NONEXISTENCE OF TRAVELING WAVE SOLUTIONS, EXACT AND SEMI-EXACT TRAVELING WAVE SOLUTIONS FOR DIFFUSIVE LOTKA-VOLterra SYSTEMS OF THREE COMPETING SPECIES

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L.-C. Hung dedicates this work to Xian Liao
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Abstract. In reaction-diffusion models describing the interaction between the invading grey squirrel and the established red squirrel in Britain, Okubo et al. ([19]) found that in certain parameter regimes, the profiles of the two species in a wave propagation solution can be determined via a solution of the KPP equation. Motivated by their result, we employ an elementary approach based on the maximum principle for elliptic inequalities coupled with estimates of a total density of the three species to establish the nonexistence of traveling wave solutions for Lotka-Volterra systems of three competing species. Applying our estimates to the May-Leonard model, we obtain upper and lower bounds for the total density of a solution to this system. For the existence of traveling wave solutions to the Lotka-Volterra three-species competing system, we find new semi-exact solutions by virtue of functions other than hyperbolic tangent functions, which are used in constructing one-hump exact traveling wave solutions in [2]. Moreover, new two-hump semi-exact traveling wave solutions different from the ones found in [1] are constructed.

1. Introduction. In the present paper, we study the following Lotka-Volterra system of three competing species in the entire space $\mathbb{R}$:

$$(LV) \begin{cases}
  u_t = d_1 u_{xx} + u \left( \lambda_1 - c_{11} u - c_{12} v - c_{13} w \right), \\
  v_t = d_2 v_{xx} + v \left( \lambda_2 - c_{21} u - c_{22} v - c_{23} w \right), \\
  w_t = d_3 w_{xx} + w \left( \lambda_3 - c_{31} u - c_{32} v - c_{33} w \right),
\end{cases} \quad x \in \mathbb{R}, \quad t > 0, \quad (1)$$

where $u(x,t)$, $v(x,t)$ and $w(x,t)$ represent the density of the three species $u$, $v$ and $w$ respectively; $d_i$, $\lambda_i$, $c_{ii}$ ($i = 1, 2, 3$), and $c_{ij}$ ($i, j = 1, 2, 3, i \neq j$) are the

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diffusion rates, the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates, respectively. These constants are all assumed to be positive.

The problem as to which species will survive in a competitive system is of importance in ecology. In order to tackle this problem, we use traveling wave solutions, which are solutions of the form

\[(u(x, t), v(x, t), w(x, t)) = (U(z), V(z), W(z)), \quad z = x - \theta t,\]

where \(\theta\) is the propagating speed of the traveling wave. Inserting (2) into (1) yields the following system of ordinary differential equations, where we still use \(u, v\) and \(w\) to denote \(U, V\) and \(W\),

\[
\begin{align*}
&d_1 u_{zz} + \theta u_z + u(\lambda_1 - c_{11} u - c_{12} v - c_{13} w) = 0, \\
&d_2 v_{zz} + \theta v_z + v(\lambda_2 - c_{21} u - c_{22} v - c_{23} w) = 0, \quad z \in \mathbb{R}, \\
&d_3 w_{zz} + \theta w_z + w(\lambda_3 - c_{31} u - c_{32} v - c_{33} w) = 0.
\end{align*}
\]

We note here that the unknown constant \(\theta\) needs to be determined while solving the above system.

In [17, 18], a singular perturbation approach is employed to establish existence of traveling wave solutions for \((LV)\). For existence of standing wave solutions, we refer to [7, 8, 9, 10].

In addition to the wave solutions mentioned above, a natural question is to study the existence of traveling waves which consist of a forward front \(v\), a backward front \(u\), and a pulse \(w\) in the middle (see Figure 1). Due to the hump-like profile of \(w\), we call a wave of this type a one-hump wave. Under certain assumptions on the parameters, existence of solutions of such type was established by finding exact traveling wave solutions as well as numerical simulations in [1, 2]. Also, the study in those papers indicates that one-hump waves are of importance to be the blocks for building more complicated waves. In [2], the problem of incorporating \(w\) into the simpler system for \((u, v)\) alone was considered. Numerical evidence indicates that the invasion of \(w\) could dramatically change the inter-specific competitive behavior between \(u\) and \(v\). By using the numerical tracking method AUTO, a global branch of one-hump traveling wave solutions was shown to bifurcate from an exact traveling wave solution when a certain parameter was varied. This indicates that, when some parameter is slightly perturbed around an exact solution, traveling wave solutions of \((LV)\) persist.

In this paper, we continue the study in [1, 2] to find new exact and semi-exact traveling wave solutions. In [2], exact one-hump solutions of \((LV)\) are represented in terms of polynomials in hyperbolic tangent functions with degree 2. It is shown in this paper, that some new semi-exact one-hump solutions can also be constructed using functions other than hyperbolic tangent functions. Moreover we show, in addition to the ones obtained in [1], that there exist more types of semi-exact two-hump solutions.

The long-term goal of related study is to find rather precise conditions for the existence of traveling wave solutions. Therefore, on the other hand, we are also interested in the opposite question:

**Q1:** Under what conditions does there fail to exist traveling wave solutions of \((LV)\) with positive \(u, v\) and \(w\)?

As far as the authors’ knowledge, such nonexistence results are rather few in the literature. Based on the maximum principle for elliptic inequalities coupled
with estimates of a total density of the three species, we develop in this paper an approach to establish nonexistence results for traveling wave solutions of \((LV)\) when the diffusion coefficients of the three species are identical.

In an ecosystem, how to estimate the total biomass is an important issue. To study this problem, it is natural to consider the total density \(\eta_1 u + \eta_2 v + \eta_3 w\), where \(\eta_1, \eta_2\) and \(\eta_3\) are suitable constants which transfer the population densities \(u, v\) and \(w\) into the ones with the same unit of mass. Thus the total biomass can be expressed as the integration of the total density. The total density is not only important in calculating the total mass, but also crucial in the proof of our nonexistence results. The idea is to obtain estimates for the total density which force one of the three species to be zero. To our knowledge, there are few results only important in calculating the total mass, but also crucial in the proof of our nonexistence results. The idea is to obtain estimates for the total density which force one of the three species to be zero. To our knowledge, there are few results in the literature regarding the problem as to how to estimate the total density of the three species in \((LV)\). It remains unknown how to deal with the general total density \(\eta_1 u + \eta_2 v + \eta_3 w\). However, we are able to estimate the quantity \(u + v + w\) for the case of equal diffusion coefficients. The method is to derive some differential inequalities for \(u + v + w\) and to use the maximum principle for elliptic inequalities. This suffices to establish nonexistence results in some cases. For the general situation, we are interested in the following question:

**Q2:** Find better optimized estimates for the total density of the three species \(\eta_1 u + \eta_2 v + \eta_3 w\) in \((LV)\), where \(\eta_1, \eta_2\) and \(\eta_3\) are certain non-negative constants. Moreover, we are able to arrive at a rough estimate for \(\eta_1 u + \eta_2 v + \eta_3 w\) through Proposition 1 (see Section 2) together with the following inequalities

\[
\min\{\eta_1, \eta_2, \eta_3\} (u + v + w) \leq \eta_1 u + \eta_2 v + \eta_3 w \leq \max\{\eta_1, \eta_2, \eta_3\} (u + v + w).
\]

Throughout this paper, we shall always consider \((3)\) with the following two types of boundary conditions at \(\pm\infty\):

- \((u, v, w)(-) = \left(\frac{\lambda_1}{c_1}, 0, 0\right), \quad (u, v, w)(\infty) = \left(0, \frac{\lambda_2}{c_2}, 0\right);
- \((u, v, w)(-) = \left(\frac{\lambda_1}{c_1}, 0, 0\right), \quad (u, v, w)(\infty) = \left(\frac{\lambda_2}{c_2}, 0, 0\right).

