Type IIB $R^4 H^{4g-4}$ Conjectures

Nathan Berkovits
Instituto de Física Teórica, Univ. Estadual Paulista
Rua Pamplona 145, São Paulo, SP 01405-900, Brasil

Cumrun Vafa
Lyman Laboratory of Physics, Harvard University
Cambridge, MA 02138, USA

We propose $SL(2, Z)$ (and $SL(3, Z)$) invariant conjectures for all $R^4 H^{4g-4}$ couplings of Type IIB strings on $R^{10}$ (and $R^8 \times T^2$), generalizing conjectures of Green and Gutperle (and Kiritsis and Pioline) for the $R^4$ coupling. A strong check for our conjectures is that on $T^2$ at weak coupling, they reproduce the multiloop scattering amplitudes which had been previously computed using $N = 2$ strings in the $N = 4$ topological formalism. Applications to $(p, q)$ string production in a background $H$ field, generalizing Schwinger’s computation for pair production in constant $F$ field, are suggested.
1. Introduction

Non-trivial string duality conjectures often lead to perturbative predictions for certain amplitudes. Such amplitudes are usually very special and receive corrections only at specific genera. A well known case of this is the $R^4$ coupling in Type IIB theory which has been argued to only receive perturbative corrections at tree-level and one-loop. Green and Gutperle conjectured that the $R^4$ term appears in the effective action multiplied by the manifestly SL(2,Z)-invariant Eisenstein function $E_{3/2}(\tau)$, and their conjecture is supported by various types of evidence, in particular by the match with the genus 0 and genus 1 amplitudes which are explicitly known. The success of the $R^4$ conjecture naturally leads one to look for generalizations, but in the absence of explicit multiloop calculations, it is difficult to choose between different proposals. Also, it is not apriori clear what kinds of amplitudes one should concentrate on.

More than three year ago, we showed that $R^4H^{4g-4}$ (or $R^4F^{4g-4}$) terms can be computed at genus $g$ for the Type IIB (or Type IIA) superstring compactified to six dimensions on any hyper-Kahler manifold. Like the better known four-dimensional $R^2F^{2g-2}$ terms, these six-dimensional terms can be expressed in terms of ($N = 4$) topological string computations, which are in turn equivalent to the partition function of the $N = 2$ string on the corresponding four manifold. The topological reformulation of the amplitudes allows one to find methods to compute them explicitly, as was done in when the four manifold is $T^2 \times R^2$, i.e. when considering Type II strings compactified on $T^2$. The amplitudes thus obtained involved Eisenstein functions of various degrees (as a function of complex/kaehler structure of $T^2$). Unlike the four-dimensional $R^2F^{2g-2}$ terms, the $R^4H^{4g-4}$ (or $R^4F^{4g-4}$) terms survive in the large volume limit to give non-zero contributions for the uncompactified superstring. Furthermore, the $R^4$ term at genus one has precisely the same index structure as the $R^4$ term multiplying $E_{3/2}(\tau)$ in the conjecture of.

As will be discussed in this paper, the structure of the $R^4H^{4g-4}$ terms leads us to conjecture that they are multiplied by the manifestly SL(2,Z)-invariant Eisenstein function $E_{g+1/2}(\tau)$ in the uncompactified Type IIB low-energy effective action. More precisely, we

---

1 It should be noted that there is a possible contradiction in the literature concerning the two-loop $R^4$ contribution.
conjecture that there is a term in the Type IIB effective action on $R^{10}$ which in the Einstein gauge takes the form

$$S = N_g \int d^{10}x \sqrt{\text{det}g}$$

$$\sum_{p=2-2g}^{2g-2} (-1)^p R^4(H^+)^{2g-2+p}(H^-)^{2g-2-p} \sum'_{m,n} \tau_2^{g+\frac{1}{2}} \frac{1}{(m+n\tau)^{\frac{1}{2}+p}(m+n\bar{\tau})^{\frac{1}{2}-p}}$$

where $H^+ = \tau_2^{-\frac{1}{2}}(H_{RR} - \tau H_{NS-NS})$, $H^- = \tau_2^{-\frac{1}{2}}(H_{RR} - \bar{\tau} H_{NS-NS})$ and $N_g$ is an overall normalization constant.

Furthermore, we conjecture that for compactification on $T^2$, the eight-dimensional $R^4H^{4g-4}$ terms are multiplied by a manifestly SL(3,$\mathbb{Z}$)-invariant version of the Eisenstein function which generalizes the $R^4$ conjecture of Kiritsis and Pioline[7]. The conjectured form of the amplitude in this case essentially follows from extending $SL(2,\mathbb{Z})$-invariant Eisenstein functions obtained by summing over a 2d lattice to $SL(3,\mathbb{Z})$-invariant functions obtained by summing over 3d lattice points, and is very natural (and perhaps unique). Our conjecture in this case implies that $R^4H^{4g-4}$ terms only get perturbative contributions at genus 0 and genus g, just as in $R^{10}$. Moreover, the genus g contribution is itself an Eisenstein function of the Kahler structure of $T^2$ and precisely coincides with the corresponding string computation of [12] for compactification on $T^2$ for all g. This we consider strong evidence for our conjecture.

The organization of this paper is as follows: In section 2, we review the results in [10] for six-dimensional topological amplitudes $R^4H^{4g-4}$ (or $R^4F^{4g-4}$) which arise upon compactification of Type IIB (or Type IIA) on a hyper-Kahler manifold. We also discuss the corresponding computation of $R^2F^{2g-2}$ terms on Calabi-Yau threefolds, in part to contrast it with the computations of $R^4H^{4g-4}$ terms. In section 3, we review the topological computations of [12] when the four manifold is $T^2 \times \mathbb{R}^2$, and describe their implications for scattering amplitudes of Type II upon compactification on $T^2$ down to $D = 8$. In section 4, we conjecture the non-perturbative structure of Type IIB $R^4H^{4g-4}$ terms in $D = 8$ and $D = 10$, and describe various types of evidence for our conjecture. In section 5, we discuss a paradox concerning the non-perturbative structure of Type IIA $R^4F^{4g-4}$ terms and suggest a possible resolution. In section 6, we discuss possible implications of these results for “pair creation” of $(p,q)$ strings (motivated by the implications of $R^2F^{2g-2}$ terms for Schwinger’s pair creation). In the concluding section, we discuss possible implications of our conjecture for the $\nabla^n R^4$ conjectures of Russo[9] and for the $F^{2g+2}$ conjectures coming from M(atrix) theory [13].
2. Review of Topological Amplitudes

In reference [10], we proved that certain six-dimensional superstring scattering amplitudes can be expressed as topological computations on the hyper-Kahler compactification manifold. Although our proof used the modified Green-Schwarz formalism where spacetime-supersymmetry is manifest and twisting is natural, it should be straightforward to reproduce our proof using the Ramond-Neveu-Schwarz formalism. Like their four-dimensional counterparts, the six-dimensional topological amplitudes involve the scattering of gravitons and Ramond-Ramond fields, and to understand their structure, it will be useful to first review the four-dimensional case.

2.1. Review of four-dimensional \( R^2 F^2 g - 2 \) terms

Four-dimensional \( R^2 F^2 g - 2 \) terms in the effective action of the Type II superstring compactified on a Calabi-Yau manifold can be computed by scattering two gravitons and \( 2g - 2 \) chiral graviphotons \([II]\). For the Type IIB (or Type IIA) superstring, the vertex operator for each chiral graviphoton carries +3/2 left-moving charge and +3/2 (or −3/2) right-moving charge with respect to the left and right-moving U(1) generators of the N=2 c=9 superconformal field theory representing the compactification.

At \( g \) loops, one needs \( 3g - 3 \) left and right-moving picture-changing operators if the \( 2g - 2 \) graviphotons are all chosen in the \((-\frac{1}{2}, -\frac{1}{2})\) picture. By U(1) conservation of the Type IIB (or Type IIA) superstring, the only contributing part of the picture-changing operators is \( e^{\phi_L} G^-_L \) and \( e^{\phi_R} G^-_R \) (or \( e^{\phi_L} G^+_L \) and \( e^{\phi_R} G^+_R \)), where \( \phi_{L/R} \) comes from fermionizing the \( \beta_{L/R} \) ghosts and \( G^\pm_{L/R} \) are the fermionic generators of the N=2 c=9 algebra. The spacetime-dependent part of the computation is trivial, leaving only a correlation function over \( G^\pm_L \)’s and \( G^\pm_R \)’s (or \( G^-_L \)'s and \( G^+_R \)'s) which is the N=2 topological computation for the “B-model” (or “A-model”).

