Research Article

Null Controllability of a Nonlinear Age Structured Model for a Two-Sex Population

Amidou Traoré 1, Okana S. Sougué, 1 Yacouba Simporé 1,2 and Oumar Traoré 1,3

1Laboratoire LAMI, Université Joseph Ki ZERBO, 01 BP 7021 Ouaga 01, Burkina Faso
2DeustoTech, Fundacion Deusto Avda. Universidades, 24 48007, Bilbao, Basque Country, Spain
3Département de Mathématiques de la Décision, Université Ouaga 2, 12 BP 417 Ouaga 12, Burkina Faso

Correspondence should be addressed to Amidou Traoré; amidoutraore70@yahoo.fr

Received 12 December 2020; Accepted 3 March 2021; Published 29 March 2021

Academic Editor: Victor Kovtunenko

Copyright © 2021 Amidou Traoré et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper is devoted to study the null controllability properties of a nonlinear age and two-sex population dynamics structured model without spatial structure. Here, the nonlinearity and the coupling are at the birth level. In this work, we consider two cases of null controllability problem. The first problem is related to the extinction of male and female subpopulation density. The second case concerns the null controllability of male or female subpopulation individuals. In both cases, if $A$ is the maximal age, a time interval of duration $A$ after the extinction of males or females, one must get the total extinction of the population. Our method uses first an observability inequality related to the adjoint of an auxiliary system, a null controllability of the linear auxiliary system, and after Kakutani’s fixed-point theorem.

1. Introduction

In practice, this study takes place in the fight against malaria. Malaria is a serious disease (in 2017, there were 219 million cases in the world [1]), and our work takes its importance in the strategy to fight against it.

In fact, malaria is a vector-borne disease transmitted by an infective female anopheles mosquito. A malaria control strategy in Brazil or Burkina Faso consists of releasing genetically modified male mosquitoes (precisely sterile males) in the nature. This can reduce the reproduction of mosquitoes since females mate only once in their life cycle.

In the theoretical framework, very few authors have studied control problems of a two-sex structured population dynamics model.

The control problems of coupled systems of population dynamics models take an intense interest and are widely investigated in many papers. Among them, we have [2–4] and the references therein. In fact, in [2], the authors studied a coupled reaction-diffusion equations describing interaction between a prey population and a predator one. The goal of the above work is to look for a suitable control supported on a small spatial subdomain which guarantees the stabilization of the predator population to zero. In [4], the objective was different. More precisely, the authors consider an age-dependent prey-predator system and they prove the existence and uniqueness of an optimal control (called also “optimal effort”) which gives the maximal harvest via the study of the optimal harvesting problem associated with their coupled model. In [5], He and Ainseba study the null controllability of a butterfly population by acting on eggs, larvas, and female moths in a small age interval.

In [3], the authors analyze the growth of a two-sex population with a fixed age-specific sex ratio without diffusion. The model is intended to give an insight into the dynamics of a population where the mating process takes place at random choice and the proportion between females and males is not influenced by environmental or social factors but only depends on a differential mortality or on a possible transition from one sex to the other (e.g., in sequential hermaphrodite species). Simpore and Traore study in [6] the null controllability of a nonlinear age, space, and two-
sex structured population dynamics model. They first study an approximate null controllability result for an auxiliary cascade system and prove the null controllability of the non-linear system by means of Schauder’s fixed-point theorem.

Unlike the model treated in [6], we consider a nonlinear cascade system with two different fertility rates and without space variable. The fertility rate of the male \( \lambda \) and the fertility rate \( \beta \) of the female depend on the total population of the fertile males.

**Remark 1.** Here, we are talking about the null controllability as an extinction of the population, but we can see in the sense of exact controllability to the trajectories since there is an equivalence between the null controllability and exact controllability to the trajectories in the linear case.

### 2. Model and Main Results

In this paper, we study the null controllability of an infinite dimensional nonlinear coupled system describing the dynamics of a two-sex structured population. Let \( (m, f) \) be the solution of the following system:

\[
\begin{align*}
\frac{d}{dt} m_t + m_t + \mu_m m &= \chi_{\Theta_1} v_m \text{ in } Q, \\
\frac{d}{dt} f_t + f_t + \mu f &= \chi_{\Theta_2} v_f \text{ in } Q, \\
m(a, 0) &= m_0(a) = f_0(a) \text{ in } Q_A, \\
\int_0^T m(a, t) dt &= (1 - \gamma) \int_0^A \beta(a, M) f(a, t) da \text{ in } Q_T, \\
\int_0^T f(a, t) dt &= \gamma \int_0^A \beta(a, M) f(a, t) da \text{ in } Q_T, \\
M &= \int_0^A \lambda(a) m(a, t) da \text{ in } Q_T,
\end{align*}
\]

where \( T \) is a positive number, \( Q = (0, A) \times (0, T), \Theta = (0, a_2) \times (0, T), \Theta_1 = (a_1, a_2) \times (0, T), \) and \( \Theta_2 = (b_1, b_2) \times (0, T). \) Here, \( 0 < a_1 < a_2 \leq A, \quad 0 < b_1 < b_2 \leq A, \quad Q_A = (0, A), \) and \( Q_T = (0, T). \)

We denote the density of males and females of age \( a \) at time \( t \), respectively, by \( m(a, t) \) and \( f(a, t) \). Moreover, \( \mu_m \) and \( \mu_f \) denote, respectively, the natural mortality rate of males and females. The control functions are \( v_m \) and \( v_f \) and depend on \( a \) and \( t \). In addition, \( \chi_{\Theta_1} \) and \( \chi_{\Theta_2} \) are the characteristic functions of the support of the control \( v_m \) and \( v_f \), respectively.

We have denoted by \( \beta \) the positive function describing the fertility rate that depends on \( a \) and also on \( t \) are given, respectively, by \( m(0, t) = (1 - \gamma)N(t) \) and \( f(0, t) = \gamma N(t) \) where

\[
N(t) = \int_0^A \beta(a, M) f(a, t) da.
\]

We assume that the fertility rate \( \beta \) and the mortality rate \( \mu, \lambda \) satisfy the demographic properties:

\[
\begin{align*}
(\lambda_1) \quad &\mu_m(a) \geq 0, \quad \mu_f(a) \geq 0 \quad \text{a.e. } a \in (0, A), \\
(\lambda_2) \quad &\mu_m \in L^1_{\text{loc}}(0, A), \quad \mu_f \in L^1_{\text{loc}}(0, A), \\
(\lambda_3) \quad &\int_0^A \mu_m(a) da = +\infty, \quad \int_0^A \mu_f(a) da = +\infty,
\end{align*}
\]

We further assume that the birth function \( \beta \) and the fertility function \( \lambda \) verify the following hypothesis:

\[
\begin{align*}
(\lambda_4) \quad &\beta(a, p) \in C([0, A] \times R), \\
(\lambda_5) \quad &\beta(a, p) \geq 0 \quad \text{for every } a, p \in [0, A] \times R,
\end{align*}
\]

The assumption \( \beta(a, 0) = 0 \) for \( a \in (0, A) \) means that the birth rate is zero if there are no fertile male individuals.

We can now state the main results. If \( (a_1, a_2) \subset (b_1, b_2) \), we have the following theorem:

**Theorem 2.** Let us assume that the assumptions \( (\lambda_1) - (\lambda_4) \) hold true. If \( a_1 < b \) for every time \( T > a_1 + A - a_2 \) and for every \( (m_0, f_0) \in \left(L^2(Q_A)\right)^2 \), there exists \( (v_m, v_f) \in L^2(\Theta) \times L^2(\Theta) \) such that the associated solution \( (m, f) \) of system (1) verifies

\[
m(a, T) = f(a, T) = 0 \quad \text{a.e. } a \in (0, A).
\]

**Theorem 3.** Let us assume that the assumptions \( (\lambda_1) - (\lambda_4) \) hold true. We have the following:

\[(1) \quad \text{Let } v_f = 0. \text{ For any } q > 0, \text{ for every time } T > A - a_2 \text{ and for every } (m_0, f_0) \in \left(L^2(Q_A)\right)^2, \text{ there exists a control } v_m \in L^2(\Theta) \text{ such that the associated solution } (m, f) \text{ of system } (1) \text{ verifies}
\]

\[
m(a, T) = 0 \quad \text{a.e. } a \in (q, A),
\]

where \( \Theta = (0, a_2) \times (0, T). \)
Remark 4. The first condition of \((H_3)\) is not necessary for Theorem 3-(1).

We use the technique of \([6, 7]\) combining final-state observability estimates with the use of characteristics to establish the observability inequalities necessary for the null controllability property of the auxiliary systems. Roughly, in our method, we first study the null controllability result for an auxiliary cascade system. Afterwards, we prove the null controllability result for system (1) by means of Kakutani’s fixed-point theorem.

The remainder of this paper is as follows: in Section 2, we describe the model and give the main results. Then, we study the existence and uniqueness of a positive solution for the model in Section 3. Section 4 is devoted to the proof of Theorem 2 and Theorem 3. Some illustrations of numerical simulations are given in Section 5.

3. Well-Posedness Result

In this section, we study the existence of positive solution of the model. For this, we assume that the so-called demographic conditions \((H_1), (H_2), (H_3),\) and \((H_4)\) are verified. Moreover, here, we suppose that

\[
\beta(a, p) = \beta_1(a) \beta_2(p) \text{ for all } (a, p) \in (0, A) \times \mathbb{R},
\]

\[
(H_3) \left\{ \begin{array}{l}
\beta(a, p) = \beta_1(a) \beta_2(p) \text{ for all } (a, p) \in (0, A) \times \mathbb{R}, \\
\beta_1, \beta_2 \in L^1(0, A) \\
\end{array} \right.
\]

(9)

holds true.

Thus, we have the following result.

Theorem 5. Assume that \((H_1) - (H_3)\) hold. For every \((m_0, f_0) \in (L^2(0, A))^2\) and \((v_m, v_f) \in (L^2(T))^2\), system (1) admits a unique solution \((m, f) \in (L^2((0, A) \times (0, T)))^2\) and we have the following estimations:

\[
\left\{ \begin{array}{l}
\|m\|_{L^2(0,A) \times (0,T)} \leq K \left( \|f_0\|_{L^2(0,T)} + \|m_0\|_{L^2(0,T)} + \|v_m\|_{L^2(T)} + \|v_f\|_{L^2(T)} \right), \\
\|f\|_{L^2(0,A) \times (0,T)} \leq C \left( \|m_0\|_{L^2(0,T)} + \|v_f\|_{L^2(T)} \right), \\
\end{array} \right.
\]

(10)

where \(K\) and \(C\) are positive constants.

Moreover, suppose that

\[
m_0, f_0 \geq 0 \text{ a.e } (0, A), \\
v_m, v_f \geq 0 \text{ a.e } Q;
\]

then, \((m, f)\) is also positive.

