Node Weighted Scheduling

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ABSTRACT
This paper proposes a new class of online policies for scheduling in input-buffered crossbar switches. For a system with arrivals, our policies achieve the optimal throughput, with very weak assumptions on the arrival process. For a system without arrivals, our policies drain all packets in the system in the minimal amount of time (providing an online alternative to the batch approach based on Birkhoff-VonNeumann decompositions). Policies in our class are not constrained to be work conserving in every time slot; it may be possible to add edges to the schedule.

Most algorithms for switch scheduling take an edge based approach; in contrast, we focus on scheduling (a large enough set of) the most congested ports. This alternate approach allows for lower-complexity algorithms, and also requires a non-standard technique to prove throughput-optimality. One algorithm in our class, Maximum Vertex-weighted Matching (MVM) has worst-case complexity similar to Max-size Matching, and in simulations shows better delay performance than Max-(edge)weighted-Matching (MWM).

1. INTRODUCTION
A commonly used switching fabric in high speed packet switches (e.g., Internet routers) is a crossbar with input queues (IQ) to hold packets during times of congestion. An $N_1 \times N_2$ input-buffered crossbar switch contains $N_1$ input ports and $N_2$ output ports. The crossbar is constrained to schedule a matching i.e., it can send at most one packet from any input port, and receive at most one packet at any output port in a single time slot. The switch scheduling problem is to determine which matching is to be used in every time slot.

Most algorithms on switch scheduling take an edge based approach, attempting to schedule either a maximal/maximum set of edges, or those with the largest queues. In this paper we design policies that look only at the weight of the ports in the switch; queues on the individual edges matter only to the extent that they are non-zero. Intuitively, our policies ensure scheduling of a large enough set of heavy ports in the system. By looking at port weights, we are able to characterize a new class of policies that have lower worst case complexities and are potentially simpler to implement.

To analyze our algorithms, we use a node-based analysis technique. We show that our class of policies is throughput optimal, i.e., they result in stable queues at all admissible loads. We prove throughput optimality using a novel, non-standard Lyapunov function: the maximum total queue at any port. In addition, our policies also achieve minimum clearance time, i.e., given an initial loading on the switch and no further arrivals, they remove all the packets in the minimum possible time. These policies do not require a priori knowledge of arrival rates.

1.1 Main Results
The focus of this paper is on the design and analysis of policies that determine schedules based upon the total queues at the nodes/ports of the switch. We will construct a class of such policies that are both throughput and clearance-time optimal. Throughout this paper, we will use “node” and “port” interchangeably. We will also use “weight” and “queue” interchangeably; they refer to the cumulative queue at the node. For an input port, the queue is the total number of packets waiting to be transferred from the port; for an output port the queue is the number of packets waiting to be transferred to the port. Finally, a matching $M$ is said to
match a node \(i\) if it contains some edge touching \(i\). We now describe the classes of node-weighted policies we investigate.

**Critical Port:** Given a node-weighted bipartite graph, a port \(i\) is *critical* if its weight is no smaller than any other port. A matching \(M\) is a critical port matching if it matches every critical port. A scheduling policy is a critical port policy if it produces a critical port matching in every time slot.

**Maximum Vertex-weighted Matching (MVM):** A matching \(M\) is an MVM if the total weight of the nodes it matches is higher than (or equal to) the total weight of nodes matched by any other matching \(\tilde{M}\). The *MVM scheduling policy* is one which schedules an MVM in every time slot.

**Lazy Heaviest Port First (LHPF):** The threshold \(l(M)\) of a matching \(M\) is the lowest positive integer such that \(M\) matches all ports with weight greater than or equal to \(l(M)\). So, for example, a perfect matching has threshold 1. \(M\) is an LHPF matching if it has the lowest threshold among all possible matchings. We call this the *optimal* threshold. An LHPF policy is one that produces an LHPF matching in every time slot.

The main result of this paper is that any LHPF policy is throughput-optimal (Thm. 1). The proof of this result uses a novel Lyapunov function: the weight of the heaviest port. We also show that a policy is clearance time optimal iff it is a critical port policy (Prop. 1). Given any queue configuration, a critical port matching always exists; we also provide a simple way to find it. This enables us to develop a “slot-by-slot” algorithm for the clearance problem, as opposed to existing batch policies \([10, 20, 31]\) based on Birkhoff-VonNeumann decompositions.

We call our class “lazy” because a LHPF matching may not even be maximal; in particular, it may not match any extra nodes below the optimal threshold (beyond what it needs to satisfy those above the threshold). To clarify our classes of policies we give a simple example in Fig. 2.

Consider a \(4 \times 4\) IQ switch with edge weights and corresponding port weights as shown in the Fig. 2. There is only one critical port (port \(a\)) with weight 10. So a critical port policy must at least schedule port \(a\). Now let us consider a LHPF matching. It is clear that the size of a matching can at most be three. It follows that not all the ports on the output side can be matched, in particular the threshold must be strictly greater than 3. Hence any matching that at least schedules ports \(a, i\) and \(j\) is LHPF. For example II, III and IV are LHPF matchings. There is a unique MWM (III) in the graph. There are many MVMs in this graph. III and IV are both MVM.

Fig. 1 shows how the different policy classes relate to each other. It is well known that MWM \([14, 28, 27, 30, 29]\) and MVM \([19]\) are throughput optimal. It is also known that the MVM policy is a maximum-size matching policy \([10]\), but that not all MSM policies are throughput optimal \([18, 15]\). For the policy classes defined in this paper, critical policies need not be throughput optimal. Lemma 3 shows that any LHPF policy is also a critical port policy. Theorem 1 shows that any LHPF policy is throughput optimal. Corollary 2 shows that any MVM is an LHPF matching. In Section 5.1 we provide an example to show that the clearance time of popular existing policies (like MWM, MSM and Greedy weighted maximal matching (GMM)) may be as large as twice the optimal.

