The Lagrange-Poincaré equations for interacting Yang-Mills and scalar fields

S. N. Storchak*
NRC “Kurchatov Institute” – IHEP,
Protvino, Moscow Region, 142281, Russia

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Abstract

A special case of the Lagrange-Poincaré equations for the gauge field interacting with a scalar field is obtained. For description of the dynamics on the configuration space, the adapted coordinates are used. After neglecting the group variables the obtained equations describe the evolution on the gauge orbit space of the principal fiber bundle which is related to the system under the consideration.

1 Introduction

The behavior of systems with symmetry is determined by internal dynamics, which is often hidden, which presents significant difficulties in the case of the usual description of evolution. In the theory of reduction for mechanical systems with symmetry, this problem is solved using the Lagrange-Poincaré equations. Due to symmetry, the configuration space of mechanical systems can be regarded as the total space of the principal fiber bundle associated with the system. The Lagrange-Poincare equations are given by two equations: the “horizontal” equation which belongs to the kernel of the 1-form connection (naturally emergent in such systems) and the “vertical” equation related to the motion along the orbit of the principal fiber bundle.

In case of the projection onto the base manifold (the orbit space of the principal fiber bundle), the horizontal equation describes the internal dynamics of the system. This dynamics is determined by the mechanical system

*E-mail adress: storchak@ihep.ru
that arises from the original system as a result of the reduction. In mechanics, the interrelation between the original system and the reduced one is well studied due to Marsden-Weinstein reduction theory. But internal dynamics is also the main object of research in gauge theories — infinite-dimensional dynamic systems that are invariant with respect to the action of a group of gauge transformations. Here, the true configuration space (the configuration space of physically observable variables) is the orbit space of the action of the gauge group. The main problem for these systems is that it is not possible to describe the local dynamics on the gauge orbit space in terms of the gauge-invariant variables. It is currently unknown how to do this in satisfactory way.

The generally accepted method of describing local dynamics in orbit space is to use a special coordinate system in the principal fiber bundle. The coordinates of such a system are known by the name of the adapted coordinates and are defined using local sections of the bundle. The sections are given by local surfaces (submanifolds) in the space of gauge fields. The local surfaces themselves are determined by equations that cannot be explicitly resolved, so parametric representations of the surfaces cannot be obtained. As a result, when introducing coordinates into the principal bundle, we are forced to deal with constrained variables (or dependent variables) as coordinates in this approach. In spite of this, the approach is widely used, for example, when quantizing gauge fields by the path integral method. Studies of the classical evolution of gauge fields with the use of adapted coordinates for local descriptions of the dynamics have practically not been conducted.

In this paper our goal is to obtain the Lagrange-Poincare equations for the gauge system formed from the Yang-Mills field interacting with the scalar field. We are based on our works where we have considered the mechanical system of two particles given on the product manifold consisting of the Riemannian manifold and the manifold represented by the vector space. It was assumed that the system under consideration is invariant with respect to the group action. The resulting reduced mechanical system was given on the corresponding associated bundle which serves as the base space of the principal bundle related to the system. The geometry of this special mechanical system is analogous to the gauge system we consider in the present article. So it can be regarded as the model system for our problem.

The paper will be organized as follows. Section 2 is an introduction to our paper, where we recall our previous work from arXiv, where the mechanical system of two interacting particles was investigated. In Section 3 we explain how the adapted coordinates can be determined for the gauge interacting system formed from the Yang-Mills field and a scalar field. These
coordinates correspond to the coordinates in the mechanical system. This provides the basis for using the Lagrange-Poincaré equations obtained earlier for the mechanical system, in deriving analogous equations for the gauge system under the study. In Section 3, we derive such equations for the gauge system using functional expressions for the terms of the Lagrange–Poincaré equations obtained earlier for the mechanical system. Details of derivations of the Lagrange-Poincaré equations are considered in Appendix.

2 Mechanical system of two interacting particles

In our previous works [10, 11], we considered a special finite-dimensional mechanical system with the following Lagrangian:

\[ L = \frac{1}{2} G_{AB}(Q) \dot{Q}^A \dot{Q}^B + \frac{1}{2} G_{mn} \dot{f}^m \dot{f}^n - V(Q, f). \]  

(1)

The configuration space of this system is the product manifold \( \mathcal{P} \times \mathcal{V} \). It was assumed that \( \mathcal{P} \) is a smooth finite-dimensional Riemannian manifold (without the boundary) and \( \mathcal{V} \) is a finite-dimensional vector space. So, \((Q^A, f^n), A = 1, \ldots, N_P\) and \(n = 1, \ldots, N_V\), are the coordinates of a point \((p, v) \in \mathcal{P} \times \mathcal{V}\) in some local chart. Also, it was assumed that a smooth isometric free and proper action of the compact group Lie \( \mathcal{G} \) on \( \mathcal{P} \times \mathcal{V} \) was given. We dealt with the right action on \( \mathcal{P} \times \mathcal{V} \): \((p, v)g = (pg, g^{-1}v)\). In coordinates, this action is written as follows:

\[ \tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^n = D^n_m(g)f^m. \]

Here \( D^n_m(g) \equiv D^m_n(g^{-1}) \), and by \( D^n_m(g) \) we denote the matrix of the finite-dimensional representation of the group \( \mathcal{G} \) acting on the vector space \( \mathcal{V} \).

For our metric

\[ ds^2 = G_{AB}(Q)dQ^A dQ^B + G_{mn}df^m df^n, \]  

(2)

the Killing vector fields

\[ K_\alpha(Q, f) = K^B_\alpha(Q) \frac{\partial}{\partial Q^B} + K^p_\alpha \frac{\partial}{\partial f^p} \]

have the following components: \( K^B_\alpha(Q) = \frac{\partial \tilde{Q}^n}{\partial a^\alpha} \bigg|_{a=\epsilon} \) and \( K^p_\alpha(f) = (\tilde{J}_\alpha)_m^n f^m \).

(The generators \( \tilde{J}_\alpha \) of the representation \( \tilde{D}_m^n(a) \) are such that \([\tilde{J}_\alpha, \tilde{J}_\beta] = \tilde{c}^\gamma_{\alpha \beta} \tilde{J}_\gamma\), where \( \tilde{c}^\gamma_{\alpha \beta} = -\tilde{c}^\gamma_{\beta \alpha} \).)
In the following, we will also use the condensed notation for indices: \( \hat{A} \equiv (A, p) \). So, for example, the components of the Killing vector fields will be written as \( \hat{K}_\mu^A = (K_\mu^A, K_\mu^p) \).

From the general theory \([1]\) it is known that in our case \( P \times V \) can be regarded as a total space of the principal fiber bundle

\[
\pi' : P \times V \to P \times G \, V, \n\]

that is, \( \pi' : (p, v) \to [p, v] \), where \([p, v]\) is the equivalence class with respect to the relation \((p, v) \sim (pg, g^{-1}v)\).

Due to this fact it is possible to express the coordinates \((Q^A, f^\alpha)\) of the point \((p, v)\) in terms of the principal fiber bundle coordinates. The method of performing this for the typical principal bundle \( P(M, G) \) is well-known \([7, 12–16]\). In approach close to ours was considered in \([9]\) for the abelian gauge theory. It consists of using the local sections \( \tilde{\sigma}_i \) of our bundle, \( \pi_{\Sigma_i} \cdot \tilde{\sigma}_i = \text{id} \). But to define \( \tilde{\sigma}_i \), it is necessary to use the sections \( \sigma_i \) of the principal fiber bundle \( P(M, G) \):

\[
\tilde{\sigma}_i([p, v]) = (\sigma_i(x), a(p)v) = (\tilde{p}, \tilde{v}) \in P \times V, \n\]

where \( a(p) \) is the group element defined by \( p = \sigma_i(x)a(p) \).

The adapted coordinates on \( P(M, G) \) are defined by means of the choice of the special local sections \( \sigma_i \). The sections are determined by the local submanifold \( \Sigma_i \) of \( P \), given by the equation \( \{\chi^\alpha(Q) = 0, \alpha = 1, \ldots, N_\alpha\} \). The coordinates of the points on the local submanifold \( \Sigma_i \) will be denoted by \( Q^A \), they are such that \( \{\chi^\alpha(Q^*) = 0\} \). That is, \( Q^A \) are dependent coordinates. In other words, the special section \( \sigma_i \) is defined as the map \( \sigma_i : U_i \to \Sigma_i : \pi_{\Sigma_i} \cdot \sigma_i = \text{id}_{U_i} \).

We note that there exists a local isomorphism between trivial principal bundle \( \Sigma_i \times G \to \Sigma_i \) and \( P(M, G) \): \([5, 6, 9]\)

\[
\varphi_i : \Sigma_i \times G \to \pi^{-1}(U_i), \n\]

which allows us to introduce a local coordinates on \( P(M, G) \). In coordinates we have:

\[
\varphi_i : (Q^*, a^\alpha) \to Q^A = F^A(Q^*, a^\alpha), \n\]

where \( Q^* \) are the coordinates of a point given on the local surface \( \Sigma_i \) and \( a^\alpha \) – the coordinates of an arbitrary group element \( a \). This element carries the point, taken on \( \Sigma_i \), to the point \( p \in P \) which has the coordinates \( Q^A \).

An inverse map \( \varphi_i^{-1} \):

\[
\varphi_i^{-1} : \pi^{-1}(U_i) \to \Sigma_i \times G, \n\]
has the following coordinate representation:

$$\varphi_i^{-1} : Q^A \to (Q^B, a^\alpha(Q)).$$

Here the group coordinates $a^\alpha(Q)$ of a point $p$ are the coordinates of the group element which connects, by means of its action on $p$, the surface $\Sigma_i$ and the point $p \in \mathcal{P}$. These group coordinates are given by the solutions of the following equation:

$$\chi^\beta(F^A(Q, a^{-1}(Q))) = 0. \quad (3)$$

The coordinates $Q^B$ are defined by the equation

$$Q^B = F^B(Q, a^{-1}(Q)). \quad (4)$$

In the same way as for the principal bundle $P(M, G)$, there exist a local isomorphisms of the principal fiber bundle $P(P \times G V, G)$ and the trivial principal bundles $\tilde{\Sigma}_i \times G \to \tilde{\Sigma}_i$, where now the local surfaces $\tilde{\Sigma}_i$ are the images of the sections $\tilde{\sigma}_i$.

In this case we have the following coordinate functions of the chart $s$:

$$\tilde{\varphi}_i^{-1} : \pi^{-1}(\tilde{U}_i) \to \tilde{\Sigma}_i \times G, \text{ or in coordinates,}$$

$$\tilde{\varphi}_i^{-1} : (Q^A, f^m) \to (Q^A(Q), \tilde{f}^n(Q), a^\alpha(Q)).$$

Here $Q^A$ and $f^m$ are the coordinates of a point $(p, v) \in \mathcal{P} \times V$, $Q^A(Q)$ is given by (4) and $\tilde{\varphi}_i$ maps $\tilde{\sigma}_i \times G \to \pi^{-1}(\tilde{U}_i)$:

$$\tilde{\varphi}_i : (Q^B, \tilde{f}^n, a^\alpha) \to (F^A(Q^*, a), \tilde{D}_m^a(a) \tilde{f}^n).$$

Thus, we have determined the special local bundle coordinates $(Q^A, \tilde{f}^n, a^\alpha)$, also called the adapted coordinates, in the principal fiber bundle $\pi : \mathcal{P} \times V \to \mathcal{P} \times G V$.

The replacement of the coordinate basis $(\partial/\partial Q^B, \partial/\partial a^\alpha)$ for a new basis $(\partial/\partial Q^A, \partial/\partial \tilde{f}^m, \partial/\partial a^\alpha)$ is performed as follows:

$$\frac{\partial}{\partial f^m} = D_m^a(a) \frac{\partial}{\partial \tilde{f}^m},$$

$$\frac{\partial}{\partial Q^B} = \frac{\partial Q^A}{\partial Q^B} \frac{\partial}{\partial Q^A} + \frac{\partial a^\alpha}{\partial Q^B} \frac{\partial}{\partial a^\alpha} + \frac{\partial \tilde{f}^m}{\partial Q^B} \frac{\partial}{\partial \tilde{f}^m}$$

$$= \tilde{F}_B^C \left( N_C^A(Q^*) \frac{\partial}{\partial Q^A} + \chi^\mu_C(\Phi^{-1})_{\mu}^\beta \tilde{v}_\beta^\alpha(a) \frac{\partial}{\partial a^\alpha} - \chi^\mu_C(\Phi^{-1})_{\mu}^\nu(\tilde{J}_\nu)_{\nu}^m \tilde{f}^p \frac{\partial}{\partial \tilde{f}^m} \right). \quad (5)$$
Here \( \tilde{F}_B^C \equiv F_B^C(F(Q^*,a),a^{-1}) \) is an inverse matrix to the matrix \( F_B^A(Q^*,a) \), \( \chi_C^\alpha \equiv \frac{\partial \psi^\alpha(Q)}{\partial Q^\mu}|_{Q=Q^*} \), \( (\Phi^{-1})_\mu^\alpha(Q^*) \equiv (\Phi^{-1})_\mu^\alpha(Q) \) – the matrix which is inverse to the Faddeev–Popov matrix:

\[
(\Phi)^\alpha_\mu(Q) = K^A_\mu(Q) \frac{\partial \chi^A(Q)}{\partial Q^A}.
\]

The matrix \( \tilde{v}_\beta^\alpha(a) \) is inverse of the matrix \( \tilde{u}_\beta^\alpha(a) \).\(^1\)

The operator \( N^A_C \), defined as

\[
N^A_C(Q) = \delta^A_C - K^A_\alpha(Q)(\Phi^{-1})^\alpha_\mu(Q)\chi^\mu_C(Q),
\]

is the projection operator \( (N^A_B N^B_C = N^A_C) \) onto the subspace which is orthogonal to the Killing vector field \( K^A_\alpha(Q) \equiv \frac{\partial}{\partial q^\alpha} \). \( N^A_C(Q^*) \) is the restriction of \( N^A_C(Q) \) to the submanifold \( \Sigma \):

\[
N^A_C(Q^*) \equiv N^A_C(F(Q^*,e)) \quad N^A_C(Q^*) = F^A_B(Q^*,a)N^A_B(F(Q^*,a))\tilde{F}^A_M(F^*_M(a),a)
\]

\( e \) is the unity element of the group.

