CREATIVE PROOFS IN COMBINATIONS

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ABSTRACT. In this article, we present four issues and provide a creative and concise proof for each of them. The four issues are:

(1) Inequality \(\frac{1}{\sqrt{n\pi}} < \left(\frac{2n}{\pi}\right) < \frac{1}{\sqrt{n\pi}}\)

(2) A special case of Jonathan Wilde’s problem

(3) Combination series

(4) A feature of powerful numbers.

1. INTRODUCTION

In the second section, by proving a theorem, we prove the following equation:

\[
\left(\sum_{k=1}^{n} \left(\frac{f(k)}{2k-1}\right)\right)^2 + (2n + 1)(f(n))^2 = 1, \quad f(k) := \prod_{i=1}^{k} \left(1 - \frac{1}{2i}\right)
\]

The special case of the Wallis integral that we use to prove equation (1.1) is as follows:

\[
\int_{0}^{\frac{\pi}{2}} \cos^{2k}(x) \, dx = \prod_{i=1}^{k} \left(1 - \frac{1}{2i}\right) \times \frac{\pi}{2}
\]

And then using the theorem, we infer the following inequality:

\[
\frac{1}{\sqrt{n\pi} + \frac{\pi}{2}} < \left(\frac{2n}{\pi}\right) < \frac{1}{\sqrt{n\pi}}
\]

We use the inequality in Jonathan Wilde’s problem.

Jonathan Wilde’s problem.

Definition 1.1. \(a(n)\) is the number of ways to draw \(n\) circles in the affine plane.

Jonathan Wilde stated problem \(a(n)\) by setting conditions:

Two circles must be disjoint or meet in two distinct points (tangential contacts are not permitted), and three circles may not meet at a point.

The sequence was proposed by Jonathan Wild, a professor of music at McGill University, who found the values \(a(1) = 1, a(2) = 3, a(3) = 14, a(4) = 173\), and, jointly with Christopher Jones, \(a(5) = 16951\). It is sequence A250001 in the On-Line Encyclopedia of Integer Sequences (OEIS) [1].

We prove a special case of this problem in this article, which we express in the following definition:

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Definition 1.2. \( B(n) \) is the number of ways to draw \( n \) circles in the affine plane, so that no two circles are neither intersecting nor tangent to each other.

We express some of the values of \( B(n) \) in sequence:

\[
B(0) = 1, \ B(1) = 1, \ B(2) = 2, \ B(3) = 4, \ B(4) = 9, \ B(5) = 20, \ldots
\]

In the third section, we state two formulas for \( B(n) \).

In theorems 3.4 and 3.10 we count the number of arrangements of \( n \) circles on a plane so that no two circles are neither intersecting nor tangent to each other. The simplest solution we can use to prove the theorems is positive integer partitions.

In estimating \( B(n) \) in another way, we use inequality (1.2).

We consider three series from the book *Principles And Techniques In Combinatorics* [5], which we prove in general in the fourth section and express them in this section.

We present the two series in question as follows:

\[
(1.3) \quad \sum_{r=0}^{n} \frac{1}{r+1} \binom{n}{r} = \frac{1}{n+1} \left( 2^{n+1} - 1 \right)
\]

\[
(1.4) \quad \sum_{r=0}^{n} \frac{(-1)^{r}}{r+1} \binom{n}{r} = \frac{1}{n+1}
\]

Now we express the third series. Putnam proved the following series in 1962 [5]:

\[
\sum_{r=0}^{n} r^{2} \binom{n}{r} = n(n+1)2^{n-2}
\]

And two Chinese teachers, Wei Guozhen and Wang Kai, in 1988, showed that [5]:

\[
(1.5) \quad \sum_{r=1}^{n} r^{k} \binom{n}{r} = \sum_{i=1}^{k} S(k, i) P_{n}^{i} 2^{n-i}
\]

Where \( k \leq n \) and \( S(k, i) \) are the Stirling numbers of the second kind.

We prove the generalization of the series (1.5) using derivative formulas.

There is a property of sequence \( (1^{n}, 2^{n}, 3^{n}, \ldots) \) that we prove in the fifth section.

For each \( n \) belonging to natural numbers, if we continue the difference of the sentences \( n \) steps, we will reach a fixed sequence of \( n! \).

