Chiral Homology of Elliptic Curves and Zhu’s Algebra

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Abstract. We study the chiral homology of elliptic curves with coefficients in a quasiconformal vertex algebra $V$. Our main result expresses the nodal curve limit of the first chiral homology group in terms of the Hochschild homology of the Zhu algebra of $V$. A technical result of independent interest regarding the equivalence between the associated graded with respect to Li’s filtration and the arc space of the $C_2$-algebra is proved.

1. Introduction

1.1. Spaces of conformal blocks play a central role in the mathematical approach to conformal field theory based on vertex algebras, and are the point of contact in a fruitful interaction between representation theory and the geometry of moduli spaces.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, $\hat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus CK$ the associated affine Kac-Moody algebra, $V$ a $\hat{\mathfrak{g}}$-module, and $X$ a smooth complex algebraic curve. We denote by $\hat{\mathfrak{g}}_{\text{out}}$ the Lie algebra of meromorphic $\mathfrak{g}$-valued functions on $X$ with possible pole at a point $x \in X$. If we choose a coordinate $t$ at $x$ then we may use it to expand in Laurent series and obtain a morphism $\hat{\mathfrak{g}}_{\text{out}} \to \hat{\mathfrak{g}}$. The space of conformal blocks is then (the dual of) the vector space of coinvariants

$$H(X, x, \mathfrak{g}, V) = \frac{V}{\hat{\mathfrak{g}}_{\text{out}} \cdot V},$$

which is well defined independently of the choices made. Spaces of conformal blocks appear in connection with moduli spaces of $G$-bundles over $X$ in the guise of nonabelian theta functions.

The construction (1.1) blends the notions of Lie algebra homology and de Rham cohomology of an algebraic variety. This perspective finds natural expression in Beilinson-Drinfeld’s definition of chiral homology. A vertex algebra, and more generally a chiral algebra, may be interpreted as a Lie algebra within a certain category of $D$-modules over the Ran space of an algebraic curve. The chiral homology is then defined as the de Rham cohomology of the Chevalley-Eilenberg complex of this Lie algebra. The space of conformal blocks is recovered as the zeroth chiral homology.

Higher chiral homology groups are of interest in the geometric Langlands program, specifically in connection with the construction of Hecke eigensheaves. Indeed the chiral Hecke algebra $A_k(\mathfrak{g})$ is defined as a certain vertex algebra extension of the simple affine vertex algebra $V_k(\mathfrak{g})$ at level $k \in \mathbb{Z}_+$, and Hecke eigensheaves with eigenvalue specified by a local system $E$ are realised in the higher chiral homology groups of a twist of $A_k(\mathfrak{g})$ by $E$. In this context Beilinson and Drinfeld ask [7, 4.9.10] whether the higher chiral homology groups of $V_k(\mathfrak{g})$ vanish.

Following the seminal work of Zhu [40] spaces of conformal blocks of elliptic curves, and especially their behaviour over families degenerating to a nodal curve, have come to play an important role in the representation theory of vertex algebras. Let $X_q$ denote the elliptic curve

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\(C/\mathbb{Z} + \mathbb{Z}\tau\) where \(q = e^{2\pi i \tau}\), let \(V\) be a quasiconformal vertex algebra and let \(A_V\) denote the associated chiral algebra on \(X_q\). The space of conformal blocks is given by

\[
H_{ch}^0(X_q, A_V) \cong \frac{V}{V_0 V + V_0 V},
\]

where \(\wp = \wp(z, q)\) is the Weierstrass elliptic function. Zhu observed that the specialisation of (1.2) at \(q = 0\) recovers the zeroth Hochschild homology of a certain associative algebra \(\text{Zhu}(V)\) whose representation theory is strongly connected with that of \(V\). The modular nature of graded dimensions of regular vertex algebras is ultimately explained by these facts.

The main results of this paper concern the relationship between chiral homology of elliptic curves and Hochschild homology of the Zhu algebra in higher degrees. We specialise Beilinson-Drinfeld’s general definition of chiral homology to develop explicit complexes computing the chiral homology groups of the elliptic curve \(X_q\) with coefficients in \(A_V\). We then focus attention on \(H_{ch}^1(X_q, A_V)\) for \(i = 0, 1\), deriving a small three term complex \(A^1(q)\) which computes these groups. We recover the presentation (1.2) of \(H_{ch}^0\) and similarly present \(H_{ch}^1\) as an explicit subquotient of a sum of tensor powers of \(V\) involving elliptic functions.

We then use \(A^1(q)\) to analyse the behaviour of \(H_{ch}^1\) in the \(q \to 0\) limit. What occurs is roughly speaking the following: a large subcomplex \(B^1\) decouples from \(A^1(q)\) in the \(q \to 0\) limit, and the quotient \(A^1(0)/B^1\) computes the Hochschild homology of \(\text{Zhu}(V)\). This remarkable decoupling is ultimately due to the elliptic function identity

\[
8\pi^2 q \frac{d\wp}{dq} = 2\wp + \wp',
\]

which is a consequence of the heat equation for theta functions.

Before stating the main result we recall the notions of singular support and associated scheme of a vertex algebra. Li introduced a filtration on a vertex algebra \(V\) whose associated graded \(A\) is naturally a \(\mathbb{Z}_+\)-graded commutative algebra (indeed a Poisson vertex algebra). The spectrum of \(A\) is known as the singular support of \(V\) and is denoted \(\text{SS}(V)\). The algebra \(A\) is generated by its component of degree 0, which is just Zhu’s \(C^2\)-algebra, and whose spectrum is known as the associated scheme \(X_V\) of \(V\). Thus there is a natural embedding \(\text{SS}(V) \hookrightarrow JX_V\), where in general \(JY\) denotes the arc space of the scheme \(Y\).

1.2. Theorem. Let \(V\) be a quasiconformal vertex algebra. If the natural embedding of the singular support of \(V\) into the arc space of its associated scheme is an isomorphism, i.e.,

\[
\text{SS}(V) \cong JX_V,
\]

then

\[
\lim_{q \to 0} H_{ch}^1(X_q, A_V) \cong \text{HH}_1(\text{Zhu}(V)).
\]

The meaning of the left hand side of (1.5) is to be understood in terms of the specialisation to \(q = 0\) of the complex \(A^1(q)\). See Section 1.3 for precise statements. It remains to discuss for which vertex algebras \(V\) the condition (1.4) holds.

For chiral envelopes the condition (1.4) is valid. This class includes the affine vertex algebras \(V^k(g)\), the Heisenberg vertex algebra and the universal Virasoro vertex algebras Vir. This condition is also satisfied by the universal affine \(W\)-algebras \(W^k(g, f)\).

For the class of Virasoro minimal models Vir_{p,p'}, which are rational vertex algebras, we show that (1.4) holds in some cases, namely \((p, p') = (2, 2k + 1)\), but demonstrably fails to hold in other cases. Indeed for \(V = \text{Vir}_{2, 2k+1}\) the graded dimensions of both sides of (1.4) coincide with
the function
\[
\prod_{m \geq 1, m \not\equiv 0, \pm 1 \mod (2k+1)} \frac{1}{1 - q^m}
\]
which famously appears in Gordon’s generalisation of the Rogers-Ramanujan identity. In Section 16 we prove the following

**Theorem (16.13).** Let \( V = \text{Vir}_{p,p'} \) be the Virasoro minimal model with central charge
\[
c = c_{p,p'} = 1 - 6 \frac{(p - p')^2}{pp'}.
\]
If \( (p, p') = (2, 2k+1) \) where \( k \geq 1 \), then the natural embedding \( SS(V) \hookrightarrow JX_V \) is an isomorphism of schemes. In particular
\[
\lim_{q \to 0} H^1_{ch}(X_q, \mathcal{A}_V) = 0.
\]
If \( p, p' \geq 3 \) then the embedding is not an isomorphism. In all cases, however, the reduced schemes of \( SS(V) \) and \( JX_V \) are isomorphic, both consisting of a single closed point.

Regarding simple affine vertex algebras we prove the following result.

**Theorem (16.21).** Let \( V \) be the simple affine vertex algebra \( V_k(\mathfrak{sl}_2) \) at positive integral level \( k \in \mathbb{Z}^+ \). The natural embedding \( SS(V) \hookrightarrow JX_V \) is an isomorphism of schemes. In particular
\[
\lim_{q \to 0} H^1_{ch}(X_q, \mathcal{A}_V) = 0.
\]

Our proof of this theorem uses a result of Meurman-Primc \[36] on PBW-type bases of integrable \( \hat{\mathfrak{sl}}_2 \)-modules.

In general the question of for which vertex algebras \( V \) the condition \([1.4]\) holds appears to be a subtle and interesting one. We find it particularly interesting that Rogers-Ramanujan type \( q \)-series identities appear in the comparison of the graded dimension of \( V \) with that of \( JRX_V \). As a tentative first step towards a better understanding of the relationship between \( \text{gr}^F(V) \) and \( JRX_V \) in general we construct a Gröbner basis of \( JRX_V \) for \( V = V_1(\mathfrak{sl}_2) \) in Section 17.

1.3. We briefly explain why condition \([1.4]\) appears in Theorem \([1.2]\) the key point is the following vanishing condition on Koszul homology of arc spaces. Let \( A \) be a commutative \( k \)-algebra and let \( \tau : \Omega^1_{A/k} \to A \) be a derivation. We have the associated Koszul complex \( K^\bullet = K_A^\bullet \) defined by \( K_A^n := \text{Sym} \Omega^n_{A/k}[1] \). Now let \( A^0 \) be a commutative algebra of finite type, \( X = \text{Spec} A^0 \) and let \( JA^0 \) be the coordinate ring of the arc space \( JX \), that is Spec \( JA^0 = JX \). The algebra \( JA^0 \) comes equipped with a canonical derivation and it turns out that the corresponding complex \( K_JA^0 \) is acyclic away from degree 0. Moreover in Section \([14]\) we prove the following

**Theorem (14.12).** Let \( A = \bigoplus_{n \in \mathbb{Z}^+} A^n \) be a \( \mathbb{Z}^+ \)-graded commutative algebra with a derivation \( \tau \) of degree +1, and let \( (K_A^\bullet, \tau) \) be the Koszul complex associated with \( A \) as above. We assume \( A \) is generated by \( A^0 \) as a differential algebra, and that \( A^0 \) is an algebra of finite type. Then \( H_{-1}(K_A^\bullet, \tau) = 0 \) if and only if \( A \cong JA^0 \).

The subcomplex \( B^\bullet \subset A^\bullet(q = 0) \) acquires a filtration induced by the Li filtration on \( V \). We identify the associated graded complex \( \text{gr}^F(B^\bullet) \) with the Koszul complex \( (K^\bullet_{SS(V)}, \tau_T) \) associated as above with the singular support of \( V \). Thus condition \([1.4]\) arises naturally as a sufficient condition for vanishing of \( H_{-1}(B^\bullet) \).

1.4. Although self contained and relatively short, this article uses a variety of somewhat involved tools from different fields. Here is a brief summary.
a We use the theory of chiral algebras and chiral homology as developed in [7]. We obtain the complex $A^\bullet(q)$ presented in Section 11, which computes chiral homology of the elliptic curve $X_q$ in low degrees, by a straightforward process of translation to linear algebra of Beilinson-Drinfeld’s complex for chiral homology with supports. The main ingredients here are the use of the marked point (the polarization) of the elliptic curve and a well defined étale coordinate. The support is taken to be the marked point, with the vacuum module insertion.

b We use Totaro’s theorem on cohomology rings of configuration spaces of a manifold $X$. This is explained in Section 10. We also use explicit representatives of cohomology classes in the case of $X = X_q$, an elliptic curve. These are described in terms of elliptic functions.

c We use elliptic function identities, such as (1.3), to study the behaviour of $A^\bullet(q)$ in the nodal curve limit $q \to 0$, specifically to find the subcomplex $B^\bullet \subset A^\bullet(q = 0)$.

d We use Li’s filtration to endow $B^\bullet$ with a filtration and we consider the corresponding spectral sequence. The associated graded of $B^\bullet$ is identified with the Koszul complex of the singular support of $V$ as explained above.

e We use a description of the multiplication in the Zhu algebra $\text{Zhu}(V)$ as a nodal curve limit $q \to 0$ of the operation $a(\zeta) b := \text{res}_z \zeta(z, q) a(z) b$. In this way we identify the quotient $A^\bullet(q = 0)/B^\bullet$ with the Bar complex computing the Hochschild homology of $\text{Zhu}(V)$.

f We use a result of Bruschek, Mourtada and Schepers on the Hilbert series of certain arc spaces, and their relation with Rogers–Ramanujan identities, to identify which minimal models $\text{Vir}_{p,p'}$ satisfy condition (1.4) and which ones do not.

g We use Gröbner basis techniques, and a result of Meurman and Primc, to compute the Hilbert series of the arc space of the associated variety of the simple affine vertex algebra $V_k(sl_2)$.

1.5. Although the ultimate objective of the techniques proposed in this article is to answer and generalise Beilinson and Drinfeld’s question [7, 4.9.10], the article is written in such a way that readers better acquainted with the theory of vertex algebras than that of chiral algebras may understand the main statements and proofs. The complex constructed in Section 8 does not use the chiral algebra formalism and is canonically associated to a vertex algebra and an elliptic curve. Of course in order to check that this complex indeed computes chiral homology, one needs to compare with Beilinson and Drinfeld’s construction. The reader acquainted with chapter 4 of [7] or chapter 20 of [21] will find this comparison self-evident.

Although we use Weierstrass’s $\wp$-function and its integral $\zeta$, their explicit Fourier expansions are not technically needed. We use the differential equation (1.3) and some explicit algebraic equations satisfied by these functions in Section 8 as well as their properties as $q \to 0$. It would be interesting to have an explicit purely algebraic formulation of our results here in terms of sections of $O_X(2)$ instead of explicit elliptic functions.

1.6. Here are some subjects that are not treated on this article and that we plan to address in the following articles of this series.

a) Chiral homology of the general elliptic curve $X_q$, not only in the limit $q \to 0$. Zhu’s technique for handling $\mathcal{H}_{0,\text{ch}}^\bullet$ is based on the construction of explicit representatives of homology classes, as traces on $V$-modules. These representatives are shown to be flat sections of the connection on the space of conformal blocks, i.e., Zhu shows that the trace functions satisfy certain explicit differential equations. Beilinson and Drinfeld endow their chiral complexes with Gauss-Manin connections along the moduli space of curves, hence we have at our disposal differential equations generalising those considered by
Zhu. What remains is to develop higher analogues of \textit{traces of modules}. We expect that extensions of $V$-modules provide examples of non trivial classes in chiral homology.

b) Chiral homology in degree 2 and higher. To compute higher chiral homologies we use an explicit description of the de Rham cohomology classes of configuration spaces of an elliptic curve. It is surprising that until quite recently even the Betti numbers where not available (see for example [18, 35, 39]). It is relatively easy to find a quotient complex of the chiral chain complex isomorphic to the Bar complex of Zhu($V$). However proving acyclicity of the kernel becomes more difficult due to the more involved combinatorics of spaces of configurations of more than 3 points.

c) Chiral homology of higher genus curves. The sheaves of conformal blocks (at least in the rational case) can be extended to the boundary of the moduli spaces [42]. We are not aware of a similar result for chiral homology. With the tools we develop in hand we expect to be able to reduce the computation of chiral homologies of vertex algebras to the case of elliptic curves.

1.7. \textbf{Remark.} The original question of Beilinson and Drinfeld for affine Kac-Moody vertex algebras at integral level has been answered by Dennis Gaitsgory [22] using a theorem of Teleman about the geometry of the affine Grassmannian. In the case of the universal affine Kac-Moody algebra, Sam Raskin is able to show that (the $q = 0$ limit of) chiral homology coincides with Hochschild homology by studying directly the Beilinson-Drinfeld Grassmannian over the nodal curve [37].

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2. \textbf{Vertex Algebras}

2.1. For background we refer to the text [28]. The formal delta function is defined to be $\delta(z, w) = \sum_{n \in \mathbb{Z}} z^{-n-1}w^n$, and the formal residue $\text{res}_z f(z) = f(z) = \sum_{n \in \mathbb{Z}} f_n z^n$ is defined by $\text{res}_z f(z) = f_{-1}$.

2.2. A vertex algebra is a vector space $V$ equipped with a vacuum vector $1$ and a collection of bilinear products indexed by integers. The $n^{\text{th}}$ such product of $a, b \in V$ is denoted $a(n)b$. These products are to satisfy the quantum field property

$$a(n)b = 0 \text{ for } n \gg 0,$$

the Borcherds identity

$$\sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a(n+j)b)(m+n-j)c = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} [a(m+n-j)b(k+j)c - (-1)^{n-k}b(n+k-j)\delta(n+j)c],$$

and the unit identity

$$1(a) = \delta_{n,-1}a \text{ for } n \in \mathbb{Z} \text{ and } a(0)1 = \delta_{n,-1}a \text{ for } n \in \mathbb{Z}_{\geq -1}.$$
The translation operator $T \in \text{End}(V)$ is defined by $T a = a(-2)1$.

In terms of quantum fields the Borcherds becomes the Jacobi identity

$$[a(x)b(w)c = \text{res}_z (a(z)b(w)c i_{z,w} - b(w)a(z)c i_{w,z} ) \delta(x, z - w).$$

(2.3)

The following are some useful consequences of the definitions. The translation invariance condition

$$[Ta](z) = \partial_z a(z), \quad \text{equivalently} \quad [Ta](n) = -na(n - 1).$$

(2.4)

The skew-symmetry formula

$$b(z)a = e^{i\partial_z} a(-z)b, \quad \text{equivalently} \quad b(n)a = - \sum_{j \in \mathbb{Z}_+} (-1)^{n+j} T^{(j)}(a(n+j)b).$$

(2.5)

The commutator formula

$$[a(z), b(w)] = \sum_{j \in \mathbb{Z}_+} [a(j)b(w)\partial_w^j \delta(z, w), \quad \text{equivalently} \quad [a(m), b(n)] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} [a(j)b(m+n-j).$$

(2.6)

2.3. For later convenience we recall the $f$-product notation

$$a_{(f)}b = \text{res}_z f(z)a(z)b.$$

When necessary we abuse this notation, writing for example $a_{(x^f(x))}b$ rather than introducing $g(x) = xf(x)$ and writing $a_{(g)}b$. As a consequence of the skew-symmetry identity we have the following lemma.

2.4. Lemma. Let $f \in k((x))$. Then

$$a_{(f(x))}b + b_{(f(-x))}a = \sum_{j \in \mathbb{Z}_+} \frac{(-1)^j}{(j + 1)!} T^{j+1}(a(x^{j+1}f(x))b), \quad \text{for all } a, b \in V.$$

(2.7)

2.5. Motivated by this identity we introduce the symbol

$$\int \{a_{(f)}b\} = \sum_{j \in \mathbb{Z}_+} \frac{(-1)^j}{(j + 1)!} T^j(a(x^{j+1}f(x))b).$$

(2.8)

We remark that the $f(x) = x^a$ case of this product already appeared in Borcherds’ paper [8] where it was denoted $a \times_n b$.

2.6. Let $V$ be a vertex algebra. The quotient $V/TV$ is well known to carry the structure of a Lie algebra with bracket given by $[a, b] = a(0)b$. The quotient $R_V = V/V(-2)V$ is known as Zhu’s $C_2$-algebra. It is naturally a Poisson algebra with the commutative product given by $a \cdot b = a(-1)b$ and the Poisson bracket by $\{a, b\} = a(0)b$.

2.7. Following [21] we denote $O = \mathbb{C}[[t]]$ and $K = \mathbb{C}((t))$ topological algebras, and we consider the Lie algebras $\text{Der}_0 \mathcal{O} \subset \text{Der} \mathcal{O} \subset \text{Der} \mathcal{K}$, defined by $\text{Der}_0 \mathcal{O} = \bigoplus_{n \geq 0} \mathcal{C} L_n$, $\text{Der} \mathcal{O} = \bigoplus_{n \geq -1} \mathcal{C} L_n$ and $\text{Der} \mathcal{K} = \bigoplus_{n \in \mathbb{Z}} \mathcal{C} L_n$ where $L_n = -t^{n+1} \partial_t$. The group $\text{Aut} \mathcal{O}$ of continuous automorphisms of $\mathcal{O}$ has Lie algebra $\text{Der}_0 \mathcal{O}$.

The Virasoro algebra $\text{Vir} = \text{Der} \mathcal{K} + \mathbb{C} C$ is the universal central extension of $\text{Der} \mathcal{K}$. Explicitly

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C.$$

A conformal vector of a vertex algebra $V$ is a vector $\omega \in V$ whose associated quantum field $L(z) = \omega(z) = \sum L_n z^{-n-2}$ furnishes $V$ with a representation of $\text{Vir}$ such that

- $C$ acts by a constant, called the central charge of $V$. 


