Ramanujan and Eckford Cohen totients from Visible Point Identities

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Abstract

We define an extension of Ramanujan’s trigonometric function to arbitrary dimensions, and give the Dirichlet series generating function. The extension was first given by Eckford Cohen long ago. This links directly to visible point vector (vpv) identities in the author’s papers, and possibly to lattice sums in Physics and Chemistry presented by Baake et al. New generating functions and summations are given here, generalizing the Ramanujan function, Euler totient and the Jordan totient functions, based on visible lattice point ideas.

Key words: arithmetic functions; related numbers; inversion formulas, power series (including lacunary series), convergence and divergence of infinite products, other combinatorial number theory, lattice points in specified regions, \(\zeta(s)\) and \(L(s,\chi)\), elementary theory of partitions, trigonometric and exponential sums, Combinatorial identities

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1 Introduction

The function made well known by Ramanujan [48] for positive integers \(k\), and \(n\),

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\[ c_k(n) = \sum_{(j,k)\neq 1 \atop 0 < j < k} \cos \left( \frac{2\pi nj}{k} \right) = \sum_{(j,k)\neq 1 \atop 0 < j < k} e^{\frac{2\pi nj}{k}} \] (1.1)

has been widely studied and referenced. See for example Hardy [39], Hardy and Wright [40], Sivaramakrishnan [49], and Apostol [10]. The paper by Cohen [35] contains versions of some of the formulas of the current paper, couched in the language of arithmetical functions of the 1950s. The perspective offered in the current paper shows a natural approach to these identities, with its genesis in the visible point vector identities. In this note we extend the function as follows

\[ c_k(n_1, n_2) = \sum_{(j_1, j_2, k) \neq 1 \atop 0 \leq j_1 < k} \cos \left( \frac{2\pi (j_1 n_1 + j_2 n_2)}{k} \right) = \sum_{(j_1, j_2, k) \neq 1 \atop 0 \leq j_1 < k} e^{\frac{2\pi i(j_1 n_1 + j_2 n_2)}{k}}, \] (1.2)

\[ c_k(n_1, n_2, n_3) = \sum_{(j_1, j_2, j_3, k) \neq 1 \atop 0 \leq j_1 < k} \cos \left( \frac{2\pi (j_1 n_1 + j_2 n_2 + j_3 n_3)}{k} \right) = \sum_{(j_1, j_2, j_3, k) \neq 1 \atop 0 \leq j_1 < k} e^{\frac{2\pi i(j_1 n_1 + j_2 n_2 + j_3 n_3)}{k}}, \] (1.3)

and generally
These functions are very natural to consider as their generating Dirichlet series exhibit a simplicity unexpected of a function requiring so many parameters. On the other hand, there seems an inherently natural aspect to considering the set of visible lattice points in any radial region of multidimensional Euclidean space. We shall see in the following sections that \( c_k(n_1, n_2, \ldots, n_m) \) always reduces easily to a one dimensional \( c_k(n) \) type function.

2 Dirichlet series generating functions

In this section we start with a definition then give the generating function for the new Ramanujan function, and some consequences.

**Definition 2.1** Let us define the operator

\[
\sum_{(j_1, \ldots, j_m, k) = 1}^{m_k} \cos \left( 2\pi \left( \frac{j_1 n_1 + j_2 n_2 + \cdots + j_m n_m}{k} \right) \right)
\]

\[
\sum_{(j_1, \ldots, j_m, k) = 1}^{m_k} \exp \left( 2\pi i \left( \frac{j_1 n_1 + j_2 n_2 + \cdots + j_m n_m}{k} \right) \right)
\]

so that

\[
c_k(n_1, n_2, \ldots, n_m) = \sum_{m_k} \cos \left( 2\pi \left( \frac{j_1 n_1 + j_2 n_2 + \cdots + j_m n_m}{k} \right) \right).
\]

It is well known that [40, pp. 139-143]
\[
\frac{\sigma_{-s}(n)}{\zeta(s+1)} = \sum_{k=1}^{\infty} \frac{c_k(n)}{k^{s+1}}, \quad \Re s > 0,
\]

where the left side contains the sum of \(s\)-th powers of the divisors of \(n\), and the Riemann zeta function. If the Riemann hypothesis is true then (2.3) is valid for \(\Re s > -\frac{1}{2}\). We show that

**Theorem 2.2** If \(m\) is a positive integer then,

\[
\frac{\sigma_{-s-1+m}((n_1, n_2, \ldots, n_m))}{\zeta(s+1)} = \sum_{k=1}^{\infty} \frac{c_k(n_1, n_2, \ldots, n_m)}{k^{s+1}}, \quad \Re s > 0,
\]

\[
\exp\left(\sum_{k|(n_1, \ldots, n_m)} k^{m-1}z^k\right) = \prod_{k=1}^{\infty} (1 - z^k)^{-c_k(n_1, \ldots, n_m)/k}, \quad |z| < 1.
\]

Like (2.3), (2.4) is probably also true for \(\Re s > -\frac{1}{2}\). It is easy to prove many results similar to (2.4) and (2.5). We shall prove theorem 2.2 in the next section. Features of the Ramanujan function such as multiplicativity generalize in the new versions.

**Corollary 2.3** \(c_k(n_1, n_2, \ldots, n_m)\) is multiplicativce in \(k\).

The paper by Apostol and Zuckermann deals with the functional equation \(F(mn)F((m, n)) = F(m)F(n)f((m, n))\) which many of the common arithmetical functions satisfy, and indeed so do our new functions \(c_k(n_1, n_2, \ldots, n_m)\), in the form

\[
c_{mn}(n_1, n_2, \ldots, n_{\lambda})c_{(m,n)}(n_1, n_2, \ldots, n_{\lambda}) = c_{m}(n_1, n_2, \ldots, n_{\lambda}) \sum_{k=1}^{\infty} \frac{c_k(n_1, n_2, \ldots, n_m)}{k} = 0 f((m, n)).
\]

**Corollary 2.4** If \(m\) is a positive integer,

\[
\sum_{k=1}^{\infty} \frac{c_k(n_1, n_2, \ldots, n_m)}{k} = 0.
\]

Following the original paper by Ramanujan on \(c_n(k)\) this comes from the fact that the “sum of powers of divisors” function on left of (2.4) is a finite Dirichlet series, and at \(s = 0\) in that equation depends on the well known convergence of
\[ \sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0. \] (2.8)

3 Proof of Theorem 2.2

Theorem 2.2 depends on lemmas 3.1 and 3.2. The latter is an analytic restatement of the former.

