Bianchi type $VI_1$ cosmological model with wet dark fluid in scale invariant theory of gravitation

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Abstract
In this paper, we have investigated Bianchi type $VI_h$, $II$ and $III$ cosmological model with wet dark fluid in scale invariant theory of gravity, where the matter field is in the form of perfect fluid and with a time dependent gauge function (Dirac gauge). A non-singular model for the universe filled with disorder radiation is constructed and some physical behaviors of the model are studied for the feasible $VI_h(h = 1)$ space-time.

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1 Introduction
Einstein’s general theory of gravitation has been successful in describing gravitational phenomena. His theory has been served as
a basis for different models of the universe. The homogeneous isotropic expanding model based on general relativity appears to provide a grand approximation to the observed large scale properties of the universe. However, since Einstein first published his theory of gravitation, several modifications have been proposed from time to time which seek to incorporate certain desirable features lacking in the original theory. One of the modifications he himself pointed out that Mach’s principle is not substantiated by general relativity. So, there have been considerable attempts made to generalize the general theory of relativity by incorporating Mach’s principle and other desired features which are lacking in the original theory.

There has been considerable interest in scale invariant theory owing to the scaling behavior exhibited in high-energy particle scattering experiments, (Callan et al. 1970). However, such theories are considered to be valid only in the limit of high energies or vanishing rest masses. This is because in elementary particle theories, rest masses are considered constants, and the scale invariance is generally valid only when the constant rest mass condition is relaxed. It is found from the literature that there are two prominent generalizations of Einstein theory. Firstly, in an attempt to unify electromagnetism with gravitation, Weyl (1922) generalized Riemannian geometry by allowing lengths to change under parallel displacement. The theory being unphysical was soon rejected, wherein a mathematical technique known as gauge transformation was introduced. Eddington (1924) pointed out that, the gauge transformation represents a change of units of measurement and hence gives a general scaling of the physical system. Secondly, Dirac (1973,1974) rebuilt the Weyl’s unified theory by introducing the notion of two metrics and an additional gauge function $\beta(x^i)$. A scale invariant variation principle was proposed from which gravitational and electromagnetic field equations can be derived. It is concluded that an arbitrary gauge function is necessary in all scale invariant theories. It is found that the scale invariant theory of gravitation agrees with general relativity up to the accuracy of observations made of up to now. Dirac (1973, 1974), Hoyle and Narlikar (1974) and Canuto et al. (1977) have studied several aspects of the scale invariant theories of gravitation. But Wesson’s (1981a,b) formulation is so far
best to describe all the interactions between matter and gravitation in scale free manner.

In the scale invariant theory of gravitation, Einstein equations have been written in a scale-independent way by performing the conformal or scale transformation as

$$\tilde{g}_{ij} = \beta^2(x^k)g_{ij}$$

(1)

where the gauge function $\beta(0 < \beta < 1)$, in its most general formulation, is a function of all space-time coordinates. Thus, using the conformal transformation of the type given by equation (1), Wesson (1981a,b) transforms the usual Einstein field equations into

$$G_{ij} + 2\frac{\beta_{ij}}{\beta} - 4\frac{\beta_i\beta_j}{\beta^2} + (g^{ab}\frac{\beta_{a}\beta_{b}}{\beta^2} - 2g^{ab}\frac{\beta_{ij}}{\beta^2})g_{ij} + \Lambda g_{ij} + \Lambda_0 \beta^2 g_{ij} = -\kappa T_{ij}$$

(2)

where

$$G_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij}$$

(3)

Here, $G_{ij}$ is the conventional Einstein tensor involving $g_{ij}$. Semicolon and comma respectively denote covariant differentiation with respect to $g_{ij}$ and partial differentiation with respect to coordinates. $R_{ij}$ is the Ricci tensor, and $R$ is the Ricci scalar. The cosmological term $\Lambda g_{ij}$ of Einstein theory is now transformed to $\Lambda_0 \beta^2 g_{ij}$ in scale invariant theory with dimensionless cosmological constant $\Lambda_0$. $G$ and $\kappa$ are the Newtonian and Wesson’s gravitational parameters respectively. $T_{ij}$ is the energy momentum tensor of the matter field and $\kappa = 8\pi G/c^4$. A particular feature of this theory is that no independent equation for $\beta$ exists.

