Generalized Hamacher Aggregation Operators Based on Linear Diophantine Uncertain Linguistic Setting and Their Applications in Decision-Making Problems

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ABSTRACT

Hamacher aggregation operators are more flexible and more dominant to determine the interrelationships between any number of attributes. The goal of this manuscript is to elaborate on the principle of linear Diophantine uncertain linguistic sets and explored their useful Hamacher operational laws. The existing notions of intuitionistic uncertain linguistic sets, Pythagorean uncertain linguistic sets, and q-rung orthopair uncertain linguistic sets have certain applications in different fields. Unfortunately, these theories have their limitations related to the truth and falsity grades. To eradicate these limitations, the theory of linear Diophantine uncertain linguistic sets with the addition of reference parameters is massive flexible than the existing drawbacks. This notion removes the restrictions of prevailing methodologies, and the decision-maker can freely choose the grades without any limitations. This structure also categorizes the problem by changing the physical sense of reference parameters. Moreover, by using the investigated linear Diophantine uncertain linguistic information and Hamacher aggregation operators, we explored the linear Diophantine uncertain linguistic generalized Hamacher averaging operator and linear Diophantine uncertain linguistic generalized Hamacher hybrid averaging operator. Additionally, a multi-attribute decision-making (MADM) procedure is buildup based on the investigated operators under the linear Diophantine uncertain linguistic information. Certain numerical examples are illustrated by using initiated operators to determine the dominance and flexibility of explored operators. To find the consistency and supremacy of the presented operators, we compare the proposed work with certain prevailing operators and discussed their geometrical expressions to show that the introduced operators in this manuscript are extensively powerful and more useful than the prevailing drawbacks.

INDEX TERMS

Generalized Hamacher aggregation operators, linear Diophantine uncertain linguistic sets, multi-attribute decision-making problems.

I. INTRODUCTION

The principle of the intuitionistic fuzzy set (IFS) was developed by Atanassov [1], by including the non-membership grade (NMG) \(v_{\text{NMG}}\) as the generalization of the fuzzy set (FS) [2]. The theory of IFS has extensive effectiveness and is more dominant as compared to FS, which deals with two-dimension information in the form of membership grade (MG) \(u_{\text{MG}}\) and NMG \(v_{\text{NMG}}\) at a time. The prominent characteristic of IFS is that the \(0 \leq u_{\text{MG}} + v_{\text{NMG}} \leq 1\). Certain scholars have used it in the environment of distinct fields, for instance, Beg and Rashid [3] utilized the principle of intuitionistic hesitant fuzzy sets. Moreover, Mahmood et al. [4] explored the improved intuitionistic...
hesitant fuzzy sets and discussed their application. Kumari and Mishra [5] initiated the parametric measures based on IFSs. Liu et al. [6] developed the variable-based hybrid approach by using the interval-valued IFSs. The similarity measures under the right-angle triangle based on IFSs were initiated by Garg and Rani [7]. Xue and Deng [8] explored the decision-making troubles based on measures under the granular uncertainty for IFSs. Thao [9] explored the entropy and diverges measures based on IFSs and discussed their application in supplier chain management. Jana and Pal [10] evaluated the bipolar intuitionistic fuzzy soft sets and their applications. Meng and He [11] developed the geometric interaction aggregation operators for IFSs.

The principle of IFS has achieved much attention from distinct scholars, but in certain situations, the principle of IFS has been neglected. For instance, if an individual provides information in the form of 0 ≤ u_{MG}^q + v_{NMG}^q ≤ 1, then the principle of IFS cannot be working dominantly. For managing such complicated sorts of troubles, the flexible theory of Pythagorean FS (PFS) was initiated by Yager [12]. The prominent characteristic of PFS is that the 0 ≤ u_{MG}^q + v_{NMG}^q ≤ 1 is that 0.6^2+0.5^2 = 0.36+0.25 = 0.66 < 1. Certain scholars have used it in the environment of distinct fields, for instance, Garg [13] initiated the linguistic PFSs, Wei and Wei [14] explored certain measures based on PFSs using cosine function, Xiao and Ding [15] proposed the diverges measures for PFSs, Ullah et al. [16] initiated certain distance measures for complex PFSs, Li and Lu [17] developed the similarity and distance measures for PFSs, Garg [18] investigated the improve the accuracy function based on interval-valued PFSs, Yang and Hussain [19] explored the entropy measures based on PFSs.

Still, the principle of PFS has been neglected in certain situations, for instance, if an individual provides information in the form of 0.9 for MG u_{MG} and 0.8 NMG v_{NMG} with a rule that is u_{MG}^q + v_{NMG}^q = 0.9^q+0.8^q = 0.81+0.64 = 1.45 > 1, then the principle of PFS cannot be working dominantly. For managing such complicated sorts of troubles, the flexible theory of q-rung orthopair fuzzy sets (QROFS) was initiated by Yager [20]. The prominent characteristic of QROFS is that the 0 ≤ u_{MG}^q + v_{NMG}^q ≤ 1. q ≥ 1. The principle of IFS and PFS is the particle cases of the QROFS by using the value of q = 1 and q = 2. Certain scholars have used it in the environment of distinct fields, for instance, Ali [21] initiated another view of QROFSs, Liu and Wang [22] proposed aggregation operators for QROFSs, Peng and Liu [23] initiated the information measures based on QROFSs, Wang et al. [24] developed certain measures based on QROFSs using cosine function, Ali and Mahmood [25] investigated the Macaulay symmetric mean operators based on QROFSs, Liu et al. [26] initiated the cosine similarity and distance measures for QROFSs, Liu and Wang [27] proposed Archimedean Bonferroni mean operators for QROFSs, Lin et al. [28] developed certain Heronian mean operators based on linguistic QROFSs, Khan et al. [29] initiated the knowledge measures by using QROFSs.

Still, the principle of QROFS has been neglected in certain situations, for instance, if an individual provides information in the form of 1 for MG u_{MG} and 0.1 NMG v_{NMG} with a rule that is u_{MG}^q + v_{NMG}^q = 1^q+0.1^q > 1 for any value of q, then the principle of QROFS cannot be working dominantly. For managing such complicated sorts of troubles, the flexible theory of linear Diophantine FS (LDFS) was initiated by Riaz and Hashmi [30]. The prominent characteristic of LDFS is that the 0 ≤ u_{MG}^q + v_{NMG}^q ≤ 1. The principle of IFSs, PFS, and QROFS are the particle cases of the LDFO that is 0 ≤ 0.1*1+0*0.1 = 0.1 ≤ 1. Certain scholars have used it in the environment of distinct fields, for instance, Riaz et al. [31] developed the linear Diophantine fuzzy rough sets and their applications, Kamaci [32] explored the algebraic structure based on linear Diophantine fuzzy sets, Ayub et al. [33] explored the linear Diophantine fuzzy relation and their application in decision-making.

In numerous situations, the principle of FS has been failed, due to its structure, for instance, if an individual faces information in the shape of very good, good, normal, weak, very weak, then the principle of FS has cannot working dominantly. To manage such sorts of awkward situations, the principle linguistic variable (LV) was explored by Zadeh [34]. Moreover, Herrera and Martinez [35] modified the principle of LV is to explore the theory of a 2-tuple linguistic set (2-TLS), and Xu [36] investigated the uncertain LV (ULV). Liu et al. [37] initiated the Heronian mean operators for intuitionistic uncertain linguistic sets, Xu [38] developed the intuitionistic fuzzy aggregation operators, Xu and Yager [39] proposed certain geometric aggregation operators for IFSs. The principle of LDFO has been neglected in certain situations, for instance, if an individual provides information in the form of MG, NMG, and uncertain linguistic terms, then the principle of LDFO has cannot working dominantly. For managing such complicated sorts of troubles, the flexible theory of linear Diophantine uncertain linguistic set is initiated in this manuscript. Based on the above analysis, the main theme of the elaborated approaches in this study are discussed below:

1. To elaborate on the principle of LDULS and their algebraic laws.
2. To develop generalized Hamacher aggregation operators and discussed their certain important cases.
3. To investigate a MADM technique by using proposed operators.
4. To illustrate certain examples under on the proposed operators by using LDULNs.
5. To discuss the supremacy of the elaborated operators is also presented with the help of comparative analysis and geometrical expressions.

