Categorical tori

Nora Ganter

Abstract

We give explicit and elementary constructions of the 2-group extensions of a torus by the circle and discuss an application to loop group extensions. Examples include extensions of maximal tori and of the tori associated to the Leech and Niemeier lattices.

1. Introduction

By a categorical group, or 2-group, we mean a monoidal groupoid \((G, \bullet, 1)\) with weakly invertible objects. If \(G\) is a Lie groupoid, one requires the monoidal structure to be (locally) smooth in an appropriate sense (see Construction 2.2). In this situation, one speaks of a Lie 2-group.

In recent years, there has been a steadily growing interest in categorical groups. These play as important a role in string theory as groups do in particle physics \([BH11]\). A number of prominent groups seem to be most naturally understood via their categorical refinements. Most famously, the infinite dimensional groups \(\text{String}(n)\) come from compact Lie 2-groups \([SP11]\). Evidence of 2-groups turns up in Moonshine \([Gan09]\). Weyl groups, as well as some of the sporadic groups, including the Monster, are known or conjectured to be the isomorphism classes of categorical groups (see Section 5). It is hoped that exploring the, previously ignored, categorical nature of these groups will help to gain a better understanding of them.

The purpose of this paper is to describe a basic class of examples of categorical groups, namely all central extensions (in the sense of \([SP11]\)) of Lie 2-groups of the form

\[
\text{pt} \times U(1) \rightarrow \mathcal{T} \rightarrow T,
\]

where \(T\) is a compact torus, and \(\text{pt} \times U(1)\) is the one-object groupoid with \(\text{aut} (\text{pt}) = U(1)\). We will refer to such a \(\mathcal{T}\) as a categorical torus. Categorical tori turn out to quite easy to construct, and they have relevance to all the topics mentioned above (see Section 5).

The theory of categorical groups is closely related to that of loop groups, and we will explore this point in Section 6.1. Our approach is to work from 2-groups to loop groups. In particular, our construction of \(\mathcal{T}\) does not involve loops or any other infinite dimensional objects. Via a machine called transgression, this yields a simple description of central extensions of the loop group \(\mathcal{L}T\).

What we hope to get across is that categorical groups can be fairly simple to construct and easy to work with and that some of the complicated looking features of loop groups are merely the shadow of rather obvious phenomena on the categorical side. This is the first in a series of papers describing the character theory and the symmetry 2-groups of categorical tori.

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1.1 Acknowledgments

It is a pleasure to thank David Roberts for very helpful conversations and correspondence. The idea for Construction 2.1 came from a conversation with him, and I understand that he will also write about it elsewhere. The idea to look for crossed module extensions of tori, picked up in Section 6.2, is also due to David. Many thanks go to Matthew Ando for very inspiring conversations, to Konrad Waldorf for patiently answering a long list of emails full of technical questions and to Arun Ram, who helped with some of the references.

2. Constructions of categorical tori

To build a categorical torus, we need a finite dimensional lattice\(^1\) which we denote \(\Lambda^\vee\), and an integer-valued bilinear form \(J\) on \(\Lambda^\vee\). Up to isomorphism, our constructions will depend only on the even symmetric bilinear form

\[ I(m, n) = J(m, n) + J(n, m). \]

From \(\Lambda^\vee\), we construct the Lie algebra

\[ t := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R} \]

and the torus

\[ T := \Lambda^\vee \otimes_{\mathbb{Z}} U(1). \]

We will also write \(J\) for the bilinear form \(J \otimes \mathbb{R}\) on \(t\). The exponential map is

\[ \exp: t \rightarrow T \]

\[ m \otimes r \mapsto m \otimes e^{2\pi ir}, \]

and we make the identification

\[ \Lambda^\vee = \ker(\exp) \subseteq t. \]

We write \(t/\Lambda^\vee\) for the action groupoid of \(\Lambda^\vee\) on \(t\). We have a canonical equivalence of Lie groupoids\(^2\)

\[ p: t/\Lambda^\vee \cong T, \]

where \(T\) is viewed as groupoid with only identity arrows.

We will give three equivalent constructions of the categorical torus associated to \((\Lambda^\vee, J)\). The first is as a strict monoidal Lie groupoid.

**Construction 2.1.** Let \(\mathcal{T}\) be the Lie groupoid

\[ \mathcal{T} := (t/\Lambda^\vee) \times (\text{pt}/U(1)). \]

So, \(\mathcal{T}\) has objects \(t\) and arrows \(t \times \Lambda^\vee \times U(1)\), the source of \((x, m, z)\) is \(x\), and the target of \((x, m, z)\) is \(x + m\). We equip \(\mathcal{T}\) with the following multiplication

\[ x \cdot y = x + y \quad \text{on objects} \]

\[ (x, m, z) \cdot (y, n, w) = (x + y, m + n, z \cdot w \cdot \exp(-J(m, y))) \quad \text{on arrows}. \]

The unit object is \(0\), and associativity and unit isomorphisms are identities. This makes \((\mathcal{T}, \bullet)\) a strict monoidal Lie groupoid, i.e., a group object in the category of Lie-groupoids.

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1 By “lattice”, we mean a free \(\mathbb{Z}\)-module.

