The geometry of the curve graph of a right-angled Artin group

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Abstract. We develop an analogy between right-angled Artin groups and mapping class groups through the geometry of their actions on the extension graph and the curve graph respectively. The central result in this paper is the fact that each right-angled Artin group acts acylindrically on its extension graph. From this result we are able to develop a Nielsen–Thurston classification for elements in the right-angled Artin group. Our analogy spans both the algebra regarding subgroups of right-angled Artin groups and mapping class groups, as well as the geometry of the extension graph and the curve graph. On the geometric side, we establish an analogue of Masur and Minsky’s Bounded Geodesic Image Theorem and their distance formula.

1. Introduction

1.1. Overview. In this article, we study the geometry of the action of a right-angled Artin group $A(\Gamma)$ on its extension graph $\Gamma^e$. The philosophy guiding this paper is that a right-angled Artin group $A(\Gamma)$ behaves very much like the mapping class group $\operatorname{Mod}(S)$ of a hyperbolic surface $S$ from the perspective of the geometry of the action of $A(\Gamma)$ on $\Gamma^e$, compared with the action of $\operatorname{Mod}(S)$ on the curve graph $C(S)$. The analogy between right-angled Artin groups and extension graphs versus mapping class groups and curve graphs is not perfect and it notably breaks down in several points, though it does help guide us to new results.

The results we establish in this paper can be divided into algebraic results and geometric results. From the algebraic point of view, we discuss the role of the extension graph and of the curve graph in understanding the subgroup structure of right-angled Artin groups and mapping class groups respectively. From the geometric point of view, we discuss not only the intrinsic geometry of the extension graph and curve graph, but also the geometry of the canonical actions of the right-angled Artin group and the mapping class group respectively.

The central observation of this paper is that from the point of view of coarse geometry, the extension graph can be thought of as the Cayley graph of the right-angled Artin group equipped with star length rather than with word length. Roughly speaking, the star length of an element $w \in A(\Gamma)$ is the smallest $k$ for which $w = u_1 \cdots u_k$, where each $u_i$ is contained in the subgroup generated by the star of a vertex in $\Gamma$.

Inspired by an analogous fact relating word length in the mapping class group with distance in the curve graph, we are able to refine distance estimates in $\Gamma^e$ by developing a theory of subsurface projections and proving a distance formula which recovers the
syllable length in $A(\Gamma)$. Roughly, the syllable length of $w \in A(\Gamma)$ is the smallest $k$ for which $w = v_1^{n_1} \cdots v_k^{n_k}$, where each $v_i$ is a vertex in $\Gamma$ and $n_i \in \mathbb{Z}$.

In light of the preceding remarks, an alternative title for this article could be “The geometry of the star metric on right-angled Artin groups”. In the interest of clarity and brevity, we will not state the results one by one here in the introduction. In the next subsection we have included a tabular summary of the results in this paper, together with references directing the reader to the discussion of the corresponding result.

1.2. **Summary of results.** The following two tables summarize the main results of this article. The results are recorded with parallel results in mapping class group theory in order to emphasize the analogy between the two objects. For each result, either a reference will be given or the reader will be directed to the appropriate statement in this article.

| Summary of Geometric Results | |
|-----------------------------|-----------------------------|
| **$A(\Gamma)$** | **Mod($S$)** |
| Extension graph $\Gamma^e$ | Curve graph $C(S)$  |
| $\Gamma^e$ is quasi–isometric to an electrification of Cayley($A(\Gamma)$) (Theorem 15) | $C(S)$ is quasi–isometric to an electrification of Cayley($\text{Mod}(S)$) (23) |
| Extension graphs fall into exactly two quasi–isometry classes (Theorem 23) | Curve graphs are quasi–isometrically rigid (26) |
| $\Gamma^e$ is a quasi–tree (20) | $C(S)$ is $\delta$–hyperbolic (23) |
| The action of $A(\Gamma)$ on $\Gamma^e$ is acylindrical (Theorem 30) | The action of $\text{Mod}(S)$ on $C(S)$ is acylindrical (8) |
| Loxodromic–elliptic dichotomy for nonidentity elements (Section 7) | Nielsen–Thurston classification (25) |
| Each loxodromic element has a unique pair of fixed points on $\partial \Gamma^e$ (Lemma 47) | Each pseudo-Anosov has a unique pair of fixed points on $\partial C(S)$ (25) |
| Vertex link projection (Section 11) | Subsurface projection (23, 24) |
| Bounded Geodesic Image Theorem (Theorem 55) | Bounded Geodesic Image Theorem (24) |
| Distance formula coarsely measures syllable length in $A(\Gamma)$ (Section 13) | Non–annular distance formula coarsely measures Weil–Petersson distance in Teichmüller space (24 and 10) |
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### Summary of Algebraic Results

| $A(\Gamma)$ | $\text{Mod}(S)$ |
|-------------|-----------------|
| **Extension graph** $\Gamma^e$ | **Curve graph** $\mathcal{C}(S)$ |
| Induced subgraphs of $\Gamma^e$ give rise to right-angled Artin groups of $A(\Gamma)$ (Section 2.4 and [20]) | Induced subgraphs of $\mathcal{C}(S)$ give rise to right-angled Artin subgroups of $\text{Mod}(S)$ (Subsection 2.3 and [21]) |
| An embedding $A(\Lambda) \to A(\Gamma)$ gives rise to an embedding $\Lambda \to (\Gamma^e)_k$ (Section 2.4 and [20]) | An embedding $A(\Lambda) \to \text{Mod}(S)$ gives rise to an embedding $\Lambda \to \mathcal{C}(S)_k$ (Section 2.4 and [19]) |
| $\Gamma^e$ can be recovered from the intrinsic algebra of $A(\Gamma)$ (Section 3) | $\mathcal{C}(S)$ can be recovered from the intrinsic algebra of $\text{Mod}(S)$ (Section 3) |
| Cyclically reduced elliptic elements of $A(\Gamma)$ are supported in joins (Theorem 35) | Reducible mapping classes stabilize sub–curve graphs ([6]) |
| Injective homomorphisms from right-angled Artin groups to right-angled Artin groups and to mapping class groups preserve elliptics but not loxodromics (Section 8) | Injective homomorphisms from mapping class groups to right-angled Artin groups and to mapping class groups preserve elliptics but not loxodromics ([1]) |
| Powers of loxodromic elements generate free groups (Theorem 46) | Powers of pseudo-Anosov elements generate free groups (Proposition 45) |
| Purely loxodromic subgroups are free (Theorem 52) | One–ended purely pseudo-Anosov subgroups fall in finitely many conjugacy classes per isomorphism type ([7]) |
| Powers of pure elements generate right-angled Artin groups (Theorem 43) | Powers of mapping classes with connected supports generate right-angled Artin groups ([21]) |
| Automorphism group of $\Gamma^e$ is uncountable (Theorem 65) | Automorphism group of $\mathcal{C}(S)$ is $\text{Mod}(S)$ ([16]) |

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2. **Preliminaries**

2.1. **Graph–theoretic terminology.** Throughout this paper, a *graph* will mean a one-dimensional simplicial complex. In particular, graphs have neither loops nor multi–edges. If there is a group action on a graph, we will assume that the action is a right–action.
Let $X$ be a graph. The vertex set and the edge set of $X$ are denoted by $V(X)$ and $E(X)$, respectively. We define the **opposite graph** $X^{\text{opp}}$ of $X$ by the relations $V(X^{\text{opp}}) = V(X)$ and $E(X^{\text{opp}}) = \{(V(X)) \setminus E(X)\}$. For two graphs $X$ and $Y$, the **join** of $X$ and $Y$ is defined as the graph $X \ast Y = (X^{\text{opp}} \sqcup Y^{\text{opp}})^{\text{opp}}$. A graph is called a **join** if it is the join of two non-empty graphs. A subgraph which is a join is called a **subjoin**.

For $S \subseteq V(X)$, the **subgraph of $X$ induced by $S$** is a subgraph $Y$ of $X$ defined by the relations $V(Y) = S$ and $E(Y) = \{e \in E(X) \mid$ the endpoints of $e$ are in $S\}$. In this case, we also say $Y$ is an **induced subgraph of $X$** and write $Y \leq X$. For two graphs $X$ and $Y$, we say that $X$ is **$Y$–free** if no induced subgraphs of $X$ are isomorphic to $Y$. In particular, we say $X$ is **triangle–free** if no induced subgraphs of $X$ are triangles (squares, respectively).

We say that $A \subseteq V(X)$ is a **clique** in $X$ if every pair of vertices in $A$ are adjacent in $X$. The **link** of a vertex $v$ in $X$ is the set of the vertices in $X$ which are adjacent to $v$, and denoted as $\text{Lk}(v)$. The **star** of $v$ is the union of $\text{Lk}(v)$ and $\{v\}$, and denoted as $\text{St}(v)$. By a clique, a link or a star, we often also mean the subgraphs induced by them.

Unless specified otherwise, each edge of a graph is considered to have length one. For a metric graph $X$, the distance between two points in $X$ is denoted as $d_X$, or simply $d$.

### 2.2. Extension graphs.

Let $G$ be a group and $A \subseteq G$. The **commutation graph of $A$** is the graph having the vertex set $A$ such that two vertices are adjacent if the corresponding group elements commute. If $A$ is a set of cyclic subgroups of $G$, the commutation graph of $A$ will mean the commutation graph of the set $\{x_\alpha : \alpha \in A\}$ where for each $\alpha \in A$ we choose a generator $x_\alpha$ for $\alpha$.

Suppose $\Gamma$ is a finite graph. The **right–angled Artin group on $\Gamma$** is the group presentation

$$A(\Gamma) = \langle V(\Gamma) \mid [a,b] = 1 \text{ for each } \{a,b\} \in E(\Gamma) \rangle.$$  

In [20], the authors defined the **extension graph** $\Gamma^e$ as the commutation graph of the vertex–conjugates in $A(\Gamma)$. More precisely, the vertex set of $\Gamma^e$ is $\{v^g : v \in V(\Gamma), g \in A(\Gamma)\}$ and two distinct vertices $v^g$ and $v^h$ are adjacent if and only if they commute in $A(\Gamma)$. There is a natural right–conjugation action of $A(\Gamma)$ on $\Gamma^e$ defined by $v^h \mapsto v^{hg}$ for $v \in V(\Gamma)$ and $g, h \in A(\Gamma)$.

### 2.3. Curve graphs and curve complexes.

Let $S = S_{g,n}$ be a connected, orientable surface of finite genus $g$ and with $n$ punctures. We will assume that $2g + n - 2 > 0$, so that $S$ admits a complete hyperbolic metric of finite volume. We denote the **mapping class group** of $S$ by $\text{Mod}(S)$. Recall that this group is defined to be the group of isotopy classes of orientation–preserving homeomorphisms of $S$.

By a **simple closed curve** on $S$, we mean the isotopy class of an essential (which is to say nontrivial and non-peripheral in $\pi_1(S)$) closed curve on $S$ which has a representative with no self–intersections. Observe that the three–times punctured sphere $S_{0,3}$ admits no simple closed curves. For each simple closed curve $\alpha$, we denote by $T_\alpha$ the Dehn twist along $\alpha$.

Let $S \notin \{S_{0,3}, S_{0,4}, S_{1,1}\}$. We define the **curve graph** $\mathcal{C}(S)$ of $S$ as follows: the vertices of $\mathcal{C}(S)$ are simple closed curves on $S$, and two (distinct) simple closed curves are
adjacent in $\mathcal{C}(S)$ if they can be disjointly realized. In other words, two isotopy classes $[\gamma_1]$ and $[\gamma_2]$ are connected by an edge if there exist disjoint representatives in those isotopy classes. Thus, the curve graph of $S$ can be thought of the commutation graph of the set of Dehn twists in the mapping class group $\text{Mod}(S)$.

The curve complex of $S$ is the flag complex of the curve graph of $S$. In other words, simple closed curves $\{[\gamma_0], \ldots, [\gamma_k]\}$ span a $k$-simplex in the curve complex if and only if they span a complete subgraph of $\mathcal{C}(S)$. Throughout this paper, we will be only using the curve graph of $S$ unless explicitly noted to the contrary.

The curve graph of $S$ needs to be defined differently in the case $S = S_{0,3}, S_{0,4}, S_{1,1}$. When $S = S_{0,3}$, we define $\mathcal{C}(S)$ to be empty. In the other two cases, observe that no two simple closed curves can be disjointly realized. In these cases, we define two simple closed curves to be adjacent in $\mathcal{C}(S)$ if they have representatives which intersect a minimal number of times. Note that for $S_{0,4}$ this means two intersections, and for $S_{1,1}$ this means one intersection.

We will not be using any properties of curve graphs in the proofs of our results in this paper. They will mostly serve to guide our intuition about extension graphs.

2.4. Right-angled Artin subgroups. The goal of this subsection is to note that $\Gamma^e$ and $\mathcal{C}(S)$ classify right-angled Artin subgroups of right-angled Artin groups and mapping class groups respectively, and that they do so in essentially the same way. The reader will be directed to the appropriate references for proofs.

For a possibly infinite graph $X$, we define the graph $X_k$ as follows (see [20] and also [19]). The vertices of $X_k$ are in bijective correspondence with the nonempty cliques of $X$. Two vertices $v_K$ and $v_L$ corresponding to cliques $K$ and $L$ are adjacent if $K \cup L$ is also a clique. Note that $\mathcal{C}(S)_k$ can be regarded as a multi-curve graph of $S$ in the sense that each vertex corresponds to an isotopy class of a multi-curve consisting of pairwise non-isotopic loops and two distinct multi-curves are adjacent if they do not intersect. For two groups $H$ and $G$, we write $H \leq G$ if there is an embedding from $H$ into $G$.

**Theorem 1 ([20]).** Let $\Lambda$ and $\Gamma$ be finite graphs.

1. If $\Lambda \leq \Gamma^e$, then $A(\Lambda) \leq A(\Gamma)$. More precisely, suppose $\phi$ is an embedding of $\Lambda$ into $\Gamma^e$ as an induced subgraph. Then the map

$$\phi_N : A(\Lambda) \to A(\Gamma)$$

defined by

$$v \mapsto \phi(v)^N$$

is injective for sufficiently large $N$.

2. If $A(\Lambda) \leq A(\Gamma)$, then there exists an embedding from $\Lambda$ into $(\Gamma^e)_k$ as an induced subgraph.

The corresponding result for mapping class groups is the following:

**Theorem 2 ([19] and [21]).** Let $\Lambda$ be a finite graph and $S = S_{g,n}$ where $2g + n - 2 > 0$.

1. If $\Lambda \leq \mathcal{C}(S)$, then $A(\Lambda) \leq \text{Mod}(S)$. More precisely, suppose $\phi$ is an embedding of $\Lambda$ into $\mathcal{C}(S)$ as an induced subgraph. Then the map

$$\phi_N : A(\Lambda) \to \text{Mod}(S)$$
defined by  
\[ v \mapsto T_{\phi(v)}^N \]

is injective for sufficiently large \( N \).

(2) If \( A(\Lambda) \leq \text{Mod}(S) \), then there exists an embedding from \( \Lambda \) into \( (\mathcal{C}(S))_k \) as an induced subgraph.

3. Intrinsic algebraic characterization of \( \mathcal{C}(S) \) and \( \Gamma^e \)

3.1. Maximal cyclic subgroups. In this section we would like to show that the intrinsic algebraic structure of a mapping class group \( \text{Mod}(S) \) and of a right-angled Artin group \( A(\Gamma) \) is sufficient to recover the curve graph \( \mathcal{C}(S) \) and the extension graph \( \Gamma^e \) respectively.

Recall that the mapping class group has a finite index subgroup \( \text{PMod}(S) \), a pure mapping class group, which consists of mapping classes \( \psi \) such that if \( \psi \) stabilizes a multicurve \( C \) then \( \psi \) stabilizes \( C \) component–wise and restricts to the identity or to a pseudo-Anosov mapping class on each component of \( S \setminus C \).

**Lemma 3.** Let \( G \leq \text{PMod}(S) \) be a cyclic subgroup which satisfies the following conditions:

1. The centralizer of \( G \) in \( \text{PMod}(S) \) contains a maximal rank abelian subgroup (among all abelian subgroups of \( \text{PMod}(S) \)).
2. There exist two maximal rank abelian subgroups \( A, A' \) in the centralizer of \( G \) such that \( A \cap A' \) contains \( G \) with finite index.

Then there is a simple closed curve \( c \subseteq S \) and a nonzero \( k \in \mathbb{Z} \) such that \( G = \langle T_c^k \rangle \).

**Proof.** Condition (1) on \( G \) guarantees that a generator of \( G \) is supported on a maximal multicurve on \( S \), as is true of any maximal rank abelian subgroup of \( \text{PMod}(S) \). Condition (2) guarantees that there are two maximal multicurves \( C_1, C_2 \) on \( S \) which contain the support of \( G \) and whose intersection \( C_1 \cap C_2 \) consists of exactly one curve. \( \square \)

**Proposition 4.** Let \( T \) be the set of the maximal cyclic subgroups of \( \text{PMod}(S) \) satisfying the conditions of Lemma 3. Then \( \mathcal{C}(S) \) is isomorphic to the commutation graph of \( T \).

**Proof.** Define a map \( \mathcal{C}(S) \) to the commutation graph of \( T \) by

\[ \phi : c \mapsto \langle T_c^k \rangle, \]

where \( k = k(c) \) is the smallest positive integer for which \( T_c^k \in \text{PMod}(S) \). Such a \( k \) exists since \( \text{PMod}(S) \) has finite index in \( \text{Mod}(S) \). Since distinct isotopy classes of curves give rise to distinct Dehn twists and since two Dehn twists commute if and only the corresponding curves are disjoint, the map \( \phi \) is well–defined. If two Dehn twists do not commute then they virtually generate a nonabelian free group, so that the map \( \phi \) preserves non–adjacency as well as adjacency. By Lemma 3 \( \phi^{-1} \) is defined and is surjective. Thus \( \phi \) is an isomorphism. \( \square \)

In order to get an analogous result for right-angled Artin groups, we need to put some restrictions on \( \Gamma \). The reason for this is that a vertex generator (or its conjugacy class, more precisely) is not well–defined. This is because a general right-angled Artin group
A(\Gamma) has a very large automorphism group, and automorphisms may not preserve the conjugacy classes of vertex generators.

**Lemma 5.** Let \( \Gamma \) be a connected, triangle– and square–free graph. Let \( 1 \neq g \in A(\Gamma) \) be a cyclically reduced element whose centralizer in \( A(\Gamma) \) is nonabelian. Then there exists a vertex \( v \in \Gamma \) and a nonzero \( k \in \mathbb{Z} \) such that \( g = v^k \).

