A Cut Discontinuous Galerkin Method for Coupled Bulk-Surface Problems

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Abstract We develop a cut Discontinuous Galerkin method (cutDGM) for a diffusion-reaction equation in a bulk domain which is coupled to a corresponding equation on the boundary of the bulk domain. The bulk domain is embedded into a structured, unfitted background mesh. By adding certain stabilization terms to the discrete variational formulation of the coupled bulk-surface problem, the resulting cutDGM is provably stable and exhibits optimal convergence properties as demonstrated by numerical experiments. We also show both theoretically and numerically that the system matrix is well-conditioned, irrespective of the relative position of the bulk domain in the background mesh.

1 Introduction

In recent years, the analysis and numerical solution of coupled bulk-surface partial differential equations (PDE) have gained a large interests in the fields of computational engineering and scientific computing. Indeed, a number of important phenomena in biology, geology and physics can be described by such PDE systems. A prominent use case are flow and transport problems in porous media when large-scale fracture networks are modeled as 2D geometries embedded into a 3D bulk domain [14, 27]. Another important example is the modeling of cell motility where reaction-diffusion systems on the cell membrane and inner cell are coupled to describe the active reorganization of the cytoskeleton [34, 38]. Coupled bulk-surface PDEs arise also naturally when modeling incompressible multi-phase flow problems with surfactants [15, 18, 19, 33].

The numerical solution of coupled bulk-surface systems poses several challenges even for modern computational methods. First, one faces a system of coupled PDEs...
on domains of different topological dimensionality, which needs to be accommodated by the numerical method at hand. Second, extremely complex surface geometries naturally appear in many realistic application scenarios, e.g., when complex fracture networks in porous media models are considered, and thus fast and robust mesh generation becomes a challenge. Moreover, the simulation of complex droplet systems shows that, even if the initial surface geometry is relatively simple, it might evolve significantly over time and thus can undergo large or even topological changes. For traditional discretization methods, a costly remeshing of the computational domain is then the only resort, and the question of how to transfer the computed solution components between different meshes efficiently and accurately becomes an urgent and challenging matter.

As a potential remedy to these challenges, the so-called cut finite element method (CutFEM) has gained a large interest in recent years, see [5] for a review. The basic idea is to decouple the description of the geometry as much as possible from the underlying approximation spaces by embedding the geometry of the domain into a fixed background mesh which is also used to construct the finite element spaces for the surface and bulk approximations. In order to obtain a stable method, independent of the position of the geometry in the background mesh, and to handle the potential small cut elements in the analysis, certain stabilization terms are added that provide control of the local variation of the discrete functions. In this work we extend ideas from CutFEM framework developed over the last half a decade to synthesize a novel cut discontinuous Galerkin method (cutDGM) for coupled bulk-surface PDEs.

1.1 Earlier work

The development of the cut finite element framework was initiated by the seminal papers [2, 3] considering the weak imposition of boundary conditions for the Poisson problem on unfitted meshes. Shortly after, the idea was picked up by a number of authors to formulate cut finite element methods for the Stokes type problems[4, 6, 21, 22, 28, 29], the Oseen problem [30, 42] and number of related fluid problems, see [40] for a comprehensive overview.

Prior to the arrival of CutFEMs, unfitted discontinuous Galerkin methods have successfully been employed to solve boundary and interface problems on complex and evolving domains [1, 39], including two-phase flows [25, 32, 41]. In unfitted discontinuous Galerkin method, troublesome small cut elements can be merged with neighbor elements with a large intersection support by simply extending the local finite element basis from the large element to the small cut element. As the inter-element continuity is enforced only weakly, the coupling of the these extended basis functions to additional elements incident with the small cut elements does not lead to an over-constrained system, as it would happen if globally continuous finite element functions were employed. Consequently, unfitted discontinuous Galerkin methods provide an alternative stabilization mechanism to ensure the well-posedness and well-conditioning of the discretized systems. Thanks to their favorable conservation
and stability properties, unfitted discontinuous Galerkin methods remain an attractive alternative to continuous CutFEMs, but some drawbacks are the almost complete absence of numerical analysis except for \cite{26, 31}, the implementational labor to reorganize the matrix sparsity patterns when agglomerating cut elements, and the lack of natural discretization approaches for PDEs defined on surfaces.

For PDEs defined on surfaces, the idea of using the finite element space from the embedding bulk mesh was already formulated and analyzed in \cite{36}, and then further extended to high-order methods \cite{17} and evolving surface problems \cite{23, 37}. A stabilized cut finite element for the Laplace-Beltrami problem were introduced in \cite{7} where the additional stabilization cures the resulting system matrix from being ill-conditioned, as an alternative to diagonal preconditioning used in \cite{35}. Finally, after the initial work \cite{12} on fitted finite element discretizations of coupled bulk-surface PDEs, only a few number of corresponding unfitted (continuous) finite element schemes have been formulated, see \cite{11, 20, 24}.

\subsection*{1.2 Contribution and outline of the paper}

In this work, we formulate a novel cut discontinuous Galerkin method for the discretization of coupled bulk-surface problems on a given bounded domain $\Omega$. The strong and weak formulation of a continuous prototype problem are briefly reviewed in Section 2. Motivated by our earlier work \cite{9}, we introduce a cut discontinuous Galerkin method for bulk-surface PDEs in Section 3. The method employs discontinuous piecewise linear elements on a background mesh consisting of simplices in $\mathbb{R}^d$. The boundary $\Gamma$ of the computational domain $\Omega$ is represented by a continuous, piecewise approximation of distance functions associated with $\Gamma$. For both the discrete bulk and surface domain, the active background meshes consist of those elements with a non-trivial intersection with the respective domain. Utilizing the general stabilization framework developed for continuous CutFEMs, we add certain, so-called ghost penalty stabilization in the vicinity of the embedded surface to ensure that the overall cutDGM is stable and its system matrix is well-conditioned. The exact mechanism is further elucidated in Section 4, where short proofs of the coercivity of the bilinear forms introduced in Section 3 are given. We also demonstrate that the condition number of the (properly rescaled) system matrix scales like $O(h^{-2})$. All theoretical results hold with constants independent of the position of the domain relative to the background mesh.

While a full a priori analysis of the proposed method is beyond the limited scope of this work, we perform a convergence rate study in Section 5 instead, demonstrating the optimal approximation properties of the formulated cutDGM. Finally, we also demonstrate that the employed CutFEM stabilizations are essential for the geometrically robust convergence and conditioning properties of the method.
1.3 Basic notation

Throughout this work, $\Omega \subset \mathbb{R}^d$ denotes an open and bounded domain with smooth boundary $\Gamma = \partial \Omega$. For $U \in \{\Omega, \Gamma\}$ and $s \in \mathbb{R}$, let $H^s(U)$ be the standard Sobolev spaces defined on $U$. As usual, we write $(\cdot, \cdot)_s,U$ and $\| \cdot \|_s,U$ for the associated inner products and norms. If there is no confusion, we occasionally write $(\cdot, \cdot)_U$ and $\| \cdot \|_U$ for the inner products and norms associated with $L^2(U)$, with $U$ being a measurable subset of $\mathbb{R}^d$. Finally, any norm $\| \cdot \|_{P,h}$ used in this work which involves a collection of geometric entities $P_h$ should be understood as broken norm defined by $\| \cdot \|_{2,P_h} = \sum_{P \in P_h} \| \cdot \|_{2,P}$ whenever $\| \cdot \|_{P}$ is well-defined, with a similar convention for scalar products $(\cdot, \cdot)_{P,h}$. Finally, it is understood that the notation $\| \cdot \|_{P,h \cap U}$, for any given set $U \subset \mathbb{R}^d$ means to sum up over the corresponding cut parts; that is, $\| \cdot \|_{2,P_h \cap U} = \sum_{P \in P_h} \| \cdot \|_{2,P \cap U}$.

