Scattered and paracompact order topologies

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1. Statement of main results

Write $|M|$ for the cardinal number (the size) of a set $M$. While an infinite set $S$ carries $2^{[S]}$ (mutually non-homeomorphic) topologies, $S$ cannot be equipped with more than $2^{[S]}$ order topologies. This is trivial in view of the definition that $\tau$ is an order topology on $S$ when $\tau$ is generated by a linear ordering $\preceq$ of $S$. (A subbasis of $\tau$ is formed by all rays $\{x \in S \mid x \prec y\}$ and $\{x \in S \mid x \succ y\}$ with $y \in S$.) In the following, a linearly ordered space is a space whose topology is an order topology. Note that every linearly ordered space is hereditarily normal. According to the title we consider only linearly ordered spaces which are scattered. (A Hausdorff space is scattered if and only if every nonempty subspace contains an isolated point.) If $X$ is such a space then for the weight $w(X)$ of $X$ we always have $w(X) = |X|$. (Because $w(Y) \leq |Y|$ trivially holds when the space $Y$ is linearly ordered. And it is a nice exercise to verify $|Y| \leq w(Y)$ for all scattered Hausdorff spaces $Y$.)

In the following we focus on scattered linearly ordered spaces which are either metrizable or compact. Since size and weight coincide, a compact and metrizable scattered space must be countable. Of course, every countable compact Hausdorff space is completely metrizable. Furthermore, a scattered metrizable space must be completely metrizable. The following theorem implies that every infinite set $S$ can be equipped with $2^{[S]}$ complete metrics which generate mutually non-homeomorphic scattered order topologies on $S$.

**Theorem 1.** For every cardinal $\kappa \geq \aleph_0$ there exist $2^{\kappa}$ mutually non-homeomorphic scattered and (completely) metrizable, linearly ordered spaces of size $\kappa$.

On the other hand, the following theorem implies that every uncountable set $S$ can be equipped with $2^{[S]}$ mutually non-homeomorphic scattered and compact order topologies.

**Theorem 2.** For every cardinal $\kappa > \aleph_0$ there exist $2^{\kappa}$ mutually non-homeomorphic compact, scattered, linearly ordered spaces of size $\kappa$.

The statement in Theorem 2 is unprovable for $\kappa = \aleph_0$ because, due to Mazurkiewicz and Sierpiński [9], up to homeomorphism there exist precisely $\aleph_1$ compact, countable Hausdorff spaces. (They all are linearly ordered and scattered.) And not only $\aleph_1 < 2^{\aleph_0}$ is consistent with ZFC set theory. It is also consistent with ZFC that there exist $2^{\aleph_0}$ cardinal numbers $\kappa$ with $\aleph_0 < \kappa < 2^{\aleph_0}$. The last statement is worth mentioning to put important consequences of Theorem 1 and Theorem 2 in perspective. Under the restriction that either $\kappa = \aleph_0$ or $\kappa \geq 2^{\aleph_0}$ the following consequence of Theorem 1 is well-known and has been proved for $\kappa \geq 2^{\aleph_0}$ in the realm of pathwise connected spaces, see [2].

**Corollary 1.** For every cardinal $\kappa \geq \aleph_0$ there exist $2^{\kappa}$ mutually non-homeomorphic complete metric spaces of size $\kappa$.

Under the restriction $\kappa \geq 2^{\aleph_0}$ the following consequence of Theorem 2 has already been proved in the realm of connected, linearly ordered spaces, see [4].
Corollary 2. For every cardinal \( \kappa > \aleph_0 \) there exist \( 2^\kappa \) mutually non-homeomorphic compact Hausdorff spaces of size \( \kappa \).

Remark. Not only in Theorem 2, but also in Corollary 2 the amount \( 2^\kappa \) is maximal. (Because every compact Hausdorff space of size \( \kappa \) is homeomorphic to a closed subspace of Hilbert cube \([0,1]^\kappa\).)

2. Ultrametrics and linear orderings

In proving Theorem 1 we will consider scattered, ultrametric spaces only. (A metric \( d \) on \( X \) is an ultrametric if and only if \( d(x,z) \leq \max\{d(x,y),d(y,z)\} \) for all \( x,y,z \in X \).) By considering complete ultrametric spaces in the proof of Theorem 1 we need not deal with linear orderings in view of the following proposition.

Proposition 1. The topology of a space is an order topology if it is generated by a complete ultrametric.

One way to verify Proposition 1 is to combine two well-known results. By [1] 6.3.2.f the topology of any strongly zero-dimensional metrizable space is an order topology. And by [1] 7.3.F a metrizable space is strongly zero-dimensional if and only if its topology is generated by some ultrametric. (As a noteworthy consequence, the property strongly zero-dimensional can be included in Theorem 1. In Theorem 2 the property strongly zero-dimensional and totally disconnected are equivalent for compact Hausdorff spaces.)

In particular, by applying [1] 6.3.2.f and [1] 7.3.F, Proposition 1 is true even without assuming completeness. In order to keep this paper self-contained, in the following we write down an elementary and direct proof of Proposition 1.

Let us call a strict linear ordering of a set \( S \) a DE-ordering when there is a maximum and a minimum and when each point in \( S \setminus \{\max S\} \) has an immediate successor and each point in \( S \setminus \{\min S\} \) has an immediate predecessor. Then the corresponding order topology is discrete. (The abbreviation DE refers to discrete with end points.) First of all we verify the following statement.

\[(2.1) \quad \text{Every nonempty set can be equipped with a DE-ordering.}\]

Proof. Of course, if \( S \neq \emptyset \) is finite then every linear ordering of \( S \) is a DE-ordering. Assume that \( S \) is infinite and that \( S \) is equipped with any well-ordering such that \( S \) has a maximum. Then create a linearly ordered set \( \tilde{S} \) from the well-ordered set \( S \) by replacing each point in \( S \setminus \{\min S, \max S\} \) with a copy of the naturally ordered set \( \mathbb{Z} \). Furthermore, replace \( \min S \) with a copy of the naturally ordered set \( \mathbb{N} \) and replace \( \max S \) with a copy of the naturally ordered set \( \mathbb{Z} \setminus \mathbb{N} \). So we obtain a set \( \tilde{S} \) equipped with a linear ordering which clearly is a DE-ordering. This is enough to verify (2.1) since \( S \) and \( \tilde{S} \) are equipollent sets.

Now in order to prove Proposition 1 let \( X \) be a space whose topology is generated by an ultrametric \( d \). For every \( n \in \mathbb{N} \) put

\[ \mathcal{P}_n := \{ \{ x \in X \mid d(x,a) < 2^{-n} \} \mid a \in X \}. \]
Since distinct ultrametrical balls with identical radii must be disjoint, for every \( n \in \mathbb{N} \) the family \( P_n \) is a partition of the set \( X \) consisting of open-closed sets. Since \( B_1 \subset B_2 \) or \( B_1 \supset B_2 \) whenever \( B_1, B_2 \) are non-disjoint balls in \( X \), if \( n > m \) then the partition \( P_n \) is finer than the partition \( P_m \). Clearly, \( \bigcup_{n=1}^{\infty} P_n \) is a basis of the space \( X \).

