Quantum capacity of channel with thermal noise

Xiao-yu Chen
Lab. of quantum information, China Institute of Metrology, Hangzhou, 310018, China

Abstract

The quantum capacity of thermal noise channel is studied. The extremal input state is obtained at the postulation that the coherent information is convex or concave at its vicinity. When the input energy tends to infinitive, it is verified by perturbation theory that the coherent information reaches its maximum at the product of identical thermal state input. The quantum capacity is obtained for lower noise channel and it is equal the one shot capacity.

One of the most important issues of classical information theory is the Shannon formula, which is the capacity of an additive white Gaussian noise channel. It is achieved when the input is a Gaussian noise source with power constraint \[ C = \log_2 (1 + \frac{S}{N}) \],

where \( S \) is the power of the source and \( N \) is the power of the noise, the bandwidth \( W \) should be multiplied when it is considered. The formula has guided the design of the practical communication system for decades.

Correspondingly, in quantum information theory, although a lot of works have been done \[ 3 \], such a formula is remained to be discovered. The Shannon formula comes from the Shannon noisy coding theorem, the later gives the capacity of any noisy channel:

\[ C = \sup_X I(X;Y), \]

where the supremum is taken over all inputs \( X \), \( I(X;Y) \) is the Shannon mutual information and \( Y \) is the output. The counterpart of mutual information in quantum information theory is the coherent information (CI) \( I_c(\rho,E) = S(E(\rho)) - S(\rho^{RQ}) \) \[ 4 \] \[ 5 \]. Here \( S(\varrho) = -\text{Tr} \log_2 \varrho \) is the von Neumann entropy, \( \rho \) is the input state, the application of the channel \( E \) resulting the output state \( E(\rho) \); \( \rho^{RQ} = (E \otimes I)(|\psi\rangle \langle \psi|) \), \( |\psi\rangle \) is the purification of the input state \( \rho \). The quantum channel capacity is

\[ Q = \lim_{n \to \infty} \sup_{\rho_n} \frac{1}{n} I_c(\rho_n,E^\otimes n). \]

The righthand side of above formula was firstly proved to be the upper bound of quantum channel capacity \[ 6 \]. The equality was proved at the postulation of hashing inequality \[ 3 \]; the hashing inequality was lately established \[ 7 \]. The quantum capacity of a noisy quantum channel is the maximum rate at which coherent information can be transmitted through the channel and recovered with arbitrarily good fidelity. For quantum information channel can be supplemented by one- or two-way classical channel, thus quantum capacities should be defined with these supplementary resources. We here deal with the quantum capacity without any supplementary classical channel \[ 3 \].

Quantum capacity exhibits a kind of nonadditivity \[ 8 \] that makes it extremely hard to deal with. Until now, quantum capacity has not been carried out except for quantum erasure channel \[ 9 \]. We in this paper will deal with the quantum capacity of thermal noise quantum channel (which is addressed as Gaussian quantum channel before \[ 10 \]). The general description of the channel is to map the state \( \rho \) to another state \( E(\rho) \), where \( E \) is a trace preserving completely positive map. The map \( E \) has a Krauss operator sum representation. That is \( E(\rho) = \sum_i A_i \rho A_i^\dagger \) with \( \sum_i A_i^\dagger A_i = I \). For additive quantum Gaussian channel, it is quite simple to choose \( A_\alpha \) to be proportional to the displacement operator \[ 11 \] \[ 10 \] \[ D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] \]. The output state will be

\[ E(\rho) = \frac{1}{\pi N} \int d^2 \alpha \exp(- |\alpha|^2/N) D(\alpha) \rho D^\dagger(\alpha). \]
for the simplest situation of thermal noise channel, where \( N \) is the average photon number of the output state if the input is the vacuum.