**Proof.** Let \( g \) satisfy the hypotheses of the lemma. By the Centralizer Theorem, we have that \( \text{supp}(g) \) is contained in a subjoin of \( \Gamma \), and that the full centralizer of \( g \) is also supported on a subjoin of \( \Gamma \). Because \( \Gamma \) has no triangles and no squares, every subjoin of \( \Gamma \) is contained in the star of a vertex of \( \Gamma \). So, we may write \( g = v^k \cdot g' \), where \( v \) is a vertex of \( \Gamma \), where \( k \in \mathbb{Z} \setminus \{0\} \), and where \( \text{supp}(g') \subseteq \text{Lk}(v) \). Observe that if \( g' \) is not the identity then the centralizer of \( g' \) in \( \langle \text{St}(v) \rangle \) is abelian. It follows that \( g = v^k \).

**Definition 6.** Let \( G \) be a group and \( T \) be the set of maximal cyclic subgroups of \( G \) which have nonabelian centralizers. Then the abstract extension graph \( G^e \) of \( G \) is defined as the commutation graph of \( T \).

Observe that if \( \Gamma \) is a triangle–free graph without any degree–one or degree–zero vertex, then the centralizer of each vertex is nonabelian. From this we can characterize powers of vertex–conjugates as follows.

**Proposition 7.** Suppose \( \Gamma \) is a finite, connected, triangle– and square–free graph without any degree–one or degree–zero vertex. Then for each finite-index subgroup \( G \) of \( A(\Gamma) \), we have \( G^e \cong \Gamma^e \).

**Proof.** For each vertex \( v \in V(\Gamma) \) and \( g \in A(\Gamma) \), we let \( n(v, g) = \inf\{n > 0 : (v^g)^n \in G\} \). Put \( A = \{(v^g)^{n(v, g)} : v \in V(\Gamma), g \in A(\Gamma)\} \subseteq A(\Gamma) \). By Lemma 5, we see that \( G^e \) is the commutation graph of \( A \). It is immediate that \( \phi : \Gamma^e \to G^e \) defined by \( \phi(v^g) = (v^g)^{n(v, g)} \) is a graph isomorphism.

**3.2. Consequences for commensurability.** A general commensurability classification for right-angled Artin groups is currently unknown. The discussion in the previous subsection allows us to establish some connections between a commensurable right-angled Artin groups and their extension graphs. The following is almost immediate from the discussion in the preceding subsection, combined with the fact that if \( G \leq A(\Gamma) \) is a nonabelian subgroup and \( G' \leq G \) has finite index, then \( G' \) is also nonabelian:

**Corollary 8.** Let \( \Gamma \) and \( \Lambda \) be connected, triangle– and square–free graphs, and suppose that neither \( \Gamma \) nor \( \Lambda \) has any degree one vertices. If \( A(\Gamma) \) is commensurable with \( A(\Lambda) \) then \( \Gamma^e \cong \Lambda^e \).

**Proof.** If \( A(\Gamma) \) and \( A(\Lambda) \) are abelian then they must both be cyclic, in which case the conclusion is immediate. Otherwise, we can just apply Proposition 7 to suitable finite index subgroups of \( A(\Gamma) \) and \( A(\Lambda) \).

**Example 9.** Let \( C_n \) denote the cycle on \( n \) vertices. Since the girths of \( C_n^e \) is \( n \), we see that \( A(C_m) \) is not commensurable to \( A(C_n) \) for \( m \neq n \geq 1 \).
One could analogously conclude that if \( \text{Mod}(S) \) and \( \text{Mod}(S') \) are commensurable then \( \mathcal{C}(S) \cong \mathcal{C}(S') \). In fact, if \( \text{Mod}(S) \) and \( \text{Mod}(S') \) are even quasi–isometric to each other then \( S = S' \), by the quasi–isometric rigidity of mapping class groups (see \[3\] and \[15\]).

4. Electrified Cayley graph

In \[23\], Masur and Minsky proved that an electrified Cayley graph (defined below) of \( \text{Mod}(S) \) is quasi–isometric to the curve graph. Here, this result will be placed on a more general setting and applied to the action of right-angled Artin groups on extension graphs.

4.1. General setting. Let \( G \) be a group with a finite generating set \( \Sigma \) and let \( Y = \text{Cayley}(G, \Sigma) \). Suppose \( G \) acts simplicially and cocompactly on a graph \( X \), which is not necessarily connected. Let \( A \) be a (finite) set of representatives of vertex orbits.

Put \( H_\alpha = \text{Stab}_G(\alpha) \) for \( \alpha \in A \). We let \( \hat{Y} \) be the electrification of \( Y \) with respect to the disjoint union of right-cosets \( \bigsqcup_{\alpha \in A} H_\alpha \backslash G \). This means that \( \hat{Y} \) is the graph with \( V(Y) = (\bigsqcup_{\alpha \in A} H_\alpha \backslash G) \bigsqcup G \) and \( E(Y) = E(Y) \bigsqcup \{\{g, C\} : g \in C \in \bigsqcup_{\alpha \in A} H_\alpha \backslash G\} \).

Note our convention that even when \( H_\alpha = H_\beta \), we distinguish the cosets of \( H_\alpha \) an \( H_\beta \) as long as \( \alpha \neq \beta \). The graph \( \hat{Y} \) carries a metric such that the edges from \( Y \) have length 1 and the other edges have length 1/2. Let \( T = \cup_{\alpha \in A} H_\alpha \cup \Sigma \). For \( g \in G \), define the \( T \)-length of \( g \) as \( \|g\|_T = \min\{k : g = g_1 g_2 \ldots g_k \text{ where } g_i \in T\} \). For convention, we set \( \|1\|_T = 0 \).

Lemma 10. 

(1) The metric space \((G, \| \cdot \|_T)\) is quasi-isometric to \((\hat{Y}, d)\).

(2) If \( \text{diam}_X(A) < \infty \), then \( X \) is connected and \((G, \| \cdot \|_T)\) is quasi-isometric to \((X, d)\).

Remark. All the quasi-isometries in the following proof will be \( G \)-equivariant.

Proof. (1) is obvious from the existence of a continuous surjection from \( \text{Cayley}(G, T) \) onto \( \hat{Y} \) which restricts to the natural inclusion on the vertex set.

For (2), let us fix \( \alpha_0 \in A \) and define \( \psi : G \to \hat{Y} \) by \( \psi(g) = \alpha_0 g \). It is obvious that the image is quasi-dense. Put

\[
M = \max(\{d(\alpha_0, \alpha_0 s) : s \in \Sigma\} \cup \{2 \text{diam}_X(A)\}).
\]

For \( \alpha \in A \) and \( h \in H_\alpha \), we have \( d(\alpha_0, \alpha_0 h) \leq d(\alpha_0, \alpha) + d(\alpha, \alpha_0 h) = d(\alpha_0, \alpha) + d(\alpha h, \alpha_0 h) \leq 2 \text{diam}_X(A) \leq M \). So we see that \( d(\alpha_0, \alpha_0 g) \leq M\|g\|_T \) for each \( g \in G \).

In particular for every \( \alpha, \beta \in A \) and \( x, y \in G \), we have \( d(\alpha x, \beta y) \leq d(\alpha x, \alpha_0 x) + d(\alpha_0 x, \alpha_0 y) + d(\alpha_0 y, \beta y) \leq 2 \text{diam}_X(A) + M\|yx^{-1}\|_T \leq \infty \). Hence \( X \) is connected.

Choose a (finite) set \( B \) of representatives of edge orbits for \( G \)-action on \( X \) and write \( B = \{\{\alpha_i x_i, \beta_i y_i\} : i = 1, 2, \ldots\} \) where \( \alpha_i, \beta_i \in A \) and \( x_i, y_i \in G \). Set

\[
M' = \max(\|y_i x_i^{-1}\|_T : i = 1, 2, \ldots).
\]

Consider an edge in \( X \) of the form \( \{\alpha, \beta h\} \) where \( \alpha, \beta \in A \) and \( h \in G \). There exists \( \{\alpha x, \beta y\} \in B \) and \( g \in G \) such that \( \alpha = \alpha x g \) and \( \beta h = \beta y g \). Since \( xg \in H_\alpha \) and
We may assume \( \Sigma \). Proof. Let us define a graph \( Y \) with \( V(Y) = V(X) \) by the following condition: for \( x, y \in G \) and \( \alpha, \beta \in A \), the two vertices \( \alpha x, \beta y \) are declared to be adjacent if there exists \( g \in G \) such that \( \alpha x = \alpha g \) and \( \beta y = \beta g \).

**Lemma 11.** If \( \cup_{\alpha \in A} H_\alpha \) generates \( G \), then \( (G, \| \cdot \|_T) \) is quasi-isometric to \( (X^{(0)}, d') \).

Proof. We may assume \( \Sigma \subseteq \cup_{\alpha \in A} H_\alpha \). Fix \( \alpha_0 \in A \). We will prove \( \|g\|_T - 1 \leq d'(\alpha_0, \alpha_0 g) \leq \|g\|_T + 1 \) for each \( g \in G \). Note that if \( h \in G \) and \( \alpha \in A \cap Ah \), then \( h \in H_\alpha \). Now consider \( g \in G \) and set \( \ell = d'(\alpha_0, \alpha_0 g) \). We can choose \( h_1, h_2, \ldots, h_\ell \) such that \( \alpha_0 = Ah_1, \alpha_0 g = Ah_\ell \cdots h_2 h_1 \) and \( h_i \in H_{\alpha_i} \) where \( \alpha_i \in A \) for \( i = 1, 2, \ldots, \ell \). We see that \( g(h_\ell \cdots h_2 h_1)^{-1} \in H_{\alpha_0} \). Then \( g = g(h_\ell \cdots h_2 h_1)^{-1} \cdot h_\ell \cdots h_2 h_1 \) has \( T \)-length at most \( \ell + 1 \).

Conversely, suppose we have written \( g = h_k \cdots h_2 h_1 \) where for each \( i \) we have \( \alpha_i \in A \) and \( h_i \in H_{\alpha_i} \). Then each consecutive pair in the sequence

\[
(\alpha_0, \alpha_1, \alpha_2 h_1, \ldots, \alpha_k h_{k-1} \cdots h_2 h_1, \alpha_0 g)
\]

is contained in the image of \( A \) by some element of \( G \). This shows that \( d'(\alpha_0, \alpha_0 g) \leq \|g\|_T + 1 \).

Let us define a graph \( X' \) with \( V(X') = V(X) \) by the following condition: for \( x, y \in G \) and \( \alpha, \beta \in A \), the two vertices \( \alpha x, \beta y \) are declared to be adjacent if there exists \( g \in G \) such that \( \alpha x = \alpha g \) and \( \beta y = \beta g \).

**Lemma 12.**

1. For \( \alpha, \beta \in V(X') = V(X) \), we have \( d_{X'}(\alpha, \beta) = d'_X(\alpha, \beta) \).
2. If \( \cup_{\alpha \in A} H_\alpha \) generates \( G \), then \( X' \) is connected.

Proof. Obvious from the definition and Lemma 11.

So in the above sense, the graph \( X' \) is a “connected model” for \( X \). This will become useful in Section 11 when we metrize the extension graph of a disconnected graph.

**Corollary 13.** If \( X \) is connected and \( \cup_{\alpha \in A} H_\alpha \) generates \( G \), then the following metric spaces are all quasi-isometric to each other:

\[
(G, \| \cdot \|_T), (\hat{Y}, d), (X, d), (X^{(0)}, d'), (X', d).
\]

4.2. **Star length.** The following is now immediate from Corollary 13.

**Lemma 14.** ([23, Lemma 3.2]). Let \( S \) be a surface. Fix a generating set \( \Sigma \) for \( \text{Mod}(S) \) and a set of representatives \( A \) of vertex orbits in the curve graph \( \mathcal{C}(S) \). Let \( \hat{Y} \) be the electrification of \( \text{Cayley}(\text{Mod}(S), \Sigma) \) with respect to \( \coprod_{\alpha \in A} \text{Stab}(\alpha) \setminus \text{Mod}(S) \) and put \( T = \cup_{\alpha \in A} \text{Stab}(\alpha) \cup \Sigma \). Then the following metric spaces are quasi-isometric to each other:

\[
(\text{Mod}(S), \| \cdot \|_T), (\mathcal{C}(S), d), (\hat{Y}, d).
\]

Since \( \mathcal{C}(S) \) is \( \delta \)-hyperbolic, it follows that \( \text{Mod}(S) \) is weakly-hyperbolic relative to \( \{\text{Stab}(\alpha) : \alpha \in A\} \).
Let $\Gamma$ be a finite connected graph. Consider the (right-)conjugation action of $A(\Gamma)$ on $X = \Gamma^e$. Each vertex of $\Gamma^e$ is in the orbit of a vertex in $\Gamma$ and for $v \in V(\Gamma)$ we have $\text{Stab}(v) = \langle \text{St } v \rangle$. The star length of $g \in A(\Gamma)$ is the minimum $\ell$ such that $g$ can be written as the multiplication of $\ell$ elements in $\cup_{v \in V(\Gamma)} \langle \text{St } v \rangle$. We write the star length $g$ as $\|g\|_\star$. From Corollary 13, we have the following.

**Theorem 15.** Let $\hat{\mathcal{Y}}$ be the electrified Cayley graph of $A(\Gamma)$ with respect to the generating set $V(\Gamma)$ and the collection of cosets $\bigsqcup_{v \in V(\Gamma)} \langle \text{St } v \rangle \setminus A(\Gamma)$. Then the metric spaces $(A(\Gamma), \| \cdot \|_\star), (\Gamma^e, d)$ and $(\hat{\mathcal{Y}}, d)$ are quasi-isometric to each other.

In [20], the authors proved that $\Gamma^e$ is a quasi-tree. So we have,

**Corollary 16.** The group $A(\Gamma)$ is weakly hyperbolic relative to $\{ \langle \text{St } v \rangle : v \in V(\Gamma) \}$.

5. **Combinatorial group theory of right-angled Artin groups**

5.1. **Reduced words.** Let $\Gamma$ be a finite graph. Each $g \in A(\Gamma)$ can be represented by a word, which is a sequence $s_1s_2 \cdots s_\ell$ where each $s_i \in V(\Gamma) \cup V(\Gamma)^{-1}$ for $i = 1, 2, \ldots, \ell$. Each term $s_i$ of the sequence is called a letter of the word. The word length of $g$ is the smallest length of a word representing $g$, and denoted as $|g|$. A word $w$ is reduced if the length of $w$ is the same as the word length of the element which $w$ represents. A word is often identified with the element of $A(\Gamma)$ which the word represents. An element of $A(\Gamma)$ will be sometimes assumed to be given as a reduced word, if the meaning is clear from the context. If $w_1, w_2, \ldots, w_k$ are words, we denote by $w_1 \cdot w_2 \cdots w_k$ the word obtained by concatenating $w_1, w_2, \ldots, w_k$ in this order. If two words $w$ and $w'$ are equal as words, then we write $w \equiv w'$.

**Definition 17.** A sequence $(g_1, g_2, \ldots, g_\ell)$ of elements in $A(\Gamma)$ is called reduced if $|g_1g_2 \cdots g_\ell| = |g_1| + |g_2| + \cdots + |g_\ell|$. In this case, we write $x \sim (g_1, g_2, \ldots, g_\ell)$ where $x = g_1g_2 \cdots g_\ell \in A(\Gamma)$.

Note that $g \sim (\cdots, p, q, \cdots)$ implies that $g \sim (\cdots, pq, \cdots)$. Also, if $g \sim (g_1, g_2, \ldots, g_\ell)$ and each $g_i$ is represented by a reduced word, then $g_1 \cdot g_2 \cdots g_\ell$ is a reduced word $g \in A(\Gamma)$.

The support of a word $w$ is the set of vertices $v$ such that $v$ or $v^{-1}$ is a letter of $w$. The support of a group element $g$ is the support of a reduced word for $g$. We define the support of a sequence $(g_1, g_2, \ldots, g_s)$ of elements in $A(\Gamma)$ as the sequence $(\text{supp}(g_1), \text{supp}(g_2), \ldots, \text{supp}(g_s))$.

If a word $w'$ is a subsequence of another word $w'$, then we write $w' \preceq_0 w$. In particular, if $w'$ is a consecutive subsequence of $w$, then we say $w'$ is a subword of $w$ and write $w' \preceq w$. We define analogous terminology for group elements.

**Definition 18.** Let $g, h \in A(\Gamma)$ and $A \subseteq V(\Gamma)$.

1. If $g \sim (x, h, y)$ for some $x, y \in A(\Gamma)$, then we say $h$ is a subword of $g$ and write $h \preceq g$.

2. If a reduced word for $h$ is a subsequence of some reduced word for $g$, then we say $h$ is a subsequence of $g$ and write $h \preceq_0 g$. 
(3) If \( g \sim (x, y) \), then we say \( x \) is a prefix of \( g \) and \( y \) is a suffix of \( g \). We write \( x \lesssim_p g \) and \( y \lesssim_s g \).

(4) We write \( \tau(g; A) = h \) if (i) \( h \lesssim_s g \), (ii) \( \text{supp}(h) \subseteq A \) and (iii) if \( h' \) is another element of \( A(\Gamma) \) satisfying (i) and (ii) and \( |h| \leq |h'| \), then \( h = h' \).

(5) We write \( \iota(g; A) = h \) if (i) \( h \lesssim_p g \), (ii) \( \text{supp}(h) \subseteq A \) and (iii) if \( h' \) is another element of \( A(\Gamma) \) satisfying (i) and (ii) and \( |h| \leq |h'| \), then \( h = h' \).

5.2. More on star length. Suppose \( \Gamma \) is a finite connected graph. We have defined the star length of a word in Section 4. By a star-word, we mean a word of star length one. If \( g \) and \( g' \) are two elements of \( A(\Gamma) \), we say that \( \Gamma^g\!\!_{g'} \) is the conjugate of \( \Gamma^g \) by \( g' \).

By regarding \( V(\Gamma) \) as a set of vertex–orbit representatives, we have the notion of the covering distance on \( \Gamma^e \) as defined in Section 4. More concretely, let \( x \) and \( y \) be vertices of \( \Gamma^e \). Suppose \( \ell \) is the smallest nonnegative integer such that there exist conjugates \( \Gamma_1, \Gamma_2, \ldots, \Gamma_{\ell} \) of \( \Gamma \) in \( \Gamma^e \) satisfying (i) \( x \in \Gamma_1 \) and \( y \in \Gamma_{\ell} \) and (ii) \( \Gamma_i \cap \Gamma_{i+1} \neq \emptyset \) for \( i = 1, 2, \ldots, \ell - 1 \). Then \( \ell \) is the covering distance between \( x \) and \( y \) and denoted as \( d'(x, y) \).

As we have seen in Lemma 11 we have a quasi-isometry between \( (\Gamma^e, d') \) and \( (A(\Gamma), \| \cdot \|_*) \). We can see more precise estimates as follows, which are immediate consequences of the proof of Lemma 11.