We remark that the first is studied in [2] while in [1] the second is investigated. When \(w\) is absent in \((1)\), the resulting two-species system for \(u\) and \(v\) can be scaled as

\[
\begin{cases}
  u_t = u_{xx} + u (1 - u - a_1 v), & x \in \mathbb{R}, \quad t > 0, \\
  v_t = dv_{xx} + \lambda v (1 - a_2 u - v),
\end{cases}
\]

where \(d, \lambda, a_1\) and \(a_2\) are positive parameters. To understand the complicated phenomena exhibited by the three-species, the study of the two-species problem \((5)\) is essential. It is easy to see that \((5)\) has four equilibria: \(e_1 = (0, 0), e_2 = (1, 0), e_3 = (0, 1)\) and \(e_4 = (u^*, v^*)\), where \((u^*, v^*) = \left(\frac{1-a_2}{1-a_1 a_2}, \frac{1-a_1}{1-a_1 a_2}\right)\) is the intersection of the two straight lines \(1 - u - a_1 v = 0\) and \(1 - a_2 u - v = 0\) whenever it exists. When the domain is bounded, the asymptotic behavior of solutions \((u(x, t), v(x, t))\) for \((5)\) with initial conditions \(u(x, 0), v(x, 0) > 0\) can be classified into four cases.

**Theorem 1.1** ([4]). Let \((u(x, t), v(x, t))\) be the solution of \((5)\) with the entire space \(\mathbb{R}\) replaced by a bounded domain in \(\mathbb{R}\) under the zero Neumann boundary conditions. Then for initial conditions \(u(x, 0), v(x, 0) > 0\), we have

\[(i)\] \(a_1 < 1 < a_2 \Rightarrow \lim_{t \to \infty} (u(x, t), v(x, t)) = (1, 0);\]

\[(ii)\] \(a_2 < 1 < a_1 \Rightarrow \lim_{t \to \infty} (u(x, t), v(x, t)) = (0, 1);\]

\[(iii)\] \(a_1, a_2 > 1 \Rightarrow (1, 0)\) and \((0, a)\) are locally stable equilibria.
(iv) $a_1, a_2 < 1$ \implies \lim_{t \to \infty} (u(x, t), v(x, t)) = (u^*, v^*)$.

The traveling wave solution of (5), namely
\[ (u(x, t), v(x, t)) = (u(z), v(z)), \quad z = x - \theta t, \] where $\theta$ represents the wave velocity of the traveling wave, satisfies
\[ \begin{cases} u_{zz} + \theta u_z + u (1 - u - a_1 v) = 0, \\ d v_{zz} + \theta v_z + \lambda v (1 - a_2 u - v) = 0, \quad z \in \mathbb{R}, \\ (u, v)(-\infty) = \mathbf{e}_i, \quad (u, v)(+\infty) = \mathbf{e}_j, \end{cases} \tag{7} \]
where it is assumed that the wave tends to some equilibria at infinities and $i, j = 1, 2, 3, 4$. We remark that the determination of $\theta$ is part of the task in solving system (7). For exact solutions of (7), we refer to \[6, 20, 21\]. The solution $(u(z), v(z))$ of (7) is called an $(\mathbf{e}_i, \mathbf{e}_j)$-wave. In this paper, we only concentrate on the $(\mathbf{e}_2, \mathbf{e}_3)$-wave and solve the problem
\[ \begin{cases} u_{zz} + \theta u_z + u (1 - u - a_1 v) = 0, \\ d v_{zz} + \theta v_z + \lambda v (1 - a_2 u - v) = 0, \quad z \in \mathbb{R}, \\ (u, v)(-\infty) = \mathbf{e}_2, \quad (u, v)(+\infty) = \mathbf{e}_3, \end{cases} \tag{8} \]
Problem (P) for the special case where $d = \lambda = 1, a_1 + a_2 = 2$ and $a_1 < 1, a_2 > 1$ was considered by Okubo et al. (\cite{19}). They added the two equations in (8) together to obtain
\[ q_{zz} + \theta q_z + q (1 - q) = 0, \tag{9} \]
where $q = u + v$ and $q(\pm\infty) = 1$. Then the maximum principle (see Lemma 2.1 in Section 2) yields $q = 1$ or $u + v = 1$ for all $z \in \mathbb{R}$. Substituting $v = 1 - u$ into the first equation in P gives
\[ u_{zz} + \theta u_z + (1 - a_1) u (1 - u) = 0, \tag{10} \]
which is the Fisher equation. It is well-known that the minimal wave speed for (10) is $\theta_{min}^u = 2 \sqrt{1 - a_1}$, i.e. $\theta \geq \theta_{min}$. On the other hand, it is readily seen that $v$ satisfies
\[ v_{zz} + \theta v_z + (a_2 - 1) v (1 - v) = 0, \tag{11} \]
and the minimal wave speed for (11) is $\theta_{min}^v = 2 \sqrt{a_2 - 1}$, which coincides with $\theta_{min}^u$ since $a_1 + a_2 = 2$.

For other $(\mathbf{e}_i, \mathbf{e}_j)$-waves, the reader is referred to \[6\]. For cases (i) and (iii) in Theorem 1.1 (i.e. monostable and bistable cases), Kan-on (\cite{11, 12}), Fei and Carr (\cite{5}), Leung, Hou and Li (\cite{15}), and Leung and Feng (\cite{14}) proved the existence of $(\mathbf{e}_2, \mathbf{e}_3)$-waves using different approaches. As Q2, we can ask a similar question:

**Q3:** Suppose that a solution of (P) exists for some $d, a_1, a_2$ and $\lambda$. Then is $u + v$ larger than 1, smaller than 1, or equal to 1 when $d = \lambda = 1$?

The paper is organized as follows. Q1, Q2 and Q3 are investigated in Section 2. When the diffusion coefficients are all identical, some partial answers are obtained. As an example, we show our results can be applied to the May-Leonard model (\cite{16}). Section 3 is devoted to reconstructing the exact solution presented in [2] without the aid of Mathematica. By using the exact solution obtained there, we show that the nonexistence results in Section 2 could fail to hold when some parameters vary slightly. As a continuation of the study in [2] and [1], more semi-exact solutions of (3) are constructed respectively in Section 4 and Section 5. Finally, several remarks, including biological meaning of the results in Section 2, are discussed in Section 6.
2. Nonexistence and a priori estimates of \( u + v + w \). As mentioned in the introduction, we consider in this section solutions of (3) with two types of boundary conditions:

\[
(u, v, w)(-\infty) = \left(\frac{\lambda_1}{c_{11}}, 0, 0\right), \quad (u, v, w)(\infty) = \left(0, \frac{\lambda_2}{c_{22}}, 0\right). \tag{12}
\]

\[
(u, v, w)(-\infty) = \left(\frac{\lambda_1}{c_{11}}, 0, 0\right), \quad (u, v, w)(\infty) = \left(\frac{\lambda_1}{c_{11}}, 0, 0\right). \tag{13}
\]

Before introducing our main results in this section, we prove the following useful lemma.

