Energy-momentum tensor is nonsymmetric for spin-polarized photons

Xiang-Bai Chen\textsuperscript{a,b,*} and Xiang-Song Chen\textsuperscript{a†}

\textsuperscript{a}School of Physics and MOE Key Laboratory of Fundamental Quantities Measurement, Huazhong University of Science and Technology, Wuhan 430074, China
\textsuperscript{b}Department of Applied Physics, Konkuk University, Chungju 380-701, Korea

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It has been assumed for a century that the energy-momentum tensor of the photon takes a symmetric form, with the renowned Poynting vector assigned as the same density for momentum and energy flow. Here we show that the symmetry of the photon energy-momentum tensor can actually be inferred from the known difference between the diffraction patterns of light with spin and orbital angular momentum, respectively. The conclusion is that the symmetric expression of energy-momentum tensor is denied, and the nonsymmetric canonical expression is favored.

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Energy, momentum, and angular momentum are among the most fundamental quantities in physics. It is awkward that these quantities can often arouse controversy and confusion. In hadron physics, for example, there is no universally accepted scheme to analyze the quark-gluon origin of the nucleon momentum and spin\textsuperscript{[1–3]}. Even for the familiar photon (or electromagnetic field), the expression of momentum $\vec{P}$ and angular momentum $\vec{J}$ is problematic under close look. One often sees the mixed use of two expressions:

\begin{equation}
\vec{J} = \int d^3 x \vec{x} \times (\vec{E} \times \vec{B}) = \int d^3 x \vec{x} \times \vec{E} \frac{\partial}{\partial t} \vec{A} + \int d^3 x \vec{x} \times \vec{A} = \vec{L} + \vec{S} , \tag{1b}
\end{equation}

Eq. (1a) contains the renowned Poynting vector $\vec{E} \times \vec{B}$, which is derived as the electromagnetic momentum density in common textbooks. It is however Eq. (1b) that separates the intrinsic spin $\vec{S}$ from the extrinsic orbital angular momentum $\vec{L}$. For a free field, it can be easily shown that Eqs. (1a) and (1b) give the same conserved total $\vec{J}$, and therefore are often regarded as being identical. However, the momentum and angular momentum densities are after all different in the two expressions. Our aim is to show that one expression must be wrong at the density level, and can actually be inferred from the known experiments (see Fig. 1 and our explanations below).

Eqs. (1a) and (1b) correspond to different expressions of the angular momentum tensor:

\begin{equation}
\mathcal{M}^{\lambda \mu \nu} = \frac{1}{2} \left[ x^\mu \Theta^{\lambda \nu} - x^\nu \Theta^{\lambda \mu} , \right. \tag{2a}
\end{equation}

\begin{equation}
\mathcal{M}^{\lambda \mu \nu} = x^\mu T^{\lambda \nu} - x^\nu T^{\lambda \mu} + F^{\lambda \nu} A^\mu - F^{\lambda \mu} A^\nu . \tag{2b}
\end{equation}

Here Eq. (2a) gives the total angular momentum with an orbital-like expression, by using the symmetric energy-momentum tensor

\begin{equation}
\Theta^{\mu \nu} = F^{\mu \rho} F^\rho_\nu + \frac{1}{4} g^{\mu \nu} F^2 = \Theta^{\nu \mu} , \tag{3}
\end{equation}

while in Eq. (2b) the explicit $x$-dependent part is only the orbital contribution, constructed with the nonsymmetric canonical energy-momentum tensor

\begin{equation}
T^{\mu \nu} = -F^{\mu \rho} \partial^\nu A_\rho + \frac{1}{4} g^{\mu \nu} F^2 \neq T^{\nu \mu} . \tag{4}
\end{equation}

The two angular momentum tensors are both conserved:

\begin{equation}
\partial_\lambda \mathcal{M}^{\lambda \mu \nu} = \partial_\lambda \mathcal{M}^{\lambda \nu \mu} = 0 , \tag{5}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{diffraction.png}
\caption{Single-slit diffraction patterns of light with spin $s$ and orbital angular momentum $l$. (a) $l = +1$, (b) $l = -1$, (c) $s = +1$, (d) $s = -1$. (a) and (b) are quoted from Fig. 5 of Ref. [5]. Nonzero $l$ leads to distortion of the diffraction fringes, while nonzero $s$ does not. By a careful analysis, this simple but critical difference can tell that the energy-momentum tensor cannot be symmetric for spin-polarized photons.}
\end{figure}
and give the same angular momentum in Eqs. (1):

\[ J^k = \frac{1}{2} \epsilon_{ijk} \int d^3 x \mathcal{M}^{0ij} = \frac{1}{2} \epsilon_{ijk} \int d^3 x M^{0ij}. \]  

(6)

Similarly, the two different energy-momentum tensors are both conserved and give the same 4-momentum:

\[ \partial_\mu \Theta^{\mu \nu} = \partial_\mu T^{\mu \nu} = 0, \]

(7)

\[ P^\nu = \int d^3 x \Theta^{0 \nu} = \int d^3 x T^{0 \nu}. \]

(8)

We will explain the concrete experimental evidence that the elegant expressions \( \Theta^{\mu \nu} \) and \( \mathcal{M}^{\mu \nu} \), despite their popularity, are really wrong. In Ref. [4], we already gave a hint by examining the energy-flow component for polarized electrons the nonsymmetric canonical energy-momentum tensor is favored over the symmetric one, and proposed an experimental test. The energy-flow component is nevertheless unable to discriminate \( \Theta^{\mu \nu} \) from \( T^{\mu \nu} \) for the photon [4]. In this paper, we look at the more delicate momentum-flow component.

