A survey of Gersten’s conjecture

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Abstract

This article is the extended notes of my survey talk of Gersten’s conjecture given at the workshop “Bousfield classes form a set: a workshop in memory of Tetsusuke Ohkawa” at Nagoya University in August 2015. In the last section, I give an explanation of my recent work of motivic Gersten’s conjecture.

Introduction

This article is the extended notes of my survey talk of Gersten’s conjecture in [Ger73] given at the workshop “Bousfield classes form a set: a workshop in memory of Tetsusuke Ohkawa” at Nagoya University in August 2015. (The slide movie of my talk at the workshop is [Moc15].) In this article, we provide an explanation of a proof of Gersten’s conjecture in [Moc16a] by emphasizing the conceptual idea behind the proof rather than its technical aspects. In the last section, I give an explanation of my recent work of motivic Gersten’s conjecture in [Moc16b].

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1 What is Gersten’s conjecture?

We start by recalling the following result. For \( n = 0, 1 \), the results are classical and for \( n = 2 \), it was given by Spencer Bloch in [Blo74] by utilizing the second universal Chern class map in [Gro68].

**Proposition 1.1.** For a smooth variety \( X \) over a field, we have the canonical isomorphisms

\[
\text{CH}^n(X) \simeq H_{2n}(X, \mathcal{K}_n)
\]

for \( n = 0, 1 \), where \( \mathcal{K}_n \) is the Zariski sheafication of the \( K \)-presheaf \( U \mapsto K^n(U) \).

On the other hand, there are the following spectral sequences. To give a precise statement, we introduce some notations. For a noetherian scheme \( X \), we write \( M_X \) for the category of coherent sheaves on \( X \). There is a filtration

\[
0 \subset \cdots \subset M^2_X \subset M^1_X \subset M^0_X = M_X
\]

by the Serre subcategories \( M^i_X \) of those coherent sheaves whose support has codimension \( \geq i \).

**Proposition 1.2.**

1. (Quillen spectral sequence [Qui73], [Bal09].) For a noetherian scheme \( X \), we have the canonical equivalence

\[
K(X) \simeq H_\cdot(Zar(X); K(\cdot)).
\]

In particular, there exists the strongly convergent spectral sequence

\[
E_1^{p,q}(X) = \bigoplus_{x \in X^p} K_{p-q}(k(x)) \Rightarrow G_{p-q}(X),
\]

where \( X^p \) is the set of points codimension \( p \) in \( X \). Moreover if \( X \) is regular separated, then we have the canonical isomorphism

\[
E_\infty^{p,-p}(X) \simeq \text{CH}^p(X).
\]

2. (Brown-Gersten-Thomason spectral sequence [BG73], [TT90].) For a noetherian scheme of finite Krull dimension \( X \), we have the canonical equivalence

\[
\mathbb{K}(X) \simeq H_{2n}(X; \mathbb{K}(-)).
\]

In particular, there exists the strongly convergent spectral sequence

\[
E_1^{p,q}(X) = H_{2n}(X; \mathbb{K}_q) \Rightarrow \mathbb{K}_{p-q}(X)
\]

where \( \mathbb{K}_q \) is the Zariski sheafification of the presheaf \( U \mapsto \pi_0 K(U) \) of non-connective \( K \)-theory.
To regard the isomorphisms as the isomorphisms of $E_2$-terms of the spectral sequences above, Gersten gave the following observation in [Ger73]. For simplicity, for a commutative noetherian ring $A$ with 1, we write $M_A$ and $M^n_A$ for $M_{\text{Spec } A}$ and $M^n_{\text{Spec } A}$ respectively.

**Proposition 1.3.** The following conditions are equivalent:

1. For any regular separated noetherian scheme $X$, we have the canonical isomorphism between $E_2$-terms of Quillen and Brown-Gersten-Thomason spectral sequences

   $$E_2^{p,q}(X) \cong 1 E_2^{p-q}(X).$$

   In particular the isomorphisms hold for $X$ and any non-negative integer $n$.

2. For any commutative regular local ring $R$ of dimension, $E_1$-terms of Quillen spectral sequence for $\text{Spec } R$ yields an exact sequence

   \begin{align*}
   0 & \to K_n(R) \to K_n(\frac{R}{\mathfrak{m}}) \to \bigoplus_{\mathfrak{p} \mid \mathfrak{m}} K_{n-1}(k(\mathfrak{p})) \to \bigoplus_{\mathfrak{p} \mid \mathfrak{m}} K_{n-2}(k(\mathfrak{p})) \to \cdots .
   \end{align*}

3. For any commutative regular local ring $R$ and natural number $1 \leq p \leq \dim R$, the canonical inclusion $M^1_R \hookrightarrow M^{p-1}_R$ induces the zero map on $K$-theory

   $$K(M^1_R) \to K(M^{p-1}_R)$$

   where $K(M^1_R)$ denotes the $K$-theory of the abelian category $M^1_R$.

Here is Gersten’s conjecture:

**Conjecture 1.4 (Gersten’s conjecture).** The conditions above are true for any commutative regular local ring.

**Historical Note**

The conjecture has been proved in the following cases:

**The case of general dimension**

1. If $A$ is of equi-characteristic, then Gersten’s conjecture for $A$ is true. We refer to [Qui73] for special cases, and the general cases [Pan03] can be deduced from limit argument and Popescu’s general Néron desingularization [Pop86]. (For the case of commutative discrete valuation rings, it was first proved by Sherman [She78]).

2. If $A$ is smooth over some commutative discrete valuation ring $S$ and satisfies some condition, then Gersten’s conjecture for $A$ is true. [Blo86].

3. If $A$ is smooth over some commutative discrete valuation ring $S$ and if we accept Gersten’s conjecture for $S$, then Gersten’s conjecture for $A$ is true. [GL87]. See also [RS90].

**The case for Grothendieck groups**

Gersten’s conjecture for Grothendieck groups has several equivalent forms. Namely we can show the following three conditions are equivalent. See [CF68], [Lev85], [Dut93] and [Dut95].

1. (Gersten’s conjecture for Grothendieck groups). For any commutative regular local ring $R$ and natural number $1 \leq p \leq \dim R$, the canonical inclusion $M^1_R \hookrightarrow M^{p-1}_R$ induces the zero map on Grothendieck groups

   $$K_0(M^1_R) \to K_0(M^{p-1}_R).$$
2. **(Generator conjecture).** For any commutative regular local ring \( R \) and any natural number \( 0 \leq p \leq \dim R \), the Grothendieck group \( K_0(\mathcal{M}_R^p) \) is generated by cyclic modules \( R/(f_1, \ldots, f_p) \) where the sequence \( f_1, \ldots, f_p \) forms an \( R \)-regular sequence.

3. **(Claborn and Fossum conjecture).** For any commutative regular local ring \( R \), the Chow homology group \( CH_k(\text{Spec} R) \) is trivial for any \( k < \dim R \).

For historical notes of the generator conjecture, please see [Moc13a].

The case for singular varieties

Gersten’s conjecture for non-regular rings is in general false in the literal sense of the word and several appropriate modified versions are studied by many authors. See [DHM85], [Smo87], [Lev88], [Bal09], [HM10] and [Moc13a]. See also [Mor15] and [KM16].

Other cohomology theories

1. For torsion coefficient \( K \)-theory, Gersten’s conjecture for a commutative discrete valuation ring is true. [Gii86], [GL00].
2. For Gersten’s conjecture for Milnor \( K \)-theory, see [Ker09], [Dah15].
3. For Gersten’s conjecture for Witt groups, see [Par82], [OP99], [BW02] and [BGPW03].
4. For an analogue of Gersten’s conjecture for bivariant \( K \)-theory, see [Wal00].
5. In the proof of geometric case of Gersten’s conjecture by Quillen in [Qui73], he introduced a strengthening of Noether normalization theorem. There are several variants of Noether normalization theorem in [Oja80], [GS88], [Gab94] and [Wak98] and by utilizing them, there exists Gersten’s conjecture type theorem for universal exactness (see [Gra85]) for Cousin complexes (see [Har66]) of certain cohomology theories. For axiomatic approaches of these topics, see [BO74], [C-THK97]. See also [Gab93], [C-T95], [Lev08] and [Lev13].
6. For an analogue of Gersten’s conjecture for Hochschild coniveau spectral sequences, see [BW16].
7. For an analogue of Gersten’s conjecture for infinitesimal theory, see [DHY15].
8. For Gersten’s complexes for homotopy invariant Zariski sheaves with transfers, see [Voc00], [MVW06 Lecture 24]. See also [SZ03]. For injectivity result for pseudo pretheory, see [FS02].

Logical connections with other conjectures

1. Some conjectures imply that Gersten’s conjecture for a commutative discrete valuation ring is true. [She89].
2. Parshin conjecture in [Bei84] implies Gersten’s conjecture of motivic cohomology for a localization of smooth varieties over a Dedekind ring. (see [Gei04].) It is also known that Tate-Beilinson conjecture (see [Tat65], [Bei87] and [Tat94]) implies Parshin conjecture. (see [Gei98].)

