LA VRENTIEV GAP FOR SOME CLASSES OF GENERALIZED ORLICZ FUNCTIONS

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Abstract. In the present paper we find optimal conditions separating the regular case from the one with Lavrentiev gap for the borderline case of double phase potential and related general classes of integrands. We present new results on density of smooth functions.

1. Introduction

During last decade the resurgence of interest in different general growth models has been experienced. Along with by now almost classical variable $p(x)$-integrand, presented in hundreds of papers and several books, see [7, 6, 11], different properties of other models were considered. The essential feature of these model is possible the presence of a Lavrentiev gap and related with this lack of regularity, non-density of smooth functions in the corresponding energy space and others.

Positive recent results in this direction are sufficiently many and varied. For example, Colombo and Mingione in [5] obtained the regularity results for double-phase potential model \( \Phi(x, t) = \frac{1}{p}t^p + \frac{1}{q}a(x)t^q \) if \( \frac{2}{p} \leq 1 + \frac{\alpha}{\sigma} \) and \( a \in C^{0,\gamma} \). Moreover, bounded minimizers are automatically \( W^{1,q} \) if \( a \in C^{0,\gamma} \) and \( q \leq p + \alpha \), see the paper [3] by Baroni, Colombo and Mingione. The sharpness of this results was showed by the authors of this paper and Lars Diening in [1] by constructing the examples of Lavrentiev gap for this model. The other model is weighed \( p \)-energy \( \Phi(x, t) = \frac{1}{p}a(x)t^p \). If \( a \) itself is a Muckenhoupt weight, then it is well known that smooth functions are dense, so \( W^{1,\Phi(\cdot)}(\Omega) = H^{1,\Phi(\cdot)}(\Omega) \). For other results on the density in the context of weighted Sobolev spaces with even variable exponents \( \Phi(x, t) = a(x)t^{p(x)} \), we refer to [15, 16]. The gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains were obtained by Byun and Oh in [4]. Skrypnik and Voitovych recently proved pointwise continuity of solutions for a general class of elliptic and parabolic equations with nonstandard growth conditions using the De Giorgi-Ladyzhenskaya-Ural’tseva classes, see [14]. More models and the extensive list of the references on the generalized Orlicz functions could be found in the book by Harjulehto and Hästö [9]. In the present paper we study the integrands of the form

\[
\Phi_{p,\alpha,\beta}(x, t) \sim \frac{1}{p}t^p \log^{-\beta}(e + t) + a(x)\frac{1}{p}t^p \log^{\alpha}(e + t)
\]

and in particular for \( p = 2 \) and

\[
\Phi_{\alpha,\beta}(x, t) = \Phi_{2,\alpha,\beta}(x, t) \sim \frac{1}{2}t^2 \log^{-\beta}(e + t) + a(x)\frac{1}{2}t^2 \log^{\alpha}(e + t).
\]

where \( a \) is a non negative bounded weight. The regularity properties of the integrand of this type for \( \Phi_{0,1}(x, t) \) were studied by Baroni, Colombo, Mingione in [2] where it was called “the borderline case of double phase potential”. In particular they obtained the \( C^{0,\gamma}_{\text{loc}} \) regularity result for the minimizers provided that the weight \( a(x) \) is log-Hölder continuous (with some \( \gamma \)) and more strong result (any \( \gamma \in (0, 1) \)) for the case of vanishing log-Hölder continuous weight. In comparison with these results we obtain regularity results in the sense of density of smooth functions even if the weight is not continuous. The main result is contained in Theorem 7. It gives the full description of

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the checkerboard-type geometry for the borderline case of the double-phase potential \( \Phi_{\alpha,\beta} \). More precisely, we give the necessary and sufficient conditions on the parameters \( \alpha, \beta \) for the density of smooth functions.

The crucial point for the study of regularity for these classes of problems is possible Lavrentiev gap. The first example in this direction is the famous Zhikov’s checkerboard example for variable exponent, see [17]. This example became the guiding principle for other models. In 1995 Zhikov [18, Example 3.1] considered the double phase potential, later generalized by Esposito, Leonetti and Mingione in [8] to the case of higher dimensions and less regular weights. The general procedure to construct examples for Lavrentiev gap was presented by the authors of this paper and Lars Diening in [1].

We study the corresponding energy, which given by the integral functional

\[
F(u) = \int_{\Omega} \Phi_{p,\alpha,\beta}(x,t) \, dx
\]

and closely related functionals

\[
G(u) = F(u) + \int_{\Omega} b \cdot \nabla u \, dx
\]

We provide examples of the Lavrentiev gap for \( \Phi_{\alpha,\beta}(x,t) \) using one-saddle point construction, which is similar to the initial one from Zhikov’s checker-board examples. The energy \( F \) defines a generalized Sobolev-Orlicz space \( W^{1,\Phi}(\cdot,\Omega) \) and its counterpart \( W^{1,\Phi}_0(\cdot,\Omega) \) with zero boundary values, see Section 2 for the precise definition of the spaces. Then the above Lavrentiev gap can be also written as

\[
E_1 := \inf \{ G(W^{1,\Phi_0}(\cdot,\Omega)) | (\cdot,\Omega) \} < \inf \{ G(H^{1,\Phi_0}(\cdot,\Omega)) | (\cdot,\Omega) \} := E_2,
\]

where \( H^{1,\Phi_0}(\cdot,\Omega) \) is the closure of \( C^\infty_0(\cdot,\Omega) \) functions in \( W^{1,\Phi_0}(\cdot,\Omega) \).

In the second part of the paper we study more general integrands of double phase potential type

\[
\Phi(x,t) := \varphi(t) + a(x)\psi(t).
\]

The main result is formulated in the Theorems 17, 18.

### 2. Energy and Generalized Orlicz Spaces

In this section we introduce the necessary function spaces, the so called generalized Orlicz and Orlicz-Sobolev spaces.

We assume that \( \Omega \subset \mathbb{R}^d \) is a Lipschitz domain of finite measure. Later in our applications we will only use \( \Omega = (-1,1)^2 \).

We say that \( \phi : [0, \infty) \to [0, \infty] \) is an Orlicz function if \( \phi \) is convex, left-continuous, \( \phi(0) = 0 \), \( \lim_{t \to 0} \phi(t) = 0 \) and \( \lim_{t \to \infty} \phi(t) = \infty \). The conjugate Orlicz function \( \phi^* \) is defined by

\[
\phi^*(s) := \sup_{t \geq 0} \left( st - \phi(t) \right).
\]

In particular, \( st \leq \phi(t) + \phi^*(s) \).

In the following we assume that \( \Phi : \Omega \times [0, \infty) \to [0, \infty] \) is a generalized Orlicz function, i.e. \( \Phi(x,\cdot) \) is an Orlicz function for every \( x \in \Omega \) and \( \Phi(\cdot, t) \) is measurable for every \( t \geq 0 \). We define the conjugate function \( \Phi^* \) point-wise, i.e. \( \Phi^*(x,\cdot) := (\Phi(x,\cdot))^* \).

We further assume the following additional properties:

(a) We assume that \( \Phi \) satisfies the \( \Delta_2 \)-condition, i.e. there exists \( c \geq 2 \) such that for all \( x \in \Omega \) and all \( t \geq 0 \)

\[
\Phi(x,2t) \leq c \Phi(x,t).
\]
(b) We assume that $\Phi$ satisfies the $\nabla_2$-condition, i.e. $\Phi^*$ satisfies the $\Delta_2$-condition. As a consequence, there exist $s > 1$ and $c > 0$ such that for all $x \in \Omega$, $t \geq 0$ and $\gamma \in [0, 1]$ there holds
\[
\Phi(x, \gamma t) \leq c \gamma^s \Phi(x, t).
\]

(c) We assume that $\Phi$ and $\Phi^*$ are proper, i.e. for every $t \geq 0$ there holds $\int_\Omega \Phi(x, t) \, dx < \infty$ and $\int_\Omega \Phi^*(x, t) \, dx < \infty$.

We assume that
\[
-c_0 + c_1 |t|^{p_-} \leq \Phi(x, t) \leq c_2 |t|^{p_+} + c_0,
\]
where $1 < p_- \leq p_+ < \infty$, $c_0 \geq 0$, $c_1, c_2 > 0$.

