On equivariant oriented cohomology of Bott-Samelson varieties

Hao Li and Changlong Zhong

Abstract. For any Bott-Samelson resolution of the flag variety, and any torus equivariant oriented cohomology, we compute the restriction formula of certain basis of equivariant oriented cohomology of Bott-Samelson variety determined by the projective bundle formula. As an application, we show that the equivariant oriented cohomology of Bott-Samelson variety embeds into the equivariant oriented cohomology of $T$-fixed points, and the image can be characterized by using the Goresky-Kottwitz-MacPherson (GKM) description. Furthermore, we compute the push-forward of the basis onto equivariant oriented cohomology of flag variety, and their restriction formula.

Contents

1. Introduction 1443
2. Equivariant oriented cohomology theory 1445
3. Bott-Samelson varieties 1448
4. Restriction to $T$-fixed points 1452
5. Push-forward to cohomology of flag varieties 1459
References 1463

1. Introduction

Let $G/B$ be a flag variety. For each $w$ in the Weyl group $W$, and a reduced decomposition $w = s_i \cdots s_i$, one defines the variety (see Definition 3.1)

$$\hat{X}_w = P_{i_1} \times B P_{i_2} \times B \cdots \times B P_{i_l} / B.$$ 

Here $P_{i_j}$ is the minimal parabolic subgroup corresponding to the simple root $\alpha_{i_j}$. Multiplication of all the coordinates defines a canonical map $q_t : \hat{X}_w \to G/B$, which is proper and birational over the Schubert variety $X(w)$ of $w$. This is called a Bott-Samelson resolution of $X(w)$. These resolutions play an important role in Schubert calculus and representation theory.
Let $T$ be a split maximal torus in Borel subgroup $B$ of $G$. One has a natural $T$-action on the flag variety. We are interested in $h_T(X_{I_w})$, where $h_T$ is an (equivariant) oriented cohomology theory in the sense of Levine-Morel. Examples of $h$ include the Chow group (singular cohomology) and K-theory. For any $h$, it is proved in [CZZ12, CZZ19, CZZ14] that, after fixing a reduced decomposition $I_w$ for each $w \in W$, the push-forward $(q_{I_w})_*(1)$ in $h_T(G/B)$ of the fundamental class defines a basis of $h_T(G/B)$ over the base ring $h_T(pt)$. This enables the authors of loc. it. to construct the algebraic replacement of $h_T(G/B)$, and provides a standard setting for generalized Schubert calculus. For further study on equivariant oriented cohomology of $T$-varieties following this method, please refer to [DZ20, GZ20, LZZ16, CNZ19, Z20].

Let us consider $h_T(X_I)$ for a general sequence $I = (i_1, ..., i_l)$. The set $X_I^T$ of $T$-fixed points of $X_I$ is in bijection with the power set of $[l] = \{1, 2, ..., l\}$. Denote by $j : X_I^T \to X_I$ the canonical embedding. Our main result is the following:

**Theorem 1.1.** (Corollary 4.4) For any sequence $I$, the pull-back to $T$-fixed points $j^* : h_T(X_I) \to h_T(X_I^T)$ is injective.

Furthermore, we show that elements in the image of $j^*$ satisfy the Goresky-Kottwitz-MacPherson (GKM) description (see Theorem 4.5). Indeed, in the case where the sequence $I = (i_1, ..., i_l)$ consists of distinct $i_j$'s, we prove that the GKM description uniquely characterizes the image (Theorem 4.6).

Let us mention the idea of the proof briefly. Since $X_I$ is constructed as a tower of $\mathbb{P}^1$-bundles, there are canonically defined algebra generators $\eta_j \in h_T(X_I)$ corresponding to each parabolic subgroup $P_{i_j}$ in $X_I$. Each $\eta_j$ satisfies certain quadratic relation. Therefore, for each subset $L$ of $[l]$, denoting by $\eta_L$ the product of $\eta_j$ with $j \in L$, then $\{\eta_L \mid L \in [l]\}$ forms a basis of $h_T(X_I)$.

We compute the restriction $j^*(\eta_L)$ explicitly (Theorem 4.3). The computation uses the characteristic map $c : h_T(pt) \to h_T(X_I)$ induced by the map sending a character $\lambda$ of $T$ to the first Chern class of the associated line bundle over $X_I$. We then use the explicit formula of $j^*(\eta_L)$ to prove Theorem 1.1, and use the GKM description to characterize the image of $j^*$.

As another application of the computation of $j^*(\eta_L)$, we also compute the push-forward of $\eta_L$ via the canonical map $q_I : X_I \to G/B$. We show that the push-forward $(q_I)_*(\eta_L)$ coincides with the Bott-Samelson class corresponding to the sequence $I \setminus L$.

For future applications, one would apply the restriction formula (Theorem 4.3) and the push-forward formula (Theorem 5.4) in the study of motivic Chern (mC) classes in K-theory. MC classes are certain K-theory classes associated to constructible subsets of $T$-varieties. For details, please refer [AMSS17, RTV15, RTV17]. They are closely related with the K-theoretic stable basis of Springer resolutions, defined by Maulik-Okounkov [MO12, O15] and studied in [SZZ17, SZZ19]. Indeed, Mihalcea has some recent work on the relationship between push-forward of MC classes of Bott-Samelson varieties and the Kazdan-Lusztig
basis of Hecke algebra. The authors hope to apply the computation of this paper to understanding this relationship.

The paper is organized as follows: In Section 2, we recall necessary notions of equivariant oriented cohomology theory, formal group algebra, and the characteristic map $c$. In Section 3, we recall some basic facts about Bott-Samelson varieties. In Section 4, we compute the restriction formula (Theorem 4.3) which was used to prove the injectivity of the pull-back map $j^*$ and the GKM description (Theorem 4.5). In Section 5, we compute the push-forward of the basis $\{\eta_i\}$ onto $h_T(G/B)$.

Acknowledgments: The second author would like to thank Leonardo Mihalcea and Rebecca Goldin for helpful conversations.

2. Equivariant oriented cohomology theory

In this section, we define some notation, and collect some basic notions and facts about equivariant oriented cohomology theory.

Let $G$ be a split semisimple linear algebraic group over a field $k$, with rank $n$. Let $T$ be a split maximal torus of $G$ and $B \subset G$ be a Borel subgroup. Let $\Sigma$ be the set of roots of $G$, and $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple roots corresponding to $B$. Let $P_i$ be the minimal parabolic subgroup corresponding to the simple root $\alpha_i$. The Weyl group $W$ of $G$ is generated by $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ where $s_{\alpha_i}$ is the reflection corresponding to $\alpha_i$. Note that $W$ can be identified with $N_G(T)/T$. Sometimes we will understand $s_i = s_{\alpha_i}$ as a lifting of an element in $G$. We denote the group of characters of $T$ by $\Lambda$. For each positive integer $l$, denote $[l] = \{1, 2, \ldots, l\}$.

Let $F$ be a formal group law over the commutative ring $R$. Examples include the additive formal group law $F_a = x + y$ over $\mathbb{Z}$, and the multiplicative formal group law $F = x + y - \beta xy$ over $\mathbb{Z}[\beta, \beta^{-1}]$.

**Definition 2.1.** Let $R[\Lambda] = R[x_\lambda | \lambda \in \Lambda]$ be the power series ring. Let $J_F$ be the closure of the ideal generated by $x_0$ and $x_{\lambda+\mu} - F(x_\lambda, x_\mu)$, $\lambda, \mu \in \Lambda$. We define the formal group algebra $R[\Lambda]/J_F$ to be the quotient $R[\Lambda]/J_F = R[x_\lambda]/J_F$.

It is proved in [CPZ13, Corollary 2.13] that $R[\Lambda]/F$ is non-canonically isomorphic to the formal power series ring with $n$ variables. For simplicity, we denote $S = R[\Lambda]/F$. Note that by definition, $x_{-\lambda}$ is the formal inverse of $x_\lambda$, that is, $F(x_\lambda, x_{-\lambda}) = 0$. Since any formal group law $F$ is always of the form $F(x, y) = x + y + a_{11}xy + \text{higher order terms}$, $a_{11} \in R$, so it is not difficult to see that $x_{-\lambda} = -x_\lambda + (x_\lambda)^2 f(x_\lambda)$ for some $f(t) \in R[[t]]$. Therefore, $x_{-\lambda}$ is an invertible element in $S$.

**Example 2.2.** (1) Let $F_a$ be an additive formal group law, then we have a ring isomorphism $R[\Lambda]/F_a \cong S_G(\Lambda)^\wedge$, $x_\lambda \mapsto \lambda$. 

where \( S_F(\Lambda) \) is the symmetric algebra of \( \Lambda \) and the completion is done at the augmentation ideal.

(2) Let \( R[\Lambda] \) be the group algebra \( \{ \sum_j a_j \epsilon^j | a_j \in R, \lambda_j \in \Lambda \} \). Then we have isomorphism

\[
R[\Lambda]_{F,m} \cong R[\Lambda]^\wedge, \quad \lambda_j \mapsto \beta^{-1}(1 - \epsilon^j),
\]

where the completion \(^\wedge\) is done at the augmentation ideal.

