Derivations on Triangular Banach Algebras of Order Three

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Abstract. In this paper, we define some new notions of triangular Banach algebras and we investigate the derivations on these algebras.

1. Introduction

Let \( A, B \) and \( C \) be algebras. Consider the triangular matrix of order three

\[
\mathcal{T} = \begin{bmatrix}
A & M & P \\
B & N & C
\end{bmatrix},
\]

where \( M \) is left \( A \)-module, right \( B \)-module, \( N \) is left \( B \)-module, right \( C \)-module, and \( P \) is left \( A \)-module, and right \( C \)-module. The matrix \( \mathcal{T} \) is become an algebra by usual adding and product of \( 3 \times 3 \) matrix, via

\[
\begin{bmatrix}
a_1 & m_1 & p_1 \\
b_1 & n_1 & c_1
\end{bmatrix} + \begin{bmatrix}
a_2 & m_2 & p_2 \\
b_2 & n_2 & c_2
\end{bmatrix} = \begin{bmatrix}
a_1 + a_2 & m_1 + m_2 & p_1 + p_2 \\
b_1 + b_2 & n_1 + n_2 & c_1 + c_2
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
a_1 & m_1 & p_1 \\
b_1 & n_1 & c_1
\end{bmatrix} \begin{bmatrix}
a_2 & m_2 & p_2 \\
b_2 & n_2 & c_2
\end{bmatrix} = \begin{bmatrix}
a_1a_2 & a_1m_2 + m_1b_2 & a_1p_2 + \mu(m_1 \otimes n_2) + p_1c_2 \\
b_1b_2 & b_1n_2 + n_1c_2 & c_1c_2
\end{bmatrix},
\]

where \( \mu : M \otimes N \to P \) is a left \( A \)-module and right \( C \)-module homomorphism. Now, suppose \( A, B \) and \( C \) are Banach algebras, and \( M \) is left Banach \( A \)-module, right Banach \( B \)-module, \( N \) is

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left Banach $B$-module, right Banach $C$-module, and $P$ is left Banach $A$-module, and right Banach $C$-module. Then the triangular algebra $T$ is a Banach algebra by the following norm

$$
\|[\begin{array}{ccc}
a & m & p \\
b & n & c
\end{array}]\| = \|a\|_A + \|m\|_M + \|p\|_P + \|b\|_B + \|n\|_N + \|c\|_C.
$$

We identify this algebra by $T = A \oplus_1 M \oplus_1 P \oplus_1 B \oplus_1 N \oplus_1 C$.

Recently, some results regarding homomorphisms and generalized homomorphisms on order three ring matrices obtained by Xing in [1]. This paper motivated us that we study the derivations on these rings. Derivations of $2 \times 2$ triangular matrix algebras studied by Forrest and Marcoux in [2]. In this paper, by using the ideas of Forrest and Marcoux, we study the derivations on triangular matrix algebras.

2. Main Results

We start our work with the following easy but essential Proposition which play important role in characterizations of derivations on triangular matrix algebras of order three.

**Proposition 2.1.** Let $A$, $B$ and $C$ be unital Banach algebras, and let $D : T \rightarrow T$ be a derivation. Then there exist derivations $D_A : A \rightarrow A$, $D_B : B \rightarrow B$, $D_C : C \rightarrow C$, the linear mappings $\tau_M : M \rightarrow M$, $\tau_P : P \rightarrow P$, $\tau_N : N \rightarrow N$, and elements $m_D \in M$, $p_D \in P$ and $n_D \in N$ such that the following statements hold:

1. $D\left(\begin{array}{ccc}
\epsilon_A & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) = \left[\begin{array}{cc}
0 & m_D \\
0 & 0 \\
0 & 0
\end{array}\right]$. 

2. $D\left(\begin{array}{ccc}
a & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) = \left[\begin{array}{ccc}
D_A(a) & am_D & ap_D \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]$. 

3. $D\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & \epsilon_B & 0 \\
0 & 0 & 0
\end{array}\right) = \left[\begin{array}{cc}
0 & -m_D \\
0 & 0 \\
0 & 0
\end{array}\right]$. 

4. $D\left(\begin{array}{ccc}
0 & 0 & 0 \\
b & 0 & 0 \\
0 & 0 & 0
\end{array}\right) = \left[\begin{array}{cc}
0 & -m_Db \\
0 & 0 \\
D_B(b) & bn_D \\
0 & 0
\end{array}\right]$. 

