SEXTIC VARIETY AS GALOIS CLOSURE VARIETY OF SMOOTH CUBIC

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Abstract. Let $V$ be a nonsingular projective algebraic variety of dimension $n$. Suppose there exists a very ample divisor $D$ such that $D^n = 6$ and $\dim H^0(V, \mathcal{O}(D)) = n + 3$. Then, $(V, D)$ defines a $D_6$-Galois embedding if and only if it is a Galois closure variety of a smooth cubic in $\mathbb{P}^{n+1}$ with respect to a suitable projection center such that the pull back of hyperplane of $\mathbb{P}^n$ is linearly equivalent to $D$.

1. Introduction

The purpose of this article is to generalize the following assertion (cf. [13, Theorem 4.5]) to $n$-dimensional varieties.

Proposition 1.1. Let $C$ be a smooth sextic curve in $\mathbb{P}^3$ and assume the genus is four. If $C$ has a Galois line, then the group $G$ is isomorphic to the cyclic or dihedral group of order six. Moreover, $G$ is isomorphic to the latter one if and only if $C$ is obtained as the Galois closure curve of a smooth plane cubic $E$ with respect to a point $P \in \mathbb{P}^2 \setminus E$, where $P$ does not lie on the tangent line to $E$ at any flex.

Before going into the details, we recall the definition of Galois embeddings of algebraic varieties and the relevant results. In this article a variety, a surface and a curve will mean a nonsingular projective algebraic variety, surface and curve, respectively.

Let $k$ be the ground field of our discussion, we assume it to be an algebraically closed field of characteristic zero. Let $V$ be a variety of dimension $n$ with a very ample divisor $D$; we denote this by a pair $(V, D)$. Let $f = f_D : V \hookrightarrow \mathbb{P}^N$ be the embedding of $V$ associated with the complete linear system $|D|$, where $N + 1 = \dim H^0(V, \mathcal{O}(D))$. Suppose $W$ is a linear subvariety of $\mathbb{P}^N$ satisfying $\dim W = N - n - 1$ and $W \cap f(V) = \emptyset$. Consider the projection $\pi_W$ from $W$ to $\mathbb{P}^n$, i. e., $\pi_W : \mathbb{P}^N \twoheadrightarrow \mathbb{P}^n$. Restricting $\pi_W$ onto $f(V)$, we get a surjective morphism $\pi = \pi_W : f : V \twoheadrightarrow \mathbb{P}^n$.

Let $K = k(V)$ and $K_0 = k(\mathbb{P}^n)$ be the function fields of $V$ and $\mathbb{P}^n$ respectively. The covering map $\pi$ induces a finite extension of fields $\pi^* : K_0 \hookrightarrow K$ of degree $\deg f(V) = D^n$, which is the self-intersection number of $D$. We denote by $K_W$ the Galois closure of this extension and by $G_W = Gal(K_W/K_0)$ the Galois group of $K_W/K_0$. By [1] $G_W$ is isomorphic to the monodromy group of the covering $\pi : V \twoheadrightarrow \mathbb{P}^n$. Let $V_W$ be the $K_W$-normalization of $V$ (cf. [3, Ch.2]). Note that $V_W$ is determined uniquely by $V$ and $W$. 
Definition 1.2. In the above situation we call $G_W$ and $V_W$ the Galois group and the Galois closure variety at $W$ respectively (cf. [14]). If the extension $K/K_0$ is Galois, then we call $f$ and $W$ a Galois embedding and a Galois subspace for the embedding respectively.

Definition 1.3. A variety $V$ is said to have a Galois embedding if there exist a very ample divisor $D$ satisfying that the embedding associated with $|D|$ has a Galois subspace. In this case the pair $(V, D)$ is said to define a Galois embedding.

If $W$ is a Galois subspace and $T$ is a projective transformation of $\mathbb{P}^N$, then $T(W)$ is a Galois subspace of the embedding $T \cdot f$. Therefore the existence of Galois subspace does not depend on the choice of the basis giving the embedding.

Remark 1.4. If a variety $V$ exists in a projective space, then by taking a linear subvariety, we can define a Galois subspace and Galois group similarly as above. Suppose $V$ is not normally embedded and there exists a linear subvariety $W$ such that the projection $\pi_W$ induces a Galois extension of fields. Then, taking $D$ as a hyperplane section of $V$ in the embedding, we infer readily that $(V, D)$ defines a Galois embedding with the same Galois group in the above sense.

