FURTHER REMARKS ON DERIVED CATEGORIES OF ALGEBRAIC STACKS

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Abstract. Let $X$ be an algebraic stack with quasi-affine diagonal of finite type over a field $k$ of characteristic 0. We extend the well-known equivalence $\mathbf{D}^+(\mathbf{QCoh}(X)) \simeq \mathbf{D}^b_{qc}(X)$ to unbounded derived categories. We also prove that if $X$ is smooth over $k$, then $\mathbf{D}^b_{qc}(X)$ is compactly generated. We accomplish the former using the descendable algebras of Mathew. We also establish related results in positive and mixed characteristics.

1. Introduction

Let $X$ be a quasi-compact and quasi-separated scheme. Then

(a) $\mathbf{D}_{qc}(X)$, the unbounded derived category of $\mathcal{O}_X$-modules whose cohomology sheaves belong to the abelian category of quasi-coherent sheaves $\mathbf{QCoh}(X)$, is compactly generated by perfect complexes; and

(b) if $X$ has affine diagonal or is noetherian, then the functor $\Psi_X : \mathbf{D}(\mathbf{QCoh}(X)) \to \mathbf{D}(X)$ from the unbounded derived category of quasi-coherent $\mathcal{O}_X$-modules to the unbounded derived category of $\mathcal{O}_X$-modules is fully faithful with image $\mathbf{D}^b_{qc}(X)$.

In this generality, (a) was established in [BB03]—also see [Nee96]; and (b) is essentially due to [ATJLL97] (separated case) and [Har66] (noetherian case)—also see [Stacks, Tags 08DB & 09T4].

Now let $X$ be a quasi-compact and quasi-separated algebraic stack. In reasonable situations (e.g., affine diagonal), there is a bounded below equivalence $\mathbf{D}^+(\mathbf{QCoh}(X)) \simeq \mathbf{D}^b_{qc}(X)$ [HNR19, Thm. C.1]. In these reasonable situations, if (a) holds, then so does (b) [HNR19, Thm. 1.2]. For many stacks in positive characteristic, however (e.g., $X = BG_a$), the functor $\mathbf{D}(\mathbf{QCoh}(X)) \to \mathbf{D}^b_{qc}(X)$ fails to be full and $\mathbf{D}^b_{qc}(X)$ is not compactly generated [HNR19 Thms. 1.1 & 1.3].

Local quotient descriptions of algebraic stacks are a natural way to establish (a) (and consequently (b)) for algebraic stacks. Such descriptions are not easy to come by. Indeed, they usually rely on Sumihiro-type results for quotient stacks [Sum74, Bri15] or abstract criteria such as [AHR20, AHR19]; also see [Lie04, §2.2.4].

The goal of this article is to establish some useful instances of (a) and (b) that do not rely on the existence of local quotient descriptions. In the smooth case, for example, we can prove the following result.

Theorem A. Let $k$ be a field of characteristic 0. Let $X$ be an algebraic stack that is quasi-compact with quasi-affine diagonal. If $X$ is smooth over $\text{Spec} \, k$, then

(1) $\mathbf{D}(\mathbf{QCoh}(X)) \simeq \mathbf{D}_{qc}(X)$;

(2) $\mathbf{D}^b(\mathbf{Coh}(X)) \simeq \text{Perf}(X) \simeq \mathbf{D}_{qc}(X)^c$; and

(3) $\mathbf{D}_{qc}(X)$ is compactly generated by perfect complexes.
We have not seen Theorem A in the literature before, though suspect it to be folklore to experts (also see Theorem 2.1 for refinements). We also develop a new method to prove (b) which avoids (a) and use it to establish the following.

**Theorem B.** Let \( k \) be a field of characteristic 0. Let \( X \) be a noetherian algebraic stack over \( \text{Spec} \, k \). If \( X \) has quasi-affine diagonal, then \( \mathbb{D}(\text{QCoh}(X)) \simeq \mathbb{D}_{qc}(X) \).

A particularly useful Corollary of Theorem B is the following result. This was previously only known under étale-local linearizability of the action (e.g., [Sum74, Bri15, HNR19]).

**Corollary.** Let \( k \) be a field of characteristic 0. If \( U \) is a noetherian Deligne–Mumford stack with an action of an affine algebraic group \( G \) over \( \text{Spec} \, k \), then \( \mathbb{D}(\text{QCoh}^G(U)) \simeq \mathbb{D}_{qc}([U/G]) \).

Theorem B admits various generalizations (see §8), which are essentially of maximal generality. For example, we can also prove the following non-noetherian and mixed characteristic variant.

**Theorem C.** Let \( X \) be a quasi-compact algebraic stack with affine diagonal. If \( X \) has finite cohomological dimension, then \( \mathbb{D}(\text{QCoh}(X)) \simeq \mathbb{D}_{qc}(X) \).

Recall that a quasi-compact and quasi-separated algebraic stack \( X \) has cohomological dimension \( \leq d \) if \( H^i(X, F) = 0 \) for all \( i > d \) and quasi-coherent sheaves \( F \) (see [HR17, §2] and [HR15 Thms. B & C]). Algebraic spaces and algebraic \( \mathbb{Q} \)-stacks with affine stabilizers have finite cohomological dimension. In positive characteristic, having finite cohomological dimension and affine stabilizers is equivalent to having linearly reductive stabilizers. In mixed characteristic it is more complicated, but these criteria persist for stacks with finitely presented inertia (e.g., noetherian). Some of these subtleties are discussed in detail in [AHR19, App. A].

We will give two proofs of Theorem A. The first proof will be a direct calculation. The second proof will employ derived functors of small products in Grothendieck abelian categories. This approach will also be used to establish Theorem B. This idea for schemes goes back to [Kel98, App. A] and for stacks to [Lie04, HX09, Kri09].

In order to understand derived functors of small products for algebraic stacks, we use the *descendable* morphisms of Mathew [Mat16] (see §7). A key technique that we develop is cohomological calculations of \( R\text{Hom}_{\mathcal{O}_X}(M, N) \), where \( M \) and \( N \) are quasi-coherent \( \mathcal{O}_X \)-modules, and \( M \) is *countably* presented. In particular, \( R\text{Hom}_{\mathcal{O}_X}(M, N) \) is not necessarily quasi-coherent. These calculations are new, even for varieties of finite type over a field.

But we also provide a direct proof of the Corollary above without this general machinery (see §5). We also prove a comparison between quasi-coherent and lisse-étale cohomology (Theorem 8.3), which generalizes a well-known result on the bounded below derived category (e.g., [HNR19 Prop. 2.1]).

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2. **Direct proof of Theorem A**

An algebraic stack \( X \) is affine-pointed if for every field \( k \), every morphism \( x: \text{Spec} \, k \to X \) is affine. If \( X \) has quasi-affine diagonal, then it is affine-pointed [HR19, Lem. 4.7]. Surprisingly, it is an open question whether affine stabilizers implies affine-pointed [HR19 Qstn. 1.8].

We will prove the following mild refinement of Theorem A.

**Theorem 2.1.** Let \( X \) be a noetherian and affine-pointed algebraic stack of finite cohomological dimension. If \( X \) is locally regular of finite Krull dimension, then

\[ \mathbb{D}(\text{QCoh}(X)) \simeq \mathbb{D}_{qc}(X). \]
(1) $D(\text{QCoh}(X)) \simeq D_{\text{qc}}(X)$;
(2) $D^b(\text{Coh}(X)) \simeq \text{Perf}(X) \simeq D_{\text{qc}}(X)^c$;
(3) $D_{\text{qc}}(X)$ is compactly generated by perfect complexes; and
(4) There exists an integer $n$ such that for all $M \in \text{Coh}(X)$, $N \in D_{\text{qc}}(X)$, and $r \in \mathbb{Z}$ there is a quasi-isomorphism:

$$\tau_{-r}^\leq R\text{Hom}_{\mathcal{O}_X}(M, N) \simeq \tau_{-r}^\leq R\text{Hom}_{\mathcal{O}_X}(M, \tau_{-r-n}^\leq N).$$

Proof. We first prove (4). Now since $X$ is locally regular of finite Krull dimension, there exists an integer $d \geq 0$ such that for all $M \in \text{Coh}(X)$, $R\text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X) \in D^b_{\text{Coh}}(X) \subseteq \text{Perf}(X)$, and $X$ has finite cohomological dimension, so there is an integer $t \geq 0$ such that

$$\tau_{-r}^\leq R\Gamma(X, L) \simeq \tau_{-r}^\leq R\Gamma(X, \tau_{-r-t}^\leq L)$$

for all integers $r, t \geq t$, and $L \in D_{\text{qc}}(X)$ [HR17 Thm. 2.6(i)]. Hence,

$$\tau_{-r}^\leq R\text{Hom}_{\mathcal{O}_X}(M, N) \simeq \tau_{-r}^\leq R\Gamma(X, R\text{Hom}_{\mathcal{O}_X}(M, N))$$

$$\simeq \tau_{-r}^\leq R\Gamma(X, \tau_{-r-t}^\leq (R\text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X) \otimes_{\mathcal{O}_X}^L N))$$

$$\simeq \tau_{-r}^\leq R\Gamma(X, \tau_{-r-t}^\leq (R\text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X) \otimes_{\mathcal{O}_X}^L \tau_{-r-t-d}^\leq N))$$

$$\simeq \tau_{-r}^\leq R\Gamma(X, R\text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X) \otimes_{\mathcal{O}_X}^L \tau_{-r-t-d}^\leq N)$$

$$\simeq \tau_{-r}^\leq R\text{Hom}_{\mathcal{O}_X}(M, \tau_{-r-t-d}^\leq N).$$

Taking $n = t + d$ gives the claim.

Since $X$ is of finite cohomological dimension, $\text{Perf}(X) \simeq D_{\text{qc}}(X)^c$ [HR17 Rem. 4.6]. Also $X$ is locally regular, so $D^b_{\text{Coh}}(X) \simeq \text{Perf}(X)$; hence, if $M \in \text{Coh}(X)$, then $M[n] \in D_{\text{qc}}(X)^c$ for all $n \in \mathbb{Z}$. By [HNR19 Thm. C.1], $D^b(\text{QCoh}(X)) \simeq D^b_{\text{Coh}}(X)$. In particular, $D^b_{\text{Coh}(X)}(\text{QCoh}(X)) \simeq D^b_{\text{Coh}}(X)$. The equivalence $D^b(\text{Coh}(X)) \simeq D^b_{\text{Coh}(X)}(\text{QCoh}(X))$ follows from a standard argument. Putting these all together, we have $D^b(\text{Coh}(X)) \simeq \text{Perf}(X) \simeq D_{\text{qc}}(X)^c$. This proves (2).

We will now prove that the set $\text{Coh}(X)$ compactly generates $D_{\text{qc}}(X)$. This will establish (3). Let $N \in D_{\text{qc}}(X)$ and assume that $\text{H}^r(N) \neq 0$. Now $\tau_{-r-n}^\leq N \in D^+_{\text{Coh}}(X)$, so it is quasi-isomorphic to a bounded below complex $Q^* = (\cdots \to Q^d \xrightarrow{\partial^d} Q^{d+1} \to \cdots)$ of quasi-coherent sheaves on $X$ [HNR19 Thm. C.1]. By assumption, $\text{H}^r(Q^*) \neq 0$, so there is a non-zero surjection of quasi-coherent sheaves $\text{ker}(\partial^d) \rightarrow \text{H}^r(Q^*)$. Now choose a coherent $\mathcal{O}_X$-module $M$ together with a morphism $M \rightarrow \ker(\partial^d)$ such that the composition $M \rightarrow \ker(\partial^d) \rightarrow \text{H}^r(Q^*)$ is non-zero. Indeed, every quasi-coherent sheaf on $X$ is a direct limit of its coherent subsheaves [LM99 Prop. 15.4] (the separated diagonal hypothesis can be safely ignored). It follows immediately that there is a non-zero morphism $M[r] \rightarrow \tau_{-r-n}^\leq N$ in $D_{\text{qc}}(X)$. By (1), we have a non-zero morphism $M[r] \rightarrow N$ in $D_{\text{qc}}(X)$. The claim follows.

