The Hawking–Page phase transitions in the extended phase space in the Gauss–Bonnet gravity

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In this paper, the Hawking–Page phase transitions between the black holes and the thermal radiation in the anti-de Sitter (AdS) space are studied with the Gauss–Bonnet term in the extended phase space, in which the cosmological constant plays the role of an effective thermodynamic pressure. The Gauss–Bonnet term exhibits its effects via introducing the corrections to the black hole entropy and Gibbs free energy, and these two aspects correspondingly determine the lower and upper bounds of the Gauss–Bonnet coupling constant \( \alpha \). The global phase structures, especially the phase transition temperatures \( T_{HP} \) and the Gibbs free energies \( G \) of the Schwarzschild–AdS and charged and rotating AdS black holes, are systematically investigated in order, with both analytical and numerical methods. It is found that \( T_{HP} \) increases at large charges and angular momenta and decreases with \( \alpha \). In the extended phase space, the Schwarzschild–AdS black holes have positive minimal temperatures, leading to the terminal points in the coexistence lines for any non-vanishing \( \alpha \). In contrast, the minimal temperatures of the charged and rotating AdS black holes are allowed to reach zero, and there are only terminal points in the coexistence lines for negative \( \alpha \).

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I. INTRODUCTION

Black hole thermodynamics is one of the most profound branches in modern physics, which indicates that a black hole is not simply a mathematical singularity, but should be regarded as a complicated physical system with temperature and entropy [1]. It shows deep relationship among thermodynamics, classical gravity, and quantum mechanics, and thus paves the way to our final understanding of quantum gravity [2].

However, in the first law of black hole thermodynamics, the lack of the \( p-V \) term makes it still somehow different from traditional thermodynamics. Introducing an effective pressure is equivalent to adding a new dimension in the thermodynamic phase space, so such a theory is usually named as “black hole thermodynamics in the extended phase space” or “black hole chemistry” [3–9]. In this framework, black hole thermodynamics is studied in the asymptotic anti-de Sitter (AdS) space with a negative cosmological constant \( \Lambda \). If \( \Lambda \) is allowed to change, it plays the role of a positive varying thermodynamic pressure instead of a fixed background,

\[
p = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2},
\]

where \( l \) is the curvature radius of the AdS space, and the conjugate variable of \( p \) can be effectively defined as the black hole thermodynamic volume. By this means, the missing \( p-V \) term appears in black hole thermodynamics, but it proves to be \( \tilde{V} \, dp \), not the usual work term \( -p \, dV \). Therefore, the black hole mass should be identified as its enthalpy rather than internal energy.

In the extended phase space, many remarkable similarities between black holes and non-ideal fluids were discovered, e.g., phase transitions, critical exponents, and equations of corresponding states. These interesting observations aroused a large number of successive works, and almost all aspects of black hole physics were reinspected, such as the van der Waals black hole [10], super-entropic black hole [11], superfluid black hole [12], reentrant phase transition [13], heat engine [14], throttling process [15], reverse isoperimetric inequality [16], microscopic structure [17], and holographic entanglement entropy [18] (see Ref. [19] for reviews of recent progresses and the references therein). Currently, the research topics are mainly focusing on the black hole thermodynamics in various modified gravity theories [20–28].

One of the most promising modified gravity theories is the Gauss–Bonnet (GB) gravity (also referred to as Einstein–GB gravity), which offers the leading order correction to the Einstein gravity. The GB term \( G \) is exactly the second order term in the Lagrangian of the most general Lovelock gravity. Therefore, although \( G \) itself is quadratic in curvature tensors, the equations of gravitational fields are still of second order and naturally avoid ghosts. The GB gravity possesses many important physical properties and has been heavily studied in gravitation [29–34] and cosmology [35–39], also with emphasis in the extended phase space [40–45].

In a four-dimensional manifold, the GB term merely reduces to a topological invariant, \( \int d^4x \sqrt{-g} G = \chi \), with \( \chi \) denoting the Euler characteristic of the manifold. At this point, the GB term cannot have any dynamic effect in four dimensions, so it does not influence space-time structure, horizon area, global charges, and their conjugate potentials. Therefore, the GB term is usually studied in extra-dimensional physics. However, there is an exception. Albeit the GB term is irrelevant to dynamics, it does affect the thermodynamics of gravitational fields

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in four-dimensional space-time. The basic reason lies in the fact that, beyond the Bekenstein–Hawking formula \[1\], black hole entropy receives a contribution from the GB term \[46\]. In this sense, the first law of black hole thermodynamics, Smarr relation, and all the issues related to entropy will be modified. Consequently, black holes can exhibit much richer thermodynamic phenomena, especially in their phase structures and phase transitions.

Among various black hole phase transitions, one of the most significant is the Hawking–Page (HP) phase transition originally studied between a Schwarzschild–AdS black hole and the thermal radiation in the AdS space \[47\]. The black hole thermodynamics in the AdS space is quite different from those in the asymptotic Minkowski or de Sitter space. In the AdS space, large black holes have positive heat capacities and are thus thermodynamically stable, so they can be in equilibrium with the thermal radiation. Below a certain temperature, there is no black hole solution anymore, and the HP phase transition happens in the black hole–thermal radiation system. This phase transition was later widely investigated \[48–54\], for example, in the charged AdS \[i.e., Kerr–Newman–AdS (RN–AdS)\] black holes \[55, 56\]. Moreover, it was reinterpreted as a confinement–deconfinement phase transition in the dual quark–gluon plasma in gauge theories \[57, 58\]. The relevant studies in the extended phase space can also be found in Refs. \[59–62\].

The aim of this paper is to study the HP phase transitions in the GB gravity in four-dimensional space-time, the most general case. To our knowledge, this issue is not carefully considered. To our knowledge, this issue is not carefully considered. To our knowledge, this issue is not carefully considered.

This paper is organized as follows. In Sect. II, we discuss the HP phase transition and the GB term. In Sect. III, the HP phase transitions of the Schwarzschild–AdS, RN–AdS, and KN–AdS black holes without and with the GB term are systematically investigated in order. We conclude in Sect. IV. In this paper, we work in the natural system of units and set \(c = G = h = k_B = 1\).

II. BLACK HOLE THERMODYNAMICS IN THE EXTENDED PHASE SPACE

In this section, we outline the thermodynamic properties of the KN–AdS black holes in the extended phase space and discuss the HP phase transition and the GB term in more detail.

A. Thermodynamics of the KN–AdS black holes

The KN–AdS black hole is the most general black hole solution in four-dimensional AdS space. In the Boyer–Lindquist-like coordinates, its metric reads

\[
\begin{align*}
\text{d}s^2 &= -\frac{\Delta_r}{\rho^2} \left( dt - a \frac{\sin^2 \theta}{\Xi} \, d\phi \right)^2 + \frac{\rho^2}{\Delta_r} \, dr^2 \\
&\quad + \frac{\rho^2}{\Delta_r} \, d\theta^2 + \frac{\sin^2 \theta \Delta_\theta}{\rho^2} \left( a \, dt - \frac{r^2 + a^2}{\Xi} \, d\phi \right)^2,
\end{align*}
\]

where

\[
\begin{align*}
\rho^2 &= r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{r^2}, \\
\Delta_r &= \left( r^2 + a^2 \right) \left( 1 + \frac{a^2}{r^2} \right) - 2mr + q^2, \\
\Delta_\theta &= 1 - \frac{a^2}{r^2} \cos^2 \theta,
\end{align*}
\]

and \(m, q, a\) character the mass \(M\), charge \(Q\), and angular momentum \(J\) of the KN–AdS black hole,

\[
M = \frac{m}{\Xi^2}, \quad Q = \frac{q}{\Xi}, \quad J = aM = \frac{am}{\Xi^2}.
\]

The event horizon radius \(r_+\) can be determined as the largest root of \(\Delta_r = 0\), by which the black hole mass is expressed as \(M = \sqrt{(r_+^2 + a^2)(r_+^2 + l^2) + q^4 l^2} / (2r_+l^2\Xi^2)\). To avoid naked singularity, \(r_+\) must be positive, and this sets the lower bound of the KN–AdS black hole mass, \(2M^2 > \sqrt{4J^2 + Q^2 + Q^2}\).

