The Hierarchy of Hamiltonians for a Restricted Class of Natanzon Potentials

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Abstract
The restricted class of Natanzon potentials with two free parameters is studied within the context of Supersymmetric Quantum Mechanics. The hierarchy of Hamiltonians is indicated, where the first members of the superfamily are explicitly evaluated and a general form for the superpotential is proposed.

I. Introduction
The set of potentials known as Natanzon potentials have numerous applications in several branches of physics, [1]-[11]. An important point to stress is that the two classes of Natanzon potentials, the hypergeometric and the confluent, include all the potentials whose Schroedinger equation is analytically and exactly solvable, [2]. They have motivated several works concerning the mathematical and algebraic aspects of their structure and solutions, [2]-[8].

In particular, there are studies within the context Supersymmetric Quantum Mechanics formalism. Cooper et al, [9], for instance, investigated the relationship between shape invariance and exactly analytical solvable potentials and showed that the Natanzon potential is not shape invariant although it has analytical solutions for the associated Schroedinger equation. Lévai et al, [10], have determined phase-equivalent potentials for a class of Natanzon potentials employing the formalism of supersymmetry.

However, the hierarchy of the Hamiltonians corresponding to Natanzon potentials has not been determined yet. Cooper et al in ref. [11] attempted to this possibility but the hierarchy was not computed, since they were only interested in the first two members of the
hierarchy, the two partner Hamiltonians, in order to check the shape invariance. In this work we explore this point. The superalgebra is used to construct the hierarchy of Hamiltonians of the restricted class of Natanzon potentials, with two free parameters. The first few members of the superfamily are explicitly evaluated and a general form for the superpotential is proposed by induction. Comments about exactly solvable potentials and their relationship with supersymmetry are given in the conclusions.

II. Supersymmetric Quantum Mechanics Formalism

In the formalism of Supersymmetric Quantum Mechanics there are two operators $Q$ and $Q^+$, that satisfy the algebra

$$\{Q, Q\} = \{Q^+, Q^+\} = 0, \quad \{Q, Q^+\} = H_{SS} \tag{1}$$

where $H_{SS}$ is the supersymmetric Hamiltonian. The usual realisation of the operators $Q$ and $Q^+$ is

$$Q = a^- \sigma^- = \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix}, \quad Q^+ = a^+ \sigma^+ = \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix} \tag{2}$$

where $\sigma^\pm$ are written in terms of the Pauli matrices and $a^\pm_1$ are bosonic operators written in terms of the superpotential $W_1(r)$:

$$a^\pm_1 = \left( \mp \frac{d}{dr} + W_1(r) \right). \tag{3}$$

With this realisation the supersymmetric Hamiltonian $H_{SS}$ is given by

$$H_{SS} = \begin{pmatrix} a^+_1 a^-_1 & 0 \\ 0 & a^-_1 a^+_1 \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \tag{4}$$

where $H_1$ and $H_2$ are supersymmetric partner Hamiltonians. There is a direct relationship between these Hamiltonians and their spectra of eigenfunctions and eigenvalues, namely,

$$\Psi_{n+1}^{(1)} \propto a^+_1 \Psi_n^{(2)}, \quad \Psi_n^{(2)} \propto a^-_1 \Psi_{n+1}^{(1)}$$

$$E_{n+1}^{(1)} = E_n^{(2)} \tag{5}$$

with $E_0^{(1)} = 0, \ (n = 0, 1, \ldots)$.

The superpotential and the ground state eigenfunction are related by

$$W_1(r) = -\frac{d}{dr} \log(\Psi_0^{(1)}). \tag{6}$$
Substituting $a_1^\pm$ as defined in (3) into $H_1$ given by (4) we end up with the Riccati equation satisfied by $W_1$,

$$W_1^2 - \frac{dW_1}{dr} = V_1(r) - \epsilon_0^{(1)}.$$  

(7)

This structure can be repeated for $H_2$, i.e., $H_2$ can be factorized again in terms of its ground state

$$H_2 = a_2^+ a_2^-$$

(8)

with

$$a_2^\pm = \left( \mp \frac{d}{dr} + W_2(r) \right)$$

(9)

where the superpotential $W_2$ can be written in terms of the ground state eigenfunction of $H_2$

$$W_2(r) = -\frac{d}{dr} \log(\Psi_0^{(2)}).$$

(10)

Repeating this process $n$ times, we get a whole family of Hamiltonians, related by supersymmetry, [11], [12]

$$H_n = a_n^+ a_n^-$$

(11)

with

$$a_n^\pm = \left( \mp \frac{d}{dr} + W_n(r) \right).$$

(12)

The supersymmetry enables us to relate all the $n$ members of the hierarchy, as made for the first two members, see fig.1,

$$\Psi_n^{(1)} \propto a_1^+ a_2^+ \cdots a_n^+ \Psi_0^{(n+1)}$$

$$E_n^{(1)} = E_0^{(n+1)}$$

(13)

where the eigenfunctions must be normalizable.

