ISOMETRIES BETWEEN FINITE GROUPS

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Abstract. We prove that if \( H \) is a subgroup of index \( n \) of any cyclic group \( G \) then \( G \) can be isometrically embedded in \((H^n, d_{\text{Ham}})\), thus generalizing previous results of Carlet (1998) for \( G = \mathbb{Z}_{2^n} \) and Yildiz-Özger (2012) for \( G = \mathbb{Z}_{p^k} \) with \( p \) prime. Next, for any positive integer \( q \) we define the \( q \)-adic metric \( d_q \) in \( \mathbb{Z}_q^n \) and prove that \((\mathbb{Z}_q^n, d_q)\) is isometric to \((\mathbb{Z}_q^n, d_{\text{RT}})\) for every \( n \), where \( d_{\text{RT}} \) is the Rosenbloom-Tsfasman metric. More generally, we then demonstrate that any pair of finite groups of the same cardinality are isometric to each other for some metrics that can be explicitly constructed. Finally, we consider a chain \( \mathcal{C} \) of subgroups of a given group and define the chain metric \( d_{\mathcal{C}} \) and chain isometries between two chains. Let \( G, K \) be groups with \( |G| = q^n, |K| = q \) and let \( H < G \). Using chains, we prove that under certain conditions, \((G, d_{\mathcal{C}}) \simeq (K^n, d_{\text{RT}})\) and \((G, d_{\mathcal{C}}) \simeq (H\langle G, H \rangle, d_{\text{BRT}})\) where \( d_{\text{BRT}} \) is the block Rosenbloom-Tsfasman metric which generalizes \( d_{\text{RT}} \).

1. Introduction

Historical background. The Hamming metric \( d_{\text{Ham}} \) is the most classic and commonly used metric in coding theory, typically in codes defined over finite fields. Since the 90’s, the Lee metric \( d_{\text{Lee}} \) was also considered on the rings \( \mathbb{Z}_m \). The Gray map is an isometry between \((\mathbb{Z}_4, d_{\text{Lee}})\) and \((\mathbb{Z}_2 \times \mathbb{Z}_2, d_{\text{Ham}})\). This map naturally extends to an isometry from \( \mathbb{Z}_4^n \) to \( \mathbb{Z}_2^{2n} \). In a famous paper from 1994, Hammons et al. used the Gray isometry to explain the formal duality exhibited by some pairs of binary non-linear codes such as Kerdock and Preparata codes and Goethals and Goethals-Delsarte codes (previously, Nechaev obtained some similar results [11]).

Few years later, Salagean-Mandache proved that except for the known case \( p = n = 2 \) it is not possible to construct a metric \( d \) in \( \mathbb{Z}_p^n \) such that \((\mathbb{Z}_p^n, d)\) is isometric to \((\mathbb{Z}_p^n, d_{\text{Ham}})\) for any prime \( p \), where \( d_{\text{Ham}} \) is the Hamming metric [13]. This result was then extended by Suéli Costa and collaborators showing the non-existence of isometries from \( \mathbb{Z}_{m^n} \) to a Hamming space \( X^n, |X| = m \) (see [12] for \( m = p \) prime, [10] for arbitrary \( m \)).

In another direction, Carlet [3] generalized the Gray map to an embedding between \( \mathbb{Z}_{2^k} \) and \( \mathbb{Z}_{2^{k-1}} \) preserving distances. This map naturally extends coordinatewise to \((\mathbb{Z}_{2^k})^n \) and \( \mathbb{Z}_{2^{k-1}} \). A couple of years later, Yildiz-Özger [15] proved that \( \mathbb{Z}_{p^k} \), with \( p \) an odd prime, can be isometrically embedded into \( \mathbb{Z}_p^{k-1} \) with the Hamming metric for any \( k > 1 \). From a more general point of view, Greferath and Schmidt [7] further generalized the Gray map to an embedding from an arbitrary finite chain ring \( R \) with the homogeneous metric to the residue field \( F = R/m \) with the Hamming metric. More precisely, \((R, d_{\text{Ham}}) \simeq (F_{q^{m-1}}, d_{\text{Ham}})\) where \( q = |F| \) and \( m = \text{length}(R) \). They used their map to construct interesting non-linear binary

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codes. More recently, Firer and D’Oliveira ([4], [5]) showed that up to a decoding equivalence, any metric space can be isometrically embedded into a hypercube with the Hamming metric.

The goal of this work is to better understand isometries and isometric embeddings between finite groups (typically finite fields or finite rings for their applications in coding theory). We will give new explicit isometries and isometric embeddings from cyclic groups and also provide a general procedure to obtain isometries between arbitrary groups.

Outline and results. Now, we briefly summarize the structure and results of the paper. In Section 2, we first recall some basic preliminaries on metric spaces, $G$-invariant metrics and isometries. If $G$ is a group acting on a metric space $(X, d)$, we define the associated symmetry group and their $G$-representations. In Proposition 2.4 we show that given $(X, d)$ with a $G$-representation, there is a bijection $\varphi : X \rightarrow G$ inducing a group structure on $X$ and a metric $d_G$ on $G$ such that $\varphi$ is a group isomorphism and $\varphi : (X, d) \rightarrow (G, d_G)$ is an isometry.

In the next section we give a simple group-theoretical proof of the known result that there are no cyclic representations of a Hamming space $X^n$ for $X$ of prime cardinality (see Proposition 3.3). In particular, there is no isometry between $\mathbb{Z}_p^n$ and $(\mathbb{Z}_p^n, d_{Ham})$.

In Section 4 we consider isometric embeddings, i.e. injective maps between metric spaces preserving distances. We generalize the result of Yildiz-Özger asserting that $\mathbb{Z}_{p^k}$, $p$ prime, can be isometrically embedded into $\mathbb{Z}_p^{k-1}$ for any $k > 1$ with the Hamming metric. In Theorem 4.3 we generalize this result by proving that for any $m$ and any subgroup $H$ of $\mathbb{Z}_m$ of index $n$, $\mathbb{Z}_m$ can be isometrically embedded into $H^n$ with the Hamming metric. This allows to isometrically embed a ring into rings of different characteristics as noted in Remark 4.7.

In Example 4.8 we consider the subgroups of $\mathbb{Z}_{12}$. In Remark 4.9 we show that the isometric embedding of $\mathbb{Z}_{2n}$ into $\mathbb{Z}_2^n$ with the Hamming metric recovers the Lee metric on $\mathbb{Z}_{2n}$.

In Section 5, for any $q, n \in \mathbb{N}$ with $q \geq 2$, we define the $q$-adic metric $d_q$ on $\mathbb{Z}_{q^n}$. The $RT$-metric was introduced by Rosenbloom and Tsfasman in [13] and has since then proven to be a quite useful metric in coding theory. In Theorem 5.2 we give a short and direct proof that $\mathbb{Z}_{q^n}$ with the $q$-adic metric is isometric to $(\mathbb{Z}_q)^n$ with the $RT$-metric, that is $(\mathbb{Z}_{q^n}, d_q) \simeq (\mathbb{Z}_q^n, d_{RT})$.

In the next section we show that any isometry between subgroups can be extended to the ambient groups (see Theorem 6.4). This implies that any pair of group of the same size (and hence all) are isometric (see Corollary 6.3). So, for instance, $\mathbb{Z}_2^3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{D}_4$ and $\mathbb{Q}_8$ are all mutually isometric.

In Section 7, we consider metrics on chain of subgroups and chain isometries. If $G$ has a chain $\mathcal{C}$ of subgroups, in Definition 7.1 we introduce the associated chain metric $d_\mathcal{C}$. In Remark 7.2 we show how the $q$-adic metric and the $RT$-metric can be naturally considered as chain metrics. In Definition 7.5 we define the notion that two chain of subgroups of the same length of two groups of the same size be isometric, that we call chain isometry. To say that two groups are chain isometric gives more information than merely saying that they are isometric, since this implies that every step of the chains are isometric to each other (see 7.4). In Theorem 7.11, using geometric chains (see 7.9) we generalize Theorem 5.2 to groups not necessarily cyclic. More precisely, if $H < G$, with $|G| = q^n$ and $|H| = q$ then $(G, d_\mathcal{C}) \simeq (H^n, d_{RT})$ where $d_\mathcal{C}$ is the chain metric associated to some chain of length $n$ with initial term $H$. The most general result will be obtained in the next section.
Finally, in Section 8, we consider the block Rosenbloom-Tsfasman metric $d_{BRT}$ which generalizes the $RT$-metric (see Definition 8.1). In Theorem 8.2 we prove that given a proper subgroup $H$ of a group $G$ and a chain $\mathcal{C}$ with initial term $H$ we have that $G$ with the metric $d_{\mathcal{C}}$ induced by the chain is isometric to $H^{[G:H]}$ with the block $RT$-metric, i.e.

$$(G, d_{\mathcal{C}}) \simeq (H^{[G:H]}, d_{BRT}).$$

2. Invariant metrics on groups

In this paper $X$ will always denote a finite set and $G$ a finite group. We begin by recalling some standard definitions. A function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is called a metric on $X$ if it is definite positive, symmetric and satisfies the triangular inequality. That is, (a) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$, (b) $d(x, y) = d(y, x)$ and (c) $d(x, y) \leq d(x, z) + d(z, y)$ hold for all $x, y, z \in X$. The pair $(X, d)$ is called a metric space. If $d$ takes values in $\mathbb{N}$ and $\text{Im}(d) \subseteq \{0, n\}$ with $n = |X|$ we say that $d$ is integral and that $(X, d)$ is an integral metric space.

Given a function $f : X \rightarrow Y$ between sets and a metric $d$ on $Y$, one can define the pullback metric of $f$ on $X$ by

$$d_f(x, x') = d(f(x), f(x')), \quad x, x' \in X.$$  

(2.1)

Two metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ are said to be isometric, and it is denoted by $(X_1, d_1) \simeq (X_2, d_2)$, if there is an isometry between $X_1$ and $X_2$. That is, there is a bijection $\varphi : X_1 \rightarrow X_2$ such that for every $x, y \in X_1$ we have

$$d_1(x, y) = d_2(\varphi(x), \varphi(y)).$$

(2.2)

In other words, $d_1$ is the pullback metric of $d_2$.

