum_{I=1}^{N} \frac{N_I}{2\pi} B^I \wedge dC^I \right) + p \frac{N_1 N_2 \ldots N_{d+1}}{(2\pi)^{d+1}} C^1 \wedge \ldots \wedge C^{d+1} \right) \cdot \omega_{d+1}^{\text{Lower}}$$

with $p \in \mathbb{Z}_{N_1 \ldots d,d+1} = \mathbb{Z}_2$, with all of $N_I = 2$ thanks to $(Z^C_2)$, such that the cocycle $\omega_{d+1}^{\text{Top}}$ is known as the Top Type. (For this Top Type cocycle, see References therein [38] and [13,14,18,21,22].) These Top Type TQFTs are non-abelian topological orders in nature, while other Lower types $\omega_{d+1}^{\text{Lower}}$ of cocycle twists alone produce abelian topological orders. Combine both the Lower and Top Types of cocycle twists as in Eq. (3.3) also produce additional non-abelian topological orders.

By non-Abelian topological orders, we mean that some of the following properties are matched:

- The $\dim \mathcal{H}_{S^d;\sigma_1,\sigma_2,\sigma_3\ldots} = \mathbb{Z}(S^d \times S^1; \sigma_1, \sigma_2, \sigma_3, \ldots) = \text{GSD}$ on a sphere $S^d$ with operator insertions (or the insertions of anyonic particle/string excitations on $S^d$) show the following behavior:
  (1) $\mathbb{Z}(S^d \times S^1; \sigma_1, \sigma_2, \sigma_3, \ldots) = \dim \mathcal{H}_{S^d;\sigma_1,\sigma_2,\sigma_3\ldots} \propto k^n$ will grow exponentially as $k^n$ for a certain set of large $n$ number of insertions, for some number $k$. These anyonic particles are called non-Abelian anyons or non-Abelian quasiparticles. The anyonic string that causes this behavior can be called non-Abelian string [13,43].
  (2) $\mathbb{Z}(S^d \times S^1; \sigma_1, \sigma_2, \sigma_3) = \dim \mathcal{H}_{S^d;\sigma_1,\sigma_2,\sigma_3} \geq 1$ for a certain set of three insertions as $(\sigma_1, \sigma_2, \sigma_3)$.

- Prepare a certain configuration (with a degenerate Hilbert space $\dim \mathcal{H}_{T^d} > 1$, e.g. as the previous remark), the adiabatic braiding process of gapped excitations will rotate the ground state-vector $|\Psi_{\text{initial}}\rangle$ in the Hilbert space into a new final state-vector $|\Psi_{\text{final}}\rangle$ not parallel nor up to a phase respect to the initial state $|\Psi_{\text{initial}}\rangle$. Namely, $|\Psi_{\text{final}}\rangle = U_{\text{braid}} |\Psi_{\text{initial}}\rangle \neq e^{i\theta_{\text{abelian}}} |\Psi_{\text{initial}}\rangle$. This means the $U_{\text{braid}}$ matrix is generically non-commutative thus non-abelian.

- The $\dim \mathcal{H}_{T^d} = \mathbb{Z}(T^d \times S^1) = \text{GSD}$, for a discrete gauge theory of a gauge group $G$ on a spatial torus $T^d$, behaves as $\text{GSD} < |G_g|^d$. The GSD is reduced to a smaller number than the Abelian GSD. This criterion however works only for a 1-form gauge theory (here for the 1-form $C$ field). See the detail explanation in the footnote 7 of [22]. In Table 3, we give these examples of abelian TQFTs v.s. non-abelian TQFTs in 2+1D and in 3+1D.

| $\dim \mathcal{H}_{T^d}$ | Abelian TQFT GSD$_{T^d} = |G_g|^d$ | Non-abelian TQFT GSD$_{T^d} < |G_g|^d$ |
|--------------------------|-------------------------------|-------------------------------|
| $2+1D$                   | $(2^4)^d = 64$                | 22                           |
| $3+1D$                   | $(2^4)^d = 4096$              | 1576                          |

Table 3: Data of zero energy modes counting below the TQFT energy gap, as $\dim \mathcal{H}_{T^d} = \text{GSD}_{T^d}$ from [13,22].

3.1.2 Non-abelian tensor gauge theory with gapless modes, massive particles and gapped non-abelian anyonic strings in 3+1D (4d), and gapped non-abelian anyonic branes in 4+1D (5d)

Below let us write down the 3+1D (4d) and 4+1D (5d) models explicitly that have non-abelian anyonic excitations. Both are non-abelian tensor gauge theory with gapless modes. Both have massive particles.
(from the end of worldline of $C$ field). The 3+1D (4d) model has gapped anyonic strings (the end of worldsheet of $B$ field). The 4+1D (5d) model has gapped anyonic branes (the end of worldvolume of $B$ field). Here are the 3+1D (4d) and 4+1D (5d) models with schematic path integrals written in generality:

