A SCHREIER DOMAIN TYPE CONDITION

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Abstract. We study the integral domains $D$ satisfying the following condition: whenever $I \supseteq AB$ with $I, A, B$ nonzero ideals, there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$.

In [6], Cohn introduced the notion of Schreier domain. A domain $D$ is said to be a Schreier domain if (1) $D$ is integrally closed and (2) whenever $I, J_1, J_2$ are principal ideals of $D$ and $I \supseteq J_1J_2$, then $I = I_1I_2$ for some principal ideals $I_1, I_2$ of $D$ with $I_i \supseteq J_i$ for $i = 1, 2$. The study of Schreier domains was continued in [13] and [17] (where a domain was called a pre-Schreier domain if it satisfies condition (2) above). In [8] and [3], an extension of the class of pre-Schreier domains was studied. A domain $D$ was called a quasi-Schreier domain if whenever $I, J_1, J_2$ are invertible ideals of $D$ and $I \supseteq J_1J_2$, then $I = I_1I_2$ for some (invertible) ideals $I_1, I_2$ of $D$ with $I_i \supseteq J_i$ for $i = 1, 2$.

In this paper we study the domains satisfying a Schreier-like condition for all nonzero ideals. Since this class of domains turns out to be rather narrow, we use an ad hoc name for it.

Definition 1. We call a domain $D$ a sharp domain if whenever $I \supseteq AB$ with $I, A, B$ nonzero ideals of $D$, there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$.

If the domain $D$ is Noetherian or Krull, then $D$ is sharp if and only if $D$ is a Dedekind domain (Corollaries [2] and [12]). In Proposition [3] we show that a sharp domain is pseudo-Dedekind. In particular, a sharp domain is a completely integrally closed GGCD domain. The ring $E$ of entire functions is pseudo-Dedekind but not sharp (Example [5]). Recall (cf. [10] and [2]) that a domain $D$ is called a pseudo-Dedekind domain (the name used in [10] was generalized Dedekind domain) if the $v$-closure of each nonzero ideal of $D$ is invertible. Also, recall from [2] that a domain $D$ is called a generalized GCD domain (GGCD domain) if the $v$-closure of each nonzero finitely generated ideal of $D$ is invertible. The definition of the $v$-closure is recalled below. In Proposition [5] we show that a valuation domain is sharp if and only if the value group of $D$ is a complete subgroup of the reals.

The main results of this paper are Theorems [11] and [15]. In Theorem [11] we show that the localizations of a sharp domain at the maximal ideals are valuation domains with value group a complete subgroup of the reals. In particular, a sharp domain is a Prüfer domain of dimension $\leq 1$. A key point in proving Theorem [11] is the fact that if $D$ is a sharp domain and $x, y \in D - \{0\}$ such that $xD \cap yD = xyD$, then $xD + yD = D$ (Proposition [10]). The converse of Theorem [11] is not true (Example [15]). In Theorem [15] we prove the converse of Theorem [11] for the domains of finite character (i.e., domains whose every nonzero element is contained in only finitely

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many maximal ideals). The problem whether a sharp domain is necessarily of finite character is left open. A countable sharp domain is a Dedekind domain (Corollary 17).

For reader’s convenience, we recall the following facts. Let $D$ be a domain with quotient field $K$ and $I$ a nonzero fractional ideal of $D$. The $v$-closure of $I$ is the fractional ideal $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{ x \in K \mid xI \subseteq D \}$, and $I$ is called a $v$-ideal if $I = I_v$. The $t$-closure of $I$ is the fractional ideal $I_t$ which is the union of the $v$-closures of the finitely generated nonzero subideals of $I$. Moreover, $I$ is called a $t$-ideal if $I = I_t$. In general, we have $I \subseteq I_t \subseteq I_v$. A nonzero prime ideal $P$ of $D$ is called $t$-prime if $P = P_v$. For basic facts and terminology not recalled in this paper, our references are [10] and [11]. Throughout this paper, all rings are domains, that is, commutative, unitary and without zero-divisors.

We begin with a characterization of the sharp domains. If $I, H$ are ideals of a domain $D$, we denote by $I : H$ the ideal $\{ x \in D \mid xH \subseteq I \}$.

