Ground State Solutions of $p$-Laplacian Singular Kirchhoff Problem Involving a Riemann-Liouville Fractional Derivative

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Abstract. The purpose of this paper is to study the existence and multiplicity of solutions to the following Kirchhoff equation with singular nonlinearity and Riemann-Liouville Fractional Derivative:

\[
\begin{aligned}
&\left( a + b \int_0^T |D_\alpha^p u(t)|^p dt \right)^{p-1} D_\alpha^p \left( \Phi_p(a D_\alpha^p u(t)) \right) \\
&= \lambda_1(t) + f(t, u(t)), \quad t \in (0, T); \\
&u(0) = u(T) = 0,
\end{aligned}
\]

where $a \geq 1$, $b$, $\lambda > 0$, $p > 1$ are constants, $\frac{1}{p} < \alpha \leq 1$, $0 < \gamma < 1$, $g \in C([0, 1])$ and $f \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$.

Under appropriate assumptions on the function $f$, we employ variational methods to show the existence and multiplicity of positive solutions of the above problem with respect to the parameter $\lambda$.

1. Introduction

The theory of fractional calculus may be used to the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes has had a growing development in fields as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology (see e.g. [4], [18], [19]). In fact, the subject of fractional differential equations has been gaining more importance and attention in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one may refer to monographs [24, 28] and papers [1, 2, 25–27], and references cited therein. In particular, in the qualitative theory of fractional differential equations, the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions has attracted great attention.

In the sequel of the above-mentioned works with the new approach to the theory of fractional differential equations, here, in this paper, we investigate the existence of multiple solutions of the Kirchhoff fractional differential equation.
problem involving Riemann-Liouville fractional derivative and singular nonlinearity:

\[
(P_\lambda): \begin{cases}
\left(a + b \int_0^T |\mu D^\alpha_t (u(t))|^p dt\right)^{\frac{1}{p}} = \frac{\lambda u(t)}{l} + f(t, u(t)), \ t \in (0, T); \\
u(0) = u(T) = 0,
\end{cases}
\]

where \( a \geq 1, \ b \lambda > 0, \ p > 1 \) are constants. \( \frac{1}{p} < \alpha \leq 1, \ 0 < \gamma < 1 < p < r \) and \( g \in C([0, T]) \), and \( \Phi_p : \mathbb{R} \rightarrow \mathbb{R} \) is the \( p \)-laplacian defined by

\[\Phi_p(s) = |s|^{p-2}s \ (s \neq 0), \ \Phi_p(0) = 0.\]

While \( f \in C([0, T] \times \mathbb{R}, \mathbb{R}) \) is positively homogeneous of degree \( r - 1 \) that is \( f(x, tu) = t^{r-1}f(x, u) \) holds for all \( (x, u) \in [0, T] \times \mathbb{R} \). Put \( F(x, s) := \int_0^s f(x, t) \, dt \) and satisfying suitable growth conditions. Precisely, we assume the following:

- (H1) \( F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is homogeneous of degree \( r \) that is

\[\forall x \in [0, T], \ u \in \mathbb{R}, \quad F(x, tu) = t^r F(x, u) \quad (t > 0).\]

- (H2) \( F^\pm(x, u) = \text{max}(\pm F(x, u), 0) \neq 0 \) for all \( u \neq 0 \).

Note that, from (H1), \( f \) leads to the so-called Euler identity

\[uf(x, u) = rF(x, u)\]

and

\[|F(x, u)| \leq K|u|^r \quad \text{for some constant } K > 0. \tag{1.1}\]

Problem \((P_\lambda)\) is related to the stationary version of the Kirchhoff equation presented by Kirchhoff in 1883 [23] given by the equation

\[\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx\right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}\]

which extends the classical d’Alembert’s wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in the above equation have the following meanings: \( L \) is the length of the string, \( h \) is the area of the cross-section, \( E \) is the young modulus of the material, \( \rho \) is the mass density and \( \rho_0 \) is the initial tension. A feature of problem (1.2) is that the equation contains a nonlocal coefficient \( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \), which depends on the average \( \frac{1}{L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \). Nonlocal effect also finds its applications in biological systems. Moreover, problem \((P_\lambda)\) has a solid theoretical significance and a sharp physical background. For instance, this problem describes the surface tension of the height of a thin liquid film on a solid surface in lubrication approximation (see [6, 13]).

A parabolic version of problem (1.2) can be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with \( u \) being its population density. Alternatively, the movement of particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria), which gives rise to nonlocal parabolic equations. We refer to [30] for details.

Recently, the study of the fractional elliptic equations with regular nonlinearities and without a Kirchhoff coefficient has been attracted lot of interest by researchers in nonlinear analysis. The fractional boundary value problem using variational methods has been studied in [3, 5, 7, 16, 17, 20, 21, 31, 32] with references therein. Also, existence and multiplicity results for the Kirchhoff equations with regular nonlinearities are
shown an always increasing interest. We refer the interested readers to [10–12, 29].

Motivated by above results, in the present paper, we show the existence and multiplicity of nontrivial, non-negative solutions of the Kirchhoff fractional problem involving Riemann-Liouville fractional Derivative and singular nonlinearity (P_{\lambda}).

We give below the precise statements of results that we will prove.

**Theorem 1.1.** Assume that the hypothesis (H_1)-(H_3) are satisfied. Then, there exists \( \Lambda_0 \) such that for all \( \lambda \in (0, \Lambda_0) \), problem possesses at least two nontrivial positive solutions.

This paper is organized as follows: In Section 2, some definitions and properties on the fractional calculus are presented. The section 3 is devoted to proof some lemmas in preparation for the proof of our main result. While, existence of two solutions (Theorem 1.1) will be presented in sections 4 and 5.

### 2. Preliminaries

In this section, we give some background theory on the concept of fractional calculus, in particular the Riemann-Liouville operators and results which will used throughout this paper. Let us start by introduce the definition of the fractional integral in the sense of Riemann-Liouville.

**Definition 2.1.** (See [1]) Let \( [a, b] \) be a finite interval on the real axis \( \mathbb{R} = (-\infty, \infty) \) and \( u \) be a real-valued function defined almost everywhere (a.e.) on \( (a, b) \). The Riemann-Liouville left-sided and right-sided fractional integrals of a function \( u \)

\[
\alpha \mathcal{I}^\alpha_t u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) \, ds \quad (t \in (a, b])
\]

and

\[
\alpha \mathcal{I}_{b-}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} u(s) \, ds \quad (t \in [a, b))
\]

respectively. Here \( \Gamma \) is the familiar Gamma function.

