DEGENERACY OF THE CHARACTERISTIC VARIETY

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Abstract. The characteristic variety plays an important role in the analysis of the solution space of partial differential equations and exterior differential systems. This article studies the linear span of this variety, measuring its dimension via an integrable extension of the original system. In the PDE case with locally constant characteristic variety, this extension yields a recursive version of Guillemin normal form, inducing a sequence of foliations on integral manifolds.

Contents

1. Context 2
2. Notation 3
3. Main theorems 8
4. Involutivity of the eikonal system 10
5. Guillemin normal form 13
6. Elementary extension 18
7. Prolonged elementary extension 22
8. Discussion 26
References 27

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1. Context

This article investigates the linear span of the characteristic variety of an involutive exterior differential system using established tools of the discipline, such as eikonal systems, Guillemin normal form, and integrable extensions. In particular, we pose the question “what does degeneracy of the characteristic variety tell us about solutions of the exterior differential system?” Despite the increasingly sophisticated application of commutative algebra to the subject, this simple question has apparently been neglected in the body of late 20th-century work on exterior differential systems.\footnote{The most important studies of the characteristic variety are [Gui68], [GQS70], [Gab81], and [Mal], none of which consider degeneracy. The most thorough single overview is Chapter V of the book [BCG+90]; however, this chapter’s Theorem 3.13 incorrectly equates $S^\perp$ and $\langle \Xi \rangle$. This article arose from an attempt to state and prove a correct version of that theorem. The inclusion of that incorrect theorem appears to be a random error of the drafting and editing process: the theorem is not used elsewhere in the book, no justification is provided, and a counterexample appears in the example on Page 276 (Page 235 in the online version). However, the incorrect theorem is foreshadowed in a non-technical comment at the bottom of Page 184 (the middle of Page 159 in the online version). Based on conversations with the living authors in 2013, it appears that the error had gone unreported by other readers. That a wholly incorrect statement persisted for so long in a standard reference is strong evidence that the characteristic variety deserves more careful study.}

Projective varieties are studied over $\mathbb{C}$, and the main theorem can be put in a weak form as

**Corollary.** Suppose an involutive differential ideal $I$ on a manifold $M$ has maximal integral elements of projective dimension $n−1$, complex characteristic variety $\Xi$ of projective dimension $\ell−1$, and projective Cauchy system $S$ of dimension $n−\nu−1$. Let the complex linear space $\langle \Xi \rangle$ have projective dimension $L−1$. Then $0 \leq \ell \leq L \leq \nu \leq n$ and

(i) $0 = \ell$ if and only if $I$ is Frobenius;

(ii) $L = \nu$ if and only if the Guillemin symbol algebras, which are parametrized by $\Xi$, contain no common nilpotent subalgebra (see Main Theorem 3.2);

(iii) $\nu = n$ if and only if $(M, I)$ is free of Cauchy retractions;

(iv) Every ordinary integral manifold is foliated by submanifolds of projective dimension $n−L−1$ defined by $\langle \Xi \rangle = 0$ (see Main Theorem 3.5).

The case $L = \nu = n$ shall be called “elementary,” which corresponds to $\Xi$ being a non-degenerate variety (see Main Theorem 3.1).

A stronger and more precise statement of the results requires significant conceptual ballast, and Section 2 rapidly conveys notations and definitions for various objects associated with an exterior differential system. The terminology here is meant to be familiar and reasonably consistent with [BCG+90], diverging only when necessary for a clearer formulation of results. Experts fluent in this language should jump to the Main Theorems in Section 3 now.
2. Notation

An exterior differential system \((M, \mathcal{I})\) consists of a smooth manifold \(M\) of finite dimension \(m\) and an ideal \(\mathcal{I}\) in the total exterior algebra \(\Omega^\bullet(M)\) such that \(d\mathcal{I} \subseteq \mathcal{I}\) and such that in each degree \(p\), the \(p\)-forms in the ideal, \(\mathcal{I}_p = \mathcal{I} \cap \Omega^p(M)\), form a finitely generated \(C^\infty(M)\)-module. For convenience, we assume that \(\mathcal{I}_0 = 0\). Optionally, we sometimes specify an independence condition as an \(n\)-form \(\omega \in \Omega^n(M)\) that is not allowed to vanish on solutions. The category of exterior differential systems includes all smooth systems of differential equations expressed in local coordinates in jet space.

An integral element of \(\mathcal{I}\) at \(x \in M\) is a linear subspace \(e \subseteq T_x M\) such that \(\varphi|_e = 0\) for all \(\varphi \in \mathcal{I}\). The space of \(n\)-dimensional integral elements is labeled \(\text{Var}_n(\mathcal{I}) \subseteq \text{Gr}_n(TM)\). There is a maximal \(n\) for which \(\text{Var}_n(\mathcal{I})\) is locally non-empty, which is the case of interest. If an independence condition \(\omega\) is specified, we also require \(\omega|_e \neq 0\).

There is an open, dense subset \(\text{Var}_n^o(\mathcal{I}) \subseteq \text{Var}_n(\mathcal{I})\) defined as the smooth subbundle of \(\text{Gr}_n(TM)\) that is cut out by smooth functions. These are the Kähler-ordinary elements. A single connected component of \(\text{Var}_n^o(\mathcal{I})\) is called \(M^{(1)}\) after \(M\) is redefined to be the open set over which \(M^{(1)}\) is a smooth bundle. Let \(s\) denote the dimension of each fiber of the projection \(M^{(1)} \to M\), so \(s = n(m - n) - s\) is the corresponding codimension of \(T_e M^{(1)}\) in \(T_e \text{Gr}_n(T_x M)\). Such a space \(M^{(1)}\) is called the (ordinary) prolongation of \(M\), and it admits a prolonged ideal \(\mathcal{I}^{(1)}\) generated adding the pullback of \(\mathcal{I}\) to the tautological contact system \(J\) on \(\text{Gr}_n(TM)\).

An integral manifold of \(\mathcal{I}\) is an immersion \(\iota : N \to M\) such that \(\iota^*(\varphi) = 0\) for all \(\varphi \in \mathcal{I}\). If an independence condition \(\omega\) is specified, we require that \(\iota^*(\omega) \neq 0\). That is, a maximal integral manifold is a submanifold all of whose tangent spaces are maximal integral elements, so \(\iota^*(TN) \subset \text{Var}_n(\mathcal{I})\). A maximal integral manifold is called ordinary if \(\iota^*(TN) \subset M^{(1)}\), in which case the immersion \(\iota^{(1)} : N \to M^{(1)}\) defined by \(\iota^{(1)} : y \mapsto \iota_*(T_y N) \in M^{(1)}\) is called the prolongation of \(\iota : N \to M\). The prolonged integral manifold \(\iota^{(1)} : N \to M^{(1)}\) is an integral manifold of the prolonged system \((M^{(1)}, \mathcal{I}^{(1)})\). The overall goal is to construct all ordinary integral manifolds of \((M, \mathcal{I})\) through the careful study of the prolongation \(M^{(1)}\).

Given an integral element \(e' \in \text{Var}_{n-1}(TM)\), we consider its space of integral extensions, called the polar space,

\[ H(e') = \{ v : e = e' + \langle v \rangle \in \text{Var}_n(\mathcal{I}) \} \subset TM \]

and the polar equations comprise its annihilator,

\[ H^\perp(e') = \{ e' \cdot \varphi : \varphi \in \mathcal{I}_n \} \subset T^*M. \]

Let \(r(e') = \dim H(e') - \dim e' - 1\), called the polar rank, so \(\text{codim} H^\perp(e') = n + r\). Note that \(r(e') = -1\) means that \(e'\) admits no extensions, and \(r(e') = 0\)
means that \( e' \) admits a unique extension. Suppose that \( e \in M^{(1)} \), so that \( r(e') = 0 \) for an open set of \( e' \in \text{Gr}_{\alpha-1}(e) \). (It cannot be positive on an open set, for then the dimension \( n \) would not be maximal.) As the rank of a system of linear equations, \( r : \mathcal{P}e^* \rightarrow \mathbb{N} \) is lower semi-continuous on \( M^{(1)} \), but it can increase on a Zariski-closed set:

**Definition 2.1.** For any \( e \in M^{(1)} \), the **characteristic variety** of \( e \) is

\[
\Xi_e = \{ \xi \in \mathcal{P}e^* \otimes \mathbb{C} : r(\xi^\perp) > 0 \} \subset \mathcal{P}e^* \otimes \mathbb{C}.
\]

(Throughout, we work with complex varieties unless otherwise noted.) As a projective variety, let \( \dim \Xi_e = \ell - 1 \) and \( \deg \Xi_e = s_\ell \); both are locally constant on \( M^{(1)} \). When \( (M, \mathcal{I}) \) is involutive (which has many equivalent definitions; see [BCG+90]), \( \ell \) is the Cartan integer and \( s_\ell \) is the last non-zero Cartan character. If \( (M, \mathcal{I}) \) is analytic and involutive, then the Cartan–Kähler theorem guarantees integral manifolds parameterized by \( s_\ell \) functions of \( \ell \) variables.

To study \( \Xi_e \) simultaneously for all \( e \in M^{(1)} \) in an invariant manner, recall that the Grassmannian space \( \text{Gr}_n(TM) \) admits a canonical projective bundle \( \gamma \), which has fiber \( \gamma_e = \mathcal{P}e \otimes \mathbb{C} \), and a canonical dual bundle \( \gamma^* \), which has fiber \( \gamma_e^* = \mathcal{P}e^* \otimes \mathbb{C} \). Since \( M^{(1)} \) is a submanifold of \( \text{Gr}_n(TM) \), it admits restricted bundles \( V = \gamma|_{M^{(1)}} \) and \( V^* = \gamma^*|_{M^{(1)}} \) with fibers

\[
V_e = \mathcal{P}e \otimes \mathbb{C}, \quad V_e^* = \mathcal{P}e^* \otimes \mathbb{C}
\]

respectively. Bases of \( V_e^* \) are useful, so let \( \mathcal{F} \) denote the right principal \( \text{PGL}(n) \) bundle over \( M^{(1)} \) whose fiber over \( e \in M^{(1)} \) is \( \mathcal{F}_e = \{ u : V_e \iso \mathbb{P}^{n-1} \} \). That is, a basis \( u^1, \ldots, u^n \) of \( V_e^* \) is an element \( u \) of \( \mathcal{F}_e \). Note that, for any integral manifold \( \iota : N \rightarrow M \) of \( \mathcal{I} \) with prolongation \( \iota^{(1)} : N \rightarrow M^{(1)} \), the pullback bundle \( \iota^{(1)*}\mathcal{F} = \{ u \circ \iota^{(1)} \} \) is the usual (complexified and projectivized) coframe bundle \( \mathcal{F}_N \) over \( N \).