The rest of this manuscript is summarized in the following ways: In section 2, we review the main idea of LDFO, ULs, and their important laws which are very useful for the investigated works. In section 3, we elaborated the principle
of linear Diophantine uncertain linguistic sets and elaborated their useful Hamacher operational laws. In section 4, by using the elaborated linear Diophantine uncertain linguistic information and Hamacher aggregation operators, we explored the linear Diophantine uncertain linguistic generalized Hamacher averaging operator and linear Diophantine uncertain linguistic generalized Hamacher hybrid averaging operator are discovered. In section 5, a multi-attribute decision-making (MADM) procedure is buildup based on the investigated operators under the linear Diophantine uncertain linguistic information. Certain numerical examples are illustrated using the elaborated operators to determine the dominance and flexibility of explored operators. To find the consistency and supremacy of the elaborated operators, we compared the proposed work with certain prevailing operators and discussed their geometrical expressions to show that the elaborated operators in this manuscript are extensively powerful and more useful than the prevailing drawbacks. In section 6, we presented the conclusion of this study.

II. PRELIMINARIES

The flexible theory of linear Diophantine FS (LDFS) was initiated by Riaz and Hashmi [30]. The prominent characteristic of LDFS is that the \( 0 \leq \alpha_{AMG} + \beta_{VAMG} \leq 1 \). The principle of IFSs, PFS, and QROFS are the particle cases of LDFS. In this study, we review the main idea of LDFS, ULS, and their important laws which are very useful for the investigated works. Additionally, the symbol \( X \) is expressed the fixed set in all studies.

Definition 1 [30]: A LDFS \( A_{LD} \) is elaborated by:

\[
A_{LD} = \{(q, (u_{AMG}(q), v_{ANG}(q)), (\alpha_{AMG}, \beta_{ANG})) / q \in X \}
\]

(1)

With the rules of \( 0 \leq \alpha_{AMG} + \beta_{VAMG} \leq 1 \), \( u_{AMG}(q), v_{ANG}(q), \alpha_{AMG}, \beta_{ANG} \in [0, 1] \) and \( 0 \leq \alpha_{AMG} + \beta_{ANG} \leq 1 \). The symbol \( \xi_{LD}(q)_{A_{LD}} = 1 - \alpha_{AMG} - \beta_{VAMG} \) expresses the refusal grade. Simply \( A_{LD} = ((u_{AMG}(q), v_{ANG}(q)), (\alpha_{AMG}, \beta_{ANG})) \) is called linear Diophantine fuzzy number (LDFN).

Definition 2 [30]: For any two LDFNs \( A_{LD} = (u_{AMG}, v_{ANG}), (\alpha_{AMG}, \beta_{ANG}) \), \( i = 1, 2 \), then (2)–(5), as shown at the bottom of the page.

Definition 3 [34]: A LTS is elaborated by:

\[ S = \{ \Xi_{1}, \Xi_{2}, \ldots, \Xi_{\Gamma} \} \]

(6)

where \( \Gamma \) must be odd and keep the resulting circumstances:

1. If \( \Gamma > \Gamma' \), then \( \Xi_{\Gamma} > \Xi_{\Gamma'} \);
2. The pessimistic operator \( \text{reg} (\Xi_{\Gamma}) = \Xi_{\Gamma} \) with a condition \( \Gamma + \Gamma' = \Gamma - 1 \);
3. If \( \Gamma \geq \Gamma' \), max (\( \Xi_{\Gamma}, \Xi_{\Gamma'} \)) = \( \Xi_{\Gamma} \), and if \( \Gamma \leq \Gamma' \), min (\( \Xi_{\Gamma}, \Xi_{\Gamma'} \)) = \( \Xi_{\Gamma} \).

Let \( \tilde{S} = [\Xi_{a_{1}}, \Xi_{b_{1}}] \) where \( \Xi_{a_{1}}, \Xi_{b_{1}} \in S \), \( \Xi_{a_{1}} \) and \( \Xi_{b_{1}} \) are the lower and the upper limits, respectively. We call \( \tilde{S} \) the uncertain linguistic variable.

Definition 4 [36]: For any two ULVs \( \tilde{S}_{1} = [\Xi_{a_{1}}, \Xi_{b_{1}}] \) and \( \tilde{S}_{2} = [\Xi_{a_{2}}, \Xi_{b_{2}}] \):

\[
\tilde{S}_{1} \otimes \tilde{S}_{2} = [\Xi_{a_{1}} + \Xi_{b_{1}} \Xi_{a_{2}} + \Xi_{b_{2}}] = [\Xi_{a_{1a_{2}}} + \Xi_{a_{1b_{2}}} + \Xi_{b_{1a_{2}}} + \Xi_{b_{1b_{2}}} ]
\]

(7)

\[
\tilde{S}_{1} \ominus \tilde{S}_{2} = [\Xi_{a_{1}} - \Xi_{b_{1}} \Xi_{a_{2}} - \Xi_{b_{2}}] = [\Xi_{a_{1a_{2}}} - \Xi_{a_{1b_{2}}} - \Xi_{b_{1a_{2}}} + \Xi_{b_{1b_{2}}} ]
\]

(8)

\[
(\tilde{S}_{1})^{\lambda} = [\Xi_{(a_{1})^{\lambda}, (b_{1})^{\lambda}}]
\]

(9)

Based on the above information, we elaborate on the new principle of LDULS and its laws.

III. LINEAR DIOPHANTINE UNCERTAIN LINGUISTIC SETS

The principle of LDLSF has been neglected in certain situations, for instance, if an individual provides information in the form of MG, NMG, and uncertain linguistic terms, then the principle of LDLSF has not working dominantly. For managing such complicated sorts of troubles, the flexible theory of linear Diophantine uncertain linguistic set and their important laws are initiated in this manuscript.

Definition 5: A LDULS \( A_{LDULS} \) is elaborated by:

\[
A_{LDULS} = \{ \{ q, [\Xi_{\theta(q)}, \Xi_{\tau(q)}], (u_{AMG}(q), v_{ANG}(q)), (\alpha_{AMG}, \beta_{ANG}) \} / q \in X \}
\]

(11)

With the rules of \( 0 \leq \alpha_{AMG} + \beta_{VANG} \leq 1 \), \( u_{AMG}(q), v_{ANG}(q), \alpha_{AMG}, \beta_{ANG} \in [0, 1] \) and \( 0 \leq \alpha_{AMG} + \beta_{ANG} \leq 1 \). The symbol \( \xi_{LD}(q)_{A_{LD}} = 1 - \alpha_{AMG} - \beta_{VANG} \) expresses the refusal grade with \( \Xi_{\theta(q)}, \Xi_{\tau(q)} \in \tilde{S} \). Simply \( A_{LD} = ([\Xi_{\theta(q)}, \Xi_{\tau(q)}], (u_{AMG}(q), v_{ANG}(q)), (\alpha_{AMG}, \beta_{ANG})) \) expresses the linear Diophantine uncertain linguistic number (LDULN).

