A Relativistic Quark Model for Mesons with an Instanton–Induced Interaction

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We present new results of a relativistic quark model based on the Bethe–Salpeter equation in its instantaneous approximation. Assuming a linearly rising confinement potential with an appropriate spinorial structure in Dirac space and adopting a residual interaction based on instanton effects, we can compute masses of the light mesons up to highest observed angular momenta with a natural solution of the \( U_A(1) \) problem. The calculated ground states masses and the radial excitations describe the experimental results well. In this paper, we will also discuss our results concerning numerous meson decay properties. For processes like \( \pi^+/K^+ \to e^+\nu_e\gamma \) and \( 0^- \to \gamma\gamma \) at various photon virtualities, we find a good agreement with experimental data. We will also comment on the form factors of the \( K_{\ell3} \) decay and on the decay constants of the \( \pi \), \( K \) and \( \eta \) mesons. For the sake of completeness, we will furthermore present the electromagnetic form factors of the charged \( \pi \) and \( K \) mesons as well as a comparison of the radiative meson decay widths with the most recent experimental data.

I. INTRODUCTION

After years of research on the problem of bound states in QCD, there are still lots of open questions. Since it is not possible to apply a perturbative treatment of QCD in the low–energy region around \( \approx 1 \text{GeV} \), one has to rely on effective theoretical descriptions of hadrons. Integrating out the quark degrees of freedom leads to approaches such as Chiral Perturbation Theory or the Nambu–Jona-Lasinio model, but these ideas fail in the description of higher lying resonances and radial excitations. These states can be described (even in a non–relativistic treatment at least qualitatively) by models including quark confinement (see e.g. [1]).

In this paper, we discuss some new results on light meson spectra and decays in the framework of a relativistic quark model that has been presented in some previous publications (see [2] – [10]). In particular, we want to update our results on mass spectra and electroweak decay properties, consistently calculated with a parameter set that gives a global description of the complete meson spectrum. At the same time, we shall discuss new results from an alternative description of confinement. Our model is based on the Bethe–Salpeter equation in its instantaneous approximation and provides an excellent description of the masses of the complete meson spectrum including highest angular momenta and radial excitations. We will study these spectra and compare our results not only with the latest Particle Data Group (PDG) compilation [19] but also with alternative new interpretations of meson resonances as \( q\bar q \) states or exotics. The model that is presented here can also be applied to various meson decay processes and shows a good overall agreement compared to the experimental data. We will investigate the pseudoscalar decay constants and their relation to the decays of \( J^\pi = 0^- \) mesons into two photons at some selected photon virtualities. The electromagnetic structure of the charged \( \pi \) and \( K \) mesons will be discussed by presenting their form factors, calculated in the parameter sets used in this paper. Since we want to update our former publications, we will also briefly resume the status of the electromagnetic decay widths in our model. Furthermore, we compute form factors for the processes \( \pi^+/K^+ \to e^+\nu_e\gamma \) and \( K^+ \to \pi^0 e^+\nu_e \) (the so–called \( K_{\ell3} \) decay).

We have organized this article as follows: Section 1 gives a synopsis of our model and introduces the potentials adopted in the subsequent evaluations. Section 2 is devoted to a discussion of the parameters and their effects on the resulting meson spectra. We will present calculations on various decay processes in section 3 before we conclude with summarizing remarks in section 4.

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II. A RELATIVISTIC QUARK MODEL ON THE BASIS OF THE INSTANTANEOUS BETHE–SALPETER EQUATION

Our results will be presented in the framework of a relativistic quark model based on the Bethe–Salpeter (BS) equation (see [11]) for a $qq\bar{q}$ bound state of four-momentum $P$ and mass $M$ with $M^2 = P^2$:

$$\chi^P(p) = -i S_F^P(\eta_1 P + p) \left[ \int \frac{d^4p'}{(2\pi)^4} K(P; p, p') \chi^P(p') \right] S_F^P(-\eta_2 P + p) \quad ;$$

(1)

see fig. 1 for a diagrammatic representation of this equation. Here, $S_F^P(\pm \eta P + p)$ denotes the full quark propagator where $i = 1$ indicates the quark, $i = 2$ the antiquark and $p$ is the relative momentum between quark and antiquark. The coefficients $\eta_i$ satisfying the condition $\eta_1 + \eta_2 = 1$ fix a special choice of coordinates; in the following considerations, we will set $\eta_1 = \eta_2 = \frac{1}{2}$ for the sake of simplicity. The BS amplitude $\chi^P$ is defined in coordinate space as the time–ordered product of two (anti–)quark field operators:

$$\chi_{\alpha\beta}^P(x_1, x_2) := \langle 0 \left| T \bar{\psi}_\alpha^i(x_1) \psi_\beta^i(x_2) \right| P \rangle = e^{-ip \cdot x_1} \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x_1 - x_2)} \chi_{\alpha\beta}^P(p) \quad ,$$

(2)

where $\alpha$ and $\beta$ stand for Dirac, flavour and colour indices. The function $K(P; p, p')$ in the BS equation represents the four–dimensional irreducible kernel including all interactions between the $q\bar{q}$ pair. Neither the full propagator $S_F^P(\pm P/2 + p)$ nor the interaction kernel $K(P; p, p')$ are sufficiently well known and have to be fixed by appropriate phenomenological assumptions. In our model, we adopt the so–called instantaneous approximation for the kernel that was originally proposed by Salpeter (see [12]). It can be formulated covariantly via

$$K(P; p, p') = V(p_\perp, p'_\perp) \quad ,$$

(3)

where we have introduced components of the relative momentum $p = p_\parallel + p_\perp$ parallel and perpendicular to the meson momentum $P$ by

$$p_\parallel := \frac{p \cdot P}{\sqrt{P^2}} \quad \text{and} \quad p_\perp := p - \frac{p \cdot P}{\sqrt{P^2}} \quad .$$

(4)

In the rest frame of the meson where $P = (M, 0)$, we find $p_\parallel = (p^0, 0)$ and $p_\perp = (0, \vec{p})$ yielding finally

$$K(P; p, p')|_{P=(M,0)} = V(\vec{p}, \vec{p}') \quad$$

(5)

for the instantaneous interaction kernel. This formally covariant formulation allows to transform any solution $\chi^P$ of the BS equation that is found in the meson rest frame with $P = (M, 0)$ into a solution for non–vanishing meson momenta. This is an important point since it turned out to be crucial for a satisfying description of e.g. the pion form factor already at moderate $Q^2$ that indeed the correct Lorentz boost is applied to the BS amplitudes of the $\pi$ meson (see [13]).

The second model assumption states that the quark propagators in the BS equation can suitably be approximated by free propagators according to

$$S_F^P(p) \approx i \frac{p \pm m_i}{p^2 - m_i^2} = i \left( \frac{\Lambda_+^i(\bar{p})}{p^0 - \omega_i + i\epsilon} + \frac{\Lambda_-^i(\bar{p})}{p^0 + \omega_i - i\epsilon} \right) \gamma^0 \quad ,$$

(6)

where $m_i$ is the effective constituent mass (either for nonstrange or strange flavours in the case of light mesons) with $\omega_i = \sqrt{\bar{p}^2 + m_i^2}$ the energy of the (anti–)quark $i$. The projectors

$$\Lambda_\pm^i(\bar{p}) := \frac{1}{2} \pm \frac{H_i(\bar{p})}{2\omega_i} \quad$$

(7)

with the standard single particle Dirac Hamiltonian $H_i(\bar{p}) = \gamma^0 (\bar{p} \gamma^0 + m_i)$ distinguish states of positive and negative energy of (anti–)quark $i$.

With these assumptions and since the $p^0$ dependence of the interaction kernel vanishes in the instantaneous approximation, the $p^0$ integration of eq. 5 in the meson’s rest frame can be performed analytically via the residue theorem thus leading to the so–called Salpeter equation (see [12]):
Here, the Salpeter amplitude defined by $\Phi(\vec{p}) := \int \frac{d^3p'}{(2\pi)^3} \chi^P(p,\vec{p})|_{P=\vev{M,\bar{0}}}$ depends only on the space–like components of the meson’s relative momentum $p$.

In previous papers, it has been shown how to formulate the Salpeter equation as an eigenvalue problem

$$\mathcal{H}\Psi(\vec{p}) = M\Psi(\vec{p})$$

with $\Psi(\vec{p}) := \Phi(\vec{p})\gamma^0$ and $M$ the mass of the the $q\bar{q}$ bound state considered. For more details, we refer the reader to \cite{BD} and \cite{H} where also the numerical treatment of this eigenvalue equation is discussed.

With an adequate potential ansatz, it is thus possible to calculate meson mass spectra on the basis of the Salpeter equation. Since we wish a proper description not only of the Regge trajectories

$M^2 \propto J$ but also of the intriguing scalar sector and the characteristic pseudoscalar splittings, we do not consider potentials derived from (flavour independent) one–gluon–exchange diagrams but we adopt the following to describe the underlying quark dynamics:

- A linear confinement potential with $V_C(x) = a_c + b_c \cdot x$ in coordinate space and an appropriate spinorial structure $\Gamma \otimes \Gamma$ in Dirac space acting like

$$\int \frac{d^3p'}{(2\pi)^3} V_C(\vec{p},\vec{p}')\Phi(\vec{p'}) = \int \frac{d^3p'}{(2\pi)^3} V_C \left( \vec{p} - \vec{p}' \right) \frac{1}{2} \left[ \gamma^0 \Phi(\vec{p}') \gamma^5 + \gamma^5 \gamma^0 \Phi(\vec{p}') \gamma^5 \right]$$

in momentum space. The confinement offset $a_c$ and its slope $b_c$ are free parameters of our model. Various spin dependencies have been investigated. Below, we will discuss two variants that both yield a stable solution of the Salpeter equation and at the same time reproduce the states on the Regge trajectories correctly.