Proof of Theorem 5. Let \(p\) be fixed in \(L^2(0, T), h\) and \(h'\) be fixed in \(L^2(Q),\) and consider the following system:

\[
\begin{align*}
m_t + m + \mu m &= h \text{ in } Q, \\
f_t + f + \mu f &= h' \text{ in } Q, \\
m(a, 0) &= m_0(a) \text{ in } Q_A, \\
m(0, t) &= (1 - \gamma) \int_0^A \beta \left( a, \int_0^A \lambda(a) p(a, t) \right) f(a, t) da \text{ in } Q_T, \\
f(0, t) &= \gamma \int_0^A \beta \left( a, \int_0^A \lambda(a) p(a, t) \right) f(a, t) da \text{ in } Q_T,
\end{align*}
\]

(12)

For every \(f_0 \in L^2(0, A)\) and \(h' \in L^2(Q),\) the following system

\[
\begin{align*}
f_t + f + \mu f &= h' \text{ in } Q, \\
f(a, 0) &= f_0(a) \text{ in } Q_A, \\
f(0, t) &= \gamma \int_0^A \beta \left( a, \int_0^A \lambda(a) p(a, t) \right) f(a, t) da \text{ in } Q_T
\end{align*}
\]

(13)

admits a unique positive solution in \(L^2(Q),\) see [8, 9], and one has

\[
\|f\|_{L^2(Q)} \leq C \left( \|f_0\|_{L^2(0,A)} + \|h'\|_{L^2(0,T)} \right),
\]

(14)

where \(C\) is a positive constant and independent of \(p\) because \(\beta \in L^\infty((0, T) \times (0, A)).\)

Now, \(f\) and \(h'\) are being known; the system

\[
\begin{align*}
m_t + m + \mu m &= h \text{ in } Q, \\
m(a, 0) &= m_0(a) \text{ in } Q_A, \\
m(0, t) &= (1 - \gamma) \int_0^A \beta \left( a, \int_0^A \lambda(a) p(a, t) \right) f(a, t) da \text{ in } Q_T
\end{align*}
\]

(15)
admits a unique positive system in $L^2(Q)$, and we have the following estimation:

$$\|m\|_{L^2(Q)}^2 \leq K \left( \|f_0\|_{L^2(0,A)}^2 + \|m_0\|_{L^2(0,A)}^2 + \|h\|_{L^2(Q)}^2 + \|h'\|_{L^2(Q)}^2 \right),$$

(16)

where $K$ is a positive constant and independent of $p$ because $eta \in L^\infty((0, T) \times (0, A))$.

Let us define $\Phi : L^2_1(Q) \rightarrow L^2(Q)$, $\Phi(p) = m(p)$ where $m(p)$ is the unique solution of the system (15).

Multiplying (18) by $w$ and integrating over $(0, A) \times (0, t)$, and using Young’s inequality, we get

$$\frac{1}{2} \|w(t)\|_{L^2(0,A)}^2 + \int_0^t \int_0^A (\gamma_0 + \mu_0)w^2(s, a)dsda + \int_0^t \int_0^A (|\beta_2(B_1) - \beta_2(B_2)|\beta_1(a)f(p)da + (f(p) - f(q))\beta_2(B_2)\beta_1(a)da)ds \leq C^2 \|\lambda\|_{L^\infty} \int_0^t \left( \left| \int_0^A p(s, a)da - \int_0^A q(s, a)da \right| \right)^2 ds$$

$$+ \int_0^t \int_0^A (f(p) - f(q))\beta_2(B_2)\beta_1(a)da + \int_0^t \left( \beta_1(a)f(p(s))da \right)^2 ds$$

$$\leq C^2 \|\lambda\|_{L^\infty} \int_0^t \int_0^A (p(s) - q(s))^2 \left( \int_0^A \beta_1(a)f(p(s))da \right)^2 ds$$

$$+ \|\beta_1\|_{L^2_{(0,A)}}^2 \|\beta_2\|_{L^2_{(0,A)}}^2 \int_0^t \int_0^A (f(p) - f(q))^2 ds.$$

(19)

Hence, for every $\gamma_0 > 0$, there is a constant $C = \max \{2C^2 A \|\lambda\|_{L^\infty}^2; 2\|\beta_2\|_{L^2_{(0,A)}}^2\} \|\beta_2\|_{L^2_{(0,A)}}^2$) such that

$$\|w(t)\|_{L^2(0,A)}^2 \leq C \left( \int_0^t \int_0^A (p(s) - q(s))^2 \left( \int_0^A \beta_1(a)f(p(s))da \right)^2 ds + \int_0^t \int_0^A (f(p(s)) - f(q(s)))^2 ds \right).$$

(20)

For any $p, q \in L^2_1(Q)$, we set

$$B_1(a, t) = \int_0^A \lambda(a)p(t, a)da,$$

$$B_2(a, t) = \int_0^A \lambda(a)q(t, a) da \text{ a.e.} \in (0, A) \times (0, T),$$

(17)

and $w = (m(p) - m(q))e^{-\gamma_0 t}$ where $\gamma_0$ is a positive parameter that will be chosen later; $w$ is a solution of

$$\begin{cases}
\omega_t + \omega_a + (\gamma_0 + \mu_0)w = 0 \text{ in } Q,
\omega(a, 0) = 0 \text{ in } Q_A,
\omega(0, t) = (1 - \gamma)e^{-\gamma_0 t} \times \int_0^A (|\beta_2(B_1) - \beta_2(B_2)|\beta_1(a)f(p) + (f(p) - f(q))\beta_2(B_2)\beta_1(a)da) in Q_T,
\end{cases}$$

(18)

Now, set $F = (f(p) - f(q))e^{-\delta t}$ where $\delta$ is a positive parameter that will be chosen later. Then, $F$ solves the following auxiliary system:

$$\begin{cases}
F_+ + (\delta + \mu_0)F = 0 \text{ in } Q,
F(a, 0) = 0 \text{ in } Q_A,
\omega(0, t) = \gamma \int_0^A e^{\gamma_0 t/2}(|\beta_2(H_1) - \beta_2(H_2)|\beta_1(a)f(p) + (f(p) - f(q))\beta_2(H_2)\beta_1(a)da in Q_T.
\end{cases}$$

(21)

Similarly as above, we have

$$\delta \int_0^t \int_0^A F(a, s)^2 ds \leq C \left( \int_0^t \left( \int_0^A p(s) - q(s)^2 \left( \int_0^A \beta_1(a)f(p(s))da \right)^2 ds + \int_0^t \int_0^A F(a, s)^2 ds \right) \right).$$

(22)

Hence, there is a positive constant $C'$ such that

$$\int_0^t \int_0^A F(a, s)^2 ds \leq C' \int_0^t \int_0^A (p(s) - q(s))^2 \left( \int_0^A \beta_1(a)f(p(s))da \right)^2 ds.$$

(23)

Setting $Y(t) = \int_0^A \beta_1(a)f(p)da \text{ a.e. in } (0, T)$, $Y$ solves the following system:

$$\begin{cases}
Y_t = \int_0^A \beta_1(a)h(t, a)da + \int_0^A \beta_1(a)h(t, a)da - \int_0^t \mu(a)\beta_1(a)f(a, t)da \text{ in } (0, T),
Y(0) = \int_0^A \beta_1(a)f_0(a)da.
\end{cases}$$

(24)
Multiplying (24) by $Y$, integrating over $(0, t)$, and using Young’s inequality, we get

$$
Y^2(t) \leq Y^2(0) + \int_0^t Y^2(s)ds + \int_0^t \left( \int_0^A \beta_1(a) h'(a, s)da \right)^2 ds
+ \int_0^A \beta_1(a) f(a, s)da + \int_0^A \mu_f(a) \beta_1(a)f(a, s)da \right)^2 ds
\leq Y^2(0) + \int_0^t Y^2(s)ds + \int_0^t \left( \int_0^A \beta_1(a) h'(a, s)da \right)^2 ds
+ \int_0^A \beta_1(a) f(a, s)da + \int_0^A \mu_f(a) \beta_1(a)f(a, s)da \right)^2 ds
+ \int_0^t \left( \int_0^A \beta_1(a) \mu_f(a)f(a, s)da \right)^2 ds.
$$

(25)

So,

$$
Y^2(t) \leq \left( \int_0^A \beta_1(a)f_0(a)da \right)^2 + \int_0^T Y^2(t) dt
+ 3\int_0^T \left( \int_0^A \beta_1(a) h'(a, t)da \right)^2 dt
+ 3\int_0^T \left( \int_0^A \beta_1(a) f(a, t)da \right)^2 dt
+ 3\int_0^T \left( \int_0^A \beta_1(a) \mu_f(a)f(a, t)da \right)^2 dt.
$$

(26)

Let us set $\tilde{f} = e^{-\lambda_0 t} f$. Then, from (13), $\tilde{f}$ satisfies the following system:

$$
\begin{cases}
\tilde{f}_t + \tilde{f}_a + \left( \lambda_0 + \mu_f \right) \tilde{f} = e^{-\lambda_0 t} h' \text{ in } Q,
\tilde{f}(a, 0) = f_0(a) \text{ in } Q_A,
\tilde{f}(0, t) = \gamma \int_0^A \beta(a, \int_0^A \lambda(a) p(a, t) \tilde{f}(a, t) da) \text{ in } Q_T.
\end{cases}
$$

(27)

Multiplying the first equation of (27) by $\tilde{f}$, integrating on $Q$, and using the inequality of Young, we get

$$
\int_0^T \left( \lambda_0 + \mu_f(a) \right) \tilde{f}^2(a, t) da dt \leq \frac{1}{2} \| f_0 \|_{L^2(Q)}^2 + \frac{1}{2} \| h' \|_{L^2(Q)}^2 + \frac{1}{2} \left\| \tilde{f} \right\|_{L^2(Q)}^2 + \frac{1}{2} \int_0^T \tilde{f}_a^2(0, t) dt.
$$

(28)

Using the inequality of Cauchy-Schwarz and choosing $\lambda_0 = (3/2) + a_2^2$, we obtain

$$
\int_0^T \int_0^A \mu_f(a) \tilde{f}^2(a, t) da dt \leq \frac{1}{2} \left( \| f_0 \|_{L^2(Q)}^2 + \| h' \|_{L^2(Q)}^2 \right).
$$

(29)

So,

$$
\int_0^T \int_0^A \mu_f(a) f^2(a, t) da dt \leq \frac{3(3+2\beta_1^2)}{2} \left( \| f_0 \|_{L^2(Q)}^2 + \| h' \|_{L^2(Q)}^2 \right).
$$

(30)

Using (26), (30), and against Young’s inequality, we have

$$
Y^2(t) \leq \| \beta_1 \|_{L^\infty} A \| f_0 \|_{L^2(Q)}^2 + \| \beta_1 \|_{L^\infty} A \| f \|_{L^2(Q)}
+ 3\| \beta_1 \|_{L^\infty} A \| h' \|_{L^2(Q)}^2 + 3\| \beta_1^2 \|_{L^\infty} A \| f \|_{L^2(Q)}
+ 3C \| \beta_1 \|_{L^\infty} \| \beta_1 \|_{L^1(Q)} \| f_0 \|_{L^2(Q)}
+ 3C \| \beta_1 \|_{L^\infty} \| \beta_1 \|_{L^1(Q)} \| h' \|_{L^2(Q)}.
$$

(31)

From (14), we have just proved the existence of a positive constant $C$ such that

$$
Y^2(t) \leq C \left( \| f_0 \|_{L^2(Q)}^2 + \| h' \|_{L^2(Q)}^2 \right).
$$

(32)

The estimate (32) means also that $Y \in L^\infty(0, T)$. Combining (20), (23), and (32), we get the following estimate:

$$
\| (\Phi(p) - \Phi(q))(t) \|_{L^2(\Lambda)} \leq \sigma \int_0^T \| p(s) - q(s) \|_{L^2(\Lambda)} ds,
$$

(33)

where $\sigma$ is a positive constant.

