Analysis on almost Abelian Lie groups:

Groups, subgroups and quotients.

Marcelo Almora Rios, HMC
Zhirayr Avetisyan, Department of Mathematics, UCSB
Katalin Berlow, CMU
Isaac Martin, UU
Gautam Rakholia, UCSB
Kelley Yang, USC
Hanwen Zhang, UCSB
Zishuo Zhao, UCSB

Abstract

The subject of investigation are real almost Abelian Lie groups with their Lie group theoretical aspects, such as the exponential map, faithful matrix representations, discrete and connected subgroups, quotients and automorphisms. The emphasis is put on explicit description of all technical details.
1 Introduction

In the present paper we consider only real Lie groups and Lie algebras. An almost Abelian Lie algebra is a non-Abelian Lie algebra $L$ that contains a codimension one Abelian Lie subalgebra, and an almost Abelian group is a Lie group with an almost Abelian Lie algebra. Among the most prominent representatives of this class are the 3-dimensional Heisenberg group $H$, the group of affine transformations on the real line $\mathbf{ax} + \mathbf{b}$, and the isometry group of the plane $\mathbb{E}(2)$. Almost Abelian groups provide a far stretching generalization of the Heisenberg group in many respects, and are a promising context for developing methods of non-commutative analysis on solvable Lie groups. The first steps in the programme of analysis in almost Abelian spaces were made in the recent papers [Ave16] and [Ave16], where almost Abelian Lie algebras were studied and their structure explicitly described. The next step on this path is the study of almost Abelian groups from the Lie group theory perspective, which the present work is mainly devoted to.

The following results are obtained in this paper. Let $G$ stand for an almost Abelian group. The exponential map $\exp$ on a simply connected $G$ is described explicitly, and two conditions are given which are equivalent to the injectivity of $\exp$ (exponentiality of $G$). Two faithful matrix representations are introduced for simply connected $G$, and the centre $Z(G)$ is described. The full automorphism group $\text{Aut}(G)$ and the inner automorphism group $\text{Inn}(G)$ are given explicitly for a connected $G$. Discrete normal subgroups of a simply connected $G$ are studied, and conditions are found for two discrete normal subgroups to be related by an automorphism of $G$. This provides a necessary and sufficient condition for two connected $G$ with the same Lie algebra to be isomorphic, and thus a full classification of connected almost Abelian groups. Necessary and sufficient condition is found for a connected $G$ to admit a faithful matrix representation, and one such representation is given explicitly whenever one exists. Connected subgroups $H \subset G$ of connected $G$ are described, and a condition is established that is equivalent to the closedness of $H$ in $G$.

Properties of the exponential map

M. Almora, Z. Avetisyan, K. Berlow

Consider the real finite-dimensional almost Abelian Lie algebra $\mathcal{A}(\mathbb{N}) = \mathcal{A}_{\mathbb{R}}(\mathbb{N})$ corresponding to a finite dimensional multiplicity function $\mathbb{N} : \mathbb{C} \times \mathbb{N} \to \mathbb{N}$ (here $\sigma_{\mathbb{R}} \simeq \mathbb{C}$) as in [Ave18] given in its faithful matrix representation

$$\mathcal{A}(\mathbb{N}) \simeq \mathbb{R}^d \times \mathbb{R} \ni (v,t) \mapsto \begin{pmatrix} 0 & 0 \\ v & t J(\mathbb{N}) \end{pmatrix} \in \text{End}(\mathbb{R}^{d+1}),$$

(1)

where $J(\mathbb{N})$ is the Jordan canonical form associated with $\mathbb{N}$. Below we will establish technical facts that will together yield the necessary and sufficient condition for the exponential map to be a diffeomorphism. It is well known that a solvable real simply connected Lie group fails to be exponential if and only if it contains a copy of $\widetilde{\mathbb{E}}_+ (2)$ - the universal cover of the identity component of the Euclidean motion group in $\mathbb{R}^2$. Here we will reestablish this fact in a much more explicit way for the particular case of an almost
Abelian simply connected Lie group, and find an equivalent condition in terms of the spectrum of the adjoint representation.

**Lemma 1** The exponential map of the matrix Lie algebra \( \mathcal{A}(\mathbb{R}) \) is given by

\[
\exp \begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{e^{tJ(\mathbb{R})} - 1}{tJ(\mathbb{R})}v & e^{tJ(\mathbb{R})} \end{pmatrix}.
\]

**Proof:** Let us first show that

\[
\begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix} ^n = \begin{pmatrix} 0 & 0 \\ [tJ(\mathbb{R})]^{n-1}v & [tJ(\mathbb{R})]^n \end{pmatrix}, \quad \forall n \in \mathbb{N}, \tag{2}
\]

by induction on \( n \). For \( n = 1 \) we observe that

\[
\begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix} ^1 = \begin{pmatrix} 0 & 0 \\ [tJ(\mathbb{R})]^0v & [tJ(\mathbb{R})]^1 \end{pmatrix}.
\]

Now assuming that \( \text{(2)} \) is true for some \( n = k \) we see that equality continues to hold for \( n = k + 1, \)

\[
\begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix} ^{k+1} = \begin{pmatrix} 0 & 0 \\ [tJ(\mathbb{R})]^{k-1}v & [tJ(\mathbb{R})]^k \end{pmatrix} \begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ [tJ(\mathbb{R})]^{k+1}v & [tJ(\mathbb{R})]^{k+1} \end{pmatrix}, \tag{3}
\]

which completes the induction. Finally, by series expansion of the exponential, we see that

\[
\exp \begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 \\ v & tJ(\mathbb{R}) \end{pmatrix}^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 \\ [tJ(\mathbb{R})]^{n-1}v & [tJ(\mathbb{R})]^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{e^{tJ(\mathbb{R})} - 1}{tJ(\mathbb{R})}v & e^{tJ(\mathbb{R})} \end{pmatrix} \tag{4}
\]

gives us the desired result. \( \square \)

For the next fact note that the Lie algebra of \( E_+(2) \) is \( \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \).

**Lemma 2** If the almost Abelian Lie algebra \( \mathcal{A}(\mathbb{R}) \) contains a subalgebra \( L \subseteq \mathcal{A}(\mathbb{R}) \) isomorphic to \( \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \) then the corresponding simply connected almost Abelian Lie group \( G \) is not exponential.

**Proof:** Let \( \varphi : \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \to L \) be a Lie algebra isomorphism, and let \( H \subseteq G \) be the connected Lie subgroup with associated Lie algebra \( L \) as given by Theorem 5.20 in \[Hal15\]. Let \( \exp_1 : \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \to \widetilde{E}_+(2) \) be the exponential map on \( E_+(2) \), and \( \exp_2 : L \to H \) be the exponential map on \( H \). Assume towards a contradiction that \( \exp_2 \) is injective. Since \( \exp_2 \) is injective, \( H \) is simply connected. Thus since \( H \) and \( \widetilde{E}_+(2) \) are both simply connected, by Theorem 5.6 in \[Hal15\], there is a Lie group isomorphism \( \Phi : \widetilde{E}_+(2) \to H \) such that \( \Phi(\exp_1(X)) = \exp_2(\varphi(X)) \) for all \( X \in \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \). Since we know that \( \exp_1 : \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \to \widetilde{E}_+(2) \) is not injective, let \( X, Y \in \mathcal{A}(\mathbb{R}^1 \times \mathbb{R}) \) such that \( X \neq Y \) and \( \exp_1(X) = \exp_1(Y) \). Then \( \Phi(\exp_1(X)) = \Phi(\exp_1(Y)) \) and therefore \( \exp_2(\varphi(X)) = \exp_2(\varphi(Y)) \). Since \( \exp_2 \) is injective, this implies that \( \varphi(X) = \varphi(Y) \) in spite of \( X \neq Y \), thus contradicting the assumption of \( \varphi \) being an isomorphism.
This contradiction proves that \( \exp_t \) and therefore also \( \exp : \mathfrak{A}(\mathbb{N}) \to G \) cannot be injective, and \( G \) is not exponential. \( \square \)

Recall that we have identified \( \sigma_R \) with \( C \) in the following way. Every monic irreducible polynomial \( p \in \sigma_R \) is identified with its unique real root \( x_p \) if it is first order, and with one of the conjugate pair of complex roots \( x_p \) and \( \bar{x}_p \) (say, the one in the upper half-plane) if it is second order.

**Lemma 3** The simply connected Lie group \( G \) with almost Abelian Lie algebra \( \mathfrak{A}(\mathbb{N}) \) fails to be exponential if and only if \( \operatorname{supp} \mathbb{N} \) contains a polynomial \( p \) with non-zero imaginary root \( x_p \).

**Proof:** Let us perform some preliminary computations first. From [Ave18]

\[
J(\mathbb{N}) = \bigoplus_{p \in \operatorname{supp} \mathbb{N}} \bigoplus_{n=1}^{\infty} J(p,n). \tag{5}
\]

Thus, the exponential of the Jordan canonical form above can similarly be decomposed as

\[
e^t J(\mathbb{N}) = \bigoplus_{p \in \operatorname{supp} \mathbb{N}} \bigoplus_{n=1}^{\infty} e^{t J(p,n)}. \tag{6}
\]

Recall that every Jordan block has the form \( J(p,n) = x_p 1_n + N_n \) where \( N_n \) is the upper shift matrix of dimension \( n \) and \( x_p = a + ib \) is the root to the monic, irreducible polynomial \( p \in \operatorname{supp} \mathbb{N} \) with \( a, b \in \mathbb{R} \).

