Dark Energy from Ratio Gravity

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The theory of Ratio Gravity (RG) proposes that the curvature of 3+1 spacetime originates from a deformation of a Cross Ratio, where similar mathematical structure to general relativity emerges. This paper studies RG using the framework of the Newman-Penrose spin formalism. After proposing the general formalism, we move on to study a homogeneous and isotropic universe with RG. It is noted that the theory contains a component of dynamical dark energy with novel equation of state.

Keywords: Cosmology, Gravity, General Relativity

I. INTRODUCTION

Dark energy (DE) is one of the major mystery in cosmology [1,2]. A great number of dark energy models are built since its discovery [3].

Cross ratio is also known as the anharmonic ratio. It has been used in theoretical physics such as in conformal field theory [4] and scattering amplitudes [5]. In the present work, we study a theory of gravity known as Ratio Gravity (RG) [6] as the first principle of gravity, and explore its explanation of DE. A short introduction of the previous work is given in section 2.

We reformulate the RG theory by using the Newman Penrose (NP) formalism of general relativity [7]. The correspondence between the NP formalism and RG theory is described in section 3, which demonstrates the connection between general relativity and RG. This shows evidence that the Cross Ratio may be the underlying mathematical structure of spacetime and gravity.

In section 4, we make use of the study of cosmological structure by NP formalism, then solve the equations of RG theory for a simple model. The Friedmann equation and the equation of state of DE, w(z), in RG are obtained.

The results show that DE is dynamical with the equation of state approximately -1 in the present era, which is consistent to the widely accepted ΛCDM model. Furthermore, w(z) approaches zero as z approaches high z regime. This is an interesting feature, since recent observations show some evidences that dark energy may have similar dynamics [8]. An observational best-fit polynomial plot for w(z), with interesting w(z) = −1 crossing [1] is compared to the plot of w(z) of RG, with good agreement. The expansion history of the simple model shows good match to observation.

Furthermore, the proportional constant of Ω_b and Ω_M is originated from the context of RG in this simple model. Finally, we discuss the potential extensions of the theory.

II. OVERVIEW OF THE PREVIOUS WORK

Here we outline the foundation of the previous work about Ratio Gravity [6], which is needed for the formulation of this paper. Throughout this paper, we will use the definitions and conventions of the spinor formalism of general relativity developed by Newman and Penrose [7]. An intensive introduction for the spinor formalism is covered in chapter 13 of [14].

Cross Ratio (Ratio) is postulated to play a fundamental role in physics. In the context of gravity, the principle can be stated as: Gravity is described by the general deformation of the same cross-ratio. Such Cross Ratio is a fixed and fundamental mathematical structure, which consists of many representations - the general deformation. In RG theory, the Cross Ratio is not used as a technical tool as in the previous application of Cross Ratio in theoretical physics [4,5]. Instead, it is used to describe the mathematical structure of gravity.

As a result, the arbitrariness of the same cross ratio leads to 4 degrees of freedom because of only 4 parameters for 3 moving poles over Riemann sphere. The theory proposed to represent the Ratio by an integratable system with 3 regular singular points, because the hypergeometric differential equation with 3 regular singular points is the corresponding representation of the cross ratio [15].

The main idea of Ratio Gravity is that the transformation of the Ratio manifests the variety of representations such that each representation is correspondent to related physical event/observation.

For the realization of gravity, Ratio Gravity postulate that gravitational curvature spinors (which are described by the Newman and Penrose spinor formalism of general relativity [7]) are originated by two sets of Galois generator operators (Gal operator) of underlying differential equation from Ratio defined as following [1]: J,M operators, called as curvature operators. They generate 2 sets of 2-by-2 traceless matrices (J_{ac}, M_{acf}) or simply (J,M) as curvature matrices by operating on a 2-by-2 invertible matrix Y (fundamental solution matrix) [6].

