MINIMAL STATE SPACE REALIZATION, STATIC OUTPUT FEEDBACK AND MATRIX COMPLETION

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Abstract. We here characterize the minimality of realization of arbitrary linear time-invariant dynamical systems through (i) intersection of the spectra of the realization matrix and of the corresponding state submatrix and (ii) moving the poles by applying static output feedback.

1. Introduction

We first recall in the concept of state space realization. Let $F(s)$ be a $p \times m$-valued rational function, analytic at infinity, i.e. $\lim_{s \to \infty} F(s)$ exists. Then, $F(s)$ admits a state space realization

$$F(s) = C(sI - A)^{-1}B + D$$

with $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{p \times n}$ and $D \in \mathbb{C}^{p \times m}$, namely, $L \in \mathbb{C}^{(n+p) \times (n+m)}$.

Denote by $\nu$ the McMillan degree of $F(s)$, see e.g. [9, 6.5-9], [14, Remark 6.7.4], i.e. the minimal dimension of the corresponding state space realization, thus $n \geq \nu$. A realization is called minimal whenever $n = \nu$, see e.g. [14, Definition 6.5.9].

The issue of minimality of realization is fundamental, see e.g. [4, Sections 2.5, 2.6.2], [9, Section 2.4, Theorem 6.2-3], [11, Sections 6.2, 6.3]. Typically, one is first interested in the question of whether or not a given realization is minimal and if not, to find ways to extract out of the given non-minimal realization, a minimal one. For a survey of works addressing the second question see [3]. We here focus on the first problem.

For a system $F(s)$ admitting state space realization (1.1), we shall find it convenient to consider the associated system, $\hat{F}(s)$, with zero at infinity ("strictly proper" in engineering jargon) i.e.

$$\hat{F}(s) := F(s) - F(\infty) = C(sI - A)^{-1}B.$$

Thus, the corresponding $(n + p) \times (n + m)$ realization matrix is of the form

$$\hat{L} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$
Note that in both realizations, $L$ in (1.1) and $\hat{L}$ in (1.3) the associated controllable (observable) subspace is identical. In particular, minimality of these realizations is equivalent.

Recall that applying a static output feedback, see e.g. [9, Section 3.1], to an input-output (associated) system $y(s) = \hat{F}(s)u(s)$ means taking $u = Ky + u'$ with $u'$ an auxiliary input and a (constant) $m \times p$ matrix $K$. The resulting closed loop system is $y(s) = \tilde{F}_c(s)u'(s)$ with $F_c(s) = (I_p - \hat{F}K)^{-1}\hat{F}$. The corresponding closed loop realization matrix $L_{cl}$ is

$$L_{cl} = \begin{bmatrix} A_{cl} & B \\ C & 0 \end{bmatrix} \quad A_{cl} := A + BK. \tag{1.4}$$

For a (possibly rectangular) system $F(s)$ admitting state space realization (1.1), we shall also find it convenient to consider the squared realization matrix $L_{sq}$ without altering $A$ and thus preserving $n$. The naive (=inflating) version is obtained as follows: If $m > p$ by simply adding rows of zeros to the bottom $L$ until it is $(n + m) \times (n + m)$, and if $p > m$ by adding columns of zeros to the right side of $L$ until it is $(n + p) \times (n + p)$. This approach preserves the associated controllable and observable subspaces. In particular, if in force, minimality of realization is preserved. As mentioned, the dimensions of $L_{sq}$ here are as the larger dimension of the original $L$.

In Lemma 3.1 we show that if a rectangular realization $\hat{L}$ (see (1.3)) of an associated system is minimal, one can construct a reduced dimension square realization $L_{sq}$ of dimensions which are at most, the smaller dimension of $L$, while preserving $A$ and the minimality of the realization. Moreover, the new $B$ and $C$ matrices are of full rank.

We can now state our main results.

**Theorem 1.1.** Let $L \in \mathbb{C}^{(n+p)\times(n+m)}$ be a realization of a $p \times m$-valued rational function $F(s)$, see (1.1).