Counterexample for non-commutative discrete valuation rings (Due to Kazuya Kato)

In Gersten’s conjecture, the assumption of commutativity is essential. Let \( D \) be a skew field finite over \( \mathbb{Q}_p \), \( A \) its integer ring and \( a \) its prime element. As the inner automorphism of \( a \) induces non-trivial automorphism on its residue field, we have \( x \in A^\times \) with \( y = axa^{-1}x^{-1} \) is non vanishing in its residue field, a fortiori in \( K_1(A) = (A^\times)^{ab} \). On the other hand \( y \) is a commutator in \( D^\times \). Hence it turns out that the canonical map \( K_1(A) \to K_1(D) = (D^\times)^{ab} \) is not injective.

2 Idea of the proof

My idea of how to prove Gersten’s conjecture has come from weight argument of Adams operations in [GS87] and [GS99]. In my viewpoint, difficulty of solving Gersten’s conjecture consists of ring theoretic side and homotopy theoretic side. We will explain ideas about how to overcome each difficulty.
Ring theoretic side

Combining the results in [GS87] and [IT90], for a commutative regular local ring \( R \), there exists Adams operations \( \{ \varphi_k \}_{k \geq 0} \) on \( \mathcal{K}_R^0 \) and we have the equality

\[
\varphi_k([R/(f_1, \ldots, f_p)]) = k^p[R/(f_1, \ldots, f_p)],
\]

where a sequence \( f_1, \cdots, f_p \) is an \( R \)-regular sequence. Thus roughly saying, the generator conjecture says that for each \( p \), \( \mathcal{K}_R^0 \) is spanned by objects of weight \( p \) and Gersten’s conjecture could follow from weight argument of Adams operations. We illustrate how to prove that the generator conjecture implies Gersten’s conjecture for \( K_0 \) without using Adams operations.

**Proof.** Let a sequence \( f_1, \cdots, f_p \) be an \( R \)-regular sequence. Then there exists the short exact sequence

\[
0 \to R/(f_1, \cdots, f_{p-1}) \xrightarrow{f_p} R/(f_1, \cdots, f_p-1) \to R/(f_1, \cdots, f_p) \to 0
\]

in \( \mathcal{K}_R^{p-1} \). Thus the class \([R/(f_1, \cdots, f_p)]\) in \( \mathcal{K}_R^p \) goes to

\[
[R/(f_1, \cdots, f_p)] - [R/(f_1, \cdots, f_{p-1})] = 0
\]

in \( \mathcal{K}_R^{p-1} \).

**Strategy 1.**

We will establish and prove a higher analogue of the generator conjecture.

Inspired from the works [Iwa59], [Ser59], [Bou64], [Die86] and [Gra92], we establish a classification theory of modules by utilizing cubes in [Moc13a], [Moc13b] and [MY14]. In this article, we will implicitly use these theory and simplify the arguments in [Moc13a] and [Moc16a].

Homotopy theoretic side

Roughly saying, we will try to compare the following two functors on \( K \)-theory. We denote the category of bounded complexes on \( \mathcal{M}_R^{p-1} \) by \( \text{Ch}_b(\mathcal{M}_R^{p-1}) \).

\[
\mathcal{M}_R^p \to \text{Ch}_b(\mathcal{M}_R^{p-1}),
\]

\[
R/(f_1, \cdots, f_p) \mapsto \begin{cases} 
R/(f_1, \cdots, f_p) & \sim R/(f_1, \cdots, f_p) \\
\downarrow f_p & \sim 0 \\
R/(f_1, \cdots, f_p) & \sim 0 \\
\downarrow \text{id} & \sim 0 \\
R/(f_1, \cdots, f_p) & \sim 0
\end{cases}
\]

The functors above shall be homotopic to each other on \( K \)-theory by the additivity theorem. A problem is that the functors above are not \( 1 \)-functorial!! We need to a good notion of \( K \)-theory for higher category theory or need to discuss more subtle argument for such exotic functors.

**Strategy 2.**

We give a modified definition of algebraic \( K \)-theory in a particular situation and establish a technique of rectifying lax functors to \( 1 \)-functors and by utilizing this definition and these techniques, we will treat such exotic functors inside the classical Waldhausen \( K \)-theory.

3 Strategy of the proof

Let \( A \) be a commutative noetherian ring with \( 1 \) and let \( I \) be an ideal of \( A \) with codimension \( Y = \text{Spec} A \). Let \( \mathcal{M}_A^I \) be a full subcategory of \( \mathcal{M}_A^0 \) consisting of those modules \( M \) supported on \( Y = \text{Spec} I \) and \( \mathcal{M}_A^{\text{red}} \) be a full subcategory of \( \mathcal{M}_A^1 \) consisting of those modules \( M \) such that \( IM \) are trivial. We call a module in \( \mathcal{M}_A^{\text{red}} \) a reduced module with respect to \( I \). Let \( \mathcal{D}_A \) be the category of finitely generated projective \( A \)-modules. For a commutative regular local ring \( R \), let \( J \) be an ideal generated by \( R \)-regular sequence \( f_1, \cdots, f_p \) such that \( f_i \) is an prime element for any \( 1 \leq i \leq p \).

First notice the following isomorphisms:
Since $R$ is Cohen-Macaulay, the ordered set of all ideals of $R$ that contains an $R$-regular sequence of length $p$ with usual inclusion is directed. Thus $\mathcal{M}^p_R$ is the filtered limit $\colim_I$ where $I$ runs through any ideal generated by any $R$-regular sequence of length $p$. Thus the isomorphism $\mathbf{I}$ follows from cocontinuity of $K$-theory. The isomorphism $\mathbf{II}$ follows from the resolution theorem and regularity of $R$. Finally the isomorphism $\mathbf{III}$ follows from the d\'evissage theorem. Let $I_S = \{ f_s \}_{s \in S}$ be an $R$-regular sequence such that $f_s$ is a prime element of $R$ for any $s \in S$. We call such a sequence $I_S$ a prime regular sequence. For a commutative noetherian ring $A$ with 1 and for an ideal $I$ of $A$, let $\mathcal{M}_A^I(1)$ and $\mathcal{M}_A^I(1)_{\text{red}}$ be the full subcategory of $\mathcal{M}_A^I$ and $\mathcal{M}_A^I_{\text{red}}$ respectively consisting of those $A$-modules $M$ with projdim$_A M \leq 1$. We fix an element $s \in S$. Since the inclusion functor $\mathcal{P}_{R/\{s\}} \hookrightarrow \mathcal{M}^{S-1}_R$ factors through $\mathcal{M}_{R/\{s\}}(1)$, the problem reduce to the following:

For any prime regular sequence $I_S = \{ f_s \}_{s \in S}$ of $R$ and any element $s \in S$, the inclusion functor $\mathcal{P}_{R/\{s\}} \hookrightarrow \mathcal{M}^{S-1}_R$ induces the zero map

$$K(\mathcal{P}_{R/\{s\}}) \to K(\mathcal{M}^S_R)$$

on $K$-theory.

For simplicity we set $B = R/\{s\} R$ and $g = f_s$. We let $\operatorname{Ch}_b(\mathcal{M}_B(1))$ denote the category of bounded complexes on $\mathcal{M}_B(1)$. (We use homological index notation.) Let $\mathcal{C}$ be the full subcategory of $\operatorname{Ch}_b(\mathcal{M}_B(1))$ consisting of those complexes $x$ such that $x_i = 0$ unless $i = 0$, 1 and $x_0$ is free $B$-modules and the bounded map $d^i : x_1 \to x_0$ is injective and $H_0 x := \text{Coker}(x_1 \xrightarrow{d^i} x_0)$ is annihilated by $g$. We can show that $\mathcal{C}$ is an idempotent exact category such that the inclusion functor $\eta : \mathcal{C} \hookrightarrow \operatorname{Ch}_b(\mathcal{M}_B(1))$ is exact and reflects exactness and the functor $H_0 : \mathcal{C} \to \mathcal{P}_{B/\{s\}}$ is exact. Thus we obtain the commutative diagram

$$\begin{array}{ccc}
K(\mathcal{C}) & \xrightarrow{K(\eta)} & K(\operatorname{Ch}_b(\mathcal{M}_B(1)); \text{qis}) \\
\downarrow K(H_0) & & \downarrow I \\
K(\mathcal{P}_{B/\{s\}}) & \xrightarrow{K(\eta)} & K(\mathcal{M}_B(1))
\end{array}$$

where qis is the class of all quasi-isomorphisms in $\operatorname{Ch}_b(\mathcal{M}_B(1))$ and the map $I$ which is induced from the inclusion functor $\mathcal{M}_B(1) \hookrightarrow \operatorname{Ch}_b(\mathcal{M}_B(1))$ is a homotopy equivalence by Gillet-Waldhausen theorem. We will prove that

1. The map $K(H_0)$ is a split epimorphism in the stable category of spectra (see $\mathbb{L}_4$ and $\mathbb{L}_5$).

2. The map $K(\eta)$ is the zero map in the stable category of spectra. (See $\mathbb{L}_5$)

Assertion 1. corresponds with Strategy 1 and assertion 2. corresponds with Strategy 2 in the previous section.