Example 1.3. Consider the sequence of power numbers for power 3:

Table 1. \( a_{k} = k^{3} \)

\[
\begin{align*}
a_{1} & , \ a_{2} & , \ a_{3} & , \ a_{4} & , \ a_{5} & , \ a_{6} & , \ \cdots \\
1 & , \ 8 & , \ 27 & , \ 64 & , \ 125 & , \ 216 & , \ \cdots \\
7 & , \ 19 & , \ 37 & , \ 61 & , \ 91 & , \ 127 & , \ \cdots \\
12 & , \ 18 & , \ 24 & , \ 30 & , \ 36 & , \ \cdots \\
6 & , \ 6 & , \ 6 & , \ 6 & , \ 6 & , \ \cdots \\
3! & , \ 3! & , \ 3! & , \ 3! & , \ 3! & , \ \cdots
\end{align*}
\]
After three steps of difference of sentences, we reached the fixed sequence $3!$.

## 2. Binomial inequality

**Theorem 2.1.**

$$\lim_{n \to \infty} \left( 1 - \sum_{k=1}^{n} \frac{(f(k))^2}{2k - 1} \right) = \lim_{n \to \infty} (2n + 1)(f(n))^2 = \frac{2}{\pi}$$

We express $f(k)$ as a function in the following form:

$$f(k) := \prod_{i=1}^{k} \left( 1 - \frac{1}{2i} \right).$$

**Proof.** The proof has two parts.

In the first part, we prove the following limit:

$$\lim_{n \to \infty} \left( 1 - \sum_{k=1}^{n} \frac{(f(k))^2}{2k - 1} \right) = \frac{2}{\pi}$$

And in the second part we prove:

$$1 - \sum_{k=1}^{n} \frac{(f(k))^2}{2k - 1} = (2n + 1)(f(n))^2.$$

**Proof of the first part.**

The following integral is known as the Wallis integral:

$$\int_{0}^{\frac{\pi}{2}} \cos^{2k}(x) \, dx = \prod_{i=1}^{k} \left( 1 - \frac{1}{2i} \right) \times \frac{\pi}{2}$$

We rewrite the Wallis integral using the limit definition:

$$\lim_{b \to 0^+} \int_{b}^{\frac{\pi}{2}} \cos^{2k}(x) \, dx = f(k) \times \frac{\pi}{2}$$

Now consider the following integral:

$$g(b) = \int_{b}^{\frac{\pi}{2}} \sqrt[2]{1 - \cos^2(x)} \, dx$$

A solution for calculating the limit of the above expression is as follows:

$$\lim_{b \to 0^+} \int_{b}^{\frac{\pi}{2}} \sqrt[2]{\sin^2(x)} \, dx = \lim_{b \to 0^+} \left[ -\cos(x) \right]_{b}^{\frac{\pi}{2}} = 1$$

We rewrite integral 2.1 and calculate its limit in the following form:

$$\lim_{b \to 0^+} g(b) = \lim_{b \to 0^+} \int_{b}^{\frac{\pi}{2}} (1 - \cos^2(x))^\frac{1}{2} \, dx \quad \left( 0 < b \leq \frac{\pi}{2} \right)$$

According to the set limits we have:

$$| - \cos^2(x) | < 1$$
Using binomial expansion we can express:

\[
\lim_{b \to 0^+} g(b) = \lim_{b \to 0^+} \int_b^{\pi/2} \left( 1 + \sum_{k=1}^{\infty} \left( \frac{1}{k} \right) \left( -\cos^2(x) \right)^k \right) dx
\]

We rewrite the limit in the following form:

\[
\lim_{b \to 0^+} g(b) = \lim_{b \to 0^+} \left( \frac{\pi}{2} - b \right) + \left( \sum_{k=1}^{\infty} \left( \frac{1}{k} \right) (-1)^k \left( \lim_{b \to 0^+} \int_b^{\pi/2} \cos^{2k}(x) dx \right) \right)
\]

We calculate the value of \(\left( \frac{1}{k} \right)\) separately below:

\[
\left( \frac{1}{k} \right) = \frac{(-1)^{k-1}}{2k-1} \times \frac{1 \times 3 \times \cdots \times (2k-1)}{2 \times 4 \times \cdots \times (2k)} = \frac{(-1)^{k-1}}{2k-1} f(k)
\]

Now we place this value in the limit:

\[
\lim_{b \to 0^+} g(b) = \left( \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{2k-1}}{2k-1} f(k) \times \left( f(k) \times \frac{\pi}{2} \right) \right)
\]

