Approximation and duality problems of refracted processes

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Abstract

For given two standard processes with no positive jumps, we construct, using the excursion theory, a Markov process whose positive and negative motions have the same law as the two processes. The resulting process is a generalization of Kyprianou–Loeffen’s refracted Lévy processes. We discuss approximation problem for our refracted processes coming from Lévy processes by removing small jumps and taking the limit as the removal level tends to zero. We also discuss conditions for refracted processes to have dual processes.

1 Introduction

Let $X$ and $Y$ be $\mathbb{R}$-valued standard processes with no positive jumps. We want to study a $\mathbb{R}$-valued standard process $U$ whose positive and negative motions have the same law as $X$ and $Y$, respectively. If such a process $U$ exists, we call $U$ a refracted process (of $X$ and $Y$). We want to give its precise definition in a good generality and study approximation and duality problems for the process.

As an earlier result, Kyprianou–Loeffen [9] studied a unique strong solution of the stochastic differential equation

$$U_t - U_0 = X_t - X_0 + \delta_U \int_0^t 1_{\{U_s < 0\}} ds, \quad t \geq 0,$$

for a spectrally negative Lévy process $X$ and a positive constant $\delta_U$. They called $U$ the refracted Lévy process. This process can be regarded in our terminology as a refracted process of $X$ and $Y$ with $Y_t \overset{d}{=} X_t + \delta_U t$. Noba–Yano [11] generalized Kyprianou–Loeffen’s refracted Lévy processes when $X$ and $Y$ have different Laplace exponents and $X$ has no Gaussian parts. When $X$ has unbounded variation paths and has no Gaussian parts, they used the excursion theory to construct refracted Lévy processes from the law of stopped process and the excursion measure $n_U^X$ away from 0 satisfying the following: for all non-negative measurable functional $F$,

$$n_U^X[F(U)] = n_U^X \left[ \mathbb{E}_{X_{T_0}^0} \left[ F(w \circ Y^0) \right] \bigg| \bigwedge_{w = k_{T_0}^X} X; 0 < T_0^- \leq T_0 \right], \quad (1.2)$$

where we denote by $n_U^X$ an excursion measure of $X$ away from 0, by $Y^0$ the stopped process of $Y$ at 0, by $T_0^-$ the first hitting time to $(-\infty, 0]$, by $T_0$ the first hitting time to 0, by $\bigwedge_{w = k_{T_0}^X}$ the concatenation of two càdlàg functions and by $k_{T_0}^X$ the killing operator at $T_0^-$. We always understand $\mathbb{E}_{X_{T_0}^0} \left[ F(w \circ Y^0) \right] \bigg| \bigwedge_{w = k_{T_0}^X} X = F(X)$ on $\{T_0^- = T_0\}$.

1
In this paper, we use the excursion theory to construct a refracted process $U$ from given two standard processes $X$ and $Y$ with no positive jumps. For a non-negative constant $c_0$ and a negative function $\psi$, we define the excursion measure of $U$ away from 0 in the following: for non-negative measurable functional $F$,

$$n_0^U[F(U)] = c_0 n_0^Y[F(Y); T_0^- = 0] + n_0^X \left[ \mathbb{E}^{Y_0}_{\psi(X_{T_0^-}, X_{T_0^-})} [F(\omega \circ Y^0)] \mid \omega = k_{-X^0}; 0 < T_0^- \leq T_0^- \right].$$

The resulting process $U$ constructed from $n_0^U$ via the excursion theory satisfies the following:

For $x < 0$, \(\{U_t\}_{t \leq T_0^-} (\text{under } \mathbb{P}_x^U) \overset{d}{=} \{Y_t\}_{t \leq T_0^-} (\text{under } \mathbb{P}_x^Y),\) \hfill (1.4)

For $x > 0$, \(\{U_t\}_{t < T_0^-}, U_{T_0^-} \) (under $\mathbb{P}_x^U$) $\overset{d}{=} \left( \{X_t\}_{t < T_0^-}, \psi(X_{T_0^-}, X_{T_0^-}) \right)$ (under $\mathbb{P}_x^X$). \hfill (1.5)

We call $\psi$ the landing function because it indicates the landing point at the first hitting time $T_0^-$ of $U$.

One of our main problems is approximation. We assume that our new refracted process $U$ comes from two Lévy processes $X$ and $Y$. We will then prove that $U$ is the limit in distribution on the càdlàg function space of the sequence $\{U^{(n)}\}_{n \in \mathbb{N}}$ of refracted processes coming from the drifted compound Poisson processes constructed from $X$ and $Y$ by removing small jumps and by adding drifts. Noba–Yano [11] studied this problem in the special case of no Gaussian part of $X$, where our landing functions did not appear. In our setting, our landing functions play an important role: even when $U$ has a trivial landing function $\psi(x, y) = y$, the approximating process $U^{(n)}$ must involve a suitable landing function.

The other is duality. Let $X$ and $Y$ be standard processes with no positive jumps and let $\hat{X}$ and $\hat{Y}$ be dual processes of $X$ and $Y$, respectively. Let $U$ be the refracted process of $X$ and $Y$ and let $\hat{U}$ be the refracted process of $\hat{X}$ and $\hat{Y}$. We will then obtain the necessary and sufficient condition that the refracted processes $U$ and $\hat{U}$ are in duality in terms of a certain identity involving excursion measures and landing functions. To prove duality of $U$ and $\hat{U}$, we require that their excursion measures are transformed into each other by time reversal. For this purpose, we utilize landing functions in order to adapt the jumps at the switching time between $X$ and $Y$.

We give an example of a refracted process possessing a dual. We construct it from two spectrally negative stable processes, where we will make a computation to find a suitable landing function.

The organization of the present paper is as follows. In Section 2 we propose some notation and recall preliminary facts about scale functions of standard processes with no positive jumps. In Section 3 we give the precise definition of our new refracted processes. In Section 4 we study the approximation problem. In Section 5 we give the definition of duality. In Section 6 we study the duality problem. In Section 7 we give an example of refracted processes in duality using stable processes.
2 Preliminary

Let $\mathbb{R} \cup \{\partial\}$ denote the one-point compactification of $\mathbb{R}$. Let $\mathbb{D}$ denote the set of functions $\omega : [0, \infty) \rightarrow \mathbb{R} \cup \{\partial\}$ which are càdlàg and satisfy

$$\omega(t) = \partial \quad t \geq \zeta(\omega)$$

(2.1)

where $\zeta(\omega) = \inf\{t > 0 : \omega(t) = \partial\}$. Let $\mathcal{B}(\mathbb{D})$ denote the class of Borel sets of $\mathbb{D}$ equipped with the Skorokhod topology. For $\omega \in \mathbb{D}$, denote

$$T_x^-(\omega) := \inf\{t > 0 : \omega(t) \leq x\},$$

(2.2)

$$T_x^+(\omega) := \inf\{t > 0 : \omega(t) \geq x\},$$

(2.3)

$$T_x(\omega) := \inf\{t > 0 : \omega(t) = x\}.$$  

(2.4)

For $\omega, \omega_1, \omega_2 \in \mathbb{D}$ and $s, t \in [0, \infty)$, we adopt the following notation:

$$\rho_x \omega(t) = \begin{cases} 
\omega(T_x - t), & t < T_x < \infty, \\
x, & t \geq T_x, \\
\partial, & t \geq 0, T_x(\omega) = \infty,
\end{cases}$$

(2.5)

$$k_s \omega(t) = \begin{cases} 
\omega(t), & t < \zeta(\omega), \\
\partial, & t \geq \zeta(\omega),
\end{cases}$$

(2.6)

$$\omega_1 \circ \omega_2(t) = \begin{cases} 
\omega_1(t), & t < \zeta(\omega_1), \\
\omega_2(t - \zeta(\omega_1)), & t \geq \zeta(\omega_1),
\end{cases}$$

(2.7)

$$\theta_s \omega(t) = \omega(t + s).$$

(2.8)

We have introduced in [10] the generalized scale functions of standard processes with no positive jumps. Let $T$ be an interval of $\mathbb{R}$ and set $a_0 = \sup T$ and $b_0 = \inf T$. We assume that the process $(Z, \mathbb{P}_Z)$ considered in this paper is a $T$-valued standard process with no positive jumps satisfying the following conditions:

(A1) $(x, y) \mapsto \mathbb{E}_x^Z[e^{-T_y}] > 0$ is a $\mathcal{B}(T) \times \mathcal{B}(T)$-measurable function.

(A2) $Z$ has a reference measure $m_Z$ on $T$, i.e. for $q \geq 0$ and $x \in T$, the measure $R_Z^{(q)}1_{(\cdot)}(x)$ is absolutely continuous with respect to $m_Z(\cdot)$ where

$$R_Z^{(q)} f(x) := \mathbb{E}_x^Z\left[\int_0^\infty e^{-qt} f(Z_t) dt\right]$$

(2.9)

for non-negative measurable function $f$. Here and hereafter we use the notation $\int_a^b = \int_{(b,a) \cap \mathbb{R}}$. In particular, $\int_a^a = \int_{[b,a) \cap \mathbb{R}}$.

By [5, Theorem 18.4], there exist a family of processes $\{L^x\}_{x \in T}$ with $L^x = \{L_t^x\}_{t \geq 0}$ for $x \in T$ which we call local times such that the following conditions hold: for all $q > 0$, 

\( x \in \mathbb{T} \) and non-negative measurable function \( f \)

\[
\int_0^t f(Z_s) ds = \int_T f(y) L_t^{Z,y} m_Z(dy), \quad \text{a.s.} \tag{2.10}
\]

\[
R_{Z}^{(q)} f(x) = \int_T f(y) \mathbb{E}_Z \left[ \int_0^\infty e^{-qt} dL_t^{Z,y} \right] m_Z(dy). \tag{2.11}
\]

We have the following two cases:

- **Case 1.** If \( x \in \mathbb{T} \) is regular for itself, this \( L^{Z,x} \) is the continuous local time at \( x \) \[3\] pp.216. Note that \( L^{Z,x} \) has no ambiguity of multiple constant because of (2.10) or (2.11).

- **Case 2.** If \( x \in \mathbb{T} \) is irregular for itself, we have

\[
L_t^{Z,x} = l_t^{Z,x} \# \{0 \leq s < t : Z_s = x\}, \quad \text{a.s.} \tag{2.12}
\]

for some constant \( l_t^{Z,x} \in (0, \infty) \).

