CENTRALIZERS IN 3-MANIFOLD GROUPS

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ABSTRACT. Using the Geometrization Theorem we prove a result on centralizers in fundamental groups of 3-manifolds. This result had been obtained by Jaco and Shalen and by Johannson using different techniques.

1. Introduction

In this paper we will study centralizers in fundamental groups of 3-manifolds. By a 3–manifold we will always mean a compact, orientable, connected, irreducible 3-manifold with empty or toroidal boundary.

Let \( \pi \) be a group. The centralizer of an element \( g \in \pi \) is defined to be the subgroup

\[
C_{\pi}(g) := \{ h \in \pi \mid gh = hg \}
\]

Determining centralizers is an important step towards understanding a group. The goal of this note is to give a new proof of the following theorem.

**Theorem 1.1.** Let \( N \) be a 3-manifold. We write \( \pi = \pi_1(N) \). Let \( g \in \pi \). If \( C_{\pi}(g) \) is non-cyclic, then one of the following holds:

1. there exists a JSJ torus or a boundary torus \( T \) and \( h \in \pi \) such that \( g \in h\pi_1(T)h^{-1} \) and such that

\[
C_{\pi}(g) = h\pi_1(T)h^{-1},
\]

2. there exists a Seifert fibered component \( M \) and \( h \in \pi \) such that \( g \in h\pi_1(M)h^{-1} \) and such that

\[
C_{\pi}(g) = h\pi_1(M)(h^{-1}gh)h^{-1}.
\]

If \( N \) is Seifert fibered, then the theorem holds trivially, and if \( N \) is hyperbolic, then it follows from well-known properties of hyperbolic 3-manifold groups (we refer to Section 3.1 for details). If \( N \) is neither Seifert fibered nor hyperbolic, then by the Geometrization Theorem \( N \) has a non-trivial JSJ decomposition, in particular \( N \) is Haken, and in that case the theorem was proved by Jaco and Shalen [6, Theorem VI.1.6] and independently by Johannson [7, Proposition 32.9].
In this note we will give an alternative proof of Theorem 1.1 for 3-manifolds with non-trivial JSJ decomposition using the Geometrization Theorem proved by Perelman. Our proof involves basic facts about fundamental groups of Seifert fibered spaces and hyperbolic 3-manifolds and it consists of a careful study of the fundamental group of the graph of groups corresponding to the JSJ decomposition.

In order to determine centralizers of 3-manifolds it thus suffices to understand centralizers of Seifert fibered spaces. For the reader’s convenience we recall the results of Jaco–Shalen and Johannson. Let $N$ be a Seifert fibered 3-manifold with a given Seifert fiber structure. Then there exists a projection map $p: N \to B$ where $B$ is the base orbifold. We denote by $B' \to B$ the orientation cover, note that this is either the identity or a 2-fold cover. Following [6] we refer to $p^{-1}_*(\pi_1(B'))$ as the canonical subgroup of $\pi_1(N)$. If $f$ is a regular fiber of the Seifert fibration, then we refer to the subgroup of $\pi_1(N)$ generated by $f$ as the fiber subgroup. Recall that if $N$ is non-spherical, then the fiber subgroup is infinite cyclic and normal. (Note that the fact that the fiber subgroup is normal implies in particular that it is well-defined, and not just up to conjugacy.)

Remark. Note that the definition of the canonical subgroup and of the fiber subgroup depend on the Seifert fiber structure. By [10, Theorem 3.8] (see also [9] and [6, II.4.11]) a Seifert fibered 3-manifold $N$ admits a unique Seifert fiber structure unless $N$ is either covered by $S^3$, $S^2 \times \mathbb{R}$, or the 3-torus, or $N = S^1 \times D^2$ or if $N$ is an $I$-bundle over the torus or the Klein bottle.

The following theorem, together with Theorem 1.1, now classifies centralizers of non-spherical 3-manifolds.

**Theorem 1.2.** Let $N$ be a non-spherical Seifert fibered 3-manifold with a given Seifert fiber structure. Let $g \in \pi = \pi_1(N)$ be a non-trivial element. Then the following hold:

1. if $g$ lies in the fiber group, then $C_\pi(g)$ equals the canonical subgroup,
2. if $g$ does not lie in the fiber group, then the intersection of $C_\pi(g)$ with the canonical subgroup is abelian, in particular $C_\pi(g)$ admits an abelian subgroup of index at most two,
3. if $g$ does not lie in the canonical subgroup, then $C_\pi(g)$ is infinite cyclic.

The first statement is [6, Proposition II.4.5]. The second and the third statement follow from [6, Proposition II.4.7]. Using Theorems 1.1 and 1.2 one can now immediately obtain results on root structures.
and the divisibility of elements in 3-manifold groups. We refer to [1, Section 4] for details.

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## 2. Graphs of groups

In this section we summarize some basic definitions and facts concerning graphs of groups and their fundamental groups. We refer to [2, 3, 11] for missing details.

