A LOCAL WEIGHTED AXLER-ZHENG THEOREM IN $\mathbb{C}^n$

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ABSTRACT. The well-known Axler-Zheng theorem characterizes compactness of finite sums of finite products of Toeplitz operators on the unit disk in terms of the Berezin transform of these operators. Subsequently this theorem was generalized to other domains and appeared in different forms, including domains in $\mathbb{C}^n$ on which the $\overline{\partial}$-Neumann operator $N$ is compact. In this work we remove the assumption on $N$, and we study weighted Bergman spaces on smooth bounded pseudoconvex domains. We prove a local version of the Axler-Zheng theorem characterizing compactness of Toeplitz operators in the algebra generated by symbols continuous up to the boundary in terms of the behavior of the Berezin transform at strongly pseudoconvex points. We employ a Forelli-Rudin type inflation method to handle the weights.

1. INTRODUCTION

1.1. History. In the theory of Bergman space operators on the open unit disk $\mathbb{D}$, Axler-Zheng theorem [AZ98] provides an important characterization of compactness of a large class of operators in terms of their Berezin transforms. Specifically this theorem states that if $S$ is a finite sum of finite products of Toeplitz operators on the Bergman space $A^2(\mathbb{D})$ whose symbols are in $L^\infty(\mathbb{D})$, then $S$ is compact if and only if the Berezin transform of $S$, $BS(z) \to 0$ as $|z| \to 1$. This theorem has been extended by Suarez [Suá07] to include all operators in the Toeplitz algebra in the unit ball in $\mathbb{C}^n$. Englis [Eng99] extended the Axler-Zheng theorem to irreducible bounded symmetric domains and the unit polydisk. Mitkovski, Suarez and Wick [MSW13] proved a weighted version of Suarez’s result on the unit ball in $\mathbb{C}^n$. Using the techniques of several complex variables, Čučković and Şahutoğlu [ČS13] proved a version of the Axler-Zheng theorem on smooth bounded pseudoconvex domains on which the $\overline{\partial}$-Neumann operator is compact. The use of the $\overline{\partial}$ techniques required that the operators in their theorem belong to the algebra $\mathcal{P}(\overline{\Omega})$ which is the norm closed algebra generated by $\{T_\phi : \phi \in C(\overline{\Omega})\}$. Recently, in her Master’s thesis [Kre14], Kreutzer generalized Čučković and Şahutoğlu’s result in a more abstract setting.

In this paper our aim is to extend the previous result of Čučković and Şahutoğlu in two ways: Firstly, we want to remove the hypothesis of the compactness of the $\overline{\partial}$-Neumann operator on $\Omega$. We also want to consider weighted Bergman spaces. Our main theorem gives a local version of the Axler-Zheng theorem for a wide class of domains in $\mathbb{C}^n$. The novelty of our approach is to use the inflation of the domain argument pioneered by Forelli-Rudin and Ligocka [FR75, Lig89].
second important ingredient is the B-regularity of the inflated domain which will give us the compactness of \( \overline{\partial} \), thus replacing the assumption on the compactness of the \( \overline{\partial} \)-Neumann operator. As a corollary we obtain a weighted version of the Axler-Zheng theorem for strongly pseudoconvex domains, which itself is a new result.

1.2. Preliminaries. Let \( \Omega \) be a \( C^1 \)-smooth bounded pseudoconvex domain in \( \mathbb{C}^n \) with a defining function \( \rho \). We denote the boundary of \( \Omega \) by \( b\Omega \). Let \( L^2(\Omega, (-\rho)^r) \) denote the square integrable functions on \( \Omega \) with respect to the measure \((-\rho)^r dV\) where \( dV \) denotes the Lebesgue measure, \( r \geq 0 \), and

\[
A^2(\Omega, (-\rho)^r) = \left\{ f \in L^2(\Omega, (-\rho)^r) : f \text{ is holomorphic} \right\}.
\]

Since \( A^2(\Omega, (-\rho)^r) \) is a closed subspace of \( L^2(\Omega, (-\rho)^r) \) a bounded orthogonal projection

\[
P_r : L^2(\Omega, (-\rho)^r) \to A^2(\Omega, (-\rho)^r),
\]

called Bergman projection) exists. \( P_r \) is an integral operator of the form

\[
P_r(f)(z) = \int_{\Omega} K^r(z, \xi) f(\xi)(-\rho)^r dV
\]

for \( f \in L^2(\Omega, (-\rho)^r) \). The integral kernel \( K^r(z, \xi) \) is called the Bergman kernel and the normalized Bergman kernel \( k^r(z, \xi) \) is defined as \( k^r(z, \xi) = \frac{K^r(z, \xi)}{\sqrt{K^r(z, z)}} \). When \( r = 0 \) we drop the superscript \( r \); that is, \( K = K_\Omega \) denotes the unweighted Bergman kernel and \( k_z \) denotes the unweighted normalized Bergman kernel. For a bounded operator \( T \) on \( A^2(\Omega, (-\rho)^r) \), the Berezin transform \( B_r T \) of \( T \) is defined as

\[
B_r T(z) = \langle Tk^r_z, k^r_z \rangle_r
\]

where and \( \langle \cdot, \cdot \rangle_r \) is the inner product on \( A^2(\Omega, (-\rho)^r) \).

For \( \phi \in L^\infty(\Omega) \), the weighted Toeplitz operator \( T^r_\phi \) and the weighted Hankel operator \( H^r_\phi \) are defined as follows

\[
T^r_\phi = P^r M_\phi
\]
\[
H^r_\phi = (I - P^r) M_\phi
\]

where \( M_\phi : A^2(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r) \) denotes the multiplication by \( \phi \).

We use \( \mathcal{T}(\Omega, (-\rho)^r) \) to denote the norm closed subalgebra of bounded linear operators on \( A^2(\Omega, (-\rho)^r) \) generated by the set of Toeplitz operators \( \{ T^r_\phi : \phi \in C(\overline{\Omega}) \} \). For \( \phi \in L^\infty \) we define \( B_r \phi = B_r T_\phi \).

In this paper we look at weighted Hankel and Toeplitz operators on various domains and various weighted spaces. Whenever we need to clarify where these operators are defined, we will use appropriate subscripts and superscripts. In particular, when we need to emphasize the underlying domain we will write \( P^\Omega, K^\Omega(z, \xi), H^\Omega_\phi \), and \( T^\Omega_\phi \), where the Bergman spaces are unweighted. When we have weighted spaces and we need to indicate the domain and the weight we will write \( P^{\Omega,r}, K^\Omega(z, \xi), H^{\Omega,r}_\phi \), and \( T^{\Omega,r}_\phi \).
1.3. **Main Result.** We start with the following two definitions that capture the local structure of the main theorem. To motivate the following definition, if \( f_j \to f \) weakly in \( A^2(\Omega) \) then for any point \( p \in b\Omega \) and \( r > 0 \) one can show that \( f_j \to f \) weakly in \( A^2(\Omega \cap B(p, r)) \) where \( B(p, r) \) is the open ball centered at \( p \) with radius \( r \).

