A multiplicity result for nonlocal problems involving nonlinearities with bounded primitive

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Abstract. The aim of this paper is to provide the first application of Theorem 3 of [2] in a case where the dependence of the underlying equation from the real parameter is not of affine type. The simplest particular case of our result reads as follows:

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero continuous function such that

$$\sup_{\xi \in \mathbb{R}} |F(\xi)| < +\infty$$

where $F(\xi) = \int_0^\xi f(s)ds$. Moreover, let $k : [0, +\infty] \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be two continuous and non-decreasing functions, with $k(t) > 0$ for all $t > 0$ and $h^{-1}(0) = \{0\}$.

Then, for each $\mu$ large enough, there exist an open interval $A \subseteq \inf \mathbb{R} F, \sup \mathbb{R} F$ and a number $\rho > 0$ such that, for every $\lambda \in A$, the problem

$$\begin{cases}
-k \left( \int_0^1 |u'(t)|^2 dt \right) u'' = \mu h \left( \int_0^1 F(u(t))dt - \lambda \right) f(u) & \text{in } [0, 1] \\
u(0) = u(1) = 0
\end{cases}$$

has at least three solutions whose norms in $H_0^1(0, 1)$ are less than $\rho$.

In [2], we established the following result:

THEOREM A. - Let $X$ be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $\Psi : X \times I \to \mathbb{R}$ a continuous function satisfying the following conditions:

(a$_1$) for each $x \in X$, the function $\Psi(x, \cdot)$ is concave;

(a$_2$) for each $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ is $C^1$, sequentially weakly lower semicontinuous, coercive, and satisfies the Palais-Smale condition;

(a$_3$) there exists a continuous concave function $h : I \to \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Psi(x, \lambda) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Psi(x, \lambda) + h(\lambda)) .$$

Then, there exist an open interval $A \subseteq I$ and a positive real number $\rho$, such that, for each $\lambda \in J$, the equation

$$\Psi_\lambda'(x, \lambda) = 0$$

has at least three solutions in $X$ whose norms are less than $\rho$. 

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A consequence of Theorem A is as follows:

**THEOREM B.** - Let $X$ be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Assume that

$$\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda \Psi(x)) = +\infty$$

for all $\lambda \in I$, and that there exists a continuous concave function $h : I \to \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda \Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda \Psi(x) + h(\lambda)).$$

Then, there exist an open interval $A \subseteq I$ and a positive real number $\rho$ such that, for each $\lambda \in A$, the equation

$$\Phi'(x) + \lambda \Psi'(x) = 0$$

has at least three solutions in $X$ whose norms are less than $\rho$.

In appraising the literature, it is quite surprising to realize that, while Theorem B has been proved itself to be one of the most frequently used abstract multiplicity results in the last decade, it seems that there is no article where Theorem A has been applied to some $\Psi$ which does not depend on $\lambda$ in an affine way. For an up-dated bibliographical account related to Theorem B, we refer to [3].

The aim of this paper is to offer a first contribution to fill this gap.

To state our results, let us fix some notation.

For a generic function $\psi : X \to \mathbb{R}$, we denote by $\text{osc}_X \psi$ the (possibly infinite) number

$$\sup_{x \in X} \psi - \inf_{x \in X} \psi.$$

In the sequel, $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary. We consider the space $H^1_0(\Omega)$ equipped with the norm

$$\|u\| = \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}.$$

If $I \subseteq \mathbb{R}$ is an interval, with $0 \in I$, and $g : \Omega \times I \to \mathbb{R}$ is a function such that $g(x, \cdot)$ is continuous in $I$ for all $x \in \Omega$, we set

$$G(x, \xi) = \int_0^\xi g(x, t)dt$$

for all $(x, \xi) \in \Omega \times I$.