**Lemma 3.1 (see Campbell [30])** Consider an infinite region raying out of the origin in any Euclidean vector space. The set of all lattice point vectors from the origin in that region is precisely the set of positive integer multiples of the visible point vectors (vpv’s) in that region.

This lemma underlies all of the author’s cited papers [24] to [32] and is fundamental in obtaining vpv identities. A clearer picture of lemma 3.1 is gained from fig 1. It shows part of the 2-D first quadrant visible-from-origin lattice points as the dots. All other lattice points in the part quadrant are shown as crosses. By lemma 3.1, the coordinates of the crosses are the positive integer (scalar) multiples of the coordinates of the dots; this being true in fact for the dots and crosses enclosed in any radial region from the origin. In the study of quasicrystals Baake et al [16] for example, diffraction patterns and sectioning gives rise to Fourier inversions of lattice patterns such as the visible points depicted here. The identities in section 3 here are an “easy” way into many arithmetical sums not unlike those from Ramanujan [48].

**fig. 1.**
An analytic restatement of lemma 3.1 is given in the

Lemma 3.2 If \((a_k)\) is an arbitrary sequence and \(q_i\) are variables chosen so that the following functions are all defined,

\[
\sum_{k=1}^{\infty} \left( a_k \prod_{h=1}^{m} \frac{1-q_h}{1-q_h^{1/k}} \right) = S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{m} (q_1^{j_1} \ldots q_m^{j_m})^{1/k} \right) \quad \text{where} \quad S_k = \sum_{j=1}^{\infty} a_{jk}.
\]

(3.1)

Proof of Lemma 3.2. The case with \(m = 1\) was given in Campbell [24] to [32], leading in [31] to proofs of (2.3) and the case \(m = 1\) of (2.5). The cases of (3.1) with \(m = 1, 2, 3\), are as follows, and correspond to lemma 3.1 in 2-D, 3-D, and 4-D space:

\[
\sum_{k=1}^{\infty} a_k \frac{1-q_1}{1-q_1^{1/k}} = S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{1} q_1^{j_1/k} \right),
\]

(3.2)

\[
\sum_{k=1}^{\infty} a_k \frac{1-q_1}{1-q_1^{1/k}} \frac{1-q_2}{1-q_2^{1/k}} = S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{2} (q_1^{j_1/k} q_2^{j_2/k})^{1/k} \right),
\]

(3.3)

\[
\sum_{k=1}^{\infty} a_k \frac{1-q_1}{1-q_1^{1/k}} \frac{1-q_2}{1-q_2^{1/k}} \frac{1-q_3}{1-q_3^{1/k}} = S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{3} (q_1^{j_1/k} q_2^{j_2/k} q_3^{j_3/k})^{1/k} \right),
\]

(3.4)

The proof of each of (3.2) to (3.4) is almost a priori when seen as interpreting lemma 3.1 as it is depicted in a generalized version of fig 1. To spell this out, we see that the left side of (3.1) is

\[
\sum_{k=1}^{\infty} \left( a_k \prod_{h=1}^{m} \frac{1-q_h}{1-q_h^{1/k}} \right)
= \sum_{k=1}^{\infty} \left( a_k \prod_{h=1}^{m} \sum_{j=1}^{k-1} q_h^{j/k} \right)
= \sum_{k=1}^{\infty} \left( a_k \sum_{j=1}^{k-1} (q_1^{j_1} \ldots q_m^{j_m})^{1/k} \right)
= S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{m} (q_1^{j_1} \ldots q_m^{j_m})^{1/k} \right) \quad \text{where} \quad S_k = \sum_{j=1}^{\infty} a_{jk}.
\]

The last step here depends on application of lemma 3.1 in \(m\)-space.
The vpv identities (visible point vector) published by the author, such as for example in Campbell [30] or [29] are all particular cases of lemma 2.2.

**Proof of Theorem 2.2.** In lemma 3.2 set \( q_k = \exp(2\pi in_k) \), then set \( a_k \) equal to respectively \( k^{-s}, z^k \). The resulting identities are (2.4) and (2.5).

### 4 A new Jordan Totient generating function, and some related results

The consequent identities from (3.2) were explored in some detail in the author’s papers [24] to [32]. Borwein and Borwein [23, ex 10, pp.327] had some infinite products resembling those of the author, and some of the Dirichlet series implied from (3.2) were given in Lossers [45], Sivaramakrishnan [49], and Chandrasekharan [34]. Most books on arithmetical functions since Ramanujan’s original paper [48] have included a treatment of the function (1.1) and give at least the result (2.3) cited above. However, (2.4) and (2.5) are not in the literature, and the identities (3.3) and (3.4) for example, lead us to a class of Dirichlet series and infinite products not hitherto considered. These results are near the surface and seem worth closer study. Lemma 3.2 is based on a summation over lattice points in an \( m + 1 \) dimensional hyperpyramid. (see Campbell [29]) The methods for lattice sums given in Ninham et al [47] and Glasser and Zucker [38] using Mobius and Mellin inversions are also applicable to results from this lemma. Of course this has been recently given a more systematic treatment by Baake et al [16] to [18] and his colleagues in their investigations of quasicrystal lattice structures and associated Fourier inversions.