By this remark, for the study of Galois subspaces, it is sufficient to consider the case where $V$ is normally embedded.

We have studied Galois subspaces and Galois groups for hypersurfaces in [9], [10] and [11] and space curves in [13] and [15]. The method introduced in [14] is a generalization of the ones used in these studies.

Hereafter we use the following notation and convention:

- $\text{Aut}(V)$: the automorphism group of a variety $V$
- $\langle a_1, \cdots, a_m \rangle$: the subgroup generated by $a_1, \cdots, a_m$
- $D_{2m}$: the dihedral group of order $2m$
- $|G|$: the order of a group $G$
- $\sim$: the linear equivalence of divisors
- $I_m$: the unit matrix of size $m$
- $X \ast Y$: the intersection cycle of cycles $X$ and $Y$ in a variety.
- $(X_0: \cdots : X_m)$: a set of homogeneous coordinates on $\mathbb{P}^m$
- $g(C)$: the genus of a smooth curve $C$
- For a mapping $\varphi: X \longrightarrow Y$ and a subset $X' \subset X$, we often use the same $\varphi$ to denote the restriction $\varphi|_{X'}$.

2. Results on Galois embeddings

We state several properties concerning Galois embedding without the proofs, for the details, see [14]. By definition, if $W$ is a Galois subspace, then each element $\sigma$ of $G_W$ is an automorphism of $K = K_W$ over $K_0$. Therefore it induces a birational transformation of $V$ over $\mathbb{P}^n$. This implies that $G_W$ can be viewed as a subgroup of $\text{Bir}(V/\mathbb{P}^n)$, the group of birational transformations of $V$ over $\mathbb{P}^n$. Further we can say the following:
**Representation 1.** Each birational transformation belonging to $G_W$ turns out to be regular on $V$, hence we have a faithful representation

$$\alpha : G_W \hookrightarrow \text{Aut}(V). \quad (1)$$

**Remark 2.1.** Representation 1 is proved by using transcendental method in [14], however we can prove it algebraically by making use of the results [7, Ch. I, 5.3, Theorem 7] and [2, Ch. V, Theorem 5.2].

Therefore, if the order of $\text{Aut}(V)$ is smaller than the degree $d$, then $(V, D)$ cannot define a Galois embedding. In particular, if $\text{Aut}(V)$ is trivial, then $V$ has no Galois embedding. On the other hand, in case $V$ has an infinitely many automorphisms, we have examples such that there exist infinitely many distinct Galois embeddings, see Example 4.1 in [14].

When $(V, D)$ defines a Galois embedding, we identify $f(V)$ with $V$. Let $H$ be a hyperplane of $\mathbb{P}^N$ containing $W$ and put $D' = V \cdot H$. Since $D' \sim D$ and $\sigma^*(D') = D'$, for any $\sigma \in G_W$, we see $\sigma$ induces an automorphism of $H^0(V, \mathcal{O}(D))$. This implies the following.

**Representation 2.** We have a second faithful representation

$$\beta : G_W \hookrightarrow \text{PGL}(N+1, k). \quad (2)$$

In the case where $W$ is a Galois subspace we identify $\sigma \in G_W$ with $\beta(\sigma) \in \text{PGL}(N+1, k)$ hereafter. Since $G_W$ is a finite subgroup of $\text{Aut}(V)$, we can consider the quotient $V/G_W$ and let $\pi_G$ be the quotient morphism, $\pi_G : V \longrightarrow V/G_W$. 

**Proposition 2.2.** If $(V, D)$ defines a Galois embedding with the Galois subspace $W$ such that the projection is $\pi_W : \mathbb{P}^N \longrightarrow \mathbb{P}^n$, then there exists an isomorphism $g : V/G_W \longrightarrow \mathbb{P}^n$ satisfying $g \cdot \pi_G = \pi$. Hence the projection $\pi$ is a finite morphism and the fixed loci of $G_W$ consist of only divisors.

Therefore, $\pi$ turns out to be a Galois covering in the sense of Namba [6].

**Lemma 2.3.** Let $(V, D)$ be the pair in Proposition 2.2. Suppose $\tau \in G$ has the representation

$$\beta(\tau) = [1, \ldots, 1, e_m], \quad (m \geq 2)$$

where $e_m$ is an $m$-th root of unity. Let $p$ be the projection from $(0 : \cdots : 0 : 1) \in W$ to $\mathbb{P}^{N-1}$. Then, $V/(\tau)$ is isomorphic to $p(V)$ if $p(V)$ is a normal variety.

