A perfect Morse function on the moduli space of flat connections

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Let $X$ be a compact surface of genus $g$, and let $M_g$ be the moduli space of flat SU(2) connections on $X$ having holonomy $-I$ around a single puncture $p$. Let $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ be the usual generators for $\pi_1(X \setminus p)$, and define a real-valued function on $M_g$ by assigning to a flat connection the trace of its holonomy around $a_g$. This paper will give an elementary, direct proof that this function is a perfect Bott-Morse function, that is, one whose Morse inequalities are equalities. This leads to a new derivation of the well-known Harder-Narasimhan formula for the Betti numbers of the moduli space.

The Harder-Narasimhan formula has previously been calculated by several methods. Most of them involved the deep theorem of Narasimhan and Seshadri [23] identifying flat connections with stable holomorphic bundles. For example, Harder and Narasimhan [14] applied the Weil conjectures, counting algebraic vector bundles on curves over finite fields to deduce the result. Another approach, due to Atiyah and Bott [1], used two-dimensional Yang-Mills theory to construct $M_g$ as the symplectic quotient of an infinite-dimensional affine space by the gauge group.

More recently, the Bott-Morse function discussed here was studied by Jeffrey and Weitsman. It is straightforward to determine the critical points of this function (cf. [1.5]), and hence to compute one side of the Morse inequalities. Jeffrey and Weitsman noted that the expression thus obtained equals the Harder-Narasimhan formula. If the latter is regarded as known, this constitutes an a posteriori proof that the function is perfect.

This paper will give an a priori proof that the function is perfect, using only relatively elementary facts from finite-dimensional symplectic geometry. It therefore provides a down-to-earth proof of the Harder-Narasimhan formula, avoiding high technology such as the Weil conjectures or Yang-Mills theory. It is much closer in spirit to the original computation of the Betti numbers, due to Newstead [13, 24].

A few years ago, Mark Hoyle pointed out to the author that the a posteriori argument remains valid for the space of flat SU(2) connections whose holonomy around $p$ lies in any fixed conjugacy class (except that of $I$ when there are reducibles). In this case the Betti numbers are easily determined from the Harder-Narasimhan formula. The present a priori argument extends without change to this space. Even better, it applies more generally to the space of flat SU(2) connections on $X$ with any finite number of punctures, provided only that the conjugacy classes of the holonomies around the punctures are fixed so that there are no reducibles. Meanwhile Hoyle has obtained formulas for the Betti numbers in this case too and shown that the a posteriori argument extends [13]. The results of this paper therefore provide an independent proof of his formulas.

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A natural question for further investigation is how much can be said for other Lie groups than SU(2). One could replace the trace with any real-valued smooth class function. The proof of the main theorem does not carry over in full, but it might imply some partial results. Furthermore, the a posteriori question should be tractable.

The organization of the paper is simple. Section 1 reviews what little background material is needed: the construction of the moduli space as a space of representations of $\pi_1(X)$, and some results of Goldman and Frankel. The main theorem is stated, and the Poincaré polynomial of the moduli space is derived. Section 2 describes the spectral sequence, canonically associated to a Bott-Morse function $f: M \to \mathbb{R}$, which abuts to $H^*(M, \mathbb{Q})$. Section 4 is devoted to the proof of the main theorem. The basic strategy is to prove that all differentials vanish in the spectral sequence by combining the perfection of moment maps with a parity argument. Section 4 explains how to generalize the main theorem to the case of a surface with many punctures, recovering the formulas of Hoyle. Finally, §5 explains why the methods of §§2 and 3 immediately show that the integral cohomology is torsion-free, and yield information about the U(2) case as well.

Unless otherwise specified, all cohomology is with rational coefficients.

This work was announced in lectures given in Odense and Barcelona [26]. I wish to apologize for the delay in publication.

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1 The moduli space of flat SU(2) connections

The moduli space $M_g$ which interests us can be defined simply as follows. Let $\mu_g: \text{SU}(2)^{2g} \to \text{SU}(2)$ be given by $(A_1, B_1, \ldots, A_g, B_g) \mapsto -\prod_i [A_i, B_i]$. Then $I$ is a regular value of $\mu_g$. (This was first shown by Igusa [16]; it is also a special case of Proposition 4.1 below.) Hence $\mu_g^{-1}(I)$ is smooth; moreover $\text{SU}(2)/\pm I$ acts freely on it by conjugation, so $M_g = \mu_g^{-1}(I)/\text{SU}(2)$ is a smooth compact $6g-6$-dimensional manifold.

At any point $\rho \in M_g$, the tangent space is naturally isomorphic to the first cohomology of the complex

$$\text{su}(2) \to \text{su}(2)^{2g} \to \text{su}(2),$$

where the former map is the derivative of conjugation at $(A_i, B_i)$, and the latter is the derivative of $\mu_g$. Choose any representative for $\rho$ in $\mu_g^{-1}(I)$ and, by abuse of notation, denote it again by $\rho$. Then $\text{ad} \rho$ defines a representation of $\pi_1(X)$ on $\text{su}(2)$, and the cohomology mentioned above is none other than the group cohomology $H^1(\pi_1(X), \text{ad} \rho)$.