Definition 6: For any two LDULNs

\[
A_{LD-i} = [[\Xi_{A_{-i}}, \Xi_{A_{-i}}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i})], i = 1, 2
\]

(12)

\[
\lambda A_{LD-i} = \left[ (1-(1-u_{AMG-i}(q))^\lambda, (v_{ANG-i}(q))^\lambda), (1-(1-\alpha_{AMG-i}(q))^\lambda, (\beta_{ANG-i}(q))^\lambda) \right]
\]

(13)

\[
A_{LD-i} = \left[ (u_{AMG-i}(q))^\lambda, 1-(1-v_{ANG-i}(q))^\lambda, (\alpha_{AMG-i}(q))^\lambda, 1-(1-\beta_{ANG-i}(q))^\lambda \right]
\]

(14)
\( A_{LD-1} \oplus A_{LD-2} \)
\[
\left(\begin{array}{c}
\Xi A_{-\theta_1(\theta)+A-\theta_2(\theta)}, \\
\Xi A_{-t_1(\theta)+A-t_2(\theta)}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right)
\]
(12)

\( A_{LD-1} \otimes A_{LD-2} \)
\[
\left(\begin{array}{c}
\Xi A_{-\theta_1(\theta) \times A-\theta_2(\theta)}, \\
\Xi A_{-t_1(\theta) \times A-t_2(\theta)}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right)
\]
(13)

\( \lambda_{LD-1} \)
\[
\left(\begin{array}{c}
\Xi A_{-\theta_1(\theta) \times A-\theta_2(\theta)}, \\
\Xi A_{-t_1(\theta) \times A-t_2(\theta)}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right)
\]
(14)

\( A_{LD-1}^\lambda = \left(\begin{array}{c}
\Xi A_{-\theta_1(\theta) \times A-\theta_2(\theta)}, \\
\Xi A_{-t_1(\theta) \times A-t_2(\theta)}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
u_{AMG-1}^i u_{AMG-2}, \\
v_{AMG-1}^i v_{AMG-2}, \\
\alpha_{AMG-1}^i \alpha_{AMG-2}, \\
\beta_{AMG-1}^i \beta_{AMG-2}
\end{array}\right)
\]
(15)

**Theorem 1:** For any two LDULNs

\( A_{LD-i} = \left(\begin{array}{c}
\Xi A_{-\theta_i(\theta)}, \\
\Xi A_{-t_i(\theta)}
\end{array}\right), (u_{AMG-i}, v_{AMG-i}), (\alpha_{AMG-i}, \beta_{AMG-i}), \) \( i = 1, 2, \) then

1. \( A_{LD-1} \circ A_{LD-2} = A_{LD-2} \circ A_{LD-1}; \)
2. \( A_{LD-1} \circ A_{LD-2} = A_{LD-2} \circ A_{LD-1}; \)
3. \( \lambda (A_{LD-1} \circ A_{LD-2}) = \lambda A_{LD-1} \circ \lambda A_{LD-2}, \) \( \lambda > 0; \)
4. \( \lambda_1 A_{LD-1} \circ \lambda_2 A_{LD-1} = (\lambda_1 + \lambda_2) A_{LD-1}, \) \( \lambda_1, \lambda_2 > 0; \)
5. \( A_{LD-1}^\lambda \circ A_{LD-2}^\lambda = A_{LD-1}^{\lambda + \lambda_2}, \) \( \lambda_1, \lambda_2 > 0; \)
6. \( A_{LD-1}^\lambda \circ A_{LD-2}^\lambda = (A_{LD-1} \circ A_{LD-2})^\lambda, \) \( \lambda_1 > 0. \)

**IV. HAMACHER OPERATIONS AND THEIR OPERATORS FOR LINEAR DIOPHANTINE UNCERTAIN LINGUISTIC SETS**

To determine the elaborated Hamacher laws for investigated LDULNs, we revise the basics of Hamacher laws which are discussed below.

\[
T_\zeta (x, y) = \frac{xy}{\zeta + (1-\zeta)(x+y-xy)}, \quad \zeta > 0
\]
(16)

\[
S_\zeta (x, y) = \frac{x+y-xy-(1-\zeta)xy}{1-(1-\zeta)xy}, \quad \zeta > 0
\]
(17)

For \( \zeta = 1, \) the Hamacher laws are converted for Algebraic laws such that

\[
T (x, y) = xy
\]
(18)

\[
S (x, y) = x+y-xy
\]
(19)

For \( \zeta = 2, \) the Hamacher laws are converted for Einstein laws such that

\[
T (x, y) = \frac{xy}{1+(1-x)(1-y)}
\]
(20)

\[
S (x, y) = \frac{x+y}{1+xy}
\]
(21)

Based on the above analysis, we contract certain operational laws which are very important for the elaborated operators.

For any two LDULNs

\( A_{LD-i} = \left(\begin{array}{c}
\Xi A_{-\theta_i(\theta)}, \\
\Xi A_{-t_i(\theta)}
\end{array}\right), (u_{AMG-i}, v_{AMG-i}), \)

\( (\alpha_{AMG-i}, \beta_{AMG-i}), \) \( i = 1, 2, \)

and \( \zeta > 0, \lambda \geq 0, \) then (22), as shown at the bottom of the page, (23) and (24), as shown at the bottom of the next page, and (25), as shown at the bottom of page 6.

**Theorem 2:** For any two LDULNs

\( A_{LD-i} = \left(\begin{array}{c}
\Xi A_{-\theta_i(\theta)}, \\
\Xi A_{-t_i(\theta)}
\end{array}\right), (u_{AMG-i}, v_{AMG-i}), \)

\( (\alpha_{AMG-i}, \beta_{AMG-i}), \) \( i = 1, 2, \) \( \zeta > 0, \)

1. \( A_{LD1} \circ b A_{LD2} = A_{LD2} \circ b A_{LD1}; \)
2. \( A_{LD1} \circ b A_{LD2} = A_{LD2} \circ b A_{LD1}; \)

\[
\lambda A_{LD-1} = \left(\begin{array}{c}
\Xi A_{-\theta_1(\theta)}, \\
\Xi A_{-t_1(\theta)}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
\frac{1+(\zeta-1)\lambda^*}{u_{AMG-1}(\theta)} - \frac{1-\nu_{AMG-1}(\theta)}{\lambda^*}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
\frac{1+(\zeta-1)\lambda^*}{u_{AMG-1}(\theta)} + (\zeta-1)\frac{1-\nu_{AMG-1}(\theta)}{\lambda^*}
\end{array}\right),
\]
\[
\left(\begin{array}{c}
\xi (v_{AMG-1}(\theta))^\lambda
\end{array}\right),
\]
\[
\left(\begin{array}{c}
\frac{1+(\zeta-1)\lambda^*}{1-v_{AMG-1}(\theta)} + (\zeta-1)(v_{AMG-1}(\theta))^\lambda
\end{array}\right),
\]
\[
\left(\begin{array}{c}
\frac{1+(\zeta-1)(\alpha_{AMG-1})^\lambda-(1-\alpha_{AMG-1})^\lambda}{(1+(\zeta-1)(\beta_{AMG-1})^\lambda+(\zeta-1)(\beta_{AMG-1})^\lambda)}
\end{array}\right)
\]
(22)
3. \( \lambda (A_{LD_1} \oplus hA_{LD_2}) = \lambda A_{LD_1} \oplus hA_{LD_2} \), \( \lambda > 0 \);
4. \( \lambda_1 A_{LD_1} \oplus h_2 A_{LD_1} = (\lambda_1 + \lambda_2) A_{LD_1}, \lambda_1, \lambda_2 > 0 \);
5. \( A_{LD_1}^{\lambda_1} \oplus hA_{LD_1}^{\lambda_2} = A_{LD_1}^{\lambda_1 + \lambda_2}, \lambda_1, \lambda_2 > 0 \);
6. \( A_{LD_1}^{\lambda_1} \oplus hA_{LD_2}^{\lambda_2} = (A_{LD_1} \oplus hA_{LD_2})^{\lambda_1}, \lambda_1 > 0 \);

**Proof:** The proof of Eq. (1) and Eq. (2) is straightforward. We only prove that Eq. (3), the Eq. (4) to Eq. (6) are similar. \( \lambda (A_{LD_1} \oplus hA_{LD_2}), \) as shown at the bottom of page 7.

