MULTI-SPIKES SOLUTIONS FOR A SYSTEM OF COUPLED ELLIPTIC EQUATIONS WITH QUADRATIC NONLINEARITY

ZHONGWEI TANG AND HUAFEI XIE*

School of Mathematical Sciences
Beijing Normal University, Beijing 100875, China

(Communicated by Dong Ye)

Abstract. This paper is devoted to study the following systems of coupled elliptic equations with quadratic nonlinearity
\[
\begin{align*}
-\varepsilon^2 \Delta v + P(x)v &= \mu vw, & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta w + Q(x)w &= \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N,
\end{align*}
\]
which arises from second-harmonic generation in quadratic optical media. We assume that the potential functions \(P(x)\) and \(Q(x)\) are positive functions and have a strict local maxima at \(x_0\). Applying the finite dimensional reduction method, for any integer \(1 \leq k \leq N + 1\), we prove the existence of positive solutions which have \(k\) local maximum points that concentrate at \(x_0\) simultaneously when \(\varepsilon\) is small.

1. Introduction and main results. In this paper, we study the following system of coupled elliptic equations with quadratic nonlinearity
\[
\begin{align*}
-\varepsilon^2 \Delta v + P(x)v &= \mu vw, & x \in \mathbb{R}^N, \\
-\varepsilon^2 \Delta w + Q(x)w &= \frac{\mu}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N,
\end{align*}
\]
where \(\varepsilon\) is a small parameter, \(P(x)\) and \(Q(x)\) are positive continuous potentials, \(2 \leq N < 6, \mu > 0\) and \(\mu > \gamma\).

System (1) arises from nonlinear optic theory. The cubic nonlinear Schrödinger equation
\[
\begin{align*}
\frac{i}{\varepsilon} \frac{\partial \psi}{\partial z} + r \nabla^2 \psi + \chi |\psi|^2 \psi &= 0
\end{align*}
\]
is the basic equation describing the formation and propagation of optical solutions in Kerr-type materials [11, 28]. Here \(\psi\) is a slowly varying envelope of electric field, the real-valued parameter \(r\) and \(\chi\) represent the relative strength and sign of dispersion/diffraction and nonlinearity, respectively, and \(z\) is the propagation distance coordinate. The Laplacian operator \(\nabla^2\) can either be \(\frac{\partial^2}{\partial 	au^2}\) for temporal solitons where \(\tau\) is the normalized retarded time, or \(\nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}\) where \(x = (x_1, \ldots, x_N)\) is the direction orthogonal to \(z\). Solitary wave solutions to (2) and its generations have been proved in, for example [7, 26].

2000 Mathematics Subject Classification. Primary: 35J60; Secondary: 35J20.

Key words and phrases. Quadratic nonlinearity, concentrating solutions, finite dimensional reduction method.

The first author is supported by NSFC (11571040, 11671331).

* Corresponding author.
Experimental physical scientists obtained a powerful source of coherent light which allow them enter a new nonlinear level of optical research when the lasers were invented in 1960s. Second Harmonic Generation (SHG) were discovered when the optical material has a $\chi^{(2)}$ (i.e. quadratic) nonlinear response instead of conventional Kerr $\chi^{(3)}$ material for which (2) is based on (see\[6, 8\]). Recent progress in materials with high second-order nonlinearities, including polymeric electro-optical waveguide, has stimulated experimental efforts to increase indirectly effective $\chi^{(3)}$ nonlinearities taking advantages of cascaded second-order effects. Suppose that we consider a strong parametric interaction of three stationary quasi-plan monochromatic waves with frequencies $\omega_i (i = 1, 2, 3)$, there is no walk-off between harmonic waves, the frequencies of interacting waves are matched exactly ($\omega_1 + \omega_2 = \omega_3$), and corresponding wave vectors are almost matched $k_1 \omega_1 + k_2 \omega_2 - k_3 \omega_3 = \Delta k \ll k_i$ where $k_i = |K_i|$ and wave vectors. Then with some conventional normalization and the assumption that $\omega_1 = \omega_2 = \frac{\omega_3}{2}$, we can obtain the following system of type-I SHG (see\[8\])

\[
\begin{cases}
  i \frac{\partial v}{\partial z} + r \Delta v - v + vw^* = 0, & x \in \mathbb{R}^N, \\
  i \sigma \frac{\partial w}{\partial z} + s \Delta w - \alpha w + \frac{v^2}{2} = 0, & x \in \mathbb{R}^N,
\end{cases}
\]

where $v$ is a renormalized slowly varying complex envelope of wave with frequency $\omega_1$, $w$ is the one with frequency $\omega_3$, $\sigma, \alpha > 0$, and $r, s = \pm 1$. In the spatial soliton case $r = s = 1$, while the temporal case all four combinations for $r, s = \pm 1$ are possible. The physically realistic spatial dimensions are $N = 1$ or $N = 2$. Then the chirp-free two-wave (symbiotic) solitons can be found as real-valued solutions of the steady state ($\frac{\partial}{\partial z} = 0$) equation:

\[
\begin{align*}
  &\Delta v - v + vw = 0, & x \in \mathbb{R}^N, \\
  &\Delta w - \alpha w + \frac{v^2}{2} = 0, & x \in \mathbb{R}^N, \\
  &\lim_{|x| \to \infty} v(x) = \lim_{|x| \to \infty} w(x) = 0.
\end{align*}
\]

In the case of $N = 1$, the existence of a non-trivial ground state solution of (3) was proved in \[33\] by using a variational approach. Multi-pulse solutions of (3) for $N = 1$ were first observed in numerical simulations (see\[33\]), and the existence of multi-pulse solutions was analytically proved by using singular perturbation theory in \[32\].

In this paper, we will consider a more general problem

\[
\begin{cases}
  i \frac{\partial \psi_1}{\partial t} = -\varepsilon^2 \Delta \psi_1 + V_1(x) \psi_1 - \mu_1 |\psi_1| \psi_2, & (x, t) \in \mathbb{R}^N \times \mathbb{R}^+, \\
  i \frac{\partial \psi_2}{\partial t} = \varepsilon^2 \Delta \psi_2 + V_2(x) \psi_2 - \frac{\mu_2}{2} |\psi_2|^2 - \gamma |\psi_2|^2, & (x, t) \in \mathbb{R}^N \times \mathbb{R}^+.
\end{cases}
\]

In order to obtain solitary wave solutions to the system (4), we set $\psi_1(x, t) = v(x) e^{i\mu_1 t}$, $\psi_2(x, t) = w(x) e^{i\mu_2 t}$, and system (4) is transformed to an elliptic system given by (1) when $\mu_1 = \mu_2$ and $P(x) = V_1(x) - \mu_1$, $Q(x) = V_2(x) - \mu_2$. When $\varepsilon = 1$, $\mu_1 = 1$, $\mu_2 > 0$ and $\gamma = 0$, the existence of ground state solution to (4) was proved in \[34\].

We want to mention that recently Wang and Zhou in \[31\] obtain an unbounded sequence of non-radial vector solutions of synchronized type to (1) when $\varepsilon = 1$, $\mu_1 = \mu_2 = \mu > 0$ and $\gamma$ is arbitrary constant.
Before presenting our main assumptions and main results, we want to mention that instead of the fewer study for system (4), the following $\chi^{(3)}$ nonlinear Schrödinger system

\[
\begin{aligned}
 i\frac{\partial \psi_1}{\partial t} &= -\varepsilon^2 \Delta \psi_1 + P(x)\psi_1 - \mu_1 |\psi_1|^2 \psi_1 - \beta |\psi_2|^2 \psi_1, \\
 i\frac{\partial \psi_2}{\partial t} &= -\varepsilon^2 \Delta \psi_2 + Q(x)\psi_2 - \mu_2 |\psi_2|^2 \psi_2 - \beta |\psi_2|^2 \psi_2,
\end{aligned}
\]  

(5)

has absorbed a lot of investigation. For example, the existence of solitary waves to (5) has been explored by many authors in recent years, see [2, 3, 9, 10, 13, 16, 21, 22, 23, 24, 27, 29, 30] and their references therein.

Now we are ready to present our main assumptions on $P(x), Q(x)$, we assume that:

(H) $P(x) \in C(\mathbb{R}^N)$, $\inf_{x \in \mathbb{R}^N} P(x) > 0$, there exist positive constants $L_1$ and $\theta_1$ such that $|P(x) - P(y)| \leq L_1 |x - y|^\theta_1$ for all $x, y \in \mathbb{R}^N$, and $\exists \delta > 0$ and $x_0 \in \mathbb{R}^N$ such that $P(x) < P(x_0)$ for $x \in B_\delta(x_0) \setminus \{x_0\}$;

($Q$) $Q(x) \in C(\mathbb{R}^N)$, $\inf_{x \in \mathbb{R}^N} Q(x) > 0$, there exist positive constants $L_2$ and $\theta_2$ such that $|Q(x) - Q(y)| \leq L_2 |x - y|^\theta_2$ for all $x, y \in \mathbb{R}^N$, and $\exists \delta > 0$ and $x_0 \in \mathbb{R}^N$ such that $Q(x) < Q(x_0)$ for $x \in B_\delta(x_0) \setminus \{x_0\}$;

($H$) $P(x_0) = Q(x_0) = 1$. Our main results roughly speaking are as follows.