The final result is that the low-energy effective action contains a local \( g \)-loop contribution given by the N=2 D=4 superspace expression

\[
\int d^4x \int d^2\theta_L d^2\theta_R (W_{\alpha\beta} W^{\alpha\beta})^g f_g \tag{2.1}
\]

where \( W_{ab} = (\sigma^{\mu\nu})_{\alpha\beta} F_{\mu\nu} + (\theta_L \sigma^{\mu\nu})_\alpha (\theta_R \sigma^{\mu\nu})_\beta R_{\mu\nu\rho\kappa} + \ldots \), \( F_{\mu\nu} \) denotes the graviphoton field strength, and \( f_g \) is the topological amplitude at genus \( g \) which depends on the moduli of Calabi-Yau compactification. Integration over \( \theta_{L,R} \) is easily seen to give \( R^2 F^{2g - 2} \) terms contracted in various ways, as well as other terms which are related by supersymmetry. In
four-dimensional Einstein gauge, i.e. \( S = \int d^4x \sqrt{\text{det}g} (R + F^2 + ...) \), it is easy to check there are no \( e^\phi \) factors in front of the \( R^2 F^{2g-2} \) term at genus \( g \). This is explained by the fact that the dilaton in Einstein gauge sits in a tensor multiplet, which cannot appear in the chiral superspace action of (2.1). Therefore, the \( R^2 F^{2g-2} \) term gets no perturbative or non-perturbative contributions except at genus \( g \).

2.2. Topological amplitudes in six dimensions

As shown in [10], there is a six-dimensional analog of the four-dimensional amplitudes which involves the genus \( g \) scattering of four gravitons and \( 4g - 4 \) Ramond-Ramond fields. In 6 dimensions, the Lorentz group is most conveniently described using the spin group which is \( SU(4) \). We will denote \( SU(4) \) indices by \( a, b = 1, ..., 4 \), which can describe either chiral or anti-chiral spinors. Moreover, spinors carry an internal \( SU(2) \) index denoted by \( j, k = \pm \) which comes from the \( SU(2) \) of the hyper-Kahler manifold. Bispinor Ramond-Ramond field strengths \( M_{ab}^{jk} \) carry both left and right-moving version of these \( SU(4) \) and \( SU(2) \) indices. Note that for the Type IIB (or Type IIA) superstring, \( b \) has the same (or opposite) chirality as \( a \). So in vector notation, \( M_{ab}^{jk} \) describes field-strengths with an odd (or even) number of vector indices and we will focus primarily on the three-form \( H \) (or two-form \( F \)).

Now, suppose that we consider \( 4g - 4 \) Ramond-Ramond vertex operators and have all of them carry +1 left-moving charge and −1 right-moving charge with respect to the left and right-moving \( U(1) \) generators of the \( N=2 \ c=6 \) superconformal field theory representing the compactification. This implies that the \( SU(2) \) indices on the Ramond-Ramond field strengths \( M_{ab}^{jk} \) are all chosen to be in the directions \( j = + \) and \( k = + \) where, for later convenience, we choose notation such that the right-moving \( SU(2) \) index on \( M_{ab}^{jk} \) has the opposite sign of the right-moving \( U(1) \) charge.

If all Ramond-Ramond vertex operators are chosen in the \((-\frac{1}{2}, -\frac{1}{2})\) picture, one needs \( 4g - 4 \) picture-changing operators and by \( U(1) \) conservation, only the \( e^{\phi_L} G_{L}^- \) and \( e^{\phi_R} G_{R}^+ \) terms in the picture-changing operators contribute. As before, the spacetime-dependent

\[ ^2 \text{Note that superstring arguments alone can only prove the absence of contributions below genus } g \text{ (where there are not enough picture-changing operators to absorb the } U(1) \text{ charge), but cannot prove the absence of contributions above genus } g. \]

\[ ^3 \text{For } T^4, \text{ the internal directions will involve an } SO(4) = SU(2) \times SU(2). \text{ If we are dealing with } K3, \text{ one of the internal } SU(2)'s \text{ is broken by the holonomy of } K3 \text{ and only one survives.} \]
part of the computation is trivial, leaving only a correlation function involving \( G_L^- \)'s and \( G_R^+ \)'s, which is a topological computation on the hyper-Kahler compactification manifold.

The result is that the \( g \)-loop six-dimensional low-energy effective action contains a term proportional to the N=2 D=6 superspace expression

\[
\int d^6x \int d^4\theta_L^+ d^4\theta_R^+ \left( W^{++}_{a_1b_1} W^{++}_{a_2b_2} W^{++}_{a_3b_3} W^{++}_{a_4b_4} e^{a_1a_2a_3a_4} e^{b_1b_2b_3b_4} \right) \tag{2.2}
\]

where \( W^{++}_{\alpha\beta} = M^{++}_{ab} + (\theta^+ L \sigma^{\mu\nu})_a (\theta^+ R \sigma^{\mu\nu})_b R_{\mu\nu\rho\kappa} + \ldots \) and \( f_g \) is the topological \( N = 4 \) partition function at genus \( g \) and instanton number \((2g - 2, 2g - 2)\), which is the same as the \( N = 2 \) string partition function at genus \( g \) and instanton number \((2g - 2, 2g - 2)\) on the corresponding hyper-Kahler manifold. Note that by U(1) conservation in the \( g \)-loop computation, this amplitude vanishes when the genus is less than \( g \).

Although the above computation breaks the internal SU(2) invariance to a U(1) subgroup, the full SU(2) is easily restored by introducing the “harmonic” variables \( u_j \) and \( \bar{u}_j \) satisfying \( u_j \bar{u}_k = \epsilon_{jk} \) \([4]\). Actually, since we are considering bispinors, we need to introduce two harmonic variables, \((u^L_j, \bar{u}^L_j)\) and \((u^R_j, \bar{u}^R_j)\). By defining \( \hat{M}_{ab} = u^L_k u^R_k M^{jk}_{ab} \), we can now repeat the above calculation for arbitrary values of \( u^L_j \) and \( u^R_j \), where \( \hat{M}_{ab} \) now includes all the field strengths which are related to each other by the SU(2) internal rotation.

The spacetime dependence of the computation is still trivial, and the compactification dependent topological computation \( f_g(\bar{u}^L_j, \bar{u}^R_j) \) is now a polynomial of degree \( 4g - 4 \) in \( \bar{u}^L_j \) and \( \bar{u}^R_j \). As shown in \([10]\),

\[
f_g(\bar{u}^L_j, \bar{u}^R_j) = \sum_{n_L, n_R = 2-2g}^{2g-2} F_{g}^{n_L, n_R} (\bar{u}^L_+)^{2g-2+n_L} (\bar{u}^L_-)^{2g-2-n_L} (\bar{u}^R_+)^{2g-2+n_R} (\bar{u}^R_-)^{2g-2-n_R} \tag{2.3}
\]

where \( F_{g}^{n_L, n_R} \) computes the \( g \)-loop partition function of (left,right) instanton number \((n_L, n_R)\) for the self-dual \( N = 2 \) string propagating on the hyper-Kahler four manifold.

Knowing the scattering amplitude for any value of \( u^L_j \) and \( u^R_j \) allows one to construct the SU(2)-invariant amplitude by integrating over \( u^L_j \) and \( u^R_j \) as in \([14]\). So the complete SU(2)-invariant scattering amplitude is given by the superspace expression

\[
\int d^6x \int du^L \int du^R \int d^4\theta_L^+ d^4\theta_R^+ \left( \hat{W}_{a_1b_1} \hat{W}_{a_2b_2} \hat{W}_{a_3b_3} \hat{W}_{a_4b_4} e^{a_1a_2a_3a_4} e^{b_1b_2b_3b_4} \right) f_g(\bar{u}^L_j, \bar{u}^R_j) \tag{2.4}
\]
where \( \hat{W}_{\alpha\beta} = \hat{M}_{ab} + (\theta_L \sigma^{\mu\nu})_a (\theta_R \sigma^{\mu\nu})_b R_{\mu\nu\rho\kappa} + \ldots \), and \( \int d\mu L \int d\nu R \) is defined by

\[
\int d\mu L \int d\nu R \int \frac{du}{L} \int \frac{du}{R} f_{j_1 \ldots j_N} (k_1 \ldots k_N) \sum_{n=1}^{m_1 \ldots m_N} u^L_{j_1} u^L_{k_1} u^R_{m_1} \ldots u^L_{j_N} u^R_{k_N} u^R_{m_N}
\]

(2.5)

Expanding (2.4) in components for the Type IIB (or Type IIA) superstring gives \( R^4 H^{4g-4} \) terms (or \( R^4 F^{4g-4} \) terms), as well as various other terms related by supersymmetry.

