ON THE EXISTENCE OF VORTEX-WAVE SYSTEMS TO INVISCID GSQG EQUATION

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Abstract. We study the existence of different vortex-wave systems for inviscid gSQG flow, where the total circulation are produced by point vortices and vortices with compact support. To overcome several difficulties caused by the singular formulation and infinite kinetic energy, we introduce a modified reduction method. Several asymptotic properties of the system are also given.

1. Introduction

1.1. Historical discussion. In this paper, we construct families of solutions to the vortex wave system

\[
\begin{aligned}
\frac{d\theta}{dt} + u \cdot \nabla \theta &= 0 \\
\frac{dx_i}{dt} &= \nabla^\perp G_s * \theta(x_i, t) + \sum_{j \neq i} \kappa_j \nabla^\perp G_s(x_i, x_j), \quad i = 1, \ldots, l
\end{aligned}
\]

(1.1)

where \( G_s(x, y) = \frac{c_{2,s}}{|x-y|^{2s}} \) is the fundamental solution of \((-\Delta)^s\) in \( \mathbb{R}^2 \), \( 0 < s < 1 \), here \( c_{2,s} \) is constant defined by (1.5). \((a_1, a_2)^\perp = (a_2, -a_1)\) denotes clockwise rotation through \( \frac{\pi}{2} \), and \( G_s * \theta \) is the Newton potential of \( \theta \) defined by

\[
G_s * \theta(x, t) := \int_{\mathbb{R}^2} G_s(x, y)\theta(y, t)dy.
\]

Let us explain system (1.1) briefly. The first equation is a transport equation for the background vorticity \( \theta(x, t) \), which means that the background vorticity is transported by the velocity 'generated' by itself (the term \( \nabla^\perp G_s * \theta \) ), and \( l \) point vortices (the term \( \sum_{i=1}^l \kappa_i \nabla^\perp G_s(\cdot - x_i) \) ). The second equation expresses the fact that each point vortex \( x_i(t) \) moves by the velocity 'generated' by the background vorticity (the term \( \nabla^\perp G_s * \theta(x_i, t) \)) and the other \( l - 1 \) vortices (the term \( \sum_{j \neq i} \kappa_j \nabla^\perp G_s(x_i, x_j) \)). If \( \kappa_i = 0, i = 1, \ldots, l \), then the system reduces to the vorticity form of the modified or generalized surface quasi-geostrophic equation, which has been extensively studied; see [1, 12] for example. If the background vorticity vanishes, then the system becomes the Kirchhoff-Routh equation, which is a model describing the motion of \( l \) concentrated vortices: see [15, 17, 23] for example.

The vortex-wave system was first introduced by Marchioro and Pulvirenti in [7] to describe the motion of a planar ideal fluid in which the vorticity consists of a continuously distributed part and a finite number of concentrated vortices. In the whole plane,
they consider the following vortex-wave system

\[
\begin{cases}
\partial_t \omega + v \cdot \nabla \omega = 0, \\
\frac{dx_i}{dt} = \nabla^\perp G * \omega(x_i, t) + \sum_{j=1, j \neq i}^N \kappa_j \nabla^\perp G(x_i - x_j), \quad i = 1, \ldots, N, \\
v = \nabla^\perp G * \omega + \sum_{j=1}^k \kappa_j \nabla^\perp G(x - x_j).
\end{cases}
\]

(1.3)

where \( G(x, y) = -\frac{1}{2\pi} \ln |x - y| \). By constructing Lagrangian paths, Marchioro and Pulvirenti [7] proved a theorem of existence for initial background vorticity belonging to \( L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \). The existence and uniqueness of solutions of the non-stationary vortex-wave system in the whole plane \( \mathbb{R}^2 \) have been extensively studied over the past decades; see [3, 13, 14, 16, 18] for example. However, as far as we know, few results are known for steady solutions to this system. The only two papers [10, 11] for steady solutions of this system are available.

In [10], The author first studies the vortex wave system for Euler equation, which is the extension of the results in [9] to vortex wave systems. Then in [11], the author extended the results of [10] to the case of multiple vortices.

In this paper, our purpose is to construct the steady solution of the vortex-wave system for the inviscid gSQG equation. In this case the Kirchhoff–Routh function (defined by (1.7) in the next section) plays an essential role. To be more precise, we prove that for any given non-degenerate critical point of the Kirchhoff–Routh function, there exists a family of steady solutions to the vortex-wave system that shrinks to this point.

It is worth mentioning that the construction in [10, 11] was based on the vorticity method, which was first established by Arnold [20] and further developed by many authors [4, 6, 5, 21, 22, 8]. The advantage of the method in [10, 11] is that solutions concentrating at a given strict local minimum point of the Kirchhoff–Routh function can be constructed. However, the vorticity method is no longer applicable for saddle point cases.

The method we use in this paper to construct steady solutions is called is the modified finite-dimensional reduction method. Using the modified finite-dimensional reduction method, we are able to construct solutions concentrating at a given non-degenerate critical point of the Kirchhoff–Routh function, even if this is a saddle point. Moreover, one can intuitively see the concrete form of the solution, which is helpful for analyzing the nature of the solution; see [1] for example.

1.2. Notation. In the whole plane, Green’s function for \((-\Delta)^s, 0 < s < 1\), can be written as

\[
G_s(x, y) = \frac{c_{2,s}}{|x - y|^{2-2s}}, \quad x, y \in \mathbb{R}^2,
\]

(1.4)

where

\[
c_{2,s} = \pi^{-1} 2^{-2s} \frac{\Gamma \left( \frac{2-2s}{2} \right)}{\Gamma(s)}.
\]

(1.5)

The operator \((-\Delta)^{-s}\) in \( \mathbb{R}^2 \) is the standard inverse of the fractional laplacian and is given by the expression

\[
(-\Delta)^{-s} \theta(x) = \int_{\mathbb{R}^2} G_s(x, y) \theta(y) dy.
\]

(1.6)
Let $p$ be a positive integer, $\kappa_i \in \mathbb{R}$, $i = 1, \ldots, p$. Define the generalized Kirchhoff-Routh function as

$$W_p(x_1, \ldots, x_p) = \sum_{i=1}^{p} (U x_{i1} + \frac{1}{2} \omega |x_i|^2) - \sum_{j \neq i, 1 \leq i, j \leq p} \kappa_j G_s(x_i, x_j),$$

(1.7)

where $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2$, $U \in \mathbb{R}$ is the constant linear velocity and $\omega \in \mathbb{R}$ is the constant angular velocity.

1.3. Vortex-wave system in the plane. We consider an incompressible steady flow in $\mathbb{R}^2$. The evolution of the velocity field $u = (u_1, u_2)$ is described by the modified or generalized surface quasi-geostrophic equations

$$\begin{cases}
\partial_t \theta + u \cdot \nabla \theta = 0, & \text{in } \mathbb{R}^2 \times (0, T) \\
u = \nabla^\perp \psi, \psi = (-\Delta)^{-s} \theta, & \text{in } \mathbb{R}^2 \times (0, T),
\end{cases}$$

(1.8)

where $0 < s < 1$ and $(a_1, a_2)^\perp = (a_2, -a_1)$. Here $\theta$ is the active scalar being transported by the velocity field $u$ generated by $\theta$ and $\psi$ is the stream function.

By (1.8), we have

$$\psi(x, t) = (-\Delta)^{-s} \theta(x, t) := \int_{\mathbb{R}^2} G_s(x, y) \theta(y, t) dy, \quad x \in \mathbb{R}^2. \quad (1.9)$$

If the vorticity is a Delta measure (also called a point vortex) located at $x \in \mathbb{R}^2$, i.e., $\theta = \delta_x$, then formally the velocity field it induces is

$$\nabla^\perp (-\Delta)^{-s} \delta_x = \nabla^\perp G_s(x, \cdot), \quad (1.10)$$