\[
\begin{align*}
Z_{3\text{-}1\text{d}}^{\text{sym\text{-}tw}} & := \int \left( \prod_{I=1}^{N} [DA_{I,\mu\nu}] [DB_{I}] [DC_{I}] \right) \exp(i \int_{M+1}^{D+1} \left( \sum_{I=1}^{N} (|\hat{E}^c_{,I}|^2 - |\hat{B}^c_{,I}|^2) \right) (d^{d+1}x) \\
& + \sum_{I=1}^{N} N_{I} B^I \wedge dC^I + p \frac{N_{1}N_{2}N_{3}N_{4}}{(2\pi)^3 N_{1234}} C^1 \wedge C^2 \wedge C^3 \wedge C^4 \cdot \omega_4^{\text{Lower}}(\{C_{I}\}). \quad (3.4)
\end{align*}
\]

\[
\begin{align*}
Z_{4\text{-}1\text{d}}^{\text{sym\text{-}tw}} & := \int \left( \prod_{I=1}^{N} [DA_{I,\mu\nu}] [DB_{I}] [DC_{I}] \right) \exp(i \int_{M+1}^{D+1} \left( \sum_{I=1}^{N} (|\hat{E}^c_{,I}|^2 - |\hat{B}^c_{,I}|^2) \right) (d^{d+1}x) \\
& + \sum_{I=1}^{N} N_{I} B^I \wedge dC^I + p \frac{N_{1}N_{2}N_{3}N_{4}N_{5}}{(2\pi)^4 N_{12345}} C^1 \wedge C^2 \wedge C^3 \wedge C^4 \wedge C^5 \cdot \omega_5^{\text{Lower}}(\{C_{I}\}). \quad (3.5)
\end{align*}
\]

Here $p \in Z_{N_{1}, \ldots, d, d+1} = Z_2$ and $N_{1} = N_{2} = N_{3} = N_{4} = N_{5} = \cdots = 2$. For models of Euclidean/Lorentz version in Sec. 2.1, we have

\[
(|\hat{E}^c_{,I}|^2 - |\hat{B}^c_{,I}|^2) := |\hat{E}^c_{,\mu\nu\xi|}^2 = \left( (\partial_\mu - ig_{c}C_{\mu,I}) A_{\nu,\xi,I} - (\partial_\nu - ig_{c}C_{\nu,I}) A_{\mu,\xi,I} \right) \left( (\partial_\mu + ig_{c}C_{\mu,I}) A^{\nu,\xi,I} - (\partial_\nu + ig_{c}C^{\nu,I}) A^{\mu,\xi,I} \right),
\]

where $\hat{E}^c, \hat{B}^c$ should be the covariant version of $E, B$ in Eq. (2.21) and Eq. (2.22). For models of anisotropic version in Sec. 2.2, we have $|\hat{E}^c_{,I}|^2 - |\hat{B}^c_{,I}|^2$ given by Eq. (2.60) and Eq. (2.61), and $[DA_{I,\mu\nu}]$ should be replaced by $[DA_{I,0}] [DA_{I,ij}]$.

### 3.2 Symmetric Higher-Rank Tensor Gauge Theory and Gapless Phase

We discuss that the symmetric higher-rank tensor gauge theory with an action alone $\int_{M+1}^{D+1} |\hat{F}^c_{,\mu\nu\xi}|^2 (d^{d+1}x)$ in $d + 1$ dimensions (at least $d + 1 \geq 3 + 1$) can contain gapless phase with massless modes.

#### 3.2.1 Degrees of freedom for gapless modes

**Degrees of freedom ($\equiv \text{dof}$) counting for gapless modes:**

1. U(1) gauge theory of Sec. 1.1.1:
   
   Let us recall the standard dof counting for the gapless (or massless) modes, e.g., the photons, in the spin-1 Maxwell theory in 3+1D (4d) in Sec. 1.1.1. Let a single photon moves in the speed of light say with a momentum $p_z$ along an arbitrary $z$ direction.

   Let us first count the photon dof physically. Naively the photon can either oscillate along the longitudinal direction (1 dof along $p_z$) and the transverse direction (2 dof perpendicular to $p_z$). However, the longitudinal mode does not make sense because the photon cannot oscillate forward faster than the speed of light. Thus there are only 2 transverse modes: say along with the $p_x$ and $p_y$.

---

12 Readers should beware that the definitions of $B$ and $B$ are fundamentally different. We use $B$ for the any-symmetric tensor field (here $(d - 2)$-form gauge field), while we use $B, \tilde{B}$, $\hat{B}$, $\tilde{B}$, etc., to represent the magnetic tensor field from the abelian field strength $F_{\mu\nu\xi}$ or its anisotropic version. Similarly, we use $\hat{B}^c$, etc., to represent the magnetic tensor field from the nonabelian field strength $\hat{F}^c_{\mu\nu\xi}$. 