### 2.1. Graphs

A graph $Y$ consists of a set $V = V(Y)$ of vertices and a set $E = E(Y)$ of edges, and two maps $E \to V \times V: e \mapsto (o(e), t(e))$ and $E \to E: e \mapsto \tau$, subject to the following condition: for each $e \in E$ we have $\bar{e} = e, \tau \neq e$, and $o(e) = t(\bar{e})$. We sometimes also denote $e$ by $e^{-1}$. Throughout this paper, all graphs are understood to be connected and finite (i.e., their vertex sets and edge sets are finite).

### 2.2. The fundamental group of a graph of groups

Let $Y$ be a graph. A graph $G$ of groups based on $Y$ consists of families $\{G_v\}_{v \in V(Y)}$ and $\{G_e\}_{e \in E(Y)}$ of groups satisfying $G_e = G_{\bar{e}}$ for every $e \in E(Y)$, together with a family $\{\varphi_e\}_{e \in E(Y)}$ of monomorphisms $\varphi_e: G_e \to G_{t(e)}$ ($e \in E(Y)$). We will refer to $Y$ as the underlying graph of $G$.

Let $G$ be a graph of groups based on a graph $Y$. We recall the construction of the fundamental group $G = \pi_1(G)$ from [11, I.5.1]. First, consider the path group $\pi(G)$ which is generated by the groups $G_v$ ($v \in V(Y)$) and the elements $e \in E(Y)$ subject to the relations

$$e\varphi_e(g)e = \varphi_{\bar{e}}(g) \quad (e \in E(Y), g \in G_e).$$

By a path in $Y$ from a vertex $v$ to a vertex $w$ we mean a sequence $(e_1, e_2, \ldots, e_n)$ where $o(e_1) = v, t(e_i) = o(e_{i+1}), i = 1, \ldots, n-1$ and $t(e_n) = w$.

By a path in $G$ from a vertex $v$ to a vertex $w$ we mean a sequence

$$(g_0, e_1, g_1, e_2, \ldots, e_n, g_n),$$

of elements in $E$ where $(e_1, \ldots, e_n)$ is a path of length $n$ in $Y$ from $v$ to $w$ and where $g_0 \in G_v$ and where $g_i \in G_{t(e_i)}$ for $i = 1, \ldots, n$. We write $l(\gamma) = n$ and call it the length of $\gamma$. We say that the path $\gamma$ represents the element

$$g = g_0 e_1 g_1 e_2 \cdots e_n g_n$$

of $\pi(G)$. 
Let now $w$ be a fixed vertex of $\mathcal{Y}$. We will refer to a path from $w$ to $w$ as a loop based at $w$. The fundamental group $\pi_1(\mathcal{G}, w)$ of $\mathcal{G}$ (with base point $w$) is defined to be the subgroup of $\pi(\mathcal{G})$ consisting of elements represented by loops based at $w$. If $w' \in V(\mathcal{Y})$ is another base point, and $g$ is an element of $\pi(\mathcal{G})$ represented by a path from $w'$ to $w$, then $\pi_1(\mathcal{G}, w') \to \pi_1(\mathcal{G}, w) \colon t \mapsto g^{-1}tg$ is an isomorphism. By abuse of notation we write $\pi_1(\mathcal{G})$ to denote $\pi_1(\mathcal{G}, w)$ if the particular choice of base point is irrelevant.

Now let $v \in V$. Pick a path $g$ from $v$ to $w$. Then the map $G_v \to \pi_1(\mathcal{G}, w)$ given by $t \mapsto g^{-1}tg$ defines a group morphism which is injective (see again [11, I.5.2, Corollary 1]). In particular the vertex groups define subgroups of $\pi_1(\mathcal{G}, w)$ which are well-defined up to conjugation. Given a graph of groups $\mathcal{G}$ and a base vertex $w$ it is always understood that for each vertex $v$ we picked once and for all a path from $v$ to $w$.

We will later on make use of the following operations on paths. Given a path $p$ in $\mathcal{G}$ from $v_1$ to $v_2$ we write $o(p) = v_1$ and $t(p) = v_2$. Given two paths
\[
p := (g_0, e_1, g_1, e_2, \ldots, e_n, g_n), \quad q := (h_0, f_1, h_1, f_2, \ldots, f_m, h_m),
\]
with $t(p) = o(q)$ we define
\[
p \ast q := (g_0, e_1, g_1, e_2, \ldots, e_n, g_n \cdot h_0, f_1, h_1, f_2, \ldots, f_m, h_m)
\]
which is a path from $o(p)$ to $t(q)$. Furthermore, given a path
\[
p := (g_0, e_1, g_1, e_2, \ldots, e_n, g_n)
\]
we define the inverse path to be
\[
p^{-1} := (g_n^{-1}, e_n^{-1}, \ldots, g_1^{-1}, e_1^{-1}).
\]
Note that $p^{-1}$ is a path from $t(p)$ to $o(p)$.

2.3. Reduced paths. A path $(g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$ in $\mathcal{G}$ is reduced if it satisfies one of the following conditions:

1. $n = 0$,

2. $n > 0$ and $g_i \notin \varphi_{e_i}(G_{e_i})$ for each index $i$ such that $e_{i+1} = e_i$.

Given $g \in \pi(\mathcal{G})$ we define its length $l(g)$ to be the length of a reduced path representing it. Note that this is well-defined (see [11, p. 4]), i.e. any $g$ is represented by a reduced path and the definition is independent of the choice of the reduced path. Also note that
\[
l(g) = \min \{l(p) \mid p \text{ a path which represents } g\}.
\]
Note that $l(g) = 0$ if and only if $g$ lies in $G_v$ for some $v \in V$.