Throughout this paper, we assume that the root datum of \( G \) together with the formal group law \( F \) satisfy the regularity condition of [CZZ12, Definition 4.4]. For example, this is satisfied if 2 is regular in \( R \). Please consult loc.it for more details. In particular, \( x_\alpha \) is regular in \( S \), for any root \( \alpha \) of \( G \). The Weyl group action on \( \Lambda \) induces an action of \( \mathbb{A} \) on \( R[\Lambda]_{F,m} \) by \( \Delta \).

**Lemma 2.3.** [CPZ13, Corollary 3.4] For any \( v, w \in W \), any root \( \alpha \) of \( G \) and \( p \in S \), we have

\[
\frac{vs_\alpha w(p) - vw(p)}{x_{\alpha(\alpha)}} \in S.
\]

**Proof.** According to [CPZ13, Corollary 3.4], we know that \( s_\alpha w(p) - w(p) \) is uniquely divisible by \( x_\alpha \). In other word,

\[
\frac{vs_\alpha w(p) - w(p)}{x_\alpha} \in S.
\]

Then

\[
v(\frac{vs_\alpha w(p) - w(p)}{x_\alpha}) = \frac{vs_\alpha w(p) - vw(p)}{x_{\alpha(\alpha)}} \in S.
\]

\( \square \)

In particular, taking \( w = v = e \), we see that \( x_\alpha |(p - s_\alpha(p)) \). We can then define the Demazure operator \( \Delta_\alpha : S \to S \) by

\[
\Delta_\alpha(p) = \frac{p - s_\alpha(p)}{x_\alpha}.
\]   \hspace{1cm} (1)

**Remark 2.4.** By direct calculation, we have the following formulas: for \( p, q \in S \),

\[
\Delta_\alpha(p) = -\Delta_{-\alpha}(p) \quad \text{(2)}
\]

\[
\Delta_\alpha(pq) = \Delta_\alpha(p)q + p\Delta_\alpha(q) - \Delta_\alpha(p)\Delta_\alpha(q)x_\alpha. \quad \text{(3)}
\]

We follow [CZZ14, §2] on the assumption of equivariant oriented cohomology, however, we only consider the case when the group is fixed to be the torus \( T \). Roughly speaking, it is an additive contravariant functor \( h_T \) from the category of smooth quasi-projective \( T \)-varieties to the category of commutative rings with units, satisfying the following axioms: existence of push-forwards for projective morphisms, existence of total equivariant characteristic class for
$T$-equivariant bundle, Quillen’s formula, etc. [CZZ14, §2]. Moreover, there exists a formal group law $F$ over $R = h_T(pt)$ such that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are locally free sheaves of rank one, then

\[
c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = F(c_1(\mathcal{L}_1), c_1(\mathcal{L}_2)).
\]

It is proved in [CZZ14, Theorem 3.3] that

\[
S = R[\Lambda]_F \cong h_T(pt), \quad x_\lambda \mapsto c_1(\mathcal{L}_\lambda),
\]

where $\mathcal{L}_\lambda$ is the associated line bundle. As an immediate consequence, we see that if the variety $X$ is finite set of points of the form $\text{Spec}(k)$ (with trivial $T$-action), then

\[
h_T(X) = F(X; S),
\]

where the latter is the set of all maps from $X(k)$ to $S$. It has a $S$-basis $f_x, x \in X$, and is a ring with product defined by

\[
f_x \cdot f_y = \delta_{x,y} f_x, \quad \text{and unit } \sum_{x \in X} f_x.
\]

By functoriality, if $p : X \to Y$ is a $T$-equivariant map between two finite discrete sets of points on which $T$ acts trivially, then

\[
p^*(f_y) = \sum_{x \in f^{-1}(y)} f_x, \quad p_*(f_x) = f_{p(x)}.
\]

We recall the definition of the characteristic map. Let $X$ be a $T$-variety on which $B$ acts on the right, and the $T$ and $B$ actions commute. Moreover, suppose the quotient $X/B$ exists and $X \to X/B$ is a $T$-equivariant principal bundle. Following [CPZ13, §10.2], we can define a ring homomorphism

\[
e : S = h_T(pt) \to h_T(X/B), \quad x_\lambda \mapsto c_1(\mathcal{L}_\lambda).
\]

It is called the characteristic map.

Let $\alpha$ be a simple root with corresponding minimal parabolic subgroup $P_\alpha$. Consider the fiber product $X' = X \times^B P_\alpha$, then $X'$ is a $T$-equivariant principal $P_\alpha$-bundle over $X/B$. Denote $p : X'/B \to X/B$, and there is a zero section

\[
\sigma : X/B \to X'/B, x \mapsto (x, 1).
\]

As in [CPZ13, §10.5], we have

\[
h_T(X'/B) \cong h_T(X/B)[\xi]/(\xi^2 - y \xi), \quad \xi = \sigma_*(1), \quad y = p^* \sigma^* \xi.
\]

The following properties can be proved similarly as their non-equivariant versions in [CPZ13, §10].

**Lemma 2.5.** Denote $c : S \to h_T(X/B)$ and $c' : S \to h_T(X'/B)$. For each $\lambda \in \Lambda$, denote the associated line bundles on $X/B$ and $X'/B$ by $\mathcal{L}_\lambda$ and $\mathcal{L}'_\lambda$, respectively.

1. We have $\sigma^* \xi = c_1(\mathcal{L}_-\alpha) = c(x_-\alpha)$.
2. $y = p^* \sigma^* \xi = p^* c(x_-\alpha)$.
3. For any $u \in S$, we have

\[
\sigma^* c'(u) = c(u), \quad c'(u) = p^* c(s_\alpha(u)) + p^* c(\Delta_-\alpha(u)) \cdot \xi.
\]
Note that in [CPZ13, §10], the projective bundle theorem plays a key role. The equivariant projective bundle theorem for equivariant $\mathbb{P}^1$-bundle is proved in [CZZ14, Lemma 4.6], which then can be used to prove Lemma 2.5.

**Lemma 2.6.** If $X = B, X' = P_\alpha$, there are two $T$-fixed points in $P_\alpha/B$, indexed by $e, s_\alpha \in W$, whose embeddings are denoted by $\sigma_e, \sigma_\alpha : \text{pt} \to P_\alpha/B$. Then $\sigma_e^* c'(u) = u, \sigma_\alpha^* c'(u) = s_\alpha(u)$.

**Proof.** We have $c(u) = u, p\sigma_e = \text{id}, p\sigma_\alpha = \text{id}$, and $\sigma_e$ coincides with $\sigma$ in (5), so the first identity of Lemma 2.5.(3) implies $\sigma_e^* c'(u) = c(u)$. On the other hand, applying $\sigma_\alpha^* \sigma_e$ on the the second identity of Lemma 2.5.(3), and using $\sigma_e^* (\xi) = \sigma_e^* (\sigma_e)$, we get that $\sigma_\alpha^* c'(u) = s_\alpha(u)$. $\square$

### 3. Bott-Samelson varieties

In this section, we collect some facts about Bott-Samelson varieties.

**Definition 3.1.** For any sequence $I = (i_1, i_2, ..., i_l)$ with $1 \leq i_j \leq n$, we define the variety $\hat{X}_I$ to be

$$\hat{X}_I = P_{i_1} \times B P_{i_2} \times B ... \times B P_{i_l}/B,$$

which is the orbit space in $P_{i_1} \times P_{i_2} \times ... \times P_{i_l}$ under $B^I$-action defined by

$$(g_1, \ldots, g_l)(b_1, \ldots, b_l) = (g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{i_l-1}^{-1} g_i b_i),$$

where $b_i \in B$ and $g_j \in P_{i_j}$. Here the right $B$-action is given by right multiplication on the second coordinate. If $I = \emptyset$, then we set $\hat{X}_I = \text{pt}$. The variety $\hat{X}_I$ is called the Bott-Samelson variety corresponding to $I$. It has an obvious $T$-action by left multiplication on the first coordinate. We denote $T$-fixed points on $\hat{X}_I$ by $e_I$.

Since $P_{i_l}/B \cong \mathbb{P}^1$, so we have a sequence of $\mathbb{P}^1$-bundles:

$$\cdots \bar{x}_{i_{l-1}} \sigma_{i_{l-1}} \hat{X}_{(i_1, i_2, ..., i_{l-1})} \bar{x}_{i_{l-2}} \cdots \bar{x}_{i_1} \hat{X}_{i_1} \xrightarrow{\sigma_{i_1}} \text{pt},$$

where $\sigma_i, 1 \leq i \leq l$ are the zero sections. Multiplication of all factors of $\hat{X}_I$ induces a map

$$q_I : \hat{X}_I \to G/B.$$

Denote by $\pi_I : G/B \to G/P_{i_l}$ the canonical map, and denote $I' = (i_1, ..., i_{l-1})$. We then have the following transverse Cartesian diagram:

$$\cdots \bar{x}_{i_{l-1}} \sigma_{i_{l-1}} \hat{X}_{(i_1, i_2, ..., i_{l-1})} \bar{x}_{i_{l-2}} \cdots \bar{x}_{i_1} \hat{X}_{i_1} \xrightarrow{\sigma_{i_1}} \text{pt}$$

$$\xrightarrow{q_I} G/B \xrightarrow{\pi_I} G/P_{i_l}.$$

So we have the base-change formula

$$(q_I)_* p^* = \pi_I^* \sigma_{i_l}(q_I)_*.$$
The operator
\[ \pi^*_{\alpha_i}(\pi_{\alpha_i})_* : h_T(G/B) \to h_T(G/B) \]
is called the push-pull operator.

