Theory of Diffusion Controlled Growth

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We present a new theoretical framework for Diffusion Limited Aggregation and associated Dielectric Breakdown Models in two dimensions. Key steps are understanding how these models interrelate when the ultra-violet cut-off strategy is changed, the analogy with turbulence and the use of logarithmic field variables. Within the simplest, Gaussian, truncation of mode-mode coupling, all properties can be calculated. The agreement with prior knowledge from simulations is encouraging, and a new superuniversality of the tip scaling exponent is both predicted and confirmed.

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Diffusion Limited Aggregation (DLA) has accumulated an enormous literature since Witten and Sander first introduced their simulation model of a rigid cluster growing by the accretion of dilute diffusing particles. The importance of the model is that it encompasses a range of problems where growth or interfacial advance is governed by a conserved gradient flux, that is the local interfacial velocity is given by

\[ v_n \propto |\partial_n \phi|^\eta, \quad \nabla^2 \phi = 0, \quad \phi_{\text{interface}} \approx 0 \tag{1} \]

where for DLA \( \eta = 1 \). The generalisation to a range of positive \( \eta \) was introduced by Niemeyer, Pietronero and Wiesmann to model dielectric breakdown patterns, but in this letter we exploit it to support proposed equivalences between models with significantly different ultra-violet cut-off mechanism. Theoretical interest has been fuelled by the fractal and multifractal scaling properties of the clusters produced, with controversial claims and counter-claims of anomalous scaling, and by the longstanding absence here resolved of an overall theoretical framework to understand the problem.

The presence of a cut-off lengthscale \( a \) below which the physics dictates smooth growth is a crucial ingredient of DLA; it is known that otherwise infinitely sharp cusps develop in the interface within finite time. In DLA this cutoff is fixed and set by the size of accreting particles, but there are other problems where it is set in a more subtle dynamical way by the surface boundary conditions on the diffusion field. In dendritic solidification this comes about through competition between surface energy and diffusion kinetics (with \( \eta = 1 \)), leading to

\[ a \propto |\partial_n \phi|^{-m} \tag{2} \]

with \( m = 1/2 \) at least for those tips not in retreat. In terms of \( m \), simple DLA corresponds to \( m = 0 \), and in the theory below in two dimensions we will map onto the case where \( a \) is such that each growing tip has fixed integrated flux, corresponding to \( m = 1 \).

It is central to fractal (and multifractal) behaviour in DLA that the measure given by the diffusion flux \( |\partial_n \phi| \) onto the interface has singularities, such that the integrated flux onto the growth within distance \( r \) of a singular point is given by

\[ \mu(r) \sim r^\alpha. \tag{3} \]

Applying this phenomenology to the scaling around growing tips, we can establish an equivalence between models at different \( \eta \) and \( m \) by requiring that the relative advance rates of different growing tips are matched. Consider two competing tips labelled 1, 2, for two growths with the same overall geometry but growing governed by parameters \( \eta, m \) and \( \eta', m' \) respectively. For tip 1 we will have tip radius \( a_1 \) and flux density \( j_1 \) which are matched between the two different models by

\[ a_1^{d-1}/a_1^\alpha = j_1 a_1^{d-1}/a_1^\alpha \]

and similarly for tip 2, whilst the two tips are interrelated by \( a_2' j_2'^{m'} = a_2' j_2^{m'} \) and similarly for the unprimed quantities. If we insist that their advance velocities are in the same ratio in both models this requires \( (j_1/j_2)^\eta = (j_1'/j_2')^{\eta'} \), which forces the parameter relation

\[ \frac{1 + m(1 + \alpha - d)}{\eta} = \frac{1 + m'(1 + \alpha - d)}{\eta'}. \tag{4} \]

For the two models to be equivalent in the relative velocities of all tips requires their parameters be related as above, where \( \alpha \) is the singularity exponent associated with growing tips which we take to be the same as we are matching the geometry at scales above the cut-offs.

