Chiral Deformations of Conformal Field Theories

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Abstract

We study general perturbations of two-dimensional conformal field theories by holomorphic fields. It is shown that the genus one partition function is controlled by a contact term (pre-Lie) algebra given in terms of the operator product expansion. These models have applications to vertex operator algebras, two-dimensional QCD, topological strings, holomorphic anomaly equations and modular properties of generalized characters of chiral algebras such as the $W_{1+\infty}$ algebra, that is treated in detail.
1. Introduction

In this paper we consider chiral deformation of two-dimensional conformal field theories. By a chiral deformation we understand a field theory with an action of the form

\[ S = S_0 + \int d^2 z A(z), \]  

where \( S_0 \) indicates the action of the undeformed conformal field theory and \( A(z) \) is a holomorphic field of arbitrary (integer) spin. There are various questions and problems posed by this class of models that we will try to address in this paper. But let us first indicate some motivations to consider these field theories.

1.1. \((\partial \varphi)^3\) theory and two-dimensional QCD

We were originally motivated by the following simple two-dimensional quantum field theory. Consider a real bosonic scalar field \( \varphi(z, \bar{z}) \) on a two-dimensional Euclidean space-time with the topology of a torus and with the following cubic interaction

\[ S = \int (\partial \varphi \bar{\partial} \varphi + \lambda (\partial \varphi)^3). \]  

Here it is important that the interaction term \((\partial \varphi)^3\) is a holomorphic spin 3 field. This rather uncommon interacting scalar field theory turns out to be interesting from several points of view.

First, as pointed out by Douglas [5], if we choose the coupling constant \( \lambda = 1/N \) this model appears as an effective string field theory for the two-dimensional QCD string on a target space torus. Two-dimensional \( U(N) \) Yang-Mills theory in the large \( N \) limit has been studied in detail by Gross and Taylor [2]. They have shown that the partition function has a string interpretation in terms of maps of Riemann surfaces to the target space-time. That is, the partition function has the characteristic form

\[ Z(\tau, N) = \exp \sum_g N^{2-2g} F_g(\tau), \]  

where \( \tau \) is the (complexified) area of the space-time surface and the contributions \( F_g(\tau) \) ‘count’ the maps of a genus \( g \) string world-sheet to the target space-time. (Roughly, \( F_g(\tau) = \sum_n F_{g,n} q^n \) where \( n \) is the degree of the map and \( q = e^{2\pi i \tau} \)).

The general description of the counting functions \( F_g \) is rather complicated but can be completely understood in terms of holomorphic maps [3]. (See also [4], where a closely
related formalism using harmonic maps is used.) However, in the case that the target-space has the topology of a torus, the combinatorics becomes much more straightforward and can be summarized by the fact that the string field theory takes the extremely simple cubic form given above, with string coupling constant \( \lambda \) given by \( 1/N \). This remarkable simplification is very much dependent on the equivalence of the two-dimensional bosonic scalar field \( \phi \) with a Dirac spinor \((b, c)\). In terms of these fermions the \((\partial \phi)^3\) action simply reads

\[
S = \int \left( b \partial c + \lambda b \partial^2 c \right).
\]

This is a quadratic action, which account for the solvability of the model. In fact, similar actions have also appeared in the \( c = 1 \) matrix model \[6\].

This free field theory representation of the QCD string partition function gives a very simple and elegant formula for the string-loop genus expansion as a generalized conformal character

\[
Z(\tau, N) = \text{Tr} \left( q^{L_0} e^{H/N} \right), \quad H = \oint b \partial^2 c.
\]

It has been noticed that the expansion coefficients \( F_g \) that appear in the perturbation theory in the coupling constant have rather peculiar modular properties. They are so-called quasi-modular forms \[7, 8, 9\]. This raises the interesting issue of what the modular properties of theories of the \((\partial \phi)^3\) type are, in particular how the coupling constant \( \lambda \) transforms. This is one of the questions we will answer in generality in this paper.

### 1.2. Kodaira-Spencer theory

A second motivation for considering the \((\partial \phi)^3\) model comes from topological string theory. In fact, the above model is in many respects a two-dimensional analogue of the six-dimensional Kodaira-Spencer field theory that has been introduced as the effective field theory of a topological string of type B on a Calabi-Yau three-fold \(X\), see \[10\]. (In this way, the QCD string can be regarded as a topological string of type A, related by mirror symmetry of \(T^2\).) In the Calabi-Yau case we are dealing with a six-dimensional Lagrangian, of the form

\[
\int_X \partial \varphi \wedge \bar{\partial} \varphi + \lambda \partial \varphi \wedge \partial \varphi \wedge \partial \varphi,
\]

where the field \( \varphi \) can be seen as a \((1,1)\) form and the holomorphic three-form is used to make sense of the cubic interaction. This quantum field theory is supposed to calculate the instanton sum on the mirror manifold. Because it has a natural string field theory interpretation, the obvious problems of this Lagrangian should be cured using the string regularization. In two-dimensions there is a unique Calabi-Yau manifold, the torus or elliptic curve. Its mirror manifold is again an elliptic curve, and the instanton sum is
given by our $(\partial \varphi)^3$ model. See [8] for more details on mirror symmetry for elliptic curves in relation to the counting functions of holomorphic maps.

The two-dimensional model also shares with the Kodaira-Spencer theory the property that it is superficially non-renormalizable, while finite in some natural regularization. We will see that in the two-dimensional model this can be understood in the following way. At the expense of introducing contact terms, that we will carefully analyze, the chiral interactions can be written as contour integrals of the type

$$\oint dz (\partial \varphi)^3.$$  \hspace{1cm} (1.7)

These contour integrals can be chosen to be non-intersecting, which eliminates all divergences. In this two-dimensional case this regularization is much more straightforward than in the six-dimensional model, where it is supposed to come from string theory. In fact, one of our motivations was to understand to which extent the six-dimensional theory has an equally well-defined perturbation theory.

The six-dimensional Kodaira-Spencer model is (partially) solvable through the so-called holomorphic anomaly equation [11]. We are therefore also interested in the holomorphic properties of the above model in terms of the modulus $\tau, \bar{\tau}$ of the torus. We will derive an analogue equation for the $\tau$-derivative of the partition function, which is a generalization of the holomorphic anomaly equations derived in six dimensions.

1.3. Chiral algebras and generalized characters

The $(\partial \varphi)^3$ model can be seen as just a particular example of a large class of field theories that can be constructed by deforming a given conformal field theory with an arbitrary chiral operator. Such an operator has conformal dimensions $(h,0)$ and is therefore not marginal. Hence the conformal symmetry will be broken. Since the operator carries spin, the deformation is also not Lorentz/rotation invariant.

Some examples of perturbations by fields of non-zero spin can be found in certain models in two-dimensional statistical physics. Here rotational symmetry breaking is of course less of a problem. A rather famous example is the chiral Potts model [12], that can be considered as a deformation of a minimal CFT by an operator of conformal dimensions $(\frac{7}{5}, \frac{2}{5})$ and thus of spin one, see [13].

One can also think of these chirally deformed models as field theories coupled to generalized (constant) background higher spin gauge fields. This makes the subject of interest in the context of higher spin analogues of chiral quantum gravity, so-called $W$-gravity [14].

