FOURIER BASES AND A DISTANCE PROBLEM OF ERDŐS

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Abstract. We prove that no ball admits a non-harmonic orthogonal basis of exponentials. We use a combinatorial result, originally studied by Erdős, which says that the number of distances determined by \( n \) points in \( \mathbb{R}^d \) is at least \( C_d n^{\frac{1}{d}} + \epsilon_d, \epsilon_d > 0 \).

Introduction and statement of results

Fourier bases. Let \( D \) be a domain in \( \mathbb{R}^d \), i.e., \( D \) is a Lebesgue measurable subset of \( \mathbb{R}^d \) with finite non-zero Lebesgue measure. We say that \( D \) is a spectral set if \( L^2(D) \) has orthogonal basis of the form \( E_\Lambda = \{ e^{2\pi ix \cdot \lambda} \}_{\lambda \in \Lambda} \), where \( \Lambda \) is an infinite subset of \( \mathbb{R}^d \). We shall refer to \( \Lambda \) as a spectrum for \( D \).

We say that a family \( D + t, t \in T \), of translates of a domain \( D \) tiles \( \mathbb{R}^d \) if \( \cup_{t \in T} (D + t) \) is a partition of \( \mathbb{R}^d \) up to sets of Lebesgue measure zero.

Conjecture. It has been conjectured (see [Fug]) that a domain \( D \) is a spectral set if and only if it is possible to tile \( \mathbb{R}^d \) by a family of translates of \( D \).

This conjecture is nowhere near resolution, even in dimension one. It has been the subject of recent research, see for example [JoPe2], [LaWa], and [Ped].

In this paper we address the following special case of the conjecture. Let \( B_d = \{ x \in \mathbb{R}^d : |x| \leq 1 \} \) denote the unit ball. We prove that

Theorem 1. An affine image of \( D = B_d, d \geq 2 \), is not a spectral set.

If \( A \) is a (possibly unbounded) self-adjoint operator acting on some Hilbert space, then we may define \( \exp(-\sqrt{-1}T A) \) using the Spectral Theorem. We say that two (unbounded) self-adjoint operators \( A \) and \( B \) acting on the same Hilbert space commute if the bounded unitary operators \( \exp(-\sqrt{-1}tA) \) and \( \exp(-\sqrt{-1}tB) \) commute for all real numbers \( s \) and \( t \). See, for example, [ReSi] for more details on the needed operator theory. As an immediate consequence of [Fug] and Theorem 1 we have:

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Corollary. There do not exist commuting self-adjoint operators $H_j$ acting on $L^2(B_d)$ such that $H_j f = -\sqrt{-1} \partial f / \partial x_j$ for $f$ in the domain of the unbounded operator $H_j$ and $1 \leq j \leq d$. The derivatives $\partial / \partial x_j$ act on $L^2(B_d)$ in the distribution sense.

In other words, there do not exist commuting self-adjoint restrictions of the partial derivative operators $-\sqrt{-1} \partial / \partial x_j$, $j = 1, \ldots, d$, acting on $L^2(B_d)$ in the distribution sense.

The two-dimensional case of Theorem 1 was proved by Fuglede in [Fug]. Our proof uses the following combinatorial result. See for example [AgPa], Theorem 12.13.

Theorem 2. Let $g_d(n), d \geq 2$, denote the minimum number of distances determined by $n$ points in $\mathbb{R}^d$. Then

\begin{equation}
(*) \quad g_d(n) \geq C_d n^{\frac{2}{d-2}}.
\end{equation}

Remark. The study of the problem addressed in Theorem 2 was initiated by Erdős. He proved that $g_2(n) \geq C n^{\frac{1}{2}}$. See [Erd]. Moser proved in [Mos] that $g_2(n) \geq C n^{\frac{3}{4}}$. More recently, Chung, Szeremedi, and Trotter proved that $g_2(n) \geq C \frac{n^{\frac{3}{4}}}{\log^c(n)}$ for some $c > 0$. See [CST]. Theorem 2 above is proved by induction using the $g_2(n) \geq C n^{\frac{3}{4}}$ result proved by Clarkson et al. in [C].

As the reader shall see, Theorem 1 does not require the full strength of Theorem 2. We just need the fact $g_d(n) \geq C_d n^{\frac{1}{2}+\epsilon}$, for some $\epsilon > 0$.

It is interesting to contrast the case of the ball with the case of the cube $[0,1]^d$. It was proved in [IoPe1], (and, independently, in [LRW]; for $d \leq 3$ this was established in [JoPe2]), that $\Lambda$ is a spectrum for $[0,1]^d$, in the sense defined above, if and only if $\Lambda$ is a tiling set for $[0,1]^d$, in the sense that $[0,1]^d + \Lambda = \mathbb{R}^d$ without overlaps. It follows that $[0,1]^d$ has lots of spectra. The standard integer lattice $\Lambda = \mathbb{Z}^d$ is an example, though there are many non-trivial examples as well. See [IoPe1] and [LaSh].

