Extended quantum diffusion approach to reactions of astrophysical interests

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The quantum diffusion approach is extended to low energy fusion (capture) reactions of light- and medium-mass nuclei. The dependence of the friction parameter on bombarding energy is taken into account. A simple analytic expression is obtained for the capture probability at extreme sub-barrier energies. The calculated cross-sections are in a good agreement with the experimental data. The fusion excitation functions calculated within the quantum diffusion and WKB approaches are compared and presented in the astrophysical S-factor representation.

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I. INTRODUCTION

Fusion reactions at energies near and below the Coulomb barrier have been an object of extensive experimental and theoretical studies in the past decades \[1\]. Indeed, the heavy-ion fusion allows us to extend the periodic table beyond the elements that can not be synthesized using neutrons and light charged particles. The fusion of light- and medium-mass nuclei plays an important role in the evolution of massive stars where the behavior of fusion excitation function at extreme sub-barrier energies determines the reaction rates. For example, towards the end of stellar life-cycle the elements up to the iron can be synthesized. These reactions drive the nucleosynthesis and generates the energy in novae, supernovae, and close binary stars \[2\]. In Refs. \[6\]–\[27\], the fusion reactions involving light nuclei at low energies were investigated both experimentally and theoretically. The recent developments in fusion reactions both in experiment and theory are presented in Refs. \[2\]–\[4\] and references therein.

For light and medium-mass nuclei the fusion is governed by the penetrability of colliding nuclei through the Coulomb and centrifugal barrier (so called capture). If the collision occurs at energies permitting very large angular momentum, there is a possibility that the formed dinuclear system decays after the capture stage. However, at energies near and below the Coulomb barrier, the contribution of large angular momenta to fusion can be disregarded. Therefore, the description of fusion of these nuclei is reduced to the description of the capture of projectile by target-nucleus. To study the capture (fusion) process in heavy-ion reactions, the quantum diffusion approach, based on the quantum master-equation for the reduced density matrix, has been suggested in Refs. \[28\]–\[31\]. In this approach the collisions of nuclei are treated in terms of a single collective variable: the relative distance \( R \) between the colliding nuclei. The coupling of the relative motion to the excitation of various channels, such as non-collective single-particle excitations, low-lying collective modes (dynamical quadrupole and octupole excitations of the target and projectile) lead to the fluctuation and dissipation effects. Hence, many quantum-mechanical and non-Markovian effects, accompanying the passage through the potential barrier, are considered in our formalism. The nuclear deformation effects are taken into account through the dependence of the nucleus-nucleus potential on the deformations and mutual orientations of the colliding nuclei \[31\]–\[32\].

As shown in Refs. \[29\]–\[31\], our model successfully describes the capture (fusion) cross section in heavy-ion collisions at energies near and below the Coulomb barrier. In the present work we extend our approach to describe the capture (fusion) of light and medium-mass nuclei at energies well below the Coulomb barrier. Our aim is to calculate the fusion cross sections for nuclei of interest and importance for stellar burning. So, the approach is applied to low-energy fusion reactions with carbon, oxygen, and silicon nuclei.

II. FORMALISM OF THE QUANTUM DIFFUSION APPROACH

The capture cross section is the sum of the partial capture cross sections \[29\]–\[32\]:

\[
\sigma_{\text{cap}}(E_{\text{c.m.}}) = \sum_{J} \sigma_{\text{cap}}(E_{\text{c.m.}}, J)
\]

\[
= \pi \hbar^{2} \sum_{J} (2J + 1) \frac{d\theta_{1} \sin(\theta_{1})}{d\theta_{2} \sin(\theta_{2})} P_{\text{cap}}(E_{\text{c.m.}}, J, \theta_{1}, \theta_{2})
\]

