Global strong solvability of a quasilinear subdiffusion problem

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Abstract

We prove the global strong solvability of a quasilinear initial-boundary value problem with fractional time derivative of order less than one. Such problems arise in mathematical physics in the context of anomalous diffusion and the modelling of dynamic processes in materials with memory. The proof relies heavily on a regularity result about the interior Hölder continuity of weak solutions to time fractional diffusion equations, which has been proved recently by the author. We further establish a basic $L_2$ decay estimate for the special case with vanishing external source term and homogeneous Dirichlet boundary condition.

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1 Introduction and main results

Let $T > 0$, $N \geq 2$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. The main purpose of this paper is to prove the global strong solvability of the time fractional quasilinear problem

\begin{equation}
\begin{aligned}
\partial_t^\alpha (u - u_0) - D_i (a_{ij}(u) D_j u) &= f, \quad t \in (0, T), \ x \in \Omega, \\

u &= g, \quad t \in (0, T), \ x \in \Gamma, \\
u|_{t=0} &= u_0, \ x \in \Omega,
\end{aligned}
\end{equation}

where we use the sum convention. Here $\Gamma = \partial \Omega$, $Du$ denotes the gradient of $u$ with respect to the spatial variables and $\partial_t^\alpha$ stands for the Riemann-Liouville fractional derivation operator with respect to time of order $\alpha \in (0, 1)$; it is defined by

$$
\partial_t^\alpha v(t, x) = \partial_t \int_0^t g_{1-\alpha}(t - \tau) v(\tau, x) \, d\tau, \quad t > 0, \ x \in \Omega,
$$

where $g_{\beta}$ denotes the Riemann-Liouville kernel

$$
g_{\beta}(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \ \beta > 0.
$$

The functions $f = f(t, x)$, $g = g(t, x)$, and $u_0 = u_0(x)$ are given data.
During the last decade there has been an increasing interest in time fractional diffusion equations like (1) and special cases of it. An important application is the modelling of anomalous diffusion, see e.g. the surveys [15], [16]. In this context, equations of the type (1) are termed subdiffusion equations as the time order $\alpha$ is less than one. While in normal diffusion (described by the heat equation or more general parabolic equations), the mean squared displacement of a diffusive particle behaves like $\text{const} \cdot t$ for $t \to \infty$, in the time fractional case this quantity grows as $\text{const} \cdot t^\alpha$, for which there is evidence in a diverse number of systems, see [15] and the references therein. In the case of equation (1) the diffusion coefficients are allowed to depend on the unknown $u$.

Another context where equations of the type (1) and variants of them appear is the modelling of dynamic processes in materials with memory. An example is given by the theory of heat conduction with memory, see [19] and the references therein. Another application is the following special case of a model for the diffusion of fluids in porous media with memory, which has been introduced in [1]:

$$\partial_t^\alpha (q - q_0) - \text{div} (\kappa(q) Dq) = f, \quad t \in (0, T), \quad x \in \Omega,$$

$$q = g, \quad t \in (0, T), \quad x \in \Gamma,$$

(2)

Here $\alpha \in (0, 1)$, $q = q(t, x)$ denotes the pressure of the fluid, $\kappa = \kappa(q)$ stands for the permeability of the porous medium, and $f$ is related to external sources in the equation of balance of mass.

Model (2) is obtained by combining the latter equation with a modified version of Darcy’s law for the mass flux $J$ which reads

$$J = -\partial_t^{1-\alpha} (\kappa(q) Dq),$$

and by assuming that the (average) mass of the fluid is proportional to the pressure. We refer to [9], where a more general model is discussed.

We next describe our main result concerning (1). Letting $p > N + \frac{2}{\alpha}$ we will assume that

(Q1) $f \in L_p([0, T]; L_p(\Omega)), \quad g \in B^{\alpha(1-\frac{1}{p})} ([0, T]; L_p(\Gamma)) \cap L_p([0, T]; B^{2-\frac{1}{p}} pp (\Gamma)), \quad u_0 \in B^{\alpha - \frac{2}{p}} pp (\Omega),$ and $u_0 = g|_{t=0}$ on $\Gamma$;

(Q2) $A = (a_{ij})_{i,j=1,...,N} \in C^1(\mathbb{R}; \text{Sym}\{N\}),$ and there exists $\nu > 0$ such that $a_{ij}(y)\xi_i \xi_j \geq \nu|\xi|^2$

for all $y \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

Here $\text{Sym}\{N\}$ denotes the space of $N$-dimensional real symmetric matrices. For $s > 0$ and $1 < p < \infty$ the symbols $H^s_p$ and $B^{s}_{pp}$ refer to Bessel potential (Sobolev spaces for integer $s$) and Sobolev-Slobodeckij spaces, respectively.

The main result concerning the problem (1) reads as follows.

**Theorem 1.1** Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with $C^2$-smooth boundary $\Gamma$. Let $\alpha \in (0, 1)$, $T > 0$ be an arbitrary number, $p > N + \frac{2}{\alpha}$, and suppose that the assumptions (Q1) and (Q2) are satisfied. Then the problem (1) possesses a unique strong solution $u$ in the class

$$u \in H^\alpha_p([0, T]; L_p(\Omega)) \cap L_p([0, T]; H^2_p(\Omega)).$$

Note that short-time existence of strong or classical solutions to problems like (1) can be established by means of maximal regularity for the linearized problem and the contraction mapping...
principle. This has been known before, see e.g. [4], [27], and for the semilinear case [5]. For results on maximal regularity for fractional evolution equations we further refer to [3], [12], [19], and [26]. The novelty here is that \( T > 0 \) can be given arbitrarily large without assuming any smallness condition on the data. In some papers (global) generalized solutions are constructed for quasilinear subdiffusion problems, see [7] and [10]. These results are based on the theory of accretive operators.

The crucial step in our proof of the global existence result is an estimate of the Hölder norm of \( u \) on parabolic subdomains \([0, \delta] \times \bar{\Omega}\) which is uniform with respect to \( \delta \in (0, T] \). In a very recent work ([23], see also [25]), the author was able to prove interior Hölder regularity for weak solutions of time fractional diffusion equations of the form

\[
\partial_t^\alpha (u - u_0) - D_i (a_{ij}(t, x) D_j u) = f, \quad t \in (0, T), \quad x \in \Omega,
\]

with \( \alpha \in (0, 1) \) and merely bounded and measurable coefficients, see Theorem 2.2 below. Using this result, we derive conditions which are sufficient for Hölder continuity up to the parabolic boundary. Here we do not aim at high generality but we are content with finding some simple conditions which are also necessary when studying the quasilinear problem \((1)\) in the setting of maximal \( L_p \)-regularity. Because of the latter it is natural and also not so difficult to use the method of maximal \( L_p \)-regularity to achieve the goal.

In this paper we also prove a basic decay estimate for the \( L_2(\Omega) \)-norm of the solution \( u \) to \((1)\) in the special case when \( f = g = 0 \). It is shown that for the global strong solution of \((1)\) we have in this case

\[
|u(t, .)|_{L_2(\Omega)}^2 \leq \frac{c|u_0|_{L_2(\Omega)}^2}{1 + \mu t^\alpha}, \quad t \geq 0,
\]

where \( c = c(\alpha) \) and \( \mu = \mu(\nu, N, \Omega) \) are positive constants. A polynomial decay estimate has already been known in the linear case for problems of the type

\[
\partial_t^\alpha (u - u_0) - D_i (a_{ij}(x) D_j u) = 0, \quad t \in (0, T), \quad x \in \Omega,
\]

\[
u = 0, \quad t \in (0, T), \quad x \in \Gamma,
\]

\[
u|_{t=0} = u_0, \quad x \in \Omega,
\]

see [17] Cor. 4.1; for the special case \( a_{ij} = \delta_{ij} \) we also refer to [14] p. 11–13]. By means of an eigenfunction expansion for \( u \), there it is shown that in general \( |u(t, .)|_{L_2(\Omega)} \) decays as \( 1/t^{n} \) as \( t \to \infty \), which is optimal w.r.t. the exponent and stronger than \( 1 \), where we have this behaviour for the square of \( |u(t, .)|_{L_2(\Omega)} \). In particular we do not have exponential decay as in the case \( \alpha = 1 \).