The partition functions of such deformed models can be considered as generalized characters of the chiral algebra underlying the conformal field theory. If $R$ is a represent-
tation of a vertex operator algebra $V$ with a basis $H^i$ of commuting Noether charges, then one can define generalized characters as

$$
\chi_R(\tau, s) = \text{Tr}_R \left( q^{L_0 - \frac{c}{24}} e^{s_i H^i} \right). \tag{1.8}
$$

with $q = e^{2\pi i \tau}$ and $s_i$ coordinates on the “Cartan subalgebra” of the chiral algebra. These characters carry a representation of the modular group $PSL(2, \mathbb{Z})$. By general arguments, for a rational conformal field theory, where the irreducible representations $R_I$ are finite in number, we have a transformation rule of the form

$$
\chi_I(\tau', s') = \sum_J M_{IJ} \chi_J(\tau, s), \tag{1.9}
$$

where $\tau' = (a\tau + b)/(c\tau + d)$. One of our aims in this paper will be to determine how the transformed parameters $s'_i$ are expressed in terms of the variables $s_i$ and $\tau$ under a modular transformation. One of our conclusions will be that the variables $s_i$ do not have canonical modular properties, but certain polynomials in them will transform canonically.

A particular model where all this can be seen in great detail is the $W_{1+\infty}$ algebra. The representation theory of this algebra has been intensely studied, see e.g. [15, 16]. In §4 we will treat the $c = 1$ free field theory realization of $W_{1+\infty}$.

Finally we mention that instead of looking at holomorphic fields and characters of chiral algebras, one can also consider $N = 2$ superconformal field theories and their elliptic genera [17]. These objects behave very much like characters of holomorphic CFT’s. Our results will then apply to the so-called “refined elliptic genus” introduced in [18].

2. Chiral Algebras

We first make a few general comments about chiral algebras of two-dimensional conformal field theories. For more details about vertex operator algebras see e.g. [19, 20].

2.1. Vertex operator algebras

Consider a general unitary conformal field theory and let $V$ denote the space of chiral operators, that is, holomorphic but not necessarily primary fields $A(z)$ of conformal weight $(\nu, 0)$. This space $V$ is an infinite-dimensional vector space, naturally graded by the weight $\nu \in \mathbb{N}$ of the operators. It always contains the identity $1$, the unique field of weight zero, and the stress tensor $T$ of weight two, together with all its descendents.
On this space of chiral operators we have an action of the translation operator

$$\partial := \frac{\partial}{\partial z} = L_{-1}$$

that raises the weight of an operator by one. We will consider in this paper mainly the quotient space

$$W = V/\partial V.$$  \hspace{1cm} (2.2)

One can think of the map $V \to W$ as associating to a chiral current $A(z) \in V$ its Noether charge $Q(A) \in W$, with

$$Q(A) = \oint \frac{dz}{2\pi i} A(z).$$  \hspace{1cm} (2.3)

If $A$ has the usual mode expansion $A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-h}$, then this charge is given by $Q(A) = A_{h-1}$. Note that these charges are the zero modes on the $z$-plane, not the zero modes $A_0$ on the cylinder with coordinate $\log z$.

The operator product expansion of two fields $A, B \in V$, denoted here as

$$A(z) \cdot B(w) \sim \sum_{n=-\infty}^{\infty} (z-w)^{-n} (AB)_n(w),$$  \hspace{1cm} (2.4)

gives $V$ the structure of a vertex operator algebra. Vertex operator algebras can be completely axiomatically defined in terms of the infinite set of operator products $(\cdot \cdot \cdot)_n$ and the action of the derivative $\partial$ \cite{Li}. For bosonic fields the operator products have the symmetry property

$$(AB)_n = (-1)^n (BA)_n,$$  \hspace{1cm} (2.5)

and the conformal weight of the product $(AB)_n$ is given by $h_A + h_B - n$.

As is well-known, the first order product $(\cdot \cdot \cdot)_1$ induces a Lie bracket on the coset space $W$

$$[A, B]_1 := (AB)_1 \pmod{\partial}.$$  \hspace{1cm} (2.6)

We denote this bracket here with a suffix 1 to stress the fact that it is related to the first order pole in the operator product. The Jacobi identity only holds up to terms of the form $\partial(\cdot \cdot \cdot)$, so $W$ and not $V$ is a Lie algebra. This is the familiar Lie algebra generated by the corresponding conserved Noether charges

$$[Q(A), Q(B)] = Q([A, B]_1).$$  \hspace{1cm} (2.7)
2.2. Pre-Lie algebra structure

In the following we will only consider “abelian” chiral algebras where the first order Lie bracket $[\cdot, \cdot]_1$ on $W$ is trivial. That is, we will assume that, possibly after a suitable restriction to a “Cartan subalgebra,” the vertex operator algebra $V$ has the property

$$[AB]_1 \in \ker (\partial), \quad \forall A, B \in V. \quad (2.8)$$

This assumption has an important consequence, that will play a crucial role in the rest of the paper. It allows us to define a new product $W \times W \to W$, namely

$$\nabla_B A := \partial^{-1} [AB]_1 \quad (2.9)$$

It is straightforward to check that this expression is well-defined on equivalence classes, \textit{i.e.} modulo derivatives. We have written this product as a covariant derivative, since we will see that it satisfies all the properties of a flat, torsion-free linear connection for $W$, if one thinks of $W$ as the space of vector fields on some manifold.

We should make one remark here. As it stands, the definition of the product $\nabla_B A$ is incomplete. It is only well-defined if central terms are absent. That is, the identity operator 1, with the complicating property $\partial 1 = 0$, should not appear in $\nabla_B A$. Since the conformal weight of $\nabla_B A$ is given by $h_A + h_B - 2$, this problem only occurs in the case that both $A$ and $B$ are spin one currents. We therefore restrict ourselves to fields of spin $h \geq 2$. We will introduce the spin one fields at a later stage in §4.4.

The product $\nabla_B A$ is neither symmetric nor antisymmetric. In fact, the symmetric part gives the quadratic residue $[AB]_2$ on $W$

$$(AB)_2 = \nabla_A B + \nabla_B A. \quad (2.10)$$

This is the famous commutative but non-associative product that features in the construction of the Griess algebra — the fundamental module for the Monster group \cite{20}.

The antisymmetric part of $\nabla_A B$ gives rise to a second order Lie bracket $[\cdot, \cdot]_2$ on $W$

$$[A, B]_2 = \nabla_A B - \nabla_B A, \quad (2.11)$$

which should be distinguished from the more familiar first order bracket $(2.6)$. Equation (2.11) can be equivalently read as saying that the “connection” $\nabla$ is torsion-free. Since we will always assume that the first order Lie bracket vanishes on $W$, no confusion can arise and we will drop the subscript 2 from now on. The Jacobi identity for the bracket
[\cdot, \cdot] follows again directly from the general Jacobi identity of vertex operator algebras that gives the relation
\[ [\nabla_A C, \nabla_B C] = \nabla_{[A,B]} C. \]
(2.12)
This can be interpreted as saying that the connection \( \nabla \) is flat, i.e. the operator \( \nabla_A : W \to W \) satisfies
\[ [\nabla_A, \nabla_B] = \nabla_{[A,B]}. \]
(2.13)
We stress again that the second order Lie bracket on \( W \) is only well-defined if the first order bracket vanishes.