where \( \hbar^{2} = \hbar^{2}/(2\mu E_{\text{c.m.}}) \) is the reduced de Broglie wavelength, \( \mu = m_{0} A_{1} A_{2}/(A_{1} + A_{2}) \) is the reduced mass (\( m_{0} \) is the nucleon mass), and the summation occurs over possible values of angular momentum \( J \) at a given bombarding energy \( E_{\text{c.m.}} \). Knowing the potential of the interacting nuclei for each orientation defined by the angles \( \theta_{i} (i = 1, 2) \), one can calculate the partial capture probability \( P_{\text{cap}}(E_{\text{c.m.}}, J, \theta_{1}, \theta_{2}) \) which is the probability to
Here, we use the propagator \( P(t) = 0 \) to the final state \((R, P)\) at time \( t = 0 \) to the final state \((R, P)\) at time \( t \) \((R\) is defined with respect to the position \( R_0 \) of the Coulomb barrier and \( P \) is the conjugate momentum):

\[
P_{\text{cap}} = \lim_{t \to \infty} \int_{-\infty}^{R_{\text{in}}} dR \int_{-\infty}^{\infty} dP \, G(R, P, t| R_0, P_0, 0) = \lim_{t \to \infty} \frac{1}{2} \text{erfc}\left(\frac{-R_{\text{in}} + R(t)}{\sqrt{\Sigma_{RR}(t)}}\right). \tag{2}
\]

Here, we use the propagator

\[
G = \pi^{-1} |\det \Sigma|^{-1/2} \exp(-q^T \Sigma^{-1} q), \tag{3}
\]

where \( q^T = [q_R, q_P] \), \( q_R(t) = R - \bar{R}(t), \) \( q_P(t) = P - \bar{P}(t), \) \( \bar{R}(t = 0) = R_0, \) \( \bar{P}(t = 0) = P_0, \) \( \Sigma_{kk}(t) = 2g_k(t)q_k(t), \) \( \Sigma_{kk'}(t = 0) = 0 \) and \( k, k' = R, P, \) obtained in Ref. [33] for a local inverted oscillator which replaces the real nucleus-nucleus potential in the variable \( R \). The frequency \( \omega \) of this local inverted oscillator with an internal turning point \( R_{\text{in}} \) is defined from the condition of equality of the classical actions of approximated and real potential barriers of the same height at given \( E_{\text{c.m.}} \) and \( J \). Note that this procedure leads to the frequency depending on \( E_{\text{c.m.}} \) and \( J \). This local replacement of the real potential by the inverted oscillator with energy-dependent frequency is well justified for heavy-ion reactions at energies near and below the Coulomb barrier [29, 32, 54, 33].

As at \( t \to \infty \) the internal turning point \( R_{\text{in}} \ll \bar{R}(t), \) the capture cross section is defined by the ratio of the mean value of the collective coordinate \( \bar{R}(t) \) and its variance \( \Sigma_{RR}(t) \). For the explicit expressions for \( \bar{R}(t) \) and \( \Sigma_{RR}(t) \) we refer to our previous studies in Refs. [29, 32, 33]. Using the Hamiltonian of the system, which includes the collective subsystem, the environment (which mimics the internal excitations) and the coupling between the collective subsystem and the environment, a system of non-Markovian Langevin equations for the collective coordinates was derived. These equations of motion for the collective subsystem satisfy the quantum fluctuation - dissipation relations and contain the influence of quantum, dissipative and non-Markovian effects on the collective motion [28, 53]. The expressions for the \( \bar{R}(t) \) and \( \Sigma_{RR}(t) \) are

\[
\bar{R}(t) = A_t R_0 + B_t P_0.
\]

\[
\Sigma_{RR}(t) = \frac{4\hbar^2 \lambda \epsilon \gamma^2}{\pi} \int_0^t d\tau' B_{\tau'} \int_0^t d\tau'' B_{\tau''} \int_0^\infty d\Omega \frac{\Omega}{\Omega^2 + \gamma^2} \times \coth \frac{\hbar \Omega}{2T} \cos[\Omega(\tau' - \tau'')],
\]

\[
B_t = \frac{1}{\mu} \sum_{i=1}^3 \beta_i (s_i + \gamma) e^{s_i t}, \quad A_t = \sum_{i=1}^3 \beta_i [s_i (s_i + \gamma) + 2 \hbar \lambda e \gamma |s_i|]. \tag{4}
\]

