SOLUTION OF THE VACUUM KERR-SCHILD PROBLEM†

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ABSTRACT

The complete solution of the vacuum Kerr-Schild equations in general relativity is presented, including the space-times with a curved background metric. The corresponding result for a flat background has been obtained by Kerr.

The Kerr-Schild pencils\(^1\) of metrics

\[ \tilde{g}_{ab} = g_{ab} + V l_a l_b \]  \hspace{1cm} (1)

have been in the forefront of research in general relativity for some time. A classic example of these space-times is the Kerr metric. All the Kerr-Schild vacuum space-times with a flat parent metric \(g_{ab}\) are given by beautiful geometrical relations\(^2\). The solution of the flat problem (known as Kerr’s theorem) reveals a close relationship with complex surfaces in three-dimensional homogeneous spaces. Kerr-Schild congruences in Minkowski space-time have extensively been studied\(^3\)–\(^6\). Work on Kerr-Schild space-times in the generic case when \(g_{ab}\) has a nonvanishing curvature, \(l\) is a null vector and \(V\) a function, has certainly been motivated by the prospects of extending Kerr’s complex analytic description to curved space-times\(^7\)–\(^10\).

In this Letter we present the complete solution of the vacuum Kerr-Schild problem. The Kerr-Schild equations follow from (1) and the vacuum Einstein equations. They are:

\[ D l_a = 0 , \]  \hspace{1cm} (2)

\[ \nabla_b [\nabla_a (V l_c l^b) + \nabla_c (V l_a l^b) - \nabla^b (V l_a l_c)] = 0 \]  \hspace{1cm} (3)

\[ DDV + (\nabla^a l_a)DV + 2V (\nabla^b l_d)\nabla^{[b} l_d] = 0 . \]  \hspace{1cm} (4)

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Here $\nabla$ is the covariant derivative annihilating $g_{ab}$. The vector

$$D = l^a \nabla_a = \partial / \partial r$$

(5)

is tangent to a null geodesic congruence$^8$, with $r$ the affine parameter.

We rewrite these equations in a Newman-Penrose (NP) notation$^{12}$, choosing $l$ a vector of the null tetrad. The geodesic condition (2) becomes $\kappa = 0$. We adopt the gauge with $\epsilon = 0$, $\pi = \alpha + \bar{\beta}$, and integrate a closed subset of the ‘radial’ field equations$^{12}$ for the affine parameter dependence:

$$\rho = -\frac{1}{2r}(1 + \cos \eta \ C), \quad \sigma = -\frac{\sin \eta}{2rC}$$

(6)

$$\Psi_0 = -\frac{\sin 2\eta}{4r^2}.$$  

(7)

Here C is a complex phase factor

$$C = \frac{r^\cos \eta - iB}{r^\cos \eta + iB}.$$  

(8)

$B$ and $\eta$ are real integration functions. A further integration function has been eliminated by suitably fixing the origin of the affine parameter. The real potential can be written as

$$V = V_0 \frac{r^\cos \eta}{r^2 \cos \eta + B^2}.$$  

(9)

The complex tetrad vector $m$ has the $r$ dependence

$$m = \frac{1}{2B}(1 - \frac{1}{C})\left[iQ_1^j r^{\cos \eta - \sin \eta - 1} - Q_2^j r^{\cos \eta + \sin \eta - 1}\right] \frac{\partial}{\partial x^j} \quad j = 1, 2, 3$$  

(10)

where $V_0, Q_1^j$ and $Q_2^j$ are real integration functions. With this choice, the tetrad has been uniquely fixed.

The real function $B$ controls the amount of divergence and rotation of the null congruence with tangent $l$, respectively. $B$ gives rise to the imaginary part of the spin coefficient $\rho$. When $B = 0$, we have $C = 1$, and the congruence is curl-free. Similarly, for large values of the affine parameter $r$, the phase factor $C$ approaches the unit value, and the rotation dies out. The parameter $\eta$ governs the shear. For $\eta = 0$ or $\eta = 180^\circ$, the congruence is shear-free. When both $B = 0$ and $\eta = 0$, the rays are exactly spherical:
\( \rho = -1/r \). For \( \eta = 90^\circ \), the rays become cylindrical, \( \rho = -1/2r \). At \( \eta = 180^\circ \), there is no expansion. Our Theorem below shows that the general shearing class does not contain the shear-free case as a smooth limit.