A metric $d$ on $X$ can be naturally extended to a metric $d^n$ on $X^n$ in the following way

$$d^n(x, y) = \sum_{i=1}^{n} d(x_i, y_i)$$

(2.3)

with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X^n$. For instance, the Hamming metric $d_{Ham}$ on $X$ extends to $X^n$ giving the most popular metric in coding theory

$$d^n_{Ham}(x, y) = \sum_{i=1}^{n} d_{Ham}(x_i, y_i) = |\{1 \leq i \leq n : x_i \neq y_i\}|.$$  

Sometimes, the extended metric $d^n$ is also called $d$. This is a particular case of the product metric. If $(X_i, d_i)$, $i = 1, \ldots, n$ are metric spaces then the product metric $d_{\pi} = d_1 \times \cdots \times d_n$ on $X = X_1 \times \cdots \times X_n$ is given by

$$d_\pi(x, y) = d_1(x_1, y_1) + \cdots + d_n(x_n, y_n)$$

for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

A map $w : X \rightarrow \mathbb{R}$ is called a weight function if $w(x) \geq 0$ for all $x \in X$ and $w(x) = 0$ for exactly one element $x$ of $X$. If $w$ takes integral values and moreover $\text{Im}(w) \subseteq \{0, N\}$ for some $N \in \mathbb{N}$ we will say that $w$ is an integral weight. The pair $(X, w)$ is called a weight space or integral weight space if $w$ is integral. Given a metric space $(X, d)$ and $a \in X$ we can canonically define a weight function $w_a$ by

$$w_a(x) = d(x, a), \quad x \in X.$$
If \(|X| = n\), there are \(n\) different weight functions as above. For instance, if \(X\) is a finite set and \(x_0\) is a fixed element, the \textit{Hamming weight} is given by

\[
(2.4) \quad w(x) = d_{Ham}(x, x_0) = \begin{cases} 
1 & \text{if } x \neq x_0, \\
0 & \text{if } x = x_0.
\end{cases}
\]

If \((X, d)\) is a metric space with integral weight function \(w\), the \textit{weight distribution} of \((X, d)\) is the set of weight frequencies \(\{A_0, A_1, \ldots, A_N\}\) where \(A_i = \#\{x \in X : w(x) = i\}\). The \textit{weight enumerator polynomial} of \((X, d)\) is defined by

\[
(2.5) \quad W_{(X, d)}(t) = \sum_{x \in X} t^{w(x)} = \sum_{i=0}^{N} A_i t^i.
\]

Let \((X_1, d_1)\), \((X_2, d_2)\) be two metric spaces such that \(0 \in X_1, X_2\) and consider the product space \(X = X_1 \times X_2\) with the product metric \(d_1 \times d_2\). Notice that we get

\[
W_{(X, d)}(t) = W_{(X_1, d_1)}(t) + W_{(X_2, d_2)}(t) + \sum_{(x_1, x_2) \in X_1^* \times X_2^*} t^{w((x_1, x_2))}
\]

where \(X_i^*\) denotes \(X_i \setminus \{0\}\) for \(i = 1, 2\).

All metrics and weights considered in this paper will be integral.

\textbf{G-invariant metrics}. We are interested in the particular case in which \(X = G\) is a group. The metric \(d\) is called right (resp. left) \textit{translation invariant} if for any \(g, g', h \in G\) we have

\[
d(gh, g'h) = d(g, g')
\]

(resp. \(d(hg, hg') = d(g, g')\)). If \(G\) is abelian both notions coincide and \(d\) is called \textit{translation invariant}. There is a distinguished weight function \(w(x) = d(x, e)\), where \(e\) is the identity element of \(G\). Also, if \((G, w)\) is a weight space, one can define a metric \(d\) on \(G\) in the following manner. If \(G\) is abelian,

\[
d(x, y) = w(x - y)
\]

for every \(x, y \in G\). If \(G\) is not abelian, \(d(x, y) = w(xy^{-1})\) defines a metric provided that \(w(x^{-1}) = w(x)\) for every \(x \in G\).

Let \(S_X\) denotes the permutation group of \(X\). If \(G\) acts on \(X\) we have \(G \leq S_X\).

\textbf{Definition 2.1}. Let \((X, d)\) be a metric space and \(G \leq S_X\). We say that \((X, d)\) is \textit{\(\sigma\)-invariant} for \(\sigma \in G\), denoted \(d^\sigma = d\), if

\[
d(\sigma(x), \sigma(y)) = d(x, y)
\]

for all \(x, y \in X\). Further, \((X, d)\) is called \textit{\(G\)-invariant} if \(d\) is \(\sigma\)-invariant for every \(\sigma \in G\). The \textit{symmetry group} of \((X, d)\) is defined by

\[
(2.6) \quad \Gamma(X, d) = \{ \sigma \in S_X : d^\sigma = d \}.
\]

We will say that \((X, d)\) has a \textit{\(G\)-representation} if there is a group \(G \leq \Gamma(X, d)\) which is regular; that is, \(|G| = |X|\) and the action of \(G\) is transitive.

Notice that if \(X = G\) and \(d\) is right translation invariant then \(d\) is \(G_R\)-invariant, where \(G_R\) is the right regular representation. Furthermore, \((X, d)\) is \(G_R\)-invariant if and only if \(G_R \leq \Gamma(X, d)\). Similarly, the above facts hold for \(d\) a left translation invariant metric and the left regular representation \(G_L\).
Remark 2.2. (i) Let \( f : X \to Y \) be a bijective map from \( X \) to a \( G \)-invariant metric space \((Y, d)\). Then, the action \( \sigma_Y \) of \( G \) on \( Y \) can be transferred to \( X \) in such a way that the pullback metric \( d_f \) becomes \( G \)-invariant. In fact, defining the action of \( G \) on \( X \) by \( \sigma_X(x) = f^{-1}(\sigma_Y(f(x))) \) we have
\[
\begin{align*}
    d_f(\sigma_X(x), \sigma_X(x')) &= d_f(f^{-1}(\sigma_Y(f(x))), f^{-1}(\sigma_Y(f(x')))) \\
    &= d(\sigma_Y(f(x)), \sigma_Y(f(x'))) = d(f(x), f(x')) = d(x, x'),
\end{align*}
\]
where we have used that \( d \) is \( G \)-invariant.

Example 2.3. Let \((X, d_H)\) be the Hamming space with \(|X| = n\). Then \( \Gamma(X, d_H) \simeq S_n \) and, hence, \((X, d_H)\) has a \( G \)-representation for every group of order \( n \), as a consequence of Cayley’s Theorem.

We now show that given a \( G \)-representation on a metric space \((X, d)\), the group \( G \) inherits a metric and the set \( X \) inherits a group structure.

Proposition 2.4. Suppose that the metric space \((X, d)\) has a \( G \)-representation. Then, there is a bijection \( \varphi : X \to G \) which induces a group structure on \( X \) and a metric \( d_G \) on \( G \) such that \( \varphi \) is a group isomorphism and \( \varphi : (X, d) \to (G, d_G) \) is an isometry. Moreover, \( d_G = d_{\varphi^{-1}} \) is translation invariant, that is \( d_G(g_1h, g_2h) = d_G(g_1, g_2) \) for every \( g_1, g_2, h \in G \).

Proof. Fix an element \( x_0 \in X \). Since \( G \) acts regularly on \( X \), \( G \) acts transitively on \( X \) and \(|G| = |X|\). Thus, for each \( x \in X \) there is a unique \( g = g_x \in G \) such that \( g(x_0) = x \). Hence, we can define the map
\[
    \varphi : X \to G, \quad x \mapsto g_x.
\]
This gives a group structure on \( X \) by considering the product
\[
    xy = g_y(x).
\]
It is easy to see that \( x_0 \) is the identity element in \( X \) and
\[
    x^{-1} = \varphi^{-1}(g_{x^{-1}}) = g_x^{-1}(x_0).
\]
Therefore, \( \varphi \) is a group homomorphism and hence an isomorphism.

Now, \( d \) induces the metric \( d_G \) in \( G \) by
\[
    d_G(g_x, g_y) = d(x, y).
\]
Clearly, \((X, d)\) and \((G, d_G)\) are isometric since \( d_G(\varphi(x), \varphi(y)) = d(x, y) \), by definition. It only remains to show that \( d_G \) is translation invariant. For \( g_x, g_y, h \in G \) we have
\[
    d_G(g_x \cdot h, g_y \cdot h) = d(h(g_x(x_0)), h(g_y(x_0))) = d(h(x), h(y)) = d(x, y) = d_G(g_x, g_y)
\]
as we wanted to see. \(\square\)

We now illustrate the above proposition, showing that the \( G \)-representations of a set strongly depend on the chosen metric and on the symmetry of the group. For clarity we will sometimes use the graph of distances of a finite metric space \((X, d)\). If \(|X| = n\), the graph of distances of \( X \) is the weighted complete graph \( K_n \) where each edge \( xy \) has weight \( d(x, y) \).
**Example 2.5.** Consider the set $X = \{x, y, w, z\}$. Since $X$ has 4 elements, any $G$-representation of $X$ has only two possibilities: $G \simeq \mathbb{Z}_4$ or $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Consider some metrics on $X$ given by the following graphs of distances

![Figure 1. $d_1$](image1)

![Figure 2. $d_2$](image2)

![Figure 3. $d_3$](image3)

We now show that the number of $G$-representations of $(X, d)$ depends on the chosen metric.

(i) Consider the metric $d_1$ and let

$$G_1 = \langle \rho = (xywz) \rangle \quad \text{and} \quad G_2 = \langle \tau_1 = (xy)(wz), \tau_2 = (xz)(yw) \rangle$$

be the groups defined by the permutations $\rho$ and $\tau_1, \tau_2$, respectively. Note that $G_1 \simeq \mathbb{Z}_4$, $G_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and that they act transitively on $X$. Further, notice that $(X, d_1)$ is $G_i$-invariant, that is $G_i \leq \Gamma(X, d_1)$, for $i = 1, 2$.

Fix $x$ as the identity element in $X$ and define $\varphi_1 : X \to G_1$ as follows

$$x \mapsto e, \quad y \mapsto \rho = (xywz), \quad w \mapsto \rho^2 = (xw)(yz), \quad z \mapsto \rho^3 = (xzwy).$$

Also, define $\varphi_2 : X \to G_2$ by

$$x \mapsto e, \quad y \mapsto \tau_1 = (xy)(wz), \quad w \mapsto \tau_1 \tau_2 = (xw)(yz), \quad z \mapsto \tau_2 = (xz)(yw).$$

By Proposition 2.4 there are isometries $(X, d_1) \simeq (G_1, d_{G_1})$ and $(X, d_1) \simeq (G_2, d_{G_2})$. Note that under the isomorphisms $G_1 \simeq \mathbb{Z}_4$ and $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ the metrics $d_{G_1}$ and $d_{G_2}$ correspond to the Lee metric $d_{Lee}$ on $\mathbb{Z}_4$ and to the Hamming metric $d_{Ham}$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$. That is

$$(X, d_1) \simeq (\mathbb{Z}_4, d_{Lee}) \quad \text{and} \quad (X, d_1) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2, d_{Ham}).$$

In particular, by transitivity, we have recovered the known isometry between $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by the Gray map.

(ii) Consider now the metric $d_2$. Notice that $(X, d_2)$ is $G_2$-invariant but it is not $G_1$-invariant. In this case, $(X, d_2)$ has only one $G$-representation with $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) Finally, observe that when the metric $d_3$ is considered, the metric space $(X, d_3)$ has no $G$-representations at all because the group of symmetries of $(X, d)$ is trivial (none of the groups can preserve the distances).

\[\diamond\]

**Note.** From now on, if $G$ is a group, $(G, d)$ will denote a metric space where the distance $d$ is $G$-invariant and $G$ acts by right translations, i.e. we identify $G$ with its right regular representation $G_R \leq S_G$. 