Let $\hat{L} \in \mathbb{C}^{(n+p)\times(n+m)}$ be a realization of the $p \times m$-valued associated system $\hat{F}(s)$ see (1.2). A static output feedback $u = Ky + u'$ (with $K \in \mathbb{C}^{m \times p}$ constant) is applied to $F(s)$ so that the closed loop system matrix is as in (1.4).

Let also $L_{sq}$ be a realization of the corresponding squared system $F_{sq}(s)$.

For $A$ in (1.1) or in (1.3), for some $q \in [1, n]$ denote

$$\text{spect}(A) = \{\lambda_1, \ldots, \lambda_q\}$$

with $\lambda_j$ distinct.

The following are equivalent.

(i) The realization of $F(s)$ is minimal.

(ii) There exists a (constant) $K \in \mathbb{C}^{m \times p}$ so that (1.4),

$$\text{spect}(A) \cap \text{spect}(A_{cl}) = \emptyset. \tag{1.5}$$

(iii) Consider the realization $L$ of $F(s)$ in (1.1) with $m = p$ and assume that $B$ and $C$ are of a full rank. (If this was not the case take the reduced dimension squared counterpart $L_{sq}$ and $F_{sq}(s)$ in item 5 in Lemma 3.1).
Then,
\begin{equation}
q \bigcap_{j=1}^{q} \text{spect} \left( \frac{A}{D} (\lambda_j - \epsilon)I_p \right) \bigcap \text{spect}(A) = \emptyset,
\end{equation}
for some sufficiently small \( \epsilon > 0 \) and \( \lambda_j \in \text{spect}(A) \).

(iv) Consider the realization \( L \) of \( F(s) \) in (1.1) with \( m = p \) (if \( m \neq p \) take the squared counterpart \( L_{sq} \) and \( F_{sq}(s) \)). Then,
\begin{equation}
\bigcap_{D \in \mathbb{C}^p \times p} \text{spect} \left( \frac{A}{D} \right) \bigcap \text{spect}(A) = \emptyset.
\end{equation}

The outline of the paper is as follows. In Section 2 we provide motivation for our results and in Section 3 we present the necessary background. In Section 4 we prove Theorem 1.1.

2. Motivation

In this section we offer a perspective to our main results.

1. Output feedback

Although typically stated differently, the following is well known, see e.g. [4, Section 2.4.3], [9, Section 4.2], [14, Chapter 7].

Observation 2.1. Given an associated system \( \hat{F}(s) \) and its realization \( \hat{L} \) see (1.2) and (1.3).

The loop is closed by applying a state feedback gain to a (Luenberger) observed state.

The realization \( \hat{L} \) is minimal if and only if the closed loop poles may be located anywhere in the complex plane.

In engineering jargon, minimal realization enables one to place the poles of a closed loop system anywhere in the complex plane through a dynamic output feedback.

The simplicity of static output feedback has made it very attractive. However, exploring its properties turned out to be challenging, see e.g. [9, Section 3.1], [5], [6], and [15].

Condition (1.5) in Theorem 1.1 may be viewed as establishing a precise connection between minimal realization and static output feedback.

2. Matrix completion

Matrix completion (a.k.a. extension) problems have been of interest in the past 60 years. Many of them can be cast in the following framework: A part of a matrix is prescribed, can one complete the missing part so that the full matrix will possess certain properties, typically spectral. For a nice survey, see [2]. The case where the upper triangular part is prescribed was addressed in [1] (and not cited in [2]).

As a special case, assume that \( L = (\frac{A}{C} \frac{B}{*}) \), where \( A, B, C \) are prescribed and * stands for a square unprescribed part. Characterizing all possible characteristic polynomials of \( L \) is known to be difficult, see comment following [2, Theorem 46].

