Scaling laws and fluctuations in the statistics of word frequencies

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Abstract. In this paper we combine statistical analysis of large text databases and simple stochastic models to explain the appearance of scaling laws in the statistics of word frequencies. Besides the sublinear scaling of the vocabulary size with database size (Heaps’ law), here we report a new scaling of the fluctuations around this average (fluctuation scaling analysis). We explain both scaling laws by modeling the usage of words by simple stochastic processes in which the overall distribution of word-frequencies is fat tailed (Zipf’s law) and the frequency of a single word is subject to fluctuations across documents (as in topic models). In this framework, the mean and the variance of the vocabulary size can be expressed as quenched averages, implying that: i) the inhomogeneous dissemination of words cause a reduction of the average vocabulary size in comparison to the homogeneous case, and ii) correlations in the co-occurrence of words lead to an increase in the variance and the vocabulary size becomes a non-self-averaging quantity. We address the implications of these observations to the measurement of lexical richness. We test our results in three large text databases (Google-ngram, English Wikipedia, and a collection of scientific articles).

1. Introduction

Fat-tailed distributions [1–3], allometric scaling [4, 5], and fluctuation scaling [6–8] are the most prominent examples of scaling laws appearing in complex systems. Statistics of words in written texts provide some of the best studied examples: it shows a fat-tailed distribution of word frequencies (Zipf’s law) [9] and a sublinear growth (as in allometric scalings) of the number of distinct words as a function of database size (Heaps’ law) [10, 11]. The connection between these two scalings is known at least since Mandelbrot [12], and has been further investigated in recent years [13–15], especially for large databases [16], finite text sizes [17, 18], and more general scaling distributions [19, 20]. In this paper we report the existence of a third type of scaling in natural language: fluctuation scaling. It appears when investigating the fluctuations around the Heaps’ law, i.e., the variation of the vocabulary over different texts of the same size. We show that this scaling results from topical aspects of written text which are ignored in the usual connection between Zipf’s and Heaps’ law.

The importance of looking at the fluctuations around Heaps’ law is that this law is widely used to predict the size of vocabularies [21], e.g., (i) to optimize the
memory allocation in inverse indexing algorithms [22]; (ii) to estimate the vocabulary of a language [23, 24]; (iii) to compare the vocabulary richness of documents with different lengths [25–27]. Beyond linguistic applications, scalings of the number of unique items as a function of database size similar to Heaps’ law have been observed in other domains, e.g. the species-area relationship in ecology [28, 29], collaborative tagging [30], network growth [31], and in the statistics of chess moves [32]. These scaling laws have been analyzed from the general viewpoint of innovation dynamics [33] and sampling problems [34]. Our results allow for the quantification of uncertainties in the estimation of these scaling laws and lead to a rethinking of the statistical significance of previous findings.

We use as databases three different collections of texts: i) all articles of the English Wikipedia [35], ii) all articles published in the journal PlosOne [36], and iii) the Google-ngram database [23], a collection of books published in 1520–2008 (each year is treated as a separate document). See Appendix A for details on the data collection.

The manuscript is divided as follows. Section 2 reports our empirical findings with focus on the deviations from a Poisson null model. Section 3 shows how these deviations can be explained by including topicality, which plays the role of a quenched disorder and leads to a non-self averaging process. The consequences of our findings to applications, e.g. vocabulary richness, are discussed in Sec. 4. Finally, Sec. 5 discusses the implications of our main results to other complex systems.

2. Empirical Scaling Laws

The most-prominent case of scaling in the context of language is Zipf’s law [9] which states that the frequency, \( F_r \), of the \( r \)-th most frequent word scales as

\[
F_r \propto r^{-\alpha} \quad \text{for} \quad r \gg 1
\]

where the frequency of word \( r \) is defined as the fraction of times it occurs in the whole database. Another well-studied example of scaling in language concerns the vocabulary growth known as Heaps’ law [10, 11], which states that the number of different words, \( N \), scales sublinearly with the total number of words, \( M \), i.e.

\[
N(M) \propto M^\lambda \quad \text{for} \quad M \gg 1
\]

with \( 0 < \lambda < 1 \). As a third case, we consider here the problem of the vocabulary growth for an ensemble of texts, and study the scaling of fluctuations by looking at the relation between the standard deviation, \( \sigma_M = \sqrt{\mathbb{V}[N(M)]} \), and the mean value, \( \mu_M = \mathbb{E}[N(M)] \), computed over the ensemble of texts with the same textlength \( M \). In other systems, Taylor’s law [6]

\[
\sigma_M \propto \mu_M^\beta \quad \text{for} \quad \mu_M \gg 1
\]

with \( 1/2 \leq \beta \leq 1 \) is typically observed [8].

The connection between scalings (1) and (2) (Zipf’s and Heaps’ law) can be revealed assuming the usage of each word \( r \) is governed by an independent Poisson process with
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a given frequency $F_r$. In this description, the number of different words, $N$, becomes a stochastic variable for which we can calculate the expectation value $E[N(M)]$ and the variance $V[N(M)]$ over the realizations of the Poisson process. The probability that the word with rank $r$ appears at least once in a text of length $M$ is $1 - e^{-MF_r}$ and therefore

$$E[N(M)] \equiv \mu_M = \sum_r 1 - e^{-MF_r},$$

$$V[N(M)] \equiv \sigma_M^2 \equiv E[N(M)^2] - E[N(M)]^2 = \sum_r e^{-MF_r} - e^{-2MF_r}. \quad (5)$$

Assuming a power-law rank-frequency distribution, $F_r = cr^{-\alpha}$ we recover a scaling in the vocabulary growth for $M \gg 1$, i.e. $E[N(M)] \propto M^\lambda$, with a simple relation between the scaling exponents: $\alpha = \lambda^{-1} [37]$.