Since $X$ is an Eilenberg-Mac Lane space, $H^2(\pi_1(X), \mathbb{R}) = H^2(X, \mathbb{R}) = \mathbb{R}$. Combining the cup product with the symmetric form $\langle a, b \rangle = -\frac{1}{2} \text{tr} ab$ gives a non-degenerate antisymmetric map

$$H^1(\text{ad} \rho) \otimes H^1(\text{ad} \rho) \xrightarrow{\cup} H^2(\text{ad} \rho \otimes \text{ad} \rho) \xrightarrow{\langle , \rangle} H^2(\mathbb{R}) = \mathbb{R},$$

which determines a non-degenerate 2-form on $M_g$.  

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**Theorem (Goldman).** This 2-form is closed.

Thus $M_g$ becomes a symplectic manifold. Goldman’s original proof of this theorem used infinite-dimensional quotients in the style of Atiyah and Bott. Since then, purely algebraic, finite-dimensional proofs have been provided by Karshon [19] and Weinstein [27]; see also Guruprasad et al. [12].

Now let $f : M_g \to [-1, 1]$ be given by $(A_i, B_i) \mapsto \frac{1}{2} \text{tr} A_g$, which is well-defined since the trace is conjugation-invariant. Then $U(1)$ acts on $f^{-1}(-1, 1)$ as follows. If $A_g \neq \pm I$, then there is a unique homomorphism $\phi : U(1) \to SU(2)$ such that $A_g \in \phi(\{\text{Im} z > 0\})$. Let $\lambda \cdot (A_i, B_i) = (A_1, B_1, \ldots, A_g, B_g \cdot \phi(\lambda))$.

**Proposition (Goldman).** This action preserves the symplectic form, and it has moment map $-i \arccos f$.

See Kirwan [20] for the definition of a moment map, and Goldman [10] and Jeffrey-Weitsman [17] for a proof.

If the $U(1)$-action were global on $M_g$, the following result [8] would immediately imply that $\arccos f$ was perfect.

**Theorem (Frankel).** The moment map of a $U(1)$-action on a compact symplectic manifold (times $i$) is a perfect Bott-Morse function.

In the present case, however, the $U(1)$-action on $f^{-1}(-1, 1)$ does not extend over $f^{-1}(\pm 1)$, and $\arccos f$ is not even differentiable there. Nevertheless, the following theorem is true, and will be proved in §3.

**Main Theorem.** The map $f$ is a perfect Bott-Morse function on $M_g$.

To see what this means concretely, let us identify the critical submanifolds and their indices, assuming for the moment that $f$ is a Bott-Morse function.

**The critical submanifolds of $f$.** First, let $S_1 = f^{-1}(-1)$. As the absolute minimum of $f$, $S_1$ is of course a critical submanifold. It is exactly the locus where $A_g = -I$; hence $B_g$ may be arbitrary, and the product of the first $g-1$ commutators must be $-I$, so $S_1 = (\mu_g^{-1}(I) \times SU(2))/SU(2)$. The natural projection $\pi : S_1 \to M_{g-1}$ makes $S_1$ into an $SU(2)$-bundle over $M_{g-1}$. This is an adjoint, not a principal bundle, so it may have a section without being trivial. Indeed, $B_g = I$ determines such a section; hence the Euler class vanishes and so by the Gysin sequence $H^*(S_1) = V \oplus \tau V$, where $V = \pi^*H^*(M_{g-1})$ and $\tau \in H^3(S_1)$ is the Poincaré dual of the locus where $B_g = I$. As the absolute minimum, $S_1$ of course has index 0.

Exactly the same is true of $S_3 = f^{-1}(1)$, except that it is the absolute maximum of $f$, the locus where $A_g = I$, and hence has index equal to its codimension, which is 3.

Within $f^{-1}(-1, 1)$, on the other hand, the critical points of $f$ coincide with those of $\arccos f$, the moment map for the $U(1)$-action. They are therefore exactly the fixed points of that action, and hence are represented by $2g$-tuples $(A_i, B_i) \in SU(2)^{2g}$ that are conjugate
to \((A_1, B_1, \ldots, A_g, B_g \cdot \phi(\lambda))\) for all \(\lambda \in U(1)\). It is straightforward to check that these are all conjugate to \(2g\)-tuples such that

\[
A_g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B_g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

and the remaining \(A_i\) and \(B_i\) are diagonal. Hence the only other critical value is \(\frac{1}{2} \text{tr} A_g\), which is 0, and the corresponding critical set \(S_2\) is a \(2g - 2\)-torus. Because there is an involution \(A_g \mapsto -A_g\) on \(M_g\) changing the sign of \(f\), the index of \(S_2\) must be half the rank of its normal bundle, or \(2g - 2\).

Incidentally, \(S_1\) and \(S_3\) are empty when \(g = 1\), and everything is empty when \(g = 0\), but these special cases will not affect our arguments.

The main theorem therefore implies the so-called Harder-Narasimhan formula.

(1.6) Corollary. The Poincaré polynomial of \(M_g\) is

\[
P_t(M_g) = \frac{(1 + t^3)^{2g} - t^{2g}(1 + t)^{2g}}{(1 - t^2)(1 - t^4)}.
\]

Proof. Since \(M_0\) is empty, \(P_t(M_0) = 0\) as desired. It follows from the theorem and the discussion above that

\[
P_t(M_g) = P_t(S_1) + t^3 P_t(S_3) + t^{2g-2} P_t(S_2)
\]

\[
= (1 + t^3)^2 P_t(M_{g-1}) + t^{2g-2}(1 + t)^{2g-2}.
\]

This gives a recursion for \(P_t(M_g)\), which is satisfied by the formula. ✠

2 The spectral sequence of a Bott-Morse function

Let \(M\) be a compact manifold and \(f : M \to \mathbb{R}\) a Bott-Morse function, that is, a smooth function whose critical set is a disjoint union of submanifolds on whose normal bundle the Hessian is nondegenerate. For \(f\) to be perfect is clearly equivalent to the vanishing of all differentials in the spectral sequence described below.

