ON THE CENTEREDNESS OF SATURATED IDEALS

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Abstract. We show that Kunen’s saturated ideal over $\mathcal{N}_1$ is not centered. We also evaluate the extent of saturation of Laver’s saturated ideal in terms of $(\kappa, \lambda, < \nu)$-saturation.

1. INTRODUCTION

In [5], Kunen established

**Theorem 1.1** (Kunen [5]). Suppose that $j : V \rightarrow M$ is a huge embedding with critical point $\kappa$. Then there is a poset $P$ such that $P \ast \dot{S}(\kappa, j(\kappa))$ forces that $\mathcal{N}_1$ carries a saturated ideal.

This theorem has been improved in some ways. One is due to Foreman and Laver [3]. They established

**Theorem 1.2** (Foreman–Laver [3]). Suppose that $j : V \rightarrow M$ is a huge embedding with critical point $\kappa$. Then there is a poset $P$ such that $P \ast \dot{R}(\kappa, j(\kappa))$ forces that $\mathcal{N}_1$ carries a centered ideal.

Centeredness is one of the strengthenings of saturation. See Section 2 for the definition of centered ideal. Foreman and Laver introduced the poset $\dot{R}(\kappa, \lambda)$ to obtain the centeredness, while Kunen used the Silver collapse $\dot{S}(\kappa, \lambda)$. In their paper, it is claimed without proof that the ideal in Theorem 1.1 is not centered. We aim to give a proof of this claim in greater generality. Indeed, we will show that

**Theorem 1.3.** Suppose that $j : V \rightarrow M$ is a huge embedding with critical point $\kappa$ and $f : \kappa \rightarrow \text{Reg} \cap \kappa$ satisfies $j(f)(\kappa) \geq \kappa$. For regular cardinals $\mu < \kappa \leq \lambda = j(f)(\kappa) < j(\kappa)$, there is a $P$ such that $P \ast \dot{S}(\lambda, j(\kappa))$ forces $\mu^+ = \kappa$ and $\lambda^+ = j(\kappa)$ and $\mathcal{P}_\kappa(\lambda)$ carries a saturated ideal that is not centered.

We can regard an ideal over $\mathcal{P}_\kappa(\lambda)$ as an ideal over $\kappa$. If we put $f = \text{id}$, $\lambda = \kappa$ and $\mu = \mathcal{N}_0$, then $P$ and the ideal in Theorem 1.3 are the same as those in Theorem 1.1.

We also study Laver’s saturated ideal. Laver introduced Laver collapse $\dot{L}(\kappa, \lambda)$ to get a model in which $\mathcal{N}_1$ carries a strongly saturated ideal. He established

**Theorem 1.4** (Laver [6]). Suppose that $j : V \rightarrow M$ is a huge embedding with critical point $\kappa$. Then there is a poset $P$ such that $P \ast \dot{L}(\kappa, j(\kappa))$ forces that $\mathcal{N}_1$ carries a saturated ideal.

We don’t know whether Laver’s saturated ideal is centered or not. But we study the extent of saturation of this ideal in terms of $(\kappa, \lambda, < \nu)$-saturation.

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Theorem 1.5. Suppose that $j$ is a huge embedding with critical point $\kappa$ and $f: \kappa \to \text{Reg} \cap \kappa$ satisfies $j(f)(\kappa) \geq \kappa$. For regular cardinals $\mu < \kappa \leq \lambda = j(f)(\kappa) < j(\kappa)$, there is a $P$ such that $P \ast L(\lambda, j(\kappa))$ forces that $\mu^+ = \kappa$, $\lambda^+ = j(\kappa)$ and $\mathcal{P}_\kappa(\lambda)$ carries a strongly saturated ideal $I$, that is, $I$ is $(j(\kappa), j(\kappa), < \lambda)$-saturated and $I$ is $(j(\kappa), \lambda, \lambda)$-saturated but not $(j(\kappa), j(\kappa), \lambda)$-saturated.

The structure of this paper is as follows: In Section 2, we recall the basic facts of forcing. We also recall Silver collapses. Section 3 is devoted to the proof of Theorem 1.3. In Section 4, we recall Laver collapse and we give the proof of Theorem 1.5.

