VISIBLE LATTICE POINTS AND THE EXTENDED LINDELÖF HYPOTHESIS

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Abstract. We consider the number of visible lattice points under the assumption of the Extended Lindelöf Hypothesis. We get a relation between visible lattice points and the Extended Lindelöf Hypothesis. And we also get a relation between visible lattice points over $\mathbb{Q}(\sqrt{-1})$ and the Gauss Circle Problem.

1. Introduction

Let $K$ be a number field and let $\mathcal{O}_K$ be its ring of integers. We consider an $m$-tuple of ideals $(a_1, a_2, \ldots, a_m)$ of $\mathcal{O}_K$ as a lattice points in $\text{Frac}(\mathcal{O}_K)^m$. When $\mathcal{O}_K = \mathbb{Z}$ they are ordinally lattice points. We say that a lattice point $(a_1, a_2, \ldots, a_m)$ is visible from the origin, if $a_1 + \cdots + a_m = \mathcal{O}_K$.

There are many results about the number of visible lattice point from 1800’s. In the case $K = \mathbb{Q}$, D. N. Lehmer prove that the density of the set of visible lattice points in $\mathbb{Q}^m$ is $1/\zeta(m)$ in 1900 [3]. And in general case, B. D. Sittinger proved the number of visible lattice points $(a_1, a_2, \ldots, a_m)$ in $K^m$ with $\Re a_i \leq x$ for all $i = 1, \ldots, m$ is

$$\frac{c^m}{\zeta_K(m)} x^m + (\text{Error term}),$$

where $\zeta_K$ is the Dedekind zeta function over $K$ and $c$ is a positive constant depending only on $K$ [4].

Let $V_m(x, K)$ denote the number of visible lattice points $(a_1, a_2, \ldots, a_m)$ with $\Re a_i \leq x$ for all $i = 1, \ldots, m$. When $K = \mathbb{Q}$, $V_m(x, \mathbb{Q})$ means the number of visible lattice points in $(0, x]^m$. And we let $E_m(x, K)$ denote its error term, i.e. $E_m(x, K) = V_m(x, K) - (cx)^m/\zeta_K(m)$.

In the case $K = \mathbb{Q}$, we proved that the exact order of $E_m(x, \mathbb{Q})$ is $x^{m-1}$ for $m \geq 3$ [10]. But we do not know about the exact order of $E_m(x, K)$. In this paper, we consider better upper order of $E_m(x, K)$ under the situation that the Extended Lindelöf Hypothesis is true. The statement of our main theorem is the following.

**Theorem.** If we assume the Extended Lindelöf Hypothesis, we get

$$E_m(x, K) = O(x^{m-1/2+\varepsilon})$$

for all algebraic number field $K$ and for all $\varepsilon > 0$.

As a result, we can think of considering the number of visible lattice points as considering the Extended Lindelöf Hypothesis. And we show that the number of visible lattice points in $\mathbb{Q}(\sqrt{-1})^m$ are associated with the Gauss Circle Problem.

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2. The Extended Lindelöf Hypothesis

The Dedekind zeta function $\zeta_K$ over $K$ is considered as a generalization of the Riemann zeta function and $\zeta_K$ is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a} \in \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s},$$

with the sum taken over all nonzero ideals of $\mathcal{O}_K$.

Riemann proposed that all non-trivial zeros of the Riemann zeta function is on the line $\Re(s) = 1/2$ in his paper [2]. The Extended Riemann Hypothesis over algebraic number field is known as a generalization of the Riemann hypothesis. The statement of the Extended Riemann Hypothesis is "for all algebraic number field $K$ all non-trivial zeros of the Dedekind zeta function is on the line $\Re(s) = 1/2$".

One of other important hypotheses in analytic number theory is the Lindelöf Hypothesis. As well as the Riemann Hypothesis, this hypothesis can be generalized over algebraic number fields. The Extended Lindelöf Hypothesis can be written as follows.

**Extended Lindelöf Hypothesis.** For $\sigma \geq 1/2$

$$\zeta_K(\sigma + it) = O(t^\varepsilon)$$

for every $\varepsilon > 0$.