Here, \( \Sigma_{RR}(0) = 0, A_0 = 1, \) and \( B_0 = 0. \) In Eqs. (4), \( \beta_1 = [(s_1 - s_2)(s_1 - s_3)]^{-1}, \beta_2 = [(s_2 - s_1)(s_2 - s_3)]^{-1} \) and \( \beta_3 = [(s_3 - s_1)(s_3 - s_2)]^{-1}, \) and \( s_i \) are the real roots \( s_1 > s_2 \geq s_3 \) of the following equation

\[
(s + \gamma) (s^2 - \epsilon^2) + 2 \hbar \lambda e \gamma s = 0. \tag{5}
\]

The parameters \( \gamma, \epsilon \) and \( \lambda \) determine the characteristics of the system. The values of \( \gamma^{-1} \) is the memory time of dissipation of relative motion energy by the internal subsystem or is the inverse bandwidth of the internal subsystem excitations. The non-Markovian effects appear in the calculations through \( \gamma. \) The instantaneous dissipation corresponds to taking \( \gamma \to \infty. \) The parameter \( \epsilon \) defines the initial frequency of the collective subsystem and \( \lambda \) determines the average coupling strength of the collective subsystem with internal excitations. To set these parameters [28, 53], we use the asymptotic values of the friction coefficient

\[
\lambda = -(s_1 + s_2) \tag{6}
\]

and potential frequency

\[
\omega = \epsilon \left(\frac{(s_1 + \gamma)(s_2 + \gamma)}{(s_1 + \gamma)(s_2 + \gamma) - 2 \hbar \lambda e \gamma} \right)^{1/2}. \tag{7}
\]

Note, that \( \omega \) takes into account the renormalization of the initial frequency due to the coupling to the internal excitations. So, in the asymptotic limit \( t \to \infty, \) the friction \( \lambda \) and frequency \( \omega \) are related to the parameters \( \gamma, \epsilon, \lambda, \) and the roots \( s_1, 2, \) of Eq. (5). Setting the values of \( \lambda, \omega, \) and \( \gamma, \) we determine the dynamics of the system. The use of asymptotic values of \( \lambda \) and \( \omega \) is justified, since the characteristic time of reaching them is much shorter than the characteristic time of capture.

Equations (2), (4), (6), and (7) lead to the analytic expression for the capture probability:

\[
P_{\text{cap}} = \frac{1}{2} \text{erfc}\left[\frac{\pi s_1 (\gamma - s_1)}{2 \hbar \mu (\epsilon^2 - s_1)} \right]^{1/2} \frac{\mu \epsilon^2 R_0/s_1 + P_0}{|\gamma \ln(\gamma/s_1)|^{1/2}}. \tag{8}
\]

In the derivation of Eq. (8) the limit of low temperatures \( (T \to 0) \) was used, which is suitable for sub-barrier
fusion. Note, that the friction $\lambda$ and internal excitation width $\gamma$ are related. If the coupling with internal degrees of freedom is disregarded, $\lambda \to 0$, then the limit $\gamma \to \infty$ results in the Markovian dynamics. In the case of

$$\frac{\lambda}{\omega} \ln(\gamma) \to \text{const}$$

(9)

at $\lambda \to 0$, the well-known quantum-mechanical barrier transmission probability is obtained

$$P_{\text{exp}} \sim \exp[-2\pi(V_b - E_{\text{c.m.}})/\hbar \omega].$$

### III. NUCLEUS-NUCLEUS POTENTIAL

In the case of collision of deformed nuclei the effective nucleus-nucleus potential reads as:

$$V = V_N + V_C + \frac{\hbar^2 J(J+1)}{2\mu R^2},$$

(10)

where $V_N$, $V_C$, and the last summand stand for the nuclear, Coulomb, and centrifugal potentials, respectively [37]. The potential depends on the relative distance $R$ between the center of mass of two interacting nuclei, masses $A_i$, charges $Z_i$ and radii $r_i$ of the nuclei ($i = 1, 2$), the orientation angles $\theta_i$ of the deformed (with the quadrupole deformation parameters $\beta_2^{(i)}$) nuclei and angular momentum $J$. For deformed nuclei, the static quadrupole deformation parameters are taken from Ref. [36]. For the nuclear part of potential,