**Proposition 2.** A domain $D$ is sharp if and only if for every two nonzero ideals $I, H$ we have $I = [I : (I : H)](I : H)$.

*Proof.* $(\Rightarrow)$. Let $I, H$ be nonzero ideals of $D$. Set $A = I : (I : H)$ and $B = I : H$. Note that $AB \subseteq I$, $A = I : B$ and $I : A = I : (I : H) = I : H = B$. As $D$ is sharp, there exists a factorization $I = A'B'$ with $A', B'$ ideals such that $A' \supseteq A$ and $B' \supseteq B$. Then $A \subseteq A' \subseteq I : B' \subseteq I : B = A$, so $A = A'$. Similarly, we get $B = B'$. $(\Leftarrow)$. Let $I, A, B$ be nonzero ideals of $D$ such that $AB \subseteq I$. By our assumption, we get $I = [I : (I : A)](I : A)$. Note that $A \subseteq I : (I : A)$ and $B \subseteq I : A$. •

**Corollary 3.** A Dedekind domain is a sharp domain.

*Proof.* Let $D$ be a Dedekind domain and $I, H$ nonzero ideals of $D$. Since $I : H$ is an invertible ideal, it follows easily that $I : (I : H) = I(I : H)^{-1}$. Hence $[I : (I : H)](I : H) = I(I : H)^{-1}(I : H) = I$. Apply Proposition 2. •

**Proposition 4.** Every sharp domain is pseudo-Dedekind. In particular, a sharp domain is a completely integrally closed GGCD domain.

*Proof.* Let $D$ be a sharp domain, $A$ a nonzero ideal of $D$ and $0 \neq b \in A$. By Proposition 2 $bD = [bD : (bD : A)](bD : A)$. It follows that $bD : A = bA^{-1}$ is an invertible ideal, so $A^{-1}$ and $A_v$ are invertible ideals. Thus $D$ is pseudo-Dedekind. By [10] Corollaries 1.4 and 1.5, a pseudo-Dedekind domain is a completely integrally closed GGCD domain. •

We show that for a pseudo-Dedekind domain $D$ it suffices to test the condition in Definition 3 only for ideals $I$ with $I_v = D$.

**Proposition 5.** A pseudo-Dedekind domain $D$ is sharp if and only if for all nonzero ideals $I, A, B$ of $D$ such that $I \supseteq AB$ and $I_v = D$, there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$.

*Proof.* We prove the nontrivial implication. Let $I, A, B$ be nonzero ideals of $D$ such that $I \supseteq AB$. Then $I_v \supseteq A_vB_v$ and $I_v, A_v, B_v$ are invertible ideals, because
In particular, a sharp valuation domain has dimension \( \leq b \) follows from Proposition 4. We prove that \( I \) is pseudo-Dedekind. A pseudo-Dedekind domain is a GGCD domain, cf. [16, Corollary 1.5], and a GGCD domain is quasi-Schreier, cf. [3, Proposition 2.3]. So Proposition 6.

For a valuation domain \( D \), the following assertions are equivalent:

(a) \( D \) is sharp.

(b) \( D \) is pseudo-Dedekind.

(c) the value group of \( D \) is a complete subgroup of the reals.

In particular, a sharp valuation domain has dimension \( \leq 1 \).

Proof. (b) \( \Leftrightarrow \) (c) is given in [3] at the bottom of pages 325 and 327 and (a) \( \Rightarrow \) (b) follows from Proposition 3. We prove that (b) and (c) imply (a). By Corollary 3, we may assume that the value group of \( D \) is the whole group of real numbers. By Proposition 5, \( D \) is sharp, because the maximal ideal is the only proper ideal of \( D \) whose \( v \)-closure is \( D \). The “in particular” assertion follows from the well-known fact that a valuation domain has dimension \( \leq 1 \) if and only if its value group is a subgroup of the reals (see [18, page 45]).

Proposition 7. If \( D \) is a sharp domain, then every fraction ring \( D_S \) of \( D \) is also a sharp domain.

Proof. Let \( I, A, B \) be nonzero ideals of \( D \) such that \( ID_S \supseteq ABD_S \). Then \( H = ID_S \cap D \supseteq AB \). As \( D \) is sharp, we get \( H = A'B' \) with \( A', B' \) ideals of \( D \) such that \( A' \supseteq A \) and \( B' \supseteq B \). Then \( ID_S = HDS = A'B'D_S \).

