QUANTUM HOLONOMIES IN (2+1)-DIMENSIONAL GRAVITY

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We describe an approach to the quantization of (2+1)-dimensional gravity with topology \( \mathbb{R} \times T^2 \) and negative cosmological constant, which uses two quantum holonomy matrices satisfying a \( q \)-commutation relation. Solutions of diagonal and upper-triangular form are constructed, which in the latter case exhibit additional, non-trivial internal relations for each holonomy matrix. This leads to the notion of quantum matrix pairs. These are pairs of matrices with non-commuting entries, which have the same pattern of internal relations, \( q \)-commute with each other under matrix multiplication, and are such that products of powers of the matrices obey the same pattern of internal relations as the original pair. This has implications for the classical moduli space, described by ordered pairs of commuting \( SL(2, \mathbb{R}) \) matrices modulo simultaneous conjugation by \( SL(2, \mathbb{R}) \) matrices.

1. Introduction

It is known that the phase space of (2+1)-dimensional gravity with topology \( \mathbb{R} \times T^2 \) and negative cosmological constant \( \Lambda \) is described by the space of gauge equivalence classes of flat \( SO(2,2) \) (de Sitter) connections on the torus, or equivalently by the space of conjugacy classes of homomorphisms from \( \pi_1(T^2) \) to \( SO(2,2) \). Since the fundamental group of \( T^2 \) is generated by two classes \( \gamma_1 \) and \( \gamma_2 \), subject to the relation \( \gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1} \cdot \gamma_2^{-1} = 1 \), the phase space may be identified with pairs of commuting \( SO(2,2) \) elements \( (S_1, S_2) \), identified up to simultaneous conjugation by the same group element \( (S_1, S_2) \sim (g^{-1}S_1g, g^{-1}S_2g) \) for any \( g \in SO(2,2) \). These matrix pairs are the holonomies of the flat connection along the two generators of the fundamental group. Using the isomorphism \( SO(2,2) \cong SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R}) \) in Ref. 2 the Poisson algebra of elements \( (U_1, U_2) \) of \( SL(2, \mathbb{R}) \) was calculated.

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It follows from the Poisson brackets of the connection at points where the curves intersect. The intersection number between $\gamma_1$ and $\gamma_2$ is taken to be $+1$, and $\gamma_1 \cdot \gamma_2$ has intersection number $-1$ with $\gamma_1$ and $+1$ with $\gamma_2$. The phase space was then described in terms of the six gauge-invariant (normalized) traces $T^{\pm}_i$, $i = 1, 2, 3$, where $T^{\pm}_i = \frac{1}{2} \text{tr} U^{\pm}_i$, $i = 1, 2, 3$, which satisfy the non-linear cyclical Poisson bracket algebra

$$\{T^{\pm}_i, T^{\pm}_j\} = \mp \frac{\sqrt{-\Lambda}}{4} (\epsilon_{ijk} T^{\pm}_k - T^{\pm}_i T^{\pm}_j), \quad \epsilon_{123} = 1$$

(1)

where the superscript $\pm$ refers to the two copies (real and independent) of $\text{SL}(2, \mathbb{R})$. The six holonomies $T^{\pm}_i$ of (1) provide an overcomplete description of the spacetime geometry of $\mathbb{R} \times T^2$. To see this, consider the cubic polynomials

$$F^{\pm} = 1 - (T^{\pm}_1)^2 - (T^{\pm}_2)^2 - (T^{\pm}_3)^2 + 2 T^{\pm}_1 T^{\pm}_2 T^{\pm}_3.$$  (2)

which have vanishing Poisson brackets with all of the traces $T^{\pm}_i$, are cyclically symmetric in the $T^{\pm}_i$, and vanish classically by the $\text{SL}(2, \mathbb{R})$ Mandelstam identities; setting $F^{\pm} = 0$ removes the redundancy.

The Poisson algebra (1) and its generalization to more complicated spatial topologies can be quantized for any value of the cosmological constant $\Lambda$. For a generic topology, one obtains an abstract quantum algebra $5, 6$. For genus 1 with $\Lambda < 0$, the quantum theory has been worked out explicitly $7$.