**Lemma 2.1** (Maximum principle). For any constant \( c \in \mathbb{R} \) and any positive function \( \rho(z) \in C^1(\mathbb{R}) \), if \( \phi = \phi(z) \in C^2(\mathbb{R}) \) satisfies the following elliptic inequalities

\[
\begin{cases}
\phi''(z) + c \phi'(z) + \rho(z) \left(1 - \phi(z)\right) \geq 0 \ (\text{resp.} \leq 0), \ z \in \mathbb{R}, \\
\phi(\pm \infty) \leq 1 \ (\text{resp.} \geq 1).
\end{cases}
\]

Then \( \phi(z) \leq 1 \ (\text{resp.} \geq 1) \) for all \( z \in \mathbb{R} \). When the inequalities are replaced by equalities, i.e.

\[
\begin{cases}
\phi''(z) + c \phi'(z) + \rho(z) \left(1 - \phi(z)\right) = 0, \ z \in \mathbb{R}, \\
\phi(\pm \infty) = 1,
\end{cases}
\]

we have \( \phi(z) = 1 \) for all \( z \in \mathbb{R} \).

**Proof.** Suppose that there exists \( z_0 \in \mathbb{R} \) with \( \phi(z_0) > 1 \). Then for some \( z_1 \in \mathbb{R} \), we have \( \phi''(z_1) \leq 0, \phi'(z_1) = 0, \) and \( \phi(z_1) > 1 \). Accordingly, \( \phi''(z_1) + c \phi'(z_1) + \rho(z_1) \left(1 - \phi(z_1)\right) < 0 \), which is a contradiction. The alternative case can be shown in a similar manner. By combining the results of the two cases considered above, we obtain \( \phi(z) = 1 \) for all \( z \in \mathbb{R} \) for the equality case.

In order to prove the nonexistence result, we need to estimate \( u + v + w \) first.

**Proposition 1** (A priori estimates of \( u + v + w \)). Assume that \( d_1 = d_2 = d_3 = 1 \) and the following \([H1] \sim [H4]\) hold:

- \([H1]\) \( c_{12} + c_{21} \geq \lambda_1 + \lambda_2 \) (respectively, \( c_{12} + c_{21} \leq \lambda_1 + \lambda_2 \));
- \([H2]\) \( c_{13} + c_{31} \geq \lambda_1 + \lambda_3 \) (respectively, \( c_{13} + c_{31} \leq \lambda_1 + \lambda_3 \));
- \([H3]\) \( c_{23} + c_{32} \geq \lambda_2 + \lambda_3 \) (respectively, \( c_{23} + c_{32} \leq \lambda_2 + \lambda_3 \));
- \([H4]\) \( c_{ii} \geq \lambda_i \ (i = 1, 2, 3) \) (respectively, \( c_{ii} \leq \lambda_i \ (i = 1, 2, 3) \)).

Suppose that system (3) with boundary conditions either (12) or (13) has a positive solution \((u(z), v(z), w(z))\) (a solution with \( u(z) > 0, v(z) > 0 \) and \( w(z) > 0 \) for all \( z \in \mathbb{R} \)). Then \( u(z) + v(z) + w(z) \leq 1 \) (respectively, \( u(z) + v(z) + w(z) \geq 1 \)).

**Proof.** Using Lemma 2.1, the proof is elementary. Suppose that (3), with either (12) or (13), has a positive solution \((u(z), v(z), w(z))\). Let \( p(z) = u(z) + v(z) + w(z) \), then the addition of the three equations in (3) gives (in what follows we write for
notational convenience $p' = p'(z), p'' = p''(z)$

\[ 0 = p'' + \theta p' + (\lambda_1 u + \lambda_2 v + \lambda_3 w) - c_{11} u^2 - c_{22} v^2 - c_{33} w^2 \\
- u v (c_{12} + c_{21}) - u w (c_{13} + c_{31}) - v w (c_{23} + c_{32}) \leq p'' + \theta p' + (\lambda_1 u + \lambda_2 v + \lambda_3 w) - \lambda_1 u^2 - \lambda_2 v^2 - \lambda_3 w^2 \\
- (\lambda_1 + \lambda_2) u v - (\lambda_1 + \lambda_3) u w - (\lambda_2 + \lambda_3) v w \]

\[ = p'' + \theta p' + (\lambda_1 u + \lambda_2 v + \lambda_3 w) (1 - u - v) \]

\[ = p'' + \theta p' + (\lambda_1 u + \lambda_2 v + \lambda_3 w) (1 - p). \] (14)

Thus, $p$ satisfies

\[ \begin{cases} p'' + \theta p' + (\lambda_1 u + \lambda_2 v + \lambda_3 w) (1 - p) \geq 0, \\
p(\pm \infty) \leq 1. \end{cases} \] (15)

By means of Lemma 2.1, we see that $p \leq 1$ and the proof is completed. The alternative case can be shown in a similar manner.

From the biological point of view, the interpretation of Proposition 1 is that, whenever the birth rates $\lambda_i (i = 1, 2, 3)$ are relatively larger (respectively, smaller) than the intra-specific competition rates $c_{ii} (i = 1, 2, 3)$, the total density $u + v + w$ is greater (respectively, smaller) than or equal to 1 (after a natural scaling of the system).

Hereafter, we will always use $p = u + v + w$ throughout this paper. As an immediate consequence of Proposition 1, we have

**Corollary 1** (Constant amount of $u + v + w$). Assume that $d_1 = d_2 = d_3 = 1$ and the following [H1] holds:

- [H1] $c_{12} + c_{21} = \lambda_1 + \lambda_2$;
- [H2] $c_{13} + c_{31} = \lambda_1 + \lambda_3$;
- [H3] $c_{23} + c_{32} = \lambda_2 + \lambda_3$;
- [H4] $c_{ii} = \lambda_i (i = 1, 2, 3)$.

Suppose that system (3) with boundary conditions either (12) or (13), has a positive solution $(u(z), v(z), w(z))$ (a solution with $u(z) > 0$, $v(z) > 0$ and $w(z) > 0$ for all $z \in \mathbb{R}$). Then $u(z) + v(z) + w(z) = 1$.

For the scaled total density $p = u + v + w$, the problem of whether $p > 1, p < 1$, or $p = 1$ is of interest and is essential, since under certain conditions on the parameters it follows from Corollary 1 that $p = 1$. In this case, the number of equations under consideration is reduced from three to two due to the fact that $w$ is related to $u$ and $v$ by $w = 1 - u - v$. As a consequence, the situation among the three species dramatically changes. Therefore, this case can be regarded as a critical case for the competition among the three species.

We are now in a position to show the main result in this section.

**Theorem 2.2** (Nonexistence). Assume that $d_1 = d_2 = d_3 = 1$ and [H1] holds.

- [H1] $c_{12} + c_{21} = \lambda_1 + \lambda_2$;
- [H2] $c_{13} + c_{31} \geq \lambda_1 + \lambda_3$;
- [H3] $c_{23} + c_{32} \geq \lambda_2 + \lambda_3$;
- [H4] $c_{13} < \lambda_1, c_{23} < \lambda_2$;
- [H5] $c_{11} = \lambda_1, c_{22} = \lambda_2$. 
Then system (3) with boundary conditions either (12) or (13), has no positive solution \((u(z), v(z), w(z))\) with \(u(z), v(z), w(z) > 0\), for all \(z \in \mathbb{R}\).