• $L_0$ acts semisimply on $V$ with non negative integral eigenvalues and finite dimensional eigenspaces,
• $L_{-1}$ coincides with $T$.
A quasiconformal vertex algebra is a vertex algebra furnished with a representation of $\text{Der} \mathcal{O}$ such that
• For all $b \in V$ one has
\[ [L_m, b(n)] = \sum_{j \geq 0} \binom{m+1}{j} [L_{j-1} b](m+n+1-j) \]
(cf. equation (2.4)),
• $L_0$ acts semisimply on $V$ with integral eigenvalues, and $\text{Der}_{>0} \mathcal{O}$ acts locally nilpotently on $V$,
• $L_{-1}$ coincides with $T$.
Let $V$ be a (quasi)conformal vertex algebra and $a \in V$ an eigenvector of $L_0$. The eigenvalue of $a$ is known as its conformal weight and is denoted $\Delta(a)$.

2.8. We recall the definitions of the enveloping algebra $U(V)$ and the Zhu algebra $A(V)$ of a quasiconformal vertex algebra $V$. Firstly one defines the Lie algebra
\[ \text{Lie}(V) = V[t, t^{-1}]/(T + \partial_V)(t, t^{-1}) \]
with the Lie bracket $[at^m, bt^n] = \sum_{j \in \mathbb{Z}} \binom{m}{j} (a(j)b) t^{m+n-j}$. Next one equips $\text{Lie}(V)$ with a $\mathbb{Z}$-grading by putting $\text{deg}(at^m) = m + 1 - \Delta(a)$ and extends the grading to the universal enveloping algebra $U(\text{Lie}(V))$. Using the grading it is possible to form a degreewise completion of $U(\text{Lie}(V))$ in which the equality (2.1) of infinite sums makes sense. Finally one obtains the topological algebra $U(V)$ as the quotient of this completion by the relations (2.1) and (2.2). The category of $V$-modules is naturally equivalent to the category of smooth $U(V)$-modules.

Now let $M$ be a $V$-module. Then the space of invariants $M^{U(V)>0}$ is immediately seen to carry an action of the algebra
\[ A(V) = U(V)_0/(U(V) \cdot U(V)_{>0})_0. \]
Zhu proved that the functor $M \mapsto M^{U(V)>a}$ from the category of $V$-modules to the category of $A(V)$-modules induces a bijection between the sets of isomorphism classes of irreducible $A(V)$-modules and irreducible positive energy $V$-modules.

In fact Zhu introduced $A(V)$ in terms of the following very different presentation. For any $a, b \in V$ put
\[ a \circ b = \text{res}_w w^{-2}(1+w)^{\Delta(a)} a(w)b dw \quad \text{and} \quad a * b = \text{res}_w w^{-1}(1+w)^{\Delta(a)} a(w)b dw. \]
Then $V \circ V \subset V$ turns out to be an ideal with respect to the operation $*$, which in turn descends to an associative product on the quotient $V/V \circ V$. Furthermore $A(V) \cong V/V \circ V$ as associative algebras.

2.9. Geometrically $\mathcal{O}$ represents the algebra of functions defined on an infinitesimal disc $D = \text{Spec} \mathcal{O}$. A generator of the unique maximal ideal $t \mathcal{O}[t] \subset \mathcal{O}$ represents a coordinate on $D$ vanishing at its unique closed point. Let us denote by $\text{Coord}$ the set of coordinates on $D$; this set is naturally an $\text{Aut} \mathcal{O}$-torsor and may be identified with $\mathbb{C}^* t + t^2 \mathbb{C}[t]$. Thus for any $f(t) \in \mathbb{C}^* t + t^2 \mathbb{C}[t]$ there exists a unique element $g \in \text{Aut} \mathcal{O}$ such that $g \cdot t = f(t)$. For such $f$ we furthermore denote by $f_z$ the series defined by $f_z(t) = f(z + t) - f(z)$.

Now let $V$ be a quasiconformal vertex algebra. The restriction of the $\text{Der} \mathcal{O}$-action on $V$ to $\text{Der}_0 \mathcal{O}$ can be exponentiated to define an action of $\text{Aut} \mathcal{O}$. Given $f(t) \in \mathbb{C}^* t + t^2 \mathbb{C}[t]$ and $g \in \text{Aut} \mathcal{O}$ as above, we denote by $R(f)$ the image of $g$ under the map $\text{Aut} \mathcal{O} \to \text{End}(V)$. The
2.10. Let $V$ be a quasiconformal vertex algebra and let $\phi(z)$ denote the formal power series $e^{2\pi iz} = 1$. In [46] Zhu introduced the modified vertex operation $Y[-, z]$ on $V$ defined by

$$Y[a, z] = Y(e^{2\pi izL_0}a, \phi(z)).$$

Let us write $R = R(\phi)$. Then since $\phi_z(t) = e^{2\pi iz}\phi(t)$ we have $R(\phi_z) = R(\phi)e^{-2\pi izL_0}$, and (2.9) yields

$$Y[a, z] = R^{-1}Y(Ra, z)R.$$

So in fact $R : (V, Y[-, z]) \to (V, Y(-, z))$ is a vertex algebra isomorphism. This was used by Huang to uncover the following presentation of $A(V)$ [27]. Put

$$f(z) = 2\pi i \cdot \frac{e^{2\pi iz} - 1}{e^{2\pi iz} - 1} = z^{-1} + \pi i - \frac{\pi^2}{3}z - \frac{\pi^4}{45}z^3 + \cdots,$$

$$g(z) = (2\pi i)^2 \cdot \frac{e^{2\pi iz}}{(e^{2\pi iz} - 1)^2} = z^{-2} + \frac{\pi^2}{3}z + \frac{\pi^4}{15}z^2 + \cdots,$$

and

$$\text{Zhu}(V) = V/V(g)V \text{ with product } a \cdot b = a(f)b.$$
The right action is tautological and the left action of $D_X$ is given in local coordinates by the formula
\[ \tau(g \otimes s) = \tau(g) \otimes s + g \sum_{i=1}^{\dim(Y)} \tau(y_i \circ f) \otimes \frac{\partial s}{\partial y_i}. \]

The pullback of the left $D_Y$-module $M$ is
\[ f^*(M) = D_f \otimes_{f^*D_Y} f^*(M), \]
and the functor $f^*$ is right exact. The pushforward of the right $D_X$-module $M$ is
\[ f_*(M) = f_*(M \otimes_{D_X} D_f). \]

Since it involves the composition of a right exact functor with a left exact one, the functor $f_*$ is neither right nor left exact in general and the definition of its derived functors in general requires derived categories.

The pushforward along the closed embedding of a smooth subvariety (which is the only image functor we require in fact) has the following local description. Let $i : X \to Y$ be a closed embedding whose image is defined locally by the equations $y_i = 0$ for $i = k + 1, \ldots, \dim(Y)$ and let $M$ be a right $D_X$-module. Then
\[ i_*(M) = i_*(M) \otimes_{\mathbb{C}[\partial_{y_1}, \ldots, \partial_{y_{\dim(Y)}}, \partial_\mathbb{C}]} \mathbb{C}[\partial_1, \partial_2] \]
with the action of $\partial_y$ given by $\partial_x \otimes 1$ for $i = 1, \ldots, k$, by $1 \otimes \partial_y$ for $i = k + 1, \ldots, \dim(Y)$, and the action of functions by restriction to $i(X)$. The most important example of all is the diagonal embedding $\Delta : X \to X^2$. Then
\[ \Delta_* (M) = \Delta_* (M) \otimes_{\mathbb{C}[\partial_1, \partial_2]} \mathbb{C}[\partial_1, \partial_2] \]
where the action of $\mathbb{C}[\partial]$ on $\mathbb{C}[\partial_1, \partial_2]$ is by $\partial = \partial_1 + \partial_2$.

The other operation we require is the $\ast X$ construction \[23\] pp. 97. For $X$ a divisor in $Y$ and $\mathcal{F}$ an $O_Y$-module, the $O_Y$-module $\mathcal{F}(\ast X)$ is defined to be
\[ \mathcal{F}(\ast X) = \lim_{\to} \text{Hom}_{O_Y}(I^n, \mathcal{F}), \]
where $I$ is an ideal of definition of $X$. This construction is independent of the choices made, and works in both the algebraic and analytic categories. In the algebraic category we have $M(\ast X) \cong j_* j^* M$ and so, following \[7\], we use the latter notation. In Section 3 we pass from the algebraic to analytic context, and it pays to note that the image under the analytification functor of $j_* j^* \mathcal{F}$ is $\mathcal{F}(\ast X)$ and not $(j^m)_*(j^{m*})^* \mathcal{F}$, which is very much larger.

3.2. Let $X$ be a smooth complex algebraic curve. Let $\{L_i\}_{i \in I}$ be a collection of right $D_X$-modules parametrised by the finite set $I$, and $M$ another right $D_X$-module. A chiral $I$-operation \[7\] 3.1] from $\{L_i\}_{i \in I}$ to $M$ is by definition a morphism of right $D_X$-modules
\[ j_*(L_i) \to \Delta_* M, \]
where $j = j^{(I)}$ denotes the open embedding into $X^I$ of
\[ U^{(I)} = \{(x_i)_{i \in I} \in X^I | x_i \neq x_i \text{ for } i_1 \neq i_2\} \]
and $\Delta = \Delta^{(I)}$ denotes closed embedding into $X^I$ of the diagonal
\[ \Delta^{(I)} = \{(x_i = x)_{i \in I} | x \in X\}. \]

The vector space of chiral $I$-operations \[88\] is denoted $P^\text{ch}(\{L_i\}_{i \in I}, M)$. 
3.4 expand upon this in Section 6.4 below, after some necessary preliminaries.

Then we write

\[ U^{[J/I]} = \{(x_j)_{j \in J} \in X^J | x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\} \]

and \( j^{[J/I]}: U^{[J/I]} \to X^J \) for the natural inclusion. We also write \( \Delta(\pi) = \Delta^{(J/I)} \) for the natural inclusion \( X^I \to X^J \) with image

\[ \Delta^{(J/I)} = \{(x_j)_{j \in J} \in X^J | x_{j_1} = x_{j_2} \text{ if } \pi(j_1) = \pi(j_2)\}. \]

Then \( U^{[J/I]} \) is the complement of the union of all diagonals in \( X^J \) that are transversal to \( \Delta^{(J/I)} \).

The Grothendieck residue morphism is defined, for any right \( D_X \)-module \( M \), to be the natural surjection

\[ \text{res} : j_! j^*(\omega_X \boxtimes M) \to \frac{j_! j^*(\omega_X \boxtimes M)}{\omega_X \boxtimes M} \cong \Delta_*(M). \]

This is the archetypal example of a chiral operation, i.e., \( \text{res} \in P^{\text{ch}}(\{\omega_X, M\}, M) \).

3.3. Chiral operations may be composed and in this way the category of right \( D_X \)-modules becomes what is known as a pseudo-tensor category \[7, 1.1\]. In such a category, which generalises the notion of symmetric monoidal category, it is possible to define the notion of algebra over an operad. Thus a (non unital) chiral algebra over \( X \) is formally defined to be an algebra over the Lie operad in the pseudo-tensor category of right \( D_X \)-modules \[7, Section 3.3\]. In more concrete terms a chiral algebra over \( X \) is a right \( D_X \)-module \( A \) together with a chiral operation \( \mu \in P^{\text{ch}}(\{A, A\}, A) \) satisfying analogues of the usual skew-symmetry and Jacobi identities. We expand upon this in Section 3.4 below, after some necessary preliminaries.

3.4. Let \( X \) be a smooth complex algebraic curve and \( V \) a quasiconformal vertex algebra. We recall the construction from \( V \) of a chiral algebra over \( X \) \[21\] (see also \[45\]). Let \( x \) be a smooth \( \mathbb{C} \)-point of \( X \), and let \( O_x \cong O \) denote its local ring. The set of pairs \((x, t_x)\) consisting of a point \( x \in X \) and a coordinate \( t_x \) at \( x \) (in the sense of Section 2.9) is the set of \( \mathbb{C} \)-points of a scheme \( \text{Coord}_X \), which is furthermore an \( \text{Aut} O \)-torsor over \( X \). Applying the associated bundle construction to the \( \text{Aut} O \)-module \( V \) yields vector bundle

\[ V = \text{Coord}_X \times_{\text{Aut} O} O \]

over \( X \). The bundle \( V \) carries a connection \( \nabla : V \to V \otimes \Omega_X \), thus a left \( D_X \)-module structure. The connection is defined relative to a choice of local coordinate by

\[ \nabla_{\partial_x} = \partial_x + T, \]

but is independent of this choice. The right \( D_X \)-module obtained from \( V \) by side changing is denoted \( A \). It carries the natural structure of a chiral algebra \[21, Theorem 19.3.3\].

The chiral operation \( \mu \) of \( A \) is determined by its restrictions to \( D^2_z \) for \( x \in X \), and these restrictions can be written in terms of the vertex operation and the residue map \( (3.4) \) as follows. Let \( z \) be a coordinate at \( x \) and \((z_1, z_2)\) the coordinate on \( D^2_z \) induced by \( z \). The restriction of the chiral operation

\[ \mu : j_* j^*(A \boxtimes A) \to \frac{j_* j^*(\omega \boxtimes A)}{\omega \boxtimes A} \]

to \( D^2_z \) is given by

\[ \mu (f(z_1, z_2)adz_1 \boxtimes bdz_2) = f(z_1, z_2)dz_1 \boxtimes a(z_1 - z_2)bdz_2 \mod (\text{reg}) \]

The Jacobi identity satisfied by \( \mu \), i.e., the vanishing of the composition \( (6.3) \) below, corresponds at the level of the vertex algebra \( V \) to the Borcherds identity \[24\], while skew-symmetry of \( \mu \) corresponds to the skew-symmetry identity \[24\].
The formula (3.7) can be expressed in terms of $\lambda$-bracket notation as follows. We perform the identification

$$\Gamma(D^x, \Lambda, A) = \Gamma(D_x, A) \otimes_{\mathbb{C}[\partial]} \mathbb{C}[\partial_1, \partial_2]$$

where $\partial$ acts on $\Gamma(D_x, A)$ as by $\partial_z, \partial_i = \partial_{z_i}$, and $\partial = \partial_1 + \partial_2$. Then

$$\mu(f(z_1, z_2)adz_1 \otimes bdz_2) = res_{z_1 = z_2} e^{\langle z_1 - z_2 \rangle} f(z_1, z_2) a(z_1 - z_2)b \otimes 1,$$

where the formal residue symbol $res_{z_1 = z_2}$ indicates to write $z_1$ as $(z_1 - z_2) + z_2$, expand the expression in positive powers of $z_1 - z_2$ and extract the coefficient of $(z_1 - z_2)^{-1}$. The notation $\bar{\partial}_i$ indicates to remove all powers of $\partial_i$ to the right hand side of the tensor product symbol.

A choice of coordinate $z$ at a point $x \in X$ now induces identifications

$$\Gamma(D^x, A) \otimes A_x \mu \longrightarrow A_x$$

$$\Gamma(X, j^* A) \\ V((z))dz \otimes V \longrightarrow V,$$

where the morphism in the lower line is simply the $f$-product

$$af(z)dz \otimes b \mapsto res_z f(z) a(z)b dz = a(f)b.$$

4. Conformal Blocks

4.1. The inclusion $D^x \hookrightarrow X \setminus x$ induces a map

$$\Gamma(X, A(*x)) \rightarrow \Gamma(D^x, A),$$

and one may consider the vector space of coinvariants

$$\Gamma(D^x, A) \otimes A_x \mu \longrightarrow A_x.$$

The dual of this space is known as the space of conformal blocks associated with $X, x$ and $V$. The set of conformal blocks of an elliptic curve is the central object of Zhu’s paper [46].

4.2. The conformal vertex algebra $V$ carries an obvious increasing filtration $A^\nabla V = \bigoplus_{\nabla \leq \Delta} V_\nabla$, which is evidently stable under the linear automorphism $R(f)$ associated with any change of coordinate $f$ as in Section 2.9. Therefore $A^\nabla V$ induces a filtration $A^\nabla V$ of $V$ by vector subbundles of finite rank. It follows immediately from the construction of $V$ that the successive quotients are direct sums of tensor powers of the tangent bundle, i.e.,

$$0 \rightarrow A^{\nabla -1} V \rightarrow A^{\nabla} V \rightarrow (\Theta^\nabla_X)_{\oplus dim V_\nabla} \rightarrow 0,$$

and so the associated graded takes the form

$$gr^A V = \bigoplus_{\Delta \in \mathbb{R}^+} gr^A_\Delta V \quad \text{with} \quad gr^A_\Delta V \cong (\Theta^\nabla_X)^{\oplus dim V_\nabla}.$$

Now we refine the discussion to the case of $X$ a smooth elliptic curve and $x$ its marked point. We fix $\tau \in \mathcal{H}$ where $\mathcal{H}$ denotes the complex upper half plane, and we define $X$ as the complex analytic variety $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. The marked point $0 = [\Lambda] \in X$ and the addition induced from that in $\mathbb{C}$ gives $X$ the structure of an elliptic curve. Of course $X$ is actually algebraic, and the function

$$z \mapsto [u : v : w] = [\varphi(z) : \varphi'(z) : 1], \quad 0 \mapsto [0 : 1 : 0]$$
descends to an embedding $X \subset \mathbb{CP}^2$, presenting it as the zero set of a homogeneous cubic. We recall all facts that will be used about elliptic functions, including the definition of the Weierstrass function $\wp(z)$, in Section 4 below.

Under the analytification map $X^{an} \to X$ the space of coinvariants $\Gamma(X, \mathcal{A}^{(x)})$ corresponds to

$$H(X, x, V) = \frac{\mathcal{A}_x}{\Gamma(X, \mathcal{A}^{(x)}) \cdot \mathcal{A}_x}.$$ 

For the rest of this section we continue with the analytic topology on $X$, omitting the superscript.

Since $X$ is an elliptic curve we have $\Theta_X \cong \mathcal{O}_X$, and so $gr^AV \cong V \otimes \mathcal{O}_X$ as $\mathcal{O}_X$-modules. In fact as $\mathcal{O}_X$-modules we have $V \cong A \cong V \otimes \mathcal{O}_X$. Indeed the standard coordinate $z$ of $\mathbb{C}$ induces compatible trivialisations of $V_x$ at all points $x \in X$, so the assignment

$$a \otimes (dz)^\Delta \mapsto (z, a)$$

(where on the right hand side $z$ denotes, by abuse of notation, the formal coordinate induced by $z$ at $x$) defines a morphism $V_\Delta \otimes \mathcal{O}_X \to A^\Delta V$. It follows that (4.2) is split, and so we have $A \to V \otimes \mathcal{O}_X$ as $\mathcal{O}_X$-modules. Thus we obtain identifications

$$\Gamma(X, \mathcal{A}^{(0)}) \cdot \mathcal{A}_x \cong (V \otimes \Gamma(X, \mathcal{O}_X(0))) \cdot V \cong (V_f| f \in \Gamma(X, \mathcal{O}_X(0))_\cdot)$$

The space of global sections $\Gamma(X, \mathcal{O}_X(0))$ is the space of meromorphic elliptic functions with possible pole at 0, and the latter is spanned over $\mathbb{C}$ by the constant function 1, the Weierstrass function $\wp(z)$ and all derivatives of the latter. Since $a_{\omega}(b) = -(Ta)_\omega(b)$, we have in fact

$$H(X, 0, V) \cong \frac{V}{V(0)V + V(\wp)V}.$$ 

The space defined in (4.4) coincides with the space of conformal blocks of [46] in the following sense. Let $V, Y(-, z)$ be a quasiconformal vertex algebra and $V, Y[-, z]$ Zhu’s modified vertex algebra structure. Then the dual of $H(X, 0, (V, Y[-, z]))$ is Zhu’s conformal block, specialised at the elliptic curve $X$. This is the content of [46], Proposition 5.2.1 parts (3) and (4). Since $R : (V, Y[-, z]) \to (V, Y(\wp, -\wp))$ is an isomorphism of vertex algebras, the two spaces of conformal blocks are naturally isomorphic.

5. Preliminaries on Homology

5.1. We recall a few standard notational conventions. Let $A^\bullet$ be a (cohomological) complex in an abelian category, then $A^\bullet[k]$ signifies the complex $C^\bullet$ shifted $k$ places to the left, i.e., $C^n[k] = C^{n-k}$ with $d_A[k] = (-1)^kd_A$. The cone on a morphism $f : A^\bullet \to B^\bullet$ of complexes is defined as $\text{Cone}(f) = A^\bullet[1] \oplus B^\bullet$ with differential $d(a, b) = (-da, db + f(a))$.