3. Terms in the Eight-Dimensional Effective Action

Although \( f_g(\bar{u}_L^j, \bar{u}_R^j) \) is unknown when the compactification manifold is \( K3 \), it is known \([12]\) up to an overall constant for all \( g \) when the manifold is \( R^2 \times T^2 \), i.e. when the superstring is compactified to eight dimensions on \( T^2 \). So by ‘Lorentz-covariantizing’ the six-dimensional indices of (2.4) to eight-dimensional indices, one can find explicit expressions for \( g \)-loop terms in the eight-dimensional low-energy effective action of the Type II superstring.

To ‘Lorentz-covariantize’, one first rewrites (2.4) in six-dimensional vector notation by replacing \( \hat{W}_{ab} \) with \( \sum_n \hat{W}_{\mu_1 \ldots \mu_n} \Gamma^{\mu_1 \ldots \mu_n}_{ab} \) and using

\[
\epsilon^{a_1 a_2 a_3 a_4} \epsilon^{b_1 b_2 b_3 b_4} = \delta^{[a_1}_{[b_1} \delta^{a_2}_{b_2} \delta^{a_3}_{b_3} \delta^{a_4]}_{b_4]}
\]

to get traces of \( \Gamma \) matrices. It is easy to check that the expression only involves contractions of vector indices and contains no six-dimensional \( \epsilon \)-tensors. One now simply replaces all six-component vector indices with eight-component vector indices.

The only subtlety is that the \( u^L_+ u^R_+ \) and \( u^L_- u^R_- \) pieces of \( \hat{M}_{ab} \) come from eight-dimensional fields containing indices in the 7 or 8 directions. For example, in the Type IIB (or Type IIA) superstring \( M_{ab}^+ \Gamma_{\mu\nu\rho} \) (or \( M_{ab}^+ \Gamma_{\mu\nu} \)) comes from a four-form (or three-form) in eight dimensions with one component in the \((7+i8)\) direction. For this reason, we shall restrict our attention for the rest of this paper to fields coming from the \( u^L_+ u^R_+ \) and \( u^L_- u^R_- \) pieces of \( \hat{M}_{ab} \), which contain the same number of indices in six and eight dimensions. For terms involving these fields, the expression obtained by replacing six-component with eight-component indices is manifestly Lorentz-invariant in eight dimensions.

For compactification of Type II strings on \( T^2 \), the massless bosonic fields are described by the Riemann tensor \( R_{\mu\nu\rho\sigma} \), a real triplet of three-form field-strengths \( H^{(jk)}_{\mu\nu\rho} \) where \( j, k = \pm \) are SU(2) indices, a complex triplet of two-form field-strengths \( F^{(jk)}_{\mu\nu} \), a self-dual
and anti-self-dual four-form field-strength $F_{\mu\nu\rho\sigma}^\pm$, and seven scalars consisting of the $T^2$ Kahler modulus $\sigma = \sigma_1 + i\sigma_2$, the $T^2$ complex modulus $\rho = \rho_1 + i\rho_2$, the eight-dimensional dilaton $\lambda_8$, and two Ramond-Ramond scalars. In terms of these fields, \begin{equation}
abla B : \hat{M}_{ab} = u_L^L u_R^R \Gamma_{ab}^{\mu\nu} H_{\mu\nu\rho}^{(++)} + u_L^L u_R^R \Gamma_{ab}^{\mu\nu} H_{\mu\nu\rho}^{(--)} + \ldots, \end{equation}
\begin{equation}
\nabla A : \hat{M}_{ab} = u_L^L u_R^R \Gamma_{ab}^{\mu\nu} F_{\mu\nu}^{(++)} + u_L^L u_R^R \Gamma_{ab}^{\mu\nu} F_{\mu\nu}^{(--)} + \ldots ,\end{equation}
where the terms in ... will be ignored. Note that $H_{\mu\nu\rho}^{(++)}$, $F_{\mu\nu}^{(++)}$, and $F_{\mu\nu}^{(--)}$ are NS-NS fields which do not appear in $\hat{M}_{ab}$.

In eight-dimensional Einstein gauge, i.e.
\begin{equation}
S = \int d^8x \sqrt{\det g} \left( R + F_{\mu\nu}^{(jk)} + F_{\mu\nu}^{(--)} + H_{\mu\nu\rho}^{(jk)} H_{\mu\nu\rho}^{(--)} \right),
\end{equation}
the topological computation of [12] found
\begin{equation}
f_g(\bar{u}_j^L, \bar{u}_j^R) = \lambda_8^{2g-2} \sigma_2^g \frac{\epsilon^4}{g}
\end{equation}
where $\lambda_8 = \sigma_2^{1/2} e^\phi$ is the eight-dimensional coupling constant, $\sigma_2$ is the volume of $T^2$, and (up to an overall constant)
\begin{equation}
F_g(\bar{u}_j^L, \bar{u}_j^R) = \sum' \left( \frac{\bar{u}_j^L \bar{u}_j^R}{m + n\sigma} + \frac{\bar{u}_j^L \bar{u}_j^R}{m + n\sigma} \right)^4 g_{mn} \lambda_8^{2g-4} \lambda_8^{2g-4}\end{equation}
where $\sum'$ means to sum over all integers $m$ and $n$ except when $m = n = 0$. The $e^{\frac{2}{g}(g-1)\phi}$ dependence of $f_g$ can be understood by rescaling
\begin{equation}
g_{\mu\nu} \rightarrow e^{-2\phi/3} g_{\mu\nu}, \quad F_{\mu\nu}^{(jk)} \rightarrow e^{-\phi/3} F_{\mu\nu}^{(jk)}, \quad H_{\mu\nu\rho}^{(jk)} \rightarrow e^{-2\phi/3} H_{\mu\nu\rho}^{(jk)}
\end{equation}
which rescales the eight-dimensional Einstein gauge action to string gauge and rescales
\begin{equation}
e^{\frac{2}{g}(g-1)\phi} \sqrt{\det g} \frac{R^4 (H^{4g-4} + F^{4g-4})}{\sqrt{\det g} \frac{R^4 (H^{4g-4} + F^{4g-4})}} \end{equation}

The topological computation of (5.3) was called the ‘A-model’ in [12]. For compactification on $T^2 \times R^2$, the N=2 U(1) generator splits as $J = J_1 + J_2$ where $J_1$ comes from $T^2$ and $J_2$ comes from $R^2$. The topological computation for the ‘B-model’ comes from flipping the sign of the right-moving $J_1$ with respect to the right-moving $J_2$. This flip is just a $T$-duality transformation on one of the circles in $T^2$, so the ‘B-model’ computation is related to the ‘A-model’ computation by replacing $\sigma$ with $\rho$ in (5.3) and $\sigma_2$ with $\rho_2$ in (5.2). Since the ‘B-model’ computation vanishes as $\sigma_2 \rightarrow \infty$, it does not survive in ten dimensions. In eight dimensions, the scattering amplitudes associated with the ‘B-model’ involve $R^4 F^{4g-4}$ (or $R^4 H^{4g-4}$) terms in the Type IIB (or Type IIA) superstring effective action.
3.1. $SL(2, Z) \times SL(2, Z)$ invariance

For compactification on $T^2$, the perturbative low-energy effective action is invariant under $SL(2, Z) \times SL(2, Z)$ transformations. As usual, it is useful to think of the Kahler and complex moduli as $SL(2, R)/SO(2)$ variables, $c^i_j$ and $\tilde{c}^i_j$ ($j = \pm$ and $I = 1$ or 2). These variables satisfy

$$M_{IJ} \equiv c^+_i (c^-_j) = \frac{1}{\sigma_2} \left( \begin{array}{cc} 1 & \sigma_1 \\ \sigma_1 & |\sigma|^2 \end{array} \right),$$

and

$$\tilde{M}_{IJ} \equiv \tilde{c}^+_i (\tilde{c}^-_j) = \frac{1}{\rho_2} \left( \begin{array}{cc} 1 & \rho_1 \\ \rho_1 & |\rho|^2 \end{array} \right),$$

where $M_{IJ} \rightarrow \Lambda^I_{K} \Lambda^J_{L} M_{KL}$ and $\tilde{M}_{IJ} \rightarrow \tilde{\Lambda}^I_{K} \tilde{\Lambda}^J_{L} \tilde{M}_{KL}$ under $SL(2, Z) \times SL(2, Z)$ transformations parameterized by $\Lambda^I_{J}$ and $\tilde{\Lambda}^I_{J}$. Here $\sigma$ denotes the Kahler class of $T^2$ and $\rho$ denotes its complex structure.

