Space-time as a deformable continuum

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Abstract. Space-time may be thought of as a physical continuum endowed with properties
similar to the ones of material threedimensional continua. In this view a non-trivial metric
tensor can be considered to be the sum of the Minkowski metric plus an appropriate strain
tensor. The global symmetry of the universe can be seen as the effect of a spontaneous strained
state due to the presence of a texture defect. Consistently with this approach the Lagrangian
of space time is obtained adding to the scalar curvature, acting as a kinetic term, a potential
term depending on the strain and modeled on the one of the elasticity theory, extended to four
dimensions. The theory is applied to the fit of the luminosity dependence of type Ia supernovae
on the redshift. A result is obtained slightly better than the one of the $\Lambda$CDM theory.

1. Introduction
The vision of the cosmos we have, after one century of General Relativity (GR), is essentially
a dual one: the universe is assumed to be made of two basic ingredients, space-time and
matter/energy. The situation is perfectly well expressed by the Einstein equations:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

The left hand side of the equations (the "marble side", according to Einstein) describes
space-time and its properties, the right hand side (the "wooden side", in Einstein’s words again) is
matter/energy. We think we are familiar with the right hand side, because it contains the stuff
our bodies and everything is made of, but, despite its cleaner mathematical description, the left
hand side has a rather ambiguous status. Space-time has indeed to be real, since it produces
real and fundamental effects, but is different from matter to which it couples giving rise to the
gravitational interaction. Actually the attitude of the scientific community towards space-time
oscillates between considering it more or less as an useful mathematical device, and treating it as
a peculiar field interacting geometrically with any other field and trying to quantize it. Coming to
the observations, we see that, despite the tremendous advances of scientific cosmology during the
20th century, recent years have cumulated a number of facts which remain not fully understood
or, at least, which can be explained introducing ”dark” components of the universe: both
dark matter, producing gravitational effects, but not interacting with electromagnetism, and
dark energy, producing the accelerated expansion of the universe without any self-gravitational
behaviour. The use of ”dark” components may just be the anticipation of something real that
will soon be discovered but is also the occasion for developing a host of more or less conjectural
theories, where the internal mathematical consistency is the main requirement, whereas the link
to experiment and observation is some times impossible or at least dubious.
The subject of this paper will be space-time with its nature and properties. The approach we have adopted, in order to avoid an excessive use of conjectures, has been to move from some theory which is already known for situations that could look like space-time with its geometrical properties. Actually space-time is indeed a physical entity, it is continuous (as far as we treat it classically) and displays geometrical properties. The starting point has then been the description of ordinary three-dimensional material continua with their internal strains and stresses. The existing theory of these ordinary continua is not per se relativistic so we had to generalize it to four dimensions and the Lorentzian signature. This generalization has been done in the first part of this work. Then, after introducing also the concept of structural defect, again in analogy to ordinary material continua, our theory has been applied to the cosmic scale and the evolution of the universe. The result, tested on the luminosity curve of type Ia supernovae, has been good and even a little bit better than the one obtained by the mostly used ΛCDM theory.

2. N-dimensional "elastic" continua

Our approach to the description of general N-dimensional continua is summarized in fig.1. We start from an \((N+n)\)-dimensional Minkowskian space\(^1\) which will be used to embed the manifolds we want to use and describe; let \(X^a\) be the coordinates we use to locate positions in the embedding manifold, with the label \(a\) ranging from 1 to \(N+n\). Our embedding space \(\{X\}\) contains two different \(N\)-dimensional spaces. One, which is called the reference manifold \([1]\), is flat; the second, named natural manifold, is curved. Each manifold is equipped with appropriated coordinates: let us call \(\xi^a\) the ones of the reference manifold, \(\{\xi\}\), and \(x^\mu\) the coordinates on the natural manifold, \(\{x\}\): Greek indices range from 1 to \(N\). The two \(N\)-dimensional Riemannian manifolds are geometrically defined by a set of conditions allowing for the dimensional reduction, such as

\[
f_i(X^1, X^2, ..., X^{(N+n)}) = 0
g_{ij} = \frac{\partial f_i}{\partial X^a} \frac{\partial f_j}{\partial X^b} \eta_{ab}
\]

for the reference manifold and

\[
h_i(X^1, X^2, ..., X^{(N+n)}) = 0
\]

for the natural one; in both cases it is \(n = 1, 2, ..., n\).