If the ultrametric \( d \) is complete then, of course, \( \bigcap_{n=1}^{\infty} B_n \neq \emptyset \) for every chain \( B_1 \supset B_2 \supset B_3 \supset \cdots \) of balls \( B_n \in P_n \).

For every \( x \in X \) and for every \( n \in \mathbb{N} \) let \( B_n(x) \) denote the unique open-closed set \( B \) with \( x \in B \in P_n \). Then \( B_n(x) \supset B_m(x) \) whenever \( n \leq m \).

We define by induction for every \( n \in \mathbb{N} \) a DE-ordering \( \prec \) of the set \( P_n \) in the following way. Firstly define a DE-ordering \( \prec_1 \) of the set \( P_1 \) by virtue of (2.1).

If for \( n \in \mathbb{N} \) a DE-ordering \( \prec_n \) of the set \( P_n \) is already defined then define a DE-ordering \( \prec_P \) of the set \( \{ A \in P_{n+1} \mid A \subset P \} \) for every \( P \in P_n \) and for distinct \( A, B \in P_{n+1} \) put \( A \prec_{n+1} B \) either when \( A, B \subset P \in P_n \) and \( A \prec_P B \) or when \( A \subset P \) and \( B \subset Q \) and \( P, Q \in P_n \) and \( P \prec_n Q \). Obviously, \( \prec_{n+1} \) is a DE-ordering of the set \( P_{n+1} \).

Now define \( x \prec y \) for distinct \( x, y \in X \) if and only if \( B_n(x) \prec_n B_n(y) \) for some \( n \in \mathbb{N} \). Then \( x \prec y \) for distinct \( x, y \in X \) if and only if for some index \( n \) we have \( B_m(x) \prec_m B_m(y) \) for every index \( m \geq n \). Consequently, the relation \( \prec \) is a strict linear order of the set \( X \). We claim that

\[(2.2) \text{ the topology of } X \text{ is finer than the order topology of } \prec \]

and that if the ultrametric \( d \) is complete then

\[(2.3) \text{ the topology of } X \text{ is coarser than the order topology of } \prec .\]

In order to verify (2.2) it is enough to show that the rays

\[ \{ x \in X \mid x \prec a \} \text{ and } \{ x \in X \mid a \prec x \} \]

are open sets in the space \( X \) for every \( a \in X \). Let \( R_a = \{ x \in X \mid x \prec a \} \) for \( a \in X \) and let \( y \in R_a \) and choose \( n \in \mathbb{N} \) with \( B_n(y) \neq B_n(a) \). Then \( B_n(y) \prec_n B_n(a) \) and hence \( x \prec a \) for every \( x \in B_n(y) \) (since \( B_n(x) = B_n(y) \) for every \( x \in B_n(y) \). Thus we have \( B_n(y) \subset R_a \) if \( y \in R_a \) and \( B_n(y) \neq B_n(a) \). In other words, every point in \( R_a \) is an interior point of \( R_a \). Hence \( R_a \) is open in the space \( X \) for every \( a \in X \). Similarly, \( \{ x \in X \mid a \prec x \} \) is open in the space \( X \) for every \( a \in X \).

In order to verify (2.3) under the completeness assumption it is enough to show that all sets in the basis \( \bigcup_{n=1}^{\infty} P_n \) are open with respect to the order topology of \( \prec \). Let \( m \in \mathbb{N} \) and \( V \in P_m \). Assume firstly that \( V \) is neither the maximum nor the minimum of the DE-ordered set \( (P_n, \prec_m) \) and let \( U \) resp. \( W \) be the immediate predecessor resp. successor of \( V \) in the linearly ordered set \( (P_m, \prec_m) \). For every \( n \geq m \) define balls \( U_n, W_n \) such that \( U_m = U \) and \( W_m = W \) and that \( U_{n+1} \) is the \( \prec_{n+1} \)-maximum of \( \{ B \in P_{n+1} \mid B \subset U_n \} \) and that \( W_{n+1} \) is the \( \prec_{n+1} \)-minimum of \( \{ B \in P_{n+1} \mid B \subset W_n \} \). Then for \( \bigcap_{n=m}^{\infty} U_n = \{ u \} \) and \( \bigcap_{n=m}^{\infty} W_n = \{ w \} \) we obviously have \( V = \{ x \in X \mid u \prec x \prec w \} \). In a similar way we obtain \( V = \{ x \in X \mid x \prec a \} \) for some \( a \in X \) if \( V \) is the \( \prec_m \)-minimum of \( P_m \) and \( V = \{ x \in X \mid a \prec x \} \) for some \( a \in X \) if \( V \) is the \( \prec_m \)-maximum of \( P_m \).

This concludes the proof of Proposition 1.

**Remark.** If the ultrametric \( d \) is not complete then (2.3) is not necessarily true. For one cannot rule out the situation that \( \{ x \} \in P_n \) for some \( x \in X \) and \( n \in \mathbb{N} \) and that
Clearly, two spaces \( X \) and \( N \) are not homeomorphic if \( X \) is isolated in the space \( X \) but not isolated with respect to the order topology. However, by considering special DE-orderings it is not difficult to accomplish (2.3) without assuming that \( d \) is complete.

3. Signature sets for metrizable spaces

In general topology there are two natural ways to verify that spaces are not homeomorphic. The first way is to apply connectedness arguments. This way is out of the question when we deal with totally disconnected spaces. The second way is to use Cantor derivatives. Let \( \Omega \) denote the class of all ordinals, whence \( \{0\} \cup \mathbb{N} \subset \Omega \). If \( X \) is a Hausdorff space and \( \xi \in \Omega \) and \( A \subset X \) then \( A(\xi) \) is the \( \xi \)-th derivative of the point set \( A \). \( (A(0) = A \) and \( A' \) is the set of all limit points of \( A \) and if \( \alpha \in \Omega \) then \( A(\alpha + 1) = A(\alpha)' \) and if \( \lambda > 0 \) is a limit ordinal then \( A(\lambda) := \bigcap \{ A(\alpha) \mid 0 < \alpha < \lambda \} \). Naturally, if \( 0 \neq \xi \in \Omega \) then \( A(\xi) \) is closed.

Furthermore, \( X \) is scattered if and only if \( X(\alpha) = \emptyset \) for some \( \alpha \in \Omega \).

**Lemma 1.** Let \( Z \) be a Hausdorff space with \( Z'' = \emptyset \) and let \( H \) be a Hausdorff space. Then for the product space \( Z \times H \) we have \( (Z \times H)(\xi + 1) = Z' \times H(\xi) \cup Z \times H(\xi + 1) \) for every ordinal number \( \xi \).