$$V_N = \int \rho_1(r_1)\rho_2(R-r_2)F(r_1-r_2)dr_1dr_2,$$

(11)

the double-folding formalism is used, where $F(r_1 - r_2) = C_0[F_{\text{in}}(r_1) + F_{\text{ex}}(1 - \rho_0(r_1))]$ is the density-depending effective nucleon-nucleon interaction and $\rho_0(r) = \rho_1(r) + \rho_2(R-r)$, $F_{\text{in,ex}} = F_{\text{in,ex}} + \int_{\min}^{\max} (N_1-Z_1)(N_2-Z_2)$. Here, $\rho_0(r_1)$ and $N_i$ are the nucleon densities and neutron numbers of the light and the heavy nuclei of the dinuclear system. Our calculations are performed with the following set of parameters: $C_0 = 300$ MeV fm$^3$, $f_{\text{in}} = 0.09$, $f_{\text{ex}} = -2.59$, $f'_{\text{in}} = 0.42$, $f'_{\text{ex}} = 0.54$ and $\rho_0 = 0.17$ fm$^{-3}$ [37]. The densities of the nuclei are taken in the two-parameter symmetric Woods-Saxon form with the nuclear radius parameter $r_0=1.15$ fm and the diffuseness parameter $a=0.47-0.56$ fm depending on the charge and mass numbers of the nucleus [37].

The Coulomb interaction of two quadrupole deformed nuclei reads as

$$V_C = \frac{Z_1Z_2e^2}{R} + \left(\frac{9}{20\pi}\right)^{1/2} \frac{Z_1Z_2e^2}{R^3} \sum_{i=1,2} r_i^2 \beta_2^{(i)} \left[1 + \frac{2}{7} \left(\frac{5}{\pi}\right)^{1/2} \beta_2^{(i)} \right] P_2(\cos \theta_i),$$

(12)

where $P_2(\cos \theta_i)$ is the Legendre polynomial.

The calculated potentials with respect to their barriers $V_b$ are presented in Fig. 1 for two reactions with spherical nuclei $^{16}$O+$^{208}$Pb and $^{16}$O+$^{16}$O at $J = 0$. With increasing angular momentum, the positions of the potential barrier $R_b$ and the minimum $R_m$ merge, and at certain $J$ the potential pocket disappears. This is a natural limitation of $J$ that contribute to the capture (fusion). The large Coulomb repulsion in the case of $^{16}$O+$^{208}$Pb leads to a steep decline of the potential, compared to that in the case of $^{16}$O+$^{16}$O. So, at the fixed $E_{\text{c.m.}} - V_b < 0$, two colliding nuclei approach closer to reach smaller $R_{\text{ext}}$ in the case of heavier system.

### IV. EXTENSION OF THE APPROACH

#### A. Energy-dependent friction and internal excitation bandwidth

The formalism, introduced in Sect. II, implies that the friction $\lambda$ does not depend on $E_{\text{c.m.}}$. The use of the constant friction seems to be valid in case of fusion of rather heavy nuclei at energies near and below (up to 5-6 MeV) the Coulomb barrier. However, in the reactions with medium-mass and light nuclei, and/or at extreme sub-barrier energies, the dependence of the friction on $E_{\text{c.m.}}$ cannot be ignored. This remark can be easily understood from Fig. 2 where the comparison of the dependencies of the external turning point $R_{\text{ext}}$ on energy is shown for the reactions $^{16}$O+$^{208}$Pb and $^{16}$O+$^{16}$O. The value of $R_{\text{ext}}$ at given $E_{\text{c.m.}}$ indicates the degree of the overlap of nuclear density profiles, which is responsible for the nuclear friction. For the $^{16}$O+$^{16}$O reaction, the value of $R_{\text{ext}}$ drastically increases with decreasing $E_{\text{c.m.}} - V_b$. 

