The Single Flavor Color Superconductivity in a Magnetic Field

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We investigate the single flavor color superconductivity in a magnetic field. Because of the absence of the electromagnetic Meissner effect, forming a nonspherical CSC phase, polar, A or planar, does not cost energy of excluding magnetic flux. We found that these nonspherical phases do occupy a significant portion of the phase diagram with respect to magnetic field and temperature and may be implemented under the typical quark density and the magnetic field inside a neutron star.

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A cold quark matter will become a color superconductor at sufficiently high baryon density. In the core region of a compact star, the baryon density is expected to be several times higher than that of a normal nuclear matter. The quarks may be released from hadrons and form a quark matter of $\mu \sim 400 - 500$ MeV, providing an opportunity to the color superconductivity (CSC).

While the pairing force is maximized in the s-wave channel, the antisymmetry of the wave function requires the Cooper pairing between different quark flavors. But the mass of strange quarks and the charge neutrality induces a substantial Fermi momentum mismatch among different flavors and reduces the phase space available for pairing. A number of exotic 2 flavor or 3 flavor CSC phases have been proposed without reaching a consensus solution. The single flavor CSC (pairing within each flavor) becomes a potential candidate even at disadvantage of a reduced pairing strength. The dominant angular momentum of the single flavor Cooper pair is one. Like the superfluidity of 3He, there are a number of different pairing states and we shall focus in this letter the four of them: the spherical color-spin locked (CSL) state and nonspherical polar, A and planar ones. Without a magnetic field, the CSL pairing is energetically most favored, even when the angular momentum mixing effect is taken into account.

The energy balance among different single flavor CSC phases will be offset in a magnetic field, which is present in a compact star and could exceed $10^{15}$ G in magnitude. Only the CSL phase shield the magnetic field. The electromagnetic Meissner effect is absent for nonspherical states (polar, A or planar). Cooling a normal quark matter to the CSL will costs an extra amount work to expel out the magnetic flux. Being free from such a penalty, nonspherical phases may show up at a sufficiently high magnetic field. Obtaining the phase diagram of a single flavor CSC with respect to temperature and magnetic field is the main scope of the present letter.

The structure of the Meissner effect in a single flavor pairing is determined by the pattern of its symmetry breaking. The condensate of a diquark operator takes the form

$$\Phi = <\bar{\psi}_c \Gamma^c \lambda^c \psi>$$  \hspace{1cm} (1)

where $\psi$ is the quark field, $\psi_{C} = \gamma_{2} \psi^*$ is its charge conjugate, $\lambda^c$ with $c = 2, 5, 7$ is an antisymmetric Gell-Mann matrices and $\Gamma^c$ is a $4 \times 4$ spinor matrix. We may choose $\Gamma^5 = \Gamma^7 = 0$ for the polar and A phases, $\Gamma^2 = 0$ for the planar phase but none of $\Gamma^c$'s vanishes for CSL phase. The condensate of CSL breaks the gauge symmetry $SU(3)_c \times U(1)_{em}$ completely. A nonspherical condensate, however, breaks the gauge symmetry partially and the Meissner effect is incomplete. Among the residual gauge group, there exists a U(1) transformation, $\psi \rightarrow e^{-\frac{i}{2}\lambda_8 \theta - iqA} \psi$ with $q$ the electric charge of $\psi$, $\theta = -2\sqrt{3}q\phi$ for the polar and A phases and $\theta = 4\sqrt{3}q\phi$ for the planar phase. The corresponding gauge field, $A_{\mu}$ is identified with the electromagnetic field in the condensate. It is related to the electromagnetic field $A$ and the 8-th component of the color field $A^8$ in the normal phase through a rotation

$$A_{\mu} = A_{\mu} \cos \gamma - A_{\mu}^8 \sin \gamma$$
$$V_{\mu} = A_{\mu} \sin \gamma + A_{\mu}^8 \cos \gamma$$  \hspace{1cm} (2)

where $\tan \gamma = -2\sqrt{3}q(e/g)$ for polar and A, and $\tan \gamma = 4\sqrt{3}q(e/g)$ for planar with $g$ the QCD running coupling constant. The 2nd component of $V = 0$ because of the Meissner effect and thereby imposes a constraint inside a nonspherical CSC, $A_{\mu}^8 = -A_{\mu} \tan \gamma$, which implies the relation