Finally, [HNR19 Thm. 1.2] informs us that (3) implies (1). □

Proof of Theorem [4] Since $X$ has quasi-affine diagonal, it is affine pointed [HR19 Lem. 4.7]; in particular, it has affine stabilizers. Since $k$ has characteristic 0, it now follows that $X$ has finite cohomological dimension [HR19 Thms. B & C]. Now $X$ is smooth and of finite type, so there is a smooth covering $p : \text{Spec} A \rightarrow X$, where $\text{Spec} A$ is smooth over $\text{Spec} k$. It follows immediately that $\text{Spec} A$ is regular of finite Krull dimension. The result now follows from Theorem [2.1] □
3. Derived functors of small products in abelian categories

Let $\mathcal{A}$ be an abelian category. Let $\Lambda$ be a set. Suppose that $\Lambda$-indexed products exist in $\mathcal{A}$. It is convenient to view the product as a functor

$$\prod_{\lambda \in \Lambda} : \mathcal{A}^\Lambda \to \mathcal{A}$$

The product category $\mathcal{A}^\Lambda$ is naturally abelian and the functor $\prod_{\lambda \in \Lambda}$ is readily seen to be left-exact.

If $\mathcal{A}$ has enough injectives, then so too does the product category $\mathcal{A}^\Lambda$. Hence, we can consider the right-derived functors of $\prod_{\lambda \in \Lambda}$, which we denote via $\prod_{\lambda \in \Lambda}^{(p)}$. We say that $\mathcal{A}$ satisfies $\text{AB4}^*_{n}(\Lambda)$ if $\prod_{\lambda \in \Lambda}^{(p)} \equiv 0$ for all $p > n$. If $\text{AB4}^*_{n}(\Lambda)$ is satisfied for all sets $\Lambda$, then we say that $\mathcal{A}$ satisfies $\text{AB4}^*_{n}$. This notion was introduced and studied in [Roo06]. Just as in the recent work [Ant18], we will be principally concerned with the condition $\text{AB4}^*_{n}(\omega)$. A number of results related to those of this section are also established in [HX09].

Example 3.1. It is a well-known theorem that Grothendieck abelian categories admit small products [Stacks, Tag 07D8] and have enough injectives. We can also consider the unbounded derived category $\mathcal{D}(A)$ of $\mathcal{A}$. Since $\mathcal{A}$ is Grothendieck abelian, $\mathcal{D}(A)$ also has small products. These are formed by taking termwise products of K-injective resolutions [Stacks, Tag 07D9]. Let $\Lambda$ be a set. If $\{A_\lambda\}_{\lambda \in \Lambda}$ is a set of objects of $\mathcal{A}$, then

$$H^p\left(\prod_{\lambda \in \Lambda} A_\lambda[0]\right) \cong \begin{cases} 0 & p < 0; \\ \prod_{\lambda \in \Lambda} A_\lambda & p \geq 0. \end{cases}$$

Furthermore, let $n \geq 0$. Assume that $\mathcal{A}$ satisfies $\text{AB4}^*_{n}(\Lambda)$. If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a set of objects of $\mathcal{D}(A)$, then for every $r \in \mathbb{Z}$ we have

$$(3.2) \quad \tau^{\geq r} \prod_{\lambda \in \Lambda} N_\lambda \cong \tau^{\geq r} \prod_{\lambda \in \Lambda} \tau^{\geq r-n} N_\lambda.$$  

This is similar to [Stacks, Tag 07K7], but we reproduce the argument here. Also see [Ant18, Prop. 8.14]. Let $N_\lambda \to I^\bullet_\lambda$ be a quasi-isomorphism to a K-injective complex of $\mathcal{A}$-injectives. Consider the exact sequence of complexes:

$$0 \to L^r_\lambda[-r + n] \to \sigma^{\geq r-n-1} I^\bullet_\lambda \to \tau^{\geq r-n-1} I^\bullet_\lambda \to 0,$$

where $\sigma$ denotes the brutal truncation and $L^r_\lambda \in \mathcal{A}$. Taking products in $\mathcal{D}(A)$ we obtain a distinguished triangle:

$$\prod_{\lambda \in \Lambda} L^r_\lambda[-r + n] \to \prod_{\lambda \in \Lambda} \sigma^{\geq r-n-1} I^\bullet_\lambda \to \prod_{\lambda \in \Lambda} \tau^{\geq r-n-1} I^\bullet_\lambda \to \prod_{\lambda \in \Lambda} L^r_\lambda[-r + n + 1]$$

Now

$$\prod_{\lambda \in \Lambda} \sigma^{\geq r-n-1} I^\bullet_\lambda \cong \sigma^{\geq r-n-1} \prod_{\lambda \in \Lambda} I^\bullet_\lambda.$$  

This implies that

$$\tau^{\geq r-n} \prod_{\lambda \in \Lambda} \sigma^{\geq r-n-1} I^\bullet_\lambda \cong \tau^{\geq r-n} \prod_{\lambda \in \Lambda} N_\lambda.$$  

But $\mathcal{A}$ satisfies $\text{AB4}^*_{n}(\Lambda)$, so

$$\tau^{\geq r} \prod_{\lambda \in \Lambda} L^r_\lambda[-r + n] \cong \tau^{\geq r} \left(\prod_{\lambda \in \Lambda} L^r_\lambda[0][-r + n]\right) \cong 0.$$
It follows that
\[ \tau^{>r} \prod_{\lambda \in A} N_\lambda \simeq \tau^{>r} \prod_{\lambda \in A} \tau^{>r-n-1} N_\lambda. \]

We also have a distinguished triangle in \( D(A) \):
\[ \prod_{\lambda \in A} \mathcal{H}^{>n}(N_\lambda)[n - r] \to \prod_{\lambda \in A} \tau^{>r-n} N_\lambda \to \prod_{\lambda \in A} \tau^{>r-n-1} N_\lambda \to \prod_{\lambda \in A} \mathcal{H}^{>n}(N_\lambda)[n - r + 1]. \]

As before, \( \tau^{>r} \prod_{\lambda \in A} \mathcal{H}^{>n}(N_\lambda)[n - r] \simeq 0 \). It follows now that
\[ \tau^{>r} \prod_{\lambda \in A} N_\lambda \simeq \tau^{>r} \prod_{\lambda \in A} \tau^{>r-n} N_\lambda. \]

**Example 3.3.** Let \( X \) be an algebraic stack. Then the abelian category \( \text{QCoh}(X) \) is Grothendieck abelian. If \( X \) is noetherian, this is reasonably straightforward, and follows immediately from [LMB] Prop. 15.4. In general, this is [Stacks, Tag 0781].

**Example 3.4.** Let \( \mathcal{C} \) be a site. Let \( \mathcal{O}_\mathcal{C} \) be a sheaf of rings on \( \mathcal{C} \). Then \( \text{Mod}(\mathcal{C}, \mathcal{O}_\mathcal{C}) \) is Grothendieck abelian. Let \( u : \mathcal{C} \to \text{Sets} \) be a point of \( \mathcal{C} \). If \( \{C_\lambda\}_{\lambda \in A} \) is a set of sheaves of \( \mathcal{O}_\mathcal{C} \)-modules, then for all \( p \geq 0 \) there is a natural isomorphism
\[ \left( \prod_{\lambda \in A} C_\lambda \right)_u \simeq \lim_{(V,v)} \prod_{\lambda \in A} \mathcal{H}^p(V, C_\lambda), \]
where the limit is over a cofinal system of neighbourhoods \( (V, v) \) of \( u \). This follows by making the natural modifications to the argument given in [Roo06] Prop. 1.6.

**Example 3.5.** Let \( F : A \to A' : G \) be an adjoint and exact pair of functors between Grothendieck abelian categories. Since \( F \) is exact, \( RG : D(A') \to D(A) \) has a left adjoint, so preserves products. Hence, if \( \{M_\lambda\}_{\lambda \in A} \) is a set objects of \( A \), then
\[ RG(\prod_{\lambda \in A} M_\lambda[0]) \simeq \prod_{\lambda \in A} RG(M_\lambda[0]). \]

But \( \prod_{\lambda \in A} \) preserves injectives, so \( \prod_{\lambda \in A}^{(p)} G(M_\lambda) \simeq G(\prod_{\lambda \in A}^{(p)} M_\lambda) \). In particular,
\begin{enumerate}
  \item if \( \prod_{\lambda \in A} G(M_\lambda) \neq 0 \) for some \( n \), then \( A' \) does not satisfy AB4*\( n \)(\( A \)); and
  \item if \( G \) is conservative and \( A \) satisfies AB4*\( n \)(\( A \)) for some \( n \), then the same is true of \( A' \).
\end{enumerate}

We now give some key examples. While we have not seen these in the literature before, we expect at least some of them to have been known in some cases to experts.

**Example 3.6.** Let \( X \) be an affine-pointed algebraic stack of finite cohomological dimension. If \( X \) is locally regular of finite Krull dimension, then \( \text{QCoh}(X) \) is Grothendieck abelian. Let \( M \in \text{Coh}(X) \). Observe that \( D^+(\text{QCoh}(X)) \) is conservative and \( A \) satisfies AB4*\( n \)(\( A \)) for some \( n \), then the same is true of \( A' \).

**Example 3.7.** Let \( X \) be a quasi-compact algebraic stack with affine diagonal. If \( X \) has the compact resolution property [HR17] §7, then \( \text{QCoh}(X) \) satisfies AB4*\( n \) for some \( n > 0 \). For example, algebraic stacks of finite cohomological dimension with the resolution property (i.e., every finite type quasi-coherent sheaf is a quotient of a finite rank vector bundle) have the compact resolution property. The argument given in Example 3.6 works with only minor changes, see [HR17] Lem. 7.6.
Example 3.8. Let \( k \) be a field of characteristic \( p > 0 \). Then \( \text{QCoh}(B\mathbb{G}_{a,k}) \) does not satisfy \( AB4^*n(\omega) \) for any \( n > 0 \) [Nee11]. This holds, more generally, for poorly stabilized algebraic stacks [HNR19 §4]. This condition is equivalent to having a point of characteristic \( p > 0 \) and a subgroup of the stabilizer at that point isomorphic to \( \mathbb{G}_a \) (e.g., \( B\mathbb{G}_{a,\mathbb{A}} \) but not \( B\mathbb{G}_{a,\mathbb{R}} \)).

Example 3.9. If \( X = (|X|, \mathcal{O}_X) \) is a ringed space, where \( |X| \) is spectral (e.g., noetherian) of Krull dimension \( d \), then it is sufficient to prove that for every open neighborhood \( V \) of \( x \) there is a cofinal system of neighborhoods \( V \) of \( x \) with \( H^q(V, F) = 0 \) for all sheaves of \( \mathcal{O}_X \)-modules \( F \). Now \( |X| \) has Krull dimension \( d \), so \( V \) has Krull dimension \( < d \) for every open \( V \subseteq |X| \). It follows that \( H^q(V, F) = 0 \) for all \( q > d \) [Sch92] (also see [Stacks, Tag 0A3G]) for every quasi-compact open \( V \subseteq |X| \). But \( |X| \) is spectral, so the quasi-compact open subsets form a basis, and we have the claim.

Example 3.10. If \( X \) is a quasi-compact and quasi-separated Deligne–Mumford stack of Krull dimension \( d \) (e.g., noetherian), then \( \text{Mod}(X) \) satisfies \( AB4^*d \). To see this, choose an \( \acute{e}tale \) cover \( p: U \to X \) by a quasi-compact and quasi-separated scheme \( U \) of Krull dimension \( d \). Then we have an adjoint pair of functors:

\[
p_1: \text{Mod}(U_{\acute{e}t}) \leftrightarrows \text{Mod}(X_{\acute{e}t}) \colon p^{-1},
\]

where both \( p_1 \) and \( p^{-1} \) are exact, and \( p^{-1} \) is conservative. By Example 3.5, we are reduced to the situation where \( X = U \) is a scheme with \( |X| \) spectral of Krull dimension \( d \). Now argue just as in Example 3.9 but this time use quasi-compact \( \acute{e}tale \) morphisms \( V \to X \) as the basis.

Example 3.11. Let \( X \) be a quasi-compact and quasi-separated algebraic stack and let \( \Lambda \) be a set. Let \( p: X' \to X \) be a flat and affine morphism.

1. If \( \text{QCoh}(X) \) satisfies \( AB4^*n(\Lambda) \), then \( \text{QCoh}(X') \) satisfies \( AB4^*n(\Lambda) \).

2. If \( \text{QCoh}(X') \) satisfies \( AB4^*n(\Lambda) \) and \( p \) is finite, finitely presented, and surjective, then \( \text{QCoh}(X) \) satisfies \( AB4^*n(\Lambda) \).

For (1): we have the adjoint pair \( p^*: \text{QCoh}(X) \leftrightarrows \text{QCoh}(X'): p_* \), where \( p^* \) is exact and \( p_* \) is conservative (\( p \) is affine). By Example 3.5, the claim follows.