When $n \geq 2$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{(x, \xi) \in \Omega \times \mathbb{R}} \frac{|f(x, \xi)|}{1 + |\xi|^q} < +\infty,$$
for some $q$ with $0 < q < \frac{n+2}{n-2}$ if $n \geq 3$ and $0 < q < +\infty$ if $n = 2$. When $n = 1$, we denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for each $r > 0$, the function $x \to \sup_{|t| \leq r} |f(x, t)|$ belongs to $L^1(\Omega)$.

If $f \in \mathcal{A}$, for each $u \in H^1_0(\Omega)$, we set

$$J_f(u) = \int_{\Omega} F(x, u(x)) dx.$$ 

The functional $J_f$ is $C^1$ and its derivative is compact. Moreover, we set

$$\alpha_f = \inf_{H^1_0(\Omega)} J_f,$$

$$\beta_f = \sup_{H^1_0(\Omega)} J_f$$

and

$$\omega_f = \beta_f - \alpha_f.$$ 

Clearly, when $f$ does not depend on $x$, we have

$$\alpha_f = \text{meas}(\Omega) \inf_{\mathbb{R}} F$$

and

$$\beta_f = \text{meas}(\Omega) \sup_{\mathbb{R}} F.$$ 

Our main result reads as follows:

**THEOREM 1.** - Let $f, g \in \mathcal{A}$ be such that

$$\sup_{(x, \xi) \in \Omega \times \mathbb{R}} \max \{|F(x, \xi)|, G(x, \xi)\} < +\infty$$

and

$$\sup_{u \in H^1_0(\Omega)} \left| \int_{\Omega} F(x, u(x)) dx \right| > 0.$$ 

Then, for every pair of continuous and non-decreasing functions $k : [0, +\infty[ \to \mathbb{R}$ and $h : ]-\omega_f, \omega_f[ \to \mathbb{R}$, with $k(t) > 0$ for all $t > 0$ and $h^{-1}(0) = \{0\}$, for which the number

$$\theta^* = \inf \left\{ \frac{1}{2} K \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) - \int_{\Omega} G(x, u(x)) dx : u \in H^1_0(\Omega), \int_{\Omega} F(x, u(x)) dx \neq 0 \right\}$$

is non-negative, and for every $\mu > \theta^*$, there exist an open interval $A \subseteq ]\alpha_f, \beta_f[$ and a number $\rho > 0$ such that, for every $\lambda \in A$, the problem

$$\begin{cases} -k (\int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = \mu h (\int_{\Omega} F(x, u(x)) dx - \lambda) f(x, u) + g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

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has at least three weak solutions whose norms in $H^1_0(\Omega)$ are less than $\rho$.

Clearly, a weak solution of the above problem is any $u \in H^1_0(\Omega)$ such that

$$k \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right) \int_{\Omega} \nabla u(x) \nabla v(x) \, dx =$$

$$= \mu h \left( \int_{\Omega} F(x, u(x)) \, dx - \lambda \right) \int_{\Omega} f(x, u(x)) v(x) \, dx + \int_{\Omega} g(x, u(x)) v(x) \, dx$$

for all $v \in H^1_0(\Omega)$.

So, the weak solutions of the problem are exactly the critical points in $H^1_0(\Omega)$ of the functional

$$u \rightarrow \frac{1}{2} K(\|u\|^2) - \int_{\Omega} G(x, u(x)) \, dx - \mu H \left( \int_{\Omega} F(x, u(x)) \, dx - \lambda \right).$$

The problem that we are considering is a nonlocal one. We refer to the very recent paper [1] for a relevant discussion and an up-dated bibliography as well.

