AN EXTENSION OF THE METHOD OF CHARACTERISTIC TO A SYSTEM OF PARTIAL DIFFERENTIAL OPERATORS – AN APPLICATION TO THE WEYL EQUATION WITH EXTERNAL FIELD BY “SUPER HAMILTONIAN PATH-INTEGRAL METHOD”

ATSUSHI INOUE

Dedicated to the memory of Professor Masaya Yamaguti

Abstract. A system of PDOs (=Partial Differential Operators) has two non-commutativities, (i) one from \([\partial_q, q] = 1\) (Heisenberg relation), (ii) the other from \([A, B] \neq 0\) for \(A, B\) being matrices in general. Non-commutativity from Heisenberg relation is nicely controlled by using Fourier transformations (i.e. the theory of ΨDOs=pseudo-differential operators).

Here, we give a new method of treating non-commutativity \([A, B] \neq 0\), and explain by taking the Weyl equation with external electro-magnetic potentials as the simplest representative for a system of PDOs. More precisely, we construct a Fourier Integral Operator with “matrix-like phase and amplitude” which gives a parametrix for that Weyl equation. To do this, we first reduce the usual matrix valued Weyl equation on the Euclidian space to the one on the superspace, called the super Weyl equation. Using analysis on superspace, we may associate a function, called the super Hamiltonian function, corresponding to that super Weyl equation. Starting from this super Hamiltonian function, we define phase and amplitude functions which are solutions of the Hamilton-Jacobi equation and the continuity equation on the superspace, respectively. Then, we define a Fourier integral operator with these phase and amplitude functions which gives a good parametrix for the initial value problem of that super Weyl equation. After taking the Lie-Trotter-Kato limit with respect to the time slicing, we get the desired evolutionary operator of the super Weyl equation. Bringing back this result to the matrix formulation, we have the final result. Therefore, we get a quantum mechanics with spin from a classical mechanics on the superspace which answers partly the problem of Feynman.

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1. Introduction and Result

1.1. The method of characteristics. It is well-known that the following initial value problem on the region $\Omega$ in $\mathbb{R}^{1+m}$

\[
\begin{cases}
\frac{\partial}{\partial t} u(t, q) + \sum_{j=1}^{m} a_j(t, q) \frac{\partial}{\partial q_j} u(t, q) = b(t, q) u(t, q) + f(t, q), \\
u(t, q) = \underline{u}(q),
\end{cases}
\]

is solved by the method of characteristics. Denoting the solution of the characteristic equation

\[
\begin{cases}
\frac{d}{dt} q_j(t) = a_j(t, q(t)), \\
q_j(t) = \underline{q}_j \quad (j = 1, \ldots, m),
\end{cases}
\]

by

\[q(t) = q(t, \underline{t}; \underline{q}) = (q_1(t), \ldots, q_m(t)) \in \mathbb{R}^m,\]

we get

Proposition 1.1. Let $a_j \in C^1(\Omega : \mathbb{R})$ and $b, f \in C(\Omega : \mathbb{R})$. For any point $(\underline{t}, \underline{q}) \in \Omega$, we assume that $\underline{u}$ is $C^1$ in a neighbourhood of $\underline{q}$.

Then, in a neighbourhood of $(\underline{t}, \underline{q})$, there exists uniquely a solution $u(t, q)$. More precisely, putting

\[U(t, \underline{q}) = e^{\int_{\underline{t}}^{t} \frac{d\tau}{\tau} B(\tau, q)} \left\{ \int_{\underline{t}}^{t} ds e^{-\int_{\underline{t}}^{s} \frac{d\tau}{\tau} B(\tau, q)} F(s, \underline{q}) + \underline{u} \right\},\]

that solution is represented by

\[u(t, \underline{q}) = U(t, y(t, \underline{t}; \underline{q}))\]

where $B(t, \underline{q}) = b(t, q(t, \underline{t}; \underline{q}))$, $F(t, \underline{q}) = f(t, q(t, \underline{t}; \underline{q}))$ and $\underline{q} = y(t, \underline{t}; \underline{q})$ is a inverse function derived from $\underline{q} = q(t, \underline{t}; \underline{q})$.

Problem: Is it possible to extend the method of characteristics to the system of PDOs?

1.2. Weyl equation. We take the Weyl equation as the simplest representative of a system of PDOs and we give an answer of the following problem.

Problem: Find if possible, an explicit representation of $\psi(t, q) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^2$ satisfying

\[
\begin{cases}
ith \frac{\partial}{\partial t} \psi(t, q) = \mathbb{H}(t) \psi(t, q), \\
\psi(t, q) = \psi(q),
\end{cases}
\]

(W)
where, \( t \) is an arbitrarily fixed initial time and

\[
\mathbb{H}(t) = \mathbb{H}(t, q, \frac{\hbar}{i} \frac{\partial}{\partial q}) = \sum_{k=1}^{3} c \sigma_k \left( \frac{\hbar}{i} \frac{\partial}{\partial q_k} - \frac{\varepsilon}{c} A_k(t, q) \right) + \varepsilon A_0(t, q).
\]

(1.1)

In the above, the Pauli matrices \( \{\sigma_j\} \) is represented by

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Remark: Though the meaning of explicit representation is not so clear, but it’s meaning will be clarified in the sequel.

**Claim:** We define a **good parametrix** for the initial value problem for the Weyl equation with a given external time-dependent electro-magnetic field \( (A_0(t, q), A_1(t, q), A_2(t, q), A_3(t, q)) \) by **Super Hamiltonian Path-Integral Method**. Here, essential is to introduce “the Hamiltonian mechanics corresponding to the Weyl equation”, and to define Fourier Integral Operators using quantities based on the Hamilton-Jacobi equation.

1.3. **Important Remark.** In general, if one wants to study the precise properties of the solution of PDO, one uses the properties of corresponding classical mechanics defined by the symbol of PDO to mention the fine structure of the solution of PDOs. But this theory of \( \Psi \)DOs isn’t applicable directly to a system of PDOs. For example, one doesn’t know **how one defines the Hamilton flow for a system of PDOs**. Therefore, one needs to apply the technique of \( \Psi \)DOs after diagonalizing the given system.

In this paper, we treat the aforementioned **two non-commutativities on equal footing** by introducing odd variables, therefore, we treat matrices as they are.

Our object of this paper is **not** to prove the well-posedness of \( (W) \) as a symmetric hyperbolic system. Rather, we treat a system of PDOs **as it is**, in other word, we never concern with the properties of characteristic roots of the given system, but we **define the Hamilton flow** for that system after reformulating it on the superspace. (As is well-known, we may apply the so-called energy methods to a symmetric hyperbolic system without regarding characteristic roots.)

1.4. **Superspace setting and Result.** By introducing odd variables to decompose the matrix structure, we first reduce the usual matrix valued Weyl equation \( (W) \) on the Euclidian space \( \mathbb{R} \times \mathbb{R}^3 \) to the one on the superspace \( \mathbb{R} \times \mathbb{R}^{3|2} \), called **the super Weyl equation** \( (SW) \) on the superspace \( \mathbb{R} \times \mathbb{R}^{3|2} \):

\[
\begin{cases}
    i\hbar \frac{\partial}{\partial t} u(t, x, \theta) = \mathcal{H}(t, x, \frac{\hbar}{i} \frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}) u(t, x, \theta), \\
    u(t, x, \theta) = u(x, \theta).
\end{cases}
\]

\( (SW) \)

Remark: The most important thing here is that every quantities appeared above \( (SW) \) are **like scalars though non-commutative**!

**Claim:** There exists the classical mechanics corresponding to the (super) Weyl equation and that a parametrix of it is constructed as a Fourier integral operator using phase and amplitude functions defined by that classical mechanics. (We call this a **good parametrix** because not only it gives a parametrix but also its dependence on the quantities from classical mechanics is explicit.)

Therefore, the (super) Weyl equation is regarded as to be obtained by **quantizing that classical mechanics after Feynman’s procedure**. Because that (super) Weyl equation is “of first order” both in “even and odd variables”, we need to modify Feynman’s argument from Lagrangian to Hamiltonian
For variables, elementary and real analysis on super-smooth function $s$, please consult, $[10, 13, 14, 18, 22]$. We get

\[ \begin{array}{c}
\begin{aligned}
&= \text{mechanics which produces both boson and fermion equally by “quantization”.}
\end{aligned}
\end{array} \]

Correspondence from the Fermi oscillator to the classical object $s$ on a non-commutative algebra. On the equation and see also Inoue and Maeda [19] for the heat equation with the scalar curvature term.

The precise meaning of the above claim is formulated as follows:

**Theorem 1.2.** Let $\{A_j(t, q)\}_{j=0}^3 \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})$ satisfy, for any $k = 0, 1, 2, \cdots$,

\[ \|A_j\|_{k, \infty} = \sup_{t, q, |\gamma| = k} |(1 + |q|)^{|\gamma| - 1} \partial_q^{|\gamma|} A_j(t, q)| < \infty \quad \text{for} \quad j = 0, \cdots, 3. \quad (1.2) \]

For $|t - \xi|$ sufficiently small, we have a good parametrix for (SW) represented by

\[ U(t, \xi) u(x, \theta) = (2\pi \hbar)^{-3/2} \int_{\mathbb{R}^3} d\xi d\pi D^{1/2}(t, \xi; x, \theta, \xi, \pi) e^{i\hbar^{-1} S(t; x, \theta, \xi, \pi)} F(u(\xi, \pi)), \]

for $u(x, \theta) \in \mathcal{D}_{\mathbb{R}}$ and is extended to $L^2_{\mathbb{R}}(\mathfrak{g}^{3/2})$. Here, $S(t; x, \theta, \xi, \pi)$ and $D(t; x, \theta, \xi, \pi)$ satisfy the Hamilton-Jacobi equation and the continuity equation, respectively:

\[ \begin{aligned}
&= (H-J) \begin{cases}
\frac{\partial S}{\partial t} + \mathcal{H} \left( t, x, \frac{\partial S}{\partial x}, \theta, \frac{\partial S}{\partial \theta} \right) = 0,
S(t; x, \theta, \xi, \pi) = \langle x | \xi \rangle + \langle \theta | \pi \rangle,
\end{cases} \quad \text{and} \quad (C) \begin{cases}
\frac{\partial D}{\partial t} + \frac{\partial D}{\partial x} \left( D \frac{\partial \mathcal{H}}{\partial \xi} \right) + \frac{\partial}{\partial \theta} \left( D \frac{\partial \mathcal{H}}{\partial \pi} \right) = 0,
D(t; x, \theta, \xi, \pi) = 1.
\end{cases}
\end{aligned} \]

**Remark.** The assumption (1.2) guarantees the suitable estimates for phase and amplitude functions.

Using the identification maps

\[ \begin{aligned}
\xi : L^2(\mathbb{R} : C^2) &\rightarrow L^2_{\mathbb{R}}(\mathfrak{g}^{3/2}) \quad \text{and} \quad \psi : L^2_{\mathbb{R}}(\mathfrak{g}^{3/2}) \rightarrow L^2(\mathbb{R} : C^2),
\end{aligned} \]

we get

**Corollary 1.3.** Let $\{A_j(t, q)\}_{j=0}^3 \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})$ satisfy (1.2). For $|t - \xi|$ sufficiently small, we have a good parametrix for (W) represented by

\[ U(t, \xi) \psi(q) = \psi(2\pi \hbar)^{-3/2} \int_{\mathbb{R}^3} d\xi d\pi D^{1/2}(t, \xi; x, \theta, \xi, \pi) e^{i\hbar^{-1} S(t; x, \theta, \xi, \pi)} F(\psi)(\xi, \pi) \bigg|_{x = q}. \]

**Remarks.** (0) The result of this paper is announced in Inoue [18]. Though the notion of superanalysis on Fréchet-Grassmann algebra seems not so familiar even today, we have no space to present a part of elements of superanalysis in this paper. If the reader is strange to the notion such as even and odd variables, elementary and real analysis on super-smooth functions, please consult, [10, 13, 14, 18, 22]. We hope the procedure employed here will help to study the propagation of singularities of a system of PDOs and others, after developing theories of PDO and FIO’s to those on superspaces.

(1) The problem how to regard the spin system following the Feynman’s principle, is posed in p.355 of the book of Feynman & Hibbs [6]. J.L. Martin [30, 31] gave an idea of how to make the correspondence from the Fermi oscillator to the classical objects on a non-commutative algebra. On the other hand, without knowing Martin’s results, Berezin & Marinov [2] tried to construct the classical mechanics which produces both boson and fermion equally by “quantization”.

We answer a part of the problem of Feynman using superanalysis, by taking the Weyl equation as the typical and the simplest example of the spin system.

Feynman proposed that the solution of Schrödinger equation

\[ i\hbar \frac{\partial u}{\partial t} = - \frac{\hbar^2}{2M^2} \sum_{i=1}^{m} \frac{\partial^2 u}{\partial q_i^2} + V(t, q) u \]

is represented by

\[ u(t, q) = \int_{\mathbb{R}^m} dq' F(t, q, q') u(0, q') \]
with
\[ F(t, q, q') = \int_{\mathcal{P}_{t,q,q'}} [d\gamma] \exp \left[ i\hbar^{-1} \int_0^t ds L(s, \gamma(s), \dot{\gamma}(s)) \right], \]
\[ L(t, q, q) = \frac{q^2}{2M^2} + V(t, q), \]
\[ \mathcal{P}_{t,q,q'} = \{ \gamma \in AC([0, t] : \mathbb{R}^m) \mid \gamma(0) = q', \gamma(t) = q \}, \]
with \( d\gamma \) = notorious Feynman measure on \( \mathcal{P}_{t,q,q'} \).

Feynman-Hibbs [6] posed a question whether this derivation of quantum mechanics from classical quantities is applicable to spin systems, such as Dirac and Pauli equations. Since there exists no non-trivial Feynman measure \( [d\gamma] \) on \( \mathcal{P}_{t,q,q'} \), we try to give a mathematical meaning to the above expression. Under certain condition on \( V \), Fujiwara [7] defines
\[ F(t, 0)u(q) = \int_{\mathbb{R}^m} dq' D(t, 0, q, q')^{1/2}e^{i\hbar^{-1}S(t, 0, q, q')}u(q') \]
with
\[ S(t, 0, q, q') = \inf_{\gamma \in \mathcal{P}_{t,q,q'}} \int_0^t ds L(s, \gamma(s), \dot{\gamma}(s)), \quad D(t, 0, q, q') = \det \left( \frac{\partial^2 S(t, 0, q, q')}{\partial q_0 \partial q'_{0j}} \right), \]
and he proves that \( F(t, 0) \) gives a good parametrix. Moreover, Fujiwara [8] gives a kernel solution.

Since the above procedure is based on the Lagrangian formulation, we need to reformulate it in the Hamiltonian setting as Inoue [15], called Hamiltonian path-integral method. After reformulating a certain first order system of PDOs in the superspace, we claim to apply the above Hamiltonian path-integral procedure to that first order system of PDOs. But this paper gives a partial answer to the Feynman’s problem, because we have not yet constructed an “explicit integral representation” of the fundamental solution itself. To do this, we need to prepare more elaborated composition formulas of FIOs as in [8].

(2) We may extend our procedure to the case where the electro-magnetic potentials are valued in \( 2 \times 2 \) matrices, if the quantity \( \tilde{A}_0 \) defined below is real valued and the condition (1.2) is satisfied for all \( \{A_j^{[*]}\} \).

In fact, by decomposing
\[ A_j(t, q) = A_j^{[0]}\mathbf{i}_2 + A_j^{[1]}\mathbf{1} \times \sigma_1 + A_j^{[2]}\mathbf{1} \times \sigma_2 + A_j^{[3]}\mathbf{1} \times \sigma_3 \quad \text{for} \quad j = 0, \cdots, 3, \quad \text{with} \quad A_j^{[*]} = A_j^{[*]}(t, q) \in \mathbb{R}, \]
we have
\[ \sigma_j A_j(t, q) + A_0(t, q)\mathbf{i}_2 = \sigma_j \tilde{A}_j(t, q) + \tilde{A}_0(t, q)\mathbf{i}_2, \]
where
\[ \tilde{A}_1 = A_1^{[0]} + A_0^{[1]} + i(A_2^{[3]} - A_3^{[2]}), \quad \tilde{A}_2 = A_2^{[0]} + A_0^{[2]} + i(A_3^{[1]} - A_1^{[3]}), \quad \tilde{A}_3 = A_3^{[0]} + A_0^{[3]} + i(A_1^{[2]} - A_2^{[1]}), \quad \tilde{A}_0 = A_0^{[0]} + A_0^{[1]} + A_0^{[2]} + A_0^{[3]} \].

(3) Even if the electro-magnetic field is time-independent, that is, \( \mathbb{E}(t) = \mathbb{E} \), and the existence of a self-adjoint realization of \( \mathbb{H} \) in \( L^2(\mathbb{R}^3 : \mathbb{C}^2) \) is assured, our result above is new in the following sense. It is well-known that \( e^{-i\hbar^{-1}t\mathbb{E}} \tilde{\psi} \) gives a solution of (W) by Stone’s theorem. Moreover, by the kernel theorem of Schwartz, there exists a distribution \( \mathbb{E}(t, q, q') \) such that \( e^{-i\hbar^{-1}t\mathbb{E}} \tilde{\psi}(q) = \langle \mathbb{E}(t, q, \cdot), \psi(\cdot) \rangle \), where \( \langle \cdot, \cdot \rangle \) is the duality between \( \mathcal{D}' \) and \( \mathcal{D} \). Therefore, our result here answers partly the problem how one may express the kernel of operator \( e^{-i\hbar^{-1}t\mathbb{E}} \) using “classical quantities”.

(4) Especially, in case $\varepsilon = 0$, $U(t, 0)$ gives an explicit solution of (SW) with
\[ S_{\varepsilon=0}(t, 0; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) = \langle \bar{x}|\xi, \bar{\theta}, \bar{\pi} \rangle H(t) + \langle \bar{x}|\xi, \bar{\theta}, \bar{\pi} \rangle = \langle \bar{x}|\xi, \bar{\theta}, \bar{\pi} \rangle. \]

From the right-hand side of above, we define a Hamiltonian as follows (more precisely, the Weyl symbol should be derived):
\[ H(q, p) = e^{-i\hbar^{-1}q_p} \left( a + b \right) e^{i\hbar^{-1}q_p} = ap + bq. \]

The classical mechanics (or bicharacteristic) associated to this Hamiltonian is given by
\[ \begin{align*}
    \dot{q}(t) &= H_p = a, \\
    \dot{p}(t) &= -H_q = -b \\
\end{align*} \]
with the initial data \( \left( q(0), p(0) \right) = \left( \hat{q}, \hat{p} \right) \) (2.2)
which is readily solved as
\[ q(s) = \hat{q} + as, \quad p(s) = \hat{p} - bs. \]

From above Proposition, putting $t = 0$, we get readily that
\[ U(t, q, p) = U(0, q, p)e^{-i\hbar^{-1}(bqt + 2^{-1}abt^2)}. \]

As the inverse function of $\tilde{q} = q(t, q, p)$ is given by $q = y(t, \tilde{q}) = \hat{q} - at$, we get
\[ u(t, \tilde{q}) = U(t, \tilde{q})e^{-i\hbar^{-1}(bqt - 2^{-1}abt^2)}. \]

We remark that there is no flavor of classical mechanics of this expression because we use only a part $q(\cdot)$ of bicharacteristics ($q(\cdot), p(\cdot)$).

Another point of view from Hamiltonian path-integral method: Put
\[ S_0(t, q, p) = \int_0^t ds \left[ \dot{q}(s)p(s) - H(q(s), p(s)) \right] = -bqt - 2^{-1}abt^2, \]
\[ S(t, \tilde{q}, p) = \left( q_p + S_0(t, q, p) \right)|_{q=y(t, \tilde{q})} = \tilde{q}p - apt - b\tilde{q}t + 2^{-1}abt^2. \]
Then, the classical action $S(t, \tilde{q}, p)$ satisfies the Hamilton-Jacobi equation.
\[ \begin{align*}
    \frac{\partial}{\partial t} S + H(\tilde{q}, \partial_3 S) &= 0, \\
    S(0, \tilde{q}, p) &= \tilde{q}p. \\
\end{align*} \]

On the other hand, the van Vleck determinant (though scalar in this case) is calculated as
\[ D(t, \tilde{q}, p) = \frac{\partial^2 S(t, \tilde{q}, p)}{\partial q \partial p} = 1. \]
This quantity satisfies the continuity equation:

\[
\begin{aligned}
\frac{\partial}{\partial t} D + \frac{\partial}{\partial \theta} (DH_p) &= 0 \quad \text{where} \quad H_p = \frac{\partial H}{\partial p} (\tilde{q}, \frac{\partial S}{\partial \theta}), \\
D(0, \tilde{q}, p) &= 1.
\end{aligned}
\]

As an interpretation of Feynman’s idea, we regard that the transition from classical to quantum mechanics is to study the following quantity or the one represented by this procedure (the term “quantization” is not so well-defined mathematically):

\[
u(t, \eta) = (2\pi\hbar)^{-1/2} \int_{\mathbb{R}} dp D^{1/2}(t, \eta, p) e^{i\hbar^{-1}S(t, \eta, p)} \tilde{u}(p).
\]

(2.3)

That is, in our case at hand, we should study the quantity defined by

\[
u(t, \eta) = (2\pi\hbar)^{-1/2} \int_{\mathbb{R}} dp e^{i\hbar^{-1}S(t, \eta, p)} \tilde{u}(p)
\]

\[
= (2\pi\hbar)^{-1} \int_{\mathbb{R}^2} dp dq e^{i\hbar^{-1}(S(t, \eta, p) - q \cdot \bar{q})} \tilde{w}(q) = \tilde{w}(\eta) e^{i\hbar^{-1}(-\bar{\eta} \cdot \eta + 2a t^2)}.
\]

Therefore, we may say that this second construction (2.3) gives the explicit connection between the solution (2.1) and the classical mechanics given by (2.2). We feel the above expression “good” because there appear two classical quantities \(S(t, \eta, p)\) and \(D(t, \eta, p)\) explicitly and we regard this procedure as an honest follower of Feynman’s spirit.

**Claim:** Applying superanalysis, we may extend the second argument above to a system of PDOs e.g. quantum mechanical equations with spin such as Dirac, Weyl or Pauli equations, (and if possible, any other system of PDOs), after interpreting these equations as those on superspaces.

2.2. Our procedure. (I) We identify a “spinor” \(\psi(t, q) = \psi_1(t, q, \psi_2(t, q)) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^2\) with an even supersmooth function \(u(t, x, \theta) = u_0(t, x) + u_1(t, x)\theta_1 \theta_2 : \mathbb{R} \times \mathbb{R}^{3|2} \rightarrow \mathbb{C}_ev\). Here, \(\mathbb{R}^{3|2}\) is the superspace and \(u_0(t, x), u_1(t, x)\) are the Grassmann continuation of \(\psi_1(t, q), \psi_2(t, q)\), respectively. (The reason why we don’t identify \(\psi(t, q)\) with \(u(t, x, \theta) = u_0(t, x) + u_1(t, x)\theta\) for one odd variable \(\theta\), is clarified in [13].)

(II) We represent matrices \(\{\sigma_j\}\) as (even) operators acting on \(u(t, x, \theta)\) such that

\[
\begin{aligned}
\sigma_1 \left( \theta, \frac{\partial}{\partial \theta} \right) &= \theta_1 \theta_2 - \frac{\partial^2}{\partial \theta_1 \partial \theta_2}, \\
\sigma_2 \left( \theta, \frac{\partial}{\partial \theta} \right) &= i \left( \theta_1 \theta_2 + \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \right), \\
\sigma_3 \left( \theta, \frac{\partial}{\partial \theta} \right) &= 1 - \theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}.
\end{aligned}
\]

(2.4)

(III) Therefore, we may introduce the differential operator which corresponds to \(\mathbb{H}(t, q, -i\hbar \partial_q)\):

\[
\mathcal{H}(t, x, \frac{\hbar}{i} \frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}) = \sum_{j=1}^{3} \sigma_j \left( \theta, \frac{\partial}{\partial \theta} \right) \left( \frac{\hbar}{i} \frac{\partial}{\partial x_j} - \frac{\varepsilon}{c} A_j(t, x) \right) + \varepsilon A_0(t, x).
\]

(2.5)

It yields the superspace version of the Weyl equation represented by

\[
\begin{aligned}
\frac{i\hbar}{\partial t} u(t, x, \theta) &= \mathcal{H}(t, x, \frac{\hbar}{i} \frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}) u(t, x, \theta), \\
u(t, x, \theta) &= \mathcal{U}(t, x, \theta).
\end{aligned}
\]

(2.6)

(IV) Using the Fourier transformation on superspace \(\mathbb{R}^{m|n}\), we have the “complete Weyl symbol” of (2.5) as ordinary case. In our case, we put \(k = \hbar, n = 2\) and \(v(\theta) = v_0 + v_1 \theta_1 \theta_2, \ w(\pi) = w_0 + w_1 \pi_1 \pi_2\):

\[
(F_\theta v)(\pi) = \hbar \int_{\mathbb{R}^{0|2}} d\theta e^{-i\hbar^{-1}(\theta|\pi)} v(\theta) = \hbar v_1 + \hbar^{-1} v_0 \pi_1 \pi_2,
\]

\[
(F_\pi w)(\theta) = \hbar \int_{\mathbb{R}^{0|2}} d\pi e^{i\hbar^{-1}(\pi|\theta)} w(\pi) = \hbar w_1 + \hbar^{-1} w_0 \theta_1 \theta_2.
\]
Moreover, we put

\[ \sigma_1(\theta, \pi) = \theta_1 \theta_2 + \hbar^{-2}\pi_1 \pi_2, \]
\[ \sigma_2(\theta, \pi) = i(\theta_1 \theta_2 - \hbar^{-2}\pi_1 \pi_2), \]
\[ \sigma_3(\theta, \pi) = -i\hbar^{-1}(\theta_1 \pi_1 + \theta_2 \pi_2). \]  

Moreover, we put
\[ \mathcal{H}(t, x, \xi, \theta, \pi) = \sum_{j=1}^{3} c_\sigma_j(\theta, \pi)(\xi - \frac{\xi}{c} A_j(t, x)) + \varepsilon A_0(t, x). \]  

(V) As \( \mathcal{H} \) is even, we may consider the classical mechanics corresponding to \( \mathcal{H}(t, x, \xi, \theta, \pi) \):

\[ \begin{aligned}
\frac{d}{dt} x_j &= \frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \xi_j} = c\sigma_j(\theta, \pi), \\
\frac{d}{dt} \xi_j &= -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial x_j} = \sum_{k=1}^{3} \varepsilon\sigma_k(\theta, \pi) \frac{\partial A_k(t, x)}{\partial x_j} - \varepsilon \frac{\partial A_0(t, x)}{\partial x_j},
\end{aligned} \]  

(2.6)ev

\[ \begin{aligned}
\frac{d}{dt} \theta_1 &= -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \pi_1} = -c\hbar^{-2}(\xi_1 - i\eta_2)\pi_2 - i\hbar^{-1}\eta_3 \theta_1, \\
\frac{d}{dt} \theta_2 &= -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \pi_2} = c\hbar^{-2}(\xi_1 - i\eta_2)\pi_1 - i\hbar^{-1}\eta_3 \theta_2, \\
\frac{d}{dt} \pi_1 &= -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \theta_1} = -c(\xi_1 + i\eta_2)\theta_2 + i\hbar^{-1}\eta_3 \pi_1, \\
\frac{d}{dt} \pi_2 &= -\frac{\partial \mathcal{H}(t, x, \xi, \theta, \pi)}{\partial \theta_2} = c(\xi_1 + i\eta_2)\theta_1 + i\hbar^{-1}\eta_3 \pi_2.
\end{aligned} \]  

(2.6)od

In the above, we put
\[ \eta_j(t) = \xi_j(t) - \frac{\xi}{c} A_j(t, x(t)) \quad \text{for} \quad j = 1, 2, 3, \]  

and at time \( t = T \) the initial data are given by
\[ (x(T), \xi(T), \theta(T), \pi(T)) = (x, \xi, \theta, \pi). \]  

Then, we have the following existence theorem.

**Theorem 2.1.** Let \( \{A_j(t, q)\}_{j=1}^{3} \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R}). \)

(0) For any \( T > 0 \) and any initial data \( (x, \xi, \theta, \pi) \in \mathbb{R}^{6} = \mathcal{T}^*\mathbb{R}^{3} \), there exists a unique solution \( (x(t), \xi(t), \theta(t), \pi(t)) \) of (2.6)ev and (2.6)od on \( [-T, T] \).

(1) The solution \( (x(t), \xi(t), \theta(t), \pi(t)) \) of (2.6)ev and (2.6)od on \( [-T, T] \) is “s-smooth” in \( (t, x, \xi, \theta, \pi) \).

That is, smooth in \( t \) for fixed \( (x, \xi, \theta, \pi) \) and supersmooth in \( (x, \xi, \theta, \pi) \) for fixed \( t \).

(2) Assume, moreover, that \( \{A_j(t, q)\}_{j=1}^{3} \) satisfy (1.2).

