CHARACTERIZATIONS OF $H^1_{\Delta_N}(\mathbb{R}^n)$ AND $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ VIA WEAK FACTORIZATIONS AND COMMUTATORS

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Abstract. This paper provides a deeper study of the Hardy and BMO spaces associated to the Neumann Laplacian $\Delta_N$. For the Hardy space $H^1_{\Delta_N}(\mathbb{R}^n)$ (which is a proper subspace of the classical Hardy space $H^1(\mathbb{R}^n)$) we demonstrate that the space has equivalent norms in terms of Riesz transforms, maximal functions, atomic decompositions, and weak factorizations. While for the space $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ (which contains the classical BMO($\mathbb{R}^n$)) we prove that it can be characterized in terms of the action of the Riesz transforms associated to the Neumann Laplacian on $L^\infty(\mathbb{R}^n)$ functions and in terms of the behavior of the commutator with the Riesz transforms. The results obtained extend many of the fundamental results known for $H^1(\mathbb{R}^n)$ and BMO($\mathbb{R}^n$).

1. Introduction and Statement of Main Results

The spaces $H^1(\mathbb{R}^n)$ and BMO($\mathbb{R}^n$) are fundamental function spaces in harmonic analysis. The work of Fefferman and Stein, [9], provides a duality relationship between $H^1(\mathbb{R}^n)$ and BMO($\mathbb{R}^n$). And, further provides characterizations of these spaces in terms of maximal functions, square functions, and Riesz transforms. While the work of Coifman, Rochberg and Weiss, [4], provides a connection between weak factorization of the Hardy spaces, commutators with Riesz transforms and BMO($\mathbb{R}^n$). The main goals of this paper are to provide similar connections for $H^1$ and BMO spaces adapted to a particular linear differential operator.

There is a substantial literature related to $H^1$ and BMO spaces adapted to a linear operator $L$ on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup $e^{-tL}$ on $L^2(\mathbb{R}^n)$ with a kernel $p_t(x,y)$ satisfying an upper bound. That is, operators $L$ for which the kernel of the semigroup $p_t(x,y)$ there exists positive constants $m$ and $\epsilon$ such that for all $x, y \in \mathbb{R}^n$ and for all $t > 0$:

$$ |p_t(x,y)| \leq \frac{Ct^{\frac{m}{2}}}{(t^{\frac{1}{m}} + |x - y|)^{n+\epsilon}}. $$

JL’s research supported by ARC DP 160100153. BDW’s research supported in part by National Science Foundation DMS # 0955432 and #1560955.

2010 Mathematics Subject Classification: Primary: 42B35, 42B25.

Key words: Hardy space, Neumann Laplacian, semigroup, Gaussian bounds, commutators, Riesz transforms, Littlewood–Paley, area function, radial maximal function associated with operators.
In [1], Auscher, Duong, and McIntosh defined a Hardy space $H^1_L(\mathbb{R}^n)$ associated with such operators $L$ as the class of all functions $f \in L^1(\mathbb{R}^n)$ for which $S_L(f) \in L^1(\mathbb{R}^n)$, where $S_L(f)$ is Littlewood–Paley area function defined as follows.

\begin{equation}
S_L(f)(x) = \left( \int_0^\infty \int_{|y-x|<t} |Q_t f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},
\end{equation}

with $Q_t = tLe^{-tL}$. The $H^1_L(\mathbb{R}^n)$ norm of $f$ is defined as $||f||_{H^1_L(\mathbb{R}^n)} = ||S_L(f)||_{L^1(\mathbb{R}^n)}$.

In [7, 8], Duong and Yan defined the function space $\text{BMO}_L(\mathbb{R}^n)$ associated with an operator $L$. They then go on to prove that if $L$ has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$ and the kernel $p_t(x,y)$ of the semigroup $e^{-tL}$ satisfies the upper bound (1.1), then the space $\text{BMO}_L(\mathbb{R}^n)$ is the dual space of Hardy space $H^1_L(\mathbb{R}^n)$ in which $L^*$ denotes the adjoint operator of $L$. This gives a generalization of the duality of $H^1(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$ of Fefferman and Stein [9]. Later, the theory of function spaces associated with operators has been developed and generalized to many other different settings, see for example [2, 6, 11–13].

The choice of $L = \Delta$ gives rise to the spaces to the classical spaces $H^1(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$. While the choice of the semigroup $e^{-tL}$ is the Poisson semigroup $e^{-t\sqrt{\Delta}}$ (here $m = 2$), given by

\begin{equation}
e^{-t\sqrt{\Delta}}f(x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)dy, \quad t > 0, \quad \text{where} \quad p_t(x) = \frac{c_n t}{(t^2 + |x|^2)^{\frac{n+1}{2}}},\end{equation}

yields the spaces $H^1_{\sqrt{\Delta}}(\mathbb{R})$ and $\text{BMO}_{\sqrt{\Delta}}(\mathbb{R})$ coincide with the classical Hardy space and $\text{BMO}$ space, respectively (see [1] and [7]).

In [5], Deng, Duong, Sikora, and Yan further considered the comparison of $\text{BMO}_L(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$. By considering the Neumann Laplacian $L = \Delta_N$, they obtained that

$$\text{BMO}(\mathbb{R}^n) \subsetneq \text{BMO}_{\Delta_N}(\mathbb{R}^n).$$

Recently, in [19] Yan introduced a class of $H^p_L(\mathbb{R}^n)$ for a range of $p \leq 1$ by using the Littlewood–Paley area function $S_L(f)$. In particular, Yan showed that

$$H^p_{\Delta_N}(\mathbb{R}^n) \subsetneq H^p(\mathbb{R}^n), \quad \frac{n}{n+1} < p \leq 1.$$

The main goal of this paper is to carry out a deeper study of the spaces $H^1_{\Delta_N}(\mathbb{R}^n)$ and $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$. Interestingly, we show that these spaces behave in an analogous fashion as the standard Hardy space $H^1(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$.

We first explicitly compute the Riesz transforms $R_N = \nabla \Delta_N^{-\frac{1}{2}}$ associated to the Neumann Laplacian. Because of the close connection between the Laplacian and Neumann Laplacian, we find in Proposition 2.2 that the Riesz transforms associated to the Neumann Laplacian are given by an additive perturbation of the standard Riesz transforms.
Our first main result shows that, similar to the classical Hardy space, the space $H^1_{\Delta_N}(\mathbb{R}^n)$ can be characterized by the radial and non-tangential maximal functions, by the Riesz transforms, and by atoms, all of which are defined in terms of the Neumann Laplacian $\Delta_N$. To be more precise, we denote by $H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)$ the Hardy space defined via the radial maximal function associated with $\Delta_N$, and analogously by $H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n)$ and $H^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)$ the Hardy spaces via non-tangential maximal functions, Riesz transforms and atoms, respectively. Then we have the following characterizations.

**Theorem 1.1.** Let all notation be the same as above. We have

$$H^1_{\Delta_N}(\mathbb{R}^n) = H^1_{\Delta_N,\text{max}}(\mathbb{R}^n) = H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n) = H^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)$$

and with equivalent norms

$$\|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \approx \|f\|_{H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)} \approx \|f\|_{H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n)} \approx \|f\|_{H^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)}$$

Here $f_{\pm,e}$ is the even extension of the restriction of $f$ from $\mathbb{R}^n_{\pm}$. Namely, $f \in H^1_{\Delta_N}(\mathbb{R}^n)$ if and only if $f_{+,e} \in H^1(\mathbb{R}^n)$ and $f_{-,e} \in H^1(\mathbb{R}^n)$.

For more details on these Hardy spaces and the norms we refer to Section 3. We also obtain a Fefferman–Stein decomposition of $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ in terms of the action of the Riesz transforms associated to the Neumann Laplacian on $L^\infty(\mathbb{R}^n)$ functions.

**Corollary 1.2.** The following are equivalent for a function $b$:

(i) $b \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$;
(ii) There exists $b_0, b_1, \ldots, b_n \in L^\infty(\mathbb{R}^n)$ such that $b = b_0 + \sum_{j=1}^n R_{N,j}^* b_j$, where $R_{N,j}^*$ is the adjoint operator of $R_{N,j}$.

We then further show the connection between $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$, $H^1_{\Delta_N}(\mathbb{R}^n)$, commutators of functions in $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ and Riesz transforms $R_N$ relative to $\Delta_N$, and a weak factorization of the space $H^1_{\Delta_N}(\mathbb{R}^n)$. In particular, our second main result is the following theorem.

**Theorem 1.3.** For $1 \leq l \leq n$, let $\Pi_l(h, g) = h \cdot R_{N,l}(g) - g \cdot R_{N,l}(h)$, where $R_{N,l} = \frac{\partial}{\partial x_l} \Delta_N^{-\frac{1}{2}}$ is the $l$-th Riesz transform associated to the Neumann Laplacian and $R_{N,l}^*$ is the adjoint operator of $R_{N,l}$. Then for any $f \in H^1_{\Delta_N}(\mathbb{R}^n)$ there exists sequences $\{\lambda_j^k\} \in \ell^1$ and functions $g_j^k, h_j^k \in L^\infty(\mathbb{R}^n)$ with compact supports such that $f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_l(g_j^k, h_j^k)$. Moreover, we have that:

$$\|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \approx \inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty |\lambda_j^k| \|g_j^k\|_{L^2(\mathbb{R}^n)} \|h_j^k\|_{L^2(\mathbb{R}^n)} : f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_l(g_j^k, h_j^k) \right\}.$$
We then obtain the following new characterization of \( \text{BMO}_{\Delta N}(\mathbb{R}^n) \) in terms of the commutators with the Riesz transforms associated to \( \Delta_N \).

**Theorem 1.4.** Suppose \( b \in \cup_{p \geq 1} L^p_{\text{loc}}(\mathbb{R}^n) \).

If \( b \) is in \( \text{BMO}_{\Delta N}(\mathbb{R}^n) \), then for \( 1 \leq l \leq n \), the commutator

\[
[b, R_{N,l}](f)(x) = b(x)R_{N,l}(f)(x) - R_{N,l}(bf)(x)
\]

is a bounded map on \( L^2(\mathbb{R}^n) \), with operator norm

\[
\|[b, R_{N,l}] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\| \leq C\|b\|_{\text{BMO}_{\Delta N}(\mathbb{R}^n)}.
\]

Conversely, for \( 1 \leq l \leq n \), if \([b, R_{N,l}]\) are bounded on \( L^2(\mathbb{R}^n) \) then \( b \) is in \( \text{BMO}_{\Delta N}(\mathbb{R}^n) \) and \( \|b\|_{\text{BMO}_{\Delta N}(\mathbb{R}^n)} \leq C\|[b, R_{N,l}] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\|\).

We point out that Theorem 1.3 and Theorem 1.4 can be extended to work for \( L^p(\mathbb{R}^n) \) when \( 1 < p < \infty \).

For \( 0 < \alpha < n \), the fractional operator \( \Delta_{\alpha/2}^N \) of the operator \( \Delta_N \) is defined by

\[
\Delta_{\alpha/2}^N f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t\Delta_N}(f)(x)\frac{dt}{t^{1-\alpha/2}}.
\]

**Theorem 1.5.** If \( b \) is in \( \text{BMO}_{\Delta N}(\mathbb{R}^n) \), then for \( 1 < \alpha < n \), the commutator

\[
[b, \Delta_{\alpha/2}^N](f)(x) = b(x)\Delta_{\alpha/2}^N(f)(x) - \Delta_{\alpha/2}^N(bf)(x)
\]

is a bounded map from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) with operator norm

\[
\|[b, \Delta_{\alpha/2}^N] : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)\| \leq C\|b\|_{\text{BMO}_{\Delta N}(\mathbb{R}^n)},
\]

where \( 1 < p < \frac{n}{\alpha} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \).