From what we said above, it is clear that our proof of Theorem 1 is based on the use of Theorem A. This is made possible by the following proposition:

**PROPOSITION 1.** - Let $X$ be a non-empty set and let $\gamma : X \rightarrow [0, +\infty[$, $J : X \rightarrow \mathbb{R}$ be two functions such that $\gamma(x_0) = J(x_0) = 0$ for some $x_0 \in X$. Moreover, assume that $J$ takes at least four values. Finally, let $\varphi : ] - \text{osc}_X J, \text{osc}_X J[ \rightarrow [0, +\infty[$ be a continuous function such that

$$\varphi^{-1}(0) = \{0\}$$

and

$$\min \left\{ \liminf_{t \to (\text{osc}_X J)^+} \varphi(t), \liminf_{t \to (\text{osc}_X J)^-} \varphi(t) \right\} > 0.$$  \hspace{2cm} (2)

Put

$$\theta = \inf_{x \in J^{-1}[\inf_X J, \sup_X J[ \setminus \{0\}]} \frac{\gamma(x)}{\varphi(J(x))}.$$  

Then, for each $\mu > \theta$, we have

$$\sup_{\lambda \in [\inf_X J, \sup_X J]} \inf_{x \in X} (\gamma(x) - \mu \varphi(J(x) - \lambda)) < \inf_{x \in X} \sup_{\lambda \in [\inf_X J, \sup_X J]} (\gamma(x) - \mu \varphi(J(x) - \lambda)).$$

**PROOF.** First, we make some remarks on the definition of $\theta$. Since $J$ takes at least four values, the set $J^{-1}[\inf_X J, \sup_X J[ \setminus \{0\}]$ is non-empty. So, if $x \in J^{-1}[\inf_X J, \sup_X J[ \setminus \{0\}]$, we have $J(x) \in ] - \text{osc}_X J, \text{osc}_X J[ \setminus \{0\}$ (recall that $\inf_X J \leq 0 \leq \sup_X J$), and so $\varphi(J(x)) > 0$. Hence, $\theta$ is a well-defined non-negative real number. Now, fix $\mu > \theta$. Since $\varphi$ is continuous, we have

$$\inf_{\lambda \in [\inf_X J, \sup_X J]} \varphi(J(x) - \lambda) = 0.$$  

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for all \( x \in X \). Hence
\[
\inf_{x \in X} \sup_{\lambda \in [\inf_X J, \sup_X J]} (\gamma(x) - \mu \varphi(J(x) - \lambda)) = \inf_{x \in X} \left( \gamma(x) - \mu \inf_{\lambda \in [\inf_X J, \sup_X J]} \varphi(J(x) - \lambda) \right)
\]
\[
= \inf_X \gamma = 0 .
\]
(3)

Now, since \( \mu > \theta \), there is \( x_1 \in X \) such that
\[
\gamma(x_1) - \mu \varphi(J(x_1)) < 0 .
\]
So, again by the continuity of \( \varphi \), for \( \epsilon, \delta > 0 \) small enough, we have
\[
\gamma(x_1) - \mu \varphi(J(x_1) - \lambda) < -\epsilon
\]
for all \( \lambda \in [-\delta, \delta] \). On the other hand, (1) and (2) imply that
\[
\nu := \inf_{\lambda \in [\inf_X J, \sup_X J]} \varphi(-\lambda) > 0 .
\]
(5)

From (4) and (5), recalling that \( \gamma(x_0) = J(x_0) = 0 \), it clearly follows
\[
\sup_{\lambda \in [\inf_X J, \sup_X J]} \inf_{x \in X} (\gamma(x) - \mu \varphi(J(x) - \lambda)) \leq \max\{-\epsilon, -\mu \nu\} < 0
\]
and so the conclusion follows in view of (3). \( \triangle \)

**REMARK 1.** - It is clear that if a \( \varphi : [-\text{osc}_X J, \text{osc}_X J] \to [0, +\infty] \) satisfies (1) and is convex, then it is continuous and satisfies (2) too.

A joint application of Theorem 1 and Proposition 1 gives

**THEOREM 2.** - Let \( X \) be a separable and reflexive real Banach space and let \( \eta, J : X \to \mathbb{R} \) be two \( C^1 \) functionals with compact derivative and \( \eta(0) = J(0) = 0 \). Assume also that \( J \) is bounded and non-constant, and that \( \eta \) is bounded above.