We have a criterion that $(V, D)$ defines a Galois embedding.

**Theorem 2.4.** The pair $(V, D)$ defines a Galois embedding if and only if the following conditions hold:

1. There exists a subgroup $G$ of $\text{Aut}(V)$ satisfying that $|G| = D^n$.
2. There exists a $G$-invariant linear subspace $\mathcal{L}$ of $H^0(V, \mathcal{O}(D))$ of dimension $n+1$ such that, for any $\sigma \in G$, the restriction $\sigma^*|_{\mathcal{L}}$ is a multiple of the identity.
3. The linear system $\mathcal{L}$ has no base points.
It is easy to see that $\sigma \in G_W$ induces an automorphism of $W$, hence we obtain another representation of $G_W$ as follows. Take a basis $\{f_0, f_1, \ldots, f_n\}$ of $H^0(V, \mathcal{O}(D))$ satisfying that $\{f_0, f_1, \ldots, f_n\}$ is a basis of $\mathcal{L}$ in Theorem 2.4. Then we have the representation

$$\beta_1(\sigma) = \begin{pmatrix} \lambda_\sigma & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \lambda_\sigma & \cdots \\ 0 & \cdots & \cdots & M_\sigma \end{pmatrix}.$$  

(3)

Since the projective representation is completely reducible, we get another representation using a direct sum decomposition:

$$\beta_2(\sigma) = \lambda_\sigma \cdot 1_{n+1} \oplus M'_\sigma.$$  

Thus we can define

$$\gamma(\sigma) = M'_\sigma \in PGL(N-n, k).$$

Therefore $\sigma$ induces an automorphism on $W$ given by $M'_\sigma$.

**Representation 3.** We get a third representation

$$\gamma : G_W \rightarrow PGL(N-n, k).$$  

(4)

Let $G_1$ and $G_2$ be the kernel and image of $\gamma$ respectively.

**Theorem 2.5.** We have an exact sequence of groups

$$1 \rightarrow G_1 \rightarrow G \xrightarrow{\gamma} G_2 \rightarrow 1,$$

where $G_1$ is a cyclic group.

**Corollary 2.6.** If $N = n+1$, i.e., $f(V)$ is a hypersurface, then $G$ is a cyclic group.

This assertion has been obtained in [11]. Moreover we have another representation.

Suppose that $(V, D)$ defines a Galois embedding and let $G$ be a Galois group at some Galois subspace $W$. Then, take a general hyperplane $W_1$ of $\mathbb{P}^n$ and put $V_1 = \pi^*(W_1)$. The divisor $V_1$ has the following properties:

(i) If $n \geq 2$, then $V_1$ is a smooth irreducible variety.

(ii) $V_1 \sim D$.

(iii) $\sigma^*(V_1) = V_1$ for any $\sigma \in G$.

(iv) $V_1/G$ is isomorphic to $W_1$.

Put $D_1 = V_1 \cap H_1$, where $H_1$ is a general hyperplane of $\mathbb{P}^N$. Then $(V_1, D_1)$ defines a Galois embedding with the Galois group $G$ (cf. Remark [14]). Iterating the above procedures, we get a sequence of pairs $(V_i, D_i)$ such that

$$(V, D) \supset (V_1, D_1) \supset \cdots \supset (V_{n-1}, D_{n-1}).$$  

(5)

These pairs satisfy the following properties:
(a) $V_i$ is a smooth subvariety of $V_{i-1}$, which is a hyperplane section of $V_{i-1}$, where $D_i = V_{i+1}$, $V = V_0$ and $D = V_1$ ($1 \leq i \leq n - 1$).

(b) $(V_i, D_i)$ defines a Galois embedding with the same Galois group $G$.

**Definition 2.7.** The above procedure to get the sequence (5) is called the Descending Procedure.

Letting $C$ be the curve $V_{n-1}$, we get the next fourth representation.

**Representation 4.** We have a fourth faithful representation

$$\delta : G_W \hookrightarrow \text{Aut}(C),$$

where $C$ is a curve in $V$ given by $V \cap L$ such that $L$ is a general linear subvariety of $\mathbb{P}^N$ with dimension $N - n + 1$ containing $W$.