2 For background on Lie-groupoids, we refer the reader to [Moe02] and [Ler10].
The second construction re-interprets the data of Construction 2.1 as a compact Lie 2-group in the sense of Schommer-Pries [SP11].

**Construction 2.2.** Let $\mathcal{T}$ be the Lie groupoid

$$\mathcal{T} = T \times \text{pt} / \text{U}(1),$$

i.e., $\mathcal{T}$ has objects $T$ and arrows $T \times \text{U}(1)$, source and target are projection onto the first factor and composition of arrows is multiplication in the second factor. Then we have an equivalence of Lie groupoids

$$p \times \text{id} : \mathcal{X} \xrightarrow{\sim} \mathcal{T}.$$

Note that $p \times \text{id}$ does not possess a continuous inverse equivalence. So, one interprets the data of Construction 2.1 as those of a 2-group object in a suitable localization of the bicategory of Lie-groupoids, namely the bicategory of bibundles $\text{Bibun}$. Different communities have different language for the 1-morphisms in $\text{Bibun}$. Depending on your taste, you may think of multiplication on $\mathcal{T}$ as the zig-zag, span, orbifold map, or anafunctor

$$\mathcal{T} \times \mathcal{T} \xleftarrow{\sim} \mathcal{X} \xrightarrow{\ast} \mathcal{X} \xrightarrow{\sim} \mathcal{T},$$

or interpreted it as a bibundle.

Our third construction is as a multiplicative bundle gerbe in the sense of [Bry08] and [CJM+05].

**Construction 2.3.** Let $\mathcal{I}$ be the trivial bundle gerbe over $T$. Recall that a multiplication on $\mathcal{I}$ is a stable isomorphism

$$\mu : pr_1^*\mathcal{I} \otimes pr_2^*\mathcal{I} \longrightarrow m^*\mathcal{I}$$

of bundle gerbes over $T \times T$, where the $pr_i$ are the projections to the factors and $m$ is multiplication in $T$. Such a stable isomorphism of trivial bundle gerbes is the same as a line bundle, and we take $\mu$ to be

$$L^J = \mathbb{C} \times t \times t / \sim$$

with

$$(z, x + m, y + n) \sim (z \cdot \exp(J(m, y)), x, y),$$

for $(m, n) \in \Lambda^\vee \times \Lambda^\vee$. So, $L^J$ is the line bundle with multipliers

$$f_{(m,n)} : t \times t \longrightarrow U(1)$$

$$(x, y) \longmapsto \exp(J(m, y))$$

for $(m, n) \in \Lambda^\vee \times \Lambda^\vee$. Finally, we need to specify the associativity isomorphism

$$\alpha : m_{12}^*L^J \otimes pr_1^*\mathcal{I} \cong m_{23}^*L^J \otimes pr_2^*\mathcal{I}$$

over $T \times T \times T$, where $pr_{ij}$ is the projection onto factors $i$ and $j$ and $m_{ij}$ is multiplication of these two factors, e.g., $m_{12} = m \times \text{id}$. We take $\alpha$ to be the canonical isomorphism resulting from the fact that its source and target have identical multipliers

$$f_{(k,m,n)} : t \times t \times t \longrightarrow U(1)$$

$$(x, y, z) \longmapsto \exp(J(k, y) + J(k, z) + J(m, z))$$

for $(k, m, n) \in \Lambda^\vee \times \Lambda^\vee \times \Lambda^\vee$. 
Remark 2.4. Construction 2.3 takes the sum of two bilinear forms, $J_1 + J_2$ to the tensor product of multiplicative bundle gerbes.

By [Wal12, Thm. 3.2.5], the data of a multiplicative bundle gerbe over $T$ are equivalent to those of an extension of $T$ by pt $\mathcal{U}(1)$. In the case of Construction 2.3 we may think of the objects of $\mathcal{I}$ as pairs $(t, L)$ with $t \in T$ and $L$ a hermitian line, and of the arrows $(t, L) \rightarrow (t, L')$ as the unitary isomorphisms from $L$ to $L'$. Then $\mathcal{I}$ is a monoidal groupoid with multiplication

$$(s, L_1) \bullet (t, L_2) = (s \cdot t, L_{s,t}^J \otimes L_1 \otimes L_2).$$

The associativity isomorphisms for $\bullet$ are encoded in $\alpha$, $\alpha_{r,s,t} : L_{r,s,t}^J \otimes L_{s,t}^J \cong L_{r,s}^J \otimes L_{s,t}^J$.

**Proposition 2.5.** The three constructions yield equivalent 2-group extensions of $T$.

**Proof.** It is clear that Construction 2.1 and Construction 2.2 are equivalent. To see their equivalence with Construction 2.3, let $F$ be the functor

$$F : \mathfrak{T} \longrightarrow \mathcal{I}$$

$$x \mapsto (\exp(x), \mathbb{C})$$

on objects

$$(x, m, z) \mapsto z$$

on arrows.

