Remez-Type Inequality for Smooth Functions

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Abstract

The classical Remez inequality bounds the maximum of the absolute value of a polynomial $P(x)$ of degree $d$ on $[-1, 1]$ through the maximum of its absolute value on any subset $Z$ of positive measure in $[-1, 1]$. Similarly, in several variables the maximum of the absolute value of a polynomial $P(x)$ of degree $d$ on the unit ball $B^n \subset \mathbb{R}^n$ can be bounded through the maximum of its absolute value on any subset $Z \subset Q^n_1$ of positive $n$-measure $m_n(Z)$. In [12] a stronger version of Remez inequality was obtained: the Lebesgue $n$-measure $m_n$ was replaced by a certain geometric quantity $\omega_{n,d}(Z)$ satisfying $\omega_{n,d}(Z) \geq m_n(Z)$ for any measurable $Z$. The quantity $\omega_{n,d}(Z)$ can be effectively estimated in terms of the metric entropy of $Z$ and it may be nonzero for discrete and even finite sets $Z$.

In the present paper we extend Remez inequality to functions of finite smoothness. This is done by combining the result of [12] with the Taylor polynomial approximation of smooth functions. As a consequence we obtain explicit lower bounds in some examples in the Whitney problem of a $C^k$-smooth extrapolation from a given set $Z$, in terms of the geometry of $Z$.

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1 Introduction

The classical Remez inequality ([10], see also [6]) reads as follows:

**Theorem 1.1** Let $P(x)$ be a polynomial of degree $d$. Then for any measurable $Z \subset [-1, 1]$

$$\max_{[-1,1]}|P(x)| \leq T_d\left(\frac{4 - m}{m}\right)\max_Z|P(x)|, \quad (1.1)$$

where $m = m_1(Z)$ is the Lebesgue measure of $Z$ and $T_d(x) = \cos(d \arccos(x))$ is the $d$-th Chebyshev polynomial.

In several variables a generalization of Theorem 1.1 was obtained in [2]:

**Theorem 1.2** Let $B \subset \mathbb{R}^n$ be a convex body and let $\Omega \subset B$ be a measurable set. Then for any real polynomial $P(x) = P(x_1, \ldots, x_n)$ of degree $d$ we have

$$\sup_{B}|P| \leq T_d\left(\frac{1 + (1 - \lambda)^{\frac{1}{n}}}{1 - (1 - \lambda)^{\frac{1}{n}}}\right)\sup_{\Omega}|P|. \quad (1.2)$$

Here $\lambda = \frac{m_n(\Omega)}{m_n(B)}$, with $m_n$ being the Lebesgue measure on $\mathbb{R}^n$. This inequality is sharp and for $n = 1$ it coincides with the classical Remez inequality.

It is clear that Remez inequality of Theorems 1.1 and 1.2 cannot be verbally extended to smooth functions: such function $f$ may be identically zero on any given closed set $Z$, and non-zero elsewhere. In the present paper we show that adding a “remainder term” (expressible through the bounds on the derivatives of $f$) provides a generalization of the Remez inequality to smooth functions. Our main goal is to study the interplay between the geometry of the “sampling set” $Z$, the bounds on the derivatives of $f$, and the bounds for the extension of $f$ from $Z$ to the ball $B^n$ of radius 1 centered at the origin in $\mathbb{R}^n$. To state our main “general” result we need some definitions:

**Definition 1.1** For a set $Z \subset B^n \subset \mathbb{R}^n$ and for each $d \in \mathbb{N}$ the Remez constant $R_d(Z)$ is the minimal $K$ for which the inequality $\sup_{B^n}|P| \leq K \sup_Z|P|$ is valid for any real polynomial $P(x) = P(x_1, \ldots, x_n)$ of degree $d$. 

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For some $Z$ the Remez constant $R_d(Z)$ may be equal to $\infty$. In fact, $R_d(Z)$ is infinite if and only if $Z$ is contained in the set of zeroes $Y_P = \{ x \in \mathbb{R}^n, | P(x) = 0 \}$ of a certain polynomial $P$ of degree $d$. See [3] for a detailed discussion.

**Definition 1.2** Let $f : B^n \rightarrow \mathbb{R}$ be a $k$ times continuously differentiable function on $B^n$. For $d = 0, 1, \ldots$, the approximation error $E_d(f)$ is the minimum over all the polynomials $P(x)$ of degree $d$ of the absolute deviation $M_0(f - P) = \max_{x \in B^n} |f(x) - P(x)|$.