In 1918, R. Backlund proved that the Lindelöf Hypothesis is equivalent to the statement that the number of zeros of the Riemann zeta function $\zeta(s)$ in the strip $\{s = \sigma + it \mid 1/2 < \sigma, T \leq t \leq T + 1\}$ is $o(\log T)$ as $T$ tends to $\infty$ [7]. On the other hand, the Riemann Hypothesis stated that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ is on the line $\Re(s) = 1/2$, so this hypothesis implies the Lindelöf Hypothesis.

As well as this result, the Extended Lindelöf Hypothesis can be followed from the Extended Riemann Hypothesis. Thus the following theorem holds.

**Theorem 2.1.** If the Extended Riemann Hypothesis holds, then for all $\sigma \geq 1/2$

$$\zeta_K(\sigma + it) = O(t^\varepsilon)$$

for every $\varepsilon > 0$.

From 1900’s many results were shown under the situation that the Riemann Hypothesis is true. We assumed the Extended Lindelöf Hypothesis in this paper, hence we can get same results with assuming the Extended Riemann Hypothesis. We consider a relation between these hypotheses and visible lattice points from the origin in following sections.

3. Preparation for proof of our main theorem

In this section, we prepare for showing the main theorem. We consider the number of ideals of $\mathcal{O}_K$ with their ideal norm is less than or equal to $x$. First we need following lemma about complex analysis.
Lemma 3.1. We have

\[ \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{s} \, ds = \begin{cases} O\left( \frac{x^2}{T \log x^{-1}} \right) & \text{if } 0 < x < 1, \\ \frac{1}{2} + O\left( \frac{1}{T} \right) & \text{if } x = 1, \\ 1 + O\left( \frac{x^2}{T \log x} \right) & \text{if } x > 1. \end{cases} \]

We can prove this lemma by using contour integrals. (For the details of the proof of this result, please see Lemma 4 in section 11 of [9].) We apply this lemma to consider the number of ideals of \( \mathcal{O}_K \) with their ideal norm less than or equal to \( x \). As shown below, we can compute it with smaller error term by assuming the Extended Lindelöf Hypothesis.

Theorem 3.2. Let \( j_K(x) \) be the number of ideals of \( \mathcal{O}_K \) with their ideal norm less than or equal to \( x \). Assume the Extended Lindelöf Hypothesis. Then for every \( \varepsilon > 0 \), we have

\[ j_K(x) = cx + O(x^{1/2+\varepsilon}), \]

where

\[ c = \frac{2^{r_1}(2\pi)^{r_2} h R}{w \sqrt{|d_K|}}, \]

and:
- \( h \) is the class number of \( K \),
- \( r_1 \) and \( r_2 \) is the number of real and complex absolute values of \( K \) respectively,
- \( R \) is the regulator of \( K \),
- \( w \) is the number of roots of unity in \( \mathcal{O}_K^* \),
- \( d_K \) is discriminant of \( K \).

Proof. It suffices to show that \( j_K(x) = cx + O(x^{1/2+\varepsilon}) \) for all half integer \( x = n + 1/2 \), where \( n \) is a positive integer, because it holds for any real number \( y \in [n, n+1) \) that \( j_K(x) = j_K(y) \).

We consider the integral

\[ \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta_K(s) \frac{x^s}{s} \, ds. \]

The series \( \zeta_K(s) = \sum_a \frac{1}{\mathfrak{f} \mathfrak{a}^s} \) is absolutely and uniformly convergent on compact subsets on \( \Re(s) > 1 \). Therefore we can interchange the order of summation and integral in above equation to obtain

\[ \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta_K(s) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \sum_a \frac{1}{\mathfrak{f} \mathfrak{a}^s} \frac{x^s}{s} \, ds \]

\[ = \sum_a \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\mathfrak{f} \mathfrak{a}^s} \frac{x^s}{s} \, ds. \]
Since \( x \) was chosen to be a half integer, there are no terms with \( \Re a = x \) in the above sum. From this, we get

\[
\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta_K(s) \frac{x^s}{s} \, ds = \sum_{\Re a < x} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{\Re a^s} \, ds + \sum_{\Re a > x} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{\Re a^s} \, ds.
\]