To make $F$ a monoidal equivalence, we need a natural isomorphism

$$\phi : F(-) \bullet F(-) \Longrightarrow F(- \bullet -).$$

This is a map of the form

$$\phi : \text{ob}(\mathfrak{T} \times \mathfrak{T}) \longrightarrow \text{mor}(\mathcal{I} \times \mathcal{I})$$

$$(x, y) \longmapsto \left(\exp(x + y), L_{\exp(x), \exp(y)}^J \rightarrow \mathbb{C}\right),$$

or, equivalently, a trivialization of $L^J$ over $t \times t$. We have such a trivialization by construction of $L^J$. One checks that $(F, \phi)$ is a monoidal equivalence from $\mathfrak{T}$ to $\mathcal{I}$. \hfill $\square$

**Definition 2.6.** We will write $J^t$ for the bilinear form

$$J^t(m, n) = J(n, m)$$

on $\Lambda^t$. We say that $J$ is symmetric if $J = J^t$ and that $J$ is skew symmetric if $J = -J^t$. A symmetric bilinear form is called even if

$$I(m, m) \in 2\mathbb{Z} \quad \text{for } m \in \Lambda^t.$$  \hfill (1)

**Proposition 2.7.** The bilinear forms $J$ and $J^t$ yield equivalent 2-group extensions of $T$.

**Proof.** A monoidal equivalence from $(\mathfrak{T}, \bullet, J)$ to $(\mathfrak{T}, \bullet, J^t)$ is given by the functor

$$F : \mathfrak{T}_J \longrightarrow \mathfrak{T}_{J^t}$$

$$t \overset{\text{id}}{\longrightarrow} t$$

on objects,

$$(x, m, z) \longmapsto (x, m, z \cdot \exp(J(x, m)))$$

on arrows,

together with the natural transformation

$$\phi : F(-) \bullet_J F(-) \Longrightarrow F(- \bullet_{J^t} -)$$

$$t \times t \longrightarrow t \times \Lambda^t \times U(1)$$

$$(x, y) \longmapsto (x + y, 0, \exp(J(x, y))).$$

\hfill $\square$
Corollary 2.8. (i) If \( I \) is an even symmetric bilinear form on \( \Lambda^\vee \), then the multiplicative bundle gerbe associated to \( I \) possesses a square root.

(ii) If \( B \) is a skew symmetric integral bilinear form on \( \Lambda^\vee \), then \( B \) yields a trivial 2-group extension of \( T \).

Proof. (i) follows from the fact that every even symmetric bilinear form \( I \) can be written in the form \( I = J + J^t \) for an integer-valued bilinear form \( J \). For instance, fix a basis \((b_1, \ldots, b_r)\) of \( \Lambda^\vee \) and set

\[
J(b_i, b_j) = \begin{cases} \frac{1}{2} I(b_i, b_i) & \text{if } i = j, \\ I(b_i, b_j) & \text{if } i < j, \\ 0 & \text{else.} \end{cases}
\]

(ii) follows from the fact that, similarly, every skew symmetric bilinear form \( B \) can be written in the form \( B = J - J^t \) for an integer-valued bilinear form \( J \).

\[\square\]

Corollary 2.9. Let \( J \) be an integer-valued bilinear form on \( \Lambda^\vee \). Then, up to equivalence over \( T \), the 2-group \((\mathbb{T}, \cdot, j)\) only depends on the even bilinear form

\[ I(m, n) = J(m, n) + J(n, m). \]

Proof. Let \( J_1 \) and \( J_2 \) be two integer-valued bilinear forms on \( \Lambda^\vee \), and assume that

\[ J_1(m, n) + J_1(n, m) = J_2(m, n) + J_2(n, m). \]

Then \( J_1 - J_2 \) is skew symmetric. By Corollary 2.8, the multiplicative bundle gerbe obtained from \( J_1 - J_2 \) is trivial. Using Remark 2.4, we conclude that the multiplicative bundle gerbes obtained from \( J_1 \) and \( J_2 \) are isomorphic.

\[\square\]

3. The example of the circle

Let \( t = \mathbb{R} \) and \( \Lambda^\vee = \mathbb{Z} \). Any even symmetric bilinear form on \( \mathbb{Z} \) is an integer multiple of

\[ I(m, n) = 2mn. \]

For this \( I \), there is a unique choice of \( J \), namely

\[ J(m, n) = mn. \]

The basic circle extension \( U(1) \) of the circle group \( U(1) \) consists of the following data:

(i) the trivial bundle gerbe \( I \) over \( U(1) \),

(ii) the line bundle \( L \) on \( U(1) \times U(1) = \mathbb{R}^2 / \mathbb{Z}^2 \)

defined by the multipliers

\[ f_{(m,n)}: \mathbb{R}^2 \longrightarrow U(1) \]

\[ (x, y) \longmapsto e^{2\pi i m y} \]

for \((m, n) \in \mathbb{Z}^2\),

(iii) the canonical isomorphism

\[ \alpha: m_{12}^* L \otimes pr_{12}^* L \cong m_{23}^* L \otimes pr_{23}^* L \]

\[5\]
For $k \in \mathbb{Z}$, the $k$th circle extension $U(1)_k$ of $U(1)$ is obtained by replacing the multipliers with $\exp(kmy)$.