Since some neat examples of (3.2) have been examined in other papers, we now consider the next simplest equation (3.3) with \( q_1, q_2 \) replaced respectively by \( e^{xz}, e^{yz} \). We have therefore the analysis

\[
\sum_{k=1}^{\infty} a_k \left( \sum_{A=0}^{k-1} \sum_{B=0}^{k-1} e^{(Ax+By)z/k} \right) = S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{2}^{\infty} e^{(j_1 x + j_2 y)z/k} \right),
\]

\[
\Leftrightarrow \sum_{k=1}^{\infty} a_k \left( \sum_{A=0}^{k-1} \sum_{B=0}^{k-1} \sum_{C=0}^{k-1} (Ax + By)^{z+c}/C!k^c \right) = S_1 + \sum_{k=2}^{\infty} \left( S_k \sum_{2}^{\infty} \sum_{C=0}^{\infty} (j_1 x + j_2 y)^{z+c}/C!k^c \right),
\]

\[4.1\]

\[4.2\]
⇔ \left( \sum_{k=1}^{\infty} k^2 a_k \right) + \sum_{C=0}^{\infty} \sum_{k=1}^{\infty} \sum_{0 \leq A < k \atop 0 \leq B < k \atop A + B \neq 0} (Ax + By)^c a_k z^c \frac{1}{k^c C!} \\
= S_1 + \sum_{C=0}^{\infty} \sum_{k=2}^{\infty} \left( S_k \sum_{0 \leq j_1 + j_2 < k} \frac{(j_1 x + j_2 y)^c}{k^c} \right) z^c \frac{1}{C!},

(4.3)

where, equating coefficients of like powers of \( z \) gives us the summation formulae

\[
\sum_{k=1}^{\infty} k^2 a_k = S_1 + \sum_{k=2}^{\infty} S_k \sum_{0}^{2} 1,
\]

(4.4)

\[
\sum_{k=1}^{\infty} \sum_{0 \leq A < k \atop 0 \leq B < k \atop A + B \neq 0} (Ax + By)^a_k \frac{1}{k} = \sum_{k=2}^{\infty} \left( S_k \sum_{0}^{2} (j_1 x + j_2 y)^c \right),
\]

(4.5)

\[
\sum_{k=1}^{\infty} \sum_{0 \leq A < k \atop 0 \leq B < k \atop A + B \neq 0} (Ax + By)^2 a_k \frac{1}{k^2} = \sum_{k=2}^{\infty} \left( S_k \sum_{0}^{2} (j_1 x + j_2 y)^2 \right),
\]

(4.6)

and so on. If this process from (4.1) to (4.6) is followed in an analogous manner for the general series (3.1), we have

**Corollary 4.1** For positive integers \( n \),

\[
\sum_{k=1}^{\infty} \sum_{0 \leq A < k \atop 0 \leq B < k \atop A + B \neq 0} (Ax + By)^n a_k \frac{1}{k^n} = \sum_{k=2}^{\infty} \left( S_k \sum_{0}^{2} (j_1 x + j_2 y)^n \right).
\]

(4.7)

In (4.4) the term

\[
\sum_{0}^{2} 1 = \sum_{0}^{2} \left( \frac{j_1 + j_2}{k} \right)^0 = \varphi_0(2; k)
\]

is the number of non-negative integer solutions of \((a, b, k) = 1\) for \( a \) and \( b \) both less than \( k \) with \( a + b \neq 0 \). Suggestive from this, the following is a multidimensional totient function quite distinct from those of section 1.

**Definition 4.2** For all positive integers \( k, m, \) and \( t+1 \), let

\[
\varphi_t(m; k) = \sum_{m \atop k} \left( \frac{j_1 + \cdots + j_m}{k} \right)^t.
\]

(4.8)
If $t$ is zero and $m = 1$ we would have the Euler totient function. If $t$ is zero and $m = 2$ we have again the function $\varphi_0(2; k)$. We shall see soon that $\varphi_0(m; k)$ is the Jordan totient function $J_m(k)$. The Dirichlet series generating functions for $\varphi_t(m; k)$ are found in a similar fashion to our derivation of corollary 4.1, and setting $x = y = 1$. If this analogous process is followed we arrive at the

**Theorem 4.3** For positive integers $t$ and suitably chosen functions $a_i$,

$$\sum_{k=1}^{\infty} k^m a_k = S_1 + \sum_{k=2}^{\infty} S_k \varphi_t(m; k), \quad (4.9)$$

whilst for $t = 0$ we have

$$\sum_{k=1}^{\infty} k^m a_k = S_1 + \sum_{k=2}^{\infty} S_k \varphi_0(m; k). \quad (4.10)$$

Therefore if $a^k = k^{-z}$ we have

**Corollary 4.4**

$$\frac{\zeta(s - m)}{\zeta(s)} = 1 + \sum_{k=2}^{\infty} \frac{\varphi_0(m; k)}{k^s}, \quad R s > 1. \quad (4.11)$$

So it is clear that $\varphi_0(m; k) = J_m(k)$, the Jordan totient function. This function is well known as the number of ordered sets of $k$ elements chosen from a complete residue system (mod $r$) such that the greatest common divisor of each set is prime to $r$. It is also well known (see Sivaramakrishnan [49]), and easily deduced from writing left side of (4.11) as a product over prime $s$, that

$$J_m(k) = k^m \prod_{p|k} (1 - p^{-m}). \quad (4.12)$$

This function is therefore perhaps more simply defined as the number of non-negative integer solutions of $(a_1, a_2, \ldots, a_n, k) = 1$ with all of the $a$’s less than $k$, and $a_1 + a_2 + \cdots + a_n \neq 0$. If in (4.9) we let $a^k = z^k k^{-1}$ then

**Corollary 4.5**

$$\frac{1}{1 - z} \prod_{k=2}^{\infty} \left( \frac{1}{1 - z^k} \right)^{J_m(k)/k} = \exp \left\{ \sum_{k=1}^{\infty} k^{m-1} z^k \right\}, \quad |z| < 1. \quad (4.13)$$