5. $D\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \epsilon_C & 0
\end{array}\right) = \left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & -n_D \\
0 & 0
\end{array}\right]$. 

6. $D\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & c
\end{array}\right) = \left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & -p_Dc \\
0 & 0
\end{array}\right]$. 

$D_C(c)$. 

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$D_C(c)$.
(7) $D\left( \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & \, \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & \tau_M(m) & \mu(m \otimes n_D) \\ & 0 & 0 \\ & & 0 \end{bmatrix}$.

(8) $D\left( \begin{bmatrix} 0 & 0 & p \\ 0 & 0 & \, \\ 0 & \, & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & \tau_P(p) \\ & 0 & 0 \\ & & 0 \end{bmatrix}$.

(9) $D\left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & \, & n \\ 0 & \, & \, \end{bmatrix} \right) = \begin{bmatrix} 0 & \mu(-m_D \otimes n) \\ & 0 & \tau_N(n) \\ & & 0 \end{bmatrix}$.

Proof. We prove just cases (1), (2) and (7), and other cases have similar proof. Let

$D\left( \begin{bmatrix} e_A & 0 & 0 \\ 0 & 0 & \, \\ 0 & \, & 0 \end{bmatrix} \right) = \begin{bmatrix} \alpha & m' & p' \\ \beta & n' & \gamma \end{bmatrix}$. Since $D$ is a derivation, then we have

$D\left( \begin{bmatrix} e_A & 0 & 0 \\ 0 & 0 & \, \\ 0 & \, & 0 \end{bmatrix} \right) = \begin{bmatrix} \alpha & m' & p' \\ \beta & n' & \gamma \end{bmatrix} + \begin{bmatrix} \alpha & m' & p' \\ \beta & n' & \gamma \end{bmatrix} = \begin{bmatrix} 0 & m & p \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p_D & p \end{bmatrix}$.

For (2), existence of $D_A$ by easy calculation is clear, assume that $D\left( \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & \, \\ 0 & \, & 0 \end{bmatrix} \right) = \begin{bmatrix} D_A(a) & m' & p' \\ \beta & n' & \gamma \end{bmatrix}$. Then by (1) we have

$D\left( \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & \, \\ 0 & \, & 0 \end{bmatrix} \right) = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} D_A(a) & m' & p' \\ \beta & n' & \gamma \end{bmatrix} \begin{bmatrix} e_A & 0 & 0 \\ 0 & p_D & p \end{bmatrix}$

$= \begin{bmatrix} D_A(a) & a m_D & a p_D \\ 0 & 0 & 0 \end{bmatrix}$. 


(7) Suppose that \( D \left( \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right]. \) Then (1) and (3) imply

\[
D \left( \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} e_A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \tau_M(m) & p' \\ \beta & n' & \gamma \end{bmatrix} + \begin{bmatrix} 0 & m_D & p_D \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} \alpha & \tau_M(m) & p' \\ 0 & 0 & 0 \end{bmatrix},
\]

(2.1)

and

\[
D \left( \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -m_D \\ 0 & n_D \end{bmatrix} + \begin{bmatrix} \alpha & \tau_M(m) & p' \\ \beta & n' & \gamma \end{bmatrix} \begin{bmatrix} e_B & 0 \\ 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & \tau_M(m) & \mu(m \otimes n_D) \\ 0 & 0 & 0 \end{bmatrix},
\]

(2.2)

thus, by (2.1) and (2.2), we desert the result. □

By collecting of obtained results in Proposition 2.1 we have the following.

**Corollary 2.2.** Let \( D : T \rightarrow T \) be a derivation, then there exist derivations \( D_A : A \rightarrow A, \) \( D_B : B \rightarrow B, D_C : C \rightarrow C, \) the linear mappings \( \tau_M : M \rightarrow M, \tau_P : P \rightarrow P, \tau_N : N \rightarrow N, \) and elements \( m_D \in M, p_D \in P, \) and \( n_D \in N \) such that

\[
D \left( \begin{bmatrix} a & m & p \\ b & n & c \end{bmatrix} \right) = \begin{bmatrix} D_A(a) & am_D - m_Db + \tau_M(m) & ap_D - p_Dc + \mu(-m_D \otimes n) + \mu(m \otimes n_D) + \tau_P(p) \\ D_B(b) & bn_D - n_Dc + \tau_N(n) \end{bmatrix},
\]

for any \( \begin{bmatrix} a & m & p \\ b & n & c \end{bmatrix} \in T. \)

We introduced the mappings \( \tau_M, \tau_P \) and \( \tau_N \) on \( M, P \) and \( N, \) respectively. Now, we have the following lemma.

