Accretion of mineralized thin wall-like structures via localized growth along their edges is observed in physical and biological systems ranging from molluscan and brachiopod shells to carbonate–silica composite precipitates. To understand the shape of these mineralized structures, we develop a mathematical framework that treats the thin-walled shells as a smooth surface left in the wake of the growth front that can be described as an evolving space curve. Our theory then takes an explicit geometric form for the prescription of the velocity of the growth front curve, along with compatibility relations and a closure equation related to the nature of surface curling. Solutions of these equations capture a range of geometric precipitate patterns seen in abiotic and biotic forms across scales. In addition to providing a framework for the growth and form of these thin-walled morphologies, our theory suggests a new class of dynamical systems involving moving space curves that are compatible with non-Euclidean embeddings of surfaces.

1. Introduction

The conformations of low-dimensional physical systems, ranging from polymers to elastic sheets to deposition fronts of crystalline or amorphous phases, are often mathematically described by a smooth geometry at a continuum level. For instance, a one-dimensional (1D)
space curve is used to represent a polymer undergoing Brownian dynamics [1], a two-dimensional (2D) surface models an interface where two bulk phases coexist in three dimensions [2] or membranes that are embedded in and move through the third dimension [3]. These geometric representations are valid when out-of-plane deformations occur on length scales much larger than the thickness of the filament or membrane (and when the curvatures are also relatively small), such as when a cell membrane or a graphene sheet with a thickness $a$ deforms at a wavelength $\lambda$ where $\lambda \gg a$ [4–7].

The physics of such systems is determined not only by their intrinsic dimensionality $d$, but also by the dimension of the embedding domain $D$. For example, in the Frenet–Serret frame of a smooth closed curve [8–10], the dynamics of a ring polymer confined to a plane ($d = 1$, $D = 2$) requires again only one physical condition for a velocity along the curve normal, whereas the dynamics of a ring polymer in space ($d = 1$, $D = 3$) would be determined by two velocity components along the curve normal and curve binormal. Similarly, the interface growth dynamics of a crystal in space ($d = 2$, $D = 3$) is fully determined by one physical condition, i.e. the deposition velocity along the direction normal to the interface [11]. At equilibrium, the physical conditions are provided by the Euler–Lagrange (EL) equations that minimize the corresponding free energy. For non-equilibrium systems, the EL equations are replaced by appropriate dynamical equations. Either of these must be consistent with the $D - d$ physical relations that are needed to fully specify the state of the closed geometry of a system.

Here we develop a geometrical theory for the constrained accretive growth of a non-planar smooth surface with material deposition along its curvilinear edges. The theory follows that introduced by us in [12] for the controlled growth and form of bioinspired coprecipitation patterns of carbonate and silica [13–16]. Our formulation addresses the deposition of a surface that is laid down by a closed space curve ($d = 1$, $D = 3$), for which two physical conditions are needed to determine the dynamics and form. The first condition corresponds to the growth velocity along the curve normal (the growth direction). For the second condition, surface smoothness demands that a velocity component along the surface normal be prohibited. Instead, the time variation of the curve normal can have a component along the surface normal, which specifies an extrinsic curvature for the curling of the front. It is this curvature that needs to be dictated by a second condition for a smooth surface growing at its edge. These two physical conditions complement the geometric compatibility conditions for a smooth surface, i.e. the Codazzi–Mainardi equations [8] and the Gauss theorem egregium [8], which we express in a frame co-moving with the curvilinear front. This leads to equations for the dynamics of a space curve ‘constrained’ to a smooth surface that it leaves behind in its wake. Thus, the prescribed dynamics are fundamentally different from that of a curve freely evolving in space via the normal and binormal velocities, and with compatibility conditions given by the Frenet equations for space curves [8–10].

The theory of constrained accretive surface growth that we present here is relevant to many natural systems all of which have one dimension that is much smaller than the other two: centimetre- to metre-scale molluscan and brachiopod shells recording local shape changes during accretion [17–21], and chemical precipitates, such as micrometre- to millimetre-scale carbonate–silica composite walls [13–16] or chemical gardens of thin-walled millimetre- to metre-scale membraneous tubes including underwater hydrothermal vents [22,23]. Mathematically, these structures exhibit several common properties: growth is strongly localized along an interface of the emerging high-aspect-ratio wall that can be approximated as a 2D smooth surface, and the resulting structures achieve simply connected yet intricate surface geometries. Under the smooth surface assumption, these unifying characteristics imply universal mathematical constraints (due to geometric compatibility) imposed on the growth dynamics and final form of infinitesimally thin surfaces. To quantify the growth and form of these effectively 2D systems, current theoretical approaches usually limit the analysis to prescribed geometries or single-valued surface height functions [18,21,24,25]. Here, by deploying a self-consistent covariant formulation, we provide a geometrical theory that describes a range of precipitating structures and captures their complex morphologies independent of absolute scale. Since our theory is properly covariant, it is not limited to prescribed geometries and thus complements previous work in this area [18,21,24,25].
in terms of a space curve (shown in bold) with a position vector \( \mathbf{X}(s, t) \), where \( s \) is the position along the curve and \( s_2 \) is proportional to the time coordinate \( t \). The tangent vector of the curve \( \partial \mathbf{X}/\partial s \), the surface normal \( \mathbf{N} \), and the growth direction \( \mathbf{N} \) form an orthonormal triad. (b) The geodesic curvature \( \kappa_g \) is the curvature of a line with respect to a geodesic \( (\kappa_g = 0) \) on a surface. The boundary curve acquires a finite normal curvature \( \kappa_N \) when, e.g., a plane is folded into a cone. (c) In the absence of the twist of the orthonormal triad along the curve, a finite geodesic torsion \( \tau_g \) distinguishes a space curve (e.g., at point \( P \)) from a section of a plane curve with \( \tau_g = 0 \) (e.g., at point \( P \) on the light plane). The second normal curvature \( \kappa_{N2} \) characterizes the curling of the surface at the front (black curve on the light plane). (Online version in colour.)

In \( \S 2 \), we detail our theory by first formulating the geometry and dynamics of a curvilinear front that leaves behind a smooth surface. In \( \S 2a \), we further derive the geometric compatibility equations in a frame co-moving with the growth front. In \( \S 2(b) \), we introduce the mathematical closure relations required for a well-posed problem. In \( \S 3 \), we show the range of morphologies by solving the complete set of equations for the geometrical dynamics of edge-driven surface growth. In \( \S 4 \), we conclude with a brief discussion and potential future directions.

2. Theory of edge-driven growth of a smooth surface

Our theory considers the growth and form of a 2D non-planar smooth surface in three-dimensional Euclidean space driven by localized growth along its free curvilinear edge. Then the temporal wake (history) of the curve (figure 1a bold curves) constitutes the surface as it is laid out in time (figure 1a). Defining \( t \) as the time variable, \( U_l \) as the local Lagrangian growth speed, and \( \mathbf{n} \) as the growth direction, the equation of motion for the position vector field of the boundary curve \( \mathbf{X} \) is given by (figure 1a)

\[
\frac{d\mathbf{X}}{dt} = \mathbf{n} U_l. \tag{2.1}
\]

In equation (2.1), surface smoothness demands that the growth direction \( \mathbf{n} \) is a tangent vector to the surface. This eliminates a velocity component parallel to the local surface normal \( \mathbf{N} \), defined in figure 1a. Furthermore, although there can be a velocity component along the tangent of the curve \( \partial \mathbf{X}/\partial s \) (\( s \): arc length coordinate along the curve), this does not change the shape of the curve or the surface. Indeed, dropping the tangential growth component reflects a gauge invariance for closed curves [11]. So, we define the growth direction \( \mathbf{n} \) to be orthogonal to both \( \partial \mathbf{X}/\partial s \) and \( \mathbf{N} \) (figure 1a). We note that by adding a tangent speed \( V \) along \( \partial \mathbf{X}/\partial s \) to the right-hand side of equation (2.1), i.e., of the form \( \mathbf{n} U_l + (\partial \mathbf{X}/\partial s) V \), it is straightforward to generalize our framework to the motion of open curves that leave behind smooth surfaces.