**Proof.** Let \( E[\xi] \) denote the equation \( (Z \times H)(\xi + 1) = Z' \times H(\xi) \cup Z \times H(\xi + 1) \). Similarly as in calculus, the product formula \( (X \times Y)' = X' \times Y \cup X \times Y' \) is true for arbitrary Hausdorff spaces \( X, Y \). (For \( (x, y) \) is isolated in \( X \times Y \) if and only if \( x \) is isolated in \( X \) and \( y \) isolated in \( Y \).) This has two consequences. Firstly, \( E[\xi] \) is true for \( \xi = 0 \). Secondly if \( E[\xi] \) holds for \( \xi = \alpha \) then \( E[\xi] \) holds for \( \xi = \alpha + 1 \). The set \( Z' \times H(\xi) \cup Z \times H(\xi + 1) \) is the union of the disjoint sets \( Z' \times H(\xi) \) and \( (Z \setminus Z') \times H(\xi + 1) \) since \( H(\xi + 1) \subset H(\xi) \).

Therefore, if \( \lambda > 0 \) is a limit ordinal and \( E[\xi] \) is true for every \( \xi < \lambda \) then \( (Z \times H)(\lambda) = \bigcap_{0 < \xi < \lambda} Z' \times H(\xi) \cup \bigcap_{0 < \xi < \lambda} (Z \setminus Z') \times H(\xi + 1) = Z' \times H(\lambda) \cup (Z \setminus Z') \times H(\lambda) = Z \times H(\lambda) \) and hence \( (Z \times H)(\lambda + 1) = ((Z \times H)(\lambda))' = (Z \times H(\lambda))' = Z' \times H(\lambda) \cup Z \times H(\lambda + 1) \) by applying the product formula, whence \( E[\xi] \) is true for \( \xi = \lambda \), q.e.d.

If \( X \) is a Hausdorff space and \( \kappa \) an uncountable cardinal then let \( C_\kappa(X) \) denote the set of all points \( x \in X \) such that \( |U| \geq \kappa \) for every neighborhood of \( x \) and \( |U| = \kappa \) for some neighborhood of \( x \). (One may call the members of \( C_\kappa(X) \) the \( \kappa \)-condensation points of \( X \).) Define a signature set \( \Sigma[\kappa; X] \) by

\[
\Sigma[\kappa; X] := \{ \alpha \in \Omega \setminus \{0\} \mid (X(\alpha) \setminus X(\alpha + 1)) \cap C_\kappa(X) \neq \emptyset \}.
\]

Of course, the class \( \Sigma[\kappa, X] \) is always a set and two spaces \( X_1 \) and \( X_2 \) cannot be homeomorphic if \( \Sigma[\kappa; X_1] \neq \Sigma[\kappa; X_2] \) for some cardinal \( \kappa > \aleph_0 \).

For the proof of Theorem 1 we also need another signature set. For any Hausdorff space \( X \) let \( \Gamma(X) \) be the set of all points \( x \in X \) such that no neighborhood of \( x \) is compact (or, equivalently, the closure of an open neighborhood of \( x \) is never compact). Furthermore, define

\[
\Sigma(X) := \{ k \in \mathbb{N} \mid (X(k) \setminus X(k + 1)) \cap \Gamma(X) \neq \emptyset \}
\]

Note that we regard \( \mathbb{N} \) to be defined in the classical way, i.e. \( \mathbb{N} \) does not contain 0. Clearly, two spaces \( X_1 \) and \( X_2 \) cannot be homeomorphic if \( \Sigma(X_1) \neq \Sigma(X_2) \).
4. Countable Polish spaces

Let \( c = 2^{\aleph_0} \) denote the cardinality of the continuum. The size of a perfect completely metrizable space cannot be smaller than \( c \) (cf. [1] 4.5.5). Consequently, all completely metrizable spaces of size smaller than \( c \) are scattered. Furthermore, a Polish space is scattered if and only if it is countable. In view of this fact and by virtue of Proposition 1 the special case \( \kappa = \aleph_0 \) in Theorem 1 is settled by the following observation.

(4.1) There exist \( c \) mutually non-homeomorphic closed and countable subspaces of the Polish space \( \mathbb{R} \setminus \mathbb{Q} \).

Notice that Proposition 1 can be applied because the topology of every space provided by (4.1) can be generated by a complete ultrametric. This is a consequence of the observation that the topology of the Baire space \( \mathbb{R} \setminus \mathbb{Q} \) is generated by a very natural complete ultrametric. (Declare \( 2^{-n} \) as the distance of distinct irrationals \( a \) and \( b \) when \( n \) is the smallest index of distinct quotients in the continued fractions of \( a \) and \( b \).)

Remark. While for both \( X = \mathbb{R} \) and \( X = \mathbb{R} \setminus \mathbb{Q} \) the order topology on \( X \) generated by the natural ordering of the elements of \( X \) coincides with the Euclidean topology on \( X \), such a coincidence is not true for subsets of \( X \). Even worse, while such a coincidence holds for closed subsets of the connected space \( \mathbb{R} \), it does not necessarily hold for closed subsets of the totally disconnected Baire space \( \mathbb{R} \setminus \mathbb{Q} \). (For example, \( \{-e\} \cup \{e^{-n} \mid n \in \mathbb{N}\} \) is an infinite, closed, discrete subspace of \( \mathbb{R} \setminus \mathbb{Q} \) whose order topology is compact. See also Proposition 2 below.) Therefore, Proposition 1 is essential for deriving the case \( \kappa = \aleph_0 \) in Theorem 1 from (4.1).

The statement (4.1) equals the classic solution of the enumeration problem concerning countable Polish spaces due to Mazurkiewicz and Sierpiński (see [2] Lemma 4.10). In order to keep this paper self-contained we will prove (4.1) and hence Theorem 1 for \( \kappa = \aleph_0 \) in this section. And by proving a bit more we will gain a deeper insight in connection with the following observation.

(4.2) It is unprovable that \( \mathbb{R} \) contains \( c \) mutually non-homeomorphic closed and countable subspaces.

The observation (4.2) is true because by [9] there exist precisely \( \aleph_1 \) compact and countable subspaces of \( \mathbb{R} \) up to homeomorphism and therefore (see [5] Theorem 8.1) the real line \( \mathbb{R} \) has precisely \( \aleph_1 \) closed and countable subspaces up to homeomorphism. Motivated by comparing (4.1) and (4.2) we are now going to prove the following proposition which implies (4.1) and hence settles the case \( \kappa = \aleph_0 \) in Theorem 1. Throughout this section, if \( X \subset \mathbb{R} \) then \( \overline{X} \) denotes the space which equals the closure of \( X \) in the real line \( \mathbb{R} \) equipped with the Euclidean topology, and \( \hat{X} \) denotes the space which equals the set \( X \) equipped with the order topology generated by the natural ordering of the reals in \( X \).

**Proposition 2.** There exists a family \( \mathcal{Y} \) of mutually non-homeomorphic countable subspaces of \( \mathbb{R} \) such that \( |\mathcal{Y}| = c \) and \( \overline{Y} \setminus \mathbb{Q} = Y \) for every \( Y \in \mathcal{Y} \) and all spaces in the family \( \{ \overline{Y} \mid Y \in \mathcal{Y} \} \cup \{ \hat{Y} \mid Y \in \mathcal{Y} \} \) are homeomorphic.