In terms of mode expansions and Noether charges we simply have
\[ Q(\nabla_A B) = [\nabla_A, Q(B)], \quad \nabla_A = \oint \frac{dz}{2\pi i} z A(z) = A_{h-2}. \]
(2.14)

We also note here that in the special case of the stress tensor \( T \) (of spin two) and an arbitrary field \( A \) (of spin \( h \)) we have
\[ \nabla_A T = A, \quad \nabla_T A = (h - 1)A. \]
(2.15)

The algebraic structure of a vector space \( W \) with a product \( \nabla \) satisfying the relations (2.11) and (2.12) is sometimes referred to as a pre-Lie algebra [21], since by definition the commutator of the products gives a Lie bracket. Note that this is a stronger notion than a Lie-algebra: a pre-Lie algebra is always also a Lie algebra.

Any pre-Lie algebra \( W \) acts in two different ways on itself as a Lie algebra. First, there is the obvious adjoint action
\[ \text{ad}_A : B \to [A, B]. \]
(2.16)
Secondly, there is the fundamental action
\[ \nabla_A : B \to \nabla_A B. \]
(2.17)
A simple (and canonical) example of a pre-Lie algebra is the Lie algebra of vector fields on \( \mathbb{R}^n \) with \( \nabla \) the trivial connection (or more general a manifold with a flat metric). If we trivialize a vector field \( A \) on \( \mathbb{R}^n \) in flat coordinates \( x^i \) as \( A = A^i \partial_i \), then we can treat the components as functions:
\[ (\nabla_A B)^j = A^i \partial_i B^j. \]
(2.18)
3. Chiral Conformal Perturbation Theory

We now turn to the more general problem we want to address in this paper, the discussion of chiral deformations of conformal field theories.

3.1. Modular invariance

Consider the partition function \( Z(\tau, \bar{\tau}) \) of a general unitary conformal field theory with (abelian) chiral algebra \( V \) on a torus or elliptic curve \( E \), with modulus \( \tau \in \mathbb{H} \) in the upper-half plane. We want to deform the action \( S_0 \) of the conformal field theory by adding to it a term of the form

\[
\int_E d^2z \, A(z)
\]

with \( A(z) \in V \) a chiral current. Since total derivatives integrate to zero on a compact space — a property we will carefully preserve in the regularization procedure — we can consider \( A \) to be actually an equivalence class in the quotient space \( W = V/\partial V \) of fields modulo total derivatives. We will refer to \( W \) as the Cartan algebra.

If \( \Phi_i \) is basis for \( W \), the general form of the chiral perturbation of the action will take the form

\[
S = S_0 - \int \frac{d^2z}{2\pi \tau_2} t^i \Phi_i(z).
\]

(3.2)

We can think of the \( t^i \) as constant background gauge fields. The term \( \tau_2 = \text{Im} \, \tau \) is added to ensure proper modular weights for the coupling constants \( t^i \). In fact, under a modular transformation, that acts on the modulus \( \tau \) by fractional linear transformation

\[
\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2, \mathbb{Z}),
\]

(3.3)

we have the following transformation rule of the linear coordinate \( z \in E \):

\[
z \to z' = \frac{z}{c\tau + d}.
\]

(3.4)

Consequently a chiral field \( \Phi_i(z)(dz)^{h_i} \) of conformal weight \( h_i \) transforms as

\[
\Phi_i(z) \to (c\tau + d)^{h_i} \Phi_i(z')
\]

(3.5)

and so also has modular weight \( h_i \). Since the action (3.2) should be modular invariant, the coupling constant \( t^i \) dual to \( \Phi_i \) transforms as a modular forms of weight \( -h_i \),

\[
t^i \to (c\tau + d)^{-h_i} t^i.
\]

(3.6)
In more fancy terms: the family of perturbed field theories parametrized by the variables $t^i$ forms a non-trivial vector bundle over the genus one moduli space $\mathcal{M}_1 = \mathbb{H}/\text{PSL}(2, \mathbb{Z})$.

3.2. Contact terms

In order to make rigorous sense of the deformed model in terms of perturbation theory around the original undeformed conformal field theory, one has to make sense of the following generating functional of correlation functions

$$Z[t] = \langle \exp \int \frac{d^2z}{2\pi \tau_2} t^i \Phi_i(z) \rangle. \quad (3.7)$$

When the exponential is expanded, we encounter terms of the form

$$\langle \cdots \int \frac{d^2z}{2\pi \tau_2} A(z) \int \frac{d^2w}{2\pi \tau_2} B(w) \cdots \rangle \quad (3.8)$$

for some fields $A(z), B(w)$ and there will be singularities in the integrand for coinciding position variables $z = w$. We will have to prescribe how to integrate over these poles.

Our principal value prescription will be the one proposed by Douglas in ref. [5]. In his proposal one writes

$$A(z) = \overline{\partial} C, \quad C(z, \overline{z}) = (\overline{z} - z) A(z), \quad (3.9)$$

and subsequently applies Stokes’ theorem to the integrals, where little disks are cut out around the positions of the operator insertions. Since the operator $C$ is not single-valued, we pick up both a period contribution from the multi-valuedness and a residue contribution from the poles at the punctures. These contributions can however be explicitly evaluated in terms of the operator product coefficients, with the result

$$\int \frac{d^2z}{2\pi \tau_2} A(z) \cdot B(w) = \int_0^1 \frac{dz}{2\pi} A(z) \cdot B(w) + \frac{1}{2\tau_2} (w \cdot (AB)_1(w) + (AB)_2(w)). \quad (3.10)$$

The last two contributions on the right-hand side can be interpreted as contact terms due to first and second order poles in the operator product respectively. Since in our formulas all operators are integrated in the end, we can consistently work modulo $\partial$ and freely perform a partial integration in the variable $w$. That allows us to replace the second and third term on the right-hand side of (3.10) by the term

$$c(A, B) = \frac{1}{2\tau_2} \left[(AB)_2 - \partial^{-1}(AB)_1\right] = \frac{1}{2\tau_2} \nabla_A B. \quad (3.11)$$
Here we used definition (2.9) of the pre-Lie structure of the Cartan algebra $W$. According to the discussion in §2 the contact term (B.11) is well-defined modulo total derivatives, i.e. makes sense on the quotient space $W = V/\partial V$. It has no obvious symmetry properties under interchange of the two arguments.

More precisely, if we introduce the following short-hand notation for surface and contour integrals respectively

$$\int A = \int \frac{dz}{2\pi \tau_2} A(z), \quad \oint A = \int_0^1 \frac{dz}{2\pi} A(z), \quad (3.12)$$

we find that the following relation is valid within correlation functions

$$\int A \int B = \oint A \int B + \int c(A, B) + \ldots \quad (3.13)$$

Here $c(A, B)$ indicates the contact term between the fields $A$ and $B$ and the ellipses represent similar terms if extra fields like $B$ are present. We notice that all contact terms disappear in the limit $\tau_2 \to \infty$.