Lemma 19. Let \( v \in V(\Gamma) \), \( x, y \in V(\Gamma^e) \) and \( g \in A(\Gamma) \).

(1) \( \|g\|_* - 1 \leq d'(v, v^g) \leq \|g\|_* + 1 \).

(2) \( d'(x, y) \leq d(x, y) \leq \text{diam}(\Gamma)d'(x, y) \).

(3) \( \|g\|_* - 1 \leq d(v, v^g) \leq \text{diam}(\Gamma)(\|g\|_* + 1) \).

Remark. Note that the connectivity assumption for \( \Gamma \) is not used for the covering distance and Lemma 19 (1). This observation will become critical later.

Lemma 20. Let \( x, y, z \in A(\Gamma) \).

(1) If \( x \lesssim_0 y \), then \( \|x\|_* \leq \|y\|_* \).

(2) For \( \ell = \|x\|_* \), there exist star-words \( x_1, x_2, \ldots, x_\ell \) such that \( x \sim (x_1, x_2, \ldots, x_\ell) \).

(3) Suppose \( xy \sim (x, y) \) and \( \|y\|_* \geq \|z\|_* + 2 \). Then \( xyz \sim (x, yz) \).

Proof. (1) If \( y = y_1 \cdots y_k \) for some star-words \( y_i \), then \( x = x_1 \cdots x_k \) for some \( x_i \lesssim_0 y_i \).

(2) is obvious from (1).

(3) Assume \( \|x, yz\| \) is not reduced. Write \( y \sim (y_1, y_2), z \sim (y_2^{-1}, z_1) \) such that \( yz \sim (y_1, z_1) \).

For some \( p \in V(\Gamma) \cup V(\Gamma)^{-1} \), we can write \( x \sim (x_0, p), z_1 \sim (p^{-1}, z_2) \) and \( [p, y_1] = 1 \). Then \( \|y\|_* \leq \|y_1\|_* + \|y_2\|_* \leq 1 + \|z\|_* \). \( \square \)

We briefly explain the notion of a dual van Kampen diagram \( \Delta \) for a word \( w \) representing a trivial element in \( A(\Gamma) \); see [12, 17, 18] for more details. The diagram \( \Delta \) is obtained from a van Kampen diagram \( X \subseteq S^2 \) for \( w \) by taking the dual complex of \( X \) and removing a small disk around the vertex corresponding to the exterior region of \( X \subseteq S^2 \). Topologically, \( \Delta \) is a disk with properly embedded arcs drawn. The boundary of \( \Delta \) is oriented and divided into segments which are labeled by letters of \( w \) so that if one follows the orientation of \( \partial \Delta \) and reads the labels, one gets \( w \) with a suitable choice of the basepoint. Each arc \( \alpha \) joins two segments on \( \partial \Delta \) whose labels are inverse to each other. Such an arc \( \alpha \) is labeled by the vertex that is the support of the two letters. If two arcs intersect, then their vertices are adjacent in \( \Gamma \). We often identify a
letter with the corresponding segment if the meaning is clear. See Figure 11 for a sketch of an example.

The **syllable length** of a word $g$ is the minimum $\ell$ such that $g$ can be written as the multiplication of $\ell$ elements in $\cup_{v \in V(\Gamma)}(v)$. We write the syllable length of $g$ as $\|g\|_{\text{syl}}$. The syllable length of a word can be estimated from a decomposition into star–words in the following sense.

**Lemma 21.** For $g \in A(\Gamma)$ and $x = \|g\|_{\ast}$, we have

$$\|g\|_{\text{syl}} = \inf \{ \|h_i\|_{\text{syl}} : g = h_1h_2 \cdots h_x \text{ and } \|h_i\|_{\ast} = 1 \}.$$  

**Proof.** Let $\ell = \|g\|_{\text{syl}}, r = \|g\|_{\ast}$ and $\ell'$ be the right-hand side of the equation. It is obvious that $\ell \leq \ell'$. Write $g = x_1^{e_1}x_2^{e_2} \cdots x_\ell^{e_\ell}$ where $x_i$ are vertices of $\Gamma$ and $e_i \neq 0$. Consider star–words $h_1, \ldots, h_r$ such that $g \sim (h_1, \ldots, h_r)$. There exists a dual van Kampen diagram $\Delta$ of the word $x_1^{e_1}x_2^{e_2} \cdots x_\ell^{e_\ell} \cdot h_1^{-1} \cdots h_r^{-1}$. Let $A_i = \{j : \text{ there is an arc from } x_i^{e_i} \text{ to } h_j^{-1} \text{ in } \Delta \}$. We may choose $(h_1, h_2, \ldots, h_r)$ and $\Delta$ such that $\sum_{j=1}^r |A_i|$ is minimal.

If a pair of neighboring arcs starting at $x_i^{e_i}$ are joined to a letter in $h_j$ and to a letter in $h_{j'}$ for some $j < j'$, then the interval between those two letters contains letters in $\text{Lk}(x_i) \cup \text{Lk}(x_i)^{-1}$. In this case, we replace $h_j$ by $h_jx_i^{\text{sgn}(e_i)}$ and $h_{j'}$ by $x_i^{-\text{sgn}(e_i)}h_{j'}$. Note that $\sum_i |A_i|$ did not increase. By repeating this “left-greedy” process and using the minimality, we see that $|A_i| = 1$ for $i = 1, 2, \ldots, l$. This shows that for each $j$, there exists $k > 0$ and $i_1 < i_2 < \cdots < i_k$ such that $h_j \sim (x_1^{e_1}, x_2^{e_2}, \ldots, x_\ell^{e_\ell})$. This gives a partition of $\{x_1^{e_1}, x_2^{e_2}, \ldots, x_\ell^{e_\ell}\}$ into $r$ disjoint sets, and hence, $\sum_{j=1}^r \|h_j\|_{\text{syl}} \leq l$. \hfill $\square$

Let us record a lemma that will be used later in this paper. The proof is a simple exercise.

**Lemma 22.** Let $\Gamma$ be a finite, triangle– and square–free graph. Suppose $y \in V(\Gamma)$ and $g \in \langle \text{St}(y) \rangle \subseteq A(\Gamma)$.

1. If $|\text{supp}(g) \cap \text{Lk}(y)| \geq 2$, then $\Gamma \cap \Gamma^g = \{y\}$.
2. If $\text{supp}(g) \cap \text{Lk}(y) = \{a\}$ and $\|y\|_{\text{syl}} > 1$, then $\Gamma \cap \Gamma^g = \{a, y\}$.
3. If $\text{supp}(g) \cap \text{Lk}(y) = \emptyset$ and $g \neq 1$, then $\Gamma \cap \Gamma^g = \text{St}(y)$.
4. If $h \in A(\Gamma)$ is not a star–word, then $\Gamma \cap \Gamma^g = \emptyset$.

### 5.3. Quasi–isometries of right-angled Artin groups with the star length metric

In this subsection, we give a quasi–isometry classification of right-angled Artin groups equipped with the star length metric. It turns out that this classification is much coarser than the quasi–isometry classification of right-angled Artin groups with the word metric [4][5]. In particular, there are only two quasi–isometry classes. We will denote by $T_x$, the simplicial tree which has countable valance at each vertex.

**Theorem 23.** Let $\Gamma$ be a connected finite simplicial graph and let $A(\Gamma)$ be the associated right-angled Artin group, equipped with the star length metric. Then $A(\Gamma)$ is quasi–isometric to exactly one of the following:

1. A single point.
(2) The countable regular tree $T_\infty$.

Proof. By Theorem 15, the group $A(\Gamma)$ equipped with the star length metric is quasi-isometric to $\Gamma^e$. Recall that $\Gamma^e$ has finite diameter if and only if $A(\Gamma)$ splits as a nontrivial direct product or if $A(\Gamma) \cong \mathbb{Z}$.

Suppose $\text{diam}(\Gamma^e) = \infty$. We have a quasi-isometry $\phi : \Gamma^e \to T$ for some tree $T$. Every vertex of $\Gamma^e$ has a quasi–dense orbit under the $A(\Gamma)$ action. Furthermore, it is easy to check that for $v \in V(\Gamma^e)$, the graph $\Gamma^e \setminus \text{St}(v)$ has infinitely many components $C_1, C_2, \ldots$ of infinite diameter.

If we remove a ball $B_K(v)$ of radius $K$ about a vertex $v \in \Gamma^e$, then the minimal distance between $C_i \setminus B_K(v)$ and $C_j \setminus B_K(v)$ grows like $K$ for $i \neq j$. So, if $K$ is chosen to be much larger than the quasi–isometry constants for a quasi–isometry $\phi : \Gamma^e \to T$, we see that $\Gamma^e \setminus B_K(v)$ has infinitely many vertices which must be sent to pairwise distinct vertices of a bounded subset of $T$ under $\phi$. If $T$ is locally finite, any bounded subset of $T$ is finite, a contradiction. Thus, for each vertex $w$ of $T$, there is a uniform $M$ which is independent of $w$ such that the ball of radius $M$ about $w$ has infinitely many vertices. In particular, such an $M$–ball contains at least one vertex of infinite degree. For each vertex of finite valence, choose a path to a nearest vertex of infinite valence. Such a path has length at most $M$. Hence, contraction of such paths will yield a quasi-isometry from $T$ to $T_\infty$.

In Subsection 11.1 we will define an appropriate notion of distance on disconnected extension graphs, with respect to which even totally disconnected extension graphs will be quasi–isometric to $T_\infty$.

6. Acylindricity of the $A(\Gamma)$ action on $\Gamma^e$

Let $\Gamma$ be a finite graph. The goal of this section is to show that the conjugation action of $A(\Gamma)$ on $\Gamma^e$ is acylindrical.

Definition 24. An isometric action of a group $G$ on a path-metric space $X$ is called acylindrical if for every $r > 0$, there exist $R, N > 0$ such that whenever $p$ and $q$ are two elements of $X$ with $d(p, q) \geq R$, the cardinality of the set $\Delta(p, q; r) = \{ g \in G : d(p, g.p) \leq r \text{ and } d(q, g.q) \leq r \}$ is at most $N$.

All the real variables ($r, s, t, R, N, M$ and so forth) will be assumed to take integer values throughout this section. We say a vertex set $A$ is adjacent to another vertex set $B$ if each vertex of $A$ is adjacent to every vertex of $B$; in particular, $A$ and $B$ are disjoint.

Definition 25. Let $\beta = (s, x, y, v_1, v_2, \ldots, v_s)$ where $s > 0$, $x, y \in A(\Gamma)$ and $v_1, v_2, \ldots v_s \in V(\Gamma)$. We say the sequence $\alpha = (g_1, g_2, \ldots, g_s, h_1, h_2, \ldots, h_s) \in A(\Gamma)^{2s}$ is a cancellation sequence for $g \in A(\Gamma)$ with respect to $\beta$ if the following four conditions hold.

(i) $g = g_1g_2 \cdots g_sh_1h_2 \cdots h_s$.
(ii) $\supp(g_i), \supp(h_i) \subseteq \text{St}(v_i)$ for each $i$.
(iii) For each $1 \leq i < j \leq s$, we have that $\supp(h_i)$ is adjacent to $\supp(g_j)$. 

(iv) \( x \sim (x', w, g_s^{-1}, g_{s-1}^{-1}, \ldots, g_1^{-1}) \) and \( y \sim (h_s^{-1}, h_{s-1}^{-1}, \ldots, h_1^{-1}, w^{-1}, y') \) and for some \( x', y', w \in A(\Gamma) \).

Furthermore, we say \( \alpha \) is maximal if the lexicographical order of

\[ (|g_1|, |g_2|, \ldots, |g_s|, |h_s|, |h_{s-1}|, \ldots, |h_1|) \]

is maximal among the cancellation sequences for \( g \) w.r.t. \( \beta \).

If there is a cancellation sequence for \( g \) w.r.t. \( \beta \), then it is obvious that a maximal one exists since the complexity is bounded above by \(|x|, |x|, \ldots, |x|, |y|, |y|, \ldots, |y|\).

Roughly speaking, the item (2) below means that \( (g_1, \ldots, g_s) \) is “left-greedy” and \( (h_1, h_2, \ldots, h_s) \) is “right-greedy”. The proofs are straightforward.

**Lemma 26.** Suppose \( \alpha = (g_1, g_2, \ldots, g_s, h_1, h_2, \ldots, h_s) \) is a maximal cancellation sequence for some \( g \) w.r.t. some \( \beta \). Let \( 1 \leq a < b \leq s \).

(1) If \( (g'_1, g'_2, \ldots, g'_s, h'_1, h'_2, \ldots, h'_s) \) is another cancellation sequence for \( g \) w.r.t. \( \beta \) and \( |g_i| \leq |g'_i|, |h_i| \leq |h'_i| \) for each \( i \), then \( g_i = g'_i \) and \( h_i = h'_i \).

(2) Suppose that for some \( u \in A(\Gamma) \), either

(i) \((g_1, \ldots, g_{a-1}, g'_a, g_{a+1}, \ldots, g_{b-1}, g'_b, g_{b+1}, \ldots, g_s, h_1, h_2, \ldots, h_s)\) is another cancellation sequence for \( g \) w.r.t. \( \beta \) where \( g'_a \sim (g_a, u) \) and \( g_b \sim (u^{-1}, g'_b) \), or

(ii) \((g_1, \ldots, g_s, h_1, \ldots, h_{a-1}, h'_a, h_{a+1}, \ldots, h_{b-1}, h'_b, h_{b+1}, \ldots, h_s)\) is another cancellation sequence for \( g \) w.r.t. \( \beta \) where \( h_a \sim (h'_a, u) \) and \( h_b \sim (u^{-1}, h'_b) \).

Then \( u = 1 \).

For \( t > 0 \), we define \( B_t = \{ g \in A(\Gamma) : \|g\|_* \leq t \} \) and \( B'_t = \{ g \in A(\Gamma) : \|g\|_* < t \} \). These “balls” are infinite sets in general.

**Lemma 27.** Let \( s, t > 0 \) and \( M \geq s + t + 2 \). Suppose \( x, y \in A(\Gamma) \) satisfy that \( \|x\|_* \geq M \) or \( \|y\|_* \geq M \). If \( g \in B_s \cap x^{-1}B_t y^{-1} \), then there exist vertices \( v_1, v_2, \ldots, v_s \) of \( \Gamma \) and a cancellation sequence for \( g \) with respect to \((s, x, y, v_1, v_2, \ldots, v_s)\). Moreover, we have \( \|w\|_*, \|x\|_*, \|y\|_* \geq M - s - t \) where \( w \) is as in Definition \([25] \).

Roughly speaking, the above lemma implies that if \( x \) or \( y \) is long and \( g \) and \( xgy \) are both short in the star lengths, then \( x \) and \( y \) are both long and \( g \) must be “completely cancelled” in \( xgy \).

**Proof of Lemma \([27] \)** Without loss of generality, let us assume \( \|x\|_* \geq M \). There exist \( v_1, v_2, \ldots, v_s \) in \( V(\Gamma) \) and \( w_i \in \langle \text{St}(v_i) \rangle \) such that \( g \sim (w_1, w_2, \ldots, w_s) \). We can write \( xgy \sim (z_1, z_2, \ldots, z_t) \) for some star-words \( z_1, z_2, \ldots, z_t \).

Let \( \Delta \) be a dual van Kampen diagram for the following word

\[ x \cdot w_1 \cdot w_2 \cdots w_s \cdot y \cdot z_t^{-1} \cdot z_{t-1}^{-1} \cdots z_1^{-1}, \]

where \( x, y, z_t \) are represented by reduced words. Note that no two letters from

\[ w_1 \cdots w_2 \cdots w_s \]

are joined by an arc in \( \Delta \). We call the interval on \( \partial \Delta \) reading \( x \cdot w_1 \cdot w_2 \cdots w_s \cdot y \) as \( \partial_1 \) and the closure of \( \partial \Delta \setminus \partial_1 \) as \( \partial_2 \).

We have \( x \sim (x', w, x'') \) such that the letters of \( x', w \) and \( x'' \) are joined to \( \partial_2, y \) and \( g \), respectively. We may assume that \( x \) is written as the reduced word \( x' \cdot w \cdot x'' \). Then
Choose a cancellation sequence

Let words.

Remark.

Now, (i), (ii) and (iv) are obvious from the construction. Let $1 \leq i < j \leq s$. Each letter of $g_j$ is joined to a letter in $x$ by an arc separating $h_i$ from $y$; hence we have (iii).

Remark. (1) In the above proof, we have $w_i \sim (g_i, h_i)$. However, we do not assume for maximal $\alpha$ that each $g_i \cdot h_i$ is a reduced concatenation. This will be critical in the proof of Lemma 28.

(2) Write $x \sim (p, p')$ where $\|p\|_\ast = s + 2$. Then no letters of $p$ is joined to a letter of $g$ by Lemma 20 (3). Hence $g_s^{-1}g_{s-1}^{-1} \cdots g_1^{-1} \preceq_p p'$. Similarly if we write $y \sim (q', q)$ so that $\|q'\|_\ast = s + 2$, then $h_s^{-1}h_{s-1}^{-1} \cdots h_1^{-1} \preceq_p q'$. Hence, the cardinality of $B_s \cap x^{-1}B_t y^{-1}$ is at most the number of the choices for $g_1, g_2, \ldots, g_s \preceq_p (p')^{-1}$ and $h_1, h_2, \ldots, h_s \preceq_p (q')^{-1}$, and so bounded above by $(2^{|p'| + |q'|})^s$. The goal of Lemma 29 is to find another upper-bound depending only on $\Gamma, s$ and $t$.

We define the support of a sequence of words as the sequence of the supports of those words.

Lemma 28. Let $\beta = (s, x, y, v_1, v_2, \ldots, v_s)$ as in Definition 25 and $t > 0$. Suppose $\|x\|_\ast \geq s + t + 2$ or $\|y\|_\ast \geq s + t + 2$. Let $P_i, Q_i \subseteq \text{St}(v_i)$ for each $i = 1, 2, \ldots, s$ such that $Q_i$ is adjacent to $P_j$ for each $1 \leq i < j \leq s$. Then there exists at most one element $g$ in $x^{-1}B_t y^{-1}$ such that $(P_1, P_2, \ldots, P_s, Q_1, Q_2, \ldots, Q_s)$ is the support of a maximal cancellation sequence for $g$ with respect to $\beta$.