Beesham (1986a,b,c), Reddy and Venkateswaralu (1987), Mohanty and Mishra (2001,2003), Mishra (2004,2008), Mishra and Sahoo (2012a,b), have investigated several aspects of scale invariant theory. However, Bianchi type $VI_h(h = 1)$ space-time with wet dark fluid has not been considered, so far, in the scale invariant theory of gravitation. Hence, in this paper, we have shown that Bianchi type $VI_h(h = 1)$ cosmological model governed by the equation of state $p_{WDF} = (\rho_{WDF}c^2)/3$ is compatible whereas Bianchi type $VI_h(h = -1)$, $II$ and $III$ are not compatible. In section 2, the concept of dark energy (DE) is discussed in detail. In section 3, we have set up the field equations of scale invariant theory
of gravitation with wet dark fluid for Bianchi type $V_{II}$, $II$ and $III$ space-time. In section 4, we have obtained a cosmological model for feasible Bianchi type $V_{I}$ represented by explicit exact solution of the field equations, with the help of energy momentum tensor for a perfect fluid wherein we confine ourselves to the equation of state $p_{WDF} = (\rho_{WDF}c^2)/3$ in order to overcome the underdeterminacy for determining three unknowns involved in a system of two field equations. In section 5, we have discussed some physical properties of the model. In section 6, we have discussed some discussions on the solution of dark energy and concluding remarks are given in section 7. Finally lists of references are mentioned at the end.

2 Dark energy

The nature of dark energy component of the universe (Riess et al. 1998; Perlmutter et al. 1999) remains one of the deepest mysteries of cosmology. There is certainly no lack of candidates: a remnant cosmological constant, quintessence, k-essence, phantom energy. Modifications of the friedmann equation such as Cardassian expansion (Hayward, 1967) as well as what might be derived from brane cosmology have also been used to explain the acceleration of the universe.

Recently, there has been considerable interest in cosmological model with ”Dark Energy” (DE) in general relativity because of the fact that our universe is currently undergoing an accelerated expansion which has been confirmed by host of observations, such as type I supernovae (SNeIa) (Riess et al.1998; Perlmutter et al. 1999; Bahcall et al. 1999), Sloan Digital Sky Survey (SDSS) (Tegmark et al. 2004), Wilkinson Microwave Anisotropy Probe (WMAP) (Bennet et al. 2003; Nolta et al. 2008; Hinshaw et al. 2009). Based on these observations, cosmologists have accepted the idea of dark energy. Cosmologists have proposed many candidates for dark energy to fit the current observations such as cosmological constant, tachyon, quintessence, phantom and so on. Current studies to extract the properties of a dark energy component of the universe from observational data focus on the determination of its equation of state $w(t) = p/\rho$, which is not necessarily constant. The methods for
restoration of the quantity $w(t)$ from expressional data have been developed (Sahni and Starobinsky 2006), and an analysis of the experimental data has been conducted to determine this parameter as a function of cosmological time (Sahni et al. 2008). Recently the parameter $w(t)$ has been calculated with some reasoning which reduced to some simple parameterization of the dependences by some authors (Weller and Albrecht 1997; Huterer and Turner 2001; Krauss et al. 2007; Linden and Virey 2008; Usmani et al. 2008; Chen et al. 2009). These observations provide us a clear outline of the universe: it is flat and full of undamped form of energy density pervading the Universe. The undamped energy called ”Dark Energy” (DE) with negative pressure attributes to about 74 percent of the total energy density. The remaining 26 percent of the energy density consists of matter including about 22 percent dark matter density and about 4 percent baryon matter density. So, understanding the nature of DE is one of the most challenging problems in modern astrophysics and cosmology. Recent cosmological observations contradict the matter dominated universe with decelerating expansion indicating that our universe experiences accelerated expansion.

There is a new candidate for dark energy: Wet Dark Fluid (WDF). This model is in the spirit of generalised Chaplygin gas (GCG), where a physically motivated equation of state is offered with properties relevant for the dark energy problem. Here the motivation stems from an empirical equation of state to treat water and aqueous solutions. The equation for WDF is

$$p_{WDF} = \gamma(\rho_{WDF} - \rho^*)$$

It is motivated by the fact that it is a good approximation for many fluids including water, in which the internal attraction of the molecules makes negative pressure possible. The parameters $\gamma$ and $\rho^*$ are taken to be positive and we restrict ourselves to $0 \leq \gamma \leq 1$. Note that if $c_\alpha$ denotes the adiabatic sound speed in WDF, then $\gamma = c_\alpha^2$ (Babichev et al. 2004).

To find the WDF energy density, we use the energy conservation equation

$$\dot{\rho}_{WDF} + 3H(\rho_{WDF} + p_{WDF}) = 0$$
From equation of state (4) and using $3H = \frac{\dot{v}}{v}$ in equation (5), we get

$$\rho_{WDF} = \frac{\gamma}{1 + \gamma} p^* + \frac{k}{\nu(1 + \nu)}$$

(6)

where $k$ is the integration constant and $\nu$ is the volume expansion. WDF naturally includes two components, a piece that behaves as a cosmological constant as well as pieces those red shifts as a standard fluid with an equation of state $p = \gamma \rho$.