2 Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\Gamma$ equipped with an outward pointing normal field $n_\Gamma$ and signed distance function $\rho$; that is, $\rho$ satisfies $\rho(x) = \pm \text{dist}(x, \Gamma)$ with the distance being strictly negative if $x \in \Omega$ and positive otherwise. It is well known that for some positive $\delta_0$ small enough and any $\delta$ with $0 < \delta < \delta_0$, every point $x$ in the tubular neighborhood $U_\delta(\Gamma) = \{ x \in \mathbb{R}^d : |\rho(x)| < \delta \}$ has a uniquely defined closest point $p(x)$ on $\Gamma$ satisfying $x = p(x) + \rho(x)n(p(x))$, see, e.g, [16, Sec. 14.6]. For any function $v_\Gamma \in C^1(\Gamma)$, the tangential gradient $\nabla_\Gamma v_\Gamma$ is defined by

$$\nabla_\Gamma v_\Gamma = P\nabla v_\Gamma,$$

with $P(x) = I - n_\Gamma(x) \otimes n_\Gamma(x)$ denoting the projection of $\mathbb{R}^d$ onto the tangential space at point $x \in \Gamma$. As model for a coupled bulk-surface problem, we consider the problem: given functions $f_\Omega$ and $f_\Gamma$ on $\Omega$ and $\Gamma$, respectively, and positive constants $c_\Omega, c_\Gamma$, find functions $u_\Omega : \Omega \to \mathbb{R}$ and $u_\Gamma : \Gamma \to \mathbb{R}$ such that

$$-\Delta u_\Omega + u_\Omega = f_\Omega \quad \text{in} \ \Omega,$$
$$\partial_n u_\Omega = c_\Gamma u_\Gamma - c_\Omega u_\Omega \quad \text{on} \ \Gamma,$$
$$-\Delta_\Gamma u_\Gamma + u_\Gamma = f_\Gamma - \partial_n u_\Omega \quad \text{on} \ \Gamma,$$

where $\Delta_\Gamma$ is the Laplace-Beltrami operator on $\Gamma$ defined by

$$\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma.$$ 

Following [11, 12], we can derive a weak formulation by multiplying (2a) with a test function $v_\Omega \in H^1(\Omega)$ and using Green’s formula to obtain
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\[ (\nabla u_\Omega, \nabla v_\Omega)_{\Omega} - (\partial_t u_\Omega, v_\Omega)_{\Gamma} + (u_\Omega, v_\Omega)_{\Omega} = (f, v_\Omega)_{\Omega}, \]

which together with the coupling condition (2b) leads to

\[ (\nabla u_\Omega, \nabla v_\Omega)_{\Omega} + (u_\Omega, v_\Omega)_{\Omega} + (c_\Omega u_\Omega - c_\Gamma u_\Gamma, v_\Omega)_{\Gamma} = (f_\Omega, v_\Omega)_{\Omega}. \]

Next, taking \( v_\Gamma \in H^1(\Gamma) \), a similar treatment of (2c) yields

\[ (\nabla u_\Gamma, \nabla v_\Gamma)_{\Gamma} + (u_\Gamma, v_\Gamma)_{\Gamma} - (c_\Omega u_\Omega - c_\Gamma u_\Gamma, v_\Gamma)_{\Gamma} = (f_\Gamma, v_\Gamma)_{\Gamma}. \]

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Now replacing \( v_\Omega \) with \( c_\Omega v_\Omega \) in (5) and \( v_\Gamma \) with \( c_\Gamma v_\Gamma \) in (6) and summing up the two equations motivates us to introduce the following forms to describe the bulk, surface and coupling related parts of the overall bilinear form \( a(\cdot, \cdot) \):

\[
\begin{align*}
  a_\Omega(u_\Omega, v_\Omega) &= (\nabla u_\Omega, \nabla v_\Omega)_{\Omega} + (u_\Omega, v_\Omega)_{\Omega}, \\
  a_\Gamma(u_\Gamma, v_\Gamma) &= (\nabla u_\Gamma, \nabla v_\Gamma)_{\Gamma} + (u_\Gamma, v_\Gamma)_{\Gamma}, \\
  a_\Omega\Gamma(u, v) &= (c_\Omega u_\Omega - c_\Gamma u_\Gamma, c_\Omega v_\Omega - c_\Gamma v_\Gamma)_{\Gamma}.
\end{align*}
\]

As final ingredient, we define the bulk function spaces \( V_\Omega = H^1(\Omega) \), the surface function space \( V_\Gamma = H^1(\Gamma) \) and the total space \( V = V_\Omega \times V_\Gamma \), and introduce also the short-hand notation \( u = (u_\Omega, u_\Gamma) \in V \) and \( v = (v_\Omega, v_\Gamma) \in V \). Then the variational problem for the coupled bulk-surface PDE (2) is to seek \( u \in V \) such that \( \forall v \in V \)

\[ a(u, v) = l(v), \]

where the bilinear form \( a(\cdot, \cdot) \) and linear form \( l(\cdot) \) are given by

\[ \begin{align*}
  a(u, v) &= c_\Omega a_\Omega(u_\Omega, v_\Omega) + c_\Gamma a_\Gamma(u_\Gamma, v_\Gamma) + a_\Omega\Gamma(u, v), \\
  l(v) &= c_\Omega(f_\Omega, v_\Omega)_{\Omega} + c_\Gamma(f_\Gamma, v_\Gamma)_{\Gamma}.
\end{align*} \]

Using the natural energy norm \( ||v|| = \sqrt{a(v, v)} \), it follows immediately that the bilinear form \( a \) is coercive with respect to \( ||\cdot|| \) and that both forms \( a \) and \( l \) are continuous, and thus the Lax-Milgram theorem ensures the existence of a unique solution to the weak problem (10), see also [12].

3 A cut discontinuous Galerkin method for bulk-surface problems

The main idea in the cut discontinuous Galerkin discretization of the bulk-surface PDE (10) is now to embedd the domain \( \Omega \) into an easy-to-generate 3d background mesh in an unfitted manner. The approximation spaces for the discrete bulk and surface solution components are then given by suitable restrictions of the discontinuous finite element functions defined on background mesh to the bulk and surface domains, respectively. We start with describing the relevant computational domains
and related geometric quantities before we turn to the definition of the cut finite element spaces and the final discrete formulation.