According to 2.2 it can be stated that:

\[
1 = \frac{\pi}{2} \left( 1 - \sum_{k=1}^{\infty} \frac{(f(k))^2}{2k-1} \right)
\]

Therefore:

\[
\left( 1 - \sum_{k=1}^{\infty} \frac{(f(k))^2}{2k-1} \right) = \frac{2}{\pi}
\]

**Proof of the second part.**

We prove this part using the method of mathematical induction, now consider the two sequences as follows:

\[
a_n = \left( 1 - \sum_{k=1}^{n} \frac{(f(k))^2}{2k-1} \right), \quad b_n = (2n+1)(f(n))^2
\]

Using induction we will show that:

\[
a_n = \left( 1 - \sum_{k=1}^{n} \frac{(f(k))^2}{2k-1} \right) = (2n+1)(f(n))^2 = b_n.
\]

**Induction base:**

\[
a_1 = b_1 = \frac{3}{4}.
\]

**Induction assumption:**

\[
a_n = b_n
\]

And we try to get the correctness of induction by considering the assumption of induction correctly:

\[
a_{n+1} = b_{n+1}.
\]

By performing calculations it can be obtained that:

\[
a_{n+1} = a_n - \frac{(f(n+1))^2}{2n+1}
\]
According to the induction assumption we have:
\\[ a_{n+1} = b_n - \frac{(f(n+1))^2}{2n+1} = (2n+1)(f(n))^2 - \frac{(f(n+1))^2}{2n+1} \]

Instead of \((f(n+1))^2\) we put the following relation:
\\[ (f(n+1))^2 = \left( \frac{2n+1}{2(n+1)} \times f(n) \right)^2 \]

So we will have:
\\[ a_{n+1} = (2n+1)(f(n))^2 - \frac{(2n+1)(f(n))^2}{(2n+2)^2} = (2n+1)(f(n))^2 \left( 1 - \frac{1}{(2n+2)^2} \right) \]

Rewrite the phrase as follows:
\\[ a_{n+1} = (2n+1)(f(n))^2 \left( \frac{2n+1}{2n+2} \right) \left( \frac{2n+3}{2n+2} \right) = \left( \frac{2n+1}{2n+2} \right)^2 f(n) \times (2n+3) \]

Therefore:
\\[ a_{n+1} = (2n+3)(f(n+1))^2 = (2n+1+1)(f(n+1))^2 \]

The sentence of induction is imposed:
\\[ a_{n+1} = b_{n+1}. \]

Based on the two parts of the proof, we will have:
\\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \frac{2}{\pi}. \]

**Remark 2.2.** According to the stated theorem, the following equation will be obtained:
\\[ \left( \sum_{k=1}^{n} \frac{(f(k))^2}{2k-1} \right) + (2n+1)(f(n))^2 = 1. \]

**Corollary 2.3.**
\\[ \sum_{k=1}^{\infty} \frac{f(k)(2k)!}{(2k-1)(2k+1)!} = \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{\pi}{2} \]
\\[ \sum_{k=1}^{\infty} \frac{f(k)}{2k-1} = 1 \]

**Corollary 2.4.** The following inequality holds for every \(n\) belonging to natural numbers:
\\[ (2.3) \quad \frac{1}{\sqrt{n\pi} + \frac{\pi}{2}} < \frac{(2n)}{2^{2n}} < \frac{1}{\sqrt{n\pi}}. \]

**Proof.** Consider the following sequence:
\\[ a_n = (2n+1)(f(n))^2 \]

Sequence \(a_n\) is a descending sequence, so according to theorem 2.1 we can say:
\\[ a_n = (2n+1)(f(n))^2 > \frac{2}{\pi} \]
So we will have:

\[(f(n))^2 > \frac{2}{2n+1}\]

Inequality can be rewritten in another form:

\[f(n) > \frac{1}{\sqrt{n\pi + \frac{\pi}{2}}}\]

To prove the other part of the inequality, we consider the following sequence:

\[b_n = (2n)(f(n))^2\]

The limit of sequence can be obtained in the following way:

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n - \lim_{n \to \infty} (f(n))^2
\]

We got the limit of the sequence \(a_n\) in theorem 2.1 now to get the limit of the sequence \((f(n))^2\), we get \((f(n))^2\):

\[(f(n))^2 = \left(\frac{1 \times 3 \times \cdots \times (2n - 1)}{2 \times 4 \times \cdots \times (2n)}\right)^2\]

\[(f(n))^2 = \left(\frac{(1 \times 3 \times \cdots \times (2n - 1))(2 \times 4 \times \cdots \times (2n))}{2^{2n}(n!)^2}\right)^2\]

\[(f(n))^2 = \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 = \left(\frac{2n}{2n}\right)^2\]