Let us define the metric $d$ on $L^2(Q)$ by setting

$$
d(h_1, h_2) = \left( \int_0^T \| h_1(t) - h_2(t) \|_{L^2(\Lambda)}^2 \exp \{-2\sigma t\} dt \right)^{1/2}, \text{ for } h_1, h_2 \in L^2(Q).
$$

(34)

We have

$$
d(\Phi(p), \Phi(q))^2 = \int_0^T \| (\Phi(p) - \Phi(q))(t) \|_{L^2(\Lambda)}^2 \exp \{-2\sigma t\} dt
\leq \sigma \int_0^T \exp \{-2\sigma t\} \int_0^T \| (p(s) - q(s)) \|_{L^2(\Lambda)}^2 ds dt.
$$

(35)
Using the Fubini theorem, we conclude that

\[
\begin{align*}
    d(\Phi(p), \Phi(q))^2 &= \int_0^T \| (\Phi(p) - \Phi(q))(t) \|_{L^2((0,A))}^2 \exp \{-2\sigma t\} dt \\
    &\leq \int_0^T \|p(t) - q(t)\|_{L^2((0,A))}^2 \int_t^T \sigma e^{-2\sigma s} ds \\
    &\leq \frac{1}{2} d(p, q)^2.
\end{align*}
\]

(36)

Then, \( \Phi \) is a contraction on the complete metric space \( L^2(Q) \) into itself. Using Banach’s fixed-point theorem, we conclude the existence of a unique fixed-point \( m \). Moreover, \( m \) is nonnegative. Hence, the unique couple \((m, f)\) is the unique solution to our problem (1).

### 4. Null Controllability Results

For the sequel, the hypothesis \((H_3)\) is not necessary. As a consequence, the uniqueness and the positivity of the solution of system (1) are not guaranteed.

We first establish an observability inequality to show the controllability of a linear system. Then, by a fixed-point method, we show the controllability of the model.

#### 4.1. Null Controllability of an Auxiliary Coupled System

This section is devoted to the study of an auxiliary system obtained from system (1).

Let \( p \) be a \( L^2(Q_T) \) function, we define the auxiliary system given by

\[
\begin{align*}
    m_t + m_a + \mu_m m &= \chi_{\Theta_0} v_m \text{ in } Q, \\
    f_t + f_a + \mu_f f &= \chi_{\Theta_2} v_f \text{ in } Q, \\
    m(a, 0) &= m_0(a) f(a, 0) = f_0(a) \text{ in } Q_A, \\
    m(0, t) &= (1 - \gamma) \int_0^A \beta(a, p) f(a, t) da \text{ in } Q_T, \\
    f(0, t) &= \gamma \int_0^A \beta(a, p) f(a, t) da \text{ in } Q_T.
\end{align*}
\]

(37)

Let \( p \) be fixed in \( L^2(Q_T) \); for \((m_0, f_0) \in (L^2(Q_A))^2 \) and \((v_m, v_f) \in L^2(\Omega_1) \times L^2(\Omega_2) \), system (37) admits a unique solution \((m, f) \in (L^2(Q))^2 \), see Section 3.

The system above is null approximately controllable. Indeed, we have the following result:

**Theorem 6.** Let us assume that assumptions \((H_1)-(H_2)\) hold. For every \( T > a_1 + A - a_2 \), for every \( \kappa, \theta > 0 \), and for every \((m_0, f_0) \in (L^2(Q_A))^2 \), there exists a control \((v_m, v_f)\) such that the solutions \( m \) and \( f \) of system (37) verify

\[
\begin{align*}
    \|m(\cdot, T)\|_{L^2((0,A))} &\leq \kappa, \\
    \|f(\cdot, T)\|_{L^2((0,A))} &\leq \theta.
\end{align*}
\]

(38)

The adjoint system of (37) is given by

\[
\begin{align*}
    -n_t - n_a + \mu_n n &= 0 \quad \text{in } Q, \\
    -l_t - l_a + \mu_l l &= (1 - \gamma) \beta(a, p) n(0, t) + \gamma \beta(a, p) l(0, t) \quad \text{in } Q, \\
    n(a, T) &= n_T(a), l(a, T) = l_T(a) \quad \text{in } Q_A, \\
    n(A, t) &= 0, l(A, t) = 0 \quad \text{in } Q_T.
\end{align*}
\]

(39)

The main idea in this part is to establish an observability inequality of (39) that will allow us to prove the approximate null controllability of (37).

The basic idea for establishing this inequality is the estimation of nonlocal terms. For that, let us start by formulating a representation of the solution of the adjoint system by the method of characteristic and semigroup.

For every \((n_T, l_T) \in (L^2(Q_A))^2 \), under the assumptions \((H_1)\) and \((H_2)\), coupled system (39) admits a unique solution \((n, l)\). Moreover, integrating along the characteristic lines, the solution \((n, l)\) of (39) is as follows:

\[
\begin{align*}
    n(t) &= \begin{cases} 
        \frac{\pi_1(a + T - t)}{\pi_1(a)} n_T(a + T - t) & \text{if } T - t \leq A - a, \\
        0 & \text{if } A - a < T - t,
    \end{cases} \\
    l(t) &= \begin{cases} 
        \frac{\pi_2(a + T - t)}{\pi_2(a)} l_T(a + T - t) + \int_t^T \frac{\pi_2(a + s - t)}{\pi_2(a)} \beta(a + s - t, p(s)) ((1 - \gamma) n(0, s) + \gamma l(0, s)) ds & \text{if } T - t \leq A - a, \\
        \frac{\pi_2(a + A - a)}{\pi_2(a)} \beta(a + A - a, p(s)) ((1 - \gamma) n(0, s) + \gamma l(0, s)) ds & \text{if } A - a < T - t,
    \end{cases}
\end{align*}
\]

(40)

(41)
where $\pi_1(a) = e^{-\int_a^b \rho_\eta(r) dr}$ and $\pi_2(a) = e^{-\int_a^b \rho_\eta(r) dr}$.

Suppose that the assumptions (H1), (H2), (H3), and (H4) are fulfilled; then, we have the following result.

**Theorem 7.** Under the assumptions of Theorem 2, if $(a_1, a_2) \in (b_1, b_2)$, there exists a constant $C_T > 0$ independent of $p$ such that the couple $(n, l)$ solution of (39) verifies the following inequality:

$$\int_0^n n^2(a, 0) da + \int_0^n \tilde{l}^2(a, 0) da \leq C_T \left( \int_0^n n^2(a, t) dt + \int_0^n \tilde{l}^2(a, t) dt \right).$$

(42)

For the proof of Theorem 7, we state the following estimations of the nonlocal terms.

**Proposition 8.** Under the assumptions of Theorem 2, there exists $C > 0$ such that

$$\int_0^{T-\eta} n^2(0, t) dt \leq C \int_0^{T-a_2} n^2(a, t) dt.$$

(43)

where $a_1 < \eta < T$.

In particular, for every $\eta > 0$, if $a_1 = 0$ and $n_T(a) = 0$ a.e $a \in (0, \eta)$, there is $C_{\eta, T} > 0$ such that

$$\int_0^n n^2(0, t) dt \leq C_{\eta, T} \int_0^{T-a_2} n^2(a, t) dt.$$  

Moreover, if the first condition of (H3) hold, we have the inequality

$$\int_0^{T-\eta} \tilde{l}^2(0, t) dt \leq C \int_0^{T-a_2} \tilde{l}^2(a, t) dt,$$

(45)

for every $\eta$ such that $b_1 < b$ and $b_1 < \eta < T$.

**Proof of Proposition 8.** The state $n$ of (39) verifies

$$\begin{cases} 
        -\frac{\partial n}{\partial t} - \frac{\partial n}{\partial a} + \mu n = 0 \text{ in } (0, a_2) \times (0, T), \\
        n(a, T) = n_T \text{ in } (0, a_2).
\end{cases}$$

(46)

We denote by $\tilde{n}(a, t) = n(a, t) e^{-\int_\eta^{T-\eta} \mu_n(s) ds}$. Then, $\tilde{n}$ satisfies

$$\frac{\partial \tilde{n}}{\partial t} + \frac{\partial \tilde{n}}{\partial a} = 0 \text{ in } (a_1, a_2) \times (0, T).$$

(47)

Proving the inequality

$$\int_0^{T-\eta} \tilde{n}^2(0, t) dt \leq C \int_0^{T-a_2} \tilde{n}^2(a, t) dt$$

(48)

leads to getting inequality (43).

Indeed, we have

$$\int_0^{T-\eta} n^2(0, t) dt = \int_0^{T-a_2} \tilde{n}^2(a, t) dt$$

$$\leq C \int_0^{T-a_2} \tilde{n}^2(a, t) dt$$

$$= C \int_0^{T-a_2} e^{-2\int_0^{T-a_2} \mu_n(s) ds} n^2(a, t) dt$$

$$\leq C \int_0^{T-a_2} n^2(a, t) dt.$$

We consider the following characteristic trajectory $\gamma(\lambda) = (T - \lambda, T + t - \lambda)$. If $\lambda = T$, the backward characteristic starts from $(0, t)$. If $T < a_1$, the trajectory $\gamma(\lambda)$ never reaches the observation region $(a_1, a_2)$. So, we choose $T > a_1$.

Without loss of generality, let us assume that $\eta < a_2 < T$.

**Step 1.** Estimation of $n(0, t)$, $t \in (0, T - \eta)$

(i) Estimation for $t \in (0, T - a_2)$

We denote by

$$w(\lambda) = \tilde{n}(T - \lambda, T + t - \lambda), \quad \lambda \in (T - a_2, T).$$

(50)

Then, $\partial w/\partial \lambda = 0$ for all $\lambda \in (T - a_2, T)$. In particular, $w(\lambda)$ is constant for all $\lambda \in (T - a_2, T)$. Since $\eta < a_2$, we have

$$w(t) = \frac{1}{a_2 - \eta} \int_{T-a_2}^{T-a_2} w(\lambda) d\lambda.$$  

(51)

Therefore,

$$w(T) = \tilde{n}(0, t) = \frac{1}{a_2 - \eta} \int_{T-a_2}^{T-a_2} \tilde{n}(T - \lambda, T + t - \lambda) d\lambda$$

$$= \frac{1}{a_2 - \eta} \int_{T-a_2}^{T-a_2} \tilde{n}(s, t + s) ds.$$  

(52)

Using the fact that

$$\left( \int_{T-a_2}^{T-a_2} w(\lambda) d\lambda \right)^2 \leq (a_2 - \eta) \int_{T-a_2}^{T-a_2} w^2(\lambda) d\lambda$$

(53)

and integrating with respect to $t$ over $(0, T - a_2)$, we get

$$\int_0^{T-a_2} n^2(0, t) dt \leq C \int_0^{T-a_2} \int_0^{T-a_2} n^2(s, t + s) dt ds$$

$$= C \int_0^{T-a_2} \int_0^{T-a_2} n^2(s, u) du ds.$$  

(54)
Finally,
\[
\int_0^{T-a_2} \tilde{n}^2(0, t) dt \leq C \int_0^{T-a_2} \tilde{n}^2(a, t) da dt. \tag{55}
\]