Our method of proof is as follows. We shall argue that if $B_d$ were a spectral set, then any corresponding spectrum $\Lambda$ would have the property $\# \{ \Lambda \cap B_d(R) \} \approx R^d$, where $B_d(R)$ denotes a ball of radius $R$ and $f(R) \approx g(R)$ means that there exist constants $c \leq C$ so that $c f(R) \leq g(R) \leq C f(R)$ for $R$ sufficiently large. On the other hand, we will show that the number of distinct distances between the elements of $\{ \Lambda \cap B_d(R) \}$ is $\approx R$. Theorem 2 implies that if $R$ is sufficiently large, this is not possible.

Kolountzakis ([Kol]) recently proved that if $D$ is any convex non-symmetric domain in $\mathbb{R}^d$, then $D$ is not a spectral set. Theorem 1 is a step in the direction of proving that if $D$ is a convex domain such that $\partial D$ has at least one point where the Gaussian curvature does not vanish, then $D$ is not a spectral set. This, in its turn, would be a step towards proving the conjecture of Fuglede mentioned above.

Orthogonality

For a domain $D$ let

$Z_D = \{ \xi \in \mathbb{R} : \widehat{\chi}_D(\xi) = 0 \}.$


Consider a set of exponentials $E_\Lambda$. Observe that
\[
\hat{\chi}_D(\lambda - \lambda') = \int_D e_\lambda(x)\overline{e_{\lambda'}(x)} \, dx.
\]
It follows that the exponentials $E_\Lambda$ are orthogonal in $L^2(D)$ iff
\[
\Lambda - \Lambda \subseteq Z_D \cup \{0\}.
\]

**Proposition 1.** If $E_\Lambda$ is an orthogonal subset of $L^2(D)$ then there exists a constant $C$ depending only on $D$ such that
\[
\#(\Lambda \cap B_d(R)) \leq C R^d
\]
for any ball $B_d(R)$ of radius $R$ in $\mathbb{R}^d$.

**Proof.** Since $\hat{\chi}_D$ is continuous and $\hat{\chi}_D(0) = |D|$ it follows that
\[
\inf\{||\xi|| : \hat{\chi}_D(\xi) = 0 \} = r > 0.
\]
If $\xi_1, \ldots, \xi_n$ are in $\Lambda \cap B_d(R)$ then the balls $B(\xi_j, r/2)$ are disjoint and contained in $B_d(R + r/2)$. Since $r$ only depends on $D$ the desired inequality follows.

To study the exact possibilities for sets $\Lambda$ so that $E_\Lambda$ is orthogonal it is of interest to us to compute the set $Z_D$. We will without loss of generality assume that $0 \in \Lambda$. We again compare the sets $Z_D$ for the cases where $D$ is the cube and the ball.

Let $Q_d = [0, 1]^d$ be the cube in $\mathbb{R}^d$. The zero set $Z_Q$ for $\hat{\chi}_Q$ is the union of the hyperplanes $\{x \in \mathbb{R}^d : x_i = z\}$, where the union is taken over $1 \leq i \leq d$, and over all non-zero integers $z$.

Let $B_d = \{x \in \mathbb{R}^d : ||x|| \leq 1\}$ be the unit ball in $\mathbb{R}^d$. The zero set $Z_{B_d}$ for $\hat{\chi}_B$ is the union of the spheres $\{x \in \mathbb{R}^d : ||x|| = r\}$, where the union is over all the positive roots $r$ of an appropriate Bessel function.

For the cube $Q_d$ it is easy to find a large set $\Lambda \subseteq Z_{Q_d} \cup \{0\}$ so that $\Lambda - \Lambda \subseteq Z_{Q_d} \cup \{0\}$. For example, we may take $\Lambda = Z^d$. In the case of the ball $B_d$, we will show that only relatively small sets $\Lambda \subseteq Z_{B_d} \cup \{0\}$ satisfy $\Lambda - \Lambda \subseteq Z_{B_d} \cup \{0\}$.

**Proof of Theorem 1**

We shall need the following result.

**Theorem 3.** Suppose that $D$ is a spectral set and that $\Lambda$ is a spectrum for $D$ in the sense defined above, where $D$ is a bounded domain. There exists an $r > 0$ so that any ball of radius $r$ contains at least one point from $\Lambda$.