We can show that if we take $k > 0$, this fluid will not violate the strong energy condition $p + \rho = 0$. Thus, we get

$$p_{WDF} + \rho_{WDF} = (1 + \gamma) \rho_{WDF} - \gamma p^* = (1 + \gamma) \frac{k}{\nu^{1+\gamma}} \geq 0$$

(7)

Holman and Naidu [2005] observed that their model is consistent with the most recent SNIa data, the WMAP results as well as the constraints coming from measurements of the power spectrum. Hence, they considered both, the case where the dark fluid is smooth (i.e. only the CDM component cluster gravitationally) as well as the case where the dark fluid also clusters.

In a homogeneous universe, it is possible to infer the time evolution of the cosmic expansion from observations along the past light cone, since the expansion rate is a function of time only. In the inhomogeneous case, however, the expansion rate varies both with time and space. Therefore, if the expansion rates inferred from observations of supernovae are larger for low redshifts than higher redshifts, this must be attributed to cosmic acceleration in a homogeneous universe.

Nojiri and Odintsov (2011) developed the cosmological reconstruction method in terms of cosmological time. Using the freedom in the choice of scalar potentials and of the modified term function, which depends on geometrical invariants, such as curvature and Gauss-Bonnet term, they arrived to master differential equations whose solutions solve the problem. They explicitly considered the reconstruction in scalar tensor theory, Brans-Dicke gravity, the k-essence model, F(R) theory and Lagrangian multiplier F(R) theory. Special attention were paid to late-time dynamics of the effective quintessence/phantom dark energy of arbitrary nature: fluid, particle model or modified gravity. The advantage of the approach proposed in this work is very general character. The developed
reconstruction scheme proposes the way to change the properties of any particular theory in a desirable way. Motivated with the work of Nojiri and Odintsov (2011), Bamba et al. (2012), We have constructed Bianchi type $VI_1$ cosmological model in scale invariant theory with WDF.

3 Metric and field equations

3.1: Bianchi type $VI_h$:

We consider the Bianchi type $VI_h$ metric with a Dirac gauge function $\beta = \beta(ct)$ of the form

$$ds^2_W = \beta^2 ds^2_E$$

with

$$ds^2_E = -c^2 dt^2 + A^2 dx^2 + B^2 e^{2x} dy^2 + C^2 e^{2hx} dz^2$$

The metric potentials $A$, $B$ and $C$ are functions of $t$ only. $h$ is constant ($-1, 0, +1$) and $c$ is the velocity of light. $ds^2_W$ and $ds^2_E$ respectively represent the intervals in Wesson and Einstein theory. Further, $x^i, i = 1, 2, 3, 4$ respectively denote for $x, y, z$ and $t$ only. Here, we have taken an attempt to build cosmological model in these space-times with a perfect fluid and WDF having the energy momentum tensor of the form

$$T_{ij}^m = (p_{WDF(m)} + \rho_{WDF(m)} c^2) U_i U_j + p_{WDF(m)} g_{ij}$$

(10)

together with

$$g_{ij} U^i U^j = -1$$

(11)

where $U^i$ is the four velocity vector of the fluid, $p_{WDF(m)}$ and $\rho_{WDF(m)}$ are respectively the proper isotropic pressure and energy density of the matter in WDF.

The non-vanishing components of conventional Einstein’s tensor (3) for the metric (9) are:

$$G_{11} = \frac{A^2}{c^2} \left( \frac{B_{14}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - h \frac{c^2}{A^2} \right)$$

(12)
\[ G_{22} = \frac{B^2 e^{2x}}{c^2} \left( \frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} - h^2 \frac{c^2}{A^2} \right) \] (13)

\[ G_{33} = \frac{C^2 e^{2h x}}{c^2} \left( \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{c^2}{A^2} \right) \] (14)

\[ G_{44} = -\left( \frac{A_4 B_4}{AB} + \frac{B_4 C_4}{BC} + \frac{C_4 A_4}{CA} - (1 + h + h^2) \frac{c^2}{A^2} \right) \] (15)

\[ G_{14} = -(1 + h) \frac{A_4}{A} + \frac{B_4}{B} + h \frac{C_4}{C} \] (16)

The suffix 4 after a field variable denotes exact differentiation with respect to time \( t \).