Let $L^0(\Omega)$ denote the set of measurable function on $\Omega$ and $L_{\text{loc}}^1(\Omega)$ denote the space of locally integrable functions. We define the generalized Orlicz norm by
\[
\|f\|_{L^\Phi(\Omega)} := \inf \left\{ \gamma > 0 : \int_\Omega \Phi(x, |f(x)/\gamma|) \, dx \leq 1 \right\}.
\]

Then generalized Orlicz space $L^\Phi(\Omega)$ is defined as the set of all measurable functions with finite generalized Orlicz norm
\[
L^\Phi(\Omega) := \{ f \in L^0(\Omega) : \|f\|_{L^\Phi(\Omega)} < \infty \}.
\]

For example the generalized Orlicz function $\Phi(x, t) = t^p$ generates the usual Lebesgue space $L^p(\Omega)$.

The $\Delta_2$-condition of $\Phi$ and $\Phi^*$ ensures that our space is uniformly convex. The condition that $\Phi$ and $\Phi^*$ are proper ensure that $L^\Phi(\Omega) \hookrightarrow L^1(\Omega)$ and $L^\Phi^*(\Omega) \hookrightarrow L^1(\Omega)$. Thus $L^\Phi(\Omega)$ and $L^\Phi^*(\Omega)$ are Banach spaces.

We define the generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ as
\[
W^{1,\Phi}(\Omega) := \{ w \in W^{1,1}(\Omega) : \nabla w \in L^\Phi(\Omega) \},
\]
with the norm
\[
\|w\|_{W^{1,\Phi}(\Omega)} := \|w\|_{L^1(\Omega)} + \|\nabla w\|_{L^\Phi(\Omega)}.
\]

In general smooth functions are not dense in $W^{1,\Phi}(\Omega)$. We define $H^{1,\Phi}(\Omega)$ as
\[
H^{1,\Phi}(\Omega) := \text{closure of } C_0^\infty(\Omega) \cap W^{1,\Phi}(\Omega) \text{ in } W^{1,\Phi}(\Omega).
\]
See [7] and [9] for further properties of these spaces.

We also introduce the corresponding spaces with zero boundary values as
\[
W^{1,\Phi}_0(\Omega) := \{ w \in W^{1,1}_0(\Omega) : \nabla w \in L^\Phi(\Omega) \}
\]
with the norm
\[
\|w\|_{W^{1,\Phi}_0(\Omega)} := \|\nabla w\|_{L^\Phi(\Omega)},
\]
and
\[
H^{1,\Phi}_0(\Omega) := \text{closure of } C_0^\infty(\Omega) \text{ in } W^{1,\Phi}_0(\Omega).
\]

The space $W^{1,\Phi}_0(\Omega)$ is exactly those functions, which can be extended by zero to $W^{1,\Phi}([\mathbb{R}^d])$ functions.

Let us define our energy $\mathcal{F} : W^{1,\Phi}(\Omega) \to \mathbb{R}$ by
\[
\mathcal{F}(w) := \int_\Omega \Phi(x, |\nabla w(x)|) \, dx.
\]

In the language of function spaces $\mathcal{F}$ is a semi-modular on $W^{1,\Phi}(\Omega)$ and a modular on $W^{1,\Phi}_0(\Omega)$.

**Definition 1.** An integrand $\Phi(x, t)$ is said to be regular in the domain $\Omega$ if for all $u \in W^{1,\Phi}_0(\Omega)$ with $\mathcal{F}(u) < \infty$ there exists a smooth sequence $u_\varepsilon \in C_0^\infty(\Omega)$ such that
(a) $u_\varepsilon \to u$ in $W^{1,1}_0(\Omega)$;
(b) $\lim_{\varepsilon \to 0} \int_\Omega \Phi(x, \nabla u_\varepsilon) \, dx = \int_\Omega \Phi(x, \nabla u) \, dx$. 


Direct from the Definition 1 follows that if the integrand $\Phi(x, t)$ is regular, then $E_1 = E_2$, so there is no Lavrentiev gap. This is equivalent to $H^1_{0, \Phi(x)}(\Omega) = W^1_{0, \Phi(x)}(\Omega)$. Indeed, the reverse implication (from $H = W$ to regularity) is obvious. On the other hand, from Definition 1 by Scheffe’s theorem it follows that $\Phi(x, |\nabla u|)$ converges to $\Phi(x, |\nabla u|)$ in $L^1(\Omega)$. From this, convexity of $\Phi(x, \cdot)$ and the $\Delta_2$ condition it follows that $\Phi(x, |\nabla u| - |\nabla v|) \leq \Phi(x, |\nabla u|) + \Phi(x, |\nabla v|)$ is equiintegrable, therefore it converges to zero. Thus $u_\varepsilon$ is a sequence of $C^\infty_0(\Omega)$ functions approximating $u$ in $W^1_{0, \Phi(x)}(\Omega)$, and so $H^1_{0, \Phi(x)}(\Omega) = W^1_{0, \Phi(x)}(\Omega)$.

We use the following lemma due to Zhikov.

**Lemma 2** ([18], Lemma 2.1). Let $\Omega$ be the star-shaped with respect to the origin domain. Assume, that there exist functions $\Phi_x(x, t)$ such that $\Phi_x$ are convex with respect to $t$ and measurable with respect to $x$ and let the following relations hold.

(a) $\Phi_x(x, 0) = 0$;

(b) $c_1 \Phi_x(x, t) \leq \Phi_x(x, t) + c_3$ for $x \in \Omega$, $t \leq M \varepsilon^{-\frac{d}{p}}$;

(c) $\Phi_x(x, t) \leq c_2 \Phi_x(x, t) + c_3$ for $|x - y| \leq 2k\varepsilon$, $t \in \mathbb{R}^+$,

with $k, M, c_1, c_2, c_3 > 0$ and $0 < \varepsilon < \varepsilon_0$. Then the integrand $\Phi(x, t)$ is regular.

The function $\omega(x)$ is the modulus of continuity for the weight $a(x)$ if

$$|a(x) - a(y)| \leq \omega(|x - y|) \quad x, y \in \mathbb{R}^d, \quad |x - y| \leq \frac{1}{4}.$$  

By $B_r(x)$ we denote the ball of radius $r$ with center at the point $x$.

**Corollary 3.** Let

$$\Phi(x, t) := \varphi(t) + a(x)\psi(t),$$

with Orlicz functions $\varphi(t)$, $\psi(t)$. Assume that the weight $a(x)$ is non-negative, bounded and has the modulus of continuity

$$\omega(\varepsilon) \leq k_0 \frac{\varphi(t)}{\psi(t)}, \quad 1 \leq t \leq \varepsilon^{-d}. \tag{8}$$

Then the integrand $\Phi_{\alpha, \beta}$ is regular. In particular, $C^\infty_0(\Omega)$ is dense in $W^1_{0, \Phi(x)}(\Omega)$.

**Proof.** We set

$$\Phi_x(x, t) := \varphi(t) + a_x(x)\psi(t)$$

with

$$a_x(x) = \min \{a(y), \quad y \in \bar{\Omega} \cap B_r(x)\}.$$  

From the definition of $a_x$ it follows that

$$a_x(x) \leq a(y) \quad \text{if} \quad |x - y| < \varepsilon,$$

and thus condition (c) of Lemma 2 is verified.

Now, using (8) we get

$$\Phi(x, t) \leq \varphi(t) + a_x(x)\psi(t) + \omega(\varepsilon)\psi(t) \leq \Phi_x(x, t) + k_0 \varphi(t) \leq (k_0 + 1)\Phi_x(x, t)$$

for $1 \leq t \leq \varepsilon^{-d}$. Since $\Phi(x, t) \leq k_1$ for $t \leq 1$, we see that the condition (b) of Lemma 2 is fulfilled. It is easy to see that condition (a) also holds.

**Remark 4.** Let us mention, that Corollary 3 holds for any $d$ independently on the dimension. Note, that the further results are for $p = d = 2$.  