**Proof.** Following the proof of Proposition 1, we see that \(p \leq H_4\) where

\[
q = \frac{1}{2} \theta q' + u \lambda_1 - c_{31} u - c_{21} v - c_{23} w, \quad \frac{1}{2} \theta q' + u \lambda_2 - c_{21} u - c_{22} v - c_{23} w, \quad \frac{1}{2} \theta q' + c_{31} u - c_{21} v - c_{23} w.
\]

and

\[
0 = q'' + \theta q' + v (\lambda_2 - c_{21} u - c_{22} v - c_{23} w)\]

Using \(w \leq 1 - u - v\), the first and second equations in (3) become respectively

\[
0 = u'' + \theta u' + u (\lambda_1 - c_{11} u - c_{12} v - c_{13} w)
\]

\[
\geq u'' + \theta u' + u (\lambda_1 - c_{11} u - c_{12} v - c_{13} (1 - u - v))
\]

\[
= u'' + \theta u' + u (\lambda_1 - c_{13} - (c_{11} - c_{13}) u - (c_{12} - c_{13}) v),
\]

(16)

and

\[
0 = v'' + \theta v' + v (\lambda_2 - c_{21} u - c_{22} v - c_{23} w)
\]

\[
\geq v'' + \theta v' + v (\lambda_2 - c_{21} u - c_{22} v - c_{23} (1 - u - v))
\]

\[
= v'' + \theta v' + v (\lambda_2 - c_{23} - (c_{21} - c_{23}) u - (c_{22} - c_{23}) v).
\]

(17)

Let \(q(z) = u(z) + v(z)\). Combining (16) and (17) gives

\[
0 \geq q'' + \theta q' + u (\lambda_1 - c_{13}) + v (\lambda_2 - c_{23}) - (c_{11} - c_{13}) u^2 - (c_{22} - c_{23}) v^2
\]

\[
- u v (c_{12} - c_{13} + c_{21} - c_{23})
\]

\[
\geq q'' + \theta q' + u (\lambda_1 - c_{13}) + v (\lambda_2 - c_{23}) - (\lambda_1 - c_{13}) u^2 - (\lambda_2 - c_{23}) v^2
\]

\[
- u v (\lambda_1 - c_{13} + \lambda_2 - c_{23})
\]

\[
= q'' + \theta q' + ((\lambda_1 - c_{13}) u + (\lambda_2 - c_{23}) v) (1 - u - v),
\]

(18)

where \([H1]\) and \([H5]\) have been used. Thus, \(q\) satisfies

\[
\begin{cases}
q'' + \theta q' + ((\lambda_1 - c_{13}) u + (\lambda_2 - c_{23}) v) (1 - q) \leq 0,
q(\pm \infty) = 1.
\end{cases}
\]

(19)

By means of \([H4]\), it follows from Lemma 2.1 that \(q \geq 1\). We conclude that \(w \leq 0\) since \(p \leq 1\) and \(q \geq 1\). This is a contradiction and the proof is completed.

Suppose that \([H1], [H5]\) (in Theorem 2.2), \(\lambda_3 \geq \lambda_1\), and \(\lambda_3 \geq \lambda_2\) hold. Clearly, \([H2]\) and \([H4]\) (in Theorem 2.2) imply that \(c_{31} \geq \lambda_3\). Thus, under \([H2]\) and \([H4]\), \(c_{13} = \lambda_1\) and \(c_{31} \geq \lambda_3\) hold. Ecologically, this means that without the presence of \(v, u\) is relatively stronger than \(w\) in the inter-specific competition between \(u\) and \(w\). Similarly, \(c_{32} \geq \lambda_3\) is true when \([H3]\) (in Theorem 2.2) and \([H4]\) are satisfied. Then \(c_{32} \geq \lambda_3\) together with \(c_{23} < \lambda_2\) ecologically implies that, without the presence of \(v\), \(u\) is relatively stronger than \(w\) in the inter-specific competition between \(v\) and \(w\). Consequently, in this case the extinction of \(u\) under the competition among \(u, v\), and \(w\) can be expected and the nonexistence result in Theorem 2.2 under the the hypotheses \([H1] \sim [H5]\) ecologically makes sense.

We remark that it follows from Theorem 2.2 that under certain hypotheses, system (3) admits no solutions with profiles similar to those investigated in [1] and [2].

**Corollary 2** (Degenerate solutions). Assume that \(d_1 = d_2 = d_3 = 1\) and \([H1] \sim [H6]\) hold:

\[
\begin{align*}
&[H1] \ c_{12} + c_{21} = \lambda_1 + \lambda_2; \\
&[H2] \ c_{13} + c_{31} = \lambda_1 + \lambda_3; \\
&[H3] \ c_{23} + c_{32} = \lambda_2 + \lambda_3; \\
&[H4] \ c_{13} < \lambda_1 \text{ and } c_{23} < \lambda_2 \text{ or } \lambda_1 - c_{13} = \lambda_2 - c_{23} < 0;
\end{align*}
\]
Proposition 1 and Theorem 2.2 that

Proof. First of all, due to $[H1] \sim [H3]$ and $[H6]$, it follows from the proof of Proposition 1 and Theorem 2.2 that $p = u + v + w = 1$ and $q = u + v$ satisfies

\[
\begin{cases}
q'' + \theta q' + ((\lambda_1 - c_{13}) u + (\lambda_2 - c_{23}) v) (1 - q) = 0, \\
q(\pm \infty) = 1.
\end{cases}
\] (20)

According to $[H4]$, we shall divide the proof into two cases. For the first case where $c_{13} < \lambda_1$, $c_{23} < \lambda_2$ hold, Lemma 2.1 gives $q = u + v = 1$. This proves (i) and (iii). Substituting $v = 1 - u$ and $w = 0$ into the first equation in (3) gives

\[
u'' + \theta u' + (\lambda_1 - c_{12}) u (1 - u) = 0,
\]
and $u(-\infty) = 1, u(\infty) = 0$. Therefore, (ii) follows and the minimal speed of the wave is given by $2\sqrt{\lambda_1 - c_{12}}$ ([13]), where $\lambda_1 - c_{12} > 0$ by $[H5]$.

For the second case where $\lambda_1 - c_{13} = \lambda_2 - c_{23} < 0$ hold, once again we have (20). Using $\lambda_1 - c_{13} = \lambda_2 - c_{23}$, (20) is rewritten as

\[
\begin{cases}
q'' + \theta q' - (c_{13} - \lambda_1) q (1 - q) = 0, \\
q(\pm \infty) = 1.
\end{cases}
\] (22)

We show $q = 1$. To this end, multiplying the equation in (22) by $q'$ and integrating with respect to $z$ yield

\[
\int_{-\infty}^{\infty} q' q'' dz + \theta \int_{-\infty}^{\infty} (q')^2 dz - (c_{13} - \lambda_1) \int_{-\infty}^{\infty} q' q (1 - q) dz = 0.
\] (23)

Then

\[
\frac{1}{2} (q')^2 \bigg|_{z=-\infty}^{z=\infty} + \theta \int_{-\infty}^{\infty} (q')^2 dz - (c_{13} - \lambda_1) \int_{1}^{1} q (1 - q) dq = 0.
\] (24)

This gives $\theta \int_{-\infty}^{\infty} (q')^2 dz = 0$. As a result, either $\theta = 0$ or $\int_{-\infty}^{\infty} (q')^2 dz = 0$. The latter case yields $q$ is a constant and hence $q = 1$. For the case $\theta = 0$, the problem (22) becomes

\[
\begin{cases}
q'' - (c_{13} - \lambda_1) q (1 - q) = 0, \\
q(\pm \infty) = 1.
\end{cases}
\] (25)

We prove $q = 1$ in this case, too. Indeed, let $q' = \rho$ and $\rho' = (c_{13} - \lambda_1) q (1 - q)$. Then

\[
\frac{dq}{d\rho} = \frac{\rho}{(c_{13} - \lambda_1) q (1 - q)},
\]
which gives the relation between $\rho$ and $q$ as

\[
\frac{\rho^2}{2} = (c_{13} - \lambda_1) \left( \frac{q^2}{2} - \frac{q^3}{3} + k \right),
\] (27)
where the constant \( k \) can be determined by \( (\rho, q) = (0, 1) \) and it is easy to see that 
\[
k = -\frac{1}{6}.
\]
Thus, we have 
\[
e^{-\frac{\rho}{2}t} = \frac{1}{6} (\lambda_1 - c_{13}) (q - 1)^2 (2q + 1).
\]
Since \( c_{13} > \lambda_1 \), the only possibility for the last equality to be true is \( q = 1 \). The other desired results can be shown in a similar manner as the first case. This completes the proof. \( \square \)