Using the comoving coordinates \((0,0,0,c)\), the non-vanishing components of the field equations (2) can now be written explicitly for the metric (8) as

\[ G_{11} = -\kappa p_{WDF(m)} A^2 - \frac{A^2}{c^2} \left[ 2 \left( \frac{B_4}{B} + \frac{C_4}{C} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \] (17)

\[ G_{22} = -\kappa p_{WDF(m)} B^2 e^{2x} - \frac{B^2 e^{2x}}{c^2} \left[ 2 \left( \frac{A_4}{A} + \frac{C_4}{C} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \] (18)

\[ G_{33} = -\kappa p_{WDF(m)} C^2 e^{2h x} - \frac{B^2 e^{2h x}}{c^2} \left[ 2 \left( \frac{A_4}{A} + \frac{B_4}{B} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \] (19)

\[ G_{44} = -\kappa \rho_{WDF(m)} c^4 + \left[ 2 \left( \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right) \frac{\beta_4}{\beta} - 4 \frac{\beta_{44}}{\beta} + 3 \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \] (20)

\[ G_{14} = 0 \Rightarrow \frac{B_4}{B} + h \frac{C_4}{C} = (1 + h) \frac{A_4}{A} \Rightarrow A^{1+h} = k_1 BC^h \] (21)

where \( k_1 \) is an integrating constant.

Equation (2) and equations (17) - (21) suggest the definition vacuum pressure \( p_{WDF(v)} \) and vacuum density \( \rho_{WDF(v)} \) that involve
neither the Einstein tensor of conventional theory nor the properties of conventional matter (Wesson, 1981 a,b). These two quantities can be obtained as

\[ 2 \left( \frac{h + 1}{h} \frac{A_4}{A} + \frac{h - 1}{h} \frac{B_4}{B} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 = \kappa_{\text{p}} \rho_{\text{DF}}(v) c^2 \]

(22)

\[ 2 \left( \frac{2h + 1}{h} \frac{A_4}{A} + \left( \frac{1}{h} \right) \frac{B_4}{B} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 = \kappa_{\text{p}} \rho_{\text{DF}}(v) c^2 \]

(23)

\[ 2 \left( \frac{A_4}{A} + \frac{B_4}{B} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 = \kappa_{\text{p}} \rho_{\text{DF}}(v) c^2 \]

(24)

\[ 2 \left( \frac{2h + 1}{h} \frac{A_4}{A} + \left( \frac{2}{h} \right) \frac{B_4}{B} \right) \frac{\beta_4}{\beta} - 4 \frac{\beta_{44}}{\beta} + 3 \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 = -\kappa_{\text{p}} \rho_{\text{DF}}(v) c^4 \]

(25)

In case where there is no matter and gauge function is a constant function it can be easily obtained from Eqns. (22)-(25) that

\[ c^2 \rho_{\text{DF}}(v) = -\frac{c^4 \lambda_{\text{GR}}}{8 \pi G} = -p_{\text{DF}}(v) \]

Here \( \lambda_{\text{GR}} = \Lambda_0 \beta^2 \) = constant, is the cosmological constant in general relativity. Also, \( p_{\text{DF}}(v) \) being dependent on the constants \( \lambda_{\text{GR}}, c \) and \( G \), is uniform in all directions and hence isotropic in nature.

The cosmological model with the equation of state is rare in literature and is known as \( \rho \)-vacuum model or false vacuum model or degenerate vacuum model. The corresponding model in static case is well known de-Sitter model.

It is evident from the aforesaid equations that \( p_{\text{DF}}(v) \) being isotropic, is consistent only when

\[ A = k_2 B, \quad \text{as} \quad \beta_4 \neq 0 \]

(26)

where \( k_2 \) is an integrating constant.

Making use of the consistency condition (26), the vacuum pressure \( p_{\text{DF}}(v) \) and vacuum energy density \( \rho_{\text{DF}}(v) \) can be obtained as

\[ p_{\text{DF}}(v) = \frac{1}{\kappa c^2} \left[ 4 \left( \frac{A_4}{A} \right) \frac{\beta_4}{\beta} - 2 \frac{\beta_{44}}{\beta} - \frac{\beta^2}{\beta^2} + \Lambda_0 \beta^2 c^2 \right] \]

(27)
\[
\rho_{WDF(v)} = -\frac{1}{\kappa c^4} \left[ \left( \frac{6 A_4}{A} \right) \frac{\beta_4}{\beta} - 4 \frac{\beta_{44}}{\beta^2} + 3 \frac{\beta_4^2}{\beta^2} - \Lambda_0 \beta^2 c^2 \right]
\]  

(28)

where \( p_{WDF(v)} \) and \( \rho_{WDF(v)} \) carry the properties of vacuum in WDF of conventional physics. The definitions of above two properties is natural as regards to the scale invariant properties of vacuum. So, the total pressure and energy density in WDF can be defined as:

\[
p_{WDF(t)} \equiv p_{WDF(m)} + p_{WDF(v)}
\]

(29)

\[
\rho_{WDF(t)} \equiv \rho_{WDF(m)} + \rho_{WDF(v)}
\]

(30)

Using the components of Einstein Tensor (12) - (16) and the results obtained in equations (26) - (28) with the aforesaid of (29) - (30), the field equations (17) - (21) can be written in the following explicit form:

\[
2 \frac{A_{44}}{A} + \frac{A_4^2}{A^2} - h \frac{c^2}{A^2} = -\kappa p_{WDF(t)} c^2
\]

