ON RATIONAL MAPS FROM THE PRODUCT OF TWO GENERAL CURVES

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Abstract. This paper treats the dominant rational maps from the product of two general curves to nonsingular projective surfaces. Combining the result in [3] we prove that the product of two very general curves of genus \( g \geq 7 \) and \( g' \geq 3 \) does not admit dominant rational maps of degree \( > 1 \) if the image surface is non-ruled. We also treat the case of the 2-symmetric product of a curve.

1. Introduction

In [7] it is proven that the proper subfields of the function field \( K(X) \) of a very general smooth complex surface \( X \) of \( \mathbb{CP}^3 \) of degree \( \geq 5 \) are of pure transcendental type, provided that \( K(X) \supset C \). Another way of stating is that if \( F : X \rightarrow S \) is a dominant map which is not birational, then either \( S \) is a point, a projective line or a rational surface. Let \( X \) be smooth complex projective variety of general type. The dominant rational maps of finite degree \( X \rightarrow Y \) to smooth varieties of general type, up to birational equivalence of \( Y \) form a finite set. The proof follows from the approach of Maehara [8], combined with the results of Hacon and McKernan [5], of Takayama [9], and of Tsuji [10].

Motivated by this finiteness theorem for dominant rational maps on a variety of general type, and by the results obtained in [3] and [7], we study the case of the product of two very general smooth curves \( X = C \times D \) of genus \( g_C \) and \( g_D \) respectively. The 2-symmetric product of \( C \), \( X = C_2 \), is also treated. The product case has previously been studied in [3]. It was proved there that if \( g_C \geq 7 \), \( g_D \geq 2 \), and \( S \neq C \times D \) is of general type then a dominant rational map \( F : C \times D \rightarrow S \) does not exist. Here we complete the analysis by considering surfaces \( S \) of Kodaira dimension \( \text{kod}(S) = 0 \) and \( 1 \). We have mainly to deal with elliptic surfaces. Our main result is

**Theorem 1.1.** (≡ Theorem 3.1) Let \( C \) and \( D \) be very general curves of genus \( g_C \geq 7 \) and \( g_D \geq 3 \) respectively. Let \( F : C \times D \rightarrow S \) be a dominant rational map of degree \( > 1 \) where \( S \) is a smooth projective surface. Then \( S \) is a ruled surface, that is \( \text{kod}(S) = -\infty \).

As in [6, 7] Hodge theory and deformation theory are the two main methods used to handle our problem. The main new technical obstacle comes out from the fact that the fundamental group \( C \times D \) is not abelian. It is not hard to see that the map of the fundamental group \( \pi_1(C \times D) \rightarrow \pi_1(S) \) has to be surjective. However when \( S \) has Kodaira dimension 1, we cannot infer directly neither that the elliptic surface \( S \) has bounded topology nor that its
moduli space has bounded dimension. To overcome this problem we begin by proving that the first homology group of $S$, $H_1(S, \mathbb{Z})$, vanishes. Then we use the result of the deformation of curves on elliptic surfaces with multiple fibers adapted in [7] to obtain a contradiction. The main ingredient we use to show that $H_1(S, \mathbb{Z}) = 0$ is Theorem 2.7 which provides some restriction on the Hodge structure of certain abelian covering of $C \times D$. Theorem 2.7 seems to be of independent interest and to deserve further development.

The last section deals with the case of the 2-symmetric product of a curve. This case is simpler, by using a slightly improvement on the deformation of curves on elliptic surfaces $S$ we can prove (we do not attempt to find the optimal genus)

**Theorem 1.2.** (=Theorem 4.5) Let $C$ be a general curve of genus $\geq 10$. If $f : C_2 \rightarrow S$ is a dominant rational map of degree $> 1$ then $S$ is a rational surface.

In this paper we work on the field of complex numbers.

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2. **Product of curves**

The first two sections consider the product of two very general curves $C \times D$ of genus $g \geq 3$ and the dominant rational map $C \times D \rightarrow S$ where $S$ is a smooth projective surface with $0 \leq \text{kod}(S) \leq 1$. It is easy to show

1. $p_g(S) = q(S) = 0$. This follows by the irreducibility of the transcendental Hodge structure of $H^2$ of the general product and by the fact that the canonical map of $C \times D$ is an embedding (see also Lemma 2.7 and Lemma 4.1 in [3] or Subsection 2.3 below).

2. $\pi : S \rightarrow \mathbb{P}^1$ has an elliptic fibration.

3. The map of the fundamental group is surjective. The proof is the same as in the case of the symmetric product (see Proposition 4.3).

Since the fundamental group $C \times D$ is not abelian, the fundamental group of $S$ need not to be abelian. But we will show the following.