The set of critical values of \(f\) is finite, so the critical set of \(f\) is a disjoint union of submanifolds \(S_1, S_2, \ldots, S_n \subset M\) such that \(f\) is constant on each \(S_i\) and \(f(S_i) < f(S_j)\) for \(i < j\). Choose \(x_j \in \mathbb{R}\) satisfying \(x_0 < f(S_1) < x_1 < f(S_2) < x_2 < \cdots < f(S_n) < x_n\) and let \(U_j = f^{-1}(x_0, x_j)\), so that \(M\) is filtered by open sets \(\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = M\). The complex of singular chains on \(M\) is then filtered by the support of the chain:

\[
0 = C_*(U_0) \subset C_*(U_1) \subset \cdots \subset C_*(U_n) = C_*(M).
\]

Moreover, this filtration is preserved by the differential.

Taking duals yields a cofiltration on the group of cochains, which is also preserved by the differential. This is exactly the raw material needed to construct a spectral sequence,
and the whole machine runs smoothly. Except that everything is dualized because of the cofiltration, it is just as described by Bott & Tu [7]. As in their equation (14.3), there is a short exact sequence

\[ 0 \to B \to A \longrightarrow A \to 0 \]

where \( A = \bigoplus_j C^*(U_j) \) and \( \pi : C^*(U_j) \to C^*(U_{j-1}) \) is the restriction of cochains. This leads to an exact couple

\[ \bigoplus H^*(U_j) \longrightarrow \bigoplus H^*(U_j) \]

\[ \bigoplus H^*(U_j, U_{j-1}) \]

whose derived couples abut to \( H^*(M) \). On the other hand, the Morse lemma implies that up to homotopy, \( U_j \) is a CW complex obtained from \( U_{j-1} \) by attaching, along its boundary, the disc bundle \( E_j \) associated to the negative normal bundle consisting of the negative definite subspaces of the Hessian of \( f \) on \( S_j \). So by excision, \( H^*(U_j, U_{j-1}) \cong H^*(E_j, \partial E_j) \).

On the other hand, suppose that \( S_j \) and \( E_j \) are orientable for each \( j \), and choose orientations. Then the cup product with the Thom class induces the Thom isomorphism \( H^*(E_j, \partial E_j) \cong H^*(S_j) \). The spectral sequence then can be regarded as a sequence of differentials \( d_1, d_2, \ldots \) where \( d_1 : \bigoplus H^*(S_j) \to \bigoplus H^*(S_j) \) and \( d_{i+1} : H^*(d_i) \to H^*(d_i) \).

By the definition of the differentials of an exact couple, the first differential \( d_1 \) is the direct sum of maps \( H^*(S_j) \to H^*(S_{j+1}) \) induced by the upper part of the following diagram.

\[
\begin{array}{ccccccccc}
H^*(S_j) & \to & H^*(U_j, U_{j-1}) & \to & H^*(U_j) & \to & H^*(U_{j+1}, U_j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^*(U_{j+1}, U_{j+1}) & \to & H^*(U_{j+1}) & \to & H^*(S_{j+2})
\end{array}
\]

Here all the maps are induced by inclusion, except the diagonal arrows, which are Thom isomorphisms.

An element of \( H^*(S_j) \) is thus in the kernel of the first differential if and only if it maps to \( 0 \in H^*(U_{j+1}, U_j) \). By exactness of the middle row, it comes from some element of \( H^*(U_{j+1}) \), and its image in \( H^*(S_{j+2}) \) is the value of the second differential.

The higher differentials can now be described in the same manner, applying the argument of the previous paragraph inductively.

The contents of this section go back to Bott [5].

### 3 Proof of the main theorem

Let \( M_g \) and \( f : M_g \to \mathbb{R} \) be as in §5. To prove the main theorem, we will show first that \( f \) is a Bott-Morse function, then that it is perfect. The former task is accomplished with three lemmas; the latter will occupy the remainder of the section.
Lemma. For $g \geq 2$, the map $\mu_g^{-1}(I) \to SU(2) \times SU(2)$ given by $(A_i, B_i) \mapsto (A_g, B_g)$ is a submersion at the locus where $A_g = \pm I$.

Proof. This means that the infinitesimal map $\ker D\mu_g \to su(2) \times su(2)$ induced by projection of $SU(2)^2$ on the last two factors is surjective when $A_g = \pm I$. Direct computation shows that in this case,

$$D\mu_g(a_1, \ldots, b_g) = D\mu_{g-1}(a_1, \ldots, b_{g-1}) + D\mu_1(a_g, b_g).$$

But as mentioned before, $I$ is a regular value of $\mu_{g-1}$. Hence $D\mu_{g-1}$ is surjective, so there exist $(a_i, b_i) \in \ker D\mu_g$ having any desired values for $a_g$ and $b_g$.

Lemma. There is an $SU(2)$-equivariant diffeomorphism between a neighborhood of $S_1 \subset \mu_g^{-1}(I)$ and a neighborhood of

$$\mu_g^{-1}(I) \times \{-I\} \times SU(2) \subset \mu_{g-1}^{-1}(I) \times SU(2) \times SU(2),$$

identifying the map of the previous lemma with the projection on the last two factors. Likewise for $S_3$ if $\{I\}$ is substituted for $\{-I\}$.