In Case 1, let \( \eta^{Z,x} \) denote the inverse local time of \( L^{Z,x} \). Let \( n_t^{Z,x} \) be an excursion measure away from \( x \) which is associated with \( L^{Z,x} \) (See \[7\]). Then, for all \( q > 0 \), we have

\[
- \log \mathbb{E}_0^Z \left[ e^{-q\eta^{Z,x}(1)} \right] = \delta^{Z,x}_y + n^{Z,x}_t [1 - e^{-qT_x}] \tag{2.13}
\]

for a non-negative constant \( \delta^{Z,x}_y \) called the *stagnancy rate*. We thus have

\[
\mathbb{E}_x^Z \left[ \int_0^\infty e^{-qt} dL_t^{Z,x} \right] = \mathbb{E}_x^Z \left[ \int_0^\infty e^{-q\eta^{Z,x}(s)} ds \right] = \frac{1}{\delta^{Z,x}_y + n^{Z,x}_t [1 - e^{-qT_x}]]. \tag{2.14}
\]

In Case 2, we define \( n_t^{Z,x} = \frac{1}{l_t^{Z,x}} \mathbb{P}_x^{Z,x} \) where \( \mathbb{P}_x^{Z,x} \) denotes the law of \( Z \) started from \( x \) and stopped at \( x \). Then we have

\[
\mathbb{E}_x^Z \left[ \int_0^\infty e^{-qt} dL_t^{Z,x} \right] = l_t^{Z,x} \sum_{i=0}^\infty \left( \frac{\mathbb{E}_x^Z \left[ e^{-qT_x} \right]}{\mathbb{E}_x^Z [1 - e^{-qT_x}]} \right)^i \frac{l_t^{Z,x}}{\mathbb{E}_x^Z [1 - e^{-qT_x}]} = \frac{1}{n_x^Z [1 - e^{-qT_x}].} \tag{2.15}
\]

In \[10\] Definition 3.1, the author has introduced the \( q \)-scale function of \( Z \) as, for \( q \geq 0 \) and \( x, y \in \mathbb{T} \),

\[
W_{Z}^{(q)}(x, y) = \begin{cases} \frac{1}{n_x^Z [e^{-qT_x}; T_x^+ < \infty]}, \quad &x > y, \\ 0, \quad &x \leq y. \end{cases} \tag{2.16}
\]

Let us fix \( b, a \in \mathbb{T} \) with \( b < a \). We need the following results.

**Theorem 2.1** ([10] Theorem 3.4). For \( q \geq 0 \) and \( x \in (b, a) \), we have

\[
\mathbb{E}_x^Z \left[ e^{-qT_a^+}; T_a^+ < T_b^- \right] = \frac{W_{Z}^{(q)}(x, b)}{W_{Z}^{(q)}(a, b)}. \tag{2.17}
\]


For \( q \geq 0, \ x \in (b, a) \) and non-negative measurable function \( f \), we define
\[
\mathcal{F}^{(q,b,a)} \ f(x) := \mathbb{E}^x_z \left[ \int_{0}^{T^+_a \wedge T^+_b} e^{-qt} f(Z_t) dt \right].
\] (2.18)

Then, for \( q \geq 0 \), we have
\[
\mathcal{F}^{(q,b,a)} \ f(x) = \int_{\mathcal{T}} f(y) \mathbb{E}^x_\mathcal{T} \left[ \int_{0}^{T^+_a \wedge T^+_b} e^{-qt} dL^y_t \right] m_\mathcal{T}(dy).
\] (2.19)

**Theorem 2.2** ([10] Theorem 3.6). For \( q \geq 0 \) and \( x, y \in (b, a) \), we have
\[
\mathbb{E}^x_\mathcal{T} \left[ e^{-qt}; T^+_a \leq T^+_b \right] = \frac{W^{(q)}_\mathcal{T}(x, b) W^{(q)}_\mathcal{T}(a, y) - W^{(q)}_\mathcal{T}(x, y)}{W^{(q)}_\mathcal{T}(a, b)}.\] (2.20)

**Lemma 2.3** ([10] Lemma 3.5 and [11] Lemma 6.1). For \( q \geq 0 \) and \( x \in (b, a) \), we have
\[
\mathbb{E}^x_\mathcal{T} \left[ e^{-qt}; T^+_a \leq T^+_b \right] = n^x_\mathcal{T} \left[ e^{-qt}; T^+_a < T^+_b \right] \mathbb{E}^x_\mathcal{T} \left[ \int_{0}^{T^+_b \wedge T^+_a} e^{-qt} dL^y_t \right]
\]
\[
= \frac{n^x_\mathcal{T} \left[ e^{-qt}; T^+_a < \infty \right]}{\delta^x_q + n^x_\mathcal{T} \left[ 1 - e^{-qt} 1_{\{T^+_a = \infty, T^+_b = \infty\}} \right]}.
\] (2.21)

### 3 Refracted processes

In this section, we construct a refracted process from two \( \mathbb{R} \)-valued standard processes with no positive jumps \( X \) and \( Y \) using the excursion theory.

Let \( a_0, a_1, b_0 \) and \( b_1 \) be real numbers with \( -\infty \leq b_0 \leq b_1 < 0 < a_1 \leq a_0 \leq \infty \). Let \( \mathcal{T}_x \) be an interval with \( \sup \mathcal{T}_x = a_0 \) and \( \inf \mathcal{T}_x = b_0 \). Let \( \mathcal{T}_Y \) be an interval with \( \sup \mathcal{T}_Y = a_1 \) and \( \inf \mathcal{T}_Y = b_0 \). We let \( \mathcal{T} := \mathcal{T}_X \cup \mathcal{T}_Y \). Let \( X \) and \( Y \) be \( \mathcal{T}_X \) and \( \mathcal{T}_Y \)-valued standard processes with no positive jumps, respectively. We assume \( X \) (resp. \( Y \)) satisfying the following conditions:

(B1) \( (x, y) \to \mathbb{E}^X_x [e^{-T_x}] > 0 \) (resp. \( (x, y) \to \mathbb{E}^Y_x [e^{-T_y}] > 0 \)) is a \( B(\mathcal{T}_X) \times B(\mathcal{T}_X) \) (resp. \( B(\mathcal{T}_Y) \times B(\mathcal{T}_Y) \))-measurable function.

(B2) We assume that \( \lim_{y \to x} \mathbb{E}^Y_y [e^{-T_y}] = 1 \) for all \( x \in \mathcal{T}_X \cap (0, \infty) \) (resp. \( \lim_{y \to x} \mathbb{E}^Y_y [e^{-T_y}] = 1 \) for all \( x \in \mathcal{T}_Y \cap (-\infty, 0] \)).

(B3) If \( a_0 \notin \mathcal{T}_X \), we assume that \( \lim_{x \to a_0} \mathbb{E}^X_x [e^{-T_y}] = 0 \) for all \( y \in \mathcal{T}_X \) (resp. If \( b_0 \notin \mathcal{T}_Y \), we assume that \( \lim_{x \to b_0} \mathbb{E}^Y_x [e^{-T_y}] = 0 \) for all \( y \in \mathcal{T}_Y \)).

(B4) \( X \) (resp. \( Y \)) has a reference measure \( m_X \) on \( \mathcal{T}_X \) (resp. \( m_Y \) on \( \mathcal{T}_Y \)).
We define local times \( \{L^x_x\}_{x \in T_x} \) and \( \{L^y_y\}_{y \in T_y} \), excursion measures \( \{n^x_x\}_{x \in T_x} \) and \( \{n^y_y\}_{y \in T_y} \), and scale functions \( \{W^{(q)}_x\} \) and \( \{W^{(q)}_y\} \) of \( X \) and \( Y \) in the same way as \( Z \)'s in Section 2 respectively.

Let \( \psi : (0, \infty) \times (-\infty, 0) \to (-\infty, 0) \) be a measurable function satisfying
\[
\text{\( n^x_0 \left[ 1 - e^{-T_0} E^Y_x [e^{-T_0}] ; 0 < T_0 < T_0 \right] < \infty, \) (3.1)}
\]
where \( J_X = \psi(X_{T_0}) \). Let \( c_0 \geq 0 \) be a constant. We define the law of stopped process \( P^{(c_0)}_x \) for \( x \in T \setminus \{0\} \) and the excursion measure \( n^U_0 \) away from 0 by
\[
E^U_x [F(U^0)] = \begin{cases} 
E^X_x [f(Y^0)], & x \in T \cap (-\infty, 0), \\
E^X_x [E^{Y^0}_{J_X} [F(w \circ Y^0)] \big|_{w = k_{-x}}; T_0^- \leq T_0], & x \in T \cap (0, \infty), 
\end{cases}
\]
\[
n^U_0 [F(U)] = c_0 n^Y_0 [F(Y); T_0^- = 0] + n^0_0 \left[ E^{Y^0}_{J_X} [F(w \circ Y^0)] \big|_{w = k_{-x}}; 0 < T_0^- \leq T_0 \right] \quad (3.3)
\]
for all non-negative measurable functional \( F \) (if \( P^X_0 [T_0 > 0] = 1 \) or \( P^Y_0 [T_0 > 0] = 1 \), we assume that \( c_0 = 0 \)). We write \( X^0 \) and \( Y^0 \) for the stopped processes of \( X \) and \( Y \) upon hitting zero, respectively. By means of the excursion theory, we can construct from \( n^U_0 \) and \( \{P^{(c_0)}_x \}_{x \in T \setminus \{0\}} \), a \( T \)-valued right continuous strong Markov process without stagnancy at 0 (See, e.g., [13]).

**Remark 3.1.** The condition \( c_0 = 0 \) is necessary when \( P^X_0 [T_0 > 0] = 1 \). Indeed, when \( P^X_0 [T_0 > 0] = 1 \) and \( c_0 > 0 \), the measure \( n^U_0 \) does not satisfy the condition [13, pp.323, (vi)] and then \( n^U_0 \) is not an excursion measure.

**Lemma 3.2.** The refracted process \( U \) is a Feller process. So \( U \) is a standard process.

**Proof.** Let \( C_0 = \overline{C_0} \) denote the set of continuous functions \( f \) from \( T \) to \( \mathbb{R} \) such that \( f(x) \to 0 \) as \( x \downarrow b_0 \) when \( b_0 \notin T \) and as \( x \uparrow a_0 \) when \( a_0 \notin T \). For \( f \in C_0 \), we write \( \|f\| = \sup_{x \in \mathbb{R}} |f(x)| \). It is sufficient to verify the following conditions:

(i) For all \( q > 0 \), \( R^{(q)}_U \) is a map from \( C_0 \) to \( C_0 \).

(ii) For all \( f \in C_0 \), \( \lim_{q \uparrow \infty} \left\| q R^{(q)}_U f - f \right\| = 0 \).