Because \( [x_p 1_n, N_n] = 0 \), the exponential of each Jordan block can also be expressed as

\[
e^{tx_p} = e^{t(a+ib)} = e^{ta}(\cos(tb) + i\sin(tb)),
\]

which for \( b \neq 0 \) in the real matrix representation gives

\[
e^{tx_p} = e^{ta} \begin{pmatrix} \cos(tb) & -\sin(tb) \\ \sin(tb) & \cos(tb) \end{pmatrix}. \tag{7}
\]

Now \( G \) is not exponential iff

\[
\exists (v_1, t_1), (v_2, t_2) \in \mathfrak{A}(\mathbb{N}) \text{ s.t. } (v_1, t_1) \neq (v_2, t_2), \text{ exp}(v_1, t_1) = \exp(v_2, t_2). \tag{8}
\]

From Lemma [1] we see that \( \exp(v_1, t_1) = \exp(v_2, t_2) \) iff

\[
e^{t_1 J(\mathbb{N})} - 1 \begin{pmatrix} v_1 \\ t_1 J(\mathbb{N}) \end{pmatrix} = e^{t_2 J(\mathbb{N})} - 1 \begin{pmatrix} v_2 \\ t_2 J(\mathbb{N}) \end{pmatrix} \text{ and } e^{t_1 J(\mathbb{N})} = e^{t_2 J(\mathbb{N})}. \tag{9}
\]

Observe that \( e^{t_1 J(\mathbb{N})} = e^{t_2 J(\mathbb{N})} \) iff

\[
\bigoplus_{p \in \operatorname{supp} \mathbb{N}} \bigoplus_{n=1}^{\infty} e^{t_1 J(p,n)} = \bigoplus_{p \in \operatorname{supp} \mathbb{N}} \bigoplus_{n=1}^{\infty} e^{t_2 J(p,n)}. \tag{10}
\]
which is equivalent to
\[ e^{(t_1 - t_2) J(p,n)} = I_n, \quad \forall p \in \text{supp} \mathbb{A}, \quad \forall n \in \mathbb{N}, \]
where matrices on two sides are over the field \( \mathbb{R}(x_p) \). Using (6) and (7) we can rewrite this as
\[ e^{(t_1 - t_2)x_p} \left( I_n + (t_1 - t_2) N_n + \frac{(t_1 - t_2)^2}{2!} N_n^2 + \cdots + \frac{(t_1 - t_2)^{n-1}}{(n-1)!} N_n^{n-1} \right) = I_n, \]
which holds if and only if
\[ t_1 = t_2 \quad \text{or} \quad n = 1, \quad a = 0, \quad e^{(t_1 - t_2)b} = 1. \]
Consider the case \( t_1 = t_2 \). Then the first condition of (9) gives
\[ e^{t J(\mathbb{A})} - 1 = 0. \]
Therefore (8) is equivalent to
\[ \exists t \in \mathbb{R} \quad \text{s.t.} \quad \det \left[ \frac{e^{t J(\mathbb{A})} - 1}{t J(\mathbb{A})} \right] (v_1 - v_2) = 0. \]
From (5), (6) and (7) we find that
\[ \det \left[ \frac{e^{t J(\mathbb{A})} - 1}{t J(\mathbb{A})} \right] = \prod_{p \in \text{supp} \mathbb{A}} \prod_{n=1}^{\infty} \det \left[ \frac{e^{t J(p,n)} - 1}{t J(p,n)} \right]^{N(p,n)} = \prod_{p \in \text{supp} \mathbb{A}} \prod_{n=1}^{\infty} \det \left[ \frac{e^{tx_p} - 1}{tx_p} \right]^{N(p,n)} . \]
If \( x_p \in \mathbb{R} \) then
\[ \det \left[ \frac{e^{tx_p} - 1}{tx_p} \right] = \frac{e^{tx_p} - 1}{tx_p} > 0, \]
whereas if \( x_p = a + ib \not\in \mathbb{R} \) then
\[ \det \left[ \frac{e^{tx_p} - 1}{tx_p} \right] = \frac{(e^{ta} \cos(tb) - 1)^2 + (e^{ta} \sin(tb))^2}{t^2(a^2 + b^2)}. \]
Therefore
\[ \det \left[ \frac{e^{t J(\mathbb{A})} - 1}{t J(\mathbb{A})} \right] = 0 \quad \Leftrightarrow \quad \exists p \in \text{supp} \mathbb{A} \quad \text{s.t.} \quad \frac{tx_p}{2\pi} \in \mathbb{Z}. \]
Thus (3) is equivalent to
\[ \exists p \in \text{supp} \mathbb{A} \quad \text{s.t.} \quad 0 \neq x_p \in i\mathbb{R} \quad \text{or} \quad \forall p \in \text{supp} \mathbb{A}, \quad x_p \in i\mathbb{R}, \quad \forall n > 1, \quad N(p,n) = 0. \]
Note that if \( x_p = 0 \) and \( n = 1 \) for all Jordan blocks then \( J(\mathbb{A}) = 0 \), which is impossible since \( \mathbb{A}(\mathbb{A}) \) is non-Abelian. Therefore
\[ \forall p \in \text{supp} \mathbb{A}, \quad x_p \in i\mathbb{R}, \quad \forall n > 1, \quad N(p,n) = 0 \quad \Rightarrow \quad \exists p \in \text{supp} \mathbb{A} \quad \text{s.t.} \quad 0 \neq x_p \in i\mathbb{R} \]
and hence (3) is equivalent to
\[ \exists p \in \text{supp} \mathbb{A} \quad \text{s.t.} \quad 0 \neq x_p \in i\mathbb{R}, \]
Lemma 4 If \( \text{supp}_N \) contains a polynomial \( p \) with non-zero imaginary root \( x_p \) then there exists a Lie subalgebra \( L \subset \mathfrak{a}(N) \) which is isomorphic to \( \mathfrak{a}(1 \times i^1) \).

Proof: Suppose that \( \exists p \in \text{supp}_N \) such that \( x_p = ib \) with \( 0 \neq b \in \mathbb{R} \) and \( N(p, n) > 0 \) for some \( n \in \mathbb{N} \). Fix an \( \alpha \in N(p, n) \) and let \( \{ \xi_{\alpha}^n(p, n) \}_{n=1}^N \) be the standard basis in the Jordan block \( (p, n, \alpha) \) as in \([Ave18]\).

Let \( W = \mathbb{C}\{\xi_{\alpha}^n(p, n)\} \) as an \( \mathbb{R} \)-vector space. By Corollary 3 in \([Ave18]\), \( W \) is an \( \text{ad}_{\xi_{\alpha}} \)-invariant subspace, and the restriction \( \text{ad}_{\xi_{\alpha}}|_W = x_p = ib \), which is \( \mathbb{R} \)-projectively similar to \( i \) on \( \mathbb{C} \). Thus the Lie subalgebra \( L \rtimes e_0 \mathbb{R} \subset \mathfrak{a}(N) \) is nothing else but \( \mathfrak{a}(1 \times ib) \), which by Proposition 11 in \([Ave16]\) is isomorphic to \( \mathfrak{a}(1 \times i^1) \). \( \square \)

Finally we are ready to formulate the main result of this paper.

Proposition 1 A simply connected almost Abelian Lie group with Lie algebra \( \mathfrak{a}(N) \) fails to be exponential if and only if \( \text{supp}_N \) contains a non-zero purely imaginary number, which is equivalent to the existence of a Lie subalgebra isomorphic to \( \mathfrak{a}(1 \times i^1) \).

Proof: Follows directly by combining Lemma 2, Lemma 3 and Lemma 4. \( \square \)

Matrix representations of simply connected almost Abelian groups

Z. Avetisyan, K. Berlow, G. Rakholia, Z. Zhao

In this paper we study the kernel of the exponential map on simply connected almost Abelian Lie groups and establish faithful matrix representations for them. We work with a fixed real finite dimensional almost Abelian Lie algebra \( \mathfrak{a}(N) \) in the faithful matrix representation \([1]\). Denote

\[
T_N = \{ t \in \mathbb{R} \mid e^{tJ(N)} = 1 \} \subset \mathbb{R}, \\
X_N = \{ |\omega| \mid \omega \in \mathbb{R}, \, \text{supp}_N \subset \omega \mathbb{Z} \}.
\]

Lemma 5 For a given finite real multiplicity function \( N \), \( T_N \neq \{0\} \) if and only if \( N(p, n) = 0 \) for all \( p \in \text{supp}_N \) and \( n > 1 \) and \( X_N \neq \emptyset \), in which case

\[
T_N = \frac{2\pi}{\omega_0} \mathbb{Z}, \quad \omega_0 = \max X_N.
\]

Proof: Performing calculations very similar to the proof of Lemma 3 we see that

\[
e^{tJ(N)} = 1 \quad \Leftrightarrow \quad \left[ t = 0 \quad \text{or} \quad \forall p \in \text{supp}_N, x_p = ib_p \in i\mathbb{R}, \, e^{ib_p} = 1, \, N(p, n) = 0, \, \forall n > 1 \right],
\]

6
which means that

\[ T_R \neq \{0\} \iff \exists t \neq 0 \text{ s.t. } \forall p \in \text{supp} R, x_p = ib_p \in iR, e^{itb_p} = 1, R(p, n) = 0, \forall n > 1. \]

Let us show that provided \( R(p, n) = 0, \forall p \in \text{supp} R, \forall n > 1 \), we have

\[ \exists t \neq 0 \text{ s.t. } \forall p \in \text{supp} R, x_p = ib_p \in iR, e^{itb_p} = 1 \iff X_R \neq \emptyset. \]

Indeed, if \( t \neq 0 \) then the condition \( e^{itb_p} = 1 \) can be written as

\[ x_p \in \frac{2\pi}{t}Z, \forall p \in \text{supp} R, \]

which implies that

\[ \frac{2\pi}{|t|} \in X_R. \]

Conversely, let \( 0 \leq \omega \in X_R \). The possibility \( \omega = 0 \) is excluded, since in that case \( \text{supp} R = \{0\} \), which together with \( n = 1 \) would imply that \( J(R) = 0 \), i.e., that the Lie algebra is Abelian. Thus \( \omega > 0 \), and setting \( t = 2\pi/\omega \) we check that \( e^{tJ(R)} = 1 \), i.e., \( t \in T_R \).

Finally, \( T_R \subset R \) is the kernel of the homomorphism \( R \ni t \mapsto e^{tJ(R)} \in \text{End}(R^d) \), and is therefore a discrete subgroup of the form

\[ T_R = t_0Z, \quad t_0 = \min \{ |t| \mid 0 \neq t \in T_R \}. \]

Since \( \omega \in X_R \) is equivalent to \( 2\pi/\omega \in T_R \), we have that

\[ t_0 = \frac{2\pi}{\omega_0}, \quad \omega_0 = \max X_R, \]

which completes the proof. \( \square \)

Now we are prepared to work with matrix representations.

**Proposition 2** For a finite multiplicity function \( R \) let

\[ G \doteq \left\{ \begin{pmatrix} 1 & 0 \\ v & e^{tJ(R)} \end{pmatrix} \right\}_{(v, t) \in R^d \oplus R}. \]

Then \( G \) is a connected Lie group with Lie algebra \( \mathfrak{g}A(R) \), and it is simply connected if and only if \( T_R = \{0\} \).

