Local 4/5-Law and Energy Dissipation Anomaly in Turbulence

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Abstract

A strong local form of the “4/3-law” in turbulent flow has been proved recently by Duchon and Robert for a triple moment of velocity increments averaged over both a bounded spacetime region and separation vector directions, and for energy dissipation averaged over the same spacetime region. Under precisely stated hypotheses, the two are proved to be proportional, by a constant 4/3, and to appear as a nonnegative defect measure in the local energy balance of singular (distributional) solutions of the incompressible Euler equations. Here we prove that the energy defect measure can be represented also by a triple moment of purely longitudinal velocity increments and by a mixed moment with one longitudinal and two transverse velocity increments. Thus, we prove that the traditional 4/5- and 4/15-laws of Kolmogorov hold in the same local sense as demonstrated for the 4/3-law by Duchon-Robert.
1 Introduction

Recently, Duchon and Robert [1] have established an energy balance relation for distributional solutions of the three-dimensional (3D) incompressible Euler equations. Their balance relation contains a “defect” or “anomaly” term, with an interesting connection to turbulence theory. Since the work of Duchon-Robert provides the point of departure of the present paper, it is appropriate to describe their theorems briefly here. For our purposes, there are three main results:

First, if \( u \in L^3([0,T] \times T^3) \) is a weak solution of the incompressible Euler equations on the 3-torus \( T^3 \), then it is proved in [1] (Proposition 2) that the following local balance holds in the sense of distributions:

\[
\partial_t \left( \frac{1}{2} |u|^2 \right) + \nabla \cdot \left[ \left( \frac{1}{2} |u|^2 + p \right) u \right] = -D(u). \tag{1.1}
\]

Here \( D(u) \) is a defect distribution which for classical solutions vanishes identically, implying local energy conservation. Duchon and Robert also establish various expressions for the defect term. In particular, they have shown that

\[
D(u) = D \lim_{\varepsilon \to 0} \frac{1}{4} \int_{T^3} \nabla \phi_\varepsilon(\ell) \cdot \delta u(\ell) |\delta u(\ell)|^2 d^3\ell \tag{1.2}
\]

where \( D \lim \) means limit in the sense of distributions on \([0,T] \times T^3\), with \( \phi \in C_0^\infty(T^3) \), even, nonnegative with unit integral, \( \phi_\varepsilon(x) = \varepsilon^{-3} \phi(x/\varepsilon) \), and \( \delta u(x,t; \ell) = u(x + \ell, t) - u(x, t) \).

This expression is remarkable because it is closely connected with an exact result in turbulence theory, the so-called “Kármán-Howarth-Monin relation”:

\[
\nabla \cdot \langle \delta u(\ell) |\delta u(\ell)|^2 \rangle = -4\bar{\varepsilon}. \tag{1.3}
\]

See [1], section 6.2.1. In this relation, \( \bar{\varepsilon} = \nu \langle |\nabla u|\nu |^2 \rangle \) is the mean energy dissipation for a Navier-Stokes solution \( u^\nu \), which is assumed to remain finite as viscosity \( \nu \) tends to zero. We see that this result is essentially equivalent to the statement that the Duchon-Robert defect satisfy \( \langle D \rangle = \bar{\varepsilon} \).
For a general distributional solution of Euler, there need be no connection of the defect with viscous dissipation, nor need it even be true that $D(u) \geq 0$. However, a second theorem (Proposition 4) in [1] states that, if $u^\nu$ is a Leray solution of incompressible Navier-Stokes equation for viscosity $\nu$ and if $u^\nu \to u$ strong in $L^3([0,T] \times \mathbb{T}^3)$ as $\nu \to 0$, then

$$D(u) = D-\lim_{\nu \to 0} \nu|\nabla u^\nu|^2 + D(u^\nu)$$

(1.4)

Since it is well-known that $D(u^\nu) \geq 0$ for Leray solutions, thus also $D(u)$ is a nonnegative distribution, i.e. a Radon measure ([3], Example 12.5). This theorem may be paraphrased as saying that strong viscosity solutions of the incompressible Euler equations are also dissipative solutions.