With these canonical bundles in place, \( \Xi \) is a global object over \( M^{(1)} \) when considered as a subvariety of \( V^* \). More precisely, define the **characteristic sheaf** of \( \mathcal{I} \), denoted \( \mathcal{M} \), as the sheaf over \( V^* \) defined by the homogeneous condition that the linear system \( H^\perp(\xi^\perp) \) has submaximal rank at \( \xi \in V_e^* \). The characteristic variety is the support of \( \mathcal{M} \).

Another way to see \( \Xi_e \) is to view \( T_eM^{(1)}_x \) as a subspace of \( e^\perp \otimes e^* \), which is canonically identified with \( T_e \text{Gr}_n(T_xM) \). Specifically, let

\[
W_e = \mathcal{P}e^\perp \otimes \mathbb{C}, \quad A_e = \mathcal{P}T_eM^{(1)}_x \otimes \mathbb{C}.
\]

The space \( A_e \) is called the (complexified and projectivized) **tableau** of \( \mathcal{I} \), and it is defined as the kernel of a linear map \( \sigma \), called the **symbol**:

\[
0 \rightarrow A_e \rightarrow W_e \otimes V_e^* \xrightarrow{\sigma} A_e^\perp \rightarrow 0.
\]
For any hyperplane $\xi^\perp \subset e$, the condition $r(\xi^\perp) > 0$ is equivalent to the condition $\ker \sigma_\xi \neq \emptyset$, where $\sigma_\xi : W_e \rightarrow A_e^\perp$ is the restricted symbol map $\sigma_\xi : z \mapsto \sigma(z \otimes \xi)$. So, the projective variety of rank-one elements of the tableau, $C_e = \{z \otimes \xi \in W_e \otimes V_e^* : \sigma_\xi(z) = 0\}$, is the incidence correspondence for $\ker \sigma$ over $\Xi_e$, as in Figure 1.

Definition 2.6. The Cauchy retractions\(^2\) of $I$ comprise the subspace $g = \{v \in TM : v \not\in \mathcal{I} \subset T \mathcal{I}\} \subset TM$. The ideal generated by $g^\perp$ is the smallest Frobenius ideal containing the algebraic generators of $\mathcal{I}$. (See [Gar67] and Section 6.4 of [IL03].) Let $S_e = P(e \cap g) \otimes \mathbb{C} \subset V_e$. Let $\nu-1$ denote the projective rank of the annihilator subbundle $S^\perp \subset V^*$. Let $\langle \Xi \rangle$ denote the linear subbundle of $V^*$ whose fiber $\langle \Xi \rangle_e$ is the span of $\Xi_e$. Let $L-1 = \dim \langle \Xi \rangle_e$. It is easy to verify that $\langle \Xi \rangle \subset S^\perp$. Permanently reserve the following index ranges, where $1 \leq \ell \leq L \leq \nu \leq n \leq m$:

$$
\begin{align*}
\lambda, \mu &= 1, \ldots, \ell \\
g, \varsigma &= \ell+1, \ldots, n \\
i, j &= 1, \ldots, L \\
\alpha, \beta &= L+1, \ldots, n \\
k, l &= 1, \ldots, \nu \\
a, b &= \nu+1, \ldots, m
\end{align*}
$$

(2.7)

If $(u^k)$ is a basis of $V_e^*$ with dual basis $(u_k)$ for $V_e$ and if $(w_a)$ is a basis of $W_e$, then an element $\pi \in W_e \otimes V_e^*$ may be written as a matrix $\pi = \pi^a_k (w_a \otimes u^k)$, and the symbol relations $0 = \sigma^\tau(\pi^0_k), \tau = 1, \ldots, t$. For a dense, open subset of these bases, all $s$ generators of the subspace $A_e$ appear in the matrix $\pi$ according to the Cartan characters, in the first $s_1$ entries of column 1, the first $s_2$ entries of column 2, and so on up to the first $s_\ell$ entries of column $\ell$. Set $s_\theta = 0$ for $\theta > \ell$. A basis $(u^k)$ of $V_e^*$ is called generic if the sequence $(s_1, s_2, \ldots, s_\ell)$ is lexicographically maximized. A stronger condition is “$u^k \not\in \Xi_e$ for all $k$,” in which case the basis $(u^k)$ of $V_e^*$ is called regular.

2 These are typically called Cauchy characteristics, but because this article focuses on the relation between the characteristics $\xi \in \Xi$ and the retractions-née-characteristics $v \in S$, we hope to avoid confusion through this name change.
Figure 2. A tableau in coordinates, with Cartan characters $s_1 \geq s_2 \geq \cdots \geq s_\ell$. The upper-left shaded entries are independent generators. The lower-right entries depend on them via $\pi^a_i B^{a,\lambda}_{i,b} \pi^b_\lambda$, summed as in (2.8).

For each $e \in M^{(1)}$, the symbol relations can be reduced as a minimal system of equations of the form

$$(2.8) \quad \{0 = \pi^a_k - B^{a,\lambda}_{k,b} \pi^b_\lambda\}_{s_k < a}$$

where $B^{a,\lambda}_{k,b} = 0$ unless $\lambda < k$ and $b \leq s_\lambda$ and $s_k < a$, as discussed in Chapter IV, §5 of [BCG+90]. The symbol relations (2.8) can be used to define an element $^3$ of $\text{End}(W_e) \otimes \text{End}(V^*_e)$:

$$(2.9) \quad \sum_{a \leq s_k} \delta^\lambda_a \delta^b_k (w_a \otimes w^b) \otimes (u^k \otimes u_\lambda) + \sum_{a > s_k} B^{a,\lambda}_{k,b} (w_a \otimes w^b) \otimes (u^k \otimes u_\lambda).$$

Then, for each $\phi \in V^*_e$, there is a homomorphism $B(\phi) : V_e \to \text{End}(W_e)$ defined by (2.9). In Chapter V of [BCG+90], only the second summand of Equation (2.9) is used, and the domain of $B(\phi)$ is restricted to the annihilator of $\{u_\lambda\}$, but the identity part is useful for us in Section 5. The endomorphism $B(\phi)(v) \in \text{End}(W_e)$ is most interesting when restricted to a particular subspace,

$$(2.10) \quad W^1_e(\phi) = \{z \in W_e : z \otimes \phi + J^a_\phi (w_a \otimes w^\phi) \in A_e, \text{ for some } J\}.$$  

$^3$ Despite the complicated indexing, (2.9) is just the dual of (2.8). For example, one often encounters a linear condition like $\langle dy^a - p^a_i dx^i \rangle$, and describes a solution as $\langle \frac{\partial}{\partial x^i} + p^i_a \frac{\partial}{\partial y^a} \rangle$. 
In [Gui68], Guillemin proved that involutivity implies that $B(\varphi)(v)|_{W_1^e(\varphi)}$ is an endomorphism of $W_1^e(\varphi)$, and that these endomorphisms commute for all $v \in V_e$.

The next several definitions are new (or at least, not found in the standard references), but they allow us to formulate the main theorems clearly.

**Definition 2.11.** An exterior differential system $(M, I)$ is called **elementary** if and only if $\langle \Xi \rangle |_{e} = V^*_{e}$ for all $e \in M^{(1)}$.

One can see whether $(M, I)$ is elementary by examining its characteristic sheaf. In the language of commutative algebra, recall that an algebraic ideal admits a saturation ideal, which is the largest ideal defining the same variety. The saturation of an ideal is a basic tool in computational algebraic geometry, using Gröbner bases with tools such as Macaulay2. (See [BM93] and Exercise 5.10 on Page 125 of [Har77].) The same terminology applies to a sheaf such as $M$ with local coordinates parameterizing the fibers of $V^*$. From that perspective, “elementary” means sat$(M)_1 = \emptyset$, so sat$(M)$ contains no linear functions, meaning that $\Xi$ is defined only by higher-degree polynomials. Since sat$(M)_1$ plays an important role, we emphasize and relabel it in Definition 2.12.

**Definition 2.12.** Let $X^1_e$ denote the linear subbundle of $V$ with fiber

$$X^1_e = \langle \Xi \rangle |_{e} = \{ v \in \mathbb{P}e : v \cdot \xi = 0 \ \forall \xi \in \Xi_e \} = (\text{sat } M_e)_1 \subset V_e.$$  

Next, we use $X^1$ to construct a new exterior differential system on $M^{(1)}$. Let $\omega^1, \ldots, \omega^m$ be a frame on $M$, and lift it to give 1-forms $\omega^1, \ldots, \omega^m$ on $M^{(1)}$ via the pull-back of the projection $M^{(1)} \to M$. (We omit writing the pull-back.) Fix a particular element $e \in M^{(1)}$, and suppose that our coframe of $M$ is generic and adapted so that $\{\omega^a\}$ span $e^\perp$.

Recall that the prolonged system $I^{(1)}$ on $M^{(1)}$ takes the form of a restricted contact system:

$$(2.13)\begin{cases}
0 = h^\tau(P), & \forall \tau = 1, \ldots, t \\
0 = \theta^a = \omega^a - P^a_k \omega^k, & \forall a = n+1, \ldots, m
\end{cases}$$

where the $(m-n)n$ numbers $P^a_k$ provide coordinates of nearby elements in $\text{Gr}_n(TM)$ and the $t$ functions $h^\tau$ describe the smooth submanifold $M^{(1)} \subset \text{Gr}_n(TM)$ of dimension $m+s$. Their derivatives $0 = dh^\tau = \frac{\partial h^\tau}{\partial P} dP$ provide the symbol map $\sigma$ defining the tableau.