**Proof:** That \( G \) is a connected Lie group is clear from the definition. For \( \forall (u, s) \in R^d \oplus R \) let

\[ (-1, 1) \ni \tau \mapsto \begin{pmatrix} 1 & 0 \\ v(\tau) & e^{t(\tau)J(R)} \end{pmatrix} \in G \]

be a smooth curve with

\[ (v(0), t(0)) = (0, 0), \quad (v'(0), t'(0)) = (u, s). \]
Then
\[ \frac{d}{d\tau} \begin{pmatrix} 1 & 0 & 0 \\ v(\tau) & e^{t(\tau)J(\mathbb{N})} & 0 \\ 0 & 0 & e^{t(\tau)} \end{pmatrix} \bigg|_{\tau=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{sJ(\mathbb{N})} \\ u & sJ(\mathbb{N}) & 0 \end{pmatrix} \in \mathfrak{gA}(\mathbb{N}), \]
which proves that \( \mathfrak{gA}(\mathbb{N}) \) is the Lie algebra of \( G \). Finally, by construction \( G \) is diffeomorphic to \( \mathbb{R}^d \times (\mathbb{R}/T_{\mathbb{N}}) \), which is simply connected iff \( T_{\mathbb{N}} \) is trivial. \( \square \)

If the Lie group above is simply connected then it is a faithful matrix representation for the simply connected almost Abelian Lie group with Lie algebra \( \mathfrak{gA}(\mathbb{N}) \). But there is a simple modification that yields a faithful matrix representation for every simply connected almost Abelian Lie group. Note that
\[ \mathfrak{gA}(\mathbb{N}) \ni \begin{pmatrix} 0 & 0 & 0 \\ v & tJ(\mathbb{N}) & 0 \\ 0 & 0 & t \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\mathbb{N})} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \quad (10) \]
is a faithful matrix representation of \( \mathfrak{gA}(\mathbb{N}) \) given in representation (1).

**Proposition 3** For a finite multiplicity function \( \mathbb{N} \) let
\[
G_I \triangleq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\mathbb{N})} & 0 \\ 0 & 0 & e^t \end{pmatrix} \bigg| (v, t) \in \mathbb{R}^d \oplus \mathbb{R} \right\}, \quad G_{II} \triangleq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\mathbb{N})} & 0 \\ t & 0 & 1 \end{pmatrix} \bigg| (v, t) \in \mathbb{R}^d \oplus \mathbb{R} \right\}.
\]
Then both \( G_I \) and \( G_{II} \) are simply connected Lie groups with Lie algebras isomorphic to \( \mathfrak{gA}(\mathbb{N}) \).

**Proof:** That \( G_I \) and \( G_{II} \) are Lie groups is clear from definition. The map
\[
\mathbb{R}^d \oplus \mathbb{R} \ni (v, t) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\mathbb{N})} & 0 \\ 0 & 0 & e^t \end{pmatrix} \in G_I
\]
is a diffeomorphism, which proves that \( G_I \) is simply connected. For \( \forall (u, s) \in \mathbb{R}^d \oplus \mathbb{R} \) let
\[
(-1, 1) \ni \tau \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v(\tau) & e^{t(\tau)J(\mathbb{N})} & 0 \\ 0 & 0 & e^{t(\tau)} \end{pmatrix} \in G_I
\]
be a smooth curve with
\[
(v(0), t(0)) = (0, 0), \quad (v'(0), t'(0)) = (u, s).
\]
Then
\[
\frac{d}{d\tau} \begin{pmatrix} 1 & 0 & 0 \\ v(\tau) & e^{t(\tau)J(\mathbb{N})} & 0 \\ 0 & 0 & e^{t(\tau)} \end{pmatrix} \bigg|_{\tau=0} = \begin{pmatrix} 1 & 0 & 0 \\ u & e^{sJ(\mathbb{N})} & 0 \\ 0 & 0 & e^s \end{pmatrix},
\]
which through faithful representation (10) shows the isomorphism between \( \mathfrak{gA}(\mathbb{N}) \) and the Lie algebra of...
Finally, the map
\[
G_I \ni \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\aleph)} & 0 \\ 0 & 0 & e^t \\
\end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ v & e^{tJ(\aleph)} & 0 \\ 0 & 0 & e^t \\
\end{pmatrix} \in G_{II}
\]
is easily checked to be a Lie group isomorphism, which shows that the above statements hold for the Lie group \(G_{II}\) as well. \(\square\)

Thus a simply connected almost Abelian Lie group is a semidirect product \(G = \mathbb{R}^d \rtimes \mathbb{R}\), which is consistent with the Lie algebra being a semidirect product \(\mathfrak{A}(\aleph) = \mathbb{R}^d \rtimes \mathbb{R}\). In order to notationally distinguish between a Lie algebra element in \(\mathbb{R}^d \rtimes \mathbb{R}\) and a Lie group element in \(\mathbb{R}^d \rtimes \mathbb{R}\) we will use \((v, t) \in \mathbb{R}^d \rtimes \mathbb{R}\) for the former and \([v, t] \in \mathbb{R}^d \rtimes \mathbb{R}\) for the latter. Lemma \(1\) can now be written as
\[
\exp(v, t) = \begin{pmatrix} e^{tJ(\aleph)} - 1 \\ v, t \end{pmatrix}, \quad \forall (v, t) \in \mathfrak{A}(\aleph).
\]

**Remark 1** It follows that on the Abelian Lie subalgebra \(\ker J(\aleph) \oplus \mathbb{R}\) the exponential map is
\[
\exp(v, t) = [v, t], \quad \forall (v, t) \in \ker J(\aleph) \oplus \mathbb{R}.
\]

Discrete normal subgroups and quotients of simply connected almost Abelian groups

Z. Avetisyan, I. Martin, Z. Zhao

In this paper we will describe explicitly the discrete normal subgroups \(N\) of a simply connected almost Abelian Lie group \(G\). Then we will derive a necessary and sufficient condition for two quotient groups \(G/N\) to be isomorphic.

We start by describing the centre of a simply connected almost Abelian Lie group. Recall from [Ave16] and [Ave18] that the centre of an almost Abelian Lie algebra \(\mathfrak{A}(\aleph)\) is
\[
Z(\mathfrak{A}(\aleph)) = \ker J(\aleph),
\]
and denote
\[
T_\aleph \doteq \bigl\{ t \in \mathbb{R} \bigm| e^{tJ(\aleph)} = 1 \bigr\} \subset \mathbb{R}.
\]

**Proposition 4** The centre of the simply connected almost Abelian Lie group \(G\) with Lie algebra \(\mathfrak{A}(\aleph)\) is given by
\[
Z(G) = \exp\left[Z(\mathfrak{A}(\aleph))\right] \times T_\aleph = \exp\left[Z(\mathfrak{A}(\aleph)) \times T_\aleph\right]
\]
\[
= \left\{ [u, s] \in \mathbb{R}^d \rtimes \mathbb{R} \bigm| u \in \ker J(\aleph), \quad e^{sJ(\aleph)} = 1 \right\}.
\]
The preimage of the identity component of the centre through the exponential map is
\[
\exp^{-1}[Z(G)_0] = Z(\mathfrak{A}(N)).
\]

Proof: Let us use the faithful matrix representation
\[
G = \mathbb{R}^n \rtimes \mathbb{R} \ni [v, t] = \begin{pmatrix}
1 & 0 & 0 \\
v & e^{J(N)} & 0 \\
0 & 0 & e^t
\end{pmatrix}
\]
provided by Proposition 3. Suppose that \([u, s] \in Z(G)\). Then the following must be satisfied,
\[
[v, t][u, s] = \begin{pmatrix}
1 & 0 & 0 \\
v & e^{J(N)} & 0 \\
0 & 0 & e^t
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
u & e^{J(N)} & 0 \\
0 & 0 & e^t
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
v + e^{J(N)}u & e^{(t+s)J(N)} & 0 \\
0 & 0 & e^t
\end{pmatrix}
\]
This is equivalent to \(v + e^{J(N)}u = u + e^{sJ(N)}v\) or
\[
(e^{J(N)} - 1)u = (e^{sJ(N)} - 1)v, \quad \forall [v, t] \in G.
\]
Setting \(v = 0\) we have that \((e^{J(N)} - 1)u = 0\) which forces \(J(N)u = 0\) or \(u \in \ker J(N)\), as desired. But if \(u\) is such then \((e^{sJ(N)} - 1)v = 0\) for all \(v\), which means that \(e^{sJ(N)} = 1\). The first statement of the proposition now follows from Remark 4. If \(\exp(v, t) = [u, s] \in Z(G)_0\) then \(t = s = 0\) and \(v = u\), as desired. □

Now let us proceed to the discrete normal subgroups \(N \subset G\) of a simply connected almost Abelian Lie group.

**Proposition 5** Every discrete normal subgroup \(N \subset G\) of a simply connected almost Abelian Lie group \(G\) with Lie algebra \(\mathfrak{A}(N)\) is a free group of rank \(k \leq \dim \ker J(N) + 1\) generated by \(\mathbb{R}\) linearly independent elements
\[
[v_1, t_1], \ldots, [v_k, t_k] \in Z(G) \subset G = \mathbb{R}^d \rtimes \mathbb{R}.
\]

Proof: It is well known that every discrete normal subgroup of a connected Lie group is in fact central (e.g., [Hall]). Thus it suffices to find discrete subgroups of \(Z(G)\). Notice that for every \([v, t], [u, s] \in Z(G)\),
\[
[v, t][u, s] = [v + u, t + s],
\]
so that the restriction of the obvious homeomorphism \(f : G \to \mathbb{R}^{d+1}\) to \(Z(G)\) is also an injective Lie group homomorphism
\[
f|_{Z(G)} : Z(G) \to \mathbb{R}^{d+1}.
\]
Therefore every discrete subgroup \(N \subset Z(G)\) is mapped to a discrete subgroup \(f(N) \subset \mathbb{R}^{d+1}\). As a discrete subgroup of \(\mathbb{R}^{d+1}\), \(f(N)\) is a free Abelian group generated by \(\mathbb{R}\) linearly independent elements
\(\nu_1, \ldots, \nu_k \in \mathbb{R}^{d+1}\), and their span satisfies
\[
\mathbb{R}^{k} \subseteq \mathbb{R} \{ f(Z(G)) \},
\]
which implies that
\[
k \leq \dim \mathbb{R} \{ f(Z(G)) \} \leq \dim \ker J(\mathbb{N}) + 1.
\]
Setting \([v_i, t_i] = f^{-1}(\nu_i)\) for \(i = 1, \ldots, k\) completes the proof. \(\square\)

Now that we have a description of discrete normal subgroups \(N \subseteq G\) of a simply connected almost Abelian Lie group, and since every connected almost Abelian Lie group can be written as a quotient \(G/N\) for a corresponding \(N\), we have effectively covered all connected almost Abelian Lie groups. Next we want to know for which distinct discrete normal subgroups \(N, M \subset G\) the quotient groups \(G/N\) and \(G/M\) are isomorphic. Below is a pretty quantitative answer to this question. Denote by \(q_N : G \rightarrow G/N\) and \(q_M : G \rightarrow G/M\) the canonical quotient homomorphisms, and by \(\text{Hom}^*(G/N, G/M)\) the set of all Lie group isomorphisms \(G/N \rightarrow G/M\).