### 3.1 Computational domains

Assume that $\mathcal{T}^h$ is a quasi-uniform\(^1\) background mesh with global mesh size $h$ consisting of shape-regular elements $\{T\}$ which cover $\Omega$. Let $\rho^h$ be a continuous, piecewise linear approximation of the distance function $\rho$ and define the discrete surface $\Gamma^h$ as the zero level set of $\rho^h$,

$$\Gamma^h = \{x \in \Omega : \rho^h(x) = 0\}$$

and correspondingly, the discrete bulk domain is given by

$$\Omega^h = \{x \in \Omega : \rho^h(x) < 0\}.$$  \hspace{1cm} (14)

Note that $\Gamma^h$ is a polygon consisting of flat faces with a piecewise defined constant exterior unit normal $n$. We assume that:

- $\Gamma^h \subset U_{\delta_0}(\Gamma)$ and that the closest point mapping $p : \Gamma^h \to \Gamma$ is a bijection for $0 < h \leq h_0$.
- The following estimates hold

$$\|\rho\|_{L^\infty(\Gamma^h)} \lesssim h^2, \quad \|n - n^h \circ p\|_{L^\infty(\Gamma)} \lesssim h.$$ \hspace{1cm} (15)

These properties are, for instance, satisfied if $\rho^h$ is the Lagrange interpolant of $\rho$.

Starting from the background mesh $\mathcal{T}^h$, we define the active (background) meshes for discretization of the bulk and surface problem by

$$\mathcal{T}^h_\Omega = \{T \in \mathcal{T}^h : T^\circ \cap \Omega^h \neq \emptyset\},$$

$$\mathcal{T}^h_\Gamma = \{T \in \mathcal{T}^h : T \cap \Gamma^h \neq \emptyset\},$$

respectively. Here, $T^\circ$ denotes the topological interior of an element $T$ and thus $\mathcal{T}^h_\Omega$ does not contain any element which intersects only with the boundary $\Gamma^h$ but not with the interior $\Omega^h$. Clearly, $\mathcal{T}^h_\Gamma \subset \mathcal{T}^h_\Omega$. For the actives meshes $\mathcal{T}^h_\Omega$ and $\mathcal{T}^h_\Gamma$, the corresponding sets of interior faces are denoted by

$$\mathcal{F}^h_\Omega = \{F = T^+ \cap T^- : T^+ , T^- \in \mathcal{T}^h_\Omega\},$$

$$\mathcal{F}^h_\Gamma = \{F = T^+ \cap T^- : T^+ , T^- \in \mathcal{T}^h_\Gamma\}.$$ \hspace{1cm} (18)

Note that by extracting $\mathcal{F}^h_\Gamma$ from $\mathcal{F}^h_\Omega$ instead of $\mathcal{F}^h$, we automatically pick a unique element from $\mathcal{F}^h$ in the case that $\Gamma^h \cap T$ coincides with an interior face of the

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\(^1\) Quasi-uniformity is mainly assumed to simplify the overall presentation.
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background mesh $\mathcal{T}^h$. Additionally, we will also need the set of interior faces of the active bulk mesh $\mathcal{T}_{\Omega}^h$ which belong to elements intersected by the discrete surface $\Gamma^h$,

$$\mathcal{F}_{\Omega}^{h,g} = \{ F = T^+ \cap T^- : T^+ \in \mathcal{T}_{\Gamma}^h \lor T^- \in \mathcal{T}_{\Gamma}^h \}. \quad (20)$$

This set of faces will be instrumental in defining certain stabilization forms, also known as ghost penalties, hence the superscript $g$. As usual, face normals $n^+_F$ and $n^-_F$ are given by the unit normal vectors which are perpendicular on $F$ and are pointing exterior to $T^+$ and $T^-$, respectively.

For the surface approximation $\Gamma^h$, corresponding collection of geometric entities can be generated by considering the intersection of $\Gamma^h$ with individual elements of the active mesh, i.e., we define the set of surface faces and their edges by

$$\mathcal{K}^h = \{ K = \Gamma^h \cap T : T \in \mathcal{T}_{\Gamma}^h \}, \quad (21)$$

$$\mathcal{E}^h = \{ E = K^+ \cap K^- : K^+, K^- \in \mathcal{K}^h \}. \quad (22)$$

To each interior edge $E$ we associate the co-normals $n^+_E$ given by the unique unit vector which is coplanar to the surface element $K^\pm$, perpendicular to $E$ and points outwards with respect to $K^\pm$. Note that while the two face normals $n^+_F$ only differ by a sign, the edge co-normals $n^+_E$ do lie in genuinely different planes. The various set of geometric entities are illustrated in Figure 1.

Fig. 1: Computational domains for the bulk-surface problem. (Left) Active mesh used to define the approximation space for the bulk solution. Faces on which ghost penalty stabilization are defined are plotted as dashed faces. (Right) Corresponding computational domain set-up for the discretization of the surface.
3.2 The cut discontinuous Galerkin method

We start with defining the discrete counterparts of the function spaces \( V_\Omega \) and \( V_\Gamma \) to be the broken polynomial spaces consisting of piecewise linear, but not necessarily globally continuous functions defined on the respective active meshes:

\[
V^h_\Omega = \bigoplus_{T \in \mathcal{T}^h_\Omega} P_1(T), \quad V^h_\Gamma = \bigoplus_{T \in \mathcal{T}^h_\Gamma} P_1(T), \quad V^h = V^h_\Omega \times V^h_\Gamma. \tag{23}
\]

For the formulation of the cut discontinuous Galerkin method, we also need the notation of average and fluxes of piecewise defined functions. More precisely, assume that \( \sigma \) and \( w \) are, possibly vector-valued, elementwise defined functions on \( \mathcal{T}^h \) which are smooth enough to admit a two-valued trace on all faces. Then the standard and face normal weighted average fluxes are given by

\[
\langle \sigma \rangle |_F = \frac{1}{2} (\sigma_F^+ + \sigma_F^-), \tag{24}
\]

\[
\langle n_F \cdot \sigma \rangle |_F = \frac{1}{2} n_F^+ \cdot (\sigma_F^+ + \sigma_F^-) = \frac{1}{2} (n_F^+ \cdot \sigma_F^+ - n_F^- \cdot \sigma_F^-), \tag{25}
\]

while the jump across an interior face \( F \in \mathcal{F}^h \) is defined by

\[
[w]|_F = w_F^+ - w_F^-, \tag{26}
\]

with \( w(x)^\pm = \lim_{t \to 0^\pm} w(x - tn_F^\pm) \). In the case of vector-valued functions, the jump is taken componentwise. As the co-normal vectors \( n_F^\pm \) are generally not collinear, the standard and co-normal weighted average fluxes for a piecewise discontinuous, possibly vector-valued function \( \sigma \) on \( \mathcal{F}^h \) is defined by

\[
\langle \sigma \rangle |_E = \frac{1}{2} (\sigma_E^+ + \sigma_E^-), \tag{27}
\]

\[
\langle n_E \cdot \sigma \rangle |_E = \frac{1}{2} (n_E^+ \cdot \sigma_E^+ - n_E^- \cdot \sigma_E^-), \tag{28}
\]

respectively. Similarly, the jump across an interior face \( E \in \mathcal{E}^h \) is given by

\[
[w]|_E = w_E^+ - w_E^-. \tag{29}
\]