- A flavour dependent two–body force from an instanton induced interaction (abbr.: III), following an idea of ’t Hooft (see \cite{H} and references therein):

$$\int \frac{d^3p'}{(2\pi)^3} V_{II}(\vec{p},\vec{p}')\Phi(\vec{p}') = 4G(g, g') \int \frac{d^3p'}{(2\pi)^3} R_{\Lambda}(\vec{p},\vec{p}') \left( \frac{i}{2} \text{tr} \left( \Phi(\vec{p}') \right) + \frac{i}{2} \text{tr} \left( \Phi(\vec{p}') \gamma_5 \right) \right)$$

Here, $R_{\Lambda}$ represents a regularizing function and $G(g, g')$ is a flavour matrix, i.e. a summation over flavour indices is understood. We treat the coupling strengths $g$ (nonstrange sector), $g'$ (nonstrange/strange sector) and the finite effective range $\Lambda = \Lambda_{\text{III}}$ as free parameters.

The latter feature enables us to describe properly the $\pi – K – \eta – \eta'$ mass splittings; without an explicit flavour dependent residual interaction, their masses would be partly degenerate.

Let us finally introduce the meson–quark–antiquark vertex function $\Gamma^P$ (or the amputated BS amplitude) which is defined by

$$\Gamma^P(p) := \left[ S_{P \frac{1}{2}}^F \left( \frac{P}{2} + p \right) \right]^{-1} \chi^P(p) \left[ S_{\frac{1}{2}}^F \left( \frac{P}{2} + p \right) \right]^{-1}$$

It depends only on variables $p_\perp$ of a three–dimensional subspace and therefore reduces in the rest frame of the meson to

$$\Gamma^P(p)|_{P=\vev{M,\bar{0}}} = -i \int \frac{d^3p'}{(2\pi)^3} V(\vec{p},\vec{p}')\Phi(\vec{p'}) =: \Gamma(\vec{p})$$

which can be seen by inserting $\Gamma^P(p)$ in the BS equation with an instantaneous interaction kernel. After simultaneously computing the mass spectra and the associated Salpeter amplitudes with eq. \cite{BD}, it is therefore possible to reconstruct the BS amplitudes $\chi^P$ in the meson’s rest frame with eqs. \cite{BD} and \cite{H}.
It has been shown in [3] that — for a pure boost defined by $P = \Lambda p \tilde{P}$ with $\tilde{P} = (M, \vec{0})$ — the rest frame BS amplitude $\chi^P$ is linked via $\chi^P(p) = S_{\Lambda p} \chi^\Lambda(\Lambda^{-1} p) S^{-1}_{\Lambda p}$ to the amplitude $\chi^P$ for any on–shell momentum $P (P^2 = M^2)$ of the $q\bar{q}$ bound state considered; here, $S_{\Lambda p}$ is a matrix acting on the spinor indices of $\chi^P$ that obeys $S_{\Lambda p} \gamma\mu S^{-1}_{\Lambda p} = (\Lambda p)_\mu \gamma\nu$. A similar relation holds for the vertex function $\Gamma^P$ so that we regard our model as fully relativistic (and not only ‘relativized’) as it fulfills the general prescriptions for Lorentz boosts. This can be seen as a consequence of the covariant formulation of our ansatz which is in fact possible in spite of the use of the instantaneous kernels in the BS equation as has been shown above.

III. PARAMETERS AND MASS SPECTRA

As discussed in the previous section, the relativistic quark model presented here contains some free parameters: the effective constituent quark masses $m_n$ and $m_s$, the confinement parameters $a_c$ and $b_c$ (together with an appropriate spin structure), and the couplings $g$ and $g'$ for 't Hooft’s instanton–induced interaction with an effective range $\Lambda_{\text{II}}$. We want to stress that the latter residual interaction only acts on mesons with $J = 0$ as has been stated in [3]. Accordingly we can apply the following scheme for parameter fixing:

1. Choose the quark masses in a physically reasonable range, i.e. $m_n \approx 300 \ldots 400\text{MeV}$ and $m_s \approx 500 \ldots 600\text{MeV}$.
2. Assume an appropriate spin structure that does not conflict with the condition of numerical stability and that provides the correct position of states on the Regge trajectories.
3. Fit the confinement offset and slope to the mesons with $J \neq 0$ and do some fine tuning of the constituent quark masses.
4. Observe the mass spectrum in the scalar and pseudoscalar sector for $g, g' \neq 0$ and fix these coupling constants to the $\pi–K$–$\eta$–$\eta'$ mass splittings.

It is crucial for the description of the light meson sector with $J = 0$ that a flavour dependent force lifts the degeneracies which would otherwise occur in models only assuming confinement and/or one–gluon–exchange potentials. Therefore we include 't Hooft’s instanton–induced interaction in the parameter sets of our two model variants, see table I. They mainly differ in the form assumed for the spin dependence of the confinement force: in model $A$, a combination of a scalar and a timelike vector structure is adopted as in ref. [6] whereas model $B$ employs a Fierz invariant and $\gamma_5$ invariant spin dependence also investigated by Böhm et al. (see [14]) as well as by Gross and Milana (see [15]). A more detailed discussion (especially focussing the radial excitations) will be presented in a separate contribution [16].

In figs. (1) and (2), the effect of the instanton induced interaction $V_{\text{II}}$ is shown for the light pseudoscalar mesons with the parameters of both models. For $g = g' = 0$, the pseudoscalar mesons are bound by confinement only. Therefore the $\pi$ and $\eta$ are degenerate in this limit since no mixing is induced in the isoscalar sector; in a calculation without 't Hooft’s interaction, the $\eta$ is therefore a pure nonstrange state (i.e. $|n\bar{n}\rangle = (|u\bar{u}\rangle + |d\bar{d}\rangle)/\sqrt{2}$). With increasing coupling $g$, the pion mass is lowered to its experimental value while $M_\eta$ grows since the underlying effective interaction has an opposite sign for isovector and isoscalar states (see [3] for details). With fixed $g$, the nonstrange–strange coupling $g'$ is used to bring simultaneously the masses of the $K$, $\eta$ and $\eta'$ mesons to their experimental values. We stress that a non–vanishing coupling $g'$ not only yields the correct $\eta$–$\eta'$ mass splitting but also induces the expected nonstrange–strange mixing in the isoscalar sector, see also table III for explicit results.

The resulting parameter sets are shown in table IV. Note that the numerical values for the confinement offset and slope, the 't Hooft couplings and the constituent quark masses are comparable for both models although they differ significantly in their confinement spin structure. Let us now briefly discuss the resulting spectra in the isovector, isodublet and isoscalar sectors:

- Both models yield excellent Regge trajectories $M^2 \propto J$, see fig. (3) for the light isovector mesons up to $J = 6$. The complete spectrum with all its radial excitations for light mesons with isospin $I = 1$ is shown in fig. (4) and table V.

For the $a_0(980)$, we observe a remarkable difference between the models $A$ and $B$: in contrast to model $A$, the spin structure of the parameter set $B$ obviously allows an interpretation of the $a_0$ meson as a $q\bar{q}$ state. This is somewhat puzzling since this state is often considered to be a $KK$ molecule — an assignment that would be consistent with the fact that we indeed can not describe the $a_0(980)$ with the parameters of model $A$. We refer to [3] for a detailed discussion of the scalar meson spectrum and its interpretation in the framework of the parameter set $A$. 
The complete kaonic spectrum is shown in fig. 8 and table III. We observe a very good agreement with the established experimental data for angular momenta from $J = 0$ to $J = 5$ with one exception: as in the isovector sector, the $0^+$ state is lowered significantly in model $B$ compared to model $A$. Let us briefly comment on this $K_0^0$ ground state: compared to the PDG value of $M_{K^0_0} \approx 1430$ MeV (see 14), we find a significant lowering of the mass in model $B$ compared to model $A$. Indeed, there is an indication from a recent $K$–matrix analysis for the scalar nonet that the $IP^+ = \frac{1}{2}1^0^+$ ground state actually might be around 1200 MeV (see 21). Although there are good reasons to compare our results to the “bare poles” of such an analysis where the effects of decay–channel couplings are at least partially taken into account, we have to refrain from a detailed discussion until such an analysis has indeed been performed in all meson sectors.

Furthermore, we note that the calculated states in the $K_1^-$ spectrum are each twice degenerate; the present interactions do not distinguish the $S = 0$ and the $S = 1$ states and therefore do not show the experimentally observed splitting. However, we want to mention that an alternative procedure for the regularization of ’t Hooft’s instanton–induced force separates these two states; the reason is that the strict $J = 0$ selection rule for this interaction is relaxed if the regularization is applied before evaluating the occurring matrix elements. This effect, although strongly suppressed, yields the correct splittings in the $K_1^-$ sector, but we will not discuss further the implications of this slightly modified approach for the residual interaction in our model.

In the PDG listings (see 19), it is stated that the masses of the $K_3$ and the $K_4$ need confirmation; they are therefore omitted in the summary tables. We stress that our calculated masses for these states fit perfectly in a linear Regge plot for the $K_J$ mesons — in contrast to the experimental $K_3$ and $K_4$ data shown in fig. 8. We thus support the statement that these have to be considered with caution.

In fig. 9 and table IV, the results for the isoscalar mass spectra are presented and compared to experimental data. Similar to the pattern in the isovector and isodublet sector, we find a remarkable downward shift of the scalar states in model $B$ compared to model $A$. We thus arrive at two alternative interpretations of the scalar isoscalar $q\bar{q}$ spectrum: in model $A$, we might identify the lowest calculated $IJ^{\pi\pi} = 0^{++}$ state (mainly flavour singlet) either with the broad structure $f_0(400 - 1200)$ or with the $f_0(980)$ meson and the second state at $\approx 1500$ MeV as a member of the flavour octet: accordingly, this would leave either the $f_0(980)$ or the $f_0(400 - 1200)$ and the $f_0(1370)$ as non–$q\bar{q}$ state. Let us note that if we furthermore regard the $\gamma\gamma$ decay width and the strong decay width of our lowest $f_0$ state in model $A$, we would slightly prefer the $f_0(980)$ to be the $q\bar{q}$ candidate (see 11 and 17). In model $B$, we do account for a very low–lying scalar state (“σ meson”), the next excitations to be identified with states around 1250 MeV and 1550 MeV. As mentioned above, model $B$ also accounts for the $a_0(980)$ meson as a $q\bar{q}$ state whereas this can not be confirmed with the parameters of model $A$. Clearly, on the basis of the spectrum alone this interpretation can be preliminary at best, especially in view of the complexity in this sector that arises from strong decay channel couplings and possible mixtures with gluonic or other exotic states. A more detailed discussion which also includes numerical results on the strong decay widths of these states will be given in a forthcoming paper (see 17).

