A high-order electromagnetic gyrokinetic model

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Abstract

A high-order extension is presented for the electromagnetic gyrokinetic formulation in which the parallel canonical momentum is taken as one of phase space coordinates. The high-order displacement vector associated with the guiding-center transformation should be considered in the long wavelength regime. This yields additional terms in the gyrokinetic Hamiltonian which lead to modifications to the gyrokinetic Poisson and Ampère equations. In addition, the high-order piece of the guiding-center transformation for the parallel canonical momentum should be also kept in the electromagnetic model. The high-order piece contains the Baños drift effect and further modifies the gyrokinetic Ampère equation.

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I. INTRODUCTION

The gyrokinetic models are widely used for studies of low frequency microturbulence in strongly magnetized plasmas. The standard gyrokinetic model is formulated for perturbations with small amplitude ($\epsilon \varphi / T \sim \epsilon_\delta \ll 1$) and short wavelength ($k_\perp \rho \sim 1$) (gyrokinetic ordering)[1–3], where $\varphi$ is the electrostatic potential, $k_\perp$ the perpendicular wavenumber and $\rho$ the Larmor radius. The gyrokinetic ordering can be interpreted as the slow flow condition $\epsilon \rho \cdot \nabla_\perp \varphi / T \sim V_{E} / V_{th} \ll 1$ with the $\mathbf{E} \times \mathbf{B}$ drift velocity $V_{E}$ and the thermal velocity $V_{th}$. Although this slow flow condition is also satisfied in the long wavelength regime ($k_\perp \rho \ll 1, \epsilon \varphi / T \sim 1$), the standard gyrokinetic quasi-neutrality equation which is truncated to $O(\epsilon_\delta)$ is not always valid in the long wavelength regime. For the long wavelength component of $\varphi$, the polarization term with $\varphi$ can go to the higher order and then the other higher order terms should be kept in the Poisson or quasi-neutrality equation. The high-order displacement vector associated with the guiding-center transformation gives the other higher order terms and should be kept in the long wavelength regime[4]. Since the guiding-center model is constructed up to high order enough by Littlejohn[5, 6], it is no necessary to recalculate the guiding-center model. Although large electric field is considered in the original Littlejohn guiding-center model, related terms are neglected in the standard gyrokinetic model. Another guiding-center model with large electric field is found in [7]. When the large electric field is neglected, the high-order displacement vector is related to the nonuniformity of magnetic field only. Therefore, the high-order terms may be important for the components whose wavelengths are comparable to that of the magnetic field.

The high-order contributions are not considered at the gyro-center transformation stage in the standard gyrokinetic formulation since they are negligible for the short wavelength perturbations. The gyro-center models with the high-order contributions for general electromagnetic perturbations are found in [8] in which the parallel velocity $v_\parallel$ is an independent variable. The calculation of the $v_\parallel$ formulation with magnetic perturbations is rather cumbersome. This is because the gyro-center transformation becomes complicated due to the vector potential perturbation at the symplectic part of the fundamental 1-form or the phase space Lagrangian. There is no such complication in the electrostatic limit and the high-order gyrokinetic model is constructed easily[4]. The simplification is achieved even in the electromagnetic case by taking the parallel canonical momentum $p_\parallel$ as an independent variable.
instead of \( v_{||} \). If only the shear-Alfvénic fluctuations \( (A_{||}) \) are considered, the use of \( p_{||} \) deletes \( A_{||} \) from the symplectic part. In this paper we extend the high-order gyrokinetic model in the electrostatic limit to the electromagnetic one in terms of \( p_{||} \).

This paper is organized as follows. The well-known guiding-center model with \( p_{||} \) is briefly explained in Sec. II. In Sec. III, we derive the gyrokinetic Hamiltonian with additional terms related to the high-order pieces in the guiding-center transformation. The gyrokinetic Poisson and Ampère equations are obtained systematically through the functional derivatives of the derived gyrokinetic Hamiltonian in Sec. IV. Finally a summary is given in Sec. V.