2. Preliminaries

In this section, we recall some definitions. We use [4] as a reference for set theory in general.

Our notation is standard. We use $\kappa, \lambda, \mu$ to denote a regular cardinal unless otherwise stated. We also use $\nu$ to denote a cardinal, possibly finite, unless otherwise stated. We write $|\kappa, \lambda|$ for the set of all ordinals between $\kappa$ and $\lambda$. By Reg, we mean the set of all regular cardinals. For $\kappa < \lambda$, $E^\kappa_\lambda$ and $E^\lambda_\kappa$ denote the set of all ordinals below $\lambda$ of cofinality $\geq \kappa$ and $< \kappa$, respectively.

Throughout this paper, we identify a poset $P$ with its separative quotient. Thus, $p \leq q \iff \forall r \leq p(r||q) \iff p \Vdash q \in \dot{G}$, where $\dot{G}$ is the canonical name of $(V, P)$-generic filter.

We say that $P$ is $\kappa$-centered if there is a sequence of centered subsets $\langle C_\alpha \mid \alpha < \kappa \rangle$ with $P = \bigcup_{\alpha < \kappa} C_\alpha$. A centered subset is $C \subseteq P$ such that every $X \in [C]^\omega$ has a lower bound in $P$. We call such sequence a centering family of $P$. It is easy to see that the $\kappa$-centeredness implies the $\kappa^+$-c.c.

We say that $P$ is well-met if $\prod X \in P$ for all $X \subseteq P$ with $X$ has a lower bound. If $P$ is well-met, the $\kappa$-centeredness of $P$ is equivalent to the existence of $\kappa$-many filters that cover $P$. Note that every poset that we will deal with in this paper is well-met.

For a filter $F \subseteq Q$, by $Q/F$, we mean the subset $\{ q \in Q \mid \forall p \in F(p||q) \}$ ordered by $\leq_{Q}$. For a given complete embedding $\tau : P \to Q$, $P \ast (Q/\tau^{\ast}\dot{G})$ is forcing equivalent with $Q$. We also write $Q/\dot{G}$ for $Q/\tau^{\ast}\dot{G}$ if $\tau$ is obvious from context. If the inclusion mapping $P \to Q$ is complete, we say that $P$ is a complete suborder of $Q$, denoted by $P \leq Q$.

In this paper, by ideal, we mean normal and fine ideal. For an ideal $I$ over $\mathcal{P}_\kappa$, $\mathcal{P}(\mathcal{P}_\kappa, \lambda)/I$ is $\mathcal{P}(\kappa^+) \setminus I$ ordered by $A \leq B \iff A \setminus B \in I$. We say that $I$ is saturated and centered if $\mathcal{P}(\kappa^+)/I$ has the $\kappa^{++}$-c.c. and $\mathcal{P}(\kappa^+)/I$ is $\kappa^+$-centered, respectively.

The ideal $I$ over $\kappa^+$ is $(\alpha, \beta, < \gamma)$-saturated if $\mathcal{P}(\kappa^+)/I$ has the $(\alpha, \beta, < \gamma)$-c.c. in the sense of Laver [6]. Whenever $I$ is $(\lambda^+, \lambda^+, \lambda)$-saturated, we say that $I$ is strongly saturated.

For a stationary subset $S \subseteq \lambda$, $P$ is $S$-layered if there is a sequence $\langle P_\alpha \mid \alpha < \lambda \rangle$ of complete suborders of $P$ such that $|P_\alpha| < \lambda$ and there is a club $C \subseteq \lambda$ such that $P_\alpha = \bigcup_{\beta < \alpha} P_\beta$ for all $\alpha \in S \cap C$. We say that $I$ is layered and if $\mathcal{P}(\mathcal{P}_\kappa, \lambda)/I$ is $S$-layered for some stationary $S \subseteq E^\lambda_\kappa$.

For cardinals $\kappa < \lambda$ ($\lambda$ is not necessary regular), Silver collapse $S(\kappa, \lambda)$ is the set of all $p$ with the following properties:

- $p \in \prod_{\gamma \in [\kappa^+, \lambda) \cap \text{Reg}}^{< \kappa} p \gamma$.