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3. Hamming spaces are not isometric to cyclic groups

Due to the relevance shown by the Gray map $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2^2$ in coding theory, people were concerned whether there is a generalization of this isometry sending $\mathbb{Z}_{p^n}$ to $(\mathbb{Z}_p)^n$, with $p$ prime. As already mentioned in the Introduction, Salagean-Mandache proved ([14]) that, except for the known case $p = n = 2$, it is impossible to construct a metric $d$ in $\mathbb{Z}_{p^n}$ such that $(\mathbb{Z}_{p^n}, d)$ is isometric to $(\mathbb{Z}_p^n, d_{Ham})$. Sueli Costa and collaborators deal with the existence of isometries of a Hamming space $X^n$, $|X| = m$ (see [12] for $m = p$ prime, [10] for arbitrary $m$). They proved that there are no $G$-representations of the Hamming space $X^n$ with $G$ a cyclic group, except for the Gray map and the trivial case $n = 1$; that is, we have

**Theorem 3.1** ([10]). Let $(X^n, d_{Ham})$ be a Hamming space, with $|X| = m$. If $(m, n) \neq (2, 2)$ and $n > 1$, there does not exist any cyclic group $G$ and any metric $d$ on $G$ such that $(G, d)$ is isometric with $(X^n, d_{Ham})$.

We will give an alternative simple proof of this result, using group theory, in the case that $|X| = p$ is prime.

**Lemma 3.2.** If $G$ is a finite group containing two subgroups $H \simeq \mathbb{Z}_p^k$ and $K \simeq \mathbb{Z}_{p^\ell}$, with $p$ prime and $k, \ell \in \mathbb{N}$, then the order of $G$ is divisible by $p^{k+\ell-1}$.

**Proof.** By the Sylow's Theorems it is enough to consider only the case when $G = P$ is a $p$-group. Therefore, $P$ contains subgroups $H$ and $K$ isomorphic to $\mathbb{Z}_p^k$ and $\mathbb{Z}_{p^\ell}$ respectively. Then we have that

$$|P| \geq |HK| = \frac{|H||K|}{|H \cap K|} \geq \frac{p^k \cdot p^\ell}{p} = p^{k+\ell-1},$$

where we have used that $|H \cap K| = 1$ or $p$, since $H \cap K$ is cyclic of order $p$. Since $|P|$ is a power of $p$ and $|P| \geq p^{k+\ell-1}$, then $|P|$ is divisible by $p^{k+\ell-1}$. $\square$

We now restate Theorem 3.1 in terms of representations for spaces of prime cardinality.

**Proposition 3.3.** Let $(X^n, d_{Ham})$ be the Hamming space, with $|X| = p$ prime. If $(p, n) \neq (2, 2)$ and $n > 1$, then there is no cyclic representation of $(X^n, d_{Ham})$. In particular, there is no isometry between $\mathbb{Z}_{p^n}$ and $(\mathbb{Z}_p^n, d_{Ham})$.

**Proof.** The Hamming space $X^n$ has a cyclic representation if and only if the symmetry group has an element of order $p^n$, such that the subgroup generated by it acts regularly. The symmetry group of the Hamming space is

$$\Gamma(X^n, d_{Ham}) \simeq S_p \wr S_n = (S_p)^n \rtimes S_n,$$

where $\wr$ denotes wreath product, and hence

$$|\Gamma(X^n, d_{Ham})| = (p!)^n n!$$

Suppose there exists a cyclic representation. We have that $\mathbb{Z}_{p^n} \not\subset \Gamma(X^n, d_{Ham})$ and also that $\mathbb{Z}_p \not\subset \Gamma(X^n, d_{Ham})$. Thus, by Lemma 3.2, $p^{2n-1}$ must divide $|\Gamma(X^n, d_{Ham})| = (p!)^n n!$. On the other hand, note that

$$\nu_p((p!)^n n!) = n\nu_p(p!) + \nu_p(n!) = n\nu_p(p) + \nu_p(n!)= n + \nu_p(n!).$$
Now, suppose that \( n = n_0 + n_1p + n_2p^2 + \cdots + n_r p^r \) is the \( p \)-adic expansion of \( n \), and let \( s_p(n) = n_0 + n_1 + \cdots + n_r \). Then, the Legendre formula for the \( p \)-adic valuation of \( n! \) implies that \( \nu_p(n!) = \frac{n-s_p(n)}{p-1} \). Then we have that

\[
\nu_p((p!)^n n!) = n + \frac{n-s_p(n)}{p-1} \leq n + n - 1 = 2n - 1.
\]

Moreover, the equality holds in (3.1) if and only if \( p = 2 \) and \( n = 2^k \) for some \( k \).

It only remains to prove the case \( p = 2 \). It is enough to show that if

\[
g \in \Gamma(X^n, d_{H_{Ham}}^m) \simeq \mathbb{Z}_2^m \times S_n
\]
then its order satisfies \( |g| < 2^n \). We recall Landau’s function \( G(n) = \max \{ \text{ord}(\sigma) : \sigma \in S_n \} \) and the known bound \( G(n) \leq e^{\frac{n}{2}} \). Now, if \( g = (t, s) \) with \( t \in \mathbb{Z}_2^n \) and \( s \in S_n \) then we have \( |g| \leq |t||s| \leq 2e^{\frac{n}{2}} \), by the bound on Landau’s function. In particular if \( n > 2 \),

\[
|g| \leq 2e^{\frac{n}{2}} < 2^n,
\]
and hence there is no element of order \( 2^n \) in \( \Gamma(X^n, d_{H_{Ham}}^m) \).

\[\square\]

4. ISOMETRIC EMBEDDINGS

We begin with the following definition. 

**Definition 4.1.** A map \( \varphi : (X_1, d_1) \to (X_2, d_2) \) between metric spaces is an *isometric embedding* if it is injective and preserves distances. That is, for every \( x, y \in X_1 \)

\[
d_1(x, y) = d_2(\varphi(x), \varphi(y)).
\]

As we previously mentioned, for any fixed \( m \), the cyclic group \( \mathbb{Z}_m \) cannot be isometric to any Hamming space \((X^n, d_{H_{Ham}}^m)\) where \( m = |X^n| \). However, there are isometric embeddings of \( \mathbb{Z}_p^k \) into the Hamming space \( \mathbb{Z}_p^{k-1} \) with \( p \) prime due to Carlet, Yildiz-Özger and Greferath thus generalizing the Gray map. Namely, we have the following result.

**Theorem 4.2** ([3], \( p = 2 \); [13], [22] any prime). Let \( p \) be a prime and \( k > 1 \), then there exists an isometric embedding from \((\mathbb{Z}_p^k, d)\) to \((\mathbb{Z}_p^{k-1}, d_{H_{Ham}}^{k-1})\).

In this section we will generalize the previous result to any cyclic group. More precisely, we will see that, for any \( m \in \mathbb{N} \), it is always possible to isometrically embed \( \mathbb{Z}_m \) into a Hamming space \( X^n \) with \( m < |X^n| \) for some \( n \), i.e.

\[
\mathbb{Z}_m \hookrightarrow (X^n, d_{H_{Ham}}^n).
\]

Let \( G = \mathbb{Z}_m = \{0, 1, \ldots, m-1\} \) and \( H \) be a subgroup of index \( n \), i.e. \( n = [G : H] \). Consider \( v \in H^n \) and \( \rho \in S_n \). We define the map (well-defined since \( H \) is a group)

\[
\Psi_{v, \rho} : \mathbb{Z}_m \to H^n, \quad t \mapsto \Psi_t(\rho) \cdot v
\]
where \( \rho \cdot v \) is the action \( \rho(v_1, \ldots, v_n) = (v_{\rho(1)}, \ldots, v_{\rho(n)}) \) and

\[
\Psi_t(x) = \frac{x^t - 1}{x - 1} = x^{t-1} + \cdots + x + 1.
\]

In the previous notations, we have the following.
Lemma 4.3. Let $H$ be a subgroup of $G = \mathbb{Z}_m$ of index $n$. Suppose $H = \langle h \rangle$ and $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the $i$-th coordinate. If $v = he_i$ and $\rho$ is an $n$-cycle then $\Psi_{v,\rho}$ is injective.

Proof. For $0 \leq t \leq m$, if $t = qn + r$ with $0 \leq r \leq n$, then

$$\Psi_{v,\rho}(t) = (q + 1) \sum_{k=1}^{r} h e_{\rho^k(i)} + q \sum_{k=r+1}^{n} h e_{\rho^k(i)}.$$ 

Now, if $0 \leq s \leq m$, if $s = q'n + r'$ with $0 \leq r' \leq n$, we can see that $\Psi_{v,\rho}(s) = \Psi_{v,\rho}(t)$ if and only if $q = q'$, and $r = r'$, that is, only if $s = t$. Therefore $\Psi_{v,\rho}$ is injective. \qed

We now generalize Theorem 4.2 to $\mathbb{Z}_m$, with $m$ any positive integer.

Theorem 4.4. Let $H$ be a subgroup of $G = \mathbb{Z}_m$ of index $n$. Consider $v = he_i \in H^n$ with $H = \langle h \rangle$, $1 \leq i \leq n$, and $\rho \in S_n$ an $n$-cycle. Then we have the isometric embedding

$$(4.1) \quad \Psi_{v,\rho} : (\mathbb{Z}_m, d) \rightarrow (H^n, d_{Ham})$$

where $d$ is the translation invariant metric with associated weight given by

$$w_{\rho}(x) = \begin{cases} t & \text{if } t \leq n, \\ n & \text{if } n \leq t \leq m - n, \\ n - t & \text{if } m - n \leq t \leq m - 1. \end{cases}$$

Proof. Consider $\tilde{\Psi}_{v,\rho} : \mathbb{Z}_m \rightarrow H^n \rtimes S_n$ given by $t \mapsto (\Psi_t(\rho) : v, \rho t)$ and notice that it is a homomorphism. In fact, given $t, s \in \mathbb{Z}_m$ we have

$$(4.3) \quad \tilde{\Psi}_{v,\rho}(t)\tilde{\Psi}_{v,\rho}(s) = (\Psi_t(\rho)v, \rho t) (\Psi_s(\rho)v, \rho s) = (\rho^s \Psi_t(\rho)v + \Psi_s(\rho)v, \rho t \rho^s) = ((\rho^s)^{t+s-1} + \cdots + \rho + 1)v + ((\rho^{s-1} + \cdots + \rho + 1)v, \rho^{t+s}) = (\Psi_{t+s}(v), \rho t^{s}) = \Psi_{v,\rho}(t + s).$$

Further, if $t + s \equiv u \pmod{m}$, then $\Psi_{v,\rho}(t + s) = \Psi_{v,\rho}(u)$ and $\rho t^{s} = \rho^u$, and hence we get $\tilde{\Psi}_{v,\rho}(s + t) = \tilde{\Psi}_{v,\rho}(u)$.

Note that $\Psi_{v,\rho} = \pi \circ \tilde{\Psi}_{v,\rho}$, so we have the following diagram

$$(4.4) \quad \mathbb{Z}_m \xrightarrow{\Psi_{v,\rho}} H^n \rtimes S_n \xrightarrow{\pi} H^n.$$ 

Now, $\Psi_{v,\rho}$ is 1-1 by Lemma 4.3 and hence $\tilde{\Psi}_{v,\rho}$ is also injective.