By Lemma 3.1 for two sums,

\[
\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta_K(s) \frac{x^s}{s} \, ds = \sum_{\Re a < x} \left( 1 + O \left( \frac{x^2}{T \Re a^2 \log x / \Re a} \right) \right) + O \left( \sum_{\Re a > x} \frac{x^2}{T \Re a^2 \log \Re a / x} \right)
\]

\[
= j_K(x) + O \left( \frac{x^2}{T} \sum_{\Re a < x} \frac{1}{\Re a^2 \log \Re a / x} + \frac{x^2}{T} \sum_{\Re a > x} \frac{1}{\Re a^2 \log \Re a / x} \right).
\]

Now we estimate how fast the above sums grow. Since \( x = n + 1/2 \)

\[
\left| \sum_{\Re a < x} \frac{1}{\Re a^2 \log x / \Re a} \right| \leq \left( \log \left( \frac{x}{x - \frac{1}{2}} \right) \right)^{-1} \sum_{\Re a < x} \frac{1}{\Re a^2}
\]

\[
= O \left( x \sum_{\Re a < x} \frac{1}{\Re a^2} \right),
\]

because \( \log \left( 1 - \frac{1}{2x} \right) = O(x^{-1}) \) for \( x > 1 \). The sum is bounded by \( \zeta_K(2) \), so we estimate

\[
\left| \sum_{\Re a < x} \frac{1}{\Re a^2 \log x / \Re a} \right| = O(x).
\]

By similar estimate the sum over \( \Re a > x \) is \( O(x) \). Hence

\[
\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta_K(s) \frac{x^s}{s} \, ds = j_K(x) + O \left( \frac{x^3}{T} \right).
\]

We can select large \( T \) so that the error term is sufficiently small.

Next we consider the integral

\[
\frac{1}{2\pi i} \int_C \zeta_K(s) \frac{x^s}{s} \, ds,
\]

where \( C \) is \( C_4 C_3 C_2 C_1 \) in the following figure.
The integral on \( C_1 \) is the integral which we considered, so we will estimate other integrals.

First we consider the integral over \( C_2 \) as

\[
\left| \frac{1}{2\pi i} \int_{C_2} \zeta_K(s) \frac{x^s}{s} ds \right| = \left| \frac{1}{2\pi i} \int_2^{1/2} \zeta_K(\sigma + iT) \frac{x^{\sigma+iT}}{\sigma + iT} d\sigma \right| \\
\leq \frac{1}{2\pi} \int_{1/2}^2 |\zeta_K(\sigma + iT)| \frac{x^\sigma}{|\sigma + iT|} d\sigma.
\]

We assume the Extended Lindelöf Hypothesis, so we have \( \zeta_K(\sigma + iT) = O(T^\varepsilon) \),

\[
\left| \frac{1}{2\pi i} \int_{C_2} \zeta_K(s) \frac{x^s}{s} ds \right| = O\left( \int_{1/2}^2 T^\varepsilon \frac{x^\sigma}{|\sigma + iT|} d\sigma \right) \\
= O\left( \frac{x^{1/2}}{T^{1-\varepsilon}} \right).
\]

As well as before, if we select sufficiently large \( T \), the error term will be very small.

Next we calculate the integral over \( C_3 \) as

\[
\left| \frac{1}{2\pi i} \int_{C_3} \zeta_K(s) \frac{x^s}{s} ds \right| = \left| \frac{1}{2\pi i} \int_T^{1/2} \zeta_K \left( \frac{1}{2} + it \right) \frac{x^{1/2+it}}{\frac{1}{2} + it} i dt \right| \\
\leq \frac{1}{2\pi} \int_{1/2}^T \left| \zeta_K \left( \frac{1}{2} + it \right) \right| \frac{x^{1/2}}{\frac{1}{2} + it} dt.
\]

We assume the Extended Lindelöf Hypothesis, so we have \( \zeta_K \left( \frac{1}{2} + it \right) = O(t^\varepsilon) \),

\[
\left| \frac{1}{2\pi i} \int_{C_3} \zeta_K(s) \frac{x^s}{s} ds \right| = O\left( \int_{1/2}^T T^\varepsilon \frac{x^{1/2}}{\left| \frac{1}{2} + it \right|} dt \right) \\
= O(x^{1/2}T^\varepsilon \log T).
\]
If we select $T = x^3$, then the order of this error term is $O(x^{1/2+\varepsilon})$.

