Deviation from Tribimaximal mixing using $A_4$ flavour model with five extra scalars

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Abstract

A modified neutrino mass model with five extra scalars is constructed using $A_4$ discrete symmetry group. The resultant mass matrix is able to give necessary deviation from Tribimaximal mixing which reproduces the current neutrino masses and mixing data with good accuracy. The model gives testable prediction for the future measurements of the neutrinoless double-beta decay parameter $|m_{\beta\beta}|$. The analysis is consistent with latest cosmological bound $\Sigma m_i \leq 0.12$ eV.

Keywords: $A_4$ symmetry, scalars, Tribimaximal mixing, normal hierarchy, inverted hierarchy, neutrinoless double-beta decay

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I. INTRODUCTION

In the last few decades, the neutrino experiments have confirmed neutrino oscillations and mixings through the observation of solar and atmospheric neutrinos, indicating their masses thereby providing an important solid clue for a new physics beyond the Standard Model (SM) of particle physics. At present, the neutrino oscillation experiments \[1–7\] have measured the oscillation parameters viz: mass squared differences ($\Delta m_{21}^2$ and $\Delta m_{31}^2$) and mixing angles ($\theta_{12}$, $\theta_{23}$ and $\theta_{13}$) to a good accuracy. The bounds on the absolute neutrino masses scale are also greatly reduced by direct neutrino mass experiments \[8\], the neutrinoless double beta decay experiments ($0\nu\beta\beta$) \[9–11\] and cosmological observation \[12\]. However, the current data is still unable to explain several key issues such as the octant of $\theta_{23}$, the neutrino mass ordering, CP violating phase, etc.

The oscillation data reveals certain pattern of neutrino mixing matrix. Out of the several approaches to explain the observed pattern, the Tribimaximal mixing (TBM) \[13, 14\] used to be very favourable. The $A_4$ flavour symmetry model proposed by Altarelli and Feruglio \[15, 16\] can accommodate TBM mixing scheme in a neutrino mass model. The TBM mixing pattern has the form

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}.$$ 

However, the recently observed non-zero $\theta_{13}$ disfavours TBM and leads to the modification of several mass models constructed with TBM \[17–22\]. As a result, the neutrino mixing patterns like TM1 \[23\] and TM2 \[24, 25\] which are proposed with slight deviation from TBM, gain momentum. Currently, they can predict the observed pattern and mixing angles with good consistency.

In this present work, we propose a model with five extra SM singlet scalars to explain neutrino parameters in their experimental ranges. The present model is constructed in the basis where charged lepton is diagonal. The deviation from TBM and non-zero value of $\theta_{13}$ are obtained as a consequence of specific Dirac mass matrix which is constructed using antisymmetric part arising from the product of two $A_4$ triplets. Here, we present a detailed analysis on the neutrino oscillation parameters and its correlation among themselves and
with neutrinoless double-beta decay parameter $|m_{\beta\beta}|$.

The paper is organized as follows: In Section 2, we present the description of the model with the particle contents in the underlying symmetry group. In Section 3, we give the detailed numerical analysis and the results in terms of correlation plots. Section 4 deals with summary and conclusion. The Appendix provides brief description of $A_4$ discrete group.

II. DESCRIPTION OF THE MODEL

We extend the SM by adding five extra scalars namely $\xi_1$, $\xi_2$, $\xi_3$, $\phi_T$ and $\phi_S$ which are transformed as $1$, $1''$, $1'$, $3$ and $3$ respectively under $A_4$ group. The SM lepton doublet $l$ are assigned to the triplet representation under $A_4$, right-handed charged lepton $e^c$, $\mu^c$, $\tau^c$, and right-handed neutrino field $\nu^c$ are assigned to the $A_4$ representations $1$, $1''$, $1'$ and $3$, respectively. The right-handed neutrino field $\nu^c$ contributes to the effective neutrino mass matrix through Type-I see-saw mechanism. The $A_4$ symmetry is supplemented by additional $Z_3 \times Z_2$ group to restrict additional terms otherwise allowed by $A_4$ symmetry. The transformation properties of the fields used in the model are given in Table I.

| Fields | $l$ | $e^c$ | $\mu^c$ | $\tau^c$ | $\nu^c$ | $H_{u,d}$ | $\phi_S$ | $\phi_T$ | $\xi_1$ | $\xi_2$ | $\xi_3$ |
|--------|-----|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $A_4$  | 3   | 1     | 1''    | 1'     | 3      | 1      | 3      | 1      | 1      | 1      | 1      |
| $Z_3$  | $\omega^2$ | $\omega$ | $\omega$ | $\omega$ | 1      | 1      | $\omega$ | 1      | $\omega$ | $\omega$ | 1      |
| $Z_2$  | 1   | -1    | -1     | -1     | 1      | 1      | -1     | 1      | 1      | 1      | 1      |
| $SU(2)_L$ | 2   | 1     | 1      | 1      | 1      | 2      | 1      | 1      | 1      | 1      | 1      |

TABLE I: Transformation properties of various fields under $A_4 \times Z_3 \times Z_2 \times SU(2)_L$ group.