Denote $\Psi_{v,\rho}$ by $\Psi$ and let $d_{\Psi}$ be the pull-back metric in $\mathbb{Z}_m$ of the Hamming metric in $H^n$, that is

$$d_{\Psi}(a, b) = d_{Ham}(\Psi(a), \Psi(b)).$$

Hence $\Psi$ preserves the metric $d_{\Psi}$ by definition.
We now prove that \( d_\Phi \) is translation invariant. Note that \( H^n \times S_n \subseteq S^n_H \times S_n \), since \( H \subset S_H \) by Cayley’s Theorem, and \( \Gamma(H^n, d^n_{Ham}) \cong S^n_H \times S_n \). Thus, we have

\[
H^n \times S_n \subseteq \Gamma(H^n, d^n_{Ham}).
\]

In this way, for every \( a, b, c \in \mathbb{Z}_m \) we have

\[
d_\Phi(a + c, b + c) = d^n_{Ham}(\Psi(a + c), \Psi(b + c)) = d^n_{Ham}(\rho^c \Psi(a) + \Psi(c), \rho^c \Psi(b) + \Psi(c))
\]

where in the second equality we have used (4.5) and that \( \Psi(a + c) = \rho^c \Psi(a) + \Psi(c) \), deduced from (4.3).

Finally, we check the weights. For \( t \in \mathbb{Z}_m \) we have

\[
w_\Phi(t) = d^n_{Ham}(\Psi(t), 0) = w_{Ham}(\Psi(t)) = w_{Ham}( (\rho^{t-1} + \cdots + 1) \cdot h e_i ).
\]

Thus, considering \( t = qn + r \), with \( 0 \leq r \leq n \), we arrive at

\[
w_\Phi(t) = w_{Ham}( (q + 1) \sum_{k=1}^r h e_{\rho^k(i)} + q \sum_{k=r+1}^n h e_{\rho^k(i)} ),
\]

from which expression (4.2) readily follows. \( \square \)

In the situation of the previous theorem, there are \( \phi(m) n! \) different isometric embeddings, where \( \phi \) is the Euler totient function. Indeed, there are \( n \) vectors \( e_i \), \( (n-1)! \) different \( n \)-cycles \( \rho \) and \( \phi(m) \) different generators \( h \) of \( H \) of index \( n \) in \( \mathbb{Z}_m \). However, all these maps have the same associated metric.

**Remark 4.5.** Let \( G = \mathbb{Z}_{p^k} \) with \( p \) prime and for \( 1 \leq i \leq k-1 \) consider the subgroup \( H_i = \mathbb{Z}_{p^i} \) of index \( n_i = p^{k-i} \). By the previous theorem, there is an isometric embedding

\[
\mathbb{Z}_{p^k} \hookrightarrow (\mathbb{Z}_{p^i})^{p_{k-i}}, d_{Ham})
\]

determined by \( \Psi_{e_i, \rho} \) for any \( e_i \in H_i^{n_i} \) and any \( n_i \)-cycle \( \rho \) in \( S_{n_i} \).

In particular, if we take \( H_1 = \mathbb{Z}_p \), \( v = e_1 = (1, 0, \ldots, 0) \) and \( \rho = (12 \cdots n) \) then the weight \( w_\Phi \) becomes the extended Lee weight over \( \mathbb{Z}_{p^k} \) and we recover the isometric embedding \( \mathbb{Z}_{p^k} \hookrightarrow (\mathbb{Z}_{p^i})^{p_{k-i}} \) previously given by Yildiz-Özger ([15]).

**Remark 4.6.** Consider \( G = \mathbb{Z}_m \). One could want \( H^n \) to be of the least possible size such that \( \Psi \) is close to be a bijective embedding (i.e. an isometry). In this case, we must choose \( H \) minimizing \( |H^n| \). On the other hand, if we want to minimize the dimension of the Hamming space of the embedding, we should choose \( H \) to be the subgroup of maximum cardinality.

For instance, let \( G = \mathbb{Z}_{p_1^{k_1} p_2^{k_2}} = \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \) where \( p_1 < p_2 \) are different primes. By Theorem 4.3 and choosing \( H = \mathbb{Z}_{p_1^{k_1-1} p_2^{k_2}} \) in order to minimize the size of the embedding space, we have the isometric embedding

\[
\mathbb{Z}_{p_1^{k_1} p_2^{k_2}} \hookrightarrow ((\mathbb{Z}_{p_1^{k_1-1} p_2^{k_2}})^{p_1}, d_{Ham})
\]

On the other hand, we can apply the same theorem to \( \mathbb{Z}_{p_1^{k_1}} \) and \( \mathbb{Z}_{p_2^{k_2}} \) separately and then concatenate the spaces obtaining the isometric embedding

\[
\mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \hookrightarrow ((\mathbb{Z}_{p_1^{k_1}})^{p_1} \times (\mathbb{Z}_{p_2^{k_2}})^{p_2}, d_{Ham})
\].
Thus, and suppose that the triangle inequality, the other conditions being straightforward. Let \( d \)

\[
\text{Definition 5.1.}
\]

Remark 4.9. It is possible to isometrically embed a ring into rings of different characteristics. In fact, if \( G = \mathbb{Z}_{pq} \) with \( p, q \) primes, by Theorem 4.4 we have \( \mathbb{Z}_{pq} \hookrightarrow ((\mathbb{Z}_p)^q, d^{\text{Ham}}) \) and \( \mathbb{Z}_{pq} \hookrightarrow ((\mathbb{Z}_q)^p, d^{\text{Ham}}) \).

Example 4.8. Let \( G = \mathbb{Z}_{12} \), we can consider the four subgroups \( H_1 \cong \mathbb{Z}_2 \), \( H_2 \cong \mathbb{Z}_3 \), \( H_3 \cong \mathbb{Z}_4 \) and \( H_4 \cong \mathbb{Z}_6 \) with corresponding indices \( n_1 = 6 \), \( n_2 = 4 \), \( n_3 = 3 \) and \( n_4 = 2 \). Thus, by Theorem 4.4 we have \( \mathbb{Z}_{12} \hookrightarrow (H_i^{n_i}, d_{\text{Ham}}) \) for \( i = 1, 2, 3, 4 \). By (4.2), we have the following weight distributions

\[
\begin{array}{c|cccccccccccc}
 t & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 w_1(t) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 \\
 w_2(t) & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 3 & 2 & 1 \\
 w_3(t) & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\
 w_4(t) & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
\end{array}
\]

The corresponding weight enumerators are

\[
\begin{align*}
W_{(\mathbb{Z}_{12},d_1)}(t) & = t^6 + 2t^5 + 2t^4 + 2t^3 + 2t^2 + 2t + 1, \\
W_{(\mathbb{Z}_{12},d_2)}(t) & = 5t^4 + 2t^3 + 2t^2 + 2t + 1, \\
W_{(\mathbb{Z}_{12},d_3)}(t) & = 7t^3 + 2t^2 + 2t + 1, \\
W_{(\mathbb{Z}_{12},d_4)}(t) & = 9t^2 + 2t + 1.
\end{align*}
\]

(4.6)

Note that the associated metrics \( d_i \) obtained are all different and that the metric \( d_1 \) is just the Lee metric.

Remark 4.9. In general, considering different subgroups \( H \) of \( G \), the isometric embeddings provided by Theorem 4.4 give rise to different metrics. In the particular case that \( G = \mathbb{Z}_{2n} \), the isometric embedding

\[
\mathbb{Z}_{2n} \hookrightarrow ((\mathbb{Z}_2)^n, d_{\text{Ham}})
\]

recovers the Lee metric on \( \mathbb{Z}_{2n} \) since the associated weight function \( w \) is given by \( (w(i))^{2n-1}_{i=0} = (0, 1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1) \).

5. ISOMETRIES BETWEEN \( \mathbb{Z}_{q^n} \) AND \( \mathbb{Z}_q^n \)

Here we will prove that the groups \( \mathbb{Z}_{q^n} \) and \( \mathbb{Z}_q^n \) are isometric for positive integers \( n \) and \( q \) with \( q \geq 2 \) by using metrics different from the Hamming metric. Namely, the \( RT \)-metric in \( \mathbb{Z}_q^n \) and the \( q \)-adic metric on \( \mathbb{Z}_{q^n} \) that we now define.

Definition 5.1. Let \( n, q \in \mathbb{N} \) with \( q \geq 2 \). The \( q \)-adic metric \( d_q \) in \( \mathbb{Z}_{q^n} \) is given by

\[
d_q(x, y) = \min_{0 \leq i \leq n} \{ i : q^{-i} | x - y \}.
\]

(5.1)

Indeed, \( d_q \) is a translation invariant metric. To check that it is a metric it is enough to show the triangle inequality, the other conditions being straightforward. Let \( x, y, z \in \mathbb{Z}_{q^n} \) and suppose that \( i = d(x, z), j = d(z, y) \) and \( k = d(x, y) \). Then \( q^n-1 | x - z \) and \( q^n-j | z - y \). Thus,

\[
q^{n-\max\{i,j\}} | (x - z) + (z - y) = x - y.
\]
Hence we have \( k \leq \max\{i, j\} \) and therefore \( d(x, y) \leq d(x, z) + d(z, y) \). In fact, \( d_q \) is an ultrametric. Finally, we have \( d_q(x + z, y + z) = d_q(x, y) \) by definition, hence \( d_q \) is translation invariant. Notice that alternatively we have

\[
d_q(x, y) = \lceil \log_q(\text{ord}(x - y)) \rceil
\]

where \( \text{ord} \) denotes the order of an element in the group. In particular, if \( q = p \) is prime we simply get \( d_p(x, y) = \log_p(\text{ord}(x - y)) \).

We recall the definition of the *Rosenbloom-Tsfasman metric*, originally defined over \( \mathbb{F}_q^n \) ([13]), hence for \( q \) a prime power. However, this metric can be defined over \( G^n \) for any group \( G \). We now define the \( RT \)-metric on \( \mathbb{Z}_q^n \) for any pair of integers \( n, q \) with \( q \geq 2 \) as follows:

\[
d_{RT}(x, y) = \max_{1 \leq i \leq n} \{ i : x_i - y_i \neq 0 \}.
\]

(5.2)

Note that \( d_{RT} \) is translation invariant by definition. It is known that it coincides with the poset metric \( d_P \) on \( \mathbb{Z}_q^n \) given by the chain poset \( P \) defined by \( 1 \preceq 2 \preceq \cdots \preceq n \) (see [9]).