Furthermore, the KN–AdS black hole entropy is obtained by the Bekenstein–Hawking formula as one quarter of the event horizon area \(A\),

\[
S = \frac{A}{4} = \frac{\pi(r_+^2 + a^2)}{\Xi}.
\]

Solving \(r_+\) from \(S\) and using Eqs. (1) and (3), we can reexpress the KN–AdS black hole mass as a function of the thermodynamic quantities, \(S, p, J,\) and \(Q\),

\[
M = \sqrt[4]{\frac{S}{4\pi} \left[ \left( 1 + \frac{\pi Q^2}{S} + 8\rho S \right)^2 + \frac{4\pi^2 J^2}{S^2} \left( 1 + \frac{8\rho S}{3} \right) \right]}.
\]

Differentiating Eq. (5) yields the first law of black hole thermodynamics in the extended phase space,

\[
dM = T \, dS + V \, dp + \Omega \, dJ + \Phi \, dQ,
\]
where $T$, $V$, $\Omega$, and $\Phi$ are the Hawking temperature, thermodynamic volume, angular velocity, and electric potential of the KN–AdS black holes respectively,

$$
T = \left( \frac{\partial M}{\partial S} \right)_{p,J,Q} = \frac{1}{8\pi M} \left[ \left( \frac{\pi Q^2}{S} + \frac{8pS}{3} \right) \left( 1 - \frac{\pi Q^2}{S} + \frac{8pS}{3} \right) - \frac{4p^2J^2}{S^2} \right], 
$$

$$
V = \left( \frac{\partial M}{\partial p} \right)_{s,J,Q} = \frac{2S^2}{3\pi M} \left( \frac{\pi Q^2}{S} + \frac{8pS}{3} + \frac{2p^2J^2}{S^2} \right),
$$

$$
\Omega = \left( \frac{\partial M}{\partial J} \right)_{s,p,Q} = \frac{\pi J}{MS} \left( 1 + \frac{8pS}{3} \right),
$$

$$
\Phi = \left( \frac{\partial M}{\partial Q} \right)_{s,p,J} = \frac{Q}{2M} \left( 1 + \frac{\pi Q^2}{S} + \frac{8pS}{3} \right).
$$

Moreover, in Eq. (6), the $p$–$V$ term has the form of $V\,dp$ but not $-p\,dV$, so the KN–AdS black hole mass $M$ should be essentially identified as its enthalpy instead of internal energy. Furthermore, the Smarr relation, as the integral form of Eq. (6), can be obtained by a scaling argument, $M = 2TS - 2pV + 2\Omega J + \Phi Q$.

### B. HP phase transition

With quantum effects taken into account, a black hole not only absorbs but also emits energy to external environment via the Hawking radiation mechanism [63]. The exchange of energies and particles will establish the thermal equilibrium at a fixed temperature between a stable black hole (with positive heat capacity) and the thermal radiation in the AdS space.

On the one hand, as we have seen in Sect. II.A, the black hole mass should be regarded as enthalpy in the extended phase space, so the thermodynamic potential of interest turns out to be the Gibbs free energy,

$$
G = G(T, p, J, Q) = M - TS.
$$

On the other hand, since the gravitational potential of the AdS space increases at large distances, acting as a box of finite volume, the total energy of thermal radiation is finite. As the particle number of thermal radiation is not finite, the total energy of thermal radiation is not only absorbs but also emits energy to external environment. Consequently, above $T_{\text{HP}}$, the configuration of the black hole with negative Gibbs free energy is thermodynamically preferred; below $T_{\text{HP}}$, the thermal radiation phase with vanishing Gibbs free energy is preferred and is stable against collapse to a black hole. This counterintuitive observation indicates that the thermal radiation in the AdS space behaves more like a solid rather than an ordinary gas.

### C. GB term

The action of the KN–AdS black hole in the GB gravity reads

$$
\frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} + \alpha G),
$$

with

$$
G := R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2
$$

being the GB term, where $R_{\mu\nu\lambda\rho}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar, $F_{\mu\nu}$ is the electromagnetic field tensor, and $\alpha$ is the GB coupling constant.

With the GB term, the KN–AdS black hole entropy can be attained via an integral over the event horizon of the Ricci scalar of the two-dimensional induced metric [46],

$$
S = \frac{1}{4} \int d\theta d\phi \sqrt{h} (1 + \alpha \tilde{R}),
$$

where $\tilde{h} = (r_+^2 + a^2)\sin^2 \theta/\Xi$ is the determinant of the induced metric and $\tilde{R}$ is the Ricci scalar, which can be directly extracted from Eq. (2),

$$
\tilde{R} = \frac{1}{l^2[2r_+^2 + a^2(1 + 2\cos \theta)]^3} \times
$$

$$
\{16l^2r_+^4 - 5a^2(1 + 6\cos \theta)\}
$$

$$
- 8a^2r_+^2[4r_+^2(1 + 3\cos \theta) + l^2(1 + 6\cos \theta)]
$$

$$
- 6a^4[l^2(4 + 8\cos \theta) + r_+^2(7 + 20\cos \theta)]\}.
$$

A straightforward integral shows that the correction to the black hole entropy in Eq. (13) is neatly $4\pi \alpha$, independent of the horizon radius and shape. This is not surprising, as it is a natural result of the Gauss–Bonnet theorem applied to the two-dimensional horizon, so the integral of $\tilde{R}$ simply corresponds to its Euler characteristic. Therefore,

$$
S = \frac{A}{4} + 4\pi \alpha = \frac{\pi(r_+^2 + a^2)}{\Xi} + 4\pi \alpha.
$$

From Eq. (14), a black hole may still have entropy $4\pi \alpha$, when its mass tends to zero. This ambiguity can be removed by the redefinition of black hole entropy via introducing higher order curvature corrections [64]. Furthermore, the Gibbs free energy of the KN–AdS black hole should also receive a correction with the GB term,

$$
G(T, p, J, Q, \alpha) = G(T, p, J, Q) - 4\pi \alpha T.
$$
Equations (14) and (15) reflect the fundamental effects of the GB term on black hole thermodynamics. In extra-dimensional physics without compactification, the GB coupling constant $\alpha$ is proportional to the inverse string tension with positive coefficient [65], so it is always positive. However, in four dimensions, as the GB term is a topological invariant and does not affect space-time, $\alpha$ is free to be chosen both positive and negative. Actually, it was pointed out that only if $\alpha$ is allowed to be negative can there be a reentrant phase transition [62]. Furthermore, the positivity of entropy in Eq. (14) sets a lower bound of $\alpha$ of the phase transition happens, there is also an upper bound $T_\alpha$ and $M_\alpha$ for intuitive physical comprehension. If possible, in order to present a clear mathematical derivation of the Schwarzschild–AdS and KN–AdS black holes, analytical solutions are given in terms of the horizon radius $r_+$. By this means, the HP phase transition temperature $T_{HP}$ can be determined. Second, we solve $S$ from Eqs. (17) and (18) as a function of $T$ and substitute it into Eq. (11) to obtain the black hole Gibbs free energy at arbitrary temperature and pressure. Compared with the vanishing Gibbs free energy of the thermal radiation in the AdS space, the global phase structures of the HP phase transitions can finally be achieved. In brief, we utilize the black hole entropy $S$ as the intermediate variable in all calculations.