(31)

\[
2 \frac{A_{44}}{A} + \frac{A_4^2}{A^2} - h^2 \frac{c^2}{A^2} = -\kappa p_{WDF(t)} c^2
\]

(32)

\[
2 \frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \frac{c^2}{A^2} = -\kappa p_{WDF(t)} c^2
\]

(33)

\[
3 \frac{A_4^2}{A^2} - (1 + h + h^2) \frac{c^2}{A^2} = \kappa \rho_{WDF(t)} c^4
\]

(34)

As the total pressure is isotropic in nature, we obtain from eqns. (31)-(34) that

\[
h(h - 1) \frac{c^2}{A^2} = 0; (h^2 - 1) \frac{c^2}{A^2} = 0
\]

(35)

These equations simultaneously hold good for \( h = 1 \). Otherwise it leads to unphysical situation i.e., either the velocity of light vanishes or the metric potential A is infinitely large. So, this theory is not feasible when the space time is governed by Bianchi type \( VI_h(h = -1) \) and \( VI_h(h = 0) \) metrics whereas it is feasible for Bianchi type \( VI_h(h = 1) \).
3.2: Bianchi type II and III metric:

With a similar approach we have also determined the field equations for Bianchi type II and III space-times with a perfect fluid and WDF.

The field equations of Bianchi type II non-diagonal metric
\[
ds_E^2 = -c^2 dt^2 + R^2 dx^2 + S^2 dy^2 + 2S^2 x dy dz + (S^2 x^2 + R^2) dz^2 \tag{36}
\]
where \( R \) and \( S \) are functions of cosmic time \( t \) only and \( c \) is the velocity of light
are:
\[
2 \frac{R_{44}}{R} + \frac{R_4^2}{R^2} + \frac{c^2}{4R^2} = -\kappa \rho_{WDF(t)} c^2 \tag{37}
\]
\[
2 \frac{R_{44}}{R} + \frac{R_4^2}{R^2} - 3 \frac{c^2}{4R^2} = -\kappa p_{WDF(t)} c^2 \tag{38}
\]
\[
3 \frac{R_4^2}{R^2} - \frac{c^2}{4R^2} = -\kappa \rho_{WDF(t)} c^2 \tag{39}
\]
Now, eqns. (37)-(39) yield
\[
\frac{c^2}{R^2} = 0 \tag{40}
\]
which leads to inconsistency for both the cases i.e. \( R \) is very large or \( c = 0 \). Neither of the cases are acceptable under geometrical and physical ground. Thus, scale invariant theory of gravitation with perfect fluid and WDF is not compatible with Bianchi type II space-time.

The Bianchi type III space-time defined as:
\[
ds_E^2 = -c^2 dt^2 + e^{2A} dx^2 + e^{2(B+x)} dy^2 + e^{2D} dz^2 \tag{41}
\]
where \( A \) and \( B \) are functions of cosmic time \( t \) only and \( c \) is the velocity of light.
The introduction of a new time variable \( T \) defined by
\[
dT = \frac{a}{e^{A+B+D}} dt \tag{42}
\]
transforms the metric into
which describes the space-time in general theory of relativity. Now, the field equations of Bianchi type III space-time in WDF can be written as:

\[ e^{-2(3A + k_1 + k_2)}[2A_{44} - 3A_4^2] = -\kappa p_{WDF(t)}c^2 \quad (44) \]

\[ e^{-2(3A + k_1 + k_2)}[2A_{44} - 3A_4^2 - c^2e^{2(2A + k_1 + k_2)}] = -\kappa p_{WDF(t)}c^2 \quad (45) \]

\[ e^{-2(3A + k_1 + k_2)}[3A_4^2 - c^2e^{2(2A + k_1 + k_2)}] = -\kappa \rho_{WDF(t)}c^4 \quad (46) \]

which again leads to inconsistency for both the cases i.e. \( A \) is negative or very large or \( c = 0 \). Neither of these cases are acceptable and no solution to the field equations can be obtained in scale invariant theory in WDF, which is mainly governed by a gauge function involving a non-zero velocity of light. Thus the scale invariant theory with WDF is not compatible in Bianchi type III space-time. [Mohanty and Mishra, 2003]

4 Solutions for feasible Bianchi type \( VI_1 \) space-time

Now, we have two field equations with three unknowns viz., \( p_{WDF(t)} \), \( \rho_{WDF(t)} \) and \( A \) for \( h = 1 \). For the complete determinacy one extra condition is needed. We therefore, consider the equation of state

\[ p_{WDF(t)} = \frac{\rho_{WDF(t)}c^2}{3} \quad (47) \]