By using the elaborated linear Diophantine uncertain linguistic information and Hamacher aggregation operators, we explored the linear Diophantine uncertain linguistic generalized Hamacher averaging operator and linear Diophantine uncertain linguistic generalized Hamacher hybrid averaging operator are discovered.

\[
A_{LD_1}^\lambda = \begin{cases} 
\frac{1 + (\xi - 1) \ast (1 - u_{AMG_1}(q))}{1 + (\xi - 1) \ast v_{ANG_1}(q)} + (\xi - 1) \left[ u_{AMG_1}(q) \right]^{\lambda}, \\
\frac{1 + (\xi - 1) \ast (1 - v_{ANG_1}(q))}{1 + (\xi - 1) \ast (1 - u_{AMG_1}(q))} + (\xi - 1) \left[ 1 - v_{ANG_1}(q) \right]^{\lambda} \\
\frac{1 + (\xi - 1) \ast (1 - v_{ANG_1}(q))}{1 + (\xi - 1) \ast (1 - u_{AMG_1}(q))} + (\xi - 1) \left[ 1 - u_{AMG_1}(q) \right]^{\lambda} \\
\end{cases}
\]

**Definition 7:** For any collection of LDULNs \( A_{LD-i} = \left( [\Xi_{A-i}, \Xi_{A-i}], (u_{AMG-i}, v_{ANG-i}) \right), \) \( i = 1, 2, \ldots, n \), \( \zeta > 0, \) then linear Diophantine uncertain linguistic generalized Hamacher weighted averaging (LDULGHWA) operator is initiated by:

\[
LDULGHWA (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) = \left( \sum_{i=1}^{n} \omega_i A_{LD-i}^\lambda \right)^\lambda
\]

where \( \lambda > 0, \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is the weight vector of \( (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}), \) with a rule that is \( \omega_i \in [0, 1], \) \( \sum_{i=1}^{n} \omega_i = 1. \)

Based on the above analysis, we discuss
the monotonicity, idempotency, and boundedness to improve the quality of the proposed works.

**Property 1:** For any collection of LDULNs \((A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})\) and \((B_{LD-1}, B_{LD-2}, \ldots, B_{LD-n})\),

1. If \(A_{LD-i} \leq B_{LD-i}\) for all \(i = 1, 2, \ldots, n\) then \(LDULGHWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq LDULGHWA(B_{LD-1}, B_{LD-2}, \ldots, B_{LD-n})\).

2. \(A_{LD-i} = A, i = 1, 2, \ldots, n\) then \(LDULGHWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) = A\).

3. If the LDULGHWA operator lies between the max and min operators, then \(min(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq LDULGHWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq max(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})\).

**Definition 8:** For any collection of LDULNs \(A_{LD-i} = \left(2\sum_{a=1}^{n} a_{AMG-i}, v_{ANG-i}\right), i = 1, 2, \ldots, n, \right)\), \(\zeta > 0\), then linear Diophantine uncertain linguistic generalized Hamacher ordered weighted averaging (LDULGHOWA) operator is initiated by:

\[
LDULGHOWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) = \left(2\sum_{a=1}^{n} \omega a_{AMG-i}^{\frac{1}{\zeta}} \omega_{AMG-i}, v_{ANG-i}\right)^{\frac{1}{\zeta}} \tag{27}
\]

where \(\lambda > 0, \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T\) is the weight vector of \((A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})\), \(\omega_i \in [0, 1], \sum_{i=1}^{n} \omega_i = 1\), and \((\sigma_1, \sigma_2, \ldots, \sigma_n)\) is a permutation of \((1, 2, \ldots, n)\) s.t. \(\sigma_{i-1} \geq \sigma_i\) for all \(i\). Based on the above analysis, we discuss the monotonicity, idempotency, and boundedness to improve the quality of the proposed works.

**Property 2:** For any collection of LDULNs \((A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})\) and \((B_{LD-1}, B_{LD-2}, \ldots, B_{LD-n})\),

1. If \(A_{LD-i} \leq B_{LD-i}\) for all \(i = 1, 2, \ldots, n\) then \(LDULGHOWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq LDULGHOWA(B_{LD-1}, B_{LD-2}, \ldots, B_{LD-n})\).

2. \(A_{LD-i} = A, i = 1, 2, \ldots, n\) then \(LDULGHOWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) = A\).

3. If LDULGHOWA operator lies between the max and min operators, then \(min(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq LDULGHOWA(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq max(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})\).

Additionally, by using the value of parameters, we will discuss certain special cases of the elaborated operators.

1. For \(\lambda = 1\), then the LDULGHOWA operator is converted for LDULWA operator, such that (28), as shown on the bottom of page 8.

2. For \(\zeta = 1\), then the LDULWA operator is converted for LDULWA operator, such that (29), as shown on the bottom of page 8.

3. For \(\zeta = 2\), then the LDULWA operator is converted for LDULWA operator, such that

**ELDULWA** \((A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})\)

\[
\left[\sum_{i=1}^{n} a_{AMG-i}^{\frac{1}{\zeta}} \omega_{AMG-i}, v_{ANG-i}\right]_{\frac{1}{\zeta}} = \left[\sum_{i=1}^{n} (1+u_{AMG}, \omega_{AMG-i}), \omega_{AMG-i}, v_{ANG-i}\right]_{\frac{1}{\zeta}} \tag{30}
\]

\[
A_{LD-1}, \otimes_{\zeta} A_{LD-2} = \left[\sum_{i=1}^{n} A_{LD-i} \otimes_{\zeta} B_{LD-i}\right]_{\frac{1}{\zeta}} \tag{25}
\]
$$\lambda (A_{LD_1} \oplus h A_{LD_2})$$

$$= \left( \begin{array}{c}
\sum_{i=1}^{n} A_{-\theta_i (q)} \ln \left( \frac{\left( 1 - \alpha_{AMG_1, uAMG_2 (q)} \right) \beta_{ANG_1, vANG_2 (q)}}{\left( 1 - \alpha_{AMG_1, uAMG_2 (q)} \right) \beta_{ANG_1, vANG_2 (q)}} \right) \\
\sum_{i=1}^{n} A_{-\tau_i (q)} \ln \left( \frac{\left( 1 - \alpha_{AMG_1, uAMG_2 (q)} \right) \beta_{ANG_1, vANG_2 (q)}}{\left( 1 - \alpha_{AMG_1, uAMG_2 (q)} \right) \beta_{ANG_1, vANG_2 (q)}} \right)
\end{array} \right)$$