**Definition 1.** Let \( r \geq 0 \) and \( \Omega \) be a \( C^2 \)-smooth bounded pseudoconvex domain in \( C^n \) with a defining function \( \rho \). Furthermore, let \( \{f_j\} \subset A^2(\Omega, (-\rho)^r) \) be a sequence and \( f \in A^2(\Omega, (-\rho)^r) \). We say that \( \{f_j\} \) converges to \( f \) weakly about strongly pseudoconvex points if

1. \( f_j \to f \) weakly in \( A^2(\Omega, (-\rho)^r) \) as \( j \to \infty \),
2. in case \( \Gamma_{\Omega} \), the set of the weakly pseudoconvex points in \( b\Omega \), is non-empty, there exists an open neighborhood \( U \) of \( \Gamma_{\Omega} \) such that \( \|f_j - f\|_{L^2(U \cap \Omega, (-\rho)^r)} \to 0 \) as \( j \to \infty \).

We note that on strongly pseudoconvex domains, sequences converging weakly about strongly pseudoconvex points and weakly convergent sequences coincide.

**Definition 2.** Let \( r, \Omega, \) and \( \rho \) be as above. Furthermore, let \( T : A^2(\Omega, (-\rho)^r) \to A^2(\Omega, (-\rho)^r) \) be a bounded linear operator. We say that \( T \) is compact about strongly pseudoconvex points if \( T f_j \to T f \) in \( A^2(\Omega, (-\rho)^r) \) whenever \( f_j \to f \) weakly about strongly pseudoconvex points.

**Remark 3.** As shown in Proposition 13 below, it is interesting that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

With the help of these two definitions, we state our main result as follows.

**Theorem 4.** Let \( r \) be a nonnegative real number, \( \Omega \) be a \( C^2 \)-smooth bounded pseudoconvex domain in \( C^n \) with a defining function \( \rho \), and \( T \in \mathcal{T}(\overline{\Omega}, (-\rho)^r) \). Then \( T \) is compact about strongly pseudoconvex points on \( A^2(\Omega, (-\rho)^r) \) if and only if \( \lim_{z \to p} B_r T(z) = 0 \) for any strongly pseudoconvex point \( p \in b\Omega \).

If \( \Omega \) is a strongly pseudoconvex domain then we have the following corollary.

**Corollary 5.** Let \( r \) be a nonnegative real number, \( \Omega \) be a \( C^2 \)-smooth bounded strongly pseudoconvex domain in \( C^n \) with a defining function \( \rho \), and \( T \in \mathcal{T}(\overline{\Omega}, (-\rho)^r) \). Then \( T \) is compact on \( A^2(\Omega, (-\rho)^r) \) if and only if \( \lim_{z \to p} B_r T(z) = 0 \) for any \( p \in b\Omega \).

**Remark 6.** In the case of the unit ball \( B^n \) in \( C^n \) and \( \rho(z) = |z|^2 - 1 \), we partially recover [MSW13, Theorem 1.1]. Unlike the arguments on the unit ball, the proof of Corollary 5 does not require any explicit form for the weight or the weighted Bergman kernel.

2. **Proof of Theorem 4**

In this section, before we prove Theorem 4, we present some propositions and lemmas that encapsulate the technical details of the proof.

**Proposition 7.** Let \( \Omega \) be a \( C^2 \)-smooth bounded pseudoconvex domain in \( C^n \) and \( \{T_j\} \) be a sequence of operators compact about strongly pseudoconvex points that converge to \( T \) in the operator norm. Then \( T \) is compact about strongly pseudoconvex points.
Proof. Let \( \{f_j\} \) be a sequence in \( A^2(\Omega, (-\rho)^r) \) that converges to 0 weakly about strongly pseudo-convex points. Since \( f_j \to 0 \) weakly there exists \( C > 0 \) such that
\[
\sup \{ \|f_j\| : j = 1, 2, 3, \ldots \} \leq C.
\]
Then for any \( k \) we have
\[
\|T f_j\| \leq \|(T - T_k)f_j\| + \|T_k f_j\| \leq C \|T - T_k\| + \|T_k f_j\|.
\]
Let \( \varepsilon > 0 \) be given. Since \( T_j \to T \) in the operator norm, we choose \( k_\varepsilon \) such that \( \|T - T_{k_\varepsilon}\| \leq \varepsilon \). Then
\[
\limsup_{j \to \infty} \|T f_j\| \leq C \varepsilon + \limsup_{j \to \infty} \|T_{k_\varepsilon} f_j\| \leq C \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary we conclude that \( T f_j \to 0 \). That is, \( T \) is compact about strongly pseudo-convex points. \( \Box \)

One of the key ideas in the proof is to use an inflated domain over \( \Omega \) to understand the weighted Bergman spaces. For this purpose, unless stated otherwise, for the rest of the paper, \( \Omega \) will be a bounded pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary, \( \rho \) will be a defining function for \( \Omega \), and
\[
(7.1) \quad \Omega^p_r = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : z \in \Omega \text{ and } \rho(z) + |w_1|^{2p/r} + \cdots + |w_p|^{2p/r} < 0\}
\]
where \( p \) is a positive integer and \( r \) is a real number such that \( 0 < r \leq p \). For a function \( f \in A^2(\Omega, (-\rho)^r) \), we let \( F(z, w) = f(z) \) be the trivial extension of \( f \) to \( \Omega^p_r \). It easily follows from an iterated integral argument that \( F \in A^2(\Omega^p_r) \).

The following proposition is interesting in its own right as it gives a relationship between the Bergman kernels of the inflated domain and base.

**Proposition 8.** Using the notation above
\[
K^p_r(\zeta, \xi) = c_{p,r} K^p_r(z, 0; \xi, 0)
\]
where \( c_{p,r} = \int_{[\bar{w}_1]^{2p/r} + \cdots + [\bar{w}_p]^{2p/r} < 1} dV(\bar{w}) \) and \( K^p_r(z, \xi) \) is the weighted Bergman kernel of \( \Omega \) with weight \( (-\rho)^r \).

Proof. We will follow a standard inflation argument (see for instance [FR75, Lig89]). Since \( \Omega^p_r \) is a Hartogs domain with base \( \Omega \), the Bergman kernel of \( \Omega^p_r \) can be written as
\[
K_{\Omega^p_r}(z, w; \xi, \eta) = K_{\Omega^p_r}(z, 0; \xi, 0) + \sum_{|j| \geq 1} K_j(z, \xi) w^j \eta^j
\]
where \( j \) is a multiindex with nonnegative entries. Then for \( f \in A^2(\Omega, (-\rho)^r) \) and \( z \in \Omega \) we have (\( F \) below is the trivial extension of \( f \))
\[
(8.1) \quad f(z) = \int_{\Omega^p_r} K_{\Omega^p_r}(z, 0; \xi, 0) F(\xi, \eta) dV(\xi, \eta) + \sum_{|j| \geq 1} \int_{\Omega^p_r} K_j(z, \xi) w^j \eta^j F(\xi, \eta) dV(\xi, \eta).
\]
However, the integrals under the sum on the right hand side above all vanish.