We now discuss some implications of our work from an algorithmic perspective. We prove simple properties of LHPF policies which can be used as a source for algorithms to find LHPF matchings in node-weighted graphs. This is similar to augmenting-path characterizations which provide algorithms for edge-weighted matching problems (like maximum cardinality, maximum-edge-weight etc.). We provide a way to modify simple but non-throughput-optimal policies (like edge-based greedy, or maximal matching) into throughput optimal ones via post-processing. We elaborate on this procedure in Section 4.

The tradeoff between delay and implementation complexity has been studied in \([23, 20, 19]\). In general, a lower complexity scheduler will result in higher delays. There are simple algorithms like maximal matching and GMM, which empirically perform well in most cases \([11, 12, 25]\), but are difficult to analyze. In fact they are not even throughput optimal in some cases. When used in conjunction with LHPF (via post-processing), they should have both good delay and throughput. Note that one of the members of LHPF class is the MVM algorithm which can be shown empirically to have delay performance very close to the well known delay-optimal MWM-\(\alpha\) \([14, 28, 27]\) policies at a lower complexity \([19]\) of \(O(N^{2.5})\) as compared to \(O(N^3)\) for MWM \([13]\). LHPF class contains policies which are simpler to implement than the MVM and hence are potential candidate for a low complexity delay efficient scheduler with theoretical guarantees on the throughput. Additionally, these policies are clearance-time optimal.

### 1.2 Related Work

Throughput optimal policies can be classified broadly into Backlog-aware policies, which require the knowledge of the backlogs at every time slot and those which are Backlog...
Backlog independent policies instead use the knowledge of the arrival rates \[11\] to construct a randomized or periodic scheduling rule precisely matched for the input rates. Such scheduling offers arbitrarily low per-time slot computation complexity at the cost of large delays (shown to be \(O(N)\) where \(N\) is the size of switch).

Backlog aware policies can be further classified into those which are \textit{frame} based or \textit{batch} based and the \textit{online} policies.

The \textit{frame} based policies are considered in \[31\] \[10\] \[20\] and are based on the principle of iteratively clearing the backlog in minimum time. The throughput optimality of these policies is restricted to Bernoulli i.i.d. traffic in \[31\] \[10\]. In \[20\], prior knowledge of the statistics of arrival process is required to be able to select the \textit{frame} size appropriately so as to achieve throughput optimality. Minimum clearance time policies have been applied to stabilize networks in \[23\] \[22\].

\textit{Batch} based policies \[10\] are similar to the \textit{frame} based policies except that the \textit{frame} size is dependent on the traffic arrival pattern and the scheduling algorithm used.

In this paper we restrict our attention to the development of \textit{online} algorithms, which attempt to schedule traffic by computing a matching every time slot. One such policy is the famous MWM policy which computes the maximum weight matching and is known to be throughput optimal. The proof for stability can be provided either in the fluid limits \[4\] or in the stochastic sense \[29\]. But essentially it hinges on a quadratic Lyapunov function and ensuring that the drift is negative.

The Maximum Size Matching (MSM) policy schedules the maximum size matching and hence maximizes the instantaneous throughput in each time slot. However it is known that if ties are broken randomly, MSM does not achieve \(100\%\) throughput for all admissible Bernoulli traffic patterns \[18\] \[15\]. It is possible that if the ties are broken carefully, a special MSM might be stable. Among the class of MSM policies, there are two policies that have been proposed in the literature to be throughput optimal: MVM and MWM-0+. MVM is known to be throughput optimal \[19\]. The proof of throughput optimality in \[19\] uses the fact that a MVM on a graph \(G\) is a MWM on a graph \(G'\), where edge weights have been selected carefully. The technique to prove throughput optimality of MVM is essentially the same as that for MWM. The proof provided in this paper serves as a alternate, since MVM is a member of the LHFP class of policies.

\textbf{MWM-0+}: At each time slot, consider all matchings which have maximal size. Among these choose one which has maximum weight, with weight function \(\log\). Break ties arbitrarily. This is conjectured to be throughput optimal in \[27\].

It is useful to also consider online scheduling according to maximal matches, which are matchings where no new edges can be added without sharing a node with an already matched edge. Maximal matchings can be found with \(O(N^2)\) operations and the computation is easily parallelizable to \(O(N)\) complexity \[31\]. Greedy weighted maximal matching (GMM) is a scheduler that tries to schedule the heavy edges. The GMM policy has been analyzed for the general class of networks with interference constraints \[6\] where it is shown that they achieve full throughput in a network that satisfies the local pooling condition. In simple terms, the local pooling condition means that a vector \(\lambda\) in the capacity region cannot dominate another vector \(\mu\) in the capacity region in all the coordinates. This result can be generalized \[11\] \[12\] \[2\] to show that GMM achieves at least a certain fraction of the capacity region given by the local pooling factor. Although our Lyapunov function looks similar to that in \[6\] \[11\] \[12\] \[2\] it is based on node weights as opposed to weights on the individual edges in the graph. Moreover, we can show that the LHFP class of policies are not even required to be maximal in every time-slot whereas the policies considered in \[11\] \[12\] \[2\] are.

The general research on the delay analysis of scheduling policies has progressed in the following main directions:

- **Heavy traffic regime using fluid models**: Fluid models have typically been used to either establish stability of the system or to study the workload process in the heavy traffic regime. It has been shown in \[20\] that the MWM policy minimizes the workload process for a generalized switch. Furthermore, \[27\] proves multiplicative state space collapse of a family of scheduling algorithms related to MWM in the heavy traffic regime and conjectures an optimal algorithm MWM-0+.

- **Stochastic Bounds using Lyapunov drifts**: This method is developed in \[16\] \[8\] \[21\] and is used to derive upper bounds on the average queue length for these systems. However, these results are order results and provide only a limited characterization of the delay of the system. For example, it has been shown in \[20\] that the bounds in \[16\] \[20\] are \(O(N)\) bounds and hence not very useful. It is also shown that it is possible to achieve \(O(\log N)\) delay.