Although we have not strictly proved the equivalence of the models related above, we have shown that any such relationship must follow Eq. (4) and we will assume in the rest of this letter that this equivalence holds. All such models are then classifiable in terms of a convenient reference such as \( \eta_0 \), the equivalent \( \eta \) when \( m = 0 \), corresponding to the original Dielectric Breakdown Model. For example dendritic solidification with \( \eta = 1 \) and \( m = 1/2 \) corresponds to \( \eta_0 = 2^{2/(\alpha - 2)} \): it is thus not equivalent to DLA, but to another member of the DBM class. Another puzzle resolved by our classification is a recent study showing conflicting scaling between DLA and different limits of a ’laplacian growth’ model. In the present terminology the latter model corresponds to \( m = -1 \) and its two limits of low and high coverage of the
for \( \eta > 0 \) (the Mullins-Sekerka instability), whereas for scales of \( \theta \) less than \( K^{-1} \) the equation drives smooth behaviour (corresponding locally to the case \( \eta = -1 \)). This cutoff on a scale of \( \theta \), the cumulative integral of flux, corresponds in terms of tip radii and flux densities to \( a_j \approx K^{-1} \), that is an \( m = 1 \) cutoff law. Thus the parameter \( \eta \) in Eq. (6) is more specifically \( \eta_\alpha = \alpha \eta_0 \), using Eq. (4) with \( d = 2 \).

We have made a numerical test of Eq. (5) and the equivalence (4), with disorder supplied only through the initial condition, by applying them to the case of growth along a channel with periodic boundary conditions (cylinder). For this case analyticity of the conformal map requires that \( z(\theta) = i\theta + \sum_{k \geq 0} z_k e^{-ik\theta} \) and the overall advance rate of the interface diverges as a power of \( \eta > 0 \) as \( K \to \infty \).

The most important result of our numerical study of Eq. (4) is that this clearly does self-organise into statistical scaling behaviour, given disorder from only the initial conditions. However the numerical results are also remarkable, as we obtain \( \alpha \approx 0.74 \pm 0.02 \) with no significant dependence on \( \eta_\alpha \) in the range studied. This not only agrees reasonably with the value \( \alpha = D_{\text{eff}} = 1 \) known from large direct simulations of DLA, but also appears to imply a deeper universality which we will see is replicated in our analytic theory below.

We now turn to a theoretical analysis of Eq. (5), for which a primary requirement is that we must obtain results explicitly independent of the cut-off as \( K \to \infty \). This is hard because we have already seen that the mean advance rate of the interface diverges as a power of \( K \), and on fractal scaling grounds one would expect the same divergence to appear in the rate of change of simple variables such as \( z^k \) or \( \psi_k \). One can of course take ratios of rates of change and look to order terms such that divergences cancel, but to make this work we have been forced to introduce yet another change of variables,

\[
-\frac{1}{\partial \theta} \exp \left( -\lambda(\theta) \right) = \exp \left( \sum_{k>0} \lambda_k e^{-ik\theta} \right),
\]

which corresponds to Fourier decomposing the logarithm of the flux density. The key to the success of these vari-
FIG. 2: Cumulative contribution to the mean growth velocity plotted against wavevector as $k^\eta_1$ with logarithmic scales. The data are (bottom to top) for $\eta_1 = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ and all exhibit a common power law slope $1/\alpha - 1 \approx 0.35 \pm 0.04$ per the guidelines shown.

ables is that they decompose the flux density itself multiplica-

tively and, as we shall see, quite naturally capture its multi-

fractional behaviour. In terms of these ‘logarithmic

variables’ the equation of motion becomes

\[
\frac{\partial \lambda_k}{\partial t} = - \sum_{j \leq k} (k - j) \lambda_{k-j} P(j) \left( e^{y(\lambda+\lambda)} \right)_j + 2k \left( e^{y(\lambda+\lambda)} \right)_k
\]

(8)

where subscripts on bracketed expressions imply the tak-

ing of a Fourier component, by analogy with $\lambda_k$. The

advance rate of the mean interface is given in these vari-

ables by $\frac{\partial \eta}{\partial t} = \left( e^{y(\lambda+\lambda)} \right)_0$.