The holonomies of (1) can be represented classically as

$$T^{\pm}_1 = \cosh \frac{r^{\pm}_1}{2}, \quad T^{\pm}_2 = \cosh \frac{r^{\pm}_2}{2}, \quad T^{\pm}_3 = \cosh \frac{(r^{\pm}_1 + r^{\pm}_2)}{2},$$

(3)

where $r^{\pm}_{1,2}$ are also real, global, time-independent (but undetermined) parameters which, from (1), satisfy the Poisson brackets

$$\{r^{\pm}_1, r^{\pm}_2\} = \mp \sqrt{-\Lambda}, \quad \{r^{\pm}_{1,2}, r^{\pm}_{1,2}\} = 0.$$  (4)

In this case the cubic polynomials (2) are identically zero, and quantization $b$ is achieved by replacing Eqs. (4) with the commutators

$$[\hat{r}^{\pm}_1, \hat{r}^{\pm}_2] = \mp i \hbar \sqrt{-\Lambda}, \quad [\hat{r}^{\pm}_{1,2}, \hat{r}^{\pm}_{1,2}] = 0.$$  (5)

Previous quantizations $7, 9$ have concentrated entirely on the traces $T^{\pm}_i$ and their representation Eq. (3). Here we observe that we may regard the quantized traces $\hat{T}^{\pm}_i$ as traces of operator-valued holonomy matrices $\hat{T}^{\pm}_i = \frac{1}{2} \text{tr} \hat{U}^{\pm}_i$, $i = 1, 2,$

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$a$There is an analogous discussion for $\Lambda$ positive or zero. $11$ For example, for $\Lambda$ positive, the spinor group of the de Sitter group $\text{SO}(3, 1)$ is $\text{SL}(2, C)$, and the two $\pm$ copies refer to complex conjugates.

$b$Direct quantization of the algebra (1) gives an algebra related to the Lie algebra of the quantum group $\text{SU}(2)_q$, $28$ where $q = \exp (4i \theta)$, $\tan \theta = -\frac{\sqrt{-\Lambda}}{\hbar}$. A (scaled) representation of the operators (4) leads to the commutators $[\hat{r}^{\pm}_1, \hat{r}^{\pm}_2] = \pm 8i \theta$, which differ from (4) by terms of order $\hbar^3$. 


\[ T_3^{\pm} = \frac{1}{2} \text{tr}(\hat{U}_1^{\pm}\hat{U}_2^{\pm}), \]

where (for the (+) matrices, dropping the superscript) the matrices $\hat{U}_i$ have, for example, the diagonal form

\[ \hat{U}_i = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \exp \left( \frac{i \sigma_3}{2} \right) \]

(6)

where $\sigma_3$ is one of the Pauli matrices. Now, from Eq. (5) and the identity

\[ e^{\hat{X}} e^{\hat{Y}} = e^{\hat{Y}} e^{\hat{X}} e^{[\hat{X},\hat{Y}]} , \]

valid when $[\hat{X},\hat{Y}]$ is a $c$-number, one finds that the matrices (6) satisfy, by both matrix and operator multiplication, the $q$–commutation relation:

\[ \hat{U}_1 \hat{U}_2 = q \hat{U}_2 \hat{U}_1, \quad \text{with} \]

(7)

\[ q = \exp \left( -\frac{i\hbar \sqrt{-\Lambda}}{4} \right) \]

(8)

i.e. a deformation of the classical equation stating that the holonomies commute.

Equations of the form (7) appear abundantly in the quantum group and quantum geometry literature, as Weyl relations or $q$–commutators, or as the defining relation for the quantum plane (10), but normally the symbols $\hat{U}_1$ and $\hat{U}_2$ stand for scalar operators, as opposed to $2 \times 2$ matrices with operator entries.

Our approach is based on the fundamental equation (7). Instead of representing the algebra of traces, Eq. (4), we find representations of matrices $\hat{U}_1$ and $\hat{U}_2$ satisfying Eq. (4) that generalize the choices (6), for a general $q$–parameter. This constitutes a new approach to quantization that is consistent with previous approaches for this model (5, 7, 9), namely a deformation of classical holonomies that consequently satisfy a $q$–commutation relation. The gauge-invariance of the traces is replaced by the gauge-covariance of Eq. (7) under the replacements $\hat{U}_i \rightarrow g^{-1} \hat{U}_ig$, $i = 1, 2$ for $g \in \text{SL}(2, \mathbb{R})$ an ordinary, i.e. not operator-valued, matrix. We argue that working directly with the matrices $U_i$, rather than with the indirect information contained in their traces, gives a clearer insight into the structure of the phase space, both classically and after quantization.