$$= \left( \begin{array}{c}
\left[ A_{-\theta_1 (q)}, A_{-\tau_1 (q)} \right] \\
\frac{(1+\left( 1-\xi \right) u_{AMG_1 (q)} + u_{AMG_2 (q)} - u_{AMG_1 (q)} u_{AMG_2 (q)})^{\lambda} - (1-\left( 1-\xi \right) u_{AMG_1 (q)} u_{AMG_2 (q)})^{\lambda}}{(1+\left( 1-\xi \right) \left( 1-\left( 1-\xi \right) u_{AMG_1 (q)} u_{AMG_2 (q)} \right))^{\lambda} + (1-\left( 1-\xi \right) (1-\left( 1-\xi \right) u_{AMG_1 (q)} u_{AMG_2 (q)}))^{\lambda}} \\
\left( 1+\left( 1-\xi \right) \left( 1-\left( 1-\xi \right) \alpha_{AMG_1, \alpha_{AMG_2}} \right)^{\lambda} - (1-\left( 1-\xi \right) \alpha_{AMG_1, \alpha_{AMG_2}})^{\lambda} \right) \\
\left( 1+\left( 1-\xi \right) \left( 1-\left( 1-\xi \right) \alpha_{AMG_1, \alpha_{AMG_2}} \right)^{\lambda} + (1-\left( 1-\xi \right) (1-\left( 1-\xi \right) \alpha_{AMG_1, \alpha_{AMG_2}}))^{\lambda} \right) \\
\left( 1+\left( 1-\xi \right) \left( 1-\left( 1-\xi \right) \beta_{ANG_1, \beta_{ANG_2}} \right)^{\lambda} \right) \\
\left( 1+\left( 1-\xi \right) \left( 1-\left( 1-\xi \right) \beta_{ANG_1, \beta_{ANG_2}} \right)^{\lambda} \right) \\
\left( 1+\left( 1-\xi \right) \left( 1-\left( 1-\xi \right) \beta_{ANG_1, \beta_{ANG_2}} \right)^{\lambda} \right)
\end{array} \right)$$
4. For  \( \lambda \to 0 \), then the LDULGHWA operator is converted for HLDULGWA operator, such that

\[
\text{HLDULGWA} (A_{LD,-1}, A_{LD,-2}, \ldots, A_{LD,-n}) = \left(\begin{array}{c}
\sum_{i=1}^{n} \omega_{A} \theta_{i}(\theta) \cdot \sum_{i=1}^{n} \omega_{A} \tau_{i}(\theta) \\
\zeta \prod_{i=1}^{n} (u_{\text{AMG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (1-u_{\text{AMG}}(\theta)))^{\omega_{i}} + (\zeta-1) \prod_{i=1}^{n} (u_{\text{AMG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (v_{\text{ANG}}(\theta)))^{\omega_{i}} - \prod_{i=1}^{n} (1-v_{\text{ANG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (\beta_{\text{ANG}}(\theta)))^{\omega_{i}} - \prod_{i=1}^{n} (1-\beta_{\text{ANG}}(\theta))^{\omega_{i}} \\
\end{array}\right)
\]

\[(31)\]

5. For  \( \zeta = 1 \), then the HLDULGWA operator is converted for LDULGWA operator, such that (32), as shown at the bottom of the next page.

6. For  \( \zeta = 2 \), then the HLDULGWA operator is converted for ELDDLGWA operator, such that (33), as shown at the bottom of the next page.

7. For  \( \lambda = 1 \), then the LDULGHOWA operator is converted for HLDULOWA operator, such that (34), as shown at the bottom of the next page.

8. For  \( \zeta = 1 \), then the HLDULOWA operator is converted for LDULOWA operator, such that (35).

\[
\text{LDULOWA} (A_{LD,-1}, A_{LD,-2}, \ldots, A_{LD,-n}) = \left(\begin{array}{c}
\sum_{i=1}^{n} \omega_{A} \theta_{i}(\theta) \cdot \sum_{i=1}^{n} \omega_{A} \tau_{i}(\theta) \\
\prod_{i=1}^{n} (1+(\zeta-1) (1-u_{\text{AMG}}(\theta)))^{\omega_{i}} - \prod_{i=1}^{n} (1-u_{\text{AMG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (v_{\text{ANG}}(\theta)))^{\omega_{i}} + (\zeta-1) \prod_{i=1}^{n} (v_{\text{ANG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (\beta_{\text{ANG}}(\theta)))^{\omega_{i}} - \prod_{i=1}^{n} (1-\beta_{\text{ANG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (1-u_{\text{AMG}}(\theta)))^{\omega_{i}} + (\zeta-1) \prod_{i=1}^{n} (1-u_{\text{AMG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (\beta_{\text{ANG}}(\theta)))^{\omega_{i}} - (\zeta-1) \prod_{i=1}^{n} (\beta_{\text{ANG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (1-\beta_{\text{ANG}}(\theta)))^{\omega_{i}} + (\zeta-1) \prod_{i=1}^{n} (1-\beta_{\text{ANG}}(\theta))^{\omega_{i}} \\
\prod_{i=1}^{n} (1+(\zeta-1) (1-\beta_{\text{ANG}}(\theta)))^{\omega_{i}} - (\zeta-1) \prod_{i=1}^{n} (1-\beta_{\text{ANG}}(\theta))^{\omega_{i}} \\
\end{array}\right)
\]

\[(35)\]

9. For  \( \zeta = 2 \), then the HLDULOWA operator is converted for ELDDLWGA operator, such that (36), as shown at the bottom of the next page.

10. For  \( \lambda \to 0 \), then the LDULGOWA operator is converted for HLDULOWA operator, such that (37), as shown at the bottom of page 10.

11. For  \( \zeta = 1 \), then the HLDULGWGA operator is converted for LDULGWGA operator, such that (38), as shown at the bottom of page 10.

12. For  \( \zeta = 2 \), then the HLDULGWGA operator is converted for ELDDLWGA operator, such that (39), as shown at the bottom of page 10.

Definition 9: For any collection of LDULNs  \( A_{LD,-i} = (\sum_{A} \theta_{i} \cdot \sum_{A} \tau_{i}(\theta), u_{\text{AMG}}(\theta), v_{\text{ANG}}(\theta), 1) \), then  \( i = 1, 2, \ldots, n, \)
then the LDULGHHWA operator is initiated by:

$$
LDULGHHWA (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) = \Theta_h \left( \gamma_i \right)
$$

(40)

where, $\gamma_i$ is the largest term of the $\gamma_i$ value $\gamma_i = no_{i}A_{LD-i} = \left( \prod_{i=1}^{n} (A_{AMG} \cdot i) \cdot (v_{ANG-i} \cdot \tilde{v}_{ANG-i}) \right)_{i=1}, \tilde{\beta}_{ANG-i}, 1, 2, \ldots, n \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}$, $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$. $n$ is a balancing coefficient. $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation of $(1, 2, \ldots, n)$ s.t. $\sigma_{i-1} \geq \sigma_i$ for all $i$. Based on the above analysis, we discuss the monotonicity, idempotency, and boundedness to improve the quality of the proposed works.

Property 3: For any collection of LDULNs $(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})$ and $(B_{LD-1}, B_{LD-2}, \ldots, B_{LD-n})$.

1. If $A_{LD-i} \leq B_{LD-i}$ for all $i = 1, 2, \ldots, n$ then $LDULGHHWA (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \leq LDULGHHWA (B_{LD-1}, B_{LD-2}, \ldots, B_{LD-n})$.

LDULWGA $(A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) = \left( \prod_{i=1}^{n} (\prod_{i=1}^{n} (A_{AMG} \cdot i) \cdot (v_{ANG-i} \cdot \tilde{v}_{ANG-i}) \right)_{i=1}, \tilde{\beta}_{ANG-i}, 1, 2, \ldots, n \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}$.
2. \( A_{LD-i} = A, i = 1, 2, \ldots, n \) then \( LDULGHWA \) (\( A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n} = A \)).

3. If \( LDULGHWA \) operator lies between the max and min operators, then \( \min (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-i}) \leq \\frac{LDULGHWA (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})}{LDULGHWA (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n})} \leq \max (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-i}). \)

**Theorem 3:** For any collection of the LDULNs \( A_{LD-i} = ([\mathcal{S}_{A-i}, \mathcal{S}_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i})) \), \( i = 1, 2, \ldots, n \), then (41), as shown at the bottom of the next page.