Since the Inverse Problem of Galois Theory over $k(x)$ is affirmative ([4]), we can prove the following.

**Remark 2.8.** Giving any finite group $G$, there exists a smooth curve and very ample divisor $D$ such that $(C, D)$ defines a Galois embedding with the Galois group $G$.

### 3. Statement of results

Let $V$ be a variety of dimension $n$. We say that $V$ has the property $(\mathfrak{p}_n)$ if

1. there exists a very ample divisor $D$ with $D^n = 6$, and
2. $\dim H^0(V, O(D)) = n + 3$.

An example of such a variety is a smooth $(2, 3)$-complete intersection, where $D$ is a hyperplane section. In particular, in case $n = 1$, $V$ is a non-hyperelliptic curve of genus four and $D$ is a canonical divisor. In case $n = 2$, $V$ is a $K3$ surface such that there exists a very ample divisor $D$ with $D^2 = 6$. However, the variety with the property $(\mathfrak{p}_n)$ is not necessarily the complete intersection, see Remark 3.10 below.

We will study the Galois embedding of $V$ for the variety with the property $(\mathfrak{p}_n)$. Clearly the Galois group is isomorphic to the cyclic group of order six or $D_6$. In the latter case we say that $(V, D)$ defines a $D_6$-embedding or, more simply $V$ has a $D_6$-embedding.

**Theorem 3.1.** Assume $V$ has the property $(\mathfrak{p}_n)$. If $V$ has a $D_6$-embedding, then $V$ is obtained as the Galois closure variety of a smooth cubic $\Delta$ in $\mathbb{P}^{n+1}$ with respect to a suitable projection center.

Next we consider the converse assertion. Let $\Delta$ be a smooth cubic of dimension $n$ in $\mathbb{P}^{n+1}$. Take a non-Galois point $P \in \mathbb{P}^{n+1} \setminus \Delta$. Note that, for a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, the number of Galois points is at most $n + 2$. The maximal number is attained if and only if $X$ is projectively equivalent to the Fermat variety (cf. [11]).

Define the set $\Sigma_P$ of lines as

$$\Sigma_P = \{ \ell \mid \ell \text{ is a line passing through } P \text{ such that } \ell \ast \Delta \text{ can be expressed as } 2P_1 + P_2, \text{ where } P_i \in \Delta (i = 1, 2) \text{ and } P_1 \neq P_2 \}.$$

The closure of the set $\bigcup_{\ell \in \Sigma_P} \ell$ is a cone, we denote it by $C_P(\Delta)$. Then we have the following.

**Lemma 3.2.** The cone $C_P(\Delta)$ is a hypersurface of degree six.
We can express as $\Delta \ast C_P(\Delta) = 2R_1 + R_2$, where $R_1$ and $R_2$ are different divisors on $\Delta$.

**Definition 3.3.** We call $P$ a good point if

1. $R_2$ is smooth and irreducible in case $n \geq 2$, or
2. $R_2$ consists of six points in case $n = 1$.

**Proposition 3.4.** If $P$ is a general point for $\Delta$, then $P$ is a good point.

To some extent the converse assertion of Theorem 3.1 holds as follows.

**Theorem 3.5.** If $\Delta_P$ is a Galois closure variety of a smooth cubic $\Delta \subset \mathbb{P}^{n+1}$, where the projection center $P$ is a good point, then $\Delta_P$ is a smooth $(2,3)$-complete intersection in $\mathbb{P}^{n+2}$ with $D_6$-embedding.

**Remark 3.6.** In the assertion of Theorem 3.5 the construction of the Galois closure is closely related to the one in [8, Tokunaga]. In case $n = 2$, the Galois closure surface is a $K3$ surface.

Applying the Descending Procedure to the variety of Theorem 3.5, we get the following.

**Proposition 3.7.** If a variety $V$ is a smooth $(2,3)$-complete intersection and has a $D_6$-embedding, then there exists the following sequence of varieties $V_i$, where $V_i$ has the same properties as $V$ does, i.e.,

1. $V_i$ is a subvariety of $V_{i-1}$ ($i \geq 1$), where $V_0 = V$.
2. $V_i$ is also a smooth $(2,3)$-complete intersection of hypersurfaces in $\mathbb{P}^{n+2-i}$, $0 \leq i \leq n - 1$.
3. $V_i$ has the property $\left(\mathbb{P}_{n-i}\right)$,
4. $V_i$ has a $D_6$-embedding.