II. GUIDING-CENTER TRANSFORMATION

The fundamental 1-form for a charged particle with mass \( m \) and electric charge \( q \) in an equilibrium magnetic field with small electromagnetic perturbations \( \phi, A_{||} \) is written in the particle phase space as

\[
\gamma = \left[ qA_0(x) + \epsilon_\delta qA_{||}(x,t)\hat{b} + mv \right] \cdot dx - \left[ \epsilon_\delta q\phi(x,t) + \frac{m}{2}|v|^2 \right] dt,
\]

(1)

where \( x \) and \( v \) are the particle position and velocity, respectively, and \( \hat{b} = \mathbf{B}_0/|\mathbf{B}_0| \) is the unit vector along the equilibrium magnetic field \( \mathbf{B}_0 \equiv \nabla \times \mathbf{A}_0 \). Only the shear Alfvén perturbation is considered here. Changing the velocity variables to \((v_\perp, v_{||}, \theta)\) gives

\[
\gamma = \left[ qA_0(x) + \epsilon_\delta qA_{||}(x,t)\hat{b} + mv_{||}\hat{b} + mv_\perp\hat{c} \right] \cdot dx
\]

\[
- \left[ \epsilon_\delta q\phi(x,t) + \frac{mv^2_{||}}{2} + \frac{mv^2_\perp}{2B_0}B_0 \right] dt,
\]

(2)

where \( \hat{c} = -\sin \theta \hat{e}_1 - \cos \theta \hat{e}_2 \) is the unit vector along the velocity vector perpendicular to \( \hat{b} \), and \( \hat{e}_1 \) and \( \hat{e}_2 \) are unit vectors spanning the perpendicular plane. The symplectic part of the above 1-form has the time dependency through \( A_{||} \). Therefore, the phase space transformation to remove fast gyromotion from the 1-form becomes more complicated compared to the electrostatic case. There is an easy solution for this. When, as one of phase space coordinates, we use the parallel canonical momentum,

\[
p_{||} \equiv mv_{||} + qA_{||},
\]

(3)
$A_\parallel$ disappears from the symplectic part as,
\[
\gamma = \left[ qA_0(x) + p_\parallel \hat{b} + mv_\perp \hat{c} \right] \cdot dx
- \left[ \epsilon_\delta q\phi(x, t) + \frac{(p_\parallel - \epsilon_\delta qA_\parallel)^2}{2m} + \frac{mv_\perp^2}{2B_0B_0} \right] dt. \tag{4}
\]
The time dependent perturbations appear only in the Hamiltonian. The $O(\epsilon_\delta^0)$ part given by
\[
\gamma_0 = \left[ qA_0(x) + p_\parallel \hat{b} + mv_\perp \hat{c} \right] \cdot dx
- \left[ \frac{p_\parallel^2}{2m} + \frac{mv_\perp^2}{2B_0B_0} \right] dt \tag{5}
\]
is the usual unperturbed 1-form except $mv_\parallel \rightarrow p_\parallel$. Therefore, the standard guiding-center transformation can be applied in order to remove the gyrophase dependence from the 1-form and gives the following guiding-center 1-form \cite{3},
\[
\Gamma = qA^* \cdot dX + \frac{m}{q} \mu d\xi
- \left[ \frac{P_\parallel^2}{2m} + \mu B_0 \right] dt, \tag{6}
\]
where $Z = (X, P_\parallel, \mu, \xi)$ are the guiding-center coordinates, $A^*$ is the modified vector potential given by
\[
A^* = A_0 + \frac{P_\parallel}{q} \hat{b} - \frac{m}{q^2} \mu W, \tag{7}
\]
and $W = (\nabla \hat{e}_1) \cdot \hat{e}_2 + (\hat{b} \cdot \nabla \times \hat{b}) \hat{b}/2$.

III. GYROKINETIC HAMILTONIAN

Now we consider the perturbations which have still the gyrophase dependence. In order to remove the remaining gyrophase dependence, the gyro-center transformation will be performed. As shown in the previous section, the perturbations only appear in the Hamiltonian. Hence, no modification to the symplectic part is needed. In this case the gyro-center transformation becomes the simple canonical transformation and only the Hamiltonian is modified. The perturbed Hamiltonian in terms of the guiding-center coordinates is formally represented by
\[
h(Z, t) = q \left[ \phi(T_{GC}^{-1}x, t) - \frac{T_{GC}^{-1}P_\parallel}{m} A_\parallel(T_{GC}^{-1}x, t) \right]
+ \frac{q^2}{2m} A^2(T_{GC}^{-1}x, t), \tag{8}
\]
where $T_{GC}^{-1}x$ and $T_{GC}^{-1}P_\parallel$ denote the particle position and particle parallel momentum in the guiding-center phase space, respectively. In the standard formulation $T_{GC}^{-1}x$ is approximated
by $T_{GC}^{-1}x = X + \rho$. When we consider the long wavelength regime, however, the higher-order displacement vector should be retained in $T_{GC}^{-1}x$ as

$$T_{GC}^{-1}x = X + \rho + \rho_B.$$  

(9)