Then, for every sequentially weakly lower semicontinuous and coercive \( C^1 \) functional \( \psi : X \to \mathbb{R} \) whose derivative admits a continuous inverse on \( X^* \) and with \( \psi(0) = 0 \), for every convex \( C^1 \) function \( \varphi : [-\text{osc}_X J, \text{osc}_X J] \to [0, +\infty] \), with \( \varphi^{-1}(0) = \{0\} \), for which the number
\[
\hat{\theta} = \inf_{x \in J^{-1}(\mathbb{R}\{0\})} \frac{\psi(x) - \eta(x)}{\varphi(J(x))}
\]
is non-negative, and for every \( \mu > \hat{\theta} \) there exist an open interval \( A \subseteq [\inf_X J, \sup_X J] \) and a number \( \rho > 0 \) such that, for each \( \lambda \in A \), the equation
\[
\psi'(x) = \mu \varphi'(J(x) - \lambda)J'(x) + \eta'(x)
\]
has at least three solutions whose norms are less than \( \rho \).
PROOF. We apply Theorem A taking $I = [\inf_X J, \sup_X J]$ and

$$
\Psi(x, \lambda) = \psi(x) - \eta(x) - \mu \varphi(J(x) - \lambda)
$$

for all $(x, \lambda) \in X \times I$.

Clearly, $\Psi$ is $C^1$ in $X$, continuous in $X \times I$ and concave in $I$. By Corollary 41.9 of [4], the functionals $\eta, J$ are sequentially weakly continuous. Hence, for each $\lambda \in I$, the functional $\Psi(\cdot, \lambda)$ is sequentially weakly lower semicontinuous. Moreover, it is coercive, since $\psi$ is so and $\sup_{x \in X} \max \{|J(x)|, \eta(x)| < +\infty$. Moreover, it is clear that, for each $\lambda \in I$, the derivative of the functional $\eta(\cdot) + \varphi(J(\cdot) - \lambda)$ is compact (due to the assumptions on $\eta$ and $J$ and to the fact that $\varphi'$ is bounded on the compact interval $[\inf_X J, \sup_X J - \lambda]$), and so, by Example 38.25 of [4], the functional $\Psi(\cdot, \lambda)$ satisfies the Palais-Smale condition.

Now, to realize that condition $(a_3)$ is satisfied, we use Remark 1 and Proposition 1 with $\gamma = \psi - \eta$, observing that $\hat{\gamma} = \gamma$ since the range of $J$ is an interval. Then, we see that all the assumptions of Theorem A are satisfied, and the conclusion follows in view of the chain rule.

It is worth noticing the following consequence of Theorem 2:

THEOREM 3. - Let $X$ be a separable and reflexive real Banach space, let $J : X \to \mathbb{R}$ be a non-constant bounded $C^1$ functional with compact derivative and $J(0) = 0$, and let $\psi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous and coercive $C^1$ functional whose derivative admits a continuous inverse on $X^*$ and with $\psi(0) = 0$. Assume that there exists $\mu > 0$ such that

$$
\inf_{x \in X} (\psi(x) - \mu(e^{J(x)} - 1)) < 0 \leq \inf_{x \in X} (\psi(x) - \mu J(x)) .
$$

Then, there exist an open interval $A \subseteq [\mu e^{-\sup_X J}, \mu e^{-\inf_X J}]$ and a number $\rho > 0$ such that, for each $\lambda \in A$, the equation

$$
\psi'(x) = \lambda e^{J(x)} J'(x)
$$

has at least three solutions whose norms are less than $\rho$.

