Constraints on $\tan \beta$ in SUSY SU(5) GUT

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Abstract

On the evaluation of $\tan \beta$ is revisited by the method of the renormalization group with the criteria of the bottom-tau unification in the SUSY SU(5) GUT model. Among the conventional supersymmetric theories, an energy-scale constant $M_{SUSY}$ is introduced by-hand to decouple the lower energy-scale region from the effects of the supersymmetric particles. Provided that $M_{SUSY}$ really exists, only small $\tan \beta \simeq 2$ is allowed theoretically. While, medium value as $\tan \beta = 6 \sim 25$ is also supported if $M_{SUSY}$ is vanished away from the model.

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1. INTRODUCTION

1) - What are unified?
As one of the general features of the SU(5) GUT models, both Yukawa coupling constants of the bottom-quark and the tau-lepton are unified at GUT scale. Additionally, in several SU(5) models including the SUSY SU(5) GUT model[1], the evidence for the unification of gauge coupling constants is numerically provided[2].

2) - What is $M_{SUSY}$?
In the conventional analysis[2], the gauge unification in the SUSY SU(5) GUT model is achieved by introducing an energy-scale constant $M_{SUSY}$ by-hand to decouple the lower energy-scale region from the effects of the supersymmetric particles. Thus, $M_{SUSY}$ is regarded as the threshold between the standard model(SM) and SUSY. The value of $M_{SUSY}$ depends on the strong coupling constant $\alpha_3$ very sensitively[2]. For instance, while $M_{SUSY} \simeq 1$(TeV) for $\alpha_3 = 0.106$ is given[4] in 1991, $M_{SUSY} \leq 100$(GeV) is obtained for the recent value[5] of $\alpha_3 = 0.1185$. If $M_{SUSY}$ were really less than $100$(GeV), we should have already detected such supersymmetric particles in the region of $\sim 100$(GeV) experimentally, however we have not yet. This fact may allow us to think of $M_{SUSY}$ as a negligible parameter, or, at least, the authors should discuss on the possibility of the models without $M_{SUSY}$. Note that such model without $M_{SUSY}$ means the pure supersymmetric theory without including the conventional standard model.

3) How much $\tan \beta$ is optimal?
$\tan \beta$ is evaluated by the two different ways of the likelihood analysis at two-loop level in this paper. One of the two ways is on the unification of the gauge coupling constants, and another is of the bottom-tau’s Yukawa coupling constants. The sequence to optimize $\tan \beta$ is as follows:
In the first, $M_{SUSY}$ and the GUT scale $M_X$ are fixed by the $\chi^2$-fitting on the unification of the gauge($\chi^2_\beta$) coupling constants. Once $M_{SUSY}$ and $M_X$ are fixed, corresponding $\tan \beta$ is derived numerically to realize the bottom-tau unification at $M_X$ scale.
Additionally, the analysis on the model without $M_{SUSY}$ is also carried out in the same manner.
II. ANALYSIS OF $\tan \beta$ WITH $M_{SUSY}$

On the optimization of $\tan \beta$ is discussed by the renormalization group equations (RGEs) of the gauge and Yukawa coupling constants at two-loop level in the SM and SUSY. As the Yukawa coupling constants, heavier three of the ordinal Fermions (top-quark, bottom-quark, and tau-lepton) are taken into consideration. First of all, the explicit values of the parameters in this paper are referred from the latest issue of the Particle Data Group.

On the boundary conditions of the RGEs, they are described further below. The conditions of the gauge coupling constants are given on Z-boson’s mass shell $M_Z = 91.184$ (GeV) in the MS-scheme as:

$$\alpha_1 = \frac{5}{3} \frac{\alpha_{MS}}{1 - \sin^2 \theta_{MS}},$$

(1)

$$\alpha_2 = \frac{\alpha_{MS}}{\sin^2 \theta_{MS}},$$

(2)

as their explicit values are:

$$\alpha_{MS}^{-1} = 127.934 \pm 0.027,$$

(3)

$$\sin^2 \theta_{MS} = 0.23117 \pm 0.00016,$$

(4)

$$\alpha_3 = 0.1185 \pm 0.0020.$$  

(5)

Moreover, the definition of $\tan \beta$ at $M_Z$ is given as usual:

$$\tan \beta(M_Z) \equiv v_2(M_Z)/v_1(M_Z),$$

(6)

where, $v_1$ and $v_2$ imply the vacuum expectation values (VEVs) of two Higgs doublets, and as their explicit value:

$$v(M_Z) = (v^2_1(M_Z) + v^2_2(M_Z))^{1/2} = 246\text{ (GeV)},$$

(7)

is fixed. For simplicity, mere $\tan \beta$ means $\tan \beta(M_Z)$ in this paper.