We say that $s = (g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$ is cyclically reduced if $s$ is reduced and if one of the following holds:
(1) $n = 0$, or
(2) $e_1 \neq \overline{e_n}$, or
(3) $e_1 = \overline{e_n}$ but $g_ng_0$ is not conjugate to an element in $\operatorname{Im}(\varphi_{e_n})$.

Note that a reduced loop $s = (g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$ is cyclically reduced if and only if the element it represents has minimal length in its conjugacy class in the path group $\pi(G)$.

We say that $g \in \pi_1(G, w)$ is cyclically reduced if there exists a cyclically reduced loop which represents it. It is straightforward to see that $g$ is cyclically reduced if and only if any reduced loop representing it is cyclically reduced. Also note that if $g$ is cyclically reduced, then $l(g^n) = n \cdot l(g)$.

Any element $g$ of the path groups $\pi(G)$ is conjugate in $\pi(G)$ to a cyclically reduced element $s$, we can thus define $cl(g) = l(s)$. Note that this is independent of the choice of $s$. Note that if $g$ is cyclically reduced, then a straightforward argument shows that $l(g^n) = n \cdot l(g)$. In particular given any $g$ we have $cl(g^n) = n \cdot cl(g)$.

3. Fundamental groups of 3-manifolds

In the next two sections we cover properties of fundamental groups of hyperbolic 3-manifold groups and of Seifert fibered spaces, before we return to the study of 3-manifold groups in general.

3.1. Fundamental groups of hyperbolic 3-manifolds. Let $N$ be a 3-manifold. We say that $N$ is hyperbolic if the interior admits a complete metric of finite volume and constant sectional curvature equal to $-1$.

Throughout this section we write

$$U := \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} \right\} \text{ with } \varepsilon \in \{ -1, 1 \} \text{ and } a \in \mathbb{C} \subset \operatorname{SL}(2, \mathbb{C}) \right\} \subset \operatorname{SL}(2, \mathbb{C}).$$

Note that $U$ is an abelian subgroup of $\operatorname{SL}(2, \mathbb{C})$. Recall that $A \in \operatorname{SL}(2, \mathbb{C})$ is called parabolic if it is conjugate to an element in $U$. We say that $A$ is loxodromic if $A$ is diagonalizable with eigenvalues $\lambda, \lambda^{-1}$ such that $|\lambda| > 1$. We recall the following well known proposition.

**Proposition 3.1.** Let $N$ be a hyperbolic 3-manifold. Then the following hold:

(1) There exists a faithful discrete representation $\rho: \pi_1(N) \to \operatorname{SL}(2, \mathbb{C})$.

(2) Let $g \in \pi_1(N)$, then $\rho(g)$ is either parabolic or loxodromic.

(3) An element $g \in \pi_1(N)$ is conjugate to an element in a boundary component if and only if $\rho(g)$ is parabolic.
Let $T$ be a boundary torus, then there exists a matrix $P \in \text{SL}(2, \mathbb{C})$ such that $P \rho(\pi_1(T)) P^{-1} \subset U$.

Let $g \in \pi_1(N)$. Then $C_g(\pi_1(N))$ is either infinite cyclic or a free abelian group of rank two. The latter case occurs precisely when $g$ is conjugate to an element in a boundary component $T$ and in that case $C_g(\pi_1(N))$ is a conjugate of $\pi_1(T)$.

We include the proof of the proposition for completeness’ sake.

**Proof.**

1. A hyperbolic 3-manifold $N$ admits a faithful discrete representation $\pi_1(N) \to \text{Isom}(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$. Thurston (see [12, Section 1.6]) showed that this representation lifts to a faithful discrete representation $\pi_1(N) \to \text{SL}(2, \mathbb{C})$.

2. This follows immediately from considering the Jordan transform of $\rho(g)$ and from the fact that the infinite cyclic group generated by $\rho(g)$ is discrete in $\text{SL}(2, \mathbb{C})$.

3. This is well-known.

4. This statement follows easily from the fact that $\pi_1(T) \subset \text{SL}(2, \mathbb{C})$ is a discrete subgroup isomorphic to $\mathbb{Z}^2$.

5. By (1) we can view $\pi = \pi_1(N)$ as a discrete, torsion-free subgroup of $\text{SL}(2, \mathbb{C})$. Note that the centralizer of any non-trivial matrix in $\text{SL}(2, \mathbb{C})$ is abelian (this can be seen easily using the Jordan normal form of such a matrix). Now let $g \in \pi \subset \text{SL}(2, \mathbb{C})$ be non-trivial. Since $\pi$ is torsion-free and discrete in $\text{SL}(2, \mathbb{C})$ it follows easily that $C_\pi(g)$ is in fact either infinite cyclic or a free abelian group of rank two. It now follows from [13, Proposition 5.4.4] (see also [10, Corollary 4.6] for the closed case) that there exists a boundary component $S$ and $h \in \pi_1(N)$ such that

$$C_\pi(g) = h\pi_1(S)h^{-1}.$$ 

\[\square\]

Given a group $\pi$ we say that an element $g$ is *divisible by an integer* $n$ if there exists an $h \in \pi$ with $g = h^n$. We say $g$ is *infinitely divisible* if $g$ is divisible by infinitely many integers. The following lemma is an immediate consequences of Proposition 3.1 (5).