We should stress that this method is different from the frequently-used ones in the literature (i.e., express everything as a function of the horizon radius $r_+$). The difference is not evident for the Schwarzschild–AdS and RN–AdS black holes with simple spherical horizons, but is essential for the KN–AdS black holes without spherical horizons. In this circumstance, the calculations via $r_+$ are usually rather tedious and even problematic, and the method via $S$ will prove systematic and efficient.

### III. HP PHASE TRANSITIONS IN THE GB GRAVITY

In this section, we study the HP phase transitions of the Schwarzschild–AdS, RN–AdS, and KN–AdS black holes in the GB gravity in order. In each case, we discuss the relevant issues first without and then with the GB term respectively. For the simple Schwarzschild–AdS and RN–AdS black holes, analytical solutions are given if possible, in order to present a clear mathematical description. However, for the complicated KN–AdS black holes, only numerical results are shown diagrammatically for intuitive physical comprehension.

The basic strategy of our calculations consists of two steps. First, at the HP phase transition point, we substitute Eqs. (17) and (18) into the criterion in Eq. (12) to obtain the black hole entropy $S$ in terms of $p$, $J$, $Q$, and $\alpha$. By this means, the HP phase transition temperature $T_{HP}$ can be determined. Second, we solve $S$ from Eqs. (17) and (18) as a function of $T$ and substitute it into Eq. (11) to obtain the black hole Gibbs free energy at arbitrary temperature and pressure. Compared with the vanishing Gibbs free energy of the thermal radiation in the AdS space, the global phase structures of the HP phase transitions can finally be achieved. In brief, we utilize the black hole entropy $S$ as the intermediate variable in all calculations.

We should stress that this method is different from the frequently-used ones in the literature (i.e., express

\[
M = \sqrt{\frac{S - 4\pi\alpha}{4\pi}} \left\{ \left[ 1 + \frac{\pi Q^2}{S - 4\pi\alpha} + \frac{8p}{3} (S - 4\pi\alpha) \right]^2 + \frac{4\pi^2J^2}{(S - 4\pi\alpha)^2} \left( 1 + \frac{8p}{3} (S - 4\pi\alpha) \right) \right\},
\]

\[
T = \frac{1}{8\pi M} \left\{ \left[ 1 + \frac{\pi Q^2}{S - 4\pi\alpha} + \frac{8p}{3} (S - 4\pi\alpha) \right] \left( 1 + \frac{\pi Q^2}{S - 4\pi\alpha} + \frac{8p}{3} (S - 4\pi\alpha) \right) - \frac{4\pi^2J^2}{(S - 4\pi\alpha)^2} \right\}.
\]

### A. HP phase transitions of the Schwarzschild–AdS black holes

The HP phase transitions of the Schwarzschild–AdS black holes in the GB gravity have been explored in the literature. However, there still remains some subtlety to be clarified, so we discuss them both for completeness and as a demonstration of our calculational method. The same procedure will be applied to the more complicated black hole solutions, and the results below will be taken for comparison in Sects. III B and III C.

First, we begin our discussions without the GB term. For the Schwarzschild–AdS black holes, Eqs. (17) and (18) reduce to

\[
M = \sqrt{\frac{S}{4\pi}} \left( 1 + \frac{8pS}{3} \right),
\]

\[
T = \frac{1}{8\pi M} \left( 1 + \frac{8pS}{3} \right) (1 + 8pS).
\]
Substituting these expressions into Eq. (12), we obtain

\[ S = \frac{3}{8p} \]  

(21)

From Eqs. (19)–(21), the HP phase transition temperature is

\[ T_{\text{HP}} = \sqrt{\frac{8p}{3\pi}}. \]  

(22)

Naturally, this is just the equation of coexistence line in the \( T-p \) phase diagram, as shown in Fig. 1. Since \( p \) has no bound in Eq. (22), there is no terminal point in the coexistence line, and the HP phase transitions can happen at all pressures, without a critical point. Therefore, it is more like a solid–liquid phase transition, rather than a liquid–gas phase transition. Interestingly, the thermal radiation phase even plays the role of a solid, as it lies below the coexistence line [19].

Furthermore, from Eqs. (19) and (20), we can solve \( S \) in terms of \( T \) and \( p \),

\[ S = \frac{1}{8p^2} \left( \pi T^2 - p \pm T \sqrt{\pi^2 T^2 - 2\pi p} \right). \]  

(23)

where ± correspond to large and small black holes respectively. The \( S-T \) curves are plotted in Fig. 2. As the heat capacity at constant pressure is \( C_p = T(\partial S/\partial T)_p \), from the slopes of the \( S-T \) curves, we see that large black holes are thermodynamically stable with positive \( C_p \), but small black holes are unstable with negative \( C_p \) and thus cannot establish the equilibrium with the thermal radiation. Moreover, from Eq. (23), the Schwarzschild–AdS black hole temperature must have a positive minimum,

\[ T_0 = \frac{2p}{\pi}. \]

Actually, \((T_0, S(T_0)) = (\sqrt{2p/\pi}, 1/(8p))\) is just the meeting point of the \( S-T \) curves for large and small black holes in Fig. 2.

Substituting Eqs. (19), (20), and (23) into (11), the Gibbs free energies of large and small Schwarzschild–AdS black holes at arbitrary temperature and pressure are obtained as

\[ G(T, p) = \frac{-4p^2 + 7\pi p T^2 - 2\pi^2 T^4 \pm 5\pi T^3 \sqrt{\pi^2 T^2 - 2\pi p} + 2\pi T^3 \sqrt{\pi^2 T^2 - 2\pi p}}{24p^2(\pi T + \sqrt{\pi^2 T^2 - 2\pi p})}. \]  

(24)
The $G - T$ curves are shown in Fig. 3. The two branches of the curves correspond to large and small black holes and meet with a cusp at $(T_0,G(T_0)) = (\sqrt{2p}/\pi, 1/(12\sqrt{2}\pi p))$. We observe that both Gibbs free energies of large and small black holes decrease with temperature. For the unstable small black holes, the $G - T$ curves are concave and will never reach the $T$-axis ($G$ only tends to 0 when $T \to \infty$), so there is no HP phase transition. On the contrary, for the stable large black holes, their $G - T$ curves are convex and will cross the $T$-axis at the HP phase transition temperatures $T_{\text{HP}}$. With $p$ increasing, $T_{\text{HP}}$ moves rightward, indicating that $T_{\text{HP}}$ increases at high pressures, as shown in Fig. 1. Setting $G = 0$ in Eq. (24), we recover the result of $T_{\text{HP}}$ in Eq. (22). Below $T_{\text{HP}}$, the Gibbs free energy of the thermal radiation is lower than that of the large black hole; above $T_{\text{HP}}$, the Gibbs free energy of the large black hole becomes negative and is thus lower than that of the thermal radiation. As a result, there is a discontinuity in the first order derivatives of the Gibbs free energies of the black hole–thermal radiation system, corresponding to a first order phase transition at $T_{\text{HP}}$.

\[ G(T,p,\alpha) = -4p^2 + 7\pi p T^2 - 2\pi^2 T^4 + 5p T \sqrt{\pi^2 T^2 - 2\pi p} - 2\pi T^3 \sqrt{\pi^2 T^2 - 2\pi p} - 4\pi \alpha T. \]  

Therefore, $G$ decreases with $\alpha$, leading to a lower $T_{\text{HP}}$.