In a neighborhood of $e$, we may apply the independence condition $\omega = \omega^1 \wedge \cdots \wedge \omega^n$ and write the degree-2 generators of $I^{(1)}$ using the tableau $0 = \sigma(\pi^a_k)$ as

$$(2.14) \quad d\theta^a \equiv \pi^a_k \wedge \omega^k = \pi^a_i \wedge \omega^i + \pi^a_\alpha \wedge \omega^\alpha \mod \{\theta^b\}$$
For each \( i = 1, \ldots, L \), fix \( \xi^i \in \Xi_e \) and extend it to a local section of \( \Xi \) such that \( \{ \xi^i \} \) forms a basis of \( \langle \Xi \rangle \) in a neighborhood of \( e \). Because the coframe \( \omega^k \) is generic, it must be that \( \xi^i = H^i_{\beta} \omega^\beta + K^i_{\beta} \omega^\beta \) for some invertible \( L \times L \) matrix \( H \). Apply a change of coframe to \( M^{(1)} \) depending on \( e \) so that \( H^i_{\beta} \omega^\beta \mapsto \omega^i \). It can be arranged that the resulting coframe is still generic. (A particular method of changing the coframe this way is the linear projection described in Section 5.) Re-label \( P, K, \) and \( \pi \) using this new coframe. Near any \( e \in M^{(1)} \), consider the system

\[
\begin{align*}
0 &= h^\tau(P), & \forall \tau = 1, \ldots, t \\
0 &= \theta^a = \omega^a - \left( P^a_{\beta} - P^a_{i} K^i_{\beta} \right) \omega^\beta, & \forall a = n+1, \ldots, m \\
0 &= \xi^i = K^i_{\beta} \omega^\beta, & \forall i = 1, \ldots, L
\end{align*}
\]

Therefore, using the coframe \( (\xi^i, \omega^\alpha, \theta^a, \cdots) \) on \( M^{(1)} \) and the symbol \( \sigma(\pi^a_k) = 0 \), the derivatives of system (2.15) take the form

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{d\theta^a}{d\xi^i} \equiv \left( \pi^a_{\alpha} - \pi^a_{i} K^i_{\alpha} \right) \land \omega^\alpha, & \text{mod } \{ \theta^b, \xi^j \} \\
\frac{d\xi^i}{d\xi^i} \equiv K^i_{\beta} \land \omega^\alpha, & \text{mod } \{ \theta^b, \xi^j \}
\end{array}
\right.
\end{align*}
\]

**Definition 2.17.** Let \( \text{elem}(\mathcal{I}) \) denote the linear Pfaffian system defined locally on \( M^{(1)} \) that is generated by Equations (2.15) and (2.16) with independence condition \( \omega^{L+1} \land \cdots \land \omega^n \neq 0 \).

Note that this system is generally not well-defined on \( M \) because the coefficients \( K^i_{\beta} \) vary with \( e \in M^{(1)} \). The system \( \text{elem}(\mathcal{I}) \) is said to descend to \( M \) if all vertical vector fields (the kernel of \( TM^{(1)} \to TM \)) are Cauchy retraction of \( \text{elem}(\mathcal{I}) \). Moreover, the system \( \text{elem}(\mathcal{I}) \) must be defined on the complexification of \( M^{(1)} \), since \( \Xi \) is a complex variety.

Let \( \text{elem}^0(\mathcal{I}) = \mathcal{I} \), and recursively define \( \text{elem}^k(\mathcal{I}) = \text{elem}(\text{elem}^{k-1}(\mathcal{I})) \).

We can now state the main theorems.

### 3. Main Theorems

**Main Theorem 3.1.** Let \( (M, \mathcal{I}) \) be an involutive exterior differential system with no Cauchy retractions. The following are equivalent:

(i) The ideal \( \mathcal{I} \) is elementary, meaning \( \langle \Xi \rangle_e = V_e^* \) for all \( e \in M^{(1)} \);

(ii) \( (\text{sat } M)_1 = \emptyset \);

(iii) The system \( \text{elem}(\mathcal{I}) \) on \( M^{(1)} \) is Frobenius (in particular, irrelevant);

(iv) The system \( \text{elem}(\mathcal{I}) \) on \( M^{(1)} \) descends to \( M \).

(v) If the Guillemin symbol endomorphism \( B(\varphi)(v)|_{W^1(\varphi)} \) is nilpotent for all \( \varphi \), then \( B(\varphi)(v) = 0 \).

**Main Theorem 3.2.** Let \( (M, \mathcal{I}) \) be an involutive exterior differential system. The following are equivalent:

(i) \( \langle \Xi \rangle_e = S^\perp_e \) for all \( e \in M^{(1)} \);
(ii) \((\text{sat } \mathcal{M})_1 = S\);

(iii) The system \(\text{elem}(\mathcal{I})\) on \(M^{(1)}\) is Frobenius;

(iv) The system \(\text{elem}(\mathcal{I})\) on \(M^{(1)}\) descends to \(M\);

(v) If the Guillemin symbol endomorphism \(B(\varphi)(v)|_{W^1(\varphi)}\) is nilpotent for all \(\varphi\), then \(B(\varphi)(v) = 0\).

It is interesting that statements (iii), (iv), and (v) ignore Cauchy retractions entirely. This suggests that they may be useful when studying “intrinsic” equivalence of Lie pseudogroups in the sense of mutual coverings and Bäcklund transformations. The intrinsic nature of statement (v) is not very surprising, but the intrinsic nature of statement (iii) suggests a new invariant of \((M, \mathcal{I})\), which is the subject of the next corollary.

**Corollary 3.3.** For any exterior differential system \((M, \mathcal{I})\), there exists some \(\varepsilon \leq n\) such that the ideal \(\text{elem}^{\varepsilon}(\mathcal{I})\) is Frobenius. The minimum such \(\varepsilon\) is called the elementary depth of \(\mathcal{I}\). Moreover, for any \(e \in M^{(1)}\), there is a flag

\begin{equation}
V_e = X_e^0 \supset X_e^1 \supset X_e^2 \supset \cdots \supset X_e^\varepsilon = S_e
\end{equation}

where \((X_e^k)^\perp\) is the span of the characteristic variety of \(\text{elem}^{k-1}(\mathcal{I})\).

In the case that \(\mathcal{I}\) is already Frobenius, \(\varepsilon = 0\), for Frobenius ideals are identical to their prolongation and have no characteristic variety, so \((M, \mathcal{I})\) Frobenius trivially implies \(\text{elem}^1(\mathcal{I}) = \mathcal{I}^{(1)} = \mathcal{I} = \text{elem}^0(\mathcal{I})\) is Frobenius.

The elementary system may be pulled back to maximal ordinary integral manifolds, and there it is Frobenius, as given by Main Theorem 3.5.

**Main Theorem 3.5.** Suppose that \((M, \mathcal{I})\) is an involutive exterior differential system. For every maximal ordinary integral manifold \(\iota : N \to M\) and every \(y \in N\), there are unique submanifolds \(\Lambda \subset D \subset N\) such that \(T_y \Lambda = S_N\) and \(T_y D = X_N^1\). That is, every ordinary integral element \(\iota^{(1)} : N \to M^{(1)}\) is locally foliated by manifolds \(D\) integral to \(\text{elem}(\mathcal{I})\), and each such \(D \subset N\) is foliated by manifolds \(\Lambda\) integral to \(g^\perp\).

The qualifier “locally” is required in Main Theorem 3.5 because the eikonal system does not guarantee global solutions. Main Theorem 3.5 does not imply that \(\text{elem}(\mathcal{I})\) is Frobenius as an ideal on \(M^{(1)}\), nor does it even imply that \(\text{elem}(\mathcal{I})\) is involutive. At most, it yields Corollary 3.6.

**Corollary 3.6.** If \((M, \mathcal{I})\) is an analytic involutive exterior differential system, then some prolongation of \(\text{elem}(\mathcal{I})\) over \(M^{(1)} \otimes \mathbb{C}\) is involutive.

The strongest possible version of Corollary 3.6 would be the following conjecture.

**Conjecture 3.7.** Suppose that \((M, \mathcal{I})\) is an analytic involutive exterior differential system, considered over \(\mathbb{C}\). Then \(\text{elem}(\mathcal{I})\) is involutive on \(M^{(1)}\), and the integral manifold \(D\) from Main Theorem 3.5 is ordinary.
As seen in Section 6, this conjecture holds in the case that the involutive exterior differential system \((M^{(1)}, \mathcal{I}^{(1)})\) represents a PDE in local jet-space coordinates such that the span of the characteristic variety is locally constant. A general proof of Conjecture 3.7 eludes the author in light of significant technical obstacles discussed in Section 7, but it would imply a beautifully recursive version of Main Theorem 3.5.

Main Theorem 3.8. Suppose that Conjecture 3.7 holds. If \((M, \mathcal{I})\) is an analytic involutive exterior differential system over \(\mathbb{C}\), then every ordinary integral manifold \(N\) of \((M, \mathcal{I})\) is foliated locally by submanifolds \(N ⊃ D^1 ⊃ D^2 ⊃ \cdots ⊃ D^d = \Lambda\) where \(TD^k = X^k\).

Moreover, each \(X^k\) admits a decomposition \(X^k = U^{k+1} \oplus Y^{k+1} \oplus X^{k+1}\) where the characteristic variety of \(\text{elem}^k(\mathcal{I})\) spans \((U^{k+1} \oplus Y^{k+1})^*\) and admits a finite branched cover over \((U^{k+1})^*\).

When it holds, Main Theorem 3.8 can be seen as a recursive version of Guillemin normal form, in the sense that the Guillemin symbols of \(\text{elem}^k(\mathcal{I})\) form commutative algebras on \((Y^{k+1} + X^{k+1})\) in the usual way (see Theorem 5.12).

One other important case does not require any recursion.

Corollary 3.9. Suppose that \((M, \mathcal{I})\) is involutive and \(\ell = n - 1\). (For example, if it is determined.) Then exactly one of the following must hold:

(i) \(\ell = L = \nu < n\), in which case \((M, \mathcal{I})\) admits Cauchy retractions to an elementary involutive system in dimension \(n - 1\);

(ii) \(\ell = L < \nu = n\), in which case each maximal ordinary integral manifold locally admits a foliation by curves annihilated by \(\Xi|_N\);

(iii) \(\ell < L = \nu = n\), in which case each maximal ordinary integral manifold locally admits a complete system of characteristic coordinates.

The remainder of this article proves these theorems (and a few others) in a piecemeal manner, first using the eikonal system in Section 4 to guarantee that bases adapted to \(\langle \Xi \rangle\) can be extended to frames on \(N\), then adapting Guillemin normal form in Section 5 to express \(X^1\) in terms of the symbol, and finally exploring the integrable extension \(\text{elem}(\mathcal{I})\) in Section 6.

4 Recall that an exterior differential system is called determined if \(\dim A^\perp = \dim W_e\) and \(\Xi \neq V^*_e\), equivalently if \(\ell = n - 1\) and \(s_1 = s_2 = \cdots = s_{n-1} = \dim W_e\), as discussed in Section 1.4 of [Yan87].
\[ \Sigma_N = \Sigma_{\iota_*(TN)}, \] which may be considered via the immersion \( \iota \) as a projective sub-variety of \( T^*N \).