Similarly, for (2), we have an adjoint pair \( p_*: \text{QCoh}(X') \leftrightarrows \text{QCoh}(X): p^* \), where \( p_* \) is exact and \( p^* \) is exact and conservative [HR17 Cor. 4.15]. The claim follows from Example 3.5 again.

Example 3.12. A variant of Example 3.11 is the following. Let \( j: U \subseteq X \) be a quasi-compact open immersion of algebraic stacks. Assume that \( X \) is quasi-compact with affine diagonal or noetherian and affine-pointed. If \( \text{QCoh}(X) \) is \( AB4^*n(\Lambda) \), then so is \( \text{QCoh}(U) \). Indeed, let \( \{N_\lambda\}_{\lambda \in \Lambda} \) be a set of quasi-coherent sheaves on \( U \).

By [HNR19 Prop. 2.1] it follows immediately that there is a quasi-isomorphism in \( \mathcal{D}(\text{QCoh}(U)) \):

\[
\prod_{\lambda \in \Lambda} N_\lambda \simeq j^* \mathbb{R}j_\ast \text{QCoh}_+, (\prod_{\lambda \in \Lambda} N_\lambda) \simeq j^* (\prod_{\lambda \in \Lambda} \mathbb{R}j_\ast \text{QCoh}_+, N_\lambda).
\]

The right hand side is bounded above, so the claim follows.

Example 3.13. Consider a cartesian square of algebraic stacks:

\[
\begin{array}{ccc}
U' & \xrightarrow{j'} & X' \\
\downarrow f & & \downarrow f \\
U & \xrightarrow{j} & X
\end{array}
\]

that are all quasi-compact with affine diagonal or noetherian and affine-pointed. Assume that
(1) \( j \) is a quasi-compact open immersion with finitely presented complement \( i: Z \to X \); and

(2) \( f \) is concentrated, flat, and \( f_Z: X' \times_X Z \to Z \) is an isomorphism.

That is, the square is a flat Mayer–Vietoris square, in the sense of [HR22, Defn. 1.2]. This is satisfied, for example, when \( f \) is a representable étale neighbourhood of \( Z \) (i.e., \( f \) is étale and \( f_Z \) is an isomorphism). We now claim that if \( X' \) and \( U' \) satisfy AB4\(^*\)n(\( A \)) for some \( n \), then \( X \) satisfies AB4\(^*\)m(\( A \)) for some \( m \). To see this, we let \( \{ M_\lambda \}_{\lambda \in \Lambda} \) be a set of quasi-coherent sheaves on \( X \). We may form the Mayer–Vietoris triangle (see point (i) of the proof of [HR22, Thm. 4.4]) in \( D^+_q(X) \):

\[
\begin{array}{ccc}
M_\lambda & \longrightarrow & Rj_\ast j^\ast M_\lambda \oplus Rf_\ast f^\ast M_\lambda \\
& & \longrightarrow Rf_\ast f^\ast Rj_\ast j^\ast M_\lambda \\
& & \longrightarrow M_\lambda[1].
\end{array}
\]

By [HNR19] Prop. 2.1, this induces the following distinguished triangle in \( D(\text{QCoh}(X)) \):

\[
\begin{array}{ccc}
M_\lambda & \longrightarrow & Rj_{\text{QCoh}, \ast} j^\ast M_\lambda \oplus Rf_{\text{QCoh}, \ast} f^\ast M_\lambda \\
& & \longrightarrow Rf_{\text{QCoh}, \ast} f^\ast Rj_{\text{QCoh}, \ast} j^\ast M_\lambda \\
& & \longrightarrow M_\lambda[1].
\end{array}
\]

We now take the product over \( \lambda \in \Lambda \) in \( D(\text{QCoh}(X)) \) to obtain the following triangle:

\[
\prod_{\lambda \in \Lambda} M_\lambda \to \prod_{\lambda \in \Lambda} (Rj_{\text{QCoh}, \ast} j^\ast M_\lambda \oplus Rf_{\text{QCoh}, \ast} f^\ast M_\lambda) \to \prod_{\lambda \in \Lambda} Rf_{\text{QCoh}, \ast} f^\ast Rj_{\text{QCoh}, \ast} j^\ast M_\lambda \to \prod_{\lambda \in \Lambda} M_\lambda[1].
\]

Since pushforwards commute with products, we see that \( \prod_{\lambda \in \Lambda} M_\lambda \) belongs to a triangle whose other terms are

\[
\prod_{\lambda \in \Lambda} (Rj_{\text{QCoh}, \ast} j^\ast M_\lambda \oplus Rf_{\text{QCoh}, \ast} f^\ast M_\lambda) \simeq Rj_{\text{QCoh}, \ast} \left( \prod_{\lambda \in \Lambda} j^\ast M_\lambda \right) \oplus Rf_{\text{QCoh}, \ast} \left( \prod_{\lambda \in \Lambda} f^\ast M_\lambda \right)
\]

and

\[
\prod_{\lambda \in \Lambda} Rf_{\text{QCoh}, \ast} f^\ast Rj_{\text{QCoh}, \ast} j^\ast M_\lambda \simeq Rf_{\text{QCoh}, \ast} \prod_{\lambda \in \Lambda} f^\ast Rj_{\text{QCoh}, \ast} j^\ast M_\lambda.
\]

Since \( f \) and \( j \) are concentrated, they have cohomological dimension bounded by some \( r \). It follows that both of the above objects of \( D(\text{QCoh}(X)) \) have cohomological support bounded by \( r + n \). It follows immediately that \( \prod_{\lambda \in \Lambda} M_\lambda \) has cohomological support bounded by \( r + n \). Hence, \( \text{QCoh}(X) \) satisfies AB4\(^*\)(\( r + n \))(\( A \)).

Combining the examples above with the étale devissage results of [HR18], one easily obtains the following.

**Proposition 3.14.** Let \( X \) be an algebraic stack. Assume

(1) \( X \) has quasi-finite and affine diagonal; or

(2) \( X \) is noetherian with quasi-finite and separated diagonal; or

(3) \( X \) is a noetherian Deligne–Mumford \( \mathcal{Q} \)-stack.

Then \( \text{QCoh}(X) \) satisfies AB4\(^*\)n for some \( n > 0 \).

We conclude this section with a derived variant of Example 3.5.

**Example 3.15.** Let \( \mathcal{A} \) be a Grothendieck abelian and let \( \mathcal{M} \subseteq \mathcal{A} \) be a weak Serre subcategory that is closed under coproducts [HNR19 App. A]. Consider the functor

\[
\Psi: D(\mathcal{M}) \to D^+_\mathcal{M}(\mathcal{A}).
\]

Assume:

(1) \( \mathcal{M} \) is Grothendieck abelian; and

(2) \( \Psi^+: D^+(\mathcal{M}) \to D^+_\mathcal{M}(\mathcal{A}) \) is an equivalence.
Then $D_M(A)$ is well-generated [HNR19 Thm. A.3], so admits small products. Also, $\Psi$ preserves coproducts, so the functor $\Psi: D(M) \to D_M(A)$ admits a right adjoint $\Phi: D_M(A) \to D(M)$ [NeeD1 Prop. 1.20]. In particular, $\Phi$ preserves products. These conditions are satisfied for the inclusion $\text{QCoh}(X) \subseteq \text{Mod}(X)$, where $X$ is an algebraic stack that is quasi-compact with affine diagonal or is noetherian and affine-pointed [HNR19 Thm. C.1].

Now since $\Phi$ is right adjoint to a (right) $t$-exact functor, then $\Phi$ is left $t$-exact. In particular, $\Phi$ restricts to a functor $\Phi^+: D_M^+(A) \to D^+(M)$. Let $M \in D(M)$ and form the distinguished triangle:

$$ M \longrightarrow \Phi(\Psi(M)) \longrightarrow C(M) \longrightarrow M[1], $$

Now assume that $M$ is bounded below. Then so too is $C(M)$ and applying $\text{RHom}_M(C(M), -)$ to this triangle produces the triangle

$$ \text{RHom}_M(C(M), M) \to \text{RHom}_M(C(M), \Phi(\Psi(M))) \to \text{RHom}_M(C(M), C(M)) \to \text{RHom}_M(C(M), M)[1]. $$

By full faithfulness of $\Psi^+$, the composition

$$ \text{RHom}_M(C(M), M) \to \text{RHom}_M(C(M), \Phi(\Psi(M))) \cong \text{RHom}_{D_M(A)}(\Phi(C(M)), \Psi(M)) $$

is a quasi-isomorphism. Hence, $C(M) = 0$ and $M \to \Phi(\Psi(M))$ is a quasi-isomorphism for all $M \in D^+(M)$. To show that $\Psi$ and $\Phi$ restrict to adjoint equivalences on $D^+$, it remains to prove that $N \in D_M^+(A)$ implies that $\Phi(N) \neq 0$. But this is clear: $N \cong \Psi(M)$ for some $M \in D^+(M)$ and so $0 \simeq \Phi(N) \cong \Phi(\Psi(M)) \simeq M$. Hence, $N \simeq \Phi(M) \cong 0$.

Let $\Lambda$ be a set. If $M$ satisfies $\text{AB}4^* n(\Lambda)$ for some $n$, then for $-\infty < a \leq b < \infty$ and a subset $\{N_\lambda\}_{\lambda \in \Lambda} \subseteq D^{[a,b]}_M(A)$, we have that $\prod_{\lambda \in \Lambda} D_M^{[a,b]}(A) N_\lambda \in D^{[a,b]}_M(A)$. Indeed, $\prod_{\lambda \in \Lambda} N_\lambda \simeq \Phi(\prod_{\lambda \in \Lambda} M_\lambda) \cong \Phi(\prod_{\lambda \in \Lambda} \Phi(N_\lambda))$.

But $\Phi$ is $t$-exact on $D_M^+(A)$, so $\Phi(N_\lambda) \in D^{[a,b]}(M)$. The claim follows from Example 3.1. Conversely, if $\{M_\lambda\}_{\lambda \in \Lambda} \subseteq M$ and $\tau^{-n} [\prod_{\lambda \in \Lambda} \Phi(M_\lambda[0])] \simeq 0$, then $\prod_{\lambda \in \Lambda} M_\lambda = 0$ for all $s > n$ in $M$. In particular, if this is true for all families $\{M_\lambda\}_{\lambda \in \Lambda}$, then $M$ satisfies $\text{AB}4^* n(A)$. Indeed,

$$ \prod_{\lambda \in \Lambda} M_\lambda[0] \simeq \Phi(\prod_{\lambda \in \Lambda} \Phi(M_\lambda[0])) \simeq \Phi(\prod_{\lambda \in \Lambda} \Phi(M_\lambda[0])) = \Phi(\prod_{\lambda \in \Lambda} \Psi(M_\lambda[0])). $$

But $\prod_{\lambda \in \Lambda} \Psi(M_\lambda[0]) \in D^+_M(A)$, so $\Phi$ is $t$-exact. The claim follows.

### 4. Left completeness of $t$-structures on a triangulated category

Let $\mathcal{T}$ be a triangulated category. Consider an inverse system of objects in $\mathcal{T}$:

$$ \cdots \xrightarrow{t_{r+2}} t_{r+1} \xrightarrow{t_{r+1}} t_r \xrightarrow{\theta_r} t_{r-1} \xrightarrow{\theta_{r-1}} \cdots. $$

If $\mathcal{T}$ has countable products, then the homotopy limit of this system is the (unique, up to non-unique isomorphism) object that fits into the distinguished triangle:

$$ \text{holim} t_r \longrightarrow \prod_r t_r \xrightarrow{1-[\text{shift}]} \prod_r t_r \longrightarrow \text{holim} t_r[1]. $$
Example 4.1. If $A$ is a Grothendieck abelian category. Consider an inverse system of objects $\{t_r\}$ in $D(A)$ as above. If $A$ satisfies AB4$^*$($n(\omega)$) for some $n$, then for each $p \in \mathbb{Z}$ we have
\[
\tau^\geq_p \text{holim} t_r \simeq \tau^\geq_p \text{holim} \tau^\geq_{p-n-1} t_r
\]
This is immediate from (3.2) and the defining triangle for $\text{holim}$. This also follows from Example 4.1. Moreover, if $(4.2)$ holds.

For each $x \in T$ and $p \in \mathbb{Z}$, applying $\mathbb{R}\text{Hom}(x,-)$ to this triangle results in the Milnor exact sequence [Stacks, Tag 0919]:
\[
0 \to \lim_{\leftarrow r} \mathbb{H}^{p-1}(t_r) \to \mathbb{H}^p(\text{holim} t_r) \to \lim_{\leftarrow r} \mathbb{H}^p(t_r) \to 0.
\]
One easily deduces from this that the homotopy limit of a constant system associated to $t \in T$ (i.e., $t_r = t$ and $\theta_r = \text{Id}$ for all $r$) is just $t$ and that the homotopy limit of a general system only depends upon its tail.