In particular we see that for \( \Phi_{p,\alpha,\beta} \) defined in (1) all conditions of Corollary 3 are fulfilled provided that the weight \( a(x) \) has the modulus of continuity
\[
\omega(r) \leq \frac{k_0}{\log^{\alpha+\beta/(r-1)}}, \quad \text{if } r \leq \frac{1}{4}.
\]

From the point of view of functional spaces we are interested mostly in the Zygmund classes
\[ L^p \log^\gamma L, \quad 1 \leq p < \infty, \quad \gamma \in \mathbb{R}, \]
which were studied, for example by Iwaniec and Sbordone in [10, Section 18]. These classes correspond to \( \varphi(t) = t^p \log^\gamma(e + t) \). These Orlicz classes are convex if \( \gamma \geq 1 - p \). This issue is not important for us, since we are only interested in the behaviour of the integrand for large values of \( t \), hence we can always replace it by an appropriate convex Orlicz substitution. The norm, defined as
\[
\|f\|_{L^p \log^\gamma L} := \left( \int_{\Omega} |f|^p \log^\alpha(e + |f|/\|f\|_p) \, dx \right)^{1/p}
\]
is equivalent to the Luxemburg norm. The following Hölder-type inequality is valid
\[
\|AB\|_{L^p \log^\gamma L} \leq C_{\alpha,\beta}(a,b)\|A\|_{L^p \log^\gamma L}\|B\|_{L^p \log^\gamma L}
\]
for
\[
\frac{1}{c} = \frac{1}{a} + \frac{1}{b}, \quad \frac{\gamma}{c} = \frac{\alpha}{a} + \frac{\beta}{b}.
\]
And for
(9) \[ \varphi(t) = \frac{1}{p} |t|^p \log^\gamma(e + |t|), \quad \varphi^*(t) \approx \frac{1}{p} |t|^p \log \frac{e^p}{|t|}. \]
Also
\[
(a(x)|t|^p \log^\gamma(e + |t|))^+ \approx a(x)^{1-p'} |t|^p \log \frac{e^p}{a(x)} \left( e + \frac{|t|}{a(x)} \right).
\]

3. Borderline case of double phase potential

In this section we consider the integrand \( \Phi_{\alpha,\beta} \) defined in (2) in planar domains. We describe its regularity for the checkerboard geometry. Further in this section we drop \( \alpha, \beta \) from the notation \( \Phi_{\alpha,\beta} \) and write
\[
\Phi(x,t) = \varphi(t) + a(x)\psi(t), \quad \varphi(t) = t^2 \log^\beta(e + t), \quad \psi(t) = t^2 \log^\alpha(e + t).
\]
So instead of \( W^{1,\Phi_{\alpha,\beta}(\cdot)}(\Omega) \) we write simply \( W^{1,\Phi(\cdot)}(\Omega) \), and the same convention is used for other spaces.

We denote \( \Omega = (-1,1)^2 \) and use the notation from [1].

**Definition 5** (Checkerboard setup). Let
\[
x = (x_1, x_2), \quad x_1, x_2 \in \mathbb{R}.
\]
We define \( u_2, A_2 \) and \( b_2 \) on \( \mathbb{R}^2 \) by
\[
u_2 := \frac{1}{2} \text{sgn}(x_2) \theta \left( \frac{|x_2|}{|x_1|} \right),
\]
\[
A_2 := \theta \left( \frac{|x_1|}{|x_2|} \right) \frac{1}{\sigma_1} |x|^{-1} \begin{pmatrix} 1 & -1 \\ 0 & x_1 \\ 0 & 0 \end{pmatrix},
\]
\[
b_2 := \text{div} A_2,
\]
where \( \sigma_1 = 2 \) is the “surface area” of the 1-dimensional sphere and \( \theta \in C_0^\infty((0,\infty)) \) is such that \( 1_{(\frac{1}{2},\infty)} \leq \theta \leq 1_{(\frac{1}{4},\infty)}, \quad \|\theta\|_\infty \leq 6. \)
The matrix divergence is taken rowwise, i.e. for matrix $A = \{A_{ij}\}$ we define $(\text{div } A)_i = \partial_j A_{ij}$. That is,

$$b_2 = \nabla^1 v, \quad v = \frac{1}{2} \text{sgn}(x_1) \theta\left(\frac{|x_1|}{|x_2|}\right).$$

The following properties of functions $u_2$, $b_2$, $A_2$ were proved in [1].

**Proposition 6.** There holds

(a) $u_2 \in L^\infty(\mathbb{R}^2) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$,
(b) $A_2 \in W^{1,1}_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$,
(c) $b_2 \in L^1_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$.

(d) The following estimates hold

$$|\nabla u_2| \lesssim |x_2|^{-1} \mathbb{1}_{\{2|x_2| \leq |x_1| \leq 4|x_2|\}} \approx |x_1|^{-1} \mathbb{1}_{\{2|x_2| \leq |x_1| \leq 4|x_2|\}}$$

$$|b_2| \lesssim \frac{|x_2|}{x_1}^{-1} \mathbb{1}_{\{2|x_2| \leq |x_1| \leq 4|x_2|\}} \approx |x_1|^{-1} \mathbb{1}_{\{2|x_2| \leq |x_1| \leq 4|x_2|\}}.$$

(e) $|\nabla u_2| \cdot |b_2| = 0$.

(f) $\int_{\partial \Omega}(b_2 \cdot v)u_2 \, dS = 1$.

We denote

$$C_+ = \{x : |x_1| < x_2\} \cap \Omega,$$

$$C_- = \{x : |x_1| < -x_2\} \cap \Omega.$$

The weight $a(x)$ as defined as

$$a(x) = \begin{cases} 
1, & \text{if } |x_1| < |x_2| \\
0, & \text{if } |x_1| \geq |x_2|.
\end{cases}$$

The Figure 1 shows the function $u$, the $(2,1)$-component of $A$ and a possible weight $a(x)$. It jumps from 0 to 1 and in the filled regions is smooth transition.

![Weight a](image1)

![Function u](image2)

![Function v](image3)

**Figure 1.** One saddle point

**Theorem 7.** For the integrand $\Phi = \Phi_{\alpha,\beta}$ the equality $H^1_{0,\Phi(\cdot)} = W^1_{0,\Phi(\cdot)}$ is valid when $\min(\alpha, \beta) \leq 1$.

If $\alpha > 1$ and $\beta > 1$, then $H^1_{0,\Phi(\cdot)} \neq W^1_{0,\Phi(\cdot)}$ and there exists $b \in L^{\Phi(\cdot)}(\Omega)$ such that there is Lavrentiev gap for $G(\cdot)$ defined by (3):

$$\inf G(W^1_{0,\Phi(\cdot)}) < \inf G(H^1_{0,\Phi(\cdot)}).$$

Moreover, in this case the codimension of $H^1_{0,\Phi(\cdot)}$ in $W^1_{0,\Phi(\cdot)}$ is one.

The proof of this theorem is split into several lemmata.

**Lemma 8.** If $\alpha > 1$ and $\beta > 1$ then $u_2 \in W^1_{\Phi(\cdot)}(\Omega)$ and $b_2 \in L^{\Phi(\cdot)}(\Omega)$. 

Proof. Clearly

\[ \{ \|\nabla u\|_2 \neq 0 \} \subset \{ a = 0 \}, \quad \{ \|b_2\| \neq 0 \} \subset \{ a = 1 \}. \]

Then by Proposition 6 we get

\[
\int_{\Omega} \Phi^*(x, |b_2|) \, dx \lesssim \int_{\Omega \cap \{a=1\}} |b_2|^2 \log^{-\alpha}(e + |b_2|) \, dx
\]

\[
\lesssim \int_0^2 \frac{dt}{t \log^\alpha(e + t)} < \infty
\]

provided \( \alpha > 1 \). And

\[
\int_{\Omega} \Phi(x, |\nabla u|_2) \, dx \lesssim \int_{\Omega \cap \{a=0\}} |\nabla u|^2 \log^{-\beta}(e + |\nabla u|_2) \, dx
\]

\[
\lesssim \int_0^2 \frac{dt}{t \log^\beta(e + t)} < \infty,
\]

provided \( \beta > 1 \).

Let \( D_h = C_+ \cap \{ 0 < x_2 < h \} \).

**Lemma 9.** If \( \alpha > 1 \) and \( u \in W^{1, \Phi(\cdot)}(\Omega) \) then it is continuous in \( \overline{C_\pm} \) with modulus of continuity

\[ \omega(t) \leq C(\alpha) \|\nabla u\|_{L^\psi(\cdot)(C_\pm)} \log^{\frac{1-\alpha}{\alpha}}(1/t), \quad t < 1/e. \]

Moreover,

\[ \omega(t) \log^{\frac{2}{1+\alpha}}(1/t) \to 0 \quad \text{as} \quad t \to 0. \]

**Proof.** We start with the well-known estimate of \( u \) in terms of Riesz potential and Hölder inequality

\[ |u(x) - (u)_{D_h}| \lesssim \int_{D_h} \frac{|
abla u(y)|}{|x - y|} \, dy \lesssim \|\nabla u\|_{L^{\psi}(\cdot)(D_h)} \|x - \cdot\|_{L^{\psi}(\cdot)(D_h)}^{-1}, \quad x \in \overline{D_h}. \]