**Theorem 1.3** Let $f : B^n \rightarrow \mathbb{R}$ be a $k$ times continuously differentiable function on $B^n$, and let a subset $Z \subset B^n$ be given. Put $L = \max_{x \in Z} |f(x)|$. Then

$$\max_{x \in B^n} |f(x)| \leq \inf_d [R_d(Z)(L + E_d(f)) + E_d(f)].$$

(1.3)

**Proof:** Let for a fixed $d$ $P_d(x)$ be the polynomial of degree $d$ for which the best approximation of $f$ is achieved: $E_d(f) = \max_{x \in B^n} |f(x) - P(x)|$. Then $\max_{x \in Z} |P(x)| \leq L + E_d(f)$. By definition of the Remez constant $R_d(Z)$ we have $\max_{x \in B^n} |P(x)| \leq R_d(Z)(L + E_d(f))$. Returning to $f$ we get $\max_{x \in B^n} |f(x)| \leq R_d(Z)(L + E_d(f)) + E_d(f)$. Since this is true for any $d$, we finally obtain $\max_{x \in B^n} |f(x)| \leq \inf_d [R_d(Z)(L + E_d(f)) + E_d(f)].$ □

In this paper we produce, based on Theorem 1.3, explicit Remez-type bounds for smooth functions in some typical situations.

## 2 Bounding $R_d(Z)$ via Metric Entropy

It is well known that the inequality of the form (1.1) or (1.2) may be true also for some sets $Z$ of measure zero and even for certain discrete or finite sets $Z$. Let us mention here only a couple of the most relevant results in this direction: in [4, 9, 13] such inequalities are provided for $Z$ being a regular grid in $[-1, 1]$. In [7] discrete sets $Z \subset [-1, 1]$ are studied. In this last paper the invariant $\phi_Z(d)$ is defined and estimated in some examples, which is the best constant in the Remez-type inequality of degree $d$ for the couple $(Z \subset [-1, 1])$. 


In [12] (see also [1]) a strengthening of Remez inequality was obtained: the Lebesgue $n$-measure $m_n$ was replaced by a certain geometric quantity $\omega_{n,d}(Z)$, defined in terms of the metric entropy of $Z$, and satisfying $\omega_{n,d}(Z) \geq m_n(Z)$ for any measurable $Z \subset Q^n_1$. So we have the following proposition, which combines the result of Theorem 3.3 of [12] with the well-known bound for Chebyshev polynomials (see [6]):

**Proposition 2.1** For each $Z \subset B^n$ and for any $d$ the Remez constant $R_{n,d}(Z)$ satisfies

$$R_{n,d}(Z) \leq T_d \left( \frac{1 + (1 - \lambda)^\frac{1}{n}}{1 - (1 - \lambda)^\frac{1}{n}} \right) \leq \left( \frac{4n}{\lambda} \right)^d, \quad (2.1)$$

where $\lambda = \omega_{n,d}(Z)$.

In what follows we shall omit the dimension $n$ from the notations for $\omega_{d}(Z) = \omega_{n,d}(Z)$. It was shown in [12] that in many cases (but not always!) the bound of Proposition 2.1 is pretty sharp. In the present paper we recall the definition of $\omega_{d}(Z)$ and estimate this quantity in several typical cases, stressing the setting where $Z$ is fixed, while $d$ changes.

### 2.1 Definition and properties of $\omega_{d}(Z)$

To define $\omega_{d}(Z)$ let us recall that the covering number $M(\epsilon, A)$ of a metric space $A$ is the minimal number of closed $\epsilon$-balls covering $A$. Below $A$ will be subsets of $\mathbb{R}^n$ equipped with the $l^\infty$ metric. So the $\epsilon$-balls in this metric are the cubes $Q^n_\epsilon$.

For a polynomial $P$ on $\mathbb{R}^n$ let us consider the sub-level set $V_\rho(P)$ defined by $V_\rho(P) = \{ x \in B^n, |P(x)| \leq \rho \}$. The following result is proved in ([11]):