The paper is organized as follows. In Section 2, we collect the background for the Neumann Laplacian and the associated Riesz transforms. In Section 3 the related Hardy and BMO spaces associated to \( \Delta_N \) are studied and their basic properties are collected. In particular, we demonstrate a collection of equivalent norms for \( H^1_{\Delta N}(\mathbb{R}^n) \), Theorem 3.12, and show the Fefferman-Stein decomposition of \( \text{BMO}_{\Delta N}(\mathbb{R}^n) \) holds, Corollary 1.2. Finally, in Section 4 we provide the proof of Theorems 1.3 and 1.4. Throughout this paper, the letter “\( C \)” will denote, possibly different, constants that are independent of the essential variables.

## 2. The Neumann Laplacian and the Associated Riesz Kernels

We now recall some notation and basic facts introduced in [5, Section 2]. For any subset \( A \subset \mathbb{R}^n \) and a function \( f : \mathbb{R}^n \to \mathbb{C} \) by \( f|_A \) we denote the restriction of \( f \) to \( A \).
Next we set \( \mathbb{R}_n^n = \{(x', x_n) \in \mathbb{R}^n : x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0 \} \). For any function \( f \) on \( \mathbb{R}^n \), we set

\[
 f_+ = f|_{\mathbb{R}_n^n} \quad \text{and} \quad f_- = f|_{\mathbb{R}^n}.
\]

For any \( x = (x', x_n) \in \mathbb{R}^n \) we set \( \tilde{x} = (x', -x_n) \). If \( f \) is any function defined on \( \mathbb{R}_n^n \), its even extension defined on \( \mathbb{R}^n \) is

\[
(2.1) \quad f_e(x) = f(x), \quad \text{if} \ x \in \mathbb{R}_n^n; \quad f_e(x) = f(\tilde{x}), \quad \text{if} \ x \in \mathbb{R}_-^n.
\]

2.1. The Neumann Laplacian. We denote by \( \Delta_n \) the Laplacian on \( \mathbb{R}^n \). Next we recall the Neumann Laplacian on \( \mathbb{R}_+^n \) and \( \mathbb{R}_-^n \).

Consider the Neumann problem on the half line \((0, \infty)\) (see [15, (7), page 59 in Section 3.1]):

\[
(2.2) \begin{cases}
 w_t - w_{xx} = 0 & \text{for} \ 0 < x < \infty, 0 < t < \infty, \\
 w(x, 0) = \phi(x), \\
 w_x(0, t) = 0.
\end{cases}
\]

Denote this corresponding Laplacian by \( \Delta_{1,N_+} \). According to [15, (7), Section 3.1], we see that

\[
w(x, t) = e^{-t \Delta_{1,N_+}}(\phi)(x).
\]

For \( n > 1 \), we write \( \mathbb{R}_+^n = \mathbb{R}_-^{n-1} \times \mathbb{R}_+ \). And we define the Neumann Laplacian on \( \mathbb{R}_+^n \) by

\[
\Delta_{n,N_+} = \Delta_{n-1} + \Delta_{1,N_+},
\]

where \( \Delta_{n-1} \) is the Laplacian on \( \mathbb{R}_-^{n-1} \) and \( \Delta_{1,N_+} \) is the Laplacian corresponding to (2.2). Similarly we can define Neumann Laplacian \( \Delta_{n,N_-} \) on \( \mathbb{R}_-^n \).

In the remainder of the paper, we skip the index \( n \), we denote by \( \Delta \) the Laplacian on \( \mathbb{R}^n \), denote the Neumann Laplacian on \( \mathbb{R}_+^n \) by \( \Delta_{N_+} \), and Neumann Laplacian on \( \mathbb{R}_-^n \) by \( \Delta_{N_-} \).

The Laplacian and Neumann Laplacian \( \Delta_{N_\pm} \) are positive definite self-adjoint operators. By the spectral theorem one can define the semigroups generated by these operators \( \{\exp(-t\Delta), t \geq 0\} \) and \( \{\exp(-t\Delta_{N_\pm}), t \geq 0\} \). By \( p_t(x, y) \), \( p_t\Delta_{N_+}(x, y) \) and \( p_t\Delta_{N_-}(x, y) \) we denote the heat kernels corresponding to the semigroups generated by \( \Delta \), \( \Delta_{N_+} \) and \( \Delta_{N_-} \), respectively. Then we have

\[
p_t(x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}.
\]

From the reflection method (see [15, (9), page 60 in Section 3.1]), we get

\[
p_t\Delta_{N_+}(x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}} \left( e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}} \right), \quad x, y \in \mathbb{R}_+^n;
\]

\[
p_t\Delta_{N_-}(x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}} \left( e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}} \right), \quad x, y \in \mathbb{R}_-^n.
\]
For any function $f$ on $\mathbb{R}^n_+$, we have
\[
\exp(-t\Delta_{N_+})f(x) = \exp(-t\Delta)f(x)
\]
for all $t \geq 0$ and $x \in \mathbb{R}^n_+$. Similarly, for any function $f$ on $\mathbb{R}^n_-$,
\[
\exp(-t\Delta_{N_-})f(x) = \exp(-t\Delta)f(x)
\]
for all $t \geq 0$ and $x \in \mathbb{R}^n_-$. 

Now let $\Delta_N$ be the uniquely determined unbounded operator acting on $L^2(\mathbb{R}^n)$ such that
\[
(\Delta_N f)_+ = \Delta_{N_+} f_+ \quad \text{and} \quad (\Delta_N f)_- = \Delta_{N_-} f_-
\]
for all $f : \mathbb{R}^n \to \mathbb{R}$ such that $f_+ \in W^{1,2}(\mathbb{R}^n_+)$ and $f_- \in W^{1,2}(\mathbb{R}^n_-)$. Then $\Delta_N$ is a positive self-adjoint operator and
\[
(\exp(-t\Delta_N)f)_+ = \exp(-t\Delta_{N_+})f_+ \quad \text{and} \quad (\exp(-t\Delta_N)f)_- = \exp(-t\Delta_{N_-})f_-.
\]
The heat kernel of $\exp(-t\Delta_N)$, denoted by $p_{t,\Delta_N}(x,y)$, is then given as:
\[
p_{t,\Delta_N}(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \left( e^{-\frac{|x_n-y_n|^2}{4t}} + e^{-\frac{|x_n+y_n|^2}{4t}} \right) H(x_ny_n),
\]
where $H : \mathbb{R} \to \{0,1\}$ is the Heaviside function given by
\[
H(t) = 0, \quad \text{if } t < 0; \quad H(t) = 1, \quad \text{if } t \geq 0.
\]

Let us note that

(α) All the operators $\Delta, \Delta_{N_+}, \Delta_{N_-},$ and $\Delta_N$ are self-adjoint and they generate bounded analytic positive semigroups acting on all $L^p(\mathbb{R}^n)$ spaces for $1 \leq p \leq \infty$;

(β) Suppose that $p_{t,L}(x,y)$ is the kernel corresponding to the semigroup generated by one of the operators $L$ listed in (α). Then the kernel $p_{t,\Delta_N}(x,y)$ satisfies Gaussian bounds:
\[
|p_{t,L}(x,y)| \leq C \frac{e^{-c|x-y|^2}}{t^{n/2}},
\]
for all $x,y \in \Omega$, where $\Omega = \mathbb{R}^n$ for $\Delta, \Delta_N$; $\Omega = \mathbb{R}^n_+$ for $\Delta_{N_+}$ and $\Omega = \mathbb{R}^n_-$ for $\Delta_{N_-}$.

Next we consider the smoothness property of the heat kernel for $\Delta_N, \Delta_{N_+},$ and $\Delta_{N_-}$.

**Proposition 2.1.** Suppose that $L$ is one of the operators $\Delta_{N_+}, \Delta_{N_-}$ and $\Delta_N$. Then for $x,x',y \in \mathbb{R}^n_+$ (or $\mathbb{R}^n_-$) with $|x - x'| \leq \frac{1}{2} |x - y|$, we have
\[
|p_{t,L}(x,y) - p_{t,L}(x',y)| \leq C \frac{|x - x'|}{\sqrt{t}} \frac{\sqrt{t}}{(\sqrt{t} + |x - y|) (\sqrt{t} + |x - y|)^{n+1}};
\]
symmetrically, for $x,y,y' \in \mathbb{R}^n_+$ (or $\mathbb{R}^n_-)$ with $|y - y'| \leq \frac{1}{2} |x - y|$, we have
\[
|p_{t,L}(x,y) - p_{t,L}(x,y')| \leq C \frac{|y - y'|}{\sqrt{t}} \frac{\sqrt{t}}{(\sqrt{t} + |x - y|) (\sqrt{t} + |x - y|)^{n+1}}.
\]
Proof. Suppose $x, y \in \mathbb{R}^n_+$. Then for $i = 1, \ldots, n - 1$, we have
\[
\frac{\partial}{\partial x_i} p_{t, \Delta N_+}(x, y) = -\frac{(x_i - y_i)}{2t} \left( \frac{1}{4\pi t} \right)^\frac{1}{2} e^{-\frac{|x' - y'|^2}{4t}} \left( e^{-\frac{|x_n - y_n|^2}{4t}} + e^{-\frac{|x_n + y_n|^2}{4t}} \right).
\]
Moreover,
\[
\frac{\partial}{\partial x_n} p_{t, \Delta N_+}(x, y) = -\frac{1}{(4\pi t)^\frac{1}{2}} e^{-\frac{|x' - y'|^2}{4t}} \left( e^{-\frac{|x_n - y_n|^2}{4t}} \frac{(x_n - y_n)}{2t} + e^{-\frac{|x_n + y_n|^2}{4t}} \frac{(x_n + y_n)}{2t} \right).
\]
Then we obtain that
\[
\left| \nabla_x p_{t, \Delta N_+}(x, y) \right|^2 = \sum_{i=1}^{n-1} \left| \frac{\partial}{\partial x_i} p_{t, \Delta N_+}(x, y) \right|^2 + \left| \frac{\partial}{\partial x_n} p_{t, \Delta N_+}(x, y) \right|^2
\leq \sum_{i=1}^{n-1} \frac{(x_i - y_i)^2}{4t^2} \left( \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} \left( e^{-\frac{|x_n - y_n|^2}{4t}} \frac{(x_n - y_n)}{2t} + e^{-\frac{|x_n + y_n|^2}{4t}} \frac{(x_n + y_n)}{2t} \right) \right)
+ 2 \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} \left( e^{-\frac{|x_n - y_n|^2}{4t}} \frac{(x_n - y_n)}{2t} \right)
+ 2 \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} \left( e^{-\frac{|x_n + y_n|^2}{4t}} \frac{(x_n + y_n)}{2t} \right)
\leq C \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{t^2} \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} + 2 \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} \left( e^{-\frac{|x_n - y_n|^2}{4t}} \frac{(x_n + y_n)}{2t} \right)^2
\leq C \frac{|x - y|^2}{t^2} \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} + 2 \frac{1}{(4\pi t)^n} e^{-\frac{|x' - y'|^2}{2t}} \left( e^{-\frac{|x_n - y_n|^2}{4t}} \frac{(x_n + y_n)}{2t} \right)^2
\leq C \frac{t}{(t + |x - y|^2)^{n+2}}.
\]
Hence, it is easy to verify that
\[
\left| \nabla_x p_{t, \Delta N_+}(x, y) \right| \leq C \frac{\sqrt{t}}{(\sqrt{t} + |x - y|)^{n+2}}
\]
and similarly we can obtain that
\[
\left| \nabla_y p_{t, \Delta N_+}(x, y) \right| \leq C \frac{\sqrt{t}}{(\sqrt{t} + |x - y|)^{n+2}},
\]
which implies that
\[
|p_{t, \Delta N_+}(x, y) - p_{t, \Delta N_+}(x', y)| \leq C \frac{|x - x'|}{(\sqrt{t} + |x - y|) (\sqrt{t} + |x - y|)^{n+1}} \sqrt{t}
\]
for $x, x', y \in \mathbb{R}^n_+$ with $|x - x'| \leq \frac{1}{2} |x - y|$, and
\[
|p_{t, \Delta N_+}(x, y) - p_{t, \Delta N_+}(x, y')| \leq C \frac{|y - y'|}{(\sqrt{t} + |x - y|) (\sqrt{t} + |x - y|)^{n+1}} \sqrt{t}
\]
for $x, x', y \in \mathbb{R}^n_+$ with $|y - y'| \leq \frac{1}{2} |x - y|$. We can obtain similar estimates for the heat semigroup of $\Delta_{N_+}$ and $\Delta_N$. □
2.2. The Riesz Kernels Associated to the Neumann Laplacian. A fundamental object in our study are the Riesz transforms associated to the Neumann Laplacian. Recall that the Riesz transforms associated to the Neumann Laplacian are given by: \( R_N = \nabla \Delta_N^{-\frac{1}{2}} \). We collect the formula for these kernels in the following proposition.