Condition (1.7) in Theorem 1.1 can be seen as answering a more modest question: Under what conditions can one complete \( (\frac{A}{C} \frac{B}{*}) \) with \( D \) so that the spectra of the resulting \( L \) and \( A \) will not (or will always) intersect.

3. Pole placement and matrix completion

The fact that problems matrix completion and pole placement through feedback, are linked is well known. See e.g.
The equivalence of (1.4) and (1.7) in Theorem 1.1 falls into this category.

4. PBH test for minimality  Consider a rational function $F(s)$ and its realization as in (1.1) or (1.2). As already mentioned, the issue of minimality of realization is fundamental.

Adopting Kailath’s terminology, The Popov-Belevitch-Huaat (PBH) Rank Tests, [9, Theorem 6.2-6] say that: A pair $(A, B)$ is controllable, if and only if

(2.1) $\text{rank} (\lambda I - A \ B) = n \quad \forall \lambda \in \mathbb{C}.$

A pair $(A, C)$ is observable, if and only if

(2.2) $\text{rank} (\lambda I - A \ C) = n \quad \forall \lambda \in \mathbb{C}.$

It is clear that without loss of generality one can confine the search in (2.1) and (2.2) to $\lambda \in \text{spect}(A)$.

We here examine the adaptation of these tests in two aspects:

(i) To minimality of realization (without independently testing for controllability and observability).

(ii) To consider the spectrum of $L_{sq}$, a square realization matrix.

To this end recall that a realization is minimal if and only if it is both controllable and observable, see e.g. [4, Theorem 2.33], [9, Theorem 6.2-3] [14, Definition 6.5.3, Theorem 27].

From a combination of the PBH Rank Tests in (2.1) and (2.2) it follows says that if a realization is minimal, then

(2.3) $\min_{\lambda \in \mathbb{C}} \text{rank} \left( \frac{\lambda I - A \ B}{s^2 - \alpha^2} \right) = n + \min (\text{rank}(B), \text{rank}(C)).$

The following example illustrates the fact that formulating the converse to (2.3) is more delicate. Consider the $2 \times 1$ rational function of McMillan degree two

$$F(s) = \begin{pmatrix} \frac{2\beta s^2}{s^2 - \alpha^2} \\ \frac{\gamma(s + 2\alpha)}{s + \alpha} \end{pmatrix}, \quad 0 \neq \alpha, \beta, \gamma \in \mathbb{R}.$$  

Its realization is

$$L = \begin{pmatrix} 0 & \beta \\ -\alpha & \gamma \\ 0 & \beta \\ \gamma & 0 \end{pmatrix}.$$  

Namely, $n = 2$, $m = 1$ and $p = 2$. Although this realization is minimal, (2.3) does not hold:

$$\min_{\lambda \in \mathbb{C}} \text{rank} \left( \frac{\lambda I - A \ B}{0} \right) = \text{rank} \left( \frac{\lambda I - A \ B}{0} \right)_{|\lambda = \alpha} = \text{rank} \left( \begin{pmatrix} -\alpha & 0 & \beta \\ 0 & \alpha & \gamma \\ 0 & \gamma & 0 \end{pmatrix} \right) = 2.$$  

(Indeed, the non-zero vector $\begin{pmatrix} -\beta \\ -\alpha \end{pmatrix}$ is in the nullspace of the rightmost matrix). On the other hand

$$n + \min (\text{rank}(B), \text{rank}(C)) = 2 + 1 = 3.$$
One may view (1.7) in Theorem 1.1 as a correct extension of the PBH Rank tests to the realization matrix $L$.

5. PBH test for minimality of scalar systems

There is a clear gap between scalar and matrix-valued rational functions.

Proposition 2.2. Let $L \in \mathbb{C}^{(n+p) \times (n+m)}$ be a realization of a $p \times m$-valued rational function $F(s)$, see (1.1).

Let $\hat{L} \in \mathbb{C}^{(n+p) \times (n+m)}$ be a realization of the $p \times m$-valued associated system $\hat{F}(s)$ see (1.2).