On the Higgs quartic coupling constant $\lambda$, this is precisely defined at $M_{SUSY}$ as:

$$\lambda(M_{SUSY}) = \pi \left[\frac{3}{5} \alpha_1(M_{SUSY}) + \alpha_2(M_{SUSY})\right] \cos^2 2\beta(M_{SUSY}),$$

(8)

however, it is very difficult to derive $\lambda$ from this definition. As $\lambda$, the authors make use of an approximate, and easier alternative definition at $M_Z$:

$$\lambda \approx \pi \left[\frac{3}{5} \alpha_1(M_Z) + \alpha_2(M_Z)\right] \cos^2 2\beta(M_Z).$$

(9)
Note that the numerical difference between the definitions Eq. (8) and Eq. (9) is very small as about 0.1%, and \( \lambda \) is decoupled to the Yukawa coupling constants among the RGEs at one-loop level, therefore, all the numerical results in this paper are effectually independent of the choice of the definition of \( \lambda \).

Boundary conditions of the Yukawa coupling constants are given at each on-shell energy-scales as:

\[
\alpha_q(m_q) = \frac{m_q^2(m_q)}{2\pi v^2(m_q)},
\]

where, the suffix \( q \) implies the flavor of Fermions as \( q = t, b, \) or \( \tau, \) respectively. Their explicit values are as follows:

\[
\begin{align*}
\begin{align*}
 m_t(m_t) &= 174.3 \pm 5.1\text{(GeV)}, \\
 m_b(m_b) &= 4.2 \pm 0.2\text{(GeV)}, \\
 m_\tau(m_\tau) &= 1.77699 \pm 0.00029\text{(GeV)}.
\end{align*}
\end{align*}
\]

When assuming the existence of \( M_{SUSY} \), corresponding boundary conditions are given as:

\[
\begin{align*}
\alpha_i^{SUSY}(M_{SUSY}) &= \alpha_i^{SM}(M_{SUSY}) , (i = 1, 2, 3), \\
\alpha_t^{SUSY}(M_{SUSY}) &= \frac{m_t^2(M_{SUSY})}{2\pi v^2(M_{SUSY})}\frac{1}{\sin^2\beta(M_{SUSY})} \alpha_t^{SM}(M_{SUSY}) , \\
\alpha_{b,\tau}^{SUSY}(M_{SUSY}) &= \alpha_{b,\tau}^{SM}(M_{SUSY})\frac{1}{\cos^2\beta(M_{SUSY})} \alpha_{b,\tau}^{SM}(M_{SUSY}),
\end{align*}
\]

where, the variables with a prime(‘) mean the redefinition of them in the SM region to keep their continuity between the regions of the SM and SUSY. With such redefinitions, the graphs in this paper is continuous even if crossing on the \( M_{SUSY} \) border.

The conditions at GUT energy-scale \( M_X \) are described in the following subsections.
A. On the gauge coupling constants

The average value $\alpha_g$ among $\alpha_1, \alpha_2,$ and $\alpha_3$ is introduced around the GUT scale $M_X$. $\chi_g^2$-fitting function is defined as the squared sum of each differences between $\alpha_g$ and $\alpha_i (i = 1 \sim 3)$ as follows:

$$\chi_g^2 = \sum_{i=1}^{3} \left( \frac{\alpha_i - \alpha_g}{\delta \alpha_i} \right)^2,$$

(17)

where, $\delta \alpha_i$ means the uncertainty width of $\alpha_i$, respectively. the GUT scale $M_X$ is defined as the energy-scale where $\chi_g^2$ becomes the minimum. In order to make $\chi_g^2$ the minimum, $M_{SUSY}$ is fine-tuned. As the results, we find $\chi_g^2 \simeq 0$ can be achieved by tuning $M_{SUSY}$ at any $\tan \beta$ in the ranges of the errorbars of the other parameters. Fine-tuned values of $M_{SUSY}$ are shown as functions of $\tan \beta$ with the uncertainty width of $\alpha_3$ in Fig. 1. We find the derived $M_{SUSY}$’s value depends on only the strong coupling constant $\alpha_3$ rather than the other parameters.