Now let $C$ be a tensor category, i.e., an abelian symmetric monoidal category. The tensor product of the complexes $A^\bullet, B^\bullet$ in $C$ is the complex $(A^\bullet \otimes B^\bullet)^n = \bigoplus_{i+j=n} A^i \otimes B^j$ with $d(a \otimes b) = (da) \otimes b + (-1)^{|a|}a \otimes (db)$. The symmetric algebra $\text{Sym}(A^\bullet) = \bigoplus_{n \geq 0} \text{Sym}_n(A^\bullet)$ on $A^\bullet$ is the sum of the quotients $\text{Sym}_n(A^\bullet) = (A^\bullet)^{\otimes n}/\Sigma_n$ by symmetric group actions in which $a \otimes b$ is identified with $(-1)^{|a||b|}b \otimes a$. The product in the symmetric algebra is given by $(a \otimes b) \cdot c = a \otimes b \otimes c$, etc.

5.2. Let $\mathfrak{g}$ be a Lie algebra over a field $k$ and $U(\mathfrak{g})$ its universal enveloping algebra. The augmentation morphism is the morphism of algebras $U(\mathfrak{g}) \to k$ that sends $\mathfrak{g} \subset U(\mathfrak{g})$ to 0. The homology of $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module $M$ is by definition the torsion

$$H_n(\mathfrak{g}, M) = \text{Tor}_n^{U(\mathfrak{g})}(k, M).$$ 

Lie algebra homology is computed by the (reduced) Chevalley complex defined as follows. Let the linear map $\delta : \mathfrak{g}^* \to \Lambda^2(\mathfrak{g})$ denote the transpose of the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. Then $\delta$ may
be extended uniquely to a derivation of the exterior algebra \( \text{Sym}(g^*[-1]) \). The Jacobi identity on \([\cdot, \cdot]\) implies that \( \delta^2 = 0 \). There is a perfect pairing between the exterior algebras \( \text{Sym}(g[1]) \) and \( \text{Sym}(g^*[-1]) \) extending that between \( g \) and \( g^* \), and the dual \( d \) of \( \delta \) with respect to this perfect pairing gives \( \text{Sym}(g[1]) \) the structure of a complex. The differential \( d : g \to k \) vanishes, so as a complex \( \text{Sym}(g[1]) \) splits as the direct sum of \( k \) concentrated in degree 0 and a complex \( C^*(g) \) concentrated in negative degrees. This complex \( (C^*(g), d) \) is the reduced Chevalley-Eilenberg complex of \( g \). Explicitly \( d \) is given by

\[
d(x_1 \wedge \ldots \wedge x_n) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n.
\]

For \( n \geq 1 \) we have \( H^{-n}(C^*(g), d) \cong H_n(g, k) \) the homology of \( g \) with coefficients in the trivial \( g \)-module \( k \).

If \( \mathcal{C} \) is any symmetric monoidal category then the definitions of Lie algebra and of reduced Chevalley complex can be formulated in \( \mathcal{C} \). That is to say for \( g \) an algebra in \( \mathcal{C} \) over the Lie operad one may directly write down \( C(g) \) as a complex in \( \mathcal{C} \) with differential \([\cdot, \cdot]\).

5.3. Let \( A \) be an associative unital algebra over the field \( k \). An \((A, A)\)-bimodule is the same thing as a left module over the algebra \( A^e = A \otimes_k A^{op} \). The Hochschild homology of \( A \) with coefficients in the bimodule \( M \) is then the torsion

\[
\text{HH}_n(A, M) = \text{Tor}^A_n(A, M).
\]

We denote the Hochschild homology \( \text{HH}_n(A, A) \) of the bimodule \( A \) simply by \( \text{HH}_n(A) \). Hochschild homology is computed by the Bar complex defined as follows. The Bar complex is the free product \( A * k[\varepsilon] \) of unital algebras, made into an \((A, A)\)-bimodule in the obvious way, and made into a differential graded algebra by putting \( \deg(A) = 0 \), \( \deg(\varepsilon) = -1 \), \( da = 0 \) for all \( a \in A \), and \( d\varepsilon = 1_A \). The Bar complex is a resolution of \( A \) by free \( A^e \)-modules. The Hochschild homology \( \text{HH}_n(A, M) \) is thus presented as the homology of the complex with \( A \otimes_k M \) in degree \(-n \), and differentials as follows:

\[
d(a_0 \otimes \ldots \otimes a_n \otimes m) = a_1 \otimes \ldots \otimes a_n \otimes ma_0 \\
+ \sum_{i=1}^n (-1)^i a_0 \otimes \ldots \otimes a_{i-2} \otimes a_{i+1} \otimes \ldots \otimes a_n \\
- (-1)^n a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes m.
\]

5.4. Let \( Y \) be a smooth complex variety of complex dimension \( n \). The de Rham cohomology groups of the right \( \mathcal{D}_Y \)-module \( M \) are by definition the cohomology groups of the object

\[
R\Gamma_{\mathcal{D}_Y}(Y, M) = R\Gamma(Y, \mathcal{D}_Y(M)) = R\Gamma(Y, M \otimes^L_{\mathcal{D}_Y} \mathcal{O}_Y)
\]

of the derived category of the category of sheaves of \( \mathbb{C} \)-vector spaces on \( Y \).

The left \( \mathcal{D}_Y \)-module \( \mathcal{O}_Y \) may be resolved as \( \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \text{Sym}(\Theta_Y[1]) \), and the object \( M \otimes^L_{\mathcal{D}_Y} \mathcal{O}_Y \) thus represented by the complex of sheaves

\[
0 \to M \otimes \Theta_Y \to \cdots \to M \otimes \Theta_Y \to M \otimes \Theta_Y \to \mathcal{O}_Y \to 0
\]

(nonzero in degrees \(-n \) through 0) with differentials

\[
d(m \otimes \xi_1 \wedge \cdots \wedge \xi_k) = \sum_i (-1)^i (m \xi_i) \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_k \\
+ \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_k.
\]
6. Chiral Homology

6.1. Following [7] we denote by $S$ the category whose objects are finite nonempty sets and whose morphisms are surjective functions.

6.2. The Ran space Ran$(X)$ of a topological space $X$ is the set of non empty finite subsets of $X$, equipped with the strongest topology under which the obvious functions $X^I \to \text{Ran}(X)$, where $I$ is any finite nonempty set, become continuous. To work with the Ran space of a variety $X$ using tools from geometry the notions of "left sheaf on $X^S$" and "right $D$-module on $X^S$" are introduced as technical substitutes for the notions of "sheaf of vector spaces on Ran$(X)$" and "right $D$-module on Ran$(X)$", respectively.

A left sheaf $F$ on $X^S$ [7, 4.2.1] consists of a sheaf $F_{X^I}$ of vector spaces on $X^I$ for each finite set $I$ and a morphism $\theta^{(e)} : \Delta_+^{(e)}(M_{X^I}) \to M_{X^J}$ for each surjection $\pi : J \to I$, subject to the compatibility conditions

\begin{equation}
\theta^{(\pi_1 \pi_2)} = \theta^{(\pi_2)} \circ \Delta_+^{(\pi_2)}(\theta^{(\pi_1)}) \quad \text{and} \quad \theta^{(\text{id}_I)} = \text{id}.
\end{equation}

Similarly a right $D$-module $M$ on $X^S$ [7, 3.4.10] consists of a right $D_{X^I}$-module $M_{X^I}$ for each finite set $I$ and a morphism $\theta^{(e)} : \Delta_*^{(e)}(M_{X^I}) \to M_{X^J}$ (where now $\Delta_*^{(e)}$ denotes pushforward of $D$-modules) for each surjection $\pi : J \to I$, satisfying [7, 1].

Let $M$ be a right $D_{X^I}$-module. The assignment $M_{X^I} = \Delta_+^{(1)} M$, $\theta^{(e)} = \text{id}_{M_{X^J}}$ defines a right $D$-module on $X^S$. We denote this assignment by $\Delta^{(S)}_+$.

Let $\{L_i\}_{i \in I}$ be a finite nonempty set of right $D$-modules on $X^S$. One defines [7, 3.4.10] a new right $D$-module $\bigotimes_{i \in I}^{\text{ch}} L_i$ on $X^S$ by putting

\begin{equation}
\left( \bigotimes_{i \in I}^{\text{ch}} L_i \right)_{X^J} = \bigoplus_{\pi : J \to I} j_*^{[\pi]} \left( j_\pi^* (L_i)_{X^{\pi^{-1}(i)}} \right).
\end{equation}

A permutation of $I$ induces a permutation of the factors of the direct sum on the right hand side of (6.2) and thus natural commutativity isomorphisms between differently ordered tensor products. In this way the category of right $D$-modules on $X^S$ becomes a tensor category.

6.3. Let $\mathcal{A}$ be a chiral algebra on $X$. Then its image $\Delta^{(S)}_+ \mathcal{A}$ is a Lie algebra in the category of right $D$-modules on $X^S$. The Chevalley-Cousin complex $C(\mathcal{A})$ of $\mathcal{A}$ is by definition the Chevalley complex of $\Delta^{(S)}_+ \mathcal{A}$ as in Section 5.

6.4. It is clear that the component $C^{-n}(\mathcal{A})$ of the Chevalley complex in cohomological degree $-n$ contributes to $C(\mathcal{A})_{X^I}$ only if $\# I \geq n$. Thus $C(\mathcal{A})_X = C^{-1}(\mathcal{A})_X = \mathcal{A}$. On the other hand $C(\mathcal{A})_{X^2}$ receives the contributions $C^{-1}(\mathcal{A})_{X^2} = \Delta_+ \mathcal{A}$ and

\begin{equation}
C^{-2}(\mathcal{A})_{X^2} = \left( (\mathcal{A} \otimes^{\text{ch}} \mathcal{A})_{X^2} \right)_{\Sigma_2} \cong j_* j^* (\mathcal{A} \otimes \mathcal{A}).
\end{equation}

We remark that the coinvariants by the action of $\Sigma_2$ appear here due to the passage from the tensor algebra to the symmetric algebra. In general the effect of passing from the tensor algebra to the symmetric algebra is to identify certain components of (6.2) so that the complex $C(\mathcal{A})_{X^I}$ is presentable as a sum, not over the set of surjections $\pi : I \to J$, but rather over the set $Q(I)$ of equivalence relations on $I$. See [7, 3.4.11]. In total then $C(\mathcal{A})_{X^2}$ is the complex

\begin{equation}
\sum_{k \neq l} j_* j^* (\mathcal{A}[1] \boxtimes \mathcal{A}[1]) \to \Delta_+ \mathcal{A}[1],
\end{equation}

where the morphism is nothing but the chiral product $\mu$. Similarly $C(\mathcal{A})_{X^3}$ is

\begin{equation}
\sum_{k \neq l} j_* j^* (\mathcal{A}[1]^\otimes) \to \bigoplus_{k \neq l} j_* j^* (\mathcal{A}[1] \boxtimes \Delta_+ \mathcal{A}[1]) \to \Delta_+^{(3)} \mathcal{A}[1],
\end{equation}

$\Delta_+^{(3)} \mathcal{A}[1]$.
where \( j^{[k,\ell]} \) denotes the open embedding associated as in \( \text{(3.5)} \) with the surjection \( \{1, 2, 3\} \to \{1, 2\} \) that sends \( k \) and \( \ell \) to 2 and sends the remaining element of \( \{1, 2, 3\} \) to 1. The morphisms are built from the chiral operation \( \mu \), and to say that \( \mu \) satisfies the Jacobi identity means nothing other than to say that the composition \( \text{(6.3)} \) vanishes. Thus \( C(A)_{X^3} \) is a complex, and in general \( C(A)_{X^I} \) is a complex of right \( \mathcal{D}_{X^I} \)-modules for each finite set \( I \).

6.5. The functor \( R\Gamma_{\text{DR}} \) of de Rham cohomology for \( \mathcal{D} \)-modules on \( X^S \) is defined, as for varieties, as the derived composition

\[
R\Gamma_{\text{DR}}(X^S, M) := R\Gamma(X^S, \text{DR}(M)),
\]

of DR and \( \Gamma(X^S, -) \) \([7, 4.2.6(iv)]\). Let \( M \) be a \( \mathcal{D} \)-module or complex of \( \mathcal{D} \)-modules on \( X^S \), then \( \text{DR}(M) \) is defined by the assignment

\[
\text{DR}(M)_{X^I} := \text{DR}(M_{X^I}),
\]

and may be regarded as an object of the derived category of \( ! \)-sheaves on \( X^S \) or else, making use of the explicit representative \( \text{(5.2)} \), as a complex of \( ! \)-sheaves on \( X^S \).

Let \( F \) be a \( ! \)-sheaf on \( X^S \). Then

\[
I \mapsto \Gamma(X^I, F_{X^I})
\]

defines an \( S^{\text{op}} \)-diagram in \( \text{Vect}_k \):

\[
\Gamma(X, F_X) \longrightarrow \Gamma(X^2, F_{X^2}) \longrightarrow \Gamma(X^3, F_{X^3}) \longrightarrow \cdots
\]

By definition \( \Gamma(X^S, F) \) is the colimit of this diagram, that is, the initial object among all objects which receive a compatible system of morphisms from the diagram. The derived functor \( R\Gamma(X^S, -) \), since it is the derived functor of a colimit, is naturally described as a homotopy colimit. We omit the definition of homotopy colimit (which requires the language of simplicial sets) because in this article it will not play a role. See \([\text{??}]\) for a definition.

6.6. Let \( A \) be a chiral algebra on \( X \). By definition \([7, 4.2.11]\) the chiral homology of \( A \) is the de Rham cohomology of the Chevalley-Cousin complex \( C(A) \). That is

\[
H_{\text{ch}}^n(X, A) = H^{-n}(C^{\text{ch}}(X, A))
\]

where

\[
C^{\text{ch}}(X, A) = R\Gamma_{\text{DR}}(X^S, C(A)).
\]

7. Chiral Homology with Coefficients

7.1. Let \( A \) be a chiral algebra, then a chiral \( A \)-module is a \( \mathcal{D}_X \)-module \( M \) together with an action

\( \mu_M : j_*j^*(A \boxtimes M) \to M \) satisfying a natural analogue of \( \text{(6.3)} \). As usual one may extend the chiral algebra structure from \( A \) to \( A \oplus M \) by \( \mu_M \) and by declaring the product of two elements of \( M \) to be zero. In fact \( A \oplus M \) becomes a graded chiral algebra once we declare \( \deg(A) = 0 \) and \( \deg(M) = +1 \).

Following \([7, 4.2.19]\) we define \( C(A, M) \) to be the component of degree +1 of the Chevalley-Cousin complex of the graded chiral algebra \( A \oplus M \). The chiral homology of \( A \) with coefficients in \( M \) is the de Rham cohomology of \( C(A, M) \). That is

\[
H_{\text{ch}}^n(X, A, M) = H^{-n}(C^{\text{ch}}(X, A, M))
\]

where

\[
C^{\text{ch}}(X, A, M) = R\Gamma_{\text{DR}}(X^S, C(A, M)).
\]
Now let \( i : \ast \to X \) be the embedding of a point with image \( x \in X \), and \( j \) the open embedding of the complement \( U_x = X \setminus \{x\} \). Proposition 4.4.3 of [7] implies a quasi-isomorphism

\[
C^{\text{ch}}(X, \mathcal{A}) \cong C^{\text{ch}}(X, \mathcal{A}, \tilde{\mathcal{A}}_x),
\]

where the chiral \( \mathcal{A} \)-module \( \tilde{\mathcal{A}}_x \) is by definition the cone on the canonical morphism \( \mathcal{A} \to j_\ast j^* \mathcal{A} \).

For us \( \mathcal{A} \) will be the chiral algebra associated with a quasiconformal vertex algebra, so \( \tilde{\mathcal{A}}_x \) is quasi-isomorphic to \( i_\ast \mathcal{A}_x \).

7.2. Now let \( \mathcal{M} \) be an \( \mathcal{A} \)-module supported at the point \( x \in X \) (in this article \( \mathcal{M} = i_\ast \mathcal{A}_x^\lambda \)). Let \([R] \in Q^+(I)\) be a pointed equivalence relation on \( I \), represented by the set \( R = \overline{\mathbb{R}} \cup \{r_0\} \).

See Sections 8.2 and 8.8 below. We write \( U^{(R)} = (X \setminus \mathbb{R}) \times \{r_0\} \subset X^R \). Since the components of \( C(\mathcal{A}, \mathcal{M})_X \) are supported on affine subvarieties of \( X^I \) of the form \( U^{(R)} \), the \( S^\text{op} \)-diagram \( C^{\text{ch}}(X, \mathcal{A}, \mathcal{M}) \) reduces to

\[
I \mapsto \bigoplus_{[R] \in Q^+(I)} \Gamma(U^{(R)}; \text{DR}[\mathcal{A}[1] ^{[\mathbb{Q}^\mathbb{R}] \otimes \mathcal{M}})).
\]

Compare with [7] 4.2.19.

8. Chiral Homology of Elliptic Curves

8.1. Now we pass from the general theory summarised in the previous sections to the specific case of \( X \) a smooth elliptic curve and \( \mathcal{A} \) the chiral algebra on \( X \) associated with a quasiconformal vertex algebra \( V \). We work out, at the level of global sections, the chiral chain complexes with and without coefficients.

Let \( \mathbb{C} \) be the complex plane and \( z \) its standard global coordinate. From now on \( X \) will denote the elliptic curve \( \mathbb{C}/\mathbb{Z} + \mathbb{Z} \tau \) with marked point 0. The vector field \( \partial / \partial \tau \) on \( \mathbb{C} \) induces a global vector field on \( X \) which we denote \( \xi \). Under the embedding \( \mathbb{C} \to \mathbb{P}^1 \) into projective space \( \mathbb{C} \) corresponds to the algebraic vector field \( u \partial / \partial \tau \).

We have an isomorphism of algebras \( \Gamma(X, \mathcal{D}_X) \cong \mathbb{C}[\lambda] \) where \( \lambda \) represents \( -\xi \), and in general for any finite set \( I \) an isomorphism of algebras \( \Gamma(X^I, \mathcal{D}_{X^I}) \cong \mathbb{C}[\lambda_i]_{i \in I} \) where \( \lambda_i \) represents \( -\pi_i^*(\xi) \).

Let \( M \) be a right \( \mathcal{D}_X \)-module, then the space of global sections \( \Gamma(X^I, M) \) becomes a \( \mathbb{C}[\lambda_i]_{i \in I} \)-module. Now let \( M \) be a right \( \mathcal{D}_X \)-module, so that \( \Gamma(X, M) \) is a \( \mathbb{C}[\lambda] \)-module and \( \Gamma(X^2, \Delta_* M) \) is a \( \mathbb{C}[\lambda_1, \lambda_2] \)-module. There is a natural \( \mathbb{C}[\lambda] \)-action on \( \mathbb{C}[\lambda_1, \lambda_2] \) given by putting \( \lambda = \lambda_1 + \lambda_2 \). One then has as in Section 3.1

\[
\Gamma(X^2, \Delta_* M) \cong \Gamma(X, M) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}[\lambda_1, \lambda_2].
\]

Let \( I \) be a finite set. We write

\[
\Gamma_I := \Gamma(X^I, j_\ast j^* \mathcal{O}_{X^I})
\]

where \( j : U^{(I)} \to X^I \) is the embedding \( [7,4] \). Now let \( V \) be a quasiconformal vertex algebra and \( \mathcal{A} \) the chiral algebra on \( X \) associated with \( V \). As in Section 4 we have a trivialisation of \( \mathcal{A} \) and hence identifications

\[
\Gamma(X^I, j_\ast j^* \mathcal{A}^{\otimes I}) \cong V^{\otimes I} \otimes \Gamma_I.
\]

The action of \( \mathbb{C}[\lambda_i]_{i \in I} \) on \( \Gamma(X^I, j_\ast j^* \mathcal{A}^{\otimes I}) \) is identified with its action on \( V^{\otimes I} \otimes \Gamma_I \) defined by

\[
\lambda_i = -\pi_i^*(\xi) = \partial_{\xi_i} + T^{(i)}.
\]
The chiral operation now yields the diagram

\[ \Gamma(X^2, j^* A^2) \xrightarrow{\Gamma_{\mathfrak{G}^2}} \Gamma(X, \Delta A) \]

\[ (V \otimes^k \Gamma_2) \otimes_{\mathbb{C}[\lambda_1, \lambda_2]} \mathbb{C}^{[\lambda_1, \lambda_2]} \xrightarrow{(V \otimes^k \Gamma_1) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}^{[\lambda_1, \lambda_2]},} \]

where, by \( \mathbb{R} \), the map of the lower row is

\[ f \otimes (a^1 \otimes a^2) \otimes_{\mathbb{C}[\lambda_1, \lambda_2]} g(\lambda_1, \lambda_2) \mapsto \left( \text{res}_{z_1 = z_2} f \cdot e^{i_1(z_1 - z_2)} a^1(z_1 - z_2) a^2 \right) \otimes_{\mathbb{C}[\lambda]} g(\lambda_1, \lambda_2). \]

Here the notation \( \tilde{\lambda}_i \) means that all copies of \( \lambda_1 \) obtained upon expansion of the exponential are to be read as elements of the right hand factor of the tensor product.