In which \( (\sigma_1, \sigma_2, \ldots, \sigma_n) \) is a permutation of \( (1, 2, \ldots, n) \) s.t \( \sigma_{i-1} \geq \sigma_i \) for all \( i, \xi > 0 \):

\[
\begin{align*}
\mu_i &= (1 + (\xi - 1)(1 - \tilde{u}_{AMG-i}))^\lambda + \left(\xi^2 - 1\right) \tilde{u}_{AMG-i}, \\
\eta_i &= (1 + (\xi - 1)(1 - \tilde{u}_{AMG-i}))^\lambda - \tilde{u}_{AMG-i}, \\
\gamma_i &= (1 + (\xi - 1)\tilde{v}_{ANG-i})^\lambda + \left(\xi^2 - 1\right)(1 - \tilde{v}_{ANG-i})^\lambda, \\
\delta_i &= (1 + (\xi - 1)\tilde{v}_{ANG-i})^\lambda - (1 - \tilde{v}_{ANG-i})^\lambda,
\end{align*}
\]

\[
\tilde{u}_{AMG-i} = \frac{\xi (u_{AMG-i})^{\alpha_i}}{[1 + (\xi - 1)(1 - u_{AMG-i})]^{\alpha_i} + (\xi - 1)(u_{AMG-i})^{\alpha_i}},
\]

\[
\tilde{v}_{ANG-i} = \frac{(1 + (\xi - 1)\tilde{v}_{ANG-i})^{\beta_i} + (\xi^2 - 1)(1 - \tilde{v}_{ANG-i})^\lambda}{(1 + (\xi - 1)\tilde{v}_{ANG-i})^{\beta_i} + (\xi - 1)(1 - \tilde{v}_{ANG-i})^\lambda},
\]

\[
\alpha_{\mu_i} = (1 + (\xi - 1)(1 - \tilde{u}_{AMG-i}))^\lambda - \tilde{u}_{AMG-i},
\]

\[
\beta_{\eta_i} = (1 + (\xi - 1)(1 - \tilde{u}_{AMG-i}))^\lambda - \tilde{u}_{AMG-i},
\]

\[
\alpha_{\gamma_i} = (1 + (\xi - 1)\tilde{v}_{ANG-i})^\lambda + (\xi^2 - 1)(1 - \tilde{v}_{ANG-i})^\lambda, \\
\beta_{\delta_i} = (1 + (\xi - 1)\tilde{v}_{ANG-i})^\lambda - (1 - \tilde{v}_{ANG-i})^\lambda,
\]

\[
\tilde{\alpha}_{AMG-i} = \frac{\xi (\alpha_{AMG-i})^{\alpha_i}}{[1 + (\xi - 1)(1 - \alpha_{AMG-i})]^{\alpha_i} + (\xi - 1)(\alpha_{AMG-i})^{\alpha_i}},
\]

\[
\tilde{\beta}_{ANG-i} = \frac{(1 + (\xi - 1)\beta_{ANG-i})^{\beta_i} + (\xi^2 - 1)(1 - \beta_{ANG-i})^\lambda}{(1 + (\xi - 1)\beta_{ANG-i})^{\beta_i} + (\xi - 1)(1 - \beta_{ANG-i})^\lambda}.
\]

**HLDULOWGA** \( (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \)

\[
= \left[ \begin{array}{l}
\mathcal{S}_{\sum_{i=1}^n o_i A - \theta_i(\xi)}, \mathcal{S}_{\sum_{i=1}^n o_i A - \tau_i(\xi)} \\
\xi \prod_{i=1}^n (u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i} \\
\prod_{i=1}^n (1 + (\xi - 1)(1 - u_{AMG_i}(\psi_{\sigma_i})))^{\alpha_i} + (\xi - 1)(u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i}, \\
\prod_{i=1}^n (1 + (\xi - 1)(u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i} - \prod_{i=1}^n (1 - v_{ANG_i}(\psi_{\sigma_i}))^{\alpha_i})^{\alpha_i}, \\
\prod_{i=1}^n (1 + (\xi - 1)(1 - \tilde{u}_{AMG_i}))^{\lambda} - \prod_{i=1}^n (1 - \tilde{u}_{AMG_i})^{\lambda}, \\
\prod_{i=1}^n (1 + (\xi - 1)(1 - \alpha_{AMG_i}))^{\alpha_i} + (\xi - 1)(\alpha_{AMG_i})^{\alpha_i}, \\
\prod_{i=1}^n (1 + (\xi - 1)(1 - \tilde{v}_{ANG_i}))^{\alpha_i} + (\xi - 1)(1 - \tilde{v}_{ANG_i})^{\alpha_i} \\
\prod_{i=1}^n (\alpha_{AMG_i})^{\alpha_i} - \prod_{i=1}^n (1 - \beta_{ANG_i})^{\alpha_i}
\end{array} \right]
\]

**LDULOWGA** \( (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \)

\[
= \left[ \begin{array}{l}
\mathcal{S}_{\sum_{i=1}^n o_i A - \theta_i(\xi)}, \mathcal{S}_{\sum_{i=1}^n o_i A - \tau_i(\xi)} \\
(1 - \prod_{i=1}^n (u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i}, \\
\prod_{i=1}^n (1 - u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i}, \\
1 - \prod_{i=1}^n (1 - v_{ANG_i}(\psi_{\sigma_i}))^{\alpha_i}, \\
(\prod_{i=1}^n (\alpha_{AMG_i})^{\alpha_i}, 1 - \prod_{i=1}^n (1 - \beta_{ANG_i})^{\alpha_i})
\end{array} \right]
\]

**ELDULOWGA** \( (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n}) \)

\[
= \left[ \begin{array}{l}
\mathcal{S}_{\sum_{i=1}^n o_i A - \theta_i(\xi)}, \mathcal{S}_{\sum_{i=1}^n o_i A - \tau_i(\xi)} \\
2 \prod_{i=1}^n (u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i} \\
\prod_{i=1}^n (2 - u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i} + \prod_{i=1}^n (u_{AMG_i}(\psi_{\sigma_i}))^{\alpha_i}, \\
\prod_{i=1}^n (1 + v_{ANG_i}(\psi_{\sigma_i}))^{\alpha_i} + \prod_{i=1}^n (1 - v_{ANG_i}(\psi_{\sigma_i}))^{\alpha_i}, \\
\prod_{i=1}^n (1 + \beta_{ANG_i})^{\alpha_i} - \prod_{i=1}^n (1 - \beta_{ANG_i})^{\alpha_i}, \\
\prod_{i=1}^n (2 - \alpha_{AMG_i})^{\alpha_i} + \prod_{i=1}^n (\alpha_{AMG_i})^{\alpha_i} \\
\prod_{i=1}^n (2 - \alpha_{AMG_i})^{\alpha_i} + \prod_{i=1}^n (\alpha_{AMG_i})^{\alpha_i}
\end{array} \right]
\]
A brief algorithm of the MADM process is illustrated in the following section.

A. ALGORITHM
The algorithm of MADM based on LDU information and using investigated operators is proposed as follows:

Step 1: In this step, we collect the information about alternatives given by the decision-makers. The decision-makers gave their opinion about alternatives in the form of LDU values which leads to the formation of the decision matrix.

Step 2: If there exist attributes of cost type, we normalize the decision matrix by taking the complement of each triplet in the matrix. If not, then we use the following investigated LDUHWA operators to aggregate the data given in the decision matrix.