Finally we estimate the integral over $C_4$ as

$$\left| \frac{1}{2\pi i} \int_{C_4} \zeta_K(s) \frac{x^s}{s} \, ds \right| = \frac{1}{2\pi i} \int_{1/2}^{2} \left| \zeta_K(\sigma - iT) \frac{x^{\sigma - iT}}{\sigma - iT} \right| d\sigma \leq \frac{1}{2\pi} \int_{1/2}^{2} \frac{1}{|\sigma - iT|} \frac{x^\sigma}{|\sigma - iT|} d\sigma.$$

We assume the Extended Lindelöf Hypothesis, so we have $\zeta_K(\sigma - iT) = O(T^\varepsilon)$,

$$\left| \frac{1}{2\pi i} \int_{C_4} \zeta_K(s) \frac{x^s}{s} \, ds \right| = O \left( \frac{x^\sigma}{|\sigma - iT|} \right) = O \left( \frac{x^2}{T^{1-\varepsilon}} \right).$$

As well as before, if we select sufficiently large $T$, we can make the error term very small.

By the Cauchy residue theorem we get

$$\frac{1}{2\pi i} \int_{C} \zeta_K(s) \frac{x^s}{s} \, ds = \rho x,$$

where $\rho$ is the residue of $\zeta_K(s)$ at $s = 1$. But it is known that $\rho = c$. (For the proof of this result, please see Theorem 5 in Section 8 of [8].)

By using all result above, we reach

$$j_K(x) + O \left( \frac{x^3}{T} \right) = cx + O \left( \frac{x^2}{T^{1-\varepsilon}} \right) + O(x^{1/2}T^\varepsilon \log T) + O \left( \frac{x^2}{T^{1-\varepsilon}} \right).$$

When we select $T = x^3$, this becomes

$$j_K(x) = cx + O(x^{1/2+\varepsilon}).$$

This proves the theorem.

We estimated $j_K(x)$ with assuming the Extended Lindelöf Hypothesis in Theorem 3.2. In Lemma 3.3 we consider the sum $\sum_{\mathfrak{a} \leq x} \mu(\mathfrak{a}) j_K \left( \frac{x}{\mathfrak{a}} \right)^m$, where $\mu(\mathfrak{a})$ is the Möbius function defined as

$$\mu(\mathfrak{a}) \overset{def}{=} \begin{cases} 1 & \text{if } \mathfrak{a} = 1, \\ (-1)^s & \text{if } \mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s, \text{ where } \mathfrak{p}_1, \ldots, \mathfrak{p}_s \text{ are distinct prime ideals,} \\ 0 & \text{if } \mathfrak{a} \subset \mathfrak{p}^2 \text{ for some prime ideal } \mathfrak{p}. \end{cases}$$

As we show in next section, this sum has a crucial role in computing visible lattice points.
Lemma 3.3. If we assume the Extended Lindelöf Hypothesis, we get
\[
\sum_{\frac{x}{\mathfrak{m}a} \leq x} \mu(a) j_K \left( \frac{x}{\mathfrak{m}a} \right)^m = \frac{e^m}{\zeta_K(m)} x^m + O(x^{m-1/2+\varepsilon})
\]
for all \( \varepsilon > 0 \).

Proof. We can show this lemma from last Theorem 3.2.