![FIG. 1: The nucleus-nucleus potentials calculated at $J = 0$ for the reactions $^{16}$O+$^{208}$Pb (solid line) and $^{16}$O+$^{16}$O (dashed line). The coordinate $R$ is defined relative to the position $R_b$ of the Coulomb barrier.](image-url)
which leads to a strong reduction of the friction with respect to the $^{16}\text{O}+^{208}\text{Pb}$ reaction. At fixed $E_{\text{c.m.}} - V_b$, the value of $R_{\text{ext}}$ is much closer to the position of the corresponding Coulomb barrier for heavy system.

To include the bombarding energy dependence of friction in our model, we refer to the studies of Refs. [38, 39], where the friction,

$$\lambda(R) = \lambda_b \left( \frac{\nabla V_N(R)}{\nabla V_N(R_b)} \right)^2, \quad (13)$$

proportional to the square of nuclear force, was suggested for fusion and deep inelastic reactions. This form of $\lambda(R)$ takes into account the overlap of nuclear surfaces on which the friction strength depends. To determine the normalization parameter $\lambda_b$, we use our previous studies [29, 32], where the fusion cross section of heavy nuclei at energies near and below (up to 4-5 MeV) the Coulomb barrier was well described with constant friction coefficient $h\lambda_b = h\lambda(R = R_b) = 2$ MeV. The calculated dependencies of the friction on $R$ are shown in Fig. 3 for the reactions $^{16}\text{O}+^{208}\text{Pb}$ and $^{16}\text{O}+^{16}\text{O}$. One can see the rapid decrease of the friction with increasing $R$. Note that the calculated capture cross sections are rather insensitive to the value of $\lambda_b$. For example, the variation of this parameter by 2 times leads to the change of the results of the calculations by less then 5%.

In accordance with Eq. (9) the internal excitation bandwidth $\gamma$ is related to the friction. We take the same relation also in the case of coordinate-dependent friction coefficient $\lambda(R)$:

$$\gamma(R) = \gamma_0 \exp \left[ \frac{\hbar \omega(R)}{\lambda(R)} \right]. \quad (14)$$

In the case of constant friction $\hbar \lambda = 2$ MeV, the best agreement with the experimental data is archived at constant internal excitation width $\hbar \gamma = 32$ MeV for the reactions with heavy nuclei $^{29,32}$. Thus, we choose $\gamma_0$ to have $\hbar \gamma (R = R_b) = 32$ MeV. Note that at deep sub-barrier energies the results of calculations are almost insensitive to $\gamma_0$ (see subsection IV.D). In the limit $\lambda \rightarrow 0$, Eq. (14) results in $\frac{1}{2} \ln(\gamma) \rightarrow k_1$ as in Eq. (9).

The value of $k_1$ in Eq. (14) is a parameter to be adjusted, and may vary for different reactions. However, our calculations show a certain universality of this parameter for all considered reactions. The perfect agreement with the experimental cross sections is archived if the values of $\gamma$, $\omega$, and $\lambda$ are calculated at $R = R_{\text{ext}}$ and the value of $k_1$ is defined as

$$k_1 = \frac{\alpha}{\sqrt{\mu \omega_b^2}}, \quad (15)$$

where $\alpha = \frac{2}{3}$ MeV$^{1/2}$ fm$^{-1}$ and $\omega_b = \omega(R = R_b)$ is the frequency at the barrier position $R_b$.

So, in our extended model we use the values of friction and internal excitation bandwidth which are calculated at $R = R_{\text{ext}}$: $\lambda(R_{\text{ext}})$ and $\gamma(R_{\text{ext}})$. Thus, the bombarding energy dependence of $\gamma$ and $\lambda$ are included through their dependence on $R_{\text{ext}}$.