Consider a light beam propagating along the \( z \) axis, around which the beam is rotationally symmetric. Suppose that \( T^{\mu \nu} \) is the true energy-momentum tensor of the light beam. The components \( T^{zx} \) and \( T^{zy} \) describe the flow of \( P^x \) and \( P^y \) along the \( z \) direction. An often ignored but vital fact is that \( T^{zx} \) and \( T^{zy} \) are measurable locally. Especially, we will see that for the sake of telling the symmetry of \( T^{\mu \nu} \), it suffices to make a rough estimation of the momentum flux by looking at the diffraction fringes after passing through a small aperture. Hence, the energy-momentum tensor does not have the usually assumed arbitrariness. This warns us that \( \Theta^{\mu \nu} \) and \( T^{\mu \nu} \) cannot be both correct, and can in principle be discriminated by comparing the measured density of momentum flow with that calculated via \( \Theta^{\mu \nu} \) and \( T^{\mu \nu} \). In what follows, we present a clever way to experimentally distinguish \( \Theta^{\mu \nu} \) from \( T^{\mu \nu} \), with no need to know the detailed profile or wavefunction of the beam.

The technique is to look at a very elucidating quantity

\[ K_z = \int M^{xzx} dxdy = \int (xT^{zx} - yT^{zz}) dxdy. \]  

(9)

Here the integration is over the beam cross section, say, in the \( x-y \) plane. If \( T^{\mu \nu} = \Theta^{\mu \nu} \), then \( K_z \) measures the flow of the total angular momentum \( J_z \) across the \( x-y \) plane. But if \( T^{\mu \nu} = T^{\mu \nu} \), then \( K_z \) measures only the flow of the orbital angular momentum \( L^z \) across the \( x-y \) plane.

By Eq. (9), the angular-momentum flow \( K_z^2 \) is the \textit{moment} of the momentum flow \( T^{xz} \). To get a nonzero \( K_z^2 \), the integration in Eq. (9), the momentum flow \( T^{zx} \) and \( T^{zy} \) must display a circular behavior around the \( z \) axis. For example, to give a positive \( K_z^2 \), the dominating configuration must be \( T^{zy} > 0 \) for \( x > 0 \) and \( T^{zy} < 0 \) for \( x < 0 \) (and similarly, \( T^{zx} < 0 \) for \( y > 0 \) and \( T^{zx} > 0 \) for \( y < 0 \)). In this case, therefore, (remembering the physical meaning of momentum flow,) the photons passing though the region with \( x > 0 \) must have a net positive \( P^y \), while those through \( x < 0 \) must have a net negative \( P^y \). (Similarly, the diffracted photons through \( y > 0 \) must have a net negative \( P^x \), while those through \( y < 0 \) must have a net positive \( P^x \).) Such net momentum would manifest in diffraction pattern of light: the diffraction fringe would shift towards \(+y (-y)\) direction for \( x > 0 \) \((x < 0)\), or shift towards \(+x (-x)\) direction for \( y < 0 \) \((y > 0)\), leaving a distorted diffraction pattern.

If \( T^{\mu \nu} = \Theta^{\mu \nu} \), then the above distortion would be observed as long as the beam carries a nonzero \( J^2 \), no matter of spin and/or orbital origin. On the other hand, such a distortion would only be observed for a beam with nonzero \( L^z \) (whatever \( S^z \) is) if \( T^{\mu \nu} = T^{\mu \nu} \). Therefore, the patterns of the simple single-slit diffraction of light with spin and orbital angular momentum can tell concretely whether \( T^{\mu \nu} \) can be symmetric or not.

In fact, these diffraction patterns are already known: For nonzero \( L^z \) one does observe the expected distortion (see, e.g., Fig. 5 of Ref. [2], which is quoted here in Fig. 1), while circularly polarized light carrying nonzero \( S^z \) but zero \( L^z \) does not display any similar distortion (see Fig. 1). We therefore conclude that for spin-polarized photons the symmetric expression of energy-momentum tensor is excluded by experiment, and the nonsymmetric canonical expression is favored.

We close this paper with the following remarks:

(i) The same analysis can be performed for the electron, by studying the diffraction patterns of spin-polarized beam and the recently realized electron beam with orbital angular momentum [6 3], and the same conclusion can be expected.

(ii) From our illustration, spin-polarization does not produce a circular momentum, but it does produce a circular energy flow [4]. In the canonical expression, momentum and energy flow are two different quantities.

(iii) The canonical expressions are in general gauge-dependent and need gauge-invariant revision. Such revision is subject to certain theoretical uncertainties [3], but experiments would ultimately remove such uncertainties.

(iv) In the interacting case, the two expressions in Eqs. (1) no longer give the same angular momentum, and those in Eq. (8) do not give the same momentum, either. For a strongly interacting system the difference can even be huge [2].

(v) Given the experimentally selected nonsymmetric canonical energy-momentum-momentum tensor \( T^{\mu \nu} \), one has to seriously consider whether Nature would choose to use \( T^{\mu \nu} \) for gravitational coupling, and thus abandon the Einstein’s theory (which requires a symmetric energy-momentum tensor); or, Nature would favor Einstein’s theory and permits both the symmetric \( \Theta^{\mu \nu} \) and the nonsymmetric \( T^{\mu \nu} \), probably with the latter describing only the inertial energy-momentum tensor.

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