Under $SO(2) \times SO(2)$ transformations, $c^\pm_I \rightarrow e^{i\theta} c^\pm_I$ and $\tilde{c}^\pm_I \rightarrow e^{i\tilde{\theta}} \tilde{c}^\pm_I$, so one can choose an $SO(2)$ gauge in which

$$c^+_I = \sigma_2^{-1/2} (1, \sigma), \quad c^-_I = \sigma_2^{-1/2} (1, \bar{\sigma}), \quad \tilde{c}^+_I = \rho_2^{-1/2} (1, \rho), \quad \tilde{c}^-_I = \rho_2^{-1/2} (1, \bar{\rho}).$$

The relevant Ramond-Ramond field strengths are defined to be invariant under $SL(2, Z) \times SL(2, Z)$ transformations, however they transform under $SO(2) \times SO(2)$ transformations for the Type IIB (or Type IIA) superstring as

$$H^{\pm\pm}_{\mu\nu\rho} \rightarrow e^{i\theta} H^{\pm\pm}_{\mu\nu\rho}, \quad F^{\pm(+)}_{\mu\nu} \rightarrow e^{i\tilde{\theta}} F^{\pm(+)}_{\mu\nu}$$

(or $H^{\pm\pm}_{\mu\nu\rho} \rightarrow e^{i\tilde{\theta}} H^{\pm\pm}_{\mu\nu\rho}, \quad F^{\pm(+)}_{\mu\nu} \rightarrow e^{i\tilde{\theta}} F^{\pm(+)}_{\mu\nu}$).

The eight-dimensional Einstein gauge action obtained by ‘covariantizing’ (2.4) is invariant under these transformations since it can be written as

$$S = \int d^8x \int du^L du^R \sqrt{det s g} \lambda_8^{2g-2} \hat{M}^{4g-4}$$

$$\sum_{m^I, m^J} \left( \frac{u^L_{m^I, \pm} u^R_{m^I, \pm}}{m^I c^+_I} + \frac{\bar{u}^L_{m^I, \pm} \bar{u}^R_{m^I, \pm}}{m^I c^-_I} \right)^{4g-4} |m^I c^+_I|^{2g-4}$$

where $\hat{M}_{ab}$ is defined as in (3.1), and the index contractions on $R^4 \hat{M}^{4g-4}$ are determined using the method discussed earlier. Invariance is manifest if $m^I \rightarrow (\Lambda^{-1})^I_{J} m^J$ under $SL(2, Z) \times SL(2, Z)$ transformations and $(u^L_{m^I, \pm}, u^R_{m^I, \pm}) \rightarrow (e^{\pm i(\theta+\bar{\theta})} u^L_{m^I, \pm}, e^{\pm i(\theta-\bar{\theta})} u^R_{m^I, \pm})$ under $SO(2) \times SO(2)$ transformations. Similar techniques have previously been used in [13].
Performing the integrations over the $u^L$ and $u^R$ variables, one obtains for the Type IIB superstring
\[ S = \int d^8 x \sqrt{\text{det}g} \, \lambda^2_8 \sum_{p=2-2g}^{2g-2} R^4 (H^{(++)})^{2g-2+p} (H^{(-)})^{2g-2-p} \]  
\[ \sum_{m,n} \frac{\sigma^g}{(m+n\sigma)^{g+p}(m+n\bar{\sigma})^{g-p}}. \]  
Note that in terms of the D=10 three-form and five-form field strengths, comparison of the D=8 and D=10 kinetic terms implies that
\[ H^{(\pm\pm)}_{\mu\nu\rho} = \sigma^2_2 \tau_2^{-1/3} (H^{R-R}_{\mu\nu\rho} - \tau_1 H^{NS-NS}_{\mu\nu\rho}) \mp i\sigma^4_2 \tau_2^{2/3} H^{R-NS}_{\mu\nu\rho89}. \]  
For the Type IIA superstring, the only difference from (3.11) is that $(H^{(++)})^{2g-2+p} (H^{(-)})^{2g-2-p}$ is replaced with $(F^{(+--)})^{2g-2+p} (F^{(--)})^{2g-2-p}$ where, in terms of the D=10 Ramond-Ramond two-form and four-form field strengths,
\[ F^{\pm(+-)}_{\mu\nu} = \sigma^2_2 \tau_2^{-1/6} F^{R-R}_{\mu\nu} \pm i\sigma^7_2 \tau_2^{1/3} F^{R-NS}_{\mu\nu89}. \]

4. Type IIB $R^4 H^{4g-4}$ Conjectures

4.1. Eight-dimensional conjecture

For the Type IIB (or Type IIA) superstring compactified on $T^2$, it has been conjectured that the $SL(2,Z) \times SL(2,Z)$ symmetry of the previous subsection is extended non-perturbatively to an $SL(3,Z) \times SL(2,Z)$ symmetry where the $SL(3,Z)$ extends the Kahler (or complex) moduli of $T^2$. In this subsection, we conjecture an $SL(3,Z) \times SL(2,Z)$-invariant non-perturbative extension of (3.10) for the case of $R^4 H^{4n-4}$ terms in the Type IIB superstring effective action. When $n=1$, our conjecture coincides with that of [4] and [7], and when $n > 1$, it generalizes their Type IIB conjecture in a natural way.

Under $SL(3,Z) \times SL(2,Z)$ transformations, the seven scalars of the Type IIB superstring on $T^2$ split into a quintuplet which can be thought of as $SL(3,R)/SU(2)$ variables $C^{(jk)}_{\alpha}$ ($\alpha = 1$ to 3 and $j,k = \pm$), and a doublet which can be thought of as $SL(2,R)/SO(2)$ variables $c^{(j)}_I$. $c^{(j)}_I$ is defined as in (3.8) and (3.7). $C^{(jk)}_{\alpha}$ is defined to satisfy [7]:
\[ \mathcal{M}_{\alpha\beta} \equiv C^{(jk)}_{\alpha} C^{(jk)}_{\beta} = \]  
(4.1)
\[ = \lambda_8^{4/3} \left( \begin{array}{ccc} 1 & \tau_1 & B_R + \tau_1 \sigma_1 \\ \tau_1 & |\tau|^2 & \tau_1 B_R + |\tau|^2 \sigma_1 \\ B_R + \tau_1 \sigma_1 & \tau_1 B_R + |\tau|^2 \sigma_1 & (B_R + \tau_1 \sigma_1)^2 + \lambda_8^{-2} \sigma_2^{-1}|\sigma|^2 \end{array} \right) \]

where \( \tau = \tau_1 + i\lambda_8^{-1} \sigma_2^{-\frac{1}{2}} \), \( \tau_1 \) and \( B_R \) are the two Ramond-Ramond scalars (\( \tau_1 \) is already a scalar in 10 dimensions and \( B_R \) is the constant expectation value of the RR B-field on \( T^2 \)), and \( M_{\alpha\beta} \rightarrow \Lambda_\alpha^\gamma \Lambda_\beta^\delta M_{\gamma\delta} \) under \( SL(3, Z) \) transformations parameterized by \( \Lambda_\alpha^\beta \).

Under \( SU(2) \) transformations, \( C_\alpha^{(jk)} \rightarrow \Omega_{\alpha}^i \Omega_{\mu}^k C_\alpha^{(lm)} \), so one can choose an \( SU(2) \) gauge in which

\[ C_\alpha^{(++)} = \lambda_8^{-1/3} \sigma_2^{-1/2} (0, 1, \sigma), \quad C_\alpha^{(--)} = \lambda_8^{-1/3} \sigma_2^{-1/2} (0, 1, \sigma), \quad C_\alpha^{(+-)} = -i \lambda_8^{2/3} (1, \tau_1, B_R + \tau_1 \sigma_1). \]

The triplet of three-form field strengths are defined to be invariant under \( SL(3, Z) \times \) \( SL(2, Z) \) transformations, however they transform under \( SU(2) \times SO(2) \) transformations as \( H_{\mu\nu\rho}^{(jk)} \rightarrow \Omega_{\alpha}^i \Omega_{\mu}^k H_{\mu\nu\rho}^{(lm)} \).