A very nice and useful consequence of this sequence is the following well-known example.

Example 4.3. Let $T = D(A)$, where $A$ is a ring. Then applying (4.2) with $x = A$ we obtain an exact sequence of $A$-modules for all $p \in \mathbb{Z}$:
\[
0 \to \lim_{\leftarrow r} \mathbb{H}^{p-1}(t_r) \to \mathbb{H}^p(\text{holim} t_r) \to \lim_{\leftarrow r} \mathbb{H}^p(t_r) \to 0.
\]
It follows that for all $p \in \mathbb{Z}$:
\[
\tau^p \text{holim} t_r \simeq \tau^p \text{holim} (\tau^\geq_{p-1} t_r).
\]
This also follows from Example 4.1. Moreover, if $\tau^p \text{holim} t_r \simeq 0$, then $\text{holim} \tau^p t_r \simeq 0$. Indeed, the above shows that $\tau^p \text{holim} (\tau^p t_r) \simeq 0$ and $\tau^{<p} \text{holim} (\tau^p t_r) \simeq 0$. It remains to check what happens in degree $p + 1$. But now we have a commutative diagram with exact rows:
\[
0 \to \lim_{\leftarrow r} \mathbb{H}^{p+1}(t_r) \to \mathbb{H}^{p+1}(\text{holim} t_r) \to \lim_{\leftarrow r} \mathbb{H}^{p+1}(t_r) \to 0
\]
\[
\begin{array}{c}
0 \to \lim_{\leftarrow r} \mathbb{H}^{p+1}(\tau^p t_r) \\
\downarrow \\
0 \to \lim_{\leftarrow r} \mathbb{H}^{p+1}(\tau^{p+1} t_r)
\end{array}
\]
Now the right vertical arrow is an isomorphism and bottom left entry is zero. By the Snake Lemma, the middle vertical map is surjective. But the middle entry in the top row is zero and the claim follows.

Now assume that $T$ has a $t$-structure. We say that $T$ is:
- left separated if $t \in T$ satisfies $\tau^\geq_{-r} t \simeq 0$ for all $r \in \mathbb{Z}$, then $t \simeq 0$;
- left faithful if for each $t \in T$, any induced morphism:

\[
t \to \text{holim} \tau^\geq_{-r} t
\]
is an isomorphism (note that we only need to check it for one such morphism);
- left complete if it is left separated and given an inverse system

\[
\cdots \to t_{r+1} \xrightarrow{\theta_{r+1}} t_r \to \cdots
\]
in $T$, where $\tau^\geq_{-r} t_{r+1} \simeq t_r$ for all $r \in \mathbb{Z}$ (i.e., a Postnikov pretower), there exists $t \in T$ and compatible maps $\psi_r: t \to t_r$ such that $\tau^\geq_{-r} t \simeq t_r$ for all $r \in \mathbb{Z}$.

We now make some useful remarks.
Remark 4.4. Certainly, left faithful implies left separated. A brief argument also shows that left complete implies left faithful.

Remark 4.5. In the triangulated category literature, what we call left faithful is sometimes called left complete.

Remark 4.6. If $\mathcal{C}$ is a stable $\infty$-category with a $t$-structure, then there is also a notion of its left completion [HA Prop. 1.2.1.17]. This produces another stable $\infty$-category $\hat{\mathcal{C}}$ with a $t$-structure, together with a $t$-exact functor $f: \mathcal{C} \to \hat{\mathcal{C}}$, which induces an equivalence $\hat{\mathcal{C}}^{\geq 0} \cong \mathcal{C}^{\geq 0}$. Explicitly, $\hat{\mathcal{C}} = \lim_n \mathcal{C}^{\geq -n}$. This can be viewed as the subcategory of $\text{Fun}(N(\mathbb{Z}), \mathcal{C})$ with objects those functors $F: N(\mathbb{Z}) \to \mathcal{C}$ such that $F(n) \in \mathcal{C}^{\geq n}$ and if $m \leq n$ in $\mathbb{Z}$, then the induced map $F(m) \to F(n)$ induces an equivalence $\tau^{\geq n}F(m) \cong F(n)$. Lurie then defines $\mathcal{C}$ to be left complete if $f$ is an equivalence. If $\mathcal{C}$ admits countable products, then the functor $f: \mathcal{C} \to \hat{\mathcal{C}}$ admits a right adjoint $g: \hat{\mathcal{C}} \to \mathcal{C}$. Explicitly, $g(F) = \text{holim}_n F(n)$. Note that if $c \in \mathcal{C}$, then $g(f(c)) = \text{holim}_n \tau^{\geq n}c$. Similarly, if $F \in \hat{\mathcal{C}}$, then $f(g(F))$ is the functor $n \mapsto \tau^{\geq n}(\text{holim}_s F(s))$. Then $f$ is fully faithful (resp. an equivalence) if and only if the homotopy category $h(\mathcal{C})$, which is triangulated, is left faithful (resp. left complete). To see this, we note that $f$ is fully faithful if and only if the adjunction $c \to g(f(c)) = \text{holim}_n \tau^{\geq n}c$ is an equivalence for all $c \in \mathcal{C}$. Similarly, if $f$ is fully faithful, then it is an equivalence if and only if $g$ is conservative. Now it suffices to check these equivalences on the homotopy category. Observe that the functor $\pi: \mathcal{C} \to h(\mathcal{C})$ preserves $\text{holim}_n$\footnote{It suffices to show that $\pi$ preserves countable products. Let $\{y_n\}_{n \in \mathbb{Z}}$ belong to $\mathcal{C}$. By assumption, there is a $y \in \mathcal{C}$ that represents the functor $x \mapsto \prod_n \text{Hom}(x, y_n)$ from $\mathcal{C}^{\geq 0} \to \text{D}(\mathbb{C})$. It suffices to prove that $\pi(y)$ represents the cohomological functor $z \mapsto \prod_n \text{Hom}(\pi(y), z, \pi(y_n))$ from $h(\mathcal{C})^{\geq 0} \to \text{Ab}$. This is clear: the relationship here is just taking $\text{Holim}_(-)$, which preserves products.}. In particular, the full faithfulness claim is now clear. If $h(\mathcal{C})$ is left complete, let $F \in \hat{\mathcal{C}}$ be such that $g(F) = 0$. Let $c_n = \pi(F(n))$; then $0 = \pi(g(F)) = \text{holim}_n c_n$. By left completeness, $c_n = 0$ for all $n$. The converse is clear.

Remark 4.7. If $t \in T$ and $\tau^{\leq s}t \cong 0$ for some $s \in \mathbb{Z}$, then any induced morphism

$$t \mapsto \text{holim}_r \tau^{\geq -rt}$$

is an isomorphism. Indeed, as the homotopy limit of an inverse system only depends upon the tail, we may replace $\{\tau^{\geq -rt}\}_r$ with the constant system $\{\tau^{\leq s}t\}_r$. The claim follows.

Lemma 4.8. Let $\mathcal{A}$ be a Grothendieck abelian category.

1. $\text{D}(\mathcal{A})$ is left separated with respect to the standard $t$-structure.
2. If $\mathcal{A}$ satisfies AB4*$n(\omega)$, then $\text{D}(\mathcal{A})$ is left complete with respect to the standard $t$-structure.

Proof. Claim (1) is trivial. For claim (2): by (1), we know $\text{D}(\mathcal{A})$ is left separated. Now consider an inverse system in $\text{D}(\mathcal{A})$

$$\cdots \to A_{r+1} \overset{g_{r+1}}{\longrightarrow} A_r \to \cdots,$$

with $\tau^{\geq -r}A_{r+1} \cong A_r$. Let $\hat{A} = \text{holim}_r A_r$. Then Example 4.1 gives isomorphisms:

$$\tau^{\geq p}\hat{A} \cong \tau^{\geq p}(\text{holim}_r A_r) \cong \tau^{\geq p}(\text{holim}_r \tau^{\geq p-n-1} A_r) \cong \tau^{\geq p-n-1} A_{r-n} \cong A_p.$$ 

The result follows. \hfill $\square$
We now have the following theorem.

**Theorem 4.9.** Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{M} \subseteq \mathcal{A}$ be a weak Serre subcategory that is closed under coproducts. Consider the functor

$$\Psi : D(\mathcal{M}) \to D_{\mathcal{M}}(\mathcal{A}).$$

Assume that

1. $\mathcal{M}$ is Grothendieck abelian;
2. $\Psi^+: D^+(\mathcal{M}) \to D^+(\mathcal{A})$ is an equivalence; and
3. $D_{\mathcal{M}}(\mathcal{A})$ is left faithful.

Then the following assertions hold.

(a) If $D(\mathcal{M})$ is left faithful, then $\Psi$ is fully faithful.

(b) If $\mathcal{M}$ satisfies AB4$^\star(\omega)n$ for some $n > 0$, then $\Psi$ is an equivalence and $D_{\mathcal{M}}(\mathcal{A})$ is left complete with respect to the standard $t$-structure.

**Proof.** In Example 3.15, we have established that $\Psi$ admits a right adjoint $\Phi$ and that both $\Psi$ and $\Phi$ restrict to an equivalence on $D^+$. We first establish (a). It suffices to prove that if $M \in D(\mathcal{M})$, then the naturally induced morphism $M \to \Phi\Psi(M)$ is an isomorphism. Now $D_{\mathcal{M}}(\mathcal{A})$ is left faithful, so $\Psi(M) \simeq \varprojlim_r \tau_{\geq -r} M$. But $\Phi$ preserves homotopy limits (it is a right adjoint) and $\Psi$ is $t$-exact (it is the derived functor of the exact inclusion $\mathcal{M} \subseteq \mathcal{A}$); hence,

$$\Phi\Psi(M) \simeq \varprojlim_r \Phi\Psi(\tau_{\geq -r} M) \simeq \varprojlim_n \tau_{\geq -r} M.$$

But $D(\mathcal{M})$ is left faithful, so it follows that $M \to \Phi\Psi(M)$ is an isomorphism.

We now prove (b). By Lemma 4.8, $D(\mathcal{M})$ is left complete and so left faithful. It follows from (a) that $\Psi$ is fully faithful. Hence, it remains to prove that if $A \in D_{\mathcal{M}}(\mathcal{A})$ and $\Phi(A) = 0$, then $A = 0$. By assumption, $D_{\mathcal{M}}(\mathcal{A})$ is also left faithful, so we have an equivalence $A \simeq \varprojlim_r \tau_{\geq -r} A$ in $D_{\mathcal{M}}(\mathcal{A})$. Applying $\Phi$ to this equivalence, we obtain an equivalence $\Phi(A) \simeq \varprojlim_r \tau_{\geq -r} A$. Now $\Phi$ is $t$-exact on $D_{\mathcal{M}}^+(\mathcal{A})$. In particular, if $p \in \mathbb{Z}$, Example 4.11 gives

$$0 \simeq \tau_{\geq p} \Phi(A) \simeq \tau_{\geq p} \varprojlim_r \Phi(\tau_{\geq -r} A) \simeq \tau_{\geq p} \varprojlim_r \tau_{\geq p-n-1} \Phi(\tau_{\geq -r} A) \simeq \tau_{\geq p} \varprojlim_r \Phi(\tau_{\geq p-n-1} A) \simeq \Phi(\tau_{\geq p} A).$$

It follows that $\tau_{\geq p} A \simeq 0$ for all $p \in \mathbb{Z}$. Since $D_{\mathcal{M}}(\mathcal{A})$ is left separated, $A = 0$. □

We immediately obtain the following corollary.

**Corollary 4.10.** Let $X$ be an algebraic stack. Assume

1. $X$ is quasi-compact with affine diagonal; or
2. $X$ is noetherian and affine-pointed.

If $\text{QCoh}(X)$ satisfies AB4$^\star(\omega)n$ for some $n$, then $\text{D}(`\text{QCoh}(X)) \to D_{\text{qc}}(X)$ is an equivalence.

Combining Corollary 4.10 with Example 3.6 gives our second proof of Theorems [1] and 2.1.
Remark 4.11. Hogadi–Xu [HX09] Thm. 1.5 prove a result related to Theorem 4.10. The conclusion is similar and it has the additional assumption that $A$ is also AB4$^\omega n$. Our argument is a simple variant of theirs. They use this result to prove their Theorem 1.4 (our Corollary 4.10(1)). They do not appear to establish that $\text{Mod}(X)$ satisfies AB4$^\omega n(\omega)$, however. While we were unable to find a counterexample, that $\text{Mod}(X)$ satisfies AB4$^\omega n(\omega)$ in this generality appears to be non-obvious.

