HIGHER-ORDER SYMMETRIC DUALITY FOR MULTIOBJECTIVE PROGRAMMING WITH CONE CONSTRAINTS

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Abstract. In this work, a pair of higher-order symmetric dual multiobjective optimization problems is formulated. Weak, strong and converse duality theorems are established under suitable assumptions. Some examples are also given to illustrate our main results. Furthermore, certain deficiencies in the formulations and the proof of the work of Kassem [Applied Mathematics and Computation, 209 (2009), 405–409] are pointed out.

1. Introduction. Symmetric duality for nonlinear optimization problems was first introduced by Dorn [8] in the sense that the dual of the dual is recast in the form of the primal. Subsequently, first-order symmetric duality for differentiable optimization problems has been studied by many researchers like Dantzig et al. [6], Bazaraa and Goode [4], Chandra and Kumar [5] and Mond and Weir [15].

Motivated by the concept of second- and higher-order duality in nonlinear programming problems introduced by Mangasarian [13], several researchers have worked extensively in this field, due to the computational advantage that it provides better tighter bounds for the primal problem than the first-order duality. For example, Yang et al. [20] discussed higher-order symmetric duality of multiobjective optimization problems under invexity conditions. Padhan and Nahak [16] established higher-order duality results for a pair of Wolfe type and Mond-Weir type higher-order multiobjective symmetric dual problems under higher-order invexity and higher-order pseudoinvexity assumptions. Agarwal et al. [1] presented strong duality theorem for a pair of Mond-Weir type nondifferentiable multiobjective higher-order symmetric dual problems over arbitrary cones under higher-order $K$-$F$-convexity assumptions. Ahmad [2] formulated a unified higher-order symmetric dual for a nondifferentiable multiobjective programming problem and presented

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duality results under suitable assumptions. Suneja and Louhan [18] established weak, strong and converse duality results for a pair of Wolfe type higher-order symmetric dual problems over cones under the assumption of higher-order cone-invexity. They also obtained weak, strong and converse duality results for the pair of Mond-Weir type higher-order symmetric dual problems. Kassem [12] studied invexity. They also obtained weak, strong and converse duality results for a pair of Wolfe type higher-order symmetric dual models, and establish various types of duality results under suitable assumptions. Suneja and Louhan [18] established modified pair of higher-order symmetric dual models, and establish various types of duality results under appropriate assumptions. This fills in some gaps in the work of Kassem [12].

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided. New concepts of higher-order strong $K$-pseudoinvexity and higher-order strict $K$-pseudoinvexity are presented. Moreover, an example is constructed to show the relationship between higher-order strong $K$-pseudoinvexity and higher-order strict $K$-pseudoinvexity. In Section 3, we introduce a new pair of higher-order symmetric dual models. Under the assumption of higher-order generalized cone-invexity and other suitable conditions, weak, strong and converse duality theorems of higher-order symmetric dual problems are also derived. Furthermore, an example is provided to illustrate the results of strong duality theorem in this section. Finally, we draw some conclusions in Section 4.

2. Preliminaries. Let $\mathbb{R}^n$ denote $n$-dimensional Euclidean space with the non-negative orthant $\mathbb{R}^n_+$. Let $K$ be a closed convex cone in $\mathbb{R}^k$. The positive dual cone $K^+$ of $K$ is defined as $K^+ = \{z \in \mathbb{R}^k : x^Tz \geq 0 \text{ for all } x \in K\}$.

A general multiobjective optimization problem can be expressed in the following form:

\[(MOP) \quad K - \min \quad f(x) = (f_1(x), f_2(x), \ldots, f_k(x))\]
\[\text{s.t.} \quad x \in S = \{x \in C : -g(x) \in Q\},\]

where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and each component is differentiable; $\emptyset \neq C \subseteq \mathbb{R}^n$ is convex; $K$ and $Q$ are closed convex cones in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively.

The solution discussed in this paper is defined in the sense of efficiency as given below:
Definition 2.1. A point $\bar{x} \in S$ is said to be an efficient solution of (MOP), if there exists no other $x \in S$ such that
$$f(x) - f(\bar{x}) \not\in -K \setminus \{0\}.$$  

The following Fritz John type necessary optimality conditions for efficient solutions from [17] will be needed.