B. On the Yukawa coupling constants

When the SU(5) GUT scenario is presumed, Yukawa coupling constants of the bottom-quark and tau-lepton are unified from each other at $M_X$. Their $\chi_Y^2$-fitting function is defined as the following equations:

$$\chi_Y^2 = \sum_{q=b,\tau} \left( \frac{\alpha_q - \alpha_Y}{\delta \alpha_q} \right)^2,$$

(18)

$$\alpha_Y = (\alpha_b + \alpha_\tau) / 2 ,$$

(19)

where, $q$ implies $b$ or $\tau$, and $\delta \alpha_q$ means the uncertainty width of $\alpha_q$, respectively. With the uncertainty width $\delta = 0.2$(GeV) of $m_b$, $\chi_Y^2$ at $M_X$ scale as functions of $\tan \beta$ are shown in Fig. 2. The behavior of $\chi_Y^2$ is not sensitive on the variance of $m_t$ or $\alpha_3$. Also shown in Fig. 2, only small $\tan \beta \simeq 2$ is allowed by the likelihood analysis with the $\chi_Y^2$ function, and this minimal point of $\chi_Y^2$ is stable despite of the uncertainty of the parameters like $m_b$, $m_t$, or $\alpha_3$. 

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III. ANALYSIS OF $\tan\beta$ WITHOUT $M_{SUSY}$

As mentioned above, on the model without $M_{SUSY}$ is discussed in this section. Therefore, the RGEs in this section are pure SUSY. On the boundary conditions of the gauge coupling constants, the VEVs of Higgs bosons, and $\tan\beta$, they are given at $M_Z$ in the same way of the model with including $M_{SUSY}$ as described in the previous section as from Eq. (3) to Eq. (6). The boundary conditions of the Yukawa coupling constants are given at the energy-scales of each on-shell masses as follows:

$$\alpha_t(m_t) = \frac{m_t^2(m_t)}{2\pi v^2(m_t) \sin^2 \beta(m_t)},$$

and

$$\alpha_{b,\tau}(m_{b,\tau}) = \frac{m_{b,\tau}^2(m_{b,\tau})}{2\pi v^2(m_{b,\tau}) \cos^2 \beta(m_{b,\tau})}.$$  

The definitions of $\chi^2_g$ and $\chi^2_Y$ at $M_X$ are the same ones Eq. (17) and Eq. (18) in the previous section, however, in this section, $\chi^2_g$ depends on $\tan\beta$ or other parameters directly in contrast to the previous section, because the decoupling boundary $M_{SUSY}$ is vanished away from the model.

The dependence of $\tan\beta$ on $\chi^2_g$ with the uncertainty width of $\alpha_3$ is shown in Fig. 3. As Fig. 3 shows, two minimal points exist at small and large $\tan\beta$. The variance of $\chi^2_Y$ at $M_X$ is shown with each uncertainty width of $m_b, m_\tau$, or $\alpha_3$ in Fig. 4a, 4b, or 4c, respectively. As shown in these graphs, usually $\chi^2_Y$ has three local minima.

IV. DISCUSSIONS

A. On the model with $M_{SUSY}$

Since $\chi^2$ of the gauge coupling constants($\chi^2_g$) can be always equal to zero by fine-tuning of $M_{SUSY}$, there exists no effective constraint on $\tan\beta$ from the unification of the gauge coupling constants. $\tan\beta$ is evaluated as 1.6 by the criteria of the bottom-tau unification in this model with $M_{SUSY}$. $M_{SUSY}$ is estimated about 2(TeV) to unify the gauge coupling constants at two-loop level. (As a reference, $M_{SUSY} \ll M_Z$ is derived at one-loop level.) Nevertheless, such small $\tan\beta \simeq 2$ is difficult to be allowed by $g-2$ experiments, because small $\tan\beta$ makes the masses of sparticles quite smaller than expected. The evolutions
B. On the model without $M_{\text{SUSY}}$

The value of $\tan \beta$ is possibly settled at 2, $6 \sim 25$, or 50 in accordance with three minima of $\chi^2_Y$. However, large $\tan \beta \simeq 50$ is denied theoretically, because $\chi^2_g$ and $\chi^2_Y$ cannot be minimal simultaneously, i.e., they are exclusive from each other as shown in Fig. 5. On the small $\tan \beta \simeq 2$ choice, both of $\chi^2_g$ and $\chi^2_Y$ are enough small, however, this is excluded by the recent $g-2$ experiments[7] as mentioned previously. Large $\tan \beta \simeq 50$ is also not probable if $b \rightarrow s\gamma$ experiment[8] is true. Moreover, the authors are dubious whether these two local minima of $\chi^2_Y$ at small and large $\tan \beta$ are physical ones or not, because they make the RGEs divergent at one-loop level, therefore, the evolution of the Yukawa coupling constants seems strange even at two-loop level, for instance, as shown in Fig. 7 at $\tan \beta = 1.9$. Medium $\tan \beta$ as $6 \sim 25$ is good for $\chi^2_Y$ of the bottom-tau unification, however, $\chi^2_g$ of the gauge unification cannot be enough small at two-loop level. Thus, precise unification of the gauge coupling constants cannot occur with medium $\tan \beta$ as $6 \sim 25$. The evolutions of the gauge or Yukawa coupling constants at $\tan \beta = 6.0$ are shown in Fig. 8a or 8b, respectively. These discussions are summarized in the Table II.