Denote by \( c_i : S \to h_T(\hat{X}_t) \) the characteristic map. The following proposition describes the \( R \)-algebra structure of equivariant oriented cohomology of Bott-Samelson varieties.

**Proposition 3.2.** [CPZ13, §11.3] We have the following presentation
\[ h_T(\hat{X}_t) \cong h_T(pt)[\eta_1, \eta_2, \ldots, \eta_l]/\langle \eta_j^2 - y_j \eta_j | j = 1, \ldots, l \rangle, \]
where
\[ y_j = p^*_j c_{(i_1, \ldots, i_{j-1})}(X_{-\alpha_{i_j}}), \quad \eta_j = p^*_j (\sigma_j)_*(1), \]
with \( p^*_j \) the pull-back from \( h_T(X_{(i_1, \ldots, i_j)}) \) to \( h_T(\hat{X}_t) \).

For ordinary oriented cohomology, this theorem is proved in [CPZ13]. The idea of the proof is to apply the projective bundle formula to the sequence of \( \mathbb{P}^1 \)-bundle (7). One can check that all the arguments hold in the equivariant setting, which can be used to prove Proposition 3.2.

For each subset \( L \subseteq [l] \), define
\[ \eta_L = \prod_{i \in L} \eta_i \in h_T(\hat{X}_t). \]

Since in Proposition 3.2, the \( y_j \) does not belong to the coefficient ring \( h_T(pt) \), the presentation of \( h_T(X) \) is not satisfactory. To get a polynomial presentation of it, we follow the idea in [CPZ13, Theorem 11.4].

**Lemma 3.3.** For any sequence \( l = (i_1, \ldots, i_l) \), we have
\[ c_l(u) = \sum_{L \subseteq [l]} \theta_{L,L}(u) \eta_L, \quad u \in S, \]
where \( \theta_{L,L} = \theta_1 \cdots \theta_l \) with \( \theta_j = \begin{cases} \Delta_{-\alpha_{i_j}}, & \text{if } j \in L, \\ s_{i_j}, & \text{otherwise.} \end{cases} \)

**Proof.** We prove it by induction on \( l \). If \( l = 1 \), from Lemma 2.5, we have
\[ c_{i_1}(u) = p^* c_{s_{i_1}}(u) + p^* c_{(\Delta_{-\alpha_{i_1}})}(u) \cdot \eta_1. \]
Note that the characteristic map \( c_{s_{i_1}} : S \to h_T(pt) \) is the identity map. So it holds.

Now assume the conclusion holds for \( l' := (i_1, \ldots, i_{l-1}) \). Denote the canonical projection from \( \hat{X}_t \) to \( \hat{X}_{l'} \) by \( p \). By Lemma 2.5 we have
\[ c_l(u) = p^* c_{l'}(s_{i_1}(u)) + p^* c_{l'}(\Delta_{-\alpha_{i_1}}(u)) \cdot \eta_l = \sum_{L \subseteq [l]} \theta_{L-1,L}(s_{i_1}(u)) \eta_L + \sum_{L \subseteq [l-1]} \theta_{L-1,L}(\Delta_{-\alpha_{i_1}}(u)) \eta_L \cdot \eta_l = \sum_{L \subseteq [l]} \theta_{L,L}(u) \eta_L. \]
**Proposition 3.4.** [CPZ13, Theorem 11.4] The ring $h_T(\hat{X}_I)$ is a quotient of the polynomial ring $S[\eta_1, \eta_2, \ldots, \eta_l]$ modulo the relations

$$\eta_j^2 = \sum_{L \subseteq [j-1]} \theta_{j-1,L}(x_{-\alpha_j}) \eta_L \eta_j, \quad j \in [l].$$

**Proof.** Denote $K = (i_1, \ldots, i_{j-1})$ and $p : \hat{X}_I \to \hat{X}_K$. By definition of $y_j$ and Lemma 3.3, we have

$$y_j = p^*c_K(x_{\alpha_j}) = p^* \left( \sum_{L \subseteq [j-1]} \theta_{j-1,L}(x_{-\alpha_j}) \eta_L \right) = \sum_{L \subseteq [j-1]} \theta_{j-1,L}(x_{-\alpha_j}) \eta_L.$$

The statement then follows from the fact that $\eta_j^2 = y_j \eta_j$. \hfill \Box

**Corollary 3.5.** The $S$-module $h_T(\hat{X}_I)$ is free with basis $\{\eta_L | L \in \mathcal{P}_l\}$.

**Example 3.6.** For $SL(4)$ whose simple roots are $\alpha_1, \alpha_2, \alpha_3$, let us consider Bott-Salmeleon $\hat{X}_I = P_{\alpha_1} \times B P_{\alpha_2} \times B P_{\alpha_3} / B$. Then $h_T(\hat{X}_I)$ is a polynomial algebra generated by $\eta_1, \eta_2, \eta_3$ with following quotient relations:

- $\eta_1 = x_{-\alpha_3} \eta_1$
- $\eta_2 = x_{-\alpha_1 - \alpha_2} \eta_1 + \frac{x_{-\alpha_1} - x_{\alpha_1 - \alpha_2}}{x_{-\alpha_1}} \eta_1 \eta_2$
- $\eta_3 = x_{\alpha_1 - \alpha_2 - \alpha_3} \eta_1 + \frac{x_{\alpha_3 - \alpha_2} - x_{\alpha_2 - \alpha_3}}{x_{-\alpha_1}} \eta_1 \eta_3 + \frac{x_{\alpha_3} - x_{\alpha_1 + \alpha_2 - \alpha_3}}{x_{-\alpha_1 - \alpha_2}} \eta_2 \eta_3$
- $\eta_4 = \frac{x_{\alpha_3 - \alpha_2 - \alpha_3} - x_{\alpha_3 - \alpha_1 - \alpha_2}}{x_{\alpha_2} - x_{-\alpha_1 - \alpha_2}} \eta_1 \eta_2 \eta_3$.

Let us consider some geometry information on $\hat{X}$, and its $T$-fixed points. We fix some notations first. For any $L \subseteq [l]$, define

$$(\hat{X}_I)_L = \{[g_1, g_2, \ldots, g_l] \in \hat{X}_I | g_j \in B \text{ if } j \notin L, \text{ and } g_i \notin B \text{ if } j \in L \} \subseteq \hat{X}_I,$$

and

$$v^L_I = \prod_{k \in L \setminus [j]} s_{i_k}, \quad v^L_l = \prod_{k \in L} s_{i_k}.$$

The following lemma will be used in the proof of Theorem 4.6.

**Lemma 3.7.** Let $I = (i_1, \ldots, i_l)$ be a sequence such that $i_j$ are all distinct. Let $L \subseteq [l]$, then $v^L_{j-1}(x_{\alpha_j})$, $j \in L^c$ are all distinct. In particular, $v^L_{j-1}(x_{-\alpha_j})$, $j \in L^c$ are all distinct.

**Proof.** Suppose $j_1, j_2 \in L^c$ and $j_1 < j_2$. Then $L \cap [j_1] \subseteq L \cap [j_2]$. There are two cases.

- **Case 1:** $L \cap [j_2] = L \cap [j_2]$. Then

$$v^L_{j-1}(x_{\alpha_{i_{j_1}}}) = (\prod_{k \in L \setminus [j_1]} s_{i_k})(x_{\alpha_{i_{j_1}}}).$$
They are not equal since \( \alpha_{i_j} \neq \alpha_{i_j'} \).

Case 2. \( L \cap [j_1] \subseteq L \cap [j_2] \). Denote \( M = (L \cap [j_2]) \setminus (L \cap [j_1]) \). Then

\[
\begin{align*}
\nu_{j_1-1}^L(\alpha_{i_{j_1}}) &= \left( \prod_{k \in L \cap [j_1]} s_k \right)(\alpha_{i_{j_1}}), \\
\nu_{j_2-1}^L(\alpha_{i_{j_2}}) &= \left( \prod_{k \in L \cap [j_1]} s_k \right)(\alpha_{i_{j_2}}).
\end{align*}
\]

By definition of the Weyl group action,

\[
\left( \prod_{k' \in M} s_{k'} \right)(\alpha_{i_{j_2}}) = \alpha_{i_{j_2}} + \sum_{k' \in M} c_{k'} \alpha_{i_{k'}} \quad c_{k'} \in \mathbb{Z},
\]

which is different from \( \alpha_{i_{j_1}} \), since the set \( \{\alpha_{i_{j_1}}, \alpha_{i_{j_2}}\} \cup \{\pm \alpha_{i_{k'}} | k' \in M\} \) is linearly independent. Thus \( \nu_{j_1-1}^L(\alpha_{i_{j_1}}) \) and \( \nu_{j_2-1}^L(\alpha_{i_{j_2}}) \) are not equal to each other. \( \square \)

The following lemma recalled from [W04, Proposition 2.6] provides some geometric information on the Bott-Samelson variety, which is useful for our computation.

