Anisotropic $H_{\text{div}}$-norm error estimates for rectangular $H_{\text{div}}$-elements

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Abstract

For the discretisation of $H_{\text{div}}$-functions on rectangular meshes there are at least three families of finite elements, namely Raviart-Thomas-, Brezzi-Douglas-Marini- and Arnold-Boffi-Falk-elements. In order to prove convergence of a numerical method using them, sharp interpolation error estimates are important. We provide them here in an anisotropic setting for the $H_{\text{div}}$-norm.

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1 Introduction

The discretisation of $H_{\text{div}}$ using piecewise polynomials is known for a long time, see e.g. [7] for one of the first papers, and they are used to discretise a variety of problems. For the analysis of these numerical methods we need estimates of the interpolation error. Especially on anisotropic meshes a finely tuned estimate, that incorporates the anisotropy, is important. In [1] anisotropic $L^p$-interpolation error estimates for Raviart-Thomas elements and the interpolation operator $\mathcal{I}$ on simplicial meshes

$$\|u - \mathcal{I}u\|_{L^p(T)} \lesssim \sum_{|\alpha|=k+1} h^\alpha \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} u\|_{L^p(K)} + h_T^{k+1} \|D^{k+1} \text{div } u\|_{L^p(T)}$$

(1)

is presented, where $\alpha$ is a multiindex, $h$ are the lengths of an anisotropic simplex $T$, $h_T$ its diameter and $D^k$ denotes the sum of the absolute values of all the derivatives of order $k$. A similar result is given in [3] for the Brezzi-Douglas-Marini element. They are not completely anisotropic due to the final term which for simplices and $\text{div } u \neq 0$ cannot be neglected.

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On rectangles and more general d-dimensional parallelotopes the situation is different. Here \[8\] provides several anisotropic interpolation error estimates for the \(L^p\)-norm error based on Poincaré-inequalities. One of optimal order with minimal assumptions on the regularity of \(u\) is

\[
\|u - \mathcal{I}u\|_{L^p(K)} \lesssim \sum_{|\alpha|=k+1} h^\alpha \|\partial_\alpha^{\alpha_1} \partial_\alpha^{\alpha_2} u\|_{L^p(K)}.
\]

(2)

Compared with estimate (1) this estimate is completely anisotropic.

In all these publications no anisotropic estimate for \(\|\text{div}(u - \mathcal{I}u)\|_{L^2(\Omega)}\) is given. We will present here such an estimate for rectangular elements in 2d. The generalisation to 3d and beyond is straightforward.

Notation: We denote vector valued functions with a bold font. \(L^p(D)\) with the norm \(\|\cdot\|_{L^p(D)}\) is the classical Lebesque space of function integrable to the power \(p\) over a domain \(D \subset \mathbb{R}^2\) and \(W^{\ell,p}(D)\) the corresponding Sobolev space of (weak) derivatives up to order \(\ell\). Furthermore, we write \(A \lesssim B\) if there exists a generic constant \(C > 0\) such that \(A \leq C \cdot B\).

## 2 Interpolation error estimates

Let us denote by \(\hat{K} := [0,1]^2\) the reference square, by \(Q_{p,q}(\hat{K})\) the space of polynomials with degree \(p\) and \(q\) in the two dimensions over \(\hat{K}\) and \(Q_k(\hat{K}) := Q_{k,k}(\hat{K})\). Furthermore, let \(P_k(\hat{K})\) be the space of polynomials of total degree \(k\) over \(\hat{K}\).

The basic ingredient of an interpolation error estimate for \(\text{div} u\) is the commuting diagram property. Let \(V_h\) be a discrete space over \(\hat{K}\), \(\mathcal{I}\) an interpolator into \(V_h\) and \(\Pi\) the \(L^2\)-projection into \(\text{div} V_h\). Then the commuting diagram property is, see [5, Remark 2.5.2]

\[
\text{div} \mathcal{I}u = \Pi \text{div} u.
\]

(3)

We will use this property first in an abstract way and apply it to the three families of finite elements afterwards.

Having [3] we obtain for \(w := \text{div} u\)

\[
\|\text{div}(u - \mathcal{I}u)\|_{L^p(K)} = \|\text{div} u - \Pi \text{div} u\|_{L^p(K)} = \|w - \hat{\Pi}w\|_{L^p(K)},
\]

and the estimation becomes that of the \(L^2\)-projection into \(\text{div} V_h\). A very useful tool in proving anisotropic interpolation error estimates is the technique presented in [2]. For it to apply we only need a set of functionals with some properties.