Proof. Suppose $g$ is such an element. Define $x_0 = x$ and inductively, $p_i = \tau(x_{i-1}; P_i), x_i = x_{i-1}^{-1}p_i^{-1}$ for $i = 1, 2, \ldots, s$. Similarly, put $y_0 = y$ and $q_i = \tau(y_{i-1}; Q_i), y_i = q_{i-1}^{-1}y_{i-1}$ for $i = 1, 2, \ldots, s$. We will prove the conclusion by showing that $g = p_1^{-1}p_2 \cdots p_{s-1}^{-1}q_1q_2 \cdots q_s^{-1}$. Choose a cancellation sequence $\alpha = (g_1, g_2, \ldots, g_s, h_1, h_2, \ldots, h_s)$ for $g$ with respect to $(s; x, y; v_1, v_2, \ldots, v_s)$ with support $(P_1, P_2, \ldots, P_s, Q_1, Q_2, \ldots, Q_s)$. Let $z$ be a reduced word representing $xgy$ and $\Delta$ be a dual van Kampen diagram for the following word:

$x \cdot g_1 \cdot h_1 \cdots g_s \cdot h_s \cdot y \cdot z^{-1}.$
We use an induction on \( i = 1, 2, \ldots, s \) to prove that there is a one-to-one correspondence between the letters of \( g_i \) and those of \( p_i \) such that each corresponding pair are joined by an arc in \( \Delta \), and hence, \( g_i = p_i^{-1} \). By the inductive hypothesis and the maximality of \( p_i \), each letter of \( g_i \) is joined to a letter of \( p_i \). Suppose a letter \( u \) of \( p_i \) is not joined to any of the letters of \( g_i \). We may assume \( u \in V(\Gamma) \) without loss of generality. Let \( w \) be as in Definition \( \text{25} \). If \( u \) is joined to a letter in \( z \) then every letter of \( w \) would be adjacent to \( u \). Since \( \|w\|_s \geq 2 \) by Lemma \( \text{27} \), this is impossible. So we only have the following two cases.

Case 1. \( u \) is joined to a letter \( u^{-1} \) of \( g_j \) for some \( i < j \leq s \) by an arc, say \( \gamma \); see Figure \( \text{2} \)(a).

We may assume every letter appearing after \( u \) in \( p_i \) is joined to a letter in \( g_i \); namely, \( u \cdot g_i \leq g_i, p_i \). Choose an arbitrary letter \( u' \) between the last letter of \( g_i \) and the letter \( u^{-1} \) of \( g_j \). The arc \( \gamma' \) originating from \( u' \) ends at a letter of \( x \) before \( u \) or at a letter of \( y \). Then \( \gamma' \) intersects with \( \gamma \) and \( \{u', u\} = \emptyset \). We can enlarge \( g_i \) by setting \( g_i' = (g_i, u^{-1}) \) and \( g_j' = u g_j \). Note that \( \text{supp}(g_i') \subseteq P_i \) and \( \text{supp}(g_j') \subseteq P_j \) and hence, the condition (iii) of Definition \( \text{25} \) is satisfied. This contradicts to the left-greediness of \( (g_1, g_2, \ldots, g_s) \).

Case 2. \( u \) is joined to a letter of \( y \); see Figure \( \text{2} \)(b).

For each \( j > i \), the arc \( \gamma \) intersects with every arc originating from \( g_j \) and hence, the vertex \( u \) is adjacent to \( P_j \). Since \( u \leq P_i \), the vertex \( u \) is adjacent to \( Q_1, Q_2, \ldots, Q_{i-1} \). We set \( g_i' = (g_i, u^{-1}) \) and \( h_i' = (u, h_i) \); see Figure \( \text{2} \)(c). Note that \( \text{supp}(g_i') = P_i = \text{supp}(g_i) \) and \( \text{supp}(h_i') \leq \{u\} \cup Q_i \). The condition (iii) of Definition \( \text{25} \) is obvious. This again contradicts to the maximality of the sequence \( \alpha \).

We conclude that \( g_i = p_i^{-1} \) for \( i = 1, 2, \ldots, s \). We also see that \( h_i = q_i^{-1} \) by symmetry.

\[ \square \]

Lemma 29. Let \( s, t > 0 \) and \( x, y \in A(\Gamma) \) such that \( \|x\|_s \geq s + t + 2 \) or \( \|y\|_s \geq s + t + 2 \). Then the cardinality of \( B_s \cap x^{-1}B_t y^{-1} \) is at most \( |V(\Gamma)|^s (2^{|V(\Gamma)|})^{2s} \).

Proof. In Lemmas \( \text{27} \) the number of the possible choices for \( v_1, v_2, \ldots, v_s \) is bounded by \( |V(\Gamma)|^s \). Also, the number of ways of choosing \( P_1, P_2, \ldots, P_s, Q_1, Q_2, \ldots, Q_s \) in Lemma \( \text{28} \) is at most \( (2^{|V(\Gamma)|})^{2s} \). \[ \square \]

Theorem 30. The action of \( A(\Gamma) \) on \( \Gamma^e \) is acylindrical.

Proof. Let us fix a vertex \( v \) of \( \Gamma \) and let \( r > 0 \) be given. Put \( D = \text{diam}(\Gamma), s = r + 2D + 1, R = D(2s + 5) \) and \( N = |V(\Gamma)|^s (2^{|V(\Gamma)|})^{2s} \). Suppose \( p \) and \( q \) are two vertices of \( \Gamma^e \) such that \( d(p, q) \geq R \). Then there exist \( w', w \in A(\Gamma) \) such that \( d(p, v^w) \leq D \) and \( d(q, v^w) \leq D \). Without loss of generality, we may assume \( w' = 1 \). By Lemma \( \text{19} \), we have \( \|w\|_s \geq d(v, v^w)/D - 1 \geq (R - 2D)/D - 1 \geq 2s + 2 \). For every \( g \in \Delta(p, q; r) \), we have \( \|g\|_s \leq d(v, v^g) + 1 \leq d(p, p^q) + d(p^q, v^g) + 1 \leq r + 2D + 1 = s \) and similarly, \( \|w^g\|_s \geq d(v, v^{w^g}) + 1 = d(v^w, v^{w^g}) + 1 \leq d(v, q) + d(q, q^g) + d(q, v^g) + 1 \leq s \). Hence, \( \Delta(p, q; r) \subseteq w^{-1}B_s w \cap B_s \). Since \( \|w\|_s \geq 2s + 2 \), Lemma \( \text{29} \) implies that the set \( \Delta(p, q; r) \) has at most \( N \) elements. \[ \square \]
Nielsen–Thurston classification

We have shown that the action of the right-angled Artin group on the extension graph $\Gamma^e$ is acylindrical. We will now discuss several consequences of the acylindricity of the action.

Recall that if a group $G$ acts on a metric space $(X, d)$ by isometries, we may consider the translation length (also called stable length) of an element $g \in G$. This quantity is defined for an arbitrary choice of $x \in X$ as following.

$$\tau(g) = \lim_{n \to \infty} \frac{1}{n}d(x, g^n(x)).$$

This limit always exists and independent of $x$. The following proposition appears as Lemma 2.2 of [8].

**Proposition 31.** Let $G$ be a group acting acylindrically on a $\delta$–hyperbolic graph $X$. Then for each nonidentity $g \in G$, either $g$ is elliptic, in which case $\tau(g) = 0$ and one (and hence every) orbit of $g$ is bounded, or $g$ is loxodromic, in which case $\tau(g) \geq \epsilon > 0$. The constant $\epsilon$ depends only on the acylindricity parameters of the action and the hyperbolicity constant of $X$.

The principal content of the previous proposition is that acylindrical actions of groups on hyperbolic graphs do not have parabolic elements. The previous proposition gives us a way to formulate a Nielsen–Thurston classification for nonidentity elements of a right-angled Artin group. The following tables summarize the Nielsen–Thurston classification for mapping classes.
Nontrivial mapping class $ψ$

| $\mathcal{C}(S)$ type | Nielsen–Thurston classification | Curve complex characterization | Intrinsic algebraic characterization (in $\text{Out}(\pi_1(S))$) |
|------------------------|---------------------------------|-------------------------------|-------------------------------------------------|
| Elliptic               | Finite order                    | Every orbit is finite         | Some nonzero power of $ψ$ is the identity       |
| Elliptic               | Infinite order reducible        | There exists a finite orbit   | Some nonzero power of $ψ$ preserves             |
|                        |                                 | and an infinite orbit         | the conjugacy class of a nonperipheral,          |
|                        |                                 |                               | nontrivial isotopy class in $\pi_1(S)$          |
| Loxodromic             | Pseudo-Anosov                   | Stable length is nonzero      | No power of $ψ$ preserves any nonperipheral,    |
|                        |                                 |                               | nontrivial isotopy class in $\pi_1(S)$          |

The following table summarizes the analogous Nielsen–Thurston classification for nonidentity elements of right-angled Artin groups:

| $Γ^e$ type | Extension graph characterization | Intrinsic algebraic characterization (in $A(Γ)$) |
|------------|---------------------------------|-------------------------------------------------|
| Elliptic   | Every orbit is bounded          | Support of $g$ is contained in a nontrivial subjoin of $Γ$ |
| Loxodromic | Stable length is nonzero        | Support of $g$ is not contained in a subjoin of $Γ$ |

Now let us give a proof of the intrinsic algebraic characterization of loxodromic and elliptic elements of $A(Γ)$.

**Lemma 32.** Let $(a_1,a_2,\ldots,a_ℓ,a_1)$ be an edge–loop in $Γ^{\text{opp}}$ such that $\|a_1a_2\cdots a_ℓ\|_s > 1$. Then for arbitrary $e_1, e_2, \ldots, e_ℓ \neq 0$, we have that $\|(a_1^{e_1}a_2^{e_2}\cdots a_ℓ^{e_ℓ})^n\|_s > n$.

**Proof.** Write $g = (a_1^{e_1}a_2^{e_2}\cdots a_ℓ^{e_ℓ})^n = w_1w_2\cdots w_k$ where $k = \|g\|_s$ and $\|w_i\|_s = 1$ for each $i$. Note that $g$ has only one reduced word representation. Since $\{a_1, a_2, \ldots, a_ℓ\}$ is not contained in one star, $\|w_i\|_{\text{syl}} \leq ℓ - 1$ for each $i$. Hence, $nℓ = \|g\|_{\text{syl}} \leq \sum_i \|w_i\|_{\text{syl}} \leq (ℓ - 1)k$. We have $k \geq nℓ/(ℓ - 1) > n$. □

**Lemma 33.** Let $g \in A(Γ)$ be a cyclically reduced element such that the support of $g \in A(Γ)$ is not contained in a join. Then for each $n > 0$, we have $\|g^{2n\|V(Γ)\|^2}\|_s \geq n$.

**Proof.** Put $M = 2(|V(Γ)| - 1)^2$. Find an edge–loop $(a_1,a_2,\ldots,a_ℓ,a_1)$ in $Γ^{\text{opp}}$ such that $\text{supp}(g) = \{a_1,a_2,\ldots,a_ℓ\}$. We can further require that $ℓ \leq M$, since there exists an edge–loop in $Γ$ of length at most $M$ which visits all the vertices in the connected component of $Γ^{\text{opp}}$. Note that $a_1$’s may be redundant. We will regard $g^M$ as the reduced word obtained by concatenating $M$ copies of a reduced word for $g$. We can choose $M$ occurrences of $a_1^{±1}$ in $g^M$, all corresponding to the same letter of $g$. Since $[a_1,a_2] \neq 1$, we can choose $M - 1$ occurrences of $a_2^{±1}$, alternating with the previously chosen $M$ occurrences of $a_1^{±1}$. We can then similarly choose $M - 2$ occurrences of $a_3^{±1}$ alternating with the previously chosen $M - 1$ occurrences of $a_2^{±1}$, and continue. Since
\[ M - \ell + 1 \geq 1, \text{ it is easy to deduce that } a_1^{\pm 1} a_2^{\pm 2} \cdots a_{\ell}^{\pm 1} \leq_0 g^M. \] By Lemma 20 (1) and 32, we have \( \|g^{Mn}\|_* \geq n. \) Note that \( 2|V(\Gamma)|^2 > M. \)

**Lemma 34.** Let \( g \in A(\Gamma) \) be a cyclically reduced element. Then \( \{\|g^n\|_*\}_{n \in \mathbb{N}} \) is bounded if and only if the support of \( g \) is contained in a join.

**Proof.** Assume that the support of \( g \) is contained in a join \( J = J_1 * J_2 \leq \Gamma. \) Let us choose \( v_i \in V(J_i) \) so that \( J_i \subseteq \text{St}(v_{3-1}) \) for \( i = 1, 2. \) We can write \( g = g_1g_2 \) so that \( g_i \in A(J_i) \leq \langle \text{St}(v_{3-1}) \rangle \) for each \( i. \) Then \( \|g^n\|_* \leq \|g^n_1\|_* + \|g^n_2\|_* \leq 2 \) for every \( n. \)

The opposite direction is obvious from Lemma 33.

**Theorem 35.** Let \( 1 \neq g \in A(\Gamma) \) be cyclically reduced. Then \( g \) is elliptic if and only if the support of \( g \) is contained in a join of \( \Gamma. \)

**Proof.** An element \( 1 \neq g \in A(\Gamma) \) is elliptic if and only if for every vertex \( v \) of \( \Gamma^e, \) the orbit \( \{v^g\}_{n \in \mathbb{Z}} \) is bounded. We have the estimate that for any \( g \in A(\Gamma), \) the distance \( d_{\Gamma^e}(v, v^g) \) is coarsely equal to the star length \( \|g\|_* \). It follows that each orbit \( \{v^g\}_{n \in \mathbb{Z}} \) is bounded then the star lengths \( \|g^n\|_* \) are universally bounded. By Lemma 34, we see that this occurs if and only if \( \text{supp}(g) \) is contained in a join in \( \Gamma. \)

It is interesting to note a fundamental difference between the action of \( \text{Mod}(S) \) on \( C(S) \) and the action of \( A(\Gamma) \) on \( \Gamma^e. \) Observe that if \( \psi \in \text{Mod}(S) \) is nontrivial and elliptic, then \( \psi \) has a finite orbit in \( C(S). \) If \( g \in A(\Gamma) \) is elliptic, it may not have a finite orbit, even though each orbit is bounded in \( \Gamma^e. \)

**Proposition 36.** Consider

\[ A(\Gamma) = F_2 \times F_2 = \langle a, b \rangle \times \langle c, d \rangle. \]

The element \( abcd \in F_2 \times F_2 \) is elliptic and has no finite orbits in \( \Gamma^e. \)

**Proof.** Every nontrivial element of \( A(\Gamma) \) is elliptic, since \( \Gamma \) splits as a join. Actually, \( \Gamma^e \) has finite diameter in this case. By adding a degree-one vertex \( v \) to \( a, \) one obtains a graph \( \Lambda \) which does not split as a join and thus satisfies \( \text{diam}(\Lambda^e) = \infty. \) It is clear from the Centralizer Theorem that if \( g \in A(\Lambda) \) and \( n \neq 0 \) then \( (abcd)^n \) does not stabilize \( v^g. \) Thus, it suffices to show that for each nonzero \( n, \) the element \( (abcd)^n \) does not stabilize any conjugate of any of the vertices \( \{a, b, c, d\}. \)

Let \( w \in \{a, b, c, d\}, \) and let \( g \in A(\Lambda). \) The centralizer of the vertex \( w^g \in \Lambda^e \) is generated by \( \text{St}(w^g) \subseteq \Lambda^e. \) However, the element \( abcd \) and all of its nonzero powers are not supported in the star of any vertex of \( \Lambda^e. \)

8. **Injective homomorphisms and types**

8.1. **Type preservation under injective homomorphisms.** In this subsection we consider the behavior of elliptic and loxodromic elements under elements of \( \text{Mono}(G, H), \) where \( G \) and \( H \) are allowed to be right-angled Artin groups and mapping class groups, and where Mono denotes the set of injective homomorphisms from \( G \) to \( H. \) Namely, we consider \( f \in \text{Mono}(G, H) \) and \( 1 \neq g \in G. \) Since \( g \) is either elliptic or loxodromic, we consider whether \( f(g) \) is also elliptic or loxodromic.

The following table summarizes type preservation under injective homomorphisms:
| Source       | Target          | Mod($S$)                             |
|--------------|-----------------|--------------------------------------|
| $A(\Gamma)$  | Elliptics preserved, loxodromics not preserved | Elliptics preserved, loxodromics not preserved |
| Mod($S$)     | Usually no injective homomorphisms             | Elliptics preserved, loxodromics not preserved |

The case where Mod($S$) is the source and $A(\Gamma)$ is the target is handled by the following result (see [21] and the references therein):

**Proposition 37.** Let $S$ be a surface with genus $g \geq 3$ or $g = 2$ with at least two punctures, let $G < \text{Mod}(S)$ be a finite index subgroup, and let $A(\Gamma)$ be a right-angled Artin group. Then there is no injective homomorphism $G \to A(\Gamma)$.

**Theorem 38.** Let $f \in \text{Mono}(G, H)$, where $G$ and $H$ are right-angled Artin groups or mapping class groups, and let $1 \neq g \in G$ be elliptic. Then $f(g)$ is elliptic.

**Proof.** Observe that without loss of generality, we may replace $g$ with any positive power. In particular, whenever $G$ is a mapping class group, we may assume that $g$ is a pure mapping class. Reducible pure mapping classes and elliptic elements in $A(\Gamma)$ are characterized by the fact that their centralizers are not (virtually) cyclic. Since $f$ is injective and $g$ is elliptic, the centralizer of $f(g)$ in $H$ will not be virtually cyclic. It follows that $f(g)$ is also elliptic. \hfill $\square$

For the non-preservation of loxodromics, we have the following examples, the first of which is due to Aramayona, Leininger and Souto:

**Theorem 39 ([1]).** There exists an injective map between two mapping class groups under which a pseudo-Anosov mapping class is sent to a Dehn multitwist.

For the next proposition, let $P_4$ denote the path on four vertices and let $S_2$ denote a surface of genus two.

**Proposition 40.** There exists an injective map $A(P_4) \to \text{Mod}(S_2)$ whose image contains a pseudo-Anosov mapping class and under which a loxodromic element is mapped to a reducible element.

**Proof.** Label the vertices of $P_4$ in a row by $\{a, b, c, d\}$ as in Figure 3(a). We consider a chain of four simple closed curves on $S_2$ which fills $S_2$. The curves should then be labelled in order by $\{B, D, A, C\}$; see Figure 3(b).