Example 8. The ring \( E \) of entire functions is pseudo-Dedekind but some localization of \( E \) is not pseudo-Dedekind, cf. [16, Example 2.1]. By Proposition 7, \( E \) is not a sharp domain.

Proposition 9. If \( D \) is a sharp domain and \( P \) is a t-prime ideal of \( D \), then \( DP \) is a valuation domain whose value group is a complete subgroup of the reals. In particular, in a sharp domain every t-prime ideal of \( D \) has height one.

Proof. By Proposition 2, \( D \) is a GGCD domain. By [2] page 218, [14, Corollary 4.3] and Proposition 7, \( DP \) is a sharp valuation domain. Apply Proposition 6.

Recall that two nonzero elements \( x, y \) of a domain \( D \) are called \( v \)-coprime if \( (xD + yD)_v = D \) (equivalently \( xD \cap yD = xyD \), equivalently \( xD : yD = xD \)).

Proposition 10. Let \( D \) be a sharp domain and \( x, y \) two nonzero \( v \)-coprime elements. Then \( xD + yD = D \).

Proof. We have \( (x, y)^2 \subseteq (x^2, y) \), so \( (x^2, y) \) is \( AB \) with \( A, B \) ideals such that \( A, B \supseteq (x, y) \). Note that \( (x^2, y) : (x, y) = (x, y) \). Indeed, if \( a \in (x^2, y) : (x, y) \), then
ax = bx^2 + cy for some b, c ∈ D, so c ∈ xD : yD = xD, hence a = bx + (c/x)y belongs to (x, y). From (x^2, y) = AB, we get A ⊆ (x^2, y), hence A = (x, y) = (x^2, y), similarly, we get B = (x, y). Then (x^2, y) = (x, y)^2. So y = fx + gy^2 for some f, g ∈ D, hence f ∈ yD : xD = yD, thus 1 = (f/y)x + gy, that is, xD + yD = D.

Theorems 11 and 15 are the main results of this paper.

**Theorem 11.** If D is a sharp domain, then DM is a valuation domain with value group a complete subgroup of the reals, for each maximal ideal M of D. In particular, a sharp domain is a Prüfer domain of dimension ≤ 1.

**Proof.** By Proposition 7 we may assume that D is quasi-local with nonzero maximal ideal M. Suppose that the height of M is ≥ 2. By Proposition 4, D is a quasi-local GGCD domain, hence a GCD domain, cf. [2, Corollary 1]. By Proposition 5, M is not a t-ideal, so M1 = D. Since D is a GCD domain, there exist two v-coprime elements x, y ∈ M (see the paragraph before Theorem 4.8 in [1]). But this contradicts Proposition 10. It remains that M has height one, hence it is a t-prime, cf. [11, Proposition 6.6]. Now apply Proposition 9 to conclude. The “in particular” assertion is clear.

According to [12], a TV domain is a domain in which every t-ideal is a v-ideal. It is well known that Noetherian domains and Krull domains are TV domains.

**Corollary 12.** If D is a sharp TV domain, then D is a Dedekind domain. In particular, if a sharp domain is Noetherian or Krull, then it is a Dedekind domain.

**Proof.** Let D be a sharp TV domain. By Theorem 11, D is a Prüfer domain, so every nonzero ideal of D is a t-ideal, hence a v-ideal, because D is a TV domain. Since D is also a pseudo-Dedekind domain (cf. Proposition 1), it follows that every nonzero ideal of D is invertible. Thus D is a Dedekind domain.

The converse of Theorem 11 is not true. Recall [10, page 434] that a domain D is said to be almost Dedekind if DM is a discrete (Noetherian) valuation domain for each maximal ideal M of D. We exhibit an almost Dedekind domain which is not a sharp domain (not even pseudo-Dedekind).