Step 3: In this step, we compute the expected values of the aggregated information using the below formula. For any LDU

\[
A_{LD} = \left( \left[ A_{LD-\theta(q)}, A_{LD-\tau(q)} \right], \left( u_{AMG} (q), v_{ANG} (q) \right), \left( \alpha_{AMG}, \beta_{ANG} \right) \right),
\]

then the expected value (EV) is elaborated by:

\[
E \left( A_{LD} \right) = \mathbb{E} \left( \left[ \left( 1-\theta(q) \right) + \left( 1-\tau(q) \right) \right] \cdot \frac{1}{4} \left( \varphi_{AMG} (q) + v_{ANG} (q) + \alpha_{AMG} + \beta_{ANG} \right) \right)
\]  

(42)

For any LDU

\[
A_{LD} = \left( \left[ A_{LD-\theta(q)}, A_{LD-\tau(q)} \right], \left( u_{AMG} (q), v_{ANG} (q) \right), \left( \alpha_{AMG}, \beta_{ANG} \right) \right),
\]

then accuracy value (AV) is elaborated by:

\[
\varphi_{LD} = \varphi \left( A_{LD} \right) \cdot \frac{1}{4} \cdot \varphi_{AMG} \left( q \right) + \varphi_{ANG} \left( q \right) + \alpha_{AMG} + \beta_{ANG}
\]  

(43)

For any two LDU

\[
A_{LD} = \left( \left[ A_{LD-\theta(q)}, A_{LD-\tau(q)} \right], \left( u_{AMG} (q), v_{ANG} (q) \right), \left( \alpha_{AMG}, \beta_{ANG} \right) \right),
\]

\[
LDULGHWA \left( A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n} \right) =
\]

\[
\frac{(1+\zeta)}{2} \left[ \frac{1}{2} \left( \sum_{i=1}^{n} a_{\beta_{ANG-i}} (\theta_i) \right) + \sum_{i=1}^{n} a_{\beta_{ANG-i}} (\tau_i) \right],
\]

where \( \omega_1 = 1 \), then

\[
A_{LD-1} = \left( \left[ A_{LD-\theta(q)}, A_{LD-\tau(q)} \right], \left( u_{AMG} (q), v_{ANG} (q) \right), \left( \alpha_{AMG}, \beta_{ANG} \right) \right),
\]

then accuracy value (AV) is elaborated by:

\[
\varphi_{LD} = \varphi \left( A_{LD} \right) \cdot \frac{1}{4} \cdot \varphi_{AMG} \left( q \right) + \varphi_{ANG} \left( q \right) + \alpha_{AMG} + \beta_{ANG}
\]  

(43)

For any two LDU

\[
A_{LD} = \left( \left[ A_{LD-\theta(q)}, A_{LD-\tau(q)} \right], \left( u_{AMG} (q), v_{ANG} (q) \right), \left( \alpha_{AMG}, \beta_{ANG} \right) \right),
\]

\[
LDULGHWA \left( A_{LD-1}, A_{LD-2}, \ldots, A_{LD-n} \right) =
\]
and

\[
B_{LD} = \left( [\Xi_{B-\theta(q)}, \Xi_{B-\tau(q)}], (u_{BMG}(q), v_{BNG}(q)), (\alpha_{BMG}, \beta_{BNG}) \right),
\]

then

1. If \( E_{ALD} < E_{BLD} \) then \( A_{LD} < B_{LD} \).
2. If \( E_{ALD} > E_{BLD} \) then \( A_{LD} > B_{LD} \).
3. If \( E_{ALD} = E_{BLD} \) then,
   1) If \( \varphi_{ALD} < \varphi_{BLD} \) then \( A_{LD} < B_{LD} \).

\[
LDULGHWWA (A_{LD-1}) = \left[ \Xi_{\Sigma_{\lambda=1}^{k} \alpha_{\lambda,\beta_{\lambda}}(\theta_{\lambda})}, \Xi_{\Sigma_{\lambda=1}^{k} \alpha_{\lambda,\beta_{\lambda}}(\theta_{\lambda})} \right]
\]

\[
\begin{pmatrix}
\begin{bmatrix}
\zeta \left( \frac{1}{\mu_{1}+\left( \xi^{2}-1 \right) \eta_{1} } \right) \\
\frac{1}{\lambda} - \left( \frac{1}{\mu_{1}+\left( \xi^{2}-1 \right) \eta_{1} } \right) \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right) \\
\frac{1}{\lambda} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right) \\
\frac{1}{\lambda} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right) \\
\frac{1}{\lambda} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right)
\end{bmatrix}
\end{pmatrix}
\]

\[
LDULGHWWA (A_{LD-1}, A_{LD-2}, \ldots, A_{LD-k}) =
\begin{pmatrix}
\begin{bmatrix}
\zeta \left( \frac{1}{\mu_{1}+\left( \xi^{2}-1 \right) \eta_{1} } \right) \\
\frac{1}{\lambda} - \left( \frac{1}{\mu_{1}+\left( \xi^{2}-1 \right) \eta_{1} } \right) \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right) \\
\frac{1}{\lambda} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right) \\
\frac{1}{\lambda} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right) \\
\frac{1}{\lambda} \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} \right)
\end{bmatrix}
\end{pmatrix}
\]

**B. ILLUSTRATED EXAMPLE**

An example of technology commercialization is adapted from [39] where the selection of the most favorable software...
LDULGHWWA (A_{LD−1}, A_{LD−2}, \ldots, A_{LD−k+1})
= LDULGHWWA (A_{LD−1}, A_{LD−2}, \ldots, A_{LD−k}) \oplus_h (\omega_{k+1} \gamma_{k+1})
TABLE 1. Original decision matrix.

|   | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $A_1$ | $[0.0, 0.2]$, $(0.41, 0.22)$, $(0.32, 0.22)$, $(0.43, 0.23)$, $(0.33, 0.23)$ |
| $A_2$ | $[0.5, 0.2]$, $(0.51, 0.22)$, $(0.32, 0.22)$, $(0.53, 0.23)$, $(0.33, 0.23)$ |
| $A_3$ | $[0.8, 0.5]$, $(0.81, 0.51)$, $(0.52, 0.52)$, $(0.83, 0.53)$, $(0.5, 0.5)$ |
| $A_4$ | $[0.9, 0.8]$, $(0.91, 0.81)$, $(0.92, 0.82)$, $(0.93, 0.83)$, $(0.5, 0.5)$ |

TABLE 2. Aggregated values by using the proposed operator.

| Method | Aggregated values |
|---|---|
| $A_1$ | $[0.5624, 0.3624]$, $(0.8828, 0.9308)$ |
| $A_2$ | $[0.6477, 0.3654]$, $(0.8828, 0.8268)$ |
| $A_3$ | $[0.8738, 0.6477]$, $(0.7706, 0.7706)$ |
| $A_4$ | $[0.9423, 0.8738]$, $(0.7706, 0.7706)$ |

TABLE 3. Score Values of the aggregated operators.

|   | Score values |
|---|---|
| $A_1$ | 0.99409 |
| $A_2$ | 0.4675 |
| $A_3$ | 0.60538 |
| $A_4$ | 0.16311 |

TABLE 4. Ranking values of the information in Table 3.

|   | Ranking Analysis |
|---|---|
| Proposed operators | $A_3 \succeq A_2 \succeq A_4 \succeq A_1$ |

By using different values of the parameters $\zeta$ and $\lambda$, we obtained the same ranking results in the form of Table 5. We briefly explained the motivation of the proposed work. The concepts of intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs), q-rung orthopair fuzzy sets (QROFSs), and linear Diophantine fuzzy sets have numerous applications in various fields of real life, but these theories have their limitations related to the membership and non-membership grades. To eradicate these restrictions, we introduce the novel concept of linear Diophantine uncertain linguistic set (LDULS) with the addition of reference parameters and uncertain linguistic terms. The proposed model of LDULS is more efficient and flexible rather than other approaches due to the use of reference parameters and ULVs. LDULS also categorizes the data in MADM problems by changing the physical sense of reference parameters and ULVs. LDULS also categorizes the data in MADM problems by changing the physical sense of reference parameters and ULVs. This set covers the spaces of existing structures and enlarges the space for membership and non-membership grades with the help of reference parameters and ULVs. The motivation of the proposed model is given step by step in the whole manuscript. Now we discuss some important objectives of this paper.