**Lemma 3.8.**

1. The set \( \bar{X}_T^T \) of \( T \)-fixed points in \( X_T \), consists of \( 2^l \) points

\[
\{g_1, g_2, \ldots, g_l\}
\]

where \( g_j \in \{e, s_i\} \). Here we think of \( s_i \) as in \( W \cong N_G(T)/T \) and pick a preimage for \( s_i \) in \( N_G(T) \subset G \). Consequently, we have bijection of sets from the power set \( P_l := \mathcal{P}([l]) \) to \( \bar{X}_T^T \),

\[
L \mapsto \text{pt}_L := [g_1, \ldots, g_l], \quad g_j = \begin{cases} s_i, & \text{if } j \in L, \\ e, & \text{if } j \not\in L. \end{cases}
\]

2. The set \( (\bar{X}_T)_L \) is \( T \)-stable, contains the fixed point \( \text{pt}_L \), and is isomorphic to the affine space of dimension \( |L| \). The variety \( \bar{X}_T \) has a decomposition \( \prod_{L \in \mathcal{P}_l} (\bar{X}_T)_L \).

3. Suppose \( L, L' \subset [l] \). then \( \text{pt}_L \in (\bar{X}_T)_{L'} \), if and only if \( L \subset L' \). The weights of the \( T \)-action on the tangent space of \( (\bar{X}_T)_{L'} \) at \( \text{pt}_L \) are

\[
\{-v_{j'}^L(\alpha_j) | j \in L' \}.
\]

**Example 3.9.** For the \( A_2 \)-case, consider \( \bar{X}_{(1, 2)} = P_1 \times B^* P_2 / B \). There are four \( T \)-fixed points, denoted by \( \{0, 0, 1, 1, 11\} \), corresponding to

\[
\{[e, e], [e, s_2], [s_1, e], [s_1, s_2]\},
\]

or

\[
\emptyset, \{2\}, \{1\}, \{1, 2\}.
\]
as subsets of [2]. The weights of the tangent spaces of $\tilde{X}_{(1,2)}$ at the four points are:

- $00 : -\alpha_1, -\alpha_2$
- $01 : -\alpha_1, \alpha_2$
- $10 : \alpha_1, -\alpha_1 - \alpha_2$
- $11 : \alpha_1, \alpha_1 + \alpha_2$

We denote the set of functions on $\mathcal{E}_I$ with values in $S$ by $F(\mathcal{E}_I; S)$. It is a free $S$-module with basis $f_L, L \in \mathcal{E}_I$ defined by $f_L(L') = \delta_{L,L'}$, and have a ring structure given by

$$f_L \cdot f_{L'} = \delta_{L,L'} f_L.$$ 

Moreover, we have

$$h_I(\mathcal{E}_I) \cong F(\mathcal{E}_I; S),$$

where the total Chern class of the tangent space at the fixed point $pt_L$, corresponds to the basis element $f_L$ up to a scalar.

Let $j^I : \tilde{X}_I \to \tilde{X}_I$ be the embedding of fixed points. For each $L \subset [l]$, denote by $j_{L}^I$ the embedding of $pt_L$ into $\tilde{X}_I$. Sometimes we will drop the superscript $I$ for simplicity. Then

$$j^*(f) = \sum_{L \subset [l]} j_{L}^*(f) f_L, \quad f \in h_I(\tilde{X}_I).$$

Denote

$$x_{I,L} = \prod_{1 \leq j \leq l} v_j^I(x_{-\alpha_j}). \quad (9)$$

We have

**Lemma 3.10.** For any $L \subset [l]$, we have $j^* j_{L}^*(f_L) = x_{I,L} f_L$.

**Proof.** This follows easily from [CZZ14, §2.A8] and Lemma 3.8 concerning the weights of the tangent space of $\tilde{X}_I$ at the point $L$. \hfill $\square$

**Example 3.11.** Following Example 3.9, with $I = (\alpha_1, \alpha_2)$, we have

$$x_{I,00} = x_{-\alpha_1} x_{-\alpha_2}, \quad x_{I,10} = x_{\alpha_1} x_{-\alpha_1 - \alpha_2},$$
$$x_{I,01} = x_{-\alpha_1} x_{\alpha_2}, \quad x_{I,11} = x_{\alpha_1} x_{\alpha_1 + \alpha_2}.$$ 

4. **Restriction to T-fixed points**

In this section, we compute the restriction formula of the $\eta_I$ basis. We first compute the restriction formula of the image of the characteristic map.

**Lemma 4.1.** Let $L$ be a sequence of length $l$, and $c_I : S \to \tilde{X}_I$ be the characteristic map, then

$$j^* c_I(u) = \sum_{L \subset [l]} v_j^I(u) f_L.$$
Proof. We prove it by induction on the length $l$ of $I$. If $I = (i_1)$, then it follows from Lemma 2.6.

Now assume it holds for all sequences of length $\leq l - 1$, and assume $I = (i_1, \ldots, i_l)$. Denote $I' = (i_1, \ldots, i_{l-1})$ and $\sigma : \hat{X}_{I'} \to \hat{X}_I$ the zero section. By induction assumption, for each $L' \subset [l - 1]$, we have

$$(j'_{L'})^*c_{I'}(u) = v'_{L-1}(u). \quad (10)$$

Concerning $L \subset [l]$, we have two cases:

Case 1: $l \in L$. In this case, $pt \notin \sigma(\hat{X}_{I'})$, so

$$(j'_L)^* \circ \sigma_* = 0. \quad (11)$$

Moreover, we have the following commutative diagram

$$
\begin{array}{ccc}
pt & \xrightarrow{j'_L} & \hat{X}_I \\
\downarrow & & \downarrow p \\
\hat{X}_{I'} & \xrightarrow{j'_{L'}} & \hat{X}_{I'},
\end{array}
$$

that is, $p \circ j'_L = j'_{L'}$, so

$$(j'_L)^* \circ p^* = (j'_{L'})^*. \quad (12)$$

Denote $\xi = \sigma_* (1)$, then by Lemma 2.5, we have

$$(j'_L)^* c_I (u) = (j'_L)^*[p^* c_{I'}(s_i(u)) + p^* c_{I'}(\Delta_{-d_i})(u) \cdot \xi] = (j'_L)^* c_{I'}(s_i(u)) + (j'_L)^* p^* c_{I'}(\Delta_{-d_i})(u) \cdot (j'_L)^*(\sigma_* (1)) \equiv (j'_{L'})^* c_{I'}(s_i(u)) \equiv v'_{L}'(u).
$$

Here the identity $\equiv$ follows from (11) and (12), and $\equiv$ follows from (10).

Case 2: $l \notin L$. In this case, we can view $L \subset [l - 1]$, so we have commutative diagrams:

$$
\begin{array}{ccc}
pt & \xrightarrow{j'_L} & \hat{X}_I \\
\downarrow & & \downarrow p \\
\hat{X}_{I'} & \xrightarrow{j'_{L'}} & \hat{X}_{I'},
\end{array}, \quad
\begin{array}{ccc}
pt & \xrightarrow{j'_L} & \hat{X}_I \\
\downarrow & & \downarrow \sigma \\
\hat{X}_{I'} & \xrightarrow{j'_{L'}} & \hat{X}_{I'},
\end{array}
$$

so $p \circ j'_L = j'_{L'}$ and $\sigma \circ j'_{L'} = j'_L$. The latter implies that

$$(j'_L)^* \sigma_* (1) = (j'_L)^* \sigma^* \sigma_* (1) = (j'_L)^* c_{I'}(x_{-d_i}). \quad (13)$$

Therefore,

$$(j'_L)^* (c_I (u)) = (j'_L)^*[p^* c_{I'}(s_i(u)) + p^* c_{I'}(\Delta_{-d_i})(u) \cdot \xi].$$
Example 3.9. There are four torus-fixed points, denoted by \( \sigma_i \) for \( i = 1, 2 \).

Example 4.2. The proof is finished.

Similarly, denote \( \lambda \) has weight \( 0 \) where the last identity follows from the fact that the tangent space of \( P_1 \) at 0 has weight \(-\alpha_1 \). Hence,

\[
\eta_i(p_2) = p_2^* \mathcal{H}^s(\sigma_i, 1) = (p_2^* \mathcal{H}^s(\sigma_i, 1)) = (p_2^* \mathcal{H}^s(\sigma_i, 1) \mathcal{H}^s(f_0) = \mathcal{H}^s(f_0 + f_{11} + f_{10} + f_{01}).
\]

Denote \( \mathcal{C}_1 : S \to h_t(P_1 / B) \).