1) The proof of (i)

First, we prove that \( R^{(q)}_U f \) is continuous. We let \( x \in T \). By the construction of \( U \) and (B2), it is easy to check that \( \lim_{q \uparrow x} E^U_y [e^{-qT_y}] = \lim_{q \downarrow x} E^U_x [e^{-qT_x}] = 1 \). We fix \( x \in T \). For
Therefore we have \( \lim_{x \to y} |R_U^{(q)} f(x) - R_U^{(q)} f(y)| \). 

(3.3)

For \( y > x \), we have

\[
\begin{align*}
\lim_{y \uparrow x} & \left| R_U^{(q)} f(x) - R_U^{(q)} f(y) \right| \\
\leq & \lim_{y \uparrow x} \left[ R_U^{(q)} f(x) - E_y^{e_{-qT_x}} R_U^{(q)} f(x) \right] + \lim_{y \uparrow x} \left[ E_y^U \left[ \int_0^{T_x} e^{-qt} f(U_t) dt \right] \right] = 0.
\end{align*}
\]

(3.4)

Second, we prove that \( \lim_{x \uparrow a_0} R_U^{(q)} f(x) = 0 \) when \( a_0 \notin \mathbb{T} \) and \( \lim_{x \downarrow b_0} R_U^{(q)} f(x) = 0 \) when \( b_0 \notin \mathbb{T} \). We assume that \( a_0 \notin \mathbb{T} \). By the assumption (B3), for all \( x \in (0, a_0) \),

\[
\lim_{y \uparrow a_0} E_y^U \left[ e^{-qT_x} \right] = \lim_{y \uparrow a_0} E_y^X \left[ e^{-qT_x} \right] = 0.
\]

Since \( f \in C_0 \), for all \( \epsilon > 0 \), there exists \( \delta \in (0, a_0) \) such that \( \sup_{x \in (\delta, a_0)} |f(x)| < \epsilon \). So we have

\[
\begin{align*}
\lim_{x \uparrow a_0} \left| R_U^{(q)} f(x) \right| & \leq \lim_{x \uparrow a_0} \left( \int_{0}^{T_x} e^{-qt} |f(X_t)| dt \right) + \int_{0}^{\infty} e^{-qt} \|f\| dt \\
& \leq \frac{\epsilon}{q} + \lim_{x \uparrow a_0} \left( \int_{0}^{T_x} e^{-qt} \|f\| dt \right).
\end{align*}
\]

(3.7)

(3.8)

Therefore we have \( \lim_{x \uparrow a_0} \left| R_U^{(q)} f(x) \right| = 0 \). In the same way, we have \( \lim_{x \downarrow b_0} \left| R_U^{(q)} f(x) \right| = 0 \) when \( b_0 \notin \mathbb{T} \).

2) The proof of (i)

By classical arguments, it is sufficient to prove \( \lim_{q \uparrow \infty} \left| q R_U^{(q)} f(x) - f(x) \right| = 0 \) for \( x \in \mathbb{T} \). Fix \( x \in \mathbb{T} \). For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon, \quad x, y \in \mathbb{T}.
\]

(3.9)

We define

\[
T_\delta = \inf \{ t > 0 : |U_t - x| \geq \delta \}.
\]

(3.10)

Then we have

\[
\begin{align*}
\left| q R_U^{(q)} f(x) - f(x) \right| & \leq q \mathbb{E}_x \left[ \int_{0}^{T_\delta} e^{-qt} |f(U_t) - f(x)| dt \right] + q \mathbb{E}_x \left[ \int_{T_\delta}^{\infty} e^{-qt} |f(U_t) - f(x)| dt \right] \\
& \leq \epsilon \mathbb{E}_x \left[ 1 - e^{-qT_\delta} \right] + 2 \|f\| \mathbb{E}_x \left[ e^{-qT_\delta} \right].
\end{align*}
\]

(3.11)

(3.12)

By the dominated convergence theorem, we have

\[
\limsup_{q \uparrow \infty} \left| q R_U^{(q)} f(x) - f(x) \right| \leq \epsilon.
\]

(3.13)

and so we have \( \lim_{q \uparrow \infty} \left| q R_U^{(q)} f(x) - f(x) \right| = 0 \). The proof is completed. \( \square \)
4 Approximation problem for refracted processes coming from Lévy processes

Let $U$ be the refracted process constructed by $X$, $Y$, $ψ$ and $c_0$ as Section 3. In this section, we assume that $X$ and $Y$ are spectrally negative Lévy processes and we shall construct a sequence $\{U^{(n)}\}_{n \in \mathbb{N}}$ of refracted processes coming from compound Poisson processes which converges to $U$ in distribution. In [11, Section 8], Noba–Yano studied this approximation problem only when $σ_X = 0$ and $ψ(x, y) = y$. So this section is a generalization of [11, Section 8].

(C0) Let $X$, $Y$ be spectrally negative Lévy processes which have Laplace transforms
\[
Ψ_X(λ) = χ_X λ + \frac{σ_X^2}{2} λ^2 - \int_{(−∞,0)} (1 - e^{λy} + λy 1_{(-1,0)}(y)) Π_X(dy), \quad λ ≥ 0, \quad (4.1)
\]
\[
Ψ_Y(λ) = χ_Y λ + \frac{σ_Y^2}{2} λ^2 - \int_{(−∞,0)} (1 - e^{λy} + λy 1_{(-1,0)}(y)) Π_Y(dy), \quad λ ≥ 0 \quad (4.2)
\]
for some constants $χ_X, χ_Y ∈ \mathbb{R}$, $σ_X, σ_Y ≥ 0$ and some Lévy measures $Π_X, Π_Y$, respectively. We let $Φ_X(θ) = \inf\{λ > 0 : Ψ_X(λ) > θ\}$ and $Φ_Y(θ) = \inf\{λ > 0 : Ψ_Y(λ) > θ\}$. We assume that reference measures $m_X, m_Y$ are Lebesgue measures and let the excursion measures $\{n^X_x\}_{x ∈ \mathbb{R}}$ and $\{n^Y_y\}_{y ∈ \mathbb{R}}$ of $X$ and $Y$ be those in Section 3 satisfying the following, respectively: for $x ∈ \mathbb{R}$ and $q > 0$,
\[
n^X_x[1 - e^{-qT_x}] = \frac{1}{Φ_X(q)}, \quad n^Y_y[1 - e^{-qT_y}] = \frac{1}{Φ_Y(q)}. \quad (4.3)
\]
Let $ψ$ be a continuous landing function which has the following condition:
There exist $k, l > 0$ such that $ψ(x, y) ≥ l(y - x)$, for $x − y < k$. \quad (4.4)
(Note that (4.4) implies (3.1).) Let $c_0$ be a non-negative constant such that $c_0 = 0$ when $σ_X = 0$ or $σ_Y = 0$.

(C1) Let $\{ε^n_x\}_{n ∈ \mathbb{N}}$ and $\{ε^n_y\}_{n ∈ \mathbb{N}}$ be sequences of strictly positive numbers satisfying
\[
\lim_{n ↑ ∞} ε^n_x = \lim_{n ↑ ∞} ε^n_y = 0. \quad (4.5)
\]
When $c_0 > 0$ (and consequently $σ_Xσ_Y > 0$), we assume that
\[
\lim_{n ↑ ∞} \frac{ε^n_y}{ε^n_x} = \frac{σ_Y^2}{σ_X^2} c_0. \quad (4.6)
\]
For $n ∈ \mathbb{N}$, we define

\[
Ψ_{X^{(n)}}(λ) = χ_X λ - \frac{σ_X^2}{(ε^n_x)^2} \left( 1 - e^{λ(−ε^n_x)} + λ(−ε^n_x) \right) - \int_{(−∞,−ε^n_x)} (1 - e^{λy} + λy 1_{(-1,−ε^n_x)}(y)) Π_X(dy) \quad (4.7)
\]
\[
= δ_X(λ) - \int_{(−∞,0)} (1 - e^{λy}) Π_X^{(n)}(dy) \quad (4.8)
\]
where
\[
\delta_{X^{(n)}} = \chi_X + \frac{\sigma_X^2}{\epsilon_n^2} + \int_{(-\epsilon_n, \epsilon_n)} (-y)\Pi_X(dy) \quad (4.9)
\]
\[
\Pi_{X^{(n)}} = 1_{(-\infty, -\epsilon_n]}(y)\Pi_X + \frac{\sigma_X^2}{\epsilon_n^2}\delta_{-\epsilon_n}. \quad (4.10)
\]

Let $X^{(n)}$ be a compound Poisson process with positive drift which has Laplace exponent $\Psi_{X^{(n)}}$. We let $\Phi_{X^{(n)}}$ denote the right inverse of $\Psi_{X^{(n)}}$. We note that $\Psi_{X^{(n)}}(\lambda) \to \Psi_X(\lambda)$ for all $\lambda \geq 0$, so that we have $X^{(n)} \to X$ in law on $\mathbb{D}$. Furthermore, we have $\lim_{n \to \infty} \Phi_{X^{(n)}}(\lambda) = \Phi_X(\lambda)$ for all $\lambda \geq 0$. More precisely, by [11 pp.210], we see that there exists a coupling of $X^{(n)}$’s such that $X^{(n)} \to X$ uniformly on compact intervals almost surely. We define $\Psi_{Y^{(n)}}, \delta_{Y^{(n)}}, \Pi_{Y^{(n)}}, \Phi_{Y^{(n)}}$ and $Y^{(n)}$ in the same way as those for $X$.