**Proof.** This is a special case of [IoPe2]. See also [Beu], [Lan], and [GrRa].

It is a consequence of Theorem 3 that if $D$ is a spectral set then there exists a constant $C > 0$ such that if $\Lambda$ is a spectrum for $D$ then $\#(\Lambda \cap B_d(R)) \geq C R^d$ for any ball $B_d(R)$.
of radius $R$ provided that $R$ is sufficiently large. Combining this with Proposition 1 we see that $\# \{ \Lambda \cap B_d(R) \} \approx R^d$.

Suppose $\Lambda$ is a spectrum for the unit ball $B_d$ centered at the origin in $\mathbb{R}^d$. Let $B_d(R)$ be a ball of radius $R$. Since $\# \{ \Lambda \cap B_d(R) \} \approx R^d$ it follows from Theorem 2 that

$$\# \{ |\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \cap B_d(R) \} \geq C R^{\frac{3d}{3d-2}}.$$ (**) 

Now, since $\hat{\chi}_{B_d}$ is an analytic radial function, it follows that if $f$ is given by $f(|\xi|) = \hat{\chi}_{B_d}(\xi)$, then the number of zeros of $f$ in the interval $[-R, R]$ is bounded above by a multiple of $R$. In fact an explicit calculation shows that $\hat{\chi}_{B_d}(\xi) = |\xi|^\frac{d}{2} J_d(2\pi|\xi|)$, where $J_\nu$ denotes the usual Bessel function of order $\nu$. See, for example, [BCT, p. 265].

If $\lambda, \lambda' \in \Lambda$ then

$$f(|\lambda - \lambda'|) = \hat{\chi}_{B_d}(\lambda - \lambda') = 0.$$

Combining the upper bound on the number of zeros of $f$ in $[-R, R]$ with the lower bound (***) we derived from Theorem 2 above we have

$$C' R \geq \# \{ |\lambda - \lambda'| : \lambda, \lambda' \in \Lambda \cap B_d(R) \} \geq C R^{\frac{3d}{3d-2}}.$$

Since $1 < \frac{3d}{3d-2}$ this leads to a contradiction by choosing $R$ sufficiently large. This completes the proof of Theorem 1.

References

[AgPa] P. Agarwal and J. Pach, Combinatorial Geometry, Wiley-Interscience Series, 1995.
[Beu] A. Beurling, Local harmonic analysis with some applications to differential operators, Some Recent Advances in the Basic Sciences, Academic Press 1 (1966).
[BCT] L. Brandolini, L. Colzani, and G. Travaglini, Average decay of Fourier transforms and integer points in polyhedra, Ark. Mat. 35 (1997), 253-275.
[CST] F. Chung, E. Szeremedi, and W. Trotter, The number of distinct distances determined by a set of points in the Euclidean plane, Discrete and Computational Geometry 7 (1992).
[C] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for for arrangements of curves and surfaces, Discrete and Computational Geometry 5 (1990).
[Erd] P. Erdős, On sets of distances of $n$ points, American Mathematical Monthly 53 (1946).
[Fug] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. 16 (1974), 101-121.
[GrRa] K. Gröchenig and H. Razafinjatovo, On Landau’s necessary density conditions for sampling and interpolation of band-limited functions, J. London. Math. Soc. 54 (1996), 557-565.
A. Iosevich and S. Pedersen, *Spectral and tiling properties of the unit cube*, Internat. Math Reseach Notices 16 (1998), 819-828.

A. Iosevich and S. Pedersen, *How large are the spectral gaps?*, Pacific J. Math. (to appear) (1998).

P. E. T. Jorgensen and S. Pedersen, *Spectral pairs in Cartesian coordinates*, J. Fourier Anal. Appl. (to appear) (1998).

P. E. T. Jorgensen and S. Pedersen, *Orthogonal harmonic analysis of fractal measures*, ERA Amer. Math. Soc. 4 (1998), 35-42.

M. Kolountzakis, *Non-symmetric convex domains have no basis of exponentials*, (preprint) (1999).

J. Lagarias, J. Reed, and Y. Wang, *Orthonormal bases of exponentials for the n-cube*, (preprint) (1998).

J. Lagarias and P. Shor, *Keller’s cube tiling conjecture is false in high dimensions*, Bull. Amer. Math. Soc. 27 (1992), 279-283.

J. Lagarias and Y. Wang, *Spectral sets and factorizations of finite abelian groups*, J. Funct. Anal. 145 (1997), 73-98.

H. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. 117 (1967), 37-52.

H. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics (1994).

L. Moser, *On different distances determined by n points*, American Mathematical Monthly 59 (1952).

S. Pedersen, *Spectral sets whose spectrum is a lattice with a base*, J. Funct. Anal. 141 (1992), 496-509.

M. Reed and B. Simon, *Modern Mathematical Physics, Vol. I: Functional Analysis*, Academic Press, 1972.

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