First of all, from the definition of \( p_2^* \) and (4), we know

\[
(p_2^* \mathcal{H}^s(f_0) = f_0 + f_{10} + f_{11} + f_{01}.
\]

Moreover, since \( j_i^0 \) coincides with \( \sigma_1 \) and \( j_i^1(\mathcal{C}_1) \neq \sigma_i(\mathcal{C}_1) \), so \( j_i^1 \mathcal{H}^s(\sigma_i, 1) = \mathcal{H}^s(\sigma_i, 1) \),\( j_i^1 \mathcal{H}^s(\sigma_i, 1) = \mathcal{H}^s(\sigma_i, 1) \mathcal{H}^s(f_0) = \mathcal{H}^s(f_0 + f_{11} + f_{10} + f_{01}).
\]

where the last identity follows from the fact that the tangent space of \( P_1 / B \) at 0 has weight \(-\alpha_1 \). Hence,

\[
(j_i^1)^* \eta_i = j_i^1 \mathcal{H}^s(\sigma_i, 1) = p_2^* \mathcal{H}^s(\sigma_i, 1) = (p_2^* \mathcal{H}^s(\sigma_i, 1)) = (p_2^* \mathcal{H}^s(\sigma_i, 1) \mathcal{H}^s(f_0) = \mathcal{H}^s(f_0 + f_{11} + f_{10} + f_{01}).
\]

Before computing the restriction formula of \( \eta_i \), we first consider an example.

**Example 4.2.** Consider the case of \( A_2 \). Let \( \{ \alpha_1, \alpha_2 \} \) be the set of simple roots. We consider the Bott-Samelson variety \( X_I = P_1 \times B P_2 / \mathcal{P}_2 \) for \( I = (1, 2) \). Following Example 3.9, there are four torus-fixed points, denoted by \( \mathcal{P}_2 = \{ 00, 01, 10, 11 \} \).

Similarly, denote \( (P_1 / B)^t \) by \( \mathcal{P}_1 = \{ 0, 1 \} \). Denote \( j_i^t : \mathcal{E}_i \hookrightarrow X_I \) and \( j_i : \mathcal{P}_1 \hookrightarrow (P_1 / B)^t \). Consider the following commutative diagram:

\[
P_1 \times B P_2 / \mathcal{P}_2 = \{ 00, 01, 10, 11 \}.
\]

Here \( \sigma_i \) are the zero sections, \( p_i^t \) is induced by the projection map \( p_2 \), so it maps 00, 01 to 0, and 10 and 11 to 1. Moreover, by definition, \( j_0^t = \sigma_1 \) and \( \sigma_2 j_0^t = j_0^1 \) for \( i = 0, 1 \). We have

\[
\eta_1 = p_2^* \mathcal{H}^s(\sigma_1, 1), \quad \eta_2 = (\sigma_2, 1),
\]

and

\[
h_t((X_I)^t) = S(f_0, f_{10}, f_{11}), \quad h_t((P_1 / B)^t) = S(f_0, f_1).
\]

Denote \( \mathcal{C}_1 : S \to h_t(P_1 / B) \).

First of all, from the definition of \( p_2^* \) and (4), we know

\[
(p_2^* \mathcal{H}^s(f_0) = f_0 + f_{10} + f_{11} + f_{01}.
\]

Moreover, since \( j_0^t \) coincides with \( \sigma_1 \) and \( j_0^1(\mathcal{C}_1) \neq \sigma_1(\mathcal{C}_1) \), so \( j_0^1 \mathcal{H}^s(\sigma_1, 1) = 0 \) and

\[
(j_0^1)^* \mathcal{H}^s(\sigma_1, 1) = (j_0^1)^* \mathcal{H}^s(\sigma_1, 1) = \sigma_1^s(\sigma_1, 1) = x_{-\alpha_1} f_0.
\]

where the last identity follows from the fact that the tangent space of \( P_1 / B \) at 0 has weight \(-\alpha_1 \). Hence,

\[
(j_0^1)^* \eta_1 = (j_0^1)^* p_2^* \mathcal{H}^s(\sigma_1, 1) = (p_2^* \mathcal{H}^s(\sigma_1, 1)) = (p_2^* \mathcal{H}^s(\sigma_1, 1) \mathcal{H}^s(f_0) = \mathcal{H}^s(f_0 + f_{11} + f_{10} + f_{01}).
\]

(14)
We then compute \((j^I)^*(\eta_2)\), by using the identity
\[
(j^I)^*(\eta_2) = \sum_{x \in p_3} (j^I_x)^*(\eta_2) f_x.
\]
Since \(0,1,11 \not\in \sigma_2(P_1/B)\), we have \((j^I_{01})^*(\eta_2) = (j^I_{11})^*(\eta_2) = 0\). From Lemma 2.5, we know that \(\sigma_2(\sigma_2)_s(1) = c_1(x_{-\alpha_2})\). So
\[
(j^I_{00})^*(\eta_2) = (j^I_{30})^*(\sigma_2)_s(1) = (j^I_{00})^*(\sigma_2)_s(1) = (j^I_{01})^*(c_1(x_{-\alpha_2})) = x_{-\alpha_2},
\]
where \# follows from Lemma 4.1. Similarly, from \(j^I_{10} = \sigma_2 j^I_1\), we have
\[
(j^I_{10})^*(\eta_2) = (j^I_{10})^*(\sigma_2)_s(1) = (j^I_1)^*(\sigma_2)_s(1) = (j^I_1)^*(c_1(x_{-\alpha_2})) = s_1(x_{-\alpha_2}) = x_{-\alpha_1-\alpha_2}.
\]
Therefore,
\[
(j^I)^*(\eta_2) = x_{-\alpha_2} f_{00} + x_{-\alpha_1-\alpha_2} f_{10}. \tag{16}
\]
Now we compute the restriction formula of \(\eta_L\).

**Theorem 4.3.** Let \(I\) be a sequence of length \(l\). For any two subsets \(L, M \subset [l]\) denote \(L^c = [l] \setminus L\) and
\[
a_{L,M} = \prod_{k \in L} v_{k-1}^M (x_{-\alpha_{i_k}}).
\]
Then
\[
 j^*(\eta_L) = \sum_{M \subset L^c} a_{L,M} f_M.
\]

**Proof.** We first consider \(L = \{k\}\), and prove the following identity
\[
 j^*(\eta_k) = \sum_{M \subset L^c} v_{k-1}^M (x_{-\alpha_k}) f_M.
\]
Write \(I_j = (i_1, \ldots, i_j)\) for \(j = k\) and \(j = k - 1\). Firstly, we compute \((j^I_k)^*(\sigma_k)_s(1)\) for each \(M \subset \{k\}\), with \(\sigma_k : \hat{X}_{i_{k-1}} \to \hat{X}_{i_k}\). If \(k \in M\), then the point \(j^I_k(\text{pt}) \not\in \sigma_k(\hat{X}_{i_{k-1}})\), so
\[
(j^I_k)^*(\sigma_k)_s(1) = 0.
\]
If \(k \not\in M\), then \(M \subset [k-1]\), \(v_{k-1}^M = v_{k-1}^M\), and we have the following commutative diagram
\[
\text{pt} \xrightarrow{j_{k-1}^I} \hat{X}_{i_{k-1}} \xrightarrow{\sigma_k} \hat{X}_{i_k}.
\]
Therefore
\[(j^*_M)^* (\sigma_k)_* (1) = (j^*_{M-1})^* \sigma_k^* (\sigma_k)_* (1)\] (17)

\[\text{Lem. 2.5} \]
\[(j^*_{M-1})^* c_{i_{k-1}}(x_\alpha_k) = v_{k-1}^m (x_\alpha_k).\] (18)

\[\text{Lem. 4.1} \]
\[v_{k-1}^m (x_\alpha_k).\] (19)

Here \(c_{i_{k-1}}\) is the characteristic map on \(X_{i_{k-1}}\). Then we consider the following commutative diagram

\[
\begin{array}{ccc}
(X_{i})^T & \xrightarrow{j^i} & \hat{X}_I \\
\downarrow{p^i} & & \downarrow{p} \\
(X_{i_{k}})^T & \xrightarrow{j_{k}} & \hat{X}_{i_{k}}.
\end{array}
\]

We have
\[
(j^i)^* (\eta_k) = (j^i)^* p^* ((\sigma_k)_* (1)) = p^* (j^{i_k})^* (\sigma_k)_* (1) = p^* [\sum_{M \subseteq [k]} (j^*_{M})^* (\sigma_k)_* (1) f_M] = p^* [\sum_{M \subseteq [k-1]} v_{k-1}^m (x_\alpha_k) f_M] = \sum_{M \subseteq [k-1]} v_{k-1}^m (x_\alpha_k) \sum_{M' \subseteq [k+1, l]} f_{M \cup M'} = \sum_{M \subseteq ([l] \setminus \{k\})} v_{k-1}^m (x_\alpha_k) f_M.
\]

So the case \(L = \{k\}\) is proved.

Now for a general subset \(L \subseteq [l]\), we have
\[
j^* (\eta_L) = \prod_{k \in L} j^* (\eta_k) = \prod_{k \in L} \sum_{M \subseteq ([l] \setminus \{k\})} v_{k-1}^m (x_\alpha_k) f_M = \sum_{M \subseteq [l] \setminus \{k\}} \prod_{k \in L} v_{k-1}^m (x_\alpha_k) f_M.
\]

\[\square\]

For \(I\) of length \(l\) and \(L \subseteq [l]\), note the difference between
\[a_{[l] \setminus L} = \prod_{1 \leq k \leq l} v_{k-1}^l (x_\alpha_k), \quad x_{I \setminus L} = \prod_{1 \leq k \leq l} v_k^l (x_\alpha_k).
\]

They are only related when \(L = [l]\), in which case we have
\[a_{[l] \setminus [l]} = \prod_{1 \leq k \leq l} v_{k-1}^l (x_\alpha_k) = \prod_{1 \leq k \leq l} v_k^l (x_\alpha_k), \quad x_{I \setminus [l]} = \prod_{1 \leq k \leq l} v_k^l (x_\alpha_k).
\]
Corollary 4.4. The map $j^* : h_T(X_f) \to h_T(X_f^T)$ is an injection.

