A REMARK ON LETZTER-MAKAR-LIMANOV INVARIANTS

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1. Introduction

Let $X$ be an irreducible affine algebraic curve over $\mathbb{C}$ with normalization $\tilde{X}$, and let $D(X)$ be the ring of (global) differential operators on $X$. Due to general results of Smith and Stafford (see [SS]), it is known that $D(X)$ is Morita equivalent to $D(\tilde{X})$ if (and only if) the normalization map $\pi : \tilde{X} \to X$ is injective. In the special case when $X$ is rational and $\tilde{X}$ is isomorphic to the affine line $\mathbb{A}^1$, there exist non-isomorphic curves with isomorphic rings of differential operators (the first examples of this kind were found in [L]). To distinguish non-isomorphic rings $D(X)$ in the Morita equivalence class of $D(\mathbb{A}^1)$, G. Letzter and L. Makar-Limanov [LM] introduced a certain ring-theoretic invariant which they called ‘codim $D(X)$’. Subsequently, a complete classification of rings of differential operators on affine curves was given (see [K], [BW1], and [BW4] for a detailed exposition). This showed that for the above class of curves the algebras $D(X)$ were determined (up to isomorphism) by a single non-negative integer $n$. The relation between this new invariant $n$ and the Letzter-Makar-Limanov one turned out to be very simple (cf. Remark in [BW1]): codim $D(X) = 2n$. However, since the invariants in hand seemed to be of different nature, this relation looked somewhat mysterious: it did not follow a priori from the definitions, but rather was the result of comparing the theorems of [K] and [BW1] with some explicit calculations in [LM].

In this short note we will give a natural interpretation of the Letzter-Makar-Limanov (LM) invariants in the spirit of noncommutative projective geometry of M. Artin et al. (see [A], [AZ], and [SV] for a general overview of the subject). As a result, we will prove the above relation in a very simple, purely homological fashion. We will introduce and work with LM-invariants in a slightly more general setting (than that of [LM]) and will mostly use the notation of [BW3]. Apart from [AZ], the latter reference also contains a review of all the definitions and results from noncommutative geometry needed for the present paper.

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2. Geometric Interpretation of LM-invariants

Let $A := \mathbb{A}^1(\mathbb{C})$ be the first complex Weyl algebra; $Q := \text{Frac}(A)$, its field of fractions; $M$, a finitely generated rank 1 torsion-free right module over $A$; $D := \text{End}_A(M)$, the endomorphism ring of $M$. Since $A$ is a simple Noetherian hereditary algebra, $M$ is automatically a progenerator in the category $\text{Mod}(A)$ of right $A$-modules, and the ring $D$ is Morita equivalent to $A$.

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As usual, we may (and will) identify $M$ with an ideal of $A$ (possibly fractional), and $D$ with the subring of $Q$:

$D = \{ q \in Q : qM \subseteq M \} .

Let $w = (w_1, w_2)$ be a nonzero non-negative vector in $\mathbb{R}^2$, and let $\{Q^w(w)\}$ denote the standard Dixmier filtration on $Q$ of weight $w$ (see [D]). We will equip all the subspaces of $Q$ with induced filtrations. In particular, we set

$$A_k(w) := A \cap Q^k(w) \quad \text{and} \quad D_k(w) := D \cap Q^k(w) ,$$

and write $GQ(w) := \bigoplus_{k \in \mathbb{Z}} Q^k/Q^k-1$, $GA(w) := \bigoplus_{k \in \mathbb{Z}} A_k/A_{k-1}$, and $GD(w) := \bigoplus_{k \in \mathbb{Z}} D_k/D_{k-1}$ for the associated graded rings. The natural inclusions $A_k \subseteq Q_k$, $D_k \subseteq Q_k$ induce then the embeddings of graded algebras $GA(w) \hookrightarrow GQ(w)$, $GD(w) \hookrightarrow GQ(w)$. We will identify $GA(w)$ and $GD(w)$ with their images in $GQ(w)$ and note that $GD(w)$ does not depend on the choice of representation of $M$ as an ideal in $Q$.