**Theorem 2.1.** We have $H_1(S, \mathbb{Z}) = 0$, that is $H^2(S, \mathbb{Z})$ is torsion free and $q = 0$.

To prove Theorem 2.1 we first study the Hodge structure of some abelian covering of $C \times D$. We will obtain Theorem 2.7 that allows us to prove Theorem 2.1.
2.1. Hodge structure on abelian étale covering of curves. In this subsection we let $C$ and $D$ be curves of genus $g_C \geq 3$ and $g_D \geq 3$ respectively.

**Definition 2.2.** We call a curve $C$ pairwise torsion free (PTF) if for any pair of distinct points $A, B \in C$ with $A \neq B$, $nA \neq nB$ for any nonzero integers $n$ i.e. $\mathcal{O}_C(A - B)$ is not a torsion element of Pic$^0(C)$.

**Remark 2.3.** A PTF curve $C$ is not hyperelliptic, and its genus $g_C \geq 3$. From the computation of the moduli of the Hurwitz schemes of a curve having a map to $\mathbb{P}^1$ with 2 total ramification points, we see that the moduli of curves which are not PTF depends on $2g - 1$ parameters. Then a very general curve of genus $\geq 3$ is a PTF curve.

We now prove two simple algebraic lemmas that will be helpful in studying the infinitesimal variation of Hodge structure.

**Lemma 2.4.** Let $C$ be a PTF curve of genus $g \geq 3$, and let $L$ and $M$ be torsion line bundles in Pic$^0(C)$. Let $\alpha \in H^0(C, \omega_C \otimes L)$ and $\beta \in H^0(C, \omega_C \otimes M)$ be non trivial sections. Assume that either $L \neq M$ or $L = M$ with $\alpha \neq \lambda \beta$ for any $\lambda \in \mathbb{C}$. Then there exists $\eta \in H^1(C, T_C)$ such that the cup-product $\eta \cdot \alpha \neq 0 \in H^1(C, \mathcal{O}_C(L))$ and $\eta \cdot \beta = 0 \in H^1(C, \mathcal{O}_C(M))$.

**Proof.** We consider the following subspaces $H_\beta$ and $H_\alpha$ of $H^0(C, \omega_C^2)$:

$$H_\beta = \beta \cdot H^0(C, \omega_C \otimes M^{-1}); \quad H_\alpha = \alpha \cdot H^0(C, \omega_C \otimes L^{-1}).$$

If we can prove that $H_\alpha$ is not contained in $H_\beta$, then by the duality we can find an element $\eta \in H^1(T_C)$ the space dual to $H^0(C, \omega_C^2)$ such that for the map $\eta : H^0(C, \omega_C^2) \to \mathbb{C}$ one has $\ker(\eta) \not\supset H_\alpha$, but $\ker(\eta) \supset H_\beta$.

Let $E$ and $F$ be the divisor $\alpha$ and $\beta$ of degree $2g - 2$, respectively. Let $G$ be an effective divisor of the maximal degree (with multiplicity) contained in $E$ and $F$. Since $\alpha$ and $\beta$ are not proportional $\deg G < 2g - 2$. Moreover, since $C$ is a PTF curve $\deg G < 2g - 3$. In fact if $\deg G = 2g - 3$ then there exist two distinct points $P$ and $Q$ such that $G + P \equiv \omega_C(L)$ and $G + Q \equiv \omega_C(M) L \otimes M^{-1} \cong P - Q$. Since $L$ and $M$ are torsion, $L \otimes M^{-1}$ is also a torsion line bundle and we get a contradiction. Next we consider the following exact sequence:

$$0 \to G(LM)^{-1} \to \omega_C(L^{-1}) \oplus \omega_C(M^{-1}) \xrightarrow{(\alpha, \beta)} \omega_C^2(-G) \to 0$$

It follows that $H_\alpha + H_\beta$ is the image of the cohomology map

$$H^0(\omega_C(L^{-1})) \oplus H^0(\omega_C(M^{-1})) \xrightarrow{(\alpha, \beta)} H^0(\omega_C^2(-G)) \subset H^0(\omega_C^2)$$

and we can identify $H_\beta \cap H_\alpha$ and $H^0(C, G(LM)^{-1})$. Since $\deg G(LM)^{-1} = \deg G \leq 2g - 4$, and $C$ is not hyperelliptic we obtain

$$\dim H_\alpha \cap H_\beta \leq g - 2 < \min(\dim H_\alpha, \dim H_\beta)$$

which proves our lemma.