**Lemma 2.1.** There exist \(d := \dim(\text{div} V_h)\) functionals \(F_i\) and an integer \(\ell \in \mathbb{N}_0\), such that for all \(p \in [1,\infty)\) and \(1 \leq i \leq d\)

\[
F_i \in (W^{\ell,p}(\hat{K}))',
\]

(4a)

\[
F_i(\hat{\Pi}w - w) = 0 \text{ for all } w \in C(\hat{K}),
\]

(4b)

\[
w \in \text{div} V_h \text{ and } F_i(w) = 0 \text{ for all } i \in \{1,\ldots,d\} \text{ implies } w = 0.
\]

(4c)
Theorem 2.2. Everything needed to apply the general result (6).

The commuting diagram property (3) can be shown with integrations by parts and using \(P\) where \(\frac{1}{p} + \frac{1}{q} = 1\). Thus we have (4).

\[\|F_i(w)\| \leq \|q_i\|_{L^p(\hat{K})} \|w\|_{L^p(\hat{K})} \lesssim \|w\|_{W^{\ell,p}(\hat{K})},\]

where \(\frac{1}{p} + \frac{1}{q} = 1\). Thus we have (4). The consistency (4) follows directly by definition of the functionals and the \(L^2\)-projection. Finally, for \(w \in \text{div} \ V_h \) we obtain

\[F_i(w) = 0, \quad i \in \{1, \ldots, \hat{d}\} \quad \Leftrightarrow \quad \hat{\Pi}w = 0 \quad \Leftrightarrow \quad w = 0,\]

which is (4).

Following the technique of [2], shown therein for Lagrange and Scott-Zhang interpolation, and using Lemma 2.3, we obtain the anisotropic interpolation error estimate for \(\text{div} \ u\). In the case of \(\mathcal{P}_\ell(\hat{K}) \subset \text{div} \ V_h \) and assuming \(\text{div} \ u \in W^{\ell+1,p}(\hat{K})\) it can be written as

\[\|\text{div}(u - \hat{\mathcal{I}}u)\|_{L^p(\hat{K})} = \|\text{div} u - \hat{\Pi} \text{div} u\|_{L^p(\hat{K})} \lesssim \sum_{|\alpha| = \ell+1} \|\partial_x^\alpha \partial_y^\beta \text{div} u\|_{L^p(\hat{K})},\]

where \(\alpha\) is a multiindex, and for \(\mathcal{Q}_\ell(\hat{K}) \subset \text{div} \ V_h\) we have the sharper estimate

\[\|\text{div}(u - \hat{\mathcal{I}}u)\|_{L^p(\hat{K})} \lesssim \|\partial_x^\ell \text{div} u\|_{L^p(\hat{K})} + \|\partial_y^\ell \text{div} u\|_{L^p(\hat{K})}.\]

2.1 Raviart-Thomas elements

The Raviart-Thomas space over \(\hat{K}\) is given by, see [3],

\[RT_k(\hat{K}) := \mathcal{Q}_{k+1,k}(\hat{K}) \times \mathcal{Q}_{k,k+1}(\hat{K}).\]

Note, that it holds

\[(\mathcal{Q}_k(\hat{K}))^2 \subset RT_k(\hat{K}) \subset (\mathcal{Q}_{k+1}(\hat{K}))^2\]

and \(\text{div} \ RT_k(\hat{K}) = \mathcal{Q}_k(\hat{K})\).

The interpolation operator \(\hat{\mathcal{I}}_{RT} : (C(\hat{K}))^2 \rightarrow RT_k(\hat{K})\) is given for any \(v \in (C(\hat{K}))^2\) by

\[\int_{\hat{F}} (\hat{\mathcal{I}}_{RT} \mathbf{v} - \mathbf{v}) \cdot \mathbf{n} = 0, \quad \forall \mathbf{n} \in \mathcal{P}_k(\hat{F}), \forall \hat{F} \subset \partial \hat{K},\]

\[\int_{\hat{K}} (\hat{\mathcal{I}}_{RT} \mathbf{v} - \mathbf{v}) \cdot \mathbf{q} = 0, \quad \forall \mathbf{q} \in \mathcal{Q}_{k-1,k}(\hat{K}) \times \mathcal{Q}_{k,k-1}(\hat{K}),\]

where \(\mathcal{P}_k(\hat{F})\) is the space of polynomials of total degree \(k\) on a face \(\hat{F}\) of \(\hat{K}\).