Now, \( T^*N \times \mathbb{R} \) is identical to the jet space \( J^1(N, \mathbb{R}) \) and carries a canonical contact 1-form \( \Upsilon \) that may be expressed in local jet coordinates \((y^1, \ldots, y^n, z, p_1, \ldots, p_n)\) as \( \Upsilon = dz - p_k dy^k \). Let \( \psi : \Sigma_N \to T^*N \) denote the inclusion defining \( \Sigma_N \). Since each fiber is a projective variety, \( \Sigma_N \) is defined locally by functions \( F^\lambda(y, p) \) that are homogeneous polynomials in \( p \). The \textit{eikonal system} of \( \Sigma \), denoted by \( \mathcal{E}(\Sigma_N) \), is the Pfaffian system on \( \Sigma_N \times \mathbb{R} \) that is differentially generated by \( \psi^*(\Upsilon) \) with independence condition \( dy^1 \land \cdots \land dy^n \).

The purpose of the eikonal system is to obtain specific results of the following form:

**Lemma 4.1.** Suppose that \((M, \mathcal{I})\) is involutive, that \( \Sigma \subset V^* \) is a projective variety, that \( \iota : N \to M \) is an ordinary integral manifold of \((M, \mathcal{I})\), and that the eikonal system \((\Sigma_N, \mathcal{E}(\Sigma_N))\) is involutive. Then, for any \( \xi_0 \) in the fiber \( \Sigma_{N,y} \) over \( y \), there is at least one hypersurface \( H \subset N \) such that \( (T_yH)^\perp = \ker \xi_0 \) and such that \( (T_zH)^\perp \in \Sigma_{N,z} \) for all \( z \in H \). Moreover, such hypersurfaces are parameterized according to the Cartan characters of \( \mathcal{E}(\Sigma_N) \).

For various projective varieties \( \Sigma \) that one might choose to study, establishing the involutivity of \( \mathcal{E}(\Sigma_N) \) may be of wildly varying difficulty. In the case \( \Sigma = S^1 \), the theorem is nearly trivial:

**Theorem 4.2.** For any ordinary integral manifold \( N \), the eikonal system of restricted Cauchy retractions, \( \mathcal{E}(S^1_N) \), is involutive with Cartan characters \( s_1 = s_2 = \cdots = s_\nu = 1 \).

**Proof.** The Cauchy retractions \( S \subset TM \) are closed under bracket, so they form an integrable distribution. That is, \( g^\perp \subset T^*M \) is a Frobenius system on \( M \). Therefore, for any integral manifold \( \iota : N \to M \) of \((M, \mathcal{I})\), we have that \( S^1_N = \iota^*(g^\perp) \) is a Frobenius system as well. Therefore, we may choose coordinates \((y^1, \ldots, y^n)\) on \( N \) such that \( S^1_N \subset T^*N \) is the span of \( dy^1, dy^2, \ldots, dy^n \). In other words, \( \varphi = p_k dy^k \) is in \( S^1_N \) if and only of \( p_\nu+1 = \cdots = p_n = 0 \), so \( S^1_N \) is defined by these \( n - \nu \) functions, and \( TS^1_N \) is defined by \( dp_{\nu+1} = \cdots = dp_n = 0 \). Therefore, the eikonal system has generating 2-form

\[
\psi^*(d\Upsilon) = -dp_1 \land dy^1 - \cdots - dp_\nu \land dy^n.
\]

This is involutive with Cartan characters \( s_1 = s_2 = \cdots = s_\nu = 1 \). \( \square \)

In the case \( \Sigma = \Xi \), the theorem is very deep and difficult. It is known as “the integrability of characteristics,” as summarized in Theorem 4.4.

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5Properly, it ought to be called the \textit{involutivity} of characteristics, since the characteristic hypersurfaces are unique only in the case that \( \mathcal{I} \) has Cartan integer \( \ell = 1 \).
**Theorem 4.4** (Guillemin–Quillen–Sternberg, Gabber). *For any ordinary integral manifold $N$ of an involutive exterior differential system $(M,I)$, the eikonal system of the characteristic variety, $\mathcal{E}(\Xi_N)$, is involutive. At smooth points in $\Xi \times \mathbb{R}$, the Cartan characters are $s_1 = s_2 = \cdots = s_L = 1$.*

Cartan showed many examples of Theorem 4.4 in [Car11] and probably at a 1911 lecture “Sur les caractéristiques de certains systèmes d’équations aux dérivées partielles” whose abstract appears immediately after [Car11] in Volume 2 of his collected works. The first complete proof in the PDE case appears in [GQS70], and a general algebraic proof appears in [Gab81]. Reexaminations of these proofs appear in [Mal] and Chapter V of [BCG+90].

For our present purposes, we are concerned with the case of $\mathcal{E}(\langle \Xi \rangle_N)$, which one expects to lie neatly between the easy case of $\mathcal{E}(S^\perp N)$ and the difficult case of $\mathcal{E}(\Xi_N)$. We avoid proving involutivity from scratch, instead using the difficult case as a crutch, with the following lemma.

**Lemma 4.5.** If $\mathcal{E}(\Sigma_N)$ is involutive, then $\mathcal{E}(\langle \Sigma \rangle_N)$ is involutive.

**Proof.** Since we are only concerned with the case $\Sigma = \Xi$, we use notation consistent with Section 1, but no peculiar properties of the characteristic variety are used. Let $\Sigma_e$, $\langle \Sigma \rangle_e$, and $N$ have dimension $\ell$, $L$, and $n$ respectively, and recall the index ranges reserved in Equation (2.7).

Since $\dim \langle \Sigma \rangle_{N,y} = L$, we may choose linearly independent $\xi_0^i, \ldots, \xi_L^i \in \Sigma_y$ and also (because $\mathcal{E}(\Sigma_N)$ is involutive) local extensions $\xi_0^i, \ldots, \xi_L^i \in \Sigma_N$ such that $d\xi^i \equiv 0$ mod $\xi^i$ for each $i = 1, \ldots, L$. That is, we choose $L$ linearly independent characteristic hypersurfaces defined by local functions $y^i : N \rightarrow \mathbb{R}$ such that $d\xi^i = \xi^i$. Complete $(y^1, \ldots, y^L)$ to a local coordinate system $(y^1, \ldots, y^n)$ on $N$, and let $(p_1, \ldots, p_n)$ be the corresponding symplectic coordinates on $T^*_y N$. Note that completing the coordinate system is possible because $\mathcal{E}(\mathbb{P}T^*N)$ is itself trivially involutive.

In our chosen coordinates, $\langle \Sigma \rangle_{N,y}$ is merely the subspace of $T^*_y N$ defined by the $n-L$ functions $0 = p_\alpha$, so $T \langle \Sigma \rangle_N$ is defined by $dp_\alpha = 0$. Therefore, the eikonal system of $\langle \Sigma \rangle_N$ has generating 2-form

\[
d\Upsilon = -dp_i \wedge dy^i - dp_\alpha \wedge dy^\alpha
\equiv -dp_i \wedge dy^i \mod \{dp_\alpha\}
\]

(4.6)

This is involutive with Cartan characters $s_1 = s_2 = \cdots = s_L = 1$. \hfill \Box

Using Lemma 4.1 for $\Xi_N$, $\langle \Xi \rangle_N$, and $S^\perp_N$ sequentially to build a full coordinate system, we obtain:

**Corollary 4.7.** Suppose that $(M,I)$ is an involutive exterior differential system, and that $N$ is a maximal ordinary integral manifold. Then $N$ admits a coordinate system $(y^1, \ldots, y^n)$ such that $dy^1, \ldots, dy^\ell \in \Xi_N$, such that $dy^1, \ldots, dy^L \in \langle \Xi \rangle_N$ and such that $dy^1, \ldots, dy^n \in S^\perp_N$. For generic smooth
points in $\Xi_N$, the choice of such coordinates depends on $\ell$ functions of $\ell$ variables, $L - \ell$ functions of $L$ variables, $\nu - L$ functions of $\nu$ variables, and $n - \nu$ functions of $n$ variables.

5. Guillemin normal form

Guillemin normal form of the tableau $A$ plays an essential role in the proofs of the main theorem. A comment on our approach: The literature contains two notable versions of Guillemin normal form. The first, seen in [Gui68, GQS70] and discussed Chapter VIII §6 of [BCG+90], is essentially coordinate-free and implies commutativity of the symbol maps on certain non-characteristic subspaces of $V$ using a linear projection of the characteristic variety. The second is the iterative method described in Section 1.1 of [Yan87], which explicitly uses a chosen coframe of $V$, but allows one to state a commutativity condition on both characteristic and non-characteristic subspaces. To state the results most elegantly, and to identify some subtleties, we use the mixture of these two perspectives that is developed in [Smi14]. See that article for further discussion of the lemmas in this section.

For any $e \in M^{(1)}$, consider the projective space $V_e^* = \mathbb{P}e^*$ of dimension $n-1$. Let $X_e^* \subset V_e^*$ be a linear subspace of dimension $n-L-1$ such that $\langle \Xi \rangle_e \cap X_e^* = \emptyset$. Similarly, we may choose a linear subspace $Y_e^* \subset \langle \Xi \rangle_e$ of dimension $L - \ell - 1$ such that $Y_e^* \cap \Xi_e = \emptyset$. If $L = \ell$, we allow $Y_e^* = \emptyset$. Let $U_e^* \subset \langle \Xi \rangle_e$ be a linear subspace of dimension $\ell - 1$ such that $U_e^* \cap Y_e^* = \emptyset$. So, $V_e^*$ decomposes$^6$ as $U_e^* \oplus Y_e^* \oplus X_e^*$. The notation is meant to be suggestive, as equating $X_e^* \cong (X_e^1)^*$ is equivalent to splitting the exact sequence $\emptyset \to \langle \Xi \rangle_e \to V_e^* \to X_e^* \to \emptyset$ for $X_e^1 \equiv (\Xi_e^1)^*$ as in Definition 2.12.

Let the covectors $u^i, \ldots, u^L$ be a basis for $U_e^*$, let $u^{L+1}, \ldots, u^n$ be a basis for $Y_e^*$, and let $u^{L+1}, \ldots, u^n$ be a basis for $X_e^*$, so any $\phi \in V_e^*$ can be decomposed as

$$\phi = \phi_k u^k + \phi_\lambda u^\lambda + \phi_i u^i + \phi_\alpha u^\alpha$$

using the index ranges reserved in Equation (2.7). Let $(u_k)$ denote the basis of $V_e$ dual to $(u^k)$, so $u_k(\phi) = \phi_k$ and $u^k(v) = v^k$ for any $v = v^k u_k \in V_e$.