**Theorem 1.1.** Suppose (H), ($Q$) and ($H$) hold. Then for any positive integer $k \leq N + 1$, problem (1) has a $k$-spikes non-radial solution, and all spikes concentrate near $x_0$.

Before giving the more precise statement of our main results, we present some notations first. Let us denote $U$ be the unique ground state of

\[
\begin{aligned}
 -\Delta u + u &= u^2, \\
 u &> 0, \\
 u(0) &= \max_{x \in \mathbb{R}^N} u(x), \\
 u &\in H^1(\mathbb{R}^N).
\end{aligned}
\]  

(6)

Hereafter, for any function $K(x) > 0$, we denote

\[
\langle u, v \rangle_{\varepsilon,K} = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla u \nabla v + K(x)uv),
\]

and

\[
\|u\|_{\varepsilon,K} = \left[\int_{\mathbb{R}^N} \varepsilon^2 |\nabla u|^2 + K(x)u^2 \right]^{\frac{1}{2}}.
\]

Let

\[
H^1_{\varepsilon,K} = \{u \in H^1(\mathbb{R}^N) : \|u\|_{\varepsilon,K} < \infty\}.
\]

We define $H$ be the product space $H^1_{\varepsilon,P}(\mathbb{R}^N) \times H^1_{\varepsilon,Q}(\mathbb{R}^N)$ with the norm

\[
\| (u, v) \|_\varepsilon^2 = \| u \|_{\varepsilon,P}^2 + \| v \|_{\varepsilon,Q}^2.
\]

Now we assume that $(v_{\varepsilon}, w_{\varepsilon})$ is a solutions of (1) and $x_{\varepsilon} \in \mathbb{R}^N$ with $x_{\varepsilon} \to x_0 \in \mathbb{R}^N$ as $\varepsilon \to 0$. We take

\[
(\tilde{v}_{\varepsilon}(y), \tilde{w}_{\varepsilon}(y)) = (v_{\varepsilon}(\varepsilon y + x_{\varepsilon}), w_{\varepsilon}(\varepsilon y + x_{\varepsilon})).
\]
Then one can easily check that \((\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)\) satisfies

\[
\begin{cases}
-\Delta \tilde{v} + P(\varepsilon y + x_\varepsilon) \tilde{v} = \mu \tilde{w}, & x \in \mathbb{R}^N, \\
-\Delta \tilde{w} + Q(\varepsilon y + x_\varepsilon) \tilde{w} = \frac{\mu^2}{2} \tilde{v}^2 + \gamma \tilde{w}^2, & x \in \mathbb{R}^N.
\end{cases}
\] (7)

Suppose that \(\{(\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)\}_\varepsilon\) are bounded in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\). Then as \(\varepsilon \to 0\), \((\tilde{v}_\varepsilon, \tilde{w}_\varepsilon)\) converges in some sense to \((v, w)\) which satisfies

\[
\begin{cases}
-\Delta v + v = \mu vw, & x \in \mathbb{R}^N, \\
-\Delta w + w = \frac{\mu^2}{2} v^2 + \gamma w^2, & x \in \mathbb{R}^N.
\end{cases}
\] (8)

So we find that for \(x\) near \(x_\varepsilon\),

\[
(v_\varepsilon, w_\varepsilon) \approx \left(v\left(\frac{x - x_\varepsilon}{\varepsilon}\right), w\left(\frac{x - x_\varepsilon}{\varepsilon}\right)\right),
\]

where \((v, w)\) solves (8). Thus the system (8) in some sense is the limit system of (1).

Note that \((V, W) := (\alpha U, \beta U)\) solves (8) provided that \(\mu > \gamma\) and

\[
\begin{cases}
\alpha = \frac{1}{\mu} \sqrt{\frac{2(\mu - \gamma)}{\mu}}, \\
\beta = \frac{1}{\mu}.
\end{cases}
\]

We will use \((V, W) := (\alpha U, \beta U)\) as the main component to construct the solutions of (1). We remark that the system (8) possesses a symmetry that if \((V, W)\) is a solution of (8), so is \((-V, W)\).

Let \(k\) be any positive integer, we denote

\[
D^{\varepsilon, \delta}_{k} = \{y = (y_1, \cdots, y_k) \in (\mathbb{R}^N)^k : y_i \in B_{\frac{\delta}{2}}(x_0), \left|\frac{y_i - y_j}{\varepsilon}\right| \geq |\ln \varepsilon|^\frac{1}{2}, i \neq j, i, j = 1, 2, \cdots, k\}
\]

and

\[
E_{\varepsilon, y} := \left\{(v, w) \in H : \left\langle (v, w), \left(\frac{\partial V_{\varepsilon, y_i}}{\partial y_{i,l}}, \frac{\partial W_{\varepsilon, y_l}}{\partial y_{l,i}}\right)\right\rangle_{\varepsilon} = 0, i = 1, \cdots, k, l = 1, \cdots, N \right\}.
\]

Define

\[
V_{\varepsilon, y} = \sum_{j=1}^{k} V_{\varepsilon, y_j}, \quad W_{\varepsilon, y} = \sum_{j=1}^{k} W_{\varepsilon, y_j},
\]

where \(y \in D^{\varepsilon, \delta}_{k}, V_{\varepsilon, y_j}(x) = V\left(\frac{x - y_j}{\varepsilon}\right), \text{ and } W_{\varepsilon, y_j}(x) = W\left(\frac{x - y_j}{\varepsilon}\right)\).

Now we are ready to state our main result:

**Theorem 1.2.** Under the assumptions of Theorem 1.1, for any positive integer \(k \leq N + 1\), there exists \(\varepsilon_0 > 0\), such that for each \(\varepsilon \in (0, \varepsilon_0)\), the problem (1.1) has a solution \((v_\varepsilon, w_\varepsilon)\) of the form

\[
(v_\varepsilon, w_\varepsilon) = (V_{\varepsilon, y} + \varphi_\varepsilon, W_{\varepsilon, y} + \psi_\varepsilon),
\]
where \((\varphi_\varepsilon, \psi_\varepsilon) \in E_{\varepsilon, Y}\) such that
\[
||| (\varphi_\varepsilon, \psi_\varepsilon) |||_\varepsilon \leq C \left( \varepsilon^{\frac{N}{2} + \theta_1 - \tau} + \varepsilon^{\frac{N}{2} + \theta_2 - \tau} + \varepsilon^{\frac{N}{2}} \sum_{j=1}^{k} |P(y_j)| - 1 \right)^{1-\tau} \\
+ \varepsilon^{\frac{N}{2}} \sum_{j=1}^{k} |Q(y_j)| - 1 \right)^{1-\tau} + \varepsilon^{\frac{N}{2}} \sum_{i \neq j} e^{-(1-\tau) |y_i - y_j|} ,
\]
where \(\theta_1, \theta_2\) are the numbers defined in assumptions \((H_P),(H_Q)\). \(\tau\) is a positive constant which will be specified in Lemma 2.1.

Our main strategy to prove Theorem 1.2 is the Lyapunov-Schmidt reduction method which has been applied successfully in many papers, see e.g. [1, 12, 14, 15, 17, 18, 19, 20].

This paper is organized as follows. In section 2, we will establish some preliminary estimates. In section 3, we will carry out a reduction procedure and then give the proof of our main results. In Appendix, we present some basic estimates for the functional corresponding to problem (1).

2. Some preliminary estimates. In this section, we will present some preliminary estimates which are the main ingredient for the proof of our main results in next section.

Set the variational functional corresponding to (1) by
\[
I_\varepsilon(v, w) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \varepsilon^2 |\nabla v|^2 + P(x) v^2 + \varepsilon^2 |\nabla w|^2 + Q(x) w^2 \right) - \frac{\mu}{2} \int_{\mathbb{R}^N} v^2 w - \frac{\gamma}{3} \int_{\mathbb{R}^N} w^3 .
\]

We define for every \(y \in D_k^{\varepsilon, \delta}\) and \((\varphi, \psi) \in E_{\varepsilon, Y}\)
\[
J_\varepsilon(y, \varphi, \psi) = I_\varepsilon(V_{\varepsilon, Y} + \varphi, W_{\varepsilon, Y} + \psi).
\]

Thus by a standard argument (see Cao, Noussair and Yan [12]), one can easily show that \((V_{\varepsilon, Y} + \varphi, W_{\varepsilon, Y} + \psi)\) is a critical point of \(I_\varepsilon(v, w)\) if and only if \((y, \varphi, \psi)\) is a critical point of \(J_\varepsilon(y, \varphi, \psi)\). Now we fix \(y \in D_k^{\varepsilon, \delta}\) first and we want to find a critical point \(J_\varepsilon(y, \varphi, \psi)\) with respect to \((\varphi, \psi)\). In the following of this section, without leading confusion we will omit the dependence of \(y\) for the functional \(J_\varepsilon\) and we denote \(J_\varepsilon(\varphi, \psi)\) by \(J_\varepsilon(\varphi, \psi)\) instead.