5. Direct Proof of the Corollary in §4

Let $\pi: X \to Y$ be an affine and faithfully flat morphism of algebraic stacks. We say that $\pi$ is a \textit{globally retracted cover} if $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ has a $\mathcal{O}_Y$-module retraction. Derived variants of these were considered by Elagin [Ela11], where it was proved that globally retracted covers allow one to do descent in the unbounded derived category. This is also related to the work of Balmer on descent in triangulated categories [Bal12]. The natural context to work in is that of \textit{descendable} morphisms, which are discussed in the next section. Since the Corollary in §4 is so simply stated and useful, we thought it might be a good idea to provide a direct proof of it.

Globally retracted covers are not easy to find. We will locate some interesting and useful ones, however. Our interest in them stems from the following elementary result.

Lemma 5.1. Let $\pi: X \to Y$ be a globally retracted cover. Let $\Lambda$ be a set. If $\text{QCoh}(X)$ satisfies AB4$^n(\Lambda)$ for some $n > 0$, then so does $\text{QCoh}(Y)$.

Proof. Let $Q = \text{coker}(\mathcal{O}_Y \to \pi_*\mathcal{O}_X)$. Then we have a split exact sequence:

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \pi_*\mathcal{O}_X \longrightarrow Q \longrightarrow 0.$$ 

Fix a retraction $r: \pi_*\mathcal{O}_X \to \mathcal{O}_Y$. In particular, the sequence above remains exact after the application of any additive functor to another abelian category. Hence, if $\{M_\lambda\}_{\lambda \in \Lambda}$ is a set of quasi-coherent sheaves on $Y$, then we obtain a set of split of exact sequences

$$0 \longrightarrow M_\lambda \longrightarrow \pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} M_\lambda \longrightarrow Q \otimes_{\mathcal{O}_Y} M_\lambda \longrightarrow 0,$$

each of which is split by $r_\lambda: \pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} M_\lambda \to M_\lambda$. This family of exact sequences corresponds to a split exact sequence in the product category $\text{QCoh}(Y)^\Lambda$. Since $\pi$ is affine and flat, we also have a projection isomorphism $\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} M_\lambda \simeq \pi_*\pi^*M_\lambda$ in $\text{QCoh}(Y)$. It follows that for all $s > 0$ that we have an injection in $\text{QCoh}(Y)$:

$$\prod_{\lambda \in \Lambda}^{(s)} M_\lambda \hookrightarrow \prod_{\lambda \in \Lambda}^{(s)} (\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} M_\lambda) \simeq \prod_{\lambda \in \Lambda}^{(s)} \pi_*\pi^*M_\lambda \simeq \pi_* \left( \prod_{\lambda \in \Lambda}^{(s)} \pi^*M_\lambda \right).$$

The final equivalence follows from Example 4.5 applied to the adjoint pair $(\pi^*, \pi_*).$ The result follows. \hfill \Box

We now go about finding useful examples.

Example 5.2. If $k$ is a field of characteristic 0, then $\text{Spec } k \to B\text{GL}_{s,k}$ is globally split $\forall s > 0$.

The above example can be generalized to linearly reductive group schemes.

Example 5.3. Let $f: Y' \to Y$ be a morphism of algebraic stacks. If $\pi: X \to Y$ is a globally split cover, then $\pi': X' = X \times_Y Y' \to Y'$ is a globally split cover.
Proof of the Corollary in §1. Since $G$ is an affine group scheme over $k$, it admits an embedding $G \hookrightarrow \text{GL}_{s,k}$ for some $s > 0$. Then $[U/G] \simeq [V/\text{GL}_{s,k}]$, where $V \simeq U \times (\text{GL}_{s,k}/G)$ with the diagonal action of $\text{GL}_{s,k}$. Then $V$ is noetherian and Deligne–Mumford. It follows by combining Examples 5.2 and 5.3 that $V \to [V/\text{GL}_{s,k}] \simeq [U/G]$ is globally retracted. By Proposition 5.11, $V$ satisfies $AB4^{*}n$ for some $n > 0$. By Lemma 5.1, $[U/G]$ satisfies $AB4^{*}n$. By Corollary 4.10 we have the result. \hfill \square

6. DESCENDABLE MORPHISMS

Here we review the elegant discussion of descendable morphisms, due to \textcite{Mat16}, which appears in \textcite{Aok21} §2 & §4. This is closely related to the work of Balmer for separable algebras \textcite{Bal13}.

Let $S$ be a triangulated category. A collection of objects of $S$ is closed if it is closed under finite coproducts and direct summands. If $S$ is a collection of objects of $S$, let $S$ denote its closure (i.e., the smallest closed collection of objects of $S$ that contains it). If $S_1, S_2$ are closed collections of objects of $S$, define $S_1 \ast S_2$ to be the closure of the collection

$\{s \in S : s \text{ sits in a distinguished triangle } s_1 \to s \to s_2 \to s_1[1], \text{ where } s_i \in S_i\}$.

Let $B$ be an abelian category and let $H : S \to B$ be a homological functor. If $s \in T$, define its $H$-amplitude, $\text{amp}_H(s)$, to be the smallest connected interval of $\mathbb{Z}$ with $H(s) = 0$ if $i \notin \text{amp}_H(t)$.

We have the following trivial lemma.

\textbf{Lemma 6.1.} Let $H : S \to B$ be a homological functor. Let $S_1, \ldots, S_k$ be a collection of objects of $S$. If $s \in S_1 \ast \cdots \ast S_k$, then

$$\text{amp}_H(s) \subseteq \bigcup_{i=1}^{k} \bigcup_{y \in S_i} \text{amp}_H(y).$$

\textbf{Example 6.2.} Let $A$ be a Grothendieck abelian category. We have the homological functor $\mathcal{H} : D(A) \to A$ that sends $M$ to its 0th cohomology group. The $\mathcal{H}$-amplitude, we just refer to as amplitude.

Now suppose that $(S, \otimes)$ is a tensor triangulated category (i.e., $S$ is a triangulated category, is symmetric monoidal, and the $\otimes$ is compatible with triangles and shifts). Let $A$ be an $S$-algebra. Then $A$ is descendable if the smallest thick tensor ideal of $S$ containing $A$ is $S$.

Form the distinguished triangle:

$$K_A \xrightarrow{\delta_A} 1_S \xrightarrow{\eta_A} A \xrightarrow{\delta_A} K_A[1]$$

We say that $A$ is descendable of index $\leq d$ if $\phi_A^{\otimes d} : K_A^{\otimes d} \to 1_S$ is 0. For each non-negative integer $i$, form a distinguished triangle:

$$(6.3) \quad K_A^{\otimes i} \xrightarrow{\phi_A^{\otimes i}} 1_S \xrightarrow{n_A^i} Q_A^{i+1} \xrightarrow{\delta_A^{i+1}} K_A^{\otimes (i+1)}[1].$$

It follows that if $A$ is descendable of index $\leq d$, then $1_S$ is a summand of $Q_A^{d}$. We now have a morphism of triangles:

$$K_A \xrightarrow{\phi_A^{(i+1)}} 1_S \xrightarrow{\eta_A^{i+1}} Q_A^{i+1} \xrightarrow{\delta_A^{i+1}} K_A^{\otimes (i+1)}[1]$$

$$\text{id}_{K_A^{\otimes i} \otimes \phi_A^{i}} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$K_A^{\otimes i} \xrightarrow{\phi_A^{(i+1)}} 1_S \xrightarrow{\eta_A^{i+1}} Q_A^{i+1} \xrightarrow{\delta_A^{i+1}} K_A^{\otimes (i+1)}[1].$$
From the octahedral axiom \cite{Nee01} Prop. 1.4.6, we now obtain the following commutative diagram with distinguished rows and columns:

\[
\begin{array}{ccc}
  0 & \rightarrow & K_A^{[i]} \otimes A[-1] \\
  1_S[-1] & \rightarrow & Q_A^{i+1}[-1] \\
  1_S[-1] & \rightarrow & Q_A'[-1] \\
  0 & \rightarrow & K_A^{[i]} \otimes A
\end{array}
\]

This gives the distinguished triangle:

\[
K_A^{[i]} \otimes A \rightarrow Q_A^{i+1} \rightarrow Q_A' \rightarrow K_A^{[i]} \otimes A[1].
\]

In particular, if \( A \) is descendable of index \( \leq d \) and \( s \in S \), then

\[
s \in (s \otimes K_A^{[d-1]} \otimes A) * (s \otimes K_A^{[d-2]} \otimes A) * \cdots * (s \otimes A).
\]

Hence, if \( A \) is descendable of index \( \leq d \), then \( A \) is descendable. The converse also holds (\cite{Bal13} Prop. 3.15, \cite{Mat16} Prop. 3.26, and \cite{BS17} Lem. 11.20), but we will not need it for our main results.

**Example 6.6.** Let \( R \) be a ring. Let \( M \) be a countably presented \( R \)-module. If \( M \) is flat over \( R \), then

\[
\text{Ext}^r_R(M, N) = 0
\]

for all \( r > 1 \) and \( R \)-modules \( N \) \cite{Laz69} Thm. 3.2. Now let \( A \) be a faithfully flat and countably presented \( R \)-algebra. Then \( R \) is a flat and countably presented \( R \)-module. Let \( L = \text{coker}(R \rightarrow A) \); then \( L \) is also a flat and countably presented \( R \)-module (\( L \otimes_R A \) is a direct summand of \( A \otimes_R A \)). In particular, \( L^{[2]} \) is a flat and countably presented \( R \)-module. But \( K_A = L[-1] \), so

\[
\text{Hom}_R(K_A^{[2]}, R) \simeq \text{Ext}^2_R(L^{[2]}, R) \simeq 0.
\]

Hence, \( A \) is descendable of index \( \leq 2 \).

**Example 6.7.** Let \( X = B_{\mathbb{F}_2}(\mathbb{Z}/2\mathbb{Z}) \). Consider \( \mathcal{S} = \mathcal{D}_{qc}(X) \). Let \( \pi_1 : \text{Spec } \mathbb{F}_2 \rightarrow X \) be the standard finite étale covering. Then \( A = R\pi_1_* \mathcal{O}_{\text{Spec } \mathbb{F}_2} \) (the ring of regular functions on \( \mathbb{Z}/2\mathbb{Z} \)) is not descendable in \( \mathcal{D}_{qc}(X) \). Indeed, it is easily calculated that \( K_A \simeq \mathcal{O}_X \). It follows from Equation (6.5) that if \( A \) is descendable of index \( \leq d \), then \( \mathcal{O}_X \) belongs to \( \overline{A} \subseteq \mathcal{D}_{qc}(X)^c \). But this is impossible: \( \mathcal{O}_X \notin \mathcal{D}_{qc}(X)^c \) [HR17] Rem. 4.6].

We now have a key lemma.

**Lemma 6.8.** Let \( (\mathcal{S}, \otimes) \) be a tensor triangulated category. Let \( H : \mathcal{S} \rightarrow \mathcal{B} \) be a homological functor. Let \( A \) be a descendable \( \mathcal{S} \)-algebra of index \( \leq d \). If \( s \in \mathcal{S} \), then

\[
\text{amp}_H(s) \subseteq \bigcup_{i=1}^{d} \text{amp}_H(s \otimes K_A^{[i-1]} \otimes A).
\]

**Proof.** Combine Lemma 6.1 with Equation (6.5). \( \square \)
If X is an algebraic stack of finite cohomological dimension, let \( \text{cd}(X) \) be its cohomological dimension. Let \( \pi : X \to Y \) be a concentrated morphism of quasi-compact and quasi-separated algebraic stacks; that is, \( X \times_Y \text{Spec} \ A \) has finite cohomological dimension for all affine schemes \( \text{Spec} \ A \) and morphisms \( \text{Spec} \ A \to Y \). \cite{HR17, §2}. Let

\[
\text{cd}(\pi) = \max\{\text{cd}(X \times_Y V) : V \text{ is an affine object of } Y_{\mathrm{lis-ét}}\}
\]

It follows from \cite[Lem. 2.2]{HR17} that \( \text{cd}(\pi) < \infty \).