The Yukawa Langrangian terms for the leptons which are invariant under $A_4 \times Z_3 \times Z_2 \times SU(2)_L$ transformation, are given in the equation:

$$\begin{align*}
-L_Y = & \frac{Y_e}{\Lambda} (l\phi_T)_1 H_d e^c + \frac{Y_\mu}{\Lambda} (l\phi_T)_1 H_d \mu^c + \frac{Y_\tau}{\Lambda} (l\phi_T)_1 H_d \tau^c \\
+ & y_1 \xi_1 (lH_u \nu^c)_1 + y_2 \xi_2 (lH_u \nu^c)_1 + y_3 \xi_3 (lH_u \nu^c)_1 \\
+ & y_a \phi_S (lH_u \nu^c)_A + \frac{y_b}{\Lambda} \phi_S (lH_u \nu^c)_S + \frac{1}{2} M_N (\nu^c \nu^c) + h.c.,
\end{align*}$$

(1)

where $\Lambda$ is the model cutoff high scale. The Yukawa mass matrices can be derived from Eq. (1) by using the vacuum expectation value given in Table II. The charged lepton mass
matrix thus obtained, is diagonal and has the form

\[ M_L = \frac{v_d v_T}{\Lambda} \begin{pmatrix} Y_e & 0 & 0 \\ 0 & Y_\mu & 0 \\ 0 & 0 & Y_\tau \end{pmatrix}. \tag{2} \]

The Majorana neutrino mass matrix has the structure

\[ M_R = \begin{pmatrix} M_N & 0 & 0 \\ 0 & 0 & M_N \\ 0 & M_N & 0 \end{pmatrix}. \tag{3} \]

The Dirac mass matrix is in the form

\[ M_D = \begin{pmatrix} 2a + c & -a + b + d & -a - b + e \\ -a - b + d & 2a + e & -a + b + c \\ -a + b + e & -a - b + c & 2a + d \end{pmatrix}, \tag{4} \]

where \( a = y_b v_u v_s / \Lambda, \quad b = y_a v_u v_s / \Lambda \) and \( c, d, e = y_i v_u u_i / \Lambda, \quad i = 1, 2 \) and 3.

The effective neutrino mass matrix is obtained by using Type-I see-saw mechanism,

\[ m_\nu = M_D^T M_R^{-1} M_D \tag{5} \]

\[ = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}, \tag{6} \]
where

\[
\begin{align*}
  m_{11} &= \frac{1}{M_N}((2a + c)^2 + 2(a + b - d)(a - b - e)) \\
  m_{12} &= m_{21} = \frac{1}{M_N}(-3a^2 + b^2 + 2cd + e^2 + b(-d + e) + a(6b - 2c + d + e)) \\
  m_{13} &= m_{31} = \frac{1}{M_N}(-3a^2 + b^2 + d^2 + 2ce + b(-d + e) + a(-6b - 2c + d + e)) \\
  m_{22} &= \frac{1}{M_N}((−a + b + d)^2 - 2(a + b - c)(2a + e)) \\
  m_{23} &= m_{32} = \frac{1}{M_N}(6a^2 - 2b^2 + c^2 + 2de + b(-d + e) + a(-2c + d + e)) \\
  m_{33} &= \frac{1}{M_N}2((−a + b + c)(2a + d) + (a + b - e)^2)).
\end{align*}
\]