The smaller exponent (\( \alpha/2 \) instead of \( \alpha \)) in the quasilinear case is due to the different method. Note that the aforementioned method is no longer applicable in the quasilinear case. Our proof of \((1)\) is based on energy estimates; it makes use of the fundamental convexity inequality \((9)\) for the Riemann-Liouville fractional derivative, see Theorem 2.4 below. Our method applies to a more general class of quasilinear problems, including e.g. the time fractional \( p \)-Laplace equation, and it extends to the (more natural) weak setting. This will be elaborated in a forthcoming paper.

The paper is organized as follows. In Section 2 we collect some known results on a priori estimates for linear time fractional diffusion equations and recall the fundamental convexity identity \((9)\) for the fractional derivative. Section 3 is devoted to Hölder regularity up to \( t = 0 \).
for weak solutions of (3). Regularity up to the full parabolic boundary is then established in Section 4. Using these estimates, the global existence result, Theorem 1.1, is proved in Section 5. Finally, we derive the described decay estimate in Section 6.

2 Preliminaries

We first fix some notation. For \( T > 0 \) and a bounded domain \( \Omega \subset \mathbb{R}^N \) with boundary \( \Gamma \) we put \( \Omega_T = (0, T) \times \Omega \) and \( \Gamma_T = (0, T) \times \Gamma \). The Lebesgue measure in \( \mathbb{R}^N \) will be denoted by \( \lambda_N \).

The boundary \( \Gamma \) is said to satisfy the property of positive geometric density, if there exist \( \beta \in (0, 1) \) and \( \rho_0 > 0 \) such that for any \( x_0 \in \Gamma \), any ball \( B(x_0, \rho) \) with \( \rho \leq \rho_0 \) we have that \( \lambda_N(\Omega \cap B(x_0, \rho)) \leq \beta \lambda_N(B(x_0, \rho)) \), cf. e.g. [6, Section I.1].

By \( y^+ := \max\{y, 0\} \) we denote the positive part of \( y \in \mathbb{R} \).

In the following we collect some known results from the linear theory which are basic to the investigation of (1).

Let \( T > 0 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with \( N \geq 2 \). We consider the linear time fractional diffusion equation

\[
\partial_t^\alpha (u - u_0) - D_i(a_{ij}(t, x)D_j u) = f, \quad t \in (0, T), \quad x \in \Omega.
\]  

(6)

We assume that

(H1) \( A \in L_\infty(\Omega_T; \mathbb{R}^{N \times N}) \), and

\[
\sum_{i,j=1}^N |a_{ij}(t, x)|^2 \leq \Lambda^2, \quad \text{for a.a. } (t, x) \in \Omega_T.
\]

(H2) There exists \( \nu > 0 \) such that

\[
a_{ij}(t, x)\xi_i\xi_j \geq \nu |\xi|^2, \quad \text{for a.a. } (t, x) \in \Omega_T, \quad \text{and all } \xi \in \mathbb{R}^N.
\]

(H3) \( u_0 \in L_\infty(\Omega); f \in L_r([0, T]; L_q(\Omega)) \), where \( r, q \geq 1 \) fulfill

\[
\frac{1}{\alpha r} + \frac{N}{2q} = 1 - \kappa,
\]

and

\[
r \in \left[ \frac{1}{\alpha(1 - \kappa)}, \infty \right], \quad q \in \left[ \frac{N}{2(1 - \kappa)}, \infty \right], \quad \kappa \in (0, 1).
\]

Following [23] and [25] we say that a function \( u \) is a weak solution of (6) in \( \Omega_T \), if \( u \) belongs to the space

\[
S_u := \{ v \in L_{2/(1-\alpha), w}([0, T]; L_2(\Omega)) \cap L_2([0, T]; H^1_2(\Omega)) \text{ such that } g_{1-\alpha} * v \in C([0, T]; L_2(\Omega)), \text{ and } (g_{1-\alpha} * v)|_{t=0} = 0 \},
\]

and for any test function

\[
\eta \in \dot{H}^{1, 1}_2(\Omega_T) := H^{1, 1}_2([0, T]; L_2(\Omega)) \cap L_2([0, T]; \dot{H}^1_2(\Omega)) \quad (\dot{H}^1_2(\Omega) := C_0^\infty(\Omega) H^1_2(\Omega))
\]
with \( \eta|_{t=T} = 0 \) there holds

\[
\int_0^T \int_\Omega \left( -\eta [g_{1-\alpha} * (u - u_0)] + a_{ij} D_j u D_i \eta \right) dx dt = \int_0^T \int_\Omega f \eta \ dx dt. \tag{7}
\]

Here \( L_p, \omega \) stands for the weak \( L_p \) space and \( f_1 \ast f_2 \) means the convolution on the positive halfline with respect to time, that is \( (f_1 \ast f_2)(t) = \int_0^t f_1(t - \tau) f_2(\tau) \ d\tau, \ t \geq 0. \)

Existence of weak solutions of (6) in the class \( S_\alpha \) has been shown in [28]. For example, assuming \( (H1), (H2), u_0 \in L_2(\Omega), \) and \( f \in L_2(\Omega_T), \) the corresponding Dirichlet problem has a unique solution in \( S_\alpha. \) Global boundedness of weak solutions has been obtained in [24], the result can be stated as follows, cf. [24 Corollary 3.1].

**Theorem 2.1** Let \( \alpha \in (0,1), T > 0 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N (N \geq 2) \) with \( \Gamma \) satisfying the property of positive density. Let further the assumptions \( (H1)-(H3) \) be satisfied. Suppose that \( u \in S_\alpha \) is a weak solution of (6) such that \( |u| \leq K \) a.e. on \( \Gamma_T \) (in the sense that \( (u - K)_+ - (u - K)_- \in L_2([0,T]; H^1_0(\Omega)) \)) for some \( K \geq |u_0|_{L_\infty(\Omega)}. \) Then \( u \) is essentially bounded in \( \Omega_T \) and

\[
|u|_{L_\infty(\Omega_T)} \leq C(1 + K),
\]

where the constant \( C = C(\alpha, r, q, T, N, \nu, \Omega, |f|_{L_r(\omega)}). \)

An interior Hölder estimate for bounded weak solutions of (6) has been proved recently in [23], see also [25]. For \( \beta_1, \beta_2 \in (0,1) \) and \( Q \subset \Omega_T \) we put

\[
[u]_{C^{\beta_1, \beta_2}(Q)} := \sup_{(t,x), (s,y) \in Q \atop (t,x) \neq (s,y)} \left\{ \frac{|u(t,x) - u(s,y)|}{|t-s|^{\beta_1} + |x-y|^{\beta_2}} \right\}.
\]

Then the interior regularity result reads as follows, cf. [23 Theorem 1.1].