Using the change of variables \( \tilde{w}_j = \frac{w_j}{(-\rho(z))^{p/r}} \) one can compute that

\[
\int_{|w_1|^{2p/r} + \cdots + |w_p|^{2p/r} < -\rho(z)} dV(w) = (-\rho(z))^r \int_{|\tilde{w}_1|^{2p/r} + \cdots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w}).
\]

We denote

\[
c_{p,r} = \int_{|\tilde{w}_1|^{2p/r} + \cdots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w}).
\]

Then using (8.1), (8.2), and (8.3) we get

\[
f(z) = \int_{Q^p} K_{\Omega^p}(z,0;\xi,0) F(\xi,\eta) dV(\xi,\eta) = c_{p,r} \int_{Q} K_{\Omega^p}(z,0;\xi,0) f(\xi)(-\rho(\xi))' dV(\xi).
\]

Therefore, \( c_{p,r} K_{\Omega^p}(z,0;\xi,0) = K^r(z,\xi). \)

For a \( C^2 \)-smooth function \( \rho \) around a point \( P \in \mathbb{C}^n, X = (x_1, \ldots, x_n) \in \mathbb{C}^n \), and \( Y = (y_1, \ldots, y_n) \in \mathbb{C}^n \), we define the complex Hessian of \( \rho \) at \( P \) as

\[
H_\rho(P;X,Y) = \sum_{j,k=1}^n \frac{\partial^2 \rho(P)}{\partial z_j \partial \bar{z}_k} x_j y_k.
\]

Furthermore, we use the notation \( H_\rho(P;X) = H_\rho(P;X,X) \).

**Lemma 9.** Let \( \Omega \) be a \( C^2 \)-smooth bounded pseudoconvex domain in \( \mathbb{C}^n \), \( z_0 \in \mathcal{B} \) be a strongly pseudoconvex point, and \( \Omega^p \) be defined as in (7.1). Then there exists \( s > 0 \) such that \( (z,w) \in \mathcal{B} \) is strongly pseudoconvex for \( |z-z_0| < s \) and \( w_k \neq 0 \) for all \( 1 \leq k \leq p \).

**Proof.** Let \( \tilde{\rho}(z,w) = \rho(z) + \lambda(w) \) where \( \lambda(w) = |w_1|^{2p/r} + \cdots + |w_p|^{2p/r} \) and \( p \geq r \) an integer. Then \( \tilde{\rho} \) is a \( C^2 \)-smooth function. Assume that \( Q = (z,w) \in \mathcal{B} \) is near \( z_0 \) and \( X \) is a complex tangential vector to \( \mathcal{B} \) at \( Q \). Then \( X \) can be written as \( X = X_n + X_p \) where \( X_n \) and \( X_p \) are the components of \( X \) in the \( z \) and \( w \) variables, respectively. Then

\[
H_{\tilde{\rho}}(Q;X) = H_{\rho}(z,X_n) + H_{\rho}(Q;X_n,X_p) + H_{\rho}(Q;X_p,X_n) + H_{\lambda}(w;X_p).
\]

However, \( H_{\tilde{\rho}}(Q;X_n,X_p) = H_{\tilde{\rho}}(Q;X_p,X_n) = 0 \) as \( z \) and \( w \) are decoupled in \( \tilde{\rho} \). Then

\[
H_{\tilde{\rho}}(Q;X) = H_{\rho}(z,X_n) + H_{\lambda}(w;X_p).
\]

Let \( \pi \) denote the projection from a neighborhood of \( b\Omega \) in \( \mathbb{C}^n \) onto \( \mathbb{C}^n \). Then \( X_n = X_t + X_v \) where \( X_t \) is a tangential vector to \( b\Omega \) at \( \pi z \) and \( X_v \) is a vector complex normal to \( b\Omega \) at \( \pi z \). Then

\[
H_{\rho}(z,X_n) = H_{\rho}(z,X_t) + H_{\rho}(z,X_t,X_v) + H_{\rho}(z,X_v,X_t) + H_{\rho}(z,X_v).
\]

We note that the complex Hessian \( H_{\rho} \) changes continuously and \( w \to 0 \) as \( z \to z_0 \) (here we assume that \( (z,w) \in \mathcal{B} \)). Furthermore, \( X_v \to 0 \) as \( z \to z_0 \) (as the complex normal to \( b\Omega \) at \( z_0 \) is parallel to the complex normal to \( \mathcal{B} \) at \( (z_0,0) \)). Then, using the fact that \( z_0 \) is a strongly pseudoconvex
Let \( \Omega \)

\[ H_\rho(z; X_u) \geq \frac{H_\rho(\pi z; X_t)}{2} > 0 \]

for \(|z - z_0| < s \) and \( X_t \neq 0 \). Also \( H_\lambda(w; X_p) > 0 \) whenever \( X_p \neq 0 \) and \( w_k \neq 0 \) for all \( k \) as \( \lambda \) is strictly plurisubharmonic whenever \( w_k \neq 0 \) for all \( k \). Therefore, \( H_\rho(Q; X) > 0 \) for \( Q = (z,w) \in b\Omega^p_r \) such that \(|z - z_0| < s \) and \( w_k \neq 0 \) for all \( k \).

The following corollary follows from the previous lemma together with the fact that \( \Omega^p_r \) has C\(^2\)-smooth boundary for \( 0 < r \leq p \).

**Corollary 10.** Let \( \Omega \) be a C\(^2\)-smooth bounded pseudoconvex domain in \( \mathbb{C}^n \), \( z_0 \in b\Omega \) be a strongly pseudoconvex point, and \( \Omega^p_r \) be defined as in (7.1). Then there exists \( \varepsilon > 0 \) such that \( B((z_0,0),\varepsilon) \cap \Omega^p_r \) is pseudoconvex.

Next we will prove some statements about compactness of single Toeplitz and Hankel operators.

**Lemma 11.** Let \( \phi \in L^\infty(\Omega) \), \( \{f_j\} \) be a bounded sequence in \( A^2(\Omega,(-\rho)^r) \) and \( F_j \) be the trivial extension of \( f_j \) to \( \Omega^p_r \) for each \( j \) where \( \Omega^p_r \) be defined as in (7.1). Assume that \( \{H^{\Omega^p_r}_\phi F_j\} \) is convergent in \( L^2(\Omega^p_r) \). Then \( \{H^{\Omega^p_r}_{\phi^r} F_j\} \) is convergent in \( L^2(\Omega,(-\rho)^r) \).

**Proof.** We will abuse the notation and denote the trivial extension of \( \phi \) to \( \Omega^p_r \) by \( \phi \). We assume that \( \{H^{\Omega^p_r}_{\phi^r} F_j\} \) is convergent (and hence Cauchy). Let

\[ G_j(z,w) = (H^{\Omega^p_r}_{\phi^r} F_j)(z,w) \]

and \( g_j(z) = G_j(z,0) \). Then \( G_j \) is holomorphic in \( w \) because

\[ \frac{\partial G_j}{\partial w_k} = \frac{\partial}{\partial w_k}(I - p^{\Omega^p_r})(F_j \phi) = \frac{\partial (F_j \phi)}{\partial w_k} = 0 \]

for all \( j \) and \( 1 \leq k \leq p \). We note that \( \frac{\partial (F_j \phi)}{\partial w_k} = 0 \) as \( F_j \phi \) is independent of \( w_k \). Then \( |G_j(z,w) - G_k(z,w)|^2 \) is subharmonic in \( w \) and using the mean value property for subharmonic functions together with (8.2) and (8.3) one can show that