Let also $L_{sq}$ be a realization of the corresponding squared system $F_{sq}(s)$.

For $A$ in (1.1) or in (1.3), for some $q \in [1, n]$ denote

$$\text{spect}(A) = \{\lambda_1, \ldots, \lambda_q\}$$

with $\lambda_j$ distinct.

Consider the following statements

(i) For any $D \in \mathbb{C}^{p \times p}$

$$(2.4) \quad \text{spect} (\begin{bmatrix} A & B \\ C & D \end{bmatrix}) \bigcap \text{spect}(A) \neq \emptyset.$$  

(ii) There exists a $D \in \mathbb{C}^{p \times p}$ so that

$$(2.4) \quad \text{spect} (\begin{bmatrix} A & B \\ C & D \end{bmatrix}) \bigcap \text{spect}(A) \neq \emptyset.$$  

(iii) Consider the realization $L$ of $F(s)$ in (1.1) with $m = p$ (if $m \neq p$ take the squared counterpart $L_{sq}$ and $F_{sq}(s)$). For each $j = 1, \ldots, q$ there exists $D_j \in \mathbb{C}^{p \times p}$ so that

$$\lambda_j \notin \text{spect} (\begin{bmatrix} A & B \\ C & D_j \end{bmatrix}).$$

(iv) The realization is minimal.

Then,

$$(i) \implies (ii) \implies (iii) \implies (iv)$$

If $F(s)$ is a scalar rational function, namely $m = p = 1$, then $(iv) \implies (i)$.

Indeed, the fact that $(i) \implies (ii) \implies (iii)$ is straightforward. The equivalence of $(iii)$ and $(iv)$ is established in Theorem 1.1.

The fact that for scalar systems $(iv) \implies (i)$ was first proved (in an elaborate way) in [12] Theorem 4.1. We next illustrate a straightforward way of showing that.

Without loss of generality one can take $A$ to be in its Jordan canonical form. For example,

$$L = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 \\ 0 & \lambda_2 & 1 & 0 & 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & \lambda_4 & 1 & 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 & 1 & 0 & b_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_7 & 0 & b_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_8 & b_8 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & D \end{bmatrix}.$$
Using the PBH tests, controllability implies, see e.g. (2.1), that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are distinct and $b_3, b_5, b_7, b_8$ are non-zero. Observability implies, see e.g. (2.2), that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are distinct and $c_1, c_4, c_6, c_8$ are non-zero. Namely, minimality of realization means that

$$L = \begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & b_3 \\
0 & 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & b_5 \\
0 & 0 & 0 & 0 & 0 & \lambda_3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & b_7 \\
c_1 & * & * & c_4 & * & c_6 & * & c_8 \\
\end{pmatrix},$$

where $*$ stands for “don’t care” and $b_3 b_5 b_7 b_8 \neq 0$, $c_1 c_4 c_6 c_8 \neq 0$. However, this is exactly the condition for having $(L - \lambda_j I_9)$ of a full rank for $j = 1, 2, 3, 4$. (Else, there are rows or columns of zeroes). To sum-up, in the context of scalar systems minimality of realization and condition (2.4) are equivalent.

The following example illustrates the gap between these conditions for matrix-valued rational functions.

**Example 2.3.** Consider the scalar rational functions $f_1(s) = \frac{2+s}{s}$ and $f_2(s) = \frac{s}{s-2}$.

The corresponding minimal realizations are

$$L_1 = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \quad L_2 = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $0 = \text{spec}(A_1) \notin \text{spec}(L_1)$ and $2 = \text{spec}(A_2) \notin \text{spec}(L_2)$.