Here are a few words of explanation about these choices: gerbes on $U(1)$ are classified, up to stable isomorphism, by their Dixmier-Douady class in

$$H^3(U(1); \mathbb{Z}) = 0.$$ 

So, any bundle gerbe over $U(1)$ is trivializable, and we might as well start with the trivial bundle gerbe $\mathcal{I}$. Line bundles on $U(1) \times U(1)$ are classified, up to isomorphism, by their first Chern class in

$$H^2(U(1) \times U(1); \mathbb{Z}).$$

This is isomorphic to the group of skew symmetric bilinear forms on $\mathbb{Z}^2$, which is infinite cyclic, generated by the determinant. To construct a line bundle $L$ with Chern class $c_1(L) = \det$, we use Chern-Weil theory:

\[
\Omega^2_{U(1) \times U(1)} \to dx \wedge dy \\
\Omega^1_{U(1) \times U(1)} \to xdy \to mdy \\
\mathcal{T}_{U(1) \times U(1)} \to \exp(my) \to 1 \\
\mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{R}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2
\]

Figure 1. Chern-Weil theory: the multipliers can be read off from this cocycle in the truncated Čech-Deligne double complex for $U(1) \times U(1)$.

Note that this argument builds $L$ as a line bundle with connection, given by the 1-form

$$\omega_{\mathbb{R} \times \mathbb{R}} = xdy$$

with curvature

$$d\omega_{\mathbb{R} \times \mathbb{R}} = dx \wedge dy.$$ 

Similarly, the bundle $L^{\otimes k}$ with multipliers $\exp(kmy)$ has connection $kxdy$ and Chern class $k \cdot \det$.

Remark 3.1. This example is not new. A construction of the multiplicative gerbe $U(1)$ (but with a different connection) already turns up in [BM96, Sec.3]. I understand that there is also
unpublished work of Murray, Roberts, Stevenson and Vozzo, constructing the underlying non-trivial equivariant gerbe on $U(1)$.

4. The classification

Let $T$ be a compact torus with Lie algebra $t$ and coweight lattice $\Lambda^\vee = \ker(\exp)$. Up to equivalence, the 2-group extensions of $T$ by $pt \// U(1)$ are classified by

$$H_3^{gp}(T; U(1)) \cong H^4(BT; \mathbb{Z}),$$

where the left-hand side is Lie group cohomology as in [SP11] [WW13]. There are a number of different, but equivalent definitions of Lie group cohomology. We choose to work with the Čech-simplicial double complex $\check{C}^\ast(BT; \mathbb{Z})$ as in the classification of multiplicative bundle gerbes in [Bry08] and [CJM+05, Prop 5.2]. The goal of this section is to analyze the degree four part of the composite of isomorphisms

$$S^\ast(\Lambda) \cong H^\ast(BT; \mathbb{Z}) \cong \check{H}^\ast(BT; \mathbb{Z}) \cong \check{H}^{\ast-1}(BT; U(1)), \quad (2)$$

where $S^\ast(\Lambda)$ is the symmetric algebra of the weight lattice $\Lambda = \text{Hom}(\Lambda^\vee, \mathbb{Z})$.

Weights are given degree 2, so that the degree four part is

$$S^2\Lambda = (\Lambda \otimes \Lambda)/S_2.$$

We may think of elements of $S^2\Lambda$ as homogeneous polynomials of degree 2 in the weights, and we have the symmetrization map

$$S^2(\Lambda) \longrightarrow \text{Bil}(\Lambda^\vee, \mathbb{Z})$$

$$\lambda \mu \longmapsto \lambda \otimes \mu + \mu \otimes \lambda,$$

identifying $S^2(\Lambda)$ with the group of even symmetric bilinear forms on $\Lambda^\vee$. So, (2) establishes a group homomorphism from the even symmetric bilinear forms on $\Lambda^\vee$ to the isomorphism classes of multiplicative bundle gerbes over $T$.

**Theorem 4.1.** Let $I$ be an even symmetric bilinear form on $\Lambda^\vee$, and let $J$ be an integral bilinear form on $\Lambda^\vee$ satisfying $I = J + J^\dagger$. If we apply Construction 2.3 to $(\Lambda^\vee, J)$, then the resulting multiplicative bundle gerbe is classified by $I$.

**Proof.** Let $\lambda \in \Lambda$ be a weight of $T$ with character $e^\lambda$, and write

$$L_\lambda = ET \times_T C_\lambda$$

for the line bundle on $BT$ classified by $Be^\lambda$. The first isomorphism in (2) goes back to Borel, and is defined as

$$S^\ast(\Lambda) \xrightarrow{\cong} H^\ast(BT; \mathbb{Z})$$

$$\lambda \longmapsto c_1(L_\lambda), \quad (3)$$

where on the left-hand side, the weights have degree 2.

To define the second isomorphism in (2), let $ET$ be a contractible free $T$-space, $BT = ET/T$, and and recall that

$$ET \times_{BT} \cdots \times_{BT} ET \cong ET \times T \times \cdots \times T.$$
So, the maps

\[ ET \rightarrow BT \]
\[ ET \times t \rightarrow ET \times_{BT} ET \]

form a hypercover of \( BT \) whose \( Č \)ech double complex can be identified with \( Č^*(BT_\bullet;\mathbb{Z}) \). Under this identification, the cup product becomes

\[ (f \cup g)(x_0, \ldots, x_{r+s}) = (pr_1^* f)(x_0, \ldots x_r) \cdot (pr_2^* g)(x_r, \ldots x_{r+s}), \]

where \( r \) and \( s \) are the \( Č \)ech degrees of \( f \) and \( g \), \( pr_1 \) is the projection onto the first \( \text{deg}_{\text{simp}}(f) \) factors and \( pr_2 \) is the projection onto the last \( \text{deg}_{\text{simp}}(g) \) factors.