PROOF. From (6), it clearly follows that

$$
0 \leq \inf_{x \in J^{-1}(\mathbb{R}\setminus\{0\})} \frac{\psi(x) - \mu J(x)}{e^{J(x)} - J(x) - 1} < \mu .
$$

Consequently, we can apply Theorem 2 with $\eta = \mu J$ and $\varphi(t) = e^t - t - 1$, so that $\mu > \hat{\theta}$. Then, there exist an open interval $B \subseteq [\inf_X J, \sup_X J]$ and a number $\rho$ such that, for each $\nu \in B$ the equation

$$
\psi'(x) = \mu(e^{J(x)-\nu} - 1)J'(x) + \mu J'(x) = \mu e^{-\nu} e^{J(x)} J'(x)
$$

has at least three solutions whose norms are less than $\rho$. Therefore, the conclusion follows taking

$$
A = \{\mu e^{-\nu} : \nu \in B\} ,
$$
and the proof is complete.

**Proof of Theorem 1.** Let us apply Theorem 2 taking

\[ X = H_0^1(\Omega), \]

\[ J = J_f, \]

\[ \eta = J_g, \]

\[ \varphi = H \]

and

\[ \psi(u) = \frac{1}{2}K(\|u\|^2) \]

for all \( u \in X \).

Since \( f, g \in A \), the functionals \( J_f, J_g \) are \( C^1 \), with compact derivative. Since \( K \) is \( C^1 \), increasing and coercive, the functional \( \psi \) is sequentially weakly lower semicontinuous, \( C^1 \) and coercive. Let us show that \( \psi' \) has a continuous inverse on \( X^* \) (identified to \( X \), since \( X \) is a real Hilbert space). To this end, note that the continuous function \( t \rightarrow tk(t^2) \) is increasing in \([0, +\infty[\) and onto \([0, +\infty[\). Denote by \( \sigma \) its inverse and consider the operator

\[ T(v) = \begin{cases} 
\frac{\sigma(\|v\|)}{\|v\|}v & \text{if } v \neq 0 \\
0 & \text{if } v = 0 
\end{cases} \]

Since \( \sigma \) is continuous and \( \sigma(0) = 0 \), the operator \( T \) is continuous in \( X \). For each \( u \in X \setminus \{0\} \), since \( k(\|u\|^2) > 0 \), we have

\[ T(\psi'(u)) = T(k(\|u\|^2)u) = \frac{\sigma(k(\|u\|^2)|u|)}{k(\|u\|^2)|u|}k(\|u\|^2)u = \frac{\|u\|}{k(\|u\|^2)|u|}k(\|u\|^2)u = u , \]

as desired. Clearly, the assumptions on \( h \) imply that \( \varphi \) is non-negative, convex, with \( \varphi^{-1}(0) = \{0\} \). So, all the assumptions of Theorem 2 are satisfied, and the conclusion follows. \( \triangle \)

We conclude pointing out the following sample of application of Theorem 1 which is made possible by the fact that \( h \) is assumed to have the required properties on \( ]-\omega_f, \omega_f[ \) only.

**EXAMPLE 1.** - Let \( f : \mathbb{R} \to \mathbb{R} \) be a non-zero function belonging to \( A \), with \( \sup_{\mathbb{R}} |F| < +\infty \) and let \( k : [0, +\infty[ \to \mathbb{R} \) be a continuous and non-decreasing function, with \( k(t) > 0 \) for all \( t > 0 \).

Then, for each \( \mu \) large enough, there exist an open interval \( A \subseteq ]\inf_{\mathbb{R}} F, \sup_{\mathbb{R}} F[ \) and a number \( \rho > 0 \) such that, for every \( \lambda \in A \), the problem

\[ \begin{cases} 
- k \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \mu \frac{\int_{\Omega} F(u(x))dx - \lambda}{(\inf_{\mathbb{R}} F)^2 - (\int_{\Omega} F(u(x))dx - \lambda)^2} f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega 
\end{cases} \]

has at least three weak solutions whose norms in \( H_0^1(\Omega) \) are less than \( \rho \).
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