Thus, the metric (2) of the original manifold \( P \times V \) in a new coordinate basis is given by

\[
\tilde{G}_{AB}(Q^*,\tilde{f},a) = \begin{pmatrix}
G_{CD}(P_\perp)_B^C(P_\perp)^D_B & 0 & G_{CD}(P_\perp)_A^C K^D_B \tilde{u}_\nu^\alpha \\
0 & G_{mn} & G_{mp} K^p_v \tilde{u}_\nu^\alpha \\
G_{CD}(P_\perp)_A^C K^D_B \tilde{u}_\nu^\alpha & G_{mp} K^p_v \tilde{u}_\nu^\alpha & d_{mn} \tilde{u}_\nu^\alpha \tilde{u}_\beta^\nu
\end{pmatrix}
\]

(6)

where \( G_{CD}(Q^*) \equiv G_{CD}(F(Q^*,e)) \):

\[
G_{CD}(Q^*) = F^M_C(Q^*,a)F^N_D(Q^*,a)G_{MN}(F(Q^*,a)),
\]

\( (P_\perp)_B^A \) is the projection operator on the tangent plane to the submanifold \( \Sigma \).

It is given by

\[
(P_\perp)_B^A = \delta^A_B - \chi^\alpha_\beta (\chi^\top)^{-1}\alpha^\beta A
\]

\( (\chi^\top)^\beta_A \) is a transposed matrix to the matrix \( \chi^\beta_B \):

\[
(\chi^\top)_\mu^\alpha = G^{AB} \gamma_{\mu\nu} \chi^\nu_B \quad \gamma_{\mu\nu} = K^A_B G_{AB} K^B_\nu
\]

\( ((P_\perp)_B^A \) has the following properties: \( (P_\perp)_B^A N^C_A = (P_\perp)_B^C N^C_A = (P_\perp)_B^C \) \( d_{\mu\nu}(Q^*,\tilde{f}) \tilde{u}_\alpha^\nu(a)(a) \tilde{u}_\beta^\nu(a) \) in (5) is the metric on \( G \)-orbit through the point \( (p,v) \):

\[
d_{\mu\nu}(Q^*,\tilde{f}) = K^A_\mu(Q^*) G_{AB}(Q^*) K^B_\nu(Q^*) + K^m_\mu(\tilde{f}) G_{mn} K^n_\nu(\tilde{f})
\]

\[
\equiv \gamma_{\mu\nu}(Q^*) + \gamma_{\mu\nu}'(\tilde{f}).
\]

\(^1\tilde{u}_\beta^\alpha(a) \) (and \( u_\beta^\alpha(a) \)) are the coordinate representations of the auxiliary functions given on the group \( G \).
In our works [10, 11], the Lagrange-Poincaré equations were derived using the so-called the horizontal lift basis on the total space of the principal fiber bundle. The new basis consists of the horizontal and vertical vector fields and can be determined by using the “mechanical connection” which exists [1] in case of the reduction of mechanical systems with a symmetry.

In the principal fiber bundle \( \mathcal{P}(\mathcal{G} \times G, \mathcal{V}) \), in coordinates \((Q^* A, \tilde{f}^n, a^\alpha)\), the connection \( \tilde{\omega} = \tilde{\omega}^\alpha \otimes \lambda_\alpha \) (\(\{\lambda_\alpha\}\) is the basis in the Lie algebra of the group \( \text{Lie} \mathcal{G} \)) is given by the following expression:

\[
\tilde{\omega}^\alpha = \tilde{\rho}^\alpha_\alpha(a) \left( d_\alpha^\mu K^D_\mu (Q^*) G_{DA}(Q^*) dQ^A + d_\alpha^\mu K^q_\mu (\tilde{f}) G_{qn} d\tilde{f}^n \right) + u_\beta^\alpha(a) da^\alpha,
\]

where \( d_\alpha^\mu = d_\alpha^\mu (Q^*, \tilde{f}) \). And the matrix \( \tilde{\rho}^\alpha_\alpha(a) \) is inverse to the matrix \( \rho^\beta_\alpha(a) = \bar{u}_\nu^\alpha(a) \rho^\nu_\beta(Q) \) of the adjoint representation of the group \( G \).

In terms of the (“gauge”) potentials \( A_\alpha B^\alpha \) and \( A_\alpha^m \), together with a new notation: \( \tilde{A}_\alpha^\alpha(B, Q, \tilde{f}, a) \), the connection can be rewritten as

\[
\tilde{\omega}^\alpha = \tilde{\omega}^\alpha_B(Q^*, \tilde{f}, a) dQ^B + \tilde{\omega}^\alpha_m(Q^*, \tilde{f}, a) d\tilde{f}^m + u_\beta^\alpha(a) da^\alpha,
\]

or using the condensed notations of indices like

\[
\tilde{\omega}^\alpha = \tilde{\omega}^\alpha_B(Q^*, \tilde{f}, a) dQ^B + u_\beta^\alpha(a) da^\alpha.
\]

In coordinates \((Q^* A, \tilde{f}^n, a^\alpha)\), the horizontal lift basis \((H_A, H_p, L_\alpha)\) is given by the vector fields

\[
H_M(Q^*, \tilde{f}, a) = \left[ N^T_M \left( \frac{\partial}{\partial Q^\alpha} - \omega^\alpha_T L_\alpha \right) + N^m_M \left( \frac{\partial}{\partial \tilde{f}^m} - \omega^m_m L_\alpha \right) \right],
\]

\[
H_m(Q^*, \tilde{f}, a) = \left( \frac{\partial}{\partial \tilde{f}^m} - \omega^m_m L_\alpha \right),
\]

and also by the left-invariant vector field \( L_\alpha = A_\alpha^{\mu} \frac{\partial}{\partial a^\mu} \) which is obtained from the Killing vector field \( K_\alpha(Q) \). Note that \( L_\alpha \) commutes with the horizontal vector fields \( H_A \) and \( H_p \).

In the definition of \( H_M \), new components of the projection operator

\[
N^\alpha_C = (N^A_C, N^A_m, N^m_A, N^m_p),
\]

were used. They are

\[
N^A_m = 0, \quad N^A_m = -K^m_\alpha (\Phi^{-1})^\alpha_\mu \lambda^A = -K^m_\alpha A^A, \quad N^m_p = \delta^m_p.
\]
The operator $N^A_B$ satisfy the following property: $N^A_B N^B_C = N^A_C$.

In a new coordinate basis $(H_A, H_p, L_\alpha)$, the metric tensor (8) is represented as

$$\tilde{G}_{AB}(Q^*, \tilde{f}, u) = \begin{pmatrix} \tilde{G}_{AB}^H & \tilde{G}^{Hm} \nabla \tilde{G}_{m}\nabla 0 \\ 0 & \tilde{d}_{\alpha\beta} \end{pmatrix} \equiv \begin{pmatrix} \tilde{G}_{AB}^H & 0 \\ 0 & \tilde{d}_{\alpha\beta} \end{pmatrix},$$

where $\tilde{d}_{\alpha\beta} = \rho_{\alpha}^\nu \rho_{\beta}^\nu d_{\nu\nu}$. The components of the “horizontal metric” $\tilde{G}_{AB}^H$ depending on $(Q^A, f^m)$ are defined as follows:

$$\tilde{G}_{AB} = G_{AB} - \Pi^A \Pi^B G_{AB} = G_{AB} - \Pi^A \Pi^B G_{AB} - \Pi^A \Pi^B G_{AB} - \Pi^A \Pi^B G_{AB} - \Pi^A \Pi^B G_{AB},$$

$$\tilde{G}^{Hm} = -G_{AB} \Pi^B G_{pm},$$

$$\tilde{G}_{mA} = -G_{mq} \Pi^q G_{DA},$$

$$\tilde{G}_{mn} = G_{mn} - G_{mr} K_{\alpha}^\mu d_{\alpha}^\beta K_{\beta}^\mu G_{pm}.$$  

In the coordinate basis $(H_A, H_p, L_\alpha)$, the original Lagrangian $\mathcal{L}$ has the following representation:

$$\tilde{\mathcal{L}} = \frac{1}{2} \left( \tilde{G}_{AB}^H \omega^A \omega^B + \tilde{G}_{Ap}^H \omega^A \omega^p + \tilde{G}_{pA}^H \omega^A \omega^p + \tilde{G}_{p\alpha}^H \omega^A \omega^\alpha + \tilde{d}_{\mu\nu} \omega^\mu \omega^\nu \right) - V, \quad (9)$$

where the new time-dependent variables $\omega^A, \omega^p$ and $\omega^\alpha$, which are associated with velocities, are given by

$$\omega^A = \left( P_\perp^A \right) B \frac{dQ^B}{dt}, \quad \omega^p = \frac{d\tilde{f}^p}{dt},$$

$$\omega^\alpha = u_\alpha \frac{d\psi^m}{dt} + \tilde{\xi}_E \frac{dQ^E}{dt} + \tilde{\psi}_m \frac{d\tilde{f}^m}{dt}. \quad (10)$$

The Lagrangian (9) was used in [10][11] for derivation of the Lagrange-Poincaré equations. This was done using the Poincaré variational principle. The following equations were obtained:

$$N^A_B \frac{d\omega^B}{dt} + N^A_{RB} \tilde{\Gamma}^B_{AB} \omega^B \omega^M + G^{EF} N^E_{RN} \left[ \mathcal{F}_{QR} \omega^A \tilde{Q}_p \omega^A + \frac{1}{2} \left( \mathcal{D}_R d^{\mu} \right) \left( p_{\mu} p_{\sigma} + V_{\sigma} \right) \right] = 0,$$

$$N^R_B \frac{d\omega^B}{dt} + \frac{d\omega^R}{dt} + N^R_{AB} \tilde{\Gamma}^B_{AB} \omega^A \omega^B + G^{EF} N^E_{RN} \left[ \mathcal{F}_{QR} \omega^A \tilde{Q}_p \omega^A + \frac{1}{2} \left( \mathcal{D}_R d^{\mu} \right) \left( p_{\mu} p_{\sigma} + V_{\sigma} \right) \right] = 0,$$

$$\frac{dp_{\mu}}{dt} + c_{\mu\nu} d^{\mu} \left( p_{\nu} p_{\mu} - c_{\mu\nu} \mathcal{F}_{\nu} \omega^A \tilde{Q}_p \omega^A \right) = 0.$$
In these equations $p_\alpha = \gamma_{\alpha\rho}\dot{\omega}^\rho$, the curvature tensor $\mathcal{F}^a_{SP}$ of the connection $\mathcal{L}_p^a$ is given by $\mathcal{F}^a_{SP} = \mathcal{L}_{S,P}^a - \mathcal{L}_{S,P}^a + \mathcal{L}_{S,P}^a \mathcal{L}_{P,S}^a$. The tensors $\mathcal{F}^a_{Ep}$ and $\mathcal{F}^a_{Cm}$ are defined in a similar way.

The covariant derivative $\mathcal{D}_R(d\sigma(Q^*,\dot{f}))$ are given by

$$\mathcal{D}_R(d\sigma) = \partial_Q d\sigma + \mathcal{L}_p d\sigma + \mathcal{L}_p (\mathcal{D}_R d\sigma).$$

The Christoffel symbols $\Gamma_{BM}^a$, $\Gamma_{qB}^a$ and $\Gamma_{pq}^a$ for the horizontal (degenerate) metric $G_{BM}^a$ are defined by means of the equalities:

$$\Gamma_{BM}^a = \Gamma_{qB}^a = \Gamma_{pq}^a$$

where

$$\Gamma_{BMD} = \frac{1}{2}(\Gamma_{BD,M} + \Gamma_{MD,B} - \Gamma_{BM,D}).$$

And $\Gamma_{qB}$ and $\Gamma_{pq}$ have an analogous definitions.

Taking into account the following properties:

$$K^E_{ABC} = 0, \quad K^R_{Q,R} = 0, \quad N_F^T \Gamma_{BM} = \Gamma_{BMF},$$

$$N_F^T \Gamma_{qB} = \Gamma_{qBF}, \quad N_F^T \Gamma_{pq} = \Gamma_{qF},$$

and the invariance of the potential $V(Q^*,\dot{f})$ under the action of the group $G$, this means that $N_F^R V_R = V_F$, we can rewrite the Lagrange-Poincaré equations in the following way:

$$N_A^B \left( \frac{\omega^A}{dt} + G^{AR} \Gamma_{BM}^a \omega_B^a \omega_N^a \right) + G^{AR} \left[ \mathcal{F}^a_{QR} \omega_Q^a p_\alpha + \frac{1}{2} (\mathcal{D}_R d\sigma)p_\alpha + V_R \right] = 0. \quad (11)$$

$$N_A^a \left[ \frac{d\omega^A}{dt} + G^{AR} \Gamma_{BM}^a \omega_B^a \omega_N^a \right] + \frac{d\omega^a}{dt} + G^{m} \left[ \mathcal{F}^a_{BM} \omega_B^a \omega_N^a + \mathcal{F}^a_{Qm} \omega_Q^a p_\alpha + \frac{1}{2} (\mathcal{D}_m d\sigma)p_\alpha + V_m \right] = 0. \quad (12)$$

$$\frac{dp_\beta}{dt} + \epsilon^\sigma_{\mu\beta} d\sigma p_\sigma - \epsilon^\sigma_{\sigma\beta} d\sigma \omega^\rho p_\nu = 0. \quad (13)$$

These equations will be used for derivation of the Lagrange-Poincaré equations in gauge theories.
3 Adapted coordinates in configuration space of the gauge system with interaction

Our aim is to extend the methods we have used for the finite-dimensional mechanical system with symmetry to the gauge system which describe the dynamics of Yang-Mills field interacting with the scalar field. The standard relativistically invariant Lagrangian for this system is singular (we can not determine the Hamiltonian using the Legendre transformation) in contrast to what we had for the model mechanical system. The problem is related to presence of the redundant variable $A_0^a$ in the Lagrangian. Therefore, by setting $A_0^a = 0$ in the Lagrangian we obtain the Lagrangian which is free of this problem. Note, that the same can be performed by suitable gauge transformation

So, the Lagrangian (Lagrange density) we will consider is the following

$$
- \frac{1}{2g_0^2} k_{\alpha \beta} (\partial_0 A^\alpha_i) (\partial_0 A^{i \beta}) + \frac{1}{2} G_{ab} (\partial_0 f^a) (\partial_0 f^b)
- \frac{1}{4g_0^2} k_{\alpha \beta} F^\alpha_{ij} F^{i \beta \gamma} + \frac{1}{2} G_{ab} (\nabla_i f^a) (\nabla^i f^b) - V_0(A, f).
$$

Here $k_{\alpha \beta} = c^\tau_{\mu \alpha} c^{\mu \tau}_{\beta}$ is the Cartan–Killing metric on the group $G$, $V_0$ is some gauge-invariant potential. $g_0$ is a coupling constant.