Proposition 6 Let \(G\) be a simply connected Lie group and \(N, M \subset G\) two discrete normal subgroups. Then
\[
\text{Hom}^*(G/N, G/M) = \{ \Phi_{NM} = q_M \circ \Phi \circ q_N^{-1} \mid \Phi \in \text{Aut}(G), \quad \Phi(N) = M \}.
\]

Proof: We first prove that
\[
\exists \Phi_{NM} \in \text{Hom}^*(G/N, G/M) \quad \text{s.t.} \quad \Phi_{NM} \circ q_N = q_M \circ \Phi
\]

\[
\Leftrightarrow \quad \Phi(N) = M, \quad \forall \Phi \in \text{Aut}(G). \quad (11)
\]

Let \(\Phi_{NM}\) as above be given. Then
\[
q_M \circ \Phi(n) = \Phi_{NM} \circ q_N(n) = 1, \quad \forall n \in N,
\]
whence \(\Phi(n) \in M, \forall n \in N\), and thus \(\Phi(N) \subset M\). But also
\[
q_N(\Phi^{-1}(m)) = \Phi_N^{-1} \circ q_M \circ \Phi \circ (\Phi^{-1}(m)) = \Phi_{NM}^{-1} \circ q_M(m) = 1, \quad \forall m \in M,
\]
so that \(\Phi^{-1}(m) \in N, \forall m \in M\), and thus \(\Phi^{-1}(M) \subset N\). We conclude that \(\Phi(N) = M\). Conversely, assume that \(\Phi(N) = M\). Then
\[
q_M \circ \Phi(q_N^{-1}(1)) = q_M(\Phi(N)) = q_M(M) = 1,
\]
so that \(\Phi_{NM} = q_M \circ \Phi \circ q_N^{-1} : G/N \rightarrow G/M\) is well defined. This completes the proof of (11). It remains to show that every isomorphism \(\Psi \in \text{Hom}^*(G/N, G/M)\) arises as \(\Psi = \Phi_{NM}\) for a unique \(\Phi \in \text{Aut}(G)\). To see this let \(d\Psi\) be the corresponding Lie algebra automorphism (say, Theorem 3.28 in [Hal15]). Then since \(G\) is simply connected and has the same Lie algebra as \(G/N\) and \(G/M\), there exists a unique \(\Phi \in \text{Aut}(G)\) such that \(d\Phi = d\Psi\). Consider the following two Lie group homomorphisms,
\[
\Psi \circ q_N : G \rightarrow G/N, \quad q_M \circ \Phi : G \rightarrow G/M.
\]

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By Proposition 3.30 in [Hal15],
\[ d(Ψ \circ q_N) = dΨ \circ dq_N = dΨ = dΦ = dq_M \circ dΦ = d(q_M \circ Φ). \]

But then by uniqueness in Theorem 5.6 of [Hal15] it follows that \( Ψ \circ q_N = q_M \circ Φ \), and that \( Φ \) is unique with this property. The assertion is proven. \( \square \)

In particular, two quotient groups \( G/N \) and \( G/M \) are isomorphic if and only if \( \text{Hom}^\ast(G/N, G/M) \neq \emptyset \).

**Automorphisms of almost Abelian Lie groups**

Z. Avetisyan, K. Berlow, I. Martin

In this paper we will find an explicit description of the automorphism group \( \text{Aut}(G) \) of a connected almost Abelian Lie group \( G \), with each automorphism given as a diffeomorphism in global group coordinates. For this purpose we will first combine Proposition 7, Proposition 8, Proposition 9 and Proposition 10 from [Ave16] into a single convenient description of automorphisms of an almost Abelian Lie algebra.

**Proposition 7** The automorphism group \( \text{Aut}(\mathfrak{aA}(\mathcal{N})) \subset \text{End}(\mathbb{R}^d \times \mathbb{R}) \) of a real almost Abelian Lie algebra \( \mathfrak{aA}(\mathcal{N}) = \mathbb{R}^d \times \mathbb{R} \) takes the form

\[
\text{Aut}(\mathfrak{aA}(\mathcal{N})) = \begin{cases} 
\begin{pmatrix} x \\
y \\
t \\
w \\
\end{pmatrix} \mapsto \begin{pmatrix} x \\
αΔ_{22} - β_2γ_2 & Δ_{12} & γ_1 & φ_{01} \\
0 & Δ_{22} & γ_2 & 0 \\
0 & β_2 & α & 0 \\
0 & η & ρ & φ_{11} \\
\end{pmatrix} \end{cases}, \begin{array}{c}
α, β_2, γ_1, γ_2, Δ_{12}, Δ_{22} ∈ \mathbb{R}, \\
αΔ_{22} - β_2γ_2 \neq 0, η, ρ ∈ \mathbb{R}^{d-2}, \\
φ_{01} ∈ \text{Hom}(\mathbb{R}^{d-2}, \mathbb{R}), \\
φ_{11} ∈ \text{Aut}(\mathbb{R}^{d-2}) \\
\end{array}
\] (12)

if \( \mathfrak{aA}(\mathcal{N}) = \mathbb{H} ⊕ \mathbb{R}^{d-2} \) is a central extension of the Heisenberg algebra and

\[
\text{Aut}(\mathfrak{aA}(\mathcal{N})) = \begin{cases} 
\begin{pmatrix} Δ \\
γ \\
0 \\
α \\
\end{pmatrix} \end{cases}, \begin{array}{c}
α ∈ \text{Dil}(\mathcal{N}), \\
γ ∈ \mathbb{R}^d, \\
Δ ∈ \text{Aut}(\mathbb{R}^d), \\
Δ J(\mathcal{N}) = α J(\mathcal{N})Δ \\
\end{array}
\] (13)

otherwise.

**Remark 2** If we apply formula (13) to the Lie algebra \( \mathbb{H} ⊕ \mathbb{R}^{d-2} \) then we will obtain only the subgroup consisting of those automorphisms corresponding to \( β_2 = 0 \) in formula (13).

We begin with the case of a simply connected \( G \), where there is a bijective correspondence between Lie algebra automorphisms and Lie group automorphisms. On several occasions we will make use of the following elementary fact.

**Remark 3** If \( A, B \) and \( C \) are square matrices such that \( AB = BC \) then for every entire holomorphic function \( F ∈ \text{Hol}(\mathbb{C}) \) one has \( F(A)B = BF(C) \).

This can be easily checked term by term in the Taylor expansion.

Let \( \mathcal{H} = \exp(\mathbb{H}) \) stand for the Heisenberg group.
Proposition 8 If $G$ is a simply connected almost Abelian Lie group with Lie algebra $\mathfrak{a}(\mathbb{R})$ then

$$\text{Aut}(G) = \left\{ \begin{array}{l}
\begin{bmatrix} x \\ y \\ t \\ w \end{bmatrix} \rightarrow \Phi \left[ \begin{array}{c}
[\alpha \Delta_{22} - \beta_2 \gamma_2] x + \Delta_{12} y + \gamma_1 t + \beta_2 \gamma_2 y + \frac{1}{2} \Delta_{22} \beta_2 y^2 + \phi_0 (w) \\
\Delta_{22} y + \gamma_2 t \\
\beta_2 y + \alpha t \\
\eta y + pt + \phi_{11} (w)
\end{array} \right] \\
& \in \text{Aut}(\mathfrak{a}(\mathbb{R}))
\end{array} \right\}$$

(14)

if $G = H \times \mathbb{R}^{d-2}$ is a central extension of the Heisenberg group and

$$\text{Aut}(G) = \left\{ [v, t] \rightarrow \Phi \left[ \frac{e^{\alpha J(N)}}{\alpha J(N)} \gamma + \Delta v, \alpha t \right] \right\} \quad d\Phi |_{(0,0)} = \left( \begin{array}{c}
0 \\
\phi_0 (w) \\
\eta \\
\phi_{11} (w)
\end{array} \right) \quad \in \text{Aut}(\mathfrak{a}(\mathbb{R}))$$

(15)

otherwise.

Proof: Central extensions of the Heisenberg group are exponential, and we can use the bijectivity of the exponential map to switch from Lie algebra automorphisms to Lie group automorphisms. Namely, if

$$d\Phi \left( \begin{array}{c}
x \\
y \\
t \\
w
\end{array} \right) = \left( \begin{array}{c}
[\alpha \Delta_{22} - \beta_2 \gamma_2] x + \Delta_{12} y + \gamma_1 t + \phi_0 (w) \\
\Delta_{22} y + \gamma_2 t \\
\beta_2 y + \alpha t \\
\eta y + pt + \phi_{11} (w)
\end{array} \right)$$

then

$$\Phi \left( \begin{array}{c}
\exp \left( \begin{array}{c}
0 \\
x \\
y \\
w
\end{array} \right) \\
0 \\
0 \\
0
\end{array} \right) = \Phi \left( \begin{array}{c}
1 \\
0 \\
y \\
w
\end{array} \right)$$

(15)

which yields the desired assertion. For the generic case let us first show that the map

$$[v, t] \rightarrow \Phi \left[ \frac{e^{\alpha J(N)}}{\alpha J(N)} \gamma + \Delta v, \alpha t \right]$$
is bijective by checking that its inverse is given by

$$\phi^{-1}([u, t]) = \begin{pmatrix} e^{\frac{\mathcal{J}(N)}{\gamma} - 1} \Delta^{-1} \gamma + \Delta^{-1} v, t \end{pmatrix}.$$  

Indeed, 

$$\Phi^{-1} \circ \Phi[v, t] = \begin{pmatrix} e^{\frac{\mathcal{J}(N)}{\alpha} J(N)} - 1 \Delta^{-1} \gamma + \Delta^{-1} \alpha t, J(N) \gamma + \Delta v, \alpha t \end{pmatrix} = [v, t],$$

where we used $\alpha J(N) \Delta = \Delta J(N)$ and Remark 3. Next we establish that the same map is a Lie group homomorphism,

$$\Phi[v, t] \cdot \Phi[u, s] = \begin{pmatrix} e^{\frac{\mathcal{J}(N)}{\alpha} J(N)} - 1 \Delta^{-1} \gamma + \Delta^{-1} \alpha t, J(N) \gamma + \Delta v, \alpha t \end{pmatrix} \cdot \begin{pmatrix} e^{\frac{\mathcal{J}(N)}{\alpha} J(N)} - 1 \Delta^{-1} \gamma + \Delta^{-1} \alpha s, J(N) \gamma + \Delta v, \alpha s \end{pmatrix}$$

$$= \begin{pmatrix} e^{\alpha(t+s) J(N)} - 1 \Delta^{-1} \gamma + \Delta \left(v + e^{\mathcal{J}(N)} u, \alpha t + \alpha s \right) \end{pmatrix} = \Phi[v + e^{\mathcal{J}(N)} u, t + s] = \Phi([v, t] \cdot [u, s]).$$

Finally, for every $(u, s) \in \mathbb{R}^d \times \mathbb{R} = \mathcal{A}(N)$ let $(-1, 1) \ni \tau \mapsto [v(\tau), t(\tau)] \in G$ be a smooth curve such that $[v(0), t(0)] = [0, 0]$ and $(v'(0), t'(0)) = (u, s)$. Then

$$d\Phi(u, s) = \frac{d}{d\tau} \Phi[v(\tau), t(\tau)]|_{\tau=0} = \frac{d}{d\tau} \begin{pmatrix} e^{\alpha(t+s) J(N)} - 1 \Delta^{-1} \gamma + \Delta v(\tau), \alpha t(\tau) \end{pmatrix} = (\Delta u + s\gamma, \alpha s),$$

which completes the proof. □

Remark 4 Again, if we apply formula (13) to a central extension $G = H \times \mathbb{R}^{d-2}$ of the Heisenberg group then we will exactly recover those automorphisms with $\beta_2 = 0$ in formula (14).