The situation above is illustrated as follows:

\[
\begin{array}{cccccccc}
\mathbb{P}^{n+2} & \rightarrow & \mathbb{P}^{n+1} & \rightarrow & \mathbb{P}^4 & \rightarrow & \mathbb{P}^3 \\
\mathbb{P}^n & \rightarrow & \mathbb{P}^{n-1} & \rightarrow & \mathbb{P}^2 & \rightarrow & \mathbb{P}^1, \\
V & \supset & V_1 & \supset & \cdots & \supset & V_{n-2} & \supset & V_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\end{array}
\]

where $\rightarrow$ is a point projection, $\downarrow$ is a triple covering, $V_{n-2}$ and $V_{n-1}$ are a $K3$ surface and a sextic curve, respectively.

Here we present examples.

**Example 3.8.** Let $\Delta$ be the smooth cubic in $\mathbb{P}^3$ defined by

\[ F(X_0, X_1, X_2, X_3) = X_0^3 + X_1^3 + X_2^3 + X_0^2X_3 + X_1X_2^2 + X_3^3. \]  

Let $\pi_P$ be the projection from $P = (0 : 0 : 0 : 1)$ to the hyperplane $\mathbb{P}^2$. Taking the affine coordinates $x = X_0/X_3$, $y = X_1/X_3$ and $z = X_2/X_3$, we get the defining equation of the affine part

\[ f(x, y, z) = x^3 + y^3 + z^3 + x^2 + y + 1. \]
Put $x = at, \ y = bt$ and $z = ct$. Computing the discriminant $D(f)$ of $f(at, bt, ct) = (a^3 + b^3 + c^3)t^3 + a^2t^2 + bt + 1$ with respect to $t$, we obtain

$$D(f) = -(31a^6 - 18a^5b - a^4b^2 + 58a^3b^3 - 18a^2b^4 + 31b^6 + 54ac^3 - 18a^2bc^3 + 58b^3c^3 + 27c^6).$$

(8)

This yields the branch divisor of $\pi_P : \Delta \rightarrow \mathbb{P}^2$. From (7) and (8) we infer that the defining equation of $2R_1$ is

$$F(X_0, X_1, X_2, X_3) = 0 \quad \text{and} \quad (X_0^3 + 2X_1X_3 + 3X_3^2)^2 = 0,$$

and that of $R_2$ is

$$F(X_0, X_1, X_2, X_3) = 0 \quad \text{and} \quad 4X_0^2 - X_1^2 + 2X_1X_3 + 3X_2^2 = 0.$$

It is not difficult to check that $R_2$ is smooth and irreducible, hence $P$ is a good point for $\Delta$. By taking a double covering along this curve $[S], we get the $K3$ surface $\Delta_P$ defined by $F = 0$ and $X_0^3 = 4X_0^3 - X_1^2 + 2X_1X_3 + 3X_2^2$, which is a $(2,3)$-complete intersection. The Galois line is given by $X_0 = X_1 = X_2 = 0$.

How is the Galois closure variety when the projection center is not a good point? Let us examine the following example.

**Example 3.9.** For a projection with some center $P \in \mathbb{P}^3 \setminus \Delta$, the Galois closure surface $\Delta_P$ is not necessarily a $K3$ surface. Indeed, let $\Delta$ be the smooth cubic defined by

$$F(X_0, X_1, X_2, X_3) = X_0^3 + X_1^3 + X_2^3 + X_0X_2^2 - X_3^2.$$  

(9)

Clearly the point $P = (0 : 0 : 0 : 1)$ is not a Galois one. Taking the same affine coordinates as in Example 3.8, we get the defining equation of the affine part

$$f(x, y, z) = x^3 + y^3 + z^3 + x - 1.$$

Put $x = at, \ y = bt$ and $z = ct$. Computing the discriminant $D(f)$ of $f(at, bt, ct) = (a^3 + b^3 + c^3)t^3 + at - 1$ with respect to $t$, we obtain

$$D(f) = -(31a^3 + 27b^3 + 27c^3)(a^3 + b^3 + c^3).$$

(10)