Therefore, using the Stirling approximation, the limit of \((f(n))^2\) can be obtained as follows:

\[
\lim_{n \to \infty} (f(n))^2 = \lim_{n \to \infty} \left(\frac{1}{\sqrt{n\pi 2^{2n}}}\right)^2 = 0
\]

In this case, the limit of \(b_n\) will be obtained:

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \frac{2}{\pi}
\]

So we will have:

\[b_n = 2n(f(n))^2 < \frac{2}{\pi}\]

We rewrite this inequality in another form:

\[f(n) < \frac{1}{\sqrt{n\pi}}.
\]

3. No. of arrangements of n circles in the plane

**Definition 3.1.** A partition of a positive integer \(n\) is a set of positive integers whose sum is \(n\). Since the ordering is immaterial, we may regard a partition of \(n\) as a finite nonincreasing sequence \(n_1 \geq n_2 \geq \cdots \geq n_l\) of positive integers such that \(\sum_{i=1}^{l} n_i = n\). So if \(n = n_1 + n_2 + \cdots + n_l\) is a partition of \(n\), we say that \(n\) is partitioned into \(l\) parts of sizes \(n_1, n_2, \ldots, n_l\) respectively.

We denote the number of partitions of \(n\) where \(n\) is a natural number by \(p(n)\).

**Definition 3.2.** \(p_1(n, k)\) is the number of partitions of \(n\) whose largest size is \(k\).
**Definition 3.3.** The function $p_2(n, k)$ is a function of counting the number of states $n$ of the circle, so that the maximum $k - 1$ of the circle is inside at least one circle.

It is clear from the definition of $p_2(n, k)$ and $p_1(n, k)$ that for any natural number such as $n$:

$$p_1(n, 1) = p_2(n, 1) = 1$$

**Notation 1.** Since the question in question is equivalent to partitions of positive integers, so we use the notation to match the circles on the partitions.

In a circle where there is no circle:

$$1 \equiv \bigcirc = B^1(1 - 1) = B^1(0)$$

In general, if there are $n$ circles in a circle, there are $B(n - 1)$ states for that circle.

To get the number of states that the circles are next to each other based on the MP principle, we multiply them:

$$1 + 1 \equiv \bigcirc \bigcirc = B(0)B(0) = B^2(0).$$

Thus, according to the partitions of integers, we can express:

$$n = n_1 + n_2 + \cdots + n_l = B(n_1 - 1)B(n_2 - 1)\cdots B(n_l - 1). \tag{3.1}$$

**Theorem 3.4.** For each number belonging to natural numbers such as $n$, the function $B(n)$ is equal to:

$$B(n) = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} (B^{k_1}(0)B^{k_2}(1)\cdots B^{k_n}(n - 1)).$$

**Proof.** Given (3.1) the sum of the states of the partitions is equal to $B(n)$. Therefore, using the equation $k_1 + 2k_2 + \cdots + nk_n = n$, we can express that:

$$B(n) = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} (B^{k_1}(1 - 1)B^{k_2}(2 - 1)\cdots B^{k_n}(n - 1)).$$

Now we try to express another formula for $B(n)$ by obtaining $p_2(n, k)$.

**Theorem 3.5.** $p_2(n, k)$ for every $n$ belongs to natural numbers and for every $k$ belongs to natural numbers where $1 < k \leq n$:

$$n = k \left\lfloor \frac{n}{k} \right\rfloor + r, \quad 0 \leq r \leq k - 1$$

$$p_2(n, k) = \left( \left\lfloor \frac{n}{k} \right\rfloor - 1 \right) \left( B^i(k - 1) \sum_{j=1}^{k-1} p_2(n - ik, j) \right) + B^i \left( k - 1 \right) B(r)$$

**Proof.** According to the division algorithm, if we assume

$$\left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n$$

\[1\] The Multiplication Principle (MP)
It can be stated that:

\[ n \equiv B(n - 1)B(0) = p_2(n, n) \]
\[ 1 + (n - 1) \equiv B(n - 2)B(1) = p_2(n, n - 1) \]
\[ \vdots \]
\[ (n - \left\lfloor \frac{n}{2} \right\rfloor) + \left\lfloor \frac{n}{2} \right\rfloor \equiv B \left( \left\lfloor \frac{n}{2} \right\rfloor \right) B \left( n - \left\lfloor \frac{n}{2} \right\rfloor \right) = p_2 \left( n, \left\lfloor \frac{n}{2} \right\rfloor \right) \]