**Lemma 2.5.** Let $C$ be a curve of genus $g \geq 3$. Let $L \in \text{Pic}^0(C)$, $\alpha \in H^0(C, \omega_C(L))$, $\alpha \neq 0$. Then

1. for any $\gamma \in H^1(C, \mathcal{O}_C(L))$, there is an element $\eta \in H^1(C, T_C)$ such that $\eta \cdot \alpha = \gamma$;
(2) there is an element \( \eta \in H^1(T_C) \) such that the map \( H^0(C, \omega_C(L)) \xrightarrow{\eta} H^1(C, \mathcal{O}_C(L)) \) is an isomorphism.

**Proof.** Let \( E \) be again the divisor of \( \alpha \) and consider the exact sequence:

\[
0 \to T_C \xrightarrow{\alpha} \mathcal{O}_C(L) \to \mathcal{O}_E(L) \to 0.
\]

Since \( h^1(\mathcal{O}_E(L)) = 0 \) it follows that \( H^1(T_C) \xrightarrow{\alpha} H^1(C, \mathcal{O}_C(L)) \) is surjective. This proves the first part of the lemma. For the second part we consider the coboundary map:

\[
\partial : H^0(C, \mathcal{O}_E(L)) \to H^1(C, T_C).
\]

We have that \( h^0(\mathcal{O}_E(L)) = \deg E = 2g - 2 \). Therefore \( \Gamma_\alpha = \partial(H^0(\mathcal{O}_E(L))) \) has dimension \( 2g - 3 \) if \( L = \mathcal{O}_C \) or \( 2g - 2 \) if \( L \neq \mathcal{O}_C \). We remark that

\[
\eta \in \Gamma_\alpha \iff \alpha \in \ker \eta.
\]

So we have to show that \( \bigcup_\alpha \Gamma_\alpha \neq H^1(C, T_C) \). In fact every element in \( H^1(T_C) \setminus \bigcup_\alpha \Gamma_\alpha \) defines an isomorphism \( H^0(C, \omega_C(L)) \xrightarrow{\eta^{-1}} H^1(C, \mathcal{O}_C(L)) \).

This will be done by a dimension count passing to the associated projective spaces. Consider the projective space \( \mathbb{P} = \mathbb{P}H^1(T_C) \), \( \dim \mathbb{P} = 3g - 4 \), we let \( \mathbb{P}_\alpha \subset \mathbb{P} \) be the sub-projective space associated to \( \Gamma_\alpha \). We have to show that \( \bigcup_\alpha \mathbb{P}_\alpha \neq \mathbb{P} \). Let \( \mathbb{P}_L \) be the projective space associated to \( H^0(C, \omega_C(L)) \). Since \( H^0(C, \omega_C(L)) = g - 1 \) if \( L \) is not trivial and \( g \) if it is trivial, we have that \( \dim \mathbb{P}_L = g - 2 \) and \( g - 1 \) respectively. Now consider the incidence correspondence \( \mathcal{I} \subset \mathbb{P} \times \mathbb{P}_L : \)

\[
\mathcal{I} = \{ ((\eta), (\alpha)) \in \mathbb{P} \times \mathbb{P}_L : \eta \cdot \alpha = 0 \}.
\]

Let \( \pi_i \) for \( i = 1, 2 \) be the projections. Since \( \pi_2 \) is surjective and the fibers are the \( \mathbb{P}_\alpha \) we get \( \dim \mathcal{I} = 3g - 5 \). Set \( Y_L = \pi_1(\mathcal{I}) \subset \mathbb{P} \). We notice that

\[
Y_L = \bigcup_{(\alpha) \in \mathbb{P}_L} \mathbb{P}_\alpha = \{ (\zeta) : \zeta \in \Gamma_\alpha, (\alpha) \in \mathbb{P}_L \}.
\]

Therefore we obtain \( \dim Y_L \leq 3g - 5 \) in all cases (and the equality must hold since the \( Y_L \) can be defined by the vanishing of a determinant). This proves the lemma. \( \square \)

Now we set our notation. We let \( p > 1 \) be a prime number, \( L_1 \) be a line bundle on \( C \) and \( L_2 \) be a line bundle on \( D \), such that \( L_1^p = \mathcal{O}_C \) and \( L_2^p = \mathcal{O}_D \). We assume that \( L_1 \) and \( L_2 \) are not trivial, but the case \( L_1 = \mathcal{O}_C \) could be considered and it is simpler. Let \( f_1 : C' \to C \) and \( f_2 : D' \to D \) be the étale covering associated to \( L_1 \) and \( L_2 \). Set \( X = C' \times D' \). We have:

\[
H^{2,0}(X) \cong H^{1,0}(C') \otimes H^{1,0}(D') \quad H^{0,2}(X) \cong H^{0,1}(C') \otimes H^{0,1}(D')
\]

\[
H^{1,0}(C') \cong \oplus_{i=0}^{p-1} H^0(C, \omega_C(L_1^i)) \quad H^{0,1}(C') \cong \oplus_{i=0}^{p-1} H^1(C, \mathcal{O}_C(L_1^{-i}))
\]

\[
H^{1,0}(D') \cong \oplus_{i=0}^{p-1} H^0(D, \omega_D(L_2^i)) \quad H^{0,1}(D') \cong \oplus_{i=0}^{p-1} H^1(D, \mathcal{O}_D(L_2^{-i})).
\]