To derive the self-consistent equations of motion for the spatial configuration and temporal evolution of the boundary curve from equation (2.1), we need to define the differential geometric variables that determine the dynamic configuration of the curve. To that end, we first define \( u^i \equiv \{(\sigma, t)i = 1, 2\} \) where \( \sigma \) is a fixed parametrization along the curve, and \( t \) is the time variable. Then, \( \mathbf{X} = \mathbf{X}(\sigma, t) \), and \( \mathbf{X}_i \equiv \partial_{\sigma} \mathbf{X}, \mathbf{X}_{ik} \equiv \partial_{\sigma} \partial_{\sigma} \mathbf{X} \) are defined as first and second derivatives of \( \mathbf{X} \), respectively, where \( \partial/\partial t = d/dt \) due to the parametrization of \( \mathbf{X}(\sigma, t) \). In the following, all letter indices take the values 1, 2, corresponding to the coordinates \( \sigma \) and \( t \), respectively, and Einstein summation convention (sum over repeated indices) is used. Because \( \sigma \) is time independent, the
Table 1. Definitions of the scalar and vector geometric variables. The independent variables \( \sigma, t \) and the dependent variables listed here determine the configuration of the boundary curve in space and time.

| variable      | definition                                      | description                        |
|---------------|-------------------------------------------------|------------------------------------|
| \( \sigma \)  | \( \sigma \in [0, \sigma_{\text{max}}] \)       | fixed coordinate along the boundary curve |
| \( t \)       | \( t \geq 0 \)                                  | time                               |
| \( g \equiv g_{11} \) | \( (\partial X / \partial \sigma)^2 \)       | metric of the boundary curve       |
| \( g \equiv g_{22} \) | \( (\partial X / \partial t)^2 \)             | metric of the orthogonal curve along \( \hat{n} \) |
| \( U \)       | growth speed                                    |                                    |
| \( g_{ij} \)  | \[ \begin{array}{cc} g & 0 \\ 0 & \nu \end{array} \] | metric tensor                      |
| \( \hat{g} \)  | \[ \begin{array}{cc} \frac{1}{g} & 0 \\ 0 & \frac{1}{\nu} \end{array} \] | inverse of the metric tensor       |
| \( ds \)      | \( \sqrt{\hat{g}} \)                          | local arc length                   |
| \( \partial X / \partial s \) | \( \sqrt{g}^{-1} \partial X / \partial \sigma \)   | unit vector along the boundary curve (figure 1a) |
| \( \hat{n} \)  | \( \hat{n} = U^{-1}(\partial X / \partial t) \) | unit vector along the growth direction (figure 1a) |
| \( \kappa_g \) | \( \hat{n} \cdot \partial^2 X / \partial s^2 \) | geodesic curvature (figure 1b)     |
| \( \kappa_N \) | \( \hat{N} \cdot \partial^2 X / \partial s^2 \) | normal curvature (figure 1b)       |
| \( \tau_g \)  | \( \hat{N} \cdot \partial \hat{n} / \partial s \) | geodesic torsion (figure 1c)       |
| \( \kappa_{N2} \) | \( \hat{N} \cdot \partial \hat{n} / \partial t \) | second normal curvature (figure 1c) |

Mixed partial derivatives satisfy the equality

\[
\frac{\partial}{\partial \sigma} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial \sigma}.
\] (2.2)

Next, we introduce the differential geometric variables to evaluate the temporal dynamics of a growing non-planar surface at its curve front. The elements of the 2 × 2 metric tensor (first fundamental form) and their inverse are defined as

\[
g_{ij} \equiv X_i \cdot X_j, \quad \hat{g}^{ij} \equiv (g_{ij})^{-1}.
\] (2.3)

The second derivatives \( X_{ik} \) necessitate the definition of the Christoffel symbols \( \Gamma^l_{ik} \) and the coefficients of the second fundamental form \( L_{ik} \) as

\[
X_{ik} \equiv \Gamma^l_{ik} X_l + L_{ik} \hat{N}.
\] (2.4)

Equation (2.4) then yields

\[
\Gamma^l_{ik} = X_{ik} \cdot X_m \hat{g}^{ml}, \quad L_{ik} = X_{ik} \cdot \hat{N}.
\] (2.5)

Evaluating equations (2.4), each of the Christoffel symbols \( \Gamma^l_{ik} \) and the coefficients of the second fundamental form \( L_{ik} \) are expressed in terms of the six dependent scalar variables (see figure 1b, c and table 1); the metric of the curve \( \sqrt{\hat{g}} \), the geodesic curvature \( \kappa_g \), the normal curvature \( \kappa_N \), the geodesic torsion \( \tau_g \), the second normal curvature \( \kappa_{N2} \), and the growth speed \( U \):

\[
\begin{align*}
\Gamma^1_{11} &= \frac{\partial \sqrt{\hat{g}}}{\partial s}, & \Gamma^2_{11} &= \frac{\hat{g}}{U \sqrt{\hat{g}}} \kappa_g, & \Gamma^1_{22} &= -\frac{U}{\sqrt{\hat{g}}} \frac{\partial U}{\partial t}, & \Gamma^2_{22} &= \frac{1}{U} \frac{\partial U}{\partial t} \\
\Gamma^1_{12} &= \Gamma^2_{21} = -U \kappa_g, & \Gamma^2_{12} &= \Gamma^2_{21} = \frac{\sqrt{\hat{g}}}{U} \frac{\partial U}{\partial s}
\end{align*}
\] (2.6)

and

\[
L_{11} = g \kappa_N, \quad L_{22} = U^2 \kappa_{N2}, \quad L_{12} = L_{21} = \sqrt{\hat{g}} U \tau_g.
\] (2.7)
(a) Geometric compatibility equations

The fundamental theorem of surface geometry [8,9] demands that a necessary condition for the first and second fundamental forms to be consistent with a surface is the satisfaction of geometric compatibility conditions that relate $\Gamma_{jk}^l$, $L_{ij}$, and their derivatives. This ensures the existence of a smooth surface (with continuous third derivatives of $X$) in 3-space. These compatibility conditions are the well-known Codazzi–Mainardi and Gauss equations, which lead to three independent dynamical equations for $\kappa_g$, $\kappa_N$, and $\tau_g$. We will separately determine a dynamical equation of motion for $\sqrt{g}$ (see equation (2.11) and table 2). This leaves us requiring two more equations for closure that need input from physical chemistry. These two relations, for the second-normal curvature $\kappa_N$ and the edge-curve velocity $U$, are motivated by a combination of experimental growth mechanisms as well as symmetry relations. Altogether, this leads to six equations for the six dependent scalar variables $\sqrt{g}(s,t)$, $\kappa_g(s,t)$, $\kappa_N(s,t)$, $\tau_g(s,t)$, $\kappa_{N,2}(s,t)$, $U(s,t)$ and constitute a self-consistent model for the form of smooth surfaces driven by growth along their free edge.