In dealing with the maximization of the CI, there is a useful lemma in classical information theory which gives necessary and sufficient conditions for the global maximum of a convex function of probability distributions in terms of the first partial derivatives. The lemma was extended to quantum information theory \[12\] in evaluating the capacities of bosonic Gaussian channels. Let \( F \) be a convex function on the set of density operators which contains \( \rho(0) \) and \( \rho \), the necessary and sufficient condition for \( F \) achieves maximum on \( \rho(0) \) is that the convex function \( F((1-t)\rho(0)+tp) \) of the real variable \( t \) achieves maximum at \( t = 0 \) for any \( \rho \). That is

\[
\frac{d}{dt}|_{t=0}F((1-t)\rho(0)+tp) \leq 0 \quad \text{Generally speaking, CI } I_c(\rho,\mathcal{E}) \text{ is not a global convex function of its input state } \rho. \text{ Without lose of generality, let us suppose it is convex } \[13\] \text{at the vicinity of some } \rho(0), \text{ the necessary and sufficient condition that } \rho(0) \text{ is the maximal state will be}
\]

\[
\frac{d}{dt}|_{t=0}I_c((1-t)\rho(0)+tp,\mathcal{E}) \leq 0 . \tag{5}
\]

where \( \rho \) is at the vicinity of \( \rho(0) \). The derivative will be

\[
-\text{Tr}(\mathcal{E}(\rho)-\mathcal{E}(\rho(0)))\log\mathcal{E}(\rho(0))+\text{Tr}(\rho^{RQ'}-\rho^{(0)})\log\rho^{(0)}.
\]

If \( \mathcal{E} \) is a trace preserving completely positive Gaussian operation, then for a gaussian input state \( \rho(0) \), the output state \( \mathcal{E}(\rho(0)) \) and the joint state \( \rho^{RQ'} \) will be Gaussian. Hence their logarithms are quadratic polynomials in the corresponding canonical variables\[12\]. The derivative will be zero under the constraints of the first and second moments. Where the trace preserving property of \( \mathcal{E} \) is also used. The conclusion is that for input states with the same first and second moments, Gaussian input state achieves the maximum of CI for a given trace preserving completely positive Gaussian channel. The same conclusion can be applied to CI \( I_c(\rho_n,\mathcal{E}^{\otimes n}) \), the maximum will be arrived when the input state is Gaussian.

We then turn to Gaussian input state. Every operators \( A \in \mathcal{B}(\mathcal{H}) \) is completely determined by its quantum characteristic function \( \chi_A(z) := \text{Tr}[AW(z)] \), where \( W(z) = \exp[-iz^TR] \) are Weyl operators and \( R = (X_1,P_1,X_2,\cdots,P_n) \), with \( \{X_k,P_l\} = i\delta_{kl} \). The density operator \( \rho_n \) is called Gaussian, if its characteristic function \( \chi_\rho(z) \) has the form \( \chi_\rho(z) = \exp(i\eta^Tz-\frac{1}{2}z^T\gamma z) \). One can show that the first moment \( \eta = \text{Tr}\rho_n R \), the second moment \( \gamma = 2\text{Tr}(R-\eta)\rho_n(R-\eta)^T+iJ_n \), where

\[
J_n = \bigoplus_{k=1}^n J, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]  

The \( 2n \times 2n \) real symmetric matrix \( \gamma \) is usually called the correlation matrix (CM) of the state \( \rho_n \). The first moment represents the displacement of the state, it is irrelevant to the problem of entanglement as well as channel capacity so that dropped. The completely positive map on the input state \( \rho_n \) will be \( \rho_n^{RQ'} = (\mathcal{E} \otimes 1)(|\psi_n\rangle \langle \psi_n|) \).