The injective map $A(P_4) \to \text{Mod}(S_2)$ is given by taking the vertex generator labelled by a lower case letter $x$ to a sufficiently high power of a Dehn twist about the curve labelled by the upper case letter $X$. Since $\{A, B, C, D\}$ fill $S_2$, there is a pseudo-Anosov mapping class in the image of $A(P_4)$. Since $A$ and $D$ intersect exactly once, they cannot fill $S_2$ (in fact they fill a torus with one boundary component). The element $ad \in A(P_4)$ is loxodromic since the distance between $a$ and $d$ in $P_4$ is three, and the image of $ad$ is supported on a torus with one boundary component. It follows that the image of $ad$ is reducible and hence elliptic in Mod($S_2$). \hfill $\square$
In [20], the authors showed that for any graph $\Gamma$, there is a surface $S$ such that $\Gamma^e$ embeds in $C(S)$. The previous proposition has the following corollary, which shows that embeddings from $\Gamma^e$ into curve graphs may not be well-behaved from a geometric standpoint:

**Corollary 41.** There exists an embedding $P^e_4 \to C(S_2)$ which is not metrically proper.

**Proof.** Consider the embedding of $A(P_4)$ into $\text{Mod}(S_2)$ as given in Proposition 40. We obtain a map from $P^e_4$ to $C(S)$ as follows: send the vertices $\{a, b, c, d\}$ to the curves $\{A, B, C, D\}$ respectively. We have that sufficiently high powers of Dehn twists about those curves generate a copy of $A(P_4) < \text{Mod}(S_2)$. We can obtain the double of $P_4$ over the star of any vertex by applying the corresponding power of a Dehn twist to the curves $\{A, B, C, D\}$, and adding the new curves to the collection $\{A, B, C, D\}$. The map from the double of $P_4$ to $C(S)$ is given by sending a vertex to the corresponding curve.

We identify the vertex generators $\{a, b, c, d\}$ of $A(P_4)$ with sufficiently high powers of Dehn twists about the curves $\{A, B, C, D\}$. We then consider the orbits of these curves inside of $C(S_2)$ under the action of $A(P_4)$. By definition, this gives us an embedding of $P^e_4$ into $C(S_2)$. Specifically, let

$$\phi : A(P_4) \to \text{Mod}(S_2)$$

be the embedding above which sends a vertex generator $v$ to the Dehn twist power $T^M_{\gamma(v)}$. The map $P^e_4 \to C(S_2)$ is given by

$$v^g \mapsto \phi(g^{-1})(\gamma(v)).$$

Write $w = ad \in A(P_4)$. We have seen that this element is loxodromic. It follows that for any vertex $v \in P^e_4$, we have that the distances $d(v, v^w)$ tend to infinity. However, the curves $A$ and $D$ do not fill $S_2$, so that any mapping class written as a product of twists about $A$ and $D$ will be reducible, and each of its orbits in $C(S_2)$ will be bounded. It follows that for any vertex $v \in P^e_4$, the distances between the images of $v$ and $v^w$ in $C(S_2)$ will remain bounded as $n$ tends to infinity. \hfill \Box

### 8.2. Subgroups generated by pure elements.

Whereas it is not true that loxodromic elements are sent to pseudo-Anosov mapping classes under injective maps between right-angled Artin groups and mapping class groups, one can always find faithful representations of right-angled Artin groups into mapping class groups which
map loxodromics to pseudo-Anosov mapping classes on the subsurface filled by the support of the loxodromic.

In fact, more is true. Recall that an element \( 1 \neq g \in A(\Gamma) \) is pure if, after cyclic reduction, it cannot be written as a product of two commuting subwords. Put \( X = \{g_1, \ldots, g_k\} \subseteq G \) and denote by \( \text{Comm}(X) \) the commutation graph of \( X \). We evidently get a surjective map

\[
A(\text{Comm}(X)) \to \langle g_1, \ldots, g_k \rangle \leq G,
\]

given by sending the vertex of \( \text{Comm}(X) \) labelled by \( g_i \) to the element \( g_i \) (see [20] for more discussion of both of these definitions). In this subsection we establish the following result:

**Proposition 42.** Given a graph \( \Gamma \), there exists a surface \( S \) and a faithful homomorphism

\[
\phi : A(\Gamma) \to \text{Mod}(S)
\]

such that if \( 1 \neq g \in A(\Gamma) \) is pure then \( \phi(g) \) is pseudo-Anosov on a subsurface of \( S \).

Proposition [42] will then imply the following result, which is the analogue of the primary result of [21] for right-angled Artin groups:

**Theorem 43.** Let \( X = \{g_1, \ldots, g_k\} \) be an irredundant collection of nonidentity, pure elements of \( A(\Gamma) \). Then for all sufficiently large \( N \), we obtain an isomorphism

\[
A(\text{Comm}(X)) \cong \langle g_1^N, \ldots, g_k^N \rangle \leq A(\Gamma).
\]

Here, the list \( X \) is called irredundant if no two elements of \( X \) share a nonzero common power (or equivalently generate a cyclic subgroup of \( A(\Gamma) \)). The following result is due to Clay, Leininger and Mangahas in [11] (compare with [21]):

**Theorem 44.** Let \( X = \{f_1, \ldots, f_k\} \subseteq \text{Mod}(S) \) be an irredundant collection of pseudo-Anosov mapping classes on connected subsurfaces of \( S \). Suppose furthermore that for \( i \neq j \), there are no inclusion relations between \( \text{supp}(f_i) \) and \( \text{supp}(f_j) \), and that no component of \( \partial \text{supp}(f_i) \) coincides with any component of \( \partial \text{supp}(f_j) \) whenever \( \text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset \). Put \( \Gamma = \text{Comm}(X) \) and let \( v_i \in V(\Gamma) \) be the vertex labeled by \( f_i \). Then for all sufficiently large \( N \), the map

\[
\phi : A(\text{Comm}(X)) \to \langle f_1^N, \ldots, f_k^N \rangle \leq \text{Mod}(S)
\]

defined by \( v_i \mapsto f_i^N \) is an isomorphism. Furthermore, if \( 1 \neq w \in A(\Gamma) \) is cyclically reduced, then \( \phi(w) \) is a pseudo-Anosov mapping class on each component of the subsurface \( \text{Fill}(X_0) \subseteq S \) where \( X_0 = \{\text{supp}(f_i) : 1 \leq i \leq k \text{ and } v_i \in \text{supp}(w)\} \).

Here, \( \text{Fill}(X_0) \) denotes the smallest subsurface (up to isotopy) filled by

\[
\bigcup_{S_i \in X_0} S_i.
\]

The subsurface \( \text{Fill}(X_0) \) is given, up to isotopy, by taking the set–theoretic union of the subsurfaces in \( X_0 \) and filling in any nullhomotopic boundary curves.
Proof of Proposition 42. Taking a sufficiently large surface \( S \), we can find a collection of elements \( X \) of \( \text{Mod}(S) \) for which \( \text{Comm}(X) \cong \Gamma \) and which satisfy the hypotheses of Theorem 44 (this fact is proved in [11]. The reader may also consult [20] for another discussion). Let \( 1 \neq w \in A(\Gamma) \) be pure. After cyclically reducing \( w \), we have that \( \text{supp}(w) \subseteq \Gamma \) is not a join. Identifying \( \text{supp}(w) \) with \( X_0 \), the purity of \( w \) translates to the connectedness of \( \text{Fill}(X_0) \). It follows that \( \phi(w) \) is pseudo-Anosov on a connected subsurface of \( S \).

\[ \square \]

Proof of Theorem 43. Let \( X = \{ g_1, \ldots, g_k \} \subseteq A(\Gamma) \) be an irredundant collection of nonidentity pure elements of \( A(\Gamma) \), and let \( \phi : A(\Gamma) \to \text{Mod}(S) \) be an injective homomorphism coming from Theorem 44. We have that \( \phi(g_i) \) is pseudo-Anosov mapping class on a connected subsurface of \( S \) for each \( g_i \in X \). One can either check that these partial pseudo-Anosov mapping classes again satisfy the conditions of Theorem 44, or one can appeal to the primary result of [21] to see that for all sufficiently large \( N \), we obtain an isomorphism

\[ A(\text{Comm}(X)) \cong \langle g_1^N, \ldots, g_k^N \rangle \leq A(\Gamma), \]

as claimed.

\[ \square \]

9. North–south dynamics on \( \partial \Gamma^e \)

The following is a standard result about pseudo-Anosov mapping classes:

**Proposition 45.** Let \( \{ \psi_1, \ldots, \psi_k \} \) be pseudo-Anosov elements of \( \text{Mod}(S) \) with the property that for \( i \neq j \), we have that \( \psi_i \) and \( \psi_j \) share no common power. Then for all sufficiently large \( N \), we have

\[ \langle \psi_1^N, \ldots, \psi_k^N \rangle \cong F_k. \]

**Sketch of proof.** One approach is as follows: we have that for \( i \neq j \), the mapping classes \( \psi_i \) and \( \psi_j \) stabilize pairs of measured laminations \( \mathcal{L}_i^+ \) and \( \mathcal{L}_j^+ \). Furthermore, neither of \( \mathcal{L}_i^+ \) coincides with either of \( \mathcal{L}_j^+ \). The action of a pseudo-Anosov \( \psi_i \) on \( \mathbb{P}\mathcal{ML}(S) \) is by north–south dynamics, with a source at \( \mathcal{L}_i^- \) and a sink at \( \mathcal{L}_i^+ \).

The boundary of \( \mathcal{C}(S) \) is identified with the space of ending laminations \( \mathcal{E}\mathcal{L}(S) \subseteq \mathbb{P}\mathcal{ML}(S) \), and the north–south dynamics restricts to \( \mathcal{E}\mathcal{L}(S) \). A straightforward ping-pong argument shows that sufficiently high powers of the mapping classes in question generate the expected free group.

\[ \square \]

One does not need to consider the boundary of \( \mathcal{C}(S) \) to prove the previous proposition. The north–south dynamics on \( \mathbb{P}\mathcal{ML}(S) \) suffices. The reason we mention the boundary of \( \mathcal{C}(S) \) is because we will be using the boundary of \( \Gamma^e \) to prove the following analogue:

**Theorem 46.** Let \( \lambda_1, \ldots, \lambda_k \) be loxodromic elements of \( A(\Gamma) \) with the property that for \( i \neq j \), we have that \( \lambda_i \) and \( \lambda_j \) share no common power. Then for all sufficiently large \( N \), we have

\[ \langle \lambda_1^N, \ldots, \lambda_k^N \rangle \cong F_k. \]
The proof of Theorem 46 is identical to that of Proposition 45. The only point which needs to be checked is that a loxodromic \( \lambda \in A(\Gamma) \) truly acts on \( \partial \Gamma^e \) by north–south dynamics. Observe that Theorem 46 follows immediately from Theorem 43. It is not the result we wish to analogize, but rather the method of proof:

**Lemma 47.** Let \( \lambda \in A(\Gamma) \) be a loxodromic element. Then there exists a unique pair of points \( p^\pm \subseteq \partial \Gamma^e \) such that for any compact subset \( K \subseteq \partial \Gamma^e \setminus \{p^+\} \) and for any open subset \( U \subseteq \partial \Gamma^e \) containing \( p^+ \), there exists an \( N \) such that \( \lambda^N(K) \subseteq U \). Furthermore, if \( \lambda' \in A(\Gamma) \) is another loxodromic which shares no common power with \( \lambda \), then no fixed boundary point of \( \lambda \) coincides with any fixed boundary point of \( \lambda' \).

We will first gather the necessary facts and then prove Lemma 47. Theorem 46 will follow immediately.

**Lemma 48.** Let \( \lambda \in A(\Gamma) \) be loxodromic. Then the centralizer of \( \lambda \) is cyclic.

**Proof.** This follows from the Centralizer Theorem (see [27] or [2]) since after cyclic reduction, the support of \( \lambda \) is not contained in a subjoin of \( \Gamma \).

It follows that two loxodromics commute if and only if they share a common power. We say that two sequences of vertices \( \{v_n\}_{n \geq 1} \) and \( \{w_n\}_{n \geq 1} \) in \( \Gamma^e \) diverge if for all \( M \neq 0 \) there is an \( N \) such that for all \( m \geq N \), we have
\[
d_{\Gamma^e}(v_m, \{v_n\}) \geq M \quad \text{and} \quad d_{\Gamma^e}(w_m, \{v_n\}) \geq M.
\]

Following [9], let us write \( a \sim_M b \) to mean \( |a - b| \leq M \).

**Lemma 49.** Let \( g, h \in A(\Gamma) \) be any nontrivial elements and let \( v \) be a vertex of \( \Gamma^e \). Suppose that the sequences \( \{v^{g^n}\}_{n \geq 1} \) and \( \{v^{h^n}\}_{n \geq 1} \) do not diverge. Then the diameter of the set \( \{v^{g^n h^{-n}} : n \geq 1\} \) is finite.

**Proof.** Let \( M \) be a constant which witnesses the failure of divergence of \( \{v^{g^n}\} \) and \( \{v^{h^n}\} \), and \( L = \text{diam}(\Gamma)(\|h\| + 1) \). By Lemma 19 and the triangle inequality, for every \( n \) we have
\[
M \geq d(v^{g^n}, v^{h^n}) = L d(v^{g^n}, v^{h^n}) = M d(v^{g^n}, v^{g^n h}) = d(v, v^{g^n h^{-n}}).
\]

This establishes the claim.

Observe that the proof of Lemma 49 is more or less the same as the standard proof that if \( g \) and \( h \) are elements of a finitely generated group \( G \) and if the sets \( \{g^n\} \) and \( \{h^n\} \) in Cayley(\( G \)) fail to diverge, then there exists a nonzero \( N \) such that \( [g^N, h] = 1 \) (see [9, p.467]). The complicating factor in the case of the extension graph is that \( \Gamma^e \) is not locally finite, so that bounded diameter subsets need not be finite. This problem is circumvented by the following fact:

**Lemma 50.** Let \( \lambda \in A(\Gamma) \) be a loxodromic element and \( g \) be an element of \( A(\Gamma) \) which is not contained in the centralizer of \( \lambda \). Then as \( n \) tends to infinity, we have that
\[
\|\lambda^n g \lambda^{-n}\| \to \infty.
\]
If $\lambda'$ is a loxodromic element of $A(\Gamma)$ which does not share a common nonzero power with $\lambda$ then $\lambda'$ is not contained in the centralizer of $\lambda$ and thus satisfies the hypotheses on $g$ in Lemma 50.

Proof. Suppose for every $n \geq 0$ we have $\|\lambda^n g \lambda^{-n}\| < t < \infty$. Recall our notation $B_t = \{h \in A(\Gamma): \|h\| < t\}$. Choose $M > 0$ such that $\|\lambda^M\| = \|\lambda^{-M}\| > 2t + 2$. For every $n \geq 0$ we see that $\lambda^n g \lambda^{-n} \in B_t \cap \lambda^{-M} B_t \lambda^M$. Lemma 29 implies that $\lambda^n g \lambda^{-n} = \lambda^m g \lambda^{-m}$ for some $m > n > 0$. This contradicts to the fact that $[\lambda, g] \neq 1$. \hfill $\square$

Lemma 51. Let $\lambda \in A(\Gamma)$ be loxodromic. Then there exists a unique pair $p^\pm \subseteq \partial \Gamma^e$ with respect to which $\lambda$ acts by north–south dynamics.

Proof. Let $v \in \Gamma^e$ be an arbitrary vertex. Since $\lambda$ is loxodromic, we have that $d(v, v^{\lambda^n})$ tends to infinity. Since $\Gamma^e$ is a quasi–tree, it follows that $v^{\lambda^n}$ converges to a point $p^+ \in \partial \Gamma^e$. Observe that $A(\Gamma)$ acts on $\Gamma^e$ by isometries, even though $p^+$ generally depends on the choice of $v$. We define $p^-$ to be $p^+$ for $\lambda^{-1}$.

Let $\gamma$ be a quasi–geodesic ray emanating from $v$ whose endpoint $p'$ is different from $p^\pm$. Then hyperbolicity implies that for all sufficiently large $N$, the quasi–geodesic $\gamma_N$ which travels from $v$ to $v^{\lambda^N}$ and then follows the corresponding translate of $\gamma$, the endpoint of $\gamma_N$ is strictly closer to $p^+$ than $p'$.

We are now ready to prove Lemma 47.

Proof of Lemma 47. Let $\lambda_1$ and $\lambda_2$ be non–commuting loxodromics. We have already shown that for each $i$, the loxodromics $\lambda_i$ act on $\partial \Gamma^e$ by north–south dynamics. It suffices to show that none of the pairs of endpoints $p_i^+$ and $p_2^+$ coincide.

Without loss of generality, suppose $p = p_1^+ = p_2^+$. Then for any vertex $v \in \Gamma^e$, the quasi–geodesics $\{v^{\lambda_1^p}\}$ and $\{v^{\lambda_2^p}\}$ must fellow travel and converge to $p$. Combining Lemmas 49 and 50 we see that $\lambda_1$ and $\lambda_2$ must commute, a contradiction. \hfill $\square$

Theorem 46 follows immediately.

10. Purely loxodromic subgroups of $A(\Gamma)$

In [7] (cf. [13]), Bowditch proves that if $G \leq \text{Mod}(S)$ is a finitely presented, one–ended, purely pseudo-Anosov subgroup of the mapping class group $\text{Mod}(S)$, then there are finitely many isomorphic subgroups $\{G_1, \ldots, G_k\}$ of $\text{Mod}(S)$ and an element $\psi \in \text{Mod}(S)$ such that $G_i^\psi = G_j$ for some $1 \leq i \leq k$. In other words, finitely presented, one–ended, purely pseudo-Anosov subgroups of $\text{Mod}(S)$ fall into finitely many conjugacy classes of subgroups in $\text{Mod}(S)$ for each isomorphism type.

Finitely presented, one–ended subgroups of right-angled Artin groups are also quite restricted. In [22], Leininger proves that any finitely presented, one–ended subgroup of a right-angled Artin group contains a nontrivial element whose support, up to conjugacy, is contained in a nontrivial join. In light of our characterization of loxodromic and elliptic elements of right-angled Artin groups, Leininger’s result implies that there are no finitely presented, one ended, purely loxodromic subgroups of right-angled Artin groups. By analogy to purely pseudo-Anosov subgroups of $\text{Mod}(S)$, a purely loxodromic subgroup of $A(\Gamma)$ is a subgroup $G$ for which every $1 \neq g \in G$ is loxodromic in $A(\Gamma)$.
We will prove the following result, which completely characterizes purely loxodromic subgroups of right-angled Artin groups:

**Theorem 52.** A purely loxodromic subgroup of a right-angled Artin group is free.

Observe that we put no further hypotheses on the purely loxodromic subgroup. In particular, we do not assume that it is finitely generated.

**Proof of Theorem 52.** Let \( G \) be a purely loxodromic subgroup of \( A(\Gamma) \). The proof proceeds by induction on the number of vertices of \( \Gamma \). The crucial observation is that if \( v \in V(\Gamma) \) and if \( g \in A(\Gamma) \) then the assumption that \( G \) is purely loxodromic implies

\[
G \cap \langle \text{St}(v) \rangle^g = \langle 1 \rangle,
\]

since every cyclically reduced element of \( A(\Gamma) \) whose support is contained in a nontrivial join is elliptic.