The obtained Lagrangian looks as if it represents the motion of two “particle” in the product space $\mathcal{P} \times \mathcal{V}$ in the potential

$$
V[A, f] = \int d^8x \left[ \frac{1}{2} k_{\alpha \beta} F^\alpha_{ij} F^{\beta \gamma ij} (x) - \frac{1}{4} G_{ab} (\nabla f)^a_i (x) (\nabla f)^b_i (x) + V_0 \right].
$$

Further, in the formulas, we omit the coupling constant $g_0$, absorbing it in $k_{\alpha \beta}$ since in the final expressions, the coupling constant can be easily restored.
One of the space, \( \mathcal{P} \), is an infinite-dimensional Riemannian manifold. The gauge fields \( A^a_i \) can be regarded as points of this manifold. And the other space, \( \mathcal{V} \), is the space of functions with the values in the vector space \( V \). Also, we are given an action of the group, the group of the gauge transformations, on the product space. This is analogous to what we have in reduction problem for dynamical system with symmetry in mechanics, which was considered in the previous section. Here we are interested in description of internal dynamics given on the gauge orbit space.

The reduction theory for the gauge-invariant dynamical systems follows from the result obtained in \([3,6,17,18]\), and in other works, where the geometric approach to the gauge fields was developed.

First of all, the gauge fields \( A^a_i(x) \) are regarded as coordinate representations of connections defined on the principal fiber bundle \( P(M,G) \) over the compact manifold \( M \). Then, in order to have a smooth free and proper action of the gauge group on the space of connections \( \mathcal{P} \), one must consider the irreducible connections. (The isotropy subgroup of these connections coincides with \( Z(G) \), the center of gauge group \( G \).) The gauge transformation group must be group \( \tilde{G} = G/Z(G) \). Moreover, the connections and the gauge transformation functions must belong to classes of Sobolev functions \( H^k \) and \( H^{k+1} \), respectively, with \( k \geq 3 \). Only in this case one leads to the principal fiber bundle defined by \( \pi: \mathcal{P} \to \mathcal{P}/\tilde{G} = M \).

The function space \( \mathcal{V} \) of the matter fields \( f^b(x,t) \) consists of the sections of the associated bundle \( \Gamma(P \times_\mathcal{G} V) \). (These sections also must be from \( H_k \).)

In our case, \( \mathcal{P} \times \mathcal{V} \) is the original configuration space of the gauge system with the Lagrangian (14), and the gauge orbit space \( \mathcal{P} \times \mathcal{V} \), the base of the principal fiber bundle \( \pi': \mathcal{P} \times \mathcal{V} \to (\mathcal{P} \times \mathcal{V})/\tilde{G} = \mathcal{P} \times \tilde{G} \mathcal{V} \), is the configuration space of the physically observable quantities.

From the quadratic part of the Lagrangian (14) it follows that the Riemannian metric of the original configuration space is flat. It can be presented as follows:

\[
\begin{align*}
\text{ds}^2 &= G_{(a,i,x)(\beta,j,y)} \delta A^{(a,i,x)} \delta A^{(\beta,j,y)} + G_{(a,x)(b,y)} \delta f^{(a,x)} \delta f^{(b,y)},
\end{align*}
\]

where

\[
G\left(\frac{\delta}{\delta A^{(a,i,x)}}, \frac{\delta}{\delta A^{(\beta,j,y)}}\right) = G_{(a,i,x)(\beta,j,y)} = k_{\alpha \beta} \delta_{ij} \delta^3(x - y)
\]

is the metric on \( \mathcal{P} \) and the metric on \( \mathcal{V} \) is

\[
G\left(\frac{\delta}{\delta f^{(m,x)}}, \frac{\delta}{\delta f^{(n,y)}}\right) = G_{(m,x)(n,y)} = G_{mn} \delta^3(x - y).
\]

\[\text{Using compact manifolds needed to ensure the boundedness of the action functional [3,21].}\]
In these formulae we have used the extended notation for indices by which
\( A^{(\alpha,i,x)} \equiv A^{\alpha i}(x) \) and \( f^{(m,x)} \equiv f^m(x) \). Note that the use of such notations helps in the generalization of formulas obtained in the finite-dimensional case to the corresponding formulas in field theories.

From the gauge invariance of the Lagrangian (and the metric) it follows that the Killing vectors of the original metric are

\[
K_{(a,y)} = K^{(\mu,i,x)}_{(a,y)} \frac{\delta}{\delta A^{(\mu,i,x)}} + K^{(b,x)}_{(a,y)} \frac{\delta}{\delta f^{(b,x)}},
\]

where components of this vector field are given by

\[
K^{(\mu,i,x)}_{(a,y)} (A) = \left[ \left( \delta^A_\mu \partial^i (x) + c^\mu_\nu A^{\nu i}(x) \right) \delta^3(x-y) \right] \equiv \left[ D^\mu_\alpha (A(x)) \delta^3(x-y) \right]
\]

(here \( \partial_i (x) \) is a partial derivative with respect to \( x^i \)), and

\[
K^{(b,x)}_{(a,y)} (f) = (\bar{J}_a)^b_c f^c(x) \delta^3(x-y).
\]

We can determine the coordinates in the principal bundle for the gauge system under study just the same way as was done for a mechanical system with symmetry in a finite-dimensional space. The local sections \( \Sigma \) of the principal fiber bundle \( \mathcal{P} (\mathcal{M}, \tilde{G}) \), which are necessary for determination of the bundle coordinates in the total space \( \mathcal{P} \times \mathcal{V} \) of the bundle \( \pi' \), will be defined by means of the Coulomb gauge condition (or the Coulomb gauge):

\( \partial_i A^{\alpha i} = 0 \). The gauge potentials that will satisfy this equation (dependent variables) will be denoted by \( A^*_\alpha^\mu \). Note that dependent variables are typically used when quantizing gauge fields [6–9, 22, 23].

As was shown in previous section, for transition from the original coordinate \(( A^\alpha, f^\mu )\) given on \( \mathcal{P} \times \mathcal{V} \) to the adapted coordinates \(( A^*_\alpha^\mu, f^b, a^\mu )\) of the principal fiber bundle it is required that the group coordinates \( a^\alpha (A) \) of the "point" \( A \) should be known. In mechanics, they are obtained as a solution of the equation (3):

\[
\chi^\beta (F^A(Q,a^{-1}(Q))) = 0.
\]

For the Coulomb gauge, this equation is as follows:

\[
\partial^i (x) \left[ \rho^\alpha_\beta (a(x)) A^\beta_i (x) - \rho^\alpha_\nu (a(x)) u^\nu_\sigma (a(x)) \frac{\partial a^\sigma (x)}{\partial x^i} \right] = 0.
\]

Then, the coordinates \( Q^* \) of the corresponding point on a submanifold \( \Sigma \) are determined by the corresponding group transformation:

\[
Q^{*A} = F^A (Q, a^{-1}(Q)).
\]

In gauge theories, we have the following gauge transformation:

\[
A^\alpha_i (x) = \rho^\alpha_\beta (a^{-1}(x)) A^*_\beta_i (x) + u^\alpha_\mu (a(x)) \frac{\partial a^\mu (x)}{\partial x^i}.
\]
With the obtained $a^\alpha(x)$, $f^a$ is expressed in terms of $\tilde{f}^a$ as follows: $f^a(x) = D^a_\alpha(a(x))\tilde{f}^a(x)$. Thus, the initial coordinates $(A^a_\alpha(x), f^a(x))$ on $P \times V$ are transformed into adapted bundle coordinates $(A^a_\alpha(x), \tilde{f}^a(x), a^\alpha(x))$.

To obtain a new coordinate representation of the original Riemannian metric, we must transform the coordinate vector fields. The “vector fields” transformation formula is a straightforward generalization of the corresponding formula from the finite-dimensional case:

$$
\frac{\delta}{\delta A^{(\alpha,i,x)}} = \tilde{F}^{(\mu,k,u)}_{(\alpha,i,x)} \left( N^{(\nu,p,v)}_{(\mu,k,u)}(A^x) \frac{\delta}{\delta A^{(\nu,p,v)}} + N^{(m,y)}_{(\mu,k,u)} \frac{\delta}{\delta f^{(m,y)}} \right) + \chi^{(\mu',v)}_{(\mu,k,u)}(A^x) (\Phi^{-1})^{(\beta,u')}_{(\mu',x)}(A^x) \tilde{v}^{(\sigma,p)}_{(\beta,u)}(a) \frac{\delta}{\delta a^{(\sigma,p)}} ,
$$

where we have denoted by $\tilde{F}$ the matrix which is inverse to the matrix $F^{(\mu,k,u)}_{(\alpha,i,x)}$ defined as follows

$$
F^{(\alpha,i,x)}_{(\beta,j,y)}[A, a] = \frac{\delta A^{(\alpha,i,x)}}{\delta A^{(\beta,j,y)}} = \rho_{\beta}^\alpha(g^{-1}(x)) \delta\delta^i\delta^3(x - y).
$$

$\tilde{F}$ satisfies the relation:

$$
F^{(\alpha,i,x)}_{(\beta,j,y)} \tilde{F}^{(\beta,j,y)}_{(\epsilon,k,z)} = \delta_\alpha^\beta \delta_i^\epsilon \delta^3(x - z).
$$

Also, we have

$$
\frac{\delta}{\delta f^{(m,x)}} = D^m_n(a(x)) \frac{\delta}{\delta f^{(m,x)}}.
$$

In formula (15), by $N^{(\nu,p,v)}_{(\mu,k,u)}$, which is equal to

$$
N^{(\alpha,i,x)}_{(\beta,j,y)} = \delta^{(\alpha,i,x)}_{(\beta,j,y)} - K^{(\alpha,i,x)}_{(\mu,z)}(\Phi^{-1})^{(\mu,z)}_{(\nu,u)} \chi^{(\nu,u)}_{(\beta,j,y)},
$$

we have denoted the projection operator onto the subspace which is orthogonal to the component of the Killing vector field $K_{(\alpha,y)}$ which is related to $P$.

The projection operator $N^{(m,y)}_{(\mu,k,u)}$ is equal to

$$
N^{(m,y)}_{(\mu,k,u)} = -K^{(m,y)}_{(\alpha,z)}(\Phi^{-1})^{(\alpha,z)}_{(\beta,v)} \chi^{(\beta,v)}_{(\mu,k,u)}.
$$

The Faddeev–Popov matrix $\Phi$ is defined as follows

$$
\Phi^{(\nu,y)}_{(\mu,z)}[A] = K^{(\alpha,i,x)}_{(\mu,z)} \chi^{(\nu,y)}_{(\alpha,i,x)}.
$$

For the Coulomb gauge, we have

$$
\chi^{(\nu,y)}_{(\alpha,i,x)} = \delta^\nu_\alpha \left[ \partial_i(y) \delta^3(y - x) \right].
$$
Therefore, the matrix $\Phi$ (restricted to the gauge surface) is equal to

$$
\Phi_{(\nu,\sigma)}^{(\nu,\sigma)[A^*]} = \left[ (\delta^\nu_\mu \partial^2(y) + c_{\eta\rho} A^{\nu\eta}(y) \partial^\rho(y) ) \delta^3(y - z) \right]
$$

or

$$
\Phi_{(\mu,\nu)}^{(\mu,\nu)[A^*]} = \left[ (\mathcal{D}[A^*] \cdot \partial)_{\mu}^\nu(y) \delta^3(y - z) \right].
$$

An inverse matrix $\Phi^{-1}$ can be determined by the equation

$$
\Phi_{(\mu,\nu)}^{\nu}(\Phi^{-1})_{(\nu,\mu)}(y, u) = \delta^\nu_\sigma \delta^3(y - u).
$$

That is, it is the Green function for the Faddeev–Popov operator:

$$
\left[ \partial_l(y) \mathcal{D}_{\mu}^{\nu}(A(y)) \right] (\Phi^{-1})_{(\nu,\mu)}(y, u) = \delta^\nu_\sigma \delta^3(y - u).
$$

(The boundary conditions of this operator depend on a concrete choice of a base manifold $M$. By a second group of variables, the Green function $\Phi^{-1}$ satisfies the following equation:

$$
\left[ -\mathcal{D}_{\lambda}^{\sigma}(A(z)) \partial_l(z) \right] (\Phi^{-1})_{(\nu,\mu)}(y, z) = \delta^\nu_\sigma \delta^3(y - z).
$$

Notice that in the formula (15), the matrix $\Phi^{-1}$, as well as the other terms of the projector $N$, is given on the gauge surface $\Sigma$.