We denotes the high-order displacement vector by $\rho_B$ which is in general defined by

$$\rho_B \equiv - \left( G^X_2 - \frac{1}{2} G_1 \cdot dG_1^X \right).$$  

(10)

where $G_n$ is the vector field generating the guiding-center transformation at $n$th order, and $G_n \cdot d = G_n^j \partial_j$. The usual gyroradius vector is given by $\rho = -G_1^X$. The explicit representation of $\rho_B$ is found in [4, 5, 11]. Considering $\rho_B$ in $T_{GC}^{-1}x$, we may expand the potentials as

$$\phi(T_{GC}^{-1}x) \simeq \phi(X + \rho) + \rho_B \cdot \nabla \phi(X + \rho),$$  

(11)

$$A_{||}(T_{GC}^{-1}x) \simeq A_{||}(X + \rho) + \rho_B \cdot \nabla A_{||}(X + \rho).$$  

(12)

Similarly, although in the standard model $T_{GC}^{-1}p_{||}$ is simply replaced by the lowest order term $P_{||}$, we retain here the higher order term for $T_{GC}^{-1}p_{||}$

$$T_{GC}^{-1}p_{||} = P_{||} - G_{1||},$$  

(13)

where $G_{1||}$ is the $p_{||}$ component of $G_1$ given by

$$G_{1||} = \frac{m\mu}{q} (a_1 : \nabla \hat{b} + \hat{b} \cdot \nabla \times \hat{b}) - P_{||} \cdot (\hat{b} \cdot \nabla \hat{b}),$$  

(14)

with $a_1 = -(\hat{a} \cdot \hat{c})/2, \hat{a} = \hat{b} \times \hat{c}$. Then, the perturbed Hamiltonian in the guiding-center phase space is written as

$$h(Z,t) = q \left[ \phi(T_{GC}^{-1}x) - \frac{P_{||} - G_{1||}}{m} A_{||}(T_{GC}^{-1}x) \right] + \frac{q^2}{2m} A_{||}^2(T_{GC}^{-1}x)$$

$$= q \left[ \phi(T_{GC}^{-1}x) - \frac{P_{||}}{m} A_{||}(T_{GC}^{-1}x) \right] + \frac{q}{m} G_{1||} A_{||} + \frac{q^2}{2m} A_{||}^2(T_{GC}^{-1}x).$$  

(15)

The phase space transformation to the gyro-center coordinates $Z = (\bar{X}, \bar{P}_{||}, \bar{\mu}, \bar{\xi})$ is performed to remove gyrophase dependence from the above perturbed Hamiltonian. The lowest order gyro-center Hamiltonian is simply $\bar{H}_0 = \bar{P}_{||}^2/2m + \bar{\mu} B_0$. The perturbed Hamiltonian is given by

$$\bar{h} = q(\langle \psi(\bar{X} + \bar{\rho}) \rangle + \langle \bar{\rho}_B \cdot \nabla \psi(\bar{X} + \bar{\rho}) \rangle)$$

$$+ \frac{q}{m} \langle G_{1||}(\bar{Z}) A_{||}(\bar{X} + \bar{\rho}) \rangle + \frac{q^2}{2m} \langle A_{||}^2(\bar{X} + \bar{\rho}) \rangle - \frac{q}{2} \langle \{S_1, \bar{\psi} \} \rangle,$$  

(16)
where $\langle \cdot \rangle$ denotes the gyrophase average and $\psi$ is the generalized potential defined by

$$\psi(\vec{Z}, t) \equiv \phi(\vec{Z}, t) - \frac{\vec{P}}{m} A_\parallel(\vec{Z}, t).$$  \hspace{1cm} (17)$$

The scalar function generating the first order gyro-center transformation is

$$S_1 = \frac{q}{\Omega} \int \tilde{\psi} d\xi,$$  \hspace{1cm} (18)

where $\tilde{\psi} = \psi(\vec{X} + \vec{\rho}) - \langle \psi(\vec{X} + \vec{\rho}) \rangle$ is the oscillatory part of $\psi$. The nonlinear term of $\psi$ is usually approximated by

$$\langle \{ S_1, \tilde{\psi} \} \rangle \simeq \frac{q^2}{m \Omega} \frac{\partial (\tilde{\psi}^2)}{\partial \mu} = \frac{q^2}{m \Omega} \frac{\partial}{\partial \mu} (\langle \psi^2 \rangle - \langle \psi \rangle^2).$$  \hspace{1cm} (19)