The normal subgroup $\text{Inn}(G) \subset \text{Aut}(G)$ of inner automorphisms contains $\Phi_g \in \text{Aut}(G)$ such that $\Phi_g(h) = ghg^{-1}$ for some $g \in G$ and all $h \in G$.

Corollary 1 If $G$ is a simply connected almost Abelian Lie group with Lie algebra $\mathcal{A}(N)$ then

$$\text{Inn}(G) = \left\{ [v, t] \mapsto \begin{pmatrix} e^{\mathcal{J}(N)} - 1 \Delta^{-1} \gamma + \Delta v, t \end{pmatrix} \middle| \gamma \in J(N) \mathbb{R}^d, \Delta = e^{\mathcal{J}(N)}, s \in \mathbb{R} \right\}.$$  

Proof: That $\Phi_g \in \text{Inn}(G)$ means that $\Phi_g(h) = ghg^{-1}$, for $g \in G$, $\forall h \in G$. Let $g = [u, s]$ and $h = [v, t]$, so that

$$\Phi_{[u, s]}[v, t] = [u, s][v, t][u, s]^{-1} = \begin{pmatrix} e^{\mathcal{J}(N)} u - \left(e^{\mathcal{J}(N)} - 1\right) u, t \end{pmatrix} = \begin{pmatrix} e^{\mathcal{J}(N)} - 1 \Delta^{-1} \gamma + \Delta v, t \end{pmatrix},$$

where 

$$\Delta = e^{\mathcal{J}(N)}, \quad \gamma = -J(N)u,$$

precisely as asserted. □

We turn now to the case of more general connected almost Abelian Lie group $G/N$ where $G$ is simply connected and $N \subset G$ is a discrete central subgroup. Denote by $q_N : G \to G/N$ the canonical quotient homomorphism. By Proposition 5 we know that

$$\text{Aut}(G/N) = \left\{ \Phi_N = q_N \circ \Phi \circ q_N^{-1} \middle| \Phi \in \text{Aut}(G), \Phi(N) = N \right\}.$$
We will describe the condition $\Phi(N) = N$ more explicitly using Proposition 8. The following simple fact will come in handy.

**Lemma 6** If $e^{\alpha J(N)} = 1$ then $\forall \alpha \in L_0 \oplus W$ where $L_0$ is indecomposable and $W = \ker J(N)$, and

$$
\frac{e^{\alpha J(N)} - 1}{\alpha J(N)} = t \left[ 0_{L_0} \oplus 1_W \right], \quad \forall \alpha \in \text{Dil}(N).
$$

**Proof:** Note that

$$
\frac{e^{\alpha J(N)} - 1}{\alpha J(N)} = \frac{e^{\alpha t J(N)} - 1}{\alpha t J(N)}.
$$

If $t = 0$ then

$$
\frac{e^{\alpha J(N)} - 1}{\alpha t J(N)} = 1
$$

and the assertion is clear. If $t \neq 0$ then $t \in T_H \neq 0$, and by Lemma 5

$$
\frac{e^{\alpha J(N)} - 1}{\alpha t J(N)} = [0_{L_0} \oplus 1_W],
$$

as desired. □

Now fix a central discrete subgroup $N \subset G$ and let by Proposition 8 $N$ be generated by $\{[x_i, 0, 0, w_i] \}_{i=1}^k$ if $G = H \times \mathbb{R}^{d-2}$ and $\{[v_i, t_i] \}_{i=1}^k$ otherwise.

**Proposition 9** In terminology of Proposition 8 an automorphism $\Phi \in \text{Aut}(G)$ satisfies $\Phi(N) = N$ if and only if

$$
\begin{pmatrix}
\alpha \Delta_{22} - \beta_2 \gamma_2 & \phi_{01} \\
0 & \phi_{11}
\end{pmatrix}
\begin{pmatrix}
x_1 & \ldots & x_k \\
w_1 & \ldots & w_k
\end{pmatrix}
= \begin{pmatrix}
x_1 & \ldots & x_k \\
w_1 & \ldots & w_k
\end{pmatrix}
\cdot A,
\quad A \in \text{GL}(\mathbb{Z}, k)
$$

for $G = H \times \mathbb{R}^{d-2}$ and

$$
\begin{pmatrix}
\Delta & \gamma_W \\
0 & \alpha
\end{pmatrix}
\begin{pmatrix}
v_1 & \ldots & v_k \\
t_1 & \ldots & t_k
\end{pmatrix}
= \begin{pmatrix}
v_1 & \ldots & v_k \\
t_1 & \ldots & t_k
\end{pmatrix}
\cdot A,
\quad A \in \text{GL}(\mathbb{Z}, k)
$$

otherwise. Here $\gamma_W = [0_{L_0} \oplus 1_W] \gamma$ as per Lemma 6.

**Proof:** If $G = H \times \mathbb{R}^{d-2}$ then for every $\Phi \in \text{Aut}(G)$ the condition $\Phi(N) \subset N$ can be expressed as the statement that for every fixed $1 \leq i_0 \leq k$, the image $\Phi([x_{i_0}, 0, 0, w_{i_0}])$ is an integer linear combination of $\{[x_i, 0, 0, w_i] \}_{i=1}^k$. In matrix language of Proposition 8 this can be written as

$$
\Phi
\begin{bmatrix}
x_{i_0} \\
0 \\
0 \\
w_{i_0}
\end{bmatrix}
= \begin{pmatrix}
\alpha \Delta_{22} - \beta_2 \gamma_2 & \phi_{01} (w_{i_0}) \\
0 & 0 \\
0 & 0 \\
\phi_{11} (w_{i_0})
\end{pmatrix}
\begin{bmatrix}
x_{i_0} \\
0 \\
0 \\
w_{i_0}
\end{bmatrix}
= \begin{pmatrix}
\alpha \Delta_{22} - \beta_2 \gamma_2 & 0 & 0 & \phi_{01} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \phi_{11}
\end{pmatrix}
\begin{bmatrix}
x_{i_0} \\
0 \\
0 \\
w_{i_0}
\end{bmatrix}
$$
it follows that

\[ \Phi(N, M) \]

for two discrete central subgroups

Pursuant to the aims of Proposition 6, in this paper we want to derive necessary and sufficient conditions

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almost Abelian groups revisited

Discrete normal subgroups and quotients of simply connected

Combining these statements for all \( i = 1, \ldots, k \) we obtain the formula (10) with \( A \) being a \( k \times k \) matrix with integer entries. Following the same logic for \( \Phi^{-1}(N) \subset N \) we will obtain a similar formula where the matrix \( A^{-1} \) figures and is supposed to have integer coefficients. But \( \Phi(N) = N \) is equivalent to \( \Phi(N) \subset N \) and \( \Phi^{-1}(N) \subset N \), which holds if and only if both \( A \) and \( A^{-1} \) have integer entries, i.e., \( A \in \text{GL}(Z, k) \), as desired. If \( G \neq \mathcal{H} \times \mathbb{R}^{d-2} \) then by Proposition 4 we see that \( e^{t_i J(\alpha)} = 1 \) for all \( i = 1, \ldots, k \). Thus by Remark 5 and Lemma 8 the condition \( \Phi(N) \subset N \) becomes

\[
\Phi \begin{pmatrix} v_{i0} \\ t_{i0} \end{pmatrix} = \begin{pmatrix} e^{\alpha t_{i0} J(\gamma)} + \Delta v_{i0} \\ \alpha t_{i0} \end{pmatrix} = \begin{pmatrix} \Delta \gamma w \\ 0 \alpha \end{pmatrix} \begin{pmatrix} v_{i0} \\ t_{i0} \end{pmatrix} = \begin{pmatrix} v_1 & \ldots & v_k \\ t_1 & \ldots & t_k \end{pmatrix} \begin{pmatrix} A_{1i0} \\ \vdots \\ A_{ki0} \end{pmatrix}, \quad A_{i0} \in \mathbb{Z}, \quad i = 1, \ldots, k.
\]

Combining these statements for all \( i_0 = 1, \ldots, k \) we obtain the formula (17) with \( A \) being a \( k \times k \) matrix with integer entries. The rest of the argument follows as before. \( \square \)

**Remark 5** Let \( N \subset G \) be a discrete central subgroup. Since all \( \Phi \in \text{Inn}(G) \) act trivially on \( N \subset Z(G) \), it follows that \( \Phi(N) = N \) is satisfied automatically.

**Discrete normal subgroups and quotients of simply connected almost Abelian groups revisited**

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Pursuant to the aims of Proposition 6 in this paper we want to derive necessary and sufficient conditions for two discrete central subgroups \( N, M \subset G \) to be related by an automorphism \( \Phi \in \text{Aut}(G) \) of the simply connected almost Abelian Lie group \( G \). We begin with preparatory steps with a discrete central subgroup \( N \subset G \) given in terms of a set of generators \([v_1, t_1], \ldots, [v_k, t_k]\) according to Proposition 8. Every other set of generators \([u_1, s_1], \ldots, [u_k, s_k]\) of \( N \) is related to the original one by

\[
\begin{pmatrix} u_1 & \ldots & u_k \\ s_1 & \ldots & s_k \end{pmatrix} = \begin{pmatrix} v_1 & \ldots & v_k \\ t_1 & \ldots & t_k \end{pmatrix} \cdot A, \quad A \in \text{GL}(Z, k).
\]

According to Lemma 5 there exists \( t_0 \in T_N \) and \( n_1, \ldots, n_k \in \mathbb{Z} \) such that \( t_i = n_i t_0, i = 1, \ldots, k \).

**Lemma 7** There exists a change of generators \( A \in \text{GL}(Z, k) \) such that

\[
\begin{pmatrix} u_1 & u_2 & \ldots & u_k \\ t_0 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & \ldots & v_k \\ t_1 & t_2 & \ldots & t_k \end{pmatrix} \cdot A.
\]
**Proposition 10** Two discrete central subgroups the generators \( v_1, v_2, \ldots, v_k \) ordered by non-decreasing block dimension

**Proof:** This can be achieved easily by column operations justified with Bezout’s identity. □

In what follows we will assume that a discrete central subgroup \( N \subset G \) is given by a set of generators in the more economic form \([v_1, t_1], [v_2, 0], \ldots, [v_k, 0]\). In terminology of formula (16) in [Ave18],

\[
\ker J(N) = \bigoplus_{n=1}^{\infty} \mathbb{R}e_{x}^{n}(X, n),
\]

or in other words, the vectors \( v_1, \ldots, v_k \in \mathbb{R}^d \) written in the standard basis \( e_x^m(p, n) \) may have non-zero entries only in the rows corresponding to the topmost elements of the Jordan blocks with eigenvalue zero.