From the above $f_1(s)$ and $f_2(s)$ we now construct the following matrix valued rational function,

$$F(s) = \begin{pmatrix} \frac{2+s}{s} & 0 \\ 0 & \frac{s}{s-2} \end{pmatrix}.$$

Its minimal realization is given by

$$L = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that not only (2.4) is no longer true, in fact (2.5) no longer true, in fact

$$\{0, 2\} = \text{spec}(A) \subset \text{spec}(L).$$

Moreover, if one generalizes this system to

$$F(s) = \begin{pmatrix} \frac{2+s}{d_2} & d_2 \\ \frac{s}{d_3} & \frac{s}{s-2} \end{pmatrix},$$

where $d_2, d_3$ are arbitrary constants, the corresponding minimal realization is given by

$$L = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & d_2 \\ 0 & 1 & d_3 & 1 \end{pmatrix}.$$

Since,

$$(L - 0 \cdot I_4) \begin{pmatrix} d_2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = 0 \quad (L - 2 \cdot I_4) \begin{pmatrix} -d_3 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0,$$

it follows that (2.5) holds for all $d_2, d_3$. $\square$
3. BACKGROUND - TRUNCATED REALIZATION

We here provide the background for obtaining a square truncated realization \( L_{sq} \) out of a rectangular \( L \).

For a given associated system \( \hat{F}(s) \) in (1.2), and its realization \( \hat{L} \) in (1.3), we here explore \( \hat{L} \) the realization of

\[
\hat{F}(s) := T_c \hat{F}(s) T_b
\]

with \( T_b \in \mathbb{C}^{m \times \hat{m}} \) and \( T_c \in \mathbb{C}^{\hat{p} \times p} \) of a full rank. Namely,

\[
\hat{L} = \begin{pmatrix}
A & B \\
C & 0_{\hat{p} \times \hat{m}}
\end{pmatrix},
\]

with \( BT_b = \hat{B} \in \mathbb{C}^{n \times \hat{m}} \) and \( T_c C = \hat{C} \in \mathbb{C}^{\hat{p} \times n} \). Specifically, we are interested in preserving the associated controllable and observable subspaces. It is well known that a sufficient condition for that is having \( T_b \) and \( T_c \) square non-singular, see e.g. [4, Lemmas 2.25, 2.29], [9, Theorem 6.2-4], [11, Theorem 6.1.4], [14, Theorem 27].

Hence, we here focus on the case where \( m + p > \hat{m} + \hat{p} \), so \( \hat{F}(s) \) is a square, truncated version of \( \hat{F}(s) \). To this end we need some background.

We now address some quantities associated with a given matrix \( A \in \mathbb{C}^{n \times n} \). We start by introducing two notions: We shall call the controllability rank of \( A \), denoted by \( \beta(A) \), the minimal rank of a matrix \( B \in \mathbb{C}^{n \times m} \) so that the pair \( A, B \) is controllable, see (2.1). In analogy, we shall call the observability rank of \( A \), denoted by \( \gamma(A) \), the minimal rank of \( C \in \mathbb{C}^{p \times n} \) so that the pair \( A, C \) is observable, see (2.2).

Next recall that the geometric multiplicity of \( \lambda \in \text{spect}(A) \) is the number of linearly independent eigenvectors corresponding to this \( \lambda \), see e.g. [7, Definition 1.4.3]. For a given matrix \( A \in \mathbb{C}^{n \times n} \), we shall denote by \( \alpha(A) \) the largest geometric multiplicity among its eigenvalues.

Recall also that a matrix \( A \) is called non-derogatory if and only if \( \alpha(A) = 1 \). (This is equivalent to having the minimal and the characteristic polynomials equal, see e.g. [7 Definitions 1.4.4, 3.2.4.1], [8 Corollary 4.4.18]). This is closely related to companion form, see e.g. [7 Theorem 3.3.15] and in electrical engineering terminology controller form see e.g. [9 Section 2.3]. For a nice treatment see [4 Section 2.2.4], [11 Lemma 6.1.1].

**Lemma 3.1.** Given \( \hat{L} \) in (1.3) a realization of an associated system and the respective reduced dimension realization \( \hat{L} \) in (3.1).

1. One can always choose \( \hat{B} \) (\( \hat{C} \)) to be of a full rank, while preserving the controllable (observable) subspace.