The CM of the Schmidt purification \( |\psi_n\rangle \) is \[12\]

\[
\gamma_\psi = \begin{bmatrix} \gamma & \beta \\ \beta^T & \gamma \end{bmatrix}, \tag{7}
\]

where \( \beta = -\beta^T = J_n\sqrt{-(J_n^{-1}\gamma)^2-I_{2n}} \) are purely off-diagonal. It should be noticed that the symplectic eigenvalues of \( \gamma_\psi \) are the square root of the eigenvalues of \( -(J_{2\times n}^{-1}\gamma)^2 \), where \( J_{2\times n} = J_n \oplus (-J_n) \) are chosen to produce the off-diagonal \( \beta \). The application of trace preserving Gaussian channel will result the state \( \rho_n^{RQ'} \) with CM \[13\]

\[
\gamma'_\psi = \begin{bmatrix} M_n^T\gamma M_n+N_n & M_n^T\beta \\ \beta^T M_n & \gamma \end{bmatrix}, \tag{8}
\]

where \( M_n = M_n^{\oplus n}, N_n = N_n^{\oplus n} \).

For thermal noise quantum channel, we have \( M_1 = I_2, N_1 = 2NI_2 \). Then \( \gamma'_\psi \) will be

\[
\gamma'_\psi = \begin{bmatrix} \gamma + 2NI_2 & \beta \\ \beta^T & \gamma \end{bmatrix}. \tag{9}
\]
The energy of the input state $\rho_n$ is $E_n = \sum_i (\pi_i + 1/2) = \text{Tr}\rho_n \sum_i (a_i^\dagger a_i + 1/2) = (\gamma)/4$, where we set the unit of the energy such that $h\omega = 1$. Under the energy constraint $\text{Tr}\gamma = 4E_n$, which state will achieve the maximum of the CI? We suppose the state having maximum CI is $\rho_n(0)$ with its CM $\gamma(0) = 2E\mathbf{1}_{2n}$, where $E = E_n/n$ is the average energy of each mode of input state. We need to prove that

$$-\text{Tr}((\rho'_n) - (\rho_n(0)))) \log \rho_n(0) + \text{Tr}(\rho'^{RQ}_n - \rho^{RQ}_n(0)) \log \rho^{RQ}_n(0) \leq 0,$$

(10)

where $\rho'_n = \mathcal{E}^{\otimes n}(\rho_n)$, $\rho^{\otimes n}_n = \mathcal{E}^{\otimes n}(\rho_n(0))$. The density operator $\rho_n(0)$ now is the direct product of $\rho_1(0)$, $\rho^{\otimes n}_n = \rho^{\otimes n}_1(0)$, we have $\rho'_n(0) = \rho^{\otimes n}_1(0)$ and $\rho'^{RQ}_n = \rho^{RQ}_1 \otimes \mathbf{1}_{2n}$ as well. The CM of $\rho_n(0)$ now is $\gamma_n(0) + 2N\mathbf{1}_{2n} = 2(E + N)\mathbf{1}_{2n}$, thus $\rho'_n(0) = \bigotimes_i (1 - v_i^2) v_i a_i^\dagger a_i$ is a thermal state with $v^2 = (E + N - 1/2)/(E + N + 1/2)$. We have $-\text{Tr}((\rho'_n - \rho^{RQ}_n(0))) \log \rho'_n(0) = -\text{Tr}((\rho'_n - \rho^{RQ}_n(0))) \sum_i a_i^\dagger a_i \log v^2 = 0$ under the energy constraint $\text{Tr}\rho_n \sum_i (a_i^\dagger a_i + 1/2) = E_n$, where $\text{Tr}\rho'_n \sum_i (a_i^\dagger a_i + 1/2) = \frac{1}{4} \text{Tr}(\gamma(2N\mathbf{1}_{2n})$ and $\rho'_n \rho^{RQ}_n(0) = 1$ are used. The density operator $\rho_n^{RQ}$ could be diagonalized by some unitary transformation $U_1$, the corresponding symplectic transformation

$$S_1 = \begin{bmatrix} \cosh r\mathbf{I}_2 & -\sinh r\mathbf{J}_2 \\ \sinh r\mathbf{J}_2 & \cosh r\mathbf{I}_2 \end{bmatrix}$$