Now let us suppose some ignorance of the initial condi-

tions and describe the system in terms of a joint proba-

bility distribution over the $\lambda_k$, and let us denote aver-

ages over this [unknown] distribution by $\langle \ldots \rangle$. We

can in principle determine the distribution through its

moments, whose evolution we now compute. For sim-

plicity we assume translational invariance with respect to

$\theta$, so that only moments of zero total wavevector need

be considered, of which the lowest gives: $\frac{\partial}{\partial t} \left( \langle \lambda_k \lambda_{\bar{k}} \rangle \right) = \left\langle - \sum_{j \leq k} (k - j) P(j) \left( \lambda_{k-j} \lambda_{\bar{k}} e^{y(\lambda+\lambda)} \right)_j + 2k \left( \lambda_{k} \lambda_{\bar{k}} e^{y(\lambda+\lambda)} \right) \right\rangle$ (c. conj.). All of the higher moments lead to the same form of averages on the RHS, $\langle \text{multinomial}(\lambda, \bar{\lambda}) e^{y(\lambda+\lambda)} \rangle$, and all of these terms are conveniently expressed in terms of cumulants $\bar{\lambda}$, using the identities $\langle X e^W \rangle / \langle e^W \rangle = \langle X e^W \rangle$ etc.

The key helpful feature is that the expressions we require all

naturally divide by one factor of $\langle e^{y(\lambda+\lambda)} \rangle = \frac{2}{\alpha} \langle \zeta_0 \rangle$, which is what we sought in order to remove divergences.

To obtain tractable results we need to introduce some

closure approximation(s) and we present here the sim-

plest, neglecting all cumulants higher than the second,
equivalent to assuming a joint Gaussian distribution (of zero

mean) for $\lambda$. This is entirely characterised by its sec-

ond moments $S(k) = \langle \lambda_k \lambda_{\bar{k}} \rangle$ which by Eq. (8) we find evolve according to $\partial S(k) / \partial (\zeta_0) = -k S(k) - y^2 k S(k)^2 - 2y^2 \sum_{j \leq k} j S(j) S(k) + 2y k S(k)$. This in turn approaches a stable steady state solution where

\[ S(k) = \frac{2y - 1}{y^2} k^{-1}, \quad k \text{ odd}; \quad S(k) = 0, \quad k \text{ even}. \] (9)

The absence of even $k$ is readily interpreted in terms of the

dominance of one major finger and one major fjord.

Within the Gaussian approximation and its predicted

variances (9) we can now compute all [static] properties

diffusion controlled growth, in a channel and (see later

discussion) also in radial geometry. The multifractal

spectrum of the harmonic measure follows from comput-
ing the general moment $\langle \langle \partial \eta / \partial t \rangle - \tau \rangle \rangle = \langle e^{(\lambda+\lambda)\tau/2} \rangle = \exp \left( \tau^2 / 4 \sum_k S(k) \right) \approx K^{q(\tau)-1-\tau}$, leading to

\[ q(\tau) = 1 + \tau + \frac{\eta_1}{2} \tau^2 \] (10)

and it is easy to see that any closure scheme based on keeping cumulants of $\lambda$ up to some finite order leads to a polynomial truncation of $q(\tau)$. From the Legendre Transform of the inverse function $\tau(q)$ we obtain the corresponding spectrum of singularities,

\[ f(\alpha) = 2 - \frac{1}{\alpha} + \frac{1}{2} \left( \eta_1 + \frac{1}{\eta_1} \right) \left( 2 - \alpha - \frac{1}{\alpha} \right) \] (11)

which in Fig. 3 is compared to measured data for DLA

[15], which later measurements [20] reinforce. For the

region of active growth $\alpha \leq 1 \ (q \geq 0)$ the theory is quan-
titatively accurate. At $\alpha = 1$ it conforms to Makarov’s

theorem [21], and in contrast to the Screened Growth

Model [22] it does this without adjustment. For $\alpha > 1$

the spectrum only qualitatively the right shape, and for

such screened regions our equations based on tip scaling

may not hold.