From eqns. (31)- (34), we obtained

\[ A = (c^2t^2 + d_1t + d_2)^{1/2} \quad (48) \]

where \( d_1 \) and \( d_2 \) are integrating constants.
Without loss of generality, we take \( k_1 = k_2 = 1 \) in eqns. (21) and (26). Subsequently, we have

\[ A = B = C = (c^2t^2 + d_1t + d_2)^{1/2} \quad (49) \]
Now the total pressure $p_{WDF(t)}$ and total energy density $\rho_{WDF(t)}$ can be calculated as

$$p_{WDF(t)} = \frac{\rho_{WDF(t)} c^2}{3} = \frac{1}{4\kappa c^2} \left[ \frac{(d_1^2 - 4d_2c^2)}{(c^2t^2 + d_1t + d_2)^2} \right]$$  \hspace{1cm} (50)$$

where the reality condition demands that $d_1^2 > 4d_2c^2$.

Now, considering Dirac gauge in the form $\beta = \frac{1}{c}t$, the pressure and energy density corresponding to vacuum case in WDF can be calculated as

$$p_{WDF(v)} = -\frac{1}{\kappa c^2} \left[ \frac{2(2c^2t + d_1)}{t(c^2t^2 + d_1t + d_2)} - \frac{\Lambda_0 - 5}{t^2} \right]$$  \hspace{1cm} (51)$$
$$\rho_{WDF(v)} = \frac{1}{\kappa c^4} \left[ \frac{3(2ct^2 + d_1)}{t(c^2t^2 + d_1t + d_2)} - \frac{\Lambda_0 - 5}{t^2} \right]$$  \hspace{1cm} (52)$$

The matter pressure $p_{WDF(m)}$ and matter energy density $\rho_{WDF(m)}$ can be calculated as

$$p_{WDF(m)} = \frac{1}{\kappa c^2} \left[ \frac{(d_1^2 - 4d_2c^2)}{4(c^2t^2 + d_1t + d_2)^2} - \frac{2(2c^2t + d_1)}{t(c^2t^2 + d_1t + d_2)} - \frac{\Lambda_0 - 5}{t^2} \right]$$  \hspace{1cm} (53)$$
$$\rho_{WDF(m)} = \frac{1}{\kappa c^4} \left[ \frac{3(d_1^2 - 4d_2c^2)}{4(c^2t^2 + d_1t + d_2)^2} - \frac{3(2c^2t + d_1)}{t(c^2t^2 + d_1t + d_2)} + \frac{\Lambda_0 - 5}{t^2} \right]$$  \hspace{1cm} (54)$$

The Bianchi type $VI_1$ model in scale invariant theory with WDF is given by the eqns. (8) and (9) and the metric in this case takes the following form

$$ds^2_W = \frac{1}{c^2t^2} \left[ -c^2dt^2 + \left( c^2t^2 + d_1t + d_2 \right) \left( dx^2 + e^{2x} (dy^2 + dz^2) \right) \right]$$  \hspace{1cm} (55)$$

Using the transformation $t = e^T$, the metric can be put in the following form

$$ds^2_W = -dT^2 + Q^2(T) \left[ dx^2 + e^{2x} (dy^2 + dz^2) \right]$$  \hspace{1cm} (56)$$

where $Q(T) = \left( 1 + \frac{d_1}{c^2} e^{-T} + \frac{d_2}{c^4} e^{-2T} \right)^{\frac{1}{2}}$. 

13
5 Some physical properties of the model

In this section, we investigated some physical properties of the model obtained in the previous section in the transformed coordinate system; however the behavior of the physical quantities remain alike even though there is a shift i.e. $t(0, 1, \infty) \rightarrow T(-\infty, 0, \infty)$. The new time coordinate being stretched covers the time region from past to future completely. Thus a clear picture of the model (56) in the new time coordinate can be obtained.

The scalar expansion of the model (56)

$$
\Theta = U^i_{;i} = 3 \frac{Q_T}{Q}
$$

which gives the rate of expansion or contraction of the model is found to be

$$
\Theta(T) = \frac{-3}{2} \left( \frac{d_1 e^{-T} + 2d_2 e^{-2T}}{c^2 + d_1 e^{-T} + d_2 e^{-2T}} \right).
$$

So, $\Theta(0) = \frac{-3}{2} \left( \frac{d_1 + 2d_2}{c^2 + d_1 + d_2} \right)$ and $\Theta \rightarrow 0$ as $T \rightarrow \infty$

If $d_1$ and $d_2$ are both non-zero then they are of opposite sign and at $T = \ln 2 + \ln \left( -\frac{d_2}{d_1} \right)$, the model gets contracted. Hence, the model contracts without admitting any singularity during evolution.