Hence in the general case for this range \( \left\lfloor \frac{n}{k} \right\rfloor \leq k \leq n \):
\[ p_2(n, k) = B(n - k)B(k - 1). \]

And now suppose:
\[ 2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \]

In this case, based on the number \( k \), we will have:
\[
(n - k) + k \equiv B^1(k - 1) \sum_{j=1}^{k-1} p_2(n - k, j) \\
(n - 2k) + k + k \equiv B^2(k - 1) \sum_{j=1}^{k-1} p_2(n - 2k, j) \\
\vdots \\
(n - \left( \left\lfloor \frac{n}{k} \right\rfloor - 1 \right) k) + k + \cdots + k \equiv B^\left\lfloor \frac{n}{k} \right\rfloor - 1(k - 1) \sum_{j=1}^{k-1} p_2 \left( n - \left( \left\lfloor \frac{n}{k} \right\rfloor - 1 \right) k, j \right) \\
(n - \left\lfloor \frac{n}{k} \right\rfloor k) + k + k + \cdots + k \equiv B^\left\lfloor \frac{n}{k} \right\rfloor (k - 1)B(r).
\]

**Remark 3.6.** According to theorem 3.5 it can be stated:
If \( \left\lfloor \frac{n}{k} \right\rfloor = 1 \) or \( \frac{n}{2} \leq k \leq n \) then:
\[ p_2(n, k) = B(n - k)B(k - 1) \]

And if \( \left\lfloor \frac{n}{k} \right\rfloor = 2 \) then:
\[ p_2(n, k) = B(k - 1)(p_2(n - k, k - 1) + \cdots + p_2(n - k, 1)) + B^2(k - 1)B(r) \]

**Remark 3.7.** For \( k = 2 \) we will have:
\[ p_2(n, 2) = \left( \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} B^i(2 - 1) \sum_{j=1}^{2-1} p_2(n - 2i, j) \right) + B^\left\lfloor \frac{n}{2} \right\rfloor (2 - 1)B(r) \]
So we will have:
\[ p_2(n, 2) = \left\lfloor \frac{n}{2} \right\rfloor. \]

**Theorem 3.8.** \( p_1(n, k) \) for every \( n \) belongs to natural numbers and for every \( k \) belongs to natural numbers where \( 1 < k \leq n \):
\[ n = k \left\lfloor \frac{n}{k} \right\rfloor + r, \quad 0 \leq r \leq k - 1 \]
\[ p_1(n, k) = \left( \left\lfloor \frac{n}{k-1} \right\rfloor - 1 \right) \sum_{i=1}^{k-1} \sum_{j=1}^{n} p_1(n - ik, j) + p(r). \]

**Proof.** According to theorem 3.5, this theorem is clear. \(\square\)

**Remark 3.9.** For \(k = 2\) we will have:
\[ p_1(n, 2) = \left\lfloor \frac{n}{2} \right\rfloor. \]

**Theorem 3.10.** For each number belonging to natural numbers such as \(n\), the function \(B(n)\) is equal to:
\[ B(n) = \sum_{k=1}^{n} p_2(n, k). \]

**Proof.** According to theorem 3.5 it is clear that \(B(n)\) is equal to the sum of all states \(p_2(n, k)\). \(\square\)

For each \(k > n\), \(p_2(n, k) = p_1(n, k) = 0\).

**Table 2.** \(p_2(n, k)\)

| \(k\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 3 | 2 | 2 | 4 | 8 | 10 |
| 4 | 4 | 4 | 8 | 16 |
| 5 | 9 | 9 | 18 |
| 6 | 20 | 20 |
| 7 | 49 |

**Table 3.** \(p_1(n, k)\)

| \(k\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 3 | 1 | 1 | 2 | 3 | 4 |
| 4 | 1 | 1 | 2 | 3 |
| 5 | 1 | 1 | 2 |
| 6 | 1 | 1 |
| 7 | 1 |
\( p_1(n, 1) = p_2(n, 1) = 1 \)

\( p_1(n, 2) = p_2(n, 2) = \left\lfloor \frac{n}{2} \right\rfloor \)