In Fig. 1 we show empirical data of real texts for the scaling relations (1)-(3) and compare them with predictions from the Poisson null model in Eqs. (4,5). The Poisson null model correctly elucidates the connection between the scaling exponents in Zipf’s and Heaps’ law, but it suffers from two severe drawbacks. First, it is of limited use for a quantitative prediction of the vocabulary size for individual articles as it systematically overestimates its magnitude, see Fig. 1(b,e,h). Second, it dramatically underestimates the expected fluctuations of the vocabulary size yielding a qualitatively different behavior in the fluctuation scaling: whereas the Poisson null model yields an exponent $\beta \approx 1/2$ expected from central-limit-theorem-like convergence [8], the three empirical data [Fig. 1(c,f,i)] exhibit a scaling with $\beta \approx 1$. This implies that relative fluctuations of $N$ around its mean value $\mu$ for fixed $M$ do not decrease with larger text size (the vocabulary growth, $N(M)$, is a non-self-averaging quantity) and remain of the order of the expected value. Furthermore, we find that in all three databases the fluctuation scaling approximately gives a quantitative relation between $\mu_M$ and $\sigma_M$:

$$\sigma_M \approx 0.1\mu_M. \quad (6)$$

Heaps’ law can also be constructed by considering the vocabulary growth of a single text as a curve $N(M)$ for $M = 1, 2, ..., M_{\text{max}}$, where $M_{\text{max}}$ is the length of the text. This construction was employed in Fig. 1(e,f) and leads to the same results reported above. In Fig. 1(f) we show that anomalous fluctuation scaling in the vocabulary growth is preserved if shuffling the word order of individual texts. This illustrates that in contrast to usual explanations of fluctuation scaling in terms of long-range correlations in time-series [8], here, the observed deviations from the Poisson null model are mainly due to fluctuations among different texts.

In the following, we argue that these observations can be accounted for by considering the topical aspects of written language, i.e. instead of treating word-frequencies as fixed, we will consider them to be topic-dependent ($F_r \mapsto F_r(\text{topic})$).
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Figure 1. Scaling of Zipf’s law (1), Heaps’ law (2), and fluctuation scaling (3). Each row corresponds to one of the three databases used in our work. (a,d,g) Zipf’s law: Rank-frequency distribution $F_r$ considering the full database (the double power-law nature of the curves is apparent [19]). (b,e,h) Heaps’ law: the number of different words, $N$, as a function of textlength, $M$, for each individual article in the corresponding database (black dots). (c,f,i) Fluctuation scaling: standard deviation, $\sigma_M$, as a function of the mean, $\mu_M$, for the vocabulary $N(M)$ conditioned on the textlength $M$. Poisson (dark line) shows the expectation from the Poisson null model, Eqs. (4,5), assuming the empirical rank-frequency distribution from (a,d,g), respectively. (Data: $\mu, \sigma$) (pale line) shows the mean, $\mu_M$, and standard deviation, $\sigma_M$, of the data $N(M)$ within a running window in $M$ (see Appendix A for the details on the procedure). Additionally, (e,f) show the results (Data: $\mu, \sigma$) when the word order for each individual article is shuffled (dashed-dotted) illustrating that the results are not due to temporal correlations within the text. For comparison we show in (c,f,i) the scalings $\sigma_M \propto \mu_M^{1/2}$ and $\sigma_M \propto \mu_M$ (dotted lines).
3. Topicality in the Vocabulary Growth

3.1. Topicality

The frequency of an individual word varies significantly among different texts meaning that its usage cannot be described alone by a single global frequency [38–40]. For example, consider the usage of the (topical) word “network” in all articles published in the journal PlosOne. It has an overall rank $r^* = 428$ and a global frequency, $F_{r^*} = 2.9 \times 10^{-4}$, see Fig. 2(a). The local frequency obtained from each article separately varies over more than one decade in Fig. 2(b).

One popular approach to account for the heterogeneity in the usage of single words are topic models [41]. The basic idea is that the variability across different documents can be explained by the existence of (a smaller number of) topics. In the framework of a generative model it assumes i) that individual documents are composed of a mixture of topics (indexed by $j = 1, \ldots, T$), represented by the probabilities $P_{\text{doc}}(\text{topic} = j)$; and ii) that the frequency of each word is topic-dependent, i.e. $F_r(\text{topic} = j)$, which together leads to a different effective frequency in each document, $F_{r, \text{doc}} = \sum_{j=1}^{T} P_{\text{doc}}(j) F_r(j)$. One particularly popular variant of topic models is Latent Dirichlet Allocation (LDA) [42], which assumes that the topic composition $P_{\text{doc}}(\text{topic})$ of each document is drawn from a Dirichlet distribution such that only few topics contribute to each document. Given a database of documents, LDA infers the topic-dependent frequencies, $F_r(\text{topic})$, from numerical maximization of the posterior likelihood of the generative model [43]. As an illustration, in Fig. 2(c) we show $F_{r^*}(\text{topic})$ obtained using LDA for the word “network” in the PlosOne database. As expected from a meaningful topic model, we see that the conditional frequencies vary over many orders of magnitude, and that the global frequency $F_{r^*}$ is governed by few topics.