**Proof.** In the following put \( \mathbb{N}^* = \mathbb{N} \setminus \{1\} \), whence \( \mathbb{N}^* = \{ k \in \mathbb{Z} \mid k \geq 2 \} \). For \( n \in \mathbb{N} \) let \( K_n \) be a compact, countable, well-ordered subset of \( [\pi + 2n, \pi + 2n + 1] \setminus \mathbb{Q} \) such that \( K_n^{(n)} = \{ \max K_n \} = \{ \pi + 2n + 1 \} \). (The set \( K_n \) may be defined as an appropriate order-isomorphic copy of the canonically ordered set of all ordinals \( \alpha \leq \omega^n \). It is also straightforward to construct \( K_1, K_2, K_3, \ldots \) recursively without using ordinal numbers.)
In order to prove Proposition 2, let \( r_0, r_1, r_2, \ldots \) be a strictly decreasing sequence of rational numbers in \([\pi, \pi+1]\) with \( \inf \{ r_m \mid m \in \mathbb{N} \} = \pi \). For every \( m \in \mathbb{N} \) let \( \xi_m(1), \xi_m(2), \xi_m(3), \ldots \) be a strictly decreasing sequence of irrational numbers in \([r_m, r_{m-1}]\) with \( \inf \{ \xi_m(k) \mid k \in \mathbb{N} \} = r_m \). For every \( n \in \mathbb{N} \) define a discrete subset \( E_n \) of \([\pi + 2n, \pi + 2n + 2 \setminus \mathbb{Q}\) via \( E_n := \{ 2n+1 + \xi_m(k) \mid m, k \in \mathbb{N} \} \).

For every infinite subset \( S \) of \( \mathbb{N}^* \) put
\[
G_S := \bigcup_{n \in S} (K_n \cup E_n).
\]

For every \( n \in \mathbb{N} \) we have \( \overline{E_n} \setminus E_n = \{ 2n+1+r_m \mid m \in \mathbb{N} \} \cup \{ \pi + 2n + 1 \} \) and hence the only irrational limit point of \( E_n \) is \( \pi + 2n + 1 = \max K_n \). So for every infinite subset \( S \) of \( \mathbb{N}^* \) we have \( G_S = \overline{G_S} \setminus \mathbb{Q} \) since \( \overline{G_S} = G_S \cup \bigcup_{n \in S} \overline{E_n} \). We claim that the family
\[
\mathcal{Y} := \{ G_S \mid S \subset \mathbb{N} \land |S| = \aleph_0 \}
\]
is as desired. Obviously, we always have \( \Gamma(G_S) = \{ \max K_n \mid n \in S \} \). Therefore, since the point \( \max K_n \) lies in \( (K_n \setminus \{ \max K_n \})^{(m)} \) but not in \( (K_n \setminus \{ \max K_n \})^{(m+1)} \) if and only if \( m = n \), we must always have \( \Sigma(G_S) = S \). Hence two spaces \( G_S \) and \( G_T \) are never homeomorphic for distinct infinite sets \( S, T \subset \mathbb{N}^* \).

We conclude the proof by showing that for some space \( W \) both spaces \( \overline{Y} \) and \( \hat{Y} \) are homeomorphic to \( W \) for each \( Y \in \mathcal{Y} \). Let \( S \) be an arbitrary infinite subset of \( \mathbb{N}^* \). The order-type of the naturally ordered set \( \{ -x \mid x \in \overline{E_n} \} \) is \( \omega^2 + 1 \) while the order type of \( K_n \) is \( \omega^{\omega} + 1 \). Hence by a zipper argument (in a Hilbert’s hotel kind of way) it is plain to find a homeomorphism from the compact space \( K_n \cup \overline{E_n} \) onto the compact space \( K_n \) for every \( n \geq 2 \). Consequently, since each compact building block \( K_n \cup \overline{E_n} \) is open in the space \( \overline{G_S} = \bigcup_{n \in S} (K_n \cup \overline{E_n}) \), the space \( \overline{G_S} \) is homeomorphic to \( V_S := \bigcup_{n \in S} K_n \).

The naturally ordered set \( V_S \) is well-ordered without a maximum and \( V_S^{(k)} \neq \emptyset \) for every \( k \in \mathbb{N} \) but \( \bigcap_{k=1}^{\infty} V_S^{(k)} = \emptyset \). Hence \( V_S \) is order-isomorphic to the well-ordered set \( W \) of all ordinal numbers smaller than \( \omega^\omega \). Therefore, since for every closed subset \( A \) of \( \mathbb{R} \) the linearly ordered space \( \hat{A} \) is identical with the subspace \( A \) of the real line, the space \( \overline{G_S} \) is homeomorphic to the space \( W \) equipped with the canonical order topology. Finally it is evident that the linearly ordered space \( \hat{G}_S \) is homeomorphic to the subspace \( \overline{G_S} \) of \( \mathbb{R} \) and hence to the space \( W \), q.e.d.

5. Ultrametric hedgehogs

Let us call a space completely ultrametrizable if and only if its topology is generated by some complete ultrametric. (It is worth mentioning that a space must be completely ultrametrizable if its topology is generated by some ultrametric and also by some complete metric, see [7] Lemma 3. Therefore, since every scattered metrizable space is completely metrizable, in view of Proposition 1 it would be enough to consider ultrametrics instead of complete ultrametric. But the ultrametrics we will consider in the proof of Theorem 1 are very natural and immediately recognized as complete metrics.)

Clearly, if \( X_1 \) and \( X_2 \) are completely ultrametrizable spaces then the product space \( X_1 \times X_2 \) and the topological sum of the two spaces are completely ultrametrizable as well. (Note that if \( d_i \) is an ultrametric on \( X_i \) then the standard maximum metric on \( X_1 \times X_2 \) with respect to \( d_1, d_2 \) is an ultrametric.)
For an index set \( I \neq \emptyset \) let \( X_i \) be an infinite metric space and \( b_i \) a point in \( X_i \) for every \( i \in I \). The metric of \( X_i \) is denoted by \( d_i \) and we assume that \( d_i \) is an ultrametric. Fix \( \alpha \) such that \( \alpha \neq \bigcup_{i \in I} I \times X_i \) and put
\[
\mathcal{H}[(X_i, b_i)_{i \in I}] = \{o\} \cup \bigcup_{i \in I} \{i\} \times (X_i \setminus \{b_i\}) .
\]

Define a function \( d \) from \( \mathcal{H}[(X_i, b_i)_{i \in I}]^2 \) into \( \mathbb{R} \) as follows:

Put \( d(o, o) = 0 \), and for each \( i \in I \) put
\[
\begin{align*}
      & d((i, x), o) = d(o, (i, x)) = d_i(b_i, x) \quad \text{whenever} \quad b_i \neq x \in X_i , \quad \text{and for each} \quad i \in I \quad \text{put} \\
      & d((i, x), (i, y)) = d_i(x, y) \quad \text{whenever} \quad x, y \in X_i \setminus \{b_i\} , \quad \text{and for distinct} \quad i, j \in I \quad \text{put} \\
      & d((i, x), (j, y)) = \max\{d_i(b_i, x), d_j(b_j, y)\} \quad \text{whenever} \quad b_i \neq x \in X_i \quad \text{and} \quad b_j \neq y \in X_j .
\end{align*}
\]

After having thoroughly checked that in all possible scenarios any triangle is isosceles and any non-equilateral triangle has one side shorter than the other sides, we see that \( d \) is an ultrametric on the set \( \mathcal{H}[(X_i, b_i)_{i \in I}] \). Trivially, the subspace \( \{o\} \cup (X_i \setminus \{b_i\}) \) of \( \mathcal{H}[(X_i, b_i)_{i \in I}] \) is an isometric copy of \( X_i \) for every \( i \in I \). (So if the sets \( X_i \) are mutually disjoint then every basic space \( X_i \) can be identified with the space \( \{o\} \cup (X_i \setminus \{b_i\}) \) in order to achieve \( \mathcal{H}[(X_i, b_i)_{i \in I}] = \bigcup_{i \in I} X_i \) and \( X_i \cap X_j = \{o\} \) for all distinct \( i, j \in I \).)