---
directions. Then their linear combinations \( p_x + ip_y \) and \( p_x - ip_y \) are effectively the two eigenvectors for the spin angular momentum operator \( S_z \) of the photon. Namely, there is a map of 2 dof:

\[
p_x \pm ip_y \implies \langle \hat{S}_z \rangle = \pm 1 \implies |S, S_z \rangle = |1, \pm 1\rangle.
\]

These are in fact also the 2 dof from the 2 helicity from the 2 transverse modes

\[
\hat{S}_z \cdot \hat{P}_z = \pm 1.
\]

Let us also count the dof mathematically. We see that a photon field is a 4-vector \( A_\mu \) with \( \mu = 0, 1, 2, 3 \). We can choose the gauge on \( \eta \), such that Eq. (1.5) \( A_0 \rightarrow A_0 + \frac{1}{g} \partial_0 \eta \) = 0, which gives 1 dof out of 4 dof. Furthermore, the gauge transformation \( A_0 \rightarrow A_0 + \frac{1}{g} \partial_0 \eta \) can still hold if \( \eta \rightarrow \eta + \eta'(x_j) \) where \( \eta'(x_j) \) is a pure function of space indices \( x_j \). Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most \( 4 - 1 - 1 = 2 \) dof, which agrees with the physics story given earlier.

2. SU(N) YM gauge theory of Sec. 1.1.2:

Similarly, the dof counting of gluon field \( A_\alpha^\mu \) is similar to the previous counting. We have 2 dof for each \( A_\alpha^\mu \) and there are \( N^2 - 1 \) independent of glouns with index \( \alpha \) in the adjoint representation. However, as we know due to the color confinement [44], the gapless dof of gluon field \( A_\alpha^\mu \) do not appear at the low energy, but only above the confinement energy gap scale \( \geq \Lambda_{\text{YM}} \). At low energy \( < \Lambda_{\text{YM}} \), we only observe a confinement energy gap without massless modes.

3. Euclidean or Lorentz invariant non-abelian higher-rank tensor gauge theory of Sec. 2.1:

Ref. [29] finds that the compact abelian symmetric tensor gauge theory is unstable to confinement in 2+1D, but it becomes stable to be gapless-ness with deconfinement in 3+1D. We thus focus on 3+1D (or above) for its richer physics in a gapless phase.

Let us count the dof in 3+1D. We see that the rank-2 symmetric tensor field is \( A_{\mu \nu} \) with \( \mu, \nu = 0, 1, 2, 3 \) naively with 10 dof. We can choose the gauge on \( \eta^\nu \) such that Eq. (2.14) \( A_{\mu 0} \rightarrow A_{\mu 0} + \frac{1}{g} \partial_0 \eta^\nu = 0 \), which kills 1 dof out of 10 dof. Furthermore, the gauge transformation \( A_{\mu 0} \rightarrow A_{\mu 0} + \frac{1}{g} \partial_0 \partial_0 \eta^\nu \) = 0 can still hold if \( \eta^\nu \rightarrow \eta^\nu + \eta'(x_j) + t \cdot \eta''(x_j) \) where the two additional redundant dof \( \eta'(x_j) \) and \( \eta''(x_j) \) are pure functions of space indices \( x_j \). Since the shifted function has a linear \( t \) dependence, we have \( \partial_0 \partial_0 \eta^\nu \rightarrow \partial_0 \partial_0 \partial_0 \eta^\nu \). Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most \( 10 - 1 - 2 = 7 \) dof.

Let us also count the dof in \( d + 1 \)D. The rank-2 symmetric tensor field is \( A_{\mu \nu} \) naively with \( \frac{(d+2)(d+1)}{2} \) dof. We can choose the gauge on \( \eta^\nu \) such that Eq. (2.14) \( A_{00} \rightarrow A_{00} + \frac{1}{g} \partial_0 \partial_0 \eta^\nu = 0 \), which reduces 1 dof. Furthermore, the gauge transformation \( A_{00} \rightarrow A_{00} + \frac{1}{g} \partial_0 \partial_0 \partial_0 \eta^\nu = 0 \) can still hold if \( \eta^\nu \rightarrow \eta^\nu + \eta'(x_j) + t \cdot \eta''(x_j) \) since the shifted function (with 2 dof) has a linear \( t \) dependence, we have \( \partial_0 \partial_0 \partial_0 \eta^\nu \rightarrow \partial_0 \partial_0 \partial_0 \partial_0 \eta^\nu \). Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most \( \frac{(d+2)(d+1)}{2} - 1 - 2 = \left(\frac{(d+2)(d+1)}{2} - 3\right) \) dof.

4. Anisotropic non-abelian higher-rank tensor gauge theory for space and time of Sec. 2.2.

Again due to 3+1D may becomes stable to be gapless-ness with deconfinement [29], let us count the dof in 3+1D or in \( d + 1 \)D. The rank-2 symmetric tensor field is \( A_{ij} \) naively with \( \frac{(d+1)d}{2} \) dof and \( A_0 \) for 1 dof. We can choose the gauge on \( \eta^i \) such that Eq. (2.57) \( A_0 \rightarrow A_0 + \frac{1}{g} \partial_0 \eta^i = 0 \), which reduces 1 dof. Furthermore, the gauge transformation \( A_0 \rightarrow A_0 + \frac{1}{g} \partial_0 \eta^i = 0 \) can still hold if \( \eta^i \rightarrow \eta^i + \eta'(x_j) \) since the shifted function (with 1 dof) has a linear \( t \) dependence, we have \( \partial_0 \eta^i \rightarrow \partial_0 \partial_0 \partial_0 \eta^i \). Therefore the real physical degrees of freedom (dof) for gapless modes of photon can be at most \( \frac{(d+1)d}{2} + 1 - 1 - 1 = \left(\frac{(d+1)d}{2} - 1\right) \) dof. In 3+1D, we have 5 dof.

