Classification of Cosmological Trajectories

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Abstract

In the context of effective Friedmann equation we classify the cosmologies in multi-scalar models with an arbitrary scalar potential \(V\) according to their geometric properties. It is shown that all flat cosmologies are geodesics with respect to a conformally rescaled metric on the ‘augmented’ target space. Non-flat cosmologies with \(V = 0\) are also investigated. It is shown that geodesics in a ‘doubly-augmented’ target space yield cosmological trajectories for any curvature \(k\) when projected onto a given hypersurface.

Key words:

1 Introduction

Scalar fields have come to play a central role in modern cosmology. Current astronomical observations indicate that our universe is undergoing an acceleration sourced by dark energy [1]. As prospective dark energy candidates, a wild variety of scalar field dark energy models have recently been proposed, including quintessence [2], k-essence [3][4], and ghost (phantom)[5]. However, the analysis on properties of dark energy from the recent observations mildly favor models with state equation parameter \(\omega\) crossing -1 in the near past[5][6] which could not be realized only by a single scalar[7]. Consequently, study of multi-scalar fields becomes one of the most intriguing subjects[8].

On the other hand, the Friedmann equation forms the starting point for almost all investigations in cosmology. Over the past few years possible correc-

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tions to the Friedmann equation have been derived or proposed in a num-
ber of different contexts, generally inspired by braneworld investigation[9][10].
These modification are often of a form that involves the total energy density
$\rho$. In[8], multi-scalar coupled to gravity is studied in the context of con-
ventional Friedmann cosmology. It is found that the cosmological trajectories can
be viewed as geodesic motion in an ‘augmented’ target space. In this paper,
taking the $N$ scalar fields as coordinates of a Riemannian target space with
metric $G_{ij}$ we present a unified framework to investigate the geometric prop-
erties in a general cosmological background characterized by(6). The General
Relativity, Randall-Sundrum[12], and Gauss-Bonnet[11] scenarios correspond
to $n = 1, n = 2$ and $n = 2/3$, respectively.

We start the formalism in the flat universe in the context of modified Fried-
mann equation. It is shown that solutions of gravity coupled to $N$ scalar field
with $V = 0$ can be viewed as null geodesics in an ‘augmented’ target space
of dimension $N + 1$ with a Lorentzian signature metric. However, for $V \neq 0$,
the cosmological trajectories are geodesics only with respect to a conformally
rescaled metric; the geodesic is timelike if $V > 0$ and spacelike if $V < 0$.
Investigation of purely kinetic energy driving cosmology is of interest in its
own right. For example, purely kinetic k-essence can serve as a unified model
for dark matter and dark energy[4]. Alternative time-coordinate is introduced
in this case. It turns out that cosmological solutions with pure kinetic term
is again null geodesic when $k = 0$. For non-flat cosmologies, the trajectories
in ‘augmented’ target space are neither geodesic nor with global consistent
causal property. However, they can be viewed as projections of geodesics in a
‘doubly-augmented’ target space.

The organization of the paper is as follows. In the next section we present the
multi-scalar solutions in the context of modified Friedmann equation. In Sec.
3 and 4 we develop the interpretation of multi-scalar cosmologies as geodesic
motion with two different choice of metric. In Sec. 5 we introduce a light cone
in which the universe undergoes an acceleration. This paper is ended in Sec.
6 with some conclusions.

2 Equations of Motion

We start by reviewing the equations of motion for multi-scalar. We shall study
cosmologies in Friedmann-Robertson-Walker(FRW) background spacetime:

$$ds^2 = -dt^2 + S(t)^2 d\Sigma_k^2,$$

(1)

where the function $S(t)$ is the scale factor, and $\Sigma_k^2$ represents the 3-dimensional
spatial sections of constant curvature $k$. Here $k$ is normalized to take values
0, ±1.

The Lagrangian density for multi-scalar is

\[ L_\phi = -\frac{1}{2} g^{\mu\nu} G_{ij} \partial_\mu \phi^i \partial_\nu \phi^j - V(\phi). \] (2)