III. Natanzon Potential and the Hierarchy of Hamiltonians

The restricted class of Natanzon potentials having two parameters and given in terms of the variable $y(r)$ is,

$$V(r) = \{-\lambda^2 v(v+1) + \frac{1}{4}(1 - \lambda^2)[5(1 - \lambda^2)y^4 - (7 - \lambda^2)y^2 + 2]\}(1 - y^2),$$

(14)
where the variable function \( y(r) \) satisfies

\[
dy/dr = (1 - y^2)[1 - (1 - \lambda^2)y^2].
\] (15)

The dimensionless free parameters \( v \) and \( \lambda \) measure the depth and the shape of the potential, respectively.

We write the Schroedinger equation for this potential, [2], [3], in dimensionless units

\[
r = bx = (2mv_0/h^2)^{1/2}x
\]

\[
[-d^2/dr^2 + V(r)]\Psi_n(r) = \epsilon_n\Psi_n(r)
\] (16)

where \( V(x) = v_0V(r), \epsilon_n = E_n/v_0 \).

The analytic solutions for the energy eigenfunctions are given by,

\[
\Psi_n = (1 - \lambda^2)^{\lambda_n/2}[g(y)]^{-(2\mu_n+1)/4}C_n^{\mu_n+1/2}(\lambda y/[g(y)]^{1/2})
\] (17)

where \( g(y) = 1 - (1 - \lambda^2)y^2 \). The factor \( C_n^{(a)}(x) \) is a Gegenbauer polynomial when \( n \) is a non-negative integer, which is our case. The corresponding energy eigenvalues are given by

\[
\epsilon_n = -\mu_n^2\lambda^4, \quad \mu_n > 0,
\] (18)

where

\[
\mu_n\lambda^2 = [\lambda^2(v + 1/2)^2 + (1 - \lambda^2)(n + 1/2)^2]^{1/2} - (n + 1/2).
\] (19)

Notice the relationship between the energy levels which will be extensively used in what follows,

\[
(\mu_n^2 - \mu_{n-1}^2)\lambda^2 = (-2n - (2n + 1)\mu_n + (2n - 1)\mu_{n-1}).
\] (20)

In order to construct the superfamily we firstly factorize the Natanzon potential, calling \( V(r) = V_1(r) = V_\pm(r) + \epsilon_0^{(1)}, [5], \) whose Schroedinger equation is

\[
H_1 - \epsilon_0^{(1)} = a_1^+a_1^- + d/dr + W_1(r)
\] (21)

where \( \epsilon_n^{(1)} = \epsilon_n \). The superpotential \( W_1(r) \) is evaluated from the knowledge of the ground-state eigenfunction of \( V(r) \), \( W_1(r) = -d/dr\log(\Psi_0^{(1)}) \), where \( \Psi_0^{(1)} = \Psi_n \), given by [7]. It satisfies the Riccati equation and it is given by

\[
W_1(r) = \frac{1}{2}(1 - \lambda^2)y(y^2 - 1) + y\mu_0\lambda^2.
\] (22)
The superpartner Hamiltonian satisfies the equation
\[ H_2 - \epsilon^{(1)}_0 = a_1^- a_1^+ \] (23)
which is written in terms of \( V_2(r) \) like
\[ W_1^2 + \frac{dW_1}{dr} = V_2(r) - \epsilon^{(1)}_0 \] (24)
where
\[ V_2(r) = \{-\mu_0^2 \lambda^4 + \mu_0 \lambda^2 + \frac{1}{4} (1 - \lambda^2) [-7(1 - \lambda^2)y^4 + (9 - 3\lambda^2 - 8\mu_0 \lambda^2)y^2 - 2]} \} (1 - y^2). \] (25)
To construct the next member of the superfamily, we factorize the Schroedinger equation for \( V_2 \). It gives
\[ H_2 - \epsilon^{(2)}_0 = a_2^+ a_2^- \quad a_2^\pm = \mp \frac{d}{dr} + W_2(r) \] (26)
where \( W_2(r) \) satisfies the associated Riccati equation,
\[ W_2^2 - \frac{dW_2}{dr} = V_2(r) - \epsilon^{(2)}_0 . \] (27)
\( \epsilon^{(2)}_0 \) is the energy ground state of the potential \( V_2(r) \) and it is such that \( \epsilon^{(2)}_0 = \epsilon^{(1)}_1 \), see fig. 1.
The superpotential \( W_2 \) is given by \( W_2(r) = -\frac{d}{dr} \log(\Psi^{(2)}_0) \), where \( \Psi^{(2)}_0 = a_1^- \psi^{(1)}_1 \), i.e.,
\[ W_2(r) = \frac{3}{2} (1 - \lambda^2)y(y^2 - 1) + y\mu_1 \lambda^2 - \frac{d}{dr} \log(f_1) \] (28)
where
\[ f_1(y) = 1 + a_{11} y^2 \quad a_{11} = (\mu_0 - \mu_1)\lambda^2 - 1. \] (29)
The new superpartner of \( H_2 \) is given by
\[ W_2^2 + \frac{dW_2}{dr} = V_3(r) - \epsilon^{(2)}_0 \] (30)
where
\[ V_3(r) = \{-\mu_1^2 \lambda^4 + \mu_1 \lambda^2 + \frac{1}{4} (1 - \lambda^2) \left( -27(1 - \lambda^2)y^4 + (33 - 15\lambda^2)y^2 - 6 \right) + \] 
\[ + \frac{2a_{11} g(y)}{f_1(y)} \left( 1 + (-9 + 6\lambda^2 - 2\mu_1 \lambda^2)y^2 + 8(1 - \lambda^2)y^4 \right) + 8a_{11}^2 y^2 (1 - y^2)(\frac{g(y)}{f_1(y)})^2 \} (1 - y^2). \] (31)
and \( g(y) = 1 - (1 - \lambda^2) y^2 \). Thus, factorizing the Hamiltonian for this potential we have