See a summary in Table 4.
3.2.2 Dispersions of gapless modes

Now we determine the dispersions of gapless modes, as the scaling of energy $\omega_E(k) \propto k^z$ and the momentum $k$ to the dispersion’s dynamical exponent $z$. For the dimensional analysis, below we denote the scaling of the spatial dimensions as $[L]$. 

| gapless dof | Euclidean/Lorentz invariant of Sec. 2.1 | Anisotropic for space and time of Sec. 2.2 |
|------------|----------------------------------------|--------------------------------------------|
| 3+1D       | 7                                      | $\frac{(d+1)d}{2} - 3$                      |
| $d+1D$     | $\propto c_v k$                        | $\propto c_v k$                            |

Table 4: We collect the data of the degrees of freedom (dof) and dispersion relation for gapless modes, and the electric E and magnetic B tensors in terms of the spatial rotational SO(3) representation.

1. Obviously both Maxwell and Yang-Mills theories have photons and gluons with linear dispersion relation $\omega_E(k) \propto ck$, where $c$ is known as the speed of light constant (alternatively, the effective speed of light in the quantum matter material). This is easy to derive based on the scaling between $E^2 = c^2 B^2$, where the ratio gives the (square of) the speed of light.

2. Euclidean or Lorentz invariant non-abelian higher-rank tensor gauge theory of Sec. 2.1:

   We see that from Eq. (2.14), Eq. (2.21) and Eq. (2.22),

   $$
   \begin{align*}
   [A_{\mu\nu}] & \sim \frac{1}{g} [\partial_{\mu} \partial_{\nu} (\eta_{\nu}(x))] \sim [L]^{-2}, \\
   [E_{ij}] & \sim \{ \partial_{0} A_{ij} - \partial_{i} A_{0j} \} \sim [L]^{-3}, \\
   [B_{\ell k}] & \sim \left[ \frac{1}{2} \epsilon^{ij} (\partial_{i} A_{jk} - \partial_{j} A_{ik}) \right] \sim [L]^{-3}. \\
   [E_{ij}]^2 & \sim (B_{\ell k})^2, \text{ as } [\partial_{0}]^2 \sim (v_c)^2 [\partial_{i}]^2.
   \end{align*}
   \tag{3.6}
   $$

   So $[t] \sim [x] \sim [L]$, thus it is a linear dispersion relation $\omega_E(k) \propto v_c k$ with $z = 1$.

3. Anisotropic non-abelian higher-rank tensor gauge theory for space and time of Sec. 2.2:

   From Eq. (2.57) and Eq. (2.51),

   $$
   \begin{align*}
   [A_{0}] & \sim \frac{1}{g} [\partial_{0} (\eta_{\nu}(x))] \sim \frac{1}{g} [\partial_{0} (\eta_{\nu}(x))] \sim [L]^{-1}, \\
   [A_{ij}] & \sim \frac{1}{g} [\partial_{\mu} \partial_{\nu} (\eta_{\nu}(x))] \sim [L]^{-2}, \\
   [E_{ij}] & \sim \left[ - \partial_{0} A_{ij} + \partial_{i} A_{0j} \right] \sim [L]^{-3}, \\
   [B_{ij}] & \sim \epsilon_{ijm} \partial_{m} A_{mj} \sim [L]^{-3}.
   \end{align*}
   \tag{3.7}
   $$

   So $[t] \sim [x] \sim [L]$, it is still a linear dispersion relation $\omega_E(k) \propto v_c k$ with $z = 1$.

See a summary in Table 4. In fact, for other tensor gauge models, there are other possibilities of gapless modes with different dispersion of $\omega_E(k) \propto k^z$ with $z = 1, 2, 3$, etc., e.g., see Ref. [29,33].
4 Discussions: Fracton, Embeddon and Foliation

4.1 Fracton

The name Fracton is introduced in [45, 46]. Famous examples include the Chamon quantum glass model [47, 48], the Haah’s cubic code [49] and others. The characterizations and physics definitions of fracton orders are that the excitations (including gapless or gapped excitations) have restricted mobility when acted by local operators [6]. The fractonic excitations have either of the following:

1. The excitations cannot move without creating additional excitations (other fractonic excitations).
2. The excitations can move only in sub-dimensions or certain directions. These excitations are also known as sub-dimensional particles.

It is easy to see that the theory in Eq. (2.20) and Eq. (2.56), indeed have the property (1). By looking at one term, for instance, we obtain

\[
\Phi(x, y)\left(\Phi(x + \Delta x, y) - \Phi(x, y + \Delta y)ight) - \Phi(x + \Delta x, y)\Phi(x, y + \Delta y) - \Phi(x, y)\Phi(x + \Delta x, y) + igA_{xy}(x, y)\Phi^2(x, y)
\]

in a discretized manner. To minimize the action by the variational principle, say this \(\Phi(x, y)\Phi(x + \Delta x, y)\Phi(x, y + \Delta y)\Phi(x, y)\) term, it means that the \(U(1)^{(d+1)}_x\) or \(U(1)^{(d)}_x\) vector global symmetry charge should be arranged in a higher-moment quadrupole manner. Indeed, the property that the dipole should not be created but the quadrupole can be created, is noticed and analyzed in various Pretko’s work [36].