**Lemma 2.2.** If $\bar{x}$ is an efficient solution of (MOP), then there exist $\bar{\lambda} \in K^+$ and $\bar{\mu} \in Q^+$ with $(\bar{\lambda}, \bar{\mu}) \neq 0$ such that
$$\left[ \sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) \right] + \sum_{i=1}^{m} \bar{\mu}_j \nabla g_j(\bar{x}) \geq 0, \forall x \in C,$$
$$\sum_{i=1}^{m} \bar{\mu}_j g_j(\bar{x}) = 0.$$

In the rest of this paper, we assume that $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ ($i = 1, 2, \ldots, k$), $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable functions; $K$, $C_1$, $C_2$ are closed convex cones with nonempty interiors in $\mathbb{R}^k$, $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and for each $i = 1, 2, \ldots, n$ is the positive dual cone of $C_i$. Let $\eta_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\eta_2 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ be vector valued functions. Fix $e = (e_1, e_2, \ldots, e_i, \ldots, e_k)^T \in int K$, in which $e_i$ is the $i$-th component of $e$.

The following definitions are based on Definition 2.4 of Padhan and Nahak [16].

**Definition 2.3.** Let $\eta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ be a vector valued function. The vector valued function $f(x, \cdot)$ is said to be higher-order strongly $K-$pseudoinvex at $y \in \mathbb{R}^m$ with respect to $h$ and $\eta$, if for all $v \in \mathbb{R}^m$ and $p \in \mathbb{R}^m$,
$$-\eta(v, y)^T \nabla_y f(x, y) + \nabla_p h(x, y, p)e \not\in int K.$$

$$\Rightarrow f(x, v) - f(x, y) - [h(x, y, p) - p^T \nabla_p h(x, y, p)]e \not\in K \setminus \{0\}.$$

**Definition 2.4.** Let $\eta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ be a vector valued function. The vector valued function $f(x, \cdot)$ is said to be higher-order strictly $K-$pseudoinvex at $y \in \mathbb{R}^m$ with respect to $h$ and $\eta$, if for all $v \in \mathbb{R}^m$ and $p \in \mathbb{R}^m$,
$$-\eta(v, y)^T \nabla_y f(x, y) + \nabla_p h(x, y, p)e \not\in int K.$$

$$\Rightarrow f(x, v) - f(x, y) - [h(x, y, p) - p^T \nabla_p h(x, y, p)]e \not\in -K \setminus \{0\}.$$

**Remark 1.** (i) It is easy to see that when $K = \mathbb{R}_+$ and $h(x, y, p) \equiv 0$, both higher-order strong $K-$pseudoinvexity and higher-order strict $K-$pseudoinvexity reduces to pseudoinvexity as considered in Chandra and Kumar [5].

(ii) If $K$ is a pointed convex cone, then higher-order strong $K-$pseudoinvexity of $f(x, \cdot)$ at $y \in \mathbb{R}^m$ implies higher-order strict $K-$pseudoinvexity of $f(x, \cdot)$ at $y \in \mathbb{R}^m$ with respect to the same functions $h$ and $\eta$, but the converse fails. This can be seen via the following example.

**Example 2.1.** Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \left( \begin{array}{c} x_1 + x_2^2 + y_1 + y_2 \\ x_1 + x_2^2 - y_1 - y_2 \end{array} \right),$$
$$h(x, y, p) = p_1(1 + 2y_1) + p_2(1 + \sin(2y_2)),$$
$$\eta(v, y) = \left( \begin{array}{c} v_1 - y_1 \\ v_2 - y_2 \end{array} \right).$$
respectively. Let $K = \mathbb{R}_+^2$ and $e = (1,1)^T \in \text{int} K$.

Then $f(x, \cdot)$ is higher-order strictly $K$–pseudoinvex at $y = (0,0)^T$ with respect to $h$ and $\eta$, because for each $v \in \mathbb{R}^2$ and $p \in \mathbb{R}^2$

$$-\eta(v,y)^T[\nabla_y f(x,y) + \nabla_p h(x,y,p)e] = \left(\begin{array}{c} -2v_1 - 2v_2 \\ 0 \end{array}\right) \notin \text{int} K$$

$$\Rightarrow f(x,v) - f(x,y) - (h(x,y,p) - p^T \nabla_p h(x,y,p))e = \left(\begin{array}{c} v_1 + v_2 \\ -v_1 - v_2 \end{array}\right) \notin -K \setminus \{0\}.$$ 