**Lemma 4.1.** We assume that $\sigma_Y > 0$. Then for all $q > 0$ and all bounded continuous function $f$, we have
\[
\lim_{n \to \infty} \frac{\sigma_Y^2}{\epsilon_n^2} \int_0^\infty R^{(q)}_{X^{(n)}}(y) f(-v) \, dv = n_0^Y \left[ \int_0^{T_0} e^{-qt} f(Y_t) \, dt; T_0^- = 0 \right]. \quad (4.11)
\]

**Proof.** By the definition of $\{Y^{(n)}\}_{n \in \mathbb{N}}$, we have that for all $q > 0$,
\[
\lim_{n \to \infty} n_0^{Y^{(n)}} \left[ \int_0^\infty e^{-qt} g(Y_t) \, dt \right] = \lim_{n \to \infty} R^{(q)}_{Y^{(n)}}(0) = R^{(q)}_Y(0) = \frac{n_0^Y \left[ \int_0^\infty e^{-qt} g(Y_t) \, dt \right]}{n_0^Y \left[ \int_0^\infty e^{-qt} dt \right]} \quad (4.12)
\]
and
\[
\lim_{n \to \infty} R^{(q)}_{Y^{(n)}}(u) = R^{(q)}_Y(u) \text{ for } u < 0 \text{ and for } q = f_1(-\infty, 0) \text{ or } f_1(0, \infty). \quad (4.12)
\]
By [11 Lemma 3.5] (which can easily be extended to the case of positive Gaussian component) and by $\lim_{n \to \infty} \Phi_{Y^{(n)}}(\lambda) = \Phi_Y(\lambda)$ on for all $\lambda \geq 0$, we have, for all $q > 0$,
\[
\lim_{n \to \infty} n_0^{Y^{(n)}} \left[ \int_0^\infty e^{-qt} f(Y_t) \, dt \right] = n_0^Y \left[ \int_0^\infty e^{-qt} f(Y_t) \, dt \right]. \quad (4.13)
\]
and thus by (4.12), we have $\lim_{n \to \infty} n_0^{Y^{(n)}} \left[ \int_0^\infty e^{-qt} dt \right] = n_0^Y \left[ \int_0^\infty e^{-qt} dt \right]$. Again by (4.12), we obtain
\[
\lim_{n \to \infty} n_0^{Y^{(n)}} \left[ \int_0^\infty e^{-qt} f(Y_t) \, dt \right] = n_0^Y \left[ \int_0^\infty e^{-qt} f(Y_t) \, dt \right]. \quad (4.14)
\]
By [11 Theorem 3.3] (which can easily be extended to the case of positive Gaussian component), we have
\[
\begin{align*}
&n_0^Y \left[ \int_0^{T_0} e^{-qt} f(Y_t) \, dt; T_0^- = 0 \right] = n_0^Y \left[ e^{-qT_0} \Phi^{Y^{(n)}}_{X^{(n)}} \left( \int_0^{T_0} e^{-qt} f(Y_t) \, dt \right); 0 < T_0^- < T_0 \right] \quad (4.15) \\
&= n_0^Y \left[ \int_0^{T_0} e^{-qt} f(Y_t) \, dt; T_0^- = 0 \right] + n_0^Y \left[ e^{-qT_0} \Phi^{Y^{(n)}}_{X^{(n)}} \left( \int_0^{T_0} e^{-qt} f(Y_t) \, dt \right); 0 < T_0^- < T_0 \right] \quad (4.16) \\
&= n_0^Y \left[ \int_0^{T_0} e^{-qt} f(Y_t) \, dt; T_0^- = 0 \right] + \int_0^\infty dv \int_{(-\infty,0)} R^{(q)}_{X^{(n)}}(y) e^{-qY^{(n)}} \Pi_Y(du - v). \quad (4.17)
\end{align*}
\]
Let $Y^{(n)}$ be the refracted process constructed by Theorem 4.2. By a simple argument, we can see that the left hand side of (4.22) coincides with that of (4.18).

By (4.14), (4.17), (4.20) and (4.21), we obtain

\[ R^{(q)}_{k,T_0} Y^{(n)} f(u) e^{-\Phi_{Y^{(n)}}(q) v} \Pi_Y (du - v) \]

By the same argument as that of the proof of [11, Theorem 8.4], we have

\[ \lim_{n \to \infty} \int_{(-\infty,0 \wedge (-\epsilon_n^Y + v))} R^{(q)}_{k,T_0} Y^{(n)} f(u) e^{-\Phi_{Y^{(n)}}(q) v} \Pi_Y (du - v) = \int_{(-\infty,0)} R^{(q)}_{k,T_0} Y^{(n)} f(u) e^{-\Phi_Y(q) v} \Pi_Y (du - v) \]

By (4.14), (4.17), (4.20) and (4.21), we obtain

\[ \lim_{n \to \infty} \frac{\sigma_n^Y}{\epsilon_n^Y} \int_{0}^{\epsilon_n^Y} R^{(q)}_{k,T_0} Y^{(n)} f(u - \epsilon_n^Y) e^{-\Phi_{Y^{(n)}}(q) v} dv = n_0 Y \left[ \int_{0}^{T_0} e^{-qt} f(Y_t) dt; T_0^- = 0 \right] . \]

By a simple argument, we can see that the left hand side of (4.23) coincides with that of (4.11), which leads to the desired conclusion.

(C2) Let $\{\psi^{(n)}\}_{n \in \mathbb{N}}$ be a sequence of functions satisfying

\[ \psi^{(n)}(x,y) = \psi(x,y)1_{\{x-y > \epsilon_n^X\}} - \frac{\sigma_n^Y}{\sigma_n^X} c_0 x 1_{\{x-y = \epsilon_n^X\}} \]

for all $x > 0$, $y < 0$ and $n \in \mathbb{N}$ where we understand $\frac{\sigma_n^Y}{\sigma_n^X} c_0 = 0$.

**Theorem 4.2.** Let $X^{(n)}$ and $Y^{(n)}$ be those in (C1) and let $\psi^{(n)}$ be that in (C2). Let $U^{(n)}$ be the refracted process constructed by $X^{(n)}$, $Y^{(n)}$, $\psi^{(n)}$ and $c_0 = 0$. Then, for all $q > 0$, $x \in \mathbb{R}$ and bounded continuous function $f$, we have

\[ \lim_{n \to \infty} R^{(q)}_{U^{(n)}} f(x) = R^{(q)}_{U} f(x) . \]

**Proof.** i) We prove (4.24) for $x = 0$. For this purpose we shall prove that

\[ \lim_{n \to \infty} n_0^{U^{(n)}} \left[ \int_{0}^{T_0} e^{-qt} f(U^{(n)}_t) dt \right] = n_0^{U} \left[ \int_{0}^{T_0} e^{-qt} f(U_t) dt \right] . \]
for all \( q > 0 \) and bounded continuous function \( f \). By [11, Lemma 3.5] and \( \lim_{n \to \infty} \Phi_{X(n)}(\lambda) \leq \Phi_X(\lambda) \) for all \( \lambda \geq 0 \), we have

\[
\lim_{n \to \infty} n_0^{U(n)} \left[ \int_0^{T_0^-} e^{-qt} f(U_t^{(n)}) dt \right] = \lim_{n \to \infty} n_0^{X(n)} \left[ \int_0^{T_0^-} e^{-qt} f(X_t^{(n)}) dt \right] = n_0^X \left[ \int_0^{T_0^-} e^{-qt} f(X_t) dt \right] = n_0^U \left[ \int_0^{T_0^-} e^{-qt} f(U_t) dt \right].
\] (4.26)

By the definition of \( n_0^{U(n)} \) and [11, Theorem 3.3], we have

\[
n_0^{U(n)} \left[ \int_{T_0^-}^{T_0} e^{-qt} f(U_t^{(n)}) dt \right] = n_0^{X(n)} \left[ \int_{T_0^-}^{T_0} e^{-qt} f(U_t^{(n)}) dt \right] = \int_{T_0^-}^{T_0} dv \int_{(-\infty,0)} R_{k_t^0+Y(n)}^{(q)} f(\psi(v, u)) e^{-\Phi_X(n)(q)v} \Pi_X(du - v) = \int_{T_0^-}^{T_0} dv \int_{(-\infty,0)} R_{k_t^0+Y(n)}^{(q)} f(\psi(v, u)) e^{-\Phi_X(n)(q)v} \Pi_X(du - v) + \int_{T_0^-}^{T_0} dv \int_{(\infty,0)} R_{k_t^0+Y(n)}^{(q)} f(\psi(v, u)) e^{-\Phi_X(n)(q)v} \Pi_X(du - v)
\] (4.29)

Let us compute the limit of (II). We have

\[
(II) = \int_{(-\infty,0)} \Pi_X(du) 1_{\{u < -\epsilon_n^X\}} \int_{(0)}^{\epsilon_n^X} R_{k_t^0+Y(n)}^{(q)} f(\psi(v, u + v)) e^{-\Phi_X(n)(q)v} dv
\] (4.30)

To use the dominated convergence theorem, we dominate the integrand as

\[
\left| 1_{\{u < -\epsilon_n^X\}} \int_{(0)}^{\epsilon_n^X} R_{k_t^0+Y(n)}^{(q)} f(\psi(v, u + v)) e^{-\Phi_X(n)(q)v} dv \right| \leq \|f\|_q \int_{(-\infty,0)} \left| e^{-\Phi_X(n)(q)v} \right| \Pi_X(du) + \int_{(-\infty,0)} \left| e^{-\Phi_X(n)(q)v} \right| \Pi_X(du) dv
\] (4.31)

where \( \Phi_X(n)(q) = \inf_{n \in \mathbb{N}} \Phi_{X(n)}(q) \) and \( \Phi_Y(n)(q) = \inf_{n \in \mathbb{N}} \Phi_{Y(n)}(q) \). By [4.3], we have

\[
\Phi_X(n)(q) \leq \|f\|_q \int_{(-\infty,0)} e^{-\Phi_X(n)(q)v} (1 - 1_{\{u < -k\}} e^{\Phi_Y(n)(q)} du) dv
\] (4.32)

and

\[
\Phi_Y(n)(q) \leq q \Phi_X(n)(q) (1 - 1_{\{u < -k\}} e^{\Phi_Y(n)(q)} du) \in L^1(\Pi_X). \] (4.33)
By (4.38) and the dominated convergence theorem, we have
\[
\lim_{n \uparrow \infty} \int_{(-\infty,0)} \Pi_X(du) \int_{-u}^{-\infty} R^{(q)}_{k_T Y} f(\psi(v,u+v)) e^{-\Phi_X(v)u} dv = \int_0^\infty dv \int_{(-\infty,0)} R^{(q)}_{k_T Y} f(\psi(v,u)) e^{-\Phi_X(v)u} \Pi_X(du - v). \tag{4.40}
\]

By the definition of \( n^U \) and [11, Theorem 3.3], we have
\[
\begin{align*}
(4.40) &= n \int \left[ e^{-q T_0} \mathbb{E}_v(X_{T_0}^{\infty} \cdots X_{T_0}^\sigma) \left( \int_0^{T_0} e^{-q t} f(Y_t) dt \right) ; 0 < T_0^- < T_0 \right] \\
&= n \int_0^{T_0} e^{-q t} f(U_t) dt ; 0 < T_0^- < T_0. \tag{4.41}
\end{align*}
\]

Let us compute the limit of (I). Let \( c_1 = \frac{\sigma^2}{\sigma^2} c_0. \) By the definition of \( \psi^{(n)} \), we have
\[
(I) = \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^\infty R^{(q)}_{k_T Y} f(\psi(v,1)) e^{-\Phi_X(v)u} dv. \tag{4.43}
\]