We can use equation (3.13) to recursively eliminate all surface integrals in terms of contour integrals. It is instructive to work this out explicitly for correlators with a small number of operators. Using the identities of the previous section one finds that everything can be expressed in terms of the second order operator product $(\cdot \cdot)_2$. For example, the two-point function satisfies

$$\langle \int A \int B \rangle = \langle \oint A \oint B \rangle + \frac{1}{2\tau_2} \langle \oint (AB)_2 \rangle , \quad (3.14)$$

and similarly the three-point function satisfies

$$\langle \int A \int B \int C \rangle = \langle \oint A \oint B \oint C \rangle$$

$$+ \frac{1}{2\tau_2} \langle \oint A \oint (BC)_2 + cyclic \rangle$$

$$+ \frac{1}{8\tau_2^2} \langle \oint (A(BC)_2 + cyclic) \rangle. \quad (3.15)$$

3.3. Reparametrization of coupling constants

We can now reformulate our deformation problem as follows. The most general action with chiral interactions we want to consider is of the type

$$S = S_0 - \int t^i \Phi_i. \quad (3.16)$$
We have seen in the previous subsection that these chiral interactions can be rewritten in terms of contour integrals at the expense of introducing contact terms. In that way the deformed Lagrangian $S$ is rewritten in terms of a deformed Hamiltonian $H$, where we integrate chiral fields over a space-like contour,

$$H = H_0 + \oint s^i \Phi_i.$$  

(3.17)

The Hamiltonians $\oint \Phi_i$ are the conserved charges or zero modes of the chiral currents $\Phi_i$ on the cylinder. They should not be confused with the charges introduced in (2.3). If a chiral field $A$ of spin $h$ has a mode expansion $A(w) = \sum_n A_n w^{-n-h}$ on the plane, with coordinate $w = e^{2\pi iz}$, then the corresponding Hamiltonian is given by

$$\oint \frac{dz}{2\pi} A = i(2\pi i)^{h-1} A_0.$$  

(3.18)

By the familiar contour deformation argument, these Hamiltonians commute,

$$\left[ \oint \Phi_i, \oint \Phi_j \right] = 0,$$  

(3.19)

because of our assumption (2.8) that the chiral algebra is abelian. There is consequently no ambiguity in the choice of contours. They can be chosen to be non-intersecting in some arbitrary time ordering. Therefore, in Hamiltonian perturbation theory the model is perfectly well-defined. The partition function can be computed in the operator formalism as a trace in the Hilbert space $\mathcal{H}$ of the conformal field theory,

$$Z[\tau, \bar{\tau}; s] = \left\langle \exp \oint s^i \Phi_i \right\rangle$$

$$= Tr_{\mathcal{H}} \left[ q^{L_0 - c/24} \bar{q}^{-\bar{c}/24} \exp \oint s^i \Phi_i \right],$$  

(3.20)

with $q = e^{2\pi i \tau}$. The partition function $Z[\tau, \bar{\tau}; s]$ should be considered in the limit $\bar{\tau} \to -i\infty$ as a generalized character of the chiral algebra $V$.

Note that the interactions that we have added have weights greater than 2 and are strictly speaking nonrenormalizable. However, because of holomorphicity, they are effectively integrated only over one-dimensional cycles. Therefore, they do not give rise to the expected divergences of nonrenormalizable interactions.

In our notation we already anticipated that the parameters $t^i$ in the Lagrangian and the parameters $s^i$ in the Hamiltonian will differ. Indeed, due to the effect of the contact

Because of the conformal anomaly $c$, this becomes for the stress-tensor $\oint T = -2\pi L_0 + \frac{c}{12}$.  

12
terms, the coupling constants \( s^i \) will in general be some non-trivial function of the so-called canonical coordinates \( t^i \) that appear in the Lagrangian \([10]\)

\[
s^i = s^i[t^j]. \tag{3.21}
\]

The fact that contact terms induce a reparametrization of the space of coupling constants is a familiar phenomenon in conformal field theory \([22]\). It is precisely in the appearance of contact terms that the superficial non-renormalizability of the model (re)emerges. Indeed, from (3.11) we see that the weight of the contact term \( c(A, B) \) is \( h_A + h_B - 2 \). Since we have to add this higher spin field to the Lagrangian with a non-zero coupling constant, this gives a cascade of terms of higher and higher dimension. For example, two spin 3 fields can produce a spin 4 field in their contact term, which on its term can produce a spin 5 field, etc. etc.

Another way to see the necessity of a reparametrization of the coupling constants, is that the variables \( s^i \) have a priori no obvious modular properties. We explicitly have broken the modular invariance of the model by picking a preferred cycle (in this case the \( a \)-cycle, \( z \) real, a constant time-slice) on the torus. Equivalently, in the Hamiltonian formalism, by a choice of a time-direction, we break global diffeomorphism invariance.

### 3.4. A simple example: the stress-tensor deformation

This reparametrization effect is perhaps most familiar in the case of a perturbation by the stress-tensor, which is simply a deformation of the metric,

\[
\delta S = -t \int T(z). \tag{3.22}
\]

This can be related to a shift in the Hamiltonian

\[
\delta H = s \oint T(z) = -2\pi s L_0 \tag{3.23}
\]

for a particular function \( s(t) \) as follows. (See also \([10]\) where this example is discussed in the general context of deforming the complex structure of a Calabi-Yau manifold.) The deformation (3.22) translates into a deformation of the \( \overline{\partial} \)-operator of the form

\[
\overline{\partial}_\mu = \overline{\partial} + \mu \partial \tag{3.24}
\]

with Beltrami differential \( \mu = -t/2\tau_2 \). If we write the complex coordinates \( z = x_1 + \tau x_2, \overline{z} = x_1 + \overline{\tau} x_2 \), so that

\[
\partial = \frac{\overline{\tau} \partial_1 - \partial_2}{\overline{\tau} - \tau}, \quad \overline{\partial} = \frac{-\tau \partial_1 + \partial_2}{\tau - \tau}, \tag{3.25}
\]
we see that this corresponds to a deformation of the modulus $\tau_\mu$ given by

$$\tau_\mu = \tau + \frac{2i\tau_2\mu}{1 - \mu}. \quad (3.26)$$

(In these formulas the complex-conjugate is left unchanged, $\tau_\mu = \tau$.) Since the variable $s$ is given by $s = -i(\tau_\mu - \tau)$, we find $s = 2\tau_2\mu/(1 - \mu)$. Therefore the coupling constants $s$ and $t$ are related as

$$\frac{s}{2\tau_2} + 1 = \left(1 - \frac{t}{2\tau_2}\right)^{-1}. \quad (3.27)$$

We will have a chance to verify this relation in a moment. The variable $t$ (or $\mu$) is the so-called canonical coordinate, centered at $\tau$, and the variable $s$ (or $\tau_\mu$) can be thought of as the canonical coordinate centered at $\tau = i\infty$ [10]. We will generalize this point of view to arbitrary coupling constants. In the limit $\tau_2 \to \infty$ all contact terms disappear, and we find that in a perturbation around that point the coupling constants are simply identical, $s^i = t^i$, so $s^i$ is indeed the canonical coordinate at infinity.

### 3.5. Differential equations and recursion relations

We now wish to calculate the relation between the two sets of coupling constants $s^i$ and $t^i$ that follows from the equality of the partition sums

$$\langle \exp \int t^i \Phi_i \rangle = \langle \exp \int s^i \Phi_i \rangle \quad (3.28)$$

The relation between the coupling constants can in principle be solved by considering the more general generating function of two sets of variables

$$Z[s, t] = \exp \left( \int s^i \Phi_i + \int t^i \Phi_i \right). \quad (3.29)$$

We want to find the relation between $s^i$ and $t^i$ such that

$$Z[s, 0] = Z[0, t]. \quad (3.30)$$

We will demonstrate that, as a consequence of the contact term relation (3.13), the generalized partition function $Z[s, t]$ satisfies a set of differential equations, that allows us to solve the dependencies.