As noted in \[19\], the MVM policy combines the benefit a maximum size algorithm, with those of a maximum weight algorithm, while lending itself to simple implementation in hardware. In MVM, each weight is a function of queue lengths (sum of all edges that touch a node) and hence it has an advantage of both the maximum size matchings with high instantaneous throughput while guaranteeing high throughput, even when the arrival traffic is non-uniform. We have in fact characterized a class of policies much larger than the MVM policy and potentially lower complexity and equivalent performance benefits.

### 2. PRELIMINARIES

**Switches**: This paper is about scheduling in (the standard) input-buffered crossbar switches, which we now briefly describe. An \(N_1 \times N_2\) input-buffered crossbar switch contains \(N_1\) input ports and \(N_2\) output ports. The system operates in discrete time slots. In each slot, packets may arrive at the input ports; each packet has an output port it needs to be transferred to. Packets have to be transferred from inputs to outputs, under the following constraint: in any one time slot each input port can send at most one packet to at most
one output port, and each output port can receive at most one packet from at most one input port. The scheduling problem is to determine how to transfer packets subject to these constraints.

**Notation:** Switch scheduling can be modeled as the problem of finding matchings in bipartite graphs, one in every time slot. Consider $G(s)$ the graph at slot $s$. $G(s)$ is a bipartite graph with input ports on one side and output ports on the other. As mentioned in the introduction, we will use “nodes” and “ports” interchangeably. There is an edge $(i, j)$ in $G(s)$ if and only if there is at least one packet at input $i$ that has output $j$ as its destination. The scheduling algorithm finds a matching $M(s)$ in $G(s)$; then, for every edge $(i, j) \in M(s)$ one packet is then transferred from $i$ to $j$. These packets are then considered to have left the system. A scheduling policy is a rule to pick the matching $M(s)$, in every slot $s$, based on the state of the system. For any input port $i$, $q_i(s)$ denotes the total number of packets at $i$. Similarly, for any output port $j$, $q_j(s)$ denotes the total number of packets in the system (i.e., all inputs) that are waiting to be transferred to $j$. We will not need to refer to the queues on individual edges. We will however often refer to the total queue at a port as the “weight” of that port; “heavy” ports have more packets in their queues than “lighter” ports.

We now state a couple of well-known results, from [17, 24] which we will use in the proofs of this paper.

**Lemma 1 (Hall’s Condition).** Let $G$ be any bipartite graph, with the two partitions being $V_1$ and $V_2$. Let $S_1 \subset V_1$ be any subset of one partition. Then, there exists a matching in $G$ that matches every node in $S_1$ if and only if for every further subset $S \subset S_1$, we have that $|\mathcal{N}(S)| \geq |S|$. Here the neighborhood $\mathcal{N}(S)$ is all nodes in $V_2$ that have an edge to some node in $S$.

**Lemma 2.** Let $G$ be any bipartite graph, with the two partitions being $V_1$ and $V_2$. Let $S_1 \subset V_1$, and suppose there exists a matching $M_1$ that matches all nodes in $S_1$. Similarly, let $S_2 \subset V_2$, and there exist $M_2$ that matches all nodes in $S_2$. Then there exists a matching $M$ that matches all nodes in both $S_1$ and $S_2$.

Note that in Lemma 2 $M$ may not match the nodes in the two sets to each other; just that each node in $S_1 \cup S_2$ will be matched to some node in the graph.

**Graph-theoretic preliminaries:**

We now formally define the terms we will use. All are standard, except for the definition of “absorbing paths”. Throughout, we consider a node-weighted graph. The length of a path is the number of edges it contains. The weight $w(M)$ of a matching $M$ is the total weight of all the nodes it matches. For any two matchings $M_1$ and $M_2$, the symmetric difference, denoted by $M_1 \triangle M_2$, is the set of edges in one of the two matchings, but not in both. It is well known that $M_1 \triangle M_2$ is always the node-disjoint union of paths and even-length cycles. Finally, given a matching $M$ and path $P$, the set $M \oplus P = M - (M \cap P) + (M^c \cap P)$ denotes the edges obtained by “flipping” the edges in $P$. We now define the two scenarios of our interest where the resulting set $M \oplus P$ is also a matching.

Given a matching $M$, and any node $i$ not matched by $M$,

1. An augmenting path from $i$ is any odd-length path $P$ whose every alternate edge is in $M$, has $i$ as one endpoint, and ends at an unmatched node (say $j$). Note that now $M \oplus P$ matches every node $M$ does, and in addition matches $i$ and $j$ as well. Thus its weight is $w(M \oplus P) = w(M) + w_i + w_j$, which is strictly bigger than $w(M)$.

2. An absorbing path from $i$ is any even-length path $P$ whose every alternate edge is in $M$, has $i$ as one endpoint, and whose last endpoint – say $j$ – has weight $w_j < w_i$.

Note that now $M \oplus P$ matches every node $M$ does except $j$, which is replaced by $i$. Thus it has strictly higher weight: $w(M \oplus P) = w(M) + w_i - w_j > w(M)$.

Fig. 3 illustrates the idea of augmenting and absorbing paths. $a-i-b$ is an absorbing path from $a$ since it is an even-length path ending in a node with smaller weight. $a-i-b-j$ is an augmenting path from $a$ since it is an odd-length path and ends in an unmatched node $j$.

3. CLEARANCE TIME AND CRITICAL PORT POLICIES

In the clearance time problem, the queues in the system have an initial loading, and there are no arrivals. We are interested in scheduling so as to minimize the clearance time, which is the time before every packet in the initial loading has exited the system. In the following, $q_i(s)$ denotes the remaining packets at port $i$ immediately after time slot $s$, and $q_i(0)$ the initial loading.