(11)

will diagonalize the CM of $\rho_n^{RQ}$, that is $S_1\gamma(0) S_1^T = \gamma(1)$, meanwhile $S_1J \otimes (-J) S_1^T = J \otimes (-J)$. Here $\tanh 2r = \sqrt{4E^2 - 1}/(2E + N)$. The diagonalized CM $\gamma(1) = \text{diag}\{\gamma_A, \gamma_A, \gamma_B, \gamma_B\}$. The density operator $\rho_n^{RQ}$ is a direct product of $\rho_1^{RQ}$. The unitary transformation diagonalizes $\rho_n^{RQ}$ will be $U_n = U_1^{\otimes n}$. The corresponding symplectic transformation will be

$$S_n = \begin{bmatrix} \cosh r\mathbf{I}_n & -\sinh r\mathbf{J}_n \\ \sinh r\mathbf{J}_n & \cosh r\mathbf{I}_n \end{bmatrix}.$$ 

(12)

Thus $U_n\rho_n^{RQ} U_1^{\dagger} = \rho^{\otimes n}_n$, where $\rho_{AB} = (1 - v_A) a_a^\dagger \otimes (1 - v_B) b_b$ is a thermal state, with $b, b^\dagger$ being the annihilation and creation operators of 'reference' R system which is introduced in the purification, and $v_j = (\gamma_j - 1)/(\gamma_j + 1), (j = A, B)$. Hence $\text{Tr}(\rho^{RQ}_n - \rho^{RQ}_n(0)) \log \rho^{RQ}_n = \text{Tr}(U_n\rho^{RQ}_n U_1^{\dagger} - \rho^{\otimes n}_n) \log \rho^{\otimes n}_n = \text{Tr}(U_n\rho_n^{RQ} U_1^{\dagger} - \rho^{\otimes n}_n) \sum_i (a_i^\dagger a_i \log v_a + b_i^\dagger b_i \log v_B)$. After the unitary transformation, the density operator $U_n\rho_n^{RQ} U_1^{\dagger}$ is an operator function of the creation and annihilation operators $a_i, a_i^\dagger, b_i$ and $b_i^\dagger$. The CM of density operator $U_n\rho_n^{RQ} U_1^{\dagger}$ will be $S_n \gamma \otimes S_n^T$, denote it as

$$\gamma_U = \begin{bmatrix} \gamma_{AA} & \gamma_{AB} \\ \gamma_{BA} & \gamma_{BB} \end{bmatrix},$$

(13)