Note that these first two isomorphisms are actually isomorphisms of graded rings. To determine the image of \( \lambda \mu \) in \( H^4(BT_\bullet;\mathbb{Z}) \), we determine the images of \( \lambda \) and \( \mu \) and then take the cup product. The first Chern class of \( L_\lambda \) in \( Č \)ech hypercohomology is given by the multipliers

\[ ET \times t \rightarrow U(1) \]
\[ (\eta, x) \mapsto \exp(\lambda(x)). \]

Hence, \( \lambda \) maps to the degree \((1,1)\) cocycle

\[ t \times \Lambda^\vee \rightarrow \mathbb{Z} \]
\[ (x, m) \mapsto \lambda(m) \]

in the \( Č \)ech-simplicial double complex. Given two weights, \( \lambda \) and \( \mu \), their cup product is represented by the cocycle

\[ t^2 \times (\Lambda^\vee)^2 \times (\Lambda^\vee)^2 \rightarrow \mathbb{Z} \]
\[ \left( \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} m \\ n \end{array} \right), \left( \begin{array}{c} k \\ l \end{array} \right) \right) \mapsto \lambda(k) \cdot \mu(n) \]

in \( Č^2(BT_2;\mathbb{Z}) \). The last isomorphism in \( (2) \) is the inverse of the connecting homomorphism for the short exact sequence of presheaves

\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \underline{U(1)} \rightarrow 0. \]

It is easy to read off that Construction 2.3 associates to \( J = \mu \otimes \lambda \) the multiplicative bundle gerbe corresponding to the \( U(1) \)-valued \( Č \)ech-simplicial 3-cocycle

\[ (1, \lambda(m)\mu(y), 1, 1). \]

The image of this cocycle under the connecting homomorphism is indeed the integral 4-cocycle

\[ (1, 1, \lambda(k)\mu(n), 1, 1), \]

see the picture:
This concludes the proof. \[\square\]

**Remark 4.2.** In particular, we have seen that the underlying gerbe of any multiplicative gerbe on \( T \) is trivial. This is consistent with work of Waldorf [Wal10, Prop.2.10], who identifies the forgetful map

\[
\{\text{multiplicative bundle gerbes on } G\} / \cong \longrightarrow \{\text{bundle gerbes on } G\} / \cong
\]

with the inverse transgression

\[
\tau : H^4(BG; \mathbb{Z}) \longrightarrow H^3(G; \mathbb{Z})
\]

in the Leray-Serre spectral sequence for the universal bundle \( EG \to BG \). For a compact, connected Lie group \( G \), this map was calculated (in all degrees) by Chern and Simons. In the relevant degree, their result is summarized by the commuting diagram

\[
\begin{array}{ccc}
H^4(BG; \mathbb{R}) & \xrightarrow{\tau} & H^3(G; \mathbb{R}) \\
\downarrow \text{Chevalley-Eilenberg isomorphism} & & \downarrow \text{Chevalley-Eilenberg isomorphism} \\
(S^2g^*)^{Ad} & \longrightarrow & (\Lambda^3g^*)^{Ad} \\
I & \longrightarrow & -\nu
\end{array}
\]
where $\nu$ is the Cartan 3-form associated to $I$,

$$\nu(x, y, z) = I([x, y], z).$$

Since the Lie bracket on a torus is zero, it follows that the Dixmier-Douady class of the underlying bundle gerbe of any multiplicative bundle gerbe on $T$ vanishes.

4.1 Connections
If $J = \lambda \otimes \mu$, then the Chern-Weil discussion analogous to that of Section 3 produces the connection $\nabla$ on $L^J$ with connection 1-form

$$\omega_{t \times t} = \lambda d\mu$$

and curvature 2-form

$$\kappa_{t \times t} = d\lambda \wedge d\mu.$$ 

For arbitrary $J$, we introduce the maps

$$J^t(x) : t \to \mathbb{R} \; \; \; \; \; v \mapsto J(x, v).$$

Then $\nabla$ is defined by the 1-form

$$\omega(x, y) = d(J^t(x))_y.$$ 

The pair $(L^J, \nabla)$ makes a multiplicative bundle gerbe with connection (in the sense of [Wal10]) out of the trivial bundle gerbe $I$ on $T$ (with the remaining data trivial).

4.2 Symmetries
In many interesting examples, we have a finite group $\Gamma$ of linear isometries of $(\Lambda^\vee, I)$. In this case, $I$ may be interpreted as a $\Gamma$-invariant cohomology element,

$$I \in H^4(BT; \mathbb{Z})^\Gamma,$$

and the action of $\Gamma$ on $T$ preserves the multiplicative bundle gerbe classified by $I$ up to isomorphism. As a consequence, we obtain a 2-group extension of $\Gamma$, namely the automorphism 2-group of the categorical torus. There are two variations worth exploring, depending on whether or not we require our symmetries to preserve the connection. We will come back to this topic at a different occasion.