**Theorem 2.1** (Vitushkin’s bound) For $V = V_\rho(P)$ as above

$$M(\epsilon, V) \leq \sum_{i=0}^{n-1} C_i(n, d)(\frac{1}{\epsilon})^i + m_n(V)(\frac{1}{\epsilon})^n, \quad (2.2)$$

with $C_i(n, d) = C'_i(n)(2d)^{(n-i)}$. For $n = 1$ we have $M(\epsilon, V) \leq d + \mu_1(V)(\frac{1}{\epsilon})$, and for $n = 2$ we have

$$M(\epsilon, V) \leq (2d - 1)^2 + 8d(\frac{1}{\epsilon}) + \mu_2(V)(\frac{1}{\epsilon})^2.$$
For $\epsilon > 0$ we denote by $M_{n,d}(\epsilon)$ (or shortly $M_d(\epsilon)$) the polynomial of degree $n - 1$ in $\frac{1}{\epsilon}$ as appears in (2.2):

$$M_d(\epsilon) = \sum_{i=0}^{n-1} C_i(n,d)(\frac{1}{\epsilon})^i.$$  \quad (2.3)

In particular,

$$M_{1,d}(\epsilon) = d, \quad M_{2,d}(\epsilon) = (2d - 1)^2 + 8d(\frac{1}{\epsilon}).$$

Now for each subset $Z \subset B^n$ (possibly discrete or finite) we introduce the quantity $\omega_d(Z)$ via the following definition:

**Definition 2.1** Let $Z$ be a subset in $B^n \subset \mathbb{R}^n$. Then $\omega_d(Z)$ is defined as

$$\omega_d(Z) = \sup_{\epsilon > 0} \epsilon^n [M(\epsilon, Z) - M_d(\epsilon)].$$ \quad (2.4)

The following results are obtained in [12]:

**Proposition 2.2** The quantity $\omega_d(Z)$ for $Z \subset B^n$ has the following properties:

1. For a measurable $Z$ $\omega_d(Z) \geq m_n(Z)$.

2. For any set $Z \subset B^n$ the quantities $\omega_d(Z)$ form a non-increasing sequence in $d$.

3. For a set $Z$ of Hausdorff dimension $n - 1$, if the Hausdorff $n - 1$ measure of $Z$ is large enough with respect to $d$, then $\omega_d(Z)$ is positive.

4. Let $G_s = \{x_1 = -1, x_2, \ldots, x_s = 1\}$ be a regular grid in $[-1,1]$. Then $\omega_d(G_s) = \frac{2(s-d)}{s-1}$.

Let $Z_r = \{1, \frac{1}{2r}, \frac{1}{3r}, \ldots, \frac{1}{kr}, \ldots\}$. In this case $\omega_d(Z_r) \sim \frac{1}{(r+1)^{d-1}} \frac{1}{r^d}$.

Let $Z(q) = \{1, q, q^2, q^3, \ldots, q^m, \ldots\}$, $0 < q < 1$. Then $\omega_d(Z(q)) \sim \frac{q^d}{\log(q)}$.

We need the following result, which, although in the direction of the results in [12], was not proved there explicitly. Let $S$ be a connected smooth curve in $B^2$ of the length $\sigma$. Define $\epsilon_0$ as the maximal $\epsilon$ such that for each $\delta \leq \epsilon$ we have $M(\delta, S) \geq \frac{l(S)}{2\delta}$. The parameter $\epsilon_0$ is a kind of “injectivity radius” of the
curve $S$, and for any curve of length $\sigma$ inside the unit ball $B^2$ it cannot be larger than $\frac{1}{\sigma}$. Write $\epsilon_0$ as $\epsilon_0 = \frac{1}{l\sigma}$, $l \geq 1$. The computation below essentially compares the length of $S$ with the maximal possible length of an algebraic curve of degree $d$ inside $B^2$, which is of order $d$. So it is convenient for any given $d$ to write $\sigma$ as $\sigma = md$.

**Proposition 2.3** In the notations above, $\omega_d(S)$ satisfies

$$\omega_d(S) \geq \frac{1}{2l}(1 - \frac{24}{m}).$$

(2.5)

In particular, for the length of $S$ larger than $24d$, $\omega_d(S)$ is strictly positive.

**Proof:** By definition,

$$\omega_d(S) = \sup_{\epsilon} \epsilon^2[M(\epsilon, S) - M_d(\epsilon)] = \sup_{\epsilon} \epsilon^2[M(\epsilon, S) - (2d - 1)^2 - 8d(\frac{1}{\epsilon})].$$

Substituting here $\epsilon_0 = \frac{1}{l\sigma}$ we get

$$\omega_d(S) \geq \left(\frac{1}{lmd}\right)^2\left[\frac{l(md)^2}{2} - (2d - 1)^2 - 8lmd^2\right] =$$

$$= \frac{1}{2l}(1 - \frac{2}{m}\left[\left(\frac{2d - 1}{d}\right)^2 + 8\right]) \geq \frac{1}{2l}(1 - \frac{24}{m}).$$