Now, we construct an explicit isometry between the groups \( \mathbb{Z}_q^n \) and \( \mathbb{Z}_q^n \) with the previous metrics. Let \( q \geq 2 \) and \( n \) be positive integers and consider the function

\[
\varphi : \mathbb{Z}_q^n \to \mathbb{Z}_q^n \quad \varphi(a_1, a_2, \ldots, a_n) \mapsto a_1q^{n-1} + a_2q^{n-2} + \cdots + a_{n-1}q + a_n \quad (\text{mod } q^n).
\]

(5.3)

One can check that its inverse

\[
\varphi^{-1} : \mathbb{Z}_q^n \to \mathbb{Z}_q^n
\]

is given by the \( q \)-base expansion, namely

\[
\begin{align*}
0 & \mapsto 0000\ldots000 \\
1 & \mapsto 0000\ldots001 \\
& \vdots \\
q - 1 & \mapsto 0000\ldots0(q - 1) \\
q & \mapsto 0000\ldots010 \\
q + 1 & \mapsto 0000\ldots011 \\
& \vdots \\
q^2 - 1 & \mapsto 0000\ldots0(q - 1)(q - 1) \\
q^2 & \mapsto 0000\ldots100 \\
q^2 + 1 & \mapsto 0000\ldots101 \\
& \vdots \\
q^n - 1 & \mapsto (q - 1)(q - 1)(q - 1)(q - 1)\cdots(q - 1)(q - 1)
\end{align*}
\]

(5.5)

We now show that \( \varphi \) as in (5.3) preserves distances.

**Theorem 5.2.** For any \( n, q \in \mathbb{N} \) with \( q \geq 2 \) the map \( \varphi : (\mathbb{Z}_q^n, d_{RT}) \to (\mathbb{Z}_q^n, d_q) \) as in (5.3) is an isometry.

**Proof.** To see that \( \varphi \) is an isometry between metric groups we must show that \( \varphi \) preserves distances and that the involved metrics are translation invariant.
Let \( x, y \in \mathbb{Z}^n_q \) and suppose that \( d_{RT}(x, y) = k \). This means that \( x_k \neq y_k \) and \( x_i = y_i \) for \( i = k + 1, \ldots, n \). On the other hand,

\[
d_q(\varphi(x), \varphi(y)) = \min_{0 \leq i \leq n} \{ i : q^{n-i} | \varphi(x) - \varphi(y) \}
\]

where, by (5.3), we have that

\[
\varphi(x) - \varphi(y) = (x_1 - y_1)q^{n-1} + (x_2 - y_2)q^{n-2} + \cdots + (x_k - y_k)q^{n-k} \pmod{q^n},
\]

with \( x_k - y_k \neq 0 \). Thus

\[
d_q(\varphi(x), \varphi(y)) = k = d_{RT}(x, y)
\]

and hence \( \varphi \) preserves distances. Finally, we have previously observed that both \( d_{RT} \) and \( d_q \) are translation invariant and the result thus follows. \( \square \)

Notice that, by (5.1), (5.2) and Theorem 5.2, the weight enumerators are

\[
W_{(\mathbb{Z}^n_q, d_q)}(t) = W_{(\mathbb{Z}^n_q, d_{RT})}(t) = \sum_{i=0}^{n} (q^i - q^{i-1}) t^i = (q - 1) \sum_{i=0}^{n} q^{i-1} t^i.
\]

Example 5.3. We now illustrate the previous theorem showing that the groups \( \mathbb{Z}_2^3 \) and \( \mathbb{Z}_8 \) are isometric. We take the \( d_{RT} \) metric on \( \mathbb{Z}_2^3 \) and the 2-adic metric \( d_2 \) on \( \mathbb{Z}_8 \). In this case, the map \( \varphi^{-1} : \mathbb{Z}_8 \to \mathbb{Z}_2^3 \) in (5.3) is given by

\[
0 \mapsto (0,0,0), \quad 2 \mapsto (0,1,0), \quad 4 \mapsto (1,0,0), \quad 6 \mapsto (1,1,0), \\
1 \mapsto (0,0,1), \quad 3 \mapsto (0,1,1), \quad 5 \mapsto (1,0,1), \quad 7 \mapsto (1,1,1).
\]

The graphs of distances of the groups are as follows:

One can easily check that the map \( \varphi \) preserves distances.
Also, note that the associated weight functions \( w_{\mathcal{RT}} : \mathbb{Z}_2^3 \rightarrow [0,3] \) and \( w_2 : \mathbb{Z}_8 \rightarrow [0,3] \) are given by
\[
w_{\mathcal{RT}}(x) = \begin{cases} 
0 & \text{if } x = (0,0,0), \\
1 & \text{if } x = (1,0,0), \\
2 & \text{if } x = (a,1,0), \\
3 & \text{if } x = (a,b,1), 
\end{cases}
\]
and
\[
w_2(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } x = 4, \\
2 & \text{if } x = 2,6, \\
3 & \text{if } x = 1,3,5,7, 
\end{cases}
\]
with \( a, b \in \mathbb{Z}_2 \). The weight enumerators are thus
\[
W_{\mathcal{RT}}(t) = W_{\mathcal{RT}}(t) = 4t^3 + 2t^2 + t + 1.
\]

6. Extending isometries of subgroups

In this section we show that any isometry between metric subgroups can be extended to an isometry between the ambient groups with extended metrics. More precisely, we have the following

**Theorem 6.1.** Let \( G_1 \) and \( G_2 \) be two finite groups with \( |G_1| = |G_2| \) and let \( H_1 \subseteq G_1 \), \( H_2 \subseteq G_2 \) be non-trivial proper subgroups with \( |H_1| = |H_2| \). Then, any isometry between \( H_1 \) and \( H_2 \) can be extended to a isometry between \( G_1 \) and \( G_2 \).

**Proof.** Suppose that \( (H_1,d_1) \cong (H_2,d_2) \) and let \( \tau : H_1 \rightarrow H_2 \) be the isometry. Now, for \( i = 1,2 \), we extend the metrics \( d_i \) of \( H_i \) to metrics \( \tilde{d}_i \) of \( G_i \) as follows:
\[
\tilde{d}_i(x,y) := \begin{cases} 
\max\{d_i(u,v)\} + 1 & \text{if } x = y \notin H_i, \\
d_i(x,y) & \text{if } x,y \in H_i.
\end{cases}
\]
Clearly \( \tilde{d}_i(x,y) = 0 \) if and only if \( x = y \) and \( \tilde{d}_i(x,y) = \tilde{d}_i(y,x) \). We must check that \( \tilde{d}_i \) satisfies the triangular inequality. Let \( x, y, z \in G_i \). If \( x - y, x - z, z - y \in H_i \) it follows from the triangular inequality from \( d_i \). Now, if one of \( x - z \) or \( z - y \) is not in \( H_i \), say \( x - z \), then
\[
\tilde{d}_i(x,z) = \max_{u,v \in H_i} \{d_i(u,v)\} + 1
\]
and we have \( \tilde{d}_i(x,z) = \tilde{d}_i(x,y) \) if \( x - y \notin H_i \) or \( \tilde{d}_i(x,z) \geq \tilde{d}_i(x,y) \) if \( x - y \in H_i \), and the claim follows.

Now, suppose that \( m = |G_1| = |G_2| \) and \( h = |H_1| = |H_2| \). Let \( T_1 \) be a complete set of representatives of the right cosets of \( H_i \) in \( G_i \) for \( i = 1,2 \). Consider any bijection \( \rho : T_1 \rightarrow T_2 \) and define the map
\[
G_1 \xrightarrow{\eta} G_2
\]
where \( g_1, g_2, \ldots, g_m \) are the elements of \( T_1 \). It is clear that \( \eta \) is bijective.

Note that \( x, y \) belong to the same coset of \( H_1 \) if and only if \( \eta(x), \eta(y) \) belong to the same coset of \( H_2 \). Therefore we conclude that
\[
\tilde{d}_1(x,y) = d_1(x,y) = d_2(\eta(x),\eta(y)) = \tilde{d}_2(\eta(x),\eta(y))
\]
and hence \( (G_1,\tilde{d}_1) \cong (G_2,\tilde{d}_2) \), as we wanted to see. \( \square \)
Remark 6.2. In the previous proof, the isometry $\eta$ given by (6.2) is not unique, since $\eta = \eta_\rho$ depends on the bijection $\rho$ between the complete set of representatives of right cosets $T_1$ on $G_1$ and $T_2$ on $G_2$ chosen. However, two such metrics differ by a distance preserving map. That is, if $\rho$ and $\rho'$ are two bijections from $T_1$ to $T_2$ then there is some $f \in \Gamma(G, \tilde{d}_2)$ such that $\tilde{d}_2(f(x), f(y)) = \tilde{d}_2(\eta_\rho(x), \eta_\rho(y)) = \tilde{d}_2(\eta_{\rho'}(x), \eta_{\rho'}(y))$. In fact, if $f = \eta_\rho \circ \eta_{\rho'}^{-1}$ then

$$\tilde{d}_2(f(x), f(y)) = \tilde{d}_2(\eta_\rho(\eta_{\rho'}^{-1}(x)), \eta_\rho(\eta_{\rho'}^{-1}(y))) = \tilde{d}_2(\eta_{\rho'}^{-1}(x), \eta_{\rho'}^{-1}(y)) = \tilde{d}_2(x, y)$$

and hence $f$ preserves distances.

A direct consequence of this result is that every pair of groups of the same size are isometric. For groups of prime cardinality, the isometry is trivial in the sense that both metrics are Hamming metrics.

Corollary 6.3. Let $G_1$ and $G_2$ be groups of the same cardinality. Then, there exists an isometry $\phi : (G_1, d_1) \rightarrow (G_2, d_2)$ where $d_1$ and $d_2$ are certain metrics in $G_1$ and $G_2$. Moreover, if $|G_1|$ is not prime then $d_i \neq d_{Ham}$ for $i = 1, 2$.

Proof. Suppose $m = |G_1| = |G_2|$. If $m$ is not prime, consider a prime $p$ dividing $m$. Then, there are non-trivial proper subgroups $H_1 < G_1$ and $H_2 < G_1$ with $p = |H_1| = |H_2|$. Considering the Hamming metric in both $H_1$ and $H_2$ it is clear that these subgroups are isometric, that is $(H_1, d_{Ham}) \simeq (H_2, d_{Ham})$. From Theorem 6.1 this isometry lifts to an isometry

$$(G_1, d_1) \simeq (G_2, d_2)$$

where the metric $d_i = \tilde{d}_{Ham}$ for $i = 1, 2$ and $\tilde{d}$ is as in (6.1). That is, $d_i(x, x) = 0$,

$$(6.3) \quad d_i(x, x + h) = 1 \quad \text{if } h \in H \setminus \{0\} \quad \text{and} \quad d_i(x, x + g) = 2 \quad \text{if } g \in G \setminus H,$$

for any $x \in G_1$ and $i = 1, 2$.

If $m = p$, then $G_1 \simeq G_2 \simeq \mathbb{Z}_p$ and they are trivially isometric with the Hamming metrics. This completes the proof. □

Remark 6.4. By Corollary 6.3 for any $m, n \geq 2$ there exist, for instance, non-trivial isometries $\mathbb{Z}_m^n \simeq (\mathbb{Z}_m)^n$, $\mathbb{F}_q^n \simeq (\mathbb{F}_q)^n$, $\mathbb{D}_m \simeq \mathbb{Z}_{2m}$, etc.

Let $H \subset G$ a non-trivial proper subgroup and $d$ a metric in $H$. In the sequel we will denote by

$$(6.4) \quad \tilde{d} = Ext_H^d(d)$$

(or simply $Ext_H(d)$ when $G$ is understood) the metric in $G$ induced by the extension given in Theorem 6.1 (see (6.1)). We will call this the extended metric of $d$ from $H$ to $G$.