**Lemma 6.9.** Let \( \pi : U \to X \) be a quasi-compact and quasi-separated morphism of algebraic stacks. Let \( H : D_{\text{qc}}(X) \to \mathcal{B} \) be a cohomological functor. Assume that \( R\pi_*\mathcal{O}_U \) is descendable of index \( \leq d \). If \( \pi \) is concentrated and \( M \in D_{\text{qc}}(X) \), then

\[
\text{amp}_H(M) \leq \bigcup_{i=1}^d \text{amp}_H(\pi_*L\pi^*(M \otimes_{\mathcal{O}_X} K_{R\pi_*\mathcal{O}_U}^{\otimes i-1})).
\]

**Proof.** Combine Lemma 6.8 with the projection formula \cite[Cor. 4.12]{HR17}, which shows that \( R\pi_*L\pi^*(M \otimes_{\mathcal{O}_X} K_{R\pi_*\mathcal{O}_U}^{\otimes i-1}) \cong (M \otimes_{\mathcal{O}_X} K_{R\pi_*\mathcal{O}_U}^{\otimes i-1}) \otimes_{\mathcal{O}_X} R\pi_*\mathcal{O}_U \). \( \square \)

The following example improves upon Example 6.7.

**Example 6.10.** Let \( \pi : U \to X \) be a concentrated morphism of algebraic stacks that is descendable of index \( \leq d \). If \( U \) has finite cohomological dimension, then so does \( X \). Indeed, if \( W \) is an algebraic stack, consider the homological functor \( H_W : D_{\text{qc}}(W) \to \text{Ab} \) that sends \( M \) to \( H^0(W, M) \). It follows immediately that \( H_X(R\pi_*(-)) = H_U(-) \). Lemma 6.9 implies that if \( M \in \text{QCoh}(X) \), then

\[
\text{amp}_{H_X}(M) \leq \bigcup_{i=1}^d \text{amp}_{H_U}(L\pi^*(M \otimes_{\mathcal{O}_X} K_{R\pi_*\mathcal{O}_U}^{\otimes i-1})) \subseteq (\infty, (d-1)(\text{cd}(\pi) + 1)].
\]

Hence, \( \text{cd}(X) \leq (d-1)(\text{cd}(\pi) + 1) \).

Our interest in descendable morphisms is because of the following.

**Proposition 6.11.** Let \( \pi : U \to X \) be a concentrated, faithfully flat, and finitely presented morphism of algebraic stacks. Let \( \Lambda \) be a set. Assume that

1. \( R\pi_*\mathcal{O}_U \) is descendable in \( D_{\text{qc}}(X) \);
2. \( \text{QCoh}(U) \) satisfies \( AB^4n(\Lambda) \) for some \( n \); and
3. \( D^+(\text{QCoh}(U)) \) and \( D^+(\text{QCoh}(X)) \) are equivalent.

Then \( \text{QCoh}(X) \) satisfies \( AB^4m(\Lambda) \) for some \( m \).

Unfortunately, we are not yet in a position to prove Proposition 6.11. We will first need to establish some cohomological estimates for \( R\pi_*\mathcal{O}_U \).

7. Descendable morphisms of stacks

In this section, we establish that a wide class of morphisms of algebraic stacks are descendable.

**Theorem 7.1.** Let \( \pi : U \to X \) be a concentrated, faithfully flat, and finitely presented morphism of algebraic stacks. If \( X \) has finite cohomological dimension, then \( R\pi_*\mathcal{O}_U \) is descendable of index \( \text{cd}(X) + 3 \) in \( D_{\text{qc}}(X) \) when

1. \( \pi \) is tame (e.g., representable); or
2. \( \pi \) has affine stabilizers and \( X \) has equicharacteristic; or
3. \( X \) is noetherian.
The conditions [11–13] arise in Theorem [7.1] because of the poor approximation properties of concentrated morphisms with infinite stabilizers in mixed characteristic [HR15 §2]. Example [6.7] shows that $X$ having finite cohomological dimension is essential in Theorem [7.3].

We will prove Theorem [7.1] by establishing global vanishing results for certain complexes of sheaves, which we now define.

Now let $N$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the homological functor $\Hom_{\mathcal{O}_X}(-, N) : \mathbb{D}_{qc}(X) \to \text{Ab}^\circ$. By the discussion in [56] we obtain a notion of $\Hom(-, N)$-amplitude. We define the $\Hom$-amplitude of $M$ to be

$$\text{amp}_{\Hom}(M) = \bigcup_N \text{amp}_{\Hom}(-, N)(N).$$

Example 7.2. Because of the contravariance, the $\Hom$-amplitude can be confusing. We offer the following simple example. Let $X = \mathbb{P}_k^1$, where $k$ is a field, and let $\pi : X \to \text{Spec} \ k$ be the structure map. Then $R\pi_*\mathcal{O}(-2)$ has $\Hom$-amplitude $[-1, -1]$ on $\text{Spec} \ k$. Indeed, a standard calculation shows that $R\pi_*\mathcal{O}(-2) \simeq \mathcal{O}_{\text{Spec} \ k}[-1]$. While this has cohomological amplitude $[1, 1]$ in $\mathbb{D}(k)$, the $\Hom$-amplitude is defined in the opposite category, so there is a reversing of degrees (i.e., homological grading).

We also define the $\mathcal{K}$-amplitude as:

$$\text{amp}_{\mathcal{K}\text{om}}(M) = \bigcup_V \text{amp}_{\mathcal{K}\text{om}}(M_V),$$

where $V$ ranges over all affine objects of $X_{\text{lis-\acute{e}t}}$. The following result follows easily from arguments similar to those in [Stacks 0B66].

Lemma 7.3. Let $X$ be an algebraic stack. Let $P \in \mathbb{D}_{qc}(X)$.

1. If $P$ has $\mathcal{K}$-amplitude $[a, b]$, then $P$ is locally quasi-isomorphic to a complex of projective $\mathcal{O}_X$-modules supported only in cohomological degrees $[-b, -a]$.
2. If $P$ has $\mathcal{K}$-amplitude $[a, b]$ and $P'$ has $\mathcal{K}$-amplitude $[a', b']$, then $P \otimes_{\mathcal{O}_X} P'$ has $\mathcal{K}$-amplitude contained in $[a + a', b + b']$.
3. If $\pi : U \to X$ is faithfully flat and $\pi^* P$ has $\mathcal{K}$-amplitude $[a, b]$, then so does $P$.
4. If $Q$ is a direct summand of $P$, and $P$ has $\mathcal{K}$-amplitude $[a, b]$, then $Q$ has $\mathcal{K}$-amplitude contained in $[a, b]$.

We now have our amplitude results.

Theorem 7.4. Let $X$ be a quasi-compact and quasi-separated algebraic stack. If $X$ has finite cohomological dimension, then there exists a non-negative integer $d_X$ with the following property: if $Q \in \mathbb{D}_{qc}(X)$ has $\mathcal{K}$-amplitude $[a, b]$, then it has $\Hom$-amplitude contained in $[a, b + d_X]$.

Theorem 7.5. Let $\pi : U \to X$ be a flat and finitely presented morphism of algebraic stacks. If $\pi$ is concentrated, then there is a non-negative integer $e_\pi$ such that if $P$ has $\mathcal{K}$-amplitude $[a, b]$ on $U$, then $R\pi_* P$ has $\mathcal{K}$-amplitude contained in $[a - e_\pi, b + 1]$.

Corollary 7.6. Let $\pi : U \to X$ be a faithfully flat and finitely presented morphism of algebraic stacks. If $\pi$ is concentrated, then for all non-negative integers $i$, $\mathcal{K}_{R\pi_* \mathcal{O}_U}^{\otimes i}$ has $\mathcal{K}$-amplitude contained in $[-i(e_\pi + 1), -i + 1]$.

Remark 7.7. If $\pi$ is affine, then we can take $e_\pi = 0$. If $\pi$ has affine diagonal, then we can take $e_\pi = \text{cd}(\pi) + 2$. Also, if $X$ has affine diagonal, then $d_X = \text{cd}(X) + 2$.

With these results in hand, we can now prove Proposition 6.11.
Proof of Proposition 6.11. Let $\Lambda$ be a set. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a set of quasi-coherent $\mathcal{O}_X$-modules. Let $A = \mathcal{R}_{\pi_*}\mathcal{O}_U$, which is descendable of index $\leq d$. Consider the homological functor $F: \mathcal{D}_{qc}(X) \to \mathcal{Q Coh}(X)$ that sends $M \in \mathcal{D}_{qc}(X)$ to $\mathcal{R}^0(\prod\mathcal{D}_{\omega_i}(X)(N_\lambda \otimes_{\mathcal{O}_X} M))$. By Lemma 6.9

$$\text{amp}_F(\mathcal{O}_X) \subseteq \bigcup_{i=1}^d \text{amp}_F(\mathcal{R}_{\pi_*}\mathcal{L}\pi^*(K_A^{\otimes i-1})).$$

But the projection formula [HR17, Cor. 4.12] implies that

$$\prod_{\lambda \in \Lambda} (N_\lambda \otimes_{\mathcal{O}_X} \mathcal{R}_{\pi_*}\mathcal{L}\pi^*(K_A^{\otimes i-1})) \simeq \prod_{\lambda \in \Lambda} \mathcal{R}_{\pi_*}(\pi^*N_\lambda \otimes_{\mathcal{O}_U} \mathcal{L}\pi^*K_A^{\otimes i-1})$$

$$\simeq \mathcal{R}_{\pi_*} \left( \prod_{\lambda \in \Lambda} \pi^*N_\lambda \otimes_{\mathcal{O}_U} \mathcal{L}\pi^*K_A^{\otimes i-1} \right).$$

By Corollary 7.6 $K_A^{\otimes i-1}$ has $\mathcal{H om}$-amplitude contained in $[-(\mathcal{e}_\pi + 1)(i - 1), 0]$. It follows from Lemma 7.3(1) that the amplitude of $\pi^*N_\lambda \otimes_{\mathcal{O}_U} \mathcal{L}\pi^*K_A^{\otimes i-1}$ is contained in $[0, (\mathcal{e}_\pi + 1)(i - 1)]$. By Example 3.15 applied to $\mathcal{Q Coh}(U) \subseteq \mathcal{M od}(U)$, the amplitude of $\prod_{\lambda \in \Lambda} \pi^*N_\lambda \otimes_{\mathcal{O}_U} \mathcal{L}\pi^*K_A^{\otimes i-1}$ is contained in $[0, (\mathcal{e}_\pi + 1)(i - 1) + n]$. It follows that $\mathcal{R}_{\pi_*} \left( \prod_{\lambda \in \Lambda} \pi^*N_\lambda \otimes_{\mathcal{O}_U} \mathcal{L}\pi^*K_A^{\otimes i-1} \right)$ has amplitude contained in $[0, (\mathcal{e}_\pi + 1)(i - 1) + n + \text{cd}(\pi)]$. Putting everything together,

$$\text{amp}_F(\mathcal{O}_X) \subseteq [0, (\mathcal{e}_\pi + 1)(d - 1) + n + \text{cd}(\pi)].$$

Now take $m = (\mathcal{e}_\pi + 1)(d - 1) + n + \text{cd}(\pi)$. Then $\tau^{>m} \prod_{\lambda \in \Lambda} \mathcal{D}_{\omega_i}(X) N_\lambda \simeq 0$. Since we have the equivalence $\mathcal{D}^+(\mathcal{Q Coh}(X)) \simeq \mathcal{D}^+_\mathcal{Q}(X)$, the result follows from Example 3.15 applied to $\mathcal{Q Coh}(X) \subseteq \mathcal{M od}(X)$.

Proof of Theorems 7.1, 7.4, 7.5 and Corollary 7.6. We prove these results simultaneously via a bootstrapping process. We have the following assertions that we will establish. Step 4 is the most difficult. If $\gamma: T \to S$ is a morphism, for each $r \geq 1$, let $(T/S)^{r}$ be the $r$th fiber product of $T$ over $S$. Let $\gamma^{r}: (T/S)^{r} \to S$ be the induced morphism. Let $A = \mathcal{R}_{\pi_*}\mathcal{O}_U$.

1. Theorem 7.5 for $\pi^r$ and $P = \mathcal{O}(U/X)^r$, $r \leq i$ $\implies$ Corollary 7.6 for $\pi$ and $i$.
2. Let $p: W \to U$ be a smooth surjection from an affine scheme $W$. Theorem 7.1 for $p + \text{Theorem 7.5}$ for $\pi$, $\pi \circ p = \text{Theorem 7.5}$ for $\pi$.
3. Let $p: V \to X$ be a faithfully flat, finitely presented, and concentrated morphism. Theorems 7.1 + 7.5 for $\rho + \text{Theorem 7.5}$ for $V = \text{Theorem 7.1}$ for $X$.
4. Theorem 7.5 for $\pi^r$, $r \leq \text{cd}(X) + 3$ $\implies$ Theorem 7.1 for $\pi$.