(ii) Estimation for \( t \in (T - a_2, T - \eta) \)

We define
\[
w(\lambda) = \tilde{n}(T - \lambda, T + t - \lambda), \quad \lambda \in (T - a_2, T). \tag{56}
\]

Then, \( \partial w / \partial \lambda = 0 \) for all \( \lambda \in (T - a_2, T) \). In particular, \( w(\lambda) \) is constant for all \( \lambda \in (T - a_2, T) \). Thus, we have
\[
w(t) = \frac{1}{\eta - a_1} \int_{T-t}^{T-a_2} w(\lambda) d\lambda = \frac{1}{\eta - a_1} \int_{T-t}^{T-a_2} \tilde{n}(s, t + s) ds. \tag{57}
\]

Integrating with respect to \( t \) over \( (T - a_2, T - \eta) \), we get
\[
\int_{T-a_2}^{T-\eta} \tilde{n}^2(0, t) dt \leq C(\eta, a_1) \int_{a_1}^{T-a_2} \tilde{n}(s, t + s) ds. \tag{58}
\]

Finally,
\[
\int_{T-a_2}^{T-\eta} \tilde{n}^2(0, t) dt \leq C(\eta, a_1) \int_0^{T-a_2} \tilde{n}(a, t) da dt. \tag{59}
\]

Combining (55) and (59), we obtain
\[
\int_0^{T-\eta} \tilde{n}^2(0, t) dt \leq C(\eta, a_1, a_2) \int_0^{T-a_2} \tilde{n}(a, t) da dt, \tag{60}
\]

leading to (43).

Remark that \( \lim_{\eta \to a_1} (1/(\eta - a_1)) = +\infty \) then, \( \lim_{\eta \to a_2} C(\eta, a_1, a_2) = +\infty \).

Suppose now that \( a_1 = 0 \). From the above, we have for all \( q > 0 \), the existence of a constant depending on \( q \) such that
\[
\int_0^{T-q} n^2(0, t) dt \leq C(q, a_2) \int_0^{T-a_2} n^2(a, t) da dt. \tag{61}
\]

Moreover, if \( n_T(a) = 0 \) in \((0, q)\), we have \( w(t) = 0 \) in \((T - q, T)\). Then, \( w = 0 \). So \( n(0, t) = 0 \) in \((T - q, T)\).

Finally, if \( n_T(a) = 0 \) in \((0, q)\), we have the following inequality:
\[
\int_0^{T} n^2(0, t) dt \leq C(q, a_2) \int_0^{T} \int_0^{a_1} n^2(a, t) da dt. \tag{62}
\]

Step 2. Estimation of \( I(0, t), t \in (0, T - \eta) \)

Considering \( \nu = \min \{b, b_2\} \), the state \( l \) of the adjoint system can be rewritten as
\[
\begin{aligned}
&- \frac{\partial l}{\partial t} - \frac{\partial l}{\partial a} + \lambda l = 0 \text{ in } (0, \nu) \times (0, T), \\
&l(a, T) = l_r \text{ in } (0, \nu).
\end{aligned} \tag{63}
\]

We denote by \( \tilde{l}(a, t) = l(a, t)e^{-\int_{\nu}^{\nu} \nu(s) ds} \). Then, \( \tilde{l} \) satisfies \((\partial \tilde{l}/\partial t) + (\partial \tilde{l}/\partial a) = 0 \) in \((b_1, \nu) \times (0, T)\). (174)

(i) Estimation for \( t \in (0, T - \nu) \)

Defining \( w \) as
\[
w(\lambda) = \tilde{l}(T - \lambda, T + t - \lambda), \lambda \in (T - \nu, T - b_1), \tag{64}
\]

then, \( w(\lambda) \) is constant for all \( \lambda \in (T - \nu, T - b_1) \) and we have \( w(t) = 1/(\nu - \eta) \int_{\nu}^{\nu} w(\lambda) d\lambda \). Likewise step 1, we obtain
\[
\int_{T-\nu}^{T-\eta} \tilde{l}^2(0, t) dt \leq C(\eta, \nu) \int_{\Theta_2} \tilde{l}^2(a, t) da dt. \tag{65}
\]

(ii) Estimation for \( t \in (T - \nu, T - \eta) \)

We denote by
\[
w(\lambda) = \tilde{l}(T - \lambda, T + t - \lambda), \lambda \in (T - \eta, T - b_1). \tag{66}
\]

As above, \( w(\lambda) \) is constant for all \( \lambda \in (T - \eta, T - b_1) \) and we set \( w(t) = 1/(\eta - b_1) \int_{\eta}^{b_2} w(\lambda) d\lambda \). Then,
\[
\int_{T-\nu}^{T-\eta} \tilde{l}^2(0, t) dt \leq C(\eta, b_1) \int_{\Theta_1} \tilde{l}^2(a, t) da dt. \tag{67}
\]

Combining (65) and (67), the inequality (45) follows.

**Proposition 9.** Let us assume the assumptions \((H_1) - (H_3)\). For every \( T > sup \{a_1, A - a_2\} \), there exists \( C_T > 0 \) such that the solution \((n, l)\) of system (37) verifies the following inequality:
\[
\int_0^A n^2(a, 0) da \leq C_T \int_{\Theta_1} n^2(a, t) da dt. \tag{68}
\]

Note that for every \( T > sup \{a_1, A - a_2\} \), there exists \( a_0 \in (a_1, a_2) \) such that \( n(a, 0) = 0 \) for all \( a \in (a_0, A) \). This is a consequence of the following lemma.

**Lemma 10.** Let us suppose that \( T > sup \{a_1, A - a_2\} \). Then, there exists \( a_0 \in (a_1, a_2) \) such that \( T > A - a_0 > A - a \) for all \( a \in (a_0, A) \). Therefore, \( n(a, 0) = 0 \) for all \( a \in (a_0, A) \).

**Proof of Lemma 10.** Suppose that \( T > A - a_2 \); then, there exists \( \kappa > 0 \) (we choose \( \kappa \) such that \( \kappa < a_2 - a_1 \)) \( T > A - a_2 + \)
\[ T > A - (a_2 - \kappa), \text{ and we denote } a_0 = a_2 - \kappa. \text{ Then, } T > A - a_0 > A - a \text{ for all } a \in (a_0, A). \] Finally, from (40) for all \((a, t)\) such that \(T - t > A - a\), we get \(n(a, 0) = 0\) for all \(a \in (a_0, A)\).

**Proof of Proposition 9.** From the Lemma 10, we have to prove the following inequality:

\[
\int_0^{a_1} n^2(a, 0)da \leq C_{t, 0} \int_0^T \int_{a_1}^{a_2} n^2(a, t)da dt. \quad (69)
\]

We set \(\hat{n}(a, t) = e^{-\int_0^t \mu_0(s)ds} n(a, t)\). Then, from the first equation of (39), \(\hat{n}\) satisfies

\[
\frac{\partial \hat{n}}{\partial t} + \frac{\partial \hat{n}}{\partial a} = 0 \text{ in } (0, A) \times (0, T). \quad (70)
\]

Inequality (47) is a consequence of the following estimation:

\[
\int_0^{a_1} \hat{n}^2(a, 0)da \leq C \int_0^T \int_{a_1}^{a_2} \hat{n}^2(a, t)da dt. \quad (71)
\]

Indeed, we have

\[
\int_0^{a_1} n^2(a, 0)da = \int_0^{a_1} e^{2\int_0^a \mu_0(s)ds} \hat{n}^2(a, 0)da \\
\leq e^{2\|\mu_0\|_1} \int_0^{a_1} \hat{n}^2(a, 0)da \\
\leq C e^{2\|\mu_0\|_1} \int_0^T \int_{a_1}^{a_2} n^2(a, t)da dt. \quad (72)
\]

We consider in this proof the characteristics \(\gamma(\lambda) = (a + \lambda, \lambda)\). For \(\lambda = 0\), the characteristics start from \((a, 0)\). We have two cases.

**Case 1.** \(T < a_2\).

Two situations can arise:

(i) \(b_0 = a_2 - T < a_1 < a_0\), in this situation, we split the interval \((0, a_0)\) as

\[
(0, a_0) = (0, b_0) \cup (b_0, a_1) \cup (a_1, a_0) \quad (73)
\]

(ii) \(a_1 < b_0 < a_0\), in this situation, we split the interval \((0, a_0)\) as

\[
(0, a_0) = (0, a_1) \cup (a_1, a_0) \quad (74)
\]

**Case 2.** \(T \geq a_2\).

In this case, we split the interval \((0, a_0)\) as

\[
(0, a_0) = (0, a_1) \cup (a_1, a_0). \quad (75)
\]

We make the proof for the second case. For the proof in the first case, see [6].

Upper bound on \((0, a_1)\):

For \(a \in (0, a_1)\), we set \(w(\lambda) = \hat{n}(T + a - \lambda, T - \lambda), \lambda \in (a_1, T)\). We prove easily that \(w\) is a constant, see the proof of Proposition 8. Then, we set

\[
w(t) = \frac{1}{a_2 - a_1} \int_{t-a_1}^{t-a_1} w(\lambda)d\lambda. \quad (76)
\]

Integrating with respect to \(a\) over \((0, a_1)\), we get

\[
\int_0^{a_1} \hat{n}^2(a, 0)da \leq C \int_0^T \int_{a_1}^{a_2} \hat{n}^2(a + \alpha, a)dada. \quad (77)
\]

Finally, we obtain

\[
\int_0^{a_1} \hat{n}^2(a, 0)da \leq C \int_0^T \int_{a_1}^{a_2} \hat{n}^2(a, t)da dt. \quad (78)
\]

Upper bound on \((a_1, a_0)\):

For \(a \in (a_1, a_0)\), we set \(w(\lambda) = \hat{n}(T + a - \lambda, T - \lambda), \lambda \in (a_0, T)\) and

\[
w(t) = \frac{1}{a_2 - a_0} \int_{t-a_0}^{t-a_1} w(\lambda)d\lambda. \quad (79)
\]

Making as above and integrating with respect to \(a\) over \((a_1, a_0)\), it follows that

\[
\int_0^{a_1} \hat{n}^2(a, 0)da \leq C \int_0^T \int_{a_1}^{a_2} \hat{n}^2(a, t)da dt. \quad (80)
\]

The inequalities (78) and (80) give the desired result.

We also need the following estimate for the proof of Theorem 7.

**Proposition 11.** Let us assume the assumptions \((H_1) - (H_2)\), let \(b_1 < a_0 < b\) and \(T > b_1\).

Then, there exists \(C_T > 0\) such that the solution \(l\) of (39) verifies

\[
\int_0^{a_1} l^2(a, 0)da \leq C_T \int_{a_1}^{a_2} l^2(a, t)da dt. \quad (81)
\]

**Proof of Proposition 11.** We suppose that \(\beta = 0\ in \ (0, b), \) the function \(l\) verifies

\[
\begin{cases}
-\frac{\partial l}{\partial t} - \frac{\partial l}{\partial a} + \mu_j l = 0 \text{ in } (0, b) \times (0, T), \\
l(a, T) = l_T \text{ in } (0, b).
\end{cases} \quad (82)
\]

Proceeding as in the proof of Proposition 9, we get the desired result.
For the proof of Theorem 7, we start with the following lemma.