The field equations in the NP form may be grouped in three sets. The first set of equations is a coupled system of linear inhomogeneous equations for the affine parameter dependence of the quantities \( \tau, \pi, \alpha, \beta, \Psi_1 \) and for their complex conjugates:

\[
\begin{align*}
D\tau &= \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) + \Psi_1 \\
D\pi &= 2\rho\pi + 2\sigma\bar{\pi} + \Psi_1 \\
D\alpha &= \rho(\pi + \alpha) + \sigma(\bar{\pi} - \bar{\alpha}) \\
D\beta &= \bar{\rho}\beta + \sigma(2\pi - \bar{\beta}) + \Psi_1 \\
D\Psi_1 &= 4\rho\Psi_1 + (\bar{\delta} + \pi - 4\alpha)\Psi_0.
\end{align*}
\]

The \( \Psi_1 \) equations are complemented by two equations, linear in the unknown functions:

\[
\delta \left( \frac{\Psi_0}{2\sigma} \right) + \frac{\Psi_0}{\sigma} \delta \ln \phi - 2\sigma \delta \ln \phi - \Psi_1 = \frac{\Psi_0}{2\sigma}(\tau - \bar{\alpha} - \beta) - 2\sigma(\bar{\tau} - \alpha - \bar{\beta}) + \bar{\tau}\sigma - \tau\rho \quad (12)
\]

\[
\delta\rho - \bar{\delta}\sigma = \rho\bar{\pi} + \sigma(\pi - 4\alpha) + (\rho - \bar{\rho})\tau - \Psi_1. \quad (13)
\]

where \( \phi = \sqrt{V} \).

**Theorem:** For a generalized vacuum-vacuum Kerr-Schild pencil, either of the following conditions holds: The parameter \( \eta \) assumes one of the special values given by

\[
\sin \eta = 0, \pm 1, \pm 2^{-\frac{1}{2}}. \quad (14)
\]

or, alternatively, the spin coefficient quantities \( \rho, \sigma \) and \( \Psi_0 \) depend only on the affine parameter \( r \).

The theorem can be proven\(^{11} \) by taking the \( D \) derivative of Eq. (12), and eliminating the unknown spin coefficients by use of the NP commutators, Eq. (13) and the equations of the \( \Psi_1 \) system. This yields that \( \delta\eta = 0 \) and, unless \( \eta \) takes any of the exceptional values, the integration functions are restricted by \( \delta V_0 = \delta B = 0 \). In the generic case, \( \rho - \bar{\rho} \neq 0 \), the commutator equations imply also \( \Delta \phi = 0 \) for any real function \( \phi \) with \( D\phi = \delta\phi = 0 \). Hence the integration functions \( B, \eta \) and \( V_0 \) are constants. The curl-free fields, \( B = 0 \), will be discussed in Ref. 11.
It follows that the Kerr solution with $\eta = 0$ will not emerge as a smooth limiting case of the shearing Kerr-Schild metrics.

For fields of arbitrary deformation parameter, $\alpha$ and $\Psi_1$ may be expressed algebraically in terms of $\pi, \tau$ and their complex conjugates, by the respective Eqs. (12) and (13). The $\Psi_1$ system becomes a quartet of coupled equations for the spin coefficients $\tau, \pi$ and their complex conjugates:

$$D\tau = (2\rho - \frac{\Psi_0}{2\sigma})\tau + \sigma(2\bar{\tau} - \pi) + (\rho + \frac{\Psi_0}{2\sigma})\bar{\pi}$$
$$D\pi = (2\rho + \frac{\Psi_0}{2\sigma})\pi + \bar{\sigma}\tau + (\bar{\rho} - \frac{\Psi_0}{2\sigma})\bar{\tau}$$

along with the complex conjugate equations. The fundamental solution of the $\Psi_1$ set is given in Table 1. The general solution is a linear combination of the four fundamental solution vectors with real coefficients.