In particular, for $m > 24$, i.e. for the length of $S$ larger than $24d$, the quantity $\omega_d(S)$ is strictly positive. □

### 3 Bounding Smooth Functions

Let $f : B^n \to \mathbb{R}$ be a $k$ times continuously differentiable function on $B^n$. For $l = 0, 1, \ldots, k$ put $M_l(f) = \max_{B^n}\|d^lf\|$, where the norm of the $l$-th differential of $f$ is defined as the sum of the absolute values of all the partial derivatives of $f$ of order $l$. To simplify notations, we shall not make specific assumptions on the continuity modulus of the last derivative $d^k f$. Now we use Taylor polynomials of an appropriate degree between 0 and $k - 1$ in order to bound from above the approximation error $E_d(f)$, $d = 0, 1, \ldots, k$. Applying Theorem 1.3, we obtain the following result:
Proposition 3.1 Let \( f : B^n \to \mathbb{R} \) be a \( k \) times continuously differentiable function on \( B^n \), with \( M_l(f) = \max_{B^n} \|d^lf\| \), \( l = 0, 1, \ldots, k \), and let a subset \( Z \subset B^n \) be given. Put \( L = \max_{x \in Z} |f(x)| \). Then

\[
M_0(f) = \max_{x \in B^n} |f(x)| \leq \min_{d=0,1,\ldots,k-1} [R_d(Z)(L+E_d^T(f))+E_d^T(f)], \tag{3.1}
\]

where \( E_d^T(f) = \frac{1}{(d+1)!}M_{d+1}(f) \) is the Taylor remainder term of \( f \) of degree \( d \) on the unit ball \( B^n \).

Proof: We restrict infimum in Theorem 1.3 to a smaller set of \( d \)'s, and replace \( E_d(f) \) with a larger quantity \( E_d^T(f) \). □

In general we cannot get an explicit answer for the minimum in Proposition 3.1, unless we add more specific assumptions on the set \( Z \) and the sequence \( M_d(f) \). However, this proposition provides an explicit and rather sharp information in the case where the set \( Z \) is "small". Let us pose the following question: for a fixed \( s = 1, \ldots, k - 1 \) and a given set \( Z \subset B^n \) is it possible to bound \( M_0(f) = \max_{x \in B^n} |f(x)| \) through \( L = \max_{x \in Z} |f(x)| \) and \( M_{s+1}(f) \) only, without knowing bounds on the derivatives \( d^lf \), \( l \leq s \) ?

Proposition 3.2 If \( R_s(Z) < \infty \) then \( M_0(f) \leq R_s(Z)(L+E_s^T(f))+E_s^T(f) \) with \( E_s^T(f) = \frac{1}{(s+1)!}M_{s+1}(f) \). If \( R_s(Z) = \infty \) then \( M_0(f) \) cannot be bounded in terms of \( L \) and \( M_l(f), l \geq s + 1 \).

Proof: In case \( R_s(Z) < \infty \) the required bound is obtained by restricting the minimization in (3.1) to \( d = s \) only. If \( R_s(Z) = \infty \) then already polynomials of degree \( s \) vanishing on \( Z \) cannot be bounded on \( B^n \). □

Now we can apply explicit calculations of \( \omega_d(Z) \) in Section 2 above to get explicit inequalities relating the geometry of \( Z \), the values of \( f \) on this set, and the bounds on the derivatives of \( f \). We shall restrict ourselves to the case of \( Z \) being a curve in the plane, as considered in Proposition 2.3. Other situations presented in Proposition 2.2 can be treated in the same way. Let \( S \) be a connected smooth curve in \( B^2 \) of the length \( \sigma \), and the injectivity radius \( \epsilon_0 \). For \( d \leq \frac{\sigma}{21} - 1 \) put \( \kappa_d = \frac{1}{27}(1 - \frac{24}{m}) \), in notations of Proposition 2.3.

Proposition 3.3 Let \( f : B^2 \to \mathbb{R} \) be a \( k \) times continuously differentiable function on \( B^2 \), with \( M_l(f) = \max_{B^2} \|d^lf\| \), \( l = 0, 1, \ldots, k \), and \( S \subset B^2 \)
be a curve with the length \( \sigma \), and with the injectivity radius \( \epsilon_0 \). Put \( L = \max_{x \in S} |f(x)| \). Then for each \( s \leq \frac{\sigma}{2} - 1 \) we have

\[
M_0(f) \leq (\frac{8}{\kappa_s})^n(L + E^T_s(f)) + E^T_s(f), \tag{3.2}
\]
with \( E^T_s(f) = \frac{1}{(s+1)!}M_{s+1}(f) \) and \( \kappa_s = \frac{1}{2^n}(1 - \frac{24}{m}) > 0 \). For each \( s \) there are curves \( S_s \subset B^2 \) of the length at least \( 2s \) such that \( M_0(f) \) cannot be bounded in terms of \( L \) and \( M(f) \), \( l \geq s + 1 \).

