Neutrino Mass Eigenvalues for Different Scheme within Four Flavor Neutrino Framework

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Abstract

In this paper, we discuss neutrino mass eigenvalues for four flavor neutrino mixing. An extra mass states, in four flavor mixing and possible various combination of CP violating Majorana phases effects the neutrino mass eigenvalues. We have considered the effective Majorana mass \(m_e\), related for \((\beta\beta)_{0\nu}\) decay. In calculation, we consider two different neutrino mass order, normal and inverted. We find the limits for neutrino mass eigenvalue \(m_i\) in the different neutrino mass spectrum and the sum of all four neutrino masses is \(\sum \equiv m_1 + m_2 + m_3 + m_4 \sim 1.17eV\) which is relevant for cosmological observations and explain the different neutrino oscillations data.

Keywords: Neutrino Scheme, Atmospheric neutrino, Solar neutrino

1 Introduction

The Dirac and Majorana nature of a particle has been a puzzle for last decades. Any form of the kinematical test has not been capable to determine the particle’s nature. Our only chance is the experimental results of Neutrinoless Double Beta Decay via the decay of two-electron without the emissions of two neutrinos \cite{1} which eventually confirm the Majorana nature of the neutrinos. This type of decay violates the lepton number by 2 units. In order to confirm the type of nature of a particle it can probe the physics beyond the Standard Model like the compositeness, violation of equivalence principle, absolute neutrino mass scale and the neutrino mass ordering \cite{2-5}. Only three flavor of neutrinos are in the standard neutrino oscillation with mass-squared differences of order \(10^{-4}\) and \(10^{-3}\) eV\(^2\)\cite{8}. The three neutrino framework has been found to be effective in explaining numerous sorts of experiments.
However, the results of the LSND[6] and MiniBooNE[7] experiments point to the presence of one or two neutrino states with a considerable mass of eV scale.[6, 7]. This additional neutrino termed as "sterile neutrino". The effective mass in neutrinoless double beta decay [9-11] and the oscillation probabilities in different baseline experiments are significantly changes due to the presence of sterile neutrinos. Three scales of neutrino mass-squared differences $\Delta m^2_{\text{solar}} \ll \Delta m^2_{\text{atm}} \ll \Delta m^2_{\text{LSND}}$ were found by atmospheric and solar neutrino oscillation experiments, as well as LSND collaboration, which needed four neutrinos with defined mass to explain these results.

All these neutrino oscillation data could be explained successfully by four neutrinos with definite mass by S.M. Bilenky et al. [12, 13, 14, 15, 16]. The preference of a (2+2) mass spectrum over a (3+1) spectrum is due to the LSND finding [12, 13] which was found to be compatible with null results from accelerator [17] and reactor [18] disappearance. Both atmospheric and solar oscillations might be influenced by sterile neutrinos. Indeed, The sterile neutrino has larger contribution in the solar or atmospheric oscillations or in both for (2+2) mixing schemes while in (3+1) scheme has the three active conventional neutrinos and one additional sterile neutrino which has larger mass than the others. Thus four-neutrino models have been studied by many authors [19, 20, 21, 22, 36, 37, 38].

This paper is organized as follows. In Sec. 2, we briefly describe the nearly degenerate masses in different scheme for both mass order. In Sec. 3 we summarize the effective electron neutrino mass in different scheme for normal and Inverted mass order. Results and Conclusions are briefly discussed in Sec. 4 and Sec. 5 respectively.

## 2 Degenerate masses in Different Scheme

The mixing of four neutrinos [23-24], which allows to accommodate all the three existing neutrinos with an extra neutrino $\nu_s$ in four flavour neutrino mixing. Here $\nu_s$ is a sterile neutrino state. The presence of one sterile neutrino, in four flavour neutrino mixing provides the different possibilities for the mass spectrum of four neutrinos. The neutrino mass-squared differences associated with solar, atmospheric and LSND for six possibility for normal and Inverted mass order are shown in Fig. (1) and Fig.(2). These six possible mass spectra are broadly classified into two classes of schemes. The class A, which is also called (3+1) scheme, and class B, which is termed as (2+2) scheme for both mass order are shown in Fig. (1) and Fig. (2), respectively. The two nearly degenerate pairs of neutrino mass eigenstates $\Delta m^2_{\text{solar}}$ and $\Delta m^2_{\text{atm}}$ separated by the LSND scale $\Delta m^2_{\text{LSND}} = 1eV^2$ are described by (2+2) scheme. The combination of these two pairs of degenerate masses $\Delta m^2_{\text{solar}}$ and $\Delta m^2_{\text{atm}}$ are described by scheme 5 and scheme 6 of class B as shown in Fig. (2). The solar and atmospheric data are explained by the mixing of $\nu_2, \nu_1$ and $\nu_3, \nu_4$. The additional neutrino in four flavor mixing has a significant contribution in the solar or atmospheric or in both the oscillation. (3+1) scheme are assumed the solar and atmospheric oscillation are given by the three active neutrinos and the addition neutrino is sterile one which is considered to be heavier than the three active neutrinos. This addition of sterile neutrino provides a new independent neutrino mass squared difference, $\Delta m^2_{41}$, and mixing matrix is elaborated from $3 \times 3$ to $4 \times 4$ by incorporating three new mixing angles, $\theta_{14}, \theta_{24}, \theta_{34}$ and two additional CP
Figure 1: Neutrino mass spectrum with Degenerate masses for Normal mass order

Class A (3+1) Schemes for Normal Mass Order

Class B (2+2) Schemes for Normal Mass Order
Figure 2: Neutrino mass spectrum with Degenerate masses for Inverted mass order
phases $\delta_{14}$ and $\delta_{24}$. The large mass eigenvalue from the coupling of $\nu_e$ and $\nu_\mu$ via small mixings with $\nu_s$, separated from others are capable for explain the LSND data. Earlier, (3+1) scheme has ruled out due to its incompatibility with the LSND data and the results of the CDHS and BUGEY experiments [12, 13]. However, the shifting of LSND allowed region from the new LSND experiment [2] results opens a small channel for (3+1) scheme. Another one is (1+3) scheme, in which a massive neutrino is lighter than three neutrinos. This scheme is disfavoured by the upper bound on the neutrino masses that is smaller than 1eV, and by the upper bound on effective neutrino mass in neutrinoless double beta decay if neutrino are Majorana particles [25, 26].