If $\ell < L$, then $\Xi_e \neq \langle \Xi \rangle_e$, and one may further assume that $\Xi_e \cap U_e^* = \emptyset$ and $\Xi_e \cap Y_e^* = \emptyset$, in which case this basis is also regular, meaning $u^k \not\in \Xi_e$ for all $k$. However, if $\ell = L$, then $\Xi_e = \langle \Xi \rangle_e$, as sets, so $Y_e^* = \emptyset$ and $E_e^* = \langle \Xi \rangle_e = \Xi_e$. It is therefore impossible for this basis to be regular. There are two ways of proceeding: either perturb $U_e^*$ by a small angle to be non-characteristic, or

$^6$Here, we are using $Y_e^* \oplus X_e^*$ as a particularly nice example of a maximal non-intersecting subspace, which would be called $\Omega$ on Page 379 (Page 324 in the online edition) of [BCG+90]. While the particular choice of $X_e^*, Y_e^*$ and $U_e^*$ is not canonical, the desired lemmas hold for any such decomposition. One could express the linear projections in a completely invariant manner using additional language from commutative algebra, but the notation of bases and frames is useful for us.
take care that the desired lemmas require genericity but not regularity. We take the latter approach.

For a dense open subset of the bases \((w_a)\) of \(W_e\), the generators of \(A_e\) appear in the first \(s_1\) entries of column 1, the first \(s_2\) entries of columns 2, etc., of the matrix \(\pi = \pi_k^a(w_a \otimes u^k)\), so the symbol relations take the form of Equation (2.8). Recall that the symbol coefficients define a map

\[(2.9 \text{ bis}) \quad B(\varphi)(v) : z \mapsto \sum_{a \leq s_k} w_a \delta_k^a \delta_b^k z^b v^k \varphi_{a} + \sum_{a > s_k} w_a B_{a,b}^{\lambda} z^b v^k \varphi_{a}.
\]

**Lemma 5.2.** If \(\xi \in \Xi_e\), \(v \in V_e\), and \(z \in \ker \sigma_\xi \subseteq W_e\), then

\[(5.3) \quad B(\xi)(v)z = \xi(v)z.
\]

Despite the neatness of Lemma 5.2, we do not really want to deal with \(\Xi_e\) directly; rather, it is better to deal with \(U_e^* \cong \mathbb{P}^{\ell-1}\), noting that the linear projection \(\Xi_e \rightarrow U_e^*\) is a finite branched cover. Thus, every \(\varphi \in U_e^*\) represents some finite number of corresponding \(\xi \in \Xi_e\).

For each basis element \(u^k\) of \(V_e^*\), let

\[(5.4) \quad W_e^-(u^k) = \left\{ z = w_a z^a : z^a = 0 \ \forall \ a \leq s_k \right\} \quad \text{and} \quad W_e^+(u^k) = \left\{ z = w_a z^a : z^a = 0 \ \forall \ a > s_k \right\}.
\]

So that \(W_e = W_e^-(u^k) \oplus W_e^+(u^k)\) and \(W_e^-(u^l) \supset W_e^-(u^2) \supset \cdots \supset W_e^-(u^n)\) because \(s_1 \geq s_2 \cdots \geq s_n\). Of course, for \(q > \ell\), we have \(W_e^-(u^q) = \emptyset\). For
each \( \lambda \), consider also the subspace
\[
A_e^- (u^\lambda) = \left\{ \pi = B(u^\lambda)(\cdot)z, \ z \in W_e^- (u^\lambda) \right\} \subset A_e
\]
The symbol relations (2.8) imply that the coefficients \( \pi^a_{\lambda} \) of \( \pi \in A_e^- (u^\lambda) \) are
determined uniquely by the choice of \( z \in W_e^- (u^\lambda) \), so \( A_e^- (u^\lambda) \) and \( W_e^- (u^\lambda) \)
are isomorphic via the projection onto the \( u^\lambda \) column.

Using this basis and isomorphism, there is a decomposition
\[
A_e = \bigoplus_{\lambda=1}^\ell A_e^- (u^\lambda) \cong \bigoplus_{\lambda=1}^\ell W_e^- (u^\lambda).
\]
Specifically, if \( \pi = \pi^a_{\lambda} (w_a \otimes u^k) \in A_e \), then let
\[
z_\lambda = \sum_a z^a_{\lambda} w_a \in W, \text{ for } z^a_{\lambda} = \begin{cases} \pi^a_{\lambda}, & a \leq s_\lambda \& \lambda \leq \ell \\ 0, & \text{otherwise} \end{cases}
\]
So, the decomposition (5.6) yields
\[
\pi = \sum_{\lambda} \pi_\lambda = \sum_{\lambda} B(u^\lambda)(\cdot)z_\lambda.
\]

Since \( \dim W_e^- (u^\lambda) = s_\lambda \), this is a more precise version of the statement that,
for a generic flag, the tableau matrix has \( s_1 \) generators in the first column, \( s_2 \)
in the second column, and so on until the final \( s_\ell \) generators in the \( \ell \) column.

The complete linear and quadratic conditions of involutivity are provided
by Theorem 5.9, which is an adaptation of the construction described in
Chapter 1 of [Yan87] and thus a re-expression of Guillemin normal form.
Compare it to Theorem 7.1 in [Gui68].

**Theorem 5.9 (Involutivity Criteria).** Let \( A \) denote an tableau given in a
generic basis of \( V^* \) by with symbol relations (2.8), as in Figure 2. Write \( B^\lambda_k \)
for \( B(u^\lambda)(u_k) \). The tableau \( A \) is involutive if and only if there exists a basis
of \( W \) such that
\begin{enumerate}
  \item \( B^a_{k,b} = 0 \) for all \( a > s_\lambda \);
  \item \( (B^\lambda_k B^\mu_l - B^\mu_l B^\lambda_k)^a_b = 0 \) for all \( b \), \( \lambda < l < k \) and \( \lambda \leq \mu < k \), and
    all \( a > s_l \).
\end{enumerate}
In particular, \( B(u^\lambda)(v) \) is an endomorphism of \( W^- (u^\lambda) \) such that for all \( v, \tilde{v} \in (U^*)^\perp \),
\[
[B(u^\lambda)(v), B(u^\lambda)(\tilde{v})] = 0.
\]

Because its notational intricacies are useless for the Main Theorems here,
we remove the discussion of Theorem 5.9 to a separate article, [Smi14]. Also,
compare Corollary 5.10 to Theorem A in [Gui68].
Corollary 5.10 (Guillemin). If $A$ is involutive, then $A|_U$ (the projection of $A$ to $W \otimes U^*$) is involutive.

The map $B(\varphi)$ makes sense for any $\varphi \in U^*_e$, not just the basis elements, and the spaces $W^-(u^\lambda)$ can be generalized for any $\varphi \in U^*_e$ in the following way: Let $\varphi = \varphi_{\lambda} u^\lambda$, and let $\lambda = \min\{\lambda : \varphi_{\lambda} \neq 0\}$. Define the space

$$W^-(\varphi) = W^-(u^\lambda).$$

Then condition (i) of Theorem 5.9 reveals $B(\varphi) = \sum_\lambda \varphi_{\lambda} B(u^\lambda)$, so involutivity implies that $B(\varphi)(v)$ is an endomorphism of $W^-(\varphi)$; however, the commutativity property is more subtle because of the ordering of condition (ii).

Recall the space $W^1_e(\varphi)$ from (2.10) studied by Guillemin. The spaces $W^-(\varphi)$ and $W^1_e(\varphi)$ have the following relationship.

**Lemma 5.11.** For any $\varphi \in U^*_e$,

$$W^1_e(\varphi) = \left\{ z \in W^-(\varphi) : \left( \sum_\lambda \varphi_{\lambda} B_{\mu} z \right)^b = \varphi_{\mu} z^b, \forall a > s_{\mu}, \forall \mu \leq \ell \right\}.$$

**Theorem 5.12** (Guillemin). For every $\varphi \in U^*_e$ and $v \in V_e$, the restricted homomorphism $B(\varphi)(v)|_{W^1_e(\varphi)}$ is an endomorphism of $W^1_e(\varphi)$, with

$$B(\varphi)(v)z = (\varphi_{\lambda} v^\lambda)z + (J^a_{\varphi} v^a)w_a = \varphi(v)z + J(v) = \pi v,$$

where $\pi = B(u^\lambda)(\cdot)z$. Moreover, for all $v, \tilde{v} \in V_e$,

$$[B(\varphi)(v), B(\varphi)(\tilde{v})]|_{W^1_e(\varphi)} = 0.$$

One important distinction between Theorems 5.9 and 5.12 is the space $W^-(u^\lambda)$ versus $W^1_e(u^\lambda)$. Note also that the usual statement of this theorem, as in Proposition 6.3 in Chapter VIII of [BCG+90] and Lemma 4.1 [Gui68] restricts $v, \tilde{v}$ to the subspace $(U^*_e)^\perp \cong Y_e \oplus X_e$, but this is unnecessary because of our inclusion of the identity term in (2.9).

Theorems 5.9 and 5.12 allow a converse of Lemma 5.2 in the form of Corollary 5.15.

**Corollary 5.15.** Suppose that $(M, \mathcal{I})$ is an involutive exterior differential system. Fix $\varphi \in U^*_e$ and suppose that $z \in W^-\varphi(\varphi)$ such that $z$ is an eigenvector of $B(\varphi)(v)$ for every $v \in V_e$. Then there is an $\xi \in \Xi_e$ over $\varphi \in U^*_e$ such that $z \in W^1_e(\varphi)$, so $z \otimes \xi \in A_e$.

Corollary 5.15 is a bit more subtle than it might first appear. It is similar to the construction in the usual proof of Theorem 5.12, but that construction requires $z \in W^1_e(\varphi)$ a priori. The key is Lemma 5.11. See [Smi14] for details.
Corollary 5.15 deserves a warning: The specification of $\xi$ over $\varphi$ is not unique, as the variety $\Xi$ may have multiple components and multiplicity.

**Lemma 5.16.** Suppose that $(M, \mathcal{I})$ is an involutive exterior differential system with a coframe $u$ on $V$ as described above. For any $v \in V_e$, the following are equivalent:

(i) $v \in X_e^1$;
(ii) $v^i = u^i(v) = 0$ for all $i = 1, \ldots, L$;
(iii) $B(\varphi)(v)|_{W^1_e(\varphi)}$ is nilpotent (possibly trivial) for all $\varphi \in U_e^*$;

*Proof.* Recall that $X_e^1 = \langle \Xi \rangle_e^1 = (U_e^* \oplus Y_e^*)^\perp$. The equivalence of statements (i) and (ii) is immediate in our chosen basis for $V_e^*$.

Fix $v \in X_e^1$, and suppose that $\zeta_\varphi(v)$ is an eigenvalue of $B(\varphi)(v)|_{W^1_e(\varphi)}$ for some $\varphi \in U_e^*$. The commutativity property of Theorem 5.12 holds, so the eigenspace of $\zeta_\varphi(v)$ contains an eigenvector $z$ that is shared among the sets $\{B(\varphi)(v)|_{W^1_e(\varphi)} : \tilde{v} \in V_e\}$. Therefore, Equation (5.3) holds, and $\zeta_\varphi(v)z = \xi(v)z$. By the assumption that $v \in X_e^1 = \langle \Xi \rangle_e^1$, we have $\xi(v) = 0$, so the corresponding eigenvalue $\xi_\varphi(v)$ is zero.