**Proposition 2.2.** Denote by \( R_{N,j}(x, y) \) the kernel of the \( j \)-th Riesz transform \( \frac{\partial}{\partial x_j} \Delta_N^{-\frac{1}{2}} \) of \( \Delta_N \). Then for \( 1 \leq j \leq n-1 \) and for \( x, y \in \mathbb{R}^n_+ \) we have:

\[
R_{N,j}(x, y) = -C_n \left( \frac{x_j - y_j}{|x - y|^{n+1}} + \frac{x_j - y_j}{(|x' - y'|^2 + |x_n + y_n|^2)^{\frac{n+1}{2}}} \right)
\]

and

\[
R_{N,n}(x, y) = -C_n \left( \frac{x_n - y_n}{|x - y|^{n+1}} + \frac{x_n + y_n}{(|x' - y'|^2 + |x_n + y_n|^2)^{\frac{n+1}{2}}} \right),
\]

where \( C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{(\pi)^{\frac{n+1}{2}}} \).

Similar expressions also hold for \( R_{N,j}(x, y), j = 1, \ldots, n \), when \( x, y \in \mathbb{R}^n_- \).

**Proof.** Working from the definition of the square root of \( \Delta_N \), i.e.,

\[
\Delta_N^{-\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty e^{-t \Delta_N} \frac{dt}{\sqrt{t}},
\]

we have that for \( 1 \leq j \leq n-1 \):

\[
R_{N,j}(x, y) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{\partial}{\partial x_j} \int_0^\infty p_{t, \Delta_N}(x, y) \frac{dt}{\sqrt{t}}
= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{\partial}{\partial x_j} \left( \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{\sqrt{t}} + \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2+|x_n+y_n|^2}{4t}} \frac{dt}{\sqrt{t}} \right)
= -\frac{\Gamma\left(\frac{n+1}{2}\right)}{(\pi)^{\frac{n+1}{2}}} \left( \frac{x_j - y_j}{|x - y|^{n+1}} + \frac{x_j - y_j}{(|x' - y'|^2 + |x_n + y_n|^2)^{\frac{n+1}{2}}} \right).
\]

For \( j = n \) and for \( x, y \in \mathbb{R}^n_+ \) we again observe:

\[
R_{N,n}(x, y) = \frac{\sqrt{\pi}}{2} \frac{\partial}{\partial x_n} \int_0^\infty p_{t, \Delta_N}(x, y) \frac{dt}{\sqrt{t}}
= -\frac{\Gamma\left(\frac{n+1}{2}\right)}{(\pi)^{\frac{n+1}{2}}} \left( \frac{x_n - y_n}{|x - y|^{n+1}} + \frac{x_n + y_n}{(|x' - y'|^2 + |x_n + y_n|^2)^{\frac{n+1}{2}}} \right).
\]

□

We next make the observation that kernels \( R_{N,j}(x, y) \) are Calderón–Zygmund kernels.

**Proposition 2.3.** Denote by \( R_N(x, y) \) the kernel of the vector of Riesz transforms \( \nabla \Delta_N^{-\frac{1}{2}} \). Then:

\[
R_N(x, y) = \left( R_{N,1}(x, y), \ldots, R_{N,n}(x, y) \right) H(x_n y_n),
\]

(2.9)
with $H(t)$ the Heaviside function defined in (2.5). Moreover, we have that
\[ |R_N(x, y)| \leq C_n \frac{1}{|x - y|^n}, \]
and
\[ |R_N(x, y) - R_N(x_0, y)| + |R_N(y, x) - R_N(y, x_0)| \leq C \frac{|x - x_0|}{|x - y|^{n+1}} \]
for $x, x_0, y \in \mathbb{R}_+^n$ (or $x, x_0, y \in \mathbb{R}_-^n$) with $|x - x_0| \leq \frac{1}{2}|x - y|$.

**Proof.** We first claim that for $j = 1, \ldots, n$, and $x, y \in \mathbb{R}_+^n$ (or $x, y \in \mathbb{R}_-^n$)
\[ |R_{N,j}(x, y)| \leq C_n \frac{1}{|x - y|^n}. \]
In fact, from Proposition 2.2, it is direct that for $1 \leq j \leq n - 1$,
\[ \frac{|x_j - y_j|}{(|x'|^2 + |x_n + y_n|^2)^{\frac{n+1}{2}}} \leq \frac{|x_j - y_j|}{(|x'|^2 + |x_n - y_n|^2)^{\frac{n+1}{2}}} \leq \frac{1}{|x - y|^n} \]
and for $j = n$,
\[ \frac{|x_n + y_n|}{(|x'|^2 + |x_n + y_n|^2)^{\frac{n+1}{2}}} \leq \frac{1}{(|x'|^2 + |x_n - y_n|^2)^{\frac{n}{2}}} \leq \frac{1}{|x - y|^n} \]
where we use the fact that $x, x_0, y \in \mathbb{R}_+^n$ (or $x, x_0, y \in \mathbb{R}_-^n$) and hence $x_j + y_j > |x_j - y_j|$ for $1 \leq j \leq n$.

Similarly, by considering the estimates for the terms $\frac{\partial}{\partial x_j} R_{N,j}(x, y)$ and $\frac{\partial}{\partial y_j} R_{N,j}(x, y)$, we obtain that
\[ |R_{N,j}(x, y) - R_{N,j}(x_0, y)| + |R_{N,j}(y, x) - R_{N,j}(y, x_0)| \leq C \frac{|x - x_0|}{|x - y|^{n+1}} \]
for $x, x_0, y \in \mathbb{R}_+^n$ (or $x, x_0, y \in \mathbb{R}_-^n$). with $|x - x_0| \leq \frac{1}{2}|x - y|$.

2.3. The Kernels of Fractional operators Associated to the Neumann Laplacian. For $0 < \alpha < n$, denote by $K(x, y)$ the kernel of the classical fractional operator $\Delta^{-\alpha/2}$, which is defined by
\[ \Delta^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t\Delta}(f)(x) \frac{dt}{t^{1-\alpha/2}}. \]
We know that
\[ K(x, y) = \frac{C_{n,\alpha}}{|x - y|^{n-\alpha}}, \]
where $C_{n,\alpha} = \frac{\Gamma(\frac{n}{2} - \frac{\alpha}{2})}{\frac{n}{2} \Gamma(\frac{n}{2})} \frac{1}{\pi^{2\alpha}}$. It is well known that when $b \in \text{BMO}(\mathbb{R}^n)$, the commutator $[b, \Delta^{-\alpha/2}]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. See [3].
Proposition 2.4. Denote by $K_N(x, y)$ the kernel of the fractional operator $\Delta_N^{-\alpha/2}$. Then $x, y \in \mathbb{R}^n_+$ we have:

$$K_N(x, y) = K(x, y) + \tilde{K}_N(x, y)$$

with

$$\tilde{K}_N(x, y) := C_{n, \alpha} \frac{1}{(|x' - y'|^2 + |x_n + y_n|^2)^{\frac{\alpha}{2} - \frac{n}{2}}}.$$ 

Similar expressions for $K_N(x, y)$ when $x, y \in \mathbb{R}_n^-$ also hold.

Proof. For $x, y \in \mathbb{R}_n^+$, working from the fraction of the square root of $\Delta_N$ we have that:

$$K_N(x, y) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p_{t, \Delta_N}(x, y) \frac{dt}{t^{1-\alpha/2}}$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t^{1-\alpha/2}} + \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{|x'-y'|^2}{4t}} e^{-\frac{|x_n+y_n|^2}{4t}} \frac{dt}{t^{1-\alpha/2}}$$

$$= C_{n, \alpha} \left( \frac{1}{|x-y|^{n-\alpha}} + \frac{1}{(|x'-y'|^2 + |x_n + y_n|^2)^{\frac{\alpha}{2} - \frac{n}{2}}} \right)$$

$$= K(x, y) + \tilde{K}_N(x, y).$$

where we set

$$\tilde{K}_N(x, y) = C_{n, \alpha} \frac{1}{(|x' - y'|^2 + |x_n + y_n|^2)^{\frac{\alpha}{2} - \frac{n}{2}}}.$$ 

\[\square\]

3. Characterization and Properties of $H^1_{\Delta_N}(\mathbb{R}^n)$ and BMO$_{\Delta_N}(\mathbb{R}^n)$

3.1. Fundamental Properties of BMO$_{\Delta_N}(\mathbb{R}^n)$. We now recall the definition and some fundamental properties of BMO$_{\Delta_N}(\mathbb{R}^n)$ from [5].

Define

$$\mathcal{M} = \left\{ f \in L_{1, \text{loc}}(\mathbb{R}^n) : \exists d > 0 \text{ s.t. } \int_{\mathbb{R}^n} \frac{|f(x)|^2}{1 + |x|^{n+d}} \, dx < \infty \right\}.$$ 

Definition 3.1 ([5, Definition 2.2]). We say that $f \in \mathcal{M}$ is of bounded mean oscillation associated with $\Delta_N$, abbreviated as BMO$_{\Delta_N}(\mathbb{R}^n)$, if

$$\|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} = \sup_{B(y, r)} \frac{1}{|B(y, r)|} \int_{B(y, r)} |f(x) - \exp(-r^2 \Delta_N) f(x)| \, dx < \infty,$$

where the supremum is taken over all balls $B(y, r)$ in $\mathbb{R}^n$. The smallest bound for which (3.1) is satisfied is then taken to be the norm of $f$ in this space, and is denoted by $\|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)}$. 

Definition 3.2 ([5, Definition 2.1]). A function \( f \) on \( \mathbb{R}^n_+ \) is said to be in \( \text{BMO}_r(\mathbb{R}^n_+) \) if there exists \( F \in \text{BMO}(\mathbb{R}^n) \) such that \( F|_{\mathbb{R}^n_+} = f \). If \( f \in \text{BMO}_r(\mathbb{R}^n_+) \), then we set
\[
\|f\|_{\text{BMO}_r(\mathbb{R}^n_+)} = \inf \left\{ \|F\|_{\text{BMO}(\mathbb{R}^n)} : F|_{\mathbb{R}^n_+} = f \right\}.
\]

Definition 3.3 ([5, Page 270]). For any function \( f \in L^1_{\text{loc}}(\mathbb{R}^n_+) \), define
\[
\|f\|_{\text{BMO}_e(\mathbb{R}^n_+)} = \|f_e\|_{\text{BMO}(\mathbb{R}^n)}.
\]

where \( f_e \) is defined in (2.1). We denote by \( \text{BMO}_e(\mathbb{R}^n_+) \) the corresponding Banach space.

Similarly we can define the spaces \( \text{BMO}_r(\mathbb{R}^n_-) \) and \( \text{BMO}_e(\mathbb{R}^n_-) \).

Proposition 3.4 ([5, Proposition 3.1]). The spaces \( \text{BMO}_r(\mathbb{R}^n_+) \) and \( \text{BMO}_e(\mathbb{R}^n_+) \) coincide, and their norms are equivalent. Similar result holds for \( \text{BMO}_r(\mathbb{R}^n_-) \) and \( \text{BMO}_e(\mathbb{R}^n_-) \).