V. SUMMARY AND CONCLUSION

In the SU(5) GUT scenario, $\tan \beta$ is evaluated by making use of the RGEs of the gauge and Yukawa coupling constants. Only small $\tan \beta \simeq 2$ is allowed in the model including $M_{\text{SUSY}}$, and $\tan \beta$ is estimated as 2 or 6 to 25 in the model without $M_{\text{SUSY}}$, theoretically. However, small $\tan \beta \simeq 2$ is difficult to be allowed by the analysis of the recent $g-2$ experiment[7]. As the result, only medium $\tan \beta$ as $6 \sim 25$ is acceptable in the SUSY SU(5) GUT model without $M_{\text{SUSY}}$ if the bottom-tau unification is prior to the unification among the gauge coupling constants.
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### TABLE I: Possibility of $\tan \beta$ with $M_{SUSY}$.

| $\tan \beta$ | small ($\simeq 2$) | medium ($6 \sim 25$) | large ($\simeq 50$) |
|---------------|---------------------|-----------------------|----------------------|
| $\chi^2_g$    | $\bigcirc$          | $\bigcirc$            | $\bigcirc$           |
| $\chi^2_Y$    | $\bigcirc$          | $\times$              | $\times$             |
| $g - 2$       | $\times$            | $\bigcirc$            | $\bigcirc$           |

### TABLE II: Possibility of $\tan \beta$ without $M_{SUSY}$.

| $\tan \beta$ | small ($\simeq 2$) | medium ($6 \sim 25$) | large ($\simeq 50$) |
|---------------|---------------------|-----------------------|----------------------|
| $\chi^2_g$    | $\bigcirc$          | $\bigtriangle$        | $\bigcirc$ ($\times$) |
| $\chi^2_Y$    | $\bigcirc$          | $\bigcirc$            | $\times$ ($\bigcirc$) |
| $g - 2$       | $\times$            | $\bigcirc$            | $\bigcirc$           |
| $b \to s\gamma$ | $\bigcirc$       | $\bigcirc$            | $\bigtriangle$        |

*These two scores are exclusive from each other.*
FIG. 1: $M_{SUSY}$ as functions of $\tan \beta$ for $\alpha_3 \pm \delta = 0.1185 \pm 0.0020$. 
FIG. 2: $\chi^2$ of the Yukawa coupling constants($\chi^2_Y$) at GUT scale $M_X$ for $m_b \pm \delta = 4.2 \pm 0.2$(GeV) with $M_{SUSY}$. 
FIG. 3: $\chi^2$ of the gauge coupling constants ($\chi^2$) at $M_X$ for $\alpha_3 \pm \delta = 0.1185 \pm 0.0020$ without $M_{SUSY}$. 
(a) $m_b \pm \delta = 4.2 \pm 0.2 \text{ (GeV)}$

Fig. 4a
$m_t \pm \delta = 174.3 \pm 5.1$ (GeV)

Fig. 4b
(c) $\alpha_3 \pm \delta = 0.1185 \pm 0.0020$

![Graph](image)

**Fig. 4c**

FIG. 4: $\chi^2_Y$ at $M_X$ for each uncertainty width of (a)$m_b$, (b)$m_t$, and (c)$\alpha_3$ without $M_{SUSY}$, respectively.
FIG. 5: Opposite behavior of $\chi^2_Y$ vs $\chi^2_g$ around large $\tan\beta \simeq 50$ at $M_X$ without $M_{SUSY}$. 
(a) gauge couplings

\[ \alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1} \]

\[ \tan \beta = 1.6 \]

energy scale (GeV)

Fig. 6a
(b) Yukawa couplings

\[ \tan \beta = 1.6 \]

\[ \alpha_t, \alpha_b, \alpha_\tau \]

energy scale (GeV)

FIG. 6b

FIG. 6: The evolutions of the (a)gauge and (b)Yukawa coupling constants at \( \tan \beta = 1.6 \) with \( M_{\text{SUSY}} \).
Fig. 7

FIG. 7: The evolution of the Yukawa coupling constants at tan $\beta=1.9$ without $M_{SUSY}$. 
(a) gauge couplings

\[ \tan \beta = 6.0 \]

Fig. 8a
FIG. 8b

The evolutions of the (a)gauge and (b) Yukawa coupling constants at $\tan \beta = 6.0$ without $M_{SUSY}$. 