ON POSITIVE AND ALMOST ALTERNATING LINKS

KAZUHIKO INOUE

Abstract. In this paper, we show that a link which has a positive and almost alternating diagram is alternating, besides that a positive and non-alternating Montesinos link has an almost positive-alternating diagram.

1. Introduction

A link is a disjoint union of circles embedded in $S^3$, and a knot is consist of one circle. A diagram of a link is a generic projection of a link on $S^2$ with over/under information for each double point. A diagram is alternating if the over-crossings and under-crossings appear alternately along every component of the diagram, and a link is alternating if it has an alternating diagram. A link diagram is almost alternating if one crossing change makes it into an alternating diagram, and a link is almost alternating if it has an almost alternating diagram and no alternating diagram.

A diagram is positive if the sign of every crossing is positive. (A negative diagram is the mirror image of a positive diagram.) A link is positive if it has a positive diagram, and a link is positive and alternating if it has a positive diagram and an alternating diagram. A link is positive-alternating if it has a positive and alternating diagram. Nakamura showed that every positive and alternating link has a positive-alternating diagram ([11]). (We call a positive-alternating diagram, PA-diagram, and a positive and alternating link, PA-link for short.) So our concern is a positive and almost alternating link, that is to say a link which has a positive diagram and almost alternating diagram and has no alternating diagram.

In section 3, we show the following:

Theorem 3.1 Let $L$ an oriented link. If $L$ has a positive and almost alternating diagram then $L$ is alternating.

Besides we know that every positive and almost alternating knot has an almost positive-alternating diagram with up to eleven crossings. Furthermore Jong and Kishimoto showed that every positive knot up to genus two is positive-alternating or almost positive-alternating([7]). A diagram is almost positive-alternating if one crossing change makes it into a PA-diagram. We say such diagram almost PA-diagram. In this paper we show the following:

Proposition 4.1 Every positive Montesinos link has an almost positive-alternating diagram.

This paper is organized as follows: In Section 2, we briefly review almost alternating links and positive alternating links. In Section 3, we prove Theorem 3.1. In Section 4, we characterize positive Montesinos links and prove Proposition 4.1.
2. Preliminary

First we shortly introduce some definitions. A diagram $D$ is said to be equivalent to a diagram $D'$ if they both represent same link. A diagram is said to be reduced if there exists no crossing such that the diagram is separated by splicing the crossing as shown in Figure 1(1). A diagram is said to be II-reduced if there are no obvious removal Reidemeister-II move i.e. the link contains no 2-tangle as shown in Figure 1(2) (See [12]).

![Figure 1. A reducible diagram and a II-reducible diagram](image)

A flype is an isotopy move applied on a sub tangle of the form $[±1] + t$, and it fixes the endpoints of the sub tangle. See Figure 2. A flype preserves the alternating structure of a diagram. ([8])

![Figure 2. A flype](image)

We distinguish a positive tangle from a positive diagram as the following. A tangle is positive if it is as shown in Figure 3(1) and negative as shown in (2). The sign of a crossing point is $+$ if it is as shown in Figure 3(3) and $-$ as shown in (4). A diagram is positive (resp. negative) if every crossing point in the diagram has the same sign $+$ (resp. $-$).

![Figure 3. A positive(negative) tangle and the sign of a crossing point](image)

Next we introduce some results about almost alternating links. First any alternating link involving a trivial link has an almost alternating diagram. Moreover any alternating link has infinite almost alternating diagrams(see [2]). For example we can make infinite almost alternating diagrams from a trefoil knot as shown in Figure 4. Every diagram turns into an alternating diagram if we change the crossing point $d$.

**Theorem 2.1.** Every positive and almost alternating knot is almost positive-alternating with up to eleven crossings ([4]).
Theorem 2.2. Positive knots up to genus two are positive-alternating or almost positive-alternating (\cite{7}).

Theorem 2.3. Non-alternating Montesinos links are almost alternating (\cite{1}).

Theorem 2.4. Any reduced alternating diagram of a positive alternating link is positive-alternating (\cite{11}).

Then our concern at the moment is the following question.
Question: How is the diagram of a positive and almost alternating link?
In section 3, we show that a positive and almost alternating link does not have a positive and almost alternating diagram for the partial answer to the question above.

3. Main Theorem

Theorem 3.1. Let $L$ be an oriented link. If $L$ has a positive and almost alternating diagram then $L$ is alternating.