8.2. Let \( I \) be finite set. The set of equivalence relations on \( I \) is denoted by \( Q(I) \) and the set of equivalence classes consisting of exactly \( k \) equivalence classes is denoted by \( Q(I, k) \). If \( \pi_R : I \to R \) is a surjection with \( |R| = k \) then we denote by \([R] \in Q(I, k)\) the corresponding equivalence relation and by \( I_r \subset I \) the preimage \( \pi_R^{-1}(r) \) of \( r \in R \). If \( \pi_{R'} : I \to R' \) is another surjection then we have \([R] = [R']\) if there exists a bijection \( R \cong R' \) such that the following diagram commutes

\[ \begin{array}{ccc} I & \xrightarrow{\sim} & R' \\ \downarrow & & \downarrow \pi_{R'} \\
R & \xrightarrow{\pi_R} & R' \end{array} \]

We say that \([R] \in Q(I, k)\) and \([S] \in Q(I, \ell)\), where \( k + \ell = |I| + 1 \), are complementary if the graph with vertices \( R \cup S \) and one edge for each \( i \in I \) connecting \( \pi_S(i) \) with \( \pi_R(i) \) is a connected tree. The notion of complementarity depends on \([R], [S]\) and not on their representatives \( R, S \).

8.3. We denote by \( k[I] \) the polynomial \( k \)-algebra on generators \( \lambda_i \) indexed by \( i \in I \). A surjection \( \pi : I \to R \) induces a morphism \( \varphi : k[I] \to k[R] \) defined by \( \varphi(\lambda_i) = \sum_{i \in I_r} \lambda_i \), and hence a \( k[R] \)-module structure on \( k[I] \). If \([R] = [S] \in Q(I)\) then the \( k[S] \)-module and \( k[R] \)-module structures on \( k[I] \) are identified by \( k[R] \cong k[S] \).

8.4. Lemma. Let \([R], [S] \in Q(I)\) be complementary. There exists a natural isomorphism of \( k[R] \otimes_k k[S] \) modules

\[ k[R] \otimes_{k[I]} k[S] \cong k[I]. \]

This isomorphism does not depend on the choice of representatives \( R, S \).

Proof. The product of the canonical maps \( k[R], k[S] \to k[I] \) gives a map \( k[R] \otimes k[S] \to k[I] \) which factors through the quotient since

\[ \sum_{r \in I_r} \lambda_i = \sum_{i \in I} \lambda_i = \sum_{s \in I_s} \sum_{i \in I_s} \lambda_i. \]

The result follows from \([2] \) Proposition 1.3.2.a]. \( \square \)

8.5. We may use Lemma 8.3 to rewrite (8.1) as

\[ f \otimes (a^1 \otimes a^2) \otimes g(\lambda_2) \mapsto \left( \text{res}_{z_1 = z_2} e^{i_1(z_1 - z_2)} f \otimes a^1(z_1 - z_2) a^2 \right) \otimes g(\lambda_1 + \lambda_2). \]

Let \( V \) be a vector space endowed with a linear endomorphism \( T \). We equip \( V \) with a \( k[s] \)-module structure by \( \lambda = T \). Recall that \( V[1] \) denotes the complex in which \( V \) is placed in degree \(-1\). We consider the complex \( V[1]^{\otimes I} \) with its natural \( k[I] \)-module structure in which \( \lambda_i \)
acts by \(-\lambda\) in the \(i\)th factor of the tensor product. For \(I \cong J\) we have a natural isomorphism \(V[1] \otimes I \cong V[1] \otimes J\).

We fix a finite set \(I\) and an equivalence relation \([R] \in Q(I)\) and we put

\[
(8.2) \quad \Gamma_{I,[R]} = \left(\Gamma_R \otimes V[1] \otimes R, k[I]\right) = \frac{\Gamma_R \otimes V[1] \otimes R \otimes k[I]}{\langle -(T^{(r)} + \partial_{x_r}) + \sum_{i \in I} \lambda_i \rangle_{r \in R}},
\]

where \(T^{(r)}\) denotes the action of \(T\) on the \(r\)th factor of the tensor product. Clearly \(\Gamma_{I,[R]}\) is a \(k[I]\)-module by multiplication on the right hand factor. Now let \([S] \in Q(I)\) be an equivalence relation complementary to \([R]\). Using \((8.4)\) we may rewrite \((8.2)\) as

\[
(8.3) \quad C_{I,[R]} = \left(\Gamma_R \otimes V[1] \otimes R \otimes k[S]\right) = \frac{\Gamma_R \otimes V[1] \otimes R \otimes k[S]}{\langle -(\sum_r T^{(r)} + \partial_{x_r}) + \sum_s \lambda_s \rangle}.
\]

This structure is independent of the choices of \(R\) and \(S\) up to natural isomorphism.

8.6. Let \(I\) be a finite set. We define the graded \(k[I]\)-module \(C_I\) to be

\[
(8.4) \quad C_I = \bigoplus_{k \geq 1} C_I^{-k} = \bigoplus_{k \geq 1} \left( C_{I,[R]} \right)_{[R] \in Q(I,k)},
\]

where \(C_{I,[R]}\) was defined in \((8.2)\) or equivalently in \((8.3)\).

8.7. Now let \(V\) be a vertex algebra, and \(T\) its translation operator. We consider the graded \(k[I]\)-module \(C_I\) associated with \(V\) as above. We use the vertex operation of \(V\) to turn \(C_I\) into a complex. The nonzero components of the differential are of the form \(d_{[R],[R']} : C_{I,[R]} \to C_{I,[R']}\) where \([R] \in Q(I,k+1)\) and \([R'] \in Q(I,k)\) is obtained from \([R]\) by identifying two of its equivalence classes. Let \(R = R \cup \{r_0,r_1\}\) and \(R' = R \cup \{r_1\}\) with the map \(R \to R'\) given by sending both \(r_0\) and \(r_1\) to \(r_1\). Let \(S'\) be complementary to \(R'\). Then \(S'\) can be written as \(S' = S \cup \{s_0,s_1\}\) and its quotient \(S = S \cup \{s_1\}\), with \(S' \to S\) sending both \(s_0\) and \(s_1\) to \(s_1\), is complementary to \(R\). Let \(S = \{s_0, \ldots, s_0\}\). We define the \(C_{I,[R]} \to C_{I,[R']}\) component of the differential to be

\[
(8.5) \quad f \otimes \left( \bigotimes_{r \in R} a^r \right) \otimes g(\lambda_{s_1}, \ldots, \lambda_{s_0}, \lambda_{s_1}) \mapsto
\]

\[
\left( \bigotimes_{r \in R} a^r \right) \otimes \left( \operatorname{res}_{z_{r_0} = z_{r_1}} f \cdot e^{\lambda_{r_0} - z_{r_1}} a^{r_0}(z_{r_0} - z_{r_1}) a^{r_1} \right) \otimes g(\lambda_{s_1}, \ldots, \lambda_{s_0}, \lambda_{s_0} + \lambda_{s_1}).
\]

8.8. Now let \(M\) be a positive energy \(V\)-module graded by integer conformal weights. This is a strong restriction on \(M\), but in the end we shall only be interested in the case of \(M = V\). We have the \(\mathcal{A}\)-module \(\mathcal{M}\), constructed by the usual localisation procedure, and the \(\mathcal{A}\)-module

\[
\mathcal{M}_0 = \operatorname{coker}(\mathcal{M} \to j_* j^* \mathcal{M})
\]

where \(j\) denotes the embedding into \(X\) of the complement of the point 0.

For a finite (possibly empty) set \(I\) we write

\[
\tilde{\Gamma}_I = \Gamma(\tilde{X}^I, j_* j^* \mathcal{O}_{\tilde{X}^I})
\]

(so for example for \(I = \emptyset\) we have \(\tilde{X}^I = *\) and \(\tilde{\Gamma}_I \cong \mathbb{C}\)). In fact the group structure of the elliptic curve \(X\) can be used to write a bijection \(\Gamma_I \to \Gamma_{I,0}\), namely \(f \mapsto f\) where \(f(\{x_i\}) = f(\{x_i - x_s\})\). However we will not use this fact.

We define a pointed equivalence relation on a set \(I\) to be an equivalence relation on \(I\) together with a choice of one of the equivalence classes that comprises it, which is referred to as the marked equivalence class. We denote by \(Q^*(I)\) the set of pointed equivalence relations, and by \(Q^*(I,k)\) the subset of those composed of exactly \(k\) equivalence classes.
Now let $I$ be a nonempty finite set and let $[R] \in Q^*(I,k)$. Let $R = \mathbb{R} \cup \{r_0\}$ where $r_0$ is the marked equivalence class. We give $\Gamma_{\mathbb{R}} \otimes V[1] \otimes \mathbb{R} \otimes M$ a $\mathbb{C}[R]$-module structure by letting $\lambda_r$ act by $\partial_{x^r} + T^{(r)}$ for $r \neq r_0$ and $\lambda_{r_0}$ act by $T^{(r_0)} \equiv T^{(M)}$. Now we put

$$\hat{C}_I[|R|] = \left( \Gamma_{\mathbb{R}} \otimes V'[1] \otimes \mathbb{R} \otimes M \right) \otimes_{\mathbb{C}[R]} \mathbb{C}[I] \equiv \left( \Gamma_{\mathbb{R}} \otimes V'[1] \otimes \mathbb{R} \otimes M \right) \otimes_{\mathbb{C}[|I|]} \mathbb{C}[|I|].$$

The differential is defined in terms of its components $d_{[R],[R']}^I$ as in Section 8.7 except that now we must divide into two cases according to whether or not $R \to R'$ identifies the marked equivalence class of $R$ with another class.

If neither of the two equivalence classes identified by $R \to R'$ is the marked one, then $d_{[R],[R']}$ is given by

$$f \otimes \left( \bigotimes_{r \in R} a^{r'} \right) \otimes m \otimes g(\lambda_{s_1}, \ldots, \lambda_{s_n}, \lambda_{s_1}) \mapsto \left( \bigotimes_{r \in R} a^{r'} \right) \otimes \left( \text{res}_{z_{r_0} = z_{r_1}} f \cdot e^{\lambda_{r_0} (z_{r_0} - z_{r_1})} a^{r_0} (z_{r_0} - z_{r_1}) a^{r_1} \right) \otimes m \otimes g(\lambda_{s_1}, \ldots, \lambda_{s_n}, \lambda_{s_0} + \lambda_{s_1}).$$

Suppose now that $R \to R'$ identifies $r_1$ with the marked equivalence class $r_0$. Then $d_{[R],[R']}$ is given by

$$f \otimes \left( \bigotimes_{r \in R} a^{r'} \right) \otimes m \otimes g(\lambda_{s_1}, \ldots, \lambda_{s_n}, \lambda_{s_1}) \mapsto \left( \bigotimes_{r \in R} a^{r'} \right) \otimes \left( \text{res}_{z_{r_1} = 0} f \cdot e^{\lambda_{r_0} z_{r_1}} a^{r_1} (z_{r_1}) m \right) \otimes g(\lambda_{s_1}, \ldots, \lambda_{s_n}, \lambda_{s_0} + \lambda_{s_1}).$$

8.9. Let $I$ be a finite set. We define the graded $k[I]$-module $\hat{C}_I$ to be

$$\hat{C}_I = \bigoplus_{k \geq 1} \hat{C}_I^{-k} = \bigoplus_{k \geq 1} \bigoplus_{[R] \in Q^*(I,k)} \hat{C}_I[|R|],$$

where $\hat{C}_I[|R|]$ was defined in (8.6) and (8.7).

8.10. For any finite set $I$ we have an augmentation map $k[I] \to k$ which sets $\lambda_i$ to 0 for all $i \in I$. In this way we regard $k$ as a $k[I]$-module. We define

$$D_I = \text{Sym}(\text{Cone}(\text{Id}_{k[I]})).$$

Clearly the degree 0 component of $D_I$ is naturally isomorphic to $k[I]$, hence $D_I$ is a $k[I]$-module. Indeed $D_I$ is a resolution by free $k[I]$-modules of $k$.

Let $X$ be an elliptic curve as above. For a right $\mathcal{D}_X$-module $M$ one has

$$\Gamma(X^I, \text{DR}(M)) \cong \Gamma(X^I, M) \otimes_{\mathbb{C}[I]} D_I.$$

If $M^*$ is a complex of right $\mathcal{D}_X$-modules then $\Gamma(X^I, \text{DR}(M^*))$ is computed by the standard hypercohomology spectral sequence whose $E_1$-page is

$$E_2^{p,q} = H^p(\Gamma(X^I, M^p) \otimes_{\mathbb{C}[I]} D_I^q).$$

In the case of $X$ an elliptic curve, $x = 0$, and $\mathcal{A}$ the chiral algebra associated with the quasiconformal vertex algebra $V$, the specialisation of the $S^\text{op}$-diagram is

$$I \mapsto \hat{C}_I \otimes_{\mathbb{C}[I]} D_I,$$

where $\hat{C}_I$ is the complex of $\mathbb{C}[I]$-modules $[8.8]$. 

8.11. We now write out $\hat{C}_I \otimes_{\mathbb{C}[I]} D_I$, and the associated spectral sequence, somewhat explicitly for the case $I = \{1, 2, 3\}$. Firstly $\# Q^*(I) = 10$ and so $\hat{C}_I$ has 10 components: one in degree 0 of the form

$$P^0 := M \otimes \mathbb{C}[\lambda_0],$$

six in degree $-1$ of the form

$$P^{-1} := \frac{\hat{\Gamma}_1 \otimes (V \otimes M) \otimes \mathbb{C}[\lambda_1, \lambda_0]}{\langle \partial_1 + T^{(1)} + \lambda_1 \rangle},$$

and three in degree $-2$ of the form

$$P^{-2} := \frac{\hat{\Gamma}_2 \otimes (V \otimes V \otimes M) \otimes \mathbb{C}[\lambda_1, \lambda_2, \lambda_0]}{\langle \partial_1 + T^{(1)} + \lambda_1 \rangle_{i=1,2}}.$$

The complex $\hat{C}_I \otimes_{\mathbb{C}[I]} D_I$ carries a $\Sigma_3$-action, and the coinvariants are given by

$$(\hat{C}_I^0)_{\Sigma_3} = A^0, \quad (\hat{C}_I^{-1})_{\Sigma_3} = A^{-1}, \quad \text{and} \quad (\hat{C}_I^{-2})_{\Sigma_3} = (A^{-2})_{\Sigma_3},$$

where $A^* = P^* \otimes_{\mathbb{C}[I]} D_I$ and the action of $\Sigma_2$ on the last factor here corresponds to exchange of the two factors of $V$ in the definition of $P^{-2}$ above.

8.12. The nontrivial entries $E_{i,q}^2$ of the spectral sequence (8.9) associated with $I = \{1, 2, 3\}$ (or in general with any finite set) are confined to the region in which $p \leq q \leq 0$. Therefore in total degree $i = 0, -1$ the spectral sequence converges at the page $E_2$, and furthermore

$$H^i(\hat{C}_I \otimes_{\mathbb{C}[I]} D_I)_{\Sigma_3} \cong H^i(H^0(P^* \otimes_{\mathbb{C}[I]} D_I) \cong H^i(P^* \otimes_{\mathbb{C}[I]} \mathbb{C}).$$

8.13. Now let us put $M = V$. As noted in Section 7 the chiral homology with coefficients in $M$ coincides with the plain chiral homology of $A$. On the other hand [7, Lemma 4.2.10] asserts that $H^i_{\text{ch}}(X, A)$ is computed correctly by truncating its defining $S_{\text{op}}$-diagram at $n + 2$, that is by restricting to the subcategory of finite sets of cardinality at most $n + 2$. Thus we obtain $H^i_{\text{ch}}(X, A)$ for $i = 0, 1$ as a quotient of $H^{-i}(P^* \otimes_{\mathbb{C}[I]} \mathbb{C})$ by coequalising those morphisms in the $S_{\text{op}}$-diagram associated with surjections $I \to \{1, 2\}$. These morphisms coincide it turns out, and are injective. From this and the final remarks of the preceding section it follows that

$$H^i_{\text{ch}}(X, A) \cong H^{-i}(P^* \otimes_{\mathbb{C}[I]} \mathbb{C}),$$

for $i = 0, 1$.

To summarise, the chiral homology $H^i_{\text{ch}}(X, A)$ for $i = 0, 1$ coincides with the cohomology in degree $-i$ of the complex $h(P^*)$, viz.,

$$h(P^0) = V, \quad h(P^{-1}) = \frac{\hat{\Gamma}_1 \otimes (V \otimes V)}{\langle \partial_1 + T^{(1)} \rangle}, \quad h(P^{-2}) = \frac{\hat{\Gamma}_2 \otimes (V \otimes V \otimes V)}{\langle \partial_1 + T^{(1)} \rangle_{i=1,2}},$$

with differentials as follows. The differential $h(P^{-1}) \to h(P^0)$ is given by

$$(8.10) \quad f(x)a \otimes m \mapsto \text{res}_x f(x)a(x)m = a_{(f)} m,$$

and the differential $h(P^{-2}) \to h(P^{-1})$ is given by

$$(8.11) \quad f(x,y)a \otimes b \otimes m \mapsto \text{res}_y f(x,y)a \otimes b(y)m \quad - \text{res}_y f(y,x)a(y-x) b \otimes m - \text{res}_y f(y,x)b \otimes a(y)m.$$
The differentials are well defined in the quotients. Indeed $[Ta](x) = \partial_x a(x)$ and
\[
\text{res}_x [(\partial_x f(x))a(x)m + f(x)(\partial_x a(x)m)] = 0.
\]
Similarly one sees that
\[
(\partial_x f(x,y))a \otimes b \otimes m + f(x,y)(Ta) \otimes b \otimes m, \quad \text{and} \quad (\partial_y f(x,y))a \otimes b \otimes m + f(x,y)a \otimes (Tb) \otimes m
\]
are annihilated by the differential \((8.11)\).

8.14. The permutation of $I = \{1, 2, 3\}$ which swaps two elements corresponds, in the colimit, to identification of sections
\[
f(x,y) \cdot a \otimes b \otimes m \quad \text{and} \quad -f(y,x) \cdot b \otimes a \otimes m,
\]
We verify that indeed
\[
f(x,y) \cdot a \otimes b \otimes m + f(y,x) \cdot b \otimes a \otimes m
\]
is mapped to a total derivative in $h(P^{*-1})$, by the skew-symmetry identity \((2.5)\). After recalling some background material on elliptic functions we present an even more explicit description of the complex $h(P^*)$ in Section \[11\] below.

9. Elliptic Functions

9.1. We recall the Eisenstein series $G_k \in \mathbb{C}[[q]]$ defined for $k \geq 1$ to be
\[
G_k = (2\pi i)^k \left( \frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n=1}^{\infty} \frac{n^{k-1}}{1-q^n} \right)
\]
for $k$ even, and $0$ for $k$ odd. Here $B_n$ are the Bernoulli numbers, defined by $t/(e^t - 1) = \sum_{n=1}^{\infty} B_n t^n/n!$. For $k \geq 4$ the Eisenstein series $G_k$ is a modular form of weight $k$, while $G_2$ is a quasimodular form of weight $2$.

We also recall the Weierstrass elliptic function $\wp$. For us it will be convenient to put
\[
\wp(z, q) = z^{-2} + \sum_{k=0}^{\infty} (2k+1)G_{2k+2} z^{2k},
\]
which differs from the standard normalisation by the additive constant $G_2$. The Weierstrass quasielliptic function $\zeta$ is similarly defined to be
\[
(9.1) \quad \zeta(z, q) = z^{-1} - \sum_{k=0}^{\infty} G_{2k+2} z^{2k+1}.
\]
We clearly have the relation $\wp(z) = -\partial_z \zeta(z)$. Note that $\wp$ is an even function of $z$ and $\zeta$ an odd function of $z$.