When \( c_1 = 0 \), we have (4.43) = 0. When \( c_1 > 0 \), we have
\[
\begin{align*}
\lim_{n \uparrow \infty} (4.43) &= \lim_{n \uparrow \infty} c_0 c_1 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^\infty R^{(q)}_{k_T Y} f(\psi(v,1)) dv \\
&= c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^\infty R^{(q)}_{k_T Y} f(\psi(v,1)) dv. \tag{4.44}
\end{align*}
\]

By the change of variables, we have
\[
\begin{align*}
c_0 c_1 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^\infty R^{(q)}_{k_T Y} f(\psi(v,1)) dv &= c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^{c_1 \epsilon_n^X} R^{(q)}_{k_T Y} f(\psi(v,1)) dv. \tag{4.45}
\end{align*}
\]

We prove
\[
\begin{align*}
\lim_{n \uparrow \infty} c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^{c_1 \epsilon_n^X} R^{(q)}_{k_T Y} f(\psi(v,1)) dv &= \lim_{n \uparrow \infty} c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^Y R^{(q)}_{k_T Y} f(\psi(v,1)) dv. \tag{4.46}
\end{align*}
\]

Let \( M_Y(q) = \sup_{n \in \mathbb{N}} \Phi_Y^{(n)}(q) \times (1 \vee \sup_{n \in \mathbb{N}} \frac{c_1 \epsilon_n^X}{\epsilon_n^X}). \) We have
\[
\begin{align*}
&\left| c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^{c_1 \epsilon_n^X} R^{(q)}_{k_T Y} f(\psi(v,1)) dv - c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} \int_0^Y R^{(q)}_{k_T Y} f(\psi(v,1)) dv \right| \\
&\leq c_0 \frac{\sigma^2}{(\epsilon_n^X)^2} |c_1 \epsilon_n^X - \epsilon_n^Y| \left( \sup_{0 \leq t \leq (c_1 \epsilon_n^X) \vee \epsilon_n^Y} \mathbb{E}_Y^{(n)} \left[ \int_0^{T_0} e^{-q t} dt \right] \right) \\
&\leq c_0 \frac{\sigma^2}{q} \left| \frac{c_1 \epsilon_n^X}{\epsilon_n^Y} - 1 \right| \frac{1 - e^{-M_Y(q) \epsilon_n^Y}}{\epsilon_n^Y}. \tag{4.48}
\end{align*}
\]
By the definition of $c_1$, we have
\[
\frac{c_0 \sigma_Y^2 \|f\|}{q} \left| c_1 e_n^X - \frac{1 - e^{-M_Y(q)v_n}}{e_n^X} \right| \frac{c_0 \sigma_Y^2 \|f\|}{q} \times 0 \times M_Y(q) = 0, \quad \text{as } n \uparrow \infty.
\]
(4.50)
So we have (4.46). By (4.43), (4.44), (4.45), (4.46) and Lemma 4.1, we have
\[
\lim_{n \to \infty} \frac{\sigma_Y^2}{(e_n^X)^2} \int_0^{e_n^X} R_{k^{(q)}_{T_0^+,Y(n)}} f(\psi(n)(v, v - e_n^X)) e^{-\Phi_Y(q)v} dv = c_0 n_Y \int_{T_0^-} e^{-qt} f(Y_0) dt; T_0^-, \quad (4.51)
\]
\[
=c_0 n_U \int_{T_0^-} e^{-qt} f(U_0) dt; T_0^-, \quad (4.52)
\]
\[
=n_U \int_{T_0^-} e^{-qt} f(U_0) dt; T_0^-, \quad (4.53)
\]
By (4.28), (4.32), (4.42) and (4.53), we obtain (4.25).

ii) We prove (4.24) for $x \neq 0$. For $x < 0$, we obtain (4.24) by i) and the same argument as that of the proof of [11, Theorem 8.4]. We now prove (4.24) for $x > 0$. We divide
\[
R_{U(n)}^{(q)} f(x) = E_x^U \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(U_0) dt \right] + E_x^U \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(U_0) dt \right] + E_x^U \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(U_0) dt \right]
\]
(4.54)
and we can divide $R_{U(n)}^{(q)} f(x)$ similarly. By the definition of $U$ and [8, Theorem 3.2], we have the following:
\[
E_x^U \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(U_0) dt \right] = E_x^X \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(X_0) dt \right],
\]
(4.55)
\[
E_x^U \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(U_0) dt \right] = E_x^X \left[ e^{-qT_0} R_{k^{(q)}_{T_0^+,Y(n)}} f(\psi(n)(X_{T_0^+,n}, X_{T_0^+}^-)) \right],
\]
(4.56)
\[
E_x^U \left[ \int_{T_0^-}^{T_0^+} e^{-qt} f(U_0) dt \right] = E_x^X \left[ e^{-qT_0} \Phi_Y(q) \psi(n)(X_{T_0^+,n}, X_{T_0^-}) \right] R_{U(n)}^{(q)} f(0),
\]
(4.57)
where we understand $\psi(0, 0) = 0$. We have similar identities also for $U^{(n)}$. By the dominated convergence theorem, by the uniformly convergent coupling, by [11, Lemma 8.3], i) and by $\lim_{n \to \infty} \Phi_Y(n) = \Phi_Y$, it is sufficient to prove that
\[
T_0^- (X^{(n)}) \to T_0^- (X) \quad \text{and} \quad \psi(n)(X_{T_0^-,n}^{(n)}; X_{T_0^-,n}^{(n)}; X_{T_0^-,n}^{(n)}; X_{T_0^-,n}^{(n)}) \to \psi(X_{T_0^-,n}^{-}; X_{T_0^{-}}(X)), \quad (4.58)
\]
hold as $n \uparrow \infty$ almost surely.

First, we prove (4.58) on $A := \{T_0^-(X) = \infty\} \cup \{T_0^-(X) < \infty, X_{T_0^-,n}^{-}(X) < 0\}$. By the same argument as that of the proof of [11, Theorem 8.4], we have
\[
T_0^-(X) = T_0^-(X^{(n)}) \quad \text{for large } n \text{ on } A \quad (4.59)
\]
13
and
\[
\lim_{n \to \infty} X^{(n)}_{T_0^- (X^n)_-} = X_{T_0^- (X)_-} \quad \text{and} \quad \lim_{n \to \infty} X^{(n)}_{T_0^- (X^n)} = X_{T_0^- (X)} \quad \text{on} \ A. \quad (4.60)
\]

By (4.60) and the definition of \( \psi^{(n)} \), we obtain (4.58) on \( A \).

Second, we prove (4.58) on \( A^c = \{ T_0^- (X) < \infty, X_{T_0^- (X)} = 0 \} \). Let \( \epsilon > 0 \) and let us argue on \( A^c \). Set \( I_\epsilon := [T_0^- (X) - \epsilon, T_0^- (X) + \epsilon] \) and \( \epsilon' := \left( \inf_{t \in [0, T_0^- (X) - \epsilon]} X_t \right) \wedge \left( \inf_{t \in I_\epsilon} X_t \right) \).

Then there exists \( N(\epsilon) > 0 \) such that for all \( n > N(\epsilon) \), we have
\[
\sup_{t \in [0, T_0^- (X) + \epsilon]} |X^{(n)}_t - X_t| < \epsilon'. \quad (4.61)
\]

By (4.61), (4.4) and the definition of \( \psi^{(n)} \), for \( n > N(\epsilon) \), we have
\[
T_0^- (X) - \epsilon < T_0^- (X^{(n)}) < T_0^- (X) + \epsilon, \quad (4.62)
\]
\[
\psi^{(n)}(X^{(n)}_{T_0^- (X^{(n)})_-}, X^{(n)}_{T_0^- (X^{(n))}}) < 2(l \vee c_1) \left( \sup_{t \in I_\epsilon} X_t - \inf_{t \in I_\epsilon} X_t \right). \quad (4.63)
\]

By (4.62) and (4.63), we have (4.58) on \( A^c \).

The proof is therefore completed.

**Corollary 4.3.** Under the same assumption of Theorem 4.2, the process \( (U^{(n)}, \mathbb{P}^{U^{(n)}}_x) \) converges in distribution to \( (U, \mathbb{P}^U_x) \) for all \( x \in \mathbb{R} \).

The proof of Corollary 4.3 can be obtained in the same way as that of [11, Theorem 8.1 and 8.5] using scale functions of \( U \) and \( U^{(n)} \), so we omit it.

## 5 Preliminary facts about duality

In this section, we recall the definition of duality.

Let \( \mathbb{T} \), \( Z \) and \( m_Z \) be the same as those in Section 2. We assume that the process \( (\hat{Z}, \mathbb{P}_x^\hat{Z}) \) considered in this paper is a \( \mathbb{T} \)-valued standard process with no negative jumps satisfying the following conditions:

(C1) \( (x, y) \mapsto \mathbb{E}_x^\hat{Z}[e^{-Ty}] > 0 \) is a \( \mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T}) \)-measurable function.

(C2) \( \hat{Z} \) has a reference measure \( m_Z \) on \( \mathbb{T} \).

**Definition 5.1** (See e.g., [4]). We say that \( Z \) and \( \hat{Z} \) are in duality (relative to \( m_Z \)) if for \( q > 0 \), non-negative measurable functions \( f \) and \( g \), we have
\[
\int_{\mathbb{T}} f(x) R_Z^{(q)} g(x) m_Z(dx) = \int_{\mathbb{T}} R_{\hat{Z}}^{(q)} f(x) g(x) m_Z(dx). \quad (5.1)
\]
Theorem 5.2 (See e.g., [4] or [12]). We suppose \( Z \) and \( \hat{Z} \) be in duality relative to \( m_Z \). Then, for each \( q > 0 \), there exists a function \( r^{(q)}_Z : \mathbb{T} \times \mathbb{T} \to [0, \infty) \) such that

(i) \( r^{(q)}_Z \) is \( \mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T}) \)-measurable.

(ii) \( x \mapsto r^{(q)}_Z(x, y) \) is \( q \)-excessive and finely continuous for each \( y \in \mathbb{T} \).

(iii) \( y \mapsto r^{(q)}_Z(x, y) \) is \( q \)-coexcessive and cofinly continuous for each \( x \in \mathbb{T} \).

