A nonexistence result for CMC surfaces in hyperbolic 3-manifolds

William H. Meeks III       Alvaro K. Ramos*

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Abstract

We prove that a complete hyperbolic 3-manifold of finite volume does not admit a properly embedded noncompact surface of finite topology with constant mean curvature greater than or equal to 1.

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1 Introduction.

We continue the study of properly immersed surfaces of constant mean curvature $H$ in hyperbolic 3-manifolds of finite volume that began with the works of Collin, Hauswirth, Mazet and Rosenberg [2, 4] in the minimal case, and was extended to the $H \in (0, 1)$ case by the authors [6].

In this paper we prove:

Theorem 1.1. A complete hyperbolic 3-manifold of finite volume does not admit a properly embedded noncompact surface of finite topology with constant mean curvature $H \geq 1$.

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Theorem 1.1 contrasts with [6, Proposition 4.8], where it is shown that, for any \( H \geq 1 \) and any noncompact hyperbolic 3-manifold \( N \) of finite volume, there exists a complete, properly immersed annulus with constant mean curvature \( H \). Therefore, the hypothesis of embeddedness in Theorem 1.1 is necessary. Moreover, in [1], together with Adams, we proved that for any \( H \in [0, 1) \) and any surface \( S \) of finite negative Euler characteristic, there exists a hyperbolic 3-manifold of finite volume with a proper embedding of \( S \) with constant mean curvature \( H \). Furthermore, there are examples of closed surfaces in hyperbolic 3-manifolds of finite volume for any \( H \geq 1 \); namely geodesic spheres and tori and Klein bottles in its cusp ends.

The work of the first author with Tinaglia [7] allows one to replace the hypothesis of properness in Theorem 1.1 by the weaker assumption of completeness, since [7] shows that any complete, embedded, finite topology surface of constant mean curvature \( H \geq 1 \) in a complete hyperbolic 3-manifold is proper. Finally, there remains the question of whether or not there exist properly embedded surfaces of infinite topology and constant mean curvature \( H \geq 1 \) in hyperbolic 3-manifolds of finite volume.

2 Proof of Theorem 1.1.

Theorem 1.1 follows directly from next lemma.

**Lemma 2.1.** A complete hyperbolic 3-manifold of finite volume \( N \) does not admit a proper embedding of \( A = \mathbb{S}^1 \times [0, \infty) \) with constant mean curvature \( H \geq 1 \).

**Proof.** After passing to the oriented two-sheeted cover of \( N \), we may assume without loss of generality that \( N \) is orientable.

Arguing by contradiction suppose that \( E \subset N \) is the image of a proper embedding of \( A \) as stated in the lemma. Since \( E \) is proper and \( N \) is an orientable hyperbolic 3-manifold of finite volume, there exists some cusp end \( C \) of \( N \) with the following properties:

1. \( \partial E \cap C = \emptyset \).
2. \( \partial C \) is a flat torus \( \mathcal{T}(0) \) which intersects \( E \) transversely in a finite collection of pairwise disjoint simple closed curves.
3. \( E \cap C \) contains a unique noncompact component \( \Delta \), which is a planar domain.
Since \( \Delta \) is connected and \( \partial \Delta \) separates \( \partial E \) from the end of \( E \), it follows that 
\( \partial \Delta \subset \mathcal{T}(0) \) contains a unique simple closed curve \( \gamma \subset \partial \Delta \) which generates the first homology group \( H_1(E) \). Moreover, any other boundary component of \( \partial \Delta \) is homotopically trivial in \( E \), and hence, homotopically trivial in \( N \). In particular, 
\( i_\ast(\pi_1(\Delta)) \) is either trivial or an infinite cyclic subgroup of \( \pi_1(C) \), where \( i: \Delta \to C \) is the inclusion map and \( i_\ast: \pi_1(\Delta) \to \pi_1(C) \) is the induced map on fundamental groups, after choosing a base point on \( \gamma \).