Let $\tau \in \mathcal{H}$ the upper half complex plane and put $q = e^{2\pi i \tau}$. Then the two series above may be viewed as Laurent series expansions of meromorphic functions $\wp(z)$ and $\zeta(z)$ with poles for $z$ in the lattice $\Lambda_\tau = \mathbb{Z}1 + \mathbb{Z}\tau$. We have
\[
\wp(z+1,q) = \wp(z + \tau, q) = \wp(z, q),
\]
in other words $\wp$ is an elliptic function, and
\[
(9.2) \quad \zeta(z+1,q) = \zeta(z, q), \quad \text{while} \quad \zeta(z + \tau, q) = \zeta(z, q) - 2\pi i,
\]
so $\zeta$ is said to be a quasielliptic function.
The following identities are valid:
\[
\zeta(z, q) = 2\pi i \left( \frac{e^{2\pi iz}}{e^{2\pi iz} - 1} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \left[ e^{2\pi inz} - e^{-2\pi inz} \right] \right),
\]
and
\[
\wp(z, q) = (2\pi i)^2 \left( \frac{e^{2\pi iz}}{(e^{2\pi iz} - 1)^2} + \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \left[ e^{2\pi inz} + e^{-2\pi inz} \right] \right),
\]
and from them it is clear that
\[
\zeta(z, q = 0) = 2\pi i e^{2\pi iz} e^{2\pi iz} - 1 - \pi i \quad \text{and} \quad \wp(z, q = 0) = (2\pi i)^2 \frac{e^{2\pi iz}}{(e^{2\pi iz} - 1)^2}.
\]

9.2. Lemma. The following identity holds
\[
(9.3) \quad - (2\pi i)^2 q \frac{d\zeta}{dq} = \zeta \wp + \frac{1}{2} \varphi'.
\]

Proof. The relation may be proved by differentiating the identity \( \wp(z + \tau, q) - \wp(z, q) = 0 \) with respect to \( \tau \) and using (9.2). See [43]. \( \square \)

9.3. Differentiating (9.3) with respect to \( z \) yields the identity
\[
(2\pi i)^2 q \frac{d\varphi}{dq} = \zeta \varphi' - \varphi^2 + \frac{1}{2} \varphi''.
\]

9.4. In this section we introduce some elliptic functions of three variables and some useful identities among them. Firstly it is clear that the function
\[
\zeta(x, y, z) = \zeta(x - y) + \zeta(y - z) + \zeta(z - x)
\]
is elliptic in the three variables \( x, y \) and \( z \).

It is useful to define
\[
\mathfrak{3}(x, y, z) = \varphi(x - y) [\zeta(x - y) + \zeta(y - z) + \zeta(z - x)] + \frac{1}{2} \varphi'(x - y).
\]
Surprisingly this function is cyclically symmetric in its three variables.

9.5. Lemma. The function \( \mathfrak{3}(x, y, z) \) is cyclically symmetric in the variables \( x, y \) and \( z \).

Proof. The Weierstrass function \( \sigma \) is defined by
\[
\sigma(z) = z \prod_{\omega \in \Lambda \setminus \{0\}} \left( 1 - \frac{z}{\omega} \right) e^{z/\omega + \frac{1}{2}(z/\omega)^2},
\]
and satisfies
\[
\frac{\sigma'(z)}{\sigma(z)} = \zeta(z) + G_2 z.
\]

Now we recall the identity [33, pp. 243]
\[
\varphi(u) - \varphi(v) = -\frac{\sigma(u + v)\sigma(u - v)}{\sigma^2(u)\sigma^2(v)}.
\]
Taking the logarithmic derivative (in \( u \)) of this identity gives
\[
\frac{\varphi'(u)}{\varphi(u) - \varphi(v)} = \frac{d}{du} \left[ \log \sigma(u + v) + \log \sigma(u - v) - 2 \log \sigma(u) \right]
= \zeta(u + v) + \zeta(u - v) - 2\zeta(u).
Combining this with a similar logarithmic derivative in \( v \) yields
\[
\frac{\varphi'(u) - \varphi'(v)}{\varphi(u) - \varphi(v)} = 2\zeta(u + v) - 2\zeta(u) - 2\zeta(v),
\]
which establishes the desired cyclic symmetry. \(\square\)

9.6. The cyclic symmetry of \( \mathcal{Z}(x, y, z) \) is made manifest by the identity asserted in Lemma 9.7 below. To state the identity we need to recall the Jacobi theta functions \[12, \text{Chapter V}\]
\[
\theta(z, \tau) = -i \cdot \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1)/2} e^{2\pi i (n+1/2)z},
\]
\[
\theta_1(z, \tau) = \sum_{n \in \mathbb{Z}} q^{(n+1)/2} e^{2\pi i (n+1/2)z},
\]
\[
\theta_2(z, \tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2} e^{2\pi inz},
\]
\[
\theta_3(z, \tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} e^{2\pi inz}.
\]
and the Jacobi elliptic function \[12, \text{pp. 100}\]
\[
\text{sn}(z, \tau) = \frac{1}{\pi \cdot \theta_1(0, \tau) \cdot \theta_3(0, \tau) \cdot \theta_2(z, \tau)} \cdot \theta(z, \tau).
\]
One has the following identity relating the Weierstrass and Jacobi elliptic functions \[12, \text{pp. 102}\]
(9.4)
\[
\text{sn}(z, \tau)^2 = \frac{1}{\wp(z, \tau) - e(\tau)},
\]
where the half period value \( e(\tau) = \wp(\tau/2, \tau) \) can in fact be expressed explicitly as
\[
e(\tau) = -8\pi^2 \sum_{n=1}^{\infty} \frac{n q^{n/2}}{1 - q^n}.
\]

9.7. Lemma. The functions
(9.5)
\[
\mathcal{Z}(x, y, z) - e(\tau)\zeta(x, y, z) \quad \text{and} \quad -\frac{1}{\text{sn}(x - y) \text{sn}(y - z) \text{sn}(z - x)}
\]
differ by a constant.

Proof. Equation (9.4), or alternatively Jacobi’s formula \( \theta'(0, \tau) = \pi \cdot \theta_1(0, \tau) \cdot \theta_2(0, \tau) \cdot \theta_3(0, \tau) \) (here \( \theta' \) denotes the derivative with respect to the first variable), implies that the function \( \text{sn}(z)^{-1} \) has a simple pole at \( z = 0 \) with residue 1.

Consequently the right hand side of (9.5), as a function of \( x \), has a simple pole at \( x = y \) with residue \( \text{sn}(y - z)^{-2} \) (here we have used that \( \text{sn}(z) \) is an odd function of \( z \)). The left hand side \( \mathcal{Z}(x, y, z) \) of (9.5), as a function of \( x \), has a simple pole at \( x = y \) with residue \( \wp(\tau - y) \).

The identity (9.4) shows that the difference, call it \( \beta \), between the two sides of (9.5) is regular at \( x = y \). Similarly \( \beta \) is regular at \( x = z \), and hence is an elliptic function of \( y - z \). By Lemma 9.3 and the manifest symmetry of the right hand side of (9.5), \( \beta \) is cyclically symmetric in \( x, y, z \).

As a cyclically symmetric function independent of \( x \), \( \beta \) is constant as claimed. \(\square\)
10. Configuration Spaces of Elliptic Curves

10.1. In this section we let $X$ be a smooth complex projective algebraic curve, and we denote by $H^\bullet(X^n)$ the cohomology ring of $X^n$. We let $\pi_i : H^\bullet(X) \to H^\bullet(X^n)$ denote the pullback morphism associated with the projection $X^n \to X$ to the $i$th component, and similarly $\pi_{ij} : H^\bullet(X^2) \to H^\bullet(X^n)$ for $i \neq j$. We let $\Delta \in H^2(X^2)$ denote the class of the diagonal $\Delta \subset X^2$. We recall the subvariety $U^{(n)} \subset X^n$ defined as the complement of all diagonal divisors.

A special case of a theorem of Totaro [41, Theorem 4] (see also [9]) asserts that $H^\bullet(U^{(n)})$ is isomorphic to the cohomology of the differential graded algebra

$$H^\bullet(X^n) \otimes \mathbb{C}[G_{i,j} | i,j=1,\ldots,n] / J,$$

with differential $d$ defined by $d(G_{i,j}) = \pi_{ij}(\Delta)$, where $J$ is the ideal generated by the elements

$$G_{ij} - G_{ji},
G_{ij}G_{ik} + G_{jk}G_{ji} + G_{kj}G_{ij},
[\pi_i(x) - \pi_j(x)]G_{ij}.$$

We now pass to the case $X$ an elliptic curve. We recall the notation $\hat{X} = X \setminus 0$, and we write $\hat{U}^{(n)}$ for the complement of all diagonal divisors in $X^n$. The group structure of the elliptic curve $X$ permits one to write down, for any $n \geq 0$, an isomorphism of varieties

$$(10.1) U^{(n+1)} \to X \times \hat{U}^{(n)}$$

given by

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0, x_1 - x_0, \ldots, x_n - x_0).$$

By (10.1) it is possible to infer the Betti numbers of $U^{(n)}$ from those of $U^{(n)}$. The Betti numbers $h^k(U^{(n)}) = \dim_{\mathbb{C}}(H^k(U^{(n)}))$ are easily computed for small values of $n$ using the presentation above and a computer algebra system such as SAGE [40] (for a general computation see [18, 35, 39]). We are particularly interested in the top Betti number $h^n(U^{(n)})$, we compute:

| $n$ | $h^n(U^{(n)})$ |
|-----|----------------|
| 0   | 1              |
| 1   | 2              |
| 2   | 2              |
| 3   | 18             |
| 4   | 79             |
| 5   | 432            |

Let $Y$ be a smooth variety of dimension $n$. At the level of complexes of vector spaces one has the equality

$$(10.2) H^\bullet(Y)[n] = H^\bullet(R\Gamma_{\text{DR}}(\omega_Y)),$$

and so $\dim_{\mathbb{C}}(H^k(R\Gamma_{\text{DR}}(\omega_Y))) = h^{k+n}(Y)$. Now we let $I = \{1, 2, \ldots, n\}$, we write

$$\hat{\Gamma}_n = \hat{\Gamma}_I = \Gamma(\hat{X}^n, j^*j^*\mathcal{O}_{X^n})$$

where $j$ is the embedding $\hat{U}^{(n)} \to \hat{X}^n$, and we put $Y = \hat{U}^{(n)}$. The sheaf $\omega_Y$ is a free $\mathcal{O}_Y$-module of rank 1 with generator $\nu = dx_1 \wedge \cdots \wedge dx_n$ where $dx_i = \pi^*_i(dx)$ is the pullback of the global 1-form $dz$ on $X$ (as in Section 4). The action (3.1) of $\Theta_Y$ on $\omega_Y$ is given by

$$(f\nu) \cdot g\partial_{x_i} = -\partial_{x_i}(fg)\nu,$$

and so

$$H^0(R\Gamma_{\text{DR}}(j^*j^*\omega_{\hat{X}^n})) \cong H^0(R\Gamma_{\text{DR}}(\omega_Y)) \cong \hat{\Gamma}_n / \left\langle \partial_{x_i}\hat{\Gamma}_n \right\rangle_{i=1,\ldots,n}. $$

By (10.2) if follows that

$$F_n := \hat{\Gamma}_n / \left\langle \partial_{x_i}\hat{\Gamma}_n \right\rangle_{i=1,\ldots,n}.$$
10.2. We construct bases of the vector spaces $\mathcal{F}_n$ inductively. If we view $f(x_1, \ldots, x_{n+1}) \in \Gamma_{n+1}$ as a function of $x_{n+1}$ with possible poles at $x_1, \ldots, x_n$ and 0 then by Liouville’s theorem we have

$$f(x_1, \ldots, x_{n+1}) = c(x_1, \ldots, x_n) + \sum_{i=0}^{n} \alpha_i(x_1, \ldots, x_n)\zeta(x_{n+1} - x_i)$$

(10.3)

for some collection of functions $c, \alpha_i, \beta_{i,k} \in \Gamma_n$. Here $x_0$ stands for 0, for notational simplicity, and $\psi^{(k)}(z)$ denotes $\frac{1}{k!}\partial_z^k \psi(z)$. In the quotient $\mathcal{F}_n$ we have

$$\beta(x_1, \ldots, x_n)\psi^{(k)}(x_i - x_{n+1}) = -\frac{1}{k}\partial_{x_n+1} \left( \beta(x_1, \ldots, x_n)\psi^{(k-1)}(x_i - x_{n+1}) \right),$$

so in the second summation of (10.3) all terms beyond $k = 0$ can be discarded. In the first summation of (10.3) the condition $\sum_{i=0}^{n} \alpha_i = 0$ must hold by the residue theorem, and so the sum can be replaced by one of the form

$$\sum_{i=0}^{n-1} \gamma_i(x_1, \ldots, x_n)\zeta(x_i, x_{i+1}, x_{n+1}),$$

with a corresponding modification of $c$. Next we note that the difference

$$\beta(x_1, \ldots, x_n)(\psi(x_i - x_{n+1}) - \psi(x_j - x_{n+1})) = \partial_{x_{n+1}} (\beta(x_1, \ldots, x_n)\zeta(x_i, x_j, x_{n+1}))$$

is a total derivative. So we may write

$$f(x_1, \ldots, x_{n+1}) = c(x_1, \ldots, x_n) + \sum_{i=0}^{n-1} \gamma_i(x_1, \ldots, x_n)\zeta(x_i, x_{i+1}, x_{n+1})$$

(10.4)

$$+ \beta(x_1, \ldots, x_n)\psi(x_{n+1}).$$

It is straightforward to show that modification of any of $c, \beta, \alpha_i$ in (10.3) by a total derivative modifies $f$ by a total derivative.

10.3. Lemma. A set of generators of $\mathcal{F}_{n+1}$ is furnished by the functions $c, \beta\psi(x_n - x_{n+1}) \gamma_i(x_1, x_{i+1}, x_{n+1})$, where $c, \beta, \gamma_i, i = 0, \ldots, n - 1$ run over any set of generators of $\mathcal{F}_n$.

10.4. Let $b_n = h^n(\hat{U}^{(n)})$. From Lemma 10.3 it follows that $b_n \leq (n+1)b_{n-1}$. In fact the construction of Section 10.2 may be refined to yield $b_n \leq (n+1)b_{n-1} - b_{n-2}$ in fact.

10.5. Lemma. The classes in $\mathcal{F}_2$ of the five functions

$$1, \zeta(0, x_1, x_2), \phi(x_1), \phi(x_1)\zeta(0, x_1, x_2), \text{ and } \phi(x_1)\phi(x_2)$$

constitute a basis.

Proof. Applying the construction of Lemma 10.3 to the basis $\{1, \phi(x_1)\}$ of $\mathcal{F}_1$ yields the set of functions

$$1, \zeta(0, x_1, x_2), \phi(x_1), \phi(x_1)\zeta(0, x_1, x_2), \phi(x_1)\phi(x_2).$$

We may omit $\phi(x_2)$ since it is equivalent to $\phi(x_1)$ modulo a total derivative. Since $h^2(\hat{U}^{(2)}) = 5$ the remaining functions form a basis of $\mathcal{F}_2$. The lemma follows immediately. □
11. Explicit Differentials in Low Degree

11.1. We now combine the material of Sections 8, 9 and 10 to write more explicitly the complex $A^* = A^*(q) = h(\mathcal{P}^*)$ which computes $H^i_A(X, A)$ for $i = 0, 1$. Recall

$$A^{-n} = \frac{V \otimes n \otimes \hat{\Gamma}_n}{\langle \partial x_i + T^{(j)} \rangle_{i=1, \ldots, n}}$$

for $n = 0, 1, 2$. Explicitly $A^0 \cong V$. For $n = 1, 2$ the relations in $A^{-n}$ may be used to ‘trade’ a total derivative in $\hat{\Gamma}_n$ for a copy of $T$ acting on one of the factors of $V \otimes n$. Hence $A^{-1}$ is a quotient of the vector space $\mathbb{C} \langle [1], [\varphi(x)], [3(x, y, 0)], [\varphi(x)\varphi(y)] \rangle \otimes V \otimes 2$, indeed $A^{-1} \cong (V/TV) \otimes V \otimes (V \otimes V)$, while $A^{-2}$ is a quotient of the vector space

$$\mathbb{C} \langle [1], [\zeta(x, y, 0)], [\varphi(x)], [3(x, y, 0)], [\varphi(x)\varphi(y)] \rangle \otimes V \otimes 3.$$ 

Since the differential $d : A^{-1} \to A^0$ is given by (8.10) it follows that the $0^{th}$ cohomology of $A^*$ is

$$H^0(A^*) = \frac{V}{V(0)V + V(\omega)V} = H(X, 0, V),$$

the space of coinvariants (11.4). In Lemma 11.2 below we use (5.11) to explicitly compute the differential $d : A^{-2} \to A^{-1}$.

11.2. Lemma. We have the following explicit differentials:

(11.1) \[ d (1 \cdot a \otimes b \otimes m) = a \otimes (b(0)m - b(a(0)m), \]

(11.2) \[ d (\varphi(x - y) \cdot a \otimes b \otimes m) = -a(\varphi) b \otimes m \]
\[ + \varphi(x) \sum_{j \in \mathbb{Z}^+} \left( T^{(j)} a \otimes b(j)m - T^{(j)} b \otimes a(j)m \right), \]

(11.3) \[ d (\varphi(x) \cdot a \otimes b \otimes m) = -b \otimes a(\varphi)m \]
\[ + \varphi(x) \right( a \otimes b(0)m - a(0)b \otimes m \right), \]

(11.4) \[ d (-\zeta(x, y, 0)) \cdot a \otimes b \otimes m) = a(\varsigma) b \otimes m - b \otimes a(\varsigma)m - a \otimes b(\varsigma)m \]
\[ + \varphi(x) \int \{a(0)b\} \otimes m \]
\[ - \varphi(x) \sum_{j \in \mathbb{Z}^+} \frac{1}{j + 1} \left( T^{(j)} a \otimes b(j + 1)m + T^{(j)} b \otimes a(j + 1)m \right), \]

(11.5) \[ d (3(x, y, 0)) \cdot a \otimes b \otimes m) = - \varphi(x) \int \{a(\varphi)b\} \otimes m \]
\[ + \varphi(x) \sum_{j \in \mathbb{Z}^+} \frac{1}{j + 1} \left( T^{(j)} a \otimes b(x_j + 1)m + T^{(j)} b \otimes a(x_j + 1)m \right) \]
\[ + (2\pi i)^2 \int \frac{dq}{dq} \left( a(\varsigma)b \otimes m - a \otimes b(\varsigma)m - b \otimes a(\varsigma)m \right), \]

(11.6) \[ d (\varphi(x)\varphi(y) \cdot a \otimes b \otimes m) = \varphi(x) \int \{a(\varphi)b\} \otimes m \]
\[ + \varphi(x) \right( a \otimes b(\varphi)m - b \otimes a(\varphi)m + a(\varphi)b \otimes m - b(\varphi)a \otimes m \]
\[ + (2\pi i)^2 \int \frac{dq}{dq} a(\varphi)b \otimes m. \]
Proof. Let \( f(t) \) be an elliptic function meromorphic with possible pole at \( t = 0 \). Then by (11.1), the differentials of \( f(x) \cdot a \otimes b \otimes m, f(y) \cdot a \otimes b \otimes m \) and \( f(x - y) \cdot a \otimes b \otimes m \) are, respectively,

\[
f(x) a \otimes b(0)m - \sum_{j \in \mathbb{Z}_+} \partial^{(j)} f(x) a(j)b \otimes m - b \otimes a(j)f m,
\]

(11.7)

\[
a \otimes b(f) m - f(x) a(0)b \otimes m - f(x)b \otimes a(0)m,
\]

and

\[
\sum_{j \in \mathbb{Z}_+} (-1)^j \partial^{(j)} f(x) a \otimes b(j)m - a(f) b \otimes m - \sum_{j \in \mathbb{Z}_+} \partial^{(j)} f(-x) b \otimes a(j)m.
\]

From (11.7) we get (11.1) immediately. We also get (11.2) and (11.3) easily, using the relation \( \partial x + T^{(1)} = 0 \) and also using that \( \psi(t) \) is an even function of \( t \).

By (11.7) the differential of \( \zeta(x) - \zeta(y) - \zeta(x - y) \cdot a \otimes b \otimes m \) is

\[
a_{(\zeta)} b \otimes m - b \otimes a_{(\zeta)} m - a \otimes b_{(\zeta)} m + \sum_{j \geq 1} (-1)^j \partial^{(j)} \zeta(x)a \otimes b(j)m - \sum_{j \geq 1} (-1)^j \partial^{(j)} \zeta(x)b \otimes a(j)m,
\]

here we have used that \( \zeta(t) \) is an odd function of \( t \). Using Lemma 2.4 the expression above is reduced to (11.4).

We now compute the differential of \( \mathcal{F}(x, y, 0) a \otimes b \otimes m \) using Lemma 9.5 to make the calculations more comfortable. The first term \( \text{res}_{y = 0} f(x, y) a \otimes b(y)m \) of (8.11) becomes

(11.8)

\[
a \otimes b_{(\psi)} m + \frac{1}{2} a \otimes b_{(\psi)} m + \sum_{j \geq 1} (-1)^j \partial^{(j)} \zeta(x)a \otimes b_{(\psi)} m
\]

The second term \( \text{res}_{y = x} f(y, x) a(y - x)b \otimes m \) becomes

(11.9)

\[
a_{(\psi)} b \otimes m + \frac{1}{2} a_{(\psi)} b \otimes m - \sum_{j \geq 1} \partial^{(j)} \zeta(x)a_{(\psi)} b \otimes m
\]

The third term \( \text{res}_{y = 0} f(y, x) b \otimes a(y)m \) becomes

(11.10)

\[-b \otimes a_{(\psi)} m + \frac{1}{2} b \otimes a_{(\psi)} m + \sum_{j \geq 1} \partial^{(j)} \zeta(-x)b \otimes a_{(\psi)} m.
\]

By Lemma 9.2 equation (11.3) becomes

(11.11)

\[-(2\pi i)^2 q \frac{d}{dq} a_{(\psi)} b \otimes m + \psi(x) \int \{ a_{(\psi)} b \} \otimes m.
\]

Similar reductions are performed on (11.8) and (11.10). Collecting we obtain (11.5).