To fix notation, we will write the connection $\nabla$ in terms of our basis $\Phi_i$ as

$$\nabla_i \Phi_j = c_{ij}^k \Phi_k. \quad (3.31)$$
Consider now the linear first-order differential operators $L_i^{(s)}, L_i^{(t)}$ with

$$
L_i^{(s)} = \sum_{j,k} c_{ij}^k s^j \frac{\partial}{\partial s^k}, \quad L_i^{(t)} = \sum_{j,k} c_{ij}^k t^j \frac{\partial}{\partial t^k}.
$$

They form a representation of the Lie algebra $W$, as introduced in §2,

$$
[L_i, L_j] = f_{ij}^k L_k, \quad f_{ij}^k = c_{ij}^k - c_{ji}^k.
$$

We now claim that the partition function $Z[s, t]$ satisfies the following linear differential equation, that we will call the “master equation”

$$
\frac{\partial Z}{\partial t^i} = \left[ \frac{\partial}{\partial s^i} + \frac{1}{2\tau_2} \left( L_i^{(s)} + L_i^{(t)} \right) \right] Z.
$$

Let us first try to explain the derivation of this relation in words. Differentiating the generating function $Z$ with respect to $t^i$ “brings down” the surface integral $\int \Phi_i$. According to the fundamental contact term relation (3.13) we can write this as a contour integral $\oint \Phi_i$, i.e. a differentiation with respect to $s^i$, plus additional terms coming from first and second order poles in the operator product of the field $\Phi_i$ with the various other fields. This contact term has the form (3.11) and is valid both for the surface and contour integrals. More explicitly, in terms of a particular term in the generating function $Z$,

$$
\langle \int \Phi_i \prod_{m \in M} \int \Phi_m \prod_{n \in N} \oint \Phi_n \rangle = \langle \oint \Phi_i \prod_{m \in M} \int \Phi_m \prod_{n \in N} \oint \Phi_n \rangle
$$

$$
+ \sum_{j \in M} \frac{1}{2\tau_2} \langle c_{ij}^k \int \Phi_k \prod_{m \in M-j} \int \Phi_m \prod_{n \in N} \oint \Phi_n \rangle
$$

$$
+ \sum_{j \in N} \frac{1}{2\tau_2} \langle c_{ij}^k \oint \Phi_k \prod_{m \in M} \int \Phi_m \prod_{n \in N-j} \oint \Phi_n \rangle.
$$

Here $M, N$ are two subsets of indices. By an argument familiar from the theory of two-dimensional topological gravity the effect of this contact term algebra is represented on the generating function $Z$ by the differential operators $L_i^{(s), (t)}$.

The master equation (3.34) can be used to eliminate the variables $t^i$ in favor of $s^i$ once the structure of the pre-Lie algebra $W$ is given. We will illustrate this with a concrete model in the next section, but as a warming-up, let us first reconsider the perturbation
with the stress-tensor $T$ with two couplings $t$ and $s$ discussed in §3.2. Since we have the simple relation $\nabla_T T = T$, in this case the master equation reads

$$\frac{\partial}{\partial t} Z = \left( \frac{\partial}{\partial s} + \frac{1}{2\tau_2} s \frac{\partial}{\partial s} + \frac{1}{2\tau_2} t \frac{\partial}{\partial t} \right) Z.$$  

(3.36)

If we introduce new variables $a = 1 + \frac{s}{2\tau_2}$, $b = 1 - \frac{t}{2\tau_2}$, the master equation reduces simply to

$$\left( a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} \right) Z = 0,$$  

(3.37)

so that $Z[a, b] = Z[a/b]$. Together with the appropriate initial conditions this tells us that $Z[a, 0] = Z[0, b]$, where the couplings $a, b$ are related via $a = 1/b$ or

$$\frac{s}{2\tau_2} + 1 = \left( 1 - \frac{t}{2\tau_2} \right)^{-1}.$$  

(3.38)

This is indeed the relation we found in (3.27).

As this example shows clearly, at this point it is advantageous to introduce a differently normalized set of coupling constants $a^i, b^i$ defined by

$$a^i = \frac{s^i}{2\tau_2} + \delta^{i,T}, \quad b^i = -\frac{t^i}{2\tau_2} + \delta^{i,T}.$$  

(3.39)

Here $i = T$ labels the coupling to the stress-tensor. If we make use of the identity

$$\nabla_A T = A, \quad \forall A \in W,$$  

(3.40)

one can verify that after this shift the constant terms in (3.34) disappear and in terms of the new coupling constants $a^i, b^i$ we have the simplified equation

$$\left( L_i^{(a)} + L_i^{(b)} \right) Z[a, b] = 0,$$  

(3.41)

with an expansion around $a^k = b^k = \delta^{k,T}$.

Written like this, the master equation has a simple interpretation. Remember that the pre-Lie algebra $W$ carries, besides the adjoint representation, a second, fundamental representation (2.17) of the underlying Lie algebra. With this representation, the above equation simply states that the partition function $Z : W \times W \to \mathbb{C}$ is an invariant function.
4. The $c = 1$ Model

We will now turn to the example that motivated the above discussion: the $c = 1$ bosonic field $\phi$ with interactions of the form $(\partial \phi)^n$. Here it is most convenient to use the equivalent fermionic formulation in terms of a spin $\frac{1}{2} (b, c)$ system or Dirac fermion. (For the partition function we should remember to integrate in the end over the spin structures if we wish to obtain the bosonic partition function.)

4.1. $W_{1+\infty}$ algebra

The free boson or Dirac fermion forms a $c = 1$ representation of the $W_{1+\infty}$ vertex operator algebra that is generated as an algebra by the local chiral fields \[ \Phi^{p,q}(z) = \partial^p b \partial^q c \] of spin $h = p + q + 1$. For a given spin $h \geq 1$, only one particular (rather complicated) linear combination of these operators is actually a primary field. However, since we will only be interested in the algebra $W$ of operators modulo total derivatives, we can represent the unique primary field $\Phi^n$ of weight $h = n + 1$ by the class of operators\footnote{In order to eliminate possible confusing notation, we will raise/lower indices in this section compared with the previous section.}

\[ \Phi^n(z) = -b \partial^n c = (-1)^{n-1} \partial^n bc \pmod{\partial}. \] (4.2)

In terms of the bosonic field $\phi$ this field is represented by

\[ \Phi^n(z) = \frac{1}{n+1} (-i \partial \phi)^{n+1} \pmod{\partial}. \] (4.3)

For the moment we do not want to consider the $U(1)$ current $\Phi^0 = -bc$ and therefore only study deformations using the $W_\infty$-piece, generated by the currents $\Phi^n$ with $n \geq 1$ of spin $h = n + 1 \geq 2$.