**Lemma 2.3.** Let \( D : T \rightarrow T \) be a derivation, then the following statements hold

1. \( \tau_M(am) = D_A(a)m + a\tau_M(m). \)
2. \( \tau_M(mb) = D_B(b)m + \tau_M(m)b. \)
3. \( \tau_P(ap) = D_A(a)p + a\tau_P(p). \)
4. \( \tau_P(pc) = D_C(c)p + \tau_P(p)c. \)
(5) $\tau_N(bn) = D_B(b)n + br_N(n)$.
(6) $\tau_N(nc) = nD_C(c) + r_N(n)c$.

Proof. We only prove (1), other cases are similar. By Proposition 2.1 and Corollary 2.2, we have

$$D\left(\begin{bmatrix} 0 & am & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = D\left(\begin{bmatrix} a & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \tau_M(m) \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} D_A(a) & am_D & ap_D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_A(a)m + a\tau_M(m) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tau_M(am) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

In the following theorem, we do not need to assume that $A$, $B$ and $C$ are unital, and our proof is algebraic.

Theorem 2.4. Let $D_A : A \to A$, $D_B : B \to B$, and $D_C : C \to C$, be continuous derivations and $\tau_M : M \to M$, $\tau_P : P \to P$, and $\tau_N : N \to N$ be continuous linear mappings. If the linear mappings $\tau_M, \tau_P$, and $\tau_N$ satisfy in cases (1)-(6) of the Lemma 2.3 and

$$\tau_P(\mu(m \otimes n)) = \mu(\tau_M(m) \otimes n) + \mu(m \otimes \tau_N(n)),$$

then $D : \mathcal{T} \to \mathcal{T}$ defined by

$$D\left(\begin{bmatrix} a & m & p \\ b & n & c \end{bmatrix}\right) = \begin{bmatrix} D_A(a) & \tau_M(m) & \tau_P(p) \\ D_B(b) & \tau_N(n) & D_C(c) \end{bmatrix},$$

for any $\begin{bmatrix} a & m & p \\ b & n & c \end{bmatrix} \in \mathcal{T}$, is a continuous derivation.
Proof. Continuity of $D$ by its definition is clear. 

$$D = egin{bmatrix} a_1 & m_1 & p_1 \\ b_1 & n_1 & c_1 \end{bmatrix} \begin{bmatrix} a_2 & m_2 & p_2 \\ b_2 & n_2 & c_2 \end{bmatrix}$$

$$= D \begin{bmatrix} a_1 a_2 & a_1 m_2 + m_1 b_2 & a_1 p_2 + \mu(m_1 \otimes n_2) + p_1 c_2 \\ b_1 b_2 & b_1 n_2 + n_1 c_2 & c_1 c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \tau_A(a) & \tau_M(a_1 m_2 + m_1 b_2) & \tau_P(a_1 p_2 + \mu(m_1 \otimes n_2) + p_1 c_2) \\ \tau_B(b_1 b_2) & \tau_N(b_1 n_2 + n_1 c_2) & \tau_C(c_1 c_2) \end{bmatrix}.$$

Conversely,

$$= \begin{bmatrix} a_1 & m_1 & p_1 \\ b_1 & n_1 & c_1 \end{bmatrix} \begin{bmatrix} D_A(a_2) & \tau_M(m_2) & \tau_P(p_2) \\ D_B(b_2) & \tau_N(n_2) & D_C(c_2) \end{bmatrix}$$

$$+ \begin{bmatrix} D_A(a_1) & \tau_M(m_1) & \tau_P(p_1) \\ D_B(b_1) & \tau_N(n_1) & D_C(c_2) \end{bmatrix} \begin{bmatrix} a_2 & m_2 & p_2 \\ b_2 & n_2 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 D_A(a_2) & a_1 \tau_M(m_2) + m_1 D_B(b_2) & a_1 \tau_P(p_2) + \mu(m_1 \otimes \tau_N(n_2)) + p_1 D_C(c_2) \\ b_1 D_B(b_2) & b_1 \tau_N(n_2) + n_1 D_C(c_2) & c_1 D_C(c_2) \end{bmatrix}$$

$$+ \begin{bmatrix} D_A(a_1) a_2 & D_A(a_1) m_2 + \tau_M(m_1) b_2 & D_A(a_1) p_2 + \mu(\tau_M(m_1) \otimes n_2) + \tau_P(p_1) c_2 \\ D_B(b_1) b_2 & D_B(b_1) n_2 + \tau_N(n_1) c_2 & D_C(c_1) c_2 \end{bmatrix}.$$