We set

\[
V_{i,j} \cong H^0(C, \omega_C(L_1^i)) \otimes H^0(D, \omega_D(L_2^j)),
\]

\[
V'_{i,j} \cong H^1(C, \mathcal{O}_C(L_1^i)) \otimes H^1(D, \mathcal{O}_D(L_2^j)).
\]
For the sake of notation we will use the above isomorphism as an identification. Then often we will omit the pull-backs, for instance we write \( H^0(C, \omega_C(L_1)) \) instead of \( f_1^*H^0(C, \omega_C(L_1)) \), etc.

(2) \( H^{2,0}(X) = \oplus_{ij=0}^{p-1} V_{i,j}, \quad H^{0,2}(X) = \oplus_{ij=0}^{p-1} V'_{i,j}. \)

One has \( V_{i,j} = V_{i+p,j} = V_{i,j+p} \) and by the complex conjugation \( \overline{V_{i,j}} = V'_{-i,-j} = V'_{p-i,p-j} \).

Then we get the following proposition.

Proposition 2.6. Let \( C \) and \( D \) be very general curves of genus \( \geq 3 \). Let \( \Lambda \subset H^2(C' \times D') \) be a Hodge substructure. If \( \Lambda^{2,0} \neq 0 \) then for some index \( a,b \), we have \( \Lambda^{2,0} \supset V_{a,b} \).

Proof. We remark that the deformations of \( C' \) and \( D' \) that preserves the covering, correspond to the deformations of \( C \) and \( D \), and therefore their infinitesimal deformation space is \( T = H^1(T_C) \oplus H^1(T_D) \).

We shall use a basic result from the infinitesimal variation of the Hodge structure (cf. \([11]\)). If we have a pair of the infinitesimal deformation, \( \psi, \phi \in T \), then \( \psi \cdot \phi \cdot \Lambda^{2,0} \subset \Lambda^{0,2} \). This follows because \( C \) and \( D \) are generic and \( \Lambda \) deforms infinitesimally in any direction of \( T \). The infinitesimal Hodge structure theory gives that the above maps are obtained by the cup product. In particular we will consider \( \psi = \eta \in H^1(T_C) \subset T \) and \( \psi = \zeta \in H^1(T_D) \subset T \).

Our first aim is to show that \( \Lambda^{2,0} \) contains some decomposable element. For any \( \Gamma \in \Lambda^{2,0} \subset H^{2,0}(X) \), we write \( \Gamma = \sum \gamma_{i,j}, \gamma_{i,j} \in V_{i,j} \). First we will show that there is an element \( \Gamma \in \Lambda^{2,0} \) such that the components \( \gamma_{i,j} \in V_{i,j} \) are all decomposable: \( \gamma_{i,j} = \alpha_{i,j} \wedge \beta_{i,j} \). (Since our element are forms we use the wedge product instead of \( \otimes \)). Starting with any \( \Omega = \sum \omega_{i,j} \neq 0 \) and take an index \( i,j \) such that \( \omega_{i,j} \neq 0 \). Write

\[
\omega_{i,j} = \sum_{k=1}^{s} \alpha_k \wedge \beta_k.
\]

We assume that \( \alpha_k \) and \( \beta_k \) are independent, that is the rank of tensor \( \omega_{i,j} \) to be \( s \). If \( s = 1 \) there is noting to do for the index \( i,j \). Assume \( s > 1 \). In particular \( \alpha_1 \) and \( \alpha_2 \) are not proportional. Now by Lemma 2.3 we find \( \eta \in H^1(T_C) \) such that \( \eta \cdot \alpha_1 = 0 \) and \( \eta \cdot \alpha_2 \neq 0 \). And by Lemma 2.5 we have \( \zeta \in H^1(T_D) \) such that \( \zeta : H^0(D, \omega_D(L_2^s)) \to H^1(D, \mathcal{O}_D(L_2^s)) \) is an isomorphism. Since the Hodge structure \( \Lambda \) must deform with \( C \times D \), \( \Theta = \zeta \cdot (\eta \cdot \Omega) = \eta \cdot (\zeta \cdot \Omega) \in \Lambda^{0,2} \). The infinitesimal variation of Hodge structure is given by the cup-product, and the cup product commutes with the decomposition (2) we get that \( \Theta_{i,j} = \zeta \cdot (\eta \cdot \omega_{i,j}) \). Then

(3) \( \Theta_{i,j} = \sum_{k=1}^{s} \eta \cdot \alpha_k \wedge \zeta \cdot \beta_k \).