with $\gamma_{AA} = \cosh^2 r(\gamma + 2N\mathbf{1}_{2n}) - \sinh^2 r J_n \gamma J_n + \sinh r \cosh r J_n \gamma J_n + 2N\mathbf{1}_{2n}) J_n + \sinh r \cosh r J_n \gamma J_n + 2N\mathbf{1}_{2n})$. From the definition of CM, one can get that $\text{Tr}U_n\rho_n^{RQ} U_1^{\dagger} \sum_i (a_i^\dagger a_i + 1/2) \log v_a + (b_i^\dagger b_i + 1/2) \log v_B = (\text{Tr}\gamma_{AA} \log v_A + \text{Tr}\gamma_{BB} \log v_B)/4, \text{Tr}\gamma_{AA} = \text{cov} 2(\text{Tr}\gamma) + 4nN \cosh^2 r - \sinh 2r \text{Tr}(\gamma_{AA}^T) - 2n \sqrt{4E^2 - 1} \mathbf{I}_{2n}$, $\text{Tr}\gamma_{BB} = \text{cov} 2(\text{Tr}\gamma) + 4nN \sinh^2 r - \sinh 2r \text{Tr}(\gamma_{BB}^T) - 2n \sqrt{4E^2 - 1} \mathbf{I}_{2n}$. When the energy is constrained, $\text{Tr}\gamma = \text{Tr}\gamma(0)$, we have $\text{Tr}(\rho_n^{RQ} - \rho^{RQ}_n(0)) \log \rho^{RQ}_n(0) = -\frac{1}{4} \text{Tr}(v_A v_B) \sinh 2r \text{Tr}(\gamma_{AA}^T - \mathbf{I}_{2n}) - 2n \sqrt{4E^2 - 1})$. For $v_j = (\gamma_j - 1)/(\gamma_j + 1) < 1$, thus $-\log(v_A v_B) > 0$. What left to be proved is that at the constraint of $\text{Tr}\gamma = 4nE_n = 4E_n, \text{Tr}(\gamma_{AA}^T - \mathbf{I}_{2n})$ reaches its maximum when $\gamma = \gamma(0) = 2E\mathbf{1}_{2n}$. We start with any given $\gamma$ with $\text{Tr}\gamma = 4E_n, \gamma$ can be symplectically diagonalized to $S \gamma^{ST} = \text{diag}\{\gamma_1, \gamma_1, \gamma_2, \gamma_2, \ldots, \gamma_n, \gamma_n\}$. The symplectical transformation $S$ can be written as $R_1 D R_2$, where $R_1, R_2$ are rotations and $D = \text{diag}\{d_1, 1/d_1, d_2, 1/d_2, \ldots, d_n, 1/d_n\}$ is the squeezing operation. The rotation does not change the trace of the CM, the squeezing operation reduces the trace of the CM in the diagonalizing process. Hence $\text{Tr}\gamma \geq 2\sum_i \gamma_i = 1, \gamma' = k \cdot \text{diag}\{\gamma_1, \gamma_1, \gamma_2, \gamma_2, \ldots, \gamma_n, \gamma_n\}$ so that $\text{Tr}\gamma = \text{Tr}\gamma'$. We have $\text{Tr}(\gamma_{AA}^T - \mathbf{I}_{2n}) = 2\sum_i \sqrt{\gamma_i^2 - 1} \leq \text{Tr}(\gamma_{AA}^T - \mathbf{I}_{2n}) - 2n \sqrt{4E^2 - 1}$. For the diagonal CM $\gamma'$ with the energy constraint $\text{Tr}\gamma' = 4E_n$, it is
easy to elucidate that when all the diagonal elements are equal, \( \text{Tr} \sqrt{-(J_n^{-1} \gamma)^2 - I_{2n}} = 2 \sum_i \sqrt{\gamma_i^2 - 1} \) reaches its maximal value \( 2n\sqrt{4E^* - 1} \). We have proved that
\[
\text{Tr}(\rho_n^{RQ} - \rho_{n(0)}) \log \rho_{n(0)}^{RQ} \leq 0, \tag{14}
\]

together with \( \text{Tr}(\rho_n^{RQ} - \rho_{n(0)}) \log \rho_{n(0)}^{RQ} = 0 \). Hence \( \rho_{n(0)} \) is the extremal state that maximizes CI as far as CI is convex at the vicinity of \( \rho_{n(0)} \). Similarly, if CI is concave at the at the vicinity of \( \rho_{n(0)}, \rho_{n(0)} \) is the extremal state that minimizes CI. CI is the difference of two convex function, quantum mutual information and the entropy of the input state. This will provide the other way of obtaining the extremal state.

The next part of this paper is to give evidence of CI really reaching its maximal at \( \rho_{n(0)} \) if the input energy is strong enough. The calculation is based on perturbation theory. Suppose the input state \( \rho_n \) have a complex characteristic function \( \chi_n(\mu) = \text{Tr}(\rho_n D(\mu)) = \chi_n(0)(\mu)(1 + \varepsilon f(\mu, \mu^*)), \) where \( \chi_n(0)(\mu) = \exp[-(N_{\mu} + \frac{1}{2}) |\mu|^2] \) is the complex characteristic function of \( \rho_{n(0)} \), with \( N_{\mu} = E - \frac{1}{2} \) being the average photon number of the thermal input \( \rho_1(0) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \). The perturbation item \( f(\mu, \mu^*) \) can be expanded with power of \( \mu \) and \( \mu^* \). The condition that the first and second moments of \( \rho_1 \) are equal to that of \( \rho_{n(0)} \) leads to all the item of the linear and square power in the expansion of \( f(\mu, \mu^*) \) being 0. The cubic item \( \mu^3 \mu^* \) as well as other odd power items will have no contribution in the first order perturbation. So the first none zero contribution will be the fourth power. The other even power items can be neglected comparing with the fourth power as the input energy become strong enough. So we suppose
\[
f(\mu, \mu^*) = \sum_{i,j \geq 1} c_{ij} |\mu_i\mu_j|^2. \tag{15}\]