Proposition 3.5 ([5, Proposition 4.2]). The Neumann BMO space \( \text{BMO}_{\Delta_N}(\mathbb{R}^n) \) can be described in the following way:
\[
\text{BMO}_{\Delta_N}(\mathbb{R}^n) = \{ f \in \mathcal{M} : f_+ \in \text{BMO}_r(\mathbb{R}^n_+) \text{ and } f_- \in \text{BMO}_r(\mathbb{R}^n_-) \}.
\]

As a consequence of the results from [5] listed above, we obtain that \( f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n) \) if and only if \( f_+ \in \text{BMO}(\mathbb{R}^n) \). A final key fact that plays a role in our analysis is the duality between \( \text{BMO}_{\Delta_N}(\mathbb{R}^n) \) and \( H^1_{\Delta_N}(\mathbb{R}^n) \).

Proposition 3.6 ([5, Corollary 4.3]). The dual space of \( H^1_{\Delta_N}(\mathbb{R}^n) \) is \( \text{BMO}_{\Delta_N}(\mathbb{R}^n) \).

3.2. Properties of \( H^1_{\Delta_N}(\mathbb{R}^n) \). In this subsection, we provide a deeper study of the space \( H^1_{\Delta_N}(\mathbb{R}^n) \).

We first provide several equivalent characterizations of \( H^1_{\Delta_N}(\mathbb{R}^n) \). To do so, we need the following definitions of the Hardy space associated to \( \Delta_N \) in terms of the radial maximal function, the non-tangential maximal function, the Riesz transforms, and atoms. As one might expect, these definitions all turn out to be equivalent as shown below in Theorem 3.12.

Definition 3.7. We define \( H^1_{\Delta_N,\text{max}}(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) : f^+_{\Delta_N} \in L^1(\mathbb{R}^n) \} \) with the norm \( \|f\|_{H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)} = \|f^+_{\Delta_N}\|_{L^1(\mathbb{R}^n)} \), where \( f^+_{\Delta_N}(x) = \sup_{t>0} |\exp(-t^2\Delta_N)f(x)| \).

Definition 3.8. We define \( H^1_{\Delta_N,\text{nt}}(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) : f^*_{\Delta_N} \in L^1(\mathbb{R}^n) \} \) with the norm \( \|f\|_{H^1_{\Delta_N,\text{nt}}(\mathbb{R}^n)} = \|f^*_{\Delta_N}\|_{L^1(\mathbb{R}^n)} \), where \( f^*_{\Delta_N}(x) = \sup_{|x-y|<t} |\exp(-t^2\Delta_N)f(y)| \).

Definition 3.9. We define
\[
H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : \frac{\partial}{\partial x_l} \Delta_N^{-\frac{1}{2}} f \in L^1(\mathbb{R}^n) \text{ for } 1 \leq l \leq n \right\}
\]
with the norm \( \|f\|_{H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + \sum_{l=1}^n \left\| \frac{\partial}{\partial x_l} \Delta_N^{-\frac{1}{2}} f \right\|_{L^1(\mathbb{R}^n)} \).
Next we define the atoms for $H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)$, which we adapt from a very recent result of Song and Yan [14].

**Definition 3.10.** Given $M \in \mathbb{N}$. We say that a function $a(x) \in L^\infty(\mathbb{R}^n)$ is an $H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)$-atom, if there exist a function $b$ in the domain of $\Delta_N^M$ and a ball $B \subset \mathbb{R}^n$ such that

(i) $a = \Delta_N^M b$;
(ii) $\text{supp} \Delta_N^k b \subset B$, $k = 0, 1, \ldots, M$;
(iii) $\|r_B^2 \Delta_N^k b\|_{L^\infty(\mathbb{R}^n)} \leq r_B^{2M} |B|^{-1} \cdot 2^n$, $k = 0, 1, \ldots, M$.

**Definition 3.11.** We say that $f = \sum_j \lambda_j a_j$ is an atomic representation of $f$ if $\{\lambda_j\} \in \ell^1$, each $a_j$ is an $H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)$ atom, and the sum converges in $L^2(\mathbb{R}^n)$. Set

$$\tilde{H}^1_{\Delta_N,\text{atom}}(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : f \text{ has an atomic representation} \}$$

with the norm $\| f \|_{\tilde{H}^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)}$ given by

$$\inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \text{ is an atomic representation} \right\}.$$

The space $H^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)$ is defined as the completion of $\tilde{H}^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)$ with respect to this norm.

We now collection the equivalence of all these definitions and moreover provide a link between $H^1(\mathbb{R}^n)$ and $H^1_{\Delta_N}(\mathbb{R}^n)$.

**Theorem 3.12.** Let all the notation be as above. Then,

$$H^1_{\Delta_N}(\mathbb{R}^n) = H^1_{\Delta_N,\text{max}}(\mathbb{R}^n) = H^1_{\Delta_N,\text{loc}}(\mathbb{R}^n) = H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n) = H^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)$$

and they have equivalent norms

$$\| f \|_{H^1_{\Delta_N}(\mathbb{R}^n)} \approx \| f \|_{H^1_{\Delta_N,\text{max}}(\mathbb{R}^n)} \approx \| f \|_{H^1_{\Delta_N,\text{Riesz}}(\mathbb{R}^n)} \approx \| f \|_{H^1_{\Delta_N,\text{atom}}(\mathbb{R}^n)} \approx \| f \|_{H^1_{\Delta_N,\text{loc}}(\mathbb{R}^n)}.$$

Namely, $f \in H^1_{\Delta_N}(\mathbb{R}^n)$ if and only if $f_{+,e} \in H^1(\mathbb{R}^n)$ and $f_{-,e} \in H^1(\mathbb{R}^n)$.

**Proof.** We recall that the Hardy space associated with $\Delta_N$ is defined as the set of functions $\{ f \in L^1(\mathbb{R}^n) : \| S_{\Delta_N}(f) \|_{L^1(\mathbb{R}^n)} < \infty \}$ in the norm of $\| f \|_{H^1_{\Delta_N}} = \| S_{\Delta_N}(f) \|_{L^1(\mathbb{R}^n)}$, where $S_{\Delta_N}(f)(x) = \left( \int_0^\infty \int_{|y-x|<t} |Q_t f(y)|^2 \frac{dy dt}{t^n} \right)^{\frac{1}{2}}$, and $Q_t = t^2 \Delta_N \exp(-t^2 \Delta_N)$.

We now consider the operator $Q_t = t \Delta_N \exp(-t \Delta_N) = -t \frac{d}{dt} \exp(-t \Delta_N)$ for any $t > 0$ (see [8, (3.5) in Section 3.1]). Then we have

$$Q_t f(x) = t^2 \Delta_N \exp(-t^2 \Delta_N) f(x) = \int_{\mathbb{R}^n} -\frac{t}{2} \frac{\partial}{\partial t} p_{t^2 \Delta_N}(x,y) f(y) \, dy.$$
From the definition of $p_{t,\Delta N}(x, y)$, see (2.4), we have that for any $x \in \mathbb{R}^n$,

$$t^2 \Delta_N \exp(-t^2 \Delta_N) f(x) = \int_{\mathbb{R}^n} -\frac{t}{2} \frac{\partial}{\partial t} p_{t^2, \Delta N}(x, y) f(y) dy$$

$$= \int_{\mathbb{R}^n} -\frac{t}{2} \frac{\partial}{\partial t} p_{t^2}(x, y) f_{+e}(y) dy$$

$$= t^2 \Delta \exp(-t^2 \Delta) f_{+e}(x).$$

Similarly, for any $x \in \mathbb{R}^n$, we have $t^2 \Delta_N \exp(-t^2 \Delta_N) f(x) = t^2 \Delta \exp(-t^2 \Delta) f_{-e}(x)$.

Moreover, by a change of variable,

$$t^2 \Delta_N \exp(-t^2 \Delta_N) f(x) = \int_{|x-y|<t, y \in \mathbb{R}^n} |t^2 \Delta_N \exp(-t^2 \Delta_N) f(y)|^2 \frac{dy dt}{t^n}$$

$$= 2 \int_{|x-y|<t, y \in \mathbb{R}^n} |t^2 \Delta \exp(-t^2 \Delta) f_{+e}(y)|^2 \frac{dy dt}{t^n}$$

$$= \frac{1}{2} \left( \int_{|x-y|<t} |t^2 \Delta \exp(-t^2 \Delta) f_{+e}(y)|^2 \frac{dy dt}{t^n}$$

$$+ \int_{|x-y|<t} |t^2 \Delta \exp(-t^2 \Delta) f_{-e}(y)|^2 \frac{dy dt}{t^n} \right),$$

which implies that $S_{\Delta N}(f)(x) \leq \frac{\sqrt{t}}{2} \left( S(f_{+e})(x) + S(f_{-e})(x) \right)$. Conversely,

$$S(f_{+e})(x)^2 = \int_{|x-y|<t} |t^2 \Delta \exp(-t^2 \Delta) f_{+e}(y)|^2 \frac{dy dt}{t^n}$$

$$= 2 \int_{|x-y|<t, y \in \mathbb{R}^n} |t^2 \Delta \exp(-t^2 \Delta) f_{+e}(y)|^2 \frac{dy dt}{t^n}$$

$$\leq 2S_{\Delta N}(f)(x)^2.$$
Similarly, \( \exp(-t^2 \Delta_N) f(x) = \int_{\mathbb{R}^n} p_t \Delta_N(x, y) f(y) \, dy = \int_{\mathbb{R}^n} p_t \Delta_N(x, y) f_+(y) \, dy \)

\[ = \int_{\mathbb{R}^n} p_t \Delta_N(x, y) f_+(y) \, dy = \exp(-t^2 \Delta) f_+(x). \]

Similarly, \( \exp(-t^2 \Delta_N) f(x) = \exp(-t^2 \Delta) f_-(x) \) for any \( t \geq 0 \) and \( x \in \mathbb{R}^n \). Thus,

\[ \sup_{t>0} |\exp(-t^2 \Delta_N) f(x)| = \sup_{t>0} |\exp(-t^2 \Delta) f_+(x)| \quad \text{for any } \quad x \in \mathbb{R}^n_+; \]

\[ \sup_{t>0} |\exp(-t^2 \Delta_N) f(x)| = \sup_{t>0} |\exp(-t^2 \Delta) f_-(x)| \quad \text{for any } \quad x \in \mathbb{R}^n_. \]

Again, by a change of variable, we have that

\[ \exp(-t^2 \Delta_N) f(x) = -\exp(-t^2 \Delta) f_+(\bar{x}) \quad \text{for any } t > 0, \quad x \in \mathbb{R}^n_+; \]

\[ \exp(-t^2 \Delta_N) f(x) = -\exp(-t^2 \Delta) f_-(\bar{x}) \quad \text{for any } t > 0, \quad x \in \mathbb{R}^n_. \]

Then, for any \( f \in H^1_{\Delta^N, \max} (\mathbb{R}^n) \), from (3.4) and (3.5) we can obtain that

\[ \|f\|_{H^1_{\Delta^N, \max} (\mathbb{R}^n)} = \int_{\mathbb{R}^n_+} \left| f_\Delta^+(x) \right| \, dx + \int_{\mathbb{R}^n_-} \left| f_\Delta^-(x) \right| \, dx \]

\[ = \int_{\mathbb{R}^n_+} \sup_{t>0} |\exp(-t^2 \Delta_N) f(x)| \, dx + \int_{\mathbb{R}^n_-} \sup_{t>0} |\exp(-t^2 \Delta_N) f(x)| \, dx \]

\[ = \int_{\mathbb{R}^n_+} \sup_{t>0} |\exp(-t^2 \Delta) f_+(x)| \, dx + \int_{\mathbb{R}^n_-} \sup_{t>0} |\exp(-t^2 \Delta) f_-(x)| \, dx \]

\[ = \frac{1}{2} \left( \int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t^2 \Delta) f_+(x)| \, dx + \int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t^2 \Delta) f_-(x)| \, dx \right) \]

\[ = \frac{1}{2} \left( \|f_+\|_{L^1(\mathbb{R}^n)} + \|f_-\|_{L^1(\mathbb{R}^n)} \right) \]

\[ = \frac{1}{2} \left( \|f_+\|_{H^1(\mathbb{R}^n)} + \|f_-\|_{H^1(\mathbb{R}^n)} \right) ; \]

where \( f^+ = \sup_{t>0} |p_t \ast f(x)| \) is the classical maximal function as defined in (3) in §2.4. Thus (3.6) yields that \( f \in H^1_{\Delta^N, \max} (\mathbb{R}^n) \) if and only if \( f_+ \in H^1(\mathbb{R}^n) \) and \( f_- \in H^1(\mathbb{R}^n) \).