- There is a $\xi < \kappa$ with $\forall \gamma \in \text{dom}(p) \text{dom}(p(\gamma)) \subseteq \xi$.
Lemma 2.1.  
1. \( S(\kappa, \lambda) \) is \( \kappa \)-closed.
2. If \( \lambda \) is inaccessible, then \( S(\kappa, \lambda) \) has the \( \lambda \)-c.c.
3. If \( \mu < \lambda \) then \( S(\kappa, \mu) \preceq S(\kappa, \lambda) \).

The following lemma will be used in a proof of Theorem 1.3.

Lemma 2.2.  
1. If \( \operatorname{cf}(\delta) < \kappa \) then \( \operatorname{Coll}(\kappa, \delta) \) has an anti-chain of size \( \delta^+ \).
2. If \( \operatorname{cf}(\delta) > \kappa \) and \( \delta^\kappa = \delta \) then \( S(\kappa, \delta) \) forces \( (\delta^+)^V \geq \kappa^+ \).

Proof. (2) follows by a standard cardinal arithmetic. Indeed, The assumption shows \( |S(\kappa, \delta)| = \delta \). Therefore \( S(\kappa, \delta) \) has the \( \delta^+ \)-c.c. We only check about (1). By \( \operatorname{cf}(\delta) < \kappa \), we can fix an increasing sequence of regular cardinals \( \{ \delta_i \mid i < \operatorname{cf}(\delta) \} \) which converges to \( \delta \). For \( f \in \prod_{i < \operatorname{cf}(\delta)} \delta_i \), define \( p_f \in S(\kappa, \delta) \) by \( \operatorname{dom}(p_f) = \{ \delta_i \mid i < \operatorname{cf}(\delta) \} \) and \( p_f(\delta_i) = \{(0, f(i))\} \). It is easy to see that \( f \neq g \) implies \( p_f \perp p_g \). Therefore \( \{p_f \mid f \in \prod_{i < \operatorname{cf}(\delta)} \delta_i \} \) is an anti-chain of size \( (\delta^{\operatorname{cf}(\delta)})^+ \geq \delta^+ \), as desired.

Remark 2.3. A similar proof shows the following analogue of Lemma 2.2.
1. If \( \operatorname{cf}(\delta) < \kappa \) then \( \operatorname{Coll}(\kappa, \delta) \) has an anti-chain of size \( \delta^+ \).
2. If \( \operatorname{cf}(\delta) \geq \kappa \) and \( \delta^{\kappa} = \delta \) then \( \operatorname{Coll}(\kappa, \delta) \) forces \( (\delta^+)^V \geq \kappa^+ \).

3. PROOF OF THEOREM 1.3

Let \( j : V \to M \) be a huge embedding with critical point \( \kappa \). Fix \( \mu < \kappa \). We also fix \( f : \kappa \to \kappa \cap \text{Reg} \) with \( \kappa \leq j(f)(\kappa) = \lambda \). We may assume that \( f(\alpha) \geq \alpha \) for all \( \alpha \).

Let \( \langle P_\alpha \mid \alpha \leq \kappa \rangle \) be the \( < \mu \)-support iteration such that
\[
\begin{align*}
P_0 &= S(\mu, \kappa), \\
P_{\alpha+1} &= \begin{cases} 
P_\alpha \ast S^P_{\alpha} \cap V_\alpha(f(\alpha), \kappa) & \text{if } \alpha \text{ is good} \\
P_\alpha & \text{otherwise}.
\end{cases}
\end{align*}
\]

Here, we say that \( \alpha \) is good if \( P_\alpha \cap V_\alpha \preceq P_\alpha \) has the \( \alpha \)-c.c., \( \alpha \) is inaccessible, and \( \alpha \geq \mu \). The set \( P_\alpha \ast S^P_{\alpha} \cap V_\alpha(\alpha, \kappa) \) is the set of all \( (p, q) \) such that \( p \in P_\alpha \) and \( q \) is a \( P_\alpha \cap V_\alpha \)-name for an element of \( S^P_{\alpha} \cap V_\alpha(\alpha, \kappa) \).