Theorem 3.2 and the binomial theorem lead to
\[
\sum_{\frac{x}{\mathfrak{m}a} \leq x} \mu(a) j_K \left( \frac{x}{\mathfrak{m}a} \right)^m = \sum_{\frac{x}{\mathfrak{m}a} \leq x} \mu(a) \left( \frac{cx}{\mathfrak{m}a} + O \left( \left( \frac{x}{\mathfrak{m}a} \right)^{1/2+\varepsilon} \right) \right)^m
\]
\[= (cx)^m \sum_{\frac{x}{\mathfrak{m}a} \leq x} \mu(a) + O \left( \sum_{\frac{x}{\mathfrak{m}a} \leq x} \left( \frac{x}{\mathfrak{m}a} \right)^{m-1/2+\varepsilon} \right).\]

By the fact \( \sum_{a} \frac{\mu(a)}{\mathfrak{m}a^m} = \frac{1}{\zeta_K(m)} \), we get
\[
\sum_{\frac{x}{\mathfrak{m}a} \leq x} \mu(a) j_K \left( \frac{x}{\mathfrak{m}a} \right)^m = \frac{e^m}{\zeta_K(m)} x^m - (cx)^m \sum_{\frac{x}{\mathfrak{m}a} > x} \mu(a) \frac{1}{\mathfrak{m}a^m} + O \left( \sum_{\frac{x}{\mathfrak{m}a} \leq x} \left( \frac{x}{\mathfrak{m}a} \right)^{m-1/2+\varepsilon} \right).
\]

Now we estimate how fast above first sum grows. From Theorem 3.2 we can estimate \( j_K(x) - j_K(x-1) = O(x^{1/2+\varepsilon}) \), so we have
\[
\sum_{\frac{x}{\mathfrak{m}a} > x} \frac{\mu(a)}{\mathfrak{m}a^m} = O \left( x^m \int_{x}^{\infty} y^{1/2+\varepsilon} \frac{dy}{y^m} \right)
\]
\[= O(x^{3/2+\varepsilon}).\]

Next we estimate how fast above second sum grows. As well as first sum, \( j_K(x) - j_K(x-1) = O(x^{1/2+\varepsilon}) \) holds, so we have
\[
\sum_{\frac{x}{\mathfrak{m}a} \leq x} \left( \frac{x}{\mathfrak{m}a} \right)^{m-1/2+\varepsilon} = O \left( x^{m-1/2+\varepsilon} \left( 1 + \int_{y=1}^{x} \frac{y^{1/2+\varepsilon}}{y^{m-1/2+\varepsilon}} dy \right) \right)
\]
\[= \begin{cases} O(x^{m-1/2+\varepsilon}) & \text{if } m \geq 3, \\
O(x^{3/2+\varepsilon} \log x) & \text{if } m = 2. \end{cases}
\]

Hence we get final estimate.
\[
\sum_{\frac{x}{\mathfrak{m}a} \leq x} \mu(a) j_K \left( \frac{x}{\mathfrak{m}a} \right)^m = \frac{e^m}{\zeta_K(m)} x^m + O(x^{m-1/2+\varepsilon}).
\]

This proves this lemma. \( \square \)
4. The proof of the main theorem

When we assume the Extended Lindelöf Hypothesis, we can improve order of error term of some estimates in the last section. In this section we will show the main theorem by using results shown.

**Theorem 4.1.** If we assume the Extended Lindelöf Hypothesis, we get

\[ E_m(x, K) = O(x^{m-1/2+\varepsilon}) \]

for all algebraic number field \( K \) and for all \( \varepsilon > 0 \).

**Proof.** From Theorem 3.2, we know that

\[ j_K(x) = cx + O(x^{1/2+\varepsilon}) \]

We use this approximation formula to consider the error term \( E_m(x, K) \) by following B. D. Sittinger way \( [1] \).