We hope to explore (2) in a new setting in terms of the spacetime embedding in the next subsection.

4.2 Embedding to Embeddon

Let us construct another infinite series of new theories similar to Eq. (2.47) and Eq. (2.62), by the idea of embedding. We can devise new kinds of path integral as:

\[
\hat{\mathcal{L}} = \left(\prod_{I=1}^{N} [\mathcal{D}A_{I,\mu\nu}] [\mathcal{D}B_I] [\mathcal{D}C_I] \right) \exp\left( i \int_{M^{d+1}} \left( \sum_{I=1}^{N} |\hat{F}^{c,I}_{\mu\nu}\xi|^2 \right) + \int_{M_{\text{sub}}^{n-M}} \left( \frac{2}{2\pi} \sum_{I=1}^{N} B_I dC_I + \#C_I \land C_J \land \cdots \land dC_K \right) \right)
\]

Here we introduce a notation \(\hat{\mathcal{L}}\) means that there is no volume form of the ambient space. We introduce the lower dimensional\(n\) anti-symmetric TQFT on the submanifold \(M_{\text{sub}}^{n-M} \subset M^{d+1}\). For example, the submanifold \(M_{\text{sub}}^{n-M} \subset M^{d+1}\) is a darken slice in Fig. 5. We thus have a constraint \(N \leq n < d + 1\) for the twisted topological term \(\omega_n(C_I)\) that we can add from Table 1 and Table 2, here \(\omega_n \in H^n((\mathbb{Z}_2)^N, \mathbb{R}/\mathbb{Z})\) for a lower-dim twisted cohomology group.
Figure 5: Interpretations of the embeddon: (1) Quantization of spacetime embedding [50]. (2) The anyonic objects (e.g. particles/strings/branes) live in the embedded manifold \( M^{n}_{\text{sub-M}} \subset M^{d+1} \) in \( nd \), thus the anyonic objects are embedded inside the sub-dimensions.

Physically, what we are doing is to embed an anti-symmetric TQFT in a lower dimensional space, into a higher dimensional space where the symmetric tensor gauge theory resides. Mathematically, this can be done by gauging the sub-sector of the Eq. (2.45)’s \( G_g = [\mathbb{Z}_2^N] \). The anyonic objects (e.g. particles/strings/branes) described by \( \omega_n \) live in \( nd \), thus the anyonic objects are embedded inside the sub-dimensions. We shall name those embedded objects in the quantum theory as embeddon.

4.3 Foliation

We design new kinds of path integral as another infinite series of new theories similar to Eq. (2.47) and Eq. (2.62), similar to Eq. (4.3) but different in spirit:

\[
\int \left( \prod_{I=1}^{N} [DA_{I,\mu\nu}] [DB_I] [DC_I] \right) \exp \left( \int \sum_{I=1}^{N} |\hat{F}_{\mu\nu}^I|^2 \right) + \left( \frac{2}{2\pi} \sum_{I=1}^{N} B_I dC_I + \#C_I \wedge C_J \wedge \cdots \wedge dC_K \left( \wedge \cdots \wedge dx' \right) \right). \tag{4.4}
\]

Here \( \left( \wedge \cdots \wedge dx' \right) \) is the volume of the ambient space. We gain another theory by a sequence of embedding of submanifolds into a new picture of foliation. By this stacking of embedded TQFT picture along any codimension direction (in the dual \( d-n \)-dimensions in space), we can also easily reproduce the exponential growth of ground state degeneracy (GSD) associated to the \( d \)-dimensional space (counting zero energy modes) and thus exponential growth of the ground state subspace \( \text{dim}(\mathcal{H}_d) \).
For the physics picture, our theory thus potentially corresponds to the exponential growth of GSD counting respect to the system size in Ref. [51] and the foliation picture [52] on a wider class of manifolds. We may also construct the sub-dimensional sectors with BF theory or Chern-Simons theory in \( nd \) with \( n \leq d + 1 \), either via the embedding in Sec. 4.2 or the foliation in Sec. 4.3, potentially bridge to another recent work [53]. Our theory can also combine the foliation (in this subsection) and the anisotropic properties (Sec. 2.2), likely reproduce a field theory interpretation for a lattice construction similar to [54].

5 Conclusions: Quantization, Feynman diagram, Lattice Model, and Dark Matter

1. We have constructed a new family of hybrid classes between symmetric higher-rank tensor gauge theories and anti-symmetric tensor topological field theories (TQFTs) in any dimension. In condensed matter community, it is believed that the compact abelian symmetric rank-2 tensor field theory in Eq. (2.20) without matter field, in 3+1D (4d) or above dimensions, describes a gapless deconfined phase [29]. Thus our theory in 3+1D (4d) could describe a unitary mixture phase including the gapless sectors (which can live with or without Euclidean, Poincaré or anisotropic symmetry, from compact abelian symmetric tensor gauge theories, thus possibly can be relativistic or non-relativistic), and the gapped topological order phases (with anti-symmetric tensor TQFTs at low energy).