For the Type IIB superstring, the Ramond-Ramond \( H_{\mu\nu\rho} \) fields appear in \( \tilde{M}_{ab} \) as in (3.1). To make \( SU(2) \) invariance manifest, it is useful to define a new harmonic field

\[ \tilde{M}_{ab}(u_j^L, u_j^R) = \Gamma_{ab}^{\mu\nu\rho} H_{\mu\nu\rho}^{(jk)} u_j^L u_j^R \]

which now contains the NS-NS three-form field strength. Note that \( H_{\mu\nu\rho}^{(+-)} \) is imaginary since the reality condition is \( (H_{\mu\nu\rho}^{(jk)})^* = \epsilon_{jl} \epsilon_{km} H_{\mu\nu\rho}^{(jk)} \). In terms of the D=10 three-form field strengths, \( H_{\mu\nu\rho}^{(+-)} = i \sigma_2^{1/3} T_2 \sqrt{\det g} N_{\mu\nu\rho} \), and \( H_{\mu\nu\rho}^{(\pm\pm)} \) are defined as in (3.12).

Our \( SL(3, Z) \times SL(2, Z) \)-invariant conjecture for \( R^4 H^{4g-4} \) terms in the eight-dimensional Einstein gauge Type IIB low-energy effective action is

\[ S = N_g \int d^8x \int du^L \int du^R \sqrt{\det g} R^4 M^{\frac{3}{2}g-4} \]

\[ \sum_{m^1, m^2, m^3} \left( m^\alpha C_\alpha^{(jk)} u_j^L u_j^R \right)^g (m^\alpha C_\alpha^{(jk)} C_\beta^{(jk)} m^\beta)^{3g+\frac{3}{2}} \]

where \( N_g \) is an overall constant and \( \sum' \) means to sum over all integers \( m^1, m^2 \) and \( m^3 \) except when \( m_1 = m_2 = m_3 = 0 \). Note that when \( g = 1 \), this conjecture coincides with the D=8 \( R^4 \) conjecture of [7]. Before checking how this conjecture fits with the perturbative computation in eight dimensions given in (3.14), let us discuss what this conjecture implies for the uncompactified Type IIB effective action in 10 dimensions.
4.2. Ten-dimensional conjecture

To obtain our conjecture for ten-dimensional $R^4 H^{4g-4}$ terms, one takes the large volume limit $\sigma_2 \to \infty$, keeping $\tau_2 \equiv \lambda_8^{-1} \sigma_2^{-\frac{1}{2}}$ fixed. In this large volume limit, the terms in (4.4) involving $m^3 \neq 0$ vanish, giving

$$ S = N_g \int d^8 x \int du^L \int du^R \sqrt{det g} \left( \sigma_2^2 \tau_2 \right)^{(2g+1)/6} R^4 \tau \left( \sigma_2^2 \tau_2 \right)^{4g-4} $$

(4.5)

$$ \sum_{m^1, m^2}^\prime \left( m^I D_I^{(jk)} \tilde{u}_j^L \tilde{u}_k^R \right)^{4g-4} (m^I D_I^{(jk)} D_J^{(jk)} m^J)^{-3g+\frac{1}{2}} $$

where $I = 1$ to 2 and

$$ D_I^{(++)} = D_I^{(--)} = \tau_2^{-\frac{1}{2}} (0, \tau_2), \quad D_I^{(+--)} = -i \tau_2^{-\frac{1}{2}} (1, \tau_1). \quad (4.6) $$

It is convenient to define $H^\pm_{\mu \nu \rho} = \sigma_2^{-1/3} \tau_2^{-1/6} \left[ \frac{1}{2} (H^{(++)}_{\mu \nu \rho} + H^{(--)}_{\mu \nu \rho}) \pm H^{(+-)}_{\mu \nu \rho} \right]$, which can be expressed in terms of the $D = 10$ NS-NS and R-R three-forms as $H^+_{\mu \nu \rho} = \tau_2^{-\frac{1}{2}} (H^{R-R}_{\mu \nu \rho} - \tau H^{NS-NS}_{\mu \nu \rho})$ and $H^-_{\mu \nu \rho} = \tau_2^{-\frac{1}{2}} (H^{R-R}_{\mu \nu \rho} + \tau H^{NS-NS}_{\mu \nu \rho})$. Rescaling to ten-dimensional Einstein gauge (where the classical action after compactification on $T^2$ is $S = N_g \int d^8 x \sqrt{det g} \sigma_2 \tau_2^2 \left( R + H^+_{\mu \nu \rho} H^-_{\mu \nu \rho} + \ldots \right)$, one obtains

$$ S = N_g \int d^8 x \int du^L \int du^R \sqrt{det g} \sigma_2 \tau_2^2 \left( R^4 \left( \sigma_2^2 \tau_2 \right)^{4g-4} \right) $$

(4.7)

$$ \sum_{m^1, m^2}^\prime \left( m^I D_I^+ \tilde{v}_L^\mp \tilde{v}_R^+ - m^I D_I^- \tilde{v}_L^\mp \tilde{v}_R^- \right)^{4g-4} (m^I D_I^+ D_J^- m^J)^{-3g+\frac{1}{2}} $$

where $D_I^+ = \tau^{-\frac{1}{2}} (1, \tau), \quad D_I^- = \tau^{-\frac{1}{2}} (1, \tau), \quad \tilde{v}_L^\mp = 2^{-\frac{1}{2}} (\tilde{u}_L \pm \tilde{u}_L) \quad \text{and} \quad \tilde{v}_R^\pm = 2^{-\frac{1}{2}} (\tilde{u}_R \pm \tilde{u}_R) . \quad (4.8) $
For the term with \( p = 0 \) in (4.8), the coefficient multiplying \( R^4H^4 \) is proportional to the Eisenstein function

\[
E_{g+\frac{1}{2}}(\tau) = \frac{1}{2\zeta(2g + 1)} \sum_{m,n} \frac{\tau_2^{g+\frac{1}{2}}}{(m + n\tau)^{g+\frac{1}{2}}(m + n\tau)^{g+\frac{1}{2}}}. \tag{4.9}
\]

Furthermore, the coefficients for \( p > 0 \) are proportional to \( \tau_2^{-p}(\tau_2^2 \frac{\partial}{\partial \tau})^p E_{g+\frac{1}{2}}(\tau) \) while the coefficients for \( p < 0 \) are proportional to \( \tau_2^p(\tau_2^2 \frac{\partial}{\partial \tau})^{-p} E_{g+\frac{1}{2}}(\tau) \). At large values of \( \tau_2 \),

\[
E_{g+\frac{1}{2}}(\tau) \rightarrow \tau_2^{g+\frac{1}{2}} + \gamma_{g+\frac{1}{2}} \tau_2^{-g} + O(e^{-2\pi\tau_2}), \tag{4.10}
\]

where \( \gamma_{g+\frac{1}{2}} = \sqrt{\frac{\Gamma(g)\zeta(2g)}{\Gamma(g+\frac{1}{2})\zeta(2g+1)}} \). After rescaling to D=10 string gauge,

\[
(\tau_2^{g+\frac{1}{2}} + \gamma_{g+\frac{1}{2}} \tau_2^{-g}) \sqrt{\det_{10G}} R^4H^4 \rightarrow (\tau_2^2 + \gamma_{g+\frac{1}{2}} \tau_2^{2-2g}) \sqrt{\det_{10G}} R^4H^4, \tag{4.11}
\]

so (4.8) only gets perturbative contributions at tree-level and at genus \( g \).