**Lemma 3.2.** Let $\pi \subset \text{SL}(2, \mathbb{C})$ be a discrete torsion-free group. Then $\pi$ does not contain any non-trivial elements which are infinitely divisible.

Let $\pi$ be a group. We say that a subgroup $H \subset \pi$ is *division closed* if for any $g \in \pi$ and $n > 0$ with $g^n \in H$ the element $g$ already lies in $H$. The following lemma is an immediate consequence of Proposition 3.1 (2) and (5) and from the observation that $A \subset \text{SL}(2, \mathbb{C})$ is parabolic
(respectively loxodromic) if and only if a non-trivial power of $A$ is parabolic (respectively loxodromic).

**Lemma 3.3.** Let $N$ be a 3-manifold such that the interior of $N$ is a hyperbolic 3-manifold of finite volume. Let $T$ be a boundary component of $N$. Then $\pi_1(T) \subset \pi_1(N)$ is division closed.

Let $\pi$ be a group. We say that a subgroup $H$ is malnormal if $gHg^{-1} \cap H$ is trivial for any $g \not\in H$. The following lemma is well-known.

**Lemma 3.4.** Let $N$ be a hyperbolic 3-manifold.

1. Let $T$ be a boundary torus. Then $\pi_1(T) \subset \pi_1(N)$ is malnormal.
2. Let $T_1$ and $T_2$ be distinct boundary tori. Then for any $g \in \pi_1(N)$ we have $\pi_1(T_1) \cap g\pi_1(T_2)g^{-1} = \{e\}$.

### 3.2. Fundamental groups of Seifert fibered manifolds.

Let $N$ be a Seifert fibered space with regular fiber $c$. First note that if $T$ is a boundary torus, then the Seifert fibration restricted to $T$ induces a product structure. It follows that $c \in \pi_1(T)$ and that $c$ is indivisible in $\pi_1(T) \cong \mathbb{Z}^2$.

The following results summarize the key properties of fundamental groups of Seifert fibered spaces which are relevant to our discussion.

**Theorem 3.5.** Let $N$ be a Seifert fibered 3-manifold with regular fiber $c$. Then there exists an $s \in \mathbb{N}$ with the following property: If $T$ is a boundary component, and if $g \not\in \pi_1(T)$ but some power of $g$ lies in $\pi_1(T)$, then there exists $d \leq s$ such that $g^d = c$ or $g^d = c^{-1}$.

**Proof.** Let $N$ be a Seifert fibered 3-manifold with boundary. Let $s$ be the maximum order of a singular fiber of the fibration. Let $T$ be a boundary component, and let $g \not\in \pi_1(T)$ such that some power of $g$ lies in $\pi_1(T)$. We denote by $p : N \to B$ the projection to the base orbifold. We denote by $b$ the boundary curve of $B$ corresponding to $T$. Note that $p(g) \not\in \langle b \rangle$ but a power of $p(g)$ lies in $\langle b \rangle$. It follows easily from [6, Remark II.3.1] that $p(g)$ is of finite order. In particular $g$ corresponds to a singular fiber, and then it follows from the definition of $s$ that there exists a $d \leq s$ such that $g^d = c$ or $g^d = c^{-1}$. \hfill \qed

**Lemma 3.6.** Let $N$ be a Seifert fibered 3-manifold with regular fiber $c$ and let $T$ be a boundary component. Let $g \in \pi_1(T)$ which is not a power of $c$, then $C_g(\pi_1(N)) = \pi_1(T)$.

**Proof.** We denote by $p : N \to B$ the projection to the base orbifold. Note that $p(g) \in \pi_1(B)$ is non-trivial. It follows easily from [6, Remark II.3.1] that $C_{p(g)}(\pi_1(B))$ is the group generated by the boundary curve of $N$ corresponding to $T$. It follows easily that $C_g(\pi_1(N)) = \pi_1(T)$. \hfill \qed
The following lemma is also well-known. It can be proved in a similar fashion as Lemma 3.6 by considering the equivalent problem in the fundamental group of the base manifold.

**Lemma 3.7.** Let \( N \) be a Seifert fibered 3-manifold. Denote by \( c \in \pi_1(N) \) the element represented by a regular fiber.

1. Let \( T \) be a boundary torus and \( g \in \pi_1(N) \setminus \pi_1(T) \), then \( \pi_1(T) \cap g\pi_1(T)g^{-1} = \langle c \rangle \).
2. Let \( T_1 \) and \( T_2 \) be distinct boundary tori. Then for any \( g \in \pi_1(N) \) we have \( \pi_1(T_1) \cap g\pi_1(T_2)g^{-1} = \langle c \rangle \).

We conclude with the following lemma.

**Lemma 3.8.** Let \( N \) be a non-spherical Seifert fibered manifold. Then \( \pi_1(N) \) does not contain non-trivial elements which are infinitely divisible.