2. If the pair \( A, B \) is controllable, one can always choose \( \hat{B} \in \mathbb{C}^{n \times \hat{m}} \) so that the pair \( A, \hat{B} \) is controllable with \( \hat{m} \) arbitrary in the range \([\beta(A), m]\).
   In particular, if \( \hat{m} = \beta \) then \( \hat{B} \) is of a full rank.

   If the pair \( A, C \) is observable, one can always choose \( \hat{C} \in \mathbb{C}^{\hat{p} \times n} \) so that the pair \( A, \hat{C} \) is observable with \( \hat{p} \) arbitrary in the range \([\gamma(A), p]\).
   In particular, if \( \hat{p} = \gamma \) then \( \hat{C} \) is of a full rank.
3. One can always choose $\hat{B}, \hat{C}$ to satisfy both 1. and 2.

4. For a given matrix $A \in \mathbb{C}^{n \times n}$ denote by $\alpha(A)$ the largest geometric multiplicity among its eigenvalues. $\beta(A)$ and $\gamma(A)$, the controllability and observability ranks, respectively, are equal and given by $\alpha(A)$.

5. If a realization $\hat{L}$ is minimal, for arbitrary $r$ in the range $[\alpha(A), \min(m, p)]$ one can take $\hat{m} = \hat{p} = r$ with $\hat{L} = L_{sq}$ a minimal realization.

In particular, $\hat{B}$ and $\hat{C}$ are of a full rank whenever $r = \alpha(A)$.

Proof 1. If $p > \hat{p} = \text{rank}(C)$ let $t_{\hat{p}+1}, \ldots, t_n \in \mathbb{C}^n$ a basis for the null-space of $C$, i.e. they are linearly independent and $Ct_j = 0$. Let now $t_1, \ldots, t_{\hat{p}} \in \mathbb{C}^n$ a basis for the orthogonal complement of the null-space of $C$. Then, taking $T_\varepsilon = [t_1: \cdots : t_{\hat{p}}]$ satisfies the requirements.

The construction of $T_\varepsilon$ is analogous and thus omitted.

2. Assume first that $(A, B)$ and $(A, \hat{B})$ are controllable. From (2.1) one has that $\text{rank}(\lambda I - A \hat{B} 0_{n \times (m - \hat{m})}) = \text{rank}(\lambda I - A \hat{B}) = \text{rank}(\lambda I - A B) = n \quad \forall \lambda \in \mathbb{C}$.

Thus, see e.g. [7, 0.4.6(c)], there exist $V_l \in \mathbb{C}^{n \times n}$ and $V_r \in \mathbb{C}^{(n+m) \times (n+m)}$ (the subscripts stand for “left” and “right”) both non-singular so that

$$(\lambda I - A \hat{B} 0_{n \times (m - \hat{m})}) = V_l (\lambda I - A B) V_r \quad \forall \lambda \in \mathbb{C}. $$

From the structure, one can take $V_l = I_n$ and $V_r = \text{diag}\{I_n, \hat{V}_r\}$, with $\hat{V}_r \in \mathbb{C}^{m \times m}$ non-singular. Take $T_b$ to be the first $\hat{m}$ columns of $\hat{V}_r$.

The minimality of $\beta$ implies that if $\hat{m} = \beta$ then $\hat{B}$ must be of a full rank.

The construction of $T_b$ is analogous and thus omitted.

3. One can apply first $T$ from item 2. and if still necessary, then apply $T$ from item 1.

4. We start by showing that $\beta(A) = \alpha(A)$.

By assumption, there exists $\lambda \in \text{spec}(A)$ so that

$$\text{rank}(\lambda I_n - A) = n - \alpha.$$ 

Hence, if $\alpha > \text{rank}(B)$, the controllability condition (2.1) can not hold. Thus, $\beta(A) \geq \alpha(A)$.