We remark that the rank cannot increase, moreover since \( \eta \cdot \alpha_1 = 0 \), \( \Theta \) has rank \( \leq s - 1 \):

\[
\Theta_{i,j} = \sum_{k=2}^{s} \eta \cdot \alpha_k \wedge \zeta \cdot \beta_k.
\]
Since $\zeta$ is an isomorphism the vectors $\zeta \cdot \beta_k$ are all independent and $\eta \cdot \alpha_2 \neq 0$, and then we obtain that $\Theta_{i,j} \neq 0$. Now we use the complex conjugation. We define $\tilde{\Omega} = \Xi \in \Lambda^{2,0}$, moreover $\tilde{\Omega}_{p-i,p-j} :$

$$\tilde{\Omega}_{p-i,p-j} = \sum_{k=2}^{s} (\eta \cdot \alpha_k) \wedge (\zeta \cdot \beta_k).$$

is non trivial of rank $\leq s - 1$. As the formula (3) shows the rank of the component of our tensor cannot increase under the cup product action. We can repeat the above operation for any index $i', j'$. Finally we find $0 \neq \Omega' \in \Lambda^{2,0}$ such that $\Omega' = \sum \alpha_{ij} \wedge \beta_{ij}$.

Next we would like to show that we can find a decomposable non trivial element $\alpha_{ij} \wedge \beta_{ij} \in \Lambda^{2,0}$. We start this time from $\Omega' = \sum \alpha_{ij} \wedge \beta_{ij}$. If its rank is one then we have done. We may assume the rank is $r \geq 2$, and then that for two pairs of different indices $ij$ and $i'j'$ $\alpha_{ij} \wedge \beta_{ij} \neq 0$ and $\alpha_{i'j'} \wedge \beta_{i'j'} \neq 0$. We have $i \neq i'$ or $j \neq j'$. By the symmetry of the hypothesis on $C$ and $D$ we may assume $i \neq i'$. By Lemma 2.4 we can find $\eta \in H^1(T_C)$ such that $\eta \cdot \alpha_{ij} \neq 0$ and $\eta \cdot \alpha_{i'j'} = 0$, and can find $\zeta \in \Lambda^1(T_C)$ such that $\zeta \cdot \beta_{ij} \neq 0$. It follows then $\Theta = \zeta \cdot (\eta \cdot \Omega) \in \Lambda^{0,2}$ : $\Theta_{i,j} = \zeta \cdot (\eta \cdot \Omega')_{i,j} = \eta \cdot \alpha_{ij} \wedge \zeta \cdot \beta_{ij} \neq 0$ and $\Theta_{i',j'} = 0$. Taking $\Omega'' = \Xi^*$ we get a non trivial element in $\Lambda^{2,0}$ with rank $0 \leq r' < r$. Repeating the operation we can find an element

$$0 \neq \alpha \wedge \beta \in V_{a,b} \cap \Lambda^{2,0}.$$ 

Finally we see that $\eta \cdot \alpha \wedge \zeta \cdot \beta$ belongs to $\Lambda^{0,2}$ for all $\eta \in H^1(T_C)$ and $\zeta \in H^1(T_D)$. Using the first part of Lemma 2.4 we get that all the decomposable elements in $V_{a,b}$ are in $\Lambda^{0,2}$. It concludes $V_{a,b} \subset \Lambda^{0,2}$. Therefore $V_{p-a,p-b} \subset \Lambda^{2,0}$.

Now we can prove the following.

**Theorem 2.7.** Let $C$ and $D$ are very general curves of genus $g \geq 3$. Let $f_1 : C' \rightarrow C$ and $f_2 : D' \rightarrow D$ as above where $f_i$ for $i = 1, 2$ are étale covering of prime order $p$. Let $X = C' \times D'$. Assume that $\Lambda \subset H^2(X)$ is a Hodge substructure such that $\Lambda^{2,0} \neq 0$ ($\Lambda^{2,0} \subset H^{2,0} = H^0(X,K_X)$). Let $|\Lambda^{2,0}|$ be the corresponding sublinear series of $H^0(X,K_X)$. Then the image of the rational map $X = C' \times D' \dashrightarrow |\Lambda^{2,0}|$ has dimension 2.