Proof. It follows from Theorem 4.3 that

$$j^*(\eta_L) = \sum_{M \subseteq \mathcal{L}} a_{L,M} f_M.$$ 

So if we order $\{j^*(\eta_L) | L \subseteq [l], \{f_M | M \subseteq [l]\}$ by inclusion of subsets $L' \subseteq L$, then the transition matrix from $f_M$ to $j^*(\eta_L)$ will be skew-triangular. Moreover, the entries on the skew-diagonal will be

$$a_{L,L'} = \prod_{k \in L} u_{k-1}^{L'}(x_{-\sigma_k}),$$

which is regular in $S$. Therefore, $j^*$ is injective.

Theorem 4.5. Let $I$ be a sequence of length $l$. Then

$$\text{im} j^* \subseteq \{ \sum_{L \subseteq [l]} a_L f_L | a_L = \frac{a_{L_1} - a_{L_2}}{u_{k-1}^{L_1}(x_{-\sigma_k})} \} \subseteq S, \forall L_1, L_2 \text{ such that } L_1 = L_2 \cup \{k\}.$$ Here $\cup$ denotes the disjoint union.

Proof. Denote the right hand side by $\Psi$. We first show that $\Psi$ is a ring. It is clearly additively closed. For the multiplication, consider

$$f = \sum_{L \subseteq [l]} a_L f_L, \quad g = \sum_{L \subseteq [l]} b_L f_L \in \Psi,$$

then

$$f g = \sum_{L,L' \subseteq [l]} \delta_{L,L'} a_L b_L f_L = \sum_{L \subseteq [l]} a_L b_L f_L.$$ 

For any $L_1, L_2$ such that $L_1 = L_2 \cup \{k\}$, by definition we have $u_{k-1}^{L_1} = u_{k-1}^{L_2}$, so $u_{k-1}^{L_1}(x_{-\sigma_k}) = u_{k-1}^{L_2}(x_{-\sigma_k})$. Therefore,

$$a_{L_1} b_{L_1} - a_{L_2} b_{L_2} = (a_{L_1} - a_{L_2}) b_{L_1} - (b_{L_2} - b_{L_1}) a_{L_2},$$

is divisible by $u_{k-1}^{L_1}(x_{-\sigma_k})$. We have $fg \in \Psi$.

We then show that $\text{im} j^* \subseteq \Psi$. Since $j^*$ is multiplicative, it suffices to show

$$j^*(\eta_m) = \sum_{L \subseteq [l] \setminus \{m\}} u_{m-1}^{L}(x_{-\sigma_m}) f_L$$

belongs to the RHS. Suppose $L_1 = L_2 \cup \{k\}$. Clearly $k \neq m$. If $k > m$, then by definition we have $u_{m-1}^{L_1} = u_{m-1}^{L_2}$. Thus $u_{m-1}^{L_1}(x_{-\sigma_m}) = u_{m-1}^{L_2}(x_{-\sigma_m})$, which implies that $j^*(\eta_m) \in \Psi$.

If $k < m$, denote

$$L_1 \cap [m-1] = \{j_1 < j_2 < \cdots j_t < k < j_{t+1} < \cdots < j_s\},$$

$$L_2 \cap [m-1] = \{j_1 < j_2 < \cdots j_t < k < j_{t+1} < \cdots < j_s\},$$

(in other words, $k$ is omitted in $L_2$). Then

$$u_{m-1}^{L_1}(x_{-\sigma_m}) - u_{m-1}^{L_2}(x_{-\sigma_m}) = s_{j_1} s_{j_2} \cdots s_{j_t} s_k s_{j_{t+1}} \cdots s_{j_s},$$

where $s_{j_i}$ are the elementary symmetric polynomials in $\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_t}, \sigma_k, \sigma_{j_{t+1}}, \ldots, \sigma_{j_s}$. Therefore, $j^*(\eta_m) \in \Psi$. The proof is complete.
Let $\Phi$ be a subset of $|s|$. For any $\eta \in \Psi$, we know that $\eta \cong |s|$, we have that $\alpha \cong \nu_k \cong |s|$. Therefore, $f \in \text{im} \Phi$. Assume the conclusion holds for any $\eta \in \Psi$ that can be written as a linear combination of $f_L$ with $|L| \leq t$.

Let $L_0$ be a subset of $|s|$ of cardinality $t$. For any $k \in L_0$, we have $\alpha \cong \nu_k \cong |s|$. By [CZZ19, Lemma 2.7], we know that $\alpha \cong \nu_k \cong |s|$. Therefore, $f = \sum_{L \subseteq [k]} a_L f_L \in \Psi$, with $a_L = 0$ unless $L = \emptyset$.

By Theorem 4.3, we know $f = \sum_{L \subseteq [k]} a_L f_L \in \Psi$, with $a_L = 0$ unless $L = \emptyset$. The proof is finished.
Remark 4.7. Let $T_j$ be the subtorus of rank 1 corresponding to $\alpha_j$, i.e., $T_j = (\ker \alpha_j)^0$ where $\alpha_j$ is viewed as a character $T \to k^*$. If $I = (i_1, ..., i_l)$ is a sequence such that $i_j$ are all distinct, it is not difficult to see that for any $1 \leq k \leq l$,

$$X^T_{i_k} = \{[g_1, ..., g_l] | g_j B \in \{B, s_j B\} \forall j \neq k\},$$

and

$$X^T_{i_k'} = \{[g_1, ..., g_l] | g_j B \in \{B, s_j B\} \forall j\}$$

if $T'$ is any subtorus of corank 1 different from $T_{i_j}$, $j = 1, ..., l$. In other words, for any subtorus of corank 1, the irreducible component of the invariant subvariety has dimension at most one. This corresponds to the so-called Goresky-Kottwitz-MacPherson (GKM) condition. In other words, in this case, the Bott-Samelson variety is a GKM space. This corresponds to the conclusion of Theorem 4.6.

On the other hand, if $P_{i_j}$ are not distinct, the space $\hat{X}_I$ will not be GKM. For instance, if $I = (1, 2, 1)$, the $T_1$-fixed subspace contains the following subset

$$\{[g_1, e, g'_1] | g_1, g'_1 \in P_1\},$$

so the dimension condition is not satisfied. Indeed, it follows from the proof of Theorem 4.6 that in this case, the inclusion in Theorem 4.5 is strict. For more detailed discussion of GKM spaces, see [GKM98, GHZ06].

5. Push-forward to cohomology of flag varieties

In this section, we compute the push-forward of the basis $\eta_k$ along the canonical map $q_I : \hat{X}_I \to G/B$, which generalizes the computation of Bott-Samelson classes in [CZZ14].

Recall that the set of $T$-fixed points of $G/B$ is in bijection to $W$, so we have

$$h_T((G/B)^T) \cong \bigoplus_{w \in W} S.$$

Denote by $f_w \in h_T(W)$ the basis element corresponding to $w \in W$. Denote $i : W \to G/B$ to be the embedding, and denote $pt_w = (i|_{\mathfrak{e}})_*(1) \in h_T(G/B)$. Let $\pi_i : G/B \to G/P_i$ be the canonical map, and denote

$$A_i = \pi_i^* \circ (\pi_i)_* : h_T(G/B) \to h_T(G/B).$$

For any sequence $I$, denote by $I^{rev}$ the sequence obtained by reversing $I$.

Proposition 5.1. [CZZ14, Lemma 7.6] For any sequence $I$, we have

$$(q_I)_* (1) = A_{I^{rev}}(pt_c).$$

The following is an easy generalization of Proposition 5.1.

Theorem 5.2. Let $I$ be a sequence of length $l$ and $1 \leq k \leq l$. Denote by $I_k$ the subsequence of $I$ obtained by removing the $k$-th term from $I$. Then $(q_I)_*(\eta_k) = A_{I_k^{rev}}(1).$
Proof. Denote the sequence by \( I = (i_1, \ldots, i_l) \). For any \( k \leq l \), denote

\[
P_{i_1} \times^B P_{i_2} \times^B \ldots \times^B P_{i_k} / B \xrightarrow{q_k} G / B
\]

\[
P_{i_1} \times^B P_{i_2} \times^B \ldots \times^B P_{i_{k-1}} / B \xrightarrow{q_{k-1}} G / B.
\]

Note that \( q_l = q_1 \) and \( q_k \circ \sigma_k = q_{k-1} \). Denote by \( p \) the composition of \( p_{k+1}, \ldots, p_l \). By using the base change formula from diagram (8), we have

\[
(q_l)_*\eta_k = (q_l)_* p^*((\sigma_k)_*(1)) = (q_1)_* p^*((\sigma_k)_*(1)) = (q_1)_* p_1^* p_{i_1 - 1}^* \cdots p_{i_k + 1}^* (\sigma_k)_*(1).
\]

\[
= \pi_{\alpha_{i_1}} (\sigma_{i_1})_*(q_{l-1})_* p_1^* p_{i_1 - 1}^* \cdots p_{i_k + 1}^* (\sigma_k)_*(1) = (\pi_{\alpha_{i_1}} (\sigma_{i_1})_*)(\pi_{\alpha_{i_1 - 1}} (\pi_{\alpha_{i_1 - 1}})_*) \cdots (\pi_{\alpha_{i_k + 1}} (\pi_{\alpha_{i_k + 1}})_*)(q_k)_*(\sigma_k)_*(1) = A_{i_1} A_{i_1 - 1} \cdots A_{i_k + 1} (q_{k-1})_*(1) = A_{i_1} A_{i_1 - 1} \cdots A_{i_k + 1} A_{i_k - 1} \cdots A_{i_k} (p_{i_k}) = A_{i_k}^{rev}(p_{i_k}).
\]

To compute \((q_l)_*\eta_L\) for general \( L \subset [l] \), we need the following lemma.

