On the Existence of Universally Decodable Matrices

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Abstract

Universally decodable matrices (UDMs) can be used for coding purposes when transmitting over slow fading channels. These matrices are parameterized by positive integers \( L \) and \( N \) and a prime power \( q \). The main result of this paper is that the simple condition \( L \leq q + 1 \) is both necessary and sufficient for \((L, N, q)\)-UDMs to exist. The existence proof is constructive and yields a coding scheme that is equivalent to a class of codes that was proposed by Rosenbloom and Tsfasman. Our work resolves an open problem posed recently in the literature.

Index terms — Universally decodable matrices, UDMs, coding for slow fading channels, rank condition, Rosenbloom-Tsfasman codes.
1 Introduction

Let $L$ and $N$ be positive integers, let $q$ be a prime power, let $[M] \triangleq \{0, \ldots, M-1\}$ for any positive integer $M$, and let $[M] \triangleq \{\}$ for any non-positive integer $M$. While studying slow fading channels, Tavildar and Viswanath [2,3] introduced a communication system which works as follows. An information (column) vector $x_\ell \in \mathbb{F}_q^N$, $\ell \in [L]$, where $A_0, \ldots, A_{L-1}$ are $L$ matrices over $\mathbb{F}_q$ and of size $N \times N$. Upon sending $x_\ell$ over the $\ell$-th channel we receive $y_\ell \in (\mathbb{F}_q \cup \{?\})^N$, where the question mark denotes an erasure. The channels are such that the received vectors $y_0, \ldots, y_{L-1}$ can be characterized as follows: there are integers $v_0, \ldots, v_{L-1}$ (that can vary from transmission to transmission), $0 \leq v_\ell \leq N$, $\ell \in [L]$, such that the first $v_\ell$ entries of $y_\ell$ are non-erased and agree with the corresponding entries of $x_\ell$ and such that the last $N - v_\ell$ entries of $y_\ell$ are erased.

Based on the non-erased entries we would like to reconstruct $u$. The obvious decoding approach works as follows: construct an $N \times N$-matrix $A$ that stacks the $v_0$ first rows of $A_0$, ..., the $v_{L-1}$ first rows of $A_{L-1}$. Since $u$ is arbitrary in $\mathbb{F}_q^N$, a necessary condition for successful decoding is that $\sum_{\ell \in [L]} v_\ell \geq N$. Because we would like to be able to decode successfully for all $L$-tuples $(v_0, \ldots, v_{L-1})$ that satisfy this necessary condition, we must guarantee that the matrix $A$ has full rank for all possible $L$-tuples $(v_0, \ldots, v_{L-1})$ with $\sum_{\ell \in [L]} v_\ell = N$. This will automatically also guarantee that the matrix $A$ has full rank for all possible $L$-tuples $(v_0, \ldots, v_{L-1})$ with $\sum_{\ell \in [L]} v_\ell \geq N$. Matrices that fulfill this condition are called $(L, N, q)$-universally decodable matrices (UDMs). A more precise definition and some examples are given in the next section. It is trivially verified that $(L, N, q)$-UDMs always exist when $N = 1$, so throughout the rest of this paper it is assumed that $N \geq 2$.

Given the definition of $(L, N, q)$-UDMs, there are several immediate questions. For what values of $L$, $N$, and $q$ do such matrices exist? What are the properties of these matrices? How can one construct such matrices? The authors in [2] provide a construction for these matrices for the cases $L = 3$, any $N$, and $q = 2$. For the case $L = 4$, any $N$, and $q = 3$, Doshi [4,2] conjectures that a particular construction yields $(L=4, N, q=3)$-UDMs. The existence and construction of $(L, N, q)$-UDMs for the general case is proposed as an open problem in [2]. This is the take-off point for the present paper. Ganesan and Boston [1] showed that a necessary condition for $(L, N, q)$-UDMs to exist is $L \leq q + 1$ and conjectured that this condition is also sufficient. In this paper we complete the resolution of this problem on the existence of $(L, N, q)$-UDMs.

The paper is structured as follows. In Sec. 2 we properly define UDMs and prove the necessary condition. Sec. 3 discusses the explicit construction that proves the sufficiency part and Sec. 4 contains some concluding remarks. Sec. A contains the proof of Cor. 8 and Sec. B collects some results on Hasse derivatives which are the main tool for the proof of our UDMs construction.