The following theorem is essentially a restatement (perhaps, in a slightly more general form) of one of the main results of [LM] (see loc. cit., Proposition 2.4, and [SS], Sections 3.11, 3.12).

**Theorem 1.** For each nonzero $w \in \mathbb{R}^2$, we have $GD(w) \subseteq GA(w)$ in $GQ(w)$; the quotient space $LM(w) := GA(w)/GD(w)$ is finite-dimensional, its dimension being independent of $w$.

In what follows we will denote the number $\dim_{\mathbb{C}} LM$ by $p_D$ and call it the *LM-invariant* of the ring $D$; since $p_D$ is independent of $w$, we will also assume $w = (1, 1)$ for simplicity of further considerations.

Let $A := \bigoplus_{k \in \mathbb{Z}} A_k$ and $D := \bigoplus_{k \in \mathbb{Z}} D_k$ be the Rees algebras (homogenizations) of $A$ and $D$ with respect to the filtrations $\{A^w\}$ and $\{D^w\}$. Then $A$ is isomorphic to the algebra of ‘noncommutative polynomials’ $\mathbb{C}[x, y, z]$ with generators $x, y, z$, all having degree $1$, $z$ commuting with $x$ and $y$, and $xy - yx = z^2$. On the other hand, $GD$ and $D$ are both graded connected Noetherian (and hence, locally finite) algebras. This can be shown easily first for $GD$ by adapting the techniques of [SS] (specifically, the proof of Theorem 3.12, loc. cit.), and then for $D$ by a standard lifting argument (see, for example, [B], Appendix III, Proposition 1.29).

Now, following [AZ], we will think of $A$ and $D$ as the homogeneous coordinate rings of *noncommutative* projective schemes $X_A := \text{Proj}(A)$ and $X_D := \text{Proj}(D)$. By definition, these are the Serre quotients $\text{Coh}(X_A) := \text{GrMod}(A)/\text{Tors}(A)$ and $\text{Coh}(X_D) := \text{GrMod}(D)/\text{Tors}(D)$ of the categories of (finitely generated graded right) modules modulo the torsion subcategories (consisting of finite-dimensional modules) taken with distinguished objects $O_{X_A} := A \in \text{Coh}(X_A)$ and $O_{X_D} := D \in \text{Coh}(X_D)$. By analogy with the commutative case, the objects of $\text{Coh}(X_A)$ and $\text{Coh}(X_D)$ are referred to as *coherent sheaves*, with $O_{X_A}$ and $O_{X_D}$ playing the role of ‘structure sheaves’ on $X_A$ and $X_D$ respectively. Note that $X_A$ is the quantum projective plane $\mathbb{P}^2_q$ defined in [LH] and [BW].

The following observation is a simple consequence of Theorem 1.

**Lemma 1.** The Hilbert polynomial of the graded algebra $D$ is given by the formula

$$P_D(k) = \frac{1}{2}(k+1)(k+2) - p_D ,$$

where $p_D$ is the LM-invariant of the ring $D$. 

so that \( p_D = 1 - P_D(0) \). By analogy with the commutative case, we may thus interpret \( p_D \) as (minus) the arithmetic genus of the ‘projective surface’ \( X_D \).

**Proof.** Being finite, the graded vector space \( LM \) is bounded from above. Therefore, by Theorem 1, choosing \( k \gg 0 \) we can write the exact sequence of finite-dimensional graded vector spaces:

\[
0 \to GD_{\leq k} \to GA_{\leq k} \to LM \to 0.
\]

Whence we have

\[
\dim LM = \dim GA_{\leq k} - \dim GD_{\leq k} = \dim \bigoplus_{i=-\infty}^{k} A_i/A_{i-1} - \dim \bigoplus_{i=-\infty}^{k} D_i/D_{i-1} = \sum_{i=-\infty}^{k} (\dim A_i - \dim A_{i-1}) - \sum_{i=-\infty}^{k} (\dim D_i - \dim D_{i-1}) = \dim A_k - \dim D_k \quad \text{for all } k \gg 0.
\]

By definition, \( P_D(k) = \dim D_k \) for \( k \gg 0 \), and \( \dim A_k = (k+1)(k+2)/2 \) for all \( k \geq 0 \). Combined together these give formula (2.3), and therefore finishes the proof of the lemma.