1. The theory of LDULS is more generalized than IFSs, PFSs, QROFSs, LDFSs, and ULVs.

2. If we choose the information in the form of $(0.5, 0.6)$, then by using the condition of IFSs that is the sum of both terms is limited to the unit interval, but $0.5+0.6 = 1.1 > 1$, the theory of IFS has been failed for coping with such sorts of issues, the theory of LDULS is very comfortable to resolve the above issues. For this, we choose the reference parameters such as $(0.1, 0.2)$, then by using the condition of LDULS is that $0.1+0.5+0.2+0.6 = 0.05+0.12 = 0.17 < 1$. We clarify that the IFS is the special case of the proposed LDULS.

3. If we choose the information in the form of $(0.8, 0.9)$, then by using the condition of PFSs that is the sum of the square of both terms is limited to the unit interval, but $0.8^2+0.9^2 = 0.64+0.81 = 1.45 > 1$, the theory of PFS has been failed for coping with such sorts of issues, the theory of LDULS is very comfortable to resolve the above issues. For this, we choose the reference parameters such as $(0.2, 0.2)$, then by using the condition of LDULS is that $0.2+0.8+0.2+0.9 = 0.16+0.18 = 0.34 < 1$. We clarify that the PFS is the special case of the proposed LDULS.

4. If we choose the information in the form of $(0.1, 0.1)$, then by using the condition of QROFSs that is the sum of the q-powers of both terms is limited to the unit interval, but $1+1 = 2 > 1$, the theory of QROFS has been failed for coping with such sorts of issues, the theory of LDULS is very comfortable to resolve the above issues. For this, we choose the reference parameters such as $(0.0, 0.1)$, then by using the condition of LDULS is that $0.0+1+0.1+1 = 0+0.1 = 0.1 < 1$. We clarify that the theory of QROFS is the special case of the proposed LDULS.
5. If we choose the information in the form of $((s_1, s_2), (0.5, 0.3), (0.5, 0.4))$, then by using the condition of IFSs, PFSs, q-ROFSs, and LDFS have been failed, for coping with such sorts of issues, the theory of LDULSs is a very proficient and reliable technique to resolve it. From the above analysis, the theory of IFSs, PFSs, QROFSs, and LDFSs is the special case of the proposed LDULSs. In real-life problems, we come across many situations where we need to quantify the uncertainty existing in the data to make optimal decisions. To illustrate the significance of Linear Diophantine Uncertain Linguistic sets, we give an example. Suppose XYZ company decides to set up biometric-based attendance devices (BBADs) in all its offices spread all over the country. For this, the company consults an expert who gives the information regarding (i) models of BBADs, (ii) production dates of BBADs, and (ii) the Price of BBADs. The company wants to select the most optimal model of BBADs with its production date simultaneously. Here, the problem is three-dimensional, namely, the model of BBADs and the production date of BBADs. This type of problem cannot be modeled accurately using traditional fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, and Linear Diophantine fuzzy set theory cannot tackle with triplet the dimensions simultaneously. The best way to represent all the information provided by the expert is by using Linear Diophantine uncertain linguistic set theory. The membership and non-membership terms in Linear Diophantine uncertain linguistic set may be used to give company’s decision regarding the model of BBADs, the reference parameters may be used to represent the company’s judgment in respect of production date of BBADs, and the uncertain linguistic terms may be used to represent company’s judgment in respect of price level (ups and downs) of BBADs.

### C. COMPARATIVE ANALYSIS

As shown above, we founded the ranking results by using the elaborated operator and applied them by interval-valued linear Diophantine uncertain linguistic types of information to find the effectiveness and proficiency of the discovered approaches. Additionally, to improve the quality of the proposed approach, we compare the discovered approaches with some existing approaches [30], [38], [37].

1. Riaz and Hashmi [30] elaborated the aggregation operators based on LDFSs. In [30], the authors combined the aggregation operators with LDFSs and determine the best optimal to show the dominance of the elaborated operators, but the theory proposed in [30] based on LDFS is the special case of the proposed operators based on LDULSs. Therefore, if we choose the proposed types of information, then the theory of Riaz and Hashmi [30] is not able to resolve it, because the elaborated approach is more general than the prevailing ideas in [30].

2. Aggregation operators based on intuitionistic uncertain linguistic sets (IULSs) were elaborated by Xu [38], which is the mixture of the aggregation operators with IULSs. In [38], the authors combined the aggregation operators with IULSs and determine the best optimal to show the dominance of the elaborated operators, but the theory proposed in [38] based on IULS is the special case of the proposed operators based on LDULSs. Therefore, if we choose the proposed types of information, then the theory of Xu [38] is not able to resolve it, because the elaborated approach is more general than the prevailing ideas in [38].

3. Liu et al. [37] elaborated the Heronian mean (HM) operators based on IULSs. In [37], the authors combined the HM operators with IULSs and determine the best optimal to show the dominance of the elaborated operators, but the theory proposed in [37] based on IULS is the special case of the proposed operators based on LDULSs. Therefore, if we choose the proposed types of information, then the theory of Liu et al. [37] is not able to resolve it, because the elaborated approach is more general than the prevailing ideas in [37].

In the future, the principle of Hamacher aggregation operators for complex IFS [40], q-rung orthopair fuzzy graph [41], complex picture fuzzy sets [42], and m-polar fuzzy sets [43] to extend for the presented works.

From the above theories, we obtained the result that is the elaborated operator based on a new IV-LDULS is extensively useful and more dominant to manage awkward and inconsistent information in genuine issues.

### VI. CONCLUSION

As the structure of Hamacher aggregation operators is powerful and massive suitable tool to cope with awkward and complicated information in realistic issues, so keeping in view the importance of Hamacher aggregation operators in the field of fuzzy logic and in the field of decision sciences many scholars have employed the theory of Hamacher aggregation operators.
operators in the environment of different fields [40]–[43]. In this manuscript the notion of Hamacher aggregation is applied in the environment of linear Diophantine uncertain linguistic sets and is applied in decision making problems. Based on the above analysis the main points of the presented work are discussed below:

1. The principle of linear Diophantine uncertain linguistic sets is explored with its useful Hamacher operational laws.
2. A MADM procedure is a build up based on the investigated operators under the linear Diophantine uncertain linguistic information.
3. Certain numerical examples are illustrated by using initiated operators to determine the dominance and flexibility of explored operators.
4. The consistency and supremacy of the presented operators in the proposed work with certain prevailing operators is discussed.

But there are some issues, when an intellectual gives information in the shape of “yes”, “abstinence”, and “no”, then the principle of linear Diophantine uncertain linguistic set is enabled to manage with it. For managing with such sorts of issues, in future we will be exploring the ideas of picture linear Diophantine uncertain linguistic sets, spherical linear Diophantine uncertain linguistic sets, and T-spherical linear Diophantine uncertain linguistic sets. Further, we will extend the proposed operators for spherical fuzzy sets [44], complex spherical fuzzy sets [45], [46], and their modifications [47] to improve the quality of the research work.

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