Similarly, the evolution of $l$ point vortices can be described by the ODE system

$$\frac{dx_i}{dt} = \sum_{j=1, j \neq i}^{l} \kappa_j \nabla_{x_i}^\perp G_s(x_j, x_i), \quad i = 1, \ldots, l, \quad (1.11)$$

where $\kappa_i$ is the vorticity strength of the $i$-th point vortex. The case $s = 1$ corresponds to the classical point vortex Eulerian interaction. A general review about the $l$-vortex problem and vortex statics can be found in [2] for the Newtonian interaction and [19] for gSQG interactions. We are concerned with periodic solutions for which the configuration of vortices is instantaneously moving as a rigid body, so that

$$\frac{dx_i}{dt} = (0, U) - \omega x_i^\perp, \quad (1.12)$$

where $U \in \mathbb{R}$ is the constant linear velocity and $\omega \in \mathbb{R}$ is the constant angular velocity. Such solutions are known as relative equilibria or vortex crystals. Using the notation $W_p$ and (1.12), (1.11) reduces to the algebraic system

$$\nabla_{x_i} W_p = 0, \quad i = 1, \ldots, l. \quad (1.13)$$

We say that a critical point $b_0 = (\bar{b}_1, \ldots, \bar{b}_p) \in \mathbb{R}^{2p}$ of $W_p$ is non-degenerate if it satisfying $\text{deg}(\nabla W_p, b_0) \neq 0$.
Now we consider the mixed problem, which is the vorticity consists of $k$ continuously distributed part $\theta$ and $l$ point vortices $x_i$ with strength $\kappa_i$, $i = k + 1, \ldots, k + l$. Then the evolution of $\theta$ and $x_i$ will obey the equations

\[
\begin{align*}
\frac{d\theta}{dt} + \nabla^\perp (\psi + \sum_{i=k+1}^{k+l} G_s(x_i, \cdot)) \cdot \nabla \theta &= 0, \\
\frac{dx_i}{dt} &= \nabla^\perp (\psi + \sum_{j=k+1, j \neq i}^{k+l} G_s(x_j, \cdot))(x_i), i = k + 1, \ldots, k + l,
\end{align*}
\]

(1.14)

which is called the vortex-wave system in the plane.

In this paper, we confined ourselves to the relative equilibria case. More precisely, we look for traveling solutions to (1.14) by requiring that

\[
\theta(x_1, x_2, t) = \theta_0(x_2 - Ut),
\]

(1.15)

for some profile function $\theta_0(x_1, x_2)$ defined on $\mathbb{R}^2$. In this case, the first equation in (1.14) can be rewritten as the stationary problem

\[
(\nabla^\perp (\psi + \sum_{i=k+1}^{k+l} \kappa_{i+k} G_s(x_i, \cdot)) - U e_2) \cdot \nabla \theta_0 = 0,
\]

(1.16)

Similarly, we look for rotating solutions $\theta(x, t)$ of (1.14) by requiring that

\[
\theta(x, t) = \theta_0(Q_{\omega t} x), \quad x \in \mathbb{R}^2,
\]

(1.17)

where

\[
Q_{\omega t} = \begin{bmatrix}
\cos(\omega t) & -\sin(\omega t) \\
\sin(\omega t) & \cos(\omega t)
\end{bmatrix}.
\]

(1.18)

Then the first equation in (1.14) becomes

\[
(\nabla^\perp (\psi + \sum_{i=1}^{l} \kappa_{i+k} G_s(x_i, \cdot)) + \omega x^\perp) \cdot \nabla \theta_0 = 0,
\]

(1.19)

For the sake of abbreviation and simplicity we shall unify (1.16) and (1.19) as follows

\[
(\nabla^\perp (\psi + \sum_{i=1}^{l} \kappa_{i+k} G_s(x_i, \cdot)) + \omega x^\perp - U e_2) \cdot \nabla \theta_0 = 0,
\]

(1.20)

Then (1.14) become the following system:

\[
\begin{cases}
(\nabla^\perp (\psi + \sum_{i=k+1}^{k+l} \kappa_i G_s(x_i, \cdot)) + \omega x^\perp - U e_2) \cdot \nabla \theta = 0, \\
(U, 0) + \omega x_i + \nabla (\psi + \sum_{j=k+1, j \neq i}^{k+l} \kappa_j G_s(x_j, \cdot))(x_i) = 0, i = k + 1, \ldots, k + l, \\
\psi = (-\Delta)^{-s} \theta.
\end{cases}
\]

(1.21)

1.4. Main result. Our main result in this paper is the following:

**Theorem 1.1.** Let $k, l, p$ be positive integers such that $k + l = p$, and $\kappa_i, i = 1, \ldots, p$, be $p$ real numbers such that $\kappa_i \neq 0$. Suppose that $b_0 = (\bar{b}_1, \ldots, \bar{b}_p)$ is a non-degenerate critical point of $\mathcal{W}_p$ defined by (1.7), where $\bar{b}_i \in \mathbb{R}^2$ and $\bar{b}_i \neq \bar{b}_j$ if $i \neq j$. Then there
exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, (1.21) has a family of solution pairs $(\psi^\varepsilon, \theta^\varepsilon)$ satisfying in the sense of measures

$$(-\Delta)^s(\psi^\varepsilon + \sum_{i=k+1}^{p} \kappa_i G_s(b_i, \cdot)) \rightharpoonup \sum_{j=1}^{k} \kappa_j \delta(x - \bar{b}_j) + \sum_{i=k+1}^{p} k_i \delta(x - \bar{b}_i) \quad \text{as} \ \varepsilon \to 0, \quad (1.22)$$

and

$$(-\Delta)^s \psi^\varepsilon = \theta^\varepsilon, \quad \text{supp} \ \theta^\varepsilon \subset \bigcup_{j=1}^{k} B_{C\varepsilon}(\bar{b}_j), \quad (1.23)$$

where $\theta^\varepsilon$ is $C^1(\mathbb{R}^2)$ and $C > 0$ is a constant.

**Remark 1.2.** It is worth mentioning that in [10, 11], the vorticity obtained are in the shape of patches, while in theorem 1.1, we can construct vortices with certain smoothness.

**Remark 1.3.** As we shall see later in section 3, the non-degenerate nature of the critical point of $W_p$ in theorem 1.1 guarantees that the dimension of the kernel space of linear operator $L$ in (3.6) is fixed.

If $b_0 = (\bar{b}_1, ..., \bar{b}_p)$ in Theorem 1.1 has some symmetry assumption, then maybe this point will degenerate, so let’s consider two degenerate cases.

For simplicity, we will concentrate on the most elementary solutions to (1.16) and (1.19). For (1.16) we consider the traveling vortex-wave systems, namely the solution with $p = 2, k = 1, l = 1$, and

$$\begin{align*}
\psi &= \frac{\Gamma(2 - s)}{4\pi \Gamma(s)} \frac{\kappa}{d^{3 - 2s}}, \\
\omega &= 0, \quad (1.24)
\end{align*}$$

where $d > 0$.

In the case of rotating solutions, we consider the rotating polygon with equal masses, that is, for $p \geq 2, p = k + 1, l = 1$,

$$\begin{align*}
\psi &= \frac{\Gamma(2 - s)}{2\pi \Gamma(s)} \frac{1}{\rho^{2 - 2s} \pi (1 - \cos \left(\frac{2\pi j}{k+1}\right))^{1-s}}, \\
\omega &= 0, \quad (1.25)
\end{align*}$$

where $\rho > 0$.

Our next result states the existence of a traveling solution to (1.21).

**Theorem 1.4.** Consider the traveling vortex-wave systems given by (1.24). Then for $\varepsilon > 0$ small, there is a solution $\theta^\varepsilon$ of (1.21) of the form (1.15) such that $\theta^\varepsilon_0$ is $C^1(\mathbb{R}^2)$, in the sense of measures

$$\theta^\varepsilon = (-\Delta)^s \psi^\varepsilon \Rightarrow \kappa \delta(x - b_1) \quad \text{as} \ \varepsilon \to 0, \quad (1.26)$$

and

$$\text{supp} \ \theta^\varepsilon_0 \subset B_{C\varepsilon}(b_1), \quad (1.27)$$

where $C > 0$ is a constant.
Similarly, we obtain rotating concentrated solutions near the vertices of the rotating polygon solution

**Theorem 1.5.** Consider the traveling vortex-wave systems given by (1.25). Then for \( \varepsilon > 0 \) small, there is a solution \( \theta^\varepsilon \) of (1.21) of the form (1.17) such that \( \theta_0^\varepsilon \) is \( C^1(\mathbb{R}^2) \), in the sense of measures

\[
\theta_0^\varepsilon = (−Δ)^s ψ^\varepsilon → ∑_{j=1}^{k} κδ(x − b_j) \quad \text{as } ε → 0,
\]

and

\[
\text{supp } \theta^\varepsilon ⊂ ∪_{j=1}^{k} B_{Cε}(b_j),
\]

where \( C > 0 \) is a constant.