Finally we compute the image of \( \psi(x) \psi(y) a \otimes b \otimes m \). The first and last terms of (8.11) together yield

\[
\psi(x) \left( a \otimes b_{(\psi)} - b \otimes a_{(\psi)} m \right).
\]
To evaluate the second term of (8.11) we apply Lemma 9.5 as follows:

\[ \varphi(x)\varphi(y) = -\partial_x (\varphi(y)\zeta(x)) \]

\[ = \partial_x \left( \varphi(y) [\zeta(x-y) - \zeta(x) + \zeta(y)] + \frac{1}{2} \varphi'(y) - \frac{1}{2} \varphi'(y) - \varphi(y)\zeta(y) - \varphi(y)\zeta(x-y) \right) \]

\[ = \partial_x \left( \varphi(y) [\zeta(x-y) - \zeta(x) + \zeta(y)] + \frac{1}{2} \varphi'(y) + \varphi(y)\varphi(x-y) \right) \]

\[ = \varphi'(x-y)\zeta(x-y) - \varphi(y)(x-y)^2 + \frac{1}{2} \varphi''(x-y) \]

\[ + \varphi'(x-y) [\zeta(y) - \zeta(x)] + \varphi(x-y) [\varphi(x) + \varphi(y)] . \]

Now the second term of (8.11) becomes

\[ a(x')b \otimes m - a(x')b \otimes m + \frac{1}{2} a(x')b \otimes m + \varphi(x) a(x')b \otimes m \]

\[ + \varphi(x) \sum_{j \in \mathbb{Z}_+} \frac{1}{j+1} (-T(j)) a(x_{j+1})b \otimes m + \varphi(x) \sum_{j \in \mathbb{Z}_+} (-T(j)) a(x_{j+1})b \otimes m. \]

Using Lemma 9.2, equation (2.8) and Lemma 2.4, the expression above reduces to

\[ (2\pi i)^2 q \frac{d}{dq} a(x) b \otimes m + \varphi(x) a(x) b \otimes m + \varphi(x) \int \{ a(x')b \} \otimes m - \varphi(x) h(x) a \otimes m. \]

Collecting we obtain (11.6) as the differential of \( \varphi(x)\varphi(y)a \otimes b \otimes m. \)

\[ \square \]

12. Passage to the \( q = 0 \) Limit

12.1. It turns out that the chiral homologies \( \mathcal{H}_i^{ch}(X, A) (i = 0, 1) \) of the elliptic curve \( X \) simplify somewhat in the limit where \( X \) degenerates to a nodal cubic. To see this we put \( q = 0 \) in the complex \( A^*(q) \) of the previous section. We introduce subgroups \( B^{-n} \subset A^{-n} \) for \( n = 0, 1, 2 \) defined as follows

\[ B^0 = \langle [1] \cdot V_{(0)} V \rangle , \]

(12.1)

\[ B^{-1} = \langle [1] \cdot V_{(0)} V \otimes V + [1] \cdot V \otimes V_{(0)} V + [\varphi(x)] \cdot V \otimes V \rangle , \]

\[ B^{-2} = \langle [\varphi(x)] \cdot V \otimes V \otimes V + [3(x, y, 0)] \cdot V \otimes V \otimes V + [\varphi(x)\varphi(y)] \cdot V \otimes V \otimes V \rangle . \]

12.2. Lemma. The \( B^i \) defined by (12.1) constitute a subcomplex \( B^* \subset A^*(q = 0) \).

Proof. First we demonstrate \( dB^{-1} \subset B^0 \). It is obvious that \( d([\varphi] \cdot V \otimes V) \subset B^0 \). Now

\[ d([1] \cdot a \otimes b_{(0)}) m = a(0)(b_{(0)}m) = a(0)(\text{res}_z \varphi(z)b(z)m) \]

\[ = \text{res}_z \varphi(z) [(a(0), b(z))m + b(z)a(0)m] \]

\[ = (a(0)b_{(0)})m + b_{(0)}(a(0)m) \in B^0 \]
(using here that \([a(0), b(z)] = [a(0)b(z)]\). On the other hand
\[
d([1] \cdot a(\varphi) b \otimes m) = (a(\varphi)b)(0)m = \text{res}_x \text{res}_w \varphi(x)[a(x)b(w)m
\]
\[
= \text{res}_x \text{res}_z \text{res}_w \varphi(x) (a(z)b(w)mi_{z,w} - b(w)a(z)mi_{z,w}) \delta(x, z - w)
\]
\[
= \text{res}_z \text{res}_w \sum_{j \in \mathbb{Z}_+} \left( (-1)^j (a(z)b(w)m(-w)^j \partial^{(j)} \varphi(z) - b(w)a(z)zm^j \partial^{(j)} \varphi(-w)) \right)
\]
\[
= \sum_{j \in \mathbb{Z}_+} (-1)^j (a(\varphi^{(j)})(b(j)m) - b(\varphi^{(j)})a(j)m)) .
\]
Since \( (Ta)(f)b = -a(\partial f)b \) we see that \( d([1] \cdot a(\varphi) b \otimes m) \in B^0 \). So far we have not actually used
the condition \( q = 0 \).

The fact that \( dB^{-2} \subset B^{-1} \) is a consequence of the explicit formulas of Lemma 11.2. More
precisely the terms of the form \( \frac{\partial f}{\partial x} (\cdots) \) vanish at \( q = 0 \), and all remaining terms manifestly
lie within \( B^{-1} \). \( \square \)

12.3. Now let us put \( Q^\bullet = A^\bullet / B^\bullet \) and consider the long exact sequence in
cohomology associated
with \( 0 \to B^\bullet \to A^\bullet \to Q^\bullet \to 0 \), namely
\[
\xymatrix{ H^0(B^\bullet) & H^0(A^\bullet) \ar[ll]_{d: B^\bullet \to A^\bullet} \ar[l] \ar[r] & H^0(Q^\bullet) \ar[r] & 0 }
\]
\[ (12.2) \]
\[
\xymatrix{ H^{-1}(B^\bullet) & H^{-1}(A^\bullet) \ar[ll]_{d: B^\bullet \to A^\bullet} \ar[l] \ar[r] & H^{-1}(Q^\bullet) \ar[r] & \cdots \ar[r] & H^{-2}(Q^\bullet) \ar[r] & \cdots } 
\]
Clearly \( d : B^{-1} \to B^0 \) is surjective. Therefore we have an isomorphism
\[
H^0(A^\bullet) \to H^0(Q^\bullet)
\]
and a surjection
\[
H^{-1}(A^\bullet) \to H^{-1}(Q^\bullet)
\]
whose kernel is a quotient of \( H^{-1}(B^\bullet) \). In the next section we compute \( H^{-1}(B^\bullet) \) using the
spectral sequence associated with a filtration.

12.4. Comparing the Laurent series \( f \) and \( g \) of \( (2.10) \) with the specialisations of the Weierstrass
functions \( \zeta \) and \( \varphi \) and at \( q = 0 \), we see that
\[
\zeta(z, q = 0) = f(z) - \pi i \quad \text{and} \quad \varphi(z, q = 0) = g(z).
\]
By the second of these relations we have
\[
Q^0 = A^0 / B^0 \cong V / V(g)V = \text{Zhu}(V)
\]
and
\[
Q^{-1} = A^{-1} / B^{-1} \cong \frac{V \otimes V}{\left[V(g)V \otimes V + V \otimes V(g)V\right]} \cong \text{Zhu}(V) \otimes \text{Zhu}(V).
\]
as vector spaces. The morphism \( Q^{-1} \to Q^0 \) is given by
\[
a \otimes b \mapsto a(0)b = [a, b] = a \ast b - b \ast a,
\]
This morphism coincides with the differential of the Bar resolution of Zhu$(V)$. It follows that
\[ H^0(Q^\bullet) \cong HH_0(Zhu(V)). \]
On the other hand $H^{-1}(Q^\bullet)$ is the quotient of $Q^{-1}$ by the images of $[1] \cdot V \otimes V \otimes V$ and $[\cdot] \cdot V \otimes V \otimes V$. According to \[2, Proposition 2.6.1\] one has
\[ a \otimes b(0)c - a(0)b \otimes c - b \otimes a(0)c = a \otimes [b, c] - [a, b] \otimes c - b \otimes [a, c], \]
and according to \[11.3\] the image of $[\cdot] \cdot a \otimes b \otimes c$ is
\[ a_{(\cdot)}b \otimes c - b \otimes a_{(\cdot)c} - a \otimes b_{(\cdot)c}. \]
By the first relation of \[12.3\] the expression \[12.5\] reduces to
\[ a \cdot b \otimes c - b \otimes a \cdot c - a \otimes b \cdot c - \pi i \{[a, b] \otimes c - b \otimes [a, c] - a \otimes [b, c]\}. \]
Clearly the terms \[12.4\] are contained within the span of the terms \[12.6\], so we have proved
\[ H^{-1}(Q^\bullet) \cong HH_1(Zhu(V)). \]

13. Li’s Filtrations

13.1. Let $V$ be a vertex algebra. The Li filtration \[34\] $F^\bullet V$ is the decreasing filtration on $V$ defined by putting $F^p V$ to be the span of the vectors
\[ a^1(-n_1-1) \cdots a^r(-n_r-1)b, \]
where $r \geq 0$, $a^1, \ldots, a^r, b \in V$, and $n_1, \ldots, n_r \in \mathbb{Z}_+$, such that $\sum_j n_j \geq p$.

The product $a \cdot b = a(-1)b$ induces a commutative associative algebra structure on the associated graded $gr^F V$ with 1 as unit. Following \[2\] the singular support of $V$ is by definition the scheme
\[ SS(V) = \text{Spec} (gr^F V). \]
The translation operator $T$ turns $gr^F V$ into a differential algebra, so $SS(V)$ comes equipped with a vector field.

Clearly $gr^F_0 V$ coincides with Zhu’s $C_2$-algebra $R_V$ of Section 2.6. By the universal property of the arc space (see Section 12 below) there is a canonical morphism $JR_V \to gr^F V$ of differential algebras. Lemma 4.2 of \[34\] implies that this morphism is surjective. Geometrically this says that $SS(V)$ is naturally a subscheme of the arc space $JX_V$ of the associated variety $X_V = \text{Spec} R_V$.

The arc space of a Poisson algebra carries a canonical Poisson vertex algebra structure and the inclusion is compatible with this structure.

13.2. Now suppose that $V$ is (quasi)conformal. In \[34\] Li introduced a second filtration on $V$. Let $\{a^1[i] \in I\}$ be a strong generating set of $V$. The standard filtration $G^\bullet V$, relative to the choice of strong generating set, is the increasing filtration on $V$ defined by putting $G^p V$ to be the span of the vectors
\[ a^{n_1}(-n_1) \cdots a^n(-n_r)1 \]
where $r \geq 0$, $a^1, \ldots, a^r \in V$, and $n_1, \ldots, n_r \in \mathbb{Z}_{>0}$, such that $\sum_j (a^j) \leq p$.

For a (quasi)conformal vertex algebra both the Li filtration and the standard filtration are compatible with the conformal weight grading. Let $F^p V_\Delta = V_\Delta \cap F^p V$ and $G^p V_\Delta = V_\Delta \cap G^p V$. According to \[2, Proposition 2.6.1\] one has
\[ F^p V_\Delta = G_{\Delta-p} V_\Delta \]
for all $\Delta, p \in \mathbb{Z}$. The equalities \[13.1\] induce an isomorphism
\[ gr^G V \to gr^F V \]
of commutative differential algebras. (even of Poisson vertex algebras). Although both sides of (13.2) are naturally \(\mathbb{Z}_+\)-graded, the isomorphism does not respect the grading.

13.3. From equation (11.2) we see that any element of \([1] \cdot V(\varphi) V \otimes V\) is equal, modulo a boundary, to an element of \([\varphi] \cdot V \otimes V\). Similarly from equation (11.3) we see that any element of \([1] \cdot V \otimes V(\varphi) V\) is equal, modulo a boundary, to an element of \([\varphi] \cdot V \otimes V\). Thus it suffices to analyse the kernel of the restriction of \(d : B^{-1} \rightarrow B^0\) to \([\varphi] \cdot V \otimes V\).

We extend the standard filtration to a filtration on \(B^\bullet\) by putting

\[
G_p B^k = \bigoplus_{p_i \leq p} \left[ f \right] \cdot G_{p_1} V \otimes \cdots \otimes G_{p_k} V
\]

In \(\text{gr}^G V\) all terms of

\[a(\varphi) b = a(-2)b + G_2 \cdot a(0)b + 3G_4 \cdot a(2)b + \cdots\]

beyond the first vanish, and we have the map \(\text{gr}^G B^{-1} \rightarrow \text{gr}^G B^0\) explicitly as

\[\left[ \varphi \right] \cdot a \otimes b \mapsto [1] \cdot a(-2)b.\]

The image of \([3] \cdot a \otimes b \otimes m\), which at \(q = 0\) is

\[-[\varphi] \cdot \int \left\{ a(\varphi) b \right\} \otimes m - [\varphi] \cdot \sum_{j \in \mathbb{Z}_+} \frac{1}{j+1} \left( T(j) a \otimes b (z^{j+1} \varphi) m + T(j) b \otimes a (z^{j+1} \varphi) m \right),\]

similarly becomes

\[-[\varphi] \cdot (a(-1)b \otimes m + a \otimes b(-1)m + b \otimes a(-1)m).\]

in \(\text{gr}^G B^{-1}\).

Finally the image of \(\varphi(x) \varphi(y) a \otimes b \otimes m\) at \(q = 0\) is

\[[\varphi] \cdot \int \left\{ a(\varphi') b \right\} \otimes m + [\varphi] \cdot \left( a \otimes b(y) m - b \otimes a(\varphi) m + a(\varphi') b \otimes m - b(\varphi) a \otimes m \right).\]

In \(\text{gr}^G B^{-1}\) the integral term becomes

\[[\varphi] \cdot \left( a(\varphi') b - \frac{1}{2} T(a(z^{1} \varphi') b) \right) \otimes m \equiv [\varphi] \cdot \left[ -2a(-2)b + T(a(-1)b) \right] \otimes m,
\]

while the latter term becomes

\[[\varphi] \cdot \left( a \otimes b(-2)m - b \otimes a(-2)m + [a(-2)b - b(-2)a] \otimes m \right) \equiv [\varphi] \cdot \left( a \otimes b(-2)m - b \otimes a(-2)m + [2a(-2)b - T(a(-1)b)] \otimes m \right),
\]

so that the total is

\[[\varphi] \cdot \left( a \otimes b(-2)m - b \otimes a(-2)m \right).\]

Thus \(H^{-1}(B^\bullet)\) is the kernel of the map \(V \otimes V \rightarrow V\) defined by

(13.3) \[a \otimes b \mapsto (Ta) \cdot b,\]

modulo the subspace generated by the elements

(13.4) \[a \cdot b \otimes m - a \otimes b \cdot m - b \otimes a \cdot m,\]

(13.5) \[a \otimes (Tb) \cdot m - b \otimes (Ta) \cdot m.\]
13.4. Associated with the increasing filtration $G_pB^\bullet$ is the spectral sequence in which the first page is $E^{p,q}_1 = H^{p+q}(gr^{-p}B^\bullet)$ and the differential of bidegree $(+1,0)$ is induced by the connecting morphisms of

$$0 \to G_{p-1}V \to G_pV \to G_pV \to 0.$$ 

Since the filtration is exhaustive and bounded below, we have convergence

$$(13.6) \quad E^{p,q}_1 \Rightarrow H^{p+q}(B^\bullet).$$

14. Kähler Differentials and the Arc Space

14.1. Let $A$ be a commutative algebra over the field $k$. The space $\Omega_{A/k}$ of Kähler differentials is the free $A$-module generated by symbols $df$ for $f \in A$, modulo the relations $d\alpha = 0$ for $\alpha \in k$, $d(f + g) = df + dg$ and $d(fg) = f \cdot dg + g \cdot df$. It is well known that

$$\text{HH}_0(A) = A \quad \text{and} \quad \text{HH}_1(A) \cong \Omega_{A/k},$$

where the latter isomorphism is induced by $a \otimes b \mapsto a \cdot db$. The action $a(f \cdot dg) = (af) \cdot dg$ furnishes $\Omega_{A/k}$ with the structure of an $A$-module. There is an isomorphism

$$(14.1) \quad \text{Hom}_A(\Omega_{A/k}, A) \cong \text{Der}_k(A),$$

in which the homomorphism $\iota_\tau$ associated with the derivation $\tau$ is defined by $\iota_\tau(f \cdot dg) = f\tau(g)$.

Let us fix $\tau \in \text{Der}_k(A)$ and consider the exterior algebra

$$(14.2) \quad K^A = \text{Sym}_A(\Omega_{A/k}[1])$$

of the $A$-module $\Omega_{A/k}$. The morphism $\iota_\tau : \Omega_{A/k} \to A$ of $A$-modules uniquely extends, much as in Section 5.2, to a differential on $K^A$, which we also denote $\iota_\tau$.

14.2. Now we pass to the general case of $A = \bigoplus_{n \in \mathbb{Z}_+} A^n$ be a $\mathbb{Z}_+$-graded commutative algebra, and we suppose the derivation $\tau$ to be of degree $+1$. The $A$-module $\Omega_{A/k}$ acquires a natural $\mathbb{Z}_+$-grading by declaring $\deg(a \cdot db) := \deg(a) + \deg(b) + 1$. This grading extends to $K^\bullet := K_A^A$ and is preserved by $\iota_\tau$. Therefore $K^\bullet$ becomes a $\mathbb{Z}_{\geq 0}$-graded complex of $A$-modules and its homology $K^\bullet(\iota_\tau)$ becomes a $\mathbb{Z}_{\geq 0}$-graded $A$-module. We use the lower index to denote homological degree and the upper index to denote the additional $\mathbb{Z}_{\geq 0}$-grading. We are interested in the vanishing of the homology in strictly negative homological degree.

The degree 0 component of $K^\bullet$ is easily seen to be $A^0[0]$, from which it follows that

$$H^0(K^\bullet, \iota_\tau) = A^0[0]$$

as well. Consequently, for any $\mathbb{Z}_+$-graded algebra $A$, the degree 0 component of the homology of $K^\bullet$ vanishes in negative homological degree.

14.3. The canonical morphism $A \to \Omega_{A/k}$, sending $a$ to $da$, extends uniquely to a derivation of $K^\bullet$, known as the de Rham differential and denoted $d$. The Lie derivative $\text{Lie}_\tau = [d, \iota_\tau]$ (given explicitly in formula (13.1) above) enhances $K^\bullet$ to a complex of differential $A$-modules. With respect to the $\mathbb{Z}_{\geq 0}$ grading $\iota_\tau$, $d$ and $\text{Lie}_\tau$ are homogeneous of degree $0$, $+1$ and $+1$ respectively.

14.4. Let us assume from now on that each component $A^n$ of $A$ is of finite type as an $A^0$-module. Then each subcomplex $K^\bullet_n$ is a complex of $A^n$-modules of finite type (as well as being concentrated in cohomological degrees $-n, \ldots, 0$). We also assume from now on that $(A, \iota)$ is generated as a differential algebra by $A^0$. It follows easily that $(K, \text{Lie}_\tau)$ is generated as a differential algebra
by its subalgebra $\tilde{A} \cong \text{Sym}_{A^0}(\Omega_{A^0/k}[1])$ of total degree 0. Writing $A^+ = \bigoplus_{j > 0} A^j$, it is clear that

\begin{equation}
H_0(K_\bullet) = A/A^+ \cong A^0.
\end{equation}

14.5. Let $X = \text{Spec } A^0$ and $JX = \text{Spec } JA^0$ be its arc space. The algebra $JA^0$ carries a canonical derivation $\partial$ which is universal in the sense that for any algebra morphism $A^0 \to R$ to an algebra $R$ endowed with a derivation $\tau$, there exists a unique morphism $(JA^0, \partial) \to (R, \tau)$ of differential algebras such that the diagram

\begin{equation}
\begin{array}{ccc}
JA^0 & \longrightarrow & R \\
\downarrow & & \downarrow \\
A^0 & \longrightarrow & R
\end{array}
\end{equation}

commutes. In fact $JA^0$ is freely generated by $A^0$ as a differential algebra, and is naturally $\mathbb{Z}_+$-graded with $\partial$ of degree +1. Therefore we have a morphism $JA^0 \to A$ of graded commutative unital differential algebras which, by our assumption on $A$, is surjective. We also have an obvious surjection

$$(J\tilde{A}, \partial) \to (K, \text{Lie}_\tau)$$

of differential algebras.