Finally, Duchon and Robert, under an additional hypothesis, establish an even simpler form of the defect distribution. With $\omega$ unit Haar measure on $S^2$, they define the function

$$S(u, \ell)(x, t) := \frac{1}{\ell} \int_{S^2} d\omega(\ell) \delta u_L(x, t; \ell) |\delta u(x, t; \ell)|^2.$$  

(1.5)

in $L^1([0,T] \times \mathbb{T}^3)$. Here $\delta u_L(x, t; \ell) = \ell \cdot \delta u(x, t; \ell)$ is the longitudinal velocity increment. Assuming that the following limit exists

$$S(u)(x, t) := D-\lim_{\ell \to 0} S(u, \ell)(x, t)$$

(1.6)

Duchon and Robert [1], Section 5 show that

$$S(u) = -\frac{4}{3} D(u).$$

(1.7)

This is a rigorous form of another well-known relation in turbulence theory, sometimes called the “4/3-law”:

$$\langle \delta u_L(\ell) |\delta u(\ell)\rangle^2 \sim -\frac{4}{3} \bar{\epsilon} \ell.$$  

(1.8)

See [4]. It is well-known that the Kármán-Howarth-Monin relation reduces to the 4/3-law under conditions of local isotropy. This is achieved here by the angle average over the sphere in [1].3
Such relations as (1.3) and (1.8) in turbulence theory go back to the original work of A. N. Kolmogorov [4]. However, Kolmogorov in fact proved a relation involving only longitudinal velocity increments, the so-called “4/5-law”:

$$\langle [\delta u_L(\ell)]^3 \rangle \sim -\frac{4}{5}\varepsilon \ell$$

(1.9)

This was established from the Navier-Stokes equations, under conditions of statistical homogeneity and local isotropy and with the assumption that energy dissipation remains finite in the zero-viscosity limit. It is our purpose here to establish an expression for the Duchon-Robert energy dissipation anomaly $D(u)$ with exactly the form of Kolmogorov’s law. Our proof yields another expression which is related to the so-called “4/15-law”:

$$\langle [\delta u_L(\ell)|\delta u_T(\ell)|^2 \rangle \sim -\frac{4}{15}\varepsilon \ell$$

(1.10)

Here $\delta u_T(\ell) = \hat{i} \cdot \delta \mathbf{u}(\ell)$ is a transverse velocity increment, with $\hat{i}$ any unit vector orthogonal to $\hat{\ell}$. This relation is known to hold under the same conditions as the 4/5-law. The rigorous derivation of the 4/5- and 4/15-laws given here, under precisely stated assumptions, yields a result with a wider domain of validity than that of some previous rigorous derivations, such as that of Nie and Tanveer [6]. In particular, the form of the 4/5-law established here—like the Duchon-Robert version of the 4/3-law—is local, in the sense that it relates third-order moments of velocity increments and viscous dissipation averaged over the same bounded spacetime region, not necessarily large.

In the following section we prove our main theorem. In a final discussion section we discuss briefly its physical significance and a compare it with previous results. Let us now make just a few remarks on notations: The symbol $L^p$ will be used below for $L^p([0,T] \times \mathbb{T}^3)$. If $F(x,t)$ is any spacetime distribution, we denote $F^\varepsilon(x,t) = (\varphi^\varepsilon \ast F)(x,t)$, where $\varphi \in C_0^\infty(\mathbb{T}^3)$ and * is space convolution. Thus, $F^\varepsilon$ remains, for each fixed $x$, a distribution in $t$. Often below, as above, we omit the variables $(x,t)$ where their presence is clear from the context.
The Main Theorem

We prove the following:

Theorem 1 Let \( u \in L^3([0,T] \times \mathbb{T}^3) \) be a weak solution of the incompressible Euler equations on the 3-torus \( \mathbb{T}^3 \). Let \( \varphi(\ell) \) be any \( C^\infty \) function with compact support, nonnegative with unit integral, spherically symmetric, and let \( \varphi^\varepsilon(\ell) = \varepsilon^{-3}\varphi(\ell/\varepsilon) \). Finally, define longitudinal and transverse velocity increments as

\[
\delta u_L(x, t; \ell) = (\hat{\ell} \otimes \hat{\ell}) u(x + \ell, t), \quad \delta u_T(x, t; \ell) = (1 - \hat{\ell} \otimes \hat{\ell}) u(x + \ell, t),
\]

(2.1)

Then, the following functions in \( L^1([0,T] \times \mathbb{T}^3) \)

\[
D_{\varepsilon}^L(u) = \frac{3}{4} \int_{\mathbb{T}^3} d^3 \ell \left\{ \nabla \varphi^\varepsilon(\ell) \cdot \delta u_L(\ell) |\delta u_L(\ell)|^2 + \frac{2}{\ell} \varphi^\varepsilon(\ell) \delta u_L(\ell) |\delta u_T(\ell)|^2 \right\},
\]

(2.2)

and

\[
D_{\varepsilon}^T(u) = \frac{3}{8} \int_{\mathbb{T}^3} d^3 \ell \left\{ \nabla \varphi^\varepsilon(\ell) \cdot \delta u_T(\ell) |\delta u_T(\ell)|^2 - \frac{2}{\ell} \varphi^\varepsilon(\ell) \delta u_L(\ell) |\delta u_T(\ell)|^2 \right\},
\]

(2.3)

both converge in the sense of distributions as \( \varepsilon \to 0 \) to \( D(u) \), where the latter is the defect distribution in the local energy balance for \( u \),

\[
\partial_t \left( \frac{1}{2}|u|^2 \right) + \nabla \cdot \left( \frac{1}{2}|u|^2 + p \right) u = -D(u),
\]

(2.4)

established earlier by Duchon-Robert.

We shall prove this theorem in several steps.

The idea of the proof is to consider separate balance equations for the longitudinal and transverse components of the energy. We define first longitudinal and transverse velocities relative to a vector \( \ell \):

\[
u_L(x, t; \ell) = (\hat{\ell} \otimes \hat{\ell}) u(x + \ell, t), \quad u_T(x, t; \ell) = (1 - \hat{\ell} \otimes \hat{\ell}) u(x + \ell, t).
\]

(2.5)

Of course, \( u_L + u_T = u \). We define also mollified versions

\[
u_X^\varepsilon(x, t) = \int_{\mathbb{T}^3} d^3 \ell \varphi^\varepsilon(\ell) u_X(x, t; \ell), \quad X = L, T
\]

(2.6)
It is easy to see that these satisfy the equations

$$\partial_t u_X^\varepsilon + \nabla \cdot (u \otimes u_X)^\varepsilon = -\nabla \cdot \Pi_X^\varepsilon, \quad X = L, T$$

(2.7)
distributionally in time, where

$$\Pi_X^\varepsilon(x, t) = \int_{T^3} d^3\ell \, \varphi^\varepsilon(\ell)(\hat{\ell} \otimes \hat{\ell})p(x + \ell, t), \quad \Pi_T^\varepsilon(x, t) = \int_{T^3} d^3\ell \, \varphi^\varepsilon(\ell)(1 - \hat{\ell} \otimes \hat{\ell})p(x + \ell, t). \quad (2.8)$$

These equations can be simplified by the observation that

$$\nabla \cdot \Pi_X^\varepsilon(x, t) = \nabla p_X^\varepsilon(x, t), \quad X = L, T$$

(2.9)
where $p_X^\varepsilon, p_T^\varepsilon$ are scalar functions defined by

$$p_X^\varepsilon(x, t) = \int_{T^3} d^3\ell \, \varphi_X^\varepsilon(\ell)p(x + \ell, t), \quad X = L, T$$