This is new, and related to many results in the author’s papers cited, and also
to a product in Borwein and Borwein [23, ex 10, pp.327]. The right side of (4.13) is easily reduced by using the well known result

$$\sum_{k=0}^{\infty} k^{m-1} z^k = \sum_{j=0}^{m-1} S_{m-1}^{(j)} z^j \frac{1 - z^n}{1 - z} , \quad z \neq 1,$$

where \( S_{m-1}^{(j)} \) are the Stirling numbers of the second kind. Hence we have the

**Theorem 4.6** If \( m \) is a positive integer

$$\frac{1}{1 - z} \prod_{k=2}^{\infty} \left( \frac{1}{1 - z^k} \right)^{I_m(k)/k} = \exp \left\{ \sum_{j=0}^{m-1} S_{m-1}^{(j)} \frac{j! z^j}{(1 - z)^{j+1}} \right\} , \quad |z| < 1. \quad (4.15)$$

The Stirling numbers of the second kind are of importance in combinatorial number theory. The Stirling number \( S(m) \) is the number of ways of partitioning a set of \( m \) elements into \( n \) non-empty sets. In light of (4.13), formula (4.15) corresponds to a limiting case of the hyperpyramid lattice vpv identity first given in Campbell [30],

$$\prod_{(a_1,a_2,\ldots,a_n)=1 \atop a_1,a_2,\ldots,a_n \geq 1} \left( \frac{1}{1 - x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}} \right)^{1 \atop a_1^1 a_2^2 \cdots a_n^n} = \exp \left\{ \sum_{k=1}^{\infty} \prod_{i=1}^{n-1} \left( \sum_{j=1}^{k-1} \frac{x^j}{j^b_j} \right) \frac{x^k}{k^b_k} \right\}$$

where for \( i = 1, 2, \ldots, n; \ |x_i| < 1, \ b_i \) is a complex number with \( \sum_{i=1}^{n} b_i = 1. \)

5 Further multidimensional formulae

In Campbell [8] we proved the

**Theorem 5.1** If \((a_i), (b_i)\) are arbitrary sequences chosen so that, together with choice of \( x \), the following functions are defined then

$$\sum_{k=1}^{n} a_k \frac{1 - \exp(b_k x)}{1 - \exp(b_k x/k)} = \left( \sum_{k=1}^{n} a_k \right) + \sum_{m=2}^{n} \sum_{k=1}^{[n/m]} \sum_{j,m=1 \atop 0<j<k} a_{mk} \exp(b_{mk} j x/k) , \quad (5.1)$$

where \( [n] \) denotes the greatest integer in \( n. \)
This theorem has led to many new results on vpv identities, namely:

a) the so-called “companion identities”, where a group of interrelated identities divide up space radially in a “raying from origin” region. These were often infinite products in the examples given.

b) new results on Dirichlet series. These included generating functions for Ramanujan type trigonometric functions $c_k(n)$.

c) new results involving Jacobi theta functions.

d) new results on Diophantine equations.

In this section we give further results like (5.1) and some corresponding results as a consequence. Theorem 5.1 is a 2-D vpv summation formula, and it was seen in Campbell [24] to [32] that vpv identities can arise in any dimension. This is true in any particular radial region of space with apex at the origin. We now give the 3-D version of theorem 5.1.

**Theorem 5.2** If $(a_i), (b_i), (c_i)$, are each arbitrary sequences chosen so that, together with choice of $x$, the following functions are defined then

\[
\sum_{k=1}^{n} a_k \left( \frac{1 - \exp(b_k x)}{1 - \exp(b_k x/k)} \frac{1 - \exp(c_k x)}{1 - \exp(c_k x/k)} \right) = \left( \sum_{k=1}^{n} a_k \right) + \sum_{k=1}^{[n/2]} \sum_{2} a_{2k} \exp \left( b_{2k} j_1 + c_{2k} j_2 \frac{x}{2} \right) \\
+ \sum_{k=1}^{[n/3]} \sum_{2, 3} a_{3k} \exp \left( b_{3k} j_1 + c_{3k} j_2 \frac{x}{3} \right) \\
+ \sum_{k=1}^{[n/4]} \sum_{2, 4} a_{4k} \exp \left( b_{4k} j_1 + c_{4k} j_2 \frac{x}{4} \right) \\
\vdots \\
+ \sum_{k=1}^{[n/n]} \sum_{2, n} a_{nk} \exp \left( b_{nk} j_1 + c_{nk} j_2 \frac{x}{n} \right) 
\]

where $[n]$ denotes the greatest integer in $n$.

Most of the 3-D vpv infinite products found in Campbell [10] are also corollaries of theorem 5.2. For example, if $n \to \infty, a_k = z^k/k, b_k = (k/x) \log x, c_k = (k/x) \log y$ we have the

**Corollary 5.3** If $a$ and $b$ are non-negative integers and $c$ positive integers such that $(a, b, c) = 1$ then
\[
\prod_{a,b,c} (1 - x^a y^b z^c)^{-1/c} = \left(\frac{(1 - xz)(1 - yz)}{(1 - z)(1 - xyz)}\right)^{1/(n-1)}
\] (5.3)

valid for every one of \(|x|, |xz|, |yz|, |xyz| < 1\).