Define \( P = P_\kappa \). For every \( p \in P \) and good \( \alpha \), we may assume that \( p(\alpha) \in P_\alpha \cap V_\alpha \)-name. This \( P \) is called an universal collapse. \( P \) has the following properties:

Lemma 3.1.  
1. \( P \) is \( \mu \)-directed closed and has the \( (\kappa, \kappa, < \mu) \)-c.c.
2. \( P \subseteq V_\kappa \) and \( P \models \mu^+ = \kappa \).
3. \( \kappa \) is good for \( j(P) \). In particular, \( j(P)_\kappa \cap V_\kappa = P \preceq j(P)_\kappa \).
4. There is a complete embedding \( \tau : P \ast S(\lambda, j(\kappa)) \to j(P)_{\kappa+1} \preceq j(P) \) such that \( \tau(p, \emptyset) = p \) for all \( p \in P \).

Note that \( j(P) \) has the \( j(\kappa) \)-c.c. by the hugeness of \( j \). By Lemma 3.1(2), in the extension by \( P \ast S(\lambda, j(\kappa)) \), \( \mu^+ = \kappa \), \( \lambda^+ = j(\kappa) \), and every cardinal between \( \kappa \) and \( \lambda \) are preserved.

Kunen proved Theorem 3.2 in the case of \( \kappa = \lambda \) and \( \mu = \omega \). Moreover,

Theorem 3.2. \( P \ast S(\lambda, j(\kappa)) \) forces that \( \mathcal{P}_\kappa \lambda \) carries a saturated ideal \( \mathcal{I} \).

Proof. The same proof as in [2 Section 7.7] works.
The ideal which we call “Kunen’s saturated ideal” is this $\hat{I}$. Studying the saturation of $\hat{I}$ will be reduced to that of some quotient forcing. Indeed, Theorem 1.3 follows from Lemmas 3.3 and 3.4. We only give a proof of Lemma 3.4. In [1], Foreman, Magidor and Shelah proved Lemma 3.3 in the case of $\kappa = \lambda$.

**Lemma 3.3.** $P * \dot{S}(\lambda, j(\kappa))$ forces $\mathcal{P}(\mathcal{P}_\kappa \lambda)/\hat{I} \simeq j(P)/\hat{G} * \hat{H}$. Here, $\hat{G} * \hat{H}$ is the canonical name for generic filter.

**Proof.** The same proof as in [1] Claim 7 works. \[\square\]

**Lemma 3.4.** $P * \dot{S}(\lambda, j(\kappa))$ forces $j(P)/\hat{G} * \hat{H}$ is not $\lambda$-centered.

**Proof.** Note that $\{ \alpha < j(\kappa) \mid \alpha \text{ is good} \}$ is unbounded in $j(\kappa)$. We fix a good $\alpha > \kappa$. It is enough to prove that $j(P)_{\alpha+1}/\hat{G} * \hat{H}$ is not $\lambda$-centered in the extension.

We show by contradiction. Suppose the existence of a centering family $\langle \dot{C}_\xi \mid \xi < \lambda \rangle$ of $j(P)_{\alpha+1}/\hat{G} * \hat{H}$ is forced by some condition. We may assume that each $\dot{C}_\xi$ is forced to be a filter. To simplify notation, we assume $P * \dot{S}(\lambda, j(\kappa))$ forces the existence of such a centering family. By the $\kappa$-c.c. of $P$, for every $\langle q, \dot{q} \rangle \in P * \dot{S}(\lambda, j(\kappa)), P \forces \dot{q} \in \dot{S}(\lambda, \beta)$ for some $\beta < j(\kappa)$. For each $q \in j(P)_{\alpha+1}$, let $\rho(q)$ be defined by the following way:

For $\xi < \lambda$, let $A^r_\xi \subseteq \dot{P}(\lambda, j(\kappa))$ be a maximal anti-chain such that, for every $r \in A^r_\xi$, $r$ decides $q \in \dot{C}_\xi$. $\rho(q)$ is the least ordinal $\beta < j(\kappa)$ such that $P \forces \dot{q} \in \dot{S}(\lambda, \beta)$ for every $\langle p, \dot{q} \rangle \in \bigcup_c A^r_c$.