We expand \(J_\varepsilon(\varphi, \psi)\) as follows:
\[
J_\varepsilon(\varphi, \psi) = J_\varepsilon(0, 0) + l_\varepsilon(\varphi, \psi) + \frac{1}{2} L_\varepsilon(\varphi, \psi) + R_\varepsilon(\varphi, \psi),
\]
where
\[
l_\varepsilon(\varphi, \psi) = \int_{\mathbb{R}^N} \left( \varepsilon^2 \nabla V_{\varepsilon, Y} \nabla \varphi + P(x) V_{\varepsilon, Y} \varphi + \varepsilon^2 \nabla W_{\varepsilon, Y} \nabla \psi + Q(x) W_{\varepsilon, Y} \psi \right) \\
- \mu \int_{\mathbb{R}^N} V_{\varepsilon, Y} W_{\varepsilon, Y} \varphi - \frac{\mu}{2} \int_{\mathbb{R}^N} V_{\varepsilon, Y}^2 \psi - \gamma \int_{\mathbb{R}^N} W_{\varepsilon, Y}^2 \psi \\
= \int_{\mathbb{R}^N} (P(x) - 1) V_{\varepsilon, Y} \varphi + (Q(x) - 1) W_{\varepsilon, Y} \psi \\
- \mu \int_{\mathbb{R}^N} (V_{\varepsilon, Y} W_{\varepsilon, Y} - \sum_{j=1}^{k} V_{\varepsilon, y_j} W_{\varepsilon, y_j}) \varphi - \frac{\mu}{2} \int_{\mathbb{R}^N} (V_{\varepsilon, Y}^2 - \sum_{j=1}^{k} V_{\varepsilon, y_j}^2) \varphi.
\]
and 

\[ L_\varepsilon(\varphi, \psi) = \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla \varphi|^2 + P(x)\varphi^2 - \mu \psi \varepsilon \nu, \nu) \varphi \psi \]

\[ + \int_{\mathbb{R}^N} (\varepsilon^2 |\nabla \psi|^2 + Q(x)\psi^2 - 2\gamma \psi \varepsilon \nu, \nu \psi) - 2\mu \int_{\mathbb{R}^N} \psi \varepsilon \nu, \nu \psi, \]

and 

\[ R_\varepsilon(\varphi, \psi) = -\frac{\mu}{2} \int_{\mathbb{R}^N} [(V_{\varepsilon, \nu} + \varphi)^2 (W_{\varepsilon, \nu} + \psi) - V_{\varepsilon, \nu}^2 W_{\varepsilon, \nu} - 2V_{\varepsilon, \nu} W_{\varepsilon, \nu} \varphi \psi] - V_{\varepsilon, \nu}^2 \psi - W_{\varepsilon, \nu} \varphi^2 - 2W_{\varepsilon, \nu} \varphi \psi \]

\[ - \frac{\gamma}{3} \int_{\mathbb{R}^N} [(W_{\varepsilon, \nu} + \psi)^3 - W_{\varepsilon, \nu}^3 - 3W_{\varepsilon, \nu}^2 \psi - 3W_{\varepsilon, \nu} \psi^2]. \]

In order to find a critical point \((\varphi, \psi) \in E_{\varepsilon, \nu}\) for \(J_\varepsilon(\varphi, \psi)\), we need to discuss each term in the expansion. It is easy to check that

\[ \int_{\mathbb{R}^N} (\varepsilon^2 \nabla \varphi, \nabla \psi + P(x)\varphi \psi - \mu \psi \varepsilon \nu, \nu \varphi) \]

\[ + \int_{\mathbb{R}^N} (\varepsilon^2 \nabla \psi, \nabla \varphi + Q(x)\psi \psi - 2\gamma \psi \varepsilon \nu, \nu \psi) - \mu \int_{\mathbb{R}^N} \psi \varepsilon \nu, \nu \psi \]

is bounded bi-linear functional in \(E_{\varepsilon, \nu}\). Thus, there is a bounded linear operator \(L\) from \(E_{\varepsilon, \nu}\) to \(E_{\varepsilon, \nu}\), such that

\[ (L_\varepsilon(v, w), (\varphi, \psi)) = \int_{\mathbb{R}^N} (\varepsilon^2 \nabla v, \nabla \varphi + P(x)v \varphi - \mu \psi \varepsilon \nu, \nu \varphi) \]

\[ + \int_{\mathbb{R}^N} (\varepsilon^2 \nabla w, \nabla \psi + Q(x)w \psi - 2\gamma \psi \varepsilon \nu, \nu \psi) - \mu \int_{\mathbb{R}^N} \psi \varepsilon \nu, \nu \psi, (v, w), (\varphi, \psi) \in E_{\varepsilon, \nu}. \]

By the above analysis, we have the following results. First we give the non-degeneracy property of \((V, W)\).

**Lemma 2.1.** (see [31]) For any \(\mu > 0\) and \(\mu > \gamma\), \((V, W)\) is non-degenerate for the system (8) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) in the sense that the kernel is given by \(\text{span}\{\eta(\mu, \gamma) \partial^i \partial^j U | i, j \in \{1, \ldots, N\}\}\), where \(\eta(\mu, \gamma) \neq 0\).

**Proof.** We will use the same arguments as in [5] to prove this lemma. We firstly consider the weight eigenvalue problem in \(\lambda : -\Delta \phi + \phi = \lambda U \phi\) which has a sequence of eigenvalues \(1 = \lambda_1 < \lambda_2 = \lambda_3 = \cdots = \lambda_{N+1} = 2 < \lambda_{N+2} < \cdots \) with associated eigenfunctions \(\phi_k\) satisfying \(\int_{\mathbb{R}^N} U \phi_i \phi_j dx = 0\) for \(\phi_i \neq \phi_j\). For \(i = 2, 3, \ldots, N + 1\), we may take \(\frac{\partial U}{\partial x_i} \). Now for \(\mu > \lambda\), linearization of equation (8) at \((V, W)\) gives us

\[
\begin{cases}
-\Delta \Phi + \Phi = (a \Phi + b \Psi) U, & x \in \mathbb{R}^N, \\
-\Delta \Psi + \Psi = (b \Phi + c \Psi) U, & x \in \mathbb{R}^N,
\end{cases}
\]
By a direct calculation, for any $\Phi$ and $\Psi$, we have
\[ (\Phi - \tau \Psi) + (\Phi - \tau \Psi) = 2U(\Phi - \tau \Psi). \]
Thus $\Phi - \tau \Psi = \sum_{i=2}^{N+1} \alpha_i \phi_i$. Returning to the equation for $\Psi$, we get
\[ \Delta \Psi + \Psi = (b\tau + c)U \Psi + bU \sum_{i=2}^{N+1} \alpha_i \phi_i. \]

Set $\Psi = \sum_{i=1}^{\infty} \gamma_i \phi_i$ and $f(\mu, \gamma) = b\tau + c$. Noting that $\mu > \gamma$, then $f(\mu, \gamma) = 2\frac{\gamma - \mu}{\mu} < 1 < \lambda_k$ for any $k$. Using orthogonality we see easily that $\gamma_i = 0$, for $i \neq 2, \cdots, N + 1$, and $\gamma_i = \frac{b \lambda_i}{2 - f(\mu, \gamma)}$ for $i = 2, \cdots, N + 1$. Thus, the kernel at $(V, W)$ is given by $\text{span} \{(\frac{b \lambda_i}{2 - f(\mu, \gamma)} \phi_i | i = 2, \cdots, N + 1\}$, a $N$-dimensional space. Since $b \neq 0, c - 2 \neq 0$, we may take $\eta(\mu, \gamma) = \tau \frac{b}{2 - f(\mu, \gamma)} + 1$ and $\eta(\mu, \gamma) \neq 0$. 