2 Universally Decodable Matrices

The notion of universally decodable matrices (UDMs) was introduced by Tavildar and Viswanath [2]. Before we give the definition of UDMs, let us agree on some notation. For any positive integer $N$, we let $I_N$ be the $N \times N$ identity matrix, and we let $J_N$ be the $N \times N$ matrix where all entries are zero except for the anti-diagonal entries that are equal to one; i.e., $J_N$ contains the rows of $I_N$ in reverse order. For any positive integer $L$ and any
non-negative integer \( N \) we define the set
\[
\Upsilon^=_{L} \triangleq \left\{ (v_0, \ldots, v_{L-1}) \mid 0 \leq v_\ell \leq N, \ell \in [L], \sum_{\ell \in [L]} v_\ell = N \right\}.
\]

Throughout the rest of this paper, indices start at 0 and not at 1. We let \([M]_{n,k}\) denote the \((n,k)\)-th entry of the matrix \(M\) and we let \([v]_n\) denote the \(n\)-th entry of the vector \(v\).

**Definition 1** Let \( N \) and \( L \) be some positive integers and let \( q \) be a prime power. The \( L \) matrices \( A_0, \ldots, A_{L-1} \) over \( \mathbb{F}_q \) and of size \( N \times N \) are \((L,N,q)\)-UDMs, or simply UDMs, if for every partition of \( N \) into \( L \) non-negative summands \((v_0, v_1, \ldots, v_{L-1}) \in \Upsilon^=_{L} \), the following condition is satisfied: the \( N \times N \) matrix composed of the first \( v_0 \) rows of \( A_0 \), the first \( v_1 \) rows of \( A_1 \), \ldots, the first \( v_{L-1} \) rows of \( A_{L-1} \) has full rank. We call this condition the UDMs condition. \( \square \)

**Example 2** Let \( N \) be any positive integer, let \( q \) be any prime power, and let \( L \equiv 2 \). Let \( A_0 \equiv I_N \) and let \( A_1 \equiv J_N \). It can easily be checked that \( A_0, A_1 \) are \((L=2,N,q)\)-UDMs. Indeed, let for example \( N \equiv 5 \). We must check that for any non-negative integers \( v_1 \) and \( v_2 \) such that \( v_1 + v_2 = 5 \) the UDMs condition is fulfilled. E.g. for \((v_1, v_2) = (3,2)\) we must show that the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\] has rank 5, which can easily be verified. \( \square \)

**Example 3** In order to give the reader a feeling how UDMs might look like for \( L > 2 \), we give here a simple example for \( L = 4 \), \( N = 3 \) and \( q = 3 \), namely
\[
A_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

One can verify that for all \((v_0, v_1, v_2, v_3)\) such that \( v_0 + v_1 + v_2 + v_3 = 4 \) (there are 20 such four-tuples) the UDMs condition is fulfilled and hence the above matrices are indeed UDMs. For example, for \((v_0, v_1, v_2, v_3) = (0,0,3,0)\), \((v_0, v_1, v_2, v_3) = (0,0,1,2)\), and \((v_0, v_1, v_2, v_3) = (1,1,0,1)\) the UDMs condition means that we have to check if the matrices
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
0 & 1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 2 & 1
\end{pmatrix}
\] have rank 3, respectively, which is indeed the case. Before concluding this example, let us remark that the above UDMs are the same UDMs that appeared in [4] and [2, Sec. 4.5.4]. \( \square \)

**Theorem 4** If \( N \geq 2 \) then a necessary condition for \((L,N,q)\)-UDMs to exist is \( L \leq q + 1 \).
Proof: We use the notation \( A_i[j] \) to denote the \( j \)-th row of \( A_i \), with \( j \in [N] \). Suppose the \( N \times N \) matrices \( A_0, \ldots, A_{L-1} \) are UDMs. Let \( S_{N-2} \) be the \((N-2)\) dimensional subspace spanned by \( \{A_0[0], A_0[1], \ldots, A_0[N-3] \} \). The quotient space \( Q \triangleq \mathbb{F}_q^N / S_{N-2} \) is isomorphic to a 2-dimensional subspace of \( \mathbb{F}_q^N \). Given the set of \( L \) vectors \( A \triangleq \{A_0[N-2], A_1[0], A_2[0], \ldots, A_{L-1}[0] \} \), form the corresponding set of cosets \( C \triangleq \{a + S_{N-2} \mid a \in A \} \) in \( Q \). Now by the UDMs condition with \( v_0 = N-2 \), any two elements of \( C \) must be linearly independent. Thus, if a particular coset is in \( C \), \( q-2 \) other cosets from \( Q \) cannot be in \( C \). So we partition \( Q \) into sets containing \( q-1 \) cosets, with the \( q-1 \) cosets in each partition being linearly dependent and with at most one coset from a partition being a member of \( C \). The cardinality of \( Q \setminus \{0\} \) is \( q^2 - 1 \). We obtain the necessary condition on the number of partitions