Therefore $D$ is a derivation. \qed 

Note that if $\mu(M \otimes N) = 0$, then the mapping $D : T \longrightarrow T$ defined in the above theorem is a derivation.

We denote the space of all continuous left $A$-module morphisms and right $B$-module morphisms on $M$ by $\text{Hom}_{A,B}(M)$, if $A = B$, we write $\text{Hom}_{A}(M)$. Similarly we define $\text{Hom}_{A,C}(P)$ and $\text{Hom}_{B,C}(N)$.

Consider the continuous mappings $\tau_M : M \longrightarrow M, \tau_P : P \longrightarrow P$ and $\tau_N : N \longrightarrow N$. We say these maps are generalized Rosenblum operators, if there exist derivations $D_A : A \longrightarrow A, D_B : B \longrightarrow B$ and $D_C : C \longrightarrow C$ such that

$$\tau_M(amb) = D_A(a) mb + a \tau_M(m) b + am D_B(b),$$

$$\tau_P(apc) = D_A(a) pc + a \tau_P(p) c + ap D_C(c),$$
and

$$\tau_{M}(bnc) = D_{B}(b)nc + b\tau_{M}(n)c + bnD_{C}(c),$$

for every $$a \in A, b \in B, c \in C, m \in M, p \in P$$ and $$c \in C$$. We denote the generalized Rosenblum operator on $$M$$ specified by $$x \in A$$ and $$y \in B$$, by $$\tau^{x,y}_{M}$$, and similarly, we denote the generalized Rosenblum operators on $$P$$ and $$N$$ specified by $$x \in A$$, $$y \in B$$, and $$z \in C$$ by $$\tau^{x,z}_{P}$$ and $$\tau^{y,z}_{N}$$, respectively.

By $$Z(A), Z(B)$$ and $$Z(C)$$, we mean the center of the Banach algebras $$A$$, $$B$$ and $$C$$, respectively. Let $$x \in Z(A), y \in Z(B)$$ and $$z \in Z(C)$$, the operators $$\tau^{x,y}_{M}, \tau^{x,z}_{P}$$ and $$\tau^{y,z}_{N}$$ called central Rosenblum operators on $$M$$, $$P$$ and $$N$$, respectively. These operators defined as follows

$$\tau^{x,y}_{M}(m) = my - xm, \tau^{x,z}_{P}(p) = pz - xp, \tau^{y,z}_{N}(n) = nz - yn.$$  

The space of all central Rosenblum operators on $$M$$, $$P$$ and $$N$$ denoted by $$ZR_{A,B}(M)$$, $$ZR_{A,C}(P)$$ and $$ZR_{B,C}(N)$$, respectively.

**Lemma 2.5.** Let $$T$$ be a triangular Banach algebra of order three defined as above. Then

(i) $$ZR_{A,B}(M) \subseteq \text{Hom}_{A,B}(M).$$

(ii) $$ZR_{A,C}(P) \subseteq \text{Hom}_{A,C}(P).$$

(iii) $$ZR_{B,C}(N) \subseteq \text{Hom}_{B,C}(N).$$

**Proof.** It follows by a same reasoning as proof of Lemma 2.6 of [1].

**Lemma 2.6.** Let $$\varphi \in \text{Hom}_{A,B}(M), \theta \in \text{Hom}_{A,C}(P)$$ and $$\psi \in \text{Hom}_{B,C}(N)$$ such that

$$\theta(\mu(m \otimes n)) = \mu(\varphi(m) \otimes n) + \mu(m \otimes \psi(n)). \quad (2.3)$$