(iv) For all non-negative function \( f \),

\[
R^{(q)}_Z f(x) = \int f(y) r^{(q)}_Z(x, y) m_Z(dy), \quad R^{(q)}_{\hat{Z}} f(y) = \int f(x) r^{(q)}_Z(x, y) m_Z(dx). \tag{5.2}
\]

By [12, Proposition of Section V.1], if \( Z \) and \( \hat{Z} \) are in duality relative to \( m_Z \), we normalize families of local times \( \{L^Z_{x, t}\}_{x \in \mathbb{T}} \) and \( \{L^{\hat{Z}}_{x, t}\}_{x \in \mathbb{T}} \) which are the same as those in Section 2 by

\[
\mathbb{E}^Z_x \left[ \int_0^\infty e^{-qt} dL^Z_{t, y} \right] = r^{(q)}_Z(x, y), \quad \mathbb{E}^\hat{Z}_y \left[ \int_0^\infty e^{-qt} dL^{\hat{Z}}_{t, x} \right] = r^{(q)}_Z(x, y). \tag{5.3}
\]

for all \( q > 0 \).

Lemma 5.3 ([6, Lemma 4.16]). We assume that \( Z \) and \( \hat{Z} \) have the following conditions:

- \( Z \) and \( \hat{Z} \) are in duality relative to \( m_Z \).
- \( Z \) and \( \hat{Z} \) are recurrent processes.
- \( Z_0 = Z_{T_x} = x \), \( n^Z_x \)-a.s. (This condition is equivalent to the counterpart of \( n^{\hat{Z}}_x \).)

Then we have

\[
n^{\hat{Z}}_x[\cdot] = n^Z_x[\rho_x(\cdot)]. \tag{5.4}
\]

When \( Z \) and \( \hat{Z} \) are in duality, we always use the local times defined by [12] Proposition of Section V.1. In other cases, we use the normalization of the local times in Section 2. We let scale functions \( \{W^{(q)}_Z\}_{q \geq 0} \) and \( \{W^{(q)}_{-\hat{Z}}\}_{q \geq 0} \) be those in Section 2. Then we have the following lemma:

Theorem 5.4 ([10, Theorem 4.5]). If \( Z \) and \( \hat{Z} \) are in duality relative to \( m_Z \), then we have

\[
W^{(q)}_Z(x, y) = W^{(q)}_{-\hat{Z}}(-y, -x), \quad x, y \in (b_0, a_0). \tag{5.5}
\]

If \( \mathbb{T} \) is open, then the converse is also true.
6 Duality problem of refracted processes

In this section, we obtain the necessary and sufficient condition that the refracted processes $U$ and $\hat{U}$ are in duality in terms of an identity involving excursion measures and landing functions.

We assume that $\mathbb{T}$ is an open set. Let $X$ and $Y$ be recurrent standard processes which are same as those in Section 3. We assume that 0 is irregular for itself for $X$ and $Y$ or 0 is regular for itself for $X$ and $Y$. Let $\hat{X}$ and $\hat{Y}$ be $\mathbb{T}_X$-valued and $\mathbb{T}_Y$-valued standard processes with no negative jumps which satisfy the following conditions:

(B1) $(x,y) \to \mathbb{E}^\hat{X}_x[e^{-T_y}] > 0$ (resp. $(x,y) \to \mathbb{E}^\hat{Y}_x[e^{-T_y}] > 0$) is a $\mathcal{B}(\mathbb{T}_X) \times \mathcal{B}(\mathbb{T}_X)$ (resp. $\mathcal{B}(\mathbb{T}_Y) \times \mathcal{B}(\mathbb{T}_Y)$)-measurable function.

(B2) We assume that $\lim_{y \uparrow x} \mathbb{E}^\hat{X}_y[e^{-T_y}] = 1$ for all $x \in \mathbb{T}_X \cap [0, \infty)$ (resp. $\lim_{y \uparrow x} \mathbb{E}^\hat{Y}_y[e^{-T_y}] = 1$ for all $x \in \mathbb{T}_Y \cap (-(\infty,0))$).

(B3) We assume that $\lim_{x \searrow 0} \mathbb{E}^\hat{X}_x[e^{-T_y}] = 0$ for all $y \in \mathbb{T}_X$ (resp. $\lim_{x \searrow 0} \mathbb{E}^\hat{Y}_x[e^{-T_y}] = 0$ for all $y \in \mathbb{T}_Y$).

(B4) $\hat{X}$ (resp. $\hat{Y}$) has a reference measure $m_X$ on $\mathbb{T}_X$ (resp. $m_Y$ on $\mathbb{T}_Y$).

In addition we assume the following conditions:

- $X_0 = X_{T_0^-} = 0$, $n^X_0$-a.s., $Y_0 = Y_{T_0^-} = 0$, $n^Y_0$-a.s.
- $X$ and $\hat{X}$ (resp. $Y$ and $\hat{Y}$) are in duality relative to $m_X$ (resp. $m_Y$).

We take the local times $\{L^X_x\}_{x \in \mathbb{T}_X}$, $\{L^Y_x\}_{x \in \mathbb{T}_Y}$, $\{\hat{L}^X_x\}_{x \in \mathbb{T}_X}$ and $\{\hat{L}^Y_x\}_{x \in \mathbb{T}_Y}$, the excursion measures $\{n^X_x\}_{x \in \mathbb{T}_X}$, $\{n^Y_x\}_{x \in \mathbb{T}_Y}$, $\{\hat{n}^X_x\}_{x \in \mathbb{T}_X}$ and $\{\hat{n}^Y_x\}_{x \in \mathbb{T}_Y}$, and the scale functions $\{W^X_{-x}\}_{q \geq 0}$, $\{W^Y_{-x}\}_{q \geq 0}$, $\{\hat{W}^X_{-x}\}_{q \geq 0}$ and $\{\hat{W}^Y_{-x}\}_{q \geq 0}$ as those in Section 3. As the landing functions, let $\psi : (0, \infty) \times (-(\infty,0) \to (0, \infty)$ be a measurable function satisfying (3.1) and $\hat{\psi} : (-(\infty,0) \times (0, \infty) \to (0, \infty)$ be a measurable function satisfying

$$n^Y_0 \left[1 - e^{-T_0^+} \mathbb{E}^X_{\psi(Y_0^+, -Y_0^+)}[e^{-T_0^+}] \right] ; 0 < T_0^+ < T_0 \right] < \infty. \tag{6.1}$$

Let $\mathbb{P}^*_x$ and $n^U_0$ be those in Section 3. By the excursion theory, we can construct a $\mathbb{T}$-valued right continuous strong Markov processes $U$ from $n^U_0$ and $\{\mathbb{P}^*_x\}_{x \in \mathbb{T} \setminus \{0\}}$. Let $\tilde{c}_0 \geq 0$ and $\tilde{c}_1 > 0$ be constants. We define the law of stopped process $\mathbb{P}^*_x$ for $x \neq 0$ and an
excursion measure \( n_0^U \) away from 0 by the following identities:

\[
\mathbb{P}_x \left[ F(\hat{U}^0) \right] = \begin{cases} \\
\mathbb{E}_x \left[ F(\hat{X}^0) \right], & x > 0, \\
\mathbb{E}_x \left[ \mathbb{E}_x \left[ F(\hat{X}^0) \right] \bigg| \omega = k_{x_0} \right], & x < 0,
\end{cases}
\]

\[ n_0^U \left[ F(\hat{U}) \right] = \hat{c}_0 n_0^\psi \left[ F(\hat{X}); T_0^+ = 0 \right] 
+ \hat{c}_1 n_0^\psi \left[ \mathbb{E}_x \left[ F(\hat{X}) \right] \bigg| \omega = k_{x_0} \right] \]

for all positive measurable functional \( F \). By the excursion theory, we can construct a \( \mathbb{T} \)-valued right continuous strong Markov processes \( \hat{U} \) from \( n_0^U \) together with \( \{\mathbb{P}_x^U\}_{x \in \mathbb{T}} \).

We may and do assume \( c_0 = \hat{c}_0 = \hat{c}_1 = 1 \) without loss of generality. Let us explain the reason. We discuss positivity of \( c_0 \). By Lemma 5.3, the excursion measures \( n_0^U \) and \( n_0^\psi \) need to satisfy \( n_0^U \cdot [\cdot]; T_0^- = 0 \) = \( c_0 n_0^Y \cdot [\cdot]; T_0^- = 0 \) = \( c_2 \hat{c}_1 n_0^\psi \cdot [\cdot]; T_0^+ = T_0 \) = \( c_2 n_0^\psi \cdot [\cdot]; T_0^+ = T_0 \). So \( c_0 \) needs to be equal to \( c_2 \hat{c}_1 \) unless \( n_0^\psi \cdot [\cdot]; T_0^+ = T_0 \) is the zero measure. When \( n_0^\psi \cdot [\cdot]; T_0^+ = T_0 \) is the zero measure, so is \( n_0^Y \cdot [\cdot]; T_0^- = 0 \) by Lemma 5.3 which allows us to take \( c_0 > 0 \). For the same reason, we may assume that \( 1 = c_2 \hat{c}_1 \). By changing the normalization of \( m_Y \), \( n_0^Y \) and \( n_0^U \), we may assume \( c_0 = c_2 = 1 \) without loss of generality, which yields \( \hat{c}_0 = \hat{c}_1 = 1 \).

We define \( m_U = m_X|_{[0, \infty)} + m_Y|_{(-\infty, 0)} \). Then we have the following theorem:

**Theorem 6.1.** If \( n_0^X, n_0^Y, \psi \) and \( \hat{\psi} \) satisfy

\[ n_0^X \left[ h(X_{T_0^-}, \psi(X_{T_0^-}, X_{T_0^-})); 0 < T_0^- < T_0 \right] = n_0^Y \left[ h(\hat{\psi}(Y_{T_0^-}, Y_{T_0^-})); 0 < T_0^- < T_0 \right] \]

for all non-negative measurable function \( h \), or equivalently,

\[ n_0^X \left[ h(\hat{X}_{T_0^+}, \psi(\hat{X}_{T_0^+}, \hat{X}_{T_0^+})); 0 < T_0^+ < T_0 \right] = n_0^Y \left[ h(\hat{\psi}(Y_{T_0^+}, \hat{Y}_{T_0^+})); 0 < T_0^+ < T_0 \right] \]

for all \( h \), then \( U \) and \( \hat{U} \) are in duality relative to \( m_U \). The converse is also true.