We put $Q = j(P)_{\alpha} \cap V_\alpha$. Let $C \subseteq j(\kappa)$ be a club generated by $\beta \mapsto \sup \{ \rho(q) \mid q \in Q * S^Q(\alpha, \beta) \}$. Since $j(\kappa)$ is inaccessible, we can find a strong limit cardinal $\delta \in C \cap E_{\alpha+1}^{j(\kappa)} \cap E_{\alpha}^{j(\kappa)} \setminus (\alpha + 1)$.

By the $\kappa$-c.c. of $P$ and Lemma 2.2 (2), $P * \dot{S}(\lambda, \delta) \forces (\delta^+)^V \geq \lambda^+$. We will discuss in the extension by $P * \dot{S}(\lambda, \delta)$. Let $\dot{G} * \dot{H}_\delta$ be the canonical $P * \dot{S}(\lambda, \delta)$-name for a generic filter.

By Lemma 2.2 (1), $P * \dot{S}(\lambda, \delta) \forces S^V(\alpha, \delta)$ has an anti-chain of size $(\delta^+)^V \geq \lambda^+$. By Lemma 3.1 (4), this $S^V(\alpha, \delta)$-anti-chain defines an anti-chain in $Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$ in the extension. Thus, $Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$ is forced not to have the $\lambda^+\text{-c.c.}$, and thus not to be $\lambda$-centered.

On the other hand, a name of a centering family of $j(P)_{\alpha+1}/\hat{G} * \hat{H}$ defines a centering family of $Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$ in the extension as follows.

**Claim 3.5.** $P * \dot{S}(\lambda, \delta)$ forces that $Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$ is $\lambda$-centered.

**Proof of Claim.** For every $q \in \bigcup_\beta \dot{Q} \subseteq Q * S^Q(\alpha, \beta)$, by $\rho(q) < \delta$, the statement $q \in \dot{C}_\xi$ has been decided by $P * \dot{S}(\lambda, \delta)$ for all $\xi < \lambda$. Let $G * H$ be an arbitrary $(V, P * \dot{S}(\lambda, j(\kappa)))$-generic filter. Note that $G * H_\delta = G * H \cap (P * \dot{S}(\lambda, \delta))$ is $(V, P * \dot{S}(\lambda, \delta))$-generic. Let $D_\xi$ be defined by

$$\langle p, \dot{q} \rangle \in D_\xi \text{ if and only if } \langle p, \dot{q} \mid \beta \rangle \in \dot{C}_\xi \text{ forced by } G * H_\delta \text{ for every } \beta < \delta.$$ 

It is easy to see that $D_\xi$ is a filter over $Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$. We claim that $\{D_\xi \mid \xi < \lambda \}$ covers $Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$. For each $\langle p, \dot{q} \rangle \in Q * S^Q(\alpha, \delta)/\hat{G} * \hat{H}_\delta$, in $V[G][H_\delta]$, there is a $\xi$ such that $\langle p, \dot{q} \rangle \in \dot{C}_\xi^{G * H}$. Then $\langle p, \dot{q} \mid \beta \rangle \in \dot{C}_\xi^{G * H}$ for every $\beta < \delta$, since $C^{G * H}$ is a filter. In particular, $\langle p, \dot{q} \mid \beta \rangle \in \dot{C}_\xi$ is forced by $G * H_\delta$ for every $\beta < \delta$. By the definition of $D_\xi$, $\langle p, \dot{q} \rangle \in D_\xi$ in $V[G][H_\delta]$, as desired. \[\square\]
This claim shows the contradiction. The proof is completed. □

Let us give a proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemmas 3.3 and 3.4, we have \[\mathcal{P}(P, \lambda)/\hat{I}\] is not \(\lambda\)-centered. Thus, \(\hat{I}\) is not centered ideal in the extension. □

Lastly, we list other saturation properties of \(\hat{I}\).