The Inclusion-Exclusion Principle shows that

\[ V_m(x, K) = j_K(x)^m - \sum_{p_1} j_K \left( \frac{x}{\mathfrak{p}_1} \right)^m + \sum_{p_1, p_2} j_K \left( \frac{x}{\mathfrak{p}_1 \mathfrak{p}_2} \right)^m - \cdots \]

where \( p_1, p_2, \ldots \) denote distinct prime ideals with \( \mathfrak{p}_i \leq x \). By the definition of Möbius function \( \mu(a) \) we can rewrite this sum simpler

\[ V_m(x, K) = \sum_{\mathfrak{a} | a \leq x} \mu(a) j_K \left( \frac{x}{\mathfrak{a}} \right)^m. \]

Using the Lemma 3.3 we get

\[ V_m(x, K) = \frac{c^m}{\zeta_K(m)} x^m + O(x^{m-1/2+\varepsilon}). \]

This proves the main theorem.

\[ \square \]

In 2010, B. D. Sittinger showed following theorem about visible lattice points over algebraic number field \( K \) without assuming the Extended Lindelöf Hypothesis.

**Theorem 4.2.** When \( n = [K : \mathbb{Q}] \)

\[ V_m(x, K) = \frac{c^m}{\zeta_K(m)} x^m + \begin{cases} O(x^{m-1/n}) & \text{if } m \geq 3, \\ O(x^{2-1/n} \log x) & \text{if } m = 2, \end{cases} \]

where \( c \) is same constant as before.

Considering Sittinger’s result \( [1] \), we can improve the order of \( E_m(x, K) \) for all algebraic number field \( K \) with \([K : \mathbb{Q}] \geq 3\) under the situation that the Extended Lindelöf Hypothesis is true.
5. A relation with Gauss Circle Problem

From Theorem 4.1 in the last section, we can consider the relation between the number of visible lattice points from the origin over \(K\) and the Extended Lindelöf Hypothesis. In this section, we consider about some relation between the number of visible lattice points from the origin over \(\mathbb{Q}(\sqrt{-1})\) and the Gauss Circle Problem. It is well known that the Gauss Circle Problem can be written as follows.

**Gauss Circle Problem.** Let \(N(r)\) be the cardinality of the set \(\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 \leq r\}\). Then

\[
N(r) = \pi r + O(r^{1/4+\varepsilon}),
\]

for every \(\varepsilon > 0\).

The original statement of the Gauss Circle Problem is not \(r\) but \(r^2\), but we change its statement slightly to consider a relation with the number of visible lattice points from the origin over \(K = \mathbb{Q}(\sqrt{-1})\). Because \(\mathcal{O}_K = \mathbb{Z}[\sqrt{-1}]\) is PID, all ideals of \(\mathbb{Z}[\sqrt{-1}]\) can be written \((x + y\sqrt{-1})\) and their ideal norm is \(x^2 + y^2\). Considering the number of units in \(\mathbb{Z}[\sqrt{-1}]\), \(N(r) = 4j_K(r)\) holds. We can show the following theorem in a way similar to the proof of Theorem 4.1. As we remark later, we can assume that \(N(r) - \pi r = O(r^\alpha)\), where \(1/4 < \alpha < 1/3\).

**Theorem 5.1.** When \(K = \mathbb{Q}(\sqrt{-1})\) and \(1/4 < \alpha < 1/3\),

\[
N(r) = \pi r + O(r^\alpha)
\]

is equivalent to for all \(m \geq 2\)

\[
E_m(x, K) = \begin{cases} 
O(x^{m-1+\alpha}) & \text{if } m \geq 3, \\
O(x^{1+\alpha} \log x) & \text{if } m = 2.
\end{cases}
\]

**Proof.** If we assume \(N(r) = \pi r + O(r^\alpha)\) we can get statement about \(E_m(x, K)\) in this theorem by same argument of Theorem 4.1 and the proof of Lemma 3.3 with \(1/2 + \varepsilon\) replaced by \(\alpha\).