However, $f(x, \cdot)$ is not higher-order strongly $K$–pseudoinvex at $y = (0,0)^T$ with respect to the same functions $h$ and $\eta$, because for $\tilde{v} = (1,0)^T \in \mathbb{R}^2$

$$-\eta(\tilde{v},y)^T[\nabla_y f(x,y) + \nabla_p h(x,y,p)e] = \left(\begin{array}{c} -2 \\ 0 \end{array}\right) \notin \text{int} K,$$

we have

$$f(x,\tilde{v}) - f(x,y) - (h(x,y,p) - p^T \nabla_p h(x,y,p))e = \left(\begin{array}{c} 1 \\ -1 \end{array}\right) \notin K.$$

3. Higher-order symmetric duality. We now formulate the following higher-order symmetric dual pair, where $e = (e_1, e_2, \ldots, e_k)^T \in \text{int} K$ is fixed.

(MP) $K - \min \ f(x,y) + [h(x,y,p) - p^T \nabla_p h(x,y,p)]e$

s.t. $-\left[ \sum_{i=1}^{k} \lambda_i \nabla_y f_i(x,y) + \nabla_p h(x,y,p) \right] \in C_2^+$, 

(1)

$$\left[\sum_{i=1}^{k} \lambda_i \nabla_y f_i(x,y) + \nabla_p h(x,y,p) \right] \ge 0,$$

(2)

$$p^T \left[ \sum_{i=1}^{k} \lambda_i \nabla_y f_i(x,y) + \nabla_p h(x,y,p) \right] \ge 0,$$

(3)

$$x \in C_1, y \in \mathbb{R}^m, \lambda \in \text{int} K^+, \lambda^T e = 1, p \in \mathbb{R}^m,$$

(4)

(MD) $K - \max \ f(u,v) + [g(u,v,r) - r^T \nabla_r g(u,v,r)]e$

s.t. $\left[ \sum_{i=1}^{k} \lambda_i \nabla_u f_i(u,v) + \nabla_r g(u,v,r) \right] \in C_1^+$, 

(5)

$$\left[\sum_{i=1}^{k} \lambda_i \nabla_u f_i(u,v) + \nabla_r g(u,v,r) \right] \le 0,$$

(6)

$$r^T \left[ \sum_{i=1}^{k} \lambda_i \nabla_u f_i(u,v) + \nabla_r g(u,v,r) \right] \le 0,$$

(7)

$$u \in \mathbb{R}^n, v \in C_2, \lambda \in \text{int} K^+, \lambda^T e = 1, r \in \mathbb{R}^n.$$ 

(8)

Remark 2. (i) If $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, K = \mathbb{R}_+, h(x,y,p) \equiv 0, g(u,v,r) \equiv 0, p = 0$ and $r = 0$, then the pair (MP) and (MD) reduce to the pair (P) and (D) introduced by Chandra and Kumar [5].
Then, Proof. By contradiction, suppose that the above symmetric dual models are equivalent to (MP) $K - \min f(x, y) + [h(x, y, p) - p^T \nabla_p h(x, y, p)] \epsilon$

\[ \begin{aligned}
\text{s.t. } & - \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + e_i \nabla_p h(x, y, p)] \in C_2^+,& (9) \\
y^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + e_i \nabla_p h(x, y, p)] & \geq 0, & (10) \\
p^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + e_i \nabla_p h(x, y, p)] & \geq 0, & (11) \\
x & \in C_1, y \in \mathbb{R}^m, \lambda \in \text{int} K^+, p \in \mathbb{R}^m, & (12)
\end{aligned} \]

(MD) $K - \max f(u, v) + [g(u, v, r) - r^T \nabla_v g(u, v, r)] \epsilon$

\[ \begin{aligned}
\text{s.t. } & \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + e_i \nabla_v g(u, v, r)] \in C_1^+, & (13) \\
u^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + e_i \nabla_v g(u, v, r)] & \leq 0, & (14) \\
r^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + e_i \nabla_v g(u, v, r)] & \leq 0, & (15) \\
u \in \mathbb{R}^n, v \in C_2, \lambda \in \text{int} K^+, r \in \mathbb{R}^n, & (16)
\end{aligned} \]

respectively. Therefore, in the following, we discuss the higher-order symmetric duality between the pair (MP) and (MD).

First, we present weak duality results, which state that the objective value of any feasible solution of (MP) is not less than those of (MD) under some appropriate conditions, such as higher-order strong cone-pseudoinvexity and higher-order strict cone-pseudoinvexity.