\[ J_{ac}(Y) = J_{ac}Y, M_{acf}(Y) = M_{acf}Y, \]

where

\[ J_{ac}(Y) := e^{\beta dt}(B_{ab}D_{cd} - B_{cd}D_{ab})Y, \]

\[ B_{ab} := Y_{ac}Y_{ac}^{-1}, \]

\[ D_{ab} := Y^{-1}_{ab}Y_{cd}Y_{cd}^{-1} \]
\[ M_{bd}(Y) := \epsilon^{ac}(B_{ab}D_{cd} - B_{cd}D_{ab})Y \]  
\[ (a, c \text{ and } d, e \text{ are spinor indexes from } 0 \text{ to } 1 \text{ and from } 0 \text{ to } 1 \text{ respectively}; \epsilon^{bd} \text{ is the usual skew spinor metric for spinor formalism that is used to raise or lower the spinor index}) \]

Matrices \( J, M \) are identified as the algebras of Galois differential group of Ratio (Hypergeometric differential equation representation) \( SL_2 \). \( B_{ab} \) are four 2-by-2 matrices indexed by \( ab \); \( D_{cd} \) are four 2-by-2 matrix-differential operators indexed by \( cd \), which obeys Leibniz’s rule for derivation:

\[ D_{ab}(XY) = (D_{ab}X)Y + X(D_{ab}Y), \]

where \( X, Y \) are differentiable 2-by-2 matrices.

The NP formalism defines spinor over complex number, while the equations of RG theory are written by spinor-indexed 2-by-2 matrices or 2-by-2 matrix operators, so the basic object is in 2-by-2 matrix structure over complex number.

RG leverages the Galois differential theory as the tool for deforming the Ratio, that the curvature operators by Gal operator are defined. In particular, Ratio needs an equation for deforming the Ratio, that the curvature operators by Gal group is \( SL_2 \) constant matrix for \( D \) operator (i.e. \( D(C) = 0 \)).

\[ DZ = BZ, \sigma(Z) = ZC, \]

where \( Z := \sigma(Y) \) (In fact, \( C \) is diagonal and in \( SL_2 \)). So the new differential equation and solution are obtained.

The following is a simple illustration for the correspondence of Ratio and differential equation

\[ (z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \]

Hypergeometric differential equation (3 regular singular poles)

### III. USE OF NEWMAN PENROSE SPINOR FORMALISM

In this section, the well-known Newman Penrose formalism \([8]\) is used. We explain how RG theory establishes the connection to NP formalism such that the mathematical structure like Bianchi identities emerges from RG theory.

In NP formalism, there are two parts of equations: NP equations and the Bianchi identities. The NP equations provides geometric properties (through spin coefficients \( \Gamma_{abcd} \)) from curvature spinors, which is postulated to be originated by curvature matrices in dyad:

\[ -J_{ab} = -\epsilon_{acde}d\Gamma_{d,e} + \epsilon_{bcde}d\Gamma_{a,c} + \epsilon_{a,bcd}d\Gamma_{e,d} + \epsilon_{a,bce}d\Gamma_{d,c} - \epsilon_{abcde}d\Gamma_{d,e} + \epsilon_{a,bce}d\Gamma_{d,c} - \epsilon_{a,bcd}d\Gamma_{e,d} \]

\[ -M_{abcd} = \epsilon_{acde}d\Gamma_{d,e} + \epsilon_{bcde}d\Gamma_{a,c} + \epsilon_{a,bcd}d\Gamma_{e,d} + \epsilon_{a,bce}d\Gamma_{d,c} - \epsilon_{abcde}d\Gamma_{d,e} + \epsilon_{a,bce}d\Gamma_{d,c} - \epsilon_{a,bcd}d\Gamma_{e,d} \]

\[ X_{ab} = -J_{ab} \epsilon, \]

\[ \Phi_{abcd} = -M_{abcd} \epsilon, \]

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2 In this paper, we use the convention \( \epsilon_{ab} \) defined as \[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

As an example, one may write differential equation, e.g. \( y'' - y = 0 \), in matrix form (i.e. \( Y \) is the fundamental solution matrix) and the Galois group is \( SL_2 \). By defining a Gal operator that satisfies \( \sigma(Y) = CY \), where \( C \) is \( SL_2 \) constant matrix for \( D \) operator (i.e. \( D(C) = 0 \)), then, \( D \cdot \sigma(Y) = \sigma \cdot D(Y) \) implies

\[ DZ = BZ, \sigma(Z) = ZC, \]

where \( Z := \sigma(Y) \) (In fact, \( C \) is diagonal and in \( SL_2 \)). So the new differential equation and solution are obtained.