1.5. **Outline of the paper.** This paper is organized as follows. In section 2 we construct a approximate solution to the elliptic equation (2.2), we use it to prove Theorems 1.1 and the form of the solution at main order. In Section 3, we mainly introduce the linearized operator for the main equation (2.2), follow by the non-degenerate property of the linearized operator near the ground state solution to (3.7). A solvability theory for the projected linearized equation is developed in section 3, then the projected non-linear problem is solved using the compression mapping principle. Some calculations of nonlinear problems are given in section 4, after which we complete the proof of the theorem 1.1. In Section 5, we complete the proof of Theorem 1.4 and 1.5 through the analysis similar to the previous sections.

2. **Approximate solutions**

A natural way of obtaining solutions to the stationary problem (1.21) is to locally impose that

\[
θ(x) = f(ψ(x) + ∑_{i=k+1}^{p} κ_i G_s(x_i, x) + Ux_1 + ω|x|^2)
\]

so that (1.21) locally becomes the elliptic equation

\[
(−Δ)^s ψ = f(ψ(x) + ∑_{i=k+1}^{p} κ_i G_s(x_i, x) + Ux_1 + ω|x|^2) \quad \text{in } \mathbb{R}^2,
\]

Using this observation, in order to prove Theorem 1.1, we need to find a family of solution \( ψ^\varepsilon(x) \) to the equation

\[
\begin{cases}
(−Δ)^s ψ = f(ψ(x) + ∑_{i=k+1}^{p} κ_i G_s(b_i, x) + Ux_1 + ω|x|^2) \quad \text{in } \mathbb{R}^2, \\
ψ(x) → 0, \quad \text{as } |x| → ∞,
\end{cases}
\]

where

\[
f(α) = ε^{(2−2s)γ−2} ∑_{j=1}^{k} (α − ε^{2s−2}λ_j)^γ X_B(0) \quad α ∈ \mathbb{R},
\]
and $\lambda_j$ will be suitably chosen later on, $b_i \in \mathbb{R}^2, i = 1, \ldots, p$. We assume that $1 < \gamma < \frac{2 + 2s}{2 - 2s}$. The number $\delta > 0$ such that the balls $B_\delta(b_i)$ are disjoint and the points $b_1, \ldots, b_p$ are close enough to $b_1, \ldots, b_p$.

Corresponding to (2.2), we consider the following equation:

\[
\left\{ \begin{array}{l}
(-\Delta)^s W = (W - 1)_+^{1+} \quad \text{in} \mathbb{R}^2, \\
W(y) \to 0 \quad \text{as} \quad |y| \to \infty,
\end{array} \right.
\]

where $0 < s < 1$ and $1 < \gamma < \frac{2 + 2s}{2 - 2s}$. In [12], it was proved that there exists a unique radial solution $W(y)$, which has the precise asymptotic behavior

\[
W(y) = M_\gamma c_{2,s} |y|^{-(2-2s)(1 + o(1))} \quad \text{as} \quad |y| \to \infty,
\]

where $M_\gamma = \int_{\mathbb{R}^n} (W - 1)_+^\gamma dy > 0$ and $c_{2,s}$ is given by (1.5). Set

\[
\psi_j = \varepsilon^{2s-2} \mu_j^{-\frac{2s}{\gamma - 1}} W \left( \frac{x - b_j}{\varepsilon \mu_j} \right),
\]

which satisfies

\[
(-\Delta)^s \psi_j = \varepsilon^{(2-2s)\gamma-2} (\psi_j - \varepsilon^{2s-2} \mu_j^{-\frac{2s}{\gamma - 1}})_+^\gamma.
\]

Therefore, we need verify the following function is a good approximate solution to equation (2.2)

\[
\Psi_0(x) = \varepsilon^{2s-2} \sum_{j=1}^k \mu_j^{-\frac{2s}{\gamma - 1}} W \left( \frac{x - b_j}{\varepsilon \mu_j} \right),
\]

where $\mu_j$ are positive constants. As $\varepsilon \to 0$, we see that

\[
(-\Delta)^s \Psi_0(x) \to \sum_{j=1}^k \kappa_j \delta(x - b_j),
\]

Here, we fix $\mu_j > 0$ such that

\[
M_\gamma \mu_j^{2(\frac{1}{1-s} - \frac{\gamma}{\gamma - 1})} = \kappa_j, \quad j = 1, \ldots, k
\]

which is possible, if we assume that $\gamma \neq \frac{1}{1-s}$.

Next we define that

\[
S(\psi) := (-\Delta)^s \psi - \varepsilon^{(2-2s)\gamma-2} \sum_{j=1}^k (\psi + \sum_{i=k+1}^p \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} - \varepsilon^{2s-2} \lambda_j)_+^\gamma \chi_{B_\delta(b_j)}
\]

By direct calculation, for $x \in B_\delta(b_n), n \in \{1, \ldots, k\}$,

\[
S(\Psi_0) = \varepsilon^{-2} \mu_n^{-\frac{2s}{\gamma - 1}} \left[ \left( W \left( \frac{x - b_n}{\varepsilon \mu_n} \right) - 1 \right)_+^{\gamma} + \sum_{j \neq n} \left( W \left( \frac{x - b_j}{\varepsilon \mu_j} \right) - \mu_j^{-\frac{2s}{\gamma - 1}} \right)_+^{\gamma} + \right.
\]

\[
- \left( W \left( \frac{x - b_n}{\varepsilon \mu_n} \right) + \sum_{i=k+1}^p \varepsilon^{2s-2} \mu_n^{-\frac{2s}{\gamma - 1}} \left( \kappa_i G_s(x, b_i) + U x_1 + \omega \frac{|x|^2}{2} \right) 
\]

\[
+ \sum_{j \neq n} \left( \mu_n^{-\frac{2s}{\gamma - 1}} W \left( \frac{x - b_j}{\varepsilon \mu_j} \right) - \lambda_n \mu_n^{-\frac{2s}{\gamma - 1}} \right)_+^{\gamma} \right].
\]
Let \( \lambda_n \) be chosen such that
\[
\varepsilon^{2-2s} \frac{2s}{\mu_n^{2s-1}} \left( \sum_{i=k+1}^{p} \kappa_i G_s(b_n, b_i) + U b_{n1} + \omega \frac{|b_n|^2}{2} \right) + \sum_{j \neq n} \left( \frac{\mu_n}{\mu_j} \right)^{2s} W \left( \frac{b_n - b_j}{\varepsilon \mu_j} \right) - \lambda_n \mu_n^{2s-1} = -1.
\]

As \( \varepsilon \to 0 \) and using the asymptotic behavior \((2.5)\) of \( W \), we get that
\[
\lambda_n = \mu_n^{2s-1} + O \left( \varepsilon^{2-2s} \right), \quad n = 1, \ldots, k, \tag{2.11}
\]

With the choice of \( \lambda_n \), we see that the error of approximation created by \( \Psi_0 \) has the following estimate
\[
\varepsilon^2 S(\Psi_0) = O(\varepsilon^{3-2s}) \sum_{j=1}^{k} \chi_{B_{C_\varepsilon}(b_j)}. \tag{2.12}
\]

That implies that \( \Psi_0 \) is a good approximate solutions for equation \((2.2)\). The proof of Theorem 1.1 consists in finding a solution of equation \((2.2)\) as a suitable small perturbation of \( \Psi_0 \).

3. The linear operator and reduction

In this section, our aim is to find a solution \( \Psi = \Psi_0 + \tilde{\Phi} \) for equation \((2.2)\), so \( \tilde{\Phi} \) needs to satisfies the following equation
\[
(-\Delta)^s \tilde{\Phi} = \varepsilon^{(2-2s)\gamma - 2} \sum_{j=1}^{k} \left[ (\Psi_0 + \tilde{\Phi} + \sum_{i=k+1}^{p} \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} - \varepsilon^{2s-2} \lambda_j \right]_+
\]
\[
- (\Psi_0 + \sum_{i=k+1}^{p} \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} - \varepsilon^{2s-2} \lambda_j \right]_+ \chi_{B_{\varepsilon}(b_i)}
\]
\[
- \varepsilon^{-2} \mu_n^{2s-1} \left[ \left( W \left( \frac{x - b_n}{\varepsilon \mu_n} \right) - 1 \right) \gamma_+ + \sum_{j \neq n} \left( W \left( \frac{x - b_j}{\varepsilon \mu_j} \right) - \mu_j^{2s-1} \mu_n^{2s-1} \right) \right]_+
\]
\[
- \left( W \left( \frac{x - b_n}{\varepsilon \mu_n} \right) + \varepsilon^{-2s} \mu_n^{2s} \sum_{i=k+1}^{p} \kappa_i G(x, b_i) + U x_1 + \omega \frac{|x|^2}{2} \right)
\]
\[
+ \sum_{j \neq n} \left( \frac{\mu_n}{\mu_j} \right)^{2s} W \left( \frac{x - b_j}{\varepsilon \mu_j} \right) - \lambda_n \mu_n^{2s-1} \right]_+
\].