Proof. This follows immediately from the previous lemma using the equivariant version of the tubular neighborhood theorem for submanifolds [2, I Thm. 2.1.1] and the inverse function theorem.

Lemma. The function $f$ is a Bott-Morse function.

Proof. By definition, we need to show that the Hessian on the normal bundle to every critical submanifold is nondegenerate. By Frankel's result, Theorem [3], $\arccos f$ is a Bott-Morse function on $f^{-1}(-1, 1)$, and hence so is $f$. Therefore the Hessian of $S_2$ is nondegenerate. In a neighborhood of $S_1$, $f$ is locally the composite with the diffeomorphism from Lemma 3.2 of the trace map on the first $SU(2)$ factor. But if $SU(2)$ is identified with the 3-sphere of unit quaternions, then the trace map is simply a linear projection, whose Hessian at $\pm I$ is certainly nondegenerate. Hence the Hessian of $f$ is nondegenerate on $S_1$. The case of $S_3$ is similar.

The symplectic cut $\hat{M}_g$. It remains to show that $f$ is perfect. As a first step towards this goal, we construct a slight modification of $M_g$ on which the U(1)-action becomes global and hence the moment map is perfect. Take the 2-sphere $S^2$ with the standard U(1)-action given by rotation, and normalize the moment map so that its image is $[-\frac{2}{3}, \frac{2}{3}]$. Then let $\hat{M}_g$ be the symplectic quotient of $f^{-1}(-1, 1) \times S^2$ by the diagonal U(1)-action. This is a simple example of a symplectic cut of $M_g$, as introduced by Lerman [21]. As a set, it is just $f^{-1}[-\frac{2}{3}, \frac{2}{3}]$ with the U(1)-orbits in $f^{-1}(-\frac{2}{3})$ and $f^{-1}(\frac{2}{3})$ collapsed to points. However, the symplectic cut construction shows that it is a symplectic manifold and that the residual U(1)-action is Hamiltonian. Since this action is globally defined, its moment map $\hat{f}: \hat{M}_g \to [-\frac{2}{3}, \frac{2}{3}]$ is perfect. It again has three critical submanifolds $\hat{S}_1, \hat{S}_2, \text{and} \hat{S}_3$.  

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Of these, \( \hat{S}_2 = S_2 \) obviously. However, \( \hat{S}_1 \) and \( \hat{S}_3 \) are \( S^2 \times S^2 \)-bundles over \( M_{g-1} \); this follows from Lemma 3.2.

Filter \( M_g \) by the open subsets \( U_0 = f^{-1}(-\frac{3}{2}, -\frac{3}{2}), U_1 = f^{-1}(-\frac{3}{2}, -\frac{1}{2}), U_2 = f^{-1}(-\frac{3}{2}, \frac{1}{2}), \) and \( U_3 = f^{-1}(-\frac{3}{2}, \frac{3}{2}) \), so that

\[
\emptyset = U_0 \subset U_1 \subset U_2 \subset U_3 = M_g.
\]

The machinery of \( \mathbb{S} \) then produces a spectral sequence abutting to \( H^*(M_g) \). We need to show that all the differentials are zero. Since there are only three nonzero terms in the cochain filtration, only the first two differentials can possibly be nonzero.

As shown in \( \mathbb{S} \), the first differential is the direct sum of the maps \( H^*(S_1) \to H^*(S_2) \) and \( H^*(S_2) \to H^*(S_3) \) induced by the upper route in the diagram (2.1). We will consider these maps separately.

(3.5) The map \( H^*(S_1) \to H^*(S_2) \). As seen in (1.3), \( H^*(S_1) = V \oplus \tau V \), where \( \tau \in H^3 \) is Poincaré dual to the locus where \( B_g = I \), and \( V = \pi^*H^*(M_{g-1}) \). The map \( H^*(S_1) \to H^*(S_2) \) accordingly splits into two maps \( V \to H^*(S_2) \) and \( \tau V \to H^*(S_2) \); we will show that each of these vanishes.

Let \( U_1' = f^{-1}(-\frac{2}{3}, -\frac{1}{2}) \). Then \( U_1' \subset U_1 \), but there is also a natural inclusion \( U_1' \subset \hat{U}_1 \), where \( \hat{U}_1 = f^{-1}(-\frac{3}{2}, -\frac{1}{2}) \). It then follows from Lemma 3.2 that the hexagon in the diagram below commutes. Here the maps of the second column are the Thom isomorphisms, which in this case are simply induced by retractions, and those of the first column are induced by the fiber bundle projections.

\[
\begin{array}{ccc}
H^*(S_1) & \longrightarrow & H^*(U_1) \\
\downarrow & & \downarrow \\
H^*(M_{g-1}) & \longrightarrow & H^*(U_1') \longrightarrow H^*(U_2', U_1') \longrightarrow H^*(S_2) \\
\downarrow & & \downarrow \\
H^*(\hat{S}_1) & \longrightarrow & H^*(\hat{U}_1)
\end{array}
\]

On the other hand, if \( U_2' = f^{-1}(-\frac{3}{2}, \frac{1}{2}) \), then by excision \( H^*(U_2', U_1') = H^*(U_2, U_1) \). The map \( V \to H^*(S_2) \) in question is therefore the upper route in the diagram. However, the map \( H^*(\hat{S}_1) \to H^*(S_2) \) induced by the lower route is a component of \( \hat{d}_1 \), the differential for \( \hat{M}_g \), which vanishes since \( \hat{f} \) is perfect. Hence the map \( V \to H^*(S_2) \) must vanish.