(i) Then, we have, for \( t, \xi \in [-T, T] \), and \( k = |\alpha + \beta| = 0, 1, 2, \cdots, \)

\[ \begin{aligned}
|\pi_B \partial_x^\alpha \partial_\xi^\beta (x(t, \xi, \theta, \pi) - \xi)| &= 0, \\
|\pi_B \partial_x^\alpha \partial_\xi^\beta (\xi(t, \xi, \theta, \pi) - \xi)| &\leq \varepsilon |t - \xi| |\delta_{|\alpha + \beta|}| A_0|_{|\alpha| + 1, \infty}.
\end{aligned} \]  

(2.12)

(ii) Let \( |x - \xi| \leq 1 \). If \( |\alpha + \beta| = 1 \) and \( k = |\alpha + \beta| = 0, 1, 2, \cdots, \) there exist constants \( C^{(k)} \) (with \( C^{(0)} = 0 \)) independent of \( (t, \xi, \theta) \) such that

\[ \begin{aligned}
|\pi_B \partial_x^\alpha \partial_\xi^\beta \partial_\theta^\gamma \partial_\pi^\delta (\theta(t, \xi, \theta, \pi) - \theta)| &\leq C^{(k)} |t - \xi|^{(1/2)(1 - (1 - k)_+)} \\
|\pi_B \partial_x^\alpha \partial_\xi^\beta \partial_\theta^\gamma \partial_\pi^\delta (\pi(t, \xi, \theta, \pi) - \pi)| &\leq C^{(k)} |t - \xi|^{(1/2)(1 - (1 - k)_+)}. 
\end{aligned} \]  

(2.13)
(iii) Let $|t - \xi| \leq 1$. If $|a + b| = 2$ and $k = |\alpha + \beta| = 0, 1, 2, \ldots$, there exist constants $C_2^{(k)}$ independent of $(t, \xi, \theta)$ such that
\begin{align}
&|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (x(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| \leq C_2^{(k)} |t - \xi|^{1+1/(2)(1-1-k)+},
&|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (\xi(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \xi)| \leq C_2^{(k)} |t - \xi|^{1+1/(2)(1-1-k)+}.
\end{align}

(iv) Let $|t - \xi| \leq 1$. If $|a + b| = 3$ and $k = |\alpha + \beta| = 0, 1, 2, \ldots$, there exist constants $C_3^{(k)}$ independent of $(t, \xi, \theta)$ such that
\begin{align}
&|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (\theta(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \theta)| \leq C_3^{(k)} |t - \xi|^{3/2+1/(2)(1-1-k)+},
&|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (\xi(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \xi)| \leq C_3^{(k)} |t - \xi|^{3/2+1/(2)(1-1-k)+}.
\end{align}

(v) Let $|t - \xi| \leq 1$. If $|a + b| = 4$ and $k = |\alpha + \beta| = 0, 1, 2, \ldots$, there exist constants $C_4^{(k)}$ independent of $(t, \xi, \theta)$ such that
\begin{align}
&|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (x(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| \leq C_4^{(k)} |t - \xi|^{5/2+1/(2)(1-1-k)+},
&|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (\xi(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \xi)| \leq C_4^{(k)} |t - \xi|^{5/2+1/(2)(1-1-k)+}.
\end{align}

Remark. In the following, we denote the solution $x(t) = (x_j(t))$ by $x(t, \xi) = (x_j(t, \xi))$ with $x_j(t) = x_j(t, \xi) = x_j(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi})$, etc. if necessary.

On the other hand, we have

\textbf{Theorem 2.2.} Let $\{A_j(t, q)\}_{j=0}^\infty \in C^\infty(\mathbb{R} \times \mathbb{R}^3; \mathbb{R})$ satisfy (1.2).
For $x(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \theta(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \xi(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \pi(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi})$ solutions of (2.6)\textsubscript{ev} and (2.6)\textsubscript{od} obtained in Theorem 2.1, there exists a constant $0 < \delta \leq 1$ such that the followings hold:

(i) For any fixed $(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), |t - \xi| < \delta$, the mapping
\begin{equation}
\mathfrak{R}^{3|2} \ni (\xi, \bar{\theta}) \mapsto (x = x(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \theta = \theta(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi})) \in \mathfrak{R}^{3|2}
\end{equation}
gives a supersmooth diffeomorphism. We denote the inverse mapping defined by
\begin{equation}
\mathfrak{R}^{3|2} \ni (\bar{x}, \bar{\theta}) \mapsto (y = y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \omega = \omega(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi})) \in \mathfrak{R}^{3|2},
\end{equation}
which is supersmooth in $(\bar{x}, \xi, \bar{\theta}, \bar{\pi})$ for fixed $(t, \xi)$.

(ii) Let $|a + b| = 0$. We have
\begin{equation}
\begin{cases}
|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta (y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| = 0, \\
|y(0, t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi})| \leq C_2 |t - \xi|^{1 + |\xi[0]|}.
\end{cases}
\end{equation}

(iii) Let $|a + b| = 1$. For $k = |\alpha + \beta|$, there exists a constant $\tilde{C}_1^{(k)}$ such that
\begin{equation}
|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (\omega(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \theta)| \leq \tilde{C}_1^{(k)} |t - \xi|^{1/(2)(1-1-k)+}.
\end{equation}

(iv) Let $|a + b| = 2$. For $k = |\alpha + \beta|$, there exists a constant $\tilde{C}_2^{(k)}$ such that
\begin{equation}
|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| \leq \tilde{C}_2^{(k)} |t - \xi|^{1/(2)(1-1-k)+}.
\end{equation}

(v) Let $|a + b| = 3$. For $k = |\alpha + \beta|$, there exists a constant $\tilde{C}_3^{(k)}$ such that
\begin{equation}
|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (\omega(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \theta)| \leq \tilde{C}_3^{(k)} |t - \xi|^{3/2+1/(2)(1-1-k)+}.
\end{equation}

(vi) Let $|a + b| = 4$. For $k = |\alpha + \beta|$, there exists a constant $\tilde{C}_4^{(k)}$ such that
\begin{equation}
|\pi_B \partial_x^\alpha \partial_{\bar{\xi}}^\beta \partial_{\bar{\theta}}^\gamma (y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| \leq \tilde{C}_4^{(k)} |t - \xi|^{5/2+1/(2)(1-1-k)+}.
\end{equation}
Moreover, decomposing\[ S_0(t, \bar{t}; x, \xi, \theta, \pi) = \int_0^t ds \{ \langle \dot{x}(s) | \xi(s) \rangle + \langle \dot{\theta}(s) | \pi(s) \rangle - \mathcal{H}(s, x(s), \xi(s), \theta(s), \pi(s)) \}, \quad (2.24) \]
and\[ S(t, \bar{t}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) = \left( \langle \bar{x} | \bar{\xi} \rangle + \langle \bar{\theta} | \bar{\pi} \rangle + S_0(t, \bar{t}; x, \xi, \theta, \pi) \right) |_{\bar{x}=y(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi})}, \quad (2.25) \]

**Theorem 2.3.** \( S(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) \) satisfies the following Hamilton-Jacobi equation:
\[
\begin{align*}
&\left\{ \frac{\partial}{\partial t} S + \mathcal{H} \left( t, \bar{x}, \frac{\partial S}{\partial x}, \bar{\theta}, \frac{\partial S}{\partial \theta} \right) = 0, \\
&S(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) = \langle \bar{x} | \bar{\xi} \rangle + \langle \bar{\theta} | \bar{\pi} \rangle.
\end{align*}
\]

Moreover, decomposing
\[ S(t, s; x, \xi, \theta, \pi) = S_B(t, s; x, \xi) + S_{\bar{b}}(t, s; x, \xi, \theta, \pi) \quad (2.27) \]
with
\[
S_B(t, s; x, \xi) = S(t, s; x, \xi, 0, 0) = S_{10}(t, s; x, \xi), \quad S_{\bar{b}}(t, s; x, \theta, \xi, \pi) = \sum_{|c| + |d| = \text{even} \geq 2} S_{c,d}(t, s; x, \xi) \theta^c \pi^d
\]
\[ = S_{10} \theta \theta_2 + \sum_{j=1}^2 S_{c_j d_k} \theta^j \pi^d + S_{01} \pi_1 \pi_2 + S_{11} \theta \theta_2 \pi_1 \pi_2, \quad (2.28) \]
where \( 0 = (0, 0), \bar{1} = (1, 1), c_1 = (1, 0) = d_1, c_2 = (0, 1) = d_2 \in \{0, 1\}^2, \)

we get the following estimates: For any \( \alpha, \beta, \) there exist constants \( C_{\alpha \beta} > 0 \) such that
\[
|\partial_{\xi}^\alpha \partial_{\bar{x}}^\beta (S_{00}(t, s; x, \xi) - \langle \xi(x) \rangle) | \leq C_{\alpha \beta} (1 + |x|)^{(1 - |\alpha|) + \delta_{0\beta} |t - s|} \\
|\partial_{\xi}^\alpha \partial_{\bar{x}}^\beta (S_{10}(t, s; x, \xi)) | \leq C_{\alpha \beta} \pi_1 |t - s|, \\
|\partial_{\xi}^\alpha \partial_{\bar{x}}^\beta (S_{c_j d_k}(t, s; x, \xi) - 1) | \leq C_{\alpha \beta} \pi_1 |t - s| \quad \text{for } j = 1, 2, \\
|\partial_{\xi}^\alpha \partial_{\bar{x}}^\beta (S_{01}(t, s; x, \xi)) | \leq C_{\alpha \beta} \pi_1 |t - s|, \\
|\partial_{\pi}^\beta \partial_{\bar{x}}^\alpha (S_{11}(t, s; x, \xi)) | \leq C_{\alpha \beta} \pi_1 |t - s|.
\]

Defining (sdet denotes the super-determinant)
\[ D(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) = \text{sdet} \left( \begin{array}{cc} \frac{\partial^2 S}{\partial \xi \partial \bar{x}} & \frac{\partial^2 S}{\partial \xi \partial \bar{\theta}} \\ \frac{\partial^2 S}{\partial \bar{x} \partial \bar{\theta}} & \frac{\partial^2 S}{\partial \bar{x} \partial \bar{\pi}} \end{array} \right) = A^2(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}), \quad (2.30) \]
we get

**Theorem 2.4.** \( D(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) \) or \( A(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) \) satisfies the following continuity equation:
\[
\begin{align*}
&\left\{ \frac{\partial}{\partial t} D + \frac{\partial}{\partial \bar{x}} \left( \frac{\partial D}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{\theta}} \left( \frac{\partial D}{\partial \bar{\theta}} \right) = 0, \\
&D(t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) = 1.
\end{align*}
\]

Or, we have
\[
\begin{align*}
A(t, s; x, \xi, \theta, \pi) &= A(t, s; x, \xi, \theta, \pi) + \frac{1}{2} A(\partial_x \mathcal{H}_{\xi_j} + \partial_{\theta_k} \mathcal{H}_{\pi_k}) = 0, \\
A(s, s; x, \xi, \theta, \pi) &= 1.
\end{align*}
\]

Here, the argument of \( D \) or \( A \) is \( (t, \bar{t}; x, \xi, \bar{\theta}, \bar{\pi}) \), those of \( \partial_x \mathcal{H} \) and \( \partial_\pi \mathcal{H} \) are \( (x, \bar{x}, \bar{\theta}, \bar{\theta}, \bar{\pi}) \), respectively.

Decomposing\[ A(t, s; x, \xi, \theta, \pi) = \sum_{|c| + |d| = \text{even} \geq 0} A_{c,d}(t, s; x, \xi) \theta^c \pi^d = A_B(t, s; x, \xi) + \mathcal{D}_B(t, s; x, \xi, \theta, \pi), \quad (2.33) \]
as before, we get the following: If $|t - s|$ is sufficiently small, we have

\[
|\partial_x^2 \partial_{\xi}^2 (A_{00}(t, s; x, \xi) - 1)| \leq C_{\alpha \beta} |t - s|,
\]

\[
|\partial_x^2 \partial_{\xi}^2 A_{01}(t, s; x, \xi)| \leq C_{\alpha \beta} |t - s|,
\]

\[
|\partial_x^2 \partial_{\xi}^2 A_{1j}(t, s; x, \xi)| \leq C_{\alpha \beta} |t - s| \quad \text{for } j = 1, 2
\]

(234)

In the following argument of this section, we change the order of variables and rewrite them from $(t, \xi; x, \theta, \pi)$ to $(t, \xi; x, \theta, \pi)$, and then to $(t, s; x, \theta, \xi, \pi)$.

(VII) We define an operator

\[
(\mathcal{U}(t,s)u)(x, \theta) = c_{3,2} \int \int_{\mathbb{R}^{3/2} \times \mathbb{R}^{3/2}} d\xi d\pi A(t, s; x, \theta, \xi, \pi) e^{ih^{-1}S(t,s; x, \theta, \xi, \pi)} F u(\xi, \pi).
\]

(235)

On the other hand, we show readily

\[
\hat{\mathcal{H}}(t) = \mathcal{H}(t) (2.36)
\]

where $\hat{\mathcal{H}}(t)$ is a (Weyl type) pseudo-differential operator with symbol $\mathcal{H}(t, x, \theta, \xi, \pi)$, that is,

\[
(\hat{\mathcal{H}}(t)u)(x, \theta) = c_{3,2} \int \int_{\mathbb{R}^{3/2} \times \mathbb{R}^{3/2}} d\xi d\pi dy d\omega e^{ih^{-1}((x-y)\xi + (\theta - \omega)\pi)}
\]

\[
\times \mathcal{H}\left(t, \frac{x + y}{2}, \frac{\theta + \omega}{2}, \xi, \pi\right) u(y, \omega). (2.37)
\]

Claim: The operator (2.35) is a good parametrix for (2.6).

[good parametrix] We call (2.35) as a good parametrix because it has the following properties:

(a) This (2.35) gives a parametrix of the problem (2.6), which has the explicit dependence on the classical quantities. That is, the Bohr correspondence is exemplified even with spin structure.

(b) As the infinitesimal generator of that parametrix, we have an operator $\hat{\mathcal{H}}(t)$ which corresponds to the the Weyl quantization for the symbol $\mathcal{H}(t, x, \theta, \xi, \pi)$, that is, a certain symmetry is preserved naturally.

(c) By Trotter-Kato’s time-slicing method, products of (2.35) yield a fundamental solution as is presented in the following theorem.

**Theorem 2.5.** Let $\{A_j(t, q)\}_{j=0}^3 \in C^\infty (\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})$ satisfy (1.2).

(1) There exists a positive number $\delta$ such that if $|t - s| < \delta$ then $\mathcal{U}(t, s)$ is a well defined bounded operator in $L^2_{ss}(\mathbb{R}^{3/2})$. Moreover, if $|t - r| + |r - s| < \delta$, then there exists a constant $C$ such that

\[
\|\mathcal{U}(t, r)\mathcal{U}(r, s) - \mathcal{U}(t, s)\|_{B(\mathcal{E}^2_{ss}(\mathbb{R}^{3/2}), \mathcal{E}^2_{ss}(\mathbb{R}^{3/2}))} \leq C(|t - r|^2 + |r - s|^2).
\]

(2.38)

(2) Let $\Delta$ be an arbitrary subdivision of the interval $[s, t]$ or $[t, s]$ for any $t, s \in \mathbb{R}$, such that

\[
\Delta : s = t_0 < t_1 < \cdots < t_N = t \quad \text{or} \quad \Delta : s = t_0 > t_1 > \cdots > t_N = t
\]

with $|\Delta| = \max_{1 \leq i \leq N} |t_i - t_{i-1}|$. We put

\[
\mathcal{U}_\Delta(t, s) = \mathcal{U}(t_N, t_{N-1})\mathcal{U}(t_{N-1}, t_{N-2}) \cdots \mathcal{U}(t_1, t_0).
\]

Then, $\mathcal{U}_\Delta(t, s)$ converges when $|\Delta| \to 0$ to an unitary operator $\mathcal{E}(t, s)$ in the uniform operator topology in $L^2_{ss}(\mathbb{R}^{3/2})$. More precisely, there exist constants $\gamma_1, \gamma_2$ such that

\[
\|\mathcal{E}(t, s) - \mathcal{U}_\Delta(t, s)\|_{B(\mathcal{E}^2_{ss}(\mathbb{R}^{3/2}), \mathcal{E}^2_{ss}(\mathbb{R}^{3/2}))} \leq \gamma_1 |\Delta| e^{\gamma_2 |\Delta|}.
\]

(2.39)
(3) (i) $\mathbb{R}^2 \ni (t, s) \rightarrow \mathcal{E}(t, s) \in \mathbb{B}(\mathcal{L}_{SS}^2(\mathbb{R}^{3|2}), \mathcal{L}_{SS}^2(\mathbb{R}^{3|2}))$ is continuous and satisfies $\mathcal{E}(t, r)\mathcal{E}(r, s) = \mathcal{E}(t, s)$.

(ii) For $u \in \mathcal{C}_{SS, 0}(\mathbb{R}^{3|2})$, we have
\[
\begin{cases}
  i\hbar \frac{d}{dt}\mathcal{E}(t, s)u = \hat{\mathcal{H}}(t)\mathcal{E}(t, s)u, \\
  \mathcal{E}(s, s)u = u.
\end{cases}
\] (2.40)

**Remark:** The reason for preparing complicated estimates in Theorems 2.1–2.4, is to apply the known $L^2$-bounded theorem to our FIO.

On the other hand, remarking that
\[
\mathcal{E}(t, s)\psi = \mathcal{H}(t)\mathcal{E}(t, s)\psi,
\] (2.41)
and putting that
\[
\mathcal{U}(t, s)\psi = \mathcal{U}(t, s)\mathcal{E}(t, s)\psi, \quad \mathcal{U}_\Delta(t, s) = \mathcal{U}(t, s)\mathcal{E}(t, s)\psi, \quad \mathcal{E}(t, s)\psi = \mathcal{E}(t, s)\mathcal{E}(t, s)\psi,
\] (2.42)
we have

**Theorem 2.6.** Let $\{A_j(t, q)\}_{j=0}^3 \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})$ satisfy (1.2).

(1) There exists a positive number $\delta$ such that if $|t-s| < \delta$ then $\mathcal{U}(t, s)$ is well defined bounded operator in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$. Moreover, if $|t-r| + |r-s| < \delta$, then there exists a constant $C$ such that
\[
\|\mathcal{U}(t, r)\mathcal{U}(r, s) - \mathcal{U}(t, s)\|_{\mathcal{B}(L^2(\mathbb{R}^3 : \mathbb{C}^2), L^2(\mathbb{R}^3 : \mathbb{C}^2))} \leq C(|t-r|^2 + |r-s|^2).
\] (2.43)

(2) For any $(t, s) \in \mathbb{R}^2$, $\mathcal{U}_\Delta(t, s)$ converges when $|\Delta| \rightarrow 0$ to an unitary operator $\mathcal{E}(t, s)$ in the uniform operator topology in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$. More precisely, there exist constants $\gamma_1, \gamma_2$ such that
\[
\|\mathcal{E}(t, s) - \mathcal{U}_\Delta(t, s)\|_{\mathcal{B}(L^2(\mathbb{R}^3 : \mathbb{C}^2), L^2(\mathbb{R}^3 : \mathbb{C}^2))} \leq \gamma_1|\Delta|e^{\gamma_2|\Delta|}.
\] (2.44)

(3) (i) $\mathbb{R}^2 \ni (t, s) \rightarrow \mathcal{E}(t, s) \in \mathbb{B}(L^2(\mathbb{R}^3 : \mathbb{C}^2), L^2(\mathbb{R}^3 : \mathbb{C}^2))$ is continuous and satisfies $\mathcal{E}(t, r)\mathcal{E}(r, s) = \mathcal{E}(t, s)$.

(ii) For $\psi \in C^\infty_0(\mathbb{R}^3 : \mathbb{C}^2)$, we have
\[
\begin{cases}
  i\hbar \frac{d}{dt}\mathcal{E}(t, s)\psi = \hat{\mathcal{H}}(t)\mathcal{E}(t, s)\psi, \\
  \mathcal{E}(s, s)\psi = \psi.
\end{cases}
\] (2.45)

**Problem:** Construct a kernel representation of $\mathcal{E}(t, s)$ (i.e. a fundamental solution). If we could do so properly, then we would give an answer of the problem posed by Feynman in p.355 of [6]. To do so, we need to develop the theory of FIO on superspace more precisely. For example, we must extend the $\#$-product formula for two FIP with different phases which is done for FIO on ordinary Euclidian space.

3. Classical Mechanics: Proofs of Theorems 2.1–2.5.

3.1. Hamiltonian flows.
3.1.1. Proof of Theorem 2.1(Existence). We rewrite (2.6) as

\[
\frac{d}{dt} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = i \hbar^{-1} \mathcal{X}(t) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},
\]

(3.1)

where

\[
\mathcal{X}(t) = \begin{pmatrix} -\eta_3(t) & 0 & 0 & 0 \\ 0 & -i \hbar^{-1}(\eta_1(t) - i\eta_2(t)) & 0 \\ 0 & i \hbar(\eta_1(t) + i\eta_2(t)) & \eta_3(t) & 0 \\ -i \hbar(\eta_1(t) + i\eta_2(t)) & 0 & 0 & \eta_3(t) \end{pmatrix}.
\]

(3.2)

Moreover, defining \( \sigma_j(t) = \sigma_j(\theta(t), \pi(t)) \), we have, by simple calculations,

\[
\frac{d}{dt} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 2\hbar^{-1} \mathcal{Y}(t) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},
\]

(3.3)

where

\[
\mathcal{Y}(t) = \begin{pmatrix} 0 & -\eta_3(t) & \eta_2(t) \\ \eta_3(t) & 0 & -\eta_1(t) \\ -\eta_2(t) & \eta_1(t) & 0 \end{pmatrix}.
\]

(3.4)

Now, we start our existence proof. We decompose variables, using degree, as follow:

\[
x_j(t) = \sum_{\ell=0}^{\infty} x_j^{[2\ell]}(t), \quad \xi_j(t) = \sum_{\ell=0}^{\infty} \xi_j^{[2\ell]}(t), \quad \theta_k(t) = \sum_{\ell=0}^{\infty} \theta_k^{[2\ell+1]}(t), \quad \pi_k(t) = \sum_{\ell=0}^{\infty} \pi_k^{[2\ell+1]}(t).
\]

(3.5)

Then, we get, for \( m = 0, 1, 2, \cdots \),

\[
\begin{aligned}
\frac{d}{dt} x_j^{[2m]}(t) &= c \mathcal{O}_j^{[2m]}(t) \quad \text{with} \quad \mathcal{O}_j^{[2m]}(t) = \mathcal{O}_j^{[0]}(t), \\
\frac{d}{dt} \xi_j^{[2m]}(t) &= \varepsilon \sum_{\ell=1}^{m} \sum_{k=1}^{3} \sigma_k^{[2\ell]} \frac{\partial A_k^{[2m-2\ell]}}{\partial x_j} - \varepsilon \frac{\partial A_j^{[2m]}}{\partial x_j} \quad \text{with} \quad \frac{\partial x_j^{[2m]}(t)}{\partial \xi_j^{[2m]}(t)} = \frac{x_j^{[2m]}(t)}{\xi_j^{[2m]}(t)},
\end{aligned}
\]

(3.6)

and

\[
\begin{aligned}
\frac{d}{dt} \begin{pmatrix} \sigma_1^{[2m+1]} \\ \sigma_2^{[2m+1]} \\ \sigma_3^{[2m+1]} \end{pmatrix} &= i \hbar^{-1} \sum_{\ell=0}^{m} \mathcal{X}^{[2\ell]}(t) \begin{pmatrix} \theta_1^{[2\ell+1-2\ell]} \\ \theta_2^{[2\ell+1-2\ell]} \\ \pi_1^{[2\ell+1-2\ell]} \\ \pi_2^{[2\ell+1-2\ell]} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \theta_1^{[2m+1]} \\ \theta_2^{[2m+1]} \\ \pi_1^{[2m+1]} \\ \pi_2^{[2m+1]} \end{pmatrix} = \begin{pmatrix} \theta_1^{[2m]} \\ \theta_2^{[2m]} \\ \pi_1^{[2m]} \\ \pi_2^{[2m]} \end{pmatrix},
\end{aligned}
\]

(3.7)

and

\[
\frac{d}{dt} \begin{pmatrix} \sigma_1^{[2m]} \\ \sigma_2^{[2m]} \\ \sigma_3^{[2m]} \\ \sigma_4^{[2m]} \end{pmatrix} = \sum_{\ell=0}^{m-1} 2\hbar^{-1} \mathcal{Y}^{[2\ell]}(t) \begin{pmatrix} \sigma_1^{[2m-2\ell]} \\ \sigma_2^{[2m-2\ell]} \\ \sigma_3^{[2m-2\ell]} \\ \sigma_4^{[2m-2\ell]} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \sigma_1^{[2m]} \\ \sigma_2^{[2m]} \\ \sigma_3^{[2m]} \\ \sigma_4^{[2m]} \end{pmatrix} = \begin{pmatrix} \sigma_1^{[2m]} \\ \sigma_2^{[2m]} \\ \sigma_3^{[2m]} \\ \sigma_4^{[2m]} \end{pmatrix}.
\]

(3.8)

Here, \( \mathcal{X}^{[2\ell]}(t) \), \( \mathcal{Y}^{[2\ell]}(t) \), \( \sigma^{[2\ell]}(t) \) are the degree of \( 2\ell \) parts of \( \mathcal{X}(t) \), \( \mathcal{Y}(t) \), \( \sigma(t) \), respectively:

\[
\eta_k^{[2\ell]}(t) = \xi_k^{[2\ell]}(t) - \frac{\varepsilon}{c} A_k^{[2\ell]}(t, x),
\]

\[
A_k^{[2\ell]}(t, x) = \sum_{|\alpha| \leq 2\ell} \frac{1}{\ell_1 + \ell_2 + \ell_3 + \ell_4} \partial^{\alpha}_{\ell_1 + \ell_2 + \ell_3 + \ell_4} A_{k, \alpha}(t, x^0) \cdot (x_1^{\alpha_1})^{[2\ell_1]}(x_2^{\alpha_2})^{[2\ell_2]}(x_3^{\alpha_3})^{[2\ell_3]},
\]

\[
\frac{\partial A_k^{[2\ell]}(t, x)}{\partial x_j} = \sum_{|\alpha| \leq 2\ell} \frac{1}{\ell_1 + \ell_2 + \ell_3 + \ell_4} \partial^{\alpha}_{\ell_1 + \ell_2 + \ell_3 + \ell_4} A_{k, \alpha}(t, x^0) \cdot (x_1^{\alpha_1})^{[2\ell_1]}(x_2^{\alpha_2})^{[2\ell_2]}(x_3^{\alpha_3})^{[2\ell_3]}.
\]
and

\[
\begin{align*}
\sigma_1^{[2m]} &= \sum_{\ell=0}^{m-1} \left( \theta_1^{[2\ell+1]} \theta_2^{[2m-2\ell-1]} + h^{-2} \pi_1^{[2\ell+1]} \pi_2^{[2m-2\ell-1]} \right), \\
\sigma_2^{[2m]} &= i \sum_{\ell=0}^{m-1} \left( \theta_1^{[2\ell+1]} \theta_2^{[2m-2\ell-1]} - h^{-2} \pi_1^{[2\ell+1]} \pi_2^{[2m-2\ell-1]} \right), \\
\sigma_3^{[2m]} &= -ih^{-1} \sum_{\ell=0}^{m-1} \left( \theta_1^{[2\ell+1]} \pi_1^{[2m-2\ell-1]} + \theta_2^{[2\ell+1]} \pi_2^{[2m-2\ell-1]} \right).
\end{align*}
\]

[0] From (3.6) with \( m = 0 \), we get

\[
\frac{d}{dt} x_j^{[0]}(t) = 0 \quad \text{and} \quad \frac{d}{dt} \xi_j^{[0]}(t) = -\varepsilon \frac{\partial A_0^{[0]}(t, x^{[0]})}{\partial x_j} = -\varepsilon \partial_q A_0^{[0]}(t, x^{[0]}) \quad \text{for} \quad j = 1, 2, 3.
\]

Therefore, for any \( t \in \mathbb{R} \),

\[
x_j^{[0]}(t) = x_j^{[0]} \quad \text{and} \quad \xi_j^{[0]}(t) = \xi_j^{[0]} - \varepsilon \int_{-\infty}^{t} ds \partial_q A_0^{[0]}(s, x^{[0]}) \quad \text{for} \quad j = 1, 2, 3.
\]

[1] We put these into (3.7) with \( m = 0 \), to have

\[
\frac{d}{dt} \begin{pmatrix} \theta_1^{[1]} \\ \theta_1^{[1]} \\ \pi_1^{[1]} \\ \pi_2^{[1]} \end{pmatrix} = i \hbar^{-1} \Xi^{[0]}(t) \begin{pmatrix} \theta_1^{[1]} \\ \theta_1^{[1]} \\ \pi_1^{[1]} \\ \pi_2^{[1]} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \theta_1^{[1]} \\ \theta_1^{[1]} \\ \pi_1^{[1]} \\ \pi_2^{[1]} \end{pmatrix} = \begin{pmatrix} \theta_1^{[1]} \\ \theta_1^{[1]} \\ \pi_1^{[1]} \\ \pi_2^{[1]} \end{pmatrix}.
\]

(3.9)

Here, \( \Xi^{[0]}(t) \) is a \( 4 \times 4 \)-matrix whose arguments depend on \((t, t, x^{[0]}, \xi^{[0]}, \partial^\beta A_0, \partial_q^\beta A ; |\alpha| = 0, |\beta| \leq 1)\) with values in \( \mathbb{C} \). Or more precisely, \( \Xi^{[0]}(t) \) has components given by

\[
\eta_j^{[0]}(t) = \xi_j^{[0]}(t) - \frac{\varepsilon}{c} A_j(t, x^{[0]}) = \xi_j^{[0]} - \varepsilon \int_{-\infty}^{t} ds \partial_q A_0^{[0]}(s, x^{[0]}) - \frac{\varepsilon}{c} A_j^{[0]}(t, x^{[0]})
\]

As (3.9) is the linear ODE in \( (\mathbb{R}^{1,3})^4 \) with smooth coefficients in \( t \), there exists a unique global (in time) solution, which has the following dependence. Putting \( A = (A_1, A_2, A_3) \), we have

\[
\begin{align*}
\theta_1^{[1]}(t) &= \theta_1^{[1]}(t, x^{[0]}, \xi^{[0]}, \partial^\beta A_0, \partial_q^\beta A ; |\alpha| = 0, |\beta| \leq 1), \text{ linear in } \theta_2^{[1]}, \pi^{[1]}, \\
\pi_1^{[1]}(t) &= \pi_1^{[1]}(t, x^{[0]}, \xi^{[0]}, \theta^{[1]}, \pi^{[1]}, \partial^\beta A_0, \partial_q^\beta A ; |\alpha| = 0, |\beta| \leq 1), \text{ linear in } \theta_1^{[1]}, \pi^{[1]}.
\end{align*}
\]

(3.10)

[2] For (3.6) with \( m = 1 \), we have

\[
\begin{align*}
\frac{d}{dt} x_j^{[2]} &= \varepsilon \sum_{k=1}^{3} \sigma_k^{[2]} \frac{\partial A_0^{[0]}}{\partial x_j}, \\
\frac{d}{dt} x_j^{[2]} &= \varepsilon \sum_{k=1}^{3} \sigma_k^{[2]} \frac{\partial A_0^{[0]}}{\partial x_j} - \varepsilon \frac{\partial A_0^{[2]}}{\partial x_j} \quad \text{with} \quad \begin{pmatrix} x_j^{[2]}(t) \\ \xi_j^{[2]}(t) \end{pmatrix} = \begin{pmatrix} \xi_j^{[2]} \\ \xi_j^{[2]} \end{pmatrix}.
\end{align*}
\]

Then, using (3.8) with \( m = 1 \) and (3.10), we have, for \( j = 1, 2, 3 \),

\[
\begin{align*}
\sigma_1^{[2]} &= \theta_1^{[1]} \theta_2^{[1]} + h^{-2} \pi_1^{[1]} \pi_2^{[1]}, \\
\sigma_2^{[2]} &= i(\theta_1^{[1]} \theta_2^{[1]} - h^{-2} \pi_1^{[1]} \pi_2^{[1]}), \quad \text{and} \\
\sigma_3^{[2]} &= -ih^{-1}(\theta_1^{[1]} \pi_1^{[1]} + \theta_2^{[1]} \pi_2^{[1]})
\end{align*}
\]

\[
\begin{align*}
A_0^{[2]}(x) &= \sum_{k=1}^{3} \partial_{q_k} A_0(x^{[0]} x_k^{[2]}), \\
\frac{\partial A_0^{[2]}}{\partial x_j} &= \sum_{k=1}^{3} \partial_{q_k} A_0(x^{[0]} x_k^{[2]}).
\end{align*}
\]

Therefore, we have, for \( j = 1, 2, 3 \),

\[
\begin{align*}
x_j^{[2]}(t) &= x_j^{[2]}(t, x^{[2]}, \xi^{[0]}, \theta^{[1]}, \pi^{[1]}, \partial^\beta A_0, \partial_q^\beta A ; 0 \leq \ell \leq 1, |\alpha| = 0, |\beta| \leq 1), \\
\xi_j^{[2]}(t) &= \varepsilon \sum_{k=1}^{3} \sigma_k^{[2]} x_j^{[2]}(t, x^{[2]}, \xi^{[0]}, \theta^{[1]}, \pi^{[1]}, \partial^\beta A_0, \partial_q^\beta A ; 0 \leq \ell \leq 1, |\alpha| \leq 1, |\beta| \leq 2).
\end{align*}
\]
This gives the existence proof (Proof of Theorem 2.1). Remarking that at each degree, the solution of (3.1).