Conversely, choose $v \in V_e$ such that $B(\varphi)(v)|_{W^1_e(\varphi)}$ is nilpotent for all $\varphi \in U_e^*$ representing $\xi \in \Xi$. Then every eigenvalue of $B(\varphi)(v)|_{W^1_e(\varphi)}$ is zero. Fixing a particular $\varphi$, if $z$ is a mutual eigenvector of $\{B(\varphi)(v)|_{W^1_e(\varphi)} : \tilde{v} \in V_e\}$, then $\xi(v) = 0$ for all $\xi \in \Xi_e$ over $\varphi \in U_e^*$. Since this holds for all $\varphi \in U_e^*$, we have $v \in \langle \Xi \rangle_e^1 = X_e^1$. \[\square\]

**Lemma 5.17.** Suppose that $(M, \mathcal{I})$ is an exterior differential system equipped with a basis $(u^k)$ of $V_e^*$ and $(w_a)$ of $W_e$ such that the coefficients $B_{k,b}^{\alpha,\lambda}$ describing $A$ satisfy condition (i) of Theorem 5.9. The following are equivalent:

(i) $v \in S_e$;
(ii) $B(\varphi)(v)$ is the trivial endomorphism for all $\varphi \in U_e^*$, and
(iii) $B(\varphi)(v)|_{W^1_e(\varphi)}$ is the trivial endomorphism for all $\varphi \in U_e^*$.

*Proof.* Now, $v \in S_e$ if and only if $\pi v = 0$ for all $\pi \in A_e$. The decomposition (5.8) means this is equivalent to $\pi v = 0$ for all $\pi \in A_-(u^\lambda)$ for all $\lambda$. By the isomorphism $A_-(u^\lambda) \cong W_-(u^\lambda)$, this is equivalent to $B(u^\lambda)(\cdot)z = 0$ for all $z \in W_-(u^\lambda)$ for all $\lambda$, which is clearly equivalent to $B(\varphi)(\cdot)z = 0$ for all $z \in W_-(\varphi)$ for all $\varphi \in U_e^*$. Hence, (i) and (ii) are equivalent. Moreover, (ii) implies (iii), as $W_-(\varphi) \subset W_-(\varphi)$.

Suppose (iii) holds for $v$. Note that $W_-(u^\ell) = W^1(u^\ell)$, so $B(u^\ell)(v) = 0$. If the Cartan characters are all equal, $s_1 = s_2 = \cdots = s_\ell$, then the claim (ii) follows trivially. Therefore, suppose that $\lambda$ is maximal such that $s_\lambda > s_\ell$. We consider the symbol endomorphisms $B(u^\lambda)(u_\ell)$ and $B(u^\lambda)(v)$. Using $0 < s_\ell < s_\lambda$, we may consider the following block-decomposition of $B(u^\lambda)(u_\ell)$:

\begin{equation}
B(u^\lambda)(u_\ell) = \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}.
\end{equation}
(The \( C, D \) here are merely block matrices, not the objects previously labeled by those glyphs.) Note that \( z \in W^1(u^\lambda) \) implies that \( z \in \ker B(u^\lambda)(u_\ell) = \ker(C, D) \) by Lemma 5.11, so the assumption (iii) implies that \( z \in B(u^\lambda)(v) \).

Now, for any \( \varphi = u^\lambda + \tau u^\ell \) in the \( \mathbb{P}^1 \) spanned by \( u^\lambda \) and \( u^\ell \), we apply Lemma 5.11 similarly. In particular, \( z \in \ker(C, D) \) implies \( z \in \ker B(u^\lambda)(v) \). If \( C = 0 \) and \( D = 0 \), then \( W^1(u^\lambda) = W^-(u^\lambda) \), so \( B(u^\lambda)(v) = 0 \) by assumption. If \( C \) or \( D \) is non-zero, then varying \( \tau \) makes the kernel of \( B(u^\lambda)(v) \) span all of \( W^-(u^\lambda) \), so \( B(u^\lambda)(v) = 0 \).

Repeat this argument, decreasing \( \lambda \) until \( B(u^1)(v) = 0 \). Hence, (ii) holds.

The fact that (iii) implies (ii) is actually the key to Main Theorems 3.1 and 3.2; without it, the condition of Lemma 5.16 regarding \( W_e^1(\varphi) \) and the condition of Lemma 5.17 regarding \( W_e^-(\varphi) \) are incomparable.

Finally, the choice of basis \( (u^1, \ldots, u^n) \) just described in a single fiber \( V_e^* \) may be extended to a local section \( u : M^{(1)} \to F \). There is still some freedom in selecting the basis \( (u^k) \), as there is always freedom in choosing complementary subspaces and sections of exact sequences. There is also the usual freedom in extending a particular basis \( (u^k) \) of \( V_e^* \) to a local section \( u : M^{(1)} \to F \). In any case, the coframe is generic and adapted to \( V \supset X^1 \supset S \).

6. Elementary extension

In this section, we study the ideal \( \text{elem}(I) \) and prove Main Theorems 3.1, 3.2, and 3.5. The construction of \( \text{elem}(I) \) is similar to the notion of an integrable extension as in [BG] and Definition 6.5.3 of [IL03].

Let \( \varpi : M^{(1)} \to M \) denote the bundle projection. We have established a \( \mathbb{C}^{m+s} \)-valued coframe of \( M^{(1)} \) comprised of

\[
(u^i)_{i=1,\ldots,L}, \quad (u^\alpha)_{\lambda=L+1,\ldots,n}, \quad (\theta^a)_{a=n+1,\ldots,m}, \quad \text{and} \quad (\pi^a_{\lambda})_{a \leq s_\lambda}.
\]

So, \( du^i \equiv \eta^i_\alpha \wedge u^\alpha \mod \{\theta^b, w^j\} \) for some forms \( \eta^i_\alpha \) that may be written explicitly as

\[
\eta^i_\alpha \equiv H^i_{\alpha,\beta} u^\beta + \sum_{b \leq s_\mu} H^i_{\alpha,\mu} \pi^b_{\mu} \mod \{\theta^b, w^j\},
\]

where the \( H \)-coefficients are determined by the choice of coframe.

With respect to this coframe, the complexified prolongation system \( \mathcal{I}^{(1)}_C \) on \( M^{(1)} \) is generated by

\[
\begin{cases}
\theta^a, \\
d\theta^a \equiv \pi^a_i \wedge u^i + \pi^a_{\mu} \wedge u^\alpha \mod \{\theta^b\}
\end{cases}
\]

with independence condition \( u^1 \wedge \cdots \wedge u^n \neq 0 \). The elementary system \( \text{elem}(I) = \mathcal{I}^{(1)}_C + \langle \Xi \rangle \), whose definition implicitly requires complexification,
is generated as

\begin{equation}
\begin{aligned}
\theta^a, \\
u^i, \\
\text{d}\theta^a \equiv \pi^a_i \wedge u^i + \pi^a_\alpha \wedge u^\alpha \mod \{\theta^b\}, \\
\text{d}u^i \equiv \eta^i_\alpha \wedge u^\alpha \mod \{\theta^b, u^j\}
\end{aligned}
\end{equation}

with independence condition \( u^{L+1} \land \cdots \land u^n \neq 0 \). This is the same system described casually in Section 1, but now our coframe of \( M^{(1)} \) is adapted to the problem. See Figure 4.

If \( \text{elem}(\mathcal{I}) \) were itself involutive, then the decomposition \( V^* = U^* \oplus X^* \oplus Y^* \) could be repeated for \( \text{elem}(\mathcal{I}) \). However, a proof of Conjecture 3.7 eludes the author, one obstruction being Conjecture 7.11, discussed below. Instead, we can prove a slightly weaker version:

**Lemma 6.5.** Suppose that \((M, \mathcal{I})\) is an involutive exterior differential system. Then the system \( \text{elem}(\mathcal{I}) \) on \( M^{(1)} \) admits a smooth family of maximal integral manifolds of dimension \( n-L \).

**Proof of Lemma 6.5 and Main Theorem 3.5.** Suppose that \( \text{elem}(\mathcal{I}) \) is not Frobenius, for the claim is trivial in that case. Because \( \text{elem}(\mathcal{I}) \) contains the involutive ideal \( \mathcal{I}^{(1)} \), we have

\[ \text{Var}_k(\text{elem}(\mathcal{I})) \subset \text{Var}_k(\mathcal{I}^{(1)}) \]

for all \( k \), so the maximal dimension of ordinary integral elements of \( \text{elem}(\mathcal{I}) \) cannot be greater than the maximal dimension for \( \mathcal{I}^{(1)} \), namely \( n \). Moreover, since \( \text{elem}(\mathcal{I}) \) contains \( L \) additional generating 1-forms that are independent of \( \mathcal{I}^{(1)} \), the maximal dimension of integral elements of \( \text{elem}(\mathcal{I}) \) is at most

\[ n-L \].
Using the independence condition $u^{L+1} \wedge \cdots \wedge u^n \neq 0$, the maximal integral elements $f \in \Var_{n-L}(\elem(\mathcal{I}))$ may be written as

$$f = \left( \pi^a_\lambda - \sum_{a \leq s_\lambda} Q^a_{\lambda,\alpha} u^a \right)^\perp \subset T_e M^{(1)},$$

where the coefficients $Q^a_{\lambda,\alpha}$ define $Q \in A_e \otimes X^*_e$ and are subject to the 2-form conditions from (6.4),

$$\begin{align*}
\sum_{b \leq s_\lambda} B^a_{\alpha,b} Q^b_{\lambda,\beta} &= \sum_{b \leq s_\lambda} B^a_{\beta,b} Q^b_{\lambda,\alpha}, \\
H^i_{\alpha,\beta} + \sum_{b \leq s_\lambda} H^i_{\alpha,b} Q^b_{\lambda,\beta} &= H^i_{\beta,\alpha} + \sum_{b \leq s_\lambda} H^i_{\beta,b} Q^b_{\lambda,\alpha}, \quad \forall \alpha, \beta.
\end{align*}$$

Let $E_e$ denote the subspace of $A_e \otimes X^*_e$ defined by the condition (6.7). The bundle $E$ over $M^{(1)}$ is the tableau of $\elem(I)$, which is discussed further in Lemma 7.2.