For the second component, the map \( \tau V \to H^*(S_2) \), this argument no longer works. Instead, we resort to a parity argument.

There is a symplectomorphism \( \rho : M_g \to M_g \) induced by a half-twist of the \( g \)th handle of the surface \( X \). Explicitly, it is given by \( A_g \mapsto A_g B_g A_g^{-1} B_g^{-1} A_g^{-1} \) and \( B_g \mapsto A_g B_g^{-1} A_g^{-1} \), with all other coordinates remaining fixed. This fixes \( f \) and hence preserves all of the sets discussed above. It acts as an involution on \( S_1 \), preserving the projection to \( M_{g-1} \) and the locus where \( B_g = I \), but reversing orientation. It therefore takes \( \tau \) to \( -\tau \), so it acts as \(-1\) on \( \tau V \). On the other hand, it fixes \( S_2 \), so it acts trivially on \( H^*(S_2) \).

A more subtle point is that \( \rho^* \) commutes with the Thom isomorphism \( H^*(U_2, U_1) \to H^*(S_2) \). This follows from the Morse lemma provided that \( \rho \) preserves the orientation of
the negative normal bundle. Of course, $\rho$ will not preserve every Riemannian metric, so it
may not preserve the corresponding negative normal bundle. But it does commute with the
$U(1)$-action: this can be verified directly, or deduced from the fact that it preserves $f$ and
the symplectic form. If a suitable metric is chosen—for example, one equivalent near $S_2$ to
a $U(1)$-invariant flat product metric on the normal bundle, via the equivariant symplectic
tubular neighborhood theorem—then its negative normal bundle is the $-1$-weight space for
the $U(1)$-action on $TM|_{S_2}$. This subbundle, and its natural orientation coming from the
$U(1)$-action, are therefore preserved by $\rho$.

Any element of $\tau V$ is therefore acted on as $-1$ in the first entry, and hence in the last
entry, of the sequence of natural maps

$$H^*(S_1) \to H^*(U_1) \to H^*(U_2, U_1) \to H^*(S_2).$$

Its image in $H^*(S_2)$ must therefore be zero.

**3.6) The map $H^*(S_2) \to H^*(S_3)$.** This map is handled exactly like the previous one.

The fact that $\hat{f}$ is perfect implies the vanishing of the component $H^*(S_2) \to \tau V$. Indeed,
there is a diagram

$$H^*(S_2) \to H^*(U_2, U_1) \to H^*(U_2) \to H^*(U_2') \to H^*(M_{g-1}) \to H^*(S_3).$$

where $U_2' = f^{-1}(\frac{2}{3}, \frac{3}{2})$, $U_2'^3 = \hat{f}^{-1}(\frac{2}{3}, \frac{3}{2})$, and the final column consists of push-forwards.

It commutes because the map $H^*(U_2') \to H^*(S_3)$ can be interpreted as restriction to $f^{-1}(\frac{4}{3})$,
which is a sphere bundle over $S_3$, followed by push-forward by projection, and similarly for
$H^*(U_2') \to H^*(S_3)$. But the push-forward $H^*(S_3) \to H^*(M_{g-1})$ is projection on the factor
$\tau V$.

On the other hand, the parity argument implies the vanishing of the component $H^*(S_2) \to \tau V$. Indeed, we have already seen that $\rho^*$ commutes with all of the arrows between $H^*(S_2)$ and
$H^*(U_3, U_2)$ in the above diagram. However, the Thom isomorphism $H^*(U_3, U_2) \to H^*(S_3)$ does not commute with $\rho^*$. Rather, the two composites differ by a sign. This is
because $\rho$ reverses the orientation of $S_3$, but not of the open set $U_3$; after all, it is a sym-
plectomorphism. It therefore reverses the orientation of the normal bundle to $S_3$, so it acts
as $-1$ on the Thom class. Since any class in $H^*(S_2)$ is acted on trivially by $\rho$, its image in
$H^*(S_3)$ must therefore be acted on as $-1$, so its component in $\tau V$ must vanish.

**3.7) The map $H^*(S_1) \to H^*(S_3)$.** At this point we have shown that the first
differential vanishes. The second differential is therefore the map $H^*(S_1) \to H^*(S_3)$ given
by taking $j = 1$ in the description of 3.4 and it suffices to show that this also vanishes.

This time, since both $H^*(S_1)$ and $H^*(S_3)$ split as $V \oplus \tau V$, the differential splits into four
components.

Of these, the component $V \to \tau V$ vanishes by the perfect argument. Simply note that,
since $H^*(U_3, U_2) = H^*(U_3', U_2')$ by excision, a prime may be added to every set in the middle
row of \((2.1)\) without changing the differential. Then graft on the hexagonal diagrams of \((3.5)\) and \((3.6)\) and argue as before.

On the other hand, the component \(V \to V\) vanishes by the parity argument. Indeed, suppose that a class in \(H^*(S_1)\) belongs to the component \(V\), so that it is invariant under \(\rho^*\). Then the same is true of its image in \(H^*(U_1)\). Since the differential \(H^*(S_1) \to H^*(S_2)\) vanishes, this lifts to a class \(u \in H^*(U_2)\). Now \(u\) may not be invariant under \(\rho^*\), but the equivariance of the middle row of \((2.1)\) implies that \(\rho^* u = u + v\), where \(v\) is a class in the image of \(H^*(U_2, U_1)\). Since the differential \(H^*(S_2) \to H^*(S_3)\) vanishes, the image of \(v\) in \(H^*(U_3, U_2)\) must be zero. Hence \(u\) maps to a class in \(H^*(U_3, U_2)\) invariant under \(\rho^*\). Its image in \(H^*(S_3)\) is therefore anti-invariant as in \((3.6)\), so its component in \(V\) vanishes.