\[ |g_j(z) - g_k(z)|^2 \leq \frac{1}{c_{p,r}(-\rho(z))^r} \int_{|w_1|^{2p/r} + \cdots + |w_p|^{2p/r} < -\rho(z)} |G_j(z,w) - G_k(z,w)|^2 dV(w) \]

for \( j, k = 1,2,\ldots \) and \( z \in \Omega \). By integrating over \( \Omega \) we get

\[ c_{p,r} \|g_j - g_k\|_{L^2(\Omega,(-\rho)^r)}^2 \leq \|G_j - G_k\|_{L^2(\Omega^p_r)}^2 \]

for \( j, k = 1,2,\ldots \). Then \( \{g_j\} \) is a Cauchy sequence in \( L^2(\Omega,(-\rho)^r) \) (and hence convergent) because \( \|G_j - G_k\|_{L^2(\Omega^p_r)} \to 0 \) as \( j, k \to \infty \).
Let $h_j(z) = P_{\Omega^p}(\phi F_j)(z, 0)$. Then
\[c_{r,p}||h_j||^2_{L^2(\Omega, (-\rho)^r)} \leq ||P_{\Omega^p}(\phi F_j)||^2_{L^2(\Omega^p)} \leq ||\phi F_j||^2_{L^2(\Omega^p)} = c_{r,p}||\phi f_j||^2_{L^2(\Omega, (-\rho)^r)} < \infty\]
for each $j$. Hence, $h_j \in A^2(\Omega, (-\rho)^r)$ and $(I - P_{\Omega^r})h_j = 0$ for all $j$. We get equality between the last terms above because $F_j$ and $\phi$ are independent of $w$. Now
\[(I - P_{\Omega^r})g_j = (I - P_{\Omega^r}) \left( \phi f_j - P_{\Omega^p}(\phi F_j)(., 0) \right) = (I - P_{\Omega^r}) \left( \phi f_j \right) - (I - P_{\Omega^r}) \left( h_j \right) = H_{\phi, r}^{\Omega} f_j.\]
Therefore, the sequence $\{H_{\phi, r}^{\Omega} f_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$. \hfill \Box

**Lemma 12.** Let $r$ be a nonnegative real number and $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function $\rho$. Assume that $\phi \in C(\overline{\Omega})$ such that $\phi(z) = 0$ if $z$ is a strongly pseudoconvex point in $b\Omega$. Then $T_{\phi, r}^\Omega$ is compact about strongly pseudoconvex points on $A^2(\Omega, (-\rho)^r)$.

**Proof.** Let $\{f_j\}$ be a sequence in $A^2(\Omega, (-\rho)^r)$ that (without loss of generality) converges to 0 weakly about strongly pseudoconvex points. Then $f_j \to 0$ weakly as $j \to \infty$ and there is a neighborhood $U$ of weakly pseudoconvex points in $b\Omega$ such that
\[\|f_j\|_{L^2(U \cap \Omega, (-\rho)^r)} \to 0 \text{ as } j \to \infty.\]
Using the uniform boundedness principle and the fact that $f_j \to 0$ weakly we conclude that the sequence $\{f_j\}$ is bounded in $A^2(\Omega, (-\rho)^r)$. Furthermore, Cauchy estimates together with Montel’s Theorem (and the fact that $f_j \to 0$ weakly) imply that $\{f_j\}$ converges to zero uniformly on compact subsets of $\Omega$. Using the fact that $\phi = 0$ on strongly pseudoconvex points, one can show that $\phi f_j \to 0$ in $A^2(\Omega, (-\rho)^r)$. Therefore, $T_{\phi, r}^\Omega f_j \to 0$ in $A^2(\Omega, (-\rho)^r)$. That is, $T_{\phi, r}^\Omega$ is compact about strongly pseudoconvex points on $A^2(\Omega, (-\rho)^r)$. \hfill \Box

Let $\Omega$ be a domain in $\mathbb{C}^n$. Then $z \in b\Omega$ is said to have a holomorphic (plurisubharmonic) peak function if there exists a holomorphic (plurisubharmonic) $f$ that is continuous on $\overline{\Omega}$ such that $f(z) = 1$ and $|f(w)| < 1$ ($f(w) < 1$) for $w \in \overline{\Omega} \setminus \{z\}$.

Next we show that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

**Proposition 13.** Let $r$ be a nonnegative real number, $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function $\rho$, and $\phi \in C(\overline{\Omega})$. Then $H_{\phi}^{\Omega^r} : A^2(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$ is compact about strongly pseudoconvex points.

**Proof.** We will prove more (see Corollary 14 below). First of all, for any $\phi \in C(\overline{\Omega})$ there exists $\{\phi_j\} \subset C^1(\overline{\Omega})$ such that $\phi_j \to \phi$ uniformly on $\overline{\Omega}$ as $j \to \infty$. Furthermore, $\{H_{\phi_j}^{\Omega^r}\}$ converges to $H_{\phi}^{\Omega^r}$ in the operator norm and, by Proposition 7 if $H_{\phi_j}^{\Omega^r}$ is compact about strongly pseudoconvex point for
every \( j \) then so is \( H_\phi^r \). Therefore, for the rest of the proof we will assume that \( \phi \in C^1(\overline{\Omega}) \). Secondly, the proof for \( r = 0 \) does not require the inflation argument in the next paragraph and hence it is easier than the case \( r > 0 \). Since both proofs are similar, except for the inflation argument, in the rest of the proof, we will assume that \( r > 0 \).

Let \( z_0 \in b\Omega \) be a strongly pseudoconvex point. Then, by Corollary 10, the domain \( B((z_0,0),\varepsilon) \cap \Omega^p_r \) is pseudoconvex for small \( \varepsilon \). Let \( \varepsilon > 0 \) be such that \( X_0 = b\Omega \cap B(z_0,\varepsilon) \subset \mathbb{C}^n \) consists of strongly pseudoconvex points. Let us define
\[
Y = b\Omega^p_r \cap \overline{B(z_0,\varepsilon)} \cap \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^p : w_k = 0 \text{ for some } 1 \leq k \leq p\}
\]
\[
X_j = b\Omega^p_r \cap \overline{B(z_0,\varepsilon)} \cap \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^p : |w_k| \geq 1/j \text{ for all } 1 \leq k \leq p\}
\]
for \( j = 1,2,3,\ldots \). Then \( X_0 \) is B-regular as any point in \( X_0 \) has a holomorphic (hence plurisubharmonic) peak function on \( \Omega \subset \mathbb{C}^n \). The same function (by extending it trivially) is also a plurisubharmonic peak function on \( \Omega^p_r \subset \mathbb{C}^{n+p} \). Hence, \( X_0 \) is B-regular as a compact set in \( \mathbb{C}^{n+p} \). Furthermore, Lemma 9 implies that we can shrink \( \varepsilon \), if necessary, so that \( X_j \)'s are composed of strongly pseudoconvex points for \( j \geq 1 \). Hence, \( X_j \) is B-regular for every \( j = 0,1,2,\ldots \).