$$B^8 = -B \tan \gamma$$  \hspace{1cm} (3)

between the color and the ordinary magnetic fields. Expressing the gauge coupling

$$\bar{\psi} \gamma_{\mu} (eqA_{\mu} + \frac{1}{2}A_{\mu}^8 \lambda_8) \psi$$  \hspace{1cm} (4)

in terms of $A_{\mu}$ and its orthogonal partner $V_{\mu}$, we extract
the electric charges with respect to $\mathcal{A}$ in color space,

$$Q = \begin{cases} \frac{3qg}{\sqrt{g^2 + 12q^2 e^2}} \text{diag.}(0, 0, 1) & \text{for polar and } \mathcal{A} \\ \frac{3qg}{\sqrt{g^2 + 48q^2 e^2}} \text{diag.}(1, 1, -1) & \text{for planar.} \end{cases}$$

(5)

The thermal equilibrium in a magnetic field $H$ is determined by minimizing the Gibbs free energy density, 

$$\mathcal{G} = \Gamma - BH$$

(6)

where $\Gamma$ is the thermodynamical potential in the grand canonical ensemble. Ignoring the induced magnetization due to the normal current, we have

$$\Gamma = \frac{1}{2}B^2 + \frac{1}{2} \sum_{i=1}^{8} (B_i^2)^2 - p$$

(7)

where $p$ is the pressure at $B = 0$, maximized with respect the gap parameter in the case of the CSC phase. The minimization with respect to $B$ and $B^i$ in other CSC phases is subject to the constraint imposed by the Meissner effect. For a hypothetical quark matter of one flavor only, we find that

$$\mathcal{G} = \begin{cases} -p_n - \frac{1}{2}H^2, & \text{for normal phase} \\ -p_{\text{CSL}}, & \text{for CSL} \\ -p_i - \frac{1}{2}H^2 \cos^2 \gamma_i, & \text{for } i = \text{polar, A, planar} \end{cases}$$

(8)

after the minimization. As will be shown below,

$$p_n < p_{\mathcal{A}} < p_{\text{polar}} < p_{\text{planar}} < p_{\text{CSL}}.$$ 

(9)

The phase corresponding to minimum among $\mathcal{G}$’s above wins the competition and transition from one phase to another is first order below $T_c$.

The situation becomes more subtle when quarks of different flavors coexist even though pairing is within each flavor. Different electric charges of different quark flavors imply different mixing angles which may not be compatible with each other. Consider a quark matter of $u$ and $d$ flavors with each flavor in a nonspherical CSC phase with different mixing angles. Eq. (9) imposes two constraints, which are consistent with each other only if $B = B^8 = 0$. Then we end with an effective Meissner shielding [8], making it fail to compete with the phase of both flavors in CSL states. On the other hand, one may relax the constraints by assuming that the basis underlying the CSC phase of $u$ quarks differ from that underlying the CSC of $d$ quarks by a color rotation. Consequently the constraint (3) for each flavor yields $B^8 = -B \tan \gamma^u$ and $\mathcal{B}^8 = -B \tan \gamma^d$. If both flavors stay in the polar or planar phases, which allows $B^{1-3}$ to penetrate in, an orthogonal transformation

$$B^{a8} = B^8 \cos \beta - B^3 \sin \beta$$

$$B^{a3} = B^8 \sin \beta + B^3 \cos \beta$$

(10)

could compromise both constraints. Such a transformation, however, cannot be implemented in an adjoint representation of the color $SU(3)$ and therefore, the mutual rotation of color basis is not an option. The phases of the two flavor quark matter ($u,d$) without Meissner effects, which can compete with (CSL,CSL), include (polar,planar), (polar(normal), normal(polar)), ((A(normal), normal(A)) and (normal, normal). Notice the coincidence of the mixing angle of the polar phase of $u$ quarks and that of the planar phase of the $d$ quarks. Also the normal phase does not impose any constraint on the gauge field and can coexist with any nonspherical CSC.