3.2. General treatment

In this section we show how topicality can be included in the analysis of the vocabulary growth. The simplest approach is to consider again that the usage of each word is governed by an independent Poisson process, but this time to consider that frequencies are not fixed but are themselves random variables that vary among texts.

In this setting, the random variable representing the vocabulary size, $N$, for a text of length $M$ can be written as

$$N(M) = \sum_r I\left[n_r(M, F_r)\right], \quad (7)$$

in which $n_r$ is the integer number of times the word $r$ occurs in a Poisson process of length $M$ with frequency $F_r$ and $I[x]$ is an indicator-type function, i.e. $I[x = 0] = 0$ and $I[x \geq 1] = 1$. The calculation of the expectation value now consists of two parts: i) the average over realizations $i$ of the Poisson processes $n_r^{(i)}(M, F_r^{(j)})$ for a given realization $j$ of the set of frequencies $F_r^{(j)}$; and ii) the average over all possible realizations $j$ of the
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Figure 2. Variation of frequencies due to topicality in the PlosOne database. (a) Rank-frequency distribution considering the full data with the location of the word “network” (dotted line) with \( F_{r^*} = 428 \approx 2.9 \times 10^{-4} \). (b) Distribution \( P(F_r) \) of the local frequency \( F_r \), obtained from each article separately for the word “network” with the global frequency from (a) (dotted) and the effective frequency \( F_{r,doc} = \sum_{t=1}^{T} P_{doc}(t) F_r(t) \) from (c+d) (solid). (c) Topic-dependent frequencies \( F_{r^*}(\text{topic}) \) inferred from LDA with \( T = 20 \) topics for the word “network” with global frequency from (a) as comparison (dotted). (d) One realization for the topic composition, \( P_{doc}(\text{topics}) \), of a single document drawn from a Dirichlet distribution.

In this framework expectation values correspond to quenched averages (denoted by subscript \( q \))

\[
\mathbb{E}_q [N(M)] = \langle N(M) \rangle_{i,j} = \sum_r \langle I \left[ n_r^{(i)}(M, F_r^{(j)}) \right] \rangle_{i,j} = \sum_r 1 - \langle e^{-MF_r^{(j)}} \rangle, \quad (8)
\]

where we used

\[
\langle I \left[ n_r^{(i)}(M, F_r^{(j)}) \right] \rangle_i = 1 - P(n_r = 0; M, F_r^{(j)}) = 1 - e^{-MF_r^{(j)}}. \quad (9)
\]

The last equation corresponds to the probability of word \( r \) not occurring for a Poisson process of duration \( M \) with frequency \( F_r^{(j)} \), as in Eq. (4). For simplicity, hereafter \( \langle \ldots \rangle \equiv \langle \ldots \rangle_j \) (the average over realizations of sets of frequencies \( F_r^{(j)} \)).

Using the inequality between arithmetic and geometric mean, i.e.

\[
e^{\langle \ln x \rangle} = \langle x \rangle_{\text{geometric}} \leq \langle x \rangle_{\text{arithmetic}} = \langle e^{\ln x} \rangle,
\]

we obtain that

\[
\mathbb{E}_q [N(M)] = \sum_r 1 - \langle e^{-MF_r} \rangle \leq \sum_r 1 - e^{-M(F_r)} = \mathbb{E}_a [N(M)]. \quad (11)
\]

The right hand side correspond to the result of the Poisson null model (with fixed \( F_r = \langle F_r \rangle \)), see Eq. (4), and can be interpreted as an annealed average (denoted by subscript \( a \)). This implies that the heterogeneous dissemination of words across different texts leads to a reduction of the expected size of the vocabulary, in agreement with the first deviation of the Poisson null model reported in Fig. 1(b,e,h).
For the quenched variance we obtain (see Appendix B)

$$\forall q [N(M)] \equiv \mathbb{E}_q [N(M)^2] - \mathbb{E}_q [N(M)]^2$$

$$= \sum_r \langle e^{-MF_r} \rangle - \langle e^{-MF_r} \rangle^2 + \sum_r \sum_{r' \neq r} \text{Cov}[e^{-MF_r}, e^{-MF_{r'}}]$$

where \( \text{Cov}[e^{-MF_r}, e^{-MF_{r'}}] \equiv \langle e^{-MF_r} e^{-MF_{r'}} \rangle - \langle e^{-MF_r} \rangle \langle e^{-MF_{r'}} \rangle \). Comparing to the Poisson case in Eq. (5), we see that the quenched average yields an additional term containing the correlations of different words. In general, this term does not vanish and is responsible for the anomalous fluctuation scaling with \( \beta = 1 \) observed in real text, explaining the second deviation from the Poisson null model reported in Fig. 1(c,f,i).

### 3.3. Specific ensembles

In this section we compute the general results from Eqs. (8,13) for particular ensembles of frequencies \( F_{r}^{(j)} \) and compare them to the empirical results. To the best of our knowledge, a generally accepted parametric formulation of such an ensemble has so far not been justified by systematic statistical analysis, which is why we propose two nonparametric approaches explained in the following.

In the first approach we construct the ensemble \( F_{r}^{(j)} \) directly from the collection of documents, i.e. the frequency \( F_{r}^{(j)} \) corresponds to the frequency of word \( r \) in document \( j \), such that

$$\langle e^{-MF_r} \rangle = \frac{1}{D} \sum_{j=1}^{D} e^{-MF_r^{(j)}}, \quad (14)$$

where \( D \) is the number of documents in the data, see Fig. 2(b).