Let \( [a, b] \) be a finite or infinite interval of the real axis \( \mathbb{R} = (-\infty, \infty) \). We denote by \( L^p(a, b) \) the set of those Lebesgue complex-valued measurable functions \( u \) on \( [a, b] \) for which \( \|u\|_p < \infty \), where

\[
\|u\|_p = \left( \int_a^b |u(t)|^p \, dt \right)^{1/p} \quad (1 \leq p < \infty)
\]

and

\[
\|u\|_\infty = \text{ess sup}_{a \leq x \leq b} |u(x)|.
\]
If \( u \in L^1(a, b) \), then \( _aI_t^\alpha u \) and \( _bI_t^\alpha u \) are defined a.e. on \((a, b)\).

Let \([a, b] (-\infty \leq a < b \leq \infty)\) and \( m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} \). We denote by \( C^m[a, b] \) a space of functions \( u \) which are \( m \) times continuously differentiable on \([a, b] \) with the norm

\[
\|u\|_{C^m} = \max_{t \in [a, b]} |u^{(k)}(t)| \quad (m \in \mathbb{N}_0).
\]

In particular, for \( m = 0 \), \( C^0[a, b] \equiv C[a, b] \) is the space of continuous functions \( u \) on \([a, b] \) with the norm

\[
\|u\|_C = \max_{t \in [a, b]} |u(t)|.
\]

When \([a, b]\) is a finite interval and \( 0 \leq \gamma < 1 \), we introduce the weighted space \( C_\gamma[a, b] \) of functions \( u \) given on \([a, b]\), such that the function \((t - a)^\gamma u(t) \in C[a, b]\), and

\[
\|u\|_{C_\gamma} = \|(t - a)^\gamma u(t)\|_C, \quad C_0[a, b] = C[a, b].
\]

**Definition 2.2.** (See [1]) The Riemann-Liouville left-sided and right-sided fractional derivatives of a function \( u \)

\[
_aD_t^\alpha u(t) = _aD_t^\alpha u(t) = \left( _aD_t^\alpha a\right) (t) = \left( _aD_t^\alpha u(t)\right)
\]

and

\[
_bD_t^\alpha u(t) = _bD_t^\alpha u(t) = \left( _bD_t^\alpha b\right) (t) = \left( _bD_t^\alpha u(t)\right)
\]

of order \( \alpha \in \mathbb{R}^+ \cup \{0\} \) are defined by

\[
_aD_t^\alpha u(t) = \left( \frac{d}{dt}\right)^n\left( _aI_t^{n-\alpha} u(t)\right) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s) \, ds \quad (n = [\alpha] + 1; \ t > a)
\]

and

\[
_bD_t^\alpha u(t) = \left( -\frac{d}{dt}\right)^n\left( _bI_t^{n-\alpha} u(t)\right) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} u(s) \, ds \quad (n = [\alpha] + 1; \ t < b),
\]

respectively, where \([\alpha]\) means the integral part of \( \alpha \).

**Remark 2.3.** From [24, pp. 2-3], if \( u \) is absolutely continuous on \([a, b]\), then the fractional derivatives \( _aD_t^\alpha u \) and \( _bD_t^\alpha u \) exist almost everywhere on \([a, b]\) and can be represented in the forms

\[
_aD_t^\alpha u(t) = _aI_t^{1-\alpha} u'(t) + \frac{u(a)}{(t-a)^{\alpha-1}}
\]

and

\[
_bD_t^\alpha u(t) = -_bI_t^{1-\alpha} u'(t) + \frac{u(b)}{(b-t)^{\alpha-1}}.
\]

In particular, we have

\[
_aD_t^\alpha u(t) = _aI_t^{1-\alpha} u'(t) \quad \text{and} \quad _bD_t^\alpha u(t) = -_bI_t^{1-\alpha} u'(t) \quad (u(a) = 0 = u(b)).
\]
The left-sided and right-sided Caputo fractional derivatives $\frac{c}{a}D_t^\alpha u(t)$ and $\frac{c}{b}D_t^\alpha u(t)$ of order $\alpha \in \mathbb{R}^+ \cup \{0\}$ with, here, $0 < \alpha < 1$ are defined by

$$\frac{c}{a}D_t^\alpha u(t) = aD_t^\alpha [u(t) - u(a)]$$

and

$$\frac{c}{b}D_t^\alpha u(t) = bD_t^\alpha [u(t) - u(b)],$$

respectively.

We find from (2.8)–(2.12) that, if $u$ is absolutely continuous on $[a, b]$, $u(a) = u(b)$, and $0 < \alpha < 1$, then the Riemann-Liouville fractional integrals and the Caputo fractional derivatives coincide:

$$\frac{c}{a}D_t^\alpha u(t) = a^{-1-\alpha}u'(t) \quad \text{and} \quad \frac{c}{b}D_t^\alpha u(t) = -b^{-1-\alpha}u'(t).$$

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration.

**Remark 2.4.** ([24, Lemma 2.3]) If $\alpha, \beta \in \mathbb{R}^+$, then the equations

$$(\frac{c}{a}D_t^\alpha \frac{c}{b}D_t^\beta u)(t) = (\frac{c}{a}D_t^\alpha \frac{c}{b}D_t^\beta u)(t) \quad \text{and} \quad (\frac{c}{b}D_t^\beta \frac{c}{a}D_t^\alpha u)(t) = (\frac{c}{b}D_t^\beta \frac{c}{a}D_t^\alpha u)(t)$$

are satisfied at almost every point $t \in [a, b]$ for $f(t) \in L^p(a, b) \ (1 \leq p \leq \infty)$. If $\alpha + \beta > 1$, then the relations in (2.14) hold at any point of $[a, b]$.

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration.

**Remark 2.5.** ([24, Lemma 2.4]) If $\alpha \in \mathbb{R}^+$ and $f(t) \in L^p(a, b) \ (1 \leq p \leq \infty)$, then the following equalities

$$(\frac{c}{a}D_t^\alpha \frac{c}{a}D_t^\alpha u)(t) = f(t) \quad \text{and} \quad (\frac{c}{b}D_t^\alpha \frac{c}{b}D_t^\alpha u)(t) = f(t)$$

hold almost everywhere on $[a, b]$.