**Theorem 2.2** Let \( \alpha \in (0,1), T > 0 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N (N \geq 2) \). Let the assumptions \( (H1)-(H3) \) be satisfied and suppose that \( u \in S_\alpha \) is a bounded weak solution of (6) in \( \Omega_T. \) Then there holds for any \( Q \subset \Omega_T \) separated from \( \Gamma_T \) by a positive distance \( d, \)

\[
[u]_{C^{\frac{\alpha}{2}, \frac{\alpha}{2}}(Q)} \leq C\left( |u|_{L_\infty(\Omega_T)} + |u_0|_{L_\infty(\Omega)} + |f|_{L_r([0,T]; L_q(\Omega))} \right)
\]

with positive constants \( \epsilon = \epsilon(\Lambda, \nu, \alpha, r, q, N, \text{diam} \Omega, \inf_{(\tau,v) \in Q} \tau) \) and \( C = C(\Lambda, \nu, \alpha, r, q, N, \text{diam} \Omega, \lambda_{N+1}(Q), d). \)

**Remark 2.1** The statement of Theorem [23, Remark 6.1] can be extended to the case where the right-hand side of equation (6) has the form

\[
\sum_{k=1}^{k_f} f_k - \sum_{k=1}^{k_g} D_{i_k} g_{i_k},
\]

with \( f_k \in L_{r_k}([0,T]; L_{q_k}(\Omega)), \ k = 1, \ldots, k_f, \) \( \sum_{i=1}^N (g_{i_k})^2 \in L_{r(k)}([0,T]; L_{q(k)}(\Omega)), \ k = 1, \ldots, k_g, \) and all pairs of exponents \( (r_k, q_k) \) and \( (r(k), q(k)) \), respectively, are subject to the condition in (H3). This follows from [23, Remark 6.1], see also [25].
Next we are concerned with maximal $L_p$-regularity for the corresponding problem in non-
divergence form,

$$
\partial_t^\alpha (u - u_0) - a_{ij}(t, x)D_i D_j u = f, \quad t \in (0, T), \quad x \in \Omega,
$$

$$
u = g, \quad t \in (0, T), \quad x \in \Gamma,
$$

$$
u|_{t=0} = u_0, \quad x \in \Omega,
$$

where we again use the sum convention. The following result is a special case of [27, Theorem
3.4] on linear initial-boundary value problems in the context of parabolic Volterra equations.

**Theorem 2.3** Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^2$-boundary $\Gamma$, and $N \geq 2$. Let $\alpha \in (0, 1)$ and $p > \frac{1}{\alpha} + \frac{N}{2}$. Suppose that $A = (a_{ij})_{i,j=1,\ldots,N} \in C([0, T] \times \Omega; \text{Sym}\{N\})$, and there exists $\nu > 0$ such that $a_{ij}(t, x)\xi_i \xi_j \geq \nu|\xi|^2$ for all $(t, x) \in [0, T] \times \Omega$ and $\xi \in \mathbb{R}^N$. Then the problem (8) has a unique solution $u$ in the class

$$
Z := H^\alpha_p([0, T]; L_p(\Omega)) \cap L_p([0, T]; H^2_p(\Omega)) \hookrightarrow C([0, T] \times \Omega)
$$

if and only if the following conditions are satisfied.

(i) $f \in L_p([0, T]; L_p(\Omega))$, $g \in Y_D := B_{pp}^{\alpha(1-\frac{N}{2p})}([0, T]; L_p(\Gamma)) \cap L_p([0, T]; B_{pp}^{2-\frac{N}{p}}(\Gamma))$, and $u_0 \in Y_\gamma := B_{pp}^{2-\frac{N}{p}}(\Omega)$;

(ii) $u_0 = g|_{t=0}$ on $\Gamma$.

In this case one has an estimate of the form

$$
|u|_Z \leq C(|f|_{L_p(\Omega)} + |g|_{Y_D} + |u_0|_{Y_\gamma}),
$$

where $C$ only depends on $\alpha, p, N, T, \Omega, A$.

We conclude this preliminary part with an important convexity inequality for the Riemann-Liouville fractional derivation operator, which will be needed in Section 6.

**Theorem 2.4** Let $\alpha \in (0, 1)$, $T > 0$ and $\mathcal{H}$ be a real Hilbert space with inner product $(\cdot|\cdot)_\mathcal{H}$. Suppose that $v \in L_2([0, T]; \mathcal{H})$ and that there exists $x \in \mathcal{H}$ such that $v - x \in \mathcal{H}$ and $\mathcal{H} := \{g \ast w : w \in L_2([0, T]; \mathcal{H})\}$. Then

$$
(v(t), \frac{d}{dt}((g_{1-\alpha} \ast v)(t))_\mathcal{H} \geq \frac{1}{2} \frac{d}{dt}((g_{1-\alpha} \ast |v|^2_\mathcal{H})(t) + \frac{1}{2} g_{1-\alpha}(t)|v(t)|^2_{\mathcal{H}}, \quad \text{a.a. } t \in (0, T).
$$

**Proof.** This follows from Theorem 2.1, Proposition 2.1, and Example 2.1 in [22].

\[\square\]

### 3 Regularity up to $t = 0$

The objective of this and the following section is to find conditions on the data which ensure Hölder continuity up to the parabolic boundary for weak solutions of the linear time fractional diffusion equation (5). As already mentioned in the introduction we do not aim at great generality but at results which are sufficient for the quasilinear problem to be studied.

We first discuss regularity up to $t = 0$. 

6
Theorem 3.1 Let $\alpha \in (0,1)$, $T > 0$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$). Let the assumptions (H1)-(H3) be satisfied. Let further $\Omega' \subset \Omega$ be an arbitrary subdomain and assume that

$$u_0|_{\hat{\Omega}} \in B_{pp}^{2-\frac{2}{\alpha}}(\hat{\Omega}) \quad \text{with} \quad p > \frac{1}{\alpha} + \frac{N}{2},$$

for some $C^2$-smooth domain $\hat{\Omega}$ such that $\Omega' \subset \hat{\Omega} \subset \Omega$ and $\Omega'$ is separated from $\partial \hat{\Omega}$ by a positive distance $d$. Then, for any bounded weak solution $u$ of $(\tilde{\Omega})$ in $\Omega_T$, there holds

$$[u]_{C^{0,\alpha}([0,T] \times \partial \Omega')} \leq C \left( |u|_{L^\infty(\Omega_T)} + |u_0|_{L^\infty(\Omega)} + |u_0|_{B_{pp}^{2-\frac{2}{\alpha}}(\hat{\Omega})} + |f|_{L^p([0,T];L^q(\Omega))} \right)$$

with positive constants $\epsilon = \epsilon(\Lambda, \nu, \alpha, p, r, q, N, \text{diam} \Omega)$ and $C = C(\Lambda, \nu, \alpha, p, r, q, N, d, \text{diam} \Omega, T, \lambda_N(\Omega'))$.

Proof. The basic idea of the proof is to extend $u$ to $[-1, T] \times \Omega$ such that $u$ is Hölder continuous on $[-1,0] \times \partial \Omega'$ and to apply Theorem [27].