Next we will apply a similar idea to \( Y \) in lower dimensions. Let us define \( Y_1 = \cup_{m=1}^p Y_1^m \) where
\[
Y_1^m = b\Omega^p_r \cap \overline{B(z_0,\varepsilon)} \cap \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^p : w_k = 0 \text{ for } k \neq m\}.
\]
We can write \( Y_1^m \) as the union of \( X_0 \) together with the compact sets
\[
b\Omega^p_r \cap \overline{B(z_0,\varepsilon)} \cap \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^p : |w_m| \geq 1/j, w_k = 0 \text{ for } k \neq m\}
\]
for \( j = 1,2,3,\ldots \). However, we can think of the sets above as subsets in \( \mathbb{C}^n \times \mathbb{C} \) and (by Lemma 9) they are composed of strongly pseudoconvex points. Hence, they are B-regular. Then [Sib87, Proposition 1.9] implies that each \( Y_1^m \) is B-regular as it is a countable union of B-regular sets. Hence, applying Sibony’s proposition again, we conclude that \( Y_1 \) is compact. Similarly, we can define \( Y_2 \subset Y \) as a countable union of compact sets where all but at most two \( w_k \)'s are equal to 0. Using the same reasoning above adopted for \( Y_2 \) we can conclude that \( Y_2 \) is B-regular. In a similar fashion, we can define \( Y_l \) for \( 1 \leq l \leq p-1 \) and prove that all of them are B-regular. Hence \( Y = (\cup_{l=1}^p Y_l) \cup X_0 \) is B-regular. Then
\[
b(\Omega^p_r \cap B(z_0,\varepsilon)) \subset Y \cup (\cup_{j=0}^\infty X_j) \cup bB(z_0,\varepsilon)
\]
is B-regular (satisfies Property (P) in Catlin’s terminology) and, hence, the \( \overline{\partial} \)-Neumann operator on \( \Omega^p_r \cap B(z_0,\varepsilon) \) is compact (see [Str10, Theorem 4.8] and [Cat84]). Then \( H^{\Omega^p_r \cap B(z_0,\varepsilon)}_\phi \) is compact (see [Str10, Proposition 4.1]) and Lemma 11 implies that \( H^{\Omega^p_r \cap B(z_0,\varepsilon),r}_\phi \) is compact.

Next we will use local compact solution operators to show that \( H^{\Omega^p_r \cap B(z_0,\varepsilon)}_\phi \) is compact about strongly pseudoconvex points. Let \( \{f_j\} \subset A^2(\Omega, (-\rho)^r) \) be a sequence weakly convergent about strongly pseudoconvex points. Then there exists an open neighborhood \( U \) of the set of weakly pseudoconvex points in \( b\Omega \) such that
i. $\{f_j\}$ is weakly convergent,

ii. $\|f_j - f_k\|_{L^2(U \cap \Omega, (-\rho)^r)} \to 0$ as $j, k \to \infty$.

Let us choose $\{p_k : k = 1, \ldots, m\} \subset \partial \Omega \setminus U$ and $\varepsilon_k > 0$ (for $k = 1, \ldots, m$) such that

i. $\partial \Omega \setminus U \subset \cup_{k=1}^m B(p_k, \varepsilon_k)$

ii. $H_{\phi}^{k,r} = H_{\phi}^{B(p_k, \varepsilon_k) \cap \Omega, r}$ is compact on $A^2(B(p_k, \varepsilon_k) \cap \Omega, (-\rho)^r)$ for $k = 1, \ldots, m$.

Let us choose a strongly pseudoconvex domain $\Omega_{-1} \Subset \Omega$ and smooth cut-off functions $\chi_{-1} \in C_0^\infty(\Omega_{-1}), \chi_0 \in C_0^\infty(U)$, and $\chi_k \in C_0^\infty(B(p_k, \varepsilon))$ for $k = 1, \ldots, m$ such that $\sum_{k=1}^m \chi_k \equiv 1$ on $\overline{\Omega}$.

Let $H_{\phi}^{-1,r} = H_{\phi}^{\Omega_{-1}, r}, H_{\phi}^{0,r} = H_{\phi}^{L^2(\Omega, r)}$, and $g_j = \sum_{k=1}^m \chi_k H_{\phi}^{k,r} f_j$. We note that $H_{\phi}^{-1,r}$ is compact as $\Omega_{-1} \Subset \Omega$ is strongly pseudoconvex (and $\rho < 0$ on the closure of $\Omega_{-1}$); $\{H_{\phi}^{0,r} f_j\}$ is convergent as $\{f_j\}$ is convergent in $L^2(U \cap \Omega, (-\rho)^r)$; and by the previous part of this proof, $H_{\phi}^{k,r}$ is compact for each $k = 1, \ldots, m$. Therefore, $\{g_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$. Furthermore,

$$\overline{\partial} g_j = f_j \overline{\partial} \phi + \sum_{k=1}^m \left( \overline{\partial} \chi_k \right) H_{\phi}^{k,r} f_j.$$  

Then $\left\{ \sum_{k=1}^m \left( \overline{\partial} \chi_k \right) H_{\phi}^{k,r} f_j \right\}$ is a convergent sequence of $\overline{\partial}$-closed $(0,1)$-forms as both $\overline{\partial} g_j$ and $f_j \overline{\partial} \phi$ are $\overline{\partial}$-closed. Let $Z^r : L^2_{(0,1)}(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$ be a bounded linear solution operator to $\overline{\partial}$ (see [Hör65]). Let

$$h_j = g_j - Z^r \sum_{k=1}^m \left( \overline{\partial} \chi_k \right) H_{\phi}^{k,r} f_j.$$  

Then $\{h_j\}$ is convergent and $\overline{\partial} h_j = f_j \overline{\partial} \phi$. So by taking projection on the orthogonal complement of $A^2(\Omega, (-\rho)^r)$ we get $(I - P^r) h_j = H_{\phi}^r f_j$. Therefore, $\{H_{\phi}^r f_j\}$ is convergent. \qed

Using the proof of the proposition above we get the following corollary.

**Corollary 14.** Let $r$ be a nonnegative real number and $\Omega$ be a $C^2$-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function $\rho$. Assume that $\Omega$ satisfies property (P) of Catlin (or $B$-regularity of Sibony). Then

i. $\overline{\partial}$ has a compact solution operator on $K^2_{(0,1)}(\Omega, (-\rho)^r)$, the weighted $\overline{\partial}$-closed $(0,1)$-forms,

ii. $H_{\phi}^r : A^2(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$ is compact for all $\phi \in C(\overline{\Omega})$.

**Proof.** Since ii. follows from i. we will only prove i. By a theorem Diederich and Fornæss [DF77] there exists a $C^2$-smooth defining function $\rho_1$ and $0 < \eta < 1$ such that $-(\rho_1)^\eta$ is a strictly plurisubharmonic exhaustion function for $\Omega$. Since $\rho_1$ and $\rho$ are comparable on $\overline{\Omega}$ it is enough to prove that $\overline{\partial}$ has a compact solution operator on $K^2_{(0,1)}(\Omega, (-\rho_1)^r)$.

Let $s = r/\eta \geq 0$ and $q$ be an integer such that $s \leq q$. We define

$$\Omega^q_s = \{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^q : -(\rho_1(z))^\eta + \lambda(w) < 0 \}$$  

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where \( \lambda(w) = |w_1|^{2q/s} + \cdots + |w_q|^{2q/s} \). Then \(-\rho_1 + \lambda\) is a bounded \( C^2\)-smooth plurisubharmonic function and \( \Omega_s^q \) is pseudoconvex. Furthermore, the first part of the proof of Proposition 13 shows that \( \Omega_s^q \) satisfies property (P).