The order of mass spectrum can be normal mass order ($m_4 \gg m_3 > m_2 > m_1$ and $\Delta m_{LSND}^2 = \Delta m_{41}^2$) or inverted mass order ($m_3 < m_1 < m_2 \ll m_4$ and $\Delta m_{LSND}^2 = \Delta m_{43}^2$). The pairs of degenerate masses $\Delta m_{solar}^2$ and $\Delta m_{atm}^2$ with six possible scheme for Normal mass order are describe as [12];

Class A,

\[\begin{align*}
\text{Scheme 1} & \quad \Delta m_{21}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{31}^2 = \Delta m_{atm}^2 \\
\text{Scheme 2} & \quad \Delta m_{32}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{31}^2 = \Delta m_{atm}^2 \\
\text{Scheme 3} & \quad \Delta m_{41}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{22}^2 = \Delta m_{atm}^2 \\
\text{Scheme 4} & \quad \Delta m_{32}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{42}^2 = \Delta m_{atm}^2
\end{align*}\]  

Class B,

\[\begin{align*}
\text{Scheme 5} & \quad \Delta m_{43}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{21}^2 = \Delta m_{atm}^2 \\
\text{Scheme 6} & \quad \Delta m_{52}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{43}^2 = \Delta m_{atm}^2
\end{align*}\]

and the pairs of degenerate masses $\Delta m_{solar}^2$ and $\Delta m_{atm}^2$ with six possible scheme for Inverted mass order are describe as;

Class A,

\[\begin{align*}
\text{Scheme 1} & \quad \Delta m_{13}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{23}^2 = \Delta m_{atm}^2 \\
\text{Scheme 2} & \quad \Delta m_{21}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{23}^2 = \Delta m_{atm}^2 \\
\text{Scheme 3} & \quad \Delta m_{42}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{21}^2 = \Delta m_{atm}^2 \\
\text{Scheme 4} & \quad \Delta m_{12}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{41}^2 = \Delta m_{atm}^2
\end{align*}\]  

Class B,

\[\begin{align*}
\text{Scheme 5} & \quad \Delta m_{42}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{13}^2 = \Delta m_{atm}^2 \\
\text{Scheme 6} & \quad \Delta m_{13}^2 = \Delta m_{solar}^2 ; \quad \Delta m_{42}^2 = \Delta m_{atm}^2
\end{align*}\]

Where,

$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$
3 Neutrino Mass Eigenvalue in Different Scheme

In this section, we deals with four flavor framework for (3+1) and (2+2) schemes by incorporating the sterile neutrino of eV range and the mixing of this sterile neutrino with three neutrinos is light. By adding one sterile neutrinos [27], there is an increment in mixing angles and CP violating phases in the PMNS matrix $U_{4x4}$ which is given by,

$$U = R_{34}R_{24}R_{14}R_{23}R_{13}R_{12}P,$$

(3)

where the matrices $R_{ij}$ are rotations in ij space,

$$R_{14} = \begin{pmatrix} c_{14} & 0 & 0 & s_{14}e^{i\delta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14}e^{-i\delta} & 0 & 0 & c_{14} \end{pmatrix} \quad \text{and} \quad R_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{pmatrix},$$

where $s_{ij} = \sin\theta_{ij}, c_{ij} = \cos\theta_{ij}$. The diagonal matrix P contains the three Majorana phases $\alpha, \beta$ and $\gamma$:

$$P = diag(1, e^{i\alpha}, e^{i\beta}, e^{i\gamma})$$

(4)

Note that there are in total three Dirac CP-violating phase $\delta_{ij}$. P is constructed in such a way that only Majorana phases show up in the effective mass expression. The explicit form of $U$ for (3+1) and (2+2) schemes are define as;

$$U_{(3+1)} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} & U_{\mu4} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} & U_{\tau4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix} \quad \text{and} \quad U_{(2+2)} = \begin{pmatrix} U_{s1} & U_{s2} & U_{s3} & U_{s4} \\ U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} & U_{\mu4} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} & U_{\tau4} \end{pmatrix},$$

(5)

In the presence of one sterile neutrino, four flavour neutrino mixing the electron neutrino is combination of mass eigenstate, $\nu_{i}$ with eigenvalue $m_{i}$

$$\nu_{\alpha} = \sum U_{\alpha j} \nu_{j} \quad j = 1, 2, 3, 4 \quad \text{and} \quad \alpha = e, \mu, \tau, s.$$  

(6)

Here $U_{ei}$ are the elements of the $4 \times 4$ mixing matrix, which relates the flavour states with the mass eigenvalues. The $\beta\beta_{0\nu}$ decay rate is determined by the effective Majorana mass of the electron neutrino $m_{e}$. Under the assumption of four flavour neutrino mixing of neutrino, the effective Majorana neutrino mass $|m_{e}|_{4\nu}$ [28] is
The effective Majorana neutrino mass \( |m_{e}|_{\nu} \) for \((3+1)\) scheme is given by:

\[
|m_{e}|_{\nu} = |m_{e}|_{1}^2 \ |U_{e1}|^2 m_1 + |m_{e}|_{2}^2 \ e^{i\alpha} m_2 + |m_{e}|_{3}^2 \ e^{i\beta} m_3 + |m_{e}|_{4}^2 \ e^{i\gamma} m_4 |
\]