We assume that all functional relations are smooth enough for all subsequent purposes; this means that it is always possible to go from one set of coordinates to the other without troubles. Physically we may think to obtain the natural manifold deforming the reference one, which means that the natural manifold will be considered as a strained version of the reference one. In the case of a material support the deformation process will imply the presence of stresses either externally or internally generated. The above ideal process can be formally described introducing a displacement vector field in the flat embedding manifold: the \(u\)'s in fig. 1. The flatness of the embedding allows for global definitions of vectors. If \(r\) points at an event in \(\{\xi\}\) and \(r'\) localizes an event in \(\{x\}\) it will be

\[
u = r' - r
\]

The displacement vector field may be expressed either in terms of the \(\xi\)'s or of the \(x\)'s since the end points of the arrow are on corresponding points of the two manifolds. An actual strain is present when the \(u\)-field is non-uniform. This fact is better seen considering corresponding line elements on the two manifolds and using Eqs. (4), (2) and (3). On the natural manifold we have:

\[
ds'^2 = \eta_{ab} X'^a X'^b |_{\text{nat}} \rightarrow g_{\mu\nu} dx^\mu dx^\nu
\]

\(^1\) In what follows it will always be \((N+1)\)-dimensional, but there are curved manifolds which require more than 1 extradimensions in order to be embedded in a flat manifold.
Figure 1. Embedding of the reference and the natural manifolds in an $N + 1$ - dimensional flat manifold. $X_a$ are the Cartesian coordinates in the embedding manifold; $\xi_\mu$ are the coordinates on the reference (sub)-manifold ($N$ - dimensional), $x_\nu$ are the coordinates in the natural (generally curved) (sub)-manifold. $u$ represents the displacement vector from points of the reference manifold to points of the natural manifold.

and on the reference manifold it is

$$ds^2 = \eta_{ab}X^aX^b|_{\text{ref}} \rightarrow \eta_{\mu\nu}dx^\mu dx^\nu.$$  \hspace{1cm} (6)

The symbol $\eta$ labels the Minkowski metric tensor expressing the flatness both of the embedding ($N + 1$)-dimensional and the reference $N$-dimensional manifold. The natural manifold is in general curved and its metric tensor is $g$. For convenience everything has been expressed in terms of the $x$’s. Comparing (5) with (6) it is easy to see that

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\varepsilon_{\mu\nu}$$  \hspace{1cm} (7)

being

$$\varepsilon_{\mu\nu} = \frac{1}{2}(\eta_{\mu\rho}\frac{\partial u^a}{\partial x^\rho} + \eta_{\nu\rho}\frac{\partial u^b}{\partial x^\rho} + \eta_{ab}\frac{\partial u^a}{\partial x^\rho} \frac{\partial u^b}{\partial x^\sigma})$$  \hspace{1cm} (8)

the strain tensor of the natural manifold.

3. ”Elasticity” and the role of defects

If it is globally possible to write

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$  \hspace{1cm} (9)

we have a diffeomorphism and we can verify that it is also

$$R^\lambda_{\mu\nu\rho} = 0.$$  \hspace{1cm} (10)
We have a defect whenever an entire region of the reference manifold corresponds to a point (or other lower dimensional variety) in the natural manifold, or viceversa. $R^\lambda_{\mu\nu\rho}$ is the Riemann curvature tensor and eq. (10) corresponds to De Saint Venant’s integrability condition for (9). In practice this means that the curvature of the natural manifold cannot be felt from within (using intrinsic coordinates, i.e. the $x$’s); this is the typical situation of a pure and global elastic deformation. In the case of space-time the non-trivial part of the metric tensor, which, in our approach, corresponds to the strain tensor of the manifold, contains the gravitational interaction, so we see that a real gravitational field exists only when the displacement $u$ field is singular. The presence of some singularity in $u$ prevents (9) to hold globally and expresses a well known result of general relativity. The relevance of singularities in the displacement field lends the opportunity to introduce in our theory another ingredient of the classical theory of material continua: the defects. It is again convenient to have a look first to a graphic schematization of what a defect is; it can be found in fig.2. We say that a defect is present whenever a whole region of the reference manifold corresponds to a (less than $N$)-dimensional variety in $\{x\}$, or viceversa. This definition of defects is consistent with the one given by Volterra [2] at the beginning of the 20th century while studying elastic and plastic deformations in solids.