The terms with $\vec{\rho}_B$ and $G_1^{\parallel}$ are only important in the long wavelength regime. Hence, the perturbed Hamiltonian may be approximated as

$$\tilde{h}(\vec{X}, \vec{P}_\parallel, \mu, t) = q(\langle \psi(\vec{X} + \vec{\rho}) \rangle + \langle \vec{\rho}_B \rangle \cdot \nabla \psi(\vec{X}) + \frac{q}{m} \langle G_1^{\parallel}(\vec{Z}) \rangle A_\parallel(\vec{X})$$

$$+ \frac{q^2}{2m} \langle A_\parallel^2(\vec{X} + \vec{\rho}) \rangle - \frac{q^2}{2B_0} \frac{\partial}{\partial \mu} (\langle \psi^2 \rangle - \langle \psi \rangle^2).$$  \hspace{1cm} (20)

where $\langle \vec{\rho}_B \rangle$ and $\langle G_1^{\parallel} \rangle$ are, respectively, given by

$$\langle \vec{\rho}_B \rangle = - \left\{ \frac{\mu B_0}{m \Omega^2} \frac{1}{2} (\nabla \cdot \hat{b} ) \hat{b} + \frac{U^2}{\Omega^2} \hat{b} \cdot \nabla \hat{b} + \frac{3 \mu B_0}{2m \Omega^2} \nabla \perp \log B_0 \right\},$$  \hspace{1cm} (21)

and

$$\langle G_1^{\parallel} \rangle = \frac{m \mu}{q} \hat{b} \cdot \nabla \times \hat{b}.$$  \hspace{1cm} (22)

IV. FIELD EQUATIONS

The gyrokinetic field equations are easily obtained through the field-theoretical treatment\[12\]. The Poisson equation is given by

$$\epsilon_0 \nabla^2 \phi(\vec{r}) = - \sum_{sp} \int d^6\vec{Z} \vec{\tilde{J}} \vec{F} \frac{\delta \tilde{h}(\vec{Z})}{\delta \phi(\vec{r})},$$  \hspace{1cm} (23)

where $\epsilon_0$ is the permittivity of vacuum, $\vec{\tilde{J}} = B^*_\parallel/m^2$, $B^*_\parallel \equiv \hat{b} \cdot \vec{B}^*$, $\vec{B}^* \equiv \nabla \times \vec{A}^*$, and the summation is taken over species. Similarly, the gyrokinetic Ampère equation is given by

$$\frac{1}{\mu_0} \nabla \perp A_\parallel(\vec{r}) = \sum_{sp} \int d^6\vec{Z} \vec{\tilde{J}} \vec{F} \frac{\delta \tilde{h}(\vec{Z})}{\delta A_\parallel(\vec{r})},$$  \hspace{1cm} (24)
where \( \mu_0 \) is the permeability of vacuum. It is noted that the lowest order Hamiltonian does not have the potential perturbations and therefore only the functional derivatives of the perturbed Hamiltonian appear in the field equations. Taking the functional derivatives of the perturbed Hamiltonian (20), we have

\[
\frac{\delta \bar{h}(\bar{Z})}{\delta \varphi(r)} = q \langle \delta^3(\bar{X} + \bar{\rho} - r) \rangle + q \langle \bar{\rho}_B \rangle \cdot \bar{\nabla} \delta^3(\bar{X} - r) - \frac{q^2}{B_0} \frac{\partial}{\partial \bar{\mu}} (\langle \psi(\bar{X} + \bar{\rho}) \delta^3(\bar{X} + \bar{\rho} - r) \rangle - \langle \psi(\bar{X} + \bar{\rho}) \rangle \langle \delta^3(\bar{X} + \bar{\rho} - r) \rangle)
\]

(25)

and

\[
\frac{\delta \bar{h}(\bar{Z})}{\delta A_{\parallel}(r)} = -\frac{q \bar{P}_{\parallel}}{m} \langle \delta^3(\bar{X} + \bar{\rho} - r) \rangle - \frac{q \bar{P}_{\parallel}}{m} \langle \bar{\rho}_B \rangle \cdot \bar{\nabla} \delta^3(\bar{X} - r) + \frac{q}{m} \langle G_1^P \rangle \delta^3(\bar{X} - r)
\]

\[
+ \frac{q^2}{m A_{\parallel}(\bar{X} + \bar{\rho}) \delta^3(\bar{X} + \bar{\rho} - r)}
\]

\[
+ \frac{q^2 \bar{P}_{\parallel}}{m B_0} \frac{\partial}{\partial \bar{\mu}} (\langle \psi(\bar{X} + \bar{\rho}) \delta^3(\bar{X} + \bar{\rho} - r) \rangle - \langle \psi(\bar{X} + \bar{\rho}) \rangle \langle \delta^3(\bar{X} + \bar{\rho} - r) \rangle)
\]  