Therefore, we get, for \( k = 1, 2, \)

\[
\begin{align*}
\theta_k^{[3]}(t) &= \theta_k^{[3]}(t, x^{[2]}), \\
\pi_k^{[3]}(t) &= \pi_k^{[3]}(t, x^{[2]}),
\end{align*}
\]

Moreover, we have easily

\[
\begin{align*}
\eta_j &= \sum_{k=1}^{3} \varepsilon \sigma_k(\theta_j, \pi_j) B_{jk}(t, x) - \varepsilon \frac{\partial A_0(t, x)}{\partial x_j},
\end{align*}
\]

This gives the existence proof (Proof of Theorem 2.1). Remarking that at each degree, the solution of (3.6) and (3.7) is defined uniquely, we have the uniqueness of the solution of (2.6)\(_{ev}\) and (2.6)\(_{od}\).

Moreover, we have easily

**Corollary 3.1.** Let \((x(t), \xi(t), \theta(t), \pi(t)) \in C^1(\mathbb{R} : \mathcal{T}^*\mathcal{R}^{3|2})\) be a solution of (2.6)\(_{ev}\) and (2.6)\(_{od}\). Then, it satisfies

\[
\frac{d}{dt} \mathcal{H}(t, x(t), \xi(t), \theta(t), \pi(t)) = \frac{\partial \mathcal{H}}{\partial t}(t, x(t), \xi(t), \theta(t), \pi(t)).
\]

Using (2.10) and putting

\[B_{jk}(t, x) = \frac{\partial A_k(t, x)}{\partial x_j} - \frac{\partial A_j(t, x)}{\partial x_k},\]

we rewrite (2.6)\(_{ev}\) as

\[
\begin{align*}
\frac{d}{dt} x_j &= c \sigma_j(\theta, \pi), \\
\frac{d}{dt} \eta_j &= \sum_{k=1}^{3} \varepsilon \sigma_k(\theta_j, \pi_j) B_{jk}(t, x) - \varepsilon \frac{\partial A_0(t, x)}{\partial x_j}.
\end{align*}
\]

**Corollary 3.2.** Let \(\{A_j(t, q)\}_{j=0}^{3} \in C^\infty(\mathbb{R} \times \mathbb{R}^3 : \mathbb{R})\) satisfy (1.2). There exists a unique solution \((\tilde{x}(t), \tilde{\eta}(t), \tilde{\theta}(t), \tilde{\pi}(t)) \in C^1(\mathbb{R} : \mathcal{T}^*\mathcal{R}^{3|2})\) of (3.6)\(_{ev}\) and (2.6)\(_{od}\) with initial data

\[(\tilde{x}(\xi), \tilde{\eta}(\xi), \tilde{\theta}(\xi), \tilde{\pi}(\xi)) = (x, \eta, \theta, \pi) \quad \text{where} \quad \eta_j = \xi_j - \frac{\varepsilon}{c} A_j(t, x).
\]

Moreover, they are related to \((x(t), \xi(t), \theta(t), \pi(t))\) as

\[
\begin{align*}
x_j(t, \xi, \theta, \pi) &= \tilde{x}_j(t, \xi, \theta, \pi) - \frac{\varepsilon}{c} A_j(t, \xi, \theta, \pi), \\
\xi_j(t, \xi, \theta, \pi) &= \tilde{\eta}_j(t, \xi, \theta, \pi) - \frac{\varepsilon}{c} A_j(t, \xi, \theta, \pi) + \frac{\varepsilon}{c} A_j(t, \tilde{x}(t, \xi, \theta, \pi) - \frac{\varepsilon}{c} A_j(t, \xi, \theta, \pi)), \\
\theta_k(t, \xi, \theta, \pi) &= \tilde{\theta}_k(t, \xi, \theta, \pi) - \frac{\varepsilon}{c} A_j(t, \xi, \theta, \pi), \\
\pi_k(t, \xi, \theta, \pi) &= \tilde{\pi}_k(t, \xi, \theta, \pi) - \frac{\varepsilon}{c} A_j(t, \xi, \theta, \pi).
\end{align*}
\]
3.1.2. Smoothness: Proof of Theorem 2.1 continued. For notational simplicity, we represent \( x(t, t; \xi, \theta, \pi) \) as \( x(t) \) or \( x \), etc. We investigate the smoothness of \( (x(t), \xi(t), \theta(t), \pi(t)) \) with respect to the initial data \( (\xi, \theta, \pi) \). In the following, we put \((1 - k)_+ = \max(0, 1 - k)\) for \( k \geq 0 \).

**s-smoothness:** In order to prove the smoothness w.r.t. the initial data, we differentiate (2.6)_{ev} and (2.6)_{od} formally w.r.t. \((\xi, \theta, \pi)\), which gives us the following differential equation:

\[
\frac{d}{dt} J^{(1)}(t) = H^{(2)}(t) J^{(1)}(t) \quad \text{with} \quad J^{(1)}(0) = I.
\]

(3.12)

Here

\[
J^{(1)}(t) = \begin{pmatrix}
\partial_x x & \partial_x \xi & \partial_x \theta & \partial_x \pi \\
\partial_\xi x & \partial_\xi \xi & \partial_\xi \theta & \partial_\xi \pi \\
\partial_\theta x & \partial_\theta \xi & \partial_\theta \theta & \partial_\theta \pi \\
\partial_\pi x & \partial_\pi \xi & \partial_\pi \theta & \partial_\pi \pi
\end{pmatrix}, \quad \partial_x x = \begin{pmatrix}
\partial_x x_1 \\
\partial_x x_2 \\
\partial_x x_3
\end{pmatrix}, \quad \text{etc.}
\]

(3.13)

with arguments \((t, x; \xi, \theta, \pi)\) and

\[
H^{(2)}(t) = \begin{pmatrix}
\partial_x \xi \partial_\xi H & \partial_x \theta \partial_\xi H & -\partial_x \theta \partial_\xi H & -\partial_x \pi \partial_\xi H \\
\partial_x \theta \partial_\xi H & -\partial_x \theta \partial_\xi H & \partial_x \theta \partial_\xi H & \partial_x \pi \partial_\xi H \\
\partial_x \theta \partial_\xi H & -\partial_x \theta \partial_\xi H & -\partial_x \pi \partial_\xi H & \partial_x \pi \partial_\xi H \\
\partial_x \pi \partial_\xi H & \partial_x \theta \partial_\pi H & -\partial_x \theta \partial_\pi H & -\partial_x \pi \partial_\pi H
\end{pmatrix}
\]

(3.14)

where

\[
\partial_x \partial_\xi H = \begin{pmatrix}
\partial_x \partial_\xi \xi \partial_\xi H & \partial_x \partial_\xi \theta \partial_\xi H & \partial_x \partial_\xi \pi \partial_\xi H \\
\partial_x \partial_\theta \xi \partial_\xi H & \partial_x \partial_\theta \theta \partial_\xi H & \partial_x \partial_\theta \pi \partial_\xi H \\
\partial_x \partial_\pi \xi \partial_\xi H & \partial_x \partial_\pi \theta \partial_\xi H & \partial_x \partial_\pi \pi \partial_\xi H
\end{pmatrix}, \quad \text{etc.}
\]

with arguments \((t, x(t), \xi(t), \theta(t), \pi(t))\). Remarking that each component of \( H^{(2)}(t) \) is differentiable w.r.t. \( t \) for fixed \((\xi, \theta, \pi)\) and proceeding as in the proof of the first part of Theorem 2.1, we get the unique global (in time) solution of (3.12). On the other hand, taking the difference quotient of (2.6)_{ev} and (2.6)_{od} w.r.t. the small perturbation of the initial data, making that perturbation tends to 0 and remarking that each component of \( H^{(2)}(t) \) is continuous w.r.t. \((\xi, \theta, \pi)\), we may prove that the solution of (2.6)_{ev} and (2.6)_{od} is in fact differentiable w.r.t. \((\xi, \theta, \pi)\) and satisfies (3.12). (This process is well-known for proving the continuity of the solution of ODE w.r.t. the initial data.)

Furthermore, for each positive integer \( k \geq 1 \) and \( \ell \geq 2 \), putting

\[
J^{(k)}(t) = \left( \partial_x^a \partial_\xi^b \partial_\theta^c \partial_\pi^d \right)_{|a + b| + |a + b| = k} x \quad \text{and} \quad H^{(t)}(t) = \left( \partial_x^a \partial_\xi^b \partial_\theta^c \partial_\pi^d \partial_\xi H \right)_{|a + b| + |a + b| = \ell}
\]

(3.15)

respectively, we have the following differential equation for \( k \geq 2 \):

\[
\frac{d}{dt} J^{(k)}(t) = H^{(2)}(t) J^{(k)}(t) + R^{(k)}(t) \quad \text{with} \quad J^{(k)}(0) = 0
\]

(3.16)

where \( R^{(k)}(t) = \sum_{p=2}^{k} \sum_{k_1 + \cdots + k_p} c_{p,k} H^{(p+1)}(t) J^{(k_1)}(t) \otimes \cdots \otimes J^{(k_p)}(t) \).

Here, \( c_{p,k} \) are suitable constants. It is inductively proved that the each component of \( R^{(k)}(t) \) is continuous w.r.t. \((\xi, \theta, \pi)\) and differentiable w.r.t. \( t \). As above, this equation has the unique solution and therefore the solution of (2.6)_{ev} and (2.6)_{od} is in fact \( k \)-times differentiable w.r.t. \((\xi, \theta, \pi)\).
Therefore, we get the \( s \)-smoothness of the solution of (2.6)_{ev} and (2.6)_{od} w.r.t. \((t, \xi; \bar{\xi}, \pi)\):

\[
\begin{align*}
\{ x(t) &= \sum_{|a|+|b|=0,2,4} x_{ab}(t) \partial^a \partial^b H \quad \text{where} \quad x_{ab}(t) = \partial^b \partial^a x(t; t; \xi, \xi, 0, 0), \\
\xi(t) &= \sum_{|a|+|b|=0,2,4} \xi_{ab}(t) \partial^a \partial^b H \quad \text{where} \quad \xi_{ab}(t) = \partial^b \partial^a \xi(t; t; \xi, \xi, 0, 0), \\
\theta(t) &= \sum_{|a|+|b|=1,3} \theta_{ab}(t) \partial^a \partial^b \pi \quad \text{where} \quad \theta_{ab}(t) = \partial^b \partial^a \theta(t; t; \xi, 0, 0), \\
\pi(t) &= \sum_{|a|+|b|=1,3} \pi_{ab}(t) \partial^a \partial^b \pi \quad \text{where} \quad \pi_{ab}(t) = \partial^b \partial^a \pi(t; t; \xi, 0, 0),
\end{align*}
\]

with \( a = (a_1, a_2), \ b = (b_1, b_2) \in \{0,1\}^2 \).

**Estimates:** We remark, by the structure of \( \mathcal{H}(t, \xi, \theta, \pi) \), the following:

\[
\partial_x, \partial_x, \mathcal{H} = 0, \quad \partial_{\xi}, \partial_{\xi}, \mathcal{H} = 0, \quad (3.18)
\]

\[
\partial_x, \partial_x, \mathcal{H} = \varepsilon \partial_x, \partial_x, A_0 - \varepsilon \sum_{k=1}^{3} \sigma_k \partial_x, \partial_x, A_k \\
= \varepsilon \partial_x, \partial_x, A_0 - \varepsilon \sum_{|a|+|b|=2} (\text{linear combination of } \partial_x, \partial_x, A_0) \theta^a \partial^b \pi, \quad (3.19)
\]

\[
\partial_{\theta}, \partial_{\xi}, \mathcal{H} = \varepsilon \partial_{\theta}, \partial_{\xi}, \pi = \varepsilon \sum_{|a|+|b|=1} \text{const} \theta^a \partial^b \pi, \quad (3.20)
\]

\[
\partial_{\pi}, \partial_{\xi}, \mathcal{H} = \varepsilon \partial_{\pi}, \partial_{\xi}, \pi = \varepsilon \sum_{|a|+|b|=1} \text{const} \theta^a \partial^b \pi, \quad (3.21)
\]

\[
\partial_{\theta}, \partial_x, \mathcal{H} = -\varepsilon \sum_{\ell=1}^{3} \frac{\partial x_{\ell}}{\partial k} \frac{\partial A_l}{\partial x_j} = \varepsilon \sum_{|a|+|b|=1} (\text{linear combination of } \partial_x, A_0) \theta^a \partial^b \pi, \quad (3.22)
\]

\[
\partial_{\pi}, \partial_x, \mathcal{H} = -\varepsilon \sum_{\ell=1}^{3} \frac{\partial x_{\ell}}{\partial k} \frac{\partial A_l}{\partial \pi} = \varepsilon \sum_{|a|+|b|=1} (\text{linear combination of } \partial_x, A_0) \theta^a \partial^b \pi, \quad (3.23)
\]

\[
\partial_{\theta}, \partial_{\pi}, \mathcal{H} = -i \hbar^{-1}(c \xi_3 - \varepsilon A_3) \delta_{kl} = -i \hbar^{-1} c \eta_3 \delta_{kl} = -\partial_{\eta}, \partial_{\pi}, \mathcal{H}, \quad (3.24)
\]

\[
\partial_{\pi}, \partial_{\pi}, \mathcal{H} = -h^{-2}(c(\xi_1 - i \xi_2) - \varepsilon (A_1 - i A_2)) = -h^{-2} c(\eta_1 - i \eta_2) = -\partial_{\eta}, \partial_{\pi}, \mathcal{H}, \quad (3.25)
\]

\[
\partial_{\theta}, \partial_{\theta}, \mathcal{H} = -c(\xi_1 + i \xi_2) + \varepsilon (A_1 + i A_2) = -c(\eta_1 + i \eta_2) = -\partial_{\eta}, \partial_{\theta}, \mathcal{H} \quad (3.26)
\]

for any \( i, j = 1, 2, 3 \) and \( k, l = 1, 2 \).

On the other hand, by (3.17), we must estimate, for any \( \alpha, \beta \),

\[
\pi_B \partial_{\xi}^\alpha \partial_{\pi}^\beta \partial_{\theta}^\alpha \partial_{\pi}^\beta \partial_{\xi}^\alpha \xi_j(t) \quad \text{for} \quad |a+b| = 0, 2, 4,
\]

and

\[
\pi_B \partial_{\xi}^\alpha \partial_{\pi}^\beta \partial_{\theta}^\alpha \partial_{\pi}^\beta \theta_k, \quad \pi_B \partial_{\xi}^\alpha \partial_{\pi}^\beta \partial_{\pi}^\alpha \partial_{\xi}^\beta \pi_k \quad \text{for} \quad |a+b| = 1, 3.
\]

Since it is obvious that body parts of other terms are 0.

**The case** \( |a+b| = 0 \): From (2.12), we have \( \pi_B \partial^x_j = \pi_B c \sigma_j(\theta, \pi) = 0 \), which implies \( \pi_B (x_j - \bar{x}_j) = 0 \).

By (2.12), we have, for \( |a+b| \geq 1 \),

\[
\partial_{\xi}^\alpha \partial_{\pi}^\beta \partial_{\xi}^\alpha \xi_j(t) = \sum_{|\alpha-\alpha'|+|\beta|+|\beta'| \geq 1} \left( \begin{array}{c} \alpha \ \\
\beta \end{array} \right) \left( \begin{array}{c} \beta' \ \\
\beta' \end{array} \right) \left( \partial_{\xi}^{\alpha-\alpha'} \partial_{\pi}^{\beta-\beta'} x_k(t) \cdot \partial_{\xi}^{\alpha'} \partial_{\xi}^{\beta'} \mathcal{H}_{\xi_k \xi_j} + \partial_{\xi}^{\alpha-\alpha'} \partial_{\pi}^{\beta-\beta'} \pi_k(t) \cdot \partial_{\xi}^{\alpha'} \partial_{\xi}^{\beta'} \mathcal{H}_{\pi_k \xi_j} \right).
\]
with argument of \( \mathcal{H}_{xk}\xi_j \), etc. being \((x(t), \xi(t), \theta(t), \pi(t))\). On the other hand, as \( \mathcal{H}_{xk}\xi_j = 0 = \mathcal{H}_{\xi_k}\xi_j \) and body parts of \( \frac{\partial^\alpha}{\partial x_\xi} \xi_j \), we have, 
\[
\frac{d}{dt} \frac{\partial^\alpha}{\partial x_\xi} \xi_j (t, \xi; \xi_0, \xi_0, 0, 0) = 0 \quad \text{and therefore} \quad \pi_\beta \frac{\partial^\alpha}{\partial x_\xi} \xi_j (x - x_j) = 0.
\]
Analogously, as we get 
\[
\frac{d}{dt} \frac{\partial^\alpha}{\partial x_\xi} \xi_j (t, \xi; \xi_0, \xi_0, 0, 0) = -\varepsilon\frac{\partial^\alpha}{\partial x_\xi} \xi_j A_0 (\xi_0)
\]
we have 
\[
|\pi_\beta \frac{\partial^\alpha}{\partial x_\xi} \xi_j (\xi_j (t, \xi; \xi_0, \xi_0, 0, 0), \xi_j) | \leq \varepsilon |t| + |t| |\delta_0| \|\mathbf{A}_0\|_0 |\alpha | + 1, \infty
\]
These give (2)-(1) of Theorem 2.1.

The case \( |a + b| = 1 \): For notational simplicity, we denote by 
\[
\partial_\xi \theta = -\partial_\xi \mathcal{H}_\pi = -\partial_\xi x \cdot \mathcal{H}_\pi - \partial_\xi \xi \cdot \mathcal{H}_\pi - \partial_\xi \theta \cdot \mathcal{H}_\pi - \partial_\xi \pi \cdot \mathcal{H}_\pi \quad \text{etc}
\]
which is the abbreviation of 
\[
\partial_\xi \theta = \partial_\xi \mathcal{H}_\pi = -\partial_\xi x \cdot \mathcal{H}_\pi - \partial_\xi \xi \cdot \mathcal{H}_\pi - \partial_\xi \theta \cdot \mathcal{H}_\pi - \partial_\xi \pi \cdot \mathcal{H}_\pi \quad \text{etc}
\]
From above, we have, 
\[
\frac{d}{dt} \mathcal{H}^{(0)(1)}_1 (t) = \mathcal{H}^{(0)(2)}_1 (t) \mathcal{H}^{(0)(1)}_1 (t) \quad \text{with} \quad \mathcal{H}^{(0)(1)}_1 (0) = I
\]
where 
\[
\mathcal{H}^{(0)(1)}_1 (t) = \left( \frac{\partial \theta}{\partial \pi} \frac{\partial \theta}{\partial \pi} \right) (t, \xi; \xi_0, 0, 0), \quad \mathcal{H}^{(0)(2)}_1 (t) = \left( \frac{-\mathcal{H}_\theta}{-\mathcal{H}_\pi} \frac{-\mathcal{H}_\pi}{-\mathcal{H}_\theta} \right) (\xi_0 (t), \xi_0 (t), 0, 0).
\]
More explicitly, a part of (3.30) with the argument \((t, \xi; \xi_0, 0, 0, 0)\) abbreviated, is rewritten as 
\[
\frac{d}{dt} \left( \frac{\partial \theta}{\partial \pi} \frac{\partial \theta}{\partial \pi} \right) = \begin{pmatrix}
-ich^{-1} \eta_1 \\
-ich^{-2} (\eta_1 - i \eta_2) \\
ich^{-1} \eta_3
\end{pmatrix} \begin{pmatrix}
\frac{\partial \theta}{\partial \pi} \\
\frac{\partial \theta}{\partial \pi} \\
\frac{\partial \theta}{\partial \pi}
\end{pmatrix}
\]
(3.31)
Applying \( \begin{pmatrix} \hbar & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix} \) to both sides of (3.31) and taking the body parts, we have 
\[
\frac{d}{dt} \mathcal{H}_2 \frac{\partial \theta}{\partial \pi} \frac{\partial \theta}{\partial \pi} (t) = \mathcal{H}_2 \frac{\partial \theta}{\partial \pi} \frac{\partial \theta}{\partial \pi} (t)
\]
where 
\[
\mathcal{H}_2 \frac{\partial \theta}{\partial \pi} \frac{\partial \theta}{\partial \pi} (t, \xi; \xi_0, 0, 0, 0), \quad \mathcal{H}_2 (t) = \begin{pmatrix}
-ich^{-1} \eta_1 \\
-ich^{-2} (\eta_1 - i \eta_2) \\
ich^{-1} \eta_3
\end{pmatrix} \begin{pmatrix}
-ich^{-1} \eta_1 \\
-ich^{-2} (\eta_1 - i \eta_2) \\
ich^{-1} \eta_3
\end{pmatrix} (t, \xi; \xi_0, 0, 0).
\]
We prepare the following simple lemma:

**Lemma 3.3.** Let \( \mathcal{H} \) be a Hilbert space over \( \mathbb{C} \) with scalar product and norm denoted by \( \langle , , \rangle \) and \( \| \cdot \| \), respectively. Let \( A(t) \in C([0, T] : \mathcal{B}(\mathcal{H})) \) with \( \Re (A(t)v, v) = 0 \) for any \( v \in \mathcal{H} \). If \( u(t) \in C^1([0, T] : \mathcal{H}) \) satisfy 
\[
\dot{u}(t) = A(t) u(t) + F(t),
\]
then, we have 
\[
\|u(t)\|^2 = \|u(0)\|^2 + 2 \int_0^t ds \Re (F(s), u(s)).
\]
Moreover, we get 
\[
\|u(t)\|^2 \leq e^t \|u(0)\|^2 + e^t \int_0^t ds e^{-s} \|F(s)\|^2.
\]
Now, applying this lemma to (3.32) with $H = \mathbb{C}^2$, $A(t) = X^1_\omega(t)$, $u(t) = Z^1_\omega(t)$ and $F(t) = 0$, we have

\[
\left| \hbar \frac{\partial^2}{\partial t \partial h} (t, \xi, \tilde{\xi}) + \frac{\partial^2}{\partial \theta \partial \pi} (t, \xi, \tilde{\xi}) \right|^2 = \hbar^2 \delta_{1k} \quad \text{for } k = 1, 2. \tag{3.33}
\]

Analogously, as

\[
\frac{d}{dt} Z^1_{2, \omega}(t) = X^1_\omega(t) - Z^1_{2, \omega}(t) \quad \text{with} \quad Z^1_{2, \omega}(t) = \left( \frac{\hbar \theta}{\delta \xi} \right) (t, \xi, \tilde{\xi}, 0, 0), \tag{3.34}
\]

we have

\[
\left| \hbar \frac{\partial^2}{\partial t \partial h} (t, \xi, \tilde{\xi}) + \frac{\partial^2}{\partial \theta \partial \pi} (t, \xi, \tilde{\xi}) \right|^2 = \hbar^2 \delta_{2k} \quad \text{for } k = 1, 2. \tag{3.35}
\]

By the same fashion, we have

\[
\left| \hbar \frac{\partial^2}{\partial t \partial h} (t, \xi, \tilde{\xi}) + \frac{\partial^2}{\partial \theta \partial \pi} (t, \xi, \tilde{\xi}) \right|^2 = \hbar^2 \delta_{2k} \quad \text{for } k = 1, 2. \tag{3.36}
\]

This gives the proof of (ii) with $|a + b| = 1$, $k = |\alpha + \beta| = 0$ (here, we abused the subscript $k$).

We put

\[
J^{(k)}_1(t) = \left( \frac{\partial^2}{\partial t \partial h} \right) (t, \xi, \tilde{\xi}) \quad \text{with} \quad J^{(k)}_1(0) = 0,
\]

and

\[
H^{(k)}(t) = \left( \begin{array}{c}
\hbar^2 \delta_1 \\
\hbar^2 \delta_2 \\
\hbar^2 \delta_3 \\
\hbar^2 \delta_4
\end{array} \right) (t, \xi, \tilde{\xi}, 0, 0)
\]

Then, we have

\[
\frac{d}{dt} J^{(k)}_1(t) = H^{(k)}(t) J^{(k)}_1(t) + J^{(k)}_1(t)
\]

where

\[
J^{(k)}_1(t) = \sum_{\ell=1}^{k} J^{(\ell)}(t) J^{(k-\ell)}_1(t) = H^{(k)}(t) J^{(0)}_1(t) + \cdots. \tag{3.38}
\]

For example, when $k = 1$, we have

\[
\frac{d}{dt} J^{(1)}_1(t) = X^1_\omega(t) Z^1_{2, \omega}(t) + Y^1_{2, \omega}(t) Z^1_{2, \omega}(t)
\]

where

\[
\begin{aligned}
X^1_{2, \omega}(t) &= \left( \begin{array}{c}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \tilde{\xi}} \\
\frac{\partial}{\partial \theta}
\end{array} \right) \left( \begin{array}{c}
H_{\theta \pi} \\
H_{\theta \pi} \\
H_{\theta \pi} \\
H_{\theta \pi}
\end{array} \right) (t, \xi, \tilde{\xi}, 0, 0), \\
Y^1_{2, \omega}(t) &= \hbar^{-1} \left( \begin{array}{c}
0 \\
1 \\
0 \\
0
\end{array} \right) \left( \begin{array}{c}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \tilde{\xi}} \\
\frac{\partial}{\partial \theta}
\end{array} \right) \left( \begin{array}{c}
H_{\theta \pi} \\
H_{\theta \pi} \\
H_{\theta \pi} \\
H_{\theta \pi}
\end{array} \right) (t, \xi, \tilde{\xi}, 0, 0) \left( \begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array} \right).
\end{aligned}
\]

Using above estimate, we have

\[
\left| Y^1_{2, \omega}(t) Z^1_{2, \omega}(t) \right| \leq \tilde{C}^{(1)}_1,
\]

where $\tilde{C}^{(1)}_1$ depending on $\|A\|_{1, \infty} = \sup_{j=1,2,3} \|A_j\|_{1, \infty}, \|A_0\|_{2, \infty}$ (dependence on $\varepsilon, c, h, T$ won’t be clearly mentioned hereafter). By Lemma 3.4, we get

\[
\left| Z^1_{2, \omega}(t) \right| \leq C^{(1)}_1 |t - \tilde{t}|^{1/2}.
\]
Calculating analogously, we get
\[ |F_1^{(1)}(t)| \leq \tilde{C}_1^{(1)} \quad \text{and} \quad |\dot{F}_1^{(1)}(t)| \leq C_1^{(1)}|t - \bar{t}|^{1/2} \quad \text{for} \quad |t - \bar{t}| \leq 1. \]

By induction w.r.t. \( k \), because of the first term of the right-hand side of (3.38) having the bounded body part, there exists constant \( \tilde{C}_1^{(k)} \) s.t.
\[ |F_1^{(k)}(t)| \leq \tilde{C}_1^{(k)} \quad \text{for} \quad |t - \bar{t}| \leq 1. \] (3.40)
Therefore, using Lemma 3.4, we have,
\[ |\dot{F}_1^{(k)}(t)| \leq C_1^{(k)}|t - \bar{t}|^{1/2} \quad \text{for} \quad |t - \bar{t}| \leq 1, \quad k \geq 1. \]
In the above, constants \( \tilde{C}_1^{(k)} \) and \( C_1^{(k)} \) are independent of \((t, \xi, \theta)\) (this saying will be abbreviated if it is no need to stress this).

We proved (ii) with \( |a + b| = 1 \).