4.3. Evidence for Type IIB conjecture

The most important evidence for our conjecture comes from explicit agreement of the eight-dimensional conjecture with the genus \( g \) computation in (3.10). To compare (4.4) with (3.10), it is useful to split the sum \( \sum_{m_1,m_2,m_3} \) in (4.4) into \( \sum_{m_1} \) where \( m_2 = m_3 = 0 \), and \( \sum_{m_1} \sum_{m_2,m_3} \) where one sums over all values except \( m_2 = m_3 = 0 \). The first sum contributes

\[
S = N_g \int d^8x \int du^L \int du^R \sqrt{\det_{sG}} R^4M \tag{4.12}
\]

\[
\sum_{m_1} (m^1 C_1^{(+-)} \bar{u}_{(+)}^L \bar{u}_{(-)}^R)^{4g-4} (-m^1 C_1^{(-+)} C_1^{(+)} m^1)^{-3g+\frac{1}{2}}
\]

\[
= \int d^8x \int du^L \int du^R \sqrt{\det_{sG}} \lambda_8^{-(4g+2)/3} R^4M \sum_{m_1} (m^1)^{-2g-1}
\]

where we used that \( C_1^{(++)} = C_1^{(-)} = 0 \). Integrating over \( u \) gives the contribution

\[
S = N_g \int d^8x \sqrt{\det_{sG}} \lambda_8^{-(4g+2)/3} \sum_{m^1} (m^1)^{-2g-1} \tag{4.13}
\]

\[
R^4 \sum_{q=0}^{2g-2} c_q (H^{(++)}H^{(-)})^{2g-2-q}(H^{(+-)})^{2q}
\]
where $c_q$ are constants which can be easily computed. This is easily seen to be a tree-level contribution since the $\lambda_8^{3(4g+2)/3}$ dependence differs from the $g$-loop $\lambda_8^{(2g-2)/3}$ dependence of (3.2) by a factor of $\lambda_8^{-2g}$.

To evaluate the contribution of the second sum $\sum_{m^1} \sum_{m^2,m^3}'$, it is convenient to perform a Poisson resummation on $m_1$. Following [7], this can be done by writing

$$
\sum_{m^1} \sum_{m^2,m^3}' (m^\alpha C_\alpha^{(jk)} \bar{u}_j^L \bar{u}_k^R \lambda_8^{-2g} (m^\alpha C_\alpha^{(jk)} C_\beta^{(jk)} m^\beta)^{-3g+\frac{3}{2}}
$$

$$
= \frac{\pi^{3g-\frac{3}{2}}}{\Gamma(3g-\frac{3}{2})} \int_0^\infty \frac{dt}{t^{3g-\frac{3}{2}}} \sum_{m^1} \sum_{m^2,m^3}' (m^\alpha C_\alpha^{(jk)} \bar{u}_j^L \bar{u}_k^R \lambda_8^{-2g} (m^\alpha C_\alpha^{(jk)} C_\beta^{(jk)} m^\beta)^{-3g+\frac{3}{2}}
$$

$$
= \frac{\pi^{3g-\frac{3}{2}}}{\Gamma(3g-\frac{3}{2})} \int_0^\infty \frac{dt}{t^{3g-\frac{3}{2}}} \sum_{m^1} \sum_{m^2,m^3}' (m^\alpha C_\alpha^{(jk)} \bar{u}_j^L \bar{u}_k^R \lambda_8^{-2g} (m^\alpha C_\alpha^{(jk)} C_\beta^{(jk)} m^\beta)^{-3g+\frac{3}{2}}
$$

$$
\left(\frac{1}{2\pi i} \frac{d}{dn} C_1^{(jk)} \bar{u}_j^L \bar{u}_k^R + m^Y C_Y^{(jk)} \bar{u}_j^L \bar{u}_k^R \lambda_8^{-2g} (m^\alpha C_\alpha^{(jk)} C_\beta^{(jk)} m^\beta)^{-3g+\frac{3}{2}}
$$

$$
\left. \exp\left(-\frac{\pi}{t} m^Y C_Y^{(jk)} C_Z^{(jk)} m^Z - \pi t \lambda_8^{-4/3} n^2 - 2\pi i n (m_2 \tau_1 + m_3 (B_R + \tau_1 \sigma_1)) \right) \right). \quad (4.14)
$$

where $Y, Z = 2$ or 3.

Splitting (4.14) into the $n = 0$ and $n \neq 0$ parts, it is straightforward to show that when $n \neq 0$, the contribution to $S$ is of order $O(e^{-\tau_2})$ and is therefore non-perturbative. The contribution to $S$ when $n = 0$ is proportional to

$$
S = \int d^8 x \int du^L \int du^R \sqrt{det g} \lambda_8^{-2/3} R^4 \hat{M}^{Ag-4} \quad (4.15)
$$
\[
\sum_{m^2,m^3} \left( m^Y C_{Y}^{(+)} \bar{u}_+^L u_+^R + m^Y C_{Y}^{(-)} \bar{u}_-^L u_-^R \right)^{4g-4} (m^Y C_{Y}^{(+)} C_{Z}^{(-)} m^Z)^{-3g+2}.
\]

Comparing \( C_Y^{(\pm)} \) with \( c_+^\pm \), it is easy to check that (4.13) agrees with (3.10) if one ignores the NS-NS three-form \( H^{(+-)}_{\mu\nu\rho} \) and keeps the R-R three-forms. So the conjecture precisely reproduces the \( g \)-loop computation of (3.10) and the only other perturbative contribution to \( R^4 H^{4g-4} \) terms is at tree-level.

There are at least two arguments why a perturbative non-renormalization theorem for \( R^4 H^{4g-4} \) terms would not be surprising. One argument comes from superstring amplitude computations which imply by \( U(1) \) conservation that \( R^4 (H^{(++)}_{\mu\nu\rho})^{4g-4} \) terms cannot receive corrections below genus \( g \). This does not disagree with (4.4) since (4.4) predicts tree-level contributions only if there are an equal number of \( H^{(++)}_{\mu\nu\rho} \) and \( H^{(---)}_{\mu\nu\rho} \) fields.

The second argument for a non-renormalization theorem comes from the structure of the duality group and the perturbative decoupling of Ramond-Ramond zero modes. Since the duality group is \( SL(3, Z) \times SL(2, Z) \), it is reasonable to assume that any duality-invariant amplitude is proportional to the factorized product \( f(T)g(U) \) where \( T \) are the \( SL(2, R)/SO(2) \) moduli and \( U \) are the \( SL(3, R)/SU(2) \) moduli. Since the \( g \)-loop Type IIB \( R^4 H^{4g-4} \) amplitude is independent of the \( T \) moduli, \( f(T) = 1 \). So the full non-perturbative amplitude only depends on the \( U \) moduli, which include the two Ramond-Ramond scalars and the string coupling constant. Although only proven for \( R^4 \) terms \( ^2 \), it seems probable that any \( SL(3,Z) \)-invariant expression which is perturbatively independent of the Ramond-Ramond moduli contains only a finite number of perturbative contributions.

5. **Paradox for Type IIA \( R^4 F^{4g-4} \) Term**

In the \( g \)-loop topological computation, Type IIA \( R^4 F^{4g-4} \) terms have precisely the same structure as Type IIB \( R^4 H^{4g-4} \) terms. Nevertheless, we have been unable to find a natural \( SL(3, Z) \times SL(2, Z) \)-invariant extension of (3.10) except when \( g = 1 \). When \( g = 1 \), (3.10) only depends on the Kahler moduli and is independent of \( \lambda_8 \) and the complex moduli. Therefore, the genus 1 expression in (3.10) is already \( SL(3, Z) \times SL(2, Z) \)-invariant and needs no modification. But for \( g > 1 \), (3.10) depends on \( \lambda_8 \) which mixes with the complex moduli under \( SL(3, Z) \) transformations.

\footnote{Of course, the amplitude could be proportional to a sum of such products \( \sum_i f_i(T)g_i(U) \), but in this case, each term in the sum would have to be separately invariant under \( SL(3, Z) \times SL(2, Z) \) transformations.}
Note that N=2 D=8 supersymmetry only implies decoupling of \( T \) and \( U \) moduli for terms involving eight derivatives (such as \( R^4 \) terms) but does not imply decoupling for terms with twelve derivatives or more. This is because eight-derivative superspace actions \([2]\) must be of the form \( \int d^8x (D_+)^8(D_-)^8 f(W) \) or \( \int d^8x \int du (D_+)^8(\bar{D}_+)^8 g(L_{++++}) \) where \( D_\pm \) and \( \bar{D}_\pm \) are N=2 D=8 supersymmetric derivatives, \( W \) is a chiral superfield whose lowest component is the \( T \) modulus, and \( L_{jklm} \) is a linear superfield whose lowest components are the \( U \) moduli. But twelve-derivative superspace actions can be of the form \( \int d^8x \int du (D_+)^8(D_-)^8(\bar{D}_+)^8 f(W)g(L_{++++}) \).