**Remark 2.6.** ([24, Lemma 2.4]) The fractional integration operators $\frac{c}{a}I_t^\alpha$ and $\frac{c}{b}I_t^\alpha$ with $\alpha \in \mathbb{R}^+$ are bounded in $L^p(a, b) \ (1 \leq p \leq \infty)$:

$$\|\frac{c}{a}I_t^\alpha u\|_p \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|u\|_p \quad \text{and} \quad \|\frac{c}{b}I_t^\alpha u\|_p \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|u\|_p.$$

In the same way, we give another classical result on the boundedness of the left fractional integral from $L^p(a, b)$ to $C_0(a, b)$ which completes Remark 2.6 in the case $0 < \frac{1}{p} < \alpha < 1$.

**Remark 2.7.** ([9, Property 4]) Let $0 < \frac{1}{p} < \alpha < 1$ and $q = \frac{p}{p-1}$. Then, for any $u \in L^p(a, b)$, $\frac{c}{a}I_t^\alpha u$ is Hölder continuous on $(a, b]$ with exponent $\alpha - \frac{1}{p} > 0$, that is, there exists a constant $M \in \mathbb{R}^+$ such that

$$\left|\frac{c}{a}I_t^\alpha u(t_2) - \frac{c}{a}I_t^\alpha u(t_1)\right| \leq M (t_2 - t_1)^{\alpha - 1/p}$$

for any $a < t_1 < t_2 < b$. Moreover, $\lim_{t \to a} \frac{c}{a}I_t^\alpha u(t) = 0$. Consequently, $\frac{c}{a}I_t^\alpha u$ can be continuously extended by 0 at $t = a$. Finally, for any $u \in L^p(a, b)$, $\frac{c}{a}I_t^\alpha u \in C_0(a, b)$, and

$$\|\frac{c}{a}I_t^\alpha u\|_p \leq \frac{(b-a)^{\alpha - 1/p}}{\Gamma(\alpha)((\alpha - 1)q + 1)^{1/p}} \|u\|_p.$$

The following formula which is often called fractional integration by parts will also be required.
Moreover, if \( v \) (see [22, Remark 3.1])

\[
P \geq 1, \quad q \geq 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha
\]

(and \( p \neq 1 \neq q \) in the case \( 1/p + 1/q = 1 + \alpha \)). Then, for all \( u \in L^p(a, b) \) and all \( v \in L^q(a, b) \), we have

\[
\int_a^b s I_0^\alpha u(t) \cdot v(t) \, dt = \int_a^b u(t) \cdot s I_0^\alpha v(t) \, dt,
\]

and

\[
\int_a^b u(t) \, s D_0^\alpha v(t) \, dt = v(t) \mid_{t=0}^{t=b} + \int_a^b v(t) \, s D_0^\alpha u(t) \, dt.
\]

Moreover, if \( v(a) = v(b) = 0 \), then we have

\[
\int_a^b u(t) \, s D_0^\alpha v(t) \, dt = \int_a^b v(t) \, s D_0^\alpha u(t) \, dt.
\]

3. Nehari manifold and fibering map analysis

To show the existence of solutions to the problem \((P_\lambda)\), we will use critical point theory (see, e.g., [22]). We begin by introduce some notations and results which will be used. The set of all functions \( u \in C^\infty([0,1], \mathbb{R}) \) with \( u(0) = u(1) = 0 \) is denoted by \( C_0^\infty([0,1], \mathbb{R}) \). For \( \alpha \in \mathbb{R}^+ \), we define the fractional derivative space \( E_{0}^{\alpha, p} \) as the closure of \( C_{0}^\infty([0,1], \mathbb{R}) \) with the norm

\[
\|u\|_{\alpha, p} = \left( \|u\|_p^p + \| C_0^{\alpha} D_0^\alpha u \|_p^p \right)^{\frac{1}{p}}.
\]

We summarize some properties for the space \( E_{0}^{\alpha, p} \) in the following remark.

Remark 3.1. (see [22, Remark 3.1])

(i) The space \( E_{0}^{\alpha, p} \) is the space of functions \( u \in L^p[0,1] \) having an \( \alpha \)-order Caputo fractional derivative \( C_0^{\alpha} D_0^\alpha u \in L^p[0,1] \) and \( u(0) = u(1) = 0 \).

(ii) For any \( u \in E_{0}^{\alpha, p} (0 < \alpha < 1) \), since \( u(0) = 0 \), we have (see (2.11))

\[
C_0^{\alpha} D_0^\alpha u(t) = D_0^\alpha u(t) \quad (t \in [0,1]),
\]

(iii) The space \( E_{0}^{\alpha, p} \) is a reflexive and separable Banach space.

Lemma 3.2. [22, Proposition 3.2] Let \( 0 < \alpha \leq 1 \) and \( 1 < p < \infty \). Then, for all \( u \in E_{0}^{\alpha, p} \), we have

\[
\|u\|_p \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|C_0^{\alpha} D_0^\alpha u\|_p.
\]

Moreover, if \( \alpha > \frac{1}{p} \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{q}}}{\Gamma(\alpha)((\alpha - 1)p + 1)^{\frac{1}{p}}} \|C_0^{\alpha} D_0^\alpha u\|_p.
\]
Incorporating (3.2) and (3.3) into the norm (3.1), we can consider the space $E_0^{α, p}$ with respect to the form

$$\|u\|_{α, p} = \|bD_1^α u\|_p,$$  \hfill (3.5)

in the following analysis.

**Lemma 3.3.** (see [22, Proposition 3.3]) Let $0 < α ≤ 1$ and $1 < p < ∞$. Assume that $α > \frac{1}{p}$ and the sequence $\{u_n\}$ converges weakly to $u$ in $E_0^{α, p}$, i.e., $\{u_n\} → u$. Then, $\{u_n\} → u$ in $C([0, 1])$, that is, $\|u_n - u\|_∞ → 0$ as $n → ∞$.