In order to explain smallness of active neutrino masses, we consider the heavy neutrino with masses \( (M_N) \sim \mathcal{O}(10^{15} \text{ GeV}) \). When \( d = e \) and \( b = c = 0 \), the mass matrix \( m_\nu \) obtained in Eq. (5), takes the form

\[
m_\nu = \frac{1}{M_N} \begin{pmatrix}
4a^2 + 2(a - e)^2 & -3a^2 + 2ae + e^2 & -3a^2 + 2ae + e^2 \\
-3a^2 + 2ae + e^2 & (-a + e)^2 - 2a(2a + e) & 6a^2 + 2ae + 2e^2 \\
-3a^2 + 2ae + e^2 & 6a^2 + 2ae + 2e^2 & (a - e)^2 - 2a(2a + e)
\end{pmatrix}, \quad (7)
\]

with mass eigenvalues \( m_1 = (3a - e)^2/M_N \), \( m_2 = (4e^2)/M_N \) and \( m_3 = -(3a + e)^2/M_N \). However, in this case, the mass matrix \( m_\nu \) can be diagonalized by \( U_{TBM} \) and this possibility would violate the currently observed neutrino oscillation data especially the non-zero values of \( \theta_{13} \). Therefore, we need to consider the non-zero values for \( b, c, d \) and \( e \) to get \( \theta_{13} \neq 0 \). Hence, the unitary matrix that can diagonalize the resultant mass matrix, should deviate from the \( U_{TBM} \).

III. NUMERICAL ANALYSIS AND RESULTS

As the light neutrino mass matrix \( m_\nu \) obtained in Eq. (5) is in the basis where charged lepton mass matrix is diagonal, the Pontecorvo-Maki-Nakagawa-Sakata leptonic mixing matrix \( (U_{PMNS}) \) which is necessary for the diagonalization of \( m_\nu \) becomes a unitary matrix \( U \).
| Parameters          | Best fit±1σ | 2σ       | 3σ       |
|--------------------|-------------|----------|----------|
| $\theta_{12}/^\circ$ | 34.3±1.0    | 32.3-36.4 | 31.4-37.4 |
| $\theta_{13}/^\circ$(NO) | 8.53$^{+0.13}_{-0.12}$ | 8.27-8.79 | 8.20-8.97 |
| $\theta_{13}/^\circ$(IO) | 8.58$^{+0.12}_{-0.14}$ | 8.30-8.83 | 8.17-8.96 |
| $\theta_{23}/^\circ$(NO) | 49.26±0.79 | 47.35-50.67 | 41.20-51.33 |
| $\theta_{23}/^\circ$(IO) | 49.46$^{+0.60}_{-0.57}$ | 47.35-50.67 | 41.16-51.25 |
| $\Delta m^2_{21}[10^{-5}eV^2]$ | 7.50$^{+0.22}_{-0.20}$ | 7.12-7.93 | 6.94-8.14 |
| $|\Delta m^2_{31}|[10^{-3}eV^2]$(NO) | 2.55$^{+0.02}_{-0.03}$ | 2.49-2.60 | 2.47-2.63 |
| $|\Delta m^2_{31}|[10^{-3}eV^2]$(IO) | 2.45$^{+0.02}_{-0.03}$ | 2.39-2.50 | 2.37-2.53 |
| $\delta/^\circ$(NO) | 194$^{+24}_{-22}$ | 152-255 | 128-359 |
| $\delta/^\circ$(IO) | 284$^{+26}_{-28}$ | 226-332 | 200-353 |

TABLE III: The global-fit result for neutrino oscillation parameters \[26\].

\[
\begin{align*}
\text{FIG. 1: Correlation plots between the model parameters for normal hierarchy (NH).}
\end{align*}
\]

Therefore, the light neutrino mass matrix $m_\nu$ is diagonalized as:

\[
m_\nu = U^* m_{\text{diag}} U^\dagger
\]
FIG. 2: Correlation plots between the model parameters for inverted hierarchy (IH).

where $m_{\text{diag}} = \text{diag}(m_1, m_2, m_3)$ is the light neutrino mass matrix in the diagonal form.

The three neutrino mass eigenvalues can be written as,

$$m_{\text{diag}} = \begin{cases} 
\text{diag}(m_1, \sqrt{m_1^2 + \Delta m_{21}^2}, \sqrt{m_1^2 + \Delta m_{31}^2}) & \text{in normal hierarchy (NH)} \\
\text{diag}(\sqrt{m_3^2 + \Delta m_{31}^2}, \sqrt{m_3^2 + \Delta m_{32}^2 + \Delta m_{21}^2}, m_3) & \text{in inverted hierarchy (IH)} 
\end{cases}$$

The upper bound on the sum of neutrino masses ($\sum m_i = m_1 + m_2 + m_3$) obtained by the Planck is 0.12 eV [12].