Proof 3.2. By the assumption above $L$ has an almost alternating diagram, so we see $L$ is alternating or almost alternating. Our claim is that every positive and almost alternating diagram of $L$ is equivalent to an alternating diagram. First of all if a diagram $D$ is reducible then $D$ is equivalent to an alternating diagram as shown in Figure 5. If $D$ is $\Pi$-reducible we can see this is equivalent to an alternating diagram in a similar fashion. Therefore we can assume that $D$ is both reduced and $\Pi$-reduced.

Figure 5. A reducible almost alternating diagram

In general positive and almost alternating diagrams are as shown in Figure 6, where the diagram in every shaded portion is positive-alternating.

Figure 6. positive and almost alternating diagram
The shaded portion in the rightmost figure is equivalent to a disk and we denote this region by $A$. Note that there does not happen the case as shown in Figure 7, because this diagram is not positive.

![Figure 7. This diagram is not positive](image)

Assume disk $A$ separates into disk $A_1$ and disk $A_2$. Since each diagram in $A_1$ and $A_2$ is alternating, then the diagram $D$ is equivalent to an alternating diagram $D'$. (See Figure 8.) Hence $L$ is alternating.

![Figure 8.](image)

Next we prove that disk $A$ actually separates into disk $A_1$ and disk $A_2$. We name five crossing points outside of $A$, $\alpha$, $\alpha'$, $\beta$, $\beta'$, $d$ as shown in Figure 9. Besides we also name the strand which passes through $\alpha$ and enters into $A$, $\overline{\alpha}$, similarly the strand which passes through $\beta$ and enters into $A$, $\overline{\beta}$.

![Figure 9.](image)

When the strand which passed under the strand $\overline{\alpha}$ at $\alpha$ crosses next strand, there can be three cases as shown in Figure 10 (1) $\sim$ (3).
Since the diagram $D$ is positive and alternating in region $A$ therefore in any case the next strand passes under this strand from the right side to the left side as shown in Figure 11(1). We name these crossing points $p_1, p_2, \ldots$ and also name the arc from $\alpha$ to $p_1$, $p_0$, from $p_1$ to $p_2$, $p_1$, similarly $p_2, p_3, \ldots$ and so on. On the other hand we consider the strand which passes over $\beta$ as shown in Figure 11(2).

In the case where a strand crosses a loop or a strand crosses by itself, we regard as shown in Figure 12.

Finally there are two sequences of arcs in $A$ and they are both oriented. This is such as shown in Figure 15(1). If $\overline{p}_m$ and $\overline{q}_n$ cross each other then $\overline{p}_m$ passes over $\overline{q}_n$ from the left side to the right side as shown in Figure 15(2).

We name this crossing point $c$, then there is a polygon with vertices $\alpha, d, \beta, q_1, q_2, \ldots, q_n, c, p_m, p_{m-1}, \ldots, p_2, p_1$. And two arcs $\overline{p}_m, \overline{q}_n$ enter this polygon as shown in Figure 14. This is the contradiction to The Jordan curve theorem(5).

Theorem 3.3. (Jordan curve theorem)
Let $C$ be the image of the unit circle, that is $C = \{(x, y); x^2 + y^2 = 1\}$ under an injective continuous mapping $\gamma$ into $\mathbb{R}^2$. Then $\mathbb{R}^2 \setminus C$ is disconnected and consists of two component.

Moreover if $\overline{p}_m$ or $\overline{q}_n$ crosses some arc in $\{\overline{p}_i\}$ or $\{\overline{q}_j\}$ then next it crosses the same arc and enter this polygon again. Because each $\overline{p}_i$ is an arc from under crossing to over crossing and each $\overline{q}_j$ is an arc from over crossing to under crossing. After all we can see that $\overline{p}_m$ and $\overline{q}_n$ never cross each other in $A$.

For this reason $A$ must separate into $A_1$ and $A_2$ hence $D$ is equivalent to an alternating diagram $D'$. This completes the proof of Theorem 3.1.
From the theorem above we know that a positive and almost alternating link has no positive and almost alternating diagram. Furthermore we think the question again. How is the diagram of a positive and almost alternating link? And we give a partial result of this question in section 4.