We can construct a smooth family of maximal integral manifolds in the following way:

Fix $e \in M^{(1)}$ and consider the family of ordinary integral manifolds $\iota : N \to M$, with $y \in N$ and $\iota_*(T_y N) = e$. Cartan’s test for involutivity of $\mathcal{I}$ guarantees that this family is smooth, parameterized by $s_\ell$ functions of $\ell$ variables. For each such $N$, choose $L$ independent elements of $\Xi_{N,y} \subset T_y^* N$ and use the eikonal system of $\Xi_N$ to build $L$ independent characteristic hypersurfaces through $y \in N$. The intersection of these hypersurfaces is a submanifold $D \subset N$ of dimension $n - L$. The submanifold $D$ is unique in the sense that it does not depend on the particular choice of $L$ characteristic hypersurfaces, because $\xi|_{TD} = 0$ for all $\xi \in \Xi_N$. Of course, $\xi|_{TD} = 0$ also implies that $\iota^{(1)}|_D : D \to M^{(1)}$ is an integral manifold of $\elem(\mathcal{I})$ through $X^1_e$.

To be explicit, suppose $\hat{e} = \iota^{(1)}_*(T_y N) \subset T_e M^{(1)}$ is given as

$$\hat{e} = \left( u_i + \sum_{a \leq s_\lambda} P^a_{\lambda,i} \pi^a_\lambda, u_\alpha + \sum_{a \leq s_\lambda} P^a_{\lambda,\alpha} \pi^a_\lambda \right),$$

where the coefficients define a section $P : N \to A \otimes V^*$. The subspace $\iota_*(T_y D)$ satisfies $\theta^a = 0$ and $d\theta^a = 0$ because $N$ is integral to $\mathcal{I}^{(1)}$. This implies the first condition in Equation (6.7) is satisfied for $Q^a_{\lambda,\alpha} = P^a_{\lambda,\alpha}$. (Compare to Lemma 7.2.) Note that Lemma 4.5 implies that $du^i \wedge u^i$ pulls back to 0 on $N$, as our choice of coframe $u : N \to \mathcal{F}_N$ determines a function $u_i : (\Xi)_N \to \mathbb{C}$ solving the eikonal system. Therefore, the coefficients $Q^a_{\lambda,\alpha} = P^a_{\lambda,\alpha}$ also satisfy the second condition in Equation (6.7). That is, for any ordinary integral manifold $\iota : N \to M$ with corresponding $P : N \to A^{(1)} \subset A \otimes V^*$, setting $Q = P|_X$ at $y \in N$ yields an infinitesimal solution to
(6.7), and this solution extends to an integral manifold $\nu^{(1)}|_D : D \to M^{(1)}$ of $\text{elem}(\mathcal{I})$.

The submanifold $\Lambda$ in Main Theorem 3.5 is the usual foliation for Cauchy retractions, as in Corollary 4.7.

Proof of Corollary 3.6. In Lemma 6.5, we are working with a linear Pfaffian ideal over an analytic manifold with algebraic fiber of locally constant rank, all over $\mathbb{C}$, so the Cartan–Kuranishi prolongation theorem implies that some prolongation of $\text{elem}(\mathcal{I})$ is involutive or empty. By Lemma 6.5, it is not empty. (See Theorem 4.2 in Chapter VI and Proposition 3.9 in Chapter VIII of [BCG+90] and the discussion therein.)

Lemma 6.9. Suppose that $(M, \mathcal{I})$ is an involutive exterior differential system. The system $\text{elem}(\mathcal{I})$ on $M^{(1)}$ descends to $M$ if and only if $\text{elem}(\mathcal{I})$ is Frobenius.

Proof. Suppose that $\text{elem}(\mathcal{I})$ descends to $M$; that is, suppose that if $\varpi(e) = \varpi(\tilde{e}) = x \in M$, then $X^i_M = X^i_{\tilde{M}}$ as subspaces of $\mathbb{P}T_xM \otimes \mathbb{C}$; call this subspace $X_x$, which has projective dimension $n-L-1$. Let $\omega^1, \ldots, \omega^n$ be a coframe of $M$ near $x$ that is generic for $\mathcal{I}$ and such that

$$X^\perp_x = \langle \omega^1, \ldots, \omega^L, \omega^{n+1}, \ldots, \omega^n \rangle.$$  

Using $\varpi$ to pull back this coframe to $M^{(1)}$ (and omitting writing $\varpi^*$), the linear Pfaffian system $\mathcal{I}^{(1)}$ is generated by

$$\begin{cases} 
\theta^a = \omega^a - P^a_i \omega^i - P^a_{\alpha} \omega^\alpha, \\
\omega^i, \\
d\theta^a \equiv \pi^a_i \wedge \omega^i + \pi^a_{\alpha} \wedge \omega^\alpha \mod \{ \theta^b \}, \\
d\omega^i \equiv \eta^i_{\alpha} \wedge \omega^\alpha \mod \{ \theta^b, \omega^i \}.
\end{cases}$$

with independence condition $\omega^{L+1} \wedge \cdots \wedge \omega^n \neq 0$. This system is Frobenius if and only if $\eta^i_{\alpha} \wedge \omega^\alpha \equiv \pi^a_{\alpha} \wedge \omega^\alpha \equiv 0$. (The forms $\pi^a_{\alpha}$ and $\eta^i_{\alpha}$ here are not identical to those from Equation (6.4), since the coframe is different, but they play similar roles, so we use similar notation.) Because $\omega^i$ is basic with respect to $\varpi$, it must be that $\eta^i_{\alpha} = H^{i}_{\alpha,j} \omega^j + H^{i}_{\alpha,\beta} \omega^\beta$. Because $\text{elem}(\mathcal{I})$ admits maximal integral manifolds by Lemma 6.5, the independence condition implies that the “torsion” terms $H^{i}_{\alpha,\beta}$ are symmetric, so $\eta^i_{\alpha} \wedge \omega^\alpha \equiv 0$. Comparing Equations (6.10) and (6.12), we see that $X^\perp_x = \langle \theta^a, \omega^i \rangle = \langle \omega^a, \omega^i \rangle$, so $P^a_{\alpha} = 0$. Differentiate and use $d\omega^i \equiv 0$ to obtain $\pi^a_{\alpha} \wedge \omega^\alpha \equiv d\omega^a$, which must vanish because the coframe $(\omega^k)$ is basic and integral manifolds exist.
Conversely, suppose that $\text{elem} (\mathcal{I})$, generally written in the form of Equation (6.4), is Frobenius. It suffices to show that the generators of $\text{elem} (\mathcal{I})$ are basic with respect to $\varpi$: $M_1 \to M$, since this is equivalent to the condition that the Cauchy reductions of $\text{elem} (\mathcal{I})$ contain $\ker \varpi$. Of course, the 1-forms $\theta^a$ and $u^i$ are semi-basic, meaning that they annihilate the vertical subspace $\ker \varpi$. The Frobenius condition is $d\theta^a \equiv du^i \equiv 0 \mod \{\theta^b, \xi^j\}$, so these generators are basic. \[\square\]

**Proof of Main Theorems 3.1 and 3.2.** Lemma 6.9 shows that (iii) and (iv) are equivalent. Statements (i) and (ii) are dual, and these trivially imply statements (iii) and (v) by dimension count. Of course, in the case of Main Theorem 3.1, the Frobenius system $\text{elem} (\mathcal{I})$ is actually the “irrelevant” differential ideal, whose integral manifolds have dimension zero.

Suppose that statement (iii) holds. Then the tableau of $\text{elem} (\mathcal{I})$ is empty, so $\pi^a_0 = 0$. In particular, $v = v^a u_a \in X^1_\mathcal{E}$ implies that $v \wedge d\theta^a \equiv \pi^a_0 = 0$, so $v \in S_\mathcal{E}$. This is statement (ii).

Suppose that statement (v) holds, and suppose that $v \in V_\mathcal{E}$. By Lemmas 5.16 and 5.17, we have that $v \in S_\mathcal{E}$ if and only if $v \in X^1_\mathcal{E}$, which is (ii). \[\square\]

**Remark 6.13.** A Warning: Lemma 6.9, Main Theorem 3.1, and Main Theorem 3.2 do not require or imply that $\Xi$ is constant in each fiber of $M^{(1)}$. Even if $\text{elem} (\mathcal{I})$ is Frobenius, the $(\pi^a_0)$ portion of the tableau (6.3) may vary over $M^{(1)}$. Conversely, even if $\Xi$ is locally constant, the system $\text{elem} (\mathcal{I})$ may fail to descend to $M$ if $d\theta^a \not\equiv 0 \mod \{\theta^b, \xi^j\}$.

**7. Prolonged elementary extension**

Finally, we want to try to understand the case when $\mathcal{I}$ is not elementary, so $\text{elem} (\mathcal{I})$ is not Frobenius. The main question is “How can we compute $\text{elem} (\mathcal{I})$?” For an involutive exterior differential system with Cartan integer $\ell = \dim \Xi + 1 > 1$ and Cartan character $s_\ell = \deg \Xi > 1$, the (nonlinear) characteristic variety $\Xi$ is difficult to compute and parametrize. One might expect that selecting $L$ “random” elements of $\Xi$ to generate $\langle \Xi \rangle$—and therefore $\text{elem} (\mathcal{I})$—would also be difficult. If $\mathcal{M}$ is known, then computer algebra systems allow computation of sat($\mathcal{M}$) using Gröbner bases. But, it would be preferable to bypass the computation of $\Xi$ and $\mathcal{M}$ entirely, since $X^1$ is a linear subspace of $V$ defined by linear symbol relations, $B_\lambda^\Lambda$. Moreover, can we hope to compute $\text{elem}^k (\mathcal{I})$ from $B_\lambda^\Lambda$ directly for all $k \geq 2$? The remaining results offer a possible approach to these questions.
Recall the skewing maps $\delta$, which define tableau prolongation and are essential to the study of involutivity via Spencer cohomology:

$$
0 \to A^{(1)} \to A \otimes V^* \xrightarrow{\delta} W \otimes \wedge^2 V^* \to H^2(A) \to 0,
$$

$$
0 \to A^{(2)} \to A^{(1)} \otimes V^* \xrightarrow{\delta} W \otimes \wedge^3 V^* \to H^3(A) \to 0,
$$

$$
\vdots
$$

$$
0 \to A^{(n-1)} \to A^{(n-2)} \otimes V^* \xrightarrow{\delta} W \otimes \wedge^n V^* \to H^n(A) \to 0.
$$

(7.1)

Involutivity of the linear Pfaffian system $(M^{(1)}, \mathcal{I}^{(1)})$ is equivalent to $H^\rho(A) = 0$ for all $\rho \geq 2$. This condition for a formal tableau is sometimes called “formally integrable,” but since our tableau comes from a linear Pfaffian system, there is no distinction. See Theorem 5.16 in Chapter IV of [BCG+90].