However, the mixture of such gapless phase and gapped phase is only the effective field theory (EFT) description, at least in the ultraviolet high energy (UV) or intermediate energy, but not yet to the lattice cutoff scale.

What are the infrared low energy (IR) fates and quantum dynamics of our theories? First of all, the continuous non-abelian gauge structure of our theories is \( \mathbb{Z}_2^C \ltimes (U(1)_{x_{(d+1)}}) \) in Eq. (2.30), a
higher-moment continuous $U(1)_{x(d+1)}$ twisted by the discrete gauged $\mathbb{Z}_2^C$. So one may expect that the gapless and deconfinement of compact abelian symmetric higher-rank tensor gauge theories should be maintained. We may speculate the gapped anti-symmetric tensor TQFT sector would not severely interfere the gapless sector, but only introduce additional massive degrees of freedom of anyonic particles (in 2+1D)/strings (in 3+1D)/branes (in 4+1D), etc. as massive high-energetic excitations above certainly TQFT energy gap $\Delta E$.

To confirm the speculation, we need to know the quantum dynamics of our theories. It may be challenging. To make an analogy, writing down a path integral form of Yang-Mills theory [5] is still quite far from understanding its quantum dynamics and confinement mechanism [44]. Let us make a few remarks before coming back to the quantization issue in the Remark 6.

2. Causes of non-abelian higher-moment global symmetry:
It is easy to see that the non-abelian global symmetry is due to the non-commutative nature of $(\mathbb{Z}_2^C \ltimes \left(U(1) \times U(1)_{x(d+1)}\right))$ displayed in Eq. (2.10) and Eq. (2.11). Our non-abelian-ness is however not due to a non-abelian Lie group and its non-commutative Lie algebra as the familiar case like Yang-Mills isopsin theory [5].

3. Causes of non-abelian gauge structure:
There are two ingredients in our construction to cause the non-abelian gauge structure.

- The first ingredient is due to the “continuous gauge group-analogous structure” is Eq. (2.30) $\left[\mathbb{Z}_2^C \ltimes \left(U(1)_{x(d+1)}\right)\right]$ once we gauge both $U(1)_{x(d+1)}$ and $\mathbb{Z}_2^C$. Even without any topological twisted term $\omega_{d+1}$, our theory is still a non-abelian tensor gauge theory (say in Eq. (2.47) and others).

- The second ingredient is due to the topological twisted term $\omega_{d+1}$ involving the Top Type explained in Sec. 3.1.1.

4. Mathematical ingredients for the field theories:
Let us discuss the mathematical interpretations of field contents in the abelian gauged fractonic matter theory Eq. (2.20) and the non-abelian gauge theory Eq. (2.47).

- The complex scalar $\Phi$ is a section (or a cross section) of field bundle which is a complex line bundle over $M^{d+1}$.

- The $A$ is a rank-2 symmetric tensor gauge field, see the explicit math formulation and generalization in [41,42,60,61]. (This is rather different, in contrast, from the ordinary 1-form gauge field which is a gauge connection of the principal U(1)-bundle. The ordinary 1-form gauge field can be also the gauge connection of the complex line bundle of $\Phi$.)

- The $B$ is an anti-symmetric tensor $(d-1)$-form gauge field. The $B$ can also be mapped to the $\mathbb{Z}_2$-valued $(d-1)$-cochain, in that case $B \in H^{d-1}(M^{d+1}, \mathbb{Z}_2)$, mapping from $M^{d+1}$ to the higher classifying space of $\mathbb{Z}_2$: $M^{d+1} \to B^{d-1}\mathbb{Z}_2$. The $B$ can also be regarded as the Lagrange multiplier to set $C$ to be locally flat.\(^{13}\)

\(^{13}\)In the homotopy theory, the classifying space of $BG_\alpha$ [24] of a group $G_\alpha$ is the quotient of a weakly contractible space $EG_\alpha$ (i.e., a topological space whose all homotopy groups are trivial) by a proper free action of $G_\alpha$. The classifying space $BG_\alpha = K(G_\alpha,1)$ is the Eilenberg-MacLane space. For an abelian $G_\alpha$, the higher classifying space $B^{d-1}G_\alpha$ is the Eilenberg-MacLane space $K(G_\alpha,d-1)$.
The $C$ is a 1-form gauge field. The $C$ can also be regarded as the $\mathbb{Z}_2$-valued 1-cochain, in that case $C \in H^1(M^{d+1}, \mathbb{Z}_2)$, mapping from $M^{d+1}$ to the classifying space of $\mathbb{Z}_2$: $M^{d+1} \to B\mathbb{Z}_2$. It becomes the $\mathbb{Z}_2$-valued 1-cocycle under the constraint of $B$’s EOM.

5. Fracton, Embeddon, and Foliation: We explain how our theories can incorporate the phenomenon of Fracton in Sec. 4.1, almost directly given from Ref. [36]. We introduce the spacetime embedding in terms of field theory setup and the Embeddon in Sec. 4.2, and how it bridges to build up the foliation in Sec. 4.3.