**Lemma 6.2.** If (6.4) is true, then we have

\[ n_0^U \cdot [\cdot] \overset{d}{=} n_0^\hat{U}(\rho_0(\cdot)) \, . \]
Proof. By (1.3) and Lemma 5.3, for non-negative measurable functional $F$, we have

\[ n_0^X [F(U); T_0^- = T_0] = n_0^X [F(X); T_0^- = T_0] \]

\[ + n_0^X \left[ \mathbb{E}_{\psi(X_{t_0^-}, X_{T_0^+})}^\rho [F(\omega \circ Y^0)] \mid w = k_{T_0} X ; 0 < T_0^- < T_0 \right] \]

\[ + n_0^Y [F(Y); T_0^- = 0] \]

\[ = n_0^X [F(\rho_0 \hat{X}); T_0^+ = 0] \]

\[ + n_0^Y \left[ \mathbb{E}_{\psi(X_{t_0^-}, X_{T_0^+})}^\rho [F(\omega \circ Y^0)] \mid w = k_{T_0} \rho_0 \theta T_0 \hat{X} ; 0 < T_0^+ < T_0 \right] \]

\[ + n_0^Y [F(\rho_0 \hat{Y}); T_0^+ = T_0] \]  \hspace{1cm} (6.7)

By (6.5), Lemma 5.3 and Fubini’s theorem, we have

\[ n_0^X \left[ \mathbb{E}_{\psi(X_{t_0^+}, X_{T_0^+})}^\rho [F(\omega \circ Y^0)] \mid w = k_{T_0} \rho_0 \theta T_0 \hat{X} ; 0 < T_0^+ < T_0 \right] \]

\[ = n_0^Y \left[ \int \mathbb{E}_{\psi(X_{t_0^+}, X_{T_0^+})}^\rho [\hat{X}^0 \in d\omega] \mathbb{E}_{\psi(X_{t_0^+}, X_{T_0^+})}^\rho [F(k_{T_0} \rho_0 \omega \circ Y^0)] ; 0 < T_0^+ < T_0 \right] \]  \hspace{1cm} (6.8)

\[ = n_0^Y \left[ \int \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [\hat{X}^0 \in d\omega] \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(k_{T_0} \rho_0 \omega \circ Y^0)] ; 0 < T_0^+ < T_0 \right] \]  \hspace{1cm} (6.9)

\[ = n_0^Y \left[ \int \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(\omega \circ Y^0)] \mathbb{P}_{\hat{X}^0} \left[ k_{T_0} \rho_0 \hat{X} \in d\omega \right] ; 0 < T_0^- < T_0 \right] \]  \hspace{1cm} (6.10)

By the strong Markov property and Lemma 5.3, we have

\[ n_0^X \left[ \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(\omega \circ Y^0)] ; 0 < T_0^- < T_0 \right] \]

\[ = n_0^Y \left[ \int F(\omega) \mathbb{P}_{\hat{X}^0} \left[ k_{T_0} \rho_0 \hat{X} \in d\omega \right] ; 0 < T_0^- < T_0 \right] \] \hspace{1cm} (6.11)

\[ = n_0^Y \left[ \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(\rho_0 \hat{X} \circ Y^0)] ; 0 < T_0^- < T_0 \right] \] \hspace{1cm} (6.12)

\[ = n_0^Y \left[ \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(\rho_0 \hat{X} \circ Y^0)] ; 0 < T_0^- < T_0 \right] \] \hspace{1cm} (6.13)

\[ = n_0^Y \left[ \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(\rho_0 \hat{X} \circ Y^0)] ; 0 < T_0^- < T_0 \right] \] \hspace{1cm} (6.14)

By (6.5), we have

\[ n_0^\hat{Y} [F(\rho_0 \hat{Y}); T_0^+ = T_0] \]

\[ + n_0^\hat{X} \left[ \mathbb{E}_{\psi(Y_{t_0^-}, Y_{T_0^+})}^\rho [F(\rho_0 \omega \circ \hat{X}^0)] \mid \omega = k_{T_0} \hat{Y} ; 0 < T_0^+ < T_0 \right] \]

\[ + n_0^\hat{X} [F(\rho_0 \hat{X}); T_0^+ = 0] \]

\[ = n_0^\hat{Y} [F(\rho_0 \hat{Y})] \]

\[ = n_0^\hat{X} [F(\rho_0 \hat{X})] \]  \hspace{1cm} (6.16)

\[ = n_0^\hat{Y} [F(\rho_0 \hat{Y})] \]  \hspace{1cm} (6.17)
So we obtain (6.16).

**Lemma 6.3.** For all \( q > 0 \) and \( x \in \mathbb{T} \), the measure \( R_U^{(q)}1_x(x) \) is absolutely continuous with respect to \( m_U(\cdot) \).

**Proof.** Let \( A \) be a set in \( \mathcal{B}(\mathbb{T}) \) which satisfies \( m_X(A \cap [0, \infty)) = 0 \) and \( m_Y(A \cap (-\infty, 0)) = 0 \). It is sufficient to prove that \( \mathbb{E}_0^U \left[ \int_0^\infty e^{-qt}1_A(U_t)dt \right] = 0 \). By the compensation theorem of excursion point processes, we have

\[
qn_0^U \left[ 1 - e^{-qT_0} \right] \mathbb{E}_0^U \left[ \int_0^\infty e^{-qt}1_A(U_t)dt \right] = n_0^U \left[ \int_0^{T_0} e^{-qt}1_A(U_t)dt \right]
\]

\[
= n_0^X \left[ \int_0^{T_0} e^{-qt}1_A(X_t)dt \right] + n_0^X \left[ \mathbb{E}_0^X \left[ \int_0^{T_0^+} e^{-qt}1_A(Y_t)dt \right] \right] + n_0^Y \left[ \int_0^{T_0} e^{-qt}1_A(Y_t)dt; T_0^- = 0 \right].
\]

By the assumption of \( A \), we have

\[
n_0^X \left[ \int_0^{T_0} e^{-qt}1_A(X_t)dt \right] = qn_0^X \left[ 1 - e^{-qT_0} \right] \mathbb{E}_0^X \left[ \int_0^\infty e^{-qt}1_A(\mathbb{R}) (X_t)dt \right] = 0,
\]

\[
\mathbb{E}_0^X \left[ \int_0^{T_0^+} e^{-qt}1_A(Y_t)dt \right] \leq \mathbb{E}_0^Y \left[ \int_0^\infty e^{-qt}1_A(\mathbb{R}^-) (Y_t)dt \right] = 0.
\]

and

\[
n_0^Y \left[ \int_0^{T_0} e^{-qt}1_A(Y_t)dt; T_0^- = 0 \right] \leq qn_0^Y \left[ 1 - e^{-qT_0} \right] \mathbb{E}_0^Y \left[ \int_0^\infty e^{-qt}1_A(\mathbb{R}^-) (Y_t)dt \right] = 0
\]

So we obtain \( \mathbb{E}_0^U \left[ \int_0^\infty e^{-qt}1_A(U_t)dt \right] = 0 \). 

We want to find suitable normalization of local times of \( U \). By [5] Theorem 18.4, we let local times \( \{L^{U,x}_{t}\}_{x \in \mathbb{T}\setminus\{0\}} \) of \( U \) be those in Section 2. We set \( n_0^{U,t} = n_0^U \) and let \( n_x^{U,t} \) for \( x \in \mathbb{T}\setminus\{0\} \) be the excursion measure associated to \( L^{U,x}_{t} \). Then there exists the positive function \( c(x) \) such that \( c(0) = 1 \) (by the definition of \( U^0 \)) and for all non-negative functional \( F \):

\[
n_x^{U,t} \left[ F(\{U_t\}_{t<T_0^-}) \right] = c(x)n_x^X \left[ F(\{X_t\}_{t<T_0^-}) \right], \quad x \in \mathbb{T} \cap [0, \infty),
\]

\[
n_x^{U,t} \left[ F(\{U_t\}_{t<T_0^+}) \right] = c(x)n_x^Y \left[ F(\{Y_t\}_{t<T_0^+}) \right], \quad x \in \mathbb{T} \cap (-\infty, 0].
\]
Then we have \( c(x) = 1 \) \( m_U \)-a.e. Indeed, for all \( q > 0, x, y \in \mathbb{T} \cap [0, \infty) \) and non-negative measurable function \( f \), we have

\[
\mathbb{E}^U_x \left[ \int_0^{T^+_0} e^{-qt} dL^U_{t}^{y} \right] = \frac{1}{c(y)} \mathbb{E}^X_x \left[ \int_0^{T^+_0} e^{-qt} dL^X_{t}^{y} \right] \tag{6.26}
\]

and

\[
\int_{\mathbb{T} \cap [0, \infty)} f(y) \mathbb{E}^U_x \left[ \int_0^{T^+_0} e^{-qt} dL^U_{t}^{y} \right] m_U(dy) = \mathbb{E}^U_x \left[ \int_0^{T^+_0} e^{-qt} f(U_t)dt \right] \tag{6.27}
\]

\[
= \mathbb{E}^X_x \left[ \int_0^{T^+_0} e^{-qt} f(X_t)dt \right] \tag{6.28}
\]

\[
= \int_{\mathbb{T} \cap [0, \infty)} f(y) \mathbb{E}^X_x \left[ \int_0^{T^+_0} e^{-qt} dL^X_{t}^{y} \right] m_U(dy). \tag{6.29}
\]

So \( c(x) = 1 \) on \( \mathbb{T} \cap [0, \infty) \) \( m_U \)-a.e. Similarly, \( c(x) = 1 \) on \( \mathbb{T} \cap (-\infty, 0) \) \( m_U \)-a.e. We now set \( L^{U,x} = c(x)L^{U,x,0} \) and \( n_x^U = \frac{1}{c(x)}n_x^{U,0} \). This local times satisfy (2.10) and (2.11) since \( c(x) = 1 \) \( m_U \)-a.e.

In the same way, let the excursion measures \( \{n^\hat{U}_x\}_{x \in \mathbb{T}} \) of \( \hat{U} \) be those in Section 2 satisfying the following conditions;

\[
n^\hat{U}_x \left[ F(\{\hat{U}_t\}_{t \leq T^+_0}) \right] = n^X_x \left[ F(\{\hat{X}_t\}_{t \leq T^+_0}) \right], \quad x \in \mathbb{T} \cap [0, \infty), \tag{6.30}
\]

\[
n^\hat{U}_x \left[ F(\{\hat{U}_t\}_{t > T^+_0}) \right] = n^Y_x \left[ F(\{\hat{Y}_t\}_{t > T^+_0}) \right], \quad x \in \mathbb{T} \cap (-\infty, 0]. \tag{6.31}
\]

We let the scale functions \( \{W^{(q)}_U\}_{q \geq 0} \) and \( \{W^{(q)}_{-U}\}_{q \geq 0} \) be those in (2.16).