## 4 Split epimorphism theorem

In this section, we will give a brief proof of assertion that $K(H_0)$ is a split epimorphism in the stable category of spectra.

Let $\mathcal{D}$ be the full subcategory of $\operatorname{Ch}_b(\mathcal{M}_B(1))$ consisting of those complexes $x$ such that $x_i = 0$ unless $i = 0$, 1 and the bounded map $d^i : x_1 \to x_0$ is injective and $H_0 x := \text{Coker}(x_1 \xrightarrow{d^i} x_0)$ is in $\mathcal{M}_B^{gB}(1)$. We can show that $\mathcal{D}$ is an exact category such that the inclusion functor $\mathcal{D} \hookrightarrow \operatorname{Ch}_b(\mathcal{M}_B(1))$ is exact and reflects exactness and we can also show that the functor $H_0 : \mathcal{D} \to \mathcal{M}_B^{gB}(1)$ is exact. Thus we obtain the commutative square below

$$\begin{array}{ccc}
K(\mathcal{C}) & \xrightarrow{K(\eta)} & K(\mathcal{D}) \\
\downarrow K(H_0) & & \downarrow K(H_0) \\
K(\mathcal{P}_{B/\{s\}}) & \xrightarrow{K(\eta)} & K(\mathcal{M}_B^{gB}(1))
\end{array}$$
where the horizontal maps are induced from the inclusion functors. Since the functor $H_0: \mathcal{D} \to \mathcal{A}_{B, \text{red}}(1)$ admits a section which is defined by sending an object $x$ in $\mathcal{A}_{B, \text{red}}(1)$ to the complex $[0 \to x]$ in $\mathcal{D}$, the right vertical map in the diagram above is a split epimorphism. Moreover the inclusion functors $\mathcal{D}_{B/gB} \hookrightarrow \mathcal{A}_{B, \text{red}}(1)$ and $\mathcal{C} \hookrightarrow \mathcal{D}$ induce equivalences of triangulated categories on bounded derived categories respectively. (Compare [Moc13a, 2.21] and [Moc16a, 2.1.1].) Thus the horizontal maps in the diagram above are homotopy equivalences and the left vertical map is also a split epimorphism in the stable category of spectra.

5 Zero map theorem

In this section we will give an outline of the proof of assertion that $K(\eta)$ is the zero map in the stable category of spectra. Let $\mathcal{B}$ be the full subcategory of $\text{Ch}_0, \mathcal{A}_B(1)$ consisting of those complexes $x$ such that $x_i = 0$ unless $i = 0$ or $i = 1$. Let $\eta: \mathcal{D} \to \mathcal{A}_B(1)$ (i = 0, 1) be an exact functor defined by sending an object $x$ in $\mathcal{D}$ to $x_1$ in $\mathcal{A}_B(1)$.

By the additivity theorem, the map $\eta_1 \times \eta_2: i_S \mathcal{B} \to i_S \mathcal{A}_{B}(1) \times i_S \mathcal{A}_{B}(1)$ is a homotopy equivalence. Let $j: \mathcal{B} \to \text{Ch}_0(\mathcal{A}_B(1))$ be the inclusion functor.

We wish to define two exact ‘functors’ $\mu_1$, $\mu_2: \mathcal{C} \to \mathcal{B}$ which satisfy the following conditions:

1. We have the equality
   \[ \eta_1 \times \eta_2 \mu_1 = \eta_1 \times \eta_2 \mu_2. \] (11)

2. There are natural transformations $\eta \to j \mu_1$ and $0 \to j \mu_2$ such that all components are quasi-isomorphisms.

Then we have the equalities
\[ K(\eta) = K(j \mu_1) = K(j)K(\eta_1 \times \eta_2)^{-1}K(\eta_1 \times \eta_2 \mu_1) = K(j)K(\eta_1 \times \eta_2)^{-1}K(\eta_1 \times \eta_2 \mu_2) = K(j \mu_2) = 0. \]

To define the ‘functors’ $\mu_i$ (i = 1, 2), we analyze morphisms in $\mathcal{C}$.

5.1 Structure of $\mathcal{C}$

Definition 5.1. (Compare [Moc16a, 1.1.3].) For a pair of non-negative integers $(n,m)$, we write $(n,m)_B$ for the complex of the form
\[
\begin{pmatrix}
B^{\oplus n} & B^{\oplus m} \\
\downarrow & \downarrow \\
\bigg( gE_n & 0 \\
0 & E_m \bigg)
\end{pmatrix}
\] in $\mathcal{C}$ where $E_k$ is the $k \times k$ unit matrix.

Lemma 5.2.

(1) (Compare [Moc16a, 1.2.10., 1.2.13.]) Let $n$ be a positive integer. For any endomorphism $a: (n,0)_B \to (n,0)_B$, the following conditions are equivalent.

(i) $a$ is an isomorphism.

(ii) $a$ is a quasi-isomorphism.

(2) (Compare [Moc13a, 2.17.]) An object in $\mathcal{C}$ is projective. In particular, $\mathcal{C}$ is a semi-simple exact category.

(3) (Compare [Moc16a, 1.2.15.]) For any object $x$ in $\mathcal{C}$, there exists a pair of non-negative integers $(n,m)$ such that $x$ is isomorphic to $(n,m)_B$.

Proof. (1) We assume condition (ii). In the commutative diagram below
\[
\begin{array}{ccc}
B^{\oplus n} & \xrightarrow{gE_n} & B^{\oplus n} \\
\downarrow a_1 & & \downarrow a_0 \\
B^{\oplus n} & \xrightarrow{gE_n} & H_0((n,0)_B)
\end{array}
\]

first we will prove that $a_0$ is an isomorphism. Then $a_1$ is also an isomorphism by the five lemma. By taking determinant of $a_0$, we shall assume that $n = 1$. Then assertion follows from Nakayama’s lemma.

(2) Let $t: y \to z$ be an admissible epimorphism in $\mathcal{C}$ and let $f: x \to z$ be a morphism in $\mathcal{C}$. Then since $H_0(x)$ is a projective $B/gB$-module, there exist a homomorphism of $B/gB$-modules $\sigma: H_0(x) \to H_0(y)$ such that
H_0(t)\sigma = H_0(f)$. Since $x$ is a complex of free $B$-modules, there exists a morphism of complexes $s': x \to y$ such that $H_0(s') = \sigma$ and $ts'$ is chain homotopic to $f$ by [Weig94] Comparison theorem 2.2.6. Namely there is a map $h: x_0 \to z_1$ such that $(f-ts')_0 = d^f h$ and $(f-ts')_1 = hd^s$.

\[
\begin{array}{c}
\begin{array}{c}
 x_1 \\ (f-ts')_1
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
 x_2 \\ d^f
\end{array}
\begin{array}{c}
 h \\ \downarrow
\end{array}
\begin{array}{c}
 z_1 \\
 d^s
\end{array} \\
\begin{array}{c}
 x_0 \\
 (f-ts')_0
\end{array}
\end{array}
\]

Since $x_0$ is projective, there is a map $u: x_0 \to y_1$ such that $t_1 u = h$. We set $s_1 := s'_1 + ud^s$ and $s_0 := s'_0 + d^s u$. Then we can check that $s$ is a morphism of complexes of $B$-modules and $f = ts$.

(3) By considering $x \otimes_B B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we notice that $x_1$ and $x_0$ are same rank. Thus we shall assume $x_1 = x_0 = B^{\oplus m}$.

First we assume that $x$ is acyclic. Then the boundary map $d^x: x_1 \to x_0$ is invertible and

\[
\begin{bmatrix}
 x_1 \\
 \downarrow d^x \\
 x_0
\end{bmatrix} \begin{bmatrix}
 d^x \\
 \downarrow \text{id}
\end{bmatrix} \begin{bmatrix}
 B^{\oplus m} \\
 \downarrow \text{id}
\end{bmatrix}
\]

gives an isomorphism between $x$ and $(0,m)_B$. Thus we obtain the result in this case.