By analogy to the classical hedgehog of spininess \( |I| \) (see [1] or [10]) we call \( \mathcal{H}[(X_i, b_i)_{i \in I}] \) an ultrametric hedgehog and we call the basic spaces \( X_i \) \( (i \in I) \) the spines and the point \( o \) the body of the hedgehog.

It is evident that the hedgehog-metric \( d \) is complete if and only if all spine-metrics \( d_i \) are complete. The weight of \( \mathcal{H}[(X_i, b_i)_{i \in I}] \) is not smaller than \( \max\{|I|, \aleph_0\} \) because each spine contains an open set disjoint from \( \{o\} \). If \( w(X_i) = \kappa \geq \aleph_0 \) for every \( i \in I \) then \( w(\mathcal{H}[(X_i, b_i)_{i \in I}]) = \max\{|I|, \kappa\} \) because each spine contains a dense set of size \( \kappa \).

6. Proof of Theorem 1

**Proposition 3.** For every cardinal \( \kappa > \aleph_0 \) there exists a complete ultrametric space \( Y_{\kappa} \) of size \( \kappa \) such that \( Y_{\kappa}' = C_\kappa(Y_{\kappa}) \) and \( |C_\kappa(Y_{\kappa})| = 1 \) and hence \( Y_{\kappa} \setminus C(Y_{\kappa}) \) is a discrete subspace of \( Y_{\kappa} \).

**Proof.** Consider the set \( Z := \{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\} \) and declare \( \max\{x, y\} \) as the distance between distinct points \( x, y \in Z \). It is evident that this distance defines a complete ultrametric on \( Z \) which generates the Euclidean topology restricted to \( Z \). Thus \( Z' = \{0\} \) . Let \( I \) be an index set of size \( \kappa > \aleph_0 \) and for every \( i \in I \) let \( A_i \) be identical with the complete ultrametric space \( Z \). Put \( Y_{\kappa} := \mathcal{H}[(A_i, 0)_{i \in I}] \) and let \( o \) be the body of this ultrametric hedgehog. We observe that \( Y_{\kappa}' = C_\kappa(Y_{\kappa}) = \{o\} \), q.e.d.

**Proposition 4.** For every ordinal \( \alpha > 0 \) there exists a complete ultrametric space \( Z_\alpha \) of size \( \max\{\aleph_0, |\alpha|\} \) such that \( Z_\alpha^{(\alpha)} \) is a singleton. (In particular, \( Z_\alpha \) is scattered.)

**Proof.** Of course, we construct the desired spaces by transfinite induction. Put \( Z_1 = Z \) where \( Z = \{0\} \cup \{2^{-n} \mid n < \mathbb{N}\} \) is the complete ultrametric space as in the proof of Proposition 3. Suppose firstly that for an ordinal \( \alpha > 0 \) a complete ultrametric space \( Z_\alpha \) with \( Z_\alpha^{(\alpha)} = \{z_\alpha\} \) and \( |Z_\alpha| = \max\{\aleph_0, |\alpha|\} \) is already defined. (This is true for \( \alpha = 1 \).) Then we define \( Z_{\alpha+1} \) as the product space \( Z \times Z_\alpha \) and consider it equipped with
the maximum metric which is a complete ultrametric. Clearly, \(|Z_{\alpha+1}| = \max\{|N_0, |\alpha|\} = \max\{|N_0, |\alpha + 1|\}\). By virtue of Lemma 1 we have \(Z_{\alpha+1} = \{(0, z_\alpha)\}\).

Secondly, let \(\lambda > 0\) be a limit ordinal and assume that for every ordinal \(\alpha\) with \(1 \leq \alpha < \lambda\) a complete ultrametric space \(Z_\alpha\) with \(Z^{(\alpha)}_\alpha = \{z_\alpha\}\) and \(|Z_\alpha| = \max\{|N_0, |\alpha|\}\) is already defined. Then let \(Z_\lambda\) be the complete ultrametric hedgehog \(H([Z_\alpha, z_\alpha]_{1 \leq \alpha < \lambda})\). Obviously, the body \(o\) of the hedgehog \(Z_\lambda\) lies in \(Z_\lambda^{(\beta)}\) whenever \(1 \leq \beta < \lambda\). Furthermore, \(Z^{(\beta)}_\alpha = \emptyset\) whenever \(1 \leq \alpha < \beta < \lambda\). Consequently, \(Z^{(\lambda)}_\lambda = \bigcap_{1 \leq \beta < \lambda} Z_\lambda^{(\beta)} = \{o\}\). Of course, \(|Z_\lambda| = |\lambda| = \max\{|N_0, |\lambda|\}\). This concludes the proof of Proposition 4.

Now we are ready to prove Theorem 1. The special case \(\kappa = \aleph_0\) is settled by Propositions 1 and 2. Therefore we assume \(\kappa > \aleph_0\). Let \(K := \{\alpha \in \Omega \mid \omega \leq \alpha < \kappa\}\). Then \(|K| = \kappa\) and hence \(K\) has precisely \(2^\kappa\) nonempty subspaces. Let \(Y_\kappa\) be an ultrametric space with \(|C_\kappa(Y_\kappa)| = 1\) as in the proof of Proposition 3 and for each \(\alpha \in K\) let \(Z_\alpha\) be an ultrametric space as in Proposition 4 with \(Z^{(\alpha)}_\alpha = \{z_\alpha\}\). Since \(0 \not\in K\) we may unambiguously write \(Z_0\) for the space \(Y_\kappa\) and \(z_0\) for the unique condensation point in \(Y_\kappa\). For every \(\alpha \in K\) put \(H_\alpha := H([Z_\beta, z_\beta]_{\beta \in \{0, \alpha\}})\). So \(H_\alpha\) is a complete ultrametric hedgehog with the two spines \(Y_\kappa\) and \(Z_\alpha\). Clearly, \(|H_\alpha| = \kappa\) and if \(o\) is the body of the hedgehog \(H_\alpha\) then \(H^{(\alpha)}_\alpha = C_\kappa(H_\alpha) = \{o\}\). (Note that \(|Z_\alpha| < \kappa\) for every \(\alpha \in K\).) In view of the proofs of Propositions 3 and 4 we may assume that the distance between points in \(H_\alpha\) is always smaller than 1. (Alternatively, if \((X, d)\) is a complete ultrametric space then it is evident that \(\min\{\frac{8}{9}, d(\cdot, \cdot)\}\) is a complete ultrametric which generates the topology of \((X, d)\).)