**Lemma 5.3.** For any \( L \subset [l] \), we have

\[
\eta_L = \sum_{L_1 \subset [l]} \frac{a_{L_1, L_1}}{x_{L_1, L_1}} j_*(f_{L_1}),
\]

where \( a_{L_1, L_1} \) are defined in Theorem 4.3. Note that the coefficients in this formula belong to \( Q := S[\frac{1}{x_{a}} | a \in \Sigma] \).

**Proof.** By Corollary 4.4, we know that \( j_*(\eta_L) \) becomes a basis of \( Q \otimes_S h_T(W) \). In other words, \( j^* \) induces an isomorphism

\[
\tilde{j}^*: Q \otimes_S h_T(X_T) \to Q \otimes_S h_T(W).
\]

Moreover, by Lemma 3.10, we know that \( j_* \) is the inverse of the \( j^* \) (after tensoring with \( Q \)). Therefore, \( j_*(f_{L_1}) \) is a \( Q \)-basis of \( Q \otimes_S h_T(X_T) \). Denote

\[
\eta_L = \sum_{L_1 \subset [l]} b_{L_1, L_1} j_*(f_{L_1}), \quad b_{L_1, L_1} \in Q.
\]

Then by Theorem 4.3 and Lemma 3.10, we have

\[
\sum_{L_2 \subset L_1} a_{L_2, L_2} f_{L_2} = j^*(\eta_L) = \sum_{L_1 \subset [l]} b_{L_1, L_1} j_*(f_{L_1}) = \sum_{L_1 \subset [l]} b_{L_1, L_1} x_{L_1, L_1} f_{L_1}.
\]

Therefore, \( b_{L_1, L_1} = \frac{a_{L_1, L_1}}{x_{L_1, L_1}} \). 

The following is the main result of this section, which computes the pushforward of \( \eta_L \) to the cohomology of \( G / B \).
Theorem 5.4. For any sequence \( I = (i_1, \ldots, i_l) \), we have

\[
i^*(q_I)_*(\eta_L) = \sum_{L_1 \in L^c} a_{L,L_1} \cdot \frac{v_{\Pi,I}(x_{\Pi,I})}{x_{I,L_1}} f_{v_{I,l}}, \quad x_{\Pi,I} := \prod_{\alpha < 0} x_{\alpha} \in S.
\]

Note that a priori the coefficients of \( f_{v_{I,l}} \) belong to \( S \).

Proof. Consider the following commutative diagram

\[
\begin{array}{ccc}
X_I & \xrightarrow{j} & \tilde{X}_I \\
\downarrow{q'} & & \downarrow{q} \\
W & \xrightarrow{i} & G/B.
\end{array}
\]

Note that by definition, \( q' \) maps the point corresponding to \( L \subset [l] \) to \( v^l \in W \). Therefore,

\[
(q')_* (f_L) = f_{v_L} \in h_T(W).
\]

Firstly, we have

\[
i^*(q_I)_* j_* (f_L) = i^* i_* (q'_*) (f_L) = \sum_{L_1 \in L^c} a_{L,L_1} \frac{v_{\Pi,I}(x_{\Pi,I})}{x_{I,L_1}} f_{v_{I,l}},
\]

where the last identity follows from [CZZ14, Corollary 6.4]. Consequently, by Lemma 5.3, we have

\[
i^*(q_I)_* (\eta_L) = \sum_{L_1 \in L^c} a_{L,L_1} \frac{v_{\Pi,I}(x_{\Pi,I})}{x_{I,L_1}} f_{v_{I,l}}.
\]

\[\Box\]

Remark 5.5. In case \( \eta_L = \eta_\emptyset \) or \( \eta_e \), as in Proposition 5.1 and Theorem 5.2, one can express \((q_I)_*(\eta_L)\) as the operators \( A_i \) applied on \( \text{pt}_e \). By using the method of formal affine Demazure algebra, started in [KK86, KK90] and continued in [CZZ12, CZZ19, CZZ14], one will obtain a restriction formula of \( i^*(q_I)_*(\eta_L) \). Roughly speaking, there is an algebra \( D_F \) generated by algebraic analogue of the push-pull operators \( A_i \), whose dual is isomorphic to \( h_T(G/B) \). The algebra \( D_F \) acts on \( h_T(G/B) \), via two actions (denoted by \( \cdot \) and \( \otimes \) in [LZZ16]). Both actions will give restriction formulas of \( A_f(\text{pt}_e) \). Indeed, by using the two actions, one will obtain two different, but equivalent formulas, one of which coincides with the one given by Theorem 5.4.

Corollary 5.6. Let \( I \) be any sequence of length \( l \). For any \( L \subset [l] \), denote by \( \tilde{X}_L = (\tilde{X}_I)_L \) and \( q_L : \tilde{X}_L \rightarrow G/B \). Then \( (q_I)_*(\eta_L) = (q_L)_*(1) \).

Proof. From Theorem 5.4 we have

\[
i^*(q_I)_* (\eta_L) = \sum_{L_1 \in L^c} a_{L,L_1} \frac{v_{\Pi,L_1}(x_{\Pi,L_1})}{x_{I,L_1}} f_{v_{I,l}},
\]

(20)
\[ i^*(q_{L^*})_*(1) = \sum_{L \subseteq L^*} \prod_{\alpha < 0} v^L(x_\alpha)_{L,L_i} f_{L,L_i}. \]  
\[ (21) \]

By definition
\[ x_{L_1} = \prod_{j \in J} u^{L_j}_{j}(x_{-\alpha_j}), \quad x_{L, L_1} = \prod_{j \in J} u^{L_j}_{j}(x_{-\alpha_j}), \quad a_{L, L_1} = \prod_{j \in J} u^{L_j}_{j-1}(x_{-\alpha_j}). \]

Since \( L \cap L_1 = \emptyset \), so for any \( j \in L \), \( u^{L_j}_{j} = u^{L_j}_{j-1} \), and we have
\[ x_{L, L_1} = \prod_{j \in J} u^{L_j}_{j}(x_{-\alpha_j}) \prod_{j \in J} u^{L_j}_{j}(x_{-\alpha_j}) \]
\[ = \prod_{j \in J} u^{L_j}_{j}(x_{-\alpha_j}) \prod_{j \in J} u^{L_j}_{j-1}(x_{-\alpha_j}) \]
\[ = x_{L, L_1} q_{L, L_1}. \]

Therefore, \( i^*(q_{L^*})_*(\eta_L) = i^*(q_{L^*})_*(1). \) By [CZZ14, Theorem 8.2], we know \( i^* \) is injective. So \( (q_{L^*})_*(\eta_L) = (q_{L^*})_*(1). \)

**Remark 5.7.** This corollary shows that for any \( L \subseteq [I] \), the class in \( h_T(G/B) \) determined by \( \eta_L \) coincides with the Bott-Samelson class determined by \( I_{L^*} \), in other words, for the class \( \eta_L \), the minimal parabolic subgroups \( P_{(j)} \), \( j \in L \) are ‘omitted’.

By using this result, we can derive the Chevalley formula for equivariant oriented cohomology. For each \( w \in \mathcal{W} \), we fix a reduced sequence \( I_w \), then the Bott-Samelson class \( \zeta_{I_w} \) is defined to be the push-forward class along the map \( q_{I_w} : X_{I_w} \to G/B \), i.e., \( \zeta_{I_w} = (q_{I_w})_*(1) \). It is proved in [CZZ14, Proposition 8.1] that \( \{ \zeta_{I_w} \mid w \in \mathcal{W} \} \) is a basis of \( h_T(G/B) \). Denote the characteristic maps from \( h_T(pt) \) to \( G/B \) and to \( X_{I_w} \) by \( \mathbf{c}' \) and \( \mathbf{c}_{I_w} \), respectively. By definition, \( \mathbf{c}_{I_w} = q_{I_w}^* \mathbf{c}' \).

**Corollary 5.8 (Chevalley Formula).** For any \( u \in h_T(pt) \), we have
\[ \mathbf{c}'(u) \cdot \zeta_{I_w} = \sum_{L \subseteq [I](w)} \theta_{L,L}(u) \zeta_{L,L}, \]
where \( \zeta_{L,L} = (q_{L,L})_*(1) \) and \( \theta_{L,L}(u) \) was defined in Lemma 3.3.