In this particular example the connection $\nabla$ that determines the contact terms is easily computed using the operator product expansion of the fermion bilinears, with the result

\[ \nabla^m \Phi^n = n \Phi^{m+n-1}. \] (4.4)

From this it follows that the second order operator product and Lie bracket are given by

\[ (\Phi^m \Phi^n)_2 = (m + n) \Phi^{m+n-1}, \quad [\Phi^m, \Phi^n] = (n - m) \Phi^{m+n-1}. \] (4.5)
So the underlying Lie algebra is the (positive part of the) Virasoro algebra (actually the Witt algebra) and the pre-Lie algebra $W$ is isomorphic to the space of holomorphic vector fields on the complex plane vanishing at the origin,

$$\Phi^n \sim x^n \frac{\partial}{\partial x}, \quad n \geq 1,$$

(4.6)

with the trivial connection. This representation of $W$ will be useful in the following.

4.2. The deformed model

Let us now turn to the family of perturbed conformal field theories. In terms of the fermions the most general action we want to consider is of the form

$$S = \frac{1}{\pi} \int \partial^2 z \left( b \overline{\partial}_t c + \overline{\partial} \tau \right),$$

(4.7)

with $\overline{\partial}_t$ the deformed "$\overline{\partial}$-operator" parametrized by the coupling constants $t_n, n \geq 1$ (or $b_n$ as in (3.39)) as

$$\overline{\partial}_t = \overline{\partial} - \sum_{n=1}^{\infty} \frac{t_n}{\tau^2} \overline{\partial}^n$$

$$= \overline{\partial} - \partial + \sum_{n=1}^{\infty} b_n \partial^n. \quad (4.8)$$

Equivalently, in terms of the bosonic field $\varphi$, we have an action with general $(\partial \varphi)^n$ interactions

$$S = \frac{1}{\pi} \int d^2 z \left( \frac{1}{2} \partial \varphi \overline{\partial} \varphi - V(-i \partial \varphi) \right),$$

(4.9)

with potential

$$V(x) = \sum_{n=1}^{\infty} \frac{t_n}{2 \tau^2} \frac{x^{n+1}}{n+1}$$

$$= x - \sum_{n=1}^{\infty} \frac{b_n x^{n+1}}{n+1}. \quad (4.10)$$

Interactions of this type have also appeared in [24] in the context of the $c = 1$ string. Here the following 'duality' was pointed out: Let $H_n = \oint (i \partial \varphi)^{n+1}$ and consider the map $w = i \partial \varphi(z)$. Then its (formal) inverse $z = i \partial \chi(w)$ has a mode expansion $\chi(w) = \sum_n H_n w^{-n}$. So the interchange of 'base' and 'target' manifold, interchanges the zero-modes of spin $n$ fields with the $n$-th order modes of a spin zero field.
In this case the differential operators $L_n^{(a)}$, $L_n^{(b)}$ ($n = 0, 1, \ldots$) are given by Virasoro generators\footnote{Here, by a slight misuse of notation, the Virasoro generator $L_n$ corresponds to the field $\Phi^{n+1}$ of conformal dimension $n + 2$.}

$$L_n^{(a)} = \sum_{k=1}^{\infty} k a_k \frac{\partial}{\partial a_{k+n}}, \quad L_n^{(b)} = \sum_{k=1}^{\infty} k b_k \frac{\partial}{\partial b_{k+n}},$$

(4.11)

with Virasoro algebra

$$[L_n, L_m] = (n - m) L_{n+m}.$$  

(4.12)

The master equation (3.41) now reads

$$(L_k^{(a)} + L_k^{(b)}) Z = 0, \quad k = 0, 1, \ldots,$$

(4.13)

where we should remember to expand around $a_k = b_k = \delta_{k,1}$.

**4.3. Solution of the master equation**

The master equation can be solved as follows. Introduce the holomorphic functions $a(x), b(x)$, vanishing at $x = 0$, with Taylor expansions

$$a(x) = \sum_{n=1}^{\infty} a_n x^n, \quad b(x) = \sum_{n=1}^{\infty} b_n x^n.$$  

(4.14)

On these functions the Virasoro generators $L_k$ act of course as the vector fields

$$L_k = x^{k+1} \frac{\partial}{\partial x}.$$  

(4.15)

These vector fields generate the holomorphic diffeomorphisms $f : \mathbb{C} \to \mathbb{C}$ of a neighbourhood of 0 that leave the origin fixed. Condition \footnote{Here, by a slight misuse of notation, the Virasoro generator $L_n$ corresponds to the field $\Phi^{n+1}$ of conformal dimension $n + 2$.} (4.13) now expresses the fact that the functional $Z[a, b]$ is invariant under these diffeomorphisms

$$Z[a \circ f, b \circ f] = Z[a, b].$$  

(4.16)

This fact can be used to determine the function $a$ in terms of the function $b$ directly. Let $\mathbf{1}$ be the identity map, $\mathbf{1}(x) = x$. Now choose $a = \mathbf{1}$ and $f = b^{-1}$, the inverse map. (This inverse always exists as a power series expression.) Then the above equation implies

$$Z[1, b] = Z[b^{-1}, \mathbf{1}].$$

(4.17)
So the functions $a$ and $b$ are simply each other’s inverses,

$$a(b(x)) = x. \quad (4.18)$$

This relation can be easily expanded in terms of Taylor coefficients

$$a_1 = \frac{1}{b_1}, \quad a_2 = -\frac{b_2}{b_1^2}, \quad a_3 = -\frac{b_3}{b_1^3} + \frac{2b_2^2}{b_1^5}, \quad \text{etc.} \quad (4.19)$$

The first relation has been established for a general CFT in (3.27). Note now that in terms of the coupling constants $s_n, t_n$ we have the relation

$$y = x + \sum_n s_n x^n, \quad x = y - \sum_n t_n y^n. \quad (4.20)$$

In terms of the coefficients this gives

$$s_1 = \frac{t_1}{1 - \frac{t_1}{2\tau_2}}, \quad s_2 = \frac{t_2}{(1 - \frac{t_1}{2\tau_2})^3}, \quad s_3 = \frac{t_3}{(1 - \frac{t_1}{2\tau_2})^4} + \frac{t_2^2}{\tau_2(1 - \frac{t_1}{2\tau_2})^5}, \quad \text{etc.} \quad (4.21)$$

These relations have of course also a straightforward interpretation in terms of tree level Feynman diagrams, with $n$-th order vertices labeled by $t_{n-1}$.

### 4.4. Spin one fields

Until now we have only considered deformations by fields of spin two or greater. One of the reasons for this was that definition (2.9) of the contact term product $\nabla_B A$ was ill-defined in case both $A$ and $B$ have spin one. However, it is not very difficult to include the spin one fields too, as we will now illustrate for the $c = 1$ model.

Consider the current

$$\Phi^0 = -bc = -i\partial\varphi, \quad (4.22)$$

with corresponding coupling constants $s_0, t_0$. No problems arise with the previous formalism if we consider contact terms of $\Phi^0$ with fields $\Phi^n$ with $n > 0$. We simply have

$$\nabla^0 \Phi^n = n\Phi^{n-1}, \quad \nabla^n \Phi^0 = 0, \quad n \geq 1. \quad (4.23)$$

so that in particular

$$[\Phi^0, \Phi^n] = n. \quad (4.24)$$
This implies that (taking into account the shift by one that we use in our notation) we have to add the extra generator

\[ L_{-1} = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} \]

(4.25)

to our Virasoro algebra. Note that still no central charge term in the Virasoro algebra appears.