For simplicity, it will be convenient to work with the unknown \( v \) defined by
\[
\Psi(x) := \varepsilon^{2s-2} \mu^{-\frac{2s}{s-1}} \mu \left( y \right), \quad \gamma = \frac{x}{\varepsilon \mu} \in \mathbb{R}^2, \quad b_j = \frac{b_j}{\varepsilon \mu},
\]
where \( \mu > 0 \) is constant. We find that
\[
S(\Psi) = \varepsilon^{-2s-2} \mu^{-\frac{2s}{s-1}} \left[ (-\Delta)^s \gamma - f(y, v) \right], \tag{3.1}
\]
where
\[
f(y, v) = \sum_{j=1}^{k} \left( v + \varepsilon^{2-2s} \mu^{\gamma-1} \left( \sum_{i=k+1}^{p} \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} \right) - \mu^{\gamma-1} \lambda_j \right) + \chi B_{S(\varepsilon \mu)}(b_j').
\]  

Thus (2.2) becomes the nonlinear problem
\[
\begin{cases}
(\Delta)^s v = f(y, v) \text{ in } \mathbb{R}^2, \\
v(y) \to 0 \text{ as } |y| \to \infty,
\end{cases}
\]  
and the first approximate solution (2.8) takes the form
\[
v_0(y) = \sum_{j=1}^{k} \left( \frac{\mu}{\mu_j} \right)^{\gamma-1} W \left( \frac{\mu}{\mu_j} (y - b_j') \right).
\]

We will look for a solution of the form \( v = v_0 + \phi \). Therefore, equation (3.3) is equivalent to
\[
L[\phi] = -E + N[\phi],
\]  
where
\[
L[\phi] = (\Delta)^s \phi - \sum_{j=1}^{k} V_j(y) \phi,
\]
\[
V_j(y) = \gamma \left( \left( \frac{\mu}{\mu_j} \right)^{\gamma-1} \left( W \left( \frac{\mu}{\mu_j} (y - b_j') \right) - 1 \right) \right)^{\gamma-1} \chi B_{S(\varepsilon \mu)}(b_j') \phi,
\]
\[
E = (\Delta_y)^s v_0 - f(y, v_0),
\]
\[
N[\phi] = f(y, v_0 + \phi) - f(y, v_0) - \sum_{j=1}^{k} V_j(y) \phi.
\]

In order to studies the non-degeneracy of \( W \), we consider the following problem
\[
\begin{cases}
(\Delta)^s \phi = \gamma(W - 1)^{\gamma-1} \phi \text{ in } \mathbb{R}^n, \\
\phi(y) \to 0, \text{ as } |y| \to \infty,
\end{cases}
\]  
For the above problem, the non-degeneracy of \( W \) studied in [1], which present as following:

**Proposition 3.1.** (Proposition 3.2 in [1]). If \( \phi \in L^\infty(\mathbb{R}^n) \) is a solution to (3.7), then \( \phi \) is a linear combination of \( \frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n} \).

Now we define the kernel of the operator \( L \) as following:
\[
Z_{1j}(y) = \frac{\partial W}{\partial y_1} \left( \frac{\mu}{\mu_j} (y - b_j') \right), Z_{2j}(y) = \frac{\partial W}{\partial y_2} \left( \frac{\mu}{\mu_j} (y - b_j') \right),
\]  
where
\[
y = (y_1, y_2) \in \mathbb{R}^2, \ j = 1, \ldots, k.
\]
Let
\[
E := \left\{ u : u \in H^s(\mathbb{R}^2), \int_{\mathbb{R}^2} V_j Z_{ij} \phi = 0, j = 1, \ldots, k; i = 1, 2 \right\},
\]
and
\[ F := \left\{ h : h \in H^{-s}(\mathbb{R}^2), \int_{\mathbb{R}^2} Z_{ij} h = 0, j = 1, \ldots, k, i = 1, 2 \right\}. \]

Note that \((-\Delta)^s Z_{ij} \in H^{-s}(\mathbb{R}^2)\). For any \( h \in H^s(\mathbb{R}^2) \), we define \( Q \) as follows:
\[ Qh = h + \sum_{j=1}^{k} \sum_{i=1}^{2} c_{ij} (-\Delta)^s Z_{ij}, \]
where \( c_{1j} \) and \( c_{2j} \) are the constants such that \( Qh \in F \). The operator \( Q \) can be regarded as a projection from \( H^{-s}(\mathbb{R}^2) \) to \( F \).

Let us consider the following projected linear problem,
\[
\begin{cases}
L[\phi] = h(x) + \sum_{j=1}^{k} \sum_{i=1}^{2} (c_{ij} + d_{ij}) V_j Z_{ij}, \text{ in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} V_j Z_{ij} \phi = 0, i = 1, 2; j = 1, 2, \ldots, k, \\
\phi(x) \to 0, \text{ as } x \to \infty
\end{cases}
\tag{3.9}
\]
where \( \phi \in E, h \in H^{-s} \). Here \( c_{ij} \) is the constant such that
\[ Qh = h + \sum_{j=1}^{k} \sum_{i=1}^{2} c_{ij} V_j Z_{ij} \in F, \]
and \( d_{ij} \) is the constant such that
\[ Q(\sum_{j=1}^{k} V_j \phi) = \sum_{j=1}^{k} V_j \phi + \sum_{j=1}^{k} \sum_{i=1}^{2} d_{ij} V_j Z_{ij} \in F. \]

For the proof of the existence of solution to the equation (3.9) and (3.3), we introduce the following norms:
\[ \| \phi \|_* = \sup_{y \in \mathbb{R}^2} \rho(y)^{-2s} |\phi(y)|, \quad \| h \|_{**} = \sup_{y \in \mathbb{R}^2} \rho(y)^{-2-\sigma} |h(y)|, \]
where \( 0 < \sigma < 1 \) and
\[ \rho(y) = \sum_{j=1}^{k} \frac{1}{1 + |y - b'_j|}. \tag{3.10} \]

For the rest of this section, our purpose is to solve the following the projected non-linear problem
\[
\begin{cases}
L[\phi] = -E + N[\phi] + \sum_{j=1}^{k} \sum_{i=1}^{2} (c_{ij} + d_{ij}) V_j Z_{ij}, \text{ in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} V_j Z_{ij} \phi = 0, i = 1, 2; j = 1, 2, \ldots, k, \\
\phi(y) \to 0, \text{ as } |y| \to \infty.
\end{cases}
\tag{3.11}
\]
To do this, we first need to solve the projected linear problem (3.9). For the need of later proof, a priori estimate of equation (3.9) is studies first.
Proposition 3.2. Assume that \( h \in H^s(\mathbb{R}^2) \) satisfies \( \|h\|_{**} < \infty \), for \( \varepsilon > 0 \) small and for any solution \( \phi \in E \) of (3.9), there exists \( C > 0 \) independent of \( \varepsilon \) such that

\[
\|\phi\|_* + \sum_{j=1}^{k} \sum_{i=1}^{2} (|c_{ij}| + |d_{ij}|) \leq C \|h\|_{**}.
\]  

(3.12)

Proof. First we claim that

\[
\sum_{j=1}^{k} \sum_{i=1}^{2} (|c_{ij}| + |d_{ij}|) \leq C(\|h\|_{**} + o(1)\|\phi\|_*).
\]  

(3.13)

The coefficient \( c_{ij} \) is determined by

\[
\sum_{j=1}^{k} \sum_{i=1}^{2} c_{ij} \int_{\mathbb{R}^2} V_j Z_{ij} Z_{hm} dy + \int_{\mathbb{R}^2} h Z_{hm} dy = 0, \ h = 1, 2, m = 1, \ldots, k,
\]  

(3.14)

if \( h = i, m = j \), we see that

\[
\int_{\mathbb{R}^2} V_j Z_{ij}^2 dy = C \int_{\mathbb{R}^2} \gamma(W - 1)^{\gamma-1} \left( \frac{\partial W}{\partial y_i} \right)^2 dy (1 + o(1))
\]

\[
= c_0 (1 + o(1)) > 0;
\]

if \( m \neq j, h \neq i \),

\[
\int_{\mathbb{R}^2} V_j Z_{ij} Z_{hm} = \int_{\mathbb{R}^2} (-\Delta)^{\gamma} Z_{ij} Z_{hm} dy (1 + o(1))
\]

\[
= o(1);
\]

From (3.14), we obtain

\[
|c_{ij}| \leq C \|h\|_{**}.
\]  

(3.15)

Similarly, for the coefficient \( d_{ij} \) is determined by

\[
\sum_{j=1}^{k} \sum_{i=1}^{2} d_{ij} \int_{\mathbb{R}^2} V_j Z_{ij} Z_{hm} dy + \int_{\mathbb{R}^2} V_j \phi Z_{hm} dy = 0, \ h = 1, 2, m = 1, \ldots, k,
\]  

(3.16)

Since \( \phi \in E \), it have

\[
\int_{\mathbb{R}^2} V_j \phi Z_{ij} dy = 0;
\]

and if \( m \neq j, h \neq i \)

\[
\left| \int_{\mathbb{R}^2} V_j \phi Z_{hm} dy \right| \leq o(1) \|\phi\|_*.
\]

Therefore, we get that

\[
|d_{ij}| \leq o(1) \|\phi\|_*.
\]  

(3.17)

Combining (3.15) and (3.17), we obtain (3.13).