The radial component of the vector $n$ may be obtained by applying the commutator $[\delta, \bar{\delta}]$ on $r$. Hence the operator $\Delta$, when acting on any of the functions depending only on $r$, takes the form $\Delta = -\frac{\mu - \bar{\mu}}{\rho - \bar{\rho}} D$. The second set of field equations consists of the NP Eqs. (4.2l,p,q), as well as the fifth of NP (4.5) and of the Kerr-Schild equation

$$\gamma \rho + \frac{1}{2r} \left( \frac{\rho + \bar{\rho}}{\rho - \bar{\rho}} \right) \left( \bar{\mu} - \mu \right) + \frac{1}{2} \left( \frac{\Psi_0}{\sigma} \rho + \rho - \bar{\rho} \right) \left( \bar{\mu} + \mu \right) + \Psi_2 + \bar{\Psi}_2 - (\rho + \bar{\rho})(\gamma + \bar{\gamma})$$

$$= \delta(\bar{\tau} - \pi) + \bar{\delta}(\tau - \pi) - 6\pi\bar{\pi} + 2\pi\bar{\alpha} + 2\pi\alpha - 2\tau\bar{\tau} - 2\tau\alpha - 2\bar{\tau}\bar{\alpha} + 3\tau\pi + 3\bar{\tau}\bar{\pi}.$$  \hspace{1cm} (16)

We obtain a lengthy relation from these, containing only functions the $r$-dependence of which is explicitly known. Collecting the coefficients of independent powers of $r$, one finds that nothing but the trivial solution of the $\Psi_1$ system satisfies this equation:

$$\alpha = \beta = \tau = \pi = \Psi_1 = 0.$$  \hspace{1cm} (17)

Thus the second system of equations is homogeneous and linear in $\mu, \lambda, \gamma$ and $\Psi_2$. The determinant vanishes, and we get

$$\frac{\gamma}{\rho + 1/2r} = \frac{\lambda}{\bar{\sigma}} = -\frac{\Psi_2\sigma}{\Psi_0\rho} = \frac{\mu}{\rho}.$$  \hspace{1cm} (18)

The remaining field equations constitute the third set. They further restrict the metric, leaving us with a three-parameter pencil for which the image of the Kerr-Schild map is the Kóta-Perjés' metric (44):

$$ds^2 = -\frac{r^{\cos 2\eta} + B^2}{r^{\cos \eta}} (dr^2 + r^{1-\sin \eta} dx^2 + r^{1+\sin \eta} dy^2) + \Lambda r^{\cos \eta} + B^2 (l_\alpha dx^\alpha)^2.$$  \hspace{1cm} (19)
Here \( l = \partial / \partial r \) is tangent to the Kerr-Schild congruence and \( \Lambda \) is the pencil parameter.

To complete our investigation of vacuum Kerr-Schild space-times, we consider now in turn the metrics with either of the values (14).

(a) When \( \sin \eta = 0 \), both \( \sigma \) and \( \Psi_0 \) vanish, and \( l \) is a principal null vector of the curvature. By the Goldberg-Sachs theorem\(^{12}\), these parent space-times are algebraically special, and \( \Psi_1 = 0 \). It then follows from Thompson’s Theorem 3.2 that also the ensuing space-time is algebraically special, with the Kerr-Schild congruence being a principal null congruence\(^8\). All the vacuum Kerr-Schild spacetimes generated from the flat space-time are in this class.

(b) The case with \( \cos \eta = 0 \) maps Minkowski space-time to itself.

(c) Case \( \sin \eta = 1/\sqrt{2} = k \) contains the following Kőta-Perjés metrics:

\[
ds^2 = -\frac{f^0}{f}(r^{1-k}dx^2 + r^{1+k}dy^2) + 2dr(l_adx^a) + f(l_adx^a)^2
\]

Metric (53) of Ref. 8 is given by

\[
f = \Lambda Re\left\{ \frac{x + iy^k y}{r^k + iB} \right\}, \quad f^0 = \Lambda(x + By)
\]

with \( B \) a real constant. For metric (66), \( B = x/y \) and

\[
f = \Lambda \frac{x + by}{x^2 r^{-k} + y^2 r^k}, \quad f^0 = \Lambda(x + by)/y^2
\]

Notice that we have enlisted above all the metrics in Ref. 8.

We thus find that all Kőta-Perjés metrics are explicit cases of Kerr-Schild space-times, either with a real deformation parameter or with \( \eta = 1/\sqrt{2} \).