Formally the passage from the coordinates on $\{\xi\}$ to the ones on $\{x\}$ is written
\[
dx^\mu = \omega^\mu_\alpha d\xi^\alpha.\tag{11}\]

If $\omega^\mu_\alpha = \frac{\partial x^\mu}{\partial \xi^\alpha}$ we have a diffeomorphism (even though the coordinates are on different manifolds). Otherwise $\omega^\mu_\alpha$ is a general 1-form and
\[
\oint \omega^\mu_\alpha d\xi^\alpha \neq 0. \tag{12}\]

Condition (12) is typical of defects known as dislocations; other similar conditions using for instance 2-forms lead to other kinds of defects.
In practice the presence of a defect produces two effects: a spontaneous strained state and a lower symmetry of the natural manifold. Actually in GR the two above effects are caused by matter/energy; here we introduce a new type of source, which is intrinsic to space-time.

4. Generalized Lagrangian for space-time

The concept of strain in the manifold suggests a Lagrangian slightly different from the one commonly used in GR. In fact a typical Lagrangian for classical problems has two additive terms: one contains time derivatives and is known as the kinetic term; the other is a function of the coordinates and represents the potential energy in the system. Taking our view of space-time as a deformable 4-dimensional continuum seriously, we are led to interpret the usual Einstein-Hilbert Lagrangian density, i.e. the scalar curvature $\mathcal{R}$, as the "kinetic" term, since it contains linearly second order derivatives with respect to the Lagrangian coordinates (i.e. the elements of the metric tensor); a potential energy term would be missing. Following our analogy we can easily build such potential energy term borrowing the form valid for ordinary three-dimensional elastic materials. It would be:

$$L_e = \frac{1}{2} \sigma_{\mu\nu} \varepsilon^{\mu\nu}$$

(13)

$L_e$ contains the elements of the stress tensor $\sigma_{\mu\nu}$. Stresses are the expression of the causes of the strain; in ordinary conditions they depend on forces and defects, in GR they will depend on matter/energy and again on defects. Stresses and strains are not independent from each other; in the linear theory of elasticity they are mutually proportional. In the case of space-time we cannot a priori say whether the theory is linear or not, however let us assume it is (alternatively we may always think this to be the lowest approximation order), so we shall write:

$$\sigma_{\mu\nu} = C_{\mu\nu\alpha\beta} \varepsilon^{\alpha\beta}$$

(14)

$C_{\mu\nu\alpha\beta}$ are the elements of the elastic modulus tensor which expresses the properties of the material continuum under consideration: in our case this is space-time. Eq. 14 is the tensorial version of Hooke’s law. If the material continuum is isotropic the elements of the elastic modulus tensor depend on two parameters only. As far as the natural manifold admits a tangent Minkowskian space we may assume local isotropy for space-time too, so we are allowed to write

$$C_{\mu\nu\alpha\beta} = \lambda \eta_{\mu\alpha} \eta_{\nu\beta} + \mu \eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}$$

(15)

The two independent parameters $\lambda$ and $\mu$ are known as the Lamé coefficients. Now introducing Eq. (15) into Eq. (13) we get

$$L_e = \frac{1}{2} (\lambda \varepsilon^2 + 2 \mu \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta}).$$

(16)

In order to raise and lower indices the full metric tensor (7) is used; $\varepsilon = \varepsilon_{\alpha}^\alpha$ is the trace of the strain tensor.

Finally the full action integral for space-time in presence of matter will be:

$$S = \int (R + L_e + \kappa L_m) \sqrt{-g} d^4x.$$  

(17)

$L_m$ is the usual matter term; $\kappa = 16\pi G/c^2$ is the coupling constant between matter and geometry; $g$ is the determinant of the metric tensor. Both $R$ and $L_e$ are geometry, even though applying the usual variational procedure it is possible to write the Einstein’s equations in the form
\[ G_{\mu\nu} = T_{\mu\nu} + \kappa T_{\mu\nu}. \]  

(18)

The tensor \( T_{\mu\nu} \) comes from \( L_e \) and appears as a new source of curvature (due to the presence of strain), written on the right of the equations rather than as being part of the geometry of the manifold on the left.