Next assume that $H_0(x) \neq 0$. Then since $H_0(x)$ is a finitely generated projective $B/gB$-module, there is a positive integer $n$ and an isomorphism $\sigma: (B/gB)^{\oplus n} \to H_0(x)$. Then by [Weig94] Comparison theorem 2.2.6., there exists morphisms of complexes in $\mathscr{C}$, $(n,0)_B \xrightarrow{a} x$ and $x \xrightarrow{b} (n,0)_B$ such that $H_0(a) = \sigma$ and $H_0(b) = \sigma^{-1}$. Thus by (1), $ba$ is an isomorphism. By replacing $a$ with $a(ba)^{-1}$, we shall assume that $ba = \text{id}$. Hence there exists a complex $y$ in $\mathscr{C}$ and a split exact sequence:

\[
(n,0)_B \xrightarrow{a} x \xrightarrow{y}.
\]

Since boundary maps of objects in $\mathscr{C}$ are injective, the functor $H_0$ from $\mathscr{C}$ to the category of finitely generated projective $B/gB$-modules is exact. By taking $H_0$ to the sequence [12], it turns out that $y$ is acyclic and by the first paragraph, we shall assume that $y$ is of the form $(0,m)_B$. Hence $x$ is isomorphic to $(n,m)_B$. \hfill \Box

**Remark 5.3 (Morphisms of $\mathscr{C}$).** (Compare [Moc21a] 1.2.16.) We can denote a morphism $\varphi: (n,m)_B \to (n',m')_B$ of $\mathscr{C}$ by

\[
\begin{bmatrix}
 B^{\oplus n} \oplus B^{\oplus m} \\
 \downarrow (gE_n \oplus 0) \\
 B^{\oplus n} \oplus B^{\oplus m}
\end{bmatrix}
\begin{bmatrix}
 \varphi_l \\
 \downarrow \varphi_n \\
 \varphi_0
\end{bmatrix}
\begin{bmatrix}
 B^{\oplus n} \oplus B^{\oplus m} \\
 \downarrow (gE'_n \oplus 0) \\
 B^{\oplus n} \oplus B^{\oplus m}
\end{bmatrix}
\]

with $\varphi_l = \begin{pmatrix} \varphi_{(n',n)} & \varphi_{(n',m)} \\ g\varphi_{(m',n)} & g\varphi_{(m',m)} \end{pmatrix}$ and $\varphi_n = \begin{pmatrix} \varphi_{(n,n)} \\ \varphi_{(m,m)} \end{pmatrix}$ where $\varphi_{(i,j)}$ are $i \times j$ matrices whose coefficients are in $B$. In this case we write

\[
\begin{pmatrix}
 \varphi_{(n,n)} \\
 \varphi_{(m,m)}
\end{pmatrix}
\begin{pmatrix}
 \varphi_{(n',n)} \\
 \varphi_{(m',m)}
\end{pmatrix}
\]

for $\varphi$. In this matrix presentation of morphisms, the composition of morphisms between objects $(n,m)_B \xrightarrow{\varphi} (n',m')_B \xrightarrow{\psi} (n'',m'')_B$ in $\mathscr{C}$ is described by

\[
\begin{pmatrix}
 \psi_{(n',n')} & \psi_{(n',m')} \\ \psi_{(m',n')} & \psi_{(m',m')}
\end{pmatrix}
\begin{pmatrix}
 \varphi_{(n',n')} \\
 \varphi_{(m',n')}
\end{pmatrix}
\begin{pmatrix}
 \psi_{(n'',n'')} \\
 \psi_{(m'',m'')}
\end{pmatrix}
\begin{pmatrix}
 \varphi_{(n',m')} \\
 \varphi_{(m',m')}
\end{pmatrix}
\]

\[
\begin{pmatrix}
 \psi_{(n'',n'')} \varphi_{(n',n')} + g\psi_{(m'',n'')} \varphi_{(m',n')} & \psi_{(n'',m'')} \varphi_{(n',m')} + g\psi_{(m'',m'')} \varphi_{(m',m')} \\ \psi_{(m'',n'')} \varphi_{(n',n')} + g\psi_{(m'',m'')} \varphi_{(m',n')} & \psi_{(m'',m'')} \varphi_{(m',m')} + g\psi_{(m'',n'')} \varphi_{(m',m')} \end{pmatrix}
\]

Thus the category $\mathscr{C}$ is categorical equivalent to the category whose objects are ordered pair of non-negative integers $(n,m)$ and whose morphisms from an object $(n,m)$ to $(n',m')$ are $2 \times 2$ matrices of the form [12] of $i \times j$ matrices $\varphi_{(i,j)}$ whose coefficients are in $B$ and compositions are given by the formula [14]. We sometimes identify these two categories.
5.2 Modified algebraic $K$-theory

A candidate of a pair of $\mu_i : \mathcal{C} \to \mathcal{D}$ ($i = 1, 2$) are following. We define $\mu'_1, \mu'_2 : \mathcal{C} \to \mathcal{D}$ to be associations by sending an object $(n, m)_B$ in $\mathcal{C}$ to $(n, 0)_B$ and $(0, n)_B$ respectively and a morphism

$$\varphi = (\varphi_{(n', n)} \varphi_{(n', m)} : (n, m)_B \to (n', m')_B$$

in $\mathcal{C}$ to

$$[B^n \phi_{(n', n)}]$$

if $gE_n \to gE_{n'}$ and

$$[B^n \phi_{(n', m)}]$$

respectively. Notice that they are not 1-functors.

We need to make revisions in the previous idea. We introduce a modified version of algebraic $K$-theory of $\mathcal{C}$.

**Definition 5.4** (Triangular morphisms). (Compare [Moc16a, 2.3.5.].) We say that a morphism $\varphi : (n, m)_B \to (n', m')_B$ in $\mathcal{C}$ of the form

is an upper triangular if $\varphi_{(n', n)}$ is the zero morphism, and say that $\varphi$ is a lower triangular if $\varphi_{(n', m)}$ is the zero morphism. We denote the class of all upper triangular isomorphisms in $\mathcal{C}$ by $i^\triangle$. Next we define $S^\triangle\mathcal{C}$ to be a simplicial subcategory of $S\mathcal{C}$ consisting of those objects $x$ such that $x(i \leq j)$ is a lower triangular morphism for each $i \leq j$. For $n = 0, 1, 2$, we define $S^\triangle\mathcal{C}$ to be a simplicial subcategory of $S\mathcal{C}$ consisting of those objects $x$ such that $x(i \leq j)$ is a lower triangular morphism for each $i \leq j$.

**Proposition 5.5.**

(1) (Compare [Moc16a, 2.3.5].) The inclusion functor $iS^\triangle\mathcal{C} \to iS\mathcal{C}$ is a homotopy equivalence.

(2) (Compare [Moc16a, 2.3.6].) The inclusion functor $i^\triangle S^\triangle\mathcal{C} \to iS^\triangle\mathcal{C}$ is a homotopy equivalence.

(3) (Compare [Moc16a, 2.3.7].) The associations $\mu'_1 : \mu'_2 : \mathcal{C} \to \text{Ch}_{\text{b}(\mathcal{B}(1))}$ induces simplicial maps $\mu_1, \mu_2 : i^\triangle S^\triangle\mathcal{C} \to iS\mathcal{C}$ respectively.

(4) (Compare [Moc16a, 2.3.7].) $\mu_1$ is homotopic to $\mu_2$.

**Proof.** (1) Since $\mathcal{C}$ is semi-simple by Lemma 5.3 (2), the inclusion functor $k : iS^\triangle\mathcal{C} \to iS\mathcal{C}$ is an equivalence of categories for each degree. Therefore the inclusion functor $k$ induces a weak homotopy equivalence $NiS^\triangle\mathcal{C} \to NiS\mathcal{C}$.

(2) First for non-negative integer $n$, let $i_n\mathcal{C}$ be the full subcategory of $\mathcal{C}^{|n|}$ the functor category from the totally ordered set $[n] = \{0 < 1 < \cdots < n\}$ to $\mathcal{C}$ consisting of those objects $x : [n] \to \mathcal{C}$ such that $x(i \leq j)$ is an isomorphism in $\mathcal{C}$ for any $0 \leq i \leq n - 1$. Next for integers $n \geq 1$ and $n - 1 \geq k \geq 0$, let $i^\triangle_n\mathcal{C}^{(k)}$ be the full subcategory of $i_n\mathcal{C}$ consisting of those objects $x : [n] \to \mathcal{C}$ such that $x(i \leq j)$ is in $i^\triangle$ for any $k \leq i \leq n - 1$. In particular $i^\triangle_1\mathcal{C}^{(0)} = i^\triangle_1\mathcal{C}$ and by convention, we set $i^\triangle_n\mathcal{C}^{(n)} = i^\triangle_n\mathcal{C}$.