Since the unstable small black holes have no HP phase transition and cannot be in equilibrium with the thermal radiation at all, we will omit the related discussions on them in the following.

Now, we take into account the GB term and discuss its effects on the HP phase transitions of the Schwarzschild–AdS black holes. With the GB term, the black hole mass and temperature should be modified to

\[ M = \sqrt{\frac{S - 4\pi \alpha}{4\pi}} \left[ 1 + \frac{8p}{3}(S - 4\pi \alpha) \right], \]  

\[ T = \frac{1}{8\pi M} \left[ 1 + \frac{8p}{3}(S - 4\pi \alpha) \right] \left[ 1 + 8p(S - 4\pi \alpha) \right]. \]

Therefore, at the HP phase transition point, from Eq. (12) and for large black holes, we have

\[ S = \frac{3}{16p} \left( 1 - \frac{32}{3} \pi \alpha p + \sqrt{1 - \frac{320}{3} \pi \alpha p + 1024\pi^2 \alpha^2 p^2} \right). \]

From Eqs. (25)–(27), we obtain the HP phase transition temperature as

\[ T_{\text{HP}} = \sqrt{\frac{p}{3\pi} \left[ \frac{5}{2} - 48\pi \alpha p + \frac{3}{2} \sqrt{1 - \frac{320}{3} \pi \alpha p + 1024\pi^2 \alpha^2 p^2} \right]}. \]

This result naturally reduces to Eq. (22) if $\alpha$ vanishes.

Furthermore, from Eq. (26), we can solve $S$ in terms of $T$, $p$, and $\alpha$,

\[ S = \frac{1}{8p^2} \left( \pi T^2 - p + T \sqrt{\pi^2 T^2 - 2\pi p} \right) + 4\pi \alpha, \]

so there is only a shift $4\pi \alpha$ in entropy, as expected in Eq. (14). Besides, the minimal black hole temperature, $T_0 = \sqrt{2p/\pi}$, remains unchanged in the presence of $\alpha$.

Then, from Eq. (15), the Gibbs free energies of large Schwarzschild–AdS black holes with the GB term are

\[ G(T,p,\alpha) = -4p^2 + 7\pi p T^2 - 2\pi^2 T^4 + 5p T \sqrt{\pi^2 T^2 - 2\pi p} - 2\pi T^3 \sqrt{\pi^2 T^2 - 2\pi p} - 4\pi \alpha T. \]
curves for the Schwarzschild–AdS black holes with the GB term, a crucial issue must be carefully elucidated, that is, the lower and upper bounds of the GB coupling constant $\alpha$. The bounds of $\alpha$ come from the two corrections by the GB term to the black hole entropy and Gibbs free energy: $S + 4\pi \alpha$ and $G - 4\pi \alpha T$. On the one hand, $\alpha$ cannot be too negative, otherwise the minimum of $S$ would be negative, in contradiction to the positivity of entropy; on the other hand, $\alpha$ cannot be too positive either, otherwise the maximum of $G$ would be negative, and there would be no HP phase transition. First, from Eq. (28), $S > S(T_0) = 1/(8\pi) + 4\pi \alpha > 0$, so $\alpha$ has a lower bound, $\alpha > -1/(32\pi p)$, consistent with the condition $\alpha > -r_s^2/4$ in Sect. II.B. Second, because the Gibbs free energy is a monotonically decreasing function of temperature, the maximum of $G$ should be evaluated as $G(T_0) = 1/(12\sqrt{2\pi p}) - 4\sqrt{2\pi p}$. This value must be positive to guarantee the HP phase transition, so it sets an upper bound of $\alpha$, $\alpha < 1/(96\pi p)$. Altogether, $\alpha$ has both lower and upper bounds simultaneously,

$$-\frac{1}{32\pi p} < \alpha < \frac{1}{96\pi p}. \quad (31)$$

However, we must point out that this is true only for the Schwarzschild–AdS black holes. For the RN–AdS and KN–AdS black holes, $\alpha$ has only the lower bounds but no upper ones, to be explained in Sect. III.B. Moreover, the bounds in Eq. (31) automatically satisfy the requirements that the terms inside the square roots must be positive in Eq. (28).

The bounds of $\alpha$ cause great difference between the coexistence lines of the Schwarzschild–AdS black holes without and with the GB term. From Eq. (31),

$$p < p_{\text{max}} = \frac{1}{96\pi \alpha} \quad (\text{for } \alpha > 0),$$

or

$$p < p_{\text{max}} = -\frac{1}{32\pi \alpha} \quad (\text{for } \alpha < 0). \quad (32)$$

Therefore, unless $\alpha = 0$, no matter how positive or negative it is, there is a corresponding upper bound of pressure. As a result, there must be terminal points in the coexistence lines, and the HP phase transitions can happen only below the critical pressures $p_{\text{max}}$. From Eq. (28), the coexistence lines of the Schwarzschild–AdS black holes with the GB term are plotted in Fig. 4, with different values of $\alpha$. We find that $T_{\text{HP}}$ decreases with $\alpha$ at a fixed pressure.

Last, the $G$–$T$ curves of large Schwarzschild–AdS black holes with the GB term are shown in Fig. 5, with different values of $\alpha$. According to Eq. (30), all these curves set off from the same minimal temperature $T_0 = \sqrt{2p}/\pi$, and the effect of the GB term is just to proportionally translate the $G$–$T$ curves downward when $\alpha$ increases, inducing a lower $T_{\text{HP}}$.

**B. HP phase transitions of the RN–AdS black holes**

We continue to study the HP phase transitions of the RN–AdS black holes in the GB gravity. This issue was mentioned in Ref. [62], but the corresponding discussions were absent. Here, we apply both analytical and
numerical methods to reconsider this problem.

In principle, the procedures are in parallel to those in Sect. IIIA, but this does not mean that the whole process is merely a repetition. There are intrinsic differences between the results of the Schwarzschild–AdS and RN–AdS black holes, which will be shown explicitly below.