5. Cosmological application

It is commonly accepted that the universe has a global Robertson-Walker (RW) symmetry based on space homogeneity and isotropy. Considering space-time the RW symmetry is not obvious and is not a direct consequence of the matter content, so why is it there? Applying our theory the natural candidate to fix the global symmetry of the manifold is a cosmic defect (CD) corresponding to the initial singularity or big bang. In a sense the CD gives the "container" (i.e. the space-time) a predefined "shape", or, to say better, strain field; matter then couples to the given manifold with its global symmetry, contributing additional and local curvature.

5.1. A closed Robertson-Walker space-time

A useful example of the application of our theory to cosmology may be given studying a closed RW space-time. The situation is sketched in fig.3. The bell-shaped surface is obtained from a plane by cutting a portion out of it, then sewing the rims together. The corresponding defect induces an axial symmetry when seen from the embedding three-dimensional flat space. The most appropriate coordinates are cylindrical: \( z, r \) and \( \sigma \). The picture is three-dimensional, however it represents a four-dimensional situation if we assume \( \sigma \) to be a bi-dimensional surface element. The embedding space is assumed to be Minkowskian and \( z \) is a time-like variable. In this way the reference flat manifold is the \( z = \text{constant} \) space-like Euclidean (hyper)-plane. For the natural curved manifold \( r \) and \( \sigma \) are the same as for the flat one, but it is \( z = f(r) \) being \( f \) some regular (except, possibly, at the origin) function of \( r \).

The line element on the reference manifold is

\[ dl^2 = -dr^2 - r^2 d\sigma^2, \]  

(19)

the corresponding line element on the natural manifold is

\[ ds^2 = dz^2 - dr^2 - r^2 d\sigma^2 = (f' - 1) dr^2 - r^2 d\sigma^2. \]  

(20)

\( f' \) is the derivative of \( f \) with respect to \( r \). Subtracting (19) from (20) we can read out the strain tensor in the embedding coordinates. The only non-zero component is

\[ \varepsilon_{rr} = \frac{f'^2 - 2}{2}. \]  

(21)

In order to convert everything to the natural coordinates we put

\[ \sqrt{f'^2 - 1} dr = d\tau \]  

(22)

so defining the "radial" (time-like) coordinate on the natural manifold, \( \tau \) (the cosmic time). It will also be \( r = a(\tau) \) and the line element on the natural manifold assumes the typical RW form:

\[ ds^2 = d\tau^2 - a^2 d\sigma^2. \]  

(23)

In the natural coordinates the only non-null component of the strain tensor becomes

\[ \varepsilon_{\tau\tau} = \frac{1 - \ddot{a}^2}{2}. \]  

(24)

being \( \dot{a} \) the derivative of the \( a \) function with respect to \( \tau \). The situation represented in fig.3 corresponds, for reasons of graphical clarity, to a closed space (finite \( z = \text{constant} \) sections), however the outlined method can equally well be applied to open and critical RW space-times.
Figure 3. A curved surface with a central symmetry is embedded in a three-dimensional manifold. The reference frame is a plane and the global coordinates are cylindrical. The situation reproduces, in a three-dimensional view, a closed RW space-time.

5.2. The accelerated expansion of the universe
If we wish to describe our universe we must start from the fact that it appears to be flat in space. This situation can be dealt with using the same procedure outlined in the previous subsection (details may be read in [3]). Using Cartesian coordinates for space, one obtains three equal non-zero elements of the strain tensor:

\[ \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \frac{1 - a^2}{2}. \]  

(25)

Once the global symmetry has been fixed and the strain tensor is at hands, we are able to compute the \( L_e \) to be introduced into (16). The final explicit Lagrangian density turns out to be

\[ L = -6(a\dddot{a} + \dot{a}^2) + \frac{9}{8}B\frac{(1 - a^2)^2}{a} + \kappa L_m. \]  

(26)