*Proof.* Let \( N \) be a Seifert fibered manifold. Then there exists a finite cover \( N' \) which is an \( S^1 \)-bundle over a surface \( S \) (see e.g. [5, p. 391] for details). We write \( \Gamma = \pi_1(S) \), \( \pi = \pi_1(N) \) and \( \pi' = \pi_1(N') \). If \( N \) is non-spherical then the long exact sequence in homotopy implies that there exists a short exact sequence

\[ 1 \to \mathbb{Z} \to \pi' \to \Gamma \to 1. \]

Since \( \mathbb{Z} \) and \( \Gamma \) are well-known not to admit any non-trivial infinitely divisible elements, it follows easily that \( \pi' \) does not admit a non-trivial infinitely divisible element. We write \( n = [\pi : \pi'] \). Since \( N \) is non-spherical we know that \( \pi \) is torsion-free. Note that if \( g \in \pi \) is non-trivial, then \( g^n \) lies in \( \pi' \) and it is also non-trivial. It is now easy to see that \( \pi \) can not admit a non-trivial infinitely divisible element either. \( \Box \)

3.3. 3-manifolds and graphs of groups. In this section we recall the well-known interpretation of 3-manifold groups as the fundamental group of a graph of groups. Let \( N \) be an irreducible, closed, oriented 3-manifold. Recall that the JSJ tori are a minimal collection \( \{T_1, \ldots, T_k\} \) of tori such that the complements of the tori are either atoroidal or Seifert fibered.

We denote by \( \mathcal{G}(N) \) the corresponding JSJ graph, i.e. the vertex set \( V = V(\mathcal{G}) \) of \( \mathcal{G} \) consists of the set of components of \( N \) cut along \( T_1, \ldots, T_k \) pieces and the set \( E = E(\mathcal{G}) \) of (unoriented) edges consists of the set of JSJ tori \( T_1, \ldots, T_k \). We sometimes denote the JSJ tori by \( T_e, e \in E \) and we denote the components of \( N \) cut along \( \bigcup_{e \in E} T_e \) by \( N_v, v \in V \). We equip each \( T_e \) with an orientation, we thus obtain two
canonical embeddings $i_\pm$ of $T_e$ into $N$ cut along $T_e$. We then denote by $o(e) \in V$ the unique vertex with $i_-(T_e) \in N_{o(e)}$ and we denote by $t(e) \in V$ the unique vertex with $i_+(T_e) \in N_{f(e)}$.

Suppose that $N$ has a non-trivial JSJ decomposition. Then given a Seifert fibered component $N_v$ of the JSJ decomposition of $N$ we denote by $c_v \in \pi_1(N_v)$ the group element defined by a corresponding regular fiber. Note that $c_v$ is well-defined up to inversion (see [14, Lemma 1] or [4]).

We conclude this section with the following theorem.

**Theorem 3.9.** Let $N$ be a closed, oriented 3-manifold. Denote by $G = \mathcal{G}(N)$ the corresponding JSJ graph. If $e$ is an edge such that $o(e)$ and $t(e)$ correspond to Seifert fibered spaces, then $\varphi_e^{-1}(c_{l(e)}) \neq c_{o(e)}^\pm 1$.

**Proof.** If $\varphi_e^{-1}(c_{l(e)})$ was equal to $c_{o(e)}^\pm 1$, then $N_{o(e)}$ and $N_{t(e)}$ would have Seifert fiber structures which (after an isotopy) match along the edge torus. But this contradicts the minimality of the JSJ decomposition. \hfill \qed

4. **Proof of the main results**

4.1. **Divisibility in 3-manifold groups.** We will first prove the following theorem.

**Theorem 4.1.** Let $N$ be a 3-manifold. If $N$ is not spherical, then $\pi_1(N)$ does not contain any non-trivial elements which are infinitely divisible.

**Proof.** Let $N$ be a non-spherical 3-manifold and let $x \in \pi_1(N)$ be a non-trivial element. Since the statement of theorem is independent of the choice of base point and conjugation we can without loss of generality assume that $l(x) = cl(x)$. We write $l = l(x)$.

First suppose that $l > 0$. Suppose we have $y \in \pi_1(N)$ and $n$ such that $y^n = x$. Note that $0 < cl(x) = cl(y^n) = n \cdot cl(y)$. It now follows immediately that $n \leq l = cl(x)$.

Now suppose that $l = 0$. Note that this means that $x$ lies in a vertex group $\pi_1(N_w)$. We now define

$$d := \max\{n \in \mathbb{N} \mid x = y^n \text{ for some } y \in \pi_1(N_w)\}.$$

Note that $d < \infty$ by Lemmas 3.2 and 3.8. Furthermore, given a Seifert fibered component $N_v$ we define

$$s_v := \text{maximum of the orders of the singular fibers of } N_v.$$
Finally we define \( s \) to be the maximum over all \( s_v \). If there are no Seifert fibered components, then we set \( s = 1 \). The following claim now implies the theorem.