\[
H_3 - \epsilon_0^{(3)} = a_3^+ a_3^- , a_3^\pm = \mp \frac{d}{dr} + W_3(r)
\]

where \( W_3(r) \) satisfies the Riccati equation,

\[
W_3^2 - \frac{dW_3}{dr} = V_3(r) - \epsilon_0^{(3)} .
\]

\( \epsilon_0^{(3)} \) is the energy ground state of the potential \( V_3(r) \), with \( \epsilon_0^{(3)} = \epsilon_1^{(2)} = \epsilon_2^{(1)} \), see fig. 1. The superpotential, defined by \( W_3(r) = -\frac{d}{dr} \log(\Psi_0^{(3)}) \) and \( \Psi_0^{(3)} = a_2^- a_1^- \psi_2^{(1)} \), is given by

\[
W_3 = \frac{5}{2} (1 - \lambda^2) y (y^2 - 1) + y \mu_2 \lambda^2 + \frac{d}{dr} \log(f_1) - \frac{d}{dr} \log(f_2)
\]

where

\[
f_2(y) = 1 + a_{21} y^2 + a_{22} y^4
\]

with

\[
a_{21} = 2(\mu_1 - \mu_2) \lambda^2 - 2
\]

\[
a_{22} = 1 + \frac{\lambda^2}{3} (2 - \mu_0 - 3 \mu_1 + 6 \mu_2) + \frac{\lambda^4}{3} (-4 \mu_0 + 6 \mu_1 - 2 \mu_2 - \mu_0^2 - 5 \mu_0 \mu_2 + 3 \mu_0 \mu_1 + 3 \mu_1 \mu_2).
\]

For the next member of the superfamily, we show the result of the evaluation of the superpotential \( W_4(r) = -\frac{d}{dr} \log(\Psi_0^{(4)}) \) with \( \Psi_0^{(4)} = a_3^- a_2^- a_1^- \psi_3^{(1)} \). It is given by

\[
W_4 = \frac{7}{2} (1 - \lambda^2) y (y^2 - 1) + y \mu_3 \lambda^2 + \frac{d}{dr} \log(f_1) + \frac{d}{dr} \log(f_2) - \frac{d}{dr} \log(f_3)
\]

where

\[
f_3(y) = 1 + a_{31} y^2 + a_{32} y^4 + a_{33} y^6
\]

with

\[
a_{31} = 3(\mu_2 - \mu_3) \lambda^2 + 2(\mu_0 - \mu_1) \lambda^2 - 5
\]

\[
a_{32} = 10 + \frac{\lambda^2}{3} (6 + 25 \mu_0 + 15 \mu_1 - 6 \mu_2 + 22 \mu_3) +
+ \frac{\lambda^4}{3} (2 \mu_0 + 2 \mu_0^2 + 18 \mu_1 - 6 \mu_0 \mu_1 + 18 \mu_2 + 13 \mu_0 \mu_2 -
- 3 \mu_1 \mu_2 - 6 \mu_3 - 11 \mu_0 \mu_3 - 3 \mu_1 \mu_3 + 8 \mu_2 \mu_3)
\]
\[ a_{33} = -10 + \lambda^2 (-6 + 13\mu_0 - 3\mu_1 - 12\mu_2 - 4\mu_3) + \frac{\lambda^4}{15} \left( \sum_{i=0}^{n-1} a_{n-1} y_i^{2i} + \frac{d}{dr} \log \left( \frac{\Pi_{i=0}^{n-1} f_{i-1}}{f_n} \right) \right), \quad f_0 = f_{-1} = 1 \] (42)

where \( f_n(y) \) is a 2n-order polynomial of the form

\[ f_n(y) = \sum_{i=0}^{n} a_{n-1} y_i^{2i}, \quad a_{n-1} = 1. \] (43)