We now consider the Hardy space \( H^1_{\Delta^N, \ast} (\mathbb{R}^n) \) via the non-tangential maximal function. Note that

\[ f_\Delta^N(x) = \sup_{|x-y|<t} |\exp(-t^2 \Delta_N) f(y)| \]

\[ \leq \sup_{|x-y|<t, y \in \mathbb{R}^n_+} |\exp(-t^2 \Delta_N) f(y)| + \sup_{|x-y|<t, y \in \mathbb{R}^n_-} |\exp(-t^2 \Delta_N) f(y)| \]

\[ \leq \sup_{|x-y|<t, y \in \mathbb{R}^n_+} |\exp(-t^2 \Delta) f_+(y)| + \sup_{|x-y|<t, y \in \mathbb{R}^n_-} |\exp(-t^2 \Delta) f_-(y)| \]
\[ \sum_{j=0}^{n} \| f_j \|_{L^1(\mathbb{R}^n)} \]. We have that \( B^* = \bigoplus_{j=0}^{n} L^\infty(\mathbb{R}^n) \). Let \( S \) be the subspace of \( B \) given by
\[ S = \{ (f, R_{N,1}f, \ldots, R_{N,n}f) : f \in L^1(\mathbb{R}^n) \} . \]
We have that \( S \) is a closed subspace and that \( f \rightarrow (f, R_{N,1}f, \ldots, R_{N,n}f) \) is an isometry of \( H^1_{\Delta_N}(\mathbb{R}^n) \) to \( S \). Linear functionals on \( S \) and \( H^1_{\Delta_N}(\mathbb{R}^n) \) can be identified in an obvious way,
hence any continuous linear functional on $H^1_{\Delta_N}(\mathbb{R}^n)$ can be extended by Hahn-Banach to a continuous linear functional on $B$ and can be identified with a vector of functions $(b_0, b_1, \ldots, b_n)$ with each $b_j \in L^\infty(\mathbb{R}^n)$.

We use this conclusion in the following way. Let $\ell$ be a continuous linear functional on $H^1_{\Delta_N}(\mathbb{R}^n)$. Then by Proposition 3.6 there is a function $b \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ so that:

$$\int_{\mathbb{R}^n} f(x) b(x) \, dx = \ell(f).$$

However, by the discussion above, and by restricting the extended linear functional back to $H^1_{\Delta_N}(\mathbb{R}^n)$ we have for $(f, R_{N,1}f, \ldots, R_{N,n}f) = (f_0, \ldots, f_n)$:

$$\ell(f) = \sum_{j=0}^{n} \int_{\mathbb{R}^n} f_j(x) b_j(x) \, dx.$$ 

Using the definition of the $f_j = R_{N,j}f$ we see that:

$$\ell(f) = \int_{\mathbb{R}^n} f(x) \left( b_0(x) + \sum_{j=1}^{n} R^*_{N,j} b_j(x) \right) \, dx.$$ 

This then gives the decomposition that any $b \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ can be written as:

$$b = b_0 + \sum_{j=1}^{n} R^*_{N,j} b_j$$

with $b_j \in L^\infty(\mathbb{R}^n)$.

For the converse, we simply observe that from our Theorem 3.12, we obtained that $R_N$ maps $H^1_{\Delta_N}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Hence, the boundedness of the Riesz transform $R^*_N$ from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ follows from duality of $H^1_{\Delta_N}(\mathbb{R}^n)$ with $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$. We then have that any $b$ that can be written as:

$$b = b_0 + \sum_{j=1}^{n} R^*_{N,j} b_j$$

with $b_j \in L^\infty(\mathbb{R}^n)$ must belong to $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$. □

We next note that $H^1_{\Delta_N}(\mathbb{R}^n)$ is a proper subspace of the classical $H^1(\mathbb{R}^n)$, which was proved by Yan in [19, Proposition 6.2] from the viewpoint of the semigroup generated by $\Delta_N$. And we now give a direct proof and provide a specific function $f$ which lies in $H^1(\mathbb{R}^n)$ but does not belong to $H^1_{\Delta_N}(\mathbb{R}^n)$. A related claim is made in [5, Corollary 4.3].

**Theorem 3.13 ([19, Proposition 6.2])**. $H^1_{\Delta_N}(\mathbb{R}^n) \subsetneq H^1(\mathbb{R}^n)$.

**Proof.** We first show that the containment $H^1_{\Delta_N}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ holds. This follows directly from the fact that corresponding BMO spaces norm the $H^1$ spaces, namely
that:
\[ \|f\|_{H^1_{\Delta}({\mathbb R}^n)} \approx \sup_{\|b\|_{\text{BMO}({\mathbb R}^n)} \leq 1} |\langle f, b \rangle|_{L^2({\mathbb R}^n)}. \]

An identical statement holds for \( H^1({\mathbb R}^n) \) and \( \text{BMO}({\mathbb R}^n) \). As shown in [5], \( \text{BMO}({\mathbb R}^n) \subset \text{BMO}_{\Delta}({\mathbb R}^n) \), and so we have
\[ \|f\|_{H^1({\mathbb R}^n)} \approx \sup_{\|b\|_{\text{BMO}({\mathbb R}^n)} \leq 1} |\langle f, b \rangle|_{L^2({\mathbb R}^n)} \leq \sup_{\|b\|_{\text{BMO}_{\Delta}({\mathbb R}^n)} \leq 1} |\langle f, b \rangle|_{L^2({\mathbb R}^n)} \approx \|f\|_{H^1_{\Delta}({\mathbb R}^n)}. \]

This gives the containment, \( H^1_{\Delta}({\mathbb R}^n) \subset H^1({\mathbb R}^n) \).

We now show that there exists a function \( f \in H^1({\mathbb R}^n) \) but \( f \not\in H^1_{\Delta}({\mathbb R}^n) \). For the sake of simplicity, we just consider the example in dimension 1.

Define
\[ f(x) := \frac{\chi_{[0,1]}(x)}{\sqrt{2}} - \frac{\chi_{[-1,0]}(x)}{\sqrt{2}}. \]

It is easy to see that \( f(x) \) is supported in \([-1, 1]\), and \( \int_{\mathbb R} f(x) \, dx = 0 \). Moreover, we have
\[ \|f\|_{L^2(\mathbb R)} = 1. \]

These implies that \( f \) is an atom of \( H^1(\mathbb R) \), which shows that \( f \in H^1(\mathbb R) \). From the definition of \( f \), we obtain that \( f_+(x) = \frac{\chi_{[0,1]}(x)}{\sqrt{2}} \), and the even extension is
\[ f_{+,e}(x) = \frac{\chi_{[-1,1]}(x)}{\sqrt{2}}. \]

But, then it is immediate that \( f_{+,e} \not\in H^1(\mathbb R) \) since \( \int_{\mathbb R^n} f_{+,e}(x) \, dx \neq 0 \). One can also prove this by using the equivalent definition of \( H^1(\mathbb R) \) via the radial maximal function.

Similarly we have these estimates for \( f_{-,e} \). Hence, \( f_{+,e} \not\in H^1(\mathbb R) \) and \( f_{-,e} \not\in H^1(\mathbb R) \), which, combining the result in Theorem 3.12, implies that \( f \not\in H^1_{\Delta}({\mathbb R}^n) \). \( \Box \)

Finally, we provide a description of the atoms in \( H^1_{\Delta}({\mathbb R}^n) \) that connects back to the atom in \( H^1({\mathbb R}^n) \).

**Proposition 3.14.** Suppose \( a(x) \) is an \( H^1_{\Delta}({\mathbb R}^n) \)-atom supported in \( B \subset {\mathbb R}^n \) as in Definition 3.10. Then we have
\[ \int_{\mathbb R^n} a(x) \, dx = 0. \]

Moreover, if \( B \cap \{x \in {\mathbb R}^n : x_n = 0\} \neq \emptyset \), we denote \( B_+ = B \cap {\mathbb R}^n_+ \) and \( B_- = B \cap {\mathbb R}^n_- \). Then we have
\[ \int_{B_+} a(x) \, dx = \int_{B_-} a(x) \, dx = 0. \]
Proof. First note that from Theorem 3.13, $H_{\Delta N}^1(\mathbb{R}^n) \subsetneq H^1(\mathbb{R}^n)$. Since $a(x)$ is an $H^1_{\Delta N}(\mathbb{R}^n)$ atom, we have $a(x) \in H^1(\mathbb{R}^n)$, and hence (3.8) holds, where we use [10, Corollary 6.7.7].

Second, suppose $B \cap \{x \in \mathbb{R}^n : x_n = 0\} \neq \emptyset$. Then we define $a_+(x) = a(x)|_{B_+}$ and $a_-(x) = a(x)|_{B_-}$. Since $a(x) \in H^1_{\Delta N}(\mathbb{R}^n)$, from Theorem 3.12 we obtain that both $a_{+,e}(x)$ and $a_{-,e}(x)$ are in $H^1(\mathbb{R}^n)$, which implies that

$$\int_{\mathbb{R}^n} a_{+,e}(x) \, dx = \int_{\mathbb{R}^n} a_{-,e}(x) \, dx = 0.$$ 

Next we claim that $\int_{\mathbb{R}^n} a_+(x) \, dx = 0$. In fact,

$$\int_{\mathbb{R}^n} a_{+,e}(x) \, dx = \int_{\mathbb{R}^n_+} a_{+,e}(x) \, dx + \int_{\mathbb{R}^n_-} a_{+,e}(x) \, dx = 2 \int_{\mathbb{R}^n_+} a_{+,e}(x) \, dx.$$

Hence, $\int_{\mathbb{R}^n} a_{+,e}(x) \, dx = 0$ implies that $\int_{\mathbb{R}^n} a_+(x) \, dx = 0$, i.e., $\int_{B_+} a(x) \, dx = 0$.

Similarly we obtain that $\int_{B_-} a(x) \, dx = 0$. Hence (3.9) holds. \qed

Remark 3.15. In [18], it was asked if a proper subspace of the classical Hardy space exists in which the subspace is characterized by maximal functions. This question was answered positively in [17]. Our result above, Theorem 3.12, also gives a proper subspace of the classical Hardy space where the subspace is characterized by radial maximal functions as well as non-tangential maximal functions.

4. WEAK FACTORIZATION OF THE HARDY SPACE $H^1_{\Delta N}(\mathbb{R}^n)$

In this section we turn to proving Theorem 1.3. There are two parts to this Theorem, and upper and lower bound, and we focus first on the (easier) upper bound.

Recall that, for notational simplicity, we are letting

$$\Pi_l(h, g) := h \cdot R^*_N(g) - g \cdot R_{N,l}(h),$$

where $R_{N,l} = \frac{\partial}{\partial x_l} \Delta_N^{-\frac{1}{2}}$ for $1 \leq l \leq n$. We now prove the following theorem.

Theorem 4.1. Let $g, h \in L^\infty(\mathbb{R}^n)$ with compact supports. Then for $1 \leq l \leq n$,

$$\|\Pi_l(h, g)\|_{H^1_{\Delta N}(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^2(\mathbb{R}^n)}.$$

This will be an immediate corollary of the following theorem.