This identity was given in [10] as one of four “companion identities” dividing up the first hyperquadrant in 3-D space into three radial regions and the union region of these. Of course the simplicity of statement of (5.3) is slightly disguised in the notation of theorem 5.2. Also, the cases of theorem 2.2 with \(m = 2\) are trivially obtained from theorem 5.2. Equation 3.2 is also an easy consequence of (5.2). If in theorem 5.2 we let both \(b_k\) and \(c_k\) approach unity, we get the new result,

Corollary 5.4 If \((a_k)\) is an arbitrary sequence,

\[
\sum_{k=1}^{n} a_k k^2 = \left(\sum_{k=1}^{n} a_k\right) + \left(\sum_{k=1}^{[n/2]} a_{2k} J_2(2)\right) + \left(\sum_{k=1}^{[n/3]} a_{3k} J_2(3)\right) + \cdots + \left(\sum_{k=1}^{[n/n]} a_{nk} J_2(n)\right).
\] (5.4)

An obvious example is

\[
\frac{n(n + 1)(2n + 1)}{6} = n + [n/2]J_2(2) + [n/3]J_2(3) + \cdots + [n/n]J_2(n).
\] (5.5)

It is easy to follow an analogous line of reasoning to that deriving theorem 5.2, then to obtain the analogue to corollary 5.4, which we rate here as a theorem,

Theorem 5.5 If \((a_k)\) is an arbitrary sequence and \(m\) any positive integer,

\[
\sum_{k=1}^{n} a_k k^m = \left(\sum_{k=1}^{n} a_k\right) + J_m(2) \left(\sum_{k=1}^{[n/2]} a_{2k}\right) + J_m(3) \left(\sum_{k=1}^{[n/3]} a_{3k}\right) + \cdots + J_m(n) \left(\sum_{k=1}^{[n/n]} a_{nk}\right).
\] (5.6)

We therefore see that (5.5) is a case of this. The known Dirichlet series generating function for the Jordan totient function is seen to be a limiting case of (5.6) if \(a_k = k^{-(s+m)}\) and the appropriate convergence restrictions are in place. Partial sums of this generating function are seen in
Theorem 5.6 If $m$ and $n$ are positive integers

$$n = \left( \sum_{k=1}^{n} \frac{1}{k^m} \right) + \frac{J_m(2)}{2^m} \left( \sum_{k=1}^{[n/2]} \frac{1}{k^m} \right) + \frac{J_m(3)}{3^m} \left( \sum_{k=1}^{[n/3]} \frac{1}{k^m} \right) + \cdots + \frac{J_m(n)}{n^m} \left( \sum_{k=1}^{[n/n]} \frac{1}{k^m} \right), \quad (5.7)$$

$$\sum_{k=1}^{n} k^a = \left( \sum_{k=1}^{n} \frac{1}{k^{m-a}} \right) + \frac{J_m(2)}{2^{m-a}} \left( \sum_{k=1}^{[n/2]} \frac{1}{k^{m-a}} \right) + \frac{J_m(3)}{3^{m-a}} \left( \sum_{k=1}^{[n/3]} \frac{1}{k^{m-a}} \right) + \cdots + \frac{J_m(n)}{n^{m-a}} \left( \sum_{k=1}^{[n/n]} \frac{1}{k^{m-a}} \right), \quad (5.8)$$

$$\sum_{k=1}^{n} k^m = n + [n/2]J_m(2) + [n/3]J_m(3) + \cdots + [n/n]J_m(n). \quad (5.9)$$

Corollary 5.7 If $m$ and $n$ are positive integers with $|z| < 1$

$$\sum_{k=1}^{n} z^k k^m = \frac{1 - z^n}{1 - z} + \frac{1 - z^{2n/2}}{1 - z^2} + \frac{1 - z^{3n/3}}{1 - z^3} + \cdots + \frac{1 - z^{n[n/n]}}{1 - z^n}. \quad (5.10)$$

Clearly as $z$ approaches unity in this we have (5.9), or if $n$ is increased indefinitely (with $|z| < 1$) we have essentially the logarithmic derivative of (4.13). A very interesting corollary of theorem 5.2 is found from letting $c_k$ approach zero. This results in many simple new identities. It is also interesting to compare (5.10) with (4.13) to (4.15) inferring a relation between the Jordan totient function and the Stirling numbers of the second kind. Another related yet distinct summation formula derived in the vein of the above analysis is

Theorem 5.8 If $(a_k), (b_k)$, are each arbitrary sequences chosen so that, together with choice of $x$, the following functions are defined then

$$\sum_{k=1}^{n} k a_k \frac{1 - \exp(b_k x)}{1 - \exp(b_k x/k)} = \left( \sum_{k=1}^{n} a_k \right) + \left( \sum_{k=1}^{[n/2]} \sum_{2} a_{2k} \exp((b_{2k} j_1) \frac{x}{2}) \right) + \left( \sum_{k=1}^{[n/3]} \sum_{3} a_{3k} \exp((b_{3k} j_1) \frac{x}{3}) \right) + \cdots + \left( \sum_{k=1}^{[n/n]} \sum_{n} a_{nk} \exp((b_{nk} j_1) \frac{x}{n}) \right).$$

This result with $a_k = z^k/k, b_k = k(\log x)/x, n \to \infty$ is
Corollary 5.9 If $|z|, |xz|$ are both $< 1$ and $m \varphi_k$ is the number of solutions in integers of the conditions

$$(a, m, k) = 1, 0 \leq a \leq k, 0 \leq m \leq k, a + m \neq 0, \quad (5.11)$$

for fixed $m$ and $k$, then

$$\prod_{k=1}^{\infty} \left( \frac{1}{1 - x^m z^k} \right)^{m \varphi_k / k} = \exp \left\{ \frac{1}{1 - x} \left( \frac{z}{1 - z} - \frac{xz}{1 - xz} \right) \right\}. \quad (5.12)$$

This identity and similarly derived products in higher dimensions are non-trivial limiting cases of some of the results already given in Campbell [24] to [32]. That is, for example, (5.12) is a limiting case of an identity like (5.3). We may write down the generalized version of theorems 5.1 and 5.2 as follows:

**Theorem 5.10** If $(a_k), (1b_k), (2b_k), \ldots (h b_k)$ are each arbitrary sequences of functions chosen so that, together with choice of $x$, the following series of functions are defined then

$$\sum_{k=1}^{n} \left( a_k \prod_{\lambda=1}^{h} \frac{1 - \exp(\lambda b_k x)}{1 - \exp(\lambda b_k x / k)} \right)$$

$$= \left( \sum_{k=1}^{n} a_k \right) + \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{h=2} a_{2k} \exp((1b_{2k} j_1 + \cdots + h b_{2k} j_h) \frac{x}{2})$$

$$+ \sum_{k=1}^{\lfloor n/3 \rfloor} \sum_{h=3} a_{3k} \exp((1b_{3k} j_1 + \cdots + h b_{3k} j_h) \frac{x}{3})$$

$$+ \sum_{k=1}^{\lfloor n/4 \rfloor} \sum_{h=4} a_{4k} \exp((1b_{4k} j_1 + \cdots + h b_{4k} j_h) \frac{x}{4})$$

$$\vdots$$

$$+ \sum_{k=1}^{\lfloor n/n \rfloor} \sum_{h=n} a_{nk} \exp((1b_{nk} j_1 + \cdots + h b_{nk} j_h) \frac{x}{n}). \quad (5.13)$$

We may now write down the general derivative with respect to $x$ of both sides of this theorem as follows

**Corollary 5.11** If $m$ is a positive integer; and $(a_k), (1b_k), (2b_k), \ldots (h b_k)$ are each arbitrary sequences of functions chosen so that, together with choice of $x$, the following series of functions are all defined then

$$\sum_{k=1}^{n} (a_k h P_m(k))$$
\[= \sum_{k=1}^{[n/2]} a_{2k} \sum_{h=2}^{[n/2]} \left( (1b_{2k,j_1} + \cdots + h b_{2k,j_h}) / 2 \right)^m + \sum_{k=1}^{[n/3]} a_{3k} \sum_{h=3}^{[n/3]} \left( (1b_{3k,j_1} + \cdots + h b_{3k,j_h}) / 3 \right)^m + \sum_{k=1}^{[n/4]} a_{4k} \sum_{h=4}^{[n/4]} \left( (1b_{4k,j_1} + \cdots + h b_{4k,j_h}) / 4 \right)^m + \cdots + \sum_{k=1}^{[n/n]} a_{nk} \sum_{h=n}^{[n/n]} \left( (1b_{nk,j_1} + \cdots + h b_{nk,j_h}) / n \right)^m, \]

where \( hP_m(k) \) is the coefficient of \( x^m \) in

\[\prod_{\lambda=1}^{h} \sum_{\mu=1}^{\infty} T_{\mu}(\lambda b_k x)^{\mu-1}\]

where \( T_{\mu} = -\sum_{\alpha=1}^{\mu} \binom{\mu}{\alpha} B_{\alpha} k^{\alpha-1} \)

with \( B_{\alpha} \) the Bernoulli numbers.

Theorem 5.5 results from theorem 5.10 if \( x \) approaches zero, whilst corollary 5.11 is the result of equating coefficients of powers of \( x \) from expanding the two sides of theorem 5.10 as power series in \( x \). There is no problem about this as both sides of (5.14) are finite sums. The formally complex nature of \( hP_m(k) \) means that the simplest cases of corollary 5.11 are those in which \( h = 2 \) and \( m = 1, 2, 3, \ldots \) successively. Clearly,

\[2P_1(k) = 2T_1T_2(1b_k - 2b_k), \]

\[2P_2(k) = T_1T_3((1b_k)(2b_k)^3) + T_2T_2((1b_k)^2(2b_k)^2) + T_3T_1((1b_k)^3(2b_k)) = T_1T_3((1b_k)(2b_k)((1b_k)^2 + (2b_k)^2) + (T_2)^2(1b_k)^2(2b_k)^2), \]

so applying (5.16) and then (5.17) to corollary 5.11 gives us successively the two following corollaries.

**Corollary 5.12** If \((a_k), (1b_k), (2b_k)\) are each arbitrary sequences of functions chosen so that the following series of functions are all defined then for \( n \) positive integers greater than 1,

\[\frac{1}{3} \sum_{k=1}^{n} \left( \frac{1}{k} a_{k1}b_k \right) = \sum_{k=1}^{[n/2]} \frac{1}{2} a_{2k} \sum_{2}^{[n/2]} (1b_{2k,j_1} + 2 b_{2k,j_2}) + \sum_{k=1}^{[n/3]} \frac{1}{3} a_{3k} \sum_{2}^{[n/3]} (1b_{3k,j_1} + 2 b_{3k,j_2}) \]
\[ + \sum_{k=1}^{\lceil n/4 \rceil} \frac{1}{4} a_{4k} \sum_{j_1,j_2}^{2} (1b_{4k,j_1} + 2b_{4k,j_2}) \]  
\[ : \]
\[ + \sum_{k=1}^{\lceil n/n \rceil} \frac{1}{n} a_{nk} \sum_{j_1,j_2}^{2} (1b_{nk,j_1} + 2b_{nk,j_2}). \]

**Corollary 5.13** For the same conditions as corollary 5.12,

\[ \frac{1}{2} \sum_{k=1}^{n} a_k \left\{ \left( \frac{1}{6} k^2 - \frac{1}{4} k + \frac{1}{12} \right) (1b_k^2 + 2b_k^2) \right\} \left( \frac{1}{4}(k-1)^2 1b_k 2b_k \right) = \]
\[ = \sum_{k=1}^{\lceil n/2 \rceil} \frac{1}{2} a_{2k} \sum_{j_1,j_2}^{2} (1b_{2k,j_1} + 2b_{2k,j_2}) \]
\[ + \sum_{k=1}^{\lceil n/3 \rceil} \frac{1}{3} a_{3k} \sum_{j_1,j_2}^{3} (1b_{3k,j_1} + 2b_{3k,j_2}) \]
\[ + \sum_{k=1}^{\lceil n/4 \rceil} \frac{1}{4} a_{4k} \sum_{j_1,j_2}^{4} (1b_{4k,j_1} + 2b_{4k,j_2}) \]
\[ : \]
\[ + \sum_{k=1}^{\lceil n/n \rceil} \frac{1}{n} a_{nk} \sum_{j_1,j_2}^{n} (1b_{nk,j_1} + 2b_{nk,j_2}). \]