Theorem 3.6. \(P \ast \mathcal{S}(\lambda, j(\kappa))\) forces that

1. \(\hat{I}\) is \((j(\kappa), j(\kappa), < \mu)\)-saturated.
2. \(\hat{I}\) is not \((j(\kappa), \mu, \mu)\)-saturated. In particular, \(\hat{I}\) is not strongly saturated.
3. \(\hat{I}\) is layered.
4. \(\hat{I}\) is not centered.

Proof. For (1) and (2), we refer to [9]. (3) has been proven in [1]. (4) follows by Lemma 3.4. □

Remark 3.7. Kunen’s theorem can be improved by Magidor’s trick which appeared in [7]. This shows that an almost huge cardinal is enough to show Theorem 1.1. Then we can use Levy collapse instead of Silver collapse. Remark 3.5 enables us to show the same result with Theorem 1.3 for Levy collapses. On the other hand, to obtain layeredness, we need an almost huge embedding \(j: V \to M\) with \(j(\kappa)\) is Mahlo. If \(j(\kappa)\) is not Mahlo, the ideal \(\hat{I}\) is forced to be not layered. For details, we refer to [9].

4. Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5.

First, we recall the definition and basic properties of Laver collapse \(L(\kappa, \lambda)\). \(L(\kappa, \lambda)\) is the set of all \(p\) such that

- \(p \in \prod_{\gamma \in [\kappa^+, \lambda) \cap \text{Reg}} \mathcal{P}(\gamma)\).
- There is a \(\xi < \kappa\) with \(\forall \gamma \in \text{dom}(p) (\text{dom}(p(\gamma)) \subseteq \xi)\).
- \(\text{dom}(p) \subseteq \lambda\) is Easton subset.

\(L(\kappa, \lambda)\) is ordered by reverse inclusion. It is easy to see that \(L(\kappa, \lambda)\) is \(\kappa\)-closed.

Lemma 4.1. (1) \(L(\kappa, \lambda)\) is \(\kappa\)-closed.

(2) If \(\lambda\) is Mahlo, then \(L(\kappa, \lambda)\) has the \((\lambda, \lambda, < \mu)\)-c.c. for all \(\mu < \lambda\). Therefore, \(L(\kappa, \lambda) \models \kappa^+ = \lambda\).

Proof. (1) follows by the standard argument. (2) follows by the usual \(\Delta\)-system argument. □

For \(\kappa, \lambda, \mu, j, f\) in the assumption of Theorem 1.5, let us define \(P\). Let \(\langle P_\alpha \mid \alpha \leq \kappa \rangle\) be the Easton support iteration of such that

- \(P_0 = L(\mu, \kappa)\).
- \(P_{\alpha+1} = \begin{cases} P_\alpha \ast L(\alpha, f(\alpha), \kappa) & \alpha \text{ is good} \\ P_\alpha & \text{otherwise} \end{cases}\).

Goodness of \(\alpha\) is defined as in Section 3. Define \(P = P_\kappa\). Then

Lemma 4.2. (1) \(P\) is \(\mu\)-directed closed and has the \((\kappa, \kappa, < \nu)\)-c.c for all \(\nu < \kappa\).
(2) $P \subseteq V_\kappa$ and $P \forces \mu^+ = \kappa$.  
(3) $\kappa$ is good for $j(P)$. In particular, $j(P)_\kappa \cap V_\kappa = P \subseteq j(P)_\kappa$.  
(4) There is a complete embedding $\tau : P \ast \dot{L}(\lambda, j(\kappa)) \to j(P)_{\kappa+1} \leq j(P)$ such that $\tau(p, \emptyset) = p$ for all $p \in P$.

Laver proved the following in the case of $\kappa = \lambda$. But the same proof showed

**Theorem 4.3.** $P \ast \dot{L}(\lambda, j(\kappa))$ forces that $\mathcal{P}_\kappa \lambda$ carries a strongly saturated ideal $\dot{I}$.

We have an analogue of Lemma 4.3 for Laver’s ideal.

**Lemma 4.4.** $P \ast \dot{L}(\lambda, j(\kappa))$ forces $\mathcal{P}(\mathcal{P}_\kappa \lambda)/\dot{I} \simeq j(P)/\dot{G} \ast \dot{H}$. Here, $\dot{G} \ast \dot{H}$ is the canonical name for generic filter.