5. Examples

5.1 Maximal tori
Let $G$ be a compact connected Lie group with maximal torus $T$ and Weyl group $W$. Then we have

$$H^4(BG; \mathbb{Z}) \cong H^4(BT; \mathbb{Z})^W,$$

where $\omega$ is the right-invariant Maurer-Cartan form on $G$. A look at the definitions on page 50 of [CS74] identifies $I(\omega \wedge [\omega, \omega])$ with the bi-invariant 3-form on $G$ whose restriction to $g = T_1G$ equals $12 \nu$.
and we can choose $I$ as a multiple of the Killing form. If $G$ is simple and simply connected, then we have

$$H^4(BT; \mathbb{Z})^W \cong \mathbb{Z},$$

The positive definite generator $I_{bas}$ is called the basic bilinear form on $\Lambda'$. The form $kI_{bas} \in H^4(BG; \mathbb{Z})$ classifies the 2-group denoted $\text{String}_k(G)$ in [SPI1]. The restriction of $\text{String}_k(G)$ to $T$ is equivalent to the categorical torus constructed from $(\Lambda', kI_{bas})$. This property determines $\text{String}_k(G)$ uniquely. In other words, if we are given a 2-group extension $G$ of $G$ by $\text{pt}/U(1)$, and wonder which of the $\text{String}_k(G)$ it is equivalent to, it suffices to identify its restriction to $T$. This recognition principle promises to be useful for a comparison of the many equivalent constructions of the String 2-groups.

5.2 The Leech lattice
Another interesting example is given by the Leech lattice $\Lambda' = \Lambda_{\text{Leech}} \subseteq \mathbb{R}^{24}$, together with the symmetric bilinear form on $\mathbb{R}^{24}$ making $\Lambda'$ an even unimodular lattice. Then $T = \mathbb{R}^{24}/\Lambda_{\text{Leech}}$ is the Leech torus, and the group of linear isometries of $(\Lambda', I)$ is the Conway group $\text{Co}_0$. So $I$ can be interpreted as an element

$$I \in H^4(BT, \mathbb{Z})^{\text{Co}_0}.$$  

5.3 Niemeyer lattices
Similarly, we can choose $\Lambda'$ as one of the Niemeyer lattices, e.g., $A_{24}^1$ or $A_{12}^2$, and $I$ as the symmetric bilinear form on $\mathbb{R}^{24}$ making $\Lambda'$ an even unimodular lattice. Then $T = \mathbb{R}^{24}/A_{24}^1$ (respectively, $T = \mathbb{R}^{24}/A_{12}^2$). The group of linear isometries of $(\Lambda', I)$ is the Mathieu group $\text{M}_{24}$ or $\text{M}_{12}$, and we have

$$I \in H^4(BT, \mathbb{Z})^{\text{M}_{24}} \quad \text{respectively} \quad I \in H^4(BT, \mathbb{Z})^{\text{M}_{12}}.$$  

5.4 Conway, Mathieu and Weyl 2-groups
To return to our remarks in Section 4.2, the symmetries of categorical tori form 2-group extensions of Weyl and Mathieu groups as well as Conway’s group $\text{Co}_0$. The low dimensional (co)homology of the Mathieu groups was calculated in [DSE09], the generator of the cohomology group

$$H^3_{\text{gp}}(\text{M}_{24}; U(1)) \cong \mathbb{Z}/12\mathbb{Z}$$

plays an important role in Mathieu Moonshine, see [GPRV14, p.8]. The existence of a similar cocycle for the Monster, governing Monstrous Moonshine, was conjectured by Mason. Examples of Weyl 2-groups will contain extensions of the symmetric groups. Such extensions play a key role in Kapranov’s formulation of categorified supersymmetry. These are a few of the reasons to be interested in 2-group extensions of the finite isometry groups in Section 4.2.

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This is unpublished but was presented at the Zelevinsky Conference.

11
6. Other things classified by \((N', I)\)

Besides our categorical tori, the degree 4 cohomology class

\[ I \in H^4(BT; \mathbb{Z}) \]

classifies a number of important objects. In this section, we list a few of these and offer some comments on their relationship to \(T\).

6.1 From categorical groups to loop groups

\(I\) classifies a central extension \(\widetilde{\mathcal{L}} T\) of the loop group of \(T\) by \(U(1)\). If \(T\) is a maximal torus of \(G\) and \(I\) is invariant under the Weyl group, as in Section 5.1, then \(I\) classifies a central extension \(\widetilde{\mathcal{L}} G\) of the loop group of \(G\).

The relationship between loop groups and 2-groups is well studied, see for instance [BSCS07] and, more recently, [Wal13] and [Wal12].

We will work in the context of Waldorf’s transgression-regression machine. The term ‘transgression’ in this context does not refer to the map \(\tau\) mentioned above, but to a recipe for turning multiplicative bundle gerbes with connection into central extensions of loop groups. Applied to \((I, (L^J, \nabla), \alpha)\), transgression yields the central extension of \(\mathcal{L} T\) with trivial underlying principal bundle and 2-cocycle

\[
c(\varphi, \gamma) = Hol_{(L^J, \nabla)}((\varphi, \gamma)) = \exp \left(- \int_0^1 J(f(t), \dot{g}(t))dt \right),
\]

where \(Hol\) stands for holonomy, and \((f, g)\) is a lift of \((\varphi, \gamma)\) to \(t \times t\).