**Proof.** From Proposition 2.6 we can find indices $i, j$ such that $\Lambda^{2,0} \supset V_{i,j} = f_1^*(H^0(C,\omega_C(L_1^i))) \otimes f_2^*(H^0(D,\omega_D(L_2^j)))$. It is enough to show that the image of

$$X \dashrightarrow |f_1^*(H^0(C,\omega_C(L_1^i))) \otimes f_2^*(H^0(D,\omega_D(L_2^j)))|$$

has dimensions 2. This is clear since it factorizes through

$$C \times D \dashrightarrow |H^0(C,\omega_C(L_1^i)) \otimes H^0(D,\omega_D(L_2^j))|.$$ 

Since $C$ and $D$ have both genus $\geq 3$, $h^0(C,\omega_C(L_1^i)) \geq 2$ and $h^0(D,\omega_D(L_2^j)) \geq 2$. The result follows by using the Segre embedding. \qed
For the sake of completeness, we can consider \( P(C', C) \subset H^1(C) \), and \( P(D', D) \subset H^1(D) \) respectively. These are the Hodge structures corresponding to the kernel of the norm mappings \( H^1(C') \to H^1(C) \), and \( H^1(D') \to H^1(D) \) respectively. Clearly \( P(C', C) \) corresponds to the Prym variety of the covering \( C' \to C \). The result of Theorem 2.1 and the examination of the monodromy on the torsion line bundle imply the following proposition.

**Proposition 2.8.** Let \( C, D, C' D' \) be as above. Assume that the covering \( C' \to C \) and \( D' \to D \) are non-trivial. There are exactly 4 irreducible Hodge substructures of \( H^2(C' \times D') \) with non trivial (2,0) part: \( H^1(C) \otimes H^1(D) \), \( P(C', C) \otimes H^1(D) \), \( H^1(C) \otimes P(D', D) \) and \( P(C', C) \otimes P(D', D) \).

We remark that
\[
(P(C', C) \otimes P(D', D))^2 = 0, \quad 0 < i, j < p.
\]

2.2. **Proof of Theorem 2.1.** Using Theorem 2.7 we can prove easily our main result:

**Proof.** We assume by contradiction that there is a dominant rational map
\[
f : C \times D \twoheadrightarrow S
\]
where \( C \) and \( D \) are very general curves and \( H_1(S, \mathbb{Z}) \neq 0 \). Since \( H_1(S, \mathbb{Z}) \) is a finite generated abelian group we can find a prime \( p \), and a surjection \( H_1(S, \mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z} \). Therefore we have a surjection \( \psi : \pi_1(S) \to \mathbb{Z}/p\mathbb{Z} \). Let \( S' \to S \) be the étale covering associated to \( \ker(\psi) \). By composing with \( f \) (after a suitable resolution) we get a surjection
\[
\psi' : \pi_1(C \times D) = \pi_1(C) \times \pi_1(D) \to \pi(S) \to \mathbb{Z}/p\mathbb{Z}.
\]
This gives two maps \( \pi_1(C) \to \mathbb{Z}/p\mathbb{Z} \) and \( \pi_1(D) \to \mathbb{Z}/p\mathbb{Z} \). They produce two étale covering \( C' \to C \) and \( D' \to D \) (one of the two coverings can be trivial).

By the construction we have then a dominant rational map
\[
C' \times D' \xrightarrow{f'} S'.
\]
We remark that \( \text{kod}(S') \) is 0 or 1 since \( K_{S'}^2 = 0 \). Since \( \chi(O_S) = 1 \) we have \( \chi(O_{S'}) = p \) and therefore \( \dim H^{2,0}(S') = p_g \geq p - 1 > 0 \).

Then
\[
\Lambda = f'^*(H^2(S'))
\]
is a Hodge substructure of \( H^2(C' \times D') \) with non zero (2,0) part, \( \Lambda^{2,0} \neq 0 \). By Theorem 2.7 we get that the image \( \kappa : C' \times D' \to |\Lambda^{2,0}| \) has dimension 2. On the other hand \( \Lambda^{2,0} = f'^*(H^{2,0}(S')) \), then \( \kappa \) must factorize: \( \kappa = f' \circ \gamma \) where \( \gamma \) is the canonical map of \( S' \). Since the Kodaira dimension of \( S' \) is \( \leq 1 \) we get a contradiction. □

3. **Dominant Map**

Combining Theorem 2.1 with the result in [3] we will proof our main theorem.

**Theorem 3.1.** Let \( C \) and \( D \) be very general curves of genus \( g_C \geq 7 \) and \( g_D \geq 3 \) respectively. Then there is no dominant rational map of degree \( > 1 \) from \( C \times D \) to \( S \) where \( S \) is a smooth projective surface of \( \text{kod}(S) \geq 0 \).
Proposition 4.1. Assume that $H$ is an irreducible component of the Kuranishi family of $\kappa$ and it is shown that there is no dominant rational map from $S$. So we have only to consider the case where $\text{Kod}(\kappa) = 1$, $p_g = q = 0$ and $\text{Pic}(S)$ is torsion free. Remark 3.5 in [7] shows that there is no birational map form a general curve of genus $C$ of genus $g\geq 7$ to $S$. This implies the map $F$ restricted to the fiber $C \times t$ where $t \in D$ is general cannot be birational. Therefore $F(C \times t)$ is a rational curve since $C$ is very general. Therefore $S$ is a ruled surface, this gives a contradiction. 