Then $$D_{\varphi,\theta,\psi} : T \rightarrow T$$ defined by

$$D_{\varphi,\theta,\psi}(\begin{bmatrix} a & m & p \\ b & n \\ c \end{bmatrix}) = \begin{bmatrix} 0 & \varphi(m) & \theta(p) \\ 0 & \psi(n) \\ 0 \end{bmatrix}$$

is a continuous derivation. Moreover, $$D_{\varphi,\theta,\psi}$$ is an inner derivation if and only if there exist $$x \in A, y \in B$$ and $$z \in C$$ such that $$\varphi = \tau^{x,y}_{M} \in ZR_{A,B}(M), \theta = \tau^{x,z}_{P} \in ZR_{A,C}(P)$$ and $$\psi = \tau^{y,z}_{N} \in ZR_{B,C}(N).$$
Proof. By Theorem 2.4, $D_{\varphi, \theta, \psi}$ is a continuous derivation. Now, suppose that $D_{\varphi, \theta, \psi}$ is inner. Therefore there exists 

$$
\begin{bmatrix}
  x & \alpha & \beta \\
  y & \gamma & \\
  z & \\
\end{bmatrix} \in \mathcal{T}
$$

such that

$$
D_{\varphi, \theta, \psi}
\begin{bmatrix}
  a & m & p \\
  b & n & c \\
\end{bmatrix}
\begin{bmatrix}
  x & \alpha & \beta \\
  y & \gamma & \\
  z & \\
\end{bmatrix}
- 
\begin{bmatrix}
  x & \alpha & \beta \\
  y & \gamma & \\
  z & \\
\end{bmatrix}
\begin{bmatrix}
  a & m & p \\
  b & n & c \\
\end{bmatrix}
= 
\begin{bmatrix}
  ax & a\alpha + my & a\beta + \mu(m \otimes \gamma) + pz \\
  by & b\gamma + nz & cz \\
  xa & xm + \alpha b & xp + \mu(\alpha \otimes n) + \beta c \\
  yb & yn + \gamma c & \\
  zb & \\
\end{bmatrix}
= 
\begin{bmatrix}
  ax - xa & a\alpha - \alpha b + my - xm & a\beta - \beta c + pz - xp \\
  by -yb & b\gamma - \gamma c + nz - yn & cz - zc \\
\end{bmatrix}
$$

On the other hand $D_{\varphi, \theta, \psi}
\begin{bmatrix}
  a & m & p \\
  b & n & c \\
\end{bmatrix}
= 
\begin{bmatrix}
  0 & \varphi(m) & \theta(p) \\
  0 & \psi(n) & 0 \\
\end{bmatrix}$. Thus, $ax - xa = 0$, $by - yb = 0$ and $cz - zc = 0$. It follows that $x \in \mathcal{Z}(\mathcal{A})$, $y \in \mathcal{B}$ and $z \in \mathcal{Z}(\mathcal{C})$. Moreover, we have

1. $\varphi(m) = a\alpha - \alpha b + my - zm$.
2. $\theta(p) = a\beta - \beta c + pz - xp + \mu(m \otimes \gamma) - \mu(\alpha \otimes n)$.
3. $\psi(n) = b\gamma - \gamma c + nz - yn$.

Since $\varphi \in \text{Hom}_{\mathcal{A}, \mathcal{B}}(\mathcal{M})$, $\theta \in \text{Hom}_{\mathcal{A}, \mathcal{C}}(\mathcal{P})$ and $\psi \in \text{Hom}_{\mathcal{B}, \mathcal{C}}(\mathcal{N})$, therefore we conclude that in

1. $a\alpha - \alpha b = 0$, and $\varphi(m) = my - zm = \tau_{\mathcal{M}}^{x,y}(m)$; in
2. $\theta(p) = a\beta - \beta c + \mu(m \otimes \gamma) - \mu(\alpha \otimes n) = 0$ and $\theta(p) = pz - xp = \tau_{\mathcal{P}}^{x,z}(p)$, and in
3. $b\gamma - \gamma c = 0$ and $\psi(n) = nz - yn = \tau_{\mathcal{N}}^{y,z}(n)$. 


Conversely, assume \( \varphi = \tau^x_y \in ZR_{A,B}(\mathcal{M}) \), \( \theta = \tau^x_z \in ZR_{A,C}(\mathcal{P}) \) and \( \psi = \tau^y_z \in ZR_{B,C}(\mathcal{N}) \).