**Lemma 12.** Suppose that \( T > A - a_2 + a_1 \) and \( a_1 < b \). Then, there exists \( a_0 \in (a_1, b) \) and \( \kappa > 0 \) such that

\[
T > T - (a_1 + \kappa) > A - a_0 > A - a \text{ for all } a \in (a_0, A). \tag{83}
\]

Proof of Lemma 12. Notice that the solution \( l \) of (39) is given by

\[
l(t) = \begin{cases} 
\pi_2(a + T - t) / \pi_2(a) l_T(a + T - t) + \int_t^T \pi_2(a + s - t) / \pi_2(a) \beta(s, t, p(s))((1 - \gamma)n(s) + yl(s))ds \text{ if } T - t \leq A - a, \\
\int_t^{a + \gamma - a} \pi_2(a + s - t) / \pi_2(a) \beta(s, t, p(s))((1 - \gamma)n(s) + yl(s))ds \text{ if } A - a < T - t.
\end{cases}
\]

Without loss of generality, suppose that \( a_2 = b \). Suppose that \( T > a_1 + A - a_2 \iff T - a_1 > A - a_2 \). Then, there exists \( \kappa > 0 \) (we choose \( \kappa \) such that \( 2\kappa < a_2 - a_1 \)) such that \( T - (a_1 + \kappa) > A - (a_2 - \kappa) \). We denote by \( a_0 = a_2 - \kappa \), and as \( A - a_0 > A - a \) for all \( a \in (a_0, A) \), then \( T > T - (a_1 + \kappa) > A - a_0 > A - a \) for all \( a \in (a_0, A) \).

Moreover, for \( (a, t) \) such that \( T - t > A - a \), we have

\[
l(a, t) = \int_t^{a + \gamma - a} \pi_2(a + s - t) / \pi_2(a) \beta(s, t, p(s))((1 - \gamma)n(s) + yl(s))ds.
\]

For \( t = 0 \) and \( a \in (a_0, A) \), \( T > A - a_0 > A - a \), one has

\[
l(a, 0) = \int_0^{a + \gamma - a} \pi_2(a + s) / \pi_2(a) \beta(a + s, p(s))((1 - \gamma)n(s) + yl(s))ds.
\]

Remark that as \( (a_1, a_2) \subset (b_1, b_2) \), if \( a_0 \in (a_1, a_2) \), then \( a_0 \in (b_1, b_2) \).

Now, we can prove Theorem 7.

**Proof of Theorem 7.** Let \( a_0 \) as in Lemma 12, we have

\[
\int_0^A \tilde{l}_T^2(a, 0)da = \int_0^{a_0} \tilde{l}_T(a, 0)da + \int_{a_0}^A \tilde{l}_T^2(a, 0)da. \tag{88}
\]

Using the results of Lemma 12, the assumption of \( \beta \), and the regularity of \( (\pi_2(a + s)) / (\pi_2(a)) \), we can prove the existence of a constant \( K_T \) independent of \( p \)

Therefore,

\[
l(a, 0) = \int_0^{T(a) - a} \pi_2(a + s) / \pi_2(a) \beta(a + s, p(s))(l(0, s) + n(0, s))ds \text{ for all } a \in (a_0, A). \tag{84}
\]

Moreover, we have \( b_1 \leq a_1 \leq a_1 + \kappa \). Using Proposition 8, it follows that

\[
\int_0^A \tilde{l}_T^2(a, 0)da = K_T \left( \int_0^{T(a) - a} n^2(0, t)dt + \int_0^{T(a) - a} \tilde{l}_T^2(0, t)dt \right). \tag{89}
\]

For \( \varepsilon > 0 \) and \( \theta > 0 \), we consider the functional \( I_{\varepsilon, \theta} \) defined by

\[
I_{\varepsilon, \theta}(\psi_m, \psi_f) = \frac{1}{2} \int_{\Theta_1} \psi_m^2 d\theta + \frac{1}{2} \int_{\Theta_2} \psi_f^2 d\theta + \frac{1}{2\varepsilon} \int_0^A m^2(a, T)da + \frac{1}{2\varepsilon} \int_0^A \tilde{l}_T^2(a, T)da, \tag{92}
\]
where \((m, f)\) is the solution of the following system:

\[
\begin{align*}
    m_t + m_a + \mu m &= \chi \partial_t v_m \text{ in } Q, \\
    f_t + f_a + \mu f &= \chi \partial_t v_f \text{ in } Q, \\
    m(a, 0) = m_0(a), f(a, 0) = f_0(a) \text{ in } Q_A, \\
    m(0, t) = (1 - \gamma) \int_0^A \beta(a, p)f(a, t)\, da, f(0, t) = \gamma \int_0^A \beta(a, p)f(a, t)\, da \text{ in } Q_T.
\end{align*}
\]

(93)

**Lemma 13.** The functional \(I_{\varepsilon, \theta}\) is continuous, strictly convex, and coercive. Consequently, \(I_{\varepsilon, \theta}\) reaches its minimum at a point \((v_{m\varepsilon}, v_{f\varepsilon}) \in L^2(\Theta) \times L^2(\Theta_2)\). Setting \((m_\varepsilon, f_\theta)\) the associated solution of (93) and \((n_\varepsilon, l_\theta)\) the solution of (39) with

\[
\begin{align*}
    n_\varepsilon(a, T) &= -\frac{1}{\varepsilon} m_\varepsilon(a, T), \\
    l_\theta(a, T) &= -\frac{1}{\beta} f_\theta(a, T),
\end{align*}
\]

we have

\[
\begin{align*}
    \chi \partial_t v_{m\varepsilon} &= \chi \partial_t n_\varepsilon, \\
    \chi \partial_t v_{f\varepsilon} &= \chi \partial_t l_\theta.
\end{align*}
\]

(95)

Moreover, there exist \(C_i > 0, 1 \leq i \leq 4\), independent of \(\varepsilon\) and \(\theta\) such that

\[
\begin{align*}
    \int_{\Theta_1} n_\varepsilon^2(a, t)\, dt &\leq C_1 \left( \int_0^A m_\varepsilon^2(a)\, da + \int_0^A f_\theta^2(a)\, da \right), \\
    \int_0^A m_\varepsilon^2(a, T)\, da &\leq \varepsilon C_2 \left( \int_0^A m_\varepsilon^2(a)\, da + \int_0^A f_\theta^2(a)\, da \right), \\
    \int_{\Theta_2} l_\theta^2(a, t)\, dt &\leq C_3 \left( \int_0^A m_\varepsilon^2(a)\, da + \int_0^A f_\theta^2(a)\, da \right), \\
    \int_0^A f_\theta^2(a, T)\, da &\leq \varepsilon C_4 \left( \int_0^A m_\varepsilon^2(a)\, da + \int_0^A f_\theta^2(a)\, da \right).
\end{align*}
\]

(96)

**Proof of Lemma 13.** It is easy to check that \(I_{\varepsilon, \theta}\) is coercive, continuous, and strictly convex. Then, it admits a unique minimizer \((v_\varepsilon, v_\theta)\). The maximum principle gives

\[
\begin{align*}
    \chi \partial_t v_{m\varepsilon} &= \chi \partial_t n_\varepsilon, \\
    \chi \partial_t v_{f\varepsilon} &= \chi \partial_t l_\theta.
\end{align*}
\]

(97)

where the couple \((n_\varepsilon, l_\theta)\) is the solution of the system

\[
\begin{align*}
    -\partial_t n_\varepsilon - \partial_a n_\varepsilon + \mu n_\varepsilon &= 0 \text{ in } Q, \\
    -\partial_t l_\theta - \partial_a l_\theta + \mu f_\theta &= (1 - \gamma)\beta(a, p)n_\varepsilon(0, t) + \gamma \beta(a, p)f_\theta(0, t) \text{ in } Q, \\
    n_\varepsilon(a, T) &= -\frac{1}{\varepsilon} m_\varepsilon(a, T), l_\theta(a, T) = -\frac{1}{\beta} f_\theta(a, T) \text{ in } Q_T, \\
    n_\varepsilon(a, T) = 0, l_\theta(a, T) = 0 \text{ in } Q_T.
\end{align*}
\]

(98)

Multiplying the first and the second equation of (98) by, respectively, \(m_\varepsilon\) and \(f_\theta\), integrating with respect to \(Q\) and using (97), we get

\[
\begin{align*}
    \int_{\Theta_1} n_\varepsilon^2(a, t)\, dt + \frac{1}{\varepsilon} \int_0^A m_\varepsilon^2(a, T)\, da &= -\int_0^A m_\varepsilon(a, T)\, da \\
    &\quad - (1 - \gamma) \int_0^T \int_0^A \beta(a, p)f_\theta(a, t)n_\varepsilon(0, t)\, da dt \\
    \int_{\Theta_2} l_\theta^2(a, t)\, dt + \frac{1}{\beta} \int_0^A f_\theta^2(a, T)\, da &= -\int_0^A f_\theta(a, T)\, da \\
    &\quad + (1 - \gamma) \int_0^T \int_0^A \beta(a, p)f_\theta(a, t)n_\varepsilon(0, t)\, da dt.
\end{align*}
\]

(99)

(100)

Combining (99) and (100), we obtain

\[
\begin{align*}
    \int_{\Theta_1} n_\varepsilon^2(a, t)\, dt + \frac{1}{\varepsilon} \int_0^A m_\varepsilon^2(a, T)\, da \\
    + \int_{\Theta_2} l_\theta^2(a, t)\, dt + \frac{1}{\beta} \int_0^A f_\theta^2(a, T)\, da \\
    &= -\int_0^A m_\varepsilon(a, T)\, da \\
    &\quad - \int_0^A f_\theta(a, T)\, da.
\end{align*}
\]

(101)

Using the inequality of Young, we have, for any \(\delta > 0,

\[
\begin{align*}
    \int_{\Theta_1} n_\varepsilon^2(a, t)\, dt + \frac{1}{\varepsilon} \int_0^A m_\varepsilon^2(a, T)\, da \\
    + \int_{\Theta_2} l_\theta^2(a, t)\, dt + \frac{1}{\beta} \int_0^A f_\theta^2(a, T)\, da \\
    \leq \frac{\delta}{2} \int_0^A m_\varepsilon^2(a)\, da + \frac{1}{2\delta} \int_0^A n_\varepsilon^2(a, 0)\, da \\
    + \frac{\delta}{2} \int_0^A f_\theta^2(a)\, da + \frac{1}{2\delta} \int_0^A l_\theta^2(a, 0)\, da.
\end{align*}
\]

(102)
Using observability inequality (42) and choosing $\delta = C_T$ in the previous inequality, it follows that

\[
\frac{1}{2} \int_{\Theta_2} m^2_t(a, t) da dt + \frac{1}{2} \int_{\Theta_2} f^2_t(a, t) da dt + \frac{1}{2} \int_{\Theta_2} l^2_\theta(a, t) da dt \leq C_T \left( \int_{0}^{A} m^2_0(a) da + \int_{0}^{A} f^2_0(a) da \right).
\]

This gives the desired result necessary to the proof of the main one.