To this purpose we first extend $u_0|_{\hat{\Omega}} \in B_{pp}^{2-\frac{2}{\alpha}}(\hat{\Omega})$ to a function $\hat{u}_0 \in B_{pp}^{2-\frac{2}{\alpha}}(\mathbb{R}^N)$. By [27, Theorem 3.1], the problem

$$\partial_t^\alpha (w - \hat{u}_0) - \Delta w = 0, \quad t \in (0,1), \quad x \in \mathbb{R}^N,$$

$$w|_{t=0} = \hat{u}_0, \quad x \in \mathbb{R}^N,$$

possesses a unique solution $w$ in the class

$$Z := H^\alpha_p([0,1]; L_p(\mathbb{R}^N)) \cap L_p([0,1]; H^2_p(\mathbb{R}^N)),$$

and one has an estimate of the form

$$|w|_Z \leq C_0 |\hat{u}_0|_{B_{pp}^{2-\frac{2}{\alpha}}(\mathbb{R}^N)} \leq \tilde{C}_0 |u_0|_{B_{pp}^{2-\frac{2}{\alpha}}(\hat{\Omega})}.$$

Note that by the mixed derivative theorem (cf. [21]),

$$Z \hookrightarrow H^{\alpha(1-\varsigma)}([0,1]; H^{2\varsigma}_p(\mathbb{R}^N)), \quad \varsigma \in (0,1],$$

and thus $Z \hookrightarrow BUC^\delta([0,1] \times \mathbb{R}^N)$ for some sufficiently small $\delta \in (0, \alpha/2)$. In fact, the assumption $p > \frac{1}{\alpha} + \frac{N}{2}$ ensures existence of some $\varsigma \in (0,1)$ with $\alpha(1-\varsigma) - \frac{1}{p} > \delta$ and $2\varsigma - \frac{N}{p} > \delta$.

Multiplying $u$ by a suitable smooth cut-off function $\varphi(t)$ we can construct a function $\hat{w} \in Z$ with $\hat{w}|_{t=0} = \hat{u}_0$ and $\hat{w}|_{t=1} = 0$. We then extend $u$ to $[-1, T] \times \tilde{\Omega}$ by setting $u(t,x) = \hat{w}(-t,x)$ for $t \in [-1,0)$ and $x \in \hat{\Omega}$.

Next, we shift the time by setting $\tau = t+1$. Put $\hat{u}(\tau,x) = u(\tau-1, x)$, $\tau \in (0, T+1)$, $x \in \hat{\Omega}$. Define further

$$g := \partial^\alpha_x \hat{u} - \Delta \hat{u}, \quad \tau \in (0,1), \quad x \in \hat{\Omega}.$$

Then $g \in L_p([0,1] \times \hat{\Omega})$, since $\hat{u}|_{\tau \in (0,1)} \in H^\alpha_p([0,1]; L_p(\hat{\Omega})) \cap L_p([0,1]; H^2_p(\hat{\Omega}))$ and $\hat{u}|_{\tau=0} = 0$. Furthermore we have for any test function $\eta \in \tilde{H}^{2,1}_p([0, T+1] \times \hat{\Omega})$,

$$\int_0^1 \int_\hat{\Omega} \left( - \eta_\tau (g_{1-\alpha} \ast \hat{u}) + D_j \hat{u} D_j \eta \right) dx \, d\tau = \int_0^1 \int_\hat{\Omega} g \eta dx \, d\tau - \int_\hat{\Omega} \eta (g_{1-\alpha} \ast \hat{u}) dx \bigg|_{\tau=1}.$$  

(11)
On the other hand, we have for a.a. \((\tau, x) \in (1, T + 1) \times \tilde{\Omega}\),
\[
(g_{1-\alpha} \ast \hat{u})(\tau, x) = (g_{1-\alpha} \ast u)(\tau - 1, x) + \int_0^1 g_{1-\alpha}(\tau - \sigma)\hat{u}(\sigma, x) \, d\sigma
\]
\[
= (g_{1-\alpha} \ast (u - u_0))(\tau - 1, x) + g_{2-\alpha}(\tau)u_0(x)
+ \int_0^1 g_{1-\alpha}(\tau - \sigma)(\hat{u}(\sigma, x) - u_0(x)) \, d\sigma.
\]

Set
\[
h(\tau, x) = g_{1-\alpha}(\tau)u_0(x) + \int_0^1 \hat{g}_{1-\alpha}(\tau - \sigma)(\hat{u}(\sigma, x) - u_0(x)) \, d\sigma =: h_1(\tau, x) + h_2(\tau, x),
\]
\[
\hat{a}_{ij}(\tau, x) = a_{ij}(\tau - 1, x), \text{ and } \hat{f}(\tau, x) = f(\tau - 1, x) \text{ for } (\tau, x) \in (1, T + 1) \times \tilde{\Omega}. \text{ Since } u \text{ is a weak solution of } (\ref{eq:11}) \text{ in } \Omega_T, \text{ we thus obtain after a short computation that for any } \eta \in \dot{H}^{1,1}_2([0, T+1] \times \tilde{\Omega}) \text{ with } \eta|_{\tau=T+1} = 0
\]
\[
\int_1^{T+1} \int_\tilde{\Omega} \left( -\eta_{\tau}(g_{1-\alpha} \ast \hat{u}) + \hat{a}_{ij}D_j\hat{u}D_i\eta \right) \, dx \, d\tau =
\int_1^{T+1} \int_\tilde{\Omega} \left( \hat{f} + h \right) \eta \, dx \, d\tau + \int_\tilde{\Omega} \eta(g_{1-\alpha} \ast \hat{u}) \, dx \bigg|_{\tau=1}.
\]

Adding (\ref{eq:11}) and (\ref{eq:12}) shows that \(\hat{u}\) is a weak solution of
\[
\partial_{\tau}^2 \hat{u} - D_i(b_{ij}(D_j\hat{u})) = \hat{f}, \quad \tau \in (0, T+1), \, x \in \tilde{\Omega},
\]
where
\[
b_{ij}(\tau, x) = \chi_{[0,1]}(\tau) + \chi_{(1,T+1]}(\tau)\hat{a}_{ij}(\tau, x)
\]
and
\[
\hat{f}(\tau, x) = \chi_{[0,1]}(\tau)f(\tau, x) + \chi_{(1,T+1]}(\tau)(\hat{f} + h)(\tau, x).
\]

Evidently, \(\chi_{[0,1]}(\tau)g \in L_p([0, T + 1] \times \tilde{\Omega})\) and \(\chi_{(1,T+1]}(\tau)f \in L_r([0, T + 1]; L_q(\tilde{\Omega}))\). Concerning the \(h\)-term we clearly have \(\chi_{(1,T+1]}(\tau)h_1 \in L_{\infty}([0, T + 1] \times \tilde{\Omega})\). To estimate \(\chi_{(1,T+1]}(\tau)h_2\), we employ the H"{o}lder estimate
\[
|\hat{u}(\sigma, x) - u_0(x)| = |\hat{u}(\sigma, x) - \hat{u}(1, x)| \leq C_1(1 - \sigma)^{\delta}, \quad \sigma \in [0, 1], \, x \in \tilde{\Omega},
\]
which results from the embedding \(Z \hookrightarrow BUC^{\delta}([0, 1] \times \mathbb{R}^N)\) and the construction of \(\hat{u}\). It follows that for \(1 < \tau = t + 1 \leq 1 + T\) and \(x \in \tilde{\Omega}\)
\[
|h_2(\tau, x)| \leq C_1 \int_0^1 [\hat{g}_{1-\alpha}(\tau - \sigma)](1 - \sigma)^{\delta} \, d\sigma
\]
\[
= \frac{\alpha C_1}{\Gamma(1 - \alpha)} \int_0^1 (t + \sigma)^{-1 - \alpha} \sigma^{\delta} \, d\sigma.
\]
Assuming that \( t = \tau - 1 \in (0, 1) \) we then have
\[
|h_2(\tau, x)| \leq \frac{\alpha C_1}{\Gamma(1 - \alpha)} \left( \int_0^t (t + \sigma)^{-1 - \alpha \sigma} d\sigma + \int_t^1 (t + \sigma)^{-1 - \alpha} d\sigma \right)
\]
\[
\leq \frac{\alpha C_1}{\Gamma(1 - \alpha)} \left( \int_0^t (t + \sigma)^{-1 - \alpha t} d\sigma + \int_t^1 \sigma^{-1 + \alpha} d\sigma \right)
\]
\[
\leq \frac{\alpha C_1}{\Gamma(1 - \alpha)} t^{-\alpha + \delta} \left( \frac{1}{\alpha} + \frac{1}{\alpha - \delta} \right)
\]
\[
\leq 3C_1(\tau - 1)^\delta g_{1-\alpha}(\tau - 1).
\]
This shows that \( \chi_{(1,T+1)}(\tau)h_2 \in L_{r_0}([0, T + 1]; L_\infty(\hat{\Omega})) \) for all \( r_0 < \frac{1}{\alpha - \delta} \). In particular we find some \( \hat{r} = \frac{1}{\alpha} \) such that \( \chi_{(1,T+1)}(\tau)h_2 \in L_{\hat{r}}([0, T + 1]; L_\infty(\hat{\Omega})) \).