4. Curves on an elliptic surface and symmetric products

4.1. Curves on an elliptic surface. In this subsection we will slightly improve the result of [7] on deformation on curves on elliptic surface. We will give an application of this result in the next subsection.

Let $\pi: S \to \mathbb{P}^1$ be an elliptic surface (relatively minimal) with $p_g = q = 0$ and of Kodaira dimension $1$. Let $C$ be a smooth projective curve of genus $g > 1$. Let 

$$\kappa: C \to S$$

be a birational immersion, that is the map $C \to \kappa(C)$ is birational. Let $U$ be an irreducible component of the Kuranishi family of $\kappa$.

**Proposition 4.1.** Assume that $C$ is neither a hyperelliptic nor a trigonal curve of genus $g > 2$. Then $\dim U \leq g - 2$.

**Proof.** We assume that $C$ is a general curve in the family. Since $K_S$ is nef, non-trivial and semi-ample, we have that $\deg \kappa^*(K_S) \geq 1$. The differential of $\kappa$ induces an exact sequence 

$$0 \to T_C \to \kappa^* T_S \to N \to 0.$$ 

Let $N_{\text{tors}}$ be the torsion of the normal bundle $N$ and $N' = N/N_{\text{tors}}$ be the quotient. This induces an exact sequence on $C$: 

$$0 \to T_C \to T' \to N' \to 0.$$ 

We have (\cite{1}, Chapter XXI in [2]) $\dim(U) \leq h^0(N')$, $N' \subset K_C \otimes \kappa^* K_S^{-1}$. Since the curve $C$ is not hyperelliptic, we have to consider only the case where $\deg \kappa^*(K_S) = 1$. By examination of the multiple fibers we see that $K_S$ is numerically a multiple of line bundle $\lambda$, $K_S = \rho \lambda$ where $\rho \geq 1$. Then $\deg \kappa^*(K_S) = 1$ implies $\rho = 1$, this is possible only in the following two cases, both with only two multiple fibers (cf. Remark 3.2 in [7]):

1. the case $(2, 3)$,
2. the case $(2, 4)$.

We note now that $h^0(\kappa^*(K_S)) = h^1(K_C \otimes \kappa^* K_S^{-1}) = 0$ implies $h^0(N') \leq h^0(K_C \otimes \kappa^* K_S^{-1}) = g - 2$. In the $(2, 3)$ case $S = S_{2, 3}$ has multiple fibers $F_1$ and $F_2$ where $2F_1 = F = 3F_2$. Then $F_1 = 3K_S$ and $F_2 = 2K_S$, and it implies $K_S = F_1 - F_2$. If $h^0(\kappa^*(K_S)) = 1$ then $\kappa^*(K_S) = \mathcal{O}_C(P)$ $P \in C$. On the other hand since $\kappa(C)$ is not contained in the fibers $\kappa^*(F_1) = \mathcal{O}_C(G + E + F)$ and $\kappa^*(F_2) = \mathcal{O}_C(R + S)$ where $G, E, F, R, S$ are point of $C$. Therefore
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$R + S + P \equiv G + E + F$, but since $C$ is not trigonal we would have then

$\{R, S, P\} = \{G, E, F\}$, which is impossible since the two fibers are distinct.

In the second case we have $K_S = F - F_1 - F_2 = F_1 - F_2 = 3F_2 - F_1$,

$2F_1 = F = 4F_2$. If $\kappa^*(K_S) = \mathcal{O}_C(P)$ we have $\kappa^*(F_1) = \mathcal{O}_C(A + B)$ and

$\kappa^*(F_2) = \mathcal{O}_C(Q)$. Then $A + B \equiv P + Q$, which is also impossible. □

Example 4.2. We note that the above result is sharp. Let us consider the case (2.3), that is $S = S_{2,3}$. The elliptic surface $S = S_{2,3}$ is simply connected with $p_g = q = 0$. [4]. We have $6K_S = F$ and $3K_S = F_1$ and $2K_S = F_2$. By

Poincaré duality the coset $H = \{\alpha \in H^2(S, \mathbb{Z}) : K_S \cdot \alpha = 1\}$ is not empty.

We notice that

(1) if $\alpha \in H$ then $\alpha^2 = 2s + 1$ is odd,

(2) if $\alpha \in H$ then $\alpha + K_S \in H$,

(3) if $\alpha \in H$ then $(\alpha + K_S)^2 = \alpha^2 + 2$.