Define the inner derivation \( D : \mathcal{T} \rightarrow \mathcal{T} \) specified by
\[
\begin{bmatrix}
  a & m & p \\
  b & n & c
\end{bmatrix}
\]

and since \( \psi, \theta, \psi \) is the equivalence class of \( \Phi : \text{Hom} \rightarrow \mathcal{T} \) by \( \Phi(\varphi, \theta, \psi) \).

This means that \( D_{\varphi, \theta, \psi} \) is inner.

\[ \square \]

Note that in the last part of above proof, we do not use condition \([23]\). Now, we are ready to prove the main theorem of this paper.

**Theorem 2.7.** Let \( \mathcal{T} \) be a triangular Banach algebra of order three. If \( \mathcal{H}^1(A) = 0 \), \( \mathcal{H}^1(B) = 0 \) and \( \mathcal{H}^1(C) = 0 \), then

\[
\mathcal{H}^1(\mathcal{T}) \cong \frac{\text{Hom}_{A,B}(\mathcal{M}) \oplus \text{Hom}_{A,C}(\mathcal{P}) \oplus \text{Hom}_{B,C}(\mathcal{N})}{\mathcal{Z}R_{A,B}(\mathcal{M}) \oplus \mathcal{Z}R_{A,C}(\mathcal{P}) \oplus \mathcal{Z}R_{B,C}(\mathcal{N})}.
\]

**Proof.** Define \( \Phi : \text{Hom}_{A,B}(\mathcal{M}) \oplus \text{Hom}_{A,C}(\mathcal{P}) \oplus \text{Hom}_{B,C}(\mathcal{N}) \rightarrow \mathcal{H}^1(\mathcal{T}) \) by \( \Phi(\varphi, \theta, \psi) = D_{\varphi, \theta, \psi} \), where \( D_{\varphi, \theta, \psi} \) is the equivalence class of \( D_{\varphi, \theta, \psi} \) in \( \mathcal{H}^1(\mathcal{T}) \). It easy to show that \( \Phi \) is linear. We have to show that \( \Phi \) is onto. To this end, let \( D : \mathcal{T} \rightarrow \mathcal{T} \) be a continuous derivation. Then the Corollary \([22]\) implies that there exist derivations \( D_A : \mathcal{A} \rightarrow \mathcal{A}, D_B : \mathcal{B} \rightarrow \mathcal{B}, D_C : \mathcal{C} \rightarrow \mathcal{C}, \) the linear mappings \( \tau_M : \mathcal{M} \rightarrow \mathcal{M}, \tau_P : \mathcal{P} \rightarrow \mathcal{P}, \tau_N : \mathcal{N} \rightarrow \mathcal{N}, \) and elements \( m_D \in \mathcal{M}, p_D \in \mathcal{P} \) and \( n_D \in \mathcal{N} \) such that

\[
D\left(\begin{bmatrix}
  a & m & p \\
  b & n & c
\end{bmatrix}\right) = \begin{bmatrix}
  D_A(a) & am_D - m_Db + \tau_M(m) & ap_D - p_Dc + \mu(-m_D \otimes n) \\
  b & n_D & bn_D - n_Dc + \tau_N(n)
\end{bmatrix},
\]

and since \( \mathcal{H}^1(A), \mathcal{H}^1(B) \) and \( \mathcal{H}^1(C) \) are zero, so there are \( x \in \mathcal{A}, y \in \mathcal{B} \) and \( z \in \mathcal{C} \) such that

\( D_A(a) = ax - xa = D_x(a), D_B(b) = by - yb = D_y(b) \) and \( D_C(c) = cz - zc = D_z(c). \) Define
$D_0 : T \rightarrow T$ as follows

$$D_0\left(\begin{array}{ccc}
  a & m & p \\
  b & n & c
\end{array}\right) = \begin{array}{ccc}
  D_x(a) & am_D - m_Db + \tau_M^{x,y}(m) & ap_D - p_DC + \mu(-m_D \otimes n) \\
  & & + \mu(m \otimes n_D) + \tau_P^{x,z}(p) \\
  & D_Y(b) & bm_D - n_DC + \tau_N^{y,z}(n) \\
  & D_z(c) & D_z(c)
\end{array}. $$