(i) Curve metric evolution in a moving frame

To derive an equation of motion for the metric of the curve $\sqrt{g}$, we take the time derivative of $g \equiv g_{11} = X_1 \cdot X_1$. Using equation (2.2), taking the derivative of equation (2.1) with respect to $\sigma$, and substituting the result in $\partial/\partial \sigma$ yields

$$\frac{\partial g}{\partial t} = 2 \frac{\partial X}{\partial \sigma} \cdot \frac{\partial}{\partial \sigma} (\hat{n}U). \quad (2.8)$$

Then, rewriting the spatial derivatives in terms of $\partial/\partial s$, which satisfies $\sqrt{g} \partial/\partial s = \partial/\partial \sigma$, and the orthogonality relation $\hat{N} \perp \partial X/\partial s$ transform equation (2.8) to

$$\frac{\partial g}{\partial t} = 2 gU \frac{\partial X}{\partial s} \cdot \frac{\partial \hat{n}}{\partial s} = -2 gU \frac{\partial^2 X}{\partial s^2} \cdot \hat{n}. \quad (2.9)$$

In equation (2.9) we have first used the product rule $a \cdot b' = (a \cdot b)' - a' \cdot b$ for two arbitrary vectors $a$, $b$ (where accents denote derivatives). Then, to obtain the second equality, we used $\hat{N} \perp \partial X/\partial s$ since for any unit vector $\hat{a}$,

$$\hat{a} \cdot \hat{a}' = 0 \quad (2.10)$$

always holds. Substituting $\kappa_g \equiv \hat{n} \cdot \partial^2 X/\partial s^2$, equation (2.9) becomes

$$\frac{\partial \sqrt{g}}{\partial t} = -\sqrt{g} \kappa_g U. \quad (2.11)$$

Equation (2.11) governs the time dependence of the metric, in other words, the change of the local arc length during growth.

(ii) Codazzi–Mainardi equations in a moving frame

Two of the three compatibility equations are given in the closed form as [8]

$$\frac{\partial L_{ik}}{\partial t} - \frac{\partial L_{ij}}{\partial u^k} + \Gamma_{ik}^l L_{lj} - \Gamma_{ij}^l L_{lk} = 0. \quad (2.12)$$

Equation (2.12) is trivially satisfied when $j = k$. Then, taking $j = 1, k = 2$ ($j = 2, k = 1$ multiplies equation (2.12) by $-1$), and setting $i = 1,2$ successively, returns the two compatibility equations, known as the Codazzi–Mainardi equations [8]:

$$\begin{aligned}
\frac{\partial L_{12}}{\partial \sigma} - \frac{\partial L_{11}}{\partial t} + \Gamma_{12}^1 L_{11} + \Gamma_{12}^2 L_{21} - \Gamma_{11}^1 L_{12} - \Gamma_{11}^2 L_{22} &= 0, \\
\frac{\partial L_{22}}{\partial \sigma} - \frac{\partial L_{21}}{\partial t} + \Gamma_{22}^1 L_{11} + \Gamma_{22}^2 L_{21} - \Gamma_{21}^1 L_{12} - \Gamma_{21}^2 L_{22} &= 0.
\end{aligned} \quad (2.13)$$
Table 2. Equations of motion of the geometrical variables, boundary conditions and simulation parameters. The nonlinear partial and ordinary differential equations governing the motion of the curvilinear growth site, boundary conditions for each of the simulated shapes shown in figures 3–7, and the corresponding simulation parameters are listed. The parameter \( k \) is the wavenumber of the initial perturbation (e.g. \( k = 4 \) for a fourfold vase or \( k = 6 \) for a sixfold vase.) The Gaussian curvature of the surface at the boundary curve \( \kappa_G \) is defined as \( \kappa_G = \kappa_N \kappa_{N,2} - \tau_g^2 \). For initial conditions, see equations (3.1) and (3.2), and the main text.

| variable | equation |
|----------|----------|
| \( \kappa_N \) | \( \frac{d\kappa_N}{dt} = \frac{1}{s} (U \tau_g) + \tau_g \frac{dU}{ds} + \kappa_N \left( \kappa_N - \kappa_{N,2} \right) U \), equation (2.14) |
| \( \tau_g \) | \( \frac{d\tau_g}{dt} = \frac{1}{s} (\kappa_N U) - \kappa_N \frac{dU}{ds} + 2k \tau_g U \), equation (2.15) |
| \( \sqrt{\bar{g}} \) | \( \frac{d\sqrt{\bar{g}}}{dt} = -\sqrt{\bar{g}} \kappa_g U \), equation (2.11) |
| \( \kappa_g \) | \( \frac{d\kappa_g}{dt} = \frac{1}{\sqrt{\bar{g}}} \left( \kappa_g \kappa_g + \kappa_g \kappa_{g,2} \right) U \), equation (2.23) |
| \( U \) | \( U = \alpha_1 \kappa_g + \alpha_2 \kappa_g^2 + \alpha_3 \kappa_g^3 + \eta_1 H^2 + \eta_2 \kappa_N \kappa_{N,2} + \eta_3 \tau_g^2 + \eta_4 \kappa_N \kappa_{N,3} + \eta_5 \kappa_g H^2 - \eta_6 \kappa_N \kappa_{N,2} \), equation (2.24) |
| \( H \equiv \frac{1}{2} (\kappa_N + \kappa_{N,2}) \) | \( \frac{dH}{dt} = \zeta \frac{dU}{ds} \), equation (2.25) |
| structure | position | boundary condition |
| all | \( \sigma = 0 \) | \( \frac{\partial \kappa}{\partial s} = \frac{\partial U}{\partial s} = 0 \), \( \frac{\partial \kappa}{\partial t} = 0 \) |
| all | \( \sigma = 1/k \) | \( \frac{\partial \kappa}{\partial s} = \frac{\partial U}{\partial s} = 0 \) |
| structure | parameters | equation, figure |
| vases, open shells | \( \alpha_1 = 1, \alpha_2 = 0.5, \alpha_3 = 1, \eta_1 = 1 \), \( \eta_2 = 1, \eta_3 = -1, \eta_4 = 3 \), \( \eta_5 = \eta_6 = 0, \eta_7 = 1 \) | equation (2.24), figures 3,4,5,6 |
| vases, open shells | \( \zeta = -0.4 \) | equation (2.25), figures 3,4,5,6 |
| vases | \( \delta = 0.05, m^2 = 0.1, k = 4 \) or \( k = 6 \) | equation (3.1), figures 3,4 |
| open shells | \( \delta = 0.01, k = 4 \) or \( k = 6 \) | equation (3.2), figures 5,6 |
| oscillating stems | \( \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0.5, \eta_1 = 1.6 \), \( \eta_2 = -0.3, \eta_3 = 0.5, \eta_4 = 0 \), \( \eta_5 = \eta_6 = 0, \eta_7 = 1 \) | equation (2.24), figure 7 |
| oscillating stems | \( \zeta = 0.02 \) | equation (2.25), figure 7 |
| oscillating stems | \( \delta = 0.1, m^2 = 0.1, k = 1 \) | equation (3.1), figure 7 |