Let \( \{f_j\} \) be a bounded sequence in \( K_{(0,1)}^{2}((\Omega, (-\rho_1)^r)) \). Then \( \{F_j\} \) is a bounded sequence in \( K_{(0,1)}^{2}((\Omega_s^q)) \). As shown in the first part of this proof, \( \Omega_s^q \) is a bounded (not necessarily \( C^2 \)-smooth) pseudoconvex domain with property (P). Then \( \bar{\partial} N^q_f F_j \) has a convergent subsequence in \( L^2(\Omega_s^q) \) where \( N^q_f \) is the \( \bar{\partial} \)-Neumann operator on \( L^2((\Omega_s^q)) \). By the proof of Proposition 8 and the fact that \( \bar{\partial} N^q_f F_j \) is holomorphic in \( w \), we conclude that \( \bar{\partial} N^q_f F_j(.,0) \in L^2(\Omega, (-\rho_1)^r) \). Furthermore, \( \bar{\partial} \delta N^q_f F_j(.,0) = f_j \) for all \( j \) and \( \{\bar{\partial} N^q_f F_j(.,0)\} \) has a convergent subsequence in \( L^2((\Omega, (-\rho_1)^r)) \). Therefore, \( \bar{\partial} \) has a compact solution operator \( R \bar{\partial} N^q_f E \) on \( K_{(0,1)}^{2}((\Omega, (-\rho_1)^r)) \) where \( E \) is the trivial extension operator and \( R \) is the restriction from \( \Omega_s^q \) onto \( \Omega \).

The following Lemma is essentially contained in the proof of [AE01, Proposition 1.3]. We present it here for the convenience of the reader.

**Lemma 15.** Let \( r \) be a nonnegative real number, \( \Omega \) be bounded domain in \( \mathbb{C}^n \), and \( \phi \in C(\bar{\Omega}) \). Assume that \( z_0 \in b\Omega \) has a holomorphic peak function. Then

\[
\lim_{z \to z_0} B_r T_{\phi}^\rho(z) = \phi(z_0).
\]

**Proof.** First, we prove that for any neighborhood \( U \) of \( z_0 \)

\[
(15.1) \quad \int_{\Omega \not\subset U} |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w) \to 0 \quad \text{as} \quad z \to z_0.
\]

Indeed, for given \( U \) and \( \varepsilon > 0 \) first we choose a holomorphic peak function \( g \) such that \( |g(w)| \leq \varepsilon \) for all \( w \in \Omega \setminus U \). This can be simply done by taking a high enough power of the holomorphic peak function \( g \). Then we choose \( \delta > 0 \) such that if \( |z - z_0| < \delta \) and \( z \in \Omega \) then \( |g(z)| > 1 - \varepsilon \). In this case,

\[
\int_{U} |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w) \geq \int_{U} |g(w)| |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w)
\]

\[
\geq \left| \int_{\Omega} g(w) |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w) \right|
\]

\[
- \left| \int_{\Omega \not\subset U} g(w) |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w) \right|
\]

\[
\geq |g(z)| - \int_{\Omega \not\subset U} |g(w)| |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w)
\]

\[
\geq 1 - \varepsilon - \varepsilon \int_{\Omega \not\subset U} |k^r_{z}(w)|^2 (-\rho(w))^r \, dV(w)
\]

\[
\geq 1 - 2\varepsilon.
\]
whenever $|z - z_0| < \delta$. This implies that for a given neighborhood $U$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$\int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r \, dV(w) \leq \varepsilon.$$ 

This gives (15.1).

Now for $\varepsilon > 0$, we choose a neighborhood $U$ of $z$ such that $|\phi(w) - \phi(z_0)| \leq \varepsilon$ for all $w \in U$. Then for this neighborhood $U$ and the same $\varepsilon$ we choose $\delta > 0$ such that if $|z - z_0| < \delta$ then

$$\int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r \, dV(w) \leq \frac{\varepsilon}{1 + \| \phi \|_{L^\infty}}.$$ 

In this case,

$$|B_r T_\phi(z) - \phi(z_0)| \leq \int_{\Omega} \| \phi(w) - \phi(z_0) \| \| k_z^r(w) \|^2 (-\rho(w))^r \, dV(w)$$

$$= \int_{U} \| \phi(w) - \phi(z_0) \| \| k_z^r(w) \|^2 (-\rho(w))^r \, dV(w)$$

$$+ \int_{\Omega \setminus U} \| \phi(w) - \phi(z_0) \| \| k_z^r(w) \|^2 (-\rho(w))^r \, dV(w)$$

$$\leq \varepsilon \int_{U} \| k_z^r(w) \|^2 (-\rho(w))^r \, dV(w)$$

$$+ 2 \| \phi \|_{L^\infty} \int_{\Omega \setminus U} \| k_z^r(w) \|^2 (-\rho(w))^r \, dV(w)$$

$$\leq \varepsilon + \varepsilon = 2\varepsilon.$$

This indeed concludes $\lim_{z \to z_0} B_r T_\phi(z) = \phi(z_0)$. \qed

We note that on any bounded domain, we have (see [CS14 Lemma 1])

$$T_{\phi_2}^r T_{\phi_1}^r = T_{\phi_2 \phi_1}^r - H_{\phi_2}^r H_{\phi_1}^r.$$ 

Using the fact above inductively one can prove the following lemma.

**Lemma 16.** Let $r$ be a nonnegative real number and $\Omega$ be a $C^1$-smooth bounded domain in $\mathbb{C}^n$ with a defining function $\rho$. Supposed $\phi_1, \ldots, \phi_m \in L^\infty(\Omega)$. Then

$$T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_2}^r T_{\phi_1}^r = T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1}^r - T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_3}^r H_{\phi_2}^r H_{\phi_1}^r$$

$$- T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_4}^r H_{\phi_3}^r H_{\phi_2}^r \phi_1 - \cdots - H_{\phi_m}^r H_{\phi_{m-1}}^r \cdots H_{\phi_2}^r \phi_1$$

$$= T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1}^r + S^r$$

where $S^r$ is a finite sum of finite products of operators and each product starts with a Hankel operator.

Therefore, if the symbols $\phi_1, \ldots, \phi_m$ are continuous on $\overline{\Omega}$ we can write

$$(16.1) \quad T_{\phi}^r \cdots T_{\phi_m}^r = T_{\phi_1 \cdots \phi_m}^r + S^r$$

where $S^r$ is a finite sum of finite products of operators such that each product starts with a Hankel operator with symbol continuous on $\overline{\Omega}$. 
We state the lemma below for general weights $\mu(z)$ (not only the ones of the form $(-\rho)^k$) that are nonnegative (can vanish on the boundary) and continuous on $\Omega$. The weights of this form are called admissible weights (see \cite{PW90}) and the corresponding weighted Bergman projections and kernels are well defined. We say two weights $\mu_1$ and $\mu_2$ are comparable if there exists $c > 0$ such that $c^{-1}\mu_1 < \mu_2 < c\mu_1$ on $\Omega$.