The case \(|a + b| = 2\): As before, using \( H_{x \xi} = 0 = H_{\xi \xi}, \) we get
\[ \partial_\theta \hat{x} = \partial_\theta \theta \cdot H_{\theta \xi} + \partial_\theta \pi \cdot H_{\pi \xi}, \quad \hat{x} = \partial_\pi \theta \cdot H_{\theta \xi} + \partial_\pi \pi \cdot H_{\pi \xi}. \] (3.41)
Moreover,
\[ \partial_\theta^2 \hat{x} = \partial_\theta^2 \theta \cdot H_{\theta \xi} + \partial_\theta \theta (\partial_\theta H_{\theta \xi}) + \partial_\theta^2 \pi \cdot H_{\pi \xi} + \partial_\theta \pi (\partial_\theta H_{\pi \xi}), \]
\[ \partial_\pi^2 \hat{x} = \partial_\pi^2 \theta \cdot H_{\theta \xi} + \partial_\pi \theta (\partial_\pi H_{\theta \xi}) + \partial_\pi^2 \pi \cdot H_{\pi \xi} + \partial_\pi \pi (\partial_\pi H_{\pi \xi}), \quad \partial_\theta \partial_\pi \hat{x} = \cdots, \]
with \( \partial_\theta H_{\theta \xi} = \partial_\theta \theta \cdot H_{\theta \xi} + \partial_\theta \pi \cdot H_{\pi \xi}, \partial_\pi H_{\pi \xi} = \partial_\pi \theta \cdot H_{\theta \xi} + \partial_\pi \pi \cdot H_{\pi \xi}, \) etc. (3.42)
As we have
\[ \partial_\theta^2 \theta \cdot H_{\theta \xi} \bigg|_{\theta = \pi = 0} = \partial_\pi^2 \pi \cdot H_{\pi \xi} \bigg|_{\theta = \pi = 0} = 0, \]
\[ |\pi B \partial_\theta \theta (\partial_\theta H_{\theta \xi})| \leq C_2^{(0)}, \quad \text{etc.,} \]
using estimates already obtained, we get
\[ |\pi B \partial_\theta^2 x|, \quad |\pi B \partial_\theta \partial_\pi x| \leq C_2^{(0)}|t - \bar{t}|. \]
Analogously,
\[ \partial_\theta \xi = -\partial_\theta x \cdot H_{xx} - \partial_\theta \theta \cdot H_{\theta x} - \partial_\theta \pi \cdot H_{\pi x}, \quad \partial_\pi \xi = -\partial_\pi x \cdot H_{xx} - \partial_\pi \theta \cdot H_{\theta x} - \partial_\pi \pi \cdot H_{\pi x}, \] (3.43)
and
\[ \partial_\theta^2 \xi = -\partial_\theta^2 x \cdot H_{xx} - \partial_\theta \theta (\partial_\theta H_{\theta x}) - \partial_\theta \pi (\partial_\theta H_{\pi x}) - \partial_\theta^2 \pi \cdot H_{xx} - \partial_\pi^2 \theta \cdot H_{\theta x} - \partial_\pi^2 \pi \cdot H_{\pi x}, \]
\[ \partial_\theta \partial_\pi \xi = \cdots, \quad \partial_\pi^2 \xi = \cdots, \quad \text{etc.} \] (3.44)
which implies
\[ |\pi B \partial_\theta^2 \xi|, \quad |\pi B \partial_\theta \partial_\pi \xi| \leq C_2^{(0)}|t - \bar{t}|. \]
For \( k \geq 1 \), putting
\[ \mathcal{J}_0^{(2)}(t) = \left( \frac{\partial^0 \partial^2 \partial_\theta \partial_\pi x}{\partial_\theta^2 \xi} \right)(t, \bar{t}; \bar{x}, \bar{\xi}, \bar{\tilde{x}}, 0, 0) \quad \text{with} \quad |a + b| = 2 \quad \text{and} \quad |\alpha + \beta| = k, \]
we have
\[ \mathcal{J}_0^{(2)}(t) = \mathcal{F}_2^{(k)}(t) \quad \text{with} \quad |\mathcal{F}_2^{(k)}(t)| \leq \tilde{C}_2^{(k)}|t - \bar{t}|^{1/2} \quad \text{when} \quad |t - \bar{t}| \leq 1, \]
which yields
\[ |\mathcal{J}_0^{(2)}(t)| \leq C_2^{(k)}|t - \bar{t}|^{3/2} \quad \text{when} \quad |t - \bar{t}| \leq 1. \]
The case \( |a + b| = 3 \):

\[
\mathcal{J}_1^{(3)}(t) = \left( \frac{\partial^2 \partial_x^2 \partial_\theta x \partial_\theta}{\partial_x^2 \partial_\theta^2} \right) (t, \xi^0, \zeta^0, 0, 0)^T \text{ with } |a + b| = 3 \text{ and } |\alpha + \beta| = k
\]
satisfies

\[
\mathcal{J}_1^{(3)}(t) = \mathcal{H}_1(0)^2(t) \mathcal{J}_1^{(3)}(t) + \mathcal{F}_3^{(k)}(t), \tag{3.45}
\]

with

\[
\mathcal{F}_3^{(k)}(t) = \sum_{\ell=1}^{k} \mathcal{H}_1(\ell^2)(t) \mathcal{J}_1^{(k-\ell)(3)}(t) = \mathcal{H}_1(\ell^2)(t) \mathcal{J}_1^{(0)(3)}(t) + \ldots.
\]

For example, when \( k = 0 \), we have

\[
\partial_x \partial_\theta^2 \hat{\theta} = -\partial_x^2 x \partial_\theta \mathcal{H}_\pi + \partial_\theta \partial_x (\partial_\theta \mathcal{H}_\pi) - \partial_x^2 \xi \partial_\theta \mathcal{H}_\pi + \partial_\theta (\partial_x \partial_\theta \mathcal{H}_\pi) - \partial_x \partial_\theta (\partial_\theta \mathcal{H}_\pi) - \partial_x^2 \mathcal{H}_\pi \partial_\theta - \partial_x \partial_\theta \mathcal{H}_\pi,
\]

and

\[
\partial_x \partial_\theta \partial_\theta \hat{\theta} = -\partial_x^2 x (\partial_\theta \mathcal{H}_\pi) + \partial_\theta (\partial_\theta \mathcal{H}_\pi) - \partial_x^2 (\partial_\theta \mathcal{H}_\pi) + \partial_\theta (\partial_x \partial_\theta \mathcal{H}_\pi) - \partial_x (\partial_\theta \mathcal{H}_\pi) \partial_\theta - \partial_x \mathcal{H}_\pi (\partial_\theta \partial_\theta \mathcal{H}_\pi) - \partial_\theta (\partial_\theta \mathcal{H}_\pi) - \partial_x \mathcal{H}_\pi (\partial_\theta \mathcal{H}_\pi) - \partial_\theta (\partial_\theta \mathcal{H}_\pi) - \partial_x \partial_\theta \mathcal{H}_\pi + \{**\},
\]

where \{**\} has no body part, because

\[
\{**\} = -\partial_x \partial_\theta \partial_x (\partial_\theta \mathcal{H}_\pi) - \partial_x (\partial_\theta \partial_\theta \mathcal{H}_\pi) - \partial_x^2 (\partial_\theta \mathcal{H}_\pi) + \partial_\theta (\partial_\theta \mathcal{H}_\pi) - \partial_x (\partial_\theta \mathcal{H}_\pi) \partial_\theta - \partial_x \mathcal{H}_\pi (\partial_\theta \partial_\theta \mathcal{H}_\pi) - \partial_\theta (\partial_\theta \mathcal{H}_\pi) - \partial_x \partial_\theta \mathcal{H}_\pi + \{**\}.
\]

Then, the body part of a part of \( \mathcal{F}_3^{(0)}(t) \) is estimated by

\[
|\pi_\mathcal{B} \partial_x^2 \partial_\theta \partial_\theta \mathcal{H}_\pi| \leq C|t - \mathcal{U}|, \quad \text{and therefore, } |\mathcal{F}_3^{(0)}(t)| \leq C|t - \mathcal{U}|
\]

Using Lemma 3.4 and the inequality above, we have

\[
|\mathcal{J}_1^{(0)(3)}(t)| \leq C_3^{(0)} |t - \mathcal{U}|^{3/2}.
\]

Moreover, when \( k \geq 1 \), we have

\[
|\mathcal{J}_1^{(k)(3)}(t)| \leq C_3^{(k)} |t - \mathcal{U}|^{3/2} \quad \text{and} \quad |\mathcal{J}_1^{(k)(3)}(t)| \leq C_3^{(k)} |t - \mathcal{U}|^2.
\]

The case \( |a + b| = 4 \): Let \( k = 0 \). From (3.41), we have

\[
\partial_x^2 \hat{\theta} = \partial_x^2 \theta \cdot \mathcal{H}_\theta + \partial_x \theta (\partial_x \mathcal{H}_\theta) + \partial_x^2 \pi \cdot \mathcal{H}_\pi + \partial_\pi (\partial_\pi \mathcal{H}_\pi),
\]

\[
\text{with } \partial_x \mathcal{H}_\theta = \partial_x \theta \cdot \mathcal{H}_\theta \partial_\pi + \partial_\pi \pi \cdot \mathcal{H}_\pi \partial_\theta, \partial_\pi \mathcal{H}_\pi = \partial_\pi \partial_\theta \mathcal{H}_\pi + \partial_\pi \partial_\pi \mathcal{H}_\pi.
\]

Remarking that \( \partial_\pi \mathcal{H}_\theta = \partial_\pi \partial_\pi \mathcal{H}_\pi = 0 \), we have

\[
\partial_x^2 \partial_\theta \hat{\theta} = \partial_x \partial_\theta \partial_x \mathcal{H}_\theta - \partial_\theta \partial_\theta (\partial_x \mathcal{H}_\theta) + \partial_\theta \partial_\theta (\partial_x \mathcal{H}_\theta) + \partial_\theta (\partial_\theta \partial_\theta \mathcal{H}_\theta) + \partial_\pi (\partial_\pi \partial_\pi \mathcal{H}_\pi) + \partial_\pi \mathcal{H}_\pi \partial_\pi \mathcal{H}_\pi.
\]

\[
\text{with } \partial_\theta \mathcal{H}_\pi = \partial_\theta \partial_\theta \mathcal{H}_\pi + \partial_\theta \partial_\pi \mathcal{H}_\pi \partial_\pi, \partial_\pi \mathcal{H}_\pi = \partial_\pi \partial_\theta \mathcal{H}_\pi + \partial_\pi \partial_\pi \mathcal{H}_\pi.
\]

Finally, we have

\[
\partial_x^2 \partial_\theta \partial_x \mathcal{H}_\pi = \partial_x \partial_\theta \partial_x \mathcal{H}_\pi \partial_\pi + \partial_\theta \partial_\theta \partial_\pi \mathcal{H}_\pi \partial_\pi + \partial_\theta \partial_\pi \partial_\pi \mathcal{H}_\pi \partial_\pi + \partial_\pi \partial_\theta \mathcal{H}_\pi \partial_\pi \mathcal{H}_\pi \partial_\pi + \partial_\pi \partial_\pi \mathcal{H}_\pi \partial_\pi \mathcal{H}_\pi \partial_\pi + \{**\},
\]

with \( \partial_\theta \partial_\pi \mathcal{H}_\pi \partial_\pi = \partial_\theta \partial_\theta \mathcal{H}_\pi \partial_\pi + \partial_\theta \partial_\pi \mathcal{H}_\pi \partial_\pi + \{**\} \).

Therefore, we get

\[
|\text{body part of the right hand side above}| \leq C_4^{(0)} |t - \mathcal{U}|^{3/2} \quad \text{and} \quad |\pi_\mathcal{B} \partial_x^2 \partial_\theta \partial_\theta \mathcal{H}_\pi| \leq C_4^{(0)} |t - \mathcal{U}|^{5/2}.
\]

By the same talk, we get

\[
|\pi_\mathcal{B} \partial_x^2 \partial_\theta \partial_\theta \mathcal{H}_\pi| \leq C_4^{(0)} |t - \mathcal{U}|^{5/2}.
\]
Proceeding as before, we get
\[ |\pi_B B^\beta_\xi B^\gamma_\xi B^2_\xi|^1, \quad |\pi_B B^\beta_\xi B^\gamma_\xi B^2_\xi| \leq C_4^{(k)}|t - \bar{t}|^3 \quad \text{for} \quad k = |\alpha + \beta| \geq 1. \]
Therefore, Theorem 2.1 has been proved. \(\square\)

3.1.3. Inverse map: Proof of Theorem 2.2. Supersmoothness of (2.17) is already proved.

**Existence of the inverse map.** For notational simplicity, we put
\[ X_i(x, \theta) = x_i(t, \xi; x, \xi, \xi, \pi), \quad \Theta_i(x, \theta) = \theta_i(t, \xi; x, \xi, \xi, \pi), \]
\[ Y_j(x, \theta) = y_j(t, \xi; x, \xi, \xi, \pi), \quad \Omega_m(x, \theta) = \omega_m(t, \xi; x, \xi, \xi, \pi), \]
and we consider \((t, \xi; x, \xi, \xi, \pi)\) as parameters which will not be represented explicitly.

For any fixed \((t, \xi; x, \xi, \xi, \pi)\) and any given \((x, \theta)\), we want to find \((x, \theta)\) such that
\[ \bar{x}_1 = X_i(x, \theta), \quad \bar{\theta}_i = \Theta_i(x, \theta). \]

Denoting this \((x, \theta)\) as \((Y(x, \theta), \Omega(x, \theta))\), then we should have
\[ x = Y_i(X(x, \theta), \Theta(x, \theta)), \quad \theta = \Omega_i(X(x, \theta), \Theta(x, \theta)). \]
(3.46)

By the supersmoothness proved in 1) of Theorem 2.1, we have
\[ X_i(x, \theta) = X_i,0(x) + X_i,1(x)\theta_1 + X_i,2(x)\theta_2 + X_i,3(x)\theta_1\theta_2 = \sum_{|a| \leq 2} \partial_a X_i,0(x)\theta^a, \]
\[ \Theta_k(x, \theta) = \Theta_k,0(x) + \Theta_k,1(x)\theta_1 + \Theta_k,2(x)\theta_2 + \Theta_k,3(x)\theta_1\theta_2 = \sum_{|a| \leq 2} \partial_a \Theta_k,0(x)\theta^a. \]

We assume (and prove by construction) that we may put also
\[ Y_j(x, \theta) = Y_j,0(x) + Y_j,1(x)\theta_1 + Y_j,2(x)\theta_2 + Y_j,3(x)\theta_1\theta_2 = \sum_{|b| \leq 2} \partial_b Y_j,0(x)\theta^b, \]
\[ \Omega_{\xi}(x, \theta) = \Omega_{\xi,0}(x) + \Omega_{\xi,1}(x)\theta_1 + \Omega_{\xi,2}(x)\theta_2 + \Omega_{\xi,3}(x)\theta_1\theta_2 = \sum_{|b| \leq 2} \partial_b \Omega_{\xi,0}(x)\theta^b. \]

Denoting
\[ \tilde{X} = (\tilde{X}_1, \tilde{X}_2, \tilde{X}_3), \quad \tilde{X}_B = (\tilde{X}_{1,B}, \tilde{X}_{2,B}, \tilde{X}_{3,B}) \]
with \( \tilde{X}_i = \tilde{X}_{i,B} + \tilde{X}_{i,S} = X_i,0,B(x,0) + X_i,0,S(x) = X_i(x,0) \in \mathcal{R}_{ev}, \quad \tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2) \quad \text{with} \quad \tilde{\Theta}_i = \Theta_i,0(x) = \Theta_i(x,0) \in \mathcal{R}_{od}. \)

we have
\[ Y_j(\tilde{X}, \tilde{\Theta}) = Y_j,0(\tilde{X}) + Y_j,1(\tilde{X})\tilde{\Theta}_1 + Y_j,2(\tilde{X})\tilde{\Theta}_2 + Y_j,3(\tilde{X})\tilde{\Theta}_1\tilde{\Theta}_2, \]
\[ \text{with} \quad Y_j,0(\tilde{X}), Y_j,3(\tilde{X}) \in \mathcal{R}_{ev}, \quad Y_j,1(\tilde{X}), Y_j,2(\tilde{X}) \in \mathcal{R}_{od}, \]
\[ \Omega_{\xi,0}(\tilde{X}, \tilde{\Theta}) = \Omega_{\xi,0}(\tilde{X}) + \Omega_{\xi,1}(\tilde{X})\tilde{\Theta}_1 + \Omega_{\xi,2}(\tilde{X})\tilde{\Theta}_2 + \Omega_{\xi,3}(\tilde{X})\tilde{\Theta}_1\tilde{\Theta}_2, \]
\[ \text{with} \quad \Omega_{\xi,0}(\tilde{X}), \Omega_{\xi,3}(\tilde{X}) \in \mathcal{R}_{ev}, \quad \Omega_{\xi,1}(\tilde{X}), \Omega_{\xi,2}(\tilde{X}) \in \mathcal{R}_{od}. \]

**Claim I:** From the first equation of (3.46), we construct \( Y_i,*(\tilde{X}_B) \) for each degree.

(I-0) Restricting \((x, \theta)\) to \((x, 0)\) in (3.46), we have
\[ x = Y_i,0(\tilde{X}) + Y_i,1(\tilde{X})\tilde{\Theta}_1 + Y_i,2(\tilde{X})\tilde{\Theta}_2 + Y_i,3(\tilde{X})\tilde{\Theta}_1\tilde{\Theta}_2. \]
Or more precisely, putting \( Y_i^{[-1]}(\tilde{X}) = 0 \), we have
\[ Y_i^{[p]}(\tilde{X}) = \frac{p}{2p} \sum_{q=0}^{p-1} Y_i^{[2p-2q-1]}(\tilde{X})\tilde{\Theta}_1^{2q+1} + \frac{p}{2p} \sum_{q=0}^{p-1} Y_i^{[2p-2q-1]}(\tilde{X})\tilde{\Theta}_1^{2q+1} \]
\[ - \sum_{r=0}^{p} Y_i^{[2p-2q]}(\tilde{X}) \sum_{q=0}^{p} \tilde{\Theta}_1^{2q+1} \tilde{\Theta}_2^{2q-1} \quad \text{for} \quad p = 0, 1, 2, \ldots. \]
(3.47)
(I-1) Differentiating (3.46) w.r.t. \( \theta_1 \) or \( \theta_2 \) and restricting as above, we have for each \( i = 1, 2, 3 \),

\[
\begin{pmatrix}
\theta_{1,1}(x) & \theta_{2,1}(x) \\
\theta_{1,2}(x) & \theta_{2,2}(x)
\end{pmatrix}
\begin{pmatrix}
Y_{i,1}(\tilde{x}) \\
Y_{i,2}(\tilde{x})
\end{pmatrix}
= \begin{pmatrix}
-X_{j,1}(x)\partial_{x_i}Y_i(\tilde{x}, \tilde{\theta}) \\
-X_{j,2}(x)\partial_{x_i}Y_i(\tilde{x}, \tilde{\theta})
\end{pmatrix}.
\]

Or, we have

\[
\sum_{q=0}^{p-1} \left( \begin{array}{cc}
\theta_{1,1}^{[2q]}(x) & \theta_{2,1}^{[2q]}(x) \\
\theta_{1,2}^{[2q]}(x) & \theta_{2,2}^{[2q]}(x)
\end{array} \right)
\begin{pmatrix}
Y_{i,1}^{[2p-2q-1]}(\tilde{x}) \\
Y_{i,2}^{[2p-2q-1]}(\tilde{x})
\end{pmatrix}
= \sum_{q=0}^{p-1} \left( \begin{array}{cc}
-X_{j,1}^{[2p-2q-1]}(x)\partial_{x_i}Y_i^{[2q]}(\tilde{x}, \tilde{\theta}) \\
-X_{j,2}^{[2p-2q-1]}(x)\partial_{x_i}Y_i^{[2q]}(\tilde{x}, \tilde{\theta})
\end{array} \right).
\]

(I-2) Differentiating (3.46) w.r.t. \( \omega_1 \) and \( \omega_2 \), we have

\[
-\left[ \theta_{1,1}(x)\theta_{2,2}(x) - \theta_{2,1}(x)\theta_{1,2}(x) \right] Y_{i,3}(\tilde{x}) = X_{i,3}(x)\partial_{x_i}Y_i(\tilde{x}, \tilde{\theta})
- X_{j,1}(x)X_{j,2}(x)\partial_{x_i}Y_i(\tilde{x}, \tilde{\theta})
+ \left[ -X_{j,1}(x)\theta_{1,2}(x) + \theta_{1,1}(x)X_{j,2}(x) \right] \partial_{x_i}Y_{i,1}(\tilde{x})
+ \left[ -X_{j,1}(x)\theta_{2,2}(x) + \theta_{2,1}(x)X_{j,2}(x) \right] \partial_{x_i}Y_{i,2}(\tilde{x}).
\]

Therefore, we have

\[
-\sum_{q=0}^{p} \sum_{r=0}^{q} \left[ \theta_{1,1}^{[2q-2r]}(x)\theta_{2,2}^{[2r]}(x) - \theta_{2,1}^{[2q-2r]}(x)\theta_{1,2}^{[2r]}(x) \right] Y_{i,3}^{[2p-2q]}(\tilde{x})
= \sum_{q=0}^{p} \sum_{r=0}^{q-1} X_{j,1}^{[2q]}(x)\partial_{x_i}Y_i^{[2p-2q-1]}(\tilde{x}, \tilde{\theta})
- \sum_{q=0}^{p-1} \sum_{r=0}^{q} X_{j,1}^{[2q]}(x)\partial_{x_i}Y_i^{[2p-2q-1]}(\tilde{x}, \tilde{\theta})
+ \sum_{q=0}^{p-1} \sum_{r=0}^{q} \left[ -X_{j,1}^{[2q]}(x)\theta_{1,2}^{[2r]}(x) + \theta_{1,1}^{[2q]}(x)X_{j,2}^{[2r]}(x) \right] \partial_{x_i}Y_{i,1}^{[2p-2q]}(\tilde{x})
+ \sum_{q=0}^{p-1} \sum_{r=0}^{q} \left[ -X_{j,1}^{[2q]}(x)\theta_{2,2}^{[2r]}(x) + \theta_{2,1}^{[2q]}(x)X_{j,2}^{[2r]}(x) \right] \partial_{x_i}Y_{i,2}^{[2p-2q]}(\tilde{x}).
\]

(1) For any fixed \( \bar{x}_B \), we consider the map

\[
\bar{x}_B \rightarrow F(\bar{x}_B) = \bar{x}_B + X_B(\bar{x}_B),
\]

which satisfies

\[
F(\bar{x}_B) - F(\bar{x}_B^\prime) = (\bar{x}_B - \bar{x}_B^\prime) \int_0^1 d\tau \left( I - \frac{\partial X_B}{\partial x} (\tau \bar{x}_B + (1 - \tau) \bar{x}_B^\prime) \right).
\]

As \( \partial_{x_i}X_{i,0}(x) - \delta_{ij} = 0 \), \( F(\bar{x}_B) \) is the contraction map, therefore, there exists a unique \( \bar{x}_B \) such that

\[
\bar{x}_B = X_B(\bar{x}_B).
\]

We denote this as \( x_B = Y_B(\bar{x}_B) \), that is, \( Y^{[0]}(\bar{x}_B) \), and therefore, \( Y^{[0]}(\bar{x}) \) is defined, which satisfies (3.47) with \( p = 0 \).

(2) From (3.48) with \( p = 1 \), we have, for each \( i = 1, 2, 3 \),

\[
\begin{pmatrix}
\theta_{1,1}(x) & \theta_{2,1}(x) \\
\theta_{1,2}(x) & \theta_{2,2}(x)
\end{pmatrix}
\begin{pmatrix}
Y_{i,1}^{[1]}(\tilde{x}) \\
Y_{i,2}^{[1]}(\tilde{x})
\end{pmatrix}
= \begin{pmatrix}
-X_{j,1}^{[1]}(x)\partial_{x_i}Y_i^{[1]}(\tilde{x}, \tilde{\theta}) \\
-X_{j,2}^{[1]}(x)\partial_{x_i}Y_i^{[1]}(\tilde{x}, \tilde{\theta})
\end{pmatrix}.
\]

As \( \partial_{x_i}Y_i^{[0]}(\tilde{x}, \tilde{\theta}) = \partial_{x_i}Y_i^{[0]}(\bar{x}_B) \), the right-hand side above is given by the step (1) above. On the other hand, when \( \|t - \tilde{t}\| < \delta \), \( t, \tilde{t} \in [-T, T] \), for any \( (x, \xi) \), we have

\[
\det \begin{pmatrix}
\theta_{1,1}(x, \xi) & \theta_{1,2}(x, \xi) \\
\theta_{2,1}(x, \xi) & \theta_{2,2}(x, \xi)
\end{pmatrix} \neq 0 \quad \text{because} \quad \theta_{k,j}(x) = \partial_{x_i} \theta_k(t, \tilde{t}; x, \xi, \tilde{\xi}, 0, 0).
\]

Solving (3.50), we get the degree 1 part of \( Y_{i,\ast}^{[1]}(\bar{x}_B) \), that is, \( Y_{i,\ast}^{[1]}(\bar{x}) \) for \( i = 1, 2, 3, \ast = 0, 1, 2, 3 \).

(3) Putting \( p = 0 \) in (3.49), we have

\[
-\left[ \theta_{1,1}(x)\theta_{2,2}(x) - \theta_{2,1}(x)\theta_{1,2}(x) \right] Y_{i,3}^{[0]}(\bar{x}) = X_{j,3}(x)\partial_{x_i}Y_i^{[0]}(\bar{x}, \tilde{\theta}).
\]

Therefore, we get \( Y_{i,\ast}^{[0]}(\bar{x}) \).
Returning back to (1) with \( p = 1 \) in (3.47) and then (2) with \( p = 2 \) in (3.48) and lastly (3) with \( p = 2 \) in (3.49). This process determines \( Y_{i,0}^1(\hat{X}), Y_{i,1}^1(\hat{X}), Y_{i,2}^1(\hat{X}) \) and \( Y_{i,3}^1(\hat{X}) \). Proceeding recursively, we determine \( Y_{i,*}(\hat{X}) \).

**Claim II:** Analogously as above, we determine \( \Omega_{\ell,*}(\hat{X}) \) as follows.

(II-0) Restricting \((\bar{x}, \bar{\theta})\) to \((\bar{x}, 0)\) in the second equation of (3.46), we have

\[
0 = \Omega_{\ell,0}(\hat{X}) + \Omega_{\ell,1}(\hat{X})\bar{\theta}_1 + \Omega_{\ell,2}(\hat{X})\bar{\theta}_2 + \Omega_{\ell,3}(\hat{X})\bar{\theta}_1\bar{\theta}_2.
\]

In other word, we have

\[
\Omega_{\ell,0}^{[2p+1]}(\hat{X}) = -\sum_{q=0}^{p} \Omega_{\ell,1}^{[2q]}(\hat{X})\bar{\theta}_1^{[2p-2q+1]} - \sum_{q=0}^{p} \Omega_{\ell,2}^{[2q]}(\hat{X})\bar{\theta}_2^{[2p-2q+1]}
- \sum_{q=0}^{p} \Omega_{\ell,3}^{[2p-2q+1]}(\hat{X}) \sum_{r=0}^{q} \bar{\theta}_1^{[2q-2r-1]}\bar{\theta}_2^{[2r+1]}.
\] (3.51)

(II-1) Differentiating (3.46) w.r.t. \( \bar{\theta}_1 \), we have,

\[
\begin{pmatrix}
\Theta_{1,1}(\bar{x}) & \Theta_{1,2}(\bar{x}) \\
\Theta_{2,1}(\bar{x}) & \Theta_{2,2}(\bar{x})
\end{pmatrix} \begin{pmatrix}
\Omega_{1,1}(\hat{X}) \\
\Omega_{1,2}(\hat{X})
\end{pmatrix} = \begin{pmatrix}
1 - X_{j,1}(\bar{x})\partial_{\bar{x}} \Omega_{1,1}(\hat{X}) \\
-X_{j,2}(\bar{x})\partial_{\bar{x}} \Omega_{1,2}(\hat{X})
\end{pmatrix},
\]

that is,

\[
\sum_{q=0}^{p} \begin{pmatrix}
\Theta_{1,1}^{[2q]}(\bar{x}) & \Theta_{2,1}^{[2q]}(\bar{x}) \\
\Theta_{1,2}^{[2q]}(\bar{x}) & \Theta_{2,2}^{[2q]}(\bar{x})
\end{pmatrix} \begin{pmatrix}
\Omega_{1,1}^{[2p-2q]}(\hat{X}) \\
\Omega_{1,2}^{[2p-2q]}(\hat{X})
\end{pmatrix} = \begin{pmatrix}
1-\sum_{q=0}^{p} X_{j,1}^{[2q-1]}(\bar{x})\partial_{\bar{x}} \Omega_{1,1}^{[2p-2q+1]}(\hat{X}) \\
-\sum_{q=0}^{p} X_{j,2}^{[2q-1]}(\bar{x})\partial_{\bar{x}} \Omega_{1,2}^{[2p-2q+1]}(\hat{X})
\end{pmatrix}.
\] (3.52)

Analogously, differentiating (3.46) w.r.t. \( \bar{\theta}_2 \),

\[
\sum_{q=0}^{p} \begin{pmatrix}
\Theta_{1,1}^{[2q]}(\bar{x}) & \Theta_{2,1}^{[2q]}(\bar{x}) \\
\Theta_{1,2}^{[2q]}(\bar{x}) & \Theta_{2,2}^{[2q]}(\bar{x})
\end{pmatrix} \begin{pmatrix}
\Omega_{1,1}^{[2p-2q]}(\hat{X}) \\
\Omega_{1,2}^{[2p-2q]}(\hat{X})
\end{pmatrix} = \begin{pmatrix}
-\sum_{q=0}^{p} X_{j,1}^{[2q-1]}(\bar{x})\partial_{\bar{x}} \Omega_{2,1}^{[2p-2q+1]}(\hat{X}) \\
1-\sum_{q=0}^{p} X_{j,2}^{[2q-1]}(\bar{x})\partial_{\bar{x}} \Omega_{2,2}^{[2p-2q+1]}(\hat{X})
\end{pmatrix}.
\] (3.53)

(II-2) Differentiating (3.46) w.r.t. \( \bar{\theta}_1 \) and \( \bar{\theta}_2 \), we have, for \( \ell = 1, 2, \)

\[
0 = X_{i,3}(\bar{x})\partial_{\bar{x}} \Omega_{\ell}(\hat{X}, \hat{\theta}) - X_{i,1}(\bar{x})X_{j,3}(\bar{x})\partial_{\bar{x},\bar{x}}^2 \Omega_{\ell}(\hat{X}, \hat{\theta})
+ \Theta_{1,3}(\bar{x})\Omega_{\ell,1}(\hat{X}) + \Theta_{2,3}(\bar{x})\Omega_{\ell,2}(\hat{X}) + [-X_{j,1}(\bar{x})\Theta_{1,2}(\bar{x}) + \Theta_{1,1}(\bar{x})X_{i,2}(\bar{x})]\Omega_{\ell,1}(\hat{X})
+ [-X_{j,1}(\bar{x})\Theta_{2,2}(\bar{x}) + \Theta_{2,1}(\bar{x})X_{i,2}(\bar{x})]\Omega_{\ell,2}(\hat{X}) + \Theta_{1,1}(\bar{x})\Theta_{2,2}(\bar{x}) - \Theta_{2,1}(\bar{x})\Theta_{1,2}(\bar{x})]\Omega_{\ell,3}(\hat{X}).
\]

