Markov risk mappings and risk-sensitive optimal stopping

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Abstract

We give a probabilistic setting to the Markov property for dynamic conditional risk mappings in discrete time. This is useful for recursive solutions to dynamic optimisation problems such as optimal prediction, where the recursion depends on randomness multiple steps ahead. The property holds for standard measures of risk used in practice, such as Value at Risk and Average Value at Risk. It is formulated in several equivalent versions including a representation via acceptance sets, a strong version, and a dual representation.

Key words: optimal stopping, risk measures, Markov property, Markov decision processes.

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1 Introduction and related work

The Markov property is a main tool used in stochastic optimisation and, for a Markov chain \( X = (X_t)_{t \in \mathbb{N}_0} \) taking values in a measurable space \( E \) (and defined say on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}_0}, \mathbb{P}^x) \) where \( X_0 = x, \mathbb{P}^x \)–a.s. and \( \mathbb{N}_0 = \{0, 1, \ldots\} \)) it can be formulated either analytically or probabilistically. A dynamic evaluation of risk or dynamic conditional risk mapping can also have the Markov property for the chain \( X \). In this paper we present a probabilistic formulation of the Markov property for families of dynamic conditional risk mappings. To fix ideas, the family \( \varrho = ((\rho^x_t)_{t \in \mathbb{N}_0})_{x \in E} \) of conditional linear expectations given by

\[
\rho^x_t(\cdot) = \begin{cases} 
\mathbb{E}^x \left[ \cdot \right], & t = 0, \\
\mathbb{E}^x \left[ \cdot | \mathcal{F}_t \right], & t \geq 1,
\end{cases}
\]

is trivially Markovian (for any Markov chain \( X \)), and our aim is to investigate nonlinear (that is, risk-sensitive) families \( \varrho \).

The analytic Markov property for dynamic conditional risk mappings has been explored in, for example, [1, 2, 3, 4]. It is based on so-called transition risk mappings, and is useful

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in recursive solution techniques which evaluate risk only one step ahead: 
\[ \rho^t(Z), \] 
where \( Z = f(X_{t+1}) \). To complement this, our probabilistic Markov property is useful for recursions which directly evaluate risk multiple steps ahead: 
\[ \rho^t(Z), \] 
where \( Z = f(X_{t+1}, X_{t+2}, ...) \). In optimal prediction, for example, the evaluation of risk depends on the evolution of the process after a user-selected stopping time: for instance, the problem of stopping as close as possible to the ultimate maximum of a time-homogeneous Markov chain \( X \) taking values in \( E = \mathbb{R} \) (cf. [5, 6] in the case of linear expectation):

\[
V_T^{\text{pred}}(x) := \inf_{\tau \in \mathcal{F}_{[0,T]}} \rho^0_0(X^*_T - X_\tau),
\]

(2)

where \( T \in \mathbb{N}_0, X^*_T := \max_{0 \leq s \leq T} X_s, \) and \( \mathcal{F}_{[0,T]} \) is the set of stopping times taking values in \( \{0, 1, \ldots, T\} \) (see also [7, 8] for work in continuous time). In the case of linear expectation, explicit solutions have been obtained for this problem by applying the probabilistic Markov property to represent the objective as a function of \( \tau \) and \( X_\tau \), obtaining a function \( F \) such that

\[
\mathbb{E}^x[f(X^*_T - X_\tau)] = \mathbb{E}^x[F(\tau, X_\tau)].
\]

In Section 5.1 below we solve (2) recursively for any \( \varrho \) satisfying our probabilistic Markov property, namely, that for each \( Z \) in a suitable class of real random variables we have

\[
\rho^t(Z \circ \theta_t) = \rho^0_0(Z) \quad \mathbb{P}^x\text{-a.s. for each } t \in \mathbb{N}_0,
\]

where \( \theta_t \) is the shift operator. Sufficient conditions are obtained to make the probabilistic and analytic formulations equivalent (see Proposition 2.10), thus unifying them as in the linear case. For convex risk mappings we characterise the Markov property in terms of the dual representation (see for example [9, 10, 11, 12, 13, 14]). More precisely, we show that a Markovian convex risk mapping can be characterised as a supremum over penalised linear expectations with respect to certain transition kernels, extending the dual representation of transition risk mappings beyond the coherent case studied in [3]. We also obtain sufficient conditions under which the latter structure implies the probabilistic Markov property.

This probabilistic Markov property is verified for the commonly used entropic, mean semi-deviation, VaR, AVaR and worst-case risk mappings. Related work on risk mappings under additional assumptions can be found in [15, 13, 14], and in [16] and [17] under the dual formulation. In the framework of Markov decision processes, related work includes [18, 11, 2, 3, 4] and [19, 20, 21], respectively in the fully and partially observable cases. Recently Dentcheva and Ruszczyński [22] introduced risk forms and applied them to the optimisation of partially observable two-stage systems. Our approach relates the abstract risk forms of [22] to the Markovian framework useful for solving multi-stage optimisation problems, by making the connection to the Markov property under a given family \( (\mathbb{P}^x)_{x \in E} \) of probability measures.

The paper is structured as follows. Section 2 provides the probabilistic framework, together with equivalences between versions of the Markov property, and a representation in terms of acceptance sets. Section 3 provides examples and Section 4 addresses the dual representation, while applications to optimisation problems are given in Section 5.

2 A probabilistic Markov property for risk mappings

After presenting the setup and briefly recalling necessary definitions (Section 2.1), in Sections 2.2 and 2.3 we provide our novel probabilistic setting for the Markov property and establish equivalent forms. The Markov property in terms of acceptance sets is studied in Section 2.4.
2.1 Setup and notation

Suppose we have an $E$-valued time-homogeneous Markov process $(X_t)_{t \in \mathbb{N}_0}$ with respect to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where:

- $E$ is a Polish space equipped with its Borel $\sigma$-algebra $\mathcal{E}$,
- $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is the discrete time parameter set,
- $\Omega$ is the canonical space of trajectories $\Omega = E^{\mathbb{N}_0}$,
- $X$ is the coordinate mapping, $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \in \mathbb{N}_0$,
- $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{N}_0}$ with $\mathcal{F}_t = \sigma(\{X_s : s \leq t\})$ the natural filtration generated by $X$ and $\mathcal{F} = \sigma(\bigcup_{t \in \mathbb{N}_0} \mathcal{F}_t)$.

Let $\mathcal{P}(\mathcal{F})$ denote the set of probability measures on $(\Omega, \mathcal{F})$. Unless otherwise specified, all inequalities between random variables will be interpreted in the almost sure sense with respect to the appropriate probability measure. We write $\mathcal{T}$ for the set of finite-valued stopping times and $\mathcal{T}_{[0, T]}$ for the set of stopping times taking values in $\{t, t+1, \ldots, T\}$. We denote by $b\mathbb{F}$ the space of bounded random variables on $(\Omega, \mathcal{F})$ and similarly for other $\sigma$-algebras. It will also be convenient to define $\mathcal{F}_{t,\infty} = \sigma(X_s : s \geq t)$ and $\mathcal{F}_{t,t} = \sigma(X_t)$.

In the above setup the following objects exist:

- The law $\mu_{X_0}$ of $X_0$ under $\mathbb{P}$ and a family of probability measures defined by the measurable mapping $x \mapsto \mathbb{P}^x$ from $E$ to $\mathcal{P}(\mathcal{F})$, which is a disintegration of $\mathbb{P}$ with respect to $X_0$ (see [23, p. 78]). To be precise, this family satisfies $\mathbb{P}^x(X_0 = x) = 1$ and for every $F \in \mathcal{F}$ we have
  $$\mathbb{P}(F) = \int_E \mathbb{P}^x(F) \mu_{X_0}(dx).$$

- A time-homogeneous Markov transition kernel $q^X : \mathcal{E} \times E \to [0, 1]$ such that for every $x \in E$ and $B \in \mathcal{E}$ we have $q^X(B|x) = \mathbb{P}^x(X_1 \in B)$,

- Markov shift operators $\theta_t : \Omega \to \Omega$, $t \in \mathbb{N}_0$ such that $\theta_0(\omega) = \omega$, $\theta_t \circ \theta_s = \theta_{t+s}$ and $(X_t \circ \theta_s)(\omega) = X_{t+s}(\omega)$ for each $\omega \in \Omega$ and $s, t \in \mathbb{N}_0$.

For $\tau \in \mathcal{T}$ define the random shift operator $\theta_\tau$ by

- $\theta_\tau(\omega) = \theta_{\tau(\omega)}(\omega)$,
- $= \theta_\tau(\omega)$ on $\{\tau(\omega) = t\}$.

We recall the definitions of risk mapping and conditional risk mapping (which are interchangeable via the mapping $Z \mapsto \rho(-Z)$ with the monetary conditional risk measures of [24, Def. 11.1]):

**Definition 2.1 (Risk mapping).** A risk mapping on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a function $\rho : b\mathbb{F} \to \mathbb{R}$ satisfying

- **Normalisation:** $\rho(0) = 0$,
- **Translation invariance:** $\forall Z \in b\mathbb{F}$ and $c \in \mathbb{R}$ we have $\rho(Z + c) = c + \rho(Z)$,
- **Monotonicity:** $\forall Z, Z' \in b\mathbb{F}$, we have $Z \leq Z' \implies \rho(Z) \leq \rho(Z').$

We recall that all inequalities in Definition 2.1 are interpreted in the almost sure sense where applicable.
Definition 2.2 (Conditional risk mapping). A conditional risk mapping on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with respect to a \(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\) is a function \(\rho_\mathcal{G} : b\mathcal{F} \to b\mathcal{G}\) satisfying \(\mathbb{P}\)-a.s.:

Normalisation: \(\rho_\mathcal{G}(0) = 0\),

Conditional translation invariance: \(\forall Z \in b\mathcal{F}\) and \(Z' \in b\mathcal{G}\),

\[\rho_\mathcal{G}(Z + Z') = Z' + \rho_\mathcal{G}(Z),\]

Monotonicity: \(\forall Z, Z' \in b\mathcal{F}\),

\[Z \leq Z' \implies \rho_\mathcal{G}(Z) \leq \rho_\mathcal{G}(Z').\]

Conditional risk mappings also satisfy the following property (cf. [25, Prop. 3.3], [24, Ex. 11.1.2]):

Conditional locality: for every \(Z, Z' \in b\mathcal{F}\) and \(A \in \mathcal{G}\),

\[\rho_\mathcal{G}(1_A Z + 1_A c Z') = 1_A \rho_\mathcal{G}(Z) + 1_A c \rho_\mathcal{G}(Z').\]

Definition 2.3 (Dynamic conditional risk mapping). For each \(x \in E\) a dynamic conditional risk mapping on the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)\) is a sequence \((\rho_x t)_{t \in \mathbb{N}_0}\) where

- \(\rho_x 0\) is a risk mapping,
- for each \(t \geq 1\), \(\rho_x t\) is a conditional risk mapping on \((\Omega, \mathcal{F}, \mathbb{P}_x)\) with respect to \(\mathcal{F}_t\).

We use the superscript \(x\) in \((\rho_x t)_{t \in \mathbb{N}_0}\) to indicate a dynamic conditional risk mapping on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)\).

Note that the codomain of \(\rho_x 0\) is \(\mathbb{R}\) while, for each \(t \geq 1\), the codomain of \(\rho_x t\) is \(b\mathcal{F}_t\). This setup is motivated by the fact that any \(\mathcal{F}_0\)-measurable random variable is \(\mathbb{P}_x\)-a.s. constant. For example, for each \(x \in E\), the sequence \((\rho_x t)_{t \in \mathbb{N}_0}\) given by \(1\) is a dynamic conditional risk mapping.

For a finite stopping time \(\tau\) define

\[\rho_\tau = \sum_{t \in \mathbb{N}_0} 1_{\{\tau = t\}} \rho_t,\]

noting that \(\rho_\tau : b\mathcal{F} \to b\mathcal{F}_\tau\).

2.2 Markov property

Definition 2.4 (Regularity). A collection of risk mappings \((\rho_x)_{x \in E}\) is said to be regular if for all \(Z \in b\mathcal{F}\) the map \(x \mapsto \rho_x(Z)\) is bounded and measurable.

Definition 2.5 (Markov property). The family \(\varrho := ((\rho_x t)_{t \in \mathbb{N}_0})_{x \in E}\) of dynamic conditional risk mappings satisfies the Markov property (for the chain \((X_t)_{t \in \mathbb{N}_0}\)) if

1. \((\rho_x 0)_{x \in E}\) is regular,
2. for each \(x \in E\), \(Z \in b\mathcal{F}\) and \(t \in \mathbb{N}_0\) we have

\[\rho_x t(Z \circ \theta_t) = \rho_x^{X_t}(Z) \quad \mathbb{P}_x\text{-a.s.},\]  \hspace{1cm} (3)

where \(\rho_x^{X_t}(Z)\) is interpreted as the random variable \(\omega \mapsto \rho_x^{X_t(\omega)}(Z)\).
By construction, if \( \varrho := ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E} \) is a family of dynamic conditional risk mappings then \((\rho_0^x)_{x \in E}\) is a collection of risk mappings. For convenience we often write \(\rho^x\) for \(\rho_0^x\).

In particular we have

\[
\rho^x(Z) = \rho^x(\mathds{1}_{\{x\}}(X_0)Z), \quad Z \in b\mathcal{F}, \ x \in E.
\]

Note that the linear conditional expectation \(\varrho\) satisfies this Markov property and corresponds to the risk-neutral case. Examples of \(\rho\) which are risk sensitive are presented in Section 3.

**Remark 2.6.** Note that \((3')\) could have been specified differently. For example, by relating all risk mappings \(\rho_t^x\) to the same regular collection \((\rho^x)_{x \in E}\) in \((3)\) we have imposed a time homogeneity on the measurement of risk. This is not essential, since taking a collection \(\{\rho^{x,s} : x \in E, s \in \mathbb{N}_0\}\) indexed also by time and specifying

\[
\rho_t^x(Z \circ \theta_t) = \rho^{X_{t,t}}(Z) \ \mathbb{P}^x\text{-a.s.},
\]

the family of dynamic conditional risk mappings may be time-inhomogeneous.

A regular collection \((\rho^x)_{x \in E}\) of risk mappings can also be used to construct a Markovian family \(\varrho = ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E}\) satisfying Definition 2.5 as follows. We use the fact that any bounded \(\mathcal{F}\)-measurable random variable \(Z\) can be represented as \(Z = f(X_0, X_1, \ldots)\) for some measurable and bounded function \(f : E^{\mathbb{N}_0} \to \mathbb{R}\), which follows by standard monotone class arguments, see [26 Prop. 0.2.7] or [27 Th. 2.4.4]. As it is obtained without reference to any probability measure, the equality \(Z = f(X_0, X_1, \ldots)\) holds for all (rather than almost all) \(\omega \in \Omega\) and therefore the function \(f\) is unique.

**Proposition 2.7.** Let \((\rho^x)_{x \in E}\) be regular. For each \(x \in E, t \in \mathbb{N}_0\) and \(Z = f(X_0, X_1, \ldots)\) let

\[
\rho_t^x(Z)(\omega) := \rho^{X_t(\omega)}(Z_t(X_0(\omega), \ldots, X_t(\omega))), \quad \omega \in \Omega,
\]

where

\[
Z_t(x_0, \ldots, x_t) := f(x_0, \ldots, x_t, X_1, X_2, \ldots).
\]

Then for each \(x \in E, (\rho_t^x)_{t \in \mathbb{N}_0}\) is a dynamic conditional risk mapping and the family \(\varrho = ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E}\) satisfies the Markov property.

**Proof.** Let \(x \in E, t \in \mathbb{N}_0\) and \(\omega \in \Omega\) be arbitrary. For compactness we will write \(X_{0:t}(\omega)\) for \((X_0(\omega), \ldots, X_t(\omega)) \in E^{t+1}\). Clearly \(\rho_t^x\) is normalised, so we check conditional translation invariance and monotonicity. Taking \(Z = f(X_0, X_1, \ldots) \in b\mathcal{F}\) and \(W = g(X_0, \ldots, X_t) \in b\mathcal{F}_t\), by construction we have

\[
\rho_t^x(Z + W)(\omega) = \rho^{X_t(\omega)}(Z_t(X_0(\omega), \ldots, X_t(\omega)) + W_t(X_0(\omega), \ldots, X_t(\omega)))
\]

\[
= \rho^{X_t(\omega)}(Z_t(X_0(\omega), \ldots, X_t(\omega))) + W_t(X_0(\omega), \ldots, X_t(\omega))
\]

\[
= \rho_t^x(Z)(\omega) + W(\omega).
\]

To check monotonicity let \(Z = f(X_0, X_1, \ldots)\) and \(Z' = f'(X_0, X_1, \ldots)\) be two bounded random variables such that \(Z \leq Z'\) \(\mathbb{P}^x\text{-a.s.}\). We first show that \(Z_t(X_{0:t}(\omega)) \leq Z_t'(X_{0:t}(\omega))\), \(\mathbb{P}^{X_t(\omega)}\text{-a.s.}\). Writing as usual \(\mathbb{P}^x(A|\mathcal{F}_t)\) for \(\mathbb{P}^x[1_A|\mathcal{F}_t]\) for each \(A \in \mathcal{F}\), and applying conditional locality and the Markov property, for almost all \(\omega\) we have, with a slight abuse of notation

\[
1 = \mathbb{P}^x(Z \leq Z'|\mathcal{F}_t)(\omega)
\]

\[
= \mathbb{P}^x(f(X_{0:t}(\omega), X_{t+1}, \ldots) \leq f'(X_{0:t}(\omega), X_{t+1}, \ldots)|\mathcal{F}_t)(\omega)
\]

\[
= \mathbb{P}^{X_t(\omega)}(f(X_{0:t}(\omega), X_{t+1}, \ldots) \leq f'(X_{0:t}(\omega), X_{t+1}, \ldots))
\]
\[
\mathbb{P}_{X_t(\omega)}(Z_t(X_{0:t}(\omega))) \leq Z_t'(X_{0:t}(\omega)).
\]

By the monotonicity of \(\rho_{X_t(\omega)}\) we then have that \(\mathbb{P}_{-}\) a.s.,
\[
\rho_t^x(Z)(\omega) = \rho_{X_t(\omega)}(Z_t(X_{0:t}(\omega))) \leq \rho_{X_t(\omega)}(Z_t'(X_{0:t}(\omega))) = \rho_t^x(Z')(\omega).
\]

Lastly we verify the Markov property for the family \(\varrho\). For \(Z = f(X_0, X_1, \ldots)\) we have by construction and \([1]\) that
\[
\rho_t^x(Z \circ \theta_t)(\omega) = \rho_{X_t(\omega)}(f(X_t(\omega), X_1, X_2, \ldots)) = \rho_{X_t(\omega)}(Z).
\]

\(\square\)

2.3 Equivalent forms of the Markov property

Just as for the linear conditional expectation, the Markov property for risk mappings can be stated in several equivalent forms. We begin with the strong Markov property.

**Proposition 2.8** (Strong Markov Property). If \(\varrho := ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E}\) satisfies the Markov property then for any stopping time \(\tau \in \mathcal{F}\) and \(Z \in b\mathcal{F}\) we have
\[
\rho_t^x(Z \circ \theta_\tau) = \rho^x_{X_\tau}(Z) \quad \mathbb{P}_{-}\text{-a.s.}
\]

**Proof.** Using \(\{\tau = t\} \in \mathcal{F}_t\), conditional locality and the Markov property we have \(\mathbb{P}_{-}\) a.s.:
\[
\rho_t^x(Z \circ \theta_\tau) = \sum_{t=0}^{\infty} \mathbb{1}_{\{\tau = t\}} \rho_t^x(Z \circ \theta_t) = \sum_{t=0}^{\infty} \mathbb{1}_{\{\tau = t\}} \rho^x_{X_t}(Z) = \rho^x_{X_\tau}(Z).
\]

\(\square\)

To make a connection to one-step Markov properties we will require time consistency:

**Definition 2.9.** The family \(\varrho := ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E}\) is said to be time consistent if for all \(Y, Z \in b\mathcal{F}\), \(t \in \mathbb{N}_0\) and \(x \in E\) we have
\[
\rho_{t+1}^x(Y) \leq \rho_{t+1}^x(Z) \implies \rho_t^x(Y) \leq \rho_t^x(Z).
\]

We say that a regular collection \((\rho^x)_{x \in E}\) of risk mappings is time consistent if the associated Markovian dynamic conditional risk mapping (constructed in Proposition 2.7) is time consistent.

It is well known (see e.g. [28 Prop. 1.16]) that we then have the following recursive relation: for every \(x \in E\) and \(0 \leq s \leq t\),
\[
\rho_s^x = \rho_s^x \circ \rho_t^x.
\]

As noted in [24 Exercise 11.2.2], this relation can be generalised to stopping times: for any bounded stopping times \(\tau_1 \leq \tau_2\) one has
\[
\rho_{\tau_1}^x = \rho_{\tau_1}^x \circ \rho_{\tau_2}^x. \quad (5)
\]

Since risk mappings are nonlinear in general, in the next proof we use a non-standard version of the Monotone Class Theorem (see Appendix A) which, unlike [26 Th. 0.2.3], does not appeal to vector spaces.

**Proposition 2.10.** Let \(\varrho := ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E}\) be a family of dynamic conditional risk mappings such that \((\rho^x)_{x \in E}\) is regular. Let each \(\rho_t^x\) be continuous from above and below: that is, \(\rho_t^x(Y_n) \rightarrow \rho_t^x(Y)\) a.s. for every \(t \in \mathbb{N}_0\), \(x \in E\) and monotone sequence \((Y_n)_{n \in \mathbb{N}_0}\) in \(b\mathcal{F}\) converging to \(Y \in b\mathcal{F}\). Then
(i) \( \rho \) is Markov iff for all \( k \geq 0 \) the \( k \)-step Markov property holds:
\[
\rho^k_t(f(X_{t+1}, \ldots, X_{t+k})) = \rho^{X_t}(f(X_1, \ldots, X_k)),
\]
\( \mathbb{P}^x \)-a.s. for all \( t \in \mathbb{N}_0 \), \( x \in E \) and bounded measurable functions \( f : E^k \rightarrow \mathbb{R} \).

(ii) If the family \( \rho \) is time consistent, then \( \rho \) is Markov iff the one-step Markov property holds:
for every \( t \in \mathbb{N}_0 \), \( x \in E \) and bounded measurable function \( f : E \rightarrow \mathbb{R} \) we have
\[
\rho^1_t(f(X_{t+1})) = \rho^{X_t}(f(X_1)), \quad \mathbb{P}^x \text{-a.s.}.
\] (6)

The one-step Markov property (6) corresponds to analytic formulations such as [3, Def. 6].

Remark 2.11. The assumptions of this proposition simplify in the case of convex risk mappings, for which continuity from above implies continuity from below (see proof of Corollary 4.3).

Proof. Part 1.

Assume that (6) holds for every \( x \in E \), \( t \in \mathbb{N}_0 \) and bounded measurable function \( f : E \rightarrow \mathbb{R} \) and that \( \rho \) is time consistent. We first show that for each \( k \geq 1 \), the Markov property (3) holds for all linear combinations
\[
f(x_{t+1}, \ldots, x_{t+k}) = \sum_{j=1}^n \alpha_j g_j(x_{t+1}, \ldots, x_{t+k}), \quad t \in \mathbb{N}_0, n \geq 1, \alpha_i \in \mathbb{R},
\] (7) of elementary functions \( g_j \) satisfying
\[
g_j(x_{t+1}, \ldots, x_{t+k}) = \prod_{i=1}^k 1_{A_{ij}}(x_{t+i}), \quad A_{1j}, \ldots, A_{kj} \in \mathcal{E}.
\]

We proceed by induction on \( k \). The claim is true for \( k = 1 \) since this is a special case of the one-step Markov property (6). Suppose it is also true for some \( k \geq 1 \). We have
\[
f(x_{t+1}, \ldots, x_{t+k+1}) = \sum_{j=1}^n \alpha_j g_j(x_{t+1}, \ldots, x_{t+k+1})
\]
\[
= \sum_{j=1}^n \alpha_j \left( \prod_{i=1}^{k+1} 1_{A_{ij}}(x_{t+i}) \right)
\]
\[
= \sum_{j=1}^n \prod_{j=1}^k 1_{A_{ij}}(x_{t+1}) \left( \alpha_j \prod_{i=2}^{k+1} 1_{A_{ij}}(x_{t+i}) \right).
\] (8)

By taking all possible intersections of the sets \( A_{11}, \ldots, A_{1n} \) and their complements, we can define \( N \geq n \) mutually disjoint sets \( \hat{A}_1, \ldots, \hat{A}_N \) belonging to \( \mathcal{E} \) such that
\[
\sum_{j=1}^n \prod_{j=1}^k 1_{A_{ij}}(x_{t+1}) \left( \alpha_j \prod_{i=2}^{k+1} 1_{A_{ij}}(x_{t+i}) \right) = \sum_{\ell=1}^N \prod_{j=1}^k 1_{\hat{A}_j}(x_{t+\ell}) \left( \sum_{j=1}^n \alpha_{\ell j} \prod_{i=2}^{k+1} 1_{A_{ij}}(x_{t+i}) \right),
\]
where \( \alpha_{\ell j} = \alpha_j \) if \( A_{ij} \cap \hat{A}_\ell \neq \emptyset \) and \( \alpha_{\ell j} = 0 \) otherwise. Therefore we can rewrite \( f \) in (8) as
\[
f(x_{t+1}, \ldots, x_{t+k+1}) = \sum_{\ell=1}^N \prod_{j=1}^k 1_{\hat{A}_j}(x_{t+1}) f_{\ell}(x_{t+2}, \ldots, x_{t+k+1}),
\] (9)
where the \( \tilde{A}_t \) are mutually disjoint and each \( f_t \) has the form (7). Using the local property and time consistency for \( \rho^\pi_t \), the induction hypothesis and the one-step Markov property we have

\[
\rho^\pi_t(f(X_{t+1}, \ldots, X_{t+k+1})) = \rho^\pi_t \left( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_{t+1}) f_\ell(X_{t+2}, \ldots, X_{t+k+1}) \right)
= \rho^\pi_t(\rho^\pi_{t+1} \left( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_{t+1}) f_\ell(X_{t+2}, \ldots, X_{t+k+1}) \right))
= \rho^\pi_t \left( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_{t+1}) \rho^\pi_{t+1} (f_\ell(X_{t+2}, \ldots, X_{t+k+1})) \right)
= \rho^\pi_t \left( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_{t+1}) \rho^{X_{t+1}}(f_\ell(X_1, \ldots, X_k)) \right)
= \rho^{X_t} \left( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_1) \rho^{X_1}(f_\ell(X_1, \ldots, X_k)) \right). \tag{10}
\]

Note that for every realisation \( x_t \) of \( X_t(\omega) \) we have that almost surely under \( \mathbb{P}^{x_t} \),

\[
\sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_1) \rho^{X_1}(f_\ell(X_1, \ldots, X_k)) = \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_1) \rho^{x_\ell_1}(f_\ell(X_2, \ldots, X_{k+1}))
= \rho^{x_1}( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_1) f_\ell(X_2, \ldots, X_{k+1}) ) \tag{11}
\]

Therefore, by (9)–(11) we have for almost every \( \omega \in \Omega \):

\[
\rho^\pi_t(f(X_{t+1}, \ldots, X_{t+k+1}))(\omega) = \rho^{X_t(\omega)} \left( \rho^{X_1(\omega)} \left( \sum_{\ell=1}^{N} \mathbb{1}_{\tilde{A}_\ell}(X_1) f_\ell(X_2, \ldots, X_{k+1}) \right) \right)
= \rho^{X_t(\omega)}(f(X_1, \ldots, X_{k+1})),
\]

and, by induction, the Markov property (3) holds for all functions \( f \) of the form (7).

**Part 2.**

Let \( \mathcal{H}_0 \) be the set of random variables having the form \( Z = f(X_0, \ldots, X_k) \) for some \( k \in \mathbb{N}_0 \) and some \( f \) of the form (7). Clearly \( \mathcal{H}_0 \) is closed under the operation of taking the pointwise minimum. Let

\[
\mathcal{H} := \{ Z \in b\mathcal{F} : \rho^\pi_t(Z \circ \theta_t) = \rho^{X_t}(Z) \ \mathbb{P}^{x}_t \text{-a.s. for all } x \in E, t \in \mathbb{N}_0 \}.
\]

We show that \( \mathcal{H}_0 \subset \mathcal{H} \). Suppose that \( Z = f(X_0, \ldots, X_k) \in \mathcal{H}_0 \). Then by conditional locality and Part 1 we have that for each \( \omega \in \Omega \),

\[
\rho^\pi_t(Z \circ \theta_t)(\omega) = \rho^\pi_t(f(X_t(\omega), X_{t+1}, \ldots, X_{t+k}))(\omega)
= \rho^{X_t(\omega)}(f(X_t(\omega), X_1, \ldots, X_k))(\omega) = \rho^{X_t(\omega)}(f(X_0, X_1, \ldots, X_k))(\omega),
\]

i.e. \( Z \in \mathcal{H} \).

The space \( \mathcal{H} \) is closed under monotone limits and Theorem 3.1 implies that \( \mathcal{H} \) contains all bounded \( \sigma(\mathcal{H}_0) \)-measurable functions. Since \( \sigma(\mathcal{H}_0) = \mathcal{F} \) we conclude that \( \mathcal{H} = b\mathcal{F} \). \( \square \)
2.4 Markov property in terms of acceptance sets

Particularly in the context of mathematical finance, conditional risk mappings can be characterised by their acceptance sets \( \mathcal{A}_t \) [23 Sec. 1.4.1] or [24 Sec. 4.1], where

\[
\mathcal{A}_t := \{ Y \in b\mathcal{F} : \rho_t^Y(Y) \leq 0 \text{ \( \mathbb{P}^x \)-a.s.} \},
\]

and so for completeness we also formulate the Markov property in these terms. First define another acceptance set, which will be useful in formulating the Markov property:

\[
\tilde{\mathcal{A}}_t := \{ Y \in b\mathcal{F} : \rho_t^{X_t}(Y) \leq 0 \text{ \( \mathbb{P}^x \)-a.s.} \}.
\]

Note that for any \( \mathcal{F}_{t,\infty} \)-measurable random variable \( Y = \tilde{Y} \circ \theta_t \) with the representation \( Y = f(X_t, X_{t+1}, \ldots) \) one can define \( Y \circ \theta_{-t} := \tilde{Y} = f(X_0, X_1, \ldots) \).

**Lemma 2.12.** The family \( g := ((\rho_{t,i})_{t \in \mathbb{N}_0})_{x \in E} \) is Markov if and only if \( \rho_t^Y : b\mathcal{F}_{t,\infty} \to b\mathcal{F}_{t,t} \) and for each \( x \in E, t \in \mathbb{N}_0 \) and \( Z \in b\mathcal{F} \) we have

\[
Z \circ \theta_t \in \mathcal{A}_t^Y \iff Z \in \tilde{\mathcal{A}}_t^Y. \tag{12}
\]

**Proof.** Necessity is obvious. Conversely, suppose that \( \rho_t^Y : b\mathcal{F}_{t,\infty} \to b\mathcal{F}_{t,t} \) and that the equivalence [12] holds. Fix \( Z \in b\mathcal{F}, t \in \mathbb{N}_0 \) and \( x \in E \).

Step 1. We show that

\[
\rho_t^{X_t}(Z) = \text{ess inf}\{ Y = g(X_t) \in b\mathcal{F}_{t,t} : Z - Y \circ \theta_{-t} \in \tilde{\mathcal{A}}_t^Y \}, \quad \mathbb{P}^x \text{-a.s.} \tag{13}
\]

Proof of ‘\( \geq \)’. Let \( g(y) := \rho^y(Z) \) for \( y \in E \). Then \( \mathbb{P}^y \)-a.s. we have \( g(X_t) \circ \theta_{-t} = \rho^{X_t}(Z) = \rho^y(Z) \). Since \( \rho^y(Z - g(X_t) \circ \theta_{-t}) = \rho^y(Z - \rho^y(Z)) = 0 \) we have \( \rho^{X_t}(Z - g(X_t) \circ \theta_{-t}) = 0 \), implying \( Z - g(X_t) \circ \theta_{-t} \in \tilde{\mathcal{A}}_t^Y \).

Proof of ‘\( \leq \)’. Let \( Y = g(X_t) \) belong to the set on the right-hand side of [13]. Then for \( \Omega_1 = \{ \omega \in \Omega : \rho^{X_t}(Z - Y \circ \theta_{-t}) \leq 0 \} \) we have \( \mathbb{P}^x(\Omega_1) = 1 \). Let \( \Omega_0 = \{ \omega \in \Omega : \rho^{X_t}(Z) > Y \} \). To show that \( \Omega_1 \subset \Omega_0^c \), let \( \omega \in \Omega_0 \) and \( x_t := X_t(\omega) = \omega(\xi) \). Since \( \omega \in \Omega_0 \), we have that \( \rho^{X_t(\omega)}(Z) > Y(\omega) \), which is equivalent to \( \rho^{X_t}(Z) > g(x_t) \). Then

\[
\rho^{X_t(\omega)}(Z - Y \circ \theta_{-t}) = \rho^{x_t}(1_{x_t}(X_0)(Z - Y \circ \theta_{-t})) = \rho^{x_t}(Z - g(x_t)) = \rho^{x_t}(Z) - g(x_t) > 0,
\]

i.e. \( \omega \in \Omega_0^c \). Since \( \mathbb{P}^x(\Omega_0^c) = 0 \), it follows that \( \mathbb{P}^x(\Omega_0) = 0 \), which finish the proof of the claim of Step 1.

Step 2. To finish the proof, note from [23 Prop. 1.2] (modulo a minus sign which appears because [23] considers decreasing risk mappings) that

\[
\rho_t^Y(Z \circ \theta_t) = \text{ess inf}\{ Y \in b\mathcal{F}_t : Z \circ \theta_t - Y \in \mathcal{A}_t^Y \} \leq \text{ess inf}\{ Y \in b\mathcal{F}_{t,t} : Z \circ \theta_t - Y \in \tilde{\mathcal{A}}_t^Y \} \leq \rho_t^Y(Z \circ \theta_t),
\]

where the last inequality follows from the fact that \( Y = \rho_t^Y(Z \circ \theta_t) \in b\mathcal{F}_{t,t} \) and \( Z \circ \theta_t - \rho_t^Y(Z \circ \theta_t) \in \mathcal{A}_t^Y \). Since for \( Y \in b\mathcal{F}_{t,t} \) we have from [12] that

\[
Z \circ \theta_t - Y \in \mathcal{A}_t^Y \iff Z - Y \circ \theta_{-t} \in \tilde{\mathcal{A}}_t^Y,
\]

Step 1 completes the proof.

\[\square\]

**Remark 2.13.** The above lemma implies in particular that for a Markovian risk map, if \( Z \in b\mathcal{F}_{t,\infty} \), then \( \rho_t^Z(Z) \) is \( \sigma(X_t) \)-measurable.
3 Examples

In this section we provide examples of Markovian families of dynamic conditional risk mappings. Note that the entropic and worst case risk mappings are time consistent (see [11 Prop. 6 and 29 Theorem 2.8(b)(ii)] respectively), while the mean semi-deviation risk mapping and average value at risk are not ([24 Example 11.13], [30 pages 20-21]). Below we take $Z \in b\mathcal{F}, t \in \mathbb{N}_0, x \in E$.

3.1 Composite risk mappings

Let $K \in \mathbb{N}_0$ and $g_k : \mathbb{R}^{m_k} \times E \to \mathbb{R}$ for $k = 0, \ldots, K$ be measurable functions with $m_0 = 1$ and $m_k = 2$ for $k \geq 1$. Suppose that $g_k$ is bounded on compact sets and for every $r_k \in \mathbb{R}^{m_k}$ the map $x \mapsto g_k(r_k, x)$ is bounded on $E$. Then a Markovian family $\varrho = ((\rho_t^F)_{t\in\mathbb{N}_0})_{x\in E}$ of dynamic conditional risk mappings is given by $\rho_0^F(Z) = R_K^F(Z)$ and $\rho_t^F(Z) = R_z^F(Z|\mathcal{F}_t)$ for $t \geq 1$, where

$$R^F_k(Z) = \begin{cases} \mathbb{E}^Z[g_0(Z, X_0)], & \text{if } k = 0, \\ \mathbb{E}^Z[g_k(Z, R_{k-1}^F(Z), X_0)], & \text{if } 1 \leq k \leq K, \end{cases}$$

and

$$R^F_k(Z|\mathcal{F}_t) = \begin{cases} \mathbb{E}^Z[g_0(Z, X_t)|\mathcal{F}_t], & \text{if } k = 0, \\ \mathbb{E}^Z[g_k(Z, R_{k-1}^F(Z|\mathcal{F}_t), X_t)|\mathcal{F}_t], & \text{if } k \geq 1. \end{cases}$$

This family clearly includes the linear expectation ($K = 0, g_0(z, x) = z$) and its statistical estimation properties are studied in [31].

**Lemma 3.1.** The family of dynamic conditional risk mappings $\varrho = ((\rho_t^F)_{t\in\mathbb{N}_0})_{x\in E}$ defined through (14) and (15) is Markovian.

**Proof.** The Markov property holds at $k = 0$ since

$$R_0^F(Z \circ \theta_t|\mathcal{F}_t) = \mathbb{E}^Z[g_0(Z \circ \theta_t, X_t)|\mathcal{F}_t]$$

$$= \mathbb{E}^Z[g_0(Z, X_0) \circ \theta_t|\mathcal{F}_t]$$

$$= \mathbb{E}^{X_t}[g_0(Z, X_0)] = R_0^{X_t}(Z).$$

Assuming that it holds at $k - 1$, the Markov property also holds at $k$:

$$R_k^F(Z \circ \theta_t|\mathcal{F}_t) = \mathbb{E}^Z[g_k(Z \circ \theta_t, R_{k-1}^F(Z \circ \theta_t|\mathcal{F}_t), X_t)|\mathcal{F}_t]$$

$$= \mathbb{E}^Z[g_k(Z, R_{k-1}^F(Z), X_0) \circ \theta_t|\mathcal{F}_t]$$

$$= \mathbb{E}^{X_t}[g_k(Z, R_{k-1}^F(Z), X_0)] = R_k^{X_t}(Z).$$

\[
\]
3.1.2 Mean-semideviation risk mapping

Similarly, the mean–semideviation risk mapping satisfies the Markov property since it is recovered from (15) by taking $K = 2$, $g_2(z, r, x) = z + \kappa(x) r^p$, $g_1(z, r, x) = ((z - r)^+)^p$ and $g_0(z, x) = z$ in (15), where $\kappa : E \to [0, 1]$ is measurable and $p \geq 1$ is an integer, giving

$$
\rho^x_t(Z) = \begin{cases} 
\mathbb{E}^x[Z] + \kappa(x) \left( (Z - \mathbb{E}^x[Z])^+ \right)^\frac{1}{p}, & t = 0, \\
\mathbb{E}^x[Z | \mathcal{F}_t] + \kappa(X_t) \left( (Z - \mathbb{E}^x[Z | \mathcal{F}_t])^+ \right)^\frac{1}{p}, & t \geq 1.
\end{cases}
$$

3.2 Worst-case risk mapping

The worst-case risk mapping is given by the family

$$
\rho^x_t(Z) = \begin{cases} 
\mathbb{P}^x - \text{ess sup}(Z), & t = 0, \\
\mathbb{P}^x - \text{ess sup} (Z | \mathcal{F}_t), & t \geq 1.
\end{cases}
(16)
$$

For $t \geq 1$ this is the $\mathcal{F}_t$-conditional $\mathbb{P}^x$-essential supremum of $Z$, that is, the smallest $\mathcal{F}_t$-measurable random variable dominating $Z$ almost surely with respect to $\mathbb{P}^x$ [29, Proposition 2.6].

Lemma 3.2. The family of dynamic conditional risk mappings given by (16) is Markovian.

Proof. Supposing first that $Z$ is non-negative, then using [29, Proposition 2.12] and the Markov property of the conditional expectation, we have $\mathbb{P}^x$-a.s.:

$$
\rho^x_t(Z \circ \theta_t) = \lim_{p \to \infty} \left( \mathbb{E}^x[(Z \circ \theta_t)^p | \mathcal{F}_t] \right)^\frac{1}{p}
= \lim_{p \to \infty} \left( \mathbb{E}^x[Z^p \circ \theta_t | \mathcal{F}_t] \right)^\frac{1}{p}
= \lim_{p \to \infty} \left( \mathbb{E}^{X_t}[Z^p] \right)^\frac{1}{p} = \rho^{X_t}(Z),
$$

while the case $t = 0$ establishes measurability in $x$. For general $Z \in b\mathcal{F}$ we first set $Z_c := Z + c$ with $c = \sup_{\omega} |Z(\omega)|$, then use translation invariance with respect to constants (see [29, Proposition 2.1]),

$$
\rho^x_t(Z \circ \theta_t) = \rho^x_t(Z_c \circ \theta_t) - c = \rho^{X_t}(Z_c) - c = \rho^{X_t}(Z),
$$

completing the proof. \qed

3.3 Value at Risk

The value at risk [24, Sec. 4.4 & Ex. 11.4] may be defined by the family

$$
\rho^x_t(Z) = \begin{cases} 
\text{VaR}^x_\lambda(-Z), & t = 0, \\
\text{VaR}^x_\lambda(-Z | \mathcal{F}_t), & t \geq 1,
\end{cases}
(17)
$$

where $\lambda \in (0, 1)$,

$$
\text{VaR}^x_\lambda(-Z) := \inf \{ m \in \mathbb{R} : \mathbb{P}^x(m < Z) \}
$$

and

$$
\text{VaR}^x_\lambda(-Z | \mathcal{F}_t) := \mathbb{P}^x - \text{ess inf} \{ m_t \in b\mathcal{F}_t : \mathbb{P}^x(m_t < Z | \mathcal{F}_t) \leq \lambda \}
$$

for $t \geq 1$. 

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Lemma 3.3. The family of dynamic conditional risk mappings given by \([17]\) is Markovian.

Proof. We first show that \(x \mapsto \rho^x(Z)\) is measurable. For \(y \in \mathbb{R}\) we have

\[
\{x \in E : \rho^x(Z) < y\} = \{x \in E : \inf\{m \in \mathbb{R} : \mathbb{P}^x(m < Z) \leq \lambda\} < y\} = \{x \in E : \exists m < y : \mathbb{P}^x(m < Z) \leq \lambda\} = \{x \in E : \exists m \in \mathbb{Q} : m < y : \mathbb{P}^x(m < Z) \leq \lambda\} = \bigcup_{m \in (-\infty,y) \cap \mathbb{Q}} \{x \in E : \mathbb{P}^x(m < Z) \leq \lambda\} = \bigcup_{m \in (-\infty,y) \cap \mathbb{Q}} f^{-1}_m((-\infty,\lambda]),
\]

where for each \(m \in \mathbb{R}\) the function \(f_m : E \to \mathbb{R}\) is defined by \(f_m(x) = \mathbb{P}^x(m < Z)\). Note that each \(f_m\) is measurable because for every \(A \in \mathcal{F}\) the mapping \(E \ni x \mapsto \mathbb{P}^x(A) \in \mathbb{R}\) is measurable by the measurability of \(E \ni x \mapsto \mathbb{P}^x \in \mathcal{B}(\mathcal{F})\). Thus \(\{x \in E : \rho^x(Z) < y\}\) is a measurable set.

To show that \(x \mapsto \rho^x(Z)\) is bounded note that, since \(Z\) is bounded, there exists \(M \in \mathbb{R}\) such that \(\{|Z| < M\} = \mathbb{Q}\). Thus \(-M \leq \rho^x(Z) \leq M\) for all \(x \in E\).

Next we show that \(\rho^x_t(Z \circ \theta_t) = \rho^{X_t}(Z)\) almost surely. Since \(Z\) is bounded, let \(m_t(X_0,\ldots,X_t) \in b\mathcal{F}_t\) satisfy

\[
\mathbb{P}^x(m_t(X_0,\ldots,X_t) < Z \circ \theta_t|\mathcal{F}_t) \leq \lambda.
\]

Then for almost all \(\omega \in \Omega\), by conditional locality and the Markov property we have

\[
\lambda \geq \mathbb{P}^x(m_t(X_0,\ldots,X_t) < Z \circ \theta_t|\mathcal{F}_t)(\omega) = \mathbb{P}^x(m_t(X_0(\omega),\ldots,X_t(\omega)) < Z \circ \theta_t|\mathcal{F}_t)(\omega) = \mathbb{P}^{X_t(\omega)}(m_t(X_0(\omega),\ldots,X_t(\omega)) < Z),
\]

giving \(m_t(\omega) \geq \rho^{X_t(\omega)}(Z)\). We conclude that \(\rho^x_t(Z \circ \theta_t) \geq \rho^{X_t}(Z)\) almost surely under \(\mathbb{P}^x\).

Conversely we have by the Markov property that \(\mathbb{P}^x\text{-a.s.},\)

\[
\mathbb{P}^x(\rho^{X_t}(Z) < Z \circ \theta_t|\mathcal{F}_t)(\omega) = \mathbb{P}^{X_t(\omega)}(\rho^{X_0}(Z) < Z) = \mathbb{P}^{X_t(\omega)}(\rho^{X_t(\omega)}(Z) < Z) \leq \lambda,
\]

and, since \(\omega \mapsto \rho^{X_t(\omega)}(Z)\) is bounded and \(\mathcal{F}_t\text{-measurable, we conclude that} \rho^x_t(Z \circ \theta_t) \leq \rho^{X_t}(Z)\). \(\square\)

### 3.4 Average Value at Risk

For \(\lambda \in (0,1)\) the average value at risk \([25]\) Ex. 1.10] may be defined by the following family of dynamic conditional risk mappings:

\[
\rho^x_t(Z) = \begin{cases} A\text{VaR}^x_t(-Z), & t = 0, \\ A\text{VaR}^x_{\lambda,t}(-Z), & t \geq 1, \end{cases}
\]

where

\[
A\text{VaR}^x_t(-Z) = \mathbb{E}^x \left[ \text{VaR}^x_t(-Z) + \frac{1}{\lambda}(Z - \text{VaR}^x_t(-Z))^+ \right]
\]

and

\[
A\text{VaR}^x_{\lambda,t}(-Z) = \mathbb{E}^x \left[ \text{VaR}^x_t(-Z|\mathcal{F}_t) + \frac{1}{\lambda}(Z - \text{VaR}^x_t(-Z|\mathcal{F}_t))^+ \big| \mathcal{F}_t \right]
\]

for \(t \geq 1\).
Lemma 3.4. The family of dynamic conditional risk mappings given by (18) is Markovian.

Proof. This follows from the Markov property for $\text{VaR}_X^\lambda(x|\mathcal{F}_t)$ (Lemma 3.3), since

$$\text{AVaR}_X^\lambda_t(-Z \circ \theta_t) = \text{VaR}_X^\lambda(-Z) + \mathbb{E}x[\frac{1}{\lambda}(Z \circ \theta_t - \text{VaR}_X^\lambda(-Z))^+]_{|\mathcal{F}_t}$$

$$= \text{VaR}_X^\lambda(-Z) + \mathbb{E}x[\frac{1}{\lambda}(Z - \text{VaR}_X^\lambda(-Z))^+]$$

$$= \text{AVaR}_X^\lambda(-Z).$$

4 Dual representation of convex Markovian risk mappings

In this section we characterise the dual representation of convex Markovian risk mappings. Recalling from Section 2.1 that $(q^X(x|B): B \in \mathcal{E}, x \in E)$ is the kernel associated to the Markov process $X$ under $\mathbb{P}$, we begin with the necessary definitions:

Definition 4.1. (i) $\mathcal{R}: E \times b\mathcal{E} \to \mathbb{R}$ is a transition risk mapping (cf. [1, 2, 3]) if:

- for all $f \in b\mathcal{E}$, $x \mapsto \mathcal{R}(x,f)$ is bounded and measurable,
- for all $x \in E$, $f \mapsto \mathcal{R}(x,f)$ satisfies
  - normalisation: $\mathcal{R}(x,0) = 0$,
  - monotonicity: $\mathcal{R}(x,f) \leq \mathcal{R}(x,g)$ for all $f \leq g$,
  - constant translation invariance: $\mathcal{R}(x,f+c) = \mathcal{R}(x,f) + c$ for all constants $c$.

(ii) A transition risk mapping is convex if for all $x \in E$, $f,g \in b\mathcal{E}$ and $\lambda \in [0,1]$ we have

$$\mathcal{R}(x,\lambda f + (1-\lambda)g) \leq \lambda \mathcal{R}(x,f) + (1-\lambda)\mathcal{R}(x,g).$$

Note that by Definitions 2.3 and 2.4, a transition risk mapping can be derived from a regular collection of risk mappings $(\rho^x)_{x \in E}$ by writing

$$\mathcal{R}(x,f) := \rho^x(f(X_1)) \quad \text{for } f \in b\mathcal{E}. \quad (19)$$

If the transition risk mapping $\mathcal{R}$ defined by (19) is convex and continuous from below it has the following dual representation (cf. [32, Th. 2.3]):

$$\mathcal{R}(x,f) = \sup_{Q \in \mathcal{P}(\mathcal{E}), \ Q \ll P \circ X^{-1}_1} \left( \int_{\mathcal{E}} f(y) Q(dy) - \alpha^x(Q) \right), \quad (20)$$

where the penalty functions $\alpha^x: \mathcal{P}(\mathcal{E}) \to \mathbb{R}$ are defined by

$$\alpha^x(Q) = \sup_{g \in b\mathcal{E}} (\mathbb{E}_Q[g] - \mathcal{R}(x,g)).$$

Letting $\mathcal{K}$ denote the set of kernels $q$ such that $q(\cdot|x)$ is absolutely continuous with respect to $q^X(\cdot|x)$ for every $x \in E$, we have the following proposition:
Proposition 4.2. Let \( \varrho := (\varrho^x)_{x \in E} \) be a family of dynamic conditional risk mappings such that \((\varrho^x)_{x \in E}\) is regular and each risk mapping \(\varrho^x\) is convex and continuous from below. Then \( \varrho \) satisfies the one-step Markov property \((6)\) if and only if for all \( x \in E \) and \( f \in bE \) we have

\[
\varrho^x(f(X_1)) = \sup_{q \in \mathcal{K}} \left( \int_E f(y) q(dy|x) - \alpha^x(q(\cdot|x)) \right),
\]

\[
\varrho^x_t(f(X_{t+1})) = \sup_{q \in \mathcal{K}} \left( \int_E f(y) q(dy|X_t) - \alpha^{X_t}(q(\cdot|X_t)) \right), \quad t = 1, 2, \ldots
\]

Proof. Using kernels, for all \( x \in E \) and \( f \in bE \) the representation \((20)\) can be rewritten as \((21)\). Indeed, it is clear that the right-hand side of \((21)\) is less than or equal to the right-hand side of \((20)\). For the reverse inequality, simply note that for every given \( x \in E \) and every \( Q \in \mathcal{P}(E) \) such that \( Q \ll P^x \circ X^{-1}_0 \), we can associate a kernel \( q \in \mathcal{K} \) by setting \( q(\cdot|x') = Q1\{x\}(x') \). Then Equation \((21)\) implies

\[
\sup_{q \in \mathcal{K}} \left( \int_E f(y) q(dy|X_t) - \alpha^{X_t}(q(\cdot|X_t)) \right) = \varrho^{X_t}(f(X_1)) \]

which shows that \((22)\) is equivalent to the one step Markov property \((6)\).

Note that in \((21)\) we take the supremum (rather than essential supremum) over a potentially uncountable family of kernels. Therefore the regularity of the collection \((\varrho^x)_{x \in E}\) follows from the assumptions of Proposition 4.2 rather than from \((21)\).

Corollary 4.3. Let \( \varrho := (\varrho^x)_{x \in E} \) be a time-consistent family of dynamic conditional risk mappings such that each risk mapping \( \varrho^x \) is convex and continuous from below. Then \( \varrho \) satisfies \((21)-(22)\) (for all \( x \in E \) and \( f \in bE \)) if and only if the Markov property of Definition 2.5 holds.

Proof. Combining Lemma 4.21 and Theorem 4.22 in [24], we see that a convex conditional risk mapping that is continuous from above is also continuous from below (recall the sign difference in our work). The corollary is then an application of Proposition 2.10 to Proposition 4.2.

The following example identifies a maximising kernel in \((22)\) in the case of the entropic risk mapping of Section 3.1.1.

Example 4.4 (Entropic risk). For \( q \in \mathcal{K} \) let

\[
\alpha^x(q(\cdot|x)) = \frac{1}{\gamma(x)} \int_E \ln \frac{dq(\cdot|x)}{dx}(y) q(dy|x). \tag{23}
\]

Fixing a bounded, measurable function \( f : E \to \mathbb{R} \), define the kernel \( q_{\text{op}} \) by

\[
\frac{d q_{\text{op}}(\cdot|x)}{dx}(y) = \frac{e^{\gamma(x)f(y)}}{\int_E e^{\gamma(x)f(z)} q^{X}(dz|x)}. \tag{24}
\]

It is well known [11, Rem. 9] from the non-Markovian setting that for each \( x \in E \) the function \((23)\) is the minimal penalty corresponding to the entropic risk mapping on \( bE \) and that \((24)\) defines a measure attaining the maximum in \((22)\). In order to show that \( q_{\text{op}} \) defined in this way is indeed a kernel we show that for each \( A \in \mathcal{E} \) the function \( x \mapsto q_{\text{op}}(A|x) \) is measurable.

Indeed we have \( q_{\text{op}}(A|x) = \int_E 1_A(y) \int_E e^{\gamma(x)f(y)} q^{X}(dz|x) q^{X}(dy|x) \) and measurability follows since more generally, for any jointly measurable function \( g : E \times E \to \mathbb{R} \) and any kernel \( p \), the function \( x \mapsto \int_E g(x,y) p(dy|x) \) is measurable.
5 Applications

The probabilistic Markov property can provide a convenient tool to address, for example, optimal stopping problems with costs which are measurable only after the chosen stopping time. A first example is the case of exercise lag, where we seek

\[ L^T(x) := \inf_{\tau \in \mathcal{T}_{[0,T]}} \rho^x \left( \sum_{i=0}^{\tau-1} c(X_i) + g(X_{\sigma \theta_t + \tau}) \right), \]

where functions \( c, g : E \rightarrow \mathbb{R} \) and a potentially unbounded stopping time \( \sigma \in \mathcal{T} \) represent respectively an observation cost, exercise cost and exercise lag, and \( \rho = ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E} \) is a time-consistent Markovian family of dynamic risk mappings. The strong Markov property of Proposition 2.8 then allows dynamic programming to be applied indirectly by first transforming the objective function. Indeed it then follows by the recursive property (5), conditional locality, conditional translation invariance, the identity \( X_{\sigma \theta_t + \tau} = X_{\sigma} \circ \theta_t \) and the strong Markov property that

\[
L^T(x) = \inf_{\tau \in \mathcal{T}_{[0,T]}} \rho^x \left( \sum_{i=0}^{T} c(X_i) + g(X_{\sigma \theta_t + \tau}) \right)
= \inf_{\tau \in \mathcal{T}_{[0,T]}} \rho^x \left( \sum_{i=0}^{\tau-1} \rho^x \left( \sum_{i=0}^{T} c(X_i) + 1_{\{\tau=t\}} g(X_{\sigma \theta_t + \tau}) \right) \right)
= \inf_{\tau \in \mathcal{T}_{[0,T]}} \rho^x \left( \sum_{i=0}^{\tau-1} c(X_i) + h(X_{\tau}) \right),
\]

where \( h(x) := \rho^x(g(X_{\sigma})), \) and standard dynamic programming arguments can then be applied to obtain the Wald-Bellman equations

\[
\begin{align*}
L^0(x) &= h(x), \\
L^m(x) &= h(x) \land (c(x) + \rho^x(L^{m-1}(X_1))), \quad m = 1, \ldots, T.
\end{align*}
\]

In the optimal prediction problem of the next section, use of the probabilistic Markov property enables dynamic programming to instead be applied directly.

5.1 Optimal prediction

Generalising (2), let

\[ V^{T}_{\text{pred}}(x) := \inf_{\tau \in \mathcal{T}_{[0,T]}} \rho^x(g(X^*_T - X_{\tau})), \]

where \( x \in E = \mathbb{R}, \Omega = \mathbb{R}^{\mathbb{N}_0}, T \in \mathbb{N}_0, \rho = ((\rho_t^x)_{t \in \mathbb{N}_0})_{x \in E} \) is a Markovian family of dynamic conditional risk mappings, \( X^*_T = \max_{0 \leq s \leq T} X_s \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) is bounded and measurable.

We extend this probability space to include the ‘process’ running maximum by letting \( \hat{\Omega} = (\mathbb{R} \times \mathbb{R})^{\mathbb{N}_0}. \) On this space, we have the canonical process \((X_t(\omega), M_t(\omega)) = (\hat{\omega}^1(t), \hat{\omega}^2(t)) = \hat{\omega}(t)\).

Setting \( \hat{\mathcal{F}} = (\hat{\mathcal{F}}_t)_{t \in \mathbb{N}_0} \) with \( \hat{\mathcal{F}}_t = \sigma(\{(X_s, M_s): s \leq t\}) \) and \( \hat{\mathcal{F}} = \sigma(\cup_t \hat{\mathcal{F}}_t), \) there exists (see, for example, [7, Th. 4.1.18]) a unique probability measure \( \hat{\rho}^{x,m} \) on \((\hat{\Omega}, \hat{\mathcal{F}})\) such that \((X, M)\) is a time-homogeneous Markov chain on \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\rho}^{x,m})\) with \( \hat{\rho}^{x,m}(X_0 = x, M_0 = m) = 1 \) and transition kernel \( q^{X,M} \) satisfying \( q^{X,M}(dx', dm'|x, m) = \delta_{m \vee x}(dm')q^X(dx'|x) \) for all \((x, m) \in \mathbb{R}^2. \) Note that \( \hat{\rho}^{x,m}(M_n = X^*_n \vee m) = 1 \) and, in particular, for \( m = x \) we have \( \hat{\rho}^{x,x}(M_n = X^*_n) = 1. \)

Recalling Proposition 2.7, define a regular collection of risk mappings by

\[ \rho^{x,m}(f(X_0, M_0, X_1, M_1, \ldots)) := \rho^x(f(X_0, m, X_1, X^*_n \vee m, \ldots)), \]

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Theorem 5.1. If \((\rho_t^{x,m})_{t \in \mathbb{N}_0}\) is time consistent then, for each bounded measurable function \(g : \mathbb{R} \to \mathbb{R}\), the extended value function
\[
\tilde{V}^T(x, m) := \inf_{\tau \in \mathcal{F}_{[0,T]}} \rho^{x,m}(g(M_T - X_\tau))
\]
satisfies the following modified Wald-Bellman equations:
\[
\tilde{V}^0(x, m) = g(m - x), \quad \tilde{V}^n(x, m) = \rho^{x,m}(g(M_n - x)) \land \rho^{x,m}(\tilde{V}^{n-1}(X_1, M_1)).
\]

The optimal prediction problem (2) satisfies \(V^m_{\text{pred}}(x) = \tilde{V}^n(x, x)\).

Proof. Set
\[
S^T_k = g(M_T - X_T), \quad S^T_n = \rho^{x,m}(g(M_T - X_n)) \land \rho^{x,m}(S^T_{n+1})..
\]
Following the outline of [33, Sec. 1.2], we may now proceed in five steps:

Step 1. We show that for all \(n = 0, 1, \ldots, T\) and \(k = T - n, T - n - 1, \ldots, 0\), we have
\[
S^{T-n}_{k} \circ \theta_n = S^T_{k+n}.
\]
One can easily see that the claim is true for \(k = T - n\). Further, by the Markov property and backward induction, for all \(n\) the random variable \(S^T_n\) is \(\sigma(M_n, X_n)\)-measurable (cf. Remark 2.13). Secondly, for any \(Z = \tilde{Z} \circ \tilde{b}_k \in \mathcal{B}_{T,\infty}\) we have
\[
\rho^{x,m}(Z) \circ \theta_n = \rho^{x,m}(\tilde{Z} \circ \tilde{b}_k) \circ \theta_n = \rho^{X_k, M_k}(\tilde{Z}) \circ \theta_n = \rho^{X_{k+n}, M_{k+n}}(\tilde{Z})
\]
\[
= \rho^{x,m}(\tilde{Z} \circ \tilde{b}_{k+n}) = \rho^{x,m}(Z \circ \theta_n).
\]
This and the induction hypothesis imply
\[
S^{T-n}_{k} \circ \theta_n = \rho^{x,m}(g(M_{T-n} - X_k)) \circ \theta_n \land \rho^{x,m}(S^T_{k+n}) \circ \theta_n
\]
\[
= \rho^{x,m}(g(M_{T-n} - X_{k+n})) \land \rho^{x,m}(S^T_{k+n}) \circ \theta_n
\]
\[
= \rho^{x,m}(g(M_{T-n} - X_{k+n})) \land \rho^{x,m}(S^T_{k+n}) \circ \theta_n
\]
\[
S^T_{k+n}.
\]

Step 2. Let \(\tau^T_n := \inf\{k = n, \ldots, T : S^T_k = \rho^{x,m}(g(M_T - X_k))\}\). We show that \(\tau^T_n = n + \tau^{T-n}_0 \circ \theta_n\).

Indeed,
\[
\tau^T_n = \inf\{k = n, \ldots, T : S^{T-n}_{k} \circ \theta_n = \rho^{x,m}(g(M_{T-n} - X_{k-n})) \circ \theta_n\}
\]
\[
= n + \inf\{k = 0, \ldots, T - n : S^{T-n}_{k} \circ \theta_n = \rho^{x,m}(g(M_{T-n} - X_{k})) \circ \theta_n\}
\]
\[
= n + \tau^{T-n}_0 \circ \theta_n.
\]

Step 3. We show that for \(n = T, \ldots, 0\) we have
\[
S^T_n = \rho^{x,m}(g(M_T - X_{\tau^T_n}))
\]
Note that on \( \{ \tau_{n-1}^T \geq n \} \) we have \( \tau_{n-1}^T = \tau_n^T \) (by definition of these stopping times). From this and time consistency we have

\[
\rho_{n-1}^{x,m}(g(M_T - X_{\tau_{n-1}^T})) = \mathbb{1}_{\{\tau_{n-1}^T = n\}} \rho_{n-1}^{x,m}(g(M_T - X_{n-1})) + \mathbb{1}_{\{\tau_{n-1}^T \geq n\}} \rho_{n-1}^{x,m}(g(M_T - X_{\tau_n^T})).
\]

By the induction hypothesis

\[
\rho_{n-1}^{x,m}(g(M_T - X_{\tau_{n-1}^T})) = \mathbb{1}_{\{\tau_{n-1}^T = n\}} \rho_{n-1}^{x,m}(g(M_T - X_{n-1})) + \mathbb{1}_{\{\tau_{n-1}^T \geq n\}} \rho_{n-1}^{x,m}(S_n^T). \tag{26}
\]

Note that

\[
S_n^T = \rho_{n-1}^{x,m}(g(M_T - X_{n-1})) \quad \text{on} \quad \{ \tau_{n-1}^T = n-1 \},
\]

\[
S_n^T = \rho_{n-1}^{x,m}(S_n^T) \quad \text{on} \quad \{ \tau_{n-1}^T \geq n \}.
\]

Thus, (26) implies that \( \rho_{n-1}^{x,m}(g(M_T - X_{\tau_{n-1}^T})) = S_{n-1}^T \).

**Step 4.** We prove that

\[
S_n^T = \mathcal{V}_n(X_n, M_n). \tag{27}
\]

We have

\[
S_n^T = \rho_n^{x,m}(g(M_T - X_{\tau_n^T})) = \rho_n^{x,m}(g(M_T - X_{\tau_n^T} + \tau_0 - \tilde{\tau} - \tilde{\vartheta} - \tilde{\theta})) = \rho_n^{x,m}(g(M_T - X_{\tau_n^T} + \theta_n)) = \rho_n^{x,m}(g(M_T - X_{\tau_n^T})). \tag{28}
\]

On the other hand, one can show by induction that for each \( k = T, \ldots, 0 \) and every \( \tau \in \mathcal{F}_{[k,T]} \) we have \( \rho_k^{x,m}(g(M_T - X_{\tau})) \geq S_k^T \). The claim is true for \( k = T \), and we may write

\[
\rho_{k-1}^{x,m}(g(M_T - X_{\tau})) = \mathbb{1}_{\{\tau = k-1\}} \rho_{k-1}^{x,m}(g(M_T - X_{\tau}) + \mathbb{1}_{\{\tau \geq k\}} \rho_{k-1}^{x,m}(S_k^T).
\]

By the induction hypothesis, since \( \tau \vee k \in \mathcal{F}_{[k,T]} \) we have

\[
\rho_{k-1}^{x,m}(g(M_T - X_{\tau})) \geq \mathbb{1}_{\{\tau = k-1\}} \rho_{k-1}^{x,m}(g(M_T - X_{\tau})) + \mathbb{1}_{\{\tau \geq k\}} \rho_{k-1}^{x,m}(S_k^T) \geq \mathbb{1}_{\{\tau = k-1\}} S_{k-1}^T + \mathbb{1}_{\{\tau \geq k\}} S_{k-1}^T = S_{k-1}^T.
\]

In particular for \( k = 0 \) we conclude by Step 3 that for every \( T \in \mathbb{N}_0 \), the stopping time \( \tau_T^0 \) is optimal and \( \mathcal{V}_T(x, m) = \rho^{x,m}(g(M_T - X_{\tau_T^0})) \). Combining this with (28) gives (27).

**Step 5.** We have by the previous step and the Markov property that

\[
\mathcal{V}_n(X_n, M_n) = S_n^T = \rho_n^{x,m}(g(M_T - X_n)) \wedge \rho_n^{x,m}(S_n^T) = \rho_n^{x,m}(g(M_T - X_n)) \wedge \rho_n^{x,m}(\mathcal{V}_{T-n-1}(X_{n+1}, M_{n+1})) = \rho_n^{x,m}(g(M_T - X_n)) \wedge \rho_{X_n,M_n}(\mathcal{V}_{T-n-1}(X_{1}, M_{1})).
\]

Taking \( n = 0 \) we get \( \mathcal{V}_T(x, m) = \rho^{x,m}(g(M_T - x)) \wedge \rho^{x,m}(\mathcal{V}_{T-1}(X_{1}, M_{1})) \), and the result follows by construction.
Monotone Class Theorem

For the reader’s convenience we state the monotone class theorem in the form given in [34, Th. 2.12.9] with helpful additional observations in [35, Th. 7].

Theorem A.1. Let $\mathcal{H}$ be a class of real functions on a set $\Omega$ such that $1 \in \mathcal{H}$ and let $\mathcal{H}_0$ be a subset in $\mathcal{H}$. Then, any of the following conditions yields that $\mathcal{H}$ contains all bounded functions measurable with respect to the $\sigma$-algebra generated by $\mathcal{H}_0$:

(i) $\mathcal{H}$ is a closed linear subspace in the space of all bounded functions on $\Omega$ with the norm $\|f\| := \sup_{\Omega} |f(\omega)|$ such that $\lim_{n \to \infty} f_n \in \mathcal{H}$ for every increasing uniformly bounded sequence of nonnegative functions $f_n \in \mathcal{H}$, and, in addition, $\mathcal{H}_0$ is closed with respect to multiplication (i.e., $fg \in \mathcal{H}_0$ for all functions $f, g \in \mathcal{H}_0$).

(ii) $\mathcal{H}$ is closed with respect to the formation of uniform limits and monotone limits and $\mathcal{H}_0$ is an algebra of functions (i.e., $f + g, cf, fg \in \mathcal{H}_0$ for all $f, g \in \mathcal{H}_0$, $c \in \mathbb{R}$) and $1 \in \mathcal{H}_0$.

(iii) $\mathcal{H}$ is closed with respect to monotone limits and $\mathcal{H}_0$ is a linear space containing $1$ such that $\min(f, g) \in \mathcal{H}_0$ for all $f, g \in \mathcal{H}_0$.

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