**Theorem 3.1.** (Weak duality) Let $(x, y, \lambda, p)$ and $(u, v, \lambda, r)$ be feasible solutions of (MP) and (MD), respectively. Assume that

(i) $f(\cdot, v)$ is higher-order strongly $K$-pseudoinvex at $u$ with respect to $g$ and $\eta_1$;
(ii) $-f(x, \cdot)$ is higher-order strictly $K$-pseudoinvex at $y$ with respect to $-h$ and $\eta_2$;
(iii) $\eta_1(x, u) + u \in C_1$;
(iv) $\eta_2(v, y) + y \in C_2$.

Then,

\[ f(u, v) + [g(u, v, r) - r^T \nabla_v g(u, v, r)] \epsilon \]

\[ - f(x, y) - [h(x, y, p) - p^T \nabla_p h(x, y, p)] \epsilon \notin K \setminus \{0\}. \]

Proof. By contradiction, suppose that

\[ f(u, v) + [g(u, v, r) - r^T \nabla_v g(u, v, r)] \epsilon \]

\[ - f(x, y) - [h(x, y, p) - p^T \nabla_p h(x, y, p)] \epsilon \in K \setminus \{0\}. \] (17)
By hypothesis (iv), we have

\[ [\eta_2(v, y) + y]^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + e_i \nabla_p h(x, y, p)] \leq 0, \quad (18) \]

as (9) holds.

Combining (10) and (18), we get

\[ \eta_2(v, y)^T \sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) + e_i \nabla_p h(x, y, p)] \leq 0, \]

which along with \( \lambda \) which together with \( \lambda \) yields that

\[ -\eta_2(v, y)^T [-\nabla_y f_1(x, y) - e_1 \nabla_p h(x, y, p), \ldots, -\nabla_y f_k(x, y) - e_k \nabla_p h(x, y, p)] \]

\[ = [\eta_2(v, y)^T \nabla_y f_1(x, y) + e_1 \nabla_p h(x, y, p), \ldots, \nabla_y f_k(x, y) + e_k \nabla_p h(x, y, p)] \notin \text{int} K. \]

It follows from the higher-order strict \( K \)-pseudoinvexity of \(-f(x, \cdot)\) at \( y \) with respect to \(-h\) and \( \eta_2 \) that

\[ f(x, v) \leq f(x, y) - [h(x, y, p) - p^T \nabla_p h(x, y, p)]e \notin K \setminus \{0\}. \]

(19)

According to (17) and (19), we obtain

\[ f(x, v) - f(u, v) - [g(u, v, r) - r^T \nabla_r g(u, v, r)]e \notin K. \]

Then, by the higher-order strong \( K \)-pseudoinvexity of \( f(\cdot, v) \) at \( u \) with respect to \( g \) and \( \eta_1 \), we derive

\[ -\eta_1(x, u)^T \nabla_u f_i(u, v) + e_i \nabla_r g(u, v, r), \ldots, \nabla_u f_k(u, v) + e_k \nabla_r g(u, v, r)] \in \text{int} K, \]

which together with \( \lambda \in \text{int} K^+ \) implies

\[ \eta_1(x, u)^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + e_i \nabla_r g(u, v, r)] < 0. \]

(20)

On the other hand, by (13) and hypothesis (iii), we get

\[ [\eta_1(x, u) + u]^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + e_i \nabla_r g(u, v, r)] \geq 0, \]

which along with (14) yields that

\[ \eta_1(x, u)^T \sum_{i=1}^{k} \lambda_i [\nabla_u f_i(u, v) + e_i \nabla_r g(u, v, r)] \geq 0. \]

This contradicts (20), and hence we conclude that

\[ f(u, v) + [g(u, v, r) - r^T \nabla_r g(u, v, r)]e \]

\[ - f(x, y) - [h(x, y, p) - p^T \nabla_p h(x, y, p)]e \notin K \setminus \{0\}. \]

Remark 3. (i) If conditions (i) and (ii) of Theorem 3.1 are replaced by

\( (i') f(\cdot, v) \) is higher-order strictly \( K \)-pseudoinvex at \( u \) with respect to \( g \) and \( \eta_1 \),

\( (ii') -f(x, \cdot) \) is higher-order strongly \( K \)-pseudoinvex at \( y \) with respect to \(-h\) and \( \eta_2 \).
Suppose that Theorem 3.2. (Strong duality) Let symmetric dual problem \((\text{MD}′)\) and \((\text{MD})\) respectively, the conclusion of Theorem 3.1 still holds. The proof is similar to those of Theorem 3.1.