Associated to the problem $(P_λ)$ we define the functional $E_λ : E_0^{α, p} → \mathbb{R}$ given by

$$E_λ(u) = \frac{1}{bp^2} \left( a + b\|u\|_{α, p}^p \right)^p - \frac{λs^1}{1 - γ} \int_0^T g(t)|u|^{1 - γ} \, dt - \frac{s^r}{r} \int_0^T F(t, u) \, dt - \frac{a^p}{bp^2}.$$

One can easily verify that the energy functional $E_λ(u)$ is not bounded below on the space $E_0^{α, p}$. But we will show that $E_λ(u)$ is bounded below on this Nehari manifold and we will extract solutions by minimizing the functional on suitable subsets. In order to investigate the problem $(P_λ)$, we define the constraint set

$$N_λ := \left\{ u ∈ E_0^{α, p} \setminus \{0\} : t(u)u = 0 \right\},$$

where $t(u)$ is the zero of the map $Φ_u : (0, ∞) → \mathbb{R}$ defined as

$$Φ_u(t) = E_λ(tu).$$

Now as we know that the Nehari manifold is closely related to the behaviour of the functions $Φ_u : s → E_λ(su)$ for $s > 0$ defined by

$$Φ_u(s) = \frac{1}{bp^2} \left( a + bs^p\|u\|_{α, p}^p \right)^p - \frac{λs^1}{1 - γ} \int_0^T g(t)|u|^{1 - γ} \, dt - \frac{s^r}{r} \int_0^T F(t, u) \, dt - \frac{a^p}{bp^2}.$$

Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [14]. For $u ∈ E_0^{α, p}$, we have

$$Φ_u'(s) = \left(a + bs^p\|u\|_{α, p}^p\right)^{p-1} s^{p-1} \|u\|_{α, p}^p - \frac{λs^{1-γ}}{s^{2}} \int_0^T g(t)|u|^{1-γ} \, dt - s^{r-1} \int_0^T F(t, u(t)) \, dt,$$

$$Φ_u''(s) = (p-1)s^{p-2}\|u\|_{α, p}^p \left(a + bs^p\|u\|_{α, p}^p\right)^{p-1} + bp(p-1)s^{2p-2}\|u\|_{α, p}^p \left(a + bs^p\|u\|_{α, p}^p\right)^{p-2}$$

$$+ \frac{λγ}{s^{2+1}} \int_0^T g(t)|u(t)|^{1-γ} \, dt - (r-1)s^{r-2} \int_0^T F(t, u(t)) \, dt.$$

Note that $N_λ$ contains every nonzero solution of $(P_λ)$, and $u ∈ N_λ$ if and only if

$$\left(a + b\|u\|_{α, p}^p\right)^{p-1} \|u\|_{α, p}^p - λ \int_0^T g(t)|u|^{1-γ} \, dt - \int_0^T F(t, u) \, dt = 0.$$ \hfill (3.6)

**Lemma 3.4.** Let $u ∈ E_0^{α, p}$, then $tu ∈ N_λ$ if and only if $Φ_u'(t) = 0$.

**Proof.** Let $su ∈ N_λ$. This means that

$$0 = \left(a + b\|su\|_{α, p}^p\right)^{p-1} \|su\|_{α, p}^p - λ \int_0^T g(t)|su|^{1-γ} \, dt - \int_0^T F(t, su) \, dt$$

$$= Φ_u'(s).$$

This gives the proof of the Lemma 3.4. ∎
From the lemma 3.4, we have that the elements in $\mathcal{N}_1$ correspond to stationary points of the maps $\Phi_\mu(t)$ and in particular, $u \in \mathcal{N}_1$ if and only if $\Phi_\mu'(1) = 0$. Hence, it is natural to split $\mathcal{N}_1$ into three parts corresponding to local minima, local maxima and points of inflection $\Phi_\mu(s)$ defined as follows:

$$
\mathcal{N}_1^+ = \{u \in \mathcal{N}_1 : \Phi_\mu'(1) > 0\} = \{su \in E_0^{s\rho} : \Phi_\mu(s) = 0, \Phi_\mu''(s) > 0\},
$$

$$
\mathcal{N}_1^- = \{u \in \mathcal{N}_1 : \Phi_\mu'(1) < 0\} = \{su \in E_0^{s\rho} : \Phi_\mu'(s) = 0, \Phi_\mu''(s) < 0\},
$$

$$
\mathcal{N}_1^0 = \{u \in \mathcal{N}_1 : \Phi_\mu'(1) = 0\} = \{su \in E_0^{s\rho} : \Phi_\mu(s) = 0, \Phi_\mu''(s) = 0\}.
$$

Our first result is the following

**Lemma 3.5.** $E_\lambda$ is coercive and bounded below on $\mathcal{N}_1$.

**Proof.** Let $u \in \mathcal{N}_1$. Then, using (1.1) and (3.4), we obtain

$$
\int_0^T F(t,u(t))dt \leq K \int_0^T |u'|^\rho dt \leq \frac{KT^{1+\tau(\frac{1}{\beta})}}{\beta^\rho} ||u||^\rho,
$$

and

$$
\int_0^T g(t)|u|^{1-\gamma}dt \leq ||g||_\infty \int_0^T |u|^{1-\gamma}dt \leq ||g||_\infty \frac{T^{1+(1-\gamma)(\frac{1}{\beta})}}{\beta^{1-\gamma}} ||u||^{1-\gamma}. \tag{3.8}
$$

Consequently, from (3.7) and (3.8), we obtain

$$
E_\lambda(u) = \frac{1}{bp^2} \left( a + b||u||_{s\rho}^p \right) - \frac{\Lambda}{1-\gamma} \int_0^T g(t)|u|^{1-\gamma}dt - \frac{1}{r} \int_0^T F(t,u)dt - \frac{a^p}{bp^2}
$$

$$
\geq \left( a + b||u||_{s\rho}^p \right) \left( \frac{1}{bp^2} - \frac{1}{r} \right) ||u||_{s\rho}^p - \frac{\Lambda}{1-\gamma} \int_0^T g(t)|u|^{1-\gamma}dt - \frac{a^p}{bp^2}.
$$

Since $0 < 1 - \gamma < p^2 < r$, $E_\lambda(u)$ is coercive and bounded below on $\mathcal{N}_1$. The proof of the Lemma 3.5 is now completed. □

**Lemma 3.6.** Given $u \in \mathcal{N}_1^\gamma$ (respectively $\mathcal{N}_1^\gamma$) with $u \geq 0$, for all $v \in E_0^{s\rho}$ with $v \geq 0$, there exist $\epsilon > 0$ and a continuous function $\omega$ such that for all $k \in \mathbb{R}$ with $|k| < \epsilon$ we have

$$
\omega(0) = 1 \quad \text{and} \quad \omega(u + kv) \in \mathcal{N}_1^\gamma \quad \text{(respectively $\mathcal{N}_1^\gamma$)}.
$$