The Gibbs free energies of (normal, normal) and (CSL, CSL) phases remain given by the first and the second equations of (8), but with $p_n$ and $p_{\text{CSL}}$ referring to the total pressure of $u$ and $d$ quarks. For nonspherical phases, we have

$$\mathcal{G} = -p - \frac{1}{2}H^2 \cos \gamma.$$ 

(11)

where $p$ is the total pressure of both flavors with at least one of them in a nonspherical CSC state and $\gamma$ is their common mixing angle. For normal-CSC combination, $\gamma$ refers to that of the CSC state. The charge neutrality condition is imposed in all phases, which makes the Fermi sea of $d$ quarks larger than that of $u$ quarks. The color neutrality condition is ignored owing to small energy gap associated to the single flavor pairing. The number of combinations to be examined is reduced by two criteria: 1) For two combinations of the same mixing angle, the one with higher pressure wins. 2) For two combinations of the same pressure, the one with smaller magnitude of the mixing angle wins. It follows that there are only four phases to be considered in each case of two and three flavors, which are shown in Table I. The phase diagram

| I       | II       | III      | IV       | $H_0(10^{14}G)$ | $T_c(10^{-1}MeV)$ |
|---------|----------|----------|----------|----------------|------------------|
| 2 flavor| CSL$_{u}$, CSL$_{d}$ | (polar)$_{u}$, (planar)$_{d}$ | (normal)$_{u}$, (polar)$_{d}$ | (normal)$_{u}$, (normal)$_{d}$ | 5.44 | 1.35 |
| 3 flavor| CSL$_{u}$, CSL$_{d,s}$ | (polar)$_{u}$, (planar)$_{d,s}$ | (normal)$_{u}$, (polar)$_{d,s}$ | (normal)$_{u}$, (normal)$_{d,s}$ | 1.97 | 0.49 |
of each case in ultrarelativistic limit will be determined below and their relevance to the realistic s quark mass will be discussed afterwards.

The pressure of the single flavor CSC in the absence of a magnetic field has been obtained in the literature at zero temperature within the framework of the one-gluon-exchange. We shall extend the analysis up to the transition temperature \( T_c \), which is universal for all single flavor pairings. To avoid the technical complexity of the one-gluon-exchange, we shall work with a NJL-like effective action which picks up only the dominant pairing channel of the former, the transverse pairing, in the ultra-relativistic limit. The Hamiltonian of the effective action reads:\[ \mathcal{H} = \int d^3 \mathbf{r} \left[ \bar{\psi} \gamma_i \mathbf{\nabla} \psi - \mu \gamma_0 \psi + G \bar{\psi} \gamma_i \gamma_0 \gamma_j \psi \right]. \] (12)

with \( T^i = \frac{1}{2} \lambda^i \) and \( G \) an effective coupling. Introducing the condensate \( \langle \bar{\psi} \gamma_0 \gamma_5 \psi \rangle \), we find the pressure of each flavor under mean field approximation

\[ p = - \frac{2}{\Omega} \sum_k \left[ (k - \mu - E_{k}) \right] - \frac{1}{\Omega} \sum_k \left[ k - \mu - |k - \mu| \right] 
+ \frac{2T}{\Omega} \sum_k \ln \left( 1 + e^{-\frac{|k - \mu|}{T}} \right) - \frac{9}{4G} \Delta^2 
+ \frac{4T}{\Omega} \sum_k \ln \left( 1 + e^{-\frac{k - \mu}{T}} \right), \] (13)

where \( E_k = \sqrt{(k - \mu)^2 + \Delta^2 f^2(\theta)} \) with \( \theta \) the angle between \( k \) and a prefixed spatial direction and \( \Delta \) given by the solution of the gap equation \( \left( \frac{\partial p}{\partial \Delta} \right)_\mu = 0 \). The function \( f(\theta) \) is given by

\[ f(\theta) = \begin{cases} 
1, & \text{for CSL phase} \\
\sqrt{\frac{3}{2}} (1 + \cos \theta), & \text{for planar phase} \\
\sqrt{\frac{3}{2}} \sin \theta, & \text{for polar phase} \\
\sqrt{3} \cos^2 \frac{\theta}{2}, & \text{for A phase}
\end{cases} \] (14)

Introducing \( \Delta p_s = p_s - p_n = \rho_s(T) \frac{\mu^2 \Delta^2}{2\sqrt{2}} \) with \( s \) labeling different pairing states and \( \Delta_0 \) the CSL gap at \( T = 0 \). We have \( \rho_{\text{CSL}}(0) = 1, \rho_{\text{planar}}(0) = 0.98, \rho_{\text{polar}}(0) = 0.88 \) and \( \rho_{\text{A}}(0) = 0.65 \), and \( \rho_s(T_c) = 0 \) with \( T_c = \frac{\mu}{\Delta_0} \). The function \( \rho_s(T) \) for \( 0 < T < T_c \) of various states are displayed in Fig.1, which satisfy the inequities [4].