The commuting diagram property [3] can be shown with integrations by parts and using the properties of the interpolation operator, see also [3] Proposition 2.5.2. Thus we have everything needed to apply the general result [3].

Theorem 2.2. For any \(1 \leq \ell \leq k + 1\) and \(\mathbf{u} \in (L^1(\hat{K}))^2\), such that \(\text{div} \mathbf{u} \in W^{\ell,p}(\hat{K})\) it holds

\[\|\text{div}(\mathbf{u} - \hat{\mathcal{I}}_{RT} \mathbf{u})\|_{L^p(\hat{K})} \lesssim \|\partial_x^\ell \text{div} \mathbf{u}\|_{L^p(\hat{K})} + \|\partial_y^\ell \text{div} \mathbf{u}\|_{L^p(\hat{K})}.\]
2.2 Brezzi-Douglas-Marini elements

The Brezzi-Douglas-Marini space over $\hat{K}$ is given by, see [6],

$$BDM_k(\hat{K}) := (P_k(\hat{K}))^2 \oplus \text{span}\{\text{curl} x^{k+1}y, \text{curl} xy^{k+1}\}$$

and it holds

$$(P_k(\hat{K}))^2 \subset BDM_k(\hat{K}) \quad \text{and} \quad \text{div} \ BDM_k(\hat{K}) = P_{k-1}(\hat{K}).$$

The interpolation operator $\hat{I}_{BDM} : (C(\hat{K}))^2 \to BDM_k(\hat{K})$ is given for any $v \in (C(\hat{K}))^2$ by

$$\int_{\hat{F}} (\hat{I}_{BDM}v - v) \cdot \mathbf{n} \cdot q = 0, \quad \forall q \in P_k(\hat{F}), \forall \hat{F} \subset \partial \hat{K}, \quad \text{(8a)}$$

and again the commuting diagram property (5) can be shown with integrations by parts, see also [5, Proposition 2.5.2]. We therefore obtain with (5) the following theorem.

**Theorem 2.3.** For any $1 \leq \ell \leq k$, $p \in \lbrack 1, \infty \rbrack$ and $u \in (L^1(\hat{K}))^2$, such that $\text{div} u \in W^{\ell,p}(\hat{K})$ it holds

$$\|\text{div}(u - \hat{I}_{BDM}u)\|_{L^p(\hat{K})} \lesssim \sum_{|\alpha| = \ell} \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \text{div} u\|_{L^p(\hat{K})},$$

where $\alpha$ is a multiindex of degree $\ell$.

2.3 Arnold-Boffi-Falk elements

The Arnold-Boffi-Falk space over $\hat{K}$ is given by, see [4],

$$ABF_k(\hat{K}) := Q_{k+2,k}(\hat{K}) \times Q_{k,k+2}(\hat{K})$$

and it holds

$$(Q_k(\hat{K}))^2 \subset ABF_k(\hat{K}) \quad \text{and} \quad \text{div} ABF_k(\hat{K}) = Q_{k+1}(\hat{K}) \setminus \text{span}\{x^{k+1}y^{k+1}\}.$$

The interpolation operator $\hat{I}_{ABF} : (C(\hat{K}))^2 \to ABF_k(\hat{K})$ is given for any $v \in (C(\hat{K}))^2$ by

$$\int_{\hat{F}} (\hat{I}_{ABF}v - v) \cdot \mathbf{n} \cdot q = 0, \quad \forall q \in P_k(\hat{F}), \forall \hat{F} \subset \partial \hat{K}, \quad \text{(9a)}$$

and again the commuting diagram property (5) can be shown with integrations by parts, see also [5, Proposition 2.5.2]. We therefore obtain with (5) the following theorem.

**Theorem 2.3.** For any $1 \leq \ell \leq k$, $p \in \lbrack 1, \infty \rbrack$ and $u \in (L^1(\hat{K}))^2$, such that $\text{div} u \in W^{\ell,p}(\hat{K})$ it holds

$$\|\text{div}(u - \hat{I}_{ABF}u)\|_{L^p(\hat{K})} \lesssim \sum_{|\alpha| = \ell} \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \text{div} u\|_{L^p(\hat{K})},$$

where $\alpha$ is a multiindex of degree $\ell$. 