Finally we want to comment on the $\eta(1295)$: neither in model $A$ nor in model $B$ we can interpret this as a $q\bar{q}$ bound state. Indeed, it has recently been stated in 22 that there is no experimental evidence for an $\eta(1295)$ in the reaction $p\bar{p} \rightarrow \pi^+\pi^-\pi^+\pi^-\eta$.

In summary, we find a good overall agreement both in model $A$ and $B$. The discrepancies, occurring e.g. in the $K_3$ or $K_4$ masses, can mostly be traced back to questionable $q\bar{q}$ assignments of the considered states. We find the right behaviour with respect to the linear Regge trajectories $M^2 \propto J$ up to highest angular momenta as has been shown in fig. 6. Furthermore, due to the instanton–induced effects our model shows the correct splitting in the pseudoscalar sector independent of the underlying parameter set. The effects of the different Dirac structures of the confinement force in model $A$ and $B$ and their implications for the non–relativistic reduction of the Salpeter equation will be discussed in the detailed analysis of ref. 16.

Altogether, a linearly rising confinement potential plus a residual interaction à la ’t Hooft provides a very satisfying description of the light $q\bar{q}$ spectra. Therefore we consider our approach based on the Bethe–Salpeter equation in its instantaneous approximation as a trustworthy framework for studying not only meson masses but also their characteristic decays, especially in the pseudoscalar sector.

IV. MESON DECAY PROPERTIES

In previous publications (see 1, 3, 11 and 11), we have studied various mesonic decay properties such as the widths of the decays $\pi^0, \eta, \eta' \rightarrow \gamma\gamma$ or the decay constants $f_\pi$ and $f_K$. Furthermore, electromagnetic form factors of $\pi$ and
$K$ mesons have been investigated as well as electromagnetic decays like $\rho \to \pi \gamma$ and related processes. We note that instanton–induced vertices in strong decays were also studied (see [3]); a more extensive publication in this context including quark loop contributions is in preparation [17].

We want to extend these approaches to other processes where $q\bar{q}$ bound states are involved. Firstly, we will pick up the recent discussion of the pseudoscalar decay constants (see [22] and [23]) and quote our results for the $\eta$ and the $\eta'$ meson for their nonstrange and strange content separately. We will review the two photon decay of $J^P = 0^-$ mesons and the related transition form factors at various photon virtualities in section IV B where our results for $-q_\ell^2 \to \infty$ will be linked to the pseudoscalar decay constants. We present the electromagnetic form factors of the charged $\pi$ and $K$ mesons for the sake of completeness since they have not been published yet in the parameter sets that we adopt in this paper. For the same reason, we also present new results on the electromagnetic decay widths of the processes $M \to M'\gamma$ that have already been studied in ref. [3]. Then, we will comment on the weak decays $\pi^+/K^+ \to e^+\nu_e\gamma$ before we finally investigate our results on the form factors of the so–called $K_{e3}$ decay.

Let us stress that these calculations concerning meson decay properties are done in the parameter sets $A$ and $B$ that were completely fixed with regard to the experimental meson mass spectra. We do therefore not introduce new parameters or alter the models discussed in the previous section since we aim at a global description of meson masses as well as their characteristic decays.

A. Pseudoscalar Decay Constants

In recent years, the question of pseudoscalar decay constants — especially those of the $\eta$ and $\eta'$ mesons — was discussed in numerous publications (see e.g. [22], [24], and references therein). It has become clear that due to the mixing in the isoscalar sector, the contributions of the nonstrange and the strange content (or between singlet and octet contributions in an alternate mixing scheme) have to be distinguished.

Let us start our discussion with the definition of the pseudoscalar decay constant with the axial current $A^\mu = \bar{u}\gamma_\mu\gamma_5d$ for a pion with positive charge:

$$\langle 0|A^\mu_\pi|p^+\rangle = ip_\mu f_\pi \ .$$

(14)

In a similar way, one can define the decay constant of the $K^+$ meson; these constants are related with the corresponding decay constants for the neutral mesons via $f_{M^0} = f_{M^\pm}/\sqrt{2}$ for $M = \pi, K$.

Due to the instantaneous approximation, $f_\pi$ can be derived from the pion’s Salpeter amplitude $\Phi_{(\pi)}(\vec{p})$ in the rest frame of the meson:

$$f_\pi = \frac{\sqrt{3}}{M_\pi} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ S_1^F \left( \frac{P}{2} + p \right) \Gamma (\pi) (p) S_2^F \left( -\frac{P}{2} + p \right) \gamma_0 \gamma_5 \right]$$

$$\equiv \frac{\sqrt{3}}{M_\pi} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \Phi_{(\pi)}(\vec{p}) \gamma_0 \gamma_5 \right]$$

(15)

where the trace is evaluated only on Dirac indices. It can be shown analytically that the latter expression is proportional to the difference of upper and lower component of the equal–time wave function of the pion (see [2]). Therefore, $f_\pi$ is highly sensitive on relativistic ingredients of the model in which it is calculated; in non–relativistic models, the evaluation fails typically by orders of magnitude. As has been shown in refs. [2] and [3], the electromagnetic decay widths for $\rho, \omega, \phi \to e^+e^-$ can be treated in a similar way; the numerical results for the parameter sets $A$ and $B$ of this work are comparable to the widths of model V2 in these previous publications.

In table V, the results for $f_\pi$ and $f_K$ are shown for the parameters of model A and B; obviously, we overestimate these observables by a factor of $\approx 1.5$ which is typical for models with large constituent quark masses as found here, see [3]. Although our relativistic ansatz improves dramatically the result compared to non–relativistic calculations, we find in these results a shortcoming of our model which might be related to the instantaneous approximation of the underlying Bethe–Salpeter equation and/or our refraining from “running quark masses”.

Now let us focus the $\eta$ and the $\eta'$ meson, respectively. As we have already emphasized, mixing between the nonstrange and the strange sector is crucial for the understanding of the isoscalar states. We adopt the flavour decomposition

$$|\eta\rangle = X_\eta|n\rangle + Y_\eta|s\rangle$$

$$|\eta'\rangle = X_{\eta'}|n\rangle + Y_{\eta'}|s\rangle$$

(16)

where $|n\rangle := (|u\bar{u}⟩ + |d\bar{d}⟩)/\sqrt{2}$ and $|s\rangle := |s\bar{s}\rangle$; the coefficients obey $X_M^2 + Y_M^2 = 1$ ($M = \eta, \eta'$). In table V, we compare the coefficients, extracted from our calculated Salpeter amplitudes, with experimental numbers from the
B. Two Photon Decays

In this section, we will discuss the two photon decays of pseudoscalar mesons not only for two real photons in the final state ($q_1^2 = q_2^2 = 0$) but also the $0^−$ transition form factors in the case that either one ($q_1^2 = 0, q_2^2 \neq 0$) or both ($q_1^2 \neq 0, q_2^2 \neq 0$) of the photons are virtual.

In fig. (2), the $\gamma \gamma$ decay of pseudoscalar mesons is shown, calculated in lowest order in the Mandelstam formalism (see [18]). The related matrix element for the transition $M \to \gamma \gamma$ with $M = \pi^0, \eta, \eta'$ reads

$$T_{\gamma \gamma}^M(q_1, q_2) = \frac{i\sqrt{3}}{2\pi^3} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ S_F \frac{P}{2} + p \right] \Gamma_\gamma^M(p) S_F \left( \frac{P}{2} + p - q_1 \right) \not\!q_1$$

$$+ S_F \frac{P}{2} + p \right] \Gamma_\gamma^M(p) S_F \left( \frac{P}{2} + p - q_2 \right) \not\!q_2 \right]$$

where $\Gamma_\gamma^M(p)$ is the vertex function for the pseudoscalar meson $M$ defined in eq. (12). The polarization vectors $\varepsilon_i$ obey $\varepsilon_i^2 = -1$ and $\varepsilon_i \cdot k_i = 0$ ($i = 1, 2$) for real photons. Here, the factor $\sqrt{3}$ originates from the trace over colour indices; the trace in eq. (13) is understood with respect to Dirac and flavour indices.

From Lorentz invariance, it is possible to derive another expression for this matrix element for pseudoscalar mesons

$$T_{\gamma \gamma}^M(q_1, q_2) = \alpha \epsilon_{\mu \nu \alpha \beta} \varepsilon_\gamma^\mu(q_1) \varepsilon_\gamma^\nu(q_2) q_1^\alpha q_2^\beta F_{\gamma \gamma}^M(q_1^2, q_2^2)$$

in terms of the transition form factor $F_{\gamma \gamma}^M(q_1^2, q_2^2)$; here, $\alpha$ is the fine structure constant. With this last equation, it is now possible to relate our matrix element calculated via eq. (14) with the transition form factor and the decay width defined by

$$\Gamma_{\gamma \gamma}(Q_1^2, Q_2^2) = \alpha^2 \frac{M_\gamma^2}{64\pi} \left| F_{\gamma \gamma}^M(Q_1^2, Q_2^2) \right|^2$$

with $Q_i^2 := -q_i^2$.

It is worth noting that in a similar way it is possible to calculate the two photon widths $\Gamma_{\gamma \gamma}$ in our relativistic quark model not only for $J^\pi = 0^−$ mesons but also for $J^\pi = 0^+, 2^±, 4^± \ldots$ mesons (including $c\bar{c}$ and $b\bar{b}$ bound states with an additional one–gluon–exchange potential) and their radial excitations in reasonable agreement with the (rare) experimental data (see [1]).
In table VII, we show our results for the widths of the decays \( \pi^0, \eta, \eta' \rightarrow \gamma\gamma \) into two real photons; obviously, we underestimate these widths and as for the pseudoscalar decay constants, we find a discrepancy of about a factor of 1.5 in the amplitude. Indeed, this can be understood with the results of the foregoing discussion as the instanton approximation shows up its shortcomings although the meson Salpeter amplitudes are correctly evaluated. We also present the widths for the scalar meson decays \( f_0(000) \rightarrow \gamma\gamma \) in table VII. Note that each of these decays can only be calculated in one of the parameter sets \( A \) or \( B \) due to the different \( q\bar{q} \) assignments to the mesons in the scalar sector in both models, see sect. III.