Let \( \tilde{v}_1, \ldots, \tilde{v}_k \in \mathbb{R}^q \), \( q = \text{dim} \ker J(N) \), be the vectors obtained by picking only these significant rows. We have seen in Proposition 8 that operators \( \Delta \in \text{Aut}(\mathbb{R}^d) \) with \( [\Delta, J(N)] = 0 \) play a prominent role in the structure of automorphisms of \( G \). Such an operator \( \Delta \) preserves the invariant subspace \( \ker J(N) \), and we denote the restriction of \( \Delta \) to \( \ker J(N) \) by \( \tilde{\Delta} \in \text{Aut}(\mathbb{R}^q) \). Let us now assume that Jordan blocks in \( J(N) \) are ordered by non-decreasing block dimension \( n \). Applying Proposition 7 and Lemma 2 from [Ave18], we see that \( \Delta = \tilde{\Delta} \oplus 0 \) (i.e., the matrix \( \Delta \) beyond the submatrix \( \tilde{\Delta} \) is identically zero) and \( \tilde{\Delta} \) is an arbitrary real invertible block-upper-triangular matrix with blocks corresponding to constant Jordan block dimension \( n \). That means,

\[
\tilde{\Delta} = \begin{pmatrix}
\tilde{\Delta}_{n_1 n_1} & \tilde{\Delta}_{n_1 n_2} & \cdots & \tilde{\Delta}_{n_1 n_s} \\
0 & \tilde{\Delta}_{n_2 n_2} & \cdots & \tilde{\Delta}_{n_2 n_s} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \tilde{\Delta}_{n_s n_s}
\end{pmatrix}, \quad \tilde{\Delta}_{n_i n_j} \in \text{Hom}(\mathbb{R}^{q_j}, \mathbb{R}^{q_i}), \quad n(X, n_i) = q_i, \quad i, j = 1, \ldots, s,
\]

\( q_1 + \ldots + q_s = q \), \( n_i > n_{i+1} \), \( i = 1, \ldots, s - 1 \).

The following simple observation will be useful in what follows.

**Remark 6** In terminology of [Ave18], \( \text{Dil}(N) \subset \mathbb{R}^* \) is a finite multiplicative subgroup and therefore \( \text{Dil}(N) \subset \mathbb{Z}_2 \). If \( \text{supp} R \subset \mathbb{R}^\mathbb{R} \), which by Lemma 2 is the case when \( T_R \neq \{0\} \), then necessarily \( \text{Dil}(N) = \mathbb{Z}_2 \).

**Proposition 10** Two discrete central subgroups \( N \) and \( M \) given in terms of generators \([v_1, t_1], [v_2, 0], \ldots, [v_k, 0]\) and \([u_1, s_1], [u_2, 0], \ldots, [u_k, 0]\), respectively, are related by an automorphism of \( G \) if and only if \( t_1 = \pm s_1 \) and there exist \( \tilde{\Delta} \) as above and an \( A \in \text{GL}(\mathbb{Z}, k) \) such that

\[
\tilde{\Delta} \cdot (\tilde{v}_1 \tilde{v}_2 \ldots \tilde{v}_k) = (\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_k) \cdot A \quad \text{if} \quad t_1 = 0
\]

and

\[
\tilde{\Delta} \cdot (\tilde{w} \tilde{v}_2 \ldots \tilde{v}_k) = (\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_k) \cdot A \quad \text{if} \quad t_1 \neq 0,
\]

where \( \tilde{w} \in \mathbb{R}^q \) can be chosen arbitrarily.

**Proof:** The subgroup \( N \) is mapped to the subgroup \( M \) by an automorphism \( \Phi \in \text{Aut}(G) \) if and only if the generators \([v_1, t_1], [v_2, 0], \ldots, [v_k, 0]\) are mapped to any set of generators of \( M \), which must be related.
to the original generators \([u_1, s_1], [u_2, 0], \ldots, [u_k, 0]\) through a matrix \(A \in \text{GL}(\mathbb{Z}, k)\), i.e.,

\[
\Phi \begin{pmatrix} v_1 & v_2 & \ldots & v_k \\ t_1 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \ldots & u_k \\ s_1 & 0 & \ldots & 0 \end{pmatrix} \cdot A.
\]

By Proposition 5, this amounts to

\[
\begin{pmatrix} \Delta & \gamma \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} v_1 & v_2 & \ldots & v_k \\ t_1 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \ldots & u_k \\ s_1 & 0 & \ldots & 0 \end{pmatrix} \cdot A,
\]

since even for \(H \times \mathbb{R}^{d-2}\) the coefficient \(\beta_2\) has no effect in acting on vectors from \(\ker J(\mathbb{N})\). By Remark 6, we have \(\alpha = \pm 1\) so that \(t_1 = \pm s_1\). Further,

\[
\Delta \cdot (v_1, v_2, \ldots, v_k) + (t_1\gamma, 0, \ldots, 0) = (v_1, v_2, \ldots, v_k) \cdot A,
\]

where the choice of \(\gamma \in \mathbb{R}^d\) is completely arbitrary. The assertion now follows by restricting the above equation to \(\ker J(\mathbb{N})\). □

Finding algebraic criteria under which the above conditions are satisfied is a hard problem which we will not pursue here.

As a simple side result, the structure of a discrete central subgroup \(N \subset G\) can be simplified further using automorphisms. In the above economic form of the basis for \(N\) the element \(v_1\) is arbitrary, and it need not be possible to kill \(v_1\) by any further right \(\text{GL}(\mathbb{Z}, k)\) action. Instead, we can use automorphisms of \(G\) to achieve that simplification.

**Proposition 11** For every discrete central subgroup \(N \subset G\) of a simply connected almost Abelian group \(G = \mathbb{R}^d \rtimes \mathbb{R}\) with Lie algebra \(\mathfrak{a}(\mathbb{N})\) there exists an automorphism \(\Phi \in \text{Aut}(G)\) such that the discrete central subgroup \(M = \Phi(N)\) satisfies \(M = (M \cap \ker J(\mathbb{N})) \times (M \cap T_{\mathbb{N}})\).

**Proof:** Let \(N\) be given in terms of the generators \([v_1, t_1], [v_2, 0], \ldots, [v_k, 0]\). If \(t_1 = 0\) then \(N \subset \ker J(\mathbb{N})\) and the assertion is trivial. Assume that \(t_1 \neq 0\), so that by Lemma 5 we have \(\text{supp } \mathbb{N} \subset \tau \mathbb{R}\), and therefore \(\text{Dil}(\mathbb{N}) = \mathbb{Z}_2\) [Ave18]. Choose \(\Phi\) according to Proposition 5 with \(\alpha = \text{sgn } t_1\), \(\Delta = 1\) and \(\gamma = -\frac{1}{t_1} v_1\). Then \(M = \Phi(N)\) is given by the set of generators

\[
\begin{pmatrix} 1 & -\frac{1}{t_1} v_1 \\ 0 & \text{sgn } t_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 & v_2 & \ldots & v_k \\ t_1 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} 0 & v_2 & \ldots & v_k \\ |t_1| & 0 & \ldots & 0 \end{pmatrix},
\]

whence the statement of the proposition follows. □
Connected almost Abelian groups

Z. Avetisyan, I. Martin, Z. Zhao

The goal of this paper is to describe connected (not necessarily simply connected) almost Abelian groups in terms of faithful matrix representations whenever the latter exist. Recall that a connected almost Abelian group can be written as $G/N$ where the universal cover $\tilde{G}$ is a simply connected almost Abelian group and $N \subset G$ is a discrete central subgroup. Regardless of whether $G/N$ is a matrix group, the matrix representation of $G$ can be used to produce a natural (almost global) coordinate chart on $G/N$ as follows. Consider a modification of the second faithful matrix representation of $G$ from Proposition 3 as a faithful "quotient-matrix" representation of $G/N$,

$$G/N \ni [v, t] \mod N \mapsto \begin{pmatrix}
1 & 0 & 0 \\
v & e^{t J(\infty)} & 0 \\
t & 0 & 1
\end{pmatrix} \in \text{End}(\mathbb{R}^{d+2}).$$

This representation is algebraically convenient since by Proposition 5 we know that $N$ can be seen as an additive subgroup of $\mathbb{R}^{d+1}$, and $[v, t] \mod N$ is easy to compute. In a neighbourhood of the identity the above representation coincides with the true faithful matrix representation of $G$.

Let us now turn to proper faithful matrix representations. The following provides an explicit faithful matrix representation for a quotient group $G/N$ under certain assumptions on $N$. Let

$$\mathfrak{g}(\mathbb{N}) = \mathbb{R}^{d_0} \times \mathbb{R} \oplus \mathbb{R}^{d-d_0}$$

be a decomposition of $\mathfrak{g}(\mathbb{N})$ as in [Ave16] where $\mathbb{R}^{d_0} \times \mathbb{R}$ is indecomposable. Then the simply connected group decomposes as $G = G_0 \times \mathbb{R}^{d-d_0}$. The first faithful representation of $G$ from Proposition 3 upon substitution of the decomposition $\mathbb{R}^d \ni u \mapsto v \oplus w \in \mathbb{R}^{d_0} \oplus \mathbb{R}^{d-d_0}$, gives

$$G = \mathbb{R}^{d_0} \times \mathbb{R} \times \mathbb{R}^{d-d_0} \ni [v, t, w] \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 \\
v & e^{t J(\infty)} & 0 & 0 \\
w & 0 & 1 & 0 \\
0 & 0 & 0 & e^t
\end{pmatrix} \in \text{End}(\mathbb{R}^{d+1}).$$

If we denote by $\text{diag} w$ the $(d-d_0)$-dimensional diagonal matrix composed of components of $w$ then it can be easily checked that

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
v & e^{t J(\infty)} & 0 & 0 \\
w & 0 & 1 & 0 \\
0 & 0 & 0 & e^t
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 \\
v & e^{t J(\infty)} & 0 & 0 \\
0 & 0 & e^{\text{diag} w} & 0 \\
0 & 0 & 0 & e^t
\end{pmatrix} \quad (18)$$

is a matrix Lie group isomorphism, and therefore the right hand side is another faithful matrix representation of $G$. 