**Lemma 2.2.** There is a constant $C > 0$ independent of $\varepsilon$ such that
\[ |R_\varepsilon(\varphi, \psi)| \leq C \varepsilon^{-\frac{N}{2}} \| (\varphi, \psi) \|_\varepsilon^3, \]
\[ |(R_\varepsilon'(\varphi, \psi))| \leq C \varepsilon^{-\frac{N}{2}} \| (\varphi, \psi) \|_\varepsilon^2, \]
\[ |(R_\varepsilon''(\varphi, \psi))| \leq C \varepsilon^{-\frac{N}{2}} \| (\varphi, \psi) \|_\varepsilon. \]

**Proof.** By a direct calculation, for any $(\varphi, \psi) \in E_{\varepsilon, y}$, we have
\[ |R_\varepsilon(\varphi, \psi)| \leq \frac{\mu}{2} \int_{\mathbb{R}^N} \left| (V_{\varepsilon, y} + \varphi)^2 (W_{\varepsilon, y} + \psi) - V_{\varepsilon, y} W_{\varepsilon, y} \right. \]
\[ - 2V_{\varepsilon, y} W_{\varepsilon, y} \varphi - V_{\varepsilon, y} \psi - W_{\varepsilon, y} \varphi^2 - 2V_{\varepsilon, y} \psi \varphi \]
\[ \left. + \frac{|\gamma|}{3} \int_{\mathbb{R}^N} \left| ((W_{\varepsilon, y} + \psi)^3 - W_{\varepsilon, y}^3 - 3W_{\varepsilon, y}^2 \psi - 3W_{\varepsilon, y} \psi^2) \right| \right| \]
\[ \leq C \int_{\mathbb{R}^N} (|\varphi|^2 |\psi| + |\psi|^3). \]

Let $(\tilde{\varphi}, \tilde{\psi}) = (\varphi(\varepsilon x), \psi(\varepsilon x))$. By the fact that $\inf_{x \in \mathbb{R}^N} P(x) > 0, \inf_{x \in \mathbb{R}^N} Q(x) > 0$ we see
\[ \int_{\mathbb{R}^N} (|\varphi|^2 |\psi| + |\psi|^3) = \varepsilon^N \int_{\mathbb{R}^N} (|\tilde{\varphi}|^2 |\tilde{\psi}| + |\tilde{\psi}|^3) \]
\[ \leq C \varepsilon^N \left[ \left( \int_{\mathbb{R}^N} |\nabla \tilde{\varphi}|^2 + \tilde{\varphi}^2 \right)^{\frac{3}{2}} + \left( \int_{\mathbb{R}^N} |\nabla \tilde{\psi}|^2 + \tilde{\psi}^2 \right)^{\frac{3}{2}} \right] \]
\[ \leq C \varepsilon^{-\frac{N}{2}} \| (\varphi, \psi) \|_\varepsilon^3. \]
Lemma 2.3. There is a constant 

\[ |\langle R'_\varepsilon(\varphi, \psi), (v, w) \rangle| \]

\[ \leq \frac{\mu}{2} \int_{\mathbb{R}^N} |(2W_{\varepsilon,y} + \varphi)(W_{\varepsilon,y} + \psi)v - 2W_{\varepsilon,y}W_{\varepsilon,y}v - 2W_{\varepsilon,y}\varphi v - 2V_{\varepsilon,y}\psi v| \]

\[ - |(W_{\varepsilon,y} + \varphi)^2 w - 2V_{\varepsilon,y}w - 2V_{\varepsilon,y}\varphi w| \]

\[ + \frac{3\mu}{4} \int_{\mathbb{R}^N} (3(W_{\varepsilon,y} + \psi)^2 w - 3W_{\varepsilon,y}^2 w - 6W_{\varepsilon,y}\psi w) | \]

\[ \leq C \left( \int_{\mathbb{R}^N} |\varphi \psi v| + \int_{\mathbb{R}^N} |\varphi^2 w| + \int_{\mathbb{R}^N} |\psi^2 w| \right) \]

\[ \leq C \varepsilon^{-\frac{N}{2}} \| (\varphi, \psi) \|_\varepsilon^2 \|(v, w)\|_\varepsilon \]

and

\[ |\langle R''_\varepsilon(\varphi, \psi)(v, w), (g, h) \rangle| \]

\[ = \frac{\mu}{2} \int_{\mathbb{R}^N} (2(W_{\varepsilon,y} + \varphi)hv + 2(W_{\varepsilon,y} + \varphi)gv - 2W_{\varepsilon,y}gv - 2V_{\varepsilon,y}hv \]

\[ + 2(V_{\varepsilon,y} + \varphi)gw - 2V_{\varepsilon,y}gw) - \frac{\gamma}{3} \int_{\mathbb{R}^N} 6\psi hw | \]

\[ \leq C \varepsilon^{-\frac{N}{2}} \| (\varphi, \psi) \|_\varepsilon \|(v, w)\|_\varepsilon \|(g, h)\|_\varepsilon. \]

Hence, we get the conclusion. \( \square \)

Now we give an estimate for \( l_\varepsilon \) which is the following lemma.

**Lemma 2.3.** There is a constant \( C > 0 \) which is independent of \( \varepsilon \), such that

\[ \|l_\varepsilon\| \leq C \left( \varepsilon^{\frac{N}{2} + \theta_1} + \varepsilon^{\frac{N}{2} + \theta_2} + \varepsilon^{\frac{N}{2}} \sum_{j=1}^k |P(y_j) - 1| + \varepsilon^{\frac{N}{2}} \sum_{j=1}^k |Q(y_j) - 1| \right. \]

\[ \left. + \varepsilon^{\frac{N}{2}} \sum_{j=1}^k e^{-\frac{(1-\varepsilon) |y_j - y_{j'}}}{\varepsilon} \right). \]

**Proof.** By a direct calculation, for any \( (\varphi, \psi) \in E_{\varepsilon,y} \), we have

\[ \left| \int_{\mathbb{R}^N} (P(x) - 1)V_{\varepsilon,y}\varphi \right| = \left| \sum_{j=1}^k \int_{\mathbb{R}^N} (P(x) - 1)V_{\varepsilon,y_j}\varphi \right| \]

\[ = \left| \sum_{j=1}^k \int_{\mathbb{R}^N} ((P(x) - P(y_j)V_{\varepsilon,y_j}\varphi + (P(y_j) - 1)V_{\varepsilon,y_j}\varphi \right| \]

\[ \leq C \left( \sum_{j=1}^k \left( \int_{\mathbb{R}^N} (P(x) - P(y_j))^2 V_{\varepsilon,y_j}^2 \right)^{\frac{1}{2}} + \sum_{j=1}^k \left( \int_{\mathbb{R}^N} (P(y_j) - 1)^2 V_{\varepsilon,y_j}^2 \right)^{\frac{1}{2}} \|\varphi\|_{\varepsilon,p} \right) \]

\[ \leq C \left( \sum_{j=1}^k \varepsilon^{\frac{N}{2}} |P(y_j) - 1| + \varepsilon^{\frac{N}{2} + \theta_1} \right) \|(\varphi, \psi)\|_\varepsilon. \]
Lemma 3.1. There exists Arguing by contradiction, we assume that there exist Proof. section, we intend to prove the main theorem by the Lyapunov-Schmidt reduction. The Finite-Dimensional reduction and proof of the main results. 3.\n
\begin{align*}
\left| \int_{\mathbb{R}^N} (Q(x) - 1) w_{\epsilon,y} \varphi \right| & \leq C \left( \sum_{j=1}^{k} \varepsilon^\frac{N}{2} |Q(y_j) - 1| + \varepsilon^\frac{N}{2} + \delta_2 \right) \| (\varphi, \psi) \|_{\varepsilon}.
\end{align*}

A direct computation deduce that
\begin{align*}
\left| \int_{\mathbb{R}^N} (V^2_{\epsilon,y} - \sum_{j=1}^{k} V^2_{\epsilon,y_j}) \varphi \right| & \leq C \left| \int_{\mathbb{R}^N} \sum_{i \neq j} V_{\epsilon,y_i} V_{\epsilon,y_j} \varphi \right| \\
& \leq C \sum_{i \neq j} \left( \int_{\mathbb{R}^N} V^2_{\epsilon,y_i} V^2_{\epsilon,y_j} \right)^{\frac{1}{2}} \| \varphi \|_{\varepsilon,P} \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} e^{-(1-\gamma)|y_i - y_j|} \| (\varphi, \psi) \|_{\varepsilon}.
\end{align*}

Similarly, we can get
\begin{align*}
\left| \int_{\mathbb{R}^N} (W^2_{\epsilon,y} - \sum_{j=1}^{k} W^2_{\epsilon,y_j}) \psi \right| & \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} e^{-(1-\gamma)|y_i - y_j|} \| (\varphi, \psi) \|_{\varepsilon}.
\end{align*}

Since $V = \frac{a}{p} W$, we get
\begin{align*}
\left| \int_{\mathbb{R}^N} (V_{\epsilon,y} W_{\epsilon,y} - \sum_{j=1}^{k} V_{\epsilon,y_j} W_{\epsilon,y_j}) \varphi \right| & \leq C \varepsilon^{\frac{N}{2}} \sum_{i \neq j} e^{-(1-\gamma)|y_i - y_j|} \| (\varphi, \psi) \|_{\varepsilon}.
\end{align*}

Finally, combining the above estimates gives the required estimate. \hfill \qed

3. The Finite-Dimensional reduction and proof of the main results. In this section, we intend to prove the main theorem by the Lyapunov-Schmidt reduction.

Let $L_{\varepsilon}$ be defined as in above section, we have the following estimate.