Since at most one packet can be scheduled at any given port, an obvious lower bound on the clearance time is

$$\tau \geq \max_i q_i(0)$$

(1)

It is known that this lower bound is tight, based on the following “batch” policy. We first briefly describe this policy, and then describe a more elegant slot-by-slot policy. Let $\tau^* = \max_i q_i(0)$.
Batch policy: This is based on the Birkhoff-Von Neumann theorem. Let $N = \max\{N_1, N_2\}$, and consider the $N \times N$ matrix $L$ in which, for $i \leq N_1$ and $j \leq N_2$, has entries $L(i,j) = \frac{q_{ij}(0)}{s_i}$, and $q_{ij}(0)$ is the number of packets waiting at input $i$ for output $j$ in the initial loading. All the other entries of $L$, i.e., all $L(i,j)$ for which either $i > N_1$ or $j > N_2$, are 0. It is clear that $L$ is a sub-stochastic matrix (i.e., the sum of every row and every column is less than or equal to 1). The Birkhoff-Von Neumann theorem says that any such matrix can be represented as a convex combination of (sub)permutation matrices; each (sub)permutation matrix corresponds to a matching in the switch. Furthermore, the fact that every entry of $L$ is an integer multiple of $\frac{1}{N}$ implies that a batch of at most $q^*$ such matchings will be needed. Thus the lower bound is tight, and can be achieved by this batch of matchings.

The Birkhoff-Von Neumann approach above gives us an algorithm for clearing out a given batch of packets, but it would be more practical to have a “slot-by-slot” solution: one in which the matching at each time can be easily determined from the current loading. We now show that the class of critical port policies is exactly what is needed for a slot-by-slot solution.

**Proposition 1.** A scheduling policy is clearance-time optimal, i.e., it achieves the lower bound, if and only if it is a critical port policy.

**Proof:** Suppose $\pi$ is a clearance-time optimal policy. This means that at any time slot $s < \tau^*$, every port $i$ has $q_i(s) \leq \tau^* - s$; otherwise, the port cannot be emptied by time slot $\tau^*$. Also, it is clear that all the ports with initial load $q_i(0) = \tau^*$ will now have $q_i(s) = \tau^* - s$; thus the weight of the critical ports at time slot $s$ is $\tau^* - s$. If any one of these critical ports is excluded by $\pi$ in slot $s$, it will have a total queue of $\tau^* - s$ at time slot $s + 1$, and hence cannot be drained by time $\tau^*$. Thus every clearance-time optimal policy is a critical port policy.

Conversely, suppose now that $\pi$ is a critical-port policy. It is easy to see that in any time slot the maximum load at any port will decrease by exactly one. This is because the ports with the maximum loads are the critical ports, and every one of them will be scheduled by $\pi$ in slot $s$.

**Corollary 1.** Given any set of queues, there exists a critical-port matching.

**Proof:** Given the set of queues, consider the clearance time problem with these queues as the initial loading. We know that there exists a policy, e.g., based on the Birkhoff-Von Neumann decomposition, that achieves the bound. By Lemma\[1\] this policy has to be a critical-port policy. Hence, in the first time slot it will have a critical-port matching. This implies such a matching exists for our set of queues.

We now give a procedure to find a critical-port matching, given any set of queues.

**Procedure for Critical Port Matching**

**INPUT:** A node-weighted graph, and any initial matching $M_0$ (which could be empty)

**OUTPUT:** $M^*$, a critical-port matching

- Set $l = 1$
- While there exists critical port $i$ not matched by $M_{l-1}$,
  - Find $P$, an augmenting path or absorbing path from $i$ with respect to $M_{l-1}$
  - Set $M_l = M_{l-1} \oplus P$ and increment $l = l + 1$

**Lemma 3.** Given any matching $M$, and a critical port $i$ not matched by $M$, there exists an augmenting path or alternating path $P$ from $i$.

**Proof:** By Corollary 1 there exists a matching $M^*$ that matches all critical ports. In particular, it matches $i$. Consider now the symmetric difference $M \triangle M^*$, which contains node-disjoint paths and cycles; since $i$ is not matched by $M$, $i$ will be the endpoint of a path $P$ in $M \triangle M^*$. If $P$ is of odd length, it is an augmenting path, and we are done. If $P$ is even length, let $j$ be the other endpoint of $P$. Now, $P$ begins at $i$ with an edge in $M^*$, so it ends in $j$ with an edge in $M$. Also, there is no edge in $M^*$ touching $j$, because $j$ is the endpoint in the symmetric difference. This means that $j$ cannot be a critical port, because $M^*$ matches every critical port. Since $i$ is critical, this means that $w_i > w_j$, which means that $P$ is an alternating path.

**Correctness of Procedure:** Suppose at iteration $l$, we have that $M_{l-1}$ does not match critical port $i$. Lemma 3 guarantees that an augmenting or absorbing path $P$ from $i$ will be found. Also, if $M_l = P \oplus M_{l-1}$ then $i$ will be matched by $M_l$. Thus all we need to show is that any critical port that is matched by $M_{l-1}$ remains matched by $M_l$. This is so because: if $P$ is augmenting, $M_l$ matches all nodes matched by $M_{l-1}$. If $P$ is absorbing, the node $j$ removed at the expense of $i$ is not critical, because absorbing requires that $w_i < w_j$. Thus the procedure gives us the desired critical port matching.

### 3.1 Clearance-time of other Policies

We now provide an example to show that edge weight based policies like MWM, Greedy weighted maximum matching (GMM) and MSM are not clearance time optimal.

Consider a $N \times N$ switch with the following configuration. Input Port 1 has one packet each destined for ports 1 through $N - 1$. Port 2 has $2 \geq i \geq N$ have $N - 1$ packets destined for output port $i - 1$. The clearance time $\tau^*$ for the above configuration is $N$.