Likewise, the component \(\tau V \to \tau V\) vanishes by the same parity argument, with the roles of invariance and anti-invariance interchanged.

This leaves only one of the four components, namely the map \(\tau V \to V\). Now \(\tau\) is the restriction to \(S_1\) of a global class \(\tau \in H^2(M_g)\), namely the Poincaré dual of the locus where \(B_p = I\). Furthermore, the differential \(d_2\) is a module homomorphism over \(H^*(M_g)\). Indeed, multiplication by a class such as \(\tau\) clearly commutes with the natural maps shown as horizontal and vertical arrows in \((2.1)\). It is therefore compatible with the lifting from \(H^*(U_1)\) to \(H^*(U_2)\): given a lifting \(u \in H^*(U_2)\) of a class \(w \in H^*(U_1)\), \(\tau u\) is a lifting of \(\tau w\).

4 The case of additional punctures

As promised in the introduction, the main theorem goes through for flat \(SU(2)\) connections on a surface \(X\) with more than one puncture. The proof is essentially the same. However, it is necessary to generalize the background material of \(\S\) to the case of additional punctures.

So let \(p_1, \ldots, p_n\) be distinct points in \(X\), let \(t_1, \ldots, t_n \in [0, 1]\), let \(\Gamma_1, \ldots, \Gamma_n \subset SU(2)\) be the conjugacy classes in \(SU(2)\) containing \(diag(e^{it_1}, e^{-it_1})\), and let \(M_{g,n}\) denote the moduli space of flat \(SU(2)\) connections on \(X\), or rather on \(X \setminus \{p_1, \ldots, p_n\}\), having holonomy around \(p_j\) in \(\Gamma_j\). Of course \(M_{g,n}\) depends on the choice of the \(t_j\).

In analogy with \(\S\), \(M_{g,n}\) can be described as follows. Let \(\mu_{g,n} : SU(2)^{2g} \times \prod_{j=1}^n \Gamma_j \to SU(2)\) be given by

\[
\mu_{g,n}(A_1, B_1, A_2, B_2, \ldots, A_g, B_g, C_1, \ldots, C_n) = \left( \prod_{i=1}^g [A_i, B_i] \right) \left( \prod_{j=1}^n C_j \right).
\]

Then \(M_{g,n} = \mu_{g,n}^{-1}(I)/SU(2)\).

(4.1) Proposition. For \(J \subset \{1, \ldots, n\}\), let \(\kappa_J = \frac{1}{2} \left( \sum_{j \in J} t_j - \sum_{j \not\in J} t_j \right)\). Then \(\mu_{g,n}\) has \(I \in SU(2)\) as a critical value if and only if for some \(J\), \(\kappa_J \in \mathbb{Z}\).

Proof. The derivative of \(\mu_{g,n}\) at \((A_i, B_i, C_j)\) is a map \(D\mu_{g,n} : su(2)^{2g} \oplus \bigoplus_j T_{C_j} \Gamma_j \to su(2)\).
A direct calculation shows that
\[
D\mu_{g,n}(a_i, b_i, c_j) = \text{ad} \left( \prod_{j=1}^{n} C_j \right)^{-1} D\mu_{g,0}(a_i, b_i) + D\mu_{0,n}(c_j)
\]
\[
= \sum_{i=1}^{g} \text{ad} \left( \prod_{k>i}^{n} \left[ A_k, B_k \right] \prod_{j=1}^{n} C_j \right)^{-1} \text{ad} B_i A_i \left( (\text{ad} B_i^{-1} - 1) a_i + (1 - \text{ad} A_i^{-1}) b_i \right)
\]
\[
+ \sum_{j=1}^{n} \text{ad} \left( \prod_{\ell>j}^{n} C_\ell \right)^{-1} c_j.
\]

Suppose first that \( g = 0 \). Any \( c_j \in T_{C_j} \Gamma_j \) is of the form \((1 - \text{ad} C_j^{-1}) d_j \) where \( d_j \in \mathfrak{su}(2) \). A further computation shows that
\[
D\mu_{0,n}(c_j) = \sum_{j=1}^{n-1} \left( \text{ad} \left( \prod_{k>j} C_k \right)^{-1} - 1 \right) (d_{j+1} - d_j).
\]

This fails to be surjective if and only if for all \( j \), the image of \( \text{ad} \left( \prod_{k>j} C_k \right)^{-1} - 1 \) is the same, which holds if and only if all \( \prod_{k>j} C_k \) commute, which holds if and only if all \( C_j \) commute, since \( \prod_{j=1}^{n} C_j = I \). The \( C_j \) may then be simultaneously diagonalized to \( \text{diag}(e^{\pm i\pi t_j}, e^{\mp i\pi t_j}) \). Let \( J \) be the set of \( j \) for which the plus sign holds in the first factor. Then the stated condition on the \( t_j \) holds.