Rewriting above, we have, for each \( p = 0, 1, 2, \cdots, \)

\[
-\sum_{q=0}^{p} \sum_{r=0}^{q} \Theta_{1,2}^{[2q-2r]}(\bar{x})\Theta_{2,2}^{[2r]}(\bar{x})\partial_{\bar{x}} \Omega_{\ell,3}^{[2p-2q+1]}(\hat{X})
= \sum_{q=0}^{p} X_{i,3}^{[2q]}(\bar{x})\partial_{\bar{x}} \Omega_{\ell,3}^{[2p-2q+1]}(\hat{X}, \hat{\theta})
- \sum_{q=0}^{p} \sum_{r=0}^{q} X_{i,1}^{[2q+1]}(\bar{x})X_{j,2}^{[2q-2r-1]}(\bar{x})\partial_{\bar{x},\bar{x}}^2 \Omega_{\ell,3}^{[2p-2q+1]}(\hat{X}, \hat{\theta})
+ \sum_{q=0}^{p} \Theta_{1,3}^{[2q]}(\bar{x})\Omega_{\ell,1}^{[2p-2q+1]}(\hat{X}) + \sum_{q=0}^{p} \Theta_{2,3}^{[2q]}(\bar{x})\Omega_{\ell,2}^{[2p-2q+1]}(\hat{X})
+ \sum_{q=0}^{p} X_{i,1}^{[2q+1]}(\bar{x})\Theta_{1,2}^{[2q-2r]}(\bar{x})\Theta_{1,2}^{[2p-2q+1]}(\hat{X}) + \sum_{q=0}^{p} \Theta_{1,1}^{[2q]}(\bar{x})\Theta_{1,2}^{[2p-2q+1]}(\hat{X}),
\] (3.54)

By the same procedure which we employed to determine \( Y_{i,*}(\hat{X}) \), we may define \( \Omega_{\ell,*}(\hat{X}) \).
Therefore, there exist a constant $\delta > 0$ and functions $y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \omega(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi})$ such that for $|t - \xi| < \delta$

$$
\bar{x} = y(t, \xi; x(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \xi, \theta(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \Xi),
$$

$$
\bar{\theta} = \omega(t, \xi; x(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \xi, \theta(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \Xi),
$$

and therefore

$$
\bar{x} = x(t, \Xi; y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \xi, \omega(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \Xi),
$$

$$
\bar{\theta} = \theta(t, \xi; y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \xi, \omega(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}), \Xi).
$$

**Estimates of the inverse mapping**: When $|a + b| = 0$, differentiating once the first equation of (3.56) w.r.t. $\bar{x}$ or $\bar{\pi}$, we get

$$
\delta_{jk} = \frac{\partial \bar{e}_j}{\partial \bar{x}_k} = \frac{\partial y_{jk}}{\partial \bar{x}_k} \frac{\partial x_j}{\partial \bar{x}_l} + \frac{\partial \omega_{mk}}{\partial \bar{x}_k} \frac{\partial x_j}{\partial \bar{\theta}_m},
$$

$$
0 = \frac{\partial x_j}{\partial \bar{\pi}_k} = \frac{\partial y_{jk}}{\partial \bar{x}_k} \frac{\partial x_j}{\partial \bar{\pi}_l} + \frac{\partial \omega_{mk}}{\partial \bar{x}_k} \frac{\partial x_j}{\partial \bar{\theta}_m}.
$$

Taking the body part and remarking (2.12), we get

$$
|\pi_B \partial_\bar{x} (y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| = 0 \text{ if } |t - \xi| \leq \delta.
$$

Here, we used the fact

$$
\pi_B \frac{\partial x_j}{\partial \bar{L}_k} = \pi_B \left( \frac{\partial (x_j - \bar{x})}{\partial \bar{L}_k} + \frac{\partial \bar{x}}{\partial \bar{L}_k} \right) = \delta_{j\ell} \text{ if } |t - \xi| \leq \delta,
$$

which follows from (2.12). Analogously, we have

$$
|\pi_B \partial_\bar{\pi} (y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| = 0 \text{ if } |t - \xi| \leq \delta.
$$

By the same procedure, we have

$$
|\pi_B \partial_\bar{x}^k \partial_\bar{\pi}^\ell (y(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{x})| = 0 \text{ if } k = \|a + b\| \geq 2 \text{ and } |t - \xi| \leq \delta.
$$

The case when $|a + b| = 1$, we have

$$
\delta_{\ell m} = \frac{\partial \bar{\theta}_m}{\partial \bar{\theta}_m} = \frac{\partial y_{\ell k}}{\partial \bar{\theta}_m} \frac{\partial \bar{\theta}_k}{\partial \bar{\theta}_m} + \frac{\partial \omega_{\ell k}}{\partial \bar{\theta}_m} \frac{\partial \bar{\theta}_k}{\partial \bar{\theta}_m},
$$

$$
0 = \frac{\partial \bar{\theta}_m}{\partial \bar{\pi}_m} = \frac{\partial y_{\ell k}}{\partial \bar{\pi}_m} \frac{\partial \bar{\theta}_k}{\partial \bar{\pi}_m} + \frac{\partial \omega_{\ell k}}{\partial \bar{\pi}_m} \frac{\partial \bar{\theta}_k}{\partial \bar{\pi}_m}.
$$

Taking the body part and using (2.13), we get

$$
I = (\delta_{\ell m}) = XY, \quad X = \left( \pi_B \frac{\partial \omega_k}{\partial \bar{\theta}_m} \right), \quad Y = \left( \pi_B \frac{\partial \bar{\theta}_k}{\partial \bar{\pi}_m} \right).
$$

Since

$$
\left| \pi_B \frac{\partial (\theta_k - \bar{\theta}_k)}{\partial \bar{\theta}_m} \right| \leq \tilde{C}^{(1)} \|t - \xi\|^{1/2}, \quad \pi_B \frac{\partial \bar{\theta}_m}{\partial \bar{\pi}_m} = \delta_{k\ell},
$$

if we take $|t - \xi| \leq \delta \leq (4\tilde{C}^{(1)})^{-1}$, then we have, as operators

$$
|X| = |Y^{-1}| \leq 2 \text{ if } |t - \xi| \leq \delta.
$$

Therefore, using $(X - I)Y = I - Y$, we have, as operators

$$
|X - I| \leq |I - Y| \cdot |Y^{-1}| \leq 2 \tilde{C} \|t - \xi\|^{1/2},
$$

that is,

$$
|\pi_B \partial_\bar{x} (\omega_k(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{\theta}_k)| \leq \tilde{C}^{(0)} \|t - \xi\|^{1/2}.
$$

From the second equality of (3.58) combined with (2.13),

$$
|\pi_B \partial_\bar{x} (\omega_k(t, \xi; \bar{x}, \xi, \bar{\theta}, \bar{\pi}) - \bar{\theta}_k)| \leq \tilde{C}^{(0)} \|t - \xi\|^{1/2}.
$$
Analogously, we get, when $|a + b| = 1$,

$$|\pi_B \partial_\xi^{d_3} \partial_\theta^{d_2} \partial_\pi^{d_1} (\omega_k(t; \xi, \theta, \pi) - \tilde{\omega}_k)| \leq \hat{C}_1^{(k)}|t - \bar{t}|^{(1/2)(1-1-k)_+} \quad \text{if} \quad k = |a + \beta|, \quad |t - \bar{t}| \leq \delta.
$$

Proceeding analogously as we did in proving Theorem 2.1, we have the desired results for $|a + b| \geq 2$, which are abbreviated here. The second inequality in (2.19) is given in the next subsection. $\square$

Analogously, we have

**Proposition 3.4.** (i) For any fixed $(t; \xi, \theta, \pi)$, $|t - \bar{t}| < \delta$, the mapping

$$\mathfrak{K}^{3[2]} \ni (\xi, \pi) \mapsto (\xi = \xi(t; \xi, \theta, \pi), \pi = \pi(t; \xi, \theta, \pi)) \in \mathfrak{K}^{3[2]}$$

is supersmooth. The inverse mapping defined by

$$\mathfrak{K}^{3[2]} \ni (\xi, \pi) \mapsto (\eta = \eta(t; \xi, \theta, \pi), \rho = \rho(t; \xi, \theta, \pi)) \in \mathfrak{K}^{3[2]},$$

is supersmooth in $(\xi, \theta, \pi)$ for fixed $(t, \xi)$.

(ii) Let $|a + b| = 0$. We have

$$\left\{ \begin{array}{l}
|\pi_B \partial_\xi^{d_3} \partial_\theta^{d_2} \partial_\pi^{d_1} (\eta(t; \xi, \theta, \pi) - \xi)| = 0, \\
|\eta^{[0]}(t; \xi, \theta, \pi)| \leq C_2|t - \bar{t}|(1 + |\xi^{[0]}| + |\theta^{[0]}|).
\end{array} \right. \quad (3.62)$$

(iii) Let $|a + b| = 1$. For $k = |a + \beta|$, there exists a constant $\hat{C}_1^{(k)}$ such that

$$|\pi_B \partial_\xi^{d_3} \partial_\theta^{d_2} \partial_\pi^{d_1} (\rho(t; \xi, \theta, \pi) - \pi)| \leq \hat{C}_1^{(k)}|t - \bar{t}|^{(1/2)(1-1-k)_+}.$$  

(iv) Let $|a + b| = 2$. For $k = |a + \beta|$, there exists a constant $\hat{C}_2^{(k)}$ such that

$$|\pi_B \partial_\xi^{d_3} \partial_\theta^{d_2} \partial_\pi^{d_1} (\rho(t; \xi, \theta, \pi) - \pi)| \leq \hat{C}_2^{(k)}|t - \bar{t}|^{1+(1/2)(1-1-k)_+}.$$  

(v) Let $|a + b| = 3$. For $k = |a + \beta|$, there exists a constant $\hat{C}_3^{(k)}$ such that

$$|\pi_B \partial_\xi^{d_3} \partial_\theta^{d_2} \partial_\pi^{d_1} (\rho(t; \xi, \theta, \pi) - \pi)| \leq \hat{C}_3^{(k)}|t - \bar{t}|^{3/2+(1/2)(1-1-k)_+}.$$  

(vi) Let $|a + b| = 4$. For $k = |a + \beta|$, there exists a constant $\hat{C}_4^{(k)}$ such that

$$|\pi_B \partial_\xi^{d_3} \partial_\theta^{d_2} \partial_\pi^{d_1} (\rho(t; \xi, \theta, \pi) - \pi)| \leq \hat{C}_4^{(k)}|t - \bar{t}|^{5/2+(1/2)(1-1-k)_+}.$$  

3.1.4. **Time reversing.** As the Hamilton equation (2.6)$_{ev}$ and (2.6)$_{od}$ may be solved backward in time, we denote, for $\bar{t} \leq t \leq \bar{t}$, that $x(t; \bar{t}, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}, \bar{\theta}, \bar{\xi}, \bar{\pi})$, $\theta(t; \bar{t}, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})$, $\pi(t; \bar{t}, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})$, $\eta(t; \bar{t}, \bar{x}, \theta, \pi, \pi)$ are solutions at time $t$ of (2.6)$_{ev}$ and (2.6)$_{od}$ with the initial time $t = \bar{t}$ and the initial data $(\bar{x}, \bar{\theta}, \bar{\xi}, \bar{\pi})$.

Proceeding as in previous sections, we have the following:

$$\begin{aligned}
\bar{x} &= x(t; \bar{t}, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \\
\bar{\theta} &= \theta(t; \bar{t}, \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \\
\bar{\xi} &= \xi(t; \bar{t}, \bar{x}, \bar{\theta}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \\
\bar{\pi} &= \pi(t; \bar{t}, \bar{x}, \bar{\theta}, \bar{\xi}, \bar{\pi})
\end{aligned}$$

(3.67)

For the inverse mappings, we have, if $|\bar{t} - l| < \delta$,

$$\begin{aligned}
\bar{x} &= x(l; t, \bar{x}, \xi, \theta, \pi), \\
\bar{\theta} &= \theta(l; t, \bar{x}, \xi, \theta, \pi), \\
\bar{\xi} &= \xi(l; t, \bar{x}, \theta, \pi), \\
\bar{\pi} &= \pi(l; t, \bar{x}, \theta, \pi).
\end{aligned}$$

(3.68)
Therefore, we get for $|\bar{t} - \bar{t}'| < \delta$,
\[
\begin{aligned}
y(\bar{t}, \bar{t}; \bar{x}, \bar{\xi}, 0, 0) - \bar{x} &= \int_0^1 dt \bar{\xi} \cdot \partial_{\bar{\xi}} y(\bar{t}, \bar{t}; \bar{x}, \bar{r}_0, 0, 0) \\
&\quad + \int_0^1 dt \bar{x} \cdot \partial_{\bar{x}} y(\bar{t}, \bar{t}; \bar{x}, \bar{r}, 0, 0) - \bar{x} + y(\bar{t}, \bar{t}; 0, 0, 0, 0).
\end{aligned}
\]

Proof of Theorem 2.2 continued. The second inequality in (2.19) is proved using
\[
|y^{[0]}(\bar{t}, \bar{t}; \bar{x}^{[0]}, \bar{\xi}^{[0]})) - \bar{x}^{[0]})| \leq C|\bar{t} - \bar{t}'|(|\bar{x}^{[0]}| + |\bar{\xi}^{[0]}|) + |y^{[0]}(\bar{t}, \bar{t}; 0, 0, 0, 0)|.
\]

Replacing $(\bar{t}, \bar{t})$ in the first equality of (3.69) by $(\bar{t}, \bar{t})$ and using (2.14), we have
\[
|y^{[0]}(\bar{t}, \bar{t}; 0, 0)| = |x(\bar{t}, \bar{t}; 0, \eta^{[0]}(\bar{t}, \bar{t}; 0, 0, 0, 0)| \leq C|\bar{t} - \bar{t}'|,
\]

since $\eta^{[0]}(\bar{t}, \bar{t}; 0, 0)$ is continuous in $\bar{t}, \bar{t}$. Combining these, we get the desired inequality.  

3.2. Action integral. We prepare the following lemma in a slightly general situation:

**Lemma 3.5.** Let $(x(x, \theta), \theta(x, \theta), \xi(x, \theta), \pi(x, \theta))$ be supersmooth functions of $(x, \theta) \in R^{m|n}$ satisfying
\[
\begin{aligned}
\frac{\partial u}{\partial x_j} &= \sum_{a=1}^m \frac{\partial x_a}{\partial x_j} \xi_a(x, \theta) + \sum_{b=1}^n \frac{\partial \theta_b}{\partial x_j} \pi_b(x, \theta) \quad \text{for } j = 1, 2, \ldots, m, \\
\frac{\partial u}{\partial x_j} &= \sum_{a=1}^m \frac{\partial x_a}{\partial x_j} \xi_a(x, \theta) + \sum_{b=1}^n \frac{\partial \theta_b}{\partial x_j} \pi_b(x, \theta) \quad \text{for } \ell = 1, 2, \ldots, n.
\end{aligned}
\]

Assuming that
\[
\text{sdet} \left( \begin{array}{cc} \frac{\partial \xi}{\partial x_j} & \frac{\partial \pi}{\partial x_j} \\ \frac{\partial \xi}{\partial \theta_j} & \frac{\partial \pi}{\partial \theta_j} \end{array} \right) (x_0, 0) \neq 0 \quad \text{and} \quad \bar{x} = x(\bar{t}, \bar{\theta}), \quad \bar{\theta} = \theta(\bar{t}, \bar{\theta}),
\]

we have:

(i) There exist inverse functions $y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta})$ such that
\[
x(y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta})) = \bar{x}, \quad \theta(y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta})) = \bar{\theta}.
\]

(ii) Moreover, putting $w(\bar{x}, \bar{\theta}) = u(y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta}))$, we have, for $j = 1, 2, \ldots, m$ and $\ell = 1, 2, \ldots, n$,
\[
\frac{\partial w}{\partial x_j} = \xi_j(y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta})), \quad \frac{\partial w}{\partial \theta_\ell} = \pi_\ell(y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta})).
\]

**Proof.** By the inverse function theorem, we get (i). From this, we have
\[
\begin{aligned}
\delta_{ak} &= \frac{\partial x_a}{\partial x_k}, \quad \delta_{bk} = \frac{\partial x_b}{\partial x_k}, \quad \delta_{ak} = \frac{\partial x_a}{\partial x_k}, \quad \delta_{bk} = \frac{\partial x_b}{\partial x_k}, \\
0 &= \frac{\partial x_a}{\partial \theta_m} = \frac{\partial x_a}{\partial \theta_m} + \frac{\partial \omega_\ell}{\partial \theta_m} \frac{\partial x_a}{\partial \theta_\ell}, \\
0 &= \frac{\partial \theta_b}{\partial x_k} = \frac{\partial \theta_b}{\partial x_k} + \frac{\partial \omega_\ell}{\partial x_k} \frac{\partial \theta_b}{\partial \theta_\ell}, \quad \delta_{bm} = \frac{\partial \theta_b}{\partial \theta_m} = \frac{\partial \theta_b}{\partial \theta_m} + \frac{\partial \omega_\ell}{\partial \theta_m} \frac{\partial \theta_b}{\partial \theta_\ell}.
\end{aligned}
\]
Using these, we get readily that
\[
\frac{\partial w}{\partial x_k} = \frac{\partial y_j}{\partial x_k} \frac{\partial u}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \frac{\partial u}{\partial \omega_j} + \frac{\partial \omega_j}{\partial x_k} \frac{\partial u}{\partial \omega_j} + \frac{\partial \omega_j}{\partial x_k} \frac{\partial u}{\partial \omega_j} \left| \begin{array}{c}
\varepsilon = y(\bar{x}, \bar{\theta}), \\
\bar{\theta} = \omega(\bar{x}, \bar{\theta})
\end{array} \right.
\]
\[
= \frac{\partial y_j}{\partial x_k} \left( \frac{\partial x_a}{\partial x_j} \xi_a(\bar{x}, \bar{\theta}) + \frac{\partial \theta_b}{\partial x_k} \pi_b(\bar{x}, \bar{\theta}) \right) + \frac{\partial \omega_j}{\partial x_k} \left( \frac{\partial x_a}{\partial x_j} \xi_a(\bar{x}, \bar{\theta}) + \frac{\partial \theta_b}{\partial x_k} \pi_b(\bar{x}, \bar{\theta}) \right) \left| \begin{array}{c}
\varepsilon = y(\bar{x}, \bar{\theta}), \\
\bar{\theta} = \omega(\bar{x}, \bar{\theta})
\end{array} \right.
\]
\[
= \left( \frac{\partial y_j}{\partial x_k} + \frac{\partial \omega_j}{\partial x_k} \right) \frac{\partial x_a}{\partial x_j} \xi_a(\bar{x}, \bar{\theta}) + \left( \frac{\partial \omega_j}{\partial x_k} + \frac{\partial \omega_j}{\partial x_k} \right) \frac{\partial \theta_b}{\partial x_k} \pi_b(\bar{x}, \bar{\theta}) \left| \begin{array}{c}
\varepsilon = y(\bar{x}, \bar{\theta}), \\
\bar{\theta} = \omega(\bar{x}, \bar{\theta})
\end{array} \right.
\]
\[
= \xi_j(y(\bar{x}, \bar{\theta}), \omega(\bar{x}, \bar{\theta})).
\]
Analogously, we get the second equality in (ii). \( \square \)

**Proof of Theorem 2.3.** For fixed \((\xi, \pi)\), we put
\[
\hat{S}(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) = \langle x(\xi) \rangle + \langle \theta(\pi) \rangle + S_0(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}).
\]
Then, we have, using integration by parts w.r.t. \(s\) in (2.25),
\[
\frac{\partial \hat{S}}{\partial \xi_j} = \xi_j + \int_{\bar{t}}^{\bar{t}} ds \left[ \frac{\partial \hat{S}}{\partial \xi_j} \xi_k + \frac{\partial \theta_m}{\partial \xi_j} \pi_m + \frac{\partial \pi_m}{\partial \xi_j} \right]
\]
\[
= \xi_j + \frac{\partial \hat{S}}{\partial \xi_j} \xi_k + \frac{\partial \theta_m}{\partial \xi_j} \pi_m.
\]
Analogously, we get
\[
\frac{\partial \hat{S}}{\partial \pi_j} = \frac{\partial \hat{S}}{\partial \xi_j} \xi_k + \frac{\partial \theta_m}{\partial \xi_j} \pi_m.
\]
As we have already proved that if \(|\bar{t} - \bar{L}| \leq \delta\), we have
\[
\pi_B \simeq \det \begin{pmatrix}
\frac{\partial \xi_l(\bar{t})}{\partial \theta_j} & \frac{\partial \theta_l(\bar{t})}{\partial \theta_j} \\
\frac{\partial \theta_l(\bar{t})}{\partial \xi_j} & \frac{\partial \xi_l(\bar{t})}{\partial \xi_j}
\end{pmatrix} \neq 0,
\]
we may apply the above lemma. Therefore, putting
\[
S(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) = \hat{S}(\bar{t}, \bar{L}; y(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\xi}, \omega(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\pi}),
\]
we have, by (3.70),
\[
\frac{\partial S}{\partial \xi_j} = \xi_j(y(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\xi}, \omega(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\pi}),
\]
\[
\frac{\partial S}{\partial \theta_l} = \pi_l(y(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\xi}, \omega(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \bar{\pi}), (3.71)
\]
and
\[
\frac{\partial S}{\partial \pi_l} = \frac{\partial \hat{S}}{\partial \xi_j} \xi_k + \frac{\partial \theta_m}{\partial \xi_j} \pi_m.
\]
On the other hand,
\[
\frac{\partial}{\partial \bar{t}} \hat{S}(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) = \langle x(\bar{t})(\xi(\bar{t})) \rangle + \langle \theta(\bar{t})(\pi(\bar{t})) \rangle - \mathcal{H}(\bar{t}, x(\bar{t}), \theta(\bar{t}), \xi(\bar{t}), \pi(\bar{t}))
\]
Combining these with simple calculations, we get the desired Hamilton-Jacobi equation.

Moreover, using (3.71) and (3.69), we have
\[
\frac{\partial x_i S(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})}{\partial x_i} = \eta_i(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \quad \frac{\partial x_i S(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})}{\partial x_i} = \rho_i(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}),
\]
\[
\frac{\partial x_i S(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})}{\partial x_i} = \eta_i(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}), \quad \frac{\partial x_i S(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})}{\partial x_i} = \omega_i(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}). \quad \square (3.72)
\]
From here on, we change the notation:
\[
(\bar{t}, \bar{L}; \bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi}) \rightarrow (t, s; x, \xi, \theta, \pi).
\]
Lemma 3.6. Using the decomposition (2.28), we get

\[ S_{i0}(s; x, \xi) = \langle x | \xi \rangle, \]  
(3.73)

from the Hamilton-Jacobi equation and \( S(s; x, \xi, \theta, \pi) = \langle x | \xi \rangle + \langle \theta | \pi \rangle \), we get

\[ \tilde{S}_{i0}(s; x, \xi) = 0 \]  
(3.74)

Proof. (3.73) is obtained by restricting (H-J) to \( \theta = \pi = 0 \). Integrating (H-J) w.r.t. \( d\theta_1 d\theta_2 \), we get (3.74). In fact, as we have

\[ \partial_{\theta_1} \mathcal{H} = \frac{\partial S_{x\xi}}{\partial \theta_1} \mathcal{H}_{\xi_1} + \mathcal{H}_{\theta_1} + \frac{\partial S_{x\pi}}{\partial \theta_1} \mathcal{H}_{\pi_1}, \]

\[ \partial_{\theta_2} \partial_{\theta_1} \mathcal{H} = \frac{\partial^2 S_{x\xi}}{\partial \theta_2 \partial \theta_1} \mathcal{H}_{\xi_1} + \frac{\partial S_{x\xi}}{\partial \theta_1} \partial_{\theta_2} \mathcal{H}_{\xi_1} + \partial_{\theta_2} \mathcal{H}_{\theta_1} + \frac{\partial S_{x\pi}}{\partial \theta_1} \partial_{\theta_2} \mathcal{H}_{\pi_1}, \]

where

\[ \partial_{\theta_2} \mathcal{H}_{\xi_1} = \mathcal{H}_{\theta_2 \xi_1} + \frac{\partial S_{\theta_1}}{\partial \theta_2} \mathcal{H}_{\pi_1 \xi_1}, \]

\[ \partial_{\theta_2} \mathcal{H}_{\theta_1} = \frac{\partial S_{\xi_1}}{\partial \theta_2} \mathcal{H}_{\xi_1 \theta_1} + \mathcal{H}_{\theta_2 \theta_1} + \frac{\partial S_{\theta_1}}{\partial \theta_2} \mathcal{H}_{\pi_1 \theta_1}, \]

\[ \partial_{\theta_2} \mathcal{H}_{\pi_1} = \frac{\partial S_{\xi_1}}{\partial \theta_2} \mathcal{H}_{\xi_1 \pi_1} + \mathcal{H}_{\theta_2 \pi_1} + \frac{\partial S_{\theta_1}}{\partial \theta_2} \mathcal{H}_{\pi_1 \pi_1}, \]

remarking \( \tilde{\mathcal{H}}_{ij} = 0 \), \( \mathcal{H}_{ij \xi_k} = 0 \), and restricting \( \partial_{\theta_2} \partial_{\theta_1} \mathcal{H} \) to \( \theta = \pi = 0 \), we get the desired equality. Other equalities are obtained in the same manner. \( \square \)

Since \( \tilde{\mathcal{H}} = \varepsilon A_0(t, x) \), we get readily

\[ S_{i0}(t, s; x, \xi) = \langle x | \xi \rangle - \varepsilon \int_s^t dr A_0(r, x). \]  
(3.79)

Putting

\[ u_0(t, s; x, \xi) = \tilde{\mathcal{H}}_{\pi_1 \theta_1} + S_{i0} \tilde{\mathcal{H}}_{\pi_2 \xi_1} = \tilde{\mathcal{H}}_{\pi_2 \theta_2} + S_{i0} \tilde{\mathcal{H}}_{\pi_2 \xi_1}, \]  
(3.80)

with

\[ \tilde{\mathcal{H}}_{\pi_1 \theta_1} = -i h^{-1}(c \xi_3 - \varepsilon A_3 - c \varepsilon \int_s^t dr \partial r A_0) = \tilde{\mathcal{H}}_{\pi_2 \theta_2}, \]

\[ \tilde{\mathcal{H}}_{\pi_2 \xi_1} = h^{-2}[c \xi_3 - \varepsilon A_1 - c \int_s^t dr \partial r A_0 - i(c \xi_2 - \varepsilon A_2 - c \varepsilon \int_s^t dr \partial r A_0)], \]

we get

Lemma 3.7. For \(|t - s| \leq \delta\),

\[ S_{c_1 d_1}(t, s; x, \xi) = e^{-\int_s^t dr u_0(r, s; x, \xi)} = S_{c_1 d_2}(t, s; x, \xi), \]  
(3.81)

\[ S_{c_1 d_2}(t, s; x, \xi) = S_{c_2 d_1}(t, s; x, \xi) = 0. \]  
(3.82)

Remark (3.82), we have
Lemma 3.8.

\begin{align}
S_{01,t} + S_{c_1,d_1} S_{c_2,d_2} \tilde{H}_{\pi_2 \pi_1} = 0 \quad \text{with} \quad S_{01}(s, s; x, \xi) = 0, \\
S_{11,t} + 2 w_0 S_{11} + w_1 = 0 \quad \text{with} \quad S_{11}(s, s; x, \xi) = 0, 
\end{align}

where we put

\begin{equation*}
w_1(t, s; x, \xi) = (S_{10} S_{01,x} - S_{c_1,d_1} S_{c_2,d_2} \pi_1 \theta_1 + (S_{10} S_{01,x} - S_{c_1,d_1} S_{c_2,d_2} \pi_2 \theta_2 + [S_{10} S_{c_1,d_1} S_{c_2,d_2} \pi_1 \theta_1 + S_{01,x} \pi_2 \theta_2,
\end{equation*}

\begin{equation*}
= c h^{-2} [S_{01,x} - i S_{01,x_2} + 2 S_{10} \int_s^t dr (w_0, x_1 - i w_0, x_2)] e^{-2 \int_s^t dr w_0(r, s; x, \xi)}
\end{equation*}

\begin{equation*}
+ c h^{-2} S_{10}^2 (S_{01,x} - i S_{01,x_2} - 2 i c h^{-1} (S_{10} S_{01,x_3} + \int_s^t dr w_0, x_2 e^{-2 \int_s^t dr w_0(r, s; x, \xi)}) + c (S_{01,x_1} + i S_{01,x_2}) .
\end{equation*}

Proof. To get (3.83), we used (3.82). Remark

\begin{align*}
\tilde{H}_{\pi_2 \pi_1} = c h^{-2}, \quad \pi_2 \pi_1 = - i c h^{-1}, \quad \pi_2 \theta_1, \pi_1 \theta_2 = ic, \quad \pi_1 \pi_2 \theta_3 = \pi_2 \theta_2 \pi_3 = - i c h^{-1},
\end{align*}

and (3.81), we have (3.84). \Box

Therefore, we get the representation

\begin{align}
S_{01}(t, s; x, \xi) &= - \int_s^t dr \pi_2 \pi_1 (r, s; x, \xi) e^{-2 \int_s^t dr w_0(r, s; x, \xi)} , \\
S_{11}(t, s; x, \xi) &= - \int_s^t dr w_1(r, s; x, \xi) e^{-2 \int_s^t dr w_0(r, s; x, \xi)} .
\end{align}

Combining estimates in Theorems 2.1, 2.2 and (3.72), we get

Lemma 3.9. For any \( \alpha, \beta \), there exist constants \( C_{\alpha \beta} > 0 \) such that

\begin{align}
|\partial_x^\alpha \xi^\beta (S_{01}(t, s; x, \xi) - (x, \xi))| \leq C_{\alpha \beta} (1 + |x|)^{(1+|\alpha|) + \delta_{0|\beta}|t-s|} , \\
|\partial_x^\alpha \xi^\beta S_{10}(t, s; x, \xi)| \leq C_{\alpha \beta} |t-s| , \\
|\partial_x^\alpha \xi^\beta (S_{c_1,d_1}(t, s; x, \xi) - 1)| \leq C_{\alpha \beta} |t-s| \quad \text{for} \quad j = 1, 2 , \\
|\partial_x^\alpha \xi^\beta (S_{01}(t, s; x, \xi)| \leq C_{\alpha \beta} |t-s| , \\
|\partial_x^\alpha \xi^\beta (S_{11}(t, s; x, \xi)| \leq C_{\alpha \beta} |t-s| .
\end{align}

Proof. As \( S_{10}(t, s; x, \xi) = \theta_{|\theta_1|} (t, s; x, \xi, 0, 0) = \theta_{|\theta_1|} (-t, s; x, \xi, 0, 0) \), we have the desired one from Proposition 3.4 and (3.72). Other terms are calculated similarly. \Box

3.3. Continuity equation. Defining \( D \) as in (2.30), we get Theorem 2.4 as in [13].

Using the notation introduced in Theorem 2.3, we may decompose

\begin{align}
D(t, s; x, \xi, \theta, \pi) = \sum_{|c|+|d|=\text{even} \geq 0} D_{c,d}(t, s; x, \xi) \theta^c \pi^d = D_B(t, s; x, \xi) + D_S(t, s; x, \xi, \theta, \pi) ,
\end{align}

where

\begin{align}
D_B &= D_{00} = D^{[0]} , \\
D_S(t, s; x, \xi, \theta, \pi) &= D_{10} \theta_1 \theta_2 + \sum_{|c|+|d|=2} D_{c,d}(t, s; x, \xi) \theta^c \pi^d + D_{01} \pi_1 \pi_2 + D_{11} \theta_1 \theta_2 \pi_1 \pi_2 .
\end{align}

From the continuity equation (2.31), we have

\begin{equation*}
\frac{\partial}{\partial t} D_{00} + D_{00} \frac{\partial}{\partial \xi} \tilde{H}_{\pi_2} = 0 \quad \text{with} \quad D_{00}(s, s; x, \xi) = 1 .
\end{equation*}
As
\[
\partial_{\theta_1} \mathcal{H}_{\pi_1} = \mathcal{H}_{\theta_1 \pi_1} + \frac{\partial S_{\theta_2}}{\partial \theta_1} \mathcal{H}_{\pi_2 \pi_1} + \frac{\partial S_{\theta_1}}{\partial \theta_1} \mathcal{H}_{\xi_1 \pi_1},
\]
\[
\partial_{\theta_2} \mathcal{H}_{\pi_2} = \mathcal{H}_{\theta_2 \pi_2} + \frac{\partial S_{\theta_1}}{\partial \theta_2} \mathcal{H}_{\pi_1 \pi_2} + \frac{\partial S_{\theta_2}}{\partial \theta_2} \mathcal{H}_{\xi_1 \pi_2},
\]
we have
\[
\tilde{\partial}_{\theta_1} \mathcal{H}_{\pi_k} = \partial_{\theta_1} \mathcal{H}_{\pi_k} \big|_{\theta = \pi = 0} = -\tilde{\mathcal{H}}_{\pi_1 \theta_1} - \tilde{\mathcal{H}}_{\pi_2 \pi_2} - 2S_{\theta_0} \tilde{\mathcal{H}}_{\pi_2 \pi_1} = -2w_0(t, s; x, \xi),
\]
and we get
\[
\mathcal{D}_{00}(t, s; x, \xi) = e^{2\int_x^s d\varphi(t, s; x, \xi)}.
\] (3.89)

Instead of \(\mathcal{D}\), we should study the properties of a function \(\mathcal{A} = \mathcal{D}^{1/2}\): Putting
\[
\mathcal{A}(t, s; x, \theta, \xi, \pi) = \sum_{|c| + |d| = \text{even} \geq 0} A_{cd}(t, s; x, \theta, \xi) \theta^c \pi^d
\]
\[
= A_{00} + A_{10} \theta \theta_2 + A_{c_1 d_1} \theta_1 \pi_1 + A_{c_2 d_2} \theta_2 \pi_2 + A_{11} \theta_1 \theta_2 \pi_1 \pi_2 - 2A_{\theta_0} \mathcal{H}_{\pi_2 \pi_1} = -2w_0(t, s; x, \xi),
\] (3.90)
we define each coefficient \(A_{cd}(t, s; x, \theta, \xi)\) from \(A^2 = \mathcal{D}\) as
\[
\mathcal{D}_{00} = A_{00}^2 \quad \text{with} \quad A_{00}(s, s; x, \xi) = 1, \quad \mathcal{D}_{01} = 2A_{00}A_{01}, \quad \mathcal{D}_{10} = 2A_{00}A_{10},
\]
\[
\mathcal{D}_{c_1 d_1} = 2A_{00}A_{c_1 d_1}, \quad \mathcal{D}_{c_2 d_2} = 2A_{00}A_{c_2 d_2}, \quad \mathcal{D}_{11} = 2A_{00}A_{11} + 2A_{10}A_{01} - 2A_{c_1 d_1}A_{c_2 d_2} - 2A_{c_1 d_2}A_{c_2 d_1}.
\] (3.91)

More precisely, we define \(A_{00}(t, s; q, p) = \sqrt{\mathcal{D}_{00}(t, s; q, p)}\) such that \(A_{00}(s, s; q, p) = 1\) and \(A_{cd}(t, s; q, p)\) are defined from above, and then they are Grassmann continued to \(\mathbb{R}^{3|0}\).