We let \( \Pi: \mathbb{H}^3 \to N \) be the universal cover of \( N \) and let \( B \subset \mathbb{H}^3 \) be a horoball such that \( \Pi|_B: B \to C \) is the universal cover of \( C \). Using the half-space model for \( \mathbb{H}^3 \), we assume, without loss of generality, that \( B \) is the region 
\[ B = \{(x, y, z) \in \mathbb{H}^3 \mid z \geq 1\}. \]

**Case 2.2.** \( i_\ast: \pi_1(\Delta) \to \pi_1(C) \) is trivial.

Since \( i_\ast \) is trivial, \( i: \Delta \to C \) admits a lift \( \tilde{i}: \Delta \to B \), whose image \( \tilde{\Delta} \) is a properly embedded planar domain in \( B \) with \( \partial \tilde{\Delta} \subset \partial B \). By [3, Theorem 10] (for \( H = 1 \)) and [5, Theorem 6.9] (for \( H > 1 \)), it follows that \( \tilde{\Delta} \) is asymptotic to a constant mean curvature \( H \) annulus \( A \subset B \), in the sense that a subend of \( \tilde{\Delta} \) is a graph in exponential normal coordinates over a subend of \( A \) with graphing function converging in the \( C^2 \)-norm to zero for any divergent sequence of points. Moreover, \( A \) admits a vertical axis of rotational symmetry, and there are three possibilities:

1. \( H = 1 \) and \( A \) is the end of a horosphere;
2. \( H = 1 \) and \( A \) is the end of an embedded catenoid cousin;
3. \( H > 1 \) and \( A \) is the end of a Delaunay surface.

Note that the Catenoid cousin in item 2 is embedded because any end representative of a nonembedded Catenoid cousin is never contained in a horoball.

In each case, \( A \) has bounded norm on its second fundamental form and infinite area; hence, since \( \tilde{\Delta} \) is asymptotic to \( A \) in the \( C^2 \)-norm, it follows that \( \tilde{\Delta} \) also has bounded norm of its second fundamental form \( \|A_{\tilde{\Delta}}\| \) and infinite area. Next, we use these properties to get a contradiction.

Since \( \tilde{\Delta} \) is a complete, properly embedded surface with \( \partial \tilde{\Delta} \subset \partial B \), then \( \tilde{\Delta} \) defines a mean convex region \( M \subset B \) with \( \partial M \setminus \partial B = \tilde{\Delta} \). Moreover, since \( \mathbb{H}^3 \) is a homogeneously regular manifold and \( \tilde{\Delta} \) has compact boundary and separates \( B \), the bound on \( \|A_{\tilde{\Delta}}\| \) gives the existence of a one-sided regular neighborhood in \( M \) of radius \( \delta > 0 \), see [8, Lemma 3.1]. Since \( \tilde{\Delta} \) has infinite area and \( \delta > 0 \), then \( M \) has infinite volume.
Figure 1: \( \tilde{\Delta} \) has constant mean curvature \( H \geq 1 \) and each hypersphere \( S_t^+ \) has constant mean curvature \( H_0 = \cos(\alpha) \). The plane \( L \) separates \( p \) and \( \partial \tilde{\Delta} \), \( p \) lies in the mean convex region of \( \mathbb{H}^3 \) determined by \( L \) and \( S_t^+ \) converge, when \( t \to \infty \), to \( L \).

Let \( \sigma: B \to B \) be a parabolic translation of \( \mathbb{H}^3 \) that is a covering transformation of \( \Pi \). Since \( \Delta \) is embedded, then \( \sigma(\tilde{\Delta}) \cap \tilde{\Delta} = \emptyset \); thus, either \( \sigma(M) \cap M = \emptyset \) or \( \sigma(M) \subset M \). Since \( \sigma \) is a translation, the latter is not possible and we obtain that \( \sigma(M) \cap M = \emptyset \). Hence, \( \Pi|_M: M \to \mathcal{C} \) is injective, which is a contradiction because \( \mathcal{C} \) has finite volume, and this proves Lemma 2.1 when \( i^* \) is trivial.

**Case 2.3.** \( i_*: \pi_1(\Delta) \to \pi_1(\mathcal{C}) \) is nontrivial.

In this case, \( i_*(\pi_1(\Delta)) \) is a \( \mathbb{Z} \)-subgroup of \( \mathbb{Z} \times \mathbb{Z} = \pi_1(\mathcal{C}) \), generated by \( i_*(\gamma) \). Let \( \tilde{\Delta} \subset B \) be a connected component of \( \Pi^{-1}(\Delta) \). Then \( \tilde{\Delta} \) is a complete, non-compact, properly embedded planar domain in \( B \) with \( \partial \tilde{\Delta} \subset \partial B \); in particular, it defines a mean convex region \( M \subset B \) with \( \partial M \setminus \partial B = \tilde{\Delta} \).