**Proof.** We have
\[ (q_{I_w})_*(\mathbf{c}_{I_w}(u)) = (q_{I_w})_*(\mathbf{c}_{I_w}(u) \cdot 1) = (q_{I_w})_*(q_{I_w}^* \mathbf{c}'(u)) \cdot 1 = \mathbf{c}'(u) \zeta_{I_w}, \]
where the last identity follows from the projection formula. Then Lemma 3.3 and Corollary 5.6 imply that
\[ (q_{I_w})_*(\mathbf{c}_{I_w}(u)) = \sum_{L \subseteq [I](w)} \theta_{I_w,L}(u)(q_{I_w})_*(\eta_L) \]
\[ = \sum_{L \subseteq [I](w)} \theta_{I_w,L}(u)(q_{L^*})_*(1) \]
\[ \sum_{L \in L^*} \theta_{L \longrightarrow L}(u) \zeta_{L^*}. \]

The conclusion then follows. □

References

[AM15] Aluffi, Paolo; Mihalcea, Leonardo C. Chern–Schwartz–MacPherson classes for Schubert cells in flag manifolds. *Compos. Math.* 152 (2016), no. 12, 2603–2625. MR3594289, Zbl 1375.14163, arXiv:1508.01535, doi: 10.1112/S0010437X16007685.

[AMSS17] Aluffi, Paolo; Mihalcea, Leonardo C.; Schuermann, Joerg; Su, Changjian. Shadows of characteristic cycles, Verma modules, and positivity of Chern–Schwartz–MacPherson classes of Schubert cells. Preprint, 2017. arXiv:1709.08697.

[CNZ19] Calmès, Baptiste; Nesin, Alexander; Zainoulline, Kirill. Relative equivariant motives and modules. *Canad. J. Math.* 73 (2021), no. 1, 131–159. MR4201536, Zbl1375.14163, arXiv:1508.01535, doi: 10.4153/CJM-2017-001-X.

[CPZ13] Calmès, Baptiste; Petrov, Viktor; Zainoulline, Kirill. Invariants, torsion indices and oriented cohomology of complete flags. *Ann. Sci. Éc. Norm. Supér.* (4) 46 (2013), no. 3, 405–448. MR3099981, Zbl 1323.14026, arXiv:1409.7111.

[CZZ14] Calmès, Baptiste; Zainoulline, Kirill; Zhong, Changlong. Equivariant oriented cohomology of flag varieties. *Doc. Math.* 2015, Extra vol.: Alexander S. Merkurjev’s Sixtieth Birthday, 113–144. MR3404378, Zbl 1351.14014, arXiv:1409.7111.

[CZZ12] Calmès, Baptiste; Zainoulline, Kirill; Zhong, Changlong. A coproduct structure on the formal affine Demazure algebra. *Math. Z.* 282 (2016), no. 3-4, 1191–1218. MR3473664, Zbl 1362.14024, arXiv:1209.1676, doi: 10.1007/s00209-015-1585-6.

[CZZ19] Calmès, Baptiste; Zainoulline, Kirill; Zhong, Changlong. Push-pull operators on the formal affine Demazure algebra and its dual. *Manuscripta Math.* 160 (2019), no. 1-2, 9–50. MR3983385, Zbl 1423.14271, arXiv:1312.0019, doi: 10.1007/s00229-018-1058-4.

[D74] Demazure, Michel. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup.* (4) 7 (1974), 53–88. MR0354697, Zbl 0312.14009, doi: 10.24033/asens.1261.

[DZ20] Douglass, J. Matthew; Zhong, Changlong. The Leray–Hirsch theorem of equivariant oriented cohomology of flag varieties. Preprint, 2020. arXiv:2009.05902.

[GZ20] Goldin, Rebecca; Zhong, Changlong. Structure constants in equivariant oriented cohomology of flag varieties. Preprint, 2020. arXiv:2009.03466.

[GKM98] Goresky, Mark; Kottwitz, Robert; MacPherson, Robert. Equivariant cohomology, Koszul duality and the localization theorem. *Invent. Math.* 131 (1998), no. 1, 25–83. MR1489894, Zbl 0897.22009, doi: 10.1007/s002220050197.

[GHZ06] Guillemin, Victor W.; Holm, Tara S.; Zara, Catalin. A GKM description of the equivariant cohomology ring of a homogeneous space. *J. Algebraic. Comb.* 23 (2006), no. 1, 21–41. MR2218848, Zbl 1096.53050, doi: 10.1007/s10801-006-6027-4.

[HM13] Heller, Jeremiah; Málaga–López, Jose. Equivariant algebraic cobordism. *J. Reine Angew. Math.* 684 (2013), 87–112. MR3181557, Zbl 1343.14015, arXiv:1006.5509, doi: 10.1515/crelle-2011-0004.

[H12] Humphreys, James E. Linear algebraic groups. Graduate Texts in Mathematics, 21. *Springer-Verlag, New York-Heidelberg*, 1975. xiv+247 pp. MR0396773, Zbl 0325.20039, doi: 10.1007/978-1-4684-9443-3.
[KK86] Kostant, Bertram; Kumar, Shrawan. The nil Hecke ring and cohomology of G/P for a Kac–Moody group. *Adv. in Math.* **62** (1986), no. 3, 187–237. MR0866159, Zbl 0641.17008, doi: 10.1016/0001-8708(86)90101-5. 1461

[KK90] Kostant, Bertram; Kumar, Shrawan. T -equivariant K -theory of generalized flag varieties. *J. Differential Geom.* **32** (1990), no. 2, 549–603. MR1072919, Zbl 0731.55005, doi: 10.1073/pnas.84.13.4351. 1461

[LZZ16] Lenart, Cristian; Zainoulline, Kirill; Zhong, Changlong. Parabolic Kazhdan–Lusztig basis, Schubert classes and equivariant oriented cohomology. *J. Inst. Math. Jussieu.* **19** (2020), no. 6, 1889–1929. MR4166997, Zbl 1460.14111, arXiv:1608.06554, doi: 10.1017/S1474748018000592. 1444

[MO12] Maulik, Davesh; Okounkov, Andrei. Quantum groups and quantum cohomology. *Astérisque* **408** (2019) ix+209 pp. ISBN: 978-2-85629-900-5. MR3951025, Zbl 1422.14002, arXiv:1211.1287, doi: 10.24033/ast.1074. 1444

[O15] Okounkov, Andrei. Lectures on K-theoretic computations in enumerative geometry. *Geometry of moduli spaces and representation theory*, 251–380. IAS/Park City Math. Ser., 24, Amer. Math. Soc., Providence, RI, 2017. MR3752463, Zbl 1402.19001, arXiv:1512.07363. 1444

[RTV15] Rimányi, Richárd; Tarasov, Vitaly O.; Varchenko, A. N. Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety. *J. Geom. Phys.* **94** (2015), 81–119. MR3350271, Zbl 1349.19002, arXiv:1411.0478, doi: 10.1016/j.geomphys.2015.04.002. 1444

[RTV17] Rimányi, Richárd; Tarasov, Vitaly O.; Varchenko, A. N. Elliptic and K-theoretic stable envelopes and Newton polytopes. *Selecta Math. (N.S.)* **25** (2019), no. 1, Paper No. 16, 43 pp. MR3911740, Zbl 1452.55009, arXiv:1705.09344, doi: 10.1007/s00029-019-0451-5. 1444

[SZZ17] Su, Changjian; Zhao, Gufang; Zhong, Changlong. On the K-theory stable bases of the Springer resolution. *Ann. Sci. Éc. Norm. Supér. (4)* **53** (2020), no. 3, 663–711. MR4181580, Zbl 1452.55009, arXiv:1705.09344, doi: 10.24033/asens.2431. 1444

[SZZ19] Su, Changjian; Zhao, Gufang; Zhong, Changlong. Wall-crossings and a categorification of K-theory stable bases of the Springer resolution. *Compositio Math.*, to appear. 1444

[W04] Willems, Matthieu. Cohomologie équivariante des tours de Bott et calcul de Schubert équivariant. *J. Inst. Math. Jussieu* **5** (2006), no. 1, 125–159. MR2195948, Zbl 1087.55005, doi: 10.1017/S1474748005000216. 1451

[Z20] Zainoulline, Kirill. Localized operations on T -equivariant oriented cohomology of projective homogeneous varieties. *J. Pure Appl. Algebra* **226** (2022), no. 3, Paper No. 106856. MR4295178, Zbl 07396435, arXiv:2001.00498. 1444

[Z15] Zhong, Changlong. On the formal affine Hecke algebra. *J. Inst. Math. Jussieu.* **14** (2015), no. 4, 837–855. MR3394129, Zbl 1347.20005, arXiv:1301.7497, doi: 10.1017/S1474748014000188. 1444

(Hao Li) DEPARTMENT OF MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK ALBANY, 1400 WASHINGTON AVE, ALBANY, NY 12222, USA hli29@albany.edu

(Changlong Zhong) DEPARTMENT OF MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK ALBANY, 1400 WASHINGTON AVE, ALBANY, NY 12222, USA czhong@albany.edu

This paper is available via [http://nyjm.albany.edu/j/2021/27-56.html](http://nyjm.albany.edu/j/2021/27-56.html).