All in all we see that \( \tilde{\phi} \) is of the form \( \tilde{\phi} = \sum_{i=1}^4 \tilde{f}_i \), where \( \tilde{f}_i \in L_{r_i}([0, T + 1]; L_q(\hat{\Omega})) \) with
\[
\frac{1}{\alpha r_i} + \frac{N}{2q_i} < 1, \quad i = 1, 2, 3, 4.
\]
Hence Theorem \([2.2]\) and Remark \([2.1]\) imply that \( \hat{u} \) is Hölder continuous in \([1/2, T + 1] \times \hat{\Omega}'\). This in turn yields Hölder continuity of \( u \) in \([0, T] \times \hat{\Omega}'\), and it is not difficult to see that \( u \) is subject to the estimate \( \|u\| \). 

**Remark 3.1** It follows from Remark \([2.1]\) and the proof above, that Theorem \([3.1]\) can be generalized to the case where the right-hand side of equation \( \text{(8)} \) has the form
\[
\sum_{k=1}^{k_f} f_k - \sum_{k=1}^{k_g} D_i g_k,
\]
with \( f_k \) and \( g_k \) as in Remark \([2.1]\).

### 4 Regularity up to the parabolic boundary

The following result gives conditions on the data which are sufficient for Hölder continuity on \([0, T] \times \hat{\Omega} \).

**Theorem 4.1** Let \( \alpha \in (0, 1) \), \( T > 0 \), \( N \geq 2 \), and \( \Omega \subset \mathbb{R}^N \) be a bounded domain with \( C^2 \)-smooth boundary \( \Gamma \). Let the assumptions (H1)-(H3) be satisfied. Suppose further that
\[
u 0 \in B_{pp}^{2-\frac{N}{2p}}(\Omega), \quad g \in Y_D := B_{pp}^{1/2}\left([0, T]; L_p(\Gamma)\right) \cap L_p([0, T]; B_{pp}^{2-\frac{N}{2}}(\Gamma))
\]
with \( p > \frac{1}{\alpha} + \frac{N}{2} \), and that the compatibility condition
\[
u 0 = g|_{t=0} \quad \text{on} \quad \Gamma
\]
is satisfied. Then for any bounded weak solution \( u \) of \( \text{(6)} \) in \( \Omega_T \) such that \( u = g \) a.e. on \((0, T) \times \Gamma \), there holds
\[
u u_{C^{\alpha, \epsilon}([0,T] \times \hat{\Omega})} \leq C \left( \|u\|_{L_\infty(\Omega_T)} + |u_0|_{B_{pp}^{2-\frac{N}{2p}}(\Omega)} + \|f\|_{L_v([0,T]; L_q(\Omega))} + |g|_{Y_D} \right) \tag{13}
\]
with positive constants \( \epsilon = \epsilon(\lambda, \nu, \alpha, p, r, q, N, \Omega) \) and \( C = C(\lambda, \nu, \alpha, p, r, q, N, \Omega, T) \).
Proof. By Theorem 2.3 the problem

\[ \partial_t^\alpha (v - u_0) - \Delta v = 0, \quad t \in (0, T), \quad x \in \Omega \]
\[ v = g, \quad t \in (0, T), \quad x \in \Gamma, \]
\[ v|_{t=0} = u_0, \quad x \in \Omega, \]

admits a unique strong solution \( v \) in the class

\[ v \in Z := H^\alpha_p ([0, T]; L_p(\Omega)) \cap L_p([0, T]; H^2_p(\Omega)) \]

and

\[ |v|_Z \leq C_0(|u_0|_{B^{2-\frac{2\alpha}{\alpha p}}_{pp}(\Omega)} + |g|_{Y_D}), \]

where \( C_0 \) only depends on \( \alpha, p, N, T, \Omega \). As in the proof of Theorem 3.1 we see that \( v \in C^\delta([0, T] \times \overline{\Omega}) \) for some \( \delta > 0 \). Furthermore, the mixed derivative theorem implies that

\[ D_i v \in H^{2\alpha}_p ([0, T]; L_p(\Omega)) \cap L_p([0, T]; H^1_\alpha_p(\Omega)) \rightarrow H^{2\alpha}_p ([0, T]; H^{1-\varsigma}_p(\Omega)) \]

for all \( \varsigma \in [0, 1] \). Without restriction of generality we may assume that \( p \in (\frac{1}{\alpha} + \frac{N}{2}, \frac{2}{\alpha} + N) \).

With

\[ \hat{p} := \frac{1}{2\alpha} + \frac{N}{2} - 1 > \frac{1}{\alpha} + \frac{N}{2} \]

and \( \varsigma := \frac{2}{\alpha p} - \frac{1}{\alpha p} \in (0, 1) \) we then have \( H^{2\alpha}_p ([0, T]; H^{1-\varsigma}_p(\Omega)) \rightarrow L_{\hat{p}}(\Omega_T) \), which shows that

\[ |D_i v|^2 \in L_{\hat{p}}(\Omega_T) \]

with \( \frac{1}{2\alpha} + \frac{N}{2\hat{p}} < 1 \).

Setting \( w = u - v \), \( w \) is a bounded weak solution of

\[ \partial_t^\alpha w - D_i (a_{ij} D_j w) = f + D_i (a_{ij} D_j v) - \Delta v, \quad t \in (0, T), \quad x \in \Omega, \]

and \( w = 0 \) a.e. on \( (0, T) \times \Gamma \).

Next, let \( \Omega_0 \) be an arbitrary bounded domain containing \( \Omega \) such that \( \text{dist}(\Omega, \partial \Omega_0) > 0 \). We extend \( w, f, a_{ij} \) and \( \varphi_i := D_i v \) to \( [0, T] \times \Omega_0 \) by setting \( w, f, \varphi_i = 0 \) and \( a_{ij} = \delta_{ij} \) on \( [0, T] \times (\Omega_0 \setminus \Omega) \). Then \( w \) solves

\[ \partial_t^\alpha w - D_i (a_{ij} D_j w) = f + D_i (a_{ij} \varphi_j - \varphi_i), \quad t \in (0, T), \quad x \in \Omega_0, \]

in the weak sense, and thus Theorem 3.1 and Remark 3.1 imply that \( w \) is Hölder continuous on \( [0, T] \times \overline{\Omega} \). Since \( u = v + w \), the assertion of Theorem 4.1 follows.

\[ \square \]

5 Proof of the global solvability theorem

The proof of Theorem 4.1 is divided into three parts, devoted respectively to local well-posedness, existence of a maximally defined solution, and to a priori estimates which lead to global existence.