As an immediate application of the results in this section, we consider the May-Leonard model \([16]\). Suppose that we consider a non-transitive relationship among the three competing species \( u, v \) and \( w \). That is, in pairwise competition between \( u, v \) and \( w \), we assume that \( v \) eliminates \( u \), \( u \) eliminates \( w \), but \( w \) eliminates \( v \). An example to model such a phenomenon is the May-Leonard model

\[
\begin{align*}
  u_t &= u (1 - u - \beta v - \alpha w), \\
  v_t &= v (1 - \alpha u - v - \beta w), \\
  w_t &= w (1 - \beta u - \alpha v - w),
\end{align*}
\]

where \( 0 < \alpha < 1 \) and \( 1 < \beta \) are constants. When the effect of species immigrating described by diffusion is incorporated into (28), we are led to the diffusive May-Leonard model

\[
\begin{align*}
  u_t &= u_{xx} + u (1 - u - \beta v - \alpha w), \\
  v_t &= v_{xx} + v (1 - \alpha u - v - \beta w), \\
  w_t &= w_{xx} + w (1 - \beta u - \alpha v - w).
\end{align*}
\]

Suppose that \((u, v, w) = (u, v, w)(z), z = x - \theta t, \) is a traveling wave solution of (29) with boundary conditions \((u, v, w)(-\infty) = (0, 1, 0)\) and \((u, v, w)(\infty) = (1, 0, 0)\), According to Theorem 1, when \( \alpha + \beta \leq 2 \), the total density of the three species has a lower bound, that is \( u + v + w \geq 1 \), while when \( \alpha + \beta \geq 2 \), we have an upper bound for the total density of the three species, that is \( u + v + w \leq 1 \). Due to Theorem 2.2, when \( \alpha = \beta = 1 \), there exists no traveling wave solution of (29) satisfying boundary conditions \((u, v, w)(-\infty) = (0, 1, 0)\) and \((u, v, w)(\infty) = (1, 0, 0)\).

On the other hand, we also give another example to illustrate Theorem 2.2. It is readily seen that the parameters \( \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, c_{11} = 1, c_{12} = 1, c_{13} = \frac{1}{2}, c_{21} = 2, c_{22} = 2, c_{23} = \frac{3}{2}, c_{31} = 4, c_{32} = 5 \) and \( c_{33} = 3 \) satisfy the hypotheses \([H1] \sim [H5]\) in Theorem 2.2.

To answer Q3 mentioned in the introduction, for simplicity we consider \((P)\) with \( d = 1 \), i.e.,

\[
\begin{align*}
  u'' + \theta u' + u (1 - u - a_1 v) = 0, \\
  v'' + \theta v' + \lambda v (1 - a_2 u - v) = 0, \\
  (u, v)(-\infty) = (1, 0), \\
  (u, v)(\infty) = (0, 1).
\end{align*}
\]

**Theorem 2.3.** Suppose that \((u, v)\) is a solution of problem \((*)\), then

(i) when \( \lambda < \frac{a_1 - 1}{1 - a_2} \Rightarrow u + v < 1; \)

(ii) when \( \lambda = \frac{a_1 - 1}{1 - a_2} \Rightarrow u + v = 1; \)

(iii) when \( \lambda > \frac{a_1 - 1}{1 - a_2} \Rightarrow u + v > 1. \)

**Proof.** The proof of the desired result is similar to that of Theorem 1, and is hence omitted here. \( \square \)

As a direct consequence of Theorem 2.3 when \( \lambda = 1 \), we have

**Corollary 3.** Suppose that \((u, v)\) is a solution of problem \((*)\) with \( \lambda = 1 \), then
(i) when \( a_1 + a_2 > 2 \) \( \Rightarrow u + v < 1; \)
(ii) when \( a_1 + a_2 = 2 \) \( \Rightarrow u + v = 1; \)
(iii) when \( a_1 + a_2 < 2 \) \( \Rightarrow u + v > 1. \)

3. Exact traveling wave solutions revisited. In [2], we have shown that under certain conditions on the parameters, (3) admits exact traveling wave solutions with the profiles of \( u(z) \) being increasing (respectively, \( u(z) \) being decreasing) in \( z \), \( v(z) \) being decreasing (respectively, \( v(z) \) being increasing) in \( z \), and \( w(z) \) being a pulse (i.e., \( w(\pm \infty) = 0 \) and \( w(z) > 0 \) for \( z \in \mathbb{R} \)). All the calculations there were carried out with the aid of Mathematica. In what follows, we show that the same result can be obtained without using Mathematica.

Looking for traveling wave solutions \((u(z), v(z), w(z))\) of (1), where \( z = px - \theta t \), and for simplicity restricting ourself to the case where \( d_1 = d_2 = d_3 = 1 \) and \( \lambda_1 = \lambda_2 = \lambda_3 = 1 \) lead to

\[
\begin{align*}
\begin{cases}
p^2 u_{zz} + \theta u_z + u(1 - u - c_{12}v - c_{13}w) = 0, \\
p^2 v_{zz} + \theta v_z + v(1 - c_{21}u - v - c_{23}w) = 0, \\
p^2 w_{zz} + \theta w_z + w(1 - c_{31}u - c_{32}v - w) = 0.
\end{cases} \\
(z \in \mathbb{R}, \quad (31))
\end{align*}
\]

As in [2], our ansatz for solving (31) is

\[
\begin{align*}
\begin{cases}
u(z) = k_1(1 + \tanh z), \\
u(z) = k_2(1 - \tanh z)^2, \\
u(z) = k_3(1 - \tanh^2 z),
\end{cases} \\
(32)
\end{align*}
\]

where \( k_1, k_2, \) and \( k_3 \) are positive constants to be determined. It is easy to see that since \( u(z) = k_1(1 + T(z)) \), we have

\[
u'(z) = k_1 T'(z) = k_1 (1 - T(z)^2) = k_1 (1 + T(z))(1 - T(z)) = u(z)(1 - T(z)), \]

and

\[
\begin{align*}
u''(z) &= u'(z)(1 - T(z)) + u(z)(-T'(z)) \\
&= u(z)((1 - T(z))^2 - 1 + T(z)^2) \\
&= -2 u(z) T(z)(1 - T(z)),
\end{align*}
\]

where \( T(z) = \tanh z \). Similarly, from \( v(z) = k_2(1 - \tanh z)^2 \) and \( w(z) = k_3(1 - \tanh^2 z) \) we obtain

\[
\begin{align*}
v'(z) &= -2 v(z)(1 + T(z)), \\
v''(z) &= 2 v(z)(1 + T(z))(1 + 3 T(z)) \quad (33)
\end{align*}
\]

and

\[
\begin{align*}
w'(z) &= -2 w(z) T(z), \\
w''(z) &= 2 w(z)(-1 + 3 T(z)^2), \quad (34)
\end{align*}
\]