The mutual contact terms of the spin one fields are a bit more subtle. They are of course given in terms of the second order operator product

\[ \Phi^0(z)\Phi^0(w) \sim \frac{1}{(z-w)^2}. \]

(4.26)

However, in our formalism we now have to distinguish between the situation where this second order pole is integrated over a contour or over the surface of the torus. Only in the latter case do we get a contribution in the recursion relation (3.35).

All of this combinatorics can be collected in the following addition to our master equation:

\[ \frac{\partial Z}{\partial t_0} = \left[ \frac{\partial}{\partial s_0} + \frac{1}{2\tau_2} \left( L_{-1}^{(s)} + L_{-1}^{(t)} \right) + \frac{t_0}{4\pi \tau_2} \right] Z. \]

(4.27)

After the usual shift (3.39), where we replace the variables \( s_i, t_i \) by the variables \( a_i, b_i \), the above relation reduces to the extra constraint

\[ \left[ L_{-1}^{(a)} + L_{-1}^{(b)} - \frac{\tau_2}{2\pi} b_0 \right] Z = 0. \]

(4.28)

It is not difficult to derive the solution of this condition by a similar argument as in the previous subsection. Extending the definition of the functions \( a(x), b(x) \) in (4.14) by including the constant terms \( a_0, b_0 \), we find after some algebra that relation (4.17) is generalized to

\[ Z[1, b] = \exp \left( \frac{\tau_2}{2\pi} B \right) Z[b^{-1}, 1] \]

(4.29)

with constant

\[ B = \int_0^{b^{-1}(0)} b(x)dx = \int_0^{b(0)} b^{-1}(y)dy. \]

(4.30)

It might be instructive to consider this relation in the case that only the spin one fields are included. In that case \( t_0 = s_0 \). Since \( b(x) = x - \frac{t_0}{2\tau_2} \) and \( a(x) = b^{-1}(x) = x + \frac{t_0}{2\tau_2} \), the constant \( B \) is given by \( B = t_0^2/4\tau_2^2 \). The master equation now reduces to the statement

\[ \langle \exp \int t_0 \Phi^0 \rangle = \exp \left( \frac{\tau_2 t_0^2}{8\pi \tau_2} \right) \langle \exp \int t_0 \Phi^0 \rangle \]

(4.31)
This equation is familiar for the bosonic model. Then the current \( \Phi^0 = -i \partial \varphi \) is a total derivative and integrates to zero so that \( \int \Phi^0 = 0 \). On the other hand the \( U(1) \) zero-mode \( J_0 = -i \oint \Phi^0 \) is non-vanishing and can be inserted in the partition function (with \( t_0 = 2\pi iz \))

\[
Z(z) = \text{Tr} \left( e^{2\pi i z J_0} q^{L_0 - \frac{1}{24}} \right).
\]

It gives the well-known result

\[
Z(z) = e^{-\frac{\pi z^2}{2}} Z(0),
\]

which is in accordance with equation (4.31).

4.5. Modular properties of \( W_{1+\infty} \) characters

Let us now discuss the implications of all this for the transformation rules of the \( W_{1+\infty} \) characters. We define the conserved charges

\[
H^n = \oint \Phi^n,
\]

and consider the generalized character (with spin structure \( \alpha, \beta = 0, \frac{1}{2} \))

\[
\chi(\tau, s) = \text{Tr}_F \left( y^{H_0} q^{L_0 - \frac{1}{2\tau}} e^{s_n H^n} \right).
\]

Here we also added the \( U(1) \) charge \( H_0 = i J_0 \) with \( y = e^{2\pi \beta} \). The trace is taken in the free fermion Fock space \( F \) with boundary conditions \( b(e^{2\pi i z}) = e^{2\pi i \alpha} b(z) \), \( c(e^{2\pi i z}) = e^{-2\pi i \alpha} c(z) \).

As is well-known the three even spin structures on the torus will transform into each other, while the odd one is invariant. These transformations of the spin structures are always implicitly understood in the following. Alternatively, one can also restrict the modular transformation to the subgroup \( \Gamma^0(2) \) which leaves the spin structure invariant.

Of course, this \( W_{1+\infty} \) character can be easily evaluated, since the Hamiltonians act diagonal in the fermion basis. We find [16]

\[
\chi(\tau, s) = q^{-\frac{1}{2\tau}} \prod_{p \in \mathbb{Z}_{>0} + \alpha} \left( 1 + y q^p e^{iS(2\pi ip)} \right) \left( 1 + y^{-1} q^p e^{-iS(-2\pi ip)} \right),
\]

with the notation

\[
S(p) = \sum_{n \geq 0} s_n p^n.
\]

We are interested in the modular properties of this character. Under a modular transformation

\[
\tau \rightarrow \tau' = \frac{a \tau + b}{c \tau + d}
\]
we will have

\[ \chi(\tau, s) \rightarrow \chi(\tau', s'), \] (4.39)

where we want to determine the transformation rule of the transformed variables \( s'_n \).

As we hope has become clear in §3, our philosophy is that the Hamiltonian variables \( s_n \) do not have canonical transformation properties, but the Lagrangian variables \( t_n \) in contrast do transform simply, \( \text{viz.} \) with modular weight \(- (n + 1)\)

\[ t_n \rightarrow t'_n = \frac{t_n}{(c\tau + d)^{n+1}}. \] (4.40)

The transformation properties of the coefficients \( s_n \) can now be read off from the relations (4.20). Unfortunately, we have not found an elegant closed expression (although integral formulas are easily written down) for \( s'_n \). But for the first few terms we find (with \( s_i, t_i = 0 \) for \( i = 0, 1 \)) that \( s_2 \) still has modular weight \(-3\) but that \( s_3 \) has a more complicated transformation behaviour

\[ s_2 \rightarrow \frac{s_2}{(c\tau + d)^3}, \]
\[ s_3 \rightarrow \frac{s_3}{(c\tau + d)^4} - \frac{2i\zeta s_2^2}{(c\tau + d)^5}. \] (4.41)

This implies that the expansion coefficients of the character

\[ \chi(\tau, s) = \sum \chi^{n_1, \ldots, n_k}(\tau)s_{n_1} \cdots s_{n_k} \] (4.42)

have corresponding modular properties. In fact, by generalizing the arguments of [19] one can prove that the coefficients \( \chi^{n_1, \ldots, n_k}(\tau) \) will transform as quasi-modular forms, of weight \( \sum_i (n_i + 1) \).

4.6. The holomorphic anomaly equation

As we mentioned in the §1, it is of interest to consider the holomorphic anomaly equation of [11, 10] in this context. That is, we consider the partition function \( Z[\tau, \tau'; s] \) of the perturbed model and try to derive an equation for the anti-holomorphic derivative \( \partial Z/\partial \tau \). This is most easily done in terms of perturbation theory of the bosonic model.