For each $0 \leq k \leq n - 1$, we will define $q_k : i^\triangle_n\mathcal{C}^{(k+1)} \to i^\triangle_n\mathcal{C}^{(k)}$ to be an exact functor as follows. First for any object $z$ in $i^\triangle_n\mathcal{C}^{(k+1)}$, we shall assume that all $z(i)$ are the same object, namely $z(0) = z(1) = \cdots = z(n)$. Then we define $\alpha_z$ to be an isomorphism of $z(i)$ in $\mathcal{C}$ by setting $\alpha_z := UT(z(k \leq k + 1))$. Here for the definition of the upper triangulation $UT$ of $z(k \leq k + 1)$, see Definition 5.3 below. Next for an object $x : [n] \to \mathcal{C}$ and a morphism $x \to y$ in $i^\triangle_n\mathcal{C}^{(k+1)}$, we define $q_k(x) : [n] \to \mathcal{C}$ and $q_k(\theta) : q_k(x) \to q_k(y)$ to be an object and a morphism in $i^\triangle_n\mathcal{C}^{(k)}$ respectively by setting

$$q_k(x)(i) := x(i),$$

$$q_k(x)(i + 1) := \begin{cases} \alpha_z^{-1}(x(k - 1 \leq k)) & \text{if } i = k - 1 \\ x(k \leq k + 1) & \text{if } i = k \\ x(i \leq i + 1) & \text{otherwise}, \end{cases}$$

$$q_k(\theta)(i) := \begin{cases} \alpha_z^{-1}(\theta(k)) & \text{if } i = k \\ \theta(i) & \text{otherwise}. \end{cases}$$

Obviously $q_kj_k = \text{id}$. We define $j^\mathcal{C} : j_kq_k \to \text{id}$ to be a natural equivalence by setting for any object $x$ in $i^\triangle_n\mathcal{C}^{(k+1)}$

$$j^\mathcal{C}(x)(i) := \begin{cases} \alpha_z & \text{if } i = k \\ \text{id}_x & \text{otherwise}. \end{cases}$$

Let $\gamma^\mathcal{C} := \text{Ob}\mathcal{C}$ be a variant of $\gamma = \text{Ob}\mathcal{S}$-construction. Notice that there is a natural identification $i_n\mathcal{C}^{(l)} = i_n\mathcal{S}^{\triangle\mathcal{C}}^{(l)}$ for any $0 \leq l \leq n$. We will show that $\gamma$ induces a simplicial homotopy between the maps...
Proof. \( C \in \text{object} \) be an isomorphism in \( C \). Thus we complete the proof.

Namely we have the equations:

\[ \text{Definition 5.6} \]

\[ \text{Definition 5.7}. \]

Obviously UD is an involution and an exact functor. We call UD the upside-down involution. (Compare [Moc16a, 1.2.17.].) Let \( x = [x_1 \rightarrow x_0] \) be an object in \( C \). Since \( d^x \) is a monomorphism and \( x_1 \) and \( x_0 \) have the same rank, \( d^x \) is invertible in \( B \left( \frac{1}{g} \right) \) and \( g^{-1} : x_0 \rightarrow x_1 \) is a morphism of \( B \)-modules. We define UD: \( C \rightarrow C \) to be a functor by sending an object \( [x_1 \rightarrow x_0] \) to \( [x_0 \rightarrow x_1] \) and a morphism \( \varphi : x \rightarrow y \) to

\[ \begin{bmatrix} x_0 \\ g(d^{-1}) \downarrow \\ x_1 \end{bmatrix} \xrightarrow{\varphi} \begin{bmatrix} y_0 \\ g(d'^{-1}) \downarrow \\ y_1 \end{bmatrix}. \]

Namely we have the equations:

\[ \text{UD}((n,m)_B) = (m,n)_B, \]

\[ \text{UD} \left( \begin{bmatrix} \varphi(n,m) & \varphi(n',m) \\ \varphi(m,n) & \varphi(m',n) \end{bmatrix} \right) : (n,m)_B \rightarrow (n',m')_B = \begin{bmatrix} \varphi(n',m) & \varphi(m',n) \\ \varphi(m,n) & \varphi(n,m) \end{bmatrix}. \]

Obviously UD is an involution and an exact functor. We call UD the upside-down involution.

Lemma 5.7. (Compare [Moc16a, 1.2.18.].) Let \( \varphi = \begin{bmatrix} \varphi(n,n) & \varphi(n,m) \\ \varphi(m,n) & \varphi(m,m) \end{bmatrix} : (n,m)_B \rightarrow (n,m)_B \) be an isomorphism in \( C \). Then \( \varphi_{(n,n)} \) and \( \varphi_{(m,m)} \) are invertible.

Proof. For \( \varphi_{(n,m)} \), assertion follows from Lemma 5.7 (1). For \( \varphi_{(m,m)} \), we apply the same lemma to UD(\( \varphi \)).

Definition 5.8 (Upper triangulation). (Compare [Moc16a, 2.2.5.].) Let

\[ \varphi = \begin{bmatrix} \varphi(n,n) & \varphi(n,m) \\ \varphi(m,n) & \varphi(m,m) \end{bmatrix} : (n,m)_B \rightarrow (n,m)_B \]

be an isomorphism in \( C \). By Lemma 5.7, \( \varphi_{(m,m)} \) is an isomorphism. We define UT(\( \varphi \)) : (n,m)_B \rightarrow (n,m)_B to be a lower triangular isomorphism by the formula UT(\( \varphi \)) := \( \begin{bmatrix} E_n & 0 \\ -\varphi_{(m,m)}^{-1} \varphi_{(n,m)} & E_n \end{bmatrix} \). Then we have an equality

\[ \varphi \text{UT}(\varphi) = \begin{bmatrix} \varphi(n,n) - g \varphi_{(n,m)} \varphi_{(m,m)}^{-1} \varphi(n,m) \\ 0 \end{bmatrix}. \]

(20)

We call UT(\( \varphi \)) the upper triangulation of \( \varphi \). Notice that if \( \varphi \) is upper triangular, then UT(\( \varphi \)) = id_{(n,m)_B}.
Lemma 5.9 (Exact sequences in \( \mathcal{C} \)). (Compare [Moc16a, 1.2.19.]) Let

\[
(n, 0)_B \xrightarrow{\alpha} (n', 0)_B \xrightarrow{\beta} (n'', 0)_B
\]

be a sequence of morphisms in \( \mathcal{C} \) such that \( \beta \alpha = 0 \). If the induced sequence of projective \( B/gB \)-modules

\[
H_0((n, 0)_B) \xrightarrow{H_0(\alpha)} H_0((n', 0)_B) \xrightarrow{H_0(\beta)} H_0((n'', 0)_B)
\]

is exact, then the sequence (21) is also (split) exact.

Proof. Since the sequence (22) is an exact sequence of projective \( B/gB \)-modules, it is a split exact sequence and hence we have the equality \( n' = n + n'' \) and there exists a homomorphism \( \gamma \): \( H_0((n', 0)_B) \to H_0((n'', 0)_B) \) such that \( H_0(\beta) \gamma = \text{id}_{H_0((n'', 0)_B)} \). Then by [Wei94, Comparison theorem 2.2.6.], there is a morphism of complexes of \( B \)-modules \( \gamma \): \( (n'', 0)_B \to (n', 0)_B \) such that \( H_0(\gamma) = \gamma \). Since \( \beta \gamma \) is an isomorphism by Lemma 5.2 (1), by replacing \( \gamma \) with \( \gamma(\beta \gamma)^{-1} \), we shall assume that \( \beta \gamma = \text{id}_{(n'', 0)_B} \).

Therefore there is a commutative diagram

\[
\begin{align*}
(n, 0)_B & \xrightarrow{\alpha} (n', 0)_B \xrightarrow{\beta} (n'', 0)_B \\
\text{dotted line} & \xrightarrow{\delta} (n', 0)_B \xrightarrow{\beta} (n'', 0)_B
\end{align*}
\]

such that the bottom line is exact. Here the dotted arrow \( \delta \) is induced from the universality of \( \text{Ker} \beta \).

By applying the functor \( H_0 \) to the diagram above and by the five lemma, it turns out that \( H_0(\delta) \) is an isomorphism of projective \( B/gB \)-modules and hence \( \delta \) is also an isomorphism by Lemma 5.2 (1). We complete the proof.

5.3 Homotopy natural transformations

We denote the simplicial morphism \( i^\Lambda S^7 \mathcal{C} \to \text{qis} \mathcal{S} \text{Ch}_0(\mathcal{M}) \) induced from the inclusion functor \( \eta \): \( \mathcal{C} \to \text{Ch}_0(\mathcal{M}) \) by the same letter \( \eta \). For simplicial functors \( \eta, j\mu_1, j\mu_2, 0; i^\Lambda S^7 \mathcal{C} \to \text{qis} \mathcal{S} \text{Ch}_0(\mathcal{M}) \), there is a canonical natural transformation \( j\mu_2 \to 0 \) and we wish to define a canonical natural transformation \( \eta \to j\mu_1 \). A candidate of \( \eta \to j\mu_1 \) is the following.

For any object \( (n, m)_B \) in \( \mathcal{C} \), we write \( \delta_{(n,m)} : \eta((n,m)_B) \to j\mu_1((n,m)_B) \) for the canonical projection

\[
\begin{pmatrix}
E_n & 0 \\
0 & E_m
\end{pmatrix}
\]

following nice properties. For any morphism \( \varphi = \begin{pmatrix} \varphi_{(n',n)} & \varphi_{(m',m)} \\ \varphi_{(m',n)} & \varphi_{(m',m)} \end{pmatrix} : (n, m)_B \to (n', m')_B \) in \( \mathcal{C} \),

(i) \( \delta_{(n,m)} \) is a chain homotopy equivalence,

(ii) if \( \varphi \) is lower triangular, we have the equality \( j\mu_1(\varphi) \delta_{(n,m)} = \delta_{(n',m)} \eta(\varphi) \),

(iii) if \( \varphi \) is upper triangular, there is a unique chain homotopy between \( j\mu_1(\varphi) \delta_{(n,m)} \) and \( \delta_{(n',m') \eta(\varphi)} \).