\( p_1(n, n - 1) = 1 \quad , \quad p_2(n, n - 1) = p_2(n - 1, n - 1) \)

\( p_1(n, n) = 1 \quad , \quad p_2(n, n) = B(n - 1) = \sum_{k=1}^{n-1} p_2(n - 1, k) \)

\[ p_1(n, k) = p_1(n - 1, k - 1) + p_1(n - k, k) \]

\[ p(n) = \sum_{k=1}^{n} p_1(n, k) \quad , \quad B(n) = \sum_{k=1}^{n} p_2(n, k) \]

\[ 0 < x < \frac{1}{4} \]

\[ \lim_{n \to \infty} \sum_{k=0}^{n} B(k)x^k = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{1}{1 - B(k - 1)x^k} \right) \]

A simple bounded of this question is obtained using the generating function \((1, S, S^2, \ldots)\), where \(S\) is the sum of all possible states of the question. This is the bounded of the Catalan numbers:

\[ B(n) < \frac{(2n)^n}{n+1} \]

Suppose \( x \) is a real number and \( 0 < x < \frac{1}{4} \), in this case according to \((3.2)\):

\[ B(n)x^n < \frac{(2n)^n}{2^{2n}} \frac{1}{n+1} \]

Given the inequality of \((2,3)\):

\[ B(n)x^n < \frac{1}{\sqrt{\pi}} \frac{1}{n^{1+\frac{1}{2}}} \]

Therefore:

\[ \sum_{n=0}^{\infty} B(n)x^n < 1 + \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{\pi}} \frac{1}{n^{1+\frac{1}{2}}} \right) \]

Series \( \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{2}}} \) is a Riemann series that according to the Riemann test is a convergent series, so there is a real number like \( w \) that:

\[ \sum_{n=0}^{\infty} B(n)x^n < w \quad , \quad w \in \mathbb{R} \]

By mathematical induction it will be easy to obtain that:

\[ \forall n \in \mathbb{N} \quad , \quad 2^{n-1} \leq B(n) \]

Therefore, it can be stated that:

\[ \frac{2^{2n}}{2} \leq B(n) < \frac{2^{2n}}{(n+1)\sqrt{n\pi}} \]
In which case \( B(n) = (c(n))^{2(n-1)} \) and \( \sqrt{2} \leq c(n) < 2. \)

4. Series

Theorem 4.1. The generalization of the two series (1.4) and (1.3) is as follows:

\[
\forall x \in \mathbb{R} - \{0\}, \quad \forall n \in \mathbb{N}
\]

(1)

\[
\sum_{i=0}^{n} \frac{r}{r+i}(-1)^i \binom{n}{i} x^{r+i} = - \left( \sum_{i=1}^{r} x^{r-i}(1-x)^{n+i} \left( \frac{P_r}{P_i^{n+i}} \right) \right) + \frac{1}{(n+r)}
\]

(2)

\[
\sum_{i=0}^{n} \frac{r}{r+i} \binom{n}{i} x^{r+i} = \left( \sum_{i=1}^{r} (-1)^{i-1} x^{r-i}(1+x)^{n+i} \left( \frac{P_r}{P_i^{n+i}} \right) \right) + \frac{(-1)^r}{(n+r)}
\]

Proof of the first series.

\[
S(x, r) = \sum_{i=0}^{n} \frac{r}{r+i}(-1)^i \binom{n}{i} x^{r+i}
\]

We obtain the derivative of the series:

\[
S'(x, r) = \sum_{i=0}^{n} r(-1)^i \binom{n}{i} x^{r+i-1}
\]

We rewrite the phrase in another form:

\[
S'(x, r) = rx^{r-1} \sum_{i=0}^{n} \binom{n}{i} (-1)^i x^i = rx^{r-1} \sum_{i=0}^{n} \binom{n}{i} (-x)^i
\]

Using a two-sentence series theorem, the phrase is expressed as follows:

\[
S'(x, r) = rx^{r-1}(1-x)^n
\]

Therefore:

\[
S(x, r) = \int_0^x ry^{r-1}(1-y)^n dy.
\]

Thus, by solving the obtained integral, the correctness of the series can be obtained. To solve this integral, we use the by parts method repeatedly. Thus it will be obtained:

\[
S(x, r) = \left[ - \sum_{i=1}^{r-1} y^{r-i}(1-y)^{n+i} \left( \frac{P_r}{P_i^{n+i}} \right) \left( \frac{P_r}{P_i^{n+r}} \right) (1-y)^{n+r} \right]_0^x
\]