**Proof.** We introduce the function $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ define by:

$$
\psi(t,k) = t^{p+\gamma-1} \left( a + b||u + kv||_{s\rho}^p \right)^{p-1} ||u + kv||_{s\rho}^{p-1} - \frac{\tau^\gamma}{\gamma} \int_0^T g(s)(u + kv)^{1-\gamma}ds - t^\gamma \int_0^T F(s,u + kv)ds.
$$

Hence,

$$
\psi(t,k) = (p + \gamma - 1)t^{p+\gamma-2} \left( a + b||u + kv||_{s\rho}^p \right)^{p-1} ||u + kv||_{s\rho}^{p-1} - (r + \gamma - 1)t^\gamma \int_0^T F(s,u + kv)ds,
$$

is continuous on $\mathbb{R} \times \mathbb{R}$. Since $u \in \mathcal{N}_1^- \subset \mathcal{N}_1$, we have $\psi(1,0) = 0$, and

$$
\psi(1,0) = (p + \gamma - 1) \left( a + b||u||_{s\rho}^p \right)^{p-1} ||u||_{s\rho}^{p-1} - (r + \gamma - 1) \int_0^T F(t,u)dt < 0.
$$
Therefore, applying the implicit function theorem to the function \( \psi \) at the point \((1, 0)\), we obtain the existence of a parameter \( \delta > 0 \) and a positive continuous function \( \omega \) satisfying

\[
\omega(0) = 1, \quad \omega(k)(u + kv) \in \mathcal{N}_A, \quad \forall \, k \in \mathbb{R}, \ |k| < \delta.
\]

Hence, taking \( \varepsilon > 0 \) possibly smaller enough, we get

\[
\omega(k)(u + kv) \in \mathcal{N}_A^+, \quad \forall \, k \in \mathbb{R}, \ |k| < \varepsilon.
\]

The case \( u \in \mathcal{N}_A^+ \) may be obtained in the same way. This completes the proof of the Lemma 3.6. \( \square \)

**Lemma 3.7.** There exists \( \Lambda_0 \), such that for \( 0 < \lambda < \Lambda_0 \) there exist \( s^- \) and \( s^+ \) such that

\[
\Phi_u(s_0) = \Phi_u(s_1)
\]

and

\[
\Phi_u'(s_0) < 0 < \Phi_u'(s_1);
\]

that is, \( s_0u \in \mathcal{N}_A^- \) and \( s_1u \in \mathcal{N}_A^+ \).

**Proof.** Fix \( u \in \mathbb{R}^n \). Then it follows that

\[
\Phi_u'(s) = \left( a + bs\|u\|_p \right)^{p-1} s^{p-1} \|u\|_p^p - \frac{\lambda}{s^p} \int_0^T g(t)u(t)^{1+\gamma}dt - s^{-1} \int_0^T F(t, u(t))dt
\]

and

\[
= s^{p-1} \left( \|u\|_p^p - \psi(s) \right)
\]

where \( \psi(s) = \lambda s^{1+\gamma} \int_0^T g(t)u(t)^{1+\gamma}dt + s^{-p} \int_0^T F(t, u(t))dt \). Since, the first derivative of the function \( \psi \) is given by

\[
\psi'(s) = \lambda (1 - p - \gamma) s^{p-1-\gamma} \int_0^T g(t)u(t)^{1+\gamma}dt + (r - p) s^{-p-1} \int_0^T F(t, u(t))dt,
\]

it is simple to verify that \( \psi(s) \), attains its maximum at

\[
s_{\text{max}} = \left( \frac{\lambda(1 - p - \gamma)}{(r - p)} \left( \frac{\lambda(1 - p - \gamma)}{(r - p)} \int_0^T g(t)u(t)^{1+\gamma}dt \right) \right)^{\frac{1}{1+\gamma}}.
\]

Moreover,

\[
\psi(s_{\text{max}}) = \left( \frac{r - p}{(p + \gamma - 1)} + 1 \right) \int_0^T F(t, u(t))dt
\]

and

\[
\leq \left( \frac{r + \gamma - 1}{(p + \gamma - 1)} \right) \left( \frac{\lambda(1 - p - \gamma)}{r - p} \right)^{\frac{1}{p+\gamma-1}} \left( \int_0^T g(t)u(t)^{1+\gamma}dt \right)^{\frac{1}{1+\gamma}} \left( \int_0^T F(t, u(t))dt \right)^{\frac{1}{1+\gamma}}.
\]
Consequently, from (3.7) and (3.8), one has

$$\psi(s_{\text{max}}) \leq \left( \frac{r + \gamma - 1}{p + \gamma - 1} \right) \left( \frac{\lambda}{r - p} \right) \left( \frac{\|g\|_{\infty}}{\beta^{1 - \gamma}} \right) \left( \frac{T^{1 + (1 - \gamma)(\alpha - 1)}}{\beta^{1 - \gamma}} \right) \left( \frac{K^{1 + r(\alpha - 1)}}{\beta^{r}} \right) \|u\|^p_{a,p}.$$  

Hence, \(\|u\|^p_{a,p} > \psi(s_{\text{max}})\) and so \(\Phi'_u(s) > 0\) for all

$$0 < \lambda < \frac{r - p}{1 - p - \gamma} \left( \frac{p + \gamma - 1}{r + \gamma - 1} \right) \left( \frac{\|g\|_{\infty}}{\beta^{1 - \gamma}} \right) \left( \frac{\|u\|^p_{a,p}}{T^{1 + (1 - \gamma)(\alpha - 1)}} \right) \left( \frac{K^{1 + r(\alpha - 1)}}{\beta^{r}} \right) \|u\|^p_{a,p}.$$