B. Energy-dependent frequency

We use the local inverted oscillator approximation which means that the nucleus-nucleus interaction potential at each $E_{\text{c.m.}}$ is locally replaced by the inverted oscillator with own frequency. At different $E_{\text{c.m.}}$, there are different local inverted oscillators. As mentioned in Sect. II, for the reactions with heavy nuclei at sub-barrier energies, we determine the frequency $\omega$ of the approximated oscillator from the condition of equality of the classical
actions under the barrier of the real and approximated potentials. This approximation leads to the close values of $R_{\text{ext}}$ for the real and approximated potentials. For the reactions with light- and medium-mass nuclei, the same procedure leads to completely different values of $R_{\text{ext}}$ in the cases of real and approximated potentials. Because the friction strongly depends on $R_{\text{ext}}$, this approximation becomes irrelevant. For the light- and medium-mass nuclei, we suggest to match the height and position of the barrier of the real potential with the height and position of inverted oscillator. To determine the frequency $\omega$ at sub-barrier energies, we use the following expression

$$V_b - E_{\text{c.m.}} = \frac{\mu \omega^2 (R_{\text{ext}} - R_b)^2}{2}$$

which provides the dependence of the frequency $\omega$ on $E_{\text{c.m.}}$ that is on $R_{\text{ext}}$.

C. Initial conditions and parameters

Employing Eq. (15) and the initial coordinate $R_0$ and momentum $P_0$, we calculate the capture probability $P_{\text{cap}}$. Let us consider the initial conditions and parameters used in our calculations.

If the collision of nuclei occurs at sub-barrier energies $E_{\text{c.m.}} < V_b$, the dissipation of the kinetic energy of relative motion before $R_{\text{ext}}$ is neglected. Hence, $R_0$ coincides with the external turning point, $R_0 = R_{\text{ext}}$, and $P_0 = 0$. Here, the values of $\lambda$, $\omega$, $\gamma$ are calculated at $R = R_0 = R_{\text{ext}}$, $\omega(R_{\text{ext}})$, $\gamma(R_{\text{ext}})$, and correspondingly they depend on $E_{\text{c.m.}}$.

If the capture occurs at energies $E_{\text{c.m.}}$ above the Coulomb barrier $V_b$, $R_0 = R_b$ and $P_0 = \sqrt{2 \mu E_{\text{c.m.}}} \exp(-2 \lambda \sigma \gamma \tau_{\text{int}})$. Here, the dissipation $\Delta E = E_{\text{c.m.}}[1 - \exp(-2 \lambda \sigma \gamma \tau_{\text{int}})]$ of the kinetic energy of relative motion is taken effectively into account by using the average friction coefficient $\lambda_0$ and energy-dependent interaction time estimated as $\tau_{\text{int}} = 1/\sqrt{E_{\text{c.m.}}}$. s. For the calculations of $\sigma_{\text{cap}}(E_{\text{c.m.}})$ at energies above the Coulomb barrier, we use the values of $\lambda$, $\omega$, and $\gamma$ calculated at the barrier position: $\hbar \lambda_b = \hbar \lambda(R = R_b) = 2$ MeV, $\hbar \gamma_b = \hbar \gamma(R = R_b) = 32$ MeV and $\omega_b = \omega(R = R_b) = \sqrt{\frac{1}{\mu} \frac{d^2 V}{dr^2}} |_{r=R_b}$.

D. Analytical expression for the capture at extreme sub-barrier energies

At extreme sub-barrier energies, we have the following initial conditions: $P_0 = 0$ and $R_0 = R_{\text{ext}} = Z_1 Z_2 e^2 / E_{\text{c.m.}}$. Using this $R_0$ and Eq. (15), we obtain the analytical expression

$$\omega = \sqrt{\frac{E_{\text{c.m.}}}{Z_1 Z_2 e^2 - R_b E_{\text{c.m.}}}} \left( \frac{2(V_b - E_{\text{c.m.}})}{\mu} \right)^{1/2}$$

for frequency. Because at extreme sub-barrier energies the value of friction is small and $\gamma \gg \omega$, $\omega/\lambda \gg \ln(\gamma_0)$, we derive $s_1 \approx \omega$ and

$$\ln \left( \frac{\gamma}{s_1} \right) \approx \ln(\gamma) \approx \frac{k_1}{\lambda} \omega.$$

Substituting these expressions and initial conditions into Eq. (15), we finally obtain

$$P_{\text{cap}} = \frac{1}{2} \text{rfc} \left[ \frac{\pi (V_b - E_{\text{c.m.}})}{k_1 \hbar \omega} \right].$$

Note that Eq. (18) is similar to the well known quantum-mechanical barrier transmission probability but with the replacement of the usual frequency by the effective one.