Substituting equations (2.6) and (2.7) into equations (2.13) yields the Codazzi–Mainardi equations in a frame co-moving with the front at a speed \( U \):

\[
\frac{\partial \kappa_N}{\partial t} = \frac{\partial}{\partial s} (U \tau_g) + \tau_g \frac{\partial U}{\partial s} + \kappa_N \kappa_N - \kappa_{N,2}, \tag{2.14}
\]

and

\[
\frac{\partial \tau_g}{\partial t} = \frac{\partial}{\partial s} (U \kappa_{N,2}) - \frac{\partial U}{\partial s} \kappa_N + 2U \kappa_g \tau_g. \tag{2.15}
\]

Equations (2.14) and (2.15) are the two of the three compatibility equations that govern the spatial and temporal configuration of the growth front represented by a space curve embedded in a growing smooth surface.
Figure 2. Gauss–Bonnet theorem on an infinitesimal surface patch dA. The boundary $\delta \Omega \equiv \delta dA$ consists of four curvilinear sections, labelled from (I) to (IV), and is parametrized by two coordinates $s_1 \equiv s$ and $s_2 \sim t$. The orthonormal triad is shown at the curve section (III). The positive angles are defined in the counter-clockwise direction along the integration path, and $\theta_1, \theta_2, \theta_3, \theta_4$ are right angles. (Online version in colour.)

(iii) The Gauss theorema egregium in a moving frame

The third compatibility equation for the existence of a smooth surface patch $\Omega$ relates its Gaussian curvature $\kappa_G$ to the geodesic curvature of its boundary $\delta \Omega$, $\tilde{\kappa}_g$. This relation is known in integral form as the Gauss–Bonnet theorem, which, for a simply connected surface patch $\Omega$ is given as ($\theta_i$: angles at the vertices along the boundary $\delta \Omega$; $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \pi/2$; see figure 2)

$$\int_\Omega \kappa_G dA - \oint_{\delta \Omega} \tilde{\kappa}_g d\tilde{s} + \sum_{i=1}^{4} \theta_i = 2\pi. \tag{2.16}$$

The differential formulation of equation (2.16) can be derived by defining $\sigma_1 \equiv \sigma, \sigma_2 \equiv t$. Then, the infinitesimal arc lengths along the coordinates $\sigma, t$ become $\text{ds}_1 \equiv \text{ds} = \sqrt{g} \text{d}\sigma$ and $\text{ds}_2 \equiv \text{d}t$, respectively, where, $g \equiv g_{11}$ and $\text{U}^2 = g_{22}$ from equation (2.3) (table 1). This implies that the infinitesimal area element $dA = \sqrt{g} \text{d}\sigma \text{d}t$. Traversing the boundary $\delta \text{d}A$ of $dA$ in the counter-clockwise direction (figure 2), the line elements of the four curve sections along $\delta dA$ are given as

$$d\tilde{s}_{(I)} = ds, \quad d\tilde{s}_{(II)} = U \text{d}t, \quad d\tilde{s}_{(III)} = -ds, \quad d\tilde{s}_{(IV)} = -U \text{d}t. \tag{2.17}$$

Then, the geodesic curvatures of each of the four curved sections become

$$\tilde{\kappa}_g^{(I)} = -\kappa_g, \quad \tilde{\kappa}_g^{(II)} = -\frac{\partial U}{U \partial \sigma}, \quad \tilde{\kappa}_g^{(III)} = -\kappa_g, \quad \tilde{\kappa}_g^{(IV)} = -\frac{1}{\sqrt{g}} \frac{\partial U}{\partial \sigma}. \tag{2.18}$$

By using equations (2.17) and (2.18), the integral of $\tilde{\kappa}_g$ over $\delta dA$ (the second term on the left-hand side of equation (2.16)) is rewritten as

$$\int_{\delta dA} \tilde{\kappa}_g d\tilde{s} = \int_{(I)} (\kappa_g \sqrt{g}) \text{d}\sigma' - \int_{(II)} \frac{\partial U}{\partial \sigma} \text{d}t' \left. \right|_{s_1, s_2} + \int_{(III)} (\kappa_g \sqrt{g}) \text{d}\sigma' + \int_{(IV)} \frac{\partial U}{\partial \sigma} \text{d}t' \left. \right|_{s_1, s_2} + \int_{s_1}^{s_1 + \sqrt{g} \text{d}\sigma} \left[ (\kappa_g \sqrt{g}) \text{d}t' + \left. \frac{\partial U}{\partial \sigma} \right|_{s_1, s_2} \text{d}\sigma' \right] \text{d}t' + \int_{s_2}^{s_2 + \frac{U \text{d}t}{\sqrt{g}}} \left[ \frac{\partial U}{\partial \sigma} \right. \left. \text{d}t' \right] \left. \right|_{s_1, s_2}, \tag{2.19}$$
where the accents are used to distinguish between the integration constants and the boundaries of integration. Rewriting the expressions inside the square brackets of equation (2.19) as

\[
\left( \kappa_8 \sqrt{g} \right) (s_1, s_2 + U \, dt) - \left( \kappa_8 \sqrt{g} \right) (s_1, s_2) = \left[ s_2 + U \, dt \right]_{s_1}^{s_2 + U \, dt} \frac{1}{U} \frac{\partial}{\partial t} \left( \kappa_8 \sqrt{g} \right)
\]

and dropping the accents, equations (2.19) and (2.20) together yield

\[
\oint_{\partial \mathcal{A}} \kappa_8 \, d\vec{s} = \int_{\mathcal{A}} \left[ \frac{1}{U} \frac{\partial \kappa_8}{\partial t} \sqrt{g} + \frac{1}{U} \kappa_8 \frac{\partial \sqrt{g}}{\partial t} - \frac{1}{U} \frac{\partial^2 U}{\partial s^2} \sqrt{g} \right] U \, d\sigma \, dt.
\]

For an arbitrary surface patch \( \mathcal{A} \gg dA \), equations (2.16), (2.21), (2.11) and \( \sum_{i=1}^{4} \theta_i = 2\pi \) (figure 2) result in

\[
\int_{\mathcal{A}} \left[ \kappa_G - \frac{1}{U} \frac{\partial \kappa_8}{\partial t} + \kappa_8^2 + \frac{1}{U} \frac{\partial^2 U}{\partial s^2} \right] \sqrt{g} U \, d\sigma \, dt = 0.
\]

In order for the integral in equation (2.22) to vanish on every infinitesimal surface patch \( dA \) of the finite surface \( \mathcal{A} \), the integrand must be equal to zero, that is

\[
\frac{\partial \kappa_8}{\partial t} = \frac{\partial^2 U}{\partial s^2} + (\kappa_8^2 + \kappa_G) U.
\]

Equation (2.23) is the Gauss theorem egregium, i.e. differential formulation of the Gauss–Bonnet theorem, for a simply connected smooth surface in a frame co-moving with the front.

Equations (2.14), (2.15) and (2.23) constitute the three geometric compatibility equations that must hold at the growth front of a surface to maintain its smoothness. They govern, respectively, the dynamics of the extrinsic scalar variables \( \kappa_N, \tau_8 \) that depend on the local surface orientation \( \tilde{N} \), and the intrinsic scalar variable \( \kappa_8 \) that is independent of \( \tilde{N} \) (table 1). That way, the instantaneous configuration of the growth site is coupled to the extrinsic geometry of the embedding surface determined by \( \tilde{N} \), as well as the intrinsic geometries of the surface and the curve itself that are independent of \( \tilde{N} \). To complete the formulation of the problem, equations (2.11), (2.14), (2.15) and (2.23) need to be complemented with two closure relations that set the growth speed \( U \) and the second normal curvature \( \kappa_{N,2} \), which is another extrinsic variable.