By definition

\[ \|x - \cdot\|_{L^{\psi}(\cdot)(D_h)}^{-1} \lesssim \|r^{-1}\|_{L^{\psi}(\cdot)(B_{2h}(0))} \]

\[ \lesssim \inf \left\{ \lambda > 0 : 2\pi \int_0^{2h} \psi^*(\lambda r)^{-1}r \, dr \leq 1 \right\} \]

\[ \lesssim \inf \left\{ \lambda > 0 : \int_0^{2h} (\lambda r)^{-2} \log^{-\alpha}(e + (\lambda r)^{-1})r \, dr \leq c(\alpha) \right\}. \]
Now
\[
\int_0^{2h} (\lambda r)^{-2} \log^{-\alpha} (e + (\lambda r)^{-1}) r \, dr = \lambda^{-2} \int_0^{2\lambda h} t^{-1} \log^{-\alpha} (e + t^{-1}) \, dt \\
= \lambda^{-2} \int_{(2\lambda h)^{-1}}^\infty s^{-1} \log^{-\alpha} (e + s) \, ds = \lambda^{-2} \frac{1}{\alpha - 1} \log^{1-\alpha} (e + (2\lambda h)^{-1}).
\]
And so
\[
\|x - |x|^{-1}\|_{L^{\Phi^*(\Omega)}(D_h)} \leq \left(2h \sup \left\{ \lambda > 0 : \lambda^2 \log^{1-\alpha} (e + \lambda) \leq c(\alpha) h^{-2} \right\}\right)^{-1} \\
\leq c(\alpha) \log^{1-\alpha} \frac{1}{h}.
\]
Then
\[
|u(x) - \langle u \rangle_{D_h}| \leq c(\alpha) \|\nabla u\|_{L^{\Phi^*(\Omega)}(D_h)} \log^{1-\alpha} \frac{1}{h}, \quad x \in \bar{D}_h.
\]
This proves the required continuity at the origin when it is approached from \(C_+\) and for other points of \(C_+\) the proof is by obvious modification. For \(C_-\) the reasoning is the same. \(\square\)

Let \(\alpha > 1\) and define using Lemma 9 the limit values
(10) \[
\begin{align*}
 u_+ &= \lim_{C_+ \not\rightarrow 0} u(x), \\
 u_- &= \lim_{C_- \not\rightarrow 0} u(x).
\end{align*}
\]

**Lemma 10.** Assume that \(\alpha > 1, \beta \leq 1\) and \(u \in W^{1,\Phi^*(\Omega)}\). Then \(u_+ = u_-\).

**Proof.** Indeed, assume that \(u_+ \neq u_-\). We assume without loss that \(|u_+ - u_-| = 1\). Then for any \(s \in (0, 1/4)\) we have
\[
\int_{s}^{h} |\nabla u(s, x)| \, dx \geq 1
\]
and upon integration over \(s \in (h/2, h), h \leq 1/4\), this yields
\[
\int_{\Omega_h} |\nabla u| \, dx \geq \frac{h}{2},
\]
where \(\Omega_h = \{(x_1, x_2) : \frac{h}{2} \leq x_1 \leq h, \ |x_2| < x_1\}\). Now
\[
\frac{h}{2} \leq \|\nabla u\|_{L^{\Phi^*(\Omega_h)}} \|1\|_{L^{\Phi^*(\Omega_h)}}
\]
and
\[
\|1\|_{L^{\Phi^*(\Omega_h)}} = \|1\|_{L^{\Phi^*(\Omega_h)}} \\
= \inf \left\{ \lambda > 0 : \int_{\Omega_h} \lambda^{-2} \log^\beta (e + \lambda^{-1}) \, dx \leq c(\beta) \right\} \\
= \inf \left\{ \lambda > 0 : \lambda^{-2} \log^\beta (e + \lambda^{-1}) \, dx \leq c(\beta) h^{-2} \right\} \\
\leq h \log^{\beta/2} \frac{1}{h}
\]
yield
\[
\|\nabla u\|_{L^{\Phi^*(\Omega_h)}} \geq \varepsilon \log^{-\beta/2} \frac{1}{h}
\]
with some positive constant \(\varepsilon = \varepsilon(\beta) \in (0, 1)\).

By definition of the Luxemburg norm we have
\[
\int_{\Omega_h} \frac{|\nabla u|^2}{\varepsilon^2 \log^{-\beta} \frac{1}{h}} \log^{-\beta} \left(e + \frac{|\nabla u|}{\varepsilon \log^{-\beta/2} \frac{1}{h}}\right) \, dx \geq 1.
\]
Therefore for $h \leq 2^{-j_0}$ for some $j_0 > 0$ there holds
\[
\int_{\Omega_h} |\nabla u|^2 \log^{-\beta} \left( e + \frac{|\nabla u|}{\varepsilon \log^{-\beta} \frac{1}{h}} \right) \, dx \geq \varepsilon^2 \log^{-\beta} \frac{1}{h},
\]
\[
\int_{\Omega_h} |\nabla u|^2 \log^{-\beta} \left( e + |\nabla u| \right) \, dx \geq \varepsilon^2 \log^{-\beta} \frac{1}{h}.
\]
Summing the last inequality over $h = 2^{-j}$, $j \geq j_0$, we arrive at
\[
\int_{\Omega} |\nabla u|^2 \log^{-\beta} \left( e + |\nabla u| \right) \, dx \geq \sum_{j=j_0}^{\infty} \varepsilon^2 \log^{-\beta} 2^j \geq \frac{\varepsilon^2}{\log\beta} \sum_{j=j_0}^{\infty} \frac{1}{j^\beta} = +\infty
\]
provided that $\beta \leq 1$. This proves $u_\varepsilon = u_+$. \qed

**Lemma 11.** Let $u \in W^{1,\Phi(\cdot)}(\Omega)$ and $u = 0$ in a neighbourhood of the origin. Then $u \in H^{1,\Phi(\cdot)}(\Omega)$. If $u \in W^{1,\Phi(\cdot)}_0(\Omega)$ and $u = 0$ in a neighbourhood of the origin then $u \in H^{1,\Phi(\cdot)}_0(\Omega)$.

**Proof.** By partition of unity, rotation and dilation the proof is reduced to showing the following fact. Let $\Phi(x,t) = \varphi(t) + \tilde{a}(x) \psi(t)$, where $\tilde{a}(x) = 0$ when $x_2 > 0$ and $\tilde{a}(x) = 1$ when $x_2 < 0$. Denote $Q = \{ (x_1, x_2) : |x_1| + |x_2| < 1 \}$. Then $W^{1,\Phi(\cdot)}(Q) = H^{1,\Phi(\cdot)}(Q)$ and $W^{1,\Phi(\cdot)}_0(Q) = H^{1,\Phi(\cdot)}_0(Q)$.

Let $v \in W^{1,\Phi(\cdot)}(Q)$. Denote $Q_+ = Q \cap \{ x_2 > 0 \}$, $Q_- = Q \cap \{ x_2 < 0 \}$ and by $w$ the even extension of the function $v$ from $Q_-$ to $Q_+$, that is
\[
w(x_1, x_2) = \begin{cases} v(x_1, x_2), & x_2 < 0, \\ v(x_1, -x_2), & x_2 \geq 0. \end{cases}
\]
Set $z = v - w$. The function $z$ has zero trace on $\{ x_2 = 0 \}$ and vanishes on $Q_-$. In the region $Q_+$ obviously $\bar{\Phi}(x,t) = \varphi(t)$ is independent of $x$. So there exists (by the standard mollification procedure) a sequence $z_\varepsilon \in C^\infty(Q_+)$ such that $z_\varepsilon = 0$ when $\{ x_2 < \varepsilon \}$ and $z_\varepsilon \to z$ in $W^{1,\psi(\cdot)}(Q_+)$, therefore $z_\varepsilon \to z$ in $W^{1,\Phi(\cdot)}(Q)$. On the other hand, $w \in W^{1,\psi(\cdot)}(Q)$ and thus it can be approximated by $w_\varepsilon \in C^\infty(Q)$ in $W^{1,\psi(\cdot)}(Q)$. Take $u_\varepsilon = w_\varepsilon + z_\varepsilon$. Clearly, it converges to $u$ in $W^{1,\Phi(\cdot)}(Q)$.

For $v \in W^{1,\Phi(\cdot)}_0(Q)$ the proof is the same, but $z \in W^{1,\psi(\cdot)}_0(Q_+)$, $w \in W^{1,\psi(\cdot)}_0(Q)$, so we can take approximating sequences $z_\varepsilon$ from $C^\infty_0(Q_+)$ and $w_\varepsilon$ from $C^\infty_0(Q)$. \qed

Thus the difference between $W^{1,\Phi(\cdot)}(\Omega)$ and $H^{1,\Phi(\cdot)}(\Omega)$ is in some sense concentrated at the origin (the saddle point). In the next statement we claim that this possible singularity is always removable provided that $\alpha \leq 1$.