Remark 6.1. This definition of \(\widetilde{\mathcal{L}} T\) as transgression of a categorical torus leads to a considerable simplification of the picture in the standard literature on loop groups, see [PS86, 4.8]: the expression \([4]\) is less complicated than [PS86, (4.8.3)], it is by construction invariant under \(Diff^+(SS^1)\), and there is no need to restrict ourselves to maximal tori of simply laced groups.

Lemma 6.2. The central extension \(\widetilde{\Omega} T\) defined by the 2-cocycle \(c\) restricted to the based loops \(\Omega T\) is, up to a sign, isomorphic to that in [PS86, 4.8].

Proof. Pressley and Segal denote our \(I\) by \(\langle -, - \rangle\), their \(b\) can be taken to be our \(J\), and they write \(\Lambda\) for the cocharacter lattice \(\check{T} = \text{Hom}(U(1), T)\). We first note that \(c\) describes the correct extension of this lattice

\[ N' \cong \check{T} \subset \mathcal{L} T, \]

namely

\[ c_{N'}(m, n) = (-1)^{J(m,n)}. \]
Categorical tori

The commutators in $\tilde{\mathcal{L}}\mathcal{T}$ are

$$\tilde{\varphi} \cdot \tilde{\gamma} \cdot \tilde{\varphi}^{-1} \cdot \tilde{\gamma}^{-1} = \frac{c(\varphi, \gamma)}{c(\gamma, \varphi)} = \exp \left( - \int_0^1 J(f(t), \dot{g}(t)) \, dt + \int_0^1 J(g(t), \dot{f}(t)) \, dt \right)$$

$$= \exp \left( - \int_0^1 I(f(t), \dot{g}(t)) \, dt + [J(g(t), f(t))]_0^1 \right)$$

$$= \exp \left( \int_0^1 I(\dot{f}(t), g(t)) \, dt - [J(f(t), g(t))]_0^1 \right)$$

(5)

(integration by parts). Over the identity component of $\mathcal{L}\mathcal{T}$, our extension is completely described by its Lie-algebra cocycle,

$$\omega = - \int_0^1 I(f(t), \dot{g}(t)) \, dt$$

(from (5) and [PS86, 2.1]), which agrees with the expression for $-\omega$ in [PS86, (4.2.2)]. The adjoint action of $\varphi \in \mathcal{L}\mathcal{T}$ on $\tilde{\mathcal{L}}\mathcal{T}$ can be read off from (6), it sends $(\gamma, r) \in \tilde{\mathcal{L}}\mathcal{T}$ to

$$\left( \gamma, r + \int_0^1 I(\dot{f}(t), g(t)) \, dt - J(\Delta_f, g(0)) \right),$$

where $\Delta_f = f(1) - f(0)$. If $g \in \Omega\mathfrak{t}$ is a based loop, this agrees with the expression in [PS86, (4.3.2)] with $-I$ in the role of $I$.

Waldorf’s regression machine reconstructs a Lie 2-group extension $G$ of $G$ by $\text{pt} //U(1)$ from a central extension $\tilde{\mathcal{L}}\tilde{G}$ equipped with an extra piece of data, called fusion product. Let $\mathcal{P}_1G$ be the space of paths in based at 1 in $G$, and $\pi$ the evaluation map

$$\pi: \mathcal{P}_1G \longrightarrow G$$

$$\gamma \longmapsto \gamma(1).$$

Then the arrows in the Čech groupoid

$$\mathcal{P}_1G \leftarrow \mathcal{P}_1G \times_G \mathcal{P}_1G$$

are identified with the space $\Omega G$ of based loops in $G$, and replacing $\Omega G$ with the central extension $\tilde{\Omega}G$, one obtains the pt $//U(1)$ extension of $G$

$$\mathcal{G} := (\mathcal{P}_1G \leftarrow \tilde{\Omega}G).$$

The composition of arrows in $\tilde{\Omega}G$ is the fusion product mentioned above.

Let us study this in detail for our torus. The construction of $c$ in [4] as holonomy implies that $c$ is a fusion map, i.e.,

$$c(\varphi_1 \ast \varphi_2, \gamma_1 \ast \gamma_2) \cdot c(\varphi_2 \ast \varphi_3, \gamma_2 \ast \gamma_3) = c(\varphi_1 \ast \varphi_3, \gamma_1 \ast \gamma_3)$$

(7)

for triples $(\varphi_1, \varphi_2, \varphi_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ with identical start and end points, $\varphi_1(0) = \varphi_2(0) = \varphi_3(0)$, and $\varphi_1(1) = \varphi_2(1) = \varphi_3(1)$, and likewise for the $\gamma_i$. Therefore, we have a canonical fusion product on $\tilde{\mathcal{L}}\mathcal{T}$. To compare the regression of $\tilde{\mathcal{L}}\mathcal{T}$ to Construction [2.3] we consider the commuting diagram
Here \( \exp_{[0,x]} \) denotes the path
\[
\exp_{[0,x]}: t \mapsto \exp(tx),
\]
and the second horizontal map sends \((x, m)\) to the concatenation
\[
\gamma(x,m) := \exp_{[0,x]} \circ \exp_{[0,x+m]}.
\]
Waldorf has proved that transgression followed by regression returns the original multiplicative bundle gerbe, up to isomorphism. Indeed, we have
\[
c\left(\gamma(x,m), \gamma(y,n)\right) = \exp\left(-\int_0^1 J(tx, y) dt - \int_0^1 J(x - t(x + m), -(y + n)) dt\right)
= \exp\left(-\frac{1}{2} (J(m, y) + J(m, n) - J(x, n))\right).
\]