**Proof of Theorem 6.4** Let us assume that we have (6.4) for all non-negative measurable function \( h \). By Theorem 5.4 and Lemma 6.3 it is sufficient to prove that

\[
W^{(q)}_U(x, y) = W^{(q)}_{-U}(-y, -x), \tag{6.32}
\]

for \( q \geq 0 \) and \( x, y \in \mathbb{T} \). For \( 0 \leq y < x \), we have

\[
W^{(q)}_U(x, y) = n^U_y \left[ e^{-qT^+_y}; T^+_x < \infty \right]^{-1} = n^X_y \left[ e^{-qT^+_y}; T^+_x < \infty \right]^{-1} = W^{(q)}_X(x, y) \tag{6.33}
\]

\[
= W^{(q)}_{-\hat{X}}(-y, -x) = n^-_{-x} \left[ e^{-qT^+_y}; T^+_y < \infty \right]^{-1} = n^-_{-x} \left[ e^{-qT^+_y}; T^+_y < \infty \right]^{-1} = W^{(q)}_{-\hat{U}}(-y, -x). \tag{6.34}
\]

by the definitions of \( n^U_y, n^-_{-x} \) and Theorem 5.4. Similarly, for \( y < x \leq 0 \), we have

\[
W^{(q)}_U(x, y) = W^{(q)}_Y(x, y) = W^{(q)}_{-\hat{Y}}(-y, -x) = W^{(q)}_{-\hat{U}}(-y, -x). \tag{6.35}
\]

20
When $y < 0 < x$, by (2.22), (2.16) and (2.17), we have

$$W_U^{(q)}(x, y) = W_U^{(q)}(0, y)W_U^{(q)}(x, 0)n_0^U \left[ 1 - e^{-qT_0}1_{\{T_y = \infty, T_x = \infty\}} \right]$$

(6.36)

and

$$W_{-U}^{(q)}(-y, -x) = W_{-U}^{(q)}(0, 0)n_0^\hat{U} \left[ 1 - e^{-qT_0}1_{\{T_y = \infty, T_x = \infty\}} \right].$$

(6.37)

By Lemma 6.2, (6.34), (6.35), (6.36) and (6.37), we obtain (6.32).

We assume that $U$ and $\hat{U}$ are in duality relative to $m_U$. By Lemma 5.3 and the definitions of $n_0^U$ and $n_0^\hat{U}$, we have (6.6). We have

$$n_0^X \left[ h(X_{T_0^-}, \psi(X_{T_0^-}, X_{T_0^+})); 0 < T_0^- < T_0 \right] = n_0^U \left[ h(U_{T_0^-}, U_{T_0^+}); 0 < T_0^- < T_0 \right]$$

(6.38)

and

$$n_0^Y \left[ h(\hat{\psi}(Y_{T_0^-}, Y_{T_0^-}), Y_{T_0^+}); 0 < T_0^- < T_0 \right] = n_0^\hat{U} \left[ h(\hat{\psi}(\hat{Y}_{T_0^-}, \hat{Y}_{T_0^-}), \hat{Y}_{T_0^+}); 0 < T_0^+ < T_0 \right]$$

(6.39)

$$= n_0^\hat{U} \left[ h(\hat{U}_{T_0^+}, \hat{U}_{T_0^+}); 0 < T_0^+ < T_0 \right].$$

(6.40)

By (6.6), (6.38) and (6.40), we obtain (6.4). The proof is completed.

7 An example of the duality problem

In this section, we construct refracted processes in duality from spectrally negative stable processes.

Let $X$ be a spectrally negative strictly $\alpha$-stable process whose Lévy measure is

$$\Pi_X(dx) = c_X1_{\{x < 0\}}|x|^{-\alpha-1}dx$$

(7.1)

for a constant $c_X > 0$, and $Y$ be a spectrally negative strictly $\beta$-stable process whose Lévy measure is

$$\Pi_Y(dx) = c_Y1_{\{x < 0\}}|x|^{-\beta-1}dx$$

(7.2)

where $c_Y > 0$. Then it is known that

$$\hat{X} \text{ (under } \mathbb{P}_x^X) \overset{d}{=} -X \text{ (under } \mathbb{P}^X_x)$$

(7.3)

and

$$\hat{Y} \text{ (under } \mathbb{P}_x^Y) \overset{d}{=} -Y \text{ (under } \mathbb{P}^Y_x).$$

(7.4)

We set reference measure $m_X(dx)$ as $\frac{1}{c_X}dx$ and reference measure $m_Y(dx)$ as $\frac{1}{c_Y}dx$. Let $n_0^X$ and $n_0^Y$ be those in Section 6. We want to find suitable landing functions such that $U$ and $\hat{U}$ are in duality. So we need to find $\psi$ and $\hat{\psi}$ satisfying (6.4).
Proposition 7.1. Suppose \( \alpha > \beta \). We let \( \psi(x, y) = y(x - y)^{\frac{1}{\beta - 1}} \) and \( \hat{\psi}(x, y) = y(y - x)^{\frac{1}{\alpha - 1}} \). Then \( U \) constructed from \( X, Y, \psi \) and \( c_0 = 0 \) and \( \hat{U} \) constructed from \( \hat{X}, \hat{Y}, \hat{\psi} \) and \( c_0 = 0 \) are well-defined and in duality relative to \( m_U \).

Proof. Let us prove (6.4). By [11, Theorem 3.3], we have

\[
n_0^X \left[ h(X_{0}^{-}, Y_{0}^{-}, X_{0}^{-}, Y_{0}^{-}); 0 < T_0^- < T_0 \right] = \frac{\alpha - 1}{c_X} \int_0^\infty dv \int_{(-\infty,0)} h(v, \psi(v, u)) \Pi_X(du - v) \tag{7.5}
\]

and

\[
n_0^Y \left[ h(\hat{\psi}(Y_{0}^{-}, Y_{0}^{-}, Y_{0}^{-}, Y_{0}^{-}); 0 < T_0^- < T_0 \right] = \frac{\beta - 1}{c_Y} \int_0^\infty dv \int_{(-\infty,0)} h(\hat{\psi}(v, u), u) \Pi_Y(du - v) \tag{7.7}
\]

We set \( s = \frac{u}{u+v}, t = u+v, t_1 = t^{-\alpha+1} \) and \( t_2 = t^{-\beta+1} \). Then we have \( u = st, v = t(1-s) \) and \( \left| \frac{\partial u}{\partial s} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial t} \frac{\partial v}{\partial s} \right| = t \). So we have

\[
\tag{7.6} = (\alpha - 1) \int_0^1 ds \int_0^\infty h(t(1-s), \psi(t(1-s), -st)) t^{-\alpha} dt
\]

\[
= \int_0^1 ds \int_0^\infty h(t_1^{-\frac{1}{\alpha-1}}(1-s), \hat{\psi}(t_1^{-\frac{1}{\alpha-1}}(1-s), -st_1^{-\frac{1}{\alpha-1}})) dt_1 \tag{7.10}
\]

and

\[
\tag{7.8} = (\beta - 1) \int_0^1 ds \int_0^\infty h(\hat{\psi}(-st, t(1-s)), -st) t^{-\beta} dt
\]

\[
= \int_0^1 ds \int_0^\infty h(\hat{\psi}(-st_2^{-\frac{1}{\beta-1}}, t_2^{-\frac{1}{\beta-1}}(1-s)), -st_2^{-\frac{1}{\beta-1}}) dt_2. \tag{7.12}
\]

Since \( \psi(x, y) = y(x - y)^{\frac{1}{\beta - 1}} \) and \( \hat{\psi}(x, y) = y(y - x)^{\frac{1}{\alpha - 1}} \), we have

\[
\psi(t^{-\frac{1}{\beta-1}}(1-s), -st^{-\frac{1}{\beta-1}}) = -st^{-\frac{1}{\beta-1}}, \quad s, t > 0 \tag{7.13}
\]

and

\[
\hat{\psi}(-st^{-\frac{1}{\alpha-1}}, t^{-\frac{1}{\alpha-1}}(1-s)) = t^{-\frac{1}{\alpha-1}}(1-s), \quad s, t > 0. \tag{7.14}
\]

By (7.10), (7.12), (7.13) and (7.14), we obtain (6.4).
Let us prove (3.1) and (6.1). Let $\Phi_X$ and $\Phi_Y$ be those in Section 4. By $[8$, Theorem 3.2$]$, we have

$$n_0^X \left[ 1 - e^{-T_0^-} \mathbb{E}^Y_{X_{T_0^-}^-} \mathbb{E}^Y_{X_{T_0^-}^-} \left[ e^{-T_0^-} \right]; 0 < T_0^- < T_0 \right]$$

(7.15)

$$= n_0^X \left[ 1 - e^{-T_0^-} e^{\Phi_Y(1) Y_{T_0^-}^-} \mathbb{E}^Y_{X_{T_0^-}^-} \left[ e^{-T_0^-} \right]; 0 < T_0^- < T_0 \right]$$

(7.16)

$$\leq n_0^X \left[ 1 - e^{-T_0^-} e^{\Phi_Y(1) X_{T_0^-}^-} \mathbb{E}^Y_{X_{T_0^-}^-} \left[ e^{-T_0^-} \right]; 0 < T_0^- < T_0, X_{T_0^-}^- < Y_{T_0^-}^- \leq 1 \right]$$

(7.17)

$$+ n_0^X \left[ 1 - e^{-T_0^-} e^{\Phi_Y(1) Y_{T_0^-}^-} \mathbb{E}^Y_{X_{T_0^-}^-} \left[ e^{-T_0^-} \right]; 0 < T_0^- < T_0, X_{T_0^-}^- < Y_{T_0^-}^-, X_{T_0^-}^- > 1 \right]$$

(7.18)

where the inequality (7.17) uses $\alpha > \beta$. There is a constant $q \geq 1$ such that $\Phi_X(q) \geq \Phi_Y(1)$. By $[8$, Theorem 3.2$]$ and the strong Markov property, we have

$$\left(7.17\right) \leq n_0^X \left[ 1 - e^{-qT_0^-} e^{\Phi_X(q) X_{T_0^-}^-} \mathbb{E}^Y_{X_{T_0^-}^-} \left[ e^{-T_0^-} \right]; 0 < T_0^- < T_0, X_{T_0^-}^- < Y_{T_0^-}^- \leq 1 \right]$$

(7.19)

By the property of excursion measures, we have

$$\left(7.18\right) \leq n_0^X \left[ X_{T_0^-}^- < Y_{T_0^-}^- \leq 1 \right] < \infty.$$ 

(7.20)

By (7.19) and (7.20), we obtain (3.1). Since we have (6.4) and (3.1), in the same way as the proof of Lemma 6.2, we obtain (6.1). So the refracted processes $U$ and $\tilde{U}$ are well-defined. By (6.4) and Theorem 6.1, the proof is completed.

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