**Example 13.** Let D be the almost Dedekind domain constructed in the proof of [7, Proposition 7]. We recall some properties of D proved there. The maximal ideals of D are the principal ideals (pD)i≥1 and the ideal M = (q0, q1, ..., qn,...). Here (qi)i≥0 are nonzero elements of D such that qi−1 = piqi and pi does not divide qi for all i ≥ 1. Note that M is not finitely generated, because it is the union of the strictly ascending chain of principal ideals (qD)i≥0. We claim that D is not pseudo-Dedekind, so it is not a sharp domain (cf. Proposition 6). For that, it suffices to prove that the v-ideal ∩i≥1p2i−1D equals the union of the strictly ascending chain of principal ideals p1q2D ⊂ p1q3q4D ⊂ p1p3q5q6D · · · , so it is not finitely generated. Indeed, the inclusion ⊇ is clear. Conversely, let x ∈ ∩i≥1p2i−1D. If x /∈ M, then 1 = ax + bq2n for some a, b ∈ D and n ≥ 0. But this is a contradiction, because p2n+1 divides both x and q2n. So x ∈ M, say x = cq2n for some c ∈ D and...
\[ n \geq 1. \text{ Since } x \in \cap_{i \geq 1}p_{2i-1}D \text{ and } q_{2n} \text{ is not divisible by } p_1, p_3, \ldots, p_{2n-1}, \text{ we get that } x \in p_1p_3 \cdots p_{2n-1}q_{2n}D. \]

We give a partial converse of Theorem [11]. Recall that a domain \( D \) is said to be of \textit{finite character} if every nonzero element is contained in only finitely many maximal ideals. And \( D \) is said to be \textit{h-local} if \( D \) is of finite character and every nonzero prime ideal of \( D \) is contained in a unique maximal ideal of \( D \). It is easy to see that a one-dimensional domain of finite character is \( h \)-local. The next lemma was implicit in [15 Proposition 3.1].

**Lemma 14.** Let \( D \) be a \( h \)-local domain, \( A, B \) nonzero ideals of \( D \) and \( M \in \text{Max}(D) \). Then \((A : B)D_M = AD_M : BD_M\).

**Proof.** Let \( K \) denote the quotient field of \( D \). The inclusion \( (\subseteq) \) is clear. Conversely, let \( x \in AD_M : BD_M \). We may assume that \( x \in D \). Pick \( a \in A - \{0\} \). Since \( D \) is \( h \)-local, we have \([M]D_M = K\) where \([M] = \cap\{D_N \mid N \in \text{Max}(D)\} \) and \( N \neq M \), cf. [15 Proposition 3.1]. Consequently, there exist \( y \in [M] \) and \( s \in D - M \) such that \( x/a = y/s \). So \( sx = ay \). Note that \( ayB \subseteq AD_N \) for each \( N \in \text{Max}(D) - \{M\} \). So \( sxB_Q \subseteq AD_Q \) for each \( Q \in \text{Max}(D) \), hence \( sx \in (A : B)D_M \). Thus \( x \in (A : B)D_M \). \( \bullet \)

We show that the converse of Theorem [11] is true for a domain of finite character.

**Theorem 15.** Let \( D \) be a domain of finite character such that \( D_M \) is a valuation domain with value group a complete subgroup of the reals for each \( M \in \text{Max}(D) \). Then \( D \) is a sharp domain.

**Proof.** Let \( I, A \) be nonzero ideals of \( D \). By Proposition [2] it suffices to check locally that \( (I : A)(I : (I : A)) = I \). Let \( M \) be a maximal ideal of \( D \). Since \( D \) is one-dimensional of finite character, it is \( h \)-local. By Lemma [14], we have \((I : A)(I : (I : A))D_M = (ID_M : AD_M)\) \( (ID_M : (ID_M : AD_M)) = ID_M \), where the last equality follows from Propositions [6] and [2]. \( \bullet \)

We do not know if a sharp domain is necessarily of finite character. A connected question, which is up to our knowledge not solved, is whether a pseudo-Dedekind almost Dedekind domain is necessarily a Dedekind domain. We end our paper with two results for countable domains.