Now, we consider the system

\[
\begin{aligned}
\partial_t x(p) + \partial_x f_p(p) + \mu(p)x(p) &= x_0 n_0 \text{ in } \Omega, \\
\partial_t f_p(x) + \partial_x f_p(x) + \mu_f f_p(x) &= x_0 l_0 \text{ in } \Omega, \\
m(p)(a, 0) = m_0(a), f_p(x)(a, 0) = f_0(x) \text{ in } \Omega, \\
m(p)(0, t) = (1 - \gamma) \int_0^t \beta(a, p)f(x, t) da + \int_0^t \beta(a, p)f(x, t) da \text{ in } \Omega.
\end{aligned}
\]

where $(n_0, l_0)$ is the solution of (98) that minimizes the functional $J_{x, l, \Theta}$. We have the following result:

**Lemma 14.** Under the assumptions of Theorem 2, the solutions $x_\alpha$ and $x_\beta$ verify the following inequalities:

\[
\begin{aligned}
\int_{0}^{A} m^2_\alpha(a, t) da dt + \int_{0}^{A} f^2_\alpha(a, t) da dt &\leq C \left( \int_{0}^{A} m^2_0(a) da + \int_{0}^{A} f^2_0(a) da \right), \\
\int_{0}^{A} f^2_\beta(a, T) da dt + \int_{0}^{A} l^2_\beta(a, t) da dt &\leq C \left( \int_{0}^{A} m^2_0(a) da + \int_{0}^{A} f^2_0(a) da \right).
\end{aligned}
\]

**Proof of Lemma 14.** We denote by

\[
(y_0, z_0) = \left( e^{-\lambda_T} m_0, e^{-\lambda_T} f_0 \right).
\]

The functions $y_\alpha$ and $z_\beta$ verify

\[
\begin{aligned}
\partial_t y_\alpha + \partial_x y_\alpha + (\lambda_0 + \mu_{m}) y_\alpha &= x_\Theta_1 \partial_x e^{-\lambda_T} n_0, \\
\partial_t z_\beta + \partial_x z_\beta + (\lambda_0 + \mu_f) z_\beta &= x_\Theta_2 \partial_x e^{-\lambda_T} l_0.
\end{aligned}
\]

Multiplying equality (108) and equality (109) by, respectively, $y_\alpha$ and $z_\beta$ and integrating with respect to $Q$, we get

\[
\begin{aligned}
\frac{1}{2} \int_{0}^{A} y^2_\alpha(a, T) da + \frac{1}{2} \int_{0}^{T} y^2_\alpha(a, t) dt &+ \frac{1}{2} \int_{0}^{T} \int_{0}^{A} \left( \lambda_0 + \mu_m(a) \right) y^2_\alpha(a, t) da dt \\
&\leq \frac{1}{2} \int_{0}^{A} y^2_\beta(a) da + (1 - \gamma)^2 \int_{0}^{T} \left( \int_{0}^{A} \beta(a, p) z_0^2 da \right) dt \\
&+ \int_{0}^{T} \int_{0}^{A} X_{\Theta_1} e^{-\lambda_f T} n_0 y_\alpha da dt,
\end{aligned}
\]

\[
\begin{aligned}
\frac{1}{2} \int_{0}^{A} z^2_\beta(a, T) da + \frac{1}{2} \int_{0}^{T} z^2_\beta(a, t) dt &+ \int_{0}^{T} \int_{0}^{A} \left( \lambda_0 + \mu_f(a) \right) z^2_\beta(a, t) da dt \\
&\leq \frac{1}{2} \int_{0}^{A} f^2_\alpha(a) da + (1 - \gamma)^2 \int_{0}^{T} \left( \int_{0}^{A} \beta(a, p) z_\beta^2 da \right) dt \\
&+ \int_{0}^{T} \int_{0}^{A} X_{\Theta_2} e^{-\lambda_f T} l_0 z_\beta da dt.
\end{aligned}
\]

Using the Young inequality, Cauchy-Schwarz inequality, and the fact that $\beta = L^\infty$, we prove that

\[
\begin{aligned}
(1 - \gamma)^2 \int_{0}^{T} \left( \int_{0}^{A} \beta(a, p) z_\beta^2 da \right) dt &+ \int_{0}^{T} \int_{0}^{A} X_{\Theta_1} e^{-\lambda_f T} n_0 y_\alpha da dt \\
&\leq \alpha^2 ||z_\beta||_{L^2(a, \Theta_1)}^2 + \frac{1}{2} ||y_\alpha||_{L^2(a, \Theta_1)}^2 + \frac{1}{2} ||n_0||_{L^2(a, \Theta_1)}^2,
\end{aligned}
\]

\[
\begin{aligned}
\gamma^2 \int_{0}^{T} \left( \int_{0}^{A} \beta(a, p) z_\beta^2 da \right) dt &+ \int_{0}^{T} \int_{0}^{A} X_{\Theta_2} e^{-\lambda_f T} l_0 z_\beta da dt \\
&\leq \alpha^2 ||z_\beta||_{L^2(a, \Theta_1)}^2 + \frac{1}{2} ||z_\beta||_{L^2(a, \Theta_1)}^2 + \frac{1}{2} ||l_0||_{L^2(a, \Theta_1)}^2.
\end{aligned}
\]

Therefore, choosing $\lambda_0 > (\alpha^2 + 3/2)$, we get

\[
\begin{aligned}
\frac{1}{2} \int_{0}^{A} z^2_\beta(a, T) da + \int_{0}^{T} \int_{0}^{A} (1 + \mu_f(a)) z^2_\beta(a, t) da dt \\
&\leq \frac{1}{2} \left( ||f_0||_{Q_0}^2 + ||l_0||_{L^2(a, \Theta_1)}^2 \right),
\end{aligned}
\]

Finally, applying the result of Lemma 13 to the above inequality, it follows that

\[
\begin{aligned}
\frac{1}{2} \int_{0}^{A} z^2_\beta(a, T) da + \int_{0}^{T} \int_{0}^{A} (1 + \mu_f(a)) z^2_\beta(a, t) da dt \\
&\leq C \left( \int_{0}^{A} f^2_0(a) da + \int_{0}^{A} m^2_0(a) da \right)
\end{aligned}
\]

and then the inequality (106) holds.
Likewise, we have

\[
\frac{1}{2} \int_0^T y^2_t(a,T)da + \int_0^T \int_0^A (1 + \mu_m) y^2_x(a,t)dadt \\
\leq \frac{1}{2} \|m_0\|_{L^2(Q_A)}^2 + \alpha^2 \|z_\theta\|_{L^2(Q_A)}^2 + \frac{1}{2} \|n_\epsilon\|_{L^2(\Theta)}^2.
\]

(116)

Using the above inequality, Lemma 13, and inequality (115), we obtain

\[
\int_0^A y^2_x(a,T)da + \int_0^T \int_0^A (1 + \mu_m) y^2_x(a,t)dadt \\
\leq C \left( \int_0^A f^2_0(a)da + \int_0^A m_0^2(a)da \right)
\]

(117)

and then we get the desired result.

Finally, from Lemma 13 and Lemma 14, if $(\epsilon, \theta) \to (0,0)$, we get

\[
\begin{bmatrix}
X_{\epsilon n}, n' \\
X_{\theta n'}, \theta_f
\end{bmatrix} \to \begin{bmatrix}
X_{0 n}, v_m, X_{0 f}, \gamma_f
\end{bmatrix},
\]

\[
(m_{\epsilon f}, f_\theta) \to (m, f),
\]

(118)

with $(m, f)$ solution of problem (37) and

\[
m(\cdot, T) = f(\cdot, T) = 0 \text{ a.e. } a \in (0, A).
\]

(119)

We have now the necessary ingredients for the proof of Theorem 2.

4.2. Proof of Theorem 2. In this section, we established the existence of a fixed point for the preceding auxiliary problem. Indeed, we consider that $(H_1)$ holds and we suppose to simplify that $\lambda(0) = \lambda(A) = 0$. For each $p \in L^2(Q_T)$, let us denote by $\Lambda(p) \in L^2(0, T)$ the set of all $\int_0^A \lambda(a)m(p)da$, where the couple $(m(p), f(p))$ is the solution of the following system:

\[
\begin{aligned}
\partial_t m(p) + \partial_x m(p) + \mu_m m(p) &= \epsilon X_{\theta n}, n \text{ in } Q, \\
\partial_t f(p) + \partial_x f(p) + \mu_f f(p) &= \theta X_{\theta n}, n \text{ in } Q, \\
m(p)(a,0) &= m_0(a), f(p)(a,0) = f_\theta(a) \text{ in } Q_A, \\
m(p)(0,t) &= \mu_m(a) \lambda(a) m_0(p)da + \gamma f_\theta(a) \text{ in } Q, \\
m(p)(T,t) &= \mu_m(a) \lambda(a) m_0(p)da + \gamma f_\theta(a) \text{ in } Q.
\end{aligned}
\]

(120)

and $(m(p), l(p))$ the corresponding solution of the minimizer of $J$ with $m(p)(a, T) = f(p)(a, T) = 0$ for almost every $a \in (0,A).

We have the following result.

**Proposition 15.** Under the assumptions of Theorem 2, for any $p \in L^2(Q_T)$, the solution of problem (120) satisfies

\[
|Y(t)| + \left| \frac{d}{dt} Y \right|_{L^2(Q_T)} \leq C \left( \|m_0\|_{L^2(Q_A)} + \|f_0\|_{L^2(Q_A)} \right),
\]

(121)

where $Y(t) = \int_0^A \lambda(a)m(p)da$ and the constant $C$ is independent of $p, m_0$ and $f_0$.

**Proof of Proposition 15.** Let $Y(t) = \int_0^A \lambda(a)m(p)da$. It is easy to prove that $Y$ is the solution of system

\[
\begin{aligned}
\partial_t Y + \int_0^A \mu_m(a) \lambda(a)m(p)da &= R(t) \text{ in } Q_T, \\
Y(0) &= \int_0^A \lambda(a)m_0(a)da,
\end{aligned}
\]

(122)

where

\[
R(t) = \int_0^A \lambda'(a)m(p)da + (1 - \gamma) \lambda(0) \\
\cdot \int_0^A \beta(a,p)f(p)da + \int_0^A \lambda(a)m(p)da.
\]

(123)

Using Lemma 14 and the assumptions on $\beta$ and $\lambda$, we infer that there exists $K > 0$ such that

\[
\|R\|_{L^2(Q_T)} \leq K \left( \|m_0\|_{L^2(Q_A)} + \|f_0\|_{L^2(Q_A)} \right). \quad (124)
\]

By using (122), the Young inequality and integrating on $Q_T$, we obtain

\[
\int_0^T \left| \partial_t Y \right|^2 dt \leq 2 \int_0^T |R(t)|^2 dt + 2 \int_0^T \left( \int_0^A \mu_m(a) \lambda(a)m_1(p)da \right)^2 dt.
\]

(125)

Moreover, the Cauchy-Schwarz inequality leads to

\[
\int_0^T \left( \int_0^A \mu_m(a) \lambda(a)m_1(p)da \right)^2 dt \leq \int_0^A \mu_m(a) \lambda(a)da \int_0^T \int_0^A \mu_m(a) \lambda(a)m_1^2(p)da dt.
\]

(126)