So using the notation of the previous subsection, the Type IIA \( R^4 F^{4g-4} \) term is multiplied by \( f(T)g(U) \) where \( f(T) \) is the \( SL(2, \mathbb{Z}) \)-invariant function given by the \( g \)-loop topological computation and

\[
g(U) = \lambda_8^{(2g-2)/3}(1 + h(\lambda_8, \rho, \tau_1, B_R))
\]

is some \( SL(3, \mathbb{Z}) \)-invariant function. In order that the \( R^4 F^{4g-4} \) term does not blow up in the ten-dimensional limit, \( h(\lambda_8, \rho, \tau_1, B_R) \) must go to zero as \( \lambda_8 \to 0 \). This implies that the eight-dimensional \( R^4 F^{4g-4} \) term gets no corrections below genus \( g \), as expected from \( U(1) \) conservation in the superstring computation.

By taking the \( \sigma_2 \to \infty \) limit where \( \lambda_8 = \sigma_2^{-1/2} e^{\phi} \), one finds that \( h \) does not contribute so the complete ten-dimensional \( R^4 F^{4g-4} \) term is given by

\[
S = N_g \int d^8x \int du^L du^R \sqrt{det_8g} \sigma_2^{(2g+1)/3} e^{(2g-2)\phi} R^4 \tilde{M}^{4g-4} \sum_{m^1 \neq 0} (m^1)^{-2g}
\]

where \( \tilde{M}_{ab} = (u^L_+ u^R_+ + u^L_- u^R_-)\Gamma_{ab}^{\mu\nu} F_{\mu\nu} \) and we are ignoring the Ramond-Ramond two-form coming from dimensional reduction of the D=10 four-form. Replacing all contracted eight-component vector indices with contracted ten-component vector indices and rescaling to ten-dimensional string gauge, one obtains the effective action

\[
S = N_g \int d^{10}x \int du^L du^R \sqrt{det_{10}g} \quad e^{(6g-6)\phi} R^4 \tilde{M}^{4g-4}(\bar{u}^L_+ u^R_+ + \bar{u}^L_- u^R_-)^{4g-4} \sum_{m^1 \neq 0} (m^1)^{-2g}
\]

\[
= N_g \int d^{10}x \sqrt{det_{10}g} \quad e^{(6g-6)\phi} R^4 F^{4g-4} \sum_{m^1 \neq 0} (m^1)^{-2g}
\]
up to an overall normalization factor.

If the M-theory conjecture is correct, this term should come from dimensional reduction of an eleven-dimensional term compactified on a circle of radius \( r = e^{2\phi/3} \) where the gauge field \( A_\mu \) is identified with \( g_{\mu 10}/g_{1010} \). As one scales \( r \),

\[
\sqrt{\text{det}g_{10}} \to r^{5}\sqrt{\text{det}g_{10}}, \quad R^4 \to r^{-4}R^4, \quad \hat{M}^{4g-4} \to r^{8-8g},
\]

so (5.3) scales like \( r^9 \), i.e. it blows up faster than the circle radius when \( g > 1 \) and therefore naively violates the conjecture.

One possible resolution of this paradox is that the Type IIA \( R^4F^{4g-4} \) term comes from a non-local term in the eleven-dimensional action, similar to momentum-dependent Type IIA \( R^4 \) terms. As discussed in [8], the one-loop four-graviton scattering amplitude in eleven-dimensional supergravity gets contributions from a local \( R^4 \) term and from a non-local \( s^{3/2}R^4 \) term where \( s = p_1 \cdot p_2 \) is a Mandelstam variable. After compactification on a circle of radius \( r \), the non-local term gives rise to an infinite sum of local Type IIA terms \( \sum_{k=2}^{\infty} c_k r^{2k-2} s^k R^4 \), each of which blows up faster than \( r \). Perhaps the genus \( g \) Type IIA \( R^4F^{4g-4} \) term is the first term of an infinite sum of terms, \( \sum_{k=0}^{\infty} c_k r^{g+2k} s^k R^4 F^{4g-4} \), which sums up to a non-local eleven-dimensional term in the limit \( r \to \infty \). Note that the term proportional to \( s^k \) would come from a genus \( g + k \) Type IIA term.

6. String “pair creation”: A Stringy extension of Schwinger’s Computation

As we have discussed in section 2, there are some parallels between the superpotential \( R^2F^{2g-2} \) terms obtained in the context of Calabi-Yau threefold compactifications and the \( R^4H^{4g-4} \) terms on which we have concentrated in this paper. In the context of Calabi-Yau threefolds, the \( f_g R^2F^{2g-2} \) terms were used in [16] [17] [18] (see [19] for a recent discussion) to check Strominger’s conjecture about the resolution of the conifold singularity by a light wrapped D3 brane [20]. In particular, if one considers giving a vev to \( F = \lambda \) and computes the \( R^2 \) term in the four-dimensional effective action, one gets

\[
S = \int d^4x \sqrt{\text{det}g} \ R^2 f(\lambda)
\]

5 We would like to thank Michael Green for suggesting that the paradox might be resolved by summing an infinite series of terms.
where

\[ f(\lambda) = \sum_{g} f_{g} \lambda^{2g-2}. \]

Moreover in the case of the conifold, the function \( f(\lambda) \) was related in [18] to the function computed by Schwinger for corrections to the effective action for a charged scalar in the presence of constant \( E, B \) fields. This has the interpretation of a one loop computation, as in [18], where the light charged wrapped D3 brane goes around the loop. The existence of \( R^{2} \) (instead of 1) reflects the fact that this case has two more D=4 supersymmetries than the problem considered by Schwinger. Certain non-perturbative aspects of this in connection with pair creation have been discussed in [19]. In particular, it was argued that the above expansion of \( f \) as a power series in \( \lambda \) should be viewed as an asymptotic expansion and that there would be corrections of the form \( \exp(-\frac{1}{\lambda}) \). In fact, such corrections are exactly captured by Schwinger’s computation (with a Euclidean circle instanton giving these kind of corrections).

It is natural to ask if there is a parallel situation for the context we are considering in this paper. The obvious guess is to give vev to \( H = h \) fields and consider contributions to the action of the form

\[ S = \int d^{10}x \sqrt{\text{det}_{10}g} \ R^{4} f(h) \]

where

\[ f(h) = \sum_{g} f_{g} h^{4g-4} \]

and \( f_{g} R^{4} H^{4g-4} \) are the corrections we have considered in this paper. In this case, we expect strings to be created (at least virtually) and to give corrections to \( R^{4} \). We can turn on different types of \( H \)’s, and for a generic choice of vevs for \( H_{NS-NS} \) and \( H_{R-R} \), we should expect production of all \((p,q)\) strings.

One can ask if \( f(h) \) (and its non-perturbative extensions) can be computed in a similar manner as was done in the case of Schwinger’s problem. If we could compute \( f(h) \) in a different way from perturbative superstring computations, as was done in the case of \( R^{2} F^{2} g^{-2} \) terms near a conifold, we would be able to fix the overall normalization for each \( f_{g} \), which we have not fixed in this paper. It would also combine our conjectures for all the different \( g \)’s into a single conjecture.

For concreteness, let us consider corrections involving \( H_{R-R} \) in Type IIB on \( R^{10} \). If we turn on a constant \( H_{R-R} \), say in the \( H_{012} = h \) direction, we expect that virtual D-strings would be relevant for computation of \( R^{4} \) corrections. In analogy with the conifold
problem, we could consider a limit where the D-string becomes light, which happens at strong coupling of the Type IIB string. In this limit, by the $SL(2, \mathbb{Z})$ symmetry of Type IIB, we can view this from the viewpoint of the dual D-string which now plays the role of the fundamental string, and for which $H_{R-R}$ is now mapped to $H_{NS-NS}$. In other words, we are back to perturbative Type IIB computation of $R^4$ terms in the presence of constant $H_{NS-NS}$. This, according to our conjecture, gets infinitely many contributions at genus 0 (one for each $H^{4g-4}$ term) and one contribution for each genus ($H^{4g-4}$ correction at genus $g$). And it is as difficult as the original problem. So we see that the stringy analog of Schwinger’s computation seems intrinsically stringy and, unlike the conifold case, we seem not to find a simpler problem to map it to.