We are now ready to define the discrete, discontinuous Galerkin counterparts of the bilinear forms (7), (8), and (9) and set
Now the cut discontinuous Galerkin method for the bulk-surface problem is to seek the ghost penalty enhanced bulk and surface bilinear forms where 

\[
d_h^b(v_{\Omega}, w_{\Omega}) = (\nabla v_{\Omega}, \nabla w_{\Omega})_{\mathcal{F}_{\Omega}^h \cap \Omega^h} + (v_{\Omega}, w_{\Omega})_{\mathcal{F}_{\Omega}^h \cap \Omega^h} + \Gamma_h (h^{-1}[v_{\Omega}],[w_{\Omega}])_{\mathcal{F}_{\Omega}^h} \\
- (\langle n_F \cdot \nabla v_{\Omega} \rangle, [w_{\Omega}])_{\mathcal{F}_{\Omega}^h \cap \Omega^h} - (\langle [\nabla w_{\Omega} \rangle, n_F \cdot [v_{\Omega}])_{\mathcal{F}_{\Omega}^h \cap \Omega^h},
\]

(30)

\[
d_h^b(v_T, w_T) = (\nabla v_T, \nabla w_T)_{\mathcal{F}_{T}^h} + (v_T, w_T)_{\mathcal{F}_{T}^h} + \Gamma_T (h^{-1}[v_T],[w_T])_{\mathcal{F}_{T}^h} \\
- (\langle n_F \cdot \nabla v_T \rangle, [w_T])_{\mathcal{F}_{T}^h} - (\langle [\nabla w_T \rangle, n_F \cdot [v_T])_{\mathcal{F}_{T}^h},
\]

(31)

\[
d_h^b(v, w) = (c_{\Omega} v_{\Omega} - c_T v_T, c_{\Omega} w_{\Omega} - c_T w_T)_{\mathcal{F}_{T}^h},
\]

(32)

\[
a_h(v, w) = c_{\Omega} d_h^b(v_{\Omega}, w_{\Omega}) + c_T d_h^b(v_T, w_T) + d_{\mathcal{F}_{T}^h}(v, w).
\]

(33)

Similarly, the relevant discrete linear forms are given by

\[
l_h^b(v_{\Omega}) = (f_{\Omega}, v_{\Omega})_{\Omega^h},
\]

(34)

\[
l_h^b(v_T) = (f_{T}, v_T)_{T^h},
\]

(35)

\[
l_h(v) = c_{\Omega} l_h^b(v_{\Omega}) + c_T l_h^b(v_T).
\]

(36)

Here, \( f_T \) denotes the extension of \( f_T \) to the tubular neighborhood \( U_\delta(\Gamma) \) using the closest point projection by requiring that \( f_T(x) = f_T(p(x)) \). Finally, appropriate ghost penalties for the bulk and surface part are defined by

\[
f_h^b(v_{\Omega}, w_{\Omega}) = \mu_{\Omega} h^{-1}([v_{\Omega}],[w_{\Omega}])_{\mathcal{F}_{\Omega}^h} + \tau_{\Omega} h (n_F \cdot [\nabla v_{\Omega}], n_F \cdot [\nabla w_{\Omega}])_{\mathcal{F}_{\Omega}^h}
\]

(37)

\[
f_h^b(v_T, w_T) = \mu_T h^{-2}([v_T],[w_T])_{\mathcal{F}_{T}^h} + \tau_T (n_F \cdot [\nabla v_T], n_F \cdot [\nabla w_T])_{\mathcal{F}_{T}^h}
\]

(38)

\[
l_h^b(v, w) = c_{\Omega} l_h^b(v_{\Omega}, w_{\Omega}) + c_T l_h^b(v_T, w_T)
\]

(39)

where \( \mu_{\Omega}, \mu_T, \tau_{\Omega}, \tau_T \) are positive parameters. To ease the notation, we also define the ghost penalty enhanced bulk and surface bilinear forms

\[
A_h^b(v_U, w_U) = a_h^b(v_U, w_U) + f_h^b(v_U, w_U), \quad U \in \{\Omega, \Gamma\}.
\]

(40)

Now the cut discontinuous Galerkin method for the bulk-surface problem is to seek \( u^h = (u_{\Omega}^h, u_T^h) \in V^h = V_{\Omega}^h \times V_{T}^h \) such that \( \forall v \in V^h \)

\[
A_h^b(u^h, v) := a_h^b(u^h, v) + f_h^b(u^h, v) = l_h^b(v).
\]

(41)

**Remark 1.** The defined ghost penalties are crucial to devise a geometrically robust, well-conditioned and optimally convergent discretization method, irrespective of the particular cut configuration. We note that in general, the unstabilized cutDGM suffers from three drawbacks. First, certain inverse inequalities fundamental for the analysis of DGMs do not hold any more when only the physical, cut part of the background mesh is considered. Second, cut configurations with very small cut parts can lead to an almost vanishing contribution of certain degree of freedoms in the system matrix. Third, the restriction of discontinuous finite element functions from the active mesh \( \mathcal{F}_{\Omega}^h \) to the surface \( \Gamma^h \) results in a highly linear dependent set of
functions, and thus purely surface-based “norms” are not capable of distinguishing them, which also leads to an ill-conditioned system matrix.

4 Stability properties

In this section, we investigate the stability properties of the proposed cutDGM for the coupled bulk-surface problem. In particular, we show that the ghost-penalty enhanced discrete form $A_h$ is coercive with respect to a natural discrete energy norm and that the condition number of the resulting system matrix scales as $O(h^{-2})$, irrespective of the position of $\Omega^h$ relative to the background mesh $\mathcal{T}^h$.