It should be noted that items such as \( \mu_1 \mu_2^* |\mu_1|^2 \) and \( \mu_2^2 \mu_2^* \) will also contribute to the first order perturbation, but they can also be neglected at the strong input energy as we will see below. All other items with unequal number of \( \mu \) and \( \mu^* \) will not contribute to the first order perturbation.

Let's first consider the \( |\mu_i|^4 \) item, the input state now is a direct product of perturbed first mode and other \( n - 1 \) thermal state modes. The situation is reduced to deal with the perturbation problem of \( \chi_1(\mu_i) = \chi_1(0)(\mu_i)(1 + \varepsilon |\mu_i|^4) \). We have \( \rho_1 = \rho_1(0) + \varepsilon \phi \), with \( \rho_1(0) = (1 - v_s) \sum_k v_k^2 |k\rangle \langle k| \), with \( v_s = N_s/(N_s + 1) \). The strict eigenvalues of \( \rho_1 \) are \( \lambda_k = \lambda_{k(0)} + \varepsilon \phi_k \), with \( \lambda_{k(0)} = (1 - v_s) v_k^2 \) and \( \phi_k = \lambda_k v_k^2 \). The entropy of \( \rho_1 \) can be expanded up to the second derivative as \( S(\rho_1) = S(\rho_1(0) - \frac{1}{2} \varepsilon^2 \sum_k \phi_k^2 / \lambda_{k(0)} + o(\varepsilon^3)) \), where null first and second moments of \( \phi \) are used. The calculation of the entropy of \( \rho_1' \) is straightforward, it is \( S(\rho_1') = S(\rho_1(0) - \frac{1}{2} \varepsilon^2 (N_N(\varepsilon + 1))^2 + o(\varepsilon^3)) \), with \( N' = N_s + N \). The purification of \( \rho_1(0) \) is \( \rho_1^{RQ} = \sum_{km} \sqrt{\lambda_{km} \lambda_{km}} |kk\rangle \langle mm| \), such a purification is more frequently used in literature but different from our above purification. The state \( \rho_1^{RQ} \) then is expanded in \( \varepsilon \) to the linear item \[15\]. \( \rho_1^{RQ} = \rho_1^{RQ(0)} + \varepsilon \Phi \), with \( \Phi = \frac{1}{2}(\Phi_0 + \Phi_0') \), \( \Phi_0 = (1 - v_s)^2 (1 - a_1 b_1^2 (1 - v_s) v_s^{-1/2} + a_1 b_1^2 v_s^{-1})(\rho_1^{RQ(0)}) \). It can be proved that
\[
(\mathcal{E} \otimes I) a_1^b b_1^a \rho_1^{RQ(0)} = v_s^{-j/2} a_1^a b_1^b \rho_1^{RQ(0)}. \tag{16}\]