We now illustrate the previous theorem for groups of small order $n = 4, 6, 8$.

Example 6.5 ($n = 4$). Let $G_1 = \mathbb{Z}_4$ and $G_2 = \mathbb{Z}_2^2$. It is known that $(\mathbb{Z}_4, d_{loc})$ is isometric to $(\mathbb{Z}_2^2, d_{Ham}^2)$ via the Gray map. We now show that they are isometric by using Theorem 6.1.

Consider the subgroups $H_1 = \mathbb{Z}_2 \subset \mathbb{Z}_4$ and $H_2 = \mathbb{Z}_2 \times \{0\} \subset \mathbb{Z}_2 \times \mathbb{Z}_2$, both with the Hamming metric $d_{Ham}$. These subgroups are isometric via the inclusion map $\iota : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \times \{0\}$ given by $x \mapsto (x, 0)$. By Theorem 6.1 and (6.4) we have that

$$(\mathbb{Z}_4, \tilde{d}_1 = Ext_{\mathbb{Z}_2}^{\mathbb{Z}_4}(d_{Ham})) \simeq (\mathbb{Z}_2^2, \tilde{d}_2 = Ext_{\mathbb{Z}_2 \times \{0\}}^{\mathbb{Z}_2 \times \mathbb{Z}_2}(d_{Ham})).$$
Notice that \( \text{Ext}_{\mathbb{Z}_2}^{d_{\text{Ham}}}(d_{\text{Ham}}) = d_2 \), the 2-adic metric, and \( \text{Ext}_{\mathbb{Z}_2 \times \mathbb{Z}_2}^{d_{\text{Ham}}}(d_{\text{Ham}}) = d_{\text{RT}} \), the RT-metric. In fact, by (5.1) and (5.2), we have
\[
d_2(x, y) = \min_{0 \leq i \leq 2} \{ i : 2^{2^{-i}} | x - y \} \quad \text{and} \quad d_{\text{RT}}(x, y) = \max_{1 \leq i \leq 2} \{ i : x_i - y_i \neq 0 \}
\]
respectively. Hence, by (6.1) or (6.3), for any \( x, y \in \mathbb{Z}_4 \) we have
\[
\tilde{d}_1(x, y) = 1 = d_2(x, y) \quad \text{if} \quad x - y = 2,
\]
\[
\tilde{d}_1(x, y) = 2 = d_2(x, y) \quad \text{if} \quad x - y = 1, 3,
\]
while for any \( u, v \in \mathbb{Z}_2^3 \) we get
\[
\tilde{d}_2(u, v) = 1 = d_{\text{RT}}(u, v) \quad \text{if} \quad u - v = (1, 0),
\]
\[
\tilde{d}_2(u, v) = 2 = d_{\text{RT}}(u, v) \quad \text{if} \quad u - v = (0, 1), (1, 1).
\]

We want to point out that, although \( d_2 \) and \( \tilde{d}_{\text{RT}} \) are different from \( d_{\text{Lee}} \) and \( d_{\text{Ham}} \), these metrics are correspondingly equivalent, i.e. \( d_2 \simeq d_{\text{Lee}} \) and \( d_{\text{RT}} \simeq d_{\text{Ham}} \), in a precise sense that we will not discuss here (this will be treated in another work).

**Example 6.6** \((n = 6)\). We now consider the groups \( \mathbb{Z}_6 \) and \( \mathbb{D}_3 \). By Theorem 6.1, we have two different isometries between them, taking as \( H_1, H_2 \) subgroups of order 2 or 3 respectively. Namely,
\[
(\mathbb{Z}_6, \text{Ext}_{\mathbb{Z}_2}(d)) \simeq (\mathbb{S}_3, \text{Ext}_{\langle \rho \rangle}(d)) \quad \text{and} \quad (\mathbb{Z}_6, \text{Ext}_{\mathbb{Z}_3}(d)) \simeq (\mathbb{S}_3, \text{Ext}_{\langle \tau \rangle}(d))
\]
where \( \rho \) is a 2-cycle, \( \tau \) a 3-cycle and \( d \) is the Hamming metric.

Apart from the Hamming and Lee metrics, in addition we have the metrics obtained by the previous subgroup construction. The corresponding weight functions and enumerators are given by

| \( \mathbb{Z}_6 \) | 0 | 1 | 2 | 3 | 4 | 5 | enumerator |
|---|---|---|---|---|---|---|---|
| \( w_{\text{Ham}} \) | 0 | 1 | 1 | 1 | 1 | 1 | \( 5t + 1 \) |
| \( w_{\mathbb{Z}_2} \) | 0 | 2 | 1 | 2 | 1 | 2 | \( 4t^2 + t + 1 \) |
| \( w_{\mathbb{Z}_3} \) | 0 | 2 | 1 | 2 | 1 | 2 | \( 3t^2 + 2t + 1 \) |
| \( w_{\text{Lee}} \) | 0 | 1 | 2 | 3 | 2 | 1 | \( t^3 + 2t^2 + t + 1 \) |

and

| \( \mathbb{S}_3 \) | id | (12) | (13) | (23) | (123) | (132) | enumerator |
|---|---|---|---|---|---|---|---|
| \( w_{\text{Ham}} \) | 0 | 1 | 1 | 1 | 1 | 1 | \( 5t + 1 \) |
| \( w_{\langle (12) \rangle} \) | 0 | 1 | 2 | 2 | 1 | 1 | \( 4t^2 + t + 1 \) |
| \( w_{\langle \tau \rangle} \) | 0 | 2 | 2 | 2 | 1 | 1 | \( 3t^2 + 2t + 1 \) |

where \( \tau \) is any 3-cycle.

**Example 6.7** \((n = 8)\). By Corollary 6.3, all the groups of the same size are isometric to each other. Thus, for instance, we have
\[
\mathbb{Z}_2^3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \simeq \mathbb{Z}_8 \simeq \mathbb{D}_4 \simeq \mathbb{Q}_8.
\]
In fact, note that all these groups have at least one isomorphic copy of \( \mathbb{Z}_2 \) as a subgroup. Thus, if we take any pair \( G_1, G_2 \in \{ \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{D}_4, \mathbb{Q}_8 \} \), with the trivial identifications, we then have
\[
(G_1, \text{Ext}_{\mathbb{Z}_2}^{G_1}(d_{\text{Ham}})) \simeq (G_2, \text{Ext}_{\mathbb{Z}_2}^{G_2}(d_{\text{Ham}})).
\]
The associated weights \( w_{\mathbb{Z}_2} \) are given by

| \( \mathbb{Z}_8 \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----------------|---|---|---|---|---|---|---|---|
| \( w \)       | 0 | 2 | 2 | 2 | 2 | 1 | 2 | 2 |

| \( \mathbb{D}_4 \) | \( e \) | \( \rho \) | \( \rho^2 \) | \( \rho^3 \) | \( \tau \) | \( \rho \tau \) | \( \rho^2 \tau \) | \( \rho^3 \tau \) |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( w \)       | 0   | 2   | 2   | 2   | 1   | 2   | 2   | 2   |

| \( \mathbb{Q}_8 \) | 1   | -1  | \( i \) | -\( i \) | \( j \) | -\( j \) | \( k \) | -\( k \) |
|----------------|-----|-----|------|------|------|------|------|------|
| \( w \)       | 0   | 1   | 2   | 2   | 2   | 2   | 2   | 2   |

| \( \mathbb{Z}_2^2 \) | (0,0), (1,0) | (0,1) | (0,0,1) | (1,1,0) | (0,1,1) | (1,0,1) | (1,1,1) |
|----------------|-------------|------|---------|---------|---------|---------|---------|
| \( w \)       | 0           | 1    | 2       | 2       | 2       | 2       | 2       |

| \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) | (0,0) | (1,0) | (1,1) | (1,2) | (1,3) | (0,1) | (0,2) | (0,3) |
|----------------|------|------|------|------|------|------|------|------|
| \( w \)       | 0    | 1    | 2    | 2    | 2    | 2    | 2    | 2    |

The weight enumerator is \( W(G, w_{\mathbb{Z}_2})(t) = 6t^2 + t + 1 \) where \( G \) is any group of order 8.

We now compute the weight enumerators for all the subgroups of all the groups of order 8. Isomorphic subgroups give the same metric so we consider subgroups up to isomorphism. It is clear that \( \mathbb{Z}_4 \) is a subgroup of \( G_1 \in \{ \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Q}_8, \mathbb{D}_4 \} \), that \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) is a subgroup of \( G_2 \in \{ \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{D}_4 \} \) and that \( \mathbb{Z}_2 \) is a subgroup of \( G_3 \) any of the order 8 groups. Thus, the weight enumerators for the corresponding extended metrics are as follows

\[
W(G_1, Ext_{\mathbb{Z}_4}(d_{\text{Ham}}))(t) = 4t^2 + 3t + 1,
\]
\[
W(G_1, Ext_{\mathbb{Z}_4}(d_{\text{Lee}}))(t) = 4t^3 + t^2 + 2t + 1,
\]
\[
W(G_2, Ext_{\mathbb{Z}_2}(d_{\text{Ham}}))(t) = 4t^2 + 3t + 1,
\]
\[
W(G_2, Ext_{\mathbb{Z}_2}(d_{\text{Lee}}))(t) = 4t^3 + t^2 + 2t + 1,
\]
\[
W(G_3, Ext_{\mathbb{Z}_2}(d_{\text{Ham}}))(t) = 6t^2 + t + 1.
\]

7. Chain metrics and chain isometries

In this section we will consider chain metrics and chain isometries on groups with chains of subgroups, generalizing the construction and results of the previous section.

**Definition 7.1.** Let \( G \) be a group and \( \mathcal{C} \) a chain of subgroups of \( G \)

\[
\langle 0 \rangle = H_0 \subset H_1 \subset \cdots \subset H_n = G.
\]

The **chain metric** on \( G \) associated to \( \mathcal{C} \) is defined by

\[
d_{\mathcal{C}}(x, y) = i \quad \text{if} \quad x - y \in H_i \setminus H_{i-1}
\]

for \( i = 0, \ldots, n \). Here and hereafter we use the convention \( H_{-1} = \emptyset \).

We now check that \( d_{\mathcal{C}} \) is indeed a metric. We only have to show that the triangular inequality holds. Let \( x, y, z \in G \) and suppose that \( d(x, y) = i \), \( d(x, z) = j \) and \( d(z, y) = k \). Thus, \( x - y \in H_i \setminus H_{i-1}, x - z \in H_j \setminus H_{j-1} \) and \( z - y \in H_k \setminus H_{k-1} \). We can assume that \( k \geq j \), therefore

\[
x - y = (x - z) - (y - z) \in H_k \setminus H_{k-1}.
\]
This implies that \( d(x, y) \leq k \leq d(x, z) + d(z, y) \), as we wanted to see.

The weight enumerator of \( G \) with the chain metric is given by

\[
W_G(x) = \sum_{i=0}^n (|H_i| - |H_{i-1}|) x^i.
\]

This is a direct consequence of the definition.