Now from Eq. (5), the effective Majorana neutrino mass \( |m_{e}|_{\nu} \) for \((3+1)\) scheme is given by:

\[
|m_{e}|_{\nu} = |c_{13}^2 s_{12}^2 s_{14}^2 m_1 + c_{13}^2 s_{12}^2 c_{14}^2 e^{i\alpha} m_2 + s_{13}^2 c_{14}^2 e^{i\beta} m_3 + s_{14}^2 e^{i\gamma} m_4 |
\]

and for \((2+2)\) scheme is;

\[
|m_{e}|_{\nu} = |(-c_{24} c_{23} s_{12} - c_{24} s_{23} s_{13} c_{12} - s_{24} s_{14} c_{13} c_{12})^2 m_1 + (c_{24} c_{23} c_{12} - c_{24} s_{23} s_{13} s_{12} - s_{24} s_{14} c_{13} c_{12} e^{i\alpha} m_2) + (c_{24} c_{23} c_{13} - s_{24} s_{14} s_{13})^2 e^{i\beta} m_3 + (s_{24} c_{14})^2 e^{i\gamma} m_4 |
\]

Here, \( m_1, m_2, m_3 \) and \( m_4 \) are the masses eigenvalues for vacuum. On squaring Eq. (8), we get

\[
|m_{e}|_{\nu}^2 = \left( |c_{13}^2 s_{12}^2 c_{14}^2 m_1 + c_{13}^2 s_{12}^2 c_{14}^2 e^{i\alpha} m_2 + s_{13}^2 c_{14}^2 e^{i\beta} m_3 + s_{14}^2 e^{i\gamma} m_4 | \right)^2
\]

\[
|m_{e}|_{\nu}^2 = \left( c_{13}^2 s_{12}^2 s_{14}^2 c_{14}^2 m_1^2 + c_{13}^2 s_{12}^2 s_{14}^2 e^{i\alpha} m_1 m_2 + c_{13}^2 s_{12}^2 s_{14}^2 e^{i\beta} m_1 m_3 + c_{13}^2 s_{12}^2 e^{i\gamma} m_1 m_4 \right)
\]

\[
|m_{e}|_{\nu}^2 = \left( c_{13}^4 s_{12}^2 c_{14}^4 m_1^4 \right) + \left( c_{13}^4 s_{12}^2 s_{14}^2 c_{14}^2 e^{i\alpha} m_1 m_2 \right) + \left( c_{13}^4 s_{12}^2 s_{14}^2 c_{14}^2 e^{i\beta} m_1 m_3 \right) + \left( c_{13}^4 s_{12}^2 e^{i\gamma} m_1 m_4 \right)
\]

After some algebra Eq. (11) has the form,

\[
|m_{e}|_{\nu}^2 = A^2 m_1^2 + B^2 m_2^2 + C^2 m_3^2 + D^2 m_4^2 + 2AB m_1 m_2 \cos \alpha + 2AC m_1 m_3 \cos \beta + 2AD m_1 m_4 \cos \gamma
\]

\[
+ 2BC m_2 m_3 \cos (\alpha - \beta) + 2BD m_2 m_4 \cos (\alpha - \gamma) + 2CD m_3 m_4 \cos (\beta - \gamma)
\]

With,

\[
A = |U_{e1}|^2 , \quad B = |U_{e2}|^2 , \quad C = |U_{e3}|^2 \quad \text{and} \quad D = |U_{e4}|^2
\]

The effective Majorana neutrino mass \( |m_{e}|_{\nu} \) for \((2+2)\) scheme is same as Equation (12) but the coefficients \( A, B, C, D \) are replaced by the second row of Eq. (5). Now, The effective Majorana neutrino mass \( |m_{e}|_{\nu} \) in term of mass \( m_1 \) and pairs of degenerate masses \( \Delta m_{solar}^2 \) and \( \Delta m_{atm}^2 \) define in Eq. (1) with different scheme for Normal mass order are
given by:

**Scheme 1**

\[ |m_{e\nu}|^2 = m_1^2 (A^2 + B^2 + C^2 + D^2) + B^2 \Delta m_{solar}^2 + C^2 \Delta m_{atm}^2 + D^2 \Delta m_{LSND}^2 \\
+ 2ABm_1 \sqrt{m_1^2 + \Delta m_{solar}^2 \cos \alpha + 2ACm_1 \sqrt{m_1^2 + \Delta m_{atm}^2 \cos \beta}} \\
+ 2ADm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos \gamma + 2BC \sqrt{m_1^2 + \Delta m_{solar}^2 \sqrt{m_1^2 + \Delta m_{atm}^2 \cos (\alpha - \beta)}}} \\
+ 2BD \sqrt{m_1^2 + \Delta m_{solar}^2 \sqrt{m_1^2 + \Delta m_{atm}^2 \cos (\alpha - \gamma)}} + 2CD \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos (\beta - \gamma)} \]

**Scheme 2**

\[ |m_{e\nu}|^2 = m_1^2 (A^2 + B^2 + C^2 + D^2) + B^2 (\Delta m_{solar}^2 - \Delta m_{atm}^2) + C^2 \Delta m_{atm}^2 + D^2 \Delta m_{LSND}^2 \\
+ 2ABm_1 \sqrt{m_1^2 + \Delta m_{solar}^2 - \Delta m_{atm}^2 \cos \alpha + 2ACm_1 \sqrt{m_1^2 + \Delta m_{atm}^2 \cos \beta}} \\
+ 2ADm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos \gamma + 2BC \sqrt{m_1^2 + \Delta m_{solar}^2 - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{atm}^2 \cos (\alpha - \beta)}}} \\
+ 2BD \sqrt{m_1^2 + \Delta m_{solar}^2 - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{atm}^2 \cos (\alpha - \gamma)}} + 2CD \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos (\beta - \gamma)} \]