(2.10)
with

$$\varphi_L(\ell) = \varphi(\ell) - \varphi_T(\ell), \quad \varphi_T(\ell) = 2 \int_{\ell}^{\infty} \frac{\varphi(\ell')}{\ell'} \, d\ell'. \quad (2.11)$$

Note that $\varphi_L, \varphi_T$ are compactly supported and $C^\infty$ everywhere except at 0, where they have a mild (logarithmic) singularity. To prove (2.9), we use the elementary relation $\nabla \hat{\ell} = 1 - \hat{\ell} \otimes \hat{\ell}$. A simple computation then gives, for example,

$$\nabla \cdot \Pi_L^\varepsilon(x, t) = -\int_{T^3} d^3\ell \left\{ \frac{d\varphi^\varepsilon}{d\ell}(\ell) + \frac{2}{\ell} \varphi^\varepsilon(\ell) \right\} \hat{\ell} \, p(x + \ell, t), \quad (2.12)$$

From its definition,

$$\nabla \varphi_L(\ell) = \left\{ \frac{d\varphi^\varepsilon}{d\ell}(\ell) + \frac{2}{\ell} \varphi^\varepsilon(\ell) \right\} \hat{\ell}. \quad (2.13)$$

This gives easily $\nabla \cdot \Pi_L^\varepsilon = \nabla p_L^\varepsilon$ in the sense of distributions. Because $\Pi_L^\varepsilon + \Pi_T^\varepsilon = p^\varepsilon \mathbf{1}$ and $p_L^\varepsilon + p_T^\varepsilon = p^\varepsilon$, this yields also the relation $\nabla \cdot \Pi_T^\varepsilon = \nabla p_T^\varepsilon$. Finally, we obtain the simpler equations

$$\partial_t u_X^\varepsilon + \nabla \cdot (u \otimes u_X)^\varepsilon = -\nabla p_X^\varepsilon, \quad X = L, T$$

(2.14)
for $u_L^\varepsilon$ and $u_T^\varepsilon$. 
We next observe that both \( u^ε_L \) and \( u^ε_T \) are divergence-free. In fact, a computation like that above shows that

\[
\nabla \cdot u^ε_L (x, t) = - \int_{\mathbb{T}^3} d^3 \ell \left\{ \frac{d \varphi^ε}{d \ell} (\ell) + \frac{2}{\ell} \varphi^ε (\ell) \right\} \hat{\ell} \cdot u (x + \ell, t),
\]

\[
= - \int_{\mathbb{T}^3} d^3 \ell \nabla \varphi^ε_L (\ell) \cdot u (x + \ell, t).
\]

(2.15)

If we integrate against any smooth test function \( \psi(x) \), we then get

\[
\int_{\mathbb{T}^3} d^3 x \psi(x) \nabla \cdot u^ε_L (x, t) = \int_{\mathbb{T}^3} d^3 \ell \varphi^ε_L (\ell) \nabla \cdot (\psi * u)(\ell, t).
\]

(2.16)

The latter is zero, since \( u \) is divergence-free (in the distributional sense). Thus, \( \nabla \cdot u^ε_L = 0 \).

Since \( u^ε_L + u^ε_T = u^ε \), then also \( \nabla \cdot u^ε_T = 0 \).

From the equations (2.14) for \( X = L, T \), the incompressibility conditions for \( u^ε_L \) and \( u^ε_T \), and the Euler equations for \( u \) (in distribution sense), we derive the following balance equations

\[
2 \partial_t (u \cdot u^ε_L) + \nabla \cdot [2(u \cdot u^ε_L) u + ((u_L \cdot u_L)^ε - (u_L \cdot u_L)^ε) u + 2p u^ε_L + 2p^ε_L u] = -\frac{4}{3} D^ε_L (u),
\]