On the other hand, we have

$$\Phi'_u(s) = \left( a + b\|u\|^p_{a,p} \right)^{p - 1} + \frac{p - 1}{2} \lambda \|u\|^2_{a,p} \left( a + b\|u\|^p_{a,p} \right)^{p - 2} \left( p + \gamma - 1 \right) \int_0^T g(t)|u(t)|^{1 - \gamma} dt - \frac{p - 1}{2} \lambda \|u\|^2_{a,p} \left( a + b\|u\|^p_{a,p} \right)^{p - 2} \left( p + \gamma - 1 \right) \int_0^T g(t)|u(t)|^{1 - \gamma} dt - \left( p + \gamma - 1 \right) \int_0^T F(t,u)dt.$$  

since \(1 - \gamma < p < r\), there exist \(0 < s_0 < s_{\text{max}} < s_1\) such that \(\Phi'_u(s_0) < 0\), \(\Phi'_u(s_1) > 0\). Note that \(N^0_\lambda = \emptyset\), we deduce that there exist \(s_1, s_0\) such that \(\Phi'_u(s_1) = \Phi'_u(s_0) = 0\) and \(\Phi'_u(s_1) > \Phi'_u(s_0) = 0\). Thus, \(\Phi_u\) has a local minimum at \(s = s_0\) and a local maximum at \(s = s_1\), that is \(\Phi_u\) is decreasing in \((0, s_0)\) and increasing in \((s_0, s_1)\). Hence, \(s_1 u \in N^+_\lambda\) and \(s_0 u \in N^-_\lambda\). The proof of Lemma 3.7 is now completed.

Now, we prove the following crucial Lemma:

**Lemma 3.8.** Suppose \(\lambda \in (0, \Lambda_0)\), \(N^0_\lambda = \emptyset\).

**Proof.** We proceed by contradiction to prove that \(N^0_\lambda = \emptyset\) for all \(\lambda \in (0, \Lambda_0)\). Let us suppose that there exists \(u_0 \in N^0_\lambda\). Then, it follows that

$$\Phi'_u(s) = \left( p + \gamma - 1 \right) \frac{\|u\|^p_{a,p}}{p} \left( a + b\|u\|^p_{a,p} \right)^{p - 2} \left( p + \gamma - 1 \right) \int_0^T g(t)|u(t)|^{1 - \gamma} dt - \left( p + \gamma - 1 \right) \int_0^T F(t,u)dt.$$  

Furthermore, if \(u \in N^0_\lambda\), then

$$\left( p + \gamma - 1 \right) \frac{\|u\|^p_{a,p}}{p} \leq \left( p + \gamma - 1 \right) \frac{\|u\|^p_{a,p}}{p} \left( a + b\|u\|^p_{a,p} \right)^{p - 2} \left( p + \gamma - 1 \right) \int_0^T F(t,u)dt \leq \left( p + \gamma - 1 \right) \frac{K^{1 + r(\alpha - 1)}}{\beta^{r}} \|u\|^p_{a,p} \leq \left( p + \gamma - 1 \right) \frac{K^{1 + r(\alpha - 1)}}{\beta^{r}} \|u\|^p_{a,p}.$$
and
\[
(p - r) \frac{a^p - 1(\alpha + 1)}{pT^\gamma} ||u||^p_{a,p} \leq (p - r)||u||^p_{a,p} (a + b||u||^p_{a,p})^{p-1}
\]
\[
\leq (p - r)||u||^p_{a,p} (a + b||u||^p_{a,p})^{p-1} - bp(p - 1)||u||^p_{a,p} (a + b||u||^p_{a,p})^{p-2}
\]
\[
\leq \lambda(r + \gamma - 1) \int_0^T g(t)|u(t)|^{1-\gamma} dt
\]
\[
\leq \lambda(r + \gamma - 1)||g||_\infty T^{1+(1-\gamma)(\alpha-\frac{1}{2})}\beta^{1-\gamma}||u||^{1-\gamma}_{a,p}.
\]
Consequently,
\[
\left(\frac{a^p - 1(p + \gamma - 1)\beta^p}{p(r + \gamma - 1)KT^{1+\gamma(\alpha-\frac{1}{2})}}\right)^{\frac{1}{1-\gamma}} \leq ||u||_{a,p} \leq \left(\frac{\lambda p(r + \gamma - 1)||g||_\infty T^{1+(1-\gamma)(\alpha-\frac{1}{2})}\beta^{1-\gamma}T^{\gamma}(\alpha + 1)}{a^p - 1(p - r)\beta^{1-\gamma}T(\alpha + 1)}\right)^{\frac{1}{1-\gamma}}.
\]
Therefore,
\[
\lambda \geq \Lambda_0 := \left(\frac{a^p - 1(p + \gamma - 1)\beta^p}{p(r + \gamma - 1)KT^{1+\gamma(\alpha-\frac{1}{2})}}\right)^{\frac{1}{1-\gamma}} \lambda p(r + \gamma - 1)||g||_\infty T^{1+(1-\gamma)(\alpha-\frac{1}{2})}\beta^{1-\gamma}T^{\gamma}(\alpha + 1).
\]
Hence \(N^0_\lambda = \emptyset\) for all \(\lambda \in (0, \Lambda_0)\). \(\Box\)

By Lemmas 3.5 and 3.7, we can write \(N_\lambda = N^+_\lambda \cup N^-_\lambda\) and define
\[
c^+_\lambda := \inf_{u \in N^+_\lambda} E_\lambda(u) \quad \text{and} \quad c^-_\lambda := \inf_{u \in N^-_\lambda} E_\lambda(u).
\]

4. Existence of minimizer on \(N^+_\lambda\)

In this section, we will show that the minimum of \(E_\lambda\) is achieved in \(N^+_\lambda\). Also, we show that this minimizer is also the first solution of \((P_\lambda)\).