Recall that the data belong to the following regularity classes:

\[ f \in X_T := L_p([0, T]; L_p(\Omega)), \quad u_0 \in Y_\gamma := B^{2-\frac{2\alpha}{\alpha p}}_{pp}(\Omega) \]
\[ g \in Y_D^T := B^{(1-\delta)\alpha}_{pp}([0, T]; L_p(\Gamma)) \cap L_p([0, T]; B^{2-\frac{1}{\delta}}_{pp}(\Gamma)). \]
We seek a unique solution \( u \) of (1) in the space

\[ Z^T := H^p_\Omega([0,T]; L_p(\Omega)) \cap L_p([0,T]; H^0_p(\Omega)). \]

1. **Local well-posedness.** Short-time existence and uniqueness in the regularity class \( Z^5 \) can be established by means of the contraction mapping principle and maximal \( L_p \)-regularity for an appropriate linearized problem. We proceed similarly as in [27], see also [2] and [20].

We first define a reference function \( w \) as the unique solution of the linear problem

\[
\partial_t^\alpha (w - u_0) - a_{ij}(u_0)D_i D_j w = f + a'_{ij}(u_0)D_i u_0 D_j u_0, \quad t \in (0,T), \; x \in \Omega,
\]

\[
w = g, \quad t \in (0,T), \; x \in \Omega, \quad w|_{t=0} = u_0, \; x \in \Omega,
\]

see Theorem [2.3]. Note that the condition \( p > N - \frac{2}{\alpha} \) ensures the embedding

\[ u_0 \in Y_{\gamma} = B^{2-\frac{2}{\alpha}}_{pp}(\Omega) \hookrightarrow C^1(\overline{\Omega}), \]

and thus we also have

\[ Z^T \hookrightarrow C([0,T]; Y_{\gamma}) \hookrightarrow C([0,T]; C^1(\overline{\Omega})). \]

For \( \delta \in (0,T) \) and \( \rho > 0 \) let

\[ \Sigma(\delta, \rho) = \{ v \in Z^5 : v|_{t=0} = u_0, |v - w|_{Z^5} \leq \rho \}, \]

which is a closed subset of \( Z^5 \). By Theorem [2.3] we may define the mapping \( F : \Sigma(\delta, \rho) \to Z^5 \) which assigns to \( u \in \Sigma(\delta, \rho) \) the unique solution \( v = F(u) \) of the linear problem

\[
\partial_t^\alpha (v - u_0) - a_{ij}(u_0)D_i D_j v = f + h(u, Du, D^2 u), \quad t \in (0, \delta), \; x \in \Omega,
\]

\[
v = g, \quad t \in (0, \delta), \; x \in \Omega, \quad v|_{t=0} = u_0, \; x \in \Omega,
\]

where

\[ h(u, Du, D^2 u) = (a_{ij}(u) - a_{ij}(u_0))D_i D_j u + a'_{ij}(u)D_i u_0 D_j u. \]

Observe that every fixed point \( u \) of \( F \) is a local solution of (1) and vice versa, at least for some small time interval \( [0, \delta] \).

Since \( Z^5 \hookrightarrow C([0,\delta]; C^1(\overline{\Omega})) \) we may set

\[ \mu_w(\delta) := \max\{|w(t,x) - u_0(x)| + |Dw(t,x) - Du_0(x)| : t \in [0,\delta], \; x \in \overline{\Omega}\}. \]

Evidently, \( \mu_w(\delta) \to 0 \) as \( \delta \to 0 \), due to \( w|_{t=0} = u_0 \). Letting \( u \in \Sigma(\delta, \rho) \) we then have for all \( t \in [0, \delta] \) and \( x \in \overline{\Omega} \)

\[
|u(t,x) - u_0(x)| + |Du(t,x) - Du_0(x)| \leq |u - w|_{C([0,\delta]; C^1(\overline{\Omega}))} + \mu_w(\delta) \leq M_0|u - w|_{Z^5} + \mu_w(\delta) \leq M_0 \rho + \mu_w(\delta),
\]

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where the embedding constant $M_0 > 0$ does not depend on $u$ and $\delta \in (0, T]$; the latter is true since $u - w$ belongs to the space $aZ^\delta := \{ \varphi \in Z^\delta : \varphi |_{t=0} = 0 \}$.

 yields for any $u \in \Sigma(\delta, \rho)$ the bound

$$ |u(t, x) - u_0(x)| + |Du(t, x) - Du_0(x)| \leq M_0 \rho_0 + \mu_w(T), \quad t \in [0, \delta], \ x \in \overline{\Omega}, \quad (16) $$

where we assume $\rho \in (0, \rho_0]$.

Let now $u \in \Sigma(\delta, \rho)$ and $v = F(u)$. Then $v - w \in aZ^\delta$ solves the problem

$$ \partial_t^\prime (v - w) - a_{ij}(u_0)D_iD_j(v - w) = h(u, Du, D^2u) - a_{ij}^\prime(u_0)D_iu_0D_ju_0, \quad t \in (0, \delta), \ x \in \Omega, $$

$$ v - w = 0, \quad t \in (0, \delta), \ x \in \Gamma, \quad (v - w)|_{t=0} = 0, \ x \in \Omega. $$

Consequently, it follows from Theorem [2,3] that for some constant $M_1 > 0$ which is independent of $\delta \in (0, T]$ there holds

$$ |v - w|_{Z^\delta} \leq M_1|h(u, Du, D^2u) - a_{ij}^\prime(u_0)D_iu_0D_ju_0|_{X^\delta} $$

$$ \leq M_1|(a_{ij}(u) - a_{ij}(u_0))D_iD_ju|_{X^\delta} + M_1|a_{ij}^\prime(u)D_iuD_ju - a_{ij}^\prime(u_0)D_iu_0D_ju_0|_{X^\delta}. $$

Using (15) and (16) we may estimate the first term as follows.

$$ |(a_{ij}(u) - a_{ij}(u_0))D_iD_ju|_{X^\delta} \leq \left( (A(u) - A(u_0))_{(L^\infty)^N} + |A(u_0)|_{(L^\infty)^N} \right) $$

$$ \times \left( |D^2u - D^2w|_{(X^\delta)^N} + |D^2w|_{(X^\delta)^N} \right) $$

$$ \leq M_2 (\rho + \mu_w(\delta)) (\rho + \delta^\frac{1}{2}), $$

where $M_2 > 0$ does not depend on $\delta$ and $\rho$; similarly we obtain

$$ |a_{ij}^\prime(u)D_iuD_ju - a_{ij}^\prime(u_0)D_iu_0D_ju_0|_{X^\delta} \leq M_3(\rho + \mu_w(\delta)) (\rho + \delta^\frac{1}{2}), $$

with $M_3 > 0$ being independent of $\delta$ and $\rho$; here the factor $\delta^\frac{1}{2}$ comes from the estimate $|z|_{X^\delta} \leq (\lambda_N(\Omega)\delta)^{1/p}|z|_{\infty}$. We conclude that

$$ |v - w|_{Z^\delta} \leq M (\rho + \mu(\delta))^2, \quad (17) $$

where $M$ and $\mu(\delta)$ are constants, which do not depend on $\rho$, $M$ is independent of $\delta$, and $\mu(\delta)$ is non-decreasing with $\mu(\delta) \to 0$ as $\delta \to 0$.

Next let $u_i \in \Sigma(\delta, \rho)$ and $v_i = F(u_i)$, $i = 1, 2$. Then $v_1 - v_2 \in aZ^\delta$ solves the problem

$$ \partial_t^\prime (v_1 - v_2) - a_{ij}(u_0)D_iD_j(v_1 - v_2) = h(u_1, Du_1, D^2u_1) - h(u_2, Du_2, D^2u_2), \quad t \in (0, \delta), \ x \in \Omega, $$

$$ v_1 - v_2 = 0, \quad t \in (0, \delta), \ x \in \Gamma, \quad (v_1 - v_2)|_{t=0} = 0, \ x \in \Omega, $$

hence

$$ |v_1 - v_2|_{Z^\delta} \leq M_1|h(u_1, Du_1, D^2u_1) - h(u_2, Du_2, D^2u_2)|_{X^\delta}. $$

Estimating similarly as above we obtain

$$ |v_1 - v_2|_{Z^\delta} \leq M ((\rho + \mu(\delta))|u_1 - u_2|_{Z^\delta}, \quad (18) $$
where $M$ and $\mu(\delta)$ are like those in \eqref{17}.