This yields the branch divisor of $\pi_P : \Delta \rightarrow \mathbb{P}^2$. From (9) and (10) we infer that the defining equation of $2R_1$ is $C_1 + C_2$, where $C_1$ (resp. $C_2$) is given by $X_0^3 + X_2^3 + X_3^2 = 0$ (resp. $31X_0^3 + 27X_1^3 + 27X_2^3 = 0$). Hence the defining equation of the sextic $C_P(V)$ is

$$(X_0^3 + X_1^3 + X_2^3)(31X_0^3 + 27X_1^3 + 27X_2^3) = 0.$$

Let $\Delta_P$ be the double covering of $\Delta$ branched along the divisor $R_2$, where $R_2 = R_{21} + R_{22}$ such that $R_{21}$ (resp. $R_{22}$) is given by the intersection of $F = 0$ and $X_0 - X_3 = 0$ (resp. $F = 0$ and $X_0 - 3X_3 = 0$). The $R_{2i}$ ($i = 1, 2$) is a smooth curve on $\Delta$ satisfying that $R_{2i}^2 = 3$, $R_{2i} = 12$ and $R_{21}R_{22} = 3$. We infer that $\Delta_P$ is a normal surface, therefore it is a Galois closure surface at $P$ (Definition 1.2). However, it has three singular points of type $A_1$, so that it is not a $K3$ surface. The minimal resolution of $\Delta_P$ turns out to be a $K3$ surface.

**Remark 3.10.** The variety with the property $(\mathfrak{C}_n)$ is not necessarily a $(2,3)$-complete intersection. For example, in case $n = 1$, Take $V = C$ as the Galois closure curve of a smooth cubic $\Delta \subset \mathbb{P}^2$ obtained as follows: let $T$ be a tangent line to $\Delta$ at a flex. Choose a point $P \in \Delta$ satisfying the following condition: if $\ell_P$ is a line passing through $P$ and $\ell_P \neq T$, then $\ell_P$ does not tangent to $\Delta$ at any flex. Let $C$ be the
Galois closure curve for the point projection \( \pi_P : \Delta \to P^1 \), i.e., \( \tilde{\pi} : C \to \Delta \) is a double covering, which has four branch points (see, for example [5] pp. 287–288), hence \( g(C) = 3 \). Let \( D \) be the divisor \( \tilde{\pi}^*(\ell + \Delta) \), where \( \ell \) is a line passing through \( P \). Clearly we have \( \deg D = 6 \), the complete linear system \( |D| \) has no base point and \( \dim H^0(C, O(D)) = 4 \). Let \( f : C \to C' \) be the morphism associated with \( |D| \). The double covering \( \tilde{\pi} \) factors as \( \tilde{\pi} = \tilde{\pi}' \cdot f \), where \( \tilde{\pi}' : C' \to \Delta \) is a restriction of the projection \( \mathbb{P}^3 \to \mathbb{P}^2 \). Since \( g(C') \geq 1 \), we see \( \deg C' \neq 2 \) and 3. Hence \( \deg C' = 6 \) and \( f \) is a birational morphism. Further, we have the projection \( \tilde{\pi}' : C' \to \Delta \) and \( \Delta \) is nonsingular, hence \( C' \) is smooth. Therefore \( f \) is an isomorphism. Since \( g(C) = 3 \), \( C \) is not a \((2, 3)\)-complete intersection.

4. PROOF

First we prove Theorem 3.1. The case \( n = 1 \) have been proved ([13]). So that we will restrict ourselves to the case \( n \geq 2 \).

Since \( V \) is embedded into \( \mathbb{P}^{n+2} \) associated with \( |D| \), where \( D \) is a very ample divisor with \( D^n = 6 \), we can apply the results in Section 2. By assumption \( V \) has a Galois line \( \ell \) such that the Galois group \( G = G_\ell \) is isomorphic to \( D_6 \). We can assume \( G = \langle \sigma, \tau \rangle \) where \( \sigma^3 = \tau^2 = 1 \) and \( \tau \sigma \tau = \sigma^{-1} \). Let \( \rho_1 : V \to V^\tau = V/\langle \tau \rangle \). We see \( \rho_2 : V^\tau \to V^\tau/G \cong \mathbb{P}^n \) turns out a morphism. Then, we have \( \pi = \rho_2 \rho_1 : V \to V/G \cong \mathbb{P}^n \). Note that \( \rho_2 \) is a non-Galois triple covering. By taking suitable coordinates, we can assume \( \ell \) is given by \( X_0 = X_1 = \cdots = X_n = 0 \). As we see in Section 2, we have the representation \( \beta : G \to PGL(n + 3, k) \). Since the characteristic of \( k \) is zero, the projective representation is completely reducible, hence \( \beta(\sigma) \) and \( \beta(\tau) \) can be represented as