**Remark 7.2.** It is worth noting that the \( q \)-adic metric in \( \mathbb{Z}_q^n \) and the \( RT \)-metric in \( \mathbb{Z}_q^n \) are chain metrics.

(i) Let \( G = \mathbb{Z}_q^n \) and consider the chain of subgroups \( C \) given by

\[
\langle 0 \rangle \subseteq \mathbb{Z}_q \subseteq \mathbb{Z}_q^2 \subseteq \cdots \subseteq \mathbb{Z}_q^n
\]

where we are identifying \( Z_q^i \) with \( \langle q^{n-i} \rangle = q^{n-i}Z_q^n \) for \( i = 0, \ldots, n \). In fact, since for \( x, y \in G \), we have

\[
x - y \in \langle q^{n-i} \rangle - \langle q^{n-(i-1)} \rangle \iff q^{n-i} | x - y \text{ and } q^{n-(i-1)} \not| x - y
\]

then

\[
d_C(x, y) = \min_{0 \leq i \leq n} \{ i : q^{n-i} | x - y \} = d_q(x, y)
\]

holds for any \( x, y \in G \), by (5.1) and (7.2).

(ii) Let \( G = \mathbb{Z}_q^n \) and consider the following chain of subgroups \( C \),

\[
\langle 0 \rangle \subseteq \mathbb{Z}_q \subseteq \mathbb{Z}_q^2 \subseteq \cdots \subseteq \mathbb{Z}_q^n
\]

where by abuse of notation \( \mathbb{Z}_q^i \) denotes \( \mathbb{Z} \times \{0\}^n_{i=1} \) for \( i = 1, \ldots, n \). In fact, since for \( x, y \in G \), we have

\[
x - y \in \mathbb{Z}_q^i - \mathbb{Z}_q^{i-1} \iff x_i - y_i \neq 0 \text{ and } x_i - y_i \in \mathbb{Z}_q^i
\]

\[
\iff x_i - y_i \neq 0 \text{ and } x_j - y_j = 0 \quad \text{for } j > i.
\]

Then,

\[
d_C(x, y) = \max_{1 \leq i \leq n} \{ i : x_i - y_i \neq 0 \} = d_{RT}(x, y)
\]

holds for any \( x, y \in G \), by (5.2) and (7.2).

Notice that the weight enumerators given in (5.6) are of the form (7.3).

We now exhibit another chain metric. Let \( G \) be a finite group and \( r, n \) positive integers. Consider the following chain of groups

\[
\mathcal{C} : \ G \subset G^r \subset G^{r^2} \subset G^{r^3} \subset \cdots \subset G^{r^n}
\]

where the inclusions are given by the diagonal maps \( \delta_i \). For instance, \( \delta_0 : G \to G^r \) is given by \( x \mapsto (x, x, \ldots, x) \) with \( x \) repeated \( r \)-times, \( \delta_1 : G^r \to G^{r^2} \) is given by

\[
(x, x, \ldots, x) \mapsto ((x, x, \ldots, x), (x, x, \ldots, x), \ldots, (x, x, \ldots, x)),
\]

and so on. The chain metric \( d_C \) associated to \( \mathcal{C} \) is given as follows. If \( x = (x_1, \ldots, x_r) \in G^{r^n} \) the weight function associated to \( \mathcal{C} \) is given by

\[
w_C(x) = \min_{0 \leq i \leq n} \{ i : x_j = x_i \text{ if } j \equiv k \mod r^{i-1} \}.
\]
Let \( m = r^n \). The group \( S_m \) acts on \( G^n \) by permutation of coordinates. If \( \sigma = (12 \cdots m) \in S_m \), one can check that this is equivalent to
\[
(7.6) \quad w_\mathcal{C}(x) = \min_{1 \leq i \leq m} \{i : \sigma^i(x) = x\}
\]
for \( x \neq 0 \) and \( w_\mathcal{C}(x) = 0 \) if \( x = (0,0,\ldots,0) \). The chain metric is given by \( d_\mathcal{C}(x,y) = w_\mathcal{C}(x-y) \). We call this the diagonal chain metric of \( G \), and we denote it by \( d_\Delta \).

**Example 7.3.** Take \( G = \mathbb{Z}_2 \), \( r = 2 \) and \( n = 3 \) in \((7.4)\). Then we have
\[
(0) \subset \mathbb{Z}_2 \subset \mathbb{Z}_2^2 \subset \mathbb{Z}_2^4 \subset \mathbb{Z}_2^8.
\]
The possible weights in \( \mathbb{Z}_2^8 \) are 0, 1, 2, 3, 4 given by
\[
w_\mathcal{C}(x) = \begin{cases} 0 & \text{if } x = (0,0,0,0,0,0,0,0), \\ 1 & \text{if } x = (1,1,1,1,1,1,1,1), \\ 2 & \text{if } x = (x_1,x_2,x_1,x_2,x_1,x_2,x_1,x_2) \text{ with } x_1 \neq x_2, \\ 3 & \text{if } x = (x_1,x_2,x_3,x_4,x_1,x_2,x_3,x_4) \text{ with } x_1 \neq x_3 \text{ or } x_2 \neq x_4, \\ 4 & \text{otherwise}. \end{cases}
\]
This is in coincidence with expressions \((7.5)\) and \((7.6)\). It is clear that the corresponding weight enumerator is given by
\[
\mathcal{W}(\mathbb{Z}_2^8,d_{\Delta}')(t) = 240t^4 + 12t^3 + 2t^2 + t + 1.
\]

Compare with the weight enumerator
\[
\mathcal{W}(\mathbb{Z}_2^8,d_{RT})(x) = 128t^8 + 64t^7 + 32t^6 + 16t^5 + 8t^4 + 4t^3 + 2t^2 + t + 1
\]
of \( \mathbb{Z}_2^8 \) with the \( RT \)-metric. \( \diamond \)

Let \( \mathcal{C} \) denote a chain of subgroups as in \((7.1)\) and let \( d \) be a metric in \( H_1 \). The metric in \( G \) obtained by repeated extensions is
\[
(7.7) \quad \bar{d} = \text{Ext}_\mathcal{C}(d) = \text{Ext}_{H^n}^{H_{n-1}} \circ \cdots \circ \text{Ext}_{H_1}^{H_1}(d).
\]

**Remark 7.4.** In the above situation, if in \((7.7)\) we take the Hamming metric in \( H_1 \), the extended metric turns out to be the chain metric of \( \mathcal{C} \), i.e.
\[
\bar{d}_{\text{Ham}} = d_\mathcal{C}.
\]

**Chain isometries.** We now consider isometries between whole chains of groups. Let \( \mathcal{C} \) and \( \mathcal{C}' \) be two chains of subgroups of the same length of groups \( G \) and \( G' \) respectively, say \( H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G \) and \( H'_1 \subseteq H'_2 \subseteq \cdots \subseteq H'_n = G' \).

**Definition 7.5.** We say that \( \mathcal{C} \) is isometric to \( \mathcal{C}' \), denoted \( \mathcal{C} \simeq \mathcal{C}' \), if for every \( i = 1,\ldots,n \) there are metrics \( d_i \) of \( H_i \) and \( d'_i \) of \( H'_i \) such that \((H_i,d_i) \cong (H'_i,d'_i)\). The groups \( G \) and \( G' \) are said to be chain isometric if they admit isometric chains.

That is, if two chains \( \mathcal{C} \) and \( \mathcal{C}' \) are isometric we have
\[
(7.8) \quad H_1 \preceq H_2 \preceq \cdots \preceq H_n = G \quad |\simeq| \quad H'_1 \preceq H'_2 \preceq \cdots \preceq H'_n = G'.
\]
We now show that chains of the same length and corresponding sizes are isometric.

**Lemma 7.6.** Let $G$ and $G'$ be groups with chains of subgroups $\mathcal{C}$ and $\mathcal{C}'$, respectively given by $0 \neq H = H_1 \subset H_2 \subset \cdots \subset H_n = G$ and $0 \neq H' = H'_1 \subset H'_2 \subset \cdots \subset H'_n = G'$. If $|H_i| = |H'_i|$ for $1 \leq i \leq n$ then we have the chain isometry $(G, d_G) \simeq (G', d_{G'})$.

**Proof.** Since $|H_1| = |H'_1|$ there is a bijection $\eta : H_1 \rightarrow H'_1$ inducing the trivial isometry $(H_1, d_{Ham}) \simeq (H'_1, d_{Ham})$. By applying part (b) of Theorem 6.1 we can lift this isometry to the get $(H_2, Ext_{H_1}(d_{Ham})) \simeq (H'_2, Ext_{H'_1}(d_{Ham}))$. Repeating this lifting procedure we obtain that $\mathcal{C}$ and $\mathcal{C}'$ are isometric chains with the extended metrics. □

**Example 7.7.** (i) The isometry $\mathbb{Z}_q^n \simeq (\mathbb{Z}_q)^n$ given explicitly in Section 5 can be seen as a chain isometry. In fact, the chains $\mathbb{Z}_q \subset \mathbb{Z}_q^2 \subset \cdots \subset \mathbb{Z}_q^{n-1} \subset \mathbb{Z}_q^n$ are isometric by the previous lemma.

(ii) There is a chain isometry $\mathbb{Z}_q^n \simeq \mathbb{F}_q^n$ given by the chains $\mathbb{Z}_q \subset \mathbb{Z}_q^2 \subset \cdots \subset \mathbb{Z}_q^n$ and $\mathbb{F}_q \subset \mathbb{F}_q^2 \subset \cdots \subset \mathbb{F}_q^n$. In fact, any bijection between $\mathbb{F}_q$ and $\mathbb{Z}_q$ with the Hamming metrics induces a chain isometry between $\mathbb{F}_q^n$ and $\mathbb{Z}_q^n$. One can replace $\mathbb{F}_q$ and $\mathbb{Z}_q$ by any group $C_q$ of order $q$.

**Example 7.8 (Galois fields and rings).** Let $p$ be a prime and $r_1, r_2, \ldots, r_n$ be positive integers such that $r_1 \mid r_2 \mid \cdots \mid r_n$. Consider the Galois rings $R_i = GR(p^{k_i}, r_i)$ for $i = 1, \ldots, n$. Then we have the isometric chains of rings $GR(p^{k_1}, r_1) \subset GR(p^{k_1}, r_1)^{r_1} \subset \cdots \subset GR(p^{k_1}, r_1)^{r_n}$ and, in particular taking $k = 1$, $GR(p, r_i) \simeq \mathbb{F}_{p^{r_i}}$, so this becomes $\mathbb{F}_{p^{r_1}} \subset (\mathbb{F}_{p^{r_1}})^{r_1} \subset \cdots \subset (\mathbb{F}_{p^{r_1}})^{r_n}$.

**Geometric chains.** Let $\mathcal{C}$ be a chain $0 = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = G$ with the sizes of the terms in geometric progression, that is

$$[H_i : H_{i-1}] = m \quad \text{for } i = 1, \ldots, n.$$ \hspace{1cm} (7.9)

We will call this a geometric chain.