(ii) If \(f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, K = \mathbb{R}^+, h(x, y, p) \equiv 0, g(u, v, r) \equiv 0, p = 0\) and \(r = 0\), then the weak duality theorem between the pair \((\text{MP}′)\) and \((\text{MD}′)\) (i.e., Theorem 3.1) reduces to the weak duality theorem between the pair \((P)\) and \((D)\) in [5].

In what follows, we move our attention to strong duality between the pair \((\text{MP}′)\) and \((\text{MD}′)\). This is the issue that how to get efficient solutions of higher-order symmetric dual problem \((\text{MD}′)\) from those of primal problem \((\text{MP}′)\).

Theorem 3.2. (Strong duality) Let \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})\) be an efficient solution to \((\text{MP}′)\). Suppose that

(i) the \(m \times m\) Hessian matrix \(\nabla_{pp}h(\bar{x}, \bar{y}, \bar{p})\) is positive or negative definite;
(ii) the set \(\{\nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}), i = 1, 2, \ldots, k\}\) is linearly independent;
(iii) \(\bar{x}\) is feasible for \((\text{MD}′)\), and the objective values of \((\text{MP}′)\) and \((\text{MD}′)\) are equal. Furthermore, if the conditions of weak duality theorem (Theorem 3.1) hold for all feasible solutions of \((\text{MD}′)\), then \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})\) is an efficient solution of \((\text{MD}′)\).

Proof. Since \((\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})\) is an efficient solution to \((\text{MP}′)\), it follows from Lemma 2.2 that there exist \(\alpha \in K^+, \beta \in C_2, \mu \in \mathbb{R}^+\) and \(\delta \in \mathbb{R}^+\) such that

\[
\begin{align*}
\left\{ \sum_{i=1}^{k} \alpha_i \left[ \nabla_{xx} f_i(\bar{x}, \bar{y}) + e_i \nabla_x h(\bar{x}, \bar{y}, \bar{p}) \right] - \alpha^T e \nabla_{px} h(\bar{x}, \bar{y}, \bar{p}) \cdot \bar{p} \\
+ \sum_{i=1}^{k} \lambda_i \left[ \nabla_{yx} f_i(\bar{x}, \bar{y}) + e_i \nabla_{yx} h(\bar{x}, \bar{y}, \bar{p}) \right] (\beta - \mu \bar{y} - \delta \bar{p}) \right\} (x - \bar{x}) \geq 0, \forall x \in C_1, \\
\sum_{i=1}^{k} (\alpha_i - \mu \lambda_i) \left[ \nabla_{xy} f_i(\bar{x}, \bar{y}) + e_i \nabla_y h(\bar{x}, \bar{y}, \bar{p}) \right] - \alpha^T e \nabla_{pp} h(\bar{x}, \bar{y}, \bar{p}) \cdot \bar{p} \\
+ \sum_{i=1}^{k} \lambda_i \left[ \nabla_{yy} f_i(\bar{x}, \bar{y}) + e_i \nabla_{yy} h(\bar{x}, \bar{y}, \bar{p}) \right] (\beta - \mu \bar{y} - \delta \bar{p}) = 0, \\
(\beta - \mu \bar{y} - \delta \bar{p})^T \left[ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right] = 0, i = 1, 2, \ldots, k, \\
\nabla_{pp} h(\bar{x}, \bar{y}, \bar{p}) \left[ \lambda^T e \beta - \lambda^T e \mu \bar{y} - \lambda^T e \delta \bar{p} - \alpha^T e \bar{p} \right] \\
- \delta \sum_{i=1}^{k} \lambda_i \left[ \nabla_{yy} f_i(\bar{x}, \bar{y}) + e_i \nabla_{yy} h(\bar{x}, \bar{y}, \bar{p}) \right] = 0, \\
\beta^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_{yx} f_i(\bar{x}, \bar{y}) + e_i \nabla_{yx} h(\bar{x}, \bar{y}, \bar{p}) \right] = 0, \\
\mu \bar{y}^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_{yy} f_i(\bar{x}, \bar{y}) + e_i \nabla_{yy} h(\bar{x}, \bar{y}, \bar{p}) \right] = 0, \\
\delta \bar{p}^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_{yy} f_i(\bar{x}, \bar{y}) + e_i \nabla_{yy} h(\bar{x}, \bar{y}, \bar{p}) \right] = 0, \end{align*}
\]
follows from (29) that
\[(\alpha, \beta, \mu, \delta) \neq 0. \tag{28}\]