Let us now study these results in some more detail. In ref. [27], Brodsky and Lepage presented the well–known and parameter–free interpolation formula for the pion transition form factor:

\[
F_{\gamma\gamma}^{\pi}(Q^2, 0) = \frac{6C_\pi f_\pi}{Q^2 + 4\pi^2 f_\pi^2}
\]

(22)

with the charge factor \( C_\pi := 1/(3\sqrt{2}) \) coming from the quark flavours. Although Brodsky et al. recently proposed a slightly modified version of this formula (see [28]), it works quite well in its original version and leads to the famous limit

\[
\lim_{Q^2 \rightarrow \infty} Q^2 F_{\gamma\gamma}^{\pi}(Q^2, 0) = 6C_\pi f_\pi = 2f_{\pi^0}
\]

(23)

for the form factor of the decay \( \pi^0 \rightarrow \gamma\gamma^* \) in the asymptotic region; note that here \( f_{\pi^0} := f_\pi/\sqrt{2} \approx 93\text{MeV} \). On the other hand, we recover \( \Gamma_{\gamma\gamma}^{\eta}(Q^2) := \Gamma_{\gamma\gamma}^{\eta}(Q^2 := Q_1^2, 0) \) is underestimated. We conclude that for a deeply bound particle such as the pion the instantaneous approximation shows up its shortcomings although the meson Salpeter amplitudes are correctly boosted. We also present the limits for the scalar meson decays \( f_0(400 - 1200), f_0(980), a_0(980) \rightarrow \gamma\gamma \) in table VII. Note that each of these decays can only be calculated in one of the parameter sets \( A \) or \( B \) due to the different \( q\bar{q} \) assignments to the mesons in the scalar sector in both models, see sect. III.

Finally, we want to comment on the decay of a \( J^P = 0^- \) meson into two virtual photons. We will focus on the case of identical virtuality of the outgoing photons \( Q_1^2 = Q_2^2 =: Q^2 \) but we want to emphasize that in general the whole \( Q_1^2 - Q_2^2 \) plane can be calculated in our model. A result for \( Q^2 \rightarrow \infty \) from operator product expansion yields

\[
\lim_{Q^2 \rightarrow \infty} Q^2 F_{\gamma\gamma}^{\pi}(Q^2, Q^2) = 2C_\pi f_\pi
\]

(25)

for the transition form factor of the pion (see [14] and [15]). We can confirm this result in our calculations in contrast to the limits of the foregoing discussion since now the virtuality distribution of the two outgoing photons is symmetric even at very large \( Q^2 \). Furthermore, we can give a similar relation for the form factor of the decays \( \eta, \eta' \rightarrow \gamma^*\gamma^* \) in the limit of large \( Q^2 \) from analytical considerations in the framework of our model (see the Appendix for details):

\[
\lim_{Q^2 \rightarrow \infty} Q^2 F_{\gamma\gamma}^{\eta}(Q^2, Q^2) = 2C_n f_{\eta n} + 2C_s f_{\eta s}
\]

(26)

for \( M = \eta, \eta' \); the charge factors \( C_j \) and the decay constants \( f_{Mj} \) \( (j = n, s) \) are defined as above. In figs. [13] and [14], these form factors are plotted together with their limits for large \( Q^2 \) with model \( A \) and \( B \); no experimental data exist in this kinematical region so far. Obviously, the numerically evaluated form factors in both models indeed show up the limits that we found in our analytical calculations, see the Appendix.
C. The Electromagnetic Form Factors of the Charged π and K Mesons

For the sake of completeness, we want to present the electromagnetic form factors of the $\pi^\pm$ and the $K^\pm$ in this section since they have not been published in the models $A$ and $B$ up to now. The meson form factors for the transitions $\pi^\pm(P) \to \pi^\pm(P')\gamma^*(q)$ and $K^\pm(P) \to K^\pm(P')\gamma^*(q)$ with a (spacelike) photon virtuality $q^2 = (P - P')^2 = -Q^2 < 0$ are defined as follows:

$$\langle M(P') \mid J_\mu(0) \mid M(P) \rangle = QF_M(Q^2)(P + P')_\mu$$

Here, $P(P')$ is the four–momentum of the incoming (outgoing) meson $M = \pi^\pm, K^\pm$; the factor $Q = e_1 + e_2$ denotes the meson charge. As has been shown in ref. [5], we derive this matrix element in the Bethe–Salpeter approach via

$$P$$

Here, $P$ results for the decays $\eta$ nonstrange/strange section since they have not been published in the models $A$ and $B$. As already stated in [5], we clearly underestimate the processes with a pion in the final state, especially in model $B$.

We show the form factor $Q^2 \cdot F_\pi(Q^2)$ in fig. (13). The correct behaviour of the form factor at very large $Q^2$ can be traced back to the Lorentz boost that is applied to the outgoing vertex function. The charged kaon form factor is plotted in fig. (14); we conclude that our model provides a satisfying description of the electromagnetic $\pi^\pm$ and $K^\pm$ form factors in both parameter sets. Let us finally note that the correct form factor normalization $F_M(Q^2 = 0) = 1$ is a consequence of the normalization condition of the Bethe–Salpeter equation; we therefore do not need to impose an \textit{ad hoc} normalization of the electromagnetic form factor.

D. The Electromagnetic Decay Widths for $M \to M'\gamma$

To complete our discussion on electromagnetic processes involving mesons and to update the results of ref. [5], we will briefly comment on the widths of the decays $M(P) \to M'(P')\gamma(q)$. For this purpose, we extend the matrix element in eq. (28) to processes like $\rho \to \pi\gamma$ where also $J \neq 0$ mesons occur. The related decay form factor $F_{\rho\pi}(Q^2)$ with $Q^2 = -q^2$ is then given by

$$\langle \pi(P') \mid J_\mu(0) \mid \rho(P, \lambda) \rangle = QF_{\rho\pi}(Q^2)\epsilon_{\mu\nu\alpha\beta}e^{\alpha}_{\rho}(P, \lambda)P^{\mu}\lambda^{\nu}\alpha^{\beta}\frac{1}{M_\rho}$$

Here, $\epsilon_{\rho}(P, \lambda)$ denotes the polarization vector of the $\rho$ meson with spin projection $\lambda$; analogue definitions hold for all processes that will be discussed in the following. The general decay width can be computed via

$$\Gamma_{M \to M'\gamma} = \frac{1}{2}\frac{q}{M_\rho \lambda} \sum_{M_j, M_{j'}} \left| \epsilon^{\alpha}_{\rho}(q^2, +1) \langle M'(P', J', M_{j'}) \mid J_\mu(0) \mid M(P, J, M_j) \rangle \right|^2$$

where $\epsilon_{\rho}$ is the polarization vector of the photon with three–momentum $q = q\hat{e}_z$ and the matrix element is evaluated in the rest frame of meson $M$ with $P = (M, 0)$.

We show the widths for various meson decays in table [VIII] and compare them to the latest PDG data compilation (see [1]). As already stated in [5], we clearly underestimate the processes with a pion in the final state, especially in model $B$. Obviously this is again a significant shortcoming of the instantaneous approximation for deeply bound particles such as the pion; the resulting lack of retardation effects seems to spoil the correct overlapping of the associated wave functions in our calculation. Note however for the $\rho/\omega \to \pi\gamma$ decays that an exact SU(2) flavour symmetry — i.e. $m_\omega = m_d$ — implies $\Gamma_{\rho^0 \to \pi^\pm\gamma} = \Gamma_{\rho^0 \to \pi^0\gamma} = \frac{1}{2}\Gamma_{\omega \to \pi^0\gamma}$ as can be seen from the calculated values; therefore the results for the experimental widths in table [VIII] are rather puzzling.

We stress that the excellent results for $\Gamma_{M \to \eta\gamma}$ with $M = \rho^0, \omega, \phi$ indicate that the coefficients $X_\eta$ and $Y_\eta$ for the \textit{nonstrange/strange} flavour mixing are determined well in both models, see table [I]. In addition, we find plausible results for the decays $\eta' \to \rho^0\gamma$ and $\eta' \to \omega\gamma$ with the calculated widths in both models close to the experimental findings.
Due to the \( J = 0 \) selection rule for 't Hooft's flavour dependent interaction, no flavour mixing is induced for the \( \omega, \phi \) and \( f_1 \) mesons. It is therefore not surprising that we find vanishing decay widths for some processes since e.g. a pure nonstrange state (\( \sim n\bar{n} \)) like the \( \omega \) does not couple to a pure strange state (\( \sim s\bar{s} \)) like the \( \phi \) thus yielding \( \Gamma_{\phi \rightarrow \omega \gamma} = 0 \) in both models; we do not quote these zero widths in table VII.

In the kaonic sector, we find a very good agreement of the \( \Gamma_{K^+ \rightarrow K\gamma} \) widths in model \( A \) with the PDG values; our numerical results in model \( B \) slightly underestimate the experimental data although we can describe the correct ratio between the neutral and the charged decay mode.

Let us now come back to the form factor defined in eq. (29). For the decay \( \omega \rightarrow \pi^0\gamma^* \) with the virtual photon decaying into \( \mu^+\mu^- \), there are some experimental values for the normalized form factor \( \tilde{F}_{\omega\pi}(Q^2) \). We have plotted these experimental data points and our numerical results, calculated with the parameters of the models \( A \) and \( B \). Note that for \( Q^2 > -(m_q^2 + m_{\pi}^2) \) our model will become ill-defined since we cannot guarantee confinement for timelike momentum transfers; we therefore did not compute \( F_{\omega\pi}(Q^2) \) beyond the threshold at \( \approx -0.37 \text{GeV}^2 \) of model \( A \). Comparing our calculated curves with the experimental data and with a pole fit according to

\[
\tilde{F}_{\omega\pi}(Q^2) = \frac{1}{1 + Q^2/\Lambda^2}
\]

we find that we underestimate the shape of the form factor in the timelike region; obviously, this again can be traced back to the shortcomings of our model in the case of a \( \pi \) meson in the final state as has been discussed above. However, our results are comparable for \( Q^2 > 0 \) with the pole fit extracted in the experimental study in ref. [47]. The authors found \( \Lambda_{\text{exp}} = (650 \pm 30) \text{MeV} \) while our form factors would merely coincide with a simple \( \rho \) pole ansatz, i.e. \( \Lambda_\rho \approx 770 \text{MeV} \); this discrepancy becomes obvious for \( Q^2 < 0 \).