Theorem 4.2. If $b \in \text{BMO}_{\Delta N}(\mathbb{R}^n)$, then for $1 \leq l \leq n$, the commutator

$$[b, R_{N,l}](f)(x) = b(x)R_{N,l}(f)(x) - R_{N,l}(bf)(x)$$

is a bounded map on $L^2(\mathbb{R}^n)$, with operator norm

$$\|[b, R_{N,l}] : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\| \leq C\|b\|_{\text{BMO}_{\Delta N}(\mathbb{R}^n)}.$$
Proof. Suppose \(b\) is in \(BMO_{\Delta N}(\mathbb{R}^n)\). Then according to [5, Proposition 4.2], we have that \(b_{+,e} \in BMO(\mathbb{R}^n)\) and \(b_{-,e} \in BMO(\mathbb{R}^n)\), and moreover,

\[
\|b\|_{BMO_{\Delta N}(\mathbb{R}^n)} \approx \|b_{+,e}\|_{BMO(\mathbb{R}^n)} + \|b_{-,e}\|_{BMO(\mathbb{R}^n)}.
\]

For every \(f \in L^2(\mathbb{R}^n)\), we have

\[
\|b, R_{N,I}(f)\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} [b, R_{N,I}(f)](x)^2 \, dx + \int_{\mathbb{R}^n} [b, R_{N,I}(f)](x)^2 \, dx =: I + II.
\]

For the term \(I\), note that when \(x \in \mathbb{R}^n_+\), we have

\[
[b, R_{N,I}](f)(x) = b(x)R_{N,I}(f)(x) - R_{N,I}(bf)(x) = b_{+,e}(x)R_{I}(f_{+,e})(x) - R_{I}(b_{+,e}f_{+,e})(x) = [b_{+,e}, R_{I}](f_{+,e})(x),
\]

which implies that

\[
I = \int_{\mathbb{R}^n_+} [b, R_{N,I}](f)(x)^2 \, dx = \int_{\mathbb{R}^n_+} [b_{+,e}, R_{I}](f_{+,e})(x)^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^n} [b_{+,e}, R_{I}](f_{+,e})(x)^2 \, dx
\]

\[
\leq C\|b_{+,e}\|_{BMO(\mathbb{R}^n)}^2 \|f_{+,e}\|_{L^2(\mathbb{R}^n)}^2,
\]

where \(R_{I}\) is the classical \(l\)-th Riesz transform \(\frac{\partial}{\partial x_l} \Delta^{-\frac{1}{2}}\).

For the last estimate we use the result [4, Theorem 1], which applies since we know from Proposition 2.9 that \(R_{N,I}\) is a Calderón–Zygmund kernel. Similarly we can obtain that

\[
II \leq C\|b_{-,e}\|_{BMO(\mathbb{R}^n)} \|f_{-,e}\|_{L^2(\mathbb{R}^n)}^2.
\]

Combining the estimates for \(I\) and \(II\) above, we obtain that

\[
\|b, R_{N,I}(f)\|_{L^2(\mathbb{R}^n)}^2 \leq C\|b_{+,e}\|_{BMO(\mathbb{R}^n)}^2 \|f_{+,e}\|_{L^2(\mathbb{R}^n)}^2 + C\|b_{-,e}\|_{BMO(\mathbb{R}^n)}^2 \|f_{-,e}\|_{L^2(\mathbb{R}^n)}^2
\]

\[
\leq C\|b\|_{BMO_{\Delta N}(\mathbb{R}^n)}^2 \left(\|f_{+,e}\|_{L^2(\mathbb{R}^n)}^2 + \|f_{-,e}\|_{L^2(\mathbb{R}^n)}^2\right)
\]

\[
\leq C\|b\|_{BMO_{\Delta N}(\mathbb{R}^n)}^2 \|f\|_{L^2(\mathbb{R}^n)}^2,
\]

which yields that \(\|b, R_{N,I}\| : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\| \leq C\|b\|_{BMO_{\Delta N}(\mathbb{R}^n)}\). 

\( \square \)

Proof of Theorem 4.1. By the duality result of [5], stated in Proposition 3.6, we know that \(H^1_{\Delta N}(\mathbb{R}^n) = BMO_{\Delta N}(\mathbb{R}^n)\). A simple duality computation shows for \(b \in BMO_{\Delta N}(\mathbb{R}^n)\) and for any \(g, h \in L^\infty(\mathbb{R}^n)\) with compact supports:

\[
\langle b, \Pi_l(g, h) \rangle_{L^2(\mathbb{R}^n)} = \langle b, R_{N,I}^*(g)h - R_{N,I}(h)g \rangle_{L^2(\mathbb{R}^n)} = \langle g, [b, R_{N,I}^*]h \rangle_{L^2(\mathbb{R}^n)}.
\]

Thus, from Theorem 4.2, we obtain that

\[
\left| \langle b, \Pi_l(g, h) \rangle_{L^2(\mathbb{R}^n)} \right| \leq C\|b\|_{BMO_{\Delta N}(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^2(\mathbb{R}^n)}.
\]
This, together with the duality of \( H^1_{\Delta_N}(\mathbb{R}^n) \) with \( \text{BMO}_{\Delta_N}(\mathbb{R}^n) \) shows that \( \Pi_t(g, h) \) is in \( H^1_{\Delta_N}(\mathbb{R}^n) \). And then by testing \( \Pi_t(g, h) \) against \( b \in \text{BMO}_{\Delta_N}(\mathbb{R}^n) \) functions, we find:

\[
\|\Pi_t(g, h)\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \approx \sup_{\|b\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \leq 1} \left| \langle \Pi_t(g, h), b \rangle \right|_{L^2(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^2(\mathbb{R}^n)} \sup_{\|b\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \leq 1} \|b\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^2(\mathbb{R}^n)}.
\]

\( \square \)

4.1. The Lower Bound in Theorem 1.3. The proof of the lower bound is more algorithmic in nature and follows a proof strategy developed by Uchiyama in [16]. We begin with a fact that will play a prominent role in the algorithm below. It is a modification of a related fact for the standard Hardy space \( H^1(\mathbb{R}^n) \).

**Lemma 4.3.** Suppose \( f \) is a function satisfying: \( \int_{\mathbb{R}^n} f(x) \, dx = 0 \), and \( |f(x)| \leq \chi_{B(x_0,1)}(x) + \chi_{B(y_0,1)}(x) \), where \( |x_0 - y_0| := M > 10 \). Then we have

\[
(4.1) \quad \|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq C_n \log M.
\]

**Proof.** First note that

\[
f^+_{\Delta_N}(x) = \sup_{t > 0} |e^{-t\Delta_N} f(x)| = \sup_{t > 0} \left| \int_{\mathbb{R}^n} p_t\Delta_N(x, y) f(y) \, dy \right| \leq \sup_{t > 0} \int_{\mathbb{R}^n} |p_t\Delta_N(x, y)| \, dy \leq C.
\]

Hence, we obtain that

\[
\int_{B(x_0,5)} f^+_{\Delta_N}(x) \, dx + \int_{B(y_0,5)} f^+_{\Delta_N}(x) \, dx \leq C_n.
\]

Now it suffices to estimate

\[
\int_{\mathbb{R}^n\setminus(B(x_0,5)\cup B(y_0,5))} f^+_{\Delta_N}(x) \, dx.
\]

To see this, we write it as

\[
\int_{\mathbb{R}^n\setminus(B(x_0,2M)\cup B(x_0,2M))] f^+_{\Delta_N}(x) \, dx + \int_{B(x_0,2M)\setminus(B(x_0,5)\cup B(y_0,5))} f^+_{\Delta_N}(x) \, dx =: I + II.
\]

We now estimate the term \( I \). First note that from Hölder’s regularity (2.8) of the heat kernel \( p_t\Delta_N(x, y) \), we have

\[
|p_t\Delta_N(x, y) - p_t\Delta_N(x, x_0)| \leq C \left( \frac{|y - x_0|}{\sqrt{t} + |x - x_0|} \right)^n \left( \frac{\sqrt{t}}{\sqrt{t} + |x - x_0|} \right)^{n+1}
\]

for \( |y - x_0| < \sqrt{t} \). Moreover, when \( |y - x_0| \geq \sqrt{t} \), we have

\[
|p_t\Delta_N(x, y) - p_t\Delta_N(x, x_0)| \leq |p_t\Delta_N(x, y)| + |p_t\Delta_N(x, x_0)| \leq C \frac{e^{-|x-x_0|^2/ct}}{t^{n/2}} + \frac{e^{-|y|^2/ct}}{t^{n/2}}
\]

for \( t > 0 \).
As a consequence, we obtain that

$$\leq C \left( \frac{|y - x_0|}{\sqrt{t}} \right) \frac{e^{-|x-x_0|^2/ct}}{t^{n/2}}$$

$$\leq C \left( \frac{|y - x_0|}{\sqrt{t} + |x - x_0|} \right) \frac{\sqrt{t}}{(\sqrt{t} + |x - x_0|)^{n+1}}.$$ 

Now note that from the cancellation condition of \( f \) and Hölder’s regularity of the heat kernel \( p_t(x, y) \) as above, we have

$$f_{\Delta_N}^+(x) = \sup_{t>0} \left| \int_{\mathbb{R}^n} p_{t,\Delta_N}(x, y) - p_{t,\Delta_N}(x, x_0) \right| f(y) \, dy \leq C \sup_{t>0} \int_{B(x_0,1) \cup B(y_0,1)} \left( \frac{|y - x_0|}{\sqrt{t} + |x - x_0|} \right) \frac{\sqrt{t}}{(\sqrt{t} + |x - x_0|)^{n+1}} \, dy \leq C_n \frac{|y_0 - x_0|}{|x - x_0|^{n+1}} = C_n \frac{M}{|x - x_0|^{n+1}}.$$ 

As a consequence, we obtain that

$$I \leq \int_{\mathbb{R}^n \setminus B(x_0,2M)} C_n \frac{M}{|x - x_0|^{n+1}} \, dx \leq C_n.$$ 

We now turn to the term \( II \). Note that when \( x \in B(x_0,2M) \setminus (B(x_0,5) \cup B(y_0,5)) \), we have

$$\left| \int_{\mathbb{R}^n} p_{t,\Delta_N}(x, y) f(y) \, dy \right| \leq \int_{B(x_0,1)} |p_{t,\Delta_N}(x, y)| \, dy + \int_{B(y_0,1)} |p_{t,\Delta_N}(x, y)| \, dy.$$ 

When \( t > 1 \), from the size estimate of the heat kernel \( p_{t,\Delta_N}(x, y) \), we have

$$\left| \int_{\mathbb{R}^n} p_{t,\Delta_N}(x, y) f(y) \, dy \right| \leq C \frac{1}{|x - x_0|^n} + C \frac{1}{|x - y_0|^n}.$$ 

When \( t \leq 1 \), similarly we obtain that

$$\left| \int_{\mathbb{R}^n} p_{t,\Delta_N}(x, y) f(y) \, dy \right| \leq C \frac{1}{|x - x_0|^{n+1}} + C \frac{1}{|x - y_0|^{n+1}} \leq C \frac{1}{|x - x_0|^n} + C \frac{1}{|x - y_0|^n}.$$ 

Thus,

$$II \leq \int_{B(x_0,2M) \setminus (B(x_0,5) \cup B(y_0,5))} f_{\Delta_N}^+(x) \, dx \leq C \int_{B(x_0,2M) \setminus (B(x_0,5) \cup B(y_0,5))} \frac{1}{|x - x_0|^n} + \frac{1}{|x - y_0|^n} \, dx \leq C_n \log M.$$ 

Combining all the estimates above, we obtain that

$$\| f \|_{H_{\Delta_N}^1(\mathbb{R}^n)} = \| f_{\Delta_N}^+ \|_{L^1(\mathbb{R}^n)} \leq C_n \log M.$$
Suppose $1 \leq l \leq n$. Ideally, given an $H^1_{\Delta_N}(\mathbb{R}^n)$-atom $a$, we would like to find $g, h \in L^2(\mathbb{R}^n)$ such that $\Pi_l(g, h) = a$ pointwise. While this can’t be accomplished in general, the Theorem below shows that it is “almost” true.

**Theorem 4.4.** Suppose $1 \leq l \leq n$. For every $H^1_{\Delta_N}(\mathbb{R}^n)$-atom $a(x)$ and for all $\varepsilon > 0$ there exist a large positive number $M$ and $g, h \in L^\infty(\mathbb{R}^n)$ with compact supports such that:

$$\|a - \Pi_l(h, g)\|_{H^1_{\Delta_N}(\mathbb{R}^n)} < \varepsilon$$

and $\|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^2(\mathbb{R}^n)} \leq CM^n$.