Multiplying (24) by \[\begin{bmatrix} \lambda^T \epsilon_\beta - \lambda^T e \mu \bar{y} - \lambda^T e \delta \bar{p} - \alpha^T e \bar{p} \end{bmatrix},\]
and combining (25), (26) and (27), we obtain
\[\begin{bmatrix} \lambda^T e \beta - \lambda^T e \mu \bar{y} - \lambda^T e \delta \bar{p} - \alpha^T e \bar{p} \end{bmatrix}^T \nabla_{pp} h(\bar{x}, \bar{y}, \bar{p}) \begin{bmatrix} \lambda^T e \beta - \lambda^T e \mu \bar{y} - \lambda^T e \delta \bar{p} - \alpha^T e \bar{p} \end{bmatrix} = 0. \tag{29}\]

Since the \(m \times m\) Hessian matrix \(\nabla_{pp} h(\bar{x}, \bar{y}, \bar{p})\) is positive or negative definite, it follows from (29) that
\[\lambda^T e \beta - \lambda^T e \mu \bar{y} - \lambda^T e \delta \bar{p} - \alpha^T e \bar{p} = 0. \tag{30}\]

Note that \(K\) has nonempty interiors and \(\lambda \in \text{int} K^+\), we can easily verify that \(\lambda \neq 0\). Moreover, by hypotheses (ii), we have
\[\sum_{i=1}^{k} \lambda_i \left[ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right] \neq 0. \tag{31}\]

From (24), (30) and (31), it follows that
\[\delta = 0, \tag{32}\]
and
\[\lambda^T e(\beta - \mu \bar{y}) - \alpha^T e \bar{p} = 0. \tag{33}\]

Now, we claim that \(\alpha \neq 0\). Suppose on the contrary that \(\alpha = 0\). From (33), it follows that \(\beta = \mu \bar{y}\) as \(\lambda \in \text{int} K^+\) and \(e \in \text{int} K\). Taking \(\alpha = 0, \beta = \mu \bar{y}\) and (32) in (22), we deduce
\[\mu \sum_{i=1}^{k} \lambda_i \left[ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right] = 0.\]

By (31), \(\mu = 0\), which along with \(\beta = \mu \bar{y}\) yields that \(\beta = 0\), that is, \((\alpha, \beta, \mu, \delta) = 0\), which contradicts (28). Therefore
\[\alpha \in K^+ \setminus \{0\} \text{ and } \alpha^T e > 0.\]

Next, we show \(\bar{p} = 0\). Multiplying (23) by \(\bar{\lambda}_i\), and summing up over \(i \in \{1, 2, \ldots, k\}\), we have
\[((\beta - \mu \bar{y} - \delta \bar{p}) \sum_{i=1}^{k} \lambda_i \left[ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right] = 0,\]
which together with (32) and (33) implies that
\[\bar{p}^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right] = 0,\]
as \(\lambda^T e > 0\) and \(\alpha^T e > 0\). By hypothesis (iii), we get
\[\bar{p} = 0. \tag{34}\]

Since \(\lambda^T e > 0\) and \(\bar{p} = 0\), it follows from (33) that
\[\beta = \mu \bar{y}. \tag{35}\]