(b) Closure relations

(i) Constitutive equation for edge-curve speed

We determine the local growth speed \( U \) by a power series expansion in terms of \( \kappa_8 \), its second derivative \( \partial^2 \kappa_8 / \partial s^2 \), \( \kappa_N \), \( \tau_8 \) and \( \kappa_{N,2} \). The physical motivation for this approximation is that it reproduces qualitatively the dynamics of diffusion-limited growth. This is a common accretion mechanism for physical and biological systems, ranging from crystal growth [26,27] to early lung morphogenesis [28,29], that leads to geometrical models of dendritic solidification [11] in the asymptotic limit of fast reaction and slow diffusion [30]. Additionally, the power series approximation simplifies the analysis by assuming that growth dynamics are localized to the boundary curve.

Just as dendritic growth amplifies local perturbations along the boundary curve, the surfaces resulting from our theory can also form highly intricate shapes. For reasons clarified below, we choose to truncate the expansion at the third order in powers of an inverse characteristic length scale \( 1/\ell \), where \( \ell \) sets the linear dimensions of the initial condition of the structures in the simulations. Setting the only length scale in the problem \( \ell = 1 \) in dimensionless units the series
expansion of the growth speed is given by

\[
U = -a_1\kappa_g + a_2\kappa_g^2 + a_3\kappa_g^3 + \eta_1 H^2 + \eta_21\kappa_N\kappa_{N,2} + \eta_22\tau_g^2 + \eta_3\kappa_g H^2
\]

\[
- \eta_41\kappa_g\kappa_N\kappa_{N,2} + \eta_42\kappa_g^2\tau_g^2 + \eta_5\tau_g H + \eta_6\kappa_g\tau_g H - \lambda \frac{\partial^2 \kappa_g}{\partial s^2} + O(\ell^{-4}),
\]

(2.24)

where \(H \equiv (\kappa_N + \kappa_{N,2})/2\) is the mean curvature of the surface, and the coefficients of each term are positive scalars. In equation (2.24), \(\kappa_g\) breaks the \(\mathbf{N} \rightarrow -\mathbf{N}\) symmetry and is present at all orders, i.e. \(\mathbf{N}\) points into the surface already laid down, whereas \(\mathbf{N}\) is the direction of growth. Additionally, there are no first order derivatives in the arc length coordinate \(s\) because \(U\) must remain unchanged under the transformation \(s \rightarrow -s\). In contrast with the geodesic curvature \(\kappa_g\), the extrinsic geometrical quantities \(\kappa_N\) and \(\kappa_{N,2}\) appear as even terms since they change sign under \(\mathbf{N} \rightarrow -\mathbf{N}\) transformation, under which \(U\) must remain invariant. Therefore, \(\kappa_N\), \(\tau_g\), and \(\kappa_{N,2}\) enter the expansion at \(O(\ell^{-2})\) in equation (2.24). Since \(\kappa_g^2\) and \(\partial^2 \kappa_g/\partial s^2\) are at \(O(\ell^{-3})\), the series expansion must be terminated at the third order. To that end, equation (2.24) contains all possible terms allowed by symmetry.

To understand the physical meaning of the terms in equation (2.24), we start by considering the first three terms in powers of \(\kappa_g\) and the last term \(\partial^2 \kappa_g/\partial s^2\). These are identical to those used to understand the dynamics of in-plane diffusion-limited growth for \(a_1\), \(a_2\), \(a_3\), \(\lambda > 0\) [11], where the term \(\partial^2 \kappa_g/\partial s^2\) suppresses unstable sharp kinks along the boundary curve, just as in the well-known Mullins–Sekerka instability in dendritic solidification [26]. Here the prefactor \(\lambda\) is analogous to the line tension along the growth front [11]. Furthermore, diffusion-limited growth dictates that the growth front with a large \(\kappa_N\) must propagate faster than that with a small \(\kappa_N\), imposing \(\eta_1 > 0\). Then, equation (2.24) implies that growth continues indefinitely albeit with an ever decreasing speed, e.g. while a vase with a uniform circular boundary grows, where only \(\kappa_g\) and \(\kappa_N\) are finite (figure 1b) [12].

(ii) Constitutive equation for surface second-normal curvature

The second normal curvature \(\kappa_{N,2}\) is associated with local curling along the growth direction \(\mathbf{n}\). So, \(\kappa_{N,2}\) is the extrinsic curvature of any curve locally parallel to \(\mathbf{n}\) on the surface (figure 1c). Intuitively, its evolution must vanish when \(U = 0\). A simple closure relation that satisfies this requirement is an evolution equation for the mean curvature \(H\), given as

\[
\frac{\partial H}{\partial t} = \zeta U H, \quad H \equiv \frac{1}{2}(\kappa_N + \kappa_{N,2}).
\]

(2.25)

In real units, \(\zeta\) would have dimensions of \(1/\text{length}\), i.e. it provides a curvature scale over which the growth in \(H\) occurs. Equation (2.25) is convenient because, for \(\zeta > 0\), it can locally amplify any emerging non-uniformity in \(\kappa_{N,2}\), giving rise to reinforced out-of-plane wrinkles of the surface along the boundary curve. For \(\zeta < 0\) it leads to a surface that evolves towards \(H = 0\) and locally resembles a minimal surface (\(H = 0\)) with a saddle-like conformation commonly observed in nature in a range of situations such as leaves, bivalved mollusc shells, etc. Although one can propose an infinite number of closure relations for \(\kappa_{N,2}\) that would take into account these dynamics, equation (2.25) is very simple in that it obeys the limit \(\partial \kappa_{N,2}/\partial t \rightarrow 0\) when \(U \rightarrow 0\) (equation (2.14) already ensures \(\partial \kappa_N/\partial t \rightarrow 0\) when \(U \rightarrow 0\)), it is first order in both \(\kappa_N\), \(\kappa_{N,2}\) (ignoring the curvature dependence of \(U\)), and depends only on a single control parameter \(\zeta\).

Equations (2.11), (2.14), (2.15), (2.23), (2.24) and (2.25) constitute a mathematical framework for the edge-driven surface growth when complemented by boundary conditions and initial conditions that are summarized in table 2 and discussed in the next section. Our theory models the dynamics of the curvilinear growth front in a fundamentally different way from that of conventional curves, such as vortex filaments in fluids [31–33], where only the intrinsic geometry of the curve is relevant. This is because here the configuration of the growth site is coupled with both the extrinsic and intrinsic geometries of the non-planar embedding surface at its growth front.
Previously [12], we used a similar mathematical framework but with a different closure relation instead of equation (2.25) for the accretive growth and form of thin-walled composites emerging from BaCO$_3$–SiO$_2$ coprecipitation in high pH aqueous solutions. The experimental data for the growth of these composites exhibited $U \sim t^{-1/2}$ and a growth instability with increasing $\kappa_g$, both (i) and (ii) being characteristics of diffusion-limited growth [12,13], thus making equation (2.24) useful. In earlier work [12], our closure relation for $\kappa_{N,2}$ was taken to be

$$\frac{\partial \kappa_{N,2}}{\partial t} = \gamma \frac{\partial^2 \kappa_{N,2}}{\partial s^2} + \xi \kappa_g \kappa_N U (\kappa_{N,2} - \eta_b),$$

(2.26)

where the first term on the right relaxes the curling mode along the growth front with a diffusivity $\gamma$. With a constant $\xi$ (dimensions: length), the second term (the source term) induces curling due to a coarse-grained bending parameter $\eta_b$, which was assumed to be inversely proportional to the local $pH$ of the solution. The closure relations (2.26) for $\kappa_{N,2}$ and equation (2.24) for $U$, which are both experimentally motivated, along with the geometric compatibility relations equations (2.14), (2.15), (2.23), and the equation governing the curve metric (equation (2.11)) led to the geometrical theory of accretive growth that explained a range of observed morphologies, specifically vaselike, coral-like and helical precipitates [12].