We stress that since \( W_{n+1} \) is a superpotential it checks the Riccati equation,

\[ W_{n+1}^2 - \frac{dW_{n+1}}{dr} = V_{n+1}(r) - \epsilon_0^{(n+1)} \] (44)

where \( V_{n+1}(r) \) is the superpartner potential of \( V_n \) which satisfies

\[ W_n^2 + \frac{dW_n}{dr} = V_n(r) - \epsilon_0^n. \] (45)

We have therefore a recursive relationship between \( W_{n+1} \) and \( W_n \) given by

\[ W_{n+1}^2 - \frac{dW_{n+1}}{dr} = W_n^2 + \frac{dW_n}{dr} + \epsilon_0^n - \epsilon_0^{(n+1)} \] (46)

where \( \epsilon_0^n = -\mu_{n-1}^2 \lambda^4 \) and \( \epsilon_0^{(n+1)} = -\mu_n^2 \lambda^4 \). After the substitutions we end up with the condition

\[ 2n(1 - \lambda^2)^2 y^2 (y^2 - 1) + (y^2 - 1)\lambda^2 \left( ((2n - 1)\mu_{n-1} - (2n + 1)\mu_n)(1 - (1 - \lambda^2)y^2) - 2n \right) + \left( \sum_{i=0}^{n-1} \frac{f_{i-1}}{f_{i-1}^n} \right) \left( 4\frac{f_{i-1}^n}{f_{i-1}^n} - 2\frac{f_{i-1}^n}{f_{i-1}^n} + 2(1 - \lambda^2)y(y^2 - 1) + 2y\lambda^2(\mu_n - \mu_{n-1}) \right) + \left( \sum_{i=0}^{n-1} \frac{f_{i-1}^n}{f_{i-1}^n} \right) \left( 4n(1 - \lambda^2)y(y^2 - 1) - 2\frac{f_{i-1}^n}{f_{i-1}^n} + 2y\lambda^2(\mu_n + \mu_{n-1}) \right) - \left( \sum_{i=0}^{n-1} \frac{f_{i-1}^n}{f_{i-1}^n} \right) \left( (1 - \lambda^2)y(y^2 - 1)(2n + 1) + 2y^2 \mu_n \right) + \left( \sum_{i=0}^{n-1} \frac{f_{i-1}^n}{f_{i-1}^n} \right) \left( 2(2n(1 - \lambda^2)(1 - 3y^2) - \lambda^2(\mu_n + \mu_{n-1})) \right) \frac{dy}{dr} = 0. \]
where $f' = \frac{df}{dr}$ and $f'' = \frac{d^2f}{dr^2}$.

Therefore, $f_{n+1}$ can be determined from the knowledge of $f_n$. In this way, the particular cases of $n = 1$, $n = 2$ and $n = 3$ can be checked by inspection and the resulting functions $f_1$, $f_2$ and $f_3$ perfectly agree with equations (29), (35) and (38) respectively.

IV. Conclusions

The hierarchy of Hamiltonians is studied for the restricted class of Natanzon potentials, with two parameters and a general form for the superpotential is proposed. The superalgebra drives us to the conclusion that the whole superfamily is a collection of exactly solvable Hamiltonians.

As a final remark, some aspects related to shape invariance emerge from the results presented here. The shape invariance concept introduced by Gedenshtein, [13], has motivated several discussions about the exactly solvable potentials. In ref. [11] there is an extensive explanation about this subject.

The Natanzon potential is not shape invariant in the Gedenshtein sense, [9]. However, for the restricted class analysed here, it was possible to obtain a general form for the superpotential, as shown in the previous section. Thus, we conceived a more general condition, written in terms of the whole superfamily (not in terms of two members, $V_1$ and $V_2$, as usual), to indicate if a given potential is exactly solvable or not.

We argue that a criteria of solvability of a potential can be written in terms of the superpotential. In this way, if it is possible to construct a general expression for all superpotentials in the hierarchy of Hamiltonians then the original potential is exactly solvable.
The Hulthén potential without the potential barrier term is another example of an exactly solvable potential which is not shape invariant, but for which it is possible to determine a general expression for the superpotential in the hierarchy, [14].

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