In summary, the structure of our solution is as follows. The Kerr-Schild space-times are characterized by the real deformation parameter \( \eta \). The deformation parameter vanishes for Kerr-Schild space-times the parent of which is Minkowski space-time. By our theorem, the field quantities are severely restricted unless the parameter \( \eta \) assumes either of the exceptional values 0, \( \pm 1, \pm \sqrt{2}/2 \). The metrics with arbitrary values of the deformation parameter are Kőta-Perjés metrics. For the exceptional values of \( \eta \), (a)
the class with $\eta = 0$ is algebraically special, (b) the values $\sin \eta = \pm 1$ can occur only in automorphisms of the Minkowski space-time, and (c) the class with $\sin \eta = \pm 1/\sqrt{2}$ contains the remaining Kóta-Perjés metrics. While the Kerr-Schild congruences in a Minkowski space-time form a four-parameter family\(^2\), there is no corresponding structure on a curved background. Our results have the important implication that hopes are dashed for a complex-analytic description of space-time within the framework of Kerr-Schild theory.

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\[
\pi^{(1)} = \frac{C + 1}{r} \left( C^{-3} - \frac{\cos \eta}{\sin \eta + 1} C^{-2} - \frac{\sin \eta}{\sin \eta + 1} C^{-1} + \frac{5\sin^2 \eta - 4 \sin \eta + 3}{\cos \eta (\sin \eta - 3)} + \frac{3 \sin \eta - 1}{\sin \eta + 1} \frac{2 \sin \eta}{\sin \eta - 3} C + 2 \frac{\sin \eta}{\cos \eta} C^2 \right)
\]

\[
\pi^{(2)} = i \frac{C + 1}{r} \left( C^{-3} + \frac{\cos \eta}{\sin \eta - 1} C^{-2} - \frac{\sin \eta}{\sin \eta - 1} C^{-1} + \frac{5 \cos^2 \eta - 4 \sin \eta - 8}{\cos \eta (\sin \eta + 3)} + \frac{6 \cos^2 \eta + 7 \sin \eta - 3}{\cos^2 \eta - 2 \sin \eta + 2} C - \frac{2 \sin \eta}{\cos \eta} C^2 \right)
\]

\[
\pi^{(3)} = i \frac{C - 1}{r} \left( C + 1 \right)^2 \left( -C^{-3} - \frac{\sin \eta + 1}{\cos \eta} C^{-2} + \frac{1}{\sin \eta - 1} C^{-1} - 2 \frac{\sin \eta}{\cos \eta} + \frac{3 \sin \eta - 1}{\cos \eta} C + 2 C^2 \right)
\]

\[
\pi^{(4)} = \frac{C - 1}{r} \left( C + 1 \right)^2 \left( -C^{-3} - \frac{\cos \eta}{\sin \eta + 1} C^{-2} - \frac{1}{\sin \eta + 1} C^{-1} + 2 \frac{\sin \eta}{\cos \eta} \right)
\]

\[
\tau^{(1)} = \frac{C + 1}{r} \left( \frac{\sin \eta}{\cos \eta} C^{-3} - \frac{\sin \eta}{\sin \eta + 1} C^{-2} - \frac{\sin^2 \eta + 8 \sin \eta + 3 \sin \eta - 1}{\cos \eta (\sin \eta - 3)} \frac{\sin \eta + 1}{\sin \eta + 1} C^{-1} - 3 \frac{\sin \eta}{\sin \eta + 1} + \frac{2 \sin \eta - 3}{\cos \eta} C + 2 C^2 \right)
\]

\[
\tau^{(2)} = i \frac{C + 1}{r} \left( \frac{\sin \eta}{\cos \eta} C^{-3} + \frac{\sin \eta}{\sin \eta - 1} C^{-2} - \frac{\sin \eta \cos^2 \eta - 7 \cos^2 \eta + 4 \sin \eta + 4}{\cos \eta (2 \sin \eta - 2 - \cos^2 \eta)} C^{-1} + 3 \frac{\sin \eta}{\sin \eta - 1} + \frac{2 \sin \eta + 3}{\cos \eta} C - 2 C^2 \right)
\]

\[
\tau^{(3)} = i \frac{C - 1}{r} \left( C + 1 \right)^2 \left( -\frac{\sin \eta}{\cos \eta} C^{-3} - \frac{\sin \eta}{\sin \eta - 1} C^{-2} - \frac{2 \sin \eta + 1}{\cos \eta} C^{-1} - 2 \right)
\]

\[
\tau^{(4)} = \frac{C - 1}{r} \left( C + 1 \right)^2 \left( -\frac{\sin \eta}{\cos \eta} C^{-3} - \frac{\sin \eta}{\sin \eta + 1} C^{-2} - \frac{2 \sin \eta - 1}{\cos \eta} C^{-1} + 2 \right)
\]

Table 1. The four solutions for $\pi$ and $\tau$