6. Feynman diagrams (Dyson graphs), path integral and (canonical) quantization:
A fully-fledged quantum field theory, even of Eq. (2.20) or (2.56), with matter and with or without gauge fields, by a field-theoretic quantization is still an open question. The usual canonical quantization by identifying the field variable $\Phi$ and the conjugate momentum variable may not be transparent for this quartic higher-order intrinsic interacting field theory. On the other hand, the path integrals we formulated are so far schematic. It is important to notice that the matter theory (Eq. (2.20) or (2.56)) has none quadratic (free) kinetic term, thus it is associated with the non-Gaussian integral, instead of the familiar Gaussian integral. It also suggests that the two-point correlation function (or Green’s function) as the free propagator may not exist, $\langle \Phi^\dagger(x_1)\Phi(x_2) \rangle \simeq 0$. Likely we should think about this quantum field theory starting from the three-point correlation function

\[ \langle \Phi^\dagger(x_1)\Phi^\dagger(x_2)\Phi(x_3) \rangle \]  \hspace{1cm} (5.1)

or the four-point correlation function of matter fields:

\[ \langle \Phi^\dagger(x_1)\Phi^\dagger(x_2)\Phi(x_3)\Phi(x_4) \rangle. \]  \hspace{1cm} (5.2)

However, we can at least attempt to identify the interaction terms and write down the Feynman diagrams in the perturbative sense. We do not include the Feynman rules, as it depends on whether we can use the propagators of free fields as the starting point to compute the scattering amplitudes. As we mentioned in Sec. 4.1 that the quadrupole (instead of dipole) of interacting matters at four different locations is the better configuration to think about the low energy physics of the theory — this observation indeed is consistent with Eq. (5.2). We can be afraid that the free field description of the theory (Eq. (2.20)) may not be ideal. In any case, we list the Feynman diagrams for $|(\Phi\partial_\mu\partial_\nu\Phi - \partial_\mu\Phi\partial_\nu\Phi - igA_{\mu\nu}\Phi^2)|^2$ in Figure 7.

On the other hand, there are free field descriptions for rank-2 symmetric tensor $A$ in Eq. (2.47). Thus, the propagator of $A$ read from $|\hat{F}_{\mu\nu\xi}^c|^2 = \left( (\partial_\mu - ig_\xi C_\mu)A_{\nu\xi} - (\partial_\nu - ig_\xi C_\nu)A_{\mu\xi} \right) \left( (\partial^\mu + ig_\xi C^\mu)A^{\nu\xi} - (\partial^\nu + ig_\xi C^\nu)A^{\mu\xi} \right)$, and its Feynman diagrams, may be a more tractable problem than that of Eq. (2.20). We list the Feynman diagrams for $|\hat{F}_{\mu\nu\xi}^c|^2$ in Figure 8.

In Figure 8 and in $|\hat{F}_{\mu\nu\xi}^c|^2$, it is important to notice that there is a kinetic term for $A$ in the Lagrangian $\partial A\partial A$ term. But there is no kinetic term for $C$. In fact, as we already knew the $B$ and $C$ are dual gauge fields of the topological BF theory sector for the TQFT. Thus we expect the physical observables of $C$ fields captured by the topological links of extend operators listed in Table 2 [18, 21, 23].

7. Universality of field theory: We have attempted to develop our theories based on field theories in any dimension. We hope that the universality of our field theories can capture the wider universality of a large class of lattice models. Although in the fracton literature [6, 7], many known exactly solvable lattice models are powerful, it is still important for us to re-collect the universal data out of arbitrary lattice models.

\[ ^{14}\text{Note that there is a very recent work by Seiberg [62] commenting about the quantization of the quartic matter field theory and its spectrum, however without the gauge fields.} \]
(i) The solid line, the curly spring, and the dashed line are for the propagators for the complex scalar $\Phi$, the real symmetric rank-2 tensor $A$, and the 1-form $C$ field respectively. We however should beware that the free propagator may not be an ideal concept for these highly interacting (non-Gaussian integral) field theories.

(ii) The $|\left(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi\right)|^2$ in Eq. (2.20) contains multiple terms involving the 4 points of $\Phi$ field interactions (subfigure (ii)), involving their “analogous momentums.”

(iii) The $ig(\Phi \partial_\mu \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \Phi)A_{\mu \nu} \Phi^2$ and its complex conjugation in Eq. (2.20) contains the 5 points of four $\Phi$ field and one $A_{\mu \nu}$ interactions (subfigure (iii)).

(iv) The $g^2 A_{\mu \nu}^2 \Phi^4$ in Eq. (2.20) contains the 6 points of four $\Phi$ field and two $A_{\mu \nu}$ interactions (subfigure (iv)).

Figure 8: Feynman diagrams relevant for tree-level interactions of $|\hat{F}^c_{\mu \nu \xi}|^2$. The conventions follow Fig. 7. The propagator of $A$ in Fig. 7 is from the $\partial A \partial A$ in Eq. (2.42) and Eq. (2.47).

(i) The interaction term from $i\partial ACA$ actually vanishes in Eq. (2.42), because $|\hat{F}^c_{\mu \nu \xi}|^2$ is necessarily real but $i\partial ACA$ involves an imaginary term. (ii) The interaction term from $CACA$ in Eq. (2.42).