The PMNS matrix $U$ can be parametrized in terms of neutrino mixing angles and Dirac CP phase $\delta$. Following the PDG convention [27], $U$ takes the form

$$U_{\text{PMNS}} = P \begin{pmatrix} 
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
-s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} 
\end{pmatrix} P,$$
where $\theta_{ij}$ (for $ij = 12, 13, 23$) are the mixing angles (with $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. $P = \text{diag}(e^{ia}, e^{ib}, 1)$ contains two Majorana CP phases $\alpha$ and $\beta$, while $P_\phi = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$ consists of three unphysical phases $\phi_{1,2,3}$ that can be removed via the charged-lepton field rephasing [28]. The neutrino mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ in terms of the elements of $U$
are given below:

\[
\begin{align*}
    s_{12}^2 & = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2}, &
    s_{23}^2 & = \frac{|U_{e3}|^2}{1 - |U_{e3}|^2}, &
    s_{13}^2 & = |U_{e3}|^2 \\

    J & = \text{Im} \{U_{e1}U_{\mu2}U_{e2}^*U_{\mu1}^*\} = s_{12}c_{12}s_{23}c_{23}s_{13}^2 s_{13} \sin \delta.
\end{align*}
\]
In order to show that the model is consistent with the present neutrino oscillation data, we vary the free parameters of our model $a$, $b$, $c$, $d$ and $e$ to fix the neutrino oscillation observables $\theta_{13}$, $\theta_{23}$, $\theta_{12}$, $\Delta m^2_{21}$, $\Delta m^2_{31}$ and $\delta$ to their experimental values. The experimental values of neutrino oscillation parameters used in our analysis are given in Table III. The allowed regions of the model parameters that can satisfy current oscillation data for both NH and IH are shown in Fig. 1 and Fig. 2, respectively in the form of correlation plots.

The values of $\theta_{13}$ obtained in the allowed regions of our model parameters for both NH and
FIG. 6: Variation of $\theta_{23}$ with the model parameters for IH.

IH are shown in Fig. 3 and Fig. 5, respectively while Fig. 4 and Fig. 6 give the variation of $\theta_{23}$ in NH and IH, respectively. The model predictions of the neutrino oscillation parameters in $3\sigma$ range are shown in Fig. 7 and Fig. 8. One of the important predictions of model is that solar neutrino mixing angle $\theta_{12}$ lies around 35.73° in both NH and IH cases. In NH, the Dirac CP violating phase $\delta$ is obtained in the range $201.47^\circ \leq \delta \leq 356.20^\circ$ as shown in Fig. 7 with the average model value of 230.89°. The reactor angle $\theta_{13}$ and atmospheric angle $\theta_{23}$ predicted by the model are outside $1\sigma$ range but well within $3\sigma$ range. In the case
of $\theta_{23}$, our present model prefers the higher octant ($> 45^\circ$), with the average model value of $49.32^\circ$. The average model value of $\Delta m^2_{21}$ and $\Delta m^2_{31}$ are $7.46 \times 10^{-5}$eV$^2$ and $2.56 \times 10^{-3}$eV$^2$, respectively and show good agreement with the experimental data.

The predictions in inverted hierarchy are shown in Fig. 8. In this case, the predicted values for $\theta_{13}$, $\Delta m^2_{21}$ and $\Delta m^2_{31}$ are all in the $3\sigma$ range with the average model values of $8.63^\circ$, $7.61 \times 10^{-5}$eV$^2$ and $-2.38 \times 10^{-3}$eV$^2$, respectively. The angle $\theta_{23}$ are predicted in the higher octant ($> 45^\circ$) with the average model value of $51.09^\circ$. The Dirac CP violating phase $\delta$ is predicted in the range $184.05^\circ \leq \delta \leq 186.45^\circ$ and shows a deviation from the global fit values. The current analysis is consistent with the latest cosmological bound $\Sigma m_i \leq 0.12$ eV. Thus, the presented model produces the required deviation from TBM necessary to accommodate the current neutrino oscillation data with distinctive predictions of Dirac CP violating phase $\delta$ in IH.

The additional predictions for the effective Majorana mass $|m_{\beta\beta}|$ vs Jarlskog invariant J in $3\sigma$ range for both NH and IH are shown in Fig. 9(a) and Fig. 10(a), respectively. The
FIG. 8: Correlation plots for inverted mass hierarchy (IH).

FIG. 9: Model predictions for Jarlskog invariant versus effective Majorana mass and Dirac CP violating phase versus effective Majorana mass for NH.

correlation between $\delta$ and $|m_{\beta\beta}|$ are depicted in Fig. 9(b) for NH and Fig. 10(b) for IH.
FIG. 10: Model predictions for Jarlskog invariant versus effective Majorana mass and Dirac CP violating phase versus effective Majorana mass for IH.