In our principal bundle, the orbit metric $d_{(\mu,\nu)}(\nu,\gamma)$ is determined by using the Killing vectors $K_{(\alpha,\eta)}$:

$$
d_{(\mu,\nu)}(\nu,\gamma) = K_{(\alpha,\gamma)}^{(\alpha,\beta)} G_{(\alpha,\eta)}(\beta,\delta,\alpha,\kappa) K_{(\beta,\delta,\alpha,\kappa)}^{(\alpha,\gamma)} + K_{(\alpha,\gamma)}^{(\beta,\delta,\alpha,\kappa)} K_{(\beta,\delta,\alpha,\kappa)}^{(\alpha,\gamma)}
$$

That is,

$$
d_{(\mu,\nu)}(\nu,\gamma) = \left[ k_{\nu\sigma} \delta^{\nu\sigma} \mathcal{D}_{\mu}^{\nu}(A(x)) \mathcal{D}_{\nu}^{\gamma}(A(x)) + G_{ab}(\bar{J}_\mu)_{\alpha}^{a}(\bar{J}_\nu)_{\beta}^{b} f^c(x) f^{c'}(x) \right] \delta^3(x - y)
$$

$$
= \gamma_{\mu\nu}(x, y) + \gamma'_{\mu\nu}(x, y)
$$

An “inverse matrix” to the “matrix” $d_{(\mu,\nu)}(\nu,\gamma)$ is defined by the following equation:

$$
d_{(\mu,\nu)}(\nu,\gamma) q_{(\nu,\gamma)}(\sigma,\sigma) = \delta^\gamma_{(\mu,\nu)} = \delta^\nu_{\sigma} \delta^3(z - x).
$$

In explicit form this equation is written as follows:

$$
\left[ k_{\nu\sigma} \delta^{\nu\sigma} \mathcal{D}_{\mu}^{\nu}(A(x)) \mathcal{D}_{\nu}^{\gamma}(A(x)) + G_{ab}(\bar{J}_\mu)_{\alpha}^{a}(\bar{J}_\nu)_{\beta}^{b} f^c(x) f^{c'}(x) \right] q_{(\nu,\gamma)}(\sigma,\sigma)
$$

$$
= \delta_{\sigma} \delta^3(z - x).
$$
Thus, \( d^{(\nu,y)(\sigma,z)} \) is the Green function of the operator given by the expression in square brackets. It is assumed that a certain boundary condition for the equation is chosen.

The Green function \( d^{(\nu,y)(\sigma,z)} \) and the Killing vectors are the main elements with which the “Coulomb connection” (or “mechanical connection”) is determined: \( \hat{\omega} = \hat{\omega}^\alpha \otimes \lambda_\alpha \) in the principal fiber bundle \( P(\mathcal{P} \times \mathcal{Y}, \mathcal{G}) \):

\[
\hat{\omega}^\alpha = \overline{\rho}_\alpha^\alpha (a(x)) \left( A^{(\alpha',x)}_{(\beta,j,y)} dA^{(\beta,j,y)} + A^{(\alpha',x)}_{(n,y)} df^{(n,y)} \right) + u^\alpha_n (a(x)) da^\mu (x),
\]

where the components of the connection are given by

\[
A^{(\alpha',x)}_{(\beta,j,y)} = d^{(\alpha,x)(\sigma,z)} K^{(\mu,k,v)}_{(\sigma,z)} G^{(\mu,k,v)}_{(\beta,j,y)} = k_{\mu \beta} [D_{\sigma j}^\mu (A^\ast (y))] d^{(\alpha,x)(\sigma,z)}
\]

and

\[
A^{(\alpha',x)}_{(p,y)} = d^{(\alpha,x)(\sigma,z)} K^{(\alpha,v)}_{(\sigma,z)} G^{(\alpha,v)}_{(p,y)} = d^{(\alpha,x)(\sigma,z)} (\tilde{J}_c) c \tilde{f}^c (y) G_{ap}.
\]

The following transformation of the coordinate basis in our principal bundle is connected with the replacement of the basis vector fields by the horizontal ones. This can be done with the help of horizontal projection operators, which are determined by the connection we have just defined. All this is similar to what we did in the finite-dimensional case. Therefore, we will not follow all the steps that ultimately must lead to the Lagrange-Poincaré equations in the functional space of gauge fields. Instead, for this purpose we will use the finite-dimensional equations (11), (12) and (13).

4 The Lagrange-Poincaré equations in gauge theories

The equations that we derive in this article are a special case of the Lagrange–Poincaré equations. In this article, we restrict ourselves to a particular case of the Lagrange–Poincaré equations. They can be obtained from finite-dimensional equations if we assume that the expression under the projector \( N^B_A \) in the first horizontal equation (11) is zero. In addition, we neglect those terms of the first equations that explicitly depend on Killing vectors. Then, from our assumption and the structure of the second horizontal equation (12), it follows that the terms of the second equation with the projector \( N^r_B \) are equal to zero. Thus, we will deal with the following equations:

\[
\frac{d\omega^A}{dt} + G^{ARH} \tilde{\Gamma}^{BM}_{BR} \omega^B \omega^M
\]
\[ + G^{AR} \left[ \mathcal{F}_R^\alpha \omega^Q p_\alpha + \frac{1}{2} (\mathcal{D}_R d^\sigma) p_\sigma p_\sigma + V_R \right] = 0. \tag{16} \]

\[ \frac{d\omega^\nu}{dt} + G^{rm} \left( H_{ABm}^D \omega^A \omega^B + \mathcal{F}_m^\alpha \omega^Q p_\alpha + \frac{1}{2} (\mathcal{D}_m d^\sigma) p_\alpha p_\sigma + V_m \right) = 0. \tag{17} \]

\[ \frac{dp_\beta}{dt} + c_{\mu\beta} d^\mu \sigma p_\sigma - c_{\sigma\beta} \mathcal{E}_\sigma \omega^A p_\nu = 0. \tag{18} \]

Since the Riemannian metric of the original manifold of gauge fields is flat, \( G_{AB} = \delta_{AB} \) must be used as a metric in these finite-dimensional equations. In addition, this fact must be taken into account when calculating the terms of equations are made with using the Killing relation.

In this regard, we first transform the terms of the equations so that later it was possible to replace them by the appropriate functional expressions. Terms of equations with Christoffel symbols \( \Gamma^{ABm} \), curvatures \( \mathcal{F}^\alpha \) and \( \mathcal{D}_m d^\sigma \) will be expressed using the Killing vectors, the components of the mechanical connection and the metric on the orbit. Further we will list the obtained representations for these terms.

### Christoffel symbols for the horizontal metric

\[ G^{AR} \Gamma_{BMR} = -\frac{1}{2} (\mathcal{A}^\beta_{B,M} K^A_\beta + \mathcal{A}^\beta_{M,B} K^A_\beta) - (\mathcal{A}^\beta_{M} K^A_\beta,B + \mathcal{A}^\beta_{B} K^A_\beta,M) \]

\[ + \frac{1}{2} (K^A_{\mu,D} K^D_\beta) (\mathcal{A}^\mu_{M} \mathcal{A}^\mu_{B} + \mathcal{A}^\mu_{M} \mathcal{A}^\mu_{B}). \]

\[ G^{AR} \Gamma_{BmR} = -\frac{1}{2} (\mathcal{A}^\beta_{B,m} K^A_\beta + \mathcal{A}^\beta_{m,B} K^A_\beta) - \mathcal{A}^\beta_{m} K^A_\beta,B \]

\[ + \frac{1}{2} (K^A_{\mu,D} K^D_\beta) (\mathcal{A}^\mu_{m} \mathcal{A}^\mu_{B} + \mathcal{A}^\mu_{m} \mathcal{A}^\mu_{B}). \]

\[ G^{AR} \Gamma_{mR} = -\frac{1}{2} (\mathcal{A}^\beta_{p,q} K^A_\beta + \mathcal{A}^\beta_{q,p} K^A_\beta) \]

\[ + \frac{1}{2} (K^A_{\mu,D} K^D_\beta) (\mathcal{A}^\mu_{q} \mathcal{A}^\mu_{p} + \mathcal{A}^\mu_{q} \mathcal{A}^\mu_{p}). \]

\[ G^{AR} \Gamma_{mBR} = -\frac{1}{2} (\mathcal{A}^\beta_{m,B} K^A_\beta + \mathcal{A}^\beta_{B,m} K^A_\beta) - \mathcal{A}^\beta_{m} K^A_\beta,B \]

\[ + \frac{1}{2} (K^A_{\mu,D} K^D_\beta) (\mathcal{A}^\mu_{m} \mathcal{A}^\mu_{B} + \mathcal{A}^\mu_{m} \mathcal{A}^\mu_{B}). \]

\[ G^{rm} \Gamma_{ABm} = -\frac{1}{2} (\mathcal{A}^\beta_{A,B} K^A_\beta + \mathcal{A}^\beta_{B,A} K^A_\beta) \]

\[ + \frac{1}{2} (K^A_{\mu,p} K^p_\sigma) (\mathcal{A}^\mu_{A} \mathcal{A}^\mu_{B} + \mathcal{A}^\mu_{A} \mathcal{A}^\mu_{B}). \]

\[ G^{rm} \Gamma_{pBm} = -\frac{1}{2} (\mathcal{A}^\beta_{p,B} K^A_\beta + \mathcal{A}^\beta_{B,p} K^A_\beta) - \mathcal{A}^\beta_{B} K^A_\beta,p \]
\[ G^{rm} \Gamma_{pqm} = \frac{1}{2} \left( K_{\alpha,p}^{\nu} K_{\nu,q}^{\mu} \right) (\mathcal{A}_{\mu}^{\nu} \mathcal{A}_{\nu}^{\rho} + \mathcal{A}_{\nu}^{\rho} \mathcal{A}_{\rho}^{\mu}) \]

\[ \Gamma^\nu_{\mu\rho} = -\frac{1}{2} \left( \mathcal{A}_{\mu}^{\beta} K_{\beta,q}^{\nu} + \mathcal{A}_{\nu}^{\beta} K_{\beta,p}^{\mu} \right) \]

\[ + \frac{1}{2} \left( K_{\mu,n}^{\nu} \mathcal{A}_{\nu}^{\mu} + \mathcal{A}_{\nu}^{\mu} \mathcal{A}_{\mu}^{\nu} \right). \]

Curvatures \( \mathcal{G}^{\tilde{A}} \tilde{R}_{Q\tilde{E}} \)

\[ \mathcal{G}^{\tilde{A}} \tilde{R}_{Q\tilde{E}} = -(K^S_{\varphi,Q})(d^{\alpha} \mathcal{A}_{S}^{\mu} + d^{\mu} \mathcal{A}_{S}^{\alpha}) K^{A}_{\mu} - (K^{A}_{e,B} K^{B}_{\nu})(d^\alpha \mathcal{A}_{\nu}^{\mu} + d^\mu \mathcal{A}_{\mu}^{\nu}) \]

\[ + 2 d^{\mu} K_{\mu,Q} + \epsilon_{\nu\mu} d^{\nu} \mathcal{A}_{Q}^{\mu} K^{A}_{\varphi}. \]

\[ \mathcal{G}^{AR} \tilde{R}_{q\tilde{R}} = -(K^{\nu}_{\mu,q})(d^{\alpha} \mathcal{A}_{T}^{\nu} + d^{\nu} \mathcal{A}_{T}^{\alpha}) K^{A}_{\nu} - (K^{A}_{\nu,B} K^{B}_{\varphi})(d^\alpha \mathcal{A}_{\nu}^{\mu} + d^\mu \mathcal{A}_{\mu}^{\nu}) \]

\[ + \epsilon_{\nu\mu} d^{\nu} \mathcal{A}_{q}^{\mu} K^{A}_{\varphi}. \]

\[ \mathcal{G}^{rm} \tilde{R}_{Qm} = -K^{T}_{\mu,n}(d^{\alpha} \mathcal{A}_{T}^{\beta} + d^{\beta} \mathcal{A}_{T}^{\alpha}) K^{r}_{\beta} - (K^{n}_{\nu,p,n}(d^{\alpha} \mathcal{A}_{Q}^{\nu} + d^{\nu} \mathcal{A}_{Q}^{\mu}) \]

\[ + \epsilon_{\nu\mu} d^{\nu} \mathcal{A}_{q}^{\mu} K^{r}_{\beta}. \]

\[ \mathcal{G}^{rm} \tilde{R}_{q\tilde{R}} = -K^{n}_{\mu,q}(d^{\alpha} \mathcal{A}_{T}^{\beta} + d^{\beta} \mathcal{A}_{T}^{\alpha}) K^{r}_{\beta} - (K^{n}_{\nu,p,n}(d^{\alpha} \mathcal{A}_{Q}^{\nu} + d^{\nu} \mathcal{A}_{Q}^{\mu}) \]

\[ + 2 d^{\alpha} K^{r}_{\mu,q} + \epsilon_{\nu\mu} d^{\nu} \mathcal{A}_{q}^{\mu} K^{r}_{\beta}. \]

\[ \mathcal{G}^{\tilde{A} \tilde{R}} (\mathcal{D}_{R} d^{\nu} \mathcal{A}) p_{\kappa} p_{\sigma} \]

\[ + \mathcal{G}^{AR} (\mathcal{D}_{R} d^{\nu} \mathcal{A}) p_{\kappa} p_{\sigma} = 2 \left[ (K^{A}_{\beta} K^{A}_{\mu,\nu}) d^{\beta} d^{\mu} + \epsilon_{\beta\mu} d^{\beta} d^{\mu} K^{A}_{\epsilon} \right] p_{\kappa} p_{\sigma} \]

\[ \mathcal{G}^{rm} (\mathcal{D}_{m} d^{\nu} \mathcal{A}) p_{\kappa} p_{\sigma} = 2 \left[ (K^{A}_{\beta} K^{A}_{\mu,\nu}) d^{\beta} d^{\mu} + \epsilon_{\beta\mu} d^{\beta} d^{\mu} K^{A}_{\epsilon} \right] p_{\kappa} p_{\sigma} \]

Note that before starting the transition to the functional representation in the Christoffel symbols, the partial derivatives of the connections are replaced using the following formulas:

Partial derivatives \( \mathcal{A}_{Q}^{\mu} \)
\[ \mathcal{A}_{Q,R} = -d^\alpha K^A_{\varepsilon R} G_{AB} K^B_{\mu} \mathcal{A}_Q^\mu - \mathcal{A}_R^\alpha \mathcal{A}_Q^\mu K^B_{\mu R} + d^\alpha \mathcal{A}_R^\mu K^B_{\mu R} G_{BQ} \]

\[ \mathcal{A}_{Q,p} = -d^\alpha K^T_{\varepsilon p} G_{\tau n} K^n_{\mu} \mathcal{A}_Q^\mu - \mathcal{A}_n^\alpha \mathcal{A}_p^\mu K^B_{\mu Q} \]

\[ \mathcal{A}_{p,q} = -d^\alpha K^A_{\varepsilon q} G_{\tau n} K^n_{\mu} \mathcal{A}_p^\mu - \mathcal{A}_n^\alpha \mathcal{A}_q^\mu K^B_{\mu q} + d^\mu K^m_{\mu p} G_{mp} \]

Another equivalent representation of derivatives are

\[ \mathcal{A}_{B,m}^\beta = 2d^\beta \mathcal{A}_{E}^{\mu} (K^\rho_{\mu} K^p_{\varepsilon q}) \mathcal{A}_B^\rho + c^\sigma \mathcal{A}_{E}^{\mu} K^\rho_{\mu q} \mathcal{A}_B^\rho, G_{pm} \]

To obtain a functional representation for the members of the equations, one needs to treat the indices of variables as if they were compact notations of extended indices.

\[ A \to (\alpha, i, x); \quad a \to (n, y); \quad \mu \to (\mu, u); \quad \text{etc.} \]

Recall that our basic variables are \( \omega^A \equiv \dot{Q}^A, \omega^m \equiv \dot{\tilde{f}}^m \). So, we have

\[ \omega^A(t) \to \frac{d}{dt} A^{(\alpha, i, x)}(t) \equiv \frac{d}{dt} A^{\alpha, i}(x, t) \equiv \dot{A}^{\alpha i}(x, t), \]

and similarly for \( \omega^m(t): \omega^m(t) \to \frac{d}{dt} \tilde{f}^{(n, x)}(t) \equiv \dot{\tilde{f}}^n(x, t) \).

To obtain functional representations for the terms of the equations, we made the following replacements:

\[ K^A_{\beta, B} \to K^{(\alpha, i, x)}_{(\beta, u)(\varepsilon, k, z)}, \]

where

\[ K^{(\alpha, i, x)}_{(\beta, u)(\varepsilon, k, z)} = \frac{\delta}{\delta A^{(\alpha, i, x)}} = \delta^\beta_{\varepsilon} \epsilon^\alpha_{\beta} \delta^3(x - u) \delta^3(x - z). \]

\[ K^E_{\mu, E} K^\rho_{\nu} \to K^{(\alpha, i, x)}_{(\beta, v)(\varepsilon, k, z)} K^{(\alpha)}_{(\nu, k, z)} \]

\[ K^{(\alpha, i, x)}_{(\beta, v)(\varepsilon, k, z)} = c^\alpha_{\beta} [D^j_{\nu} (A^j_A(x)) \delta^3(x - u)] \delta^3(x - v). \]

\[ K^m_{\beta, m} \to K^{(n, x)}_{(\beta, y)(m, z)} \]
\[ K^{(n,v)}_{(\beta,y)(m,z)} = (\bar{J}_\beta)^n_m \delta^3(v - z)\delta^3(v - y) \]

\[ K^n_{\beta,m} K^m_{\alpha} \rightarrow K^{(n,v)}_{(\beta,y)(m,z)} K^{(m,z)}_{(\alpha,u)} \]

\[ K^{(n,v)}_{(\beta,y)(m,z)} K^{(m,z)}_{(\alpha,u)} = (\bar{J}_\beta)^n_m (\bar{J}_\alpha)^m_q \int^q(y) \delta^3(v - y) \delta^3(v - u) \]

To get expressions in the right parts of the formulas, you need to take sum over repeated generalized indices. Sum over continuous indices means corresponding integration.

There are also the appropriate functional representations for the connections.

\[ \mathcal{A}^{(\alpha,x)}_{(\beta,j,y)} \]

\[ \mathcal{A}^{(\alpha,x)}_{(\beta,j,y)} = [D^p_\mu (A^*(y)) d^{(\alpha,x)(\mu,y)]} k_{p\beta} \]

\[ \mathcal{A}^{(\alpha,x)}_{(m,z)} \]

\[ \mathcal{A}^{(\alpha,x)}_{(m,z)} = d^{(\alpha,x)(\beta',z)} (\bar{J}_{\beta'})^n_p \int^p(x) G_{nm} \]

But in our final formulas we do not use them, despite the fact that this can be done as it does not lead to the simplification of the already rather complex expressions.

The results of our calculations - functional representations of the terms of the equations are presented in Appendix.

The first horizontal equation (16) results in the following Lagrange-Poincaré equation for the gauge system

\[ \frac{d}{dt} \omega^{(\alpha,i,x)} + G^A R H_{BMR} \omega^B \omega^M : \\
-2 \epsilon^\alpha_{\epsilon,\beta} \left[ \int d^3y \mathcal{A}^{(\beta,x)}_{(\gamma,j,y)} \omega^{(\gamma,j,y)} \omega^{(\epsilon,i,x)} \right] + c^\alpha_{\epsilon,\mu} \omega^{(\mu,x)}_{(\gamma,j,y)} \left[ D^x_\nu (A^*(x)) \mathcal{A}^{(\nu,x)}_{(e,k,z)} \omega^{(\gamma,j,y)} \omega^{(e,k,z)} \right] + 2G^A R H_{BMR} \omega^B \omega^m : \\
-2 \delta^\alpha_{\epsilon,\beta} \left[ \int d^3v \mathcal{A}^{(\beta,x)}_{(m,v)} \omega^{(m,v)} \omega^{(\epsilon,k,x)} \right] + c^\alpha_{\epsilon,\beta} \omega^{(\beta,x)}_{(e,k,z)} \left[ D^x_\nu (A^*(x)) \mathcal{A}^{(\nu,x)}_{(m,v)} \omega^{(m,v)} \omega^{(e,k,z)} \right] \]
\[ + G^{\text{AR}} \tilde{\mathcal{F}}_q \Phi^q, R^{\alpha} p_{\alpha} : \]
\[ \frac{1}{2} C_{\mu \beta} \int d^3 y d^3 z \left( A_{(p, y)}^{(x)} \right) \left( D_\nu (A^*(x)) \right) \omega^{(\mu, y)} \omega^{(q, z)} \]
\[ + G^{\text{AR}} \tilde{\mathcal{F}}_q \Phi^q, R^{\alpha} p_{\alpha} : \]
\[ - C_{\mu \nu} \int d^3 y d^3 z \left[ D_\nu (A^*(x)) \right] \omega^{(\mu, y)} p_{(x, u)} \]
\[ - C_{\mu \nu} \int d^3 y d^3 z \left[ D_\nu (A^*(x)) d^{(\mu, u)}(\nu, x) \right] \omega^{(x, z)} p_{(x, u)} \]
\[ + 2 C_{\nu \mu} \int d^3 y d^3 z \left( \omega^{(\nu, y)} p_{(x, u)} \right) \omega^{(x, z)} \]
\[ + C_{\nu \mu} \int d^3 y d^3 z \left[ D_\nu (A^*(x)) d^{(\mu, u)}(\nu, x) \right] \omega^{(x, z)} p_{(x, u)} \]
\[ + G^{\text{AR}} \tilde{\mathcal{F}}_q \Phi^q, R^{\alpha} p_{\alpha} : \]
\[ - C_{\mu \nu} \int d^3 y d^3 z \left[ D_\nu (A^*(x)) \right] \omega^{(\mu, y)} p_{(x, u)} \]
\[ + 1 \left( \partial_\nu d^{(\mu, y)}(\nu, x) \right) \omega^{(x, z)} p_{(x, u)} \]
\[ + G^{\text{AR}} \tilde{\mathcal{F}}_q \Phi^q, R^{\alpha} p_{\alpha} : \]
\[ - D_\beta (A^*(x, t)) \right) P^{\beta j}(x, t) + G_\alpha k^{\alpha} \left( \tilde{J}_\alpha \right) P^{\alpha} (x, t) = 0 \] (19)
\[+2G^{mn} \hat{\Gamma}_p B_{m} \omega^p \omega^B :\]
\[+ \frac{1}{2} (\tilde{J}_\nu)^{n}_{\nu} G_{q,n}(\tilde{J}_\mu)^{m}_{n} \tilde{f}^m(x) (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 z d^3 y d(\beta,x)(\epsilon^\prime, z^\prime) \hat{\mathcal{F}}^{(\mu, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[+ \frac{1}{2} (\tilde{J}_\mu)^{n}_{\nu} (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 z d^3 y d(\beta,x)(\epsilon^\prime, z^\prime) \hat{\mathcal{F}}^{(\mu, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[+ \frac{1}{2} c_{\epsilon \nu} (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 z d^3 y d(\beta,x)(\epsilon^\prime, z^\prime) \left[ D^{\nu \beta}_{\mu} (\hat{A}^\nu(z)) \hat{\mathcal{F}}^{(\mu, x)}_{(p,z)} \right] \omega^{(e,k,z)}_{(p,z)} \]
\[+ \frac{1}{2} c_{\epsilon \nu} (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 z d^3 y d(\beta,x)(\epsilon^\prime, z^\prime) \hat{\mathcal{F}}^{(\mu, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[- \frac{1}{2} (\tilde{J}_\beta)^{x}_{x} \int d^3 z \hat{\mathcal{F}}^{(\beta, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[+ \frac{1}{2} (\tilde{J}_\beta)^{x}_{x} \int d^3 z \hat{\mathcal{F}}^{(\gamma, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[+ \frac{1}{2} (\tilde{J}_\beta)^{x}_{x} \int d^3 z \hat{\mathcal{F}}^{(\mu, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[+ \frac{1}{2} (\tilde{J}_\beta)^{x}_{x} \int d^3 z \hat{\mathcal{F}}^{(\nu, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,z)} \]
\[\mathcal{F}_{\gamma m}^{\alpha} \omega^p :\]
\[- c^\gamma_{\mu} (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 u d^3 z d(\alpha', u)(\mu, z) \hat{\mathcal{F}}^{(\beta, x)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,\alpha', u)} \]
\[- c^\gamma_{\mu} (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 u d^3 z d(\alpha', u)(\mu, z) \hat{\mathcal{F}}^{(\nu, u)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,\alpha', u)} \]
\[+ c^\gamma_{\mu} (\tilde{J}_\beta)^{x}_{x} \tilde{f}^x(x) \int d^3 u d^3 z d(\alpha', u)(\mu, z) \hat{\mathcal{F}}^{(\nu, u)}_{(e,k,z)} \omega^{(e,k,z)}_{(p,\alpha', u)} \]
- (\tilde{J}_\mu)_n^a (\tilde{J}_\beta)_m^b \tilde{f}^m (x) \int d^3u \, d^3z \, \omega^{(\beta, x)} (n, z) \omega^{(\alpha', u)} (\mu, z) \omega^{(q, z)} p_{(\alpha', u)} \\
- (\tilde{J}_\mu)_n^a (\tilde{J}_\beta)_m^b \tilde{f}^m (x) \int d^3u \, d^3z \, \omega^{(\beta, x)} (\mu, z) \omega^{(\alpha', u)} (n, z) \omega^{(q, z)} p_{(\alpha', u)} \\
- (\tilde{J}_\mu)^{r}_m (\tilde{J}_\nu)^{r}_n \tilde{f}^{m'} (x) \int d^3u \, d^3z \, \omega^{(\alpha', u)} (\mu, z) \omega^{(\nu, x)} (\sigma, z) \omega^{(q, z)} p_{(\alpha', u)} \\
- (\tilde{J}_\mu)^{r}_m (\tilde{J}_\nu)^{r}_n \tilde{f}^{m'} (x) \int d^3u \, d^3z \, \omega^{(\alpha', u)} (\nu, z) \omega^{(\mu, x)} (\sigma, z) \omega^{(q, z)} p_{(\alpha', u)} \\
+ 2 (\tilde{J}_\beta)_q \omega^{(q, x)} \int d^3u \, d^3x \omega^{(\beta, z)} p_{(\alpha', u)} \\
+ c_{\nu \mu}^a (\tilde{J}_\beta)_m^b \tilde{f}^m (x) \int d^3u \, d^3z \, \omega^{(\mu, u)} (\beta, z) \omega^{(\nu, x)} (n, z) \omega^{(q, z)} p_{(\alpha', u)} \\
\frac{1}{2} G^{mn} (\mathcal{D}_n d^m \sigma) p_n p_\sigma : \\
+ (\tilde{J}_\mu)^{r}_n (\tilde{J}_\beta)_q ^n q \tilde{f}^m (x) \int d^3z \, d^3\beta \omega^{(\alpha', u)} (\beta, z) \omega^{(\nu, x)} (n, z) \omega^{(q, z)} p_{(\alpha', u)} \\
+ c_{\beta \mu}^k (\tilde{J}_\epsilon)_n^k \tilde{f}^n (x) \int d^3z \, d^3\beta \omega^{(\alpha', u)} (\epsilon, z) \omega^{(\nu, x)} (n, z) \omega^{(q, z)} p_{(\alpha', u)} \\
G^{r m} V_m : + G^{r m} G_{a b} \nabla^a_m (\nabla \tilde{f}^b) = 0.