For \( g > 0 \), the presence of the term \((\text{ad} B_i^{-1} - 1) a_i + (1 - \text{ad} A_i^{-1}) b_i \) in the formula above shows that if \( D\mu_{g,n} \) is not surjective, then \( A_i \) and \( B_i \) must commute for each \( i \). This brings us back to the case \( g = 0 \).

The case studied before was that of one puncture with \( t_1 = 1 \). Without loss of generality we may assume that, if we are not in this case, then each \( t_j \in (0,1) \). Indeed, those \( j \) with \( t_j = 0 \) or 1 can be eliminated from the construction above without changing it, unless there are an odd number of \( j \) with \( t_j = 1 \). In the latter case one can multiply some other \( C_j \), say \( C_1 \), by \(-I\), and change \( t_1 \) to \( 1-t_1 \).

Suppose that the \( t_j \) are so chosen that Proposition 4.1 is false, and \( I \) is a regular value. Then \( M_{g,n} \) is smooth. A symplectic form on \( M_{g,n} \) has been constructed by Guruprasad et al. [12]. Moreover, if \( f : M_{g,n} \to [-1,1] \) and a U(1)-action on \( f^{-1}(-1,1) \) are defined just as in §1, ignoring the extra \( C_j \) variables, then Audin shows [3] that this action is again symplectic with moment map \(-i \arccos f\).

(4.2) The critical submanifolds of \( f \). The critical submanifolds of \( f \) are classified just as in [13]. First, there are \( f^{-1}(-1) \) and \( f^{-1}(1) \), both of which are SU(2)-bundles over \( M_{g-1,n} \) with vanishing Euler class. These are actually empty in certain cases when \( g = 1 \), but the main theorem will still hold even then.

The remaining critical points are again fixed points of the U(1)-action. It is straightforward to check that these are all conjugate to \( 2g + n \)-tuples such that
\[
A_g = \begin{pmatrix} e^{-i\pi \kappa} & 0 \\ 0 & e^{i\pi \kappa} \end{pmatrix}, \quad B_g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and the remaining $A_i$, $B_i$, and $C_j$ are all diagonal. For $i < g$, the $A_i$ and $B_i$ are free to move in the 1-parameter subgroup of diagonal matrices, but each $C_j$ is in a fixed conjugacy class distinct from $\pm I$ and so must equal $\text{diag}(e^{\pm i\pi t_j}, e^{\pm i\pi t_j})$. Let $J$ be the set of those $j$ for which the upper sign holds. The constraint imposed by $\mu$ then implies that $\kappa = \kappa_J$, as defined in Proposition 4.1. The critical submanifolds in $f^{-1}(-1,1)$ therefore consist of a disjoint union of $2^n$ tori of dimension $2g - 2$.

(4.3) Theorem. The map $f$ is a perfect Bott-Morse function on $M_{g,n}$.

Proof. The proof is similar to that of the main theorem. We indicate only the points where it must be modified.

First of all, the three lemmas at the beginning of §3 go through unchanged, except that the formula in the proof of Lemma 3.1 gets replaced by

$$D\mu_{g,n} = D\mu_{g-1,n} + \text{ad}\left(\prod_j C_j\right)^{-1} D\mu_{1,0}.$$ 

Suppose that $f^{-1}(\pm 1) \neq \emptyset$, since the result follows directly from Theorem 1.3 otherwise. Write the set of critical points as a disjoint union of submanifolds $S_1, \ldots, S_N$ so that

$$-1 = f(S_1) < f(S_2) < \cdots < f(S_{N-1}) < f(S_N) = 1.$$ 

If there are no coincidences between the values of $\kappa_J$ obtained in (4.2) for different $J$, then $N$ will equal $2^n + 2$. Otherwise some of the $S_i$ will be disjoint unions of tori. The Morse indices of the components may vary, so the Thom class will be inhomogeneous, but this makes no difference.

Since there are $N$ critical values, only the first $N-1$ differentials in the spectral sequence can possibly be nonzero. By induction suppose all differentials before $d_k$ vanish. If $1 < i < k + i < n$, then the component $H^*(S_i) \to H^*(S_{k+i})$ of $d_k$ vanishes because $-i \arccos f$ is a moment map there. The component $H^*(S_1) \to H^*(S_{k+1})$ is handled as in (3.5), except that in the parity argument, the $\rho$-equivariance of the long exact sequence of the relative cohomology of the pair $(U_{i+1}, U_i)$ must be applied inductively as in (3.7) to show that the differential preserves the $-1$-weight space. Likewise the component $H^*(S_{N-k}) \to H^*(S_N)$ is handled as in (3.6). Finally, if $k = N-1$, then $d_k : H^*(S_1) \to H^*(S_N)$ is handled as in (3.7).

This leads to a formula for the Betti numbers of $M_{g,n}$ as soon as the Morse indices of the critical tori are computed. At this point our approach meets the route taken by Hoyle [15]. From a lengthy trigonometric computation, he deduced the following formula.

(4.4) Lemma (Hoyle). For any $J \subset \{1, \ldots, n\}$, the index of the critical torus associated to $J$ is $2g + 2n - 2|J| + 4[k_J]$.

Therefore, the Poincaré polynomials of $M_{g,n}$ satisfy the recursion

$$P_t(M_{g,n}) = P_t(S_1) + t^3 P_t(S^3) + (1 + t)^{2g-2} \sum_J t^{2(g+n-|J|+2[k_J])}.$$
This immediately implies the following.