Let us put these together. We first prove Theorem 7.5 when $\pi$ is affine. We may immediately reduce to the case where $X = \text{Spec} R$ and $U = \text{Spec} A$ are affine. Then $P$ is quasi-isomorphic to a complex of projective $R$-modules supported in cohomological degrees $[-b, -a]$ (Lemma 7.3(1)). A simple argument using brutal truncations shows that it suffices to prove the result when $P$ is supported in degree 0. Then $P$ is a direct summand of a free $A$-module. But $\mathcal{R}_{\text{Hom}}(-, N)$ sends $\oplus$ to $\prod$, so it suffices to prove the result when $P = A$. We are now reduced to Example 6.6 and we can deduce the result.

By (1), we have Corollary 7.6 for all affine $\pi$ and $i \geq 1$. It now follows from (4) that Theorem 7.1 holds when $\pi$ is affine.

We next prove that Theorem 7.5 holds when $\pi$ has affine diagonal. As before, we may reduce to the case where $X = \text{Spec} R$ and $U$ is a quasi-compact algebraic
stack with affine diagonal. Let $p: W \to U$ be a smooth surjection from an affine scheme $W$. Then $p$ and $\pi \circ p$ are affine. It follows from (2) that Theorem 7.3 holds for $\pi$. It now follows from (1) and (3) that Theorem 7.1 holds when $\pi$ has affine diagonal.

A similar argument proves that Theorems 7.5 and so Theorem 7.1 holds when $\pi$ has affine second diagonal. Repeating, we get Theorems 7.5 and 7.4 when $\pi$ has affine third diagonal. But for every algebraic stack, the third diagonal is an isomorphism, so we have it for all $\pi$.

Let $\rho: V \to X$ be a smooth surjection from an affine scheme $V$. Then Theorem 7.4 holds for $\rho$. By (2), Theorem 7.1 holds for $X$.

We now go about proving the assertions (1)-(4).

**Proof of (1).** Clearly, $K_A \in \mathcal{A}[\mathbf{1}]*\mathcal{O}_X$. By Theorem 7.5 we have that $\mathcal{A}[\mathbf{1}]$ has $\mathcal{H}(\text{om}$-amplitude $[-e_\pi - 1, 0]$. Trivially, $\mathcal{O}_X$ has $\mathcal{H}(\text{om}$-amplitude $[0, 0]$. It follows from Lemma 6.3 that $K_A$ has $\mathcal{H}(\text{om}$-amplitude contained in $[-e_\pi - 1, 0]$. Lemma 7.3 gives the trivial estimate of $[-i(e_\pi + 1), 0]$ for the $\mathcal{H}(\text{om}$-amplitude, which will be useful, but it turns out we can do much better. Tensoring the defining triangle for $K_A$ with $K_A^{\oplus i-1} \otimes \mathcal{O}_X A^{\otimes 3}$ we produce a triangle:

$$K_A^{\oplus i-1} \otimes \mathcal{O}_X A^{\otimes j+1} \to K_A^{\oplus i} \otimes \mathcal{O}_X A^{\otimes j} \to K_A^{\oplus i-1} \otimes \mathcal{O}_X A^{\otimes j} \to K_A^{\oplus i-1} \otimes \mathcal{O}_X A^{\otimes j+1}[1].$$

The pullback of this sequence along $\pi$ is split and $\pi$ is faithfully flat. Hence, $K_A^{\oplus i} \otimes \mathcal{O}_X A^{\otimes j}$ has $\mathcal{H}(\text{om}$-amplitude contained in the $\mathcal{H}(\text{om}$-amplitude of $K_A^{\oplus i} \otimes \mathcal{O}_X A^{\otimes j+1}[-1]$. Starting with $(i, 0)$ and repeating, we find that $K_A^{\oplus i} \otimes \mathcal{O}_X A^{\otimes j}$ has $\mathcal{H}(\text{om}$-amplitude contained in $A^{\otimes j}[-i]$. But $A^{\otimes j} \simeq \pi^* \mathcal{O}_{U(X)}$, which has $\mathcal{H}(\text{om}$-amplitude contained in $[-e_\pi, 1]$. Hence, $K_A^{\oplus i} \otimes \mathcal{O}_X A^{\otimes j}$ has $\mathcal{H}(\text{om}$-amplitude contained in $[-(e_\pi + i), -i + 1]$. From the trivial estimate, $K_A^{\oplus i}$ has $\mathcal{H}(\text{om}$-amplitude $[-i(e_\pi + 1), 0]$. Hence, $K_A^{\oplus i}$ has $\mathcal{H}(\text{om}$-amplitude contained in $[-i(e_\pi + 1), -i + 1]$.

**Proof of (2).** Let $V$ be an affine object of $X_{\text{lis-et}}$ and $N$ a quasi-coherent $\mathcal{O}_V$-module. Let $\gamma: Y \to X$ be a concentrated morphism of algebraic stacks. Let $\mathcal{H}_{\gamma,V,N}: \mathcal{D}(\mathcal{O}_V) \to \mathcal{A}^\mathcal{B}$ be the homological functor $H^0(V, R\mathcal{H}(\text{om})_{\mathcal{O}_V}((\mathcal{R}(\gamma,-) V, N))$. It suffices to prove that there is an $e_\pi$ such that if $P$ has $\mathcal{H}(\text{om}$-amplitude $[a, b]$, then $\text{amp}_{\mathcal{H}_{\gamma,V,N}}(P) \subseteq [a - e_\pi, b + 1]$ for all $V$ and $N$. We also have an isomorphism of homological functors:

$$\mathcal{H}_{\pi,V,N}(\mathcal{R}p_*(-)) \simeq \mathcal{H}_{\pi_\text{op},V,N}(-).$$

Now $\mathcal{R}p_* \mathcal{O}_W$ is descendable of index $\leq d = \text{cd}(U) + 3$. Let $P \in \mathcal{D}_{\text{qc}}(U)$ have $\mathcal{H}(\text{om}$-amplitude $[a, b]$; then Lemma 6.3 implies that

$$\text{amp}_{\mathcal{H}_{\pi,V,N}}(P) \subseteq \bigcup_{i=1}^d \text{amp}_{\mathcal{H}_{\pi,V,N}}(\mathcal{R}p_*(\mathcal{L}p^*(P \otimes_{\mathcal{O}_W} K_{\mathcal{R}p_* \mathcal{O}_W}^{\otimes i-1}))) = \bigcup_{i=1}^d \text{amp}_{\mathcal{H}_{\pi_\text{op},V,N}}(p^*(P \otimes_{\mathcal{O}_W} K_{\mathcal{R}p_* \mathcal{O}_W}^{\otimes i-1})).$$

By (1), we have that $K_{\mathcal{R}p_* \mathcal{O}_W}$ has $\mathcal{H}(\text{om}$-amplitude $[-(e_\pi + 1)(i - 1), 0]$. It follows from Lemma 7.3 that $p^*(P \otimes_{\mathcal{O}_W} K_{\mathcal{R}p_* \mathcal{O}_W}^{\otimes i-1})$ has $\mathcal{H}(\text{om}$-amplitude contained in $[-(e_\pi + 1)(i - 1) + a, b]$. Hence,

$$\text{amp}_{\mathcal{H}_{\pi,V,N}}(P) \subseteq [-(e_\pi + 1)(i - 1) + a - e_\pi, b + 1].$$

Taking $e_\pi = (e_\pi + 1)(\text{cd}(U) + 2) + e_{\pi_\text{op}}$ gives the claim.
Proof of (3). Let \( P \in \mathcal{D}_c(X) \) have \( \mathcal{H}om \)-amplitude \([a, b]\). Let \( H_P : \mathcal{D}_c(X) \to \mathbf{A}b \) be the homological functor \( \mathcal{H}om_{\mathcal{O}_X}(P, -) \). It suffices to find \( d_X \), independent of \( P \), such that if \( N \) is a quasi-coherent \( \mathcal{O}_X \)-module, then \( \text{amp}_{H_P}(N) \subseteq [a, b + d_X] \). Let \( d_P = \text{cd}(X) + 3 \), which is the descent index of \( \rho \). Then Lemma 6.9 implies that
\[
\text{amp}_{H_P}(N) \subseteq \bigcup_{i=1}^{d_P} \text{amp}_{H_P}(\mathcal{R}\rho_* (\rho^* N \otimes_{\mathcal{O}_V} \mathcal{L} \rho^* K^{\otimes -1}_{\mathcal{R}\rho, \mathcal{O}_V})).
\]
By adjunction:
\[
\mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{R}\rho_* (\rho^* N \otimes_{\mathcal{O}_V} \mathcal{L} \rho^* K^{\otimes -1}_{\mathcal{R}\rho, \mathcal{O}_V})) \simeq \mathcal{R}\mathcal{H}om_{\mathcal{O}_V}(\mathcal{L} \rho^* P, \rho^* N \otimes_{\mathcal{O}_V} \mathcal{L} \rho^* K^{\otimes -1}_{\mathcal{R}\rho, \mathcal{O}_V}).
\]
By (1), \( K^{\otimes -1}_{\mathcal{R}\rho, \mathcal{O}_V} \) has \( \mathcal{H}om \)-amplitude \([- (\rho_0 + 1)(i - 1), 0]\). It follows from Lemma 7.3 that \( \rho^* N \otimes_{\mathcal{O}_V} \mathcal{L} \rho^* K^{\otimes -1}_{\mathcal{R}\rho, \mathcal{O}_V} \) has amplitude in \([0, (\rho_0 + 1)(i - 1)]\). Further, \( \mathcal{L} \rho^* P \) has \( \mathcal{H}om \)-amplitude \([a, b]\) on \( V \). But \( V \) is affine, so a short inductive argument on \( \mathcal{H}om \)-amplitude shows that the \( \mathcal{H}om \)-amplitude of \( \mathcal{R}\mathcal{H}om_{\mathcal{O}_V}(\mathcal{L} \rho^* P, \rho^* N \otimes_{\mathcal{O}_V} \mathcal{L} \rho^* K^{\otimes -1}_{\mathcal{R}\rho, \mathcal{O}_V}) \) is contained in \([a, b + (\rho_0 + 1)(i - 1)]\). It follows that
\[
\text{amp}_{H_P}(N) \subseteq [a, b + (\rho_0 + 1)(d_P - 1)].
\]
Taking \( d_X = (\rho_0 + 1)(\text{cd}(X) + 2) \) gives the claim. 

Proof of (4). By Lemma 4.4 we may write \( K^{\otimes i}_X \simeq \text{hocolim} E_s \), where the \( E_s \in \mathcal{D}_c^{\leq \text{cdr}}(X) \) are pseudo-coherent. Set \( m_i = \text{cd}(X) - i + 2 \). Then,
\[
\tau^{> m_i} \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(K^{\otimes i}_X, N) \simeq \tau^{> m_i} \mathcal{H}om_{\mathcal{O}_X}(\text{hocolim} E_s, N)
\]
\[
\simeq \tau^{> m_i} \text{hocolim} \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N)
\]
\[
\simeq \tau^{> m_i} \text{hocolim} \tau^{> m_i} \mathcal{R}\Gamma(X, \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N)) \quad \text{(Example 4.3)}
\]
\[
\simeq \tau^{> m_i} \text{hocolim} \tau^{> m_i} \mathcal{R}\Gamma(X, \tau^{> -i+1} \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N)) \quad \text{(Example 4.3)}
\]
\[
\simeq \tau^{> m_i} \mathcal{R}\Gamma(X, \text{hocolim} \tau^{> -i+1} \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N)).
\]

If \( M \in \mathcal{D}(X) \) and \( p \in \mathbf{Z} \), then the \( p \)-th cohomology of \( M \) is the sheafification of the presheaf \( V \mapsto \mathcal{H}^p(V, M) \), as \( V \) ranges over the affine objects of \( X_{\text{lis-ét}} \). Now fix an affine object \( V \) of \( X_{\text{lis-ét}} \); then since \( \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, \mathcal{O}_X) \in \mathcal{D}_c(X) \) and \( V \) is affine we have
\[
\mathcal{R}\Gamma(V, \text{hocolim} \tau^{> -i+1} \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N)) \simeq \text{hocolim} \mathcal{R}\Gamma(V, \tau^{> -i+1} \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N))
\]
\[
\simeq \text{hocolim} \tau^{> -i+1} \mathcal{R}\Gamma(V, \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(E_s, N))
\]
\[
\simeq \text{hocolim} \tau^{> -i+1} \mathcal{H}om_{\mathcal{O}_V}((E_s)_V, N_V).
\]

But
\[
\tau^{> -i+1} \text{hocolim} \mathcal{H}om_{\mathcal{O}_V}((E_s)_V, N_V) \simeq \tau^{> -i+1} \mathcal{H}om_{\mathcal{O}_V}((E_s)_V, N_V)
\]
\[
\simeq \tau^{> -i+1} \mathcal{H}om_{\mathcal{O}_V}((K^{\otimes i}_X)_V, N_V)
\]
\[
\simeq 0,
\]
with the vanishing because \( K^{\otimes i}_X \) has \( \mathcal{H}om \)-amplitude \([-i(\rho_0 + 1), -i + 1]\). By Example 4.3
\[
\text{hocolim} \tau^{> -i+1} \mathcal{H}om_{\mathcal{O}_V}((E_s)_V, N_V) \simeq 0
\]
for all affine objects $V$ of $X_{\text{lis-ét}}$. It follows that $\text{holim}_{s} \tau^{>-i+1} \mathcal{R}\text{Hom}_{\mathcal{O}_{X}}(E_{s}, N) \simeq 0$ in $D(X)$. Hence, $\tau^{>\text{cd}(X)-i+3} \mathcal{R}\text{Hom}_{\mathcal{O}_{X}}(K^{O_{X}}, N) \simeq 0$. Taking $i = \text{cd}(X) + 3$, we see that 

$$\text{Hom}_{\mathcal{O}_{X}}(K^{O_{X}} \otimes \text{cd}(X)+3, \mathcal{O}_{X}) = 0.$$ 

Hence, $\pi$ is descendable of index $\text{cd}(X) + 3$. ■

This completes the proof. □

8. Applications

Using Proposition 6.11 and Theorem 7.1 we can prove the following result on the boundedness of products.