Lemma 3.1. There exists $\varepsilon_0 > 0$, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\rho > 0$ satisfying

$$
\| L_{\varepsilon}(v, w) \| \geq \rho \| (v, w) \|_{\varepsilon}, \quad \forall (v, w) \in E_{\varepsilon,y}.
$$

Proof. Arguing by contradiction, we assume that there exist $\varepsilon_n \to 0$ as $n \to \infty$, $y_n = (y_{1,n}, \ldots, y_{k,n}) \in D_{\kappa,\delta}^n$, and $(v_n, w_n) \in E_{\varepsilon_n}$ such that

$$
(L_{\varepsilon}(v_n, w_n), (\varphi, \psi)) = o_n(1) \| (v_n, w_n) \|_{\varepsilon_n} \| (\varphi, \psi) \|_{\epsilon_n}, \quad \forall (\varphi, \psi) \in E_{\varepsilon_n}. \tag{9}
$$

We may assume that $\| (v_n, w_n) \|^2_{\varepsilon_n} = \varepsilon_n^N$. Let

$$
(\bar{v}_{n,i}, \bar{w}_{n,i}) = (v_n(\varepsilon_n x + y_{i,n}), w_n(\varepsilon_n x + y_{i,n})).
$$

Then we can obtain

$$
\| (\bar{v}_{n,i}, \bar{w}_{n,i}) \| \leq C.
$$

So we may assume that there exist $v, w \in H^1(\mathbb{R}^N)$, such that as $n \to +\infty$,

$$
\bar{v}_{n,i} \to v, \text{ weakly in } H^1(\mathbb{R}^N),
$$

$$
\bar{w}_{n,i} \to w, \text{ weakly in } H^1(\mathbb{R}^N),
$$

$$
\bar{v}_{n,i} \to v, \text{ strongly in } L^2_{loc}(\mathbb{R}^N),
$$

and

$$
\bar{w}_{n,i} \to w, \text{ strongly in } L^2_{loc}(\mathbb{R}^N).
$$
We claim that \((v, w) = (0, 0)\). In fact, from (9), \((\tilde{v}_{n,i}, \tilde{w}_{n,i})\) satisfies
\[
\int_{\mathbb{R}^N} \left( \nabla \tilde{v}_{n,i} \nabla \varphi + P(\epsilon_n x + y_{i,n}) \tilde{v}_{n,i} \varphi - \mu \sum_{j=1}^{k} \tilde{W}_{n,j} \tilde{v}_{n,i} \varphi \right)
+ \int_{\mathbb{R}^N} \left( \nabla \tilde{w}_{n,i} \nabla \psi + Q(\epsilon_n x + y_{i,n}) \tilde{w}_{n,i} \psi - 2\gamma \sum_{j=1}^{k} \tilde{W}_{n,j} \tilde{w}_{n,i} \psi \right)
- \mu \int_{\mathbb{R}^N} \sum_{j=1}^{k} \tilde{V}_{n,j} \tilde{w}_{n,i} \varphi - \mu \int_{\mathbb{R}^N} \sum_{j=1}^{k} \tilde{V}_{n,j} \tilde{v}_{n,i} \psi
= o_n(1) (\varphi, \psi) \|_{\mathbb{E}_n}, \ (\varphi, \psi) \in \tilde{E}_n,
\]
where \(\tilde{V}_{n,j}(x) = V_{e_n y_{j,n}}(\epsilon_n x + y_{i,n}), \tilde{W}_{n,j}(x) = W_{e_n y_{j,n}}(\epsilon_n x + y_{i,n})\) and
\[
\tilde{E}_n = \left\{ (\varphi, \psi) : \left( \varphi \left( \frac{x - y_{i,n}}{\epsilon_n} \right), \psi \left( \frac{x - y_{i,n}}{\epsilon_n} \right) \right) \in H, \right. \\
\left. \left\langle \left( \varphi \left( \frac{x - y_{i,n}}{\epsilon_n} \right), \psi \left( \frac{x - y_{i,n}}{\epsilon_n} \right) \right), \left( \frac{\partial V_{e_n y_{j,n}}}{\partial y_{j,n,l}} \frac{\partial W_{e_n y_{j,n}}}{\partial y_{j,n,l}} \right) \right\rangle \epsilon_n = 0, \right. \\
\right. \\
\right. \\
\left. j = 1, \ldots, k, l = 1, \ldots, N \right\}.
\]
Now for every \((\varphi, \psi) \in C^\infty_0(\mathbb{R}^N) \times C^\infty_0(\mathbb{R}^N)\), we define
\[
(\tilde{\varphi}_n, \tilde{\psi}_n) = (\varphi, \psi) - \sum_{j=1}^{k} \sum_{l=1}^{N} a_{n,j,l} \left( \frac{\partial V_{n,j}}{\partial x_l}, \frac{\partial W_{n,j}}{\partial x_l} \right) \in \tilde{E}_n.
\]
for suitable \(a_{n,j,l}, j = 1, \ldots, k, l = 1, \ldots, N\) such that \((\tilde{\varphi}_n, \tilde{\psi}_n) \in \tilde{E}_n\).

Keep in mind of the exponential decay of \(\frac{\partial V_{n,j}}{\partial x_l}\) and \(\frac{\partial W_{n,j}}{\partial x_l}\), we see that for \(j \neq i\) and \(j \neq h, j = 1, \ldots, k, l = 1, \ldots, N\),
\[
\int_{\mathbb{R}^N} \left( \nabla \frac{\partial V_{n,j}}{\partial x_l} \nabla \varphi + P(\epsilon_n x + y_{i,n}) \frac{\partial V_{n,j}}{\partial x_l} \varphi \right)
+ \nabla \frac{\partial W_{n,j}}{\partial x_l} \nabla \psi + Q(\epsilon_n x + y_{i,n}) \frac{\partial W_{n,j}}{\partial x_l} \psi
= o_n(1),
\]
\[
\int_{\mathbb{R}^N} \left( \nabla \frac{\partial V_{e_n y_{j,n}}}{\partial y_l} \frac{\partial V(y)}{\partial y_l} + P(\epsilon_n x + y_{i,n}) \frac{\partial V_{e_n y_{j,n}}}{\partial y_l} \frac{\partial V(y)}{\partial y_l} + \nabla \frac{\partial W_{e_n y_{j,n}}}{\partial y_l} \frac{\partial W(y)}{\partial y_l} + Q(\epsilon_n x + y_{i,n}) \frac{\partial W_{e_n y_{j,n}}}{\partial y_l} \frac{\partial W(y)}{\partial y_l} \right)
= o_n(1).
\]
On the other hand,

\[
\int_{\mathbb{R}^N} \left( |\nabla \frac{\partial V(y)}{\partial y_l}|^2 + P(\varepsilon_n, x + y_{i,n}) |\frac{\partial V(y)}{\partial y_l}|^2 + |\nabla \frac{\partial W(y)}{\partial y_l}|^2 + Q(\varepsilon_n, x + y_{i,n}) |\frac{\partial W(y)}{\partial y_l}|^2 \right) \\
\geq C > 0.
\]

Hence, we can easily check that \( a_{i,n,l} \to 0 \) as \( n \to \infty \) for \( i \neq j \), while \( a_{i,n,l} \to a_{i,l} \) up to a subsequence. Taking \( (\tilde{\varphi}_n, \tilde{\psi}_n) \) into (10) and letting \( n \to \infty \), we obtain that

\[
\int_{\mathbb{R}^N} \left( \nabla v \nabla \varphi + v\varphi - \mu Wv\varphi \right) + \int_{\mathbb{R}^N} \left( \nabla w \nabla \psi + w\psi - 2\gamma Ww\psi \right) \\
+ \sum_{l=1}^{N} a_{i,l} \left( \int_{\mathbb{R}^N} \left( \nabla \frac{\partial V}{\partial x_l} \nabla v + \frac{\partial V}{\partial x_l} v - \mu W \frac{\partial V}{\partial x_l} v \right) \right) \\
+ \int_{\mathbb{R}^N} \left( \nabla \frac{\partial W}{\partial x_l} \nabla w + \frac{\partial W}{\partial x_l} w - 2\gamma W \frac{\partial W}{\partial x_l} w \right) \\
- \mu \int_{\mathbb{R}^N} V \frac{\partial W}{\partial x_l} w - \mu \int_{\mathbb{R}^N} V \frac{\partial V}{\partial x_l} v - \mu \int_{\mathbb{R}^N} V w\varphi - \mu \int_{\mathbb{R}^N} V v\psi = 0.
\]

From the fact that \((V, W)\) solves (8), we see that

\[
\int_{\mathbb{R}^N} \left( \nabla \frac{\partial V}{\partial x_l} \nabla v + \frac{\partial V}{\partial x_l} v - \mu W \frac{\partial V}{\partial x_l} v \right) + \int_{\mathbb{R}^N} \left( \nabla \frac{\partial W}{\partial x_l} \nabla w + \frac{\partial W}{\partial x_l} w - 2\gamma W \frac{\partial W}{\partial x_l} w \right) \\
- \mu \int_{\mathbb{R}^N} V \frac{\partial W}{\partial x_l} w - \mu \int_{\mathbb{R}^N} V \frac{\partial V}{\partial x_l} v = 0.
\]