The base case of the induction is where \( \Gamma = v \). In this case, \( A(\Gamma) \) is purely elliptic, in which case the result is vacuous. For the inductive step, let \( v \in V(\Gamma) \). Observe that we have the following free product with amalgamation description of \( A(\Gamma) \):

\[
A(\Gamma) \cong \langle \text{St}(v) \rangle \ast_{\langle \text{Lk}(v) \rangle} A(\Gamma_v),
\]

where \( \Gamma_v \) is the graph spanned by \( V(\Gamma) \setminus \{v\} \). By standard Bass–Serre Theory, we see that \( G \) acts on the corresponding Bass–Serre tree with trivial edge stabilizer, since \( G \) is purely loxodromic. In particular, there exists a (possibly infinite) collection of subgroups \( \{H_i\} \) with each \( H_i \) conjugate to \( A(\Gamma_v) \) in \( A(\Gamma) \) and \( 0 \leq r \leq \infty \) such that

\[
G \leq \ast_i H_i * F_r.
\]

Observe that if \( g \in A(\Gamma_v) \leq A(\Gamma) \) is loxodromic in \( A(\Gamma) \) then \( g \) is loxodromic as an element of \( A(\Gamma_v) \). Indeed, if \( g \) is cyclically reduced then \( \text{supp}(g) \subseteq \Gamma_v \) is not contained in a subjoin of \( \Gamma \), whence it is not contained in a subjoin of \( \Gamma_v \). Since \( H_i \) is conjugate to \( A(\Gamma_v) \) and \( \Gamma_v \) has fewer vertices than \( \Gamma \), we see that \( G \) is free by induction. \( \square \)

11. **Subsurface and vertex link projections**

In this section, we will give the necessary setup to develop the Bounded Geodesic Image Theorem and the distance formula for right-angled Artin groups.

11.1. **Distances in disconnected extension graphs.** When \( \Gamma \) is a connected graph, we have that the extension graph \( \Gamma^e \) is also connected. Equipping \( \Gamma^e \) with the graph metric, it is straightforward to discuss the distance between each pair of points. If \( \Gamma^e \) is disconnected, we cannot talk about paths between arbitrary pairs of vertices, but we would still like to discuss distances between vertices in those graphs.

Our approach is motivated by the curve graph of a once–punctured torus or of a four–punctured sphere. In both of these cases, there are simple closed curves, but there are no pairs of disjoint curves. Thus, the usual definition of the curve graph gives us a graph with infinitely many vertices and no edges.

The definition of the curve graph for these two surfaces is modified in such a way that two curves are adjacent whenever they intersect minimally. In the case of a once–punctured torus, pairs of curves which intersect exactly once are adjacent. In the
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case of the four-punctured sphere, two curves are adjacent when they intersect exactly twice.

From an algebraic point of view, recall Mod(S1,1) = ⟨a, b|aba = bab⟩. The simple closed curves on S1,1 are represented as a^g or b^g where g, h ∈ Mod(S1,1). Two vertices a^g and b^h in C(S1,1) are adjacent if a^g = a^w and b^h = b^w for some w ∈ A(Γ). This is equivalent to the condition a^g b^h a^g = b^h a^g b^h. It turns out that the curve complex in this case is the Farey graph.

Now let us consider a finite discrete graph Γ and F be the free group generated by V(Γ). We will define Γe as a graph with the vertex set {v^g : v ∈ V(Γ) and g ∈ F} such that two vertices u^g and v^h are adjacent if u^g = u^w and v^h = v^w for some w ∈ F. Note that there is a simply transitive simplicial action of F on Γe such that all the edges are in the orbit of an edge joining two vertices of Γ; see Figure 4 for the case Γ has two vertices.

Figure 4. A part of Γe.

Recall that the covering distance d' as in Section 3 is a well-defined metric on Γe even when Γ is not connected; see also Section 5.2. In the case where Γ is connected we have that for all vertices v and elements g ∈ A(Γ), the distance d(v, v^g) is coarsely given by the star length ∥g∥,* as shown in Lemma 19. If Γ is a finite discrete graph, then the distance on Γe is precisely the covering distance. Note that the star length and the syllable length coincide for words in free groups. Hence, the following is obvious from Lemma 19(1).

Lemma 53. Let Γ be a finite discrete graph. Suppose g ∈ A(Γ) and a, b ∈ V(Γe) such that a ≠ b^g. Then

\[ \max(1, \|g\|_{\text{syll}} - 1) \leq d_{Γe}(a, b^g) \leq \|g\|_{\text{syll}} + 1. \]

11.2. Subsurface and vertex link projection. We are now in a position to define subsurface projections for right-angled Artin groups. Since the vertices of Γe do not correspond to curves on a surface in any canonical way, there is some difficulty in defining a subsurface projection in the right-angled Artin group case.

Let γ ∈ C(S) be a vertex. The set of essential, nonperipheral simple closed curves on the subsurface S\γ forms a curve complex in its own right, and it is canonically identified with the subcomplex Lk(γ) of C(S). If γ' ∈ C(S) is another vertex, then γ' cannot generally be viewed as a curve on S\γ. One way to take γ' and produce a simple closed curve on S\γ which is “most like” γ' is as follows. Write G(γ, γ') for the set of geodesic paths between γ and γ' in C(S). Define the subsurface projection π_γ(γ') of γ' to S\γ by setting

\[ \pi_γ(γ') = \bigcup_{δ \in G(γ, γ')} δ \cap \text{Lk}(γ), \]
following Masur and Minsky (see [23, 24]). Thus, \( \pi_\gamma(\gamma') \) is the set of points of \( \text{Lk}(\gamma) \) which some geodesic \( \delta \) meets on its way from \( \gamma' \) to \( \gamma \). Of course there is an issue of how complicated such a set might be, which is the point of Bounded Geodesic Image Theorem.

We will define subsurface projection in right-angled Artin groups in a way which is analogous to curve complex case. In the curve complex case we have that for any vertex \( \gamma \), the subcomplex \( \text{Lk}(\gamma) \) is itself a curve complex. An analogous fact holds for vertices in the extension graph:

**Lemma 54.** Let \( v \in \Gamma \) be a vertex. Then there is a canonical identification

\[
\text{Lk}_{\Gamma^e}(v) = (\text{Lk}_\Gamma(v))^e.
\]

Here we choose \( v \in \Gamma \), since then we can discuss \( \text{Lk}_\Gamma(v) \). If \( v \) is an arbitrary vertex of \( \Gamma^e \) then \( v \) is conjugate to a unique vertex of \( \Gamma \). Note our convention that \( \text{Lk}_X(v) \) also denotes the subgraph of \( X \) induced by the vertex set \( \text{Lk}_X(v) \).

**Proof of Lemma 54.** The inclusion

\[
\text{Lk}_{\Gamma^e}(v) \supseteq (\text{Lk}_\Gamma(v))^e
\]

is clear. Suppose that \( v \) is adjacent to a vertex \( w \) in \( \Gamma^e \). By the definition of edges in \( \Gamma^e \), we have that there is an element \( g \in A(\Gamma) \) such that \( (v, w)^g \) is an edge in \( \Gamma \). Since conjugation by \( g \) fixes \( v \), we have that \( \text{supp}(g) \subseteq \text{St}(v) \). It follows that we can write

\[
g = u \cdot v^n,
\]

where \( \text{supp}(u) \subseteq \text{Lk}(v) \). Finally, conjugation by \( v \) fixes the star of \( v \). It follows that \( w \) can be written as a conjugate of a vertex in \( \text{Lk}_\Gamma(v) \) by an element of \( A(\Gamma) \) whose support is contained in \( \text{Lk}_\Gamma(v) \). Thus we get the reverse inclusion

\[
\text{Lk}_{\Gamma^e}(v) \subseteq (\text{Lk}_\Gamma(v))^e.
\]

To see that the two links are canonically identified, take \( v' \in \text{Lk}_\Gamma(v) \) and \( g = u \cdot v^n \in \langle \text{St}(v) \rangle \). We identify \( (v')^g \in \text{Lk}_{\Gamma^e}(v) \) with \( (v')^u \in (\text{Lk}_\Gamma(v))^e \). By the previous paragraph, this identification is a well-defined bijection. \( \square \)

Now let \( v, v' \in \Gamma^e \) be vertices. Write \( G(v, v') \) for the set of geodesic paths in \( \Gamma^e \) which connect \( v \) and \( v' \). We define the vertex link projection of \( v' \) to \( v \) by

\[
\pi_v(v') = \bigcup_{\delta \in G(v, v')} \delta \cap \text{Lk}(v).
\]

If \( X \subseteq \Gamma^e \) is an arbitrary set of vertices, we can also define its vertex link projection to \( v \) by

\[
\pi_v(X) = \bigcup_{v' \in X} \bigcup_{\delta \in G(v, v')} \delta \cap \text{Lk}(v).
\]
12. THE BOUNDED GEOIDESIC IMAGE THEOREM

In this section and for the rest of this paper unless otherwise noted, we will be assuming that the graph \( \Gamma \) has no triangles and no squares and is connected. This is equivalent to the requirement that \( A(\Gamma) \) contains no copies of \( \mathbb{Z}^3 \) or \( F_2 \times F_2 \) and that \( A(\Gamma) \) is freely indecomposable. We will say that a subset \( X \subseteq V(\Gamma^e) \) is connected if the subgraph of \( \Gamma^e \) spanned by \( X \) is connected. In this section, we will prove the following result:

**Theorem 55** (Bounded Geodesic Image Theorem for right-angled Artin groups). Let \( \Gamma \) be a finite connected triangle– and square–free graph.

1. There exists a constant \( M \) which depends only on \( \Gamma \) such that if \( v, v' \in \Gamma^e \) are vertices then
   \[
   \text{diam}_{Lk(v)} \pi(v') \leq M.
   \]

2. Let \( v \in \Gamma^e \) and let \( Y \subseteq \Gamma^e \) be a connected subset contained outside of a closed ball of radius two about \( v \). Then
   \[
   \text{diam}_{Lk(v)} \pi(Y) \leq M.
   \]

Furthermore, there exists a constant \( M' \) which depends only on \( \Gamma \) such that if \( v \in \Gamma^e \) and if \( X \subseteq \Gamma^e \) is a geodesic segment, ray, or bi–infinite line contained in \( \Gamma^e \setminus \text{St}(v) \), then
\[
\text{diam}_{Lk(v)} \pi(v)(X) \leq M'.
\]

In the statement of Theorem 55, the notation \( Lk(v) \) means \( Lk_{\Gamma^e}(v) \). Theorem 55 should be compared to the Bounded Geodesic Image Theorem of Masur and Minsky:

**Theorem 56** (Bounded Geodesic Image Theorem for Surfaces, [24]). There exists a constant \( M \) which depends only on \( S \) such that if \( Y \subseteq S \) is a proper, incompressible surface which is not a pair of pants and if \( \gamma \) is any geodesic segment, ray, or bi–infinite line for which \( \pi_Y(v) \neq \emptyset \) for each vertex \( v \) of \( \gamma \), then
\[
\text{diam}_{C(Y)} \pi_Y(\gamma) \leq M.
\]

In Theorem 55 (2), the additional assumption that \( Y \) misses a 2–ball about \( v \) is essential. For example, let \( \Gamma = C_5 \), the pentagon. Clearly, \( C_5 \) is triangle– and square–free. Fix a vertex \( v \in C_5 \). Label the vertices in \( Lk_{C_5}(v) \) by \( x \) and \( y \), and write \( a \) and \( b \) for the remaining vertices, with \( a \) adjacent to \( x \). Write \( E \subseteq C_5^e \) be the graph whose vertices are given by
\[
\{ u^g \mid u \in V(C_5) \text{ and } g \in \langle x, y \rangle \},
\]
and whose edges are inherited from \( C_5^e \). As in the proof of Lemma 54, \( Lk_{C_5}(v) \subseteq E \).

One can show that \( D = E \setminus v \) is connected. In fact, \( D \) is the graph \( Y_v \) which we will construct in a more general context in Subsection 12.2. In fact, the vertices of \( D \) which correspond to the conjugates of \( x \) and \( y \) are precisely the vertices of degree one of \( D \), and all the others have infinite degree. It follows that if we set \( Y \) to be the subgraph of \( D \) spanned by the vertices of infinite degree, we will obtain a connected subgraph of \( C_5 \). Furthermore, by Lemma 54 we have that \( Y \cap Lk_{C_5}(v) = \emptyset \). Finally, consider the
map \( \pi_v \) restricted to \( Y \). The distance from any vertex of \( Y \) to \( v \) is exactly two. Since \( C_5^e \) is square–free, there is a unique geodesic connecting a vertex of \( Y \) to \( v \). Thus, the map \[ \pi_v : Y \to \text{Lk}_{C_5^e}(v) \] is surjective. It follows that \( \text{diam}_{\text{Lk}(v)} \pi_v(Y) \) is not finite.

12.1. The Bounded Geodesic Image Theorem when \( \Gamma \) is a tree. The situation at hand is significantly simpler when the graph \( \Gamma \) is a tree. Recall that in this case, the extension graph \( \Gamma^e \) is also a tree. Geodesics in trees are unique, so that if \( v' \in \Gamma^e \), the set \( G(v, v') \) consists of exactly one path. Thus, the diameter of the vertex link projection of \( v' \) to \( v \) is zero.

Now suppose that \( X \) is a connected subset of \( \Gamma^e \) which avoids \( \text{St}(v) \). Choose a geodesic path \( \gamma \) from \( v \) to a point in \( X \), and let \( v_0 \) be the unique point in \( \gamma \cap \text{St}(v) \). The connectedness of \( X \) and the 0–hyperbolicity of trees implies that any geodesic path from \( v \) to a point in \( X \) must also pass through \( v_0 \). Thus, \( \pi_v(X) = v_0 \).

The requirement that \( X \cap \text{St}(v) = \emptyset \) is clearly essential, even if \( X \) is assumed to just be a geodesic segment. A geodesic segment which enters \( \text{St}(v) \) can be extended to \( v \) and then extended past \( v \) arbitrarily. Since \( \text{Lk}(v) \) has infinite diameter, no bound on the projection of \( X \) to \( \text{Lk}(v) \) can possibly hold.

12.2. A connected model for the link of \( v \). Let us fix a vertex \( v \) of \( \Gamma \). A difficulty in understanding the geometry of \( \text{Lk}(v) \subseteq \Gamma^e \) is that \( \text{Lk}(v) \) is completely disconnected whenever \( \Gamma \) has no triangles. In this subsection, we will build a connected graph \( Y_v \) which is canonically obtained from \( \Gamma \setminus v \) and which is quasi–isometric to \( \text{Lk}(v) \).

We first modify \( \Gamma \) so that every vertex \( x \) in \( \text{Lk}_\Gamma(v) \) has degree larger than one in \( \Gamma \). If \( \text{deg}(x) = 1 \), then we simply add an extra vertex \( x' \) to \( \Gamma \) such that the link of \( x' \) is \( \{x\} \). For \( x \in \text{Lk}_\Gamma(v) \) we let \( B_x \) be the set of vertices \[ \{u \in V(\Gamma \setminus v) \mid d(u, x) = 1\} = \text{Lk}_{\Gamma \setminus v}(x). \]

The set \( B_x \) can be thought of as the “boundary” of \( x \). Note that \( B_x \) is a nonempty, completely disconnected graph.

For each pair of distinct vertices \( x, y \in \text{Lk}_\Gamma(v) \) we see that \( x \) and \( y \) are not adjacent in \( \Gamma \) and also that \( x \) and \( y \) have no common neighbors. In particular, we have \( d_{\Gamma \setminus v}(x, y) > 2 \). If \( x \neq y \) are vertices in \( \text{Lk}_\Gamma(v) \), we have that \( B_x \cap B_y = \emptyset \) and that \( B_x \cap \text{Lk}_\Gamma(v) = \emptyset \).

Let \( Z_v \) be the graph obtained from \( \Gamma \setminus v \) by declaring that whenever \( z \in B_x \) and \( w \in B_y \) for a pair of distinct vertices \( x, y \in \text{Lk}_\Gamma(v) \) we have an edge \( \{z, w\} \) in \( Z_v \). Note that \( Z_v \) contains a canonical copy of \( \text{Lk}_\Gamma(v) \), since we did not add any edges between vertices of \( \text{Lk}_\Gamma(v) \). It follows that \( \langle \text{Lk}_\Gamma(v) \rangle \) can be identified with a subgroup of \( A(Z_v) \). See Figure 5(a) for an illustration of how \( Z_v \) might look.

The graph \( Y_v \) is defined to be a subgraph of \( Z_v^5 \) in which we only allow conjugation by elements of \( \langle \text{Lk}_\Gamma(v) \rangle \). In other words, \( Y_v \) is the commutation graph of \[ \{x^g \mid x \in V(Z_v) \text{ and } g \in \langle \text{Lk}_\Gamma(v) \rangle \} \subseteq A(Z_v). \]

A small part of \( Y_v \) is given in Figure 5(b).

Lemma 57. Let \( Y_v \) be as above.
(1) The graph $Y_v$ is connected.
(2) The graph $Y_v$ contains a canonical copy of $\text{Lk}_{\Gamma}(v)^e = \text{Lk}_{\Gamma^e}(v)$.
(3) The inclusion of $\text{Lk}_{\Gamma}(v)^e \to Y_v$ is a quasi–isometry, where $\text{Lk}_{\Gamma}(v)^e$ is equipped with the extension graph metric and where $Y_v$ is equipped with the graph metric.

Proof. First, let us check that $Z_v$ is connected. Suppose first that $\text{Lk}_{\Gamma}(v)$ consists of exactly one vertex. Then $v$ has degree one in $\Gamma$. It follows that $\Gamma\backslash v$ is connected and hence $Z_v$ is connected. If there is a pair of distinct vertices $x, y \in \text{Lk}(v)$, then $Z_v$ contains the join $B_x \ast B_y$. Furthermore since $\Gamma$ is connected, for each vertex $v' \in \Gamma \backslash v$ there is a path in $\Gamma \backslash v$ connecting $v'$ to $\text{St}(v)$. It follows that $Z_v$ is connected, since the image of the vertices

$$\bigcup_{x \in \text{Lk}_{\Gamma}(v)} B_x$$

in $Z_v$ spans a connected subgraph.

To build $Y_v$, we take one copy of $Z_v$ for each element of $g \in \langle \text{Lk}_{\Gamma}(v) \rangle \leq A(Z_v)$ and label the vertices of $Z_v^g$ by the appropriate conjugate in $A(Z_v)$. Then, we identify two vertices in two different copies of $Z_v$ if they have the same label. One can therefore build $Y_v$ by repeatedly doubling $Z_v$ over the stars of vertices in $\text{Lk}_{\Gamma}(v) \subseteq Z_v$, as in the construction of $\Gamma^e$ by repeated doubling along stars. It follows that $Y_v$ is connected.