**Remark.** The category of affine schemes is dual (that is, anti-equivalent) to the category of commutative rings and ring homomorphisms. In particular, two schemes are isomorphic if and only if their coordinate rings are isomorphic. Passing to noncommutative domain it seems more appropriate to associate ‘a noncommutative affine scheme’ not to an isomorphism class but to a Morita equivalence class of a given noncommutative ring (see, for example, [S] and [SV]). In other words, two Morita equivalent rings should be thought of as coordinate rings of isomorphic noncommutative spaces. From this perspective the result of Lemma 1 appears to be natural. Indeed, if we define the quantum plane \( k_q^2 \) as an ‘affine space’ associated with the Morita class of \( A \), the projective schemes \( X_D \) and \( X_A \) can be regarded as different ‘compactifications’ of \( k_q^2 \). In that case any invariant that may distinguish \( A \) and \( D \) up to isomorphism should be an object of projective (rather than affine) geometry.

### 3. The Relation \( p_D = 2n \)

First, we recall the definition of the number \( n \). As shown in [BW3], Section 4, every ideal (class of) \( M \) admits a unique extension to a coherent rank 1 torsion-free sheaf \( \mathcal{M} \) on \( \mathbb{P}^2_q \) trivial over the line at infinity. This extension is called the canonical extension of \( M \) to \( \mathbb{P}^2_q \). The invariant \( n \) associated to \( M \) can be defined cohomologically in terms of its canonical extension, namely

\[
n := \dim_{\mathbb{C}} H^1(\mathbb{P}^2_q, M(-1)).
\]

By analogy with the commutative case (see [N], Chapter 2), it is suggestive to think of (and refer to) \( n \) as the ‘second Chern class’ \( c_2(\mathcal{M}) \) of the sheaf \( \mathcal{M} \); however, in general, it seems that Chern classes have yet to be defined in the realm of noncommutative projective geometry.
Lemma 2. Let $\text{Hom}(M, M) := \bigoplus_{k \geq 2} \text{Hom}(M, M(k))$ be the (graded) endomorphism algebra of the canonical extension $M$ of $M$ to $\mathbb{P}_q^2$. Then, for $k \gg 0$, we have isomorphisms of graded vector spaces: $D_{\geq k} \cong \text{Hom}(M, M)_{\geq k}$. In particular, $\dim D_k = \dim \text{Hom}(M, M(k)) \forall k \gg 0$.

Proof. Without loss of generality, we may identify $M$ with one of the distinguished (fractional) ideals of $A$ (see [BW3], Section 5), and define its canonical extension $M$ to $\mathbb{P}_q^2$ with the help of the induced filtration $M_k = M \cap Q_k$. Then, by (2.1) and (2.2), we have $D_k = \{ q \in Q : qM_i \subseteq M_{i+k} \text{ for all } i \}$, and hence

$D_k \cong \text{Hom}_A(M, M(k))$ for all $k \in \mathbb{Z}$,

where $\text{Hom}_A$ is taken in the category $\text{GrMod}(A)$.

On the other hand,

$\text{Hom}(M, M(k)) = \lim_{\rightarrow} \text{Hom}_A(M_{\geq p}, M(k))$.

To calculate the inductive limit in (3.3) we fix some $p \gg 0$ and consider the short exact sequence in $\text{GrMod}(A)$:

$0 \rightarrow M_{\geq p} \rightarrow M \rightarrow M/M_{\geq p} \rightarrow 0$.