Namely since we have the equality

\[
\begin{pmatrix} \varphi_{(n',n)} & \varphi_{(m',m)} \\ \varphi_{(m',n)} & \varphi_{(m',m)} \end{pmatrix} = \begin{pmatrix} 0 & -\varphi_{(m',m)} \\ 0 & 0 \end{pmatrix}
\]

the map

\[
\begin{pmatrix} 0 & -\varphi_{(m',m)} \\ \varphi_{(m',m)} & 0 \end{pmatrix} : B^\oplus m \to B^\oplus m'
\]

gives a chain homotopy between \( j\mu_1(\varphi) \delta_{(n,m)} \) and \( \delta_{(n',m') \eta(\varphi)} \).

\[
\begin{pmatrix}
\varphi_{(m',m)} & 0 \\
0 & \varphi_{(m',m)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & -\varphi_{(m',m)} \\
\varphi_{(m',m)} & 0
\end{pmatrix}.
\]
Thus we establish a theory of homotopy natural transformations. Then it turns out that $\delta$ induces a simplicial homotopy natural transformation $\eta \Rightarrow_{simp} j\mu_I$. (For the definition of simplicial homotopy natural transformations, see Definition 5.13 below.) Therefore by the theory below, it turns out that there exists a zig-zag sequence of simplicial natural transformations which connects $\eta$ and $j\mu_I$. Thus finally we obtain the fact that $\eta$ is homotopic to 0. We complete the proof of zero map theorem.

The rest of this subsection, we will establish a theory of homotopy natural transformations and justify the argument above.

**Conventions.**

For simplicity, we set $\mathcal{E} = Ch_0(\mathcal{M}_B(1))$. The functor $C : \mathcal{E} \to \mathcal{E}$ is given by sending a chain complex $x$ in $\mathcal{E}$ to $C x := \text{Cone} \text{id}_x$ the canonical mapping cone of the identity morphism of $x$. Namely the degree $n$ part of $C x$ is $(C x)_n = x_{n-1} \oplus x_n$ and the degree $n$ boundary morphism $d_n : (C x)_n \to (C x)_{n-1}$ is given by $d_n^x = \begin{pmatrix} -d_{n-1}^x & 0 \\ -\text{id}_{x_{n-1}} & d_n^x \end{pmatrix}$. For any complex $x$, we define $t_x : x \to C x$ and $r_x : C C x \to C x$ to be chain morphisms by setting $(t_x)_n = \begin{pmatrix} 0 \\ \text{id}_x \end{pmatrix}$ and $(r_x)_n = \begin{pmatrix} 0 & \text{id}_{x_{n-1}} & 0 \\ 0 & 0 & \text{id}_x \end{pmatrix}$.

Let $f, g : x \to y$ be a pair of chain morphisms $f, g : x \to y$ in $\mathcal{E}$. Recall that a chain homotopy from $f$ to $g$ is a family of morphisms $\{h_n : x_n \to y_{n+1}\}_{n \in \mathbb{Z}}$ in $\mathcal{M}_B(1)$ indexed by the set of integers such that it satisfies the equality

$$d_{n+1}^y h_n + h_{n-1}^y d_n^x = f_n - g_n$$

for any integer $n$. Then we define $H : C x \to y$ to be a chain morphism by setting $H_n := (-h_{n-1}^y f_n - g_n)$ for any integer $n$. We can show that the equality

$$f - g = H 1_x$$

Conversely if we give a chain map $H : C x \to y$ which satisfies the equality $\text{(24)}$ above, it provides a chain homotopy from $f$ to $g$. We denote this situation by $H : f \Rightarrow_C g$ and we say that $H$ is a $C$-homotopy from $f$ to $g$. We can also show that for any complex $x$ in $\mathcal{E}$, $r_x$ is a $C$-homotopy from $\text{id}_x$ to 0.

Let $[f : x \to x']$ and $[g : y \to y']$ be a pair of objects in $\mathcal{E}^{[1]}$ the morphisms category of $\mathcal{E}$. A $(C)$-homotopy commutative square (from $[f : x \to x']$ to $[g : y \to y']$) is a triple $(a, b, H)$ consisting of chain morphisms $a : x \to y$, $b : x' \to y'$ and $H : C x \to y'$ in $\mathcal{E}$ such that $H t_x = g a - b f$. Namely $H$ is a $C$-homotopy from $g a$ to $b f$.

Let $[f : x \to x']$, $[g : y \to y']$ and $[h : z \to z']$ be a triple of objects in $\mathcal{E}^{[1]}$ and let $(a, b, H)$ and $(a', b', H')$ be homotopy commutative squares from $[f : x \to x']$ to $[g : y \to y']$ and from $[g : y \to y']$ to $[h : z \to z']$ respectively. Then we define $(a', b', H')(a, b, H)$ to be a homotopy commutative square from $[f : x \to x']$ to $[h : z \to z']$ by setting $(a', b', H')(a, b, H) := (a' a, b' b, H' * H)$

where $H' * H$ is a $C$-homotopy from $h a a$ to $b' b f$ given by the formula

$$H' * H := b' H + H' C a.$$  

(25)

We define $\mathcal{E}^{[1]}_h$ to be a category whose objects are morphisms in $\mathcal{E}$ and whose morphisms are homotopy commutative squares and compositions of morphisms are given by the formula $\text{(25)}$ and we define $\mathcal{E}^{[1]}_h \to \mathcal{E}^{[1]}$ to be a functor by sending an object $[f : x \to x']$ to $[f : x \to x']$ and a morphism $(a, b) : [f : x \to x'] \to [g : y \to y']$ to $(a, b, 0) : [f : x \to x'] \to [g : y \to y']$. By this functor, we regard $\mathcal{E}^{[1]}_h$ as a subcategory of $\mathcal{E}^{[1]}$.

We define $Y : \mathcal{E}^{[1]}_h \to \mathcal{E}$ to be a functor by sending an object $[f : x \to y]$ to $Y(f) := y \oplus C x$ and a homotopy commutative square $(a, b, H) : [f : x \to y] \to [f' : x' \to y']$ to $Y(a, b, H) := \begin{pmatrix} b & H \\ 0 & C a \end{pmatrix}$.

We write $s$ and $t$ for the functors $\mathcal{E}^{[1]}_h \to \mathcal{E}$ which sending an object $[f : x \to y]$ to $x$ and $y$ respectively. We define $j_1 : s \to Y$ and $j_2 : t \to Y$ to be natural transformations by setting $j_{1 f} := \begin{pmatrix} f & -H \\ -1_x & 0 \end{pmatrix}$ and $j_{2 f} := \begin{pmatrix} \text{id}_x & 0 \\ 0 & 0 \end{pmatrix}$ respectively for any object $[f : x \to y]$ in $\mathcal{E}^{[1]}_h$.

**Definition 5.10** (Homotopy natural transformations). Let $\mathcal{F}$ be a category and let $f, g : \mathcal{F} \to \mathcal{E}$ be a pair of functors. A homotopy natural transformation (from $f$ to $g$) is consisting of a family of morphisms $\{\theta_i : f_i \to g_i\}_{i \in \text{Ob} \mathcal{F}}$ indexed by the class of objects of $\mathcal{F}$ and a family of $C$-homotopies $\{\theta_0 \circ \theta_i = C \theta_i f_0\}_{i : j}$ indexed by the class of morphisms of $\mathcal{F}$ such that for any object $i$ of $\mathcal{F}$, $\theta_0 i = 0$ and...
for any pair of composable morphisms $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathcal{F}$, $\theta_{ba} = \theta_b \circ \theta_a = (g_b \theta_a + \theta_b C f_a)$. We denote this situation by $\theta : f \Rightarrow g$. For a usual natural transformation $\kappa : f \Rightarrow g$, we regard it as a homotopy natural transformation by setting $\kappa_a = 0$ for any morphism $a : i \Rightarrow j$ in $\mathcal{F}$.

Let $h$ and $k$ be another functors from $\mathcal{F}$ to $\mathcal{E}$ and let $\alpha : f \Rightarrow g$ and $\gamma : h \Rightarrow k$ be natural transformations and $\beta : g \Rightarrow h$ a homotopy natural transformation. We define $\beta \alpha : f \Rightarrow h$ and $\gamma \beta : g \Rightarrow k$ to be homotopy natural transformations by setting for any object $i$ in $\mathcal{F}$, $(\beta \alpha)_i := \beta_i \alpha_i$ and $(\gamma \beta)_i := \gamma_i \beta_i$ and for any morphism $a : i \Rightarrow j$ in $\mathcal{F}$, $(\beta \alpha)_a := \beta_a \circ \alpha_a$ and $(\gamma \beta)_a = \gamma_a \beta_a$.