Summarizing this subsection on the electromagnetic decay widths, we conclude that we find a good overall agreement with the experimental data on the level of a factor of \( \approx 1.5 \) in the amplitudes with one exception: for a pion in the final state, our calculations fail significantly.

### E. The Decays \( \pi^+ \rightarrow e^+\nu\gamma \) and \( K^+ \rightarrow e^+\nu\gamma \)

The so-called \( \pi_{\ell\gamma} \) decay, i.e. \( \pi^+(P) \rightarrow \ell^+(p_\ell)\nu(p_\nu)\gamma(q) \), with the lepton \( \ell = e, \mu \) and the analogously defined \( K_{\ell\gamma} \) decay have been studied extensively both experimentally and theoretically in the last decade. For a muon in the final state, the \( \pi_{\ell\gamma} \) decay would be dominated by Bremsstrahlung effects; however, for \( \ell = e \) this contribution is strongly helicity suppressed such that the structure dependent parts of the related amplitude can be measured. The matrix element for this process reads

\[
M_{\pi^+ \rightarrow e^+\nu\gamma} = -\frac{eG_F}{\sqrt{2}} V_{ud}^* \epsilon^{\mu*}(q) M_{\mu\gamma}^{\ell\gamma}(p, q) \bar{u}(p_\nu)\gamma^\nu(\not{\gamma} + \gamma_5) v(p_e)
\]

where \( \bar{u}(p_\nu) \) and \( v(p_e) \) denote the Dirac spinors for the neutrino and the positron (see [42], [43] and the PDG mini-review in [19]). Here, the outgoing photon is real so that for its polarization vector \( \epsilon \cdot q = 0 \) with \( q^2 = 0 \) holds. The quantity \( M_{\mu\gamma}^{\ell\gamma} \) can be formulated as the time–ordered product of the electromagnetic current \( J^{\text{el.magn.}} \) and the weak current \( J^{\text{weak}} \) and reads explicitly:

\[
M_{\mu\nu}^{\ell\gamma}(p, q) = \int d^4x \left[ \langle J^{\text{el.magn.}}(x) J^{\text{weak}}(0) \big| \pi^+(P) \rangle e^{iq\cdot x} \right. \nonumber \\
= f_\pi \left( g_{\mu\nu} - \langle \pi^+(P - q) \big| J^{\text{el.magn.}} \big| \pi^+(P) \rangle \frac{(P - q)_\nu}{(P - q)^2 - M_\pi^2} \right. \nonumber \\
- h_A \left( (P - q)_\mu q_\nu - q \cdot (P - q) g_{\mu\nu} \right) + i h_V \epsilon_{\mu\nu\alpha\beta} q^\alpha P^\beta 
\]

The first term including \( f_\pi \) is the so-called Born part; the subscripts at the form factors \( h_V \) and \( h_A \) denote their origin either in the vector (\( V \)) or in the axial vector (\( A \)) part of the weak current. Note that an additional axial form factor occurs if the outgoing photon is virtual, i.e. for the decays \( \pi^+/K^+ \rightarrow e^+\nu_e e^+e^- \); we will not study these processes here.

In eq. (33), the Born terms contain the pion form factor; this quantity has been calculated in the framework of our model, see the foregoing subsection. As it stands, the full matrix element is gauge invariant due to the inclusion of the Born terms. In the following, we will only consider the structure dependent contributions, but we will take care that in the extraction of the form factors from the full tensor \( M_{\mu\nu}^{\ell\gamma} \) no terms will occur that violate gauge invariance.
The structure dependent parts of eq. (33) can be calculated in the Mandelstam formalism. The result is analogous to the matrix element of the two photon decay, see eq. (19) and fig. 2 except that one photon line is substituted by a $W^+$ boson line with the typical $V - A$ Dirac structure and the flavour matrix $\lambda_1 - i\lambda_2$. The calculation can be extended to the $K_{\ell 2\gamma}$ decays by inserting kaon observables $M_K, F_K$ in eq. (33) and by using the corresponding matrix in flavour space.

In table IX, we present the results for the form factors $h_V$ and $h_A$ of the $\pi_{\ell 2\gamma}$ decay as well as for the $K_{\ell 2\gamma}$ decay. They are compared with the world averages of the Particle Data Group (see [19]). The form factors of the $K_{\ell 2\gamma}$ decay are known only incompletely; we quote the experimental results for the sum and the difference of $h_V$ and $h_A$. In the case of the $\pi_{\ell 2\gamma}$ decay, one has to regard the values of the axial form factor $h_A$ with care. Since usually only the ratio $\gamma := h_A/h_V$ is measured and only two direct determinations of $h_V$ are presented up to now, the PDG results for $h_A$ are determined via this ratio with the input $h_V^{\text{CVC}} = (0.0259 \pm 0.0005)/M_2^2$ from the CVC prediction. Despite this caveat, we will however quote the results of ref. [19] for a comparison of our results with experimental data. We obtain excellent results compared to the PDG averages in both models, see table IX.

F. Form Factors of the $K_{\ell 3}$ Decay

The processes $K^+ \to \pi^0 \ell^+ \nu_\ell$ and $K^0 \to \pi^0 \ell^+ \nu_\ell$ are called $K^+_{\ell 3}$ and $K^0_{\ell 3}$ decay, respectively. They are usually parameterized in terms of two form factors for which isospin invariance requires $f_\pm^{K^0}\pi^0 = f_\pm^{K^0}\pi^- /\sqrt{2} =: f_\pm$ (see the minireview in [13]):

$$
\langle \pi^0(P') | J_{\mu}^{\text{weak}} | K^+(P) \rangle = f_+^{K^0}\pi^0 (P + P')_\mu + f_-^{K^0}\pi^0 (P - P')_\mu \quad (34)
$$

$$
\langle \pi^-(P') | J_{\mu}^{\text{weak}} | K^0(P) \rangle = f_+^{K^0}\pi^- (P + P')_\mu + f_-^{K^0}\pi^- (P - P')_\mu .
$$

In fig. 3, we show the $K^0_{\ell 3}$ decay in our Bethe–Salpeter quark model picture. Note that we describe the emission of the leptonic pair by a $W^+$ boson coupling to the strange quark and decaying into $\ell^+ \nu_\ell$.

For heavy quark systems, semileptonic decays of this type can also be calculated in our model if either an additional one–gluon exchange potential (see [7]) or an appropriate extension of ’t Hooft instanton–induced force is adopted; a subsequent paper on this subject is currently in preparation.

The matrix element for the $K_{\ell 3}$ decay can be defined analogously to the expression for the electromagnetic form factor of the $\pi^0/\pi^\pm$ meson, see eq. (28). In our model, it reads explicitly

$$
\langle \pi^-(P') | J_{\mu}^{\text{weak}} | K^0(P) \rangle = -\int \frac{d^4p}{(2\pi)^4} \text{tr} \left[ \tilde{F}_{(\pi^-)}(p + \frac{q}{2})S_d^F \Gamma_{(K\mu)}(p)S_s^F (-\frac{P}{2} + p)\bar{u}\gamma_\mu sS_u^F (-\frac{P}{2} + p + q) \right] \quad (35)
$$

Note that for $0^- \to 0^-$ transitions only the vector part of the weak coupling enters $J_{\mu}^{\text{weak}}$; the axial part of the weak coupling $\gamma(1 - \gamma_5)$ contributes to the form factors; furthermore, a second term with a $W$ boson coupling to the $d$ quark trivially does not occur due to its vanishing flavour trace.

In most experiments, the $q^2 = (P' - P)^2$ dependence of the $f_\pm$ form factor are found to be consistent with a linear parametrization like

$$
f_\pm(Q^2) = f_\pm(0) \left( 1 + \lambda_\pm \cdot Q^2 \right) . \quad (36)
$$

Since $f_-$ is multiplied by the lepton mass, this form factor is difficult to measure; however, there exist some rough experimental values for the ratio $\xi := f_- / f_+$. In table IX, we show our results for the form factors at $Q^2 = 0$ and their ratio $\xi$. For $f_+$ and $f_-$, the absolute values of these form factors are not determined so that only their ratio can be compared with experimental estimations.

We show the $Q^2$ dependence of the normalized $f_+$ form factor in fig. 4. Obviously, a linear fit to the experimental data is justified only approximately; indeed, our results show remarkable non–linear shapes. Although the data are underestimated in model B, they can be acceptably described with the parameters of model A so that no general shortcoming of our model can be stated.

Let us finally note a special feature of our model: the nonstrange and the strange quark sector are distinct not only due to different constituent masses, but also because of the ’t Hooft couplings $g$ and $g'$. In the simultaneous limit $m_s \to m_n$ and $g' \to g$, the $SU_F(3)$ symmetry is restored and thus the kaon amplitude will become the pion amplitude.

The $SU_F(3)$ limit leads then to $-f_+(Q^2) \to F_+(Q^2)$ and $f_-(Q^2) \to 0$ with $F_+(Q^2)$ being the well–known pion form factor, see eq. (27); this trivial result has been checked numerically in our model.
V. SUMMARY AND OUTLOOK

In this paper, we have presented some new results of a relativistic quark model for mesons. We have briefly resumed our approach on the basis of the Bethe–Salpeter equation in its instantaneous approximation. The potential in the resulting three–dimensional reduction is a combination of a linear confinement potential plus a residual interaction à la ‘t Hooft based on instanton effects.

Our numerical calculations were done with two different parameter sets that have distinct Dirac structures for the confinement. The discussion of the complete meson spectrum shows an excellent agreement with the experimental data; the correct splittings in the pseudoscalar sector can be backtraced to the effects of our residual interaction which in turn yields remarkable splittings for scalar states. We found considerable differences between the spectra of model A and model B: the masses of all $J^P = 0^+$ ground states were significantly lowered in model B so that the assignment of possible $q\bar{q}$ states in this puzzling sector will differ in both models.