**Lemma 4.1.** If \(0 < \lambda < \Lambda_0\), then for all \(u \in N^+_\lambda, c^+_\lambda < 0\).

**Proof.** Let \(u \in N^+_\lambda\), then we have \(\phi_\lambda'(u)(1) > 0\) which gives
\[
(p - 1)||u||^p_{a,p} (a + b||u||^p_{a,p})^{p-1} + bp(p - 1)||u||^p_{a,p} (a + b||u||^p_{a,p})^{p-2} + \lambda \gamma \int_0^T g(t)|u(t)|^{1-\gamma} dt
\]
\[
- (r - 1) \int_0^T F(t, u(t)) dt > 0.
\]
Now multiplying equation (3.6) by \((r - 1)\) and subtracting from equation (4.1), we get
\[
\lambda(r + \gamma - 1) \int_0^T g(t)|u|^{1-\gamma} dt > ||u||^p_{a,p} (a + b||u||^p_{a,p})^{p-2} \left((r - p)a + b||u||^p_{a,p}(r - p^2)\right).
\]
Now using equation (3.6) and (4.2), we get

\[
E_\lambda(u) = \frac{1}{bp^2} (a + b\|u\|_{a,p}^p)^p - \frac{\lambda}{1 - \gamma} \int_0^T g(t)|u|^{1-\gamma} \, dt - \frac{1}{r} \int_0^T F(t,u) \, dt - \frac{\alpha'}{bp^2}
\]

\[
= (a + b\|u\|_{a,p}^p)^p - \frac{\lambda}{1 - \gamma} \int_0^T g(t)|u|^{1-\gamma} \, dt - \frac{1}{r} \int_0^T g(t)|u|^{1-\gamma} \, dt - \frac{\alpha'}{bp^2}
\]

\[
\leq (a + b\|u\|_{a,p}^p)^p - \frac{\lambda}{1 - \gamma} \int_0^T g(t)|u|^{1-\gamma} \, dt - \frac{1}{r} \int_0^T g(t)|u|^{1-\gamma} \, dt - \frac{\alpha'}{bp^2}
\]

\[
\leq -\beta \frac{(r-p^2)(p^2 - (1 - \gamma))}{p^2(1 - \gamma)} (a + b\|u\|_{a,p}^p)^p - \|u\|_{a,p}^p < 0
\]

since \(0 < a < 1\), and \(b > 0\). Thus,

\[
c_\lambda^+ = \inf_{u \in N_\lambda^+} E_\lambda(u) < 0 \quad \text{for all } \lambda \in (0, \Lambda_0).
\]

(4.3)

\[
\text{Theorem 4.2.} \quad \text{If } 0 < \lambda < \Lambda_0, \text{ then there exists } u_0 \in N_\lambda^+ \text{ satisfying } E_\lambda(u_0) = \inf_{u \in N_\lambda^+} E_\lambda(u).
\]

**Proof.** Since \(E_\lambda\) is bounded below on \(N_\lambda\) and so on \(N_\lambda^+\). Then, using the Ekeland variational principle \([15]\), there exist a minimizing sequence \(\{u_n\} \subset N_\lambda^+\) such that

\[
E_\lambda(u_n) \to \inf_{u \in N_\lambda^+} E_\lambda(u) \quad \text{as } n \to \infty.
\]

Since \(E_\lambda\) is coercive, \(\{u_n\}\) is bounded in \(E_0^{a,p}\). Then there exists a subsequence, still denoted by \(u_n\) and \(u_0 \in E_0^{a,p}\) such that, as \(n \to \infty\),

\[
u_n \to u_0, \quad \text{weakly in } E_0^{a,p}
\]

\[
u_n \to u_0, \quad \text{strongly in } L^q(\Omega) \quad \text{for all } 1 \leq q < p',
\]

\[
u_n \to u_0, \quad \text{a.e. in } \Omega.
\]

Now, using Lemma 3.3, we get that, as \(n \to \infty\),

\[
\int_0^T g(t)|u_n|^{1-\gamma} \, dt \leq \int_0^T g(t)|u_0|^{1-\gamma} \, dt + \int_0^T g(t) |u_n - u_0|^{1-\gamma} \, dt
\]

\[
\leq \int_0^T g(t)u_0^{1-\gamma} \, dx + Tg(t) \| u_n - u_0 \|^{1-\gamma}_{\infty}
\]

\[
= \int_0^T g(t)u_0^{1-\gamma} \, dt + o(1).
\]

Similarly

\[
\int_0^T g(t)|u_n|^{1-\gamma} \, dt \leq \int_0^T u_n^{1-\gamma} \, dt + \int_0^T g(t) |u_n - u_0|^{1-\gamma} \, dt
\]

\[
\leq \int_0^T g(t)u_n^{1-\gamma} \, dt + Tg(t) \| u_n - u_0 \|^{1-\gamma}_{\infty}
\]

\[
= \int_0^T g(t)u_n^{1-\gamma} \, dt + o(1).
\]
Thus,
\[
\int_0^T g(t)u_n^{1-\gamma} \, dt = \int_0^T g(t)u_0^{1-\gamma} \, dx + o(1).
\] (4.4)

On the other hand, by [8] there exists \( l \in L'(\mathbb{R}^N) \) such that
\[
|u_n(x)| \leq l(x), \quad \text{as } n \to \infty
\]
for any \( 1 \leq r < p' \). Therefore by Dominated convergence Theorem we have that
\[
\lim_{n \to \infty} \int_0^T F(t, u_n) \, dt = \int_0^T F(t, u_0) \, dt.
\] (4.5)

Moreover, by Lemma (3.7), there exists \( s_1 \) such that \( s_1u_0 \in \mathcal{N}_{\lambda, p}^+ \). Now, we shall prove \( u_n \to u_0 \) strongly in \( E_0^{1,p} \). Suppose otherwise, then either
\[
\|u_0\|_{p,p}^2 \leq \liminf_{n \to \infty} \|u_n\|_{p,p}^2.
\]

Thus, since \( u_n \in \mathcal{N}_{\lambda, p}^+ \), one has
\[
\lim_{n \to \infty} \Phi'_{u_n}(s_1) = \lim_{n \to \infty} \left( (a + b \|u_n\|_{L^p})^{-1} \|u_n\|_{L^p}^p - \frac{\lambda}{s_1} \int_0^T g(t)|u_n(t)|^{1-\gamma} \, dt - \frac{1}{s_1} \int_0^T F(t, u_n(t)) \, dt \right)
\]
\[
> (a + b \|u\|_{L^p})^{-1} \|u\|_{L^p}^p - \frac{\lambda}{s_1} \int_0^T g(t)|u(t)|^{1-\gamma} \, dt - \frac{1}{s_1} \int_0^T F(t, u(t)) \, dt = \Phi'_{u_0}(s_1) = 0
\]

Therefore, \( \Phi'_{u_0}(s_1) > 0 \) for \( n \) large enough. Since \( u_n \in \mathcal{N}_{\lambda, p}^+ \), we have \( s_{\max}(u_n) > 1 \). Moreover \( \Phi'_{u_0}(1) = 0 \) and \( \Phi_{u_0}(1) \) is increasing for \( s \in (0, s_{\max}(u_n)) \). This implies that \( \Phi_{u_0}(s) < 0 \) for all \( s \in (0, 1] \) and \( n \) sufficiently large. We obtain \( 1 < s_1 < s_{\max}(u_0) \). But \( s_1u_0 \in \mathcal{N}_{\lambda, p}^+ \) and
\[
E_{\lambda}(s_1u_0) = \inf_{1 < s < s_{\max}(u_0)} E_{\lambda}(su_0).
\]