The Lamé coefficient appear to be combined into the bulk modulus \( B \):

\[ B = \lambda + \frac{2}{3}\mu. \]  

(27)

The only variable is cosmic time and the only unknown function is \( a \). We need to specify the matter Lagrangian. Considering the possibility of having a number of different components, we can deduce from (26) the equation:

\[ H^2 = \frac{\dot{a}^2}{a^2} = \frac{9}{16}B\frac{(1 - a^2)^2}{a^4} + \kappa \sum_i \rho_i \frac{a_0^{3(1+w_i)}}{a^{3(1+w_i)}}. \]  

(28)
Figure 4. Fit of the luminosity data from 307 type Ia supernovae obtained applying the CD theory. Three optimization parameters have been used. The horizontal error bars are not visible at scale of the graph.

The summation index $i$ runs from 1 to the number of different matter components; the $\varrho$'s are matter/energy densities; $w$'s are equation of state parameters; the $0$ index labels present day values.

We shall apply (28) to the fit of the luminosity data from type Ia supernovae (SnIa) from which the accelerated expansion of the universe has been discovered [4]. For that purpose we have considered two components only: dust, for which it is $w = 0$, and radiation for which $w = -1/3$. The latter contribution, however, turns out to be negligible since the highest redshift factor $z$ for an SnIa is less than 1.8. The observable quantity related to (28) is the so called distance modulus [5] of the supernova:

$$m - M = 25 + 5 \log \left( (1 + z) \int_0^z \frac{dz'}{H(z')} \right)$$

(29)

The observed magnitude is $m$, the absolute magnitude is $M$; the link between $z$ and the scale factor $a$ is $a = \frac{a_0}{1 + z}$; all distances must be expressed in Mpc.

The fit of the experimental observations has been obtained by means of an optimization method applied to a set of 307 SnIa's [4]. The optimization parameters were $a_0$, $\varrho_0$ and $B$, variously combined. The result is shown in fig. 4.

The reduced $\chi^2$ of the fit is 1.017, slightly better than the value (1.019) obtained, with the same number of parameters, using the $\Lambda$CDM theory. The optimal values for the Hubble parameter $H_0$ and the present matter density $\varrho_0$ are within the range of commonly accepted ones, with big uncertainties coming from the dispersion of the actual data. The value obtained for the bulk modulus is

$$B = (3 \pm 2) \times 10^{-7} \text{ Mpc}^{-2} = (3 \pm 2) \times 10^{-52} \text{ m}^{-2}.$$  

(30)
6. Conclusion

We have exploited the existing analogy between GR and the theory of ordinary elastic continua. First of all we have generalized the theory to Riemannian arbitrary dimensional manifolds, showing that the gravitational field may be described as a strain in space-time treated as a four-dimensional continuum. The strain can be originated both by a matter/energy distribution and by defects in the texture of space-time. The definition of defect also has been extrapolated from the one valid for ordinary material continua. Going further in our generalization we have hypothesized that the strain in the four-dimensional manifold should show up also in the Lagrangian of empty space-time. The corresponding additional term has been modeled on the classical elastic potential energy. Considering the consequent action integral, in presence of matter, we have succeeded in reproducing the luminosity curve of type Ia supernovae with an accuracy slightly better than the one obtainable from the popular $\lambda$-cold-dark-matter theory. The "elastic" parameters of space-time as obtained through the optimization process for the SNIa fit are compound in a bulk modulus of classical vacuum, $B$. The value obtained by the fit, as reported in (30), is extremely small, so that its effects appear at the cosmic scale only. At the scale of galaxies or even galaxy clusters the possible signatures of the CD theory (the present theory) are negligible. In fact, our theory is, in a sense, isomorphic to GR, since it is a metric theory and space-time always admits a tangent space, then a Newtonian limit; our final equations are in practice Einstein’s equations with one more geometric source of curvature originated by the strain of the manifold. Our final description of the geometry of space-time somehow resembles bimetric theories [6], however we actually have only one metric. Our cosmic defect simply fixes the global symmetry of space-time. So far the CD theory proves to be more than a formal analogy and could be a fruitful new paradigm for gravity and space-time. The next steps will be to work out all implications of the theory, for instance concerning gravitational waves and the inhomogeneities in the CMB.

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