4.1 Norms and coercivity

A natural discrete energy norm for the forthcoming stability analysis is given by combining the individual discrete energy norms for the bulk and surface parts,

$$
\|v\|_{1,h,\Omega}^2 = \|\nabla v\|_{\Omega,h}^2 + \|v\|_{\Omega,h}^2 + \|h^{-\frac{1}{2}}|v|_\Omega\|_{\Omega,h}^2 + j^h_{\Omega}(v,\Omega),
$$

(42)

$$
\|v\|_{1,h,\Gamma}^2 = \|\nabla v\|_{\Gamma,h}^2 + \|v\|_{\Gamma,h}^2 + \|h^{-\frac{1}{2}}|v|_\Gamma\|_{\Gamma,h}^2 + j^h_{\Gamma}(v,\Gamma),
$$

(43)

with the semi-norm induced by the coupling bilinear $a^h_{\Omega\Gamma}$ to define

$$
\|v\|_h^2 = c_{\Omega}\|v\|_{1,\Omega}^2 + c_{\Gamma}\|v\|_{1,\Gamma}^2 + \|c_{\Omega}v_{\Omega} - c_{\Gamma}v_{\Gamma}\|_{\Gamma,h}^2.
$$

(44)

With these norm definitions, the coercivity of the total bilinear form $A_h$ can be easily shown once coercivity properties for the bulk and surface bilinear form are established individually. In other words, we wish to show that

$$
\|v\|_{1,\Omega}^2 \leq A_{\Omega}(v,\Omega) \quad \forall v_{\Omega} \in V_{\Omega}^h,
$$

(45)

$$
\|v\|_{1,\Gamma}^2 \leq A_{\Gamma}(v,\Gamma) \quad \forall v_{\Gamma} \in V_{\Gamma}^h,
$$

(46)

which together with the simple observation that

$$
A_h(v,v) = A_{\Omega}(v,\Omega) + A_{\Gamma}(v,\Gamma) + a^h_{\Omega\Gamma}(v,v)
$$

(47)

$$
\geq \|v\|_{1,\Omega}^2 + \|v\|_{1,\Gamma}^2 + \|c_{\Omega}v_{\Omega} - c_{\Gamma}v_{\Gamma}\|_{\Gamma,h}^2,
$$

(48)

leads us to the following proposition.

**Proposition 1.** The discrete bilinear form $A_h$ is coercive with respect to the discrete energy norm (44):

$$
\|v\|_h^2 \leq A_h(v,v), \quad \forall v \in V^h.
$$

(49)
The following two subsection are thus devoted to prove that the estimates (45) and (46) hold.

### 4.2 Coercivity of the discrete bulk form $A^h_\Omega$

A standard ingredient in the numerical analysis of discontinuous Galerkin methods is the inverse inequality

$$\|n_F \cdot \nabla v\|_F \leq C_I h^{-\frac{1}{2}} \|\nabla v\|_T,$$

(50)

which holds for discrete functions $v \in P_1(T)$. Here, the face $F$ is part of the element boundary $\partial T$ and the inverse constant $C_I = C_I(\frac{|F|}{|T|})$ depends on the ratio of the face area $|F|$ and element volume $|T|$, and thus ultimately on the shape regularity of $\mathcal{S}^h$.

Unfortunately, a corresponding inverse inequality of the form

$$\|n_F \cdot \nabla v\|_{F \cap \Omega^h} \leq C_I h^{-\frac{1}{2}} \|\nabla v\|_{T \cap \Omega^h}$$

(51)

does not hold as the ratio $\frac{|F|}{|T|}$ can become arbitrarily large, depending on the cut configuration. As a partial replacement, one might be tempted to use the simple estimate

$$\|n_F \cdot \nabla v\|_{F \cap \Omega^h} \leq \|n_F \cdot \nabla v\|_F \leq C_I h^{-\frac{1}{2}} \|\nabla v\|_T$$

(52)

instead. To fully exploit this idea, it is necessary to extend the control of the $\|\nabla v\|_{\Omega^h}$ part in natural energy norm associated with $A^h_{\Omega}$ from the physical domain $\Omega^h$ to the entire active mesh $\mathcal{S}^h_{\Omega}$. This is precisely the role of the ghost-penalty term $j^h_{\Omega}$.

**Lemma 1.** For $v \in V^0_{\Omega}$ it holds that

$$\|\nabla v\|^2_{\mathcal{S}^h_{\Omega}} \leq \|\nabla v\|^2_{\Omega^h} + j^h_{\Omega}(v, v) \lesssim \|\nabla v\|^2_{\mathcal{S}^h_{\Omega}},$$

(53)

and consequently, using (52)

$$\|h^\frac{1}{2} n_F \cdot \nabla v\|^2_{\mathcal{S}^h_{\Omega} \cap \Omega^h} \lesssim \|\nabla v\|^2_{\Omega^h} + j^h_{\Omega}(v, v),$$

(54)

with the hidden constant depending only in the shape-regularity of $\mathcal{S}^h$.

**Proof.** For a detailed proof, we refer to [3, 28, 29].

Thanks to the ghost penalty Lemma 1, we can establish the coercivity of $A^h_{\Omega}$ by simply following the standard arguments in the classical proof for symmetric interior penalty methods.
Proposition 2. The discrete bulk form $A^h_{\Omega}$ is coercive with respect to the discrete energy norm $||| \cdot |||_{h,\Omega}$; that is,

$$|||v|||^2_{h,\Omega} \lesssim A^h_{\Omega}(v,v), \quad \forall v \in V^h,$$

(55)

Proof. We follow closely the standard arguments. Setting $w_\Omega = v_\Omega$ in (30) and combining the ghost-penalty Lemma 1 and a $\varepsilon$-Cauchy-Schwarz inequality of the form $2ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ with an inverse estimate yields

$$A^h_{\Omega}(v,v) = \|\nabla v\|^2_{\Omega} - 2([n_\Gamma \cdot \nabla v])_{\partial \Omega} + \gamma_\Omega \|h^{-\frac{1}{2}}[v]\|^2_{\partial \Omega}$$

$$+ \int_{\Omega} \varepsilon \|\nabla v\|^2_{\partial \Omega} - \varepsilon \|h^{-\frac{1}{2}}(n_\Gamma \cdot \nabla v_T)\|^2_{\partial \Omega} - \varepsilon^{-1}\|h^{-\frac{1}{2}}[v]\|_{\partial \Omega}^2$$

$$+ \gamma_\Omega h^{-\frac{1}{2}}[v]\|^2_{\partial \Omega} + \frac{1}{2} \int_{\Omega} (v,v) + \|v\|^2_{\Omega}$$

$$\geq \gamma_{\Omega} (1 - \varepsilon G_{\gamma}) \|\nabla v\|^2_{\partial \Omega}$$

$$+ (\gamma_{\Omega} - \varepsilon^{-1}) \|h^{-\frac{1}{2}}[v]\|^2_{\partial \Omega} + \frac{1}{2} \int_{\Omega} (v,v) + \|v\|^2_{\Omega} \geq \|v\|^2_{h,\Omega}$$

(61)

if we chose $0 < \varepsilon \lesssim 1/(2G_{\gamma})$ small enough and $\gamma_{\Omega} > \varepsilon^{-1}$.

4.3 Coercivity of the discrete surface form $A^h_{\Gamma}$.

Next, we turn to the stability properties of the discrete surface form $A^h_{\Gamma}$. First observe that the unstabilized DG energy “norm”

$$|||v|||^2_{\Gamma} := \|\nabla_{\Gamma} v\|^2_{\Gamma} + \|v\|^2_{\Gamma} + \|h^{-\frac{1}{2}}[v]\|^2_{\partial \Gamma}$$

(62)

does not define an actual norm on $V^h_{\Gamma}$. For instance, the piecewise linear and continuous approximation $\rho^h$ of the distance function $\rho$ vanishes on $\Gamma^h$. It was shown in [9] that a proper norm can obtained if the ghost penalty term $j^h_{\Gamma}$ was added, resulting in our norm definition (43). More, precisely, the following discrete Poincaré inequality was established.