Let us consider, how MWM schedules packets in the given system. In the given system, at any time, no more than $N - 1$ input ports can be matched under the switch constraints. The maximum weight matching policy does not match input port 1 for the first $N - 2$ slots since for any output port $j$,
the heaviest unmatched nodes; every other unmatched node
unmatched nodes has no augmenting path or absorbing path.

\[ \tau \text{ in Section 1.1} \] must be smaller than the
Critical port matching. Hence the optimal threshold (define
d by Corollary 1, for any set of queues, there exists a
critical port matching. Hence the optimal threshold (defined
in Section 1.1) must be smaller than the \( \tau^* \), the weight
of the critical port. Since the LHPF policy matches all ports
above the optimal threshold, it will match all critical ports
and hence is a Critical port policy.

**Proof:** By Corollary 1 for any set of queues, there exists
a critical port matching. Hence the optimal threshold (defined
in Section 1.1) must be smaller than the \( \tau^* \), the weight
of the critical port. Since the LHPF policy matches all ports
above the optimal threshold, it will match all critical ports
and hence is a Critical port policy.

**Lemma 4.** Any LHPF matching is also a Critical port
matching, and hence any LHPF policy is also a Critical port
policy.

**Proof:** Now, by assumption, \( M \) is not LHPF. Let \( M^* \) be any LHPF
matching. It follows that the threshold of \( M^* \) is strictly
lower than that of \( M \), which can only happen if \( M^* \) sched-
ules all nodes of weight \( w \), and in particular, all nodes in
the set \( U \). Consider now the symmetric difference \( M \triangle M^* \),
which contains node-disjoint paths and cycles. Every \( i \in U \)
is matched by \( M^* \) but not by \( M \). Thus each \( i \in U \) will be
an endpoint of a path, say \( P_i \), in \( M \triangle M^* \).

Consider any such \( i \) and \( P_i \). If \( P_i \) is of odd length, it is an
augmenting path. If \( P_i \) is of even length, let \( j \) be its other
endpoint. Because \( P \) is even length, \( j \) is not matched by
\( M^* \). This means that \( w_j < w = w_i \), which implies that \( P_i \)
is an absorbing path.

**Lemma 5.** \( M \) is an LHPF if at least one of its heaviest
unmatched nodes has no augmenting path or absorbing path.

**Remarks:** Note that the condition just concerns one of
the heaviest unmatched nodes; every other unmatched node
(heaviest or otherwise) is free to have augmenting/alternating
paths. This is a reflection of the fact that LHPF matchings
need not even be maximal.

**Lemma 6** is a sufficient condition for a matching to be LHPF,
but it is not necessary. This is because if the heaviest
unmatched node is below the optimal threshold, then it is not
required to be matched to be LHPF. For example, consider
the graph in Fig. 2 Matching II, for example is a LHPF,
although there is an augmenting path \( l - a - i - c \) which
results in matching III, which is again LHPF.

**Proof:** We will prove the contrapositive, i.e. we will prove
that if \( M \) is not and LHPF then every heaviest unmatched
node will have an augmenting or absorbing path. Let \( w \) be
the weight of the heaviest node not matched by \( M \), and let \( U \)
be the set of heaviest unmatched nodes (i.e., all unmatched
nodes with weight \( w \)).

Now, by assumption, \( M \) is not LHPF. Let \( M^* \) be any LHPF
matching. It follows that the threshold of \( M^* \) is strictly
lower than that of \( M \), which can only happen if \( M^* \) sched-
ules all nodes of weight \( w \), and in particular, all nodes in
the set \( U \). Consider now the symmetric difference \( M \triangle M^* \),
which contains node-disjoint paths and cycles. Every \( i \in U \)
is matched by \( M^* \) but not by \( M \). Thus each \( i \in U \) will be
an endpoint of a path, say \( P_i \), in \( M \triangle M^* \).

Consider any such \( i \) and \( P_i \). If \( P_i \) is of odd length, it is an
augmenting path. If \( P_i \) is of even length, let \( j \) be its other
endpoint. Because \( P \) is even length, \( j \) is not matched by
\( M^* \). This means that \( w_j < w = w_i \), which implies that \( P_i \)
is an absorbing path.

**In the same way that augmenting-path characterizations
provide algorithms for edge-weighted matching problems (like
maximum cardinality, maximum-edge-weight etc.),** Lemma
6 can be used as a source for algorithms to find LHPF
matchings in node-weighted graphs; it can also be used to modify
a (potentially non-LHPF) matching to obtain an LHPF one.
We now describe a simple procedure for either of these tasks.

**Procedure for LHPF**

**INPUT:** a node-weighted graph, and any initial matching
**OUTPUT:** \( M^* \), an LHPF matching

- Set \( l = 1 \)
- At iteration \( l \),
  - IF \( M_{l-1} \) matches all nodes, set \( M^* = M_{l-1} \) and
    BREAK.
  - Pick any highest unmatched node \( i \) in \( M_{l-1} \), and
    try to find an augmenting or absorbing path \( P \)
    from \( i \).
  - IF such a \( P \) can be found, set \( M_l = M_{l-1} \oplus P \)
    and \( l = l + 1 \).
  - ELSE set \( M^* = M_{l-1} \), and BREAK loop.
For clarity, we now describe a policy that guarantees that the matching is LHPF.

Proof: Let $M$ be a MVM. If there exists some path $P$ that is either an augmenting path or an absorbing path, then the matching $M \oplus P$ will have strictly higher weight than $M$, which is a contradiction. Thus no such $P$ exists.

Now suppose that $M$ has no augmenting or alternating paths. Suppose also that it is not a MVM. Let $M^*$ be any MVM, and consider $M \Delta M^*$; it is a collection of node-disjoint cycles and paths. Consider any path $P$ in this collection. If $P$ is of odd length, it is either an augmenting path for $M^*$, or for $M$. The latter possibility is ruled out by assumption, and we just proved that the former is ruled out too: $M^*$ is an MVM, and so cannot have an augmenting path. So there are no odd length paths. This means there has to be an even length path $P_1$ in the collection whose endpoints have unequal weights; else the weights of $M$ and $M^*$ will be equal. However, depending on which endpoint is heavier, this $P_1$ is either an absorbing path for $M$ or $M^*$; again, either possibility is ruled out.