\[
\begin{pmatrix}
1 & & & 0 \\
& \ddots & & \\
& & 1 & 0 \\
0 & & -\frac{1}{2} & \omega + \frac{1}{2} \\
& & \omega + \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & & & 0 \\
& \ddots & & \\
& & 1 & 0 \\
0 & & & -1
\end{pmatrix},
\]

where \( \omega \) is a primitive cubic root of 1. Therefore, the fixed locus of \( \tau \) is given by \( f(V) \cap H \), where \( H \) is the hyperplane defined by \( X_{n+2} = 0 \). Put \( Z = f(V) \cap H \), i.e., \( Z \sim D \). Since \( Z \) is ample, it is connected. Looking at the representation \( \beta(\tau) \), we see \( Z \) is smooth, hence it is a smooth irreducible variety. Take the point \( P = (0 : \cdots : 0 : 1) \in \ell \) and an arbitrary point \( Q \) in \( V \). Let \( \ell_{PQ} \) be the line passing through \( P \) and \( Q \). Then we have \( \tau(\ell_{PQ}) = \ell_{PQ} \) and \( \tau(V) = V \). Let \( \pi_P \) be the projection from the point \( P \) to the hyperplane \( H \). Since \( Z \) is smooth, \( \pi_P(V) \) is smooth. By Lemma 2.3 \( \pi_P(V) \) is isomorphic to \( V/\langle \tau \rangle \) and we may assume \( \pi_P = \rho_1 \). Therefore we see \( V \) is contained in the cone consisting of the lines passing through \( P \) and the points in \( V \). Since \( \deg V = 6 \) and \( \deg p = 2 \), we conclude the variety \( V^\tau \) is a smooth cubic in \( \mathbb{P}^{n+1} \). This proves Theorem 3.1.
Next we prove Lemma 3.2. Let \( H_2 \) be a linear variety of dimension two and passing through \( P \). If \( H_2 \) is general, then \( \Delta \cap H_2 \) is a smooth cubic in the plane \( H_2 \cong \mathbb{P}^2 \). Thus \( C_P(\Delta) \cap H_2 \) consists of six lines, hence we have \( \deg C_P(\Delta) = 6 \).

The proof of Proposition 3.3 is as follows. Suppose \( P \) is a general point for \( \Delta \) and let \( \pi_P \) be the projection from \( P \) to the hyperplane \( \mathbb{P}^n \). Put \( B = \pi_P(R_2) \).

**Claim 4.1.** The divisor \( B \) is irreducible.

**Proof.** It is sufficient to check in a general affine part. Put \( x_i = X_i/X_0 \) (\( i = 1, \ldots, n+1 \)) and let \( f(x_1, \ldots, x_{n+1}) \) be the defining equation of an affine part \( X_0 \neq 0 \) of \( \Delta \) and \( P = (u_1, \ldots, u_{n+1}) \in A^{n+1} \). Put
\[
g(u_1, \ldots, u_{n+1}, t_0, \ldots, t_n, x) = f(u_1 + xt_0, \ldots, u_{n+1} + xt_n),
\]
where \((t_0, \ldots, t_n) \in \mathbb{P}^n\). Let \( D(g) = D(u_1, \ldots, u_{n+1}, t_0, \ldots, t_n) \) be the discriminant of \( g \) with respect to \( x \). Owing to [3] Lemma 3 and [11] Claim 1, we see \( D(g) \) is reduced and irreducible. Therefore for a general value \( u_1 = a_1, \ldots, u_{n+1} = a_{n+1}, D(a_1, \ldots, a_{n+1}, t_0, \ldots, t_n) \) is irreducible. This implies \( B \) is irreducible.