respectively. Now we insert ansatz (32) into (31) to get

\[
\begin{align*}
p^2 u_{zz} + \theta u_z + u(1 - u - c_{12}v - c_{13}w) &= u \left[ \alpha_0 + \alpha_1 T(z) + \alpha_2 T(z)^2 \right], \\
p^2 v_{zz} + \theta v_z + v(1 - c_{21}u - v - c_{23}w) &= v \left[ \beta_0 + \beta_1 T(z) + \beta_2 T(z)^2 \right], \\
p^2 w_{zz} + \theta w_z + w(1 - c_{31}u - c_{32}v - w) &= w \left[ \gamma_0 + \gamma_1 T(z) + \gamma_2 T(z)^2 \right],
\end{align*}
\]

where

\[
\begin{align*}
\alpha_0 &= 1 + \theta - k_1 - c_{12} k_2 - c_{13} k_3, \quad (35a) \\
\alpha_1 &= -2 p^2 - \theta - k_1 + 2 c_{12} k_2, \quad (35b) \\
\alpha_2 &= 2 p^2 - c_{12} k_2 + c_{13} k_3, \quad (35c)
\end{align*}
\]
\[
\begin{align*}
\beta_0 &= 1 + 2p^2 - 2\theta - c_{21}k_1 - k_2 - c_{23}k_3, \\
\beta_1 &= 8p^2 - 2\theta - c_{21}k_1 + 2k_2, \\
\beta_2 &= 6p^2 - k_2 + c_{23}k_3, \\
\gamma_0 &= 1 - 2p^2 - c_{31}k_1 - c_{32}k_2 - k_3, \\
\gamma_1 &= -2\theta - c_{31}k_1 + 2c_{32}k_2, \\
\gamma_2 &= 6p^2 - c_{32}k_2 + k_3.
\end{align*}
\] (35d)

Equating the coefficients of powers of \( T(z) \) to zero yields a system of 9 equations:

\[
\begin{align*}
\alpha_i &= 0, \quad i = 1, 2, 3, \\
\beta_i &= 0, \quad i = 1, 2, 3, \\
\gamma_i &= 0, \quad i = 1, 2, 3.
\end{align*}
\]
(36)

Let us first determine \( k_1 = \frac{1}{2} \) and \( k_2 = \frac{1}{4} \). It follows from (35a)~(35f) that \( \alpha_0 + \alpha_1 + \alpha_2 = 1 - 2k_1 \) and \( \beta_0 - \beta_1 + \beta_2 = 1 - 4k_2 \). Due to ansatz (32), we have \( k_1 = \frac{1}{2} \) and \( k_2 = \frac{1}{4} \). Since \( \alpha_0 + \alpha_1 + \alpha_2 = 0 \) and \( \beta_0 - \beta_1 + \beta_2 = 0 \), we can take out, say \( \alpha_0 = 0 \) and \( \beta_0 = 0 \) in solving system (36). Then we observe that the 6 equations \( \alpha_1 = \alpha_2 = \beta_1 = \gamma_1 = \gamma_2 = \gamma_3 = 0 \) can be regarded as a linear system which can be written in matrix form as follows:

\[
\begin{pmatrix}
2k_2 & 0 & 0 & 0 & 0 & -1 \\
-k_2 & 0 & 0 & 0 & c_{13} & 0 \\
0 & -k_1 & 0 & 0 & 0 & -2 \\
0 & 0 & -k_1 & -k_2 & -1 & 0 \\
0 & 0 & -k_1 & 2k_2 & 0 & -2 \\
0 & 0 & 0 & -k_2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
c_{12} \\
c_{21} \\
c_{31} \\
c_{32} \\
c_3 \\
\theta
\end{pmatrix}
= \begin{pmatrix}
2p^2 + k_1 \\
-2p^2 \\
-8p^2 - 2k_2 \\
2p^2 - 1 \\
0 \\
-6p^2
\end{pmatrix}.
\]
(37)

Once \( k_3, c_{12}, c_{21}, c_{31}, c_{32} \) and \( \theta \) are solved as functions of \( c_{13} \) and \( p, c_{23} \) (as a function of \( c_{13} \) and \( p \)) can be obtained by solving \( \beta_2 = 6p^2 - k_2 + c_{23}k_3 = 0 \). Furthermore, letting

\[
\mathcal{M} = \begin{pmatrix}
2k_2 & 0 & 0 & 0 & 0 & -1 \\
-k_2 & 0 & 0 & 0 & c_{13} & 0 \\
0 & -k_1 & 0 & 0 & 0 & -2 \\
0 & 0 & -k_1 & -k_2 & -1 & 0 \\
0 & 0 & -k_1 & 2k_2 & 0 & -2 \\
0 & 0 & 0 & -k_2 & 1 & 0
\end{pmatrix},
\]
(38)

we have \( \det(\mathcal{M}) = \frac{1}{18} (c_{13} - 1) \). It follows immediately that if \( c_{13} \neq 1 \), a unique solution of (37) exists. The result obtained is summarized in the following

**Theorem 3.1** (See also [2]). **System (31) has a solution of the form**

\[
\begin{align*}
u(z) &= \frac{1}{2} (1 + \tanh z), \\
v(z) &= \frac{1}{4} (1 - \tanh z)^2, \\
w(z) &= \frac{4p^2}{c_{13} - 1} (1 - \tanh^2 z)
\end{align*}
\]
provided that
\[ c_{12} = 8p^2 \left( 3 + \frac{2}{c_{13} - 1} \right), \quad c_{21} = 3 + p^2 \left( -24 - \frac{32}{c_{13} - 1} \right), \quad (39) \]
\[ c_{23} = \frac{(1 - 24p^2)(c_{13} - 1)}{16p^2}, \quad c_{32} = 8p^2 \left( 3 + \frac{2}{c_{13} - 1} \right), \quad (40) \]
\[ c_{31} = 2 - \frac{16p^2c_{13}}{c_{13} - 1}, \quad \theta = \frac{1}{2} + p^2 \left( 10 + \frac{8}{c_{13} - 1} \right), \quad (41) \]
where the two free parameters \( c_{13} \) and \( p \) satisfy
\[ c_{13} > \frac{3 + 8p^2}{3 - 24p^2}, \quad p \in (0, \frac{1}{2\sqrt{6}}). \quad (42) \]

Remark 1. Under the conditions (39)~(41), it turns out that the necessary and sufficient condition for \( c_{12}, c_{21}, c_{23}, c_{32}, c_{31}, k_3, p > 0 \) to be satisfied is given by (42).

Remark 2. If we choose \((c_{13}, p) = (3, \frac{1}{5})\) in the above theorem, then \( c_{12} = \frac{32}{25}, c_{21} = \frac{7}{5}, c_{23} = \frac{1}{5}, c_{32} = \frac{26}{27}, c_{31} = \frac{2509}{2500} \approx 1, \theta = \frac{3}{50}\) by (39)~(41), and the resulting solution is
\[
\begin{align*}
    u(z) &= \frac{1}{2} \left( 1 + \tanh z \right), \\
    v(z) &= \frac{1}{4} \left( 1 - \tanh z \right)^2, \\
    w(z) &= \frac{99}{400} \left( 1 - \tanh^2 z \right).
\end{align*}
\]
The profiles of \( u(z), v(z), \) and \( w(z) \) are shown in Figure 1.

Now we demonstrate an interesting example which shows that the nonexistence result established in Theorem 2.2, as mentioned in the introduction, can fail to hold when the parameters vary slightly.