The superscheme $\tilde{X} = \text{Spec } \tilde{A}$ is the shifted tangent bundle $T[-1]X$ of $X$, and its arc space $J\tilde{X}$ is the shifted tangent bundle $JT[-1]X = \text{Spec } \tilde{A} = T[-1]JX$ of the arc space of $X$. Let us write $Y = \text{Spec } A$, then by assumption we have an embedding $Y \hookrightarrow JX$, and therefore

$$T[-1]Y = \text{Spec } K \hookrightarrow T[-1]JX = JT[-1]X.$$

14.6. **Theorem.** Let $A = \bigoplus_{n \in \mathbb{Z}_+^n} A^n$ be a $\mathbb{Z}_+$-graded commutative algebra with a derivation $\tau$ of degree +1, and let $(K^A_\bullet, \iota_\tau)$ be the Koszul complex associated with $A$ as above. We assume $A$ is generated by $A^0$ as a differential algebra, and that $A^0$ is an algebra of finite type. Then $H_{-1}(K^A_\bullet, \iota_\tau) = 0$ if and only if $A \cong JA^0$.

**Proof.** We first prove the implication ($\Leftarrow$). We begin by considering the case $A^0 = k[x^1, \ldots, x^n]$ and $A = JA^0 = k[x^j_{i,j=1} = x_{n,j} \in \mathbb{Z}_+$ graded by deg $x^j_i = j$ and equipped with the differential $\partial$ defined by $\partial(x^j_i) = x^j_{i+1}$. Lexicographically ordering the generators $x^j_i$ by increasing $j$ yields a regular sequence in $A$, and the complex $K^A_\bullet$ is the Koszul complex associated with this regular sequence. Hence $H_n(K^A_\bullet, \iota_\tau) = 0$ for $n \neq 0$.

Now let $A^0$ be arbitrary of finite type, let $A = JA^0$, and suppose $H_{-1}(K^A_\bullet, \iota_\tau) = 0$. We now put $B^0 = A^0/(f)$ where $(f)$ is the ideal generated by some nonzero element $f$. We shall prove that $H_{-1}(K^B_\bullet, \iota_\tau) = 0$. Clearly the statement of the theorem follows since any algebra of finite type is a quotient of a polynomial algebra, and polynomial algebras are Noetherian.

We have $B = JB^0 \cong A/(\partial f) f\in \mathbb{Z}_+$. Now let $\omega \in (K^B_\bullet)^j_{-1}$ be a nonzero cycle of degree $j$ (since it is nonzero we have $j > 0$), which we write in the form

$$\omega = \sum a_i \cdot db_i.$$ 

We choose a representative $\omega = \sum a_i \cdot db_i$ of $\omega$ in $(K^A_\bullet)^j_{-1}$ and obtain

$$\iota_\tau \omega = \sum a_i \partial db_i = \sum_{k=0}^j c_k \partial^k f,$$
for some collection of elements \( c_k \in A^j \). The form \( \omega = \sum_{k=1}^j c_k \cdot d(\partial f) \) is also a representative of \( \bar{\omega} \) in \((K^A)_{-1}^j\) now with the property that
\[
\iota_\partial \omega = c_0 f
\]
where \( \deg(c_0) = j > 0 \). We may write
\[
c_0 = \sum d_i \partial e_i,
\]
for some collection of \( d_i, e_i \in A \) such that \( \deg(d_i) + \deg(e_i) = j - 1 \). It follows that
\[
\iota_\partial \left( \omega - \sum_{k=1}^j c_k \cdot d(\partial f) - \sum_i d_i \cdot \partial e_i \right) = 0.
\]
Since by assumption \( H_{-1}(K^A, \iota_\partial) = 0 \) the form inside the parentheses is exact, hence there exists \( \omega^2 \in (K^A)_{-2} \) such that
\[
\iota_\partial \omega^2 = \omega - \sum_{k=1}^j c_k \cdot d(\partial f) - \sum_i d_i \cdot \partial e_i,
\]
and consequently the image \( \bar{\omega}^2 \in (K^B)_{-1} \) of \( \omega^2 \) satisfies
\[
\iota_\partial \bar{\omega}^2 = \bar{\omega},
\]
proving that \( H_{-1}(K^B, \iota_\partial) = 0 \) as required.

Now we prove the implication \((\Rightarrow)\). By the universal property of \( JA^0 \) we have \( A = JA^0/I \) where \( I \) is a homogeneous differential ideal. Let \( K_* = K_*^A \) and \( \bar{K}_* = K_*^{JA^0} \), so that we have a surjective morphism of complexes \( \bar{K} \to K \). Let \( f \in I \) be non-zero and homogeneous of minimal possible degree. Since \((JA^0)^0 = A^0 \) it follows that \( I^0 = 0 \) and so \( j = \deg(f) > 0 \).

It follows from the minimal degree condition on \( f \) that it cannot be expressed as a linear combination of terms of the form \( a \partial b \) where either \( a \in I \) or \( b \in I \) since \( \deg(a \partial b) \) is strictly greater than both \( \deg(a) \) and \( \deg(b) \).

Since \( H_0(\bar{K}_j, \iota_\partial) = 0 \) (see (14.5)) there exists \( \omega \in \bar{K}_j^j \) such that \( \iota_\partial \omega = f \). Let \( \bar{\omega} \) be the projection of \( \omega \) to \( K_j^j \), then we have \( \iota_\tau \bar{\omega} = 0 \). By assumption \( H_{-1}(K_*, \iota_\tau) = 0 \), so \( \bar{\omega} \) is exact, and so we choose \( \omega \in K_j^j \) such that \( \iota_\tau \omega = \bar{\omega} \). Now we let \( \omega^2 \) be a preimage of \( \bar{\omega}^2 \) in \( \bar{K}_j^j \). It follows that
\[
\iota_\partial \omega^2 = \omega + \sum a_i \cdot \partial b_i
\]
where, for each \( i \), either \( a_i \in I \) or \( b_i \in I \). On the other hand \( \deg(a_i) + \deg(b_i) = j - 1 \), and applying \( \iota_\partial \) once more we obtain
\[
f = - \sum a_i \partial b_i,
\]
which contradicts our hypothesis on \( f \). Therefore \( I = 0 \) and so \( A = JA^0 \).

15. **Main Theorem**

15.1. We now return to the setting of Section 13. We use the isomorphism \((13.2)\) to rewrite the complex \( \text{gr} B^* \) in terms of \( A \) and \( \iota_T \). The quotient of \( A \oplus A \) by the terms \((13.4)\) is \( HH_1(A) \cong \Omega_{A/C} \).

The map \((13.3)\) is precisely \( \iota_T \), and the elements \((13.5)\) span the image of \( \iota_T \). Thus we have
\[
H^{-1}(\text{gr}^C B^*) \cong H_{-1}(K_*^A, \iota_T).
\]
15.2. Theorem. Let $V$ be a quasiconformal vertex algebra and let $A^\bullet(q)$ denote the complex defined in Section 14. Let $A = \bigoplus_{n \in \mathbb{Z}_+} A^n$ denote the associated graded $\text{gr}^F V$ of $V$ with respect to the Li filtration. For $q \neq 0$ we have
$$H^{-1}(A^\bullet(q)) \cong H_i^{0}(X_q, A_V),$$
and if the canonical morphism $JA^0 \to A$ is an isomorphism then we also have
$$H^{-1}(A^\bullet(q = 0)) \cong HH_1(\text{Zhu}(V)).$$

Proof. The first part was proved in Section 13.3 To prove the second part we let $K_\bullet = K\bullet^A$ be the complex (14.2) with differential $\iota_T$. By Theorem 14.6 the condition $A \cong JA^0$ guarantees (indeed is equivalent to)
$$H_{-1}(K\bullet^A, \iota_T) = 0.$$ Thus we have $H^{-1}(\text{gr}^G B^\bullet) = 0$. Now by convergence of the spectral sequence (13.6) we infer $H^{-1}(B^\bullet) = 0$, and in turn by the long exact sequence (12.2) we have
$$H^{-1}(A^\bullet(q = 0)) \cong H^{-1}(Q^\bullet).$$ Combining this with the isomorphism (12.7) yields the result. \hfill \square

15.3. Remark. In Section 10 below we show that the condition appearing in Theorem 15.2, namely that $JA^0 \to A$ be an isomorphism, is satisfied for many well known vertex algebras. It might be possible to strengthen Theorem 15.2 by removing or weakening this condition by way of an analysis not only of the $E^1$ page of the spectral sequence (13.6) but also of its $E^2$ page. The Poisson vertex algebra structure of $\text{gr}^F V$ is invisible on the $E^1$ page but in principle appears in the structure of the $E^2$ page. It would be very interesting to see if the theory of Poisson homology 11 or Poisson vertex algebra homology 15 plays some role in this connection. In [4] the Poisson homology of $A^0$ is indeed seen to be related to the representation theory of $V$.

15.4. With the notation as in Theorem 15.2 to any quasi-conformal vertex algebra we have associated a Poisson algebra $A^0 = A/(A \cdot TA)$. Let $I$ be the kernel of the surjection $JA^0 \to A$. Then it is straightforward to see that $I/(I \cdot TI)$ is a non-unital Poisson algebra. Moreover, it is a Poisson module over $A^0$. Namely, if we let $J = A^0 \oplus I/(I \cdot TI)$, $X = \text{Spec} A^0$, $Y = \text{Spec} J$, we have two Poisson affine schemes $X$ and $Y$ canonically associated to any quasi-conformal vertex algebra, together with a Poisson map $Y \to X$ and a section $X \hookrightarrow Y$. The condition in the theorem is that these two Poisson schemes are equal.

As in the previous remark, it would be interesting to see if the second page of the spectral sequence can be described in terms of the Poisson homologies of these Poisson algebras.

16. Examples

16.1. We consider the highest weight Vir-module
$$\text{Vir}^c = U(\text{Vir}) \otimes_{U(\mathfrak{w}+c)} \mathbb{C}v$$
in which $L_{n}v = 0$ for $n \geq -1$ and $C$ acts on $v$ by the constant $c$. This module carries the natural structure of a conformal vertex algebra of central charge $c$ in which the quantum field associated with $L_{-2}1$ is $L(z) = \sum L_{n}z^{-n-2}$. It follows from (13.1) that $F^p\text{Vir}^c_\Delta$ is the span of the monomials $L_{-n_1-2}\cdots L_{-n_p-2}1$ for which $\sum n_i = p$ and $\sum(n_i+2) = \Delta$. Hence as differential commutative algebras
$$\text{gr}^F \text{Vir}^c = \mathbb{C}[L_{-2}, L_{-3}, \ldots],$$
where $\text{deg}(L_{-n-2}) = n$ and $TL_{-n} = nL_{-n-1}$. In particular $R_{\text{Vir}} \cong \mathbb{C}[x]$ where $x = [L_{-2}]$, and the natural surjection $JR_{\text{Vir}} \to \text{gr}^F \text{Vir}^c$ is an isomorphism.
16.2. Let \( \mathfrak{g} \) be a finite dimensional Lie algebra over \( \mathbb{C} \) with invariant bilinear form \( \langle \cdot, \cdot \rangle \), and let \( \widehat{\mathfrak{g}} = \mathfrak{g}(t) \oplus \mathbb{C}K \) be the associated affine Lie algebra (the affine Kac-Moody algebra in case \( \mathfrak{g} \) is simple, the Heisenberg Lie algebra in case \( \mathfrak{g} \) is abelian)

\[
[at^m, bt^n] = [a, b]t^{m+n} + m(a, b)\delta_{m,-n}K.
\]

We consider the vacuum module

\[
V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes U(\mathfrak{g}[t] + \mathbb{C}K) \mathbb{C}v
\]
in which \( \mathfrak{g}[t] \) acts trivially on \( v \) and \( K \) acts on \( v \) by the constant \( k \). This module carries the natural structure of a quasiconformal vertex algebra. If \( k \neq -h^\vee \), where \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \), then \( V^k(\mathfrak{g}) \) is furthermore conformal of central charge \( c = \frac{k(h^2)}{k + h^2} \). The quantum field associated with \( a_{-1}1 \) is \( a(z) = \sum a_n z^{-n-1} \), where we have written \( a_n = at^n \).

As above we see that \( F^pV^k(\mathfrak{g})_{\Delta} \) is the span of the monomials \( a_{n_1-1}^1 \cdots a_{n_{-1}}^s1 \) for which \( \sum n_i = p \) and \( \sum (n_i+1) = \Delta \). Hence as differential commutative algebras

\[
gr^F V^k(\mathfrak{g}) = S(t^{-1}g[t^{-1}]),
\]

where \( \text{deg}(at^n) = n \) and \( T(at^{-n}) = nat^{-n-1} \). In particular \( R_{V^k(\mathfrak{g})} \cong S(\mathfrak{g}) \), and the natural surjection \( JR_{V^k(\mathfrak{g})} \rightarrow gr^F V^k(\mathfrak{g}) \) is an isomorphism.

16.3. Let \( \mathfrak{g} \) be a simple Lie algebra and \( f \in \mathfrak{g} \) a nonzero nilpotent element. We denote by \( S \) the associated Slodowy slice, defined by embedding \( f \) into an \( sl_2 \) triple \( \{e, h, f\} \subset \mathfrak{g} \) and putting \( S = \{f + x|x, e| = 0\} \subset \mathfrak{g} \). The universal affine \( W \)-algebra is constructed as a quantised Drinfeld-Sokolov reduction of \( V^k(\mathfrak{g}) \). See [29] for the construction. It was proved in [13] that \( R_{W^k(\mathfrak{g}, f)} \cong \mathbb{C}[S] \), and it was proved in [3] Theorem 4.17 that \( gr^F W^k(\mathfrak{g}, f) \cong \mathbb{C}[S] \). Thus the surjection \( JR_{W^k(\mathfrak{g}, f)} \rightarrow gr^F W^k(\mathfrak{g}, f) \) is an isomorphism (at arbitrary level \( k \): critical or non-critical).

16.4. Now we consider the Virasoro minimal models. Let \( p, p' \geq 2 \) be two coprime integers, and let

\[
c = c_{p, p'} = 1 - \frac{(p - p')^2}{pp'}.
\]

It is well known that the simple quotient \( Vir_{p, p'} \) of \( Vir^c \) is a rational vertex algebra [44] (see also [16]) known as a minimal model. Its (unnormalised) character is given by the formula [19] [31]

\[
(1.1) \quad \chi_{Vir_{p, p'}}(q) = \frac{1}{\prod_{n=1}^{\infty}(1 - q^n)} \sum_{n \in \mathbb{Z}} \left[ q^{\frac{(2pp'p' + p - p')^2}{4pp'} - \frac{(2pp'p' - p - p')^2}{4pp'}} - q^{\frac{(2pp'p'p' + p - p')^2}{4pp'} - \frac{(2pp'p'p' - p - p')^2}{4pp'}} \right].
\]

The maximal ideal \( I_{p, p'} \subset Vir^c \) is generated by a singular vector \( v_{p, p'} \) of conformal weight \( (p - 1)(p' - 1) \). In general \( v_{p, p'} \) is a linear combination of monomials \( L_{-1}^{n_1} \cdots L_{-n}^{s}1 \) of total degree \( (p - 1)(p' - 1) \), and the only one of these to survive in the quotient \( R_{Vir^c} \) is \( L_{-2}^{(p - 1)(p' - 1)/2}1 \). The coefficient of this latter monomial is nonzero [20] [43] Lemma 4.3]. The algebra \( R_{Vir_{p, p'}} \) is obtained as the quotient of \( R_{Vir^c} \cong \mathbb{C}[x] \) by the ideal generated by the image of \( v_{p, p'} \), therefore \( R_{Vir_{p, p'}} \cong \mathbb{C}[x]/(x^{(p - 1)(p' - 1)/2}) \).

16.5. Before proceeding we recall some standard material on monomial orders, Hilbert series and Gröbner bases [13], particularly in connection with arc spaces. Let \( k \) be a field of characteristic 0 and let \( R \) be the graded \( k \)-algebra of polynomials on a (finite or infinite) countable totally ordered set of variables

\[
y_0 > y_1 > y_2 > \ldots
\]
whose graded degrees are compatible with the order in the sense that \( \deg(y_{i+1}) \geq \deg(y_i) \) for all \( i \). The graded reverse lexicographic (grevlex) order on the set of monomials of \( R \) is defined as follows. We put

\[
\prod_{i} y_i^{\alpha_i} > \prod_{i} y_i^{\beta_i}
\]

if \( \sum \alpha_i \deg(y_i) > \sum \beta_i \deg(y_i) \), or else if these two sums are equal and the product \( \prod y_i^{\alpha_i} \) comes later than \( \prod y_i^{\beta_i} \) in the lexicographic ordering. For example if \( \deg(y_i) = 1 \) for \( i = 0, 1, 2 \) then \( y_i^2 > y_0y_2 \). We write \( \text{LT}(f) \) for the leading term of the polynomial \( f \) (i.e., its highest monomial in the grevlex order). The leading term ideal \( \text{LT}(I) \) of \( I \) is the ideal of \( R \) generated by \( \text{LT}(f) \) as \( f \) runs over \( I \).

From now on we assume \( \deg(x_0) > 0 \) and that \( I \subset R \) is a homogeneous ideal. Hence \((R/I)_n\) is well defined and finite dimensional. The Hilbert series of \( I \) is

\[
H_I(q) = \sum_{n=0}^{\infty} \dim (R/I)_n q^n.
\]

Clearly if \( I_1 \subset I_2 \) then we have \( \dim(R/I_2)_n \leq \dim(R/I_1)_n \) for all \( n \). Generally for two power series \( f(q) \) and \( g(q) \) we shall write \( f(q) \leq g(q) \) if \( f_n \leq g_n \) for all \( n \in \mathbb{Z}_+ \). Thus \( H_{I_2}(q) \leq H_{I_1}(q) \).

16.6. Proposition. The Hilbert series of \( I \) and \( \text{LT}(I) \) coincide.

16.7. Let \( f \in R \) and \( g_1, g_2, \ldots, g_s \in R \). We say that “\( f \) reduces to zero modulo \( \{g_i\} \)” if there exist \( a_1, a_2, \ldots, a_s \in R \) such that

- \( f - \sum a_i g_i = 0 \),
- \( \text{LT}(a_i g_i) \leq \text{LT}(f) \) for all \( i \in \{1, 2, \ldots, s\} \).

Let \( f, g \in R \) and let \( H \) be the least common multiple of their leading terms. By definition the \( S \)-polynomial of \( f \) and \( g \) is

\[
S(f, g) = \frac{H}{\text{LT}(f)} \cdot f - \frac{H}{\text{LT}(g)} \cdot g.
\]

Let \( I \subset R \) be an ideal, and \( \{g_i\} \) a countable set of generators of \( I \). Then \( \{g_i\} \) is said to be a Gröbner basis of \( I \) if \( S(g_j, g_k) \) reduces to zero modulo \( \{g_i\} \) for any \( j, k \).

16.8. Lemma. If \( \text{LT}(f) \) and \( \text{LT}(g) \) are relatively prime then \( S(f, g) \) reduces to zero modulo \( \{f, g\} \).

16.9. Proposition. If \( \{g_i\} \) is a Gröbner basis of \( I \) then \( \text{LT}(I) \) is in fact generated by the set of monomials \( \{\text{LT}(g_i)\} \).

16.10. Now let \( R \) denote the arc space

\[
Jk[x] = k[x_0, x_1, x_2, \ldots]
\]

(here \( x_n \) represents the \( n \)th derivative of \( x_0 = x \)). We equip the set of variables with the total order

\[
(x_0 > x_1 > x_2 > \ldots),
\]

and we assign \( \deg(x_i) = i + 2 \). We fix \( s \geq 2 \) and denote by \( I \) the ideal generated by \( x_0^s \) and all its derivatives. Thus \( R/I \cong J(k[x]/(x^s)) \). For clarity we write \( x(t) = \sum_{n=0}^{\infty} x_n t^n \) and define the polynomials \( [x^s]_m \) by

\[
x(t)^s = \sum_{m=0}^{\infty} [x^s]_m t^m.
\]

16.11. Proposition. Let \( R = Jk[x] \) equipped with the grevlex monomial order, and let \( I \) be the ideal defined above. Then the set \( \{[x^s]_m\}_{m \in \mathbb{Z}_+} \) is a Gröbner basis of \( I \).
16.12. This Gröbner basis may be used to write down the Hilbert series of $I$, yielding
\[(16.4)\]
\[H_I(q) = \prod_{m \geq 1, m \not\equiv 0, \pm 1 \mod (2s+1)} \frac{1}{1 - q^m}.\]

Indeed the coefficient of $q^n$ in $H_I(q)$ is easily seen to be the number of partitions $(2^{i_2}, \ldots, N^{i_N})$ of $n$ into parts of size at least 2 and such that $i_k + i_{k+1} \leq s - 1$ for $k = 2, \ldots, N - 1$. Gordon’s generalisation [1] Theorem 7.5] of the Rogers-Ramanujan identity asserts that the generating function of the number of such partitions is exactly the product on the right hand side of (16.4) above.