(4.5) Corollary. The Poincaré polynomial of $M_{g,n}$ satisfies

$$P_t(M_{g,n}) = (1 + t^3)^2 P_t(M_{g-1,n}) + (t + t^2)^{2g-2} \sum J t^{2(n+1-|J|+2|\kappa_J|)}.$$

From this, one can obtain an explicit expression for $P_t(M_{g,n})$ by substituting Hoyle’s formula for $P_t(M_{0,n})$.

(4.6) Remark. Assuming the Harder-Narasimhan formula, the recursion in the genus could also be proved by algebro-geometric methods. Here is a sketch of the argument. By a theorem of Mehta-Seshadri [22], $M_{g,n}$ may be regarded as a moduli space of vector bundles over $X$ with parabolic structure at each $p_j$. If $t_n = 1$ but the remaining $t_j$ are small, then $M_{g,n}$ is a $(\mathbb{CP}^1)^{n-1}$-bundle over $M_g$ for $g > 0$, so the recursion holds in this case. For general $t_j$, choose a path in $[0,1]^n$ connecting $(t_1, \ldots t_n)$ to the previous special case in such a way that for any point on the path, $\kappa_J \in \mathbb{Z}$ for at most one $J \subset \{1, \ldots, n\}$. Then the moduli spaces on either side of each such point are related by a blow-up and blow-down centered on a disjoint union of tori [4, 25]. It can be checked that this alters the Betti numbers in a way which preserves the recursion. Unlike the Morse approach, however, this algebraic approach gives no insight about why the recursion holds.

5 Some final remarks

Many other cheerful facts about $M_{g,n}$ can be deduced from the argument used to prove the main theorem. Let us mention two of them.

(5.1) Proposition. The integral cohomology of $M_{g,n}$ is torsion-free.

Proof. When $g = 0$, then Hoyle [15] exhibits a Hamiltonian circle action on $M_{0,n}$ whose fixed points are isolated except for two copies of $M_{0,n-1}$. By induction we may suppose these are torsion-free, and Frankel [8, Cor. 1] then shows that $M_{0,n}$ is torsion-free.

Now suppose $g > 0$ and consider again the Bott-Morse function $f$. The spectral sequence of §2 works equally well with integral cohomology. Everything in the proof of the main theorem goes through, except that the differentials over $\mathbb{Z}$ might take nonzero values in the torsion part of $H^*(S_2, \mathbb{Z})$ or $H^*(S_3, \mathbb{Z})$. But $S_2$ is a torus, so it is certainly torsion-free, and $S_1$ and $S_3$ may be assumed torsion-free by induction on $g$. $\blacklozenge$

(5.2) Proposition. Let $(\mathbb{Z}/2)^{2g}$ act on $M_{g,n}$ by

$$(\delta_i, \epsilon_i) \cdot (A_i, B_i, C_j) = ((-1)^{\delta_i} A_i, (-1)^{\epsilon_i} B_i, C_j).$$

Then the induced action on $H^*(M_{g,n}, \mathbb{Q})$ is trivial.

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Proof. The action of \((\mathbb{Z}/2)^{2g-2}\) on the first \(2g-2\) factors preserves the \(U(1)\)-action, the map \(f\) and so on. The whole proof of the main theorem can therefore be \((\mathbb{Z}/2)^{2g-2}\)-graded. But since \((\mathbb{Z}/2)^{2g-2}\) acts trivially on \(\tau\) and on \(H^*(S_2)\), by induction the whole grading is trivial. Hence \((\mathbb{Z}/2)^{2g-2}\) acts trivially on \(H^*(M_{g,n})\).

The last two factors are really no harder. After all, the choice of an ordering on the handles of \(X\) was arbitrary. For example, the argument still works if \(f\) is replaced by \(\frac{1}{2} \text{tr} A_1\), and so on.

The value of this last result is to relate the cohomology of \(M_{g,n}\) to that of the corresponding moduli space of flat \(U(2)\) connections. Let \(\tilde{M}_{g,n}\) be the moduli space of flat \(U(2)\) connections on \(X\setminus\{p_1, \ldots, p_n\}\) with holonomy around \(p_j\) in the conjugacy class \(\Gamma_j\). Just as \(M_{g,n}\) was, \(\tilde{M}_{g,n}\) may be described in terms of a map \(\tilde{\mu}_{g,n}\).

(5.3) Corollary. As rings, \(H^*(\tilde{M}_{g,n}, \mathbb{Q}) \cong H^*(U(1)^{2g}, \mathbb{Q}) \otimes H^*(M_{g,n}, \mathbb{Q})\).

Proof. There is a natural map \(U(1)^{2g} \times M_{g,n} \to \tilde{M}_{g,n}\) given by

\[(\kappa_i, \lambda_i, A_i, B_i, C_j) \mapsto (\kappa_i A_i, \lambda_i B_i, C_j).\]

Indeed, \(\tilde{M}_{g,n} = (U(1)^{2g} \times M_{g,n})/(\mathbb{Z}/2)^{2g}\), where \((\mathbb{Z}/2)^{2g}\) acts diagonally on \(U(1)^{2g}\) and \(M_{g,n}\) as above. The induced action on \(H^*(U(1)^{2g}, \mathbb{Q})\) is certainly trivial, and the induced action on \(H^*(M_{g,n}, \mathbb{Q})\) is trivial by the lemma above. The proof is completed by the result of Grothendieck \([1]\) that the rational cohomology ring of a quotient by a finite group is the invariant part of the rational cohomology.

Compare Newstead \([24]\), Harder-Narasimhan \([14]\) and Atiyah-Bott \([1]\) in the case \(n = 0\).

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