8. **Lattice models**: An open question is that whether our field theories can be regularized on the lattice as a Hamiltonian quantum mechanical model? (This question addresses a different pursuit for quantization: Instead of attempting to define our field theory as a quantum field theory, we may define our theory as a unitary many-body quantum mechanic model.) Several previous works can help, for example,
• By generalizing the Kitaev quantum double model [63], exactly solvable quantum Hamiltonian models of twisted topological gauge theories and Dijkgraaf-Witten theories are known and constructed in 2+1D [64] and in 3+1D [13, 65].

• Exactly solvable quantum Hamiltonian models of twisted fracton models, analogous to twisted topological gauge theories [64, 65], are constructed in 3+1D [58, 66], or in any dimension [67].

Since our theory is a hybrid class between symmetric higher-rank tensor gauge theory (i.e., higher-spin gauge theory) and anti-symmetric tensor TQFT of Dijkgraaf-Witten/group-cohomology types [64, 65], combining the above two versions of Hamiltonian models potentially can be related to the quantum Hamiltonians for Eq. (2.47) and Eq. (2.62).

9. **Time crystal**: We mentioned in Sec. 2.1.1 and in Eq. (2.6) about a potential application of our theories to time crystal [39, 40]. Since Eq. (2.6) shows

\[ \Phi \rightarrow \Phi e^{i\Lambda_0 \cdot x_0} = \Phi e^{i\Lambda_0 \cdot t}, \]

while \( x_0 = t \) is time coordinate. The field configuration can respect a certain periodic time constraint \( t \approx t + \frac{2\pi}{\Lambda_0} Z \). Of course, what we said is only for the possible kinematics from the vector-like global symmetry (say at a UV effective field theory). In order to know whether there is any time crystal order formed in the ground state (namely in the quantum vacua at low energy), we need to know the quantum dynamics (say as the IR fate). This is again another hard question, related to the perturbative and quantization issue in Remark 6 and the further much challenging non-perturbative dynamics [44].

10. **Dark matter**: There are several possibilities for our theories to play a fundamental physics role for the dark matter:

(1). Eq. (2.20) or (2.56), with matter and with or without gauge fields

(2). Eq. (2.47) and Eq. (2.62) with gauge fields and TQFT sectors, and their higher-moment tensor generalization.

In cosmology, the present research suggests that the total mass-energy of the Universe contains:

• 5% of ordinary matter (e.g. baryonic and leptonic) and energy,

• 27% dark matter,

• 68% of dark energy as an unknown form of energy.

Thus, dark matter constitutes 85% of the total mass of the Universe, while dark energy plus dark matter constitutes 95% of the total mass-energy content of the Universe.

Hypothetically, we can interpret our model (1) with fractonic matter \( \Phi \) of complex scalar bosons as a source of dark matter decoupling from the Standard Model particle physics. We may also interpret our model (2)’s anyonic particles or anyonic extended objects (including anyonic strings and anyonic branes, etc.), that can be embedded or foliated in the spacetime. In either case, the candidate dark matters from our model have the following properties:

1. Non-baryonic and non-leptonic.

2. Non-supersymmetry (or no need to have supersymmetry).

3. Weakly interacting massive particles (WIMPs) or extended objects as hypothetical particles for dark matter. It can be massive if we interpret dark matters as anyonic particles. But our model also provides non-particles but extended objects as dark matter candidate: Anyonic extended objects (anyonic strings/branes, etc.) in Eq. (2.47) and Eq. (2.62), e.g. in Table 2. They can be highly energetic gapped excitations that we cannot probe easily at low energy, but they affect the topological degenerate zero energy subspace, see Table 3.
4. Weakly interacting (nearly) massless or gapless matters: Models of (1) (Eq. (2.20) or (2.56)) can have massless or gapless (or nearly depending on our input on the mass) matters, decoupled (or nearly decoupled) from Standard Model sectors.

5. Cold dark matter: Our dark matter candidates are highly quantum and highly entangled states of matter (both for particle-like or extended string/brane-like objects). They exist and are robust at the absolute zero temperature without thermal effect. Although our candidates can still play a role in the warm and hot environment in the Universe.

11. Dark energy: Dark energy is an unknown energy form, hypothetically permeating all of space and time. It tends to accelerate the expansion of the Universe. Dark energy is the accepted hypothesis to explain that the Universe is expanding at an accelerating rate especially after the 1990s observations. We do not yet have a concrete model for producing such energy to expand the space from our model. However, we should note that there is an enormous amount of static and dynamical energy from the gauge structure of our models (in both Models (1) and (2)). For a simpler compact abelian tensor gauge model, the huge amount of static energy is emphasized already in [32]. Related issues on how matters interact and their relations through gravity via a Mach’s principle is revisited in [68] for the fracton context. We expect that more immense energy stored in our non-abelian models with both gapless and gapped TQFTs sectors.

Note added: The sequel of this work as companions include Ref. [41] and Ref. [42], etc. The idea of generating a larger family of theories by coupling to dynamical TQFTs is pursued and developed since 2016 in [17, 18, 21]. The theory presented here is found in July 2019. After the completion of this work, Ref. [69] appears on arXiv, suggesting a fractonic qubit lattice model with emergent matter fields and mimetic gravity. They compare such fracton models to Mimetic Dark Matter and Mimetic Tensor-Vector-Scalar Models in the high-energy physics literature.

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