First, for the RN–AdS black holes without the GB term, from Eqs. (17) and (18), we have

\[ M = \sqrt{\frac{S}{4\pi}} \left( 1 - \frac{\pi Q^2}{S} + \frac{8pS}{3} \right), \]  
\[ T = \frac{1}{8\pi M} \left( 1 + \frac{\pi Q^2}{S} + \frac{8pS}{3} \right) \left( 1 - \frac{\pi Q^2}{S} + 8pS \right). \]  

(33)  
(34)

At the HP phase transition point, substituting Eqs. (33) and (34) into (12), for large black holes, we obtain

\[ S = \frac{3}{16p} \left( 1 + \sqrt{1 + 32\pi^2 p Q^2} \right). \]  

(35)

Using Eqs. (33)–(35), we obtain the HP phase transition temperature of the RN–AdS black holes as

\[ T_{HP} = 4 \sqrt{\frac{p}{3\pi}} \frac{1 + \pi p Q^2/3 + \sqrt{1 + 32\pi^2 p Q^2}}{1 + \sqrt{1 + 32\pi^2 p Q^2}}. \]  

(36)

This result naturally reduces to Eq. (22) if \( Q \) vanishes, and again there is no bound of \( p \), so the HP phase transitions can happen at all pressures. The coexistence lines are shown in Fig. 6, with different values of \( Q \), and at a given pressure, \( T_{HP} \) is found to increase with \( Q \).

![FIG. 6: The HP phase transition temperature of the RN–AdS black holes as a function of pressure, with different values of \( Q \). The HP phase transitions can happen at all pressures, and \( T_{HP} \) increases with \( Q \) at a fixed pressure.](image_url)

From Eqs. (33) and (34), we can again solve \( S \) in terms of \( T, p, \) and \( Q \). The exact solutions are very lengthy and will not be shown here. The \( S–T \) curves for large RN–AdS black holes are shown in Fig. 7.

Here, we should emphasize that there is an obvious difference in the \( S–T \) curves between the Schwarzschild–AdS and RN–AdS black holes in Figs. 2 and 7. In the Schwarzschild–AdS case, from Eq. (23), there exists a minimal black hole temperature \( T_0 = \sqrt{2p}/\pi \). However, in the RN–AdS case, as shown above, there is no such a bound, and \( T \) is allowed to reach zero. This contrast is very important for the subsequent discussions and thus needs more explanation. From Eqs. (33) and (34), the RN–AdS black hole temperature \( T \) can be written as a function of \( S \),

\[ T = \frac{1}{4\sqrt{\pi S}} \left( 1 - \frac{\pi Q^2}{S} + 8pS \right). \]  

(37)

In addition to the two positive terms \( 1 + 8pS \) already present in the Schwarzschild–AdS case, there is a new negative term \( -\pi Q^2/S \) sourced from the charge. It is this negative term that allows \( T \) to be zero. ¹ Moreover,

¹ We should point out that, if \( Q^2 < 1/(96\pi p) \), there will be oscillatory behaviors in the \( T–S \) curves, resulting in both stable small and large black hole solutions. The phase transitions between the small black hole–large black hole–thermal radiation system (particularly with the GB term) are very complicated. The relevant discussions are beyond the scope of the present work and will be left for future research.
at zero temperature, $S$ also arrives at its minimum,

$$S(0) = \frac{\sqrt{1 + 32\pi p Q^2} - 1}{16p}. \quad (38)$$

Next, from Eqs. (33) and (34), the Gibbs free energies of the RN–AdS black holes at arbitrary temperature and pressure can be obtained straightforwardly, and again the tedious exact expressions are not shown here. The $G$–$T$ curves are plotted in Fig. 8, with different values of $Q$. From Fig. 8, we clearly observe two features: first, the HP phase transition temperature $T_{\text{HP}}$ increases with $Q$, as expected from Eq. (36); second, the RN–AdS black hole temperatures $T$ can be zero, without lower bounds, as explained above.

Below, for the HP phase transitions of the RN–AdS black holes, we repeat the above calculations, by replacing $S$ to $S - 4\pi\alpha$ in Eqs. (33) and (34). All the corresponding results do have exact solutions but are entirely omitted due to their formal complexities.

Again, before plotting the coexistence lines and the $G$–$T$ curves for the RN–AdS black holes with the GB term, the bounds of $\alpha$ must be determined with care, and we will find that the bounds notably differ from those of the Schwarzschild–AdS black holes. In the Schwarzschild–AdS case, $\alpha$ has both lower and upper bounds in Eq. (31), and there are corresponding upper bounds of $p$ in Eq. (32), resulting in the terminal points in the coexistence lines for any non-vanishing $\alpha$ in Fig. 4. In contrast, in the RN–AdS case, $\alpha$ has only a lower bound but no upper one. Hence, there is also no upper bound of $p$ for positive $\alpha$, and the coexistence lines accordingly have no terminal points. This is because from Eqs. (14) and (38), the minimum of the RN–AdS black hole entropy with the GB term, $S(0) + 4\pi\alpha$, must be positive, and this sets the lower bound of $\alpha$,

$$\alpha > \frac{1 - \sqrt{1 + 32\pi p Q^2}}{64\pi p}. \quad (39)$$

Hence, the corresponding upper bound of $p$ is

$$p < p_{\text{max}} = \frac{Q^2 + 4\alpha}{128\pi\alpha^2} \quad \text{(for } \alpha < 0), \quad (40)$$

and the bound of $\alpha$ in Eq. (39) keeps the numerator in Eq. (40) positive definite. Nevertheless, there is no upper bound of $\alpha$ for the RN–AdS black holes. Because the Gibbs free energy decreases with temperature, its maximum should be evaluated at the minimal temperature. Now, this minimal temperature is zero, as explained in Eq. (37), so no matter how large $\alpha$ is, the correction $-4\pi\alpha T$ in the Gibbs free energy always vanishes at zero temperature. This is completely different from the Schwarzschild–AdS case with the minimal temperature $T_0 = \sqrt{2p/\pi} > 0$ and the correction $-4\pi\alpha T_0 < 0$ when $\alpha > 0$. Therefore, for positive $\alpha$, the HP phase transitions can always happen for the RN–AdS black holes, and there is no upper bound of $\alpha$ or $p$, and also no terminal points in the coexistence lines.

The coexistence curves of the RN–AdS black holes are shown in Fig. 9, with $Q = 1$ and different values of $\alpha$. We see that $T_{\text{HP}}$ decreases with $\alpha$ at a given pressure, and this trend is the same as the Schwarzschild–AdS case. However, there is only one terminal point in the coexistence line for negative $\alpha$ but none for positive $\alpha$ anymore, different from the Schwarzschild–AdS case with terminal points for any non-vanishing $\alpha$ in Fig. 4.

The $G$–$T$ curves of the RN–AdS black holes are plotted in Fig. 10, with $Q = 1$, $p = 1/(48\pi)$, and different values of $\alpha$. We observe that $T_{\text{HP}}$ decreases with $\alpha$, consistent with the analysis of the coexistence lines in Fig. 9. Moreover, since there is no lower bound of $T$, all the $G$–$T$ curves set out from the same point at zero temperature, rather than from the different points at temperature $T_0$ in the Schwarzschild–AdS case in Fig. 5.

Till now, we have explained in detail the significant differences between the coexistence lines in Figs. 4 and 9 and between the $G$–$T$ curves in Figs. 5 and 10. Actually, all these differences stem from the different bounds of $\alpha$ and are again the direct consequence from the different minimal temperatures of the Schwarzschild–AdS and RN–AdS black holes.

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2 One may wonder why this lower bound cannot reduce to $-1/(32\pi p)$ for the Schwarzschild–AdS black holes in Eq. (31) if $Q = 0$. However, this is not a discrepancy at all, as for the Schwarzschild–AdS and RN–AdS black holes, their minimal entropies $S(T_0)$ and $S(0)$ are evaluated at totally different minimal temperatures, $T_0 = \sqrt{2p/\pi}$ and 0.