Lemma 2. Let $h \in (0,h_0]$ with $h_0$ small enough. Then the following estimate holds:

$$h^{-1} \|v - \lambda_{\Gamma^h}(v)\|^2_{\partial \Gamma} \lesssim \|\nabla_{\Gamma^h} v\|^2_{\Gamma^h} + \int_{\Gamma^h} (v,v) \quad \forall v \in V^h,$$

(63)

where $\lambda_{\Gamma^h}(v) = \frac{1}{|\Gamma^h|} \int_{\Gamma^h} v \, d\Gamma^h$ is the mean value of $v$ on $\Gamma^h$.

To prove that $A^h_{\Gamma}$ is in fact coercive with respect to a properly defined discrete energy norm, we need to borrow one more result from [9] which allows us to control the co-normal flux $n_E \cdot \nabla_{\Gamma^h} v$ for $v \in V^h_{\Gamma}$. 
Lemma 3. The following estimate holds
\[ h\|\nabla_{\Gamma_h} v\|^2_{\partial K_h} \leq \|\nabla_{\Gamma_h} v\|^2_{\Gamma_h} + h^d (v, v), \] (64)
for \(0 < h \leq h_0\) with \(h_0\) small enough.

Now simply replacing the crucial normal-flux estimate (54) with co-normal flux estimate from the previous Lemma (3), the proof of Lemma 2 literally transfers to the surface case, and thus we have established the following result.

Proposition 3. The discrete surface form \(A^h_\Gamma\) is coercive with respect to the discrete energy norm \(\||| \cdot |||_{h, \Gamma}\):
\[ \||| v \|||^2_{h, \Gamma} \lesssim A^h_\Gamma (v, v), \quad \forall v \in V^h. \] (65)

4.4 Condition number estimates

Following closely the presentation in [11], we now show that the condition number of the system matrix associated with a properly rescaled version of the bilinear form (11) can be bounded by \(O(h^{-2})\) independently of the position of the bulk domain \(\Omega\) relative to the background mesh \(\mathcal{T}_h\). Let \(\{\phi_{\Omega,i}\}_{i=1}^{N_\Omega}\) and \(\{\phi_{\Gamma,i}\}_{i=1}^{N_\Gamma}\) be the standard piecewise linear basis functions associated with \(\mathcal{T}^h_\Omega\) and \(\mathcal{T}^h_\Gamma\), respectively. Thus
\[ \nu^h = (\nu^h_\Omega, \nu^h_\Gamma) = \left( \sum_{i=1}^{N_\Omega} V_{\Omega,i} \phi_{\Omega,i}, \sum_{i=1}^{N_\Gamma} V_{\Gamma,i} \phi_{\Gamma,i} \right) \] (66)
for \(\nu^h \in V^h\) and expansion coefficients \(V = (V_{\Omega,i})_{i=1}^{N_\Omega} \times (V_{\Gamma,i})_{i=1}^{N_\Gamma} \in \mathbb{R}^{N_\Omega} \times \mathbb{R}^{N_\Gamma} = \mathbb{R}^N\) with \(N = N_\Omega + N_\Gamma\). It is well-known that for any quasi-uniform mesh \(\mathcal{T}^h\) consisting of \(d\)-dimensional simplices, the continuous \(\|\cdot\|_{L^2(\mathcal{T}^h)}\) norm of a finite element function \(v \in \mathcal{V}^h = \text{span}(\{\phi_i\}_{i=1}^{M})\) is related to the discrete \(\|\cdot\|_{\mathcal{V}(\mathbb{R}^M)}\) of its corresponding coefficient vector \(V\) via
\[ h^{d/2} \|V\|_{\mathbb{R}^M} \lesssim \|v^h\|_{L^2(\mathcal{T}^h)} \lesssim h^{d/2} \|V\|_{\mathbb{R}^M}. \] (67)

Note that due to the different Hausdorff dimensions of the surface and bulk domain, the discrete norms and forms for each domain scale differently with respect to the mesh size \(h\). For instance, we have clearly the Poincaré-type estimate\(^2\)
\[ \|v_{\Omega}\|_{\mathcal{V}_h} \lesssim \|v_{\Omega}\|_{h, \Omega}, \quad \forall v_{\Omega} \in V^h_{\Omega}, \] (68)
while for the surface problem, Lemma 2 shows that we have

\(^2\) This is trivial since the mass term is already included in our form.
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∥vΓ∥_{\mathcal{H}^1} \lesssim |||h^\frac{1}{2}vΓ|||_{h,T} \quad \forall vΓ \in \mathcal{V}^h_T. \quad (69)

Thus in order to pass back and forth between discrete $l^2$ and continuous, similarly
scaled $L^2$ norms on the surface and in the bulk domain, it is natural to rescale the
discrete surface functions. More precisely, the system matrix $\mathcal{A}$ we will consider is
given by the relation

$$(\mathcal{A}V, W)_{\mathbb{R}^N} = \tilde{A}_h(v_h, w_h) := A_h(v_h, h^\frac{1}{2}w_h) \quad \forall v_h, w_h \in V_h. \quad (70)$$

The system matrix $\mathcal{A}$ is a bijective linear mapping $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$. The operator
norm and condition number of the matrix $\mathcal{A}$ are then defined by

$$\|\mathcal{A}\|_{\mathbb{R}^N} = \sup_{V \in \mathbb{R}^N \setminus 0} \frac{\|\mathcal{A}V\|_{\mathbb{R}^N}}{\|V\|_{\mathbb{R}^N}} \quad \text{and} \quad \kappa(\mathcal{A}) = \|\mathcal{A}\|_{\mathbb{R}^N} \|\mathcal{A}^{-1}\|_{\mathbb{R}^N} \quad (71)$$

respectively. Following the approach in [13], a bound for the condition number can
be derived by combining (67) with suitable Poincar´e-type estimates and inverse esti-
mates relating the $L^2$ norm to the discrete energy norms. An immediate consequence
of the discrete Poincar ´e estimates for the discrete bulk and surface energy norms
given by (68) and (69), respectively, is the following Poincar´e estimate for the total
discrete energy norm:

**Lemma 4.** For $(\nu_\Omega, v_T) \in V^h = V^h_\Omega \times V^h_T$ it holds

$$\|(\nu_\Omega, v_T)\|_{\mathcal{H}^1(\Omega) \times \mathcal{H}^1_T} \lesssim \|(\nu_\Omega, h^\frac{1}{2}v_T)\|_h. \quad (72)$$

Before we turn to formulate and prove a suitable inverse inequality for the total dis-
crete energy norm, we briefly recall that we have the following inverse inequalities:

$$\|\nabla v\|_T \lesssim h^{-1}\|v\|_T, \quad \forall v \in V^h_\Omega, \quad (73)$$

$$\|v\|_F \lesssim h^{-\frac{1}{2}}\|v\|_T, \quad \forall v \in V^h_\Omega, \quad (74)$$

$$\|v\|_{T_{\Gamma \setminus F}} \lesssim h^{-\frac{1}{2}}\|v\|_T, \quad \forall v \in V^h_T. \quad (75)$$

While the first two are standard, the third one is less known and can be found in,
e.g., [7–10]. Now it is easy to show the following inverse inequality.