**Proposition 16.** If \( D \) is a countable pseudo-Dedekind Prüfer domain, then \( D \) is of finite character.

**Proof.** Assume that \( D \) is not of finite character. By [9 Corollary 7], there exists a nonzero element \( z \) and an infinite family \( (I_n)_{n \geq 1} \) of invertible proper mutually comaximal ideals containing \( z \). For each nonempty set of natural numbers \( \Lambda \), consider the \( v \)-ideal \( I_\Lambda = \cap_{n \in \Lambda} I_n \) (note that \( I_\Lambda \) contains \( z \)). As \( D \) is pseudo-Dedekind, \( I_\Lambda \) is invertible. We claim that \( I_\Lambda \neq I_{\Lambda'} \) whenever \( \Lambda, \Lambda' \) are distinct nonempty sets of natural numbers. Deny. Then there exists a nonempty set of natural numbers \( \Gamma \) and some \( k \notin \Gamma \) such that \( I_k \supseteq I_\Gamma \). Consider the ideal \( H = I_k^{-1}I_\Gamma \supseteq I_\Gamma \). If \( n \in \Gamma \), then \( I_n \supseteq I_\Gamma = I_kH \), so \( I_n \supseteq H \), because \( I_n + I_k = D \). It follows that \( I_\Gamma \supseteq H \), so \( I_\Gamma = H = I_k^{-1}I_\Gamma \). Since \( I_\Gamma \) is invertible, we get \( I_k = D \), a contradiction. Thus the claim is proved. But then it follows that \( \{I_\Lambda \mid \emptyset \neq \Lambda \subseteq \mathbb{N} \} \) is an uncountable set of
invertible ideals. This leads to a contradiction, because $D$ being countable, it has countably many finitely generated ideals. 

**Corollary 17.** If $D$ is a countable sharp domain, then $D$ is a Dedekind domain.

**Proof.** We may assume that $D$ is not a field. By Theorem 11, $D$ is a Pr"ufer domain. Now Propositions 4 and 16 show that $D$ is of finite character. Let $M$ be a maximal ideal of $D$. By Theorem 11, $D_M$ is a countable valuation domain with value group $\mathbb{Z}$ or $\mathbb{R}$, so $D_M$ is a DVR. Thus $D$ is a Dedekind domain. 

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**References**

[1] D. D. Anderson, GCD domains, Gauss' lemma, and contents of polynomials, *Non-Noetherian Commutative Ring Theory*, 1-31, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[2] D.D. Anderson and D.F. Anderson, Generalized GCD domains, Comment. Math. Univ. St. Pauli 28 (1979), 215-221.

[3] D.D. Anderson, T. Dumitrescu and M. Zafrullah, Quasi-Schreier domains II, Comm. Algebra 35 (2007), 2096-2104.

[4] D.D. Anderson and B.G. Kang, Pseudo-Dedekind domains and divisorial ideals in $R[X]_r$, J. Algebra 122 (1989), 323-336.

[5] N. Bourbaki, Commutative Algebra, Springer, Berlin, 1989.

[6] P.M. Cohn, Bezout rings and their subrings, Proc. Cambridge Philos. Soc. 64 (1968), 251-264.

[7] J. Coykendall and T. Dumitrescu, Integral domains having nonzero elements with infinitely prime divisors, Comm. Algebra 35 (2007), 1333-1339.

[8] T. Dumitrescu and R. Moldovan, Quasi-Schreier domains. Math. Reports 5 (2003), 121-126.

[9] T. Dumitrescu and M. Zafrullah, Characterizing domains of finite *-character, J. Pure Appl. Algebra 214 (2010), 2087-2091.

[10] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[11] F. Halter-Koch, *Ideal Systems: an Introduction to Multiplicative Ideal Theory*, Marcel Dekker, New York 1998.

[12] E. Houston and M. Zafrullah, Integral domains in which each t-ideal is divisorial, Mich. Math. J. 35 (1988), 291-300.

[13] S. McAdam and D.E. Rush, Schreier Rings, Bull. London Math. Soc. 10 (1978), 77-80.

[14] J.L. Mott and M. Zafrullah, On Pr"ufer v-multiplication domains, Manuscripta Math. 35 (1981), 1-26.

[15] B. Olberding, Globalizing local properties of Pr"ufer domains, J. Algebra 205 (1998), 480-504.

[16] M. Zafrullah, On generalized Dedekind domains, Mathematika 33 (1986), 285-295.

[17] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15 (1987), 1895-1920.

[18] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton 1960.
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