In the second approach we construct the ensemble from the LDA topic model [42], in which \( F_{r}^{(j)} = F_{r}(\text{topic} = j) \) corresponds to the frequency of word \( r \) conditional on the topic \( j = 1...T \), see Fig. 2(c+d). In this particular formulation each document is assumed to consist of a composition of topics which is drawn from a Dirichlet distribution, such that we get for the quenched average

$$\langle e^{-MF_r} \rangle = \int d\theta P_{\text{Dir}}(\theta|\alpha)e^{-MF_r(\theta)},$$

in which \( \theta = (\theta_1,...,\theta_T) \) are the probabilities of each topic, \( F_r(\theta) = \sum_{j=1}^{T} \theta_j F_{r}(\text{topic} = j) \), and the integral is over a \( T \)-dimensional Dirichlet-distribution \( P_{\text{Dir}}(\theta|\alpha) \) with concentration parameter \( \alpha \). We infer the \( F_{r}(\text{topic}) \) using Gensim [43] for LDA with \( T = 100 \) topics.

The results from both approaches are compared to the PlosOne database in Fig. 3. We can see in Fig. 3(a) that both methods lead to a reduction in the mean number of different words. Whereas the direct ensemble, Eq. (14), almost perfectly matches the curve of the data, the LDA-ensemble, Eq. (15), still overestimates the mean number of different words in the data. This is not surprising since due to the fewer number of topics (when compared to the number of documents) it constitutes a much more
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\( \mu_M = \mathbb{E}[N(M)] \)

\( \sigma_M = \sqrt{\mathbb{V}[N(M)]} \)

\( \text{textlength: } M \)

(a) Data
Poisson
Real Freq
LDA Freq

(b) Data
Poisson
Real Freq
LDA Freq

Figure 3. Vocabulary growth for specific topic models. (a) Average vocabulary growth and (b) fluctuation scaling in the PlosOne database (Data) and in the calculations from Eqs. (8,13) for the two topic models based on the measured frequencies in individual articles (Real Freq) and on LDA (LDA Freq), compare Eqs. (14,15). The inset in (b) (same scale as main figure) shows the individual contributions to the fluctuations in Eq. (13): \( \sum_r (e^{-MF_r} - e^{-2MF_r}) \) (dotted) and \( \sum_r \sum_{r' \neq r} \text{Cov}[e^{-MF_r}, e^{-MF_{r'}}] \) (solid), illustrating that correlations among different words lead to anomalous fluctuation scaling. The lines for LDA-Freq and Real Freq in (b) show the calculations of the corresponding topic models by replacing the assumption of Poisson usage in the derivation of Eqs. (8,13) with multinomial drawing, showing that deviations from the data for \( \mu_M < 10^0 \) are due to finite-size effects. For comparison we show the results from the Poisson null model (Poisson), Eqs. (4,5), which do not take into account topicality.

course-grained description than the direct ensemble. Additionally, the LDA-ensemble relies on a number of ad-hoc assumptions, e.g. the Dirichlet-distribution in Eq. (15) or the particular choice of parameters in the inference algorithm which were not optimized here. More importantly, both methods correctly account for the anomalous fluctuation scaling with \( \beta = 1 \) observed in the real data, see Fig. 3(b) and even yield a similar proportionality factor in the quantitative agreement with the data. The comparison of the individual contributions to the fluctuations, Eq. (13), shown in the inset of Fig. 3(b) shows that the anomalous fluctuation scaling is due to correlations in the co-occurrence of different words (contained in the term \( \text{Cov}[e^{-MF_r}, e^{-MF_{r'}}] \)). The deviation from the data for short texts, \( \mu_M < 10^0 \), is due to finite size effects, which are not captured by the Poisson assumption of word usage. To confirm this, we replaced the Poisson by a multinomial process and obtained an agreement with the empirical observations over the full range of almost four decades in \( \mu_M \), see Fig. 3(b).
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4. Applications

4.1. Adding texts

In thermodynamic terms, Heaps’ law (as other allometric scalings) implies that the vocabulary size is neither extensive nor intensive \((N(M) < N(2M) < 2N(M))\), also for \(M \to \infty\). While this can be seen as a direct consequence of Zipf’s law, our results show that the Heaps’ law depends also sensitively on the fluctuation of the frequency of specific words across different documents. To illustrate this, consider the problem of doubling the size of a text of size \(M\) to \(2M\). This can be done either by simply extending the size of the same text up to size \(2M\) (denoted by \(M' = 2 \cdot M\)) or by concatenating another text of size \(M\) (denoted by \(M' = 2 \times M\)). The Poisson model (fixed frequency or annealed average) predicts the same expected vocabulary for both procedures

\[
\mathbb{E}_a[N(2 \cdot M)] = \mathbb{E}_a[N(2 \times M)] = \sum_r 1 - e^{-2MF_r}. \tag{16}
\]

Taking fluctuations of individual frequencies among documents (quenched average) into account yields (see Appendix C for details):

\[
\mathbb{E}_q[N(2 \cdot M)] = \sum_r 1 - \langle e^{-2MF_r} \rangle \quad \text{and} \quad \mathbb{E}_q[N(2 \times M)] = \sum_r 1 - \langle e^{-MF_r} \rangle^2. \tag{17}
\]

Using Eq. (10) and the fact that \(\langle x^2 \rangle \geq \langle x \rangle^2\), we obtain the following general result

\[
\mathbb{E}_q[N(2 \cdot M)] \leq \mathbb{E}_q[N(2 \times M)] \leq \mathbb{E}_a[N(2 \times M)] = \mathbb{E}_a[N(2 \cdot M)]. \tag{18}
\]

This is consistent with the intuition that the concatenation of different texts (e.g., on different topics) leads to larger vocabulary than a single longer text. The calculations above remain true if the text is extended by a factor \(k\) (instead of 2), even for \(k \to \infty\).