**Proof.** The same proof as in [1, Claim 7] works. □

**Lemma 4.5.** Suppose that $\tau : P \to Q$ is complete embedding between complete Boolean algebras and $Q$ has the $(\kappa, \mu, \mu)$-c.c. Then $P \forces Q/\dot{G}$ has the $(\kappa, \mu, \mu)$-c.c.

**Proof.** We may assume that $P$ and $Q$ are complete Boolean algebras. Let $p \forces \{q_\alpha \mid \alpha < \kappa\} \subseteq Q/\dot{G}$ be arbitrary. For each $\alpha < \kappa$, there are $p_\alpha \leq p$ and $q_\alpha \in Q$ such that $p_\alpha \forces q_\alpha = q_\alpha$. By the $(\kappa, \mu, \mu)$-c.c. of $Q$, there is a $Z \in [\kappa]^{\kappa}$ such that $\prod_{\alpha \in Z} \tau(p_\alpha) \cdot q_\alpha \neq 0$. It is easy to see that

$$\prod_{\alpha \in Z} \tau(p_\alpha) \cdot q_\alpha = \prod_{\alpha \in Z} \tau(p_\alpha) \cdot \prod_{\alpha \in Z} q_\alpha = \tau(\prod_{\alpha \in Z} p_\alpha) \cdot \prod_{\alpha \in Z} q_\alpha.$$  

Let $r$ be a reduct of $\tau(\prod_{\alpha \in Z} p_\alpha) \cdot \prod_{\alpha \in Z} q_\alpha$. Then $r \leq \prod_{\alpha \in Z} p_\alpha \leq p$ and this forces that $\prod_{\alpha \in Z} q_\alpha \in Q/\dot{G}$ is a lower bound of $\{q_\alpha \mid \alpha \in Z\}$, as desired. □

**Lemma 4.6.** $P \ast \dot{L}(\lambda, j(\kappa))$ forces that $j(P)/\dot{G} \ast \dot{H}$ does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c.

**Proof.** Let us show $P \ast \dot{L}(\lambda, j(\kappa)) \forces j(P)/\dot{G} \ast \dot{H}$ does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c. Let $q_\alpha \in j(P) \mid \alpha > \kappa+1$ be an arbitrary such that $\text{supp}(q_\alpha) = \{\alpha\}$ for all $\alpha$. Note that $\langle \emptyset, \emptyset \rangle \in P \ast \dot{L}(\lambda, j(\kappa))$. For every $p \in P \ast \dot{L}(\lambda, j(\kappa))$, $\tau(p) \in j(P)_{\kappa+1}$ by Lemma 4.2.4. Therefore $\tau(p)$ meets with $q_\alpha$. We have $\forces \{q_\alpha \mid \alpha > \kappa + 1\} \subseteq j(P)/\dot{G} \ast \dot{H}$.

We fix an arbitrary $A$ with $\forces A \in [j(\kappa)]^{j(\kappa)}$. Let us find an $\dot{x}$ such that $\forces \dot{x} \in [\dot{A}]^{\lambda}$ and $\{q_\alpha \mid \alpha \in \dot{x}\}$ has a lower bound in $j(P)/\dot{G} \ast \dot{H}$. By the $(j(\kappa), \lambda)$-c.c. of $P \ast j(\lambda, j(\kappa))$, there is a club $C \subseteq j(\kappa)$ such that $\forces C \subseteq \text{Lim}(\dot{A})$. Since $j(\kappa)$ is Mahlo, there is an inaccessible $\alpha \in C \setminus (\lambda + 1)$. Let $\dot{x}$ be a $P \ast \dot{L}(\lambda, j(\kappa))$-name for the set $A \cap \alpha$. Since $\forces \alpha \in \text{Lim}(\dot{A})$, $\sup \dot{x} = \alpha$. Since $\lambda \leq \alpha < j(\kappa)$, $|\dot{x}| = |\alpha| = \lambda$ is forced by $P \ast \dot{L}(\lambda, j(\kappa))$. We claim that $\forces \{q_\alpha \mid \alpha \in \dot{x}\}$ witnesses. Suppose otherwise, there are $p \in P \ast \dot{L}(\lambda, j(\kappa))$ and $r \in j(P)$ such that $p \forces r \in j(P)/\dot{G} \ast \dot{H} \leq q_\alpha$ for all $\alpha \in \dot{x}$. Note that $\text{supp}(r) \cap \alpha < \beta$ for some $\beta < \alpha$. By $q \forces \sup \dot{x} = \alpha$, there are $q \leq p$ and $\gamma \in [\beta^+, \alpha)$ such that $q \forces \gamma \in \dot{x}$. Therefore $q \forces r \leq q_\gamma$, and thus, $\{\gamma\} \subseteq \text{supp}(r) \cap [\beta^+, \alpha)$. This is a contradiction. □