We choose $\alpha \in H$ such that $\alpha^2 = -3$. And we consider the line bundle

$L_r = \alpha \otimes rK_S$. Then $L_r^2 = 2r - 3$, and we have from the Riemann Roch theorem $\chi(L_r) = r - 1$ and $L_r(L_r + K_S) = 2(r - 1)$.

By the duality $h^2(L_r) = h^0(K_S - L_r)$ we have $K_S - L_r = (1 - r)K_S - \alpha$, $-\alpha \cdot F = -6 < 0$, and $F$ is semi-ample. It follows that for $g > 1 L_g$ has a

global section of arithmetic genus $g$ and the family depends upon $h^0(L_g) - 1 \geq g - 2$ parameters.

4.2. Symmetric product. This subsection is devoted to prove the following.

Proposition 4.3. Let $C$ be a very general curve of genus $g \geq 6$. Let $C_2$ be

the 2-symmetric product of $C$. Let $S$ be a surface of Kodaira dimension 1.

Then there is no dominant rational map $f : C_2 \dasharrow S$.

Proof. Assume that $f : C_2 \dasharrow S$ is a dominant rational map and let

$f^0 : X \rightarrow S$ be a regular morphism, where $f^0$ is a resolution of $f$. By

the irreducibility of the transcendental Hodge structure of $H^2(C_2)$ (see also

Subsection 2.1) and the fact that the canonical map is birational for $C_2$ we

get $p_g(S) = q(S) = 0$. Since $\text{kod}(S) = 1$ we may assume that $\pi : S \rightarrow \mathbb{P}^1$ is a

relatively minimal elliptic fibration.

We claim that the map on the fundamental groups $f^* : \pi_1(X) \rightarrow \pi_1(S)$

is surjective. In fact we know that the image $\Gamma$ of $f^*$ has a finite index. Let

$\pi : S' \rightarrow S$ be the associated covering of degree $m = [\pi_1(S) : \Gamma]$. By

the construction we can find a lift $f'' : X \rightarrow S'$ such that $\pi \circ f'' = f'$. By

the proportionality we get $\chi(O_{S'}) = m\chi(O_S) = m$. Therefore $p_g \geq m - 1$. But

again we have $p_g(S') = 0$ and then $m = 1$.

Since $\pi_1(X) \cong \pi_1(C_2) \cong H_1(C, \mathbb{Z})$, $\pi_1(S)$ is abelian. And therefore $\pi : S \rightarrow \mathbb{P}^1$ must have exactly two multiple fibers since it is non-rational [4].

We have $H^2(T_S) = 0$, and the deformation of $S$ depend upon 10 parameters

(see Proposition 2.3 in [4]). Let $C$ be a very general curve . For any point

$P$ of $C$, let $c_P : C \rightarrow C_2$ be the embedding (it is called a coordinate curve) by

$c_P(Q) = P + Q$.

Consider the composition map $f_P = f \circ c_P : C \rightarrow S$. Then for general $P$, $f_P$

is birational onto its image: Since $C$ is a very general curve in the moduli
space of curves of genus $g$, $C$ can be mapped non-trivially only on $\mathbb{P}^1$, $f_P$ is dominant and $S$ is not ruled. It follows that we have a family of birational immersions of dimension $3g - 3 + 1$ (the dimension of the moduli plus the one due to the coordinates curves $C_P \{P \in C\}$ in $C_2$). We remark that $C$ is not trigonal. Since the deformations of our surfaces depend on 10 moduli, in some fixed surface $S$ we must find at least a family 3-dimensional birational immersion. Therefore by Proposition 4.1 we have

$$3g - 2 - 10 \leq g - 2$$

$2g \leq 10$ we get $g \leq 5$. $\Box$

**Remark 4.4.** When $g = 5$ the above example 4.2 shows that there is a three dimensional family in $S_{2,3}$. Then there is a family of $3g - 2 = 13$ dimensional deformations into the family of elliptic surfaces $S_{2,3}$. It could be however that on a general $S_{2,3}$ we have no isotrivial deformation, that is deformation of constant moduli.

For the maps between $C_2$ and surfaces of general type, we have the inequality $3g - 2 - 19 \leq g - 2$ [6]: that is $g < 10$. The case $g = 9, 8$ can be also excluded. When $\text{kod}(S) = 0$ with $p_g = q = 0$, the case of Enriques surfaces can be also excluded by the dimensional count as above or by using the method in the proof of Theorem 2.1.

In conclusion we get the following.

**Theorem 4.5.** Let $C$ be general curve of genus $\geq 10$ and let $f : C_2 \dasharrow S$ be a dominant rational map of degree $\geq 2$. Then $S$ is a rational surface.

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