**Scheme 3**

\[ |m_{e\nu}|^2 = m_1^2 (A^2 + B^2 + C^2 + D^2) + B^2 (\Delta m_{LSND}^2 - \Delta m_{atm}^2) + C^2 (\Delta m_{LSND}^2 - \Delta m_{solar}^2) \\
+ D^2 \Delta m_{LSND}^2 + 2ABm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \cos \alpha} \\
+ 2ACm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \cos \beta} + 2ADm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos \gamma} \\
+ 2BC \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \cos (\alpha - \beta)}} \\
+ 2BD \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos (\alpha - \gamma)}} + 2CD \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos (\beta - \gamma)} \]

**Scheme 4**

\[ |m_{e\nu}|^2 = m_1^2 (A^2 + B^2 + C^2 + D^2) + B^2 (\Delta m_{LSND}^2 - \Delta m_{atm}^2) + C^2 (\Delta m_{LSND}^2 - \Delta m_{atm}^2 + \Delta m_{solar}^2) \\
+ D^2 \Delta m_{LSND}^2 + 2ABm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \cos \alpha} \\
+ 2ACm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 + \Delta m_{solar}^2 \cos \beta} + 2ADm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos \gamma} \\
+ 2BC \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 + \Delta m_{solar}^2 \cos (\alpha - \beta)}} \\
+ 2BD \sqrt{m_1^2 + \Delta m_{LSND}^2 - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos (\alpha - \gamma)}} + 2CD \sqrt{m_1^2 + \Delta m_{LSND}^2 \cos (\beta - \gamma)} \]
Similarly, The effective Majorana neutrino mass $|m_e|^2_{4\nu}$ in term of masses $m_3$ and pairs of degenerate masses $\Delta m_{solar}^2$ and $\Delta m_{atm}^2$ define in Eq. (2) with different scheme for Inverted mass order are given by:

### Scheme 5

$$|m_e|^2_{4\nu} = m_3^2 \left( A^2 + B^2 + C^2 + D^2 \right) + B^2 \Delta m_{atm}^2 + C^2 \left( \Delta m_{ LSND}^2 - \Delta m_{solar}^2 \right) + D^2 \Delta m_{LSND}^2 + 2ABm_1 \sqrt{m_1^2 + \Delta m_{atm}^2} \cos \alpha + 2ACm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \beta + 2ADm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \gamma \left( \Delta m_{solar}^2 \cos \alpha - \Delta m_{atm}^2 \cos \beta \right) + 2BD \sqrt{m_1^2 + \Delta m_{atm}^2} \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \left( \alpha - \gamma \right) + 2CD \sqrt{m_1^2 + \Delta m_{LSND}^2} - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \left( \beta - \gamma \right)$$

### Scheme 6

$$|m_e|^2_{4\nu} = m_3^2 \left( A^2 + B^2 + C^2 + D^2 \right) + B^2 \Delta m_{solar}^2 + C^2 \left( \Delta m_{ LSND}^2 - \Delta m_{atm}^2 \right) + D^2 \Delta m_{LSND}^2 + 2ABm_1 \sqrt{m_1^2 + \Delta m_{solar}^2} \cos \alpha + 2ACm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \beta + 2ADm_1 \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \gamma \left( \Delta m_{solar}^2 \cos \alpha - \Delta m_{atm}^2 \cos \beta \right) + 2BD \sqrt{m_1^2 + \Delta m_{atm}^2} \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \left( \alpha - \gamma \right) + 2CD \sqrt{m_1^2 + \Delta m_{LSND}^2} - \Delta m_{atm}^2 \sqrt{m_1^2 + \Delta m_{LSND}^2} \cos \left( \beta - \gamma \right)$$

For scheme 5 and scheme 6 the coefficient $A, B, C, D$ are replaced by the second row of Eq. (5).
Again, for scheme 5 and scheme 6 the coefficient effective majorana neutrino mass $m_e$ is directly proportional to the inverse of the half-life of nucleus participated.

\[ |m_e|^2_{4\nu} = m_3^2 \left(A^2 + B^2 + C^2 + D^2\right) + A^2 \left(m_{LSND}^2 - \Delta m_{atm}^2\right) + B^2 \left(m_{LSND}^2 - \Delta m_{solar}^2\right) \]
\[ + D^2 \Delta m_{LSND}^2 + 2AB \sqrt{m_3^2 + m_{LSND}^2 - \Delta m_{atm}^2} \sqrt{m_3^2 + m_{LSND}^2 - \Delta m_{solar}^2} \cos \alpha \]
\[ + 2ACm_3 \sqrt{m_3^2 + m_{LSND}^2 - \Delta m_{atm}^2} \cos \beta + 2BCm_3 \sqrt{m_3^2 + m_{LSND}^2 - \Delta m_{atm}^2} \cos \gamma \]
\[ + 2AD \sqrt{m_3^2 + m_{LSND}^2 - \Delta m_{atm}^2} \sqrt{m_3^2 + m_{LSND}^2} \cos (\alpha - \beta) \]
\[ + 2BD \sqrt{m_3^2 + m_{LSND}^2 - \Delta m_{atm}^2} \sqrt{m_3^2 + m_{LSND}^2} \cos (\alpha - \gamma) \]
\[ + 2CDm_3 \sqrt{m_3^2 + m_{LSND}^2} \cos (\beta - \gamma) \]
in the process of neutrinoless double beta decay. The effective Majorana mass is not very well understood but lower and upper bound of the effective majorana neutrino mass for normal and inverted order are 0.0089 eV $\leq m_e \leq 0.0126$ eV [29] and $0.02$ eV $\leq m_e \leq 0.05$ eV [30], respectively. The effective neutrino mass equation depends on the values of the Majorana phase $\alpha, \beta, \gamma$. We have taken $\Delta m^2_{21} = 0.002eV^2$ [30, 31], $\Delta m^2_{31} = 0.00008eV^2$ [33] and $\Delta m^2_{41} = 1.18eV^2$ [30]. The active sterile neutrino mixing angle are $\theta_{14}$, $\theta_{24}$ and $\theta_{34}$. In this work, we consider following value for sterile neutrino mixing angles [35], $\theta_{14} = 3.6^\circ$, $\theta_{24} = 4^\circ$, $\theta_{34} = 18.5^\circ$. The constraint for the cosmological observations of neutrino mass, we have taken sum of all the four masses $\sum m \equiv m_1 + m_2 + m_3 + m_4$.