It is clear that cases of our function \( \phi_t(m; k) \) from definition 4.2 are a natural occurrence in the right sides of corollaries 5.12 and 5.13 if \( 1b_k = 1 \) and \( 2b_k = 1 \). When this is the case we have the two corollaries,

**Corollary 5.14** If \( (a_k) \) is an arbitrary sequence of functions chosen so that the following series of functions are all defined, and \( \phi_m(n) \) is the \( m \)th power of the sum of the non-negative integers \( a, b \), less than \( n \) such that \((a, b, n) = 1 \) and \( a + b \neq 0 \), then for positive integers \( n > 1 \),

\[ \sum_{k=1}^{n} a_k(k-1) = \frac{1}{2} \phi_1(2) \sum_{k=1}^{\lceil n/2 \rceil} a_{2k} + \frac{1}{3} \phi_1(3) \sum_{k=1}^{\lceil n/3 \rceil} a_{3k} + \ldots + \frac{1}{n} \phi_1(n) \sum_{k=1}^{\lceil n/n \rceil} a_{nk}, \quad (5.19) \]

\[ \sum_{k=1}^{n} a_k \left( \frac{7}{12} k^2 - k + \frac{5}{12} \right) = \frac{1}{2} \phi_2(2) \sum_{k=1}^{\lceil n/2 \rceil} a_{2k} + \frac{1}{3} \phi_2(3) \sum_{k=1}^{\lceil n/3 \rceil} a_{3k} + \ldots + \frac{1}{n} \phi_2(n) \sum_{k=1}^{\lceil n/n \rceil} a_{nk}, \quad (5.20) \]
We next state some examples using this corollary. The cases are fairly obvious given the previous analysis.

**Corollary 5.15**

\[
\sum_{k=1}^{n} k(k - 1) = \frac{1}{2} \phi_1(2)[n/2] + \frac{1}{3} \phi_1(3)[n/3] + \cdots + \frac{1}{n} \phi_1(n)[n/n],
\]

(5.21)

\[
\sum_{k=1}^{n} \left( \frac{7}{12} k^2 - k + \frac{5}{12} \right) = \frac{1}{2} \phi_2(2)[n/2] + \frac{1}{3} \phi_2(3)[n/3] + \cdots + \frac{1}{n} \phi_2(n)[n/n],
\]

(5.22)

\[
\sum_{k=1}^{n} k(k - 1) = \frac{1}{2} \phi_1(2)[n/2](1+[n/2]) + \frac{1}{3} \phi_1(3)[n/3](1+[n/3]) + \cdots + \frac{1}{n} \phi_1(n)[n/n](1+[n/n]),
\]

(5.23)

\[
\sum_{k=1}^{n} k \left( \frac{7}{12} k^2 - k + \frac{5}{12} \right) = \frac{1}{2} \phi_2(2)[n/2](1+[n/2]) + \frac{1}{3} \phi_2(3)[n/3](1+[n/3]) + \cdots + \frac{1}{n} \phi_2(n)[n/n](1+[n/n]),
\]

(5.24)

These results are akin to those in Campbell [30] and Ramanujan [48]. Furthermore, if we permit \( n \) to increase indefinitely and let \( a_k = k^{-(s+2)} \) in corollaries 5.13 and 5.14 we have the Dirichlet generating functions given by

**Corollary 5.16** If \( \Re s > 1 \) then

\[
\frac{\zeta(s) - \zeta(s + 1)}{\zeta(s + 2)} = \sum_{k=2}^{\infty} \frac{\phi_1(k)}{k^{(s+3)}},
\]

(5.25)

\[
\frac{7\zeta(s) - 12\zeta(s + 1) + 5\zeta(s + 2)}{\zeta(s + 3)} = \sum_{k=2}^{\infty} \frac{\phi_2(k)}{k^{(s+3)}},
\]

(5.26)

Likewise, if \( n \) too increases indefinitely and \( a_k = z^k/k \) in corollaries 5.13 and 5.14 we have the infinite products,
Corollary 5.17 If $|z| < 1$ then

$$\prod_{k=2}^{\infty} (1 - z^k) - \phi_1(k)/k^2 = \exp \left\{ \frac{z}{(1-z)^2} \right\},$$ \hspace{1cm} (5.27)

$$\prod_{k=2}^{\infty} (1 - z^k) - \phi_2(k)/k^2 = \left( \frac{1}{1 - z} \right)^\frac{7}{12} \exp \left\{ \frac{z(12z - 5)}{12(1-z)^2} \right\}.$$ \hspace{1cm} (5.28)

Recalling (4.11) and its ensuing paragraph, it was clear that

$$\frac{\zeta(s-m)}{\zeta(s)} = 1 + \sum_{k=2}^{\infty} \frac{J_m(k)}{k^s}, \quad \Re s > 1.$$ \hspace{1cm} (5.29)

This sum when compared to corollary 5.6 shows that

Corollary 5.18 For positive integers $k$,

$$\phi_1(k) = J_2(k) - J_1(k),$$ \hspace{1cm} (5.30)

$$\phi_2(k) = \frac{7}{12} J_3(k) - J_2(k) + \frac{5}{12} J_1(k).$$ \hspace{1cm} (5.31)

6 Application of Jacobi theta series to the generalized summations.

In Campbell [28], the Jacobi theta function was applied to the finite left hand side form of the summation formula (3.2). We use now the well known terminology for the theta function

$$\Theta_1(z, q) = \Theta_1(z) = 2q^\frac{1}{2} \sum_{k=1}^{\infty} (-1)^k q^{k(k+1)} \sin(2k+1) z,$$ \hspace{1cm} (6.1)