Putting (34) and (35) in (22), and by hypothesis (iv), we have
\[\sum_{i=1}^{k} (\alpha_i - \mu \bar{\lambda}_i) \left[ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \right] = 0.\]
Then by hypothesis (ii), we get
\[ \alpha_i = \mu \bar{\lambda}, \quad i = 1, 2, \ldots, k, \] (36)
that is, \( \alpha = \mu \bar{\lambda} \), which together with \( \alpha \in K^+ \setminus \{0\} \) yields that \( \mu \neq 0 \). Therefore,
\[ \mu > 0. \] (37)
Because \( \beta \in C_2 \), it follows from (35) and (37) that
\[ \bar{y} = \frac{\beta}{\mu} \in C_2. \] (38)
Substituting (32), (34) and (35) into (21), we obtain
\[ \sum_{i=1}^{k} \alpha_i \left[ \nabla_x f_i(\bar{x}, \bar{y}) + e_i \nabla_x h(\bar{x}, \bar{y}, 0) \right]^T (x - \bar{x}) \geq 0, \quad \forall x \in C_1. \] (39)
Since \( C_1 \) is a convex cone, it follows from (39) that
\[ \sum_{i=1}^{k} \alpha_i \left[ \nabla_x f_i(\bar{x}, \bar{y}) + e_i \nabla_x h(\bar{x}, \bar{y}, 0) \right]^T x \geq 0, \quad \forall x \in C_1, \] (40)
which yields
\[ \sum_{i=1}^{k} \alpha_i \left[ \nabla_x f_i(\bar{x}, \bar{y}) + e_i \nabla_x h(\bar{x}, \bar{y}, 0) \right] \in C_1^+. \] (41)
Taking \( x = 0 \) and \( x = 2\bar{x} \) in (39), respectively, we can get
\[ \sum_{i=1}^{k} \alpha_i \left[ \nabla_x f_i(\bar{x}, \bar{y}) + e_i \nabla_x h(\bar{x}, \bar{y}, 0) \right]^T \bar{x} = 0. \] (42)
Let \( \bar{r} = 0 \). By hypothesis (iv), we can derive from (36), (37), (38), (41) and (42) that \( (\bar{x}, \bar{y}, \bar{\lambda}, 0) \) is feasible for \((MD')\). Also, hypothesis (iv) implies that the objective values of \((MP')\) and \((MD')\) are equal. The efficiency of \( (\bar{x}, \bar{y}, \bar{\lambda}, 0) \) to \((MD')\) follows from the weak duality theorem (Theorem 3.1).

**Remark 4.** Based on the proof of Theorem 3.2, we conclude that the strong duality theorem between the pair \((MP')\) and \((MD')\) (i.e., Theorem 3.2) can reduce to the strong duality theorem between the pair \((P)\) and \((D)\) in [5], when \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \ K = \mathbb{R}_+, \ h(x, y, p) \equiv 0, \ g(u, v, r) \equiv 0, \ p = 0 \) and \( r = 0 \).

Now, we give an example to illustrate strong duality theorem (Theorem 3.2).

**Example 3.1.** Let \( f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \ h : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \ g : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = \left( \begin{array}{c} x_1^2 + x_2^2 + \sin y_1 - \sin y_2 \\ x_1 + x_2 - \sin y_1 + \sin y_2 \end{array} \right),
\]
h(x, y, p) = \( p_1(1-2\cos y_1) + p_2(1-2\cos y_2) - \frac{1}{2}p_1^2 - \frac{1}{2}p_2^2 \),
g(u, v, r) = -u_1r_1 - u_2r_2 + \frac{1}{2}r_1^2 + \frac{1}{2}r_2^2.
respectively. Let $C_1 = C_2 = K = \mathbb{R}^2_+$, and $e = (1, 1)^T \in \text{int}K$. Then the pair of dual problems $(MP')$ and $(MD')$ are as follows:

$$(MP') \quad \text{min} \left( \begin{array}{c} x_1^2 + x_2^2 + \sin y_1 - \sin y_2 + \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 \\ x_1 + x_2 - \sin y_1 + \sin y_2 + \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 \end{array} \right)$$

s.t. $-\lambda_1 \left( \begin{array}{c} 1 - \cos y_1 - p_1 \\ 1 - 3 \cos y_2 - p_2 \end{array} \right) - \lambda_2 \left( \begin{array}{c} 1 - 3 \cos y_1 - p_1 \\ 1 - \cos y_2 - p_2 \end{array} \right) \geq 0,$

$$(MD') \quad \text{max} \left( \begin{array}{c} u_1^2 + u_2^2 + \sin v_1 - \sin v_2 - \frac{1}{2} r_1^2 - \frac{1}{2} r_2^2 \\ u_1 + u_2 - \sin v_1 + \sin v_2 - \frac{1}{2} r_1^2 - \frac{1}{2} r_2^2 \end{array} \right)$$

s.t. $\lambda_1 \left( \begin{array}{c} u_1 + r_1 \\ u_2 + r_2 \end{array} \right) + \lambda_2 \left( \begin{array}{c} 1 - u_1 + r_1 \\ 1 - u_2 + r_2 \end{array} \right) \leq 0,$

$u^T \left[ \begin{array}{c} \lambda_1 \left( \begin{array}{c} u_1 + r_1 \\ u_2 + r_2 \end{array} \right) + \lambda_2 \left( \begin{array}{c} 1 - u_1 + r_1 \\ 1 - u_2 + r_2 \end{array} \right) \right] \leq 0,$

$\bar{u} \in \mathbb{R}^2, v \geq 0, \lambda > 0, r \in \mathbb{R}^2.$

Take $\bar{x} = \bar{y} = \bar{p} = (0, 0)^T$, $\bar{\lambda} = (1, 1)^T$. It is easy to verify that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is an efficient solution to $(MP')$.