Thus

\[
\int_{\mathbb{R}^N} \left( \nabla v \nabla \varphi + v\varphi - \mu Wv\varphi \right) + \int_{\mathbb{R}^N} \left( \nabla w \nabla \psi + w\psi - 2\gamma Ww\psi \right) \\
- \mu \int_{\mathbb{R}^N} V w\varphi - \mu \int_{\mathbb{R}^N} V v\psi = 0.
\]

Since in (11), \((\varphi, \psi) \in C_{0}^{\infty}(\mathbb{R}^N) \times C_{0}^{\infty}(\mathbb{R}^N)\) is arbitrary, the non-degeneracy of \((V, W)\) yields that there exists \( b_l \in \mathbb{R}, l = 1, \ldots, N \), such that

\[
(v, w) = \sum_{l=1}^{N} b_l \left( \frac{\partial V}{\partial x_l}, \frac{\partial W}{\partial x_l} \right).
\]

However \((v_n, w_n) \in E_{\varepsilon_n}\) implies

\[
\int_{\mathbb{R}^N} \left( \nabla \frac{\partial V}{\partial x_l} \nabla v + \frac{\partial V}{\partial x_l} v + \nabla \frac{\partial W}{\partial x_l} \nabla w + \frac{\partial W}{\partial x_l} w \right) = 0, \ l = 1, \ldots, N.
\]

Therefore, \((v, w) = (0, 0)\), which is exactly our claim.
Now we deduce the contradiction as follows.

\[
\begin{align*}
& \int_{\mathbb{R}^N} \left( \varepsilon_n^2 |\nabla v_n|^2 + P(x) v_n^2 - \mu W_{\varepsilon_n,y_n} v_n^2 \right) \\
& + \int_{\mathbb{R}^N} \left( \varepsilon_n^2 |\nabla w_n|^2 + Q(x) w_n^2 - 2\gamma W_{\varepsilon_n,y_n} w_n^2 \right) \\
& - \mu \int_{\mathbb{R}^N} V_{\varepsilon_n,y_n} v_n - \mu \int_{\mathbb{R}^N} V_{\varepsilon_n,y_n} w_n \\
& = \| (v_n, w_n) \|_{C^1}^2 - \mu \int_{\mathbb{R}^N} W_{\varepsilon_n,y_n} v_n^2 - 2\gamma \int_{\mathbb{R}^N} W_{\varepsilon_n,y_n} w_n^2 \\
& - \mu \int_{\mathbb{R}^N} V_{\varepsilon_n,y_n} v_n - \mu \int_{\mathbb{R}^N} V_{\varepsilon_n,y_n} w_n \\
& = \| (v_n, w_n) \|_{C^1}^2 - o(\varepsilon_n^N) - o_R(1)\varepsilon_n^N.
\end{align*}
\]

Then, we have

\[o(\varepsilon_n^N) \geq \varepsilon_n^N - o(\varepsilon_n^N) - o_R(1)\varepsilon_n^N,\]

which is impossible for large \(n\) and \(R\). As a result, we complete the proof. \(\square\)

Let us denote \(\textbf{y} = (y_1, y_2, \cdots, y_k)\), we have the following lemma which is the main part in the Liapnou-Schmidt reduction argument.

**Lemma 3.2.** For \(\varepsilon\) sufficiently small, there is a \(C^1\) map

\[(\varphi_{\textbf{y}}, \psi_{\textbf{y}}) : D_{\varepsilon_k}^{\varepsilon_\delta} \to E_{\varepsilon, \textbf{y}}\]

such that

\[\langle J'_{\varepsilon} (\varphi_{\textbf{y}}, \psi_{\textbf{y}}), (g, h) \rangle = 0, \quad \forall (g, h) \in E_{\varepsilon, \textbf{y}}.\]

Moreover, there is a constant \(C > 0\) independent of \(\varepsilon\) small enough, we have

\[
\| (\varphi_{\textbf{y}}, \psi_{\textbf{y}}) \|_{C^1} \leq C \left( \varepsilon_n^{\frac{N}{2} + \theta_1 - \tau} + \varepsilon_n^{\frac{N}{2} + \theta_2 - \tau} + \varepsilon_n^{\frac{N}{2} \sum_{j=1}^k |P(y_j) - 1|^{1-\tau}} \\
+ \varepsilon_n^{\frac{N}{2} \sum_{j=1}^k |Q(y_j) - 1|^{1-\tau}} + \varepsilon_n^{\frac{N}{2} \sum_{i \neq j} e^{-\frac{(1-\tau)|y_i-y_j|}{\varepsilon}}} \right).
\]

**Proof.** We will use the contraction theorem to prove it. By the Lemma 2.3, \(l_{\varepsilon}(\varphi, \psi)\) is a bounded linear function in \(E_{\varepsilon, \textbf{y}}\). By Riesz representation theorem, we obtain that there is an \(I_{\varepsilon}\), such that

\[l_{\varepsilon}(\varphi, \psi) = \langle I_{\varepsilon}, (\varphi, \psi) \rangle_{E_{\varepsilon, \textbf{y}}}.\]

So finding a critical point for \(J_{\varepsilon}(\varphi, \psi)\) in \(E_{\varepsilon, \textbf{y}}\) is equivalent to solving

\[I_{\varepsilon} + L_{\varepsilon}(\varphi, \psi) + R'_{\varepsilon}(\varphi, \psi) = 0.\]

By Lemma 3.1, \(L_{\varepsilon}\) is invertible. Thus is equivalent to

\[(\varphi, \psi) = A(\varphi, \psi) = -L_{\varepsilon}^{-1}(I_{\varepsilon} + R'_{\varepsilon}(\varphi, \psi)).\]

Set

\[
S_{\varepsilon} = \left\{ (\varphi, \psi) : (\varphi, \psi) \in E_{\varepsilon, \textbf{y}}, \| (\varphi, \psi) \|_{C^1} \leq \varepsilon_n^{\frac{N}{2} + \theta_1 - \tau} + \varepsilon_n^{\frac{N}{2} + \theta_2 - \tau} \\
+ \varepsilon_n^{\frac{N}{2} \sum_{j=1}^k |P(y_j) - 1|^{1-\tau}} + \varepsilon_n^{\frac{N}{2} \sum_{j=1}^k |Q(y_j) - 1|^{1-\tau}} + \varepsilon_n^{\frac{N}{2} \sum_{i \neq j} e^{-\frac{(1-\tau)|y_i-y_j|}{\varepsilon}}} \right\},
\]

and
where $\tau > 0$ is small.

Then for any $(\varphi, \psi) \in S_{\varepsilon}$

$$\left\| A(\varphi, \psi) \right\| \leq C \left( \left\| \mathcal{T}_\varepsilon \right\| + \left\| R'_\varepsilon(\varphi, \psi) \right\| \right) \leq C \left\| \mathcal{T}_\varepsilon \right\| + C \varepsilon^{-\frac{N}{2}} \left\| (\varphi, \psi) \right\|_\varepsilon^2 \leq \varepsilon^{\frac{N}{2} + \theta_1 - \tau} + \varepsilon^{\frac{N}{2} + \theta_2 - \tau} + \varepsilon^N \sum_{j=1}^{k} |P(y_j) - 1|^{1-\tau} + \varepsilon^N \sum_{i \neq j}^{k} e^{-(1-\tau) \frac{|y_i - y_j|}{\varepsilon}}.$$

Then we get $A$ maps $S_\varepsilon$ to $S_\varepsilon$.

On the other hand, for any $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in S_\varepsilon$,

$$\left\| A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2) \right\| = \left\| L^{-1}_\varepsilon(R'_\varepsilon(\varphi_1, \psi_1) - R'_\varepsilon(\varphi_2, \psi_2)) \right\| \leq C \left\| (R'_\varepsilon(\varphi_1, \psi_1) - R'_\varepsilon(\varphi_2, \psi_2)) \right\| \leq C \varepsilon^{-\frac{N}{2}} \left\| t(\varphi_1, \psi_1) + (1 - t)(\varphi_2, \psi_2) \right\| \left\| (\varphi_1, \psi_1) - (\varphi_2, \psi_2) \right\| \varepsilon \leq \frac{1}{2} \left\| (\varphi_1, \psi_1) - (\varphi_2, \psi_2) \right\| _\varepsilon.$$

Then $A$ is a contraction map from $S_\varepsilon$ to $S_\varepsilon$.

Now, we are ready to prove our main theorem.