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Assume now that the discrete central subgroup satisfies \( N \subset \mathbb{R}^{d-d_0} \times T_{\mathbb{R}} \), i.e., per Proposition 5 is generated by
\[
[w_1, t_1], \ldots, [w_k, t_k] \in \mathbb{R}^{d-d_0} \times T_{\mathbb{R}}, \quad 0 \leq k \leq d - d_0 + 1.
\]
The representation on the right hand side of (15) is convenient in that it allows to reshuffle the last \( d - d_0 + 1 \) dimensions in way to separate the generators of \( N \). Namely, complete arbitrarily the above generators of \( N \) to a basis in \( \mathbb{R}^{d-d_0} \oplus \mathbb{R} \),
\[
[w_1, t_1], \ldots, [w_{d-d_0+1}, t_{d-d_0+1}] \in \mathbb{R}^{d-d_0} \oplus \mathbb{R},
\]
and consider the inverse \( P \in \text{End}(\mathbb{R}^{d-d_0+1}) \) of the matrix with columns being elements of this basis,
\[
P = \begin{pmatrix}
[w_1] & \ldots & [w_{d-d_0+1}]
\end{pmatrix}^{-1}.
\] (19)
Let \( P_\parallel \in \text{Hom}(\mathbb{R}^{d-d_0+1}, \mathbb{R}^k) \) represent the first \( k \) rows of \( P \), and \( P_\perp \in \text{Hom}(\mathbb{R}^{d-d_0+1}, \mathbb{R}^{d-d_0+1-k}) \) the remaining rows.

**Proposition 12** If the discrete central subgroup satisfies \( N \subset \mathbb{R}^{d-d_0} \times T_{\mathbb{R}} \) then the map
\[
G/N \ni [v, t, w] \mod N \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 \\
v e^{tJ(w_0)} & 0 & 0 & 0 \\
0 & 0 & e^{\frac{\text{diag } 2\pi \{P_\parallel[w, t]\}^T}{2\pi \{P_\parallel[w, t]\}^T}} & 0 \\
0 & 0 & 0 & e^{\frac{\text{diag } P_\perp[w, t]^T}{2\pi \{P_\perp[w, t]^T\}}}
\end{pmatrix} \in \text{End}(\mathbb{R}^{d+2})
\]

is a faithful matrix representation of \( G/N \).

**Proof:** In view of (15) being a faithful representation of \( G \), it suffices to show that the map
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
v e^{tJ(w_0)} & 0 & 0 & 0 \\
0 & 0 & e^{\frac{\text{diag } 2\pi \{P_\parallel[w, t]\}^T}{2\pi \{P_\parallel[w, t]\}^T}} & 0 \\
0 & 0 & 0 & e^{\frac{\text{diag } P_\perp[w, t]^T}{2\pi \{P_\perp[w, t]^T\}}}
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 \\
v e^{tJ(w_0)} & 0 & 0 & 0 \\
0 & 0 & e^{\frac{\text{diag } 2\pi \{P_\parallel[w, t]\}^T}{2\pi \{P_\parallel[w, t]\}^T}} & 0 \\
0 & 0 & 0 & e^{\frac{\text{diag } P_\perp[w, t]^T}{2\pi \{P_\perp[w, t]^T\}}}
\end{pmatrix}
\]
is a Lie group homomorphism with kernel \( N \). Checking that this is a Lie group homomorphism is straightforward. Now let \([v, t, w] \in \mathbb{R}^{d_0} \times \mathbb{R} \times \mathbb{R}^{d-d_0} = G \). Then
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
v e^{tJ(w_0)} & 0 & 0 & 0 \\
0 & 0 & e^{\frac{\text{diag } 2\pi \{P_\parallel[w, t]\}^T}{2\pi \{P_\parallel[w, t]\}^T}} & 0 \\
0 & 0 & 0 & e^{\frac{\text{diag } P_\perp[w, t]^T}{2\pi \{P_\perp[w, t]^T\}}}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
iff
\[
v = 0, \quad t \in T_{\mathbb{R}}, \quad P_\parallel[w, t]^T \in \mathbb{Z}^k, \quad P_\perp[w, t]^T = 0.
\]
The latter two conditions can be combined into \( P[w, t]^T \in \mathbb{Z}^k \oplus 0 \), which in view of (15) can be written
as
\[
\begin{pmatrix}
w \\
t
\end{pmatrix}
= \begin{pmatrix}
w_1 & \cdots & w_{d-d_0+1} \\
t_1 & \cdots & t_{d-d_0+1}
\end{pmatrix}
\begin{pmatrix}
m \\
0
\end{pmatrix}, \quad m \in \mathbb{Z}^k,
\]
which is equivalent to \([w, t]\) being generated by \([w_1, t_1], \ldots, [w_k, t_k]\) over \(\mathbb{Z}\). Thus \([v, t, w]\) is in the kernel iff \([v, t, w]\) \(\in N\), which completes the proof. \(\square\)

Below we establish a necessary and sufficient condition for \(G/N\) to be a matrix group in terms of the subgroup \(N\). We start with a little lemma.

**Lemma 8** Let \(X, Y, Z \in \text{End}(\mathbb{C}^n)\) be such that

\[
[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0, \quad Z + Z^* = 0.
\]

Then \(Z = 0\).

**Proof:** Since \(Z\) is anti-Hermitean, by the spectral theorem for Hermitean matrices it is unitarily diagonalizable with purely imaginary spectrum. Assume without loss of generality that

\[
Z = \bigoplus_{i=1}^q \lambda_i 1_{n_i}, \quad \lambda_i \in \mathbb{R}, \quad n_1 + \ldots + n_q = n.
\]

Then by Proposition 7 in [Ave18] the matrices \(X\) and \(Y\) are of the form

\[
X = \bigoplus_{i=1}^q X_i, \quad Y = \bigoplus_{i=1}^q Y_i, \quad X_i, Y_i \in \text{End}(\mathbb{C}^{n_i}).
\]

Thus \([X_i, Y_i] = \lambda_i 1_{n_i}\) and therefore \(\text{tr}[X_i, Y_i] = 0 = \lambda_i, i = 1, \ldots, q\), which shows that \(Z = 0\). \(\square\)

**Proposition 13** Let \(G = \mathbb{R}^d \rtimes \mathbb{R}\) be a simply connected almost Abelian group with Lie algebra \(L = \mathbb{R}^d \rtimes \mathbb{R} = \mathfrak{a}(\mathbb{R})\), and let \(N \subset G\) be a discrete central subgroup with generators \([v_1, t_1], \ldots, [v_k, t_k]\). Then the following two statements are equivalent:

1. \(\mathbb{R} \{[v_1, t_1], \ldots, [v_k, t_k]\} \cap [L, L] = 0\)

2. \(G/N\) has a faithful (real or complex) matrix representation

**Proof:** 2. \(\Rightarrow\) 1. Assume towards a contradiction that condition 1. is not satisfied,

\[
[(0,1), (u, 0)] = \sum_{i=1}^k \lambda_i (v_i, t_i) \neq 0, \quad u \in \mathbb{R}^d, \quad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, k,
\]

and let \(\sigma : G/N \to \text{Aut}(\mathbb{C}^n)\) be a faithful representation with \(d\sigma : L \to \text{End}(\mathbb{C}^n)\) being its derivative. Because \(\exp [L(L)] = 1\) we have that \(\exp (v_i, t_i) = [v_i, t_i]\) and thus \(\sigma[v_i, t_i] = e^{d\sigma(v_i, t_i)} = 1\), which implies by Lemma that \(d\sigma(v_i, t_i)\) is diagonalizable with spectrum in \(2\pi i \mathbb{Z}\). Moreover, since \([d\sigma(v_i, t_i), d\sigma(v_j, t_j)] = 0\) for all \(i, j = 1, \ldots, k\), there is an invertible \(P \in \text{Aut}(\mathbb{C}^n)\) such that

\[
P^{-1} d\sigma(v_i, t_i) P = i D_i, \quad D_i^* = D_i, \quad i = 1, \ldots, k.
\]
Denote

\[ X = P^{-1}d\sigma(0,1)P, \quad Y = P^{-1}d\sigma(0,u)P, \quad Z = \sum_{i=1}^{k} D_i. \]

Then the assumptions of Lemma 8 are satisfied, implying that

\[ Z = d\sigma([(0,1),(u,0)]) = 0, \quad [(0,1),(u,0)] \neq 0, \]

which contradicts the fact that \( \sigma \) is faithful.

1. \( \Rightarrow \) 2. Let now condition 1. be satisfied. By Lemma 7 we can assume without loss of generality that \( t_2 = t_3 = \ldots = t_k = 0 \). If \( L = R^{d_0} \rtimes R^{d-d_0} \) is the decomposition as before then condition 1. implies that \( v_2, \ldots, v_k \in R^{d-d_0} \). If \( t_1 = 0 \) then condition 1. also requires that \( v_1 \in R^{d-d_0} \), which shows that \( N \subset R^{d-d_0} \), and by Proposition 12 the quotient group \( G/N \) has a faithful matrix representation. If \( t_1 \neq 0 \) then applying the automorphism \( \Phi \in Aut(G) \) from Proposition 11 we obtain the discrete central subgroup \( \Phi(N) \) with generators \([0,t_1],[v_2,0],\ldots,[v_k,0]\), which now satisfies \( \Phi(N) \subset R^{d-d_0} \times T_k \). Thus by Proposition 12 the quotient group \( G/\Phi(N) \) has a faithful matrix representation. But then by Proposition 6 the automorphism \( \Phi \) induces an isomorphism between \( G/N \) and \( G/\Phi(N) \), proving that \( G/N \) has a faithful matrix representation, too. \( \square \)

Connected subgroups of a connected almost Abelian Lie group

Z. Avetisyan, I. Martin, G. Rakholia, Z. Zhao

The goal of this paper is describing all connected Lie subgroups of a connected almost Abelian Lie group. A connected almost Abelian group can be identified with the quotient group \( G/N \) where \( G \) is a simply connected almost Abelian Lie group and \( N \subset G \) is a discrete normal subgroup (see Proposition 5). The canonical quotient map \( q_N : G \to G/N \) is a Lie group homomorphism, and its derivative \( dq_N \) is an isomorphism of Lie algebras. Thus we can assume without loss of generality that the Lie algebras of both \( G \) and \( G/N \) are \( \mathfrak{a}(\aleph) \). By the Lattice Isomorphism Theorem (Theorem 20 in [DuFo04]) subgroups \( H \subset G/N \) are exactly the quotients \( H/N \) of subgroups \( H \subset G \) with \( N \subset H \subset G \). However, the complete preimage \( q_N^{-1}(H_N) \subset G \) may not be a closed subgroup, and we may have to choose a different \( H \) with \( H/N = H_N \).

We will start from a simply connected almost Abelian Lie group \( G \) with Lie algebra \( \mathfrak{a}(\aleph) = R^d \ltimes R \). By Theorem 5.20 in [Hal15] to every Lie subalgebra \( L \subset \mathfrak{a}(\aleph) \) there exists a unique connected Lie subgroup \( H_L \subset G \) for which it is the Lie algebra, and conversely, all connected Lie subgroups of \( G \) arise in this way.