(2.17)

and

\[
2 \partial_t (u \cdot u^ε_T) + \nabla \cdot [2(u \cdot u^ε_T) u + ((u_T \cdot u_T)^ε - (u_T \cdot u_T)^ε) u + 2p u^ε_T + 2p^ε_T u] = -\frac{8}{3} D^ε_T (u).
\]

(2.18)

The basic identities used to derive these equations are

\[
\int_{\mathbb{T}^3} d^3 \ell \left\{ \nabla \varphi^ε (\ell) \cdot \delta u (\ell) |\delta u_L (\ell)|^2 + \frac{2}{\ell} \varphi^ε (\ell) \delta u_L (\ell) |\delta u_T (\ell)|^2 \right\}
\]

\[
= \int_{\mathbb{T}^3} d^3 \ell \frac{\partial}{\partial \ell_k} \{ \hat{\ell}_i \hat{\ell}_j \varphi^ε (\ell) \} \delta u_i (\ell) \delta u_j (\ell) \delta u_k (\ell)
\]

\[
= - \frac{\partial}{\partial x_k} [((u_L \cdot u_L) u_k^ε - (u_L \cdot u_L)^ε u_k] + 2 u_i \frac{\partial}{\partial x_k} [((u_L u_k)^ε - u^ε_L u_k]
\]

(2.19)

and

\[
\int_{\mathbb{T}^3} d^3 \ell \left\{ \nabla \varphi^ε (\ell) \cdot \delta u (\ell) |\delta u_T (\ell)|^2 - \frac{2}{\ell} \varphi^ε (\ell) \delta u_L (\ell) |\delta u_T (\ell)|^2 \right\}
\]

\[
= \int_{\mathbb{T}^3} d^3 \ell \frac{\partial}{\partial \ell_k} \{ (\delta_{ij} - \hat{\ell}_i \hat{\ell}_j) \varphi^ε (\ell) \} \delta u_i (\ell) \delta u_j (\ell) \delta u_k (\ell)
\]

\[
= - \frac{\partial}{\partial x_k} [((u_T \cdot u_T) u_k^ε - (u_T \cdot u_T)^ε u_k] + 2 u_i \frac{\partial}{\partial x_k} [((u_T u_k)^ε - u^ε_T u_k]
\]

(2.20)
Thus, where $B$ is the ball of radius $\ell$ centered at the origin and $|B_\ell|$ is its volume. Because $\varphi^\varepsilon$ has unit integral, $\|\varphi^\varepsilon\|_\ell = 3\varepsilon$, we also deduce that $\|\varphi^\varepsilon\|_\ell = 3\varepsilon$ as $\varepsilon \to 0$. Because of the spherical symmetry, compact support and unit integral of $\varphi^\varepsilon$,

$$\int_{\mathbb{T}^3} d^3\ell \varphi^\varepsilon(\ell) \hat{\ell} \otimes \hat{\ell} = \frac{1}{3} \mathbf{1},$$

(2.21)

for sufficiently small $\varepsilon$. From this result and the definition of $u^\varepsilon_L$, it follows that

$$u^\varepsilon_L(x, t) - \frac{1}{3} u(x, t) = \int_{\mathbb{T}^3} d^3\ell \varphi^\varepsilon(\ell) \hat{\ell} \cdot \hat{\ell} [u(x + \ell, t) - u(x, t)].$$

(2.22)

Thus,

$$\|u^\varepsilon_L - \frac{1}{3} u\|_{L^3} \leq \int_{\mathbb{T}^3} d^3\ell \varphi^\varepsilon(\ell) \|u(\cdot + \ell) - u\|_{L^3}.$$ 

(2.23)

Because $u \in L^3$, it follows by a standard approximation argument that $\|u(\cdot + \ell) - u\|_{L^3} \to 0$ as $\ell \to 0$. Hence, it follows from (2.22) that $\|u^\varepsilon_L - \frac{1}{3} u\|_{L^3} \to 0$ as $\varepsilon \to 0$, as was claimed. Because $u^\varepsilon_T = u^\varepsilon - u^\varepsilon_L$, we also deduce that $\|u^\varepsilon_T - \frac{2}{3} u\|_{L^3} \to 0$.