**Proposition 7.9.** If $G$ admits a geometric chain $\mathcal{C}$ of subgroups $H = H_1 \subset \cdots \subset H_n = G$ with $H \neq 0$ then we have the isometry $(G, d_G) \simeq (H^n, d_{RT})$. 


Proof. Let $C$ be the chain $H = H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$ and consider the geometric chain $C'$ given by $H \subset H^2 \subset H^3 \subset \cdots \subset H^n$. Starting from the trivial isometry $\text{id} : H \rightarrow H$ with the Hamming metrics and applying Theorem 6.1, we get that $C$ and $C'$ are isometric chains. In particular, $G \simeq H^n$ and

$$\tilde{d}_{\text{Ham}} = d_C = d_{RT}$$

as we wanted to see. \qed

Remark 7.10. The isometries $(\mathbb{Z}_q^n, d_q) \simeq (\mathbb{Z}_n^n, d_{RT})$ and $(\mathbb{Z}_q^n, d_q) \simeq (\mathbb{F}_q^n, d_{RT})$ given in Example 7.7 are instances of geometric chains and of chain isometries given by the previous proposition.

We now show that the result in Theorem 5.2, i.e. that $(\mathbb{Z}_q^n, d_q) \simeq (\mathbb{Z}_n^n, d_{RT})$, can be generalized to any pair of groups $G$ and $H^n$ of order $q^n$, with $G$ and $H$ not necessarily cyclic.

Theorem 7.11. Let $q$ be a prime power and $G, H$ groups with $|G| = q^n$ and $|H| = q$. Then,

$$(G, d_C) \simeq (H^n, d_{RT})$$

where $d_C$ is the chain metric associated to some geometric chain of length $n$.

Proof. Since $|G| = q^n$, by the Sylow theorems we get that $G$ has a geometric chain $C$ of length $n$, say $0 \subset H_1 \subset \cdots \subset H_n = G$. By Lemma 7.9 we have that

$$(G, d_C) \simeq (H^n, d_{RT}).$$

On the other hand, since $|H_1| = |H|$ there is a bijection $\tau : H_1 \rightarrow H$ which extends to $\tau : H_1^n \rightarrow H^n$ and induces the isometry

$$(H_1^n, d_{RT}) \simeq (H^n, d_{RT}).$$

In fact,

$$d_{RT}(\tau(x), \tau(y)) = \max_{1 \leq i \leq n} \left\{ i : x_i \neq \tau(y)_i \right\} = \max_{1 \leq i \leq n} \left\{ i : x_i \neq y_i \right\} = d_{RT}(x, y).$$

This implies the result. \qed

This tell us, for instance, that there exist a metric $d$ in the generalized quaternion group $\mathbb{Q}_{2^n}$ of order $2^n$ and a metric $d'$ the dihedral group $\mathbb{D}_{2^{n-1}}$ of order $2^n$ such that

$$(\mathbb{Z}_{2^n}, d_2) \simeq (\mathbb{Q}_{2^n}, d) \simeq (\mathbb{D}_{2^{n-1}}, d') \simeq (\mathbb{Z}_2^n, d_{RT}).$$

Also, in the above list one can add all the groups $\mathbb{Z}_{2^i} \times \mathbb{Z}_{2^{n-i}}$ with some metrics $d_{(i)}$ for $i = 1, \ldots, n - 1$.

8. Block Rosembloom-Tsfasman metric

We will next extend the result for geometric chains given if the previous section for groups with arbitrary chains. For this, we must first consider a generalization of the $RT$-metric.

Definition 8.1. Let $X$ be a group and $n \in \mathbb{N}$. Given a partition $n = m_1 + \cdots + m_r$ consider $X^n = X^{m_1} \times \cdots \times X^{m_r}$. We write $x = (\tilde{x}_1, \ldots, \tilde{x}_r)$ for an element in $X^n$, where $\tilde{x}_i \in X^{m_i}$, for any $i$. We define the block Rosembloom-Tsfasman metric (or BRT-metric) on $X^n$ as

$$(8.1) \quad d_{BRT}(x, y) = \max_{1 \leq i \leq r} \left\{ i : \tilde{x}_i \neq \tilde{y}_i \right\}. $$
Note that for \( r = n \), then \( m_1 = \cdots = m_n = 1 \) and hence the BRT-metric is just the \( RT \)-metric. Also, notice that this metric can be seen as the block poset metric (see [H]) associated to the chain poset \( 1 \leq 2 \leq \cdots \leq r \).

**Theorem 8.2.** Let \( H \) be a proper subgroup of a group \( G \) and \( C \) a chain of subgroups with initial term \( H \). Then we have

\[
(G, d_C) \simeq (H^{[G:H]}, d_{BRT}).
\]

**Proof.** Suppose \( C \) is the chain \( H_1 = H \subset H_2 \subset \cdots \subset H_n = G \). Consider the group \( G' = H^{[G:H]} \). We will construct a chain \( C' \) in \( G' \) of length \( n \), say \( H'_1 = H \subset H'_2 \subset \cdots \subset H'_n = G' \), such that \( |H'_i| = |H_i| \) for all \( i = 1, \ldots, n \). Consider \( H'_2 = H^{[H_2:H_1]} \),

\[
H'_3 = (H'_2)^{[H_3:H_1]} = (H[H_2:H_1])[H_3:H_2] = H^{[H_3:H_1]}
\]

and in general for for every \( 1 \leq i \leq n \) take

\[
H'_i = H^{[H_i:H_1]}
\]

It is clear that \( |H'_i| = |H_i| \) for \( i = 1, \ldots, n \).

By Theorem 6.1, the trivial isometry \( \varphi_1 = id : (H_1, d_{Ham}) \to (H'_1, d_{Ham}) \) can be lifted to an isometry

\[
\varphi_2 : (H_2, Ext_{H_1}^{H_2}(d_{Ham})) \to (H'_2, Ext_{H_1}^{H'_2}(d_{Ham})).
\]

By iterating this process we arrive at an isometry

\[
\varphi_n : (H_n, Ext_{H_{n-1}}^{H_n}(d_{Ham})) \to (H'_n, Ext_{H_{n-1}}^{H'_n}(d_{Ham})).
\]

That is, we have

\[
(G, d_C) \simeq (H^{[G:H]}, d_{BRT}).
\]

It only remains to show that the chain metric \( d_{C'} \) is the BRT-metric. Put \( r = [G : H] \) and \( r_i = [H_i : H_{i-1}] \) for \( i = 1, \ldots, n \) (where \( H_{-1} = 0 \)). Consider the natural decomposition \( H^r = H^r_1 \times \cdots \times H^r_n \). If \( x \in H^r \) then \( x = (\tilde{x}_1, \ldots, \tilde{x}_n) \) with \( \tilde{x}_i \in H^r_i \) for any \( i \). Since

\[
d_{BRT}(x, y) = \max_{1 \leq i \leq n} \{ i : \tilde{x}_i \neq \tilde{y}_i \}
\]

one can check that for \( i = 1, \ldots, n \) we have

\[
d_{BRT}(x, y) = i \iff d_{C'}(x, y) = i.
\]

Hence the metrics coincide and the result thus follows. \( \square \)

**Example 8.3.** Let \( G = \mathbb{Z}_{q^n} \) and \( H = \mathbb{F}_q, n \geq 2 \). By the previous theorem, if we take the chains

\[
C : \quad 0 \subset \mathbb{Z}_q \subset \mathbb{Z}_{q^n} \quad \text{and} \quad C' : \quad 0 \subset \mathbb{F}_q \subset \mathbb{F}_{q^n}
\]

and we consider the decomposition \( \mathbb{F}_q^n = \mathbb{F}_q \times \mathbb{F}_q^{n-1} \) we get

\[
(\mathbb{Z}_{q^n}, d_C) \simeq (\mathbb{F}_q^n, d_{BRT}).
\]

Note that the weight function associated to \( C \) is

\[
w_C = \begin{cases}
0 & \text{if } x = 0, \\
1 & \text{if } x \in q^{n-1}\mathbb{Z}_{q^n} \setminus \{0\}, \\
2 & \text{if } x \in \mathbb{Z}_{q^n} \setminus q^{n-1}\mathbb{Z}_{q^n}.
\end{cases}
\]
Now, by properly rescaling this weight, we get the following,

\[
\tilde{w}_C = \begin{cases} 
0 & \text{if } x = 0, \\
q^{n-1} & \text{if } x \in q^{n-1}\mathbb{Z}_{q^n} \setminus \{0\}, \\
q^{n-2}(q-1) & \text{if } x \in \mathbb{Z}_{q^n} \setminus q^{n-1}\mathbb{Z}_{q^n}.
\end{cases}
\]

Thus, we get

\[(8.3) \quad (\mathbb{Z}_{q^n}, \tilde{d}_C) \simeq (\mathbb{F}_q^n, \tilde{d}_{BRT})\]

where \(\tilde{d}_{BRT}\) is a rescaled metric obtained from \(d_{BRT}\). It is easy to check that after this rescaling both keep being metric functions. In the case when \(q = p\) is prime, the metric \(\tilde{d}_C\) coincides with the homogeneous metric (see [6]) defined over the ring \(\mathbb{Z}_p^n\),

\[(8.4) \quad (\mathbb{Z}_p^n, d_{Hom}) \simeq (\mathbb{F}_p^n, \tilde{d}_{BRT}).\]

**Remark 8.4.** As in the previous example, we have the isometry \((\mathbb{Z}_{q^n}, d_{Hom}) \simeq (\mathbb{F}_q^n, \tilde{d}_{BRT})\) for \(q = p^r\). Consider the \(q\)-ary first order Reed-Muller code \(RM(1,q^{n-1})\) and let \(G\) be any generating matrix of the code whose first row is the all ones vector \((1,1,\ldots,1)\). The code \(RM(1,q^{n-1})\) lies in \(\mathbb{F}_q^n\) with the Hamming metric and right multiplication by \(G\) encode the space \(\mathbb{F}_q^n\) into \(RM(1,q^{n-1})\), that is \(RM(1,q^{n-1}) = \{xG : x \in \mathbb{F}_q^n\}\). Putting these things together we get

\[(8.5) \quad (\mathbb{F}_q^n, \tilde{d}_{BRT}) \to (RM(1,q^{n-1}), d_{Ham}^{n-1}) \hookrightarrow (\mathbb{F}_q^{n-1}, d_{Ham}^{n-1}).\]

Combining the isometry (8.3) with the embedding (8.5) we get the isometric embedding

\[(\mathbb{Z}_{q^n}, \tilde{d}_C) \hookrightarrow (\mathbb{F}_q^{n-1}, d_{Ham}^{n-1}).\]

Taking \(q = p\) prime, we obtain the following result of Greferath ([7])

\[(8.6) \quad (\mathbb{Z}_p^n, d_{Hom}) \hookrightarrow (\mathbb{F}_p^{n-1}, d_{Ham}^{n-1}).\]

In this way, similarly as in Section 4 we get isometric embeddings of the form

\[\mathbb{Z}_p^n \hookrightarrow (\mathbb{F}_p^{n-i}, d_{Ham}^{n-i})\]

for \(i = 1, \ldots, n\).

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