To simplify the equations we define

\[ (D_t^2 \phi)^i := \partial_t^2 \phi^i + \Gamma^i_{jk} \partial_t \phi^j \partial_t \phi^k, \] (3)

where \( \Gamma^i_{jk} \) is the Levi-Civita connection for the target space metric \( G \). Then we obtain the equations of motion:

\[ D_t^2 \phi^i + 3H \partial_t \phi^i - 2V a^i = 0, \] (4)

\[ a_i = -\frac{1}{2} \frac{\partial \ln |V|}{\partial \phi^i}, \] (5)

where \( H = \partial_t S/S \) is the Hubble function. We consider an effective Fridmann equation which is given by

\[ H^2 = A \rho^n - k/S^2, \] (6)

where \( A \) and \( n \) are constants. \( \rho \) is twice the energy density of scalar fields:

\[ \rho = |\partial_t \phi|^2 + 2V, \] (7)

where \( |\partial_t \phi|^2 \) is induced by the target space metric: \( |\partial_t \phi|^2 = G_{\alpha\beta} \partial_t \phi^\alpha \partial_t \phi^\beta \). Substituting Eqs.(4) to (6), we get the acceleration equation:

\[ \partial_t^2 S = AS \rho^{n-1}[2V - (3n - 1)|\partial_t \phi|^2]. \] (8)

Set

\[ S(t) = e^{\beta(t)}. \] (9)

The Friedmann equation can be written as

\[ (\partial_t \beta)^2 = A(|\partial_t \phi|^2 + 2V)^n - ke^{-2\beta}, \] (10)

the scalar field equation as

\[ D_t^2 \phi^i + 3(\partial_t \beta) \partial_t \phi^i = 0, \] (11)
and the acceleration equation as
\[
\partial_t^2 \beta + \partial_t \beta = -A(|\partial_t \phi|^2 + 2V)^{n-1}[2V - (3n - 1)|\partial_t \phi|^2].
\] (12)

3 Flat Cosmologies

In this section we develop a geometric method to describe the generic evolution of flat multi-scalar cosmologies. Define a new variable $\gamma$ by
\[
\partial_t \gamma = \left( \frac{\partial_t \beta}{\sqrt{A}} \right)^\frac{1}{n}.
\] (13)

We may consider the (N+1) variables
\[
\Phi^\mu = (\gamma, \phi^i)
\] (14)
as the elements constructing an ‘augmented’ target space. In this notion, the Friedmann equation can be written as
\[
G_{\mu\nu} \partial_t \Phi^\mu \partial_t \Phi^\nu = -2V,
\] (15)
where
\[
G_{\mu\nu} \partial_t \Phi^\mu \partial_t \Phi^\nu = -d\gamma^2 + G_{ij} d\phi^i d\phi^j
\] (16)
is a Lorentz – signature metric on the space. Therefore, the cosmological trajectory in the augmented target space can be classified by the signature of the potential $V$. It is null for $V = 0$, timelike for $V > 0$ and spacelike for $V < 0$.

Rewrite the acceleration equation (12) and scaler field equation (11) with $\gamma$ instead of $\beta$:
\[
\partial_t^2 \gamma + 3\sqrt{A}(\partial_t \gamma)^{n-1}|\partial_t \phi|^2 = 0,
\] (17)
\[
D_t^2 \phi^i + 3\sqrt{A}(\partial_t \gamma)^{n-1}\partial_t \alpha \partial_t \phi^i + a^i|\partial_t \Phi|^2 = 0,
\] (18)

where $|\partial_t \Phi|^2$ is induced by the augmented target space metric: $|\partial_t \Phi|^2 = G_{\alpha\beta} \partial_t \Phi^\alpha \partial_t \Phi^\beta$. We will find that after some special conformal transformation,
all cosmological trajectories are geodesics. Set conformal factor as

\[
\Omega = \begin{cases} \\
\sqrt{\frac{2}{V}} e^{f(\gamma)/2} & V \neq 0 \\
\sqrt{\frac{1}{V}} e^{f(\gamma)/2} & V = 0.
\end{cases}
\] (19)

Denote the new metric by \( \tilde{G} \), satisfying

\[
\tilde{G}_{\mu\nu} = \Omega^2 G_{\mu\nu}.
\] (20)

One can find

\[
\tilde{D}_t^2 \Phi^\mu = \partial_t^2 \Phi^\mu + \tilde{\Gamma}^\mu_{\nu\rho} \partial_t \Phi^\nu \partial_t \Phi^\rho \\
= D_t^2 \Phi^\mu + G^{\mu\nu} a_t |\partial_t \Phi|^2 - G^{\mu\nu} \frac{\partial f(\gamma)}{\partial \gamma} |\partial_t \Phi|^2 \partial_t \gamma \\
- \frac{\partial f(\gamma)}{\partial \gamma} \partial_t \gamma \partial_\gamma \Phi^\mu - 2a_t \partial_t \Phi^i \partial_\gamma \Phi^\mu.
\] (21)

where \( \tilde{\Gamma}^\mu_{\nu\rho} \) is the Lev-Civitra connection for the augmented target space metric \( \tilde{G} \). When \( f(\gamma) \) satisfies the condition

\[
f(t) = 6\beta(t),
\] (22)