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{q^{2k}}{1 - q^{2k}} \sin 2k \alpha \sin 2k \alpha = \frac{1}{4} \log \left( \frac{\Theta_1((\alpha + \beta), q) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta), q) \sin(\alpha + \beta)} \right).$$ \hspace{1cm} (6.2)

Application of (6.2) to (3.2) gives us the result (see Campbell [28, section 4])
Theorem 6.1 If $|q| < 1$ and $x^{1/k} \neq 1$ then

\[
\exp \left( 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{k} \frac{1-x}{1-x^{1/k}} \sin 2k\alpha \sin 2k\alpha \right)
\]

\[
= \frac{\Theta_1(\alpha + \beta) \sin(\alpha - \beta)}{\Theta_1(\alpha - \beta) \sin(\alpha + \beta)} \prod_{k=1}^{\infty} \left( \frac{\Theta_1((\alpha + \beta)k, q^k) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta)k, q^k) \sin(\alpha + \beta)} \right)^{f_k(x)/k}
\]

where for positive integers $j$ less than $k$,

\[
f_k(x) = \sum_{(j,k)=1} x^{j/k} = \sum_{k=1}^{\infty} x^{j/k}.
\]

Particular cases of this statement include some of the known arithmetical functions, as highlighted in the following examples. To begin with, for positive integers $n$ with $x = \exp(2\pi in)$ we have

Corollary 6.2 (see Campbell [10])

\[
\exp \left( 4 \sum_{k|n} \frac{q^{2k}}{1-q^{2k}} \sin 2k\alpha \sin 2k\alpha \right)
\]

\[
= \frac{\Theta_1(\alpha + \beta) \sin(\alpha - \beta)}{\Theta_1(\alpha - \beta) \sin(\alpha + \beta)} \prod_{k=1}^{\infty} \left( \frac{\Theta_1((\alpha + \beta)k, q^k) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta)k, q^k) \sin(\alpha + \beta)} \right)^{c_k(n)/k}, \quad (6.4)
\]

where $c_k(n)$ is Ramanujan’s trigonometrical function.

If in (6.3) we allow $x$ to approach unity, we have the result,

Corollary 6.3

\[
\exp \left( 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \sin 2k\alpha \sin 2k\alpha \right)
\]

\[
= \frac{\Theta_1(\alpha + \beta) \sin(\alpha - \beta)}{\Theta_1(\alpha - \beta) \sin(\alpha + \beta)} \prod_{k=1}^{\infty} \left( \frac{\Theta_1((\alpha + \beta)k, q^k) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta)k, q^k) \sin(\alpha + \beta)} \right)^{\varphi(n)/k}, \quad (6.5)
\]

where $\varphi(n)$ is the Euler totient function.

If we follow an analogous line of reasoning to that which led us to (6.3) but from starting with lemma 3.1, we obtain the new result,
Theorem 6.4 If $|q| < 1$ and all of $x_1^{1/k}, x_2^{1/k}, \ldots, x_m^{1/k} \neq 1$ then

$$\exp \left( 4 \sum_{k=1}^{\infty} \sum_{m=1}^{k} \frac{q^{2k}}{1 - q^{2k}} \frac{1 - x_1^{1/k}}{1 - x_1^{1/k}} \frac{1 - x_2^{1/k}}{1 - x_2^{1/k}} \ldots \frac{1 - x_m^{1/k}}{1 - x_m^{1/k}} \sin 2k\alpha \sin 2k\alpha \right)$$

$$= \Theta_1(\alpha + \beta) \sin(\alpha - \beta) \sin(\alpha + \beta) \prod_{k=1}^{\infty} \left( \frac{\Theta_1((\alpha + \beta)k, q^k) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta)k, q^k) \sin(\alpha + \beta)} \right)^{m f_k(x)/k} \tag{6.6}$$

where

$$m f_k(x) = \sum_{m=1}^{k} \left( x_1^j x_2^j \ldots x_m^j \right)^{1/k}. \tag{6.7}$$

We state two relevant corollaries with respect to the substance of the current work. Firstly, to obtain the corresponding result for the Jordan totient function, we need only allow all of the $x$’s to approach unity, bearing in mind our work in section 4. We therefore have

**Corollary 6.5**

$$\exp \left( 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} \sin 2k\alpha \sin 2k\alpha \right)$$

$$= \Theta_1(\alpha + \beta) \sin(\alpha - \beta) \sin(\alpha + \beta) \prod_{k=1}^{\infty} \left( \frac{\Theta_1((\alpha + \beta)k, q^k) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta)k, q^k) \sin(\alpha + \beta)} \right)^{J_m(k)/k}, \tag{6.8}$$

where $J_m(k)$ is the Jordan totient function.

Secondly, to obtain the corresponding result for our new function, $c_k(n_1, n_2, \ldots, n_m)$ we need only allow all of the $x_i$’s to approach $\exp(2\pi i n_i)$, bearing in mind our work in section 1. We therefore have

**Corollary 6.6**

$$\exp \left( 4 \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} \sin 2k\alpha \sin 2k\alpha \right)$$

$$= \Theta_1(\alpha + \beta) \sin(\alpha - \beta) \sin(\alpha + \beta) \prod_{k=1}^{\infty} \left( \frac{\Theta_1((\alpha + \beta)k, q^k) \sin(\alpha - \beta)}{\Theta_1((\alpha - \beta)k, q^k) \sin(\alpha + \beta)} \right)^{c_k(n_1, n_2, \ldots, n_m)/k}. \tag{6.9}$$

where $c_k(n_1, n_2, \ldots, n_m)$ is our new generalized Ramanujan totient function.

We have not attempted to use the full generality of corollary 3.2, and it is clear that we may return to this topic in future papers. The return to such
investigations will no doubt be hastened if it turns out that the identities
given in this paper can be shown to enumerate tilings in the sense described
by Baake et al [16] and his colleagues pertaining to quasicrystals.

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