According to Theorem 3.1, when \( c_{12} = \frac{2499}{2500} \approx 0.9996, c_{21} = \frac{1263}{1250} \approx 1.0104, c_{31} = \frac{2509}{2500} \approx 1.0036, c_{13} = \frac{2491}{2475} \approx 1.0065, c_{23} = \frac{2476}{2475} \approx 1.0004, \) and \( c_{32} = \frac{2499}{2500} \approx 0.9996, \) we have the solution
\[
\begin{align*}
    u(z) &= \frac{1}{2} \left( 1 + \tanh z \right), \\
    v(z) &= \frac{1}{4} \left( 1 - \tanh z \right)^2, \\
    w(z) &= \frac{99}{400} \left( 1 - \tanh^2 z \right),
\end{align*}
\]

Figure 1. Profiles of the solution \((u(z), v(z), w(z))\).
where $z = px - \theta t$ with $p = \frac{1}{990}$ and $\theta = \frac{1}{1000}$. So there exists a positive solution $(u, v, w)$ satisfying $(u, v, w)(-\infty) = (0, 1, 0)$ and $(u, v, w)(\infty) = (1, 0, 0)$. However, if we perturb the above parameters as $c_{12} = 0.990$, $c_{21} = 1.010$, $c_{31} = 1.001$, $c_{13} = 0.999$, $c_{23} = 0.999$, and $c_{32} = 1.001$ so that the hypotheses $[H1] \sim [H4]$ in Theorem 2.2 are satisfied, then there exists no positive solution $(u, v, w)$ satisfying $(u, v, w)(-\infty) = (0, 1, 0)$ and $(u, v, w)(\infty) = (1, 0, 0)$.

4. New exact one-hump solutions satisfying boundary conditions (12). In [2], exact traveling wave solutions of $(LV)$ are represented in terms of polynomials in hyperbolic tangent functions with degree 2. These exact traveling wave solutions have profiles with $u(z)$ being increasing in $z$, $v(z)$ being decreasing, and $w(z)$ being a pulse.

In this section, we show that exact traveling wave solutions with similar profiles can also be constructed using functions other than hyperbolic tangent functions. To be more precise, let us recall the key ideas in the previous papers. In [2], the hyperbolic tangent function tanh is used as the building block to construct the one-hump solution for three-species problem. We note that all the exact solutions in [2] are polynomials in tanh of degree 2. Those solutions lead us to the question whether there are wave solutions which can be represented by polynomials with degree more than 2 and whether there are other functions which can be used as the building blocks. The papers [20, 21] by Rodrigo and Mimura present several exact solutions for two-species case. Their results indicate that, in addition to tanh, the Weierstrass elliptic function and Jacobi elliptic function can be used to represent solutions. Although the ideas in [20, 21] are very inspiring, we need to find other methods to deal with the more complicated three-species problem. In [1], the idea to represent a wave solution in terms of a function $T(x)$ which satisfies a simple ODE with a polynomial nonlinearity was proposed. When $T(x)$ is tanh, the wave solution can be written down explicitly and is called an exact solution. When $T(x)$ can only be solved in an implicit form, the wave solution is called a semi-exact solution. The purpose of [1] is to find a two-hump solution (see Figure 5 below). Since it seems hard to construct such a solution by tanh, other $T(x)$ were used.

In this section, we show that in addition to tanh, a lot of one-hump solutions can also be constructed by other $T(x)$. Moreover, the solutions constructed are polynomials in $T(x)$ with degree great than 2. Now let us consider the solution $T(z)$ of the initial value problem

$$
\begin{align*}
\frac{dT}{dz} &= T(z)(1 - T(z))(a + T(z)), \quad z \in \mathbb{R}, \\
T(0) &= T_0,
\end{align*}
$$

(43)

where $T_0 \in (0, 1)$ and $a \in \mathbb{R}$ are constants. Suppose that (3) admits a solution of the form

$$
\begin{align*}
U(z) &= k_1 T^i(z), \\
V(z) &= k_2 (1 - T(z))^m, \\
W(z) &= k_3 T^n(z)(1 - T(z))^2,
\end{align*}
$$

(44)

where $i$, $m$ and $n$ are positive integers; $k_1$, $k_2$ and $k_3$ are positive constants. As we have done in Section 3, using ansätz (44) in (3) gives a system of algebraic equations. We then use Mathematica to solve this system as in Section 3 to obtain seven types of exact solutions. According to different $i$, $m$ and $n$, these solutions are classified into seven types as shown in Table 1. It should be noted that we are not able to apply ansätz (44) to find solutions of (3) in a systematic way. In Table 1,
restrictions on $a$ as well as the parameter dependence of the propagation speed $\theta$ for each type of solutions are shown. For simplicity, we only discuss one type of solution in more detail here. For further details on the other six types of solutions, refer to [3].

For $(i, m, n) = (2, 4, 1)$, we have the solution

\[
\begin{align*}
U(z) &= k_1 T^2(z), \\
V(z) &= k_2 (1 - T(z))^4, \\
W(z) &= k_3 T^4(z)(1 - T(z))^2,
\end{align*}
\]

(45) to (3), provided that the following relations hold:

\[
d_2 = \frac{a(7 + 5a)d_1}{-2 + a(11 + a)}, \quad d_3 = \frac{(-1 + 3a)(7 + 5a)d_1}{-13 + 3a(8 + 3a)}, \quad \theta = (-7 - 5a)d_1,
\]

(46a)

\[
\lambda_1 = 2(1 + a)(4 + 3a)d_1, \quad \lambda_2 = \frac{24a(7 + 5a)d_1}{-2 + a(11 + a)},
\]

(46b)

\[
\lambda_3 = \frac{(7 + 5a)(-15 + a)(32 + a(25 + 6a))d_1}{-13 + 3a(8 + 3a)},
\]

(46c)

\[
c_{11} = \frac{2(1 + a)(4 + 3a)d_1}{k_1}, \quad c_{12} = \frac{8d_1}{k_2}, \quad c_{13} = \frac{2(9 + 7a)d_1}{k_3},
\]

(46d)

\[
c_{21} = \frac{4(2 + a)(7 + 5a)(-1 + 5a)(2 + a)d_1}{(-2 + a)(11 + a)k_1}, \quad c_{22} = \frac{24a(7 + 5a)d_1}{(-2 + a)(11 + a)k_2},
\]

(46e)

\[
c_{23} = \frac{44a(2 + a)(7 + 5a)d_1}{(-2 + a)(11 + a)k_3},
\]

(46f)

\[
c_{31} = \frac{(5 + 3a)(7 + 5a)(-9 + a)(17 + 12a)d_1}{(-13 + 3a(8 + 3a))k_1}, \quad c_{32} = \frac{15(-1 + 3a)(7 + 5a)d_1}{(-13 + 3a(8 + 3a))k_2},
\]

(46g)

\[
c_{33} = \frac{(-1 + 3a)(7 + 5a)(47 + 27a)d_1}{(-13 + 3a(8 + 3a))k_3},
\]

(46h)

where $k_1, k_2, k_3 > 0$ are constants. The necessary and sufficient condition for $k_i$, $d_i$, $\lambda_i$, $c_{ij}$ ($i, j = 1, 2, 3$), $c_{ij}$ ($i, j = 1, 2, 3, i \neq j$) > 0, and $a \notin [-1, 0]$ in (46) is given by

\[
-11 + \sqrt{129} \quad \text{or} \quad a > \frac{-4 + \sqrt{29}}{3}.
\]

(47)
Approximately, $0.178908 < a < 0.34$ or $a > 0.491722$.