Using the continuity equation (2.31), we have the (2.32).

Remarking also
\[
\partial_{x_j} \mathcal{H}_{\xi_j} = \frac{\partial S_{\theta_2}}{\partial x_j} \mathcal{H}_{\pi_2 \xi_j} + \frac{\partial S_{\theta_1}}{\partial x_j} \mathcal{H}_{\pi_1 \xi_j},
\]
we have \(\tilde{\mathcal{H}}_{\xi_1} = 0\) and \(\tilde{\mathcal{H}}_{\xi_2} = 0\), from which we have
\[
A_{00, t} - w_0 A_{00} = 0 \quad \text{with} \quad A_{00}(s, s; x, \xi) = 1.
\] (3.92)

That is, as is desired, we have
\[
A_{00}(t, s; x, \xi) = e^{\int_s^t d\varphi(t, s; x, \xi)}.
\] (3.93)

On the other hand, for \(\{\cdots\}\) in (2.32), as
\[
\partial_{\theta_1} \{\cdots\} = A_{\theta_1 x, \pi_j} + A_{x, \theta_1} \mathcal{H}_{\xi_j} - A_{\theta_1, \theta_1} \mathcal{H}_{\pi_1} + A_{\theta_1, \theta_2} \mathcal{H}_{\pi_2} - A_{\theta_2, \theta_1} \mathcal{H}_{\pi_2}
\]
\[
+ \frac{1}{2} A_{\theta_1} [\partial_{x, \xi_j} \mathcal{H}_{\pi_1} + \partial_{x, \pi_1} \mathcal{H}_{\xi_j}] + \frac{1}{2} A_{\theta_1} [\partial_{x, \xi_j} \mathcal{H}_{\xi_1} + \partial_{x, \xi_1} \mathcal{H}_{\pi_j}],
\]
we have
\[
\partial_{\theta_2} \partial_{\theta_1} \{\cdots\} = A_{\theta_2 \theta_1 x, \pi_j} - A_{\theta_1 x, \theta_2} \mathcal{H}_{\xi_j} + A_{\theta_2 x, \theta_1} \mathcal{H}_{\xi_j} + A_{x, \theta_2} \partial_{\theta_1} \mathcal{H}_{\xi_j}
\]
\[
- A_{\theta_2, \theta_1} \mathcal{H}_{\pi_1} + A_{\theta_2, \theta_2} \mathcal{H}_{\pi_2} + A_{\theta_1, \theta_2} \mathcal{H}_{\pi_1} + A_{\theta_1, \theta_2} \mathcal{H}_{\pi_2}
\]
\[
+ \frac{1}{2} A_{\theta_2 \theta_1} [\partial_{x, \xi_j} \mathcal{H}_{\pi_1} + \partial_{x, \pi_1} \mathcal{H}_{\xi_j}] - \frac{1}{2} A_{\theta_1 \theta_2} [\partial_{x, \xi_j} \mathcal{H}_{\xi_1} + \partial_{x, \xi_1} \mathcal{H}_{\pi_j}] + \frac{1}{2} A_{\theta_2 \theta_1} [\partial_{x, \xi_j} \mathcal{H}_{\xi_1} + \partial_{x, \pi_1} \mathcal{H}_{\xi_j}] + \frac{1}{2} A_{\theta_2 \theta_1} [\partial_{x, \pi_1} \mathcal{H}_{\xi_j} + \partial_{x, \xi_j} \mathcal{H}_{\pi_1}].
\]
Using $\partial_{\theta_2}\partial_{\theta_1}[\partial_0 H_{\theta_0}] = 0$ and
\[
\partial_0 \partial \theta_1 H_{\xi_j} = \partial_{\theta_2} \theta_1 \xi_j + \partial_{S_{\theta_2}} H_{\pi_1 \theta_2 \xi_j} + \partial_{S_{\theta_2}} \left( \partial_{\theta_1} H_{\pi_1 \theta_2 \xi_j} + H_{\theta_2 \xi_j} \right),
\]
\[
\partial_0 \partial \theta_1 [\partial_{\xi_j} H_{\xi_j}] = \frac{\partial^2 S_{\theta_2}}{\partial \theta_2 \partial x_j} \left( \partial_{\theta_1} \pi_1 \xi_j + \frac{\partial S_{\theta_2}}{\partial \theta_1} H_{\pi_2 \xi_j} \right) + \frac{\partial^2 S_{\theta_2}}{\partial \theta_2 \partial x_j} \left( \partial_{\theta_2} \pi_2 \xi_j + \frac{\partial S_{\theta_2}}{\partial \theta_2} H_{\pi_1 \theta_2 \xi_j} \right),
\]
we have
\[
\partial_0 \partial \theta_1 [\cdot \cdot \cdot] \bigg|_{\theta=\pi=0} = A_{00,x_j} [\tilde{H}_{\theta_2} \theta_1 \xi_j + S_{10}(\tilde{H}_{\pi_1 \theta_1 \xi_j} + \tilde{H}_{\pi_2 \theta_2 \xi_j}) + S_{10}^2 \tilde{H}_{\pi_2 \pi_1 \xi_j}]
\]
\[
+ A_{10} w_0(t, s; x, \xi) + \frac{1}{2} A_{00} S_{10,x_j} (\tilde{H}_{\pi_1 \theta_1 \xi_j} + \tilde{H}_{\pi_2 \theta_2 \xi_j} + 2 S_{10} \tilde{H}_{\pi_2 \pi_1 \xi_j}).
\]

Therefore, we get
\[
A_{10,t} + w_0 A_{10} + w_2 = 0 \quad \text{with} \quad A_{10}(s, s; x, \xi) = 0,
\]
where
\[
w_2(t, s; x, \xi) = A_{00,x_j} [\tilde{H}_{\theta_2} \theta_1 \xi_j + S_{10}(\tilde{H}_{\pi_1 \theta_1 \xi_j} + \tilde{H}_{\pi_2 \theta_2 \xi_j}) + S_{10}^2 \tilde{H}_{\pi_2 \pi_1 \xi_j}]
\]
\[
+ \frac{1}{2} A_{00} S_{10,x_j} (\tilde{H}_{\pi_1 \theta_1 \xi_j} + \tilde{H}_{\pi_2 \theta_2 \xi_j} + 2 S_{10} \tilde{H}_{\pi_2 \pi_1 \xi_j})
\]
\[
= c(A_{00,x_j} + i A_{00,x_j}) - 2 i c^{-1} A_{00,x_j} S_{10} + c^{-2} S_{10}(A_{00,x_j} - i A_{00,x_j})
\]
\[
+ A_{00} [c^{-2} S_{10}(S_{10,x_j} - i S_{10,x_j}) - c^{-1} S_{10,x_j}].
\]

Simple but lengthy calculation yields

**Proposition 3.10.**
\[
A_{c_1 d_1,t} + S_{c_1 d_1} [A_{00,x_j} (\tilde{H}_{\pi_1 \theta_1 \xi_j} + S_{10} \tilde{H}_{\pi_2 \pi_1 \xi_j}) + A_{10} \tilde{H}_{\pi_2 \pi_1} + A_{00} S_{10,x_j} \tilde{H}_{\pi_2 \pi_1 \xi_j}] = 0,
\]
\[
A_{c_2 d_2,t} + S_{c_2 d_2} [A_{00,x_j} (\tilde{H}_{\pi_2 \theta_2 \xi_j} + S_{10} \tilde{H}_{\pi_2 \pi_1 \xi_j}) + A_{10} \tilde{H}_{\pi_2 \pi_1} + A_{00} S_{10,x_j} \tilde{H}_{\pi_2 \pi_1 \xi_j}] = 0,
\]
\[
A_{c_1 d_2} = 0 \quad \text{and} \quad A_{c_2 d_1} = 0, \quad \text{i.e.} \quad A_{c_1 d_2} = 0 \quad \text{and} \quad A_{c_2 d_1} = 0.
\]

Analogously, we have

**Proposition 3.11.**
\[
A_{01,t} + w_0 A_{01} + w_3 = 0 \quad \text{with} \quad A_{01}(s, s; x, \xi) = 0,
\]
where
\[
w_3(t, s; x, \xi) = (A_{c_1 d_1} S_{c_2 d_2} + A_{c_2 d_2} S_{c_1 d_1} - A_{00} S_{10}) \tilde{H}_{\pi_2 \pi_1}
\]
\[
+ [S_{c_1 d_1} S_{c_2 d_2} A_{00,x_j} + A_{00} (S_{c_1 d_1} S_{c_2 d_2})_{x_j} - A_{00} S_{10} S_{01,x_j} \tilde{H}_{\xi_1 \pi_2 \pi_1}].
\]

**Proposition 3.12.**
\[
A_{11,t} + w_0 A_{11} + w_4 = 0 \quad \text{with} \quad A_{11}(s, s; x, \xi) = 0,
\]
where
\[
w_4(t, s; x, \xi) = A_{01,x_j} \tilde{H}_{\xi_1 \theta_1 \pi_2} + S_{10} A_{01,x_j} (\tilde{H}_{\xi_1 \pi_1 \theta_1} + \tilde{H}_{\xi_1 \pi_2 \theta_2} + S_{10} \tilde{H}_{\xi_1 \pi_2 \pi_1})
\]
\[
+ S_{11} A_{00,x_j} (\tilde{H}_{\xi_1 \pi_1 \theta_1} + \tilde{H}_{\xi_1 \pi_2 \theta_2} + 2 S_{10} \tilde{H}_{\xi_1 \pi_2 \pi_1}) + S_{c_1 d_1} S_{c_2 d_2} A_{10,x_j} \tilde{H}_{\xi_1 \pi_2 \pi_1}
\]
\[
- S_{c_1 d_1} S_{c_2 d_2,y_j} (\tilde{H}_{\xi_1 \pi_1 \theta_1} + S_{10} \tilde{H}_{\xi_1 \pi_2 \pi_1}) - S_{c_2 d_2} A_{c_1 d_1,x_j} (\tilde{H}_{\xi_1 \pi_2 \theta_2} + S_{10} \tilde{H}_{\xi_1 \pi_2 \pi_1})
\]
\[
+ A_{10} S_{10,x_j} \tilde{H}_{\pi_1 \theta_1 \xi_1} + \tilde{H}_{\pi_2 \theta_2 \xi_1} + 2 S_{10} \tilde{H}_{\pi_2 \pi_1 \xi_1} - A_{10} (S_{c_1 d_1} S_{c_2 d_2})_{x_j} \tilde{H}_{\xi_1 \pi_2 \pi_1}
\]
\[
+ (A_{c_1 d_1} S_{c_2 d_2} + A_{c_2 d_2} S_{c_1 d_1}, S_{10,x_j} \tilde{H}_{\pi_2 \pi_1 \xi_1} - A_{c_1 d_1} S_{c_2 d_2,x_j} (\tilde{H}_{\xi_1 \pi_1 \theta_1} + S_{10} \tilde{H}_{\xi_1 \pi_2 \pi_1})
\]
\[
- A_{c_2 d_2} S_{c_1 d_1,x_j} (\tilde{H}_{\xi_1 \pi_2 \theta_2} + S_{10} \tilde{H}_{\xi_1 \pi_2 \pi_1})
\]
\[
+ \frac{1}{2} (A_{00} S_{11,x_j} + A_{01} S_{01,x_j} - A_{10} S_{10,x_j}) (\tilde{H}_{\pi_1 \theta_1 \xi_1} + \tilde{H}_{\pi_2 \theta_2 \xi_1} + 2 S_{10} \tilde{H}_{\pi_2 \pi_1 \xi_1})
\]
\[
+ [A_{10} (S_{c_1 d_1} S_{c_2 d_2})_{x_j} - A_{c_1 d_1} S_{c_2 d_2} S_{10,x_j} - A_{c_2 d_2} S_{c_1 d_1} S_{10,x_j}] \tilde{H}_{\pi_2 \pi_1 \xi_1}.
\]
Therefore, we get the estimates in (2.34).

4. Quantum part: Composition formulas of FIOs and PDOs

4.1. Super differential operators associated with $\mathcal{H}(X, \Xi)$. Given $\mathcal{H}(X, \Xi)$, we consider the super (pseudo)-differential operator of Weyl type:

$$\hat{\mathcal{H}}(X, D_X)u(X) = c_{3,2}^2 \int d\Xi dY e^{i\hbar^{-1}(X-Y)\Xi} \mathcal{H}\left(\frac{X+Y}{2}, \Xi\right) u(Y)$$

for $u \in \mathcal{F}_{ss,0}(\mathbb{R}^3)$. Here, we use the abbreviation

$$c_{m,n} = (2\pi \hbar)^{-m/2} R^{n/2} e^{i\pi n(n-2)/4}$$

where $(\hat{\mathcal{H}}\varepsilon u)(Y) = \sum \mathcal{H}_u(x, \theta, \pi) = c_{3,2} \int d\Xi dY e^{i\hbar^{-1}(X-Y)\Xi} \mathcal{H} \mathcal{H}(X, D_X)u(Y)$$

As is easily seen that $\hat{\mathcal{H}} u \in \mathcal{F}_{ss,0}(\mathbb{R}^3)$ if $u \in \mathcal{F}_{ss,0}(\mathbb{R}^3)$, we can write (4.2) as

$$(\hat{\mathcal{H}} u)(x, \theta) = \sum a (\hat{\mathcal{H}}_a u a(x) \theta a)$$

where $(\hat{\mathcal{H}}_a u a(x) = \partial \hat{\mathcal{H}}_a u)(x, \theta)$. So, applying the proof of bosonic case to $(\hat{\mathcal{H}}_a u a$) as in [26], we get $\hat{\mathcal{H}} u(X) = s - \lim \hat{\mathcal{H}}_a u(X)$ in a suitable sense.

Moreover, all integrals which appear hereafter should be considered in the above sense (called, super oscillatory integral) if the integrand is not absolutely integrable. Moreover, if the calculus under the integral sign is permitted using the above argument combined with Lax’ technique (of using integrations by parts repeatedly with $\partial_\xi e^{i\phi(x, \xi)} = i\partial_\xi \phi(x, \xi)e^{i\phi(x, \xi)}$, we do it without mentioning it.

By simple calculation, we have

**Lemma 4.1.** For $j = 1, 2, 3$ and $\nu(\theta) = \nu_0 + \nu_1 \theta_2$, putting

$$\hat{\mathcal{H}}(X, D_X)u(X) = c_{3,2}^2 \int d\pi d\theta' e^{i\hbar^{-1}(\theta-\theta')\pi} \sigma_j \left(\frac{\theta + \theta'}{2}, \pi\right) u(\theta'),$$

we get

$$(\hat{\mathcal{H}}_a u)(x, \theta) = \sum a \mathcal{H}_a u a(x) \theta a$$

**Lemma 4.2.** Let $\mathcal{H}(t, X, \Xi)$ be derived from $\mathbb{H}(t, q, -i\hbar \partial_q)$ in (W). Then, we have

$$\mathbb{H}(t, q, \frac{\hbar}{i} \frac{\partial}{\partial q}) = i \mathcal{H}_t^0: C^\infty_0(\mathbb{R}^3; \mathbb{C}^2) \to C^\infty(\mathbb{R}^3; \mathbb{C}^2) \quad \text{for each} \; t.$$
4.2. **Fourier Integral Operators associated with** $\mathcal{H}(t, X, \Xi)$. After reordering $(\bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})$ as $(\bar{x}, \bar{\xi}, \bar{\theta}, \bar{\pi})$ and denoting them by $(x, \theta, \xi, \pi)$, we consider an integral transformation $\mathcal{U}(t, s)$ on $\mathcal{C}_{SS,0}(\mathbb{R}^{3|2})$ where $\mathcal{S}(t, s; x, \theta, \xi, \pi)$ and $\mathcal{A}(t, s; x, \theta, \xi, \pi)$ are defined in §2:

$$\mathcal{U}(t, s)u(x, \theta) = (\mathcal{U}(t, s)u)(x, \theta) = c_{3,2} \int d\bar{\xi}d\bar{\pi} \mathcal{A}(t, s; x, \theta, \xi, \pi)e^{ih^{-1}\mathcal{S}(t, s; x, \theta, \xi, \pi)} \mathcal{F}u(\bar{\xi}, \bar{\pi})$$

or simply, we write it as

$$\mathcal{U}(t, s)u(X) = (\mathcal{U}(t, s)u)(X) = c_{3,2} \int d\Xi \mathcal{A}(t, s; X, \Xi)e^{ih^{-1}\mathcal{S}(t, s; X, \Xi)} \mathcal{F}u(\Xi). \tag{4.8}$$

**Theorem 4.3.** We have $\mathcal{U}(t, s)u \in \mathcal{C}_{SS}(\mathbb{R}^{3|2})$ for $u \in \mathcal{C}_{SS,0}(\mathbb{R}^{3|2})$. Moreover, there exists a constant $C$ such that

$$\|\mathcal{U}(t, s)u\| \leq 2^2(1 + C|t - s|)\|u\| \leq 2^2e^{C|t - s|}\|u\|. \tag{4.9}$$

**Proof.** Using $\mathcal{F}u(\xi, \pi) = \mathcal{h}u_1(\xi) + h^{-1}u_0(\xi)\pi_1\pi_2$, we rewrite (4.8) as

$$\mathcal{U}(t, s)u(x, \theta) = c_{3,2} \int d\xi d\pi \left( \mathcal{A}_d(t, x, \xi) + \mathcal{A}_c(t, s; x, \theta, \xi, \pi) \right)e^{ih^{-1}\mathcal{S}_d(t, s; x, \xi)}$$

$$\times \left[ \sum_{\ell = 0}^{2} \frac{(ih^{-1})^{\ell}}{\ell!} \mathcal{S}(t, s; x, \theta, \xi, \pi)^{\ell} \right] \left( \mathcal{h}u_1(\xi) + h^{-1}u_0(\xi)\pi_1\pi_2 \right). \tag{4.10}$$

Here, we remark by (2.28) that

$$\mathcal{S}(t, s; x, \theta, \xi, \pi)^{\ell} = \begin{cases} 1, & \text{when } \ell = 0, \\
\mathcal{S}_{11}\theta_1\theta_2 + \sum_{|c| = |d| = 1} \mathcal{S}_{cd}\theta^c\pi^d + \mathcal{S}_{00}\pi_1\pi_2 + \mathcal{S}_{11}\theta_1\theta_2\pi_1\pi_2, & \text{when } \ell = 1, \\
2|\mathcal{S}_{10}\mathcal{S}_{01} - \mathcal{S}_{c_1d_1}\mathcal{S}_{c_2d_2}|\theta_1\theta_2\pi_1\pi_2, & \text{when } \ell = 2, \tag{4.11} \\
0, & \text{when } \ell \geq 3. \end{cases}$$

After integrating (4.10) with respect to $d\pi$, we have

$$\mathcal{U}(t, s)u(x, \theta) = \sum_{|b| = 0, 2} v_b(t, s; x)\theta^b$$

$$= \sum_b \left[ (2\pi \mathcal{h})^{-3/2} \int d\xi \mathcal{B}_b(t, s; x, \xi)e^{ih^{-1}\mathcal{S}_b(t, s; x, \xi)}u_0(\xi) \right] \theta^b, \tag{4.12}$$

where

\begin{align*}
\mathcal{B}_{\bar{1}\bar{1}} &= h^2A_{\bar{1}\bar{1}} + ih(A_B\mathcal{S}_{\bar{1}\bar{1}} + A_{\bar{0}\bar{0}}\mathcal{S}_{\bar{0}\bar{1}} + A_{\bar{0}\bar{1}}\mathcal{S}_{\bar{0}\bar{1}} + A_{c_1d_1}\mathcal{S}_{c_2d_2} - A_{c_2d_2}\mathcal{S}_{c_1d_1}) - A_B[\mathcal{S}_{10}\mathcal{S}_{01} - \mathcal{S}_{c_1d_1}\mathcal{S}_{c_2d_2}], \\
\mathcal{B}_{\bar{0}\bar{1}} &= A_{\bar{0}\bar{1}} + ih^{-1}A_B\mathcal{S}_{\bar{0}\bar{1}}, \\
\mathcal{B}_{\bar{1}\bar{0}} &= A_{\bar{1}\bar{0}} + ih^{-1}A_B\mathcal{S}_{\bar{0}\bar{1}}, \\
\mathcal{B}_{00} &= A_B.
\end{align*} \tag{4.13}

In the above, arguments of $\mathcal{B}_{s*}$, $\mathcal{A}_{s*}$ and $\mathcal{S}_{s*}$ are $(t, s; x, \xi)$.

By using (3.86) and (2.34), we have that $\partial_x^b\partial_\xi^q \mathcal{B}_b(t, s; x_B, \xi_B) \in \mathbb{C}$ and

\begin{align*}
\left| \partial_x^b\partial_\xi^q \mathcal{B}_b(t, s; x_B, \xi_B) - 1 \right| &\leq C|t - s| \quad \text{for } b = a = 1 \text{ or } b = q = 0, \\
\left| \partial_x^b\partial_\xi^q \mathcal{B}_b(t, s; x_B, \xi_B) \right| &\leq C|t - s| \quad \text{for } b \neq a. \tag{4.14}
\end{align*}

Putting

$$\langle \mathcal{E}_b(t, s)u_0 \rangle(x_B) = (2\pi \mathcal{h})^{-3/2} \int d\xi_B \mathcal{B}_b(t, s; x_B, \xi_B)e^{ih^{-1}\mathcal{S}_b(t, s; x_B, \xi_B)}u_0(\xi_B), \tag{4.15}$$

we have
we have
\[
v_0(t, s; x) = E_{00}u_0(\xi) + E_{01}u_1(\xi)
= (2\pi \hbar)^{-3/2} \int d\xi e^{i\hbar^{-1}S(t, s; x, \xi)}[B_{00}(t, s; x, \xi)u_0(\xi) + B_{01}(t, s; x, \xi)u_1(\xi)],
\]
\[
v_1(t, s; x) = E_{10}u_0(\xi) + E_{11}u_1(\xi)
= (2\pi \hbar)^{-3/2} \int d\xi e^{i\hbar^{-1}S(t, s; x, \xi)}[B_{10}(t, s; x, \xi)u_0(\xi) + B_{11}(t, s; x, \xi)u_1(\xi)],
\]
(4.16)
By applying Theorem 2.1 of [1] to (4.15), we have
\[
\|E_{00}u_0\| \leq (1 + C|t - s|)\|u_0\|, \quad \|E_{01}u_1\| \leq C|t - s|\|u_1\|,
\]
\[
\|E_{10}u_0\| \leq C|t - s|\|u_0\|, \quad \|E_{11}u_1\| \leq (1 + C|t - s|)\|u_1\|,
\]
which implies
\[
\|U(t, s)u\|^2 = \|v_0\|^2 + \|v_1\|^2 \leq (1 + \sqrt{2}C|t - s|)^2(\|u_0\|^2 + \|u_1\|^2) = (1 + \sqrt{2}C|t - s|)^2\|u\|^2.
\]
(4.18)
Moreover, \(\pi_B(\sum_a E_{ba}(t, s)u_a(X)) \in C^\infty(\mathbb{R}^3)\), i.e., \(U(t, s)u \in \mathcal{C}_{SS}(\mathbb{R}^3, 2\). □

Remark. For \(u \in \mathcal{L}^2_{SS}(\mathbb{R}^3, 2\), \(U(t, s)u\) is defined as the limit of a Cauchy sequence \(\{U(t, s)u_k\}\) in \(\mathcal{L}^2_{SS}(\mathbb{R}^3, 2\) where \(u_k \in \mathcal{C}_{SS, 0}(\mathbb{R}^3, 2\) is a sequence converging to \(u\) in \(\mathcal{L}^2_{SS}(\mathbb{R}^3, 2\).

**Proposition 4.4.** (1) For each \(u \in \mathcal{L}^2_{SS}(\mathbb{R}^3, 2\), we have
\[
s - \lim_{|t - s| \to 0} U(t, s)u = u \quad \text{in} \quad \mathcal{L}^2_{SS}(\mathbb{R}^3, 2\).
\]
(4.19)
(2) Define \(U(s, s) = I\). For fixed \(s\), the correspondence \(t \to U(t, s)u\) gives a strongly continuous function with values in \(\mathcal{L}^2_{SS}(\mathbb{R}^3, 2\).

**Proof.** To prove (4.19), we need to claim
\[
s - \lim_{|t - s| \to 0} v_0 = u_0 \quad \text{and} \quad s - \lim_{|t - s| \to 0} v_1 = u_1.
\]
We get the desired results by the standard method applying to (4.16). □

**Remark.** In the above, Theorem 4.3 and Proposition 4.4 are proved after integrating w.r.t. \(d\pi\) and applying the standard method for pseudo-differential and Fourier integral operators on the Euclidean space \(\mathbb{R}^3\). But, this suggests us to the necessity of developing those operator theories on the superspace \(\mathbb{R}^{m|n}\).