The shear scalar $\sigma$ for the model (56) vanishes which indicates that the shape of the universe remain unchanged during evolution. Moreover, as $\frac{\sigma^2}{c^2} = 0$, the space-time is isotropized during evolution in this theory.

The Hubble parameter $H$ that determines the present rate of expansion of the universe corresponding to the metric (56) is

$$
H = \frac{Q_T}{Q} = -\frac{1}{2} \left[ \frac{d_1 e^{-T} + 2d_2 e^{-2T}}{c^2 + d_1 e^{-T} + d_2 e^{-2T}} \right]
$$

Since, $H \rightarrow \infty$ when $T \rightarrow 0$, and $H \rightarrow 0$ when $T \rightarrow \infty$, therefore it indicates that the rate of expansion is accelerated or decelerated depending on the signature of the parameters.
The deceleration parameter for the model (56) is calculated as

\[ q = -\frac{Q_{TT}Q}{Q_T^2} = -\left[ 1 + \frac{2c^2(d_1e^{-T} + 4d_2e^{-2T}) + 2d_1d_2e^{-3T}}{(d_1e^{-T} + d_2e^{-2T})^2}\right] \] (59)

It is observed that \( q(0) = \text{constant} \) and \( q \) is not defined at infinite future. Hence the model is not steady state.

The acceleration \( \dot{U}^i = 0 \) confirms that the matter particles follow geodesic in this theory. The vorticity \( W \) of the radiating fluid of the model also vanishes. Thus \( U^i \) is hypersurface orthogonal.

With proper choice of the parameters \( d_1, d_2 \) and \( \lambda_0, \rho_{WDF(m)} \) is a positive constant and \( \rho_{WDF(m)} \to 0 \) as \( t \to \infty \). Thus the universe starts evolving with constant matter density at initial epoch.

Also, \( \frac{\rho_{WDF(m)}}{\Theta^3} \to \text{constant} \) at \( T \to 0 \) and at \( T \to \infty \), \( \frac{\rho_{WDF(m)}}{\Theta^3} \to 0 \) which confirms the homogeneity nature of the space-time.

6 Discussion on the solution of dark energy

In section 3, we have derived the field equations (31)-(34). From eqns. (31) and (32), it is clear that \( h = 1 \). If we take \( c = 1 \), the set of eqns. (31)-(34) reduce to

\[ 2\frac{A_{44}}{A} + \frac{A_4^2}{A^2} - \frac{1}{A^2} = -\kappa \rho_{WDF(t)} \] (60)

\[ 3\frac{A_2^2}{A^2} - 3\frac{1}{A^2} = \kappa \rho_{WDF(t)} \] (61)

Since \( \rho_{WDF(t)} = \gamma(\rho_{WDF(t)} - \rho^*) \), the above equations for a suitable choice of \( \gamma = \frac{1}{3} \) reduces to

\[ 2AA_{44} + 2A_4^2 - \frac{\kappa \rho^*}{3}A^2 = 2 \] (62)

Now, eqn. (49) admits a solution for \( \rho^* \neq 0 \) as

\[ A = \sqrt{c_1a^{\sqrt{\alpha t}} + c_2e^{-\sqrt{\alpha t}} - \frac{2}{\alpha}} \] (63)

where \( \alpha = -\frac{\kappa \rho^*}{3} \), \( c_1 \) and \( c_2 \) are integrating constants.
The radius scale factor is an exponentially increasing function of time if \( c_1 > c_2 \). Also for \( c_2 = 0 \), it favors a de Sitter kind of universe predicting a deceleration parameter \( q = -1 \). Moreover, the hubble parameter is comes out to be constant.

In case \( \rho^* = 0 \), the result is the same as calculated in eqn. (48). Hence the role of \( \rho^* \) in getting accelerated expansion of the universe is clearly understood.

The present cosmological scenario tells us that we are living in an expanding flat and accelerating universe dominated by dark energy, whereas the remaining one-third is contributed by matter.

The deceleration parameter \( q \) is a very important vector for understanding cosmic evolution. In particular, after the emergence of the idea of an accelerated universe, the role of this parameter has become more important. This is because some recent works (Riess 2001; Amendola 2003; Padmanabhan and Roychowdhury 2003) have demonstrated that the present acceleration is a phenomenon of the recent past and was precede by a decelerating phase. So, during cosmic evolution the deceleration parameter \( q \) must have undergone a change of sign from a positive to a negative value. Eqn. (59) shows that the expression for \( q \) contains a time factor and hence a suitable change of \( c, d_1, d_2 \), its change of sign can be shown. Considering several experimental and theoretical results, one can assume that the matter energy density \( \rho_m^* \) as \( 0.33^{25} \). So, the value of \( q \) comes out to be \(-0.409\), which is in agreement with the present accepted range of \( q \) for an accelerating universe.