(26)

Substituting Eq. (25) into Eq. (23) and integrating by parts, we have the gyrokinetic Poisson equation

\[
\epsilon_0 \nabla^2 \varphi(r) = -\sum_{sp} q \left[ \int d^6 \bar{Z} \left( \bar{F} \bar{J} + \frac{q \bar{\psi}}{B_0} \frac{\partial \bar{F} \bar{J}}{\partial \bar{\mu}} \right) \delta^3(\bar{X} + \bar{\rho} - r) \right.
\]

\[
- \int d^6 \bar{Z} \delta^3(\bar{X} - r) \bar{\nabla} \cdot \bar{F} \bar{J} \langle \bar{\rho}_B \rangle \right].
\]

(27)

The Poisson equation is the same as that in the electrostatic case except that the potential on the right hand side is not \( \varphi \) but \( \psi \). The last term on the right hand side is the additional term due to \( \bar{\rho}_B \). Similarly, substituting Eq. (26) into Eq. (24) and integrating by parts, we have the gyrokinetic Ampère equation

\[
\frac{1}{\mu_0} \nabla_\perp^2 A_{\parallel}(r) = -\sum_{sp} q \left[ \int d^6 \bar{Z} \left( \frac{\bar{P}_{\parallel} - q A_{\parallel}}{m} \bar{F} \bar{J} + \frac{\bar{P}_{\parallel} q \bar{\psi}}{m B_0} \frac{\partial \bar{F} \bar{J}}{\partial \bar{\mu}} \right) \delta^3(\bar{X} + \bar{\rho} - r) \right.
\]

\[
- \int d^6 \bar{Z} \delta^3(\bar{X} - r) \left( \frac{\bar{P}_{\parallel}}{m} \nabla \cdot \bar{F} \bar{J} \langle \bar{\rho}_B \rangle + \bar{F} \bar{J} \langle \frac{G_1^P(\bar{Z})}{m} \rangle \right) \right].
\]

(28)

It is seen that besides the term with \( \bar{\rho}_B \), the term due to \( \langle G_1^P(\bar{Z}) \rangle = (m \bar{\mu}/q) \hat{b} \cdot \nabla \times \hat{b} \) appears in the Ampère equation. This shows the effect of the Baños term on the parallel current density\[13\]. The explicit form of this term may change by another choice for \( G_1^P \)[2, 14].
The appearance of this term is also shown by considering the push-forward representation of the parallel current density associated with the guiding-center transformation. The right hand side of the Ampère equation (28) must be the parallel particle current density. Hence, it can be regarded as the push-forward representation of the parallel current density which is written rigorously as

$$j_{\parallel}(r) = \sum_{sp} q \int d^3x d^3v \delta^3(x - r)$$

$$= \sum_{sp} q \int d^6Z J^{-1}_{GC} v_{\parallel} F(Z) \delta^3(T^{-1}_{GC} x - r),$$

(29)

where $T^{-1}_{GC} v_{\parallel} = U - G^{U}_{1} + \cdots$ denotes the particle parallel velocity in the guiding-center phase space and $U$ is the guiding-center parallel velocity. The gyroaverage of $\langle G^{U}_{1} \rangle$ agrees with $\langle G^{P}_{1} \rangle/m$. The additional term due to $\rho_{B}$ stems from $T^{-1}_{GC} x$ in the delta function.

V. SUMMARY

We reformulated the electromagnetic gyrokinetic model with the high-order pieces associated with the guiding-center transformation which are not considered in the standard gyrokinetic formulations. The use of parallel canonical momentum $p_{\parallel}$ instead of parallel velocity $v_{\parallel}$ deletes the magnetic perturbation $A_{\parallel}$ from the symplectic part of 1-form, and thereby makes it easier to extend the high-order gyrokinetic model to the electromagnetic one. Not only the high-order displacement vector $\rho_{B}$ but also $G^{P}_{1}$ should be retained in the electromagnetic case. We derived the gyrokinetic Hamiltonian including the high-order pieces. The field equations were easily obtained from the derived gyrokinetic Hamiltonian through the variational method. The additional term due to $\rho_{B}$ in the gyrokinetic Poisson equation is the same as the one in the electrostatic limit. $G^{P}_{1}$ as well as $\rho_{B}$ yields the additional term in the gyrokinetic Ampère equation, which contains the Baños drift effect on the parallel current density. The appearance of such term is also confirmed by simple consideration of the push-forward representation of parallel current density.
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