4. Positive and almost alternating Montesinos link

In this section we would like to study an oriented Montesinos link \( L \) denoted by \( C(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_n/\beta_n) \), \( \alpha_i/\beta_i \in \mathbb{Q} \). Any \( \alpha_i/\beta_i \) represents not only a rational number but also a rational tangle \( R_i = (\alpha_i/\beta_i) \). About rational tangles, see [10]. The standard diagram \( D \) of \( L \) denoted by \( D(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_n/\beta_n) \) is shown in Figure 15 where \( (\alpha_i/\beta_i) = R_i = R(a_{i1}, a_{i2}, \ldots, a_{im}) \). That is to say, \( D \) is the numerator of the sum of \( n \) rational tangles. For example in the case where any \( a_{ij} > 0 \) \( R_i \) is as shown in Figure 15(2) or (3).

Abe and Kishimoto showed that any non-alternating Montesinos link is almost alternating, and we have the following proposition.

Proposition 4.1. Let \( L \) be an oriented Montesinos link and be denoted by
Figure 15. The standard diagram of a Montesinos link

\[ C(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_n/\beta_n) \] where \( \alpha_i/\beta_i \in \mathbb{Q} \), and \( D \) the standard diagram of \( L \) such that \( D(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_n/\beta_n) \). If \( D \) is positive then \( L \) has an almost PA-diagram.

It is to be noted that in general if a link \( L \) has a PA-diagram, then \( L \) has also an almost PA-diagram. Because we can transform a PA-diagram \( D \) of \( L \) into an almost PA-diagram \( D' \). See Figure 5.

Before proving the proposition above, we prove two other propositions and one lemma needed later.

**Proposition 4.2.** Let \( L \) be an oriented link and \( D \) be a diagram of \( L \) such that \( D = D_1 \# D_2 \# \cdots \# D_m \) where any \( D_t \) is an alternating diagram (1 \( \leq \) \( t \) \( \leq \) \( m \)). If \( D \) is positive, then \( L \) has a PA-diagram.

**Proof 4.3.** Assume \( D = D_1 \# D_2 \# \cdots \# D_m \) is positive and \( A = D_s \# D_{s+1} \# \cdots \# D_m \) (1 \( \leq \) \( s \) \( \leq \) \( m \)) is alternating as shown in Figure 16. We consider the directions of two arcs on the left-hand side of \( A \) and the over/under informations of the leftmost crossings of \( A \) and the rightmost crossings of \( D_{s-1} \). Then we can see the four conditions as shown in Figure 17, where the symbol \( o \) (resp. \( u \)) means that an over-crossing (resp. under-crossing) appears first when we traverse the component from the end point (11). By repeating this transformation we can finally obtain a PA-diagram of \( L \) as shown in Figure 18.

\[ D = D_1 \# \cdots \# D_n \sim D_1 \# \cdots \# D_n A = \text{PA-diagram} \]

Figure 16.

\[ \begin{align*} 
(1) & \quad D_1 A \sim D_2 A \sim A_1 \\
(2) & \quad D_1 A \sim D_2 A \sim A_1 \\
(3) & \quad D_1 A \sim D_2 A \sim A_1 \\
(4) & \quad D_1 A \sim D_2 A \sim A_1 \\
(5) \quad & = \begin{array}{c} \includegraphics[width=0.1\textwidth]{condition1.png} \\
(6) \quad & = \begin{array}{c} \includegraphics[width=0.1\textwidth]{condition2.png} \\
\end{array} \\
\end{align*} \]

Figure 17. Four conditions of \( D_{s-1} \# A \)
Proposition 4.4. Let $L$ be an oriented Montesinos link, and $D$ be the standard diagram of $L$ denoted by $D(\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_n/\beta_n)$, $\alpha_i/\beta_i \in \mathbb{Q}$. Assume that $D$ is positive, $|\alpha_i/\beta_i| \geq 1$ and $\beta_i \neq 0$ for any $i$ $(1 \leq i \leq n)$. Then $D$ is alternating.

Proof 4.5. First we consider the case where the directions of left-hand side arcs of $R_1$ are parallel. In this case, naturally the directions of the right-hand side arcs of $R_n$ are also parallel as shown in Figure 19 (1). Besides it is easy to see these directions hold in the case of $R_{n-1}, R_{n-2}, \ldots, R_2$ as shown in Figure 19 (2). That is to say, the directions of the left-hand side arcs of any tangle $R_i$ are all the same as shown in Figure 19 (3). Since each tangle $R_i = R(a_{i1}, a_{i2}, \ldots, a_{im})$ is positive and alternating, we know that $a_{ij} \leq 0$ for any $j$ for any $i$ $(1 \leq j \leq m)$. Hence $\alpha_i/\beta_i < 0$ for any $i$ $(1 \leq i \leq n)$. Then $D$ is necessarily alternating.