For every \( \begin{array}{ccc}
  a & m & p \\
  b & n & c
\end{array} \in T \), we have

$$\begin{array}{ccc}
  a & m & p \\
  b & n & c
\end{array} = \begin{array}{ccc}
  x & m_D & p_D \\
  y & n_D & z
\end{array} - \begin{array}{ccc}
  x & m_D & p_D \\
  y & n_D & z
\end{array} \cdot \begin{array}{ccc}
  a & m & p \\
  b & n & c
\end{array} = \begin{array}{ccc}
  ax - xa & am_D - m_Db + my - xm & ap_D - p_DC + \mu(m \otimes n_D) + \mu(-m_D \otimes n) + pz - xp \\
  & & + \mu(m \otimes n_D) + \mu(-m_D \otimes n) + pz - xp
\end{array}. $$

It follows that $D_0$ is an inner derivation specified by $\begin{array}{ccc}
  x & m_D & p_D \\
  y & n_D & z
\end{array}$. Define $D_1 = D - D_0$.

Then $D_1$ is a derivation and we have

$$D_1\left(\begin{array}{ccc}
  a & m & p \\
  b & n & c
\end{array}\right) = \begin{array}{ccc}
  0 & \tau_M(m) - \tau_M^{x,y}(m) & \tau_P(p) - \tau_P^{x,z}(p) \\
  & 0 & \tau_N(n) - \tau_N^{y,z}(n) \\
  & & 0
\end{array}. $$

where $\tau_M - \tau_M^{x,y} = \tau_M', \tau_P - \tau_P^{x,z} = \tau_P'$ and $\tau_N - \tau_N^{y,z} = \tau_N'$. Clearly, by Lemma 2.3 and properties of $\tau_M$, $\tau_P$ and $\tau_N$, we have $\tau_M' \in \text{Hom}_{A,B}(M)$, $\tau_P' \in \text{Hom}_{A,C}(P)$ and $\tau_N' \in \text{Hom}_{B,C}(N)$. Hence, $D_D = \Phi(\tau_M', \tau_P', \tau_N')$. It means that $\Phi$ is onto. Therefore

$$H^1(T) \cong \frac{\text{Hom}_{A,B}(M) \oplus \text{Hom}_{A,C}(P) \oplus \text{Hom}_{B,C}(N)}{\ker \Phi}. $$

Suppose $(\varphi, \theta, \psi) \in \ker \Phi$. Then by Lemma 2.3 $(\varphi, \theta, \psi) \in \ker \Phi$ if and only if $D_{\varphi, \theta, \psi}$ is inner. This shows that $\ker \Phi = Z \mathcal{R}_{A,B}(M) \oplus_1 Z \mathcal{R}_{A,C}(P) \oplus_1 Z \mathcal{R}_{B,C}(N)$. □

Remark 2.8. Note that the above obtained results are true when we suppose that $B$ and $C$ have bounded approximate identity (i.e. they are non-unital). Therefore by this notation, we can write all results for unital Banach algebra $A$ and the non-unital Banach algebras $B$ and $C$ with bounded approximate identity.
Example 2.9. Suppose that \( A = B = C = \mathbb{C} \) (\( \mathbb{C} \) is the space of complex number), and \( \mathcal{M}, \mathcal{P} \) and \( \mathcal{N} \) are arbitrary Banach spaces. Then, it is known that every derivation from \( \mathbb{C} \) on itself is inner. Thus, \( H^1(A), H^1(B) \) and \( H^1(C) \) are zero. By \( L(\mathcal{M}) \), we mean the space of all linear and bounded operator from \( \mathcal{M} \) into \( \mathcal{M} \) (\( L(\mathcal{P}) \) and \( L(\mathcal{N}) \) have similar definition). Then by an easy argument, we can conclude that \( \text{Hom}_{\mathbb{C}}(\mathcal{M}) = L(\mathcal{M}) \) and \( \mathcal{Z} \mathcal{R}_{\mathbb{C}}(\mathcal{M}) = \{ \text{id}_{\mathcal{M}} : \lambda \in \mathbb{C} \} \), where \( \text{id}_{\mathcal{M}} \) is the identity map on \( \mathcal{M} \). Similarly, one can obtain same results for \( \mathcal{P} \) and \( \mathcal{N} \). Then by Theorem 2.7 we have

\[
\mathcal{H}^1(\mathcal{T}) \cong \frac{L(\mathcal{M}) \oplus_1 L(\mathcal{P}) \oplus_1 L(\mathcal{N})}{\{ \text{id}_{\mathcal{M}} : \lambda \in \mathbb{C} \} \oplus_1 \{ \text{id}_{\mathcal{P}} : \lambda \in \mathbb{C} \} \oplus_1 \{ \text{id}_{\mathcal{N}} : \lambda \in \mathbb{C} \} }.
\]