There is, however, one statement we can make. The process of nucleating strings from constant $H_{NS-NS}$ background has been considered in [21]. In particular, a Euclidean spherical instanton was constructed with action proportional to $1/h^2$, on the basis of which it was concluded that the rate of production of strings should go as $\exp(-A/h^2)$. Even though perturbative corrections to the action were not considered in [21], their result combined with our conjectures implies there should be a function $f(h_+, h_-, \tau)$ whose asymptotic expansion for small $h_+, h_-$ gives the conjecture we have stated in the introduction, and that there should be corrections of the form $\exp(-A/h^2)$ completing this function away from $h \sim 0$.

It would be interesting to develop other ways to compute the overall coefficients of the $R^4H^{4g-4}$ terms so as to sum up this series for different $H$’s. This should teach us something non-trivial about the creation of $(p, q)$ strings in background $H$ fields.

7. Concluding Remarks

In this paper, we have conjectured the non-perturbative structure of $R^4H^{4g-4}$ terms in the Type IIB low-energy effective action. The most important evidence for our conjecture is agreement with explicit $g$-loop superstring computations.

In string theory, the NS-NS $b_{\mu\nu}$ two-form usually appears with the graviton in the combination $g_{\mu\nu} + b_{\mu\nu}$. This suggests that $R^4H^{4g-4}$ terms might be related by supersymmetry to terms such as $\nabla^{4g-4}R^4$ and $R^{2g+2}$ which contain the same number of derivatives but are composed only of graviton fields. If related, our conjecture would imply that all such terms are also multiplied by the Eisenstein function $E_{g+\frac{1}{2}}(\tau)$ in the Type IIB low-energy effective action. This might relate our conjecture with at least two other conjectures in the literature.
Based on the known tree-level $\nabla^{4g-4}R^4$ term in the Type IIB effective action, Russo conjectured that the non-perturbative $\nabla^{4g-4}R^4$ is multiplied by precisely the same $E_{g+1}^{1/2}(\tau)$ function. This is gratifying since support for his conjecture (tree-level computations) comes from a completely different source than the support for our conjecture ($g$-loop computations).

Also, if $R^{2g+2}$ and $R^4H^{4g-4}$ terms are related by supersymmetry, our results may be useful for understanding certain M(atrix) model computations. These M(atrix) model computations suggest that the effective action of the open superstring contains $F^{2g+2}$ terms which can be ‘topologically’ computed at genus $g$ since they contain trivial $\alpha'$ dependence. Since the open superstring vertex operator for $F_{\mu\nu}$ is the “square-root” of the Type IIB vertex operator for $R_{\mu\nu\rho\sigma}$, it seems plausible that the topological nature of $g$-loop open superstring $F^{2g+2}$ terms are related to the topological nature of $g$-loop Type IIB $R^{2g+2}$ terms.

Acknowledgements: We would like to thank Michael Green, Hirosi Ooguri, Jorge Russo and Andrew Strominger for useful discussions. NB would also like to thank Harvard University for its hospitality and CNPq grant 300256/94-9 for partial financial support. The research of CV was supported in part by NSF grant PHY-92-18167.
References

[1] D.J. Gross and E. Witten, *Superstring Modifications of Einstein’s Equations*, Nucl. Phys. B277 (1986) 1; N. Sakai and Y. Tanii, Nucl. Phys. B287 (1987) 457.

[2] N. Berkovits, *Construction of $R^4$ Terms in N=2 D=8 Superspace*, Nucl. Phys. B514 (1998) 191, [hep-th/9709116](http://arxiv.org/abs/hep-th/9709116).

[3] R. Jengo and C.-J. Zhu, *Two Loop Computation of the Four Particle Amplitude in Heterotic String Theory*, Phys. Lett. B212 (1988) 313.

[4] M.B. Green and M. Gutperle, *Effects of D-instantons*, Nucl. Phys. B498 (1997) 195, [hep-th/9701093](http://arxiv.org/abs/hep-th/9701093).

[5] I. Antoniadis, B. Pioline and T.R. Taylor, *Calcukable exp(1/λ) Effects*, [hep-th/9707222](http://arxiv.org/abs/hep-th/9707222).

[6] M.B. Green and P. Vanhove, *D-Instantons, Strings and M-theory*, Phys. Lett. B408 (1997) 122, [hep-th/9704145](http://arxiv.org/abs/hep-th/9704145); M.B. Green and M. Gutperle and P. Vanhove, *One Loop in Eleven Dimensions*, Phys. Lett. B409 (1997) 177, [hep-th/9706175](http://arxiv.org/abs/hep-th/9706175).

[7] E. Kiritsis and B. Pioline, *On $R^4$ Threshold Corrections in IIB String Theory and (p,q) String Instantons*, Nucl. Phys. B508 (1997) 509, [hep-th/9707018](http://arxiv.org/abs/hep-th/9707018).

[8] J.G. Russo and A.A. Tseytlin, *One Loop Four Graviton Amplitude in Eleven-Dimensional Supergravity*, Nucl. Phys. B508 (1997) 245, [hep-th/9707134](http://arxiv.org/abs/hep-th/9707134).

[9] J. Russo, *An Ansatz for a Non-perturbative Four Graviton Amplitude in Type IIB Superstring Theory*, [hep-th/9707241](http://arxiv.org/abs/hep-th/9707241).

[10] N. Berkovits and C. Vafa, *N=4 Topological Strings*, Nucl. Phys. B433 (1995) 123, [hep-th/9407190](http://arxiv.org/abs/hep-th/9407190).

[11] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic Anomalies in Topological Field Theories*, Nucl. Phys. B405 (1993) 279, [hep-th/9302103](http://arxiv.org/abs/hep-th/9302103); I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological Amplitudes in String Theory*, Nucl. Phys. B413 (1994) 162, [hep-th/9307158](http://arxiv.org/abs/hep-th/9307158); M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes*, Comm. Math. Phys. 165 (1994) 311, [hep-th/9309140](http://arxiv.org/abs/hep-th/9309140).

[12] H. Ooguri and C. Vafa, *All Loop N=2 String Amplitudes*, Nucl. Phys. B451 (1995) 121, [hep-th/9505183](http://arxiv.org/abs/hep-th/9505183).

[13] K. Becker and M. Becker, *A Two-Loop Test of M(atrix) Theory*, Nucl. Phys. B506 (1997) 48, [hep-th/9705091](http://arxiv.org/abs/hep-th/9705091); M. Douglas, private communication; H. Ooguri, private communication; K. Becker, M. Becker, J. Polchinski and A. Tseytlin, *Higher Order Graviton Scattering in M(atrix) Theory*, Phys. Rev. D56 (1997) 3174, [hep-th/9706072](http://arxiv.org/abs/hep-th/9706072).
[14] A. Galperin, E. Ivanov, S. Kalitzyn, V. Ogievetskii and E. Sokatchev, *Unconstrained N=2 Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace*, Class. Quant. Grav. 1 (1984) 469.

[15] A. Kehagias and H. Partouche, *On the Exact Quartic Effective Action for the Type IIB Superstring*, hep-th/9710023.
M.B. Green, M. Gutperle and H. Kwon, *Sixteen-Fermion and Related Terms in M-Theory on $T^2$*, hep-th/9710151.
A. Kehagias and H. Partouche, *D-Instanton Corrections as $(p,q)$-String Effects and Non-Renormalization Theorems*, hep-th/9712164.

[16] C. Vafa, *A Stringy Test of the Fate of the Conifold*, Nucl. Phys. B447 (1995) 252, hep-th/9505023.

[17] D. Ghoshal, C. Vafa, *c=1 String as the Topological Theory of the Conifold*, Nucl. Phys. B453 (1995) 121.

[18] I. Antoniadis, E. Gava, K.S. Narain, T.R. Taylor, *N=2 Type II- Heterotic duality and Higher derivative F-terms*, Nucl. Phys. B455 (1995) 109.

[19] R. Gopakumar and C. Vafa, *Topological Gravity as Large N Topological Gauge Theory*, hep-th/9802016.

[20] A. Strominger, *Massless Black Holes and Conifolds in String Theory*, Nucl. Phys. B451 (1995) 96.

[21] H. F. Dowker, J. P. Gauntlett, G. W. Gibbons, G. T. Horowitz, *Nucleation of P-Branes and Fundamental Strings*, Phys. Rev. D53 (1996) 7115, hep-th/9512154.