Which implies that
\[
E_{\lambda}(s_1u_0) < E_{\lambda}(u_0) = \lim_{n \to \infty} E_{\lambda}(u_n) = c_\lambda^+
\]

which gives a contradiction. Thus, \( u_n \to u_0 \) strongly in \( E_0^{1,p} \), and \( E_{\lambda}(u_0) = \inf_{u \in \mathcal{N}_{\lambda, p}^+} E_{\lambda}(u) \). The proof of the Theorem 4.2 is now completed. \( \square \)

5. Existence of minimizer on \( \mathcal{N}_{\lambda, p}^- \)

In this section, we shall show the existence of second solution by proving the existence of minimizer of \( E_{\lambda} \) on \( \mathcal{N}_{\lambda, p}^- \).

**Lemma 5.1.** If \( 0 < \lambda < \Lambda_0 \), then for all \( u \in \mathcal{N}_{\lambda, p}^+ \), \( c_\lambda^- > 0 \).

**Proof.** Let \( v \in \mathcal{N}_{\lambda, p}^- \), then we have \( \Phi_{u_0}^-(1) \) which gives,
\[
(p + \gamma - 1)^{-1} \frac{p^{p-1} - \beta}{p} \leq (p + \gamma - 1) \|u\|_{L^p}^p (a + b\|v\|_{L^p}^p)^{p-1} + bp(p - 1)\|v\|_{L^p}^p (a + b\|v\|_{L^p}^p)^{p-2}
\]
\[
\leq (p + \gamma - 1) \frac{K T^{1/(p-\beta)} \|v\|_{L^p}^p}{p^p}.
\]
this implies
\[ \|v\|_{\lambda,p} \geq \left( \frac{(p + \gamma - 1)|\alpha|^r - 1}{p(r + \gamma - 1)K^{1+\rho(\alpha-\frac{1}{p})}} \right) \frac{1}{r} . \]

Subsequently,
\[ E_{\lambda}(v) \geq \left( a + b\|v\|_{\lambda,p} \right)^q \left( \frac{|\alpha|}{b} + \frac{1}{p} - \frac{1}{r} \|v\|_{\lambda,p}^p \right) - \lambda \left( \frac{1}{1 - \gamma} - \frac{1}{r} \right)^{\frac{1}{r}} \frac{\|v\|_{\lambda,p}^{1 - \gamma}}{(\Gamma(\alpha))^{1 - \gamma} ((\alpha - 1)p + 1) \frac{1}{r}} . \]

Moreover, by Lemma (3.7), there exists \( s_0 \) such that \( s_0 \varphi_0 \in \mathcal{N}_\lambda^- \). Now, we prove \( v_n \to v_0 \) strongly in \( E_{0,p}^q \). Suppose otherwise, then either
\[ \|v_0\|_{E_{0,p}^q} \leq \liminf_{n \to \infty} \|v_n\|_{E_{0,p}^q} . \]
Thus, since \( v_n \in \mathcal{N}_\lambda^* \), \( E_\lambda(s_0 v_0) \leq E_\lambda(v_0) \), for all \( s_0 \geq 0 \) we have
\[
E_\lambda(s_0 v_0) < \lim_{n \to \infty} E_\lambda(s_0 v_n) \leq \lim_{n \to \infty} E_\lambda(v_n) = c_\lambda^-
\]
which gives a contradiction. Thus, \( v_n \to v_0 \) strongly in \( E_{ap}^0 \) and \( E_\lambda(v_0) = \inf_{v \in \mathcal{N}_\lambda^-} E_\lambda(v) \). The proof of the Theorem 5.2 is now completed.

**Proof.** [Proof of Theorem 1.1] Now to prove the Theorem 1.1, let us start by proving the existence of non-negative solutions. First, by Theorems 4.2, 5.2, we conclude that there exist \( u_0 \in \mathcal{N}_\lambda^* \), \( v_0 \in \mathcal{N}_\lambda^- \) satisfying
\[
E_\lambda(u_0) = \inf_{u \in \mathcal{N}_\lambda^*} E_\lambda(u)
\]
and
\[
E_\lambda(v_0) = \inf_{v \in \mathcal{N}_\lambda^-} E_\lambda(v).
\]
Moreover, since \( E_\lambda(u_0) = E_\lambda(|u_0|) \) and \( |u_0| \in \mathcal{N}_\lambda^* \) and similarly \( E_\lambda(v_0) = E_\lambda(|v_0|) \) and \( |v_0| \in \mathcal{N}_\lambda^- \), so we may assume \( (u_0, v_0) \geq 0 \). By Lemma 3.4, we may assume that \( (u_0, v_0) \) are nontrivials nonnegatives solutions of problem \( (P_\lambda) \). Finally, it remain to show that the solutions found in Theorems 4.2, 5.2, are distinct. Since \( \mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^* = \emptyset \), then, \( (u_0, v_0) \) are distinct. The proof of the Theorem 1.1 is now completed.

6. Conclusion

Theorem 1.1 shows the multiplicity of positive solutions for a class of Kirchhoff fractional problem involving Riemann-Liouville fractional Derivative and singular nonlinearity by using the Nehari manifold approach and some variational techniques. This result generalises Theorems 1.1 and 1.2 in [12] which only concerns the case when the nonlinearity \( f(t, x) \) is \( (p^2 - 1) \)-superlinear in \( x \) at infinity and the nonlinearity \( f(t, x) \) is \( (p^2 - 1) \)-sublinear in \( x \) at infinity. The proof used to prove Theorem 1.1 does not modify the quasilinear operator \( \Phi_p(s) = |s|^{p-2}s \) \( s \neq 0 \), \( \Phi_p(0) = 0 \) in \( (P_\lambda) \). Therefore, this approach could be considered for more general quasilinear operators to get multiplicity results. We suggest to extend the methods developed in this paper to the more general framework of Musielak-Orlicz-Sobolev spaces for a collection of stationary problems studied in these function spaces.

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