Next we show that $p^\varepsilon_L \to \frac{1}{3} p$ and $p^\varepsilon_T \to \frac{2}{3} p$ strong in $L^{3/2}$ as $\varepsilon \to 0$. First, observe that $p = (-\Delta)^{-1}\partial_i \partial_j (u_i u_j)$, so that $p \in L^{3/2}$ by the Calderón-Zygmund inequality. From the definitions of $\phi_T$ and $p^\varepsilon_T$, it follows easily that

$$p^\varepsilon_T(x, t) = \frac{2}{3} \cdot 4\pi \int_0^\infty \ell^2 d\ell' \varphi^\varepsilon(\ell') \cdot \frac{1}{|B_\ell|} \int_{B_\ell} d^3\ell p(x + \ell, t),$$

(2.24)

where $B_\ell$ is the ball of radius $\ell$ centered at the origin and $|B_\ell|$ is its volume. Because $\varphi^\varepsilon$ has unit integral,

$$p^\varepsilon_T(x, t) - \frac{2}{3} p(x, t) = \frac{2}{3} \cdot 4\pi \int_0^\infty \ell^2 d\ell' \varphi^\varepsilon(\ell') \cdot \frac{1}{|B_\ell|} \int_{B_\ell} d^3\ell [p(x + \ell, t) - p(x, t)],$$

(2.25)

and thus

$$\|p^\varepsilon_T - \frac{2}{3} p\|_{L^{3/2}} \leq \frac{2}{3} \cdot 4\pi \int_0^\infty \ell^2 d\ell' \varphi^\varepsilon(\ell') \cdot \frac{1}{|B_\ell|} \int_{B_\ell} d^3\ell \|p(\cdot + \ell) - p\|_{L^{3/2}}.$$ 

(2.26)

Consequently, $\|p^\varepsilon_T - \frac{2}{3} p\|_{L^{3/2}} \to 0$ as $\varepsilon \to 0$, as was to be proved. Since $p^\varepsilon_L = p^\varepsilon - p^\varepsilon_T$, also $\|p^\varepsilon_L - \frac{1}{3} p\|_{L^{3/2}} \to 0$. 

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Entirely similar arguments show that 
\[ ((u_X \cdot u_X)u^\varepsilon - (u_X \cdot u_X)\varepsilon u) \to 0 \] strong in \( L^1 \) as \( \varepsilon \to 0 \), for \( X = L, T \). We leave that to the reader.

Finally we see that the lefthand side of equation (2.17) converges as \( \varepsilon \to 0 \) to the quantity
\[
\frac{4}{3} \left\{ \partial_t (\frac{1}{2}|u|^2) + \nabla \cdot \left[ (\frac{1}{2}|u|^2 + p)u \right] \right\}
\]
in the sense of distributions, and that the lefthand side of (2.18) converges to \( \frac{8}{3} \) times the same quantity in the curly brackets, also in the distribution sense. However, by the result of Duchon-Robert \[1\], that quantity in the brackets is equal to minus the defect distribution \( D(u) \). We conclude then that
\[
D_X^\varepsilon (u) \to D(u) \tag{2.27}
\]
as \( \varepsilon \to 0 \) for both \( X = L, T \), in the sense of distributions. \( \square \)

We now prove the following:

**Corollary 1** Assume that the functions \( S_L(u, \ell), S_T(u, \ell) \in L^1([0, T] \times T^3) \) defined by
\[
S_L(u, \ell) = \frac{1}{\ell} \int_{S^2} d\omega(\hat{\ell}) [\delta u_L(\ell)]^3 \tag{2.28}
\]
and
\[
S_T(u, \ell) = \frac{1}{\ell} \int_{S^2} d\omega(\hat{\ell}) \delta u_L(\ell) |\delta u_T(\ell)|^2 \tag{2.29}
\]
have limits
\[
S_X(u) = \mathcal{D} \lim_{\ell \to 0} S_X(u, \ell), \quad X = L, T. \tag{2.30}
\]
Then,
\[
S_L(u) = -\frac{4}{5} D(u), \quad S_T(u) = -\frac{8}{15} D(u). \tag{2.31}
\]