\[
\#(C) \leq \frac{q^2 - 1}{q-1}.
\]

But \( L = \#(C) \), and so the theorem follows.

As mentioned earlier, it is assumed throughout this paper that \( N \geq 2 \). In \cite{2}, a construction was given for \((L=3, N, q=2)\)-UDMs, and a construction was conjectured to yield \((L=4, N, q=3)\)-UDMs. As a corollary of the theorem above, we obtain that there do not exist any UDMs for \( q = 2, L \geq 4 \), and that there do not exist any UDMs for \( q = 3, L \geq 5 \).

3  An Explicit Construction of UDMs

In this section we present an explicit construction of \((L, N, q)\)-UDMs when \( L \leq q+1 \), cf. Th. \cite{2} and \cite{3}. Before we proceed, we need some definitions. First, whenever necessary we use the natural mapping of the integers into the prime subfield\(^1\) of \( \mathbb{F}_q \). Secondly, we define the binomial coefficient \( \binom{a}{b} \) in the usual way. Note that \( \binom{a}{b} = 0 \) for all \( a < b \).

**Definition 5** Let \( a(X) \triangleq \sum_{k=0}^{d} a_k X^k \in \mathbb{F}_q[X] \) be a polynomial and let \( \beta \in \mathbb{F}_q \). The Taylor polynomial expansion of \( a(X) \) around \( X = \beta \) is defined to be \( a(X) = \sum_{n=0}^{d} a_{\beta,n} (X - \beta)^n \in \mathbb{F}_q[X] \) for suitably chosen \( a_{\beta,n} \in \mathbb{F}_q \), \( 0 \leq n \leq d \), such that equality holds.

It can be verified that the Taylor polynomial coefficients \( a_{\beta,n} \) can be expressed using Hasse derivatives\(^2\) of \( a(X) \), i.e. \( a_{\beta,n} = a^{(n)}(\beta) = \sum_{k=0}^{d} a_k \binom{k}{n} \beta^{k-n} \). On the other hand, the coefficients of \( a(X) \) can be expressed as \( a_k = \sum_{n=0}^{d} a_{\beta,n} \binom{k}{n} (-\beta)^{n-k} \).

**Lemma 6** Let \( a(X) \triangleq \sum_{k=0}^{d} a_k X^k \in \mathbb{F}_q[X] \) be a non-zero polynomial, let \( \beta \in \mathbb{F}_q \), and let \( a(X) = \sum_{n=0}^{d} a_{\beta,n} (X - \beta)^n \in \mathbb{F}_q[X] \) be the Taylor polynomial expansion of \( a(X) \) around \( X = \beta \). The polynomial \( a(X) \) has a zero at \( X = \beta \) of multiplicity \( m \) if and only if \( a_{\beta,m} = 0 \) for \( 0 \leq n < m \) and \( a_{\beta,m} \neq 0 \).

**Proof:** Obvious.

In the following, evaluating the \( n \)-th Hasse derivative \( u^{(n)}(L) \) of a polynomial \( u(L) \) at \( L = \infty \) shall result in the value \( u_{N-1-n} \), i.e. we set \( u^{(n)}(\infty) \triangleq u_{N-1-n} \).

\(^1\)When \( q = p^s \) for some prime \( p \) and some positive integer \( s \) then \( \mathbb{F}_p \) is a subfield of \( \mathbb{F}_q \) and is called the prime subfield of \( \mathbb{F}_q \). \( \mathbb{F}_p \) can be identified with the integers where addition and multiplication are modulo \( p \).