Let $(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y})$ be the map obtained in (3.2). Define

$$F(y) = I_{\varepsilon}(V_{\varepsilon,y} + \varphi_{\varepsilon,y}, W_{\varepsilon,y} + \psi_{\varepsilon,y}), \forall (\varphi_{\varepsilon,y}, \psi_{\varepsilon,y}) \in S_\varepsilon.$$

Using the same argument used in [25], we can check that if $y$ is a critical point of $F(y)$, then $(V_{\varepsilon,y} + \varphi_{\varepsilon,y}, W_{\varepsilon,y} + \psi_{\varepsilon,y})$ is critical point of $I_{\varepsilon}$. By the Taylor expansion, we have

$$F(y) = I_{\varepsilon}(V_{\varepsilon,y}, W_{\varepsilon,y}) + I_{\varepsilon}(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y}) + \frac{1}{2} L_{\varepsilon}(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y}) + R_{\varepsilon}(\varphi_{\varepsilon,y}, \psi_{\varepsilon,y}).$$

We analyze the asymptotic behavior of $F(y)$ with respect to $\varepsilon$. From proposition 2, we have

$$I_{\varepsilon}(V_{\varepsilon,y}, W_{\varepsilon,y}) = A\varepsilon^N + C_1 \varepsilon^N \sum_{j=1}^{k} (P(y_j) - 1) + C_2 \varepsilon^N \sum_{j=1}^{k} (Q(y_j) - 1) + O \left( \varepsilon^{N + \theta_1} \right) + O \left( \varepsilon^{N + \theta_2} \right) + O \left( \varepsilon^N \sum_{i \neq j}^{k} e^{-(1-\tau) \frac{|y_i - y_j|}{\varepsilon}} \right) - O \left( \varepsilon^N \sum_{i \neq j}^{k} e^{\frac{|y_i - y_j|}{\varepsilon}} \right),$$

for some constants.
Lemma 2.3 and 3.2 give
\[ l_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon) = O \left( \varepsilon^{N+\theta_1} + \varepsilon^{N+\theta_2} + \varepsilon^{N} \sum_{j=1}^{k} |P(y_j) - 1| + \varepsilon^{N} \sum_{j=1}^{k} |Q(y_j) - 1| \right) \]
\[ + \varepsilon^{N} \sum_{i \neq j}^{k} e^{-(1-\tau)|y_i-y_j|} \left( \varepsilon^{\frac{N}{2}+\theta_1-\tau} + \varepsilon^{\frac{N}{2}+\theta_2-\tau} + \varepsilon^{N} \sum_{j=1}^{k} |P(y_j) - 1|^{1-\tau} \right) \]
\[ + \varepsilon^{N} \sum_{j=1}^{k} |Q(y_j) - 1|^{1-\tau} + \varepsilon^{N} \sum_{i \neq j}^{k} e^{-(1-\tau)|y_i-y_j|} \right). \]

By Lemma 3.1, we get
\[ L_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon, y) = O(\|l_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon, y)\|^2). \]

Lemma 2.2 gives
\[ R(\varphi_\varepsilon, \psi_\varepsilon, y) = o(\|l_\varepsilon(\varphi_\varepsilon, \psi_\varepsilon, y)\|^2). \]

Combining the above estimates yields
\[ F(y) = A\varepsilon^{N} + C_1\varepsilon^{N} \sum_{j=1}^{k} (P(y_j) - 1) + C_2\varepsilon^{N} \sum_{j=1}^{k} (Q(y_j) - 1) + O(\varepsilon^{N+\theta_1}) \]
\[ + O(\varepsilon^{N+\theta_2}) + O \left( \varepsilon^{N} \sum_{i \neq j} e^{-(1-\tau)|y_i-y_j|} \right) - O \left( \varepsilon^{N} \sum_{i \neq j} e^{-(1-\tau)|y_i-y_j|} \right) \]
\[ + O \left( \varepsilon^{\frac{N}{2}+\theta_1-\tau} + \varepsilon^{\frac{N}{2}+\theta_2-\tau} + \varepsilon^{N} \sum_{j=1}^{k} |P(y_j) - 1|^{1-\tau} \right) \]
\[ + \varepsilon^{N} \sum_{j=1}^{k} |Q(y_j) - 1|^{1-\tau} + \varepsilon^{N} \sum_{i \neq j}^{k} e^{-(1-\tau)|y_i-y_j|} \right)^2. \]

Now consider the following maximizing problem
\[ F(y) \equiv \max_{y \in D_k^\varepsilon} F(y). \]

We claim that \( y_\varepsilon \) is an interior point of \( D_k^\varepsilon \). We will prove the claim by a comparison argument.

Let \( e_j \in \mathbb{R}^N, (j = 1, \cdots, k) \) with \( |e_i-e_j| = 1 \) for \( i \neq j \). Define \( y_j := x_0 + L\varepsilon \ln \varepsilon|e_j| \). Then \( \frac{|y_i-y_j|}{\varepsilon} = L|\ln \varepsilon| \), which means that \( y = (y_1, \cdots, y_k) \in D_k^\varepsilon \) for \( \varepsilon \) small enough.

Applying the Hölder continuity of \( P(x) \) and \( Q(x) \), we derive that
\[ F(y) = A\varepsilon^{N} + C_1\varepsilon^{N} \sum_{j=1}^{k} (P(y_j) - 1) + C_2\varepsilon^{N} \sum_{j=1}^{k} (Q(y_j) - 1) + O(\varepsilon^{N+\theta_1}) \]
\[ + O(\varepsilon^{N+\theta_2}) + O \left( \varepsilon^{N} \sum_{i \neq j} e^{-(1-\tau)|y_i-y_j|} \right) - O \left( \varepsilon^{N} \sum_{i \neq j} e^{-(1-\tau)|y_i-y_j|} \right) \]
Proposition 2. \( \sum \) which implies that

where \( \alpha, \gamma \)

By using \( F(y) \leq F(y_\varepsilon) \), we deduce

\[
\sum_{j=1}^{k} (P(x_0) - P(y_{\varepsilon,j})) + \sum_{j=1}^{k} (Q(x_0) - Q(y_{\varepsilon,j})) + C \sum_{i \neq j}^{k} e^{-\frac{(1-\gamma)|y_{\varepsilon,i} - y_{\varepsilon,j}|}{\varepsilon}} \\
\leq C(\varepsilon^{\alpha_1} |\ln \varepsilon|^{\alpha_1} + \varepsilon^{\alpha_2} |\ln \varepsilon|^{\alpha_2}).
\]

Then we have

\[
\sum_{i \neq j}^{k} \frac{|y_{\varepsilon,i} - y_{\varepsilon,j}|}{\varepsilon} \geq C |\ln \varepsilon| > 0 \\
\]

which implies that \( y_\varepsilon \) is an interior point of \( D_{\varepsilon} \).

Appendix. In this section, we will give the energy expansion for the approximate solutions.

**Proposition 1** (see [4]). Suppose that \( u(x), v(x) : \mathbb{R}^N \to \mathbb{R} \) are two positive continuous radial functions satisfying

\[
u(r) \sim r^\alpha e^{-\beta r}, v(r) \sim r^\gamma e^{\eta r} (r \to +\infty),
\]

where \( \alpha, \gamma \in \mathbb{R}, \beta > 0, \eta > 0 \). Let \( y \in \mathbb{R}^N \), with \( |y| \to +\infty \). We have the following:

(i) if \( \beta < \eta \), then

\[
\int_{\mathbb{R}^N} u_y v \sim |y|^\alpha e^{-\beta |y|}.
\]

(ii) If \( \beta = \eta \), suppose for simplicity that \( \alpha \geq \gamma \), then

\[
\int_{\mathbb{R}^N} u_y v \sim \begin{cases} 
 e^{-\beta |y||y|^\alpha + \frac{1+N}{2}}, & \text{if } \gamma > -\frac{1+N}{2}, \\
 e^{-\beta |y||y|^\alpha + \frac{1+N}{2}}, & \text{if } \gamma = -\frac{1+N}{2}, \\
 e^{-\beta |y||y|^\alpha}, & \text{if } \gamma < -\frac{1+N}{2},
\end{cases}
\]

where \( u_y = u(x+y) \).

**Proposition 2.** Assume that \( (H), (H_P) \) and \( (H_Q) \) hold. Let \( y \in D_{\varepsilon}^{e,\varepsilon} \), \( V_{\varepsilon,y} = \sum_{j=1}^{k} V_{\varepsilon,y_j} \) and \( W_{\varepsilon,y} = \sum_{j=1}^{k} W_{\varepsilon,y_j} \). Then, for \( \varepsilon \) sufficiently small, we have