The proof is quite straightforward. An easy computation for a spherically symmetric test function gives
\[
\frac{4}{3} D_L^\varepsilon (u) = 4\pi \int_0^\infty d\ell \left[ \ell^3 \varphi'(\ell) S_L(u, \varepsilon \ell) + 2\ell^2 \varphi(\ell) S_T(u, \varepsilon \ell) \right] \tag{2.32}
\]
and
\[ \frac{8}{3} D^\varepsilon_T(u) = 4\pi \int_0^\infty d\ell \left[ \ell^3 \varphi'(\ell) - 2\ell^2 \varphi(\ell) \right] S_T(u, \varepsilon \ell). \] (2.33)

Taking the limit as \( \varepsilon \to 0 \), using Theorem 1, the normalization of the test function, and the above hypotheses, we see that
\[ \frac{4}{3} D(u) = -3S_L(u) + 2S_T(u) \] (2.34)

and
\[ \frac{8}{3} D(u) = -5S_T(u). \] (2.35)

Solving this linear system gives the result. \( \square \)

In the definition of \( S_T(u, \ell) \) in (2.29) we could instead have replaced \( |\delta u_T|^2 \) by the square magnitude of a transverse component \( \delta u_T = \hat{t} \cdot \delta u \), where \( \hat{t} \) is any unit vector perpendicular to \( \hat{\ell} \). Call such a quantity \( \tilde{S}_T(u, \ell) \). However, \[ |\delta u_T|^2 = |\delta u_T|^2 + |\delta u_T'|^2, \]
where \( \delta u_T' = (\hat{t} \times \hat{\ell}) \cdot \delta u \). Since both transverse components give an equal contribution in the spherical average, \( S_T(u, \ell) = 2\tilde{S}_T(u, \ell) \). Under the assumptions of the corollary, \( \tilde{S}_T(u, \ell) \) then also has a limit distribution \( \tilde{S}_T(u) \) as \( \ell \to 0 \), and
\[ \tilde{S}_T(u) = -\frac{4}{15} D(u). \] (2.36)

This is the conventional statement of the “4/15-law”.

### 3 Discussion

If we couple our Corollary 1 with Duchon-Robert’s characterization of \( D(u) \) by the inviscid limit, \( \square \), Proposition 4, then we arrive at essentially the following statement: For any “nice” spacetime region \( R \subset T^3 \times [0, T] \), let
\[ \varepsilon_R = \lim_{\nu \to 0} \frac{1}{|R|} \int_R d^3x \int_R dt \varrho(x, t) \] (3.1)

with \( \varrho(x, t) := \nu |\nabla u|^2 \) (assuming for simplicity that \( D(u^\nu) = 0 \)). If first \( \nu \to 0 \) and next \( \ell \to 0 \), then
\[ \langle [\delta u_L(\ell)]^3 \rangle_{\text{ang, } R} \sim -\frac{4}{5} \varepsilon_R \ell \] (3.2)
with $\left\langle \cdot \right\rangle_{\text{ang}, R}$ denoting an average of $(\hat{\ell}, x, t)$ over $S^2 \times R$. To be precise, the relation should hold in the sense of averaging in $x, t$ against smooth test functions $\psi(x, t)$ rather than against a characteristic function $1_R(x, t)$. The latter sense is stronger, because, if the result holds for characteristic functions of Borel sets $R$, then, by approximation arguments, it holds also for smooth functions. In fact, by the portmanteau theorem [7], it is enough to show that it holds for all bounded Borel sets $R$ such that $D(\partial R) = 0$, i.e. with no energy dissipation concentrating on the boundary. We see from these remarks that the meaning of the Corollary 1 is that the scaling observed in the Kolmogorov “4/5-law” [3] should hold with just local averaging in spacetime and an angular average over the direction of the separation vector. Note that, if the solution of the Navier-Stokes equations at finite viscosity is not regular, then the same relation would hold with the modified definition $\varepsilon(x, t) := \nu|\nabla u'|^2 + D(u')$, including the dissipation from the singularities.