Let us show

**Proof of Theorem 4.5.** Note that $j(P)$ has the $(j(\kappa), j(\kappa), \mu)$-c.c. for all $\mu < j(\kappa)$. Therefore $j(P)$ has the $(j(\kappa), \lambda, \lambda)$-c.c. By Lemma 4.5, $j(P)/\dot{G} \ast \dot{H}$ is forced to have the $(j(\kappa), \lambda, \lambda)$-c.c. By Lemma 4.4, $P \ast \dot{L}(\lambda, j(\kappa))$ forces that $\mathcal{P}(\mathcal{P}_\kappa \lambda)/\dot{I} \ast \dot{H}$ does not have the $(j(\kappa), j(\kappa), \lambda)$-c.c. By Lemma 4.3, $\dot{I}$ is forced to have the $(j(\kappa), \lambda, \lambda)$-c.c. but not the $(j(\kappa), j(\kappa), \lambda)$-c.c. □
Theorem 4.7. \( P \ast \dot{L}(\lambda, j(\kappa)) \) forces that

(1) \( \dot{I} \) is \((j(\kappa), j(\kappa), < \lambda)\)-saturated and \((j(\kappa), \lambda, \lambda)\)-saturated.
(2) \( \dot{I} \) is not \((j(\kappa), j(\kappa), \lambda)\)-saturated.
(3) \( \dot{I} \) is layered.

Proof. (1) and (2) follow from Lemma 4.6. (3) follows by the proof in [1].

Shioya improved Theorem 1.4 by using Easton collapse. Easton collapse \( E(\kappa, \lambda) \) is the Easton support product \( \prod_{\gamma \in [\kappa^+, \lambda)^{<\kappa} \cap \mathcal{S}_\kappa} E \). \( \mathcal{S}_\kappa \) is the class of all cardinal \( \gamma \) with \( \gamma^{<\gamma} = \gamma \). He showed

Theorem 4.8 (Shioya [8]). Suppose that \( j : V \to M \) is an almost-huge embedding with critical point \( \kappa \) and \( j(\kappa) \) is Mahlo. For regular cardinals \( \mu < \kappa \leq \lambda < j(\kappa) \), \( E(\mu, \kappa) \ast \dot{E}(\lambda, j(\kappa)) \) forces that \( \mu^+ = \kappa \), \( j(\kappa) = \lambda^+ \) and \( P_{\kappa, \lambda} \) carries a strongly saturated ideal.

Let \( \dot{I} \) be an \( E(\mu, \kappa) \ast \dot{E}(\lambda, j(\kappa)) \)-name for the ideal in Theorem 4.8. The similar proof shows that

Theorem 4.9. \( E(\mu, \kappa) \ast \dot{E}(\lambda, j(\kappa)) \) forces that

(1) \( \dot{I} \) is \((j(\kappa), j(\kappa), < \lambda)\)-saturated and \((j(\kappa), \lambda, \lambda)\)-saturated.
(2) \( \dot{I} \) is not \((j(\kappa), j(\kappa), \lambda)\)-saturated.
(3) \( \dot{I} \) is layered.

We conclude this paper with the following question.

Question 4.10. Does \( P \ast \dot{L}(\lambda, j(\kappa)) \) force that \( \dot{I} \) is centered? What about \( E(\mu, \kappa) \ast \dot{E}(\lambda, j(\kappa)) \)?

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