Dualizing (3.3) with $M(k)$ we have

$0 \rightarrow \text{Hom}_A(M/M_{\geq p}, M(k)) \rightarrow \text{Hom}_A(M, M(k)) \rightarrow \text{Hom}_A(M_{\geq p}, M(k)) \rightarrow \text{Ext}_A^1(M/M_{\geq p}, M(k)) \rightarrow \ldots$.

Now, since $M/M_{\geq p}$ is torsion whereas $M(k)$ is torsion-free, $\text{Hom}_A(M/M_{\geq p}, M(k))$ must be zero for all $k$. On the other hand, $\text{Ext}_A^1(M/M_{\geq p}, M(k)) = 0$ for $k \geq k_0 \gg 0$ (where $k_0$ is independent of $p$) because the algebra $A$ satisfies the $\chi$-condition of Artin-Zhang (see [AZ], Proposition 3.5(1)). It follows that

$\text{Hom}_A(M_{\geq p}, M(k)) \cong \text{Hom}_A(M, M(k)) \forall k \gg 0$.

Combining (3.2) with (3.3) and (3.5) we get the result. \hfill \square

Remark. It seems quite plausible that the entire vector spaces $D$ and $\text{Hom}(M, M)$ are isomorphic. But for our purposes the result of Lemma 2 will suffice.

To finish our calculation it remains to evaluate $\dim \text{Hom}(M, M(k))$ for $k \gg 0$ and compare the result with formula (2.1). This can be done in many different ways of which we will choose (hopefuly) the most elementary one.

First, following [KKO] (see loc. cit., Section 5.3), we introduce the dual sheaf $M^\vee := \text{Hom}(M, \mathcal{O})$ of $M$ as an object of the quotient category of left graded modules over $A$.

Lemma 3. If $M \in \text{Coh}(\mathbb{P}_q^2)$ is trivial over the line at infinity in $\mathbb{P}_q^2$ so is $M^\vee$. In that case the twisted sheaves $M(k)$ and $M^\vee(k)$ have the same Euler characteristics for all $k \in \mathbb{Z}$:

$\chi(\mathbb{P}_q^2, M^\vee(k)) = \chi(\mathbb{P}_q^2, M(k)) = \frac{1}{2}(k+1)(k+2) - n$.

\textsuperscript{1}For simplicity, we will use the same notation $\mathbb{P}_q^2 = \text{Proj}(A)$, $\mathcal{O} = \mathcal{A}$, etc. for left and right objects.
Lemma 4. Let $\mathcal{M}$ be a $\mathbb{P}^2$-module of infinite length. Then, for all $k \gg 0$, the result of the lemma follows now immediately from the obvious invariance of $H^i(\mathbb{P}^2, \mathcal{M}(k))$. Indeed, the vanishing of $H^i(\mathbb{P}^2, \mathcal{M}(k))$ depends polynomially on $k$, we may assume $k \gg 0$ in which case we have

$$(3.7) \quad \chi(\mathbb{P}^2, \mathcal{M}(k)) = \dim \text{Hom}(\mathcal{M}, \mathcal{O}(k)) \equiv H^2(\mathbb{P}^2, \mathcal{M}(-k - 3)) \, .$$

The first equality in (3.7) follows from the definition of $\mathcal{M}$ and Serre’s Vanishing theorem; the second is a consequence of Serre’s Duality (cf. [BW3], Theorem 2.4). Both theorems of Serre can be applied in our situation because, as shown in [KKO], $\mathcal{M} \in \text{Coh}(\mathbb{P}^2)$ if and only if $\mathcal{M} \in \text{Coh}(\mathbb{P}^2)$. Since we know that both $H^0(\mathbb{P}^2, \mathcal{M}(d))$ and $H^1(\mathbb{P}^2, \mathcal{M}(d))$ vanish for $d \ll 0$ (see [BW3], Lemma 10.2), the last term in (3.7) coincides with the Euler characteristic $\chi(\mathbb{P}^2, \mathcal{M}(-k - 3))$. The result of the lemma follows now immediately from the obvious invariance of $\chi(\mathbb{P}^2, \mathcal{M}(k))$ under the change of twisting $k \mapsto -k - 3$. \hfill $\square$

Remark. The result of Lemma 3 is specific for our noncommutative situation. Indeed, the vanishing of $H^1(\mathbb{P}^2, \mathcal{M}(d))$ for all $d \ll 0$ is equivalent to the fact that $\mathcal{M}$ is a bundle (in the sense of [KKO]) which is not true for extensions of ideals to $\mathbb{P}^2$ in the commutative case.