The vertical Lagrange-Poincaré equation for the gauge system is

\[
\frac{dp_\beta (x, t)}{dt} + c_{\mu \beta}^\nu \left( \int d^3y \, d^3x \omega^{(\mu, x)} (\sigma, y) p_\sigma (y, t) \right) p_\nu (x, t) \\
- c_{\nu \beta}^\mu \left( \int d^3z \omega^{(\sigma, x)} (\epsilon, z) \hat{A}^{* \epsilon k} (x, t) \right) p_\nu (x, t) - c_{\sigma \beta}^\nu \left( \int d^3y \omega^{(\sigma, x)} (\epsilon, z) \hat{f}^{n} (y, t) \right) p_\nu (x, t) = 0
\]

5 Concluding remarks

The obtained equations are represented a rather complex expressions. A possible simplification of these equations can be obtained by projection of the equations onto the orbit space of the principal bundle. This is achieved by setting the group variable to zero. This suggests that the full equations apparently describe the dynamics of the system in the excited state. A direct consequence of these equations are the equations for the relative equilibria of the dynamical system under consideration. These equations can easily be derived from the result of those equations that are obtained in the work. It should also be noted that the role played by the equations of the mechanical system. Without these equations, it would be difficult to understand in
complex expression the nature and origin of individual terms of the equation for the gauge system. It remains unclear whether it is possible to somehow simplify the resulting equation based on some kind of symmetry. It is also not clear whether the equation for relative equilibria can be used to study the problem related to symmetry breaking.

Appendix A

Functional representation for terms of the Lagrange-Poincaré equations

\[ G^{AR} H_{BMR} = -\frac{1}{2} (\omega_B^{\beta} K^\alpha_B + \omega_{M,B}^\beta K^\alpha_B) - (\omega_M^\beta K^\alpha_{\beta,B} + \omega_B^\beta K^\alpha_{B,M}) \]

\[ + \frac{1}{2} (K^\alpha_{\mu,D} K^D_{\beta}) (\omega_M^\mu \omega_B^\sigma + \omega_M^\sigma \omega_B^\mu) \]

\[ \omega_B^\beta_{,M} = -d^{\beta e} K^\alpha_{e,M} G_{A'B'} K_{\mu,B}^B A_{M}^\mu - A_{D}^\beta A_{\mu,B} K_{\mu,M}^D + d^{\beta e} K_{\mu,M}^D G_{DB} \]

The expression \( I = -\frac{1}{2} (\omega_B^\beta K^B_B + \omega_B^\beta K^A_B) K^A_B \omega_B^M = -\omega_B^\beta K^A_B \omega_B^M \omega^M \)

consists of three terms:

\( I_1 \):

\[ c_{\gamma,i} \nu \epsilon \delta_{jm} \int d^3 y \left[ D^{\gamma \mu}_{\nu} (A^*(x)) \phi^{(\mu,y)}(\epsilon,k,z) \right] \]

\[ \times \left[ D^{\beta}_{\gamma} (A^*(x)) d^{(\beta,x)}(\epsilon',z) \omega^{(\gamma,i,y)}(\epsilon,k,z) \right] \]

\( I_2 \):

\[ c_{\gamma,i} \nu \phi \int d^3 y \phi^{(\mu,y)}(\epsilon,k,z) \left[ D^{\gamma \mu}_{\nu} (A^*(x)) \phi^{(\beta,x)}(\epsilon,j,y) \right] \omega^{(\gamma,i,y)}(\epsilon,k,z) \]

\( I_3 \):

\[ -c_{\gamma,i} \delta_{jk} \int d^3 z \left[ D^{\gamma \mu}_{\nu} (A^*(x)) d^{(\beta,z)}(\mu,x) \right] \omega^{(\gamma,j,z)}(\epsilon,k,z) \]

\[ \Pi = -(\omega_M^\beta K^B_B + \omega_B^\beta K^A_M) \omega^M \omega^B = -2 \omega_M^\beta K^B_B, \omega^M \omega^B \]

\[ -2 c_{\gamma,i} \beta \left[ \int d^3 y \phi^{(\beta,z)}(\gamma,j,y) \omega^{(\gamma,i,y)} \right] \omega^{(\epsilon,i,x)} \]

\[ \Pi = \frac{1}{2} (K^A_{\mu,D} K^B_{\sigma}) (\omega_M^\mu \omega_B^\sigma + \omega_M^\sigma \omega_B^\mu) \omega^M \omega^B = (K^A_{\mu,D} K^D_{\sigma}) \omega_M^\mu \omega_B^\sigma \omega^M \omega^B \]

\[ c_{\epsilon,i} \phi \int d^3 y d^3 z \phi^{(\mu,x)}(\gamma,j,y) \left[ D^{\epsilon \mu}_{\nu} (A^*(x)) \phi^{(\nu,z)}(\epsilon,k,z) \right] \omega^{(\gamma,j,y)}(\epsilon,k,z) \]

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III can also be written in a following way:

\[ c^{α}_{βμ} \int d^3z \left[ D^{πi}_{ν} (A^α (x)) \varphi^{(ν, x)}_{(ε, k, z)} \right] ω^{(ε, k, z)} \times \int d^3y ϕ^{(μ, x)}_{(γ, j, y)} ω^{(γ, j, y)}. \]

\[ G^{AR} H^{Γ_{BM}} = I_1 + I_2 + I_3 + II + III. \]

\[ G^{AR} H^{Γ_{BmR}} = -\frac{1}{2} (\varphi^{β}_{B, m} K^A_{β} + \varphi^{β}_{m, B} K^A_{β}) - \varphi^{β}_{m} K^A_{β, B} \]
\[ + \frac{1}{2} (K^A_{μ, D} K^D_{σ})(\varphi^{σ}_{m} ϕ^{μ}_{B} + \varphi^{σ}_{m} ϕ^{μ}_{B}) \]

\[ G^{AR} H^{Γ_{BmR}} ω^{B, ω^m}, A \rightarrow (α, i, x), B \rightarrow (ε, k, z), m \rightarrow (m, v) \]

\[ G^{(α, i, x)(...) H^{Γ_{(ε, k, z)(m, v)(...) ω^{(e, k, z)} ω^{(m, v)}}}} \]

\[ ϕ^{β}_{m} = -d^{βε} K^r_{ε m} G_{rn} K^n_{μ} A^μ_{B} - A^μ_{n} A^μ_{B} K^ϕ_{mn}. \]

\[ I_1 = \frac{1}{2} (d^{βε} K^r_{ε m} G_{rn} K^n_{μ} A^μ_{B}) K^ϕ_{β} ω^{B, ω^m} : \]
\[ \frac{1}{2} (J^r_{ε})_{m} G_{rn} (J^r_{μ})\int d^3z d^3z' \left[ D^{αi}_{β} (A^α (x)) d^{(β, x)(ε', z')} \right] \varphi^{(μ, z')}_{(ε, k, z)} ϕ^{(β, ε', z')} ω^{(m, z')}. \]

\[ I_2 = \frac{1}{2} (A^μ_{α} A^μ_{B} K^ϕ_{m} K^ϕ_{β}) ω^{B, ω^m} : \]
\[ \frac{1}{2} (J^r_{μ})_m \int d^3z d^3z' \varphi^{(μ, z')}_{(ε, k, z)} \left[ D^{αi}_{β} (A^α (x)) \varphi^{(β, x)}_{(m, z')} \right] ω^{(ε, k, z)} ω^{(m, z')}. \]

\[ I_3 = \frac{1}{2} (d^{βε} K^r_{ε B} G^{AB'} K^ϕ_{μ} A^μ_{m}) K^ϕ_{β} ω^{B, ω^m} : \]
\[ \frac{1}{2} (c^{γ}_{ε ε} K^ϕ_{γ μ} δ_{kk'} \int d^3z d^3z' \left[ D^{αi}_{μ} (A^α (x)) \varphi^{(μ, z')}_{(m, z')} \right] \left[ D^{αi}_{β} (A^α (x)) d^{(β, x)(ε', z')} \right] ω^{(ε, k, z)} ω^{(m, z')}. \]

\[ I_4 = \frac{1}{2} \varphi^{β}_{m} K^ϕ_{β} ω^{m} ω^{B} : \]
\[ \frac{1}{2} (c^{γ}_{ε μ} \int d^3z d^3z' \varphi^{(μ, z')}_{(m, z')} \left[ D^{αi}_{β} (A^α (x)) \varphi^{(β, x)}_{(γ, k, z)} \right] ω^{(ε, k, z)} ω^{(m, z')}. \]
The terms without dependence on $K^3_β$ are represented by 

\[(II + III)ω^B_ω^m:\]

\[-δ^α_κδ^β_μ \int d^3v \mathcal{A}^{(β,x)}_{(m,v)} ω^{(m,v)}(v,k,x)\]

\[+\frac{1}{2}ε^α_κδ^β_μ \int d^3v d^3z \left[ \mathcal{D}_ν^{μi}(A^*(x)) \mathcal{D}_ν^{(ν,x)}(m,v) ω^{(m,v)}(v,k,z) \right]\]

\[G^{AR} H^{BnR} ω^B_ω^m = I + II + III.\]

\[G^{AR} H^Γ_{pqR}\]

\[G^{AR} H^Γ_{pqR} = -\frac{1}{2}(\mathcal{D}_μ^{β}K^A_β + \mathcal{D}_ν^{β}K^A_β)\]

\[+\frac{1}{2}(K^A_ε,D^κ_κ)(\mathcal{D}_μ^{ε}\mathcal{D}_ν^{κ} + \mathcal{D}_ν^{ε}\mathcal{D}_μ^{κ})\]

\[-\frac{1}{2}(\mathcal{D}_μ^{β}K^A_β + \mathcal{D}_ν^{β}K^A_β)ω^p_ω^q\]

has three terms.

\[I_1:\]

\[(\bar{J}_ε)_q G_{r'}^{n'}(\bar{J}_μ)_n' \int d^3y d^3z \bar{f}^{m'}(z) \left[ \mathcal{D}_μ^{β}(A^*(x))d^{(β,x)(ε,z)} \right] \mathcal{A}^{(μ,z)}_{(p,y)} ω^{(p,y)}(q,z).\]

\[I_2:\]

\[(\bar{J}_μ)_q^n G_{n'}^{p} \int d^3y d^3z \mathcal{A}^{(μ,z)}_{(p,y)} \left[ \mathcal{D}_μ^{β}(A^*(x))d^{(β,x)(n,z)} \right] ω^{(p,y)}(q,z).\]

\[I_3:\]

\[-(\bar{J}_μ)_q^n G_{np} \int d^3y \left[ \mathcal{D}_μ^{β}(A^*(x))d^{(β,x)(μ,y)} \right] ω^{(p,y)}(q,y).\]

\[II = +\frac{1}{2}(K^A_β,D^κ_κ)(\mathcal{D}_μ^{β}\mathcal{D}_ν^{κ} + \mathcal{D}_ν^{β}\mathcal{D}_μ^{κ}) + \frac{1}{2}δ^μ_κδ^ν_κ \int d^3y d^3z \left( A^{(μ,x)}_{(p,y)} \mathcal{D}_ν^{μi}(A^*(x)) \mathcal{D}_ν^{(β,x)}(q,z) \right) ω^{(p,y)}(q,z).\]

\[G^{AR} H^Γ_{pqR} ω^p_ω^q = I_1 + I_2 + I_3 + II.\]
\[
G^{\alpha m} H \Gamma_{ABm} = -\frac{1}{2} (\mathcal{A}^{\beta}_{A,B} K^r_{\beta} + \mathcal{A}^{\omega}_{B,A} K^r_{\omega}) + \frac{1}{2} (K^r_{\mu,\nu} K^r_{\nu,\mu}) (\mathcal{A}^{\mu}_{A} \mathcal{A}^{\mu}_{B} + \mathcal{A}^{\mu}_{A} \mathcal{A}^{\mu}_{B})
\]

\[
\mathcal{A}^{\beta}_{A,B} = -d^{\beta \mu} K^Q_{\mu,\beta} \mathcal{A}^{\alpha}_{Q} K^D G_{DA} - \mathcal{A}^{\beta}_{Q} K^Q_{\nu,\beta} \mathcal{A}^{\nu}_{A} + d^{\beta \epsilon} K^D_{\epsilon,B} G_{DA}
\]

\[A \to (\alpha, n, z'), \quad B \to (\gamma, k, z), \quad \beta \to (\beta, u), \quad \mu \to (\mu, v), \quad Q \to (\nu, j, y), \quad D \to (\sigma, m, y'), \quad \epsilon \to (\epsilon, t), \quad r \to (r, x).\]

The terms of \(\mathcal{A}^{\beta}_{A,B}:
\]

**I**₁) 

\[-\delta^j_k c_{\gamma \mu} d^{(\beta, u)(\mu, z)} k_{\sigma \alpha} \delta_{mn} [\mathcal{D}^{en}_{\epsilon} (A^* (z')) \mathcal{A}^{(\epsilon, z')}_{(\epsilon, j, z)}]\]

**I**₂) 

\[-c_{\gamma \mu} \mathcal{A}^{(\beta, u)}_{(\gamma, k, z)} \mathcal{A}^{(\epsilon, z')}_{(\alpha, n, z')}\]

**I**₃) 

\[c_{\gamma \epsilon} k_{\sigma \alpha} \delta_{kn} d^{(\beta, u)(\epsilon, z)} \delta^3 (z - z')\]

\[
\mathcal{A}^{\beta}_{B,A} = -d^{\beta \mu} K^Q_{\mu,\beta} \mathcal{A}^{\alpha}_{Q} K^D G_{DB} - \mathcal{A}^{\beta}_{Q} K^Q_{\nu,\beta} \mathcal{A}^{\nu}_{B} + d^{\beta \epsilon} K^D_{\epsilon,A} G_{DB}
\]

The terms of \(\mathcal{A}^{\beta}_{B,A}:
\]

**I**₄) 

\[-\delta^j_k c_{\alpha \mu} d^{(\beta, u)(\mu, z)} k_{\sigma \gamma} \delta_{mn} [\mathcal{D}^{en}_{\epsilon} (A^* (z')) \mathcal{A}^{(\epsilon, z')}_{(\epsilon, j, z)}]\]

**I**₅) 

\[-c_{\alpha \mu} \mathcal{A}^{(\beta, u)}_{(\gamma, k, z)} \mathcal{A}^{(\epsilon, z')}_{(\gamma, k, z)}\]