Also, note that \( \tilde{\Delta} \) is invariant under the parabolic covering transformation \( \theta: B \to B \) corresponding to \( i_*([\gamma]) \in \pi_1(\mathcal{C}) \). Since \( \partial \tilde{\Delta} \) is compact, \( \partial \tilde{\Delta} \) stays a finite distance \( c > 0 \) from a line \( l \subset \partial B \), invariant under \( \theta \). In order to clarify the next argument, we apply a rotation around the \( z \)-axis of \( \mathbb{H}^3 \) to assume that \( l = \{(0, y, 1) \mid y \in \mathbb{R}\} \); hence, \( \partial \tilde{\Delta} \subset \{(x, y, 1) \mid x \in (-c, c), y \in \mathbb{R}\} \). Also, after possibly reflecting through the \( xz \)-plane, we may assume that \( \{(x, y, 1) \mid x \geq \).
Let \( p = (x_1, y_1, z_1) \in \tilde{\Delta} \) be such that \( z_1 > 1 \) and \( x_1 \in (-c, c) \). Let \( q = (x_1, y_1, \frac{z_1+1}{2}) \) and let \( L \) be the tilted plane through \( q \) containing the line \( \{(c, y, 1) \mid y \in \mathbb{R}\} \). Then, \( L \) is an equidistant surface to a totally geodesic vertical plane and, when oriented with respect to the upper normal vector field, has constant mean curvature \( H_0 = \cos(\alpha) \in (0, 1) \), where \( \alpha \) is the acute, Euclidean angle between \( L \) and \( \{z = 0\} \). Note that \( L \) separates \( \partial \tilde{\Delta} \) and \( p \), and that \( p \) lies in the mean convex region \( U \) defined by \( L \) in \( \mathbb{H}^3 \).

Let \( S_0 \) be a totally geodesic surface of \( \mathbb{H}^3 \) such that \( S_0 \subset U \), with asymptotic boundary meeting the asymptotic boundary of \( L \) in a single point. Let \( S_0^+ \) denote the equidistant surface to \( S_0 \) with constant mean curvature \( H_0 \) with respect to the inner orientation. Note that we may choose \( S_0 \) so that \( S_0^+ \cap \partial B = \emptyset \), as shown in Figure 1.

Let \( M^+ \subset \mathbb{H}^3 \) denote the mean convex region defined by \( S_0^+ \). Then, there is a product foliation \( \{S_t^+\}_{t \geq 0} \) of \( U \setminus M^+ \) such that each surface \( S_t^+ \) is equidistant to a totally geodesic surface of \( \mathbb{H}^3 \) and has constant mean curvature \( H_0 \) with respect to the inward orientation; when \( t \to \infty \), the surfaces \( S_t^+ \) converge to \( L \).

Since \( p \in \tilde{\Delta} \subset B \), then \( p \not\in M^+ \). Then, the fact that \( p \in U \) implies that \( (\bigcup_{t \geq 0} S_t^+) \cap \tilde{\Delta} \neq \emptyset \). But because \( S_t^+ \cap B \) is compact for all \( t \geq 0 \), there exists a smallest \( T > 0 \) such that \( S_T^+ \cap \tilde{\Delta} \neq \emptyset \). Since our construction gives that \( \partial \tilde{\Delta} \cap U = \emptyset \), any point \( w \) in \( S_T^+ \cap \tilde{\Delta} \) is interior to both \( \tilde{\Delta} \) and \( S_T^+ \), and then \( S_T^+ \) and \( \tilde{\Delta} \) intersect tangentially at \( w \). Finally, since \( S_T^+ \cap B \subset M \), the mean curvature comparison principle implies that \( H_0 \geq H \), which is a contradiction, and this proves Lemma 2.1.

\( \square \)

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William H. Meeks, III at profmeeks@gmail.com
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Álvaro K. Ramos at alvaro.ramos@ufrgs.br
Departmento de Matemática Pura e Aplicada, Universidade Federal do Rio Grande do Sul, Brazil