Furthermore, we have investigated various meson decay modes such as the $K\ell_3$ decay and $\pi^0, \eta, \eta' \to \gamma\gamma$. The latter transitions were studied not only for real photons but also for very high virtualities. We found that our model fails if only one photon is virtual: the asymptotic limit (e.g. $Q^2 F_{\pi^0}(Q^2, 0) \to 2f_{\pi^0}$) known from perturbative QCD is not recovered. However, we can proof similar relations for $Q^2 F_{\gamma\gamma}(Q^2, Q^2)$ linking the transition form factor of the meson $\mathcal{M} = \pi^0, \eta, \eta'$ to its decay constants in the case of symmetric photon virtualities. We stress that the analytic calculations presented here are in fact model–independent. Finally, we found excellent agreement of the various form factors in the decays $\pi^+/K^+ \to \ell^+\nu\gamma$ as well as a satisfying description of the electromagnetic $\pi^+/K^+$ form factors in both parameter sets.

The relativistic quark model presented in this paper also allows further investigations; we have already mentioned an extension of ‘t Hooft’s residual interaction for heavy $q\bar{q}$ systems. A further topic is the study of strong decays in this framework. Hereby, it is of special interest that not only pure quark loops contribute to the strong decay widths, but also instanton–induced six–quark interactions will occur. Furthermore, we have studied the various implications of our quark model with respect to the concept of spontaneous breaking of the chiral symmetry; various low–energy theorems can be tested and compared with results from Chiral Perturbation Theory. A last point to mention is the study of Compton scattering off a pseudoscalar meson: it is possible to calculate the corresponding matrix elements in the framework of our Bethe–Salpeter model and compute the electromagnetic polarizabilities, even in their generalized form for virtual photons. All these various aspects are currently prepared for publication and will soon be presented in subsequent papers.

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In this Appendix, we present a model–independent factorization proof for the asymptotic limit \((Q^2 \to \infty)\) of the pseudoscalar transition form factor \(Q^2 F_{\gamma\gamma}^M(Q^2, Q^2)\) at equal photon virtualities. We stress that the following considerations are formulated on the basis of full four–dimensional Bethe–Salpeter amplitudes and quark propagators so that it will be independent of the instantaneous approximation that we adopted for our numerical evaluations.

APPENDIX
Let us start with the matrix element $T_{\gamma\gamma}^M$ in eq. (13) for the decays $M \rightarrow \gamma\gamma$ with $M = \pi^0, \eta, \eta'$ at arbitrary photon virtualities $Q_i^2 = -q_i^2$. We define $q := \frac{1}{2}(q_1 - q_2)$ and consider the decaying meson in its rest frame with $P = (M,0)$. With $q_1 = P/2 + q$ and $q_2 = P/2 - q$, we then find the following relations for $Q_1^2 = Q_2^2 = Q^2 \rightarrow \infty$:

$$q_i^2 \rightarrow -Q^2, \quad q_0^0 \rightarrow 0 \quad \Rightarrow \quad q^0_1, q^2_1 \rightarrow \frac{M}{Q}, \quad q^3_1, -q^3_2 \rightarrow \sqrt{Q^2}$$

(37)

Here, we have chosen the photon momenta to be in the direction of the $z$–axis. We now study the behaviour of the intermediate propagator in both terms of the matrix element $T_{\gamma\gamma}^M$ in eq. (19). The denominator behaves like $(P/2 + p - q_i)^2 \rightarrow -Q^2$ for asymptotic photon virtualities since terms proportional to the relative momentum $p$ will not contribute due to the vanishing vertex function for $p \rightarrow \infty$. For the same reason, only a $\frac{q_i^2}{Q^2}$ term survives in the numerator ("+" for $i = 1$, "−" for $i = 2$). Therefore we find for the complete intermediate quark propagator

$$S^F(\frac{P}{2} + p + q_i) \rightarrow \pm \sqrt{Q^2} \frac{Q^2}{Q^2} \gamma_3 \quad \text{as} \quad Q^2 \rightarrow \infty.$$  

(38)

Inserting this in the matrix element of eq. (19), we find

$$T_{\gamma\gamma}^M \rightarrow -i\sqrt{\frac{3}{2}} \sqrt{\frac{Q^2}{Q^2}} \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[ S^F(\frac{P}{2} + p)\Gamma^F_M(p)S^F(-\frac{P}{2} + p)(\gamma_2\gamma_3 \gamma_1 - \gamma_3\gamma_1 \gamma_2) \right]$$

(39)

for asymptotic virtualities. Since we are dealing with virtual photons, the polarization vectors $\epsilon_i = (0, \epsilon_i)$ do not only have transversal but also longitudinal components. However, if one of the photons is longitudinally polarized with $\epsilon_i \parallel q_i \parallel \epsilon_3$, the two terms in eq. (39) will cancel. The same happens for $\epsilon_i = \epsilon_2$ so that we conclude

$$\epsilon_2 \gamma_3 \epsilon_1 - \epsilon_1 \gamma_3 \epsilon_2 = \begin{cases} 
0 & \text{if } \epsilon_i = \epsilon_2 \text{ or } \epsilon_i \parallel q_i \\
\mp 2i\gamma_0 \gamma_5 & \text{otherwise} 
\end{cases}$$

(40)

where, as usual, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. Here and in the following, the upper sign ("−") stands for polarization vectors $\epsilon_1 = (0, 1, 0, 0)$ and $\epsilon_2 = (0, 0, 1, 0)$, while the lower sign ("+”) applies to the choice $\epsilon_1 = (0, 0, 1, 0)$ and $\epsilon_2 = (0, 1, 0, 0)$. The resulting integrand is identical to the expression for the pseudoscalar decay constants in eq. (13) in its four–dimensional generalization for $M = \pi^0, \eta, \eta'$. By comparison of the asymptotic matrix element and the definition of the decay constants $f^j_M$ ($j = \pi, n, s$) we find

$$T_{\gamma\gamma}^M \rightarrow \mp 2\sqrt{3} \sqrt{\frac{Q^2}{Q^2}} \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[ S^F(\frac{P}{2} + p)\Gamma^F_M(p)S^F(-\frac{P}{2} + p)\gamma_0\gamma_5 \right]$$

$$= \mp 2\frac{M_M\sqrt{Q^2}}{Q^2} \sum_j \bar{C}_j f^j_M$$

(41)

as $Q^2 \rightarrow \infty$. If we consider the process $\pi^0 \rightarrow \gamma\gamma$, the sum becomes trivial and contains only the pion decay constant $f_\pi$ multiplied by the factor $C_\pi = \alpha/(3\sqrt{2})$. We note that we have introduced the charge factors $C_j := \alpha C_j$ that are proportional to the factors defined in section (14B). Now we recall the relation between the matrix element of the two photon decay and the transition form factor in eq. (23). Since we have set the photon momenta as $q_i = (q_i^0, 0, 0, q_i^3)$, we can evaluate the implicit sum over the indices $\mu, \nu, \alpha$ and $\beta$. This yields

$$\mp \alpha M_M\sqrt{Q^2} F^M_{\gamma\gamma}(Q^2, Q^2) = \mp 2\frac{M_M\sqrt{Q^2}}{Q^2} \sum_j \bar{C}_j f^j_M \quad \text{as} \quad Q^2 \rightarrow \infty$$

(42)

so that we finally arrive at the asymptotic limits quoted in eqs. (23) and (26):

$$\lim_{Q^2 \rightarrow \infty} Q^2 F^{p^0}_{\gamma\gamma}(Q^2, Q^2) = \frac{2}{3} f_\pi^3$$

(43)

$$\lim_{Q^2 \rightarrow \infty} Q^2 F^{\eta\eta}_{\gamma\gamma}(Q^2, Q^2) = \frac{5}{9\sqrt{2}} f_\eta^3 + \frac{1}{9} f_\eta^3$$

(44)

$$\lim_{Q^2 \rightarrow \infty} Q^2 F^{\eta'\eta'}_{\gamma\gamma}(Q^2, Q^2) = \frac{5}{9\sqrt{2}} f_\eta^3 + \frac{1}{9} f_\eta^3$$

(45)

We want to emphasize that this proof is entirely model–independent since we only explored kinematical properties of the diagrams in fig. (4) without making any assumptions on the meson vertex function, in particular without applying the instantaneous approximation.
| Parameter                        | Model A | Model B |
|---------------------------------|---------|---------|
| 't Hooft interaction            | $g$ [GeV$^{-2}$] | 1.73    | 1.62    |
|                                 | $g'$ [GeV$^{-2}$] | 1.54    | 1.35    |
|                                 | $\Lambda_{\text{III}}$ [fm] | 0.30    | 0.42    |
| Constituent quark masses        | $m_n$ [MeV] | 306     | 380     |
|                                 | $m_s$ [MeV] | 503     | 550     |
| Confinement parameters          | $a_c$ [MeV] | -1751   | -1135   |
|                                 | $b_c$ [MeV/fm] | 2076    | 1300    |
| Spin structure                  | $\Gamma \otimes \Gamma$ | $\frac{1}{2}(\mathbf{1} \otimes \mathbf{1} - \gamma_0 \otimes \gamma_0)$ | $\frac{1}{2}(\mathbf{1} \otimes \mathbf{1} - \gamma_\mu \otimes \gamma^\mu - \gamma_5 \otimes \gamma_5)$ |