**Claim 4.2.** The divisor \( R_2 \) is irreducible and smooth.

**Proof.** Suppose \( R_2 \) is decomposed into irreducible components \( R_{21} + \cdots + R_{2r} \). Since \( B \) is irreducible, we have \( \pi_P(R_{2i}) = B \) for each \( 1 \leq i \leq r \). However, since \( \Delta \ast \ell \) has an expression \( 2P_1 + P_2 \), the \( r \) must be 1. Thus \( R_2 \) is irreducible. Since \( \Delta \ast \ell \) has an expression \( 2P_1 + P_2 \), where \( P_i \in \Delta \) (\( i = 1, 2 \)), \( \Delta \) and \( \ell \) has a normal crossing at \( P_2 \) if \( P_1 \neq P_2 \). In case \( P_1 = P_2 \), the intersection number of \( \Delta \) and \( \ell \) at \( P_1 \) is three. Since \( R_1 \supseteq P_1 \) and \( \Delta \ast C_P(\Delta) = 2R_1 + R_2 \), we see that \( R_2 \) is smooth at \( P_1 \).

This completes the proof of Proposition 3.4. The proof of Theorem 3.5 is as follows.

First note that \( P \) is not a Galois point. So we consider the Galois closure variety. The ramification divisor of \( \pi_P : \Delta \rightarrow \mathbb{P}^n \) is \( 2R_1 + R_2 \). The divisor \( R_2 \) is smooth and irreducible by assumption. Let \( \Phi \) be the equation of the branch divisor of \( \pi_P \).

As we see in Example 3.3 \((a = X_0/tX_3, b = X_1/tX_3, c = X_2/tX_3) \) the discriminant is given by the homogeneous equation of \( X_0, \ldots, X_n \), hence we infer that \( \pi_P^*(\Phi) \) has the expression as \( \Phi_2^2 \cdot \Phi_2 \), where \( \Phi_2 = 0 \) defines \( R_2 \). Since \( \deg \Phi_2 = 2 \), we can define the variety in \( \mathbb{P}^{n+2} \) by \( F = 0 \) and \( X_{n+2} = \Phi_2 \), which is smooth and turns out to be the Galois closure variety. This proves Theorem 3.5.

We go to the proof of Proposition 3.7. Let \( H \) be a general hyperplane containing the Galois line \( \ell \) for \( V \) in Theorem 3.1. Put \( V_1 = V \cap H \) and \( D_1 = D \cap H \). Since we are assuming \( n \geq 2 \), the \( V_1 \) is irreducible and nonsingular by Bertini’s theorem. Thus, we have \( \dim V_1 = n - 1 \), \( D_1^{n-1} = 6 \) and \( V_1 \) is also a smooth \((2,3)\)-complete intersection. Note that \( V_1 \sim D \) on \( V \). Thus we have the exact sequence of sheaves
\[
0 \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_V(V_1) \rightarrow \mathcal{O}_V(D_1) \rightarrow 0.
\]

Taking cohomology, we get a long exact sequence
\[
0 \rightarrow \mathcal{H}^0(V, \mathcal{O}_V) \rightarrow \mathcal{H}^0(V, \mathcal{O}_V(V_1)) \rightarrow \mathcal{H}^0(V_1, \mathcal{O}_V(D_1)) \rightarrow \mathcal{H}^1(V, \mathcal{O}_V) \rightarrow \ldots.
\]
Since \( V \) is the complete intersection, we have \( \mathcal{H}^1(V, \mathcal{O}_V) = 0 \) (cf. [2] III, Ex. 5.5). Then \( V_1 \) has the same properties as \( V \) does, i.e., \( \dim V_1 = n - 1 \), \( D_1^{n-1} = 6 \),
dim $H^0(V_1, \mathcal{O}(D_1)) = n + 2$ and $\ell$ is a Galois line for $V_1$ and the Galois group is isomorphic to $D_6$. Continuing the Descending Procedure, we get the sequence of Proposition 5.7.5.

There are a lot of problems concerning our theme, we pick up some of them.

Problems.

1. For each finite subgroup $G$ of $GL(2, k)$, does there exist a pair $(V, D)$ which defines the Galois embedding with the Galois group $G$ such that $D^n = |G|$, $\dim V = n$ and $\dim H^0(V, \mathcal{O}(D)) = n + 3$?

2. How many Galois subspaces do there exist for one Galois embedding? In case a smooth hypersurface $V$ in $\mathbb{P}^{n+1}$, there exist at most $n + 2$. Further, it is $n + 2$ if and only if $V$ is Fermat variety [11].

3. Does there exist a variety $V$ on which there exist two divisors $D_i$ ($i = 1, 2$) such that they give Galois embeddings and $D_1^n \neq D_2^n$?

For the detail, please visit our website
http://mathweb.sc.niigata-u.ac.jp/~yosihara/openquestion.html

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