Finally, for each nonempty \(L \subset K\) define \(S[L]\) as the standard metrical sum of the complete ultrametric spaces \(H_\alpha\) \((\alpha \in L)\). (The distance between any two points in a common summand remains unchanged and the distance between two points in distinct summands always equals 1.) Clearly, \(S[L]\) is a scattered and complete ultrametric space of size \(\kappa\). The \(2^\kappa\) spaces \(S[L]\) \((\emptyset \neq L \subset K)\) are mutually non-homeomorphic because we obviously have \(\Sigma[\kappa; S[L]] = L\) whenever \(\emptyset \neq L \subset K\).

7. Signature sets for non-metrizable spaces

In order to prove Theorem 2 we also work with Cantor derivatives. Again let \(\Omega\) be the canonically well-ordered class of all ordinal numbers (with \(\mathbb{N} \cup \{0\} \subset \Omega\)). Furthermore let \(\mathcal{L}\) denote the class of all limit ordinals (with \(0 \in \mathcal{L}\)). Note that any nonempty subset of the class \(\Omega\) has a well-defined supremum. Put \(\{\alpha, \beta\} := \{\xi \in \Omega \mid \alpha \leq \xi \leq \beta\}\) and \(\{\alpha, \beta\} := \{\alpha, \beta\} \setminus \{\beta\}\) and \(\alpha, \beta := \{\alpha, \beta\}\) whenever \(\alpha, \beta \in \Omega\). If we speak of the space \([\alpha, \beta]\) resp. \([\alpha, \beta]\) resp. \([\alpha, \beta]\) then we refer to the order topology of the canonical well-ordering. Note that all spaces \([\alpha, \beta]\) are compact and scattered and hereditarily normal.

If \(\alpha \in \Omega\) then put \(|\alpha| := ||0, \alpha||\). (This definition is a tautology if ordinal numbers are defined in the standard way.) Put \(o(\kappa) := \min\{\gamma \in \Omega \mid |\gamma| = \kappa\}\) for every cardinal number \(\kappa\). (If cardinal numbers are defined as initial ordinal numbers then, of course, \(o(\kappa) = \kappa\) for every \(\kappa\).) For any cardinal number \(\kappa\) let (as usual) \(\kappa^+\) denote the smallest cardinal number greater than \(\kappa\). (For example, \(\aleph_1 = \aleph_0^+\).) For cardinals \(\kappa\) and ordinals \(\alpha\) we have \(|\alpha| = \kappa\) if and only if \(o(\kappa) \leq \alpha < o(\kappa^+)\). In particular, \(\omega := o(\aleph_0)\) is the smallest infinite ordinal and \(\omega_1 := o(\aleph_1)\) is the smallest uncountable ordinal.
For \( \xi \in \Omega \) we write (as usual) \( \omega^\xi \) for the \emph{ordinal power} with basis \( \omega \) and exponent \( \xi \). So all spaces \([0, \omega^\xi]\) are compact and for \( \xi > 0 \) we have \( |[0, \omega^\xi]| = \max\{\aleph_0, |\xi|\} \). In particular, \( |[0, \omega^\xi]| = |\xi| \) for every ordinal \( \xi \geq \omega \).

As above, if \( X \) is a Hausdorff space and \( \xi \in \Omega \) and \( A \subset X \) then \( A^{(\xi)} \) is the \( \xi \)-th derivative of the point set \( A \). Clearly, \( A^{(\alpha)} \supset A^{(\beta)} \) whenever \( 0 < \alpha \leq \beta \) and for \( A \subset B \subset X \) we have \( A^{(\alpha)} \subset B^{(\alpha)} \) for every \( \alpha \in \Omega \). The following lemma is evident. (Historically, Cantor’s definition of the ordinal powers of \( \omega \) is designed precisely so that the following is true.)

\textbf{Lemma 2.} Let \( 0 \neq \xi \in \Omega \). With respect to the compact space \([0, \omega^\xi]\), for every ordinal \( \alpha > 0 \) the point sets \([0, \omega^\xi]^{(\alpha)}\) and \([0, \omega^\xi]^{(\alpha)}\) coincide and they contain the point \( \omega^\xi \) if and only if \( \alpha \leq \xi \). And \([0, \omega^\xi]^{(\xi)} = [0, \omega^\xi]^{(\xi)} = \{\omega^\xi\}\).

In the following we distinguish between \emph{regular} and \emph{singular} cardinal numbers. Singular cardinals are the cardinals which are not regular. A cardinal number \( \kappa \) is \emph{regular} if and only if \( \sup A < o(\kappa) \) for every subset \( A \) of \( \{ \alpha \in \Omega \mid \alpha < o(\kappa) \} \) with \( |A| < \kappa \). (Note that any subset of the class \( \Omega \) has a well-defined supremum in \( \Omega \).) Topologically speaking, an infinite cardinal number \( \kappa \) is regular if and only if in the compact linearly ordered space \([0, o(\kappa)]\) the first derivative of a point set \( A \) with \( |A| < \kappa \) does not contain the point \( o(\kappa) \). (In the following it is essential that \( \kappa^+ \) is regular for every \( \kappa \).)

If \((X, \preceq)\) is a linearly ordered set then let \([a, b] = \{ x \in X \mid a \leq x \leq b \}\) and \([a, b[ = \{ x \in X \mid a < x \leq b \}\) and \([a, b[ = [a, b] \setminus \{ b \}\) for \( a, b \in X \). Furthermore let \( \preceq^* \) denote the \emph{backwards linear ordering} defined by \( x \preceq^* y \) if and only if \( y < x \). In the usual sloppy way, if \( X \) is a set of ordinal numbers, then \( X^* \) is the set \( X \) equipped with the backwards linear ordering of the canonical well-ordering of \( \Omega \). If \((X, \preceq)\) and \((Y, \preceq)\) are two linearly ordered sets then the \emph{lexicographic ordering} of any nonempty subset \( Z \) of \( X \times Y \) is defined so that \((x_1, y_1)\) is smaller than \((x_2, y_2)\) when either \( x_1 < x_2 \) or when \( x_1 = x_2 \) and \( y_1 < y_2 \).

In the following let \( X \) be a \emph{scattered} Hausdorff space and \( \kappa > \aleph_0 \) be a regular cardinal number. As in Section 2 let \( C_\kappa(X) \) denote the set of all \( \kappa \)-condensation points. (So a point \( x \in X \) lies in \( C_\kappa(X) \) if and only if \( |U| \geq \kappa \) for every neighborhood \( U \) of \( x \) and \( |U| = \kappa \) for some neighborhood \( U \) of \( x \).) For example, \( C_\kappa([0, o(\kappa)]) = \{ o(\kappa) \} \) for every regular cardinal \( \kappa > \aleph_0 \). For \( x \in X \) let \( \Omega_\kappa(x) \) denote the class of all ordinals \( \alpha \) such that \( x \in A^{(\alpha)} \) for some point set \( A \subset X \) with \( |A| < \kappa \). The class \( \Omega_\kappa(x) \) is never empty since, trivially, \( 0 \in \Omega_\kappa(x) \) for every \( x \in X \). Since \( X \) is scattered, the class \( \Omega_\kappa(x) \) is a nonempty set for each \( x \in X \). Moreover, \( \Omega_\kappa(x) \subset [0, o(\kappa)] \) because \( A^{(o(\kappa))} = \emptyset \) whenever \( A \subset X \) and \( |A| < \kappa \).