### 4.3. Composition of FIOs with \(\Psi DOs\).

As \((t, s)\) is inessential in this subsection, we abbreviate it and denote \(A(X, \Xi) = A(t, s; x, \xi, \theta, \pi), S(X, \Xi) = S(t, s; x, \xi, \theta, \pi), H(X, \Xi) = H(t, X, \Xi)\) and
\[
U(A, S)u(X) = c_{3, 2} \int d\Xi A(X, \Xi)e^{i\hbar^{-1}S(X, \Xi)}u(\Xi),
\]
\[
\mathcal{H}u(X) = c_{3, 2} \int d\Xi dY e^{i\hbar^{-1}(X - Y)|\Xi|}\mathcal{H}\left(\frac{X + Y}{2}\right)u(Y).
\]
(4.20)

**Theorem 4.5.** Let \(U(A, S)\) and \(\mathcal{H}\) be given as above. There exists \(B_L = B_L(X, \Upsilon)\) such that
\[
\mathcal{H}U(A, S) = U(B_L, S).
\]
(4.21)
Moreover, \(B_L\) has the following expansion
\[
B_L = \mathcal{H}A - i\hbar\left\{\partial_\Xi \mathcal{H} \cdot \partial_X A + \frac{1}{2}\left(\partial^2_{X, \Xi, X} \mathcal{H} + \partial^2_{X, \Xi, X} S \cdot \partial^2_{X, \Xi, \Xi} \mathcal{H}\right)A\right\} + \mathcal{R}_L.
\]
(4.22)
Here, the argument of \(\mathcal{H}\) is \((X, \partial_X S)\) and that of \(S\) and \(\mathcal{R}_L\) is \((X, \Upsilon)\), and \(\mathcal{R}_L \in \mathcal{C}_{SS}(\mathbb{R}^{3|2} \times \mathbb{R}^{3|2})\) satisfies
\[
\sup |D^\alpha_X D^\beta_\Xi \mathcal{R}_L(X_B, \Upsilon_B)| \leq C_{\alpha, \beta} < \infty.
\]
(4.23)
Proof. By definition, we have
\[
\hat{H}u(A, S)u(X) = c_{3,2}^2 \int d\Xi dY d\Upsilon \hat{H}\left(\frac{X + Y}{2}, \Xi\right) A(Y, \Upsilon) e^{i\hbar^{-1}(|Y| + S(Y, \Upsilon))} \hat{u}(\Upsilon)
\]
\[
= c_{3,2}^2 \int d\Upsilon B_L(X, \Upsilon) e^{i\hbar^{-1}S(X, \Upsilon)} \hat{u}(\Upsilon).
\]
(4.24)
Here, we put
\[
B_L(X, \Upsilon) = c_{3,2}^2 \int d\Xi dY \hat{Q}(X, \Xi, Y, \Upsilon) e^{i\hbar^{-1}\psi(X, \Xi, Y, \Upsilon)}
\]
with
\[
\psi(X, \Xi, Y, \Upsilon) = \langle X - Y | \Xi \rangle + S(Y, \Upsilon) - S(X, \Upsilon).
\]
(4.25)
As det \(\left(\frac{\partial(Y, \Xi)}{\partial(\Xi, \Xi)}\right) = 1\), we may apply the formula of change of variables under integral sign, see, Theorem 3.8 of [22] or Theorem 1.14 of [10], to (4.25) getting
\[
B_L(X, \Upsilon) = c_{3,2}^2 \int d\hat{\Xi} d\hat{Y} e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})} \hat{H}\left(X + \frac{\hat{Y}}{2}, \hat{\Xi} + \hat{\partial}_X S(X, \hat{Y}, \Upsilon)\right) A(X + \hat{Y}, \Upsilon).
\]
(4.26)
By Taylor’s formula w.r.t. \(\hat{\Xi}\), we have
\[
\hat{H}\left(X + \frac{\hat{Y}}{2}, \hat{\Xi} + \hat{\partial}_X S(X, \hat{Y}, \Upsilon)\right) = \hat{H}\left(X + \frac{\hat{Y}}{2}, \hat{\partial}_X S(X, \hat{Y}, \Upsilon)\right)
\]
\[
+ \hat{\partial}_X \hat{\Xi} \hat{H}\left(X + \frac{\hat{Y}}{2}, \hat{\partial}_X S(X, \hat{Y}, \Upsilon)\right)
\]
\[
+ \hat{\partial}_X \hat{\xi}_k \int_0^1 d\tau (1 - \tau n) \partial^2_{\hat{\xi}_k \hat{\Xi}} \hat{H}\left(X + \frac{\hat{Y}}{2}, \hat{\xi}_k + \hat{\partial}_X S(X, \hat{Y}, \Upsilon)\right).
\]
(4.27)
In the above, we abbreviate summation sign and \(j \neq k\) because \(\partial^2_{\hat{\xi}_j} \hat{H} = \partial^2_{\hat{\xi}_k} \hat{H} = 0\).

Now, we remark
\[
c_{3,2}^2 \int_{\mathbb{R}^{3+2}} d\hat{\Xi} e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})} = \delta(\hat{Y}),
\]
\[
c_{3,2}^2 \int_{\mathbb{R}^{3+2}} d\hat{Y} e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})} = \delta(\hat{\Xi}),
\]
(4.32)
and
\[
\hat{\xi}_j e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})} = i\hbar \partial_X \hat{\xi}_j e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})}, \quad \hat{Y}_j e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})} = i\hbar (-1)^{\hat{\xi}_j} \partial_{\hat{\xi}_j} e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})}.
\]
(4.33)
From the first equality of (4.32), we get easily that
\[
c_{3,2}^2 \int d\hat{Y} d\hat{\Xi} e^{-i\hbar^{-1}(\hat{Y}|\hat{\Xi})} \hat{H}\left(X + \frac{\hat{Y}}{2}, \hat{\partial}_X S(X, \hat{Y}, \Upsilon)\right) A(X + \hat{Y}, \Upsilon)
\]
\[
= \hat{H}(X, \partial_X S(X, \Upsilon)) A(X, \Upsilon).
\]
(4.34)
Using the first equality of (4.33) and applying the first equality of (4.32) after integration by parts, we have

\[
c_3^2 \int d\tilde{Y} d\tilde{z} e^{-ih^{-1}(\tilde{Y}\tilde{z})} \tilde{z}_j \partial\tilde{z}_j \mathcal{H} \left( X + \frac{\tilde{Y}}{2}, \partial_X S(X, \tilde{Y}, Y) \right) A(X + \tilde{Y}, Y) \\
= -ihc_3^2 \int d\tilde{Y} d\tilde{z} e^{-ih^{-1}(\tilde{Y}\tilde{z})} \partial\tilde{z}_j \left[ \partial\tilde{z}_j \mathcal{H} \left( X + \frac{\tilde{Y}}{2}, \partial_X S(X, \tilde{Y}, Y) \right) A(X + \tilde{Y}, Y) \right] \\
= -ih \partial\tilde{z}_j \left[ \partial\tilde{z}_j \mathcal{H} \left( X + \frac{\tilde{Y}}{2}, \partial_X S(X, \tilde{Y}, Y) \right) A(X + \tilde{Y}, Y) \right] \bigg|_{\tilde{Y}=0} \\
= -ih \left\{ \partial_X A(*) \partial\tilde{z}_j \mathcal{H}(**) + \frac{1}{2} \left( \partial^2_{X, \Xi} \mathcal{H}(**) + \partial^2_{X, X, \Xi} S(*) \partial^2_{\Xi, \Xi} \mathcal{H}(**) \right) \right\} A(*) \right\}. 
\]

(4.35)

In the last line above, we put

\[ (*) = (X, Y) \quad \text{and} \quad (**) = (X, \partial_X S(X, Y)), \]

respectively. Thus, we get the main terms of (4.22) formally.

The remainder term is derived from

\[
\mathcal{R}_L(X, Y) = c_3^2 \int d\tilde{Y} d\tilde{z} e^{-ih^{-1}(\tilde{Y}\tilde{z})} \tilde{z}_j \tilde{z}_k \\
\times \left[ \int_0^1 \! d\tau \left( 1 - \tau_1 \right) \partial^2_{\Xi, \Xi} \mathcal{H} \left( X + \frac{\tilde{Y}}{2}, \tau_1 \tilde{z}_j + \partial_X S(X, \tilde{Y}, Y) \right) \right] A(X + \tilde{Y}, Y). \quad (4.36)
\]

As \[ \partial^2_{\Xi, \Xi} \mathcal{H} = 0, \] we have, for any \( \tau_1 \in (0, 1), \tilde{z} \in \mathbb{R}^3 \),

\[
\partial^2_{\Xi, \Xi} \mathcal{H} \left( X + \frac{\tilde{Y}}{2}, \tau_1 \tilde{z}_j + \partial_X S(X, \til{Y}, Y) \right) = \partial^2_{\Xi, \Xi} \mathcal{H} \left( X + \frac{\til{Y}}{2}, \partial_X S(X, \til{Y}, Y) \right) \\
+ \tau_1 \til{z}_j \partial^2_{\Xi, \Xi} \mathcal{H} \left( X + \frac{\til{Y}}{2}, \partial_X S(X, \til{Y}, Y) \right). \quad (4.37)
\]

Therefore,

\[
\mathcal{R}_L(X, Y) = \mathcal{R}_{L1}(X, Y) + \mathcal{R}_{L2}(X, Y) \quad (4.38)
\]

with

\[
\mathcal{R}_{L1}(X, Y) = c_3^2 \int d\tilde{Y} d\tilde{z} e^{-ih^{-1}(\tilde{Y}\tilde{z})} \tilde{z}_j \tilde{z}_k \frac{1}{2} \partial^2_{\Xi, \Xi} \mathcal{H}(**) A(*) \\
\mathcal{R}_{L2}(X, Y) = c_3^2 \int d\tilde{Y} d\tilde{z} e^{-ih^{-1}(\tilde{Y}\tilde{z})} \tilde{z}_j \tilde{z}_k \frac{1}{6} \tau \partial^2_{\Xi, \Xi} \mathcal{H}(**) A(*) 
\]

where we put \( (**) = \left( X + \frac{\til{Y}}{2}, \partial_X S(X, \til{Y}, Y) \right) \) and \( (*) = (X + \til{Y}, Y) \), respectively. Using (4.33) and integration by parts, we get

\[
\mathcal{R}_{L1}(*) = -\frac{1}{2} \frac{c_3^2}{4^2} \int d\tilde{Y} d\tilde{z} e^{-ih^{-1}(\tilde{Y}\tilde{z})} \tilde{z}_k (-1)^{p(\til{Y})} p(\til{z}_k) \\
\times \left[ \partial_X A(*) \partial^2_{\Xi, \Xi} \mathcal{H}(**) + \partial_Y \partial_X S(X, \til{Y}, Y) \partial^2_{\Xi, \Xi} \mathcal{H}(**) \right] \\
= - \frac{\hbar^2}{2} (-1)^{p(\til{Y})} p(\til{z}_k) \left[ \left( \frac{1}{2} A(*)^3 + \partial_X A(*) \partial^2_{\Xi, \Xi} S(*) \right) \partial^2_{\Xi, \Xi} \mathcal{H}(**) \right] \\
+ \frac{1}{2} \partial^2_{X, X, \Xi} S(*) \partial^2_{\Xi, \Xi} \mathcal{H}(**) \quad (4.39)
\]

Here, we used

\[
\partial_Y \partial_X S(X, \til{Y}, Y) \bigg|_{\til{Y}=0} = \int_0^1 \! d\tau \partial^2_{X, X, \Xi} S(X + \tau \til{Y}, Y) \bigg|_{\til{Y}=0} = \frac{1}{2} \partial^2_{X, X, \Xi} S(X, Y), \\
\partial_{\til{Y}} \partial_Y \partial_X S(X, \til{Y}, Y) \bigg|_{\til{Y}=0} = \int_0^1 \! d\tau \tau^2 \partial^3_{X, X, X, \Xi} S(X + \tau \til{Y}, Y) \bigg|_{\til{Y}=0} = \frac{1}{3} \partial^3_{X, X, X, \Xi} S(X, Y),
\]

Calculating analogously, we get

\[
\mathcal{R}_{L2}(*) = \frac{(-ih)^3}{6} (-1)^{p(\til{Y})} p(\til{z}_k) + p(\til{z}_i) p(\til{z}_i) \frac{1}{2} \partial^2_{X, X, \Xi} S(*) \partial^2_{\Xi, \Xi} \mathcal{H}(**) \quad (4.40)
\]
(II) To make the above procedure rigorous, we need to justify the usages of the changing the order of integration and those of delta functions. But, these are readily justified by using oscillatory integrals (see, Kumano-go [27]). Moreover, the estimate (4.23) is obtained easily. For example, we consider the first term of (4.39)

\[ \mathcal{A}(X)\partial^3_{x_i} \partial^3_{\alpha \beta \gamma} \mathcal{H}(X) \]

By the structure of \( \mathcal{H} \), we have terms as

\[ \mathcal{A}(X)\partial^3_{x_i, \alpha \beta \gamma} S(X) \partial^3_{\alpha \beta \gamma} \mathcal{H}(X, \partial X S(X, \Upsilon)) \]

The derivatives \( \partial^3_{\alpha \beta \gamma} \) of these terms have clearly bounded body terms by (3.86) and (2.34).

**Remark.** The main term is easily obtained from

\[ \sum_{|\alpha|=0}^{1} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} \partial_{x_i}^3 \left( \partial^3_{\alpha \beta \gamma} \mathcal{H}(X + \frac{1}{2} \hat{Y}, \hat{Z} + \partial \hat{S}(X, \hat{Y}, \Upsilon)) \cdot \mathcal{A}(X + \hat{Y}, \Upsilon) \right) \bigg|_{\hat{Y} = 0, \hat{Z} = 0} \]  \quad (4.41)

The following theorem is given for the future use.

**Theorem 4.6.** Let \( \mathcal{U}(A, S), \mathcal{H} \) be as above. Then, there exists \( \mathcal{B}_R = \mathcal{B}_R(X, \Upsilon) \) such that

\[ \mathcal{U}(A, S) \mathcal{H} = \mathcal{U}(\mathcal{B}_R, S). \]  \quad (4.42)

Moreover, \( \mathcal{B}_R \) has the following expansion:

\[ \mathcal{B}_R = \mathcal{A} \mathcal{H} - (-1)^{p(\Upsilon)} i \hbar \left\{ \partial_{\Upsilon} A \cdot \partial_{\Upsilon} \mathcal{H} + \frac{1}{2} (-1)^{p(\Upsilon)} \mathcal{A} \left( \partial^3_{\alpha \beta \gamma} \mathcal{H} + \partial^3_{\alpha \beta \gamma} S \cdot \partial^3_{\alpha \beta \gamma} \mathcal{H} \right) \right\} + \mathcal{R}_R \]  \quad (4.43)

where arguments of \( \mathcal{B}_R, \mathcal{A} \) and \( S \) are \((X, \Upsilon)\) and that of \( \mathcal{H} \) is \((-1)^{p(\Upsilon)} \partial_{\Upsilon} S(X, \Upsilon, \Upsilon)\). Furthermore, \( \mathcal{R}_R(X, \Upsilon) \) has the following from:

\[ \mathcal{R}_R(X, \Upsilon) = \mathcal{R}_{R,1}^{(1)}(X, \Upsilon) \Upsilon_i + \mathcal{R}_{R,1}^{(0)}(X, \Upsilon). \]  \quad (4.44)

**Proof.** As before, it is enough to calculate formally which yields (4.43).

(i) Remark: \( \hat{S} \) is represented as (4.5), we have

\[ \mathcal{U}(A, S) \mathcal{H} u(X) = c^3_{3, 2} \int d\Xi d\Upsilon A(X, \Xi) e^{i\hbar^{-1}(S(X, \Xi) + (Y | \Upsilon - \Xi))} \mathcal{H}^0(Y, \Upsilon) u(\Upsilon) \]

where

\[ c^3_{3, 2} \int d\Xi d\Upsilon A(X, \Xi) e^{i\hbar^{-1}(S(X, \Xi) - S(X, \Upsilon) + (Y | \Upsilon - \Xi))} \mathcal{H}^0(Y, \Upsilon). \]  \quad \text{with} \quad (4.46)

Using

\[ S(X, \Xi) - S(X, \Upsilon) = (\Xi - \Upsilon) \partial \tilde{S}(X, \hat{\Xi}, \Upsilon) = (\partial_{\Upsilon} \tilde{S}(X, \tilde{\Xi}, \Upsilon) | \Xi - \Upsilon), \]

we define a change of variables as

\[ \begin{cases} 
\hat{\Xi} = \Xi - \Upsilon, \\
\hat{Y} = Y - \partial \tilde{S}(X, \Xi - \Upsilon, \Upsilon) \end{cases} \quad \begin{cases} \Xi = \Upsilon + \hat{\Xi}, \\
Y = \hat{Y} + \partial \tilde{S}(X, \hat{\Xi}, \Upsilon). \end{cases} \]  \quad (4.47)

Rewriting (4.45), we get

\[ \mathcal{B}_R(X, \Upsilon) = c^3_{3, 2} \int d\Xi d\Upsilon A(X, \Xi) e^{i\hbar^{-1}(\hat{\Xi} | \Xi)} \mathcal{H}^0(Y + \partial \tilde{S}(X, \hat{\Xi}, \Upsilon), \Upsilon). \]  \quad (4.48)
(ii) Using Taylor’s expansion for $\mathcal{H}^0(\cdots)$ w.r.t. $\tilde{Y}$, we decompose

$$
\mathcal{H}^0(\tilde{Y} + D\tau S(X, \tilde{Z}, Y), Y) = \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y) + \tilde{Y}_j \partial_{X_j} \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y) \\
+ \tilde{Y}_k \partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y)
$$

where

$$
\partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y) = \int_0^1 d\tau_1 (1 - \tau_1) \partial_{X_k}^2 \mathcal{H}(\tau_1 \tilde{Y} + D\tau S(X, \tilde{Z}, Y), Y).
$$

(4.49)

So, we put

$$
B_R(X, Y) = I_1 + I_2 + I_3,
$$

where

$$
I_1 = e_{3,2}^3 \int d\tilde{Z} d\tilde{Y} e^{-ih^{-1}(\tilde{Y} | \tilde{Z})} A(X, \tilde{Z} + Y) \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y),
$$

(4.50)

$$
I_2 = e_{3,2}^3 \int d\tilde{Z} d\tilde{Y} e^{-ih^{-1}(\tilde{Y} | \tilde{Z})} A(X, \tilde{Z} + Y) \tilde{Y}_j \partial_{X_j} \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y),
$$

(4.51)

and

$$
I_3 = e_{3,2}^3 \int d\tilde{Z} d\tilde{Y} e^{-ih^{-1}(\tilde{Y} | \tilde{Z})} A(X, \tilde{Z} + Y) \tilde{Y}_k \partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y).
$$

(4.52)

(iii) Using (4.32), we get readily

$$
I_1 = A(X, \tilde{Z} + Y) \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y)|_{\tilde{Z}=0} = A(X, Y) \mathcal{H}^0(\partial_{\tau} S(X, Y), Y).
$$

(4.53)

Remark the second equality of (4.33), integration by parts and applying (4.32), we get

$$
I_2 = (-1)^{1-p(\tilde{Z})} i h \partial_{\tilde{Z}_j} \left[ A(X, \tilde{Z} + Y) \partial_{X_j} \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y) \right]|_{\tilde{Z}=0}
$$

$$
= (-1)^{1-p(\tilde{Z})} i h \partial_{\tilde{Z}_j} \left[ A(X, \tilde{Z} + Y) \partial_{X_j} \mathcal{H}^0(D\tau S(X, \tilde{Z}, Y), Y) \right].
$$

(4.54)

with arguments of $A$, $S$ are $(X, Y)$ and that of $\mathcal{H}^0$ is $(\partial_{\tau} S(X, Y), Y)$. Therefore, we have the main terms of (4.43) by adding $I_1 + I_2$.

(iv) Using the second equality of (4.33) twice, we get

$$
I_3 = e_{3,2}^3 \int d\tilde{Z} d\tilde{Y} e^{-ih^{-1}(\tilde{Y} | \tilde{Z})} A(X, \tilde{Z} + Y) \tilde{Y}_j \partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y)
$$

$$
= (-1)^{1-p(\tilde{Z})} i h e_{3,2}^3 \int d\tilde{Z} d\tilde{Y} e^{-ih^{-1}(\tilde{Y} | \tilde{Z})} \gamma_k \partial_{\tilde{Z}_j} A(X, \tilde{Z} + Y) \partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y)
$$

$$
+ A(X, \tilde{Z} + Y) \partial_{\tilde{Z}_j} \partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y).
$$

(4.55)

where in the last equality, the argument of $A$ is $(X, \tilde{Z} + Y)$ and that of $\partial_{X_k}^2 \mathcal{H}^0$ is $(X, \tilde{Z}, \tilde{Y}, Y)$.

Remark (4.32) once more, we have

$$
\partial_{\tilde{Z}_j} [\partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y)] = \partial_{\tilde{Z}_j} [D\tau S] \cdot \int_0^1 d\tau_1 (1 - \tau_1) \partial_{X_k}^3 \mathcal{H}^0(\tau_1 Y + D\tau S(X, \tilde{Z}, Y), Y),
$$

(4.56)

and

$$
\partial_{\tilde{Z}_j} \partial_{\tilde{Z}_k} [\partial_{X_k}^2 \mathcal{H}^0(X, \tilde{Z}, \tilde{Y}, Y)]
$$

$$
= \partial_{\tilde{Z}_j} \partial_{\tilde{Z}_k} [D\tau S] \cdot \int_0^1 d\tau_1 (1 - \tau_1) \partial_{X_k}^3 \mathcal{H}^0(\tau_1 Y + D\tau S(X, \tilde{Z}, Y), Y)
$$

$$
+ (-1)^{p(\tilde{Z})} i h \partial_{\tilde{Z}_j} [D\tau S] \cdot \partial_{\tilde{Z}_k} [D\tau S]
$$

$$
\times \int_0^1 d\tau_1 (1 - \tau_1) \partial_{X_k}^3 \mathcal{H}^0(\tau_1 Y + D\tau S(X, \tilde{Z}, Y), Y).
$$

(4.57)
Therefore, we get
\[
\mathcal{B}_R = \mathcal{H}^0 - (-1)^{\rho(\hat{\Xi})} i \hbar \{ \partial^2_{\hat{T}, X} \mathcal{H}^0 + (-1)^{\rho(\hat{\Xi})} \frac{1}{2} A \cdot \left( \partial^2_{\hat{T}, X} \mathcal{H}^0 + \partial^2_{\tilde{X}, \zeta} \mathcal{S} \cdot \partial^2_{\tilde{X}, \xi} \mathcal{H}^0 \right) \} + \mathcal{R}_R \tag{4.58}
\]
with
\[
\mathcal{R}_R(X, \Upsilon) = I_{31} + I_{32} + I_{33}
\]

where
\[
I_{31} = (-1)^{(1 + \rho(\hat{\Xi}))} h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0,
\]
\[
I_{32} = (-1)^{(1 + \rho(\hat{\Xi}))} h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} 2 \partial_{\hat{\Xi}} A \cdot \partial_{\tilde{\Upsilon}} [\partial^2_{\hat{X}, \zeta} \mathcal{H}^0],
\]
\[
I_{33} = (-1)^{(1 + \rho(\hat{\Xi}))} h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} A \partial^2_{\hat{\Xi}, \Xi, \zeta} [\partial^2_{\hat{X}, \zeta} \mathcal{H}^0].
\]

Finally, we estimate \( \mathcal{R}_R(X, \Upsilon) \) using the structure of \( \mathcal{H}^0 \). As
\[
\partial^2_{\hat{X}, \zeta} \mathcal{H}^0(X, \Xi, \Upsilon) = \partial^2_{\hat{\Xi}, \zeta} \mathcal{H}^0(X, \Xi, \Upsilon) = -c(\xi_1 + i \xi_2),
\]
using (4.49), we get
\[
\partial^2_{\hat{X}, \zeta} \mathcal{H}^0(X, \Xi, \Upsilon) = -\frac{c}{2} (\eta_1 + i \eta_2).
\]
Therefore,
\[
I_{31} = I_{311} + I_{312}
\]
with
\[
I_{311} = h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0,
\]
\[
I_{312} = h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0.
\]

By integration by parts, we get
\[
I_{311} = -\frac{i h^3}{2} c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0 + h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0
\]
where
\[
\tilde{\mathcal{S}}_{jk}(X, \Xi, 0, \tilde{\Upsilon}, \Upsilon) = \int_0^1 d\tau_2 \tau_2 (1 - \tau_2) \partial^2_{\hat{X}, \zeta, \eta} \mathcal{A}_1 \left( \tau_2 \hat{Z} + \hat{D} \mathcal{S}(X, \hat{Z}, \Upsilon) \right),
\]
and also
\[
I_{312} = h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0.
\]

Therefore, applying the same procedures to \( I_{32} \) and \( I_{33} \), we get
\[
\mathcal{R}^{(1)}_{R, \xi}(X, \Upsilon) = h^2 c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0 + \cdots,
\]
\[
\mathcal{R}^{(0)}_{R}(X, \Upsilon) = -\frac{i h^3}{2} c_{3, 2}^2 \int d\hat{\Xi} d\tilde{\Upsilon} e^{-i \hbar^{-1} (\hat{\Xi} \, \tilde{\Upsilon})} \partial^2_{\hat{\Xi}, \Xi, \zeta} A \cdot \partial^2_{\tilde{\Upsilon}, \zeta, \eta} \mathcal{H}^0 + I_{312} + \cdots. \quad \Box
\]

5. Proofs of Theorem 2.6 and Theorem 2.7.

5.1. The infinitesimal generator. Let \( u \in \mathcal{S}_{\xi, 0}(\mathcal{H}^{1/2}) \). As
\[
i \hbar \frac{\partial}{\partial t} \mathcal{A} e^{i \hbar^{-1} S} = (i \hbar \mathcal{A} - S_t \mathcal{A}) e^{i \hbar^{-1} S},
\]
using the Hamilton-Jacobi equation, the continuity equation and the composition formula (4.22), we have
\[
i \hbar \frac{\partial}{\partial t} \mathcal{U}(t, s) u(X) = c_{3, 2} \int d\Xi \left\{ \mathcal{H} \mathcal{A} - i \hbar [A_{X, \Xi} \partial_{\Xi, \zeta} + \frac{1}{2} A (H_{X, \Xi, \zeta} + S_{X, \Xi, \zeta} H_{X, \Xi, \zeta})] \right\} e^{i \hbar^{-1} S} u(\Xi)
\]
\[
= \hat{\mathcal{H}}(X, \partial_X) \mathcal{U}(t, s) u(X) - \mathcal{R}_L(t, s) u(X) \tag{5.1}
\]
Moreover, we have

\[ R_L(t, s)u = c_{3, 2} \int d\Xi R_L(t, s; X, \Xi) e^{ih^{-1}S(t, s; X, \Xi)} \hat{u}(\Xi). \]

Moreover, we have

**Proposition 5.1.**

\[ \| R_L(t, s)u \| \leq C|t - s|\| u \|. \]  

**Proof.** Using the estimates (3.86) and (2.34), we get

\[ |\pi_B \partial_x^a \partial_x^b R_L(t, s; X, \Xi)| \leq C_{a, b}|t - s|. \]

Therefore, we may proceed as we did in proving Theorem 3.3 to have (5.2). \( \square \)

5.2. **Evolutional property.** The following theorem gives one of the main estimate necessary to apply Theorem A.1 of [15] to our case.

**Theorem 5.2.** Let \( u \in \mathcal{C}_{SS, g}(\mathbb{R}^{3/2}) \). If \( |t - s| + |s - r| \) is sufficiently small, we have

\[ \| U(t, s)U(s, r)u - U(t, r)u \| \leq C(|t - s|^2 + |s - r|^2)\| u \|. \]  

(5.3)

By definition, we have

\[ U(t, r)u(X) = c_{3, 2} \int_{\mathbb{R}^{3/2}} d\Xi D^{1/2}(t, r; X, \Xi) e^{ih^{-1}S(t, r; X, \Xi)} F(u)(\Xi), \]

and

\[ U(t, s)U(s, r)u(X) = c_{3, 2} \int_{\mathbb{R}^{3/2}} dY D^{1/2}(t, s; X, Y) e^{ih^{-1}S(t, s; X, Y)} F(U(s, r)u)(Y) \]

\[ = c_{3, 2} \int_{\mathbb{R}^{3/2} \times \mathbb{R}^{3/2} \times \mathbb{R}^{3/2}} dY d\Xi D^{1/2}(t, s; X, Y) D^{1/2}(s, r; Y, \Xi) \]

\[ \times e^{ih^{-1}(S(t, s; X, Y) - (Y | Y) + S(s, r; Y, \Xi))} F(u)(\Xi). \]  

(5.4)

To prove Theorem 5.2, we follow the procedure of Taniguchi [45]. For the sake of notational simplicity, we use the following abbreviation:

\[ \phi_1(X, \Xi) = S(t, s; X, \Xi), \quad \phi_2(X, \Xi) = S(s, r; X, \Xi), \quad \phi(X, \Xi) = S(t, r; X, \Xi), \]

\[ \mu_1(X, \Xi) = D^{1/2}(t, s; X, \Xi), \quad \mu_2(X, \Xi) = D^{1/2}(s, r; X, \Xi), \quad \mu(X, \Xi) = D^{1/2}(t, r; X, \Xi), \]  

(5.5)

\[ \lambda_1 = |t - s|, \quad \lambda_2 = |s - r|, \quad \lambda = \lambda_1 + \lambda_2. \]

First of all, we prepare

**Lemma 5.3.** There exists a unique solution \((\tilde{X}, \tilde{\Xi})\), \( \tilde{X} = (\tilde{x}, \tilde{\theta}), \quad \tilde{\Xi} = (\tilde{\xi}, \tilde{\pi}) \) of

\[ \begin{cases} \tilde{X}_A = (-1)^{p(\Xi)} \partial_{\xi, A} \phi_1(\tilde{X}, \tilde{\Xi}), \\ \tilde{\Xi}_A = \partial_{X, A} \phi_2(\tilde{X}, \tilde{\Xi}), \end{cases} \]

i.e.

\[ \begin{cases} \tilde{x}_j = \partial_{\xi_j} \phi_1(\tilde{X}, \tilde{\Xi}), \\ \tilde{\xi}_j = \partial_{\xi_j} \phi_2(\tilde{X}, \tilde{\Xi}), \end{cases} \]

(5.6)

Moreover, we have, for \( a = (\alpha, \alpha), \quad b = (\beta, \beta), \)

\[ |\pi_B \partial_{\xi}^a \tilde{\Xi} - \frac{\hat{\Xi}}{X} - (X | \Xi) \| \leq C_{a, b} \lambda (1 + |x_B| + |\xi_B|)^{(1 - |\alpha + \beta|)} \]  

(5.7)

**Proof.** Putting

\[ J_{\alpha}(X, \Xi) = \phi_\alpha(X, \Xi) - (X | \Xi) \quad \text{and} \quad \tilde{X} = X + Y, \quad \tilde{\Xi} = \Xi + \Upsilon, \]

with \( X = (x, \theta), \quad Y = (y, \omega), \quad \Xi = (\xi, \pi), \quad \Upsilon = (\eta, \rho), \)

...
we rewrite (5.6) as
\[
\begin{aligned}
 y_j &= \partial_{\xi_j} J_1(x, \Xi + \Upsilon), \quad \omega_k = -\partial_{x_k} J_1(x, \Xi + \Upsilon), \\
 \eta_j &= \partial_{x_j} J_2(x + Y, \Xi), \quad \rho_k = \partial_{\theta_k} J_2(x + Y, \Xi),
\end{aligned}
\]
where
\[
y = \sum_{\ell=0}^{\infty} y^{[2\ell]}, \quad y^{[2\ell]} = \sum_{j=0}^{\ell} y^{[2j]}, \quad \omega = \sum_{\ell=0}^{\infty} \omega^{[2\ell+1]}, \quad \omega^{[2\ell+1]} = \sum_{j=0}^{\ell} \omega^{[2j+1]}, \quad \text{etc.}
\]
Defining the map \( T : (Y, \Upsilon) \rightarrow ((-1)^{\rho(\Xi)} \partial_{\Xi} J_1(x, \Xi + \Upsilon), \partial_X J_2(x + Y, \Xi)) \), we claim that there exists \( \delta_0 \), such that if \( \lambda \leq \delta_0 \), then there exists a fixed point of the map \( T \).