The inequality (105) and the fact that $\lambda \in C([0,A])$ give

\[
\int_0^T \int_0^A \mu_m(a) \lambda(a)m_1^2(p)da dt \leq K_1 \left( \|m_0\|_{L^2(Q_A)}^2 + \|f_0\|_{L^2(Q_A)}^2 \right),
\]

(127)

where $K_1 > 0$ is independent of $p, \epsilon, \theta$. Moreover, as $\lambda \in L^1(0,A)$, and using (124), it follows that...
\[
\frac{d}{dt} Y(t) \in L^2(0, T) \leq C\left(\|m_0\|_{L^2(Q_\lambda)} + \|f_0\|_{L^2(Q_\lambda)}\right).
\]  

(128)

Now, let \( \tilde{Y} = e^{-\lambda t} Y \). Then, \( \tilde{Y} \) satisfies
\[
\begin{cases}
\tilde{Y}_t + \lambda_0 \tilde{Y} + e^{-\lambda t} \int_0^T \mu_n(a) \lambda(a) m_t(p) \, da = e^{-\lambda t} R(t) \quad \text{in } Q_T, \\
\tilde{Y}(0) = \int_0^T \lambda(a) m_0(a) \, da.
\end{cases}
\]

(129)

Multiplying the first equation of (129) by \( \tilde{Y} \), integrating on \( (0, t) \), and using successively Cauchy-Schwarz and Young inequalities, we deduce that
\[
|\tilde{Y}(t)|^2 + \lambda_0 \int_0^t \tilde{Y}^2 \, dt \leq |\tilde{Y}(0)|^2 + \int_0^t \tilde{Y}^2 \, dt + \int_0^t \left( \int_0^T \mu_n(a) \lambda(a) m_t(p) \, da \right)^2 \, dt.
\]

(130)

Using the above calculations and choosing \( \lambda_0 > 2 \), we get
\[
|\tilde{Y}(t)|^2 \leq K_2 \left( \|m_0\|_{L^2(Q_\lambda)} + \|f_0\|_{L^2(Q_\lambda)} \right).
\]

(131)

The desired result comes from (128) and (131).

It is obvious that \( \Lambda(p) \) is convex, and let
\[
W(0, T) = \left\{ Y \in L^{\infty}(0, T), \|Y\|_{L^2(0, T)} \leq N; \left\| \frac{dY}{dt} \right\|_{L^2(0, T)} \leq N \right\}.
\]

(132)

with \( N = C(\|m_0\|_{L^2(Q_\lambda)} + \|f_0\|_{L^2(Q_\lambda)}). \)

We have \( W(0, T) \subset W^{1,1}(0, T) \). Moreover, the injection of \( W^{1,2}(0, T) \) into \( L^2(0, T) \) is compact, see [10], page 129.

So \( W(0, T) \) is relatively compact in \( L^2(0, T) \). From Proposition 15, we have \( \Lambda(W(0, T)) \subset W(0, T) \), and we see that \( \Lambda(W(0, T)) \) is a relatively compact subset of \( L^2(0, T) \). Let us now prove that \( A \) is uppersemicontinuous. This is equivalent to prove that for any closed subset \( K \) of \( L^2(0, T) \), the set \( \Lambda^{-1}(K) \) is compact.

The proof is similar to the proof of Proposition 14. Then, \( \Lambda(\tilde{Y}) \) converges to \( \tilde{Y} \) as \( \lambda \to 0 \). Then, \( \Lambda(\tilde{Y}) \) is compact in \( L^2(0, T) \). Moreover, we have \( \Lambda(\tilde{Y}) \subset W^{1,1}(0, T) \).

4.3. Proof of Theorem 3

4.3.1. Proof of Theorem 3-(1). In this section, we always consider the following system:
\[
\begin{align*}
& m_i + m_a + \mu_m m = x_0 v_m \quad \text{in } Q, \\
& f_j + f_a + \mu_f f = 0 \quad \text{in } Q, \\
& m(a, 0) = m_0, f(a, 0) = f_0 \quad \text{in } Q, \\
& m(0, t) = (1 - \gamma) \int_0^A \beta(a, p) f(a, 0) \, da + \gamma \int_0^A h(a, p) f(a, t) \, da \quad \text{in } Q_T,
\end{align*}
\]

(133)

for every \( p \) in \( L^2(Q_T) \). Under the assumptions of Theorem 3, the controllability problem that is to find \( v_m \in L^2(\Theta) \) such that \( (m, f) \) solution of the system (133) verifies
\[
m(., T) = 0 \quad a \in (\rho, A)
\]

(134)

is equivalent to the following observability inequality.

Proposition 16. Let us assume the assumptions \( (H_1) \) – \( (H_2) \) – \( (H_1') \), for every \( T > A - \alpha_2 \) and for any \( p > 0 \), if \( h_\gamma(a) = 0 \) a.e in \( (0, q) \), there exists \( \gamma_{\rho, T} > 0 \) such that the following inequality:
\[
\int_0^A h^2(a, 0) \, da + \int_0^A g^2(a, 0) \, da \leq C_{\rho, T} \int_0^T h^2(a, t) \, dt
\]

(135)

holds, where \( (h, g) \) is the solution of
For the proof of Proposition 16, we state the following estimate.

**Proposition 17.** Under the assumptions ($H_1$) and ($H_2$), there exists a constant $C > 0$ such that the solution $(h, g)$ of system (136) verifies

\[
\int_0^T \int_0^A g^2(a, 0) da + \int_0^T \int_0^A \left(1 + \mu_f \right) g^2(a, t) dt \leq C \int_0^T h^2(0, t) dt.
\]

Moreover, we deduce for $h_T = 0$ a.e in $(0, \Omega)$ that there exists a constant $C_{e, T} > 0$ such that

\[
\int_0^A g^2(a, 0) da \leq C_{e, T} \int_0^T h^2(0, t) dt \leq C_{e, T} \int_0^T h^2(a, t) dt.
\]

**Proof of Proposition 16.** Setting $y = e^{h^T y} g$, the function $y$ verifies

\[-\partial_y y - \partial_y \gamma + \left(\lambda_0 + \mu_f \right) y = \gamma \beta(a, p)e^{h^T y}(0, t) + (1 - \gamma) \beta(a, p)y(0, t).\]

Multiplying equality (139) by $y$ and integrating on $Q$, we obtain

\[
\frac{1}{2} \int_0^A y^2(a, 0) da + \frac{1}{2} \int_0^T y^2(0, t) dt + \int_0^T \int_0^A \left(\lambda_0 + \mu_f \right) y^2(a, t) dt
\]

\[= \int_0^T \int_0^A \left(\gamma \beta(a, p)e^{h^T y}(0, t) + (1 - \gamma) \beta(a, p)y(0, t)\right) y dadt.
\]

Using Young inequality and the condition on $\beta$, we get

\[
\int_0^T \int_0^A \left(\gamma \beta(a, p)e^{h^T y}(0, t) + (1 - \gamma) \beta(a, p)y(0, t)\right) y dadt
\]

\[\leq \frac{e^{2\lambda_0 T}}{2} \|h(0, \cdot)\|_{L^2(Q_T)} + \frac{\delta}{2} \|y\|_{L^2(Q)}^2 + \frac{\alpha_f^2}{2} \|y(0, \cdot)\|_{L^2(Q_T)}^2 + \frac{\delta}{2} \|y\|_{L^2(Q)}^2.
\]

Choosing $\delta = \alpha^2_f$, we obtain

\[
\frac{1}{2} \int_0^A y^2(a, 0) da + \int_0^T \int_0^A \left(\lambda_0 + \mu_f \right) y^2(a, t) dt
\]

\[\leq \frac{e^{2\lambda_0 T}}{2} \|h(0, \cdot)\|_{L^2(Q_T)}^2 + \alpha^2_f \|y\|_{L^2(Q)}^2.
\]

Finally, choosing $\lambda_0 > \alpha^2_f + 1$, it follows that

\[
\frac{1}{2} \int_0^A y^2(a, 0) da + \int_0^T \int_0^A \left(1 + \mu_f \right) y^2(a, t) dt
\]

\[\leq \frac{e^{2\lambda_0 T}}{2} \int_0^A h^2(0, t) dt.
\]

So,

\[
\int_0^A y^2(a, 0) da \leq e^{2\lambda_0 T} \int_0^A h^2(0, t) dt.
\]

Finally, we get

\[
\int_0^A g^2(a, 0) da \leq e^{2\lambda_0 T} \int_0^A h^2(0, t) dt.
\]

Combining the above inequality and inequality (44) of Proposition 8, for $h_T = 0$ a.e in $(0, \rho)$, we get

\[
\int_0^A g^2(a, 0) da \leq C_{e, T} \int_0^T \int_0^A h^2(a, t) dt dadt.
\]

**Proof of Proposition 17.** We use the results of Proposition 9 and Proposition 17. Indeed, by combining (68) and (146), the desired result is obtained.

Now, let $\varepsilon > 0$ and $\Omega > 0$. We consider the functional $I_\varepsilon$ defined by

\[
I_\varepsilon(y_m) = \frac{1}{2} \int_0^T \int_0^A y_m^2(a, t) dt + \frac{1}{2\varepsilon} \int_0^T \int_0^A m^2(a, T) da.
\]
where \((m,f)\) is the solution of the following system:

\[
\begin{align*}
    m_t + m_a + \mu_m m &= \chi_{\Theta_1}v_m \text{ in } Q, \\
    f_t + f_a + \mu_f f &= 0 \text{ in } Q, \\
    m(a,0) &= m_0(a)f(a,0) = f_0(a) \text{ in } Q_A, \\
    m(0,t) &= (1 - \gamma)\int_0^A \beta(a,p)f(a,t)da \text{ in } Q_T, \\
    f(0,t) &= \gamma\int_0^A \beta(a,p)f(a,t)da \text{ in } Q_T.
\end{align*}
\]  

(148)

We have the following lemma.

**Lemma 18.** The functional \(J_\varepsilon\) is continuous, strictly convex, and coercive. Consequently, \(J_\varepsilon\) reaches its minimum at one and has \(v_{m_\varepsilon} = \chi_{\Theta_1}v_m\) and there exists positive constants \(C_1, C_2\) independent of \(\varepsilon\) such that

\[
\int_0^T \int_0^A h_\varepsilon^2(a,t)da dt \leq C_1 \left( \int_0^A m_\varepsilon^2(a)da + \int_0^A f_\varepsilon^2(a)da \right) ,
\]  

(149)

\[
\int_0^A m_\varepsilon^2(a,T)da \leq \varepsilon C_2 \left( \int_0^A m_\varepsilon^2(a)da + \int_0^A f_\varepsilon^2(a)da \right) .
\]  

(150)

**Proof of Lemma 18.** The proof is similar to that of Lemma 13.

By making \(\varepsilon\) tending towards zero, we thus obtain that \(\chi_{\Theta_1}v_m \rightarrow \chi_{\Theta_1}v_m\) and \((m_\varepsilon,f_\varepsilon) \rightarrow (m,f)\), where \((m,f)\) is the solution of system (148) that verifies

\[
m(.,T) = 0 \text{ a.e in } (Q,A).
\]  

(151)

Finally, a similar function \(\Lambda\) is defined and a similar procedure is followed to get the null controllability for the nonlinear problem.