The following is a simple illustration for the correspondence of Ratio and differential equation

\[ (z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \]

3 Spin coefficients \( \Gamma_{abcd} \) defined in NP formalism as:

| \( c'd'ab \) | \( \epsilon \) | \( \kappa \) | \( \rho \) | \( \beta \) | \( \sigma \) |
|---|---|---|---|---|---|
| 00' | 0 | 1 | 0 | 1 | 1 |
| 10' | 0 | 0 | 1 | 0 | 1 |
| 01' | 1 | 0 | 0 | 1 | 0 |
| 11' | 0 | 1 | 1 | 0 | 0 |
where $\epsilon$ is the spinor metric in its matrix form and $(\Phi_{cde}, X_{ac})$ are matrix forms of curvature dyad in the NP formalism. They are the counter part of the components of Riemann tensor in the metric formulation of GR, to represent the commutation relations of covariant derivatives. The equations (7) and (8) defines NP equation in RG.

The NP equation in RG and the original NP equations are different in the way that it is written. Since original NP equations does not use matrix representation, so $\Gamma_{bd\mu}$ is written as $\Gamma_{bd\mu}^{\epsilon}$.

For Bianchi identities in the NP formalism, it describes how changes of curvature dyad and spin coefficients are connected. In RG, the Bianchi identities are indirectly connected. In RG, the Bianchi identities are indirectly originated by the spinor Bianchi equation [6] with dyad as following:

$$D_{ab}^{d}(Y) = D_{ab}^{c}M_{cde}(Y),$$  \hspace{1cm} (11)

while the Ratio ($Y$) equation for dyad $D$ differential operator is defined as

$$D_{ab}^{b}Y = \partial_{ab}^{b}Y + [\Lambda_{ab}^{b}, Y] = B_{ab}^{b}Y,$$  \hspace{1cm} (12)

where $\Lambda_{ab}^{b}$ are four traceless 2-by-2 matrices and $\partial_{ab}^{b}$ are defined as identity 2-by-2 matrix multiplying the dyad differential operators in the NP formalism (note that the original dyad differential operators are spinor operators, not metric operators). The dyad differential operators contain the geometric structure variables (e.g. scaling factor in cosmology) of the spacetime metric. $J$ and $M$ operators are the Gal operator discussed in section 2, so it is clear that they play fundamental role in describing the curvature. The dyad derivatives of curvature matrices are described by Ratio context ($B_{ab}, \Lambda_{ab}, Y$). After computing the Bianchi identities explicitly from original spinor formalism to dyad formalism (the same procedure used by the NP formalism [7], section II-Tetrad Calculus), we can connect the spin coefficients $\Gamma_{bd\mu}$ and Ratio (context) to construct Bianchi Constraints:

$$e^{eh}e^{gkr}(M_{cgh}\bar{\Gamma}_{bdh\mu bna} + M_{gch}\bar{\Gamma}_{bdh\mu cna}) -$$

$$\epsilon^{b}e^{gkr}(J_{ab}\Gamma_{bd\mu} + J_{ab}\Gamma_{bd\mu})$$

$$=-[\Lambda_{ab}^{b}, J_{ab}] - J_{ab}B_{ab}^{b} +$$

$$[\Lambda_{ab}^{c}, M_{cde}] + M_{cde}B_{ab}^{c},$$  \hspace{1cm} (13)

i.e. the geometrical construction in dyad as Bianchi identities is established as Bianchi constraints.

Physically, Ratio describes how the geometry of spacetime by mean of spin coefficients is manifested. It is even clearer if we notice Bianchi constraints are in fact algebraic. The diagram [Fig. 1] shows that RG equations connecting the related curvature dyads to spinor coefficients to Bianchi identities. The Bianchi constraints are constructed to map the Bianchi identities to RG dyads. So it forms a set of consistent equations, and, in principle, is solvable.