Corollary 2. Any MVM matching is an LHPF matching, and hence the MVM scheduling policy is an LHPF policy.

Proof: By Lemma 6 any MVM will not have an augmenting or absorbing path from any unmatched node; including any of its heaviest unmatched nodes. By Lemma 5 this implies that it is an LHPF matching.

The complexity of MVM is $O(N^{2.5})$ [19] and the policy is simple to implement in hardware. Many heuristics have been developed for MSM and they can be readily tuned to compute approximate MVMs, with the characterization of LHPF policies, which is a much bigger class, we expect that it would be much easier to develop heuristics for LHPF matchings. MSM is a special case of MWM. The proof uses the fact that a MVM is a MWM on a graph where edge weight on an edge connecting input node $i$ and output node $j$ have been chosen as follows:

$$w_{ij}(n) = \begin{cases} q_i(n) + q_j(n), & \text{if } q_i(n) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Lemma 7. If $G$ has a perfect matching (i.e. one that matches every port), then any LHPF matching also has to be perfect.

Proof: The existence of a perfect matching means that the optimal threshold for a matching is 1. Any non-perfect matching will have a higher threshold, and hence not be an LHPF.

5. THROUGHPUT OPTIMALITY OF LHPF POLICIES

In this section we show that any LHPF policy is throughput optimal. Let the system be empty at time 0. Let $a_i(n)$ denote the cumulative number of packets that have arrived at an input port $i$ up to time slot $n$. Similarly, $a_j(n)$ denotes the cumulative number of packets that have arrived in the system, destined for output port $j$ up to time slot $n$. For each edge in the matching, one packet is removed at the node-disjoint cycles and paths. Consider any path $P$ in this collection. If $P$ is of odd length, it is either an augmenting path for $M^*$, or for $M$. The latter possibility is ruled out by assumption, and we just proved that the former is ruled out too: $M^*$ is an MVM, and so cannot have an augmenting path. So there are no odd length paths. This means there has to be an even length path $P_1$ in the collection whose endpoints have unequal weights; else the weights of $M$ and $M^*$ will be equal. However, depending on which endpoint is heavier, this $P_1$ is either an absorbing path for $M$ or $M^*$; again, either possibility is ruled out.
of large numbers (SLLN): with probability one,
\[ \lim_{n \to \infty} \frac{a_i(n)}{n} = \lambda_i \quad (3) \]

For any port, input or output, let \( \lambda_i \) be the average rate of arrival of packets to port \( i \). Define
\[ e^* = \min_{i} (1 - \lambda_i) \]
The capacity region is \( \{ \lambda : \lambda_i < 1 \text{ for all } i \} \), which means that \( e^* > 0 \).

**Fluid Model**

We develop a fluid limit model following the development in [4]. Let \( q_i(t) \) denote the weight at port \( i \) and \( d_i(t) \) be the number of packets that departed from port \( i \) by time slot \( n \). Let \( h_M(t) \) be the number of slots in which matching \( M \in \mathcal{M} \) has been scheduled, where \( \mathcal{M} \) is the set of all matchings (not necessarily maximal). Then \( h_M \) is a non-decreasing function. Also note that by definition of \( G(n) \), \( M \) can schedule only non-zero edges in the system. \( M_i \) indicates if matching \( M \) schedules port \( i \). Note that \( q_i(\cdot) \) and \( d_i(\cdot) \) evolve according to the following:

\[
q_i(n) = q_i(0) + a_i(n) - d_i(n) \\
d_i(n) = \sum_{M \in \mathcal{M}} \sum_{l=1}^{n} M_i(h_M(l) - h_M(l - 1)) \\
\sum_{M \in \mathcal{M}} h_M(n) = n
\]

We define \( a_i(t) \) for a non-negative real number \( t \) by interpolating the value of \( a_i \) between time \( [t] \) and \( [t] + 1 \). We also define \( q_i(t) \) and \( d_i(t) \) in the same way by linear interpolation of the corresponding values at time \( [t] \) and \( [t] + 1 \). Then, by using the techniques of Theorem 4.1 of [4], we can show that, for almost all sample paths and for all positive sequence \( x_k \to \infty \), there exists a subsequence \( x_k \) with \( x_k \to \infty \) such that the following convergence holds uniformly over compact intervals of time \( t \):

For all \( i \),
\[
a_i(x_k t) \to \lambda_i t \\
d_i(x_k t) \to D_i(t) \\
q_i(x_k t) \to Q_i(t) \\
h_i(x_k t) \to H_i(t)
\]

(4)

(5)

(6)

The system \((D, H, Q)\) is called the fluid limit and queues evolve in the fluid limit as follows:

\[
Q_i(t) = Q_i(0) + \lambda_i(t) - D_i(t) \\
\frac{d}{dt} D_i(t) = \sum_{M \in \mathcal{M}} M_i \frac{d}{dt} H_M(t)
\]

\[
\sum_{M \in \mathcal{M}} H_M(t) = t
\]

\( D, H \) and \( Q \) are absolutely continuous functions and are differentiable at almost all times \( t \geq 0 \) (called regular times).

It follows that
\[
\frac{d}{dt} Q_i(t) = \lambda_i - \frac{d}{dt} D_i(t) = \lambda_i - \sum_{M \in \mathcal{M}} M_i \frac{d}{dt} H_M(t)
\]

(7)

The following lemma from [4] establishes the connection between the stability of the switch and the fluid model.

**Lemma 8.** A switch operating under a matching algorithm is rate stable if the corresponding fluid model is weakly stable.

**Lemma 9.** The fluid model of a switch operating under a matching algorithm is weakly stable if for every fluid model solution \( D, T, Q \) with \( Q(0)=0, Q(t)=0 \) for almost all \( t \geq 0 \).