**Theorem 8.1.** Let $X$ be an algebraic stack of finite cohomological dimension. If

1. $X$ is quasi-compact with affine diagonal or
2. $X$ is noetherian and affine-pointed,

then $\text{QCoh}(X)$ satisfies $\text{AB4}^{*}m$ for some $m$. In particular,

$$D(\text{QCoh}(X)) \simeq D_{\text{qc}}(X).$$

**Proof.** Let $\pi: U \to X$ be a smooth surjection from an affine scheme. By [HNR19, Thm. C.1], $D^{+}(\text{QCoh}(X)) \simeq D^{+}_{\text{qc}}(X)$. By Theorem 7.1, $\pi$ is descendable. Also, $\text{QCoh}(U)$ has exact products. It now follows from Proposition 6.11 that $\text{QCoh}(X)$ satisfies $\text{AB4}^{*}m$ for some $m$. For the final equivalence, apply Corollary 4.10. □

Theorems B and C are special cases of Theorem 8.1.

**Remark 8.2.** Theorem 8.1 is mainly interesting in characteristic 0. Indeed, let $p$ be a prime and let $X$ be an algebraic stack over $\text{Spec} \mathbf{F}_{p}$ that is quasi-compact with affine diagonal or noetherian and affine-pointed. Then the results of [AHHR22] (building on [AHR20, AHR19] and [HNR19]) show that the following conditions are equivalent:

1. the reduced connected component of the identity of the geometric stabilizer of every point of $X$ is linearly reductive;
2. $D_{\text{qc}}(X)$ is compactly generated;
3. $D(\text{QCoh}(X)) \to D_{\text{qc}}(X)$ is an equivalence;
4. $D(\text{QCoh}(X)) \to D_{\text{qc}}(X)$ is full.

The implication (3)$\implies$(4) is trivial. The local structure theorems of [AHHR22] gives that (1)$\implies$(2). We have from [HNR19] that (2)$\implies$(3) and (4)$\implies$(1).

We now come to the comparison of cohomology result mentioned in the Introduction. We recall the setup from [HNR19, §2]. Let $f: X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then the restriction of $(f_{\text{lis-ét}})^{\ast}: \text{Mod}(X) \to \text{Mod}(Y)$ to $\text{QCoh}(X) \subseteq \text{Mod}(X)$ factors through $\text{QCoh}(Y) \subseteq \text{Mod}(Y)$. This gives us a functor $(f_{\text{QCoh}})^{\ast}: \text{QCoh}(X) \to \text{QCoh}(Y)$. Everything is Grothendieck abelian, so we can derive these functors and obtain a diagram

$$
\begin{array}{ccc}
\Psi_{Y} \circ R(\text{f}_{\text{QCoh}})^{\ast} & \to & D(\text{QCoh}(Y)) \\
\Psi_{X} & \downarrow & \downarrow \\
D(X) & \to & D(Y),
\end{array}
$$

which comes with a natural transformation of functors $\epsilon_{f}: \Psi_{Y} \circ R(\text{f}_{\text{QCoh}})^{\ast} \Rightarrow R(\text{f}_{\text{lis-ét}})^{\ast} \circ \Psi_{X}$. 

If $\Phi_Y : D(Y) \to D_{qc}(Y)$ denotes the right adjoint to $\Psi_Y$, then we also have a natural transformation of functors

$$
\epsilon_f^\ast : R(f_{QCoh})_\ast \Rightarrow \Phi_Y \circ R(f_{lis-et})_\ast \circ \Psi_X.
$$

Now suppose that both $X$ and $Y$ are quasi-compact with affine diagonal or noetherian and affine-pointed. Then we have the following results from [HNR19 §2]:

1. if $M \in D^+(\mathcal{QCoh}(X))$, then $\epsilon_f(M)$ and $\epsilon_f^\ast(M)$ are equivalences; and
2. if $f$ is concentrated, then $\epsilon_f$ is an isomorphism.

Our application is the following.

**Theorem 8.3.** Let $f : X \to Y$ be a morphism of algebraic stacks of finite cohomological dimension. If

1. $X$ and $Y$ are quasi-compact with affine diagonal or
2. $X$ and $Y$ are noetherian and affine-pointed,

then $\epsilon_f^\ast$ is an isomorphism.

**Proof.** Given what we have established, this is straightforward: $\Psi_Y$ is conservative so it suffices to prove that $\Psi_Y(\epsilon_f^\ast)$ is an isomorphism. We may post compose this natural transformation with the adjunction $\Psi_Y \circ \Phi_Y \Rightarrow \text{id}$ to obtain:

$$
\Psi_Y \circ R(f_{QCoh})_\ast \Rightarrow \Psi_Y \circ \Phi_Y \circ R(f_{lis-et})_\ast \circ \Psi_X \Rightarrow R(f_{lis-et})_\ast \circ \Psi_X,
$$

whose composition is equal to $\epsilon_f$, which is an isomorphism by [HNR19 Cor. 2.2(2)]. Finally, $f$ is concentrated and so the restriction of $R(f_{lis-et})_\ast$ to $D_{qc}(X)$ factors through $D_{qc}(Y)$. It follows from Theorem 8.1 that $\Psi_Y \circ \Phi_Y \circ R(f_{lis-et})_\ast \circ \Psi_X \Rightarrow R(f_{lis-et})_\ast \circ \Psi_X$ is an isomorphism.

**APPENDIX A. APPROXIMATIONS**

In this brief appendix, we collect some technical lemmas that were employed in §7. We begin with noetherian results, then use the recently established absolute noetherian approximation for stacks [Ryd20] to deal with the non-noetherian case.

**Lemma A.1.** Let $R$ be a noetherian ring. Let $M$ be a countably generated $R$-module. Then every $R$-submodule of $M$ is countably generated.

**Proof.** Let $K \subseteq M$ be an $R$-submodule. Write $M = \bigcup_n M_n$ as a countable union of finitely generated $R$-modules. Then $K_n = M_n \cap K \subseteq M_n$ is finitely generated because $K$ is noetherian. It follows that $K = \bigcup_n K_n$ is countably generated.

**Lemma A.2.** Let $X$ be a noetherian algebraic stack. Let $M$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that $\Gamma(V,M)$ is a countably generated $\Gamma(V,h^i)\mathcal{O}_V$-module for all affine objects $V$ of $\mathcal{X}_{lis-et}$. Then

$$
M[0] \simeq \hocolim_s E_s[0]
$$

in $D_{qc}(X)$, where $E_s \in \text{Coh}(X)$ and $E_s \subseteq M$.

**Proof.** If $X$ is affine, then certainly $M \simeq \varprojlim s E_s$ in $\mathcal{QCoh}(X)$, where $E_s \in \text{Coh}(X)$ and $E_s \subseteq M$. In general, the argument in [LMB Prop. 15.4] extends this to noetherian algebraic stacks. Since we have inclusions $E_1 \subseteq E_2 \subseteq \cdots \subseteq M$, and filtered colimits are exact, $M[0] \simeq (\varprojlim_s E_s)[0] \simeq \hocolim_s E_s[0]$.

**Lemma A.3.** Let $X$ be a noetherian algebraic stack. Let $M \in D_{QC}(X)$ be a complex with $\Gamma(V,h^i(M))$ countably generated for every $i \in \mathbb{Z}$ and affine object $V$ of $\mathcal{X}_{lis-et}$. Then $M \simeq \hocolim_s E_s$, where $E_s \in D_{QC}(X)$. 

Proof. We prove the result by induction on \( n = |b - a| \geq 0 \). By shifting, we may assume that \( b = 0 \). By Lemma \[A.2\] and the inductive hypothesis, we may write \( \mathcal{H}^0(M) \simeq \text{hocolim}_s E^0_s \) and \( \tau^{<0} M \simeq \text{hocolim}_s F_s \), where \( E^0_s \in \text{Coh}(X) \) and \( F_s \in D^{[a,-1]}(X) \). We have a distinguished triangle:

\[
\tau^{<0} M \longrightarrow M \longrightarrow \mathcal{H}^0(M)[0] \longrightarrow (\tau^{<0} M)[1]
\]

We will build the \( E_s \) by induction on \( s \). For each \( s \) we obtain an induced map \( \theta_s : E^0_s \rightarrow \mathcal{H}^0(M)[0] \). Since the \( F_s \in D^{[a,-1]}(X) \) and \( E^0_s \in \text{Coh}(X) \), there exists \( r_s \) such that \( \theta_s \) factors through \( \theta'_s : E^0_s \rightarrow F_{r_s}[1] \) (this is standard; for example, see the proof of \[Hal14, Lem. 1.2\]). We can of course always assume that \( s \geq s' \implies r_s \geq r_{s'} \). Let \( E_s \) be the cone of \( \theta'_s \). It follows that we have a morphism of distinguished triangles:

\[
F_{r_s} \longrightarrow E_s \longrightarrow E^0_s \longrightarrow F_{r_s}[1]
\]

The morphism \( e_s \) may even be chosen so that the morphism of triangles is good, in the sense of \[Nee01, Defn. 1.3.14\]. Arguing as in \[Ric14\], we obtain a triangle:

\[
\tau^{<0} M \longrightarrow \text{hocolim}_s E_s \longrightarrow \mathcal{H}^0(M)[0] \longrightarrow (\tau^{<0} M)[1].
\]

Hence, \( \text{hocolim}_s E_s \simeq M \). \( \square \)

This last lemma is related to \[Stacks, Tag 0CRQ\], where the \( E_s \) are taken to be perfect. The following weaker variant, for stacks, is sufficient for us.

**Lemma A.4.** Let \( \pi : U \rightarrow X \) be a concentrated, flat, and finitely presented morphism of algebraic stacks. Form the distinguished triangle:

\[
K \longrightarrow \mathcal{O}_X \longrightarrow R\pi_* \mathcal{O}_U \longrightarrow K[1].
\]

Then \( K^\otimes i \simeq \text{hocolim}_s E_s \), where the \( E_s \in D^\leq_{qc}(X) \) are pseudo-coherent, in the following situations:

1. \( \pi \) is tame (e.g., representable); or
2. \( \pi \) has affine stabilizers and \( X \) has equicharacteristic; or
3. \( X \) is noetherian.

**Proof.** By absolute noetherian approximation \[Ryd20\] and tor-independent base change \[HR17, Cor. 4.13\], it suffices to establish the result in case (3). By Lemma \[A.1\] it remains to prove that \( H^0(U, R^p \pi_* \mathcal{O}_U) \) is countably generated for every affine object \( V \) of \( X_{\text{fppf}} \). This is local on \( X \), so we may assume that \( X = \text{Spec} A \) is noetherian and affine. Let \( U_* \rightarrow U \) be a smooth hypercovering, where the \( U_i = \text{Spec} B_i \) are affine. Then \( A \rightarrow B_i \) is of finite type, so \( B_i \) is a countably generated \( A \)-module. Hypercohomology says that \( H^0(U, \mathcal{O}_U) \) is computed as the cohomology of the complex with \( B_i \) sitting in degree \( i \). It follows from Lemma \[A.1\] that the cohomology of this complex is countably generated as an \( A \)-module. \( \square \)

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