In this paper, we simplify the closure relation for the second-normal curvature and use equation (2.25) instead of equation (2.26). The two equations differ in three ways: First, a natural curvature scale for curling in equation (2.26) arises from $\xi \kappa_g \kappa_N$, which is replaced by a single constant $\xi$ in equation (2.25). The factor $\xi \kappa_g \kappa_N$ was chosen to induce curling inwards at an interface with $\kappa_g < 0$ (e.g. on a cone; see figure 1b), and outwards when $\kappa_g > 0$ (e.g. on an inverse cone growing at its narrower opening). The dependence of $\xi \kappa_g \kappa_N$ on $\kappa_N$ ensured a higher curling rate at higher normal curvatures, an effect observed in the growth of helical precipitates [12]. Second, by defining a Peclét number $Pe \equiv UL_c/\gamma$ ($L_c$: time-dependent length of the edge circumference), equation (2.26) can describe diffusion-driven curling for low $Pe$, whereas equation (2.26) is strictly constrained to the limit $Pe \to \infty$ where curling happens locally. Third, equation (2.26) imposes an experimentally motivated upper limit for curling set by $\eta_b$, in contrast, equation (2.25) allows indefinite localized growth in $H$ manifested by strong undulations in $\kappa_N$ along the boundary curve or in $\kappa_{N,2}$ in the time axis.

Here, we will focus on the simulation of three classes of hypothetical morphologies that exhibit strong undulations in these two extrinsic curvatures. Our results further highlight the versatility of our geometrical approach and ease of its implementation in a 1D fixed domain spanned by the variable $\sigma$ and in time $t$.

3. Results

(a) Simulation procedure

We simulated the geometrically constrained growth of scale-free smooth surfaces at their free curvilinear boundary, specifically vase-like patterns (figures 3 and 4; electronic supplementary material, movies S1–S6), open shell-like patterns (figures 5 and 6; electronic supplementary material, movies S7–S12), and oscillating stem-like structures (figure 7; electronic supplementary material, movies S13–S15). The growth dynamics and the final form of the morphologies are based on the scalar geometric variables $\sqrt{\kappa}, \kappa_g, \kappa_N, \kappa_{N,2}, \tau_b$, governed by equations (2.11), (2.14), (2.15) and (2.23) (the geometric compatibility equations) and equations (2.24) and (2.25) (the closure relations); see table 2. These six equations constitute a closed set of nonlinear partial differential equations that is fourth order in the fixed coordinate $\sigma$ and fifth order in time $t$, subject to the four Neumann boundary conditions (table 2) and five initial conditions for $\sqrt{\kappa}, \kappa_g, \kappa_N, \kappa_{N,2}, \tau_b, U$. For the dimensionless simulation parameters given in table 2 for each class of the morphologies, we numerically solve the equations in $\sigma$ and $t$ by using the FEniCS finite-element package on Python 3.6 [34]. The initial conditions are determined by the following mathematical procedure: For vases and oscillating stems, we use the initial condition for the position vector of the front...
Figure 3. Growth and form of vase-like morphologies with fourfold symmetry. For the wave number $k = 4$ and the simulation parameters listed in Table 2, the time evolution of the vase growth is presented for three different dimensionless times, $t = 3.6$, $t = 24.4$, $t = 34.4$. The top row shows the plan view, the middle row the side view, and the bottom row the elevated view of the emergence of an undulated vase-like geometry in time. (Online version in colour.)

$X(\sigma, t)$ ($z$: height coordinate, $k$: wavenumber) at $t = 0$:

$$
X_i(\sigma, z) \equiv X(\sigma, t = 0) = \{f(\epsilon, m, z, \sigma) \cos(2\pi \sigma), -f(\epsilon, m, z, \sigma) \sin(2\pi \sigma), z\},
$$

$$
f(\epsilon, m, z, \sigma) \equiv 1 + mz + \epsilon m^2 \cos(2\pi k \sigma), \quad \sigma \in \left[0, \frac{1}{k}\right], \quad (3.1)
$$

at a height $z = 0$. The initial angle between the wall and the $xy$-plane $\beta$ is equal to $\beta = \tan^{-1}(1/m)$. A planar section in the $xy$-plane corresponds to $\beta = 0$, and, e.g. a cylinder with the long axis along the $z$-axis would have $\beta = \pi/2$. The derivatives of equation (3.1) with respect to $\sigma$ and $z$ yield the tangent vectors to the surface $\frac{\partial X}{\partial \sigma}$ and $\mathbf{n} = \sqrt{1 + m^2} \hat{\mathbf{n}}$. For open shells, the initial condition for $X(\sigma, r)$ at $t = 0$ is chosen as ($r$: radial coordinate, $k$: wavenumber)

$$
X_i(\sigma, r) \equiv X(\sigma, t = 0) = \{-r \cos(2\pi \sigma), r \sin(2\pi \sigma), \delta \sin(2\pi k \sigma)\}, \quad \sigma \in \left[0, \frac{1}{k}\right]. \quad (3.2)
$$

The derivatives of equation (3.2) with respect to $\sigma$ and $r$ yield tangent vectors to the surface $\frac{\partial X}{\partial \sigma}$ and $\mathbf{n}$. By using higher-order derivatives and other relations from differential geometry [8], the initial conditions for the variables $g, k_g, \kappa_N, \kappa_{N2}, \tau_{\delta}, U$, the position vector $X$, and the orthonormal triad $\{\frac{\partial X}{\partial s}, \hat{\mathbf{n}}, \hat{\mathbf{N}}\}$ are determined from equation (3.1) (at $z = 0$) and from equation (3.2) (at $r = 1$).
Figure 4. Growth and form of vase-like morphologies with sixfold symmetry. For the wavenumber \( k = 6 \) and the simulation parameters listed in table 2, the time evolution of the vase growth is presented for three different dimensionless times, \( t = 2.7 \), \( t = 20.7 \), \( t = 30 \). The top row shows the plan view, the middle row the side view and the bottom row the elevated view of the emergence of an undulated vase-like geometry in time. (Online version in colour.)