**Lemma 12.** Let $\alpha \leq 1$. Then there exists a sequence of functions $\eta_\varepsilon \in C^\infty(\Omega)$, $\varepsilon \to 0$, such that $\eta_\varepsilon = 1$ outside $B_\varepsilon(0)$, $\eta_\varepsilon = 0$ in a neighbourhood of the origin, $0 \leq \eta_\varepsilon \leq 1$, and
\[
\int_{\Omega} \Phi(x, |\nabla \eta_\varepsilon|) \, dx \to 0 \text{ as } \varepsilon \to 0.
\]

**Proof.** Take $0 < \varepsilon < 1/10$ and set
\[
\eta_\varepsilon(r) = \begin{cases} \frac{1}{\log(1/\varepsilon) - \log(1/r)}, & r \geq \varepsilon, \\ \frac{e^{-1/\varepsilon}}{\log(1/\varepsilon) - \log(1/\varepsilon)}, & e^{-1/\varepsilon} < r < \varepsilon, \\ 0, & r \leq e^{-1/\varepsilon}. \end{cases}
\]
Clearly it is sufficient to show that
\[
I_\varepsilon = \int_{\Omega} |\nabla \eta_\varepsilon|^2 \log(e + |\nabla \eta_\varepsilon|) \, dx \to 0 \text{ as } \varepsilon \to 0.
\]
We evaluate
\[
I_\varepsilon \lesssim \int_{e^{-1/\varepsilon}}^\varepsilon \frac{1}{r^2 \log^2(1/r) \log^2(1/\varepsilon)} \log \left( \frac{1}{r \log(1/r) \log(1/\varepsilon)} \right) r \, dr \\
\leq \frac{1}{\log^2(1/\varepsilon)} \int_{e^{-1/\varepsilon}}^\varepsilon \frac{1}{r \log(1/r)} \, dr \leq \frac{1}{\log(1/\varepsilon)}.
\]
It remains to send \( \varepsilon \) to zero. \( \square \)

**Corollary 13.** Let \( \alpha \leq 1 \). If \( u \in W^{1,\Phi(\cdot)}(\Omega) \) then \( u \in H^{1,\Phi(\cdot)}(\Omega) \). If \( u \in W_0^{1,\Phi(\cdot)}(\Omega) \) then \( u \in H_0^{1,\Phi(\cdot)}(\Omega) \).

**Proof.** Any function from \( W^{1,\Phi(\cdot)}(\Omega) \) or \( W_0^{1,\Phi(\cdot)}(\Omega) \) can be approximated by bounded functions (it is enough to consider standard level cuts). So without loss of generality we assume that \( u \in W^{1,\Phi(\cdot)}(\Omega) \) \( \|u\|_{L^\infty(\Omega)} < \infty \) (for \( u \in W_0^{1,\Phi(\cdot)}(\Omega) \) the proof is the same). Consider
\[
uu = uu_u,
\]
where \( \eta_u \) is defined in Lemma 12. Then
\[
\nabla u - \nabla \nuu = -u \nabla \eta_u + (1 - \eta_u) \nabla u
\]
and by the \( \Delta_2 \) property
\[
\int_\Omega \Phi(x, |\nabla (u - \nuu)|) \, dx \lesssim \int_\Omega \Phi(x, |\nabla \eta_u|) \, dx + \int_\Omega \Phi(x, |1 - \eta_u| |\nabla u|) \, dx.
\]
The first term on the right-hand side goes to zero by Lemma 12, and the second term tends to zero by the Lebesgue dominated convergence theorem. \( \square \)

Recall that \( u_+ \) and \( u_- \) are defined by (10) when \( \alpha > 1 \).

**Corollary 14.** Let \( \alpha > 1 \) and \( u \in W^{1,\Phi(\cdot)}(\Omega) \). If \( u_+ = u_- \) then \( u \in H^{1,\Phi(\cdot)}(\Omega) \). If \( u \in W_0^{1,\Phi(\cdot)}(\Omega) \) and \( u_+ = u_- \) then \( u \in H_0^{1,\Phi(\cdot)}(\Omega) \).

**Proof.** Set \( u(0) = u_+ = u_- \). Due to Lemma 9 we have
\[
|u(x) - u(0)| \lesssim \log \frac{M}{|x|}, \quad \text{if} \quad x \in C_+ \cup C_-.
\]
Let
\[
uu = u(0) + (u - u(0)) \eta_u.
\]
By Lemma 11 we have \( \nuu \in H^{1,\Phi(\cdot)}(\Omega) \). Without loss we can assume \( u \) to be bounded, \( \|u\| \leq M < \infty \). We claim that \( \nuu \to u \) in \( W^{1,\Phi(\cdot)}(\Omega) \), therefore \( u \in H^{1,\Phi(\cdot)}(\Omega) \). We have to check that
\[
\int_\Omega \Phi(x, |\nabla (\nuu - u)|) \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
First we prove that
\[
I_\varepsilon := \int_{C_+ \cup C_-} \Phi(x, |(u - u(0)) \nabla \eta_u|) \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Using polar coordinates we estimate
\[
I_\varepsilon \lesssim \int_{C_+ \cup C_-} (u - u(0))^2 |\nabla \eta_u|^2 \log^\alpha \left( \varepsilon + |u - u(0)||\nabla \eta_u| \right) \, dx
\]
\[
\lesssim \int_{e^{-1/\varepsilon}}^{e^\varepsilon} \frac{\log^{1-\alpha} \frac{1}{\varepsilon}}{r^2 \log^2 (1/\varepsilon)} \log^\alpha \left( \varepsilon + \frac{2M}{r \log(1/r) \log(1/\varepsilon)} \right) r \, dr
\]
\[
\lesssim \frac{1}{\log^2(1/\varepsilon)} \int_{e^{-1/\varepsilon}}^{e^\varepsilon} \frac{1}{r \log(1/r)} \, dr \lesssim \frac{1}{\log(1/\varepsilon)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
For $\beta \geq -1$

$$J_\varepsilon = \int_{\Omega \setminus (C_+ \cup C_-)} \Phi(x, |(u - u(0))| \nabla \eta_\varepsilon) \, dx \to 0 \quad \text{as } \varepsilon \to 0$$

by Lemma 12.

If $\beta < -1$ then $C(\Omega)$ with the same argument as in Lemma 9 with modulus of continuity $\log^{1/\beta} \frac{1+r}{1}$ and convergence of $J_\varepsilon \to 0$ is by the same argument as above for $I_\varepsilon$, where $\alpha$ is replaced by $-\beta$. Then

$$\int \Phi(x, |\nabla(u - u)|) \, dx \lesssim I_\varepsilon + J_\varepsilon + \int \Phi(x, (1 - \eta_\varepsilon) |\nabla u|) \, dx,$$

where the last term goes to zero by Lebesgue theorem.

For the second statement of the lemma regarding functions with zero boundary values we take $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in a neighborhood of the origin and set

$$u_\varepsilon = u(0)\eta + (u - u(0)\eta)\eta_\varepsilon.$$

By Lemma 11 we have $u_\varepsilon \in H_0^{1,\Phi(-)}(\Omega)$. Clearly,

$$\nabla u_\varepsilon - \nabla u = (\nabla u - u(0)\nabla \eta)(\eta_\varepsilon - 1) + (u - u(0)\eta)\nabla \eta_\varepsilon.$$

The first term on the right-hand side converges to zero in $L^{\Phi(-)}(\Omega)$ by the Lebesgue theorem, and the second term converges to zero by the same argument as above. Therefore, $u_\varepsilon \to u$ in $H_0^{1,\Phi(-)}(\Omega)$ and $u \in H_0^{1,\Phi(-)}(\Omega)$.

Now we prove the reverse statement.

**Proposition 15.** Let $\alpha > 1$. If $u \in H_0^{1,\Phi(-)}(\Omega)$ then $u_+ = u_-.$

**Proof.** Let $u_\varepsilon \to u$ in $W^{1,\Phi(-)}(\Omega)$, $u_\varepsilon \in C_0^\infty(\Omega)$. By Lemma 9, $u_\varepsilon$ are uniformly continuous in $C_+ \cup C_-$ and uniformly converge to $u$ on this set. So the limit function $u$ is continuous in $C_+ \cup C_-$. Hence its limit values $u_+$ and $u_-$ coincide.

Now we are ready to prove Theorem 7.

**Proof of Theorem 7.** If $-\beta \geq \alpha$ then $\Phi(x, t) \sim t^2 \log^{-\beta}(\varepsilon + t)$, and is regular by standard theory (mollifications). Further we assume that $-\beta < \alpha$.

If $\alpha \leq 1$ the statement is by Corollary 13.