The copy of $\text{Lk}_{\Gamma^e}(v)$ in $Y_v$ is given by

$$\{x^g \mid x \in \text{Lk}_{\Gamma}(v) \text{ and } g \in \langle \text{Lk}_{\Gamma}(v) \rangle \leq A(Z_v)\}.$$ 

To see that this inclusion is a quasi–isometry, we perform a further construction. Let $c_v$ be the graph obtained from $Z_v$ by collapsing $B_x$ to a single vertex $b_x$ for each $x \in \text{Lk}_{\Gamma}(v)$ and by removing any other vertices not contained in $\text{Lk}_{\Gamma}(v)$ nor in

$$\bigcup_{x \in \text{Lk}_{\Gamma}(v)} B_x.$$

Let $C_v \subseteq c_v^e$ be the subgraph spanned by the vertices

$$\{x^g \mid x \in V(c_v) \text{ and } g \in \langle \text{Lk}_{\Gamma}(v) \rangle \leq A(c_v)\}.$$
It is easy to see that $C_v$ is connected and quasi–isometric to $Y_v$, and that $C_v$ also contains a canonical copy of $\text{Lk}_{\Gamma_v}(v)$.

Observe that $C_v$ is a quasi–tree, and that each conjugate of a vertex $b_x$ is a cut point. Furthermore, the vertex $x$ generates the entire centralizer of $b_x$ inside of $\langle \text{Lk}_\Gamma(v) \rangle$. It follows that for each $g \in \langle \text{Lk}_\Gamma(v) \rangle$, we have $d_{C_v}(x, x^g) \geq \|g\|_* - 1$, where the star length is measured in the free group $\langle \text{Lk}_\Gamma(v) \rangle$.

Now write $D$ for the diameter of $c_v$. If $x \in \text{Lk}_\Gamma(v)$ and $g \in \langle \text{Lk}_\Gamma(v) \rangle$ then $d_{C_v}(x, x^g) \leq D\|g\|_* + D$ by the proof of Lemma $19$. It follows that the graph distance on $\text{Lk}_{\Gamma^c}(v) \subseteq C_v$ is quasi–isometric to the star length in $\langle \text{Lk}_\Gamma(v) \rangle$, which is what we set out to prove. □

There is a further slight complication resulting from the fact that conjugation by $v$ is generally nontrivial in $A(\Gamma)$. We may wish to add $v$ back to $Z_v$ and consider the subgraph $Y'_v$ of $Z'_v$ obtained by including all the conjugates of $Z_v$ by elements of $\langle \text{St}_\Gamma(v) \rangle$ instead of just $\langle \text{Lk}_\Gamma(v) \rangle$, and then removing the vertex $v$. Up to quasi–isometry, the resulting graph is just $Y_v$. Indeed, we have that conjugation by $v$ stabilizes $\text{Lk}_\Gamma(v)$, and in general we obtain a splitting

$$\langle \text{St}_\Gamma(v) \rangle \cong \mathbb{Z} \times \langle \text{Lk}_\Gamma(v) \rangle.$$ 

We can view the difference between $Y'_v$ and $Y_v$ thusly: for each conjugate of $Z_v$ used to obtain $Y'_v$, replace it with a $\mathbb{Z}$–worth of copies of $Z_v$, which we then glue together along the corresponding copies of $\text{Lk}_\Gamma(v)$ sitting inside of each conjugate, like an infinite stack of pancakes identified along their boundaries. It is clear that the embedding $Y_v \to Y'_v$ is a quasi–isometry.

12.3. **Vertex link projection is coarsely well–defined.** In this subsection, we will prove the first part of Theorem $55$ which implies that vertex link projection is coarsely well–defined. We will need the following lemma, the proof of which can be found in [20, Lemma 26]:

**Lemma 58.** Let $$\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$$ be a filtration of graphs such that $\Gamma_{i+1}$ is obtained from $\Gamma_i$ by attaching a copy of $\Gamma$ along the star of a vertex of $v \in \Gamma$. Suppose $x, x', z \in \Gamma_i$ are vertices such that $x$ and $x'$ lie in different components of $\Gamma_i \setminus \text{St}_{\Gamma_i}(z)$. Then the vertices $x$ and $x'$ lie in different components of $\Gamma^e \setminus \text{St}_{\Gamma^e}(z)$.

**Proof of Theorem 55 (1).** If $\Gamma$ splits as a join then the triangle– and square–freeness assumptions on $\Gamma$ imply that $\Gamma$ is a tree, so that $\Gamma^e$ is also a tree and the Bounded Geodesic Image Theorem holds trivially. So we assume that $\Gamma$ does not split as a join and $\Gamma^e$ has infinite diameter. We will also assume that $\text{Lk}_{\Gamma}(v)$ contains at least two vertices, since otherwise the conclusion is trivial.

If $d(v, v') = 1$ then $v' \in \text{Lk}_{\Gamma^e}(v)$, and if $d(v, v') = 2$ then $v' \in B_x$ for some $x \in \text{Lk}_{\Gamma^e}(v)$. By the triangle– and square–freeness of $\Gamma$, in those two cases there is a unique geodesic connecting $v$ to $v'$, so that the vertex link projection has diameter zero.

Now we assume that $d(v, v') \geq 3$. Let $\delta, \delta' \in G(v, v')$ be two distinct geodesics connecting $v$ and $v'$. We will treat $\delta$ as a reference geodesic segment between $v$ and $v'$,
and we will show that if $\delta' \cap \text{Lk}(v)$ is “too far” from $\delta \cap \text{Lk}(v)$ then $\delta'$ could not have been a geodesic segment.

Write $x$ and $x'$ for the points of $\text{Lk}(v)$ which are met by $\delta$ and $\delta'$ respectively, and consider the vertices $x$ and $x'$ as vertices in $Y_v$. Since $Y_v$ is quasi–isometric to $\text{Lk}_{\Gamma^e}(v)$, the distance between $x$ and $x'$ is comparable in $\text{Lk}_{\Gamma^e}(v)$ and in $Y_v$. Observe that if $d_{Y_v}(x, x') \geq 3 \text{diam}(Z_v)$ then there is a vertex $z \in \text{Lk}(v)$ such that $\text{St}_{Y_v}(z)$ contains neither $x$ nor $x'$ and such that $\text{St}_{Y_v}(z)$ separates $x$ from $x'$; see the proof of [20, Lemma 26]. Figure 6 gives an illustration of the fact that such a vertex $z$ exists. In this figure, the star of $x$ separates $x^y$ from $x^{y'}$. The boundaries of vertices in $\text{Lk}_{\Gamma^e}(v)$ have been collapsed to single (unlabelled) vertices. Actually, if $d_{Y_v}(x, x') \geq M \text{diam}(Z_v)$ then there are at least $M/3$ such $z$’s and this fact will be used in the proof of the second part of Theorem 55.

![Figure 6](image-url)

**Figure 6.** Distant vertices in $Y_v$ are separated by stars of intermediate vertices.

We claim that $\text{St}_{\Gamma^e}(z)$ separates $x$ from $x'$ in $\Gamma^e$, and that this suffices to prove the result. For the latter claim, suppose that the star of $z$ separates $x$ from $x'$ in $\Gamma^e$. The following is clear from triangle– and square–freeness:

$$\delta \cap \text{St}_{\Gamma^e}(w_0) = \delta' \cap \text{St}_{\Gamma^e}(w_0) = \emptyset.$$  

But then the subpath of $\delta \cdot \delta'$ from $x$ to $x'$ is contained in $\Gamma^e \setminus \text{St}(z)$ and we have a contradiction.

So, it suffices to see that $\text{St}_{\Gamma^e}(z)$ separates $x$ from $x'$ in $\Gamma^e$. Let $L_v$ be the subset of $\Gamma^e$ given by doubling over stars of vertices which lie in $\text{St}(v)$. Then $L_v$ can be filtered by finite subgraphs in a way which satisfies the hypotheses of Lemma 58. So, it suffices to see that $x$ and $x'$ are separated by $\text{St}_{L_v}(z)$. Observe that there is a natural map of graphs $L_v \setminus v \rightarrow Y_v'$. The star of $z$ in $Y_v'$ separates $x$ from $x'$, whence it follows that $\text{St}_{L_v}(z)$ separates $x$ from $x'$.

12.4. The Bounded Geodesic Image Theorem. We now give the proof of the second part of the Bounded Geodesic Image Theorem (Theorem 55).

**Proof of Theorem 55 (2).** Since $Y$ remains entirely outside of a 2–ball containing $v$, we have that if $w \in \text{Lk}_{\Gamma^e}(v)$ then $\text{St}(w) \cap Y = \emptyset$. Let $x$ and $y$ be vertices of $Y$, and let $\gamma$ be a path in $Y$ connecting $x$ and $y$. Write $\delta_x$ and $\delta_y$ for geodesics in $\Gamma^e$ connecting $v$ with $x$ and $y$ respectively. Assume that $d_{\text{Lk}_{\Gamma^e}(v)}(\pi_v(x), \pi_v(y)) > M$, where $M$ is the constant given by part (1) of Theorem 55. By the proof of part (1) of Theorem 55, there is a vertex $z \in \text{Lk}_{\Gamma^e}(v)$ whose star separates $w_x = \delta_x \cap \text{Lk}_{\Gamma^e}(v)$ from $w_y = \delta_y \cap \text{Lk}_{\Gamma^e}(v)$.
However, following $\delta_x$ followed by $\gamma$ followed by $\delta_y$ furnishes a path in $\Gamma^e \setminus \text{St}(v)$ between $w_x$ and $w_y$, a contradiction.

Now suppose that $\delta \subseteq \Gamma^e \setminus \text{St}(v)$ is a geodesic. Write $S_2(v)$ for the sphere of radius two about $v$. We may assume that $\delta \cap S_2(v) \neq \emptyset$, since otherwise we are in the case considered in the previous paragraph. Observe that $\text{diam}(S_2(v)) \leq 4$. Since $\delta$ is a geodesic, we have that $|\delta \cap S_2(v)| \leq 5$. Thus, there are at most five vertices in $\text{Lk}_{\Gamma^e}(v)$ whose links meet $\delta$. Write $M' = \max\{M, 18 \text{diam}(Z_v)\}$, where $M$ is the constant provided by part (1) of the theorem. Let $x, y \in \delta$ project to points $w_x$ and $w_y$ in $\text{Lk}_{\Gamma^e}(v)$ which are at a distance exceeding $M'$ from each other. Then there is a vertex $z \in \text{Lk}_{\Gamma^e}(v)$ such that $\text{St}_{\Gamma^e}(z)$ separates $w_x$ from $w_y$. This is again a contradiction, so that the diameter of the projection of $\delta$ is at most $M'$.

13. The distance formula

We have seen already that distance in $\Gamma^e$ is coarsely just star length in the right-angled Artin group $A(\Gamma)$. In this section, we will develop a Masur–Minsky style distance formula, which will incorporate the extra data that vertex link projections give us in order to refine distance estimates in $\Gamma^e$.

The Masur–Minsky distance formula allows one to use distances in the curve complex to coarsely recover distances in the mapping class groups. The high degree of symmetry of the extension graph does not allow such a result to be proved for right-angled Artin groups. Indeed, the automorphism group of the curve complex is, up to finite index, the mapping class group of the underlying surface. A complicating factor in the extension graph case is that powers of vertices are indistinguishable. To see this, consider a vertex $v \in V(\Gamma) = \Gamma_0$. In $\Gamma^e$, there are copies of $\Gamma$ meeting $\Gamma_0$ along the star of $v$ which correspond to conjugates of $\Gamma$ by powers of $v$. Let $v_0$ be another vertex of $\Gamma$ which does not lie in the star of $v$. We will see (Theorem 65) that there are symmetries of $\Gamma^e$ which fix $\Gamma_0$ and which permute the $v$–conjugates of $\Gamma$ arbitrarily. In particular, the extension graph cannot distinguish between the conjugates $\{v_0^k\}_{k \in \mathbb{Z}}$. It thus seems unlikely that the geometry of the extension graph can be used to recover word length in $A(\Gamma)$.

Our generalization of the Masur–Minsky distance formula to the right-angled Artin case will show that the geometry of $\Gamma^e$ can be used to recover distance in $A(\Gamma)$ up to forgetting powers of the vertex generators. In other words, we will be recovering the syllable length of words in $A(\Gamma)$.

13.1. Markings for mapping class groups. The Masur–Minsky distance formula coarsely measures distances in the Cayley graph of the mapping class group $\text{Mod}(S)$, using only the data of the curve graph $C(S)$:

**Theorem 59** ([23]). Let $S$ be a surface. There are constants $M_0, C > 0$ and a constant $K > 1$ such that for each pair $\mu, \nu$ of complete, clean markings on $S$ and for all
For the convenience of the reader, we will briefly summarize the terminology of Theorem 59. The object $\mathcal{M}(S)$ is the marking graph of $S$. A marking $\mu$ is a finite set of ordered pairs $\{(\alpha_i, \beta_i)\}$, where the curves $\{\alpha_i\}$ form a multicurve on $S$ and the $\{\beta_i\}$ are transversals of the corresponding curves $\{\alpha_i\}$, which is to say $\beta_i$ is either empty or is a subset of the annular curve graph $\mathcal{C}(\alpha_i)$ of diameter one. A transversal $\beta$ in the pair $(\alpha, \beta)$ is clean if there is a curve $\gamma \subseteq S$ such that $\beta$ is the projection of $\gamma$ to the annular surface filled by $\alpha$, and such that $\alpha$ and $\gamma$ are Farey–graph neighbors in the curve graph of $\text{Fill}(\gamma \cup \alpha)$ (which is to say they intersect a minimal nonzero number of times). A marking is clean if every transversal is clean and the curve $\gamma_i$ inducing the transversal $\beta_i$ does not intersect any $\alpha_j$ other than $\alpha_i$. The marking is complete if the curves $\{\alpha_i\}$ form a pants decomposition of $S$ and none of the transversals are empty.

If a marking is complete and clean, a transversal $\beta_i$ determines the curve $\gamma_i$ uniquely. Masur and Minsky show that if a marking is not clean, then in some appropriate sense, there is a bounded number of ways to obtain a compatible clean marking.

The marking graph $\mathcal{M}(S)$ is the graph whose vertices are complete, clean markings on $S$. Two markings are connected by an edge if they differ by one of two elementary moves, called twists and flips. To define these moves, let $\mu = \{(\alpha_i, \pi_{\alpha_i}(\gamma_i))\}$ and $\nu = \{(\alpha'_i, \pi_{\alpha'_i}(\gamma'_i))\}$ be two markings. We say that $\mu$ and $\nu$ differ by a twist if for some $\{(\alpha_j, \pi_{\alpha_j}(\gamma_j)) = (\alpha'_j, \pi_{\alpha'_j}(\gamma'_j))\}$ for all $j \neq i$, and $\{(\pi_{\alpha_i}(\gamma_i))\}$ is obtained from $\{(\pi_{\alpha_i}(\gamma_i))\}$ by a twist or half-twist about $\alpha_i$. The unclean marking $\nu'$ differs from $\mu$ by a flip if for some $i$, we have $(\alpha'_i, \pi_{\alpha'_i}(\gamma'_i)) = (\gamma_i, \pi_{\gamma_i}(\alpha_i))$. The marking $\nu'$ is then replaced by a clean marking $\nu$ which is compatible with $\nu'$.

Masur and Minsky show that the marking graph is locally finite and admits a natural proper, cocompact action of $\text{Mod}(S)$ by isometries. It follows that $\text{Mod}(S)$ is quasi–isometric to $\mathcal{M}(S)$ via the orbit map.

Finally, if $S_0 \subseteq S$ is an essential subsurface, we define the projection $\pi_{S_0}(\mu)$ of a complete marking $\mu$ on $S$. If $S_0$ is not the annular subsurface filled by some $\alpha_i$, we set $\pi_{S_0}(\mu) = \pi_{S_0}(\{\alpha_i\})$. If $S_0$ is the annular subsurface filled by $\alpha_i$, we define $\pi_{S_0}(\mu) = \beta_i$. Since $\mu$ is a complete marking, the projection is always nonempty.

One interpretation of the distance formula is as follows: projections of curves to subsurfaces of $S$ allow us to triangulate locations of markings in $\mathcal{M}(S)$, and thus via the orbit map quasi–isometry, to better understand the geometry of $\text{Mod}(S)$. Thus, subsurface projections give a good coarse notion of visual direction of $\mathcal{C}(S)$. We would like to translate this interpretation to the setting of $\Gamma^e$ and $A(\Gamma)$, with the idea that distances between points in $\text{Lk}_{\Gamma^e}(v)$ give good coarse notions of visual angle between two geodesics emanating from $v$ and passing through those two points.

13.2. The Weil–Petersson metric and projections of the pants complex. In the analogy between right-angled Artin groups and mapping class groups we have that a vertex of $\Gamma$, viewed as a generator of $A(\Gamma)$, corresponds to (a power of) a Dehn
twist about a simple closed curve on a surface $S$. We mentioned above that our distance formula, we can only recover distances in $A(\Gamma)$ after forgetting the exponents of generators used to express elements of $A(\Gamma)$. In the Masur–Minsky distance formula, one can consider projections to a restricted class of subsurfaces instead of to all subsurfaces. If one “ignores Dehn twists” in the sense that one excludes projections to annular subsurfaces, one coarsely recovers the Weil–Petersson metric on Teichmüller space. This follows from combining results of Masur–Minsky and Brock. Before we state those results, we recall some definitions. A pants decomposition of a surface $S$ is the zero–skeleton of a maximal simplex in the curve graph of $S$. The pants graph of $S$, written $P_pS_q$, is defined to be the graph whose vertices are pants decompositions of $S$ and whose edges are given by “minimal intersection”. Precisely, $P_1$ and $P_2$ are adjacent in the pants graph if $P_1$ and $P_2$ agree on all but one pair of curves, say $c_1 \in P_1$ and $c_2 \in P_2$, if $(P_1 \setminus c_1) \cup c_2$ is a pants decomposition, and if $c_1$ and $c_2$ intersect minimally.

One often says that $P_1$ and $P_2$ differ by an elementary move.

**Theorem 60** (Masur–Minsky ([24], Theorem 6.12)). There is a constant $M_0 = M_0(S)$ so that for all $M \geq M_0$, there exist constants $K$ and $C$ such that if $P_1$ and $P_2$ are pants decompositions in the pants complex $\mathbb{P}(S)$, we have

$$\frac{1}{K}d_{\mathbb{P}(S)}(P_1, P_2) - C \leq \sum_{S_0 \subseteq S} d_{S_0}(P_1, P_2) \leq Kd_{\mathbb{P}(S)}(P_1, P_2) + C,$$

where the sum is taken over all non–annular essential subsurfaces of $S$ for which $d_{S_0}(P_1, P_2) \geq M$.