To reconstruct the surface from a known time series of the geometrical variables \( \bar{g} \), \( \kappa_\bar{g} \), \( \kappa_N \), \( \kappa_{N,2} \), \( \tau_\bar{g} \), \( U \), we make use of equation (2.1), which requires knowing the time evolution of \( \hat{n} = \hat{N} \times \partial X / \partial s \) (see figure 1a), i.e.
\[
\frac{\partial \hat{n}}{\partial t} = \frac{\partial (\hat{N} \times \partial X / \partial s)}{\partial t}.
\]
The ordinary differential equations that govern the time derivatives of \( \partial X / \partial s \) and \( \hat{N} \) are given by equations (A6) and (A8), respectively, which are derived in appendix A. Thus, the dynamics and the final form of the compatible surface can be mapped from the space of dependent scalar variables \( \bar{g} \), \( \kappa_\bar{g} \), \( \kappa_N \), \( \kappa_{N,2} \), \( \tau_\bar{g} \), \( U \), to the Euclidean space \( \mathbb{R}^3 \) by (i) \( \partial X / \partial t = \hat{n} U \), (ii) \( \hat{n} = \hat{N} \times \partial X / \partial s \), (iii) equation (A6) and (iv) equation (A8).

(b) Growth and form of vases, open shells and oscillating stems

The three classes of shapes presented here, i.e. vases, open shells and oscillating stems, highlight the versatility of our theory (table 2) in capturing the growth dynamics and form of arbitrarily complex morphologies. The vases are depicted for a wave number \( k = 4 \) in figure 3; electronic supplementary material, movies S1–S3, and for \( k = 6 \) in figure 4; electronic supplementary material, movies S4–S6. The oscillating stems \((k = 1)\) are motivated by the oscillatory growth pathways of chemical garden tubes [22,23] and presented in figure 7; electronic supplementary material, movies S13–S15. Equation (3.1) indicates that, on the xy-plane, the vases and oscillating stems start growing from a unit circle (radius \( r_0 = 1 \)), along which the prefactor \( \epsilon \) in equation (3.1) induces undulating perturbations set by \( k \). Note that, in equation (2.24), the term \( \lambda \bar{a}^2 \kappa_\bar{g} / \bar{s}^2 \) penalizes the local non-uniformity in \( \kappa_\bar{g} \) along the boundary. Yet, while vases grow, because \( \lambda = 1 \).
Figure 5. Growth and form of open shell-like morphologies with fourfold symmetry. For the wave number \( k = 4 \) and the simulation parameters listed in table 2, the time evolution of the open shell growth is presented for three different dimensionless times, \( t = 0.75, t = 7.25, t = 24.25 \). The top row shows the plan view, the middle row the side view and the bottom row the elevated view of the emergence of an undulated open shell-like geometry in time. (Online version in colour.)

Inspired by molluscan and brachiopod shells that show highly undulated morphologies arising from planar surfaces, we show how such patterns can arise within our framework in figure 5; electronic supplementary material, movies S7–S9 for \( k = 4 \) and in figure 6; electronic supplementary material, movies S10–S12 for \( k = 6 \). Based on equation (3.2), the open shell-like structures start growing from a unit semicircle, and the prefactor \( \delta \) in equation (3.2) induces undulations set by the wavenumber \( k \). As with the vases, these undulations are amplified in the \( \partial X / \partial s - \hat{n} \) plane for \( \lambda = 1 \) while the open shells grow, inducing oscillations in \( \kappa_g \) through equations (2.23) and (2.24), and in turn in \( \kappa_N \) through equation (2.14). The emergence of ripples in the \( \hat{N} - \partial X / \partial s \) plane is mainly due to the fact that the initial mean curvature satisfies \( |H|_{t=0} \ll 1 \), and for \( \zeta = 0.02 \) (table 2) then \( |H| \) remains sufficiently low throughout the entire simulation time \( t_{\text{total}} = 25 \). Then, as the normal curvature \( \kappa_N \) increases, \( \kappa_{N,2} \) will also increase, albeit with an opposite sign, to satisfy a low mean curvature.
Figure 6. Growth and form of open shell-like morphologies with sixfold symmetry. For the wave number $k = 6$ and the simulation parameters listed in Table 2, the time evolution of the open shell growth is presented for three different dimensionless times, $t = 0.75$, $t = 7.25$, $t = 24.25$. The top row shows the plan view, the middle row the side view and the bottom row the elevated view of the emergence of an undulated open shell-like geometry in time. (Online version in colour.)

4. Discussion

Accretive growth through mineralization of natural solid composites are seen in many different systems: molluscan and brachiopod shells [17–21], complex carbonate–silica precipitate patterns [13–16] and chemical garden tubes [22,23]. In all these cases which involve very different absolute scales, an emergent thin-walled sheet forms convoluted shapes via growth along an edge. We can thus approximate this process as a 2D smooth surface where growth occurs through the dynamics of a space curve that coarse-grains the narrow reaction front. To quantify this, we have presented a general geometrical theory of growth and form of a 2D surface at its margin that takes into account geometric constraints based on the theory of surfaces [8], i.e. the Codazzi-Mainardi equations (equations (2.14) and (2.15)), the Gauss theorem egregium (equation (2.23)), and the evolution of the curve metric (equation 2.11). Our framework for the morphogenesis of any non-planar smooth surface complements these geometric constraints by two physical-chemical closure relations for the growth speed and local curling dynamics specific to a given system. With the goal of simulating hypothetical and aesthetic morphologies, we have proposed two simple closure relations: (i) the expansion of the growth speed $U$ as a power series of scalar geometric variables where each term obeys two reversal symmetries in the surface normal and the local arc length coordinate, $\hat{N} \rightarrow -\hat{N}$ and $s \rightarrow -s$ (equation (2.24)), (ii) a first-order ordinary differential equation for the time evolution of the mean curvature $H$ and thereby the second normal curvature $\kappa_{N,2}$ (equation (2.25)). The simulated morphologies resulting from this closed system closely
Figure 7. Growth and form of oscillating stem-like morphologies. For the simulation parameters listed in Table 2, the time evolution of the oscillating stem growth is presented for five different dimensionless times, $t = 2$, $t = 16.7$, $t = 27.9$, $t = 35.6$, $t = 41$, $t = 44.6$. The top row shows the plan view, the middle row the side view, and the bottom row the elevated view of the emergence of an oscillating stem geometry in time. (Online version in colour.)

resemble natural patterns seen in the form and periodicity of molluscan and brachiopod shells, and oscillating-stem like shapes observed in chemical gardens.

Going beyond the explanation of naturally occurring accretive morphologies, our theory also suggests links to the study of integrable systems in geometrical dynamics [33,35] associated with...
the motion of curves on non-planar surfaces that leads to the Korteweg–de Vries and modified Korteweg–de Vries hierarchies. A natural question here is one of the choice of different closure relations that lead to the construction of broader mathematical hierarchies, which might be relevant for other physico-chemical systems than the ones in our current study.

We close with a discussion of some of the limitations of our theory. The primary one is a rigorous derivation of physical closure relations that would provide a link to the microscopic mechanisms at play. For natural shell growth, biominalerization couples transport and reaction of species and their solidification at the growth front, where the length scales of the steep concentration gradients and the structure are separated. Inorganic model systems, such as carbonate-silica precipitates and chemical gardens, exhibit analogous dynamics. The challenge lies at resolving the scale separation and non-locality of species transport, as well as determining the physical laws that enable steering the gradients to guide the position, direction, and local ordering of assembly. This would entail experiments that characterize the dynamics of accretive growth at the scale of the thin wall, and use these to derive continuum-level theories for reaction and diffusion that when coarse-grained would lead to closure relations analogous to those presented here, but with coefficients that are directly related to the small scale physics, and thus reducing the number of free parameters.