For the particular case where $a = 1$ and $T(0) = \frac{1}{2}$, the change of variable $T^2(z) = \frac{1}{2}(1 + v(z))$ transforms the problem (43) into the following initial value problem for $v(z)$:

\[
\begin{cases}
\frac{dv}{dz} = 1 - v^2, & z \in \mathbb{R}, \\
v(0) = -\frac{1}{2}.
\end{cases}
\]

It is readily seen that $v(z) = \tanh z - \frac{1}{2}$. Hence, in this case the semi-exact solution (45) to (3) can be rewritten in terms of $\tanh z$ as

\[
\begin{align*}
U(z) &= \frac{1}{4} k_1 (1 + 2 \tanh z), \\
V(z) &= k_2 \left( 1 - \frac{1}{2} \sqrt{1 + 2 \tanh z} \right)^4, \\
W(z) &= \frac{1}{2} k_3 \sqrt{1 + 2 \tanh z} \left( 1 - \frac{1}{2} \sqrt{1 + 2 \tanh z} \right)^2.
\end{align*}
\]

We remark that this solution is essentially different from that presented in [2], in that $V(z)$ and $W(z)$ are not polynomials in hyperbolic tangent functions, as mentioned in the Introduction.

We give an example for illustration. If we choose $(a, k_1, k_2, k_3, d_1) = \left( \frac{1}{5}, 1, 2, 23, 3 \right)$ and $T_0 = \frac{1}{2}$ in (43), then

\[
\begin{align*}
\lambda_1 &= \frac{828}{25}, \lambda_3 = \frac{5664}{245}, c_{11} = \frac{828}{25}, c_{12} = 12, c_{13} = \frac{312}{115}, c_{31} = \frac{3072}{35}, c_{33} = \frac{3144}{1127}, \\
d_2 &= 20, \lambda_2 = 480, \theta = -24, c_{21} = 1056, c_{22} = 240, c_{23} = \frac{1936}{23}, c_{32} = \frac{450}{49}, d_3 = \frac{60}{49}
\end{align*}
\]

by (46). The resulting profiles of $T$ and $(U, V, W)$ are shown in Figure 2 and Figure 3, respectively.

![Figure 2](image-url)
5. New exact two-hump solutions satisfying boundary conditions (13). Two types of semi-exact two-hump solutions are proposed in [1]. As a continuation of the study in [1], we construct two more types of such solutions in this section. In particular, we note that all the parameter restrictions in [1] are represented in terms of rational functions, while in Theorem 5.1 and Theorem 5.2 below some of the parameter restrictions are expressed in terms of non-rational functions. For the method to obtain these semi-exact solutions, see [1].

**Theorem 5.1.** Assume that the following (48)~(55) hold.

\[
\begin{align*}
\lambda_1 &= 4 (2 + n)^2, \\
\lambda_2 &= \frac{2 \left(48 + 92 n + 60 n^2 + 13 n^3\right)}{1 + n} + \frac{2 \left(4 + 3 n\right) (2 + n) \sqrt{20 + n (20 + 9 n)}}{1 + n}, \\
\lambda_3 &= \frac{2 \left(68 + 136 n + 91 n^2 + 20 n^3\right)}{1 + n} + \frac{2 \left(5 + 4 n\right) (2 + n) \sqrt{20 + n (20 + 9 n)}}{1 + n}, \\
c_{11} &= 20 + n (24 + 7 n) - (2 + n) \sqrt{20 + n (20 + 9 n)}, \\
c_{12} &= (1 + n) \left(20 + n (24 + 7 n) - \sqrt{(2 + n)^2 (20 + n (20 + 9 n))}\right) / (1 + n), \\
c_{13} &= 10 (1 + n)^2, \\
c_{21} &= 20 (2 + n) (4 + 3 n), \\
c_{22} &= 4 (1 + n) (16 + 11 n), \\
c_{23} &= 28 (1 + n)^2, \\
c_{31} &= 20 (2 + n) (5 + 4 n), \\
c_{32} &= 16 (1 + n) (5 + 4 n), \\
c_{33} &= 54 (1 + n)^2.
\end{align*}
\]

Then (3) with (13) admits a solution of the form

\[
\begin{align*}
u(z) &= \frac{2}{3 n + \sqrt{20 + n (20 + 9 n)}} + T^2, \\
v(z) &= (1 - T^2)^2, \\
w(z) &= \frac{1}{1 + n} \left[1 + (1 + n) T^2\right] (1 - T^2)^2,
\end{align*}
\]

where \(T = T(n; z)\) is the solution of the following boundary value problem

\[
\begin{align*}
\frac{dT}{dz} &= (1 - T^2) \left[1 + (1 + n) T^2\right] (1 - T^2)^2, \\
T(n; 0) &= 0.
\end{align*}
\]
Theorem 5.2. Assume that the following (58)\textendash (65) hold.

\begin{align*}
\lambda_2 &= \frac{2 (48 + 92 n + 60 n^2 + 13 n^3)}{1 + n} - \frac{2 (4 + 3 n) \sqrt{(2 + n)^2 (20 + n (20 + 9 n))}}{1 + n} \\
\lambda_3 &= \frac{2 (68 + 136 n + 91 n^2 + 20 n^3)}{1 + n} - \frac{2 (5 + 4 n) \sqrt{(2 + n)^2 (20 + n (20 + 9 n))}}{1 + n}.
\end{align*}

\begin{align*}
c_{11} &= 20 + n (24 + 7 n) + \sqrt{(2 + n)^2 (20 + n (20 + 9 n))}. \\
(1 + n) \left( 20 + n (24 + 7 n) + \sqrt{(2 + n)^2 (20 + n (20 + 9 n))} \right) \\
c_{12} &= \frac{2 + n}{1 + n}.
\end{align*}

Then (3) with (13) admits a solution of the form

\begin{equation}
\begin{cases}
u(z) = 3 n - \sqrt{20 + n (20 + 9 n)} + T^2, \\
v(z) = (1 - T^2)^2, \\
w(z) = \frac{1}{1 + n} \left[ 1 + (1 + n) T^2 \right] (1 - T^2)^2,
\end{cases}
\end{equation}

where $T = T(n; z)$ is the solution of the following boundary value problem

\begin{equation}
\begin{cases}
\frac{dT}{dz} = (1 - T^2) \left[ 1 + (1 + n) T^2 \right], & z \in \mathbb{R}, \\
T(n; 0) = 0.
\end{cases}
\end{equation}

An example is given below to illustrate Theorem 5.1. The profiles of the solution in Theorem 5.2 are similar. When $n = 3$, Figure 4 and Figure 5 show the solution of (57) and the profiles of (56), respectively.

![Figure 4](image1.png)

**Figure 4.** Comparison of the solution to (57) with $n = 3$ and $T_0 = 0$ and the tanh function.

From Figure 6, we see that there exists a solution such that $p(z) = u(z) + v(z) + w(z)$ takes both values greater than 1 for some $z$ and values less than 1 for some other $z$. This shows the fact that the estimates in Proposition 1 fail to be true for general parameters.
Figure 5. Solution (56) with $n = 3$.

Figure 6. The profile of $u + v + w$.

6. **Concluding remarks.** Motivated by the work of Okubo et al. ([19]), we have developed an elementary approach relying on the repeated use of the maximum principle for elliptic inequalities, to establish nonexistence results for Lotka-Volterra systems of three competing species, as well as a priori estimates of total scaled species density, namely $u + v + w$. Such nonexistence results and a priori estimates have not yet been studied in the literature. We employ this approach to investigate these problems in Section 2.

In addition, exact traveling solutions for Lotka-Volterra systems of three competing species presented in [2] are expressed in terms of polynomials in hyperbolic tangent functions. Using functions other than hyperbolic tangent functions, we show that new semi-exact traveling wave solutions with similar wave profiles as in [2] can be found. Furthermore, in addition to the two types of semi-exact traveling wave solutions given in [1], new semi-exact traveling wave solutions are constructed.

Based on the maximum principle for elliptic inequalities, an elementary approach is developed to establish the estimates on the amount of the scaled density of three species, and the nonexistence result of traveling wave solutions. We believe that this approach can be applied to various problems in related studies.
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