or, equivalently,

\[
\frac{\partial f(\gamma)}{\partial \gamma} = 6\sqrt{A}(\partial_\gamma)^{n-1},
\] (23)

we find that the equation

\[
\tilde{D}_t^2 \Phi^\mu = \frac{1}{2} \frac{\partial f(\gamma)}{\partial \gamma} \partial_t \gamma \partial_\gamma \Phi^\mu - 2a_t \partial_\gamma \Phi^i \partial_\gamma \Phi^\mu
\] (24)

is equivalent to Eqs. (17) and (18) with respect to \( \mu = 0 \) and \( \mu \neq 0 \) respectively. Eq. (24) is the equation for a geodesic in a non-affine parametrization. In terms of the new time variable \( \tilde{t} \) defined by

\[
d\tilde{t} = 2\Omega^2 e^{-f(\gamma)/2} dt,
\] (25)

we have

\[
\tilde{D}_t^2 \Phi^\mu = 0.
\] (26)
This is the equation of an affinely parameterized geodesic with respect to the conformally-rescaled metric. The Friedmann equation now can be written as

\[
\tilde{G}_{\mu\nu}\partial_{t}\Phi^{\mu}\partial_{t}\Phi^{\nu} = \begin{cases} 
-s\text{sign}V & V \neq 0 \\
0 & V = 0 
\end{cases}.
\] (27)

Since null geodesics are unaffected by the conformal rescaling, the cosmological trajectories with \( V = 0 \) is null geodesic with respect to \( \tilde{G} \) as well. What is interesting is that it also shows that in flat space the the cosmological trajectories driven only by kinetic term are also geodesic with respect with another choice of metric discussed below. Whereas, for \( V \neq 0 \), the cosmological trajectories in the augmented target space are geodesics only with respect to a class of conformally-rescaled metrics which are restricted by the form of \( f(\gamma) \). Especially, to the GR case, that is, \( n = 1 \), \( f \) have the form of \( 6\sqrt{A}\gamma \) which is equivalent to what is discussed in[8].

4 Purely Kinetic Energy Driving Cosmologies

In this case the motion of scalar is discussed in an augmented space with different coordinates and metric. Define \((N+1)\) variables

\[
\Theta^{\mu} = (\beta, \phi^{\alpha})
\] (28)
as the coordinates. We take the metric as

\[
\tilde{G}_{\mu\nu}d\Theta^{\mu}d\Theta^{\nu} = -e^{-6(n-1)\beta}d\beta^{2} + G_{\alpha\beta}d\phi_{i}d\phi^{j}.
\] (29)

In this notation, The equation of motion becomes

\[
\tilde{D}_{t}^{2}\phi^{i} + 3\partial_{t}\beta\partial_{t}\phi^{i} = 0,
\] (30)

and Friedmann equation for \( V = 0 \) becomes

\[
A|\partial_{t}\phi|^{2n} - (\partial_{t}\beta)^{2} = ke^{-2\beta}.
\] (31)

From the Friedmann equation one can know that the cosmological trajectories do not have global consistent causal property. An exception is \( n = 1 \), the GR case[8]. From Eq. (31) for \( n = 1 \), the curve is null, timelike and spacelike with respect to \( k = 0,k = -1, \) and \( k = 1 \) respectively. For an arbitrary \( n \), using Friedmann equation for \( V = 0 \), the acceleration equation can be written as

\[
\tilde{D}_{t}^{2}\beta + 3(\partial_{t}\beta)^{2} = -(3n-1)ke^{-2\beta}.
\] (32)
Combined with the Eq. (30), we get a single equation

$$\tilde{D}_t^2 \Theta^\mu + 3(\partial_t \beta) \partial_t \Theta^\mu = (3n - 1)ke^{-2\beta - 6(n-1)\beta} \tilde{D}^{\mu\nu} \partial_\nu \gamma. \quad (33)$$

Introducing another time-coordinate

$$\hat{d}t = e^{-3\beta} dt, \quad (34)$$

One can obtain

$$\tilde{D}_t^2 = (3n - 1)ke^{4\beta - 6(n-1)\beta} \tilde{G}^{\mu\nu} \partial_\nu \beta. \quad (35)$$

For $k = 0$, this is the equation of an null affinely parameterized geodesic. For $k \neq 0$, the cosmological trajectory in the augmented target space is not a geodesic for the right-hand-side of Eq. (31) does not vanish. However, we will find that the trajectory can be viewed as the projection of a geodesic on a hypersurface in a ‘doubly-augmented’ target space of dimension $(N+2)$ that is foliated by hypersurfaces isometric to the augmented target space.