A more detailed account of these results, including representations, is given in Ref. (11). The matrices $\hat{U}_1$ and $\hat{U}_2$ determine a new quantum–group–like structure, which is studied from the algebraic perspective in Ref. (12). The description of the classical phase space in terms of pairs of matrices $U_i$ is given in Ref. (13). The generalization to supergroups in the context of $(2+1)$–supergravity is described in Ref. (14).

c the (-) matrices satisfy a similar relation but with $q$ replaced by $q^{-1}$
2. An algebraic solution

We give just one example of a purely algebraic solution to Eq. (7). For others see Ref. 12. Consider the upper–triangular matrices

\[ U_i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \gamma_i \end{pmatrix}, \quad i = 1, 2 \]  

(9)

which generalize Eq. (6). It can be checked that they will satisfy Eq. (7) provided their non–commuting elements satisfy the following mutual relations

\[ \alpha_1 \alpha_2 = q \alpha_2 \alpha_1, \quad \gamma_1 \gamma_2 = q \gamma_2 \gamma_1, \quad \text{and} \]

(10)

\[ \alpha_1 \beta_2 = q \beta_2 \gamma_1, \quad \beta_1 \gamma_2 = q \alpha_2 \beta_1 \]  

(11)

and the following *internal* relations for each \((i = 1, 2)\) matrix

\[ \alpha_i \gamma_i = \gamma_i \alpha_i = 1, \quad \alpha_i \beta_i = \beta_i \gamma_i \]  

(12)

Note that the mutual relations (10) are standard \(q\)--commutation relations 10, which clearly become commutative in the classical limit \(q \to 1\), whereas relations (11) have a different structure, involving three elements not two. The relations (10) also imply that, for example

\[ \alpha_1 \alpha_2^{-1} = q^{-1} \alpha_2^{-1} \alpha_1, \quad \alpha_1^{-1} \alpha_2^{-1} = q \alpha_2^{-1} \alpha_1^{-1}, \]  

(13)

and similarly for \(\gamma_1, \gamma_2\). The internal relations Eqs. (12) are a new feature, since, in the Poisson algebra of \((2+1)\)-dimensional gravity 2, only matrix elements from different holonomies have non–zero brackets and would therefore not commute on quantization. Elements of a single holonomy commute. Note that the \(q\) parameter does not appear in the internal relations (12) which therefore persist in the classical limit \(q \to 1\), when the matrices (9) commute.

The internal relations (12) are also not standard \(q\)--commutation relations, but are preserved under matrix multiplication, and in this sense they are analogous to the internal relations for quantum groups 10. For example, the product \(U_1 U_2\) is given by

\[ U_1 U_2 = \begin{pmatrix} \alpha_1 \alpha_2 & \alpha_1 \beta_2 + \beta_1 \gamma_2 \\ 0 & \gamma_1 \gamma_2 \end{pmatrix} \]  

(14)

whose internal relations are analogous to Eq. (12), by using (10)–(11) and (12). This feature is discussed in greater detail in Ref. 11.

3. The classical moduli space

The new, non–trivial *internal* relations, Eq. (12) have important implications for the classical phase space, which consists of pairs of commuting \(\text{SL}(2, \mathbb{R})\) matrices, identified up to simultaneous conjugation by elements of \(\text{SL}(2, \mathbb{R})\), since the classical counterpart to Eq. (7) is the statement that the two matrices \(U_1\) and \(U_2\)
commute. This space was studied in detail in Ref. 15. The algebraic analysis of the classical case, in terms of the eigenvalues and eigenspaces of the two matrices, does not carry over in any straightforward way to quantum matrices. For instance, an upper–triangular matrix with two distinct diagonal entries can be diagonalized as an ordinary matrix, but this is not in general true if the matrix has non–trivial internal relations between non–commuting entries. In Ref. 13 we give another parametrization of the classical phase space, which is more appropriate to the present context. It consists of sectors where both matrices are diagonalizable, but also sectors where both are non-diagonalizable but can be simultaneously conjugated into upper triangular form, as well as other sectors. A spectral analysis of commuting SL(2, \textbf{R}) matrices allows a classification of the equivalence classes, and a unique canonical form is given for each of these. In this way the moduli space becomes explicitly parametrized, and has a simple structure, resembling that of a cell complex, allowing it to be depicted. Full details are given in Ref. 13.

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