**Claim.** If there exists \( y \in \pi_1(N) \) and \( n \in \mathbb{N} \) with \( y^n = x \), then \( n \leq ds \).

Suppose we have \( y \in \pi_1(N) \) and \( n \) such that \( y^n = x \). Note that \( 0 = l(x) = cl(x) = cl(y^n) = n \cdot cl(y) \). It now follows that \( cl(y) = 0 \). If \( l(y) = 0 \), then \( y \in \pi_1(N_v) \), hence the conclusion holds trivially by the definition of \( d \). Now suppose that \( l(y) > 0 \). Then there exists a reduced path \( p = (g_0, e_1, g_1, \ldots, e_t, g_t) \) from \( w \) to a vertex \( v \) and \( z \in \pi_1(N_v) \) such that \( y \) is represented by \( p \ast z \ast p^{-1} \). Among all such pairs \((p, z)\) we pick a pair which minimizes the length of \( p \).

Since \( p \) is minimal and \( l(p) > 0 \) we see that \( g_tzg_t^{-1} \not\in \operatorname{Im}(\varphi_{e_i}) \). On the other hand \( p \ast z^n \ast p^{-1} \) represents \( y^n = x \), hence this path is reduced, which implies that \( g_tz^n g_t^{-1} \in \operatorname{Im}(\varphi_{e_i}) \). It follows that \( \operatorname{Im}(\varphi_{e_i}) \) is not division closed, using Lemma 3.3 we conclude that \( N_v \) is Seifert fibered.

We denote by \( c_v \) the regular fiber of \( N_v \). Recall that by Theorem 3.5 there exists \( r \mid s_v \) with \( g_tz^n g_t^{-1} = c_v \). It also follows from Theorem 3.5 that \( g_tz^n g_t^{-1} = c_v^m \in \operatorname{Im}(\varphi_{e_i}) \) for some \( m \). Note that \( n = mr \).

We can now apply Lemmas 3.4 and 3.7, Theorem 3.9 and the fact that \( p \) is reduced to conclude that

\[
(g_0, e_1, g_1, \ldots, e_{t-1}, g_t^{-1}\varphi_{e_i}^{-1}(c_v^m)g_t^{-1}, e_{t-1}^{-1}, \ldots, e_1^{-1}, g_0^{-1})
\]

is reduced. It follows that \( l = 1 \). Note that

\[
x = g_0\varphi_{e_i}^{-1}(c_v^m)g_0^{-1} = (g_0\varphi_{e_i}^{-1}(c_v)g_0^{-1})^m.
\]

It follows that \( m \leq d \). We also have \( r \leq s_v \leq s \). We now conclude that \( n = mr \leq ds \).

\[ \square \]

### 4.2. Commuting elements in 3-manifold groups.

**Theorem 4.2.** Let \( N \) be a 3-manifold. Let \( x, y \in \pi_1(N) \) with \( x = yxy^{-1} \). Then one of the following holds:

1. \( x \) and \( y \) generate a cyclic group in \( \pi_1(N) \), or
2. there exists a JSJ torus \( T \) such that \( x \) and \( y \) lie in a conjugate of \( \pi_1(T) \subset \pi_1(N) \), or
3. there exists a Seifert fibered component \( M \) of the JSJ decomposition such that \( x \) and \( y \) lie in a conjugate of \( \pi_1(M) \subset \pi_1(N) \).

**Proof.** Let \( N \) be a 3-manifold. Denote by \( G = G(N) \) the corresponding JSJ graph with vertex set \( V \) and edge set \( E \). We denote by \( w \in V \) the vertex which contains the base point of \( N \). We denote the vertex groups by \( G_v = \pi_1(N_v), v \in V \).
The theorem holds trivially for Seifert fibered spaces, we can therefore assume that $N$ is not a Seifert fibered space, in particular that $N$ is not spherical. Suppose we have $x, y \in \pi_1(N)$ with $x = yxy^{-1}$. By the symmetry of $x$ and $y$ we can without loss of generality assume that $cl(x) \leq cl(y)$. Note that the statement of the theorem does not change under conjugation and change of base point, we can therefore without loss of generality assume that $cl(x) = l(x).

We represent $y$ by a reduced loop $p = (h_0, f_1, h_1, \ldots, f_{i-1}, h_{i-1}, f_i, h_i)$ based at $w$. If $l = 0$, then $l(x) = 0$ as well since $l(x) = cl(x) \leq cl(y) \leq l(y) = 0$. In that case we are done by Proposition 3.1 (5). We thus henceforth only consider the case that $l \geq 1$.

After conjugating $x$ and $y$ with $h_i$ we can without loss of generality assume that $h_i = 1$. Recall that $p$ being reduced implies that for $i = 2, \ldots, l$ the following holds:

\[(4.1) \quad f_i \neq \overline{f}_{i-1} \text{ or } f_i = \overline{f}_{i-1} \text{ and } h_{i-1} \not\in \text{Im}(\varphi_{f_{i-1}}).\]

We first study the case that $l(x) = 0$, i.e. $x \in G_w$. Clearly we can assume that $x$ is non-trivial.