Next we claim that

\[
\|\phi\|_* \leq C \|h\|_{**}.
\]  

(3.18)
We prove by contradiction, assuming that there exist $\varepsilon_n \to 0$, $(\phi_n, h_n)$ is solution to (3.9) such that
\[
\|\phi_n\|_* = 1, \quad \|h_n\|_{**} \to 0 \quad \text{as } n \to \infty. \tag{3.19}
\]
First, we show that for any fixed $R > 0$, we have that
\[
\sum_{j=1}^{k} \|\phi_n\|_{L^\infty(B_R(b'_{1,n}) \cap \mathbb{R}^d)} \to 0, \tag{3.20}
\]
where $b'_{j,n} = \frac{b_{j,n}}{\varepsilon\mu}$. Indeed assume that for a subsequence $\|\phi_n\|_{L^\infty(B_R(b'_{1,n}) \cap \mathbb{R}^d)} \geq c_0 > 0$. Let
\[
\bar{\phi}_n(y) = \phi_n(y + b'_{1,n}).
\]
Then $\bar{\phi}_n$ satisfies
\[
(-\Delta)^s \bar{\phi}_n - \left(\frac{\mu}{\mu_1}\right)^{2s} \gamma \left(W\left(\frac{\mu}{\mu_1}(y)\right) - 1\right)^{-1} \bar{\phi}_n = \bar{h}_n, \tag{3.21}
\]
where
\[
\bar{h}_n(y) = h_n(y + b'_{1,n}) + \sum_{j=1}^{k} \sum_{i=1}^{2} (c_{ij,n} + d_{ij,n}) V_j(y + b'_{1}) Z_{ij}(y + b'_{1})
\]
\[
+ \left(\sum_{j=1}^{k} V_j(y + b'_{1}) - \left(\frac{\mu}{\mu_1}\right)^{2s} \gamma \left(W\left(\frac{\mu}{\mu_1}(y)\right) - 1\right)^{-1}\right) \bar{\phi}_n.
\]
We note that
\[
\left(\sum_{j=1}^{k} V_j(y + b'_{1}) - \left(\frac{\mu}{\mu_1}\right)^{2s} \gamma \left(W\left(\frac{\mu}{\mu_1}(y)\right) - 1\right)^{-1}\right) = O\left(\varepsilon^2 \gamma^{-1}\right).
\]
Using the assumption (3.19), we have $\bar{h}_n \to 0$ uniformly on compact sets. Thus passing to a subsequence, we may assume that $\bar{\phi}_n$ converge uniformly on compact sets to a function $\bar{\phi}$ with
\[
\|\bar{\phi}\|_{L^\infty(B_R(0))} \geq c_0. \tag{3.22}
\]
and it solves
\[
\begin{cases}
(-\Delta)^s \bar{\phi} - \left(\frac{\mu}{\mu_1}\right)^{2s} \gamma \left(W\left(\frac{\mu}{\mu_1}(y)\right) - 1\right)^{-1} \bar{\phi} = 0,

\bar{\phi} \to 0 \text{ as } |y| \to \infty
\end{cases}
\]
we find it satisfies the orthogonality condition
\[
\int_{\mathbb{R}^d} \left(\frac{\mu}{\mu_1}\right)^{2s} \gamma \left(W\left(\frac{\mu}{\mu_1}(y)\right) - 1\right)^{-1} \partial_i \bar{W}\left(\frac{\mu}{\mu_1}(y - b'_{j})\right) \bar{\phi} = 0,
\]
where $i = 1, 2; j = 1, 2, \ldots, k$. By the non-degeneracy in Proposition 3.1, we have $\bar{\phi} = 0$, which is a contradiction to (3.22). Next we consider the equation satisfied by $\phi_n$, that is
\[
(-\Delta)^s \phi_n = \sum_{j=1}^{k} V_j \phi_n + h_n + \sum_{j=1}^{k} \sum_{i=1}^{2} (c_{ij,n} + d_{ij,n}) V_j Z_{ij},
\]
it can be rewrite by
\[ \phi_n(y) = c_{2,s} \int_{\mathbb{R}^2} \frac{\sum_{j=1}^{k} V_j \phi_n + h_n + \sum_{j=1}^{k} \sum_{i=1}^{2} (c_{ij,n} + d_{ij,n}) V_j Z_{ij}}{|y - z|^{2-2s}} dz \]
This implies that
\[ \rho(y)^{(2-2s)} |\phi_n(y)| \leq C \left( \|h\|_\infty + \sum_{j=1}^{k} \sum_{i=1}^{2} (|c_{ij,n}| + |d_{ij,n}|) + \|\phi_n\|_{L^\infty(B_{j=1}^{k}B_R(y))} \right) \]
Thus combining (3.13), (3.20) and (3.19), we obtain that \( \|\phi_n\|_\ast \to 0 \), which is a contradiction to (3.19). So we get (3.12) from (3.13) and (3.18). \( \square \)

Now we start to solve the projected linear problem (3.9), the following proposition implies that there exists a unique solution \( \phi \) to (3.9).

**Proposition 3.3.** Assume \( h \in H^s(\mathbb{R}^2) \) satisfies \( \|h\|_\ast < \infty \), for \( \varepsilon > 0 \) small there exists a unique solution \( \phi = T(h) \in E \) of (3.9), which defines a linear operator of \( h \) and there exists \( C > 0 \) independent of \( \varepsilon \) such that
\[ \|\phi\|_\ast + \sum_{j=1}^{k} \sum_{i=1}^{2} (|c_{ij}| + |d_{ij}|) \leq C \|h\|_\ast \]

**Proof.** We define the space
\[ X := \{ \phi \in L^\infty(\mathbb{R}^2) : \|\phi\|_{2-2s-\sigma} < +\infty, \phi \in E \}, \]
which endowed with the norm
\[ \|\phi\|_X = \sup_{y \in \mathbb{R}^2} \rho(y)^{(2-2s-\sigma')} |\phi(y)|, \]
where \( \rho(y) \) is given by (3.10). Here \( 2 - 2s - \sigma' > 0 \), and \( X \) is a Banach space.

Next we want to solve the equation
\[ \phi = K(\phi) + A[h], \quad \phi \in X \]
where
\[ K[\phi] = (-\Delta)^{-s} \left( \sum_{j=1}^{k} V_j \phi + \sum_{j=1}^{k} \sum_{i=1}^{2} d_{ij} V_j Z_{ij} \right), \]
and
\[ A[h] = (-\Delta)^{-s} (h + \sum_{j=1}^{k} \sum_{i=1}^{2} c_{ij} V_j Z_{ij}). \]
Using Arzela-Ascoli’s theorem, we obtain the compactness of the operator \( K \) in \( X \) since \( \|\phi\|_X = \|\phi\|_{2-2s-\sigma'} \).

On the other hand, the equation
\[ \phi - K[\phi] = 0, \quad \phi \in X \]
only has the trivial solution from a priori estimate in Proposition 3.2. Fredholm’s alternative guarantees unique solvability of the above problem for any \( h \). \( \square \)
Next we consider the nonlinear projected problem
\[
\begin{cases}
L[\phi] = -E + N[\phi] + \sum_{j=1}^{k} \sum_{i=1}^{2} (c_{ij} + d_{ij}) V_j Z_{ij}, & \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} V_j Z_{ij} \phi = 0, \\
\phi(y) \to 0 & \text{as } |y| \to \infty.
\end{cases}
\]

(3.25)

For above problem, we have the following proposition, which shows that the nonlinear projected problem is solvable.

**Proposition 3.4.** There is $r_0 > 0$ such that for $\varepsilon > 0$ small there exists a unique solution $\phi$ to (3.25), $\|\phi\|_* \leq r_0$. Moreover it satisfies
\[
\|\phi\|_* \leq C\varepsilon^{3-2s}
\]

and $\phi$ is continuous with respect to $b$.

**Proof.** Let $T$ is in Propostion 3.3, for any $\|h\|_{**} < \infty$, $\phi = T[h]$ is unique solution to the problem (3.9), and
\[
\|T[h]\|_* \leq C\|h\|_{**}.
\]

Let $X_1 = \{\phi \in L^\infty(\mathbb{R}^2) : \|\phi\|_* < \infty, \phi \in E\}$ be endowed with $\| \cdot \|_*$. Let $A : X_1 \to X_1$, $A[\phi] = T[-E + N[\phi]]$.

Then the problem (3.25) is equivalent to the fixed point problem $\phi = A[\phi]$, we set $B = \{\phi \in X_1 : \|\phi\|_* \leq r_0\}$, where $r_0$ will be choose later. From (2.12), we have
\[
\|E\|_{**} \leq C\varepsilon^{3-2s}.
\]

We claim that, if $\phi \in B$, then
\[
|N[\phi]| \leq C|\phi|^{\min(\gamma, 2)} \chi_{\cup_{j=1}^k B_{R_0}(b_j)},
\]

where $R_0$ is a large constant. Indeed, for the definition of (3.6), we have
\[
N[\phi] = \sum_{j=1}^{k} N_j[\phi],
\]

where
\[
N_j[\phi] = \chi_{B_{\delta/(\varepsilon^2 s)}(b_j)} \left[\left(v_0 + \phi + \varepsilon^{2-2s} \mu^{2s}\gamma \sum_{i=k+1}^{p} \kappa_i G_i(b_i, x) + U x_1 + \frac{|x|^2}{2} - \mu^{2s}\gamma \lambda_j\right)_{+}^{\gamma} - \left(v_0 + \varepsilon^{2-2s} \mu^{2s}\gamma \sum_{i=k+1}^{p} \kappa_i G_i(b_i, x) + U x_1 + \frac{|x|^2}{2} - \mu^{2s}\gamma \lambda_j\right)_{+}^{\gamma} - \gamma \left(v_0 - \mu^{2s}\gamma \lambda_j\right)_{+}^{\gamma-1} \phi\right]
\]
By the definition of $v_0$ and the choice of $\lambda_j$, we obtain
\[
N_j[\phi] = \chi_{B_{\delta_j}(\xi_j)(\eta_j)} \left[ \left( \frac{\mu}{\mu_j} \right)^{2\gamma} (W\left( \frac{\mu}{\mu_j} (y - b_j') - 1 \right) + \phi + R_j \right)^\gamma
\] 
- \left( \left( \frac{\mu}{\mu_j} \right)^{2\gamma} (W\left( \frac{\mu}{\mu_j} (y - b_j') - 1 \right) + R_j \right)^\gamma
\] 
- \gamma \left( \left( \frac{\mu}{\mu_j} \right)^{2\gamma} (W\left( \frac{\mu}{\mu_j} (y - b_j') - 1 \right) \right)^{\gamma - 1} \phi],
\]
where $R_j = O(\varepsilon^{3-2s}|y - b_j|)$. Thus we deduce that
\[
|N_j[\phi]| \leq C|\phi|_{\min{\gamma;2}}^{\min{\gamma;2}}.
\]
And we find the support of $N_j[\phi]$ included in $B_{\frac{\delta_j}{2}}(b_j')$, so we obtain the claim. From the claim, we have
\[
\|N[\phi]\|_{**} \leq C\|\phi\|_{**}^{\min{\gamma;2}}.
\]
Thus for any $\phi \in B$, we see that
\[
\|A[\phi]\|_{**} \leq C(\|E\|_{**} + \|N[\phi]\|_{**}) \leq C\varepsilon^{3-2s} + C\tau_0^{\min{\gamma;2}}.
\]
We choose $r_0$ and $\varepsilon$ is small, such that $C\varepsilon^{3-2s} \leq \frac{1}{2}r_0$ and $C\tau_0^{\min{\gamma;2}} \leq \frac{1}{2}r_0$. Thus $A$ map $B$ into itself. From the expression of $N[\phi]$, we have
\[
|N[\phi_1] - N[\phi_2]| \leq C(|\phi_1|_{\min{\gamma-1,1}}^{\min{\gamma-1,1}} + |\phi_2|_{\min{\gamma-1,1}}^{\min{\gamma-1,1}})|\phi_1 - \phi_2| \chi_{\cup_{i=1}^k B_{\delta_i}(\eta_i)}.
\]
Then
\[
\|A[\phi_1] - A[\phi_2]\|_{**} \leq C(\|N[\phi_1] - N[\phi_2]\|_{**}) \leq C\tau_0^{\min{\gamma-1,1}} \|\phi_1 - \phi_2\|_{**}.
\]
We choose $r_0$ is small, so $A$ is a contraction mapping form $B$ into itself. And then nonlinear problem admits a unique solution $\phi \in B$.
\[
\|\phi\|_{**} \leq C\|E\|_{**} + C\|N[\phi]\|_{**},
\]
We notice that $\|N[\phi]\|_{**} \leq C\|\phi\|_{**}^{\min{\gamma;2}}$ and $\|\phi\|_{**} \leq r_0$, where $r_0$ is small. Thus we see that
\[
\|\phi\|_{**} \leq C\|E\|_{**} \leq C\varepsilon^{3-2s}.
\]
Due to $E$ and $V$ is continuously depends on $b = (b_1, \ldots, b_k)$ in nonlinear problem, the fixed point characterization of $\phi_b$ show that it is continuous with respect to $b$. \[\square\]

4. Proof of Theorem 1.1

In this section, we need to show the solvability of the following equation
\[
\begin{align*}
L[\phi] &= -E + N[\phi], \quad \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} V_j Z_{ij} \phi &= 0, \\
\phi(y) &\to 0 \text{ as } |y| \to \infty.
\end{align*}
\]
(4.1)

We will choose $b$, such that the constant $c_{ij}$ and $d_{ij}$ in (3.25) can be eliminated. The next result shows how to choose $b$:
Lemma 4.1. If $b$ satisfies
\[
\int_{\mathbb{R}^2} (-\Delta)^s (v_0 + \phi_b) Z_{ij} - \int_{\mathbb{R}^2} f(y, v_0 + \phi_b) Z_{ij} dy = 0,
\] (4.2)
for $i = 1, 2; j = 1, \ldots, k$; then $v_0 + \phi_b$ is a solution of (4.1)

Proof. Similar to the proof of Proposition 3.2 and Proposition 3.4, if (4.2) holds, then
\[
\sum_{m=1}^{2} \sum_{n=1}^{k} (c_{mn} + d_{mn}) \int_{\mathbb{R}^2} V_n Z_{mn} Z_{ij} dy = 0,
\] (4.3)
which implies $c_{1j} + d_{1j} = c_{2j} + d_{2j} = 0, j = 1, \ldots, k$. So $v_0 + \phi_b$ is a solution of (4.1).

In the rest of this section, we need to solve (4.2).

Lemma 4.2. The following relation
\[
\int_{\mathbb{R}^2} (-\Delta)^s (v_0 + \phi_b) Z_{ij} - \int_{\mathbb{R}^2} f(y, v_0 + \phi_b) Z_{ij} dy = 0,
\] (4.4)
i = 1, 2; j = 1, \ldots, k; is equivalent to
\[
\nabla W_p(b) = o(1)
\] (4.5)
Proof. From $\phi_b$ satisfies (3.25), we have
\[
\int_{\mathbb{R}^2} (-\Delta)^s \phi_b Z_{ij} dy = \int_{\mathbb{R}^2} \phi_b V_j Z_{ij} dy = 0.
\] (4.6)
Recall the definition of $f(y, v)$ by (3.2), we set
\[
f(y, v) = \sum_{j=1}^{k} f_j(y, v),
\] (4.7)
where
\[
f_j(y, v) = \left(v + \varepsilon^{2-2s} \mu^{\frac{2s}{2}} \sum_{i=k+1}^{p} \kappa_i G_s(b_i, x) + UX_1 + \omega \frac{|x|^2}{2} - \mu^{\frac{2s}{2} - 1} \lambda_j \right)^{\gamma} + \chi_{B_i(c_m)(b_m)}.
\]
By direct calculation and (4.6),
\[
\int_{\mathbb{R}^2} (-\Delta)^s (v_0 + \phi_b) Z_{ij} - \int_{\mathbb{R}^2} f(y, v_0 + \phi_b) Z_{ij} dy
\]
\[
= \sum_{m=1}^{k} \int_{\mathbb{R}^2} \left(\frac{\mu}{\mu_m} \right)^{\frac{2s}{2}} (W \left(\frac{\mu}{\mu_m} (y - b_m)\right) - 1)^{\gamma} - f_m(y, v_0 + \phi_b) \right) Z_{ij} dy
\]
\[
= \int_{\mathbb{R}^2} \left(\frac{\mu}{\mu_j} \right)^{\frac{2s}{2}} (W \left(\frac{\mu}{\mu_j} (y - b_j)\right) - 1)^{\gamma} - f_j(y, v_0 + \phi_b) \right) Z_{ij} dy
\]
\[
+ \sum_{m \neq j} \int_{\mathbb{R}^2} \left(\frac{\mu}{\mu_m} \right)^{\frac{2s}{2}} (W \left(\frac{\mu}{\mu_m} (y - b_m)\right) - 1)^{\gamma} - f_m(y, v_0 + \phi_b) \right) Z_{ij} dy.
\] (4.8)
By the proposition 3.4, (4.6) and the choice of \( \lambda_j \), we find that

\[
\int_{\mathbb{R}^2} \left( \left( \frac{\mu}{\mu_j} \right)^{\frac{2s}{s-1}} W \left( \frac{\mu}{\mu_j} (y - b_j') \right) - 1 \right)_+^\gamma - f_j(y, v_0 + \phi_b) \right) Z_{ij} \, dy
\]

\[
= \int_{\mathbb{R}^2} - \left( \frac{\mu}{\mu_j} \right)^{\frac{2s}{s-1}} \gamma \left( W \left( \frac{\mu}{\mu_j} (y - b_j') \right) - 1 \right)_+^\gamma - f_j(y, v_0 + \phi_b) \right) Z_{ij} \times \left( \sum_{m \neq j} \left( \frac{\mu}{\mu_m} \right)^{\frac{2s}{s-1}} W \left( \frac{\mu}{\mu_m} (y - b_m') \right) \right) dy + o(\varepsilon^{3-2s})
\]

\[
+ \varepsilon^{2-2s} \mu^{\frac{2s}{s-1}} \left( \sum_{i=k+1}^p \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} \right) \right) dy + o(\varepsilon^{3-2s})
\]

\[
= \int_{\mathbb{R}^2} \left( \frac{\mu}{\mu_j} \right)^{\frac{2s}{s-1}} \left( W \left( \frac{\mu}{\mu_j} (y - b_j') \right) - 1 \right)_+^\gamma - f_j(y, v_0 + \phi_b) \right) Z_{ij} \times \left( \sum_{m \neq j} \left( \frac{\mu}{\mu_m} \right)^{\frac{2s}{s-1}} \gamma \right) dy + o(\varepsilon^{3-2s})
\]

\[
+ \varepsilon^{2-2s} \mu^{\frac{2s}{s-1}} \left( \sum_{i=k+1}^p \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} \right) \right) dy + o(\varepsilon^{3-2s})
\]

\[
= \left( \frac{\mu}{\mu_j} \right)^{\frac{2s}{s-1}} M_{\gamma} \varepsilon \mu \frac{\partial}{\partial x_i} \left( \sum_{m \neq j} \left( \frac{\mu}{\mu_m} \right)^{\frac{2s}{s-1}} \gamma \right) \right) G_s(x, b_m)
\]

\[
+ \varepsilon^{2-2s} \mu^{\frac{2s}{s-1}} \left( \sum_{i=k+1}^p \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} \right) \right) \right) \right) dy + o(\varepsilon^{3-2s})
\]

For the last term in (4.8), we see that

\[
\sum_{m \neq j} \int_{\mathbb{R}^2} \left( \left( \frac{\mu}{\mu_m} \right)^{\frac{2s}{s-1}} W \left( \frac{\mu}{\mu_m} (y - b_m') \right) - 1 \right)_+^\gamma - f_m(y, v_0 + \phi_b) \right) Z_{ij} \, dy
\]

\[
= o(\varepsilon^{3-2s}).
\]

Recall the definition of \( W_p(b) \) in (1.7) and the choice of \( \kappa_j \) in (2.10), we obtain

\[
\int_{\mathbb{R}^2} \left[ (-\Delta)^s (v_0 + \phi_b) - f(y, v_0 + \phi_b) \right] Z_{ij} \, dy
\]

\[
= \left( \frac{\mu}{\mu_j} \right)^{\frac{2s}{s-1}} M_{\gamma} \varepsilon \mu \frac{\partial}{\partial x_i} \left( \sum_{m \neq j} \left( \frac{\mu}{\mu_m} \right)^{\frac{2s}{s-1}} \gamma \right) \right) G_s(x, b_m)
\]

\[
+ \varepsilon^{2-2s} \mu^{\frac{2s}{s-1}} \left( \sum_{i=k+1}^p \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} \right) \right) \right) \right) \right) \right) dy + o(\varepsilon^{3-2s})
\]

\[
= \varepsilon^{3-2s} \mu^{\frac{2s}{s-1}} \frac{\partial}{\partial x_i} \left( \sum_{m \neq j} \kappa_j G_s(x, b_m) + \sum_{i=k+1}^p \kappa_i G_s(b_i, x) + U x_1 + \omega \frac{|x|^2}{2} \right) \right) \right) \right) \right) dy + o(\varepsilon^{3-2s}).
\]
Therefore, we get (4.5).

**Proof of Theorem 1.1.**

*Proof.* By our assumption, from Lemma 4.2, (4.2) has a solution \( b_\varepsilon \) satisfying \( b_\varepsilon \to b_0 \) as \( \varepsilon \to 0 \).

By our construction and proposition 3.4, we find that
\[
(-\Delta)^s \psi_\varepsilon = (-\Delta)^s (\Psi_0 + \phi) \to \sum_{j=1}^k \kappa_j \delta(x - \bar{b}_j) \quad \text{as} \quad \varepsilon \to 0, \tag{4.9}
\]
and
\[
(-\Delta)^s \psi_\varepsilon = \theta_\varepsilon, \quad \text{supp} \theta_\varepsilon \subset \bigcup_{j=1}^k B_{C\varepsilon}(\bar{b}_j), \tag{4.10}
\]
where \( C > 0 \) is a constant. \( \square \)

5. The proof of theorem 1.4-1.5

Let us consider the traveling vortex-wave described by (1.24), the point vortex is at \( b_2 = (-d, 0) \) with masses \( -\kappa \), and the vorticity is concentrated at \( b_1 = (-d, 0) \) with masses \( \kappa > 0 \). Similar to the construction of theorem 1.1, we consider the special form in section 2, described as the following problem
\[
\begin{align*}
(-\Delta)^s \psi &= \varepsilon^{(2-2s)\gamma - 2}(\psi(x) - \kappa G_s(b_2, x) + Ux_1 - \varepsilon^{2s-2}\lambda)\gamma \chi_{B_{\delta}(b_1)} \quad \text{in} \quad \mathbb{R}^2, \\
\psi(x) &\to 0, \quad \text{as} \quad \varepsilon \to 0,
\end{align*}
\tag{5.1}
\]
where \( \lambda \) is chosen appropriately later.

The above problem is symmetric about the x-axis, so we naturally consider constructing a solution of the following form
\[
\psi(x_1, x_2) = \psi(x_1, -x_2) \quad \text{for all} \quad (x_1, x_2) \in \mathbb{R}^2. \tag{5.2}
\]
As with (2.8), we take as a first approximation of a solution to (5.1)
\[
\psi_0(x) = \varepsilon^{2s-2} \mu^{-\frac{2s}{\gamma - 1}} W\left(\frac{x - b_1}{\varepsilon \mu}\right), \tag{5.3}
\]
where \( \mu > 0 \) is defined as in relation (2.10) by
\[
M_\gamma \mu^{2(1-\frac{s}{\gamma - 1})} = \kappa. \tag{5.4}
\]

Let us write
\[
S_1(\psi) = (-\Delta)^s \psi - \varepsilon^{(2-2s)\gamma - 2}(\psi(x) - \kappa G_s(b_2, x) + Ux_1 - \varepsilon^{2s-2}\lambda)\gamma \chi_{B_{\delta}(b_1)}
\]
Similar to the calculation of \( S(\Psi_0) \), we choose \( \lambda = \lambda(U, d) \) with the form
\[
\lambda = \mu^{-\frac{2s}{\gamma - 1}} + O(\varepsilon^{3-2s}) > 0.
\]
With the choice of \( \lambda \), we obtain that for \( y \in B_{\delta/\mu^c}(b_1') \)
\[
S_1(\psi_0) = \varepsilon^{-2} \mu^{-\frac{2s}{\gamma - 1}} \left\{ \left( W(y - b_1') - 1 \right)\gamma - (W(y - b_1') - 1 \\
- \mu^{\frac{2s}{\gamma - 1}} - 2 \kappa G_s(b_2, y) + \varepsilon^{3-2s} \mu^{\frac{2s}{\gamma - 1} + 1} Uy_1 \right)\gamma \chi_{B_{\delta/\mu^c}(b_1')}, \tag{5.5}
\right.
\]
\[
\left. \varepsilon^{3-2s} \mu^{\frac{2s}{\gamma - 1}} \right)\gamma \chi_{B_{\delta/\mu^c}(b_1')}, \tag{5.5}
\]
where
\[ y = \frac{x}{\varepsilon \mu} = (y_1, y_2) \in \mathbb{R}^2, \quad b'_1 = \frac{b_1}{\varepsilon \mu}. \]

We will work with the parameters \( U < 0, \mu > 0 \) fixed and
\[ d \in \left( d_0, \frac{1}{d_0} \right) \quad (5.6) \]
to be adjusted, where \( d_0 > 0 \) is fixed small. By (5.5), we see that

\[ S_1(\psi_0) = o(\varepsilon^{2-2s}). \]

And we note that
\[ S_1(\psi) = \varepsilon^{-2} \mu^{-2} [(-\Delta)^s v - f_1(y, v)], \]
where
\[ f_1(y, v) = (v + U \varepsilon^{3-2s} \mu^{2s-1} y_1 - \mu^{2s-1} \kappa G_s(y_2, y) - \mu^{2s-1} \lambda) \chi_{B_{d_0}(\mu)}(b'_1). \]

Thus (5.1) becomes the nonlinear problem
\[ \begin{cases} (-\Delta)^s v = f_1(y, v) & \text{in } \mathbb{R}^2, \\ v(y) \to 0 & \text{as } |y| \to \infty. \end{cases} \quad (5.7) \]

The ansatz (5.3) takes the form
\[ v_1(y) = W(y - b'_1). \]

We look for a solution of (5.7) of the form \( v = v_1 + \phi \). Then equation (5.7) is equivalent to
\[ L_1[\phi] = -E_1 + N_1[\phi], \]
where
\[ L_1[\phi] = (-\Delta)^s \phi - V(y) \phi, \quad V(y) = \partial_c f_1(y, v_1), \]
\[ E_1 = (-\Delta)^s v_1 - f_1(y, v_1), \]
\[ N_1[\phi] = f_1(y, v_1 + \phi) - f_1(y, v_1) - \partial_c f_1(y, v_1) \phi. \]

The symmetry (5.2) states that the kernel space of the linear operator \( L_1 \) is as follows
\[ Z(y) = \frac{\partial W}{\partial y_1}(y - b'_1), \]
Let us consider the nonlinear projected problem
\[ \begin{cases} L_1[\phi] = -E_1 + N_1[\phi] + cV(y)Z(y), & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} VZ \phi = 0, \\ \phi(y) \to 0 & \text{as } |y| \to \infty. \end{cases} \quad (5.8) \]

Through a similar discussion as before, we can obtain the following properties of existence:
Proposition 5.1. Assume that \( d \) satisfies (5.6). There is \( \varepsilon_0 > 0 \) such that for \( \varepsilon > 0 \) small there exists a unique solution \( \phi = \phi_d \) to (5.8), \( \| \phi \|_* \leq r_0 \). Moreover it satisfies
\[
\| \phi_d \|_* \leq C \varepsilon^{3-2s}
\]
and \( \phi_d \) is continuous with respect to \( d \in (d_0, 1/d_0) \).

The proof of Proposition 5.1 is similar to that of Proposition 3.4. More details will be left to the reader.

We have obtained a solution \( v_d = v_1 + \phi_d \) of
\[
\begin{align*}
(-\Delta)^s v_d &= f_1(y, v_d) + c_d VZ, \text{ in } \mathbb{R}^2, \\
v_d(y) &\to 0 \text{ as } |y| \to \infty,
\end{align*}
\]
for some parameter \( c_d \).

Multiplying (5.9) by \( Z \) and integrating over \( \mathbb{R}^2 \), we have
\[
c_d \int_{\mathbb{R}^2} VZ^2 dx = \int_{\mathbb{R}^2} [(-\Delta)^s v_d - f_1(y, v_d)] Z dy.
\]
Thus \( c_d = 0 \) is reduced to
\[
\int_{\mathbb{R}^2} [(-\Delta)^s v_d - f_1(y, v_d)] Z dy = 0.
\]
And we have the following:

Proposition 5.2. If \( d \) satisfies (5.6), then
\[
\int_{\mathbb{R}^2} [(-\Delta)^s v_d - f_1(y, v_d)] Z dy = M_\gamma \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} \left( \kappa c_{2,s} (2 - 2s) \frac{1}{(2d)^{3-2s}} + U \right) + o(\varepsilon^{3-2s}),
\]
where \( U < 0 \) is a constant and \( c_{2,s} \) defined by (1.5).

Proof. Through direct calculation and Proposition 5.1, we get
\[
\begin{align*}
\int_{\mathbb{R}^2} [(-\Delta)^s v_d - f_1(y, v_d)] Z dy \\
&= \int_{\mathbb{R}^2} [(-\Delta)^s (v_1 + \phi_d) - f_1(y, v_1 + \phi_d)] Z dy \\
&= \int_{\mathbb{R}^2} [(v_1 - 1)_+ - (v_1 + U \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} + y_1 - \mu_{\gamma-1}^{2s} - \kappa G_s(y, b'_2) - \mu_{\gamma-1}^{2s} A_0) \chi_{B_{d/\varepsilon}(b'_2)}] Z dy \\
&= \int_{\mathbb{R}^2} -\gamma (v_1 - 1)_+ Z \left( U \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} + y_1 - \mu_{\gamma-1}^{2s} - \kappa G_s(y, b'_2) \right) dy + o(\varepsilon^{3-2s}) \\
&= - \int_{\mathbb{R}^2} \frac{\partial}{\partial y_1} (v_1 - 1)_+ \times \left( U \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} + y_1 - \mu_{\gamma-1}^{2s} - \kappa G_s(y, b'_2) \right) dy + o(\varepsilon^{3-2s}) \\
&= \int_{\mathbb{R}^2} (v_1 - 1)_+ dy \left( U \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} + \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} + c_{2,s} (2 - 2s) \frac{1}{(2d)^{3-2s}} + U \right) + o(\varepsilon^{3-2s}) \\
&= M_\gamma \varepsilon^{3-2s} \mu_{\gamma-1}^{2s} + (\kappa c_{2,s} (2 - 2s) \frac{1}{(2d)^{3-2s}} + U) + o(\varepsilon^{3-2s}).
\end{align*}
\]
\( \square \)
Proof of Theorem 1.4.

Proof. From Proposition 5.2, the equation $c_d = 0$ is reduce to

$$\kappa c_{2,s}(2 - 2s)\frac{1}{(2d)^{3-2s}} + U + o(1) = 0. \quad (5.12)$$

So as $\varepsilon \to 0$, we can get a solution from the above equation. $\square$

In the case of the rotating solutions in Theorems 1.5, we consider the special form in section 2, described as the following problem

$$
\begin{cases}
(-\Delta)^{\gamma} \psi = \varepsilon^{2-2s} (\psi + \kappa G_s(x, b_0) + \omega \frac{|x|^2}{2} - \varepsilon^{2s-2}\lambda_j)^{\gamma} \chi_{B_\delta(b_j)} & \text{in } \mathbb{R}^2, \\
\psi(x) \to 0 \text{ as } |x| \to \infty,
\end{cases}
\quad (5.13)
$$

The choice of $\lambda_j$ issimilar to that in Theorem 1.4 and we have $\lambda_j = \mu_j^{1-2s} + o(2-2s)$.

The point $b_0, b_1, ..., b_k$ are close to $b_0^0, b_1^0, ..., b_k^0$, which are determined by (1.25). The choice of $\delta > 0$ is fixed so that $B_\delta(b_j)$ are disjoint.

The ansatz $\Psi_0$ is the same as in (2.8) with $\mu$ as in (5.4). The proof of Theorem 1.5 is then a direct adaptation of the proof of Theorem 1.4.

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