**Lemma 5.** For $(\nu_\Omega, v_T) \in V^h = V^h_\Omega \times V^h_T$ it holds

$$\|(\nu_\Omega, h^\frac{1}{2}v_T)\|_h \lesssim h^{-1}\|(\nu_\Omega, v_T)\|_{\mathcal{H}^1(\Omega) \times \mathcal{H}^1_T}. \quad (76)$$

**Proof.** Recalling the definition of $\|\cdot\|_h$.

\begin{align*}
\|(\nu_\Omega, h^\frac{1}{2}v_T)\|^2_\mathcal{H} &= \|\nu_\Omega\|^2_{H^1(\Omega)} + |||h^\frac{1}{2}v_T|||_{h,T}^2 + |||c_\Omega\nu_\Omega - h^\frac{1}{2}c_Tv_T|||_{h,T}^2 \\
&= I + II + III,
\end{align*}

(77)

(78)
it is enough to consider the last two terms, as term I can be treated similar to II. We start with the contributions of II which are not related to $j^h_T$ and after successively applying variants of the inverse estimates type (73), (75), we get

$$\|h^{\frac{1}{2}}\nabla u_T\|_{\mathcal{T}_h} \lesssim \|\nabla u_T\|_{\mathcal{F}_h^2} \lesssim h^{-1}\|u_T\|_{\mathcal{F}_h^2},$$

(79)

$$\|h^{\frac{1}{2}}v_T\|_{\mathcal{T}_h} \lesssim \|v_T\|_{\mathcal{F}_h^2},$$

(80)

$$\|h^{-\frac{1}{2}}[h^{\frac{1}{2}}u_T]\|_{\mathcal{H}^1(T)} \lesssim h^{-1}\|u_T\|_{\mathcal{F}_h^2}.  \tag{81}$$

Turning to the contribution from $j^h_T$, we see that

$$j^h_T(h^{\frac{1}{2}}v_T, h^{\frac{1}{2}}v_T)^{\frac{1}{2}} \lesssim h^{-1}\|h^{\frac{1}{2}}v_T\|_{\mathcal{F}_h^2} + \|[h^{\frac{1}{2}}n_F \cdot \nabla v_T]\|_{\mathcal{F}_h^2} \lesssim h^{-1}\|v_T\|_{\mathcal{F}_h^2}. \tag{82}$$

Finally, we conclude the proof by estimating the remaining term III as follows,

$$III \lesssim \|\mathcal{A} v\|_{\mathcal{T}_h} + \|h^{\frac{1}{2}}c r v_T\|_{\mathcal{T}_h} \lesssim h^{-\frac{1}{2}}\|\mathcal{A} v\|_{\mathcal{F}_h^2} + \|c r v_T\|_{\mathcal{F}_h^2}. \tag{83}$$

**Theorem 1.** The condition number of the stiffness matrix satisfies the estimate

$$\kappa(\mathcal{A}) \lesssim h^{-2}, \tag{84}$$

where the hidden constant depends only on the quasi-uniformity of the background mesh $\mathcal{F}_h$ and the chosen stability parameters.

**Proof.** We need to bound $\|\mathcal{A}\|_{\mathcal{B}N}$ and $\|\mathcal{A}^{-1}\|_{\mathcal{B}N}$. To derive a bound for $\|\mathcal{A}\|_{\mathcal{B}N}$, we first use the inverse estimate (76) and equivalence (67) to find that $\forall w \in \mathcal{V}_h$,

$$\|w, h^{\frac{1}{2}}w_T\|_{\mathcal{T}_h} \lesssim h^{-1}\|w, w_T\|_{\mathcal{F}_h^2} \lesssim h^{(d-2)/2}\|W\|_{\mathcal{B}N}. \tag{85}$$

Then

$$\|\mathcal{A} V\|_{\mathcal{B}N} = \sup_{w \in \mathcal{B}N} \|\mathcal{A} V, W\|_{\mathcal{B}N} = \sup_{w \in \mathcal{B}N} \|w, h^{\frac{1}{2}}w_T\|_{\mathcal{T}_h} \lesssim h^{(d-2)/2}\|w, h^{\frac{1}{2}}w_T\|_{\mathcal{T}_h} \lesssim h^{d-2}\|V\|_{\mathcal{B}N}, \tag{86}$$

and thus by the definition of the operator norm, $\|\mathcal{A}\|_{\mathcal{B}N} \lesssim h^{d-2}$. Next we turn to the estimate of $\|\mathcal{A}^{-1}\|_{\mathcal{B}N}$. Starting from (67) and combining the Poincaré inequality (72) with the stability estimates (55) and a Cauchy Schwarz inequality, we arrive at the following chain of estimates:

$$\|V\|_{\mathcal{B}N} \lesssim h^{-d}\|(v, v_T)\|_{\mathcal{A}^{\frac{1}{2}}} \lesssim h^{-d}\|(v, v_T)\|_{\mathcal{T}_h} \lesssim h^{-d}\|V\|_{\mathcal{T}_h} \lesssim h^{-d}\|V\|_{\mathcal{B}N} \|\mathcal{A} V\|_{\mathcal{B}N}, \tag{87}$$

$$\lesssim h^{-d} \mathcal{A}^{-1}(v, v) = h^{-d}(V, \mathcal{A} V)_{\mathcal{B}N} \lesssim h^{-d}\|V\|_{\mathcal{B}N} \|\mathcal{A} V\|_{\mathcal{B}N}, \tag{88}$$

$$h^{-d} \mathcal{A}^{-1}(v, v) = h^{-d}(V, \mathcal{A} V)_{\mathcal{B}N} \lesssim h^{-d}\|V\|_{\mathcal{B}N} \|\mathcal{A} V\|_{\mathcal{B}N}, \tag{89}$$
and hence $\|V\|_{\mathbb{R}^N} \lesssim h^{-d} \|\mathcal{A}V\|_{\mathbb{R}^N}$. Now setting $V = \mathcal{A}^{-1}W$ we conclude that $\|\mathcal{A}^{-1}\|_{\mathbb{R}^N} \lesssim h^{-d}$ and combining the estimates for $\|\mathcal{A}\|_{\mathbb{R}^N}$ and $\|\mathcal{A}^{-1}\|_{\mathbb{R}^N}$ the theorem follows.

Fig. 2: Computed solutions for coupled bulk-surface PDE example. The left plot shows the approximate bulk solution $u^h_{\Omega}$ as computed on the active mesh $T^h_\Omega$, together with its restriction to the bulk domain $\Omega^h$. The right plot displays the corresponding surface solution $u^h_{\Gamma}$.