Now consider

\[p \ast x \ast p^{-1} = (h_0, f_1, h_1, \ldots, f_i, x, f_i^{-1}, \ldots, h_{i-1}^{-1}, f_1^{-1}, h_0^{-1}).\]

This path is not reduced since $yxy^{-1}$ can be represented by a path of length zero. It follows that $x \in \text{Im}(\varphi_{f_i})$. We can now represent $x = yxy^{-1}$ by the following path:

\[(4.2) \quad (h_0, f_1, h_1, \ldots, f_{i-1}, h_{i-1} \varphi_{f_i}^{-1}(x)h_{i-1}^{-1}, f_{i-1}^{-1}, \ldots, h_1^{-1}, f_1^{-1}, h_0^{-1}).\]

**Case 1:** $l = 1$, i.e. $y = (h_0, f_1, 1)$. In that case $yxy^{-1} = x$ is represented by $h_0 \varphi_{f_1}^{-1}(x)h_0^{-1}$. It follows that $x \in \text{Im}(\varphi_{f_1})$ and $x \in h_0 \text{Im}(\varphi_{f_1}^{-1})h_0^{-1}$. But if $t(f_1) = o(f_1)$ is hyperbolic this is not possible by Lemma 3.4 since the two boundary tori of $N_{t(f_1)} = N_{o(f_1)}$ corresponding to the edge $f_1$ are obviously different. If $t(f_1) = o(f_1)$ is Seifert fibered, then we can similarly exclude this case by appealing to Lemma 3.7 and Theorem 3.9.

**Case 2:** The vertex $o(f_1)$ is hyperbolic. It follows easily from (4.1) and Lemma 3.4 that the path (4.2) is reduced. Since the path represents $x$ this implies in particular that $l = 1$. We thus reduced Case 2 to Case 1.

**Case 3:** The vertex $o(f_1)$ is Seifert fibered and $\varphi_{f_1}^{-1}(x) \not\in \langle o(f_1) \rangle$. Note that Lemma 3.7 together with Theorem 3.9 and (4.1) implies that the path (4.2) is reduced, i.e. $l = 1$. We thus also reduced Case 3 to Case 1.
Case 4: The vertex $o(f_l)$ is Seifert fibered, $\varphi_{f_l}^{-1}(x) \in \langle c_{o(f_l)} \rangle$ and $l > 1$. Note that by Theorem 3.5 (2) this implies that $h_{l-1} \varphi_{f_l}^{-1}(x) h_{l-1}^{-1} \in \text{Im}(\varphi_{f_{l-1}})$. We can thus represent $x$ by

$$(h_0, f_1, \ldots, f_{l-2}, h_{l-2} \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1} \varphi_{f_l}^{-1}(x) h_{l-1}^{-1}) \cdot h_{l-2}^{-1}, f_{l-2}, \ldots, f_1^{-1}, h_0^{-1}).$$

If $o(f_{l-1})$ is hyperbolic, then the argument of Case 2 immediately shows that $l = 2$. If $o(f_{l-1})$ is Seifert fibered, then it follows from Theorems 3.5 and 3.9 and from Lemma 3.7 (2) that $h_{l-2} \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1} \varphi_{f_l}^{-1}(x) h_{l-1}^{-1}) \cdot h_{l-2}^{-1} \notin \langle c_{o(f_{l-1})} \rangle$. The argument of Case 3 immediately shows that again $l = 2$.

We now showed that $l = 2$, we thus see that $x$ equals

$$h_0 \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1} \varphi_{f_l}^{-1}(x) h_{l-1}^{-1}) \cdot h_0^{-1}.$$  

If $o(f_1) = t(f_2)$ is hyperbolic, then $x \in \text{Im}(\varphi_{f_2})$ and $x \in h_0 \text{Im}(\varphi_{f_2}) h_0^{-1}$.

It follows from Lemma 3.4 that $f_1 = \overline{f_2}$ and $h_0 \in \text{Im}(\varphi_{f_2})$. If we change the base point to $o(f_2) = t(f_1)$ we see that $x$ is represented by $\varphi_{f_2}^{-1}(x) \in G_{o(f_2)}$ and $y$ is represented by $\varphi_{f_1}(h_0) h_1 \in G_{o(f_2)}$. If on the other hand $o(f_1) = t(f_2)$ is Seifert fibered, then it follows from Theorem 3.9 that $x \notin \langle c_{t(f_2)} \rangle$. It now follows easily from Lemma 3.7 that $f_1 = \overline{f_2}$ and $h_0 \in \text{Im}(\varphi_{f_2})$. We conclude the argument as above.

We now turn to the case that $l(x) > 0$. We claim that Conclusion (1) holds. By Theorem 4.1 we can find $z \in \pi_1(N)$ which is indivisible and $n > 0$ with $x = z^n$. Without loss of generality assume that $z$ is cyclically reduced. We claim that $y$ is a power of $z$ as well. We represent $z$ by a reduced loop $q = (g_0, e_1, g_1, \ldots, e_k, g_k)$. We now consider the path $p \ast q^n \ast p^{-1}$ which is given by

$$(h_0, f_1, h_1, \ldots, f_t, h_t \cdot g_0, e_1, g_1, \ldots, e_k, g_k \cdot h_t^{-1}, f_t^{-1}, h_t^{-1}, h_t^{-1}, f_1^{-1}, h_0^{-1}).$$

This loop has to be reduced since $l > 0$ and therefore the loop is longer than the loop $q^n$ which represents the same element. We conclude that one of the following conditions hold:

1. $f_t = \overline{e_1}$ and $h_t g_0 \in \text{Im}(\varphi_{f_t})$, or
2. $e_k = f_t$ and $g_k h_t^{-1} \in \text{Im}(\varphi_{e_k})$.