Define Liapunov function
\[
V(Q(t)) = \max Q_i(t)
\]

Note that in the definition of \( V \) the maximum is taken over all ports, input and output.

**Remarks**

- The Lyapunov function used by [3] for the analysis of GMM policy also looks at the maximum queue length. The novelty of our proof is that we do not need to look at the individual queue lengths. Our Lyapunov function is based on port weights. Another difference is that while the analysis in [4] depends on the fact that GMM is a maximal matching, our proof works for all LHPF policies which are not even required to be maximal, in general.

- Our proof of stability is more subtle than the proof of stability for the MWM policy [4]. Note that the maximum weight matching in the graph remains the maximum weight matching in the corresponding fluid model. However, the ports that are critical in a given interval of time \((t, t + \delta)\) in the fluid model may not be critical on a slot by slot basis in the actual system. Hence, for example, a critical port policy may not be able to schedule all the ports that are critical in the fluid model.

**Proof Intuition**

Our proof is based on the observation that all the ports that are critical (heaviest) in the fluid limit, may not remain heaviest in the neighborhood of time \( t \), but they continue to be above a certain threshold. We show that the optimal threshold must be below this threshold and hence all ports...
that are critical in the fluid limit are scheduled in every time-slot around \( t \). We prove in Lemma 10 that a LHPF policy schedules all the ports that are critical in the corresponding fluid model and hence is throughput optimal. Note that the LHPF policy does not need to know which ports are critical in the fluid limit.

**Theorem 1.** Any LHPF policy is throughput optimal.

**Proof:** Since \( V(Q(t)) \) is a non-negative function, to show that \( V(t) = 0 \) for almost all \( t \geq 0 \), it is enough to show that, if \( t \) is a regular time and \( V(t) > 0 \) then \( V(Q(t)) \) decreases at least at a given rate.

We prove that for all regular times \( t \) such that \( V(Q(t)) > 0 \), for a system operating under any LHPF policy,

\[
\frac{d}{dt} V(Q(t)) \leq -\epsilon^*
\]

Fix time \( t \) and let \( \gamma = V(Q(t)) = \max_i Q_i(t) \). Also, define

\[ C = \{ i : Q_i(t) = \gamma \} \]

to be the set of heaviest ports at \( t \). Also, let \( \bar{\gamma} = \max_i Q_i(t) \) be the heaviest of the remaining ports. Since the number of ports is finite, \( \bar{\gamma} < \gamma \). Choose \( \beta < \gamma - 3\beta \) small enough so that (a) \( \bar{\gamma} < \gamma - \beta \), and (b) \( \beta < \frac{\epsilon}{2N+1} \). Here \( N = \max\{ N_1, N_2 \} \). Note that this implies that

\[ \left( \frac{N+1}{N} \right) (\gamma - \beta) > \gamma + \beta \]

Recall that \( Q(t) \) is absolutely continuous. This means that there exists a \( \delta \) small enough, so that at all times \( \tau \in (t, t+\delta) \) the queues satisfy the following conditions

\[ Q_i(\tau) \in (\gamma - \frac{\beta}{2}, \gamma + \frac{\beta}{2}) \quad \text{for all } i \in C \]

\[ Q_i(\tau) < \gamma - \frac{3\beta}{2} \quad \text{for all } i \notin C \]

Let \( x_{k_i} \) be a positive subsequence for which the convergence to the fluid limit holds. Consider \( l \) large enough so that

\[ \frac{x_{k_i}}{x_{k_i}(t)} - Q_i(t) < \frac{\beta}{2}. \]

Consider time slots \( T := \{ [x_{k_i}, [x_{k_i} + 1, \ldots [x_{k_i}(t+\delta))] \}. \)

The following lemma shows that all critical ports that are critical at the fixed time \( t \) in the fluid limit will be scheduled at all time slots \( n \in T \). The conditions (C1) and (C2) can be rewritten as follows for the original switching system.

\[ Q_i(n) \in [x_{k_i}(\gamma - \beta), x_{k_i}(\gamma + \beta)] \quad \text{for all } i \in C \]

\[ Q_i(n) \leq x_{k_i}(\gamma - 2\beta) \quad \text{for all } i \notin C \]

We state a lemma. We prove it immediately after the current proof.

**Lemma 10.** For all times \( n \in T \), any LHPF policy will match all ports that are in \( C \) at time \( t \) in the fluid limit.

Now, assuming that a LHPF policy indeed schedules every port \( i \in C \) at all times \( n \in T \),

\[ \sum_{i \in M} M_i(h_i([x_{k_i}(t+\delta) - h_i([x_{k_i} t])) = [x_{k_i}(t+\delta) - [x_{k_i} t] \]

Now by dividing both sides by \( x_{k_i} \) and let \( l \to \infty \), we obtain:

\[ 1 \geq \sum_{i \in M} M_i(h_i(x_{k_i}(t+\delta) - h_i(x_{k_i} t)) \]

Hence for \( \delta \to 0 \),

\[ \sum_{i \in M} M_i \frac{d}{dt} h_i(t) = \lim_{\delta \to 0} \sum_{i \in M} M_i \frac{H_i(t+\delta) - H_i(t)}{\delta} \]

\[ = \lim_{\delta \to 0} \sum_{i \in M} M_i(h_i(x_{k_i}(t+\delta)) - h_i(x_{k_i} t)) \]

\[ \to 1 \quad \text{by Eq. (10)} \]

So, by Eq. (7) it follows that, \( \forall i \in C \),

\[ \frac{dQ_i(t)}{dt} = -(1 - \lambda_i) \leq -\epsilon^* \]

Also, every port \( i \notin C \) has weight strictly lower than every port in \( C \), for the entire duration \((t, t+\delta)\). Thus it follows that

\[ \frac{d}{dt} V(Q(t)) \leq -\epsilon^* \]

This proves the theorem.