If $\alpha > 1$ and $\beta \in [0, 1]$ we first use Lemma 10 and then Corollary 14.

If $\alpha > 1$ and $\beta > 1$ we conclude the proof by application of Lemma 8 and [1, Theorems 26, 28].

We reproduce this argument for convenience of the reader. First,

$$\int_{\Omega} b_2 \cdot \nabla \varphi \, dx = 0$$

for any $\varphi \in C_0^\infty(\Omega)$ (recall that $b_2 = \nabla^4 v$), and hence for $\varphi \in H_0^{1,\Phi(-)}(\Omega)$. Second, let $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in a neighborhood of the origin. We have

$$\int_{\Omega} b_2 \cdot \nabla(\eta u_\varepsilon) \, dx = \int_{\Omega} b_2 \cdot \nabla u_\varepsilon \, dx - \int_{\Omega} b_2 \cdot \nabla((1 - \eta)u_\varepsilon) \, dx = -1$$

by Proposition 6, Lemma 11 and (11). Therefore $\eta u_\varepsilon \in W_0^{1,\Phi(-)}(\Omega)$ but it can not be approximated by smooth functions. The functional $w \mapsto \int_{\Omega} b_2 \cdot \nabla w \, dx$ is a *separating functional* — it is a nontrivial linear bounded functional on $W_0^{1,\Phi(-)}(\Omega)$, vanishing on its subspace $H_0^{1,\Phi(-)}(\Omega)$. To demonstrate the Lavrentiev gap, evaluate

$$G(u) = \int_{\Omega} \Phi(x, |\nabla u|) \, dx + \int_{\Omega} b_2 \cdot \nabla u \, dx$$

on $tu_\varepsilon$; by Proposition 6 and $\nabla_2$ property of $\Phi$ for sufficiently small $t > 0$ there holds

$$G(tu_\varepsilon) = \int_{\Omega} \Phi(x, t|\nabla u|) \, dx - t \leq \frac{t^2}{2} - t < 0.$$
On the other hand, (11) implies $G(w) \geq 0$ for any $w \in H^{1,\Phi(\cdot)}_0(\Omega)$.

Another way to show that $\eta u_2 \notin H^{1,\Phi(\cdot)}_0(\Omega)$ is by Proposition 15 since obviously $(\eta u_2)_+ \neq (\eta u_2)_-$.

To prove that the codimension of $H^{1,\Phi(\cdot)}_0(\Omega)$ in $W^{1,\Phi(\cdot)}_0(\Omega)$ is one, we note that for any $u \in W^{1,\Phi(\cdot)}_0(\Omega)$ the function $w = u - (u_+ - u_-)\eta u_2$ has the same limit values $w_+$ and $w_-:
\begin{align*}
  w_+ - w_- = u_+ - u_- - (u_+ - u_-)((u_2)_+ - (u_2)_-) = u_+ - u_- - (u_+ - u_-) = 0.
\end{align*}

Therefore $w \in H^{1,\Phi(\cdot)}_0(\Omega)$ by Corollary 14. The proof of Theorem 7 is complete. $\square$

Now we turn to the question of regularity of the solution of variational problems $\mathcal{E}_1$ and $\mathcal{E}_2$, see (4). Let us start by introducing spaces with boundary values: for $g \in H^{1,\Phi(\cdot)}(\Omega)$ we define
\begin{align*}
  H^{1,\Phi(\cdot)}_g(\Omega) := g + H^{1,\Phi(\cdot)}_0(\Omega).
\end{align*}

For $g \in W^{1,\Phi(\cdot)}(\Omega)$ we define
\begin{align*}
  W^{1,\Phi(\cdot)}_g(\Omega) := g + W^{1,\Phi(\cdot)}_0(\Omega).
\end{align*}

We can define
\begin{align*}
  h_W(g) &= \arg \min \mathcal{F}(W^{1,\Phi(\cdot)}_g(\Omega)).
\end{align*}

Formally, it satisfies the Euler-Lagrange equation (in the weak sense)
\begin{align*}
  -\Delta \Phi(\cdot) h_W := - \text{div} \left( \frac{\Phi'(x,|\nabla h_W|)}{|\nabla h_W|} \nabla h_W \right) = 0 \quad \text{in } (W^{1,\Phi(\cdot)}(\Omega))^*,
\end{align*}

where $\Phi'(x,t)$ is the derivative with respect to $t$. However, we can define also
\begin{align*}
  h_H(g) &= \arg \min \mathcal{F}(H^{1,\Phi(\cdot)}_g(\Omega)).
\end{align*}

Then
\begin{align*}
  -\Delta \Phi(\cdot) h_H := - \text{div} \left( \frac{\Phi'(x,|\nabla h_H|)}{|\nabla h_H|} \nabla h_H \right) = 0 \quad \text{in } (H^{1,\Phi(\cdot)}_g(\Omega))^*.
\end{align*}

Thus $h_W$ and $h_H$ are both $\Phi(\cdot)$-harmonic but $h_W$ is $\Phi(\cdot)$-harmonic in the sense of $W^{1,\Phi(\cdot)}$ and $h_H$ is $\Phi(\cdot)$-harmonic with respect to $H^{1,\Phi(\cdot)}$. These solutions are different in the situation of Lavrentiev gap (see [1, Theorem 30]). Indeed, let $\eta \in C^\infty(\Omega)$ be zero in the neighbourhood of the origin and be 1 in the neighbourhood of $\partial \Omega$. Then there exist sufficiently large $t$ such that for $g = t\eta u_2 \in H^{1,\Phi(\cdot)}(\Omega)$ we have $h_H(g) \neq h_W(g)$.

**Theorem 16.** Let $\alpha, \beta > 1$. Any $H$-minimizer $h_H$ is continuous in $\Omega$. Any $W$-minimizer $h_W$ that is not equal to $h_H$ is discontinuous at the origin.

**Proof.** For $H$-minimizer $h_H$ there holds $(h_H)_+ = (h_H)_-$ by Proposition 15. This and Lemma 9 give the continuity of $h_H$ on $E_R = \{x_1 = |x_2| \cap \Omega$ and $E_L = \{-x_1 = |x_2| \cap \Omega$. In $\Omega_R = \{x_1 > |x_2| \cap \Omega$ and $\Omega_L = \{-x_1 > |x_2| \cap \Omega$ the minimizer $h_H$ is a solution of $\Phi(\cdot)$-Laplace equation with $\Phi(\cdot)(x,t) = t^2 \log^{-\beta}(e + t)$ independent of $x$. This and continuous boundary data on $E_R$ and $E_L$ guarantee, by result due to Lieberman in [13], that $u$ is continuous in $\Omega_R$ and $\Omega_L$. Therefore $u$ is continuous in $\Omega$. If $h_H \neq h_W$, then $(h_W)_+ \neq (h_W)_-$.

$\square$

4. **Generalized double-phase type integrands**

We consider integrands of the type
\begin{align*}
  \Phi(x,t) := \varphi(t) + a(x)\psi(t),
\end{align*}

where $\varphi, \psi \in C^\infty(\mathbb{R})$, $\alpha > 0$, $\beta > 1$, and $a(x) > 0$ on $\Omega$.
where φ, ψ are the Orlicz functions, which satisfy \( \Delta_2 \) and \( \nabla_2 \) conditions and the non-standard growth conditions:
\[
|t|^{p_-} \leq \varphi(t) \leq c_1|t|^{p_+} + c_2, \\
|t|^{p_-} \leq \psi(t) \leq c_1|t|^{p_+} + c_2,
\]
where \( 1 < p_- < p_+ < \infty \), \( c_1 > 0 \), and the weight \( a(\cdot) \) is as in the previous section, that is
\[
a(x_1, x_2) = \begin{cases} 
1, & |x_1| < |x_2|, \\
0, & |x_1| > |x_2|.
\end{cases}
\]
We also assume that \( \varphi \leq c_3\psi + c_4 \) and
\[
\frac{\varphi(t)}{\psi(t)} \to 0 \quad \text{as} \quad t \to \infty.
\]

We recall some well-know relations for \( N \)-functions. We refer to the book [12] for the notion and the basic properties of \( N \)-functions. A real function \( \Psi : \mathbb{R}^2 \to \mathbb{R}^2 \) is called \( N \)-functions if \( \Psi(0) = 0 \) and there exists the derivative \( \Psi' \) of \( \Psi \). The function \( \Psi' \) is right continuous, non-decreasing and satisfies \( \Psi'(0) = 0 \), \( \Psi'(t) > 0 \) for \( t > 0 \) and \( \lim_{t \to \infty} \Psi'(t) = \infty \). The function \( \Psi \) satisfies \( \Delta_2 \)-condition, if there there exists \( c_1 > 0 \) such that for all \( t \geq 0 \) holds \( \Psi(t) \leq \Psi(2t) \).

By \( (\Psi')^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \) we denote the function
\[
(\Psi')^{-1}(t) := \sup \{ s \in \mathbb{R}^2 : \Psi'(s) \leq t \}.
\]
If \( \Psi' \) is strictly increasing then \( (\Psi')^{-1} \) is the inverse of \( \Psi' \). The function
\[
\Psi^*(t) := \int_0^t (\Psi')^{-1}(s) \, ds
\]
is also an \( N \)-function with
\[
(\Psi^*)'(t) = (\Psi')^{-1}(t), \quad \text{for} \quad t > 0.
\]
There also holds
\[
\Psi^*(t) = \sup_{s \geq 0} (st - \Psi(s)), \quad (\Psi^*)^* = \Psi.
\]
The following variant of Young’s inequality holds
\[
(13) \quad ts \leq \delta \Psi(t) + c_3 \Psi^*(s).
\]
The classical Young’s inequality corresponds to the case \( \delta = 1 \) and \( c_3 = 1 \). For all \( t \geq 0 \) we have
\[
(14) \quad \Psi \left( \frac{\Psi^*(t)}{t} \right) \leq \Psi^*(t) \leq \Psi \left( \frac{2\Psi^*(t)}{t} \right).
\]
Uniformly in \( t \geq 0 \)
\[
\Psi(t) \simeq \Psi'(t), \quad \Psi^* (\Psi'(t)) \simeq \Psi(t)
\]
By \( L^\Psi \) and \( W^{1, \Psi} \) we denote classical Orlicz and Sobolev-Orlicz spaces:
\[
f \in L^\Psi \quad \text{iff} \quad \int \Psi(|f|) \, dx < \infty,
\]
\[
f \in W^{1, \Psi} \quad \text{iff} \quad f, \nabla f \in L^\Psi.
\]
The space \( L^\Psi \) is normed with
\[
\|f\|_{L^\Psi(\Omega)} := \inf \{ \lambda > 0 : \int_\Omega \Psi \left( \frac{|f|}{\lambda} \right) \, dx \leq 1 \}
\]
and is complete, separable and reflexive space if both \( \Psi \) and \( \Psi^* \) satisfy \( \Delta_2 \)-condition.

As above, \( \Omega = (-1, 1)^2 \).

We obtain necessary and sufficient conditions for \( H^{1, \Phi(\cdot)}(\Omega) = W^{1, \Phi(\cdot)}(\Omega) \) in terms of \( \varphi \) and \( \psi \) for the checkerboard configuration. Thus we give a complete description of the double-phase checkerboard structure with general Orlicz functions.
\textbf{Theorem 17} (Lavrentiev gap). Let
\begin{equation}
\int_0^{\varphi(1/r)} r \, dr < \infty
\end{equation}
and
\begin{equation}
\int_0^{\psi^*(1/r)} r \, dr < \infty.
\end{equation}

Then $H^{1,\Phi}(\Omega) \neq W^{1,\Phi}(\Omega)$ and $H^{1,\Phi}_0(\Omega) \neq W^{1,\Phi}_0(\Omega)$.

Moreover there exists $b \in L^{\Phi^*(\cdot)}(\Omega)$ such that for $G(\cdot)$ defined by
\[ G(u) = \int_\Omega \Phi(x,|\nabla u|) \, dx + \int_\Omega b \cdot \nabla u \, dx \]
there is Lavrentiev gap
\[ \inf G(W^{1,\Phi}_0(\Omega)) \neq \inf G(H^{1,\Phi}_0(\Omega)). \]

\textbf{Proof.} A direct check shows that (15) implies $\nabla u^2 \in L^{\Phi^*(\cdot)}(\Omega)$ and (16) implies $b_2 \in L^{\Phi^*(\cdot)}(\Omega)$. The proof is concluded by application of [1, Theorems 26, 28]. \hfill \□

If either of the conditions of Theorem 17 is violated, then there is no Lavrentiev gap.

\textbf{Theorem 18} (No gap). The equality $H^{1,\Phi}_0(\Omega) = W^{1,\Phi}_0(\Omega)$ is valid in the following cases:
\begin{enumerate}
\item[(a)]
\begin{equation}
\int_0^{\psi^*(1/r)} r \, dr = +\infty.
\end{equation}
\item[(b)]
\begin{equation}
\int_0^{\varphi(1/r)} r \, dr = +\infty.
\end{equation}
\end{enumerate}

We start with a lemma that is a generalization of Lemma 11.

\textbf{Lemma 19.} Let $u \in W^{1,\Phi^*(\cdot)}(\Omega)$ and $u = 0$ in a neighbourhood of the origin. Then $u \in H^{1,\Phi^*(\cdot)}(\Omega)$. If $u \in W^{1,\Phi^*(\cdot)}_0(\Omega)$ and $u = 0$ in a neighbourhood of the origin then $u \in H^{1,\Phi^*(\cdot)}_0(\Omega)$.

The statement is verbatim repetition, as well as the proof, which uses only the structure of the weight $a$.

The next result is a generalization of Lemma 12.

\textbf{Lemma 20.} For the integrand $\Phi(x,t)$ defined in (12) and $d = 2$ let
\begin{equation}
\int_0^{\psi^*(1/r)} r \, dr = +\infty.
\end{equation}

Then there exists a sequence of functions $\eta_\varepsilon \in C^\infty(\bar{\Omega})$, $\varepsilon \to 0$, such that $\eta_\varepsilon = 1$ outside $B_\varepsilon(0)$, $\eta_\varepsilon = 0$ in a neighbourhood of the origin, $0 \leq \eta_\varepsilon \leq 1$, and
\[ \int_\Omega \Phi(x,|\nabla \eta_\varepsilon|) \, dx \to 0 \text{ as } \varepsilon \to 0. \]

\textbf{Proof.} Set
\begin{equation}
\eta_{r_1,r_2}(r) := \begin{cases}
0, & \text{if } r < r_1, \\
\int_{r_1}^{r} (\psi^*)' \left( \frac{c_{r_1,r_2}}{\rho} \right) \, d\rho, & \text{if } r \in (r_1,r_2), \\
1, & \text{if } r > r_2,
\end{cases}
\end{equation}

where the constant $c_{r_1,r_2}$ comes from
\begin{equation}
\int_{r_1}^{r_2} (\psi^*)' \left( \frac{c_{r_1,r_2}}{\rho} \right) \, d\rho = 1.
\end{equation}
The structure of the function \( \eta_{r_1, r_2} \) comes from the problem:
\[
\int_{r_1 < r < r_2} \psi(\|\nabla \eta\|) \, dx \to \min,
\]
\[
\eta|_{r_1} = 0,
\]
\[
\eta|_{r_2} = 1.
\]

The corresponding Euler-Lagrange equation has the form
\[
\text{div} \left( \eta \frac{\nabla \eta}{\|\nabla \eta\|} \right) = 0,
\]
so \( \eta \) should be \( \psi \)-harmonic. From the radial symmetry, \( \eta = \eta(r) \), \( \eta' \geq 0 \), so
\[
(r \psi'(\eta'))' = 0.
\]
Thus
\[
\psi'(\eta') = \frac{c}{r}, \quad c = c_{r_1, r_2} = \text{const},
\]
and
\[
\eta'(r) = (\psi')^{-1}(c/r) = (\psi^*)'(c/r).
\]

To satisfy the boundary conditions we have to require that
\[
\int_{r_1}^{r_2} (\psi^*)' \left( \frac{c}{\rho} \right) \, d\rho = 1.
\]
So the required function \( \eta = \eta_{r_1, r_2} \) is given by (20) and (21).

It remains to show that \( \nabla \eta_{r_1, r_2} \to 0 \) in \( L^{\psi(\cdot)}(\Omega) \) for some \( r_1, r_2 \to 0 \).

**Figure 3.** The function \( \eta \)

We have
\[
\int_{\Omega} \psi(\|\nabla \eta_{r_1, r_2}\|) \, dx = \int_{r_1}^{r_2} \psi \left( \left( \psi^* \right)' \left( \frac{c_{r_1, r_2}}{\rho} \right) \right) \rho \, d\rho \lesssim \int_{r_1}^{r_2} \psi^* \left( \frac{c_{r_1, r_2}}{\rho} \right) \rho \, d\rho = c_{r_1, r_2} \int_{r_1}^{r_2} \psi^* \left( \frac{c_{r_1, r_2}}{\rho} \right) \rho \, d\rho,
\]
where \( c_{r_1, r_2} \) is defined from (21). Thus,
\[
\int_{\Omega} \psi(\|\nabla \eta_{r_1, r_2}\|) \, dx \leq c_{r_1, r_2}.
\]

Note that
\[
(22) \quad \int_0^1 \psi^* \left( \frac{1}{\rho} \right) \rho \, d\rho = \infty \iff \int_0^1 (\psi^*)' \left( \frac{1}{\rho} \right) \, d\rho = \infty
\]
and for any \( r_1, r_2 \) the function \( \int_{r_1}^{r_2} (\psi^*)' \left( \frac{c}{\rho} \right) \, d\rho \) is increasing with respect to \( c \). So, if we find \( \delta \) such that
\[
\int_{r_1}^{r_2} (\psi^*)' \left( \frac{c}{\rho} \right) \, d\rho \geq 1,
\]
then \( c_{r_1, r_2} \leq \delta \).
For arbitrary small \( r_2 \) and \( \delta \), due to (22), we can find \( r_1 < r_2 \), \( r_1 = r_1(r_2, \delta) \) such that
\[
\int_{r_1}^{r_2} (\psi^*)(\frac{\delta}{\rho}) \, d\rho = \delta \int_{\frac{r_1^2}{\delta}}^{\frac{r_2^2}{\delta}} (\psi^*)(\frac{1}{\rho}) \, d\rho \geq 1.
\]
So \( c_{r_1, r_2} \leq \delta \) and thus \( \int_0^1 \psi((\nabla \eta_\varepsilon, r_2)) \, dx \leq \delta \). The proof of the lemma is concluded by taking \( \eta_\varepsilon = \eta_{r_1(\varepsilon, \varepsilon), \varepsilon} \). \( \square \)

Recall that \( \Phi^*(x, \cdot) \) is the conjugate functions of \( \Phi(x, \cdot) \).

**Lemma 21 (On duality).** If \( W^{1, \Phi^*}(\Omega) = H^{1, \Phi^*}(\Omega) \) then \( W_0^{1, \Phi}(\Omega) = H_0^{1, \Phi}(\Omega) \).

**Proof.** Suppose the contrary. Then there exists \( u \in W_0^{1, \Phi}(\Omega) \) such that \( \nabla u \) does not belong to the closure of smooth compactly supported vector-valued functions in the gradient norm in \( (L^{\Phi^*})^2 \). Hence, there exists \( g \) such that \( g \in (L^{\Phi^*})^2 \) and
\[
\int_{\Omega} g \nabla \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega),
\]
\[
\int_{\Omega} g \nabla u \, dx \neq 0.
\]
We see that the vector \( g \) is solenoidal. Therefore
\[
g = \nabla^\perp v, \quad v \in W^{1, \Phi^*}(\Omega).
\]
By the assumption of the lemma, there exists a sequence \( v_\varepsilon \in C^\infty(\Omega) \) such that \( \nabla^\perp v_\varepsilon \to g \) in \( (L^{\Phi^*})^2 \). Since
\[
0 = \int_{\Omega} \nabla^\perp v_\varepsilon \nabla u \, dx \to \int_{\Omega} g \nabla u \, dx \neq 0,
\]
we arrive at a contradiction, which concludes the proof. \( \square \)

The next statement is the counterpart of Corollary 13.

**Corollary 22.** If
\[
\int_0^1 \psi^* \left( \frac{1}{r} \right) r \, dr = +\infty,
\]
then \( H_0^{1, \Phi}(\Omega) = W_0^{1, \Phi}(\Omega) \) and \( H^{1, \Phi}(\Omega) = W^{1, \Phi}(\Omega) \).

**Proof.** Let \( \eta_\varepsilon \) be the functions from Lemma 20. Without loss of generality we can assume that \( u \in W_0^{1, \Phi}(\Omega) \cap L^\infty(\Omega) \). We set
\[
u_\varepsilon := \eta_\varepsilon u.
\]
By Lemma 19 we have \( u_\varepsilon \in H_0^{1, \Phi}(\Omega) \). Now,
\[
\nabla (u_\varepsilon - u) = u \nabla \eta_\varepsilon + (1 - \eta_\varepsilon) \nabla u.
\]
The second term converges to zero in \( L^{\Phi^*}(\Omega) \) by the Lebesgue dominated convergence theorem. The first term is bounded by \( ||u||_{\infty} |\nabla \eta_\varepsilon| \) and by the \( \triangle_2 \) condition and Lemma 20 we obtain
\[
\int_{\Omega} \Phi(x, |u \nabla \eta_\varepsilon|) \, dx \lesssim \int_{\Omega} \psi(|u \nabla \eta_\varepsilon|) \, dx \lesssim \int_{\Omega} \psi(|\nabla \eta_\varepsilon|) \, dx \to 0
\]
as \( \varepsilon \to 0 \). Hence \( \nabla u_\varepsilon \to \nabla u \) in \( L^{\Phi^*}(\Omega) \) and so \( u_\varepsilon \to u \) in \( H_0^{1, \Phi}(\Omega) \). For \( u \in W^{1, \Phi}(\Omega) \cap L^\infty(\Omega) \) the proof is the same. \( \square \)

With these preparations in mind the proof of Theorem 18 is straightforward.
Proof of Theorem 18. Case (a) is by Corollary 22.

Case (b) follows by duality Lemma 21. Indeed, $\Phi^*(x,t) = \varphi^*(t)$ if $a(x) = 0$ and $\Phi^*(x,t) \sim \psi^*(t)$ if $a(x) = 1$. Clearly, $\psi^*(t) \lesssim \varphi^*(t)$ and $\Phi^*(x,t) \lesssim \varphi^*(t)$.

For $\Phi^*(x,t)$ the function $\varphi^*$ plays the role of $\psi$ and $\psi^*$ plays the role of $\varphi$. By (18),

$$\int_0^1 (\varphi^*)^p \left( \frac{1}{r} \right) r \, dr = \int_0^1 \varphi \left( \frac{1}{r} \right) r \, dr = +\infty.$$  

By Corollary 22 applied to $\Phi^*(x,t)$ we have $H^{1,\Phi^*}(\Omega) = W^{1,\Phi^*}(\Omega)$. The proof is concluded by Lemma 21. 

REFERENCES

[1] Anna Kh. Balci, Lars Diening, and Mikhail Surnachev. New Examples on Lavrentiev Gap Using Fractals. Calc. Var. Partial Differential Equations, 59(5):180, 2020.
[2] P. Baroni, M. Colombo, and G. Mingione. Nonautonomous functionals, borderline cases and related function classes. Algebra i Analiz, 27(3):6–50, 2015.
[3] Paolo Baroni, Maria Colombo, and Giuseppe Mingione. Regularity for general functionals with double phase. Calc. Var. Partial Differential Equations, 57(2):Art. 62, 48, 2018.
[4] Sun-Sig Byun and Jehan Oh. Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains. J. Differential Equations, 263(2):1643–1693, 2017.
[5] Maria Colombo and Giuseppe Mingione. Bounded minimisers of double phase variational integrals. Ann. Inst. H. Poincaré Anal. Non Linéaire, 35(4):1037–1071, 2018.
[6] Luca Esposito, Francesco Leonetti, and Giuseppe Mingione. Sharp regularity for functionals with $(p,q)$ growth. Arch. Ration. Mech. Anal., 2017:1–66, 2017.
[7] Anna Kh. Balci, Lars Diening, and Mikhail Surnachev. New Examples on Lavrentiev Gap Using Fractals. Calc. Var. Partial Differential Equations, 59(5):180, 2020.
[8] P. Baroni, M. Colombo, and G. Mingione. Nonautonomous functionals, borderline cases and related function classes. Algebra i Analiz, 27(3):6–50, 2015.
[9] Paolo Baroni, Maria Colombo, and Giuseppe Mingione. Regularity for general functionals with double phase. Calc. Var. Partial Differential Equations, 57(2):Art. 62, 48, 2018.
[10] Sun-Sig Byun and Jehan Oh. Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains. J. Differential Equations, 263(2):1643–1693, 2017.
[11] Maria Colombo and Giuseppe Mingione. Bounded minimisers of double phase variational integrals. Ann. Inst. H. Poincaré Anal. Non Linéaire, 35(4):1037–1071, 2018.
[12] V. V. Zhikov and Mikhail Surnachev. On density of smooth functions in weighted sobolev spaces with variable exponent. Russian J. Math. Phys., 27:415–436, 2016.
[13] V. V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. Izv. Akad. Nauk SSSR Ser. Mat., 50(4):675–710, 877, 1986.
[14] V. V. Zhikov. On Lavrentiev’s phenomenon. Russian J. Math. Phys., 3(2):249–269, 1995.

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