\[
I_{\varepsilon}(V_{\varepsilon,y}, W_{\varepsilon,y}) = A\varepsilon^N + C_1 \varepsilon^N \sum_{j=1}^{k} (P(y_j) - 1) + C_2 \varepsilon^N \sum_{j=1}^{k} (Q(y_j) - 1) \\
+ O(\varepsilon^{N+\theta_1}) + O(\varepsilon^{N+\theta_2}) + O \left( \varepsilon^N \sum_{i \neq j}^{k} e^{-\frac{(1-\gamma)|y_{\varepsilon,i} - y_{\varepsilon,j}|}{\varepsilon}} \right) \\
- O \left( \varepsilon^N \sum_{i \neq j}^{k} e^{-\frac{|y_{\varepsilon,i} - y_{\varepsilon,j}|}{\varepsilon}} \right),
\]
Proof. Recall that
\[ I_\varepsilon (V_{\varepsilon,y}, W_{\varepsilon,y}) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \varepsilon^2 |\nabla V_{\varepsilon,y}|^2 + P(x)V_{\varepsilon,y}^2 + \varepsilon^2 |\nabla W_{\varepsilon,y}|^2 + Q(x)W_{\varepsilon,y}^2 \right) \]
\[ - \frac{\mu}{2} \int_{\mathbb{R}^N} V_{\varepsilon,y} W_{\varepsilon,y} - \frac{\gamma}{3} \int_{\mathbb{R}^N} W_{\varepsilon,y}^3. \]
Since \( y_j \neq y_i \) for \( j \neq i \) and \( V \) and \( W \) decay exponentially at infinity, we get
\[ I_\varepsilon (V_{\varepsilon,y}, W_{\varepsilon,y}) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} (P(x) - 1)V_{\varepsilon,y_j} + \sum_{j=1}^{k} (Q(x) - 1)W_{\varepsilon,y_j} \right) \]
\[ + \frac{3\mu}{4} \int_{\mathbb{R}^N} \sum_{j=1}^{k} V_{\varepsilon,y_j} W_{\varepsilon,y_j} + \frac{\gamma}{3} \int_{\mathbb{R}^N} \sum_{j=1}^{k} W_{\varepsilon,y_j}^3 \]
\[ - \frac{\mu}{2} \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} V_{\varepsilon,y_j} \right)^2 \left( \sum_{j=1}^{k} W_{\varepsilon,y_j} \right) - \frac{\gamma}{3} \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} W_{\varepsilon,y_j} \right)^3 \]
\[ + C \left( \epsilon^N \sum_{i \neq j} e^{- (1 - \tau) \frac{|y_i - y_j|}{\epsilon}} \right). \]

By direct computation, we obtain
\[ \int_{\mathbb{R}^N} \sum_{j=1}^{k} (P(x) - 1)V_{\varepsilon,y_j} = \sum_{j=1}^{k} \epsilon^N \int_{\mathbb{R}^N} (P(\varepsilon y + y_j) - 1) V^2 \]
\[ = \sum_{j=1}^{k} \epsilon^N \int_{\mathbb{R}^N} (P(\varepsilon y + y_j) - P(y_j)) V^2 + \sum_{j=1}^{k} \epsilon^N \int_{\mathbb{R}^N} (P(y_j) - 1) V^2 \]
\[ = O(\epsilon^{N + \theta_1}) + \epsilon^N \sum_{j=1}^{k} (P(y_j) - 1) \int_{\mathbb{R}^N} V^2. \]

Similarly, we have
\[ \int_{\mathbb{R}^N} \sum_{j=1}^{k} (Q(x) - 1)W_{\varepsilon,y_j} = O(\epsilon^{N + \theta_2}) + \epsilon^N \sum_{j=1}^{k} (Q(y_j) - 1) \int_{\mathbb{R}^N} W^2. \]

Applying the elementary computations, we derive
\[ \int_{\mathbb{R}^N} \sum_{j=1}^{k} V_{\varepsilon,y_j}^2 W_{\varepsilon,y_j} = \epsilon^N k \int_{\mathbb{R}^N} V^2 W, \]
\[ \int_{\mathbb{R}^N} \sum_{j=1}^{k} W_{\varepsilon,y_j}^3 = \epsilon^N k \int_{\mathbb{R}^N} W^3, \]
\[ \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} V_{\varepsilon,y_j} \right)^2 \left( \sum_{j=1}^{k} W_{\varepsilon,y_j} \right) = \epsilon^N k \int_{\mathbb{R}^N} V^2 W + O \left( \epsilon^N \sum_{i \neq j} e^{- \frac{|y_i - y_j|}{\epsilon}} \right). \]
and
\[ \int_{\mathbb{R}^N} \left( \sum_{j=1}^{k} W_{\varepsilon,y_j} \right)^3 = \varepsilon^N k \int_{\mathbb{R}^N} W^3 + O \left( \varepsilon^N \sum_{i \neq j} e^{-\frac{|y_i - y_j|}{\varepsilon}} \right). \]

Hence adding the above equalities together, we obtain the desired estimate. □

Acknowledgments. The motivation of doing this work comes from some valuable discussion with Professor Chunhua Wang when she was visiting Beijing Normal University. The authors want to express their great appreciation to professor Wang for her valuable comments and discussion. The authors are also grateful for the anonymous referee for very helpful suggestions and comments.

REFERENCES

1. A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Ration. Mech. Anal., 140 (1997), 285–300.
2. A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Math. Acad. Sci. Paris, 342 (2006), 453–458.
3. A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc., 75 (2007), 67–82.
4. A. Ambrosetti, E. Colorado and D. Ruiz, Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations, Calc. Var. Partial Differential Equations, 30 (2007), 85–112.
5. T. Bartsch, N. Dancer and Z. Q. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, Calc. Var. Partial Differential Equations, 37 (2010), 345–361.
6. A. V. Buryak and Y. S. Kivshar, Solitons due to second harmonic generation, Phys. Lett. A, 197 (1995), 407–412.
7. H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rationel Mech. Anal., 82 (1983), 313–345.
8. A. V. Buryak, P. Di Trapani, D. V. Skryabin and S. Trillo, Optical solitons due to quadratic nonlinearities: from basic physics to futuristic applications, Phys. Rep., 370 (2002), 63–235.
9. T. Bartsch and Z. Q. Wang, Note on ground states of nonlinear Schrödinger systems, J. Partial Differential Equations, 19 (2006), 200–207.
10. T. Bartsch, Z. Q. Wang and J. Wei, Bound states for a coupled Schrödinger system, J. Fixed Point Theory Appl., 2 (2007), 353–367.
11. T. Cazenave, Semilinear Schrödinger Equations, vol.10 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.
12. D. Cao, E. S. Noussair and S. Yan, Solutions with multiple “peaks” for nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A, 129 (1999), 235–264.
13. M. Conti, S. Terracini and G. Verzini, Nehari’s problem and competing species systems, Ann. Inst. H. Poincaré Anal. Non Linéaire, 19 (2002), 871–888.
14. M. Del Pino and P. Felmer, Multi-peak bound states for nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire, 15 (1998), 127–149.
15. E. N. Dancer and S. Yan, Multipeak solutions for a singularly perturbed Neumann problem, Pacific J. Math., 189 (1999), 241–262.
16. E. N. Dancer, J. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 953–969.
17. C. Gui and J. Wei, Multiple interior peak solutions for some singularly perturbed Neumann problems, J. Differential Equations, 158 (1999), 1–27.
18. C. Gui and J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, Canad. J. Math., 52 (2000), 522–538.
19. Y. Li, On a singularly perturbed elliptic equation, Adv. Differential Equations, 2 (1997), 955–980.
20. Y. Li and L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, Comm. Pure Appl. Math., 51 (1998), 1445–1490.
[21] C. Lin, W. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations*, 72 (1988), 1–27.

[22] Z. Liu and Z. Q. Wang, Ground states and bound states of a nonlinear Schrödinger system, *Adv. Nonlinear Stud.*, 10 (2010), 175–193.

[23] E. Montefusco, B. Pellacci and M. Squassina, Semiclassical states for weakly coupled nonlinear Schrödinger systems, *J. Eur. Math. Soc.*, 10 (2008), 47–71.

[24] B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, *Comm. Pure Appl. Math.*, 63 (2010), 267–302.

[25] E. S. Noussair and S. Yan, On positive multipeak solutions of a nonlinear elliptic problem, *J. Lond. Math. Soc.*, 62 (2000), 213–227.

[26] W. A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.*, 55 (1977), 149–162.

[27] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^N$, *Comm. Math. Phys.*, 271 (2007), 199–221.

[28] C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, vol. 39 of Applied Mathematical Sciences. Springer-Verlag, New York, 1999.

[29] R. Tian and Z. Q. Wang, Multiple solitary wave solutions of nonlinear Schrödinger systems, *Topol. Methods Nonlinear Anal.*, 37 (2011), 203–223.

[30] J. Wei and T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Ration. Mech. Anal.*, 190 (2008), 83–106.

[31] C. Wang and J. Zhou, Infinitely many solitary waves due to the second-harmonic generation in quadratic media, to be appeared in Acta Math. Sci. Ser. B (Engl. Ed.) (2020, no.1).

[32] A. C. Yew, Multipulses of nonlinearily coupled Schrödinger equations, *J. Differential Equations*, 173 (2001), 92–137.

[33] A. C. Yew, A. R. Champneys and P. J. McKenna, Multiple solitary waves due to second-harmonic generation in quadratic media, *J. Nonlinear Sci.*, 9 (1999), 33–52.

[34] L. Zhao, F. Zhao and J. Shi, Higher dimensional solitary waves generated by second-harmonic generation in quadratic media, *Calc. Var. Partial Differential Equations*, 54 (2015), 2657–2691.

Received December 2018; revised March 2019.

E-mail address: tangzw@bnu.edu.cn
E-mail address: huafeixie@mail.bnu.edu.cn