And, the effective Majorana mass $|m_{\beta\beta}|$ is given by

$$|m_{\beta\beta}| = \left| \sum_i U_{ei}^2 m_i \right|. \quad (13)$$

The upper limits of the effective Majorana mass are obtained by: Gerda $[9]$ as $|m_{\beta\beta}| < (104 - 228)$ meV corresponds to $^{76}Ge \left( T^{0\nu\beta\beta}_{1/2} > 9 \times 10^{25} \text{ yr} \right)$, CUORE $[10]$ as $|m_{\beta\beta}| < (75 - 350)$ meV corresponds to $^{130}Te \left( T^{0\nu\beta\beta}_{1/2} > 3.2 \times 10^{25} \text{ yr} \right)$ and KamLAND-Zen $[11]$ as $|m_{\beta\beta}| < (61 - 165)$ meV corresponds to $^{136}Xe \left( T^{0\nu\beta\beta}_{1/2} > 1.07 \times 10^{25} \text{ yr} \right)$. Our predicted $3\sigma$ range values of $|m_{\beta\beta}|$ are $(19.59 - 29.75)$ meV in NH and $(32.96 - 38.69)$ meV in IH. And the predicted range of $|m_{\beta\beta}|$ for both cases can be tested in future as there are planned ton-scale and next generation $0\nu\beta\beta$ experiments using $^{136}Xe$ $[29, 30]$ and $^{76}Ge$ $[31, 32]$ that can reach a sensitivity of $|m_{\beta\beta}| \sim (12 - 30)$ meV, corresponding to $T^{0\nu\beta\beta}_{1/2} \geq 10^{27} \text{ yr} \ [33, 34]$.

IV. SUMMARY AND CONCLUSION

In summary, we have presented a neutrino mass model using $A_4$ discrete symmetry. The model uses five extra SM singlets to produce the required deviation from TBM necessary to accommodate current neutrino oscillation data. The model has some characteristic predictions for neutrino oscillation parameters. The solar neutrino angle $\theta_{12}$ is centred around $35.73^\circ$ for both the mass ordering. The predicted range of the atmospheric neutrino mixing angle $\theta_{23}$ for NH is in good agreement with the experimental data, with the average model
value of 49.32°. The average model values of angle $\theta_{13}$ and $\theta_{23}$ are 8.63° and 51.09° for inverted hierarchy. The predicted range for $\delta$ for IH shows deviation from the of global fit, with the average model value of 184.88°. The presented model slightly prefers the NH data. The predicted range of $|m_{\beta\beta}|$ for both NH and IH can be tested in the near future.

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**Appendix A: $A_4$ Group**

$A_4$ is the even permutation group of 4 objects with $\frac{4!}{2}$ elements. It has four irreducible representations, namely 1, $1'$, $1''$ and 3. All the elements of the group can be generated by two elements S and T. The generators S and T satisfy the relation,

$$S^2 = (ST)^3 = T^3 = 1.$$  \hspace{2cm} (A1)

The multiplication rules of any two irreducible representations under $A_4$ are given by

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A$$

$$1 \otimes 1 = 1 \quad 1' \otimes 1' = 1''$$

$$1'' \otimes 1'' = 1' \quad 1' \otimes 1'' = 1$$

$$3 \otimes 1'/1'' = 3 \quad 1'/1'' \otimes 3 = 3$$

(A2)

where

$$1 = a_1 b_1 + a_2 b_3 + a_3 b_2$$ \hspace{2cm} (A3)

$$1' = a_3 b_3 + a_1 b_2 + a_2 b_1$$ \hspace{2cm} (A4)

$$1'' = a_2 b_2 + a_1 b_3 + a_3 b_1$$ \hspace{2cm} (A5)
\begin{align}
3_S &= \begin{pmatrix}
2a_1b_1 - a_2b_3 - a_3b_2 \\
2a_3b_3 - a_1b_2 - a_2b_1 \\
2a_2b_2 - a_1b_3 - a_3b_1
\end{pmatrix} \\
3_A &= \begin{pmatrix}
a_2b_3 - a_3b_2 \\
a_1b_2 - a_2b_1 \\
a_3b_1 - a_1b_3
\end{pmatrix}
\end{align}

The detailed studies on $A_4$ symmetry can be found in Ref \cite{35}.

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