3 Here, we choose $Q^2 > 1/(96\pi p)$, in order to avoid the intricate van der Waals-like characters (e.g., the swallowtails in the $G$–$T$ curves), which are irrelevant to the HP phase transitions.
FIG. 9: The HP phase transition temperature of the RN–AdS black holes (\(Q = 1\)) with the GB term as a function of pressure. \(T_{\text{HP}}\) decreases with \(\alpha\) at a given pressure, as same as the Schwarzschild–AdS case, but there is no terminal point in the coexistence line for positive \(\alpha\), different from the Schwarzschild–AdS case in Fig. 4.

C. HP phase transitions of the KN–AdS black holes

Finally, we present the complete picture of the HP phase transitions of the most general KN–AdS black holes in the GB gravity. Because rotation breaks the spherical symmetry of horizon topology, the calculational difficulties greatly increase. Therefore, we jump the step for the rotating Kerr–AdS black holes and proceed directly to the charged and rotating KN–AdS black holes, as their complexities are almost the same. In fact, most of the calculations deal with the algebraic equations of \(S\) of degrees higher than four that cannot be solved analytically, so all the results below are obtained numerically. In one word, everything will be illustrated diagrammatically below.

Without the GB term, the black hole mass and temperature are shown in Eqs. (5) and (7). By the same procedure as before, we obtain the HP phase transition temperature \(T_{\text{HP}}\) in terms of \(p\), \(J\), and \(Q\), as shown in Figs. 11(a) and 11(b). We find that \(T_{\text{HP}}\) increases with both \(J\) and \(Q\).

FIG. 10: The Gibbs free energies of the RN–AdS black holes as a function of temperature, with \(Q = 1\), \(p = 1/(48\pi)\), and different values of \(\alpha\). All the \(G-T\) curves start from the same point at zero temperature, entirely different from the Schwarzschild–AdS case in Fig. 5.

FIG. 11: The HP phase transition temperature of the KN–AdS black holes as a function of pressure. \(T_{\text{HP}}\) increases with both \(J\) and \(Q\), with the detailed values of \(J\) and \(Q\) listed in each panel.

For the Gibbs free energy, the \(G-T\) curves are plotted...
in Figs. 12(a) and 12(b), with different values of \( J \) and \( Q \). \(^4\) Again, we find that \( T_{HP} \) increases with both \( J \) and \( Q \) at a given pressure, and the black hole temperatures can go to zero, without a lower bound. All these features are qualitatively similar with those of the RN–AdS black holes.

Adding the GB term makes the situations more complicated, and the coexistence lines and the \( G-T \) curves are numerically plotted in Figs. 13(a) and 13(b), with \( J = 1 \), \( Q = 1 \), and different values of \( \alpha \). From these figures, \( T_{HP} \) is found to decrease with \( \alpha \), with \( J \) and \( Q \) fixed. The KN–AdS black hole temperatures can still reach zero with the GB term. As a result, there is only a lower bound of \( \alpha \) and an upper bound of \( p \), so the terminal points in the coexistence lines only exist for negative \( \alpha \). Also, all the \( G-T \) curves set off from the same point at the \( G \)-axis. In summary, all the characteristics of the KN–AdS black holes with the GB term are also very analogous to those of the RN–AdS black holes.

Last, we should make an important comment on the numerical technology in this subsection. In our calculations, we must first express \( M \) and \( T \) in terms of \( S \) and then utilize \( S \) as the intermediate variable in deriving \( T_{HP} \) and \( G \), instead of the horizon radius \( r_+ \). This difference is not evident for the Schwarzschild–AdS and RN–AdS black holes with spherical horizons, but is distinct for the KN–AdS black holes with non-spherical ones. At present, if we express \( M \) and \( T \) in terms of \( r_+ \), we have to use the variables \( q \) and \( a \) and in Eq. (3). Unfortunately, they are again related to \( \Xi = 1 - a^2/l^2 \), so \( q \) and \( a \) cannot be regarded as independent variables like \( Q \) and \( J \). Therefore, the calculations by virtue of \( r_+ \), \( q \), and \( a \) in the KN–AdS case are not only inconvenient, but also very dangerous to lead to false conclusions, e.g., without the GB term, the HP phase transitions could only happen below a critical pressure. However, as shown in Figs. 11(a) and 11(b), they can happen at all pressures, if we use \( Q \) and \( J \) as independent variables.

**IV. CONCLUSION**

The GB gravity is the minimal extension of the Einstein gravity, including the latter as the low energy and small curvature limit. In four-dimensions, the GB term is a topological invariant and is thus trivial to gravitational dynamics. However, it influences black hole thermodynamics via introducing the corrections to the black hole entropy and Gibbs free energy as \( S + 4\pi a \) and \( G - 4\pi a T \). Therefore, the GB term significantly affects one of the most important issues in black hole thermodynamics—the HP phase transition between a stable black hole and the thermal radiation in the AdS space.

In this paper, the HP phase transitions of the Schwarzschild–AdS, RN–AdS, and KN–AdS black holes in the extended phase space are systematically investigated in the GB gravity. In the extended phase space, the cosmological constant in the AdS space is effectively interpreted as a varying thermodynamic pressure \( p \). Then, the HP phase transition temperature \( T_{HP} \) as a function of \( p \) and the Gibbs free energy \( G \) as a function of \( T \) are calculated in detail. The basic conclusions of our work can be drawn as follows.

(1) The HP phase transition temperature \( T_{HP} \) is an increasing function of \( p \). Below or above \( T_{HP} \), the thermal

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\(^4\) We should state that the values of \( J \), \( Q \), and \( p \) are specially chosen, such that there are only stable large KN–AdS black holes, but no stable or unstable small ones, to avoid the irrelevant complications. Case is the same for Fig. 13(b).
radiation or large black hole phase is thermodynamically preferred, meaning that the thermal radiation is more like a solid rather than a gas in the extended phase space. For charged and rotating black holes, both charge and angular momentum enhance $T_{\text{HP}}$. If the GB term is taken into account, the GB coupling constant $\alpha$ decreases $T_{\text{HP}}$, because it induces a correction $-4\pi\alpha T$ in the Gibbs free energy, and the $G-T$ curves thus move downward and intersect the $T$-axis at lower temperatures.

(2) For the black hole temperature, in the Schwarzschild–AdS case, there is a positive minimum $T_0 = \sqrt{2p/\pi}$, which is unchanged with the GB term. However, there is no such a bound in the RN–AdS (KN–AdS) case, as there is a negative term $-\pi Q^2/S$ in Eq. (37), the RN–AdS (KN–AdS) black hole temperature is thus allowed to be zero, also in the GB gravity.

(3) The two corrections from the GB term, $S + 4\pi\alpha$ and $G - 4\pi\alpha T$, give rise to the bounds of $\alpha$. In the Schwarzschild–AdS case, there are both lower and upper bounds of $\alpha$ and the corresponding upper bounds of $p$, so there are terminal points in the coexistence lines for any non-vanishing $\alpha$ in Fig. 4. However, in the RN–AdS (KN–AdS) case, since the minimal black hole temperature is zero, there is only a lower bound of $\alpha$ and a related upper bound of $p$, so the coexistence lines end at the terminal points only for negative $\alpha$ and exist all the way for positive $\alpha$ in Fig. 9 [Fig. 13(a)].