A second limitation of our theory is that it completely ignores mechanical effects associated with the emergence of residual stresses during edge-driven accretion of 2D smooth surfaces. The residual stresses due to the emergent geometric incompatibility during in-plane and out-of-plane growth of elastic materials lead to metric frustration that is embodied in shape [36–43]. In the context of the current framework associated with edge-driven growth of thin solid structures, the true elastic energy minimum may lie at a geometrically incompatible state of the interface. The surface smoothness requirement can then arrest residual stresses within the accreted body. This is especially relevant to carbonate–silica co-precipitation, which exhibits silica-rich and carbonate-rich layers [12,16]. Silica is known to be mechanically flexible and can be exploited for directed actuation [44]. Therefore, the cross-sectional anisotropy may easily give rise to residual elastic stresses due to a potential incompatibility of the metric and the second fundamental form across layers [45]. The effect of this coupling of growth and stress accumulation is a natural question for further investigation.

Finally, our current framework ignores steric repulsion between initially distant wall sections that can intersect at a later time due to growth. Indeed, the structures presented in figures 3, 4, and 6 are not self-avoiding for longer simulation times or for sets of parameters different than in table 2. To prevent self-intersection, we need to modify the closure relations appropriately to account for non-local terms. Their origin can be easily understood by considering diffusion-limited growth as measured in carbonate-silica co-precipitation [12]: diffusion of chemical species around the interface in the background fluid yields a growth rate proportional to the diffusive flux [26,27], which diminishes when two wall sections come very close to each other, thereby locally suppressing growth. Building this into our geometrical framework would require the addition of integral contributions that suppress wall collisions. Another way to implement self-avoidance is considering the ‘nematic’ long-range order of parallel vector fields (along two families of curves, each locally parallel to \( \partial X/\partial s \) or \( \hat{n} \)) on the growing non-planar surface and penalizing the ‘isotropic’ phase corresponding to the defects in nematic ordering, which would occur at the intersection sites. Technically, this would require closing the purely geometric equations (equations (2.11), (2.14), (2.15) and (2.23)) with the EL equations (and their overdamped dynamics) for the minimization of the Landau–de Gennes (LdG) free energy of liquid crystals [46,47], in which the nematic director field must involve parallel transport of vectors on the surface in the sense of Levi–Civita [8]. The technical difficulty here is that equations (2.11), (2.14), (2.15) and (2.23) are in a Lagrangian frame, whereas the EL equations of the LdG free energy must be evaluated in an Eulerian frame. A previously developed Eulerian description of planar curve motion [33] must ideally be generalized to account for motion on non-planar smooth surfaces.
These limitations notwithstanding, by writing the surface differential-geometric compatibility equations in a dynamical setting, along with two closure relations for the growth speed and curling rate, we have developed a geometric theory of edge-driven growth of a smooth, simply connected non-Euclidean surface embedded in three dimensions. Simulations of the governing equations with appropriate initial and boundary conditions lead to morphologies that resemble a variety of natural and artificial precipitating thin-walled structures. When complemented by experimentally determined parameters in the symmetry-based closure relations, our theory has the potential to provide a quantitative theoretical understanding that paves the way for harnessing self-assembly processes to engineer complex morphologies with tailored material properties.

**Data accessibility.** The Python source codes of the simulations and also movies S1–S15 can be downloaded at https://github.com/nadirkaplan/geometrically_constrained_growth.

**Authors’ contributions.** C.N.K.: conceptualization, formal analysis, investigation, software, validation, visualization, writing—original draft, writing—review and editing; L.M.: conceptualization, formal analysis, project administration, resources, validation, writing—review and editing.

**Competing interests.** The authors declare no conflicts of interest.

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### Appendix A. Time evolution of the orthonormal triad

Here, we derive the time derivatives of the vectors \( \hat{n} \), \( \hat{N} \) and \( \partial X / \partial s \). In the simulations, these auxiliary equations are needed to map the time evolution the surface growth from the space of dependent scalar variables \( g, \kappa_g, \kappa_N, \tau_g, \kappa_{N2}, U \), to the Euclidean space \( \mathbb{R}^3 \).

As a first preliminary relation, the combination of equations (2.2), (2.11) and the relation \( \sqrt{g} \partial / \partial s = \partial / \partial \sigma \) result in the following identity between the mixed derivatives

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \frac{\partial}{\partial t} + \kappa_g U \frac{\partial}{\partial s}, \tag{A 1}
\]

A second preliminary relation expresses \( \partial \hat{N} / \partial s \) in terms of its components along \( \hat{N} \) and \( \partial X / \partial s \) since \( \partial \hat{N} / \partial s \perp \hat{N} \) based on equation (2.10). We calculate \( \partial \hat{N} / \partial s \) by taking the derivative of \( \hat{N} = \hat{N} \times \partial X / \partial s \), which becomes

\[
\frac{\partial \hat{n}}{\partial s} = \frac{\partial \hat{N}}{\partial s} \times \frac{\partial X}{\partial s} - \kappa \frac{\partial X}{\partial s}, \tag{A 2}
\]

The second term on the right-hand side of equation (A 2) is obtained by evaluating \( \hat{N} \times \partial^2 X / \partial s^2 \), where

\[
\frac{\partial^2 X}{\partial s^2} = \kappa_N \hat{N} + \kappa \hat{n}, \tag{A 3}
\]

by using the definitions of \( \kappa_N \) and \( \kappa_N \) in table 1. Since \( \partial \hat{N} / \partial s \) should lie in the plane spanned by \( \hat{N} \) and \( \partial X / \partial s \), the first term on the right-hand side of equation (A 2) must be either parallel or anti-parallel to \( \hat{N} \). Then, by using the definition of the geodesic torsion (table 1), equation (A 2) becomes

\[
\frac{\partial \hat{n}}{\partial s} = \tau_g \hat{N} - \kappa \frac{\partial X}{\partial s}. \tag{A 4}
\]

By virtue of the two preliminary relations equations (A 1) and (A 4), we can now calculate the time derivatives of the unit vectors. We first evaluate \( \partial (\partial X / \partial s) / \partial t \) by implementing equation (A 1):

\[
\frac{\partial}{\partial t} \frac{\partial X}{\partial s} = \frac{\partial}{\partial s} \frac{\partial X}{\partial t} + \kappa_g U \frac{\partial X}{\partial s}. \tag{A 5}
\]
When $\frac{\partial (\partial X/\partial t)}{\partial s}$ is evaluated by using equation (2.1) and (A 4), then equation (A 5) becomes

$$\frac{\partial}{\partial t} \frac{\partial X}{\partial s} = U_\tau \hat{N} + \frac{\partial U}{\partial s} \hat{n}. \quad (A 6)$$

The definition of the second normal curvature $\kappa_{N,2}$ (Table 1) allows us to determine one of the components of $\partial \hat{N}/\partial t$. Its second component can be extracted by dotting $\hat{N}$ into equation (A 6) and using the product rule subsequently. These steps yield

$$\frac{\partial \hat{n}}{\partial t} = U_\tau \kappa_{N,2} \hat{n} - \frac{\partial U}{\partial s} \frac{\partial X}{\partial s}. \quad (A 7)$$

Taking the scalar product of $\hat{N}$ with equations (A 6) and (A 7) and applying the product rule gives $\partial \hat{N}/\partial t$ as

$$\frac{\partial \hat{N}}{\partial t} = -U_\tau \kappa_{N,2} \hat{n} - U_\tau \frac{\partial X}{\partial s}. \quad (A 8)$$

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