5 Numerical results

5.1 Convergence rate study

Following the numerical example presented in [12], we now examine the convergence properties of the presented cutDG method for the bulk-surface problem (2). An analytical reference solution is defined by

$$u_{\Omega}(x,y,z) = c_{\Omega} e^{-x(y-1)(y-1)},$$

$$u_{\Gamma}(x,y,z) = (c_{\Gamma} + x(1 - 2x) + y(1 - 2x)) e^{-x(y-1)(y-1)},$$

with $c_{\Omega} = c_{\Gamma} = 1$, the corresponding the right-side $f = (f_{\Omega}, f_{\Gamma})$ is computed such that $u = (u_{\Omega}, u_{\Gamma})$ satisfies (2a)–(2c). Starting from a structured background mesh $\mathcal{T}_0$ for $\Omega = [-1.1, 1.1]^3$, a sequence of meshes $\{\mathcal{T}^h_\Omega\}_{h>0}$ is generated by successively refining $\mathcal{T}_0$ and extracting the relevant active background meshes for the bulk and surface problem as defined by (16)–(17). Based on the manufactured exact solution, the experimental order of convergence (EOC) is calculated by
with $E_k$ denoting the (norm-dependent) error of the numerical solution $u_k$ computed at refinement level $k$. In the present convergence study, both $\|\cdot\|_{H^1(U)}$ and $\|\cdot\|_{L^2(U)}$ for $U \in \{\Omega^b, \Gamma^h\}$ are used to compute $E_k$. For the completely stabilized cutDG method with $\gamma_D = \gamma_r = 50$, $\mu_D = \mu_r = 50$ and $\tau_D = \tau_r = 0.01$, the observed EOC reported in Table 1 (top) reveals a first-order and second-order convergence in the $H^1$ and $L^2$ norm, respectively. Note that for the bulk problem, the standard DG jump penalization term in (30) scaled with $\gamma_D$ is similar to the lowest order term in the ghost-penalty (37) scaled with $\mu_D$. Deactivating all solely ghost-penalty related stabilization by setting $\tau_D = \tau_r = \mu_r = 0$ renders the method completely unreliable and thus demonstrates the necessity to stabilize the presented DG method for the bulk-surface problem in the unfitted mesh case.

| $k$ | $\|e^k\|_{H^1(\Omega^b)}$ | EOC | $\|e^k\|_{L^2(\Omega^b)}$ | EOC | $\|e^k\|_{H^1(\Gamma^b)}$ | EOC | $\|e^k\|_{L^2(\Gamma^b)}$ | EOC |
|-----|----------------|------|----------------|------|----------------|------|----------------|------|
| 0   | 5.28 - 10^{-1} | -    | 8.60 - 10^{-2} | -    | 2.17 - 10^0   | -    | 2.73 - 10^{-1} | -    |
| 1   | 3.44 - 10^{-1} | +0.62+ 3.04 - 10^{-2} | +1.50 + 1.12 - 10^0 | +0.96 - 7.38 - 10^0 | +1.89 |
| 2   | 1.84 - 10^{-1} | +0.90 + 7.34 - 10^{-3} | +2.05 + 5.80 - 10^{-1} | +0.94 - 1.80 - 10^{-2} | +2.04 |
| 3   | 9.35 - 10^{-2} | +0.98 + 1.83 - 10^{-3} | +2.00 + 2.76 - 10^{-1} | +1.07 - 4.63 - 10^{-3} | +1.96 |
| 4   | 4.71 - 10^{-2} | +0.99 + 4.66 - 10^{-4} | +1.98 + 1.39 - 10^{-1} | +0.99 - 1.07 - 10^{-3} | +2.12 |

| $k$ | $\|e^k\|_{H^1(\Omega^b)}$ | EOC | $\|e^k\|_{L^2(\Omega^b)}$ | EOC | $\|e^k\|_{H^1(\Gamma^b)}$ | EOC | $\|e^k\|_{L^2(\Gamma^b)}$ | EOC |
|-----|----------------|------|----------------|------|----------------|------|----------------|------|
| 0   | 7.27 - 10^{-1} | -    | 1.32 - 10^{-1} | -    | 5.38 - 10^0   | -    | 1.16 - 10^{-1} | -    |
| 1   | 8.88 - 10^{-1} | -0.29 - 1.99 - 10^{-1} | -0.59 - 8.46 - 10^0 | -0.65 - 1.82 - 10^0 | -0.65 |
| 2   | 1.14 - 10^0   | -0.36 - 2.72 - 10^{-1} | -0.45 - 1.02 - 10^2 | -3.59 - 2.60 - 10^0 | -0.51 |
| 3   | 1.01 - 10^0   | +0.17 + 2.51 - 10^{-1} | +0.11 + 1.87 - 10^1 | +2.44 + 2.25 - 10^0 | +0.21 |

Table 1: Experimental order of convergence for the bulk-surface problem with DG-stabilization parameters $\gamma_D = \gamma_r = 50$. (Top) Optimal convergence rates are obtained for completely activated ghost-penalties using $\mu_D = \mu_r = 50$ and $\tau_D = \tau_r = 0.01$. (Bottom) After deactivation of the ghost penalties by setting $\mu_r = \tau_r = 0$, the convergence rate deteriorates completely and no clear trend is observable.

### 5.2 Condition number study

In the second numerical experiment, we study the sensitivity of the condition number of the system matrix defined by (70) with respect to relative positioning of $\Gamma$ within the background mesh $\mathcal{T}_h$. Starting from the set-up described in Section 5.1 and choosing refinement level $k = 1$, a family of surfaces $\{\mathcal{T}_h\}_{0 \leq \delta \leq 1}$ is generated.
by translating the unit-sphere $S^2 = \{ x \in \mathbb{R}^3 : \| x \| = 1 \}$ along the diagonal $(h,h,h)$; that is, $\Gamma_\delta = S^2 + \delta (h,h,h)$ with $\delta \in [0,1]$. Figure 3 illustrates the experimental set-up. For $\delta = l/500, l = 0, \ldots, 500$, we compute the condition number $\kappa_\delta(\mathcal{A})$ as

![Diagram](image)

Fig. 3: (Left) Principal experimental set-up to study the sensitivity of the condition number with respect to the relative $\Gamma$ position. (Right): Snapshot of an intersection configuration when moving $\Gamma$ through the background mesh. To visualize “extreme” cut configurations, the color map plots for each intersected mesh element $T$ the value of $\log(\Gamma_h \cap T / \text{diam}(T)^2)$. Thus blue-colored elements contain only an extremely small fraction of the surface.

the ratio of the absolute value of the largest (in modulus) and smallest (in modulus), non-zero eigenvalue. The resulting condition numbers are displayed in Figure 4 as a function of $\delta$. Choosing the stabilization parameters as in the convergence study for the fully stabilized cutDG method, we observe that the position of $\Gamma$ relative to the background mesh $\mathcal{R}_h$ has very little effect on the condition number. After turning off either of the bulk and surface related cutFEM stabilizations, the condition number is highly sensitive to the relative position of $\Gamma$ and clearly unbounded as a function of $\delta$.

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Fig. 4: Condition numbers plotted as a function of the position parameter $\delta$. When turning off either the surface or bulk related ghost-penalties (or both), the condition number is highly sensitive to the relative surface positioning in the background mesh. With all ghost penalties activated, the condition number is completely robust.

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