The fluctuations around the mean show a more interesting behavior as revealed by repeating the computations above for the variance. We consider the case of \(k\) texts each of length \(M\), such that \(M' = k \times M\), and focus on the terms containing correlations between different words shown to be responsible for the anomalous fluctuation scaling (see Appendix C for details):

\[
\mathbb{V}_q[N(k \times M)] \sim \sum_{r,r'} \langle e^{-MF_r} e^{-MF_{r'}} \rangle^k - \langle e^{-MF_r} \rangle^k \langle e^{-MF_{r'}} \rangle^k. \tag{19}
\]

The individual terms can be written as

\[
\sum_{r,r'} \langle e^{-MF_r} e^{-MF_{r'}} \rangle^k = \langle \sum_r e^{-MF_r(j)} \rangle^2_{j_1,...,j_k}, \tag{20}
\]

\[
\sum_{r,r'} \langle e^{-MF_r} \rangle^k \langle e^{-MF_{r'}} \rangle^k = \langle \sum_r e^{-MF_r(j)} \rangle_{j_1,...,j_k}^2, \tag{21}
\]

in which \(\langle \cdot \rangle_{j_1,...,j_k}\) denotes the averaging over the realizations \((j_1,...,j_k)\) of frequencies \(F_r(j_i)\) in each single text \(i = 1,...k\) and \(\bar{F}_r^{(k)} = \frac{1}{k} \sum_{i=1}^k F_r^{(j_i)}\) is the \(k\)-sample average frequency based on the realizations \((j_1,...,j_k)\). In the limit \(k \to \infty\): \(\bar{F}_r^{(k)} \to \langle F_r \rangle\) such that

\[
\sum_{r,r'} \langle e^{-MF_r} e^{-MF_{r'}} \rangle^k - \langle e^{-MF_r} \rangle^k \langle e^{-MF_{r'}} \rangle^k \to 0. \tag{22}
\]
for \( k \to \infty \). This implies that for \( k \gg 1 \) (adding many different texts) the fluctuations in the vocabulary across documents (and therefore the correlations between different words) vanish and normal fluctuation scaling (\( \beta = 1/2 \)) is recovered. This prediction can be tested in data. Starting from a collection of documents, we have created a new collection by concatenating \( k \) randomly selected documents (each document is used once). We then computed for each concatenated document the number of distinct words \( N \) up to size \( M \), for increasing \( M \), and computed \( E[N(M)] \) and \( V[N(M)] \). We observe a transition of the exponent \( \beta \) in the fluctuation scaling, Eq. (3), from \( \beta \approx 1 \to \beta \approx 1/2 \).

### 4.2. Vocabulary Richness

When measuring vocabulary richness we want a measure which is robust to different text sizes. The traditional approach is to use Herdan’s \( C \), i.e. \( C = \log N / \log M \) \([25–27]\). While quite effective for rough estimations, this approach has several problems. One obvious one is that it does not incorporate any deviations from the original Heaps’ law (e.g., the double scaling regime \([19]\)). More seriously, it does not provide any estimation of the statistical significance or expected fluctuations of the measure. For instance, if two values are measured for different texts one can not determine whether one is significantly larger than the other. Our approach is to compare observations with the fluctuations expected from models in the spirit of Sec. 3.2.

The computation of statistical significance requires an estimation of the probability of finding \( N \) different words in a text of length \( M \), \( P(N|M) \), which can be obtained from a given generative model (e.g., the one presented in Sec. 3.2). For a text with \( N^*, M^* \) we compute the percentile \( P(N > N^*|M^*) \), which allows for a ranking of texts with different sizes such that the smaller the percentile, the richer the vocabulary. An estimation of the significance of the difference in the vocabulary can then be obtained by comparison of the different percentile.

For the sake of simplicity, we illustrate this general approach by approximating \( P(N|M) \) by a Gaussian distribution. In this case, the percentile are determined by the mean, \( \mu_M = E[N(M)] \), and the variance, \( \sigma_M = \sqrt{V[N(M)]} \), from an underlying null model in terms of the z-score

\[
    z_{N(M)} = \frac{N(M) - \mu_M}{\sigma_M},
\]

which shows how much the measured value \( N(M) \) deviates from the expected value \( \mu_M \) in units of standard deviations (\( z_{N(M)} \) follows a standard normal distribution: \( z \sim \mathcal{N}(0,1) \)). If we take into account our quantitative result on fluctuation scaling in the vocabulary in Eq. (6), i.e. \( \sigma_M \approx 0.1 \mu_M \), we can calculate the z-score of the observation \( N(M) \) as

\[
    z_{N(M)} \approx \frac{N(M) - \mu_M}{0.1 \mu_M} = 10 \left( \frac{N(M)}{\mu_M} - 1 \right),
\]

in which we need to include the expected vocabulary growth, \( \mu_M \), from a given generative model (e.g., Heaps’ law with two scalings \([19]\)). Thus, we obtain a measure from
which we can now answer the following questions concerning the vocabulary richness:

i) for a single observation \( N(M) \) the z-score, \( z_{N(M)} \), assigns a value of lexical richness taking into account any deviation from the pure Heaps’ law via \( \mu_M \); ii) given two observations \( N_1(M_1) \) and \( N_2(M_2) \), the respective z-scores \( z_{N_1(M_1)} \) and \( z_{N_2(M_2)} \) can be directly compared for assessing which text has a higher lexical richness independent of the difference in the textlengths; and iii) we estimate the statistical significance of the difference in vocabulary by considering \( \Delta z := z_{N_1(M_1)} - z_{N_2(M_2)} \) which is distributed according to \( \Delta z \sim N(0,2) \) since \( z \sim N(0,1) \). This implies that the difference in the vocabulary richness of two texts is statistically significant on a 95%-confidence level if \( |\Delta z| > 2.77 \), i.e. in this case there is at most a 5% chance that the observed difference originates from topic fluctuations. As a rule of thumb, for two texts of approximately the same length (\( N(M) \approx \mu_M \)), the relative difference in the vocabulary has to be larger than 27.7%.

We illustrate this approach for the vocabulary richness of Wikipedia articles. As a proxy for the true vocabulary richness, we measure how much the vocabulary of each article, \( N(M) \), exceeds the average vocabulary \( N_{\text{avg}}(M) \) with the same textlength \( M \) empirically determined from all articles in the Wikipedia. In Fig. 4 we compare the measures of vocabulary richness according to Herdan’s \( C \), Fig. 4(a), and the z-score, Fig. 4(b+c). For the latter, we use Eq. (24) and calculate \( \mu_M \) from Poisson word usage by fixing Zipf’s law and assuming Gamma-distributed word-frequencies, see Appendix D for details. We see in Fig. 4(a) that Herdan’s \( C \) shows a strong bias towards assigning high values of \( C \) to shorter texts: following a line with constant \( C \) we observe for \( M \gtrsim 10 \) articles with a vocabulary below average while for \( M > 1000 \) articles with a vocabulary above average. A similar (weaker) bias is observed in Fig. 4(b) for the calculation of the z-score for the case in which we consider deviations from the pure Heaps’ law but treat frequencies of individual words as fixed, i.e. ignoring topicality. The z-score calculations including topicality in Fig. 4(c) show that we obtain a measure of vocabulary richness which is approximately unbiased with respect to the textlength \( M \) (contour lines are roughly horizontal). Furthermore, in contrast to the two other measures, we correctly assign the highest z-score to the article with the highest ratio \( N(M)/N_{\text{avg}}(M) \). Altogether, this implies that it is not only important to take into account deviations from the pure Heaps’ law but that it is crucial to consider topicality in the form of a quenched average.

5. Discussion

In summary, we have used large text databases to investigate the scaling between vocabulary size \( N \) (number of different words) and database size \( M \). Besides the usual analysis of the average vocabulary size (Heaps’ law), we measured the standard deviation across different texts with the same length \( M \). We found that the relative fluctuations (standard deviation divided by the mean) do not decay with \( M \), in contrast to simple sampling processes. We explained this observation using a simple stochastic process
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Figure 4. Measures of vocabulary richness. For 5000 randomly selected articles from the Wikipedia database (black dots), we compute the ratio between the number of different words $N(M)$ and the average number of different words $N_{\text{avg}}(M)$ (empirically determined from all articles with the same textlength $M$). We compare the predictions of different measures of vocabulary richness (solid lines): (a) Herdan’s $C$ and (b+c) $z$-score, Eq. (24), in which we calculate the expected null model, $\mu_M$, according to Eq. (D.5) with parameters $\gamma = 1.77, \tilde{r} = 7830 \ [19]$, and $a \to \infty$ (in b) or $a = 0.08$ (in c). The solid lines are contours corresponding to values of $N(M)$ that yield the same measure of vocabulary richness varying from rich (red: $C = 0.98$ and $z = 4$) to poor (purple: $C = 0.8$ and $z = -4$) vocabulary. The article with the richest vocabulary according to each measure is marked by $\times$ (red), showing that there is a clear bias towards shorter texts in (a+b).

(Poisson usage of words) in which we account for topical aspects of written text, i.e. the frequency of an individual word is not treated as fixed among different documents. This heterogeneous dissemination of words across different texts leads to a reduction of the expected size of the vocabulary and to an increase in the variance. We have further shown the implications of these findings by proposing a practical measure of vocabulary richness which allows for a comparison of the vocabulary of texts with different lengths, including the quantification of statistical significance.

Our finding of anomalous fluctuation scaling implies that the vocabulary is a non-self-averaging quantity, meaning that the vocabulary of a single text is not representative of the whole ensemble. Here we have emphasized that topicality can be responsible for this effect. While the existence of different topics is obvious for a collection of articles as broad in content as the Wikipedia, our analysis shows that we can apply the same reasoning for the Google-ngram data, in which variation in the usage of words is measured at different points in time. This offers a new perspective on language change [44] in which the difference in the vocabulary between written texts from different years can be seen as a shift in the topical content over time. Similarly, other systematic fluctuations (e.g., across different authors or in the parameters of the Zipf’s law) can play a similar role as topicality.

Beyond linguistic applications, allometric scaling [4, 5] and other sublinear scalings
similar to Heaps’ law [28–33] have been observed in different complex systems. Our results show the importance of studying fluctuations around these scalings and provide a theoretical framework for the analysis.

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Appendix A. Data

The Wikipedia database consists of the plain text of all 3,743,306 articles from a snapshot of the complete English Wikipedia [35]. The PlosOne database consists of all 76,723 articles published in the journal PlosOne which were accessible via the API at the time of the data collection [36]. The Google-ngram database is a collection of printed books counting the number of times a word appears in a given year \( t \in [1520−2008] \) [23]. We treat the collection of all books published in the the same year as a single document, yielding 393 observations for different \( t \).