So we may define a signature set with respect to the scattered space \( X \) and the regular cardinal \( \kappa \) by

\[
\Psi[X, \kappa] := \{ \sup \Omega_\kappa(x) \mid x \in C_\kappa(X) \}.
\]

Clearly, two scattered spaces \( X_1, X_2 \) cannot be homeomorphic if \( \Psi[X_1, \kappa] \neq \Psi[X_2, \kappa] \) for some regular cardinal \( \kappa \). For each regular cardinal \( \kappa > \aleph_0 \) we have \( \Psi[[0, o(\kappa)], \kappa] = \{ 0 \} \). More generally, in view of the following lemma, \( \Psi[[0, \alpha], \kappa] \subset \{ 0, o(\kappa) \} \) for every \( \alpha \in \Omega \).
8. Proof of Theorem 2

The following two lemmas are essential for the proof of Theorem 2.

**Lemma 3.** Let $\kappa \geq \aleph_1$ be a regular cardinal. For $\theta \in \Omega$ consider the space $X = [0, \theta]$. If $\gamma \in C_\kappa(X)$ then either $\Omega_\kappa(\gamma) = [0, o(\kappa)]$ or $\Omega_\kappa(\gamma) = \{0\}$.

**Proof.** Since $\gamma \in C_\kappa(X)$, if $\alpha_1 < \gamma$ and $\alpha_2 \in \Omega$ and $|[\alpha_1, \alpha_2]| < \kappa$ then $[\alpha_1, \alpha_2] \subset [0, \gamma]$. Clearly, if $\sup A \neq \gamma$ whenever $\emptyset \neq A \subset [0, \gamma]$ and $|A| < \kappa$ then $\Omega_\kappa(\gamma) = \{0\}$. So assume that there is a nonempty set $A \subset [0, \gamma]$ such that $|A| < \kappa$ and $\sup A = \gamma$. For $\xi \in \Omega$ put $U_\xi := \bigcup_{\alpha \in A} [\alpha, \alpha + \omega^\xi]$. If $\xi < o(\kappa)$ then $|[\alpha, \alpha + \omega^\xi]| = |[0, \omega^\xi]| < \kappa$ for every $\alpha \in A$ and hence $U_\xi \subset [0, \gamma]$ and hence $\sup U_\xi = \gamma$. Thus $|U_\xi| < \kappa$ and (by Lemma 2) $\gamma \in U_\xi(\xi)$ for every $\xi < o(\kappa)$ and hence $\Omega_\kappa(\gamma) = [0, o(\kappa)]$, q.e.d.

**Remark.** In Lemma 3 the case $\Psi([0, \theta], \kappa) = \{0, o(\kappa)\}$ may occur, for example if $\kappa = \aleph_1$ and $\theta = \omega_1 \cdot \omega$.

**Lemma 4.** Let $(X, \prec)$ be a linearly ordered set equipped with the order topology and assume that the space $X$ is scattered. Let $0 \neq \xi \in \Omega$ and let $\kappa$ be a regular cardinal number with $\kappa > |\omega^\xi|$. Let $x, y, z$ be three points in $X$ with $x \prec y \prec z$ so that $[x, z]_\prec$ is an order-isomorphic copy of $[0, \omega^\xi]$ and $[z, y]_\prec$ is an order-isomorphic copy of $[0, o(\kappa)]^\ast$. Then $C_\kappa(X) \cap [x, y]_\prec = \{z\}$ and $\sup \Omega_\kappa(z) = \xi$.

**Proof.** Clearly, $z$ is the only $\kappa$-condensation point of $X$ strictly between $x$ and $y$. Since $\kappa$ is regular, there is no set $A \subset [z, y]_\prec$ with $|A| \neq \kappa$ and $z \in A$. Therefore, if $0 \neq \alpha \in \Omega$ and $z \in A^{(\alpha)}$ for a point set $A$ in the space $X$ with $|A| < \kappa$ then we already have $z \in (A \cap [x, z]_\prec)^{(\alpha)}$. On the other hand, $([x, z]_\prec)^{(\xi)} = \{z\}$ by Lemma 2 and $|[x, z]| = |[0, \omega^\xi]| < \kappa$. Consequently, $\Omega_\kappa(z) = [0, \xi]$ and hence $\sup \Omega_\kappa(z) = \xi$, q.e.d.

Now we are ready to prove Theorem 2. Let $\kappa$ be a cardinal with $\kappa > \aleph_0$ and put $K = [\omega, o(\kappa)]$. Let $\mathcal{G}$ be the family of all nonempty sets $\mathcal{S}$ of successor ordinals $\alpha + 1$ with $\alpha \in K \cap \mathcal{L} \setminus \{o(\kappa)\}$. So if $\xi \in S \in \mathcal{G}$ then $|\xi| = |[0, \omega^\xi]| < \kappa$. Clearly, $|\mathcal{G}| = 2^\kappa$. For every set $S \in \mathcal{G}$ let

$$H_S := K \times \{0\} \cup \bigcup_{\xi \in S} ([\xi] \times [0, \omega^\xi] \cup \{\xi + 1\} \times [0, o(\kappa)^\ast])$$

and

$$G_S := K \times \{0\} \cup \bigcup_{\xi \in S} ([\xi] \times [0, \omega^\xi] \cup \{\xi + 1\} \times [0, o(|\xi|^\ast)^\ast])$$

be equipped with the lexicographic ordering. Illustratively, the linearly ordered set $H_S$ resp. $G_S$ is constructed from the well-ordered set $K$ by replacing $\xi$ with a copy of $[0, \omega^\xi]$ and $\xi + 1$ with a copy of $[0, o(\kappa)^\ast]$ resp. $[0, o(|\xi|^\ast)^\ast]$ for each $\xi \in S$.

Then the corresponding linearly ordered spaces $H_S$ and $G_S$ are of size $\kappa$ and it is evident that all these spaces are scattered. They are also compact since the ordering is complete with a maximum and a minimum (cf. [10] 39.7). We claim that the spaces $H_S$ ($S \in \mathcal{G}$) are mutually non-homeomorphic if $\kappa$ is regular and that the spaces $G_S$ ($S \in \mathcal{G}$) are mutually non-homeomorphic if $\kappa$ is singular.