To establish equality, recall that without loss of generality one can assume that $A$ is in its Jordan canonical form where $A = \text{diag}\{A_1, \ldots, A_\alpha\}$ with $j = 1, \ldots, \alpha(A)$ are so that each $A_j$ is a $k_j \times k_j$ non-derogatory matrix with $k_1 + \ldots + k_\alpha = n$.

Let now $1_k$ denote a $k$-dimensional vector of 1’s and let $B \in \mathbb{C}^{n \times \alpha}$ be as follows,

$$B = \begin{pmatrix} 1_{k_1} & 0 & \cdots & 0 \\ 0 & 1_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{k_\alpha} \end{pmatrix}.$$ 

A straightforward calculation reveals that (2.1) holds and thus the pair $(A, B)$ is controllable.
Proofing that  
\[ \gamma(A) = \alpha(A), \]
is quite analogous and thus omitted.

5. This follows from combining the previous items. \( \square \)

4. PROOF OF THE MAIN RESULTS

**Proof of Theorem 1.1**

(i) \( \Rightarrow \) (ii)

From Lemma 3.1 it follows that there exist full rank \( T_b \in \mathbb{C}^{m \times \alpha(A)} \) and \( T_c \in \mathbb{C}^{\alpha(A) \times p} \) so that \( \hat{B} := BT_b \in \mathbb{C}^{n \times \alpha(A)} \) and \( \hat{C} := T_cC' \in \mathbb{C}^{\alpha(A) \times n} \) are of full rank and if the realization triple \((A, B, C)\) was minimal so is \((A, \hat{B}, \hat{C})\).

Take now in (1.5) \( K = \delta T_cT_b \) where \( \delta > 0 \) is a scalar parameter. By construction \( K \in \mathbb{C}^{m \times p} \) is of rank \( \alpha(A) \).

Thus

\[ (4.1) \quad A_{cl} = A + BKC = A + \delta \hat{B}\hat{C} \]

Note now that if \( 0 \neq v_r \in \mathbb{C}^n \) is in the null-space of \( C \), observability implies that with this \( v_r \),

\[ (A_{cl} - \lambda_j I_n)v_r = (A - \lambda_j I_n)v_r \neq 0 \quad j = 1, \ldots, q. \]

Assume now that \( v_r \in \mathbb{C}^n \) is in the orthogonal complement of the null-space of \( C \) thus \( Cv_r \neq 0 \). By construction also \( BCv_r \neq 0 \). Hence for \( j = 1, \ldots, q \),

\[
\| (A_{cl} - \lambda_j I_n)v_r \| = \| (A - \lambda_j I_n) + \delta \hat{B}\hat{C} \| v_r \| \\
= \| (A - \lambda_j I_n) v_r + \delta \hat{B}\hat{C} v_r \| \\
= \| (A - \lambda_j I_n) v_r \| - \| \delta \hat{B}\hat{C} v_r \| \quad |\delta \text{ sufficiently large}| > 0.
\]

Namely, \( (A_{cl} - \lambda_j I_n)v_r = 0 \).

Similarly, if \( 0 \neq v_l \in \mathbb{C}^n \) is so that \( v_l^* B = 0 \), controllability implies that with this \( v_l \)

\[ v_l^* (A_{cl} - \lambda_j I_n) = v_l^* (A - \lambda_j I_n) \neq 0 \quad j = 1, \ldots, q. \]

Assume now that \( v_l \in \mathbb{C}^n \) is in the orthogonal complement of the null-space of \( B^* \) thus, \( v_l^* B \neq 0 \). By construction also \( v_l^* BC \neq 0 \). Hence for \( j = 1, \ldots, q \),

\[
\| v_l^* (A_{cl} - \lambda_j I_n) \| = \| v_l^* (A - \lambda_j I_n) + \delta \hat{B}\hat{C} \| \\
= \| v_l^* (A - \lambda_j I_n) + \delta v_l^* \hat{B}\hat{C} \| \\
= \| \| v_l^* (A - \lambda_j I_n) \| - \| \delta v_l^* \hat{B}\hat{C} \| \quad |\delta \text{ sufficiently large}| > 0.
\]

Namely, \( v_l^* (A_{cl} - \lambda_j I_n) \neq 0 \) so this part of the claim is established.