In different notation, we will write the estimate in the previous result as

$$\sum_{S_0 \subseteq S} d_{S_0}(P_1, P_2) = K_C d_{\mathbb{P}(S)}(P_1, P_2).$$

In the middle term of the estimate, we view pants decompositions as subsets of the curve graph of $S$ and define

$$d_{S_0}(P_1, P_2) = d_{S_0}(\pi_{S_0}(P_1), \pi_{S_0}(P_2)).$$

Thus, projections of pants decompositions to non–annular subsurfaces coarsely recovers distance in the pants graph. The pants graph is quasi–isometric to Teichmüller space with the Weil–Petersson metric:

**Theorem 61** (Brock ([10], Theorem 1.1)). The pants graph $\mathbb{P}(S)$ is quasi–isometric to Teichmüller space with the Weil–Petersson metric, via the map which takes a pants decomposition $P$ to the maximally noded surface given by collapsing the components of $P$.

In this sense, the syllable length in a right-angled Artin group can be thought of as an analogue of the Weil–Petersson metric on Teichmüller space.

13.3. **Free groups.** If $D_n$ is a completely disconnected graph, we have that the link of any vertex in $D_n$ is empty. In particular, the notions of vertices and stars of vertices coincide, so that star length and syllable length coincide. If $v$ and $v^g$ are vertices of $D_n$, we have that $\pi_{\text{Lk}(v)}(v^g) = \emptyset$. Thus, the distance formula for a free group is vacuous.
13.4. Right-angled Artin groups on trees. The first non–vacuous case of the distance formula is in the case where the graph $\Gamma$ is a tree. In this case, the extension graph $\Gamma_e$ is also a tree. Since any two vertices in a tree are connected by a unique geodesic we have that all but finitely many projections of a finite geodesic segment are nonzero, so that one need not discard any projections:

**Proposition 62.** Let $\Gamma$ be a tree and $D = \text{diam}(\Gamma)$. If $g \in A(\Gamma)$, then we have that

$$
\sum_{x \in V(\Gamma_e)} d_{\text{Lk}(x)}(\pi_x(v), \pi_x(v^g)) = 4DAD \|g\|_{\text{syl}}
$$

for some $v \in V(\Gamma)$.

Recall the notation $=_{K,C}$ denotes $K,C$–quasi–isometry. If $x$ and $y$ are two points of a tree, let us denote by $[x,y]$ the unique geodesic joining the two points.

**Proof of Proposition 62** We may assume $D \geq 2$. Put $k = \|g\|_s$. By Lemma 21, we can write $g = h_kh_{k-1}\cdots h_1$ such that $h_i \in \langle \text{St}(y_i) \rangle$ for some $y_i \in V(\Gamma)$ and $\|g\|_{\text{syl}} = \sum_i \|h_i\|_{\text{syl}}$. If $\|h_i\|_{\text{syl}} = 1$, then we may further assume $\text{supp}(h_i) = \{y_i\}$. Let us first consider the case that there exists $v \notin \text{St}(y_1) \cup \text{St}(y_k)$.

Let us denote by $\gamma$ the unique geodesic in $\Gamma_e$ connecting $v$ to $v^g$. For $x \in \Gamma_e$, put

$$
\delta(x) = d_{\text{Lk}(x)}(\pi_x(v), \pi_x(v^g)).
$$

Only the vertices on $\gamma$ contribute to the sum $\sum_{x \in V(\Gamma_e)} \delta(x)$; see Section 12.1.

For simplicity, let $g_i = h_ih_{i-1}\cdots h_1$ for $i = 1,2,\ldots,k$ and $g_0 = 1$. We define

$$
c_0 = [v,y_1], c_k = [y_k^g,v^g], c_i = [y_i^g,y_{i+1}^g] \text{ for } i = 1,2,\ldots,k-1.
$$

The concatenation $C = c_0c_1\cdots c_k$ is a path from $v$ to $v^g$. In particular, we have $c_i \subseteq C$. Since $y_{i+1}^{g_i} = y_{i+1}^{h_i+1} = y_{i+1}^{g_i}$, we have $c_i \subseteq \Gamma_g$. The minimality of $k$ and Lemma 22 (4) imply that $c_i \cap c_j = \emptyset$ for $i + 1 < j$. Using the assumption that $v \notin \text{St}(y_1) \cup \text{St}(y_k)$, we deduce that $\gamma = c_0'\cdots c_i'\cdots c_k'$ for some nontrivial geodesic segments $c_i' \subseteq c_i$. Put $c_i' = z_i^{g_i}$ where $z_i \in \Gamma$. For each $i$, let us denote the length–two subpath of $\gamma$ centered at $z_i^{g_i}$ as $(a_i^{g_i-1},z_i^{g_i},b_i^{g_i})$. By Lemma 22, either $z_i = y_i$ or $z_i \in \text{Lk}_\Gamma(y_i)$ so that we can write $h_i = z_i^{a_i}w_i$ for some $w_i \in \langle \text{Lk}(z_i) \rangle$. Note that $\|w_i\|_{\text{syl}} \leq \|h_i\|_{\text{syl}} \leq \|w_i\|_{\text{syl}} + 1$. In either of the two cases, $\delta(z_i^{g_i-1})$ is the distance between $a_i$ and $b_i^{g_i} = b_i^{w_i}$ in $\text{Lk}(z_i)^e$.

Lemma 53 implies that

$$
\|h_i\|_{\text{syl}}/3 \leq \max(1,\|w_i\|_{\text{syl}} - 1) \leq \delta(z_i^{g_i-1}) \leq \|w_i\|_{\text{syl}} + 1 \leq \|h_i\|_{\text{syl}} + 1.
$$

Let $B$ be the set of the interior vertices of $\gamma$ which is not of the form $z_i^{g_i-1}$ for $i = 1,2,\ldots,k$. Then $\delta(x) = 1$ for $x \in B$ since the length–two subpath of $\gamma$ centered at $x$ is contained in a conjugate of $\Gamma$. Lemma 19 (3) implies that

$$
0 \leq |B| \leq (d(v,v^g) - 1) - (k-2) = d(v,v^g) - k + 1 \leq (D-1)k + D + 1.
$$

From $\|g\|_s \leq \|g\|_{\text{syl}}$ we see that

$$
\|g\|_{\text{syl}}/3 \leq \sum_{x \in \Gamma_e} \delta(x) = \sum_{i=1}^k \delta(z_i^{g_i-1}) + |B| \leq (D+1)(\|g\|_{\text{syl}} + 1).
$$
Now let us assume $V(\Gamma) = \text{St}(y_1) \cup \text{St}(y_k)$. We may still choose $v \notin \{y_1, y_k\}$. If we define $c'_i$ and $z_i$ as before, it is then possible that $v = z_1$ or $v^g = z_k^{g_k-1}$; this occurs only if $\|h_1\|_{\text{syl}}$ or $\|h_k\|_{\text{syl}}$ is at most two, respectively. Hence,
\[
(\|g\|_{\text{syl}} - 4)/3 \leq \sum_{x \in \Gamma^e} \delta(x) \leq (D + 1)(\|g\|_{\text{syl}} + 1).
\]

\[ \square \]

13.5. The general triangle– and square–free case. We would now like to establish a more general version of the distance formula. We let $\Gamma$ be a triangle– and square–free graph such that $D = \text{diam}(\Gamma) \geq 2$.

**Lemma 63.** Let $g \in A(\Gamma)$ satisfy $\|g\|_{\ast} = 1$. Then there is a vertex $v \in V(\Gamma)$ such that for every geodesic $\gamma$ from $v$ to $v^g$ we have
\[
\sum_{x \in \Gamma(\gamma)} d_{\text{Lk}(x)}(\pi_x(v), \pi_x(v^g)) =_{1,1} \|g\|_{\text{syl}}.
\]

**Proof.** We may assume $\Gamma$ is not a star, since otherwise the conclusion is clear. Choose $y \in V(\Gamma)$ such that $\text{supp}(g) \subseteq \text{St}(y)$ and $\text{supp}(g) \not\subseteq \{z\}$ for every $z \neq y$. We will consider the following three cases.

**Case 1.** $\text{supp}(g) = \{y\}$.

We have $\Gamma \cap \Gamma^y = \text{St}(y)$. Let $(y, z, v)$ be a length-two path in $\Gamma$. Then $v \neq v^g \in \text{Lk}(z)$ and $\text{d}_{\text{Lk}(z)}(v, v^g) = \|g\|_{\text{syl}} = 1$.

**Case 2.** $\text{supp}(g) = \{y, a\}$ for some $a \in \text{Lk}(y)$.

We may assume $\deg(y) > 1$ for otherwise, we can switch the roles of $a$ and $y$. Put $z = y$ and choose $v \in (\text{Lk}(z) \setminus \{a\}) \cap \Gamma$. Then $v \neq v^g \in \text{Lk}(z)$ and $\text{d}_{\text{Lk}(z)}(v, v^g) = 1 = \|g\|_{\text{syl}} - 1$.

**Case 3.** $|\text{supp}(g) \cap \text{Lk}(y)| \geq 2$.

Put $z = y$ and choose $v \in \text{Lk}(z) \cap \Gamma$. Then $v \neq v^g \in \text{Lk}(z)$ and $\|g\|_{\text{syl}} - 1 \leq \text{d}_{\text{Lk}(z)}(v, v^g) \leq \|g\|_{\text{syl}}$.

In the above three cases, $\gamma = (v, z, v^g)$ is the unique geodesic from $v$ to $v^g$. Hence, the given sum consists of only one term and coarsely coincides with $\|g\|_{\text{syl}}$ using the quasi-isometry constants $K = 1 = C$.

Lemma 63 lends itself to a general distance formula for triangle– and square–free right-angled Artin groups.

**Proposition 64.** For all $1 \neq g \in A(\Gamma)$, there exists a $(10D, 10D)$–quasi–geodesic $\gamma : [0, \ell] \to \Gamma^e$ parametrized by arc length such that $\gamma(0) = v, \gamma(\ell) = v^g$ for some $v \in V(\Gamma)$ and
\[
\sum_{i=2}^{\ell-1} d_{\text{Lk}(\gamma(i))}(\gamma(i - 1), \gamma(i + 1)) =_{10D, 10D} \|g\|_{\text{syl}}.
\]

**Proof.** Let $g = h_k h_{k-1} \cdots h_1$ such that $\text{supp}(h_i) \subseteq \text{St}(y_i)$ and $\|g\|_{\text{syl}} = \sum_i \|h_i\|_{\text{syl}}$. As in the proof of Lemma 63 we may further assume the following.

(i) If $\|h_i\|_{\text{syl}} = 1$, then $\text{supp}(h_i) = \{y_i\}$.
(ii) If $\|h_i\|_{\text{syl}} \geq 2$, then $\deg(y_i) > 1$. 

Put \( g_i = h_i h_{i-1} \cdots h_1 \) and \( g_0 = 1 \).

We choose \( z_i \) and \( v_i \) as in the proof of Lemma 63. Namely, if \( \|h_i\|_{\text{syl}} = 1 \) then we choose a length-two path \((y_i, z_i, v_i)\) in \( \Gamma \). If \( \|h_i\|_{\text{syl}} \geq 2 \), then we set \( z_i = y_i \) and choose \( v_i \in \text{Lk}(z_i) \) such that \( v_i^h_i \neq v_i \). Note that \( z_i^h_i = z_i \). In (i) above, we see that \( d_{\text{Lk}(z_i)}(v_i, v_i^h) = \|h_i\|_{\text{syl}} = 1 \). In (ii), we note that \( \|h_i\|_{\text{syl}}/2 \leq \max(1, \|h_i\|_{\text{syl}} - 1) \leq d_{\text{Lk}(z_i)}(v_i, v_i^h) \leq \|h_i\|_{\text{syl}} \).

We put \( v = v_1 \). Choose a shortest path \( c_i \) in \( \Gamma \) from \( v_i \) to \( v_{i+1} \) for \( i = 1, 2, \ldots, k - 1 \) and let \( c_k \) be a shortest path in \( \Gamma \) from \( v_k \) to \( v \). Define \( \gamma \) to be the concatenation
\[
\gamma = (v_1, z_1, v_1^{g_1}) \cdot c_1^{g_1} \cdot (v_2^{g_1}, z_2^{g_1}, v_2^{g_2}) \cdot c_2^{g_2} \cdots (v_k^{g_i}, z_k^{g_i - 1}, v_k^{g_k}) \cdot c_k^{g_k}.
\]
Let \( \gamma: [0, \ell] \to \Gamma^e \) be the parametrization by arc length. In particular, \( \gamma([0, \ell] \cap \mathbb{Z}) \) is contained in the vertex set of \( \Gamma^e \). Put \( A = \{i : \gamma(i) = z_{j_i}^{g_{j_i} - 1} \text{ for some } j\} \) and \( B = \{1, 2, \ldots, \ell - 1\} \setminus A \).

From the above observations, we have
\[
\|g\|_{\text{syl}}/2 \leq \sum_{i \in A} d_{\text{Lk}(\gamma(i))}(\gamma(i) - 1, \gamma(i) + 1) \leq \|g\|_{\text{syl}}.
\]
If \( i \in B \), then \( d_{\text{Lk}(\gamma(i))}(\gamma(i) - 1, \gamma(i) + 1) \leq 1 \) since \( \{\gamma(i) - 1, \gamma(i), \gamma(i) + 1\} \subseteq \Gamma_{g_j} \) for some \( j \). So,
\[
0 \leq \sum_{i \in B} d_{\text{Lk}(\gamma(i))}(\gamma(i) - 1, \gamma(i) + 1) \leq (D + 2)k \leq (D + 2)\|g\|_{\text{syl}}.
\]
Hence,
\[
\sum_{i = 2}^{\ell - 1} d_{\text{Lk}(\gamma(i))}(\gamma(i) - 1, \gamma(i) + 1) = D + 3, 0 \geq \|g\|_{\text{syl}}.
\]

It remains to show that \( \gamma \) is a \((10D, 10D)\)-quasi-geodesic. Let us set
\[
\delta_0 = [v_1, z_1],
\]
\[
\delta_i = [z_i^{g_i - 1}, v_i^{g_i}] \cdot c_i^{g_i} \cdot [v_i^{g_i}, z_{i+1}^{g_i}] \subseteq \Gamma_{g_i} \text{ for } i = 1, 2, \ldots, k - 1,
\]
and \( \delta_k = [z_k^{g_{k-1}}, v_k^{g_k}] \cdot c_k^{g_k} \).
Then we have \( \gamma = \delta_0 \cdot \delta_1 \cdot \delta_2 \cdots \delta_{k-1} \cdot \delta_k \). Note that the length of \( \delta_i \) is between one and \( D + 2 \).

We claim that for \( 0 \leq i < j \leq k \) and \( x \in \delta_i, y \in \delta_j \) we have \( j - i - 4 \leq d'(x, y) \leq j - i + 1 \). The upper bound is clear. Assume \( d'(x, y) \leq j - i - 5 \). Since \( v_i^{g_j} \in \delta_s \subseteq \Gamma_{g_s} \) for each \( s \), we have that \( d'(v_i^{g_j}, v_i^{g_j}) \leq j - i - 3 \). Then we have \( d'(v_i^{g_j}, v_i^{g_j}) \leq j - i - 2 \). On the other hand, Lemma 19 (1) implies that \( d'(v_i^{g_j}, v_i^{g_j}) \geq \|g_j g_i^{-1}\|_{\ast} - 1 = j - i - 1 \). This is a contradiction.

Now fix \( 0 \leq p < q \leq \ell \). Let \( \gamma(p) \in \delta_i \) and \( \gamma(q) \in \delta_j \) where \( 0 \leq i < j \leq k \). We have that \( d'(\gamma(p), \gamma(q)) \geq j - i - 4 \). From the estimates on the lengths of \( \delta_i \)'s, we have
\[
j - i - 1 \leq q - p \leq (D + 2)(j - i + 1).
\]
By Lemma 19 we have
\[
j - i - 4 \leq d'(\gamma(p), \gamma(q)) \leq d(\gamma(p), \gamma(q)) \leq Dd'(\gamma(p), \gamma(q)) \leq D(j - i + 1).
\]
It follows that
\[ \frac{1}{D} d(\gamma(p), \gamma(q)) - 2 \leq j - i - 1 \leq q - p \leq (D + 2)(j - i + 1) \leq (D + 2)(d(\gamma(p), \gamma(q)) + 5). \]

The general distance formula we have given in Proposition 64 is somewhat unsatisfying in that it is not a direct analogue of the Masur–Minsky distance formula. Ideally, the sum on the left hand side would be a sum over all vertices within $\Gamma^e$, and the terms in the summation would be projected distances which exceed a certain lower threshold. At the time of the writing there are several issues which impede the authors from proving such a general result, perhaps the most serious of which is the lack of an appropriate notion of a tight geodesic.

14. Automorphism group of $\Gamma^e$

By a result of Ivanov in [16], the automorphism group of the curve graph of a non–sporadic surface is commensurable with the mapping class group of the surface. Bestvina asked the authors whether an analogous result holds for the extension graph. The answer is no:

**Theorem 65.** The isomorphism group of an extension graph of a connected, anti–connected graph contains the infinite rank free abelian group. In particular, $\text{Aut}(\Gamma^e)$ is uncountable.

Here, a graph is *anti-connected* if its complement graph is connected.

**Proof.** Let $\Gamma$ be a connected and anti–connected graph. We observe that each star of a vertex $v_0$ separates $\Gamma^e$ into infinitely many components of infinite diameter, all of which are isomorphic to each other. To see this, fix a vertex $v \in V(\Gamma)$. By assumption, $\text{St}(v) \neq \Gamma$. Observe that for each $n > 0$, we can build the graph

\[ \Gamma_n = \bigcup_{n=1}^{\infty} \text{St}(v), \]

which is obtained by identifying $n$ copies of $\Gamma$ along the star of $v$. The graph $\Gamma_n$ naturally embeds in $\Gamma^e$. Observe that $\Gamma_n \setminus \text{St}(v)$ separates $\Gamma_n$ into $n$ components, so that $\text{St}(v)$ separates $\Gamma^e$ into at least $n$ components. It follows that $\Gamma^e \setminus \text{St}(v)$ has infinitely many components. Observe that the conjugation action of $v$ permutes these components. In particular, $\Gamma^e \setminus \text{St}(v)$ has infinitely many isomorphic components. Since $\Gamma$ is anti–connected, we have that $\Gamma^e$ has infinite diameter, so that at least one component of $\Gamma^e \setminus \text{St}(v)$ has infinite diameter. By applying the conjugation action of $v$, we have that infinitely many components of $\Gamma^e \setminus \text{St}(v)$ have infinite diameter.

Pick such an infinite component $\Gamma_0$ and consider $C_0 = \{ \Gamma_0^i : i \in \mathbb{Z} \setminus \{0\} \}$. We let $f_0$ be the conjugation action by $v_0$ on $\Gamma^e$. Fix an element in $C_0$, we find the star of another vertex $v_1$ that separates out an infinite component. Around this vertex, there
is an action $f_1$ similar to the previous step fixing $\text{St}(v_1)$. Continuing this process, we see that

$$\{f_1, f_2, \ldots \} \cong \prod_{i=1}^{\infty} \mathbb{Z},$$

whence the result. □

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