**Lemma 17.** Let $\Omega$ be a domain in $\mathbb{C}^n$ and $\mu_1$ and $\mu_2$ be comparable admissible weights. Let $k_z^{\mu_j}$ be the normalized Bergman kernel corresponding to $\mu_j$ for $j = 1, 2$ and $z_0 \in b\Omega$. Then $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$ if and only if $k_z^{\mu_2} \to 0$ weakly as $z \to z_0$.

**Proof.** It is enough to show one direction. So we will showed that if $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$ then $k_z^{\mu_2} \to 0$ weakly as $z \to z_0$. Since $\mu_1$ and $\mu_2$ are equivalent measures we have $A^2(\Omega, d\mu_1) = A^2(\Omega, d\mu_2)$ and there exists $C > 1$ such that

$$\frac{\|f\|_{\mu_1}}{C} \leq \|f\|_{\mu_2} \leq C\|f\|_{\mu_1}$$

for all $f \in A^2(\Omega, d\mu_1)$. We remind the reader that for $z \in \Omega$ we have

$$K_{\mu_j}(z, z) = \sup\{||f(z)||^2 : \|f\|_{\mu_j} \leq 1\}$$

where $K_{\mu_j}$ is the Bergman kernel corresponding to $\mu_j$. Then $K_{\mu_1}$ and $K_{\mu_2}$ are equivalent on the diagonal in the sense that there exists $D = C^2 > 1$ such that

$$\frac{K_{\mu_1}(z, z)}{D} \leq K_{\mu_2}(z, z) \leq DK_{\mu_1}(z, z).$$

Now we assume that $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$. Let us fix $f \in A^2(\Omega, d\mu_1)$. Then we have

$$\frac{f(z)}{\sqrt{K_{\mu_1}(z, z)}} = \langle f, k_z^{\mu_1} \rangle_{\mu_1} \to 0 \text{ as } z \to z_0.$$

Then

$$\langle f, k_z^{\mu_2} \rangle_{\mu_2} = \frac{f(z)}{\sqrt{K_{\mu_2}(z, z)}} \to 0 \text{ as } z \to z_0.$$}

Therefore, we showed that if $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$ then $k_z^{\mu_2} \to 0$ weakly as $z \to z_0$. \hfill \Box

Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ and $z_0 \in b\Omega$. Then we call $z_0$ a **bumping point** if for any $\delta > 0$ there exists a pseudoconvex domain $\Omega_1$ such that $\{z_0\} \cup \Omega \subseteq \Omega_1 \subseteq \Omega \cup B(z_0, \delta)$.

**Lemma 18.** Let $r$ be a nonnegative real number, $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary, and $z_0 \in b\Omega$ be a bumping point. Then $k_z^r \to 0$ weakly as $z \to z_0$.

**Proof.** By Lemma 17, without loss of generality, we assume that $\rho$ denotes the negative distance to the boundary of $\Omega$.

Let us fix $f \in A^2(\Omega, (-\rho)^r)$ and choose $r_1, r_2 > 0$ so that $0 < r_1 < r_2$ and the outward unit vector $\nu$ is transversal to $B(z_0, 2r_2) \cap b\Omega$. Since $z_0$ is a bumping point we choose a bounded pseudoconvex
domain $\Omega_1$ such that
\[
\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, r_1).
\]
So even though $\Omega_1$ contains a small neighborhood of $z_0$, we have $\Omega \setminus B(z_0, r_1) = \Omega_1 \setminus B(z_0, r_1)$.

Let us choose $\chi \in C^\infty_0(B(z_0, r_2))$ such that $\chi \equiv 1$ on a neighborhood of $B(z_0, r_1)$. For $\epsilon > 0$ small we define $f_\epsilon(z) = f(z - \epsilon v)$ and $g_\epsilon = (1 - \chi)f + \chi f_\epsilon$. Then
\begin{enumerate}
  \item $f_\epsilon \in A^2(\Omega \cap B(z_0, r_2), (-\rho)^r)$ and $f_\epsilon \to f$ in $L^2(\Omega \cap B(z_0, r_2), (-\rho)^r)$,
  \item $g_\epsilon|_{\Omega \cap B(z_0, r_2)}$ is $C^\infty$-smooth and $g_\epsilon \to f$ in $L^2(\Omega, (-\rho)^r)$ as $\epsilon \to 0$.
\end{enumerate}

Let $\rho_1$ and $\text{Supp}(\overline{\partial} \chi)$ denote the negative distance to the boundary of $\Omega_1$ and the support of $\overline{\partial} \chi$, respectively. Then $\text{Supp}(\overline{\partial} \chi) \cap \Omega = \text{Supp}(\overline{\partial} \chi) \cap \Omega_1$ and $-\rho$ and $-\rho_1$ are equivalent on the support of $\overline{\partial} \chi$. Furthermore, $\overline{\partial} g_\epsilon$ is a $\overline{\partial}$-closed $(0, 1)$-form on $\Omega_1$ ($\overline{\partial} g_\epsilon$ is well defined on $\Omega_1$ as $\overline{\partial} \chi = 0$ on $B(z_0, r_1)$) for all small $\epsilon > 0$ and there exists $C > 0$ such that
\[
\|\overline{\partial} g_\epsilon\|_{L^2(\Omega_1, (-\rho_1)^r)} \leq C\|f - f_\epsilon\|_{L^2(\Omega_1 \cap B(z_0, r_2), (-\rho)^r)} \|\overline{\partial} \chi\|_{L^\infty(B(z_0, r_2))} \to 0 \text{ as } \epsilon \to 0.
\]

Next we will use Hörmander’s theorem \[\text{[Hör65]}\] with the plurisubharmonic exponential weight $-r \log(-\rho_1)$. We note that $-\log(-\rho_1)$ is plurisubharmonic because $\Omega_1$ is pseudoconvex. Then using Hörmander’s theorem we get a constant $c_1 > 0$ (depending on $\Omega_1$) and $h_\epsilon \in L^2(\Omega_1)$ such that
\[
\overline{\partial} h_\epsilon = \overline{\partial} g_\epsilon \text{ and } \|h_\epsilon\|_{L^2(\Omega_1, (-\rho_1)^r)} \leq c_1 \|\overline{\partial} g_\epsilon\|_{L^2(\Omega_1, (-\rho_1)^r)}.
\]
Furthermore, since $\overline{\partial}$ is elliptic on the interior and $\overline{\partial} g_\epsilon$ is $C^\infty$-smooth on $\Omega_1$, we have $h_\epsilon \in C^\infty(\Omega_1)$.

We define $\tilde{f}_n = g_{1/n} - h_{1/n}$. Then we have
\begin{enumerate}
  \item $\tilde{f}_n \in A^2(\Omega, (-\rho)^r)$ and $\tilde{f}_n \to f$ in $A^2(\Omega, (-\rho)^r)$,
  \item $\tilde{f}_n|_{\Omega \cap B(z_0, r_1)} \in C^\infty(\Omega \cap B(z_0, r_1)).$
\end{enumerate}

So $\{\tilde{f}_n\}$ is a sequence converging to $f$ and each member of the sequence is smooth up to the boundary of $\Omega$ on a neighborhood of $z_0$.