Dirac Large Number Hypothesis (LNH) prevent him to admit the variability of the fundamental constants involved in atomic physics. He thought of a possible change in \( G \), which led him to the differential equation. (Cetto et al. 1986)

\[
G(t) = k_1 H(t) = k_2 [H(t)]^{2} [\rho_m(t)]^{frac{12}{}} (64)
\]

where \( k_1 \) and \( k_2 \) are constants of integration, \( H(t) \) is the hubble parameter. From the above equation Dirac obtained \( G(t) \sim \frac{1}{t} \). All the three variants (early Dirac, additive creation and multiplicative creation) tells us that \( \frac{\dot{G}}{G} \), should be inversely proportional to \( t \), where \( \dot{G} \) denotes differentiation with respect to \( t \).

According to Brans-Dicke (1961), \( G \) should vary inversely with time since the scalar field \( \phi(t) \) is time increasing and \( G(t) \) is proportional
to $[\phi(t)]^{-1}$.

To the amount of variation of $G$, we found that, on the data provided by three distant quasars of red shift, $z \sim 3.5$ in favor of an increasing fine structure constant $\alpha$ (Webb at al. 2001; Murphy et al. 2002). Taking the present age of the universe as 14Gyr, it has been estimated that $\frac{\dot{G}}{G} \sim +10^{-15} yr^{-1}$ for kaluza-klein and Einstein-Yang-Mills theories whereas it is of order of $10^{-13} yr^{-1}$ for Randall-Sundrum theory. The data provided by the binary pulsar PSR 1913 + 16, is a very reliable upper bound (Damour et al. 1988), viz.

$$- (1.10 \pm 1.07) \times 10^{-1} yr^{-1} < \frac{\dot{G}}{G} < 0$$ (65)

According to Helio-seismological data (Guenther et al. 1998), the range of $\frac{\dot{G}}{G}$ is considered to be the best upper bound and is given by

$$-1.60 < 10^{-1} yr^{-1} < \frac{\dot{G}}{G} < 0$$ (66)

The data provided by observations of type Ia supernovae (Riess et al. 1998; Perlmutter 1999) gives the best upper bound of the variation of $G$ at cosmological ranges as (Gaztanaga et al. 2002)

$$-10^{-11} yr^{-1} < \frac{\dot{G}}{G} < 0 atz \sim 0.5$$ (67)

Very recently, using the data provided by the pulsating white dwarf star G117-B15A the astereoseismological bound on $\frac{\dot{G}}{G}$ is found to be

$$-2.50 \times 10^{-10} yr^{-1} \leq \frac{\dot{G}}{G} < +4.0 \times 10^{-10} yr^{-1}$$ (68)

while using the same star Biesiada and Malec (2004) has inferred that

$$\text{mod} \frac{\dot{G}}{G} < +4.0 \times 10^{-11} yr^{-1}$$ (69)

Using big bang Nucleosynthesis another recent estimate of variation of $G$ has been obtained (Copi et al. 2004) as

$$-4.0 \times 10^{-13} yr^{-1} \leq \frac{\dot{G}}{G} < +3.0 \times 10^{-13} yr^{-1}$$ (70)

Using eqn. (63), the value of $\frac{\dot{G}}{G}$ obtained as $\frac{\dot{G}}{G} \sim 10^{-12} yr^{-1}$, for some arbitrary choices of the constants. In the discussion majority
of the value of $\dot{G}$ are negative, but in some cases values of $\dot{G}$ are also found to be positive. A negative $\dot{G}$ implies a time decreasing $G$, and a positive $\dot{G}$ means $G$ growing with time.

7 Conclusions

The significance of the present work deals with the modification of gravitational and geometrical aspects of Einstein’s equations with WDF. These are 1) scale invariant theory of gravitation which describes the interaction between matter and gravitation in scale free manner; and 2) the gauge transformation, which represents a change of units of measurements and hence gives a general scaling of physical system. The nature of the cosmological model with modified gravity and with WDF that would reproduce the kinematical history and evolution of perturbation of the universe is investigated.

Here, a non-singular Bianchi type $V I_1$ cosmological model constructed here starts evolving at $T = 0$ with constant volume and expands spatially and contracts temporarily throughout the evolution. As far as matter is concerned the model does not admit either Big bang or Big crunch during evolution from initial epoch to infinite future. The model is flat with unit volume at both initial and infinite future. The total pressure and total energy density with WDF vanish. The matter density $\rho_{WDF(m)}$ in WDF vanishes for $\Lambda = 11$, but $\rho_{WDF(m)} \neq 0$ that leads to unphysical situation.

Moreover, we have attempted to study the scale invariant theory of gravity with WDF governed by Bianchi type II and III space-times with a dirac gauge function $\beta$. However in both the space-times it is found that this theory is not compatible.

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