Although the multifractal moments depend signifi-
cantly on the input parameter $\eta_1$, the tip scaling ex-
ponent $\alpha$ turns out to be independent of this and in

close agreement with our numerical results. Matching the

expected scaling of the mean velocity (as used to mea-
sure $\alpha$ above) to that of the multifractal moment with

$\tau = -(1 + \eta_1)$ leads directly to $\alpha = 2/3$ independent

of $\eta_1$. This is a remarkable success for the Gaussian Theory to

predict this hitherto unexpected result so closely.

The multifractal spectrum suggests that the Gauss-

ian approximation is good in the growth zone, so we

have computed the penetration depth as a further test.

For growth in the channel we define relative penetra-
tion depth $\Xi$ as the standard deviation of depth $\Re(z)$

along the channel, computed over the harmonic measure,
divided by the width of the channel. This leads to
FIG. 3: Multifractal spectra from the Gaussian theory ($\alpha = 2/3$), compared to measured values for DLA. Agreement is excellent for the active region $\tau \geq 0$, $\alpha \leq 1$, and there are no adjustable parameters.

$$(2\pi \Xi)^2 = \langle z^2 \rangle / 2 = \sum_{k>0} k^{-2} \langle |e_k|^2 \rangle / 2$$

where the required averages can all be computed in the approximation of Gaussian distributed $\lambda$. Using $n_1 = 2/3$ corresponding to DLA this leads to $\Xi_{\text{theory}} = 0.13$, compared to $\Xi_{\text{DLA}} = 0.14$ from direct simulations of DLA growth in a periodic channel [\ref{17}].

All of the new theory is readily extended to growth from a point seed in radial geometry. The multifractal spectrum turns out to be unchanged, in accordance with expectations from universality. The penetration depth relative to radius gives $\Xi_{\text{theory}} = 0.20$ for radial DLA, compared to our recently published extrapolation from simulations, $\Xi_{\text{DLA}} = 0.13$ [\ref{10}].

For DLA and its associated Dielectric Breakdown Models we have shown a theoretical framework which is complete in the sense that essentially all measurable quantities can be calculated. For amplitude factors such as the relative penetration depth there is no theoretical precedent. For the full spectrum of exponents the advance over the Screened Growth Model is the elimination of fitting parameters. For the exponent $\alpha_{\text{tip}}$ we have in the Gaussian approximation a striking new result that this is independent of $\eta$, which begs direct confirmation by (expensive) particle-based simulations. However for DLA in particular we have not yet improved on the best theoretical value of $\alpha_{\text{tip}}$, which remains $1/\sqrt{2} \approx 0.71$ from the Cone Angle Approximation [\ref{14}].

Within DLA and DBM we look forward to calculating more properties such as the response to anisotropy, which is fairly readily incorporated into our equations of motion. A more challenging avenue is to improve on the Gaussian approximation which we have used to obtain explicit theoretical results. Truncating at a cumulant of higher order than the second is hard, and more seriously it does not correspond to a positive (semi-)definite probability distribution. An alternative route of improvement which we are exploring is closure at the level of the full multifractal spectrum.

There are possibilities for wider application of ideas in this letter, where we have formulated DLA and DBM as a turbulent dynamics governed by a complex scalar field in $1+1$ dimensions. Decomposing this field multiplicatively (through Fourier representation of its logarithm) was the crucial step to obtain renormalisable equations and theoretical access to the multifractal behaviour, even though other representations offered equations of motion \cite{fig3} with weaker non-linearity. It is natural to speculate whether the same strategy might apply to turbulent problems more widely, where the key issue appears to be identifying suitable fields to decompose multiplicatively which are of local physical significance, and subject to closed equations of motion.

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