**Proof of Lemma 10**

Let \( C_1 \subset C \) be the set of input ports in \( C \), and \( C_2 \subset C \) the set of output ports. We will first show that all ports in \( C_1 \) can be matched, by showing that Hall's condition (given in Lemma 1) holds for this set. By symmetry, all ports in \( C_2 \) can be matched and by Lemma 2 we conclude that all ports in \( C \) can be matched.

Fix time \( n \in T \), and for any subset \( S \subset C_1 \) let \( \mathcal{N}_n(S) \) be its neighborhood at time \( n \). Suppose now that \( S \) fails Hall's condition, i.e. that \( |S| > |\mathcal{N}_n(S)| + 1 \). Now, each \( i \in S \) has \( q_i(n) > x_{k_i}(\gamma - \beta) \), by condition (C1). This means that

\[ \sum_{i \in S} q_i(n) > |S| x_{k_i}(\gamma - \beta) \geq (|\mathcal{N}_n(S)| + 1) x_{k_i}(\gamma - \beta) \]

Now, each packet in \( q_i(n), i \in S \) is destined for one node in \( \mathcal{N}_n(S) \), which means that

\[ \sum_{j \in \mathcal{N}_n(S)} q_j(n) \geq \sum_{i \in S} q_i(n) \]

(LHS and RHS may not be equal because there might be other input ports with packets for ports in \( \mathcal{N}_n(S) \)). This
threshold is \( \gamma \) and schedules all ports with weights above \( \gamma \). This means that the threshold of any LHPF matching cannot be greater than \( 1\% \) of the mean.

We simulate a \( 8 \times 8 \) switch with symmetric loading on each edge. We simulate two types of arrival processes, Bernoulli and a more bursty arrival process. Each arrival stream injects packets independently in the system. Clearly, these processes satisfy strong law of large numbers and the switch is guaranteed to be stable. The model of the bursty arrival process is described below.

**Bursty Arrival Processes:** The arrival stream is a series of active and idle periods. During the active periods, the source injects one packet into the queue in every time slot. The length of the active periods (denoted by random variable \( a \)) are distributed according the Zipf law with power exponent 1.25 and support \([1,2,3,\ldots,100]\). Heavy tailed distributions like Zipf, have been found to model the Internet traffic [2]. During the active period the source generates one packet every time-slot. The idle periods are geometrically distributed with mean \( p \). The mean arrival rate of a source can be controlled by changing the value of \( p \).

The results for Bernoulli traffic in Fig. 5 show that the delay of MVM policy is smaller than that of the MWM policy. MWM-\( \alpha \) policies have been studied in the literature [14, 27] and have been reported to incur smaller delay as the value of \( \alpha \) goes to zero. Our simulations confirm this observation and also show that the delay performance of the MVM policy is no worse than the MWM-\( \alpha \) policies even for small values of \( \alpha \).

**Bursty Arrival Processes**

Fig. 6 shows the delay for the bursty arrival process described above. The delay is significantly higher for the more bursty arrival process as compared to Bernoulli traffic. It seems that although the MVM and MWM-\( \alpha \)+ policies have different tie-breaking rule, their delay performance is actually quite similar.

### 7. DISCUSSION

This paper proposes a new class of online policies called LHPF policies for scheduling in input-buffered crossbar switches. LHPF policies are both throughput optimal for a system with arrivals, and clearance-time optimal for a system with...
out arrivals. To our knowledge, this is the first class of online policies that achieves both objectives. We also provide necessary and sufficient conditions for any policy to be clearance time optimal, and show that popular existing policies (like MWM, MSM and Greedy/GMM) can have clearance time as large as twice the optimal. A particular policy in our class, MVM, has worst-case complexity similar to MSM (which is not known to be throughput-optimal), and empirical delay performance better than MWM.

As noted in [19], the MVM policy combines the benefit of a maximum size algorithm with those of a maximum weight algorithm, while lending itself to simple implementation in hardware. In MVM, each weight is a function of queue lengths (sum of all edges that touch a node) and hence has a advantage of both the maximum size matchings with high instantaneous throughput while guaranteeing high throughput, even when the arrival traffic is non-uniform. The LHPF class of policies do not care about the weight of edges as far as the required set of nodes above the optimal threshold have been scheduled. This reduces the computational overhead for the scheduler while maintaining the throughput guarantee.

Philosophically, this paper departs from the prevalent edge-based approach to scheduling, as exemplified by MWM (schedule the heaviest queues), MSM (biggest number of queues) or Greedy. Instead we concentrate on the most congested ports. It would be interesting to see if the results of this paper generalize to other interference models (e.g. those that arise in wireless networks). In particular, ports in switches represent the scheduling constraints (at most one edge per port can be scheduled). More generally, we might concentrate on the most-congested constraints, like e.g. cliques in the conflict-graph setting. For such a setting the Lyapunov function may be the heaviest constrained set.

Our Lyapunov function is evocative of the one used by [6] for the analysis of Greedy (GMM). However, we emphasize that ours is a function of the total queues at ports, while theirs is of every single queue. Besides, the Lyapunov function of [6], other popular Lyapunov functions are also all based on individual queue lengths: sum of squares of queue lengths (for MWM) etc.

In the fluid limit, [6] can guarantee that among the set of critical queues, a maximal set of queues can be scheduled at every time slot. The GMM policy is throughput optimal only when the underlying graph satisfies a local pooling condition. Note that the GMM policy is not throughput optimal for switches. In contrast, using the node based formulation, we have been able to prove that LHPF policies are throughput optimal because they guarantee that every port that is critical in the fluid limit can be scheduled at every time slot. This is because of the special structure (bipartite) graph of a switch. More generally, it has been shown that the GMM policy achieves at least a portion of the capacity region which is given by the local-pooling factor [11] [12] [2]. It would be very interesting to see if this approach would lead to the development of simpler policies with throughput guarantees for more general class of networks; especially since MWM matching problem although throughput optimal, has exponential complexity in the general setting.

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