5 Conclusions

In the future, the aim of the neutrinoless double beta decay experiment would predict the constraint on the limit of the lightest neutrino mass $0.0007$ eV $\leq m_1 \leq 0.0008$ eV and mass order by reaching the high sensitivity of 0.001 eV. We have study the neutrino mass eigenvalue in the case of both neutrino mass order by using six different possible scheme. In table (1) and (2), by taking the effective neutrino mass, $m_e = (0.0095, 0.011, 0.0125)$ eV, within normal neutrino mass order [29] and for scheme (1-4), we have found that $m_1$ varied from (0.016–0.072) eV, $m_2$ varied from (0.019–0.073) eV, $m_3$ varied from (0.048–0.086) eV and $m_4$ varied from (1.08–1.09) eV for different allowed range of majorana phase. Since equations for scheme 5 and 6 for normal order (see eq. (24-25)) are almost the same and the only difference lies in degenerate masses $\Delta m^2_{atm}$ and $\Delta m^2_{solar}$ which produce the subtle effects into equations [30, 31, 32]. Thus we have found nearly same mass eigenvalues for both schemes i.e $m_{(2+2)} = (0.30 – 1.13)$ eV. We also find limit of the sum of all four neutrino masses $\sum_{(3+1)} \equiv m_1 + m_2 + m_3 + m_4 = (1.17 – 1.32)$ eV and $\sum_{(2+2)} \equiv m_1 + m_2 + m_3 + m_4 = (2.0 – 3.0)$ eV relevant for cosmological observations. In the case of inverted neutrino mass order [30], by assuming the value of the effective neutrino mass $m_e = (0.03 – 0.05)$ eV, we have found that $m_1$ varied from (0.045–0.278) eV, varied from (0.040–0.276) eV, $m_3$ varied from (0.019–0.274) eV and $m_4$ varied from (1.08–1.12) eV as given in table (3) and (4) and nearly same mass eigenvalues for schemes 5 and 6 i.e $m_{(2+2)} = (0.137 – 1.28)$ eV due to the subtle effect produce into the equations by the degenerate masses $\Delta m^2_{atm}$ and $\Delta m^2_{solar}$, for different allowed range of majorana phase. We find limit of the sum of all four neutrino masses $\sum_{(3+1)} \equiv m_1 + m_2 + m_3 + m_4 = (1.18 – 1.23)$ eV and $\sum_{(2+2)} \equiv m_1 + m_2 + m_3 + m_4 = (1.49 – 3.33)$ eV relevant for cosmological observations. The value of effective Majorana mass $|m_e|$ as a function of lower mass $m_1$ and sum of the neutrino masses $\sum_{(3+1)}$ in the case of (3+1) neutrino mixing with Normal and Inverted Ordering are shown in Left and right pannel of fig. (3), respectively. The inverted mass order provides the smaller window of sum of all four masses for (2+2) scheme than the (3+1) scheme with larger value of mass. Indeed, for the (3+1) scheme the mass of sterile neutrino is just simply add up to the former active 3 neutrinos while (2+2) scheme the sterile neutrino is just destroyed the whole mass spectrum. The neutrino mass eigen state is depend on choice of effective neutrino mass and allowed range of majorana phase. Hence precise determination of effective neutrino mass from beta decay experiment will gives the
Table 1: Neutrino mass eigenvalues in eV for Normal mass order.

| Majorana Phases | Effective mass $m_e$ in eV | Mass states | Scheme 1 | Scheme 2 | Scheme 3 | Scheme 4 | Scheme 5 | Scheme 6 |
|-----------------|-----------------------------|-------------|---------|---------|---------|---------|---------|---------|
| $\alpha = 0^0$, | 0.0095                      | $m_1$       | 0.01642 | 0.01697 | 0.07235 | 0.07201 | 0.30708 | 0.30708 |
| $\beta = 0^0$,  | 0.01965                     | $m_2$       | 0.01645 | 0.02034 | 0.07300 | 0.07262 | 0.30794 | 0.30794 |
| $\gamma = 0^0$ | 0.04887                     | $m_3$       | 0.04948 | 0.08636 | 0.08560 | 0.31104 | 0.31104 |
|                 |                            | $m_4$       | 1.0865  | 1.0865  | 1.0890  | 1.0888  | 1.1290  | 1.1290  |
| $\alpha = 0^0$, | 0.011                       | $m_1$       | 0.01782 | 0.01833 | 0.07251 | 0.07217 | 0.30759 | 0.30759 |
| $\beta = 0^0$,  | 0.01865                     | $m_2$       | 0.02069 | 0.02192 | 0.07318 | 0.07279 | 0.30772 | 0.30772 |
| $\gamma = 0^0$ | 0.04893                     | $m_3$       | 0.04934 | 0.05041 | 0.08671 | 0.08594 | 0.31082 | 0.31082 |
|                 |                            | $m_4$       | 1.0865  | 1.0865  | 1.0891  | 1.0889  | 1.1289  | 1.1289  |
| $\alpha = 0^0$, | 0.0125                      | $m_1$       | 0.02050 | 0.02036 | 0.07269 | 0.07235 | 0.30734 | 0.30734 |
| $\beta = 0^0$,  |                            | $m_2$       | 0.02299 | 0.02374 | 0.07338 | 0.07299 | 0.30747 | 0.30747 |
| $\gamma = 0^0$ | 0.05049                     | $m_3$       | 0.05049 | 0.05146 | 0.08712 | 0.08633 | 0.31058 | 0.31058 |
|                 |                            | $m_4$       | 1.0865  | 1.0866  | 1.0891  | 1.0890  | 1.1289  | 1.1289  |
| $\alpha = 0^0$, | 0.0095                      | $m_1$       | 0.01814 | 0.01850 | 0.07235 | 0.07201 | 0.58099 | 0.58099 |
| $\beta = 180^0$ | 0.01965                     | $m_2$       | 0.01918 | 0.01833 | 0.07251 | 0.07217 | 0.58108 | 0.58108 |
| $\gamma = 0^0$ | 0.04974                     | $m_3$       | 0.04997 | 0.05041 | 0.08671 | 0.08594 | 0.58322 | 0.58322 |
|                 |                            | $m_4$       | 1.0865  | 1.0865  | 1.0890  | 1.0888  | 1.2595  | 1.2595  |
| $\alpha = 0^0$, | 0.011                       | $m_1$       | 0.02100 | 0.02001 | 0.07251 | 0.07217 | 0.58301 | 0.58301 |
| $\beta = 180^0$ | 0.02197                     | $m_2$       | 0.02363 | 0.02342 | 0.07318 | 0.07279 | 0.58309 | 0.58309 |
| $\gamma = 0^0$ | 0.05097                     | $m_3$       | 0.05097 | 0.05120 | 0.08671 | 0.08594 | 0.58524 | 0.58524 |
|                 |                            | $m_4$       | 1.0866  | 1.0865  | 1.0891  | 1.0889  | 1.2617  | 1.2617  |
| $\alpha = 0^0$, | 0.0125                      | $m_1$       | 0.02194 | 0.02214 | 0.07269 | 0.07235 | 0.58531 | 0.58531 |
| $\beta = 180^0$ |                            | $m_2$       | 0.02432 | 0.02535 | 0.07338 | 0.07299 | 0.58540 | 0.58540 |
| $\gamma = 180^0$| 0.05122                     | $m_3$       | 0.05122 | 0.05236 | 0.08712 | 0.08633 | 0.58755 | 0.58755 |
|                 |                            | $m_4$       | 1.0865  | 1.0866  | 1.0891  | 1.0890  | 1.2642  | 1.2642  |
| $\alpha = 0^0$, | 0.0095                      | $m_1$       | 0.01814 | 0.01850 | 0.07235 | 0.07201 | 0.58099 | 0.58099 |
| $\beta = 180^0$ | 0.02124                     | $m_2$       | 0.02124 | 0.02171 | 0.07300 | 0.07262 | 0.58108 | 0.58108 |
| $\gamma = 180^0$| 0.04974                     | $m_3$       | 0.04974 | 0.05018 | 0.08636 | 0.08560 | 0.58322 | 0.58322 |
|                 |                            | $m_4$       | 1.0865  | 1.0865  | 1.0890  | 1.0888  | 1.2595  | 1.2595  |
Table 2: Neutrino mass eigenvalues in eV for Normal mass order

| Majorana Phases | Effective mass $(m_e$ in eV) | Mass states | Scheme |
|-----------------|------------------|-------------|--------|
| $\alpha = 180^\circ$ | $0.0095$ | $m_1$ | 0.01642 | 0.01697 | 0.07235 | 0.07201 | 0.58099 | 0.58099 |
| $\beta = 0^\circ$ | $m_2$ | 0.01965 | 0.02034 | 0.07300 | 0.07262 | 0.58107 | 0.58107 |
| $\gamma = 0^\circ$ | $m_3$ | 0.04887 | 0.04948 | 0.08636 | 0.08560 | 0.58322 | 0.58322 |
| | $m_4$ | 1.0865 | 1.0865 | 1.0890 | 1.0888 | 1.2595 | 1.2595 |
| | | | | | | | |
| $\alpha = 180^\circ$ | $0.011$ | $m_1$ | 0.01782 | 0.01833 | 0.07251 | 0.07217 | 0.58601 | 0.58601 |
| $\beta = 0^\circ$ | $m_2$ | 0.02069 | 0.02192 | 0.07318 | 0.07279 | 0.58309 | 0.58309 |
| $\gamma = 180^\circ$ | $m_3$ | 0.04934 | 0.05041 | 0.08671 | 0.08594 | 0.58524 | 0.58524 |
| | $m_4$ | 1.0865 | 1.0865 | 1.0891 | 1.0889 | 1.2617 | 1.2617 |
| $\alpha = 180^\circ$ | $0.0125$ | $m_1$ | 0.02050 | 0.02036 | 0.07269 | 0.07235 | 0.58539 | 0.58539 |
| $\beta = 0^\circ$ | $m_2$ | 0.02299 | 0.02374 | 0.07338 | 0.07299 | 0.58540 | 0.58540 |
| $\gamma = 180^\circ$ | $m_3$ | 0.05049 | 0.05146 | 0.08712 | 0.08633 | 0.58755 | 0.58755 |
| | $m_4$ | 1.0865 | 1.0866 | 1.0891 | 1.0890 | 1.2642 | 1.2642 |

$\alpha = 180^\circ$, $\beta = 0^\circ$, $\gamma = 180^\circ$
| Majorana Phases | Effective mass (m_e in eV) | Mass states | Scheme 1 | Scheme 2 | Scheme 3 | Scheme 4 | Scheme 5 | Scheme 6 |
|-----------------|----------------------------|-------------|---------|---------|---------|---------|---------|---------|
| α = 0°, β = 0°, γ = 0° | 0.03 | m_1 | 0.05242 | 0.04765 | 0.31268 | 0.31187 | 0.67474 | 0.67474 |
| m_2 | 0.04501 | 0.03934 | 0.31153 | 0.31072 | 0.67421 | 0.67421 |
| m_3 | 0.02703 | 0.01436 | 0.30947 | 0.30865 | 0.67326 | 0.67326 |
| m_4 | 1.0866 | 1.0864 | 1.1295 | 1.1293 | 1.2780 | 1.2780 |
| 0.04 | m_1 | 0.04863 | 0.04999 | 0.31196 | 0.30907 | 0.67510 | 0.67510 |
| m_2 | 0.04056 | 0.04211 | 0.30800 | 0.30790 | 0.67451 | 0.67451 |
| m_3 | 0.01908 | 0.02082 | 0.30873 | 0.30582 | 0.67346 | 0.67346 |
| m_4 | 1.0864 | 1.0865 | 1.1293 | 1.1285 | 1.291 | 1.291 |
| 0.05 | m_1 | 0.05406 | 0.05886 | 0.31160 | 0.30753 | 0.37923 | 0.37923 |
| m_2 | 0.04686 | 0.05234 | 0.31044 | 0.30636 | 0.37800 | 0.37800 |
| m_3 | 0.02965 | 0.03796 | 0.30837 | 0.30426 | 0.37580 | 0.37580 |
| m_4 | 1.0867 | 1.0870 | 1.1292 | 1.1281 | 1.1662 | 1.1662 |

| α = 0°, β = 180°, γ = 0° | 0.03 | m_1 | 0.05242 | 0.04765 | 0.31268 | 0.31187 | 0.67474 | 0.67474 |
| m_2 | 0.04501 | 0.03934 | 0.31153 | 0.31072 | 0.67421 | 0.67421 |
| m_3 | 0.02703 | 0.01436 | 0.30947 | 0.30865 | 0.67326 | 0.67326 |
| m_4 | 1.0866 | 1.0864 | 1.1295 | 1.1293 | 1.2780 | 1.2780 |
| 0.04 | m_1 | 0.04863 | 0.04999 | 0.31196 | 0.30907 | 0.67510 | 0.67510 |
| m_2 | 0.04056 | 0.04211 | 0.30800 | 0.30790 | 0.67451 | 0.67451 |
| m_3 | 0.01908 | 0.02082 | 0.30873 | 0.30582 | 0.67346 | 0.67346 |
| m_4 | 1.0864 | 1.0865 | 1.1293 | 1.1285 | 1.228 | 1.228 |
| 0.05 | m_1 | 0.05406 | 0.05886 | 0.31160 | 0.30753 | 0.37923 | 0.37923 |
| m_2 | 0.04686 | 0.05234 | 0.31044 | 0.30636 | 0.37800 | 0.37800 |
| m_3 | 0.02965 | 0.03796 | 0.30837 | 0.30426 | 0.37580 | 0.37580 |
| m_4 | 1.0867 | 1.0870 | 1.1292 | 1.1281 | 1.285 | 1.285 |
| α = 0°, β = 180°, γ = 180° | 0.03 | m_1 | 0.05242 | 0.04765 | 0.31268 | 0.31187 | 0.67474 | 0.67474 |
| m_2 | 0.04501 | 0.03934 | 0.31153 | 0.31072 | 0.67421 | 0.67421 |
| m_3 | 0.02703 | 0.01436 | 0.30947 | 0.30865 | 0.67326 | 0.67326 |
| m_4 | 1.0866 | 1.0864 | 1.1295 | 1.1293 | 1.2780 | 1.2780 |
| 0.04 | m_1 | 0.04863 | 0.04999 | 0.31196 | 0.30907 | 0.67510 | 0.67510 |
| m_2 | 0.04056 | 0.04211 | 0.30800 | 0.30790 | 0.67451 | 0.67451 |
| m_3 | 0.01908 | 0.02082 | 0.30873 | 0.30582 | 0.67346 | 0.67346 |
| m_4 | 1.0864 | 1.0865 | 1.1293 | 1.1285 | 1.2906 | 1.2906 |
| 0.05 | m_1 | 0.05406 | 0.05886 | 0.31160 | 0.30754 | 0.37923 | 0.37923 |
| m_2 | 0.04686 | 0.05234 | 0.31044 | 0.30637 | 0.37800 | 0.37800 |
| m_3 | 0.02965 | 0.03796 | 0.30837 | 0.30427 | 0.37580 | 0.37580 |
| m_4 | 1.0867 | 1.0870 | 1.1292 | 1.1281 | 1.1662 | 1.1662 |
Table 4: Neutrino mass eigenvalues in eV for Inverted mass order

| Majorana Phases | Effective mass (m_e in eV) | Mass states | Scheme |
|------------------|-----------------------------|-------------|--------|
|                  |                             | 1           | 2      | 3      | 4      | 5      | 6      |
| $\alpha = 180^0$, $\beta = 0^0$, $\gamma = 0^0$ | 0.03                        | $m_1$       | 0.06594 | 0.06814 | 0.29638 | 0.30574 | 0.13709 | 0.13709 |
|                  |                             | $m_2$       | 0.05981 | 0.06186 | 0.29516 | 0.30455 | 0.13408 | 0.13408 |
|                  |                             | $m_3$       | 0.04599 | 0.04526 | 0.29298 | 0.30245 | 0.13844 | 0.12844 |
|                  |                             | $m_4$       | 1.0875  | 1.0877  | 1.1252  | 1.1276  | 1.0949  | 1.0949  |
|                  |                             | 0.04        | $m_1$       | 0.07885 | 0.08259 | 0.28885 | 0.30241 | 0.13859 | 0.13859 |
|                  |                             | $m_2$       | 0.07278 | 0.07705 | 0.28759 | 0.30122 | 0.13530 | 0.13530 |
|                  |                             | $m_3$       | 0.05882 | 0.06341 | 0.28535 | 0.29909 | 0.12897 | 0.12897 |
|                  |                             | $m_4$       | 1.0886  | 1.0890  | 1.1233  | 1.1276  | 1.0958  | 1.0958  |
|                  |                             | 0.05        | $m_1$       | 0.1218  | 0.13887 | 0.26689 | 0.27993 | 0.14073 | 0.14073 |
|                  |                             | $m_2$       | 0.12097 | 0.13622 | 0.26554 | 0.27864 | 0.13702 | 0.13702 |
|                  |                             | $m_3$       | 0.11480 | 0.13138 | 0.26310 | 0.27633 | 0.12954 | 0.12954 |
|                  |                             | $m_4$       | 1.0927  | 1.0943  | 1.1178  | 1.1209  | 1.0969  | 1.0969  |
| $\alpha = 180^0$, $\beta = 0^0$, $\gamma = 180^0$ | 0.03                        | $m_1$       | 0.06594 | 0.06814 | 0.29638 | 0.30574 | 0.13709 | 0.13709 |
|                  |                             | $m_2$       | 0.05981 | 0.06186 | 0.29516 | 0.30455 | 0.13408 | 0.13408 |
|                  |                             | $m_3$       | 0.04599 | 0.04526 | 0.29298 | 0.30245 | 0.13844 | 0.12844 |
|                  |                             | $m_4$       | 1.0875  | 1.0877  | 1.1252  | 1.1276  | 1.0949  | 1.0949  |
|                  |                             | 0.04        | $m_1$       | 0.07885 | 0.08259 | 0.28885 | 0.30241 | 0.13859 | 0.13859 |
|                  |                             | $m_2$       | 0.07278 | 0.07705 | 0.28759 | 0.30122 | 0.13530 | 0.13530 |
|                  |                             | $m_3$       | 0.05882 | 0.06341 | 0.28535 | 0.29909 | 0.12897 | 0.12897 |
|                  |                             | $m_4$       | 1.0886  | 1.0890  | 1.1233  | 1.1276  | 1.0958  | 1.0958  |
|                  |                             | 0.05        | $m_1$       | 0.1218  | 0.13887 | 0.26689 | 0.27993 | 0.14073 | 0.14073 |
|                  |                             | $m_2$       | 0.12097 | 0.13622 | 0.26554 | 0.27864 | 0.13702 | 0.13702 |
|                  |                             | $m_3$       | 0.11480 | 0.13138 | 0.26310 | 0.27633 | 0.12954 | 0.12954 |
|                  |                             | $m_4$       | 1.0927  | 1.0943  | 1.1178  | 1.1209  | 1.0969  | 1.0969  |
| $\alpha = 180^0$, $\beta = 180^0$, $\gamma = 0^0$ | 0.03                        | $m_1$       | 0.06594 | 0.06814 | 0.29638 | 0.30574 | 0.13709 | 0.13709 |
|                  |                             | $m_2$       | 0.05981 | 0.06186 | 0.29516 | 0.30455 | 0.13408 | 0.13408 |
|                  |                             | $m_3$       | 0.04599 | 0.04526 | 0.29298 | 0.30245 | 0.13844 | 0.12844 |
|                  |                             | $m_4$       | 1.0875  | 1.0877  | 1.1252  | 1.1276  | 1.0949  | 1.0949  |
|                  |                             | 0.04        | $m_1$       | 0.07885 | 0.08259 | 0.28885 | 0.30241 | 0.13859 | 0.13859 |
|                  |                             | $m_2$       | 0.07278 | 0.07705 | 0.28759 | 0.30122 | 0.13530 | 0.13530 |
|                  |                             | $m_3$       | 0.05882 | 0.06341 | 0.28535 | 0.29909 | 0.12897 | 0.12897 |
|                  |                             | $m_4$       | 1.0886  | 1.0890  | 1.1233  | 1.1276  | 1.0958  | 1.0958  |
|                  |                             | 0.05        | $m_1$       | 0.1218  | 0.13887 | 0.26689 | 0.27993 | 0.14073 | 0.14073 |
|                  |                             | $m_2$       | 0.12097 | 0.13622 | 0.26554 | 0.27864 | 0.13702 | 0.13702 |
|                  |                             | $m_3$       | 0.11480 | 0.13138 | 0.26310 | 0.27633 | 0.12954 | 0.12954 |
|                  |                             | $m_4$       | 1.0927  | 1.0943  | 1.1178  | 1.1209  | 1.0969  | 1.0969  |
Figure 3: Left and right pannel respectively shows the value of effective Majorana mass $|m_e|$ as a function of lower mass $m_1$ and sum of the neutrino masses $\sum_{(3+1)}$ in the case of (3+1) neutrino mixing with Normal and Inverted Ordering. The signs combination as shown in the legend imply the signs of $e^{i\alpha}$, $e^{i\beta}$, $e^{i\gamma} = \pm 1$ for the eight possible cases in which CP is conserved.

exact picture of mass spectrum.

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