Finally, the estimates \eqref{17} and \eqref{18} show that for sufficiently small $\rho$ and $\delta$ the mapping $F$ is a strict contraction which leaves the set $\Sigma(\delta, \rho)$ invariant. Local existence and uniqueness of strong solutions to \eqref{11} follows now by the contraction mapping principle.

2. The maximally defined solution. The local solution $u \in Z^\delta$ obtained in the first part can be continued to some larger interval $[0, \delta + \delta_1] \subset [0, T]$. In fact, let $u_\delta := u|_{t=\delta} \in Y_\gamma$ and define the set

$$\Sigma(\delta, \delta_1, \rho) := \{ v \in Z^{\delta + \delta_1} : v|_{[0, \delta]} = u, |v - w|_{Z^{\delta + \delta_1}} \leq \rho \},$$

where the reference function $w \in Z^T$ is now defined as the solution of the linear problem

$$\partial_t^\alpha (w - u_0) - a_{ij}(u_\delta) D_i D_j w = f + h_1 + \chi_{(\delta, T)}(t) a_{ij}'(u_\delta) D_i u_\delta D_j u_\delta, \quad t \in (0, T), \ x \in \Omega, \quad w = g, \ t \in (0, T), \ x \in \Gamma,$$

$$w|_{t=0} = u_0, \ x \in \Omega,$$

with

$$h_1 = \chi_{[0, \delta]}(t) \left( (a_{ij}(u) - a_{ij}(u_\delta)) D_i D_j u + a_{ij}'(u_\delta) D_i D_j u_\delta \right).$$

Observe that $w|_{[0, \delta]} = u$, by uniqueness. So $\Sigma(\delta, \delta_1, \rho)$ is not empty and it becomes a complete metric space when endowed with the metric induced by the norm of $Z^{\delta + \delta_1}$.

Define next the mapping $F : \Sigma(\delta, \delta_1, \rho) \to Z^{\delta + \delta_1}$ which assigns to $\tilde{u} \in \Sigma(\delta, \delta_1, \rho)$ the solution $v = F(\tilde{u})$ of the linear problem

$$\partial_t^\alpha (v - u_0) - a_{ij}(u_\delta) D_i D_j v = f + \tilde{h}(\tilde{u}, D\tilde{u}, D^2 \tilde{u}), \quad t \in (0, \delta + \delta_1), \ x \in \Omega, \quad w = g, \ t \in (0, \delta + \delta_1), \ x \in \Gamma,$$

$$w|_{t=0} = u_0, \ x \in \Omega,$$

where

$$\tilde{h}(\tilde{u}, D\tilde{u}, D^2 \tilde{u}) = (a_{ij}(\tilde{u}) - a_{ij}(u_\delta)) D_i D_j \tilde{u} + a_{ij}'(\tilde{u}) D_i \tilde{u} D_j \tilde{u}.$$

Since $\tilde{u}|_{[0, \delta]} = u$ we have also $v|_{[0, \delta]} = u$, by uniqueness.

Observe that $h_1 = \tilde{h}(\tilde{u}, D\tilde{u}, D^2 \tilde{u})$ on $[0, \delta]$ and thus

$$|v - w|_{Z^{\delta + \delta_1}} \leq M_1 |\tilde{h}(\tilde{u}, D\tilde{u}, D^2 \tilde{u}) - a_{ij}'(u_\delta) D_i u_\delta D_j u_\delta|_{L^p([\delta, \delta + \delta_1] \times \Omega)}.$$

Further,

$$|F(\tilde{u}_1) - F(\tilde{u}_2)|_{Z^{\delta + \delta_1}} \leq M_1 |\tilde{h}(\tilde{u}_1, D\tilde{u}_1, D^2 \tilde{u}_1) - \tilde{h}(\tilde{u}_2, D\tilde{u}_2, D^2 \tilde{u}_2)|_{L^p([\delta, \delta + \delta_1] \times \Omega)},$$

for $\tilde{u}_1, \tilde{u}_2 \in \Sigma(\delta, \delta_1, \rho)$. Therefore we may estimate analogously to the first step to see that for sufficiently small $\delta_1$ and $\rho$ we have $F(\Sigma(\delta, \delta_1, \rho)) \subset \Sigma(\delta, \delta_1, \rho)$ and $F$ is a strict contraction. Hence the contraction mapping principle yields existence of a unique fixed point of $F$ in $\Sigma(\delta, \delta_1, \rho)$, which is the unique solution of \eqref{11} on $[0, \delta + \delta_1]$.

Repeating this argument we obtain a maximal interval of existence $[0, T_{\text{max}})$ with $T_{\text{max}} \leq T$, that is $T_{\text{max}}$ is the supremum of all $\tau \in (0, T)$ such that the problem \eqref{11} has a unique solution $u \in Z^\tau$.

3. A priori bounds and global well-posedness. In order to establish global existence we will show that $|u|_{Z^\tau}$ stays bounded as $\tau \nearrow T_{\text{max}}$. 

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Let \( \tau \in (0, T_{max}) \) and \( u \in Z^\tau \) be the unique solution of (1). Setting \( b_{ij}(t, x) = a_{ij}(u(t, x)) \), it is evident that \( u \) is a weak solution of
\[
\partial^\tau_t (u - u_0) - D_i(b_{ij}D_ju) = f, \quad t \in (0, \tau), \ x \in \Omega.
\]
Since \( Y_\gamma \rightarrow C(\overline{\Omega}) \) and \( Y_\partial \rightarrow C([0, \tau] \times \Gamma) \), Theorem 2.3 implies a uniform sup-bound for \( |u| \), namely
\[
|u(t, x)| \leq C_1, \quad t \in [0, \tau], \ x \in \overline{\Omega},
\]
where the constant \( C_1 \) depends only on the data \( |f|_{X^{\tau}}, |g|_{Y^{\tau}}, |u_0|_{\infty}, \Omega, T, \alpha, N, \) and \( \nu \), not on \( \tau \). It follows then from Theorem 4.1 that for some \( \varepsilon > 0 \) we have
\[
|u|_{C^\varepsilon([0, \tau] \times \overline{\Omega})} \leq C_2,
\]
where the number \( C_2 \geq 1 \) depends only on \( |f|_{X^{\tau}}, |g|_{Y^{\tau}}, |u_0|_{Y^{\tau}}, \Omega, T, \alpha, N, \) and \( \nu \), not on \( \tau \). In particular, we obtain a uniform Hölder estimate for the coefficients \( b_{ij}, i, j = 1, \ldots, N \).