**I**₆) 

\[c_{\alpha \epsilon} k_{\gamma \gamma} \delta_{kn} d^{(\beta, u)(\epsilon, z)} \delta^3 (z' - z)\]

(All terms of I must be multiplied by \(K^r_{\beta} \omega^A \omega^B\).)

\[II = \frac{1}{2} (K^r_{\beta,\mu} K^r_{\nu,\beta}) (\mathcal{A}^{\mu}_{A} \mathcal{A}^{\mu}_{B} + \mathcal{A}^{\mu}_{A} \mathcal{A}^{\mu}_{B}) : \]

\[+ \frac{1}{2} (\mathcal{J}_{\mu})_{\gamma} (\mathcal{J}_{\epsilon})_{\mu} \tilde{f}^q (x) (\mathcal{A}^{(\epsilon, x)}_{(\alpha, n, z')} \mathcal{A}^{(\mu, x)}_{(\gamma, k, z)} + \mathcal{A}^{(\mu, x)}_{(\alpha, n, z')} \mathcal{A}^{(\epsilon, x)}_{(\gamma, k, z)})\]

\[G^{\alpha m} H \Gamma_{ABm} \omega^A \omega^B = I + II = \]

\[+ c_{\gamma \mu} (\mathcal{J}_{\beta})_{\mu} \tilde{f}^p (x) k_{\sigma \alpha} \int d^3 z d^3 z' d^{(\beta, u)(\mu, z)} [\mathcal{D}^{en}_{\epsilon} (A^* (z')) \mathcal{A}^{(\epsilon, z')}_{(\epsilon, j, z)}] \omega_{(\alpha, n, z')} \omega_{(\gamma, k, z)}\]

\[+ c_{\gamma \epsilon} (\mathcal{J}_{\beta})_{\epsilon} \tilde{f}^q (x) k_{\sigma \alpha} \int d^3 z d^3 z' d^{(\beta, u)(\epsilon, z)} [\mathcal{D}^{en}_{\epsilon} (A^* (z')) \mathcal{A}^{(\epsilon, z')}_{(\epsilon, j, z)}] \omega_{(\alpha, n, z')} \omega_{(\gamma, k, z)}\]

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\[ +c^\nu_\mu k_\mu \omega (\bar{J}_\beta) \hat{f}_p (x) \int d^3 z d^3 z' \omega^{(\mu, z)}_{(n, n', z')} \omega^{(\beta, z')}_{(\nu, k, z)} \omega^{(\gamma, k, z)}_{(\nu, k, z)} \]

\[ - e^\nu_{\tau \rho} \delta_{\kappa \sigma} k_\rho (\bar{J}_\beta) \hat{f}_p (x) \int d^3 z d^3 z' \omega^{(\beta, z)}_{(\epsilon, z)} \omega^{(\gamma, k, z)}_{(\nu, k, z)} \]

\[ + (\bar{J}_\mu) m' (\bar{J}_\nu) \hat{f}_q (x) \int d^3 z d^3 z' \omega^{(\epsilon, z)}_{(n, n', z')} \omega^{(\mu, z)}_{(\gamma, k, z)} \omega^{(\epsilon, k, z)}_{(\nu, k, z)} \]

(Here the symmetry between A and B was taken into account.)

\[ G^{\text{Im}} H_{p B m} = - \frac{1}{2} (\gamma^\nu_\beta \gamma^\rho_\beta - \omega^{\beta, p} B_{\beta, p}) K^r_{\beta} - \omega^{\beta, p} K^r_{\beta, p} \]

\[ + \frac{1}{2} (K^r_{\beta, q} K^r_{\beta}) (\gamma^\rho_\beta + \omega^{\beta, p}) \]

\[ \omega^{\beta, p} = - d^{3 \epsilon} \epsilon_{\nu} G_{\epsilon, p, B_{\beta, p}} K^{\mu} - \omega^{\beta, \nu} \omega^{\mu, p} \]

\[ B \rightarrow (\epsilon, k, z), \quad p \rightarrow (p, z), \quad \beta \rightarrow (\beta, u) \]

The terms of \(- \frac{1}{2} \omega^{\beta, p} K^r_{\beta} \omega^{B, p}:

I_1:\]

\[ \quad = \frac{1}{2} (\bar{J}_\nu) x^2 G_{\epsilon, p, n} (\bar{J}_\mu) n' \hat{f}_{\nu, m} (x) (\bar{J}_\beta) \hat{f}_1 (x) \int d^3 z d^3 z' d^3 (\beta, z') \omega^{(n, z')}_{(x, k, z)} \omega^{(x, k, z)}_{(p, z')} \]

I_2:\]

\[ = \frac{1}{2} - \frac{1}{2} (\omega^{\beta, p} K^{n} K^{n, p}) \omega^{B, p} : 

+ \frac{1}{2} (\bar{J}_\nu) n (\bar{J}_\mu) \hat{f}_q (x) \int d^3 z d^3 z' \omega^{(\beta, z)}_{(n, z')} \omega^{(\mu, z')}_{(x, k, z)} \omega^{(x, k, z)}_{(p, z')} \]

I_3:\]

\[ \omega^{\beta, p} = - d^{3 \epsilon} \epsilon_{\nu} G_{\epsilon, B, B_{\beta, p}} K^{\mu} - \omega^{\beta, \nu} \omega^{\mu, p} \]

The terms of \(- \frac{1}{2} \omega^{\beta, p} K^r_{\beta} \omega^{B, p}:

I_4:\]

\[ = \frac{1}{2} + \frac{1}{2} \omega^{\beta, p} K^r_{\beta} \omega^{B, p} : 

+ \frac{1}{2} (\bar{J}_\nu) q \hat{f}_{\gamma} (x) \int d^3 z d^3 z' \omega^{(\beta, z)}_{(x, k, z)} \omega^{(x, k, z)}_{(p, z')} \]

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\[ \Pi = -\mathcal{A}_B^\beta K_{\beta,p}^r \omega^B \omega^p : \\
- (\bar{J}_\beta)_p^r \left[ \int d^3z \, \mathcal{A}^{(\beta,x)}_{A_{\epsilon,k,z}} \omega^{(\epsilon,k,z)} \right] \omega^{(p,x)} \]

\[ \Pi = \frac{1}{2} (K_{\epsilon,q}^r K_{n,q}^\nu) (\mathcal{A}_{p,q}^{\nu} \mathcal{A}_B^{\mu - \ell}) : \omega^B \omega^p : \\
\frac{1}{2} \mathcal{J}_+^r (\bar{J}_\mu)_n^r \int d^3z d^3z (\mathcal{A}_{(\epsilon,k,z)} K_{(\epsilon,k,z)} + \mathcal{A}_{(\gamma,z)} \mathcal{A}_{(\epsilon,k,z)}) \omega^{(\epsilon,k,z)} \omega^{(p,z')} \]

\[ G^m \mathcal{T}_{pBm} \omega^B \omega^p = I_1 + I_2 + I_3 + I_4 + II + III \]

\[ G^m \mathcal{T}_{pqm} = -\frac{1}{2} \left( \mathcal{A}_{p,q}^{\beta} K_{\beta,p}^r + \mathcal{A}_{q,p}^{\beta} K_{\beta,p}^r - \mathcal{A}_{p,q}^{\beta} K_{\beta,p}^r - \mathcal{A}_{q,p}^{\beta} K_{\beta,p}^r \right) + \frac{1}{2} (K_{\epsilon,q}^r K_{n,q}^\nu) (\mathcal{A}_{p,q}^{\nu} + \mathcal{A}_{n,q}^{\nu} + \mathcal{A}_{p,q}^{\nu}) \]

\[ I_1 = -\frac{1}{2} \mathcal{A}_{p,q}^{\beta} K_{\beta,p}^r \omega^p \omega^q : \\
(\bar{J}_\beta)_p^r \tilde{f}^n (x) (\bar{J}_\epsilon)_q^r G_{\gamma,n'} (\bar{J}_\mu)_n^p \int d^3z d^3y d^{(\beta,x)}(\epsilon,z) f_{\tilde{m}^n} (z) \mathcal{A}_{(p,y)}^{(\mu,z)} \omega^{(p,y)} \omega^{(q,z)} \]

\[ I_2 = -\frac{1}{2} \left( \mathcal{A}_{p,q}^{\beta} \mathcal{A}_p^{\mu} K_{\beta,p}^n \right) \omega^p \omega^q : \\
(\bar{J}_\beta)_q^n (\bar{J}_\mu)_n^p \tilde{f}^m (x) \int d^3z d^3y \mathcal{A}_{(\beta,x)}^{(\epsilon,z)} (\mu,y) \mathcal{A}_{(p,y)}^{(\mu,z)} \omega^{(p,y)} \omega^{(q,z)} \]

\[ I_3 = -\frac{1}{2} \left( \mathcal{A}_{p,q}^{\beta} \mathcal{A}_p^{\mu} K_{\beta,p}^n \right) \omega^p \omega^q : \\
- (\bar{J}_\mu)_q^p G_{np} (\bar{J}_\beta)_n^p \tilde{f}^m (x) \int d^3y d^{(\beta,x)}(p,y) \omega^{(p,y)} \omega^{(q,y)} \]

\[ II = 2 \left( \mathcal{A}_B^{\beta} K_{\beta,p}^r \right) \omega^p \omega^q : \\
-2 (\bar{J}_\beta)_p^r \left[ \int d^3y \mathcal{A}_{(p,y)}^{(q,z)} \omega^{(p,y)} \right] \omega^{(q,x)} \]

\[ III = \frac{1}{2} (K_{\epsilon,q}^r K_{n,q}^\nu) \mathcal{A}_{p,q}^{\mu} \mathcal{A}_p^{\nu} \omega^q \omega^p : \\
(\bar{J}_\beta)_m^r (\bar{J}_\nu)_n^m \tilde{f}^k (x) \int d^3z d^3y \mathcal{A}_{(q,z)}^{(\mu,z)} (\mu,y) \mathcal{A}_{(p,y)}^{(\mu,z)} \omega^{(p,y)} \omega^{(q,z)} \]

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$III$ can also be written in a following way:

$$
\left( \sum_{k} \int d^{3}y \ A^{(\nu,x)}_{(p,y)} \omega^{(p,y)} \right) \left( \sum_{m} \int d^{3}z \ \mathcal{A}^{(\mu,x)}_{(q,z)} \omega^{(q,z)} \right)
$$

$G_{mpq} = I_{1} + I_{2} + I_{3} + II + III$

$$
G^{AR} \mathcal{F}^{\alpha}_{QR} \omega^{Q} p_{\alpha'}
$$

$C_{QR}^{AR} \mathcal{F}^{\alpha}_{QR} = -(K_{S}^{-} \mathcal{A}^{\alpha}_{QR} + d^{\alpha}_{SR} + d^{\alpha}_{SR}) K_{\mu}^{A} - (K_{e,B}^{A} K_{e}^{B})(d^{\alpha} \mathcal{A}^{\alpha} + d^{\alpha} \mathcal{A}^{\alpha}) + 2d^{\alpha} K_{\mu}^{A} + c_{\mu}^{\alpha} d^{\mu} \mathcal{A}^{\alpha}

A = (\alpha, i, x), \quad Q = (\epsilon, k, z)

I_{1} = -(K_{S}^{-} \mathcal{A}^{\alpha}_{QR} + d^{\alpha} \mathcal{A}^{\alpha}) K_{\mu}^{A} \omega^{Q} p_{\alpha'}$

$$
- c_{\epsilon \nu}^{\alpha} \int d^{3}u d^{3}z d^{(\varphi,x)}(\alpha', u) \left[ \mathcal{D}_{\mu}^{\alpha}(A^{\alpha}(x)) \mathcal{A}_{(\gamma,k,z)}^{(\mu,x)} \right] \omega^{(\epsilon,k,z)} p_{(\alpha', u)}
$$

$II_{1} = -(K_{e,B}^{A} K_{e}^{B}) d^{\alpha} \mathcal{A}^{\alpha} \omega^{Q} p_{\alpha'}$

$$
- c_{\mu}^{\alpha} \int d^{3}u d^{3}z d^{(\alpha', u)}(\varphi,x) \left[ \mathcal{D}_{\nu}^{\alpha}(A^{\alpha}(x)) d^{(\varphi,x)}(\mu) \right] \omega^{(\epsilon,k,z)} p_{(\alpha', u)}
$$

$II_{2} = -(K_{e,B}^{A} K_{e}^{B}) d^{\alpha} \mathcal{A}^{\alpha} \omega^{Q} p_{\alpha'}$

$$
- c_{\nu}^{\alpha} \int d^{3}u d^{3}z d^{(\varphi,x)}(\epsilon,k,z) \left[ \mathcal{D}_{\nu}^{\alpha}(A^{\alpha}(x)) d^{(\alpha', u)}(\varphi,x) \right] \omega^{(\epsilon,k,z)} p_{(\alpha', u)}
$$

$III = 2d^{\alpha} K_{\mu}^{A}$

$$
2c_{\epsilon \nu}^{\alpha} \left[ \int d^{3}u d^{(\beta,u)}(\mu) \omega^{(\epsilon,i)} \right]
$$

IV $= c_{\mu}^{\alpha} d^{\mu} \mathcal{A}^{\alpha} K_{\nu}^{A}$

where $c_{(\alpha', u)}^{\alpha} = c_{(\nu,y)}^{\alpha} \delta^{3}(u - y) \delta^{3}(u - v)$.
\[ G_{\nu}^{\alpha'} \int d^3 u \, d^3 z \, \mathcal{A}^{(\nu, u)}_{(\kappa, k, z)} \left[ D_\varphi^{\alpha i} (A^\ast (x)) d^{(\mu, u)(\varphi, x)} \right] \omega^{(\varphi, k, z)}_{(\alpha', u)} \]

\((K^A_\mu\text{-terms: first (I) and fourth (IV))}.