Conversely, if \(E_m(x, K)\) satisfies the estimate in this theorem and \(N(r) - \pi r \neq O(x^\alpha)\) then we lead a contradiction as follows. By using same argument of the proof of Lemma 3.3 we get

\[
\sum_{\mu(a)j_K \mid x} \mu(a)j_K \left( \frac{x}{\mu(a)} \right)^m - \frac{e_m^m}{\zeta_K(m)} x^m \neq \begin{cases} 
O(x^{m-1+\alpha}) & \text{if } m \geq 3, \\
O(x^{1+\alpha} \log x) & \text{if } m = 2,
\end{cases}
\]

since we can calculate that \(c = \pi/4\) and \(N(r) - \pi r = O(r^{1/3})\) is well known result about the Gauss Circle Problem. We apply this result to estimate of \(V_m(x, K)\), then we get

\[
V_m(x, K) - \frac{e_m^m}{\zeta_K(m)} x^m \neq \begin{cases} 
O(x^{m-1+\alpha}) & \text{if } m \geq 3, \\
O(x^{1+\alpha} \log x) & \text{if } m = 2,
\end{cases}
\]

we have a contradiction. Hence we show

\[
N(r) = \pi r + O(r^\alpha).
\]

This proves Theorem 5.1. \(\square\)

This theorem means that the better order of \(E_m(x, \mathbb{Q}(\sqrt{-1}))\) we get, the better order of \(N(r) - \pi r\) we get. In 1915, E. Landau and G. H. Hardy proved that

\[
N(r) - \pi r \neq O(r^{1/4})
\]
independently [4] and [5]. And the best upper order in now is 0.3149... proved by M. N. Huxley [6], so we know the exact order of $E_m(x, Q(\sqrt{-1}))$ is less than 0.3149... and greater than 1/4.

6. Appendix

In this section we consider some further result about relatively $s$-prime lattice point. They may be a generalization of our results, but we used only visible lattice points to consider some relations with two hypotheses in this paper. Not to mention, there are some relation between relatively $s$-prime lattice point and two hypotheses.

We say that a lattice point $(a_1, a_2, \ldots, a_m)$ is relatively $s$-prime, if there exists no prime ideal $p$ such that $a_1, a_2, \ldots, a_m \subset p^s$. If $s = 1$ then relatively 1-prime lattice point is visible lattice point from the origin. Follow the definition of $V_m(x, K)$ and $E_m(x, K)$, let $V_m^s(x, K)$ denote the number of relatively $s$-prime lattice point $(a_1, a_2, \ldots, a_m)$ with $Na_i \leq x$ for all $i = 1, \ldots, m$. And we let $E_m^s(x, K)$ denote its error term, i.e. $E_m^s(x, K) = V_m^s(x, K) - (cx)^m/\zeta_K(ms)$.

A relation with the Extended Lindelöf Hypothesis as follows.

**Theorem 6.1.** If we assume the Extended Lindelöf Hypothesis, we get

$$E_m^s(x, K) = \begin{cases} O(x^{3/4+\varepsilon}) & \text{if } m = 1 \text{ and } s = 2, \\ O(x^{m-1/2+\varepsilon}) & \text{otherwise}, \end{cases}$$

for all algebraic number field $K$ and for all $\varepsilon > 0$.

And a relation with the Gauss Circle Problem as follows.

**Theorem 6.2.** When $K = Q(\sqrt{-1})$ and $1/4 < \alpha < 1/3$,

$$N(r) = \pi r + O(r^{\alpha})$$

is equivalent to for all $m \geq 2$

$$E_m^s(x, K) = \begin{cases} O(x^{m-1-\alpha}) & \text{if } m \geq 3, \text{ or } m = 2 \text{ and } s \geq 2, \\ O(x^{1+\alpha} \log x) & \text{if } m = 2 \text{ and } s = 1, \\ O(x^{(1+\alpha)/r}) & \text{if } m = 1 \text{ and } s = 2, 3, 4, \\ O(x^{\alpha}) & \text{if } m = 1 \text{ and } s \geq 5. \end{cases}$$

These theorems can be shown to change the proof of theorem about visible lattice points slightly. In a way similar to the proof of Theorem 4.1 we get a following estimate of $V_m^s(x, K)$.

$$V_m^s(x, K) = \sum_{\mathfrak{a} \leq x} \mu(\mathfrak{a}) j_K \left( \frac{x}{\mathfrak{a}^s} \right)^m.$$ 

All we have to do is considering the order of sum, but we can estimate above sum in a way similar to the proof of Lemma 4.3. Therefore we leave out the last part of proof in this paper.

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