Another recent work by Nie and Tanveer [6] has given a rigorous derivation of both the 4/5- and 4/3-laws under conditions similar to those of Duchon-Robert and of the present paper. The proof of those laws by Nie and Tanveer is for individual solutions of the Navier-Stokes equations (assumed to be strong) without any statistical averaging, and with no assumptions of homogeneity or isotropy. In [6], as here, the averaging is over spacetime and over directions of the separation vector. However, Nie and Tanveer considered the space average over the entire domain and time average over an interval $[0, T]$ with $T \to \infty$. In contrast, the present result holds for any spacetime domain, of arbitrary extent, so long as the Reynolds number is sufficiently high, and also for solutions of the Navier-Stokes equations that may be singular. In this respect, our result is stronger than that of [6]. On the other hand, our proof and that in [6] give no indication how large the Reynolds number must be taken to approach the limit, whereas Nie and Tanveer establish precise error bounds for their result.

There is a slight resemblance of our and Duchon-Robert’s local results with the “refined similarity hypothesis” that Kolmogorov postulated in 1962 [8]. Kolmogorov considered averages $\varepsilon_B$ of dissipation over balls $B \subset \mathbb{T}^3$ and the structure functions obtained by averaging velocity
increments $\delta u(x, t; \ell)$ conditioned upon the local dissipation averaged over a ball $B_\ell(x)$ of diameter $\ell$ centered at the midpoint $\frac{1}{2}(x + \ell)$. He postulated that these might be calculated by his original 1941 theory, e.g. for longitudinal differences. Hence, he obtained for unconditioned ensemble averages

$$\langle [\delta u_L(\ell)]^3 \rangle \sim C_n \langle \varepsilon_{B_\ell}^n \rangle \ell^n \quad (3.3)$$

Our scaling relation (3.2) is supposed to hold pathwise, for individual realizations. Hence, if it holds—along with a uniform and integrable bound for $\ell \to 0$— then one can infer that

$$\langle [\langle (\delta u_L(\ell))^3 \rangle_{ang,R}^n \rangle \sim \left(-\frac{4}{3}\right)^n \langle \varepsilon_{R}^n \rangle \ell^n \quad (3.4)$$

The similarity is apparent. Aside from the fact that we must integrate over time as well as space, the major and significant difference is that in the relation (3.4) the region $R$ must be fixed as $\ell \to 0$, while in Kolmogorov’s hypothesis the ball $B_\ell$ itself shrinks to zero in the limit.

Our results and that of Duchon-Robert say nothing about intermittency. Although we have proved a rigorous theorem, it is only established under various hypotheses. We regard those as plausible, but it is nevertheless of some interest to inquire about the feasibility of numerical and/or experimental tests of the local results proved here and in [1]. In [9], the 4/5-law has been verified both in a numerical simulation via a space-average over the domain and also in experimental hot-wire data. However, in the same simulation, the 4/15-law is not satisfied (S. Chen, private communication). Similar violations of the 4/15-law were observed in experimental X-wire data from an atmospheric boundary layer [10]. In both cases, anisotropy of the flow may be suspected of vitiating the result. In that case, angle-averaging as employed in the rigorous theorems may improve the agreement. On the other hand, a more recent numerical study [11] with a $1024^3$ simulation has found that the regimes of validity of both the 4/5- and 4/3-laws, in even their global form, are established only very slowly as the Reynolds number is increased. Such a slow approach to asymptotia makes a direct test of the local results, especially a verification of the numerical prefactor, rather more difficult.
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