**Example 5.11** (Homotopy natural transformations). We define $\varepsilon : s \Rightarrow t$ and $p : Y \Rightarrow t$ to be homotopy natural transformations between functors $\mathcal{E} \Rightarrow \mathcal{F}$ by setting for any object $[f : x \rightarrow y]$ in $\mathcal{E}[]$, $\varepsilon_f := f : x \rightarrow y$ and $p_f := (\text{id}_y, 0) : Y(f) = y \oplus C x \rightarrow y$ and for a homotopy commutative square $(a, b, H) : [f : x \rightarrow y] \rightarrow [f' : x' \rightarrow y'], \varepsilon_{(a, b, H)} := H : f' a \Rightarrow b f$ and $p_{(a, b, H)} := (0 \rightarrow H r) : b (\text{id}_y, 0) \Rightarrow c (\text{id}_y, 0) \left(\begin{smallmatrix} 0 & -H r \\ 0 & C a \end{smallmatrix}\right)$.

Then we have the commutative diagram of homotopy natural transformations.

![Diagram](https://via.placeholder.com/150)

Here we can show that for any object $[f : x \rightarrow y]$ in $\mathcal{E}[]$, $p_f$ and $j_{2, f}$ are chain homotopy equivalences. In particular if $f$ is a chain homotopy equivalence, then $j_{1, f}$ is also a chain homotopy equivalence.

**Definition 5.12** (Mapping cylinder functor on $\text{Nat}_h(\mathcal{E} \Rightarrow \mathcal{F})$). Let $\mathcal{F}$ be a small category. We will define $\text{Nat}_h(\mathcal{E} \Rightarrow \mathcal{F})$ the category of homotopy natural transformations (between the functors from $\mathcal{F}$ to $\mathcal{E}$) as follows. An object in $\text{Nat}_h(\mathcal{E} \Rightarrow \mathcal{F})$ is a triple $(f, g, \theta)$ consisting of functors $f, g : \mathcal{F} \rightarrow \mathcal{E}$ and a homotopy natural transformation $\theta : f \Rightarrow g$. A morphism $(a, b) : (f, g, \theta) \rightarrow (f', g', \theta')$ is a pair of natural transformations $a : f \Rightarrow f'$ and $b : g \Rightarrow g'$ such that $\theta' a = b \theta$. Compositions of morphisms is given by componentwise compositions of natural transformations.

We will define functors $S, T, Y : \text{Nat}_h(\mathcal{E} \Rightarrow \mathcal{F}) \rightarrow \mathcal{E} \Rightarrow \mathcal{F}$ and natural transformations $j_1 : S \Rightarrow Y$ and $j_2 : T \Rightarrow Y$ as follows. For an object $(f, g, \theta)$ and a morphism $(\alpha, \beta) : (f, g, \theta) \rightarrow (f', g', \theta')$ in $\text{Nat}_h(\mathcal{E} \Rightarrow \mathcal{F})$ and an object $i$ and a morphism $a : i \Rightarrow j$ in $\mathcal{F}$, we set

$$S(f, g, \theta) := f, \quad S(\alpha, \beta) := \alpha,$$

$$T(f, g, \theta) := g, \quad T(\alpha, \beta) := \beta,$$

$$Y(f, g, \theta)(i) := Y(\theta)(i) = (g_i \oplus C f_i),$$

$$Y(f, g, \theta)_a(i) := Y(f, g, \theta)(i) \left(\begin{smallmatrix} g_a & -\theta_a \\ 0 & C f_a \end{smallmatrix}\right),$$

$$Y(\alpha, \beta)_a(i) := \left(\begin{smallmatrix} \beta & 0 \\ 0 & C \alpha \end{smallmatrix}\right),$$

$$j_1(f, g, \theta)_a := j_1(\theta)_a, \quad j_2(f, g, \theta)_a := j_2(\theta)_a.$$}

In particular for an object $(f, g, \theta)$ in $\text{Nat}_h(\mathcal{E} \Rightarrow \mathcal{F})$ if for any object $i$ of $\mathcal{F}$, $\theta_i$ is a chain homotopy equivalence, then there exists a zig-zag diagram which connects $f$ to $g$, $f \xrightarrow{j_1} Y(\theta) \xleftarrow{j_2} g$ such that for any object $i$, $j_1 i$ and $j_2 i$ are chain homotopy equivalences.

**Definition 5.13** (Simplicial homotopy natural transformations). Let $\mathcal{F}$ be a simplicial object in the category of small categories which we call shortly a simplicial small category and let $f, g : \mathcal{F} \rightarrow \text{Ch}_{h}(S, \mathcal{E})$ be simplicial functors. Recall that a simplicial natural transformation (from $f$ to $g$) is a family of natural transformations $\{p_n : f_n \rightarrow g_n \}_{n \geq 0}$ indexed by non-negative integers such that for any morphism $\varphi : [n] \rightarrow [m]$, we have the equality $p_{\varphi} p_{n}$.

A simplicial homotopy natural transformation (from $f$ to $g$) is a family of homotopy natural transformations $\{\theta_n : f_n \Rightarrow g_n \}_{n \geq 0}$ indexed by non-negative integers such that for any morphism $\varphi : [n] \rightarrow [m]$, we have the equality $\theta_{\varphi} \theta_n = \theta_m \theta_{\varphi}$. We denote this situation by $\theta : f \Rightarrow \text{simp} g$.

For a simplicial homotopy natural transformation $\theta : f \Rightarrow \text{simp} g$, we will define $\mathcal{Y}(\theta) : \mathcal{F} : \rightarrow \text{Ch}_{h}(S, \mathcal{E})$ and $\mathcal{Y}(\theta)_{n} := \mathcal{Y}(\theta)(n)$, $\mathcal{Y}(\theta)(n) = j_1 \mathcal{Y}(\theta)_{n}$, $\mathcal{Y}(\theta)(n) = j_2 \mathcal{Y}(\theta)_{n}$ and $\mathcal{Y}(\theta)(n) = Y(f_n, g_n)$. In particular if for any non-negative integer $n$, any object $j$ of $\mathcal{F}_{n}$, $\theta_{n, j}$ is a chain homotopy equivalence, then there exists a zig-zag sequence which connects $f$ to $g$, $f \xrightarrow{j_1} \mathcal{Y}(\theta) \xleftarrow{j_2} g$ such that for any non-negative integer $n$ and any object $j$, $j_1 n$ and $j_2 n$ are chain homotopy equivalences.
6 Motivic Gersten’s conjecture

In this last section, I briefly explain my recent work of motivic Gernsten’s conjecture in [Moc16b].

6.1 Motivic zero map theorem

We denote the category of small dg-categories over $\mathbb{Z}$ the ring of integers by $\text{dgCat}$ and let $\ot$ be symmetric monoidal stable model categories of additive noncommutative motives and localizing motives over $\mathbb{Z}$ respectively. (See [CT12 §7].) We denote the homotopy category of a model category $\mathcal{M}$ by $\text{Ho}(\mathcal{M})$. There are functors from $\text{ExCat}$ the category of small exact categories to $\text{dgCat}$ which sending a small exact category $\mathcal{E}$ to its bounded dg-derived category $\mathcal{D}_d^b(\mathcal{E})$ (see [Kel05 §4.4]) and the universal functors $\ot$: $\text{dgCat} \to \text{Ho}(\ot)$ and $\ot$: $\text{dgCat} \to \text{Ho}(\ot)$. We denote the compositions of these functors $\text{ExCat} \to \text{Ho}(\ot)$ and $\text{ExCat} \to \text{Ho}(\ot)$ by $\ot$ and $\ot$ respectively. Then we can ask the following question:

Conjecture 6.1 (Motivic Gersten’s conjecture provisional version). For any commutative regular local ring $R$ and for any integers $1 \leq p \leq \dim R$, the inclusion functor $\ot \to \ot$ induces the zero morphism

$$M_b(\ot) \to M_b(\ot)$$

in $\text{Ho}(\ot)$ where $\# \in \{\text{add}, \text{loc}\}$.

Since in the proof of zero morphism theorem in §5 what we just using are basically the resolution theorem and the additivity theorem, by mimicking the proof of zero morphism theorem, we obtain the following result:

Proposition 6.2 (Motivic zero morphism theorem). Let $B$ be a commutative noetherian ring with 1 and let $g$ be an element in $B$ such that $g$ is a non-zero divisor and contained in the Jacobson radical of $B$. Assume that every finitely generated projective $B$-modules are free. Then the inclusion functor $\ot \to \ot$ induces the zero morphism

$$M_b(\ot) \to M_b(\ot)$$

in $\ot$ where $\# \in \{\text{add}, \text{loc}\}$. \hfill $\square$

Remark 6.3. Strictly saying, in the proof above, we shall moreover utilize the following two facts.