Remark 7 By Proposition 4 in [Ave16], either of the following two possibilities occurs:

1. \( L = W \subset R^d \) is an Abelian Lie subalgebra.
2. \( L \) is of the form

\[ L = \left\{(w+tv_0,t) \in R^d \times R \mid w \in W, \quad t \in R\right\}, \]
where \( v_0 \in \mathbb{R}^d \) is a fixed element and \( W \subset \mathbb{R}^d \) is an ad-invariant vector subspace. In this case \( L \) is Abelian if and only if \( W \subset Z(\mathfrak{A}(\mathbb{N})) \).

Accordingly, the corresponding connected Lie subgroups \( H_L \) fall into two categories.

**Proposition 14** The connected Lie subgroup \( H_L \subset G \) of the simply connected almost Abelian Lie group \( G \) with Lie algebra \( L \) as in Remark 7 is given by either of the following two forms, accordingly:

1. \[
H_L = \left\{ [w, 0] \in \mathbb{R}^d \times \mathbb{R} \mid w \in W \right\} = \exp(L)
\]

2. \[
H_L = \left\{ \left[ v + \frac{e^{tJ(v)} - 1}{J(v)} v_0, t \right] \in \mathbb{R}^d \times \mathbb{R} \mid w \in W, \quad t \in \mathbb{R} \right\} \simeq \exp(W) \cdot \mathbb{R}
\]

In the second case
\[
\exp(W) \cdot \mathbb{R} = \begin{cases} 
\exp(W) \times \mathbb{R} & \text{if } W \subset Z(\mathfrak{A}(\mathbb{N})), \\
\exp(W) \times \mathbb{R} & \text{else}.
\end{cases}
\]

**Proof:** That \( H_L \) is indeed a Lie subgroup in both cases can be checked directly using, say, the faithful matrix representation \( I \) of Proposition 3. In Case 1 the exponential map from Lemma 6 delivers the desired result immediately. For Case 2, pick an arbitrary \( (w_0 + t_0v_0, t_0) \in L \) and let \( (-1, 1) \ni \tau \mapsto (w(\tau), t(\tau)) \in W \oplus \mathbb{R} \) be a smooth curve with
\[
(w(0), t(0)) = (0, 0), \quad (w'(0), t'(0)) = (w_0, t_0) \in W \oplus \mathbb{R}.
\]

Then we have
\[
\frac{d}{d\tau} \left[ w(\tau) + \frac{e^{tJ(v)} - 1}{J(v)} v_0, t(\tau) \right] \bigg|_{\tau=0} = (w_0 + t_0v_0, t_0),
\]
showing that the Lie algebra of \( H_L \) is \( L \). Finally, an automorphism with \( \alpha = 1, \Delta = 1 \) and \( \gamma = v_0 \) from Proposition 8 can be used to establish the isomorphism between \( H_L \) and \( \exp(W) \cdot \mathbb{R} \). \( \square \)

**Remark 8** Proposition 14 easily implies, in particular, that all connected subgroups of a simply connected almost Abelian group are simply connected and closed.

**Remark 9** By Proposition 11 in [Ave13], two almost Abelian Lie subalgebras \( L_1, L_2 \subset \mathfrak{A}(\mathbb{N}) \) corresponding to ad-invariant vector subspaces \( W_1, W_2 \subset \mathbb{R}^d \) are isomorphic if and only if \( J(\mathfrak{N})|_{W_1} \) and \( J(\mathfrak{N})|_{W_1} \) are projectively similar. Since both \( H_{L_1} \) and \( H_{L_2} \) are simply connected, we have that \( H_{L_1} \simeq H_{L_1} \) if and only if \( L_1 \simeq L_2 \).

**Remark 10** By Corollary 5.7 in [Hal15], two connected subgroups \( H_{L_1}, H_{L_2} \subset G \) of a simply connected almost Abelian group \( G \), associated with Lie algebras \( L_1, L_2 \subset \mathfrak{A}(\mathbb{N}) \), respectively, are related by an automorphism \( \Phi \in \text{Aut}(G) \) if and only if the Lie algebras are related by the automorphism \( d\Pi \in \text{Aut}(\mathfrak{A}(\mathbb{N})) \).

Let us now consider subgroups \( H_N \subset G/N \) of connected almost Abelian groups \( G/N \).

**Lemma 9** Let \( G \) be a Lie group and \( N \subset G \) a normal subgroup. Then every connected subgroup \( H_N \subset G/N \) is the projection \( H_N = H/N \) of a unique connected Lie subgroup \( H \subset G \).
Proof: The quotient map \( q_N : G \to G/N \) is a surjective Lie group homomorphism, and its derivative \( dq_N : L_G :\to L_{G/N} \) is a surjective Lie algebra homomorphism. The preimage \( dq_N^{-1} L_{H_N} \) of the Lie algebra of \( H_N \) is a Lie subalgebra of \( L_G \), and thus is the Lie algebra of a unique connected subgroup \( H \subset G \) (Theorem 5.20 in [Hall15] or Proposition 5.6.5 in [RuSc13]). The image \( q_N(H) \subset G/N \) is a connected subgroup with Lie algebra \( L_{H_N} \), which by uniqueness must be \( q_N(H) = H_N \). Finally, if \( H' \subset G \) is another connected subgroup with \( q_N(H') = H_N \) then \( L_{H'} = L_H \), so that again by uniqueness \( H' = H \). □

Remark 11 Since the projection \( H/N \) of a connected subgroup \( H \subset G \) is a connected subgroup \( H/N \subset G \), we conclude that connected subgroups of \( G/N \) are exactly images \( H/N \) of connected subgroups \( H \subset G \), which were already classified above.

It remains to find when a given connected subgroup \( H/N \subset G/N \) is closed. For this purpose we will first establish a simple fact regarding the relative structure of \( H \) and \( N \).

Lemma 10 Let \( G \) be a simply connected almost Abelian group, \( N \subset G \) a discrete normal subgroup and \( H \subset G \) a connected subgroup. Then there exists a subgroup \( B \subset N \) such that \( N = (N \cap H) \times B \).

Proof: We use Proposition 11 to write \( H \) in the form \( H = \exp(W) \) or \( H = \exp(W) \times \mathbb{R} \) (direct or semidirect), with \( W \subset \mathbb{R}^d \) a vector subspace. All we need to show is that the \( N \cap H \subset N \) is a pure subgroup. Indeed, let \([v,t] \in N \) and \( q \in \mathbb{N} \) such that \([v,t]^q = [qv,qt] \in N \cap H \). Then \( qv \in W \) and thus also \( v \in W \), whence \([v,t] \in N \cap H \). Then by Corollary 28.5 in [Fuc70], \( N \cap H \) is a direct factor. □

Since
\[
\exp|_{Z(\mathfrak{g}A(N) \oplus \mathbb{R})} : Z(\mathfrak{g}A(N) \oplus \mathbb{R}) \to Z(G)_0 \times \mathbb{R}
\]
is a bijection, we can introduce its inverse
\[
\log = [\exp|_{\ker J(\mathfrak{g}A(N) \oplus \mathbb{R})}]^{-1} : Z(G)_0 \times \mathbb{R} \to Z(G)_0 \times \mathbb{R}.
\]

For every subset \( X \subset Z(G)_0 \times \mathbb{R} \) we denote by \( \overline{X} \) the connected subgroup
\[
\overline{X} = \exp[\mathbb{R}(\log(X))] , \quad \forall X \subset Z(G)_0 \times \mathbb{R}.
\]

Thus \( \overline{X} \subset G \) is a minimal Lie subgroup containing the set \( X \).

Proposition 15 Let \( G \) be a simply connected almost Abelian group, \( N \subset G \) a discrete normal subgroup and \( H \subset G \) a connected subgroup. Then the connected subgroup \( H/N \subset G/N \) is closed if and only if \( \overline{H \cap N} = H \cap \overline{N} \).

Proof: First let us note that
\[
\overline{H \cap N} \subset H \cap \overline{N}.
\]
Indeed, \( \overline{H \cap N} \subset \overline{N} \) is obvious, while \( \overline{H \cap N} \subset H \) follows from \( \mathbb{R}(\log(H \cap N)) \subset L_H \), where \( L_H \) is the Lie algebra of \( H \). Let by Lemma 11 \( N = (H \cap N) \times B \) for a subgroup \( B \subset N \). Since \( N \) is a free Abelian group, we have that \( \mathbb{R}(\log(H \cap N)) \cap \mathbb{R}(\log(B)) = 0 \), and because \( N \) is a subgroup of the Abelian Lie
By definition of quotient topology, \( H/N \subset G/N \) is closed if and only if the complete preimage \( HN \subset G \) is closed. The subgroups \( H \) and \( \overline{N} \) are connected, and so is their product \( H\overline{N} \). Since \( \overline{N} \subset G \) is central, both \( HN \) and \( H\overline{N} \) are subgroups. Being a connected subgroup, \( H\overline{N} \subset G \) is closed by Proposition \[14\]. Thus the question is reduced to whether \( H\overline{N} \subset \overline{G} \) is closed or not.

\( \overline{H \cap B} \subset \overline{B} \) is a closed Lie subgroup, hence \( \overline{B} = H \cap \overline{B} \times \overline{C} \) where \( \overline{C} \subset \overline{B} \) is a closed Lie subgroup. It follows that

\[
HN = HB = BH, \quad H\overline{N} = \overline{HB} = HC = CH,
\]

and we want to know whether \( BH \subset CH \) is closed. Again, by definition of quotient topology, this is equivalent to \( BH/H \subset CH/H \) being closed or not. Since \( B \cap H = C \cap H = \{1\} \), the homomorphisms \( B \to BH/H \) and \( C \to CH/H \) are isomorphisms, therefore \( \text{rank } BH/H = \text{rank } B \) and \( \text{dim } CH/H = \text{dim } C \), which implies that \( \text{rank } BH/H = \text{dim } B \geq \text{dim } CH/H \), and equality holds if and only if \( H \cap \overline{B} = \{1\} \). If \( H \cap \overline{B} = \{1\} \) then the homomorphism \( \overline{B} \to \overline{BH}/H \) is an isomorphism, and \( BH/H \subset CH/H = \overline{BH}/H \) is closed. On the other hand, if \( H \cap \overline{B} \neq \{1\} \) then \( \text{dim } CH/H < \text{rank } BH/H \), therefore \( BH/H \subset CH/H \) is dense (see Theorem 6.1 in \[StTa02\]). □

Acknowledgements

The work of M. Almora Rios, Z. Avetisyan, K. Berlow, I. Martin and K. Yang was supported by NSF REU grant DMS 185066. H. Zhang was supported by a UCSB CCS SURF grant. Authors G. Rakholia and Z. Zhao received support from the UCSB department of mathematics.

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