(4) For the Gibbs free energy, in the Schwarzschild–AdS case, the $G-T$ curves start from different points but at the same minimal temperature $T_0$, and $\alpha$ translates the curves downward in Fig. 5. In contrast, in the RN–AdS (KN–AdS) case, all the $G-T$ curves start from the same point at the $G$-axis in Fig. 10 [Fig. 13(b)], since the correction $-4\pi\alpha T$ vanishes at zero temperature.

In summary, the properties of the RN–AdS and KN–AdS black holes in the HP phase transitions in the GB gravity are qualitatively very similar, but obviously deviate from those of the Schwarzschild–AdS black holes. Generally speaking, all the distinctions originate from the minimal black hole temperatures. For the Schwarzschild–AdS black holes in the extended phase space, due to the effective pressure, the black hole temperature cannot reach zero. Whereas, for the RN–AdS and KN–AdS black holes, thanks to the charge and angular momentum, the black hole temperatures can be zero, bringing completely different behaviors from the Schwarzschild–AdS case, such as the different bounds of $\alpha$ and $p$ and also the different shapes of the coexistence lines. Altogether, we hope to present the whole picture of the HP phase transitions in the extended phase space in the GB gravity, as we have seen, even in the simplest Schwarzschild–AdS case, there is still some interesting issue to be explored.

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[1] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D 7, 2333 (1973).
[2] J. M. Bardeen, B. Carter, and S. W. Hawking, The four
laws of black hole mechanics, Commun. Math. Phys. 31, 161 (1973).

[3] M. M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Classical Quantum Gravity 17, 399 (2000), arXiv:hep-th/9908022[hep-th].

[4] D. Kastor, S. Ray, and J. Traschen, Entropy and the mechanics of AdS black holes, Classical Quantum Gravity 26, 195011 (2009), arXiv:0904.2765[hep-th].

[5] B. P. Dolan, The cosmological constant and the black-hole thermodynamic potentials, Classical Quantum Gravity 28, 252010 (2011), arXiv:1008.5023[gr-qc].

[6] B. P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Classical Quantum Gravity 28, 235011 (2011), arXiv:1106.6260[gr-qc].

[7] M. Cvetić, G. W. Gibbons, D. Kubizňák, and C. N. Pope, Black hole entropy and an entropy inequality for the thermodynamic volume, Phys. Rev. D 84, 024037 (2011), arXiv:1012.2888[hep-th].

[8] D. Kubizňák and R. B. Mann, P-V criticality of charged AdS black holes, J. High Energy Phys. 1207, 033 (2012), arXiv:1205.0599[gr-qc].

[9] S. Gulsekaran, D. Kubizňák, and R. B. Mann, Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, J. High Energy Phys. 1211, 110 (2012), arXiv:1208.6251[hep-th].

[10] A. Rajagopal, D. Kubizňák, and R. B. Mann, Van der Waals black hole, Phys. Lett. B 737, 277 (2014), arXiv:1408.1105[gr-qc].

[11] R. A. Hennigar, R. B. Mann, and D. Kubizňák, Entropy inequality violations from ultraspinning black holes, Phys. Rev. Lett. 115, 031101 (2015), arXiv:1411.4309[hep-th].

[12] R. A. Hennigar, R. B. Mann, and E. Tjoa, Superfluid black holes, Phys. Rev. Lett. 118, 021301 (2017), arXiv:1609.02564[hep-th].

[13] N. Altamirano, D. Kubizňák, and R. B. Mann, Reentrant phase transitions in rotating anti-de Sitter black holes, Phys. Rev. D 88, 101502(R) (2013), arXiv:1306.5756[hep-th].

[14] R. A. Hennigar, F. McCarthy, A. Ballon, and R. B. Mann, Holographic heat engines: general considerations and rotating black holes, Classical Quantum Gravity 34, 175005 (2017), arXiv:1704.02314[hep-th].

[15] Z.-W. Zhao, Y.-H. Xu, and N. Li, Throttling process of the Kerr-Newman-anti-de Sitter black holes in the extended phase space, Phys. Rev. D 98, 124003 (2018), arXiv:1805.04861[gr-qc].

[16] B. P. Dolan, D. Kastor, D. Kubizňák, R. B. Mann, and J. Traschen, Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes, Phys. Rev. D 87, 104017 (2013), arXiv:1301.5926[hep-th].

[17] S.-W. Wei and Y.-X. Liu, Insight into the microscopic structure of an AdS black hole from a thermodynamical phase transition, Phys. Rev. Lett. 115, 111302 (2015), arXiv:1502.00386[gr-qc].

[18] E. Caceres, P. H. Nguyen, and J. F. Pedraza, Holographic entanglement entropy and the extended phase structure of STU black holes, J. High Energy Phys. 1509, 184 (2015), arXiv:1507.06069[hep-th].

[19] D. Kubizňák, R. B. Mann, and M. Teo, Black hole chemistry: thermodynamics with Lambda, Classical Quantum Gravity 34, 063001 (2017), arXiv:1608.06147[hep-th].

[20] S.-B. Chen, X.-F. Liu, and C.-Q. Liu, P-V criticality of an AdS black hole in f(R) gravity, Chin. Phys. Lett. 30, 060401 (2013), arXiv:1301.3234[gr-qc].

[21] R. Zhao, H.-H. Zhao, M.-S. Ma, and L.-C. Zhang, On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes, Eur. Phys. J. C 73, 2645 (2013), arXiv:1305.3725[gr-qc].

[22] M. B. Jahani Poshteh, B. Mirza, and Z. Sherkatghanad, Phase transition, critical behavior, and critical exponents of Myers-Perry black holes, Phys. Rev. D 88, 024005 (2013), arXiv:1306.4516[gr-qc].

[23] D.-C. Zou, S.-J. Zhang, and B. Wang, Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics, Phys. Rev. D 89, 044002 (2014), arXiv:1311.7299[hep-th].

[24] H. Xu, W. Xu, and L. Zhao, Extended phase space thermodynamics for third-order Lovelock black holes in diverse dimensions, Eur. Phys. J. C 74, 3074 (2014), arXiv:1405.4143[hep-th].

[25] R. A. Hennigar, W. G. Brenna, and R. B. Mann, P-V criticality in quasi-totopological gravity, J. High Energy Phys. 1507, 077 (2015), arXiv:1508.05517[hep-th].

[26] J. Xu, L.-M. Cao, and Y.-P. Hu, P-V criticality in the extended phase space of black holes in massive gravity, Phys. Rev. D 91, 124033 (2015), arXiv:1506.03578[gr-qc].

[27] S.-L. Li, H.-D. Lyu, H.-K. Deng, and H. Wei, P-V criticality in gauged supergravities, Eur. Phys. J. C 79, 201 (2019), arXiv:1809.03471[gr-qc].

[28] A. G. Tzikas, Bardeen black hole chemistry, Phys. Lett. B 788, 219 (2019), arXiv:1811.01104[gr-qc].

[29] R.-G. Cai, Gauss-Bonnet black holes in AdS spaces, Phys. Rev. D 65, 084014 (2002), arXiv:hep-th/0109133[hep-th].

[30] M. Cvetič, S. Nojiri, and S. D. Odintsov, Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity, Nucl. Phys. B 628, 295 (2002), arXiv:hep-th/0112045[hep-th].

[31] B. Kleihaus, J. Kunz, and E. Radu, Rotating black holes in dilatonic Einstein-Gauss-Bonnet theory, Phys. Rev. Lett. 106, 151104 (2011), arXiv:1101.2868[gr-qc].