G^{AR}_{qR} \mathcal{G}^{\alpha'}_{qR} \omega^Q_{p_{\alpha'}} = I + II + III + IV.\]

\[ G^{AR}_{qR} \mathcal{G}^{\alpha'}_{qR} \omega^q_{p_{\alpha'}} \]

\(G^{AR}_{qR} \mathcal{G}^{\alpha'}_{qR} = -(K_{\mu, q}^r) (d^{\mu a} \mathcal{A}^a_{\nu} + d^{\mu \varphi} \mathcal{A}^\varphi_{\nu}) K^A_{\varphi} - (K^A_{\mu, B} K^B_{\varphi}^r) (d^{\mu \varphi} \mathcal{A}^\varphi_{\nu} + d^{\alpha'} \mathcal{A}^\alpha_{\nu}) \]

\(A = (\alpha, i, x), \quad q = (q, y), \quad \alpha' = (\alpha', u)\)

\(I_1) = -(K_{\mu, q}^r) d^{\mu a} \mathcal{A}^a_{\nu} K^A_{\varphi} \omega^q_{p_{\alpha'}} :\]

\[-(\mathcal{J}_\mu)^k \int d^3 u \, d^3 y \, d^{(\mu, y)(\alpha', u)} \left[ D_\varphi^{\alpha i} (A^\ast (x)) \mathcal{A}^{(\varphi, x)}_{(k, y)} \right] \omega^{(y, y)}_{(\alpha', u)} \]

\(I_2) = -(K_{\mu, q}^r) d^{\mu \varphi} \mathcal{A}^\varphi_{\nu} K^A_{\varphi} \omega^q_{p_{\alpha'}} :\]

\[-(\mathcal{J}_\mu)^k \int d^3 u \, d^3 y \, \mathcal{A}^{(\alpha', u)}_{(k, y)} \left[ D_\varphi^{\alpha i} (A^\ast (x)) d^{(\mu, y)(\varphi, x)} \right] \omega^{(y, y)}_{(\alpha', u)} \]

\(I_{11}) = -(K^A_{\mu, B} K^B_{\varphi}^r) d^{\alpha'} \mathcal{A}^{\alpha'}_{\nu} \omega^q_{p_{\alpha', u}} :\]

\[-c^a_{\mu, q} \int d^3 u \, d^3 y \, d^{(\alpha', u)(\nu, x)} \left[ D_\varphi^{\alpha i} (A^\ast (x)) \mathcal{A}^{(\varphi, x)}_{(q, y)} \right] \omega^{(q, y)}_{(\alpha', u)} \]

\(I_{12}) = -(K^A_{\mu, B} K^B_{\varphi}^r) d^{\alpha'} \mathcal{A}^{\alpha'}_{\nu} \omega^q_{p_{\alpha', u}} :\]

\[-c^\alpha_{\mu, q} \int d^3 u \, d^3 y \, \mathcal{A}^{(\nu, x)}_{(q, y)} \left[ D_\varphi^{\alpha i} (A^\ast (x)) d^{(\alpha', u)(\varphi, x)} \right] \omega^{(q, y)}_{(\alpha', u)} \]

\(\text{III} = c^\alpha_{\mu, q} d^{\mu \varphi} \mathcal{A}^{\alpha'}_{\nu} K^A_{\varphi} \omega^q_{p_{\alpha', u}} :\]

\[-c^\alpha_{\mu, q} \int d^3 u \, d^3 y \, \mathcal{A}^{(\nu, x)}_{(q, y)} \left[ D_\varphi^{\alpha i} (A^\ast (x)) d^{(\mu, u)(\varphi, x)} \right] \omega^{(q, y)}_{(\alpha', u)} \]

\((K^A_\mu\text{-terms: first (I) and third (III))}.

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\[ G^A_R \mathcal{F}^{\alpha'}_{q_R} \omega^Q p_{\alpha'} = I + II + III \]

\[ G^{\text{rm}} \mathcal{F}^{\alpha'}_{Q_m} \omega^Q p_{\alpha'} \]

\[ G^{\text{rm}} \mathcal{F}^{\alpha'}_{Q_m} = -K^T_{\mu,Q} (d^{\alpha'} \mathcal{A}_T^\beta + d^{\beta} \mathcal{A}_T^{\alpha'}) K^r_{\beta} - (K^n_{\nu} K^T_{\mu,n}) (d^{\alpha'} \mathcal{A}^\nu + d^{\nu} \mathcal{A}^\mu) \]
\[ + c^{\alpha'}_{\nu \mu} d^{\nu \beta} \mathcal{A}^\mu_{Q} K^r_{\beta}. \]

r = (r, x), Q = (\varepsilon, k, z), \alpha' = (\alpha', u)

\[ I_1 = -K^T_{\mu,Q} d^{\alpha'} \mathcal{A}^\nu_{Q} \omega^Q p_{\alpha'} : \]
\[ -c^{\alpha'}_{\nu \mu} (\bar{J}_\beta)^r_{m} \tilde{f}^m (x) \int d^3 u \, d^3 z \, d^{(\alpha', u)(\mu, z)} \mathcal{A}^{(\beta, z)}_{(\gamma, k, z)} \mathcal{A}^{(\epsilon, k, z)}_{(\alpha', u)} p_{(\alpha', u)} \]

\[ I_2 = -K^T_{\mu,Q} d^{\beta} \mathcal{A}^\alpha_{Q} K^r_{\beta} \omega^Q p_{\alpha'} : \]
\[ -c^{\gamma}_{\nu \mu} (\bar{J}_\beta)^r_{m} \tilde{f}^m (x) \int d^3 u \, d^3 z \, d^{(\alpha', u)(\beta, z)} \mathcal{A}^{(\nu, x)}_{(\epsilon, k, z)} \mathcal{A}^{(\epsilon, k, z)}_{(\alpha', u)} p_{(\alpha', u)} \]

\[ II_1 = -K^T_{\nu} K^r_{\mu,n} \mathcal{A}^\mu_{Q} \mathcal{A}^\nu_{Q} \omega^Q p_{\alpha'} : \]
\[ -c^{\alpha'}_{\nu \mu} (\bar{J}_\nu)^r_{m} \tilde{f}^m (x) \int d^3 u \, d^3 z \, d^{(\alpha', u)(\nu, x)} \mathcal{A}^{(\epsilon, k, z)}_{(\alpha', u)} p_{(\alpha', u)} \]

\[ II_2 = -K^T_{\nu} K^r_{\mu,n} \mathcal{A}^\nu_{Q} \omega^Q p_{\alpha'} : \]
\[ -c^{\alpha'}_{\nu \mu} (\bar{J}_\nu)^r_{m} \tilde{f}^m (x) \int d^3 u \, d^3 z \, d^{(\alpha', u)(\nu, x)} \mathcal{A}^{(\epsilon, k, z)}_{(\alpha', u)} p_{(\alpha', u)} \]

\[ III = c^{\alpha'}_{\nu \mu} d^{\beta} \mathcal{A}^\nu_{Q} K^r_{\beta} \omega^Q p_{\alpha'} : \]
\[ c^{\alpha'}_{\nu \mu} (\bar{J}_\beta)^r_{m} \tilde{f}^m (x) \int d^3 u \, d^3 z \, d^{(\mu, u)(\beta, x)} \mathcal{A}^{(\nu, x)}_{(\epsilon, k, z)} \mathcal{A}^{(\epsilon, k, z)}_{(\alpha', u)} p_{(\alpha', u)} \]

\[ G^{\text{rm}} \mathcal{F}^{\alpha'}_{Q_m} \omega^Q p_{\alpha'} = I + II + III. \]

\[ G^{\text{rm}} \mathcal{F}^{\alpha'}_{q_m} \omega^Q p_{\alpha'} \]

\[ G^{\text{rm}} \mathcal{F}^{\alpha'}_{q_m} = -K^n_{\mu,Q} (d^{\alpha'} \mathcal{A}^\beta_n + d^{\beta} \mathcal{A}^{\alpha'}_n) K^r_{\beta} - (K^n_{\nu} K^r_{\mu,n}) (d^{\alpha'} \mathcal{A}^\nu + d^{\nu} \mathcal{A}^\mu) \]
\[ + 2d^{\alpha'} K^r_{\beta} \omega^Q + c^{\alpha'}_{\nu \mu} d^{\nu \beta} \mathcal{A}^\mu_{Q} K^r_{\beta}. \]
\( r = (r, x), \quad q = (q, z), \quad \alpha' = (\alpha', u) \)

\[ I_1 = -K^\mu_{\mu,q} d^\mu \omega_q \mathcal{A}^{\alpha_1} K^\nu_{\nu} \omega p_{\alpha'} : \]

\[ -\left( \tilde{J}_\mu \right)_{\nu}^r \left( \tilde{J}_\nu \right)_{m}^r \tilde{f}^m(x) \int d^3u d^3z \mathcal{A}^{(\beta,x)} (\mu,z) \omega (q,z) p_{(\alpha',u)} \]

\[ I_2 = -K^\mu_{\mu,q} d^\mu \omega_q \mathcal{A}^{\alpha_1} K^\nu_{\nu} \omega p_{\alpha'} : \]

\[ -\left( \tilde{J}_\mu \right)_{\nu}^r \left( \tilde{J}_\nu \right)_{m}^r \tilde{f}^m(x) \int d^3u d^3z d^{(\beta,x)(\mu,z)} \mathcal{A}^{(\alpha',u)} (\mu,z) \omega (q,z) p_{(\alpha',u)} \]

\[ II_1 = -\left( K^\mu_{\mu,p} K^\nu_{\nu} \right) d^\mu \omega_q \mathcal{A}^{\alpha_1} K^\nu_{\nu} \omega p_{\alpha'} : \]

\[ -\left( \tilde{J}_\mu \right)_{\nu}^r \left( \tilde{J}_\nu \right)_{m}^r \tilde{f}^m(x) \int d^3u d^3z d^{(\alpha',u)(\mu,z)} \mathcal{A}^{(\mu,x)} (\mu,z) \omega (q,z) p_{(\alpha',u)} \]

\[ II_2 = -\left( K^\mu_{\mu,p} K^\nu_{\nu} \right) d^\mu \omega_q \mathcal{A}^{\alpha_1} K^\nu_{\nu} \omega p_{\alpha'} : \]

\[ -\left( \tilde{J}_\mu \right)_{\nu}^r \left( \tilde{J}_\nu \right)_{m}^r \tilde{f}^m(x) \int d^3u d^3z d^{(\alpha',u)(\mu,x)} \mathcal{A}^{(\mu,x)} (\mu,z) \omega (q,z) p_{(\alpha',u)} \]

\[ III = 2 d^{\alpha_2} K^\nu_{\nu} \omega p_{\alpha'} : \]

\[ 2 \left( \tilde{J}_\nu \right)_{q}^r \omega (q,z) \int d^3u d^{(\alpha',u)(\beta,x)} p_{(\alpha',u)} \]

\[ IV = c_{\nu_m} q^{\nu_m} K^\sigma_{\sigma} \omega p_{\alpha'} : \]

\[ c_{\nu_m} \left( \tilde{J}_\nu \right)_{m}^r \tilde{f}^m(x) \int d^3u d^3z d^{(\mu,u)(\beta,x)} \mathcal{A}^{(\nu,x)} (\mu,z) \omega (q,z) p_{(\alpha',u)} \]

\[ G^{\alpha_m} \mathcal{F}^{\alpha_m} \omega p_{\alpha'} = I + II + III + IV. \]

\[ \mathbf{G}^{\mathbf{AR}} (\mathcal{D}_R d^{\kappa\sigma}) \mathbf{P}_\kappa \mathbf{P}_\sigma \]

\[ \mathbf{G}^{\mathbf{AR}} (\mathcal{D}_R d^{\kappa\sigma}) \mathbf{P}_\kappa \mathbf{P}_\sigma = 2 \left[ (K^\mu_{\beta} K^A_{\mu,D}) d^{\beta\kappa} d^{\mu}\sigma + c^\kappa_{\beta\mu} d^{\beta\kappa} d^{\mu}\sigma K^A_{\epsilon} \right] \mathbf{P}_\kappa \mathbf{P}_\sigma \]

\[ A = (\alpha, i, x), \quad \beta = (\beta, v), \quad \mu = (\mu, u), \quad \kappa = (\kappa, z), \quad \sigma = (\sigma, z') \]

\[ I = 2 c^\sigma_{\mu} \int d^3z d^3x d^{(\mu,x)(\sigma,z')} \left[ D^{\mu,i}_{\beta} (A^i(x)) d^{(\beta,x)(\kappa,z)} \right] p_{(\kappa,z)} p_{(\sigma,z')} \]

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\[ II = 2 \epsilon_{\beta\mu} \int d^3 z' d^3 z (\sigma, z') \left[ \mathcal{D}_\epsilon^{\alpha i} (A^\ast (x)) d(\beta, z)(\epsilon, x) \right] p(\kappa, z) p(\sigma, z') \]

\[ G^{\text{rm}}(\mathcal{D}_m d^{\kappa\sigma}) p_\kappa p_\sigma \]

\[ G^{\text{rm}}(\mathcal{D}_m d^{\kappa\sigma}) p_\kappa p_\sigma = 2 \left[ (K^n_\beta K^r_{\mu,n}) d^{\beta\kappa} d^{\mu\sigma} + \epsilon_{\beta\mu} d^{\beta\kappa} d^{\mu\sigma} K^r_\epsilon \right] p_\kappa p_\sigma \]

\[ I = 2 (\tilde{J}_\mu)_n (\tilde{J}_\beta)_q \tilde{f}^q (x) \int d^3 z' d^3 z (\beta, x)(\kappa, z) d(\mu, z)(\sigma, z') p(\kappa, z) p(\sigma, z') \]

\[ II = 2 \epsilon_{\beta\mu} (\tilde{J}_\epsilon)_n (\tilde{J}_\kappa)_q \tilde{f}^q (x) \int d^3 z' d^3 z (\beta, z)(\epsilon, x)(\sigma, z') p(\kappa, z) p(\sigma, z') \]

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