All conditions of Theorem 3.2 are satisfied. In fact,

(i) the $2 \times 2$ Hessian matrix $\nabla_{pp} h(\bar{x}, \bar{y}, \bar{p}) = \left( \begin{array}{cc} -1 & 0 \\ 1 & -1 \end{array} \right)$ is negative definite;

(ii) the set $\{ \nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \}_{i=1}^k$ is linearly independent;

(iii) $\bar{p}^T \sum_{i=1}^k \bar{\lambda}_i [\nabla_y f_i(\bar{x}, \bar{y}) + e_i \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0 \implies \bar{p} = 0$;

(iv) $h(\bar{x}, \bar{y}, 0) = g(\bar{x}, \bar{v}, 0) = \nabla_p h(\bar{x}, \bar{y}, 0) = \nabla_p h(\bar{x}, \bar{y}, 0) = \nabla_p h(\bar{x}, \bar{y}, 0)$.

Furthermore, for any feasible solution $(x, y, \lambda, p)$ and $(u, v, \lambda, r)$ of $(MP')$ and $(MD')$, respectively, $f(\cdot, v)$ is higher-order strongly $K$-pseudoinvex at $u$ with respect to $g$ and $\eta_1(x, u) = x - u; -f(x, \cdot)$ is higher-order strictly $K$-pseudoinvex at $y$ with respect to $-h$ and $\eta_2(v, y) = v - y$. And it is clear that $\eta_1(x, u) + u = x \in C_1, \eta_2(v, y) + y = v \in C_2$.

If $\bar{r} = (0, 0)^T$, then it is obviously that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ is an efficient solution of $(MD')$.

In the end, we state the converse duality theorem whose proof follows on the lines of strong duality theorem (Theorem 3.2).

**Theorem 3.3.** (Converse duality) Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r})$ be an efficient solution to $(MD')$. Suppose that

(i) the $m \times m$ Hessian matrix $\nabla_{rr} g(\bar{u}, \bar{v}, \bar{r})$ is positive or negative definite;

(ii) the set $\{ \nabla_u f_i(\bar{u}, \bar{v}) + e_i \nabla_v g(\bar{u}, \bar{v}, \bar{r}) \}_{i=1}^k$ is linearly independent;

(iii) $\bar{r}^T \sum_{i=1}^k \bar{\lambda}_i [\nabla_u f_i(\bar{u}, \bar{v}) + e_i \nabla_v g(\bar{u}, \bar{v}, \bar{r})] = 0 \implies \bar{r} = 0$;

(iv) $g(\bar{u}, \bar{v}, 0) = h(\bar{u}, \bar{v}, 0), \nabla_v g(\bar{u}, \bar{v}, 0) = \nabla_p h(\bar{u}, \bar{v}, 0)$, $\nabla_u g(\bar{u}, \bar{v}, 0) = \nabla_p h(\bar{u}, \bar{v}, 0)$. 
Then, \((\bar{u}, \bar{v}, \bar{\lambda}, 0)\) is feasible for \((MP')\), and the objective values of \((MP')\) and \((MD')\) are equal. Also, if the assumptions of weak duality theorem (Theorem 3.1) are satisfied for all feasible solutions of \((MP')\), then \((\bar{u}, \bar{v}, \bar{\lambda}, 0)\) is an efficient solution of \((MP')\).

4. Conclusion. The study of higher-order duality has become more attractive, due to the computational advantage over first-order duality that it provides tighter bounds for the value of the objective function when approximations are used. So, it is valuable to investigate higher-order duality for multiobjective programming. In this paper we present a pair of higher-order symmetric dual models, and establish various types of duality results under appropriate assumptions. Some examples are presented to illustrate our main results. This work fills in some gaps in the work of Kassem [12], and also extends the symmetric duality results of Chandra and Kumar [5] to multiobjective programming with cone domains.

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