**TABLE I.** The parameters of the confinement force, the 't Hooft interaction and the constituent quark masses in the models $A$ and $B$. 
| Meson ($J^{PC}$) | $n$ | Model A  | Model B  |
|-----------------|-----|----------|----------|
| $\pi(0^{-+})$   | 0   | 138      | 140      |
|                 | 1   | 1357     | 1331     |
|                 | 2   | 2012     | 1826     |
|                 | 3   | 2498     | 2193     |
|                 | 4   | 2898     | 2496     |
|                 | 0   | 778      | 785      |
|                 | 1   | 1553     | 1420     |
|                 | 2   | 1605     | 1472     |
|                 | 3   | 2118     | 1891     |
|                 | 4   | 2161     | 1913     |
|                 | 5   | 2567     | 2244     |
|                 | 6   | 2608     | 2257     |
|                 | 7   | 2949     | 2538     |
|                 | 8   | 2987     | 2547     |
| $\rho(1^{-+})$  | 0   | 1633     | 1605     |
|                 | 1   | 2156     | 1997     |
|                 | 2   | 2592     | 2318     |
|                 | 0   | 1698     | 1743     |
|                 | 1   | 2157     | 2060     |
|                 | 2   | 2208     | 2091     |
|                 | 3   | 2576     | 2371     |
|                 | 4   | 2631     | 2388     |
|                 | 0   | 2279     | 2318     |
|                 | 1   | 2623     | 2545     |
| $\rho_3(3^{-+})$| 0   | 1698     | 1743     |
|                 | 1   | 2576     | 2371     |
|                 | 2   | 2631     | 2388     |
|                 | 0   | 2279     | 2318     |
| $\rho_5(5^{-+})$| 0   | 1698     | 1743     |
|                 | 1   | 2623     | 2545     |

**TABLE II.** Masses of the isovector mesons in [MeV], calculated with the parameters of the models $A$ and $B$; here, $n$ denotes the radial excitation. For a comparison with the latest experimental data of ref. [19], see fig. (7).
| Meson \((J^P)\) | \(n\) | Model \(A\) | Model \(B\) | Meson \((J^P)\) | \(n\) | Model \(A\) | Model \(B\) |
|---|---|---|---|---|---|---|---|
| \(\pi\) | 0 | 499 | 506 | \(\pi\) | 0 | 1426 | 1187 |
| | 1 | 1508 | 1470 | \(K_0^*(0^+\rangle\) | 1 | 2058 | 1788 |
| | 2 | 2159 | 1965 | \(|\rangle\) | 2 | 2561 | 2196 |
| \(K(0^-\rangle\) | 3 | 2652 | 2336 | \(K^*(0^-\rangle\) | 0 | 870 | 890 |
| | 4 | 3062 | 2644 | \(|\rangle\) | 1 | 1687 | 1550 |
| | 5 | 3420 | 2911 | \(K^*(1^-\rangle\) | 2 | 1718 | 1588 |
| | 6 | 3748 | 3151 | \(|\rangle\) | 3 | 2261 | 2018 |
| | 7 | 4074 | 3370 | \(|\rangle\) | 4 | 2289 | 2037 |
| \(K_1(1^+\rangle\) | 0 | 1353 | 1315 | \(K_2^*(2^+\rangle\) | 1 | 2019 | 1889 |
| | 1 | 1353 | 1315 | \(|\rangle\) | 2 | 2049 | 1914 |
| | 2 | 2005 | 1840 | \(K_3^*(3^-\rangle\) | 1 | 2300 | 2173 |
| | 3 | 2005 | 1840 | \(|\rangle\) | 2 | 2334 | 2192 |
| \(K_2(2^-\rangle\) | 0 | 1750 | 1709 | \(|\rangle\) | 0 | 1800 | 1828 |
| | 1 | 1750 | 1709 | \(K_3^*(3^-\rangle\) | 1 | 2300 | 2173 |
| | 2 | 2292 | 2115 | \(|\rangle\) | 2 | 2334 | 2192 |
| \(K_3(3^+\rangle\) | 0 | 2074 | 2026 | \(K_4^*(4^+\rangle\) | 1 | 2550 | 2424 |
| | 1 | 2074 | 2026 | \(|\rangle\) | 2 | 2585 | 2439 |
| | 2 | 2544 | 2362 | \(K_5^*(5^-\rangle\) | 1 | 2775 | 2649 |
| \(K_4(4^-\rangle\) | 0 | 2352 | 2299 | \(|\rangle\) | 0 | 2397 | 2400 |
| | 1 | 2352 | 2299 | \(K_5^*(5^-\rangle\) | 1 | 2775 | 2649 |
| | 2 | 2771 | 2587 | \(|\rangle\) | 2 | 2811 | 2661 |

**TABLE III.** Masses of the isodublet mesons in [MeV], calculated with the parameters of the models \(A\) and \(B\); here, \(n\) denotes the radial excitation. For a comparison with the latest experimental data of ref. [19], see fig. (8). Note that the calculated \(K_1\) states are each 2-fold degenerate for spin \(S = 0\) and \(S = 1\).
| Meson ($J^{pc}$) | n  | Model A | Model B |
|-----------------|----|---------|---------|
| $\pi(0^{-+})$   | 0  | 531     | 533     |
|                 | 1  | 975     | 950     |
|                 | 2  | 1533    | 1446    |
|                 | 3  | 1812    | 1654    |
|                 | 4  | 2177    | 1912    |
|                 | 5  | 2381    | 2118    |
|                 | 6  | 2657    | 2267    |
|                 | 7  | 2838    | 2479    |
|                 | 8  | 3054    | 2565    |
| $f_0(0^{++})$    | 0  | 984     | 665     |
|                 | 1  | 1468    | 1262    |
|                 | 2  | 1776    | 1554    |
|                 | 3  | 2113    | 1870    |
|                 | 4  | 2310    | 2006    |
|                 | 5  | 2617    | 2281    |
|                 | 6  | 2756    | 2359    |
| $f_1(1^{++})$    | 0  | 1240    | 1201    |
|                 | 1  | 1454    | 1422    |
|                 | 2  | 1876    | 1718    |
| $\omega/\phi(1^{--})$ | 0 | 778 | 785 |
| $\eta(2^{--})$  | 0  | 954     | 990     |
|                 | 1  | 1553    | 1420    |
|                 | 2  | 1605    | 1472    |
|                 | 3  | 1804    | 1674    |
|                 | 4  | 1829    | 1701    |
|                 | 5  | 2118    | 1891    |
|                 | 6  | 2161    | 1913    |
| $f_2(2^{++})$    | 0  | 1312    | 1358    |
|                 | 1  | 1495    | 1537    |
| $\eta_2(2^{--})$| 0  | 1861    | 1812    |
|                 | 1  | 2156    | 1997    |
|                 | 2  | 2163    | 1918    |
| $\omega_3/\phi_3(3^{--})$ | 0 | 1698 | 1743 |
|                 | 1  | 1899    | 1918    |
|                 | 2  | 2157    | 2060    |
|                 | 3  | 2208    | 2091    |
| $f_3(3^{++})$    | 0  | 1951    | 1926    |
|                 | 1  | 2192    | 2127    |
| $\eta_4(4^{--})$| 0  | 2223    | 2200    |
|                 | 1  | 2478    | 2398    |
|                 | 2  | 2622    | 2475    |
|                 | 3  | 2917    | 2700    |
| $f_4(4^{++})$    | 0  | 2011    | 2052    |
|                 | 1  | 2230    | 2227    |
|                 | 2  | 2402    | 2315    |
| $\omega_5/\phi_5(5^{--})$ | 0 | 2279 | 2318 |
|                 | 1  | 2514    | 2493    |
|                 | 2  | 2623    | 2345    |
|                 | 3  | 2671    | 2568    |
| $f_5(5^{++})$    | 0  | 2463    | 2444    |
|                 | 1  | 2730    | 2640    |
|                 | 2  | 2825    | 2685    |
| $h_1(1^{++})$    | 0  | 1240    | 1201    |
|                 | 1  | 1454    | 1422    |
|                 | 2  | 1876    | 1718    |
| $f_6(6^{++})$    | 0  | 2517    | 2554    |
|                 | 1  | 2766    | 2730    |
|                 | 2  | 2826    | 2755    |

TABLE IV. Masses of the isoscalar mesons in [MeV], calculated with the parameters of the models $A$ and $B$; here, $n$ denotes the radial excitation. For a comparison with the latest experimental data of ref. [19], see fig. (1).
Coefficient Model A Model B Mark III , [25] DM2 , [26] Ref. [22] Ref. [24]

$|X_\eta|$ 0.68 0.74 0.67 ± 0.05 0.647 ± 0.044 0.774 ± 0.010 0.768 ± 0.020
$|Y_\eta|$ 0.73 0.67 0.74 ± 0.10 0.771 ± 0.037 0.633 ± 0.013 0.640 ± 0.024
$|X_{\eta'}|$ 0.71 0.61 0.58 ± 0.06 0.436 ± 0.044 0.633 ± 0.013 0.640 ± 0.024
$|Y_{\eta'}|$ 0.70 0.79 1.05 ± 0.12 0.900 ± 0.021 0.774 ± 0.010 0.768 ± 0.020

**TABLE V.** The coefficients for the nonstrange/strange mixing in the $\eta$ and $\eta'$ mesons according to eq. (16), calculated with the parameters of the models $A$ and $B$. The Mark III group used an unconstrained fit in the evaluation of the coefficients while $X_M^2 + Y_M^2 = 1$ ($M = \eta, \eta'$) was demanded in the DM2 analysis. Note that the results of refs. [22] and [24] are found in the so-called one-angle mixing scheme of eq. (17), where by definition $|X_\eta| = |Y_{\eta'}|$ and $|Y_\eta| = |X_{\eta'}|$ is fixed.

| Decay Constant | Model A | Model B | PDG 2000, [19] | Ref. [22] |
|----------------|---------|---------|----------------|----------|
| $f_\pi$ [MeV] | 212     | 219     | 130.7 ± 0.46   | —        |
| $f_K$ [MeV]  | 248     | 238     | 159.8 ± 1.88   | —        |
| $f_\eta^n$ [MeV] | 142    | 161     | —              | 108.5 ± 2.6 |
| $f_\eta^s$ [MeV] | −205   | −166    | —              | −111.2 ± 5.5 |
| $f_{\eta'}^n$ [MeV] | 92     | 95      | —              | 88.8 ± 2.5  |
| $f_{\eta'}^s$ [MeV] | 166    | 176     | —              | 136.8 ± 6.4  |

**TABLE VI.** The pseudoscalar decay constants of the $\pi$, $K$, $\eta$ and $\eta'$ mesons, calculated with the parameters of the models $A$ and $B$. The results of ref. [22] originate from a phenomenological analysis of various decay ratios, e.g. $\Gamma(J/\psi \to \eta' \rho)/\Gamma(J/\psi \to \eta \rho)$; note that these numerical values are found in the so-called one-angle mixing scheme, see eq. (17).