Note though that not both conclusions can hold, otherwise $x$ would not be cyclically reduced. Now suppose that (1) holds and (2) does not hold. A straightforward induction argument now shows that $p = p' \ast q^{-1}$ for some reduced path $p'$. On the other hand, if (2) holds and (1) does not hold, then a straightforward induction argument shows that $p = q^{-1} \ast p'$ for some reduced path $p'$.

Claim. If $l(p') = 0$, then $p'$ represents the trivial element.
If \( l(p') = 0 \), then we denote by \( y' \) the element represented by \( p' \). Suppose that \( y' \) is non-trivial. In that case we have \( y'x^n(y')^{-1} = x^n \) for any \( n \), in particular \( x^n y' x^{-n} = y' \). It follows from the discussion of Cases 1, 2, 3 and 4 above that \( l(x^n) \leq 2 \) for any \( n \). Since \( x \) is cyclically reduced and \( l(x) > 0 \) this case can not occur. This concludes the proof of the claim.

If \( p' \) represents the trivial element we are clearly done. If not, then \( l(p') > 0 \) and we can do an induction argument on the length of \( p' \) to show that \( y \) is in fact a power of \( z \).

4.3. Proof of Theorem 1.1. For the reader’s convenience we recall the statement of Theorem 1.1.

**Theorem 4.3.** Let \( N \) be a 3-manifold. We write \( \pi = \pi_1(N) \). Let \( g \in \pi \). If \( C_\pi(g) \) is non-cyclic, then one of the following holds:

1. there exists a JSJ torus or a boundary torus \( T \) and \( h \in \pi \) such that \( g \in h\pi_1(T)h^{-1} \) and such that
   \[
   C_\pi(g) = h\pi_1(T)h^{-1},
   \]
2. there exists a Seifert fibered component \( M \) and \( h \in \pi \) such that \( g \in h\pi_1(M)h^{-1} \) and such that
   \[
   C_\pi(g) = hC_{\pi_1(M)}(h^{-1}gh)h^{-1}.
   \]

**Proof.** Let \( N \) be a 3-manifold and let \( g \in \pi = \pi_1(N) \). If for any \( h \in C_\pi(g) \) the group generated by \( g \) and \( h \) is cyclic, then either \( C_\pi(g) \) is cyclic, or \( g \) is infinitely divisible. Since the former case is excluded by Theorem 4.1 the latter case has to hold.

Now suppose that \( C_\pi(g) \) is not cyclic and suppose that there exist an \( h \in C_\pi(g) \) such that the group generated by \( g \) and \( h \) is not cyclic. It follows from Theorem 4.2 that one of the following three cases occurs:

1. there exists a JSJ torus \( T \) such that \( g \) lies in a conjugate of \( \pi_1(T) \subset \pi_1(N) \),
2. there exists a Seifert fibered component \( M \) of the JSJ decomposition such that \( g \) lies in a conjugate of \( \pi_1(M) \subset \pi_1(N) \),

First suppose there exists a JSJ torus \( T \) such that \( g \) lies in a conjugate of \( \pi_1(T) \subset \pi_1(N) \). Without loss of generality we can assume that \( g \in \pi_1(T) \). We first consider the case that the two JSJ components abutting \( T \) are different. We denote these two components by \( M_1 \) and \( M_2 \). By Proposition 3.1 (5) the following claim implies the theorem in this case.

**Claim.** There exists an \( i \in \{1, 2\} \) such that
\[
C_\pi(g) = C_{\pi_1(M_i)}(g).
\]
Let $h \in C_\pi(g)$. It follows easily from the proof of Theorem 4.2 that either $h \in \pi_1(M_1)$ or $h \in \pi_1(M_2)$. If $M_1$ is hyperbolic, then it follows from Lemma 3.2 and from Proposition 3.1 (5) that $h \in \pi_1(T)$. It follows that $C_\pi(g) = C_{\pi_1(M_2)}(g)$. Similarly we deal with the case that $M_2$ is hyperbolic. Finally assume that $M_1$ and $M_2$ are Seifert fibered. We denote by $c_1$ and $c_2$ the regular fibers of $M_1$ and $M_2$. If $g$ is not a power of $c_1$, then it follows from Lemma 3.6 that $C_\pi(g) = C_{\pi_1(M_2)}(g)$, similarly if $g$ is not a power of $c_2$. Recall that $c_1$ and $c_2$ are indivisible in $\pi_1(T)$ and that by Theorem 3.9 we have $c_1 \neq c_2^{\pm 1}$. It follows that $g$ is either not a power of $c_1$ or not a power of $c_2$.

The case that the torus is non-separating can be dealt with similarly. We leave this to the reader. Also, if there exists a Seifert fibered component $M$ of the JSJ decomposition such that $g$ lies in a conjugate of $\pi_1(M) \subset \pi_1(N)$ and such that $g$ does not lie in the image of a boundary torus, then it follows easily from the proof of Theorem 4.2 that

\[ C_\pi(g) = C_{\pi_1(M)}(g). \]

\[ \square \]

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