**Proof.** Let $a(x)$ be an $H^1_{\Delta_N}(\mathbb{R}^n)$-atom, supported in $B(x_0, r)$. We first consider the construction of the bilinear form $\Pi_l(h, g)$ for $1 \leq l \leq n - 1$ and the approximation to $a(x)$. To begin with, for the ball $B(x_0, r)$, we now consider the following cases: Case 1: $x_0,n \geq 0$; Case 2: $x_0,n < 0$.

We first consider Case 1. To begin with, fix $\varepsilon > 0$. Choose $M \in [100, \infty)$ sufficiently large so that $\frac{\log M}{M} < \varepsilon$. Now select $y_0 \in \mathbb{R}^n_+$ in the following way: for $1 \leq i \leq n$, choose $y_{0,i} > 0$ such that $y_{0,i} - x_{0,i} = \frac{Mr}{\sqrt{n}}$, where $x_{0,i}$ (reps. $y_{0,i}$) is the $i$th coordinate of $x_0$ (reps. $y_0$).

Note that for this $y_0$, it is clear that $B(y_0, r) \subset \mathbb{R}^n_+$ and we have $|x_0 - y_0| = Mr$. Moreover, for any $y \in B(y_0, r)$, we also have $|x_0 - y| > \frac{Mr}{2}$. We set

$$g(x) := \chi_{B(y_0, r)}(x) \quad \text{and} \quad h(x) := -\frac{a(x)}{R_{N,l}^*g(x_0)}.$$

We first claim that

$$|R_{N,l}^*g(x_0)| \geq CM^{-n}, \quad 1 \leq l \leq n - 1. \tag{4.3}$$

In fact, for $l = 1, \ldots, n - 1$, from Proposition 2.2, we have

$$R_{N,l}^*g(x_0) = \left| \int_{B(y_0, r)} R_{N,l}(y, x_0) \, dy \right|$$

$$= C_n \left| \int_{B(y_0, r)} \left( \frac{y_l - x_{0,l}}{|x_0 - y|^{n+1}} + \frac{y_l - x_{0,l}}{(|x_0' - y'|^2 + |x_{0,n} + y_n|^2)^{\frac{n+1}{2}}} \right) \, dy \right|$$

$$= C_n |y_l - x_{0,l}| \left| \int_{B(y_0, r)} \left( \frac{1}{|x_0 - y|^{n+1}} + \frac{1}{(|x_0' - y'|^2 + |x_{0,n} + y_n|^2)^{\frac{n+1}{2}}} \right) \, dy \right|$$

$$\geq CMr \int_{B(y_0, r)} \frac{1}{|x_0 - y|^{n+1}} \, dy \geq CM^{-n}.$$

As a consequence, we get that the claim (4.3) holds.

As for Case 2, we handle it in a symmetric way as follows. Fix $\varepsilon > 0$. Choose $M \in [100, \infty)$ sufficiently large so that $\frac{\log M}{M} < \varepsilon$. Now select $y_0 \in \mathbb{R}^n_+$ in the following
way: for $1 \leq i \leq n$, choose $y_{0,i} > 0$ such that $y_{0,i} - x_{0,i} = -\frac{Mr}{2}$. Note that for this $y_0$, it is clear that $B(y_0, r) \subset \mathbb{R}_+^n$ and we have $|x_0 - y_0| = Mr$. Moreover, for any $y \in B(y_0, r)$, we also have $|x_0 - y| > \frac{Mr}{2}$. We now define the functions $g$ and $h$ as in (4.2), and the following the same estimates, we can obtain that the claim (4.3) holds.

From the definitions of the functions $g$ and $h$, we obtain that $\text{supp } g(x) = B(y_0, r)$ and $\text{supp } h(x) = B(x_0, r)$. Moreover, from (4.3) we obtain that

$$\|g\|_{L^2(\mathbb{R}^n)} \approx r^\frac{n}{2} \quad \text{and} \quad \|h\|_{L^2(\mathbb{R}^n)} = \frac{1}{|R_{N,l}g(x_0)|}\|a\|_{L^2(\mathbb{R}^n)} \leq CM^{n-\frac{n}{2}}.$$

Hence $\|g\|_{L^2(\mathbb{R}^n)}\|h\|_{L^2(\mathbb{R}^n)} \leq CM^n$. Now write

$$a(x) - (h(x)R_{N,l}g(x) - g(x)R_{N,l}h(x)) = a(x)\frac{R_{N,l}^*g(x_0) - R_{N,l}^*g(x)}{R_{N,l}^*g(x_0)} - g(x)R_{N,l}h(x) =: W_1(x) + W_2(x).$$

By definition, it is obvious that $W_1(x)$ is supported on $B(x_0, r)$ and $W_2(x)$ is supported on $B(y_0, r)$.

We first turn to $W_1(x)$. For $x \in B(x_0, r)$,

$$|W_1(x)| = |a(x)|\frac{|R_{N,l}^*g(x_0) - R_{N,l}^*g(x)|}{R_{N,l}^*g(x_0)} \leq CM^n\|a\|_{L^\infty(\mathbb{R}^n)}\int_{B(y_0, r)} |R_{N,l}(y, x_0) - R_{N,l}(y, x)| dy \leq CM^n\int_{B(y_0, r)} \frac{|x - x_0|}{|x - y|^{n+1}} dy \leq C \frac{1}{Mr^n}.$$

Hence $|W_1(x)| \leq C \frac{1}{Mr^n} \chi_{B(x_0, r)}(x)$.

We next estimate $W_2(x)$. From the definition of $g(x)$, we have

$$|W_2(x)| = \chi_{B(y_0, r)}(x)|R_{N,l}h(x)| = \chi_{B(y_0, r)}(x)\frac{1}{|R_{N,l}^*g(x_0)|}\left|\int_{B(x_0, r)} R_{N,l}(x, y)a(y) dy\right| = \chi_{B(y_0, r)}(x)\frac{1}{|R_{N,l}^*g(x_0)|}\left|\int_{B(x_0, r)} R_{N,l}(x, y)a(y) dy\right|,$$

where the last equality follows from the fact that $x \in B(y_0, r) \subset \mathbb{R}_+^n$ and from the definition of the Riesz kernel $R_N(x, y)$ as in (2.9). Hence, from the cancellation property of $a_+(y)$, we get

$$|W_2(x)| = \chi_{B(y_0, r)}(x)\frac{1}{|R_{N,l}^*g(x_0)|}\left|\int_{B(x_0, r)} (R_{N,l}(x, y) - R_{N,l}(x, x_0))a_+(y) dy\right|.$$
\[ \leq C \chi_{B(y_0,r)}(x) M^n \int_{B(x_0,r)} \|a\|_{L^\infty(\mathbb{R}^n)} \frac{|y-x_0|}{|x-x_0|^{n+1}} dy \]

Combining the estimates of \( W_1 \) and \( W_2 \), we obtain that
\[ (4.4) \quad \left| a(x) - (h(x)R_{N,l}^g(x) - g(x)R_{N,l}h(x)) \right| \leq \frac{C}{M^{l-n}} \left( \chi_{B(x_0,r)}(x) + \chi_{B(y_0,r)}(x) \right). \]

Next we point out that
\[ (4.5) \quad \int \left[ a(x) - (h(x)R_{N,l}^g(x) - g(x)R_{N,l}h(x)) \right] dx \]
\[ = \int a(x)dx - \int (h(x)R_{N,l}^g(x) - g(x)R_{N,l}h(x)) dx \]
\[ = 0, \]

since \( a(x) \) has cancellation (Proposition 3.14) and the second integral equals 0 just by the definitions of \( g \) and \( h \).

Then the size estimate \((4.4)\) and the cancellation \((4.5)\), together with Lemma 4.3, imply that
\[ \left\| a(x) - (h(x)R_{N,l}^g(x) - g(x)R_{N,l}h(x)) \right\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq C \frac{\log M}{M} < C \epsilon. \]

This proves the result for \( 1 \leq l \leq n - 1 \).

We now consider the the bilinear form \( \Pi_n(g,h) \) and its approximation to \( a(x) \). Again, for the ball \( B(x_0,r) \), we now consider the following cases: Case 1: \( x_{0,n} \geq 0 \); Case 2: \( x_{0,n} < 0 \).

It suffices to consider the Case 1 since the other can be handled symmetrically. In this case, for \( x_0 \) with \( x_{0,n} \geq 0 \), choose \( y_0 \) such that \( y_{0,i} - x_{0,i} = \frac{M}{\sqrt{n}} \) for \( i = 1, \ldots, n \).

We now define the functions \( g \) and \( h \) as in \((4.2)\). This, together with Proposition 2.2, yields
\[ R_{N,l}^g(x_0) = \left| \int_{B(y_0,r)} R_{N,l}(y,x_0) dy \right| \]
\[ = C_n \left| \int_{B(y_0,r)} \frac{y_{n} - x_{0,n}}{|x_0 - y|^{n+1}} \frac{x_{0,n} + y_n}{(|x_0' - y'|^2 + |x_{0,n} + y_n|^2)^{\frac{n+1}{2}}} dy \right| \]
\[ \geq C_n \left| \int_{B(y_0,r)} \frac{y_{n} - x_{0,n}}{|x_0 - y|^{n+1}} dy \right| \]
\[ = C_n \left| y_{n} - x_{0,n} \right| \int_{B(y_0,r)} \frac{1}{|x_0 - y|^{n+1}} dy \]
\[ \geq CM^{-n}. \]

Hence, we obtain that the claim \((4.3)\) holds for these \( g \) and \( h \).
Now following the approximation as that for \( R_{N,l} \) with \( 1 \leq l \leq n - 1 \), we obtain that
\[
\left\| a(x) - (h(x)R_{N,l}^xg(x) - g(x)R_{N,l}^xh(x)) \right\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq C \frac{\log M}{M} < C\epsilon.
\]

\[ \square \]

With this approximation result, we can now prove the main Theorem 1.3, restated below for the convenience of the reader.

**Theorem 4.5.** Suppose \( 1 \leq l \leq n \). For any \( f \in H^1_{\Delta_N}(\mathbb{R}^n) \) there exists sequences \( \{\lambda_j^k\} \in \ell^1 \) and functions \( g_j^k, h_j^k \in L^\infty(\mathbb{R}^n) \) with compact supports such that
\[
f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_l(g_j^k, h_j^k).
\]

Moreover, we have that:
\[
\|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \approx \inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty |\lambda_j^k| \left\| g_j^k \right\|_{L^2(\mathbb{R}^n)} \left\| h_j^k \right\|_{L^2(\mathbb{R}^n)} : f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_l(g_j^k, h_j^k) \right\}.
\]

**Proof.** By Theorem 4.1 we have that \( \|\Pi_l(g, h)\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq C \|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^2(\mathbb{R}^n)} \), it is immediate that we have for any representation of \( f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_l(g_j^k, h_j^k) \) that
\[
\|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq C \inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty |\lambda_j^k| \left\| g_j^k \right\|_{L^2(\mathbb{R}^n)} \left\| h_j^k \right\|_{L^2(\mathbb{R}^n)} : f = \sum_{k=1}^\infty \sum_{j=1}^\infty \lambda_j^k \Pi_l(g_j^k, h_j^k) \right\}.
\]

We turn to show that the other inequality hold and that it is possible to obtain such a decomposition for any \( f \in H^1_{\Delta_N}(\mathbb{R}^n) \). By the atomic decomposition for \( H^1_{\Delta_N}(\mathbb{R}^n) \), Theorem 3.12, for any \( f \in H^1_{\Delta_N}(\mathbb{R}^n) \) we can find a sequence \( \{\lambda_j^l\} \in \ell^1 \) and sequence of \( H^1_{\Delta_N}(\mathbb{R}^n) \)-atoms \( a_j^l \) so that \( f = \sum_{j=1}^\infty \lambda_j^l a_j^l \) and \( \sum_{j=1}^\infty |\lambda_j^l| \leq C_0 \|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \).