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and again the commuting diagram property (3) can be shown with integrations by parts and a direct application of (9c) and (9d). Note that due to

\[ P_{k+1}(\hat{K}) \subset \text{div} \, ABF_k(\hat{K}) \quad \text{and} \quad Q_k(\hat{K}) \subset \text{div} \, ABF_k(\hat{K}) \]

we can use both (3) and (9) to estimate the interpolation error.

**Theorem 2.4.** For any \( 1 \leq \ell \leq k+2, p \in [1, \infty) \) and \( u \in (L^1(\hat{K}))^2 \), such that \( \text{div} \, u \in W^{\ell,p}(\hat{K}) \) it holds

\[
\| \text{div}(u - \hat{I}_{ABF}u) \|_{L^p(\hat{K})} \lesssim \sum_{|\alpha| = \ell} \| \partial^{\alpha_1}_{x} \partial^{\alpha_2}_{y} \text{div} \, u \|_{L^p(\hat{K})},
\]

and for \( 1 \leq s \leq k+1 \)

\[
\| \text{div}(u - \hat{I}_{ABF}u) \|_{L^p(\hat{K})} \lesssim \| \partial^{s}_{x} \text{div} \, u \|_{L^p(\hat{K})} + \| \partial^{s}_{y} \text{div} \, u \|_{L^p(\hat{K})}.
\]

### 3 Conclusions

Let us transform the estimates shown on the reference element back to an axi-parallel rectangle \( K \) of dimensions \( h_x \) and \( h_y \). Then, always using the highest possible derivatives, we obtain for \( p \in [1, \infty) \)

\[
\| \text{div}(u - I_{RT}u) \|_{L^p(K)} \lesssim h_x^{k+1} \| \partial^{k+1}_{x} \text{div} \, u \|_{L^p(K)} + h_y^{k+1} \| \partial^{k+1}_{y} \text{div} \, u \|_{L^p(K)},
\]

\[
\| \text{div}(u - I_{BDM}u) \|_{L^p(K)} \lesssim \sum_{|\alpha| = k} h_x^{\alpha_1} h_y^{\alpha_2} \| \partial^{\alpha_1}_{x} \partial^{\alpha_2}_{y} \text{div} \, u \|_{L^p(K)},
\]

\[
\| \text{div}(u - I_{ABF}u) \|_{L^p(K)} \lesssim \sum_{|\alpha| = k+2} h_x^{\alpha_1} h_y^{\alpha_2} \| \partial^{\alpha_1}_{x} \partial^{\alpha_2}_{y} \text{div} \, u \|_{L^p(K)},
\]

and for \( 1 \leq s \leq k+1 \)

\[
\| \text{div}(u - I_{ABF}u) \|_{L^p(K)} \lesssim h_x^{k+1} \| \partial^{k+1}_{x} \text{div} \, u \|_{L^p(K)} + h_y^{k+1} \| \partial^{s}_{y} \text{div} \, u \|_{L^p(K)}.
\]

Thus for all elements we obtain anisotropic interpolation error estimates in \( H_{\text{div}} \). Compared to Raviart-Thomas elements we see a reduction of one order for Brezzi-Douglas-Marini elements and an increase of one order for Arnold-Boffi-Falk elements, if all derivatives of \( \text{div} \, u \) were used. For the later element we have also the same estimate as for Raviart-Thomas elements.

For an affine transformation from \( \hat{K} \) to a quadrilateral \( K \) similar results follow. It is an open question, whether for non-affine transformations anisotropic interpolation error estimates can be shown.

Using as functionals \( F_i \) those conditions from the definition of the interpolation operator \( I \), we can also show anisotropic interpolation error estimates in \( L^p \), following [2] Lemma 2.13 and improve upon [2]

\[
\| u - I_{RT}u \|_{L^p(K)} + \| u - I_{ABF}u \|_{L^p(K)} \lesssim h_x^{k+1} \| \partial^{k+1}_{x} u \|_{L^p(K)} + h_y^{k+1} \| \partial^{k+1}_{y} u \|_{L^p(K)},
\]

\[
\| u - I_{BDM}u \|_{L^p(K)} \lesssim \sum_{|\alpha| = k+1} h_x^{\alpha_1} h_y^{\alpha_2} \| \partial^{\alpha_1}_{x} \partial^{\alpha_2}_{y} u \|_{L^p(K)}.
\]
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