1. For any pair of exact functors $f, g: \mathcal{E} \to \text{Ch}_0(\mathcal{F})$ from an exact category $\mathcal{E}$ to the category of bounded chain complexes on an exact category $\mathcal{F}$, if there exists a zig-zag sequence of natural equivalences which connects $f$ and $g$ whose components are quasi-isomorphisms, then $f$ and $g$ induce the same morphisms $M_b(f) = M_b(g)$ in $\ot$ where $\# \in \{\text{add}, \text{loc}\}$.

2. Let $\Delta$ be the simplicial category and let $e$ be the final object in the category of small categories and let $p: \Delta \to e$ be the projection functor. We denote the derivator associated with the model category $\ot$ by $\ot$. For a small dg category $\mathcal{A}$, there is an object $\ot(S, \mathcal{A})$ in $\ot$ where $S$ is the Segal-Waldhausen $S$-construction. We have the canonical isomorphism in $\ot$:

$$pr_1\ot(S, \mathcal{A}) = \ot(\mathcal{A})[1].$$

(See [Tab08 Notation 11.11, Theorem 11.12].)

6.2 Nilpotent invariant motives

A problem to obtain the motivic Gersten’s conjecture from Proposition 6.2 is that in the proof in §3 we utilize the dévissage theorem. (see the isomorphism [10 III].) Thus if we imitate the proof, then we need to modify statement of the conjecture and we need to construct a new stable model category of non-commutative motives which we denote by $\ot$ and which we construct by localizing $\ot$.

We briefly recall the construction of the stable model category of localizing non-commutative motives $\ot$ from [Tab08]. First notice that the category $\text{dgCat}$ carries a cofibrantly generated model structure whose weak equivalences are the derived Morita equivalences. [Tab08 Théorème 5.3]. We fix on $\text{dgCat}$ a fibrant resolution functor $R$, a cofibrant resolution functor $Q$ and a left framing $\Gamma_1$, (see [Hov99] Definition 5.2.7 and Theorem 5.2.8.) and we also fix a small full subcategory $\text{dgCat}_f \to \text{dgCat}$ such that it contains all finite dg cells and any objects in $\text{dgCat}_f$ are (Z-)flat and homotopically finitely presented (see [TV07 Definition 2.1 (3)]) and $\text{dgCat}_f$ is closed under the operations $Q$, $QR$ and $\ot$ and $\ot$. The construction below does not depend upon a choice of $\text{dgCat}_f$ up to Dwyer-Kan equivalences.
Let $\text{sdgCat}_f$ and $\text{sdgCat}_f^\bullet$ be the category of simplicial presheaves and that of pointed simplicial presheaves on $\text{dgCat}_f$, respectively. We have the projective model structures on $\text{sdgCat}_f$ and $\text{sdgCat}_f^\bullet$ where the weak equivalences and the fibrations are the termwise simplicial weak equivalences and termwise Kan fibrations respectively. (see [Hir03, Theorem 11.6.1].)

We denote the class of derived Morita equivalences in $\text{dgCat}_f$ by $\Sigma$ and we also write $\Sigma_+$ for the image of $\Sigma$ by the composition of the Yoneda embedding $h: \text{dgCat}_f \rightarrow \text{sdgCat}$ and the canonical functor $(-)_+ : \text{sdgCat}_f \rightarrow \text{sdgCat}_f^\bullet$.

Let $P$ be the canonical map $\emptyset \rightarrow h(\emptyset)$ in $\text{sdgCat}_f$ and we write $P_+$ for the image of $P$ by the functor $(-)_+$. We write $L_{\Sigma_+} \text{sdgCat}_f^\bullet$ for the left Bousfield localization of $\text{sdgCat}_f^\bullet$ by the set $\Sigma_+ \cup \{P_+\}$. The Yoneda embedding functor induces a functor

$$\mathbb{R} h : \text{Ho}(\text{dgCat}) \rightarrow \text{Ho}(L_{\Sigma_+} \text{sdgCat}_f^\bullet)$$

which associates any dg category $A$ to the pointed simplicial presheaves on $\text{dgCat}_f$:

$$\mathbb{R} h(A) : B \mapsto \text{Hom}(\Gamma_s(\Omega B), R(A))_+.$$

Let $E$ be the class of morphisms in $L_{\Sigma_+} \text{sdgCat}_f^\bullet$ of shape

$$\text{Cone}(\mathbb{R} h(A) \rightarrow \mathbb{R} h(B)) \rightarrow \mathbb{R} h(C)$$

associated to each exact sequence of dg categories

$$A \rightarrow B \rightarrow C,$$

with $B$ in $\text{dgCat}_f$ where Cone means homotopy cofiber. We write $\text{Mot}_{\text{dg}}^{\text{loc}}$ for the left Bousfield localization of $L_{\Sigma_+} \text{sdgCat}_f^\bullet$ by $E$ and call it the model category of unstable localizing non-commutative motives.

Finally we write $\text{Mot}_{\text{dg}}^{\text{loc}}$ for the stable symmetric monoidal model category of symmetric $S^1 \otimes \mathbb{I}$-spectra on $\text{Mot}_{\text{dg}}^{\text{loc}}$ (see [Hov01, §7]) and call it the model category of localizing non-commutative motives.

Next we construct the stable model category $\text{Mot}_{\text{dg}}^{\text{nilp}}$. First recall that we say that a non-empty full subcategory $\mathcal{F}$ of a Quillen exact category $\mathcal{X}$ is a topologizing subcategory of $\mathcal{X}$ if $\mathcal{F}$ is closed under finite direct sums and closed under admissible sub- and quotient objects. The naming of the term ‘topologizing’ comes from noncommutative geometry of abelian categories by Rosenberg. (See [Ros08, Lecture 2 1.1].)

We say that a full subcategory $\mathcal{F}$ of an exact category $\mathcal{X}$ is a Serre subcategory if it is an extensional closed topologizing subcategory of $\mathcal{X}$. For any full subcategory $\mathcal{Z}$ of $\mathcal{X}$, we write $\sqrt{\mathcal{Z}}$ for intersection of all Serre subcategories which contain $\mathcal{Z}$ and call it the Serre radical of $\mathcal{Z}$ (in $\mathcal{X}$).

**Definition 6.4** (Nilpotent immersion). Let $\mathcal{A}$ be a noetherian abelian category and let $\mathcal{B}$ a topologizing subcategory. We say that $\mathcal{B}$ satisfies the dévissage condition (in $\mathcal{A}$) or say that the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is a nilpotent immersion if one of the following equivalent conditions holds:

1. For any object $x$ in $\mathcal{A}$, there exists a finite filtration of monomorphisms

$$x = x_0 \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n = 0$$

such that for every $i < n$, $x_i/x_{i+1}$ is isomorphic to an object in $\mathcal{B}$.

2. We have the equality $\mathcal{A} = \sqrt{\mathcal{B}}$.

(For the proof of the equivalence of the conditions above, see [Her97, 3.1], [Gar09, 2.2].)

**Definition 6.5.** We write $\mathcal{N}$ for the class of morphisms in $\text{Mot}_{\text{dg}}^{\text{loc}}$ of shape

$$\mathbb{R} h(\mathcal{P}_d(\mathcal{B})) \rightarrow \mathbb{R} h(\mathcal{P}_d(\mathcal{A}))$$

assoicated with each noetherian abelian category $\mathcal{A}$ and each nilpotent immersion $\mathcal{B} \hookrightarrow \mathcal{A}$.

We write $\text{Mot}_{\text{dg}}^{\text{nilp}}$ the left Bousfield localization of $\text{Mot}_{\text{dg}}^{\text{loc}}$ by $\mathcal{N}$ and call it the model category of unstable nilpotent invariant non-commutative motives.

Finally we write $\text{Mot}_{\text{dg}}^{\text{nilp}}$ for the stable model category of symmetric $S^1 \otimes \mathbb{I}$-spectra on $\text{Mot}_{\text{dg}}^{\text{nilp}}$ and call it the stable model category of nilpotent invariant non-commutative motives. We denote the compositions of the following functors

$$\text{ExCat} \rightarrow \text{dgCat} \rightarrow \text{Ho}(\text{dgCat}) \rightarrow \mathbb{R} h(\text{Ho}(L_{\Sigma_+} \text{sdgCat}_f^\bullet)) \rightarrow \text{Ho}(\text{Mot}_{\text{dg}}^{\text{nilp}})$$

by $\text{M}^{\text{nilp}}$. 

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We obtain the following theorem.

**Theorem 6.6** (Motivic Gersten’s conjecture). For any commutative regular local ring $R$ and for any integers $1 \leq p \leq \dim R$, the inclusion functor $\mathcal{M}_R^{p-1} \hookrightarrow \mathcal{M}_R^p$ induces the zero morphism

$$M_{\text{nilp}}(\mathcal{M}_R^{p-1}) \to M_{\text{nilp}}(\mathcal{M}_R^p)$$

in $\text{Ho}(\mathcal{M}_{\text{nilp}})\text{dg}$. □

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