We explicitly track the implied absolute constant \( C_0 \) appearing from the atomic decomposition since it will play a role in the convergence of the approach. Fix \( \epsilon > 0 \) so that \( \epsilon C_0 < 1 \). Then we also have a large positive number \( M \) with \( \frac{\log M}{M} < \epsilon \). We apply Theorem 4.4 to each atom \( a_j^l \). So there exists \( g_j^l, h_j^l \in L^\infty(\mathbb{R}^n) \) with compact supports and satisfying \( \|g_j^l\|_{L^2(\mathbb{R}^n)} \|h_j^l\|_{L^2(\mathbb{R}^n)} \leq CM^n \) and
\[
\|a_j^l - \Pi_l(g_j^l, h_j^l)\|_{H^1_{\Delta_N}(\mathbb{R}^n)} < \epsilon \quad \forall j.
\]

Now note that we have
\[
f = \sum_{j=1}^\infty \lambda_j^l a_j^l = \sum_{j=1}^\infty \lambda_j^l \Pi_l(g_j^l, h_j^l) + \sum_{j=1}^\infty \lambda_j^l \left( a_j^l - \Pi_l(g_j^l, h_j^l) \right) := M_1 + E_1.
\]

Observe that we have
\[
\|E_1\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq \sum_{j=1}^\infty |\lambda_j^l| \left\| a_j^l - \Pi_l(g_j^l, h_j^l) \right\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq \epsilon \sum_{j=1}^\infty |\lambda_j^l| \leq \epsilon C_0 \|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)}.
\]
We now iterate the construction on the function $E_1$. Since $E_1 \in H^1_{\Delta N}(\mathbb{R}^n)$, we can apply the atomic decomposition in $H^1_{\Delta N}(\mathbb{R}^n)$, Theorem 3.12, to find a sequence $\{\lambda^2_j\} \in \ell^1$ and a sequence of $H^1_{\Delta N}(\mathbb{R}^n)$-atoms $\{a^2_j\}$ so that $E_1 = \sum_{j=1}^{\infty} \lambda^2_j a^2_j$ and

$$\sum_{j=1}^{\infty} |\lambda^2_j| \leq C_0 \|E_1\|_{H^1_{\Delta N}(\mathbb{R}^n)} \leq \varepsilon C_0^2 \|f\|_{H^1_{\Delta N}(\mathbb{R}^n)}.$$  

Again, we will apply Theorem 4.4 to each atom $a^2_j$. So there exist $g^2_j, h^2_j \in L^\infty(\mathbb{R}^n)$ with compact supports and satisfying $\|g^2_j\|_{L^2(\mathbb{R}^n)} \|h^2_j\|_{L^2(\mathbb{R}^n)} \leq C M^n$ and

$$\|a^2_j - \Pi I(g^2_j, h^2_j)\|_{H^1_{\Delta N}(\mathbb{R}^n)} < \varepsilon, \quad \forall j.$$  

We then have that:

$$E_1 = \sum_{j=1}^{\infty} \lambda^2_j a^2_j = \sum_{j=1}^{\infty} \lambda^2_j \Pi I(g^2_j, h^2_j) + \sum_{j=1}^{\infty} \lambda^2_j (a^2_j - \Pi I(g^2_j, h^2_j)) := M_2 + E_2.$$  

But, as before observe that

$$\|E_2\|_{H^1_{\Delta N}(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} |\lambda^2_j| \|a^2_j - \Pi I(g^2_j, h^2_j)\|_{H^1_{\Delta N}(\mathbb{R}^n)} \leq \varepsilon \sum_{j=1}^{\infty} |\lambda^2_j| \leq (\varepsilon C_0)^2 \|f\|_{H^1_{\Delta N}(\mathbb{R}^n)}.$$  

And, this implies for $f$ that we have:

$$f = \sum_{j=1}^{\infty} \lambda^2_j a^2_j = \sum_{j=1}^{\infty} \lambda^2_j \Pi I(g^2_j, h^2_j) + \sum_{j=1}^{\infty} \lambda^2_j (a^2_j - \Pi I(g^2_j, h^2_j))$$

$$= M_1 + E_1 = M_1 + M_2 + E_2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda^k_j \Pi I(g^k_j, h^k_j) + E_2.$$  

Repeating this construction for each $1 \leq k \leq K$ produces functions $g^k_j, h^k_j \in L^\infty(\mathbb{R}^n)$ with compact supports and satisfying $\|g^k_j\|_{L^2(\mathbb{R}^n)} \|h^k_j\|_{L^2(\mathbb{R}^n)} \leq C M^n$ for all $j$, sequences $\{\lambda^k_j\} \in \ell^1$ with $\|\{\lambda^k_j\}\|_{\ell^1} \leq \varepsilon^{k-1} C_0^k \|f\|_{H^1_{\Delta N}(\mathbb{R}^n)}$, and a function $E_K \in H^1_{\Delta N}(\mathbb{R}^n)$ with $\|E_K\|_{H^1_{\Delta N}(\mathbb{R}^n)} \leq (\varepsilon C_0)^K \|f\|_{H^1_{\Delta N}(\mathbb{R}^n)}$ so that

$$f = \sum_{k=1}^{K} \sum_{j=1}^{\infty} \lambda^k_j \Pi I(g^k_j, h^k_j) + E_K.$$  

Passing $K \to \infty$ gives the desired decomposition of $f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda^k_j \Pi I(g^k_j, h^k_j)$. We also have that:

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda^k_j| \leq \sum_{k=1}^{\infty} \varepsilon^{-1} (\varepsilon C_0)^k \|f\|_{H^1_{\Delta N}(\mathbb{R}^n)} = \frac{C_0}{1 - \varepsilon C_0} \|f\|_{H^1_{\Delta N}(\mathbb{R}^n)}.$$  

Finally, we dispense with the proof of Theorem 1.4.
Proof of Theorem 1.4. The upper bound in this theorem is contained in Theorem 4.1. For the lower bound, we first note that from Theorem 3.12, $H^1_{\Delta_N}(\mathbb{R}^n)$ has equivalent characterizations via atoms, which shows that $H^1_{\Delta_N}(\mathbb{R}^n) \cap L^\infty_{c}(\mathbb{R}^n)$ is dense in $H^1_{\Delta_N}(\mathbb{R}^n)$ with respect to the $H^1_{\Delta_N}(\mathbb{R}^n)$ norm, where we use $L^\infty_{c}(\mathbb{R}^n)$ to denote the $L^\infty$ function with compact supports.

Then using the weak factorization in Theorem 1.3 we have that for $f \in H^1_{\Delta_N}(\mathbb{R}^n) \cap L^\infty_{c}(\mathbb{R}^n)$,

$$\left| \langle b, f \rangle_{L^2(\mathbb{R}^n)} \right| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k| \left| \langle b, \Pi_l(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}^n)} \right| = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k| \left| \langle g_j^k, [b, R_{N,l}]h_j^k \rangle_{L^2(\mathbb{R}^n)} \right|.$$

Hence we have that

$$\left| \langle b, f \rangle_{L^2(\mathbb{R}^n)} \right| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k| \left\| [b, R_{N,l}] \right\|_{L^2(\mathbb{R}^n)} \left\| g_j^k \right\|_{L^2(\mathbb{R}^n)} \leq \left\| [b, R_{N,l}] \right\|_{L^2(\mathbb{R}^n)} \left\| f \right\|_{H^1_{\Delta_N}(\mathbb{R}^n)}.$$

By the duality between $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ and $H^1_{\Delta_N}(\mathbb{R}^n)$ we have that:

$$\|b\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \approx \sup_{\|f\|_{H^1_{\Delta_N}(\mathbb{R}^n)} \leq 1} \left| \langle b, f \rangle_{L^2(\mathbb{R}^n)} \right| \leq C \left\| [b, R_{N,l}] \right\|_{L^2(\mathbb{R}^n)} \left\| f \right\|_{H^1_{\Delta_N}(\mathbb{R}^n)}.$$

□

5. The fractional integrals: proof of Theorem 1.5

Suppose $b$ is in $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$. Then according to [5, Proposition 4.2], we have that $b_{+,e} \in \text{BMO}(\mathbb{R}^n)$ and $b_{-,e} \in \text{BMO}(\mathbb{R}^n)$, and moreover,

$$\|b\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \approx \|b_{+,e}\|_{\text{BMO}(\mathbb{R}^n)} + \|b_{-,e}\|_{\text{BMO}(\mathbb{R}^n)}.$$

For every $f \in L^p(\mathbb{R}^n)$, we have

$$\| [b, \Delta_{\alpha/2}^N](f) \|_{L^q(\mathbb{R}^n)}^q = \int_{\mathbb{R}^n} [b, \Delta_{\alpha/2}^N](f)(x)^q \, dx + \int_{\mathbb{R}^n} [b, \Delta_{\alpha/2}^N](f)(x)^q \, dx =: I + II.$$

For the term $I$, note that when $x \in \mathbb{R}^n_+$, we have

$$[b, \Delta_{\alpha/2}^N](f)(x) = b(x)\Delta_{\alpha/2}^N(f)(x) - \Delta_{\alpha/2}^N(bf)(x) = b_{+,e}(x)\Delta_{\alpha/2}(f_{+,e})(x) - \Delta_{\alpha/2}(b_{+,e}f_{+,e})(x) = [b_{+,e}, \Delta_{\alpha/2}^N](f_{+,e})(x),$$

and when $x \in \mathbb{R}^n_-$, we have

$$[b, \Delta_{\alpha/2}^N](f)(x) = b(x)\Delta_{\alpha/2}^N(f)(x) - \Delta_{\alpha/2}^N(bf)(x) = b_{-,e}(x)\Delta_{\alpha/2}(f_{-,e})(x) - \Delta_{\alpha/2}(b_{-,e}f_{-,e})(x) = [b_{-,e}, \Delta_{\alpha/2}^N](f_{-,e})(x).$$
which implies that
\[
I = \int_{\mathbb{R}^n_+} [b, \Delta^{-\alpha/2}](f)(x)^q \, dx = \int_{\mathbb{R}^n_+} [b_{+,e}, \Delta^{-\alpha/2}](f_{+,e})(x)^q \, dx
\]
\[
\leq \int_{\mathbb{R}^n_+} [b_{+,e}, \Delta^{-\alpha/2}](f_{+,e})(x)^q \, dx
\]
\[
\leq C\|b_{+,e}\|_{\text{BMO}(\mathbb{R}^n)}^q \|f_{+,e}\|_{L^p(\mathbb{R}^n)}^q.
\]

For the last estimate we use the result [4, Theorem 1], which applies since we know from Proposition 2.9 that $R_{N,i}$ is a Calderón–Zygmund kernel. Similarly we can obtain that
\[
II \leq C\|b_{-,e}\|_{\text{BMO}(\mathbb{R}^n)}^q \|f_{-,e}\|_{L^2(\mathbb{R}^n)}^q.
\]

Combining the estimates for $I$ and $II$ above, we obtain that
\[
\left\| [b, \Delta_N^{-\alpha/2}](f) \right\|_{L^q(\mathbb{R}^n)}^q \leq C\|b_{+,e}\|_{\text{BMO}(\mathbb{R}^n)}^q \|f_{+,e}\|_{L^p(\mathbb{R}^n)}^q + C\|b_{-,e}\|_{\text{BMO}(\mathbb{R}^n)}^q \|f_{-,e}\|_{L^p(\mathbb{R}^n)}^q
\]
\[
\leq C\|b\|_{\text{BMO}_N(\mathbb{R}^n)}^q \left( \|f_{+,e}\|_{L^p(\mathbb{R}^n)}^q + \|f_{-,e}\|_{L^p(\mathbb{R}^n)}^q \right)
\]
\[
\leq C\|b\|_{\text{BMO}_N(\mathbb{R}^n)}^q \|f\|_{L^p(\mathbb{R}^n)}^q,
\]
which yields that \( \| [b, R_{N,i}] : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \| \leq C\|b\|_{\text{BMO}_N(\mathbb{R}^n)}^q \)

Acknowledgments: The authors would like to thank the referee for careful reading of this paper and for the helpful comments and suggestions, which made this paper more accurate. The authors would like to thank Professors Xuan Thinh Duong and Dongyong Yang for helpful discussions and for Professors Lixin Yan and Liang Song for providing their paper [14], which hadn’t appeared when the authors started this project in January 2015.

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