We apply the same filtering for each database: i) we decapitalize each word (e.g. “the” and “The” are counted as the same word) and ii) we restrict ourselves to words consisting uniquely of letters present in the alphabet of the English language. This is meant as a conservative approach in order to minimize the influence of foreign words, numbers (e.g. prices), or scanning problems which are present in the raw data (for details on the preprocessing see [19]).

Due to peculiarities of the individual databases the data (Data: \( \mu, \sigma \)) in Fig. 1, i.e. the calculation of the curves \( \mu_M \) and \( \sigma_M \) conditioned on the textlength \( M \), is constructed in a slightly different way in each case. In the Wikipedia data we order all datapoints \( N(M) \) (of the full article) according to textlength \( M \) and consider 1000 consecutive datapoints (in \( M \)) from which we calculate the average value of the textlength \( M \), and the conditional mean, \( \mu_M \), and variance, \( \sigma_M \), of the vocabulary \( N \). In the PlosOne data the length of all articles is much more concentrated, which is why we consider the full trajectory \( N(M) \) with \( M = 1, 2, ... M_{\text{max}} \) for each individual article. For an arbitrary value of \( M \) we calculate \( \mu_M \) and \( \sigma_M \) from the ensemble of all articles with vocabulary \( N \) at the particular textlength \( M \). In the Google-ngram data we impose a logarithmic binning in \( M \) such that we can calculate \( \mu_M \) and \( \sigma_M \) from a finite number of samples in each bin.

Appendix B. Calculation \( \mathbb{E}_q [N(M)^2] \)

\[
\mathbb{E}_q [N(M)^2] = \langle N(M)^{(i,j)} N(M)^{(i,j)} \rangle_{i,j} \tag{B.1}
\]

\[
= \langle \sum_{r,r'} I \left[ n_r^{(i)}(M, F_r^{(j)}) \right] I \left[ n_{r'}^{(j)}(M, F_{r'}^{(j)}) \right] \rangle_{i,j} \tag{B.2}
\]
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where we used \( I[x]^2 = I[x] \), Eq. (9), and that two Poisson process of different words \((r \neq r')\) with a given set of frequencies \(F_r^{(j)}\) are independent of each other.

Appendix C. Adding Texts

In this section we show the calculation for the quenched averages of the mean and the variance of the vocabulary growth when considering a text of length \(M'\) from the concatenation of \(k\) texts of length \(M_i\) with \(M' = \sum_{i=1}^{k} M_i\). We will first focus on the case \(k = 2\), i.e. \(M' = M_1 + M_2\), from which we can easily generalize to arbitrary \(k\).

We consider the vocabulary growth, \(N(M')\), as a random variable in which we concatenate two independent realizations of the stochastic process introduced in Sec. 3.2 indicated by subscript (1) and (2) respectively:

\[
N(M' = M_1 + M_2) = \sum_r \langle I \left[ n_r^{(1)}(M_1, F_r^{(1)}) + n_r^{(2)}(M_2, F_r^{(2)}) \right] \rangle
\]

\[
= \sum_r \langle I \left[ n_r^{(1)}(M_1, F_r^{(1)}) \right] \rangle + \langle I \left[ n_r^{(2)}(M_2, F_r^{(2)}) \right] \rangle
- \langle I \left[ n_r^{(1)}(M_1, F_r^{(1)}) \right] \rangle \langle I \left[ n_r^{(2)}(M_2, F_r^{(2)}) \right] \rangle
\]

in which the word \(r\) is counted as part of the vocabulary if it appears in either of the two concatenated realizations of the stochastic process. In the same spirit as in Sec. 3.2, taking expectation values requires averaging over all realizations of the Poisson process \((i_1, i_2)\) given the frequencies \(F_r^{(j_1)}, F_r^{(j_2)}\) as well as averaging over all realizations of those frequencies \((j_1, j_2)\), which we denote by \(\langle \cdot \rangle_{i_1,i_2,j_1,j_2}\). For the individual terms appearing in \(N(M' = M_1 + M_2)\) we get

\[
\langle I \left[ n_r^{(1)}(M_1, F_r^{(j_1)}) \right] \rangle_{i_1,i_2,j_1,j_2} = 1 - \langle e^{-M_1 F_r^{(j_1)}} \rangle_{j_1} \quad \text{(C.4)}
\]

\[
\langle I \left[ n_r^{(2)}(M_2, F_r^{(j_2)}) \right] \rangle_{i_1,i_2,j_1,j_2} = 1 - \langle e^{-M_2 F_r^{(j_2)}} \rangle_{j_2} \quad \text{(C.5)}
\]

\[
\langle I \left[ n_r^{(1)}(M_1, F_r^{(j_1)}) \right] \rangle_{i_1,i_2,j_1,j_2} = \left( 1 - \langle e^{-M_1 F_r^{(j_1)}} \rangle_{j_1} \right) \left( 1 - \langle e^{-M_2 F_r^{(j_2)}} \rangle_{j_2} \right) \quad \text{(C.6)}
\]

where we used \( I[x]^2 = I[x] \), Eq. (9), and that two Poisson process of different words \((r \neq r')\) with a given set of frequencies \(F_r^{(j)}\) are independent of each other.
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Among different texts \[38\]