![Diagram](image)

**FIG. 1.** The correspondence of the NP formalism and the Ratio context.

In short, Ratio provides dyads dynamics to establish a 3+1 spacetime. It also provides the related differential operators for the RG to realize as curvature dyads, which are solved with Bianchi constraints.

In the last part of this section, we illustrate the correspondence of RG theory and general relativity in brief glance. We start with equations of the NP formalism and show them lead to RG equations.

First, we assume $J_{ab}$ and $M_{cde}$ operators generate $J_{ab}$ and $M_{cde}$ curvature matrices without requiring they are Gal operators for a moment, so we “relax” the constraint of being Gal operators. The NP equations of RG theory (7, 8) are exactly the same to NP equations of the NP formalism with direct substitution of curvature dyads to curvature matrices. Therefore, we only need to show how Bianchi identities of the NP formalism lead to RG Bianchi equation (11). The Bianchi identity of the NP formalism consists of the term, $\partial_{ab}^{b}J_{ab} - \partial_{ab}^{c}M_{cde}$, which is the the left hand side of Bianchi constraints (13) of RG equations. By introducing an invertible 2 by 2 matrix $Y$, and simply using Y equation (6) as a constraint equation, RG Bianchi equation (11) is recovered. The use of both constraints (Bianchi constraints and Y equation as constraint equation) is satisfied by the number of unknowns of RG context - all the components of $Y$, $B_{ab}$, and traceless matrices $\Lambda_{ab}$, total $4 \times (1 + 4 + 3) = 32$ unknowns.

In this brief discussion, it is required that we relax the constraints for $J_{ab}$ and $M_{cde}$ operators as defined in section 2, to illustrate the correspondence. However, as stated in previous work [6], the Gal operators are naturally related to RG Bianchi equation. So simply relaxing the constraints for $J_{ab}$ and $M_{cde}$ operators defined in section 2 is not correct approach in the context of RG theory. Instead, other forms of Gal operators as the extension of the theory are perfectly valid under the framework of the RG theory.
IV. THE SOLUTION FOR THE COSMOLOGY IN HOMOGENEOUS METRIC

To illustrate the framework described by RG in the NP formalism and its predictions, we apply it to construct cosmology with a simple solution. A widely used metric form for FRW cosmology is \( ds^2 = a(\eta)^2(d\eta^2 - dr^2 - S(r)^2(d\theta^2 + \sin^2 \theta d\phi^2)) \). The dyad differential operators associated are

\[
\partial_{00'} = \frac{\partial_0 + \partial_0}{a(\eta)^2}, \partial_{11'} = \frac{1}{2} (-\partial_0 + \partial_0), \quad \partial_{01'} = \frac{\partial_0 + i\partial_0}{\sqrt{2a(\eta)}S(r)}, \partial_{10'} = \frac{\partial_0 - i\partial_0}{\sqrt{2a(\eta)}S(r)}.
\] (14)

The condition of the spin coefficients is constructed by the cosmological principles for homogeneous spacetime, we have spin coefficients conditions \[^{18}\]:

\[
\kappa = \rho = \sigma = \epsilon + \epsilon^* = \tau + \beta + \alpha^* = 0. \quad (15)
\]

And the dependency of \( Y \) is limited to only \( \eta, r \); we parameterize, as PPV-theory-sense, only to \( D_{00'}, D_{11'} \). A solution is found by a simple model after solving the overdetermined system:

1. \( Y \) equation \[^{3}\],
2. RG Bianchi equation \[^{11}\],
3. Gal-D-commuting property:
   \( [D_{ab}, J_{cd}]Y = 0, [D_{a,b}, M_{c,d}]Y = 0 \),
4. Bianchi constraints \[^{13}\],
5. NP equation in RG \[^{7,8}\],

which we need the spin coefficients condition \( \beta = \tau = \pi = \nu = \lambda = 0 \), and condition of two constants - \( (\mu = \mu_0/2, \text{ and } C_{00'} = \text{scalar part of } B_{00'} ) \), as well as the condition of \( Y \) matrix components: \( Y_{00}, Y_{11} \ll Y_{10} \). The first three equations \( (#1 \text{ to } #3) \) are explained in previous work \[^{2}\]. The 4\(^{th}\) and 5\(^{th}\) equations are developed in this paper because of the use of the NP formalism in dyad. These additional two equations are needed when realization of the NP formalism in dyad, while, in the previous work, only spinor formalism is used. As a result, the connection to metric thru the null tetrad by the NP formalism in dyad is clear.