Consider the $(N+2)$ variables

$$\Xi^A = (\Theta^\mu, \Xi^*) \quad (36)$$

as maps from the cosmological trajectory to the ‘doubly-augmented’ target space. We adopt the metric

$$\tilde{G}_{AB} d\Xi^A d\Xi^B = \tilde{G}_{\mu\nu} d\Theta^\mu d\Theta^\nu + \tilde{G}_{**} (d\Theta^*)^2, \quad (37)$$

where

$$\tilde{G}_{**} = \frac{3n - 1}{3n - 5} ke^{(6n-10)\beta}, \quad (38)$$

We need another equation to constrain the motion of the additional variable $\Xi^*$. Take the motion equation of $\Xi^*$ as

$$\partial_t \Theta^* = e^{(10-6n)\beta}, \quad (39)$$

With this choice, one can obtain

$$\tilde{D}_t^2 \Xi^A = 0. \quad (40)$$
Thus, all cosmological trajectories driven by kinetic term is geodesics in the doubly-augmented target space. The cosmological trajectories in augmented target space are corresponding to the projection of these geodesics on a motive hypersurface whose motion is restricted by equation (39). However, for $k = 0$, the cosmological trajectories are always geodesics not only in doubly-augmented target space but also in augmented target as discussed in Sec. 2.

One can easily find that to the case of $n = 1$, the two classes of coordinates and metric in Sec. 2 and this Sec. are equivalent. It is the different effective Friedmann equations that lead to the existence of different choices to set up augment target space.

5 Acceleration Cone

From Eq.(8), it is shown that only when potential $V > 0$ could acceleration occur. In this section we only discuss the case of flat universe. In the notation of Sec.2, using Friedmann equation to eliminate $2V$ one can rewrite the acceleration equation as

$$\partial_t^2 S = 3nAS\rho^{n-1}\left[\frac{(\partial_t \gamma)^2}{3n} - |\partial_t \phi|^2\right].$$

We will see that the acceleration acquires a geometrical meaning in the augmented target space with the following acceleration metric

$$G_{\mu\nu}^{\text{acc}} d\Phi^\mu d\Phi^\nu = -\frac{1}{3n} d\gamma^2 + G_{ij} d\phi^i d\phi^j.$$  

One finds that

$$G_{\mu\nu}^{\text{acc}} \partial_t \Phi^\mu \partial_t \Phi^\nu = -\frac{1}{3n} (\partial_t \gamma)^2 + |\partial_t \phi|^2$$

Substituting it into Eq. (8) we obtain

$$\partial_t^2 S = -3nA\rho^{n-1}G_{\mu\nu}^{\text{acc}} \partial_t \Phi^\mu \partial_t \Phi^\nu$$

Thus, we come to the conclusion that a universe is accelerating when the tangent to its trajectory lies within a subcone of the lightcone defined by the acceleration metric on the augmented target space.
6 Conclusions

In this paper, we have presented an geometric analysis to classify the homogeneous isotropic cosmology in the context of effective Friedmann equation in multi-scalar models with an arbitrary scalar potential $V$. Two models of interest is discussed respectively. The first is analysis in the flat universe. In this case, the target space of $N$ scalar fields $\phi$ is augmented to a larger $(N + 1)$-dimensional Lorentzian signature space, where function of the scalar factor plays the role of time. It is found that when $V = 0$, flat cosmological trajectories correspond to null geodesic in the augmented target space. But for $V \neq 0$, the cosmological trajectories are geodesic only when a class of conformally rescaled metrics are chosen. The conformal factor greatly depend on the form of potential $V$.

The other is the universe driven by purely kinetic energy. A distinguished simple time-coordinate is chosen at the price of the relatively complicated metric. In the augmented space we take the logarithm of the scale factor as the time-coordinate. In this case, for flat universe, same conclusion is reached as the former case. While for $k \neq 0$, the cosmological trajectories are no longer geodesics. However, they can be viewed as geodesics in a doubly-augmented target space with dimension $(N + 2)$ Thus, all purely kinetic energy driving cosmological trajectories can be viewed as projection of these geodesics.

Though the choice of time-coordinate is different in these two case, the geometric properties for flat universe driven by purely kinetic energy is same. What is more, when $n = 1$, these two augmented target manifolds are equivalent.

Acknowledgments

Qi Guo thanks Prof. Paul Townsend for discussion on the paper [8]. Also she thanks Hong-Sheng Zhang, Zong-Kuan Guo, Hao Wei, Hui Li and Da-Wei Pang for helpful discussion. This project was in part supported by Chinese Academy of Sciences, by NNSFC under Grant No.10325525 and No.90403029, and also by MSTC under Grant No.TG1999075401.

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