Assuming a Gamma-distribution for the distribution of the frequency of a single word \[38\]

double power-law

Appendix D. Vocabulary Growth for Gamma-distributed frequency and a double power-law for the average rank-frequency distribution \[19\] with parameters \(\gamma\) of the distribution, given by

\[
\langle a \rangle = \frac{\Gamma(a+1)}{\Gamma(a)} \sum \frac{1}{n^{(a)}_r}\frac{1}{n^{(a)}_{r'}}.
\]

From this we can evaluate the mean and variance

\[
\mathbb{E}_q[N(M' = M_1 + M_2)] = \sum_r 1 - \langle e^{-M_1 F_r} \rangle \langle e^{-M_2 F_r} \rangle \quad (C.7)
\]

\[
\mathbb{V}_q[N(M' = M_1 + M_2)] = \sum_r \langle e^{-M_1 F_r} \rangle \langle e^{-M_2 F_r} \rangle - \langle e^{-2M_1 F_r} \rangle \langle e^{-2M_2 F_r} \rangle \quad (C.8)
\]

\[
\sum_{r,r'} \langle e^{-M_1 F_r} e^{-M_1 F_{r'}} \rangle \langle e^{-M_2 F_r} e^{-M_2 F_{r'}} \rangle - \langle e^{-M_1 F_r} \rangle \langle e^{-M_1 F_{r'}} \rangle \langle e^{-M_2 F_r} \rangle \langle e^{-M_2 F_{r'}} \rangle.
\]

Generalizing to the concatenation of an arbitrary number of \(k\) texts can be treated in the very same way, however, we will only state the result for the case of adding \(k\) texts of equal length \(M\) such that \(M' = k \times M\):

\[
\mathbb{E}_q[N(M' = k \times M)] = \sum_r 1 - \langle e^{-MF_r} \rangle^k \quad (C.9)
\]

\[
\mathbb{V}_q[N(M' = k \times M)] = \sum_r \langle e^{-MF_r} \rangle^k - \langle e^{-2MF_r} \rangle^k \quad (C.10)
\]

\[
+ \sum_{r,r'} \langle e^{-MF_r} e^{-MF_{r'}} \rangle^k - \langle e^{-MF_r} \rangle^k \langle e^{-MF_{r'}} \rangle^k.
\]

Appendix D. Vocabulary Growth for Gamma-distributed frequency and a double power-law

Assuming a Gamma-distribution for the distribution of the frequency of a single word among different texts \[38\]

\[
P_\Gamma(F_r = x; a, b) = \frac{1}{\Gamma(a)} b^{-a} x^{a-1} e^{-x/b} \quad (D.1)
\]

we can calculate the quenched average

\[
\langle e^{-MF_r} \rangle = \int dx P_\Gamma(F_r = x; a, b)e^{-Mx} = (1 + bM)^{-a}. \quad (D.2)
\]

If we assume that the distribution of frequencies for all words is given by the same shape-parameter \(a\) (e.g. \(a = 1\) corresponds to an exponential distribution) and fix the mean of the distribution, given by \(\langle F_r \rangle = ab\) we get \(\langle e^{-MF_r} \rangle = (1 + M\langle F_r \rangle/a)^{-a}\). Assuming a double power-law for the average rank-frequency distribution \[19\] with parameters \(\gamma\) and \(\bar{r}\), i.e. \(\langle F_r \rangle = Cr^{-\gamma}\) for \(r \leq \bar{r}\) and \(\langle F_r \rangle = C\bar{r}^{\gamma-1}r^{-\gamma}\) for \(r > \bar{r}\), where \(C = C(\bar{r}, \gamma)\) is the normalization constant determined by imposing \(\sum_r \langle F_r \rangle = 1\), we can calculate the
vocabulary growth according to Eq. (4) analytically in the continuum approximation by substituting $x := \langle F_r \rangle$: 

\[
\mathbb{E}_q[N(M)] = \sum_r 1 - (1 + M \langle F_r \rangle / a)^{-a} 
\]

\[
= - \int_0^1 dx \frac{dr}{dx} [1 - (1 + M x/a)^{-a}] 
\]  

(D.3)

which can be expressed in terms of the ordinary hypergeometric function $H :=_2 F_1$ [45] yielding

\[
\mathbb{E}_q[N(M)] = \tilde{r} - C + \tilde{r} \left[ H(a, -\frac{1}{\gamma}, 1 - \frac{1}{\gamma}, - \frac{CM}{a\tilde{r}}) - 1 \right] 
\]

\[
- C \left( 1 + \frac{M}{a} \right)^{-a} \left[ \frac{\Gamma(1 + a)}{\Gamma(2 + a)} H(1, 1, 2 + a, - \frac{a}{M}) - 1 \right] 
\]

\[
+ \tilde{r} \left( 1 + \frac{CM}{a\tilde{r}} \right)^{-a} \left[ \frac{\Gamma(1 + a)}{\Gamma(2 + a)} H(1, 1, 2 + a, - \frac{a\tilde{r}}{CM}) - 1 \right] \]  

(D.5)

where the vocabulary growth $\mathbb{E}_q[N(M)]$ is parametrized by $\gamma, \tilde{r}$, and $a$.

In the limit $a \to \infty$ the Gamma distribution $P_r(F_r = x; a, b)$ with given mean $\langle F_r \rangle = ab = \text{const.}$ converges to a Gaussian with $\sigma^2 = \langle F_r \rangle^2 / a \to 0$. For $a \to \infty$, $\sigma^2 \to \infty$ and we recover the Poisson null model, Eqs. (4,5), in which the individual frequencies $F_r$ are fixed (annealed average).

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