Since $X^*$ is a fixed subspace of $V^*$, let $\delta_X$ denote the restricted skewing map that imposes symmetry only on the $\otimes^{\rho+1} X^*$ component. The condition $\delta_X = 0$ is strictly weaker than $\delta = 0$. In particular, $A^{(\rho)} \otimes V^*$ projects onto $A^{(\rho)} \otimes X^*$, and the induced image $A^{(\rho+1)}|_X$ of $A^{(\rho+1)}$ satisfies $\delta_X = 0$.

Recall that the tableau of $\text{elem}(\mathcal{I})$ is the subspace $E \subset A \otimes X^*$ as given by Equation (6.7). Let $\Xi_E$ denote the characteristic variety of $E$ in $X^*$.

**Lemma 7.2.** Let $E^{(0)} = E \subset A \otimes X^*$ and let $E^{(\rho)} \subset E^{(\rho-1)} \otimes X^*$ denote the $\rho$th prolongation of the tableau $E$ of elem($\mathcal{I}$). Then, as subspaces of $A \otimes (\otimes^\rho X^*)$, we have $A^{(\rho+1)}|_X \subset E^{(\rho)} \subset \ker \delta_X$.

**Proof.** As seen in Equation (6.4) and Figure 4, the tableau of elem($\mathcal{I}$) is a subspace of $(U \oplus Y \oplus W) \otimes X^*$, but the tableau conditions (6.7) show that the $(\eta_{\alpha}^{\rho}) \in (U \oplus Y) \otimes X^*$ term depends on the $(\pi_\lambda^{\rho}) \in A$ term when using our adapted coframe (6.1). Therefore, we may consider the tableau of elem($\mathcal{I}$) to be the subspace $E \subset A \otimes X^*$ specified by those conditions. The first condition in (6.7) is $\delta_X Q = 0$, and the independence condition imposes symmetry over $\otimes^\rho X^*$ for $\rho \geq 1$.

For any $P \in A^{(1)}$, the proof of Lemma 6.5 says that the eikonal system forces $P|_X \in E$ via the restriction of $P \in A^{(1)} \subset A \otimes V^*$ to $A \otimes X^*$. Involutivity, the characteristic variety, and the eikonal system are all preserved by prolongation of $A$, so this containment is preserved as well. \qed

One problem with Conjecture 3.7 is that $E$ is fairly annoying to compute; specifically, the $\eta_{\alpha}^{\rho}$ terms in Equation (6.4) and Figure 4 depend on the local coframe chosen on $M^{(1)}$. This dependency can be ignored if the EDS $(M^{(1)}, \mathcal{I}^{(1)})$ arises from a “local PDE in jet-space” with the additional condition that the span of the characteristic variety is locally constant in the coordinates $dx^1, \ldots, dx^n$. Then we can take the adapted coframe $(u^k)$ to be closed, giving $\eta_{\alpha}^1 = 0$. 

Let $\hat{A}$ denote the formal tableau obtained the projection of $A$ to $W \otimes X^*$. Then $\hat{A}$ is given by an exact sequence

\begin{equation}
\emptyset \to \hat{A} \to W \otimes X^* \xrightarrow{\delta} \hat{A}^\perp \to \emptyset
\end{equation}

induced by the sequence (2.5). Since we have a good description of $A^\perp$, it is easy to write

\begin{equation}
\hat{A} = \{ \pi|_X, \pi \in A \} = \{ \pi^a_\alpha (w_a \otimes u^\alpha) , \pi \in A \} = \left\{ B_{a,b}^{\alpha,\lambda} (w_a \otimes u^\alpha) \right\} = \left\{ B_{a,b}^{\alpha,\lambda} (w_a \otimes u^\alpha) \right\}
\end{equation}

and

\begin{equation}
\hat{A}^\perp = \left\{ K = K^\alpha_a w^\alpha_a \otimes u_a \in (W \otimes X^*)^* : K^\alpha_a B_{a,b}^{\alpha,\lambda} = 0, \forall b \leq s_\lambda \right\}.
\end{equation}

The characteristic variety of $\hat{A}$ is

\begin{equation}
\hat{\Xi} = \left\{ \xi \in X^* : \ker(K^\alpha_a \xi w^\alpha_a) \neq 0, \forall K \in \hat{A}^\perp \right\}.
\end{equation}

The skewing map on $X^*$ defines a formal prolongation of $\hat{A}$,

\begin{equation}
0 \to \hat{A}^{(1)} \to \hat{A} \otimes X^* \xrightarrow{\delta_X} W \otimes \wedge^2 X^* \to H^2(\hat{A}) \to 0.
\end{equation}

The characteristic variety of $\hat{A}^{(1)}$ is

\begin{equation}
\hat{\Xi}^{(1)} = \left\{ \xi \in X^* : \exists \pi \in A, \delta_X(\pi|_X \otimes \xi) = 0 \right\} = \left\{ \xi \in X^* : \exists \pi \in A, \delta_X(\pi \otimes \xi) = 0 \right\}.
\end{equation}

Compare the next lemma to Corollary 5.10 and Theorem A in [Gui68], which is much harder due to a looser notion of involutivity for formal tableaux.

**Lemma 7.9.** If $(M, \mathcal{I})$ is involutive, then $H^\rho(\hat{A}) = 0$ for all $\rho \geq 2$.

**Proof.** If $(M, \mathcal{I})$ is involutive, then $H^\rho(A) = 0$ for all $\rho \geq 2$. The maps $A \to \hat{A} = A|_X$ and $V^* \to X^*$ are surjective and commute with $\delta$, so the same applies to $\hat{A}$. \qed

Lemma 7.9 says that, if we can associate $\hat{A}$ with a linear Pfaffian exterior differential system, then that system is involutive. This is useful in the local PDE case where $(\Xi)$ is locally constant, for then $E = \hat{A}$ because $u^\lambda = dx^\lambda$.

Conjecture 3.7 and Main Theorem 3.8 follow immediately.

Even in the general case, our only hope for a general result regarding $\text{elem}^2(\mathcal{I})$ is if $(\eta^a_\alpha)$ is determined by $(\pi^a_\alpha)$, so $\hat{A}$ is still worth studying.

**Lemma 7.10.** As subsets of $X^*$, we have $\Xi^{(1)}_E \subset \Xi_E \subset \hat{\Xi}^{(1)} \subset \hat{\Xi}$.

---

7It is “formal” in the sense that it did not arise a priori from a particular EDS.
Proof. The left-most and right-most inclusions are standard; prolongation can only increase the characteristic ideal of a tableau. (See the discussion leading to statement (79) in Chapter V of [BCG+90].)

Suppose that $\xi \in \Xi_E$. Then there exists $\pi \in A$ such that $\pi \otimes \xi$ is a rank-one element of $E \subset A \otimes X^*$. The first condition in (6.7) means $\delta_X Q = 0$, so $\pi|_X \in \hat{A}$ and $(\pi|_X) \otimes \xi \in \hat{A}^{(1)}$. This is rank-one, so $\xi$ lies in the characteristic variety of $\hat{A}^{(1)}$.

To see the weaker inclusion $\Xi_E \subset \hat{\Xi}$, consider the generating 2-forms (6.4) and Figure 4. If the combined matrix $(\pi^a_w \eta^w_{\alpha} \otimes u^\alpha) \in E$ is rank-one, then the upper matrix $\pi^a_w (w_{\alpha} \otimes u^\alpha) \in \hat{A}$ is rank-one over the same fiber. □

If we knew these were involutive, then the degree of $\xi$ in the variety would be seen to fall by a constant: the nullity of the projection $\pi \mapsto \pi|_X$.

The next conjecture would help establish a general equivalence between $\hat{A}$ and $E$.

**Conjecture 7.11.** Suppose that $(M, \mathcal{I})$ is an involutive exterior differential system. The characteristic sheaf of $E$ equals the characteristic sheaf of $\hat{A}$.

A weaker version would suffice if we merely want to compute $\text{elem}^2(\mathcal{I})$, regardless of its involutivity.

**Conjecture 7.12.** Suppose that $(M, \mathcal{I})$ is an involutive exterior differential system. Then $\langle \hat{\Xi} \rangle = \langle \Xi_E \rangle$ as subspaces of $X^*$.

**Remark 7.13.** On the question of involutivity for $\text{elem}(\mathcal{I})$: If one were to consider Conjecture 3.7 for a formal tableaux $A$ (as opposed to a tableau coming from a torsion free involutive exterior differential system) then studying $\hat{A} = A|_X$ itself is very difficult. Unfortunately, the only known result on involutivity of sub-tableaux is the theorem of [Gui68], which is generalized [Gui68] and is restated here as Corollary 5.10. This theorem applies to non-characteristic sub-tableaux like $A|_U$, but our sub-tableaux $A|_X$ is defined to be maximally characteristic!
8. Discussion

The main theorems are direct observations using established techniques in exterior differential systems. They fill a significant gap in the literature, but in retrospect they may not be surprising to experts who have manipulated tableaux in many examples. Notably, Cartan encountered many of these phenomena in [Car11], particularly the example beginning with paragraph 22, but apparently he did not pursue it elsewhere. To my knowledge, that is the only appearance of any similar statement in the literature.

As to the coinage “elementary,” several other names also seem appropriate. One might call these systems names like “semi-simple,” “non-parabolic,” or “primitive.” But, I believe “semi-simple” is premature without a generalized Levi decomposition theorem, “non-parabolic” is misleading without a generalized regularity theorem, and “primitive” would convolute the intricate relationship between involutive EDS and Lie pseudogroups.

However, it does seem reasonable to expect that there would be a generalization of parabolic regularity to certain non-elementary systems where \( n - L = 1 \). Involutivity of the eikonal system of \( \langle \Xi \rangle \) should guarantee the existence of a time variable corresponding to the vector subspace \( X \) of nilpotents, like in Corollary 3.9. Of course, these results would need to be loosened from \( \mathbb{C} \) to \( \mathbb{R} \) to be very useful for analysis.

I do not have a strong sense of whether the final conjectures are actually true. They seem to hold on examples I have constructed by hand using Theorem 5.9, but it is difficult to build toy systems that have sufficiently complicated characteristic varieties. I encourage you to sift through your favorite non-elementary EDS/PDEs for examples where Conjecture 7.11 holds or fails. Where it holds, it suggests that all solutions of the EDS can be found through a canonical sequence of reductions. Such a structure would provide a solvability criterion, allowing a decomposition theorem for EDS into a sequence elementary or Frobenius systems, as in Main Theorem 3.8. If it fails, then the microlocal analysis of characteristic varieties contains further mysteries that must be similarly fascinating.
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