We proceed now to the final step of our calculation.

Lemma 4. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two rank 1 bundles on $\mathbb{P}^2$ trivial over the line at infinity. Then, for all $k \gg 0$ we have

$$\dim \text{Hom}(\mathcal{M}_1, \mathcal{M}_2(k)) = \frac{1}{2}(k + 1)(k + 2) - n_1 - n_2 ,$$

where $n_1 = c_2(\mathcal{M}_1)$ and $n_2 = c_2(\mathcal{M}_2)$.

Proof. We start with a locally free resolution of $\mathcal{M}_2$ in $\text{Coh}(\mathbb{P}^2)$ (see, for example, [LB], Corollary 1):

$$0 \rightarrow \bigoplus_{i=1}^{m-1} \mathcal{O}(-k_i) \rightarrow \bigoplus_{j=1}^{m} \mathcal{O}(-l_j) \rightarrow \mathcal{M}_2 \rightarrow 0 , \quad k_i, l_j \in \mathbb{Z}_+ .$$

Shifting (3.8) by $k$ and applying the functor $\text{Hom}(\mathcal{M}_1, -)$ yields

$$0 \rightarrow \bigoplus_{i=1}^{m-1} \text{Hom}(\mathcal{M}_1, \mathcal{O}(-k_i)) \rightarrow \bigoplus_{j=1}^{m} \text{Hom}(\mathcal{M}_1, \mathcal{O}(k - l_j))$$

$$\rightarrow \text{Hom}(\mathcal{M}_1, \mathcal{M}_2(k)) \rightarrow \bigoplus_{i=1}^{m-1} \text{Ext}^1(\mathcal{M}_1, \mathcal{O}(-k_i)) \rightarrow \ldots .$$

If $k \gg 0$ the Ext term in (3.9) is zero because $\mathcal{M}_1$ is a bundle. Therefore, in that case we have

$$\dim \text{Hom}(\mathcal{M}_1, \mathcal{M}_2(k)) =$$

$$= \sum_{j=1}^{m} \dim \text{Hom}(\mathcal{M}_1, \mathcal{O}(k - l_j)) - \sum_{i=1}^{m-1} \dim \text{Hom}(\mathcal{M}_1, \mathcal{O}(k - k_i)) .$$

Proof. Since $\mathcal{M}$ is a torsion-free sheaf of rank 1, the first statement of the lemma follows from the second. On the other hand, in (3.7) we need only to prove the first equality (the second follows immediately from [BW3], Theorem 4.5). Furthermore, since $\chi(\mathbb{P}^2, \mathcal{M}'(k))$ depends polynomially on $k$, we may assume $k \gg 0$ in which case we have

$$(3.7) \quad \chi(\mathbb{P}^2, \mathcal{M}'(k)) = \dim \text{Hom}(\mathcal{M}, \mathcal{O}(k)) = H^2(\mathbb{P}^2, \mathcal{M}(-k - 3)) .$$