To compare this expression with the factor turning up in Construction \(2.1\) we note that
\[
\delta_{\text{Cech}} \left(\exp \circ \frac{J}{2}\right) ((x, m), (y, n)) = \exp \left(\frac{1}{2} (J(x, n) + J(m, y) + J(m, n))\right)
= \exp \left(\frac{1}{2} (J(x, n) + J(m, y) - J(m, n))\right)
\]
and \(\delta_{\text{simp}} \left(\exp \circ \frac{J}{2}\right) = 0\). Let \(\mathfrak{T}\) be the groupoid of Construction \(2.1\) and let \(\odot\) be the monoidal structure on \(\mathfrak{T}\) obtained like \(\bullet\), but with the factor \(\exp(-J(m, y))\) replaced by \(c(\gamma(x,m) \gamma(y,n))\). Then the map
\[
t \times t \longrightarrow t \times U(1)
\]
\[
(x, y) \mapsto \left( x + y, 0, \exp \left(\frac{1}{2} J(x, y)\right)\right)
\]
makes the identity functor a monoidal equivalence from \((\mathfrak{T}, \bullet)\) to \((\mathfrak{T}, \odot)\).
6.2 Extraspecial 2-groups
Given an even symmetric bilinear form \( I \) on \( \Lambda^\vee \), we have the integer-valued quadratic form
\[
\phi(x) = \frac{1}{2} I(x, x)
\]
on \( \Lambda^\vee \). The form \( I \) can be recovered from \( \phi \) by the identity
\[
I = \delta_{\text{simp}}(\phi),
\]
i.e.,
\[
I(m, n) = \phi(m + n) - \phi(m) - \phi(n).
\]
Let \( T[2] \) be the \( \mathbb{F}_2 \)-vectorspace of points of order 2 in \( T \). Then \( \phi \) induces a quadratic form on \( T[2] \cong \Lambda^\vee \otimes \mathbb{F}_2 \), which we will also denote
\[
\phi: T[2] \to \mathbb{F}_2.
\]
Such a \( \phi \) classifies an \textit{extraspecial 2-group} \( \hat{\mathbb{T}} \), i.e., a central extension
\[
0 \to \mathbb{F}_2 \to \hat{T}[2] \to T[2] \to 1
\]
with
\[
\hat{x} \cdot x = \phi(x) \cdot \hat{x} \cdot \hat{x}.
\]
An explicit 2-cocycle for this extension is given by \( J \otimes \mathbb{F}_2 \), for any integer-valued \( J \) on \( \Lambda^\vee \) with \( I = J + J^t \). In the situation of Section 5.2, there is a prominent subgroup of the Monster, isomorphic to
\[
\hat{T}[2] \rtimes Co_1,
\]
where \( Co_1 \) is the Conway group \( Co_1 = Co_0/\pm 1 \). This subgroup is typically the first step in the construction of the Monster, see for instance [Tit85] or [CS99].

Note that \( \hat{T}[2] \) is part of a crossed module extension
\[
0 \to \mathbb{F}_2 \to \hat{T}[2] \to T \xrightarrow{2} T \to 1.
\]
This gives rise to an extension of categorical groups
\[
\text{pt} \sslash \{ \pm 1 \} \to T_\phi \to T.
\]

6.3 Looijenga line bundles and twists of equivariant elliptic cohomology
Even symmetric bilinear forms on \( \Lambda^\vee \) also give rise to holomorphic line bundles on the moduli space
\[
\mathcal{M}_T \cong \Lambda^\vee \otimes E_\tau \cong T \times T
\]
of flat principal \( T \)-bundles on the elliptic curve \( E_\tau = \mathbb{C}/(\tau, 1) \), see [Loo77], [And00]. These \textit{Looijenga line bundles} for the coefficients of \( T \)-equivariant elliptic cohomology [Gan14] and they play an important role in the representation theory of loop groups: characters of \( \hat{\mathcal{L}}G \) are \( W \)-invariant\footnote{In this context, there is a clash of terminology: the term 2-group is used in the sense of \( p \)-group, \( p \) a prime, not in the sense of categorical group.}.
sections of the corresponding Looijenga line bundle. This phenomenon is well understood: there is a construction of the (inverse) Looijenga line bundle as a space of twisted conjugacy classes in $\tilde{\mathcal{L}}_T$, see for instance [BG96] or [EF94].

Not unexpectedly, it turns out that the Looijenga line bundle also plays a key role in the character theory of representation of categorical tori. This leads to a theta function formalism for 2-characters, resembling that of loop group characters, but developed directly from the categorical picture, without any mention of loops. We will come back to this topic at a different occasion.

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Nora Ganter nганter@unimelb.edu.au
Department of Mathematics and Statistics, The University of Melbourne, Parkville VIC 3010, Australia