Now we place the limits of the integral:

\[
S(x, r) = - \left( \sum_{i=1}^{r-1} x^{r-i}(1-x)^{n+i} \left( \frac{P_r}{P_i^{n+i}} \right) \right) - \frac{(1-x)^{n+r}}{(n+r)} + \frac{1}{(n+r)}
\]

Therefore:

\[
S(x, r) = - \left( \sum_{i=1}^{r} x^{r-i}(1-x)^{n+i} \left( \frac{P_r}{P_i^{n+i}} \right) \right) + \frac{1}{(n+r)}.
\]
Proof of the second series:

\[ M(x, r) = \sum_{i=0}^{n} \binom{n}{i} \frac{r}{r+i} x^{r+i} \]

We will do the same here as we did to prove the first series. We obtain the derivative of the series:

\[ M'(x, r) = \sum_{i=0}^{n} \binom{n}{i} x^{r+i-1} \]

We rewrite the phrase in another form:

\[ M'(x, r) = rx^{r-1} \sum_{i=0}^{n} \binom{n}{i} x^i = rx^{r-1} (1+x)^n \]

Now we express the series using integral:

\[ M(x, r) = \int_0^x \frac{ry^{r-1}}{n} (1+y)^{n+1} dy \]

Therefore the result of the integral will be obtained:

\[ M(x, r) = \left[ \sum_{i=1}^{r} (-1)^{i-1} \frac{P_{i}^{r}}{P_{i}^{n+i}} \right] + \left( -1 \right)^{r-1} \frac{n+1}{n+r} \]

We place the limits of the integral:

\[ M(x, r) = \left[ \sum_{i=1}^{r} (-1)^{i-1} x^{r-i} (1+x)^{n+i} \frac{P_{i}^{r}}{P_{i}^{n+i}} \right] - \left( -1 \right)^{r-1} \frac{n+1}{n+r} \]

Therefore:

\[ M(x, r) = \left( \sum_{i=1}^{r} (-1)^{i-1} x^{r-i} (1+x)^{n+i} \frac{P_{i}^{r}}{P_{i}^{n+i}} \right) + \left( -1 \right)^{r} \frac{n+1}{n+r} \]

\[ \square \]

Remark 4.2. The special case of two series in theorem 4.1:

\[ S(1, r) = \sum_{i=0}^{n} \frac{(-1)^i r}{i} \binom{n}{i} = \frac{1}{(n+r)} \]

\[ S(x, 1) = (1-x)^{n+1} + 1 \]

\[ M(1, r) = \sum_{i=0}^{n} \frac{r}{i} \binom{n}{i} = \left( \sum_{i=1}^{r} (-1)^{i-1} 2^{n+i} \frac{P_{i}^{r}}{P_{i}^{n+i}} \right) + \left( -1 \right)^{r} \frac{n+1}{n+r} \]

Remark 4.3. Consider the following series:

\[ \forall m, n \in \mathbb{N} \]

\[ \sum_{r=m}^{\infty} \frac{1}{\prod_{i=0}^{n}(r+i)} = \frac{1}{nn!} \left( \sum_{i=0}^{n}(r+1)^{n+i} \frac{P_{i}^{r}}{P_{i}^{n+i}} \right) + \left( -1 \right)^{r} \frac{n+1}{n+r} \]

The proof of this series is obvious according to the following relation:

\[ \frac{n}{r^{(n+r)}} = \frac{1}{(n+r-1)} - \frac{1}{r} \]
Theorem 4.4. \( \forall m \in \mathbb{R}^+, \forall n \in \mathbb{N}, \forall k \in \mathbb{N} \)

\[
\sum_{r=0}^{n} \binom{n}{r} a^{-r} r^k (bm)^r = \sum_{i=1}^{k} S(k, i) P_i^n (a + bm)^{n-i} (bm)^i.
\]

Proof. We define the \( f(x) \) function:

\( \forall x \in \mathbb{R}, f(x) := (a + be^x)^n \)

The \( f(x) \) function can be rewritten in another form:

\[
f(x) = \sum_{r=0}^{n} \binom{n}{r} a^{-r} r^k (be^x)^r
\]

We get the \( k \)-th derivative of the function \( f(x) \):

\[
f^{(k)}(x) = \sum_{r=0}^{n} \binom{n}{r} a^{-r} r^k (be^x)^r
\]