[32] P. Kanti, B. Kleihaus, and J. Kunz, Wormholes in dilatonic Einstein-Gauss-Bonnet theory, Phys. Rev. Lett. 107, 271101 (2011), arXiv:1108.3003[gr-qc].

[33] S. H. Hendi, S. Panahiyan, and B. Eslam Panah, Charged black hole solutions in Gauss-Bonnet-Massive Gravity, J. High Energy Phys. 1601, 129 (2016), arXiv:1507.06663[hep-th].

[34] D. D. Doneva and S. S. Yazadjiev, New Gauss-Bonnet black holes with curvature-induced scalarization in extended scalar-tensor theories, Phys. Rev. Lett. 120, 131103 (2018), arXiv:1711.01187[gr-qc].

[35] J. E. Lidsey and N. J. Nunes, Inflation in Gauss-Bonnet braneworld cosmology, Phys. Rev. D 67, 103510 (2003), arXiv:astro-ph/0303168[astro-ph].

[36] S. Nojiri, S. D. Odintsov, and M. Sasaki, Gauss-Bonnet dark energy, Phys. Rev. D 71, 123509 (2005), arXiv:hep-th/0504052[hep-th].

[37] T. Koivisto and D. F. Mota, Cosmology and astrophysical constraints of Gauss-Bonnet dark energy, Phys. Lett. B 644, 104 (2007), arXiv:astro-ph/0606078[astro-ph].

[38] T. Koivisto and D. F. Mota, Gauss-Bonnet quintessence: background evolution, large scale structure and cosmological constraints, Phys. Rev. D 75, 023518 (2007),
[39] B. Li, J. D. Barrow, and D. F. Mota, *The cosmology of modified Gauss-Bonnet gravity*, Phys. Rev. D **76**, 044027 (2007), arXiv:0705.3795 [gr-qc].

[40] S.-W. Wei and Y.-X. Liu, *Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes*, Phys. Rev. D **87**, 044014 (2013), arXiv:1209.1707 [gr-qc].

[41] R.-G. Cai, L.-M. Cao, L. Li, and R.-Q. Yang, *P-V criticality in the extended phase space of Gauss-Bonnet black holes in AdS space*, J. High Energy Phys. **1309**, 005 (2013), arXiv:1306.6233 [gr-qc].

[42] W. Xu, X. Hu, and L. Zhao, *Gauss-Bonnet coupling constant as a free thermodynamical variable and the associated criticality*, Eur. Phys. J. C **74**, 2970 (2014), arXiv:1311.3053 [gr-qc].

[43] D.-C. Zou, Y. Liu, and B. Wang, *Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble*, Phys. Rev. D **90**, 044063 (2014), arXiv:1404.5194 [hep-th].

[44] S. H. Hendi, S. Panahiyan, and M. Momennia, *Extended phase space of AdS Black Holes in Einstein-Gauss-Bonnet gravity with a quadratic nonlinear electrodynamics*, Int. J. Mod. Phys. D **25**, 1650063 (2016), arXiv:1503.03340 [gr-qc].

[45] Y.-G. Miao and Z.-M. Xu, *Parametric phase transition for a Gauss-Bonnet AdS black hole*, Phys. Rev. D **98**, 084051 (2018), arXiv:1806.10393 [hep-th].

[46] T. Chuan, S. F. Ross, and D. J. Smith, *On Gauss-Bonnet black hole entropy*, Classical Quantum Gravity **21**, 3447 (2004), arXiv:gr-qc/0402044 [gr-qc].

[47] S. W. Hawking and D. N. Page, *Thermodynamics of black holes in anti-de Sitter space*, Commun. Math. Phys. **87**, 577 (1983).

[48] D. Birmingham, I. Sachs, and S. N. Solodukhin, *Relaxation in conformal field theory, Hawking-Page transition, and quasinormal/nornal modes*, Phys. Rev. D **67**, 104026 (2003), arXiv:hep-th/0212308 [hep-th].

[49] R.-G. Cai, S. P. Kim, and B. Wang, *Ricci flat black holes and Hawking-Page phase transition in Gauss-Bonnet gravity and dilaton gravity*, Phys. Rev. D **76**, 024011 (2007), arXiv:0705.2469 [hep-th].

[50] P. Nicolini and G. Torrieri, *The Hawking-Page crossover in noncommutative anti-de Sitter space*, J. High Energy Phys. **1108**, 097 (2011), arXiv:1105.0188 [gr-qc].

[51] M. Eune, W. Kim, and S.-H. Yi, *Hawking-Page phase transition in BTZ black hole revisited*, J. High Energy Phys. **1303**, 020 (2013), arXiv:1301.0395 [gr-qc].

[52] A. Adams, D. A. Roberts, and O. Saremi, *Hawking-Page transition in holographic massive gravity*, Phys. Rev. D **91**, 046003 (2015), arXiv:1408.6650 [hep-th].

[53] M. Bañados, G. Düring, A. Faraggi, and I. A. Reyes, *Phases of higher spin black holes: Hawking-Page, transitions between black holes and a critical point*, Phys. Rev. D **96**, 046017 (2017), arXiv:1611.08025 [hep-th].

[54] V. G. Czinner and H. Iguchi, *Thermodynamics, stability and Hawking-Page transition of Kerr black holes from Rényi statistics*, Eur. Phys. J. C **77**, 892 (2017), arXiv:1702.05341 [gr-qc].

[55] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *Charged AdS black holes and catastrophic holography*, Phys. Rev. D **60**, 064018 (1999), arXiv:hep-th/9902170 [hep-th].

[56] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *Holography, thermodynamics, and fluctuations of charged AdS black holes*, Phys. Rev. D **60**, 104026 (1999), arXiv:hep-th/9904197 [hep-th].

[57] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2**, 505 (1998), arXiv:hep-th/9803131 [hep-th].

[58] C. P. Herzog, *A holographic prediction of the deconfinement temperature*, Phys. Rev. Lett. **98**, 091601 (2007), arXiv:hep-th/0608151 [hep-th].

[59] E. Spallucci and A. Smilagic, *Maxwell’s equal area law and the Hawking-Page phase transition*, J. Grav. **1**, 005 (2014), arXiv:1310.2186 [hep-th].

[60] A. Sahay and R. Jha, *Geometry of criticality, supercriticality and Hawking-Page transitions in Gauss-Bonnet-AdS black holes*, Phys. Rev. D **96**, 126017 (2017), arXiv:1707.03629 [hep-th].

[61] S. Mbarek and R. B. Mann, *Reverse Hawking-Page phase transition in de Sitter black holes*, J. High Energy Phys. **1902**, 103 (2019), arXiv:1804.03349 [hep-th].

[62] W. Xu, C. y. Wang, and B. Zhu, *Effects of Gauss-Bonnet term on the phase transition of a Reissner-Nordström-AdS black hole in (3+1) dimensions*, Phys. Rev. D **99**, 044010 (2019).

[63] S.W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. **43**, 199 (1975).

[64] H. Lü, A. Perkins, C. N. Pope, and K. S. Stelle, *Black holes in higher-derivative gravity*, Phys. Rev. Lett. **114**, 171601 (2015), arXiv:1502.01028 [hep-th].

[65] D. G. Boulware and S. Deser, *String generated gravity models*, Phys. Rev. Lett. **55**, 2656 (1985).