Finally, we will show weak convergence of $k_z^r$ to 0 as $z \to z_0$.
\[
\left| \langle f, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)} \right| \leq \left| \langle f - \tilde{f}_n, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)} \right| + \left| \langle \tilde{f}_n, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)} \right| \\
\leq \|f - \tilde{f}_n\|_{L^2(\Omega, (-\rho)^r)} + \frac{\|\tilde{f}_n(z)\|}{\sqrt{K_r(z, z)}}.
\]
The first term on the right hand side can be made arbitrarily small for large enough $n$, because $\|f - \tilde{f}_n\|_{L^2(\Omega, (-\rho)^r)} \to 0$ as $n \to \infty$. So for $\delta > 0$ given we choose $n_\delta$ so that $\|f - \tilde{f}_{n_\delta}\|_{L^2(\Omega, (-\rho)^r)} \leq \delta$. Then since $\tilde{f}_{n_\delta}$ is $C^\infty$-smooth on $\Omega \cap B(z_0, r_1)$ (and $K_r(z, z) \to \infty$ as $z \to z_0$) we conclude that $|\tilde{f}_{n_\delta}(z)|/\sqrt{K_r(z, z)} \to 0$ as $z \to z_0$. Hence, $\limsup_{z \to z_0} |\langle f, k_z^r \rangle| \leq \delta$ for arbitrary $\delta > 0$. Therefore, $k_z^r \to 0$ weakly as $z \to z_0$.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. In case $r = 0$, the proof of the theorem simplifies greatly as inflation and the related techniques are unnecessary. So we will prove the more difficult case, $r > 0$. 
First we assume that $T$ is compact about strongly pseudoconvex points. Let $\Omega_{r}^{p}$ be defined as in (7.1) and $z_{0} \in b\Omega$ be a strongly pseudoconvex point. Since small $C^{2}$-perturbations of strongly pseudoconvex points stay pseudoconvex, $z_{0}$ is a bumping point for $\Omega$. Then Lemma 18 implies that $k_{z}^{r} \to 0$ weakly as $z \to z_{0}$. Furthermore, there exists an open neighborhood $U$ of $z_{0}$ such that weakly pseudoconvex points are contained in $b\Omega \setminus U$; and, as in the proof of (15.1), one can show that

$$\|k_{z}^{r}\|_{L^{2}(\Omega \setminus \overline{U}, (-\rho)r)} \to 0 \text{ as } z \to z_{0}.$$ 

Therefore, $\{k_{z}^{r}\}$ converges to 0 weakly about strongly pseudoconvex points as $z \to z_{0}$. Moreover, since $T$ is compact about strongly pseudoconvex points (such operators map sequences of holomorphic functions weakly convergent about strongly pseudoconvex points to convergent sequences) we conclude that

$$B_{r}T(z) = \langle Tk_{z}^{r}, k_{z}^{r} \rangle_{A^{2}(\Omega, (-\rho)r)} \to 0$$

as $z \to z_{0}$.

Next we prove the other direction. As a first step we assume that $T$ is a finite sum of finite products of Toeplitz operators on $A^{2}(\Omega, (-\rho)r)$ with symbols continuous on $\overline{\Omega}$. Furthermore, we assume that

$$\lim_{z \to z_{0}} B_{r}T(z) = 0$$

for any strongly pseudoconvex point $z_{0} \in b\Omega$.

Lemma 16 implies that

$$(18.1) \quad T = T_{\phi}^{r} + S^{r}$$

where $\phi \in C(\overline{\Omega})$ and $S^{r}$ is a sum of operators that start with a Hankel operator with symbol continuous on $\overline{\Omega}$.

Lemma 15 implies that

$$(18.2) \quad \lim_{z \to z_{0}} B_{r}T_{\phi}^{r}(z) = \phi(z_{0})$$

as strongly pseudoconvex points have holomorphic peak functions (see, [Ran86, Theorem 1.13 in Ch VI]).

By Proposition 13 the operator $H_{\psi}^{r}$ is compact about strongly pseudoconvex points for any $\psi \in C(\overline{\Omega})$. Then $H_{\psi}^{r}k_{z}^{r} \to 0$ as $z \to z_{0}$ for any $\psi \in C(\overline{\Omega})$ because, as proven in the first part of this proof, $k_{z}^{r} \to 0$ weakly about strongly pseudoconvex points as $z \to z_{0}$. Hence, $B_{r}S^{r}(z) \to 0$ as $z \to z_{0}$. Combining this with (18.1) and (18.2) we can conclude that

$$\phi(z_{0}) = \lim_{z \to z_{0}} B_{r}T(z) = 0.$$ 

Since $z_{0}$ was an arbitrary strongly pseudoconvex point, we have $\phi = 0$ on all the strongly pseudoconvex boundary points. Then Lemma 12 and the fact that $S^{r}$ is compact about strongly pseudoconvex points imply that $T$ is compact about strongly pseudoconvex points.
Finally, we assume that $T \in \mathcal{S}(\overline{\Omega}, (-\rho)^r)$. Then, using Lemma 16 for every $\varepsilon > 0$ there exists $\phi_\varepsilon \in C(\overline{\Omega})$ and an operator $S'_{\phi_\varepsilon}$, compact about strongly pseudoconvex points, such that
\[ \| T + T'_{\phi_\varepsilon} + S'_{\phi_\varepsilon} \| \leq \varepsilon. \]
Then for $z \in \Omega$ we have
\[ \left| B_r T(z) + B_r T'_{\phi_\varepsilon}(z) + B_r S'_{\phi_\varepsilon}(z) \right| = \left| T K'_{\phi_\varepsilon} + T'_{\phi_\varepsilon} k'_z + S'_{\phi_\varepsilon} K'_z k'_z \right| \leq \| T + T'_{\phi_\varepsilon} + S'_{\phi_\varepsilon} \| \leq \varepsilon. \]
Since $B_r S'_{\phi_\varepsilon}(z) \to 0$ and $B_r T'_{\phi_\varepsilon}(z) \to \phi_\varepsilon(z_0)$ (and we assume that $B_r T(z) \to 0$ as $z \to z_0$) as $z \to z_0$ we have $|\phi_\varepsilon(z_0)| \leq \varepsilon$. That is, $|\phi_\varepsilon| \leq \varepsilon$ on strongly pseudoconvex points of $\Omega$. We choose $\psi_\varepsilon \in C(\overline{\Omega})$ such that $\psi_\varepsilon = 0$ on strongly pseudoconvex boundary points of $\Omega$ and
\[ \sup \{|\psi_\varepsilon(z) - \phi_\varepsilon(z)| : z \in \overline{\Omega}\} \leq 2\varepsilon. \]
Then Lemma 12 implies that $T'_{\phi_\varepsilon}$ is compact about strongly pseudoconvex points and
\[ \| T'_{\phi_\varepsilon} - T'_{\psi_\varepsilon} \| \leq 2\varepsilon. \]
Hence
\[ \| T + T'_{\psi_\varepsilon} + S'_{\psi_\varepsilon} \| \leq \| T + T'_{\phi_\varepsilon} + S'_{\phi_\varepsilon} \| + \| T'_{\psi_\varepsilon} - T'_{\phi_\varepsilon} \| \leq 3\varepsilon. \]
Therefore, $T$ is in the norm closure of the compact about strongly pseudoconvex points operators. Finally, Proposition 7 implies that $T$ is compact about strongly pseudoconvex points. \[ \square \]

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