Thus \( \rho_1^{RQ'} = \rho_1^{RQ} + \varepsilon \Phi' \), with \( \Phi' = \frac{1}{2}(\Phi_0' + \Phi_0'' \), \( \Phi_0 = (1 - v_s)^2 (2 - 4 a_1^a b_1^b 0 + a_1^a b_1^b 0_{\rho_1(0)}^{RQ}). \) Note that the trace and all first and second moment of \( \Phi' \) are null. The eigenstates of \( \rho_1(0) \) are \( V_1 |km\) with eigenvalues \( \lambda_{km} = (1 - v_A) v_A^k (1 - v_B) v_B^k \), where \( V_1 \) diagonalizes \( \rho_1^{RQ} \) and
\[
V_1 a_1 V_1^\dagger = a_1 \cosh r - b_1^\dagger \sinh r, \tag{17} \]
\[
V_1 b_1 V_1^\dagger = b_1 \cosh r - a_1^\dagger \sinh r, \]
with \( \tanh 2r = 2 \sqrt{N_s(N_s + 1)}/(N_s + N' + 1) \). The first order perturbation to the eigenvalue will be \( \Phi'_{km} = \langle km | V_1^\dagger \Phi' V_1 | km \rangle \) which is
\[
\Phi'_{km} = \lambda_{km} (1 - v_s)^2 (2 - 4 k \cosh^2 r + (m + 1) \sinh^2 r) / N_s \]
\[+ (k - 1) \cosh^4 r + (m + 1)(m + 2) \sinh^2 r + 4k(m + 1) \sinh^2 r \cos^2 r) / N_s. \tag{18}\]
Up to $\varepsilon^2$ item, the entropy will be $S(\rho_{1}^{RQ'}) \approx S(\rho_{1}^{RQ'}) - \frac{1}{2} \varepsilon^2 \sum_{km} \Phi_{km}^2 / \lambda_{km(0)}$. After the summation and taking the limitation of $N_s \to \infty$, the total increase of CI between the input $\rho_n$ and $\rho_n(0)$ will be

$$\lim_{N_s \to \infty} [I_c(\rho_n) - I_c(\rho_n(0))] = -\frac{1}{2} \varepsilon^2 [\frac{4}{N_s^2} - \frac{3}{2N_s^2}] < 0. \quad (19)$$

the positive part which comes from $\rho_n^{RQ'}$ state is only $\frac{3}{8}$ of the negative part which comes from $\rho_n'$. Thus $I_c(\rho_n(0))$ is maximal in this situation.

The next perturbation item is $|\mu_1\mu_2|^2$. We only need to deal with the first and second modes with $\chi_2(\mu_1, \mu_2) = \chi_2(0)(\mu_1, \mu_2)(1 + \varepsilon |\mu_1\mu_2|^2)$. While other modes are kept in thermal states and irrelavte. Here the degenerate perturbation is applied. The calculation of the entropy difference of $\rho_n'$ and $\rho_n(0)$ is easy because the perturbation operator is diagonal in degenerate subspace. The result is

$$S(\rho_n') - S(\rho_n(0)) = -\frac{\varepsilon^2}{2N^2(N^2 + 1)^2}. \quad (20)$$

The calculation of the entropy difference of $\rho_{1}^{RQ'}$ and $\rho_{2}^{RQ'}$ will encounter with non diagonal operators $a_1 b_1^+ b_2^+$ and $a_1^+ b_1 b_2$ in degenerate subspace which indicate inter-mode particle transfer, but in the degenerate subspace the total particle number of Q system (or R system) is conserved. The entropy difference can be calculated by first summing up in the degenerate subspace then the total particle number of Q system and R system. In the summation $\text{Tr} M_{k_1,m_1}^{12}$ is involved, where $M^l$ is the perturbation operator in the $l$th degenerate subspace, fortunately it is $\sum_{k_1,m_1} M_{k_1,m_1}^{21}$ by the special structure of $M^l$. The final result after the summation and taking the limitation of $N_s \to \infty$ will simply be

$$\lim_{N_s \to \infty} S(\rho_{1}^{RQ'}) - S(\rho_{2}^{RQ'}) = -\frac{3\varepsilon^2}{16N_s^4} \quad (21)$$

Still it is $\frac{3}{8}$ of the entropy difference of $\rho_n'$ and $\rho_n(0)$. Thus we have

$$\lim_{N_s \to \infty} [I_c(\rho_n) - I_c(\rho_n(0))] = -\frac{5\varepsilon^2}{16N_s^4} < 0. \quad (22)$$

In the situation of $\chi_2(\mu_1, \mu_2) = \chi_2(0)(\mu_1, \mu_2)(1 + \varepsilon |\mu_1|^2 + c |\mu_1\mu_2|^2)$, the entropy difference will be the sum of each term because the cross item is null by the null of the first and second moment of $\phi$ and $\Phi'$. The general case of $f(\mu, \mu^*)$ will be

$$\lim_{N_s \to \infty} [I_c(\rho_n) - I_c(\rho_n(0))] = -\frac{5\varepsilon^2}{16N_s^4} \left(4 \sum_i c_{ii}^2 + \sum_{i \neq j} c_{ij}^2 \right) < 0. \quad (23)$$

The conclusion is that thermal state of infinitive energy achieves the maximal of coherent information. That is

$$\lim_{n \to \infty} \max_{\rho_n} -\frac{1}{n} I_c(\rho_n, \mathcal{E}^\otimes n) = \max\{0, -\log_2 \{eN\}\}, \quad (24)$$

($e = 2.71828 \ldots$). The result is verified up to the nonzero lowest power of the inverse of the input energy in each channel use. Whether it is correct for the state without an item contributing to $N_s^{-4}$ is not known.

We obtain a local maximum, whether it is the global supremum should be verified. For lower noise channel, this can be verified. For a given channel, the CI difference is a function of $N_s$. In the two case we calculated, the CI difference is positive infinitive at $N_s \to 0$, as $N_s$ increases, it monotonically decreases to 0. After that it decreases further to negative then increases but still keeps negative and never turns to positive. At $N_s \to \infty$, it is negative as we elucidated above. Denote the zero point as $N_{5\alpha}$, then calculate the quantum mutual information $I(\rho_n(0)(N_{5\alpha}), N)$ which is the supremum of all state with equal or less energy. $I(\rho_n, N)$ is greater $I_c(\rho_n, N)$.
Hence if \( I(\rho_{1(0)}(N_{s0}), N) \leq -\log_2(eN) \), we have CI to be less than \(-n\log_2(eN)\) in the interval \([0, N_{s0}]\), thus for \( N \leq N_c \) we can safely conclude that
\[
Q = \max\{0, -\log_2(eN)\},
\]
as far as the high order perturbation and high power of \( \mu \) do not destroy the maximal property of CI at infinitive input energy, where \( N_c \) is the solution of \( I(\rho_{1(0)}(N_{s0}), N_c) = -\log_2(eN_c) \). We have \( N_c = 0.1756 \) which comes from the two-mode perturbation.

For the \( n \) uses of the thermal noise channel with \( N \leq N_c \), the supremum of the whole CI is achieve by an input of the direct product of the identical thermal noise states. It is followed that the quantum capacity of thermal noise channel is equal to the one-shot quantum capacity [12][11] of the channel. The achievable of the quantum capacity of the thermal noise channel by quantum error-correction codes had been proven [10].

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[13] CI can also be assumed cancave locally, the whole procedure will be similar.
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[15] The square item may also contribute to the entropy, but it is only \( o(1/\sqrt{N_s}) \) comparing with the linear item.
[16] The density operator can be derived from the integral \( \Delta(\sigma, \tau) = \int \prod_i \frac{d^2\mu_i}{\pi} \exp[-(N_s + \frac{1}{2})|\mu|^2 + \sigma\mu + \tau\mu^*]D(-\mu) = (1 - v_s)^2 : \exp[(1 - v_s)(\sigma - a^d)(\tau + a)] : \) by derivative on \( \sigma_i, \tau_j \) then set \( \sigma = \tau = 0 \), where the integral is carried out in order operator form. From this integral we can see that \( \mu_1\mu_2^*|\mu_i|^2 \) and \( \mu_1^2\mu_2^2 \) item will contribute to the entropy in the magnitude of \( N_s^{-5} \) and \( N_s^{-6} \) when \( N_s \to \infty \).