Assume firstly that $\kappa$ is regular and let $S \in \mathcal{G}$ and consider the space $H_S$. Clearly we have $(o(\kappa), 0) \in C_\kappa(H_S)$ and $\Omega_\kappa((o(\kappa), 0)) = \{0\}$. Obviously, $(\gamma, 0) \in C_\kappa(H_S)$ if and only if
\( \gamma = o(\kappa) \) or \( \gamma = \sup(S \cap [0, \gamma]) \) \ where \( S \cap [0, \gamma] \neq \emptyset \). If \( (o(\kappa), 0) \neq (\gamma, 0) \in C_\kappa(H_S) \) then \( \Omega_\kappa((\gamma, 0)) = [0, o(\kappa)] \) \ and hence \( \sup \Omega_\kappa((\gamma, 0)) = o(\kappa) \) \ because if \( \xi \in S \cap [0, \gamma[ \) \ then \( \{\xi + 1\} \times [0, \omega^\alpha]^* \subset \{\xi + 1\} \times [0, o(\kappa)]^* \) \ and \( \bigcup \{\{\xi + 1\} \times [0, \omega^\alpha]^* \mid \xi \in S \cap [0, \gamma[\}] \) \ < \kappa \) for arbitrarily large exponents \( \alpha < o(\kappa) \). In view of Lemma 4 we have \( C_\kappa(H_S) \setminus K \times \{0\} = \{ (\xi, \omega^\xi) \mid \xi \in S \} \) \ and \( \sup \Omega_\kappa((\xi, \omega^\xi)) = \xi \) \ for every \( \xi \in S \). Therefore,

\[ S = \Psi[H_S, \kappa] \setminus \{0, o(\kappa)\} \]

for every \( S \in \mathcal{G} \) and this settles Theorem 2 for regular \( \kappa > \aleph_0 \).

Assume now that \( \kappa \) is a singular cardinal number and let \( \mathcal{R} \) denote the set of all regular uncountable cardinals smaller than \( \kappa \). We claim that for every \( S \in \mathcal{G} \) we have

\[ S = \left( \bigcup_{\lambda \in \mathcal{R}} \Psi[G_S, \lambda] \right) \setminus \mathcal{L} = \left( \bigcup_{\lambda \in \mathcal{R}} \Psi[G_S, \lambda] \right) \setminus \left( \{0\} \cup \{ o(\lambda) \mid \lambda \in \mathcal{R} \} \right). \]

On the one hand, if \( \xi \in S \) then \( \xi \notin \mathcal{L} \) and \( |\xi|^+ \in \mathcal{R} \) \ and \( (\xi, \omega^\xi) \) is a \( |\xi|^+ \)-condensation point with \( \sup \Omega_{|\xi|^+}((\xi, \omega^\xi)) = \xi \) \ in view of Lemma 4.

On the other hand, let \( y \) be a \( \lambda \)-condensation point in \( G_S \) where \( \lambda \in \mathcal{R} \) \ and assume firstly that \( y \notin K \times \{0\} \). Then \( y \) lies in \( B_\xi := \{\xi\} \times [0, \omega^\xi] \cup \{\xi + 1\} \times [0, |\xi|^+[^* \) for some \( \xi \in S \). Since the points \( \min B_\xi = (\xi, 0) \) \ and \( \max B_\xi = (\xi + 1, 0) \) \ are isolated in the space \( G_S \), the point \( y \) must be a \( \lambda \)-condensation point in the space \( B_\xi \), whence \( \lambda \leq |B_\xi| = |\xi|^+ \).

In the case that \( \lambda = |\xi|^+ \) we must have \( y = (\xi, \omega^\xi) \) \ and hence \( \sup \Omega_\lambda(y) = \xi \in S \) \ by Lemma 4. In the case that \( \lambda < |\xi|^+ \) the point \( y \) must be the maximum resp. minimum of an isomorphic copy of \([0, \gamma]\) resp. \([0, \gamma]^* \) within the linearly ordered set \( B_\xi \) where \( \gamma \) is a \( \lambda \)-condensation point in the space \([0, \gamma]\), whence \( \sup \Omega_\lambda(y) \in \{0, o(\lambda)\} \) \ by Lemma 3.

Assume secondly that \( y = (x, 0) \) \ for \( x \in K \). If \( x \) is a \( \lambda \)-condensation point in the basic space \( K \) then it is clear that in the space \( G_S \) we also have \( \sup \Omega_\lambda(y) \in \{0, o(\lambda)\} \). If \( x \notin C_\lambda(K) \) \ then \( y \in C_\lambda(G_S) \) \ forces \( x \) \ to be the supremum of a set

\[ \hat{S} \subset \{ \xi \in S \mid \xi < x \land |\xi|^+ = \lambda \} \]

with \( |\hat{S}| < \lambda \) \ and therefore (by the same argument as for the space \( H_S \)) we must have \( \Omega_\lambda(y) = [0, o(\lambda)] \) \ and hence \( \sup \Omega_\lambda(y) = o(\lambda) \).

So in any case the ordinal number \( \sup \Omega_\lambda(y) \) \ lies in \( S \cup \{0, o(\lambda)\} \) \ if \( y \in C_\lambda(G_S) \) \ for \( \lambda \in \mathcal{R} \). This concludes the proof of Theorem 2.

9. Completions and compactifications

In this short, final section we present two nice applications of Theorem 1 and Theorem 2.

If \( X \) \ is a scattered Hausdorff space then it is plain that the set \( X \setminus X' \) \ of all isolated points is dense. Moreover, \( X \setminus X' \) \ is the intersection of all dense subsets of \( X \). In particular, \( |X \setminus X'| \) \ is the density of the scattered space \( X \). Consequently, if the scattered space \( X \) \ is metrizable then \( |X \setminus X'| = |X| \) \ because weight and density of a metric space are always identical. Therefore, from Theorem 1 we derive the following enumeration result about completions of discrete metric spaces.
Corollary 3. The topology of an infinite discrete space $S$ can be generated by $2^{|S|}$ ultrametrics $d$ such that the completions of the metric spaces $(S,d)$ are mutually non-homeomorphic scattered, ultrametric (linearly ordered) spaces (of weight and size $|S|$).

Remark. Size and weight of any scattered completion of a discrete metric space $S$ must coincide with $|S|$ since size and density of a scattered metric space are always identical.

Similarly, from the proof of Theorem 2 we can derive an enumeration theorem about compactifications of discrete spaces. While $|X \setminus X'| < |X|$ is possible for compact and scattered Hausdorff spaces $X$ (consider for example the one point compactification of space 65 in [10]), in the proof of Theorem 2 it is evident that $|X \setminus X'| = |X| = \kappa$ whenever $X \in \{H_S,G_S\}$ for $S \in \mathcal{G}$. This is clearly enough to settle the following enumeration result.

Corollary 4. Every uncountable discrete space $S$ has precisely $2^{|S|}$ scattered and linearly ordered compactifications of size (and weight) $|S|$ up to homeomorphism.

Remark. As already pointed out, the statement in Corollary 4 would be unprovable for a countably infinite discrete space $S$. However, in view of [9] it is clear that any countably infinite discrete space $S$ has precisely $\aleph_1$ countable compactifications up to homeomorphism, and they all are scattered and linearly ordered spaces. Furthermore, it is worth mentioning that any countably infinite discrete space $S$ has $2^{\aleph_0}$ mutually non-homeomorphic uncountable compactifications which all are also linearly ordered and metrizable spaces (see [8] Theorem 7).

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Theorem 1 and its proof is contained in the author’s paper [7].

Theorem 2 and its proof is contained in the author’s paper [6].