Next we consider the case where the directions of left-hand side arcs of $R_1$ are opposite. In this case, the directions of the right-hand side arcs of $R_n$ are as shown in Figure 19 (1) or (2). So for any tangle $R_i$, the directions of the right-hand side arcs are as shown in Figure 19 (3) or (4). In any case, we know that $a_{ij} > 0$ for any $j$ $(1 \leq j \leq m)$, because any $R_i$ is positive and alternating. Therefore $\alpha_i/\beta_i > 0$ for any $i$ $(1 \leq i \leq n)$ and $D$ must be positive. This completes the proof of the proposition.

In addition when we meditate upon oriented rational tangles, we can classify them into three types as shown in Figure 21. What is more we can have the next lemma.

Lemma 4.6. Let $R$ be an oriented rational tangle denoted by $(\alpha/\beta)$, where $\alpha/\beta \in \mathbb{Q}_{\neq 0}, \beta \neq 0$. If any crossing in $R$ has the same sign $+$, then the following holds.

1. If $R$ is of type I, then $\alpha/\beta < 0$.
2. If $R$ is of type II, then $\alpha/\beta > 0$.
3. If $R$ is of type III and $|\alpha/\beta| \geq 1$, then $\alpha/\beta > 0$. 
Figure 20. The case where the directions of lefthand side arcs of \( R_1 \) are opposite

\[
\begin{array}{ccc}
(1) & (2) & (3) \\
\end{array}
\]

Figure 21. Three types of oriented tangles

(4) If \( R \) is of type III and \( |\alpha/\beta| < 1 \), then \( \alpha/\beta < 0 \).

Proof 4.7. In the case where \( R \) is of type I, the oriented tangle \( R \) is naturally as shown in Figure 22(1) or (2), and in both cases \( \alpha/\beta < 0 \). If we reverse all directions, we can prove in exactly the same way. In the case where \( R \) is of type II, \( R \) is as shown in Figure 22(3) or (4), and it is easy to see in both cases \( \alpha/\beta > 0 \). Besides, when \( R \) is of type III and \( |\alpha/\beta| \geq 1 \), \( R \) is necessarily as shown in Figure 22(5), and \( \alpha/\beta > 0 \). On the contrary if \( |\alpha/\beta| < 1 \), \( R \) must be as shown in Figure 22(6), and \( \alpha/\beta < 0 \). We have thus proved the lemma.

\[
\begin{array}{ccc}
(1) & (3) & (5) \\
\end{array}
\]

Figure 22.

In fact, there are two types in type III as shown in Figure 22(5) and (6). So next we rename type III as shown in Figure 22(5) type III+, and as shown in Figure 22(6) type III-. Now we are ready to prove Proposition 4.1.

Proof 4.8. (Proof of Proposition 4.1)

First we consider the case where some \( \beta_j = 0 \). In this case the tangle \( \alpha_j/\beta_j \) is an \( \infty \)-tangle as shown in Figure 23(1), and the diagram \( D \) is like as shown in Figure 23.
where each $R_k (1 \leq k \leq j - 1, \ j + 1 \leq k \leq m)$ is an alternating tangle. So by thinking that the denominator of a tangle $R_k$ is equivalent to a diagram $D_k$, we can regard $D = D_{j+1} \cdots D_m \cdots D_1 \cdots D_{j-1}$, where any $D_k$ is alternating. Therefore if $D$ is positive then $L$ has a PA-diagram by Proposition 4.2. Hence $L$ has also an almost PA-diagram.

Next we consider the case where $\beta_j \neq 0$ for any $i \ (1 \leq i \leq n)$. In this case, from Proposition 4.2 if $|\alpha_i/\beta_i| \geq 1$ for any $i \ (1 \leq i \leq n)$ then $D$ is alternating. Thus if $D$ is non-alternating then there must be some $\alpha_j/\beta_j$ such that $|\alpha_j/\beta_j| < 1$. That is to say, there exists some tangle $R_j$ such that $R_j = R(a_{j1}, a_{j2}, \ldots, a_{jm})$, $a_{jm} = 0$. Furthermore, by Lemma 4.6 we know that there exist some (may be one) rational tangles of type $\text{III}^-$ and some (may be one) rational tangles of type $\text{II}$ or $\text{III}^+$ in $\{R_i\}$. In this condition we can transform the tangles of type $\text{II}$ as shown in Figure 22(3) into such tangles as shown in Figure 24(1) (or all the directions are opposite), and the tangles of type $\text{III}^+$ as shown in Figure 22(5) into the tangles as shown in Figure 24(2) (or all the directions are opposite).