The first equality in (3.7) follows from the definition of $\mathcal{M}'$ and Serre’s Vanishing theorem; the second is a consequence of Serre’s Duality (cf. [BW3], Theorem 2.4). Both theorems of Serre can be applied in our situation because, as shown in [KKO], $\mathcal{M} \in \text{Coh}(\mathbb{P}^2)$ if and only if $\mathcal{M} \in \text{Coh}(\mathbb{P}^2)$. Since we know that both $H^0(\mathbb{P}^2, \mathcal{M}(d))$ and $H^1(\mathbb{P}^2, \mathcal{M}(d))$ vanish for $d \ll 0$ (see [BW3], Lemma 10.2), the last term in (3.7) coincides with the Euler characteristic $\chi(\mathbb{P}^2, \mathcal{M}(-k - 3))$. The result of the lemma follows now immediately from the obvious invariance of $\chi(\mathbb{P}^2, \mathcal{M}(k))$ under the change of twisting $k \mapsto -k - 3$. \hfill $\square$
By Serre’s Vanishing theorem we may replace dimensions of each $\text{Hom}$ term in the right-hand side of (3.10) by the Euler characteristics of the corresponding twists of the dual sheaf $\mathcal{M}_1^\vee$ and then use repeatedly Lemma 3. Thus, if $k \gg 0$, we have

$$
\dim \text{Hom} (\mathcal{M}_1, \mathcal{M}_2(k)) = \sum_{j=1}^{m} \chi (\mathbb{P}^2_q, \mathcal{M}_1^\vee(k - l_j)) - \sum_{i=1}^{m-1} \chi (\mathbb{P}^2_q, \mathcal{M}_1^\vee(k - k_i))
$$

$$
= \sum_{j=1}^{m} \chi (\mathbb{P}^2_q, \mathcal{M}_1(k - l_j)) - \sum_{i=1}^{m-1} \chi (\mathbb{P}^2_q, \mathcal{M}_1(k - k_i))
$$

$$
= \left( \sum_{j=1}^{m} \chi (\mathbb{P}^2_q, \mathcal{O}(k - l_j)) - mn_1 \right) - \left( \sum_{i=1}^{m-1} \chi (\mathbb{P}^2_q, \mathcal{O}(k - k_i)) - (m - 1)n_1 \right)
$$

$$
= \left( \sum_{j=1}^{m} \chi (\mathbb{P}^2_q, \mathcal{O}(k - l_j)) - \sum_{i=1}^{m-1} \chi (\mathbb{P}^2_q, \mathcal{O}(k - k_i)) \right) - n_1
$$

$$
= \chi (\mathbb{P}^2_q, \mathcal{M}_2(k)) - n_1 = \frac{1}{2} (k + 1)(k + 2) - n_1 - n_2.
$$

Note the expression in last parentheses being equal to $\chi (\mathbb{P}^2_q, \mathcal{M}_2(k))$ follows immediately from the exact sequence (3.8). □

Finally, taking $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ and comparing Lemma 4 with formulas (2.3) and (3.1) of Lemmas 1 and 2 we arrive at our main

**Theorem 2.** Let $M$ be a finitely generated rank 1 torsion-free $A$-module whose canonical extension to $\mathbb{P}^2_q$ has the ‘second Chern class’ $n$. Let $D := \text{End}_A(M)$. Then the LM-invariant of the algebra $D$ is given by the formula $p_D = 2n$.

### 4. Generalizations

We would like to mention two natural generalizations suggested by the results of this paper. First, Lemma 4 hints at the possibility of defining a relative version of LM-invariants: given two ideal classes in $A$ with distinguished representatives $M_1$ and $M_2$ (say), we may identify $\text{Hom}_A(M_1, M_2)$ with a subspace in $Q$ (as in (2.1)), and then define the number $p_{12}$ as the codimension of the associated graded space $G\text{Hom}_A(M_1, M_2)$ in $GA$. The argument similar to that of (LM), Proposition 2.4, shows $p_{12} < \infty$, and Lemma 4 then immediately implies the relation $p_{12} = n_1 + n_2$, where $n_1 = c_2(M_1)$ and $n_2 = c_2(M_2)$.

Second, the interpretation of LM-invariants as arithmetic genera of noncommutative projective surfaces (Lemma 1) suggests an obvious generalization to higher dimensions. As in the original situation of (LM), one would expect these generalized LM-invariants to be useful in the study of rings of differential operators on singular rational algebraic varieties. An interesting problem in this direction would be to compute the LM-type invariants for the algebras of differential operators on varieties of quasi-invariants of finite reflection groups (see (BEC)).
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