As we will explain in a moment, for our purposes it is most convenient to work in terms of the Hamiltonian variables \( s_n \). So starting point is the action

\[ S = \int \frac{d^2z}{2\pi} \partial \varphi \overline{\partial \varphi} + \int \sum_n \frac{s_n}{n+1} (\partial \varphi)^{n+1}. \] (4.43)
The couplings $s_n$ will be treated perturbatively. Since the interaction terms are chiral, we will only use the holomorphic propagator given by

$$P(z) = \langle \partial \varphi(z) \partial \varphi(0) \rangle = -\varphi(z) + \frac{\pi^2}{3} E_2^*$$

$$= \partial_z^2 \log \theta_1(z) + \frac{\pi}{\tau_2}. \tag{4.44}$$

Here $\varphi(z)$ is the Weierstrass function

$$\varphi(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left( \frac{1}{(z-(m\tau+n))^2} - \frac{1}{(m\tau+n)^2} \right), \tag{4.45}$$

and $E_2^*$ is defined as

$$E_2^*(\tau, \tau) = E_2(\tau) - \frac{3}{\pi \tau_2} \tag{4.46}$$

with the Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1-q^n}. \tag{4.47}$$

Because of the zero mode contribution, the propagator has an explicit $\tau$-dependence

$$\frac{\partial}{\partial \tau} P(z) = \frac{i \pi}{2 \tau_2^2}. \tag{4.48}$$

The interaction vertices are explicitly holomorphic in $\tau$, since we have chosen to write them in terms of contour integrals. So the only non-holomorphic dependency of the perturbative expansion of the partition function arises from the propagator. Because of the simple relation (4.48), there is a graphical representation of the action of $\tau$-derivative on the Feynman graphs: it simply removes an edge. In fact, here we have to distinguish two cases. If the propagator connects two distinct vertices of order $k+1$ and $l+1$ with coupling constants $s_k$ and $s_l$, these vertices will be replaced by vertices of order $k$ and $l$ respectively. Taking into account that in the canonical $W_{1+\infty}$ normalization a vertex of order $k+1$ is weighted by a factor $1/(k+1)!$ instead of the usual symmetry factor $1/((k+1)!$, this action is represented on the partition function by the differential operator

$$\sum_{k,l \geq 1} k s_k l s_l \frac{\partial^2}{\partial s_{k-1} \partial s_{l-1}}. \tag{4.49}$$

Similarly, if the propagator begins and ends at a vertex of order $k+1$, the $\partial/\partial \tau$ will remove two outgoing edges and reduce this vertex to order $k - 1$. The corresponding differential operator is

$$\sum_{k \geq 2} k(k-1) s_k \frac{\partial}{\partial s_{k-2}} \tag{4.50}$$
Combining everything and taking into account the correct constant of proportionality, we conclude that the anti-holomorphic dependence can be summarized in the simple equation for the partition function $Z[\tau, \bar{\tau}; s]$, considered as a generating function of Feynman graphs:

$$\frac{\partial Z}{\partial \bar{\tau}} = \frac{i\pi}{2\tau^2} \left(L_{-1}^{(s)}\right)^2 Z.$$  (4.51)

Here $L_{-1}$ is the Virasoro generator that we introduced in §4.4

$$L_{-1}^{(s)} = \sum_{n \geq 0} n s_n \frac{\partial}{\partial s_n}.$$  (4.52)

This equation can be seen as a generalization of the usual holomorphic anomaly equation. If we only put the cubic coupling $s_2 = -i\lambda$ to a non-zero value (the string coupling constant) then equation (4.51) reduced to (with $s_1 = -i\tau$)

$$\frac{\partial Z}{\partial \tau} = \frac{\lambda^2 \partial^2 Z}{\tau^2 \partial \tau^2},$$  (4.53)

which is of the form given in [11].

4.7. The two-dimensional QCD string revisited

Let us now finally return to our original motivation and reconsider the implications of all this for the $(\partial \phi)^3$ model and the 2d QCD string. The expression for the torus partition function $Z(\tau, \lambda)$ in terms of the string coupling constant $\lambda = 1/N$ as it follows from the large $N$ expansion of the Yang-Mills partition function is given by [5] (see also [8] for a short derivation using branched covers)

$$Z(\tau, \lambda) = \oint \frac{dy}{2\pi i y} \prod_{p \in \mathbb{Z} \geq 0 + \frac{1}{2}} \left(1 + yq^p e^{\lambda p^2/2}\right) \left(1 + y^{-1}q^p e^{-\lambda p^2/2}\right).$$  (4.54)

We recognize the QCD string partition function as a generalized $W_{1+\infty}$ character where we added the spin three interaction term

$$\oint \Phi^2 = -\oint b \partial^2 c = \oint \frac{i}{3} (\partial \phi)^3.$$  (4.55)

and evaluate in the zero $U(1)$ charge sector.

25
One of the interesting properties of the QCD string partition function is that the coefficients $F_g(\tau)$ in the perturbative string expansion

$$Z(\tau, \lambda) = \exp \sum_{g \geq 1} \lambda^{2g-2} F_g(\tau)$$

(4.56)

have rather intricated modular properties. They are so-called quasi-modular forms, of weight $6g - 6$ [7, 8, 9]. Quasi-modular forms are polynomials in the Eisenstein series $E_2, E_4, E_6$. The Eisenstein series $E_4$ and $E_6$ are modular forms of weight 4 and 6, and generate the ring of all modular forms. The form $E_2$ is not quite modular of weight two, but has a modular anomaly

$$E_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau + d).$$

(4.57)

However, $E_2$ can be made into a proper modular form by adding an anholomorphic term and defining $E_2^*$ as in (4.46). That is to say, the string partition function should be regarded as the limit $\tau \to -i\infty$ of an expression that is no longer holomorphic in $\tau$ but that is modular invariant, if one let $\lambda$ transform with modular weight $-3$. In fact, this suggests that the “correct” string field theory lagrangian is given by

$$S = \int \frac{d^2 z}{\pi} \left( \frac{1}{2} \partial \varphi \bar{\partial} \varphi + \frac{\lambda}{6} (-i \partial \varphi)^3 \right).$$

(4.58)

or equivalently

$$S = \int \frac{d^2 z}{\pi} \left( b \bar{c} + \lambda b \partial^2 c \right).$$

(4.59)

This is in complete accordance with the philosophy of [11, 10], where it was shown that the (topological) string on a Calabi-Yau manifold has anti-holomorphic dependence. Only if we decouple the anti-holomorphic couplings, do we recover the (holomorphic) instanton counting functions. The fact that the string coupling has modular weight $-3$ is also consistent with this point of view.

Since we only recover the QCD answer in the $\tau \to -i\infty$ limit, the above action is not uniquely determined. In fact, if our starting point was a pure cubic Hamiltonian, then according to our general formulas, the corresponding Lagrangian would contain higher order terms corresponding to $(\partial \varphi)^n$ interactions with arbitrary $n > 3$. The corresponding couplings would however go to zero in the holomorphic limit.

This is important if we want to make contact with the holomorphic anomaly equation. Indeed, only with a cubic Hamiltonian (and thus a non-polynomial Lagrangian) do we recover the simple form (4.53). Vice versa, a cubic Lagrangian will give a non-polynomial
Hamiltonian which satisfies an anomaly equation of general type \((L_{31})\). So we find that the two characteristic features of the six-dimensional Kodaira-Spencer theory — a simple cubic \((\partial \phi)^3\) Lagrangian and a simple quadratic holomorphic anomaly equation — are incompatible in two-dimensions. An open question is whether the more general holomorphic anomaly equation that we derived in §4.6 also occurs in the superstring context.

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