V. RESULTS OF CALCULATIONS

Using the procedure described, we apply Eqs. (11), (16)–(18), (13), (14), and (16) to calculate the capture cross-section $\sigma_{\text{cap}}(E_{\text{c.m.}})$ for low-energy reactions with light- and medium-mass nuclei. As emphasized in [6, 8, 11, 13, 14, 22], the fusion reactions between carbon and oxygen isotopes are playing a crucial role in a wide variety of stellar burning scenarios. As the first step in that direction, we compare our calculated results with the available data.

![Graph](image-url)

FIG. 4: The calculated capture cross section (line) vs $E_{\text{c.m.}}$ for the $^{12}\text{C} + ^{12}\text{C}$ reaction compared with the available experimental data. The experimental data marked by squares, circles, stars and triangles are taken from Refs. [6, 13, 14, 22], respectively.

The results of the calculated capture cross sections and the experimental data are shown in Figs. 4–10. In all considered reactions we obtain a good agreement with the experiments. Note, that for $^{12}\text{C} + ^{12}\text{C}$ reaction the early measured data [6, 13] differ from the later ones [14, 22]. Here, the mechanism that causes the oscillations of the
cross section in the $^{12}\text{C}+^{12}\text{C}$ reaction is not considered \cite{ref22}.

Our calculated results at sub-barrier energies are rather sensitive to the coefficient $k_1$ [Eq. (15)]. However, it is uniformly determined for all reactions considered. Thus, we conclude that Eq. (15) is useful for the reactions of astrophysical interest.

At energies below the Coulomb barrier, where the cross section drops rapidly with decreasing energy, it is more convenient to discuss the astrophysical $S$-factor,

$$S(E_{\text{c.m.}}) = E_{\text{c.m.}} \sigma_{\text{fus}}(E_{\text{c.m.}}) \exp[2\pi(\eta - \eta_0)],$$

rather than the fusion excitation function. Here, $\eta(E_{\text{c.m.}}) = Z_1Z_2e^2\sqrt{\mu/(2\hbar^2E_{\text{c.m.}})}$ is the Sommerfeld parameter and $\eta_0 = \eta(E_{\text{c.m.}} = V_b)$, where $V_b$ is the Coulomb barrier height for the spherical interacting nuclei. Assuming that the capture cross section is equal to the fusion cross section, we calculate the astrophysical $S$-factor. In Figs. 11 and 12 the calculated $S$-factors versus $E_{\text{c.m.}}$ are shown for the reactions $^{12}\text{C}+^{12}\text{C}$, $^{12}\text{C}+^{16}\text{O}$, $^{12}\text{C}+^{30}\text{Si}$, $^{16}\text{O}+^{16}\text{O}$, and $^{28}\text{Si}+^{30}\text{Si}$. A good agreement of the calculated excitation function with the experimental data leads to a good description of $S$-factor as well. For the reactions under study, the $S$-factor has a maximum at $E_{\text{c.m.}} \approx \frac{3}{4}V_b$, where $V_b$ is the Coulomb barrier height for the spherical interacting nuclei. The origin of the maximum of the $S$-factor is the turning-off of the nuclear forces between the colliding nuclei with decreasing $E_{\text{c.m.}}$.

While the theory shows clear maximum, their presence in the experimental data is tenuous up to now. In the recent paper \cite{ref23} on a new measurement of the $^{12}\text{C}+^{12}\text{C}$

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**FIG. 5:** The same as in Fig. 4 but for the $^{12}\text{C}+^{16}\text{O}$ reaction. The experimental data marked by squares, circles, and stars are taken from Refs. \cite{ref7,ref8,ref21}, respectively.

**FIG. 6:** The same as in Fig. 4 but for the $^{16}\text{O}+^{16}\text{O}$ reaction. The experimental data marked by squares, circles, triangles and stars are taken from Refs. \cite{ref9,ref12}, respectively.

**FIG. 7:** The same as in Fig. 4 but for the $^{12}\text{C}+^{30}\text{Si}$ reaction. The experimental data marked by circles are taken from Ref. \cite{ref24}.