4.3.2. Proof of Theorem 3-(2). Let \(p \in L^2(Q_T)\), under the assumptions of Theorem 3, the following controllability problem finds \(v_f \in L^2(\Theta)\) such that the solution of the system verifies

\[
f(.,T) = 0 \text{ a.e in } (0,A)
\]  

(153)

is equivalent to the following observability inequality.

**Proposition 19.** Let us assume true the assumptions \((H_1) - (H_2) - (H_3)\). For any \(T > a_1 + A - a_2\), there exists \(C_T > 0\) such that

\[
\int_0^A g^2(a,0)da \leq C_T \int_{\Theta_2} g^2(a,t)da dt,
\]  

(154)

where \(g\) is a solution of the system

\[
\begin{align*}
    -g_t - g_a + \mu_g g &= \gamma \beta(a,p)g(0,t) \text{ in } Q, \\
    g(a,T) &= g_T \text{ in } Q_A, \\
    g(A,t) &= 0 \text{ in } Q_T.
\end{align*}
\]  

(155)

**Proof of Proposition 19.** Using inequality (45) of Proposition 8, the result of Proposition 11, and the representation of the solution of the system (155), we get the desired result.

To conclude, a similar function \(\Lambda\) is defined and a similar procedure is followed to get the null controllability for the nonlinear system. We omit all details because the extension is straightforward.

5. Numerical Illustrations

In this part, the idea is to highlight the numerical simulation of the nonlinear problem.

The first parts are to reduce the PDE to the finite dimensional system of the form

\[
\dot{U}_i = A_i U_i + P_i + B_i Y_i,
\]  

(156)

where \(A_i, B_i,\) and \(P_i\) are matrices, and

\[
U_i(t) = \begin{pmatrix} V_i(t) \\ W_i(t) \end{pmatrix}
\]  

(157)

is the finite dimensional state vector. Here, vector \(P_i\) is the contribution of the nonlinear part, which comes from births; \(B_i\) is the control matrix, and \(Y_i\) is the control vector.
Figure 1: Evolution of the uncontrolled system from $a = t = 0$ to $A = t = 2$.

Figure 2: Evolution of the uncontrolled system from $a = t = 0$ to $A = t = 2$.

Figure 3: Initial conditions.
Figure 4: Evolution of the uncontrolled system from $a = t = 0$ to $A = 2$ and $t = 0.5$.

Figure 5: Evolution of the controlled system from $a = t = 0$ to $A = 2$ and $t = 0.5$.

Figure 6: Evolution of the control states from $a = t = 0$ to $A = 2$ and $t = 0.5$. 
So let us consider the following system:

\[
\begin{aligned}
\frac{\partial m}{\partial t} + \partial_x m + \mu m = \nu m \quad & \text{in } [0, A] \times [0, T], \\
\frac{\partial f}{\partial t} + \partial_x f + \mu f = \nu f \quad & \text{in } [0, A] \times [0, T], \\
\end{aligned}
\]

\[
m(0, t) = (1 - \gamma) \int_0^A \lambda da \int_0^A \beta f da \quad \text{in } [0, T],
\]

\[
f(0, t) = \gamma \int_0^A \lambda da \int_0^A \beta f da \quad \text{in } [0, T],
\]

\[
m(a, 0) = m_0 \quad \text{in } [0, A],
\]

\[
f(a, 0) = f_0 \quad \text{in } [0, A],
\]

To solve (158), the age discretization is performed with a rectangular grid on \([0, A]\). For a given rectangular grid \(\mathcal{S}\) with vertex \((a_j), 1 \leq j \leq n\) and uniform step size (without loss of generality) \(\Delta a\) in \(a\)-direction, we denote by the diameter of the grid. The finite difference approximation of the aging term is given by

\[
\frac{\partial y}{\partial a}(a_j, t) = \frac{y(a_{j-1}, t) - y(a_j, t)}{\Delta a}.
\]

Let \(m^I(t)\) be the approximation of \(m(a_j, t)\) and \(V_i(t) = m^I(t)_{1 \leq i \leq n}\) and let \(f^I(t)\) be the approximation of \(f(a_j, t)\) and \(W_i(t) = f^I(t)_{1 \leq i \leq n}\). We denote by \(A_i\) the matrix of the mortality approximation and the aging approximation.

We approximate \(\int_0^A \beta(a)f(a, t)da\) by the rectangle method. In fact,

\[
\int_0^A \beta(a)f(a, t)da = \frac{A}{n} \left( \frac{1}{n} \sum_{j=1}^{n-1} \beta(a_j)f(a_j, t) + \frac{\beta(0)f(0, t) + \beta(A)f(A, t)}{2} \right).
\]

\[
\int_0^A \lambda(a)m(a, t)da = \frac{1}{n} \left( \frac{1}{n} \sum_{j=1}^{n-1} \lambda(a_j)m(a_j, t) + \frac{\lambda(0)m(0, t) + \lambda(A)m(A, t)}{2} \right).
\]

Then,

\[
f^0 = \frac{\gamma A^2}{n^2} \left( \frac{1}{n} \sum_{k=1}^{n-1} \beta(a_k)f(a_k, t) + \frac{\beta(0)f(0, t) + \beta(A)f(A, t)}{2} \right) \lambda(a_j)m(a_j, t).
\]

But as

\[
\frac{\partial f^1}{\partial a} = \frac{f^1(t) - f^0(t)}{\Delta a},
\]

then

\[
\frac{\partial f^1}{\partial a} = \frac{-\gamma A^2/n^2 \left( \frac{1}{n} \sum_{k=1}^{n-1} \beta(a_k)f(a_k, t) + \frac{\beta(0)f(0, t) + \beta(A)f(A, t)}{2} \right) \lambda(a_j)m(a_j, t)}{\Delta a},
\]

\[
\frac{\partial m^1}{\partial a} = \frac{m^1(t) - m^0(t)}{\Delta a},
\]

\[m^0 = (1 - \gamma) \frac{A^2}{n^2} \left( \frac{1}{n} \sum_{k=1}^{n-1} \beta(a_k)f(a_k, t) + \frac{\beta(0)f(0, t) + \beta(A)f(A, t)}{2} \right) \lambda(a_j)m(a_j, t) + m^I(t) = \frac{\partial m^1}{\partial a} \frac{m^1(t) - m^0(t)}{\Delta a}.
\]

Remark 20. \(f^0\) and \(m^0\) are the approximations of births.

Now, as

\[
\frac{\partial m^1}{\partial a} = \frac{m^1(t) - m^0(t)}{\Delta a},
\]

\[
\frac{\partial m^1}{\partial a} = \frac{\partial m^1}{\partial a} \frac{m^1(t) - m^0(t)}{\Delta a}.
\]

The corresponding discrete system is given for a continuous initial solution \(y_0\) by

\[
dU_i/dt = A_i U_i + P_i + B_i Y_i, U_i(0) = (y_0(a_j))_{1 \leq i \leq n}.
\]

Example 1. For the simulation, we take \(A = 2, \Delta a = 1/20, T = 0.5, \) and \(n = 20\).

Moreover, we choose the initial condition

\[
m_0(a) = 10e^{-1/4 - a},
\]

\[
f_0(a) = 15e^{-1/(4 - a)}.
\]

The fertility \(\beta\) is given by

\[
\beta(a) = e^{0.52(10a - 4)} \frac{1}{5}
\]

and the mortality rate \(\mu\) (see [12, 13]) by

\[
\mu_m(a) = \mu_f(a) = \frac{1}{10(A - a)},
\]

\[
\lambda(a) = 1.
\]

Remark 21. Note that \(\lim_{a \to A} \mu_m(a) = \lim_{a \to A} \mu_f(a) = +\infty\).

Therefore, the probability of survival \(e^\int_0^t \mu(a) ds\) at maximum age \(A\) of male and female individuals is zero.

Thus, the following numerical results were obtained.

When the initial density of males or females is zero, the evolution of the population of males or females remains zero, see Figures 1 and 2.
We notice that numerically, we have the positivity of the state and the initial conditions, see Figures 3 and 4. Also, when we tend towards $A$, the density of each population tends towards zero. Therefore, this fact reflects the reality.

When the initial density of male population or female population is zero, we have no males or females during the evolution of the population, see Figures 1 and 2. This fact shows the importance of fertility function.

Moreover, in the absence of males or females, we obtain the total extinction of females or males at time $t = A = 2$ confirming the reality.

We use ODE 45 for all the simulations of the system.

Example 2. We construct the control problem, which consists in minimizing the function, and we choose the classical Hum function

$$J_ε(U, Y) = \frac{1}{ε} \sum_{i=1}^{2} \int_0^A (U_i)^2 da + \sum_{i=1}^{2} \int_0^T \int_0^A Y_i^2 dt.$$

(172)

The approximate null controllability becomes the minimization of the functional $J_ε$, where $(U, Y)$ verifies the system

$$\frac{dU_i}{dt} = A_i U_i + P_i + B_i Y_i.$$  

(173)

In this example, we use the data of Example 1.

For the final time $T = 0.5$, the control states and the controlled states are given below.

We notice that the uncontrolled solutions are not null at the corresponding final time (see Figure 4) while the controlled states are zero at the corresponding final time (see Figure 5). However, we could not take a positivity constraint in our simulations (see Figure 6).

The CaSadi toolbox is used to simulate the control system (by the minimization of the function: Hum method).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] WHO, *World malaria report*, WHO, 2018.
[2] B. Ainseba and S. Anita, "Internal stabilizability for a reaction-diffusion problem modeling a predator-prey system," *Nonlinear analysis*, vol. 61, no. 4, pp. 491–501, 2005.
[3] M. Iannelli and J. Ripoll, "Two-sex age structured dynamics in a fixed sex-ratio population," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 6, pp. 2562–2577, 2012.
[4] C. Zhao, M. Wang, and P. Zhao, "Optimal control of harvesting for age-dependent predator-prey system," *Mathematical and Computer Modelling*, vol. 42, no. 5-6, pp. 573–584, 2005.
[5] Y. He and B. Ainseba, "Exact null controllability of the Lobesia botrana model with diffusion," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 1, pp. 530–543, 2014.
[6] Y. Simpore and O. Traore, *Null controllability of a nonlinear age, space and two-sex population dynamics structured model*, Preprint, 2020.
[7] D. Maité, "On the null controllability of the Lotka-Mckendrick system," *Mathematical Control & Related Fields*, vol. 9, no. 4, pp. 719–728, 2019.
[8] S. Anita, "Analysis and control of age-dependent population dynamics, mathematical modeling," in *Theory and Applications*, Springer-Science&Business Media, 2011.
[9] G. F. Webb, "Theory of nonlinear age-dependent population dynamics," in *Volume 89 of Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, 1985.
[10] H. Brezis, *Analyse Fonctionnelle: Théorie et Applications*, Masson, Paris, 1983.
[11] K. Deimling, *Nonlinear Functional Analysis*, Springer Verlag, Berlin, 1985.
[12] A. Traoré, B. Ainseba, and O. Traoré, "On the existence of solution of a four-stage and age-structured population dynamics model," *Journal of Mathematics Analysis and Applications*, vol. 495, no. 1, p. 124699, 2021.
[13] A. Traoré, B. Ainseba, and O. Traoré, "Null controllability of a four-stage and age-structured population dynamics model," *Journal of Mathematics*, 2021.