The calculation is greatly simplified after realizing the zeros of spin coefficients in Bianchi constraints and homogeneous of \( \Phi - \text{curvature dyad} - \text{terms such that the Ratio context is restricted to be of a simpler form. This is a simple solution so other cosmological solutions are certainly possible.} \)

As a result, the differential equations we need to describe the spin coefficients and curvature dyad are:

\[
\epsilon(\gamma + \gamma^*) - \epsilon t/2 + \gamma t/a^2 = 0, \quad (16)
\]

\[
\Phi_{22} + 2\mu_0(\mu_0 + 2Re(\gamma)) = 0, \quad (17)
\]

\[
\epsilon^* = C_{00}/2 - \Phi_{22}/(2a^2\Phi_{22}), \quad (18)
\]

where prime stands for \( \eta \) derivative, and \( \Phi_{22} := \Phi_{111'1} \). We further assume \( \mu_0, \Phi_{22} \) are real \[^{1}\] and use Einstein field equation in dyad form of the NP formalism \[^{7}\], and then an ordinary differential equation as RG-Friedmann equation is obtained by such simple model.

\[
\frac{\dot{a}}{a} = -\frac{\mu_0\mathcal{H}}{a} - \frac{\mathcal{H}^2\Omega_M}{2a^3} - \frac{1}{3}R^2_{M} + 2\mathcal{H}^2, \quad (19)
\]

where \( \mathcal{H} \) is Hubble parameter, \( \Omega_M := R_m\Omega_B, C_0 := Re(C_{00}) \), and \( R_m \) is defined as following:

\[
C_0 = R_m\mu_0. \quad (20)
\]

\( R_m \) is the proportionality-constant for ordinary matter density parameter \( (\Omega_b) \) and apparent mass density parameter \( (\Omega_M) \). (It is applicable to the matter dominated era with mass density \( \sim a^{-3} \), although one can derive the equation for radiation dominated era in exactly the same way.)

It consists of matter and dark energy (DE) terms as we desire. Although it is not complicated, it is still not apparent to us how equations \[^{19}\] can be solved analytically except for few simple scenarios. Therefore we solve it numerically in this paper \[^{7}\].

The DE part of Friedmann equation provides total 3 terms, a constant term, a dynamical term related to \( \mathcal{H}/a \), as well as a positive dynamical term related to \( \mathcal{H}^2 \). Clearly the positive term is responsible for acceleration of expansion.

The Einstein equation in spinor formalism \[^{2}\] is used to relate the curvature and energy momentum density spinor. Yet, the current theory of RG plays no role to explain this connection.

To find the equation of state of DE, \( w(z) \), we use the usual definition as:

\[
DE := \Omega_{DE}H^2_0 \exp \left[ 3 \int_0^1 \frac{1 + w(x)}{1 + x} dx \right]. \quad (21)
\]

The equation of state for the DE is found:

\[
w(z) = (6\mathcal{H}^2(z) + \mu_0(z + 1)^2\mathcal{H}'(z)) - \mu_0^2R_M - 2(1 + \mathcal{H}(z)(\mu_0 + 2\mathcal{H}')(z))) / (\mu_0^2R_M + 3\mathcal{H}(z)(\mu_0 - 2\mathcal{H}(z) + \mu_0 z)). \quad (22)
\]