\(^2\)See Sec. \cite{1} for the definition and some properties of Hasse derivatives.
Theorem 7 Let \( N \) and \( L \) be positive integers, let \( q \) be some prime power. If \( L \leq q + 1 \) then the following \( L \) matrices over \( \mathbb{F}_q \) of size \( N \times N \) are \((L,N,q)\)-UDMs:

\[
\begin{align*}
\mathbf{A}_0 & \triangleq \mathbf{I}_N, \\
\mathbf{A}_1 & \triangleq \mathbf{J}_N, \\
\mathbf{A}_2, & \ldots, \ \mathbf{A}_{L-1},
\end{align*}
\]

where \([\mathbf{A}_{\ell+2}]_{n,k} \triangleq \binom{k}{n} \alpha^{\ell(k-n)}, \ (\ell, n, k) \in [L-2] \times [N] \times [N],\]

where \( \alpha \) is any primitive element in \( \mathbb{F}_q \). Note that \( \binom{k}{n} \) is to be understood as follows: compute \( \binom{k}{n} \) over the integers and apply only then the natural mapping to \( \mathbb{F}_q \).

Proof: Follows easily from Cor. 8 and its proof, together with the paragraph after Def. 5. \( \square \)

Corollary 8 Let us associate the information polynomial \( u(L) \triangleq \sum_{k \in [N]} u_k L^k \in \mathbb{F}_q[L] \) with \( u_k \triangleq [u]_k, \ k \in [N], \) to the information vector \( u \). The construction in the above theorem results in a coding scheme where the vector \( u \) is mapped to the vectors \( \mathbf{x}_0, \ldots, \mathbf{x}_{L-1} \) with entries

\[
[\mathbf{x}_\ell]_n = u^{(n)}(\beta_\ell), \quad (\ell, n) \in [L] \times [N],
\]

where \( \beta_0 \triangleq 0, \ \beta_1 \triangleq \infty, \ \beta_{\ell+2} \triangleq \alpha^\ell, \ \ell \in [L-2]. \) (Note that because \( \alpha \) is a primitive element of \( \mathbb{F}_q \), all \( \beta_\ell, \ \ell \in [L], \) are distinct.) This means that over the \( \ell \)-th channel we are transmitting the coefficients of the Taylor polynomial expansion of \( u(L) \) around \( L = \beta_\ell \).

Proof: See Sec. A. \( \square \)

As already mentioned in Sec. 1 the construction of UDMs in Th. 7 / Cor. 8 is essentially equivalent to a class of codes presented by Rosenbloom and Tsfasman \( \cite{6} \) is essentially equivalent to a class of codes presented by Rosenbloom and Tsfasman \( \cite{6}. \)

4 Concluding Remarks

We have shown that \((L,N,q)\)-UDMs exist if and only if \( L \leq q + 1 \), and the existence proof is constructive. This completely resolves the open problem posed in \( \cite{2} \) on the existence of \((L,N,q)\)-UDMs. Many open questions remain. It is of interest to determine whether there are also other UDM constructions that are not simply reformulations of the present UDMs, and to obtain efficient decoding algorithms that exploit the structure of these matrices. Recent developments in these directions - on the uniqueness of the present constructions, how to efficiently decode, and the resolution of a conjecture in \( \cite{2} \) - will be presented at \( \cite{8} \) and in a forthcoming paper.

A Proof of Corollary 8

We have to check the UDMs condition for all \((v_0, \ldots, v_{L-1}) \in \mathbb{Y}_L^N\). Fix such a tuple \((v_0, \ldots, v_{L-1}) \in \mathbb{Y}_L^N\) and let \( \psi \) be the mapping of the vector \( u \) to the non-erased entries of

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\(3\)Note that the communication system mentioned in Sec. 1 of \( \cite{6} \) also talks about parallel channels: however, that communication system would correspond to (in our notation) sending \( L \) symbols over \( N \) channels. On the other hand, the communication system that is mentioned in Nielsen \( \cite{7, Ex. 18} \) is more along the lines of the Tavildar-Viswanath channel model \( \cite{2} \) mentioned in Sec. 8.
the vectors $y_\ell$, $\ell \in [L]$; it is clear that $\psi$ is a linear mapping. Reconstructing $u$ is therefore nothing else than applying the mapping $\psi^{-1}$ to the non-erased positions of $y_\ell$, $\ell \in [L]$. However, this gives a unique vector $u$ only if $\psi$ is an injective function. Because $\psi$ is linear, showing injectivity of $\psi$ is equivalent to showing that the kernel of $\psi$ contains only the vector $u = 0$, or equivalently, only the polynomial $u(L) = 0$.

So, let us show that the only possible pre-image of

$$[y_\ell]_n = 0 \quad (\ell \in [L], \ n \in [\nu_\ell])$$

or, equivalently, of

$$[x_\ell]_n = 0 \quad (\ell \in [L], \ n \in [\nu_\ell])$$

is $u(L) = 0$. Using the definition of $[x_\ell]_n$ this is equivalent to showing that

$$u^{(n)}(\beta_\ell) = 0 \quad (\ell \in [L] \setminus \{1\}, \ n \in [\nu_\ell]) \quad (1)$$

$$u_{N-1-n} = u^{(n)}(\beta_\ell) = 0 \quad (\ell = 1, \ n \in [\nu_\ell]) \quad (2)$$

implies that $u(L) = 0$. In a first step, Eq. (1) and Lemma 6 tell us that $\beta_\ell$, $\ell \in [L] \setminus \{1\}$, must be a root of $u(L)$ of multiplicity at least $\nu_\ell$. Using the fundamental theorem of algebra we get

$$\deg(u(L)) \geq \sum_{\ell \in [L] \setminus \{1\}} \nu_\ell = N - \nu_1 \quad \text{or} \quad u(L) = 0. \quad (3)$$

In a second step, Eq. (2) tells us that we must have $\deg(u(L)) \leq N - 1 - \nu_1$. Combining this with (3), we obtain the desired result that $u(L) = 0$.

**B Hasse Derivatives**

Hasse derivatives were introduced in [9]. Throughout this appendix, let $q$ be some prime power. For any non-negative integer $i$, the $i$-th Hasse derivative of a polynomial $a(X) \triangleq \sum_{k=0}^d a_k X^k \in \mathbb{F}_q[X]$ is defined to be$^4$

$$a^{(i)}(X) \triangleq \mathcal{D}_X^{(i)} \left( \sum_{k=0}^d a_k X^k \right) \triangleq \sum_{k=0}^d \binom{k}{i} a_k X^{k-i}.$$  

Note that when $i > k$ then $\binom{k}{i} X^{k-i} = 0$, i.e. the zero polynomial. Be careful that $\mathcal{D}_X^{(i_1)} \mathcal{D}_X^{(i_2)} \neq \mathcal{D}_X^{(i_1+i_2)}$ in general. However, it holds that $\mathcal{D}_X^{(i_1)} \mathcal{D}_X^{(i_2)} = \binom{i_1+i_2}{i_1} \mathcal{D}_X^{(i_1+i_2)}$.

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$^4$The $i$-th formal derivative equals $i!$ times the Hasse derivative: so, for fields with characteristic zero there is not a big difference between these two derivatives since $i!$ is always non-zero, however for finite fields there can be quite a gap between these two derivatives since $i!$ can be zero or non-zero.
We list some well-known properties of the Hasse derivatives:

\[
\begin{align*}
D_X^{(i)}(\gamma f(X) + \eta g(X)) &= \gamma D_X^{(i)}(f(X)) + \eta D_X^{(i)}(g(X)), \\
D_X^{(i)}(f(X)g(X)) &= \sum_{i' = 0}^{i} D_X^{(i')}(f(X)) D_X^{(i-i')}(g(X)), \\
D_X^{(i)}\left(\prod_{h \in [M]} f_h(X)\right) &= \sum_{(i_0, \ldots, i_{M-1}) \in \Upsilon_{M}^{-1}} \prod_{h \in [M]} D_X^{(i_h)}(f_h(X)), \\
D_X^{(i)}((X - \gamma)^k) &= \binom{k}{i} (X - \gamma)^{k-i},
\end{align*}
\]

where \(k\) and \(i\) are some non-negative integers, \(M\) is some positive integer, and where \(\gamma, \eta \in \mathbb{F}_q\). The fact that a \(\Upsilon\)-set appears in Def. 1 and in Eq. (4) certainly points towards the usefulness of Hasse derivatives for constructing UDMs.

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