(ii) \( \Rightarrow \) (iii)

First one can always write

\[ L_{sq} - \lambda I_{n+p} = \begin{pmatrix} A - \lambda I_n & B \\ C & D - \lambda I_p \end{pmatrix} = \begin{pmatrix} I_n & B(D - \lambda I_p)^{-1} \\ 0 & I_p \end{pmatrix} \left( \begin{pmatrix} A_{cl} - \lambda I_n & 0 \\ C & D - \lambda I_p \end{pmatrix} \right), \]

whenever \( \lambda \notin \text{spec}(D) \) and

\[ (4.2) \quad A_{cl} := A + BKC \quad \text{with} \quad K := (\lambda I_p - D)^{-1}. \]
Thus, the non-singularity of the matrix \( L - \lambda I_{n+p} \) and of the matrix \( A_{cl} - \lambda I_n \) are equivalent. Namely, \( \lambda \in \text{spec}(A_{cl}) \) if and only if \( \lambda \in \text{spec}(L_{sq}) \).

For \( j = 1, \ldots, q \) take now \( D_j = (\lambda_j - \epsilon)I_p \) with \( \lambda_j \in \text{spec}(A) \) and \( \epsilon > 0 \) sufficiently small. From (4.2) it follows that that \( A_{cl} \) is of the form (4.1) with \( \delta = \epsilon^{-1} \) thus \( \lambda_j \notin \text{spec}(A_{cl}) \) and by the above construction \( \lambda_j \notin \text{spec}(L_{sq}) \), so this part of the claim is established.

(iii) \( \implies \) (iv) Trivial.

(iv) \( \implies \) (i)

We find it more convenient to show that if the realization is not minimal, so that

\[
\exists \lambda \in \text{spec}(A) \quad \text{so that} \quad \lambda \in \bigcap_{D \in \mathbb{C}^{p \times p}} \text{spec } (\begin{pmatrix} A & B \\ C & * \end{pmatrix}).
\]

If the realization \( L \) is not controllable, from condition (2.1) it follows that there exists \( \lambda \in \mathbb{C} \) so that

\[
n - 1 \geq \text{rank}(\lambda I_n - A B).
\]

This implies that for that same \( \lambda \)

\[
n + p - 1 \geq \text{rank}(\begin{pmatrix} \lambda I_n & A \\ C & * \end{pmatrix})
\]

where \(*\) stands for “don’t care”. Namely, this \( \lambda \) is in \( \text{spec } (\begin{pmatrix} A & B \\ C & * \end{pmatrix}) \). Note now that (4.3) implies that, this \( \lambda \) is in \( \text{spec}(A) \) (else \( \text{rank}(\lambda I_n - A) = n \)). Thus (4.3) holds.

Similarly, if the realization \( L \) is not observable, from condition (2.2) it follows that there exists \( \lambda \in \mathbb{C} \) so that

\[
n - 1 \geq \text{rank}(\lambda I_n - A C)
\]

As before, (4.5) and thus this \( \lambda \) is in \( \text{spec } (\begin{pmatrix} A & B \\ C & * \end{pmatrix}) \). Note now that (4.6) implies that, this \( \lambda \) is in \( \text{spec}(A) \) (else \( \text{rank}(\lambda I_n - A) = n \)). Thus (4.3) holds and the proof is complete.

We conclude by pointing out that (for \( m = p \)) from Theorem 1.1 it follows that having

\[
\text{spec}(A) \bigcap \text{spec } (\begin{pmatrix} A & B \\ C & 0_{p \times p} \end{pmatrix}) = \emptyset,
\]

implies minimal realization. In Subsection 2.5 we showed that for \( m = p = 1 \) the converse is true as well. However, we do not know whether or not the converse holds for \( m = p \geq 2 \).

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