In order to probe the constants of the cosmological model, we preliminarily use the DE-Matter equality at \( t \approx 0.7, z \approx 0.3 \) \((H_0 \text{ is set to } 1 \text{ in this paper})\). It is found that one can set the RG-Friedmann equation to

\[^{7}\text{Im}(C_{00}) \text{ is assumed to be very large compared to } \Phi_{22}/(2a^2\Phi_{22}) \text{ and } \text{Re}(C_{00}) \text{ to justify this approximation, while this condition mainly affects the differential equation for } \text{Im}(\gamma). \]

\[^{8}\text{The initial conditions are } a(t_0) = 1 \text{ and } H(t_0) = 1, \text{ and we set } t_0 = 1. \]
be an effective equation (applicable in DE, and matter dominated eras) for \( \mu_0 \approx 0 \) limit. The effective equation is:

\[
\frac{\ddot{a}}{a} = 2H^2 - \frac{\mathcal{H}_0^2 \Omega_M}{2a^3}.
\]

(23)

With single parameter probe- \( \Omega_M \approx 1.12 \), the predictions are:

- \( w(z) \) is near -1 in low \( z \) regime; \( w(z) \) provides a crossing to line \( w=-1 \); \( w(z) \) approaches 0 for high \( z \) regime

[FIG. 2]. This is convincing comparing to the observational-best-fits from SNIa (see [3] for a review). Therefore, RG seems to be a good candidate for dynamical DE because of the fit from first principle considerations of the RG.

Furthermore, the slope of expansion history in matter era is shown to be consistent with \( \Lambda \)CDM model [FIG. 3]. It is not surprising, as \( w(z) \) approaches zero as \( z \) goes large that implies DE effect fades away as going back in time. In addition, the plot of ratio of matter term vs DE term [FIG. 4] is shown below for effective model \( (\mu_0 \approx 0) \) to illustrate the consistency of accelerating expansion regime and the shift from matter dominated era to DE era.

Finally, beside the effective model of RG, there is a parameter-set \( \Omega_M = 0.156, \mu_0 = -1.38 \) that the equation of state turns around \( z \approx 0.2 \) [FIG. 2]. In this case, \( w(z) \) is very close to -1 in low \( z \) regime while crossing from \( w < -1 \) to \( w > -1 \) then approaches zero in high \( z \). However, for such parameters, RG predicts a much longer age of universe \( \approx 2.38 \). Therefore, the validity is not obvious.

V. COMMENTS AND POTENTIAL DEVELOPMENT

By the simple physical concept of deformation on a Ratio, RG hypothesizes a simple entity to be fundamental mathematical structure of gravity.

The effective model with a simple form provides observation-consistent predictions yet only one parameter is needed. It can be considered relatively simpler, compared to \( \Lambda \)CDM for matter and DE eras, since \( \Lambda \)CDM relies on two parameters for these two eras to be explained simultaneously. Therefore, the authors suggest RG proposes a non-phenomenological approach to explain the DE dynamics by the first principle of Ratio Gravity. It is apparently justified because the parameters are linked directly to the Ratio context instead of arbitrary settings.

Since the main objective of this paper is to develop the RG framework for gravity, and establish its connection to the late-time universe, the current proposed parameters are some values chosen by hand inspired by observations.
In the future, we hope to perform a more sophisticated scan of the parameter space of the theory with the most up-to-date data.

To extend the model, one can consider more spin coefficients being non-zero, to describe a more complicated metric. Then, the related Bianchi constraints should have more equations to be solved. The curvature matrices of $J$, $M$ have numbers of functions (e.g. the diagonal elements of $J$, $M$ matrices) to be reserved to extend to richer curvature structure. We hope to explore these possibilities in the future.

It is also possible to further extend the model from the Ratio context (rather than geometry). For example, the $C_0$ constant can be relaxed first, then more dynamic components in spin coefficients or curvature matrices should enter the dynamics. We can also allow more degrees of freedom for matrix $Y$, such that more non-zero components of matrices are in place with the system to be solved.

The DE term drives the acceleration of the expansion so it is natural to ask: whether and how can the inflation era be explained from the same framework of RG? More detailed studies are needed in order to study the early universe.

Finally, to explain the CMB fluctuations, we should investigate the primordial inhomogeneity as perturbations from RG prospective. We also leave the study of these fluctuations to future work.

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