After the transformations above, we can regard that $D$ is equivalent to the diagram $D' = D(P_1, P_2, \ldots, P_n)$ as shown in Figure 24(1) where $P_k (1 \leq k \leq n)$ is a rational tangle or a 180 degree reversed rational tangle as shown in Figure 25(2) $\sim (5)$. Besides, there exists at least one tangle of type $\text{III}^-$, that is, the tangles as shown in Figure 24(4) or (5). These tangles as shown in Figure 22(2) $\sim (5)$ are all alternating and all crossings in these diagram have the same sign +. Namely, these tangles are all PA-tangles. (If a tangle is alternating and every crossing point in this diagram has the same sign $+$, we call this tangle a PA-tangle.) In addition, the depicted symbols of the tangles as shown in (2) and (3) are such as shown in (6) and those of the tangles as shown in (4) and (5) are such as shown in (7). Assume $P_n$ is the rightmost tangle of type $\text{II}$ or type $\text{III}^+$ as shown in Figure 25(2) or (3) in the diagram $D'$, then all the tangles situated on the right-hand side
of $P_s$ are of type III. Thus when we denote the sum of these tangles by $T_0$, $T_0$ must be a PA-tangle and the depicted symbols of $T_0$ is as shown in Figure 25. Thereby we can transform the diagram $D'$ as shown in Figure 26, and have a diagram $D''$ which is equivalent to $D'$ where $T_1$ is a PA-tangle.

The directions of outer arcs of $T_1$ are as same as those of $T_0$, and it is obvious that if tangle $P_{s-1}$ which is on the lefthand side of $T_1$ is like as shown in Figure 25(4) or (5) then the tangle sum $P_{s-1} + T_1$ is a PA-tangle. On the other hand, if $P_{s-1}$ is as in Figure 25(2) or (3), we can obtain a PA-tangle in a similar fashion like above. Therefore when $v$ is the number of tangles like as shown in Figure 25(2) or (3), by using $v$ time operations like above we can gain a positive diagram $D'$ as shown in Figure 27 where $T_v$ is a PA-tangle. Moreover we can gain a PA-diagram $D'''$ by changing over/under information of a crossing $d$. Hence it is clear that $L$ has an almost PA-diagram. This completes the proof of Proposition 4.1.

□

REFERENCES

[1] T. Abe and K. Kishimoto, The dealternating number and the alternation number of a closed 3-braid, J.Knot Theory Ramifications, 19 (2010), no9, 1157-1181.
[2] C. Adams, J. Brock, J. Bugbee, T. Comar, A. Huston, A. Joseph and D. Pesikoff, Almost alternating links, Topology Appl. 46 (1992), 151-165.
[3] C. Bankwitz, Über die torsionzahlen der alternierenden knoten, Math, Ann, 103, (1930), 145-161.
[4] P.R. Cromwell, Homogeneous links, J.London Math. Soc(2) 39 (1989), 535-552.
Figure 27.

[5] T. Hales, The Jordan curve theorem, formally and informally, American Mathematical Monthly, 114 (2007), 882-894.
[6] M. Hirasawa, Triviality and splittability of special almost-alternating diagrams via canonical Seifert surfaces, Topology Appl. 102 (2000), 89-100.
[7] I. D. Jong and K. Kishimoto, On positive knots of genus two,
[8] L. H. Kauffman and S. Lambropoulou, Classifying and applying rational tangles, Adv. in Appl. Math. 33 (2004), no2, 199-237.
[9] D. Kim and J. Lee, Some invariants of pretzel links, Bull. Austral. Math. Soc, vol.75 (2007), 253-271.
[10] K. Murasugi, Knot theory and its applications, translatedby Bohdan Kurpita, (2010).
[11] T. Nakamura, Positive alternating links are positively alternating, J. Knot Theory ramifications 9 (2000), no1, 107-112.
[12] T. Tsukamoto, The almost alternating diagrams of the trivial knot, J. Topol. 2(1) (2009), 77-104.

Graduate School of Mathematics, Kyusyu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan