An External Description for MIMO Systems
Sampled in an Aperiodic Way

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Abstract

An external description for aperiodically sampled MIMO linear systems has been developed. Emphasis is on the sampling period sequence, included among the variables to be handled. The computational procedure is simple and no use of polynomial matrix theory is required. This input/output description is believed to be a basic formulation for its later application to the problem of optimal control and/or identification of linear dynamical systems.

Keywords: Balancedness, Bit-string model, Combinational generator, Design rules.

1 Introduction

There are two different ways of describing dynamical systems:

(i) by means of input/output relations;

(ii) by means of state variables.

In the classical or frequency-domain approach, systems are described by transfer functions which reflect just the external or input/output properties of the system. However, this mode of description entails some difficulties concerning stability and realization [1], [2].

The modern or time-domain approach turns around the axiomatic concept of state. The method is exact in defining the notion of dynamical systems and also describes all internal couplings among the system variables [3], [4]. Nevertheless, the procedure became somewhat disappointing due to the necessity of finding state-variable models and to the implicit assumption that all state-variables are accessible for direct measurement. This assumption is justified in mechanical...
or electrical systems but it is not generally satisfied for plants in chemical, gas, paper, and other industries.

These considerations were responsible for the comeback of transfer function methods [5]-[7].

On the other hand, the enormous increase in the use of digital computers in process control has stimulated studies in the field of discrete systems for both types of representation. See [8]-[10] and also the above mentioned references. All of them are concerned with constant sampling period, which is convenient for the simplicity of implementation and mathematical treatment. However, the general case of aperiodic sampling is a priori capable of more favorable solutions to the problem of control and/or identification of dynamical systems, and it is also feasible with modern time-sharing equipment.

In this work, an input/output modeling technique for aperiodic sampling linear systems has been developed. The external description includes the sampling sequence among the variables to be handled. The system is described by input/output data according to the actual experimentation conditions. Although the multivariable case is covered, the complexity of the polynomial matrix theory is avoided.

The procedure is believed to be a basic formulation for its later application to the synthesis of linear control systems sampled in an aperiodic way, since most of these techniques for nonperiodically sampled systems rely exclusively on the state-space equations [11]-[13].

2 Basic Assumptions

Our discussion is restricted to the following:

(a) linear time-invariant multivariable dynamical systems of finite order;

(b) systems whose transfer function is a $p \times m$ matrix ($m$-inputs, $p$-outputs), where the different entries are strictly proper rational functions.

We end this preliminary section with the following statement. Statement: Let $(G_l)$ be a family of vector functions

$$G_l:|R^n| \rightarrow |R^n| \quad G_l \in C^\infty(|R^n|,|R^n|) \quad (l = 0, 1, \ldots, n)$$

$C^\infty(|R^n|,|R^n|)$ being the set of infinitely differentiable functions on $|R^n|$. If the following conditions are verified:

(a) there exists an integer $r \leq n$ such that the elements $(G_r(z))$ are linearly independent for all $z \in |R^n$

(b) there exists an integer $k > r$ such that $(G_r(z))$ depends linearly on $(G_0(z), \ldots, G_r(z))$

then, there are functions $f_0, f_1, \ldots, f_n \in C^\infty(|R^n|,|R|)$ such that the following expression holds:
∑_{l=0}^{n} f_{n-l}(z) G_l(z) = 0 \quad \forall z \in |R^n. \quad (1)

The previous result is a direct consequence of the Cramer Rule; for more details see [15].

3 External Description for Nonperiodically Sampled Linear Systems

3.1 Input/Output Modeling Technique

Let \( H(s) \) be the matrix transfer function of a linear time-invariant multivariable system.

\[
H(s) = (H_{rq}(s)) \quad (r = 1, \ldots, p), (q = 1, \ldots, m)
\]

let us rewrite \( H(s) \) as

\[
H(s) = \frac{N(s)}{d(s)}
\]

where

\[
d(s) = s^n + d_1 s^{n-1} + \ldots + d_n
\]

is the least common multiple of the denominators of the entries of \( H(s) \).

In the time domain, the impulse response \( h(t) \) can be written as

\[
h(t) = (h_{rq}(t)) = \begin{pmatrix} h_1(t) \\ \vdots \\ h_p(t) \end{pmatrix}
\]

where the \( r \)th row can also be written in matrix form by means of the triad \((A, C, B_r)\). In fact,

\[
h_r(t) = C \exp(At) B_r \quad (r = 1, \ldots, p)
\]

with \( A \) a bottom-companion matrix with last row

\[
- [d_n, d_{n-1}, \ldots, d_1]
\]

\[
C = (1, 0, \ldots, 0)_{1 \times n}
\]

\[
B_r = \begin{pmatrix} h_{r1}(0) & \cdots & h_{rm}(0) \\ \dot{h}_{r1}(0) & \cdots & \dot{h}_{rm}(0) \\ \vdots & \vdots & \vdots \\ h_{r1}^{(n-1)}(0) & \cdots & h_{rm}^{(n-1)}(0) \end{pmatrix}_{n \times m}
\]
Remark that the column vectors of $B_r$ correspond to the $n$-first Markov parameters of the scalar impulse responses $h_{rq}(t)$ ($q = 1, \ldots, m$).

It should be noticed that the triad $(A, C, B_r)$ leads us naturally to the observability canonical realization from the vector impulse response $h_r(t)$.

From this triad, we are going to define a family of vector functions $G_l; \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($l = 0, \ldots, n$) given by

$$G_j(z) = C \exp(A(z_1 + \ldots + z_j)) \quad (j = 1, \ldots, n)$$

$$G_0(z) = C$$

with

$$z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$  

Thus, the vector impulse response $h_r(t)$ can be written in terms of these functions as

$$h_r(z_1 + \ldots + z_j) = G_j(z)B_r \quad (r = 1, \ldots, p).$$

From an analytic viewpoint, the functions $G_l$ belong to $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ as composition of $C^\infty$ functions.

It has been proved [13] that there is an open interval $I$ of $\mathbb{R}$ such that the vectors $(G_0(z), G_1(z), \ldots, G_{n-1}(z))$ defined as before are linearly independent for each $z \in I \times I \times \ldots \times I = I^n$.

In this case, it is easy to see that for the new domain $I^n$ the conditions a) and b) in the previous statement hold. Hence, there will be functions $f_l(z) \in C^\infty(I^n, \mathbb{R})$ ($l = 0, \ldots, n$) such that

$$\sum_{l=0}^{n} f_{n-l}(z) G_l(z) = 0 \quad \forall z \in I^n.$$  

Thus, the functions $(f_1(z), \ldots, f_n(z))$ can be obtained by solving a compatible system of linear equations. (For simplicity $(f_0(z) = -1$).

In fact, the general form of these functions is

$$f_{n-l}(z) = \frac{\text{Det}(G'_0(z), \ldots, G'_n(z), \ldots, G'_{n-1}(z))}{\text{Det}(G'_0(z), \ldots, G'_{n-1}(z))} \quad (l = 0, \ldots, n-1)$$

(' denotes the transpose) where the numerator is the determinant obtained from the matrix $(G'_0(z), \ldots, G'_{n-1}(z))$ by replacing the $(l+1)$th column by the column vector $G'_n(z)$.

Now, we multiply both sides of (14) by

$$\exp(Az^*)B_r \quad (r = 1, \ldots, p)$$

with $z^*$ taking successively the values

$\mathbb{R}^n$.
and we get in each case

\[ C \exp(A(z_{l+1} + \ldots + z_n))B_r \]

\[ = \sum_{i=1}^{n-l} f_i(z)C \exp(A(z_{i+1} + \ldots + z_{n-i}))B_r \]

\[ + \sum_{i=0}^{l-1} f_{n-i}(z)C \exp(-A(z_{i+1} + \ldots + z_i))B_r \]

\[ (l = 1, \ldots, n), (r = l, \ldots, p). \]

Finally, we define

\[ g_{n-l}(z) = \sum_{i=0}^{l-1} f_{n-i}(z)C \exp(-A(z_{i+1} + \ldots + z_i))B_r \]

\[ = (g_{n-l}^1(z), \ldots, g_{n-l}^m(z)). \]

At this point, we can identify the components of \( z \) with the elements of the sampling period sequence. In fact,

\[ z_{n-l} = t_{k-l} - t_{k-l+1} = T_{k-l} \quad (l = 0, \ldots, n - 1) \]

where \( t_{k-l} \) are the sampling instants and \( T_{k-l} \) the length of the sampling intervals.

Thus, at an arbitrary sampling instant, say \( t_k (k \geq n) \), we can condense the preceding expressions into two sets of equations involving the functions \( f_i, g^q \) and \( h_{rq} \) as follows:

\[ \sum_{i=0}^{n} f_i h_{rq}(t_{k-i} - t_j) + g_{k-j}^q = 0 \quad (j = k, \ldots, k - n + 1) \quad (r = 1, \ldots, p), (q = l, \ldots, m). \]

Note that, at time \( t_k \), the functions \( f_i, g^q \) will depend on the sampling interval lengths \( (T_{k-n+1}, \ldots, T_k) \) and so on.

Now, we multiply (21) and (22) by \( (u^q_j) (q = 1, \ldots, m), (j = k, k-1, \ldots, 0) \), respectively, \( (u^q_j \) being the \( q \)th impulse input of the system at the sampling instant \( t_j \)) and summing all these expressions, we get
\[
\sum_{q=1}^{m} \left[ f_1 \left( \sum_{l=0}^{k-1} h_{rq}(t_{k-1} - t_l)u_l^q \right) + \ldots \right. \\
+ f_n \left( \sum_{l=0}^{k-n} h_{rq}(t_{k-n} - t_l)u_l^q \right) + \sum_{j=0}^{n-1} g_{rq}^q u_{k-j}^q \right] \\
= \sum_{q=1}^{m} \sum_{l=0}^{k} h_{rq}(t_k - t_l)u_l^q. 
\]

Then, making use of the convolution expression

\[
y_r^k = \sum_{q=1}^{m} \sum_{l=0}^{k} h_{rq}(t_k - t_l)u_l^q 
\]

\((y_r^k\) being the \(r\)th output of the system at time \(t_k\)) the above expression becomes

\[
y_r^k = \sum_{i=1}^{n} f_i y_{r-i}^k + \sum_{q=1}^{m} \sum_{j=0}^{n-1} g_{rq}^q u_{k-j}^q 
\]

which is the input output description for linear time-invariant MIMO systems sampled in an aperiodic way. Each system output at time \(t_k\) can be written as a linear combination of the same output and of the different inputs at previous instants. The expression (25) generalizes to the aperiodic case the well-known input/output representation for linear systems sampled periodically. The sequence of sampling intervals is implicit in the arguments of the functions \(f_i, g_{rq}^q\). Consequently, this freedom in the choice of the sampling instants can be used in the solution of control problems, propagation of measuring errors, parameter estimation, and related topics [11]-[14].

It is convenient to note that the functions \(f_i\) are the same for every system output, while the functions \(g_{rq}^q\) depend on the corresponding impulse response \(h_{rq}\).

### 3.2 Simplified Computation of the Functions \(f_i, g_{rq}^q\)

Companion matrices are an important example of nonderogatory matrices, which have only one (normalized) eigenvector associated with each distinct eigenvalue. This means that

(i) the Jordan canonical form is clearly simplified (there is only one Jordan block for each distinct eigenvalue);  

(ii) the similarity transformation of the given matrix to the Jordan form can be obtained in a standard way.
Thus, the computation of the Jordan canonical form for this kind of matrix is quite easy. Indeed,

\[ A = T J T^{-1} \]  

(26)

where \( J \) is the Jordan canonical form of the matrix \( A \) and \( T \) is an invertible matrix of a well-known general form [8]. In this way, (14) becomes

\[ \sum_{l=0}^{n} f_{n-l}(z) x_0 \exp(J \alpha_l) = 0 \]  

(27)

with

\[ \alpha_l = z_1 + \ldots + z_l \quad (l = 1, \ldots, n), (\alpha_0 = 0) \]  

(28)

\[ x_0 = CT. \]  

(29)

Let \((\varphi_l) (l = 1, \ldots, n)\) be the fundamental system of solutions of a \(n\)th order homogeneous linear differential equation whose characteristic polynomial is \(d(s)\).

In this case, factorizing \(Det(x_0 \exp(J \alpha_l))\) and cancelling common factors in (15), the general form of the functions \(f_i\) can be simplified to

\[ f_i = \frac{\Delta_i}{\Delta} \quad (i = 1, \ldots, n) \]  

(30)

with

\[ \Delta = Det(\varphi_l(\alpha_j)) \quad (j = 0, \ldots, n - 1), (l = l, \ldots, n) \]  

(31)

and \(\Delta_i\) is analogous but replacing the argument of the \(i\)th column by \(\alpha_n\). The functions \(g_{rq}^{ij}\) can be easily computed from (21) for the new simplified form of the functions \(f_i\).

### 3.3 Choice of the Sampling Sequence

The procedure developed imposes nonrestrictive conditions on the sampling sequence in order to guarantee the linear independence of the vectors \((G_0(z), \ldots, G_{n-1}(z))\).

Strategies to determine the set \(I^n \subset \mathbb{R}^n\) whose elements \(z\) verify the above condition can be found in [13] (in an analytic way) and in [14] (in a geometric way).

### 4 An Illustrative Example

Let \(H(s)\) be a two-input, two-output transfer function matrix

\[ H(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+2} \end{pmatrix} \]  

(32)
\[ d(s) = s^2 + 3s + 2 = (s + 1)(s + 2) \quad (n = 2) \quad (33) \]

\[ \varphi_1(t) = \exp(-t) \quad (34) \]

\[ \varphi_2(t) = \exp(-2t) \quad (35) \]

The impulse response matrix will be

\[
h(t) = \begin{pmatrix}
\exp(-t) & 2\exp(-t) \\
-\exp(-t) + \exp(-2t) & \exp(-2t)
\end{pmatrix}.
\quad (36)
\]

In this case, for an arbitrary instant \( t_k \) we know that [13] \( t_k \in (t_{k-1}, \infty) \) and so on.

Thus, choosing \( T_{k-l} = 0.8 \), \( T_k = 1.1 \), the expression (25) can be computed as follows.

According to (31) \( \Delta = 0.24743 \), \( \Delta_1 = 0.12719 \), \( \Delta_2 = -0.02014 \).

Consequently, \( f_1 = 0.51407 \) and \( f_2 = -0.08142 \). According to (10), the functions \( g_{jq}^r \) will be

\[
\begin{pmatrix}
  g_{11}^1 & g_{11}^2 \\
  g_{01}^2 & g_{12}^1 \\
  g_{12}^0 & g_{22}^1 \\
  g_{22}^2 & g_{22}^2
\end{pmatrix} = \begin{pmatrix}
  1 & -0.18121 & 2 & -0.36241 \\
  0 & -0.22207 & 1 & -0.40327
\end{pmatrix}
\quad (37)
\]

Substituting the previous expressions, we write the input/output relations.

\[
\begin{pmatrix}
  y_1^k \\
  y_2^k
\end{pmatrix} = 0.51407 \begin{pmatrix}
  y_1^{k-1} \\
  y_2^{k-1}
\end{pmatrix} - 0.08142 \begin{pmatrix}
  y_1^{k-2} \\
  y_2^{k-2}
\end{pmatrix} + \\
\begin{pmatrix}
  1 & -0.18121 & 2 & -0.36241 \\
  0 & -0.22207 & 1 & -0.40327
\end{pmatrix} \begin{pmatrix}
  u_1^k \\
  u_2^k \\
  u_{1-1}^k \\
  u_{2-1}^k
\end{pmatrix}.
\quad (38)
\]

For each new sampling instant, the functions \( f_i \), \( g_{jq}^r \) must be computed again. The lengths of the sampling intervals can be chosen in order to optimize a particular performance criterion. Further, difficulties may arise in the practical implementation of equidistant sampling as, e.g., the equidistance might be disturbed. The formulation developed above can be used to pursue the propagation and consequences of this inexactitude.

Finally, it should be mentioned that the use of well-known numerical methods for the problem of the optimization of aperiodic sampling instants leads to good results in concrete cases as it may be seen in [11]-[13]. The formulation proposed allows us to use these methods for an I/O modeling technique.
5 Conclusions

The external description developed provides a system model well adapted to the real experimentation conditions although presents the limitations inherent to the use of the transfer function.

The formulation considered emphasizes the importance of the sampling sequence against other system parameters.

The particularization to the periodic case is immediate and represents an alternative to the classic discretization methods without using the Z-transform.

The procedure is simple and no use of polynomial matrix theory is required.

This I/O modeling technique allows us to choose the sampling instants in order to improve the numerical aspects in problems such as identification, control, propagation of measuring errors, . . . , etc. Its use is merely a question of an appropriate choice of the performance criterion.

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