**FIG. 8:** The same as in Fig. 4 but for the $^{28}\text{Si}+^{30}\text{Si}$ reaction. The experimental data marked by circles are taken from Ref. \cite{ref19}.
fusion cross sections, it was found that the astrophysical S-factor exhibits a maximum around $E_{c.m.}=3.5–4$ MeV. The additional measurements of different systems at lowest bombarding energies are necessary to establish the existence of S-factor maximum. In Figs. 11 and 12, after this maximum S-factor decreases strongly with decreasing bombarding energy, which leads to a reduction of the previously predicted astrophysical reaction rates. Note also that such a strong dependence on $E_{c.m.}$, in fact, contradicts the philosophy of representing the cross section through the S-factor.

Figure 11 shows a comparison between our and WKB ($P_{cap}$ is determined within both the WKB model and the interaction potential of Eqs. (10)–(12)) S-factors for the reactions $^{12}$C+$^{12}$C ($\eta_0 = \eta(E_{c.m.} = V_b) = 5.58$) and $^{16}$O+$^{16}$O ($\eta_0 = 9.01$) (solid lines). Comparison of S-factors from the WKB model (dashed lines). The experimental data (symbols) are from Refs. [6, 9–14, 22].

VI. SUMMARY

In the collisions of light- and medium-mass nuclei at low sub-barrier energies, the external turning point is located far from the Coulomb barrier position. This means a weak overlap of nuclear surfaces and, correspondingly, small friction. To this end, we extended our quantum diffusion approach and considered the friction depending on the bombarding energy. Using the extended approach, we compared the calculated capture cross-sections with the available experimental data. In all cases we obtained a good description of the experiments. Comparing the fusion excitation functions calculated within the quantum diffusion and WKB approaches, we found that the
FIG. 12: The calculated astrophysical $S$-factor vs $E_{\text{c.m.}}$ for the reactions $^{12}\text{C}+^{16}\text{O}$ ($\eta_0 = 7.07$), $^{12}\text{C}+^{30}\text{Si}$ ($\eta_0 = 10.60$), and $^{28}\text{Si}+^{30}\text{Si}$ ($\eta_0 = 22.15$). Here, the calculated values are normalized to the experimental data \cite{7,8,11,21,24}. 

the fluctuation and dissipation increase fusion cross section at sub-barrier energies. For the reactions $^{12}\text{C}+^{12}\text{C}$, $^{12}\text{C}+^{16}\text{O}$, $^{12}\text{C}+^{30}\text{Si}$, $^{16}\text{O}+^{16}\text{O}$, and $^{28}\text{Si}+^{30}\text{Si}$, the maximum of astrophysical $S$-factor at $E_{\text{c.m.}} \approx \frac{2}{3}V_b$ was predicted. However, more experimental data at low energies is needed to confirm our predictions. Another interesting behavior of the obtained $S$-factor is that its dependence on $E_{\text{c.m.}}$ is quite strong at the collision energies below the maximum.

In the limit of weak friction, which corresponds to extreme sub-barrier energies, the analytic expression for the capture probability is obtained. This simple expression can be applied to the reactions of astrophysical interest. It determines the reaction rates from which, in turn, the astrophysical $S$-factors are derived. The strong decline of fusion cross sections at sub-barrier energies considerably reduces the stellar burning rates and, moreover, leads to severe experimental problems, inhibiting the measurements in many cases. This demands for the reliable theoretical methods, allowing us to extrapolate $\sigma_{\text{cap}}(E_{\text{c.m.}})$ into the experimentally inaccessible regions at extreme sub-barrier energies.

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\[\text{FIG. 12: The calculated astrophysical } S \text{-factor vs } E_{\text{c.m.}} \text{ for the reactions } ^{12}\text{C}+^{16}\text{O} \text{ (} \eta_0 = 7.07\text{), } ^{12}\text{C}+^{30}\text{Si} \text{ (} \eta_0 = 10.60\text{), and } ^{28}\text{Si}+^{30}\text{Si} \text{ (} \eta_0 = 22.15\text{). Here, the calculated values are normalized to the experimental data \cite{7,8,11,21,24}.}\]
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