In a multiflavor quark matter, the Fermi momentum of each flavor is displayed from each other to meet the charge neutrality requirement. For an ideal gas of (u,d) quarks and electrons at zero temperature, we find that \( k_u = 0.87 \mu \) and \( k_d = 1.09 \mu \). While for an ideal gas of (u,d,s) quarks and electrons with \( m_s << \mu \), we obtain that \( k_u = \mu, k_d = \mu + \frac{n_s^2}{4\mu} \) and \( k_s = \mu - \frac{n_s^2}{4\mu} \). The corrections brought about by nonzero temperature and/or gap parameters contribute a higher order term than \( O(\mu^2 \Delta^2) \) to the pressure and can be neglected here.

By balancing the Gibbs free energy of different phases, we obtain the phase diagram with respect to temperature and magnetic field. The two flavor and three flavor cases are shown in Fig.2, where \( H_0 \) is defined by

\[ H_0 = \frac{\mu \Delta_0}{\pi}. \] (15)

If we calibrate the effective coupling \( G \) by identifying \( \Delta_0 \) with that of the one-gluon exchange\[ \Delta_0 = 512 \pi^4 \left( \frac{2}{N_f} \right)^{\frac{3}{2}} \frac{\mu}{g^5} \exp \left( -\frac{3\pi^2}{2g} - \frac{\pi^2}{4} - \frac{9}{2} \right) \]

extrapolated to \( \mu = 500 \text{MeV} \) and \( \alpha_s = 1 \), we end up with the values of \( H_0 \) and \( T_c \) in Table I. For the three flavor case, we ignored the Fermi momentum mismatch to be consistent with the ultra-relativistic approximation.

A critical reader may challenge our ultra-relativistic treatment of s quarks in the three flavor case, which may be justified as follows: The maximum Fermi momentum mismatch supporting a cross-flavor pairing scales with the energy gap, which is much smaller than the chemical potential. The realistic value of the s quark mass (\( \approx 150 \text{MeV} \)), could induced a substantial mismatch that suppresses cross flavor pairings, but remains small enough to warrant an ultra-relativistic approximation of the pairing dynamics. Therefore we argue that the three flavor panel of Fig.2 captured the gross features of the phase diagram with the realistic s quark mass. We also admit that the approximation may be marginal and perturbation of the mass will be considered in near future.

The analysis up to now ignores the magnetization \( M = \frac{\partial \rho_s}{\partial B} \) in the absence of the Meissner effect and we attempt to justify this approximation here. The potential hazard comes from the de Hass-von Alphen (dHvA) effect stemming from the discreteness of the Landau orbits.
if the mean free path \( l \) of quarks is longer than the cyclotron radius, \( \mu/(eB) \). Even though the magnetic field in the phase diagrams is weak in the sense \((eB)^2 << \mu\) , a large magnetization may emerge through the derivative of the gap function at \( T \approx 0 \). This is also expected to be the case in a nonspherical CSC phase. Because of the nonzero charges of the pairing partners Eq.(5), the Launder orbits also impacts on the energy gap in the planar phase and a similar issue for CFL has been addressed numerically in Ref. [12].

Our analytic work reveals that these nonspherical phases occupy a significant portion of the \( H - T \) phase diagram for the plausible magnetic field. The physical implications of these possible phases to the cooling behaviors and r-mode instability of neutron stars are interesting topics deserving further investigations [13].

The nonspherical phases discussed in this paper are all homogeneous in space. A domain wall structure was suggested in Ref. [14] in the context of 2SC and CFL in a magnetic field. The mechanism involves the absence of the Meissner effect, the chiral symmetry breaking and the axial anomaly. Since the transverse pairing, which pairs quarks of opposite helicities, also breaks the chiral symmetry, it would be interesting to extend the analysis of Ref. [14] to the nonspherical phases. We have not considered the noninert phases discussed in Ref. [12]. But in any case, the importance of the nonspherical CSC in a magnetic field, revealed in this letter, will stand up.

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FIG. 2: H-T phase diagram for two flavors and three flavors.

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