**Proof:** The bound follows directly from Propositions 3.2, 2.3, and 2.1. Now take as a curve \( S_s \) a zero set of a polynomial \( y = T_s(x) \) inside \( B^2 \). Then for \( f(x, y) = K(y - T_s(x)) \) vanishing on \( S_s \), \( M_0(f) \) cannot be bounded through \( M_l(f) \), \( l \geq s + 1 \). \( \square \)

Another way to extract more explicit answer from Proposition 3.1 is to bound the norms \( M_l(f) \) of the \( l \)-th order derivatives of \( f \), for \( l = 0, 1, \ldots, k \), by their maximal value \( M = M(f) \), to substitute \( M \) instead of \( M_l(f) \) into the inequality 3.1, and to explicitly minimize the resulting expression in \( d \).

We shall fix the smoothness \( k \) and consider sets \( Z \subset B^n \) for which \( \omega(Z) = \omega_k(Z) > 0 \). In particular, let \( Z \subset B^n \) be a measurable set with \( m_n(Z) > 0 \). Then \( \omega_d(Z) \geq m_n(Z) \) for each \( d \). Sets \( Z \) in the specific classes, discussed in Section 2 above, provide additional examples. Since \( \omega_0(Z) \geq \omega_1(Z) \geq \ldots \geq \omega_k(Z) \), by Proposition 2.1 for each \( d = 0, \ldots, k - 1 \) we have \( R_d(Z) \leq (\frac{4n}{\omega(Z)})^{d} \). Let us denote \( \frac{4n}{\omega(Z)} \geq 4n \) by \( q = q(Z) \).

The following theorem provides one of possible forms of an explicit inequality, generalizing the Remez one to smooth functions:

**Theorem 3.1** Let \( f : B^n \to \mathbb{R} \) be a \( k \) times continuously differentiable function on \( B^n \), with \( M_l(f) = \max_{B^n} \|d^l f\| \leq M = M(f) \), \( l = 0, 1, \ldots, k \), and let a subset \( Z \subset B^n \) with \( \omega_{k-1}(Z) > 0 \) be given. Put \( L = \max_{x \in Z} |f(x)| \), \( q = q(Z) \geq 4n \). Then

\[
M_0(f) = \max_{x \in B^n} |f(x)| \leq 2q^d_0 L + \frac{1}{(d_0 + 1)!}M, \tag{3.3}
\]
where \( d_0 = d_0(M, L) \), satisfying \( 1 \leq d_0 \leq k - 1 \), is defined as follows: \( d_0 = 0 \) if \( L > M \), \( d_0 = k - 1 \) if \( \frac{1}{k}M \leq L \leq M \) and for \( \frac{1}{k}M \leq L \leq M \) the degree \( d_0 \) is defined by \( \frac{1}{(d_0 + 1)!}M \leq L \leq \frac{1}{d_0!}M \).
In particular, for $L > M$ the inequality takes the form

$$M_0(f) \leq L + 2M,$$  \hspace{1cm} (3.4)

while for $L \leq \frac{1}{k!}M$ we get

$$M_0(f) \leq 2q^{k-1}L + \frac{1}{k!}M.$$  \hspace{1cm} (3.5)

**Proof:** As above, $R_d(Z) \leq (\frac{4n}{m_a(Z)})^d = q^d$. By Theorem 1.3 we have

$$\max_{x \in B^n} |f(x)| \leq \inf_{d=0,1,\ldots,k} [q^d(L + E_d^T(f)) + E_d^T(f)] \leq q^d(L + \frac{1}{(d+1)!}M) + \frac{1}{(d+1)!}M.$$

Now we guess the value of $d$ which approximately minimizes the expression in the right-hand side: let $d_0 = d_0(M, L)$ be defined as follows:

- $d_0 = 0$ if $L > M$, $d_0 = k - 1$ if $L \leq \frac{1}{k!}M$, and for $\frac{1}{k!}M \leq L \leq M$ the degree $d_0$ is uniquely defined by the condition

$$\frac{1}{(d_0 + 1)!}M \leq L \leq \frac{1}{d_0!}M.$$

In each case we have $1 \leq d_0 \leq k - 1$. Substituting $d_0$ into the above expression we obtain for $L > M$ the inequality $M_0(f) = \max_{x \in B^n} |f(x)| \leq L + 2M$, while for $L \leq M$ we get $M_0(f) \leq 2q^{d_0}L + \frac{1}{(d_0 + 1)!}M$. In the case $L \leq \frac{1}{k!}M$ we get $d_0 = k - 1$, and the inequality takes the form $M_0(f) \leq 2q^{k-1}L + \frac{1}{k!}M$. \(\Box\)

**Remark** In the case $L > M$ in Theorem 3.1 we have $d_0 = 0$ and the resulting inequality (3.4) is rather straightforward. Indeed, we take one point $x_0 \in Z$. By the assumptions, $|f(x_0)| \leq L$, while $|df| \leq M$ on $B^n$. For each $x \in B^n$, we have $||x - x_0|| \leq 2$. Hence $|f(x)| \leq L + 2M$. However, for smaller $L$, i.e. for larger $d_0$ the result apparently cannot be obtained by a similar direct calculation. Compare a discussion in the next section.

## 4 Whitney Extension of Smooth Functions

There is a classical problem of Whitney (see [8] and references therein) concerning extension of $C^k$-smooth functions from closed sets. Recently a major
progress have been achieved in this problem. The following “Finiteness Principle” has been obtained, in its general form, by Ch. Fefferman in 2003: for a finite set $Z \subset B^n$ and for any real function $f$ on $Z$ denote by $||f||_{Z,k}$ the minimal $C^k$-norm of the $C^k$-extensions of $f$ to $B^n$.

There are constants $N$ and $C$ depending on $n$ and $k$ only, such that for any finite set $Z \subset B^n$ and for any real function $f$ on $Z$ we have $||f||_{Z,k} \leq C \max_{\tilde{Z}} ||f||_{\tilde{Z},k}$, with $\tilde{Z}$ consisting of at most $N$ elements.

The original proof of this result, as well as its further developments in [8] and other publications, provide rich connections between the geometry of $Z$ and the behavior of the $C^k$-extensions of $F$. Effective algorithms for the extension have been also investigated in [8]. Still, the problem of an explicit connecting the geometry of $Z$, the behavior of $f$ on $Z$, and the analytic properties of the $C^k$-extensions of $f$ to $B^n$ for $n \geq 2$ remains widely open. In one variable divided finite differences provide a complete answer (Whitney).

The following result illustrate the role of the Remez constant $R_d(Z)$ in the extension problem.

**Theorem 4.1** For a finite set $Z \subset B^n$ and for any $x \in B^n \setminus Z$ let $Z_x = Z \cup \{x\}$. Let $f_{Z,x}$ be zero on $Z$ and $1$ at $x$ and let $\tilde{f}_{Z,x}$ be a $C^k$-extensions of $f_{Z,x}$ to $B^n$. Then for each $d = 0, \ldots, k-1$ we have $M_{d+1}(\tilde{f}_{Z,x}) \geq \left(\frac{(d+1)!}{R_d(Z)+1}\right)$.

**Proof:** By Proposition 3.1 we have for the extension $\tilde{f}_{Z,x}$

$$M_0(\tilde{f}_{Z,x}) \leq \min_{d=0,1,\ldots,k-1}[R_d(Z)(L + E_d^T(f)) + E_d^T(f)],$$

where $E_d^T(\tilde{f}_{Z,x}) = \frac{1}{(d+1)!} M_{d+1}(\tilde{f}_{Z,x})$ is the Taylor remainder term of $f$ of degree $d$ on the unit ball $B^n$. In our case $M_0(\tilde{f}_{Z,x}) \geq 1$ while $L = 0$. So we obtain $1 \leq \min_{d=0,1,\ldots,k-1}(R_d(Z) + 1)\frac{1}{(d+1)!} M_{d+1}(\tilde{f}_{Z,x})$. We conclude that for each $d = 0, \ldots, k-1$ we have $M_{d+1}(\tilde{f}_{Z,x}) \geq \frac{(d+1)!}{R_d(Z)+1}$.

The results of Section 3 above can be translated into more results on extension from finite set, similar to that of Theorem 4.1. More important, Remez inequality for polynomials can be significantly improved, taking into account, in particular, a specific position of $x$ with respect to $Z$. We plan to present these results separately.
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