Now we get the \( k \)-th derivative of the function \( f(x) \) using the general Leibniz rule:\[6\]

\[
f^{(k)}(x) = \sum_{r_1+r_2+\ldots+r_n=k} \binom{k}{r_1, r_2, \ldots, r_n} (a + be^x)^{(r_1)} \cdots (a + be^x)^{(r_n)}
\]

Now if we consider, we have \( r_i \neq 0 \) for \( u \) states, so we have \( r_l = 0 \) for the remaining \((n - u)\) states, which can be expressed as: \( l, u \in \mathbb{N}_n \)

\[
f^{(k)}(x) = \sum_{u=1}^{k} \binom{k}{r_1, r_2, \ldots, r_n} (a + be^x)^{n-u} (be^x)^u \left( \binom{n}{n-u} \right)
\]

Using the Stirling numbers of the second kind, it can be stated that:

\[
f^{(k)}(x) = S(k, 1)! \left( \binom{n}{1} (a + be^x)^{n-1} be^x + \cdots + S(k, k)! \left( \binom{n}{k} (a + be^x)^{n-k} be^x \right) \right)
\]

Therefore:

\[
f^{(k)}(x) = \sum_{i=1}^{k} S(k, i) P_i^n (a + be^x)^{n-i} (be^x)^i
\]

In this case we will have:

\[
(4.1) \quad f^{(k)}(x) = \sum_{r=0}^{n} \binom{n}{r} a^{-r} r^k (be^x)^r = \sum_{i=1}^{k} S(k, i) P_i^n (a + be^x)^{n-i} (be^x)^i
\]

By limiting the function domain to \( \ln(m) \) that \( m \in \mathbb{R}^+ \), we have:

\[
f^{(k)}(\ln(m)) = \sum_{r=0}^{n} \binom{n}{r} a^{-r} r^k (bm)^r = \sum_{i=1}^{k} S(k, i) P_i^n (a + bm)^{n-i} (bm)^i.
\]

\[ \square \]

Corollary 4.5. Series\[1,4\] is actually the value of \( \ln(1) \) in the \( k \)-th derivative of the function \( f(x) \).

\[
f^{(k)}(\ln(1)) = \sum_{r=1}^{n} \binom{n}{r} r^k = \sum_{i=1}^{k} S(k, i) P_i^n 2^{n-i}.
\]
5. Powerful numbers

Theorem 5.1. In the following sequence, after $n$ steps of the difference between the sentences of the sequence, we arrive at a fixed sequence of the form $n!$.

$1^n, 2^n, 3^n, \ldots$

Proof. This theorem is equivalent to the number of surjective mappings from $\mathbb{N}_n$ to $\mathbb{N}_n$. The function of the number of surjective mappings from $\mathbb{N}_n$ to $\mathbb{N}_m$ is expressed in (5).

$$F(n, m) = \sum_{k=0}^{m} (-1)^k {m \choose k} (m - k)^n$$

Since after $n$ steps we get the answer $n!$, so we use the number $n^n$ and the numbers before it. For this reason, we rewrite the sequence in the following form:

$1^n, 2^n, \ldots, (n - 2)^n, (n - 1)^n, n^n, \ldots$

Sentences difference in the first step:

$\ldots, n^n - (n - 1)^n, (n - 1)^n - (n - 2)^n, \ldots, 2^n - 1^n$

The first step can be shown as follows:

$$A(n, 1) = n^n - (n - 1)^n$$

The second step can be expressed:

$$A(n, 2) = A(n, 1) - A(n - 1, 1) = n^n - 2(n - 1)^n - (n - 2)^n$$

For the third step:

$$A(n, 3) = A(n, 2) - A(n - 1, 2) = n^n - 3(n - 1)^n - 3(n - 2)^n - (n - 3)^n$$

And the same process can be used for the general case:

$$A(n, n) = A(n, n - 1) - A(n - 1, n - 1)$$

$$A(n, n) = \sum_{k=0}^{n} (-1)^k {n \choose k} (n - k)^n$$

It can be clearly seen that $A(n, n)$ is the same as $F(n, m)$, if $n = m$. On the other hand:

$$F(n, m) = m!S(n, m)$$

$S(n, m)$ is Stirling numbers of the second kind and $S(n, n) = 1$.

Therefore:

$$A(n, n) = F(n, n) = n!S(n, n) = n!.$$

$\square$
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