The first equation of (1) can be rewritten as
\[
\partial^\tau_t (u - u_0) - b_{ij}D_iD_ju = f + a'_{ij}(u)D_iu D_ju.
\]
By Theorem 2.3, the linear problem
\[
\partial^\tau_t (v - u_0) - b_{ij}D_iD_jv = f, \quad t \in (0, \tau), \ x \in \Omega,
\]
\[
v = g, \quad t \in (0, \tau), \ x \in \Gamma,
\]
\[
v|_{t=0} = u_0, \ x \in \Omega,
\]
has a unique solution \( v \in Z^\tau \) and there exists a constant \( M_1 > 0 \) independent of \( \tau \) such that
\[
|u - v|_{Z^\tau} \leq M_1|a'_{ij}(u)D_iu D_ju|_{X^{\tau}}
\]
\[
\leq M_1 \sum_{i,j=1}^N \max_{|u| \leq C_1} |a'_{ij}(y)||Du|_{X^{\tau}}. \tag{19}
\]
The assumption on \( p \) implies \( p > \frac{N}{2} \) and thus
\[
H^2_p(\Omega) \hookrightarrow H^{2p}_p(\Omega) \hookrightarrow C^{\varepsilon_0}(\overline{\Omega})
\]
for some \( \varepsilon_0 \in (0, \varepsilon] \). By the Gagliardo-Nirenberg inequality, there exists then \( \theta \in (0, \frac{1}{2}) \) such that
\[
|Du(t, \cdot)|_{L^{2p}(\Omega; R^N)} \leq C|u(t, \cdot)|^{\theta}_{H^2_p(\Omega)}|u(t, \cdot)|^{1-\theta}_{C^{\varepsilon_0}(\overline{\Omega})} \leq CC_2|u(t, \cdot)|^{\theta}_{H^2_p(\Omega)}, \quad t \in [0, \tau],
\]
and hence by Hölder’s and Young’s inequality
\[
||Du||_{X^{\tau}} \leq C|Du|_{L^{2p}(\Omega; R^N)}^{\theta} \leq C_3|u|_{H^2_p(\Omega)}^{\theta} \leq C_4|u|_{L^{2p}(\Omega; H^2_p(\Omega))}^{\theta} \leq C_5|u|_{L^{2p}(\Omega; H^2_p(\Omega))}^{\theta} \leq \varepsilon_1 |u|_{Z^\tau} + C_5(\varepsilon_1, C_4),
\]
for all \( \varepsilon_1 > 0 \). This together with (19) yields a bound for \( |u - v|_{Z^\tau} \) which is uniform w.r.t. \( \tau \). Since \( |v|_{Z^\tau} \) stays bounded as \( \tau \nearrow T_{max} \), it follows that \( |u|_{Z^\tau} \) enjoys the same property. Hence we have global existence. \( \square \)
6 Decay estimate

In this section we prove an $L_2$ decay estimate for solutions of \((1)\) with $f = 0$ and $g = 0$, that is we consider

\[
\partial_t^\alpha (u - u_0) - D_i (a_{ij}(u) D_j u) = 0, \quad t > 0, \quad x \in \Omega,
\]

\[
u = 0, \quad t > 0, \quad x \in \Gamma,
\]

\[u|_{t=0} = u_0, \quad x \in \Omega.\]

\[\text{(20)}\]

**Theorem 6.1** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with $C^2$-smooth boundary $\Gamma$. Let $\alpha \in (0,1)$, $p > N + \frac{2}{\alpha}$ and $u_0 \in B_{pp-\frac{2}{\alpha}}^\nu(\Omega)$ such that $u_0 = 0$ on $\Gamma$. Assume that condition (Q2) is satisfied. Then for the global strong solution $u$ of \((20)\) the function \(t \mapsto |u(t,\cdot)|_{L_2(\Omega)}^2\) is continuous on $[0, \infty)$ and we have

\[|u(t,\cdot)|_{L_2(\Omega)}^2 \leq \frac{c|u_0|_{L_2(\Omega)}^2}{1 + \mu t^\alpha}, \quad t \geq 0,
\]

with positive constants $c = c(\alpha)$ and $\mu = \mu(\nu, N, \Omega)$.

**Proof.** Let $T > 0$ be arbitrary and $u \in Z^T$ be the solution of \((20)\) on $[0, T]$. We multiply the first equation in \((20)\) by $u$, integrate over $\Omega$, and integrate by parts. Using the Dirichlet boundary condition this yields

\[I_1(t) := \int_{\Omega} \left( u \partial_t^\alpha u + a_{ij}(u) D_j u D_i u \right) dx = \int_{\Omega} u u_0 g_{1-\alpha}(t) dx =: I_2(t), \quad \text{a.a } t \in (0, T).
\]

Thanks to (Q2) and Theorem 2.4 with $H = L_2(\Omega)$ we have for a.a. $t \in (0, T)$

\[I_1(t) \geq \frac{1}{2} \partial_t^\alpha |u(t,\cdot)|_{L_2(\Omega)}^2 + \frac{1}{2} g_{1-\alpha}(t) |u(t,\cdot)|_{L_2(\Omega)}^2 + \nu |Du(t,\cdot)|_{L_2(\Omega; \mathbb{R}^N)}^2.
\]

On the other hand, Young's inequality implies that

\[I_2(t) \leq \frac{1}{2} g_{1-\alpha}(t) |u(t,\cdot)|_{L_2(\Omega)}^2 + \frac{1}{2} g_{1-\alpha}(t) |u_0|_{L_2(\Omega)}^2, \quad \text{a.a. } t \in (0, T).
\]

Combining these estimates gives

\[\partial_t^\alpha |u(t,\cdot)|_{L_2(\Omega)}^2 + 2\nu |Du(t,\cdot)|_{L_2(\Omega; \mathbb{R}^N)}^2 \leq g_{1-\alpha}(t) |u_0|_{L_2(\Omega)}^2, \quad \text{a.a. } t \in (0, T),
\]

which in turn, by Poincaré’s inequality, implies

\[\partial_t^\alpha |u(t,\cdot)|_{L_2(\Omega)}^2 + \mu |u(t,\cdot)|_{L_2(\Omega)}^2 \leq g_{1-\alpha}(t) |u_0|_{L_2(\Omega)}^2, \quad \text{a.a. } t \in (0, T),
\]

where $\mu = \mu(\nu, N, \Omega)$ is a positive constant. Setting

\[W(t) = |u(t,\cdot)|_{L_2(\Omega)}^2 \quad \text{and} \quad W_0 = W(0) = |u_0|_{L_2(\Omega)}^2
\]

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the fractional differential inequality (21) is equivalent to
\[ \partial_t^\alpha (W - W_0) + \mu W \leq 0 \quad \text{a.e. on } (0, T). \tag{22} \]

Next, let \( V \) denote the solution of the corresponding fractional differential equation, that is
\[ \partial_t^\alpha (V - V_0) + \mu V = 0 \quad \text{a.e. on } (0, T), \quad V(0) = V_0 = W_0. \tag{23} \]

By the comparison principle for linear fractional differential equations (cf. \cite{8}), we then have
\[ W(t) \leq V(t), \quad t \in [0, T]. \]

The solution of (23) is given by
\[ V(t) = V_0 E_\alpha(-\mu t^\alpha), \quad t \in [0, T], \]
where \( E_\alpha \) denotes the Mittag-Leffler function defined by
\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \]
see \cite{11} Section 4.1. Note that \( E_1(z) = e^z \). It is known that for \( \alpha \in (0, 1) \) \( E_\alpha \) is a completely monotonic function in \((\infty, 0]\) (see e.g. \cite{18}) and that there exists a constant \( c > 0 \) such that
\[ E_\alpha(-x) \leq \frac{c}{1 + x}, \quad x \geq 0, \]
see \cite{13} Formula (13)]. It follows that
\[ |u(t, \cdot)|_{L^2(\Omega)}^2 \leq |u_0|_{L^2(\Omega)}^2 E_\alpha(-\mu t^\alpha) \leq \frac{c|u_0|_{L^2(\Omega)}^2}{1 + \mu t^\alpha}, \quad t \in [0, T]. \]

Since \( T > 0 \) was arbitrary, the theorem is proved. \( \square \)

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