| Decay Width | Model A | Model B | PDG 2000, [19] |
|-------------|---------|---------|----------------|
| $\Gamma(\pi \to \gamma \gamma)$ [eV] | 4.1     | 3.42    | 7.74 ± 0.56    |
| $\Gamma(\eta \to \gamma \gamma)$ [eV] | 215     | 213     | 460 ± 40       |
| $\Gamma(\eta' \to \gamma \gamma)$ [eV] | 2320    | 1480    | 4290 ± 150     |
| $\Gamma(f_0(400-1200) \to \gamma \gamma)$ [eV] | —       | 232     | 10000 ± 6000 (*) |
| $\Gamma(f_0(980) \to \gamma \gamma)$ [eV] | 1760    | —       | 390 ± 130      |
| $\Gamma(a_0(980) \to \gamma \gamma)$ [eV] | —       | 500     | 300 ± 100 (*)  |

**TABLE VII.** The widths of the decays $0^+ \to \gamma \gamma$ ($\pi = \pm$), calculated with the parameters of the models $A$ and $B$. Experimental widths marked with (*) are quoted in ref. [19] without using them for averages, fits etc. The interpretation of the scalar mesons differ in models $A$ and $B$ such that not all widths are calculated in both parameter sets.
### Decay Width Model

**Model A** | **Model B** | **PDG 2000, [19]**
--- | --- | ---
$\Gamma(\rho^+ \rightarrow \pi^+ \gamma) [\text{keV}]$ | 35.0 | 20.6 | 68 ± 7
$\Gamma(\rho^0 \rightarrow \pi^0 \gamma) [\text{keV}]$ | 35.0 | 20.6 | 102 ± 26
$\Gamma(\rho^0 \rightarrow \eta \gamma) [\text{keV}]$ | 49.7 | 39.8 | 36 ± 12
$\Gamma(\omega \rightarrow \pi^0 \gamma) [\text{keV}]$ | 315 | 185 | 717 ± 42
$\Gamma(\omega \rightarrow \eta \gamma) [\text{keV}]$ | 5.52 | 4.42 | 5.5 ± 0.8
$\Gamma(\eta' \rightarrow \rho^0 \gamma) [\text{keV}]$ | 87.3 | 28.0 | 60 ± 5
$\Gamma(\eta' \rightarrow \omega \gamma) [\text{keV}]$ | 9.70 | 3.11 | 6.1 ± 0.8
$\Gamma(\phi \rightarrow \eta \gamma) [\text{keV}]$ | 58.1 | 34.7 | 58 ± 2
$\Gamma(\phi \rightarrow \eta' \gamma) [\text{keV}]$ | 0.01 | 0.074 | 0.30 ± 0.16
$\Gamma(K^+ \rightarrow K^0 \gamma) [\text{keV}]$ | 48.0 | 28.8 | 50 ± 5
$\Gamma(K^0 \rightarrow K^0 \gamma) [\text{keV}]$ | 102 | 70.2 | 116 ± 10
$\Gamma(h_1^+ \rightarrow \pi^+ \gamma) [\text{keV}]$ | 9.21 | 7.05 | 230 ± 60
$\Gamma(f_1(1285) \rightarrow \rho^0 \gamma) [\text{keV}]$ | 365 | 208 | 1320 ± 310

**TABLE VIII.** The widths of the decays $\mathcal{M} \rightarrow \mathcal{M}' \gamma$, calculated with the parameters of the models $\mathcal{A}$ and $\mathcal{B}$.

### Form Factor Model

**Model A** | **Model B** | **PDG 2000, [19]**
--- | --- | ---
$h_V$ | 0.014 | 0.017 | 0.017 ± 0.008
$h_A$ | 0.012 | 0.010 | 0.0116 ± 0.0016

**TABLE IX.** The form factors of the $\pi_{\ell 2\gamma}$ decay and the $K_{\ell 2\gamma}$ decay, calculated with the parameters of the models $\mathcal{A}$ and $\mathcal{B}$.

### Form Factor

| Form Factor | Model $\mathcal{A}$ | Model $\mathcal{B}$ | PDG 2000, [19] |
| --- | --- | --- | --- |
| $f_+(0)$ | −0.813 | −0.803 | — |
| $f_-(0)$ | 0.121 | 0.154 | — |
| $\xi(0)$ | −0.148 | −0.192 | −0.31 ± 0.15 |

**TABLE X.** The form factors of the $K_{\ell 3}$ decay and their ratio $\xi$ at $Q^2 = 0$, calculated with the parameters of the models $\mathcal{A}$ and $\mathcal{B}$.
FIGURES

\[ \chi^P \mathcal{M}(P) \mathcal{M} \chi^P = -i \mathcal{K} \]

FIG. 1. The Bethe–Salpeter equation for \( q\bar{q} \) bound states in a graphical notation.

\[ \gamma_1(q_1, \varepsilon_1) \gamma_2(q_2, \varepsilon_2) \gamma_1 \gamma_2 \gamma_1 \]

FIG. 2. The decay \( \mathcal{M}(P) \to \gamma_1(q_1, \varepsilon_1)\gamma_2(q_2, \varepsilon_2) \) in lowest order in the Mandelstam formalism.

\[ \pi^- e^+ \nu_e \]

FIG. 3. The decay \( K^0 \to \pi^- \ell^+ \nu_\ell \) in lowest order in the Mandelstam formalism.
\[ \eta \rightarrow g = 1.62 \text{ GeV}^{-2} \]
\[ \eta' \rightarrow g' = 0 \text{ GeV}^{-2} \]

**FIG. 4.** The effect of 't Hooft’s instanton induced interaction on the masses of the pseudoscalar mesons with the parameters of model A. Solid line: \( M_\pi \); dashed line: \( M_K \); dashed–dotted line: \( M_\eta \); dotted line: \( M_{\eta'} \); crosses denote the experimental masses from the Particle Data Group (see [19]).

**FIG. 5.** The effect of 't Hooft’s instanton induced interaction on the masses of the pseudoscalar mesons with the parameters of model B. See also caption of fig. (4).
FIG. 6. The Regge trajectory for light isovector mesons with the parameters of model $A$ and model $B$ compared to experimental masses from the Particle Data Group (see [19]).
FIG. 7. The spectrum of the light mesons with isospin $I = 1$. Left column for each $J^{\pi c}$: model $A$; middle column for each $J^{\pi c}$: experimental masses and their error bars marked by the shadowed rectangles from the Particle Data Group (see [19]); right column for each $J^{\pi c}$: model $B$. Note the difference for the $J^{\pi c} = 0^{++}$ states in the two parameter sets.
FIG. 8. The spectrum of the light mesons with isospin $I = \frac{1}{2}$. Left column for each $J^\pi$: model A; middle column for each $J^\pi$: experimental masses and their error bars marked by the shadowed rectangles from the Particle Data Group (see [19]); right column for each $J^\pi$: model B. The experimental data for the $K_3$ and $K_4$ masses need confirmation; the PDG data plotted above do not fit in a linear Regge trajectory $M_{K^*J}^2 \propto J$. Note that the calculated $K_1$ states are each 2-fold degenerate for spin $S = 0$ and $S = 1$, indicated by “2”, so that the total number of $K_1$ states is correct; see the discussion in section III.
FIG. 9. The spectrum of the light mesons with isospin $I = 0$. Left column for each $J^{pc}$: model $A$; middle column for each $J^{pc}$: experimental masses and their error bars marked by the shadowed rectangles from the Particle Data Group (see [19]); right column for each $J^{pc}$: model $B$. There are experimental hints that the $\eta(1295)$ (plotted above according to the PDG data) might not really exist, see [21].
FIG. 10. The decay widths for the processes $\pi^0, \eta, \eta' \rightarrow \gamma\gamma^*$ as a function of the momentum transfer of the virtual photon, calculated with the parameters of model $A$ and model $B$. 
\[ \eta' \rightarrow \gamma^*\gamma^* \]

\[ \eta \rightarrow \gamma^*\gamma^* \]

\[ \pi^0 \rightarrow \gamma^*\gamma^* \]

FIG. 11. The form factors of the \( \gamma^*\gamma^* \) decays at equal photon virtualities \( Q^2 := Q_1^2 = Q_2^2 \), calculated with model A, and their limits for \( Q^2 \rightarrow \infty \) according to the eqs. (25) and (26), denoted by the horizontal lines.

\[ Q^2, F(Q^2, Q^2) \text{[MeV]} \]

\[ Q^2 \text{[GeV}^2] \]

\[ Q^2 \text{[GeV}^2] \]

\[ Q^2 \text{[GeV}^2] \]

FIG. 12. The form factors of the \( \gamma^*\gamma^* \) decays at equal photon virtualities \( Q^2 := Q_1^2 = Q_2^2 \), calculated with model B, and their limits for \( Q^2 \rightarrow \infty \) according to the eqs. (25) and (26), denoted by the horizontal lines.
FIG. 13. The electromagnetic form factor $Q^2 \cdot F_\pi(Q^2)$ of the charged pion, calculated with the parameters of model $A$ and model $B$. Note that the correct shape of the form factor beyond $\approx 1\text{GeV}^2$ can be traced back to the application of the full Lorentz boost, see [5].

FIG. 14. The electromagnetic form factor $F_K^2(Q^2)$ of the charged kaon, calculated with the parameters of model $A$ and model $B$. 
FIG. 15. The normalized decay form factor $\tilde{F}_2(Q^2) = (F_{\omega\pi}(Q^2)/F_{\omega\pi}(0))^2$, calculated with the parameters of model $A$ and model $B$. The solid line is the pole fit of ref. [47] with the parameter $\Lambda = 0.65 \text{GeV}$, see eq. (31).

FIG. 16. The normalized form factor $f_+(Q^2)/f_+(0)$ of the $K_\ell 3$ decay, calculated with the parameters of model $A$ and model $B$. The solid line indicates the linear fit according to eq. (36) with the parameter $\lambda_{PDG} = (0.0276 \pm 0.0021)/M_\pi^2$, see ref. [19].