We remark that the algebras $R/I$ have also been studied in [17] in connection with principal subspaces of $\mathfrak{sl}_2$-modules.

In the case $(p, p') = (2, 2s + 1)$ for some $s \geq 1$ the minimal model character (16.1) has been shown to be simply the right hand side of (16.4) [31] [35] [32]. Evidently then, the character of $\text{Vir}_{p,p'}$ coincides with the Hilbert series of $JR_{\text{Vir}_{p,p'}}$ when $(p, p') = (2, 2s + 1)$. It follows that the surjection $JR_{\text{Vir}_{2,2s+1}} \to gr^F \text{Vir}_{2,2s+1}$ is an isomorphism.

For $p, p' \geq 3$ the character (16.1) differs from the Hilbert series (16.4) and so the surjection $JR_{\text{Vir}_{p,p'}} \to gr^F \text{Vir}_{p,p'}$ is not an isomorphism in these cases. For example for the Ising model $V = \text{Vir}_{2,4}$ the dimensions of graded pieces of $JR_V$ and $gr^F V$ agree up to conformal weight $\Delta = 8$, but disagree for $\Delta \geq 9$.

In summary, we have proved the following theorem.

16.13. **Theorem.** Let $V$ denote the Virasoro minimal model $\text{Vir}_{p,p'}$. If $(p, p') = (2, 2k+1)$ where $k \geq 1$, then the natural embedding $SS(V) \hookrightarrow JX_V$ is an isomorphism of schemes. If $p, p' \geq 3$ then the embedding is not an isomorphism. In all cases, however, the reduced schemes of $SS(V)$ and $JX_V$ are isomorphic, both consisting of a single closed point.

16.14. **Corollary.** Let $V$ denote the Virasoro minimal model $\text{Vir}_{2,2k+1}$ where $k \geq 1$, and $A_V$ the associated chiral algebra over the elliptic curve $X_q$. Then
\[\lim_{q \to 0} H^{ch}_1(X_q, A_V) = 0.\]

16.15. It is interesting to note that two different minimal models may have isomorphic associated schemes. This is the case for $\text{Vir}_{2,7}$ and the Ising model $\text{Vir}_{3,4}$, for instance. We thus obtain an embedding
\[SS(\text{Vir}_{3,4}) \hookrightarrow SS(\text{Vir}_{2,7}),\]
or equivalently a surjective morphism of the associated Poisson vertex algebras. We do not know if there is a relation between the corresponding vertex algebras which explains this morphism.

16.16. Now we consider the simple affine vertex algebras. Let $g$ be a simple Lie algebra and $k \in \mathbb{Z}_+$. The simple quotient of $V^k(g)$ is denoted $V_k(g)$ and is a rational vertex algebra. The maximal ideal of $V^k(g)$ is generated by $v_k = c_\theta (-1)^{k+1}1$, where $c_\theta$ is the highest root vector of $g$ [25]. We consider the adjoint action of $g$ on $S(g)$ and we denote by $W$ the $g$-submodule of $S(g)$ generated under this action by $c_\theta^{k+1}$. Since $v_k$ is a highest weight vector for the $g$-action on $V^k(g)$, it follows that $R_{V_k(g)}$ is the quotient of $RV^k(g) \cong S(g)$ by the commutative algebra ideal generated by $W$.

16.17. Now we pass to the special case $g = \mathfrak{sl}_2 = \mathbb{C}\langle e, h, f \rangle$. In this case $W \subset S(g)$ is the span of the vectors $\text{ad}(f)^i e^{k+1}$ for $i = 0, 1, \ldots, 2k + 2$.

16.18. Let
\[R = J(k[e, h, f]) = k[e_0, h_0, f_0, e_1, h_1, \ldots].\]
certain integer coefficients. Using the commutation relations of \( \mathfrak{sl}_3 \) and the general situation, where the sum runs over all \( i \), \( j \), \( k \) satisfying at least one of the conditions (16.6) and (16.7), we obtain the leading term of some polynomial from the set (16.6). We present the proof for the case of monomials satisfying \( \#(h_{m+1}) + \#(f_m) + \#(f_{m+1}) = k + 1 \) and omit the other three cases as they are very similar.

For given \( q, r, -r_+ \in \mathbb{Z}_+ \) satisfying \( q + r_+ + r_- = k + 1 \), we consider the monomial \( M = h_{m+1} f_m f_{m+1} \). The charge of \( M \) is \(-r_+ + r_-\), the degree is \( k + 1 \) and the weight is

\[
A = (m + 1)(k + 1) - r_-
\]

We claim that \( M \) is the leading monomial of \( X = \text{ad}(f)^q r_+ + 2(r_- + r_+) e^{k+1} f^{k+1} \). Firstly we note that \( X \) is homogeneous of charge \(-r_+ + r_-\), degree \( k + 1 \) and weight \( A \). We also observe that in general

\[
\text{ad}(f)^i e^{k+1} = \sum_{p,q,r} a_p^{(k)} e^p h^q f^r,
\]

where the sum runs over all \( p, q, r \in \mathbb{Z}_+ \) such that \( p + q + r = k + 1 \) and \( q + 2r = i \), and the \( a_p^{(k)} \) are certain integer coefficients. Using the commutation relations of \( \mathfrak{sl}_2 \) to write down a recurrence
relation for the coefficients, it is straightforward to deduce that every monomial term in the sum of \([16.3]\) appears with nonzero coefficient.

We consider an arbitrary monomial term of \(X\) and we assume it to be no lower than \(M\) in the grevlex monomial order. We therefore have \(#(e_i) = #(h_i) = 0\) for \(i \leq m\) and \(#(f_i) = 0\) for \(i \leq m - 1\). Let us write

\[
C = #(e_{\geq m+2}) + #(h_{\geq m+2}) + #(f_{\geq m+2}).
\]

The weight \(A\) of the monomial is clearly at least

\[
(m + 2)\,(#(e_{\geq m+2}) + #(h_{\geq m+2}) + #(f_{\geq m+2})) + (m + 1)\,(#(e_{m+1}) + #(h_{m+1}) + #(f_{m+1})) + m\,#(f_m)
\]

\[
= C + (m + 1)\,(#(e_{\geq m+1}) + #(h_{\geq m+1}) + #(f_{\geq m})) - #(f_m)
\]

\[
= C + (m + 1)(k + 1) - #(f_m).
\]

From this and \(A = (m + 1)(k + 1) - r_-\) we deduce

\[
#(f_m) \geq r_+ + C.
\]

Clearly \(#(f_m) > r_-\) would contradict our assumption that the monomial be no lower than \(M\), so we must have \(C = 0\) and \(#(f_m) = r_-\). Given that \(#(f_m) = r_-\) and that \(M\) does not contain the term \(e_{m+1}\), the presence of the term \(e_{m+1}\) in our monomial would also result in a contradiction, so we must have \(#(e) = #(e_{m+1}) = 0\). Now the condition on the charge implies that \(#(f) = r_- + r_+\) and hence \(#(f_{m+1}) = r_+\). So in fact the monomial we started with can be none other than \(M\) itself.

\[\Box\]

**16.21. Theorem.** Let \(k \in \mathbb{Z}_+\) and let \(V\) denote the simple affine vertex algebra \(V_k(g_2)\). The natural embedding \(SS(V) \hookrightarrow JX_\nu\) is an isomorphism.

**Proof.** The arc space \(J_{R(V)}\), with its conformal weight grading, is isomorphic as a graded ring to the quotient \(R/I\) defined in Section \([16.16]\) above, and so \(\chi_{J_{R(V)}}(q) = H_{\nu}(q)\).

It is a general fact that \(\chi_V(q) = \chi_{J_{R(V)}}(q)\). Since \(J_{R(V)} \to \text{gr}^F(V)\) is a surjection we have \(\chi_{\text{gr}^F(V)}(q) \leq \chi_{J_{R(V)}}(q)\). Proposition \([16.9]\) asserts that \(H_{\nu}(q) = H_{\nu\nu}(q)\). By Lemma \([16.20]\) we have \(H_{\nu\nu}(q) \leq H_K(q)\).

Obviously the quotient \(R/K\) possesses a linear basis consisting of all those monomials that satisfy the following four conditions for every \(m \geq 0\)

\[
#(f_{m+1}) + #(h_{m+1}) + #(f_m) \leq k, \quad #(h_{m+1}) + #(e_{m+1}) + #(f_m) \leq k, \quad
#(e_{m+1}) + #(f_m) + #(h_m) \leq k, \quad #(e_{m+1}) + #(h_m) + #(e_m) \leq k.
\]

We now define a linear morphism \(\phi : R/K \to V\) by specifying its action on monomials as follows. Write the monomial with its variables in increasing order according to \([16.5]\). Now replace \(e_i\) with \(\epsilon(-i - 1)\), \(h_i\) with \(\eta(-i - 1)\), and \(f_i\) with \(f_i\). Read the result as an element of \(V\). By comparing \([16.9]\) with \([36]\) equation (3) in the special case \(k_0 = k\) it follows immediately that the elements of \(V\) thus obtained constitute a basis. Hence \(\phi\) is a (graded) isomorphism, and we obtain \(H_K(q) = \chi_V(q)\).

Putting all these observations together yields

\[
\chi_V(q) = \chi_{\text{gr}^F(V)}(q) \leq \chi_{J_{R(V)}}(q) = H_{\nu}(q) = H_{\nu\nu}(q) \leq H_K(q) = \chi_V(q),
\]

from which it follows that \(\chi_{\text{gr}^F(V)}(q) = \chi_{J_{R(V)}}(q)\) and hence that \(J_{R(V)} \to \text{gr}^F(V)\) is an isomorphism.

\[\Box\]
16.22. Corollary. Let $V$ denote the simple affine vertex algebra $V_k(sl_2)$ where $k \in \mathbb{Z}_+$, and $\mathcal{A}_V$ the associated chiral algebra over the elliptic curve $X_q$. Then

\[ \lim_{q \to 0} H^1_{ch}(X_q, \mathcal{A}_V) = 0. \]

16.23. Remark. In the proof of Theorem 16.21 above we have used a difficult result of Meurman-Primc [36] on PBW-type bases of integrable $sl_2$-modules. It would be interesting to know if, conversely, techniques from arc spaces and Gröbner bases might be used to shed light on the structure of integrable $\hat{\mathfrak{g}}$-modules in general. As a tentative first step in this direction we construct a Gröbner basis of the ideal $I \subset R$ in the case $k = 1$ in Section 17 below.

16.24. In [6] it has been shown that if $V$ is quasi-lisse, that is, if the reduced scheme $\text{Specm}(R_V)$ has finitely many symplectic leaves, then the reduced schemes $\text{Specm}(JR_V)$ and $\text{Specm}(\text{gr}^F V)$ are isomorphic. Though as Theorem 16.13 above indicates, isomorphism of the schemes $\text{Spec}(JR_V)$ and $\text{Spec}(\text{gr}^F V)$ may fail even when the reduced schemes are a single point.

16.25. Let $V$ be the simple quotient of Zamolodchikov’s $W_3$-algebra at central charge $c = -2$. In [5] it has been shown that the surjection $JR_V \to \text{gr}^F V$ is not an isomorphism.

17. Gröbner Bases of Arc Spaces

17.1. We continue with the notation of Sections 16.16 through 16.23 above. We begin by giving a short proof of Proposition 16.11 in the case of $s = 2$.

Proof. It is easy to see that for any $N \geq 0$ we have

\[ \sum_{i=0}^{N} \left( \frac{1}{3} - \frac{i}{N} \right) x_i[x^2]_{N-i} = 0. \]

Indeed the sum simplifies to

\[ \sum_{p+q+r=N} x_p x_q x_r \left( \frac{1}{3} - \frac{p}{N} + \frac{q}{N} + \frac{r}{N} \right), \]

which vanishes since $p + q + r = N$ in each summand.

Now let $F = [x^2]_A$, $G = [x^2]_B$ and $S = S(F, G)$. If $A$ and $B$ are equal then $S = 0$ and if they are nonconsecutive then the leading terms of $F$ and $G$ are coprime and we are done by Lemma 16.3. Otherwise we distinguish two cases:

(a) $A = 2m + 1$ and $B = 2m$. We have

\[ \text{LT}(x_m[x^2]_{2m+1}) = x_m(2x_m x_{m+1}) \quad \text{versus} \quad \text{LT}(x_{m+1}[x^2]_{2m}) = x_{m+1}(x_m^2), \]

and $\text{LT}(S) = x_{m+1}^2 x_{m-1}$.

(b) $A = 2m - 1$ and $B = 2m$. We have

\[ \text{LT}(x_m[x^2]_{2m-1}) = x_m(2x_m x_{m-1}) \quad \text{versus} \quad \text{LT}(x_{m-1}[x^2]_{2m}) = x_{m-1}(x_m^2), \]

and $\text{LT}(S) = x_{m-1}^2 x_{m+1}$.

We use the sum (17.1) to show that $S$ reduces to 0 modulo $\{[x^2]_N\}$. To treat case (a) we put $N = 3m + 1$. Observe that no coefficient in the sum vanishes, because $N$ is not divisible by 3. The terms $x_m[x^2]_{2m+1}$ and $x_{m+1}[x^2]_{2m}$ together are nothing other than $S$. Every summand besides these two has leading term no higher than that of $S$. Indeed if $i < m$ then the leading term of $x_i[x^2]_{N-i}$ obviously contains $x_i$, while if $i > m + 1$ then the leading term of $[x^2]_{N-i}$ contains $x_j$ for some $j < m$. To treat case (b) we put $N = 3m - 1$ and argue similarly. \qed
17.2. Proposition. Let \( R = J(\mathfrak{sl}_2) \) and \( I \) the ideal generated by the set of terms
\[
(17.2) \quad [\text{ad}(f)^i e^2]_N \quad \text{for } i = 0, \ldots, 4 \quad \text{and } N \in \mathbb{Z}_+.
\]
Then the set \[(17.2)\] is a \( \text{Gröbner basis} \) of \( I \).

Proof. We wish to show that the \( S \)-polynomial of any pair of elements of the putative \( \text{Gröbner basis} \) reduces to zero. For some pairs, such as \([e^2]_A, [e^2]_B\), the proof of Proposition 16.11 above obviously carries over verbatim. For certain other pairs, such as \([ef-h^2]_A, [ef-h^2]_B\), the leading terms are coprime and we are done by Lemma 16.8. For all other cases we use generalisations of equation \[(17.3)\]. First we list all relevant leading terms
\[
\text{LT}([e^2]_{2m}) = e_{2m}^2, \quad \text{LT}([e^2]_{2m+1}) = 2e_{m}e_{m+1},
\]
\[
\text{LT}([eh]_{2m}) = e_{m}h_{m}, \quad \text{LT}([eh]_{2m+1}) = e_{m+1}h_{m},
\]
\[
(17.3) \quad \text{LT}([ef-h^2]_{2m}) = h_{2m}^2, \quad \text{LT}([ef-h^2]_{2m+1}) = e_{m+1}f_{m},
\]
\[
\text{LT}([hf]_{2m}) = h_{m}f_{m}, \quad \text{LT}([hf]_{2m+1}) = h_{m+1}f_{m},
\]
\[
\text{LT}([f^2]_{2m}) = f_{2m}^2, \quad \text{LT}([f^2]_{2m+1}) = 2f_{m}f_{m+1}.
\]
Notice that the only pair of such terms \([X]_A, [Y]_B\) with non coprime leading terms and for which \(A, B\) are non consecutive, is the pair \([eh]_{2m+1}, [hf]_{2m-1}\).

We consider the equation
\[
0 = \sum_{i=0}^{N} \alpha_ie_i[ef-h^2]_{N-i} + \sum_{i=0}^{N} \beta_i[h]_{N-i} + \sum_{i=0}^{N} \gamma_if_i[e^2]_{N-i}
\]
\[
= \sum_{p+q+r=N} [e_p e_q f_r (\alpha_p + \alpha_q + \gamma_r) + e_p h_q h_r (-\alpha_p + \beta_q + \beta_r)],
\]
whose most general solution, up to overall scaling, is given by putting
\[
\alpha_i = \alpha - \frac{i}{N}, \quad \beta_i = \beta + \frac{i}{N} \quad \text{and} \quad \gamma_i = \gamma - \frac{i}{N},
\]
such that the conditions
\[
2\alpha + \gamma = 1, \quad \text{and} \quad \alpha - 2\beta = 1.
\]
Now let \( F = [eh]_{2m}, G = [ef-h^2]_{2m} \) and \( S = S(F,G) \). Thus
\[
\text{LT}([e_m]_{2m}) = e_m^2h_{m}^2, \quad \text{LT}([e_m]_{2m+1}) = e_m^2h_{m+1}^2,
\]
and \( \text{LT}(S) = e_m^2f_{m} \). We put \( N = 3m \) in \[(17.4)\] and we observe that \( \text{LT}(S) \) appears as the leading term of the third summation. We have the freedom, by judicious choice of \((\alpha, \beta, \gamma)\), to ensure that the terms \( e_m[ef-h^2]_{2m} \) and \( h_m[eh]_{2m} \) appear in the first two summations with nonzero coefficients. All other terms in the summations have leading terms no greater than that of \( S \).

Now let \( F = [eh]_{2m}, G = [eh]_{2m-1} \) and \( S = S(F,G) \). Thus
\[
\text{LT}([e_{m-1}]_{2m}) = h_{m-1}^2e_{m-1}h_{m} \quad \text{versus} \quad \text{LT}([e_m]_{2m+1}) = h_{m}^2e_{m+1}h_{m+1},
\]
and \( \text{LT}(S) = e_m^2f_{m} \). We put \( N = 3m - 1 \) in \[(17.4)\]. The expression \( e_m[ef-h^2]_{2m-1} \) appears in the first summation with coefficient \( \alpha_m \). Since \( \text{LT}(e_m[ef-h^2]_{2m-1}) = -e_m(f_{m-1}e_m) \) is higher than the leading term of \( S \) we must impose \( \alpha_m = 0 \). The expression
\[
\beta_m h_{m-1}[eh]_{2m} + \beta_m h_m[eh]_{2m-1}
\]
coincides with \( S \) up to a scalar factor. The imposition of \( \alpha_m = 0 \) means that non vanishing of \[(17.3)\] is something to be checked rather than ensured by judicious choice of \((\alpha, \beta, \gamma)\). We directly compute
\[
\beta_m = \frac{1}{2} \left( \frac{3m}{3m-1} - 1 \right) \neq 0.
\]
Every summand (including all those of the third summation) besides these three has leading term no higher than that of $S$.

An exhaustive check confirms that all further cases, with one exception, may be treated by analysis of an equation of the form (17.3) exactly parallel to that used to settle one or other of the two cases presented above. The exception, which we now consider, is the unique non-consecutive case described just after (17.3). Let $F = [eh]_{2m+1}$, $G = [hf]_{2m-1}$ and $S = S(F, G)$. Thus

$$\text{LT}(f_{m-1}[eh]_{2m+1}) = f_{m-1}(e_{m+1}h_m), \quad \text{versus} \quad \text{LT}(e_{m+1}[hf]_{2m-1}) = e_{m+1}(h_mf_{m-1}),$$

and $\text{LT}(S) = f_{m-1}(e_mh_{m+1})$. We consider the equation

$$0 = \sum_{i=0}^{N} \alpha_if_i[eN][N-i] + \sum_{i=0}^{N} \beta_ie_i[hN][N-i] + \sum_{i=0}^{N} \gamma_ih_i[eN - hN][N-i]$$

$$(17.6) = \sum_{p+q+r=N} epqf_r(\alpha_r + \beta_p + \gamma_q) - \sum_{p+q+r=N} hphqhr(\gamma_p + \gamma_q + \gamma_r).$$

whose most general solution, up to overall scaling, is given by putting

$$\alpha_i = \alpha - \frac{i}{N}, \quad \beta_i = \beta - \frac{i}{N}, \quad \text{and} \quad \gamma_i = \gamma - \frac{i}{N},$$

subject to the conditions

$$3\gamma = 1, \quad \text{and} \quad \alpha + \beta + \gamma = 1.$$

We put $N = 3m$. The expression

$$\alpha_{m-1}f_{m-1}[eh]_{2m+1} + \beta_{m+1}e_{m+1}[hf]_{2m-1}$$

coincides with $S$ up to a scalar factor. However the summands $\alpha_m f_m[eN]_{2m}$, $\beta_m e_m[hN]_{2m}$ and $\gamma_m h_m[eN - hN][2m]$ all have leading terms strictly higher than the leading term of $S$. Fortunately $\alpha = \beta = \gamma = 1/3$ guarantees $\alpha_m = \beta_m = \gamma_m = 0$. Every remaining summand has leading term no higher than that of $S$, so we are done. \qed
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