To prove this claim, we decompose, for \( \alpha = 1, 2 \),
\[
J_{\alpha}(x, \Xi) = J_{\alpha,00} + J_{\alpha,10} \theta_1 \theta_2 + \sum_{j,k=1}^{2} J_{\alpha,c_j d_k} \theta^{c_j} \pi^{d_k} + J_{\alpha,01} \pi_1 \pi_2 + J_{\alpha,11} \theta_1 \theta_2 \pi_1 \pi_2
\]
where \( J_{\alpha,\ast \ast} = J_{\alpha, \ast \ast}(x, \xi) \) and \( \bar{0} = (0, 0), \bar{1} = (1, 1), c_1 = (1, 0) = d_1, c_2 = (0, 1) = d_2 \in \{0, 1\}^2 \).

**Existence:** We consider the body part of (5.8).
\[
\begin{aligned}
 y^{[0]}_j &= \partial_{\xi_j} J_1(x^{[0]}, 0, \xi^{[0]} + \eta^{[0]}, 0) = \partial_{\xi_j} J_{1,00}(x^{[0]}, \xi^{[0]} + \eta^{[0]}), \quad j = 1, 2, 3, \\
 \eta^{[0]}_j &= \partial_{x_j} J_2(x^{[0]} + y^{[0]}, 0, \xi^{[0]} + \eta^{[0]}), \quad j = 1, 2, 3,
\end{aligned}
\]
with \( J_{\alpha,00}(x^{[0]}, \xi^{[0]}) = \phi_{\alpha B}(x^{[0]}, \xi^{[0]}) - x^{[0]} \cdot \xi^{[0]} \) for \( \alpha = 1, 2 \). By Theorem 1.7 of [28], there exists \( \delta_0 \), such that if \( |t - s| + |s - r| \leq \delta_0 \), the unique existence of solutions \( y^{[0]}, \eta^{[0]} \) of this equation is guaranteed. Moreover, the estimate is also established in Theorem 1.7' of [28].

Substituting these \( y^{[0]}, \eta^{[0]} \) into (5.8), we get
\[
\begin{aligned}
 \omega^{[1]}_1 &= -\partial_{\xi_1} J_1(x^{[0]}, \theta^{[1]} + \xi^{[0]} + \eta^{[0]}), \pi^{[1]} + \rho^{[1]}), \\
 &= \sum_{j=1}^{2} J_{1,c_1 d_j}(x^{[0]}, \xi^{[0]} + \eta^{[0]})(\theta^{c_j})^{[1]} - J_{1,01}(x^{[0]}, \xi^{[0]} + \eta^{[0]})(\pi^{[1]} + \rho^{[1]}), \\
 \omega^{[1]}_2 &= -\partial_{\xi_2} J_1(x^{[0]}, \theta^{[1]} + \xi^{[0]} + \eta^{[0]}), \pi^{[1]} + \rho^{[1]}), \\
 &= \sum_{j=1}^{2} J_{1,c_2 d_j}(x^{[0]}, \xi^{[0]} + \eta^{[0]})(\theta^{c_j})^{[1]} + J_{1,01}(x^{[0]}, \xi^{[0]} + \eta^{[0]})(\pi^{[1]} + \rho^{[1]}), \\
 \rho^{[1]}_1 &= \partial_{\theta_1} J_2(x^{[0]} + y^{[0]}, \theta^{[1]} + \omega^{[1]}), \xi^{[0]}, \pi^{[1]}), \\
 &= J_{2,10}(x^{[0]} + y^{[0]}, \xi^{[0]})(\theta^{[1]} + \omega^{[1]}), + \sum_{k=1}^{2} J_{2,c_1 d_k}(x^{[0]} + y^{[0]}, \xi^{[0]})(\pi^{d_k})^{[1]}, \\
 \rho^{[1]}_2 &= \partial_{\theta_2} J_2(x^{[0]} + y^{[0]}, \theta^{[1]} + \omega^{[1]}), \xi^{[0]}, \pi^{[1]}), \\
 &= -J_{2,10}(x^{[0]} + y^{[0]}, \xi^{[0]})(\theta^{[1]} + \omega^{[1]}), + \sum_{k=1}^{2} J_{2,c_2 d_k}(x^{[0]} + y^{[0]}, \xi^{[0]})(\pi^{d_k})^{[1]},
\end{aligned}
\]
Clearly, the components of the right-hand side above are the given data.
For the part of degree 2, we have

\[
\begin{aligned}
\eta_i^{[2]} &= \partial_{\xi_i} J_i(x^{(2)}, \theta^{(1)}, \pi^{(2)}, \xi^{(2)}, \rho^{(1)}) \\
&= \partial_{\xi_i} J_{1,00}(x, \xi + \eta^{[2]} + \partial_{\xi_i} J_{1,10}(x^{[0]}, \xi^{[0]} + \eta^{[0]}) \theta_1^{[1]} \theta_2^{[1]} \\
&\quad + \sum_{j,k=1}^2 \partial_{\xi_i} J_{1,j,k}(x^{[0]}, \xi^{[0]} + \eta^{[0]})(\theta^{(j)} \pi + \rho^{(k)}) \xi^{[2]} \\
&= \partial_{x_i} J_{2,00}(x + y, \xi^{[2]} + \partial_{x_i} J_{2,10}(x^{[0]} + y^{[0]}, \xi^{[0]})(\theta_1^{[1]} + \omega_1^{[1]})(\theta_2^{[1]} + \omega_2^{[1]}) \\
&\quad + \sum_{j,k=1}^2 \partial_{x_i} J_{2,j,k}(x^{[0]} + y^{[0]}, \xi^{[0]})(\theta + \omega)^{\xi^{[2]}} \pi^{\xi^{[2]}} \\
&= \partial_{x_i} J_{1,00}(x, \xi^{[2]}) = \sum_{i=1}^m \left[ \partial_{\xi_i} \partial_{x_i} J_{1,00}(x^{[0]}, \xi^{[0]}), \xi_i^{[2]} \right] + \partial_{\xi_i} \partial_{\xi_i} J_{1,00}(x^{[0]}, \xi^{[0]}), \xi_i^{[2]} \right], \\
\end{aligned}
\]

Here, \(\partial_{\xi_i} J_{1,00}(x, \xi^{[2]}) = \sum_{i=1}^m \left[ \partial_{\xi_i} \partial_{x_i} J_{1,00}(x^{[0]}, \xi^{[0]}), \xi_i^{[2]} \right] + \partial_{\xi_i} \partial_{\xi_i} J_{1,00}(x^{[0]}, \xi^{[0]}), \xi_i^{[2]} \right], \) etc., and the right-hand sides are represented by the given data.

Then, analogously we have, for \(\ell \geq 1,\)

\[
\begin{aligned}
\omega^{[2\ell+1]} &= \partial_\pi J_1(x^{[2\ell]}, \theta^{[2\ell]}, \pi^{[2\ell+1]}), \\
\rho^{[2\ell+1]} &= \partial_\rho J_2(x^{[2\ell]}, \theta^{[2\ell]}, \pi^{[2\ell+1]}),
\end{aligned}
\]

and

\[
\begin{aligned}
y^{[2\ell+2]} &= \partial_{\xi_i} J_1(x^{[2\ell+2]}, \theta^{[2\ell+1]}, \pi^{[2\ell+2]}), \\
\eta^{[2\ell+2]} &= \partial_{x_i} J_2(x^{[2\ell+2]}, \theta^{[2\ell+1]}, \pi^{[2\ell+2]}).
\end{aligned}
\]

**Estimate:** We proceed as we did in proving Proposition 37.3.

Put \(\#\)-product \(\phi_1 \# \phi_2\) of \(\phi_1\) and \(\phi_2\) by

\[
\phi_1 \# \phi_2(X, \Xi) = \phi_1(X, \tilde{\Xi}) - \langle \tilde{X} | \tilde{\Xi} \rangle + \phi_2(\tilde{X}, \Xi),
\]

we have

**Lemma 5.4.**

\[
\begin{aligned}
\partial_{x_A} \phi(X, \Xi) &= \partial_{\Xi_A} \phi_1(X, \tilde{\Xi}), \\
\partial_{\Xi_A} \phi(X, \Xi) &= \partial_{x_A} \phi_2(\tilde{X}, \Xi) \quad \text{for} \quad A = 1, \ldots, 5.
\end{aligned}
\]

**Proof.** By the above definition of \(\phi_1 \# \phi_2\), we get

\[
\begin{aligned}
\partial_{x_A} \phi_1 \# \phi_2(X, \Xi) &= \partial_{x_A} \phi_1(X, \tilde{\Xi}) + \partial_{x_A} \tilde{\Xi}_C \partial_{\Xi_C} \phi_1(X, \tilde{\Xi}) - \partial_{x_A} \tilde{X}_C \Xi_C - \partial_{x_A} \tilde{\Xi}_C (-1)^{p(\Xi_C)} \tilde{X}_C \\
&\quad + \partial_{x_A} \tilde{X}_C \partial_{\Xi_C} \phi_2(\tilde{X}, \Xi) \\
&= \partial_{x_A} \phi_1(X, \tilde{\Xi}) + \partial_{x_A} \tilde{\Xi}_C (-1)^{p(\Xi_C)} \tilde{X}_C - \partial_{x_A} \tilde{X}_C \Xi_C - \partial_{x_A} \tilde{\Xi}_C (-1)^{p(\Xi_C)} \tilde{X}_C \\
&\quad + \partial_{x_A} \tilde{X}_C \Xi_C = \partial_{x_A} \phi_1(X, \Xi).
\end{aligned}
\]

Same holds for \(\partial_{\Xi} \phi(X, \Xi).\)

**Lemma 5.5.**

\[
\phi_1 \# \phi_2(X, \Xi) = S(t, r; X, \Xi) = \phi(X, \Xi)
\]

**Proof.** We differentiate \(\phi_1 \# \phi_2(X, \Xi)\) w.r.t. \(s\) to have

\[
\frac{\partial}{\partial s} \phi_1 \# \phi_2(X, \Xi) = 0.
\]

As \(\phi_1 \# \phi_2(X, \Xi)|_{s=r} = \phi_1 \# \phi_2(X, \Xi)|_{s=t} = S(t, r, X, \Xi),\) we get the result. More precisely, repeat the arguments in proving Lemma 5.2 of [15] with necessary modifications.

Now, we have the following:
Proposition 5.6. Under the same assumption as above, we have
\[ |\pi_B \partial_X^a \partial_Z^b (\mu_1(X, \tilde{Z}) \mu_2(\hat{X}, \tilde{Z}) - \mu(X, \tilde{Z}))| \leq C_{a,b} \lambda^2. \] (5.11)

Notation: In the following, for a small parameter \( \delta > 0 \), we denote by \( \mathcal{S}_0[\delta] \), the class of functions \( p(X, \tilde{Z}) \in \mathcal{C}_{\mathbb{S},0}(\Omega^{|1|}) \) satisfying
\[ |\pi_B \partial_X^a \partial_Z^b p(X, \tilde{Z})| \leq C_{a,b} \delta \]
with a constant \( C_{a,b} \) independent of \( \delta \).

Proof. Differentiate the first equation of (5.9) w.r.t. \( \tilde{Z} \) to have
\[ \partial_Z \partial_X \phi(X, \tilde{Z}) = \partial_Z \partial_X \phi_1(X, \tilde{Z}). \]
On the other hand, from (5.6), we have
\[ \begin{cases} \partial_Z \hat{X} = (-1)^p(\tilde{Z}) \partial_Z \partial_Z \partial_Z \partial_Z \phi_1(X, \tilde{Z}), \\ \partial_Z \tilde{Z} = \partial_Z X \partial_X \partial_X \phi_2(X, \tilde{Z}) + \partial_Z \partial_Z \phi_2(\hat{X}, \tilde{Z}). \end{cases} \]
Substituting the first equation above into the second one, we get
\[ \partial_Z \tilde{Z}[I - (-1)^p(\tilde{Z}) \partial_Z \partial_Z \partial_Z \partial_Z \phi_1(X, \tilde{Z})] = \partial_Z \partial_Z \phi_2(\hat{X}, \tilde{Z}). \]
Because of
\[ |\pi_B (-1)^p(\tilde{Z}) \partial_Z \partial_Z \partial_Z \partial_Z \phi_1(X, \tilde{Z})| \leq \delta_0 < 1, \]
we have
\[ \partial_Z \partial_X \phi(X, \tilde{Z}) = \partial_Z \partial_X \phi_2(\hat{X}, \tilde{Z}) \times [I - (-1)^p(\tilde{Z}) \partial_Z \partial_Z \partial_Z \partial_Z \phi_1(X, \tilde{Z})]^{-1} \partial_Z \partial_Z \phi_1(X, \tilde{Z}). \]
We prove that there exists \( q_1(X, \tilde{Z}) \) such that
\[ \text{sdet}(I - (-1)^p(\tilde{Z}) \partial_Z \partial_Z \partial_Z \partial_Z \phi_1(X, \tilde{Z})) = 1 + q_1(X, \tilde{Z}). \]
Moreover, by the same argument of Proposition 1.5 of [28], we have
\[ \pi_B (1 + q_1(X, \tilde{Z})) \geq (1 - \delta_0)^m > 0, \]
which yields
\[ \text{sdet} \partial_Z \partial_X \phi(X, \tilde{Z}) = \text{sdet}(\partial_Z \partial_X \phi_2(\hat{X}, \tilde{Z})) \cdot (1 + q_1(X, \tilde{Z}))^{-1} \cdot \text{sdet}(\partial_Z \partial_Z \phi_1(X, \tilde{Z})). \]
Taking the square root of both sides, and remarking the elements of the right-hand side are even, we have
\[ \mu(X, \tilde{Z}) = \mu_1(X, \tilde{Z}) \mu_2(\hat{X}, \tilde{Z}) + q_2(X, \tilde{Z}) \]
with
\[ q_2(X, \tilde{Z}) = \mu_1(X, \tilde{Z}) \mu_2(\hat{X}, \tilde{Z})[(1 + q_1(X, \tilde{Z}))^{-1/2} - 1] = -\mu_1(X, \tilde{Z}) \mu_2(\hat{X}, \tilde{Z}) \frac{q_1(X, \tilde{Z})}{\sqrt{1 + q_1(X, \tilde{Z})(1 + \sqrt{1 + q_1(X, \tilde{Z})})}}. \]
Then, we have readily that \( q_2(X, \tilde{Z}) \in \mathcal{S}_0[\lambda^2]. \)

Rewriting (5.6) by
\[ Y = \hat{X} + \tilde{Y}, \quad X = \hat{X} + \tilde{Y}, \]
we have
\[ \phi_1(X, Y) - (Y|Y) + \phi_2(Y, \tilde{Z}) - \phi(X, \tilde{Z}) = -(\tilde{Y}|\tilde{Y}) + \mathcal{R}(X, \tilde{Y}, \tilde{Y}) \]
where
\[ \mathcal{R}(X, \tilde{Y}, \tilde{Y}) = \Psi_1(X, \tilde{Y}, \tilde{Y}) + \Psi_2(X, \tilde{Y}, \tilde{Y}) \]
with
\[ \Psi_1(X, \Xi, \Upsilon) = \Psi_1(\Upsilon) = \phi_1(X, \tilde{\Xi} + \Upsilon) - \phi_1(X, \tilde{\Xi}) - \langle \tilde{X} \mid \Upsilon \rangle, \]
\[ \Psi_2(X, \Xi, \tilde{\Upsilon}) = \Psi_2(\tilde{\Upsilon}) = \phi_2(\tilde{X} + \tilde{\Upsilon}, \Xi) - \phi_2(\tilde{X}, \Xi) - \langle \tilde{\Upsilon} \mid \tilde{\Xi} \rangle. \]

Therefore, we may rewrite (5.6) as
\[ U(t, s)U(s, r)u(X) = c_{3, 2} \int_{\mathbb{R}^3 | 2} d\Xi B(X, \Xi) e^{i h^{-1} \phi(X, \Xi) \mathcal{F} u(\Xi)}, \]
with
\[ B(X, \Xi) = c_{3, 2} \int_{\mathbb{R}^3 | 2 \times \mathbb{R}^3 | 2} d\tilde{\Upsilon} d\tilde{\Xi} \mu_1(X, \tilde{\Xi} + \tilde{\Upsilon}) \mu_2(\tilde{X} + \tilde{\Xi}, \Xi) e^{i h^{-1}(\mathcal{R}(X, \Xi, \Upsilon, \tilde{\Xi}) - \langle \tilde{\Upsilon} \mid \tilde{\Xi} \rangle)}. \]
Now, we want to prove

**Proposition 5.7.**
\[ |\pi_B \partial_X^a \partial_\Xi^b (B(X, \Xi) - \mu_1(X, \tilde{\Xi}) \mu_2(\tilde{X}, \Xi))| \leq C_{a, b} \lambda^2. \]  

(5.12)

To prove this Proposition, we remark

**Lemma 5.8.** \( \Psi_1(X, \Xi, \Upsilon) \) and \( \Psi_2(X, \Xi, \tilde{\Upsilon}) \) satisfy
\[ |\pi_B \partial_X^a \partial_\Xi^b \Psi_1(X, \Xi, \Upsilon)| \leq C_{a, b} |\pi_B \Upsilon|^2, \]  

(5.13)
\[ |\pi_B \partial_X^a \partial_\Xi^b \Psi_2(X, \Xi, \tilde{\Upsilon})| \leq C_{a, b} |\pi_B \tilde{\Upsilon}|^2, \]  

(5.14)
and for \( \epsilon \neq 1 \),
\[ |\pi_B \partial_X^a \partial_Y \Psi_1(X, \Xi, \Upsilon)| \leq C_{\epsilon}, \]
\[ |\pi_B \partial_X^a \partial_Y \Psi_2(X, \Xi, \tilde{\Upsilon})| \leq C_{\epsilon}. \]

(5.15)

**Proof.** As \( \Psi_1(\Upsilon) \) is represented by
\[ \Psi_1(\Upsilon) = \Upsilon \Upsilon \int_0^1 d\tau (1 - \tau) \phi_{1, \Xi E}(X, \tilde{\Xi} + \tau \Upsilon), \]
we get readily the first inequalities in (5.13). Rewriting
\[ \partial_Y \Psi_1(\Upsilon) = \partial_\Xi \phi_1 - (-1)^p(\Upsilon) \tilde{X} = \partial_\Xi J_1 - (\tilde{X} - X) \]
\[ = \Upsilon \int_0^1 d\tau \partial_\Xi \partial_\Xi \phi_1(X, \tilde{\Xi} + \tau \Upsilon). \]
Then, we get the first inequality of (5.14) from the last expression of (5.16) and the first inequality of (5.15) from the third expression of (5.16). Similar arguments work for other inequalities. \( \square \)

Remarking
\[ e^{ih^{-1} \Psi_2} = 1 + i h^{-1} \Psi_2 \int_0^1 d\tau e^{i h^{-1} \Psi_2}, \]
we rewrite
\[ B(X, \Xi) = c_{3, 2}^2 \int_{\mathbb{R}^3 | 2 \times \mathbb{R}^3 | 2} d\tilde{\Upsilon} d\tilde{\Xi} \mu_1(X, \tilde{\Xi} + \tilde{\Upsilon}) \mu_2(\tilde{X} + \tilde{\Xi}, \Xi) e^{i h^{-1}(\Psi_1(X, \Xi, \Upsilon) - \langle \tilde{\Upsilon} \mid \tilde{\Xi} \rangle)} \]
\[ + i h^{-1} \int_0^1 d\tau B_{1, \tau}(X, \Xi) \]
(5.17)
where
\[ B_{1, \tau}(X, \Xi) = c_{3, 2}^2 \int_{\mathbb{R}^3 | 2 \times \mathbb{R}^3 | 2} d\tilde{\Upsilon} d\tilde{\Xi} \Psi_2(X, \Xi, \tilde{\Upsilon}) \mu_1(X, \tilde{\Xi} + \tilde{\Upsilon}) \mu_2(\tilde{X} + \tilde{\Xi}, \Xi) \]
\[ \times e^{i h^{-1}(\Psi_1(X, \Xi, \Upsilon) + \tau \Psi_2(X, \Xi, \tilde{\Upsilon}) - \langle \tilde{\Upsilon} \mid \tilde{\Xi} \rangle)}. \]

**Lemma 5.9.** The symbol \( B_{1, \tau}(X, \Xi) \) belongs to \( \mathcal{S}_0^0[\lambda^2] \) uniformly with respect to \( \tau \).
Proof. Remarkings

\[ \Psi_2(\bar{Y}) = \sum Y_A \hat{\Psi}_{2,A}(\bar{Y}) \quad \text{with} \quad \hat{\Psi}_{2,A}(\bar{Y}) = \int_0^1 d\tau \partial_{\bar{Y}_A} \Psi_2(\tau \bar{Y}), \]

we have

\begin{align*}
B_{1,\tau}(X, \Xi) &= \sum_{a=1}^5 B_{1,\tau,a}(X, \Xi), \\
B_{1,\tau,a}(X, \Xi) &= c_{3,2}^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\bar{Y} \, dY \, Y_A \hat{\Psi}_{2,A}(\bar{X}, \Xi, \Xi) \mu_1(\bar{X}, \bar{Y}, \Xi) \mu_2(\bar{X} + \bar{Y}, \Xi) e^{i\hbar^{-1} \Psi^\tau} \\
&= c_{3,2}^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\bar{Y} \, dY \, \bar{B}_{1,\tau,a}(X, \Xi, \bar{Y}) e^{i\hbar^{-1} \Psi^\tau}.
\end{align*}

Here, we put

\[ \Psi^\tau = \Psi_1(X, \Xi, \bar{Y}) + \tau \Psi_2(X, \Xi, \bar{Y}) - (\bar{Y} | \bar{Y}), \]

\[ \bar{B}_{1,\tau,a}(\bar{Y}, \bar{Y}, \bar{Y}) = \hat{\Psi}_{2,a}(\bar{Y}) \{ \partial_{\bar{Y}} \Psi_1(\bar{Y}) \cdot \mu_1 + i \partial_{\bar{Y}} \mu_1 \} \mu_2, \]

and we used

\[ \partial_{\bar{Y}}(\mu_1 e^{i\hbar^{-1} \Psi^\tau}) = \{ \partial_{\bar{Y}} \mu_1 + i \hbar^{-1} (\partial_{\bar{Y}} \Psi_1 - (-1)^{\mu(\bar{Y})} \mu_1) \} e^{i\hbar^{-1} \Psi^\tau}. \]

Setting

\[ \mathcal{L}_\tau = \frac{1 - i \partial_{\bar{Y}} \Psi^\tau \cdot \partial_{\bar{Y}} - i \partial_{\bar{Y}} \Psi^\tau \cdot \partial_{\bar{Y}}}{\hbar^{-1} |\partial_{\bar{Y}} \Psi^\tau|^2 + \hbar^{-1} |\partial_{\bar{Y}} \Psi^\tau|^2}, \]

we have

\[ \mathcal{L}_\tau \Psi^\tau = \Psi^\tau. \]

We apply the Lax's technique. That is, we have

\[ B_{1,\tau}(X, \Xi) = c_{3,2}^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\bar{Y} \, dY \, e^{i\hbar^{-1} \Psi^\tau} (\mathcal{L}_\tau)^{2m+3} \bar{B}_{1,\tau,a}(\bar{Y}, \bar{Y}) \]

We have

\[ |\pi_B \partial_{\bar{Y}} \partial_{\bar{Y}}^2 \partial_{\bar{Y}} \Psi_2(X, \Xi, \bar{Y})| \leq C_{a,b,c} \lambda \langle \pi_B \bar{Y} \rangle \]

On the other hand, from

\[ 1 + |\pi_B \partial_{\bar{Y}} \Psi^\tau| + |\pi_B \partial_{\bar{Y}} \Psi^\tau| \geq C \{ 1 + (|\pi_B \bar{Y} | - \lambda_1 |\pi_B \bar{Y} |) + (|\pi_B \bar{Y} | - \lambda_2 |\pi_B \bar{Y} |) \} \]

\[ \geq C (1 + |\pi_B \bar{Y} | + |\pi_B \bar{Y} |), \]

and

\[ |\pi_B \partial_{\bar{Y}} \{ \partial_{\bar{Y}} \Psi_1(\bar{Y}) \mu_1 + i \partial_{\bar{Y}} \mu_1 \} \leq C \lambda \langle \pi_B \bar{Y} \rangle, \]

we get

\[ |\pi_B (\mathcal{L}_\tau)^{2m+3} B_{1,\tau}(\bar{Y}, \bar{Y}) | \leq C \lambda^2 (1 + |\pi_B \bar{Y} | + |\pi_B \bar{Y} |)^{-2m+1} \]

This proves

\[ |\pi_B \partial_{\bar{Y}}^2 \partial_{\bar{Y}}^2 B_{1,\tau}(X, \Xi) | \leq C_{a,b,c} \lambda^2. \quad \square \]

Using the Taylor expansion, we may rewrite the first term of (5.17) as

\begin{align*}
&c_{3,2}^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\bar{Y} \, dY \, \mu_1(\bar{X}, \Xi, \bar{Y}) \mu_2(\bar{X} + \bar{Y}, \Xi) e^{i\hbar^{-1} \Psi_1(X, \Xi, \bar{Y}) - (\bar{Y} | \bar{Y})} \\
&= c_{3,2}^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} d\bar{Y} \, dY \, \mu_1(\bar{X}, \Xi, \bar{Y}) \mu_2(\bar{X} + \bar{Y}, \Xi) e^{i\hbar^{-1} \Psi_1(X, \Xi, \bar{Y}) - (\bar{Y} | \bar{Y})} + i \hbar^{-1} \int_0^1 d\tau \sum_{A=1}^5 B_{2,\tau,a}(X, \Xi) \\
&\text{with} \\
&\mathcal{L}_\tau \Psi^\tau = \Psi^\tau. \quad \square
\end{align*}
\[ \mathcal{B}_{2,\tau,A}(Y, \bar{Y}) = \{ \partial_{\bar{Y}} \Psi_1 \cdot \mu_1 + i \hbar^{-1} \partial_{\bar{Y}} \mu_1 \} \partial_{X,A} \mu_2. \]

**Lemma 5.10.** The symbol \( B_{2,\tau,A}(X, \Xi) \) belongs to \( S_0^0[\lambda^2] \) uniformly with respect to \( \tau \).

**Proof.** Set
\[
\mathcal{L}_0 = 1 - i \partial_{\bar{Y}} \Psi_0 \cdot \partial_{\bar{Y}} - i \partial_{Y} \Psi_0 \cdot \partial_{Y}, \quad \mathcal{L}_0 e^{i \hbar^{-1} \Psi_0} = e^{i \hbar^{-1} \Psi_0}
\]
and apply the Lax’s technique. We have
\[
B_{2,\tau}(X, \Xi) = c_{3,2} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} d\bar{Y} d\bar{Y} e^{i \hbar^{-1} \Psi_0} (\mathcal{L}_0^*)^{2m+2} \mathcal{B}_{2,\tau,A}(Y, \bar{Y}).
\]
As
\[
|\pi_B \partial^a_{\bar{X}} \{ \partial_{\bar{X}_A} \mu_2(\hat{X} + \tau \bar{Y}, \Xi) \}| \leq C_{\lambda^2}
\]
we get
\[
|\pi_B (\mathcal{L}_0^*)^{2m+2} \mathcal{B}_{2,\tau,A}(Y, \bar{Y})| \leq C \lambda^2 (1 + |\pi_B \bar{Y}| + |\pi_B \bar{Y}|)^{(2m+1)}
\]
This proves
\[
|\pi_B \partial^a_{\bar{X}} \partial^b_{\bar{X}} B_{2,\tau,A}(X, \bar{X})| \leq C_{a,b} \lambda^2. \quad \Box
\]

**Corollary 5.11.**
\[
|\pi_B \partial^a_{\bar{X}} \partial^b_{\bar{X}} (B(X, \Xi) - \mu(X, \Xi))| \leq C_{a,b} \lambda^2. \tag{5.18}
\]

**Proof.** Combining (5.11) and (5.12), we get the desired result.

**Proof of Proposition 5.2.** By (5.18), we get the desired result (5.3). \( \Box \)

5.3. **Proof of Theorem 2.6.** Using Theorem A.1 in [13], we have our Theorem 2.6 readily. \( \Box \)

5.4. **Proof of Theorem 2.7.** From Theorem 2.6, using identifications \( \sharp \) and \( \flat \), we get the desired results. \( \Box \)

6. **Concluding remarks**

Though in this paper, we answer one of several problems in Inoue [13], we give other problems which should be solved:

(i) Propagation of singularity: How one can extend Egorov’s theorem to the system of PDE (see [4])?

(ii) How does the Aharonov-Bohm effect and Berry’s phase appear when Schrödinger equation is replaced by the Weyl equation (see [35])? This should be studied by constructing the fundamental solution of (W) using \#-product introduced in Kumano-go [26].

**References**

[1] K. Asada and D. Fujiwara, *On some oscillatory integral transformations in \( L^2(\mathbb{R}^n) \)*, Japanese J. Math. 4(1978), pp. 299-361.
[2] F.A. Berezin and M.S. Marinov, *Particle spin dynamics as the Grassmann variant of classical mechanics*, Annals of Physics 104(1977), pp. 336-362.
[3] Y. Choquet-Bruhat, *Supergravities and Kaluza-Klein theories*, pp. 31-48, in “Topological properties and global structure of space-time” (eds. P. Bergman and V. de Sabbata), Plenum Press, New York, 1986.
[4] H.O. Cordes, *A version of Egorov’s theorem for systems of hyperbolic pseudo-differential equations*, J. Functional Analysis 48(1982), pp. 285-300.
[5] B. deWitt, *Supermanifolds*, London, Cambridge Univ. Press,1984.
[6] R. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill Book Co., New York, 1965.
[7] D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation*, J. D’Analyse Math. 35(1979), pp. 41-96.
Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo, 152, Japan

E-mail address: inoue@math.titech.ac.jp