Example 2.10. Let \( G, H \) and \( K \) be locally compact groups, and \( A = M(G), B = M(H) \) and \( C = M(K) \) be measure algebras on \( G, H \) and \( K \), respectively. By Theorem 5.6.34 (iii) of \[3\],

\[
\mathcal{H}^1(M(G)) = \mathcal{H}^1(L^1(G), M(G))
\]

and by Corollary 1.4 of \[4\], we have

\[
\mathcal{H}^1(L^1(G), M(G)) = 0.
\]

By the same reasoning one can obtain that \( \mathcal{H}^1(M(H)) = 0 \) and \( \mathcal{H}^1(M(K)) = 0 \). Let \( G_1 = G \cap H \neq \emptyset, G_2 = G \cap K \neq \emptyset \) and \( G_3 = H \cap K \neq \emptyset \). Now, consider the triangular Banach algebra

\[
\mathcal{T} = \begin{bmatrix}
M(G) & L^1(G_1) & L^1(G_2) \\
M(H) & L^1(G_3) \\
M(K)
\end{bmatrix}.
\]

Then by Theorem 2.7, we have

\[
\mathcal{H}^1(\mathcal{T}) \cong \frac{\text{Hom}_{M(G),M(H)}(L^1(G_1)) \oplus_1 \text{Hom}_{M(G),M(K)}(L^1(G_3)) \oplus_1 \text{Hom}_{M(H),M(K)}(L^1(G_2))}{\mathcal{Z} \mathcal{R}_{M(G),M(H)}(L^1(G_1)) \oplus_1 \mathcal{Z} \mathcal{R}_{M(G),M(K)}(L^1(G_3)) \oplus_1 \mathcal{Z} \mathcal{R}_{M(H),M(K)}(L^1(G_2))}.
\]

Example 2.11. Let \( G \) be a locally compact group and \( M(G) \) and \( L^1(G) \) be group algebras on \( G \). Consider

\[
\mathcal{T} = \begin{bmatrix}
M(G) & L^1(G) & L^1(G) \\
M(G) & L^1(G) \\
M(G)
\end{bmatrix}.
\]

Then

\[
\mathcal{H}^1(\mathcal{T}) \cong \frac{\text{Hom}_{M(G)}(L^1(G)) \oplus_1 \text{Hom}_{M(G)}(L^1(G)) \oplus_1 \text{Hom}_{M(G)}(L^1(G))}{\mathcal{Z} \mathcal{R}_{M(G)}(L^1(G)) \oplus_1 \mathcal{Z} \mathcal{R}_{M(G)}(L^1(G)) \oplus_1 \mathcal{Z} \mathcal{R}_{M(G)}(L^1(G))}.
\]

Example 2.12. Let \( G \) be abelian locally compact group, and

\[
\mathcal{T} = \begin{bmatrix}
L^1(G) & L^1(G) & L^1(G) \\
L^1(G) & L^1(G) \\
L^1(G)
\end{bmatrix}.
\]
Johnson’s theorem implies that $\ell^1(G)$ and $L^1(G)$ are commutative amenable Banach algebras, so
$$\mathcal{H}^1(\ell^1(G)) = 0 = \mathcal{H}^1(L^1(G)).$$
Wendel’s theorem ([5, Theorem 1]) implies that
$$\text{Hom}_{\ell^1(G), L^1(G)}(L^1(G)) = M(G) \text{ and } \text{Hom}_{L^1(G), L^1(G)}(L^1(G)) = M(G).$$
As well as,
$$Z\mathcal{R}_{\ell^1(G), L^1(G)}(L^1(G)) = \ell^1(G) \oplus L^1(G) \text{ and } Z\mathcal{R}_{L^1(G), L^1(G)}(L^1(G)) = L^1(G) \oplus L^1(G).$$
Therefore by Theorem 2.7 and Remark 2.8 we have
$$\mathcal{H}^1(T) \cong \frac{M(G) \oplus_1 M(G) \oplus_1 M(G)}{(\ell^1(G) \oplus L^1(G)) \oplus_1 (\ell^1(G) \oplus L^1(G)) \oplus_1 (L^1(G) \oplus L^1(G))}.$$